Self-similarity in turbulence and its applications

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Abstract

First, we discuss the non-Gaussian type of self-similar solutions to the Navier-Stokes equations. We revisit a class of self-similar solutions which was studied in Canonne-Planchon (1996). In order to shed some light on it, we study self-similar solutions to the 1D Burgers equation in detail, completing the most general form of similarity profiles that it can possibly possess. In particular, on top of the well-known source-type solution we identify a kink-type solution. It is represented by one of the confluent hypergeometric functions, viz. Kummer’s function $M$.

For the 2D Navier-Stokes equations, on top of the celebrated Burgers vortex we derive yet another solution to the associated Fokker-Planck equation. This can be regarded as a 'conjugate' to the Burgers vortex, just like the kink-type solution above. Some asymptotic properties of this kind of solution have been worked out. Implications for the 3D Navier-Stokes equations are suggested.

Second, we address an application of self-similar solutions to explore more general kind of solutions. In particular, based on the source-type self-similar solution to the 3D Navier-Stokes equations, we consider what we could tell about more general solutions.

Burgers equation, Navier-Stokes equations, self-similarity

1 Introduction

Self-similarity is an useful concept in handling partial differential equations particularly arising from fluid mechanics. Our motivation for this study is as follows.

An initial-boundary-value problem for the 3D Navier-Stokes equations is studied in [1] postulating that the velocity field has a self-similar form, from the very initial data to the final state. To this end they introduced Besov spaces to accommodate a singular initial velocity field which is as rough as $u(x) \propto 1/x$. The knock-on effect is that we would have similarity profiles, viz. steady solutions to the scaled Navier-Stokes equations, which are not well-localised in space. In order to shed some light on its implications, we will take
up a simpler system, the 1D Burgers equation to consider solutions in such an enlarged function space. In particular, we will be concerned with the fate of solutions which are not in $L^1$ marginally, but in some Besov space nearby.

There are two schematic themes in this paper. One is the study of non-Gaussian type solutions (loosely called 'kink-type' solutions) of scaled version of fluid dynamical equations. The other one is proposal of a new approach of constructing general solutions on the basis of particular source-type self-similar solutions.

For the first theme, non-Gaussian solutions are given explicitly with some of their properties discussed, whereas clarification of their possible significance as a dynamical system is left for further study. For the second theme a protocol for (re)building more general solutions out of self-similar solutions is exemplified using the Burgers and 2D Navier-Stokes equations. This suggests that applications to the 3D Navier-Stokes equations deserve further investigation.

This paper is organised as follows. We study self-similar profiles for the Burgers equation in Section 2, emphasising the properties of the newly identified kink-type solutions. We consider self-similar solutions to the Navier-Stokes equations, especially in two dimensions in Section 3. We address possible applications of those similar solutions, in particular about obtaining information regarding more general class of solutions in Section 4. Section 5 is devoted to summary and outlook.

2 Self-similar solutions of the Burgers equation

When it is demanded that the initial data themselves are self-similar, inevitably the initial velocity field goes singular like $u(x) \propto 1/x$. This has difficulties both at the origin and at infinity: (1) it is singular and non-integrable at the origin and (2) its decay at far distances is too slow to be integrable. For those reasons, Besov spaces were introduced to construct solutions to the initial-boundary problem in \[1\].

Assume the initial data is in a Besov space $B^0_{3,\infty}(\mathbb{R}^3)$ but not in $L^3(\mathbb{R}^3)$. The construction is concerned with a class of solutions for $t > 0$, which are in $B^0_{3,\infty}(\mathbb{R}^3)$ but not in $L^3(\mathbb{R}^3)$. The 'source-type' self-similar solutions are irrelevant here, because they are well-localised with a finite $L^3$-norm. The obvious question is: which functions can the scaled solutions possibly tend to, if they approach steady solutions at all?

To shed some light on solutions constructed in Besov spaces we consider a simpler problem of the 1D Burgers equation. For this purpose, instead of $B^0_{3,\infty}(\mathbb{R}^3)$ we consider velocity fields in $B^0_{1,\infty}(\mathbb{R}^1)$ to study an analogous problem in one spatial dimension. It is readily checked that the theory developed in \[1\] holds valid \textit{mutatis mutandis} for the 1D Burgers equation. For more recent

\footnote{They are associated with a norm defined by finite-differences; $\|u\|_{B^s_{p,q}} \equiv \left\{ \sum_{j=1}^{\infty} (2^{sj}) \|\Delta_j(u)\|_{L^p} \right\}^{1/q}$, where $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$ and $\Delta_j(u)$ represents the band-filtered velocity at frequency $2^j$.}
references on self-similar solutions, including studies with use of BMO-type spaces, see for example [2, 3, 4].

2.1 Cole-Hopf transform as Riccati substitution

We consider the Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2},$$

(1)

for an initial data $u(x,0) = u_0(x)$. It is well-known that (1) is linearisable by the so-called Cole-Hopf transform. We recall the basic results obtained in [5].

- If $\int_0^x u_0(y)dy = o(x^2)$ for large $|x|$, unique existence of solutions is guaranteed for all $t > 0$.
- If $\int_0^x u_0(y)dy = O(x^2)$ for large $|x|$, the existence of solutions is guaranteed only for finite time $0 \leq t < T$.

An instructive example for the second result is given by $u(x,t) = \frac{x}{t-\frac{1}{T}}$ [5].

To discuss forward self-similar solutions we introduce dynamic scaling transformations

$$\xi = \frac{x}{\lambda(t)}, \quad \tau = \frac{1}{2a} \log t,$$

$$u(x,t) = \frac{1}{\lambda(t)} U(\xi, \tau),$$

where $\lambda(t) = \sqrt{2at}$ denotes a scaling length and $a(>0)$ a zoom-in parameter. Applying them to (1) we find the scaled form of the Burgers equation

$$\frac{\partial U}{\partial \tau} + U \frac{\partial U}{\partial \xi} = a \frac{\partial}{\partial \xi} (\xi U) + \nu \frac{\partial^2 U}{\partial \xi^2}.$$

(2)

If a steady solution is established in the self-similar variables as $\tau \to \infty$, it satisfies

$$\frac{d^2 U}{d\xi^2} + \frac{a}{\nu} \frac{d}{d\xi} (\xi U) = \frac{d}{d\xi} \frac{U^2}{2\nu}.$$

After an integration we find the following form

$$\frac{dU}{d\xi} + \frac{a}{\nu} \xi U = \frac{U^2}{2\nu} + 2C,$$

(3)

where the prefactor of 2 is inserted in front of a constant $C$ for subsequent convenience. (In this paper $c, C, C_1, \ldots$ etc. denote constants which may be different from line to line.)

