OPEN GROMOV-WITTEN INVARIANTS AND BOUNDARY STATES

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ABSTRACT. In this note we show how Kontsevich-Soibelman algebra arisen naturally in Open Gromov-Witten theory for not compact geometries.

1. Introduction

Let $X$ be a Calabi-Yau simplectic six-manifold and let $L$ be a Maslov index zero Lagrangian submanifold of $X$. Assume that $[L] = 0 \in H_3(X, \mathbb{Z})$ and pick a four chain $K$ such that $\partial K = L$. In [11] we developed the general Open Gromov-Witten theory associated to $(X, L)$. We introduced the multi-curve chain complex and, for each $\beta \in H_1(L, \mathbb{Z})$ and $\chi \in \mathbb{Z}$, we constructed a $MC$-cycle $Z_{\beta, \chi}$ from the moduli space of pseudo-holomorphic (multi-)curves. $Z_{\beta, \chi}$ is not uniquely determined but it is well defined up to isotopy. Our theory is the natural mathematical counterpart of the partition function of the A-model Open Topological String introduced by Witten in [16], and it makes mathematically precise the relation with the perturbative Chern Simons claimed in [16]. $MC$-cycles (or more precisely what we called nice $MC$-cycles) can be considered as the mathematical definition of the configuration space of points in the point splitting perturbative Chern-Simons theory.

In this paper we consider more closely the structures arising in not compact geometries. Under standard assumptions on the geometry at infinity, the construction of [11] applies essentially the same way. The main difference consists in setting boundary conditions at infinity for the perturbation of area zero curves. This introduces new structures in the problem.

In this paper we consider the collection of $MC$-cycles $Z = (Z_{\beta, \chi})_{\beta, \chi}$ for different $\beta, \chi$. In the not compact case, up to isotopy, the vector $Z$ is determined by the choice of a boundary condition $\mathcal{B}$ on the perturbation of area zero curves. We interpret $Z$ as an element of an infinity dimensional vector space $\mathcal{S}_\mathcal{B}$, which depends on an element $\zeta \in H_1(L, \mathbb{Z})$. The class $\zeta$ depends on the boundary conditions, and we refer to it as the Chern Class.

This result can be considered as the analogous in open-closed Gromov-Witten theory of the main claim of Topological Field Theories, where the partition function on a manifold with boundary $M$ defines a vector on the space of boundary states, which is defined quantizing the theory on $\partial M \times \mathbb{R}$. In our case, $M = L$ and the relevant topological field theory is the perturbative abelian Chern-Simons theory.

Let $\Sigma = \partial L$. Consider the quantum Kontsevich-Soibelman algebra $\hat{g}$ associated to the abelian group $H_1(\Sigma, \mathbb{Z})$ equipped with the standard intersection form $\langle \cdot, \cdot \rangle$:

$$\hat{g} = \bigoplus_{\gamma \in H_1(\Sigma, \mathbb{Z})} \mathbb{Q}[q^{1 \over 2}, q^{-1 \over 2}] e_{\gamma},$$
\[ \hat{e}_{\gamma_1} \hat{e}_{\gamma_2} = q^{\langle \gamma_1, \gamma_2 \rangle} \hat{e}_{\gamma_1 + \gamma_2} \] for \( \gamma_1, \gamma_2 \in H_1(\Sigma, \mathbb{Z}) \),

with

\[ q^\frac{1}{2} = -e^{\frac{\pi i}{2}}. \]

In this paper we consider an extension \( \hat{g}^\bullet \) of the KS-algebra, which is associated to \((X, L)\). Namely the generators of \( \hat{g}^\bullet \) are labeled by pairs \((\gamma, \beta) \in H_1(\Sigma, \mathbb{Z}) \times H_2(X, L, \mathbb{Z})\) with the image of \( \gamma \) on \( H_1(L, \mathbb{Z}) \) agreeing with \( \partial \beta \). There is a natural action of \( \hat{g}^\bullet \) on \( \mathcal{H} \).

We consider another version KS-algebra \( \hat{g} \) with \( q^\frac{1}{2} = e^{\frac{\pi i}{2}} \) and \( \gamma \in \frac{1}{2} H_1(\Sigma, \mathbb{Z}) \). This algebra will play an important role in the study of the boundary conditions, and it will be related with the perturbation of the area zero annulus. We denote its generators with \( (\hat{e}_\gamma^{ann0})_\gamma \). The element \( \hat{e}_\gamma^{ann0} \) maps \( \mathcal{H} \) on \( \mathcal{H}_{\gamma + Q(\gamma)} \), where \( Q(\gamma) \) is the image of \( \gamma \) on \( H_1(L, \mathbb{Z}) \).

Assume that \( X \) has cylindrical ends, i.e., outside a compact region \( X \) can be identified with the symplectization \( \mathbb{R}^+ \times Y \) of a contact five-manifold \( Y \). In closed Gromov-Witten theory (which is the particular case of open-closed theory when \( L = \emptyset \) and \( K \) is a closed four manifold) a choice of a boundary condition consists on the choice of a perturbation of the moduli space of area zero curves on \( \mathbb{R}^+ \times Y \) which is invariant under the translation on the \( \mathbb{R} \)-direction.

The generalization to open-closed Gromov-Witten theory of the boundary condition is not straightforward. Since in this case we need to work with the moduli of multi-curves \( \mathcal{M}_{G,m} \) the set of area zero components (of a multi-curve) mapped on \( \mathbb{R}^+ \times Y \) changes when we move inside the moduli space. It is necessary to set the invariance in the \( \mathbb{R} \) direction consistently in a suitable way. Observe that, since there can be components of the multi-curve of area zero in each degree, the choice of the boundary condition affects the invariants of not zero degree also. This is in contrast with the closed case, where the ambiguity associated to the choice of the boundary condition affect only the invariants of degree zero.

We consider a particular type of boundary conditions which in particular gives a frame \( fr \) of \( \mathbb{R}^+ \times \Sigma \) invariant by translation on the \( \mathbb{R} \)-direction.

If \( B_1, B_2 \) are two such boundary conditions, the associate MC-cycles \( Z^{B_1}, Z^{B_2} \) of \( \mathcal{H} \), are related by the action of the quantum KS-algebra:

\[ Z^{B_1} = a e_\gamma^{ann0} Z^{B_2} \] (1)

for some \( a \in \mathbb{Q}(g_s) \) and \( \gamma \in \frac{1}{2} H_1(\Sigma, \mathbb{Z}) \). \( \gamma \) is determined by the frames associated to \( B_1 \) and \( B_2 \).

The scalar ambiguity \( a \) of (1) can be considered the analogous of the one appearing in closed Gromov-Witten theory. For example, in the case of area zero spheres with three marked points, it is the classical ambiguity appearing in the definition of \( K \cap K \cap K \).

We also consider compactification frames of \( L \), which are defined in terms of maximal isotropic sub-lattice \( F \subset H_1(\Sigma, \mathbb{Z}) \) (this notion is a straightforward extension of [10]). Using the multi-linking number defined in [11] we can define a wave function \( \Psi_F(x) \). Alternative this can be defined using the Chern-Simons Propagator (see [14]).

1.1. Quantum shift of the coordinates. The frame \( fr \) is strictly related to the quantum shift of the coordinated appearing in the physical literature which is critical in order to match the B-model picture where it appears in the argument of
the quantum dilogarithm. Hence, it is critical to consider this extra data in order to have a dictionary between A-model and B-model. We know explain this relation.

The definition of $MCH(M, \gamma | c)$ depends also on the Chern class $c \in H^1(L, \mathbb{Z})$ which is determinate in a purely topological way from the four chain $K$. In the compact case $\chi$ is always even, i.e., $\chi \in 2H_1(L, \mathbb{Z})$. In the not compact case this is not always the case, $\chi$ is even if the spin structure on $\Sigma$ associate to $fr$ can be extended to $L$.

The Chern class needs to be even if we want the wave function be a periodic function of $x \in \frac{H^1(L, \mathbb{R})}{H^1(L, \mathbb{Z})}$:

$$x \mapsto x + 2\pi H^1(L, \mathbb{Z}).$$

It is necessary to fix a spin structure of $L$ also in order to fix an orientation of the moduli space of (multi-)curves. If the spin structure is changed by an element $\alpha \in H^1(L, \mathbb{Z}_2)$, the orientation of the moduli space of (multi-)curves of homology class $\beta$ changes by $(-1)^{\langle \alpha, \partial \beta \rangle}$. The action on the wave function can be obtained by the shift of the coordinates

$$x \leadsto x + i\pi \alpha.$$

If we restrict our attention to the frames $fr$ compatible with the spin structure of $L$, the change of the spin structure by $\alpha \in H^1(L, \mathbb{Z}_2)$ needs to be accompanied by a twist of the frame by an element $\gamma \in H^1(\Sigma, \mathbb{Z})$ such that

$$\alpha = \gamma \mod 2H^1(\Sigma, \mathbb{Z}).$$

Hence, if $\gamma \in Ker(H_1(\Sigma, \mathbb{Z}) \to H_1(L, \mathbb{Z}))$, we obtain a change of coordinates

$$(2) \quad x \leadsto x + (i\pi + \frac{g_s}{2})\gamma \mod 2\pi H^1(L, \mathbb{Z}).$$

The quantum shift appearing in (2) is related to the metaplectic correction when the wave function is interpreted as a perturbative state for the geometric quantization of abelian flat connections (see [14]).

The shift of the spin structure is necessary in particular when a singularity on the moduli space of Lagrangians is crossed (see also Remark 1 since in the process $Ker(H_1(\Sigma, \mathbb{Z}) \to H_1(L, \mathbb{Z}))$ changes. The quantum shift (2) can be seen as part of a Konstant-Rosemberger type homomorphism mentioned in Remark 1. The most simple example of this phenomena arises in the case of Topological Vertex, where in the B-model side corresponds to the shift of the quantum dilogarithm appearing in its Fourier Transformation. (See for example section 5.10 of [1].)

Remark 1. It is natural to ask how the objects defined in this paper change when the Lagrangian manifold $L$ moves on its moduli space, in particular when we cross a singularity, what in physical literature is usually referred to as open phase transition.

The boundary conditions can be moved across the singularity without any problem. In some sense the geometry at infinity does not see the singularity.

After the boundary conditions are fixed, we would like to consider objects which, in a suitable sense, do not change when we cross the singularity, in accord with the general physical principle stating that the moduli spaces are smooth after the quantum corrections are included.

To compare the wave function in different regions of the moduli, it is necessary to implement an analytical continuation. The objects considered in this paper are defined as formal power series, hence we need to make suitable convergence assumptions in order to obtain an actual function and consider analytical continuation (this step
may involve to consider not perturbative corrections also. Moreover we need to consider the wave function as a perturbative state for the geometric quantization of abelian flat connections (see [14]). Since \( \text{Ker}(H_{1}(\Sigma, \mathbb{Z}) \to H_{1}(L, \mathbb{Z})) \) changes when the singularity is crossed, the polarization changes and it is necessary to implement a Konstant-Rosemberger type homomorphism in order to compare the wave functions in the two sides. After these considerations are taken in account, we expect that the wave function is invariant by open phase transition.

This result provide the general setting to understand the quantization of moduli of \( A \)-branes (see also remark 22).

1.2. Non Abelian Case. The \( MCH \) homology we consider in this paper are associated to decorated graphs space of fixed Euler characteristic, which is referred as the abelian case. We can use this construction in order to consider curves of a fixed genus and number of boundary components \( (g, h) \) using the following standard procedure.

Identify the tubular neighbourhood of \( L \) with a neighbourhood of the zero section of \( T^{*}L \). Consider a \( k \)-brunched covering \( \tilde{L} \) of \( L \) in \( T^{*}L \to L \). The \( MC \)-cycle of fixed euler characteristic associated to \( \tilde{L} \) is related to the moduli of fixed \( (g, h) \) of \( L \). If we apply this for arbitrary \( k \) we recover all the informations about the curves of fixed \( (g, h) \).

Alternatively it is not hard to extend the definition of \( MCH \) of [11] to consider decorated graphs of fixed genus and boundary component, providing a more direct way to determine the refined generating function.

According the proposal of Aganagic and Vafa ([2]), this result can be applied in order to obtain a mathematical theory of HOMFLY polynomials of a link \( L \) in \( S^{3} \). This apply in the particular case that \( X \) is the conifold and \( L \) is the conifold transition of the normal bundle of \( L \) in \( T^{*}S^{3} \). Actually in this case we do not need to consider brunched covering of \( L \) but it is enough to consider the conifold transition of the normal bundle of parallel copies of \( L \), i.e., \( \tilde{L} \) is a disjoint union of copies of \( L \).

2. Multi Curve Cycles

Fix an oriented three manifold \( M \) and a class \( \gamma \in H_{1}(M, \mathbb{Z}) \). Consider the objects

\[
(H, (w_{h})_{h \in H})
\]

where \( H \) is a finite set , \( \{w_{h}\}_{h \in H} \) are closed one dimensional chains on \( M \) on the homology class \( \gamma \) which are close on the \( C^{0}\)-topology. Denote by \( \mathcal{S} \mathcal{E} \mathcal{N}(\gamma)^{\dagger} \) the set of the objects \( \{H, (w_{h})_{h \in H}\} \) modulo the obvious equivalence relation: \((H, (w_{h})_{h \in H}) \cong (H', (w'_{h})_{h \in H'}) \) if there exists an identification of sets \( H = H' \) such that \( w_{h} = w'_{h} \).

Define \( \mathcal{S} \mathcal{E} \mathcal{N}(\gamma) \subset \mathcal{S} \mathcal{E} \mathcal{N}(\gamma)^{\dagger} \) as the subset obtained imposing the extra condition

\[
w_{h} \cap w_{h'} = \emptyset \text{ if } h \neq h'.
\]

The vector space \( Z_{\gamma} \) of nice \( MC \)-cycles in the homology class \( \gamma \) is the formal vector space generated by \( \mathcal{S} \mathcal{E} \mathcal{N}(\gamma) \).

In this paper we consider \( Z_{\gamma}[[g_{s}]] \) the formal power series on the formal variable \( g_{s} \) with coefficients \( Z_{\gamma} \).
An isotopy of MC-cycles is defined by a formal power series

\[ \tilde{Z} = \sum_i g_i^k r_i(H_i, (\tilde{w}_{h,i})_{h \in H_i}, [a_i, b_i]), \]

for some \( r_i \in \mathbb{Q} \), \( k_i \to \infty \). In particular \( \tilde{Z} \) defines a one parameter family of elements of \( \mathbb{Z}_t \in \mathbb{Z}_\gamma \), \( t \in [0, 1] \), which can be discontinuous on a finite number of times (we do not define \( Z_t \) if \( t \) is a discontinuity point). If for \( h, h' \in H_1 \), \( \tilde{w}_{h_1, i}, \tilde{w}_{h_2, i} \) cross transversely at a time \( t_0 \), we require that \( \tilde{Z} \) jumps according to the formula

\[ Z^{t_0} - Z^{t_0^+} = \pm g_h^{k+1} r_i(H_i, (w_{h,i})_{h \in H_1 \setminus \{h_1, h_2\}}), \]

where the sign is defined by the sign of the crossing. Denote by \( \tilde{\gamma}_\gamma \) the set of isotopies of nice MC-cycles.

