WISHE-Moisture Mode in a Vertically Resolved Model

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Abstract The Madden-Julian oscillation is a global phenomenon, a wave with the wavelength of the circumference of the Earth. A theoretical model presented by Fuchs and Raymond (2007, https://doi.org/10.1111/j.1600-0870.2007.00230.x) obtained a mode called the eastward propagating wind-induced surface heat exchange (WISHE)-moisture mode that had its largest instability for the longest wavelengths. That model imposed the first baroclinic mode vertical structure. This paper expands the Fuchs and Raymond model by developing a vertically resolved model, a model in which all the variables except the heating vertical profile are calculated. The heating profile is assumed to have a vertical structure of the first baroclinic mode. The model uses the upper boundary radiation condition and thus includes the effects of the stratosphere. The convective parametrization includes wind-induced surface heat exchange, cloud-radiation interactions, and moisture closure. The vertically resolved model produces one unstable mode whose dispersion relation characteristics closely resemble the eastward propagating WISHE-moisture mode. For the longest wavelengths it is more unstable and its phase speed is slightly larger than in the Fuchs and Raymond model. The calculated vertical structure reveals a similar vertical structure to the first baroclinic mode with small differences that to some extent agree with observations.

Plain Language Summary The Madden-Julian oscillation is the largest weather disturbance that exists on our planet. On average, it produces the most rain. It slowly moves eastward in the tropical belt bringing long-lasting rain and long-lasting winds. Historically, people used it to travel over the oceans and inhabit new areas. This paper represents the basic, most fundamental theory that explains the Madden-Julian oscillation discovered in 1971 by Madden and Julian. The interplay between the surface fluxes, due to main easterlies in the tropics, and moisture is the mechanism that drives this disturbance. Knowing the basic physical mechanism of the Madden-Julian oscillation will help us improve our numerical weather models and thus get a better weather forecast in the tropics and the rest of the world.

1. Introduction

When the Madden-Julian oscillation (MJO) was discovered in 1971 on observational data of the zonal wind in the lower and upper troposphere (Madden & Julian, 1971), most of our theoretical knowledge about the equatorially trapped waves was based on the paper by Matsuno (1966). Among other characteristics of the newly found oscillation (Madden & Julian, 1971) is that it is slowly eastward moving and that the meridional wind component is not involved with the oscillation. In his model based on the shallow water equations on an equatorial beta plane with an adiabatic atmosphere, Matsuno (1966) modeled an eastward propagating mode for the meridional wind speed $v = 0$ case that is called the Kelvin wave. The mode was neither growing nor decaying as it was not coupled with convection. This similarity between the dynamics of the free Kelvin wave from Matsuno (1966) and the newly found oscillation gave rise to a school of thought that explained the MJO in terms of convectively coupled Kelvin mode.

Madden (1986) investigated the role of the meridional wind in more detail concluding that if the data are not averaged over all seasons, the meridional wind plays an important role in the MJO. There is observational evidence that the MJO consists of rotating gyres around its convective center, best summarized in Zhang (2005). Some analytical models such as Adames and Kim (2016) and numerical models such as Hayashi and Itoh (2017) also point to the importance of the meridional wind when looking at the MJO.

Wheeler and Kiladis (1999) created a wavenumber-frequency power spectrum plot for equatorially symmetric (similar plot given as Figure 1) and antisymmetric components of the outgoing longwave radiation (OLR) based on the satellite data for the period of about 18 years (January 1979 to April 1996). This plot as well as...
many others that followed gives us an idea of what disturbances exist in the tropics. The stronger the minimum in the OLR, the stronger the coupling of the corresponding wave disturbance with moist convection. In Figure 1 the $x$ axis represents the planetary zonal wavenumber, while the $y$ axis represents the frequency. The strongest disturbance that can be seen is the MJO that propagates eastward and lives at planetary zonal wavenumbers $l = 1, 2$. Note that if $l = 1$, the disturbance has the wavelength of the circumference of the Earth. This is in agreement with the findings of Madden and Julian (1971) who saw that the newly found oscillation takes up the whole tropical band in the signal of upper level tropospheric zonal wind.

The second strongest disturbance that propagates eastward is the convectively coupled Kelvin wave (CCKW) with the strongest signal in OLR around planetary wavenumber $l = 5, 6$. The MJO's OLR signal propagates very slowly with phase speed (frequency divided by wavelength) about $c = 5$ m/s, while the CCKW propagates with phase speed $c = 15–20$ m/s.

