SINGULAR EQUIVALENCES OF MORITA TYPE WITH LEVEL, GORENSTEIN ALGEBRAS, AND UNIVERSAL DEFORMATION RINGS

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To the memory and legacy of my great-grandfather Jesús María Bermúdez.

Abstract. Let $k$ be a field of arbitrary characteristic, let $A$ be a finite dimensional $k$-algebra, and let $V$ be an indecomposable finitely generated non-projective Gorenstein-projective left $A$-module whose stable endomorphism ring is isomorphic to $k$. In this article, we prove that the universal deformation rings $R(A,V)$ and $R(\Lambda,\Omega_V)$ are isomorphic, where $\Omega_V$ denotes the first syzygy of $V$ as a left $A$-module. We also prove the following result. Assume that $A$ is also Gorenstein and that $\Gamma$ is another Gorenstein $k$-algebra such that there exists $\ell \geq 0$ and a pair of bimodules $(\Gamma X, \Lambda Y)$ that induces a singular equivalence of Morita type with level $\ell$ (as introduced by Z. Wang) between $A$ and $\Gamma$. Then the left $\Gamma$-module $X \otimes_A V$ is also Gorenstein-projective with stable endomorphism ring isomorphic to $k$, and the universal deformation ring $R(\Gamma, X \otimes_A V)$ is isomorphic to $R(A,V)$.

1. Introduction

Throughout this article, we assume that $k$ is a fixed field of arbitrary characteristic. We denote by $\hat{C}$ the category of all complete local commutative Noetherian $k$-algebras with residue field $k$. In particular, the morphisms in $\hat{C}$ are continuous $k$-algebra homomorphisms that induce the identity map on $k$. Let $\Lambda$ be a fixed finite dimensional $k$-algebra, and let $R$ be a fixed arbitrary object in $\hat{C}$. We denote by $R\Lambda$ the tensor product of $k$-algebras $R \otimes_k \Lambda$, and denote by $(R\Lambda)^e$ the enveloping $R$-algebra of $R\Lambda$, i.e., $(R\Lambda)^e = R\Lambda \otimes_R (R\Lambda)^{op}$, where $(R\Lambda)^{op}$ denotes the opposite $R$-algebra of $R\Lambda$. In particular, all $R\Lambda$-$\Lambda$-bimodules coincide with all left $(R\Lambda)^e$-modules. Note that if $R$ is an Artinian object in $\hat{C}$, then $R\Lambda$ is also Artinian (both on the left and the right sides). We denote by $R\Lambda$-$\text{mod}$ the abelian category of finitely generated left $R\Lambda$-modules and by $R\Lambda$-$\text{mod}$ its stable category. In this article, we assume all our modules to be finitely generated. Let $M$ be a left $R\Lambda$-module. We denote by $\text{End}_{R\Lambda}(M)$ (resp. by $\text{End}_{R\Lambda}(M)$) the endomorphism ring (resp. the stable endomorphism ring) of $M$. If $R$ is an Artinian object in $\hat{C}$, then we denote by $\Omega_{R\Lambda} M$ the first syzygy of $M$, i.e. $\Omega_{R\Lambda} M$ is the kernel of a projective cover $P \to M$ of $M$ over $R\Lambda$, which is unique up to isomorphism. In particular, if $N$ is a left $(R\Lambda)^e$-module, we denote by $\Omega_{R\Lambda} N$ the syzygy of $N$ as a $(R\Lambda)^e$-module. Recall that $\Lambda$ is said to be a Gorenstein $k$-algebra provided that $\Lambda$ has finite injective dimension as a left and right $\Lambda$-module (see [3]). In particular, algebras of finite global dimension as well as self-injective algebras are Gorenstein.

Let $V$ be a fixed left $\Lambda$-module. In [10, Prop. 2.1], F. M. Bleher and the author proved that $V$ has a well-defined versal deformation ring $R(\Lambda,V)$ in $\hat{C}$, which is universal provided that $\text{End}_{\Lambda}(V)$ is isomorphic to $k$. Moreover, they also proved that versal deformation rings are preserved under Morita equivalences (see [10, Prop. 2.5]). Following [21, 22], we say that $V$ is Gorenstein-projective provided that there exists an acyclic complex of projective left $\Lambda$-modules

$$P^\bullet : \cdots \to P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \to \cdots$$

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such that $\text{Hom}_\Lambda(P^*, \Lambda)$ is also acyclic and $V = \text{coker} f^0$. In particular, every projective left $\Lambda$-module is Gorenstein-projective. Following [2] and [6, §2], $V$ is said to have Gorenstein dimension zero or that it is totally reflexive provided that $V$ is reflexive (i.e. $V$ and $\text{Hom}_\Lambda(\text{Hom}_\Lambda(V, \Lambda), \Lambda)$ are isomorphic as left $\Lambda$-modules), and that $\text{Ext}^i_\Lambda(V, \Lambda) = 0 = \text{Ext}^i_\Lambda(\text{Hom}_\Lambda(V, \Lambda), \Lambda)$ for all $i > 0$. It is well-known that finitely generated Gorenstein-projective modules coincide with those that are totally reflexive (see e.g. [6, §2.4]). Following [14], $V$ is said to be (maximal) Cohen-Macaulay provided that $\text{Ext}^i_\Lambda(V, \Lambda) = 0$, for all $i > 0$.

It follows by [4, Prop. 4.1] that if $\Lambda$ is a Gorenstein $k$-algebra, then $V$ is a Gorenstein-projective left $\Lambda$-module if and only if $V$ is (maximal) Cohen-Macaulay. It is important to mention that there are examples of finite dimensional algebras $\Lambda$ and left $\Lambda$-modules $V$ that satisfy that $\text{Ext}^i_\Lambda(V, \Lambda) = 0$ for all $i > 0$, but $V$ is not a Gorenstein-projective $\Lambda$-module (see e.g. [18, Example A.3] and [33, 34]). We denote by $\Lambda$-Gproj the category of Gorenstein-projective left $\Lambda$-modules, and by $\Lambda$-Gproj its stable category. It is well-known that $\Lambda$-Gproj is a Frobenius category in the sense of [25, Chap. I, §2.1] and consequently, $\Lambda$-Gproj is a triangulated category (in the sense of [39]). Moreover, if $V$ is non-projective Gorenstein-projective, then for all $i \geq 0$, the $i$-th syzygy $\Omega^i_\Lambda V$ is also non-projective Gorenstein-projective, and $\Omega_\Lambda$ induces an autoequivalence $\Omega_\Lambda : \Lambda$-Gproj $\to \Lambda$-Gproj (see [25, Chap. I, §2.2]).

In [8, Thm. 1.2], it was proved that if $V$ is a Gorenstein-projective left $\Lambda$-module with $\text{End}_\Lambda(V) = k$, then the versal deformation ring $R(\Lambda, V)$ is universal, which generalizes [10, Thm. 2.6 (ii)]. Moreover, it was also proved that versal deformation rings of Gorenstein-projective modules are preserved under singular equivalences of Morita type between Gorenstein $k$-algebras, which generalizes [11, Prop. 3.2.6]. These singular equivalences of Morita type were introduced by X. W. Chen and L. G. Sun in [18] and then further studied by G. Zhou and A. Zimmermann in [42] as a way of generalizing the concept of stable equivalences of Morita type as introduced by M. Broué in [13].

In order to state the main result of this article, we first need to recall the following definition due to Z. Wang (see [40, Def. 2.1]) that generalizes the concept of singular equivalences of Morita type.

**Definition 1.1.** Let $\Lambda$ and $\Gamma$ be finite dimensional $k$-algebras, and let $X$ be a $\Gamma$-$\Lambda$-bimodule and $Y$ a $\Lambda$-$\Gamma$-bimodule. We say that the pair $(\Gamma X_{\Lambda, \Lambda} Y_{\Gamma}, \Gamma)$ induces a singular equivalence of Morita type with level $\ell \geq 0$ between $\Lambda$ and $\Gamma$ and say that $\Lambda$ and $\Gamma$ are singularly equivalent of Morita type with level $\ell$ if the following conditions are satisfied:

(i) $X$ is projective as a left $\Gamma$-module and as a right $\Lambda$-module;

(ii) $Y$ is projective as a left $\Lambda$-module and as a right $\Gamma$-module;

(iii) $X \otimes_{\Lambda} Y \cong \Omega_{\Gamma}^\ell \Gamma$ in $\Gamma$-mod;

(iv) $Y \otimes_{\Gamma} X \cong \Omega_{\Lambda}^\ell \Lambda$ in $\Lambda$-mod.

The goal of this article is to prove the following result.

**Theorem 1.2.** Let $\Lambda$ be a finite dimensional $k$-algebra and let $V$ be an indecomposable non-projective Gorenstein-projective left $\Lambda$-module with $\text{End}_\Lambda(V) = k$.

(i) The stable endomorphism ring $\text{End}_\Lambda(\Omega_\Lambda V)$ is isomorphic to $k$, and the universal deformation ring $R(\Lambda, \Omega_\Lambda V)$ is isomorphic to $R(\Lambda, V)$ in $\bar{C}$.

(ii) Assume that $\Lambda$ is Gorenstein and let $\Gamma$ be another finite dimensional Gorenstein $k$-algebra such that there exists $\ell \geq 0$ and a pair of bimodules $(\Gamma X_{\Lambda, \Lambda} Y_{\Gamma}, \Gamma)$ that induces a singular equivalence of Morita type with level $\ell$ as in Definition 1.1. Then $X \otimes_{\Lambda} V$ is a non-projective Gorenstein-projective left $\Gamma$-module such that $\text{End}_\Gamma(X \otimes_{\Lambda} V) = k$ and the universal deformation ring $R(\Gamma, X \otimes_{\Lambda} V)$ is isomorphic to $R(\Lambda, V)$ in $\bar{C}$.

