ON LUSTERNIK-SCHNIRELMANN CATEGORY OF SO(10)

N. IWASE, K. KIKUCHI AND T. MIYAUCHI

Abstract. Let $G$ be a compact connected Lie group and $p : E \to \Sigma A(A = \Sigma A_0)$ a principal $G$-bundle with a characteristic map $\alpha : A \to G$. We assume that there is a cone-decomposition $\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq n, F_0 = \{\ast\} \text{ and } F_n \simeq X\}$ of $G$ of length $m$. Our main theorem is as follows: we have $\text{cat}(X) \leq m + 1$, if the characteristic map $\alpha$ is compressible into $F_1$ and the Berstein-Hilton Hopf invariant $H_1(\alpha) = 0 \in [A, \Omega F_1 \ast \Omega F_1]$. We also apply it to the principal bundle $\text{SO}(9) \hookrightarrow \text{SO}(10) \to S^9$ to determine the L-S category of SO(10).

1. Introduction

In this paper, we work in the category of pointed $CW$-complex and don’t distinguish a map from its homotopy class to make the arguments simpler. The Lusternik-Schnirelmann category of a space $X$ is the least integer $n$ such that there exists an open covering $U_0, \ldots U_n$ of $X$ with each $U_i$ contractible in the space $X$. We denote this by $\text{cat}(X) = n$ and if no such integer exists, we write $\text{cat}(X) = \infty$.

Theorem 1.1 (Ganea [3]). Let $X$ be a connected space. Then there is a sequence of fibrations $F_n X \to G_n X \to X$, natural with respect to $X$ so that $\text{cat}(X) \leq n$ if and only if the fibration $G_n X \to X$ has a cross-section.

Here, $F_n X$ has the homotopy type of $E^{n+1} \Omega X = \Omega X^\ast(n+1)$ the $(n+1)$-fold join of $\Omega X$ and $G_n X$ has the homotopy type of the $\Omega X$-projective $n$-space $P^n \Omega X$ in the sense of Stasheff [12] equipped with the composition $e^X_n : P^n \Omega X \hookrightarrow P^\infty \Omega X \simeq X$, where $e^X_1$ is given by the evaluation map (see also [4]).

Let $R$ be a commutative ring and $X$ a connected space. The cup-length of $X$ with coefficients in $R$ is the least non-negative integer $k$ (or $\infty$) such that all $(k + 1)$-fold cup products vanish in the reduced cohomology $\tilde{H}^*(X; R)$. We denote this integer $k$ by $\text{cup}(X; R)$ following Iwase [6].

In 1967, Ganea introduced in [3] a homotopy invariant $\text{Cat}(X)$ for a space $X$, modifying Fox’s strong category. In the same paper, he gave the following characterization using the notion of a cone-decomposition.

Definition 1.2 (Ganea [3]). The strong category $\text{Cat}(X)$ of a connected space $X$ is 0 if $X$ is contractible and, otherwise, is equal to the least positive integer $n$.
such that there are cofibration sequences (called a cone-decomposition of length $m$)
$$\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq n, \ F_0 = \{\ast\} \text{ and } F_n \simeq X\},$$
which is often called the cone-length of $X$.

The following inequalities among these invariants are well-known
$$\cup(X; R) \leq \text{cat}(X) \leq \text{Cat}(X).$$

Let $f : \Sigma X \to \Sigma Y$ be a map. We denote $H_1(f) \in [\Sigma X, \Omega \Sigma Y \ast \Omega \Sigma Y]$ by the Berstein-Hilton Hopf invariant (see Berstein and Hilton [1]).

The purpose of this paper is to prove the following theorem. Let $G$ be a connected compact Lie group with a cone-decomposition of length $m$, that is, there are cofibration sequences
$$\{K_i \to F_{i-1} \to F_i \mid 1 \leq i \leq m\}$$
with $F_0 = \ast$ and $F_m \simeq G$. Let $G \hookrightarrow E \to \Sigma A (A = \Sigma A_0)$ be a principal bundle with a characteristic map $\alpha : A \to G$. The following is our main result.

**Theorem 1.3.** If $\alpha$ is compressible into $F_1$, $H_1(\alpha) = 0 \in [A, \Omega F_1 \ast \Omega F_1]$ and $K_m$ is a sphere, then we obtain $\text{cat}(E) \leq m + 1$.

In some application, we need to weaken the hypothesis slightly: suppose that there exists a space $F'_1 = \Sigma K'_1$ with $K'_1 \subset K_1$. Under the condition, the above theorem is extended as the following form.

**Theorem 1.4.** If $\alpha$ is compressible into $F'_1$, $H_1(\alpha) = 0$ and $K_m$ is a sphere, then we obtain $\text{cat}(E) \leq m + 1$.

This yields, we obtain the following result.

**Theorem 6.1.** $\text{cat}(\text{SO}(10)) = 21$.

In Section 2 and 3, we construct a structure map and a cone-decomposition of some spaces which play the vital role in the proof of the main theorems. In Section 4, we show the important relation between a structure map and a cone-decomposition which are constructed in Section 2 and 3. In Section 5, we prove Theorem 1.4. In Section 6, we show some applications of Theorem 1.4.

2. Structure map associated with a filtration

**Definition 2.1.** The filtered space $X$ is the space $X$ equipped with a sequence of subspaces,
$$X \supset \cdots \supset X_n \supset X_{n-1} \supset \cdots \supset \{\ast\}.$$We denote $i^X_{m,n} : X_m \to X_n$ by the inclusion map for $m < n$.

**Definition 2.2.** Suppose that the space $X$ and $Y$ are filtered by $\{X_n\}$ and $\{Y_n\}$, respectively. A filtered map $f : X \to Y$ is a filtration-preserving map, that is, $f(X_n) \subset Y_n$ for all $n$.

We denote $p^\Omega X_m$ by the map $E^m \Omega X \to P^{m-1} \Omega X$ in Theorem 1.1 and $i^{\Omega X}_{m,n} : P^m \Omega X \to P^n \Omega X$ by the inclusion map for $m < n$. 
Proposition 2.3. Let $X$ and $Y$ be filtered by $\{X_n\}$ and $\{Y_n\}$, respectively and a map $f : X \to Y$ be a filtered map. If the filtration of $X$ is a cone-decomposition of $X$, say $\{L_i \xrightarrow{h_i} X_{i-1} \xrightarrow{i_{i-1,i}^Y} X_i \mid 1 \leq i \leq n\}$, then there exist families of maps $\{f_i : X_i \to P^i\Omega Y_i \mid 0 \leq i \leq n\}$ and $\{g_i : L_i \to E^i\Omega Y_i \mid 1 \leq i \leq m\}$ such that $\{f_i\}$ and $\{g_i\}$ satisfy the following conditions.

1. The following diagram is commutative.

\[
\begin{array}{ccc}
L_i & \xrightarrow{h_i} & X_{i-1} \\
| & | & | \\
g_i & \downarrow f_{i-1} & \downarrow f_i \\
| & | & | \\
E^i\Omega Y_i & \xrightarrow{P^i\Omega Y_i} & P^i\Omega Y_i
\end{array}
\]

2. $e_i^Y \circ f_i = f \mid X_i$

Proof. We prove the proposition by induction on $i$. In the case of $i = 1$, we put $g_1 = \text{ad}(f \mid X_1)$, $f_0 = \ast$, and $f_1 = \Sigma \text{ad}(f \mid X_1)$, respectively. Then the following diagram commutes.

\[
\begin{array}{ccc}
L_1 & \xrightarrow{\ast} & \Sigma L_1 \\
| & | & | \\
g_1 & \downarrow f_0 & \downarrow f_1 \\
| & | & | \\
\Omega Y_1 & \xrightarrow{\ast} & \Sigma \Omega Y_1
\end{array}
\]

Therefore, the condition (1) is satisfied when $i = 1$. Also the condition (2) holds from the following equation. For $t \wedge x \in \Sigma L_1$,

\[
e_i^Y \circ f_1(t \wedge x) = \text{ev} \circ \Sigma \text{ad}(f \mid X_1)(t \wedge x)
= \text{ev}(t \wedge \Sigma \text{ad}(f \mid X_1)(x))
= \text{ad}(f \mid X_1)(x)(t)
= (f \mid X_1)(t \wedge x).
\]

Suppose (1) and (2) hold when $i = k - 1$. First, we construct $g_k : L_k \to E^k\Omega Y_k$ from the exact sequence:

\[
[L_k, E^k\Omega Y_k] \xrightarrow{P^k\Omega Y_k} [L_k, P^{k-1}\Omega Y_k] \xrightarrow{e_{k-1}} [L_k, Y_k].
\]

We use the equation

\[
e_{k-1}^Y \circ P^{k-1}\Omega Y_{k-1,k} \circ f_{k-1} = i_{k-1,k}^Y \circ e_{k-1}^Y \circ f_{k-1}
= i_{k-1,k}^Y \circ f \mid X_k \circ i_{k-1,k}^X
\]

and $i_{k-1,k}^X \circ h_{k-1} = 0$ by $L_k \xrightarrow{h_{k-1}} X_{k-1} \xrightarrow{i_{k-1,k}^X} X_k$ is the cofibre sequence. So, we have $e_{k-1}^Y(P^{k-1}\Omega Y_{k-1,k} \circ f_{k-1} \circ h_{k-1}) = 0 \in [L_k, Y_k]$ and there exists a map
$g_k : L_k \to E^k\Omega Y_k$ such that $\frac{\partial}{\partial Y_k}(g_k) = P^{k-1}\Omega_{i_{k-1,k}} \circ f_{k-1} \circ h_{k-1}$. Second, we construct a map $f'_k : X_k \to P^k\Omega Y_k$. We define $f'_k : X_k \to P^k\Omega Y_k$ as follows:

$$f'_k = P^{k-1}\Omega_{i_{k-1,k}} \circ f_{k-1} \cup C(g_k)$$

which makes the right square of the following diagram commutative:

