Longitudinal Doppler effect in de Sitter relativity

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Abstract

A new formula of the longitudinal Doppler effect in de Sitter expanding universe is derived combining the cosmological contribution with that of the relative geodesic motion of the source with respect to a fixed observer.

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1 Introduction

In general relativity the Doppler effect can be produced by the expansion of the universe or by the relative motion as in special relativity. The cosmological effect complies with the general redshift law [1–3] while the kinetic one is treated separately by using the methods of special relativity [4]. Thus possible interferences of these two effects cannot be pointed out without resorting to a more general theory of relativistic effects in the presence of gravity as our de Sitter relativity we proposed recently [5, 6]. This gives us the opportunity of analysing the Doppler effect globally considering simultaneously the cosmological and kinetic contributions in the de Sitter expanding universe.

In de Sitter relativity the role of inertial frames is played by any set of local charts related through isometries. In what follows we focus mainly on the comoving charts with conformal coordinates where the Maxwell equations have similar solutions as in Minkowski space-time allowing us to define correctly the photon energy and momentum [7]. Furthermore, we exploit the transformation rules of the conserved quantities for finding the observed
energies in different frames for deriving the Doppler effect in de Sitter rel-

Note that in the de Sitter manifold the energy and momentum transform as the components of a five-dimensional skew-symmetric tensor in association with the angular momentum and a new specific conserved vector we called adjoint momentum \[ \mathbf{A} \mathbf{E} \]. For this reason the formalism is different from the usual one of special relativity but the philosophy of the relative motion remains the same.

Since here we make the first step to this approach we restrict ourselves to the longitudinal Doppler effect in which the source carried by the mobile frame is translated with the distance \( d \) from its origin only along the direction of the relative velocity. Our goal is to analyse how a photon emitted at the initial time by this source is observed by a fixed observer when we know that at that time the origin of the mobile frame is passing through the origin of the fixed one with the relative velocity \( \mathbf{V} \). Thus we formulate a problem of relative motion in the presence of the de Sitter expansion able to reveal the interference between the cosmological and kinetic contributions to the Doppler effect. Applying this method we obtain a new formula of this effect such that in the particular case of \( \mathbf{V} = 0 \) we recover the cosmological effect given by Lemaître’s law \( [9, 10] \) while for \( d = 0 \) we find just the well-known formula of the Doppler effect in special relativity.

The paper is organized as follows. In the second section we briefly present the de Sitter relativity in conformal and respectively de Sitter-Painlevé local charts showing how the coordinates and conserved quantities transform under isometries. The next section is devoted to the longitudinal Doppler effect for which we derive the transformation rules of the conserved quantities leading to the final formula relating the emitted and observed frequencies. Moreover, we analyse the mentioned particular cases in which we recover the well-known results of general and special relativity. Finally we present few concluding remarks.

2  de Sitter relativity

Let us start with the de Sitter spacetime \((M, g)\) defined as the hyperboloid of radius \(1/\omega\) in the five-dimensional flat spacetime \((M^5, \eta^5)\) of coordinates \(z^A\) (labeled by the indices \(A, B, ... = 0, 1, 2, 3, 4\)) having the metric \(\eta^5 = \text{diag}(1, -1, -1, -1, -1)\). The local charts \(\{x\}\) of coordinates \(x^\mu\) \((\alpha, \mu, \nu, ... = 0, 1, 2, 3)\) can be introduced on \((M, g)\) giving the set of functions \(z^A(x)\) which
solve the hyperboloid equation,

\[ \eta_{AB} z^A(x) z^B(x) = -\frac{1}{\omega^2}. \]  

where \( \omega \) denotes the Hubble de Sitter constant since in our notations [8].

The de Sitter isometry group is just the gauge group \( SO(1,4) \) of the embedding manifold \( (M^5, \eta^5) \) that leave invariant its metric and implicitly Eq. (1). Therefore, given a system of coordinates defined by the functions \( z = z(x) \), each transformation \( g \in SO(1,4) \) defines the isometry \( x \to x' = \phi_g(x) \) derived from the system of equations \( z[\phi_g(x)] = gz(x) \). For studying the effects of these isometries we consider that the sets of local charts related through isometries play the role of systems of inertial frames similar to those of special relativity.

