PLURISUBHARMONIC AND HOLOMORPHIC FUNCTIONS
RELATIVE TO THE PLURIFINE TOPOLOGY

MOHAMED EL KADIRI, BENT FUGLEDE, AND JAN WIEGERINCK

ABSTRACT. A weak and a strong concept of plurifinely plurisubharmonic and plurifinely holomorphic functions are introduced. Strong will imply weak. The weak concept is studied further. A function $f$ is weakly plurifinely plurisubharmonic if and only if $f \circ h$ is finely subharmonic for all complex affine-linear maps $h$. As a consequence, the regularization in the plurifine topology of a pointwise supremum of such functions is weakly plurifinely plurisubharmonic, and it differs from the pointwise supremum at most on a pluripolar set. Weak plurifine plurisubharmonicity and weak plurifine holomorphy are preserved under composition with weakly plurifinely holomorphic maps.

1. Introduction

The plurifine topology $\mathcal{F}$ on $\mathbb{C}^n$ was briefly introduced in [F6] as the weakest topology in which all plurisubharmonic functions are continuous, in analogy with the H. Cartan fine topology on $\mathbb{R}^n$, in particular on $\mathbb{C} \cong \mathbb{R}^2$. For comments on this choice of “fine” topology on $\mathbb{C}^n$, see [F6]. The plurifine topology $\mathcal{F}$ is clearly biholomorphically invariant. Furthermore, $\mathcal{F}$ is locally connected, as shown in [EW1], [EW2], where also further properties of $\mathcal{F}$ are given. Much as in [ElKa], [EW2], [EW3] we begin by considering (in Definition 2.2, resp. 2.6) two concepts of plurifinely plurisubharmonic (resp. plurifinely holomorphic) functions—a strong concept defined by $\mathcal{F}$-local uniform approximation with plurisubharmonic (resp. holomorphic) functions, and a weak concept defined by restriction to complex lines. We thereby draw on the theory of finely sub- or superharmonic and finely holomorphic functions defined on finely open subsets of $\mathbb{C}$, cf. [F1], [F3], [F7]. The plurifine topology $\mathcal{F}$ on $\mathbb{C}^n$ induces on each complex line $L$ in $\mathbb{C}^n$ the Cartan fine topology on $L \cong \mathbb{C}$ (Lemma 2.1). In analogy with ordinary plurisubharmonic functions, the weakly $\mathcal{F}$-plurisubharmonic functions $f$ may be characterized by being $\mathcal{F}$-upper semicontinuous and such that $f \circ h$ is $\mathbb{R}^{2n}$-finely subharmonic (or identically $-\infty$ in some fine component of its domain of definition) for every $\mathbb{C}$-affine-linear bijection $h$ of $\mathbb{C}^n$ (Theorem 3.1).

The concepts of strongly $\mathcal{F}$-plurisubharmonic and strongly $\mathcal{F}$-holomorphic functions on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$ are obviously biholomorphically invariant. We show

2000 Mathematics Subject Classification. 31C40, 32U15

Keywords and phrases. plurifinely plurisubharmonic function, finely subharmonic function, plurifine topology

Typeset by \texttt{AMSTeX}
that the same holds for the weak concepts (Theorem 4.6), cf. [EW3]. We do not know whether the strong and the weak concepts are actually the same. The weak concepts are closed under $F$-locally uniform convergence, and seem altogether to be more useful, cf. [EW2].

The convex cone of all weakly $F$-plurisubharmonic functions on $\Omega$ is stable under pointwise infimum for lower directed families and under pointwise supremum for finite families. The above characterization of weakly $F$-plurisubharmonic functions allows us to answer questions posed by the first named author in [ElKa]. Namely, for any $F$-locally upper bounded family of weakly $F$-plurisubharmonic functions $f_\alpha$ on $\Omega$, the $F$-upper semicontinuous regularization $f^*$ of the pointwise supremum $f = \sup_\alpha f_\alpha$ is likewise weakly $F$-plurisubharmonic (Theorem 3.9), and the exceptional set $\{ f < f^* \}$ is pluripolar, as expected from a theorem of Bedford and Taylor [BT1, Theorem 7.1]. Furthermore, there is a removable singularity theorem for weakly $F$-plurisubharmonic functions (Theorem 3.7), and likewise for $F$-holomorphic functions (Corollary 3.8).

In the final Section 4 we show that the concepts of weakly $F$-plurisubharmonic map and weakly $F$-holomorphic map are biholomorphically invariant, even in a plurifine sense. In fact, composition with weakly $F$-holomorphic maps preserves weak $F$-plurisubharmonicity and weak $F$-holomorphy (Theorem 4.6).

2. Definitions and first properties of strongly and weakly $F$-plurisubharmonic and $F$-holomorphic functions

The $F$-interior (plurifine interior) of a set $K \subset \mathbb{C}^n$, $n \in \mathbb{N}$, is denoted by $K'$. It is known that every $F$-neighborhood of a point of $\mathbb{C}^n$ contains an $F$-neighborhood which is compact in the Euclidean topology—an easy consequence of [EW2, Theorem 2.3], plurisubharmonic functions being upper semicontinuous. Henceforth, topological properties not explicitly referring to the plurifine topology $F$ or the Cartan fine topology are tacitly understood to refer to the Euclidean topology. Generalizing known properties of the fine topology, cf. [F8], we have

Lemma 2.1. (a) The plurifine topology $F$ on $\mathbb{C}^n$ induces on every $\mathbb{C}$-linear subspace $L \cong \mathbb{C}^k$ of $\mathbb{C}^n$ the plurifine topology on $L$. Explicitly, for any $F$-open set $\Omega \subset \mathbb{C}^n$ the intersection $L \cap \Omega$ is $F$-open in $L$, and so is the orthogonal projection of $\Omega$ on $L$. 

(b) A set $\omega \subset \mathbb{C}^k$ is $F$-open in $\mathbb{C}^k$ if and only if $\omega \times \mathbb{C}^{n-k}$ is $F$-open in $\mathbb{C}^n$.

Proof. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ write

$$z' = (z_1, \ldots, z_k), \quad z'' = (z_{k+1}, \ldots, z_n).$$

For (a) it suffices to consider the particular subspace $L_0 = \{(z', 0'') : z' \in \mathbb{C}^k\}$ which we identify with $\mathbb{C}^k$. For any $F$-open set $\Omega \subset \mathbb{C}^n$ denote by $\omega$ the part of $\Omega$ in $L_0$. Consider a point $a' \in \omega$. According to [EW1, Theorem 2.3] there exists a plurisubharmonic function $\psi$ on $\mathbb{C}^n \cong \mathbb{C}^k \times \mathbb{C}^{n-k}$ and neighborhoods $U'$ of $a'$ in $\mathbb{C}^k$ and $U''$ of $0''$ in $\mathbb{C}^{n-k}$ such that

$$(a', 0'') \in \{(z', z'') \in U' \times U'' : \psi(z', z'') > 0\} \subset \Omega.$$

(2.1)
Define $\varphi : C^k \to [-\infty, +\infty]$ by $\varphi(z') = \psi(z', 0'')$; then $\varphi$ is plurisubharmonic and
\[ a' \in \{ z' \in U' : \varphi(z') > 0 \} \subset \omega. \] (2.2)

Thus $\omega$ is indeed an $F$-neighborhood of $a'$ in $C^k$.

For each $t \in C^{n-k}$ the translate $\Omega_t = \Omega - (0', t)$ of $\Omega$ is $F$-open in $C^n$. It follows that $\Omega_t \cap L_0$ is $F$-open in $L_0$, and so is therefore the union of the $\Omega_t \cap L_0$, that is, the projection of $\Omega$ on $L_0$.

For (b) we have just shown, in particular, that if $\Omega := \omega \times C^{n-k}$ is $F$-open in $C^n$ then $\omega$ is $F$-open in $C^k$. To establish the converse, suppose that $\omega$ is $F$-open in $C^k$ and let us prove that every point $a = (a', a'')$ of $\omega \times C^{n-k}$ is an $F$-inner point of that set. Since $\omega$ is an $F$-neighborhood of $a'$ in $C^k$ there exists (again by [EW1, Theorem 2.3]) a plurisubharmonic function $\varphi$ on $C^k$ and a neighborhood $U'$ of $a'$ in $C^k$ such that (2.2) holds. The function $\psi$ defined on $C^n$ by $\psi(z', z'') = \varphi(z')$ is plurisubharmonic—an easy and well-known consequence of the definition of plurisubharmonicity [L2, p. 306], or see [K, p. 62]. Furthermore, (2.1) holds (with $\Omega = \omega \times C^{n-k}$ and with $(a', 0'')$ replaced by $a$) for any neighborhood $U''$ of $a''$ in $C^{n-k}$. Thus $\omega \times C^{n-k}$ is indeed an $F$-neighborhood of $a$ in $C^n$. \[ \square \]

For a compact set $K \subset C^n$ we denote by $S_0(K)$ the convex cone of all restrictions to $K$ of finite continuous plurisubharmonic functions defined on open subsets of $C^n$ containing $K$, and by $S(K)$ the closure of $S_0(K)$ in $C(K, \mathbb{R})$ (the continuous functions $K \to \mathbb{R}$ with the uniform norm); then $S(K)$ is likewise a convex cone.

**Definition 2.2.** Let $\Omega$ denote an $F$-open (i.e., plurifinely open) subset of $C^n$.

(i) A function $f : \Omega \to \mathbb{R}$ is said to be $F$-cpsh if every point of $\Omega$ has a compact $F$-neighborhood $K$ in $\Omega$ such that $f|K \in S(K)$.

(ii) A function $f : \Omega \to [-\infty, +\infty]$ is said to be strongly $F$-plurisubharmonic if $f$ is the pointwise limit of a decreasing net of $F$-cpsh functions on $\Omega$.

(iii) (cf. [ElKa, Section 5] [EW2, Definition 5.1]). A function $f : \Omega \to [-\infty, +\infty]$ is said to be weakly $F$-plurisubharmonic if $f$ is $F$-upper semicontinuous and, for every complex line $L$ in $C^n$, the restriction of $f$ to the finely open subset $L \cap \Omega$ of $L$ is finely hypoharmonic.

See [F1, Definition 8.2 and §10.4] for finely hypoharmonic (resp. finely sub- or superharmonic) functions, and recall that a function $f$ is finely hypoharmonic on a finely open subset $U$ of $C$ (or of $\mathbb{R}^N$) if and only if $f$ is finely subharmonic on every fine component of $U$ in which $f \neq -\infty$. Either concept strongly or weakly $F$-plurisubharmonic is an $F$-local one (that is, has the sheaf property).

The concept of $F$-cpsh functions, defined in (i), is an auxiliary one. Every strongly $F$-plurisubharmonic function is $F$-upper semicontinuous (even $F$-continuous, see Theorem 2.4(c) and Proposition 2.5) because every $F$-cpsh function is $F$-continuous. The class of all strongly, resp. weakly, $F$-plurisubharmonic functions on $\Omega$ is clearly a convex cone which is stable under pointwise supremum of finite families. The latter class is furthermore stable under pointwise infimum for lower directed (possibly infinite) families, and closed under $F$-locally uniform convergence in view of [F1, Lemma 9.6]. For upper directed families of weakly $F$-plurisubharmonic functions, see Theorem 3.9 below.
If $f$ is strongly, resp. weakly, $\mathcal{F}$-plurisubharmonic on $\Omega$ ($\mathcal{F}$-open in $\mathbb{C}^n$) then the restriction of $f$ to $L \cap \Omega$ ($L$ a $\mathbb{C}$-linear subspace $L \cong \mathbb{C}^k$ of $\mathbb{C}^n$) has the same property in $L \cap \Omega$. This follows easily from Lemma 2.1(a) above.

For $n = 1$, $f$ is strongly, resp. weakly, $\mathcal{F}$-plurisubharmonic on $\Omega$ (finely open in $\mathbb{C}$) if and only if $f$ is finely hypoharmonic on $\Omega$. This is obvious in the weak case. In the strong case, suppose first that $f$ is finite and finely hypoharmonic on $\Omega$. By the Brelot property [F7, p. 248], every point of $\Omega$ has a compact fine neighborhood $K$ in $\Omega$ such that $f|K \in C(K, \mathbb{R})$ ($f$ being finely continuous by [F1, Theorem 9.10]). Because $f$ is finite and finely hypoharmonic in the fine interior $K'$ of $K$ we have $f \in S(K')$ according to [BH, Theorem 4.7], or see [F5, Theorem 4.7], and so $f$ is $\mathcal{F}$-cpsh on $\Omega$. For any compact set $K \subset \mathbb{C}$, every function of class $S(K)$ is finite and finely hypoharmonic on $K'$ according to [F1, Lemma 9.6]. With $K$ as in (i) this shows that $f$ indeed is finite and finely hypoharmonic on $\Omega$.

In the following two theorems we collect some properties of weakly finely plurisubharmonic functions recently obtained by the third named author in collaboration with S. El Marzguioui. By an $\mathcal{F}$-domain we understand an $\mathcal{F}$-connected $\mathcal{F}$-open set.

**Theorem 2.3.** ([EW2]) Let $f$ be a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$.

(a) If $f \not\equiv -\infty$ then $\{z \in \Omega : f(z) = -\infty\}$ has no $\mathcal{F}$-interior point.

(b) If $f \not\equiv -\infty$ then, for any $\mathcal{F}$-closed set $E \subset \{z \in \Omega : f(z) = -\infty\}$, $\Omega \setminus E$ is an $\mathcal{F}$-domain.

(c) If $f \leq 0$ then either $f < 0$ or $f \equiv 0$.

**Theorem 2.4.** ([EW3]) Let $f$ be a weakly $\mathcal{F}$-plurisubharmonic function on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$.

(a) Every point $z_0 \in \Omega$ such that $f(z_0) > -\infty$ has an $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset \Omega$ on which $f$ can be represented as the difference $f = \varphi_1 - \varphi_2$ between two bounded plurisubharmonic functions $\varphi_1$ and $\varphi_2$ defined on some open ball $B(z_0, r)$ containing $O$.

(b) $r$, $O$, and $\varphi_2$ can be chosen independently of $f$ provided that $f$ maps $\Omega$ into a prescribed bounded interval $]a, b[$.

(c) $f$ is $\mathcal{F}$-continuous.

(d) If $\Omega$ is $\mathcal{F}$ connected and $f \not\equiv -\infty$ then $\{z \in \Omega : f(z) = -\infty\}$ is a pluripolar subset of $\mathbb{C}^n$.

Assertion (d) amounts to pluripolar sets and weakly $\mathcal{F}$-pluripolar sets (in the obvious sense) being the same. The proofs of (a), (b), and (c) given below are essentially taken from [EW3].