We now distinguish two cases.
1. If $C = 0$, the problem is well-understood. It can be solved by either introducing an integrating factor $\frac{dU}{d\xi} + \frac{a}{2} \xi U = e^{-\frac{a}{2} \xi^2} \frac{d}{d\xi} \left( U e^{\frac{a}{2} \xi^2} \right)$, or linearising with a transformation $V = 1/U$ regarding it as a Bernoulli equation. Either way we find the so-called source-type solution, e.g. [6, 7]

$$U(\xi) = \frac{C' \exp \left( -\frac{a\xi^2}{2\nu} \right)}{1 - \frac{C'}{2\nu} \int_0^\xi \exp \left( -\frac{a\eta^2}{2\nu} \right) d\eta},$$

for a constant $C'$. The solution is well-localised spatially, see Figs.1, 2. The name has come from the fact that $\lim_{t \to 0} \frac{1}{\sqrt{2at}} U(\xi) = M\delta(x)$, where $\delta(\cdot)$ denotes the Dirac mass and $M \equiv \int_{-\infty}^{\infty} u_0(x) dx$. We also recall [6, 7] that for all $u_0 \in L^1(\mathbb{R})$ and $1 \leq p \leq \infty$,

$$t^{\frac{1}{2}(1-\frac{1}{p})} \left\| u(x,t) - \frac{1}{\sqrt{2at}} U(\xi) \right\|_{L^p} \to 0 \text{ as } t \to \infty,$$

showing some degree of the universality of the profile.

2. If $C \neq 0$, (3) is a Riccati equation and the method above does not work. In this case, we ought to introduce a Riccati substitution $U = -2\nu \Psi'$, to reduce it to a linear equation (see Appendix A)

$$\Psi'' + \frac{a}{\nu} \xi \Psi' + \frac{C}{\nu} \Psi = 0.$$

It is to be observed that the Cole-Hopf transform arises as a natural course of solution to the Riccati equation [8]. If the initial data $u_0 \notin L^1(\mathbb{R})$, but $u_0 \in B^{0, \infty}_1(\mathbb{R})$, the other kind of solutions, i.e. the kink-type ones will come into play. Below we will discuss those solutions to (5) in detail.

2.2 Kink-type solution

As noted already, the theory developed in [1] for the 3D Navier-Stokes equations works for initial velocity $u_0 \notin L^3(\mathbb{R})$, but $u_0 \in B^0_{2, \infty}(\mathbb{R})$. We will consider an analogous 1D problem as an illustration.

In fact, the equation (5) can be solved using confluent hypergeometric functions, viz. the Kummer’s functions. Consider

$$\Psi'' + A\xi \Psi' + B \Psi = 0,$$

where $A$ and $B$ are constants (to be set $A = \frac{a}{\nu}$ and $B = \frac{C}{\nu}$). Actually we have the following

**Proposition 2.1** The solution to (6) can be written

$$\Psi = \xi e^{-\frac{3}{4} \xi^2} \left( 1 - \frac{B}{2A} \frac{3}{2} \xi^2 \right),$$

4
where the function \( w \left( 1 - \frac{B}{2A}, \frac{3}{2}, z \right) \) satisfies the confluent hypergeometric equation

\[
zd^2w \frac{dz^2}{dz^2} + \left( \frac{3}{2} - z \right) \frac{dw}{dz} + \left( \frac{B}{2A} - 1 \right) w = 0. \tag{7}
\]

Hence the general solution is given by

\[
\Psi = C_1 \xi \exp \left( -\frac{a}{2\nu} \xi^2 \right) M_K \left( 1 - \frac{C}{2a}, \frac{3}{2}, \frac{a}{2\nu}, \xi^2 \right) + C_2 \xi \exp \left( -\frac{a}{2\nu} \xi^2 \right) U_K \left( 1 - \frac{C}{2a}, \frac{3}{2}, \frac{a}{2\nu}, \xi^2 \right),
\]

where \( M_K(\alpha, \gamma, z) \) and \( U_K(\alpha, \gamma, z) \) denote two fundamental solutions\(^2\) of \((7)\).

We note that the real parameter \( A \) does not have to be positive in the general solutions above. This will be important when we consider a backward self-similar solution below.

**Proof**

This is done by straightforward calculations. Taking \( A = a/\nu = 1 \) without loss of generality, consider \( \Psi = \xi e^{-\frac{\xi^2}{2}} \left( 1 - \frac{B}{2}, \frac{3}{2}, z \right) \), with \( z = \xi^2/2 \). Direct calculations show

\[
\partial_\xi \Psi = (1 - \xi^2) e^{-\frac{\xi^2}{2}} w + \xi^2 e^{-\frac{\xi^2}{2}} \partial_z w,
\]

and

\[
\partial_\xi \partial_\xi \Psi = (\xi^3 - 3\xi) e^{-\frac{\xi^2}{2}} w + (3\xi - 2\xi^3) e^{-\frac{\xi^2}{2}} \partial_z w + \xi^3 e^{-\frac{\xi^2}{2}} \partial_{zz} w.
\]

Thus we get

\[
\partial_\xi \partial_\xi \Psi + \xi \partial_\xi \Psi + B \Psi = e^{-\frac{\xi^2}{2}} \{ \xi^3 \partial_z w + (3\xi - 3\xi^3) \partial_z w + (B - 2) \xi w \} = e^{-\frac{\xi^2}{2}} \{ \xi^3 \partial_z w + (3 - 3\xi^2) \partial_z w + (B - 2) w \}.
\]

As \( \xi^2 = 2z \), we deduce

\[
zd^2w \frac{dz^2}{dz^2} + \left( \frac{3}{2} - z \right) \frac{dw}{dz} + \left( \frac{B}{2} - 1 \right) w. \tag{□}
\]