Mapes (2000) developed a toy model of large-scale deep convection variations in equatorial atmosphere with the imposed vertical structure of two modes, one associated with convective and one associated with stratiform heating processes. Convective inhibition (CIN) parametrization gave rise to a mode that propagated eastward with phase speed $c = 20$ m/s and exhibited the westward tilted temperature vertical structure called “boomerang” structure. Majda and Shefter (2001a, 2001b) developed a complex model with imposed vertical structure that also modeled the CCKWs. Fuchs and Raymond (2007a) and Raymond and Chiu (2007b) developed a simple analytical vertically resolved model where all the fields were calculated with only one assumption on the vertical structure—that the heating profile has the structure of the first baroclinic mode. They modeled a mode that had the greatest instability for $l = 7$, a wavelength of about 6,000 km due to anomalies in convective inhibition caused by buoyancy variations at the top of the planetary boundary layer. The mode had a boomerang temperature vertical structure and it was propagating eastward with phase speed $c = 17$ m/s. Herman et al. (2016) confirmed these theoretical findings by using radiosonde data, 3D analysis and reanalysis model output, and annual integrations with the ECMWF model on the full planet and on an aquaplanet. In addition to CCKW, Fuchs and Raymond (2007a) and Raymond and Chiu (2007b) modeled a slow moving moisture mode that was unstable due to cloud-radiation interactions (CRIIs) and negative gross moist stability at all wavelengths.

To explore the moisture mode in more detail, Fuchs and Raymond (2017) (hereafter FR17) developed a model with the linearized governing equations on an equatorial beta plane, a diabatic atmosphere with convective parametrizations of moisture closure and wind-induced surface heat exchange (WISHE), but with a simple vertical structure of the first baroclinic mode. The WISHE parametrization goes back to Emanuel.
(1987), Neelin et al. (1987), and Yano and Emanuel (1991). The moisture closure goes back to Raymond (2000). FR17 modeled a mode for the meridional wind \( v = 0 \) case that has no analog with the Matsuno (1966) mode. If the moisture closure is turned off, the mode disappears; that is, it is a moisture mode. Note that the moisture mode instability produced by negative gross moist stability that persists for all wavelengths and in a case when the meridional mode number \( n = 1 \) is different from the moisture mode itself (for more on the development of the moisture mode theory see FR17). The FR17 mode has the largest instability for planetary wavenumber \( l = 1 \) and is propagating eastward with the phase speed around \( c = 15 \text{ m/s} \) for \( l = 1 \) and around \( c = 7 \text{ m/s} \) for \( l = 2 \). The FR17 authors named that mode the eastward propagating WISHE-moisture mode as the instability of the mode was a consequence of the interplay between the moisture and surface fluxes. The mean surface zonal wind that is on average easterly in the tropics, and as such is a global phenomenon in the tropical belt, acts via surface fluxes to moisten the atmosphere to the east and to the west of the disturbance. The moistening is greater to the east of the disturbance due to the convergence produced by the mode itself. After the atmosphere reaches a certain critical value in moisture, the moisture closure, which states that the precipitation rate increases as the amount of moisture in a vertical column increases, acts and deep convection develops. The convergence in the lower troposphere and the westerly wind anomalies to the west of the precipitation maximum are then a consequence of the deep convection. As the enhanced moistening is happening on the east side of the disturbance, the disturbance moves eastward. The upper tropospheric winds are reversed in relation to the lower troposphere.

The horizontal \( x-y \) structure of the WISHE-moisture mode revealed that the maximum equivalent potential temperature anomaly is slightly to the east of the moisture anomaly, while the temperature anomaly is even further east. The authors stated that the eastward propagating WISHE-moisture mode is to its zeroth order a fundamental mechanism that governs the MJO because it has the largest instability at the longest wavelengths and it moves eastward. The fact that the phase speed did not match the observations was explained in Raymond and Fuchs (2018) where a simple linear superposition of the warm pool and the eastward propagating WISHE-moisture mode led to a mode whose combined rainfall moved with the phase speed of \( c = 5 \text{ m/s} \).

