To my son Philippe for his unbounded energy and optimism.

On the classification of Floer-type theories.

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Abstract

In this paper we outline a program for the classification of Floer-type theories, (or defining invariants of finite type for families). We consider Khovanov complexes as a local system on the space of knots introduced by V. Vassiliev and construct the wall-crossing morphism. We extend this system to the singular locus by the cone of this morphism and introduce the definition of the local system of finite type. This program can be further generalized to the manifolds of dimension 3 and 4 [S2], [S3].
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1. Introduction.

Lately there has been a lot of interest in various categorifications of classical scalar invariants, i.e. homological theories, Euler characteristics of which are scalar invariants. Such examples include the original instanton Floer homology, Euler characteristic of which, as it was proved by C. Taubes [T], is Casson’s invariant. Ozsvath-Szabo [OS] 3-manifold theory categorifies Turaev’s torsion, the Euler characteristic of their knot homologies [OS] is the Alexander polynomial. The theory of M. Khovanov categorifies the Jones polynomial [Kh] and Khovanov-Rozhansky theory categorifies the $sl(n)$ invariants [KR].

The theory that we are constructing will bring together theories of V. Vassiliev, A. Hatcher and M. Khovanov, and while describing their results we will specify which parts of their constructions will be important to us.

The resulting theory can be considered as a ”categorification of Vassiliev theory” or a classification of categorifications of knot invariants. We introduce the definition of a theory of finite type n and show that Khovanov homology theory in a categorical sense decomposes into a ”Taylor series” of theories of finite type.

The Khovanov functor is just the first example of a theory satisfying our axioms and we believe, that all theories mentioned above will fit into our template.

Our main strategy is to consider a knot homology theory as a local system, or a constructible sheaf on the space of all objects (knots, including singular ones), extend this local system to the singular locus and introduce the analogue of the ”Vassiliev derivative” for categorifications.

By studying spaces of embedded manifolds we implicitly study their diffeomorphism groups and invariants of finite type. In his seminal paper [V] Vassiliev introduced finite type invariants by considering the space of all immersions of $S^1$ into $R^3$ and relating the topology of the singular locus to the topology of its complement via Alexander duality. He resolved and cooriented the discriminant of the space and introduced a spectral sequence with a filtration, which suggested the simple geometrical and combinatorial definition of an invariant of finite type, which was later interpreted by Birman and Lin as a ”Vassiliev derivative” and led to the following skein relation.

If $\lambda$ be an arbitrary invariant of oriented knots in oriented space with values in some Abelian group $A$. Extend $\lambda$ to be an invariant of 1-singular knots (knots that may have a single singularity that locally looks like a double point $\times$), using the formula

$$\lambda(\times) = \lambda(\times) - \lambda(\times)$$

Further extend $\lambda$ to the set of $n$-singular knots (knots with $n$ double points) by repeatedly using the skein relation.

**Definition** We say that $\lambda$ is of type $n$ if its extension to $(n+1)$-singular knots vanishes identically. We say that $\lambda$ is of finite type if it is of type $n$ for some $n$.

Given the above formula, the definition of an invariant of finite type $n$ becomes similar to that of a polynomial: its $(n+1)$st Vassiliev derivative is zero.

It was shown that all known invariants are either of finite type, or are infinite linear combinations of those, e.g. in [BN1] it was shown that the $n$th coefficient of the Conway polynomial is a Vassiliev invariant of order $\leq n$. 
In this paper we are working with Khovanov homology, which will be our main example, however the latest progress in finding the combinatorial formula for the differential of the Ozsvath-Szabo knot complex [MOS], makes us hopeful that more and more examples will be coming.

For the construction of the local system it is important to understand the topological type of the base. The topology of the connected components of the complement to the discriminant in the space of knots, called chambers, was studied by A. Hatcher and R.Budney [H], [B]. They introduced simple homotopical models for such spaces. Recall that the local system is well-defined on a homotopy model of the base, so Hatcher’s model is exactly what is needed to construct the local system of Khovanov complexes.

Throughout the paper the following observation is the main guideline for our constructions: *local systems of the classifying space of the category are functors from this category to the triangulated category of complexes.*

It would be very interesting to understand the relation between the Vassiliev space of knots is the classifying space of the category, whose objects are knots and whose morphisms are knot cobordisms.

Our construction provides a *Khovanov functor* from the category of knots into the triangulated category of complexes.

This allows us to translate all topological properties of the space of knots and Khovanov local system on it into the language of homological algebra and then use the methods of triangulated categories and homological algebra to assign algebraic objects to topological ones (singular knots and links).

Recall that in his paper [Kh] M.Khovanov categorified the Jones polynomial, i.e. he found a homology theory, the Euler characteristics of which equals the Jones polynomial. He starts with a diagram of the knot and constructs a bigraded complex, associated to this diagram, using two resolutions of the knot crossing:

```
  
   /\                  /
  |   |                |   |
 /   \               /   \                      0-resolution
  \   /               \   /
   \ /                \ /                          1-resolution
      |                  |
```

The Khovanov complex then becomes the sum of the tensor products of the vector space $V$, where the homological degree is given by the number of 1’s in the complete resolution of the knot. The local system of Khovanov homologies on the Vassiliev’s space of knots can be considered as invariants of families of knots.

The discriminant of Vassiliev’s space corresponds to knots with transversal self-intersection, i.e. moving from one chamber to another we change overcrossing to undercrossing by passing through a knot with a single double point. We study how the Khovanov complex changes under such modification and find the corresponding morphism.
After defining a wall-crossing morphism we can extend the invariant to the singular locus by the cone of a morphism which is our "categorification of the Vassiliev derivative". Then we introduce the definition of a local system of finite type: the local system is of finite type $n$ if for any selfintersection of the discriminant of codimension $n$, its $n$’th cone is an acyclic complex.

The categorification of the Vassiliev derivative allows us to define the filtration on the Floer - type theories for manifolds of any dimension.

In [S4] we prove the first finiteness result:

**Theorem [S4].** Restricted to the subcategory of knots with at most $n$ crossing, Khovanov local system is of finite type $n$, $n \geq 3$ and of type zero $n = 0, 1, 2$.

This definition can be generalized to the categorifications of the invariants of manifolds of any dimension: we construct spaces of 3 and 4-manifolds by a version of a Pontryagin-Thom construction, consider homological invariants of 3 and 4-manifolds as local systems on these spaces and extend them to the discriminant.

In subsequent papers our main example will be the Heegaard Floer homology [OS], the Euler characteristic of which is Turaev’s torsion. We show that local systems of such homological theories on the space of 3 - manifolds [S1] will carry information about invariants of finite type for families and information about the diffeomorphism group. We also have a construction [S2] for the refined Seiberg-Witten invariants on the space of parallelizable 4-manifolds.

**Acknowledgements.** My deepest thanks go to Yasha Eliashberg for many valuable discussions, for inspiration and for his constant encouragement and support. I want to thank Maxim Kontsevich who suggested that I work on this project, for his attention to my work during my visit to the IHES and for many important suggestions.

I want to thank graduate students Eric Schoenfeld and Isidora Milin for reading the paper and making useful comments.

This paper was written during my visits to the IAS, IHES, MPIM and Stanford and I am grateful to these institutions for their exceptional hospitality. This work was partially supported by the NSF grant DMS9729992.
2. Vassiliev theory, invariants of finite type.

