Tarski’s classical relevant logic

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Abstract. Tarski’s classical relevant logic TR arises from his work on the foundations of the calculus of relations and on first-order logic restricted to finitely many variables. The theorems of TR are defined here as the formulas whose translations into first-order logic of binary relations can be proved using no more than four variables from the assumptions that all the relations are dense and commute under composition. Its rules are determined similarly. The vocabulary of TR is the same as the classical relevant logic CR* proposed by Meyer and Routley [28, 29]. TR properly contains CR*, since the class of model structures characteristic for TR is the class of those CR*-model structures that satisfy Dunn’s frame condition called “tagging”. There is a formula in TR \ CR* that corresponds to tagging and provides a counterexample to a theorem of Kowalski. The class of model structures characteristic for TR is the class of atom structures of atomic commutative dense relation algebras. An equation is true in every commutative dense relation algebra if and only if it can be established by a proof in first order logic restricted to four variables that the equation is true when formulas are interpreted as relations.

1 Introduction

In 1975, Alfred Tarski delivered a pair of lectures on relation algebras at the University of Campinas. At the end of his second lecture, Tarski said,

“And finally, the last question, if it is so, you could ask me a question whether this definition of relation algebra which I have suggested and which I have founded — I suggested it many years ago — is justified in any intrinsic sense. If we know

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that these are not all equations which are needed to obtain representation theorems, this means, to obtain the algebraic expression of first-order logic with two-place predicate, if we know that this is not an adequate expression of this logic, then why restrict oneself to these equations? Why not to add strictly some other equations which hold in representable relation algebras or maybe all?”

The answer is that Tarski’s definition of relation algebras axiomatizes the equations provable with 4 variables (Theorem 5 in $§9$). Since Tarski was axiomatizing the calculus of relations, he had to choose axioms that are true under the formulas-as-relations interpretation in which variables denote binary relations and operation symbols denote operations on relations. Since Meyer and Routley use the same operations as relation algebras, the formulas-as-relations interpretation can be applied to their classical relevant logic $\text{CR}^*$, as is done in Table 1. The formulas-as-relations interpretation generalizes the formulas-as-classes interpretation, but reduces to it when the base set $U$ has only one element. Interpreting formulas as binary relations on a one-element set leads to classical propositional calculus. A formula $A$ is valid (under the formulas-as-relations interpretation) if $A$ denotes a relation containing the identity relation $t$ whenever its variables are mapped to relations. Every for-
By Theorem 14 in §17, formula (41) is in the primitive vocabulary of \( R \), is not a theorem of \( R \), and is valid in a model structure if and only if the structure satisfies \( Ra^{*}bc \Rightarrow Racb \), a frame condition equivalent to tagging in the presence of the postulates for CR*-model structures in §12. Formula (41) happens to be 3-provable. Tarski had asked whether simply removing the associative law would axiomatize the 3-provable equations. The answer is no, but the 3-provable equations can be axiomatized by weakening the associative law to a 3-provable version called the semi-associative law (Theorem 4 in §9). We define Tarski’s 3-variable classical relevant logic \( CT_3 \) as the formulas provable with 3 variables, while \( CT_4 \) is set of formulas provable with 4 variables. Then (41) is in \( CT_3 \) but not \( CR^* \).

§§2–9 summarize Tarski’s work on the foundations of the calculus of relations and logic with finitely many variables. The details are taken from [37]. §2 presents first order logic \( L \), which is extended to \( L^+ \) in §3. These two formalisms are seen to be equippollent in §4. The equational formalism \( L^\times \) appears in §5 and is compared to \( L^+ \) in §6, where relation algebras RA, representable relation algebras RRA, and equational logic are reviewed.

Tarski’s 3-variable formalisms \( L^+_3 \) and \( L_3 \) are presented in §7, and are seen to be equippollent with \( L^\times \) in §8. These formalisms include the associative law, which requires four variables to prove. Weakening it to the semi-associative law produces the equational formalism \( L^w\times \), seen to be equippollent to the standardized 3-variable formalisms \( L_s \) and \( L^+_s \) in §9.

Tarski’s relevance logics are defined in §10 and characterized in §13 in terms of provability (using results from §8 and §9), sequent calculus (presented in §11), and model structures (discussed in §12). Many formulas and derived rules of inference for \( CT_3 \) are listed in §14. §15 has formulas that are in \( CT_4 \) (4-provable) but not \( CT_3 \) (3-provable), such as the associative law for fusion. §16 has formulas of \( R \) that hold only when fusion is commutative. §17 presents a formula that is in \( T_3 \), is not in \( R \), answers Dunn’s question, and disproves a theorem of Kowalski (see §18). §19 has formulas in \( CT_5 \setminus CT_4 \). §20 defines \( TR \), a proper extension of \( CT_4 \) and \( CR^* \). §21 links \( KR \) to symmetric dense relation algebras. §22 characterizes the Dunn-McColl logic \( RM \) as an extension of \( T_4 \). §23 has a concluding discussion and some questions.

2 First order logic \( L \) of binary relations

For Tarski and Givant, \( L \) is a first-order language with equality symbol \( 1 \) and exactly one binary relation symbol \( E \), while \( M^{(n)} \) is a first-order language with equality \( 1 \) and exactly \( n \) binary relation symbols, \( n \geq 0 \). We assume instead that \( 1' \) is the equality symbol of \( L \) and that \( L \) has a countable infinite set \( II \) of binary relation symbols (including \( 1' \)), but no function symbols or constants. The elements of \( II \) are called atomic predicates, and those that are distinct from \( 1' \) are also called propositional variables. The
connectives of $\mathcal{L}$ are implication $\Rightarrow$ and negation $\lnot$, and the quantifier is $\forall$. The atomic formulas of $\mathcal{L}$ are the ones of the form $xA$, where $x, y$ are variables and $A \in \Pi$ is an atomic predicate. For example, $x'y$ is an atomic formula since $1' \in \Pi$. The set of formulas of $\mathcal{L}$ is $\Phi$, defined as the intersection of every set that contains the atomic formulas and includes $\varphi \Rightarrow \psi$, $\lnot \varphi$, and $\forall x \varphi$ for every variable $x$ whenever it contains $\varphi$ and $\psi$. The set of sentences of $\mathcal{L}$ (formulas with no free variables) is $\Sigma$. The connectives $\lor$, $\land$, $\iff$, and quantifier $\exists$ are defined on formulas $\varphi, \psi \in \Phi$ by $\varphi \lor \psi = \lnot \varphi \Rightarrow \psi$, $\varphi \land \psi = \lnot (\varphi \Rightarrow \lnot \psi)$, $\varphi \iff \psi = \lnot ((\varphi \Rightarrow \psi) \Rightarrow \lnot (\psi \Rightarrow \varphi))$, and $\exists x \varphi = \lnot \forall x \lnot \varphi$ for any variable $x$.

In formulating axioms and deductive rules for $\mathcal{L}$, Tarski and Givant [37, p. 8] adopted the system $S_1$ of Tarski [36], which provides axioms for the logically valid sentences and requires only the rule $\text{MP}$ of modus ponens (to infer $B$ from $A \Rightarrow B$ and $A$). Tarski’s system $S_2$ provides axioms for the logically valid formulas (not just the sentences), and uses the rule of generalization (to infer $\forall \varphi$ from $\varphi$) as well as $\text{MP}$. The systems $S_1$ and $S_2$ in Tarski [36] were obtained by modifying a system of Quine [31, 32, 33, 34] which also uses only $\text{MP}$. Tarski’s systems avoid the notion of substitution. Henkin [10, 11] proved Gödel’s Completeness Theorem for the case in which there are relation symbols of arbitrary finite rank but no constants and no function symbols. He used $\text{MP}$ and a restricted form of generalization as rules of inference. Tarski [36, Th. 1, Th. 5] proved that his systems $S_1$ and $S_2$ are complete by deriving Henkin’s axioms and noting that both systems are semantically sound.

For every formula $\varphi \in \Phi$, the closure $[\varphi]$ of $\varphi$ is a sentence obtained by universally quantifying $\varphi$ with respect to every free variable in $\varphi$. The closure operator is determined by the following conditions: if $\varphi \in \Sigma$ is a sentence, then $[\varphi] = \varphi$, and if $x$ is the last variable (in the ordering of the variables) that occurs free in $\varphi$, then $[\varphi] = [\forall x \varphi]$. The logical axioms for $\mathcal{L}$ are (AI)–(AIX), where $\varphi, \psi \in \Phi$ and $x, y$ are variables, shown in Table 3. If $\Psi \subseteq \Sigma$, then a sentence $\varphi \in \Sigma$ is provable in $\mathcal{L}$ from $\Psi$, written $\Psi \vdash \varphi$ or $\vdash \varphi$ if $\Psi = \emptyset$, iff $\varphi$ is in the intersection of all sets that contain $\Psi$ and the axioms of $\mathcal{L}$, and are closed under $\text{MP}$. Two formulas $\varphi, \psi \in \Phi$ are provably equivalent in $\mathcal{L}$, written $\varphi \equiv \psi$, if $\{\varphi\} \vdash \psi$ and $\{\psi\} \vdash \varphi$.

3 Extending $\mathcal{L}$ to $\mathcal{L}^+$

Tarski and Givant extend $\mathcal{L}$ to $\mathcal{L}^+$ by adding two binary and two unary operators that act on relation symbols and produce new relation symbols. $\Pi^+$, the set of predicates of $\mathcal{L}^+$, is the intersection of every set containing $\Pi$ that also contains $A + B$, $\overline{A}$, $A; B$, and $A^{-1}$ whenever it contains $A$ and $B$. Predicates obtained in distinct ways are distinct, so, for example, if $A + B = C + D$ then $A = C$ and $B = D$. They add a second equality symbol =
Table 2 Axioms for $F^+$ are those of $F$ plus (DI)–(DV). Axioms for $L^\times$ and RA coincide, as do $L^w\times$ and SA. Formalisms between horizontal lines are equipollent.

\[
\begin{align*}
\text{(AI)} &- ((\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \xi) \Rightarrow (\varphi \Rightarrow \xi))) \\
\text{(AII)} &- (\neg \varphi \Rightarrow \varphi) \\
\text{(AIII)} &- (\varphi \Rightarrow (\neg \varphi \Rightarrow \psi)) \\
\text{(AIV)} &- (\forall x \forall y \varphi \Rightarrow \forall y \forall x \varphi) \\
\text{(AV)} &- (\forall x (\varphi \Rightarrow \psi) \Rightarrow (\forall x \varphi \Rightarrow \forall x \psi)) \\
\text{(AVI)} &- (\forall x \varphi \Rightarrow \varphi), \text{ where } x \text{ is not free in } \varphi \\
\text{(AVII)} &- (\exists x (x \neq y)) \\
\text{(AVIII)} &- (\forall x \varphi \Rightarrow \varphi) \Rightarrow (\varphi \Rightarrow \psi) \text{ where } \varphi \text{ is atomic, } x \text{ occurs in } \varphi, \\
\end{align*}
\]

Table 3 Axioms for first-order logic $L$

(they used $\equiv$) and a formula $A = B$, called an equation, for any predicates $A, B \in \Pi^+$. $\Sigma^\times$ is the set of equations of $L^\times$. The atomic formulas of $L^+$ are $xAy$ and $A = B$, where $x, y$ are variables and $A, B \in \Pi^+$. $\Phi^+$ is the set of formulas of $L^+$, the intersection of every set containing the atomic formulas of $L^+$ that also contains $\varphi \Rightarrow \psi$, $\neg \varphi$, and $\forall x \varphi$ for every variable $x$ whenever it contains $\varphi$ and $\psi$. $\Sigma^+$ is the set of sentences of $L^+$. Equations have no free variables, so they are sentences: $\Sigma^\times \subseteq \Sigma^+$.

The logical axioms of $L^+$ [37, p. 25] are the logical axioms of $L$ together with the sentences in Table 4, where $A, B \in \Pi^+$ and $x, y, z$ are the first three
variables. If $\Psi \subseteq \Sigma^+$, then a sentence $\varphi \in \Sigma^+$ is **provable in** $\mathcal{L}^+$ **from** $\Psi$, written $\Psi \vdash^+ \varphi$ or $\vdash^+ \varphi$ if $\Psi = \emptyset$, iff $\varphi$ is in the intersection of all sets that contain $\Psi$ and the axioms of $\mathcal{L}^+$, and are closed under MP. Two formulas $\varphi, \psi \in \Pi^+$ of $\mathcal{L}^+$ are **provably equivalent in** $\mathcal{L}^+$, written $\varphi \equiv^+ \psi$, if $\{\varphi\} \vdash^+ \psi$ and $\{\psi\} \vdash^+ \varphi$.

The **calculus of relations** may be defined as the set of sentences $A = B$ such that $\vdash^+ A = B$. One may also consider it to be the closure of this set under the propositional connectives, since Schröder and Tarski showed that every propositional combination of equations is equivalent to an equation.

## 4 Equipollence of $\mathcal{L}$ and $\mathcal{L}^+$

$\mathcal{L}$ and $\mathcal{L}^+$ are expressively and deductively equipollent. To prove this, Tarski recursively defined a **translation** (or **elimination** oning) **mapping** $G$ from formulas of $\mathcal{L}^+$ to (what turn out to be logically equivalent) formulas of $\mathcal{L}$ [37, 2.3(iii)]. $G$ eliminates operators in accordance with the definitional axioms (DI)–(DV). If $\varphi, \psi \in \Phi^+$, $x, y$ are variables, and $A, B$ are relation symbols of $\mathcal{L}^+$, then the conditions determining $G$ are shown in Table 5. From the first four conditions it follows that $G$ leaves formulas of $\mathcal{L}$ unchanged. The next result states that $\mathcal{L}$ is a subformalism of $\mathcal{L}^+$, and $\mathcal{L}$ is expressively and deductively equipollent with $\mathcal{L}^+$. Part (4) is the **main mapping theorem** for $\mathcal{L}$ and $\mathcal{L}^+$.