Note that Theorem 1.2 (i) generalizes [10, Thm. 2.6 (iv)] and answers affirmatively a question raised in [8, Rem. 5.5]. Moreover, Theorem 1.2 (ii) gives a version of [8, Thm. 1.2 (iii)] for singular equivalences of Morita type with level between Gorenstein $k$-algebras.

**Remark 1.3.** Let $\Lambda$ be a finite dimensional $k$-algebra.

(i) It is important to mention that in the proof of Theorem 1.2 (i), it is not enough to only use that $\Omega_\Lambda$ is a self-equivalence of $\Lambda$-Gproj (as mentioned above), for we also need the following result from [2,
Prop. 3.8]: $V$ is totally reflexive if and only if $\text{Ext}^i_{\Lambda}(V, \Lambda) = 0 = \text{Ext}^i_{\Lambda}(\text{Tr}_\Lambda V, \Lambda)$ for all $i > 0$, where $\text{Tr}_\Lambda V$ is the transpose of $V$ (see e.g. [5, §IV.1]).

(ii) In Example 3.6 we show that Theorem 1.2 (ii) fails provided that the conditions that $\Lambda$ is Gorenstein and that $V$ is a non-projective Gorenstein projective left $\Lambda$-module with $\text{End}_{\Lambda}(V) \cong \mathbb{k}$ are not both satisfied.

The remainder of this article is organized as follows. In §2.1, we review the precise definition of lifts, deformations, universal deformations and universal deformation rings from [10]. We also discuss some properties of singular equivalences of Morita type with level and syzygies of modules. In §3 we prove Theorem 1.2. Finally, in §4, we provide some immediate applications of Theorem 1.2 to Morita and triangular matrix $k$-algebras as well as to singular equivalences induced by homological epimorphisms (in the sense of [24]) and 2-recollements of triangulated categories (in the sense of [32]).

2. Preliminaries

Throughout this section we keep the notation introduced in §1.

2.1. Lifts, deformations, and (uni)versal deformation rings. Let $V$ be a left $\Lambda$-module and let $R$ be a fixed but arbitrary object in $\hat{\mathcal{C}}$. A lift $(M, \phi)$ of $V$ over $R$ is a finitely generated left $R$-$\Lambda$-module $M$ that is free over $R$ together with an isomorphism of $\Lambda$-modules $\phi: k \otimes_R M \to V$. Two lifts $(M, \phi)$ and $(M', \phi')$ over $R$ are isomorphic if there exists an $R$-$\Lambda$-module isomorphism $f: M \to M'$ such that $\phi' \circ (\text{id}_k \otimes_R f) = \phi$. If $(M, \phi)$ is a lift of $V$ over $R$, we denote by $[M, \phi]$ its isomorphism class and say that $[M, \phi]$ is a deformation of $V$ over $R$. We denote by $\text{Def}_\Lambda(V, R)$ the set of all deformations of $V$ over $R$. The deformation functor corresponding to $V$ is the covariant functor $\hat{F}_V: \hat{\mathcal{C}} \to \text{Sets}$ defined as follows: for all objects $R$ in $\hat{\mathcal{C}}$, define $\hat{F}_V(R) = \text{Def}_\Lambda(V, R)$, and for all morphisms $\theta: R \to R'$ in $\hat{\mathcal{C}}$, let $\hat{F}_V(\theta): \text{Def}_\Lambda(V, R) \to \text{Def}_\Lambda(V, R')$ be defined as $\hat{F}_V(\theta)([M, \phi]) = [R' \otimes_{R, \theta} M, \phi_\theta]$, where $\phi_\theta: k \otimes_{R'} (R' \otimes_{R, \theta} M) \to V$ is the composition of $\Lambda$-module isomorphisms

$$k \otimes_{R'} (R' \otimes_{R, \theta} M) \cong k \otimes_R M \xrightarrow{\phi} V.$$

Suppose there exists an object $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ of $V$ over $R(\Lambda, V)$ with the following property. For all objects $R$ in $\hat{\mathcal{C}}$ and for all deformations $[M, \phi]$ of $V$ over $R$, there exists a morphism $\psi_{R(\Lambda, V), R, [M, \phi]}: R(\Lambda, V) \to R$ in $\hat{\mathcal{C}}$ such that

$$\hat{F}_V(\psi_{R(\Lambda, V), R, [M, \phi]})(U(\Lambda, V), \phi_{U(\Lambda, V)}) = [M, \phi],$$

and moreover, $\psi_{R(\Lambda, V), R, [M, \phi]}$ is unique if $R$ is the ring of dual numbers $k[e] / e^2 = 0$. Then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ are called the versal deformation ring and versal deformation of $V$, respectively. If the morphism $\psi_{R(\Lambda, V), R, [M, \phi]}$ is unique for all $R \in \text{Ob}(\hat{\mathcal{C}})$ and deformations $[M, \phi]$ of $V$ over $R$, then $R(\Lambda, V)$ and $[U(\Lambda, V), \phi_{U(\Lambda, V)}]$ are called the universal deformation ring and the universal deformation of $V$, respectively.

In other words, the universal deformation ring $R(\Lambda, V)$ represents the deformation functor $\hat{F}_V$ in the sense that $\hat{F}_V$ is naturally isomorphic to the Hom functor $\text{Hom}_{\hat{\mathcal{C}}}(R(\Lambda, V), -)$. We denote by $F_V$ the restriction of $\hat{F}_V$ to the full subcategory of Artinian objects in $\hat{\mathcal{C}}$. Following [35, §2.6], we call the set $F_V(k[e])$ the tangent space of $F_V$, which has a structure of a $k$-vector space by [35, Lemma 2.10]. It was proved in [10, Prop. 2.1] that $F_V$ satisfies the Schlessinger’s criteria [35, Thm. 2.11], that there exists an isomorphism of $k$-vector spaces

$$(1.1) \quad F_V(k[e]) \to \text{Ext}^1_{\Lambda}(V, V),$$

and that $\hat{F}_V$ is continuous in the sense of [30, §14], i.e. for all objects $R$ in $\hat{\mathcal{C}}$, we have

$$(2.2) \quad \hat{F}_V(R) = \lim_{n} \text{Ext}^1_{\Lambda}(V, V/m^n_R),$$

where $m^n_R$ denotes the unique maximal ideal of $R$. Consequently, $V$ has always a well-defined versal deformation ring $R(\Lambda, V)$ which is also universal provided that $\text{End}_{\Lambda}(V)$ is isomorphic to $k$. It was also proved in [10,
Prop. 2.5] that versal deformation rings are invariant under Morita equivalences between finite dimensional \( k \)-algebras.

**Remark 2.1.**

(i) It follows from the isomorphism of \( k \)-vector spaces (2.1) that if \( \dim_k \text{Ext}_\Lambda^1(V, V) = r \), then the versal deformation ring \( R(\Lambda, V) \) is isomorphic to a quotient algebra of the power series ring \( k[[t_1, \ldots, t_r]] \) and \( r \) is minimal with respect to this property. In particular, if \( V \) is a left \( \Lambda \)-module such that \( \text{Ext}_\Lambda^1(V, V) = 0 \), then \( R(\Lambda, V) \) is universal and isomorphic to \( k \) (see [12, Remark 2.1] for more details).

(ii) Because of the continuity of the deformation functor as in (2.2), most of the arguments concerning \( \hat{F}_V \) can be carried out for \( F_V \), and thus we are able to restrict ourselves to discuss liftings of \( \Lambda \)-modules over Artinian objects in \( \mathcal{C} \).

(iii) Let \( R \) be an Artinian ring in \( \mathcal{C} \), let \( \iota_R : k \to R \) be the unique morphism in \( \mathcal{C} \) endowing \( R \) with a \( k \)-algebra structure, and let \( \pi_R : R \to k \) be the natural projection in \( \mathcal{C} \). Then \( \pi_R \circ \iota_R = \text{id}_k \).

(iii.a) For all projective left (resp. right) \( \Lambda \)-modules \( P \), we let \( P_R = R \otimes_{k, \iota_R} P = R \otimes_k P. \) Then \( P_R \) is a projective left (resp. right) RA-module cover of \( P \), and \( (P_R, \pi_{P,R}) \) is a lift of \( P \) over \( R \), where \( \pi_{P,R} \) is the natural isomorphism \( R \otimes_R P_R \to P_R \).

(iii.b) Let \( \alpha : P(V) \to V \) be a projective left \( \Lambda \)-module cover of \( V \) (which is unique up to isomorphism), and let \( \Omega_A V = \ker \alpha \). Then we obtain a short exact sequence of left \( \Lambda \)-modules

\[
0 \to \Omega_A V \xrightarrow{\beta} P(V) \xrightarrow{\alpha} V \to 0. \tag{2.3}
\]

Let \((M, \phi)\) be a lift of \( V \) over \( R \). Since \( P_R(V) = R \otimes_{k, \iota_R} P(V) \) is a projective left RA-module cover of \( P(V) \) by (i), and since \( \alpha \) is an essential epimorphism, there exists an epimorphism of RA-modules \( \alpha_R : P_R(V) \to M \) such that \( \phi \circ (\text{id}_k \otimes \alpha_R) = \alpha \circ \pi_{P(V),R,} \). Moreover, it follows by [12, Claim 1] that \( \alpha_R : P_R(V) \to M \) is a projective left RA-module cover of \( M \). Let \( \Omega_{RA} M := \ker \alpha_R \). Note that since \( M \) and \( P_R(V) \) are both free over \( R \), then \( \Omega_{RA} M \) is also free over \( R \), and that there exists an isomorphism of left \( \Lambda \)-modules \( \Omega_{RA}(\phi) : k \otimes_R \Omega_{RA} M \to \Omega_A V \) such that \( \pi_{P(V),R} \circ (\text{id}_k \otimes \beta_R) = \beta \circ \Omega_{RA}(\phi) \), where \( \beta : \Omega_A V \to P(V) \) and \( \beta_R : \Omega_{RA} M \to P_R(V) \) are the natural inclusions. In particular, \((\Omega_{RA} M, \Omega_{RA}(\phi))\) is a lift of \( \Omega_A V \) over \( R \).