We need also to introduce the space of MC-cycles \( \mathcal{Z}_{\gamma}[E] \) with non vanishing Chern Class \( \epsilon \). This is done as in [11].

2.1. Graph version. Fix \( \epsilon > 0 \) small enough. For \( \gamma \in H_1(M, \mathbb{Z}) \) define the set of pair of one dimensional closed chains

\[ \mathcal{P}_\gamma = \{(w, w') | [w] = [w'] = \gamma, w \cap w' = \emptyset, d(w, w') < \epsilon \}/\sim \]

where \( \sim \) is the equivalence relation generated by small isotopies of the pair of curves that do not cross each other. For \( \epsilon \) small enough \( \mathcal{P}_\gamma \) does not depend on \( \epsilon \).

We should think to an element of \( \mathcal{P}_\gamma \) as (a small perturbation of) a framed link in the homology class \( \gamma \), i.e., \( w' \) is a small translation of \( w \) in the direction of the frame.

There is a \( \mathbb{Z} \)-action on \( \mathcal{P}_\gamma \)

\[ \text{tw}_k : \mathcal{P}_\gamma \to \mathcal{P}_\gamma, \ k \in \mathbb{Z}. \]

The pair \( \text{tw}_k(w, w') \) is obtained crossing \( k \) times (with sign) the pair \( (w, w') \).

Let \( \mathcal{E}_{\text{cr}}(\gamma) \) the set of objects

\[ (E^{in}, E^{ex}, ((w_e, w'_e))_{e \in E^{in}}, (w_e)_{e \in E^{ex}}). \]

modulo the obvious equivalence relation. Let \( \mathcal{Z}^\mathbb{Z} \) the vector space of formal linear combinations with coefficients in \( \mathbb{Q}[\mathcal{Z}_\gamma] \).

We can introduce the notion of isotopy in a analogous way as for XXX. We obtain a one parameter family of elements of \( \mathcal{Z}^\mathbb{Z} \), with the jumping condition: if for an addendum \( (6) \) with coefficient \( g_h^k \), for some \( e_0 \in E^{in}, w_{e_0} \) and \( w'_{e_0} \) intersects transversally, \( Z_t \) jumps of

\[ g_h^k r(E^{in} \setminus \{e_0\}, E^{ex}, ((w_e, w'_e))_{e \in E^{in} \setminus \{e_0\}}, (w_e)_{e \in E^{ex}}). \]

There is a map

\[ \mathcal{Z} \to \mathcal{Z}^\mathbb{Z} \]

defined by

\[ (H, (w_h)_h) \to \sum_E g^{E^{in}}_h \text{Aut}(E)(E^{in}, E^{ex}, ((w_e, w'_e))_{e \in E^{in}}, (w_e)_{e \in E^{ex}}) \]

where \( (w_e, w'_e) = (w_h, w'_h) \) for \( e = \{h, h'\} \in E^{in} \), \( w_e = w_h \) for \( e = \{h\} \in E^{ex} \).
2.2. Open Gromov-Witten MC-cycle. Let \((X, L)\) be a pair given by a Calabi-Yau symplectic six-manifold \(X\) and a Maslov index zero lagrangian submanifold \(L\). We assume \([L] = 0 \in H_2(X, \mathbb{Z})\). Fix a four chain \(K\) with \(\partial K = L\).

To the four chain \(K\) it is associated a Chern Class \(c(K) \in H_1(L, \mathbb{Z})\).

**Theorem 2.** Let \(\beta \in H_2(X, L, \mathbb{Z})\). To the moduli space of pseudoholomorphic multi-curves of homology class \(\beta\) it is associated a multi-curve cycle \(Z_\beta \in \mathbb{Z}_{\partial \beta}^{-1}\) with Chern class \(c\). \(Z_\beta\) depends by the varies choices we made to define the Kuranishi structure and its perturbation on the moduli space of multi-curves. Different choices lead to isotopic MC-cycles.

The result is an extension of the main theorem of [11]. In [11] we considered MC-cycles of a fixed Euler Characteristic. In particular only a finite number of decorated graphs are involved in the construction. This point is critical in order to construct a perturbation of the moduli space of multi-curve. The extension to a full MC-cycle is made using the argument subsection 2.5. Hence, using the MC-cycles constructed in [11] (which involves only a finite number of graphs) we can construct a full MC-cycles.

2.2.1. Review of the construction of [11]. In this subsection we review the construction of the Gromov-Witten MC-cycle of [11].

A Kuranishi structure on the moduli space of pseudo-holomorphic curves in particular associate to each pseudo-holomorphic curve \(p = (\Sigma, u)\) an obstruction bundle

\[
E_p \subset C^\infty(\Sigma, u^*(TX) \otimes \Lambda^{0,1}(\Sigma)).
\]

to each pseudo-holomorphic curve \(p = (\Sigma, u)\).

The standard procedure to construct a Kuranishi structure on moduli space of stable pseudo-holomorphic curves uses an argument by induction on the singular strata. For each strata, it is used a gluing argument to extend the Kuranishi structure to a neighborhood of the substratas and after the Kuranishi structure is extended inside the strata. The method does not lead to a unique Kuranishi structure, however it determinate it up to isotopy, i.e., given two Kuranishi structures constructed in this way there exists a Kuranishi structure on the moduli spaces times \([0, 1]\) whose restrictions to 0 and 1 agree with the starting Kuranishi structures. This isotopy is constructed using an analogous inductive argument on the strata.

Denote by \(\overline{M}_{\beta, V, D, (H_v)_v}(\beta)\) the moduli of curves of genus \(g\), homology class \(\beta\), whose boundary components are labelled by \(V\), internal marked points are labelled by \(D\) and boundary marked points are labelled by \((H_v)_v\).

In [11], to a decorated graph \((G, m) \in \Phi_1\), it is associated the moduli space of multi-curves

\[
\overline{M}_{G, m} := \left( \prod_{\gamma \in \text{Comp}(G)} \overline{M}_{g_{\gamma}, D_{\gamma}, V_{\gamma}, (H_{\gamma})_{\gamma} \in \mathbb{V}_{\gamma}}(\beta_{\gamma}) \right)/\text{Aut}(G, m) \times \Delta^l
\]

where \(\Delta^l\) is the standard simplex of dimension \(l\).

Using an inductive argument similar to the one used in the construction of the Kuranishi structures for stable pseudo-holomorphic curves, we can obtain a collection of Kuranishi structures \(\overline{M}_{G, m}(G, m) \in \Phi(\beta, \leq s)\) such that:

- forgetful compatibly holds;
• the evaluation map
  \[ \text{ev} : \overline{\mathcal{M}_{G,m}} \to L^{H(G)\setminus H_i} \times X^{D(\sigma)} \]
  is weakly submersive;
• the corner faces of \( \overline{\mathcal{M}_{G,m}} \) are in bijection with the graphs \( \mathcal{G}(G,m) \)
  \[ \mathcal{G}(G,m) \leftrightarrow \{ \text{corner faces of} \ \overline{\mathcal{M}_{G,m}} \}. \]

The corner face \( \overline{\mathcal{M}_{G,m}}(G', m') \), corresponding to \( (G', m', E') \in \mathcal{G}(G,m) \),
comes with an identification of Kuranishi spaces:
\[ \overline{\mathcal{M}_{G,m}}(G', m', E') \cong \delta_{E'} \overline{\mathcal{M}_{G', m' \prime}}. \]

The Kuranishi spaces \( \delta_{E'} \overline{\mathcal{M}_{G', m' \prime}} \) are defined as fiber products. See \((11)\).

Consider collections of perturbations \( [\mathcal{G}(G,m)](G,m) \in \mathcal{G}((\beta, \leq \delta)) \) of the collection of
Kuranishi spaces \( [\overline{\mathcal{M}_{G,m}}(G,m)] \in \mathcal{G}((\beta, \leq \delta)) \) such that

• they are transversal to the zero section and small in order for constructions of
  virtual class to work;
• they are compatible with the identification of Kuranishi spaces \((9)\);
• forgetful compatibility holds;
• they are transversal when restricted to \( \delta_{E'} \overline{\mathcal{M}_{G,m}} \) for each \( E \subset (E^m(G) \setminus E_i) \cup D(G) \).

Using these perturbations we define the collections of chains \((Z_{(G,m)}^{+}(G,m)) \in \mathcal{G}((\beta, \leq \delta)) \)
\[ (10) \quad Z_{(G,m)}^{not-ab,+} = (\text{ev}_{G,m})_* ([\mathcal{G}(G,m)](0)) \in C_* (L^{H(G)} \times X^{D(G)})^{\text{Aut}(G,m)}. \]

The not abelian \( MC \)-cycle \((Z_{(G,m)}^{not-ab,+}) \) is obtained taking the fiber product
\[ (11) \quad Z_{(G,m)}^{not-ab} = Z_{G,m}^{not-ab,+} \times_{X^{D(G)+}} K^{D(G)}. \]

From \( Z_{(G,m)}^{not-ab} \) we can construct the \( MC \)-cycle and the nice \( MC \)-cycle as in \((11)\).

**Remark 3.** In \((11)\) was considered also the Kuranishi spaces
\[ (12) \quad \overline{\mathcal{M}_{G,m}} := \overline{\mathcal{M}_{G,m}} \times X^{D(G)} K^{D(G)}. \]

We define collections of perturbations \( [\mathcal{G}(G,m)] \in \mathcal{G}((\beta, \leq \delta)) \) of the collection of Ku-
ranishi spaces \( [\overline{\mathcal{M}_{G,m}}] \in \mathcal{G}((\beta, \leq \delta)) \) which satisfy suitable transversality conditions,
compatibility on the corner faces and forgetful compatibility. The associated virtual fundamental chain leads to a \( MC \)-cycle \( Z = (Z_{G,m}) \in \mathcal{G}((\beta, \leq \delta)) \).

2.3. Coherent Cycles. For \((w, w') \in \mathcal{P}_\gamma\), define \((w, w')^k \in \text{Gen}(\gamma)\) as
\[ (w, w')^k = (w_1, w_2, \ldots , w_k) \]
where \( w_i \) are closed one-chains, with \((w_i, w_j) \in \mathcal{P}_\gamma\) for each \( i, j \), and \((w_i, w_j) = (w, w')^k \) as element of \( \mathcal{P}_\gamma\). \((w, w')^k \) is well defined up to small isotopy.

Let \( Z_{\gamma}^{\circ, \circ} \) be the set of linear combinations of
\[ \exp \left( \frac{1}{2} (w, w') \right) = \sum_k \frac{1}{2^k k!} (w, w')^k \]

**Lemma 4.** \( Z_{\gamma}^{\circ, \circ} \) is a rank one modulo over \( \mathbb{Q}[[g_\gamma]] \).

Given two coherent cycles with disjoint support we can define the product
\[ (13) \quad \exp \left( \frac{1}{2} (w_1, w'_1) \right) \exp \left( \frac{1}{2} (w_2, w'_2) \right) = \exp \left( \frac{1}{2} (w_1 + w_2, w'_1 + w'_2) \right). \]
Proposition 5. To each $\beta \in H_2(X, L)$ it is associated an element $Z_\beta \in Z_\beta^{\omega, o}$ up to isotopy, in the same sense of Theorem 2.

Proof. Above we have only considered the notion of coherent nice $MC$-cycle. To adapt the proof of Theorem 2 to coherent cycles it is necessary to extend the notion of coherent cycle to (not-nice) $MC$-cycles. For this we need to adapt the definition of forgetful compatibility. Recall that the forget compatibility of $\mathcal{G}$ is defined in terms of chains with coefficients on $\mathcal{G}en(\gamma)$. We require that these chains have coefficients on nice coherent $MC$-cycles.

We remark that the $MC$-cycle constructed in $\mathcal{G}$ from the moduli space of (multi-) pseudo-holomorphic curves is coherent. Actually it satisfies a stronger property: $w_h = w_{h'}$ for each $h, h'$.

Finally the correspondence (up to isotopy) $Z \sim Z^{nice}$ can be proved for coherent $MC$-cycles using the same argument. $\square$

2.4. $MC$-cycles including the degrees. In $\mathcal{G}$ we also considered nice $MC$-cycles which keep trace of the components of multi-curves. A component will corresponds to an element of $V$ in $\mathcal{G}$ below. Here we consider a slightly different version, which forget about the Euler Characteristic of each single component and as before we consider arbitrary total Euler Characteristic at the same time.

The nice $MC$-cycles we define below arise from moduli space of multi-curves associated to graphs without closed components. (A closed component of a decorated graph $G \in \mathcal{G}$ considered in $\mathcal{G}$ is a component $c \in \text{Comp}(G)$ with $V_c = D_c = \emptyset$.) That is, we have discarded the purely closed contributions from the $MC$-cycle.

We fix the following data:

- An oriented three manifold $M$,
- a finite-rank free abelian group $\Gamma$, called topological charges,
- an homorphism of abelian groups $\partial : \Gamma \to H_1(M, \mathbb{Z})$ called boundary homomorphism.
- an homomorphism of abelian groups $\omega : \Gamma \to \mathbb{R}$ called symplectic area;
- A one dimensional homology class $c \in H_1(M, \mathbb{Z})$ called Chern-Class.

In the geometric context associated to $(X, L)$ we have

- $\Gamma = H_2(M, L, \mathbb{Z})$
- $\partial : \Gamma \to H_1(L, \mathbb{Z})$ is the usual boundary in homology.
- $\omega(\beta) = \int_{\beta} \omega$ for $\beta \in \Gamma$.
- $c = c(K) \in H_1(L, \mathbb{Z})$ is defined from the four chain $K$ (see $\mathcal{G}$).

A generator of the $MC$-cycles is defined by an array

$$ (V, H, (w_v, h)_{v \in V}, (h \in H, (\beta_v)_{v \in V}) $$

where $V$ and $H$ are finite sets, $\beta_v \in \Gamma$, $(w_v, h)_{v \in H}$ are closed integer one dimensional chain on $M$ on the homology class $\partial \beta_v$, which are close in the $C^0$-topology. $(h \in H, (\beta_v)_{v \in V})$ are closed half-integer one dimensional chains on $M$ on the homology class $c$, which are close in the $C^0$-topology.
We assume that
\begin{equation}
\beta_v \notin \text{tors}(\Gamma), \quad \| \beta_v \| \leq C_{\text{supp}} \omega_v \quad \text{for each } v \in V(G)
\end{equation}

Denote by \( \mathcal{G} \mathcal{E} n(\beta) \) the set of objects \((14)\) such that \( \sum \beta_v = \beta \), modulo the obvious equivalence relation.

Observe that \((15)\) implies that there exists \( N_{\beta} \in \mathbb{Z}_{>0} \) such that
\begin{equation}
|V| \leq N_{\beta}
\end{equation}

for each generator \((14)\).

Set \( w_h = \sum w_{v,h} + w_{h}^{\text{ann}0} \). Define \( \mathcal{G} \mathcal{E} n(\beta) \) as the subset of \( \mathcal{G} \mathcal{E} n(\beta) \) obtained imposing the extra condition
\begin{equation}
w_h \cap w_{h'} = \emptyset \quad \text{if } h \neq h'.
\end{equation}

Let \( \mathcal{Z}_{\beta} \) be the set of formal power series
\[ \sum_i g_i^k \mathcal{G} \mathcal{E} n_i \]
with \( \mathcal{G} \mathcal{E} n_i \in \mathcal{G} \mathcal{E} n(\beta) \), \( k_i \to \infty \), and \( k_i + |V_i| \geq 0 \). The last condition is related to the fact we have discarded the closed components from the \( \text{MC} \)-cycle.