**Proof.** (a) To begin with, suppose that $f$ is bounded. We may then assume that $-1 < f < 0$, for $f$ maps $\Omega$ into a bounded interval $]a, b[$, and hence $\frac{f - b}{b - a}$ maps $\Omega$
into $[-1,0[$ and is likewise weakly $\mathcal{F}$-plurisubharmonic. Let $V \subset \Omega$ be a compact $\mathcal{F}$-neighborhood of $z_0$. Since the complement $\mathcal{C}V$ of $V$ is pluri-thin at $z_0$, there exist $0 < r < 1$ and a plurisubharmonic function $\varphi$ on $B(z_0, r)$ such that

$$\limsup_{z \to z_0, z \in \mathcal{C}V} \varphi(z) < \varphi(z_0).$$

Without loss of generality we may suppose that $\varphi$ is negative on $B(z_0, r)$ and

$$\varphi(z) = -1 \text{ on } B(z_0, r) \setminus V \quad \text{and } \varphi(z_0) = -1/2.$$

Hence

$$f(z) + \lambda \varphi(z) \leq -\lambda \quad \text{for } z \in \Omega \cap B(z_0, r) \setminus V \text{ and } \lambda > 0. \quad (2.3)$$

Now define a function $u_\lambda$ on $B(z_0, r)$ by

$$u_\lambda(z) = \begin{cases} \max\{-\lambda, f(z) + \lambda \varphi(z)\} & \text{for } z \in \Omega \cap B(z_0, r) \\ -\lambda & \text{for } z \in B(z_0, r) \setminus V. \end{cases} \quad (2.4)$$

This definition makes sense because $(\Omega \cap B(z_0, r)) \cup (B(z_0, r) \setminus V) = B(z_0, r)$, and the two definitions agree on $\Omega \cap B(z_0, r) \setminus V$ in view of (2.3).

Clearly, $u_\lambda$ is weakly $\mathcal{F}$-plurisubharmonic on $\Omega \cap B(z_0, r)$ and on $B(z_0, r) \setminus V$, hence on all of $B(z_0, r)$ in view of the sheaf property, cf. [EW2]. Since $u_\lambda$ is bounded on $B(z_0, r)$, it follows from [F1, Theorem 9.8] that $u_\lambda$ is subharmonic on each complex line where it is defined. It is well known that a bounded function, which is subharmonic on each complex line where it is defined, is plurisubharmonic, cf. [Le1], or see [Le2, p. 24]. Thus, $u_\lambda$ is plurisubharmonic on $B(z_0, r)$.

Since $\varphi(z_0) = -1/2$, the set $O = \{z \in \Omega : \varphi(z) > -3/4\}$ is an $\mathcal{F}$-neighborhood of $z_0$, and because $\varphi = -1$ on $B(z_0, r) \setminus V$ it is clear that $O \subset V \subset \Omega$.

Observe now that $-4 \leq f(z) + 4\varphi(z)$ for every $z \in O$. Hence $f = \varphi_1 - \varphi_2$ on $O$, with $\varphi_1 = u_\lambda$ and $\varphi_2 = 4\varphi$, both plurisubharmonic on $B(z_0, r)$. Thus $f$ is weakly $\mathcal{F}$-plurisubharmonic on $O$, which is an $\mathcal{F}$-neighborhood of $z_0$. It follows that $f$ is $\mathcal{F}$-continuous on $O$ along with $\varphi_1$ and $\varphi_2$, provided that $f$ is bounded.

Without assuming that $f$ be bounded, $f$ remains $\mathcal{F}$-continuous on $O$ according to (c), proven below. It follows that $f$ is bounded on some $\mathcal{F}$-neighborhood $U$ of $z_0$ in $\Omega$, and we therefore have a decomposition of $f$ as required, on some $\mathcal{F}$-neighborhood (replacing the above $O$) of $z_0$ on $U \subset \Omega$.

(b) Again we may assume that $-1 < f < 0$. The set $V$ and the plurisubharmonic function $\varphi$ in the proof of (a) then do not depend on $f$, and that applies to $\varphi_2 = 4\varphi$ as well.

(c) In the remaining case where $f$ may be unbounded (cf. the proof of (a) above), note that $f$ is $\mathcal{F}$-upper semicontinuous and $< +\infty$. Choose $c, d \in \mathbb{R}$ with $d < c$. Then the set $\Omega_c = \{z \in \Omega : f(z) < c\}$ is $\mathcal{F}$-open. The function $\max\{f, d\}$ is bounded and weakly $\mathcal{F}$-plurisubharmonic on $\Omega_c$, hence $\mathcal{F}$-continuous there. The set

$$\{z \in \Omega : d < f(z) < c\} = \{z \in \Omega_c : d < \max\{f(z), d\} < c\}$$

is therefore $\mathcal{F}$-open, and hence $f$ is $\mathcal{F}$-continuous.

For (d) we refer to the proof given in [EW3, Theorem 4.1]. □
**Proposition 2.5.** Every strongly $\mathcal{F}$-plurisubharmonic function $f : \Omega \to [-\infty, +\infty]$ is weakly $\mathcal{F}$-plurisubharmonic.

**Proof.** We may assume that $f$ is even $\mathcal{F}$-cpsh. Let $(f_\nu)$ be a sequence of finite continuous plurisubharmonic functions on open sets $\Omega_\nu$ containing $K$ from Definition 2.2(i) such that $f_\nu|_K \to f|_K$ uniformly. For any complex line $L$ in $\mathbb{C}^n$, $f_\nu|_{L \cap K'}$ is finely hypoharmonic. This uses [F1, Theorem 8.7] and the fact that the intersection of any $\mathcal{F}$-open subset of $\mathbb{C}^n$ with any complex line $L$ is finely open, by Lemma 2.1. It follows by [F1, Lemma 9.6] that $f|_{L \cap K'}$ is finely hypoharmonic, and in particular finely continuous, by [F1, Theorem 9.10]. Consequently, $f$ is indeed weakly $\mathcal{F}$-plurisubharmonic. $\square$

We now pass to concepts of $\mathcal{F}$-holomorphic functions. For a compact set $K \subset \mathbb{C}^n$ we denote by $H_0(K)$ the algebra of all restrictions to $K$ of holomorphic functions defined on open subsets of $\mathbb{C}^n$ containing $K$, and by $H(K)$ the closure of $H_0(K)$ in $C(K, \mathbb{C})$ (the continuous functions $K \to \mathbb{C}$ with the uniform norm); then $H(K)$ is likewise an algebra.

**Definition 2.6.** Let $\Omega$ denote an $\mathcal{F}$-open subset of $\mathbb{C}^n$.

(i) (cf. [EW2, Definition 6.1].) A function $f : \Omega \to \mathbb{C}$ is said to be strongly $\mathcal{F}$-holomorphic if every point of $\Omega$ has a compact $\mathcal{F}$-neighborhood $K$ in $\Omega$ such that $f|_K \in H(K)$.

(ii) A function $f : \Omega \to \mathbb{C}$ is said to be weakly $\mathcal{F}$-holomorphic if $f$ is $\mathcal{F}$-continuous and if, for every complex line $L$ in $\mathbb{C}^n$, the restriction $f|_L \cap \Omega$ is finely holomorphic.

For finely holomorphic functions see [F3], [F7]. Either of the concepts strongly and weakly $\mathcal{F}$-holomorphic is an $\mathcal{F}$-local one. The class of all strongly, resp. weakly, $\mathcal{F}$-holomorphic functions on $\Omega$ is an algebra, and the latter class is closed under $\mathcal{F}$-locally uniform convergence, in view of [F3, Théorème 4]. Clearly, every strongly $\mathcal{F}$-holomorphic function is $\mathcal{F}$-continuous (on $K'$ from Definition 2.6(i), and so on all of $\Omega$).

If $f$ is strongly, resp. weakly, $\mathcal{F}$-holomorphic on $\Omega$ ($\mathcal{F}$-open in $\mathbb{C}^n$) then the restriction of $f$ to $L \cap \Omega$ ($L$ a $\mathbb{C}$-linear subspace $L \cong \mathbb{C}^k$ of $\mathbb{C}^n$) has the same property on $L \cap \Omega$. This follows easily from Lemma 2.1 above.

For $n = 1$, $f$ is strongly (resp. weakly) $\mathcal{F}$-holomorphic on $\Omega$ (finely open in $\mathbb{C}$) if and only if $f$ is finely holomorphic on $\Omega$. This is obvious in the weak case. In the strong case, suppose first that $f$ is finely holomorphic on $\Omega$. By [F3, Corollary, p. 75], every point of $\Omega$ has a compact fine neighborhood $K$ in $\Omega$ such that $f|_K \in R(K)$ ($= H(K)$ in the 1-dimensional case). Consequently, $f$ is indeed strongly $\mathcal{F}$-holomorphic on $\Omega$. Conversely, if $f$ is strongly $\mathcal{F}$-holomorphic then, for any compact set $K \subset \mathbb{C}$, every function of class $H(K)$ is finely holomorphic on $K'$, see [F3, p.63]. With $K$ as in Definition 2.6(i) this shows that $f$ indeed is finely holomorphic on $\Omega$.

**Proposition 2.7.** Every strongly $\mathcal{F}$-holomorphic function $f : \Omega \to \mathbb{C}$ is weakly $\mathcal{F}$-holomorphic, and in particular $\mathcal{F}$-continuous.
Proof. For any $K$ as in Definition 2.6(i) there exists a sequence of holomorphic functions $f_\nu$ defined on open sets containing $K$ such that $f_\nu|K \to f|K$ uniformly. For every complex line $L$ in $\mathbb{C}^n$ this shows that the finely holomorphic functions $f_\nu|L \cap K'$ converge uniformly to $f|L \cap K'$, which therefore is finely holomorphic, see again [F3, p.63]. Consequently $f|L \cap \Omega$ is finely holomorphic, and so $f$ is indeed weakly $\mathcal{F}$-holomorphic, being also $\mathcal{F}$-continuous. □

The concept of weakly $\mathcal{F}$-holomorphic function can be characterized in terms of weakly $\mathcal{F}$-pluriharmonic functions (that is, functions $f : \Omega \to \mathbb{C}$ such that $\pm \text{Re } f$ and $\pm \text{Im } f$ are weakly $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$):

**Lemma 2.8.** A function $f : \Omega \to \mathbb{C}$ is weakly $\mathcal{F}$-holomorphic if and only if $f$ and each of the functions $z \mapsto z_j f(z)$ ($j \in \{1, \ldots, n\}$) are weakly $\mathcal{F}$-plurisubharmonic on $\Omega$.

Proof. This reduces right away to the case $n = 1$ which is due to Lyons [Ly], cf. [F3, Section 3], and which asserts that a function $h : U \to \mathbb{C}$, defined on a finely open set $U \subset \mathbb{C}$, is finely holomorphic if and only if $h$ and $z \mapsto zh(z)$ are (complex) finely harmonic. □

For any $\mathcal{F}$-open set $U \subset \mathbb{C}^m$, an $n$-tuple $(h_1, \ldots, h_n)$ of strongly (resp. weakly) $\mathcal{F}$-holomorphic functions $h_j : U \to \mathbb{C}$ will be termed a strongly (resp. weakly) $\mathcal{F}$-holomorphic map $U \to \mathbb{C}^n$.

Assertion (b) of the following proposition provides two slight strengthenings of [EW2, Lemma 6.2].

**Proposition 2.9.** Let $U \subset \mathbb{C}^m$ be $\mathcal{F}$-open and let $h = (h_1, \ldots, h_n) : U \to \mathbb{C}^n$ be a strongly (resp. weakly) $\mathcal{F}$-holomorphic map.

(a) The map $h : U \to \mathbb{C}^n$ is continuous from $U$ with the $\mathcal{F}$-topology on $\mathbb{C}^m$ to $\mathbb{C}^n$ with the Euclidean topology.

(b) For any plurisubharmonic function $f$ on an open set $\Omega$ in $\mathbb{C}^n$, the function $f \circ h$ is strongly (resp. weakly) $\mathcal{F}$-plurisubharmonic on the $\mathcal{F}$-open set $h^{-1}(\Omega) = \{ z \in U : h(z) \in \Omega \} \subset \mathbb{C}^m$.

(c) For any holomorphic function $f$ on an open set $\Omega$ in $\mathbb{C}^n$, the function $f \circ h$ is strongly (resp. weakly) $\mathcal{F}$-holomorphic on the $\mathcal{F}$-open set $h^{-1}(\Omega) \subset \mathbb{C}^m$.

Proof. Assertion (a) holds because each $h_j$ (whether strongly or weakly $\mathcal{F}$-holomorphic) is $\mathcal{F}$-continuous and that the Euclidean topology on $\mathbb{C}^n$ is the product of the Euclidean topology on each of $n$ copies of $\mathbb{C}$.

For (b) with each $h_j$ strongly $\mathcal{F}$-holomorphic we begin by showing that, if the plurisubharmonic function $f$ on $\Omega$ is finite and continuous, then $f \circ h$ is even $\mathcal{F}$-cpsh (cf. Definition 2.2(i)) on $h^{-1}(\Omega)$, which is $\mathcal{F}$-open according to (a). Every point $a \in h^{-1}(\Omega)$ has a compact $\mathcal{F}$-neighborhood $K_j$ in $h^{-1}(\Omega) (\subset U \subset \mathbb{C}^m)$ such that $h_j|K_j \in H(K_j)$. Thus there exists a sequence $(h^{(\nu)}_j)_{\nu \in \mathbb{N}}$ of holomorphic functions $h^{(\nu)}_j$ on open sets $U^{(\nu)}_j$ in $\mathbb{C}^m$ containing $K_j$ such that $h^{(\nu)}_j|K_j \to h_j|K_j$ uniformly as $\nu \to \infty$. Write $K = K_1 \cap \ldots \cap K_n$ and $h^{(\nu)} = (h^{(\nu)}_1, \ldots, h^{(\nu)}_n)$ on $U^{(\nu)} = U^{(\nu)}_1 \cap \ldots \cap U^{(\nu)}_n$. Then $h_j|K_j \in C(K_j, \mathbb{C})$ and hence $h|K \in C(K, \mathbb{C}^n)$. It follows that $h(K)$ is a compact subset of $\Omega \subset \mathbb{C}^n$. Denoting by $\| \cdot \|$ the Euclidean norm on $\mathbb{C}^n$ and by $B$
the closed unit ball in $\mathbb{C}^n$, there exists accordingly $\delta > 0$ such that $h(K) + \delta B \subset \Omega$. We may assume that $\|h^\nu(z) - h(z)\| < \delta$ for any $\nu$ and any $z \in K$. Under the present extra hypothesis, $f$ is finite and uniformly continuous on the compact set $h(K) + \delta B$ containing any $h^\nu(K)$, and it follows that $f \circ h^\nu|K \to f \circ h|K$ uniformly as $\nu \to \infty$. Because $f \circ h^\nu$ is finite, continuous, and plurisubharmonic, on the open set $U^\nu \supset K$, we have $f \circ h|K \in S(K)$. By varying $a \in h^{-1}(\Omega)$ and hence the $F$-neighborhood $K$ of $a$ in $h^{-1}(\Omega)$ we infer that $f \circ h$ is $F$-cpsh on $h^{-1}(\Omega)$.