**Remark 2.2.** Let \( \Lambda \) and \( V \) be as above, let \( R \) be an object in \( \mathcal{C} \) and let \((M, \phi)\) be a lift of \( V \) over \( R \). Then the isomorphism class \([M]\) of \( M \) as an RA-module is called a \textit{weak deformation of} \( V \) over \( R \) (see e.g. [28, §5.2] and [10, Remark 2.4]). We can also define the weak deformation functor \( \hat{F}_V^w : \hat{C} \to \text{Sets} \) which sends an object \( R \) in \( \mathcal{C} \) to the set of weak deformations of \( V \) over \( R \) and a morphism \( \alpha : R \to R' \) in \( \mathcal{C} \) to the map \( \hat{F}_V^w : \hat{F}_V^w(R) \to \hat{F}_V^w(R') \), which is defined by \( \hat{F}_V^w(\alpha)([M]) = [R' \otimes_{R, \alpha} M] \). In general, a weak deformation of \( V \) over \( R \) identifies more lifts than a deformation of \( V \) over \( R \) that respects the isomorphism \( \phi \) of a representative \((M, \phi)\).

**Remark 2.3.** Assume that \( V \) is an indecomposable Gorenstein-projective left \( \Lambda \)-module with \( \End_{\Lambda}(V) = k \).

(i) It follows by [8, Thm. 1.2 (i)] that the deformation functor \( \hat{F}_V \) is naturally isomorphic to the weak deformation functor \( \hat{F}_V^w \) as in Remark 2.2. This implies that a deformation \([M, \phi]\) of \( V \) over \( R \) in \( \mathcal{C} \) does not depend on the particular choice of the \( \Lambda \)-module isomorphism \( \phi \). More precisely, if \( f : M \to M' \) is an RA-module isomorphism with \((M', \phi')\) a lift of \( V \) over \( R \), then there exists an RA-module isomorphism \( \hat{f} : M \to M' \) such that \( \phi' \circ (\text{id}_k \otimes \hat{f}) = \phi \), i.e., \([M, \phi] = [M', \phi']\) in \( \hat{F}_V(R) = \text{Def}_{\Lambda}(V, R) \).

(ii) As already noted in §1, it follows by [8, Thm. 1.2 (ii)] that the versal deformation ring is \( R(\Lambda, V) \) universal. Moreover, if \( P \) is a projective left \( \Lambda \)-module, then the versal deformation ring \( R(\Lambda, V \oplus P) \) is universal and isomorphic to \( R(\Lambda, V) \). This result follows from the fact that for all Artinian objects \( R \) in \( \mathcal{C} \), there is a bijection of set of deformations

\[
\tau_{V \oplus P,R} : F_V(R) \to F_{V \oplus P}(R) \tag{2.4}
\]

which for all lifts \((M, \phi)\) of \( V \) over \( R \), \( \tau_{P,R}([M, \phi]) = [M \oplus P, \phi \oplus \pi_{P,R}] \), where \((P_R, \pi_{P,R})\) is as in Remark 2.1 (iii.a).
2.2. Some results involving singular equivalences of Morita type with level and syzygies. Recall that \( \Lambda \) denotes a finite dimensional \( k \)-algebra and \( V \) is a finitely generated left \( \Lambda \)-module.

Remark 2.4. Assume that \( \Lambda \) has a minimal projective resolution as a \( \Lambda \)-\( \Lambda \)-bimodule (or equivalently, as a left \( \Lambda^e \)-module) given by
\[
\cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda \rightarrow 0.
\]
Let \( i \geq 1 \) be fixed. It follows that there exists a short exact sequence of \( \Lambda \)-\( \Lambda \)-bimodules
\[
0 \rightarrow \Omega^{i}_{\Lambda^e} \Lambda \xrightarrow{\iota_i} P_1 \xrightarrow{\pi_i} \Omega^{-1}_{\Lambda^e} \Lambda \rightarrow 0.
\]
Tensoring (2.6) with \( V \) over \( \Lambda \) yields an exact sequence of left \( \Lambda \)-modules
\[
\Omega^{i}_{\Lambda^e} \Lambda \otimes_{\Lambda} V \xrightarrow{\iota_i \otimes \text{id}_V} P_1 \otimes_{\Lambda} V \xrightarrow{\pi_i \otimes \text{id}_V} \Omega^{-1}_{\Lambda^e} \Lambda \otimes_{\Lambda} V \rightarrow 0.
\]
Since \( \iota_i : \Omega^{i}_{\Lambda^e} \Lambda \rightarrow P_1 \) is a section in \( \Lambda^e \)-mod, it follows that
\[
\iota_i \otimes \text{id}_V : \Omega^{i}_{\Lambda^e} \Lambda \otimes_{\Lambda} V \rightarrow P_1 \otimes_{\Lambda} V
\]
is also a section, and thus a monomorphism in \( \Lambda \)-mod. Thus we obtain a short exact sequence of left \( \Lambda \)-modules
\[
0 \rightarrow \Omega^{i}_{\Lambda^e} \Lambda \otimes_{\Lambda} V \xrightarrow{\iota_i \otimes \text{id}_V} P_1 \otimes_{\Lambda} V \xrightarrow{\pi_i \otimes \text{id}_V} \Omega^{-1}_{\Lambda^e} \Lambda \otimes_{\Lambda} V \rightarrow 0.
\]
Thus by [37, Prop. IV. 8.1 (v)], it follows that there is an isomorphism of left \( \Lambda \)-modules
\[
\Omega^{i}_{\Lambda^e} \Lambda \otimes_{\Lambda} V \cong \Omega^{i}_{\Lambda^e} V \oplus T_i^e,
\]
where \( T_i^e \) is a finitely generated projective left \( \Lambda \)-module.

Remark 2.5. Let \( \Gamma \) be another finite dimensional \( k \)-algebra, and assume that \( \Lambda \) and \( \Gamma \) are both Gorenstein. Assume that there exists \( \ell \geq 0 \) and a pair of bimodules \((\Gamma \cdot X, \Lambda \cdot Y)\) that induce a singular equivalence of Morita type with level \( \ell \) between \( \Lambda \) and \( \Gamma \) as in Definition 1.1.

(i) It follows from [36, Lemma. 3.6] that the functors
\[
X \otimes_{\Lambda} - : \Lambda \text{-mod} \rightarrow \Gamma \text{-mod} \quad \text{and} \quad Y \otimes_{\Gamma} - : \Gamma \text{-mod} \rightarrow \Lambda \text{-mod}
\]
send finitely generated Gorenstein-projective left modules to finitely generated Gorenstein-projective left modules.

(ii) By [42, Prop. 2.3] and [36, Prop. 3.7] it follows that
\[
X \otimes_{\Lambda} - : \Lambda \text{-Gproj} \rightarrow \Gamma \text{-Gproj} \quad \text{and} \quad Y \otimes_{\Gamma} - : \Gamma \text{-Gproj} \rightarrow \Lambda \text{-Gproj}
\]
are equivalences of triangulated categories that are quasi-inverses of each other.

(iii) There exist projective bimodules \( \Gamma Q_{T} \), \( \Gamma Q'_{T} \), \( \Lambda P_{A} \), and \( \Lambda P'_{A} \) such that.

(iii.a) \( X \otimes_{\Lambda} A \otimes' Q' \cong \Omega^{i}_{\Lambda^e} \Gamma \otimes Q \) as \( \Gamma \)-\( \Gamma \)-bimodules;

(iii.b) \( Y \otimes_{\Gamma} X \otimes P' \cong \Omega^{i}_{\Lambda^e} \Lambda \otimes P \) as \( \Lambda \)-\( \Lambda \)-bimodules.

Assume that \( V \) is as in Remark 2.3. Then by tensoring at both sides of (iii.b) with \( V \) over \( \Lambda \) and by using (2.8), we obtain an isomorphism of left \( \Lambda \)-modules
\[
Y \otimes_{\Gamma} X \otimes_{\Lambda} V \oplus (P' \otimes_{\Lambda} V) \cong \Omega^{i}_{\Lambda^e} V \oplus T_i^e \oplus (P \otimes_{\Lambda} V),
\]
where \( P' \otimes_{\Lambda} V \) and \( P \otimes_{\Lambda} V \) are projective left \( \Lambda \)-modules. This implies that \( \Omega^{i}_{\Lambda^e} V \) and \( Y \otimes_{\Gamma} X \otimes_{\Lambda} V \) are isomorphic indecomposable objects in \( \Lambda \text{-Gproj} \). In particular, \( \Omega^{i}_{\Lambda^e} V \) does not have projective direct summands, and thus by the Krull-Schmidt-Azumaya Theorem and by (i), we obtain that there exists a finitely generated projective left \( \Lambda \)-module \( T_i^e \) such that there is an isomorphism of left \( \Lambda \)-modules
\[
Y \otimes_{\Gamma} X \otimes_{\Lambda} V \cong \Omega^{i}_{\Lambda^e} V \oplus T_i^e.
\]

Similarly, if \( W \) is an indecomposable Gorenstein-projective left \( \Gamma \)-module, it follows that there exists a projective left \( \Gamma \)-module \( S_i \) such that there exists an isomorphism of left \( \Gamma \)-modules
\[
X \otimes_{\Lambda} Y \otimes_{\Gamma} W \cong \Omega^{i}_{\Lambda^e} W \oplus S_i.
\]
3. Proof of Theorem 1.2

Assume throughout this section that $\Lambda$ is a fixed but arbitrary finite dimensional $k$-algebra and that $V$ is an indecomposable non-projective Gorenstein-projective left $\Lambda$-module with $\text{End}_\Lambda(V) = k$.

3.1. Proof of Theorem 1.2 (i). We first need to recall the following definition from [35, Def. 1.2].

**Definition 3.1.** Let $\theta : R \to R_0$ be a morphism of Artinian objects in $\hat{\mathcal{C}}$. We say that $\theta$ is a small extension if the kernel of $\theta$ is a non-zero principal ideal $tR$ that is annihilated by the unique maximal ideal $m_R$ of $R$.

