KNOT CONCORDANCES IN $S^1 \times S^2$ AND EXOTIC SMOOTH 4-MANIFOLDS

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Abstract. It is known that there is a unique concordance class in the free homotopy class of $S^1 \times pt \subset S^1 \times S^2$. The constructive proof of this fact is given by the second author. It turns out that all the concordances in this construction are invertible. The knots $K \subset S^1 \times S^2$ with hyperbolic complements and trivial symmetry group are special interest here, because they can be used to generate absolutely exotic compact 4-manifolds by the recipe given by Akbulut and Ruberman. Here we built an absolutely exotic manifold pair by this construction, and show that this construction keeps the Stein property of the 4-manifold we start out with. By using this we establish existence of an absolutely exotic contractible Stein manifold pair.

0. Introduction

Here we will prove the following theorem which strengthens [AR]:

Theorem 1. There is a pair of compact contractible Stein manifolds $W_1, W_2$, which are homeomorphic but not diffeomorphic to each other.

As in the examples of [AR], these $W_1$ and $W_2$ are also related to each other by cork twisting along a cork $(W, f) \subset W_i, i = 1, 2$. It follows from construction that $W_1$ and $W_2$ can not be corks (compare to [AR1]). Theorem 1 will be obtained as a consequence of the following:

Theorem 2. Any knot $k$ in $S^1 \times S^2$, which is freely homotopic to $S^1 \times pt$, is invertibly concordant to $S^1 \times pt$.

1. Background

Let us recall some basic definitions about knot concordances:

Definition 1. Two knots $K_1$ and $K_2$ are said to be concordant if there is a smooth proper embedding of an annulus $F : S^1 \times [0, 1] \hookrightarrow Y \times [0, 1]$, such that its boundary is $\partial F(S^1 \times [0, 1]) = K_1 \times 0 \sqcup (-K_2) \times 1$, where $-K_2$ is the knot $K_2$ with the reversed orientation.

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Definition 2. [S] A concordance $F$ between knots $K_1$ and $K_2$ is said to be invertible if there is a concordance $F'$ from $K_2$ to $K_1$ such that $F \cup F' : S^1 \times [0,1] \to Y \times [0,1]$ is the product concordance $K_1 \times [0,1] \subset Y \times [0,1]$. In this case, we say $K_1$ is invertibly concordant to $K_2$, and say $K_2$ splits $K_1 \times [0,1]$. In particular when $Y = S^3$ and $K_1$ is the unknot then $K_2$ is called doubly slice.

Definition 3. An invertible cobordism $X$ from $M$ to $N$ is a smooth manifold with $\partial X = M \sqcup -N$, such that there is a cobordism $X'$ with $\partial X' = N \sqcup -M$ and $X \sqcup X' = M \times I$.

The following follows from computations by SnapPy [CDGW].

Proposition 3. The knot $K \subset S^1 \times S^2$ given in the Figure 1 is hyperbolic with trivial symmetry group.

By using Theorem 2 and Proposition 3 we will convert relative exotic smooth structures on compact 4-manifolds to absolutely exotic smooth structures, by using the construction given in [AR]. The goal is to generate Stein exotic examples that can not be reached by [AR].

\begin{figure}[h]
\centering
\includegraphics[width=0.25\textwidth]{knot.png}
\caption{$K \subset S^1 \times S^2$ with hyperbolic complement and trivial symmetry group}
\end{figure}

Recall the construction of absolutely exotic smooth structures on compact 4-manifolds, from relative exotic structures:

Theorem 4 ([AR]). Let $F : W \to W$ be self homeomorphism of a compact smooth 4-manifold, whose restriction to $\partial W$ is a diffeomorphism which does not extend to a self diffeomorphism of $W$. Then $W$ contains a pair of homeomorphic smooth 4-manifolds $V$ and $V'$ homotopy equivalent to $W$, but $V$ and $V'$ are not diffeomorphic to each other.
The construction of [AR] relies on finding a knot in $S^3$ which is doubly slice, hyperbolic, and with no symmetry. Here we will remind the original construction, and show that we can use invertible knot concordances in $S^1 \times S^2$ between a knot $K$ and $S^1 \times pt$, where $K$ is a hyperbolic knot with trivial symmetry group, splitting the concordance.

**Lemma 5.** Let $K$ be a knot in $S^1 \times S^2$, which is freely homotopic to $S^1 \times pt$ with exterior $X$. Then $H_*(X) = H_*(S^1 \times D^2)$.

**Proof.** Follows from Mayer-Vietoris sequence. □

**Corollary 6.** Let $K_1$ and $K_2$ be knots in the free homotopy class of the $S^1 \times pt$ in $S^1 \times S^2$, and $C$ be a concordance between $K_1$ and $K_2$, then $H_*(S^1 \times S^2 \times I - C) = H_*(S^1 \times D^2 \times I)$ and generated by the $S^1 \times pt$.