Analytical linear models are limited by their design in capturing all the characteristics of the MJO. Shi et al. (2018) and Khairoutdinov and Emanuel (2018) found compelling evidence that WISHE-moisture mode is associated with the MJO. Shi et al. (2018) simulated the MJO using Geophysical Fluid Dynamics Laboratory’s AM2.1 on an aquaplanet. Based on series of denial experiments they showed that the MJO cannot be simulated unless WISHE was included. They also found that the CRI and moisture-convection feedback were important for the phase speed of the MJO. Using the cloud-permitting System for Atmospheric Modeling (SAM) and performing the series of denial experiments, Khairoutdinov and Emanuel (2018) showed that the CRI is the driving mechanism for the MJO, while WISHE is the primary cause for its eastward propagation. Using Superparameterized Community Atmosphere Model, Ma and Kuang (2016) found that radiative-convective feedback amplified the MJO, while WISHE slowed it down. Using an atmospheric general circulation model, Kim et al. (2011) found that the MJO is strengthened with CRI turned on but weakened when WISHE is turned on.

The model from FR17 was limited by the imposed first baroclinic mode vertical structure. A series of models exist in the literature (Khouider & Majda, 2006, 2008; Kuang, 2008, Mapes, 2000; Majda & Shefter, 2001b, Majda et al., 2004) that impose a two-mode vertical structure rather than calculate the vertical structure. Fuchs and Raymond (2007a) and Raymond and Fuchs (2007b) developed a model that calculates the vertical structure of all the variables while imposing the first baroclinic mode structure on the heating source term. Fuchs et al. (2012) expanded that model by considering different vertical heating profiles and still calculating all the variables’ vertical structure. In those papers the authors were focused on modeling the CCKW and to lesser extent the moisture mode. The significance of the WISHE-moisture mechanism was unknown.

In this paper, a vertically resolved model for the eastward propagating WISHE-moisture mode with meridional speed \( v = 0 \) is presented. The goal is to build upon the simplest possible model for the MJO, the one from FR17, and to look at its vertical structure free of imposed hypotheses.

### 2. Vertically Resolved Analytical Model

The presented model is a combination of Fuchs and Raymond (2007a) and FR17. It is an analytical, linear, vertically resolved model with convective coupling. From the FR17 model that was done with the first
Table 1

| New variable | \( t \) | \( x \) | \( z \) | \( u \) | \( w \) | \( \Pi \) | \( b \) | \( q \) | \( e \) | \( S_{B,E} \) | \( B, P, R, E \) |
|--------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|
| Nondimensionalization | \( \sqrt{\beta}t \) | \( \sqrt{\beta}/cz \) | \( m_0z \) | \( u/c \) | \( m_0w/\sqrt{\beta}c \) | \( \Pi/c^2 \) | \( b/mc^2 \) | \( q/mc^2 \) | \( e/mc^2 \) | \( S_{B,E}/m_0c^2\sqrt{c/\beta} \) | \( B, P, R, E/c^2\sqrt{c/\beta} \) |

baroclinic mode structure, we know that the eastward propagating WISHE-moisture mode exists when the meridional speed \( v = 0 \). Here, the goal is to develop a vertically resolved model that couples with stratosphere in a case of \( v = 0 \) for the eastward propagating WISHE-moisture mode, similar to the one from Fuchs and Raymond (2007a). The difference between the two is in nondimensionalization and convective coupling mechanisms that in this paper focus on interplay between the surface fluxes and moisture.

The nondimensionalization is done using Matsuno and FR17 scaling \( [T] = (1/c\beta)^{1/2} \), \( [L] = (c/\beta)^{1/2} \), where \( c = \Gamma_1^{1/2}/m_0 \) is the speed of free gravity waves, \( m_0 = \pi/h \) is the vertical wavenumber, and \( h \) is the depth of the troposphere. For more details on nondimensionalization see Table 1. The system of linearized, dimensionless, Boussinesq governing equations for \( v = 0 \) is

\[
\frac{du}{dt} + \frac{d\Pi}{dx} = 0
\]

\[
yu + \frac{d\Pi}{dy} = 0
\]

\[
\frac{d\Pi}{dz} = b = 0
\]

\[
\frac{du}{dx} + \frac{dw}{dz} = 0
\]

\[
\frac{db}{dt} + w = S_B
\]

