A Stochastic Optimal Control Of Dc Pension Fund Driven By Fractional Brownian Motion With Hurst Parameter $> \alpha = 2^{-1}$

Onwukwe Ijioma

Abstract—The problem of a fund manager is to minimize the expected utility loss function, the noise generated in the dynamics of the wealth process are driven by fractional Brownian motions with long range dependence if $H > 1/2$. We replaced the classical Brownian motion by fractional Brownian motion with $H > 1/2$. We finally use time-inversion of diffusions to obtain singular equations.

II. CLASSICAL PENSION FUND PROCESS:

Merton [9] pioneered most of the pension fund models. Consider a market structure with two financial assets. The riskless asset is governed by the dynamics, whose price at time $t$ is denoted by $S_0(t)$, $S_0(t) > 0$ is

$$dS_0(t) = rS_0(t)dt$$

where $r$ is the rate of interest.

The risky asset (stock) whose price at time $t$, denoted by $S_1(t)$ is governed by the dynamics defined by

$$dS_1(t) = S_1(t)(\mu dt + \sigma dw(t))$$

where $\mu$ and $\sigma$ are parameters explaining the expected instantaneous rate of return of the risky asset and volatility of respectively. $W(t)$ is a normalized Gaussian white noise with zero mean and unit variance.

Given a fixed contribution rate $\lambda$, the invested fund proportion in risky asset at time $t$ is given by $y(t)$ and the remaining proportion $(1 - y(t))$ is invested into the riskless asset.

The wealth process $x(t)$ at time $t$ is governed by the SDE

$$dx(t) = [y(t)x(t) + (1 - y(t))rx(t) + c]dt + x(t)\sigma dw(t)$$

where $x(0) = x_0$.

The aim of the pension fund management is to maximize the expected utility at the exit from the scheme at time $T$.

Under some suitable conditions of (3), the optimal control problem is formulated as:

$$\max_{\tau} \mathbb{E}(u(x(\tau)))$$

The HJB yield the result

$$\max \left\{ v_x + \frac{1}{2} \sigma^2 x^2 v_{xx} \right\} = 0$$

Where $v(t, x)$ is the value function, $v_t, v_x, v_{xx}$ respectively denote the partial derivatives with respect to time and wealth.

The first order maximizing conditions for the optimal strategy $y^*(t)$ is

$$y^*(t) = \frac{(r-\mu)x}{\sigma^2 x^2}$$

using a function:

$$\mu(x(T)) = \frac{y'(T)}{y}, \quad y \in (-\infty, 1)$$

Onwukwe Ijioma, Department of Mathematics, Abia State University, Uturu, Nigeria
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\[ y^*(t) = \frac{\mu - r}{\sigma}, \quad \frac{1}{(1-y)^2} \] (7)

The prevailing government policies make some noises not to be Gaussian white noises, but fractional white noises with short range dependence.

III. FRACTIONAL BROWNIAN MOTION:

Fractional Brownian motion (fBm) with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process $B^H = \{ B^H_t, t \geq 0 \}$ with covariance function:

\[ R_H(s, t) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right) \] (8)

The (fBm) can be constructed from the classical Brownian motion which was proposed by Mandelbrot and Van Ness (1968), where a stochastic integral representation in terms of a standard Brownian motion was obtained.

The self-similar and long range dependence (if $H > \frac{1}{2}$) as proposed in this work and the properties of fBm make this process a useful driving noise in models arising in physics, telecommunication networks, finance and other fields.

We define self-similarity of a process $X = (X_t, t \in (-\infty, \infty))$ by means of its distribution. If $(X_{at})$ and $a^H(X_{ay})$ have identical finite-dimensional distributions for all $a > 0$, then $X$ is self-similar with parameter $H$.

Definition 1: A process $X$ is called (exactly) second-order self-similar with self-similarity parameter $H = 1 - \frac{\beta}{2}$ if for all $m = 1, 2, \ldots$, $V\text{ar}[X^{(m)}] = \sigma^2 m^{-\beta}$

And

\[ r^{(m)}_H(\cdot) = r_H(\cdot) = \frac{1}{2}((k + 1)^{2H} - 2k^{2H} + |k - 1|^{2H}), \quad k \geq 0 \] (9)

where $r_H(\cdot)$ is the autocorrelation function of $X^{(m)}$

Definition 2: A process $X$ is called (asymptotically) second-order self-similar with self-similarity parameter $H = 1 - \frac{\beta}{2}$ if for all large $k$, then $r^{(m)}_H(\cdot)$ as $m \to \infty$.

From $B^H$ is not a semi martingale and is not a Markov process if $H \neq \frac{1}{2}$ (see Rogers, L.C.G.(1997), this means that the usual stochastic calculus is not applicable for $B^H, t \geq 0$, if $H \in \left( \frac{1}{2}, 1 \right)$.