By definition, $f'_k$ satisfies the equation,

$$(f'_k \cup \Sigma g_k) \circ \nu_k = \bar{\nu}_k \circ f'_k,$$

where $\nu_k : X_k \to X_k \cup \Sigma L_k$ and $\bar{\nu}_k : P^k\Omega Y_k \to P^k\Omega Y_k \cup \Sigma E^k\Omega Y_k$ are the canonical copairings. In the exact sequence $[X_{k-1}, Y_k] \xleftarrow{i_{k-1,k}^X} [X_k, Y_k] \xrightarrow{\varphi} [\Sigma L_k, Y_k]$, we have the equation,

$$i_{k-1,k}^X (e_k^Y \circ f'_k) = e_k^Y \circ f'_k \circ i_{k-1,k}^X$$

$$= e_k^Y \circ (i_{k-1,k}^Y \circ P^{k-1}\Omega_{i_{k-1,k}} \circ f_{k-1})$$

$$= e_k^Y \circ P^{k-1}\Omega_{i_{k-1,k}} \circ f_{k-1}$$

$$= i_{k-1,k}^Y \circ f|_{X_{k-1}}$$

$$= f|_{X_k} \circ i_{k-1,k}^X$$

$$= i_{k-1,k}^X (f|_{X_k}).$$

By Theorem B. 10 of [2], there exists a map $\delta'_k : \Sigma L_k \to Y_k$ such that

$$f|_{X_k} = \nabla Y_k \circ (e_k^Y \circ f'_k \cup \delta'_k) \circ \nu_k.$$

Let us consider the following exact sequence,

$$[L_k, \Omega P^{k-1}\Omega Y_k] \xrightarrow{\Omega e_{k-1,k}^Y} [L_k, \Omega Y_k] \xrightarrow{\Delta} [L_k, E^k\Omega Y_k] \xrightarrow{\text{ad}} [\Sigma L_k, P^{k-1}\Omega Y_k] \xrightarrow{\nu_k_{i_{k-1,k}}} [\Sigma L_k, Y_k].$$
Since $\Omega e^Y_{k-1}$ has a section, there exists a map $\delta_k : \Sigma L_k \to P^{k-1}Y_k$ such that $\delta'_k = e^Y_{k-1} \circ \delta_k$. Therefore we have the following equation:

$$f|_{X_k} = \nabla_{Y_k} \circ (e^Y_k \circ f'_k \lor e^Y_{k-1} \circ \delta_k) \circ \nu_k$$

$$= \nabla_{Y_k} \circ (e^Y_k \circ f'_k \lor e^Y_k \circ f \circ i^X_{k-1,k} \circ \delta_k) \circ \nu_k$$

$$= \nabla_{Y_k} \circ (e^Y_k \lor e^Y_k) \circ (f'_k \lor f \circ i^X_{k-1,k} \circ \delta_k) \circ \nu_k$$

$$= e^Y_k \circ \nabla_{P^k\Omega Y_k} \circ (f'_k \lor f \circ i^X_{k-1,k} \circ \delta_k) \circ \nu_k.$$

We define a map $f_k$ by a map $\nabla_{P^k\Omega Y_k} \circ (f'_k \lor f \circ i^X_{k-1,k} \circ \delta_k) \circ \nu_k$, then $f_k$ satisfies the condition of (2). Since $\nu_k$ is the copairing, we have the equations

$$pr_1 \circ \nu_k \circ i^X_{k-1,k} = id_{X_k} \circ i^X_{k-1,k} = i^X_{k-1,k}$$

and $pr_2 \circ \nu_k \circ i^X_{k-1,k} = q \circ i^X_{k-1,k} = 0,$

where $pr_1 : X_k \lor \Sigma L_k \to X_k$ and $pr_2 : X_k \lor \Sigma L_k \to \Sigma L_k$ are the first and second projections, respectively. Hence we obtain the equation

$$f_k \circ i^X_{k-1,k} = \nabla_{P^k\Omega Y_k} \circ (f'_k \lor f \circ i^X_{k-1,k} \circ \delta_k) \circ \nu_k \circ i^X_{k-1,k}$$

$$= f'_k \circ i^X_{k-1,k}$$

$$= i^X_{k-1,k} \circ P^{k-1}\Omega i^X_{k-1,k} \circ f_{k-1}.$$

It follows that $f_k$ satisfies the condition of (1), too. \hfill \Box

Let $\{f_i : X_i \to P^i\Omega Y_i | 0 \leq i \leq n\}$ and $\{g_i : L_i \to E^i\Omega Y_i | 1 \leq i \leq m\}$ be the map obtained from the filtered map $f : X \to Y$ by Proposition 2.3. We denote $\nu_i : X_i \to X_k \lor \Sigma L_i$ and $\bar{\nu}_i : P^i\Omega Y_i \to P^k\Omega Y_i \lor \Sigma E^i\Omega Y_i$ by the canonical copairings.

**Proposition 2.4.** If the complex $L_i$ be a co-H-space, then the following diagram is commutative.

$$\begin{array}{ccc}
X_i & \xrightarrow{\nu_i} & X_k \lor \Sigma L_i \\
\downarrow f_i & & \downarrow f_i \lor \Sigma g_i \\
P^i\Omega Y_i & \xrightarrow{\bar{\nu}_i} & P^k\Omega Y_i \lor \Sigma E^i\Omega Y_i.
\end{array}$$

**Proof.** By the definition of $f_i,$ and by the relation between the composition and the wedge of maps, we have

$$(f_i \lor \Sigma g_i) \circ \nu_i = ((\nabla_P \circ (f'_i \lor i^Y_{i,i} \circ \delta_i) \circ \nu_i) \lor \Sigma g_i) \circ \nu_i$$

$$= (\nabla_P \circ (f'_i \lor \delta_i \circ \nu_i) \lor \Sigma g_i) \circ \nu_i$$

$$= (\nabla_P \lor \Sigma g_i) \circ (f'_i \lor \delta_i \circ \nu_i) \lor \Sigma g_i) \circ \nu_i.$$
where \( \nu_i : \Sigma L_i \to \Sigma L_i \lor \Sigma L_i \) is the co-multiplication and \( T : \Sigma L_i \lor \Sigma L_i \to \Sigma L_i \lor \Sigma L_i \) is the commutative map. So we can proceed as follows:

\[
(f_i \lor \Sigma g_i) \circ \nu_i = (\nabla_P \lor \text{id}_E) \circ (f_i' \lor \iota_{i-1}^{\Omega i} \circ \delta_i \lor \Sigma g_i) \circ (\text{id}_X \lor \nu_i) \circ \nu_i = (\nabla_P \lor \text{id}_E) \circ (f_i' \lor \iota_{i-1}^{\Omega i} \circ \delta_i \lor \Sigma g_i) \circ (\text{id}_X \lor T \circ \nu_i) \circ \nu_i = (\nabla_P \lor \text{id}_E) \circ (f'_i \lor T' \circ (\Sigma g_i \lor \iota_{i-1}^{\Omega i} \circ \delta_i)) \circ (\text{id}_X \lor \nu_i) \circ \nu_i = (\nabla_P \lor \text{id}_E) \circ (f'_i \lor T') \circ \{ (f'_i \lor \Sigma g_i) \circ \nu_i \lor \iota_{i-1}^{\Omega i} \circ \delta_i \} \circ \nu_i,
\]

where \( T' : \Sigma E^i \Omega Y_i \lor P^i \Omega Y_i \to P^i \Omega Y_i \lor \Sigma E^i \Omega Y_i \) is the commutative map and \( \text{id}_P = \text{id}_{P^i \Omega Y_i} \). By the equation (2.1), we proceed further as follows:

\[
(f_i \lor \Sigma g_i) \circ \nu_i = (\nabla_P \lor \text{id}_E) \circ (\text{id}_P \lor T') \circ \{ (\nu_i \lor f'_i \lor \iota_{i-1}^{\Omega i} \circ \delta_i) \} \circ \nu_i = (\nabla_P \lor \text{id}_E) \circ (\nu_i \lor f'_i \lor \iota_{i-1}^{\Omega i} \circ \delta_i) \circ \nu_i = (\nabla_P \lor \nabla_{E^i \Omega Y_i}) \circ (\text{id}_P \lor T' \lor \text{id}_E) \circ \nu_i \circ \nu_i \circ \iota_{i-1}^{\Omega i} \circ \delta_i \circ \nu_i = \nu_i \circ \nabla_P \circ (f'_i \lor \iota_{i-1}^{\Omega i} \circ \delta_i) \circ \nu_i = \nu_i \circ f'_i.
\]

\[\Box\]

3. Cone-Decomposition associated with projective spaces

We denote the \( k \)-skeleton of a space \( X \) by \((X)^{(k)}\) and the restriction of \( f : X \to Y \) on \((X)^{(k)}\) by \((f)^{(k)}\). By the fact that \((f)^{(k)}\) is compressible into \((Y)^{(k)}\), we use the same symbol \((f)^{(k)} : (X)^{(k)} \to (Y)^{(k)}\). And if the dimension of \( X \) is less than or equal to \( n \), then we use the same symbol \( f : X \to (Y)^{(n)} \), too.

Let \( G \) be a compact Lie group with a cone-decomposition of length \( m \), that is, there are cofibration sequences

\[
\{ K_i \xrightarrow{h_i} F_{i-1} \xrightarrow{i_{i-1,m}^F} F_i \mid 1 \leq i \leq m \},
\]

with \( F_0 = * \) and \( F_m \simeq G \). Let \( l \) be the dimension of Lie group \( G \).