2.1. The space of knots, coorientation

Vassiliev considered the space of all maps $E = f : S^1 \rightarrow \mathbb{R}^3$. This space is a space of functions, so it is an infinite-dimensional Euclidean space. It is linear, contractible, and consists of singular (D) and nonsingular (E-D) knots. The discriminant D forms a singular hypersurface in E and subdivides into chambers, corresponding to different isotopy types of knots. To move from one chamber to another one has to change one overcrossing to undercrossing, passing through a singular knot with one double point.

The discriminant of the space of knots is a real hypersurface, stratified by the number of the double points, which subdivides the infinite-dimensional space into chambers, corresponding to different isotopy types of knots.

Vassiliev resolved and cooriented the discriminant, so we can assume that all points of selfintersection are transversal, with $2^n$ chambers adjacent to a point of selfintersection of the discriminant of codimension $n$.

To study the topology of the complement to the discriminant, Vassiliev wrote a spectral sequence, calculating the homology of the discriminant and then related it to the homology of its complement via Alexander duality. His spectral sequence had a filtration, which suggested the simple geometrical and combinatorial definition of an invariant of finite type: an invariant is of type $n$ if for any selfintersection of the discriminant of codimension $(n+1)$ its alternated sum over the $2^{n+1}$ chambers adjacent to a point of selfintersection is zero.

For our constructions it will be very important to have a coorientation of the discriminant, which was introduced by Vassiliev.

**Definition.** A hypersurface in a real manifold is said to be coorientable if it has a non-zero section of its normal bundle, i.e. if there exists a continuous vector field which is not tangent to the hypersurface at any point and doesn’t vanish anywhere.

So there are two sides of the hypersurface: one where this vector field is pointing to and the other is where it is pointing from. And there are two choices of such vector field. The coorientation of a coorientable hypersurface is the choice of one of two possibilities.

For example, Mobius band in $\mathbb{R}^3$ is not coorientable.

Vassiliev shows [V] that the discriminant of the space of knots has a coorientation, the consistent choice of normal directions.

Recall that the nonsingular point $\psi \in D$ of the discriminant is a map $S^1 \rightarrow \mathbb{R}^3$, gluing together 2 distinct points $t_1, t_2$ of $S^1$, s.t. derivatives of the map $\psi$ at those points are transversal.

**Coorientation of the discriminant.** Fix the orientation of $\mathbb{R}^3$ and choose positively oriented local coordinates near the point $\psi(t_1) = \psi(t_2)$. For any point $\psi_1 \in D$ close to $\psi$ define the number $r(\psi_1)$ as the determinant:

\[
\left| \frac{\partial \psi_1}{\partial \tau}(t_1), \frac{\partial \psi_1}{\partial \tau}(t_2), \psi_1(t_1) - \psi_1(t_2) \right|
\]
with respect to these coordinates. This determinant depends only of the pair of points \( t_1, t_2 \), not on their order. A vector in the space of functions at the point \( \psi \in D \), which is transversal to the discriminant, is said to be positive, if the derivative of the function \( r \) along this vector is positive and negative, if this derivative is negative.

This rule gives the coorientation of the hypersurface \( D \) at all its nonsingular points and also of any nonsingular locally irreducible component of \( D \) at the points of selfintersection of \( D \).

The consistent choice of the normal directions of the walls of the discriminant will give the "directions" of the cobordisms (which are embedded into \( E \times I \)) between knots of the space \( E \).

Note. It is interesting to compare this construction with the result of E.Ghys [Gh], who introduced a metric on the space of knots and 3-manifolds.)

2.3. The topology of the chambers of the space of knots.

The study of the topology of the chambers of the space of knots was started by A. Hatcher [H], who found a simple homotopy models for these spaces.

The main result is based on an earlier theorem regarding the topology of the classifying space of diffeomorphisms of an irreducible 3-manifold with nonempty boundary.

In the following theorem A. Hatcher and D. McCullough answered the question posed by M. Kontsevich [K], regarding the finiteness of the homotopy type of the classifying space of the group of diffeomorphisms [HaM]:

**Theorem [HaM].** Let \( M \) be an irreducible compact connected orientable 3-manifold with nonempty boundary. Then \( BD\text{iff}(M, rel\partial) \) has the homotopy type of a finite aspherical CW-complex.

The proof of this theorem uses the JSJ-decomposition of a 3-manifold.

When applied to knot complements, the JSJ-decomposition defines a fundamental class of links in \( S^3 \), the "knot generating links" (KGL). A KGL is any \((n + 1)\)-component link \( L = (L_0, L_1, \cdots, L_n) \) whose complement is either Seifert fibred or atoroidal, such that the \( n \)-component sub-link \((L_1, L_2, \cdots, L_n)\) is the unlink. If the complement of a knot \( f \) contains an incompressible torus, then \( f \) can be represented as a 'spliced knot' \( f = J \square L \) in unique way, where \( L \) is an \((n + 1)\)-component KGL, and \( J = (J_1, \cdots, J_n) \) is an \( n \)-tuple of non-trivial long knots.

The spliced knot \( J \square L \) is obtained from \( L_0 \) by a generalized satellite construction. For any knot there is a representation of a knot as an iterated splice knot of atoroidal and hyperbolic KGLs. The order of splicing determines the "companionship tree" of \( f, G_f \), and is a complete isotopy invariant of long knots.

Given a knot \( f \in K \), denote the path-component of \( K \) containing \( f \) by \( K_f \). The topology of the chambers \( K_f \) was further studied by R. Budney The main result of his paper [Bu] is the computation of the homotopy type of \( K_f \) if \( f \) is a hyperbolically-spliced knot i.e: \( f = J \square L \) where \( L \) is a hyperbolic KGL.

The combined results can be summarized in the following theorem:
Theorem [Bu, H].

If $f = J \Box L$ where $L$ is an $(n + 1)$-component hyperbolic KGL, then

$$K_f \simeq S^1 \times \left( SO_2 \times A_f \prod_{i=1}^{n} K_{f_i} \right)$$

$A_f$ is the maximal subgroup of $B_L$ such that induced action of $A_f$ on $K^n$ preserves $\prod_{i=1}^{n} K_{L_i}$. The restriction map $A_f \to Diff(S^3, L_0) \to Diff(L_0)$ is faithful, giving an embedding $A_f \to SO_2$, and this is the action of $A_f$ on $SO_2$.

This result completes the computation of the homotopy-type of $K$ since we have the prior results:

H1 If $f$ is the unknot, then $K_f$ is contractible.

H2 If $f$ is a torus knot, then $K_f \simeq S^1$.

H3 If $f$ is a hyperbolic knot, then $K_f \simeq S^1 \times S^1$

H4 If a knot $f$ is a cabling of a knot $g$ then $K_f \simeq S^1 \times K_g$.

B5 If the knot $f$ is a connected sum of $n \geq 2$ prime knots $f_1, f_2, \cdots , f_n$ then $K_f \simeq ((C_2(n) \times \prod_{i=1}^{n} K_{f_i}) / \Sigma_f$. Here $\Sigma_f \subset S_n$ is a Young subgroup of $S_n$, acting on $C_2(n)$ by permutation of the labellings of the cubes, and similarly by permuting the factors of the product $\prod_{i=1}^{n} K_{f_i}$. The definition of $\Sigma_f \subset S_n$ is that it is the subgroup of $S_n$ that preserves a partition of $\{1, 2, \cdots , n\}$, the partition being given by the equivalence relation $i \sim j \iff K_{f_i} = K_{f_j}$.

