Quantization of the Szekeres spacetime through generalized symmetries

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We present the effect of the quantum corrections on the Szekeres spacetime, a system important for the study of the inhomogeneities of the pre-inflationary era of the universe. The study is performed in the context of canonical quantisation in the presence of symmetries. We construct an effective classical Lagrangian and impose the quantum version of its classical integrals of motion on the wave function. The interpretational scheme of the quantum solution is that of Bohmian mechanics, in which one can avoid the unitarity problem of quantum cosmology. We discuss our results in this context.

Keywords: Szekeres system; Silent universe; Quantisation; Semiclassical approach

1. Introduction

We focus on the quantisation of the Szekeres spacetime metric with the aim to study the effect of possible quantum corrections in the dynamics. This metric has the form

\[ ds^2 = -dt^2 + e^{2\alpha} dr^2 + e^{2\beta} (dy^2 + dz^2) \]

where \( \alpha \equiv \alpha(t, r, y, z) \) and \( \beta \equiv \beta(t, r, y, z) \) and represents an irrotational perfect fluid with vanishing pressure and magnetic Weyl tensor, \( p = \omega_{ab} = H_{ab} = 0 \). The interest in the silent universe lies on the fact that it can be seen as inhomogeneous solutions of Einstein equations with no symmetries which generalise Kantowski-Sachs, FRW and Tolman-Bondi spacetimes. Thus it is proper for the description of perturbations on these spacetimes.

We start by writing the field equations on the covariant variables \( (\rho, \theta, \sigma, E) \), where \( \rho = T^{\mu\nu} u_\mu u_\nu \), with \( T^{\mu\nu} \) being the energy-momentum tensor of the matter.

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\[ \theta = (\nabla_{\nu} u_{\mu}) h^{\mu \nu} \]
is the expansion rate of the observer, while \( \sigma \) and \( E \) are the shear and electric component of the Weyl tensor, \( E_{\mu}^{\nu} = E e_{\mu}^{\nu}, \sigma_{\mu}^{\nu} = \sigma e_{\mu}^{\nu} \), in which the set of \( \{u^\mu, e_{\mu}^{\nu}\} \) defines an orthogonal tetrad. The field equations then become

\[ \dot{\rho} + \theta \rho = 0, \quad (2a) \]

\[ \dot{\theta} + \frac{\theta^2}{3} + 6\sigma^2 + \frac{1}{2} \rho = 0, \quad (2b) \]

\[ \dot{\sigma} - \sigma^2 + \frac{2}{3} \theta \sigma + E = 0, \quad (2c) \]

\[ \dot{E} + 3E\sigma + \theta E + \frac{1}{2} \rho \sigma = 0, \quad (2d) \]

together with the algebraic equation which is the Hamiltonian constraint

\[ \frac{\theta^2}{3} - 3\sigma^2 + \frac{(3) R}{2} = \rho \quad (2e) \]

where \( \dot{\cdot} \) denotes the directional derivative along \( u^\mu \), the energy density is \( \rho = T_{\mu \nu} u^\mu u^\nu \), with \( T_{\mu \nu} \) being the energy-momentum tensor of the matter, the parameter \( \theta = (\nabla_{\nu} u_{\mu}) h^{\mu \nu} \) is the expansion rate of the observer, while \( \sigma \) and \( E \) are the shear and electric component of the Weyl tensor, \( E_{\mu}^{\nu} = E e_{\mu}^{\nu}, \sigma_{\mu}^{\nu} = \sigma e_{\mu}^{\nu} \), in which the set of \( \{u^\mu, e_{\mu}^{\nu}\} \) defines an orthogonal tetrad. We note that in addition to these equations, the spatial constraints ensure the integrability of the system.

In the following sections we present the quantization of this system in terms of canonical quantization in the presence of symmetries\(^6\). The starting point is the effective Lagrangian obtained in\(^7\) and we adopt the Bohmian approach for our analysis\(^8,9\) following e.g.\(^10\), since its causal character suits the context of quantum cosmology, where the notion of an external observer cannot be justified.