To define isotopies of nice-\( \text{MC} \) cycles \( \hat{Z} \) we consider objects
\begin{equation}
(V, H, (\hat{w}_{v,h})_{v \in V, h \in H}, (\hat{w}_h^{\text{ann}})_{h \in H}, (\beta_v)_{v \in V}, [a, b])
\end{equation}
where \([a, b] \subset [0, 1] \), \( \hat{w}_{v,h} : F_v \times [a, b] \to M \) for some one dimensional compact manifold \( F_v \).

Let \( \mathcal{G} \mathcal{E} n(\beta) \) be the set of objects \((18)\) modulo the obvious equivalence relation.

An isotopy of \( \text{of nice-} \text{MC} \)-cycles \( \hat{Z} \) is defined by formal power series
\[ \hat{Z} = \sum_i r_i g_i^k (V_i, H_i, (\hat{w}_{v,h,i})_{v \in V_i, h \in H_i}, (\hat{w}_h^{\text{ann}})_{h \in H_i}, (\beta_v)_{v \in V_i}, [a_i, b_i]) \]
with \( r_i \in \mathbb{Q} \), \( k_i \to \infty \), and \( k_i + |V_i| \geq 0 \).

The formal power series are considered modulo gluing of objects \((18)\).

\( \hat{Z} \) defines a one parameter family of nice \( \text{MC} \)-cycles \( (\hat{Z}^t)_t \), \( \hat{Z}^t \in \mathcal{Z}_{\beta} \) discontinuous for a finite number of times. The discontinuity at the time \( t_0 \) is obtained by the formula:
\begin{equation}
\hat{Z}^{t_0} - \hat{Z}^{t_0} = \pm r' g_i^k (V', H', (w'_{v,h})_{v \in V', h \in H'}, (w_h^{\text{ann}0'})_{h \in H'}, (\beta'_v)_{v \in V'})
\end{equation}

where

- If \( \hat{w}_{v_1,h_{1,i}} \) crosses \( \hat{w}_{v_2,h_{2,i}} \) transversely at the time \( t_0 \) we have two cases:
  - \( v_1 \neq v_2 \); \( H' = H_i \setminus \{ h_1, h_2 \} \), \( V' = V_i \setminus \{ v_1, v_2 \} \sqcup v_0 \), where \( v_0 \) is a new vertex with associated data \( \beta'_{v_0} = \beta_{v_1} + \beta_{v_2} \), \( w'_{v_0,h} = w_{v_1,h,i}^{\text{ann}} + w_{v_2,h,i}^{\text{ann}} \) for each \( h \neq h_1, h_2 \). All the other data remain the same.
  - \( v_1 = v_2 \); \( H' = H_i \setminus \{ h_1, h_2 \} \), \( V' = V_i \). All the other data remain the same.

- If \( \hat{w}_{v',h_{1,i}} \) crosses \( \hat{w}_{h_{2,i}}^{\text{ann}} \), \( H' = H_i \setminus \{ h_1, h_2 \} \), \( V' = V_i \). All the other data remain the same.

\( r' = r_i, k' = k_i + 1 \) and the sign is defined by the sign of the crossing.
2.5. **Extension to a full MC-cycle.** Assume that we have a filtration on the set of decorated graphs $G^0 \subset G^1 \subset \ldots$. Assume that

- if $G \in G^k$ then forget $eG \in G^k$ for each $e \in E^{ex}(G)$;
- if $\delta eG \in G^k$ for $e \in E^{in}(G)$, then $G \in G^k$.

Under this assumptions we can make the truncation of MC-chain complex to $G^k$. We denote by $Z_k$ the corresponding space of MC-cycles.

We have the following two fundamental Lemmas:

**Lemma 6.** Let $Z^0_k \in Z_k$, $Z^1_k \in Z_k$ and $Z_k \in \tilde{Z}_k$ isotopy between $Z^0_k$ and $Z^1_k$. Let $Z^0_{k+1} \in \tilde{Z}_{k+1}$ extending $Z^0_k$.

There exist $Z^1_{k+1} \in Z_{k+1}$ extending $Z_k$ and $Z_{k+1} \in \tilde{Z}_{k+1}$ isotopy between $Z^0_{k+1}$ and $Z^1_{k+1}$ extending $\tilde{Z}_k$.

**Lemma 7.** Let $\tilde{Z}_k \in \tilde{Z}_k$ be an isotopy of isotopies such that

- $\tilde{Z}_k \cap \{s < -S\} = Z^0_k \times \mathbb{R}_{s < -S}$ for $S > 0$
- $\tilde{Z}_k \cap \{t < -T\} = Z^1_k \times \mathbb{R}_{s > S}$ for $T > 0$
- $\tilde{Z}_k \cap \{t > T\} = Z^1_k \times \mathbb{R}_{t > T}$ for $T > 0$
- $\tilde{Z}_k \cap \{t > T\} = \tilde{Z}_k \times \mathbb{R}_{t > T}$ for $T > 0$

Let $Z^0_{k+1}, Z^1_{k+1} \in \tilde{Z}_{k+1}$ extending $Z^0_k, Z^1_k, Z^1_{k+1}$.

There exists $Z^1_{k+1} \in \tilde{Z}_{k+1}$ extending $Z^1_k$ and $Z_{k+1} \in \tilde{Z}_{k+1}$ extending $\tilde{Z}_k$ such that

- $\tilde{Z}_{k+1} \cap \{s < -S\} = Z^0_{k+1} \times \mathbb{R}_{s < -S}$ for $S > 0$
- $\tilde{Z}_{k+1} \cap \{s > S\} = Z^1_{k+1} \times \mathbb{R}_{s > S}$ for $T > 0$
- $\tilde{Z}_{k+1} \cap \{t < -T\} = Z^1_{k+1} \times \mathbb{R}_{t > T}$ for $T > 0$
- $\tilde{Z}_{k+1} \cap \{t > T\} = Z^1_k \times \mathbb{R}_{t > T}$ for $T > 0$

**Lemma 8.** Fix $J$. For each $k$, fix perturbation, etc and obtain a MC-cycle $Z_k$ up to $k$. Assume that the perturbations are small enough so that there exists $\tilde{Z}_k$ isotopy between $Z_{k+1}$ and $Z_k$ for each $k$. We can define inductively a MC-cycle $\tilde{Z}_k$ and isotopy $\tilde{Z}_k$ so that

- $\tilde{Z}_k$ extends $\tilde{Z}_{k-1}$;
- $\tilde{Z}_k$ is an isotopy between $Z_k$ and $\tilde{Z}_k$.

**Proof.** Assume that $\tilde{Z}_k$ is defined. Apply Lemma 6 to the composition of $\tilde{Z}_k$ and $\tilde{Z}_k$ to obtain $\tilde{Z}_{k+1}$ and an isotopy $\tilde{Z}_{k+1}$ between $Z_{k+1}$ and $\tilde{Z}_{k+1}$. \[\Box\]

Let us take two choices $J_0, J_1$. As above, for each $k$, fix perturbation, etc and obtain $Z^0_k, \tilde{Z}^0_k, Z^1_k, \tilde{Z}^1_k$, for each $k$. Assume also that we have defined isotopies $\tilde{Z}_k$ between $Z^0_k$ and $Z^0_k$, and isotopy of isotopies $\tilde{Z}_k$ such that

- $\tilde{Z}_k$ agrees with $\tilde{Z}^0_k$ for $s = 0$;
- $\tilde{Z}_k$ agrees with $\tilde{Z}^1_k$ for $s = 1$;
- $\tilde{Z}_k$ agrees with $\tilde{Z}_k$ for $t = 0$;
- $\tilde{Z}_k$ agrees with $\tilde{Z}_{k+1}$ for $t = 1$.

Apply Lemma 8 to obtain $\tilde{Z}^0_k, \tilde{Z}^0_k, \tilde{Z}^1_k, \tilde{Z}^1_k$. 

Lemma 9. We can define inductively a MC-cycle isotopy \( \tilde{Z}_k \) and isotopy of isotopy \( \tilde{\tilde{Z}}_k \) so that \\
\( \tilde{Z}_k \) extends \( \tilde{Z}_{k-1} \) and \\
\( \bullet \) \( \tilde{Z}_k \) agrees with \( \tilde{Z}_0 \) for \( s = 0 \); \\
\( \bullet \) \( \tilde{Z}_k \) agrees with \( \tilde{Z}_k \) for \( s = 1 \); \\
\( \bullet \) \( \tilde{Z}_k \) agrees with \( \tilde{Z}_k \) for \( t = 0 \); \\
\( \bullet \) \( \tilde{Z}_k \) agrees with \( \tilde{Z}_k \) for \( t = 1 \).

Proof. The argument is the same of Lemma 8 using Lemma 7.

\( \square \)

3. Boundary Conditions

We assume that \( X \) contains a compact domain \( X^{in} \) with boundary a smooth hypersurface \( Y \) which is a contact manifold, such that \( X \setminus X^{in} \) can be identified with \( \mathbb{R}^+ \times Y \). We write

\[
X = X^{in} \cup_Y (\mathbb{R}^+ \times Y),
\]

where \([1, +\infty) \times Y\) is the symplectization of \( Y \).

such that the translation in the \( \mathbb{R}^+ \) on \( \mathbb{R}^+ \times Y \) preserves all (?) the geometric data (such as the complex structure.)

We assume that the lagrangians submanifolds \( L \) which are cylindrical Legendrian \( \mathbb{R}^+ \times \partial X_c \). Compatible with (20), we assume that,

\[
L = L^{in} \cup_{\Sigma} (\mathbb{R}^+ \times \Sigma)
\]

where \( L^{in} = X^{in} \cap L \) and \( \Sigma = \partial L^{in} = Y \cap L \) is a Legendrian submanifold of \( \partial X_c \).

We also assume that the four chain is written as

\[
K = K^{in} \cup_T (\mathbb{R}^+ \times T)
\]

where \( K^{in} = X^{in} \cap K \) and \( T = Y \cap K \).

We make the following assumption

\( \bullet \) For any real number \( E > 0 \), we can chose \( X^{in} \) such that all the not-constant holomorphic curves with boundary on \( (X, L) \) of area less than \( E \) are mapped in \( X^{in} \).

We shall consider objects in the Novikov-Ring, and consider the objects up to \( E \).

Remark 10. A standard way to achieve the assumption above is to impose convexity properties on the pair \( (X, L) \).

As usual in non-compact geometries, in order to obtain well defined objects it is necessary to fix suitable boundary conditions at infinity. The assumptions above assure that the moduli space of pseudo-holomorphic curves of a fixed positive area live in a compact region of \( X \). However the moduli space of pseudo-holomorphic curves of area zero is always not compact if \( X \) is not compact.

The moduli space of pseudo-holomorphic curves \( \overline{\mathcal{M}}^{main}_{(g,h),(n,\bar{m})}(\beta = 0) \) in the class \( \beta = 0 \) set theoretically can be identified with \( X \times \overline{\mathcal{M}}^{main}_{(g,h),(n,\bar{m})} \) if \( h = 0 \), or \( L \times \overline{\mathcal{M}}^{main}_{(g,h),(n,\bar{m})} \) if \( h \neq 0 \), where \( \overline{\mathcal{M}}^{main}_{(g,h),(n,\bar{m})} \) is the Deligne-Mumford compactification of the moduli of the Riemannian surfaces. The obstruction bundle (7) becomes a sub-vector space

\[
E_p \subset T_p X \otimes \Lambda^{0,1}(\Sigma),
\]
where \( p \in X \) is the constant value of the pseudo-holomorphic curve \( p \).

It makes sense to consider area zero curves whose target is an open subset of \( X \). In particular we shall be interested to area zero curves with target \( \mathbb{R}^+ \times Y \).

In closed Gromov-Witten theory (which is the particular case of our problem where \( L = \emptyset \) and \( K \) is a closed four chain) a choice of a boundary condition consists on the choice of a perturbation of the moduli space of area zero curves on \( \mathbb{R}^+ \times Y \) that is invariant under the translation on the \( \mathbb{R} \)-direction. Note that this makes sense since there is an obvious action of \( \mathbb{R}^+ \) on RHS of (21), which is trivial on the second factor. Note that in the closed case the boundary condition affects only the degree zero invariant.

In the context of open-closed Gromov-Witten theory (\( L \neq \emptyset \)) we need to work with the moduli of multi-curves \( \overline{M}_{G,m} \). Since there can be area zero multi-curve components in each degree, the choice of a boundary condition affects the moduli of multi-curves of degree not zero also, in contrast with what happen in the closed case.

The generalization of the notion of boundary condition used in the context of closed Gromov-Witten theory is not straightforward since the set of area zero components of the multi-curve mapped on \( \mathbb{R}^+ \times Y \) can be different in different regions of the moduli space \( \overline{M}_G \). Hence it is necessary to set the invariance condition in the \( \mathbb{R} \)-direction consistently in a suitable sense.

3.1. Boundary Conditions. We first explain the definition of the boundary conditions for decorated graphs with \( |m| = 0 \).

Let \( G \in \mathcal{G} \) with \( \beta(G) = 0 \). Let \( \mathcal{P} = \{ P_i \} \) be an ordered partition of \( \text{Comp}(G) \).

For each \( i \), let \( H_i = \bigsqcup_{c \in P_i} H_c \). Set
\[
\mathcal{E}(\mathcal{P}) = \bigsqcup_{i} \{ (h, h') \in H_i \times H_i | h \neq h' \} / \mathbb{Z}_2.
\]

Using the partition \( \mathcal{P} \) define the decorated graph
\[
cut_{\mathcal{P}}(G) = (\text{Comp}(G), \mathcal{E}(\text{cut}_{\mathcal{P}}(G)), H(G), V(G), D(G)),
\]
obtained cutting the edges of \( G \) which do not belongs to \( \mathcal{E}(\mathcal{P}) \):
\[
\mathcal{E}^\text{in}(\text{cut}_{\mathcal{P}}(G)) = \mathcal{E}^\text{in}(G) \cap \mathcal{E}(\mathcal{P}),
\]
We have a unique decomposition of \( G \) is sub-graphs
\[
(22)
cut_{\mathcal{P}}(G) = \bigsqcup_i G_i^d.
\]
such that \( \text{Comp}(G_i) = P_i \).

Let
\[
t : \mathbb{R} \times Y \to \mathbb{R}
\]
be the projection on the \( \mathbb{R} \) factor, which we refer as the time function.

For \( \rho, \eta > 0 \), let \( Y[P, \rho, \eta] \) be the set of \( (z_h)_h \in (\mathbb{R} \times Y)^H \) such that
- \( \max_{h \in P} t(z_h) + \eta < \min_{h \in P^{i+1}} t(z_h) \) for \( 1 \leq i \leq f - 1 \);
- \( \max_{h \in P} t(z_h) < \min_{h \in P} t(z_h) + \rho \) for \( 1 \leq i \leq f \).

A boundary data \( B \) is specified by a sequence of collections of Kuranishi structures and perturbations \( \{ \overline{M}_G^{\bullet, n}, s_G^{\bullet, n} \} \) with the following properties:
- conditions of Subsection 2.2.1 hold;
- \( \{ \overline{M}_G^{\bullet, n}, s_G^{\bullet, n} \} \) are invariant under translation in the \( \mathbb{R} \)-direction;
Lemma 11. There exist boundary data. Two boundary data are isotopic.