Table 2

| Symbol | Value |
|--------|-------|
| Depth of the troposphere | \( h = 15 \text{ km} \) |
| Free gravity wave speed | \( c = \sqrt{\Gamma/m_0} = 48 \text{ m/s} \) |
| Moisture relaxation rate | \( \tilde{\alpha} = 1.16 \times 10^{-5} \text{ s}^{-1} \) |
| Cloud-radiative feedback parameter | \( \tilde{\epsilon} = 0.2 \times 10^{-5} \text{ s}^{-1} \) |
| Rossby parameter | \( \beta = 2.29 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1} \) |
| Mixing ratio lapse rate | \( \Gamma_{Qold} = gL \frac{d\theta}{d\theta_0} \) |
| Moist static stability | \( \Gamma = 0.2 \times 10^{-5} \text{ s}^{-2} \) |
| Gross moist stability | \( \Gamma = 0.1 \) |
| Moisture relaxation rate | \( \alpha = \tilde{\alpha} / \sqrt{c\beta} \) |
| Cloud-radiative feedback parameter | \( \epsilon = \tilde{\epsilon} / \sqrt{c\beta} \) |
| Wind-induced variations in surface fluxes | \( \Lambda = C \Delta qU_0/2c\sqrt{\beta}(U_0^2 + W^2)^{1/2} \) |
| Frequency | \( \Omega = 0.1 \) |
| Wavenumber | \( \kappa = k_L \) |
| Phase speed | \( \Phi = \Omega / \kappa \) |
\[ \frac{\partial q}{\partial t} + \Gamma Q w = S_Q \]  
\[ \frac{\partial e}{\partial t} + \Gamma w = S_E \]  

where \( u \) and \( w \) are the horizontal and vertical wind perturbations, \( \Pi \) is the kinematic pressure perturbation, \( b \) is the scaled buoyancy perturbation, \( q \) is the scaled mixing ratio perturbation, and \( e \) is the scaled equivalent potential temperature perturbation. \( S_B, S_Q, \) and \( S_E \) are scaled heating (buoyancy), moisture, and equivalent potential temperature sources parametrized as functions of surface winds, CRI, and moisture hereafter. \( \Gamma_B = m_0^2 c^2, \Gamma_Q, \) and \( \Gamma \) are nondimensionalized using \( \Gamma_B \) leading to the expression for moist static stability \( \Gamma = 1 + \Gamma_Q, \) a function of \( z. \) Note that \( e = b + q \) and \( S_E = S_B + S_Q \) and one equation from (5)–(7) can be eliminated.

All the variables are assumed to have \( \exp \left[ i (\kappa x - \Omega t) \right] \) and time structure, where \( \kappa \) is a dimensionless zonal wavenumber, \( \kappa = k \sqrt{\frac{c}{\beta}} \), and \( \Omega = \omega / \sqrt{\beta c} \) is a dimensionless frequency; \( k \) and \( \omega \) are dimensional wavenumber and frequency.

From equations (1) and (2) we can calculate the meridional structure for \( \Pi \) and \( u \) to have \( \exp (-\kappa y^2 / \Omega) \) form.

### 2.1. Calculating the Vertical Structure

The vertical structure of all the variables except the heating source \( S_B \) will be calculated. The only assumption that is made is that the heating source has a first baroclinic mode structure:

\[ S_B(z) = \frac{B}{2} \sin(z) \]  

where \( B = \int_0^z S_B(z) dz. \) Note that due to nondimensionalization the top of the troposphere is at \( z = \pi. \)

The assumption imposed on the heating source is oversimplified. The observations (Kiladis et al., 2005) suggest a tilted structure of heating for the MJO that could best be represented by imposing a structure that is a combination of the first and second baroclinic mode. This is beyond the scope of this paper and is left to future research.

From equations (1)–(5) it is straightforward to obtain a differential equation for the vertical wind perturbation:

\[ \frac{d^2 w(z)}{dz^2} + m^2 w(z) = m^2 S_B(z) \]  

where \( m = \kappa / \Omega. \) The assumed solution to the equation (9) is

\[ w(z) = I(z) \exp(imz) + J(z) \exp(-imz). \]  