If we drop the extra hypothesis that $f$ be finite and continuous, $f$ is the pointwise limit of a decreasing net of finite continuous plurisubharmonic functions $f_\nu$ on $\Omega$, and $f \circ h$ is then the pointwise limit of the decreasing net of functions $f_\nu \circ h$ on $h^{-1}(\Omega)$ which we have just shown are $F$-cpsh, and so $f \circ h$ is indeed strongly $F$-plurisubharmonic, cf. Definition 2.2(ii) (with $\Omega$ replaced by $h^{-1}(\Omega)$).

Next suppose instead that each $h_j$ is weakly $F$-holomorphic on $U$, and consider a complex line $L$ in $\mathbb{C}^m$; then $L \cap U$ is finely open in $L$. According to Definition 2.6(ii), $h_j|L \cap U$ is then finely holomorphic, which is the same as strongly $F$-holomorphic (see above for $n = 1$). As shown above (now with $m = 1$ and with $U$ replaced by $L \cap U$) it follows that $f \circ h|L \cap h^{-1}(\Omega)$ is strongly $F$-plurisubharmonic, which is the same as finely hypoharmonic (because the dimension is 1). According to Definition 2.2(iii) this means that $f \circ h$ indeed is weakly $F$-plurisubharmonic on $h^{-1}(\Omega)$, noting that $f \circ h$ is $F$-upper semicontinuous in view of (a) because $f$ is upper semicontinuous.

For (c), suppose first that each $h_j$ is strongly $F$-holomorphic on $U$. Proceeding as in the first part of the proof of (b) we arrange that $f \circ h^\nu|K \to f \circ h|K$ uniformly as $\nu \to \infty$; but now $f \circ h^\nu$ is holomorphic on $U^\nu$. We therefore conclude that $f \circ h|K \in H(K)$, and so $f \circ h$ is indeed strongly $F$-holomorphic according to Definition 2.6(i).

If instead each $h_j$ is weakly $F$-holomorphic on $U$ then, for every complex line $L$ in $\mathbb{C}^m$, each $h_j|L \cap U$ is again strongly $F$-holomorphic. As just established, this implies that $f \circ h|L \cap h^{-1}(\Omega)$ is strongly $F$-holomorphic, or equivalently finely holomorphic. We conclude that indeed $f \circ h$ is weakly $F$-holomorphic, according to Definition 2.6(ii), noting that $f \circ h$ is $F$-continuous in view of (a).

In the version of Proposition 2.9 with ‘weakly’ in each of the three occurrences one may allow $f$ in (b) to be just weakly $F$-plurisubharmonic (in place of plurisubharmonic), and similarly $f$ in (c) to be weakly $F$-holomorphic (in place of holomorphic), see Theorem 4.6 at the end of the paper. At this point we merely show that we may allow $f$ in (b) (of Proposition 2.9) to be strongly $F$-plurisubharmonic, and $f$ in (c) to be strongly $F$-holomorphic:

**Theorem 2.10.** Let $U \subset \mathbb{C}^m$ be $F$-open and let $h = (h_1, \ldots, h_n) : U \to \mathbb{C}^n$ be a weakly $F$-holomorphic map.

(a) The map $h : U \to \mathbb{C}^n$ is continuous from $U$ with the $F$-topology on $\mathbb{C}^m$ to $\mathbb{C}^n$ with the $F$-topology there.

(b) For any strongly $F$-plurisubharmonic function $f$ defined on an $F$-open set $\Omega$ in $\mathbb{C}^m$, the function $f \circ h$ is weakly $F$-plurisubharmonic on the $F$-open set $h^{-1}(\Omega) = \{z \in U : h(z) \in \Omega\} \subset \mathbb{C}^m$. 

(c) For any strongly \( F \)-holomorphic function \( f \) defined on an \( F \)-open set \( \Omega \) in \( \mathbb{C}^n \), the function \( f \circ h \) is weakly \( F \)-holomorphic on the \( F \)-open set \( h^{-1}(\Omega) \subset \mathbb{C}^n \).

**Proof.** For the present weakly \( F \)-holomorphic functions \( h_j \) assertion (a) is stronger than Proposition 2.9(a). We shall prove that \( h^{-1}(\Omega) \) is \( F \)-open in \( \mathbb{C}^n \) for any \( F \)-open set \( \Omega \) in \( \mathbb{C}^n \). Fix a point \( a \in h^{-1}(\Omega) \) and write \( h(a) = b \in \mathbb{C}^n \). According to [EW2, Theorem 2.3] there exist a plurisubharmonic function \( \varphi \) on an open ball \( B(b, r) \) in \( \mathbb{C}^n \) and a number \( c < \varphi(b) \) such that the basic \( F \)-neighborhood

\[
W = \{ w \in B(b, r) : \varphi(w) > c \}
\]

of \( b \) in \( \mathbb{C}^n \) is a subset of the \( F \)-open set \( \Omega \) in \( \mathbb{C}^n \). Then \( h^{-1}(W) \subset h^{-1}(\Omega) \subset U \), and

\[
h^{-1}(W) = \{ z \in U : h(z) \in B(b, r) \text{ and } (\varphi \circ h)(z) > c \}
\]

is \( F \)-open in \( \mathbb{C}^n \) because \( h : U \rightarrow \mathbb{C}^n \) is \( F \)-continuous by Proposition 2.9(a) and that \( \varphi \circ h : h^{-1}(B(b, r)) \rightarrow [-\infty, +\infty[ \) is weakly \( F \)-plurisubharmonic by Proposition 2.9(b) (applied with \( \Omega, f \) replaced by \( B(b, r), \varphi \)), and in particular \( F \)-continuous, by Theorem 2.4(c). By varying \( a \in h^{-1}(\Omega) \) we infer that indeed \( h^{-1}(\Omega) \) is \( F \)-open.

For (b) we may assume that \( f \) is even \( F \)-cpsh on \( \Omega \). Let \( K \subset \Omega \) be as in Definition 2.2(i), and let \( (f^n) \) be a sequence of finite continuous plurisubharmonic functions on open sets \( \Omega^n \supset K \) such that \( f'^\nu[K \rightarrow f]K \) uniformly as \( \nu \rightarrow \infty \). According to Proposition 2.9(a),(b) each \( f'^\nu \circ h \) is weakly \( F \)-plurisubharmonic on the \( F \)-open set \( h^{-1}(\Omega^n) \). By (a), \( h^{-1}(K') \) is \( F \)-open, and it follows that each \( f'^\nu \circ h|_{h^{-1}(K')} \) likewise is weakly \( F \)-plurisubharmonic, in particular \( F \)-upper semi-continuous; and hence so is its uniform limit \( f \circ h|_{h^{-1}(K')} \) in view of [F1, Lemma 9.6]. By varying \( K \subset \Omega \) and hence \( K' \) we conclude that indeed \( f \circ h \) is weakly \( F \)-plurisubharmonic on \( h^{-1}(\Omega) \).

Finally, the proof of (c) is quite parallel to that of (b) in view of Proposition 2.9(a),(c), using [F3, Théorème 4] in place of [F1, Lemma 9.6].}

Theorem 2.10 has two corollaries for \( m = 1 \) and \( n = 1 \), respectively. In either corollary ‘strongly’ can be replaced by ‘weakly’ according to Theorem 4.6. For \( m = 1 \) we have

**Corollary 2.11.** Let \( h_j : U \rightarrow \mathbb{C} \ (j \in \{1, \ldots, n\}) \) be finely holomorphic functions defined on a finely open set \( U \subset \mathbb{C} \), and write \( h = (h_1, \ldots, h_n) \). For any strongly \( F \)-plurisubharmonic (resp. strongly \( F \)-holomorphic) function \( f \) defined on an \( F \)-open set \( \Omega \subset \mathbb{C}^n \), the function \( f \circ h \) is finely hypoharmonic (resp. finely holomorphic) on the finely open set \( h^{-1}(\Omega) = \{ z \in U : h(z) \in \Omega \} \subset \mathbb{C} \).

**Remark 2.12.** If it can be proved that the cone of \( F \)-cpsh, resp. the algebra of strongly \( F \)-holomorphic, functions on an \( F \)-open subset \( \Omega \) of \( \mathbb{C}^n \) is closed under uniform convergence, then the proofs of Theorem 2.10(b),(c) easily show that ‘weakly’ can be replaced throughout the theorem by ‘strongly’. Indeed, with \( f \) \( F \)-cpsh (resp. strongly \( F \)-holomorphic) and \( h \) strongly \( F \)-holomorphic, let \( K \subset \Omega \) denote a compact \( F \)-neighborhood of a point \( a \in \Omega \), and let \( (f_n) \) denote a sequence
of finite continuous plurisubharmonic (resp. a sequence of holomorphic) functions, defined on open subsets $\Omega_\nu$ of $\mathbb{C}^n$ containing $K$, and such that $f_\nu \to f$ uniformly on $K$. Then $f_\nu \circ h$ is strongly $\mathcal{F}$-plurisubharmonic and even $\mathcal{F}$-cpsh (resp. strongly $\mathcal{F}$-holomorphic) on the $\mathcal{F}$-open set $h^{-1}(K') \subset h^{-1}(\Omega_\nu) \subset U \subset \mathbb{C}^m$, by Proposition 2.9 (cf. the beginning of the proof of Theorem 2.9(b)). Under the stated extra hypothesis it will follow that the uniform limit $f \circ h$ of $(f_\nu \circ h)$ on $h^{-1}(K')$ likewise is $\mathcal{F}$-cpsh (resp. strongly $\mathcal{F}$-holomorphic) on $h^{-1}(\Omega)$, by varying $K$ and hence $K'$.—If $f$ is merely strongly $\mathcal{F}$-plurisubharmonic (rather than $\mathcal{F}$-cpsh), it follows as usual that indeed $f \circ h$ is likewise strongly $\mathcal{F}$-plurisubharmonic.

In the case $n = 1$ the extra hypothesis stated in the above remark is always fulfilled in view of [F1, Lemma 9.6] (resp. [F3, Théorème 4]). We therefore have the following corollary of Theorem 2.10 for that case:

**Corollary 2.13.** Let $h : U \to \mathbb{C}$ be a strongly $\mathcal{F}$-holomorphic function defined on an $\mathcal{F}$-open set $U \subset \mathbb{C}^m$. For any finely hypoharmonic (resp. finely holomorphic) function $f$ defined on a finely open set $\Omega \subset \mathbb{C}$ the function $f \circ h$ is strongly $\mathcal{F}$-plurisubharmonic (resp. strongly $\mathcal{F}$-holomorphic) on the $\mathcal{F}$-open set $h^{-1}(\Omega) \subset \mathbb{C}^m$.

The same with ‘strongly’ replaced throughout by ‘weakly’ is simply the case $n = 1$ of Theorem 2.10 as it stands.

**Proposition 2.14.** Let now $\Omega$ be a Euclidean open subset of $\mathbb{C}^n$. For a function $f : \Omega \to [-\infty, +\infty[$ the following are equivalent:

(i) $f$ is plurisubharmonic (in the ordinary sense).

(ii) $f$ is strongly $\mathcal{F}$-plurisubharmonic and not identically $-\infty$ on any component of $\Omega$.

(iii) $f$ is weakly $\mathcal{F}$-plurisubharmonic and not identically $-\infty$ on any component of $\Omega$.

**Proof.** Every finite continuous plurisubharmonic function on $\Omega$ is of course $\mathcal{F}$-cpsh. It follows that any plurisubharmonic function on $\Omega$ is strongly $\mathcal{F}$-plurisubharmonic (being the pointwise limit of a decreasing sequence of finite continuous plurisubharmonic functions). Conversely, if $f$ is weakly $\mathcal{F}$-plurisubharmonic on $\Omega$ then $f$ is plurisubharmonic on every connectivity component $\omega$ of $\Omega$ on which $f$ is not identically $-\infty$. To see this, first observe that $f$ is $\mathcal{F}$-locally bounded from above, so every point $a \in \Omega$ has an $\mathcal{F}$-neighborhood $U \subset \Omega$ on which $f < M_a$, say. According to [EW2, Proposition 4.1] we may further arrange that there exists $\delta > 0$ such that, for every complex line $L$ passing through $a$, the intersection $U \cap L$ contains a circle about $a$ with radius at least $\delta$. By the maximum principle for finely subharmonic functions on a planar domain, [F2, Theorem 2.3], it follows that $f < M_a$ on the discs bounded by these circles, hence in particular on the ball $B(a, \delta_a)$.—For functions that are locally bounded from above in the Euclidean topology, the statement alternatively follows from [F1, Theorem 9.8(a)] in view of [Le1, Définition 1 (p. 306)].
Remark 2.15. Similarly to Proposition 2.14, a function \( f : \Omega \to \mathbb{C} \) (with \( \Omega \) Euclidean open) is holomorphic if and only if \( f \) is strongly, or equivalently weakly, \( \mathcal{F} \)-holomorphic; the ‘if part’ follows from [F3, p. 63] in view of Hartogs’ theorem.

We close this section with an application to pluripolar hulls. Recall that the pluripolar hull \( P^*_\Omega \) of a pluripolar set \( P \) relative to an open set \( \Omega \) containing \( P \) is defined as the following set (closed relatively to \( \Omega \)):

\[
P^*_\Omega = \bigcap_u \{ z \in \Omega : u(z) = -\infty \},
\]

where the intersection is taken over all plurisubharmonic functions \( u \) defined on \( \Omega \) and such that \( u|P \equiv -\infty \). A pluripolar set \( E \) has empty plurifine interior \( E' \), [EW2, Theorem 5.2]. (More generally, a polar set is a Lebesgue null set and has therefore empty fine interior.)

For any set \( E \subset \mathbb{C}^m, m \in \mathbb{N} \), and any function \( h : E \to \mathbb{C} \) we denote by \( \Gamma_h(E) = \{(z, h(z)) : z \in E\} \) the graph of \( h|E \) and by \( \Gamma_h(E)^*_{\mathbb{C}^{m+1}} \) the pluripolar hull of \( \Gamma_h(E) \).

Proposition 2.16. Let \( h \) be a weakly \( \mathcal{F} \)-holomorphic function on an \( \mathcal{F} \)-domain \( U \subset \mathbb{C}^m \).

(a) If \( h \neq 0 \), the set \( h^{-1}(0) \) of zeros of \( h \) is pluripolar in \( \mathbb{C}^m \). In particular, the graph \( \Gamma_h(U) \) of \( h \) is pluripolar in \( \mathbb{C}^{m+1} \).