Proof. We proceed by induction.

Assume that we have defined \( \{ \mathcal{M}_G^n, s_G^n \} \) for \( G \prec G_0 \).

For each \( \mathcal{P} \) we define \( \rho_{\mathcal{P}}, \zeta_{\mathcal{P}} \), Kuranishi structure \( \mathcal{M}_{G,\mathcal{P}}^n \), perturbations \( s_{G,\mathcal{P}}^n \) on the region \( Y(\mathcal{P}, \rho_{\mathcal{P}}, \zeta_{\mathcal{P}}) \) such that

1. \( \mathcal{M}_{G,\mathcal{P}}^n, s_{G,\mathcal{P}}^n \) satisfy the conditions of Subsection 2.2.1
2. \( \mathcal{M}_{G,\mathcal{P}}^n, s_{G,\mathcal{P}}^n \) are translation invariant in the \( \mathbb{R} \) direction;
3. \( \partial Y(\mathcal{P}, \rho_{\mathcal{P}}, \zeta_{\mathcal{P}}) \subset \bigcup_{\mathcal{P}_1 \prec \mathcal{P}} Y(\mathcal{P}_1, \rho_{\mathcal{P}_1}, \zeta_{\mathcal{P}_1}) \),
4. \( \mathcal{M}_{G,\mathcal{P}}^n, s_{G,\mathcal{P}}^n \) extends the Kuranishi structures and perturbations defined on the LHS of (25) by \( \mathcal{M}_{G,\mathcal{P}_1}^n, s_{G,\mathcal{P}_1}^n \) for \( \mathcal{P}_1 \prec \mathcal{P}_2 \).

Now we proceed by induction on \( \mathcal{P} \). For each \( \mathcal{P}_1 \prec \mathcal{P} \), assume that we have defined \( \rho_{\mathcal{P}_1}, \zeta_{\mathcal{P}_1} \) and \( \mathcal{M}_{G,\mathcal{P}_1}^n, s_{G,\mathcal{P}_1}^n \).

Take \( \rho_{\mathcal{P}_1} \) big enough such that (25) holds.
Take \( \eta_{\mathcal{P}_1} \) such that (23) and (24) hold for \( G \prec G_0 \), and moreover property (26) holds.
If \( |\mathcal{P}| > 1 \), condition (23) defines the Kuranishi structure on \( Y[\mathcal{P}, \rho/n, \eta/n] \).
If \( \mathcal{P} \) is the trivial partition, the Kuranishi structures defined in the preview steps and condition (26) fixes the Kuranishi structure on the RHS of (25). Extend this Kuranishi structure to a Kuranishi structure on \( M[\mathcal{P}, \rho/n, \eta/n] \) which is compatible with one defined in the preview steps on the corner faces \( \mathcal{M}_G(G', E')(G', E') \in \Theta(G) \).
In the same way we can construct the perturbation \( s_G^n \).

3.1.1. Boundary Conditions. An open partition \( \mathcal{P}^{\text{open}} \) consists of an array \( (P_0, P_1, ..., P_n) \) of disjoint sub-sets of \( \text{Comp}(G) \) with

- \( P_i \neq \emptyset \) for each \( i \),
- \( P^\text{int} \sqcup \sqcup P_1 = \text{Comp}(G) \).

There is an obvious partial ordering on the set of open partitions.
Extend to \( M \) the function \( t \) used in the preview subsection setting \( t(z) = 0 \) for each \( z \in M^\text{int} \). For \( \rho, \eta > 0 \), let \( M[\mathcal{P}, \rho, \eta] \) be the set of \( \{ z_h \} \in M^H \) such that

- \( \max_{h \in H^\text{int}} t(z_h) + \eta < \min_{h \in P_1} t(z_h), \max_{h \in H^\text{int}} t(z_h) < \rho; \)
• \( \max_{h \in P_i} t(z_h) + \eta < \min_{h \in P_{i+1}} t(z_h) \) for \( 1 \leq i \leq f - 1 \);
• \( \max_{h \in P_i} t(z_h) < \min_{h \in P_i} t(z_h) + \rho \) for \( 1 \leq i \leq f \).

We say that a collection of Kuranishi structures and perturbations \( \{ \mathcal{M}_G, s_G \}_G \) are compatible with the boundary condition specified by the boundary data \( B \) if

- the conditions of Subsection 2.2.1 hold;
- for each \( G, P, \rho \) there exists \( \eta \) such that on the region \( M[P, \rho/n, \eta/n] \) we have an identification of Kuranishi spaces
  \[ \mathcal{M}_G = \mathcal{M}_{G^{in}} \times \prod_i \mathcal{M}_{G_i} \]
  and accordingly
  \[ s^n_G = s^{n}_{G^{in}} \times \prod_i s^n_{G_i} \].

The following is proved adapting the inductive argument used in the proof of Lemma 11.

**Lemma 12.** For each boundary data \( B \), there exist Kuranishi structures and perturbations compatible with \( B \).

### 3.1.2. Extension to \( |m| > 0 \)

We now adapt the definition of boundary condition provided above to graphs \((G, m) \in \mathfrak{G}_*\) with \( |m| > 0 \). We need to extend (23), (24) in a way compatible with the boundary of the moduli space. For this we need to introduce some notation.

Consider an order set \( k = \{ k_0, k_1, ..., k_f \} \) of integer numbers with \( k_i \leq k_{i+1} \), \( 0 \leq k_i \leq l \). We assume

\[ k_0 = 0, k_f = l. \] (27)

Denote by \( \Delta^l \) the standard simplex of dimension \( l \). Set

\[ \Delta^{l,k} = \prod_{i=1}^f \Delta^{k_i-k_{i-1}}. \]

We label the boundary faces of \( \Delta^{l,k} \) in the following way:
- for each \( i \notin k \), \( \partial_i \Delta^{l,k} \);
- if \( k_j > k_{j-1} \), \( \partial_{k_j}^- \Delta^{l,k} \);
- if \( k_j < k_{j+1} \), \( \partial_{k_j}^+ \Delta^{l,k} \).

We have canonical identifications

\[ \partial_{k_j}^- \Delta^{l,k} = \partial_{k_j}^+ \Delta^{l,k} \]

whenever \( k_j > k_{j-1} \), and

\[ \partial_i \Delta^{l,k} = \Delta^{l-1,k} \]

where we have denoted \( \partial_i k = \{ k_0, ..., k_{j-1}, k_j - 1, ..., k_f - 1 \} \) if \( k_{j-1} < i < k_j \).

It is elementary to check that

\[ (\bigsqcup_k \Delta^k) / \sim \cong \Delta^l. \] (30)

where the LHS is defined making the identifications (28). The identification (30) is far to be unique.

Note that a choice of (28) for \( l \) and \( l - 1 \) together with (29) yield to an identification \( \partial_i \Delta^l = \Delta^{l-1} \). We shall consider identification (30) for different \( l \) such that
the last equation agrees with the standard embedding of $\Delta^{l-1}$ with the boundary face $\partial_i \Delta^l$.

Consider a decorated graph $(G, m)$ and a partition $\mathcal{P}$ as above. Write $m = \{E_0, E_1, ..., E_l\}$.

Set

$$cut_\mathcal{P}(G, m) = (cut_\mathcal{P}(G), cut_\mathcal{P}(m))$$

where

$$cut_\mathcal{P}(m) = \{E^{cx}(cut_\mathcal{P}(G)), E_1 \cap E(\mathcal{P}), ..., E_l \cap E(\mathcal{P})\}$$

if $m = \{E_0, E_1, ..., E_l\}$ with $E^0 = E^{cx}(G)$. Define $m^i$ from the relation

$$(31) \quad cut_\mathcal{P}(G, m) = \bigsqcup_i (G_i^m, m^i).$$

Set

$$(32) \quad \overline{M}_{G, m, \mathcal{P}, k} = \prod_{0 \leq i < f} \overline{M}_{G_i^m, m_i^k, k_{i+1}},$$

where we have used the notation $m_{[a,b]} = \{E_a, E_{a+1}, ..., E_b\}$,

if $m = \{E_0, E_1, ..., E_l\}$.

If the spaces associated to $(G^i, m^i)$ are endowed with a Kuranishi structure, relation (32) defines a Kuranishi structure on $\overline{M}_{G, m, \mathcal{P}, k}$.

A choice of the identification (30) yields to an identification

$$(33) \quad \overline{M}_{G, m} = (\bigsqcup_k \overline{M}_{G, m, \mathcal{P}, k})/\sim.$$  

We want to refine condition (23) asking that (33) becomes an identification of Kuranishi spaces compatible with the corner faces. In particular this involves a choice of an identification (30) depending on the point on moduli space $\overline{M}_G$. This identification has to be compatible at the corner faces with the identification made in the the previews inductive steps.

Now we can adapt the inductive argument of Lemma 11 in this setting. The $|\mathcal{P}| = 1$ is the same of Lemma 11. Assume $|\mathcal{P}| > 1$. The choice of an identification (30) for $\mathcal{P}' \prec \mathcal{P}$ induces an identification (30) on the region defined by the RHS of (25). Pick an extension of it on $\overline{M}_{G, \mathcal{P}}$ which is compatible with the identification inducted on the boundary strata $\overline{M}_{G, m}(G', m', E')$ for each $(G', m', E') \in \mathcal{G}(G, m)$ by the one defined in the preview steps. Use it to define the Kuranishi structure of $\overline{M}_{G, m}$ on the region $Y[\mathcal{P}, \rho/n, \eta/n]$ compatible with (33).

3.1.3. Change of Boundary Data. After we fix a boundary data $B$, the construction of section 2.2.1 applied to a perturbation of the moduli space considered on subsection 3.1.1 yield to a MC-cycle

$$Z^B \in \mathcal{Z}$$

which is well defined up to isotopy.

Now we want to consider how $Z^B$ changes when we change $B$. Fix two boundary data $B_0$ and $B_1$. In a similar way to what we have done in subsection 3.1.1, we consider the problem on $\mathbb{R} \times Y$ and define Kuranishi structures and perturbations compatible with the boundary condition $B_0$ for $t \ll 0$ and $B_1$ for $t \gg 0$. This yield to a MC-cycle

$$Z^{B_0, B_1} \in \mathcal{Z}.$$
A Kuranishi structure and perturbation on the moduli space on $X$ compatible with the boundary data $B_0$ and a Kuranishi structure on the moduli $\mathbb{R} \times Y$ compatible with with the boundary data $B_0$ and $B_1$ as above can be glued to obtain a Kuranishi structure and perturbation on the moduli space on $X$ compatible with the boundary data $B_1$. The corresponding MC-cycles satisfies a relation which we write schematically as

$$Z^{B_1} = Z^{B_0, B_1}.$$

The same considerations yield to the relation

$$Z^{B_0, B_2} = Z^{B_0, B_1} \boxplus Z^{B_1, B_2}$$

when we consider three boundary datas $B_0$, $B_1$ and $B_2$.

3.2. **Frame refined Four Chain.** In order to fix partially the boundary conditions at infinity In this section we define a refined version of the four chain. The residue ambiguity will be analogous to the one arising in Closed Gromov-Witten invariants.

3.2.1. Frames and Spin Structures. We start recalling some standard and elementary fact about frames on a compact oriented 3-manifold $M$. Fix a riemannian metric on $M$ (up to isotopy, all the objects we will consider do not dependent on the metric).

Denote by $\text{Spin}(M)$ the set of spin structures on $M$. Recall that $\text{Spin}(M)$ is a torsor on the group $H^1(M, \mathbb{Z}_2)$. Since $\text{Spin}(3) \approx S^3$ as differential manifold, from the degree of the map we obtain an isomorphism of groups

$$\text{Maps}(M, \text{Spin}(3))/\text{isot} \cong \mathbb{Z}.$$  

Moreover, since $\text{Spin}(3)$ is a $\mathbb{Z}_2$-covering of $SO(3)$, there is an exact sequence of groups

$$0 \rightarrow \text{Maps}(M, \text{Spin}(3))/\text{isot} \rightarrow \text{Maps}(M, SO(3))/\text{isot} \rightarrow H^1(M, \mathbb{Z}_2) \rightarrow 0.$$  

Denote by $\mathcal{F}(M)$ the set of orthogonal frames of $M$, i.e., the set of orthogonal trivializations of the tangent bundle of $M$. $\mathcal{F}(M)$ is a torsor on $\text{Maps}(M, SO(3))$. Denote by $\mathcal{F}(M)/\text{isot}$ the set of orthogonal frames modulo isotopies, which is a torsor on $\text{Maps}(M, SO(3))/\text{isot}$.

Since a frame in particular determines a spin structure, we have a natural map of sets

$$\mathcal{F}(M)/\text{isot} \rightarrow \text{Spin}(M).$$

From (35) we obtain the identification of sets

$$\mathcal{F}(M)/\text{isot} \cong \text{Spin}(M).$$

where $\mathbb{Z}$ acts on $\mathcal{F}(M)/\text{isot}$ throughout the isomorphism (34).

For our purpose it will be useful to consider the restriction to $\text{Maps}(M, SO(2))$ of the action of $\text{Maps}(M, SO(3))$ on $\mathcal{F}(M)$, where $SO(2)$ is embedded on $SO(3)$ from the inclusion $\mathbb{R}^2 \cong \{0\} \times \mathbb{R}^2 \subset \mathbb{R}^3$. Note that

$$\text{Maps}(M, SO(2))/\text{isot} \cong H^0(M, U(1))/H^0(M, \mathbb{R}) \cong H^1(M, \mathbb{Z})$$

From this, we obtain an action of the group $H^1(M, \mathbb{Z})$ on $\mathcal{F}(M)/\text{isot}$:

$$H^1(M, \mathbb{Z}) \times \mathcal{F}(M)/\text{isot} \rightarrow \mathcal{F}(M)/\text{isot}.$$
3.2.2. The case $\Sigma \times \mathbb{R}$. Consider the particular case $M = \Sigma \times \mathbb{R}_t$ (where $t$ denotes the coordinate on the factor $\mathbb{R}$, which we refer as time coordinate). Let $\mathfrak{g}(\Sigma \times \mathbb{R})^\mathbb{R}$ be the set of elements of $\mathfrak{g}(\Sigma \times \mathbb{R})$ invariant under translation in the $t$-direction. $\mathfrak{g}(\Sigma \times \mathbb{R})^\mathbb{R}$ can be identified with $\mathfrak{g}(T\Sigma \oplus \mathbb{R})$, where $\mathbb{R}$ denotes the trivial real bundle of dimension one over $\Sigma$.