### Theorem 1. [37, §2.3]

1. $\Phi \subseteq \Phi^+$ and $\Sigma \subseteq \Sigma^+$, [2.3(i)]
2. $G$ maps $\Phi^+$ onto $\Phi$ and $\Sigma^+$ onto $\Sigma$, [2.3(iv)(δ)]
3. $\varphi \equiv^+ G(\varphi)$ for every $\varphi \in \Phi^+$, [2.3(iv)(ε)]
4. $\Psi \vdash^+ \varphi$ iff $\{G(\psi) : \psi \in \Psi\} \vdash^+ G(\varphi)$, for all $\Psi \subseteq \Sigma^+$ and $\varphi \in \Sigma^+$. [2.3(v)]
5. $\Psi \vdash^+ \varphi$ iff $\Psi \vdash \varphi$, for all $\Psi \subseteq \Sigma$ and $\varphi \in \Sigma$. [2.3(ii)(ix)].

\[
\begin{align*}
[xA + By \iff xAy \lor xBy] & \quad \text{(DI)} \\
[x\overline{A}y \iff \overline{x}Ay] & \quad \text{(DII)} \\
[xA; By \iff \exists z (xAz \land zBy)] & \quad \text{(DIII)} \\
[xA^{-1}y \iff yAx] & \quad \text{(DIV)} \\
A = B \iff [xAy \iff xBy] & \quad \text{(DV)}
\end{align*}
\]

Table 4  **Definitional axioms for extension** $\mathcal{L}^+$
Tarski’s classical relevant logic

\[ G(xAy) = xAy \quad \text{if} \ A \in \Pi, \]
\[ G(\varphi \Rightarrow \psi) = G(\varphi) \Rightarrow G(\psi), \]
\[ G(\neg \varphi) = \neg G(\varphi), \]
\[ G(\forall x \phi) = \forall x G(\phi), \]
\[ G(xA + By) = G(xAy) \lor G(xBy), \]
\[ G(x\overline{A}y) = \neg G(xAy), \]
\[ G(xA; By) = \exists z (G(xAz) \land G(zBy)) \]
\[ \text{where } z \text{ is the first variable distinct from } x,y, \]
\[ G(xA^{-1}y) = G(yAx), \]
\[ G(A = B) = [G(xAy) \Leftrightarrow G(xBy)] \]
\[ \text{where } x,y \text{ are the first two variables.} \]

Table 5 Definition of elimination mapping \( G: \Phi^+ \rightarrow \Phi. \)

5 Equational formalism \( \mathcal{L}^\times \)

Tarski and Givant define a formalism \( \mathcal{L}^\times \) with equational axioms and deductive rules for equality. The axioms of \( \mathcal{L}^\times \) are (R1)–(R10), where \( A, B, C \in \Pi^+, \) shown in Table 6. (R1)–(R10) are the axioms for relation algebras. De-

\begin{align*}
A + B &= B + A, & (R_1) \\
A + (B + C) &= (A + B) + C, & (R_2) \\
A + (B + C) &= (A + B) + C, & (R_3) \\
A; (B; C) &= (A; B); C, & (R_4) \\
(A + B); C &= A; C + B; C, & (R_5) \\
A; 1 &= A, & (R_6) \\
(A^{-1})^{-1} &= A, & (R_7) \\
(A + B)^{-1} &= A^{-1} + B^{-1}, & (R_8) \\
(A; B)^{-1} &= B^{-1}; A^{-1}, & (R_9) \\
A^{-1}; A; B + B &= B, & (R_{10})
\end{align*}

Table 6 Axioms of equational formalism \( \mathcal{L}^\times \)

ducibility in \( \mathcal{L}^\times \) is defined as it is in equational logic. The transitivity rule Trans is to infer \( A = B \) from \( A = C \) and \( B = C \), and the replacement rule REPL is to infer \( A^{-1} = B^{-1}, \overline{A} = \overline{B}, C; A = C; B, A; C = B; C, C + A = C + B, \) and \( A + C = B + C \) from \( A = B \). See [37, §3.1] for further discussion of these rules. For any \( \Psi \subseteq \Sigma^\times \), an equation \( \varepsilon \in \Sigma^\times \) is provable in \( \mathcal{L}^\times \) from \( \Psi, \)
written $\Psi \vdash^x \varepsilon$ or $\vdash^x \varepsilon$ if $\Psi = \emptyset$, iff $\varepsilon$ is in the intersection of all sets that contain $\Psi$ and the axioms of $L^\times$, and are closed under Trans and REPL.

6 Comparing $L^\times$ with $L^+$

$L^\times$ is a subformalism of $L^+$; it has a subset of its sentences, $\Sigma^\times \subseteq \Sigma^+$, and it is easy to prove, using the completeness of Tarski’s axiomatization, that provability in $L^\times$ implies provability in $L^+$. $L^\times$ is weaker than $L$ and $L^+$ in means of expression. This is due to Korselt’s result, reported by Löwenheim [15], that no equation in $\Sigma^\times$ is logically equivalent to a sentence asserting the existence of four distinct objects, such as

$$\exists w_1 (w_1 x \lor w_1 y \lor w_1 z).$$

Korselt’s theorem was greatly generalized by Tarski [37, 3.5(viii)].

Tarski knew only that he could prove in $L^\times$ the “hundreds of theorems” of Schröder. Does he still need $L^+$? Is every equation provable in $L^+$ already provable in $L^\times$? Tarski wrote in the early 1940s, “It seems very probable that the answer to these questions is affirmative” [35, p. 169]. To make sense of these problems, define the predicate algebra to be

$$\mathfrak{P} = \langle \Pi^+, +, \cdot, -1, 1 \rangle.$$

Let $\mathfrak{A}$ be an algebra with the same similarity type as $\mathfrak{P}$, say

$$\mathfrak{A} = \langle S, +, \cdot, -1, 1 \rangle,$$

where $S$ is a set, $1' \in S$, $+$ and $\cdot$ are binary operations on $S$, and $-$ and $^{-1}$ are unary operations on $S$. An equation $A = B \in \Sigma^\times$ is true in $\mathfrak{A}$ if $h(A) = h(B)$ for every homomorphism $h : \mathfrak{P} \to \mathfrak{A}$. The algebra $\mathfrak{A}$ is a relation algebra if the equations $\text{R}(1)\text{--R}(10)$ are true in $\mathfrak{A}$ for all predicates $A, B, C \in \Pi^+$. $\mathfrak{A}$ is commutative if the equation $A; B = B; A$ is true in $\mathfrak{A}$ for all $A, B \in \Pi^+$, dense if $A \leq A; A$ is true in $\mathfrak{A}$ for every $A \in \Pi^+$, and symmetric if $A^{-1} = A$ is true in $\mathfrak{A}$ for every $A \in \Pi^+$.

For example, the relations $<$, $>$, $\leq$, and $\geq$ on the rationals $\mathbb{Q}$ are called dense linear orderings because they satisfy the equation $A \leq A; A$, which asserts that between any two rationals there is another. Along with the empty relation $\emptyset$, universal relation $\mathbb{Q}^2$, diversity relation $\neq$, and identity relation $=$, they form an 8-element commutative dense relation algebra called the Point Algebra. Belnap’s $M_0$ [3] is obtained by using only the operations $\lor$, $\land$, $\to$, and $\sim$ from Table 1 [25, Theorem 4.1].

The next lemma, a special case of the completeness theorem for equational logic, provides the link between relation algebras and deducibility in $L^\times$. 
Lemma 1. An equation is provable in $\mathcal{L}^\times$ if and only if it is true in every relation algebra, that is, if $A = B \in \Sigma^\times$ then $\vdash^\times A = B$ is equivalent to the condition that $h(A) = h(B)$ for every $\mathfrak{A} \in RA$ and every homomorphism $h : \mathfrak{P} \rightarrow \mathfrak{A}$.

Proof. Let $\Theta$ be the set of equations true in every $RA$. By the definition of $RA$, the axioms (R$_1$)--(R$_{10}$) are in $\Theta$. Next we show that $\Theta$ is closed under the rules Trans and Repl. To see this for Trans, assume $A = C$ and $B = C$ are in $\Theta$. We must show $A = B$ is in $\Theta$, so assume that we have a homomorphism $h : \mathfrak{P} \rightarrow \mathfrak{A}$ for some $\mathfrak{A} \in RA$. Since $A = C$ and $B = C$ are in $\Theta$, we have $h(A) = h(C)$ and $h(B) = h(C)$, hence $h(A) = h(B)$. Thus $A = B$ in $\Theta$. For Repl, assume $A = B$ is in $\Theta$. To show the conclusions of Repl are in $\Theta$, we assume that we have a homomorphism $h : \mathfrak{P} \rightarrow \mathfrak{A}$ for some $\mathfrak{A} \in RA$. Since $A = B \in \Theta$, we have $h(A) = h(B)$. This implies $(h(A))^{-1} = (h(B))^{-1}$, $h(A) = h(B)$, and, for any $C \in \Pi^+$, $h(A);h(C) = h(B);h(C)$, $h(C);h(A) = h(C);h(B)$, $h(A) + h(C) = h(B) + h(C)$, and $h(C) + h(A) = h(C) + h(B)$. Since $h$ is a homomorphism, $h(A^{-1}) = h(B^{-1})$, $h(A) = h(B)$, $h(A;C) = h(B;C)$, $h(C;A) = h(C;B)$, $h(A + C) = h(B + C)$, and $h(C + A) = h(C + B)$, so the conclusions of Repl are also in $\Theta$. Since $\Theta$ is a set containing the axioms of $\mathcal{L}^\times$ and is closed under the rules of inference of $\mathcal{L}^\times$, it contains the set of equations provable in $\mathcal{L}^\times$. This shows that if an equation is provable in $\mathcal{L}^\times$ then it is true in $RA$.

For the converse, we need to construct quotient algebras of $\mathfrak{P}$. Define the binary relation $\approx^\times$ on $\Pi^+$ by $A \approx^\times B$ iff $\vdash^\times A = B$ for $A,B \in \Pi^+$. Because of the rules Trans and Repl, $\approx^\times$ is a congruence relation on $\mathfrak{P}$, and so it determines a quotient homomorphism $q : \Pi^+ \rightarrow \{A/\approx^\times : A \in \Pi^+\}$ that carries each predicate $A$ to its equivalence class $A/\approx^\times$ under $\approx^\times$. The quotient algebra $\mathfrak{P}/\approx^\times$ is a relation algebra. To show this, we assume $k : \mathfrak{P} \rightarrow \mathfrak{P}/\approx^\times$ is a homomorphism and show for any axiom $A = B$ that $k(A) = k(B)$. We do just one example, say an instance $A + B = B + A$ of (R$_1$). There are $C,D \in \Pi^+$ such that $k(A) = C/\approx^\times$ and $k(B) = D/\approx^\times$, so

$$
k(A + B) = k(A) + k(B) = (C/\approx^\times) + (D/\approx^\times)
$$

$$
= (C + D)/\approx^\times
$$

$$
= (D + C)/\approx^\times
$$

$$
= (D/\approx^\times) + (C/\approx^\times)
$$

$$
= k(B) + k(A)
$$

$$
= k(B + A)
$$

$k$ is a homomorphism

quotient homomorphism

quotient homomorphism

Proofs for the other axioms are similar.

Now suppose that $A = B$ is not provable in $\mathcal{L}^\times$. We wish to show it is not in $\Theta$, i.e., there is some $\mathfrak{A} \in RA$ and some homomorphism $h : \mathfrak{P} \rightarrow \mathfrak{A}$ such that $h(A) \neq h(B)$. Let $\mathfrak{A} = \mathfrak{P}/\approx^\times \in RA$ and $h(C) = C/\approx^\times$ for every
Since $A = B$ is not provable, the equivalence classes $A/\approx^\times$ and $B/\approx^\times$ are distinct, hence $h(A) \neq h(B)$ and the equation is not true in every RA because it fails in $\mathfrak{A}$.

A relation algebra $\mathfrak{A} = \langle S, +, -; ;, -1, 1 \rangle$ is proper if there is an equivalence relation $E \in S$ such that $S$ is a set of binary relations included in $E$, $+$ is union, $-$ is complementation with respect to $E$, $;$ is relative multiplication, $-1$ is converse, and $1$ is the identity part of $E$.

A relation algebra is representable if it is isomorphic to a proper relation algebra. Let $\text{RA}$ be the class of relation algebras, and let $\text{RRA}$ be the class of representable relation algebras. Since Tarski’s axioms (R1)–(R10) hold for binary relations, $\text{RRA} \subseteq \text{RA}$. An equivalent way to ask Tarski’s question is, are $\text{RRA}$ and $\text{RA}$ the same?

If a relation algebra $\mathfrak{A}$ is representable, then for all $A, B \in \Pi^+$ and every homomorphism $h: \mathfrak{P} \to \mathfrak{A}$, $\vdash^+ A = B$ implies $h(A) = h(B)$. There is an easy proof of this using the completeness theorem for $L$ (and the assumption that $\Pi$ has countably many relation symbols, by the way). Therefore, if $\mathfrak{A}$ is a non-representable relation algebra, there are $A, B \in \Pi^+$ such that $\vdash^+ A = B$ but $\not\vdash^\times A = B$ because $h(A) \neq h(B)$ for some homomorphism $h: \mathfrak{P} \to \mathfrak{A}$, i.e., some equation is provable in $L^+$ but not $L^\times$, answering Tarski’s question in the negative.

There are countably many relation symbols in $\Pi$, so any non-representable relation algebra suffices to show $L^\times$ is weaker than $L^+$ in means of proof. The first to find an algebra in $\text{RA} \setminus \text{RRA}$ was Lyndon [16]. However, since Tarski and Givant assume $L$ has exactly one binary relation symbol besides equality, they need a non-representable relation algebra generated by a single element. The non-representable relation algebra found by McKenzie [27] has this property. Givant used it to construct an equation, simplified later by George McNulty and Tarski, that is provable in $L^+$ but not in $L^\times$ [37, 3.4(vi)] and p. 54.