Note that a tubular neighbourhood of $C$ in $S^1 \times S^2 \times I$ is an embedding of $S^1 \times D^2 \times I$ in $S^1 \times S^2 \times I$, where the image of $S^1 \times 0 \times I$ is $C$. The image of $pt \times \partial D^2 \times t$ is called a meridian of $C$, then a meridian of $C$ bounds a disk in the tubular neighbourhood of $C$, and it is homologous to zero in the exterior. Similarly we call the image of $S^1 \times pt \times t$ as a longitude.

The following lemma is adapted from [AR]. We restate it here in a slightly different way to show that we can use appropriate concordances of knots in $S^1 \times S^2$ instead of in $S^3$. The only difference here is that the gluing map $\phi$ identifies meridian to meridian not to longitude.

**Lemma 7.** Suppose that $\gamma$ is a framed knot in a closed 3-manifold $M$, and that $C$ is an invertible concordance from the $S^1 \times pt$ to the knot $K$ in $S^1 \times S^2 \times I$. Define

$$H = (M \times I - (\gamma \times D^2 \times I)) \bigcup_{\phi} (S^1 \times S^2 \times I - (C \times D^2))$$

where $\phi : T^2 \times I \to T^2 \times I$ is a diffeomorphism which sends the meridian of $K$ to the meridian of the knot $\gamma$. Then $H$ is an invertible homology cobordism from $M$ to $N$. If $\pi_1(S^1 \times S^2 \times I - C \times D^2) \cong \mathbb{Z}$, then the inclusion $\iota : M \hookrightarrow W$ induces an isomorphism on fundamental groups.
2. Proof of Theorem 2

Proof of Theorem 2. Start with a knot $K \subset S^1 \times S^2$ in the concordance class of $S^1 \times pt$. Built a concordance $F$ between $K$ and $S^1 \times pt$ as it is explained in [Y, Theorem 2]. $F$ is properly embedded in $S^1 \times S^2 \times I$, the last coordinate is the time, and we have $S^1 \times S^2$ at every level. From the construction we see the knot $K$ at the top level, and we perform genus zero cobordism, i.e. we attach bunch of bands $\{b_i\}_{i=1}^m$, turning $K$ to $S^1 \times pt$ union disjoint unknots $\{u_i\}_{i=1}^m$ linking $S^1 \times pt$.

For $1 > t_i > t_{i-1} > 0$, following levels are depicted partially in Figure 2

- $F \cap (S^1 \times S^2 \times 1) = K$
- $F \cap (S^1 \times S^2 \times t_2) = K \cup b$
- $F \cap (S^1 \times S^2 \times t_1) = u \cup (S^1 \times pt)$
- $F \cap (S^1 \times S^2 \times 0) = S^1 \times pt$

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure2.png}
\caption{The construction of the concordance}
\end{figure}

where each $u = \partial D^2$ is an unknot in $S^1 \times S^2$ which is the boundary of the corresponding 0-handle of $F$, and $b$ represents a band which is a 1-handle attached to the surface to change a crossing on the boundary knot. The $F'$ in the Definition 2 will be $-F$.

Next will be to construct a handlebody decomposition for the concordance complement $S^1 \times S^2 \times I - \nu(F \cup (-F))$ from the handlebody decomposition of the surface $F$. For a detailed discussion of handlebody decomposition of surface complement in 4-manifolds one can consult Chapter 1.4 of [A]. To construct the complement of $F$ in $S^1 \times S^2 \times I$ start from the bottom. First we see complement of $S^1 \times pt \times [0, t]$ in $S^1 \times S^2 \times [0, t]$ which is $S^1 \times D^2 \times [0, t]$. Then as $t$ increase, 0-handles of the surface $F$ appears, in the complement which corresponds carving
properly embedding of disks from $B^4$. So we have connected sum of
$m + 1$ copies of $S^1 \times B^3$. At the last step the complement gains $m$
4-dimensional 2-handles for the 1-handles $b_i$ of the surface, as in the
middle picture of Figure 3. Alternatively the reader should compare
this to $[Y]$, where the cobordism was constructed from top to bottom.
For example, Figure 3 describes the concordance from the Mazur knot
to $S^1 \times pt$

To double the concordance complement, we attach upside-down han-
dles along the dual of the 2-handles (with zero framing), and dual 3-
handles as upside-down 1-handles. We use the dual 2-handles to get
rid of the self-linking of the original 2-handle, therefore ending up with
cancelling $1/2$ and $2/3$ handle pairs. So after cancelations the comple-
ment becomes a product. In Figure 3, reader can verify that sliding
over the dual 2-handle (0-framed little circle linking the 2-handle) will
make the 2-handle trivially link the 1-handle.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.png}
\caption{The Construction of the concordance complement}
\end{figure}