Taking two derivatives of (10) and plugging them back into (9) leads to

\[ I(z) = \frac{imB}{4} \int_{z}^{\infty} \exp(-imz') \sin(z') dz' \]  
\[ J(z) = \frac{imB}{4} \int_{0}^{z} \exp(imz') \sin(z') dz' \]  

where the boundary conditions dictate the integral limits. The upper boundary condition here is the upper boundary radiation condition that assumes continuity of the variable and its derivative across the boundary \( (z = \pi) \) and the variable going to 0 at infinity. This applies only to growing modes. The solution for vertical wind perturbation is

\[ w(z) = I(z) \exp(imz) + [J(z) - I(0)] \exp(-imz) \]  

which gives us two different solutions, one for the troposphere where \( S_B = B/2 \sin(z) \) and one for the stratosphere where \( S_B = 0 \) with
w(z) = \frac{B}{2(1 - \Phi^2)} \left[ \sin(z) + \Phi \exp(-ix/\Phi) \sin(z) \right]; z \leq \pi \tag{14}

w(z) = \frac{B\Phi}{2(1 - \Phi^2)} \sin(\pi/\Phi) \exp(-imz); z > \pi \tag{15}

where \Phi = \Omega / \kappa = 1/m. Knowing the solution for w means that we know the solutions for the other variables from our governing equations and those are given in Appendix A.

2.2. Thermodynamics of the Model

The source terms in thermodynamic equations (5)–(7) are defined as

\[
\int_0^\pi S_h(z)dz = P - R = (\alpha + \epsilon) \int_0^\pi q(z)dz \tag{16}
\]

\[
\int_0^\pi S_q(z)dz = E - P = \Lambda u_s(z) - \alpha \int_0^\pi q(z)dz \tag{17}
\]

\[
\int_0^\pi S_e(z)dz = E - R = \Lambda u_s(z) + \epsilon \int_0^\pi q(z)dz. \tag{18}
\]

\(u_s = u(z = 0)\) is the surface wind perturbation and \(P, R,\) and \(E\) are scaled perturbations in the precipitation rate, radiative cooling rate, and the surface evaporation rate. The surface sensible heat flux is neglected as being small over the tropical ocean. Nondimensional parameters \(\alpha, \epsilon\) and \(\Lambda\) come from the assumptions about the precipitation rate, radiative cooling rate, and surface evaporation rate.

The precipitation rate perturbation is assumed to increase linearly as the vertically integrated mixing ratio increases

\[P = \alpha \int_0^\pi q(z)dz\]  \(\tag{19}\)

where \(\alpha\) is the moisture relaxation rate assumed to be equivalent to 1/day.

The radiative cooling rate perturbation is assumed to decrease linearly as the mixing ratio increases as a result of the associated increase in cloudiness

\[R = -\epsilon \int_0^\pi q(z)dz\] \(\tag{20}\)

where \(\epsilon\) is the cloud-radiative parameter reflecting the assumption that there is less OLR in the atmospheric column with clouds than in the one where it is clear.

The surface evaporation perturbation rate is assumed to be

\[E = \Lambda u_s\] \(\tag{21}\)

where \(\Lambda\) is the WISHE parameter. Numerical values of parameters used in this model are given in Table 2. For more details see FR17.

2.3. Dispersion Relation

From equation (16) the integrated heating source perturbation is

\[B = (\alpha + \epsilon) \int_0^\pi q(z)dz = (\alpha + \epsilon) \left[ \int_0^\pi e(z)dz - \int_0^\pi b(z)dz \right] \tag{22}\]

as \(\epsilon = b + q\). To obtain the integral of the equivalent potential temperature, equation (7) is integrated. With the help of equation (18) this leads to

\[(\epsilon + i\kappa \Phi) \int_0^\pi e(z)dz = \int_0^\pi \Gamma(z)w(z)dz + \epsilon \int_0^\pi b(z)dz - \Lambda u_s\]  \(\tag{23}\)
and combined with equation (22) to

\[ \frac{\varepsilon + i \kappa \Phi B}{\alpha + \varepsilon} = \int_0^\pi \Gamma(z)w(z)dz - i\kappa \Phi \int_0^\pi b(z)dz - \Lambda u_s. \]  \hspace{1cm} (24)

Integrating equation (5) and remembering that \( B = \int_0^\pi S_B(z)dz \) gives us

\[ \kappa \Phi \int_0^\pi b(z)dz = iB - i \int_0^\pi w(z)dz. \]  \hspace{1cm} (25)

The integral of the vertical wind perturbation is easily obtained from equation (14)

\[ \int_0^\pi w(z)dz = BF(\Phi) \]  \hspace{1cm} (26)

where \( F(\Phi) = 1 + \Phi^2 \exp(-i\pi/\Phi) \left(1 - \cos(\pi/\Phi)\right) /2 \). The surface wind is obtained from equation (4):

\[ u_s = \frac{i \partial w}{\kappa \partial z} \bigg|_{z=0} = \frac{iB}{2\kappa(1 - \Phi^2)} G(\Phi) \]  \hspace{1cm} (27)

where \( G(\Phi) = 1 + \exp(-i\pi/\Phi) \).

