Structure and regularity of the global attractor of a reaction-diffusion equation with non-smooth nonlinear term

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Abstract

In this paper we study the structure of the global attractor for a reaction-diffusion equation in which uniqueness of the Cauchy problem is not guaranteed. We prove that the global attractor can be characterized using either the unstable manifold of the set of stationary points or the stable one but considering in this last case only solutions in the set of bounded complete trajectories.

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1 Introduction

In this paper we study the structure of the global attractor of a reaction-diffusion equation in which the nonlinear term satisfy suitable growth and dissipative conditions, but there is no condition ensuring uniqueness of the Cauchy problem (like e.g. a monotonicity assumption). Such equation generates in the general case a multivalued semiflow having a global compact attractor (see [7], [14]). Also, it is known [12] that the attractor is the union of all bounded complete trajectories of the semiflow.

If we study the global attractor in more detail we can get a better understanding of the dynamics of the semiflow by restricting our attention inside the attractor. In particular, it is important to establish the relationship between the attractor and the stable and unstable manifolds of the set of stationary points. In the single-valued case, when for example the nonlinear term is a polynomial or its derivative satisfies some assumptions, it is well known [3], [4], [22] that the attractor is the unstable manifold of the set of stationary points. Moreover, if the set of stationary points is discrete, then it is the union of all heteroclinic orbits connecting the stationary points. In more particular parabolic equations the structure of the attractor has been completely understood by obtaining a list
of which stationary points are joined to each other. This the case of the famous Chafee-
Infante equation [11] or general scalar parabolic equations under suitable restrictions [8],
[9], [19], [20]. Also, in [10] similar results are obtained for retarded differential equations.

In [2] the structure of the global attractor of a scalar parabolic differential inclusion
generating a multivalued semiflow is studied, obtaining a partial description about which
pairs of stationary points are joined. As far as we know this is the only published result
about the heteroclinic connections between stationary points in the multivalued case.

Let \( F \) be the set of all complete trajectories, \( K \) be the set of all bounded complete
trajectories and \( \mathcal{R} \) the set of equilibria. We define the sets

\[
M^- (\mathcal{R}) = \{ z : \exists \gamma (\cdot) \in K, \, \gamma (0) = z, \, \text{dist}_{L^2(\Omega)} (\gamma (t), \mathcal{R}) \to 0, \, t \to +\infty \},
\]
\[
M^+ (\mathcal{R}) = \{ z : \exists \gamma (\cdot) \in F, \, \gamma (0) = z, \, \text{dist}_{L^2(\Omega)} (\gamma (t), \mathcal{R}) \to 0, \, t \to -\infty \}.
\]

We prove in this paper that the global attractor of a rather general reaction-diffusion
equation without uniqueness can be described in terms of either \( M^+ (\mathcal{R}) \) or \( M^- (\mathcal{R}) \), that
is, the unstable manifold of the set of stationary points or the stable one but considering
in this last case only solutions in the set of bounded complete trajectories.

In Section 4 it is proved that the attractor of the multivalued semiflow generated
by weak solutions in the phase space \( L^2 (\Omega) \) is the closure of \( M^- (\mathcal{R}) \). Also, \( M^+ (\mathcal{R}) \) is
contained in the attractor, and coincides with it when uniqueness takes place for regular
initial data.

In Section 5 we consider the multivalued semiflow generated by regular solutions, which
are the weak solutions which become strong after an arbitrary small time. We prove first
the existence of a global attractor in the phase space \( L^2 (\Omega) \) which is, moreover, compact
in \( H^1_0 (\Omega) \). After that we establish that it coincides with the unstable manifold of the
set of stationary points, and also with the stable one when we consider only bounded
complete solutions.

In Section 6 we consider the multivalued semiflow generated by strong solutions. We
prove first the existence of a global attractor in the phase space \( H^1_0 (\Omega) \) and that the
attractors of the regular and strong cases coincide. Finally, the same result about the
structure of the attractor as in the case of regular solutions is given.

## 2 Setting of the problem

In a bounded domain \( \Omega \subset \mathbb{R}^3 \) with sufficiently smooth boundary \( \partial \Omega \) we consider the problem

\[
\begin{aligned}
&u_t - \Delta u + f(u) = h, \quad x \in \Omega, \; t > 0, \\
&u |_{\partial \Omega} = 0,
\end{aligned}
\]

where

\[
\begin{aligned}
h &\in L^2 (\Omega), \\
f &\in C (\mathbb{R}), \\
|f(u)| &\leq C_1 (1 + |u|^3), \quad \forall u \in \mathbb{R}, \\
f (u)u &\geq \alpha |u|^4 - C_2, \quad \forall u \in \mathbb{R},
\end{aligned}
\]

with \( C_1, C_2, \alpha > 0 \).

We denote by \( A \) the operator \(-\Delta\) with Dirichlet boundary conditions, so that \( D(A) = H^2 (\Omega) \cap H^1_0 (\Omega) \). As usual, denote the first eigenvalue of \( A \) by \( \lambda_1 \).
Denote $F(u) = \int_0^u f(s)ds$. From (2) we have that $\liminf_{|u| \to \infty} \frac{f(u)}{|u|} = \infty$, and for some $D_1, D_2, \delta > 0$,

$$|F(u)| \leq D_1(1 + |u|^4), \quad F(u) \geq \delta u^4 - D_2, \quad \forall u \in \mathbb{R}. \tag{3}$$

The function $u \in L^2_{\text{loc}}(0, +\infty; H^1_0(\Omega)) \cap L^4_{\text{loc}}(0, +\infty; L^4(\Omega))$ is called a weak solution of (1) on $(0, +\infty)$ if for all $T > 0$, $v \in H^1_0(\Omega)$, $\eta \in C^0(0, T)$,

$$- \int_0^T (u, v)\eta dt + \int_0^T \left( (u, v)_{H^1_0(\Omega)} + (f(u), v) - (h, v) \right) \eta dt = 0, \tag{4}$$

where $\| \cdot \|$, $(\cdot, \cdot)$ are the norm and the scalar product in $L^2(\Omega)$. We denote by $\| \cdot \|_X$ the norm in the abstract Banach space $X$, whereas $(\cdot, \cdot)_H$ will be the scalar product in the abstract Hilbert space $H$. Also, $P(X)$ will be the set of all non-empty subsets of $X$.

It is well known [1, Theorem 2] or [6, p.284] that for any $u_0 \in L^2(\Omega)$ there exists at least one weak solution of (1) with $u(0) = u_0$ (and it may be non unique) and that any weak solution of (1) belongs to $C([0, +\infty); L^2(\Omega))$. Moreover, the function $t \mapsto \|u(t)\|^2$ is absolutely continuous and

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|^2 + \|u(t)\|^2_{H^1_0(\Omega)} + (f(u(t)), u(t)) - (h, u(t)) = 0 \text{ a.e.} \tag{5}$$

We define

$$K^+ = \{ u(\cdot) : u(\cdot) \text{ is a weak solution of (1)} \},$$

$$G : \mathbb{R}^+ \times L^2(\Omega) \to P(L^2(\Omega)),$$

$$G(t, u_0) = \{ u(t) : u(\cdot) \in K^+, \ u(0) = u_0 \}. \tag{6}$$

**Definition 1** Let $X$ be a complete metric space. The multivalued map $G : \mathbb{R}^+ \times X \to P(X)$ is a multivalued semiflow (m-semiflow) if:

1. $G(0, u_0) = u_0$, $\forall u_0 \in X$;
2. $G(t + s, u_0) \subset G(t, G(s, u_0))$, $\forall t, s \geq 0$, $\forall u_0 \in X$.

It is called strict if $G(t + s, u_0) = G(t, G(s, u_0))$, $\forall t, s \geq 0$, $\forall u_0 \in X$.

**Definition 2** The set $\Theta \subset X$ is called a global attractor of $G$, if:

1. $\Theta \subset G(t, \Theta)$, $\forall t \geq 0$ (negatively semi-invariance);
2. for any bounded set $B \subset X$,

$$\text{dist}_X(G(t, B), \Theta) \to 0, \text{ as } t \to +\infty, \tag{7}$$

where

$$\text{dist}_X(A, B) = \sup \inf_{x \in A, y \in B} \|x - y\|_X.$$

3. It is minimal, that is, for any closed set $C$ satisfying [7] it holds $\Theta \subset C$.  

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The global attractor is called invariant if $\Theta = G(t, \Theta), \forall t \geq 0$.

The map $G$ defined by (6) is a strict multivalued semiflow which possesses a global compact invariant connected attractor [7], [14], [15]. Our aim is to give a characterization of the attractor. First we shall define complete trajectories for problem (1).

**Definition 3** The map $\gamma : \mathbb{R} \to L^2(\Omega)$ is called a complete trajectory of $K^+$ if

$$\gamma(\cdot + h)|_{[0, +\infty)} \in K^+, \forall h \in \mathbb{R},$$

that is, if $\gamma|_{[\tau, +\infty)}$ is weak solution of (7) on $(\tau, +\infty), \forall \tau \in \mathbb{R}$.

In the following section (see Theorem 13) it will be shown that this is equivalent to the following:

$$\gamma(t + s) \in G(t, \gamma(s)), \forall t \geq 0, s \in \mathbb{R}.$$

Let $K$ be the set of all bounded (in the $L^2(\Omega)$ norm) complete trajectories. It is known that the global attractor of $G$ is the union of all bounded complete trajectories. We recall that the global attractor $\Theta$ is called stable if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$G(t, O_\delta(\Theta)) \subset O_\epsilon(\Theta), \forall t \geq 0.$$

**Theorem 4** [12, Theorem 3.18] Under conditions (2) the $m$-semiflow (6) has a global compact invariant attractor $\Theta \subset L^2(\Omega)$ which is connected, stable and

$$\Theta = \{\gamma(0) : \gamma(\cdot) \in K\} = \bigcup_{t \in \mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in K\}. \quad (8)$$

Let $\mathcal{R}$ be the set of all stationary points of (1), i.e., the points $u \in H^1_0(\Omega)$ such that

$$-\Delta u + f(u) = h \text{ in } H^{-1}(\Omega), \quad (9)$$

and

$$M^-(\mathcal{R}) = \{z : \exists \gamma(\cdot) \in K, \gamma(0) = z, \text{ dist}_{L^2(\Omega)}(\gamma(t), \mathcal{R}) \to 0, t \to +\infty\},$$

$$\quad M^+(\mathcal{R}) = \{z : \exists \gamma(\cdot) \in \mathcal{R}, \gamma(0) = z, \text{ dist}_{L^2(\Omega)}(\gamma(t), \mathcal{R}) \to 0, t \to -\infty\}. \quad (10)$$

It is known [22], [4, p.106], [3] that under additional conditions on $f$, like

$$f = \sum_{i=0}^{3} \alpha_i u^i, \quad \alpha_3 > 0, \quad (11)$$

$f \in C^1(\mathbb{R})$ and $\exists C_3 > 0$ such that $|f'(u)| \leq C_3 |u|^4$, $\forall u \in \mathbb{R},$

or $f \in C^1(\mathbb{R})$ and $\exists C_3 > 0$ such that $f' \geq -C_3$, $\forall u \in \mathbb{R}$,

$G$ is a single-valued semigroup, the set $\Theta$ is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$ and

$$\Theta = M^+(\mathcal{R})$$
Moreover, in [4, p.106] it is proved that

\[ \Theta = M^+(\mathcal{R}) = M^-(\mathcal{R}). \]  

\[ M^+(\mathcal{R}) \] is the unstable set of \( \mathcal{R} \). We note that under conditions \( \text{(11)} \) attraction takes place in \( H^1_0(\Omega) \). We observe that in this case an equivalent definition of the set \( M^+(\mathcal{R}) \) is the following

\[ M^+(\mathcal{R}) = \{ z : \exists \gamma(\cdot) \in \mathbb{K}, \gamma(0) = z, \; \text{dist}_{L^2(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \; t \to -\infty \}, \]

as for every complete trajectory \( \gamma(\cdot) \in \mathbb{F} \) as in \( \text{(11)} \) we have that the set \( \cup_{t \in \mathbb{R}} \gamma(t) \) is bounded, so that the inclusion \( \gamma(\cdot) \in \mathbb{K} \) follows.

The aim of our paper is to obtain something like \( \text{(12)} \) for \( K^+ \) under the general conditions \( \text{(2)} \). Moreover, taking a more regular set of solutions we will show that the equality \( \text{(12)} \) holds.

3 About some properties of complete trajectories and fixed points of \( m \)-semiflows

We will prove in this section some useful properties of fixed points and complete trajectories for abstract multivalued semiflows.

Consider a complete metric space \( X \) and let

\[ W^+ = C(\mathbb{R}^+; X). \]

Let \( \mathcal{R} \subset W^+ \) be some set of functions such that the following conditions hold:

\( (K1) \) For any \( x \in X \) there exists \( \varphi \in \mathcal{R} \) such that \( \varphi(0) = x \).

\( (K2) \) \( \varphi_\tau(\cdot) = \varphi(\cdot + \tau) \in \mathcal{R} \) for any \( \tau \geq 0 \), \( \varphi(\cdot) \in \mathcal{R} \) (translation property).

