AN ASYMPTOTIC VERSION OF A THEOREM OF KNUTH

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1. Introduction

In this note we consider the asymptotics of the number $S(d, N)$ of permutations in the symmetric group $S(N)$ which have no decreasing subsequence of length $d + 1$, in the limit where $d \geq 2$ is a fixed but arbitrary positive integer and $N \to \infty$. This is a fundamental problem in the subject of pattern avoidance in permutations, see [1] and [10, §7]. In the interest of brevity, familiarity with Young diagrams, Young tableaux, and the Robinson-Schensted-Knuth (RSK) correspondence is assumed. The reader is referred to Stanley’s survey [10] for the necessary background and further references. We adhere to the notation and terminology of [10] save for the following exceptions: the $d \times q$ rectangular Young diagram is denoted $R(d, q)$ rather than $(q^d)$, and the number of standard Young tableaux of shape $\lambda$ is denoted $\dim \lambda$ rather than $f^\lambda$. Recall that the dimension of a Young diagram may be computed from Frobenius’ formula:

\[ \dim \lambda = \frac{\Gamma(\lambda_1 + \cdots + \lambda_d + 1)}{\prod_{i=1}^d \Gamma(\lambda_i - i + d + 1)} \prod_{1 \leq i < j \leq d} (\lambda_i - \lambda_j + j - i), \]

where $d$ is any number such that $\lambda_{d+1} = 0$ and $\Gamma(z)$ is the gamma function.

The following well-known exact formula for $S(2, N)$ is attributed to Hammersley in [10], with the first published proof due to Knuth [6, §5.1.4].

**Theorem 1.1** (Knuth). Permutations with no decreasing subsequence of length 3 are counted by the Catalan numbers:

\[ S(2, N) = \dim R(2, N) = \frac{(2N)!}{N!(N+1)!}. \]

For $d > 2$, there is no known closed formula for $S(d, N)$. The main result of this note is the following asymptotic version of Knuth’s theorem.

**Theorem 1.2** (Asymptotic Knuth theorem). For $d$ fixed and $n \to \infty$,

\[ S(d, dn) \sim \dim R(d, 2n). \]

We will see below that $S(d, dn) > \dim R(d, 2n)$ as soon as $d > 2$, so that Theorem 1.2 is false for $n$ finite.

Via the RSK correspondence, an equivalent formulation of Theorem 1.2 is the following.

**Theorem 1.3.** The number of permutations in $S(dn)$ with no decreasing subsequence of length $d + 1$ is asymptotically equal, as $n \to \infty$, to the number of involutions in $S(2dn)$ with longest decreasing subsequence of length exactly $d$ and longest increasing subsequence of length exactly $2n$. 

By Frobenius’ formula, we have
\[ \dim R(d, q) = \frac{\Gamma(dq + 1)}{\prod_{i=1}^{d} \Gamma(i)} \Gamma(q + 1) \prod_{i=1}^{d} \Gamma(i(i)) \]
Thus
\[ \dim R(d, q) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=1}^{d} i(i) \right)^{\frac{1}{2}} q^{\frac{1-d^2}{2}} \]
as \( q \to \infty \) with \( d \) fixed, by Stirling’s formula. Setting \( q = 2N/d \), Theorem 1.2 together with (3) immediately implies the following.

**Corollary 1.4.** For \( d \) fixed and \( N \to \infty \),
\[ S(d, N) \sim (2\pi)^{\frac{1-d}{2}} \left( \prod_{i=1}^{d} i(i) \right)^{\frac{1}{2}} q^{\frac{1-d^2}{2}} \]

Corollary 1.4 was first obtained by A. Regev [8, Formula F.4.5.2] in 1981 by a rather different method, which will be discussed below.

### 2. Decomposition of Rectangular Tableaux

Given a Young diagram \( \mu \subseteq R(d, q) \), let
\[ \mu^* = (q - \mu_d, \ldots, q - \mu_1) \]
denote the complement of \( \mu \) relative to \( R(d, q) \). Clearly,
\[ \dim R(d, 2n) = \sum_{\mu \subseteq R(d, 2n)} (\dim \mu)(\dim \mu^*) \]
\[ = \sum_{\mu \subseteq R(d, 2n)} (\dim \mu)^2 + \sum_{\mu \subseteq R(d, 2n)} (\dim \mu)(\dim \mu^*). \]
On the other hand, by RSK, we have
\[ S(d, dn) = \sum_{\lambda, \ell(\lambda) \leq d} (\dim \lambda)^2 \]
\[ = \sum_{\mu \subseteq R(d, 2n)} (\dim \mu)^2 + \sum_{\mu \subseteq R(d, 2n)} (\dim \mu)^2 + \sum_{\nu} (\dim \nu)^2. \]
Substituting for the first group of terms in (6) using (5) and completing the square yields the following.