Robust research has been done in the recent years on new techniques to define stochastic integrals with respect to fBm.

Our interest is when $H > \frac{1}{2}$, then one can use a path-wise approach to define integrals with respect to the fBm (see Young, L.C.(1936)). Alternatively, the approach to define path-wise integrals with respect to a fBm when $H > \frac{1}{2}$ is based on fractional calculus as introduced by Feyel and Del la Pradelle(1996).

We are interested in the structure and properties of one-dimensional linear stochastic differential equation defined as:

\[ X_t = B_t^H + \int_0^t X_u d\mu(u), 0 \leq t \leq 1 \]

where $B_t^H$ is a fractional Brownian motion with Hurst parameter $H \in \left( \frac{1}{2}, 1 \right)$.

IV. PROBLEM FORMULATION:

Let $\Omega = C_0(\mathbb{R}^+, \mathbb{R})$ be the Frechet space of real-valued continuous functions on $\mathbb{R}^+$ with the initial value zero and topology of local uniform convergence.

For a fixed Hurst constant $H, \frac{1}{2} < H < 1$.

Define

\[ \Phi(s, t) = H(2H - 1)|s - t|^{2H - 2}; \quad s, t \in \mathbb{R} \] (11)

Let $f: \mathbb{R} \to \mathbb{R}$ be measurable.

Then $f \in L^2_{\Phi}(\mathbb{R})$ if $\int_{\mathbb{R}} f^2 \Phi dsdt < \infty$ (12)

Let $L^2_{\Phi}(\mathbb{R})$ be equipped with the inner product

\[ (f, g)_\Phi = \int_{\mathbb{R}} f(s)g(t)\Phi(s,t)dsdt \]

Then $L^2_{\Phi}(\mathbb{R})$ becomes a separable Hilbert space with a probability measure $B^H$ on $(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the usual Borel algebra on $\Omega$, the process is a fractional Brownian motion (fBm).

By direct computation of (11)

\[ E[B^H(t)B^H(s)] = \int_{\mathbb{R}^2} \Phi_{\Phi}(u - v)dudv \] (15)

Let $(X_t, t \geq 0)$ be the Gaussian process having the solution of

\[ X_t = B^H_t + \int_0^t X_u d\mu(u), 0 \leq t \leq 1 \] (16)

where $\mu$ is a Radon diffuse measure on $[0,1]$.

By Bochner-Minlos theorem there exist a probability measure $\mu_\Phi$ such that
\[ \int \exp(i \langle \omega, f \rangle) d\mu_\omega(\omega) = e^{-\frac{1}{2}||f||_2^2} \text{ or all } f \in S(R) \quad (17) \]

It follows from (17) that
\[ E_{\mu_\omega}[(t; f)] = 0 \quad \text{and} \quad E_{\mu_\omega}[(t; f)^2] = ||f||_1^2 \quad (18) \]

We can redefine
\[ \hat{B}_H(t) = \hat{B}_H(t, \omega) = \langle \omega, X_{[0,t]}(\cdot) \rangle \text{ as an element of } L^2(\mu_\phi) \text{ for each } t \in R, \text{ by Kolmogorov's theorem} \]

\[ \hat{B}_H(t) \text{ has a } t \text{-continuous denoted by } B_H(t). \]

From (18), it is clear that \( B_H(t) \) is a Gaussian process with
\[ E_{\mu_\phi}[\hat{B}_H(t)] = 0 \quad \text{and} \quad E_{\mu_\phi}[\hat{B}_H(s)\hat{B}_H(t)] = \frac{1}{2} \left[ |t|^2 + |s|^2 - |t-s|^2 \right] \]

(19)

It follows that \( B_H(t) \) is a fractional Brownian motion.

V. TIME-INVERSION OF DIFFUSION PROCESS.

We consider the following singular SDE:
\[ X_t = x + B^H_t + \int_0^t b(u, X_u) \, du, \quad t \geq 0 \quad (20) \]

where the function \( b(s, x) \) has a singularity at \( s = 0 \).

We now establish the connection to some diffusions \((X_s, t \geq 0)\) which is a solution to (20) as
\[ X_s = B^H_t + 2H \int_0^s \frac{X_t}{s} \, ds, \quad t \geq 0 \quad (21) \]

using time-inversion. Let the process \((X_s, t \geq 0)\) be a adapted to the filtration of \((B^H_t)\) and (20) one strong solution. By special choice of \( s \in (0, t) \), we have
\[ X_s = S^{-1/2} \int_s^t X_u \, du + \int_s^t b(u, X_u) \, du + \int_s^t dB^H_u \]  

(22)

\[ \lim_{t \to 0} \frac{X_s}{s^{1/2}} \rightarrow 0 \quad \text{and} \quad \lim_{t \to +\infty} \frac{X_s}{s^{1/2}} \exists \text{ exist a.s.} \]

(23) using (22) and (23), it follows that
\[ \frac{X_s}{s^{1/2}} - 2H \int_s^t \frac{X_u}{u^{2H+1}} \, du + \int_s^t b(u, X_u) \, du + \int_s^t dB^H_u = 0 \]

(24)

and
\[ \beta^H_t = -\lim_{t \to 0} \int_s^t dB^H_u / u^{2H}, \quad t > 0 \]

is a fBm.