Consider also some additional assumptions, which will be needed in order to obtain good properties. Namely:

\( (K3) \) Let \( \varphi_1, \varphi_2 \in \mathcal{R} \) be such that \( \varphi_2(0) = \varphi_1(s) \), where \( s > 0 \). Then the function \( \varphi(\cdot) \), defined by

\[ \varphi(t) = \begin{cases} \varphi_1(t) & \text{if } 0 \leq t \leq s, \\ \varphi_2(t-s) & \text{if } s \leq t, \end{cases} \]

belongs to \( \mathcal{R} \) (concatenation property).

\( (K4) \) For any sequence \( \varphi^n(\cdot) \in \mathcal{R} \) such that \( \varphi^n(0) \to \varphi_0 \) in \( X \), there exists a subsequence \( \varphi^{n_k} \) and \( \varphi \in \mathcal{R} \) such that

\[ \varphi^{n_k}(t) \to \varphi(t), \; \forall t \geq 0. \]

We define the multivalued map \( G : \mathbb{R}^+ \times X \to P(X) \) in the following way:

\[ y \in G(t,x) \text{ if } \exists \varphi \in \mathcal{R} \text{ such that } y = \varphi(t), \; \varphi(0) = x. \]
**Lemma 5** [13, Lemma 9] Let $(K1) - (K2)$ hold. Then $G$ is a multivalued semiflow. Moreover, if $(K3)$ is true, then $G$ is a strict $m$-semiflow.

We define now the concept of fixed point and complete trajectory for $\mathcal{R}$.

**Definition 6** The point $z \in X$ is a fixed point of $\mathcal{R}$, if $\varphi (t) \equiv z \in \mathcal{R}$. The set of all fixed points will be denoted by $\mathcal{R}_0$.

The map $\gamma : \mathbb{R} \to X$ is called a complete trajectory of $\mathcal{R}$ if

$$\gamma(\cdot + h)_{|[0, +\infty)} \in \mathcal{R}, \ \forall h \in \mathbb{R}.$$  

We will show that the fixed points of $\mathcal{R}$ coincide with the stationary points of $G$ under assumptions $(K1) - (K4)$.

**Lemma 7** Let $(K1) - (K2)$ hold. Then $z \in \mathcal{R}_0$ implies $z \in G(t, z)$ for all $t \geq 0$.

Let $(K1) - (K4)$ hold. Then $z \in \mathcal{R}_0$ if and only if $z \in G(t, z)$ for all $t \geq 0$.

**Proof.** When $(K1) - (K2)$ hold, if $z \in \mathcal{R}_0$, it is obvious that $z \in G(t, z)$ for all $t \geq 0$.

Conversely, let $(K1) - (K4)$ hold and let $z \in G(t, z)$ for all $t \geq 0$. This means that for any $T > 0$ there exists $u^T \in \mathcal{R}$ such that $u^T(T) = z$ and $u^T(0) = z$. Consider in the interval $[0, n]$ the $n$-dyadic partition $D_n := \{ j2^{-n} : j = 0, 1, 2, ..., n2^n \}$. In each interval $[j2^{-n}, (j + 1)2^{-n}], j = 0, 1, ..., n2^n - 1$, we consider $u^n_j \in \mathcal{R}$ such that $u^n_j(0) = u^n_j(2^{-n}) = z$. Then we take the concatenation of all these functions

$$u^n(t) = \begin{cases} u^n_j(t - j2^{-n}), & \text{if } j2^{-n} \leq t \leq (j + 1)2^{-n}, \ j = 0, 1, ..., n2^n - 1, \\ \varphi(t - n), & \text{if } t \geq n, \end{cases}$$

where $\varphi \in \mathcal{R}$, with $\varphi(0) = z$, is arbitrary. Thus

$$u^n(t) = z \text{ for all } t \in D_n.$$ 

By conditions $(K3) - (K4)$ we have that $u^n$ belongs to $\mathcal{R}$ and the existence of $u \in \mathcal{R}$ and a subsequence $u^{n_k}$ such that

$$u^{n_k}(t) \to u(t) \text{ in } X \text{ for all } t \geq 0.$$

Let $D = \bigcup_{n \in \mathbb{N}} D_n$. Hence, $u(t) = z$ for all $t \in D$. Since $u \in C([0, +\infty), X)$, we obtain that $u(t) = z$ for all $t \geq 0$, so that $z \in \mathcal{R}_0$. 

We will show further the relation between complete trajectories of $\mathcal{R}$ and $G$.

**Lemma 8** If $(K1) - (K4)$ hold, then the map $\gamma : \mathbb{R} \to X$ is a complete trajectory of $\mathcal{R}$ if and only if

$$\gamma(t + s) \in G(t, \gamma(s)) \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0. \quad (13)$$

When $(K1) - (K2)$ hold, then any complete trajectory of $\mathcal{R}$ satisfies $(13)$. 

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Proof. Under conditions $(K1)-(K2)$ it is obvious that any complete trajectory of $\mathcal{R}$ satisfies (13).

Assume $(K1)-(K4)$. Conversely, let $\gamma(\cdot)$ satisfy (13). Consider in the interval $[0,n]$ the $n$-dyadic partition $D_n \equiv \{j2^{-n} : j = 0, 1, 2, ..., n2^n\}$. Let $t \in \mathbb{R}$ be arbitrary. In each interval $[j2^{-n}, (j+1)2^{-n}]$, $j = 0, 1, ..., n2^n - 1$, we consider $u^n_j \in \mathcal{R}$ such that $u^n_j(0) = \gamma(\tau + j2^{-n})$, $u^n_j(2^{-n}) = \gamma(\tau + (j+1)2^{-n})$. We take the concatenation of all these functions

$$u^n(t) = \left\{ \begin{array}{ll}
u^n_j(t - j2^{-n}), & \text{if } j2^{-n} \leq t \leq (j+1)2^{-n}, j = 0, 1, ..., n2^n - 1, \\
\varphi(t-n), & \text{if } t \geq n,
\end{array} \right.$$ 

where $\varphi \in \mathcal{R}$, with $\varphi(0) = \gamma(\tau + n)$, is arbitrary.

Then $u^n \in \mathcal{R}$ by $(K3)$ and

$$u^n(t) = \gamma(t + \tau) \text{ for all } t \in D_n.$$ 

In view of $(K4)$ there exists $u \in \mathcal{R}$ and a subsequence $u^{nk}$ such that

$$u^{nk}(t) \to u(t) \text{ in } X \text{ for all } t \geq \tau.$$ 

Let $D = \bigcup_{n \in \mathbb{R}} D_n$. Hence, $u(t) = \gamma(t + \tau)$ for all $t \in D$. Since $u \in C([0, +\infty), X)$, we obtain that $u(t) = \gamma(t + \tau)$ for all $t \geq 0$, so that $\gamma(\cdot + \tau) \in \mathcal{R}$. As $\tau \in \mathbb{R}$ is arbitrary, we obtain that $\gamma$ is a complete trajectory of $\mathcal{R}$. \[\square\]

Let $\mathbb{K}$ be the set of all bounded complete trajectories of $\mathcal{R}$. Now we will establish equality (8) in the abstract setting.

Theorem 9 Assume that $(K1)-(K2), (K4)$ hold and that $G$ possesses a compact global attractor $\Theta$. Then

$$\Theta = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}\} = \bigcup_{t \in \mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}\}. \tag{14}$$

Proof. Let $\gamma(\cdot) \in \mathbb{K}$. We note that $B_{\gamma} = \bigcup_{s \in \mathbb{R}} \gamma(s) \subset G(t,B_{\gamma})$ implies (as $B_{\gamma}$ is bounded) that

$$\text{dist}_X(B_{\gamma}, \Theta) \leq \text{dist}_X(G(t,B_{\gamma}), \Theta) \to 0 \text{ as } t \to +\infty,$$

so that $B_{\gamma} \subset \Theta$.

Conversely, let $z \in \Theta$. Since $\Theta \subset G(t,\Theta)$, we have $z \in G(t_n, \Theta)$ with $t_n \to \infty$. Hence, $z = u_{n}(t_n)$, where $u_{n} \in \mathcal{R}$ and $u_{n}(0) \in \Theta$. Consider the functions $v^n_{n}(\cdot) = u_{n}(\cdot + t_n)$, which belong to $\mathcal{R}$. In view of $(K4)$ there exist $v^{0}(\cdot) \in \mathcal{R}$ with $v^{0}(0) = z$ and a subsequence (denoted again by $u_{n}$) such that $v^n_{n}(t) \to v^{0}(t)$ in $X$ for all $t \geq 0$. Since, $v^{0}(t) = \lim_{n \to \infty} u^n(t + t_n)$, we obtain that $v^{0}(t) \in \Theta$ for all $t \geq 0$. Let us take a sequence $t_{j} \to +\infty$ such that $t_{0} = 0 < t_{j} < t_{j+1}$ for any $j \in \mathbb{N}$. Consider now the sequence of functions $v^{1}_{n}(\cdot) = u_{n}(\cdot + t_{n} - t_{j})$, which belong to $\mathcal{R}$. By (7) it is clear that (up to a subsequence) $v^{1}_{n}(0)$ is convergent in $X$. As before there exist then $v^{1}(\cdot) \in \mathcal{R}$ and a subsequence (denoted again by $v_{n}$) such that $v^{1}_{n}(t) \to v^{1}(t)$ in $X$ for all $t \geq 0$. Also, $v^{1}(t) \in \Theta$ and $v^{1}(t + t_{j}) = v^{0}(t)$ for all $t \geq 0$. In this way we can define inductively a
sequence \(v^j(\cdot) \in \mathcal{R}\) such that \(v^j(t) \in \Theta\) and \(v^j(t + t_j - t_{j-1}) = v^{j-1}(t)\) for all \(t \geq 0\) and \(j \in \mathbb{N}\). We define the function \(v(t)\) by taking for all \(t \in \mathbb{R}\) the common value at \(t\) of the functions \(v^j(\cdot)\). Namely, for any \(j\) such that \(t \geq -t_j\) we put

\[
v(t) = v^j(t + t_j).
\]

Then \(v(\cdot)\) is a complete trajectory of \(\mathcal{R}\), \(v(0) = z\) and \(v(t) \in \Theta\) for all \(t \in \mathbb{R}\). Hence, \(v(\cdot) \in \mathbb{K}\).

The first equality is proved. The second one is obvious from the definition of a complete trajectory. ■

The last theorem is also true if we replace \((K4)\) by \((K3)\).

**Theorem 10** Assume that \((K1) - (K3)\) hold and that \(G\) possesses a compact global attractor \(\Theta\). Then

\[
\Theta = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}\} = \bigcup_{t \in \mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}\}.
\]

**Proof.** As in the proof of Theorem 9 we obtain that \(B_\gamma = \bigcup_{t \in \mathbb{R}} \gamma(t) \subset \Theta\) for any \(\gamma(\cdot) \in \mathbb{K}\).

We note that Lemma 8 implies that \(G\) is strict, and then by

\[
G(t, \Theta) \subset G(t, G(\tau, \Theta)) = G(t + \tau, \Theta) \to \Theta \text{ as } \tau \to +\infty
\]

we have that \(G(t, \Theta) \subset \Theta\) for any \(t \geq 0\), so that \(\Theta\) is strictly invariant.

We take an arbitrary \(z \in \Theta\). We take \(\psi^0(\cdot) \in \mathcal{R}\) such that \(\psi^0(0) = z\). Since \(\Theta\) is strictly invariant, we have \(\psi^0(t) \in G(t, z) \subset \Theta\) for all \(t \geq 0\). Let us take a sequence \(t_j \to +\infty\) such that \(t_0 = 0 < t_j < t_{j+1}\) for any \(j \in \mathbb{N}\). From \(z \in G(t_1, \Theta)\) there exists \(z_1 \in \Theta\) and \(\varphi^1(\cdot) \in \mathcal{R}\) such that \(z = \varphi^1(t_1)\) and \(\varphi^1(0) = z_1\). By \((K3)\) we can concatenate \(\varphi^1\) and \(\varphi^0\) and obtain a \(\psi^1(\cdot) \in \mathcal{R}\) satisfying \(\psi^1(t_1) = z, \psi^1(t) \in \Theta,\) for all \(t \geq 0\), and \(\psi^1(t + t_1) = \psi^0(t)\), for all \(t \geq 0\). Inductively, we can define for any \(j \geq 1\) a \(\psi^j(\cdot) \in \mathcal{R}\) satisfying \(\psi^j(t + t_j - t_{j-1}) = \psi^{j-1}(t)\) and \(\psi^j(t) \in \Theta\). For any \(t \in \mathbb{R}\) let \(\psi(t)\) be the common value of \(\psi^j(t + t_j)\) for \(t \geq -t_j\). Then \(\psi(\cdot) \in \mathbb{K}\) and \(\psi(0) = z\).