We say that two frames $\mathfrak{f}r_1, \mathfrak{f}r_2 \in \mathfrak{g}(\Sigma \times \mathbb{R})^\mathbb{R}$ are isotopic if there exists $\tilde{\mathfrak{f}} \in \mathfrak{g}(\Sigma \times \mathbb{R})$ such that

$$\tilde{\mathfrak{f}} \equiv \mathfrak{f}r_1 \text{ for } t \ll 0, \quad \tilde{\mathfrak{f}} \equiv \mathfrak{f}r_2 \text{ for } t \gg 0.$$ 

Two frames are isotopic if and only if they induce the same spin structure on $\Sigma$:

$$(\mathfrak{g}(\Sigma \times \mathbb{R})^\mathbb{R})/\text{isotopies} \cong \text{Spin}(\Sigma).$$

The group action (37) induces on $\Sigma \times \mathbb{R}$ the action of $H^1(\Sigma, \mathbb{Z})$ on $\mathfrak{g}(\Sigma \times \mathbb{R})^\mathbb{R}/\text{iso}$. Now consider pairs

$$\mathfrak{f}r = (fr_1, fr_2, fr_3) \in \mathfrak{g}(T\Sigma \oplus \mathbb{R}) \text{ and } V \in \Gamma(\Sigma, T\Sigma \oplus \mathbb{R}) \text{ is compatible with } \mathfrak{f}r \text{ in the following sense: there exist } a_1, a_2, a_3 \in \mathbb{R} \text{ such that }$$

$$V|_{\Sigma} = a_1fr_1 + a_2fr_2 + a_3fr_3.$$

The difference of two pairs $(\mathfrak{f}r_0, V_0), (\mathfrak{f}r_1, V_1)$ is defined as

$$(\mathfrak{f}r_0, V_0) - (\mathfrak{f}r_1, V_1) = [\text{graph}(\tilde{V})] \in C_3(S(T\Sigma \oplus \mathbb{R})) / \partial C_4(S(T\Sigma \oplus \mathbb{R})),$$

where $\tilde{V} = (V_t)_{t \in [0,1]}, V_t \in \Gamma(S(TM)|_\Sigma)$ and there exists an isotopy of frames $(\mathfrak{f}r_t)_{t \in [0,1]}$ with $V_t$ compatible with $\mathfrak{f}r_t$ for each $t$. Here $\text{graph}(\tilde{V})$ is considered as a map $\Sigma \times [0,1] \rightarrow S(TM)$.

There is an obvious action of the group $\text{Maps}(\Sigma, SO(2))$ on the set of pairs (38) defined by

$$A \cdot (\mathfrak{f}r, V) = (A_V \cdot \mathfrak{f}r, V)$$

where $A_V$ is the rotation in the direction $V$ with angle defined by $A \in \text{Maps}(\Sigma, SO(2))$.

It is easy to check the following formula

$$A \cdot (\mathfrak{f}r_0, V_0) - (\mathfrak{f}r_1, V_1) = ([\mathfrak{f}r_0, V_0] - (\mathfrak{f}r_1, V_1)) + PD([A]),$$

where $[A] \in \text{Maps}(\Sigma, SO(2))/\text{isot} \cong H^1(\Sigma, \mathbb{Z})$, and $PD([A]) \in H_1(\Sigma, \mathbb{Z}) = H_3^-(S(T\Sigma \oplus \mathbb{R}))$.

3.2.3. Euler Structures: the compact case. An Euler Structures on a compact oriented three manifold $M$ is a nowhere vanishing vector field on $M$. Two Euler Structures on $M$ are called homologous if they are isotopic outside a ball as nowhere vanishing vector fields. We denote by $\mathfrak{e}u(\mathfrak{e}u(M)$ the set of homology classes of Euler Structures. $\mathfrak{e}u(M)$ is a torsor on $H_1(M, \mathbb{Z})$. The Chern Class of an Euler class $\mathfrak{e}u$ is defined as $\epsilon([\mathfrak{e}u]) = V - \text{opp}(V) \in H_1(M, \mathbb{Z})$, where $\text{opp}(V)$ denotes the opposite of the vector field $V$.

We now introduce an equivalent definition of $\mathfrak{e}u(M)$. Denote by $S(TM)$ the spherical tangent bundle of $M$ and let $pr : S(TM) \rightarrow M$ be the projection. Set

$$\mathfrak{e}u(M)^\bullet := (pr_*)^{-1}([M]) \subset H_3(S(TM), \mathbb{Z}).$$
It is immediate that $\frEul(M)\uparrow$ is a torsor on
\begin{equation}
H^{-}_{3}(S(TM),\mathbb{Z}) := \text{Ker}\{pr_{*} : H_{3}(S(TM),\mathbb{Z}) \rightarrow H_{3}(M,\mathbb{Z})\}.
\end{equation}
This group can be also identified with the set of $r$-anti-invariant elements of $H_{3}(S(TM),\mathbb{Z})$, where $r : TM \rightarrow TM$ is the reverse map.

Since $H^{-}_{3}(S(TM),\mathbb{Z}) \cong H_{1}(M,\mathbb{Z})$, $\frEul(M)$ and $\frEul(M)\uparrow$ are torsors on the same group. Actually there is a natural bijection of sets
\begin{equation}
\frEul(M) \overset{\sim}{\rightarrow} \frEul(M)\uparrow
\end{equation}
compatible with the action of the group. This is defined by
\begin{equation}
V \mapsto [\text{graph}(V)] \in H_{3}(S(TM),\mathbb{Z})
\end{equation}
where $V$ is considered as a section of the fibration $S(TM) \rightarrow M$. It is immediate to check that since $[\text{graph}(V_{1})] = [\text{graph}(V_{2})]$ if $[V_{1}] = [V_{2}]$.

In $\frEul(M)\uparrow$ the difference between two Euler structures $[U_{1}], [U_{2}] \in \frEul(M)\uparrow$ is obtained by the difference of the representative
\begin{equation}
[U_{2}] - [U_{1}] = [U_{2} - U_{1}] \in H^{-}_{3}(S(TM),\mathbb{Z}).
\end{equation}
Hence the Chern Class $c(U)$ of an Euler Structure $[U] \in \frEul(M)\uparrow$ is given by
\begin{equation}
c([U]) = [U] - r_{*}[U] \in H^{-}_{3}(S(TM),\mathbb{Z}).
\end{equation}

3.2.4. Framed Euler Structures. Now consider the case that $M$ is not compact. We assume that $M$ has a collar at infinity given by $\Sigma \times \mathbb{R}^{+}$ for some (not necessary connected) surface $\Sigma$, i.e. there exists a compact manifold $M^{in}$ with $\partial M^{in} = \Sigma$ such that
\begin{equation}
M = M^{in} \sqcup_{\Sigma} (\mathbb{R}^{+} \times \Sigma).
\end{equation}

A framed Euler Structure consists in a pair $(V, \fr{fr})$, where $V$ is not vanishing vector field on $M$ whose restriction to $\Sigma$ is compatible with $\fr{fr}$ (see [39]).

We say that two framed Euler Structures $(V_{0}, \fr{fr}_{0})$, $(V_{1}, \fr{fr}_{1})$ are homologous if $V_{0}$, $V_{1}$ are isotopic outside a ball with isotopy compatible with an isotopy of frames $(\fr{fr}_{t})_{t \in [0,1]}$.

We denote by $\frfrEul(M)$ the set of homology classes of Euler Structures and with $\frfrEul(M, \fr{fr})$ the set of homologous classes of Euler Structures compatible with $\fr{fr}$. It is easy to show that $\frfrEul(M, \fr{fr})$ is a torsor on $H_{1}(M,\mathbb{Z})$.

Now we define $\frfrEul(M)\uparrow$. We say that an element $S \in C_{3}(S(TM),\mathbb{Z})$ is compatible with $\fr{fr} \in \frfrEul(TM|_{\Sigma})$ if, up to triangulation,
\begin{equation}
\partial S = \text{graph}(V),
\end{equation}
for some $V \in \Gamma(S(TM)|_{\Sigma})$ compatible with $\fr{fr}$. Here $\text{graph}(V)$ is considered as a map $\Sigma \rightarrow S(TM)$.

Denote by $C_{3}(S(TM),\mathbb{Z}|\fr{fr})$ the vector space of chains compatible with $\fr{fr}$. Consider pairs $(S, \fr{fr})$ with $\fr{fr} \in \frfrEul(TM|_{\Sigma})$ and $S \in C_{3}(S(TM),\mathbb{Z}|\fr{fr})$. We define the difference of two pairs $(S_{1}, \fr{fr}_{1})$, $(S_{0}, \fr{fr}_{0})$ as
\begin{equation}
(S_{1}, \fr{fr}_{1}) - (S_{0}, \fr{fr}_{0}) = S_{1} - S_{0} + ((V_{1}, \fr{fr}_{1}) - (V_{0}, \fr{fr}_{0})) \in H_{3}(M^{in})
\end{equation}
RIMUOVERE IN? where $V_{0}, V_{1}$ are as in [10] and we have used [40].

Define $\frfrEul(M)\uparrow$ by
\begin{equation}
\frfrEul(M)\uparrow := \{(S, \fr{fr})|\fr{fr} \in \frfrEul(TM|_{\Sigma}), S \in C_{3}(S(TM),\mathbb{Z}|\fr{fr}), \pi_{*}([S]) = [M]\}/ \sim
\end{equation}
where now \( pr_* : H_3(S(TM), \partial S(TM), \mathbb{Z}) \to H_3(M, \partial M, \mathbb{Z}) \). In \( (47) \) we consider the equivalence relation
\[
(S_0, \mathfrak{r}_0) \sim (S_1, \mathfrak{r}_1) \text{ if } (S_1, \mathfrak{r}_1) - (S_0, \mathfrak{r}_0) = 0.
\]

The action \( (41) \) induces an action of \( H_1(\Sigma, \mathbb{Z}) \) on \( \mathfrak{frEul}(M)^\bullet \). From formula \( (42) \) we obtain
\[
(2\gamma) \cdot [(S, \mathfrak{r})] = [(S + i_*(\gamma), \mathfrak{r})],
\]
where \( i : \Sigma \to M \) is the inclusion of the boundary, and on the RHS we have identified \( H_3(S(TM), \mathbb{Z}) = H_1(M, \mathbb{Z}) \). Here we have also used the canonical isomorphism \( H_1(\Sigma, \mathbb{Z}) = H_1(\Sigma, \mathbb{Z}) \).

The spin structure associated to the frame defines a surjective map \( (48) \)
\[
\mathfrak{frEul}(M)^\bullet \to Spin(\Sigma)
\]
whose fibers are torsors on \( H_1(\Sigma, \mathbb{Z}) \).

The Chern class is defined as in \( (45) \). In contrast with the compact case, the Chern class is not necessarily even. Its parity is determinate by the relative Stiefel–Whitney class
\[
w_2(M, \sigma) = c([U, \mathfrak{r}]) \mod 2H_1(M, \mathbb{Z}),
\]
where \( \sigma \) is the spin structure on \( \Sigma \) defined by \( \mathfrak{r} \).

The Chern class changes under the action of \( H_1(M, \mathbb{Z}) \) according the formula
\[
c(\gamma \cdot [U, \mathfrak{r}]) = c([U, \mathfrak{r}]) + \gamma.
\]

3.2.5. Framed Three Chains. Let \( Y \) be an oriented five manifold. Let \( \Sigma \subset Y \) be a two dimensional submanifold of \( Y \). Assume that the normal bundle \( N\Sigma \) of \( \Sigma \) is trivial. Denote by \( \hat{Y} \) the real blow-up of \( Y \) along \( \Sigma \). \( \hat{Y} \) is a five dimensional manifold whose boundary is the spherical bundle \( S(N\Sigma) \).

For \( \mathfrak{r} \in \mathfrak{fr}(N\Sigma) \) the definition of \( C_3(\hat{Y}, \mathbb{Z}[\mathfrak{r}] \) is analogous to the definition of \( C_3(S(TM), \mathbb{Z}[\mathfrak{r}] \).

A Framed Three Chain consists in a pair \( (T, \mathfrak{r}) \) with \( \mathfrak{r} \in \mathfrak{fr}(N\Sigma) \) and \( T \in C_3(\hat{Y}, \mathbb{Z}[\mathfrak{r}] \). We define the difference of two Framed Three Chains using the formula
\[
(T_1, \mathfrak{r}_1) - (T_0, \mathfrak{r}_0) = (T_1 - T_0) + ((V_1, \mathfrak{r}_1) - (V_0, \mathfrak{r}_0)) \in C_3(\hat{Y})/\partial C_4(\hat{Y}),
\]
where \( V_0, V_1 \) are as in \( (40) \) and we have used \( (49) \).

Two Framed Three Chains \( (T_0, \mathfrak{r}_0), (T_1, \mathfrak{r}_1) \) are said homologous if \( (T_1, \mathfrak{r}_1) - (T_0, \mathfrak{r}_0) = 0 \). Denote by \( \mathfrak{frThree}(Y) \) the set of homology classes of Framed Three Chains.

There is an obvious free action of \( H_3(\hat{Y}) \) on \( \mathfrak{frThree}(Y) \) by addiction. The spin structure associated to the frame defines a map of sets
\[
\mathfrak{frThree}(Y) \to Spin(N\Sigma)
\]
whose fibers are torsors on \( H_3(\hat{Y}) \).

The action of \( H^1(\Sigma, \mathbb{Z}) \) on \( \mathfrak{frThree}(Y) \) is inducted by the action of \( Maps(\Sigma, SO(2)) \) analogously to \( (37) \).
3.2.6. Frame refined Four chain. A Framed Four Chain consist in a pair
\[(K^{in}, \text{fr}) \in C_4(\hat{X}^{in}, \mathbb{Z}) \times \tilde{\fr}(\Sigma \times \mathbb{R})^R\]
with the following properties. Let \(S \in C_3(S(N_*L)), T \in C_3(\hat{Y}, \mathbb{Z})\) defined by the relation
\[\partial K^{in} = S + T.\]
We require that
\[pr_*([S]) = [L^{in}],\]
\[\partial T = \text{graph}(V),\]
\[\partial S = -\text{graph}(V),\]
for some \(V \in \Gamma(S(TL)|_\Sigma)\) compatible with \(\text{fr}\).
We say \((K^{in}, \text{fr}) \sim (K^{in'}, \text{fr}')\) if there exists \(P \in C_4(\hat{Y}), Q \in C_4(S(TL^{in}), \mathbb{Z}), R \in C_5(\hat{X}^{in})\) such that
\[T - T' - ((\text{fr}, V) - (\text{fr}', V')) = \partial P,\]
\[S - S' + ((\text{fr}, V) - (\text{fr}', V')) = \partial Q,\]
\[K^{in} - K^{in'} - P - Q = \partial R.\]
Let \(\tilde{\fr}\fr\fr(X)\) be the set of the equivalence classes of Framed Four Chains.
A framed Four Chain in particular determine a framed Euler Structure of \(L\)
\[\tilde{\fr}\fr\fr(X, L) \to \tilde{\fr}\fr\fr\fr(X)\]
\[(K^{in}, \text{fr}) \mapsto (S, \text{fr}).\]
It also determine a framed three chain of \(Y\):
\[\tilde{\fr}\fr\fr\fr(X) \to \tilde{\fr}\fr\fr\fr(Y)\]
\[(K^{in}, \text{fr}) \mapsto (T, \text{fr}).\]
Consider the fiber product of the two maps above using \((48)\) and \((50)\)
\[\tilde{\fr}\fr\fr\fr(X) \to \tilde{\fr}\fr\fr\fr\fr\fr\fr(X) \times_{\text{Spin}(\Sigma)} \tilde{\fr}\fr\fr\fr\fr\fr\fr(Y).\]
The not empty fibers of \((52)\) are torsors on \(\text{Ker}\{H_4(X) \to H_1(L)\}\). The image of \((52)\) is identified with the pre-image of zero of the map
\[\tilde{\fr}\fr\fr\fr\fr\fr\fr(L) \times_{\text{Spin}(\Sigma)} \tilde{\fr}\fr\fr\fr\fr\fr\fr(Y) \to H_3(\hat{X}).\]
\[\left[(S_1, f_{r_1}) \times [(T_2, f_{r_2})] \mapsto S_1 - T_2 - ((\text{fr}_1, V_1) - (\text{fr}_2, V_2)).\]

3.3. Area zero annulus and Euler Structures. We now compare the Euler structure associated to the Four chain with the \(MC\)-cycles associated to the perturbation of the area zero annulus. In order to make this comparison the following description of \(\fr\fr\fr\fr\fr\fr\fr(M)\) is useful. Consider the set of pairs \((Z_0, Z_1)\) where
- \(Z_0\) is a three chain on \(M \times M\) close in the \(C^0\) topology to \(\text{Diag}_M\) and transversal to \(\text{Diag}_M\);
- \(Z_1\) is a one-chain on \(M\), with
\[\partial Z_1 + Z_0 \cap \text{Diag}_M = 0,\]
where \(Z_0 \cap \text{Diag}_M\) is considered as a chain on \(\text{Diag}_M \approx M\).
For the pairs \((Z_0, Z_1)\) there is an obvious notion of isotopy. Let \(\mathcal{Eul}(M)\) be the set of pairs \((Z_0, Z_1)\) modulo isotopy. The group \(H_1(M, \mathbb{Z})\) acts on \(\mathcal{Eul}(M)\) by addition on \(Z_1\). With this action, \(\mathcal{Eul}(M)\) is a torsor on \(H_1(M, \mathbb{Z})\).