(25)

Definition 4: Let \((S)_H\) be the set of all \( \psi(\omega) = \sum_{a \in \mathcal{A}} a_a H_a(\omega) \in L^2(\mu_\phi) \) such that
\[ \|\psi\|_{H,k}^2 = \sum_{a \in \mathcal{A}} a^2 \|2N\| \alpha_k < \infty \quad \forall k \in N \quad (26) \]

where \((2N)^\gamma = \prod_{j=1}^N (2N)_{\gamma_j} \) if \( \gamma = (\gamma_1, \ldots, \gamma_m) \in I \)

Definition 5: Let \((S)_H^*\) be the set of all formal expansions:
\[ G(\omega) = \sum_{\beta \in \mathcal{B}} b^\beta H^\beta(\omega) \]  

such that
\[ ||G||_{H,-q}^2 = \sum_{\beta \in \mathcal{B}} \beta! b^\beta (2N)^{-q|\beta|} < \infty \text{ for some } q \in N \quad (27) \]

The fractional white noise \( W_H(t) \) at time \( t > 0 \) is defined by
\[ W_H(t) = \sum_{i=1}^\infty \left[ e_i(\nu) \phi(t, \nu) d\nu \right] H_{\xi(\nu)}(\omega) \]
for \( q > (\Gamma = 1.33) \) we have,
\[ ||W_H(t)||^2_{H,-q} = \sum_{i=1}^\infty e_i(\nu)^2 \left[ e_i(\nu) \phi(t, \nu) d\nu \right]^2 \times \]

(28)

(29)

By integrability of \( W_H(t) \) in \((S)_H^*\) for \( s \in [0, t] \) then
\[ \int_0^t W_H(s) \, ds = \sum_{i=1}^\infty \left[ \int_0^t e_i(\nu) \phi(u, \nu) d\nu \right] H_{\xi(\nu)}(\omega) \]

\[ = B_H(t) \quad (30) \]

If reasonable assumptions, then \( t \to B_H(t) \) is differentiable in \((S)_H^*\) and
\[ \frac{dB_H(t)}{dt} = W_H(t) \in (S)_H^* \]

justifying fractional SDE governed by
\[ dX(t) = \mu X(t) dt + \sigma X(t) dB_H(t) \quad ; \quad X(0) = x > 0 \]

where \( x, \mu \) and \( \sigma \)

are constants.

We recast the equation (31) in \((S)_H^*\):
\[ \frac{dX(t)}{dt} = \mu X(t) + \sigma X(t) \cdot W_H(t) \]

(32)

and
\[ \psi \cdot \eta \in (S)_H^* \]

And Wick calculus, we have solution of the type:
\[ X(t) = x e^{\mu t + \sigma \int_0^t W_H(s) \, ds} \]

\[ = x e^{\mu t + \sigma B_H(t)} \quad (34) \]

Using the property of Hermite polynomial, if \( f \in L_2^2(\mathcal{R}), \) we have

Lemma 1:

(1) \( F, G \in (S)_H^* \Rightarrow F \circ G \in (S)_H^* \)
(2) \( \psi, \eta \in (S)_H^* \Rightarrow \psi \circ \eta \in (S)_H^* \)
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\[
\mathbb{E}^\pi \left( \langle w, f \rangle \right) = \mathbb{E}^\pi \left( \sum_i \langle f, e_i \rangle \mathbb{E}^\pi \left( w, e_i \right) \right) 
= \mathbb{E} \left( \langle w, f \rangle - \frac{1}{2} \| f \|^2 \right) \tag{35}
\]

It follows then, that

\[
\mathbb{E}^\pi \left( \langle w, f \rangle \right) = \mathbb{E} (f^\pi) \quad \forall f \in L^2_\mathbb{H}(\mathcal{R}) \tag{36}
\]

And a solution is of the form

\[
X(t) = x \exp \left( \sigma B_\mathbb{H}(t) + \mu t - \frac{1}{2} \sigma^2 t^{2\alpha} \right) \tag{37}
\]

VI. CONCLUSION:

In this paper, the aim of the pension fund manager is to minimize the expected utility loss function. The solution of the derived diffusion problem associated with maximization of the return of the investment of the funds over a finite time horizon and the derivation of the time-inversion for the process which is verified to be a smooth solution of the recast problem as a fractional Brownian motion with Hurst parameter \( > 0.5 \).

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