TOEPLITZ OPERATORS ON THE SYMMETRIZED BIDISC

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Abstract. The symmetrized bidisc has grabbed a great deal of attention of late because of its rich structure both in the context of function theory and in the context of operator theory. Toeplitz operators on this domain have not been discussed so far. The distinguished boundary $b\Gamma$ of the symmetrized bidisc is topologically identifiable with the Mobius strip and it is natural to consider bounded measurable functions there. In this article, we show that there is a natural Hilbert space $H^2(\mathbb{G})$. We describe three isomorphic copies of this space. The $L^\infty$ functions on $b\Gamma$ induce Toeplitz operators on this space. Such Toeplitz operators can be characterized through a couple of relations that they have to satisfy with respect to the co-ordinate multiplications on the space $H^2(\mathbb{G})$ which we call the Brown-Halmos relations. A number of results are obtained about the Toeplitz operators which bring out the similarities and the differences with the theory of Toeplitz operators on the disc as well as the bidisc. We show that the Coburn alternative fails, for example. However, the compact perturbations of Toeplitz operators are precisely the asymptotic Toeplitz operators. This requires us to find a characterization of compact operators on the Hardy space $H^2(\mathbb{G})$. The only compact Toeplitz operator turns out to be the zero operator.

Although operator theory on the symmetrized bidisc has now been studied for quite some time, often there are occasions when one has to develop a result that one needs. Such is the case we encountered in the study of dual Toeplitz operators in the last section of this paper. In that section, we produce a new result about a family of commuting $\Gamma$-isometries. Just like a Toeplitz operator is characterized by the Brown-Halmos relations with respect to the co-ordinate multiplications, an arbitrary bounded operator $X$ which satisfies the Brown-Halmos relations with respect to a commuting family of $\Gamma$-isometries is a compression of a norm preserving $Y$ acting on the space of minimal $\Gamma$-unitary extension of the family of isometries. Moreover, if $X$ commutes with the $\Gamma$-isometries, then $Y$ is an extension and commutes with the minimal $\Gamma$-unitary extensions. Thus, it is a commutant lifting theorem. This result is then applied to characterize a dual Toeplitz operator.

1. Hardy space and boundary values

Ever since Brown and Halmos published their seminal paper [13] on Toeplitz operators, it has been studied on many spaces over many domains. The book by Bottcher and Silverman [12] is a veritable treasure. For the introduction to the theory for just the space $H^2(\mathbb{D})$, the survey article by Axler [6] is excellent. Over the years, research in Toeplitz operators has become a vast area. State of the art research, even just in the context of the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ is still going on, see [16], [18] and [28] and there are open problems [20]. In several variables, Toeplitz operators have been studied by several authors, see [21]

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and the references therein. Naive attempts to generalize one variable results quickly run into difficulties and innovative new ideas are required.

In this note, we are going to introduce the study of the Toeplitz operators on the Hardy space of the open symmetrized bidisc

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : |z_1| < 1 \text{ and } |z_2| < 1\}.$$ 

The novelty of this domain arises from the fact that it exhibits one-dimensional behaviour at times (e.g., the automorphism group is the same as that of the unit disc in the plane) and behaves significantly different at times (e.g., a realization formula for a function in the unit ball of $H^\infty(\mathbb{G})$ requires uncountably infinitely many "test functions", see \cite{5} and \cite{10}). The Toeplitz operators on this domain will highlight a few similarities and a lot of differences with the classical situation of Brown and Halmos as well as with later endeavours of the bidisc. It will also bring out once again the importance of the fundamental operator of a $\Gamma$-contraction introduced in \cite{8}.