(b) If \( E \) is a non-pluripolar subset of \( U \) then \( \Gamma_h(U) \) is pluripolar, and \( \Gamma_h(U) \subset \Gamma_h(E)^*_{\mathbb{C}^{m+1}} \).

With \( h \) supposed strongly \( \mathcal{F} \)-holomorphic on \( U \), Proposition was obtained in [EW3, Corollary 4.4 and Theorem 4.5], extending [EW2, Theorem 6.4], and [EEW, Theorem 3.5].

Proof of Proposition 2.16. (a) It follows from Proposition 2.9(b) that the function \( \log |h(\cdot)| \) is weakly \( \mathcal{F} \)-plurisubharmonic on \( U \). Since \( \log |h(z)| = -\infty \) for \( z \in h^{-1}(0) \), but \( \log |h(\cdot)| \not\equiv -\infty \), we conclude from Theorem 2.4(d) that \( h^{-1}(0) \) is pluripolar.

(b) The function \( (z, w) \mapsto w - h(z) \) is weakly \( \mathcal{F} \)-holomorphic and \( \not\equiv 0 \) on the \( \mathcal{F} \)-open set \( U \times \mathbb{C} \subset \mathbb{C}^{m+1} \). Again by Proposition 2.9(b) it follows that the function \( (z, w) \mapsto \log |w - h(z)| \) is weakly \( \mathcal{F} \)-plurisubharmonic and \( \not\equiv -\infty \) on \( U \times \mathbb{C} \). Since this function equals \( -\infty \) on \( \Gamma_h(E) \) we conclude that \( \Gamma_h(E) \) is pluripolar. By Josefson’s theorem [J] there exists a plurisubharmonic function \( f \) on all of \( \mathbb{C}^{m+1} \) such that \( f(z, h(z)) = -\infty \) for every \( z \in V \). It follows by Theorem 2.3(a) that \( f(z, h(z)) = -\infty \) even for every \( z \in U \), and hence \( \Gamma_h(U) \) is pluripolar in \( \mathbb{C}^{m+1} \). By definition of the pluripolar hull of \( \Gamma_h(E) \) we conclude that indeed \( \Gamma_h(U) \subset \Gamma_h(E)^*_{\mathbb{C}^{m+1}} \). \( \square \)

3. A characterization of weakly \( \mathcal{F} \)-plurisubharmonic functions

By the prefix ‘\( \mathbb{R}^{2n} \)-fine’ we denote concepts relative to the Cartan fine topology on \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). Recall that this topology is finer than the plurifine topology \( \mathcal{F} \),
It is well-known that a plurisubharmonic function \( f \) on a domain \( \Omega \subset \mathbb{C}^n \) is subharmonic when considered as a function on \( \Omega \subset \mathbb{R}^{2n} \), because the average of \( f \) over a sphere can be expressed in terms of the average of \( f \) over the circles that are intersection of the sphere with complex lines passing through the center. While this approach does not work in the fine setting, the analogous result nevertheless remains valid. Indeed, a well-known characterization of plurisubharmonic functions (see [Le2, Théorème 1 (p. 18)] or [K, Theorem 2.9.12]) may be adapted as follows. This will lead to further properties of weakly \( F \)-plurisubharmonic functions (Theorems 3.7 and 3.9).

**Theorem 3.1.** A function \( f : \Omega \to [-\infty, +\infty] \) (\( \Omega \) \( F \)-open in \( \mathbb{C}^n \)) is weakly \( F \)-plurisubharmonic if and only if \( f \circ h \) is \( \mathbb{R}^{2n} \)-finely hypoharmonic on the \( F \)-open set \( h^{-1}(\Omega) \) for every \( \mathbb{C} \)-affine bijection \( h \) of \( \mathbb{C}^n \).

For the proof of the ‘only if part’ of Theorem 3.1 we need the following

**Lemma 3.2.** Let \( u_1, u_2 \) be bounded subharmonic functions on an open set \( B \subset \mathbb{R}^n \), and consider the function \( f = u_1 - u_2 \) on \( B \). Let \( U \) be a finely open Borel subset of \( B \). Then \( f|U \) is finely subharmonic if and only if the signed Riesz measure \( \Delta f \) on \( B \) has a positive restriction to \( U \).

**Proof.** Suppose first that \( (\Delta f)|U \geq 0 \), and let us prove that \( f \) then is finely subharmonic on \( U \).

Recall that the base \( b(X) \) in \( B \) of \( X \subset B \) consists of the points of \( B \) at which \( X \) is not thin. Denote by \( \mathbb{C}X \) the complement of \( X \) relative to \( B \). For a finely open set \( U \), its regularization equals \( r(U) = \mathbb{C}b(\mathbb{C}U)) = U \cup i(\mathbb{C}U) \) where \( i(X) \) is the polar set consisting of the points of \( X \) at which \( X \) is thin. We may assume that \( U \) is regular, i.e., \( U = r(U) \) and hence an \( F_\sigma \)-set, for \( u_1 \) and \( u_2 \) are bounded, and \( \Delta u_1 \) and \( \Delta u_2 \) therefore do not charge the polar set by which \( U \) differs from \( r(U) \).

Writing \( \Delta f := \mu = \mu_+ - \mu_- \) on \( B \) we have, by hypothesis, \( \mu_-|U = 0 \). Proceeding as in the proof of the former (and easier) ‘if part’ of [F1, Theorem 8.10], consider any bounded finely open set \( V \) of compact closure \( \overline{V} \subset U \). Then \( (\mu_-)^{CV} = \mu_- \) because \( \mu_- \) is carried by \( b(\mathbb{C}V) \supset \mathbb{C}U \). For any \( x \in V \) we obtain in terms of the Green kernel \( G \) on \( B \) according to [D, Theorems 1.X.3 and 1.X.5] applied within \( B \)

\[
\int G\mu_- d\varepsilon^x_{CV} = \hat{R}_{G\mu_-}^CV(x) = G((\mu_-)^{CV})(x) = G\mu_-(x) < +\infty,
\]

\[
\int G\mu_+ d\varepsilon^x_{CV} = \hat{R}_{G\mu_+}^CV(x) \leq G\mu_+(x) < +\infty,
\]

whence by subtraction \( \int G\mu d\varepsilon^x_{CV} \leq G\mu(x) \), showing that the finely continuous function \( G\mu \) is finely hyperharmonic, and indeed (being also bounded) finely superharmonic on \( U \), [F1, Theorem 8.10 and §10.4]. By the Riesz representation theorem, \( f = -G\mu + h \) on \( B \), with \( h \) harmonic on \( B \), in particular finely harmonic on \( U \), and \( f \) is therefore likewise finely subharmonic on \( U \).
Conversely, suppose that $f|U$ is finely subharmonic. Recall the corollary in [D, 1.XI.18] that if two subharmonic functions $g_1$ and $g_2$, defined on some open set, coincide on a set $A$, then their Riesz masses satisfy $\Delta g_1 = \Delta g_2$ on the fine interior of $A$. Hence the Riesz measure $\Delta f = \Delta u_1 - \Delta u_2$ on $U$ is independent of the choice of $u_1$ and $u_2$. In fact, if $f = w_1 - w_2$ (with $w_1$ and $w_2$ subharmonic on $U$) then $u_1 + w_2 = u_2 + w_1$ on $U$, hence $\Delta u_1 + \Delta w_2 = \Delta u_2 + \Delta w_1$ on (the fine interior of) $U$, that is, $\Delta u_1 - \Delta u_2 = \Delta u_1 - \Delta u_2$ on $U$.

Since $f$ is finely subharmonic on $U$ it follows by the proof of [F1, Theorem 9.9] that every point $x \in U$ has a fine neighborhood $V_x \Subset U$ in which we can write $f = v_1 - v_2$, where $v_1$ and $v_2$ are superharmonic functions on some Euclidean neighborhood $B_0$ of $x$ in $B$. Moreover, $v_2$ is the swept-out on $B_0 \setminus V_x$ of a certain superharmonic function $\geq 0$ on $B_0$. The Riesz mass of $v_2$ is concentrated on the fine boundary of the complement of $V_x$, cf. e.g. [D, Theorem 1.XI.14(b)], hence on the fine boundary of $V_x$. It follows that the Riesz mass $\Delta f$ of $f$ is positive on $V_x$ for every $x$. By the quasi-Lindelöf property, we can find countably many $x_j \in U$ such that $U = \bigcup_{j=1}^{\infty} V_{x_j} \cup E$, where $E$ is polar. Clearly $\Delta f$ is positive on $\bigcup V_{x_i}$. Because $E$ is polar, the Riesz mass of a bounded subharmonic function does not charge $E$, so we have $\Delta u_i(E) = 0$ ($i = 1, 2$). We conclude that the measure $\Delta f$ is positive on $U$. \hfill \square

Proof of the ‘only if part’ of Theorem 3.1. The proof merely uses that $f$ is finely hypoharmonic in each variable separately. According to Theorem 2.4(a), every point $a \in \Omega$ has a bounded $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset \Omega$ on which $f$ is representable as $f = u_1 - u_2$, where $u_1$ and $u_2$ are bounded plurisubharmonic functions, defined on an open ball $B$ in $\mathbb{C}^n$ containing $O$. For every $j \in \{1, \ldots, n\}$ the distributions

$$M_{i,j} = M_{i,j}(z) = \frac{\partial^2 u_i(z_1, \ldots, z_n)}{\partial z_j \partial \bar{z}_j}, \quad i = 1, 2,$$

are well defined positive measures on $B$. Below we show that $(M_{1,j} - M_{2,j})|O \geq 0$ and hence $\Delta (u_1 - u_2) = 4 \sum_{j=1}^{n} (M_{1,j} - M_{2,j}) \geq 0$ on $O$, where $\Delta$ denotes the Laplacian on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. According to Lemma 3.2 this implies that $u_1 - u_2$ indeed is $\mathbb{R}^{2n}$-finely subharmonic on $O$, and hence actually on all of $\Omega$, by varying $a$ and $O$.

For the proof that $(M_{1,j} - M_{2,j})|O \geq 0$ it is convenient to write points of $\mathbb{C}^n$ as $(z, w)$, now with $z \in \mathbb{C}$ and $w \in \mathbb{C}^{n-1}$. Each of the above measures $M_{i,j}$ on $B$ then takes the form

$$M_i = M_i(z, w) = \frac{\partial^2 u_i(z, w)}{\partial z \partial \bar{z}}, \quad i = 1, 2,$$

and we shall prove that $(M_1 - M_2)|O \geq 0$.

For each $w \in \mathbb{C}^{n-1}$ define

$$B(w) = \{z \in \mathbb{C} : (z, w) \in B\}$$
(and similarly with $B$ replaced by other subsets of $\mathbb{C}^n$). The functions \( u_i(\cdot, w) \), \( i = 1, 2 \), induce subharmonic functions \( u_i(\cdot, w) \) on the open subset \( B(w) \) of \( \mathbb{C} \), and the distributions

\[
\mu_{i,w} = \mu_{i,w}(z) = \frac{\partial^2 u_i(z, w)}{\partial z \partial \bar{z}}, \quad i = 1, 2,
\]

are therefore positive measures on the open set \( B(w) \) (if non-empty). Being weakly \( \mathcal{F} \)-plurisubharmonic on \( O \), \( f = u_1 - u_2 \) induces the finely subharmonic function \( f(\cdot, w) = u_1(\cdot, w) - u_2(\cdot, w) \) on the finely open set \( O(w) \). According to the planar version of Lemma 3.2, applied to the induced bounded subharmonic functions \( u_i(\cdot, w) \) on \( B(w) \), \( i = 1, 2 \), the Riesz measure \( \mu_{1,w} - \mu_{2,w} \) of \( f(\cdot, w) \) is positive on the finely open set \( O(w) \subset B(w) \).

Let \( V_z, V_w \) denote Lebesgue measure on \( \mathbb{C}, \mathbb{C}^{n-1} \), respectively. For any test function \( \varphi \in C^\infty_0(B) \) we have by Fubini’s theorem

\[
\int_B \varphi \, dM_i = \int_B \frac{\partial^2 \varphi(z, w)}{\partial z \partial \bar{z}} u_i(z, w) \, dV_z \, dV_w
\]

\[
= \int_{\mathbb{C}^{n-1}} \left( \int_{B(w)} \frac{\partial^2 \varphi(z, w)}{\partial z \partial \bar{z}} u_i(z, w) \, dV_z \right) dV_w
\]

\[
= \int_{\mathbb{C}^{n-1}} \left( \int_{B(w)} \varphi(z, w) \, d\mu_{i,w}(z) \right) dV_w.
\]

Choose a compact \( \mathcal{F} \)-neighborhood \( K \) of the given point \( a \in O \subset \Omega \) so that \( K \subset O \). There exists a decreasing sequence of functions \( \varphi_k \in C^\infty_0(B) \) with \( 0 \leq \varphi_k \leq 1 \) so that \( \varphi_k = 1 \) on \( K \) and \( \varphi_k \searrow \chi_K \) (the characteristic function of \( K \)) as \( k \nearrow \infty \). Since \( M_i \) and \( \mu_{i,w} \) are locally finite positive measures and \( B \) and \( \varphi_k \) are bounded we obtain by the monotone convergence theorem

\[
M_i(K) = \int_B \chi_K \, dM_i = \lim_{k \to \infty} \int_B \varphi_k \, dM_i
\]

\[
= \lim_{k \to \infty} \int_{\mathbb{C}^{n-1}} \left( \int_{B(w)} \varphi_k(\cdot, w) \, d\mu_{i,w} \right) dV_w
\]

\[
= \int_{\mathbb{C}^{n-1}} \left( \int_{B(w)} \chi_K(w) \, d\mu_{i,w} \right) dV_w
\]

\[
= \int_{\mathbb{C}^{n-1}} \mu_{i,w}(K(w)) \, dV_w.
\]

It follows that

\[
M_1(K) - M_2(K) = \int_{\mathbb{C}^{n-1}} \left( \mu_{1,w}(K(w)) - \mu_{2,w}(K(w)) \right) dV_w \geq 0
\]

because \( \mu_{1,w} - \mu_{2,w} \geq 0 \) on \( O(w) \supset K(w) \). Thus \( M_1(K) \geq M_2(K) \) for every compact \( \mathcal{F} \)-neighborhood \( K \) of \( a \) in \( O \).
The proof of Theorem 2.4(a) shows that we may take \( O = \{ z \in B(z_0, r) : \Phi^* (z) \geq -\frac{1}{4} \} \), where \( \Phi^* \) is plurisubharmonic on the open ball \( B(z_0, r) \), and in particular upper semicontinuous there. It follows that \( O = \bigcup_{p \in \mathbb{N}} F_p \) with

\[
F_p = \{ z \in B(z_0, (1 - \frac{1}{p})r) : \Phi^* (z) \geq -\frac{1}{4} + \frac{1}{p} \},
\]

a bounded closed and hence compact subset of \( \mathbb{C}^n \). Defining \( K_p = F_p \cup K \) we find that \( K_p \) is a compact \( \mathcal{F} \)-neighborhood of \( a \). We infer that \( K_p \nearrow O \) as \( p \nearrow \infty \), and consequently

\[
M_1 (O) = \sup_{p \in \mathbb{N}} M_1 (K_p) \geq \sup_{p \in \mathbb{N}} M_2 (K_p) = M_2 (O).
\]

By Lemma 3.2, this completes the proof of the 'only if' part of Theorem 3.1. \( \square \)

For the proof of the 'if part' of Theorem 3.1 we will need the following lemma, and some results of Bedford and Taylor on slicing of currents.