B6 If a knot has a non-trivial companionship tree, then it is either a cable, in which case H4 applies, a connect-sum, in which case B5 applies or is hyperbolically spliced. If a knot has a trivial companionship tree, it is either the unknot, in which case H1 applies, or a torus knot in which case H2 applies, or a hyperbolic knot, in which case H3 applies. Moreover, every time one applies one of the above theorems, one reduces the problem of computing the homotopy-type of $K_f$ to computing the homotopy-type of knot spaces for knots with shorter companionship trees, thus the process terminates after finitely-many iterations.

For constructing a local system we need only the homotopy type of the chamber. The theorem of Hatcher and Budney provides us with a complete classification of homotopy types of chambers, corresponding to all possible knot types.
3. Khovanov’s categorification of Jones polynomial.

3.1. Jones polynomial as Euler characteristics. Skein relation.

In his paper [Kh] M. Khovanov constructs a homology theory, with Euler characteristics equal to the Jones polynomial.

He associated to any diagram $D$ of an oriented link with $n$ crossing points a chain complex $CKh(D)$ of abelian groups of homological length $(n+1)$, and proved that for any two diagrams of the same link the corresponding complexes are chain homotopy equivalent. Hence, the homology groups $Kh(D)$ are link invariants up to isomorphism.

His construction is as follows: given any double point of the link projection $D$, he allows two smoothings:

$$
\begin{array}{c}
\text{0-resolution} \\
\text{1-resolution}
\end{array}
$$

If the the diagram has $n$ double points, there are $2^n$ possible resolutions. The result of each complete smoothing is the set of circles in the plane, labled by $n$-tuples of 1’s and 0’s:

$$
CKh(\bigcirc, \ldots, \bigcirc) = V^\otimes n
$$

The cobordisms between links, i.e., surfaces embedded in $\mathbb{R}^3 \times [0, 1]$, should provide maps between the associated groups. A surface embedded in the 4-space can be visualized as a sequence of plane projections of its 3-dimensional sections (see [CS]). Given such a presentation $J$ of a compact oriented surface $S$ properly embedded in $\mathbb{R}^3 \times [0, 1]$ with the boundary of $S$ being the union of two links $L_0 \subset \mathbb{R}^3 \times \{0\}$ and $L_1 \subset \mathbb{R}^3 \times \{1\}$, , Khovanov associates to $J$ a map of cohomology groups

$$
\theta_J : Kh^{i,j}(D_0) \to Kh^{i,j+\chi(S)}(D_1), \quad i, j \in \mathbb{Z}
$$

The differential of the Khovanov complex is defined using two linear maps $m : V \otimes V \to V$ and $\Delta : V \to V \otimes V$ given by formulas:

$$
(V \otimes V \xrightarrow{m} V) \quad m : \begin{cases} 
v_+ \otimes v_- & \mapsto v_- \\
v_- \otimes v_+ & \mapsto v_+ \\
v_- \otimes v_- & \mapsto 0
\end{cases}
$$
The differential in Khovanov complex can be informally described as "all the ways of changing 0-crossing to 1-crossing".

Homological degree of the Khovanov complex in the number of 1’s in the plane diagram resolution. The sum of "quantum" components of the same homological degree $i$ gives the $i$th component of the Khovanov complex.

One can see that the $i$-th differential $d^i$ is the sum over "quantum" components, it will map one of the quantum components in homological degree $i$ to perhaps several quantum components of homological degree $i+1$.

Khovanov theory can be considered as a (1+1) dimensional TQFT. The cubes, that are used in it’s definition come from the TQFT corresponding to the Frobenius algebra defined by $V, m, \Delta$. As we will see later, our constructions will give the interpretation of Khovanov local system as a topological D-brane and will suggest to study the structure of the category of topological D-branes as a triangulated category.

We prove the following important property of the Khovanov’s complex:

**Theorem 1.** Let $k$ denote the $k$th crossing point of the knot projection $D$, then for any $k$ the Khovanov’s complex $C$ decomposes into a sum of two subcomplexes $C = C^k_0 \oplus C^k_1$ with matrix differential of the form

$$d_C = \begin{pmatrix} d_0 & d_{0,1} \\ 0 & d_1 \end{pmatrix}$$

**Proof.** Let $C^k_0$ denote the subcomplex of $C$, consisting of vector spaces, which correspond to the complete resolutions of $D$, having 0 on the $k$th place. The differential $d_0$ obtained by restricting $d$ only to the arrows between components of $C^k_0$. We define $C^k_1$ the same way, by restricting to the complete resolutions of $D$, having 1 on the $k$th place.

The only components of the differential, which are not yet used in our decomposition, are the ones which change 0-resolution on the $k$th place of $C^k_0$ to 1 on the $k$th place in $C^k_1$, we denote them $d_{0,1}$.

One can easily see from the definition of the Khovanov’s differential (which can be intuitively described as "all the ways to change 0-resolution in the $i$th component of the complex to the 1-resolution in the $(i+1)$st component"), that there is no differential mapping $i$th component of $C^k_1$ to the $(i+1)$st component of $C^k_0$.

**Mirror images and adjoints.** Taking the mirror image of the knot will dualize Khovanov complex. So if we want to invert the cobordism between two knots, we should consider the "dual" cobordism between mirror images of these knots.
3.2. Reidemeister and Jacobsson moves.

A cobordism (a surface $S$ embedded into $R^3 \times [0, 1]$) between knots $K_0$ and $K_1$ provide a morphism between the corresponding cohomology:

$$F_S : Kh^{i,j}(D_0) \to Kh^{i,j+\chi(S)}(D_1)$$

where $D_0$ and $D_1$ are diagrams of the knots $K_0$ and $K_1$ and $\chi(S)$ is the Euler characteristic of the surface.

We will distinguish between two types of cobordisms - first, corresponding to the wall crossing (and changing the type of the knot). And second, corresponding to nontrivial loops in chambers which will reflect the dependence of Khovanov homologies on the selfdiffeomorphisms of the knot, similar to the Reidemeister moves. In this paragraph we will discuss the second type of cobordisms.

By a surface $S$ in $R^4$ we mean an oriented, compact surface $S$, possibly with boundary, properly embedded in $R^3 \times [0, 1]$. The boundary of $S$ is then a disjoint union

$$\partial S = \partial_0 S \sqcup -\partial_1 S$$

of the intersections of $S$ with two boundary components of $R^3 \times [0, 1]$:

$$\partial_0 S = (S \cap R^3 \times \{0\})$$

$$-\partial_1 S = (S \cap R^3 \times \{1\})$$

Note that $\partial_0 S$ and $\partial_1 S$ are oriented links in $R^3$.

The surface $S$ can be represented by a sequence $J$ of plane diagrams of oriented links where every two consecutive diagrams in $J$ are related either by one of the four Reidemeister moves or by one of the four moves birth, death, fusion described by Carter-Saito [CS].

To each Reidemeister move between diagrams $D_0$ and $D_1$ Khovanov [Kh] associates a quasi-isomorphism map of complexes $C(D_0) \to C(D_1)$.

Given a representation $J$ of a surface $S$ by a sequence of diagrams, we can associate to $J$ a map of complexes

$$\varphi_J : C(J_0) \to C(J_1)$$

Any link cobordism can be described as a one-parameter family $D_t, t \in [0, 1]$ of planar diagrams, called a movie. The $D_t$ are link diagrams, except at finitely many singular points which correspond to either a Reidemeister move or a Morse modification. Away from these points the diagrams for various $t$ are locally isotopic. Khovanov explained how local moves induce chain maps between complexes, hence homomorphisms between homology groups. The same is true for planar isotopies. Hence, the composition of these chain maps defines a homomorphism between the homology groups of the diagrams of links.