### 2. Classical and Quantum Dynamics

In\(^7\) the Szekeres system was written in an equivalent form of a two second-order differential equations system which are equations of motion of a Lagrangian of the form

\[ L = \frac{1}{2} G_{\alpha \beta}(q(t)) \dot{q}^\alpha(t) \dot{q}^\beta(t) - V(q(t)), \quad \alpha, \beta = 0, \ldots n - 1 \quad (3) \]

where \( q(t) \) denote the degrees of freedom of the system and \( G_{\alpha \beta} \) the metric on the configuration space of variables. Adopting proper coordinates for our case, \( \rho = \frac{6}{(1-v^2)/(v^2-1)} \), \( E = \frac{\dot{v}}{u^3(v^2-1)} \), the system of the two second order equations becomes

\[ \ddot{v} - \frac{2v}{u^3} = 0, \quad (4a) \]

\[ \ddot{u} + \frac{1}{u^2} = 0 \quad (4b) \]

derivable from \( L = \dot{u} \dot{v} - \frac{\dot{v}}{u^2} \). The system\(^7\) admits two integrals of motion, quadratic in the velocities; the first is the Hamiltonian function since the system
is autonomous, while the second one is the quadratic function $I_0$ which can be constructed by the application of Noether’s theorem for contact symmetries, which in the phase space become

$$p_u p_v + \frac{u}{u^2} = h,$$  \hspace{1cm} (5a)

$$p_v^2 - 2u^{-1} = I_0$$  \hspace{1cm} (5b)

When turned to quantum operators and imposed on the wave function, according to the rules $\hat{p}_\alpha = -i \frac{\partial}{\partial q}\alpha$, $\{.,.\} \rightarrow -\frac{i}{\hbar}[.,.]$, with operator-ordering respecting the general covariance and hermiticity ensured under the inner product $\int d^nq \mu \psi_1^* \psi_2$, they lead to two eigenvalue equations

$$\left(-\partial_{uv} + \frac{u}{u^2}\right) \Psi = h\Psi,$$  \hspace{1cm} (6a)

$$\left(\partial_{uv} + \frac{2}{u}\right) \Psi = -I_0\Psi,$$  \hspace{1cm} (6b)

In the first one we can recognise the time-independent Schroödinger equation and their solution is

$$\Psi (I_0, u, v) = \frac{\sqrt{u}}{\sqrt{2 + I_0 u}} (\Psi_1 \cos f (u, v) + \Psi_2 \sin f (u, v))$$  \hspace{1cm} (7)

where

$$f (u, v) = \frac{(hu + I_0 v)\sqrt{2I_0 + I_0^2 u - 2h\sqrt{u}\arcsinh \sqrt{\frac{I_0}{u}}}}{I_0^{3/2} \sqrt{u}}, \text{ for } I_0 \neq 0, \hspace{1cm} (8)$$

$$f (u, v) = \frac{\sqrt{2}(hu^2 + 3v)}{3\sqrt{u}}, \text{ for } I_0 = 0. \hspace{1cm} (9)$$

and $\Psi_1$, $\Psi_2$ denote constants of integration. Due to the linearity of (6a), the general solution is $\Psi_{Sol} (u, v) = \sum I_0 \Psi (I_0, u, v)$.

3. Semiclassical analysis and probability

In the context of the Bohmian approach, the departure from the classical theory is determined by an additional term in the classical Hamilton-Jacobi equation, known as quantum potential $Q_V = -\frac{\Omega^2}{2\pi}$, where $\Omega$ denotes the amplitude of the wave function in polar form, $\Psi (u, v) = \Omega (u, v)e^{iS(u, v)}$. When the quantum potential is zero, the identification

$$\frac{\partial S}{\partial \theta_i} = p_i = \frac{\partial}{\partial \theta_i}$$  \hspace{1cm} (10)

is possible. If this classical definition for the momenta is retained even when $Q \neq 0$, the semiclassical solutions will differ from the classical ones.