We need to check which option, \( M_K \) or \( U_K \), is acceptable for our purpose. First we check \( M_K \). By the asymptotic formulas for \( |z| \to \infty \) (\( \Re z > 0 \)) (see Appendix B), we have as \( |\xi| \to \infty \)

\[
M_K \left( 1 - \frac{B}{2A}, \frac{3}{2}, \frac{A\xi^2}{2} \right) \approx \frac{\Gamma \left( \frac{3}{2} \right) e^{A\xi^2/2}}{\Gamma \left( 1 - \frac{B}{2A} \right)} \xi^{-\frac{1}{2} \left( 1 + \frac{B}{2A} \right)} e^{A\xi^2/2} \xi^{-\left( 1 + \frac{B}{2A} \right)}.
\]

Hence with this choice we obtain

\[
\Psi \approx \xi^{-\frac{B}{2}} \text{ and } U = -2\nu \partial_\xi \log \Psi \propto \xi^{-1},
\]

\(^2\)Standard notations for Kummer’s functions are \( M(\alpha, \gamma, z) \) and \( U(\alpha, \gamma, z) \). We add the subscript \( K \) to avoid confusion with the scaled velocity \( U \).
which is consistent with the boundary behaviour, i.e. the condition \( C \neq 0 \) above.

On the other hand, for the other solution \( U_K \) we have as \( |\xi| \to \infty \)

\[
U_K \left( 1 - \frac{B}{2A} \cdot \frac{3}{2} \cdot \frac{A\xi^2}{2} \right) \approx \left( \frac{A\xi^2}{2} \right) ^{\frac{B}{2A} - 1} \propto \xi ^{\frac{B}{2A} - 2},
\]

thus in this case we find

\[
\Psi \propto e^{-A\xi^2/2\xi} \quad \text{and} \quad U \propto -A\xi + \left( \frac{B}{A} - 1 \right) \xi^{-1}.
\]

We should discard this option because, when \( A \neq 0 \), the corresponding \( U \) does not even belong to \( B^{1,\infty}_1(\mathbb{R}^1) \), due to the presence of the linear term in \( \xi \).

We conclude that for the kink solution, we should choose

\[
\Psi = C_1 \xi \exp \left( -\frac{a}{2\nu} \xi^2 \right) M_K \left( 1 - \frac{C}{2a} \cdot \frac{3}{2} \cdot \frac{a\xi^2}{2\nu} \right).
\]

It may be in order to have a look at a specific example of the class of solutions.
Replacing \( z \to iz \), in the following identity, e.g. [9],

\[
M_K \left( 1, \frac{3}{2}, -z^2 \right) = \frac{\sqrt{\pi}}{2z} \text{erf}(z),
\]

we have

\[
M_K \left( 1, \frac{3}{2}, z^2 \right) = \frac{\sqrt{\pi}}{2iz} \text{erf}(iz) = \frac{\sqrt{\pi}}{2iz} \text{erfi}(z).
\]

Here \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \) denotes the error function and its imaginary version \( \text{erfi}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{t^2} dt \). In order to make use of the identity, we take \( C/a = 1 \).

With this a typical example can be given

\[
\Psi = C_1 \xi \exp \left( -\frac{a}{2\nu} \xi^2 \right) M_K \left( 1, \frac{3}{2}, \frac{a\xi^2}{2\nu} \right) = C_1 \sqrt{\frac{2\nu}{a}} D \left( \sqrt{\frac{a}{2\nu}} \xi \right),
\]

where \( D(x) = e^{-x^2} \int_0^x e^{t^2} dt = \frac{\sqrt{\pi}}{\sqrt{2\nu}} H[\exp(-x^2)] \) denotes Dawson’s integral and \( H[\cdot] \) the Hilbert transform. See Figs. 3, 4 in which we also assume \( a = 1 \) and \( \nu = 1/2 \) for simplicity. Note that the profile in Fig.4 is a steady solution in scaled space, which develops from the initial profile in the original variables in Fig.3 under time evolution.

It can be seen that in higher spatial dimensions \( \beta \) is generalised to

\[
\Delta \Psi + \frac{a}{\nu} (\xi \cdot \nabla) \Psi + \frac{C}{\nu} \Psi = 0.
\]

However, representation formulas for solutions to (8) are not known.

See Appendix C for a motivation or rationale for studying two different kinds of steady solutions.
2.3 Backward self-similar solutions

Generally speaking, we talk about forward self-similar solutions to study the decaying process in the late stage of evolution, whereas talk about backward self-similar solutions to study whether solutions blow up in finite time. Needless to mention, no solutions to the Burgers equation blow up if they start from well-localised smooth initial data under natural boundary conditions. Nonetheless, because the backward problem differs from the forward problem only in the sign of the parameter $a$, if one of them is obtained it is readily transferable to the other one by flipping the sign of $a$, provided that we do not bother their boundary behaviour.

In the spirit of [1] we have studied forward self-similar solutions to the Burgers equation under the setting of $B^0_{1,\infty}(\mathbb{R}^1)$. What would happen if we consider the backward self-similar solutions in the enlarged function class, or even broader one?

We can study 'possible' blow up with backward self-similarity, putting

$$u(x,t) = \frac{1}{\lambda(t)} U \left( \frac{x}{\lambda(t)} \right)$$

with the length-scale $\lambda(t) = \sqrt{2a(t_* - t)}$, where $t_*$ denotes the time of blowup. The steady equation reads

$$U \frac{\partial U}{\partial \xi} + a \left( \xi \frac{\partial U}{\partial \xi} + U \right) = \nu \frac{\partial^2 U}{\partial \xi^2},$$

to which the only smooth solution is a trivial one $U \equiv 0$ under natural boundary conditions and smoothness conditions. Relaxing those conditions, a non-trivial solution is nonetheless obtained as

$$U(\xi) = \frac{C' e^{a\xi^2/(2\nu)}}{1 - \frac{\nu}{4\pi} \int_0^{\infty} e^{a\eta^2/(2\nu)} d\eta},$$

This solution is badly-behaved at far distances\[4] $U(\xi) \propto \xi$ as $\xi \to \pm\infty$. Furthermore this has a singular point (a pole) somewhere, say at $\xi = \xi_*$, where the denominator vanishes. Thus we have $U(\xi) \propto 1/(\xi - \xi_*)$ around it and is non-integrable $U \notin L^1_{\text{loc}}$. See Fig.5.