**Definition 1.** Let $\pi$ be the symmetrization map

\begin{equation}
\pi(z_1, z_2) = (z_1 + z_2, z_1 z_2).
\end{equation}

The Hardy space $H^2(\mathbb{G})$ of the symmetrized bidisc is the vector space of those holomorphic functions $f$ on $\mathbb{G}$ which satisfy

$$\sup_{0 < r < 1} \int_{\mathbb{T} \times \mathbb{T}} |f \circ \pi(re^{i\theta_1}, re^{i\theta_2})|^2 |J(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 < \infty$$

where $J$ is the complex Jacobian of the symmetrization map $\pi$ and $d\theta_i$ is the normalized Lebesgue measure on the unit circle $\mathbb{T} = \{\alpha : |\alpha| = 1\}$ for all $i = 1, 2$. The norm of $f \in H^2(\mathbb{G})$ is defined to be

$$\|f\| = \|J\|^{-1} \left( \sup_{0 < r < 1} \int_{\mathbb{T} \times \mathbb{T}} |f \circ \pi(re^{i\theta_1}, re^{i\theta_2})|^2 |J(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 \right)^{1/2},$$

where $\|J\|^2 = \int_{\mathbb{T} \times \mathbb{T}} |J(e^{i\theta_1}, e^{i\theta_2})|^2 d\theta_1 d\theta_2$. The constant function 1 is in the space and $\|1\| = 1$.

This space has been discussed before for other purposes in \cite{10} and \cite{23}. Our first result establishes boundary values of the Hardy space functions. Let the closed symmetrized bidisc be denoted by

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1 \text{ and } |z_2| \leq 1\}.$$ 

Let $b\Gamma$ be the distinguished boundary of the symmetrized bidisc, i.e., $b\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| = |z_2| = 1\}$. Note that

$$L^2(b\Gamma) = \{f : b\Gamma \to \mathbb{C} : \int_{\mathbb{T} \times \mathbb{T}} |f \circ \pi(e^{i\theta_1}, e^{i\theta_2})|^2 |J(e^{i\theta_1}, e^{i\theta_2})|^2 d\theta_1 d\theta_2 < \infty\}.$$ 

The following theorem immediately allows us to consider boundary values of the Hardy space functions.

**Theorem 2.** There is an isometric embedding of the space $H^2(\mathbb{G})$ inside $L^2(b\Gamma)$.
Proof. Consider the subspace
\[ H^2_{\text{anti}}(\mathbb{D}^2) \overset{\text{def}}{=} \{ f \in H^2(\mathbb{D}^2) : f(z_1, z_2) = -f(z_2, z_1) \} \]
of anti-symmetric functions of the Hardy space of the bidisc
\[ H^2(\mathbb{D}^2) = \{ f : \mathbb{D}^2 \to \mathbb{C} : f(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} z_1^i z_2^j \text{ with } \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{i,j}|^2 < \infty \} \]
Suppose \( L^2_{\text{anti}}(T^2) \) be the subspace of \( L^2(T^2) \) consisting of anti-symmetric functions, i.e.,
\[ f(e^{i\theta_1}, e^{i\theta_2}) = -f(e^{i\theta_2}, e^{i\theta_1}) \text{ a.e.} \]
Define \( \tilde{U} : H^2_G \to H^2_{\text{anti}}(\mathbb{D}^2) \) by
\[ \tilde{U}(f) = \frac{1}{\|J\|} J(f \circ \pi), \text{ for all } f \in H^2_G \]
and \( U : L^2(b\Gamma) \to L^2_{\text{anti}}(\mathbb{D}^2) \) by
\[ Uf = \frac{1}{\|J\|} J(f \circ \pi), \text{ for all } f \in L^2(b\Gamma). \]
It is easy to see that \( U \) and \( \tilde{U} \) are indeed unitary operators. Also note that there is an isometry \( W : H^2_{\text{anti}}(\mathbb{D}^2) \to L^2_{\text{anti}}(T^2) \) which sends a function to its radial limit. Therefore we have the following commutative diagram:
\[
\begin{array}{ccc}
H^2_G & \xrightarrow{U^{-1} \circ W \circ \tilde{U}} & L^2(b\Gamma) \\
\downarrow & & \downarrow U \\
H^2_{\text{anti}}(\mathbb{D}^2) & \xrightarrow{W} & L^2_{\text{anti}}(T^2)
\end{array}
\]
Therefore the map that places \( H^2_G \) isometrically into \( L^2(b\Gamma) \) is \( U^{-1} \circ W \circ \tilde{U} \). \[\Box\]