7 Three-variable formalisms $L^+_3$ and $L_3$

$L^\times$ is weaker than $L^+_3$ and $L$. In fact, $L^\times$ seems to be correlated with the logic of three variables, since Korselt’s sentence uses four variables, while $G(A = B)$ contains at most three. Indeed, for every $A, B \in \Pi^+$, it is apparent
from the definition of $G$ that if neither $A$ nor $B$ contains an occurrence of $;$, then $G(A = B)$ has only the first two variables occurring in it, while if $;$ occurs in $A$ or $B$, then $G(A = B)$ has the first three variables in it. This suggests that perhaps every sentence containing only the first three variables is logically equivalent to an equation in $\Sigma^\times$. Tarski was able to show that this is actually the case. For every finite $n \geq 3$ let $\Phi^+_n$ be the set of formulas in $\Phi^+$ that contain only the first $n$ variables, and let

$$
\Phi_n = \Phi \cap \Phi^+_n, \quad \Sigma_n = \Sigma \cap \Phi^+_n, \quad \Sigma^+_n = \Sigma^+ \cap \Phi^+_n.
$$

Tarski’s theorem that every sentence in $\Sigma^+_3$ is logically equivalent to an equation in $\Sigma^\times$ suggests that $\Sigma^+_3$ could be the set of sentences of a formalism $L^+_3$, equipollent with $L^\times$ in means of proof as well as expression, and also equipollent with a subformalism $L_3$ of $L$ having $\Sigma_3$ as its set of sentences and the restriction of $G$ to $\Sigma^+_3$ as the translation mapping between $L^+_3$ of $L_3$.

Tarski’s initial proposal came in two parts. First, Tarski proposed restricting the axioms (AI)–(AIX) and the rule MP to those instances that belong to $\Sigma^+_3$. Givant found these restricted axioms were too weak and suggested replacing (AIX) with (AIX'), called the general Leibniz law, which is formulated in terms of a variant type of substitution defined by Tarski- Givant [37, pp. 66–67]:

$$
[x \Gamma y \Rightarrow (\varphi \Rightarrow \varphi[x/y])]. \quad (\text{AIX}')
$$

The variant substitution is somewhat involved, so Tarski and Givant borrowed an idea from [18] to formulate an alternate axiom schema (AIX''). For any two variables $x, y$, let $S_{xy}\varphi$ be the result of interchanging variables $x$ and $y$ throughout formula $\varphi$. The function $S_{xy}: \Phi^+ \to \Phi^+$ is determined by these rules:

$$
S_{xy}(\varphi \Rightarrow \psi) = S_{xy}(\varphi) \Rightarrow S_{xy}(\psi),
S_{xy}(\neg \varphi) = \neg S_{xy}(\varphi),
S_{xy}(\forall v \varphi) = \forall \hat{v} S_{xy}(\varphi),
S_{xy}(v Aw) = \hat{v} A\hat{w}, \text{ where } \hat{x} = y, \hat{y} = x, \text{ and } \hat{v} = v \text{ if } v \neq x, y.
$$

Givant proved that the following variant of (AIX') can be used instead of (AIX') in the axiomatization of $L^+_3$.

$$
[x \Gamma y \Rightarrow (\varphi \Rightarrow S_{xy}\varphi)]. \quad (\text{AIX''})
$$

Tarski knew by the early 1940s that (R$_4$) could not be proved with only three variables, and would have to be included in the axiomatization of $L^+_3$ by fiat. The second part of Tarski’s proposal was to include the general associativity schema:

$$
[\exists z (\exists y (\varphi[x, y] \land \psi[y, z]) \land \xi[z, y])]
$$
This schema involves the complicated substitution, but Givant proved it could be replaced by the following variant of (AX), in which the free variables of formulas \( \varphi, \psi, \xi \) are just \( x \) and \( y \),

\[
\exists z (\exists y(\varphi \land S_{yxz} \psi) \land S_{zxz} \xi) \iff \exists z (S_{yz} \varphi \land \exists x (S_{yz} \psi \land \xi)).
\]  

(AX')

The sets of sentences of \( \mathcal{L}_3^+ \) and \( \mathcal{L}_3 \) are \( \Sigma_3^+ \) and \( \Sigma_3 \), respectively, their axioms are the sentences in \( \Sigma_3^+ \) and \( \Sigma_3 \), respectively, that are instances of (AI)–(AVIII), (AIX'), or (AX), and their rule of inference is \( \text{MP} \). For simpler axiom sets, use (AIX'') and (AX') instead of (AIX') and (AX). For any \( \Psi \subseteq \Sigma_3^+ \), a sentence \( \psi \in \Sigma_3^+ \) is provable in \( \mathcal{L}_3^+ \) from \( \Psi \), written \( \psi \vdash_{\mathcal{L}_3^+} \Psi \) or \( \vdash_{\mathcal{L}_3^+} \psi \) if \( \psi = \emptyset \), if \( \psi \) is in the intersection of all sets that contain \( \Psi \) and the axioms of \( \mathcal{L}_3^+ \), and are closed under \( \text{MP} \). Similarly, for \( \psi \in \Sigma_3 \) and \( \Psi \subseteq \Sigma_3 \), we define provable in \( \mathcal{L}_3 \) from \( \Psi \), written \( \psi \vdash_\mathcal{L}_3 \Psi \).

8 Equipollence of \( \mathcal{L}^\times \), \( \mathcal{L}_3 \), and \( \mathcal{L}_3^+ \)

With Givant’s changes, Tarski’s proposal worked, and provided a three-variable restriction \( \mathcal{L}_3^+ \) of \( \mathcal{L}^+ \), and a three-variable restriction \( \mathcal{L}_3 \) of \( \mathcal{L} \), both equipollent with \( \mathcal{L}^\times \) in means of expression and proof. For the equipollence of \( \mathcal{L}_3^+ \) of \( \mathcal{L}^+ \), the appropriate translation mapping is simply the restriction of \( \mathcal{G} \) to \( \Sigma_3^+ \). The restricted \( \mathcal{G} \) maps \( \Phi_3^+ \) onto \( \Phi_3^+ [37, \text{3.8(ix)}(\delta)] \). For the equipollence of \( \mathcal{L}_3 \) and \( \mathcal{L}^\times \), Tarski and Givant define a recursive function \( \mathcal{H} : \Phi_3^+ \rightarrow \Phi_3^+ [37, \text{3.9(iii)}] \). However, concerning \( \mathcal{H} \) they said,

“The construction used here to establish these equipollence results has clearly some serious defects, ... The splintered character of the definition of the translation mapping \( \mathcal{H} \) ... the involved notion of substitution ... is another detrimental factor. As a final result, the construction is so cumbersome ... in the proofs ... that we did not even attempt to present them in full. A different construction that would remove most of the present defects would be very desirable indeed.” [37, p. 87]

For another description of \( \mathcal{H} \) see [8]. For two simpler alternative constructions see [24, Theorem 552 and pp. 548–550]. The key equipollence results, including the two main mapping theorems, are listed here.

**Theorem 2.** [37, §3.8] Formalisms \( \mathcal{L}_3 \) and \( \mathcal{L}_3^+ \) are equipollent.

1. \( \Phi_3 \subseteq \Phi_3^+ \) and \( \Sigma_3 \subseteq \Sigma_3^+ [3.8(\text{viii})(\alpha)] \),
2. \( \mathcal{G} \) maps \( \Phi_3^+ \) onto \( \Phi_3 \) and \( \Sigma_3^+ \) onto \( \Sigma_3 [3.8(\text{ix})(\delta)] \),
3. \( \varphi \equiv_{\mathcal{L}_3^+} \mathcal{G}(\varphi) \) if \( \varphi \in \Phi_3^+ [3.8(\text{ix})(\varepsilon)] \),
4. \( \psi \vdash_{\mathcal{L}_3^+} \varphi \iff \mathcal{G}(\psi) \vdash_{\mathcal{L}_3} \mathcal{G}(\varphi) \), for \( \psi \subseteq \Sigma_3^+ \) and \( \varphi \in \Sigma_3^+ [3.8(\text{xi})] \),
5. \( \psi \vdash_{\mathcal{L}_3} \varphi \iff \mathcal{G}(\psi) \vdash_{\mathcal{L}_3^+} \varphi \), for all \( \Psi \subseteq \Sigma_3 \) and \( \varphi \in \Sigma_3 \), [3.8(\text{xi})(\beta), 3.8(\text{viii})(\beta)]

**Theorem 3.** [37, §3.9] Formalisms \( \mathcal{L}^\times \) and \( \mathcal{L}_3^+ \) are equipollent.
1. \( \Sigma^x \subseteq \Sigma^+ \) [3.9(i)],
2. \( H \) maps \( \Sigma^+ \) onto \( \Sigma^x \) [3.9(iii)(\( \delta \))],
3. \( \varphi \equiv^+ \) \( H(\varphi) \) if \( \varphi \in \Phi^+_3 \) [3.9(iii)(\( \varepsilon \))],
4. \( \Psi \vdash^+ \varphi \iff H(\Psi) \vdash^x H(\varphi) \), for \( \Psi \subseteq \Sigma^+_3 \) and \( \varphi \in \Sigma^+_3 \) [3.9(vii)],
5. \( \Psi \vdash^+ \varphi \iff \Psi \vdash^x \varphi \) for all \( \Psi \subseteq \Sigma^x \) and \( \varphi \in \Sigma^x \) [3.9(ix)].

9 Equipollence of \( L^x \), \( L_s \), and \( L^+_s \)

Since Tarski and Givant included \( (AX) \) only to achieve equipollence, they defined the “(standardized) formalisms” \( L_3 \) and \( L^+_3 \), obtained by deleting \( (AX) \) from the axiom sets of \( L_3 \) and \( L^+_3 \), and said, “These standardized formalisms are undoubtedly more natural and more interesting in their own right than \( L_3 \) and \( L^+_3 \)” [37, p. 89]. They asked, would deleting \( (R_4) \) from the axioms of \( L^x \) produce a formalism equipollent with the standardized formalisms \( L_3 \) and \( L^+_3 \)? The answer is “no”, because the semi-associative law

\[
A;(B;1) = (A;B);1,
\]

where 1 = 1 + 1 is provable in \( L^+_3 \) but cannot be derived from the remaining axioms \( (R_1)-(R_3) \) and \( (R_5)-(R_{10}) \). Another example of an equation provable in \( L^+_3 \) but not from the remaining axioms is

\[
A;1 = (A;1);1.
\]

Adding either one of these as an axiom produces a formalism equipollent with standardized 3-variable logic. Therefore Tarski and Givant defined \( L^x \) as the “weakened” formalism obtained by replacing \( (R_4) \) with \( (R'_4) \) in the axiomatization of \( L^x \). The axioms of \( L^x \) are \( (R_1)-(R_3), (R'_4), (R_5)-(R_{10}) \).

This is the axiom set for the class \( SA \) of semi-associative relation algebras. For any \( \Psi \subseteq \Sigma^x \), an equation \( \varepsilon \in \Sigma^x \) is provable in \( L^x \) from \( \Psi \), written \( \Psi \vdash^x \varepsilon \) or \( \vdash^x \varepsilon \) if \( \Psi = \emptyset \), iff \( \varepsilon \) belongs to every set that contains \( \Psi \) and the axioms of \( L^x \), and is closed under \( \text{Trans} \) and \( \text{Repl} \). The equipollence of \( L^x \) with \( L_3 \) and \( L^+_3 \) is stated in the next theorem and was noted by Tarski-Givant [37, p. 89, p. 209]. Part (4) tells us, “The axioms for \( SA \) characterize the equations provable with three variables.”

**Theorem 4.** [18, Theorem 11(31)] Formalisms \( L^x, L_3, \) and \( L^+_3 \) are equipollent.

1. \( \varphi \equiv^+ \) \( G(\varphi) \equiv^+ \) \( H(\varphi) \) for all \( \varphi \in \Phi^+_3 \),
2. \( \Psi \vdash^+ \varphi \iff G(\Psi) \vdash^s G(\varphi) \iff H(\Psi) \vdash^s H(\varphi) \), for all \( \Psi \subseteq \Sigma^+_3 \) and \( \varphi \in \Sigma^+_3 \),
3. \( \Psi \vdash^+ \varphi \iff \Psi \vdash^s \varphi \), for all \( \Psi \subseteq \Sigma_3 \) and \( \varphi \in \Sigma_3 \),
4. \( \Psi \vdash^+ \varphi \iff \Psi \vdash^x \varphi \) for all \( \Psi \subseteq \Sigma^x \) and \( \varphi \in \Sigma^x \).
Tarski-Givant [37, p. 91] also define formalisms $L_n$ and $L_n^+$ for every finite $n \geq 4$, imitating the definitions of $L_3$ and $L_3^+$, but with $n$ in place of 3. Thus the sets of formulas of $L_n$ and $L_n^+$ are $\Phi_n$ and $\Phi_n^+$, the sets of sentences are $\Sigma_n$ and $\Sigma_n^+$, the sets of axioms are those instances of axiom schemata (AI)–(AVIII) and (AIX') that lie in $\Sigma_n$ and $\Sigma_n^+$, and the only rule of inference is MP.

The next theorem involves the first of these formalisms, when $n = 4$. As was noted by Tarski-Givant [37, p. 92], Theorem 5 tells us that a 3-variable sentence can be proved with 4 variables if and only if it can be proved with 3 variables together with the assumption that relative multiplication is associative, and an equation is true in every relation algebra if and only if it can be proved with 4 variables. To Tarski's question, whether the definition of RA “is justified in any intrinsic sense,” Theorem 5(3) answers, “Tarski’s axioms for RA characterize the equations provable with four variables”. We can informally express the theorem as a slogan:

“True in RA = 4-provable = 3-provable with associativity.”

Theorem 5. [22, Theorem 24]

1. $\Psi \vdash^+ \psi$ iff $\Psi \vdash^+ \psi$, for all $\Psi \subseteq \Sigma_3^+$ and $\psi \in \Sigma_3^+$,
2. $\Psi \vdash_3 \psi$ iff $\Psi \vdash_4 \psi$, for all $\Psi \subseteq \Sigma_3$ and $\psi \in \Sigma_3$,
3. $\Psi \vdash^x \psi$ iff $\Psi \vdash^+ \psi$, for all $\Psi \subseteq \Sigma^x$ and $\psi \in \Sigma^x$.