It is of interest to have another look at the backward self-similar solution. We first recast the source-type solution using Kummer’s function. Taking $\beta = 1/2$ in the following identity [11]

$$e^{-z} M_K(1, \beta + 1, z) = \beta z^{-\beta} \gamma(\beta, z),$$

where $\gamma(\beta, z) \equiv \int_0^\infty t^{\beta-1} e^{-t} dt$, we have

$$e^{-z} M_K \left( 1, \frac{3}{2}, z \right) = z^{-1/2} \int_0^{\sqrt{z}} e^{-s^2} ds.$$

\[4\] In Appendix D of [12] it was stated erroneously that $U(\xi) \to \frac{1}{|\xi|}$ as $\xi \to \pm\infty$, which should be corrected as above.
Putting \( z = \frac{a}{2\nu} \xi^2 \), we find

\[
\Psi = \xi \exp \left( -\frac{a}{2\nu} \xi^2 \right) M_K \left( 1, \frac{3}{2}, \frac{a}{2\nu} \xi^2 \right) = \sqrt{\frac{\pi \nu}{2a}} \operatorname{erf} \left( \sqrt{\frac{a}{2\nu}} \xi \right).
\]

When we replace \( a \to -a \), the right-hand side is changed as

\[
\Psi \to \frac{1}{i} \sqrt{\frac{\pi \nu}{2a}} \operatorname{erf} \left( i \sqrt{\frac{a}{2\nu}} \xi \right) = \sqrt{\frac{\pi \nu}{2a}} \operatorname{erfi} \left( \sqrt{\frac{a}{2\nu}} \xi \right),
\]

which agrees with backward self-similar solution obtained above. Note that the asymptotic formula for \( M_K(\alpha, \gamma, z) \) for \( |z| \to \infty \) does not hold valid for \( \alpha = 1 \), because \( \Gamma(1) = 0 \).

It may be interesting to consider the following question. The existence of the forward self-similar (source-type) solutions \( U \) to the 3D Navier-Stokes equations is known. Suppose we make a replacement \( a \to -a \) in such solutions, the backward profile \( U \) must be singular and/or ill-behaved at far distances, in view of the non-existence of self-similar blowup [10, 11]. We would still be interested in what kind of spatial structure the profile possesses because such a solution may be helpful in putting constraints under the replacement \( a \leftrightarrow -a \) on the sought-after forward source-type solutions.

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**Figure 1:** The initial data, Dirac delta function \( u(x) = \delta(x) \) at \( t = 0 \). Depicted as \( \delta_{\epsilon}(x) \equiv \exp(-x^2/\epsilon)/\sqrt{\pi \epsilon} \), with \( \epsilon = 1 \times 10^{-4} \). (The figures in this paper are meant to be schematic.)

**Figure 2:** The source-type solution as \( \delta_{\epsilon}(x) \equiv \exp(-x^2/\epsilon)/\sqrt{\pi \epsilon} \), with \( \epsilon = 1 \times 10^{-4} \). (The figures in this paper are meant to be schematic.)
3 Self-similar solutions of the Navier-Stokes equation

3.1 2D Navier-Stokes equations

The 2D Navier-Stokes equations is described by the vorticity equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \Delta \omega.$$ 

The dynamically-scaled form of the vorticity equation reads

$$\frac{\partial \Omega}{\partial \tau} + \mathbf{U} \cdot \nabla \Omega = \nu \Delta \Omega + a \nabla \cdot (\xi \Omega),$$

where $\Omega(\xi, \tau) = \frac{1}{2\pi} \omega(\xi, \tau)$ denotes the scaled vorticity, $\xi = \frac{x}{\sqrt{2at}}$ and $\tau = \frac{1}{2a} \log t$. Under the assumption of radial symmetry, i.e. the similarity profile $\Omega$ being a function of $\xi = |\xi|$ only, it has a solution

$$\Omega(\xi) = a \frac{\Gamma}{2\pi \nu} \exp \left( -\frac{a |\xi|^2}{2\nu} \right),$$

where $\Gamma = \int_{\mathbb{R}^2} \Omega(\xi)d\xi$ denotes the circulation. This is the celebrated Burgers vortex, which is well-localised in space and $\in L^1(\mathbb{R}^2)$.

All the radially-symmetric solutions to the 2D Fokker-Planck equations were worked out explicitly in [13]. One of them is the Burgers vortex above and the other one is lesser-localised in space. The latter solution reads

$$\Omega(\xi) = \frac{a}{4\nu} \exp \left( -\frac{a |\xi|^2}{2\nu} \right) \left( \text{Ei} \left( \frac{a |\xi|^2}{2\nu} \right) - \log \left( \frac{a}{2\nu} \right) - \gamma \right),$$

where $\text{Ei}$ is the exponential integral.
Figure 5: A blowup profile $U = e^{\xi^2} \cdot \frac{1}{1 - \int_0^\xi e^{\eta^2} d\eta}$, which behaves $\propto \frac{1}{\xi - \xi^*}$ near some $\xi^*$ and $\propto \xi$ as $|\xi| \to \infty$.

where $\text{Ei}(x) \equiv -\text{p.v.} \int_{-\infty}^x \frac{e^t}{t} dt$ denotes the exponential integral and $\gamma \approx 0.577$ the Euler’s constant. Taking $a/2\nu = 1$ for simplicity we consider

$$\Omega(\xi) = \frac{1}{2} \exp(-\xi^2) \left( \text{Ei}(\xi^2) - \gamma \right).$$

Recalling $\text{Ei}(x) = O\left(\frac{e^x}{x}\right)$ as $x \to \infty$ and $\text{Ei}(x) \approx \gamma + \log x + x$ as $x \to 0+$, we note its asymptotic properties as follows.

- It decays slowly $\Omega(\xi) \approx \frac{1}{2\xi^2}$ as $\xi \to \infty$. Hence $\Omega(\xi)$ does not belong to $L^1(\mathbb{R}^2)$, but does belong to the Besov space $B^0_{1,\infty}(\mathbb{R}^2)$.

- Near the origin it shows a mild singularity as $\Omega(\xi) \approx \log \xi$ as $\xi \to 0+$, that is, it is discontinuous in vorticity at $\xi = 0$.

- It can be verified nonetheless that the corresponding velocity is continuous there. The azimuthal component velocity is given by

$$U_\theta(\xi) = \frac{1}{\xi} \int_0^\xi \eta \Omega(\eta) d\eta = \frac{1}{4\xi} \left\{ \log \xi^2 - e^{-\xi^2} (\text{Ei}(\xi^2) - \gamma) \right\}.$$

It has the following asymptotic behaviours: $U_\theta(\xi) \approx \frac{1}{2} \log \xi$ as $\xi \to 0+$ and $U_\theta(\xi) \approx \frac{1}{4\xi} \log \xi$ as $\xi \to \infty$.