**Lemma 3.2.** Assume that $R$ is a fixed but arbitrary Artinian object in $\hat{\mathcal{C}}$.

(i) Let $M$ be a finitely generated left $R\Lambda$-module such that $k \otimes_R M \cong V$ as left $\Lambda$-modules. Then:

(i.a) $M$ is a (maximal) Cohen-Macaulay left $R\Lambda$-module, i.e., $\text{Ext}_R^i(M, R\Lambda) = 0$ for all $i > 0$;

(i.b) $M$ is reflexive as a left $R\Lambda$-module, i.e. $M$ and $\text{Hom}_{R\Lambda}(\text{Hom}_{R\Lambda}(M, R\Lambda), R\Lambda)$ are isomorphic as left $R\Lambda$-modules.

(ii) Let $U$ be a finitely generated left $R\Lambda$-module which is free over $R$ such that $k \otimes_R U \cong \Omega_AV$ as left $\Lambda$-modules. Then there exists a short exact sequence of left $R\Lambda$-modules that are free over $R$

\[(3.1) \quad 0 \to \frac{U}{\beta_R} \to \frac{P_R(V)}{\alpha_R} \to L \to 0,
\]

such that the induced short exact sequence of left $\Lambda$-modules

\[0 \to k \otimes_R U \overset{id \otimes \beta_R}{\to} k \otimes_R P_R(V) \overset{id \otimes \alpha_R}{\to} k \otimes_R L \to 0,
\]

is isomorphic to (2.3).

**Proof.** (i.a). Let $R_0$ be an Artinian object in $\hat{\mathcal{C}}$ such that $\theta : R \to R_0$ is a small extension in Definition 3.1, and assume that for all finitely generated left $R_0\Lambda$-modules such that $k \otimes_{R_0} M_0 \cong V$ as left $\Lambda$-modules, we have $\text{Ext}_R^i(M_0, R_0\Lambda) = 0$ for all $i > 0$. Consider the short exact sequence of $R$-modules

\[(3.2) \quad 0 \to tR \to R \to R_0 \to 0.
\]

Tensoring (3.2) with $k$ over $k$ yields a sequence of left $R\Lambda$-modules

\[(3.3) \quad 0 \to tR \otimes_k \Lambda \to R\Lambda \to R_0\Lambda \to 0.
\]

Applying $\text{Hom}_{R\Lambda}(M, -)$ to (3.3) yields a long exact sequence of left $R\Lambda$-modules

\[\cdots \rightarrow \text{Ext}_R^1(M, tR \otimes_k \Lambda) \rightarrow \text{Ext}_R^1(M, tR) \rightarrow \text{Ext}_R^1(M, R_0\Lambda) \rightarrow \text{Ext}_R^{i+1}(M, tR \otimes_k \Lambda) \rightarrow \cdots
\]

Since $tR \cong k$, it follows that $tR \otimes_k \Lambda \cong \Lambda$. On the other hand, by using [43, Prop. 1.8.31], the fact that $k \otimes_R M \cong V$ is maximal Cohen-Macaulay, and by using the induction hypothesis, we obtain that for all $i > 0$, $\text{Ext}_R^i(M_0, R_0\Lambda) \cong \text{Ext}_R^i(M, R_0\Lambda) = 0$ and $\text{Ext}_R^i(M, R_0\Lambda) \cong \text{Ext}_R^i(M_0 \otimes_{R_0, \theta} M_0, R_0\Lambda) = 0$.

Therefore, $\text{Ext}_R^i(M, R_0\Lambda) = 0$ for all $i > 0$.

Note that in the proof of (i.a), we only use that $\text{Ext}_R^i(V, \Lambda) = 0$ for all $i > 0$.

(i.b). Let $P_i \overset{d_i}{\rightarrow} P_0 \overset{d_0}{\rightarrow} V \to 0$ be a minimal projective presentation of $V$ as a left $\Lambda$-module. Then by using Remark 2.1 (iii.b) repeatedly, we obtain a minimal projective presentation of $M$ as a left $R\Lambda$-module

\[(3.4) \quad (P_i)_R \overset{(d_i)_R}{\rightarrow} (P_0)_R \overset{(d_0)_R}{\rightarrow} M \to 0,
\]

where $k \otimes_R (P_i)_R \cong P_i$ for $i = 0, 1$. Applying $\text{Hom}_{R\Lambda}(-, R\Lambda) = (-)^{**R\Lambda}$ to (3.4) yields the exact sequence of right $R\Lambda$-modules

\[(3.5) \quad 0 \to \text{Ext}_R^1(\text{Tr}_{R\Lambda}M, R\Lambda) \to M \to M^{**R\Lambda} \to \text{Ext}_R^2(\text{Tr}_{R\Lambda}M, R\Lambda) \to 0.
\]

On the other hand, since $k \otimes_R (P_i)_R \cong \text{Hom}_{R\Lambda}(P_i, \Lambda)$ for $i = 0, 1$, it follows that $k \otimes_R \text{Tr}_{R\Lambda}M \cong \text{Tr}_{R\Lambda}V$. Since $V$ is also totally reflexive, it follows by [2, Prop. 3.8] that $\text{Ext}_\Lambda^i(\text{Tr}_{R\Lambda}V, \Lambda) = 0$ for all $i > 0$. Thus by using
the dual arguments in the proof of (i.a) for the right RA-module $\text{Tr}_{RA}M$, we obtain $\text{Ext}^i_{RA}(\text{Tr}_{RA}M, RA) = 0$ for all $i > 0$, which together with (3.5) implies that $M \to M^{*\ast RA}$ is an isomorphism of RA-modules.

(ii). In the following, we adapt some of the arguments in the proof of [9, Prop. 2.4] to our situation. As in the proof of (i.a), let $R_0$ be an Artinian object in $\mathcal{C}$ such that $\theta : R \to R_0$ is a small extension as in Definition 3.1, and assume that for all finitely generated left $R_0\Lambda$-modules $U_0$ that are free over $R_0$ and satisfy that $k \otimes_{R_0} U_0 \cong \Omega_{A}V$ as left $\Lambda$-modules, there exists a monomorphism of left $R_0\Lambda$-modules $\beta_{R_0} : U_0 \to P_{R_0}(V)$ such that $id_k \otimes \beta_{R_0} = \beta$, where $\beta$ is the monomorphism of left $\Lambda$-modules as in (2.3). Tensoring (3.2) with $P_R(V)$ yields an exact sequence of left RA-modules

$$0 \to tR \otimes_R P_R(V) \to P_R(V) \to P_{R_0}(V) \to 0.$$ 

Since $tR \cong k$, it follows that $tR \otimes_R P_R(V) \cong P(V)$, and together with the fact that $k \otimes_R U \cong \Omega_{A}V$ is a Gorenstein-projective left $\Lambda$-module, we obtain that $\text{Ext}^1_{RA}(U, tR \otimes_R P_R(V)) \cong \text{Ext}^1_{A}(k \otimes_R U, P(V)) = 0$. On the other hand, $\text{Hom}_{RA}(U, R_0\Lambda) \cong \text{Hom}_{R_0\Lambda}(U_0, R_0\Lambda)$, where $U_0 = R_0 \otimes_R \theta U$ is a finitely generated $R_0\Lambda$-module that is free over $R_0$ and which satisfies $k \otimes_{R_0} U_0 \cong \Omega_{A}V$. Thus we obtain a short exact sequence

$$0 \to \text{Hom}_{\Lambda}(k \otimes_R U, P(V)) \to \text{Hom}_{RA}(U, P_R(V)) \to \text{Hom}_{R_0\Lambda}(U_0, P_{R_0}(V)) \to 0.$$ 

Let $\beta_{R_0} : U_0 \to P_{R_0}(V)$ be a monomorphism of left $R_0\Lambda$-modules such that $id_k \otimes \beta_{R_0} = \beta$. Then there exists $\beta_R : U \to P_R(V)$ such that $id_{R_0} \otimes \beta_R = \beta_{R_0}$. Therefore, $id_k \otimes \beta_R = \beta$ and since $\beta$ is a monomorphism of left $\Lambda$-modules, it follows by Nakayama’s Lemma that $\beta_R : U \to P_R(V)$ is also a monomorphism of left RA-modules. By letting $L = \text{coker} \beta_R$, we obtain a short exact sequence of left RA-modules as in (3.1). Note that since $U$ and $P_R(V)$ are both free over $R$, it follows that $L$ is also free over $R$. Moreover, since $k \otimes_R U \cong \Omega_{A}V$ and $k \otimes_R P_R(V) \cong V$, it follows that $k \otimes_R L \cong V$ as left $\Lambda$-modules. This finishes the proof of Lemma 3.2.

Let $R$ be a fixed Artinian object in $\mathcal{C}$ and let $(M, \phi)$ be a lift of $V$ over $R$. Then by Remark 2.1 (iii.b), we obtain that $(\Omega_{RA}M, \Omega_{RA}\phi)$ is a lift of $\Omega_{A}V$ over $R$. Thus we obtain a map between set of deformations

$$\tau_{\Omega_{A}V,R} : F_V(R) \to F_{\Omega_{A}V}(R)$$

defined as $\tau_{\Omega_{A}V,R}([M, \phi]) = [\Omega_{RA}M, \Omega_{RA}\phi]$ for all $[M, \phi] \in F_V(R)$.