In what follows we consider the comoving charts with two sets of local coordinates, the \textit{conformal} pseudo-Euclidean ones, \( \{t_c, \vec{x}_c\} \), and the 'physical' de Sitter-Painlevé coordinates, \( \{t, \vec{x}\} \). The conformal time \( t_c \) and Cartesian spaces coordinates \( x^i_c \) \((i, j, k, \ldots = 1, 2, 3)\) are defined by the functions

\[
\begin{aligned}
z^0(x_c) &= -\frac{1}{2\omega^2 t_c} \left[ 1 - \omega^2(t_c^2 - \vec{x}_c^2) \right], \\
z^i(x_c) &= -\frac{1}{\omega t_c} x^i_c, \\
z^4(x_c) &= -\frac{1}{2\omega^2 t_c} \left[ 1 + \omega^2(t_c^2 - \vec{x}_c^2) \right],
\end{aligned}
\]  

written with the vector notation, \( \vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3 \subset M^5 \). These charts cover the expanding part of \( M \) for \( t_c \in (-\infty, 0) \) and \( \vec{x}_c \in \mathbb{R}^3 \) while the collapsing part is covered by similar charts with \( t_c > 0 \). In both these cases we have the same conformal flat line element,

\[
ds^2 = \eta_{AB} dz^A(x_c) dz^B(x_c) = \frac{1}{\omega^2 t_c^2} \left( dt_c^2 - d\vec{x}_c \cdot d\vec{x}_c \right). \tag{3}
\]

In what follows we restrict ourselves to the expanding portion which is a plausible model of our expanding universe.

The de Sitter-Painlevé coordinates \( \{t, \vec{x}\} \) on the expanding portion can be introduced directly by substituting

\[
t_c = -\frac{1}{\omega} e^{-\omega t}, \quad \vec{x}_c = \vec{x} e^{-\omega t}, \tag{4}
\]

where \( t \in (-\infty, \infty) \) is the proper or cosmic time while \( x^i \) are the 'physical' Cartesian space coordinates. Then the line element reads

\[
ds^2 = (1 - \omega^2 \vec{x}^2) dt^2 + 2\omega \vec{x} \cdot d\vec{x} dt - d\vec{x} \cdot d\vec{x}. \tag{5}
\]
Notice that this chart is useful in applications since in the flat limit (when $\omega \to 0$) its coordinates become just the Cartesian ones of the Minkowski space-time.

In the charts with combined coordinates $\{t, \vec{x}_c\}$ the metric takes the Friedman-Lemaître-Robertson-Walker (FLRW) form

$$ds^2 = dt^2 - a(t)^2 d\vec{x}_c \cdot d\vec{x}_c, \quad a(t) = e^{\omega t},$$

where $a(t)$ is the scale factor of the expanding portion which can be rewritten in the conformal chart,

$$a(t_c) \equiv a[t(t_c)] = -\frac{1}{\omega t_c},$$

as a function defined for $t_c < 0$.

The classical conserved quantities under de Sitter isometries can be calculated with the help of the Killing vectors $k_{(AB)}$ of the de Sitter manifold $(M, g)$ \cite{8}. According to the general definition of the Killing vectors in the pseudo-Euclidean spacetime $(M^5, \eta^5)$, we may consider the following identity

$$K_{C}^{(AB)}dz^C = z^{A}dz^{B} - z^{B}dz^{A} = k_{(AB)}^{(CD)}dx^{\mu},$$

giving the covariant components of the Killing vectors in an arbitrary chart $\{x\}$ of $(M, g)$ as

$$k_{(AB)}^{(CD)} = \eta_{AC}^{\mu} \eta_{BD}^{\nu} k_{\mu}^{(CD)} = z_{A}^{\mu} z_{B}^{\nu} - z_{B}^{\mu} z_{A}^{\nu},$$

where $z_{A} = \eta_{AB} z^{B}$. The principal conserved quantities along the timelike geodesics have the general form $K_{(AB)}(x, \vec{P}) = \omega k_{(AB)}^{\nu} m u^{\nu}$ where $u^{\nu} = \frac{dx^{\nu}(s)}{ds}$ are the components of the covariant four-velocity that satisfy $u^{\mu} u_{\mu} = 1$.