Reverting to the original variables from (11) and discarding terms associated
with the Gaussian solution, we obtain a particular solution

\[ \omega(r, t) = \frac{1}{2a} \Omega \left( \frac{r}{\sqrt{2at}} \right) = \frac{1}{8\nu t} \exp \left( -\frac{r^2}{4\nu t} \right) \text{Ei} \left( \frac{r^2}{4\nu t} \right). \]

We have, for fixed \( t \),

\[ \omega(x, t) \approx \frac{\log r}{4\nu t} \text{ as } r \to 0^+ \]

by \( \text{Ei}(x) \approx \log x \text{ as } x \to 0^+ \). Hence the property of the vorticity being singular at the origin persists throughout time evolution. We also have

\[ \omega(x, t) \approx \frac{1}{2r^2} \text{ as } t \to 0^+. \]

The corresponding initial data is exactly scale-invariant as expected.

The azimuthal velocity field in the original variables is given by

\[ u_{\theta}(r, t) = \frac{1}{r} \int_0^r s \omega(s, t) \, ds \]

\[ = \frac{1}{r} \left\{ \log \left( \frac{r^2}{4\nu t} \right) - \exp \left( -\frac{r^2}{4\nu t} \right) \text{Ei} \left( \frac{r^2}{4\nu t} \right) + \gamma \right\}. \]

Its asymptotic properties are

\[ u_{\theta}(r, t) \approx \frac{\log r}{2r} \text{ as } t \to 0^+ \text{ and } u_{\theta}(r, t) \approx \frac{r}{8\nu t} \log r \text{ as } r \to 0^+. \]

The second non-Gaussian solution may serve as a replacement for the 'kink-type' solution to the problem. However, neither realisability nor stability of (11) is known.

### 3.2 3D Navier-Stokes equations

As already noted, in three dimensions the existence of the forward self-similar solutions is known, e.g. [1, 13], but the precise functional form of the solution is not. If the initial condition is well-localised, we know that it takes a near-Gaussian form if the dependent variable is chosen suitably, i.e. as the vorticity curl in three-dimensions.

Using the vorticity curl \( \chi = \nabla \times \omega \) the governing equations read

\[ \frac{\partial \chi}{\partial t} = \Delta (u \cdot \nabla u + \nabla p) + \nu \Delta \chi, \]

where \( u \) denotes the velocity and Under the dynamic scaling transformations

\[ \xi = \frac{x}{\sqrt{2at}}, \quad \tau = \frac{1}{2a} \log t, \quad \chi(x, t) = \frac{1}{(2at)^{3/2}} \chi(\xi, \tau), \]

\[ \text{This is same the procedure by which we get the so-called Lamb-Oseen decaying vortex from (3.2).} \]
we find
\[ \frac{\partial X}{\partial \tau} = \triangle (U \cdot \nabla U + \nabla P) + \nu \triangle X + a \nabla \cdot (\xi \otimes X). \]

A perturbative attempt of the determination of source-type solutions can be found in [13]. The leading-order approximation solution is explicitly given by a Gaussian function modulo incompressibility and the corrections due to the nonlinear term is estimated to be small. The conjugate solution at leading-order is also worked out explicitly.

## 4 Lifting of self-similar solution to more general ones

We discuss what can be learnt from studying self-similar solutions and identify open problems in this regard.

### 4.1 1D Burgers equation

When we recast (4) as
\[ U(\xi) = -2\nu \frac{\partial}{\partial \xi} \log \left( 1 - \frac{C'}{2\nu} \int_0^\xi \exp \left( -\frac{a \eta^2}{2\nu} \right) \, d\eta \right), \tag{12} \]

it is reminiscent of the structure of the celebrated Cole-Hopf transform. In other words, the source-type solution encodes the vital information as to how we may linearise this nonlinear equation.

We don’t intend to add anything new to the understanding of the Burgers equation, rather we will show how we can recover the Cole-Hopf transformation on the basis of a particular self-similar solution (viz. the source type solution).

Let us pretend that we don’t know the Cole-Hopf transformation. It’s still straightforward to derive the source-type solution, which is slightly non-Gaussian. It is then possible to recover the Cole-Hopf linearisation by replacing the self-similar heat flow with a general heat flow, which we call the lifting procedure.

For illustration we will have a look at the details of the procedure. Assume that the velocity potential \( \Phi(\xi) \) is given by
\[ \Phi(\xi) = f(\hat{\Psi}), \]

where \( f \) is a function of the self-similar source type solution of the heat equation
\[ \hat{\Psi} = \int_0^\xi \exp \left( -\frac{a \eta^2}{2\nu} \right) \, d\eta. \]

Then we have
\[ U = \partial_\xi \Phi = f'(\hat{\Psi}) \partial_\xi \hat{\Psi}. \]

Writing
\[ U(\xi) = F(\hat{\Psi}; \partial_\xi \hat{\Psi}), \]

...
we have trivially
\[ F(x;y) = yf'(x), \]
and the function \( F(x;y) \) satisfies the following scaling property
\[ F(x;\alpha y) = \alpha F(x;y), \text{ for } \forall \alpha > 0. \quad (13) \]

It should be noted that hereafter the arguments \( x, y, \) etc. in the function \( F \) do not represent spatial coordinates.

The procedure of spotting (or, recovering, in this case) more general solutions can be formalised in the following steps.