The second one is obvious from the definition of a complete trajectory. ■

**Remark 11** The map \(\gamma : \mathbb{R} \to X\) is a complete trajectory of \(G\) if \((13)\) holds. Let \(\mathbb{K}_G\) be the set of all bounded complete trajectories of \(G\). If \((K1) - (K4)\) hold, then by Lemma 8 we have \(\mathbb{K}_G = \mathbb{K}\) and equality \((13)\) is the same as

\[
\Theta = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_G\} = \bigcup_{t \in \mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_G\}.
\]

If either \((K3)\) or \((K4)\) fails to be true, then we can say only that \(\mathbb{K} \subset \mathbb{K}_G\). Nevertheless, if either \((K3)\) or \((K4)\) holds, then \((13)\) is still true. Indeed, Theorems 4, 10 imply that

\[
\Theta = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}\} \subset \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_G\}
\]

and as in the proof of Theorem 9 we obtain that \(B_\gamma = \bigcup_{t \in \mathbb{R}} \gamma(t) \subset \Theta\) for any \(\gamma(\cdot) \in \mathbb{K}_G\), so that \((13)\) holds.
If both \((K3)\) and \((K4)\) fail, then equality \((15)\) can be obtained under some assumptions on \(G\). Namely, in \([12, \text{Lemmas } 2.25 \text{ and } 2.27]\) it is shown that \((15)\) holds if either \(G\) is strict or the following condition is true: for any sequence \(\varphi^n : \mathbb{R}^+ \to X\) satisfying \((13)\), for \(s, t \geq 0\), and \(\varphi^n (0) \to \varphi_0\) in \(X\), there exists a subsequence and \(\varphi : \mathbb{R}^+ \to X\) satisfying \((13)\) for \(s, t \geq 0\) such that
\[\varphi^{n_k} (t) \to \varphi (t) \text{ for any } t \geq 0.\]

We shall apply these results to the set \(K^+\) generated by the weak solutions of \((1)\). We note that in view of Lemmas 3 and 15 in \([14]\) (see also \([12, \text{Theorems } 3.11 \text{ and } 3.18]\)) assumptions \((K1) - (K4)\) are satisfied for \(K^+\). Moreover, the sets of stationary points \(R\) of problem \((1)\) coincides with the set \(R_R = R_{K^+}\) given in Definition 6.

**Lemma 12** Let \((2)\) hold. Then \(R = R_{K^+}\).

**Proof.** Let \(u_0 \in R_{K^+}\). Then \(u (t) \equiv u_0\) belongs to \(K^+\). Therefore, \(u (\cdot)\) satisfies \((4)\), so that \((2)\) holds. Conversely, let \(v \in R\). Then it is obvious that \(v (t) \equiv v_0\) is a weak solution, so that it belongs to \(K^+\).

Then Lemmas 7, 8 and Theorem 9 imply the following result.

**Theorem 13** Let \((2)\) hold. Then the set of weak solutions \(K^+\) of \((1)\) satisfies:

1. \(z \in R\) if and only if \(z \in G(t, z)\) for all \(t \geq 0\).
2. The map \(\gamma : R \to L^2 (\Omega)\) is a complete trajectory of \(K^+\) if and only if \((13)\) holds.
3. The compact global attractor \(\Theta\) of \(G\) satisfies \((8)\).

4 Structure of the global attractor for weak solutions

In this section we will study the structure of the global attractor generated by weak solutions of equation \((1)\).

First, let us prove some regularity properties of the stationary points.

**Lemma 14** Under conditions \((2)\) the set \(R\) of solutions of the problem
\[
\begin{cases}
-\Delta u + f(u) = h, & x \in \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]  
(16)
is nonempty, compact in \(L^2 (\Omega)\), and bounded in \(H^1_0 (\Omega) \cap H^2 (\Omega)\).

**Proof.** Due to \(f(u)u \geq -C_2\), for all \(u \in \mathbb{R}\), the operator \(L = -\Delta u + f(u) : H^1_0 (\Omega) \to H^{-1} (\Omega)\) is coercive, \(-\Delta : H^1_0 (\Omega) \to H^{-1} (\Omega)\) is monotone and continuous. Also, from \((2)\) we can obtain that \(f : H^1_0 (\Omega) \to H^{-1} (\Omega)\) is strongly continuous (i.e. \(u_n \to u\) weakly in \(H^1_0 (\Omega)\) implies \(f (u_n) \to f (u)\) in \(H^{-1} (\Omega)\)). Hence, \(L\) is pseudomonotone, coercive and bounded, so that a classical theorem of Brezis (see \([23]\)) implies that \(R \neq \emptyset\). It is clear...
also that it is weakly compact in \( H_0^1(\Omega) \) and therefore compact in \( L^2(\Omega) \). We remark that \( \mathcal{R} \) is bounded in \( H_0^1(\Omega) \), as \( L : H_0^1(\Omega) \to H^{-1}(\Omega) \) is coercive. Hence, the equality

\[
-\Delta u + f(u) = h
\]

and (2), together with the continuous imbedding \( H_0^1(\Omega) \subset L^6(\Omega) \), imply

\[
\|\Delta u\|_2^2 \leq C \text{ for all } u \in \mathcal{R}.
\]

Thus, \( \mathcal{R} \) is bounded in \( H_0^1(\Omega) \cap H^2(\Omega) \).

For initial data in \( H_0^1(\Omega) \) we shall obtain the existence of more regular solutions for (1).

**Lemma 15** Assume that (2) holds. Let \( u_0 \in H_0^1(\Omega) \). Then there exists at least one weak solution \( u \) of (1) such that \( u(0) = u_0, u \in C([0, +\infty); H_0^1(\Omega)) \) and

\[
\|u(t)\|_{H_0^1(\Omega)}^2 \leq C \left(1 + \|u_0\|_{H_0^1(\Omega)}^4\right), \forall t \geq 0,
\]

\[
\int_0^{+\infty} \|u_t(s)\|^2 ds \leq C \left(1 + \|u_0\|_{H_0^1(\Omega)}^4\right),
\]

for some \( C > 0 \).

**Proof.** We take as in [6, p.281] the Galerkin approximations using the basis of eigenfunctions \( \{w_j(x), j \in \mathbb{N}\} \) of the Laplace operator with Dirichlet boundary conditions. Let \( X_m = \{w_1, \ldots, w_m\} \) and let \( P_m \) be the orthogonal projector from \( L^2(\Omega) \) onto \( X_m \). Then \( u_m(t, x) = \sum_{j=1}^m a_{j,m}(t) w_j(x) \) will be a solution of the system of ordinary differential equations

\[
\frac{d u_m}{dt} = P_m \Delta u_m - P_m f(u_m) + P_m h, \quad u_m(0) = P_m u_0.
\]

It is proved in [6, p.281] that passing to a subsequence \( u_m \) converges to a weak solution \( u \) of (1) weakly star in \( L^\infty(0, T; L^2(\Omega)) \), weakly in \( L^4(0, T; L^4(\Omega)) \) and weakly in \( L^2(0, T; H^1_0(\Omega)) \) for all \( T > 0 \). Also, \( u_{mt} \to u_t \) weakly in \( L^s(0, T; H^{-s}(\Omega)) \) for some \( s > 0 \).

Multiplying (19) by \( u_{mt} \) we get

\[
\frac{d}{dt} \left(\|u_m\|_{H_0^1(\Omega)}^2 + 2(F(u_m), 1) - 2(h, u_m)\right) + 2\|u_{mt}\|^2 = 0,
\]

so by (3),

\[
\|u_m(t)\|_{H_0^1(\Omega)}^2 + 2 \int_0^t \|u_{mt}(s)\|^2 ds
\]

\[
\leq \|u_m(0)\|_{H_0^1(\Omega)}^2 + R_1 \|u_m(0)\|_{L^4(\Omega)}^4 + 2\|h\|\|u_m(t)\| + 2\|h\|\|u_0\| + R_2.
\]

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So from the Poincaré inequality we obtain

\[ \frac{1}{2} \| u_m(t) \|_{H_0^1(\Omega)}^2 + 2 \int_0^t \| \frac{d}{ds} u_m(s) \|_{H_0^1(\Omega)}^2 \, ds \leq R_3 \| u_m(0) \|_{H_0^1(\Omega)}^4 + R_4, \]

where \( R_3 > 0. \)

By the choice of the special basis we have that \( u_m(0) \to u_0 \) in \( H_0^1(\Omega). \) Then we have

\[ \int_0^t \| \frac{d}{ds} u(s) \|_{H_0^1(\Omega)}^2 \, ds \leq \liminf_{m \to \infty} \int_0^t \| \frac{d}{ds} u_m(s) \|_{H_0^1(\Omega)}^2 \, ds \leq R_5 \left( \| u_0 \|_{H_0^1(\Omega)}^4 + 1 \right), \]

so that (18) holds and \( u_{nt} \to u_t \) weakly in \( L^2(0, T; L^2(\Omega)). \) Thus from the Ascoli-Arzelà theorem \( \{ u_m \} \) is pre-compact in \( C([0, T]; L^2(\Omega)). \) and then \( u_m \to u \) in \( C([0, T]; L^2(\Omega)). \)

On the other hand, for any \( t \geq 0 \) up to a subsequence \( u_{m_n}(t) \to a \) weakly in \( H_0^1(\Omega). \) But \( u_{m_n}(t) \to u(t) \) in \( L^2(\Omega), \) so that \( a = u(t) \) and

\[ \| u(t) \|_{H_0^1(\Omega)}^2 \leq \liminf_{m \to \infty} \| u_{m_n}(t) \|_{H_0^1(\Omega)}^2 \leq R_5 \left( \| u_0 \|_{H_0^1(\Omega)}^4 + 1 \right), \]

so that (17) holds.

As \( u \in L^\infty(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \) we have \( u \in C([0, T]; H_{0w}^1(\Omega)), \) where \( H_{0w}^1(\Omega) \) is the space \( H_0^1(\Omega) \) with the weak topology. Moreover, the equality \( \Delta u = u_t + f(u) - h \) and (2), (17), (18) imply that \( u \in L^2_{loc}(0, +\infty; D(A)). \) Thus, by standard results [21, p.102], we obtain that \( u \in C([0, +\infty), H_0^1(\Omega)). \)

Now we are ready to prove the main result of this section about the structure of the global attractor. From now on for any \( A \subset L^2(\Omega) \) we will denote by \( \overline{A} \) its closure in \( L^2(\Omega). \)

**Theorem 16** Under conditions (2) for the global attractor \( \Theta \) it holds

\[ \Theta = \overline{M^-(\mathcal{R})}. \] (20)

If, additionally, for any \( u_0 \in H_0^1(\Omega) \) problem (2) has a unique weak solution with \( u(0) = u_0, \) then \( \Theta \) is bounded in \( H_0^1(\Omega) \) and

\[ \Theta = M^+(\mathcal{R}) = M^-(\mathcal{R}). \] (21)

**Proof.** First of all \( \overline{\Theta \cap H_0^1(\Omega)} = \Theta, \) as for any \( \gamma \in \mathbb{K} \) we have that \( \gamma(t) \in H_0^1(\Omega) \) for a.a. \( t \in \mathbb{R} \) and \( \gamma : \mathbb{R} \to L^2(\Omega) \) is continuous. Let us prove that \( \Theta \cap H_0^1(\Omega) = M^-(\mathcal{R}). \)

Let \( z \in \Theta \cap H_0^1(\Omega). \) Due to Theorem 4 there exists \( \gamma \in \mathbb{K} \) such that \( \gamma(0) = z. \) Due to Lemma 15 there exist a weak solution \( u(\cdot) \) of (1) satisfying (17), (18) and \( u(0) = z. \) Then (K3) implies that

\[ \tilde{\gamma}(t) = \begin{cases} \gamma(t), & t < 0 \\ u(t), & t \geq 0 \end{cases}, \quad \tilde{\gamma}(0) = z, \]

belongs to \( \mathbb{K}. \)
Let us prove that \( \text{dist}_{L^2(\Omega)}(u(t), \mathcal{R}) \to 0 \), as \( t \to +\infty \).

Let us take arbitrary \( t_n \to \infty \). From (18),

\[
\int_{t_n-T}^{t_n} \|u_t(s)\|^2 ds \to 0, \text{ as } n \to \infty, \forall T > 0.
\]

So there exists \( t'_n \in [t_n - T, t_n] \) such that

\[
\|u_t(t'_n)\| \to 0, \quad n \to \infty.
\]

From (17) up to a subsequence \( u(t'_n) \to \tilde{u} \) weakly in \( H^1_0(\Omega) \). Then \( u(t'_n) \to \tilde{u} \) in \( L^2(\Omega) \), so that \( u(t'_n, x) \to \tilde{u}(x) \) a.e., and from (17) p.12, Lemma 1.3 we have \( f(u(t'_n)) \to f(\tilde{u}) \) weakly in \( L^2(\Omega) \).