Using a small tubular neighbourhood of \(\text{Diag}_M\), up to isotopy, we can consider \(Z_0\) as a three chain on \(TM\) close to the zero section.

If \(V\) is a vector field on \(M\) with small norm, the map

\[ V \mapsto (\text{graph}(V), 0) \]

defines an element of \(\mathcal{Eul}(M)\). Since the map is compatible with the action of \(H_1(M, \mathbb{Z})\), it yields to an isomorphism

\[ \mathcal{Eul}(M) \cong \mathcal{Eul}(M) \]

Now let \(\text{Ann}0\) be the set of area zero decorated graphs \(G\) with one only component and \(\Sigma_G\) be an annulus with one boundary marked points. The elements of \(\text{Ann}0\) are characterized as follows:

1. \(G_0\): \(|V_c| = 1, |D_c| = 0, |H_c| = 3, |E^{\text{in}}_c| = 1\);
2. \(G_1\): \(|V_c| = 2, |D_c| = 0, |H_c| = 1\);
3. \(G_2\): \(|V_c| = 1, |D_c| = 1, |H_c| = 1\).

Denote by \(\mathcal{C}_{\text{Ann}0}\) the restriction of the MC-chain complex to the graphs \(\text{Ann}0\) and \(Z_{\text{Ann}0/\text{isotopy}}\) the vector space of the corresponding MC-cycles. \(Z_{\text{Ann}0/\text{isotopy}}\) is a torsor on \(H_1(M, \mathbb{Z})\), where \(H_1(M, \mathbb{Z})\) acts by addiction on \(Z_{G_1}\).

**Lemma 13.**

\[ Z_{\text{Ann}0/\text{isotopy}} \cong \mathcal{Eul}(M) \]

**Proof.** We already observed that both the sides are torsors on \(H_1(M, \mathbb{Z})\). To prove the lemma we need to define a map between the two sets compatible with \(H_1(M, \mathbb{Z})\).

From an element \((Z_{G_0}, Z_{G_1}, Z_{G_2}) \in Z_{\text{Ann}0}\) we obtain an element of \(\mathcal{Eul}(M)\) as follows. Let \(e\) be the only internal edge of \(G_0\), and let \(pr_e : M^{H(G_0)} \to M^e \cong M^2\) be associated projection. Since \(\delta_e(pr_e(Z_{G_0}))\) is close in the \(C^0\)-topology to \(\delta_e Z_{G_0}\) and \(\delta_e Z_{G_0} + \partial Z_{G_1} + \partial Z_{G_2} = 0\), there exists \(Z'_{G_1}\) close in the \(C^0\)-topology to \(Z_{G_1}\) such that

\[ \delta_e(pr_e Z_{G_0}) + \partial Z'_{G_1} + \partial Z_{G_2} = 0. \]

Hence

\[ (pr_e(Z_{G_0}), Z'_{G_1} + Z_{G_2}) \in \mathcal{Eul}(M) \]

\[ \square \]

Now consider triples

\[ (Z_0, Z_1, Z_2) \]

where

- \(Z_0\) and \(Z_2\) are three chains on \(M \times M\) close in the \(C^0\) topology to \(\text{Diag}_M\) and transversal to \(\text{Diag}_M\);
- \(Z_1\) is a one-chain on \(M\), with

\[ \partial Z_1 + (Z_0 - Z_2) \cap \text{Diag}_M = 0. \]

There is an obvious notion of isotopy for the pairs \((Z_0, Z_1, Z_2)\).

To a triple \((Z_0, Z_1, Z_2)\) we can associate an homology class \([(Z_0, Z_1, Z_2)] \in H_1(M, \mathbb{Z})\) characterized by the following properties:
• it is invariant by isotopy;
• \([Z_0, Z_1, Z_2] = [Z_1]\) if \(Z_0 = Z_1\).

3.3.1. Perturbation of the moduli space. Let \((\overline{\mathcal{M}}_G)_{G \in \text{Ann}0}\) be the collection of Kuranishi spaces associated to the graphs \(\text{Ann}0\). They come with the following identification of Kuranishi spaces

\[
\partial \mathcal{M}_{G_1} \cong \delta_e(\overline{\mathcal{M}}_{G_0}) \sqcup \delta_{e'}(\overline{\mathcal{M}}_{G_2})
\]

where we denote by \(e\) the only internal edge of \(G_0\) and by \(e'\) the only element of \(D(G_2)\).

Let \((s^+_G)_{G \in \text{Ann}0}\) be a collection of perturbations of \((\overline{\mathcal{M}}_G)_{G \in \text{Ann}0}\) compatible with the identification of Kuranishi spaces \((56)\). \((57)\) is obtained from the inductive argument of [11], which in the particular case of \(\text{Ann}0\) reduces to the following two steps:

• choice a transversal perturbation on \(M_{G_0}\) and \(M_{G_2}\), which are transversal also when restricted to the RHS of \((56)\);
• choice an extension to \(M_{G_1}\) of the perturbation defined on its boundary by \((56)\).

In the not compact case we need to consider a boundary condition at infinity. We are going to define a particular type of boundary condition depending on the choice of a frame \(fr \in \mathfrak{g}(\mathbb{R} \times \Sigma)^\mathbb{R}\). Using \(fr\) and a complex structure, we can identify \(T_pX\) in \((21)\) with \(\mathbb{R}^3 \times \mathbb{R}^3\). We require that on \(\mathbb{R}^+ \times \Sigma\)

• the obstruction bundle \(E_p\) does not depend on \(p\),
• the perturbation \(s_p\) does not depend on \(p\),
• \(K^+\) is compatible with \(fr\).

3.3.2. Euler Structure from area zero Annulus. Let \((Z_{G_0}, Z_{G_1}, Z^+_G)\) be the collection of chains associated to the perturbation defined above. In particular \(Z^+_G \in C^0(L \times X)\) is transversal to \(L \times L\) and it is a close in the \(C^0\)-topology to Diag\(_L\) \(\subset L \times X\). We have the relation

\[
\delta_e Z_{G_0} + \partial Z_{G_1} + \delta_{e'} Z^+_{G_2} = 0
\]

where \(\delta_{e'} Z^+_{G_2} = \text{pr}_1(Z^+_{G_2} \cap (L \times L))\) with \(\text{pr}_1 : L \times X \to L\) is the projection on the first factor (we label the internal puncture of \(G_2\) with \(e'\)).

To \((Z_{G_0}, Z_{G_1}, Z^+_G)\) we can associate a triple \((Z_0, Z_1, Z_2)\) in the sense of \((55)\)

\[
(Z_{G_0}, Z_{G_1}, Z^+_G) \sim (Z_0, Z_1, Z_2).
\]

To define \((Z_0, Z_1, Z_2)\) we use a tubular neighborhood of \(L\) inside \(X\) in order to identify \(Z^+_G\) with the graph of a vector field. We then perturb \(Z_{G_1}\) as in the proof of Lemma \((53)\) to achieve the cyclic condition.

**Lemma 14.** The homology class of the triple \((Z_0, Z_1, Z_2)\) associated to \((Z_{G_0}, Z_{G_1}, Z^+_G)\) is trivial.

**Proof.** The Lemma can be considered as the analogous for annulus of the well known computation of the orbifold Euler Class of the obstruction bundle for area zeros torus with one marked point. In the case of torus we are interested to the
orbibundle over \( \mathcal{M}_{0,1} \times X \) whose fiber at the point \( ((T^2, J_{T^2}), x) \) is the tensor product \( H^0,1(T^2, J_{T^2}) \otimes TX \). The first factor is the Hodge Bundle of the torus.

In our case, the Hodge bundle of the annulus is trivial and the Euler class of \( TL \) vanishes. \((\mathcal{M}_{(0,2),(0,(0,1))}) \times L\)

To prove the Lemma we consider a particular type of perturbation of the moduli space \((\mathcal{M}_G)_{G \in \text{Ann}_0}\).

Pick a one dimensional vector sub bundle \( E^*_\delta \) of \( \Lambda^{0,1}(\Sigma) \) over the moduli space of \((\Sigma, J_\Sigma)\) which is complementary to \( \text{Im}(\tilde{\theta})\). We constrain the obstruction bundle requiring that \( E_p \) on the point \( p = (u, \Sigma) \) contains the tensor product \( E_p \supseteq T_p X \otimes E^*_\Sigma\).

Pick a not vanishing vector field \( V \) on \( L \) and a transversal section \( s' \) of \( E^*_\Sigma \). On \( \mathcal{M}_{G_1} \) consider the perturbation of the Kuranishi structure given by \( V \otimes s' \)

Requiring the compatibility with \((59)\), \((60)\) defines a perturbation of the Kuranishi structure on \( b_{\mathcal{M}_{G_1}} \). From \( s^{-1}_{G_0}(0) \) we can define an Euler structure \( V_0 \), up to small isotopy. The homology class of \( V_0 \) agrees with the homology class \( V \) or its opposite \( opp(V) \). In the same way, from \( s^{-1}_{G_2}(0) \) we obtain an Euler structure \( V_2 \) whose homology classes agrees with \( V \) or \( opp(V) \). The homology class \([\langle Z_0, Z_1, Z_2 \rangle]\) is given by \([\langle Z_0, Z_1, Z_2 \rangle] = [V_0] - [V_2]\).

Since \([\langle Z_0, Z_1, Z_2 \rangle]\) has to be independent of the perturbation (and hence of \( V \)) we conclude that \( V_0 = V_2 \) (note that if \( V_0 \) is opposite to \( V_2 \), \( V_0 - V_2 \) depends on \( V \)). \( \square \)

Set \((61)\) \( Z_{G_2} = Z^+_{G_2} \times X K. \)

The dependence of \( Z_{\text{Ann}_0} := (Z_{G_0}, Z_{G_1}, Z_{G_2}) \in Z_{\text{Ann}_0} \) on \( K \) is encoded in \((61)\).

In the not compact case, on the region \( \mathbb{R}^+ \times \Sigma \), from the construction above we obtain that \( Z_{G_1} = Z_{G_2} = 0 \) and \( Z_{G_0} \) is equal to \( (\mathbb{R}^+ \times \Sigma) \times v \) for some \( v \in \mathbb{R}^3 \) with small norm, up to an higher order error which can be canceled up to isotopy.

The four chain \( K \) defines an element \([U_K] \in \mathfrak{F} \mathfrak{e} \mathfrak{u} \mathfrak{l}(M)^\bullet \) as follows. Assuming transversality between \( L \) and \( K \), we can define a four chain \( K \) on \( X \) and set \( U_K = \partial K. \)

In the not compact case we define \([U_K] \in \mathfrak{F} \mathfrak{e} \mathfrak{u} \mathfrak{l}(M)^\bullet \) in an analogous way.

**Proposition 15.** The isotopy class of \( Z_{\text{Ann}_0} \) coincides with the homology class \([U_K]\) using \((53)\).

**Proof.** First note that, up to isotopy, \( Z_{\text{Ann}_0} \) depends on \( K \) only throughout \([U_K]\). Moreover we can define \( Z_{\text{Ann}_0} \) for any \( U \in \mathfrak{F} \mathfrak{e} \mathfrak{u} \mathfrak{l}(M)^\bullet \) not necessarily associated to a four chain \( K \).

Let \( \Delta \) be the difference between \((54)\) and \([U_K]\) as Euler Structures.

If we shift \([U_K]\) by an element of \( H_1(L, \mathbb{Z}) \) the corresponding element \((54)\) shifts by the same element. Hence \( \Delta \) does not depend on \( U_K \).

It is easy to check that \( \Delta \) agrees with the homology class of the triple \((55)\) associated to \((Z_{G_0}, Z_{G_1}, Z^+_{G_2})\). \( \square \)
3.3.3. **Abelianization and Chern Class.** In [11] we considered the abelianization of a MC-cycle. For $Ann_0$ it reduces to

$$Z_{G_0}^{ab} = \frac{1}{2}(Z_{G_0} - opp(Z_{G_0})), Z_{G_1}^{ab} = Z_{G_1}, Z_{G_2}^{ab} = Z_{G_2},$$

where $opp(Z_{G_0})$ is the chain obtained switching the half edges of $e$.

Using an isotopy to contract $Z_{G_0}^{ab}$ to zero, we can consider $Z^{ab}$, up to isotopy, as an element $[Z_{G_1} + Z_{G_2}] \in H_1(L, \mathbb{Z})$.

**Lemma 16.** $2Z_{Ann0}^{ab}$ is the Chern class of the Euler structure associate to $Z_{Ann0}$.

*Proof.* Given $Z \in Z_{Ann0}$, we can define the opposite cycle $\Sigma \in Z_{Ann0}$ as

$$Z_{G_0}^{\Sigma} = opp(Z_{G_0}), Z_{G_1}^{\Sigma} = -Z_{G_1}, Z_{G_2}^{\Sigma} = -Z_{G_2}.$$

The Euler Structure corresponding to $\Sigma$ is the opposite of the Euler Structure corresponding to $Z$. Since $2Z^{ab} = Z - Z^{\Sigma}$ the Lemma follows. □

4. **Kontsevich-Soibelman algebra from Multi-Curve-Homology**

4.1. **The case** $\Sigma \times \mathbb{R}$. Particularly important is the case of the cylinder $M = \Sigma \times [0,1]$. In this case the vector space $MCH(\mathbb{R} \times \Sigma)$ has a natural product:

$$MCH([0, 1] \times \Sigma, \gamma_1, \gamma_2) \times MCH([0, 1] \times \Sigma, \gamma_2, \gamma_3) \to MCH([0, 1] \times \Sigma, \gamma_1 + \gamma_2, \gamma_1 + \gamma_3)$$

obtained from ([13]) and the isomorphism

$$(\Sigma \times [0,1]) \sqcup_\Sigma (\Sigma \times [0,1]) \cong \Sigma \times [0,1]$$

obtained by gluing the boundary component $\Sigma \times \{0\}$ of the first cylinder to the boundary component $\Sigma \times \{1\}$ of the second cylinder, i.e., we place the first factor on the top of the second.