**Lemma 3.3.** Let \( f \) be a bounded finely subharmonic function on an \( \mathcal{F} \)-open set \( \Omega \subset \mathbb{C}^n \) and suppose that for every \( \mathbb{C} \)-affine bijection \( h \) of \( \mathbb{C}^n \) the function \( f \circ h \) is finely subharmonic on \( h^{-1}(\Omega) \). Then every \( z_0 \in \Omega \) admits a (compact) \( \mathcal{F} \)-neighborhood \( K_{z_0} \) such that \( f \) can be written as

\[
f = f_1 - f_2 \quad \text{on} \ K_{z_0},
\]

where \( f_1, f_2 \) are plurisubharmonic functions defined on a ball \( B(z_0, r) \supset K_{z_0} \).

**Proof.** As in the proof of (a) of Theorem 2.4, we can assume that \( -1 < f < 0 \), and find a compact \( \mathcal{F} \)-neighborhood \( V \) of \( z_0 \) and a negative plurisubharmonic function \( \varphi \) on a ball \( B(z_0, r) \supset V \) such that \( \varphi(z_0) = -1/2 \) and \( \varphi = -1 \) on \( B(z_0, r) \setminus V \). For every \( \lambda > 0 \) we can form the function

\[
u_\lambda (z) = \begin{cases} \max\{-\lambda, f(z) + \lambda \varphi(z)\} & \text{for} \ z \in \Omega \cap B(z_0, r), \\ -\lambda & \text{for} \ z \in B(z_0, r) \setminus V. \end{cases}
\]

It is a bounded finely subharmonic function on \( B(z_0, r) \), hence \( u_\lambda \) is subharmonic on \( B(z_0, r) \). Similarly, for every \( \mathbb{C} \)-affine bijection \( h \) of \( \mathbb{C}^n \) the function \( u_\lambda \circ h \) is finely subharmonic, hence subharmonic on \( h^{-1}(B(z_0, r)) \). From this we conclude that \( u_\lambda \) is in fact plurisubharmonic. Taking \( \lambda = 4 \), we see that

\[
u_4 (z) = f(z) + 4 \varphi(z)
\]

on the closed \( \mathcal{F} \)-neighborhood \( K_{z_0} = \{ z \in \Omega : \varphi(z) \geq -2/3 \} \subset V \cap B(z_0, r) \), and \( K_{z_0} \) is compact along with \( V \). This proves the lemma. \( \square \)
Corollary 3.4. We keep the notation as above. Then for every \( z_0 \in \Omega \), \( f \) is \( F \)-continuous on \( K_{z_0} \), hence on \( \Omega \).

We recall from [BT3] the concept of slice of an \((n-1,n-1)\)-current, now on a domain \( D \) in \( \mathbb{C}^n \). As usual we will write \( d = \partial + \overline{\partial} \) and \( d^c = i(\overline{\partial} - \partial) \) so that \( dd^c u = 2i\overline{\partial} \partial u \). Let \( T \) be an \((n-1,n-1)\)-current on \( D \).

The slice of \( T \) with respect to a hyperplane \( z_1 = a \) is the current

\[
\langle T, z_1, a \rangle(\psi) = \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{\{ z_1 - a \leq \varepsilon \} \cap \Omega} \psi(z_2, \ldots, z_n) \frac{1}{4} dd^c |z_1|^2 \wedge T.
\]

Here \( \psi \) is a \( C_0^\infty \) test form on \( z_1 = a \), extended to \( D \) independently of \( z_1 \).

Now let \( u_1, \ldots, u_{n-1} \) and \( w \) be bounded plurisubharmonic functions on \( D \), and put \( T = w dd^c u_1 \wedge \cdots \wedge dd^c u_{n-1} \). Then by [BT3, Proposition 4.1], \( \langle T, z_1, a \rangle \) exists for every \( a \in \mathbb{C} \) and

\[
\langle T, z_1, a \rangle = \frac{1}{2\pi} w(a, z') dd^c u_1(a, z') \wedge \cdots \wedge dd^c u_{n-1}(a, z').
\]

Here \( z' = (z_2, \ldots, z_n) \).

Finally, if \( F \) is holomorphic on \( D \) and \( M = \{ z \in D : F(z) = 0 \} \), then by changing variables and since only regular points of \( M \) have to be taken into account, one gets

\[
\langle w(dd^c u)^{n-1}, F, 0 \rangle = w|_M (dd^c u)^{n-1}.
\]

We write \( \varepsilon' = (\varepsilon_2, \ldots, \varepsilon_n), \varepsilon'^2 = \prod_{j=2}^n \varepsilon_j^2 \), and \( |z'| < \varepsilon' \) for \( |z_j| < \varepsilon_j, j = 2, \ldots, n \).

Lemma 3.5. Let \( \psi = \psi(z_1) \) be a test function on \( \{ z \in D : z' = 0' \} \), and let \( w \) and \( u \) be bounded plurisubharmonic functions on a bounded domain \( D \subset \mathbb{C}^n \). Then

\[
\int_{\{ z_2 = 0, \ldots, z_n = 0 \}} \psi(z_1) w(z_1, 0') dd^c u(z_1, 0')
= \lim_{\varepsilon' \downarrow 0} \frac{1}{2^{n-1} \varepsilon'^2} \int_{\{ |z'| < \varepsilon' \}} \psi(z_1) w(z) dd^c |z_2|^2 \wedge \cdots \wedge dd^c |z_n|^2 \wedge dd^c u.
\]

(3.5)

Proof. Apply slicing with respect to \( z_2 = 0 \) to the current \( T = w dd^c |z_3|^2 \wedge \cdots \wedge dd^c |z_n|^2 \wedge dd^c u \) to obtain \( \langle T, z_2, 0 \rangle = \frac{1}{2\pi} w(z_1, 0, z_3, \ldots, z_n) dd^c |z_3|^2 \wedge \cdots \wedge dd^c |z_n|^2 \wedge dd^c u \). Next in \( \{ z_2 = 0 \} \), apply slicing with respect to \( z_3 = 0 \) to the current \( w dd^c |z_4|^2 \wedge \cdots \wedge dd^c |z_n|^2 \wedge dd^c u \). Continuing in this fashion we obtain (3.5).

Proof of the ‘if part’ of Theorem 3.1. We keep our notation and proceed as follows. First we will show that \( dd^c f \geq 0 \) on the compact neighborhood \( K = K_{z_0} \) of \( z_0 \) provided by Lemma 3.3. Next we apply Lemma 3.5 to show that the restriction of \( f \) to any complex line passing through \( z_0 \) is finely subharmonic on a fine neighborhood of \( z_0 \).
Let $v$ be any plurisubharmonic function on a ball $B$ in $\mathbb{C}^n$, let $h$ be a $\mathbb{C}$-affine bijection of $\mathbb{C}^n$, and let $\varphi \in C^\infty_0(B)$ be a test function. Then the action of the Riesz measure $\Delta(v \circ h)$ on $\varphi \circ h$ can be expressed as follows

\[
4^{n-1}(n-1)! \int_{h^{-1}(B)} \varphi \circ h(z)\Delta(v \circ h)
= \int_{h^{-1}(B)} \varphi(z)dd^c(v \circ h) \wedge (dd^c\|z\|^2)^{n-1}
= \int_B \varphi(\zeta)dd^c\nu(\zeta) \wedge (dd^c\|h^{-1}(\zeta)\|^2)^{n-1}.
\]

Returning to $f$, we have by Lemma 3.2 that the Riesz measure $\Delta(f \circ h)$ is positive on $h^{-1}(K)$, hence (with $h^{-1} = g$) we obtain that

\[
dd^c f(\zeta) \wedge (dd^c\|g(\zeta)\|^2)^{n-1}
\]

is a positive measure on $K$ for every $\mathbb{C}$-affine bijection $g$ of $\mathbb{C}^n$, and by continuity also for every $\mathbb{C}$-affine map $g : \mathbb{C}^n \to \mathbb{C}^n$.

To finish the proof we want to show that $f$ restricted to a complex line $L$ passing through $z_0$ is finely subharmonic in a fine neighborhood of $z_0$ relative to $L$. We write $z = (z_1, z')$ and can assume that $z_0 = 0$ and that $L$ is given by $z' = 0'$. Because $K$ is an $\mathcal{F}$-neighborhood of $0$, there exists a bounded non-negative plurisubharmonic function $w$ defined on a ball $B_0$ about $0$ that equals $0$ on $B_0 \setminus K$, while $w(0) > 0$. Then $\{z \in B_0 : w(z) > 0\}$ is an $\mathcal{F}$-open subset of $K$ that contains $0$. On $K$ we have $f = f_1 - f_2$ where $f_1, f_2$ are plurisubharmonic on a ball containing $K$, hence on $B_0$. We apply Lemma 3.5 to $f_1$ and $f_2$ separately and subtract to obtain from (3.6) (with $g(z) = g(z_1, z') = (0, z')$)

\[
\int_{\{z_2=0, \ldots, z_n=0\}} \psi(z_1)w(z_1, 0') dd^c f(z_1, 0')
= \lim_{\varepsilon' \downarrow 0} \frac{1}{2n-1\varepsilon'^2} \int_{\{|z'| < \varepsilon'\}} \psi(z_1)w(z)dd^c|z_2|^2 \wedge \ldots \wedge dd^c|z_n|^2 \wedge dd^c f.
\]

If $\varepsilon'$ is sufficiently small then the integrals occurring in the limit on the right hand side are non-negative for every non-negative test function $\psi$ on $B_0 \cap \{z' = 0'\}$.

We conclude that the Riesz measure of $f|_L$ is positive on a neighborhood of $0$. By Lemma 3.2, $f|_L$ is finely subharmonic on this neighborhood. Varying $z_0$ over $L$ and using the sheaf-property, we find that $f|_L$ is finely subharmonic. \hfill \Box

**Corollary 3.6.** Let $f$ be a bounded weakly $\mathcal{F}$- plurisubharmonic function on an $\mathcal{F}$-domain $\Omega \subset \mathbb{C}^n$ such that $f$ admits the representation $f = f_1 - f_2$ of Lemma 3.3 on $\Omega$, and let $\chi_K$ denote the characteristic function of a compact set $K$ in $\Omega$. Then for $\mathbb{C}$-affine functions $l_1, \ldots, l_{n-1}$ on $K = K_{z_0}$ from Lemma 3.3

\[
\int_{\Omega} \chi_K(z)dd^c|l_1|^2 \wedge \ldots \wedge dd^c|l_{n-1}|^2 \wedge dd^c f \geq 0.
\]
**Proof.** This follows from (3.6) with \( g(z) = (l_1(z), \ldots, l_{n-1}(z), 0) \).

From Theorem 3.1 we derive the following two results, one about removable singularities for weakly \( \mathcal{F} \)-plurisubharmonic functions, and the other about the supremum of a family of such functions.

**Theorem 3.7.** Let \( f : \Omega \to [-\infty, +\infty[ \) be \( \mathcal{F} \)-locally bounded from above on an \( \mathcal{F} \)-open set \( \Omega \subseteq \mathbb{C}^n \), and let \( E \) be an \( \mathcal{F} \)-closed pluripolar subset of \( \Omega \). If \( f \) is weakly \( \mathcal{F} \)-plurisubharmonic on \( \Omega \setminus E \) then \( f \) has a unique extension to a weakly \( \mathcal{F} \)-plurisubharmonic function on all of \( \Omega \), and this extension \( f^* \) is given by

\[
f^*(z) = \mathcal{F}\text{-}\limsup_{\zeta \to z \atop \zeta \not\in \Omega \setminus E} f(\zeta), \quad z \in \Omega.
\]

**Proof.** The stated function \( f^* \) (the \( \mathcal{F} \)-upper semicontinuous regularization of \( f \)) equals \( f \) on the \( \mathcal{F} \)-open set \( \Omega \setminus E \) because \( f \) is \( \mathcal{F} \)-upper semicontinuous on \( \Omega \setminus E \). Furthermore, \( f^* \) is \( \mathcal{F} \)-upper semicontinuous and \( < +\infty \) on all of \( \Omega \) (finiteness because \( f \) is \( \mathcal{F} \)-locally bounded from above). By the ‘only if’ part of Theorem 3.1, for any \( \mathbb{C} \)-affine bijection \( h \) of \( \mathbb{C}^n \), \( f^* \circ h \) therefore is \( \mathbb{R}^{2n} \)-finely hypoharmonic on \( h^{-1}(\Omega \setminus E) = h^{-1}(\Omega) \setminus h^{-1}(E) \) and \( \mathcal{F} \)-upper semicontinuous \( < +\infty \) on \( h^{-1}(\Omega) \). In particular, \( f^* \circ h < +\infty \) is \( \mathbb{R}^{2n} \)-finely upper semicontinuous on \( h^{-1}(\Omega) \). Because \( E \) is pluripolar so is \( h^{-1}(E) \), which thus is \( \mathbb{R}^{2n} \)-polar. According to [F1, Theorem 9.14], \( f \circ h \) is therefore \( \mathbb{R}^{2n} \)-finely hypoharmonic on all of \( h^{-1}(\Omega) \), and so \( f^* \) is indeed weakly \( \mathcal{F} \)-plurisubharmonic on \( \Omega \), by the ‘if part’ of Theorem 3.1. Because the pluripolar set \( E \) has empty \( \mathcal{F} \)-interior, \( f^* \) is the only weakly \( \mathcal{F} \)-plurisubharmonic and hence \( \mathcal{F} \)-continuous extension of \( f \) to \( \Omega \).