The following equality

\[
u_t(t) = \Delta u(t) - f(u(t)) - h
\]

in \( H^{-1}(\Omega) \) is true for a.a. \( t \). We can take \( t'_n \) from this set of full measure and then we have

\[
\Delta u(t'_n) - h = f(u(t'_n)) + u_t(t'_n) \to f(\tilde{u}) \text{ weakly in } L^2(\Omega) \text{ and then in } H^{-1}(\Omega).
\]

From this \( u(t'_n) \to \tilde{u} \) in \( H^1_0(\Omega) \) and

\[
\Delta \tilde{u} - f(\tilde{u}) = h \text{ in } H^{-1}(\Omega), \text{ that is, } \tilde{u} \in \mathcal{R}.
\]

Let us show that up to a subsequence \( \text{dist}_{L^2(\Omega)}(u(t_n), \mathcal{R}) \to 0 \), as \( n \to \infty \). From (17) \( u(t_n) \to a \) weakly in \( H^1_0(\Omega) \). Also

\[
u(t_n) \in G(t_n-t'_n, u(t'_n))
\]

implies

\[
u(t_n) = \varphi_n(t_n-t'_n), \quad \varphi_n(0) = u(t'_n), \quad \varphi_n \in K^+.
\]

As \( u(t'_n) \to \tilde{u} \) in \( L^2(\Omega) \), \( t_n-t'_n \to \tau \in [0, T] \), by Theorem 3.11 in (12) (see also (14) Lemma 2) passing to a subsequence we have

\[
\varphi_n(t_n-t'_n) \to \varphi(\tau) \in G(\tau, \tilde{u}),
\]

that is,

\[
a \in G(\tau, \tilde{u}).
\]

We take in the previous arguments \( T_k \downarrow 0 \). Then for any \( k \geq 1 \) there exist \( \tilde{u}_k \in \mathcal{R} \), \( \tau_k \in [0, T_k] \) such that

\[
a \in G(\tau_k, \tilde{u}_k).
\]

Since \( \mathcal{R} \) is compact in \( L^2(\Omega) \), up to a subsequence \( \tilde{u}_k \to \tilde{u} \in \mathcal{R} \), as \( \tau_k \to 0 \). Thus by Theorem 3.11 in (12) we have \( a \in G(0, \tilde{u}) \) and then \( a = \tilde{u} \). By Theorem (13) \( a \in \mathcal{R} \).

So, from this we easy deduce that

\[
\text{dist}_{L^2(\Omega)}(u(t), \mathcal{R}) \to 0, \quad t \to +\infty.
\]
Hence, (20) is proved.

Now let (1) have a unique weak solution for every $u_0 \in H_0^1(\Omega)$ (for example, it is true if $(f(u) - f(v))(u - v) \geq -C|u - v|^2$, $\forall u, v \in \mathbb{R}$, for some $C > 0$).

Then for any $z \in \Theta$ we have $z = \gamma(0) = G(t, \gamma(-t))$ and $\gamma(\tau) \in H_0^1(\Omega)$ for a.a. $\tau$. So $\gamma(t) = G(t + \tau, \gamma(-\tau))$, $\forall t \geq 0$, and if we repeat for the point $\gamma(-\tau) \in H_0^1(\Omega)$ all the previous arguments, we obtain $z = \gamma(0) \in H_0^1(\Omega)$ (by Lemma 15) and $\text{dist}_{L^2(\Omega)}(\gamma(t), \mathcal{R}) \to 0$, $t \to +\infty$. Then $\Theta \subset H_0^1(\Omega)$ and $\Theta = M^+(\mathcal{R})$.

Moreover, $\Theta$ is bounded in $H_0^1(\Omega)$. Indeed, for $z \in \Theta$, $z = \gamma(0) \in H_0^1(\Omega)$ and from (17) and the uniqueness of the solution we get

$$\|z\|_{H_0^1(\Omega)}^2 \leq C \left(1 + \|\gamma(\tau)\|_{H_0^1(\Omega)}^4\right) \quad \forall \tau \leq 0.$$ 

For every $t \geq \tau$, by standard estimates from (5), $\gamma(\cdot)$ satisfies

$$\int_{\tau}^{t} \|\gamma(s)\|^2_{H_0^1(\Omega)} ds \leq \|\gamma(\tau)\|^2 + \tilde{C}(t - \tau).$$

Since $\Theta$ is bounded in $L^2(\Omega)$, for some $\tau' \in (-1, 0)$ we have $\|\gamma(\tau')\|_{H_0^1(\Omega)} \leq \tilde{K}$, where $\tilde{K}$ does not depend on $\gamma$. So, $\|z\|_{H_0^1(\Omega)}^2 \leq C(1 + \tilde{K}^4)$.

Let us prove $\Theta = M^+(\mathcal{R})$. Let $z \in \Theta$. Then $z = \gamma(0), \gamma \in \mathcal{K}$, and from the uniqueness of the solution and Lemma 15 we have

$$\|\gamma(t)\|^2_{H_0^1(\Omega)} \leq C \left(1 + \|\gamma(\tau)\|_{H_0^1(\Omega)}^4\right),$$

$$\int_{\tau}^{t} \|\gamma(t)\|^2_{H_0^1(\Omega)} ds \leq C \left(1 + \|\gamma(\tau)\|_{H_0^1(\Omega)}^4\right), \forall t \geq \tau. \quad (22)$$

$\Theta$ is bounded in $H_0^1(\Omega)$, so from (22) there exists $K > 0$ such that

$$\|\gamma(t)\|_{H_0^1(\Omega)} \leq K \forall t \leq 0,$$

$$\int_{-\infty}^{0} \|\gamma(t)\|^2 ds \leq K. \quad (23)$$

After that we can repeat the previous arguments on $(-\infty, 0)$ and obtain that

$$\text{dist}_{L^2(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \quad t \to -\infty.$$ 

The theorem is proved. \hfill \blacksquare

**Remark 17** Even in the case of uniqueness we cannot use the Lyapunov function method as in [22], because we know nothing about the boundedness of $\Theta$ in $H^2(\Omega) \cap H_0^1(\Omega)$. However, it is possible to use the Lyapunov function if the attractor is compact in $H_0^1(\Omega)$, as we will see in the next section.

**Remark 18** Under condition (2) we also have $M^+(\mathcal{R}) \subset \Theta$.  

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Lemma 19 Let (2) hold. If $\mathfrak{R} = \{ z_i \}_{i=1}^n$, then $M^\pm(\mathfrak{R}) = \bigcup_{i=1}^n M^\pm(z_i)$.

Proof. Let $y \in M^+(\mathfrak{R})$. Then there exists $\gamma \in K$ such that $\gamma(0) = y$, $\text{dist}_{L^2(\Omega)}(\gamma(t), \mathfrak{R}) \to 0$, as $t \to -\infty$. For any $\tau < 0$ the set $\Gamma_\tau = \bigcup_{t \leq \tau} z(t)$ is connected and compact (as $\Gamma_\tau \subset \Theta$).

So, $\bigcap_{\tau < 0} \Gamma_\tau$ is connected and compact. As for all $\varepsilon > 0$ there exists $T < 0$ such that $\gamma(t) \in \mathcal{O}_\varepsilon(\mathfrak{R}) \forall t \leq T$, we have $\bigcap_{\tau < 0} \Gamma_\tau \subset \mathfrak{R}$. Then

$$\bigcap_{\tau < 0} \Gamma_\tau = \{ z_{i_0} \} \subset \mathfrak{R},$$

so that

$$\gamma(t) \to z_{i_0}, \text{ as } t \to -\infty \text{ and } y \in M^+(z_{i_0}).$$

For $M^-(\mathfrak{R})$ the proof is similar. $\blacksquare$

We finish this section with a regularity result of the global attractor in the space $L^\infty(\Omega)$.

Lemma 20 Under conditions (2) and $h \in L^\infty(\Omega)$ the set $\Theta$ is bounded in $L^\infty(\Omega)$.

Proof. In fact, the arguments are the same as in [22] p.321].

Let $\varphi_+ = \max\{ \varphi, 0 \}$. It is known that for any $u \in \mathcal{D}(\tau, T; H^1_0(\Omega))$, $\eta \in C_0^\infty(\tau, T)$,

$$\int_{\tau}^{T} (u_t, u^+) \eta dt = -\frac{1}{2} \int_{\tau}^{T} \| u^+ \|^2 \eta dt. \quad (24)$$

For an arbitrary complete trajectory of (1) we have $u \in L^2(\tau, T; H^1_0(\Omega)) \cap L^4(\tau, T; L^4(\Omega))$, $u_t \in L^2(\tau, T; H^{-1}(\Omega)) + L^4(\tau, T; L^4(\Omega))$.

So, by standard regularization we find functions $u_n \in \mathcal{D}(\tau, T; H^1_0(\Omega))$ such that

$$u_n \to u \text{ in } L^2(\tau, T; H^1_0(\Omega)) \cap L^4(\tau, T; L^4(\Omega))$$

$$u_{nt} \to u_t \text{ in } L^2(\tau, T; H^{-1}(\Omega)) + L^4(\tau, T; L^4(\Omega)).$$

As $u_n^+ \to u^+$ in $L^2(\tau, T; H^1_0(\Omega)) \bigcap L^4(\tau, T; L^4(\Omega))$, we can pass to the limit in (24) and obtain that (24) is true for every solution of (1) on $[\tau, T]$. Then putting $g = f - h$ for any $M > 0$ we have

$$\frac{1}{2} \frac{d}{dt} \| (u - M)^+ \|^2 + \| (u - M)^+ \|_{H^1_0(\Omega)}^2 + \int_{\Omega} g(x, u)(u - M)^+ dx = 0.$$
From (2) and $h \in L^\infty(\Omega)$ for a.a. $x \in \Omega$ and $u \in \mathbb{R}$, 
\[ \tilde{\alpha}|u|^4 - \tilde{C}_2 \leq g(u)u \leq \tilde{C}_1|u|^4 + \tilde{C}_1, \]
where $\tilde{\alpha}$ does not depend on $u$, $x$.

If $u \leq M$, then $g(u)(u - M)^+ = 0$.
If $u > M$, then 
\[ g(x,u)(u - M)^+ = g(x,u)\frac{u(M - M)^+}{u} = g(x,u)u(1 - \frac{M}{u}) \]
\[ \geq (\tilde{\alpha}u^4 - \tilde{C}_2)(1 - \frac{M}{u}) \geq (\tilde{\alpha}M^4 - \tilde{C}_2)(1 - \frac{M}{u}) \]
and if we choose $M = (\tilde{C}_2^\frac{1}{4})^\frac{1}{\tilde{\alpha}}$, then $g(x,u)(u - M)^+ \geq 0$ a.e.

Then 
\[ \frac{d}{dt}\|(u - M)^+\|^2 + 2\|(u - M)^+\|^2_{H^1_0(\Omega)} \leq 0 \]
and for all $t > \tau$, 
\[ \|(u - M)^+(t)\|^2 \leq \|(u - M)^+(\tau)\|^2e^{-2\lambda_1(t - \tau)}. \]  \hspace{1cm} (25)

If $u(\cdot) \in \mathbb{K}$, then from (25) taking $\tau \to -\infty$ we obtain $u(x,t) \leq M$, $\forall t \in \mathbb{R}$, for a.a. $x \in \Omega$.

In the same way we will have $u(x,t) \geq M$ (using $(u - M)^-$).

Then 
\[ \text{ess sup}_{x \in \Omega} |z(x)| \leq M, \forall z \in \Theta. \]

\[ \blacksquare \]

**Remark 21** The set $M^+(\mathfrak{R})$ can be used in order to study properties of the global attractor as the fractal dimension. Let us consider an example which shows that a finite estimate of the fractal dimension of the global attractor for problem (1) is not preserved under small, but unregular perturbations (even in the single-valued case).

Let $h(x) \equiv 0$, $f_k(u) = u^3 - k^{-\frac{3}{4}}\sin(k \cdot u)$. Then for any $k \in \mathbb{Z}$, $f_k$ satisfies (2) with constants which do not depend on $k$. In this case $z = 0 \in \mathfrak{R}$, $G_k(t,u_0) = S_k(t)u_0$ is a single-valued semigroup and due to [22, p.496] $z = 0$ is a hyperbolic point if $\lambda_i \neq k^{\frac{3}{4}}$ ($\lambda_i$ are the eigenvalues of $-\Delta$), $M^+(0) \subset \Theta$ and $M^+(0)$ is a smooth manifold with dimension $n_k$, where $n_k$ is the number of eigenvalues of $S_k(t)$ in $\{ |\lambda| < 1 \}$, that is, the number of eigenvalues of $-\Delta$ which satisfy the inequality $\lambda_i < k^{\frac{3}{4}}$.

Thus, if $k \to \infty$, then for the attractors $\Theta_k$ we obtain 
\[ \dim \Theta_k \geq \dim M^+(0) = n_k \to \infty, \quad k \to \infty. \]

So, under conditions (2), we can have arbitrary large dimension of the global attractor, although $f_k(u)$ is a small perturbation of $f_0(u) = u^3$, for which it is easy to see that $\mathfrak{R} \equiv \{0\}$, so 
\[ \dim \Theta_0 = 0. \]
5 Existence and structure of the global attractor for regular solutions

We shall prove in this section that the equality

\[ \Theta = M^- (\mathcal{R}) = M^+ (\mathcal{R}) \]

holds if we consider more regular solutions than in the previous section.