The normalized gross moist stability is defined as

\[ \Gamma_M = \int_0^\pi \Gamma(z)w(z)dz \int_0^\pi w(z)dz \]  \hspace{1cm} (28)

which gives us the expression

\[ \int_0^\pi \Gamma(z)w(z)dz = \frac{BF(\Phi)}{1 - \Phi^2} \]  \hspace{1cm} (29)

Plugging in the equations (29), (27), and (25) into (24) leads to

\[ \frac{\varepsilon + i \kappa \Phi B}{\alpha + \varepsilon} = \frac{BF(\Phi)}{1 - \Phi^2} + B - \frac{BF(\Phi)}{1 - \Phi^2} - \frac{\Lambda B i}{2\kappa(1 - \Phi^2)} G(\Phi) \]  \hspace{1cm} (30)

which after some clean up leads to the dispersion relation in \( \Phi \):

\[ \kappa \Phi^3 + i\alpha \Phi^2 - \kappa \Phi - i\alpha + i(\alpha + \varepsilon)F(\Phi)(1 - \Gamma_M) - \frac{\Lambda(\alpha + \varepsilon)}{\kappa} G(\Phi) = 0 \]  \hspace{1cm} (31)

or in \( \Omega \)

\[ \Omega^3 + i\alpha \Omega^2 - \kappa^2 \Omega - i\kappa^2 \alpha + i\kappa^2(\alpha + \varepsilon)F(\Omega)(1 - \Gamma_M) - \Lambda(\alpha + \varepsilon)\kappa G(\Omega) = 0 \]  \hspace{1cm} (32)

where \( F(\Omega) = 1 + \Omega^2 \exp(-i\pi\kappa/\Omega) \left(1 - \cos(\pi\kappa/\Omega)\right) /2\kappa^2 \) and \( G(\Omega) = 1 + \exp(-i\pi\kappa/\Omega) \).

3. Results

The results for the only unstable mode from the dispersion relation (31) are presented here. This mode is unstable due to the interplay between WISHE and moisture, and the CRI. The comparison is made between the WISHE-moisture mode from FR17 and this new mode that comes from the vertically resolved model. Its vertical structure is also discussed and compared to the observations.

3.1. Dispersion Relations

Figure 2 shows the phase speed and the growth rate as a function of planetary wavenumber \( l \) (used instead of \( \kappa \) to enable the comparison with Figure 1) for the WISHE-moisture mode for dispersion relations from two different models. The blue line shows the WISHE-moisture mode from the FR17 model and the red line shows the dispersion relation for the mode that comes from the vertically resolved model with upper boundary radiation condition. The parameters are set to WISHE parameter \( \Lambda = -0.09 \) which implies mean easterlies of \(-3 \text{ m/s}\); a positive gross moist stability parameter, \( \Gamma_M = 0.1 \), and CRIs included with \( \varepsilon = 0.06 \).
For more details on gross moist stability estimation in a vertically resolved model see Fuchs and Raymond (2007a). Figure 3 shows the same thing as Figure 2 except that the gross moist stability parameter is $\Gamma_M = 0$. These parameters are chosen based on FR17 and idealized aquaplanet models such as Khairoutdinov and Emanuel (2018) that show that the interplay between moisture and WISHE as well as the CRI are important mechanisms for the MJO. As we can see from Figures 2 and 3, the gross moist stability acts only to increase the instability of the mode, but this increase is largest for shorter wavelengths. This is important because when looking for the MJO, one has to look for the mode that has the highest instability for the longest wavelengths. The CRI also act only to increase the instability for all wavelengths, but as the idealized models show their importance, the CRI are not turned off in this model. The gross moist stability however is generally positive and therefore when the vertical structure is discussed the case with $\Gamma_M = 0.1$ will be presented.