**Step 1.** Assume a self-similar profile is obtained in the form
\[ U(\xi) = F(\hat{\Psi}(\xi); \partial_\xi \hat{\Psi}(\xi)) \equiv \frac{\partial_\xi \hat{\Psi}(\xi)}{1 - \frac{1}{2\nu} \hat{\Psi}(\xi)}. \quad (14) \]

The profile \( U(\xi) \) is a near-identity transformation of the last argument of the function, i.e. that of the Gaussian function \( \partial_\xi \hat{\Psi}(\xi) \). Luckily for the Burgers equation \( F(.;.) \) is known explicitly
\[ F(x;y) = \frac{y}{1 - \frac{x}{2\nu}}, \]
as indicated by the symbol \( \equiv \) above. By definition, a particular solution is obtained by reverting to the original variables:
\[ u(x,t) = \frac{1}{\sqrt{2at}} F \left( \frac{\sqrt{2at}}{\hat{\Psi}(x,t)}; \partial_\xi \hat{\Psi}(x,t) \right) = \frac{1}{\sqrt{2at}} \frac{\partial_\xi \hat{\Psi}(\sqrt{2at})}{1 - \frac{1}{2\nu} \hat{\Psi}(\sqrt{2at})}. \]

**Step 2.** Replacing the self-similar heat solution with the general heat flow, we obtain a more general class of solutions
\[ u(x,t) = \frac{1}{\sqrt{2at}} F \left( \frac{\sqrt{2at}}{\hat{\Psi}(x,t)}; \partial_\xi \hat{\Psi}(x,t) \right) = \frac{\partial_\xi \hat{\Psi}(x,t)}{1 - \frac{1}{2\nu} \hat{\Psi}(x,t)}, \]
where
\[ \hat{\psi}(x,t) = \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{\infty} \hat{\psi}(y,0) \exp \left( -\frac{(x-y)^2}{4\nu t} \right) dy \]
denotes the general heat flow. Note that the length scales \( \sqrt{2at} \) cancel out because of (13). In the case of the Burgers equation, this last form provides the general solution.

Before closing this subsection it may be in order to emphasise the following fact. In terms of the scaled velocity potential \( \Phi(\xi) \) the exact solution is given by \( \Phi = -2\nu \log \left( 1 - \frac{1}{2\nu} \hat{\Psi} \right) \), whose leading-order approximation agrees with the scaled heat flow \( \Phi(\xi) \approx \hat{\Psi}(\xi) \). This confirms the near-identity nature of \( \Phi(\xi) \).
4.2 2D Navier-Stokes equations

In 2D incompressible flows the source-type solution is given by the Burgers vortex, which is not only near-Gaussian, but also exactly Gaussian. Because of this peculiarity, the final steady state does not contain any useful information regarding the nonlinear terms. Hence it is impossible to lift (or generalise) the final state to find a more general class of solutions and we end up with obtaining the linearised solution only. Curiously enough the better understood 2D Navier-Stokes equations defy the current approach to gain some information about their solutions.

Regarding this, it is in order to include a bit more detailed description based on kinematic relationship. The stream function corresponding to (10), with $\Gamma = 1$, is given by

$$\hat{\Psi}(\xi) = \frac{1}{4\pi} \text{Ei} \left( -\frac{a|\xi|^2}{2\nu} \right),$$

which agrees with the scaled heat flow in two dimensions. Assume that the scaled stream function $\Psi(\xi)$ is given by $\Psi(\xi) = f(\hat{\Psi}(\xi))$. It is readily derived

$$\Omega(\xi) = -\Delta \Psi = -\Delta \hat{\Psi} f'(\cdot) - \{\nabla \hat{\Psi}(\xi)\}^2 f''(\cdot).$$

Writing $\Omega(\xi) = F(\hat{\Psi}(\xi), \nabla \times \hat{\Psi}(\xi), -\Delta \hat{\Psi}(\xi))$, we find

$$F(x, y; z) = f'(\cdot)z - f''(\cdot)y^2.$$

The function $F$ satisfies the following scaling

$$F(x, \alpha y; \alpha^2 z) = \alpha^2 F(x, y; z), \text{ for } \forall \alpha > 0.$$

The only solution we know is the identity, that is, $f(x) = x$.

4.3 3D Navier-Stokes equations

We take the vorticity curl, $\chi = \nabla \times \omega$ as the basic dependent variable, whose dynamically-scaled version is denoted by $X(\xi)$. With this choice, the linearised equations have the Fokker-Planck operator In view of the critical scale-invariance of type 2 [13] it is most convenient for our analysis, as the leading-order approximation is basically given by the Gaussian function, i.e. the Gaussian function modulo incompressibility.

The steady version of the dynamically-scaled Navier-Stokes equations reads

$$\Delta^* X \equiv \Delta X + \frac{a}{\nu} \nabla \cdot (\xi \otimes X) = -\frac{1}{\nu} \Delta \mathcal{P} \left( \Delta^{-1} X \cdot \nabla \Delta^{-1} X \right). \quad (15)$$

By a formal analysis we show the following

**Proposition 4.1** The successive approximations to the solution $X$ of the equations (15) are given by a functional of

$$\{(\nabla \times)^k \hat{\Psi}(\xi) \mid k = 0, 1, 2, 3\}.$$

It is a functional rather than a function as we need to take into account nonlocal interactions due to the incompressible condition.
Here $(\nabla \times)^3 \hat{\Psi}(\xi) = \mathbb{P}MG$, defined with $M = \int X(\xi) d\xi$, $G = \left(\frac{n}{2}\pi\right)^{3/2} \exp\left(-\frac{n}{2\nu}\|\xi\|^2\right)$ and $\mathbb{P}$ solenoidal projection such that $\nabla \cdot \hat{\Psi}(\xi) = 0.$

Note that $\hat{\Psi}(\xi)$ denotes the scaled heat flow, whereas $\Psi(\xi)$ the scaled vector potential.

**Proof**

We consider the following expansion for small $\epsilon > 0$,

$$X(\xi) = \epsilon X_1 + \epsilon^2 X_2 + \epsilon^3 X_3 + \ldots,$$

deferring the justification of smallness of $\epsilon$ by its identification as the Reynolds number. We prove by mathematical induction that each $X_n$ can be represented by a combination of functionals of $\hat{\Psi}(\xi)$, including e.g. $\nabla^{-1} \hat{\Psi}$ and the derivatives of $\hat{\Psi}(\xi)$. Equating the terms with the same powers in $\epsilon$, we derive equations for $X_n$ for $n \geq 1$.

(i) We will first confirm that this is the case for $n = 1$.

To leading order at $O(\epsilon)$ we have

$$\nabla^* X_1 = 0,$$

from which it follows that $X_1 = \mathbb{P}MG.$ Indeed $X_1$ is a functional of the desired form. By definition $X_1 = (\nabla \times)^3 \Psi_1(\xi)$, we also note that the leading-order approximation satisfies $\Psi_1(\xi) = \hat{\Psi}(\xi)$.