In his paper [Ja] Jacobsson shows that there are knots, s.t. a movie as above will give a nontrivial morphism of Khovanov homology:

**Theorem [Ja]** For oriented links $L_0$ and $L_1$, presented by diagrams $D_0$ and $D_1$, an oriented link cobordism $\Sigma$ from $L_0$ to $L_1$, defines a homomorphism $\mathcal{H}(D_0) \to \mathcal{H}(D_1)$, invariant up to multiplication by -1 under ambient isotopy of $\Sigma$ leaving $\partial \Sigma$ setwise fixed. Moreover, this invariant is non-trivial.
Jacobsson constructs a family of derived invariants of link cobordisms with the same source and target, which are analogous to the classical Lefschetz numbers of endomorphisms of manifolds.

The Jones polynomial appears as the Lefschetz polynomial of the identity cobordism.

From our perspective the Jacobsson’s theorem shows that the Khovanov local system will have nontrivial monodromies on the chambers of the space of knots.

3.3. Wall-crossing morphisms.

In 3.2 we described what kind of modifications can occur in the cobordism, when we consider the "movie" consisting only of manifolds of the same topological type. These modifications implied corresponding monodromies of the Khovanov complex.

However, morphisms that are the most important for Vassiliev-type theories are the "wall-crossing" morphisms. We will define them now (locally).

Consider two complexes $A^\bullet$ and $B^\bullet$ adjacent to the generic wall of the discriminant. Recall, that the discriminant is cooriented (2.2). If $B^\bullet$ is "right" via coorientation (or "further in the Ghys metric form the unknot) of $A^\bullet$, then we shift $B^\bullet$’s grading up by one and consider $B^\bullet[1]$:

$$A^\bullet | B^\bullet[1]$$

**Note.** In general, and this will be very important for us in subsequent chapters, if the complex $K^\bullet$ is $n$ steps (via the coorientation) away from the unknot, we shift its grading up by $n$. Thus adjacent complexes will have difference in grading by one (as above), defined by the coorientation.

Now we want to understand what happens to the Khovanov complex when we change the $k$th over-crossing (in the knot diagram $D$) to an under-crossing.

We will illustrate these changes on one of the Bar-Natan’s trademark diagrams (with his permission)[BN1].

By "I" we mark the arrow, connecting components of the complex which will exchange places under wall-crossing morphisms when we change over-crossing to under-crossing for the self-intersection point 1. By "II" when we do it for point 2 and "III" when we do it for 3:
Now recall the theorem proved in (3.1): for any k, where k is the number of crossings of the diagram $D$, the Khovanov complex can be split into the sum of two subcomplexes with the uppertriangular differential.

Notice from the diagram above that when we change kth overcrossing to an undercrossing, 0 and 1-resolutions are exchanged, so $A^\bullet = A^\bullet_0 \oplus A^\bullet_1$, $B^\bullet[1] = B^\bullet_0[1] \oplus B^\bullet_1[1]$, thus for every k we can define the wall-crossing morphism $\omega$ as follows:

**Theorem 2.** The map defined as the identity on $A^\bullet_0$ and as a trivial map on $A^\bullet_1$:

$$\omega : A^\bullet_0 \xrightarrow{Id} B^\bullet_0[1]$$
$$\omega : A^\bullet_1 \xrightarrow{0} B^\bullet_1[1]$$

is the morphism of complexes.

**Proof.** From the Theorem 1 we know that for any crossing k the Khovanov complex can be decomposed as a direct sum with uppertriangular differential:
It is an easy check that the wall-crossing morphism defined as above is indeed a morphism of complexes (i.e. it commutes with the differential):

\[
\begin{pmatrix}
d_0 & d_{0,1} \\
0 & d_1
\end{pmatrix}
\]

Since we defined the morphism as 0 on \( A^1 \), the diagram above becomes the following commutative diagram:

\[
\begin{array}{c}
A^i \xrightarrow{\omega} B^i \\
\downarrow d \quad \downarrow d \\
A^i \xrightarrow{\omega} B^i
\end{array}
\]

3.4. The local system of Khovanov complexes on the space of knots.

In this paragraph we introduce the Khovanov local system on the space of knots.

**Definition.** A local system on the locally connected topological space \( M \) is a fiber bundle over \( M \), the sections of which are abelian groups. The fiber of the bundle depend continuously on the point of the base (such that the group structure on the set of fibers can be extended over small domains in the base).

Any local system on \( M \) with fiber \( A \) defines a representation \( \pi_1(M) \to Aut(A) \). To any loop there corresponds a morphism of the fibers of the bundle over the starting point of the loop. The set of isomorphism classes of local systems with fiber \( A \) are in one-to-one correspondence with the set of such representations up to conjugation. For example any representation of an arbitrary group \( \pi \) in \( Aut(A) \) uniquely (up to isomorphism) defines a local system on the space \( K(\pi, 1) \) [GM].

Morphisms of local systems are morphisms of fiber bundles, preserving group structure in the fibers. Thus introducing the continuation functions (maps between fibers) over paths in the base will define a local system over the manifold \( M \).

Next we set up the Khovanov complexes as a local system on the space of knots. If we were doing it ”in coordinates”, we would introduce charts on the chambers of the space of knots and define our local system via transition maps, starting with some ”initial” point . This would be a very interesting and realistic approach, since the homotopy models for chambers
are understood [H], [B], e.g. we would have just one chart for the chamber, containing the unknot (since that chamber is contractible), two for a torus knot, four for a hyperbolic one, etc. Then monodromies of the Khovanov local system along nontrivial loops in the chamber will be given via Jacobsson movies.

It would be also very interesting to find a unique special point in every chamber of the space \( E \) and study monodromies of the local system with respect to this point.

The candidate for such point is introduced in the works of J. O’Hara, who studied the minima of the electrostatical energy function of the knot [O’H]:

\[
E(K) = \int \int |(x - y)|^{-2} dx dy
\]

It was shown that under some assumptions and for perturbation of the above functional, its critical points on the space of knots will provide a “distinguished” point in the chamber.

The first natural question for this setup is: which nontrivial loops in the chamber \( E_K \), corresponding to the knot \( K \) are distinguished by Khovanov homologies and which are not?

However, assuming Khovanov’s theorem [Kh] (that his homology groups are invariants of the knot, independent on the choices made) and assuming also the results of Jacobsson [J], it is enough for us to introduce the continuation maps, along any path \( \gamma \) in the chamber of the space of knots.

These methods were developed by several authors (see [Hu]):

Let \( K_1 \) and \( K_2 \) be two knots in the same chamber of the space \( E \), let \( K_1 \) be generic, and let \( \gamma = \{K_t \mid t \in [0, 1]\} \) be any path of equivalent objects in \( E \) from \( K_1 \) to \( K_2 \). Then a generic path \( \gamma \) induces a chain map

\[
F(\gamma) : CKh_\ast(K_1) \longrightarrow CKh_\ast(K_2)
\]

called the “continuation” map, which has the following properties:

- **1) Homotopy** A generic homotopy rel endpoints between two paths \( \gamma_1 \) and \( \gamma_2 \) with associated chain maps \( F_1 \) and \( F_2 \) induces a chain homotopy

  \[
  H : HKh_\ast(K_1) \longrightarrow HKh_{\ast+1}(K_2)
  \]

  \[
  \partial H + H\partial = F_1 - F_2
  \]

- **2) Concatenation** If the final endpoint of \( \gamma_1 \) is the initial endpoint of \( \gamma_2 \), then \( F(\gamma_2 \gamma_1) \) is chain homotopic to \( F(\gamma_2)F(\gamma_1) \).