Assume that $[M, \phi] = [M', \phi']$ in $F_V(R)$. Then there is an isomorphism of left RA-modules $f : M \to M'$ such that $\phi' \circ (id_k \otimes f) = \phi$. In particular, we obtain an isomorphism of left RA-modules $\Omega_{RA}f : \Omega_{RA}M \to \Omega_{RA}M'$. By Remark 2.3 (i), it follows that $[\Omega_{RA}M, \Omega_{RA}\phi] = [\Omega_{RA}M', \Omega_{RA}\phi']$ in $F_{\Omega_{A}V}(R)$, which proves that $\tau_{\Omega_{A}V,R}$ is well-defined. Next let $(U, \psi)$ be a lift of $\Omega_{A}V$ over $R$. Then by Lemma 3.2 (ii), there exists a lift $(L, \psi')$ of $V$ over $R$ such that $[\Omega_{RA}L, \Omega_{RA}\psi] = [U, \psi']$. This proves that $\tau_{\Omega_{A}V,R}$ is surjective. In order to prove that $\tau_{\Omega_{A}V,R}$ is injective, we adjust (as before) some of the arguments in the proof of [9, Prop. 2.4] to our situation. Namely, assume that $[M_1, \phi_1]$ and $[M_2, \phi_2]$ are lifts of $V$ over $R$ such that $[\Omega_{RA}M_1, \Omega_{RA}\phi_1] = [\Omega_{RA}M_2, \Omega_{RA}\phi_2]$ in $F_{\Omega_{A}V}(R)$. Then for $i = 1, 2$, there exists a short exact sequence of left RA-modules

$$0 \to \Omega_{RA}M_i \to P_R(V) \to M_i \to 0.$$ 

Since by Lemma 3.2 (i.a) we have that $\text{Ext}^1_{RA}(M_i, RA) = 0$ for $i = 1, 2$, we obtain a short exact sequence of right RA-modules

$$0 \to M_i^{*\ast RA} \to (P_R(V))^{*\ast RA} \to (\Omega_{RA}M_i)^{*\ast RA} \to 0,$$

where $(P_R(V))^{*\ast RA}$ is a projective right RA-module. Since $RA$ is an Artinian $R$-algebra on both sides, it follows by Schanuel’s Lemma and the Krull-Schmidt-Azumaya Theorem that $M_i^{*\ast RA} \cong M_{i\ast RA}$ as right RA-modules and thus by Lemma 3.2 (i.b) it follows that $M_1 \cong M_1^{*\ast RA} \cong M_2^{*\ast RA} \cong M_2$ as left RA-modules. Therefore by Remark 2.3 (i) we obtain that $[M_1, \phi_1] = [M_2, \phi_2]$ in $F_V(R)$. This proves that $\tau_{\Omega_{A}V,R}$ is injective. Finally, let $\theta : R \to R'$ be a morphism between Artinian rings in $\mathcal{C}$. Then it is straightforward to prove that $\Omega_{R'}(R' \otimes_{R', \theta} M) \cong R' \otimes_{R', \theta} \Omega_{RA}M$ as left $R'$-modules. Thus by using Remark 2.3 (i) again, we obtain that $\tau_{\Omega_{A}V,R}$ is natural with respect to morphisms between Artinian objects in $\mathcal{C}$.
The continuity of the deformation functor (see Remark 2.1 (ii)) implies that for all objects \( R \) in \( \hat{C} \), there is a bijection between sets of deformations
\[
\tau_{\Omega_{\lambda}V,R} : \hat{F}_V(R) \to \hat{F}_{\Omega_{\lambda}V}(R),
\]
which is natural with respect to morphisms between objects in \( \hat{C} \). This implies that the universal deformation rings \( R(\lambda, V) \) and \( R(\lambda, \Omega_{\lambda}V) \) are isomorphic in \( \hat{C} \). This finishes the proof of Theorem 1.2 (i).

The following remark will be useful in the proof of Theorem 1.2 (ii).

**Remark 3.3.** Let \( R \) be an Artinian object in \( \hat{C} \). By using the arguments above, it follows that for all \( i \geq 1 \), there is a bijection of set of deformations
\[
(3.6) \quad \tau_{\Omega_{\lambda}V,R} : F_V(R) \to F_{\Omega_{\lambda}V}(R),
\]
which is natural with respect to morphisms between Artinian objects in \( \hat{C} \).

### 3.2. Proof of Theorem 1.2 (ii).

**Lemma 3.4.** Let \((M, \phi)\) be a lift of \( V \) over \( R \). Then for all \( i \geq 0 \), there is an isomorphism of left \( RA \)-modules
\[
(3.7) \quad \Omega^i_{(RA)^{op}}(RA) \otimes_{RA} M \cong \Omega^i_{RA}M \oplus (T_i')_R
\]
where \( \Omega^i_{RA}M \) is as in Remark 2.1 (iii.b), \( T_i' \) is as in (2.8), and \( (T_i')_R \) is the projective left \( RA \)-module as in Remark 2.1 (iii.a).

**Proof.** Consider the minimal projective resolution of \( \Lambda \) as a \( \Lambda \)-\( \Lambda \)-bimodule in (2.5). Tensoring with \( k \) yields a minimal projective resolution of \( RA \) as an \( RA \)-\( RA \)-bimodule
\[
0 \to \Omega^i_{(RA)^{op}}(RA) \to (P_i)_R \to \cdots \to (P_1)_R \to (P_0)_R \to RA \to 0,
\]
where for all \( i \geq 0 \), \( (P_i)_R \) is as in Remark 2.1 (iii.a). Let \( i \geq 0 \) be fixed but arbitrary, and consider the short exact sequence of \( RA \)-\( RA \)-bimodules
\[
(3.9) \quad 0 \to \Omega^i_{(RA)^{op}}(RA) \to (P_i)_R \to \Omega^i_{(RA)^{op}}(RA) \to 0.
\]
Tensoring (3.9) with \( M \) over \( RA \) yields an exact sequence
\[
(3.10) \quad \Omega^i_{(RA)^{op}}(RA) \otimes_{RA} M \xrightarrow{(\iota_i)_R \otimes \text{id}_M} (P_i)_R \otimes_{RA} M \xrightarrow{(\pi_i)_R \otimes \text{id}_M} \Omega^{i-1}_{(RA)^{op}}(RA) \otimes_{RA} M \to 0,
\]
where \( (P_i)_R \otimes_{RA} M \) is a projective left \( RA \)-module. Note that tensoring the morphism
\[
(\iota_i)_R \otimes \text{id}_M : \Omega^i_{(RA)^{op}}(RA) \otimes_{RA} M \xrightarrow{(\iota_i)_R \otimes \text{id}_M} (P_i)_R \otimes_{RA} M
\]
with \( \text{id}_k \) over \( R \), induces the monomorphism (2.7), and thus by Nakayama’s Lemma, we obtain that \((\iota_i)_R \otimes \text{id}_M\) is a monomorphism. Thus we obtain a short exact sequence of left \( RA \)-modules
\[
0 \to \Omega^i_{(RA)^{op}}(RA) \otimes_{RA} M \xrightarrow{(\iota_i)_R \otimes \text{id}_M} (P_i)_R \otimes_{RA} M \xrightarrow{(\pi_i)_R \otimes \text{id}_M} \Omega^{i-1}_{(RA)^{op}}(RA) \otimes_{RA} M \to 0.
\]
Note that \( \Omega^{i-1}_{(RA)^{op}}(RA) \otimes_{RA} M \) induces a lift of \( \Omega^{i-1}_{RA}M \otimes_{\Lambda} V \) over \( R \). This together with the discussion in Remark 2.1 (iii.b) implies that \( \Omega^i_{(RA)^{op}}(RA) \otimes_{RA} M \cong \Omega_{RA}(\Omega^{i-1}_{(RA)^{op}}(RA) \otimes_{RA} M) \oplus Z_i \) for some projective left \( RA \)-module \( Z_i \). Note that since (3.7) is trivially true for when \( i = 0 \), we can assume by induction that for all \( 0 \leq j < i \), \( \Omega^j_{(RA)^{op}}(RA) \otimes_{RA} M \cong \Omega^j_{RA}M \oplus (T_j')_R \), where \( T_j' \) is as in (2.8). Thus
\[
(3.11) \quad \Omega^i_{(RA)^{op}}(RA) \otimes_{RA} M \cong \Omega_{RA}(\Omega^{i-1}_{RA}M \oplus (T'_i)R) \oplus Z_i \cong \Omega^i_{RA}M \oplus Z_i.
\]
Note that the isomorphism (3.11) of left \( RA \)-modules implies that \( Z_i \) induces a lift of \( T'_i \) over \( R \), where \( T'_i \) is as in (2.8). Thus \( Z_i \) is isomorphic to \( (T'_i)_R \). This finishes the proof of Lemma 3.4.

In the following, assume further that \( \Lambda, \Gamma, \ell, X_\Lambda, \Lambda Y_\Gamma, P, P', Q \), and \( Q' \) are all as in Remark 2.5.
Remark 3.5. It follows that $X_R = R \otimes_k X$ is projective as a left $RT$-module and as a right $RA$-module, and $Y_R = R \otimes_k Y$ is projective as a left $RA$-module and as a right $RT$-module, and both are free over $R$. Note also that $X_R \otimes_{RA} Y_R \cong R \otimes_k (X \otimes_A Y)$ as $RT$-$RT$-bimodules and $Y_R \otimes_{RT} X_R \cong R \otimes_k (Y \otimes_A X)$ as $RA$-$RA$-bimodules. Moreover, we also have that
\begin{align*}
Y_R \otimes_{RT} X_R \oplus P'_R &\cong \Omega^l_R(\Lambda) \oplus P_R & \text{as } RA$-$RA$-bimodules, \\
X_R \otimes_{RA} Y_R \oplus Q'_R &\cong \Omega^l_{RA}(\Gamma) \oplus Q_R & \text{as } RT$-$RT$-bimodules,
\end{align*}
where $P_R = R \otimes_k P$ and $P'_R = R \otimes_k P'$ (resp. $Q_R = R \otimes_k Q$ and $Q'_R = R \otimes_k Q'$) are projective $RA$-$RA$-bimodules (resp. $RT$-$RT$-bimodules).