(ii) Assuming that the statement holds up to step $k(\leq n)$, we will deduce that it also holds for step $(n + 1)$. For illustration let us first take a look at, for example, $O(\epsilon^2)$ and $O(\epsilon^3)$. To next-to-leading order at $O(\epsilon^2)$ we have

$$\nabla^* X_2 = -\frac{1}{\nu} \nabla^* (\nabla^{-1} X_1 \cdot \nabla \nabla^{-1} X_1)$$

and at the third order $O(\epsilon^3)$

$$\nabla^* X_3 = -\frac{1}{\nu} \nabla^* (\nabla^{-1} X_1 \cdot \nabla \nabla^{-1} X_2 + \nabla^{-1} X_2 \cdot \nabla \nabla^{-1} X_1).$$

Likewise, at $O(\epsilon^{n+1})$ for $\forall n \in \mathbb{N}$, we have

$$\nabla^* X_{n+1} = -\frac{1}{\nu} \nabla^* \sum_{l=1}^{n} (\nabla^{-1} X_l \cdot \nabla \nabla^{-1} X_{n+1-l}),$$

or

$$X_{n+1} = -(\nu \nabla^*)^{-1} \nabla^* \sum_{l=1}^{n} (\nabla^{-1} X_l \cdot \nabla \nabla^{-1} X_{n+1-l}).$$

Here use has been made of the inverse Fokker-Planck operator is given by

$$(\nu \nabla^*)^{-1} \equiv -\int_0^\infty dse^{\nu s \nabla^*} = \int d\eta g(\xi, \eta)$$

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with
\[ g(\xi, \eta) = \frac{-1}{(2\pi \nu)^{3/2}} \text{f.p.} \int_0^\infty \frac{\sigma^2 d\sigma}{\sqrt{\sigma^2 - a}} e^{-\frac{1}{4} |\sigma \xi - \eta \sqrt{\sigma^2 - a}|^2}, \]
where f.p. denotes Hadamard’s finite part. Noting that \((\triangle^*)^{-1} \triangle P\) represents an integral operator of order zero, we see that the right-hand side of (16) is a functional of \(X_k\’s\ (1 \leq k \leq n)\). Because they are all represented as functionals of \(\hat{\Psi}(\xi)\) and its derivatives, so is \(X_{n+1}\). Hence we deduce that \(X_n\) can be expressed as a functional of \(\hat{\Psi}(\xi)\) and its derivatives for all \(n \in \mathbb{N}\). □

We note that that the nonlinear contribution \(X_n\ (n \geq 2)\) has the second order derivative of \(\Psi_1(\xi)\) at most and the \((5 - 2n)\)-th order derivative at least. At \(n = 3\) already we have \(5 - 2n < 0\) and this means that it may involve integrals of \(\Psi_1(\xi)\), hence \(F\) may be a functional, rather than a function. We also note that when the \(\epsilon\)-expansion is uniformly convergent, \(X\) itself is a functional of \(\hat{\Psi}(\xi)\) and its derivatives. To see how we can make \(\epsilon\) arbitrarily small, take for example, the next-to-leading order approximation (iteration) \[ \begin{align*}
X &= X_1 - \frac{1}{\nu} \left( \triangle^{-1} X_1 \cdot \nabla \triangle^{-1} X_1 \right) \\
&= \mathbb{P}MG - \frac{1}{\nu} \left( \triangle^{-1} \mathbb{P}MG \cdot \nabla \triangle^{-1} \mathbb{P}MG \right). \tag{17}
\end{align*} \]

Introducing the following variables for non-dimensionalisation \(\tilde{X} = X/\nu\) and \(\tilde{M} = M/\nu\), we find
\[ \tilde{X} = \mathbb{P}\tilde{M}G - \mathbb{P} \left( \triangle^{-1} \mathbb{P}\tilde{M}G \cdot \nabla \triangle^{-1} \mathbb{P}\tilde{M}G \right), \]
where \(Re = |\tilde{M}| = |M|/\nu\) denotes the Reynolds number. Identifying \(Re = \epsilon\), this corresponds to the \(O(\epsilon^2)\) approximation. We can argue similarly for the \(O(\epsilon^n)\) approximations as well for \(\forall n \in \mathbb{N}\).

On the basis of Proposition 4.1, we consider a self-similar profile in the following form\[ X(\xi) = F \left( \hat{\Psi}(\xi), \nabla \times \hat{\Psi}(\xi), (\nabla \times)^2 \hat{\Psi}(\xi); (\nabla \times)^3 \hat{\Psi}(\xi) \right) \tag{18} \]
for some functional \(F\), where \(\hat{\Psi}\) denotes the scaled heat flow, \((\nabla \times)^3 \hat{\Psi} = \mathbb{P}K\hat{G}\), \(K\) is a function of \(M = \int X(\xi)d\xi\) and \(\hat{G} = \exp(-\frac{\alpha^2}{2\nu}|\xi|^2)\). The functional \(F(x, y, z; w)\) is a near-identity transformation of the last argument \(w\) (the solenoidal Gaussian function) and satisfies
\[ F(x, \alpha y, \alpha^2 z; \alpha^3 w) = \alpha^3 F(x, y, z; w) \text{ for } \forall \alpha > 0. \tag{19} \]

The profile \[18\] is a rather strong assumption, even though we take it approximately. If \(F\) is available, at least part of solutions to the Navier-Stokes equations

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\[6\] Care should be taken of the notations; it means that \(F\) is a functional of \(\hat{\Psi}\) and its derivatives with \((\nabla \times)^3 \hat{\Psi}\) at the highest.
is reducible to those of the heat equations. Unfortunately, the precise form of \( F \) is not known at the moment, but we know that it is close to the Gaussian and how close it is \([13]\). Assuming that \( F \) is obtained, the procedure goes as follows.

**Step 1.** When the self-similar profile \( F \) is given explicitly, a particular self-similar solution is obtained as

\[
\chi(x, t) = \frac{1}{(2at)^{3/2}} F \left( \frac{x}{\sqrt{2at}}, \nabla \times \hat{\Psi} \left( \frac{x}{\sqrt{2at}} \right), (\nabla \times)^2 \hat{\Psi} \left( \frac{x}{\sqrt{2at}} \right) ; (\nabla \times)^3 \hat{\Psi} \left( \frac{x}{\sqrt{2at}} \right) \right).
\]

**Step 2.** We seek a more general class of solution on this basis

\[
\chi(x, t) = \frac{1}{(2at)^{3/2}} F \left( \hat{\psi}(x, t), (2at)^{1/2} \nabla \times \hat{\psi}(x, t), 2at (\nabla \times)^2 \hat{\psi}(x, t); (2at)^{3/2} (\nabla \times)^3 \hat{\psi}(x, t) \right).
\]

The first three arguments give rise to a factor of \((2at)^{3/2}\) in total, as a result of scaling \((19)\) and hence we have

\[
\chi(x, t) = F \left( \hat{\psi}(x, t), \nabla \times \hat{\psi}(x, t), (\nabla \times)^2 \hat{\psi}(x, t); (\nabla \times)^3 \hat{\psi}(x, t) \right).
\]

It is not known how general a class of functions such a construction can cover.