We have shown that in a conformal chart $\{t_c, \vec{x}_c\}$ the geodesic equation of a particle of mass $m$ passing through the space point $\vec{x}_c(t_c)$ at time $t_c$ is completely determined by the initial condition $\vec{x}_c(t_c) = \vec{x}_c(t_c)$ and the conserved momentum $\vec{P}$ as $\left[8, 11\right]$, \cite{10, 11},

$$x^{i}_{c}(t_c) = x^{i}_{c} + \frac{P^{i}}{\omega P^2} \times \left( \sqrt{m^2 + P^2 \omega^2 t_c^2} - \sqrt{m^2 + P^2 \omega^2 t_c^2} \right),$$

where we denote $P = |\vec{P}|$. Moreover, the other conserved quantities in an arbitrary point $(t_c, \vec{x}_c(t_c))$ of the geodesics depend only on this point and
the momentum $\vec{P}$. These are the energy $E$, angular momentum $\vec{L}$ and a specific vector $\vec{Q}$ we called the adjoint momentum. In the chart $\{t_c, \vec{x}_c\}$ these quantities have the form

$$E = \omega \vec{x}_c(t_c) \cdot \vec{P} + \sqrt{m^2 + P^2 \omega^2 t_c^2},$$  

$$L_i = \varepsilon_{ijk} x_j^i(t_c) P_k,$$  

$$Q^i = 2\omega x^i_c(t_c) E + \omega^2 P^i [\eta^2_c(t_c)^2 - \vec{x}_c(t_c)^2],$$

satisfying the obvious identity

$$E^2 - \omega^2 L^2 - \vec{P} \cdot \vec{Q} = m^2$$

corresponding to the first Casimir operator of the $so(1,4)$ algebra. In the flat limit, $\omega \to 0$ and $-\omega t_c \to 1$, we have $\vec{Q} \to \vec{P}$ such that this identity becomes just the usual mass-shell condition $p^2 = m^2$ of special relativity.

The conserved quantities $E, \vec{P}$ and the new ones,

$$K^i = -\frac{1}{2\omega} \left( \vec{P} - \vec{Q} \right), \quad \vec{R} = -\frac{1}{2\omega} \left( \vec{P} + \vec{Q} \right),$$

form a skew-symmetric tensor on $M^5$,

$$\mathcal{K}(x, \vec{P}) = \begin{pmatrix} 0 & \omega K_1 & \omega K_2 & \omega K_3 & E \\ -\omega K_1 & 0 & \omega L_3 & -\omega L_2 & \omega R_1 \\ -\omega K_2 & -\omega L_3 & 0 & \omega L_1 & \omega R_2 \\ -\omega K_3 & \omega L_2 & -\omega L_1 & 0 & \omega R_3 \\ -E & \omega R_1 & -\omega R_2 & -\omega R_3 & 0 \end{pmatrix},$$

whose elements transform under an isometry $x \to x' = \phi_g(x_c)$ as

$$\mathcal{K}_{(AB)}(x', \vec{P}') = g^C_A \cdot g^D_B \cdot \mathcal{K}_{(CD)}(x_c, \vec{P}),$$

for all $g \in SO(1,4)$. Here $g^A_B = \eta^5_{AC} \cdot g^C_D \cdot \eta^5_{BD}$ are the matrix elements of the adjoint matrix $g = \eta^5 g^5$. Thus, Eq. (17) can be written as $\mathcal{K}(x', \vec{P}') = g \mathcal{K}(x, \vec{P}) g^T$ or simpler, $\mathcal{K}' = g \mathcal{K} g^T$. As mentioned the coordinate transformation under this isometry can be obtained by solving the system

$$z[\phi_g(x_c)] = g z(x_c).$$

We have thus a specific relativity on the de Sitter space-time allowing us to study different relativistic effects in the presence of the de Sitter gravity.
3 Longitudinal Doppler effect

Let us consider two observers, the first one staying at rest in the origin $O$ of the frame \{\(t_c, \vec{x}_c\}\) and the second one staying in the origin $O'$ of a mobile frame \{\(t'_c, \vec{x}'_c\}\) moving along the $x^1_c$ axis. We adopt the synchronization condition assuming that $O'$ passes through $O$ with the velocity $\vec{V} = (V, 0, 0)$ at the initial moment

\[
t_{c0} = t'_{c0} = -\frac{1}{\omega} \rightarrow t_0 = t'_0 = 0.
\]  

(19)