**Lemma 3.1.** Suppose that the complex \( K_m \) is the sphere \( S^\ell \) and \( \ell \geq 3, m \geq 3 \). Then there is a cofibre sequence as follows:

\[
(E^m \Omega F_{m-1})^{(\ell-1)} \lor K_m \xrightarrow{\nu} (P^{m-1} \Omega F_{m-1})^{(\ell)} \to (P^m \Omega F_m)^{(\ell)}.
\]

**Proof.** First, we determine the homotopy type of the \((\ell-1)\)-skeleton of the homotopy fibre of the map \( P^{m-1} \Omega i_{m-1,m}^F : P^{m-1} \Omega F_{m-1} \to P^{m-1} \Omega F_m \). Let \( \mathcal{F} \) be
the homotopy fibre of \( P^{m-1} \Omega i_{m-1,m}^F \), we consider the following commutative diagram with rows and columns as fibrations:

\[
\begin{array}{ccc}
\Omega(E^m \Omega F_m, E^m \Omega F_{m-1}) & \xrightarrow{\Omega} & \Omega(F_m, F_{m-1}) \\
E^m \Omega F_m & \xrightarrow{p_m} & P^{m-1} \Omega F_{m-1} & \xrightarrow{e_{m-1}} & F_m \\
E^m \Omega F_{m-1} & \xrightarrow{p_m} & P^{m-1} \Omega F_{m-1} & \xrightarrow{e_{m-1}} & F_m \\
\end{array}
\]

Since \((F_m, F_{m-1})\) is \((\ell - 1)\)-connected, \((\Omega F_m, \Omega F_{m-1})\) is \((\ell - 2)\)-connected and \((E^m \Omega F_m, E^m \Omega F_{m-1})\) is \((\ell + m - 3)\)-connected. Hence \(\Omega(E^m \Omega F_m, E^m \Omega F_{m-1})\) is \((\ell + m - 4)\)-connected. By the Serre exact sequence

\[
H_{2\ell + m - 5}(\Omega(E^m \Omega F_m, E^m \Omega F_{m-1})) \rightarrow \cdots \rightarrow H_k(\Omega(E^m \Omega F_m, E^m \Omega F_{m-1})) \rightarrow H_k(\mathfrak{F}) \rightarrow H_{k-1}(\Omega(E^m \Omega F_m, E^m \Omega F_{m-1})) \rightarrow \cdots,
\]

we obtain that \(H_k(\mathfrak{F})\) is isomorphic to \(H_k(\Omega(F_m, F_{m-1}))\) for \(k \leq \ell \leq \ell + m - 3\), and hence that \(\mathfrak{F}\) is \((\ell - 2)\)-connected, \(\ell \geq 3\). On the other hand, by the Blakers-Massey’s theorem, we have \(\pi_l(F_m, F_{m-1}) \cong \pi_l(S^l)\), and hence we obtain

\[
\pi_{\ell-1}(\Omega(F_m, F_{m-1})) \cong \pi_l(F_m, F_{m-1}) \cong \pi_l(S^l) \cong \mathbb{Z}.
\]

Then by Hurewicz Isomorphism Theorem, we obtain

\[
H_{\ell-1}(\mathfrak{F}) \cong H_{\ell-1}(\Omega(F_m, F_{m-1})) \cong \pi_{\ell-1}(\Omega(F_m, F_{m-1})) \cong \mathbb{Z}.
\]

Thus \(\mathfrak{F}\) has the homology decomposition as

\[
\mathfrak{F} \cong S^{\ell-1} \cup \text{(Moore cells in dimensions } \geq \ell)\text{.}
\]

By Ganea’s fibre-cofibre construction (see Ganea [3]), we obtain a map

\[
\phi_0 : P^{m-1} \Omega F_{m-1} \cup C\mathfrak{F} \rightarrow P^{m-1} \Omega F_m,
\]

as the homotopy pushout

\[
\begin{array}{ccc}
\{\ast\} & \xrightarrow{\phi_0} & P^{m-1} \Omega F_{m-1} \\
\downarrow & & \downarrow \\
\ast & \rightarrow & P^{m-1} \Omega F_{m-1} \cup C\mathfrak{F},
\end{array}
\]

which has the homotopy type of the homotopy pullback of the diagonal

\[
\Delta : P^{m-1} \Omega F_m \rightarrow P^{m-1} \Omega F_m \times P^{m-1} \Omega F_m
\]

and the inclusion

\[
P^{m-1} \Omega F_{m-1} \times P^{m-1} \Omega F_{m} \cup P^{m-1} \Omega F_{m} \times \{\ast\} \hookrightarrow P^{m-1} \Omega F_m \times P^{m-1} \Omega F_m,
\]
(see, for example, [4, Lemma 2.1] with \((X, A) = (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1}), (Y, B) = (P^{m-1}\Omega F_m, \{\ast\})\) and \(Z = P^{m-1}\Omega F_m\)). Hence \(\mathcal{F}_0\) is given by the pullback of the trivial map
\[
\{\ast\} \to P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m
\]
and the inclusion
\[
P^{m-1}\Omega F_{m-1} \times P^{m-1}\Omega F_m \cup P^{m-1}\Omega F_m \times \{\ast\} \hookrightarrow P^{m-1}\Omega F_m \times P^{m-1}\Omega F_m
\]
which has the homotopy type of the pushout
\[
\mathcal{F} \times \Omega P^{m-1}\Omega F_m \to P^{m-1}\Omega F_{m-1}
\]
(see, for example, [4, Lemma 2.1] with \((X, A) = (P^{m-1}\Omega F_m, P^{m-1}\Omega F_{m-1}), (Y, B) = (P^{m-1}\Omega F_m, \{\ast\})\) and \(Z = \{\ast\}\)). Thus the homotopy fibre \(\mathcal{F}_0\) of \(\phi_0\) has the homotopy type of the join \(\mathcal{F} \ast \Omega P^{m-1}\Omega F_m\) and is \((\ell - 1)\)-connected, and hence \(\phi_0\) is \(\ell\)-connected. Thus we have that
\[
(P^{m-1}\Omega F_{m-1})^{(\ell)} \cup CS^{\ell-1} \simeq (P^{m-1}\Omega F_m)^{(\ell)}.
\]
We are now ready to show that \((P^m\Omega F_{m-1})^{(\ell)} \cup CS^{\ell-1} \simeq (P^m\Omega F_m)^{(\ell)}\). Since \((E^m\Omega F_m, E^m\Omega F_{m-1})\) is \((\ell + m - 3)\)-connected and \(m \geq 3\), \((E^m\Omega F_{m-1})^{(\ell - 1)} \simeq (E^m\Omega F_m)^{(\ell - 1)}\) and hence
\[
(P^m\Omega F_{m-1})^{(\ell)} \cup CS^{\ell-1} \simeq (P^m\Omega F_{m-1})^{(\ell)} \cup C(S^{\ell-1} \vee (E^m\Omega F_{m-1})^{(\ell - 1)})
\]
\[
\simeq (P^m\Omega F_m)^{(\ell)} \cup C(E^m\Omega F_{m-1})^{(\ell - 1)} \simeq (P^m\Omega F_m)^{(\ell)}.
\]
This completes the proof of Lemma 3.1.

Using Lemma 3.1, we construct cone-decompositions of \(F_m \times F_1\), \((P^m\Omega F_m)^{(\ell)}\) and \((P^m\Omega F_m)^{(\ell)} \times (\Sigma \Omega F_1)^{(\ell)}\).
First, we construct a cone-decomposition of $F_m \times F_1$: Let $K^{m,1}_i$ and $F^{m,1}_i$ be as follows.

$$K^{m,1}_i = \{K_i \times \{\ast\}\} \cup \{K_{i-1} \ast K_1\} \quad \text{for } 1 \leq i \leq m,$$

$$F^{m,1}_i = F_i \times \{\ast\} \cup F_{i-1} \times F_1 \quad \text{for } 0 \leq i \leq m,$$

$$K^{m,1}_{m+1} = K_m \ast K_1 \quad \text{and} \quad F^{m,1}_{m+1} = F_m \times F_1,$$

where $K_0$ and $F_{-1}$ are empty sets. We denote a map $\chi_i : (CK_i, K_i) \to (F_i, F_{i-1})$ by the characteristic map. We introduce the relative Whitehead product $[\chi_{i-1}, \chi_1]^r : K_{i-1} \ast K_1 \to F^{m,1}_{i-1}$ defined as follows:

$$K_{i-1} \ast K_1 = (CK_{i-1} \times K_1) \cup (K_{i-1} \times CK_1) \xrightarrow{(\chi_{i-1} \times \chi_1) \cup (\chi_{i-1} \times \chi_1)} F_{i-1} \times \{\ast\} \cup F_{i-2} \times F_1 = F^{m,1}_{i-1}.$$

Let $w^{m,1}_i : K^{m,1}_i \to F^{m,1}_i$ be the wedge of maps $(incl) \circ (h_i \times \{\ast\}) : K_i \times \{\ast\} \to F_{i-1} \times \{\ast\} \leftarrow F^{m,1}_i$ and $[\chi_{i-1}, \chi_1]^r$ for $1 \leq i \leq m$, and $w^{m,1}_{m+1} : K^{m,1}_{m+1} \to F^{m,1}_{m+1}$ be $[\chi_m, \text{id}_{CK_1}]^r$. Let $i^{m,1}_i : F^{m,1}_i \to F^{m,1}_{i+1}$ be the canonical inclusion for $0 \leq i \leq m$. Then the set of cofibration sequences

$$(3.2) \quad \{K^{m,1}_i \xrightarrow{w^{m,1}_i} F^{m,1}_{i-1} \xrightarrow{i^{m,1}_i} F^{m,1}_i \mid 1 \leq i \leq m+1\}$$

is a cone-decomposition of $F_m \times F_1$ of length $m + 1$.