To see how cancellations take place, it is helpful to consider the following example based on the next-to-leading order approximation. On the right-most side of \((17)\), the first term gives rise to \((2at)^{3/2}\) so does the second \((2at)^{1/2}\)·

\[\text{Remark}\]

In three dimensions, under the assumption of \( \Psi = f(\hat{\Psi}) \), a simpler form of \( F \) can be obtained using spatial derivatives rather than curls. Write

\[
X = \tilde{F} \left( \hat{\Psi}, D_1 \hat{\Psi}, D_2 \hat{\Psi}, D_3 \hat{\Psi} \right),
\]

where \( D_i = \partial_{\xi_i} \) for any spatial derivatives. After some algebra we find explicitly

\[
X_i = -\epsilon_{iik} \left\{ \frac{\partial s_j}{\partial \xi_k} \frac{\partial s_m}{\partial \xi_l} \frac{\partial s_n}{\partial s_m} \frac{\partial^3 f_q}{\partial \xi_k \partial \xi_l \partial \xi_m \partial s_n} + \left( \frac{\partial s_j}{\partial \xi_k} \Delta s_m + 2 \frac{\partial^2 s_j}{\partial \xi_k \partial \xi_l} \frac{\partial s_m}{\partial s_l} \right) \frac{\partial^2 f_q}{\partial \xi_k \partial s_n} + \frac{\partial \Delta s_j}{\partial s_n} \frac{\partial f_q}{\partial s_n} \right\}
\]

where we have put \( s_i = \hat{\psi}_i(\xi) \) for simplicity. Observe that the above expression reduces to \( X = -\nabla \times \Delta \hat{\Psi} \). for the identity transformation \( \frac{\partial f_q}{\partial s_j} = \delta_{qj} \).

### 5 Summary and outlook

In this paper the following things are done. As part of forward self-similar solutions to the Burgers equation we have identified a class of kink-type solutions, on top of the well-known source-type solutions. They are given by Kummer’s confluent hypergeometric function \( M_K \). We also noted ill-behaved ‘blow-up’ profiles by flipping the sign of the parameter \( a \).
We discussed the Navier-Stokes equations; in two dimensions we discussed a self-similar profile which may be regarded as a conjugate to the Burgers vortex. Some of its asymptotic properties have been studied, whereas clarification of its significance it may play, e.g. its stability as a dynamical system, requires further investigation.

We also discussed applications of the self-similar solution in three dimensions. Some properties of the self-similar profile have been analysed formally and possible lifting to more general class of solutions, at least approximately, is suggested. For the final topic, it may be worthwhile to try computing approximate solutions by numerical methods. This is also left for future study.

A Riccati equation

Consider the Riccati’s ordinary differential equation
\[
\frac{dy}{dx} = a(x)y^2 + b(x)y + c(x),
\]
where \(a(x), b(x)\) and \(c(x)\) are given functions. It is known that a substitution of the following form
\[
y = -\frac{u'(x)}{a(x)u(x)}
\]
reduces the above to a linear second-order homogeneous equation
\[
\frac{d^2u}{dx^2} - \left(\frac{a'(x)}{a(x)} + b(x)\right)\frac{du}{dx} + a(x)c(x)u = 0.
\]

B Confluent hypergeometric equation

Kummer’s confluent hypergeometric equation [9]
\[
z \frac{d^2w}{dz^2} + (\gamma - z) \frac{dw}{dz} - \alpha w = 0
\]
has two fundamental solutions \(w(z) = M_K(\alpha, \gamma, z)\) and \(U_K(\alpha, \gamma, z)\). Their asymptotic behaviours are given as follows.

For \(\Re(z) > 0\), as \(|z| \to \infty\) we have
\[
\begin{align*}
M_K(\alpha, \gamma, z) &\approx \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha-\gamma}, (\alpha \neq 1, \gamma \neq -n), \\
U_K(\alpha, \gamma, z) &\approx z^{-\alpha},
\end{align*}
\]
where \(\Gamma(\cdot)\) denotes the gamma function.

Also, as \(|z| \to 0\) we have
\[
\begin{align*}
M_K(\alpha, \gamma, z) &= 1, \\
U_K(\alpha, \gamma, z) &= \frac{\Gamma(\gamma - 1)}{\Gamma(\alpha)} z^{1-\gamma} + O(1),
\end{align*}
\]
\((\alpha \neq 1, 1 < \Re(\gamma) < 2)\).
where \( n \in \mathbb{N} \).

\[ C \quad \text{Source-kink duality} \]

The PDE, Burgers equation, allows at least in two different kinds of approximations based on ODEs via particle systems. Those particle pictures are well-known, but best stated here for motivation.

One is

1. Propagation of Wiener’s chaos, e.g. [16]

\[
\frac{dX_i}{dt} = \frac{1}{2N} \sum_{j \neq i} b(X_j - X_i) dt + dW_i,
\]

where \( dW \) denotes Brownian motion, the drift velocity \( b = \delta \) for the Burgers equation. This is related with the source-type solution.

The other one is

2. Pole decomposition, e.g. [15]

\[
u(x,t) = -2\nu \sum_{j=1}^{N} \frac{1}{x - z_j(t)},
\]

\[
\frac{dz_j}{dt} = -2\nu \sum_{k=1, k \neq j}^{N} \frac{1}{z_j - z_k},
\]

where \( z_j \) denotes the locations of poles in the complex plane. This is related with the kink-type solution, represented by Kummer’s \( M_K \).

A crude explanation why we have two different views is as follows. There are two fundamental solutions to the Fokker-Planck equation; the Gaussian function and the Dawson’s integral, which are related by the Hilbert transform to each other. One of them converges to the Dirac mass \( \delta \) and the other to a Cauchy kernel \( 1/z \), respectively in suitable limits.

\[ \text{References} \]

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