10 Tarski’s relevance logics $T_n$ and $CT_n$, $3 \leq n \leq \omega$

Let $\to$, $\sim$, and $\circ$ be operators on $\Pi^+$ defined in the predicate algebra $\Psi$ by

$$A \to B = \overline{A^{-1}}; B, \quad \sim A = \overline{A^{-1}}, \quad A \circ B = B; A.$$  

According to Dunn’s posted comments in 1992, he wanted to avoid this definition of residual $\to$, because “a ‘random’ converse is thrown in . . . that would lead to some undesirable properties in relevance logic”, but he noted many sources for interpreting De Morgan negation as converse-complementation. Fusion is order-reversed relative multiplication, but this distinction can be ignored for systems in which it is commutative. The symbols $\lor$ and $\land$ have already appeared as connectives in $L$, but will also be used here to denote operators on $\Pi^+$ defined by

$$A \lor B = A + B, \quad A \land B = \overline{A + B}.$$  

For their classical relevant logics, Routley and Meyer introduce **Boolean negation** $\neg$ (designated by a symbol already used to denote negation in $L$ but given a second meaning here) and the **Routley star** *, defined by
Tarski’s classical relevant logic

\[ \neg A = \overline{\neg A}, \quad A^* = A^{-1}. \]

These definitions produce the standard connection between $\to$ and $\circ$,

\[ \sim(A \to \sim B) = (A^{-1}; B^{-1})^\sim B; A = A \circ B, \]

and match the interpretations in Table 1. For example, if $A, B \in \Pi$, then by the definition of $G$ and cancellation of double negations,

\[ G(xA \to By) = \neg \forall z \neg \neg (zAx \Rightarrow \neg zBy) \]

\[ \equiv^+ \forall z(zAx \Rightarrow zBy). \]

By the relevance logic operators we mean $\land, \lor, \rightarrow, \sim,$ and $;$, while the classical relevant logic operators are $\land, \lor, \rightarrow, \sim, ;$, and $\ast$. The classical relevant logic operators include the operators introduced by Tarski, so the closure of $\Pi$ under the classical relevant logic operators is $\Pi^+$. On the other hand, the closure of $\Pi$ under the relevance logic operators is a proper subset of $\Pi^+$, called $\Pi^r$. For any $A, B \in \Pi^+$, let $A \leq B$ be the equation $A + B = B \in \Sigma^\times$. Suppose $3 \leq n \leq \omega$ and $\Psi \subseteq \Sigma^\times$. Tarski’s $\Psi$-based classical relevant logic of $n$ variables is

\[ \text{CT}_n^\Psi = \{ A: \Psi \vdash^+ 1 \leq A, A \in \Pi^+ \}, \]

and Tarski’s $\Psi$-based relevance logic of $n$ variables is

\[ \text{T}_n^\Psi = \Pi^r \cap \text{CT}_n^\Psi. \]

The equations in $\Psi$ are the non-logical assumptions. We omit reference to $\Psi$ when it is empty.

## 11 Sequent calculus

The sequent calculus (for proving formulas in first-order logic) and model structures (for the semantic characterization of various relevance logics) are presented here and in the next section. They are used in Theorems 9 and 10 below to characterize $\text{CT}_3$ and $\text{CT}_4$.

Assume $n \geq 3$. An $n$-sequent is an ordered pair $(\Gamma, \Delta)$ of sets $\Gamma, \Delta \subseteq \Phi_n$ of atomic formulas of $L_n$, written $\Gamma \mid \Delta$. We say $\Gamma \mid \Delta$ is an Axiom if $\Gamma \cap \Delta \neq \emptyset$ or $x \Gamma \Delta \in \Delta$ for some variable $x$. If $\Psi$ is a set of $n$-sequents, then a sequent is $n$-provable from $\Psi$ (just $n$-provable when $\Psi = \emptyset$) if it is contained in every set of $n$-sequents that includes $\Psi$ and the Axioms, and is closed under the rules of inference in Table 7. In these rules, $\Gamma, \Gamma', \Delta, \Delta' \subseteq \Phi_n$ are sets of atomic formulas, $A, B \in \Pi^+$ are predicates, and $x, y, z$ are among the
first \( n \) variables. The notation “no \( y \)” in rule ; means that \( y \neq x, z \) and \( y \) does not occur in any formula in \( \Gamma \) or \( \Delta \). The rules are taken from [20] but use notation for the classical relevant logic operators. The rules \( | \land \) and \( \land | \) are derived from the rules for \( \neg \) and \( \lor \) through the definition of \( \land \). In the notation for sequents, braces are frequently omitted. By [20, Theorem 2], the

\[
\text{Id} \quad \frac{xAy, \: \Gamma | \Delta}{xAz, \: z\Gamma y, \: \Gamma | \Delta} \\
\text{Cut} \quad \frac{xAy, \: \Gamma | \Delta; \: \Gamma', \: \Delta' \Gamma | \Delta, \: xAy}{\Gamma; \: \Gamma', \: \Delta; \: \Delta'} \\
| \lor \quad \frac{xAy, \: \Gamma | \Delta}{xA \lor By, \: \Gamma, \: \Gamma' | \Delta, \: \Delta'} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma | \Delta, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By}
\]

\textbf{Table 7} Rules of inference for the sequent calculus

\[
\begin{align*}
\text{Id} & \quad \frac{xAy, \: \Gamma | \Delta}{xAz, \: z\Gamma y, \: \Gamma | \Delta} \\
\text{Cut} & \quad \frac{xAy, \: \Gamma | \Delta; \: \Gamma', \: \Delta' \Gamma | \Delta, \: xAy}{\Gamma; \: \Gamma', \: \Delta; \: \Delta'} \\
| \lor & \quad \frac{xAy, \: \Gamma | \Delta}{xA \lor By, \: \Gamma, \: \Gamma' | \Delta, \: \Delta'} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma | \Delta, \: xA \lor By} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By} \\
| \lor & \quad \frac{\Gamma, \: xAy, \: xBy | \Delta}{\Gamma, \: xA \lor By}
\end{align*}
\]

Lemma 2. If \( A, B, C, D \in \Pi^+ \) then the sequent \( yFx | yGx \) is 3-provable, where \( F = A ; B \land C \) and \( G = (A \land \neg D); B \lor A; (B \land D; C) \).
Proof. The sequence of sequents in Table 8 shows that $yFx \mid yGx$ is 3-provable. Only the first three variables $x, y, z$ appear in any of the sequents. Sequents 1, 2, 4, and 6 are Axioms. Every other sequent is 3-provable because it is the conclusion of an instance of a rule of inference whose hypotheses are one or two previous sequents. For example, sequent 3 is 3-provable because it is the conclusion of an instance of rule $\mid ;$ whose hypotheses are the 3-provable sequents 1 and 2. The proof is a formalization of the following informal argument that, for binary relations $A, B, C, D$,

$$A; B \cap C \subseteq A; (B \cap \sim D) \cup (A \cap C; D); B.$$ 

Assume $yA; B \cap Cx$. Then $yCx$ and for some $z$, $yAz$ and $zBx$. There are two cases. In Case 1, $xDz$. Then $yC; Dz$ by $yCx$ and $xDz$, so $yA \cap C; Dz$ by $yAz$ and $yC; Dz$, and finally $y(A \cap C; D); Bx$ by $yA \cap C; Dz$ and $zBx$. In Case 2, $x \sim Dz$. Then $z \sim Dx$, so $zB \cap \sim Dx$ by $zBx$, hence $yA; (B \cap \sim D)x$ by $yAz$ and $zB \cap \sim Dx$. In either case we conclude that $yA; (B \cap \sim D) \vee (A \cap C; D); Bx$. Statement 14 asserts the 3-provability of a sequent expressing the inclusion above.

| 1.  | $yCx \mid yCx$                        | Axiom                     |
| 2.  | $xDz \mid xDz$                        | Axiom                     |
| 3.  | $yCx, xDz \mid yC; Dz$                | 1, 2, $\mid ;$           |
| 4.  | $yAz \mid yAz$                        | Axiom                     |
| 5.  | $yAz, yCx, xDz \mid y(A \cap C; D)z$  | 3, 4, $\mid \wedge$      |
| 6.  | $zBx \mid zBx$                        | Axiom                     |
| 7.  | $zBx, yAz, yCx, xDz \mid y(A \cap C; D); Bx$ | 5, 6, $\mid ;$ |
| 8.  | $zBx, yAz, yCx, zD^x \mid y(A \cap C; D); Bx$ | 7, $**$ |
| 9.  | $zBx, yAz, yCx \mid z \sim Dx, y(A \cap C; D); Bx$ | 8, $\sim, \text{def } \sim$ |
| 10. | $zBx, yAz, yCx \mid zB \sim Dx, y(A \cap C; D); Bx$ | 6, 9, $\mid \wedge$ |
| 11. | $zBx, yAz, yCx \mid yA; (B \sim D)x, y(A \cap C; D); Bx$ | 4, 10, $\mid ;$ |
| 12. | $zBx, yAz, yCx \mid yA; (B \sim D) \vee (A \cap C; D); Bx$ | 11, $\vee$ |
| 13. | $yA; Bx, yCx \mid yA; (B \sim D) \vee (A \cap C; D); Bx$ | 12, $\mid ;$, no $z$ |
| 14. | $yA; B \cap Cx \mid yA; (B \sim D) \vee (A \cap C; D); Bx$ | 13, $\wedge$ |

Table 8 A sequence of 3-provable sequents

Lemma 3. If one of the sequents $x1y \mid xEy$ and $\mid xEx$ is 3-provable, then so is the other. The same is true for $yAx \mid yBx$ and $\mid xA \rightarrow Bx$.

Proof. There are four directions. If one of the four sequents is 3-provable by some sequence of sequents, one can then add sequents, ending up with one of the other sequents, which is therefore also 3-provable. The first pair of sequents is handled in Table 9, the second in Table 10.
1. $x Ax$ Hyp.
2. $x Ay | x By$ Axiom
3. $x Ax, x Ay | x Ay$ 2, Id
4. $x Ay | x Ay$ 1, 3, Cut

1. $x Ay | x Ay$ Hyp.
2. $x Ax | x Ax$ Axiom
3. $x Ay, y A x | x Ax$ 2, Id
4. $x Ay, y A x | x Ax$ 1, 3, Cut
5. $x Ay, y A x | x Ax$ 4, ;|, no y
6. $x A y | x A y$ 1, 3, Cut
7. $x A y | x A y$ 6, |
8. $x A x | x A x$ 5, 7, Cut

Table 9 $x Ax$ is 3-provable if and only if $x Ay | x Ay$ is 3-provable

1. $y Ay | y By$ Hyp.
2. $x A y | y By$ 1, *|
3. $x A y, y A y | y By$ 2, ¬|
4. $x A y, y A y | y By$ 3, ;|, no y
5. $x A y, y A y | y By$ 4, ¬|
6. $x A y, y A y | y By$ 5, def →

1. $x A y | y By$ Hyp.
2. $x A y | y By$ 1, def →
3. $x A y, y A y | y By$ Axiom
4. $x A y, y A y | y By$ Axiom
5. $x A y, y A y | y By$ Axiom
6. $x A y, y A y | y By$ Axiom
7. $x A y, y A y | y By$ Axiom
8. $x A y, y A y | y By$ Axiom
9. $x A y, y A y | y By$ Axiom
10. $x A y, y A y | y By$ Axiom
11. $x A y, y A y | y By$ Axiom

Table 10 $y Ay | y By$ is 3-provable if and only if $x A y | y By$ is 3-provable
Model structures

A model structure is a quadruple $\mathfrak{R} = (K, R, *, I)$ consisting of a set $K$, a ternary relation $R \subseteq K^3$, a unary operation $*: K \to K$, and a subset $I \subseteq K$. The associated complex algebra of $\mathfrak{R}$ is $\mathcal{Cm}(\mathfrak{R}) = \langle \wp(K), \cup, -, ;, -1, I \rangle$, where $\wp(K)$ is the set of subsets of $K$, and the operations $\cup$, $\cdot$, $;$, and $-1$ are defined on subsets $X, Y \subseteq K$ by

- $X \cup Y = \{ x : x \in X \text{ or } x \in Y \}$,
- $\overline{X} = K \setminus X$,
- $X;Y = \{ z : Rxyz \text{ for some } x \in X, y \in Y \}$,
- $X^{-1} = \{ z^* : z \in X \}$.

A predicate $A \in \Pi^+$ is valid in $\mathfrak{R}$ if the equation $1 \leq A$ is true in $\mathcal{Cm}(\mathfrak{R})$. To see what this means, a little calculation shows that if $A \in \Pi$ then, by Theorem 2(3), $1 \leq A \iff \forall x Ax$. Any homomorphism from the predicate algebra into $\mathcal{Cm}(\mathfrak{R})$ must send $1$ to $I$, so it follows that $A \in \Pi^+$ is valid in $\mathfrak{R}$ if and only if $I \subseteq h(A)$ for every homomorphism $h : \wp \to \mathcal{Cm}(\mathfrak{R})$.

Consider the following conditions on a model structure $\mathfrak{R}$, written in the first order language of one ternary relation symbol $R$, one unary function symbol $*$, and one unary relation symbol $I$:

- $Rxyz \Rightarrow Rx^*zy$, (l-refl)
- $Rxyz \Rightarrow Rzy^*x$, (r-refl)
- $x = y \iff \exists u(Iu \land Rxuy)$, (ident)
- $\exists x(Rvwx \land Rxyz) \Rightarrow \exists uRvuz$, (semi-Pasch)
- $\exists x(Rvwx \land Rxyz) \Rightarrow \exists u(Rvuz \land Rwuy)$, (Pasch)
- $Rxyz \Rightarrow Ryzx$, (comm)
- $Rxxx$, (dense)
- $x^* = x$. (symm)

For every set $U$, let $\mathfrak{U} = \langle U^2, R, *, I \rangle$ be the model structure of pairs on $U$, where

- $U^2 = \{ (x, y) : x, y \in U \}$, (pairs)
- $R = \{ (\langle x, y \rangle, \langle y, z \rangle, \langle x, z \rangle) : x, y, z \in U \}$, (triples)
- $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in U$, (star)
- $I = \{ (x, x) : x \in U \}$. (identity)

Then $\mathfrak{R}(U)$, the algebra of binary relations on $U$, is the complex algebra of the model structure of pairs on $U$, $\mathfrak{R}(U) = \mathcal{Cm}(\mathfrak{U})$. Consider the following conditions on a model structure $\mathfrak{R}$, written in the first order language of one ternary relation symbol $R$, one unary function symbol $*$, and one unary relation symbol $I$:
The first five conditions characterize the model structures whose complex algebras are relation algebras or semi-associative relation algebras. For example, they hold in the model structure of pairs on any set. The last three conditions hold when the algebras are commutative, dense, or symmetric.

