ON SOME PECULIARITIES OF SOLVING NONSTATIONARY PROBLEMS OF QUANTUM MECHANICS

V. E. Mitroshin

Kharkov National University, Ukraine, 61077, Kharkov

(April 27, 2001; revised version June 25, 2001)

Exact solutions of several nonstationary problems of quantum mechanics are obtained. It is shown that if the initial conditions of the problem correspond to the localized-in-space particle, then it moves exactly along the classical trajectory, and the wave packet is not spread in time.

KEY WORDS: quantum mechanics, exact solutions.

PACS number(s): 03, 03.65.Ca, 03.65.Ge, 03.65.Nk

I. INTRODUCTION

The standard way of solving a nonstationary problem of quantum mechanics (in \( \hbar = m = e = 1 \) units)

\[
i \frac{\partial}{\partial t} \varphi(t, x) = (H_0(x) + H_{\text{int}}(t, x)) \varphi(t, x),
\]

with the initial condition \( \varphi(t, x)|_{t=0} = k(x) \), where \( H_0 \) is the nonperturbed Hamiltonian, and \( H_{\text{int}} \) is the time-dependent perturbation, lies in the transition to the representation of the interaction \( \varphi(t, x) = \exp(-itH_0) \psi(t, x) \), and for the new equation

\[
i \frac{\partial}{\partial t} \psi(t, x) = \exp(itH_0) H_{\text{int}}(t, x) \exp(-itH_0) \psi(t, x)
\]

powerful methods of perturbation theory are developed, the presentation of which can be found nearly in any textbook on quantum mechanics. But here we would like to draw the readers’ attention to a number of exactly solvable problems for eq.(1). It has been found that if an infinitely differentiated function with a compact carrier is chosen as an initial condition, then the wave packet not only moves along the trajectory exactly corresponding to the solution of similar problems of classical mechanics, but is also not spread in time.

II. PROCEDURE

We make use of the fact that

\[
e^{itH_0} H_{\text{int}} e^{-itH_0} = H_{\text{int}} + \sum_{n=1} \frac{1}{n!} (it)^n \text{ad}^n H_0(H_{\text{int}}),
\]

where \( \text{ad}^n H_0(H_{\text{int}}) = [H_0[H_0[...[H_0, H_{\text{int}}]...]] \) is a contracted form of \( n \)-fold commutator. It remains to choose such \( H_{\text{int}} \), at which the series (2) is either summable or stops at a certain term. For simplicity, we shall consider one-dimensional problems in order not to make calculations cumbersome. In the subsequent discussion we shall often refer to two special functions with proper names being reserved for them: the Heaviside function and the Kelly function. The Heaviside function is the function \( \chi(t) = 0 \) at \( t < 0 \) and \( \chi(t) = 1 \) at \( t \geq 0 \). The Kelly function is the function

*mitroshin@univer.kharkov.ua
\[ K_{\alpha,\beta}(x) = \text{const} \times \exp[-1/(x-\alpha)(\beta-x)] \]

at \( \alpha < x < \beta \) and equal to zero for all other argument values. This function is remarkable for its being nonzero only within a small interval \((\alpha, \beta)\) and having derivatives of any order, square integrable by the Lebesgue method.

So far as operator \( u(t, x) = \exp(itH_0) \) is a unitary operator we obtain that the function \( \varphi = \exp(itH_0)\psi(t, x) \) is being nonzero only within a interval where the function \( \psi(t, x) \) is nonzero. And now we turn to particular problems.

### III. A FREELY TRAVELLING PARTICLE

It would appear surprising to find anything new in this problem investigated so thoroughly. And yet, has the problem been posed correctly if we wish to describe the motion of the particle which acquires a velocity \( \lambda \) at the time moment \( t = 0 \)? In other words, we should solve, in fact, equation (4) with the perturbation proportional to the momentum operator, i.e., of the form \( H_\text{int} = -i\lambda\theta(t)\partial/\partial x \), where \( \theta(t) \) is a certain function of time defining how exactly the momentum was transferred to the particle, and \( \lambda \) is the constant characterizing the momentum transfer value, its dimensional representation is such that should have the dimensionality inverse to time. Since \( H_0 = -(1/2)\partial^2/\partial x^2 \) commutes with \( H_\text{int} \), then eq. (4) takes on the following form

\[ i\partial \psi(t, x)/\partial t = -i\lambda\theta(t)\partial \psi(t, x)/\partial x. \]

The general solution to this equation is

\[ \psi(t, x) = k(x - \lambda \int_0^t \theta(\tau) \, d\tau) \]

for \( t \geq 0 \), and \( k \) is any infinitely differentiated function of argument satisfying the initial requirements. In particular, any infinitely differentiated function with a compact carrier, equal identically to zero beyond a certain interval, e.g., the Kelly function, can be taken as \( k \). If the particle instantly acquires the velocity \( \lambda \), the one should put \( \theta(t) = \chi(t) \). As a result, we obtain that the probability density (in the representation of the interaction) for the particle to be at a certain point of space-time \( |\psi(x, t)|^2 = K_{\alpha,\beta}(x - \lambda t) \) is not spread in time and it proceeds exactly along the trajectory described by classical mechanics. If, however, the particle gained its velocity at uniform acceleration, i.e., \( \lambda\theta(t) = a \times t \), then \( |\psi(x, t)|^2 = K_{\alpha,\beta}^2(x - a \times t^2/2) \), this again exactly corresponds to the solution of classical mechanics, as well as \( |\varphi(x, t)|^2 = |u(t, x)\psi(x, t)|^2 \).