Under the assumption that the quantum corrections in the general solution [7] follow from the “frequency $I_0$” with the highest peak in the wave function, which is
in agreement with the so-called Hartle criterion, the quantum potential vanishes. This provides no quantum corrections and the semiclassical equations give the classical solution, i.e. the Szekeres universe remains “silent”, even at the quantum level. In the particular case \( h = 0 \) and \( \Psi_1 \to 0 \), the wave function is well behaved at \( u \to 0 \) and \( u \to \infty \). We can thus define a probability which, after a change of coordinates to \( u \to \frac{x^2}{I_0} \) becomes

\[
P = \int_{\sqrt{\frac{I_0}{\varepsilon}}}^{x^\lambda} dx \int_0^{2\pi} dv \frac{4c^2 \sin(xv)}{x(x^2 - I_0)^2}, \quad k \in \mathbb{N}. \tag{11}
\]

where the cut-off constant \( \lambda \) is introduced to exclude the case \( E = 0, \rho = 0 \). The qualitative behaviour of the probability function is given in the contour plot in Fig. 1. The plots show that for \( I_0 \to 0 \) the probability function reaches its minimum.

4. Conclusions

Our quantum analysis of the Szekeres system was based on the canonical quantization in the presence of symmetries and the results were interpreted by adopting the Bohmian mechanics approach. The starting point was an effective classical point-like Lagrangian which can reproduce the two dimensional system of second-order differential equations resulted from the initial field equations. This Lagrangian is autonomous, thus there exists a conservation law of “energy” corresponding to the Hamiltonian function. As for the extra contact symmetry, it leads to a quadratic in the momenta conserved quantity attributed to a Killing tensor of the second-rank. The two conserved quantities give two eigenequations at the quantum level, the Hamiltonian function being the Schrödinger equation.

The assumption that the wave function is peaked around its classical value leads to the lack of quantum corrections and the recovery of the classical solutions, thus leading to the conclusion that the Szekeres universe remains silent at the quantum level. Finally, for the particular case \( h = 0 \) it was shown that the probability function and relate one (unstable) exact solution with the existence of a minimum of this probability.
The classical exact solution which corresponds at the values $h = I_0 = 0$ of the integration constants is $u_A(t) = \frac{6}{25}t^2$, $v_A(t) = v_0t^{-\frac{1}{2}}$ and corresponds to an unstable critical point for the dynamical system (2). It is interesting the fact that these conditions also correspond to the extremum of the probability function, something which might be related to the existence and stability of the exact solution. This result is also in accordance with the analysis of the probability extrema in [11], where it was shown that the extrema of the probability lie on the classical values.

References

1. P. Szekeres, A Class of Inhomogeneous Cosmological Models, Commun. Math. Phys. 41, p. 55 (1975).
2. M. Ishak and A. Peel, The growth of structure in the Szekeres inhomogeneous cosmological models and the matter-dominated era, Phys. Rev. D85, p. 083502 (2012).
3. K. Bolejko and M.-N. Celerier, Szekeres Swiss-Cheese model and supernova observations, Phys. Rev. D82, p. 103510 (2010).
4. D. Vrba and O. Svitek, Modelling inhomogeneity in Szekeres spacetime, Gen. Rel. Grav. 46, p. 1808 (2014).
5. M. Bruni, S. Matarrese and O. Pantano, Dynamics of silent universes, Astrophys. J. 445, 958 (1995).
6. T. Christodoulakis, N. Dimakis, P. A. Terzis, G. Doulis, T. Grammenos et al., Conditional Symmetries and the Canonical Quantization of Constrained Minisuperspace Actions: the Schwarzschild case, J.Geom.Phys. 71, 127 (2013).
7. A. Paliathanasis and P. G. L. Leach, Symmetries and Singularities of the Szekeres System, Phys. Lett. A381, 1277 (2017).
8. D. Bohm, A Suggested interpretation of the quantum theory in terms of hidden variables. 1., Phys. Rev. 85, 166 (1952).
9. D. Bohm, A Suggested interpretation of the quantum theory in terms of hidden variables. 2., Phys. Rev. 85, 180 (1952).
10. A. Zampeli, T. Pailas, P. A. Terzis and T. Christodoulakis, Conditional symmetries in axisymmetric quantum cosmologies with scalar fields and the fate of the classical singularities, JCAP 1605, p. 066 (2016).
11. N. Dimakis, P. A. Terzis, A. Zampeli and T. Christodoulakis, Decoupling of the reparametrization degree of freedom and a generalized probability in quantum cosmology, Phys. Rev. D94, p. 064013 (2016).