The above identification theorem reveals that the isometric image of the Hardy space of the symmetrized bidisc is precisely the following space:
\[ \{ f \in L^2(b\Gamma) : U(f) \text{ has all the negative Fourier coefficients zero} \} \]
In this paper, we shall not make any distinction between these two realizations of the Hardy space of the symmetrized bidisc and \( Pr \) will stand for the orthogonal projection of \( L^2(b\Gamma) \) onto the isometric image of \( H^2_G \) inside \( L^2(b\Gamma) \). With this identification, the unitary \( \tilde{U} \) is the restriction of the unitary \( U \) to the subspace \( H^2_G \). Hence, we shall not write \( \tilde{U} \) any more. Whenever we mention \( U \), it will be clear from the context whether it is being applied on \( L^2(b\Gamma) \) or on \( H^2_G \). In the latter case, the range is \( H^2_{\text{anti}}(\mathbb{D}^2) \).

The internal co-ordinates of the (open or closed) symmetrized bidisc will be denoted by \((s,p)\). Several criteria for a member \((s,p)\) of \( C^2 \) to belong to \( G \) (or \( \Gamma \)) are known, the interested reader may see Theorem 1.1 in [8]. Let
\[ L^\infty(b\Gamma) = \{ \varphi : b\Gamma \to \mathbb{C} : \text{ there exists } M > 0, \text{ such that } |\varphi(s,p)| \leq M \text{ a.e. in } b\Gamma \} \]
For a function \( \varphi \) in \( L^\infty(b\Gamma) \), let \( M_\varphi \) be the operator on \( L^2(b\Gamma) \) defined by
\[
M_\varphi f(s, p) = \varphi(s, p) f(s, p),
\]
for all \( f \) in \( L^2(b\Gamma) \). We note that the co-ordinate multiplication operators \( M_s \) and \( M_p \) are commuting normal operators on \( L^2(b\Gamma) \).

**Definition 3.** For a function \( \varphi \) in \( L^\infty(b\Gamma) \), the multiplication operator \( M_\varphi \) is called the **Laurent operator with symbol \( \varphi \).** The compression of \( M_\varphi \) to \( H^2(G) \) is called **Toeplitz operator** and denoted by \( T_\varphi \). Therefore
\[
T_\varphi f = \Pr M_\varphi f \quad \text{for all } f \in H^2(G).
\]

We now describe an equivalent way of studying Laurent operators and Toeplitz operators on the symmetrized bidisc. Let \( L^\infty_{\text{sym}}(\mathbb{T}^2) \) denote the sub-algebra of \( L^\infty(\mathbb{T}^2) \) consisting of symmetric functions, i.e., \( f(e^{i\theta_1}, e^{i\theta_2}) = f(e^{i\theta_2}, e^{i\theta_1}) \) a.e. and \( \Pi_1 : L^\infty(b\Gamma) \to L^\infty_{\text{sym}}(\mathbb{T}^2) \) be the \(*\)-isomorphism defined by
\[
\varphi \mapsto \varphi \circ \pi
\]
where \( \pi \) is as defined in (1.1). Let \( \Pi_2 : \mathcal{B}(L^2(b\Gamma)) \to \mathcal{B}(L^2_{\text{anti}}(\mathbb{T}^2)) \) denote the conjugation map by the unitary \( U \) as defined in (1.3), i.e.,
\[
T \mapsto UTU^*.
\]

**Theorem 4.** Let \( \Pi_1 \) and \( \Pi_2 \) be the above \(*\)-isomorphisms. Then the following diagram is commutative:
\[
\begin{array}{ccc}
L^\infty(b\Gamma) & \xrightarrow{\Pi_1} & L^\infty_{\text{sym}}(\mathbb{T}^2) \\
i_1 & & \downarrow i_2 \\
\mathcal{B}(L^2(b\Gamma)) & \xrightarrow{\Pi_2} & \mathcal{B}(L^2_{\text{anti}}(\mathbb{T}^2))
\end{array}
\]
where \( i_1 \) and \( i_2 \) are the canonical inclusion maps. Equivalently, for \( \varphi \in L^\infty(b\Gamma) \), the operators \( M_\varphi \) on \( L^2(b\Gamma) \) and \( M_{\varphi|\text{sym}} \) on \( L^2_{\text{anti}}(\mathbb{T}^2) \) are unitarily equivalent via the unitary \( U \).