The WISHE-moisture mode from the dispersion relation (31) propagates eastward with phase speed $c = 20 \text{ m/s}$ for $l = 1$, only slightly faster than in FR17. For $l = 2$ it propagates significantly slower, phase speed around $c = 6 - 7 \text{ m/s}$. The observed OLR for the MJO shows that the OLR propagates with phase speed $c = 5 \text{ m/s}$ over the warm pool, while wind fields move faster. One possible explanation of how the WISHE-moisture mode interacts with the warm pool to slow down has been given by Raymond and Fuchs (2018). In that paper they linearly superimpose a mode caused by the stationary warm pool and the WISHE-moisture mode finding that the combined rainfall moves with the phase speed of $c = 5 \text{ m/s}$.

Both modes, given in Figures 2 and 3 in red and blue, are unstable for all wavelengths and have the largest instability for the planetary wavenumber $l = 1$. In the case of a stable atmosphere, that is, when the gross moist stability parameter is positive, $\Gamma_M = 0.1$, for shorter wavelengths the instability asymptotically approaches the value of $\varepsilon = 0.06 \text{ day}^{-1}$ after the planetary wavenumber $l = 3$ as expected. The instability for shorter wavelengths exists solely due to CRI in Figure 2, while it is stronger in a case of neutral atmosphere when the gross moist stability parameter is $\Gamma_M = 0$, Figure 3. The mode that comes from the vertically resolved model, shown in red in both Figures 2 and 3, is slightly more unstable for the long wavelengths than the one from FR17. There is no difference for shorter wavelengths. This implies that the interaction with the stratosphere makes the WISHE-moisture mode more unstable. The reason for it needs further investigation.

### 3.2. Vertical $x$-$z$ Structure for WISHE-Moisture Mode

FR17 model imposed a simple first baroclinic mode vertical structure with a half sine structure for the temperature and vertical velocity and half cosine structure for the zonal wind and pressure. Here we look at the
Figure 4. Vertical x-z structure of zonal and vertical wind perturbations ($u, w$), buoyancy ($b$), and pressure ($\Pi$) perturbations for the WISHE-moisture mode in a vertically resolved model for planetary wavenumber $l = 1$ for $\Lambda = -0.09$ or $u = -3 \text{ m/s}$, $\Gamma = 0.1$ and $\epsilon = 0.06$. Similarities and differences in the vertical structure when the heating is assumed to have a first baroclinic mode structure, while the vertical structure of $u$, $w$, $b$, and $\Pi$ is calculated from the vertically resolved model with the upper boundary radiation condition.

Line contours in Figures 4 and 5 show the vertical x-z structure of zonal and vertical wind perturbations ($u, w$) as well as buoyancy ($b$) and pressure ($\Pi$) perturbations for the WISHE-moisture mode in a vertically resolved model at planetary wavenumbers $l = 1$ and $l = 2$, respectively. The parameters are as in Figure 2: WISHE $\Lambda = -0.09$, the gross moist stability $\Gamma = 0.1$, and CRIs $\epsilon = 0.06$. The vertical structure of the heating, assumed to be the first baroclinic mode, is given in colored contours.

Figures 4 and 5 show a well known vertical structure for the zonal wind perturbation, where easterlies proceed the maximum heating in the lower troposphere, while the upper troposphere is dominated by westerlies.

Figure 5. As in Figure 4 except for $l = 2$. 
The MJO is a global, planetary-scale disturbance within the tropical band with a period of 30 to 90 days that slowly moves eastward. Over the warm pool the rainfall signal moves with phase speed $c = 5 \text{ m/s}$. The observations show the signal of the upper zonal wind associated with the MJO to extend over the whole circumference of the Earth. That would imply that the MJO has planetary wavenumber $l = 1$. Analysis such as the one by Wheeler and Kiladis (1999) shows that the strongest OLR signal for the MJO is at wavenumbers $l = 1, 2$. Therefore, when looking for the fundamental physical mechanism that can explain the propagation and the instability of the MJO, one should look at the physics happening on a planetary scale. The model that explains the fundamental characteristics of the MJO has to be able to produce a mode that propagates eastward and has the largest instability for the longest wavelengths.