Let $(M, \phi)$ be a lift of $V$ over $R$. Since $V$ is assumed to be indecomposable, we obtain by using Remark 3.5 together with Lemma 3.4 that there is an isomorphism of $RA$-$RA$-bimodules
\begin{equation}
(3.12)
Y_R \otimes_{RT} X_R \otimes_{RA} M \cong \Omega^l_R M \oplus (T_l)_R,
\end{equation}
where $T_l$ is as in (2.9). Note also that $X_R \otimes_{RA} M$ is free over $R$, and that there exists an isomorphism of left $\Gamma$-modules $\phi_{X_R \otimes_{RA} M} : k \otimes_R (X_R \otimes_{RA} M) \to X \otimes_{\Lambda} V$. Thus we can define a morphism between sets of deformations
\begin{equation}
(3.13)
\tau_{X \otimes_{A} V, R} : F_V(R) \to F_{X \otimes_{A} V}(R)
\end{equation}
as follows. For all deformations $[M, \phi]$ of $V$ over $R$, let $\tau_{X \otimes_{A} V, R}([M, \phi]) = [X_R \otimes_{RA} M, \phi_{X_R \otimes_{RA} M}]$. Let $(M, \phi)$ and $(M', \phi')$ be lifts of $V$ over $R$ such that $[M, \phi] = [M', \phi']$ in $F_V(R)$. It follows that there is an isomorphism of left $RT$-modules $\phi : X_R \otimes_{RA} M \to X_R \otimes_{RA} M'$. Note that by Remark 2.5 (ii), we obtain that $X \otimes_{\Lambda} V$ is an indecomposable projective $\Gamma$-module with $\text{End}_\Gamma(X \otimes_{\Lambda} V) = k$. Thus by Remark 2.3 (i), we obtain that $[X_R \otimes_{RA} M, \phi_{X_R \otimes_{RA} M}] = [X_R \otimes_{RA} M', \phi_{X_R \otimes_{RA} M'}]$ in $F_{X \otimes_{A} V}(R)$. This proves that $\tau_{X \otimes_{A} V, R}$ is well-defined. Next assume that $[N, \varphi]$ is a deformation of $X \otimes_{\Lambda} V$ over $R$. If we let $L = Y_R \otimes_{RT} N$, then $L$ is free over $L$ and there is a composition of isomorphisms between left $\Lambda$-modules which we denote by $\phi_L$ and which is given as follows:
\begin{align*}
\kappa \otimes_{R} L &= \kappa \otimes_{R} (Y_R \otimes_{RT} N) \cong (\kappa \otimes_{R} Y_R) \otimes_{\Gamma} (\kappa \otimes_{R} N) \\
&\cong Y \otimes_{\Gamma} (X \otimes_{\Lambda} V) \\
&\cong \Omega^l_{\Lambda} V \oplus T_l,
\end{align*}
where the last isomorphism follows from (2.9). In particular, $(L, \phi_L)$ is a lift of $\Omega^l_{\Lambda} V \oplus T_l$ over $R$. Thus by Remark 2.3 (ii), there exists a lift $(L', \phi_{L'})$ of $\Omega^l_{\Lambda} V$ over $R$ such that $L' \oplus (T_l)_R \cong L$ as left $RA$-modules. On the other hand, by Remark 3.3, there exists a lift $(L'', \phi_{L''})$ of $V$ over $R$ such that $\Omega^l_{RA} L'' \cong L'$ as left $RA$-modules. Therefore, $Y_R \otimes_{RT} N \cong \Omega^l_{RA} L'' \oplus (T_l)_R$ as left $RA$-modules. Note that we also have that $Y_R \otimes_{RT} X_R \otimes_{RA} L'' \cong \Omega^l_{RA} L'' \oplus (T_l)_R$ as left $RA$-modules. Thus we obtain an isomorphism of left $RA$-modules $Y_R \otimes_{RT} N \cong Y_R \otimes_{RT} X_R \otimes_{RA} L''$, which induces a composition of isomorphisms between left $RT$-modules as follows:
\begin{align*}
\Omega^l_{\Gamma} N \oplus (S_{\ell})_R &= X_R \otimes_{RA} Y_R \otimes_{RT} N \\
&\cong X_R \otimes_{RA} Y_R \otimes_{RT} X_R \otimes_{RA} L'' \\
&\cong \Omega^l_{RA}(X_R \otimes_{RA} L'') \oplus (S_{\ell})_R,
\end{align*}
where $S_{\ell}$ is as in (2.10). Thus $\Omega^l_{\Gamma} N \cong \Omega^l_{RA}(X_R \otimes_{RA} L'')$ as left $RT$-modules. By Remark 3.3 (applied to $X \otimes_{\Lambda} V$) together with Remark 2.3 (i), we obtain that $N \cong X_R \otimes_{RA} L''$ as left $RA$-modules and that $[N, \varphi] = [X_R \otimes_{RA} L'', \phi_{X_R \otimes_{RA} L''}]$ in $F_{X \otimes_{A} V}(R)$. This proves that $\tau_{X \otimes_{A} V, R}$ is surjective. Next assume that $[M, \phi]$ and $[M', \phi']$ are deformations of $V$ over $R$ such that $[X_R \otimes_{RA} M, \phi_{X_R \otimes_{RA} M}] = [X_R \otimes_{RA} M', \phi_{X_R \otimes_{RA} M'}]$. In particular, assume that there exists an isomorphism of left $RT$-modules $f : X_R \otimes_{RA} M \to X_R \otimes_{RA} M'$. Thus we obtain an isomorphism of left $RA$-modules $\text{id}_{Y_R} \otimes f : Y_R \otimes_{RT} X_R \otimes_{RA} M \to Y_R \otimes_{RT} X_R \otimes_{RA} M'$. By (3.12), we obtain that there exists an isomorphism of left $RA$-modules $\Omega^l_{RA} f : \Omega^l_{RA} M \to \Omega^l_{RA} M'$ such that $\text{id}_{Y_R} \otimes \Omega^l_{RA} f = \text{id}_{\Omega^l_{\Lambda} V}$. This together with Remark 2.3 (i) implies that $[\Omega^l_{RA} M, \Omega^l_{RA} \phi] = [\Omega^l_{RA} M', \Omega^l_{RA} \phi']$.
in $F_{\Omega V}(R)$, which together with Remark 3.3 implies that $[M, \phi] = [M', \phi']$ in $F_{V}(R)$. This proves that $\tau_{X \otimes A V, R}$ is injective. Next assume that $\theta : R \to R'$ is a morphism of Artinian objects in $\hat{C}$. Let $(M, \phi)$ a lift of $V$ over $R$. Then there is a composition of left $R'\Lambda$-modules as follows:

$$R' \otimes_{R, \theta} (X_R \otimes_{R A} M) \cong (R' \otimes_{R, \theta} X_R) \otimes_{R' \otimes_{R, \theta} R A} (R' \otimes_{R, \theta} M) \cong X_{R'} \otimes_{R' \Lambda} M',$$

where $X_{R'} = R' \otimes_k X$ and $M' = R' \otimes_{R, \theta} M$. This proves that $\tau_{X \otimes A V, R}$ is natural with respect of morphism between Artinian objects in $\hat{C}$.

The continuity of the deformation functor (see Remark 2.1 (ii)) implies that for all objects $R$ in $\hat{C}$, there is a bijection between sets of deformations

$$\tilde{\tau}_{X \otimes A V, R} : \hat{F}_{V}(R) \to \hat{F}_{X \otimes A V}(R),$$

which is natural with respect of morphisms between objects in $\hat{C}$. Consequently, we obtain that the universal deformation rings $R(\Lambda, V)$ and $R(\Lambda, X \otimes A V)$ are isomorphic in $\hat{C}$. This finishes the proof of Theorem 1.2 (ii).

In the following, we provide an example (due to Ø. Skartsætherhagen) of two finite dimensional $\kappa$-algebras $\Lambda$ and $\Gamma$ which are singularly equivalent of Morita type with level (as in Definition 1.1) and which show that Theorem 1.2 (ii) can fail if both of the conditions for $\Lambda$ and $V$ are not satisfied.

**Example 3.6.** Let $\Lambda = \kappa Q/\langle \rho \rangle$ and $\Gamma = \kappa Q'/\langle \sigma \rangle$ be the monomial $\kappa$-algebras given by the following quivers with relations:

$$Q : \alpha \xrightarrow{\beta} \gamma,$$

$$Q' : \gamma \xrightarrow{\gamma} \gamma,$$

$$\rho = \{\alpha^2, \beta \alpha\},$$

$$\sigma = \{\gamma^2\}.$$

It follows by [36, Example 7.5] that $\Lambda$ and $\Gamma$ are singularly equivalent of Morita type with level $\ell = 1$ as in Definition 1.1. Moreover, $\Lambda$ is a non-Gorenstein $\kappa$-algebra with radical square zero. It follows by [16] that $\Lambda$ is CM-free, i.e., every Gorenstein-projective left $\Lambda$-module is projective. Thus if $V$ is a Gorenstein-projective left $\Lambda$-module, then $\text{End}_\Lambda(V) = 0$ and $R(\Lambda, V) \cong \kappa$ (by Remark 2.1). On the other hand, note that $\Gamma$ is a self-injective Nakayama $\kappa$-algebra with Loewy length equal to 2. If $S_1$ denotes the simple $\Gamma$-module corresponding to the vertex of $Q'$, then $\text{End}_\Gamma(S_1) = \kappa$, and $\text{Ext}_1^\Gamma(S_3, S_3) \neq 0$. Thus by [12, Thm. 1.3 (ii)] we have that $R(\Gamma, S_3) \cong \kappa[t]/(t^2)$. This shows that Theorem 1.2 (ii) fails provided that both of the conditions $\Lambda$ being Gorenstein and that $V$ being an non-projective indecomposable Gorenstein-projective with $\text{End}_\Lambda(V) = \kappa$ are omitted in the hypothesis.

4. **Applications**

In the following, we discuss some immediate applications of the main results in this article.