Second, we construct a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}$. By lemma 3.1, we obtain a cone-decomposition of $(P^m\Omega F_m)^{(\ell)}$ of length $m$:

$$\begin{align*}
(\Omega F_{m-1})^{(\ell-1)} \to \{\ast\} \leftarrow (\Sigma \Omega F_{m-1})^{(\ell)} \\
(E^2\Omega F_{m-1})^{(\ell-1)} \to (\Sigma \Omega F_{m-1})^{(\ell)} \leftarrow (P^2\Omega F_{m-1})^{(\ell)} \\
\vdots \\
(E^{m-1}\Omega F_{m-1})^{(\ell-1)} \to (P^{m-2}\Omega F_{m-1})^{(\ell)} \leftarrow (P^{m-1}\Omega F_{m-1})^{(\ell)} \\
(E^m\Omega F_{m-1})^{(\ell-1)} \cup K_m \to (P^{m-1}\Omega F_{m-1})^{(\ell)} \leftarrow (P^m\Omega F_m)^{(\ell)}.
\end{align*}$$

Third, we construct a cone-decomposition of $(P^m\Omega F_m)^{(\ell)} \times (\Sigma \Omega F_1)^{(\ell)}$. Let $\hat{E}_i$ and $\hat{F}_i$ be as follows.

$$\hat{E}_i = \{(E^i \Omega F_{m-1})^{(\ell-1)} \times \{\ast\}\} \cup \{(E^{i-1} \Omega F_{m-1})^{(\ell-1)} \ast (\Omega F_1)^{(\ell-1)}\}$$

for $1 \leq i \leq m - 1$,

$$\hat{E}_m = \{(E^m \Omega F_{m-1})^{(\ell-1)} \cup K_m \times \{\ast\}\} \cup \{(E^{m-1} \Omega F_{m-1})^{(\ell-1)} \ast (\Omega F_1)^{(\ell-1)}\},$$

$$\hat{E}_{m+1} = \{(E^m \Omega F_{m-1})^{(\ell-1)} \cup K_m \ast (\Omega F_1)^{(\ell-1)}\},$$

$$\hat{F}_i = (P^i \Omega F_{m-1})^{(\ell)} \times \{\ast\} \cup (P^{i-1} \Omega F_{m-1})^{(\ell)} \times (\Sigma \Omega F_1)^{(\ell)}$$

for $0 \leq i \leq m - 1$,

$$\hat{F}_m = (P^m \Omega F_m)^{(\ell)} \times \{\ast\} \cup (P^{m-1} \Omega F_{m-1})^{(\ell)} \times (\Sigma \Omega F_1)^{(\ell)}$$
and
\[ \hat{F}_{m+1} = (P^m\Omega F_m)^{(\ell_1)} \times (\Sigma \Omega F_1)^{(\ell_1)}. \]

Here \( E^{-1}\Omega F_{m-1} \) and \( P^{-1}\Omega F_{m-1} \) are empty sets. We denote maps
\[ \chi' : (C((\Omega F_1)^{(\ell_1)}), (\Omega F_1)^{(\ell_1)}) \to (\Sigma(\Omega F_1)^{(\ell_1)}, \{\ast\}), \]
\[ \chi'_1 : (C(E'\Omega F_{m-1})^{(\ell_1)}, (E'\Omega F_{m-1})^{(\ell_1)}) \to ((P^i\Omega F_{m-1})^{(\ell_1)}, (P^i{-1}\Omega F_{m-1})^{(\ell_1)}) \]
for \( 0 \leq i \leq m - 1 \) and
\[ \chi'_m : (CE', E') \to ((P^m\Omega F_{m-1})^{(\ell_1)}, (P^{m-1}\Omega F_{m-1})^{(\ell_1)}) \]
by the characteristic maps, where \( E' = (E^m\Omega F_{m-1})^{(\ell_1)} \lor K_m \). Let \( \hat{w}_i : \hat{E}_i \to \hat{F}_{i-1} \) be the wedge of maps
\[(incl) \circ ((p_1^{\Omega F_{m-1}(\ell_1)} \times \{\ast\}) : (E^i\Omega F_{m-1})^{(\ell_1)} \times \{\ast\}) \to (P^{i-1}\Omega F_{m-1})^{(\ell_1)} \times \{\ast\} \]
\[ \hookrightarrow \hat{F}_{i-1} \]
and
\[ [\chi'_{i-1}, \chi']^r : (E^{i-1}\Omega F_{m-1})^{(\ell_1)} \lor (\Omega F_1)^{(\ell_1)} \to \hat{F}_{i-1} \]
for \( 1 \leq i \leq m - 1 \), \( \hat{w}_m : \hat{E}_m \to \hat{F}_{m-1} \) be the wedge of maps
\[(incl) \circ (p' \times \{\ast\}) : \{(E^m\Omega F_{m-1})^{(\ell_1)} \lor K_m\} \times \{\ast\} \to (P^{m-1}\Omega F_{m-1})^{(\ell_1)} \times \{\ast\} \]
\[ \hookrightarrow \hat{F}_{m-1} \]
and \([\chi'_{m-1}, \chi']^r\), and \( \hat{w}_{m+1} : \hat{E}_{m+1} \to \hat{F}_m \) be \([\chi'_{m-1}, \chi']^r\), where \( p' : (E^m\Omega F_{m-1})^{(\ell_1)} \lor K_m \to (P^{m-1}\Omega F_{m-1})^{(\ell_1)} \) is the map \( p' \) in Lemma \( 3.1 \). We denote \( \hat{i}_i : \hat{F}_i \to \hat{F}_{i+1} \) by the canonical inclusion for \( 0 \leq i \leq m \). Then the set of cofibration sequences
\[ \{ \hat{E}_i \xrightarrow{\hat{w}_i} \hat{F}_{i-1} \xrightarrow{\hat{i}_{i-1}} \hat{F}_i | 1 \leq i \leq m + 1 \} \]
is a cone-decomposition of \((P^m\Omega F_m)^{(\ell_1)} \times (\Sigma \Omega F_1)^{(\ell_1)} \) of length \( m + 1 \).

4. Structure map and cone-decomposition

Let a cone-decomposition of \( F_m \) be \([3.1]\) and a \( k \)-filter of \( F_m \) be \( F_k \), we apply this Proposition \( 2.3 \) to the identity map \( \text{id}_{F_m} : F_m \to F_m \). From this procedure, we obtain the structure maps \( \sigma_i : F_i \to P^i\Omega F_i \) for \( 1 \leq i \leq m \) and the maps \( g'_j : K_j \to E^j\Omega F_j \) for \( 1 \leq j \leq m \). We set \( g_j = g'_j : K_j \to (E^j\Omega F_j)^{(\ell_1)} \)
for \( 1 \leq j \leq m - 1 \) and \( g_m : K_j \to (E^m\Omega F_m)^{(\ell_1)} \sim (E^m\Omega F_{m-1})^{(\ell_1)} \hookrightarrow (E^m\Omega F_{m-1})^{(\ell_1)} \lor K_m \) the composition \( g'_m \) and the inclusion map.

Let \( \nu^{m,1}_k : F^{m,1}_k \to F^{m,1}_k \lor \Sigma K^{m,1}_k \) and \( \hat{\nu}_k : \hat{F}_k \to \hat{F}_k \lor \Sigma \hat{K}_k \) be the canonical copairings for \( 1 \leq k \leq m + 1 \). Then,
Lemma 4.1. the following diagram is commutative:

\[
\begin{array}{cccccc}
F_{m+1} & \overset{\nu_{m+1}}{\longrightarrow} & F_m & \overset{\nu_m}{\longrightarrow} & F_{m+1} & \overset{\nu_{m+1}}{\longrightarrow} \\
\downarrow g_m \ast g_1 & \quad & \downarrow \sigma_m & \quad & \downarrow \sigma_m \times \sigma_1 & \quad & \downarrow \sigma_m \times \sigma_1 \ast g_m \ast g_1 \\
\hat{E}_{m+1} & \overset{\hat{\nu}_{m+1}}{\longrightarrow} & \hat{F}_m & \overset{\hat{\nu}_m}{\longrightarrow} & \hat{F}_{m+1} & \overset{\hat{\nu}_{m+1}}{\longrightarrow} \hat{F}_{m+1} \ast \Sigma \hat{E}_{m+1}.
\end{array}
\]

Here the map \( \hat{\sigma}_m = \sigma_m \times \{ * \} \cup \sigma_m \times \sigma_1 \).

To prove this Lemma, it is necessary to show the following equation:

\[
T_1 \circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K^m_{m+1}}) \circ \nu_{m+1} = (\nu_{m+1} \cup \text{id}_{\Sigma K_m \times F_1}) \circ (\nu_m \times \text{id}_{F_1}).
\]

Here \( \nu_m : F_m \to F_m \lor \Sigma K_m \) is the canonical copairing and \( T_1 : F_{m+1}^m \cup_{F_1} (\Sigma K_m \times F_1) \lor \Sigma K^m_{m+1} \to (F_{m+1}^m \lor \Sigma K^m_{m+1}) \cup_{F_1} (\Sigma K_m \times F_1) \) is the canonical homeomorphism.

Proof. First, we show that the following diagram is commutative:

\[
\begin{array}{cccccc}
F_{m+1} & \overset{\nu_m \times \text{id}_{F_1}}{\longrightarrow} & F_{m+1} \lor \Sigma K_m \times F_1 & \overset{p_1}{\longrightarrow} & F_{m+1} \lor \Sigma K_m \times F_1 \\
\downarrow \nu_{m+1} & \quad & \downarrow \text{id}_{F_{m+1} \lor \nu'} & \quad & \downarrow \text{id}_{F_{m+1} \lor \nu'} \\
F_{m+1} \lor \Sigma K_m \times F_1 & \overset{\nu'}{\longrightarrow} & F_{m+1} \lor \Sigma K_m \times F_1 & \overset{p_1}{\longrightarrow} & F_{m+1} \lor \Sigma K_m \times F_1.
\end{array}
\]

where \( \nu' : \Sigma K_m \times F_1 = \Sigma K_m \times \Sigma K_1 = \Sigma K_m \times \Sigma F_1 \lor \Sigma K_m \times F_1 \lor \Sigma K_m \ast K_1 \) is the canonical copairing and \( p_1 \) is the map pinching \( \Sigma K_m \times F_1 \) to one point. This follow from Figure \[4\]

Therefore we have

\[
T_1 \circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K^m_{m+1}}) \circ \nu_{m+1} = T_1 \circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K^m_{m+1}}) \circ p_1 \circ (\text{id}_{F_{m+1}^m} \lor \nu') \circ (\nu_m \times \text{id}_{F_1}).
\]