The function \( u \in L^2_{\text{loc}}(0, +\infty; H^1_0(\Omega)) \cap L^4_{\text{loc}}(0, +\infty; L^4(\Omega)) \) is called a regular solution of (1) on \((0, +\infty)\) if for all \( T > 0, v \in H^1_0(\Omega) \) and \( \eta \in C_0^\infty(0, T) \) we have

\[ - \int_0^T (u, v) \eta dt + \int_0^T \left( (u, v)_{H^1_0(\Omega)} + (f(u), v) - (h, v) \right) \eta dt = 0, \]  

(26)

and

\[ u \in L^\infty (\varepsilon, T; H^1_0(\Omega)), \]  

(27)

\[ u_t \in L^2 (\varepsilon, T; L^2(\Omega)), \quad \forall \ 0 < \varepsilon < T. \]  

(28)

On the other hand, from (2) we get

\[ \int_r^T \int_\Omega |f(u)|^2 \ dx \, dt \leq K \int_r^T \left( 1 + \|u(t)\|_{H^1_0(\Omega)}^6 \right) \, dt. \]

Then the equality \( \Delta u = u_t + f(u) - h \) and (27)-(28) imply that

\[ u \in L^2 (\varepsilon, T; D(A)) \]

(29)

for any regular solution \( u \).

**Theorem 22** Let (2) hold. For any \( u_0 \in L^2 (\Omega) \) there exists at least one regular solution of (1) such that \( u(0) = u_0 \). Moreover, there exist \( R_i > 0 \) such that every regular solution with \( u(0) = u_0 \in L^2 (\Omega) \) satisfies

\[ \|u(t + r)\|^2_{H^1_0(\Omega)} \leq R_1 \left( \frac{e^{-\lambda_1 t} \|u_0\|^2 + 1}{r} + 1 + r \right) e^r, \]

(30)

\[ \|u(t)\|^2 \leq e^{-\lambda_1 t} \|u_0\|^2 + R_2, \]

(31)

\[ \int_r^T \|u_t\|^2 \, dt \leq R_3 \left( \frac{\|u_0\|^2 + 1}{r} + 1 + r \right) e^r, \]

(32)

\[ \int_r^T \|\Delta u\|^2 \, dt \leq R_4 (T - r + 1) \left( \frac{\|u_0\|^2 + 1}{r} + 1 + r \right)^3 e^{3r}, \]

(33)

for all \( 0 \leq t < t + r < +\infty \). Thus,

\[ u \in C \left( (0, +\infty), H^1_0(\Omega) \right), \]

(34)
\[
\frac{d}{dt} \|u\|_{H^1_0(\Omega)}^2 = 2 (-\Delta u, u_t) \text{ for a.a. } t > 0.
\] (35)

Moreover, the following energy equality holds
\[
E(u(t)) + 2 \int_s^t \|u_r\|^2 dr = E(u(s)), \text{ for all } t \geq s > 0,
\] (36)

where \(E(u(t)) = \|u(t)\|_{H^1_0(\Omega)}^2 + 2 (F(u(t)), 1) - 2 (h, u(t))\).

**Proof.** Let \(v_0 \in H^1_0(\Omega)\) be arbitrary. Then by Lemma 15 there exists a solution \(v(\cdot) \in C([0, +\infty), H^1_0(\Omega))\) such that
\[
\|v(t)\|_{H^1_0(\Omega)}^2 \leq C \left(1 + \|v_0\|_{H^1_0(\Omega)}^4\right), \forall t \geq 0,
\] (37)
\[
\int_0^{+\infty} \|v_t(s)\|^2 ds \leq C \left(1 + \|v_0\|_{H^1_0(\Omega)}^4\right).
\] (38)

It follows by (2) that
\[
\|f(v(t))\|^2 dt \leq \int_\Omega C_1 \left(1 + |u(t, x)|^3\right)^2 dx \leq K_1 \left(1 + \|v(t)\|_{H^1_0(\Omega)}^6\right) \leq K_2 \left(1 + \|v_0\|_{H^1_0(\Omega)}^{12}\right).
\] (39)

Hence, the equality \(\Delta v = v_t + f(v) - h\) implies that \(v \in L^2_{loc}(0, +\infty; D(A))\). Thus, by standard results [21, p.102], we obtain that
\[
\frac{d}{dt} \|v\|_{H^1_0(\Omega)}^2 = 2 (-\Delta v, v_t) \text{ for a.a. } t > 0.
\] (40)

Also, it is not difficult to show by regularization that \((F(v(t)), 1)\) is absolutely continuous and
\[
\frac{d}{dt} (F(v(t)), 1) = (v_t, f(v(t))) \text{ for a.a. } t > 0.
\] (41)

Let \(u^n_0 \in H^1_0(\Omega)\) be a sequence such that \(u^n_0 \to u_0\) in \(L^2(\Omega)\) and let \(u^n(\cdot)\) be a solution of \((1)\) with \(u^n(0) = u^n_0\) satisfying (37)-(38). We multiply \((1)\) by \(u^n_t\) and using (40), (41) we obtain
\[
2 \|u^n_t\|^2 + \frac{d}{dt} \left(\|u^n\|_{H^1_0(\Omega)}^2 + 2 (F(u^n), 1) - 2 (h, u^n)\right) = 0 \text{ for a.a. } t \in (0, T).
\] (42)

On the other hand, multiplying \((1)\) by \(u^n\) and using (2) it is standard to obtain that \(u^n\) satisfy
\[
\frac{d}{dt} \|u^n\|^2 + \lambda_1 \|u^n\|^2 + \|u^n\|_{H^1_0(\Omega)}^2 + \alpha \|u^n\|_{L^4(\Omega)}^4 \leq K_3 + \|h\|^2.
\] (43)

By Gronwall’s lemma we obtain
\[
\|u^n(t)\|^2 \leq e^{-\lambda_1 t} \|u^n_0\|^2 + \frac{1}{\lambda_1} \left(K_3 + \|h\|^2\right).
\] (44)
Thus integrating (43) over \((t, t + r)\) with \(r > 0\) we have
\[
\|u^n(t + r)\|^2 + \int_t^{t + r} \|u^n\|^2_{H^1_0(\Omega)} \, ds + \alpha \int_t^{t + r} \|u^n\|^4_{L^4(\Omega)} \, ds \\
\leq \|u^n(t)\|^2 + r \left( K_3 + \|h\|^2 \right) \\
\leq e^{-\lambda_1 t} \|u^n_0\|^2 + \left( \frac{1}{\lambda_1} + r \right) \left( K_3 + \|h\|^2 \right).
\]

Then by (3),
\[
\int_t^{t + r} \left( \|u^n\|^2_{H^1_0(\Omega)} + 2 \left( F(u^n(s)), 1 \right) - 2(h, u^n) \right) \, ds \\
\leq \int_t^{t + r} \|u^n\|^2_{H^1_0(\Omega)} \, ds + K_4 \int_t^{t + r} \int_\Omega \left( 1 + |u^n|^4 \right) \, dx \, ds + r \|h\|^2 + \int_t^{t + r} \|u^n\|^2 \, ds \\
\leq K_5 \left( e^{-\lambda_1 t} \|u^n_0\|^2 + \left( \frac{1}{\lambda_1} + r \right) \left( 1 + \|h\|^2 \right) \right) \\
\leq K_6 \left( e^{-\lambda_1 t} \|u^n_0\|^2 + r + 1 \right), \tag{46}
\]
for all \(n\) and \(t \geq 0\). Also, by (42), (3) and
\[-2(h, u^n) \geq -\frac{4}{\lambda_1} \|h\|^2 - \lambda_1 \|u^n\|^2\]
we obtain
\[
\frac{d}{dt} \left( \|u^n\|^2_{H^1_0} + 2 \left( F(u^n), 1 \right) - 2(h, u^n) \right) \\
\leq \|u^n\|^2_{H^1_0} - \lambda_1 \|u^n\|^2 \\
\leq \|u^n\|^2_{H^1_0} + 2 \left( F(u^n), 1 \right) - 2(h, u^n) + 2\tilde{D}_2 + \frac{4}{\lambda_1} \|h\|^2, \tag{47}
\]
where \(\tilde{D}_2 = \int_\Omega D_2 \, dx\).

Recall the well known uniform Gronwall lemma [22].

**Lemma 23** Let \(g, w, y\) be three positive integrable functions on \((t_0, +\infty)\) such that \(y'\) is locally integrable on \((t_0, +\infty)\) and such that
\[
\frac{dy}{dt} \leq gy + w \text{ if } t \geq t_0, \\
\int_t^{t + r} g \, ds \leq a_1, \int_t^{t + r} w \, ds \leq a_2, \int_t^{t + r} y \, ds \leq a_3 \text{ if } t \geq t_0,
\]
where \(a_i > 0\). Then
\[
y(t + r) \leq \left( \frac{a_3}{r} + a_2 \right) e^{a_1}.
\]
We apply Lemma \[23\] with \( y(s) = \|u^n(s)\|_{H^1_0(\Omega)}^2 + 2(F(u^n(s)), 1) - 2(h, u^n(s)) + M \) (where \( M > 0 \) is such that \( y(s) > 0 \), \( g(s) \equiv 1 \) and \( w(s) \equiv 2\tilde{D}_2 + \frac{2}{\lambda_1} \|h\|^2 \). Then

\[
\|u^n(t+r)\|^2_{H^1_0(\Omega)} + 2(F(u^n(t+r)), 1) - 2(h, u^n(t+r)) \leq K_7 \left( \frac{e^{-\lambda_1 t} \|u^n_0\|^2 + 1}{r} + 1 + r \right)e^r \text{ for all } 0 \leq t \leq t + r.
\]

Using (3) and

\[
2(h, u^n(t+r)) \leq \frac{2}{\lambda_1} \|h\|^2 + \frac{1}{2} \|u^n(t+r)\|^2_{H^1_0(\Omega)},
\]

we have

\[
\|u^n(t+r)\|^2_{H^1_0(\Omega)} \leq K_8 \left( \frac{e^{-\lambda_1 t} \|u^n_0\|^2 + 1}{r} + 1 + r \right)e^r \text{ for all } 0 \leq t \leq t + r.
\]

Therefore, the sequence \( u^n(\cdot) \) is bounded in \( L^\infty(\cdot; H^1_0(\Omega)) \) for all \( 0 < r < T \). Integrating (42) over \((r,T)\) and using (48), (3) we have

\[
2 \int_r^T \|u^n_t\|^2 dt + \|u^n(T)\|^2_{H^1_0} \leq \|u^n(r)\|^2_{H^1_0(\Omega)} + 2(F(u^n(r)), 1) - 2(h, u^n(r)) - 2(F(u^n(T)), 1) + 2(h, u^n(T)) \leq K_7 \left( \frac{\|u^n_0\|^2 + 1}{r} + 1 + r \right)e^r + \frac{2}{\lambda_1} \|h\|^2 + \frac{\lambda_1}{2} \|u^n(T)\|^2 + \tilde{R}
\]

\[
\leq K_9 \left( \frac{\|u^n_0\|^2 + 1}{r} + 1 + r \right)e^r + \frac{1}{2} \|u^n(T)\|^2_{H^1_0},
\]

so that \( u^n \) is bounded in \( L^2(\cdot; L^2(\Omega)) \) for all \( 0 < r < T \).