- **3) Constant** If \( \gamma \) is a constant path then \( F(\gamma) \) is the identity on chains.

These three properties imply that if \( K_1 \) and \( K_2 \) are equivalent, then \( HKh_\ast(K_1) \simeq HKh_\ast(K_2) \). (Khovanov’s theorem).

This isomorphism is generally not canonical, because different homotopy classes of paths may induce different continuation isomorphisms on Khovanov homology (Jacobsson moves).
However, since the loop is contractible, we do know that $HKh_4(K)$ depends only on $K$, so we denote this from now on by $HKh_4(K)$.

We now define the **restriction** of the Khovanov local system to finite-dimensional subspaces of the space of knots.

Note that in the original setting our complexes may have had different length. For example, the complex corresponding to the standard projection of the unknot will have length 1, however, we can consider very complicated "twisted" projections of the unknot with an arbitrary large number of crossing points. The corresponding complexes will be quasiisomorphic to the original one.

This construction resembles the definition of Khovanov homology introduced in [CK], [W]. They define Khovanov homology as a relative theory, where homology groups are calculated relative to the twisted unknots.

When considering the restrictions of the Khovanov local system to the subcategories of knots with at most $n$ crossings, we would like all complexes to be of length $n + 1$.

This can be achieved by "undoing" the local system, starting with the knots of maximal crossing number $n$ and then using the wall-crossing morphisms, define complexes of length $(n + 1)$, quasiisomorphic to the original ones, in all adjacent chambers. We continue this process till it ends, when we reach the chamber containing unknot.

Recall that Khovanov homology is defined for the knot projection (though is independent of it by Khovanov’s theorem). So we will consider a ramification of Vassiliev space, a pair, the embedding of the circle into $R^3$ and its projection on $(x,y)$-plane. Then each chamber will be subdivided into "subchambers" corresponding to nonsingular knot projections and the "sub-discriminant" will consist of singular projections of the given knot. The local system, defined on such ramification will live on the universal cover of the base, the original Hatcher chamber corresponding to knot K and morphisms of the local system between "subchambers" are given by Reidemeister moves. The composition of such moves may constitute the Jacobsson’s movie and will give nontrivial monodromies of the local system within the original chamber.

**Note.** As we will see later, if one assigns cones of Reidemeister morphisms to the walls of the "subdiscriminant", all such cones will be acyclic complexes. This statement in a different form was proved in the original Khovanov [Kh] paper.
4. The main definition, invariants of finite type for families.

4.1. Some homological algebra.

We describe results and main definitions from the category theory and homological algebra which will be used in subsequent chapters. The standard references on this subject are [GM], [Th].

By constructing the local system of (3.4) we introduced the derived category of Khovanov complexes. The properties of the derived category are summarized in the axiomatics of the triangulated category, which we will discuss in this chapter.

**Definition.** An additive category is a category \( \mathcal{A} \) such that

- Each set of morphisms \( \text{Hom}(A, B) \) forms an abelian group.
- Composition of morphisms distributes over the addition of morphisms given by the abelian group structure, i.e. \( f \circ (g + h) = f \circ g + f \circ h \) and \( (f + g) \circ h = f \circ h + g \circ h \).
- There exist products (direct sums) \( A \times B \) of any two objects \( A, B \) satisfying the usual universal properties.
- There exists a zero object \( 0 \) such that \( \text{Hom}(0, 0) \) is the zero group (i.e. just the identity morphism). Thus \( \text{Hom}(0, A) = 0 = \text{Hom}(A, 0) \) for all \( A \), and the unique zero morphism between any two objects is the one that factors through the zero object.

So in an abelian category we can talk about exact sequences and chain complexes, and cohomology of complexes. Additive functors between abelian categories are exact (respectively left or right exact) if they preserve exact sequences (respectively short exact sequences \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \)).

**Definition.** The bounded derived category \( \mathcal{D}^b(\mathcal{A}) \) of an abelian category \( \mathcal{A} \) has as objects bounded (i.e. finite length) \( \mathcal{A} \)-chain complexes, and morphisms given by chain maps with quasi-isomorphisms inverted as follows. We introduce morphisms \( f \) for every chain map between complexes \( f : X_f \rightarrow Y_f \), and \( g^{-1} : Y_g \rightarrow X_g \) for every quasi-isomorphism \( g : X_g \sim Y_g \). Then form all products of these morphisms such that the range of one is the domain of the next. Finally identify any combination \( f_1 f_2 \) with the composition \( f_1 \circ f_2 \), and \( gg^{-1} \) and \( g^{-1} g \) with the relevant identity maps \( \text{id}_{Y_g} \) and \( \text{id}_{X_g} \).

Recall that a triangulated category \( \mathcal{C} \) is an additive category equipped with the additional data:

**Definition.** Triangulated category is an additive category with a functor \( T : X \rightarrow X[1] \) (where \( X^i[1] = X^{i+1} \)) and a set of distinguished triangles satisfying a list of axioms.

The triangles include, for all objects \( X \) of the category:

1) Identity morphism

\[
X \rightarrow X \rightarrow 0 \rightarrow X[1],
\]

2) Any morphism \( f : X \rightarrow Y \) can be completed to a distinguished triangle

\[
X \rightarrow Y \rightarrow C \rightarrow X[1],
\]
3) There is also a derived analogue of the 5-lemma, and a compatibility of triangles known as the octahedral lemma, which can be understood as follows:

If we naively interpret property 1) as the difference $X - X = 0$, property 2) as $C = X - Y$, then the octahedron lemma says:

$$(X - Y) - Z = C - Z = X - (Y - Z)$$

When topological spaces considered up to homotopy there is no notion of kernel or cokernel.

The cylinder construction shows that any map $f : X \to Y$ is homotopic to an inclusion $X \to \text{cyl}(f) = Y \sqcup (X \times [0,1])/f(x) \sim (x,1)$, while the path space construction shows it is also homotopic to a fibration.

The cone $C_f$ on a map $f : X \to Y$ is the space formed from $Y \sqcup (X \times [0,1])$ by identifying $X \times \{1\}$ with its image $f(X) \subset Y$, and collapsing $X \times \{0\}$ to a point.

It can be considered as a cokernel, i.e. if $f : X \to Y$ is an inclusion, then $C_f$ is homotopy equivalent to $Y/X$.

Taking the $i$th cohomology $H_i$ of each term, and using the suspension isomorphism $H_i(\Sigma X) \cong H_{i-1}(X)$ gives a sequence

$$H_i(X) \to H_i(Y) \to H_i(Y,X) \to H_{i-1}(X) \to H_{i-1}(Y) \to \ldots$$

which is just the long exact sequence associated to the pair $X \subset Y$.

Up to homotopy we can make this into a sequence of simplicial maps, so that taking the associated chain complexes we get a lifting of the long exact sequence of homology to the level of complexes. It exists for all maps $f$, not just inclusions, with $Y/X$ replaced by $C_f$.

If $f$ is a fibration, $C_f$ can act as the “kernel” or fibre of the map. If $f : X \to \text{point}$, then $C_f = \Sigma X$, the suspension of the fibre $X$.