Let us denote \( p_2 : F_{m+1}^m \lor_{F_1} (\Sigma K_m \times F_1) \lor_{F_1} (\Sigma K_m \times F_1) \lor \Sigma K^m_{m+1} \to F_{m+1}^m \lor_{F_1} \Sigma K^m_{m+1} \lor_{F_1} (\Sigma K_m \times F_1) \lor \Sigma K^m_{m+1} \) by the map pinching the second \( \Sigma K_m \times F_1 \) to one point, \( p_3 : F_{m+1}^m \lor_{F_1} (\Sigma K_m \times F_1) \lor \Sigma K^m_{m+1} \lor_{F_1} (\Sigma K_m \times F_1) \to (F_{m+1}^m \lor \Sigma K^m_{m+1}) \lor_{F_1} \Sigma K^m_{m+1} \) by the map pinching the first \( \Sigma K_m \times F_1 \) to one point, \( \nu_0 : \Sigma K_m \to \Sigma K_m \lor \Sigma K_m \) by the canonical co-multiplication and \( T_0 : \Sigma K_m \lor \Sigma K_m \to \Sigma K_m \lor \Sigma K_m \) by the
commutative map. It is easy to check the following:

\[
T_1 \circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K_{m+1}}) \circ \nu_{m+1}^{m,1} \\
= T_1 \circ p_2 \circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K_{m} \times F_1} \lor \text{id}_{\Sigma K_{m} \times K_1}) \\
\circ ((\text{id}_{F_{m+1}} \lor \nu') \circ (\nu_m \times \text{id}_{F_1})) \\
= T_1 \circ p_2 \circ ((\text{id}_{F_{m+1}} \lor \nu') \circ (\nu_m \times \text{id}_{F_1})) \\
\circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K_{m} \times F_1} \lor \nu') \\
\circ (\text{id}_{F_{m+1}} \lor (T_0 \times \text{id}_{F_1})) \\
\circ ((\nu_m \times \text{id}_{F_1}) \lor \text{id}_{\Sigma K_{m} \times F_1} \lor (\nu_m \times \text{id}_{F_1})).
\]
Using the equations \((\text{id}_{F_m} \times \nu_0) \circ \nu_m = (\nu_m \times \text{id}_{F_m}) \circ \nu_m\) and \(T_0 \circ \nu_0 = \nu_0\) from the assumption that \(K_m\) is a co-H-space, we have

\[
T_1 \circ ((\nu_m \times \text{id}_{F_1}) \vee \text{id}_{\Sigma K_{m+1}}) \circ \nu_{m+1}^1 \\
= p_3 \circ (\text{id}_{F_{m+1}} \cup \nu' \cup \text{id}_{\Sigma K \times F_1}) \circ (\text{id}_{F_{m+1}} \cup (T_0 \times \text{id}_{F_1})) \\
\circ (\text{id}_{F_{m+1}} \cup (\nu_0 \times \text{id}_{F_1})) \circ (\nu_m \times \text{id}_{F_1}) \\
= p_3 \circ (\text{id}_{F_{m+1}} \cup \nu' \cup \text{id}_{\Sigma K \times F_1}) \\
\circ (\text{id}_{F_{m+1}} \cup (\nu_0 \times \text{id}_{F_1})) \circ (\nu_m \times \text{id}_{F_1}) \\
= p_3 \circ (\text{id}_{F_{m+1}} \cup \nu' \cup \text{id}_{\Sigma K \times F_1}) \\
\circ ((\nu_m \times \text{id}_{F_1}) \cup \text{id}_{\Sigma K \times F_1}) \circ (\nu_m \times \text{id}_{F_1}).
\]

Using the diagram (4.1), we proceed further as follows:

\[
T_1 \circ ((\nu_m \times \text{id}_{F_1}) \vee \text{id}_{\Sigma K_{m+1}}) \circ \nu_{m+1}^1 = (\nu_{m+1} \cup \text{id}_{\Sigma K \times F_1}) \circ (\nu_m \times \text{id}_{F_1}).
\]

This completes the proof of Lemma 4.2. \(\square\)

**Proof of Lemma 4.1.** The commutativity of the left square follows from Proposition 2.9 of [11]. It is obvious that the middle square is commutative. We show the equation \((\sigma_m \times \sigma_1 \vee \Sigma g_{m} \ast q_1) \circ \nu_{m+1}^1 = \check{\nu}_{m+1} \circ (\sigma_m \times \sigma_1)\). Recall that the construction of the structure map \(\sigma_m : F_m \to P^m \Omega F_m\), we can see that \(\sigma_m = \nabla P^m \Omega F_m \circ (\sigma'_m \vee \iota_{m-1,m} \circ \delta_m) \circ \nu_m\). Here \(\sigma'_m\) is the induced map from the following diagram:

\[
\begin{array}{cccccc}
K_m & \xrightarrow{h_m} & F_{m-1} & \xrightarrow{\iota_{m-1,m}} & F_m \\
\downarrow{g'_m} & & \downarrow{p_{m-1}^m \Omega F_m} & \downarrow{\iota_{m-1,m}^m \Omega F_m} & \downarrow{\sigma'_m} \\
E^m \Omega F_m & \xrightarrow{P_{m}^m \Omega F_m} & P^{m-1} \Omega F_m & \xrightarrow{P^{m-1} \Omega F_m} & P^m \Omega F_m,
\end{array}
\]

and \(\delta_m : \Sigma K_m \to P^{m-1} \Omega F_m\) is the map pulled back the difference map \(\delta'_m : \Sigma K_m \to F_m\) which is the difference between the identity map of \(F_m\) and \(e_{m}^m \circ \sigma'_m\).
So we have the equation:

\[(\sigma_m \times \sigma_1 \vee \Sigma g_m \ast g_1) \circ \nu_{m+1}^{m,1}\]

\[= \{(\nabla^{p}_m \Omega^{m}_F \circ (\sigma'_m \vee (\iota^{m}_F \circ \delta_m) \circ \nu_m) \times \sigma_1 \vee \Sigma g_m \ast g_1) \circ \nu_{m+1}^{m,1}\}
\]

\[= \{(\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F) \circ (\iota^{1}_m \circ \delta_m) \times \sigma_1) \vee \Sigma g_m \ast g_1\}
\]

\[= (\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F \circ \iota^{1}_m \circ \delta_m) \times \sigma_1) \vee \Sigma g_m \ast g_1\]

\[= (\nu_m \times \text{id} \Omega^{1}_F) \circ \iota^{1}_m \circ \delta_m) \times \sigma_1\}
\]

\[T_2 \circ \{(\sigma'_m \times \sigma_1 \vee \Sigma g_m \ast g_1) \cup ((\iota^{m}_F \circ \delta_m) \times \sigma_1)\}
\]

\[= (\nu_{m+1} \cup \text{id} \Sigma K \times \Omega^{1}_F) \circ (\nu_m \times \text{id} \Omega^{1}_F)\]

\[= (\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F \circ \iota^{1}_m \circ \delta_m) \times \sigma_1\}
\]

where \(T_2 : (\hat{F}_{m+1}^{1} \cup \Sigma \hat{E}_{m+1}^{1} \cup \Omega^{1}_F \hat{F}_{m+1}^{1} \cup \Sigma \hat{E}_{m+1}^{1} \times \Sigma \hat{E}_{m+1}^{1}) \) is the canonical homeomorphism. By Lemma 4.2, we can proceed as follows:

\[(\sigma_m \times \sigma_1 \vee \Sigma g_m \ast g_1) \circ \nu_{m+1}^{m,1}\]

\[= (\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F \circ \iota^{1}_m \circ \delta_m) \times \sigma_1) \vee \Sigma g_m \ast g_1\]

\[= (\nu_{m+1} \cup \text{id} \Sigma K \times \Omega^{1}_F) \circ (\nu_m \times \text{id} \Omega^{1}_F)\]

\[= (\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F \circ \iota^{1}_m \circ \delta_m) \times \sigma_1\}
\]

By the definitions of \(\sigma'_m\) and \(\sigma_1\), we have

\[(\sigma_m \times \sigma_1 \vee \Sigma g_m \ast g_1) \circ \nu_{m+1}^{m,1}\]

\[= (\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F \circ \iota^{1}_m \circ \delta_m) \times \sigma_1) \vee \Sigma g_m \ast g_1\]

\[= (\nu_{m+1} \cup \text{id} \Sigma K \times \Omega^{1}_F) \circ (\nu_m \times \text{id} \Omega^{1}_F)\]

\[= (\nabla^{p}_m \Omega^{m}_F \times \text{id} \Omega^{1}_F \circ \iota^{1}_m \circ \delta_m) \times \sigma_1\}
\]

\[= (\nu_{m+1} \cup \text{id} \Sigma K \times \Omega^{1}_F) \circ (\nu_m \times \text{id} \Omega^{1}_F)\]
Here \( i_1 : \hat{T}_{m+1} \to \hat{T}_{m+1} \cup \Sigma \hat{E}_{m+1} \) is the inclusion map and \( T_3 : (\hat{T}_{m+1} \cup \Sigma \hat{E}_{m+1}) \cup_{\Sigma F_1} (\hat{T}_{m+1} \cup \hat{E}_{m+1}) \to (\hat{T}_{m+1} \cup_{\Sigma F_1} F_{m+1}) \cup \Sigma \hat{E}_{m+1} \cup \Sigma \hat{E}_{m+1} \) is the canonical homeomorphism.

\[
(\sigma_m \times \sigma_1 \vee \Sigma g_m \ast g_1) \circ \nu_{m+1}^{\ast}
= (\nabla_{Pm \Omega F_m} \times \text{id}_{\Sigma F_1} \vee \nabla_{\Sigma \hat{E}_{m+1}}) \circ T_3 \circ (\tilde{\nu}_{m+1} \cup \tilde{\nu}_{m+1})
\circ \{(\sigma'_m \times \sigma_1) \cup (l_{m-1,m} \ast \delta_m) \times \sigma_1\} \circ (\nu_m \times \text{id}_{F_1})
= \tilde{\nu}_{m+1} \circ (\nabla_{Pm \Omega F_m} \times \text{id}_{\Sigma F_1})
\circ \{(\sigma'_m \vee l_{m-1,m} \ast \delta_m) \times \sigma_1\} \circ (\nu_m \times \text{id}_{F_1})
= \tilde{\nu}_{m+1} \circ \{(\sigma'_m \times \sigma_1) \cup (l_{m-1,m} \ast \delta_m) \times \nu_m \times \sigma_1\}
= \tilde{\nu}_{m+1} \circ (\sigma_m \times \sigma_1).
\]

This completes the proof. \(\square\)

5. Proof of Theorem [1.4]

In the fibre sequence \( G \hookrightarrow E \to \Sigma A \), by the James-Whitehead decomposition (see Theorem VII.(1.15) of Whitehead [14]), the total space \( E \) has the homotopy type of the space \( G \cup_{\psi} G \times CA \). Here \( \psi \) is the following composition:

\[
\psi : G \times A \xrightarrow{\text{id}_G \times \alpha} G \times G \xrightarrow{\mu} G.
\]

Since \( G \simeq F_m \) and \( \alpha \) is compressible into \( F_1' \), we can see that

\[
\psi : G \times A \simeq F_m \times A \xrightarrow{\text{id}_{F_m} \times \alpha} F_m \times F'_1 \subset F_m \times F_1 \subset F_m \times F_m \simeq G \times G \xrightarrow{\mu} G \simeq F_m
\]

and \( E \) is the homotopy push out of the following sequence:

\[
\begin{array}{c}
F_m \\
pr_1 \\
F_m \times A \\
\text{id}_{F_m \times \alpha} \\
F_m \times F_1 \\
\mu|_{F_m \times F_1} \\
F_m.
\end{array}
\]

We construct spaces and maps such that the homotopy push out of these maps dominates \( E \).