In view of Lemma 2.8 there is a similar result about removable singularities for weakly \( \mathcal{F} \)-holomorphic functions:

**Corollary 3.8.** Let \( h : \Omega \to \mathbb{C} \) be \( \mathcal{F} \)-locally bounded on \( \Omega \) (\( \mathcal{F} \)-open in \( \mathbb{C}^n \)). If \( h \) is weakly \( \mathcal{F} \)-holomorphic on \( \Omega \setminus E \) (\( E \) \( \mathcal{F} \)-closed and pluripolar in \( \mathbb{C}^n \)) then \( h \) extends uniquely to a weakly \( \mathcal{F} \)-holomorphic function \( h^* : \Omega \to \mathbb{C} \), given by

\[
h^*(z) = \mathcal{F}\text{-}\lim_{\zeta \to z \atop \zeta \not\in \Omega \setminus E} h(\zeta), \quad z \in \Omega.
\]

**Theorem 3.9.** Let \( \Omega \) denote an \( \mathcal{F} \)-open subset of \( \mathbb{C}^n \). For any \( \mathcal{F} \)-locally upper bounded family of weakly \( \mathcal{F} \)-plurisubharmonic functions \( f_\alpha \) on \( \Omega \), the least \( \mathcal{F} \)-upper semicontinuous majorant \( f^* \) of the pointwise supremum \( f = \sup_\alpha f_\alpha \) is likewise weakly \( \mathcal{F} \)-plurisubharmonic on \( \Omega \), and \( \{ z \in \Omega : f(z) < f^*(z) \} \) is pluripolar.

**Proof.** We may assume that the set \( A \) of indices \( \alpha \) is upper directed and that the net \( (f_\alpha)_{\alpha \in A} \) is increasing; furthermore that \( \Omega \) is \( \mathcal{F} \)-connected and that \( f_\alpha \not= -\infty \) for some \( \alpha \in A \). For any function \( f : \Omega \to [-\infty, +\infty[ \) which is \( \mathcal{F} \)-locally bounded from above, write

\[
f^*(z) = \mathcal{F}\text{-}\limsup_{\zeta \to z} f(\zeta), \quad \bar{f}(z) = \mathbb{R}^{2n}\text{-}\limsup_{\zeta \to z} f(\zeta).
\]
Then \( \hat{f}(z) \leq f^*(z) < +\infty \), the former inequality because the \( \mathbb{R}^{2n} \)-fine topology is finer than the \( \mathcal{F} \)-topology.

As in Theorem 3.1, let \( h : \mathbb{C}^n \to \mathbb{C}^n \) be a \( \mathbb{C} \)-affine bijection, and note that
\[
f \circ h = \sup_{\alpha}(f_{\alpha} \circ h), \quad (f \circ h)^* = \hat{f} \circ h, \quad \text{on } h^{-1}(\Omega),
\]
the latter equation because \( h \) is an \( \mathbb{R}^{2n} \)-fine homeomorphism. By Theorem 3.1, \( f_{\alpha} \circ h \) is \( \mathbb{R}^{2n} \)-finely hypoharmonic. Now \( f_{\alpha} \circ h \leq f^* \circ h \). Furthermore, \( f^* \) and hence \( f^* \circ h \) and \( \hat{f} \circ h \) are \( \mathbb{R}^{2n} \)-finely locally bounded from above. It follows by [F1, Lemma 11.2] that \( \hat{f} \circ h = (f \circ h) \) is \( \mathbb{R}^{2n} \)-finely hypoharmonic.

We proceed to show that \( \hat{f} = f^* \) on \( \Omega \), and hence that \( \hat{f} \) is \( \mathcal{F} \)-upper semicontinuous there. Invoking also Theorem 3.1 we shall thus altogether find that \( \hat{f} = f^* \) becomes \( \mathcal{F} \)-plurisubharmonic on \( \Omega \), and in particular \( \mathcal{F} \)-continuous there, by Theorem 2.4(c).

Consider a point \( z_0 \in \Omega \) such that \( f(z_0) > -\infty \). Fix \( \beta \in A \) with \( f_\beta(z_0) > -\infty \), and choose an \( \mathcal{F} \)-open \( \mathcal{F} \)-neighborhood \( U \) of \( z_0 \) so that \( U \subset \Omega \) and
\[
f_\beta(z_0) - 1 < f_\beta \leq f^* < f^*(z_0) + 1 \quad \text{on } U,
\]
noting that the weakly \( \mathcal{F} \)-plurisubharmonic function \( f_\beta \) is \( \mathcal{F} \)-continuous and that \( f^* \) is \( \mathcal{F} \)-upper semicontinuous and \( < +\infty \). Since \( f_\beta \leq f_{\alpha} \leq f \) for every \( \alpha \geq \beta \) in \( A \), any such \( f_{\alpha} \) maps \( U \) into some fixed bounded interval. According to Theorem 2.4(a),(b) there exist \( r > 0 \), an \( \mathcal{F} \)-open set \( O \) such that \( z_0 \in O \subset B(z_0, r) \), and locally bounded ordinary plurisubharmonic functions \( \varphi_\alpha \) and \( \psi \) on \( B(z_0, r) \) such that \( f_{\alpha} = \varphi_\alpha - \psi \) on \( O \) for every \( \alpha \geq \beta \) in \( A \). The net \( (\varphi_\alpha) \) is increasing, along with the given net \( (f_{\alpha}) \). The plurisubharmonic functions \( \varphi_\alpha \) and \( \psi \) are \( \mathcal{F} \)-continuous, in particular \( \mathbb{R}^{2n} \)-finely continuous. Writing \( \sup_\alpha \varphi_\alpha = \varphi \) and denoting by \( \bar{\varphi} \) the Euclidean \( \mathbb{R}^{2n} \)-subharmonic regularization of \( \varphi \) in \( B(z_0, r) \), we therefore have \( \bar{\varphi} = \varphi \) there, by Brelot’s fundamental convergence theorem, see e.g. [D, 1.XI.7]. Because \( \bar{\varphi} \leq \varphi^* \leq \bar{\varphi} \) it follows that \( \bar{\varphi} = \varphi^* \) in \( B(z_0, r) \), and consequently
\[
\hat{f} = (\varphi - \psi) = \bar{\varphi} - \psi = \varphi^* - \psi = (\varphi - \psi)^* = f^* \quad \text{on } O
\]
since \( \psi \) is \( \mathcal{F} \)-continuous and hence \( \mathbb{R}^{2n} \)-finely continuous on the \( \mathcal{F} \)-open, hence \( \mathbb{R}^{2n} \)-finely open set \( O \subset B(z_0, r) \).

Next, the set \( \{ z \in O : f(z) < f^*(z) \} = \{ z \in O : \varphi(z) < \varphi^*(z) \} \) is pluripolar, by the deep theorem of Bedford and Taylor [BT1], or see [K Theorem 4.7.6]. Writing
\[
E = \{ z \in \Omega : f(z) < f^*(z) \}, \quad e = \{ z \in \Omega : f(z) = -\infty \},
\]
we have thus found that every point \( z_0 \in \Omega \setminus e \) has an \( \mathcal{F} \)-neighborhood \( O \subset \Omega \setminus e \) for which \( O \cap E \) is pluripolar. Because \( e = \bigcap_{\alpha \in A} \{ z \in \Omega : f_{\alpha}(z) = -\infty \} \) is \( \mathcal{F} \)-closed relative to \( \Omega \), and pluripolar (some \( f_{\alpha} \) being \( \not\equiv -\infty \), we infer by the quasi-Lindelöf principle [BT2, Theorem 2.7] that indeed \( E \) is pluripolar. Finally, we have found that \( f^* \) is \( \mathcal{F} \)-plurisubharmonic on each \( \mathcal{F} \)-open set \( O \) as above (as \( z_0 \) varies), and hence on their union \( \Omega \setminus e \), by the sheaf property. Because \( f^* \) is \( \mathcal{F} \)-upper semicontinuous and \( < +\infty \) on \( \Omega \), and that \( e \) is pluripolar, we conclude from Theorem 3.7 above that indeed \( f^* \) is weakly \( \mathcal{F} \)-plurisubharmonic on all of \( \Omega \).

Taking for \( \Omega \) a Euclidean open set we obtain in particular the following
Corollary 3.10. For any family \( \{ f_\alpha \} \) of ordinary plurisubharmonic functions on a Euclidean open set \( \Omega \subset \mathbb{C}^n \) such that \( f := \sup_\alpha f_\alpha \) is locally bounded from above, the least plurisubharmonic majorant of \( f \) exists and can be expressed as the upper semicontinuous regularization of \( f \) in the Euclidean topology on \( \mathbb{C}^n \), as well as in the \( \mathcal{F} \)-topology and in the \( \mathbb{R}^{2n} \)-fine topology; that is, \( \bar{f} = f^* = \check{f} \).

The version of this involving the Euclidean topology is due to Lelong [L1], or see [L2, p. 26] or [K, Theorem 2.9.10]. Being locally bounded from above, \( f \) is in particular \( \mathcal{F} \)-locally bounded from above, and hence so is \( f^* \), which is \( \mathcal{F} \)-plurisubharmonic by Theorem 3.9. Because \( \Omega \) is Euclidean open, it follows by Proposition 2.14 that \( f^* \) even is an ordinary plurisubharmonic function. From \( f \leq f^* \leq \bar{f} \) it therefore follows that \( f^* = \bar{f} \). Similarly, \( \check{f} = \bar{f} \) in view of [F1, Theorem 9.8(a)].

The identity \( f^* = \bar{f} \) is perhaps new even in the Euclidean case.

We close this section with an alternative proof of the ‘only if part’ of Theorem 3.1. It is a bit shorter than the proof given above. On the other hand it draws substantially on the theory of functions of Beppo Levi and Deny, cf. [DL], and its connection to fine potential theory, cf. [F4]. We will need this approach again in Section 4.

Following Deny [DL] and subsequently [F4] we consider for a given Greenian domain \( D \) (denoted \( \Omega \) in [DL] and [F4]) of \( \mathbb{C}^n \sim \mathbb{R}^{2n} \) the complex Hilbert space

\[ \hat{D}^1(D), \]

the completion of \( \mathcal{D}(D) = C_0^\infty(D, \mathbb{C}) \) in the Dirichlet norm \( ||u||_1 = ||\nabla u||_{L^2(D, \mathbb{C})} \).
(For \( n \geq 2 \) we may thus take \( D = \mathbb{C}^n \). For \( n = 1 \), any bounded domain \( D \) will do.) Note that \( \hat{D}^1(D) \) is a space of distributions, [DL, Théorème 2.1 (p. 350)]. Elements of \( \hat{D}^1(D) \) may be represented by quasi-continuous functions that are finite quasi-everywhere. For an \( \mathbb{R}^{2n} \)-finely open set \( \Omega \subset D \) denote by \( \hat{D}^1(D, \Omega) \) the Hilbert subspace consisting of all \( \varphi \in \hat{D}^1(D) \) such that some (and hence any) \( \mathbb{R}^{2n} \)-quasi-continuous representative of \( \varphi \) satisfies \( \varphi = 0 \) \( \mathbb{R}^{2n} \)-quasi-everywhere on \( D \setminus \Omega \), cf. [DL, Théorème 5.1, pp. 358 f.]. The positive cone in for example \( \hat{D}^1(D, \Omega) \) is denoted by \( \hat{D}^1_+(D, \Omega) \). Let \( V_l \) denote Lebesgue measure on \( \mathbb{C}^l \), and write \( V_n = V \).

According to [F4, Théorème 11] an \( \mathbb{R}^{2n} \)-finely continuous (hence quasicontinuous) function \( f \in \hat{D}^1(D) \) is finely subharmonic quasi-everywhere (hence actually everywhere by [F1, Theorem 9.14]) on \( \Omega \), if and only if \( f < +\infty \) and the inequality sign holds in (3.7):

\[ \frac{1}{4} \int_D \nabla f \cdot \nabla \varphi dV = \sum_{j=1}^n \int_D (\partial_j f) (\partial_j \varphi) dV \leq 0 \quad (3.7) \]

for every \( \varphi \in \hat{D}^1_+(D, \Omega) \). (It suffices of course to integrate over \( \Omega \).)

Alternative proof of the ‘only if part’ of Theorem 3.1. Consider a weakly \( \mathcal{F} \)-plurisubharmonic function \( f \) on an \( \mathcal{F} \)-open set \( \Omega \subset \mathbb{C}^n \); hence \( f \) is \( \mathcal{F} \)-continuous and \( < +\infty \). We leave out the trivial case \( n = 1 \). We may assume that \( f > -\infty \) on \( \Omega \).
(otherwise replace \(f\) by \(\max\{f, -p\}\) and let \(p \to +\infty\)). It suffices to prove that \(f\) is \(\mathbb{R}^{2n}\)-finely hypoharmonic.

Write \(z = (z_1, \ldots, z_n) = (z_1, z') \in \mathbb{C}^n\). According to Theorem 2.4(a), every point \(z_0 \in \Omega\) then has an \(\mathcal{F}\)-open \(\mathcal{F}\)-neighborhood \(O \subset \Omega\) on which \(f = f_1 - f_2, f_1\) and \(f_2\) being bounded plurisubharmonic \(> -\infty\) on some open ball \(B = B(z_0, r)\) containing \(O\). In particular, \(f_1\) and \(f_2\) are \(\mathbb{R}^{2n}\)-subharmonic on \(B\). We may further assume that \(-f_i\) and \(-f_2\) are \(\mathbb{R}^{2n}\)-potentials on \(B\), for otherwise we may replace \(-f_i\) for \(i = 1, 2\) by its swept-out (relative to \(B\)) \(\hat{R}_{-f_i}\) on \(A = B(z_0, r/2)\) (and \(O\) by \(O \cap A\)). In terms of the Green kernel \(G\) on \(B\) we may therefore write \(-f_i = G\mu_i\) on \(B\) for some bounded positive measure \(\mu_i\) of compact support in \(B\). Since \(-f_i\) is bounded, its \(G\)-energy 
\[\int_O G \, d\mu_i \leq 0\]
(3.8) Here \(\nabla_1 = (\partial/\partial x_1, \partial/\partial y_1)\). Integrating (3.8) with respect to \(V_{n-1}\) leads by Fubini’s theorem to
\[\int_O \nabla_1 f(z_1, z') \cdot \nabla_1 \varphi(z_1, z') \, dV_1 \leq 0.\]

Similarly with the subscript 1 replaced by any \(j \in \{1, \ldots, n\}\). After addition this leads to
\[\int_O \nabla f \cdot \nabla \varphi \, dV \leq 0.\]

According to [F4, Théorème 11] quoted above, this shows that \(f\) indeed is \(\mathbb{R}^{2n}\)-finely subharmonic on \(O\), and hence, by varying \(z_0\), on all of \(\Omega\). \(\square\)

4. Biholomorphic invariance

The sigma-algebra \(QB\) of quasi Borel sets in \(\mathbb{C}^n\) is generated by the Borel sets and the sets of capacity 0 (see [BT2]). \(QB\) contains the finely open sets. All currents originating from wedge products of \(dd^c\) of bounded plurisubharmonic functions have measure coefficients that are Borel measures and put no mass on pluripolar sets, hence they extend naturally to \(QB\).