Then, assuming that the observer $O$ measures the parameters \((t_c, \vec{x}_c, \vec{P})\) while $O'$ observes other parameters, \((t'_c, \vec{x}'_c, \vec{P}')\), of the same particle, we may apply our general results finding that the isometry relating the coordinates of these charts, \(x_c = \phi_{g(\vec{V})}(t'_c)\), as well as the transformation rule of the conserved quantities [5]

\[
K(t_c, \vec{x}_c, \vec{P}) = g(\vec{V}) K(t'_c, \vec{x}'_c, \vec{P}') g(\vec{V})^T ,
\]  

(20)

are determined by the Lorentz transformation

\[
g(\vec{V}) = \begin{pmatrix}
\frac{1}{\sqrt{1-V^2}} & \frac{V}{\sqrt{1-V^2}} & 0 & 0 & 0 \\
\frac{V}{\sqrt{1-V^2}} & \frac{1}{\sqrt{1-V^2}} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]  

(21)

of the $SO(1, 4)$ group.

In special relativity an electromagnetic source carried by a mobile frame produces the same Doppler effect regardless its fixed position with respect of this frame since in the Minkowski space-time the energy is independent on translations. In contrast, in de Sitter relativity the energy depends on position as in Eq. (11) which means that the Doppler effect will depend on the position of the source with respect to the mobile frame.

In order to avoid complicate calculations here we restrict ourselves to the particular case of the longitudinal Doppler effect when the source is translated only along the direction of the velocity of the mobile frame \{\(t'_c, \vec{x}'_c\}\), staying at rest in the space point \((d, 0, 0)\) of this frame. The observer $O'$ and implicitly $O$ will receive signals from this source only if this remains inside the null cone in $t_{c0} = t'_{c0} = -\omega^{-1}$ such that the condition $\omega d < 1$ is mandatory [11].
We must specify that in the associated frame with de Sitter-Painlevé coordinates \( \{ t', \vec{x}' \} \) defined by Eq. (4) the source is moving because of the manifold expansion, having the coordinates \((t', d(t'), 0, 0)\) where \(d(t') = \de \omega t'\). Bering in mind that we fixed the initial moment when \(O = O'\) as in Eq. (19) we understand that \(d\) is just the physical distance of the source moving with respect to the mobile frame with the velocity

\[
v = \left. \frac{dd(t')}{dt'} \right|_{t'=0} = \omega d ,
\]

at the initial time \(t_0 = t'_0 = 0\). Thus we recover the well-known velocity-distance law (in mobile frame) that occurs naturally in the de Sitter expanding universe as in any other FLRW space-time [3].

In the conformal charts the electromagnetic potential has the same form as in the Minkowski space-time since the Maxwell equations are invariant under conformal transformations [7]. This means that a regressive plane electromagnetic wave, that has to be observed successively by \(O'\) and \(O\), may have the momentum \(\vec{k} = (-k, 0, 0)\) and energy \(k\) being proportional with

\[
A_i = \varepsilon_i e^{-ikt} - ikt\varepsilon_i
\]

where \(\varepsilon_i\) are the components of an arbitrary polarisation vector. Assuming that a photon of this type is emitted at the initial time (19) we find that, according to Eq. (11), the observer \(O'\) measures the photon energy

\[
E' = k(1 - \omega d) ,
\]

and momentum \(\vec{P}' = \vec{k}\). Moreover, he observes that \(\vec{L}' = 0\) while \(Q'^1 = -k(1 - \omega d)^2\) and \(Q'^2 = Q'^3 = 0\) as it results from Eqs. (12) and (13). Then the condition (14) is fulfilled since in this case \(m = 0\).

The conserved quantities measured by the fixed observer \(O\) can be deduced from Eq. (20). After a little calculation we obtain

\[
E = k \left[ \sqrt{1 - V} \frac{1 - \omega d}{1 + V} (1 - \omega d) - \frac{\omega^2 d^2}{2} \frac{V}{\sqrt{1 - V^2}} \right] ,
\]

\[
P^1 = -k \left[ \omega d + \sqrt{\frac{1 - V}{1 + V}} (1 - \omega d) + \frac{\omega^2 d^2}{2} \left( \frac{1}{\sqrt{1 - V^2}} - 1 \right) \right] ,
\]

\[
Q^1 = -k \left[ -\omega d + \sqrt{\frac{1 - V}{1 + V}} (1 - \omega d) + \frac{\omega^2 d^2}{2} \left( \frac{1}{\sqrt{1 - V^2}} + 1 \right) \right] .
\]
while } P^2 = P^3 = Q^2 = Q^3 = 0 \text{ and } \vec{L} = 0. \text{ These components satisfy the condition } (20) \text{ for } m = 0. 