In particular the terms with $c = 0$ define an algebra

$$MCH([0,1] \times \Sigma, c = 0) = \bigoplus_{\gamma \in H_1(\Sigma, \mathbb{Z})} MCH([0,1] \times \Sigma, \gamma|c = 0).$$

Consider the Kontsevich-Soibelman algebra associated to the homology group $H_1(\Sigma, \mathbb{Z})$ endowed with the standard intersection form $(\cdot, \cdot)$:

$$\hat{\mathfrak{g}} = \bigoplus_{\gamma \in H_1(\Sigma, \mathbb{Z})} \mathbb{Q}[|g_\gamma|] \hat{e}_\gamma$$

$$\hat{e}_\gamma \hat{e}_{\gamma'} = q^{\frac{(\gamma, \gamma')}{2}} \hat{e}_{\gamma + \gamma'}$$

for $\gamma_1, \gamma_2 \in H_1(\Sigma, \mathbb{Z})$

where

$$q^\frac{1}{2} = -e^{2\pi i}.$$

Fix a spin structure on $\Sigma$, and let $\sigma$ its quadratic differential. Consider the linear map

$$\hat{\mathfrak{g}} \to MCH([0,1] \times \Sigma, c = 0)$$

$$\hat{e}_\gamma \mapsto \sigma(\gamma) \exp((w_\gamma, w'_\gamma))$$

where $w_\gamma$ is any representative of the homology class $\gamma$ with support on $\{t\} \times \Sigma$ for some $t \in (0,1)$, and $w'_\gamma$ is a translation of $w_\gamma$ to $\{t'\} \times \Sigma$ for some $t' \neq t$ close to $t$. Observe that in this case $w^{ann} = 0$. 

Proposition 17. The linear map above defines an isomorphism of algebras

\[ MCH(\Sigma \times \mathbb{R})^{o,o} = \hat{\mathfrak{g}}. \]

Proof. Let \( w_1, w_1', w_2, w_2' \) with support on \( \{t_1\} \times \Sigma, \{t_1'\} \times \Sigma, \{t_2\} \times \Sigma, \{t_2'\} \times \Sigma \) respectively, for some \( t_1, t_1', t_2, t_2' \in [0,1] \). By definition we have

\[ \exp((w_1 + w_2, w_1' + w_2')) = \begin{cases} \hat{e}_{\gamma_2} \circ \hat{e}_{\gamma_1} & \text{if } \max\{t_2, t_2'\} < \min\{t_1, t_1'\} \\ \hat{e}_{\gamma_2 + \gamma_1} & \text{if } \max\{t_1, t_1'\} < \min\{t_2, t_2'\}. \end{cases} \]

Thus \( \hat{e}_{\gamma_2} \circ \hat{e}_{\gamma_1} \) and \( \hat{e}_{\gamma_2 + \gamma_1} \) are related by an isotopy which interchanges the order of \( t_1' \) and \( t_2 \). The Proposition follows from the fact that under this isotopy we have

\[ (w_1 + w_2, w_1' + w_2') \mapsto (w_1 + w_2, w_1' + w_2') + \frac{g_s(\gamma_1, \gamma_2)}{2}. \]

\[ \square \]

There is another version \( KS \)-algebra which plays an important role in this paper. Set

\[ MCH([0,1] \times \Sigma)^{ann} = \bigoplus_{c \in H_1(\Sigma, \mathbb{Z})} MCH([0,1] \times \Sigma, 0|c)^{o,o}, \]

which is equipped with an algebra structure induced by the gluing as before. We denote with \( \hat{\mathfrak{g}}^{ann} \) this algebra.

Denote with \( \hat{\mathfrak{g}}' \) the \( KS \)-algebra without the sign \((-1)^{\gamma_1, \gamma_2}\) and \( \gamma \in \frac{1}{2} H_1(\Sigma, \mathbb{Z}) \).

As in Proposition (17) we have

\[ \hat{\mathfrak{g}}' = MCH([0,1] \times \Sigma)^{ann}. \]

4.2. Kontsevich-Soibelman algebra associated to \((X, L)\). \( MCH(L, c) \) is a module over \( MCH(\Sigma \times [0,1]) \). The module structure is induced by the identification

\[ (\Sigma \times [0,1]) \sqcup_{\Sigma} M \cong M \]

obtained by gluing the boundary component \( \Sigma \times \{0\} \) to the boundary of \( M \). From proposition above we obtain an action of \( \hat{\mathfrak{g}} \) on \( MCH(L) \). We will keep the notation \( \hat{e}_\gamma \) for the operator on \( MCH(L|c)^{o,o} \) associated to \( \hat{e}_\gamma \in \mathfrak{g} \).

The same considerations apply to obtain an action of \( \hat{\mathfrak{g}}^{ann} \) on \( MCH(L|c)^{o,o} \).

As in (11), the space of topological charges \( \Gamma \) is defined as the relative homology:

\[ \Gamma = H_2(M, L, \mathbb{Z}). \]

Denote by

\[ Q : H_1(\Sigma, \mathbb{Z}) \to H_1(L, \mathbb{Z}) \]

the map in homology induced by the inclusion \( \Sigma \to L \). Consider the abelian group

\[ \Gamma^{\bullet} = \{(\gamma, \beta) \in H_1(\Sigma, \mathbb{Z}) \times \Gamma | \partial \beta = Q(\gamma)\} \]

endowed with the skew-symmetric pairing

\[ \langle \cdot, \cdot \rangle : \Gamma^{\bullet} \times \Gamma^{\bullet} \to \mathbb{Z} \]

\[ \langle (\gamma_1, \beta_1), (\gamma_2, \beta_2) \rangle = \langle \gamma_1, \gamma_2 \rangle, \]

where in the right side we have used the usual intersection pairing of \( H_1(\Sigma, \mathbb{Z}) \).

The quantum Kontsevich-Soibelman algebra associated to \((X, L)\) is defined by

\[ \mathfrak{g}^{\bullet} = \bigoplus_{(\gamma, \beta) \in \Gamma^{\bullet}} \mathbb{Q}[g_s] \hat{e}_{\gamma, \beta} \]
equipped with the product structure defined by
\[ \hat{e}_{\gamma_1,\beta_1} \hat{e}_{\gamma_2,\beta_2} = q^{(\gamma_1,\gamma_2)} \hat{e}_{\gamma_1+\gamma_2,\beta_1+\beta_2}. \]

4.3. Boundary States. Define the subgroup of $\Gamma$ of flavor charges as
\[ \Gamma_{\text{flavor}} = \{ \beta | \partial \beta \in \text{Im}(Q) \}. \]

Observe that $\Gamma_{\text{flavor}}$ is the image of $\Gamma_{\star}$ in $\Gamma$ by the projection $(\gamma, \beta) \mapsto \beta$.

Fix a positive real number $C_{\text{supp}} > 0$. Fix a representative $w_{\text{ann}}$ of the Chern Class. A boundary state $Z$ consists in an array $Z = (Z_\beta)_{\beta \in \Gamma_{\text{flavor}}}$ with $Z_\beta \in \text{MCH}(\partial \beta|w_{\text{ann}})$ and $Z_\beta \neq 0 \Rightarrow \| \beta \| \leq C_{\text{supp}} \omega(\beta)$.

Let $H_c$ be the vector space of boundary states with chern class $c$.

There is an obvious action of $g_{\star}$ on $H$ defined by
\[ \hat{e}_{\gamma_0,\beta_0}( (Z_\beta)_{\beta \in \Gamma_{\text{flavor}}}) = (\hat{e}_{\gamma_0} Z_\beta - \delta_0)_{\beta \in \Gamma_{\text{flavor}}}. \]

4.3.1. Novikov Ring. Denote by $g(\Lambda_0)$ the Novikov Ring with coefficients on $g$. Hence an element of $g(\Lambda_0)$ is defined by a formal sum $\sum_i a_i T^{\lambda_i}$ with $a_i \in g$, $\lim_{i \to \infty} \lambda_i = +\infty$, $\lambda_i \geq 0$.

$g(\Lambda_0)$ is equipped with an obvious structure of algebra. We have an isomorphism of moduli
\[ \mathfrak{h} = g^{\star}/\mathfrak{J}. \]

We can also consider the action of $MCH([0,1] \times \Sigma)_{\text{ann}}$:
\[ \hat{e}_{\gamma}^{\text{ann}}((Z_\beta)_{\beta \in \Gamma_{\text{flavor}}}) = (\hat{e}_{\gamma}^{\text{ann}} Z_\beta)_{\beta \in \Gamma_{\text{flavor}}}. \]
\[ Z_c \to Z_{c+2Q(\gamma)}. \]

This map is compatible with the moduli structure $\mathfrak{h}$ and $\mathfrak{h}_{\text{nov}}$ and the homomorphism of algebras (66).
4.4. **KS-algebra including the degrees.** We now extended the objects considered above to the context of section 2.4. In this context, coherent MC-cycles are defined in the same way.

4.4.1. **The case \( \Sigma \times \mathbb{R} \).** \( MCH(\mathbb{R} \times \Sigma)^{\circ,\circ} \) is an algebra for the same reason above. However, it is different from the one considered above.

The map \( \mathfrak{g}^{\mathfrak{g}} \) is not anymore an homomorphism of algebras. However we can define an homomorphism of Lie Algebras as follows.

Set

\[
\hat{e}^{\text{lie}}_{\gamma,\beta} = \frac{\hat{e}_{\gamma,\beta}}{q^2 - q^{-2}}.
\]

Set

\[
\mathfrak{g}^{\mathfrak{g}} = \bigoplus_{(\gamma,\beta) \in \Gamma} \mathbb{Q}[\mathfrak{g}][\hat{e}^{\text{lie}}_{\gamma,\beta}]
\]

equipped with the Lie bracket

\[
[\hat{e}^{\text{lie}}_{\gamma_1,\beta_1}, \hat{e}^{\text{lie}}_{\gamma_2,\beta_2}] = (-1)^{\langle \gamma_1,\gamma_2 \rangle} \frac{q^{\langle \gamma_1,\gamma_2 \rangle} - q^{-\langle \gamma_1,\gamma_2 \rangle}}{q^2 - q^{-2}} \hat{e}^{\text{lie}}_{\gamma_1 + \gamma_2,\beta_1 + \beta_2}.
\]

**Lemma 18.** The map \( \mathfrak{g}^{\mathfrak{g}} \) defines an homomorphism of Lie Algebras.

**Proof.** The reason why in the context of section 2.4 we do not get the RHS of \( \mathfrak{g}^{\mathfrak{g}} \) is that in the isotopy used in the proof of \( \mathfrak{g}^{\mathfrak{g}} \) we generate MC-cycles which have two vertices instead of one.

We claim that these elements are the same when the order of the product on the LHS is switched and hence they cancel on the bracket.

The claim follows from the fact that in the isotopy we never cross two cycles associated to the same vertex and hence the elements generated by the RHS of relation \( \mathfrak{g}^{\mathfrak{g}} \) have always one vertex. \( \square \)

**Proposition 19.** We have an isomorphism of algebras

\[
MCH(\mathbb{R} \times \Sigma)^{\circ,\circ} = U(\mathfrak{g}^{\mathfrak{g}}).
\]

**Proof.** It is immediate from the definitions and the Lemma 18. \( \square \)

We define the vector space \( \mathfrak{g}^{\mathfrak{g}} \) in a way analogous to \( \mathfrak{g}^{\mathfrak{g}} \) using MC-cycles with components of section 2.4 instead of 3. We obtain an action of \( \mathfrak{g}^{\mathfrak{g}} \) on \( \mathfrak{g}^{\mathfrak{g}} \).

5. **Compactification Frames**

So far we have considered the classes \( \beta \in \Gamma \) independently. To define the wave-function as in \[5\] we need to fix frame of \( L \).

**Definition 1.** A frame of \( L \) is a Lagrangian subvector space \( F \) of \( H_1(\Sigma, \mathbb{R}) \), such that \( F \cap H_1(\Sigma, \mathbb{Z}) \) generate \( F \) as vector space.

**Remark 20.** The lattice \( F \) is not necessarily transverse to \( K \), that is we do not assume that \( F \to H_1(L, \mathbb{Z}) \) is surjective. This is in accord with the definition in three-dimensional gauge theory \( \[15\] \).
To a frame $F$ it is associated a compactification $L_F$ of $L$ as follows. Let $H_F$ be the handlebody with $\partial H_F = \Sigma$ and $F = \text{Ker}\{H_1(\Sigma) \to H_1(H_F)\}$, i.e., $H_F$ is obtained from $\Sigma$ filling the cycles of $F$. Set
\[ L_F = L \sqcup_{\Sigma} H_F \]
where we identify the boundary $\partial L = \partial H_F = \Sigma$. Roughly, $L_F$ is the compactification of $L$ for which $F$ contractible at infinity.

**Remark 21.** In the physical literature, to define three dimensional gauge theories for not compact geometry it is necessary to fix boundary conditions at infinity (see for example [4], [5]). These boundary conditions are defined in geometric terms of a compactification of $L$ defining the set of cycles of $L$ contractible at infinity (see [4]). The last data is specified by a choice of a frame $F$. This procedure can be seen mirror to our definitions above, according to the topological string/3d gauge theory correspondence.

From the Mayer-Vietoris sequence
\[ ... \to H_1(\Sigma) \to H_1(L) \oplus H_1(H_F) \to H_1(L_F) \to H_0(\Sigma) \to ... \]
we obtain the isomorphism
\[ H_1(L_F, \mathbb{Z}) = H_1(L, \mathbb{Z})/Q(F). \]  

Fix a $F \subset H_1(M, \mathbb{Z})$ transversal to $\text{Ker}(Q)$. For each $\gamma \in H_1(L, \mathbb{Z})$ there exists a unique $F_\gamma \in F$ such that $Q(F_\gamma) = \gamma$. To simplify notation denote
\[ \hat{e}_\beta = \hat{e}_{(\beta, F_\partial \beta)} \forall \beta \in \Gamma_{flavor}, \]
\[ \hat{e}_\gamma = \hat{e}_{(0, \gamma)} \forall \gamma \in \text{Ker}(Q). \]
We stress that $\hat{e}_\beta$ depends on the choice of the frame $F$.

The factorization property [13] implies that there exists an array $(a(\beta))_{\beta \in \Gamma_{flavor}}$ (depending on $F$) with $a(\beta) \in \frac{1}{g_s} Q[[g_s]]$ such that
\[ Z = a_0 \hat{e}_{F_0} \prod_{\beta \in \Gamma_{flavor}} \exp(a(\beta)\hat{e}_\beta) Z_0. \]  

5.1. **Integrality.** We say that $(a(\beta))_{\beta \in \Gamma_{flavor}}$ of (70) satisfies the Ouguri-Vafa integrality if there exists integer numbers $N_{\beta,j} \in \mathbb{Z}$, $\beta \in H_2(X, L)$, $j \in \frac{1}{g_s} \mathbb{Z}$ with
\[ |\{N_{\beta,j} \neq 0, |\omega(\beta)| < E\}| < \infty \text{ for each } E > 0 \]
and
\[ a(\beta) = \sum_{n,j} N_{\beta,j} \frac{q^n}{n(q^2 - q^{-2})}. \]  

In the last sum $n$ run over the integers numbers such that $\frac{\beta}{n} \in \Gamma_{flavor}$.