**Theorem 6.** [19, Theorem 2.2] For any model structure \( \mathcal{R} = (K, R, *, I) \),

1. \( \text{Cm}(\mathcal{R}) \in \text{SA} \) iff \( \mathcal{R} \) satisfies (l-refl), (r-refl), (ident), (semi-Pasch).
2. \( \text{Cm}(\mathcal{R}) \in \text{RA} \) iff \( \mathcal{R} \) satisfies (l-refl), (r-refl), (ident), (Pasch).

Theorem 6 and the existence of canonical extensions entail the following representation theorem. Along with Theorem 6, it characterizes RA and SA as subalgebras of complex algebras of frames (model structures) satisfying four frame conditions.

**Theorem 7.** [19, Theorem 4.3] If \( \mathfrak{A} = \langle S, +, -, ;, \neg, 1 \rangle \in \text{SA} \) then there is a model structure \( \mathcal{R} = (K, R, *, I) \) such that

1. \( \mathfrak{A} \cong \mathfrak{A}' \subseteq \text{Cm}(\mathcal{R}) \) for some subalgebra \( \mathfrak{A}' \) of the complex algebra \( \text{Cm}(\mathcal{R}) \),
2. \( \text{Cm}(\mathcal{R}) \) is in SA,
3. \( \mathcal{R} \) satisfies (l-refl), (r-refl), (ident), and (semi-Pasch),
4. \( \text{Cm}(\mathcal{R}) \) is in RA if \( \mathfrak{A} \in \text{RA} \),
5. if \( \mathfrak{A} \in \text{RA} \) then \( \mathcal{R} \) satisfies (l-refl), (r-refl), (ident), and (Pasch).

Useful in the proofs of these last two theorems is

**Lemma 4.** Assume \( \mathcal{R} = (K, R, *, I) \) is a model structure.

1. If (l-refl) and (ident) then \( a^{**} = a \) for all \( a \in K \).
2. If (l-refl), (r-refl), and (ident), then \( v = v^* \) for all \( v \in I \).
3. Assume (l-refl), (r-refl), (ident), and (semi-Pasch). If \( u, v \in I \), \( Raua \), and \( Rava \), then \( u = v \).

**Proof (1).** Since \( a = a \) by (ident) there must be some \( u \in I \) such that \( Raua \). Applying (l-refl) twice yields \( Ra^{**}ua \), so we obtain \( a^{**} = a \) by (ident).

**Proof (2).** Assume \( v \in I \). By (ident) there is some \( u \in I \) such that \( Revu \). Then \( Ru^*vu \) by (l-refl) and \( Rev^*v \) by (r-refl). From \( Ru^*vu \) we also have \( Rv^*vu^* \) by (l-refl). From \( Rv^*vu \) and \( Ru^*vu^* \) we obtain \( v^* = u^* \) and \( v^* = u \) by (ident) since \( v \in I \). From \( v^* = u^* \) we get \( v = u \) by part (1), so from \( v^* = u \) we get \( v^* = v \), as desired.

**Proof (3).** Assume \( u, v \in I \), \( a \in K \), \( Raua \), and \( Rava \). By (l-refl) we get \( Ra^*au \) and \( Ra^*av \). From \( Ra^*au \) we get \( Ru^*a^* \) by (r-refl). Apply (semi-Pasch) to \( Rua^*a^* \) and \( R^*av \), obtaining some \( x \in K \) such that \( Ru^*xv \). Then \( Ru^*v \) by (l-refl), so \( u^* = x \) by (ident) since \( v \in I \). We therefore have \( Ru^*v \). But \( u^* \in I \) by Lemma 4(2) since \( u \in I \) by assumption, hence \( u = v \) by (ident).
The classical relevant model structures were introduced by Meyer-Routley [29, p. 184]. A CR*-model structure is a model structure $\mathfrak{A} = \langle K, R, *, I \rangle$ such that $I = \{0\}$ and for all $a, b, c, d \in K$,

- $p_1$. $R0ab$ iff $a = b$,
- $p_2$. $\exists_x (Rabx \land Rxcd)$ iff $\exists_y (Racy \land Rbxd)$,
- $p_3$. $Raaa$,
- $p_4$. $a^{**} = a$,
- $p_5$. $Rabc$ implies $Rac^*b^*$.

The classical relevant logic CR* is defined as the set of formulas valid in all CR*-model structures. These structures are commutative because

**Lemma 5.** If $\mathfrak{A} = \langle K, R, *, I \rangle$ is a model structure such that $I = \{0\}$, $p_1$, and $p_2$, then (comm).

**Proof.** Assume $Rabc$. Get $R0aa$ by $p_1$. By $p_2$, $R0aa$ and $Rabc$ imply there is some $y$ such that $R0by$ and $Ryac$. However, $R0by$ implies $b = y$ by $p_1$, so $Rbac$.

When (comm) holds, we can restate $p_5$ in two ways, by switching the order of the first two entries in the conclusion, or interchanging $a$ and $b$ in the conclusion.

$$Rabc \Rightarrow Rbc^*a^*,$$
$$Rabc \Rightarrow Rc^*ab^*.$$  \hfill (l-rot)
$$Rabc \Rightarrow Rc^*ab^*.$$  \hfill (r-rot)

In the presence of $p_1$ and $p_2$, the three postulates $p_5$, (r-rot), (l-rot) are equivalent.

**Theorem 8.** Assume $\mathfrak{A} = \langle K, R, *, I \rangle$, $I = \{0\}$, and $\mathcal{Cm}(\mathfrak{A})$ is a commutative dense relation algebra. Then $\mathfrak{A}$ is a CR*-model structure.

**Proof.** Postulate $p_1$ is what (ident) reduces to when $I = \{0\}$. For a given $a \in K$, pick a propositional variable $A \in \Pi$ and choose a homomorphism $h : \mathfrak{P} \rightarrow \mathcal{Cm}(\mathfrak{A})$ such that $h(A) = \{a\}$, $A \leq A:A$ is true in $\mathcal{Cm}(\mathfrak{A})$ since $\mathcal{Cm}(\mathfrak{A})$ is dense, so $\{a\} \subseteq \{a\};\{a\}$. By the definition of $;$, this gives us $Raaa$, so $p_3$ holds. Condition $p_4$ follows from (ident) and (l-refl) by Lemma 4(1). For $p_5$, assume $Rabc$. Then $c \in \{a\};\{b\}$ but $\{a\};\{b\} = \{b\};\{a\}$ since $\mathcal{Cm}(\mathfrak{A})$ is commutative, so $Rbac$. Then $Rb^*ca$ by (l-refl), so $Rac^*b^*$ by (r-refl). Thus $p_5$ holds. To prove $p_2$, assume $Rabx$ and $Rxcd$. By (Pasch), $Raud$ and $Rbuc$ for some $u \in K$, so $Rbuc$ by commutativity. By (l-refl) and (r-refl), $Ra^*du$ and $Rb^*c$. By (Pasch), there is some $y \in K$ such that $Ra^*yc$ and $Rdb^*y$. By (l-refl), (r-refl), and $p_4$, $Racy$ and $Rybd$. The opposite implication in $p_2$ is the same.

Jónsson-Tarski [13, Th. 4.15] proved that every relation algebra is semi-simple (isomorphic to a subdirect product of simple algebras); see Givant [9,
Furthermore, every semi-associative relation algebra $\mathfrak{A} \in \mathcal{SA}$ is semi-simple; see [18, Corollary 8(7)] or [24, Theorem 388]. By definition, $\mathfrak{A}$ is integral if $0 \neq 1$ and $x \cdot y = 0$ implies $x = 0$ or $y = 0$. By [24, Theorem 379(iii)] or [18, Theorem 7(20)] or [23, Theorem 29], $\mathfrak{A}$ is simple if and only if $0 \neq 1$, and for all $x, y \in A$, if $0 = (x;1); y$ then $x = 0$ or $y = 0$. Suppose $\mathfrak{A} \in \mathcal{SA}$ is commutative and simple. We get $0 \neq 1$ from simplicity, and if $x \cdot y = 0$, then $(x;1); y = (1;x); y = 1;(x;y) = 1;0 = 0$ by commutativity and [24, (6.188)], hence $x = 0$ or $y = 0$ by simplicity. This shows $\mathfrak{A}$ is integral, so we conclude $1'$ is an atom of $\mathfrak{A}$ by [24, Theorem 353]. Every commutative semi-associative relation algebra is isomorphic to a subdirect product of algebras in which $1'$ is an atom.

For a special case of this situation, assume we have a model structure $\mathfrak{A} = \langle K, R, ^*, I \rangle$ such that $Cm(\mathfrak{A})$ is a commutative semi-associative relation algebra, or equivalently, $\mathfrak{R}$ satisfies (l-refl), (r-refl), (ident), (semi-Pasch), and (comm). For every $u \in I$, let $K_u = K; \{u\} = \{u\}; K$. Then $\{K_u : u \in I\}$ is a partition of $K$ and $\{R \cap (K_u)^3 : u \in I\}$ is a partition of $R$. Each $K_u$ is an ideal element of $Cm(\mathfrak{R})$ and is closed under $^*$, so we may let $\mathfrak{R}_u = \langle K_u, R \cap (K_u)^3, ^*, K_u, \{u\}\rangle$. Then $Cm(\mathfrak{R})$ is isomorphic to a subalgebra of the direct product $\prod_{u \in I} Cm(\mathfrak{R}_u)$. Since validity in $\mathfrak{R}$ is equivalent to validity in $Cm(\mathfrak{R}_u)$ for every $u \in I$, it follows that, when checking whether a formula is true in every commutative semi-associative relation algebra, it suffices to check those in which the identity element is an atom. Model structures whose complex algebras are commutative dense integral relation algebras are CR$^*$-model structures (but not conversely; see Theorem 14).

### Theorem 12.10

Furthermore, every semi-associative relation algebra $\mathfrak{A} \in \mathcal{SA}$ is semi-simple; see [18, Corollary 8(7)] or [24, Theorem 388]. By definition, $\mathfrak{A}$ is integral if $0 \neq 1$ and $x \cdot y = 0$ implies $x = 0$ or $y = 0$. By [24, Theorem 379(iii)] or [18, Theorem 7(20)] or [23, Theorem 29], $\mathfrak{A}$ is simple if and only if $0 \neq 1$, and for all $x, y \in A$, if $0 = (x;1); y$ then $x = 0$ or $y = 0$. Suppose $\mathfrak{A} \in \mathcal{SA}$ is commutative and simple. We get $0 \neq 1$ from simplicity, and if $x \cdot y = 0$, then $(x;1); y = (1;x); y = 1;(x;y) = 1;0 = 0$ by commutativity and [24, (6.188)], hence $x = 0$ or $y = 0$ by simplicity. This shows $\mathfrak{A}$ is integral, so we conclude $1'$ is an atom of $\mathfrak{A}$ by [24, Theorem 353]. Every commutative semi-associative relation algebra is isomorphic to a subdirect product of algebras in which $1'$ is an atom.

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### Theorem 13 Characterizing $CT_3$ and $CT_4$

#### Theorem 9.

For every $A \in \Pi^+$, the following statements are equivalent.

1. $A \in CT_3$,
2. $\vdash_s^+ 1' \leq A$ (provable in extended standardized 3-variable logic $L_s^+$),
3. $\vdash_s^\times 1' \leq A$ (provable in the equational logic $L_w^\times$ of $\mathcal{SA}$),
4. $1' \leq A$ is true in every $\mathcal{SA}$,
5. $A$ is valid in all structures satisfying (l-refl), (r-refl), (ident), (semi-Pasch),
6. the sequent $| x:Ax$ is provable in the 3-variable sequent calculus,
7. $\vdash_s \forall x. G(x;Ax)$ (provable in standardized 3-variable logic $L_s^3$).

#### Proof.

Parts (1) and (2) are equivalent by definition. Parts (2) and (3) are equivalent by Theorem 4(4). Parts (3) and (4) are equivalent by Lemma 1 with $\mathcal{SA}$, $L_w^\times$, and $(R_4^\times)$ in place of $\mathcal{RA}$, $L_r^\times$, and $(R_4)$. Parts (4) and (5) are equivalent by Theorem 6(1). Parts (4) and (6) are equivalent by [20, Theorems 2 and 6(1)] and Lemma 3. Parts (2) and (7) are equivalent by Theorem 4(2) with $\Psi = \emptyset$ and $\varphi = (1' \leq A)$.

#### Theorem 10.

For every $A \in \Pi^+$, the following statements are equivalent.
1. $A \in \text{CT}_4$,  
2. $\vdash_4^+ 1' \leq A$ (provable in the extended 4-variable formalism $L_4^+$),  
3. $\vdash_4 1' \leq A$ (provable in the equational formalism $L_4^\times$ of RA),  
4. $1' \leq A$ is true in every RA,  
5. $A$ is valid in model structures satisfying (r-refl), (l-refl), (ident), (Pasch),  
6. the sequent $\vdash_4 x.Ax$ is provable in the 4-variable sequent calculus,  
7. $\vdash_4 \forall x.G(x.Ax)$ (provable in the 4-variable formalism $L_4$),  
8. $\vdash_3^+ 1' \leq A$ (provable in extended 3-variable logic $L_3^+$),  
9. $\vdash_3 \forall x.G(x.Ax)$ (provable in the 3-variable formalism $L_3$).

Proof. Parts (1) and (2) are equivalent by definition. Parts (2) and (3) are equivalent by Theorem 5(3). Parts (3) and (4) are equivalent by Lemma 1. Parts (4) and (5) are equivalent by Theorem 6(2). Parts (4) and (6) are equivalent by [20, Theorems 2 and 6(2)] and Lemma 3. Part (8) is equivalent to (2) by Theorem 5(1), equivalent to (3) by Theorem 3(5), and equivalent to (9) by Theorem 2(5). Finally, (7) and (9) are equivalent by Theorem 5(2).