**Proof.** To show that the above diagram commutes all we need to show is that \( UM_\varphi U^* = M_{\varphi|\text{sym}} \), for every \( \varphi \) in \( L^\infty(b\Gamma) \). This follows from the following computation: for every \( \varphi \) in \( L^\infty(b\Gamma) \) and \( f \in L^2_{\text{anti}}(\mathbb{T}^2) \),
\[
UM_\varphi U^*(f) = U(\varphi U^* f) = (\varphi \circ \pi) \frac{1}{\|J\|} J(U^* f \circ \pi) = M_{\varphi|\text{sym}}(f).
\]
\( \square \)

As a consequence of the above theorem, we obtain that the Toeplitz operators on the Hardy space of the symmetrized bidisc are unitarily equivalent to that on \( H^2_{\text{anti}}(\mathbb{D}^2) \).

**Corollary 5.** For a \( \varphi \in L^\infty(b\Gamma) \), \( T_\varphi \) is unitarily equivalent to \( T_{\varphi|\text{sym}} := P_\alpha M_{\varphi|\text{sym}}|H^2_{\text{anti}}(\mathbb{D}^2) \), where \( P_\alpha \) stands for the projection of \( L^2_{\text{anti}}(\mathbb{T}^2) \) onto \( H^2_{\text{anti}}(\mathbb{D}^2) \).

**Proof.** This follows from the fact that the operators \( M_\varphi \) and \( M_{\varphi|\text{sym}} \) are unitarily equivalent via the unitary \( \bar{U} \), which takes \( H^2(G) \) onto \( H^2_{\text{anti}}(\mathbb{D}^2) \).
\( \square \)
Remark 6. In what follows, the pair \((T_s, T_p)\) will be specially useful, where \(T_s f = M_s f\) and \(T_p f = M_p f\) for \(f\) in \(H^2(\mathbb{G})\) (no projection required because \(H^2(\mathbb{G})\) is invariant under \(M_s\) and \(M_p\)). The unitary \(U\) mentioned in the theorem above intertwines \(T_s\) with \(T_{z_1 + z_2} = M_{z_1 + z_2} H^2_{\text{anti}}(\mathbb{T}^2)\) and \(T_p\) with \(T_{z_1 z_2} = M_{z_1 z_2} H^2_{\text{anti}}(\mathbb{T}^2)\).

We end the section with one more isomorphic copy of the Hardy space. If \(\mathcal{E}\) is a Hilbert space, let \(\mathcal{O}(\mathbb{D}, \mathcal{E})\) be the class of all \(\mathcal{E}\) valued holomorphic functions on \(\mathbb{D}\). Let

\[
H^2_\mathcal{E}(\mathbb{D}) = \{ f(z) = \sum a_k z^k \in \mathcal{O}(\mathbb{D}, \mathcal{E}) : a_k \in \mathcal{E} \text{ with } \| f \|^2 = \sum \| a_k \|^2 < \infty \}.
\]

Lemma 7. There is a Hilbert space isomorphism \(U_1\) from \(H^2_{\text{anti}}(\mathbb{D}^2)\) onto the vector valued Hardy space \(H^2_\mathcal{E}(\mathbb{D})\) where

\[
\mathcal{E} = \text{span}\{ z^j_1 - z^j_2 : 1 \leq j < \infty \} \subset H^2_{\text{anti}}(\mathbb{D}^2).
\]

Moreover, this unitary \(U_1\) intertwines \(T_{z_1 z_2}\) on \(H^2_{\text{anti}}(\mathbb{D}^2)\) with the unilateral shift of infinite multiplicity \(T_z\) on \(H^2_\mathcal{E}(\mathbb{D})\).