All these cancellations happens away from the endpoints
$S^1 \times S^2 \times \{0\} - \nu(S^1 \times \{pt\})$ and $S^1 \times S^2 \times \{1\} - \nu(S^1 \times \{pt\})$ provided
attaching circles of the two handles are away from the $S^1 \times pt$. Note
that anything links to $S^1 \times pt$ can be unlinked by sliding over the middle
1-handle. So we have constructed a diffeomorphism from $S^1 \times S^2 \times I - 
\nu(F \cup (-F))$ to $S^1 \times S^2 \times I - \nu(S^1 \times pt \times I) = S^1 \times B^2 \times I$, which is
identity near the endpoints. Next, we prove that this diffeomorphism
extends to self diffeomorphism of $S^1 \times S^2 \times I$ which takes the surface
$F \cup -F$ to the product $S^1 \times \{pt\} \times I$. 
The diffeomorphism above induces a boundary diffeomorphism on \( \partial(S^1 \times B^2 \times I) = S^1 \times S^1 \times I \cup (S^1 \times B^2 \times 0 \sqcup S^1 \times B^2 \times 1) \).
When we restrict this to the partial boundary we get a diffeomorphism \( \tau : T^2 \times I \to T^2 \times I \) which is identity near \( T^2 \times \{0, 1\} \). By Lemma 3.5 of Waldhausen [W], \( \tau \) is isotopic to the identity map rel boundary hence it extends to tubular neighbourhoods of the surfaces. Therefore we have a self diffeomorphism of \( S^1 \times S^2 \times I \) takes the core of \( \nu(S^1 \times pt \times I) \) to the core of \( \nu(F \cup (-F)) \). Therefore diffeomorphic complements uniquely determine the surfaces. \( \square \)

Note that Theorem 2 also follows from 4-Dimensional Lightbulb Theorems of [L] and [G]. After doubling the concordance, we have a surface in \( S^1 \times S^2 \times I \) which is bounded by \( S^1 \times pt \times 0 \) and \( S^1 \times pt \times 1 \). By capping both boundaries with \( D^2 \times S^2 \) s, we get a surface in \( S^2 \times S^2 \) intersecting \( pt \times S^2 \) at one point, then use 4-D Lightbulb Theorems.

3. Proof of Theorem 1

By applying the technique of Section 2 to the curve \( K \subset S^1 \times S^2 \) of Figure 1, we get the Figure 4, where the curve \( K \) now looks like the standard linking circle to the 1-handle (dotted curve in the figure).

\[ \text{Figure 4. The concordance complement in } S^1 \times S^2 \times I \]
In this figure the concordance from $S^1 \times pt$ to $K$ is fully visible (the handles of its complement is visible). It follows from the construction that cancelling the handles of Figure 4 gives Figure 1 taking the dotted circle to $K$. Now we proceed as in [AR], i.e. glue the homology product cobordism $H$ obtained from Figure 4 to the boundary of the cork $W$ (as discussed in Section 1). From this we get a new contractible manifold $W_1$ containing $W$, such that performing the cork twisting $W \subset W_1$ gives us an absolutely exotic copy $W_2$ of $W_1$. Recall from Section 1 we construct $W_1 = W \cup H$ by gluing $W$ and the homology product cobordism $H$ along the longitudes of $\eta \subset W$ and $K \subset S^1 \times S^2$, which is the Figure 5. Here we are using the roping technique of [A] to draw the handlebody of $W_1$. From the discussion above, Figure 5 is equivalent to Figure 6. By zero and dot exchanges to $W$ inside $W_1$ gives $W_2$. The only remaining issue is to put Stein structure on $W_1$ and $W_2$. 

Figure 5. $W_1$

Figure 6. $W_1$
For this we will work with the picture of $W_1$, given in Figure 6. We first put Figure 1 in Legendrian position and get Figure 7, where the 2-handle has $tb = 4$. Then in both Stein handlebody picture of $W$ (i.e. before and after the zero and dot exchanges of its handles, which are the two pictures of Figure 8), we put the curve $\eta \subset \partial W$ in Legendrian position. That is, we put both handlebodies of Figure 8 in Legendrian position, and connected sum with Figure 7. This gives Figures 12 and 11, both are Stein pictures of $W_1$ and $W_2$. \hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.png}
\caption{tb=4}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{tb=−3}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure9.png}
\caption{tb=−3}
\end{figure}
Figure 11. $W_1$, as a Stein handlebody

Figure 12. $W_2$, an exotic copy of $W_1$, as a Stein handlebody
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