14 Theorems and rules of $\text{CT}_3$

To show $A \in \text{CT}_3$, it is enough by Theorem 9 to show $|x.Ax$ is 3-provable, but three variables may not all be needed. This leads to a classification of formulas and rules according to the number of variables needed to prove them. For example, the sequent $|x1'x$ is provable because it is an Axiom, and it is 1-provable because it contains only one variable. Therefore $1'$ is 1-provable. Similarly, $x.Ax | x.Ax$ is provable because it is an Axiom, and it is 1-provable because it contains only one variable. Therefore $1'$ is 1-provable. By Rule $|\neg$, the sequent $|x.Ax, x\neg Ax$ is also 1-provable. Then by Rule $|\lor$, $|x.Ax \lor Ax$ is 1-provable, hence $A \lor \neg A \in \text{CT}_3$. Therefore $A \lor \neg A$ is 1-provable. The Rule of Disjunctive Syllogism is 1-provable, in the sense that, assuming $|x.Ax \lor Bx$ and $|x.Ax \lor Bx$ are 3-provable, we do not need to introduce any additional variables to conclude that $|xBx$ is 3-provable. If one more variable besides $x$ is used, the rule is said to be 2-provable. In the next theorem, probably every formula (except (41)) is in $\text{CR}^*$, and probably every derived rule also applies to $\text{CR}^*$.

Theorem 11. Among the theorems and derived rules of $\text{CT}_3$, classified by the number of variables needed for their proofs, there are 1-provable formulas in $\text{CT}_3$,

$$A \lor \sim A \text{ (1)}$$

$$A \lor \neg A \text{ (2)}$$

$$1' \text{ (3)}$$

2-provable formulas in $\text{CT}_3$,
\[ A \rightarrow A \] (4)
\[ ((A \rightarrow A) \rightarrow B) \rightarrow B \] (5)
\[ A \rightarrow A \lor B \] (6)
\[ B \rightarrow A \lor B \] (7)
\[ A \land B \rightarrow A \] (8)
\[ A \land B \rightarrow B \] (9)
\[ A \lor B \rightarrow B \lor A \] (10)
\[ A \land B \rightarrow B \land A \] (11)
\[ (A \lor B) \lor C \rightarrow A \lor (B \lor C) \] (12)
\[ (A \land B) \land C \rightarrow A \land (B \land C) \] (13)
\[ (A \lor B) \land C \rightarrow (A \land C) \lor (B \land C) \] (14)
\[ (A \land B) \lor C \rightarrow (A \lor C) \land (B \lor C) \] (15)
\[ \sim \sim A \rightarrow A \] (16)
\[ A \rightarrow \sim \sim A \] (17)
\[ \sim (A \lor B) \rightarrow \sim A \land \sim B \] (18)
\[ \sim (A \land B) \rightarrow \sim A \lor \sim B \] (19)
\[ \sim A \land \sim B \rightarrow \sim (A \lor B) \] (20)
\[ \sim A \lor \sim B \rightarrow \sim (A \land B) \] (21)
\[ \sim A \land A \rightarrow B \] (22)
\[ \sim B \rightarrow A \] (23)
\[ A \rightarrow \sim \sim A \] (24)
\[ \sim (A \lor B) \rightarrow \sim A \land \sim B \] (25)
\[ \sim (A \land B) \rightarrow \sim A \lor \sim B \] (26)
\[ \sim A \land B \rightarrow \sim (A \lor B) \] (27)
\[ \sim A \lor B \rightarrow \sim (A \land B) \] (28)
\[ \sim A \land A \rightarrow B \] (29)

3-provable formulas in CT₃,

\[ (A \rightarrow B) \land (A \rightarrow C) \rightarrow (A \rightarrow B \land C) \] (30)
\[ (A \rightarrow C) \land (B \rightarrow C) \rightarrow (A \lor B \rightarrow C) \] (31)
\[ (A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \land C \rightarrow B \land D) \] (32)
\[ (A \rightarrow B) \land (C \rightarrow D) \rightarrow (A \lor C \rightarrow B \lor D) \] (33)
\[ (A \rightarrow B) \lor (C \rightarrow D) \rightarrow (A \land C \rightarrow B \lor D) \] (34)
\[ A \circ B \rightarrow \sim (A \rightarrow \sim B) \] (35)
\[ \sim (A \rightarrow \sim B) \rightarrow A \circ B \] (36)
\[ (A \rightarrow B) \circ A \rightarrow B \] (37)
Tarski’s classical relevant logic

\[ A \rightarrow (B \rightarrow A \circ B) \]  
\[ A \rightarrow ((B \rightarrow \neg A) \rightarrow \neg B) \]  
\[ (A \circ B) \land C \rightarrow (A \land (C \circ B^*)) \circ (B \land (A^* \circ C)) \]  
\[ (A \circ B) \land C \rightarrow (A \land \neg D) \circ B \lor A \circ (B \land (D \circ C)) \]  
\[ (A \rightarrow B) \land (C \circ D) \rightarrow ((C \land B) \circ D) \lor (C \circ (D \land \neg A)) \]  

1-provable rules of \( \text{CT}_3 \),

\[ A, B \vdash A \land B \]  
\[ A \vdash B \]  
\[ A \lor B, \neg A \vdash B \]  
\[ A \vdash A^* \]  

2-provable rules of \( \text{CT}_3 \),

\[ A \rightarrow B, B \rightarrow C \vdash A \rightarrow C \]  
\[ A \rightarrow B \vdash \neg B \rightarrow \neg A \]  
\[ A \rightarrow \neg B \vdash B \rightarrow \neg A \]  
\[ A \land B \rightarrow C, B \rightarrow C \lor A \vdash B \rightarrow C \]  
\[ A \vdash (A \rightarrow B) \rightarrow B \]  
\[ A \land D \rightarrow C, B \land \neg D \rightarrow C \vdash A \land B \rightarrow C \]  
\[ C \land E \rightarrow A, D \land E \rightarrow B \vdash (C \land D) \land E \rightarrow A \land B \]  
\[ B \land C^* \rightarrow A, E \land F^* \rightarrow D \vdash (B \land E) \land (C \land F)^* \rightarrow A \land D \]  
\[ B \land \neg C \rightarrow A, E \land \neg F \rightarrow D \vdash (B \land E) \land (C \land F) \rightarrow A \land D \]  
\[ B \land \neg C \rightarrow A \land \neg A \vdash B \rightarrow C \]  
\[ B \land C^* \rightarrow A \land \neg A \vdash B \rightarrow \neg C \]  

and 3-provable rules of \( \text{CT}_3 \),

\[ A \rightarrow B \vdash (B \rightarrow C) \rightarrow (A \rightarrow C) \]  
\[ A \rightarrow B \vdash (C \rightarrow A) \rightarrow (C \rightarrow B) \]  
\[ A \rightarrow B, C \rightarrow D \vdash (B \rightarrow C) \rightarrow (A \rightarrow D) \]  
\[ A \rightarrow B, C \rightarrow D \vdash (A \circ C) \rightarrow (B \circ D) \]  
\[ A \rightarrow (B \rightarrow C) \vdash B \rightarrow (\neg C \rightarrow \neg A) \]  

Although the Law of the Excluded Middle (2) is in \( \text{CT}_3 \), Explosion \( (A \land \neg A) \rightarrow B \) and Positive Paradox \( A \rightarrow (B \rightarrow A) \) are not, because as relations they need not contain the identity relation, for if \( \langle y, x \rangle \in A \), \( \langle x, y \rangle \notin A \), and \( \langle y, x \rangle \notin B \), then \( \langle x, x \rangle \notin (A \land \neg A) \rightarrow B \), and if \( \langle y, x \rangle \in A \), \( \langle z, y \rangle \in B \), and \( \langle z, x \rangle \notin A \), then \( \langle x, x \rangle \notin A \rightarrow (B \rightarrow A) \).
15 A non-associative commutative dense $\mathbf{SA}$

This section presents formulas that are valid for all relations but require 4 variables to prove.

**Theorem 12.** Formulas (63)–(67) are in $\mathbf{CR^*}$ and $\mathbf{CT}_4$ but not $\mathbf{CT}_3$.

$$(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B)) \quad (63)$$

$$(A \rightarrow (B \rightarrow C)) \rightarrow ((A \circ B) \rightarrow C) \quad (64)$$

$$((A \circ B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C)) \quad (65)$$

$$(A \rightarrow B) \rightarrow ((A \circ D) \rightarrow (B \circ D)) \quad (66)$$

$$(A \circ B) \circ C \rightarrow A \circ (B \circ C) \quad (67)$$

**Proof.** To show formulas (63)–(67) belong to $\mathbf{CR^*}$ it suffices to check that they are valid in all $\mathbf{CR^*}$-model structures. They can be shown to be in $\mathbf{CT}_4$ by deriving them from the axioms for relation algebras or proving them in the 4-variable sequent calculus. For example, by Theorem 10 and Lemma 3, (67) is in $\mathbf{CT}_4$ because of the following sequence of sequents.

1. $zBw | zBw$ Axiom
2. $yCz | yCz$ Axiom
3. $yCz, zBw | yC;Bw$ 1, 2, $|$;
4. $wAx | wAx$ Axiom
5. $yCz, zBw, wAx | y(C;B);Ax$ 3, 4, $|$;
6. $yCz, zB;Ax | y(C;B);Ax$ 5, $|$, no w
7. $yC;(B;A)x | y(C;B);Ax$ 6, $|$, no z
8. $y(A \circ B) \circ Cx | yA \circ (B \circ C)x$ 7, def of $\circ$

To show formulas (63)–(67) cannot be proved with three variables, we use a dense semi-associative relation algebra that is not associative. Let $\mathfrak{R}_1 = \langle K, R, *, 1' \rangle$, where $K = \{a, b, c, 1\}$, $I = \{1'\}$, $x^* = x$ for all $x \in K$, $R \subseteq K^3$, $Rxyz$ means $z \in \{x\};\{y\}$, and $;$ is defined in Table 11. Then $\mathfrak{C}m(\mathfrak{R}_1) \cong \mathfrak{E}_4(\{1,3\}) \in \mathbf{SA}$ by [19, Theorem 2.5(4)(a)]. This algebra arises by applying Lyndon’s construction to a projective geometry consisting of a single line of

| : | $\{1'\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ |
|---|---|---|---|---|
| $\{1\}$ | $\{1'\}$ | $\{a\}$ | $\{b\}$ | $\{c\}$ |
| $\{a\}$ | $\{a\}$ | $\{1',a\}$ | $\{c\}$ | $\{b\}$ |
| $\{b\}$ | $\{b\}$ | $\{c\}$ | $\{1',b\}$ | $\{a\}$ |
| $\{c\}$ | $\{c\}$ | $\{b\}$ | $\{a\}$ | $\{1',c\}$ |

**Table 11** A non-associative dense semi-associative relation algebra
order 2, i.e., one that contains 3 points. Lyndon [17, p. 24] pointed out that this case can be accommodated by deleting the triples \(aaa, bbb, ccc\) from \(R\). The resulting structure is no longer dense, but is instead the Klein 4-group.

Choose a homomorphism \(h: \mathcal{P} \rightarrow \mathcal{Cm}(\mathfrak{R}_1)\) so that \(h(A) = \{x\}, h(B) = \{y\}, \) and \(h(C) = \{z\}\), where \(x, y, z \in \{a, b, c\}\). Then

- (63) fails if \(x \neq y = z\),
- (64) fails if \(x = y \neq z\),
- (65) fails if \(x \neq y = z\),
- (66) fails if \(x \neq y\) and \(z \in \{x, y\}\),
- (67) fails if \(x = y \neq z\) or \(|\{x, y, z\}| = 3\).

16 A non-commutative dense representable RA

This section presents four formulas that are not in \(\mathbf{CT}_4\) but appear in axiomatizations of relevance logics. The axiomatization of \(R\) by Anderson-Belnap [1, §27.1.1] includes (68), (70), (71) as R12, R2, and R3, respectively. They note [1, p. 20] that Church [5] used \(A \rightarrow A\), (69), (70) and Contraction as axioms. Interpreted as relations, formulas (68)–(71) assert that certain relations commute under fusion. Since there are non-commutative dense representable relation algebras, these formulas are not even in \(\mathbf{CT}_\omega\).

**Theorem 13.** For all \(A, B, C \in \Pi\), the following four theorems of \(R\) are not in \(\mathbf{CT}_\omega\).

\[
\begin{align*}
(A \rightarrow \neg B) &\rightarrow (B \rightarrow \neg A), \quad (68) \\
(A \rightarrow (B \rightarrow C)) &\rightarrow (B \rightarrow (A \rightarrow C)), \quad (69) \\
(A \rightarrow B) &\rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)), \quad (70) \\
A &\rightarrow ((A \rightarrow B) \rightarrow B). \quad (71)
\end{align*}
\]

**Proof.** Let \(\mathfrak{R}_2 = \langle K, R, *, \{1'\} \rangle\) be the model structure determined by \(K = \{1', a, b, b^*\}, 1'^* = 1', a^* = a, (b^*)^* = b, R \subseteq K^3,\) and \(Rxyz\) holds if and only if \(z \in \{x\};\{y\}\), where \(;\) is specified in Table 12. One may check that \(\mathfrak{R}_2\) satisfies

\[
\mathfrak{R}_2 = \begin{array}{cccc}
\{1'\} & \{1\} & \{a\} & \{b\} \\
\{a\} & \{1', a, b, b^*\} & \{a, b\} & \{a\} \\
\{b\} & \{b\} & \{a\} & \{1', a, b, b^*\} \\
\{b^*\} & \{b^*\} & \{a, b^*\} & \{1', b, b^*\} & \{b^*\}
\end{array}
\]

**Table 12** The atom structure of a non-commutative dense representable relation algebra

the conditions required to conclude \(\mathcal{Cm}(\mathfrak{R}_2) \in RA\) by Theorem 6(2). Note
that $\mathfrak{K}_2$ is also dense, i.e., it satisfies $R_{xx}$. On the other hand, $\mathfrak{Cm}(\mathfrak{K}_2)$ is not commutative, so Contraposition (68), Permutation (69), Suffixing (70), and Modus Ponens (71) are invalid. These formulas are invalidated in many ways, but in rather few ways if the propositional variables are mapped to singletons and the formulas are mapped to the empty set. A complete list of such valuations has been calculated with GAP [7]. In Table 13 we list two to four assignments under which each formula evaluates to $\emptyset$ and is therefore invalid in $\mathfrak{K}_2$.

| (68) | (69) | (70) | (71) |
|------|------|------|------|
| $\{a\}$ | $\{b\}$ | $\{a\}$ | $\{a\}$ |
| $\{b\}$ | $\{b^*\}$ | $\{a\}$ | $\{a\}$ |
| $\{b^*\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |
| $\{a\}$ | $\{a\}$ | $\{a\}$ | $\{a\}$ |

Table 13 Ways that formulas (68)–(71) fail in $\mathfrak{Cm}(\mathfrak{K}_2)$

To prove Contraposition (68), Permutation (69), Suffixing (70), and Modus Ponens (71) are not in $\mathsf{CT}_\omega$ it suffices to show $\mathfrak{Cm}(\mathfrak{K}_2)$ is isomorphic to a proper relation algebra. A finite sequence is a function $f$ with domain $\text{dom}(f) = \{1, \ldots, n\}$ for some finite non-zero $n$. Let $\mathbb{Q}$ be the set of rational numbers. Let $U$ be the set of finite sequences of rational numbers. Define a binary relation $B \subseteq U \times U$ for $f, g \in U$ by $fBg$ (we say $f$ is below $g$ or $f$ comes before $g$) iff for some finite $n > 0$, $\text{dom}(f) = \{1, \ldots, n\} \subseteq \text{dom}(g)$, $f_i = g_i$ for all $i < n$, and $f_n < g_n$. Let

$$
\begin{align*}
\sigma(1') &= \{(x, x) : x \in U\}, \\
\sigma(b^*) &= B, \\
\sigma(b) &= B^{-1}, \\
\sigma(a) &= (U \times U) \setminus (\sigma(1') \cup B \cup B^{-1}), \\
\rho(X) &= \bigcup_{x \in X} \sigma(x) \text{ for all } X \subseteq K.
\end{align*}
$$

Then $\rho$ is an isomorphism from $\mathfrak{Cm}(\mathfrak{K}_2)$ onto a proper relation algebra, hence every formula in $\mathsf{CT}_\omega$ is valid in $\mathfrak{K}_2$. Since (68)–(71) are not valid in $\mathfrak{K}_2$, they are not in $\mathsf{CT}_\omega$. 
17 A formula in $T_3 \setminus CR^*$

Theorem 14 shows that (41) is not in $CR^*$ and that it is equivalent to a model structure property.