On the other hand, from (39) and (49) we get

\[
\int_r^T \int_\Omega |f(u^n)|^2 dx \, dt \leq K_{10} \int_r^T \left( 1 + \|u^n(t)\|_{H^1_0(\Omega)}^6 \right) dt \leq K_{11} (T - r) \left( \frac{\|u^n_0\|^2 + 1}{r} + 1 + r \right)^3 e^{3r}.
\]

Then the equality \( \Delta u^n = u^n_t + f(u^n) - h \) implies that

\[
\int_r^T \|\Delta u^n\|^2 dt \leq K_{12} (T - r + 1) \left( \frac{\|u^n_0\|^2 + 1}{r} + 1 + r \right)^3 e^{3r},
\]

ans then \( u^n \) is bounded in \( L^2(\cdot; D(A)) \) for all \( 0 < r < T \).
We note also that the compact embedding $H^1_0(\Omega) \subset L^2(\Omega)$ implies that for any $t > 0$ the sequence $u^n(t)$ is precompact in $L^2(\Omega)$. Hence, applying the Ascoli-Arzelà theorem we obtain, passing to a subsequence and using a diagonal argument, that there exists a function $u : [0, +\infty) \rightarrow L^2(\Omega)$ such that for all $0 < r < T$,

$$u^n \rightarrow u \text{ weakly star in } L^\infty(r, T; H^1_0(\Omega)), \tag{52}$$

$$u^n \rightarrow u \text{ in } C([r, T], L^2(\Omega)), \tag{53}$$

$$u^n \rightarrow u \text{ weakly in } L^2(r, T; D(A)), \tag{54}$$

$$u^n_t \rightarrow u_t \text{ weakly in } L^2(r, T; L^2(\Omega)). \tag{55}$$

Also, by a standard argument we obtain that for any sequence $t_n \rightarrow t_0 > 0$ we have

$$u^n(t_n) \rightarrow u(t_0) \text{ weakly in } H^1_0(\Omega). \tag{56}$$

On the other hand, by $\text{[43]}$ $u^n$ is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)) \cap L^4(0, T; L^4(\Omega))$ for all $T > 0$, and by $\text{[1]}$ and $\text{[2]}$ $u^n$ is bounded in $L^2(0, T; H^{-1}(\Omega)) + L^4(0, T; L^4(\Omega))$ for all $T > 0$. We note that $H^1_0(\Omega) \subset L^4(\Omega) \subset L^\frac{4}{3}(\Omega) \subset H^{-1}(\Omega)$ with continuous embeddings, the first one being compact. Hence, by the Compactness Theorem $\text{[17]}$,  

$$u^n \rightarrow u \text{ weakly star in } L^\infty(0, T; L^2(\Omega)), \tag{57}$$

$$u^n \rightarrow u \text{ weakly in } L^2(0, T; H^1_0(\Omega)), \tag{58}$$

$$u^n \rightarrow u \text{ weakly in } L^4(0, T; L^4(\Omega)), \tag{59}$$

$$u^n \rightarrow u \text{ strongly in } L^2(0, T; L^2(\Omega)), \tag{60}$$

$$u^n_t \rightarrow u_t \text{ weakly in } L^\frac{4}{3}(0, T; H^{-1}(\Omega)), \text{ for all } T > 0. \tag{61}$$

Again by the Ascoli-Arzelà theorem we have that

$$u^n \rightarrow u \text{ in } C([0, T], H^{-1}(\Omega)) \text{ for all } T > 0, \tag{62}$$

and for any sequence $t_n \rightarrow t_0 \geq 0$,

$$u^n(t_n) \rightarrow u(t_0) \text{ weakly in } L^2(\Omega). \tag{63}$$

The last convergence implies that $u(0) = u_0$.

From the boundedness of $u^n$ in $L^4(0, T; L^4(\Omega))$ and $\text{[2]}$ we have that $f(u^n)$ is bounded in $L^\frac{4}{3}(0, T; L^\frac{4}{3}(\Omega))$. Since $u^n(t, x) \rightarrow u(t, x)$, we have $f(u^n(t, x)) \rightarrow f(u(t, x))$ for a.a. $(t, x)$, and then $f(u^n) \rightarrow f(u)$ weakly in $L^\frac{4}{3}(0, T; L^\frac{4}{3}(\Omega))$ (see $\text{[17]}$ p.12).

In a standard way we can check then that $u(\cdot)$ is a weak solution of $\text{[1]}$. Moreover, by the previous arguments it is clear that $u$ is a regular solution.

Finally, we note that in view of $\text{[28]}$ and $\text{[29]}$ all the previous arguments leading to $\text{[44]}$, $\text{[49]}$, $\text{[50]}$ and $\text{[51]}$ are correct for any regular solution with initial value in $L^2(\Omega)$. Thus, $\text{[30]}$-$\text{[33]}$ follow. By $\text{[21]}$ p.102 we obtain that $\text{[34]}$-$\text{[35]}$ hold. The energy equality $\text{[36]}$ follows from $\text{[42]}$. \hfill \blacksquare
Let

\[ K^+_r = \{ u(\cdot) : u \text{ is a regular solution of (1)} \}. \]

We define now the map \( G_r : \mathbb{R}^+ \times L^2(\Omega) \to P(L^2(\Omega)) \) by

\[ G_r(t,u_0) = \{ u(t) : u \in K^+_r \text{ and } u(0) = u_0 \}. \]

The set \( K^+_r \) satisfies conditions (K1) – (K2), so that by Lemma 5 \( G_r \) is a multivalued semiflow.

**Remark 24** In this case we are not able to prove that the semiflow \( G_r \) is strict. The reason is that if we take \( u_1, u_2 \in K^+_r \) with \( u_2(0) = u_1(s) \) and concatenate them, that is,

\[ u(t) = \begin{cases} u_1(t) & \text{if } 0 \leq t \leq s, \\ u_2(t-s) & \text{if } s \leq t, \end{cases} \]

then \( u \) is a weak solution of (1), but we cannot state that it is regular, as properties (27)-(28) can fail now. Hence, condition (K3) is not known to be true.

We shall obtain further some properties of the semiflow \( G_r \).

**Lemma 25** Assume that (2) holds. Let \( \{u^n\} \subset K^+_r \) be a sequence such that \( u^n(0) \to u_0 \) weakly in \( L^2(\Omega) \). Then there exists a subsequence (denoted again by \( u^n \)), and a regular solution of (1) \( u \in K^+_r \) satisfying \( u(0) = u_0 \), such that for any sequence of times \( t_n \geq 0 \) such that \( t_n \to t_0 \) we have \( u^n(t_n) \to u(t_0) \) weakly in \( L^2(\Omega) \).

Also, if \( t_0 > 0 \), then \( u^n(t_n) \to u(t_0) \) strongly in \( H^1_0(\Omega) \).

Moreover, if \( u^n(0) \to u_0 \) strongly in \( L^2(\Omega) \), then for \( t_n \searrow 0 \) we get \( u^n(t_n) \to u_0 \) strongly in \( L^2(\Omega) \).

**Proof.** Arguing as in the proof of Theorem 22 we obtain the existence of a subsequence of \( u^n \) and a weak solution \( u \) of (1) with \( u(0) = u_0 \) such that the convergences (52), (53), (54), (55) hold. Hence, \( u \in K^+_r \).

It follows that if \( t_0 > 0 \) and \( t_n \to t_0 \), then \( u^n(t_n) \to u(t_0) \) strongly in \( L^2(\Omega) \) and weakly in \( H^1_0(\Omega) \). We shall prove that \( u^n(t_n) \to u(t_0) \) strongly in \( H^1_0(\Omega) \). Let \( t_n, t_0 \in (r,T) \).

Multiplying (1) by \( u^n_t \) and using (40) we obtain

\[
\frac{1}{2} \|u^n_t\|^2 + \frac{1}{2} \frac{d}{dt} \|u^n\|^2_{H^1_0(\Omega)} \leq \|f(u^n)\|^2 + \|h\|^2,
\]

and then by (2) and (30) we obtain

\[
\|u^n(t)\|^2_{H^1_0(\Omega)} \leq \|u^n(s)\|^2_{H^1_0(\Omega)} + C(t-s),
\]

\[
\|u(t)\|^2_{H^1_0(\Omega)} \leq \|u(s)\|^2_{H^1_0(\Omega)} + C(t-s), \text{ for all } r \leq s \leq t \leq T,
\]

for some \( C > 0 \). Therefore the functions \( J_n(t) = \|u^n(s)\|^2_{H^1_0(\Omega)} + Ct \), \( J(t) = \|u(s)\|^2_{H^1_0(\Omega)} + Ct \), are continuous and non-increasing in \([r,T]\). Moreover, (52) and the Compactness
Theorem [17] imply that \( J_n(t) \to J(t) \) for a.a. \( t \in (r,T) \). Take \( r < t_m < t_0 \) such that \( t_m \to t_0 \) and \( J_n(t_m) \to J_n(t_m) \) for all \( m \). Then
\[
J_n(t_n) - J(t_0) \leq J_n(t_m) - J(t_0) \leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(t_0)|,
\]
if \( t_n \geq t_m \), so that for any \( \varepsilon > 0 \) there exist \( m(\varepsilon) \) and \( N(\varepsilon) \) such that \( J_n(t) - J(t_0) \leq \varepsilon \) if \( n \geq N \). Then \( \limsup J(t_n) \leq \limsup J(t_0) \), so that \( \limsup \|u^n(t)\|_{H^1_0(\Omega)}^2 \leq \|u(t)\|_{H^1_0(\Omega)}^2 \). As \( u^n(t_n) \to u(t_0) \) weakly in \( H^1_0(\Omega) \) implies \( \liminf \|u^n(t_n)\|_{H^1_0(\Omega)}^2 \geq \|u(t)\|_{H^1_0(\Omega)}^2 \), we obtain
\[
\|u^n(t_n)\|_{H^1_0(\Omega)}^2 \to \|u(t)\|_{H^1_0(\Omega)}^2,
\]
so that \( u^n(t_n) \to u(t_0) \) strongly in \( H^1_0(\Omega) \).

Finally, let \( u^n(0) \to u_0 \) strongly in \( L^2(\Omega) \). In view of (13) we have for some \( C > 0 \),
\[
\frac{d}{dt} \|u^n\|^2 \leq C,
\]
so that the functions \( \mathcal{T}_n(t) = \|u^n(s)\|^2 + Ct, \mathcal{J}(t) = \|u(s)\|^2 + Ct \), are continuous and non-increasing for \( t \geq 0 \). Hence,
\[
\mathcal{J}_n(t_n) - \mathcal{J}(0) \leq \mathcal{J}_n(0) - \mathcal{J}(0) \to 0,
\]
so that \( \limsup \mathcal{J}(t_n) \leq \limsup \mathcal{J}(0) \) and by the same argument as before we obtain that \( u^n(t_n) \to u_0 \) strongly in \( L^2(\Omega) \).

**Corollary 26** If (2) holds, then \( K_r^+ \) satisfies condition (K4).

By standar arguments from Lemma 25 the following result follows.

**Corollary 27** Let (2) hold. Then the multi-valued semiflow \( G_r \) has compact values and the map \( u_0 \mapsto G_r(t,u_0) \) is upper semicontinuous for all \( t \geq 0 \), that is, for any neighborhood \( O(G_r(t,u_0)) \) in \( L^2(\Omega) \) there exists \( \delta > 0 \) such that if \( \|u_0 - v_0\| < \delta \), then \( G_r(t,v_0) \subset O \).

**Lemma 28** Let (2) hold. Then the ball
\[
B_0 = \{ u \in H^1_0(\Omega) : \|u\|_{H^1_0(\Omega)}^2 \leq 4eR_1 \},
\]
where \( R_1 > 0 \) is taken from (30), is a compact absorbing for \( G_r \), that is, for any set \( B \) bounded in \( L^2(\Omega) \) there exists \( T(B) \) such that
\[
G_r(t,B) \subset B_0 \text{ for } t \geq T(B).
\]

**Proof.** The fact that \( B_0 \) is absorbing follows from (30) by taking \( r = 1 \). Since \( B_0 \) is closed in \( L^2(\Omega) \) and bounded in \( H^1_0(\Omega) \), the compacity in \( L^2(\Omega) \) follows. 

Now we are ready to prove the existence of a global compact attractor.
Theorem 29 Let (2) hold. Then the multivalued semiflow $G_r$ possesses a global compact attractor $\Theta_r$. Moreover, for any set $B$ bounded in $L^2(\Omega)$ we have
\[
\text{dist}_{H^1_0(\Omega)}(G_r(t,B), \Theta_r) \to 0 \text{ as } t \to +\infty,
\]
and also that $\Theta_r$ is compact in $H^1_0(\Omega)$.

Proof. The existence of a global compact attractor $\Theta_r$ follows from Corollary 27, Lemma 28 and Theorem 4 in [18].

Let now suppose that (56) is not true. Then there exists $\varepsilon > 0$ and a sequence $y_n \in G_r(t_n, B)$ with $t_n \to +\infty$ such that
\[
\text{dist}_{H^1_0(\Omega)}(y_n, \Theta_r) > \varepsilon \text{ for all } n.
\]

There exists $y \in \Theta_r$ and a subsequence $y_{n_k}$ such that $y_{n_k} \to y$ in $L^2(\Omega)$. In view of Lemma 25 the set $G_r(t, B)$ is precompact in $H^1_0(\Omega)$ for any $t > 0$ and any bounded set in $L^2(\Omega)$. Therefore, from $y_{n_k} \in G_r(1, G_r(t_n - 1, B)) \subset G_r(1, B_0)$ for $n$ great enough, we obtain that $y_{n_k} \to y$ in $H^1_0(\Omega)$, which is a contradiction.

Finally, $\Theta_r \subset G_r(1, \Theta_r)$ implies, by the same reason, that $\Theta_r$ is precompact in $H^1_0(\Omega)$. Moreover, since $\Theta_r$ is closed in $L^2(\Omega)$, it is closed in $H^1_0(\Omega)$, as well, so that $\Theta_r$ is compact in $H^1_0(\Omega)$. ■

Remark 30 Since $G_r(t, u_0) \subset G(t, u_0)$, it is clear that $\Theta_r \subset \Theta$.

The map $\gamma : \mathbb{R} \to L^2(\Omega)$ is called a complete trajectory of $K^+_r$ if $\gamma(\cdot + h) |_{[0, +\infty)} \in K^+_r$ for any $h \in \mathbb{R}$. The set of all complete trajectories of $K^+_r$ will be denoted by $\mathbb{F}_r$. Let $\mathbb{K}_r$ be the set of all complete trajectories which are bounded in $L^2(\Omega)$, and let $\mathbb{K}^1_r$ be the set of all complete trajectories which are bounded in $H^1_0(\Omega)$.