4.1. **Morita and triangular matrix algebras.** Let $\Lambda$ and $\Gamma$ be finite dimensional $\kappa$-algebras, $B$ a $\Lambda$-$\Gamma$-bimodule, and $C$ a $\Gamma$-$\Lambda$-bimodule. We define the Morita $\kappa$-algebra

$$(4.1) \Sigma = \begin{pmatrix} \Lambda & B \\ C & \Gamma \end{pmatrix},$$

where the addition of elements of $\Sigma$ is componentwise and the multiplication is given by

$$\begin{pmatrix} \lambda & b \\ c & \gamma \end{pmatrix} \cdot \begin{pmatrix} \lambda' & b' \\ c' & \gamma' \end{pmatrix} = \begin{pmatrix} \lambda \lambda' + b \gamma' \\ c \lambda' + \gamma b' \\ c \gamma' + \gamma' \gamma' \end{pmatrix},$$

for all $\lambda, \lambda' \in \Lambda$, $\gamma, \gamma' \in \Gamma$, $b, b' \in B$ and $c, c' \in C$. We refer the reader to [23, §2.1] for a detailed description of the abelian category $\Sigma$-$\text{mod}$ as well as for a description of $\Sigma$-$\text{Gproj}$. The following result follows from [23, Cor. 4.10].

**Lemma 4.1.** Let $\Sigma$ be a Morita $\kappa$-algebra as in (4.1) which is also Gorenstein.
(i) If \( C \) is projective as a right \( \Lambda \)-module and \( B \) is projective as a left \( \Lambda \)-module, then the \( k \)-algebra \( \Lambda \) is Gorenstein.
(ii) If \( B \) is projective as a right \( \Gamma \)-module and \( C \) is projective as a left \( \Gamma \)-module, then the \( k \)-algebra \( \Gamma \) is Gorenstein.

The following result follows from Lemma 4.1 and [19, Example 4.6].

**Lemma 4.2.** Let \( \Sigma \) be as in the hypothesis of Lemma 4.1.

(i) Under the situation of Lemma 4.1 (i), if \( \Gamma \) has finite projective dimension as a \( \Sigma \)-\( \Sigma \)-bimodule, then there exist a pair of bimodules \( (\Sigma X_\Lambda, Y_\Sigma) \) that induces a singular equivalence of Morita type with level \( \ell \) between \( \Sigma \) and \( \Lambda \) (as in Definition 1.1), where \( \ell \) is equal to the projective dimension of \( \Gamma \) as a \( \Gamma \)-\( \Gamma \)-bimodule.

(ii) Under the situation of Lemma 4.1 (ii), if \( \Lambda \) has finite projective dimension as a \( \Sigma \)-\( \Sigma \)-bimodule, then there exist a pair of bimodules \( (\Sigma X_\Gamma, Y_\Sigma) \) that induces a singular equivalence of Morita type with level \( \ell \) between \( \Sigma \) and \( \Gamma \) (as in Definition 1.1), where \( \ell \) is equal to the projective dimension of \( \Lambda \) as a \( \Lambda \)-\( \Lambda \)-bimodule.

The following result is an immediate consequence of Lemmata 4.1 and 4.2 together with Theorem 1.2 (ii).

**Corollary 4.3.** Let \( \Sigma \) be as in Lemma 4.1, and let \( W \) be an indecomposable Gorenstein-projective left \( \Sigma \)-module with \( \text{End}_\Sigma(W) = k \).

(i) If \( \Sigma \) is under the situation of Lemma 4.2 (i), then \( \Lambda \) is also a Gorenstein \( k \)-algebra, \( Y \otimes_\Sigma W \) is also an indecomposable Gorenstein-projective left \( \Lambda \)-module with \( \text{End}_\Lambda(Y \otimes_\Sigma W) = k \) and the universal deformation rings \( R(\Sigma, W) \) and \( R(\Lambda, Y \otimes_\Sigma W) \) are isomorphic in \( \hat{C} \).

(ii) If \( \Sigma \) is under the situation of Lemma 4.2 (ii), then \( \Gamma \) is also a Gorenstein \( k \)-algebra, \( Y' \otimes_\Sigma W \) is also an indecomposable Gorenstein-projective left \( \Gamma \)-module with \( \text{End}_\Gamma(Y' \otimes_\Sigma W) = k \) and the universal deformation rings \( R(\Sigma, W) \) and \( R(\Gamma, Y' \otimes_\Sigma W) \) are isomorphic in \( \hat{C} \).

**Remark 4.4.** Assume next that \( \Sigma \) is as the triangular matrix \( k \)-algebra

\[
\Sigma = \begin{pmatrix}
\Lambda & B \\
0 & \Gamma
\end{pmatrix},
\]

where \( \Lambda \) has infinite global dimension, \( \Gamma \) has finite projective dimension as a \( \Gamma \)-\( \Gamma \)-bimodule equal to \( m_1 \), and that \( B \) is a \( \Lambda \)-\( \Gamma \)-bimodule with finite projective dimension equal to \( m_2 \). Then by the analogous results in [40, Claim 3.1] for \( \Sigma \), there exists a pair of bimodules \( (\Sigma X_\Lambda, Y_\Sigma) \) that induce a singular equivalence of Morita type with level \( \ell + 1 \) between \( \Sigma \) and \( \Lambda \) (as in Definition 1.1), where \( \ell = \min\{m_1, m_2\} \). Assume further that \( \Sigma \) is Gorenstein. It follows by [41, Thm. 2.2] that \( \Lambda \) and \( \Gamma \) are both Gorenstein. For a description of the objects in \( \Sigma \)-\text{Gproj} under this situation, we refer the reader to [41, Thm. 1.4].

The following result (which improves that in [38, Thm. 1.3]) follows immediately from Remark 4.4 and Theorem 1.2 (ii).

**Corollary 4.5.** Under the situation and notation in Remark 4.4, if \( W \) is an indecomposable Gorenstein-projective left \( \Sigma \)-module with \( \text{End}_\Sigma(W) = k \), then \( Y \otimes_\Sigma W \) is also an indecomposable Gorenstein-projective left \( \Lambda \)-module with \( \text{End}_\Lambda(Y \otimes_\Sigma W) = k \), and the universal deformation rings \( R(\Sigma, W) \) and \( R(\Lambda, Y \otimes_\Sigma W) \) are isomorphic in \( \hat{C} \).

### 4.2. Singular equivalences induced by homological epimorphisms.

**Remark 4.6.** Let \( \Lambda \) be a finite dimensional \( k \)-algebra and let \( J \subseteq \Lambda \) be an ideal. Following [20], we say that \( J \) is a *homological ideal* provided that the canonical map \( \Lambda \to \Lambda/J \) is a homological epimorphism (in the sense of [24]). It follows from the main result in [17] that in this situation, if we further have that \( J \) has finite projective dimension as a \( \Gamma \)-\( \Gamma \)-bimodule, then there is a singular equivalence between \( \Lambda \) and \( \Lambda/J \). In particular, if \( \Lambda \) and \( \Lambda/J \) are both Gorenstein, then the triangulated categories \( \Lambda \)-\text{Gproj} and \( \Lambda/J \)-\text{Gproj} are equivalent. Moreover, it follows from [31, Thm. 3.6] that \( \Lambda \) and \( \Lambda/J \) are singularly equivalent of Morita type with level \( \ell \) (as in Definition 1.1).
The following result follows immediately from Remark 4.6 and Theorem 1.2 (ii).

**Corollary 4.7.** Let \( \Lambda \) be a Gorenstein \( k \)-algebra and let \( J \) a homological ideal of \( \Lambda \) such that \( \Gamma = \Lambda/J \) is also Gorenstein. Let \((r\cdot X_{\Lambda}, \Lambda Y_{\Gamma})\) be a pair of bimodules that induces a singular equivalence of Morita type with level \( \ell \) (as in Definition 1.1). If \( V \) is an indecomposable Gorenstein-projective left \( \Lambda \)-module with \( \text{End}_{\Lambda}(V) = k \), then \( X \otimes_{\Lambda} V \) is also an indecomposable Gorenstein-projective left \( \Gamma \)-module with \( \text{End}_{\Gamma}(X \otimes_{\Lambda} V) = k \) and the universal deformation rings \( R(\Lambda, V) \) and \( R(\Gamma, X \otimes_{\Lambda} V) \) are isomorphic in \( \hat{\mathcal{C}} \).

In the following, we discuss an example (due to X. W. Chen) of two finite dimensional \( k \)-algebras that verify Corollary 4.7.

**Example 4.8.** Consider the basic \( k \)-algebras \( \Lambda \) and \( \Gamma \) as in Figures 1 and 2, respectively. It follows from [17, Exam. 3.5] and Remark 4.6 that \( \Lambda \) is a Gorenstein \( k \)-algebra with injective dimension 2 at both sides and that there exists a singular equivalence of Morita type with level \( \ell \) (as in Definition 1.1) between \( \Lambda \) and \( \Gamma \).

\[
Q = \begin{array}{ccc}
0' & \text{ } & 0 \\
\beta_0 & \alpha_0 & \gamma_0 \\
\gamma_1 & \beta_1 & \alpha_1 \\
2' & \beta_2 & \alpha_2 \\
\gamma_2 & \gamma_0 & 0 \\
1' & \alpha_4 & \beta_3 \\
1 & \gamma_1 & \gamma_2 \\
2 & \gamma_2 & \gamma_0 \\
\end{array}
\]

\[
\rho_0 = \{\beta_i\alpha_i, \gamma_i\alpha_i, \beta_i\gamma_i, \alpha_i\beta_i - (\gamma_i+1)\gamma_i \gamma_i+2) : i \in \mathbb{Z}/3\}.
\]

**Figure 1.** The basic \( k \)-algebra \( \Lambda_0 = kQ/(\rho_0) \).

\[
Z_3 = \begin{array}{ccc}
0 & \text{ } & 0 \\
70 & \gamma_1 & \gamma_0 \\
2 & \gamma_2 & 1 \\
\end{array}
\]

\[
\rho_1 = \{(\gamma_0, \gamma_1, \gamma_2)^6 : i \in \mathbb{Z}/3\}.
\]