**Proposition 2.1.** \( S(d, N) = \dim R(d, 2n) + E(d, n) \), where the error term is given by
\[ E(d, n) = \frac{1}{2} \sum_{\mu \subseteq R(d, 2n)} (\dim \mu - \dim \mu^*)^2 + \sum_{\nu} (\dim \nu)^2. \]
Clearly \( E(2, n) = 0 \), in agreement with Knuth’s theorem, while \( E(d, n) > 0 \) for \( d > 2 \). Nevertheless, the error term is negligible in the limit \( n \to \infty \).
3. Asymptotics of the dimension function

In order for $E(d,n)$ to be negligible, the sum $S(d, dn)$ must be dominated in the $n \to \infty$ limit by self-complementary diagrams contained in the rectangle $R(d, 2n)$. The canonical self-complementary diagram relative to $R(d, 2n)$ is the $d \times n$ rectangle $R(d, n)$. We consider the asymptotics of diagrams which deviate from $R(d, n)$ on the scale $\sqrt{n}$; this choice of scale emerges constructively in the proof of the following key Lemma.

Lemma 3.1. For any distinct real numbers $y_1 > \cdots > y_d$

satisfying

$$y_1 + \cdots + y_d = 0,$$

we have

$$\lim_{n \to \infty} C_{d, dn} \dim(n + y_1 \sqrt{n}, \ldots, n + y_d \sqrt{n}) = e^{-W(y_1, \ldots, y_d)},$$

where

$$C_{d, dn} = (2\pi)^{\frac{d}{2}} \frac{n^{d+1}}{(dn + 1)e^{dn}}$$

and

$$W(y_1, \ldots, y_d) = \frac{1}{2} \sum_{i=1}^{d} y_i^2 - \sum_{1 \leq i < j \leq d} \log(y_i - y_j).$$

Proof. Let $0 < \varepsilon < 1$, and consider deviations from $R(d, n)$ on the scale $n^\varepsilon$. By the Frobenius formula, we have

$$\dim(n + y_1 n^\varepsilon, \ldots, n + y_d n^\varepsilon) = \frac{\Gamma(dn + 1)}{\prod_{i=1}^{d} \Gamma(n + y_i n^\varepsilon + d - i + 1)} \prod_{1 \leq i < j \leq d} ((y_i - y_j)n^\varepsilon + j - i)$$

for $n$ sufficiently large.

Let us first analyze the asymptotics of the product

$$\frac{1}{\prod_{i=1}^{d} \Gamma(n + y_i n^\varepsilon + d - i + 1)}.$$

We begin by noting that $\Gamma(n + y_i n^\varepsilon + d - i + 1) \sim n^{d-i+1} \Gamma(n + y_i n^\varepsilon)$, so that

$$\frac{1}{\prod_{i=1}^{d} \Gamma(n + y_i n^\varepsilon + d - i + 1)} \sim \frac{1}{n^{d+1} \prod_{i=1}^{d} \Gamma(n + y_i n^\varepsilon)}.$$

Taking logarithms yields

$$\log\left(\frac{1}{\prod_{i=1}^{d} \Gamma(n + y_i n^\varepsilon + d - i + 1)}\right) \sim -\frac{d(d+1)}{2} \log n - \sum_{i=1}^{d} \log \Gamma(n + y_i n^\varepsilon).$$

Recall Stirling’s formula:

$$\log \Gamma(N) \sim \frac{1}{2} \log 2\pi + (N - \frac{1}{2}) \log N - N$$

for $N$ large. Thus

$$\sum_{i=1}^{d} \log \Gamma(n + y_i n^\varepsilon) \sim \frac{d}{2} \log 2\pi - dn + \sum_{i=1}^{d} (n + y_i n^\varepsilon - \frac{1}{2}) \log(n + y_i n^\varepsilon).$$
Now since
\[ \log(n + y_i n^\varepsilon) = \log n + \log(1 + y_i n^{\varepsilon-1}), \]
we have
\[ \sum_{i=1}^{d} (n + y_i n^\varepsilon - \frac{1}{2}) \log(n + y_i n^\varepsilon) = dn \log n - \frac{d}{2} \log n + \sum_{i=1}^{d} (n + y_i n^\varepsilon - \frac{1}{2}) \log(1 + y_i n^{\varepsilon-1}). \]