**Theorem 14.** Assume $A, B, C, D \in \Pi$, 

\[
F = A; B \land C, \\
G = (A \land \neg D); B \lor A; (B \land D; C),
\]

and $\mathcal{R} = \langle K, R, *, I \rangle$ is a model structure. Then

1. $F \rightarrow G \in T_3$,
2. $F \rightarrow G /\in CR^*$,
3. the equation $F \leq G$ is true in $\mathcal{Cm}(\mathcal{R})$ if and only if $\mathcal{R}$ satisfies

\[
R_d^*bc \Rightarrow R_dcb. \\
(l\text{-refl}')
\]

**Proof (1).** $F \rightarrow G \in CT_3$ by Lemma 2, Lemma 3, and Theorem 9(1)(6), so $F \rightarrow G \in T_3$ because $F \rightarrow G \in \Pi^*$.

**Proof (2).** To show $F \rightarrow G /\in CR^*$, we use a model structure whose complex algebra is a Meyer monoid that is not a relation algebra. Let $I = \{1, a, a^*\}$ and $K = \{1', a, a^*\}$ where $1^* = 1'$ and $(a^*)^* = a$. Let $\mathcal{R}_3 = \langle K, R, *, \{1'\} \rangle$ where $R \subseteq K^3$ is determined by Table 14.

| $\mathcal{R}_3$ | $\{1\}$ | $\{a\}$ | $\{a^*\}$ |
|------------------|----------|----------|-----------|
| $\{1\}$          | $\{a\}$  | $\{a^*\}$|
| $\{a\}$          | $\{1\}$  | $\{a,a^*\}$|
| $\{a^*\}$        | $\{1,a,a^*\}$| $\{a^*\}$|

Table 14 $\mathcal{Cm}(\mathcal{R}_3)$ is a Meyer monoid and not a relation algebra.

$\mathcal{R}_3$ is a $CR^*$-model structure, so everything in $CR^*$ is valid in $\mathcal{R}_3$. Tagging, $(l\text{-refl}'), (l\text{-refl})$, and $(r\text{-refl})$ all fail because $\langle a, a, a^* \rangle \in R$ but $\langle a^*, a^*, a \rangle \notin R$.

Choose a homomorphism $h: \mathcal{P} \rightarrow \mathcal{Cm}(\mathcal{R}_3)$ from the predicate algebra into the complex algebra of $\mathcal{R}_3$ such that $h(A) = h(B) = \{a\}$ and $h(C) = h(D) = \{a^*\}$. Then Axiom (R9) fails and $h(F \rightarrow G) = \emptyset$. Since $h(1') = I$ is not contained in $h(F \rightarrow G)$, $1' \leq F \rightarrow G$ is not true in $\mathcal{Cm}(\mathcal{R}_3)$, hence $F \rightarrow G$ is not valid in $\mathcal{R}_3$ and $F \rightarrow G \notin CR^*$.

**Proof (3).** Assume $\mathcal{R}$ satisfies $(l\text{-refl}')$. We wish to prove $F \leq G$ is true in $\mathcal{Cm}(\mathcal{R})$, i.e., $h(F) \subseteq h(G)$ for an arbitrary homomorphism $h: \mathcal{P} \rightarrow \mathcal{Cm}(\mathcal{R})$.

We assume $c \in h(F)$ and show $c \in h(G)$. First compute

\[
h(F) = h(A; B \land C) = h(A); h(B) \land h(C),
\]
so \( c \in h(C) \) and there are \( a \in h(A) \) and \( b \in h(B) \) such that \( Rabc \). There are two cases. First, assume \( a \in h(\sim D) \). Then

\[
a \in h(A) \cap h(\sim D) = h(A \wedge \sim D).
\]

From this and \( b \in h(B) \) we get \( c \in h((A \wedge \sim D); B) \). Since \( h(G) \) is the union of this last set with another set, we have \( c \in h(G) \), as desired.

For the second case, assume \( a \notin h(\sim D) = K \setminus (h(D))^{-1} \). Then \( a \in (h(D))^{-1} \), so \( a = d^* \) for some \( d \in h(D) \).

Then \( Rd^*bc \) since \( Rabc \), hence \( Rdcb \) by \( (l\text{-refl})' \), which gives \( b \in h(D; C) \).

By this and \( b \in h(B) \), \( b \in h(B \wedge D; C) \), hence, by \( a \in h(A) \) and \( Rabc, c \in h(A; (B \wedge D; C)) \).

Again, \( h(G) \) is the union of this set with another set, so \( c \in h(G) \), as desired.

Assume we have a relevant model structure \( \mathcal{R} = \langle K, R, *, \mathcal{I} \rangle \) in which \( (l\text{-refl})' \) fails, specifically, there are \( a, b, c \in K \) such that \( Ra^*bc \) and not \( Racb \).

We will show \( F \leq G \) is not true in \( \mathcal{Cm}(\mathcal{R}) \).

Choose a homomorphism \( h : \mathcal{P} \to \mathcal{Cm}(\mathcal{R}) \) so that propositional variables \( A, B, C, D \in I \) have these values:

\[
h(A) = \{a^*\}, \quad h(B) = \{b\}, \quad h(C) = \{c\}, \quad h(D) = \{a\}.
\]

Since \( Ra^*bc \) we have \( h(C) = \{c\} \subseteq \{a^*\}; \{b\} = h(A); h(B) = h(A; B) \), so

\[
h(F) = h(A; B \wedge C) = h(A; B) \cap h(C) = h(C) = \{c\}.
\]

Note that \( h(A \wedge \sim D) = h(A) \cap h(\sim D) = \{a^*\} \cap (K \setminus \{a^*\}) = \emptyset \), hence, independent of the value for \( B \),

\[
h((A \wedge \sim D); B) = h(A \wedge \sim D); h(B) = \emptyset; h(B) = \emptyset.
\]

(\(\alpha\))

Since \( Racb \) is false, \( b \notin \{a\}; \{c\} \), hence \( h(D; C) = h(D); h(C) = \{a\}; \{c\} \subseteq K \setminus \{b\} \). Consequently, \( h(B \wedge D; C) = h(B) \cap h(D; C) \subseteq \{b\} \cap (K \setminus \{b\}) = \emptyset \).

We therefore have

\[
h(A; (B \wedge D; C)) = h(A); h(B \wedge D; C) = \{a^*\}; \emptyset = \emptyset.
\]

(\(\beta\))

From (\(\alpha\)) and (\(\beta\)) we get \( h(G) = h((A \wedge \sim D); B) \cup h(A; (B \wedge D; C)) = \emptyset \). The inclusion \( \{c\} = h(F) \subseteq h(G) = \emptyset \) is false so \( F \leq G \) is not true in \( \mathcal{Cm}(\mathcal{R}) \).

The contrapositive of what we have just proved is that if \( F \leq G \) is true in the complex algebra of a model structure \( \mathcal{R} \), then \( \mathcal{R} \) satisfies \( (l\text{-refl})' \).

As part of the proof of [4, Lemma 6.5] it is stated that there are \( \mathbf{CR}^* \)-model structures in which tagging fails. No example is given, but \( \mathcal{R}_3 \) can be used.

18 Counterexample to a theorem of Kowalski

According to [14, Theorem 8.1], \( \mathbf{R} \) is “complete with respect to square-increasing, commutative, integral relation algebras.” However, \( \mathbf{R} \) does not
Tarski’s classical relevant logic

contain all the formulas true in this class of algebras. By Theorem 14, (41) is not a theorem of \( R \), but it is in \( T_3 \) and is therefore true in all semi-associative relation algebras, including all square-increasing, commutative, integral relation algebras. Thus (41) is a counterexample to [14, Theorem 8.1], which was obtained as an immediate consequence of [14, Theorem 7.1], that every normal De Morgan monoid is embeddable in a square-increasing, commutative, integral relation algebra. However, the complex algebra of \( K_3 \) is a counterexample, since \( K_3 \) fails to satisfy tagging, it but would have to do so if \( \text{Cm}(\mathfrak{R}_3) \) were embedded in a relation algebra. The difficulty seems to arise in the proof of [14, Lemma 5.4(1)].

19 Formulas in \( T_5 \setminus CT_4 \)

We present just two examples of formulas requiring five variables to prove. The second one (73) is a consequence of a formula that expresses DesArgues Theorem when \( A, B, \cdots \) are points and \( A; B \) is the set of points on the line passing through \( A \) and \( B \). Mikulás [30] proved that \( T_\omega \) is not finitely axiomatizable by constructing infinitely many more formulas requiring arbitrarily large numbers of variables to prove.

**Theorem 15.** [25, Theorems 8.1, 8.2] If \( A, B, C, D, E, F, G \in \Pi \), then the following two formulæ are in \( T_5 \), but not in \( CT_4 \) if the predicates are distinct.

\[
\begin{align*}
A \land (B \land C; D); (E \land F; G) & \rightarrow (A \land (B \land (C \land \neg C); D); (E \land F; G)) \\
\lor (A \land (B \land C; D); (E \land F; (G \land \neg G))) & \lor C; ((C; A \land D; E); G \land D; F \land C; (A; G \land B; F)); G
\end{align*}
\]

\[
\begin{align*}
A; B \land C; D \land E; F & \rightarrow ((A \land \neg A); B \land C; D \land E; F) \lor (A; B \land C; (D \land \neg D) \land E; F) \\
\lor (A; B \land C; D \land (E \land \neg E); F) & \lor (A; B \land C; D \land (F \land \neg F)) \\
\lor A; (A; C \land B; D \land (A; E \land B; F); (E; C \land F; D)); D
\end{align*}
\]

**Proof.** By [25, Theorem 8.1], if \( A, B, \ldots, G \) are binary relations then (72) and (73) are binary relations that contain the identity relation. In both cases a straightforward proof of this fact only refers to five objects. Two are assumed to be in the relation denoted by the left-hand side, and there are three more corresponding to the occurrences of \( ; \) in the left-hand side. The proof consists of assembling facts expressed by the right-hand side from the assumptions that five objects are related to each other by six or seven binary relations in ways described by the left-hand side. The proofs are similar to, but more
elaborate, than the proofs in Table 8 or Theorem 12. The Cut rule is not needed. Thus (72) and (73) are in CT\(_5\).

To show (72) and (73) are not in CT\(_4\), let \(\mathcal{K}_4 = \langle K, R, \ast, 0 \rangle\), where \(K = \{0, a, b, c\}\), \(x^* = x\) for every \(x \in K\), and \(R\) is determined in Table 15 (or [25, Table 6]). Then \(\mathcal{K}_4\) is a KR-model structure in which both (72) and (73) fail.

\[
\begin{array}{ccc}
0 & \{0\} & \{a\} \\
\{1\} & \{a\} & \{b, c\} \\
\{a\} & \{b\} & \{a, c\} \\
\{c\} & \{b\} & \{a, b\} \\
\end{array}
\]

Table 15 The smallest KR-model structure that invalidates (72) and (73).

if \(h : \mathcal{V} \rightarrow \mathcal{Cm}(\mathcal{K}_4)\) is a homomorphism such that \(\{a\} = h(A) = h(B) = h(E) = h(G), \{c\} = h(C) = h(F)\), and \(\{b\} = h(D)\). Such homomorphisms exist if the predicates are distinct.

The proof shows there are 3-predicate instances of (72) and (73) that fail to be in CT\(_4\). There are only two other 4-element KR-model structures that invalidate (72), but (73) is valid in both of them. There are 58 KR-model structures with 5 elements that invalidate (72) and (73).

20 TR, a proper extension of CT\(_4\) and CR\(^*\)

By Theorem 13, no formula implying commutativity occurs in CT\(_\omega\). To insure that the extension TR of CT\(_4\) includes CR\(^*\), we add axioms expressing commutativity and density. Let \(\Psi\) be the set of equations \(A \circ B = B \circ A\) and \(A \leq A \circ A\) for all \(A, B \in \Pi^+\). Define TR by

\[
\text{TR} = \{ A : \Psi \vdash^\times 1' \leq A, A \in \Pi^+ \}. \quad \text{(TR-def)}
\]

By Theorem 10, for all \(A \in \Pi^+\),

\[
A \in \text{TR} \text{ iff } \Psi \vdash^+_4 \forall_x G(xAx) \text{ iff } \Psi \vdash^\times 1' \leq A. \quad \text{(TR-char)}
\]

Thus TR is the set of formulas \(A\) which, interpreted in a set of dense relations that commute under fusion, always denotes a relation containing the identity relation, and this fact, expressed by the formula \(\forall_x G(xAx)\), can be proved in first-order logic restricted to four variables, or, equivalently, the equation \(1' \leq A\) can be proved in equational logic from Tarski’s axioms (R\(_1\))–(R\(_{10}\)) plus two axioms asserting that all relations are dense and commute with each other under relative multiplication. For example, Reductio \((A \rightarrow \neg A) \rightarrow \neg A\) and Contraction \((A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)\) are in CR\(^*\) because of the
postulate p3 \((Ra\text{a})\) on \(\text{CR}^*\)-model structures, and they are in \(\text{TR}\) because of the axioms expressing density. On the other hand, \((41)\) is in \(\text{TR}\) but not \(\text{CR}^*\) by Theorem 14, so \(\text{TR}\) is a proper extension of \(\text{CR}^*\).