### IV. A CHARGED PARTICLE IN THE ELECTRIC FIELD

\[ H_0 = -(1/2)\partial^2/\partial x^2 \]

and the electric field \( H_{\text{int}} = \lambda\theta(t) x \) included by the \( \theta(t) \)–law. Since \([H_0, H_{\text{int}}] = -\lambda\theta(t)\partial/\partial x \), then \([H_0[H_0, H_{\text{int}}]] = 0 \), and eq. (4) takes the form:

\[ i\partial \psi(t, x)/\partial t = \lambda\theta(t) x\psi(t, x) - i\lambda\theta(t)\partial \psi(t, x)/\partial x. \]

After the replacement \( \psi(t, x) = \rho(t, x) \exp\{-i\lambda x \int_0^t \theta(\tau) \, d\tau + i\lambda^2 \int_0^t \tau \theta(\tau) \, d\tau \}, \) we obtain the following equation

\[ \partial \rho(t, x)/\partial t = -\lambda\theta(t)\partial \rho(t, x)/\partial x. \]

the general solution to which is the function \( \rho(t, x) = k(x - \lambda \int_0^t \theta(\tau) \, d\tau) \). So, we have that if the Heaviside function (the electrical field is included instantly) is chosen as \( \theta(t) \), and the Kelly function is chosen at \( t = 0 \) as an initial condition, then the probability density precisely follows the classical path \(|\psi(x, t)|^2 = K_{\alpha,\beta}^2(x - \lambda \times t^2/2) \), and the wave packet is not spread in time as well as \(|\varphi(x, t)|^2 = |u(t, x)\psi(x, t)|^2 \).
V. A HARMONIC OSCILLATOR IN THE ELECTRIC FIELD

In this case, we put $H_{\text{int}} = \lambda \theta(t) x$ and $H_0 = -(1/2) \partial^2 / \partial x^2 + 1/2 x^2$. Let us calculate the commutators. As $[H_0, H_{\text{int}}] = -\lambda \theta(t) \partial / \partial x$, then $[H_0, [H_0, H_{\text{int}}]] = \lambda \theta(t) x = H_{\text{int}}$, and the series is summed to $\lambda x \theta(t) \cos t - i \lambda \theta(t) \sin t \partial / \partial x$. Then, eq. (1) is written as

$$i \partial / \partial t \psi(t, x) = \lambda x \theta(t) \cos t \psi(t, x) - i \lambda \theta(t) \sin t \frac{\partial}{\partial x} \psi(t, x)$$

and can be integrated explicitly:

$$\psi(t, x) = \kappa(x - \lambda \int_0^t \theta(\tau) \sin \tau d\tau) \times \exp\{-i \lambda x \int_0^t \theta(\tau) \cos \tau d\tau + i \lambda^2 \int_0^t \theta(\tau) \sin \tau \int_0^\tau \theta(\tau') \cos \tau' d\tau'^2|d\tau\}.$$  

If the electrical field is included instantly, $\theta(t) = \chi(t)$, then for the probability density we obtain the answer exactly corresponding to the solution of a similar problem in classical mechanics: $|\psi(x, t)|^2 = K_{\alpha, \beta}^2(x + 2\lambda \sin^2 t/2)$ as well as $|\varphi(x, t)|^2 = |u(t, x)|^2$.

One can consider the pendulum swinging under the action of force, or the pendulum acquiring the momentum at a certain moment of time, etc., all these problems have the solutions differing from the "classical solution" only by the presence of a complex phase factor, whose value is very much similar to the integral over the pendulum trajectory ("intrinsic" clock that is "ticking" as long as the perturbation acts).

VI. CONCLUSION

The given problems are far from being the only ones that have exact analytical solutions, the solutions correlating well with the solutions to similar problems of classical mechanics. And in view of this many questions arise. What is "the freely travelling particle"? In fact, once it had been given the momentum to "travel"! And what is "the particle in the electric field"? The field was "turned on" at one time, after all! And the equation $i \dot{\varphi} + 1/2 \ddot{\varphi}_{xx}$ is only an equation of free motion, it has no physical meaning in itself. The Fourier transform of this equation shows only how the equation looks like in the vector space $\{e^{-ikx}\}$, and nothing more. Being pure materialists, we impart a desirable for us "physical meaning" to them, similarly to Bernoulli who, at his time, treated the possibility of Fourier-series expansion of the function due to "world harmony". We would like to discuss the philosophy of quantum mechanics in greater detail - thousands of books have been written on the topic, so many passions were burning! But the time has not come as yet, not all the problems (passage through a barrier, interference) are solved, that might be helpful to open the mysterious meaning of the "wave-particle" dualism, the more so - it is not the right place.