Using the expansion
\[ \log(1 + y_i n^{\varepsilon-1}) = y_i n^{\varepsilon-1} - \frac{1}{2} y_i^2 n^{2\varepsilon-2} + O(n^{3\varepsilon-3}) \]
for \( n \) sufficiently large, we find that
\[ \sum_{i=1}^{d} (n + y_i n^\varepsilon - \frac{1}{2}) \log(1 + y_i n^{\varepsilon-1}) \sim \frac{n^{2\varepsilon-1}}{2} \sum_{i=1}^{d} y_i^2. \]

Putting this all together, we find that
\[ \prod_{i=1}^{d} \frac{1}{\Gamma(n + y_i n^\varepsilon + d - i + 1)} \sim \frac{e^{dn}}{(2\pi)^{\frac{d}{2}} n^{dn + \frac{d^2}{2}}} e^{-\frac{n^{2\varepsilon-1}}{2} \sum_{i=1}^{d} y_i^2} \]
as \( n \to \infty. \)

The second group of factors is much easier to handle:
\[ \prod_{1 \leq i < j \leq d} ((y_i - y_j)n^{\varepsilon} + j - i) \sim n^{\varepsilon(d+1)} \prod_{1 \leq i < j \leq d} (y_i - y_j). \]

Thus when \( \varepsilon = \frac{1}{2} \) we have
\[ \dim(n + y_1 \sqrt{n}, \ldots, n + y_d \sqrt{n}) \sim \frac{\Gamma(dn + 1)e^{dn}}{(2\pi)^{\frac{d}{2}} n^{dn + \frac{d^2}{4}}} e^{-W(y_1, \ldots, y_d)} \]
as \( n \to \infty, \) as claimed. 

\[
\square
\]

4. Riemann sum

Consider the generalized sum
\[
S(d, dn; \beta) = \sum_{(\lambda \leq dn) \atop {\ell(\lambda) \leq d}} (\dim \lambda)^\beta, \quad \beta > 0,
\]
and its presentation as a sum in parameters around the rectangle \( R(d, n) : \)
\[
S(d, dn; \beta) = \sum_{\frac{(d-1)n}{\sqrt{n}} \geq y_i \geq \frac{yd-1}{\sqrt{n}} \atop {y_i \in \mathbb{Z}^d}} \dim(n + y_1 \sqrt{n}, \ldots, n + y_d \sqrt{n})^\beta,
\]
where \( y_d := -(y_1 + \cdots + y_{d-1}). \) The lattice \( \left( \frac{1}{\sqrt{n}} \mathbb{Z} \right)^{d-1} \) partitions \( \mathbb{R}^{d-1} \) into cells
\[
[ \frac{k_1}{\sqrt{n}}, \frac{k_1 + 1}{\sqrt{n}} ] \times \cdots \times [ \frac{k_{d-1}}{\sqrt{n}}, \frac{k_{d-1} + 1}{\sqrt{n}} ] \], \quad k_i \in \mathbb{Z},
\]
of volume \( \left( \frac{1}{\sqrt{n}} \right)^{d-1}. \) Scaling by the mesh volume makes this into a Riemann sum, and by Lemma 3.1 we have
\[
\lim_{n \to \infty} \left( \frac{1}{\sqrt{n}} \right)^{d-1} C_{d, dn}^\beta S(d, dn; \beta) = \int_{\mathbb{R}^{d-1}} e^{-\beta W(y_1, \ldots, y_d)} dy,
\]
where $C_{d,dn}$ and $W$ are as in Lemma 3.1 and $\Omega_{d-1} \subset \mathbb{R}^{d-1}$ is the region

$$\Omega_{d-1} = \{(y_1, \ldots, y_{d-1}) \in \mathbb{R}^{d-1} : y_1 > \cdots > y_d := -(y_1 + \cdots + y_{d-1})\}.$$  

The details of this convergence can be checked and made rigorous using the dominated convergence theorem; we refer the reader to [7, 8, 9] for the full argument.