Thus $C_f$ acts as a combination of both cokernel and kernel, and if $f : X \to Y$ is a map inducing an isomorphism of homology groups of simply connected spaces then the sequence

$$H_i(X) \to H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to H_{i-1}(Y) \to \ldots$$

implies $H_i(C_f) = 0$. Then $C_f$ homotopy equivalent to a point. Thus we can give the following definition.

**Definition.** If $X$ and $Y$ are simplicial complexes, then a simplicial map $f : X \to Y$, defines (up to isomorphism) an object in triangulated category, called the **cone of morphism** $f$, denoted $C_f$.

$$C^\bullet \oplus C^\bullet_1$$

with differential $d_{C_f} = \begin{pmatrix} d_X & f \\ 0 & d_Y \end{pmatrix}$,

where $[n]$ means shift a complex $n$ places up.

Thus we can define the cone $C_f$ on any map of chain complexes $f : A^\bullet \to B^\bullet$ in an abelian category $A$ by the above formula, replacing $C^\bullet_X$ by $A^\bullet$ and $C^\bullet_C$ by $B^\bullet$. If $A^\bullet = A$ and $B^\bullet = B$ are chain complexes concentrated in degree zero then $C_f$ is the complex $\{ A \overset{f}{\to} B \}$. This has zeroth cohomology $h^0(C_f) = \ker f$, and $h^1(C_f) = \text{coker} f$, so combines the two (in different degrees). In general it is just the total complex of $A^\bullet \to B^\bullet$. 

So what we get in a derived category is not kernels or cokernels, but “exact triangles”

\[ A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]. \]

Thus we have long exact sequences instead of short exact ones; taking \( i \)th cohomology \( h^i \) of the above gives the standard long exact sequence

\[ h^i(A^\bullet) \rightarrow h^i(B^\bullet) \rightarrow h^i(C^\bullet) \rightarrow h^{i+1}(A^\bullet) \rightarrow \ldots \]

The cone will fit into a triangle:

\[ \begin{tikzpicture}
  \node (A) at (0,0) {A};
  \node (B) at (1,1) {B};
  \node (C) at (1,0) {C};
  \draw[->] (A) -- (B);
  \draw[->] (B) -- (C);
  \draw[->] (A) -- (C) node[midway,above] {\([1]\)};
\end{tikzpicture} \]

The “[1]” denotes that the map \( w \) increases the grade of any object by one.

4.2. Space of knots as a classifying space of the category.

In this paragraph we will construct the Khovanov functor from the category of knots into the triangulated category of Khovanov complexes.

**Definition.** The category of knots \( \mathcal{K} \) is the category, the objects of which are knots, \( S^1 \rightarrow S^3 \), morphisms are cobordisms, i.e. surfaces \( \Sigma \) properly embedded in \( \mathbb{R}^3 \times [0,1] \) with the boundary of \( \Sigma \) being the union of two knots \( K_1 \subset \mathbb{R}^3 \times \{0\} \) and \( K_2 \subset \mathbb{R}^3 \times \{1\} \).

We denote \( \mathcal{K}_n \) the subcategory of knots with at most \( n \) crossings.

(Recall that a knot’s crossing number is the lowest number of crossings of any diagram of the knot.)

Note that our cobordisms (morphisms in the category of knots) are directed via the coorientation of the discriminant of the space of knots.

Note that to reverse cobordism, we can consider the same cobordism between mirror images of the knots.

**Definition.** The nerve \( \mathcal{N}(C) \) of a category \( C \) is a simplicial set constructed from the objects and morphisms of \( C \), i.e. points of \( \mathcal{N}(C) \) are objects of \( C \), 1-simplices are morphisms of \( C \), 2-simplices are commutative triangles, 3-simplices are commutative tetrahedrons of \( C \), etc.

\[ \mathcal{N}(C) = (\underset{\text{lim}}{\text{lim}} \mathcal{N}_i(C)) \]

The geometric realization of a simplicial set \( \mathcal{N}(C) \) is a topological space, called the classifying space of the category \( C \), denoted \( B(C) \).

The following observation is the main guideline for our constructions: sheaves on the classifying space of the category are functors on that category [Wi].

Once we prove that the Vassiliev space of knots is a classifying space of the category \( \mathcal{K} \), our local system will provide a representation of the Khovanov functor.

Let \( C \) be a category and let \( \text{Set} \) be the category of sets. For each object \( A \) of \( C \) let \( \text{Hom}(A,) \) be the hom functor which maps objects \( X \) to the set \( \text{Hom}(A,X) \).
Recall that a functor $F : C \to \text{Set}$ is said to be **representable** if it is naturally isomorphic to $\text{Hom}(A, \cdot)$ for some object $A$ of $C$. A representation of $F$ is a pair $(A, \Psi)$ where
\[
\Psi : \text{Hom}(A, \cdot) \to F
\]
is a natural isomorphism.

If $E$ - the space of knots, denote $\mathcal{K}_E$ the category of knots, whose objects are points in $E$ and morphisms $\text{Mor}(x, y) = \{ \gamma : [0, 1] \to X; s.t. \gamma(0) = x, \gamma(1) = y \}$ and $\mathcal{K}_K$ - subcategory corresponding to knots of the same isotopy type $K$.

**Proposition.** The chamber $E_K$ of the space of long knots for $K$ - unknot, torus of hyperbolic knot is the classifying space of the category $\mathcal{K}_K$.

**Proof.** By Hatcher’s theorem [H] the chambers of the space of knots $E_K$, corresponding to unknot, torus or hyperbolic knot are $K(\pi, 1)$.

By definition the space of long knots is $E = \{ f : \mathbb{R}^1 \to \mathbb{R}^3 \}$, nonsingular maps which are standard outside the ball of large radius. If $f_1, f_2$ are vector equations giving knots $K_1, K_2$, then $tf_1 + (1 - t)f_2$ is a path in the mapping space, defining a knot for each value of $t$. The cobordism between two embeddings is given by equations in $\mathbb{R}^3 \times I$. All higher cobordisms can be contracted, since there are no higher homotopy groups in $E_K$. So both the classifying space of the category and the chamber of the space of knots are $K(\pi, 1)$ with the same $\pi$. They are the same as simplicial complexes.

Note, that in the case of hyperbolic knots one can choose the distinguished point in the chamber - corresponding to the hyperbolic metric on the complement to the knot.

4.3. Vassiliev derivative as a cone of the wall-crossing morphism.

To be able to construct a categorification of Vassiliev theory, we have to extend the local system, which we defined on chambers, to the discriminant of the space of knots.

Recall that according to the axiomatics of the triangulated category, described in (4.1), we assign an new object to every morphism in the category:

for a complex $X = (X^i, d^i_X)$ define a complex $X[1]$ by
\[
(X[1])^i = X^{i+1}, d_{X[1]} = -d_X
\]

For a morphism of complexes $f : X \to Y$ let $f[1] : X[1] \to Y[1]$ coincide with $f$ component-wise.