Observe that if we twist the frame by an element $\gamma_0 \in H_1(\Sigma, \mathbb{Z})$ the integers changes according to the formula
\[ N_{\beta,j} \mapsto N_{\beta,j + (\gamma_0, \partial \beta)}. \]
In particular this means that if the integrality property holds in a frame holds in each frame.
Using (71) we can rewrite (70) as
\[
Z = a_0 \hat{e}^{ann}_0 \prod_{\beta, j} (S_{\beta, j})^{N_{\beta, j}} Z_0.
\]
where
\[
S_{\beta, j} = \exp\left(\sum_n \frac{q^{n(j+\frac{1}{2})}}{n(q^n - 1)} \hat{e}_n \hat{\beta}\right).
\]

\(S_{\beta, j}\) are generators of Cluster transformations:

\[
S_{\beta, j} \hat{e}_{\gamma} S_{\beta, j}^{-1} = \begin{cases} 
\hat{e}_{\gamma} & \text{if } \langle F_{\partial \beta}, \gamma \rangle > 0; \\
\hat{e}_{\gamma} & \text{if } \langle F_{\partial \beta}, \gamma \rangle = 0; \\
\hat{e}_{\gamma} (\prod_{k=1}^{(\gamma, F_{\partial \beta})} (1 + q^{j+\frac{1}{2}} - k \hat{e}_\gamma))^{-1} & \text{if } \langle F_{\partial \beta}, \gamma \rangle < 0.
\end{cases}
\]

If the Ooguri-Vafa integrality (71) holds the vector \(Z\) satisfy the quantum equations:

\[
(73) \quad A_{\alpha} Z = 0
\]
with
\[
A_{\alpha} = \prod_{(\gamma, \beta)} \mathbb{Z}[q^{\pm \frac{1}{2}}] \hat{e}_{\gamma, \beta} \text{ for } 0 \leq \alpha \leq g.
\]

Equation (73) is obtained applying the sequence of cluster transformations (72) to the equations
\[
(\hat{e}_{\gamma} - 1) Z_0 = 0 \text{ for each } \gamma \in \text{Ker}(Q).
\]
The last equations should be considered as the quantization of the classical equations

\[
(74) \quad x_{\gamma} = 0 \text{ for each } \gamma \in \text{Ker}(Q)
\]
where for each \(\gamma \in H_1(\Sigma, \mathbb{Z})\), \(x_{\gamma} : H^1(\Sigma, \mathbb{C}) \to \mathbb{C}\) denotes the linear map defined by \(\gamma\).

Equations (74) define the sub-vector space of \(H^1(\Sigma, \mathbb{C})\)

\[
(75) \quad \frac{H^1(L, \mathbb{C})}{H^1_{\text{comp}}(L, \mathbb{C})} \subset H^1(\Sigma, \mathbb{C}).
\]
It is a standard result, and easy to check, that (75) is a Lagrangian sub-vector space of \(H^1(\Sigma, \mathbb{C})\) equipped with the symplectic form defined by the intersection pairing.

**Remark 22.** An A-brane is defined by a lagrangian sub-manifold \(L\) and an abelian flat connection of \(L\). The moduli space of A-branes is locally modelled on \(H^1(L, \mathbb{C})\), where the real part can be viewed coming from the moduli of the deformations of the Lagrangian and the imaginary part defines a flat \(U(1)\) connection on \(L\).

The vector space (72) is the tangent space of the classical moduli space of A-branes up to compact deformations. Hence the vector \(Z\) can be considered as the quantum corrected moduli obtained including the contribution of holomorphic curves to the classical moduli of A-branes. It is a subspace of the deformation quantization of the complex symplectic vector space \((H^1(\Sigma, \mathbb{C}), (\bullet, \bullet))\), which is the quantum torus with algebra of functions defined by:

\[
\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} = q^{\langle \gamma_1, \gamma_2 \rangle} \hat{e}_{\gamma_2} \hat{e}_{\gamma_1} \text{ for } \gamma_1, \gamma_2 \in H_1(\Sigma, \mathbb{Z}).
\]
5.2. **Multi-link Homomorphism.** The inclusion $L \to L_F$ induces a link map

$$\text{Link}_F : \sqcup_{\gamma \in \text{Im}(Q)} \mathcal{P}_\gamma \to \mathbb{Z}.$$ 

It has the characterizing property:

$$\text{Link}_F(w, w') = 0$$

if $(w, w') \in \mathcal{P}_\gamma$ for $\gamma \in F$, $w$ has support on $\{t\} \times \Sigma$, $w'$ has support on $\{t'\} \times \Sigma$ with $t \neq t'$.

We can now extend the Multi-Link homomorphism

$$\text{Multi-Link}_F : \mathbb{Z}_\gamma \to \mathbb{Q}[g_*]$$

defined by

$$(H, \{w_h\}_h) \mapsto \sum_{E} \frac{g_E^{\text{in}}}{\text{Aut}(E)} \prod_{e \in E^{\text{in}}} \text{Link}_F(w_e, w'_e) \prod_{e \in E^{\text{ex}}} \langle x, w_e \rangle$$

where $(w_e, w'_e) = (w_h, w_{h'})$ for $e = \{h, h'\} \in E^{\text{in}}$, $w_e = w_h$ for $e = \{h\} \in E^{\text{ex}}$.

5.3. **Wave Function.** Introduce a set of formal variables $(x_\beta)_{\beta \in \Gamma}$ with relation

$$x_\beta \cdot x_\beta = x_\beta + \beta.$$ 

Set

$$\Psi_\beta = \text{Link}_F(Z_\beta)$$

$$\Psi(x) = \sum_{\beta \in \Gamma_{\text{flavor}}} \Psi_\beta x_\beta$$

The action of $g^*$ on $\mathcal{F}$ induces an action on $\Psi$ which can be obtain from the action on the formal variables given by

$$(\mathcal{H}, \{w_h\}_h) \mapsto \exp(\langle x, \partial \beta \rangle x_\beta)$$

$$\cdot \sum_{\beta \in \Gamma_{\text{flavor}}} \Psi_\beta T^{\omega(\beta)} \exp(\langle x, \partial \beta \rangle)$$

where $x$ is a formal variable with value on $H^1(L, \mathbb{R})/2\pi H^1(L, \mathbb{Z})$.

Hence the wave function with coefficients on the Novikov Rings is given by

$$\Psi(T, x) = \exp(\langle x, \frac{1}{2} \beta \rangle) \sum_{\beta \in \Gamma_{\text{flavor}}} \Psi_\beta T^{\omega(\beta)} \exp(\langle x, \partial \beta \rangle).$$

The action of $g(\Lambda_0)$ on $\mathcal{F}_{\text{nov}}$ induces an action on $\Psi$ which can be obtain from the action on the formal variables given by

$$(\mathcal{H}, \{w_h\}_h) \mapsto \exp(\langle x, \frac{1}{2} \beta \rangle) \sum_{\beta \in \Gamma_{\text{flavor}}} \Psi_\beta T^{\omega(\beta)} \exp(\langle x, \partial \beta \rangle).$$

$$\hat{e}_{\gamma_0} : (x, \gamma) \mapsto \begin{cases} 
(x, \gamma + \gamma_0) & \text{if } \gamma_0 \in F \\
(x + g_s PD(\gamma_0), \gamma) & \text{if } \gamma_0 \in \text{Ker}(Q)
\end{cases}$$

$\hat{e}_{\gamma_0}^{\text{ann}}$ acts on the pair $(x, c)$ as

$$(x, c) \mapsto \begin{cases} 
(x, c + 2\gamma_0) & \text{if } \gamma_0 \in F \\
(x + g_s \gamma_0, c) & \text{if } \gamma_0 \in \text{Ker}(Q)
\end{cases}$$
6. Semiclassical Limit

We now define a representation of the semiclassical Kontsevich-Soibelman Lie algebra $\mathfrak{g}^{\text{scl}}$ in terms of multi-disk-homology introduced in [12] and we show that this can be considered in the natural way as the semiclassical limit of the representation defined in Section 4.4.

6.1. Multi-Disk Homology. We now define the vector space of nice multi disk cycles $Z^{\text{disk}}$.

Consider objects as

$$ (V, (w_v)_v, (\beta_v)_v) $$

where

- A finite set $V$, called the set of vertices.
- For each $v \in V$ an element $\beta_v \in H_2(X, L)$ called the homology class of $v$,
- for each $v \in V$ a curve of real dimension one $\gamma_v$, with $[\gamma_v] = [\partial \beta_v] \in H_1(L, \mathbb{Q})$.

We assume the support property $||\beta_v|| \leq C^\text{supp}\omega(\beta_v)$ for each $v \in V$.

Let $\mathcal{Gen}^\varepsilon(\beta)$ be the set of objects (79) modulo the obvious equivalence, with $\sum_v \beta_v = \beta$.

$Z^{\text{disk}}$ is defined as the formal vector space generated by $\mathcal{Gen}^\varepsilon(\beta)$.

To define isotopies of Multi-Disk cycles we consider one parameter family of the objects (79):

$$ (V, (\tilde{w}_v)_v, (\beta_v)_v, [a, b]) $$

where $\tilde{w}_v$ are one parameter family of one cycles parametrized by $[a, b]$.

An isotopy $Z^{\text{disk}}$ of $MD$-cycles is a formal sum of elements (80) with the following proprieties for a pair $v_1 \neq v_2 \in V$, $Z^{\text{disk}, t}$ jumps according to the formula

$$ Z^{\text{disk}, t}_+ - Z^{\text{disk}, t}_- = \pm \langle V \setminus \{v_1, v_2\}, (\tilde{w}_v(t_0))_v, (\beta_v)_v \rangle \cup \langle v_0, \tilde{w}_{v_1}(t_0) + \tilde{w}_{v_2}(t_0), \beta_{v_1} + \beta_{v_2} \rangle $$

where the sign is provided by the sign of the intersection $\tilde{w}_{v_1}(t_0) \cap \tilde{w}_{v_2}(t_0)$.

6.1.1. Semiclassical Kontsevich-Soibelman Algebra. To the pair $(X, L)$ we associate the semiclassical Kontsevich-Soibelman Lie algebra

$$ \mathfrak{g}^{\text{scl}} = \bigoplus_{(\gamma, \beta) \in \Gamma \times \bullet} \mathbb{Q} e_{(\gamma, \beta)} $$

with Lie bracket

$$ [e_{(\gamma_1, \beta_1)}, e_{(\gamma_2, \beta_2)}] = (-1)^{\langle \gamma_1, \beta_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2, \beta_1 + \beta_2}. $$

We fix a spin structure on $\Sigma$ and denote by $\sigma$ the associated quadratic differential.

For a $(\gamma, \beta) \in \Gamma \times \bullet$ consider the linear map

$$ e_{(\gamma, \beta)} : Z^{\text{disk}} \to Z^{\text{disk}} $$

defined on the generators (80) as

$$ e_{(\gamma, \beta)} : (V, (w_v)_v, (\beta_v)_v) \mapsto \sigma(\gamma)(V \sqcup \{v_0\}, (w_v)_v \in V \sqcup \{v_0\}, (\beta_v)_v \in V \sqcup \{v_0\}). $$

where $\beta_{v_0} = \beta$ and $w_{v_0}$ is a representative of $\gamma$ with support in $(-\infty, -T) \times \Sigma$ for $T$ big enough such that each $w_v$ have support in the complementary of $(-\infty, -T) \times \Sigma$. 

6.1.2. The case $\Sigma \times \mathbb{R}$. We now show that the Multi-Disk homology of $\Sigma \times \mathbb{R}$ is isomorphic to the enveloping algebra of $\mathfrak{g}^{scl}$.

To an element $e(\gamma_1, \beta_1) \otimes e(\gamma_2, \beta_2) \otimes \ldots \otimes e(\gamma_k, \beta_k)$ we can associated an element $(V, (w_v)_v, (\beta_v)_v) \in Z^{\text{disk}}/\text{isotopics}$ where $V = \{1, 2, \ldots, k\}$ and $w_i$ are one dimensional cycles representing the class $\gamma_i$, supported on $\Sigma \times \{t_i\}$, for $t_1 > t_2 > \ldots > t_k$.

It is easy to check that $e(\gamma_1, \beta_1) \otimes e(\gamma_2, \beta_2) - e(\gamma_2, \beta_2) \otimes e(\gamma_1, \beta_1) - e(\gamma_1 + \gamma_2, \beta_1 + \beta_2) \mapsto 0$.

Hence the homomorphism $e(\gamma_1, \beta_1) \otimes e(\gamma_2, \beta_2) \otimes \ldots \otimes e(\gamma_k, \beta_k) \mapsto \prod_i \sigma(\gamma_i)(V, (w_v)_v, (\beta_v)_v)$ defines map from the enveloping algebra $U(\mathfrak{g}^{scl}) \to Z^{\text{disk}}/\sim$.

It is easy to check that this map is an isomorphism.

6.1.3. Semiclassical states. Fix a positive real number $C^{\text{supp}} > 0$. A semiclassical boundary state $Z$ consists in an array $Z = (Z^{\text{disk}}_\beta)_\beta \in \Gamma_{\text{flavor}}$ with $Z^{\text{disk}}_\beta \in Z^{\text{disk}}_{scl}$ and $Z^{\text{disk}}_\beta \neq 0 \Rightarrow ||\beta|| \leq C^{\text{supp}}\omega(\beta)$.

Denote by $\mathfrak{y}^{scl}$ the vector space of semiclassical boundary states.

There is an obvious action of $\mathfrak{g}^{scl}$ on $\mathfrak{y}^{scl}$ defined by $\hat{e}_{\gamma_0, \beta_0}(Z^{\text{disk}}_\beta)_\beta \in \Gamma_{\text{flavor}} = (\hat{e}_{\gamma_0, \beta_0}Z^{\text{disk}}_\beta - \beta_0)_\beta \in \Gamma_{\text{flavor}}$.

6.2. Semiclassical Limit. We are now ready to define the semiclassical limit of boundary states.

**Proposition 23.** There exists a natural map $\mathfrak{scl} : \mathfrak{y}^r \to \mathfrak{y}^{scl}$ compatible with MC-isotopies.

The following identity holds:

$$\mathfrak{scl} \circ \hat{e}_{\gamma, \beta}^{\text{lie}} = e_{\gamma, \beta} \circ \mathfrak{scl},$$

where $\hat{e}_{\gamma, \beta}^{\text{lie}}$ is the operator defined in section 4.4.

**Proof.** First notice that the left side of (19) belongs to $g_*Z$ if $v_1 = v_2$. It follows that on $Z/g_*Z$ we can neglect the relative position of $w_{h,v}$ and $w_{h',v}$ for each $h, h' \in H$ and $v \in V$. Hence, up to small isotopy, on the generator (13) we can assume that $w_{h,v}$ is independent on $h \in H$. Set $w_v = w_{h,v}$. We have a linear map $Z/g_*Z \to Z^{\text{disk}}$ defined by $g_*^{-|V|}(V, H, (w_v)_v, (\beta_v)_v) \mapsto (V, (w_v)_v, (\beta_v)_v)$.

It is immediate to check that this map is compatible with (81) and (19).

The identity (83) is easy to check from the definition. \qed
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