**Proposition 4.1.** Let \(f\) be a bounded weakly \(\mathcal{F}\)-plurisubharmonic function on an \(\mathcal{F}\)-domain \(\Omega \subset \mathbb{C}^n\) such that \(f\) admits the representation \(f = f_1 - f_2\) of Lemma
3.3 on Ω, and let χK denote the characteristic function of a compact set K ⊂ Ω. Then for holomorphic functions g_1, \ldots, g_{n-1} on K = K_{z_0} from Lemma 3.3
\[ \int \chi_K(z) dd^c |g_1|^2 \wedge \cdots \wedge dd^c |g_{n-1}|^2 \wedge dd^c f \geq 0. \] (4.1)

**Proof.** Corollary 3.6 yields that (4.1) is valid for compact sets \( \tilde{K} \subset K_{z_0} \) and \( \mathbb{C} \)-affine functions \( g_i \). For arbitrary holomorphic functions \( g_j \) we have
\[ \int \chi_K(z) dd^c |g_1|^2 \wedge \cdots \wedge dd^c |g_{n-1}|^2 \wedge dd^c f = \lim_{N \to \infty} \sum_{j=1}^{N} \int \chi_{E_j^N}(z) dd^c |l_1^j N|^2 \wedge \cdots \wedge dd^c |l_{n-1}^j N|^2 \wedge dd^c f \] (4.2)
for suitable quasi Borel sets \( E_j^N \) and complex affine approximants \( l_j^N \) of \( g_k \) on \( E_j^N \) \((k = 1, \ldots, n)\). Hence the right hand side of (4.2) is indeed non-negative. □

**Theorem 4.2.** Let \( \Omega \) be \( \mathcal{F} \)-open in \( \mathbb{C}^n \) \((n \geq 2)\). Given an \( \mathcal{F} \)-continuous function \( f \in \tilde{D}^1(\mathbb{C}^n) \) with values in \([-\infty, +\infty]\), the following are equivalent:
(a) \( f \) is weakly \( \mathcal{F} \)-plurisubharmonic on \( \Omega \),
(b) for every \( \varphi \in \tilde{D}_+(\mathbb{C}^n, \Omega) \) and every \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \),
\[ \sum_{j,k=1}^{n} \lambda_j \lambda_k \int_{\Omega} (\partial_j f)(\partial_k \varphi) dV \leq 0, \]
(c) for every regular holomorphic map \( h : \omega \to \mathbb{C}^n \) \((\omega \text{ open in } \mathbb{C}^n)\), \( f \circ h \) is weakly \( \mathcal{F} \)-plurisubharmonic on \( h^{-1}(\omega) \) \((\subset \omega)\).

**Proof.** (a)⇒(b). Using the characterization of weakly \( \mathcal{F} \)-plurisubharmonic functions given in Theorem 3.1, one may adapt the proof of the ‘only if part’ of [K, Theorem 2.9.12] as follows. Suppose \( f \in \tilde{D}^1(\mathbb{C}^n) \) is weakly \( \mathcal{F} \)-plurisubharmonic on \( \Omega \), and so \( f \circ T \) is \( \mathbb{R}^{2n} \)-finely subharmonic on \( T^{-1}(\Omega) \) for any \( \mathbb{C} \)-affine bijection \( T \) of \( \mathbb{C}^n \). To prove (b) with constant \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \), take
\[ T_{\varepsilon}(z) = z_1 \lambda + \varepsilon \sum_{l=2}^{n} z_le_l, \quad \varepsilon > 0, \]
where \((e_1, \ldots, e_n)\) denotes the canonical base of \( \mathbb{C}^n \). From (3.7) we obtain (with integrations over \( \mathbb{C}^n \)), replacing \( \Omega \) and \( \varphi \), as we may, by \( T_{\varepsilon}^{-1}(\Omega) \) and \( \varphi \circ T_{\varepsilon} \in \tilde{D}_+(\mathbb{C}^n, T_{\varepsilon}^{-1}(\Omega)) \),
\[ 0 \geq \sum_{l=1}^{n} \int \partial_l (f \circ T_{\varepsilon}) \partial_l (\varphi \circ T_{\varepsilon}) dV \]
\[ = \sum_{j,k=1}^{n} \int [(\partial_j f) \circ T_{\varepsilon}] [(\partial_k \varphi) \circ T_{\varepsilon}] (\lambda_j \lambda_k + O(\varepsilon)) dV \]
\[ = | \det T_{\varepsilon} |^2 \left( \sum_{j,k=1}^{n} \int (\partial_j f)(\partial_k \varphi) \lambda_j \lambda_k dV + O(\varepsilon) \right). \]
This leads to (b) after division by $|\det T|^{2}$ when we make $\varepsilon \to 0$.

(b)$\Rightarrow$(a). Consider any $\mathbb{C}$-affine bijection $T = (T_{1}, \ldots, T_{n})$ of $\mathbb{C}^{n}$, say

$$T_{l}(z) = \sum_{j=1}^{n} c_{lj}z_{j} + d_{l}, \quad l \in \{1, \ldots, n\}, \ z \in \mathbb{C},$$

with $c_{lj}, d_{l} \in \mathbb{C}$ and $\det T \neq 0$. We obtain

$$\int \partial_{l}(f \circ T)\overline{\partial_{l}(\varphi \circ T)}dV = \sum_{j,k=1}^{n} \int [(\partial_{j}f) \circ T][(\overline{\partial_{k}\varphi}) \circ T]c_{lj}\overline{c}_{lk}dV$$

$$= |\det T|^{2} \sum_{j,k=1}^{n} \int c_{lj}\overline{c}_{lk}(\partial_{j}f)(\overline{\partial_{k}\varphi})dV \leq 0$$

by (b) with $\lambda_{j} = c_{lj}$. After division by $|\det T|^{2}$ and summation over $l$ this shows according to (3.7) and Theorem 3.1 that the $\mathcal{F}$-continuous function $f < +\infty$ indeed is $\mathcal{F}$-plurisubharmonic on $\Omega$.

(c)$\Rightarrow$(a). This is contained in Theorem 3.1 (even with $h$ in (c) just a $\mathbb{C}$-affine bijection and with $f \circ h$ just $\mathbb{R}^{2n}$-finitely subharmonic).

(a)$\Rightarrow$(c). We may assume that $f > -\infty$ on $\Omega$ (otherwise pass to $f_{p} := \max\{f, -p\}$, $p \in \mathbb{N}$, and let $p \to +\infty$). According to Theorem 2.4(a), every point $z_{0} \in h^{-1}(\Omega)$ then has an $\mathcal{F}$-open $\mathcal{F}$-neighborhood $O \subset h^{-1}(\Omega)$ on which $f = f_{1} - f_{2}$, $f_{1}$ and $f_{2}$ being bounded plurisubharmonic on some open set $D \subset \mathbb{C}^{n}$ containing $O$. In particular, $\Omega$ and $O$ are $\mathbb{R}^{2n}$-finitely open, and $f_{1}$ and $f_{2}$ are $\mathbb{R}^{2n}$-subharmonic on $D$. We may further assume that the Jacobian matrix $(\partial_{j}h_{k})$ of the regular holomorphic map $h : \omega \to \mathbb{C}^{n}$ is bounded with determinant bounded away from 0.

Denoting by $\mathcal{S}(D, O)$ the convex cone of all functions of class $\hat{D}^{1}(D)$ which are $\mathbb{R}^{2n}$-finitely superharmonic quasi-everywhere on $O$, we have by [F4, p. 129] that $-f \in \mathcal{S}(D, O)$ and hence by [F4, Théorème 11(b)]

$$\int_{O} \nabla f \cdot \nabla \varphi \ dV \leq 0 \quad \text{for } \varphi \in \hat{D}_{+}^{1}(D, O).$$

For any $\psi \in \hat{D}^{1}(\mathbb{C}^{n})$ we have (by the properties of $h$ required above) $\psi \circ h \in \hat{D}^{1}(\omega)$. According to Theorem 3.1 it suffices to show that the $\mathcal{F}$-continuous function $f \circ h$ is $\mathbb{R}^{2n}$-finitely subharmonic on $h^{-1}(O)$. For this it suffices by (3.7) to prove that, for every $j \in \{1, \ldots, n\}$,

$$\int_{h^{-1}(O)} (\partial_{j}(f \circ h))(\overline{\partial_{j}\psi})dV \leq 0 \quad \text{for every } \psi \in \hat{D}_{+}^{1}(\mathbb{C}^{n}, h^{-1}(O)),$$

and here $\psi$ may be replaced equivalently by $\varphi \circ h$ with $\varphi \in \hat{D}_{+}^{1}(D, O)$ (or just as well with $\varphi \in D_{+}(D, O)$). We take $j = 1$ and write

$$dV = (i/2)^{n} dz_{1} \wedge d\overline{z}_{1} \wedge \ldots \wedge dz_{n} \wedge d\overline{z}_{n} = (1/4)^{n} dd^{c}|z_{1}|^{2} \wedge \ldots \wedge dd^{c}|z_{n}|^{2}.$$
Then we obtain by the chain rule, writing by abuse of notation $h^{-1} = (h_1^{-1}, h_2^{-1}, \ldots, h_n^{-1})$ in terms of the inverse $h^{-1}$ of the map $h$

$$
\int_{h^{-1}(O)} (\partial_1 (f \circ h)) (\tilde{\partial}_1 (\varphi \circ h)) dd^c |z_1|^2 \land \ldots \land dd^c |z_n|^2
$$

$$= \int d(f \circ h) \land d^c (\varphi \circ h) \land dd^c |z_2|^2 \land \ldots \land dd^c |z_n|^2
$$

$$= \int d(f \circ h) \land d^c (\varphi \circ h) \land dd^c |h_2^{-1} \circ h|^2 \land \ldots \land dd^c |h_n^{-1} \circ h|^2
$$

$$= \int df \land d^c \varphi \land dd^c |h_2^{-1}|^2 \land \ldots \land dd^c |h_n^{-1}|^2
$$

$$= \int \varphi dd^c f \land dd^c |h_2^{-1}|^2 \land \ldots \land dd^c |h_n^{-1}|^2
$$

The last three lines are in $h$-coordinates. Equality (4.3) is justified by approximating $f$ and $\varphi$ in $\hat{D}^1$ with functions in $\mathcal{D}$ and applying Stokes’ theorem to the approximants. The final expression is non-positive because of Proposition 4.1, and we are done.

Now we wish to consider the case where $h$ is just some sort of plurifinely holomorphic map. Recall from the text preceding Proposition 2.9 that an $n$-tuple $(h_1, \ldots, h_n)$ of strongly/weakly $\mathcal{F}$-holomorphic functions $h_j : U \to \mathbb{C}$ ($U$ $\mathcal{F}$-open in some $\mathbb{C}^n$) is termed a strongly/weakly $\mathcal{F}$-holomorphic map (or curve if $m = 1$).

**Definition 4.3.** A strongly $\mathcal{F}$-biholomorphic map $h$ from an $\mathcal{F}$-open set $U \subset \mathbb{C}^n$ onto its image in $\mathbb{C}^n$ is an $\mathcal{F}$-homeomorphism with the property that there exists for every $z \in U$ a compact $\mathcal{F}$-neighborhood $K_z$ of $z$ in $U$ and a $C^\infty$-diffeomorphism $\Phi_z$ from an open neighborhood of $K_z$ to its image in $\mathbb{C}^n$ such that $\Phi_z | K_z = h | K_z$ and that $\Phi_z | K_z$ is a $C^1$-limit of holomorphic maps defined on open sets containing $K_z$.

**Proposition 4.4.** The composition $f \circ h$ of a weakly $\mathcal{F}$-plurisubharmonic function $f$ on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ with a strongly $\mathcal{F}$-biholomorphic map $h : U \to \Omega$ ($U$ $\mathcal{F}$-open in $\mathbb{C}^n$) is weakly $\mathcal{F}$-plurisubharmonic on $h^{-1}(\Omega)$ ($\subset \mathbb{C}^n$).

**Proof.** For $n = 1$ this is contained in [F3, Corollaire, p. 63] (in which $h$ is any finely holomorphic function on $U$). Suppose therefore that $n \geq 2$. We may assume that $\Omega$ is $\mathcal{F}$-connected and that $f \not\equiv -\infty$, and so $f$ is in particular $\mathbb{R}^{2n}$-finely subharmonic. As shown in the beginning of the alternative proof of Theorem 3.1 given at the end of Section 3 we may further suppose that $\Omega$ is bounded in $\mathbb{C}^n$ and that $f$ is bounded and of class $\hat{D}^1(D)$ for some bounded domain $D \subset \mathbb{C}^n$ containing $\Omega$. Fix $z \in U$ and let $K_z$ be a compact $\mathcal{F}$-neighborhood of $z$ in $U$ on which $h$ has the properties described in Definition 4.3. It will be sufficient to see that the expression (4.3) is non-positive if $\varphi \in \hat{D}^1_+(D, O)$ for some $\mathcal{F}$-open set $O \subset D$ with $z \in O \subset K_z$. Notice that $df \land d^c \varphi$ is a form with $L^1$ coefficients that
is supported on $K_z$. Thus let $(h_m)$ be a sequence of bi-holomorphic maps on open sets containing $K_z$ that converge in $C^1$ to $h$ on $K_z$. Then
\[
\lim_{m \to \infty} dd^c|h_{m,2}^{-1}|^2 \wedge \ldots \wedge dd^c|h_{m,n}^{-1}|^2 = dd^c|h_2^{-1}|^2 \wedge \ldots \wedge dd^c|h_n^{-1}|^2,
\]
uniformly on $K_z$. Now the expression (4.4) is non-positive when we replace $h$ by $h_m$. By Lebesgue’s dominated convergence theorem we conclude that (4.4) is also non-positive for $h$ a strongly $\mathcal{F}$-biholomorphic map. \(\square\)

It is reasonable to expect that the concept of weakly $\mathcal{F}$-plurisubharmonic function is invariant even under composition with suitable weakly $\mathcal{F}$-biholomorphic mappings. Currently, we do not know of a fine inverse function theorem for weakly $\mathcal{F}$-holomorphic maps of several variables. In fact we don’t even know if weakly $\mathcal{F}$-holomorphic functions have weakly $\mathcal{F}$-holomorphic partial derivatives. However, we can handle the special case of a map of the form
\[
G(z) = g(z_1) + (0, z_2, \ldots, z_n),
\]
where $g = (g_1, \ldots, g_n)$ is a finely holomorphic curve in $\mathbb{C}^n$, and that turns out to be sufficient.

**Theorem 4.5.** The composition $f \circ g$ of a weakly $\mathcal{F}$-plurisubharmonic (resp. weakly $\mathcal{F}$-holomorphic) function $f$ on an $\mathcal{F}$-open set $\Omega \subset \mathbb{C}^n$ with a finely holomorphic curve $g$ in $\mathbb{C}^n$ defined on a finely open set $U \subset \mathbb{C}$, is finely hypoharmonic (resp. finely holomorphic) on the finely open pre-image $g^{-1}(\Omega) \subset \mathbb{C}$.