**Figure 2.** The basic \( k \)-algebra \( \Gamma = kZ_3/(\rho_1) \).

Note also that \( \Gamma \) is a self-injective (thus Gorenstein) Nakayama \( k \)-algebra, and that \( \Lambda \) is a special biserial algebra (in the sense of [15]) of finite representation type. Thus we can describe combinatorially the indecomposable non-projective objects in \( \Lambda \)-mod by using so-called strings for \( \Lambda \); the corresponding \( \Lambda \)-modules are called string modules. The morphisms between these string \( \Lambda \)-modules can be completely described by using the results in [29]. Moreover, it follows from [4, Prop. 3.1 (b)] that \( \Lambda \text{-Gproj} = \Omega^2(\Lambda \text{-mod}) \). Using the above arguments together with the description of the irreducible morphisms between string \( \Lambda \)-modules in [15], we can identify all the indecomposable non-projective Gorenstein-projective left \( \Lambda \)-modules in the stable
Auslander-Reiten quiver of $\Lambda$. More precisely, we have that the indecomposable non-projective Gorenstein-projective left $\Lambda$-modules are given as follows:

\[
\begin{align*}
V_{i,0} &= M[\alpha_{i+1}^{-1}\gamma_{i+2}\gamma_i+1\gamma_{i+1}+2\gamma_i], \\
V_{i,2} &= M[\alpha_{i+1}^{-1}\gamma_{i+2}\gamma_i+2\gamma_i+1], \\
V_{i,4} &= M[\alpha_{i+1}^{-1}\gamma_i+2],
\end{align*}
\]

where $i \in \mathbb{Z}/3$. It is straightforward to check that for all $i \in \mathbb{Z}/3$, $\Omega V_{i,0} = V_{i+2,4}$, $\Omega V_{i,1} = V_{i+1,3}$ and $\Omega V_{i,2} = V_{i,2}$. Then it follows from \cite{29} and \cite[Lemma 5.2]{36} that for all $i \in \{0, 1, 2, 3, 4\}$, $\text{End}_{\Lambda}(V_{i,0}) \cong \mathbb{k}$, $\text{Ext}^1_{\Lambda}(V_{i,1}, V_{i,0}) = 0$ with $j \neq 3$, and $\text{Ext}^3_{\Lambda}(V_{i,3}, V_{i,3}) \cong \mathbb{k}$ as $\mathbb{k}$-vector spaces. Thus for $j \in \{0, 1, 2, 3, 4\}$, $R(\Lambda, V_{i,j})$ is universal and isomorphic to $\mathbb{k}$ for $j \neq 3$, and $R(\Lambda, V_{i,3})$ is a quotient of $\mathbb{k}[t]$. Let $i \in \mathbb{Z}/3$ be fixed and denote by $P_i$ the incomposable projective left $\Lambda$-modules corresponding to the vertex $i'$ of $Q$. Then there exists a non-splitting short sequence of left $\Lambda$-modules

\[
0 \to V_{i,3} \xrightarrow{i'} P_i \oplus V_{i,0} \xrightarrow{v} V_{i,3} \to 0.
\]

If we let $M = P_i \oplus V_{i,0}$, then $\Lambda$ defines a non-trivial left $\mathbb{k}[t]/(t^2)\Lambda$-module by letting $t$ act on $m \in M$ as $t \cdot m = (\iota \circ \pi)(m)$. Thus there exists a unique surjective $\Lambda$-algebra homomorphism $\theta : R(\Lambda, V_{i,3}) \to \mathbb{k}[t]/(t^2)$ in $\mathcal{C}$ corresponding to the deformation defined by $M$. Assume that $\theta$ is not an isomorphism. Thus there exists a surjective $\mathbb{k}$-algebra homomorphism $\theta' : R(\Lambda, V_{i,3}) \to \mathbb{k}[t]/(t^3)$ in $\mathcal{C}$ such that $\pi_{3,2} \circ \theta' = \theta$, where $\pi_{3,2} : \mathbb{k}[t]/(t^3) \to \mathbb{k}[t]/(t^2)$ is the natural projection. Let $M'$ be a left $\mathbb{k}[t]/(t^3)\Lambda$-module that defines a lift of $V_{i,3}$ over $\mathbb{k}[t]/(t^3)$ corresponding to $\theta'$. Note that $M'/t^2M' \cong M$ and $t^2M' \cong V_{i,3}$. Thus, we obtain a short exact sequence of $\mathbb{k}[t]/(t^3)\Lambda$-modules

\[
0 \to V_{i,3} \to M' \to M \to 0.
\]

Note that since $\text{Ext}^3_{\Lambda}(M, V_{i,3}) = \text{Hom}_{\Lambda}(\Omega_{\Lambda}(V_{i,0}, V_{i,3}), 0) = 0$, it follows that (4.4) splits as a sequence of left $\Lambda$-modules. Hence $M' = V_{i,3} \oplus M$ as left $\Lambda$-modules. Thus if $(v, m) \in M'$ with $v \in V_{i,3}$ and $m \in M$, it follows that the action of $t$ on $M'$ is given by $t \cdot (v, m) = (\sigma(m), t \cdot m)$, where $\sigma : M \to V_{i,3}$ is a surjective $\Lambda$-module homomorphism. Then there exists $c \in \mathbb{k}^*$ such that $\sigma = c \pi$, where $\pi$ is as in (4.3), and thus the kernel of $\sigma$ is $tM$. This implies that $\sigma(tm) = 0 = t^2m$ for all $m \in M$, and consequently, $t^2(v, m) = (\sigma(tm), t^2m) = (0, 0)$ for all $v \in V_{i,3}$ and $m \in M$, which contradicts that $t^2M' \cong V_{i,3}$. Thus $\theta$ is a $\mathbb{k}$-algebra isomorphism and $R(\Lambda, V_{i,3}) \cong \mathbb{k}[t]/(t^2)$. Therefore, if $V$ is an indecomposable Gorenstein-projective left $\Lambda$-module, then $\text{End}_{\Lambda}(V) = \mathbb{k}$ and the universal deformation ring $R(\Lambda, V)$ is isomorphic either to $\mathbb{k}$ or to $\mathbb{k}[t]/(t^2)$. On the other hand, by using \cite[Thm. 1.2]{12}, it is straightforward to see that if $V'$ is an indecomposable non-projective (thus a Gorenstein-projective) left $\Gamma$-module, then $\text{End}_{\Gamma}(V') = \mathbb{k}$ and the universal deformation ring $R(\Gamma, V')$ is isomorphic either to $\mathbb{k}$ or to $\mathbb{k}[t]/(t^2)$. This verifies Corollary 4.7.

### 4.3. Singular equivalences induced by 2-recollements

To end this section, we provide a result involving universal deformation rings of Gorenstein-projective modules over Gorenstein algebras and 2-recollements (as introduced in \cite[Def. 2]{32}).

Let $\mathcal{T}'$, $\mathcal{T}$, and $\mathcal{T}''$ be triangulated categories. Following \cite{7}, a recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is given by

\[
\begin{array}{ccc}
\mathcal{T}' & \xrightarrow{i_*} & \mathcal{T} & \xleftarrow{i^*} & \mathcal{T}'' \\
\mathcal{T}' & \xrightarrow{j_*} & \mathcal{T} & \xleftarrow{j^*} & \mathcal{T}''
\end{array}
\]

such that

- (R1) $(i^*, i_*)$, $(i^*, i_*)$, $(j_*, j^*)$ and $(j^*, j_*)$ are adjoint pairs of triangulated functors;
- (R2) $i_*$, $j_*$ and $j^*$ are full embeddings;
- (R3) $j^*i_* = 0$ (and thus we also have $i^*j_* = 0$ and $i^*j^* = 0$);
(R4) for each object $X$ in $\mathcal{T}$, there are triangles

$$
j_1j^!x \to X \to i_*i^*X \to
g_1i^*x \to X \to j_*j^*X \to
$$

where the arrows to and from $X$ are the counits and the units of the adjoint pairs respectively.

Following [32], a 2-recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$ is given by a diagram of functors of triangulated categories

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{j_1} & \mathcal{T}' \\
\downarrow{i_1} & & \downarrow{j_2} \\
\mathcal{T} & \xrightarrow{i_2} & \mathcal{T}'' \end{array}
$$

such that each of the two possible consecutive three layers form a recollement of $\mathcal{T}$ relative to $\mathcal{T}'$ and $\mathcal{T}''$.

For all finite dimensional $k$-algebras $\Lambda$, we denote by $D(\Lambda) = D(\Lambda\text{-Mod})$ the derived category of all modules over $\Lambda$ from the left side, which is a triangulated category (see e.g. [26, Chap. I]).

The following result is due to Y. Qin (see [31, Cor. 3.3]).

**Lemma 4.9.** Let $\Lambda$, $\Gamma$ and $\Sigma$ be finite dimensional $k$-algebras such that $\Gamma$ has finite projective dimension as a $\Gamma\text{-}\Gamma$-bimodule. Assume that $\mathcal{T} = D(\Sigma)$ admits a 2-recollement relative to $\mathcal{T}' = D(\Gamma)$ and $\mathcal{T}'' = D(\Lambda)$ as in (4.6). Then $\Sigma$ and $\Lambda$ are singularly equivalent of Morita type with level as in Definition 1.1.

The following result follows immediately from Lemma 4.9 and Theorem 1.2 (ii).

**Corollary 4.10.** Assume that $\Lambda$, $\Gamma$ and $\Sigma$ are as in Lemma 4.9, with $\Lambda$ and $\Sigma$ Gorenstein $k$-algebras. If $W$ is a Gorenstein-projective left $\Sigma$-module with $\text{End}_\Sigma(W) = k$, then there exists a Gorenstein-projective left $\Lambda$-module $V$ such that $\text{End}_\Lambda(V) = k$ and the universal deformation rings $R(\Sigma, W)$ and $R(\Lambda, V)$ are isomorphic in $\hat{C}$.

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