Consider now the sum

$$\sum_{\mu \vdash \lambda_{dn}} (\dim \mu)^{\alpha} (\dim \mu^*)^{\beta-\alpha}$$

where $0 \leq \alpha \leq \beta$ and as before we denote $y_d := -(y_1 + \cdots + y_{d-1})$. This can again be viewed as a Riemann sum, and by exactly the same argument we have

$$\lim_{n \to \infty} \left( \frac{1}{\sqrt{n}} \right)^{d-1} C_{d,dn}^\beta \sum_{\mu \vdash \lambda_{dn}} (\dim \mu)^{\alpha} (\dim \mu^*)^{\beta-\alpha}$$

$$= \int_{\Omega_{d-1}} e^{-\alpha W(y_1, \ldots, y_d)} e^{-(\beta-\alpha)W(-y_d, \ldots, -y_1)} dy.$$  

5. Symmetry

Note that the function $W$ has the symmetry

$$W(y_1, \ldots, y_d) = W(-y_d, \ldots, -y_1).$$

It follows that

$$\int_{\Omega_{d-1}} e^{-\beta W(y_1, \ldots, y_d)} dy = \int_{\Omega_{d-1}} e^{-\alpha W(y_1, \ldots, y_d)} e^{-(\beta-\alpha)W(-y_d, \ldots, -y_1)} dy,$$

and thus we have proved the following.

**Theorem 5.1.** For any $\beta > 0$ and $0 \leq \alpha \leq \beta$, we have

$$S(d, dn; \beta) \sim \sum_{\mu \vdash \lambda_{dn}} (\dim \mu)^{\alpha} (\dim \mu^*)^{\beta-\alpha}$$

as $n \to \infty$.

Theorem 1.2 is the special case $\beta = 2, \alpha = 1$ of this more general asymptotic equivalence.

6. Conclusion

The multidimensional integral

$$\Psi(d; \beta) = \int_{\mathbb{R}^d} e^{-\frac{\beta}{2} \sum_{i=1}^{d} x_i^2} \prod_{1 \leq i < j \leq d} |x_i - x_j|^{\beta} dx, \quad \beta > 0,$$
is known as Mehta’s integral. It is the partition function of a Coulomb gas of \( d \) identical point charges \( x_1 > \cdots > x_d \) on the real line at inverse temperature \( \beta \), with energy functional

\[
W(x_1, \ldots, x_d) = \frac{1}{2} \sum_{i=1}^d x_i^2 - \sum_{1 \leq i < j \leq d} \log(x_i - x_j).
\]

Dyson and Mehta \[3\] studied this integral and conjectured the formula

\[
\Psi(d; \beta) = (2\pi)^{\frac{d}{2}} \beta^{-\frac{d}{2}} - \beta^2 \frac{d-1}{2} \prod_{i=1}^d \frac{\Gamma(1 + i \beta)}{\Gamma(1 + i \beta^2)},
\]

which they verified for \( \beta \in \{1, 2, 4\} \) using properties of Hermite polynomials (see e.g. \[5, \S 3.5.1\] for the \( \beta = 2 \) case of this argument). Later, Bombieri observed that, for general \( \beta \), \( (18) \) can be deduced from the Selberg integral formula. See \[4\] for the interesting history of this problem.

Regev \[8, Lemma 4.3\] showed that

\[
\int_{\Omega_{d-1}} e^{-\beta W(y_1, \ldots, y_d)} \, dy = \frac{1}{\Gamma(d+1)} \sqrt{\frac{\beta}{2\pi d}} \Psi(d; \beta),
\]

and used this fact together with equation (10) above to determine the asymptotics of \( S(d, N; \beta) \) from the known form (18) of \( \Psi(d; \beta) \). In this article, we have evaluated the asymptotics of \( S(d, N) = S(d, N; 2) \) directly, without appealing to the exact value of \( \Psi(d; 2) \). Thus, we may obtain

\[
\Psi(d; 2) = (2\pi)^{\frac{d}{2}} \beta^{-\frac{d}{2}} \frac{d+1}{2} \prod_{i=1}^{d+1} \Gamma(i)
\]

by substituting the asymptotic form of \( S(d, N) \) (Corollary 1.4) in (10) and using (19). It would be interesting to know if the asymptotics of \( S(d, N; \beta) \) can be determined directly in a similar way for general \( \beta > 0 \). If so, this would yield a new and elementary verification of (18). In particular, for this purpose one may assume that \( \beta \) is an even integer, see \[3\].

Finally, in this note we have only considered the asymptotics of \( S(d, N) \) in the single scaling limit where \( N \to \infty \) with \( d \) fixed. Baik, Deift, and Johansson \[2\] have shown that

\[
S(d, N) \sim F(t)N!
\]

in the double scaling limit where \( d, N \to \infty \) at the rate \( d \sim 2\sqrt{N} + tN^{1/6} \), with \( t \in \mathbb{R} \) fixed. Here \( F(t) \) is the Tracy-Widom distribution function, see \[2\]. Since

\[
S(d, dn) = \dim R(d, 2n) + E(d, n),
\]

if the error term \( E(d, n) \) can be effectively estimated in the double scaling limit then concrete estimates for \( F(t) \) will follow.

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