Hereby we may derive the formula of the longitudinal Doppler effect in de Sitter relativity. Denoting with } \nu_0 \text{ the frequency of the emitted photon and by } \nu \text{ that measured by the observer } O \text{ we can rewrite Eq. (25) as }

\nu = \nu_0 \sqrt{1 - \frac{V}{1 + V}} \left(1 - \omega d - \frac{\omega^2 d^2}{2} \frac{V}{1 - V}\right), \tag{28}

pointing out how the observed frequency and implicitly the redshift } z = \frac{\nu_0}{\nu} - 1 \text{ are affected simultaneously by the cosmological and kinetic effects. The parameters of this equation satisfy the natural conditions } \omega d < 1 \text{ and } V < 1 \text{ but which are not enough for assuring positive frequencies, } \nu > 0, \text{ such that we must impose the supplemental restriction}

V < \frac{2(1 - \omega d)}{1 + (1 - \omega d)^2}, \tag{29}

which shows that for very remote sources, with } \omega d \sim 1, \text{ the kinetic effects are inhibited.

Two particular cases help us to study separately the cosmological and kinetic contributions. The first one is when we eliminate the kinetic effects setting } V = 0. \text{ Then we find that the photon emitted at } t_{c0} = t'_{c0} = \omega^{-1} \text{ is observed at the time } t_{cobs} = -\frac{1}{\omega} + d \text{ such that Lemaître’s law of general relativity } [9, 10],

1 + z = \frac{\nu_0}{\nu} = \frac{a(t_{cobs})}{a(t_{c0})}, \tag{30}

is fulfilled since Eq. (28) gives } \nu = \nu_0(1 - \omega d) \text{ while the scale factors defined by Eq. (7) read}

a(t_{c0}) = -(\omega t_{c0})^{-1} = 1, \tag{31}
a(t_{cobs}) = -(\omega t_{cobs})^{-1} = (1 - \omega d)^{-1}. \tag{32}

Hereby we deduce

z = \frac{1}{1 - \omega d} - 1 \simeq \omega d = v, \tag{33}

which is just the Hubble law [12] for small values of } z [3]. Note that in this case the observers } O \text{ and } O' \text{ measures the same conserved quantities mentioned above, } \vec{P} = \vec{P}' \text{ and } \vec{Q} = \vec{Q}'.

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Another particular case is when the source stays at rest in the origin of the mobile frame, with \( d = 0 \). Then we recover the genuine Doppler effect of special relativity with,

\[
\nu = \nu_0 \sqrt{1 - \frac{V}{1 + V}}, \tag{34}
\]

and \( \vec{P} = \vec{Q} \) such that Eq. (14) becomes just the dispersion relation of a photon of energy \( h\nu \) (in SI units). This is because Eq. (28) depends only on the product \( \omega d \) such that the limit \( d \to 0 \) is equivalent to the flat limit \( \omega \to 0 \).

### 4 Concluding remarks

We derived the formulas of the longitudinal Doppler effect in the de Sitter relativity restricting ourselves to the expanding portion of this manifold which is a reasonable model of our expanding universe. The result combines cosmological and kinetic terms such that we can recover the particular cases of Lemaître’s law or the Doppler effect of special relativity.

All these results are obtained in the conformal coordinates which are not convenient for astronomical interpretations as long as the measurements are performed in the de Sitter-Painlevé coordinates. This is not an impediment since all the results presented here depend exclusively on the conserved quantities which are independent on the coordinates we use. However, for a refined analysis focusing on the geodesic trajectories of the source and emitted photons we may substitute at any time the conformal coordinates according to Eqs. (4), respecting the synchronisation and initial conditions according to the general theory of the Lorentzian isometries in de Sitter relativity [5, 6].

Finally we note that the present results are only the first steps in developing the theory of the Doppler effect in the de Sitter expanding universe since the longitudinal effect is relatively simple. The challenge is to study the general case when the source has an arbitrary position generating transverse effects.

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