**Proof.** The theorem is known for $n = 1$, cf. [F3, §4 and Théorème 13(a)], so we suppose that $n \geq 2$. According to Theorem 2.10(a), $g$ is continuous from $U$ with the fine topology to $\mathbb{C}^n$ with the plurifine topology. The pre-image $g^{-1}(\Omega)$ is therefore finely open in $\mathbb{C}$, and $f \circ g$ is finely upper semicontinuous (even finely continuous). We may of course assume that $f$ is bounded, that $U$ is finely connected, and that $g$ is non-constant, for example that $g_1$ is non-constant, hence a fine-to-fine open map, [F3, p. 64].

Given a point $z_0 \in g^{-1}(\Omega) \subset U \subset \mathbb{C}$ with $g_1'(z_0) \neq 0$, cf. [F3, Corollaire 11], there exists a finely open set $O \subset \mathbb{C}$ such that $z_0 \in O \subset g^{-1}(\Omega)$ and that $g_1$ is injective on $O$ with $g_1'(z) \neq 0$ for every $z \in O$, hence $(g_1|_O)^{-1}$ is finely holomorphic on the finely open set $g_1(O)$, [F3, Théorème 13]. We may further assume after diminishing $O$ that there exists a $C^\infty$-map $\varphi : \mathbb{C} \to \mathbb{C}^n$ such that $\varphi = g$ on $O$ and hence $\partial \varphi = g'$, $\bar{\partial} \varphi = 0$ on $O$, [F3, Théorème 11(c)]. Since $\partial \varphi_1(z_0) = g_1'(z_0) \neq 0$ we may arrange (by further diminishing $O$) that $\varphi_1$ is injective on some open set $\omega \subset \mathbb{C}$ containing the closure of $O$ in $\mathbb{C}$, and hence that $\varphi_1|_{\omega}$ is a $C^\infty$-diffeomorphism of $\omega$ onto $\varphi_1(\omega)$. Likewise, we may achieve that there exists a sequence of curves $\varphi^{(\nu)}$ such that each coordinate $\varphi_j^{(\nu)}$ of $\varphi^{(\nu)}$, $j \in \{1, \ldots, n\}$, is a rational function defined on some open set $\Omega^{(\nu)} \subset \omega$ containing $O$, and that $\varphi^{(\nu)} \to \varphi$, $(\varphi^{(\nu)})' \to \varphi'$ (\(= \partial \varphi\)), uniformly on $O$ as $\nu \to \infty$, cf. [F3, Theorem 11(a)]. With this final choice of $O$ define $G : O \times \mathbb{C}^{n-1} \to \mathbb{C}^n$ by (4.5), writing now $t \in \mathbb{C}^n$ in place of $z \in \mathbb{C}^n$. 

For a given point \( z = (z_1, z') \in O \times \mathbb{C}^{n-1} \) choose a compact fine neighborhood \( L_{z_1} \) of \( z_1 \) in \( \mathbb{C} \) so that \( L_{z_1} \subset O \subset \omega \), and a number \( c > \max\{|z_2|, \ldots, |z_n|\} \). In analogy with (4.5) define

\[
\Phi_z(t) = \varphi(t_1) + (0, t_2, \ldots, t_n) \quad \text{for } t \in \omega \times \mathbb{C}^{n-1}.
\]

It is easily verified that \( \Phi_z \) is a \( C^\infty \)-diffeomorphism of \( \omega \times \mathbb{C}^{n-1} \) onto its image in \( \mathbb{C}^n \). We have \( \Phi_z = G \) on \( O \times \mathbb{C}^{n-1} \), and in particular on the compact \( \mathcal{F} \)-neighborhood \( K_z := L_{z_1} \times [-c, c]^{n-1} \) of \( z \in O \times \mathbb{C}^{n-1} \), and consequently \( G : O \times \mathbb{C}^{n-1} \to \mathbb{C}^n \) is a \( \mathcal{F} \)-biholomorphic map.

Suppose first that \( f \) is weakly \( \mathcal{F} \)-plurisubharmonic on \( \Omega \). According to Proposition 4.4, \( f \circ G \) is a weakly \( \mathcal{F} \)-plurisubharmonic map defined on the \( \mathcal{F} \)-open set \( G^{-1}(\Omega) \subset O \times \mathbb{C}^{n-1} \). Now

\[
G(t_1, 0, \ldots, 0) = g(t_1) = (g_1(t_1), g_2(t_1), \ldots, g_n(t_1)) \quad \text{for } t_1 \in O,
\]

and hence \( f \circ G(t_1, 0, \ldots, 0) = (f \circ g)(t_1) \) for \( t_1 \in O \). It follows that \( f \circ g \) is finely hypoharmonic on \( O \), hence so (by varying \( z_0 \)) on \( \{ t \in g^{-1}(\Omega) : g'_1(t) \neq 0 \} \), which differs only by a countable and hence polar set from \( g^{-1}(\Omega) \), cf. [F3, Théorème 15]. Because \( f \circ g \) is finely continuous and \( < +\infty \) on \( g^{-1}(\Omega) \subset U \) we conclude by the removable singularity theorem [F1, Theorem 9.15] that indeed \( f \circ g \) is finely hypoharmonic on \( g^{-1}(\Omega) \).

Finally, let instead \( f \) be weakly \( \mathcal{F} \)-holomorphic on \( \Omega \), in particular weakly (complex) \( \mathcal{F} \)-harmonic, by [F3, Définition 3]. As shown in the alternative proof of the ‘only if part’ of Theorem 3.1 given at the end of Section 3 we may suppose that \( \Omega \) is contained in some bounded domain \( D \subset \mathbb{C}^n \) and that \( f \) is bounded and of class \( \tilde{D}^1(D) \). Each of the functions \( z \mapsto z_j f(z) \), \( j \in \{1, \ldots, n\} \), is therefore bounded and of class \( \tilde{D}^1(D) \).

By the former part of the theorem, the bounded functions \( \pm \text{Re}(f \circ g) \) and \( \pm \text{Im}(f \circ g) \) are finely subharmonic on \( g^{-1}(\Omega) \), and hence \( f \circ g \) is (complex) finely harmonic there. Similarly, \( (z_j f) \circ g = g_j \circ (f \circ g) \) is finely harmonic on \( g^{-1}(\Omega) \subset U \subset \mathbb{C} \), \( j \in \{1, \ldots, n\} \). We may therefore further assume that \( U \) is contained in a bounded domain \( D_1 \subset \mathbb{C} \) and that \( f \circ g \) and each \( g_j \) are bounded and of class \( \tilde{D}^1(D_1) \). Some component of \( g \), say \( g_1 \), is non-constant, and it therefore again follows from [F3, Théorème 13] that every point \( z_0 \in g^{-1}(\Omega) \subset U \) with \( g'_1(z_0) \neq 0 \) has a finely open fine neighborhood \( O \subset g^{-1}(\Omega) \) on which \( g_1 \) is injective with \( g'_1 \neq 0 \); and the inverse \( h_1 := (g_1|_O)^{-1} \) is finely holomorphic on the finely open set \( g_1(O) \). So is therefore the re-parametrized curve \( g \circ h_1 : g_1(O) \to \mathbb{C}^n \). The function \( h \) on \( g_1(O) \times \mathbb{C}^{n-1} \) defined by

\[
h(z) = h(z_1, \ldots, z_n) = h_1(z_1)
\]

is weakly \( \mathcal{F} \)-holomorphic. So is therefore \( hf \), and consequently \( (hf) \circ g \) is finely harmonic on \( O \), like \( f \circ g \) above. Note that \( h \circ g = h_1 \circ g_1 \). For \( t \in O \),

\[
[(hf) \circ g](t) = [(h \circ g)(t)][(f \circ g)(t)] = [(h_1 \circ g_1)(t)][(f \circ g)(t)] = t \cdot (f \circ g)(t),
\]
and since this function \((hf) \circ g = t \cdot (f \circ g)\) of \(t \in O\) is finely harmonic it follows according to the result of Lyons [Ly], cf. [F3, §3], which was utilized in Lemma 2.8, that \(f \circ g\) is finely holomorphic on \(O\), hence (by varying \(z_0\)) quasi-everywhere on \(\{t \in g^{-1}(\Omega) : g'(t) \neq 0\}\), and indeed everywhere on \(g^{-1}(\Omega)\) by the removable singularity theorem for finely holomorphic functions, [F3, Corollaire 3].

The following extension of Theorem 4.5 from finely holomorphic curves \(g\) to \(\mathcal{F}\)-holomorphic maps \(h\) is a strengthening of Theorem 2.10(b),(c), in which \(f\) was required to be strongly \(\mathcal{F}\)-plurisubharmonic (resp. strongly \(\mathcal{F}\)-holomorphic). Likewise, Theorem 4.6 (for a weakly \(\mathcal{F}\)-plurisubharmonic function \(f\)) extends Proposition 4.4 (in which \(m = n\), and \(h\) is strongly biholomorphic).

**Theorem 4.6.** The composition \(f \circ h\) of a weakly \(\mathcal{F}\)-plurisubharmonic (resp. weakly \(\mathcal{F}\)-holomorphic) function \(f\) on an \(\mathcal{F}\)-open set \(\Omega \subset \mathbb{C}^n\) with a weakly \(\mathcal{F}\)-holomorphic map \(h : U \to \mathbb{C}^n\) (\(U\ \mathcal{F}\)-open in \(\mathbb{C}^m\)) is weakly \(\mathcal{F}\)-plurisubharmonic (resp. weakly \(\mathcal{F}\)-holomorphic) on \(h^{-1}(\Omega)\) (\(\subset \mathbb{C}^m\)).

**Proof.** According to Theorem 2.10(a), \(h\) is continuous from \(U \subset \mathbb{C}^m\) to \(\mathbb{C}^n\) with their respective plurifine topologies. It follows that \(h^{-1}(\Omega)\) is \(\mathcal{F}\)-open in \(\mathbb{C}^m\), and that \(f \circ h\) is \(\mathcal{F}\)-upper semicontinuous. Next, we restrict \(f \circ h\) to a complex line \(L\) in \(\mathbb{C}^m\), and observe that \(h|_{L \cap U}\) is a finely holomorphic curve. By Theorem 4.5, \(f \circ h\) restricted to \(L\) therefore is finely hypoharmonic (resp. finely holomorphic), and we are done. \(\square\)

**References**

[BA] Bliedtner, J., and W. Hansen, *Simplicial cones in potential theory, II (approximation theorems)*, Inventiones Math. 46 (1978), 255–275.

[BT] Bedford, E., and B.A. Taylor, *A new capacity for plurisubharmonic functions*, Acta Math. 149 (1982), 1–40.

[BT] Bedford, E., and B.A. Taylor, *Fine topology, \(\mathcal{S}\)-Silov boundary, and \((dd^c)^n\)*, J. Functional Anal. 72 (1987), 225–251.

[BT] Bedford, E., and B.A. Taylor, *Plurisubharmonic functions with logarithmic singularities*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 4, 133–171.

[DL] Deny, J., et J. L. Lions, *Les espaces de type de Beppo Levi*, Ann. Inst. Fourier (Grenoble) 5 (1953-54), 305–370.

[D] Doob, J.L., *Classical Potential Theory and Its Probabilistic Counterpart*, Grundlehren, vol. 262, Springer, Berlin, 1984.

[EEW] Edigarian, A. El Marzguioui, S., and J. Wiegerinck, *The image of a finely holomorphic map is pluripolar*, Ann. Polon. Math. 97 (2010), no. 2, 137–149.

[EL] El Kadiri, M., *Fonctions finement plurisousharmonique et topologie fine*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.(5) 27 (2003), 77-88.

[EW1] El Marzguioui, S., and J. Wiegerinck, *The pluri-fine topology is locally connected*, Potential Anal. 245 (2006), 283–288.

[EW2] El Marzguioui, S., and J. Wiegerinck, *Continuity properties of finely plurisubharmonic functions and pluripolarity*, To appear in Indiana Univ. Math. J. http://arxiv.org/abs/0906.2081.

[F1] Fuglede, B., *Finely Harmonic Functions*, Lecture Notes in Math., No. 289, Springer, Berlin, 1972.
[F2] ———, *Fonctions harmoniques et fonctions finement harmoniques*, Ann. Inst. Fourier (Grenoble) **24** (1974), no. 4, 77–91.

[F3] ———, *Sur les fonctions finement holomorphes*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 4, 57–88.

[F4] ———, *Fonctions BLD et fonctions finement surharmoniques*, Séminaire de Théorie du Potentiel No. 6, Lecture Notes in Mathematics No. 906, Springer, Berlin, 1982.

[F5] ———, *Localization in fine potential theory and uniform approximation by subharmonic functions*, J. Functional Anal. **49** (1982), 57–72.

[F6] ———, *Fonctions finement holomorphes de plusieurs variables – un essai*, Séminaire d’Analyse P. Lelong–P. Dolbeault–H. Skoda, 1983/85, pp. 133–145, Lecture Notes in Math. 1198, Springer, Berlin, 1986.

[F7] ———, *Finely holomorphic functions. A survey*, Revue Roumaine Math. Pures **33** (1988), 283–295.

[F8] ———, *Harmonic morphisms applied to classical potential theory*, Nagoya Math. J. (to appear).

[J] Josefson, B., *On the equivalence between locally polar and globally polar sets for plurisubharmonic functions on \( \mathbb{C}^n \)*, Arkiv Mat. **16** (1978), 109–115.

[K] Klimek, M., *Pluripotential Theory*, Clarendon Press, Oxford, 1991.

[La] Landkof, N.S., *Foundations of Modern Potential Theory*, Grundlehren, vol. 180, Springer, Berlin, 1972.

[Le1] Lelong, P., *Les fonctions plurisousharmoniques*, Ann. Sci. École Norm. Sup. **62** (1945), 301–328.

[Le2] ———, *Fonctions plurisousharmoniques et formes différentielles positives*, Gordon & Breach, New York, 1968.

[Ly] Lyons, T. J., *Finely holomorphic functions*, J. Functional Anal. **37** (1980), 1–18.

**Faculté des Sciences**
**Université Mohamed V**
B.P. 726, Salé-Tabriquet
11010 Salé
Morocco
elkadiri@fsr.ac.ma

**Institute for Mathematical Sciences**
**University of Copenhagen**
Universitetsparken 5
2100 Copenhagen
Denmark
fuglede@math.ku.dk

**KdV Institute for Mathematics**
**University of Amsterdam**
Science Park 904
P.O. box 94248, 1090 GE Amsterdam
The Netherlands
j.j.o.o.wiegerinck@uva.nl