Lemma 31 Let (2) hold. Then $\mathbb{K}_r = \mathbb{K}^1_r$.

Proof. It is clear that $\mathbb{K}^1_r \subset \mathbb{K}_r$. Let $\gamma(\cdot) \in \mathbb{K}_r$ and denote $B_\gamma = \bigcup_{t \in \mathbb{R}} \gamma(t)$. Then
\[
B_\gamma \subset G_r(1, B_\gamma)
\]
and (30) implies that $B_\gamma$ is bounded in $H^1_0(\Omega)$, so $\gamma(\cdot) \in \mathbb{K}^1_r$. ■

Further, we shall prove that $\Theta_r$ is the union of all points lying in a bounded complete trajectory, that is,
\[
\Theta_r = \{ \gamma(0) : \gamma(\cdot) \in \mathbb{K}_r \} = \{ \gamma(0) : \gamma(\cdot) \in \mathbb{K}^1_r \}
\]
\[
= \bigcup_{t \in \mathbb{R}} \{ \gamma(t) : \gamma(\cdot) \in \mathbb{K}_r \} = \bigcup_{t \in \mathbb{R}} \{ \gamma(t) : \gamma(\cdot) \in \mathbb{K}^1_r \}
\]
(57)

Theorem 32 Let (2) hold. Then equalities (57) hold true.
Proof. It follows from Theorem 9 Corollary 26 and Lemma 31.

We shall establish now for $G_r$ the same statements as in Lemma 12 and in the first point of Theorem 13.

Lemma 33 Let (2) hold. Then $\mathcal{R} = \mathcal{R}_{K^+_r}$.

Proof. Let $u_0 \in \mathcal{R}_{K^+_r}$. Then $u(t) \equiv u_0$ belongs to $K^+_r$. Thus $u(\cdot)$ satisfies (1), so that (2) holds. Conversely, let $v_0 \in \mathcal{R}$. In view of Lemma 14 the set of stationary points of (1) $\mathcal{R}$ is bounded in $H^2(\Omega) \cap H^1_0(\Omega)$. Then the function $v(\cdot)$ defined by $v(t) \equiv v_0$ belongs to $K^+_r$.

Corollary 34 $\mathcal{R} \subset \Theta_r$.

Lemma 35 Let (2) hold. Then $z \in \mathcal{R}$ if and only if $z \in G_r(t,z)$ for all $t \geq 0$.

Proof. If $z \in \mathcal{R}$, Lemma 33 implies that the function $v(\cdot)$ defined by $v(t) \equiv z$ belongs to $K^+_r$. Then $z \in G_r(t,z)$, for any $t \geq 0$.

Conversely, let $z \in G_r(t,z)$ for all $t \geq 0$. Since $z \in G_r(t,z) \subset G(t,z)$, Theorem 13 implies that $z \in \mathcal{R}$.

It is clear that if the map $\gamma : \mathbb{R} \to L^2(\Omega)$ is a complete trajectory of $K^+_r$, then
\begin{equation}
\gamma(t+s) \in G_r(t,\gamma(s)) \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0.
\end{equation}

Let $\gamma(\cdot)$ satisfy (58). Then $\gamma(t) \in G(t,\gamma(s))$ and by Theorem 13 we have that $\gamma(\cdot)$ is a complete trajectory for $K^+$. However, it is not clear whether $\gamma(\cdot)$ is a complete trajectory of $K^+_r$ or not, as (K3) fails, so that we cannot use Lemma 8. Nevertheless, we can obtain the following.

Lemma 36 Let (2) hold. Then the map $\gamma : \mathbb{R} \to L^2(\Omega)$ is a complete trajectory of $K^+_r$ if and only if $\gamma \in L^{\infty}_{\text{loc}}(\mathbb{R};H^1_0(\Omega)), \gamma_t \in L^2_{\text{loc}}(\mathbb{R};L^2(\Omega))$ and (58) holds.

Proof. If the map $\gamma : \mathbb{R} \to L^2(\Omega)$ is a complete trajectory of $K^+_r$, then clearly (58) and $\gamma \in L^{\infty}_{\text{loc}}(\mathbb{R};H^1_0(\Omega)), \gamma_t \in L^2_{\text{loc}}(\mathbb{R};L^2(\Omega))$ hold.

Conversely, let $\gamma \in L^{\infty}_{\text{loc}}(\mathbb{R};H^1_0(\Omega)), \gamma_t \in L^2_{\text{loc}}(\mathbb{R};L^2(\Omega))$ and (58) hold. Then by Theorem 13 we have that $\gamma(\cdot)$ is a complete trajectory for $K^+$. As $\gamma \in L^{\infty}_{\text{loc}}(\mathbb{R};H^1_0(\Omega)), \gamma_t \in L^2_{\text{loc}}(\mathbb{R};L^2(\Omega))$, it is clear that $\gamma$ is a complete trajectory of $K^+_r$.

We shall prove now that
\begin{equation}
\Theta_r = M^+_r(\mathcal{R}) = M^-_r(\mathcal{R}),
\end{equation}
where
\begin{align*}
M^+_r(\mathcal{R}) &= \{ z : \exists \gamma(\cdot) \in \mathbb{K}_r, \gamma(0) = z, \text{ dist}_{L^2(\Omega)}(\gamma(t),\mathcal{R}) \to 0, \ t \to +\infty \}, \\
M^-_r(\mathcal{R}) &= \{ z : \exists \gamma(\cdot) \in \mathbb{F}_r, \gamma(0) = z, \text{ dist}_{L^2(\Omega)}(\gamma(t),\mathcal{R}) \to 0, \ t \to -\infty \}.
\end{align*}

As in the case of weak solutions, in the definition of $M^+_r(\mathcal{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$, since every $\gamma$ as given in the definition of $M^+_r(\mathcal{R})$ belongs to $\mathbb{K}_r$ in view of (31).
Theorem 37 Under conditions \( (2) \) it holds
\[
\Theta_r = M^+_r(\mathcal{R}) = M^-_r(\mathcal{R}).
\] 
Moreover,
\[
M^-_r(\mathcal{R}) = \left\{ z : \exists \gamma(\cdot) \in \mathcal{K}_r, \quad \gamma(0) = z, \quad \text{dist}_{H^1_0(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \quad t \to +\infty \right\},
\]
\[
M^+_r(\mathcal{R}) = \left\{ z : \exists \gamma(\cdot) \in \mathcal{F}_r, \quad \gamma(0) = z, \quad \text{dist}_{H^1_0(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \quad t \to -\infty \right\}.
\] 

Proof. We can prove this theorem arguing in a rather similar way as in Theorem 16. However, we shall prove it using the method of the Lyapunov function, as in [4], [22].

Let \( z \in \Theta_r \) and let \( u \in \mathcal{K}_r \) be such that \( u(0) = z \). We note that the energy function \( E(u(t)), t > 0, \) given in (36) is nonincreasing and bounded below (which follows easily from (33)) for any \( u \in K^+_r \). Hence, \( E(u(t)) \to l, \) as \( t \to +\infty, \) for some \( l \in \mathbb{R} \).

Suppose that there exist \( \varepsilon > 0 \) and a sequence \( u(t_n), t_n \to +\infty \), such that
\[
\text{dist}_{L^2(\Omega)}(u(t_n), \mathcal{R}) > \varepsilon.
\] 
In view of Theorem 29 we have that \( \Theta_r \) is compact in \( H^1_0(\Omega) \), and then we can take a converging subsequence (denoted again \( u(t_n) \)) for which \( u(t_n) \to y \) in \( H^1_0(\Omega) \), where \( t_n \to +\infty \). Since the function \( E : H^1_0(\Omega) \to \mathbb{R} \) is continuous, we obtain that \( E(y) = l \).

We shall prove that \( y \in \mathcal{R} \). Fix \( t > 0 \). In view of Lemma 25 there exists \( v \in K^+_r \) and a subsequence of \( v_n(\cdot) = u(\cdot + t_n) \) (denoted \( v_n \) again) such that \( v(0) = y \) and \( v_n(t) \to v(t) = z \) in \( H^1_0(\Omega) \). Hence, \( E(v_n(t)) \to E(z) \) implies that \( E(z) = l \).

We note that \( v(\cdot) \) satisfies (36) for all \( 0 \leq s \leq t \), so that
\[
l + 2 \int_0^t \|v_r\|^2 \, dt = E(z) + 2 \int_0^t \|v_{r_n}\|^2 \, dt = E(v(0)) = E(y) = l.
\] 
This implies that \( v_r(r) = 0 \) for a.a. \( r \), and therefore \( y \in \mathcal{R} \). Hence, we obtain a contradiction. Thus, \( \Theta_r \subset M^-_r(\mathcal{R}) \). The converse inclusion is obvious from Theorem 32 so that \( \Theta_r = M^-_r(\mathcal{R}) \).

On the other hand, we observe that for any \( u \in \mathcal{F}_r \), equality (36) is satisfied for all \(-\infty < s \leq t \). Let \( z \in \Theta_r \) and let \( u \in \mathcal{K}_r = \mathcal{K}^1_r \) be such that \( u(0) = z \). In view of (33) the function \( (F(u(t)), 1) \) is bounded above. Hence, \( E(u(t)) \to l, \) as \( t \to -\infty, \) for some \( l \in \mathbb{R} \). As before, suppose that there exist \( \varepsilon > 0 \) and a sequence \( u(t_n), t_n \to -\infty, \) such that
\[
\text{dist}_{L^2(\Omega)}(u(-t_n), \mathcal{R}) > \varepsilon,
\] 
and we have that \( u(-t_n) \to y \) in \( H^1_0(\Omega) \), \( E(y) = l \). Also, for a fixed \( t > 0 \) there exists \( v \in K^+_r \) and a subsequence of \( v_n(\cdot) = u(\cdot - t_n) \) (denoted \( v_n \) again) such that \( v(0) = y \) and \( v_n(t) \to v(t) = z \) in \( H^1_0(\Omega) \). Hence, \( E(v_n(t)) \to E(z) \) implies that \( E(z) = l \) and therefore, arguing as before, \( y \in \mathcal{R} \), which is a contradiction. As before, we obtain that \( \Theta_r = M^-_r(\mathcal{R}) \).

Finally, let us prove that the convergence takes place in \( H^1_0(\Omega) \). Let us suppose that there exist \( \varepsilon > 0 \) and a sequence \( u(t_n), t_n \to +\infty, \) such that
\[
\text{dist}_{H^1_0(\Omega)}(u(t_n), \mathcal{R}) > \varepsilon.
\]
In view of dist\textsubscript{$L^2(\Omega)$}$(u(t_n), \mathcal{R}) \to 0$ and the compacity of $\mathcal{R}$, we can extract a subsequence $u(t_{n_k})$ such that $u(t_{n_k}) \to \overline{u} \in \mathcal{R}$ in $L^2(\Omega)$. It follows from the compacity of $\Theta_r$ in $H^1_0(\Omega)$ that in fact $u(t_{n_k}) \to \overline{u} \in \mathcal{R}$ in $H^1_0(\Omega)$, which is a contradiction. Hence, the first part of (61) holds. The second one is proved in the same way. 

6 An attractor in $H^1_0(\Omega)$. Existence and structure of the global attractor for strong solutions

In this section we shall define a semiflow in the phase space $H^1_0(\Omega)$. For this aim we introduce now a stronger concept of solution for (1).

The function $u \in L^2(0, +\infty; H^1_0(\Omega)) \cap L^4(0, +\infty; L^4(\Omega))$ is called a strong solution of (1) on $(0, +\infty)$ if for all $T > 0$, $v \in H^1_0(\Omega)$ and $\eta \in C^\infty(0, T)$ we have that (26) holds and

$$u \in L^\infty(0, T; H^1_0(\Omega)),$$

$$u_t \in L^2(0, T; L^2(\Omega)), \ \forall \ T > 0.$$  

Arguing as in Section 5 we obtain that any strong solution $u$ satisfies

$$u \in L^2(0, T; D(A)).$$

By Lemma 15 for any $u_0 \in H^1_0(\Omega)$ there exists at least one strong solution $u(\cdot)$ such that $u(0) = u_0$. Moreover, any strong solution satisfies good properties, as given in the following lemma.