Let $f : X \to Y$ be a wall-crossing morphism. The **cone of $f$** is the following complex $C(f)$:
\[
X \to Y \to Z = C(f) \to X[1]
\]
i.e.
\[
C(f)^i = X[1]^i \oplus Y^i, d_{C(f)}(x^{i+1}, y^i) = (-d_X x^{i+1}, f(x^{i+1}) - d_Y y^i)
\]

Recall, that we set up the local system on the space of knots (3.4) s.t. if the complex $X^\bullet$ is $n$ steps (via the coorientation) away from the unknot, we shift its grading up by $n$. So complexes in adjacent chambers will have difference in grading by one, defined by the coorientation.
Thus, given a bigraded complex, associated to the generic wall of the discriminant, we get two natural specialization maps into the neighbourhoods, containing \( X^\bullet \) and \( Y^\bullet \):

So with any morphism \( f \) we associate the triangle:

\[
\begin{array}{c}
\bigtriangleup \\
\text{C}_{\omega} \\
\end{array}
\]

With any commutative cube

\[
\begin{array}{c}
\bigtriangleup \\
\text{C}_{u\omega} \\
\text{C}_{u} \\
\end{array}
\]

(in the space of knots the above picture corresponds to the cobordism around the self-intersection of the discriminant of codimension two), we associate the map between cones, corresponding to the vertical and horizontal walls, and assign it to the point of their intersection:

\[
\begin{array}{ccc}
\text{C}_{u} & \longrightarrow & \text{C}_{u\omega} \\
\omega & \longrightarrow & \omega \\
\end{array}
\]

Lemma. Given four chambers as above, the order of taking cones of morphisms is irrelevant, \( \text{C}_{u\omega} = \text{C}_{\omega u} \).

Proof. see [GM].

Consider a point of selfintersection of the discriminant of codimension \( n \). There are \( 2^n \) chambers adjacent to this point. Since the discriminant was resolved by Vassiliev [V], this point can be considered as a point of transversal selfintersection of \( n \) hyperplanes in \( R^n \), or an origin of the coordinate system of \( R^n \).

Now our local system looks as follows. On chambers of our space we have the local system of Khovanov complexes, to any point \( t \) of the generic wall between chambers containing \( X^\bullet \) and \( Y^\bullet \) (corresponding to a singular knot), we assign the cone of the morphism \( X^\bullet \rightarrow Y^\bullet \) (with the specialization maps from the cone to the small neighborhoods of \( t \) containing \( X^\bullet \) and \( Y^\bullet \)). To the point of codimension \( n \) we assign the nth cone, \( 2^n \)-graded complex, etc.

Definition. The Khovanov homology of the singular knot (with a single double point) is a bigraded complex

\[
X^\bullet \oplus Y^\bullet[1] \quad \text{with the matrix differential} \quad d_{C_{\omega}} = \begin{pmatrix} d_X & \omega \\ 0 & d_Y[1] \end{pmatrix},
\]
where $X^\bullet$ is Khovanov complex of the knot with overcrossing, $Y^\bullet$ is the Khovanov complex of the knot with undercrossing and $\omega$ is the wall-crossing morphism.

In [S4] we give the geometric interpretation of the above definition.

### 4.4. The definition of a theory of finite type.

Once we extended the local system to the singular locus, it is natural to ask if such an extension will lead to the categorification of Vassiliev theory.

The first natural guess is that the theory, set up on some space of objects, which has quasi-isomorphic complexes on all chambers is a theory of order zero. Such theory will consist of trivial distinguished triangles as in (a) of the axiomatics of the triangulated category. When complexes, corresponding to adjacent chambers are quasiisomorphic, the cone of the morphism is an acyclic complex.

**Baby example of a theory of order 0.**

Let $M$ be an $n$-dimensional compact oriented smooth manifold. Consider the space of functions on $M$. This is an infinite-dimensional Euclidean space. The chambers of the space will correspond to Morse functions on $M$, the walls of the discriminant - to simple degenerations when two critical points collide, etc. Let’s consider the Morse complex, generated by the critical points of a Morse function on $M$. As it was shown by many authors, such complex is isomorphic to the CW complex, associated with $M$. Since we are calculating the homology of $M$ via various Morse functions, complexes may vary, but will have the same homology and Euler characteristics.

Then we can proceed according to our philosophy and assign cones of morphisms to the walls and selfintersections of the discriminant. Since complexes on the chambers of the space of functions are quasiisomorphic, all cones are acyclic.

Now we can introduce the main definition of a Floer-type theory being of finite type $n$:

**Main Definition.** The local system of (Floer-type) complexes, extended to the discriminant of the space of manifolds via the cone of morphism, is a local system of order $n$ if for any selfintersection of the discriminant of codimension $(n + 1)$, its $(n+1)$st cone is an acyclic complex.

How one shows that an $2n$-graded complex is acyclic? For example, if one introduces inverse maps to the wall-crossing morphisms and construct the homotopy $\mathcal{H}$, s.t.:

$$d\mathcal{H} - \mathcal{H}d = I$$

It is easy to check that the existence of such homotopy $\mathcal{H}$ implies, that the complex doesn’t have homology. Suppose $dc = 0$, i.e. $c$ is a cycle, then:

$$d\mathcal{H}c - \mathcal{H}dc = d\mathcal{H}c = I$$

**Example.** Suppose some local system is conjectured to be of finite type 3. How one would check this? By our definition, we should consider $2^3$ chambers adjacent to the every point of selfintersection of the discriminant of codimension 3, and 8 complexes, representing the
Let’s write the homotopy equation in the matrix form.
Consider dual maps $f^*, g^*, ..., w^*$. Then we get formulas for $d$ and $\mathcal{H}$ as $8 \times 8$ matrices:

$$d = \begin{pmatrix}
    d_A & f & g & 0 & a & 0 & 0 & 0 \\
    0 & d_B & 0 & h & 0 & b & 0 & 0 \\
    0 & 1 & d_D & w & 0 & 0 & e & 0 \\
    0 & 0 & 1 & d_C & 0 & 0 & 0 & c \\
    0 & 0 & 0 & 0 & d_E & k & m & 0 \\
    0 & 0 & 0 & 0 & 1 & d_F & 0 & l \\
    0 & 0 & 0 & 0 & 1 & d_H & n & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 1 & d_G
\end{pmatrix}$$

$$\mathcal{H} = \begin{pmatrix}
    d_A & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
    f^* & d_B & 0 & 0 & 0 & 0 & 0 & 0 \\
    g^* & 0 & d_D & 0 & 0 & 0 & 0 & 0 \\
    0 & h^* & w^* & d_C & 0 & 0 & 0 & 0 \\
    a^* & 0 & 0 & 0 & d_E & 0 & 0 & 1 \\
    0 & b^* & 0 & 0 & k^* & d_F & 0 & 0 \\
    0 & 0 & e^* & 0 & m^* & 0 & d_H & 0 \\
    0 & 0 & 0 & c^* & 0 & l^* & h^* & d_G
\end{pmatrix}$$

After substituting these matrices into the equation $d\mathcal{H} - \mathcal{H}d = I$ we obtain the diagonal matrix which must be homotopic to the identity matrix:
Thus the condition for the local to be of finite type $n$ can be interpreted as follows. For any selfintersection of the discriminant of codimension $n + 1$ consider $2^n$ complexes, forming a commutative cube (representatives of the local system in the chambers adjacent to the self-intersection point). Then the naive geometrical interpretation of the local system being of finite type $n$ is the following: each complex can be "split" into $n + 1$ subcomplexes, which map quasiisomorphically to $n + 1$ neighbours, at least no homologies die or being generated.
5. Knots: theories of finite type. Further directions.