The condition of \( H_1(\alpha) = 0 \) implies that

\[
(5.1) \quad \Sigma \text{ad}(\alpha) = \sigma_1|_{F_1'} \circ \alpha : A \to F'_1 \to \Sigma F'_1.
\]

We denote \( \mu_{i,j} : F_i \times F_j \to F_m \) by the restriction of \( \mu : G \times G \to G \) to \( F_i \times F_j \subset F_m \times F_m \simeq G \times G \) for \( i, j \leq m \). Then
Lemma 5.1. the following diagram is commutative:

Here the map \( \phi \) and \( \chi \) are \((i_{m,m+1}^\Omega F_\ast) (l) \circ pr_1 \) and \( \text{id}_{(p_{m+1} \Omega F_m)} \times (\Sigma \Omega \alpha)((l)) \), respectively.

Proof. It is obvious that the top left square is commutative. By the equation \( e_{F_{m+1}} F_m = e_{\Omega F_{m+1}} F_m \), the bottom left square is commutative. The commutativity of the bottom middle square follows from the equation \( \alpha \circ e_1^{F_1} = e_1^{F_1} \circ P_1 \Omega \alpha = e_1^{F_1} \circ \Sigma \alpha \). By the equation (5.1), we have the commutative diagram:

\[
\begin{array}{cccccc}
F_m \times A & \xrightarrow{\text{id}_{F_m} \times \alpha} & F_m \times F_1 & \xrightarrow{\text{id}_{F_m} \times \sigma} & F_m \times F_1 & \xrightarrow{\mu_{m,1}} F_m \\
\sigma_m \times \sigma_A & \downarrow & \sigma_m \times \sigma_A & \downarrow & \sigma_m \times \sigma_A & \downarrow \\
P_{m+1} \Omega F_m \times \Sigma \Omega A & \xrightarrow{\text{id}_{p_{m+1} \Omega F_m} \times \Sigma \alpha} & P_{m+1} \Omega F_m \times \Sigma \Omega F_1 & \xrightarrow{\text{id}_{p_{m+1} \Omega F_m} \times \Sigma \alpha'} & \hat{F}_{m+1},
\end{array}
\]

where \( \sigma_A \) is the evaluation map and \( \iota' \) is the inclusion map. Thus, the top middle square is commutative. Since \( \sigma_m \) and \( \sigma_1 \) satisfy the condition (2) of Proposition 2.3, we have \( e_{F_{m+1}} \circ \sigma_m = \text{id}_{F_m} \lor e_1^{F_1} \circ \sigma_1 = \text{id}_{F_1} \) and \( e_{F_{m+1}} \circ \sigma_m = e_{F_m} \circ \sigma_m = \text{id}_{F_m} \). Therefore the right rectangular is commutative, too. \( \square \)

Lemma 5.2. In the diagram of Lemma 5.1 there is a map \( \hat{\mu} : \hat{F}_{m+1} \rightarrow P_{m+1} \Omega F_m \) such that the right rectangular diagram is commutative.

Proof. First, we construct a map \( \hat{\mu}_k : \hat{F}_k \rightarrow P^k \Omega F_m \). Let a cone-decomposition of \( F_m \times F_1 \) be (3.2), a cone-decomposition of \( \hat{F}_{m+1} \) be (3.3) and a \( k \)-filter of \( F_m \) be \( F_m \) for all \( k \). Let us consider that the restriction of \((e_{F_m}^{F_{m-1}}(l)) \times (e_1^{F_1}(l)) \) on \( \hat{F}_k \) is

\[
(e_{F_{m-1}}^{F_{m-1}}(l)) \times \{\ast\} \cup (e_{F_{m-1}}^{F_{m-1}}(l)) \times (e_1^{F_1}(l)) : \hat{F}_k \rightarrow F_m \times F_1,
\]

then the map \( \mu_{m,1} \circ \{\} (e_{F_{m-1}}^{F_{m-1}}(l)) \times (e_1^{F_1}(l)) : \hat{F}_{m+1} \rightarrow F_m \times F_1 \rightarrow F_m \) is a filtered map. Applying this filtered map to Proposition 2.3, we obtain the map

\[
\hat{\mu}_k : \hat{F}_k \rightarrow P^k \Omega F_m
\]

for \( 0 \leq k \leq m+1 \).

Second, for \( 0 \leq k \leq m \), we assert that the equation of maps

\[
(5.2) \quad i_{m,m+1}^{\Omega F_m} \circ \sigma_m \circ \mu_{m,1}^k = i_{k,m+1}^{\Omega F_m} \circ \hat{\mu}_k \circ j_k \circ \sigma_k : F_m \rightarrow P_{m+1} \Omega F_m,
\]

where \( \mu_{m,1}^k \) = \( \mu_{k,0} \cup \mu_{k-1,1} : F_{F_k} \times \{\ast\} \cup F_{k-1} \times F_1 \rightarrow F_m \),

\[
\sigma_k = \sigma_k \times \{\ast\} \cup \sigma_{k-1} \times \sigma_1 : F_{F_k} \rightarrow \{\} \times (P^k \Omega F_m) (l) \times (P^k \Omega F_{m-1}) (l) \times (\Sigma \Omega F_1) (l)
\]
and $j_t = (P^t\Omega_{i,m-1}^F(t) \times \{*\}) \cup (P^{t-1}\Omega_{i-1,m-1}^F(t) \times \text{id}_{\Sigma(F_1)}^{(t)})$ for $1 \leq t \leq m - 1$ and $j_m = \text{id}_{F_m}$. Note that this condition is natural to cone-decompositions. This is proved by induction on $k$. The case $k = 0$ is clear, since both maps are constant maps. Assume the $k$th of (5.2). Let us consider the cofibre sequence $K^{m+1}_{k+1} \longrightarrow F^m_k \longrightarrow F_{k+1}^m$. Since $\sigma_i$ satisfy the condition (1) of Proposition [2.3] the following diagram is commutative

$$
\begin{array}{cccccc}
F_i & \xrightarrow{\sigma_i} & P^i\Omega F_i & \xrightarrow{P^i\Omega F_{i+1}} & P^i\Omega F_{i+1,m-1} & \xrightarrow{P^i\Omega F_{i+1,m-1}} & P^i\Omega F_{m-1} \\
\downarrow{i_{i+1}} & & \downarrow{\sigma_{i+1}} & & \downarrow{\sigma_{i+1}} & & \\
F_{i+1} & \xrightarrow{\sigma_{i+1}} & P^{i+1}\Omega F_{i+1} & \xrightarrow{P^{i+1}\Omega F_{i+1,m-1}} & P^{i+1}\Omega F_{i+1,m-1} & \xrightarrow{P^{i+1}\Omega F_{i+1,m-1}} & P^{i+1}\Omega F_{m-1}
\end{array}
$$

for $1 \leq i \leq m - 1$. So we have $j_{k+1} \circ \tilde{\sigma}_k \circ i_{k+1}^{m,1} = \tilde{i}_k \circ j_k \circ \tilde{\sigma}_k$. By the condition (1) of Proposition [2.3] of $\tilde{\mu}_{k+1}$, we obtain $\tilde{\mu}_{k+1} \circ i_k = i_k^{m,1} \circ \tilde{\mu}_k$.

By the induction hypothesis, we proceed further as follows:

$$
\begin{align*}
i_k^{m,1} (i_{k+1,m} \circ \tilde{\mu}_{k+1} \circ j_{k+1} \circ \tilde{\sigma}_{k+1}) &= \sigma_m \circ i_k^{m,1} \circ \tilde{\sigma}_k \\
&= i_k^{m,1} (\sigma_m \circ \tilde{\mu}_k).
\end{align*}
$$

By Theorem B. 10 of [2], there exists a map $\delta_{k+1} : \Sigma K^{m,1}_{k+1} \rightarrow P^{m,1}\Omega F_m$ such that

$$
\sigma_m \circ \mu_{k+1}^{m,1} = \nabla P^{m,1}\Omega F_m \circ (i_{k+1,m} \circ \tilde{\mu}_{k+1} \circ j_{k+1} \circ \tilde{\sigma}_{k+1} \vee \delta_{k+1}) \circ \nu_{k+1}^{m,1}.
$$

By the condition (2) of Proposition [2.3] of $\tilde{\mu}_{k+1}$, we have the equation

$$
e_{m}^{F} \circ i_{k+1,m} \circ \tilde{\mu}_{k+1} = e_{k+1}^{F} \circ \tilde{\mu}_{k+1} = \mu_{m,1} \circ \{(e_{k+1}^{F-1})^{(t)} \times \{*\} \cup (e_{k+1}^{F-1})^{(t)} \times (e_{1}^{F-1})^{(t)}\}.
$$