**Lemma 38** Let (2) hold. Then every strong solution of (1) satisfies the following properties:

$$u \in C([0, +\infty), H^1_0(\Omega)),$$

$$\frac{d}{dt} \|u\|_{H^1_0(\Omega)}^2 = 2 (-\Delta u, u_t) \text{ for a.a. } t > 0,$$

$$E(u(t)) + 2 \int_s^t \|u_r\|^2 \, dr = E(u(s)), \text{ for all } t \geq s \geq 0,$$

where $E(u(t)) = \|u(t)\|_{H^1_0(\Omega)}^2 + 2 \langle F(u(t)), 1 \rangle - 2 \langle h, u(t) \rangle$. Also, there exist $R_1, R_2 > 0$ such that

$$\int_0^t \|u_r\|^2 \, dr + \|u(t)\|_{H^1_0(\Omega)}^2 \leq R_1 \left( \|u_0\|_{H^1_0(\Omega)}^4 + 1 \right),$$

$$\int_0^t \|\Delta u^n\|^2 \, dt \leq R_2 (t + 1) \left( 1 + \|u_0\|_{H^1_0(\Omega)}^{12} \right), \text{ for all } t \geq 0.$$  

**Proof.** In view of (62)-(63) by standard results [21, p.102], we obtain that $u$ belongs to $C([0, +\infty), H^1_0(\Omega))$ and (40), (41) hold. We multiply (1) by $u_t$ and using (40), (41) we obtain

$$2 \|u_t\|^2 + \frac{d}{dt} \left( \|u\|_{H^1_0(\Omega)}^2 + 2 \langle F(u), 1 \rangle - 2 \langle h, u \rangle \right) = 0 \text{ for a.a. } t \in (0, T).$$

26
Integrating over \((s,t)\) we have
\[
2 \int_s^t \|u_r\|^2 \, dr + \|u(t)\|_{H^1_0(\Omega)}^2 + 2 (F(u(t)), 1) - 2 (h, u(t))
= \|u(s)\|_{H^1_0(\Omega)}^2 + 2 (F(u(s)), 1) - 2 (h, u(s)).
\]
Thus we obtain \((67)\). Due to \((3)\) we get
\[
2 \int_0^t \|u_r\|^2 \, dr + \|u(t)\|_{H^1_0(\Omega)}^2 \leq \left( 1 + \frac{1}{\lambda_1} \right) \|u_0\|_{H^1_0(\Omega)}^2 + \frac{\lambda_1}{2} \|u(t)\|^2 + \bar{D}_2 + 2D_1 \int_\Omega (1 + |u_0(x)|^4) \, dx + \left( \frac{2}{\lambda_1} + 1 \right) \|h\|^2,
\]
so
\[
\int_0^t \|u_r\|^2 \, dr + \|u(t)\|_{H^1_0(\Omega)}^2 \leq R_1 \left( \|u_0\|_{H^1_0(\Omega)}^4 + 1 \right), \text{ for all } t \geq 0.
\]
Finally, by
\[
\int_0^T \int_\Omega |f(u)|^2 \, dx \, dt \leq \int_0^T \left( 1 + \|u(t)\|_{H^1_0(\Omega)}^6 \right) \, dt \\
\leq T \left( 1 + \left( R_1 \left( \|u_0\|_{H^1_0(\Omega)}^4 + 1 \right) \right)^3 \right)
\]
and the equality \(\Delta u^n = u^n_t + f(u^n) - h\) we obtain that
\[
\int_0^T \|\Delta u^n\|^2 \, dt \leq R_2 (T + 1) \left( 1 + \|u_0\|_{H^1_0(\Omega)}^{12} \right).
\]

Let \(K^+_s = \{ u(\cdot) : u \text{ is a strong solution of } (1) \}\).

We define now the map \(G_s : \mathbb{R}^+ \times H^1_0(\Omega) \to P(H^1_0(\Omega))\) by
\[
G_s(t, u_0) = \{ u(t) : u \in K^+_s \text{ and } u(0) = u_0 \}.
\]

We can check easily that \(K^+_s\) satisfies conditions \((K1) - (K3)\). Then Lemma 39 implies the following.

**Lemma 39** Let \((3)\) hold. Then \(G_s\) is a strict multivalued semiflow.

We shall obtain further some properties of the semiflow \(G_s\).
Lemma 40  Assume that (2) holds. Let \( \{u^n\} \subset K^+_s \) be a sequence such that \( u^n(0) \to u_0 \) weakly in \( H^1_0(\Omega) \). Then there exists a subsequence (denoted again by \( u^n \)), and a strong solution of (1) \( u \in K^+_s \) satisfying \( u(0) = u_0 \), such that for any sequence of times \( t_n \geq 0 \) such that \( t_n \to t_0 \) we have \( u^n(t_n) \to u(t_0) \) weakly in \( H^1_0(\Omega) \).

Moreover, if \( u^n(0) \to u_0 \) strongly in \( H^1_0(\Omega) \), then for \( t_n \to t_0 \) we get \( u^n(t_n) \to u_0 \) strongly in \( H^1_0(\Omega) \).

Proof. Since obviously \( K^+_s \subset K^+_r \), it follows from Lemma 25 the existence of a regular solution \( u(t) \in K^+_r \) with \( u(0) = u_0 \) and a subsequence such that (52), (54) hold and

\[
\begin{align*}
&u^n(t_n) \to u(t_0) \text{ strongly in } H^1_0(\Omega) \text{ if } t_0 > 0, \\
&u^n(t_n) \to u(t_0) \text{ strongly in } L^2(\Omega).
\end{align*}
\]

Thus, (68) implies by a standard argument that

\[
u^n(t_n) \to u(t_0) \text{ weakly in } H^1_0(\Omega).
\]

It remains to check that \( u \) is a strong solution. In view of (68)-(69) for all \( T > 0 \) the sequence \( u^n \) is bounded in \( L^\infty(0, T; H^1_0(\Omega)) \cap L^2(0, T; D(A)) \) and \( u^n_t \) is bounded in \( L^2(0, T; L^2(\Omega)) \). Hence, \( u^n \to u \) weakly star in \( L^\infty(0, T; H^1_0(\Omega)) \), weakly in \( L^2(0, T; D(A)) \) and \( u^n_t \to u_t \) weakly in \( L^2(0, T; L^2(\Omega)) \), so that \( u \) satisfies (62)-(63) and then \( u \in K^+_s \).

Finally, if \( u^n(0) \to u_0 \) strongly in \( H^1_0(\Omega) \), then arguing in the same way as in Lemma 25 we obtain that \( u^n(t_n) \to u_0 \) strongly in \( H^1_0(\Omega) \) for \( t_n \to 0 \).

Corollary 41 Let (2) hold. Then \( K^+_s \) satisfies condition (K4).

Corollary 42 Let (2) hold. Then the multivalued semiflow \( G_s \) has compact values and the map \( u_0 \mapsto G_s(t, u_0) \) is upper semicontinuous for all \( t \geq 0 \).

We prove further the existence of a global compact attractor.

Theorem 43 Let (2) hold. Then the multivalued semiflow \( G_s \) posseses a global compact invariant attractor \( \Theta_s \).

Proof. Since \( G_s(t, u_0) \subset G_r(t, u_0) \), the ball \( B_0 \) defined in Lemma 28 is absorbing for \( G_s \). Also, the operator \( G_s(t, \cdot) \) is compact for \( t > 0 \) by Lemma 40 so that \( K = \overline{G_s(1, B_0)}_{H^1_0} \) is a compact absorbing set. Then using Corollary 42 the existence of a global compact minimal attractor \( \Theta_s \) follows from Theorem 4 in [18]. As \( G_s \) is strict, it follows from Remark 8 in [18] that \( G_s(t, \Theta_s) = \Theta_s \) for all \( t \geq 0 \).

We will prove further that in fact the attractors \( \Theta_s \) and \( \Theta_r \) coincide.

Lemma 44 Let (2) hold. Then \( \Theta_s = \Theta_r \).
Proof. Since \( G_s (t, u_0) \subset G_r (t, u_0) \), we have that \( \Theta_s \subset \Theta_r \).

Conversely, if \( z \in \Theta_r \), then by (54) we have that \( z = \gamma (0) \), where \( \gamma (\cdot) \in \mathbb{K}_r \), the set of all bounded (in \( H_0^1 (\Omega) \)) complete trajectories corresponding to \( K_r^+ \). It is easy to see that \( \gamma \mid_{[r, +\infty)} \) is a strong solution for any \( r \in \mathbb{R} \). Hence, \( z = \gamma (0) \in G_s (t_n, \gamma (-t_n)) \) for \( t_n \to +\infty \). Hence,

\[
\text{dist}_{H_0^1 (\Omega)} (z, \Theta_s) \leq \text{dist}_{H_0^1 (\Omega)} (G_s (t_n, \gamma (-t_n)), \Theta_s) \to 0,
\]

so that \( z \in \Theta_s \). ■

The map \( \gamma : \mathbb{R} \to H_0^1 (\Omega) \) is called a complete trajectory of \( G_s \) if \( \gamma (\cdot + h) \mid_{[0, +\infty)} \in K_s^+ \) for any \( h \in \mathbb{R} \). The set of all complete trajectories of \( K_s^+ \) will be denoted by \( \mathcal{F}_s \). Let \( \mathbb{K}_s \) be the set of all complete trajectories which are bounded in \( H_0^1 (\Omega) \).

Lemma 45 Let (2) hold. Then \( \mathbb{K}_s = \mathbb{K}_r^1 = \mathbb{K}_r \).

Proof. \( \mathbb{K}_s \subset \mathbb{K}_r^1 \) is obvious, and \( \mathbb{K}_r^1 \subset \mathbb{K}_s \) follows from the arguments in the proof of Lemma 43. The last equality was done in Lemma 31. ■

We shall establish now the same statements of Lemma 12 and Theorem 13 for \( G_s \).

First we can characterize the attractor \( \Theta_s \) as the union of all points lying in a bounded complete trajectory.

Lemma 46 Let (2) hold. Then we have

\[
\Theta_s = \{ \gamma (0) : \gamma (\cdot) \in \mathbb{K}_s \} = \bigcup_{t \in \mathbb{R}} \{ \gamma (t) : \gamma (\cdot) \in \mathbb{K}_s \}. \tag{70}
\]

Proof. As \( K_s^+ \) satisfies (K1) – (K4), it is a direct consequence of either Theorem 9 or 10. Also, it follows from Lemmas 44, 45 and Theorem 32. ■

Lemma 47 Let (2) hold. Then \( \mathcal{R} = \mathcal{R}_{K_s^+} \).

Proof. Let \( u_0 \in \mathcal{R}_{K_s^+} \). Then \( u (t) \equiv u_0 \) belongs to \( K_s^+ \). Thus \( u (\cdot) \) satisfies (11), so that (2) holds. Conversely, let \( v_0 \in \mathcal{R} \). In view of Lemma 14, the set of stationary points of (11) \( \mathcal{R} \) is bounded in \( H^2 (\Omega) \cap H_0^1 (\Omega) \). Then the function \( v (\cdot) \) defined by \( v (t) \equiv v_0 \) belongs to \( K_s^+ \). ■

Corollary 48 Let (2) hold. Then \( \mathcal{R} \subset \Theta_s \).

Lemma 49 Let (2) hold. Then \( z \in \mathcal{R} \) if and only if \( z \in G_s (t, z) \) for all \( t \geq 0 \).

Proof. As \( K_s^+ \) satisfies (K1) – (K4), it follows from Lemma 47 and Lemma 47 ■

Remark 50 We can prove that \( z \in G_s (t, z) \), for all \( t \geq 0 \), implies \( z \in \mathcal{R} \) also by using the Lyapunov function \( E (\cdot) \). Indeed, if \( z \in G_s (t, z) \), for any \( t \geq 0 \), then for every \( T > 0 \) there exists \( v^T (\cdot) \in K_s^+ \) such that \( v^T (T) = z \). Thus by the energy equality (67) we have

\[
E (z) + 2 \int_0^T \| v_r^T \|^2 \, dr = E (v^T (T)) + 2 \int_0^T \| v_r^T \|^2 \, dr = E (v (0)) = E (z),
\]

so that \( \int_0^T \| v_r^T \|^2 \, dr = 0 \). Therefore, \( v_r^T = 0 \) for a.a. \( t \) in \((0, T)\) and \( v^T (t) = z \) for all \( t \in [0, T] \). Hence, \( z \in \mathcal{R}_{K_s^+} = \mathcal{R} \).

29
Lemma 51 Let (2) hold. Then the map \( \gamma : \mathbb{R} \to H^1_0(\Omega) \) is a complete trajectory of \( K^+_s \) if and only if
\[
\gamma(t+s) \in G_s(t, \gamma(s)) \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0.
\]
(71)

Proof. As \((K1) - (K4)\) hold, the result follows from Lemma 8. ■

We shall prove now that
\[
\Theta_s = M^+_s(\mathcal{R}) = M^-_s(\mathcal{R}),
\]
(72)
where
\[
M^-_s(\mathcal{R}) = \left\{ z : \exists \gamma(\cdot) \in \mathcal{K}_s, \, \gamma(0) = z, \, \text{dist}_{H^1_0(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \, t \to +\infty \right\},
\]
\[
M^+_s(\mathcal{R}) = \left\{ z : \exists \gamma(\cdot) \in \mathcal{F}_s, \, \gamma(0) = z, \, \text{dist}_{H^1_0(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \, t \to -\infty \right\}.
\]

Theorem 52 Under conditions (2) equality (72) holds.

Proof. By (68) we have
\[
M^+_s(\mathcal{R}) = \left\{ z : \exists \gamma(\cdot) \in \mathcal{K}_s, \, \gamma(0) = z, \, \text{dist}_{H^1_0(\Omega)}(\gamma(t), \mathcal{R}) \to 0, \, t \to -\infty \right\},
\]
and then equality (72) follows from Lemmas 44, 45 and Theorem 37. ■

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