**Theorem 16.** The inclusion \(\text{CR}^* \subset \text{TR}\) holds and is proper because \((41)\) is in \(\text{TR}\) but not \(\text{CR}^*\).

### 21 KR, an extension of CT\(_4\)

\(\text{KR}\) is the “superclassical” logic obtained by adding the postulate \(a^* = a\) to the definition of \(\text{CR}^*\)-model structure. Under the formulas-as-relations interpretation, the equation \(A^* = A\) holds when \(A\) is a symmetric relation. The proof of the following theorem boils down to the observation that the class of atom structures of atomic symmetric dense integral relation algebras coincides with the class of \(\text{KR}\)-model structures.

**Theorem 17.** Let \(\Psi \subseteq \Sigma^\times\) be the set of equations \(A^* = A\) and \(A \leq A \circ A\), where \(A \in \Pi^+\). Then for all \(A \in \Pi^+\),

\[
A \in \text{KR} \iff \Psi \vdash^+ \forall x G(xAx) \iff \Psi \vdash^\times \forall x \leq A \quad (\text{KR-char})
\]

Theorem 17 characterizes \(\text{KR}\) as those predicates \(A \in \Pi^+\) for which the formula \(\forall x G(xAx)\) is 4-provable from the assumption that all relations are symmetric and dense, or, equivalently, \(\forall x \leq A\) is true in all symmetric dense relation algebras, or, equivalently, \(\forall x \leq A\) is derivable from the axioms \((R_1)\)–\((R_{10})\), \(A \leq A; A\), and \(A^{-1} = A\). Since relation algebras are semi-simple and simple symmetric relation algebras are commutative and integral (see the end of §12), it suffices to check only symmetric dense relation algebras in which \(\forall x \leq A\) is an atom when determining whether \(A\) is in \(\text{KR}\).

By [21, Theorem 12] (or [24, Theorem 473]), the probability that a randomly selected finite relation algebra with \(n\) atoms is integral and symmetric approaches 100% as \(n \to \infty\). Since symmetry implies commutativity, almost every finite relation algebra is commutative. The probability of being dense falls to zero, since one expects only about half of the atoms to be dense. Instead of choosing from all finite relation algebras, one may choose from just the dense ones. A randomly selected finite dense relation algebra will almost certainly be integral and symmetric, hence also commutative. Therefore the atom structure of a randomly selected finite dense relation algebra is almost certainly a \(\text{KR}\)-model structure. A slight reworking of the proof of [21, Theorem 12] shows that the number of \(\text{KR}\)-model structures with \(n\) elements is asymptotic to

\[
\kappa(n) = \frac{1}{(n-1)!} \cdot \frac{2^{(n-1)}(n-2)(n-1)}{n^6},
\]
i.e., the ratio of $\kappa(n)$ to the actual number approaches 1 as $n \to \infty$. At my first meeting with Alasdair Urquhart, in the late 1970s or a little later, we discussed how easy it is to construct finite relation algebras and relevant model structures.

For any finite dimension $d \geq 3$ chosen in advance, a randomly selected finite dense relation algebra will almost certainly be in $\mathbf{RA}_d$, the variety whose equational theory consists of all equations that are $d$-provable (see §11). Every finite subset of the equational theory of $\mathbf{RRA}$ is included in the equational theory of $\mathbf{RA}_d$ for some finite $d \geq 3$. Therefore, for any finite set of equations that are true in all representable relation algebras, a randomly selected finite dense relation algebra will almost certainly satisfy all of them [21, Theorem 15], as well as be symmetric, integral, and commutative. This suggests (but does not prove) that almost every finite dense relation algebra is representable.

22 RM, an extension of $T_4$

Interpreting formulas as relations leads naturally to classifying formulas according to what they say about relations. Some formulas express logical laws, such as $A \to A$, which asserts that the relation $A$ is a subset of itself. Other formulas amount to non-logical assumptions, i.e., assumptions about relations that may not be universally true. For example, $A \to A \circ A$ holds when $A$ is dense. By expressing fusion $\circ$ with negation $\sim$ and implication $\to$ and taking the contrapositive we get Reductio $(A \to \sim A) \to \sim A$, another formula that says $A$ is dense.

When we defined $\mathbf{TR}$, we adopted non-logical assumptions of commutativity and density and restricted the logical apparatus to four variables. The resulting logic contains all of classical relevant logic $\mathbf{CR}^*$, and actually exceeds it by one formula (41) and one postulate (l-refl'). This difference is eliminated in $\mathbf{KR}$, the result of adding $a^* = a$ to $\mathbf{CR}^*$ or adding $A^* = A$ to $\mathbf{TR}$. The key feature is the restriction to four variables. Thus $\mathbf{TR}$ has non-logical assumptions of commutativity and density, while $\mathbf{KR}$ has non-logical assumptions of symmetry and density, but in both cases we restrict the logical apparatus to four variables. This restriction is required to characterize $\mathbf{KR}$ because by Theorem 15 there are 5-provable formulas that are not in $\mathbf{KR}$.

No such restriction is needed for $\mathbf{RM}$, the Dunn-McColl logic $\mathbf{R}$-mingle, obtained by adding the mingle axiom $A \to (A \to A)$ to $\mathbf{R}$. Under the formulas-as-relations interpretation, $A \to (A \to A)$ is valid if and only if $A$ is transitive. The connectives of $\mathbf{RM}$ are $\land, \lor, \to,$ and $\sim$, the non-logical assumptions are commutativity, density, and transitivity, and the logical apparatus is not restricted. There are infinitely many variables available to prove consequences of the non-logical assumptions. Meyer [1, p. 413, Corollary 3.1]
proved that the finite normal Sugihara lattices are characteristic for $\text{RM}$. By [25, Theorem 6.2(i)], every finite normal Sugihara lattice is isomorphic to an algebra whose universe is a set $K$ of transitive dense binary relations on a set $U$, such that $K$ is closed under the operations $\wedge, \vee, \rightarrow, \sim$ in Table 1, and $K$ is commutative under fusion. This completeness and representability of normal Sugihara lattices has the following consequence.

**Theorem 18.** Let $\Psi \subseteq \Sigma^\times$ be the set of equations $A \circ B = B \circ A$ and $A = A \circ A$ where $A, B \in \Pi^\circ$. Then for all $A \in \Pi^\circ$,  

$$A \in \text{RM} \iff \Psi \vdash^+ \forall^2 G(xA x) \iff \Psi \vdash^\times 1' \leq A. \quad \text{(RM-char)}$$

Theorem 18 is restricted to $\Pi^\circ$ to exclude $1, 1, ; 1, \vdash 1, 1, \vdash 1$. Otherwise, $\Psi \vdash^\times 1' \leq A$ for every $A \in \Pi^\circ$, since $1'; \overline{1'} = \overline{1'} \vdash^\times 1' = 1'$. The consequences of $\Psi$ would include all of classical propositional logic, as was observed in [4, Lemma 2.3, Theorem 5.8]. On [4, p. 122] there is a computation intended to show that Sugihara lattices cannot be represented with binary relations with operations from Table 1. If correct, it would have ruled out Theorem 18, but it ends at $\{[i] : i \leq -2\}$, which should have been $\{[i] : i \geq -2\}$. Furthermore, [4, Lemma 5.6(2)(3)] is a claim about atoms that happens to be false but is proved for elements. It would also have precluded Theorem 18.

### 23 Conclusion and questions

Starting with $\rightarrow$, the vocabulary of relevance logic grew until, in the classical relevant logic of Meyer and Routley, it was the same as the relation algebras of Tarski. Meyer and Routley stopped one formula and one property short of Tarski, by missing (41) and $\langle 1\text{-refl'} \rangle$. Meyer and Routley included commutativity and density for good historical and logical reasons, but Tarski had no reason to restrict his axiomatization of the calculus of relations to those that are dense and commute under composition. The definition of $\text{TR}$ takes both steps. $\text{TR}$ contains everything 4-provable, including a 3-provable formula missing from $\text{CR}^*$, plus everything true in all commutative dense relation algebras. When formulas are interpreted as relations, both the logic $\text{TR}$ and the equational theory of commutative dense relation algebras may be characterized as whatever is 4-provable from commutativity and density. Similarly, $\text{KR}$ and the equational theory of symmetric dense relation algebras are characterized as whatever is 4-provable from symmetry and density. Finally, $\text{RM}$ and the equational theory of sets of transitive dense relations, closed under $\wedge, \vee, \rightarrow$, and $\sim$ from Table 1 and commuting under $\circ$, are characterized as everything in the relevant vocabulary that is $\omega$-provable from transitivity, density, and commutativity.

In the case of $\text{TR}$ and $\text{KR}$, there are two increasing chains of extensions obtained by allowing more variables, i.e., by relaxing the restrictions
on the logical apparatus while retaining the non-logical assumptions. The chains approach \( \mathsf{CT}^\omega \), where \( \Psi \) is the set of non-logical assumptions, either commutativity and density in the case of \( \mathsf{TR} \), or symmetry and density for \( \mathsf{KR} \). \( \mathsf{RM} \) already contains \( \mathsf{CT}^\omega \). These extensions are significant because if a model structure \( \mathcal{R} \) validates \( \mathsf{CT}^\omega \) then it has a complex algebra that is a simple representable relation algebra and therefore can be embedded in the complex algebra of the model structure \( \mathfrak{U} \) of pairs on some set \( U \) (see \$12), with the advantage that the ternary relation of \( \mathfrak{U} \) is \((\text{triples})\) and its Routley star is \((\text{star})\). Although (72) and (73) can fail to be valid in the atom structure of a non-representable relation algebra, such as the one in Table 15, they are valid in every model structure of pairs and are in \( \mathsf{CT}^\omega \) (in fact, \( \mathsf{CT}_5 \)), so a randomly selected dense relation algebra will almost certainly have an atom structure in which (72) and (73) are valid. They are not always valid, but almost always.

In all this, the target is \( \mathfrak{U} \), the model structure of pairs on \( U \). Although \( \mathfrak{U} \) may have never been mentioned in any of their works, one could fairly say that De Morgan, Peirce, Schröder, and Tarski were studying and axiomatizing some of its properties, and that Routley and Meyer were axiomatizing \( \mathfrak{U} \) directly, through various choices of postulates for relevant model structures. In any case, \( \mathfrak{U} \) lurks in the background—any randomly chosen finite relevant model structure is almost certain to have any particular equational property of \( \mathfrak{U} \) fixed in advance.

Perhaps \( \mathfrak{U} \) would be a good example of a relevant model structure, for in the theory of relation algebras, its complex algebra \( \mathcal{R}(\mathfrak{U}) \) is the prototypical example. One advantage is that if worlds are pairs and \( R \) is determined by \((\text{triples})\) or something similar, then \( R \) need not be explicitly mentioned in the truth condition. For example, [26] says,

“Like the semantics of modal logic, the semantics of relevance logic relativises truth of formulas to worlds. But Routley and Meyer go modal logic one better and use a three-place relation on worlds. . . . Their truth condition for \( A \to B \) on this semantics is the following:

\[ A \to B \text{ is true at a world } a \text{ if and only if for all worlds } b \text{ and } c \text{ such that } Rabc \text{ (} R \text{ is the accessibility relation) either } A \text{ is false at } b \text{ or } B \text{ is true at } c. \]

For people new to the field it takes some time to get used to this truth condition.”

If worlds are pairs \( ab \) and \( R \) is the set of triples of the form \((ab, ca, cb)\), then the truth condition becomes

\[ A \to B \text{ is true at a world } ab \text{ if and only if for all worlds } ca \text{ and } cb, \text{ either } A \text{ is false at } ca \text{ or } B \text{ is true at } cb. \]

This truth condition on pairs validates all the axioms of \( \mathsf{R} \) that do not need density or commutativity. The real worlds are the identity pairs \( aa \), while the possible worlds (where \( A \wedge \sim A \) can hold) are the diversity pairs \( ab \) with \( a \neq b \).
1. Does the conception of worlds (set-ups, situations, states, cases, trips, acts, plays, events, etc) as pairs explain or clarify any philosophical or logical phenomena?

2. “Will the real negation please stand up?” [2, p. 174]. Does Table 1 show which negation is “real”?

3. What is the logical significance of (41)? Would it be proposed as an axiom for a relevance logic? Why was it never previously considered?

4. Can an axiomatization of TR be obtained by adding (41) to an axiomatization of CR∗?

5. Can an axiomatization of CR∗ be obtained by deleting axiom (R0) from the axioms for L×?

6. Which subsets of (1)–(71) axiomatize T3, T4, CT3, CT4, and CR∗?

7. Which of the derived rules of CT3 listed in Theorem 11 are either derivable, admissible, or included by definition in R or CR∗? For example, the first two 1-provable rules are included in R by definition, and the third is admissible. What about all the others?

8. Are there any deductive rules of CT4 that require four variables?

9. Is almost every finite relation algebra representable?

10. Does the fact that a randomly selected CR∗-model structure satisfying tagging will almost certainly be symmetric and satisfy any particular finite set of equations true in every representable relation algebra have any philosophical implications?

11. Why do relevance logic and relation algebra overlap in spite of arising independently through the pursuit of completely different goals?

12. Why did Schröder’s studies and Tarski’s axiomatization stay within the confines of the 4-variable fragment of the calculus of relations?

13. Why did relevance logic develop within the 4-variable fragment of the calculus of relations? Why was one formula and one property of the 4-variable fragment missed? Why was the focus on density, commutativity, symmetry, transitivity, and no other non-logical postulates on relations?

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