By the commutative diagram

$$
\begin{array}{ccc}
F_i & \xrightarrow{\sigma_i} & (P^i\Omega F_i)^{(t)} \\
\downarrow{i_{i+1}} & & \downarrow{(e_{i+1}^{F-1})^{(t)}} \\
F_i & \xrightarrow{(e_{i+1}^{F-1})^{(t)}} & F_{m-1}
\end{array}
$$

for $i = k, k + 1 \leq m - 1$ and by the maps $\sigma_m \circ (e_{m}^{F})^{(t)}$ and $j_m$ are equal to identity maps up to homotopy, we have the equation

$$
\{(e_{k+1}^{F-1})^{(t)} \times \{*\} \cup (e_{k+1}^{F-1})^{(t)} \times (e_{1}^{F-1})^{(t)}\} \circ j_{k+1} \circ \tilde{\sigma}_{k+1} = i_{k+1}^{m,1}.
$$
Thus we obtain
\[ e_m^{F_m} \circ (\tilde{\Omega F}_{m+1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1}) = \mu_{m,1} \circ \tilde{i}_{k+1}^{m,1} = \mu_{k+1} \]
and
\[ e_m^{F_m} \circ \sigma_m \circ \mu_{k+1}^{m,1} = e_m^{F_m} \circ \nabla P^m \Omega F_m \circ (\tilde{i}_{k+1}^{m,1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1} \cup \delta_{k+1}) \circ \nu_{k+1}^{m,1} \]

\[ = \nabla F_m \circ (e_m^{F_m} \circ \tilde{\Omega F}_{m+1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1} \cup e_m^{F_m} \circ \delta_{k+1}) \circ \nu_{k+1}^{m,1} \]
\[ = \nabla F_m \circ (e_m^{F_m} \circ \sigma_m \circ \mu_{k+1} \cup e_m^{F_m} \circ \delta_{k+1}) \circ \nu_{k+1}^{m,1}. \]

Using Theorem 2.7 (1) of [9] and the multiplication \( \mu \) on \( G \simeq F_m \), the map \( e_m^{F_m} \circ \delta_{k+1} : \Sigma K_{m+1} \rightarrow F_m \) is null-homotopic. Using the following exact sequence,

\[ \cdots \rightarrow [\Sigma F_{k+1}^{m,1}, E^{m+1} \Omega F_m] \xrightarrow{\iota_{\Omega F_{m+1}}} [\Sigma K_{k+1}^{m,1}, P^m \Omega F_m] \xrightarrow{e_m^{F_m}} [\Sigma K_{k+1}^{m,1}, F_m]. \]

By the equation \( e_m^{F_m} \circ \delta_{k+1} = 0 \), there exists a map \( \delta'_{k+1} : \Sigma K_{k+1}^{m,1} \rightarrow E^{m+1} \Omega F_m \) such that \( \delta_{k+1} = \tilde{\mu}_{m+1} \circ \delta'_{k+1} \). Since \( E^{m+1} \Omega F_m \xrightarrow{\tilde{\Omega F}_{m+1}} P^m \Omega F_m \xrightarrow{\tilde{\Omega F}_{m+1}} P^{m+1} \Omega F_m \) is the cofibre sequence, we have \( \tilde{\Omega F}_{m+1} \circ \delta_{k+1} = 0 \) and

\[ \Omega F_m \circ \tilde{\Omega F}_{m+1} \circ \nabla P^m \Omega F_m \circ (\tilde{i}_{k+1}^{m,1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1} \cup \delta_{k+1}) \circ \nu_{k+1}^{m,1} \]
\[ = \nabla P^{m+1} \Omega F_m \circ (\tilde{i}_{k+1}^{m,1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1} \cup \delta_{k+1}) \circ \nu_{k+1}^{m,1} \]
\[ = \nabla P^{m+1} \Omega F_m \circ (\tilde{i}_{k+1}^{m,1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1} \cup 0) \circ \nu_{k+1}^{m,1} \]
\[ = \tilde{i}_{k+1}^{m,1} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1}. \]

From the equation (5.3), we obtain
\[ \iota_{\Omega F_{m+1}} \circ \sigma_m \circ \mu_{m+1}^{m,1} = \iota_{\Omega F_{m+1}} \circ \tilde{\mu}_{k+1} \circ \tilde{j}_{k+1} \circ \tilde{\sigma}_{k+1}. \]

Therefore we hold the statement by induction.

Finally, we construct a map \( \tilde{\mu} : \hat{F}_{m+1} \rightarrow P^{m+1} \Omega F_m \). Let us consider the exact sequence:

\[ [F_{m+1}^{m,1}, P^{m+1} \Omega F_m] \xrightarrow{e_{m+1}^{F_m}} [F_{m+1}^{m,1}, P^{m+1} \Omega F_m] \xrightarrow{q^*} [\Sigma K_{m+1}^{m,1}, P^{m+1} \Omega F_m]. \]

By the fact that the following diagrams are commutative:

\[ F_{m-1} \xrightarrow{i_{m-1}^{F_m}} F_m \xrightarrow{\sigma_m} P^m \Omega F_m \quad \hat{F}_m \xrightarrow{i_m} \hat{F}_{m+1} \]
\[ P^{m-1} \Omega F_{m-1} \xrightarrow{\tilde{\Omega F}_{m-1}} P^{m-1} \Omega F_m \quad \text{and} \quad \hat{F}_m \xrightarrow{\mu_m} \hat{F}_{m+1} \]

we have
\[ \mu_{m+1} \circ (\sigma_m \times \sigma_1) \circ i_{m+1}^{m,1} \circ \tilde{\mu}_{m+1} \circ \tilde{i}_m \circ \sigma_m \]
\[ = i_{m+1}^{F_m} \circ \mu_m \circ \sigma_m \].
and by previous inductive argument \((k = m \text{ of (5.2)})\),

\[
\begin{align*}
&= i_{m,m+1} \circ \sigma_m \circ \mu_m. \\
&= i_{m,m+1} \circ \sigma_m \circ \mu_m \circ \imath_{m+1}.
\end{align*}
\]

Hence there is a map \(\delta_{m+1} : \Sigma K_{m+1} \rightarrow P_{m+1} \Omega F_m\) such that

\[
(5.4) \quad i_{m,m+1} \circ \sigma_m \circ \mu_m = \nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \circ (\sigma_m \times \sigma_1) \lor \delta_{m+1}) \circ \nu_{m+1}.
\]

To continue calculating, we consider the map \(e : \hat{E}_{m+1} \rightarrow \Sigma K_{m+1}^{m+1}\) induced from the bottom left square of the following commutative diagram:

\[
\begin{array}{ccc}
F_{m+1} & \xrightarrow{i_{m+1}} & F_{m+1} \\
\sigma_m \downarrow & & \sigma_m \times \sigma_1 \\
F_m & \xrightarrow{i_m} & F_m \\
\hat{e}_m \downarrow & & \hat{e}_m \\
F_{m+1} & \xrightarrow{\nu_{m+1}} & \hat{E}_{m+1} \\
\end{array}
\]

where the map \(\hat{e}_m : \hat{F}_m \rightarrow F_{m+1}\) is \((e^F_m)^{(\ell)} \times \{e\} \cup (e^{F_{m-1}}_m)^{(\ell)} \times (e^{F_1}_m)^{(\ell)}\). Since \(\hat{e}_m \circ \hat{\sigma}_m\) and \((e^F_m)^{(\ell)} \times (e^{F_1}_m)^{(\ell)} \circ \sigma_m \times \sigma_1\) are homotopic to the identity maps, \(\hat{e} \circ \Sigma g_m \circ g_1\) is homotopic to the identity map of \(\Sigma K_{m+1}^{m+1}\). Then the equation \((5.4)\) is as follows:

\[
\begin{align*}
&= \nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \circ (\sigma_m \times \sigma_1) \lor \delta_{m+1}) \circ \nu_{m+1} \\
&= \nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \lor \hat{\sigma}_m \lor \hat{e}) \circ (\Sigma g_m \circ g_1) \circ \nu_{m+1} \\
&= \nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \lor \hat{\delta}_{m+1} \lor \hat{e}) \circ (\Sigma g_m \circ g_1) \circ \nu_{m+1}.
\end{align*}
\]

By Lemma \(4.1\) we proceed further:

\[
(5.4) = \nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \lor \hat{\delta}_{m+1} \lor \hat{e}) \circ \nu_{m+1} \circ (\sigma_m \times \sigma_1).
\]

Therefore we adopt \(\nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \lor \hat{\delta}_{m+1} \lor \hat{e}) \circ \nu_{m+1}\) as \(\hat{\mu}\). Then we obtain the top square is commutative. And we prove that the bottom square is commutative as follows. By the same argument of the proof of \(e_{m+1} \circ \delta_k = 0\) for \(1 \leq k \leq m\), we have the equation \(e_{m+1} \circ \delta_{m+1} = 0\). Thus we obtain

\[
\begin{align*}
e_{m+1} \circ \hat{\mu} &= e_{m+1} \circ (e_{m+1} \circ \nabla_{P_{m+1}} \Omega F_m \circ (\hat{\mu}_{m+1} \lor \hat{\delta}_{m+1} \lor \hat{e}) \circ \nu_{m+1}) \\
&= \nabla_{F_m} \circ (e_{m+1} \circ \hat{\mu}_{m+1} \lor e_{m+1} \circ \hat{\delta}_{m+1} \lor \hat{e}) \circ \nu_{m+1} \\
&= \nabla_{F_m} \circ (e_{m+1} \circ \hat{\mu}_{m+1} \lor 0) \circ \nu_{m+1} \\
&= e_{m+1} \circ \hat{\mu}_{m+1}
\end{align*}
\]

and by the condition (2) of Proposition \(2.3\) of \(\hat{\mu}_{m+1}\), we obtain

\[
e_{m+1} \circ \hat{\mu} = \mu_{m+1} \circ \{(e_{m+1}^{(\ell)}) \times (e_{m+1}^{(F_1)})\}.
\]

\[\Box\]
Thus we have the following commutative diagram:

\[
\begin{array}{cccccccccc}
F_m & \xrightarrow{pr_1} & F_m \times A & \xrightarrow{1 \times \alpha} & F_m \times F_1 & \xrightarrow{\mu_{m,1}} & F_m \\
\downarrow{\Omega F_m} & & \downarrow{\sigma \times \sigma A} & & \downarrow{\sigma \times \sigma_1} & & \downarrow{\Omega F_m} \\
\ & \ & \end{array}
\]

\[
P^{m+1}\Omega F_m \xrightarrow{\phi} (P^m\Omega F_m)^{(\ell)} \times (\Sigma \Omega A)^{(\ell)} \xrightarrow{\chi} \hat{F}_{m+1} \xrightarrow{\hat{\mu}} P^{m+1}\Omega F_m
\]

\[
\begin{array}{cccccccccc}
P_m & \xrightarrow{pr_1} & F_m \times A & \xrightarrow{1 \times \alpha} & F_m \times F_1 & \xrightarrow{\mu_{m,1}} & F_m \\
\end{array}
\]