One such model is that of FR17 where by using the moisture closure and the WISHE parametrization, the authors obtained an eastward propagating WISHE-moisture mode that had the largest instability for wavenumber $l = 1$. WISHE parametrization in the model takes advantage of the fact that zonally averaged mean surface wind over the MJO’s latitude belt is easterly and as such is a global phenomenon in the tropics. The enhanced surface fluxes in the mean easterly flow moisten the troposphere to the east of the MJO and as such are included in the equation for moisture budget. At some point the moisture content reaches the critical value and deep convection develops. This is reflected in the equations via the moisture closure included in the equation for the heating budget. The convection then enhances the easterlies to the east of the MJO, while the anomalous westerlies weaken the mean easterlies to the west of the MJO. This conceptual picture of the MJO states that the MJO’s propagation mechanism is due to WISHE, while the instability mechanism is due to the interplay between moisture and surface fluxes.

FR17 model was not a vertically resolved model as it assumed all the variables to have the first baroclinic mode structure. The goal of this paper is to expand the FR17 model by deriving the vertically resolved model to calculate, and not assume, the vertical structure of the eastward propagating WISHE-moisture mode. The only assumption that is made is that the heating has a vertical structure of the first baroclinic mode, while the rest of the variables are calculated. The upper boundary condition in the model is the radiation condition, which has the effect of including the stratosphere. The convective parametrization used includes WISHE, CRIs that come into the equation for the heating budget, and moisture closure.

The vertically resolved model with WISHE, CRI, and moisture closure in a case when the meridional wind $v = 0$ produces one unstable mode that propagates eastward and has largest instability for the planetary wavenumber $l = 1$. The interplay between the surface fluxes and moisture leads to the maximum growth rate at the longest wavelengths while CRIs enhance that instability. The phase speed of the mode is around $c = 20 \text{ m/s}$ for $l = 1$ and $c = 7 \text{ m/s}$ for $l = 2$. The mode has slightly larger instability and slightly larger phase speeds than the ones from FR17, but only for the long wavelengths. This is unlike Yano and Emanuel (1991) who expanded on the Emanuel (1987) model by adding the effect of the stratosphere and saw a significant difference between the modes in a shift of the largest instability toward longer wavelengths. The reason for this disagreement might be in the fact that the WISHE-moisture mode from FR17 already exhibited the largest instability for the planetary wavenumber $l = 1$. The vertical structure of the mode is close to that of the first baroclinic mode with temperature and vertical velocity looking like a half a sine and zonal wind and
pressure looking like half a cosine. The slight differences that occur because the model is vertically resolved (the temperature maximum proceeding the heating and vertical wind as in Kiladis et al. (2005) and the slight tilt in the temperature toward the west in the upper troposphere) to some degree agree with observations, bringing this mode closer to the realistic MJO. The stratosphere acts as a layer to which some of the wave freely propagates and dissipates. The vertical structure is not very different for the \( l = 1 \) and \( l = 2 \) cases. 

The assumption that is imposed on the heating profile in this paper is oversimplified. In reality the heating and temperature profile of the MJO are more complex than the first baroclinic mode as Kiladis et al. (2005) and Jiang et al. (2015) (Figure 13) show. 

By performing a series of denial experiments using two different numerical models, Shi et al. (2018) and Khairoutdinov and Emanuel (2018) found compelling evidence that WISHE-moisture mode is associated with MJO-like variability in aquaplanet models. FR17 and the model presented in this paper support that hypothesis. Adding non linearities, the meridional wind, more realistic imposed heating profile might be needed to explain a more detailed picture.

**Appendix A: Polarization Relations**

Following the equations (14) and (15) for \( w(z) \) in troposphere and above, we can write the same for other fields:

\[
\begin{align*}
\frac{u(z)}{2\kappa(1 - \Phi^2)} &= \left[ \cos(z) + \exp(-i\pi/\Phi) \cos(mz) \right] & z \leq \pi \\
\frac{u(z)}{2\kappa(1 - \Phi^2)} &= \sin(z) & z > \pi \\
b(z) &= \frac{iB}{2\kappa \Phi} \sin(z) - \frac{i}{\kappa \Phi} w_{\text{troposphere}}(z) & z \leq \pi \\
b(z) &= -\frac{1}{\kappa \Phi} w_{\text{stratosphere}}(z) & z > \pi \\
\Pi(z) &= \Phi w_{\text{troposphere}}(z) & z \leq \pi \\
\Pi(z) &= \Phi w_{\text{stratosphere}}(z) & z > \pi.
\end{align*}
\]

Note that it is not possible to write the explicit solutions for \( q \) and \( e \) as we do not know the shape of \( S_Q(z) \) and \( S_S(z) \); all we know is their integral.

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