5.1. Examples of combinatorially defined theories.

In the following table we give the examples of theories, which are the categorifications of classical invariants. All these theories fit into our framework and may satisfy the finiteness condition.

| Theory                  | Description                                      |
|------------------------|--------------------------------------------------|
| Jones polynomial       | $\lambda = \chi H^*(M)$                         |
| Alexander polynomial   | Khovanov homology [Kh]                           |
| $sl(n)$ invariants     | Ozsvath-Szabo knot homology [OS2]                |
| Casson invariant       | Instanton Floer homology [F]                     |
| Turaev’s torsion       | Ozsvath-Szabo 3 manifold theory [OS1]            |
| Vafa invariant         | Gukov-Witten categorification [GW]               |

Note, that the only theory which is not combinatorially defined is the original Instanton Floer homology [F]. The fact that it’s Euler characteristics is Casson’s invariant was proved by C.Taubes [T].

5.2. Generalization to dimension 3 and 4.

In our paper [S1] we generalized Vassiliev’s construction to the case of 3-manifolds. In [S2] we construct the space of parallelizable 4-manifolds and consider the parametrized version of the Refined Seiberg-Witten invariant [BF].

a). The space of 3-manifolds and invariants of finite type.

Note that all 3-manifolds are parallelizable and therefore carry spin-structures.

Following Vassiliev’s approach to classification of knots, we constructed spaces $E_1$ and $E_2$ of 3-manifolds by a version of the Pontryagin-Thom construction.

Our main results are as follows:

**Theorem [S1].** In $E_1 - D$ each connected component corresponds to a homeomorphism class of 3-dimensional framed manifold. For any connected framed manifold as above there is one connected component of $E_1 - D$ giving its homeomorphism type.

**Theorem [S1].** In $E_2 - D$ each connected component corresponds to a homeomorphism class of 3-dimensional spin manifold. For any connected spin manifold there is one connected component of $E_2 - D$ giving its homeomorphism type.

By a spin manifold we understand a pair $(M, \theta)$ where $M$ is an oriented 3-manifold, and $\theta$ is a spin structure on $M$. Two spin manifolds $(M, \theta)$ and $(M', \theta')$ are called homeomorphic, if there exists a homeomorphism $M \to M'$ taking $\theta$ to $\theta'$.

The construction of the space naturally leads to the following definition:
**Definition.** A map $I : (M, \theta) \to C$ is called a finite type invariant of (at most) order $k$ if it satisfies the condition:

$$\sum_{L' \in L} (-1)^{\#L'} I(M_{L'}) = 0$$

where $L'$ is a framed sublink of link $L$ with even framings, $L$ corresponds to the self-intersection of the discriminant of codimension $k+1$, $\#L'$ - the number of components of $L'$, $M_{L'}$ - spin 3-manifold obtained by surgery on $L'$.

We introduced an example of Vassiliev invariant of finite order. Given a spin 3-manifold $M^3$ we consider the Euler characteristic of spin 0-cobordism $W$. Denote by $I(M, \text{spin}) = (\text{sgn}(W, \text{spin}) - 1)(\text{mod} 2)$.

**Theorem [S1].** Invariant $I(M, \text{spin})$ is finite type of order 1.

The construction of the space of 3-manifolds chambers of which correspond to spin 3-manifolds is important for understanding, which additional structures one needs in order to build the theory of finite-type invariants for homologically nontrivial manifolds. It suggests that one should consider spin ramifications of known invariants.

In the following paper we will generalize our constructions and the main definition to the case of 3-manifolds. We will construct a local system of Ozsvath-Szabo homologies, extend it to the singular locus via the cone of morphism and find examples of theories of finite type.

b). **Stably parallelizable 4-manifolds.**

In this section we modify the previous construction [S1] to get the space of parallelizable 4-manifolds.

By the definition the manifold is parallelizable if it admits the global field of frames, i.e. has a trivial tangent bundle. In the case of 4-manifolds this condition is equivalent to vanishing of Euler and the second Stieffel-Whitney class. In particular signature and the Euler characteristic of such manifolds will be 0.

We will use the theorem of Quinn:

**Theorem** Any punctured 4-manifold posseses a smooth structure.

Recall also the result of Vidussi, which states that manifolds diffeomorphic outside a point have the same Seiberg-Witten invariants, so one cannot use them to detect eventual inequivalent smooth structures. Thus for the purposes of constructing the family version of the Seiberg-Witten invariants, it will be sufficient for us to consider “asymptotically flat” 4-manifolds, i.e. such that outside the ball $B_R$ of some large radius $R$ they will be given as the set of common zeros of the system of linear equations (e.g. $f_i(x_1, \ldots x_{n+4}) = x_i$ for $i = 1, \ldots n$.)

By Gromov’s h-principle any smooth 4-manifold (with all of its smooth structures and metrics) can be obtained as a common set of zeros of a system of equations in $R^N$ for sufficiently large $N$.

**Theorem [S2].** Any parallelizable smooth 4-manifold can be obtained as a set of zeros of $n$ functions on the trivial $(n+4)$-bundle over $S^n$. Each manifold will be represented by $|H^1(M, Z_2) \oplus H^3(M, Z)|$ chambers.
There is a theory which also fits into our template - Ozsvath-Szabo homologies for 3-manifolds, Euler characteristic of which is Turaev’s torsion. It would be interesting to show that this theory is also of finite type or decomposes as the Khovanov theory.

c). Ozsvath-Szabo theory as triangulated category.

In [S3] we put the theory developed by P.Ozsvath and Z. Szabo into the context of homological algebra by considering a local system of their complexes on the space of 3-manifolds and extending it to the singular locus. We show that for the restricted category the Heegaard Floer complex $\text{CF}^\infty$ is of finite type one. For other versions of the theory we will be using the new combinatorial formulas, obtained in [SW].

Recall that the categorification is the process of replacing sets with categories, functions with functors, and equations between functions by natural isomorphisms between functors. One would hope that after establishing this correspondence, homological algebra will provide algebraic structures which one should assign to geometrical objects without going into the specifics of a given theory. One can see that this approach is very useful in topological category, in particular we will be getting knot and link invariants of Ozsvath and Szabo after setting up their local system on the space of 3-manifolds.

**Note** Floer homology can be also considered as invariants for families, so it would be interesting to connect our work to the one of M.Hutchings [H]. His work can be interpreted as construction of local systems corresponding to various Floer-type theories on the chambers of our spaces. Then we extend them to the discriminant and classify according to our definition.
6.3. Further directions.

1. There is a number of immediate questions from the finite-type invariants story:
   a). What will substitute the notion of the chord diagram? What is the "basis" in the theories of finite type?
   b). What are the "dimensions" of the spaces of theories of order n?

2. What is the representation-theoretical meaning of the theory of finite type?
   a). Is it possible to construct a "universal" knot homology theory in a sense of T.Lee [L]?
   b). Is it possible to rise such a "universal" knot homology theory to the Floer-type theory of 3-manifolds?

3. There are "categorifications" of other knot invariants: Alexander polynomial [OS], HOMFLY polynomial [DGR]. These theories also fit into our setting and it will interesting to show that they decompose into the series of theories of finite type or that their truncations are of finite type.

4. The next step in our program [S3] is the construction of the local system of Ozsvath-Szabo homologies on the space of 3-manifolds introduced in [S1]. We also plan to raise Khovanov theory to the homological Floer-type theory of 3-manifolds.

5. It should be also possible to generalize our program to the study of the diffeomorphism group of a 4-manifold by considering Gukov-Witten [GW] categorification of Vafa invariant on the moduli space constructed in [S2].
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