We construct a cone-decomposition of \((P^m\Omega F_m)^{(\ell)} \times (\Sigma \Omega A)^{(\ell)}\) of length \(m + 1\):

\[
\{ \hat{E}_k' \xrightarrow{\hat{w}_k'} \hat{F}_{k-1} \xrightarrow{\hat{\gamma}_k'} \hat{F}_k \mid 1 \leq k \leq m + 1 \},
\]

by replacing \(F_1\) with \(A\) in the construction of the cone-decomposition of \((P^m\Omega F_m)^{(\ell)} \times (\Sigma \Omega F_1)^{(\ell)}\). We adopt cofibration sequences

\[
\{ E^k\Omega F_m \xrightarrow{p^k\Omega F_m} P^{k-1}\Omega F_m \xrightarrow{\ell^k_{m+1}} P^k\Omega F_m \mid 1 \leq k \leq m + 1 \}
\]
as a cone-decomposition of \(P^{m+1}\Omega F_m\) of length \(m + 1\). Let \(D\) be a homotopy pushout of \((\ell^m_{m+1})^{(\ell)} \circ pr_1\) and \(\hat{\mu} \circ (i_{P^m\Omega F_m}^{(\ell)}) \times (\Sigma \Omega A)^{(\ell)}\):

\[
(P^m\Omega F_m)^{(\ell)} \times (\Sigma \Omega A)^{(\ell)} \xrightarrow{f^-} P^{m+1}\Omega F_m
\]

Here \(f^- = \hat{\mu} \circ (i_{P^m\Omega F_m}^{(\ell)}) \times (\Sigma \Omega A)^{(\ell)}\) and \(f^- = (\ell^m_{m+1})^{(\ell)} \circ pr_1\). We construct a cone-decomposition of \(D\) as follows. By the equation \(\hat{\mu} \circ \hat{i}_m = \nabla^m_{m+1}\Omega F_m \circ (\hat{\mu}_{m+1} \cup \delta_{m+1} \cup \ell) \circ \hat{\nu}_{m+1} \cup \hat{\gamma}_m = \hat{\mu}_{m+1} \circ \hat{\nu}_m\), we can consider that the restriction of \(\hat{\mu}\) on \(\hat{F}_k\) is \(\mu_k\) and \(f^-\) is a filtered map. Since \(\hat{E}_k' \xrightarrow{\hat{w}_k'} \hat{F}_{k-1} \xrightarrow{\hat{\gamma}_k'} \hat{F}_k\) is the cofibre sequence, we have

\[
e^k_{m+1} \circ (f^-|_{\hat{F}_{k-1}} \circ \hat{w}_k') = e^k_{m+1} \circ \ell^m_{m+1} \circ f^-|_{\hat{F}_{k-1}} \circ \hat{w}_k' = e^k_{m+1} \circ f^-|_{\hat{F}_k} \circ \hat{\gamma}_k' \circ \hat{\nu}_{k-1} \circ \hat{w}_k' \]

\[
e^k_{m+1} \circ f^-|_{\hat{F}_k} \circ \hat{\nu}_k' \circ \hat{w}_k = 0.
\]

Using the fibre sequence \(E^k\Omega F_m \xrightarrow{p^k\Omega F_m} P^{k-1}\Omega F_m \xrightarrow{e^k_{m+1}} F_m\), there exists a map \(g^-_k : \hat{E}_k' \to E^k\Omega F_m\) such that the commutativity of the following diagram:

\[
\begin{array}{cccccccccc}
E^k\Omega F_m & \xrightarrow{p^k\Omega F_m} & P^{k-1}\Omega F_m & \xrightarrow{e^k_{m+1}} & F_m \\
\end{array}
\]
Therefore we have the inequalities
\[
\text{cat}(D) \leq \text{Cat}(D) \leq m + 1.
\]

Recall the horizontal top and bottom lines of the diagram (5.5). The homotopy pushout of these lines are \( G \cup \psi G \times CA \). Since dimensions of \( F_m \), \( F_1 \) and \( A \) are less than or equal to \( l \), all composition of columns in the diagram (5.5) are homotopic to identity maps. By the universal property of the homotopy pushout, we obtain a composite map \( D \to G \cup \psi G \times CA \simeq E \to D \) which is homotopic to the identity map. Thus \( D \) dominates \( E \) and we have
\[
\text{cat}(E) \leq \text{cat}(D) \leq \text{Cat}(D) \leq m + 1.
\]
6. Application of Theorem 1.4

We want to determine the L-S category of SO(10) by applying the principal bundle \( p : SO(10) \rightarrow S^9 \) to Theorem 1.4. First, we estimate the lower bound of \( \text{cat}(SO(10)) \). For the field \( k \) of characteristic 2, the ring structure of the cohomology of SO(10) is

\[
H^*(SO(10); k) \cong P_k[x_1, x_3, x_5, x_7, x_9]/(x_1^{16}, x_3^4, x_5^2, x_7^2, x_9^2),
\]

where \( \deg x_i = 1 \). Hence, we have

\[ 21 \leq \text{cup}(SO(10); k) \leq \text{cat}(SO(10)). \]

Next, we estimate the upper bound by using Theorem 1.4. We consider the cone-decomposition of SO(9). The cone-decomposition of Spin(7) is given by Iwase, Mimura and Nishimoto [7]. We denote this cone-decomposition by the following:

\[ \ast \subset F_1' \subset \Sigma \mathbb{C}P^3 \subset F_3' \subset F_4' \subset F_5' \cong \text{Spin}(7). \]

By Iwase, Mimura and Nishimoto [8], we can write the cone-decomposition of length 20

\[ \{ K_i \rightarrow F_{i-1} \rightarrow F_i \mid 1 \leq i \leq 20, F_0 = \{ \ast \} \text{ and } F_{20} = SO(9) \} \]

by using the filtration \( F_i' \) and principal bundle \( \text{Spin}(7) \hookrightarrow SO(9) \hookrightarrow \mathbb{R}P^{15} \). We find that the first filter \( F_1 \) is the space \( \Sigma \mathbb{C}P^3 \vee S^1 \). We consider the bundle \( p : SO(10) \rightarrow S^9 \) and \( p' : SU(5) \rightarrow S^9 \), and the following diagram:

\[
\begin{array}{ccc}
\Sigma \mathbb{C}P^3 & \xrightarrow{\Sigma} & SU(4) \\
\downarrow & & \downarrow \\
SU(5) & \xrightarrow{\alpha} & SO(9) \\
\downarrow & & \downarrow \\
S^8 & \xrightarrow{\alpha'} & SO(10) \\
\downarrow & p & \downarrow \\
S^9 & \xrightarrow{p'} & S^9.
\end{array}
\]

Here \( \alpha : S^8 \rightarrow SO(9) \) is a characteristic map of the bundle \( p : SO(10) \rightarrow S^9 \). By Steenrod [13], is homotopic to the characteristic map \( \alpha' : S^8 \rightarrow SU(4) \) in SO(9). Also, by Yokota [15], the suspension of the covering map \( \Sigma \gamma_3 : S^8 \rightarrow \Sigma \mathbb{C}P^3 \) which provide a cellular decomposition of the complex projective space correspond with the characteristic map \( \alpha' \). Therefore the characteristic map \( \alpha \) is compressible into \( \Sigma \mathbb{C}P^3 \subset F_1 \) and \( H_1(\alpha) = 0 \in \pi_8(\Omega \Sigma \mathbb{C}P^3 \ast \Omega \Sigma \mathbb{C}P^3) \). Hence we obtain

**Theorem 6.1.** \( \text{cat}(SO(10)) = 21 \).

**References**

[1] I. Berstein and P. J. Hilton, *Category and generalised Hopf invariants*, Illinois. J. Math. 12 (1968), 421–432.

[2] O. Cornea, G. Lupton, J. Oprea and D. Tanrég, “Lusternik-Schnirelmann Category”, Mathematical Surveys and Monographs 103, Amer. Math. Soc., Providence, 2003.

[3] T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois. J. Math., 11 (1967), 417–427.
[4] N. Iwase, *Ganea’s conjecture on LS-category*, Bull. Lon. Math. Soc., 30 (1998), 623–634.

[5] N. Iwase, *$A_\infty$-method in Lusternik-Schnirelmann category*, Topology 41 (2002), 695–723.

[6] N. Iwase, *The Ganea conjecture and recent developments on the Lusternik-Schnirelmann category* (Japanese), Sūgaku 56 (2004), 281–296.

[7] N. Iwase, M. Mimura and T. Nishimoto, *On the cellular decomposition and the Lusternik-Schnirelmann category of Spin(7)*, Topology Appl., 133 (2003), 1–14.

[8] N. Iwase, M. Mimura and T. Nishimoto, *Lusternik-Schnirelmann category of non-simply connected compact simple Lie groups*, Topology Appl., 150 (2005), 111–123.

[9] N. Oda, *Pairings and copairings in the category of topological spaces*, Publ. Res. Inst. Math. Sci., 28 (1992), 83–97.

[10] E. Spanier, “*Algebraic Topology*”, McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.

[11] D. Stanley, *On the Lusternik-Schnirelmann Category of Maps*, Canad. J. Math., 54, (2002) 608–633.

[12] J. D. Stasheff, *Homotopy associativity of H-spaces, I & II*, Trans. Amer. Math. Soc., 108 (1963), 275–292; 293–312.

[13] N. E. Steenrod, “*The Topology of Fibre Bundles*”, Princeton Mathematical Series 14, Princeton University Press, Princeton, 1951.

[14] G. W. Whitehead, “*Elements of Homotopy Theory*”, Graduate Texts in Mathematics 61, Springer Verlag, Berlin, 1978.

[15] I. Yokota, “*Groups and Topology*” (in Japanese), Shokabo, Tokyo, 1971.