Proof. Denote the vectors \((z_1^j - z_2^j)\) by \(e_j\). Then \(\{e_1, e_2, \ldots\}\) forms an orthogonal basis in \(\mathcal{E}\). Thus \(\{z^i e_j : i = 0, 1, 2, \ldots \text{ and } j = 1, 2, 3, \ldots\}\) is an orthogonal basis for \(H^2_\mathcal{E}(\mathbb{D})\). On the other hand, the space \(H^2_{\text{anti}}(\mathbb{D}^2)\) is spanned by the orthogonal set \(\{(z_1 z_2)^i (z^j_1 - z^j_2) : i \geq 0 \text{ and } j \geq 1\}\). Define the unitary operator from \(H^2_{\text{anti}}(\mathbb{D}^2)\) onto \(H^2_\mathcal{E}(\mathbb{D})\) by

\[
(z_1 z_2)^i (z^j_1 - z^j_2) \mapsto z^i e_j
\]

and then extending linearly. This preserves norms because \(T_{z_1}\) and \(T_{z_2}\) are isometries on \(H^2(\mathbb{D}^2)\) and \(T_z\) is an isometry on \(H^2_\mathcal{E}(\mathbb{D})\). It is surjective and obviously intertwines \(T_{z_1 z_2}\) and \(T_z\). \(\square\)

The content of this lemma is from \([11]\). We have included it here for the sake of completeness as well and for a more succinct presentation than in \([11]\). By virtue of the isomorphisms \(U\) and \(U_1\) described above, we have the following commutative diagram:

\[
\begin{array}{ccc}
(H^2(\mathbb{G}), T_p) & \xrightarrow{U} & (H^2_{\text{anti}}(\mathbb{D}^2), T_{z_1 z_2}) \\
\downarrow U_2 & & \downarrow U_1 \\
(H^2_\mathcal{E}(\mathbb{D}), T_z) & & (H^2_\mathcal{E}(\mathbb{D}), T_z)
\end{array}
\]

i.e., the operator \(T_p\) on \(H^2(\mathbb{G})\) is unitarily equivalent to the unilateral shift of infinite multiplicity \(T_z\) on the vector valued Hardy space \(H^2_\mathcal{E}(\mathbb{D})\) via the unitary \(U_2\). We call \(\mathcal{E}\) the co-efficient space of the symmetrized bidisc. It is a subspace of \(H^2_\mathcal{E}(\mathbb{D})\) which is naturally identifiable with the subspace of constant functions in \(H^2_\mathcal{E}(\mathbb{D})\).

2. Properties of a Toeplitz Operator

Following Brown and Halmos’s terminology in \([13]\), the multiplication operator \(M_{\phi}\) is called the Laurent operator with symbol \(\phi\).
Lemma 8. The pair \((M_s, M_p)\) is a commuting pair of normal operators and \(\sigma(M_s, M_p) = b\Gamma\).

Proof. The Laurent operators \(M_s\) and \(M_p\) are co-ordinate multiplications on \(L^2(b\Gamma)\). Hence they are normal and \(\sigma(M_s, M_p) = b\Gamma\).

If \(M\) is a bounded operator on \(L^2(\mathbb{T})\) belonging to \(\{M_z\}'\), the commutant of the operator \(M_z\) on \(L^2(\mathbb{T})\), then it is well known that there exists a function \(\varphi \in L^\infty(\mathbb{T})\) such that \(M = M_\varphi\). The following result is an analogue of this phenomenon for the symmetrized bidisc.

Theorem 9. Let \(M\) be a bounded operator on \(L^2(b\Gamma)\) which commutes with both \(M_s\) and \(M_p\). Then there exists a function \(\varphi \in L^\infty(b\Gamma)\) such that \(M = M_\varphi\).

Proof. Since \((M_s, M_p)\) is a pair of commuting normal operators and \(\sigma(M_s, M_p) = b\Gamma\), then by the spectral theorem for commuting normal operators the von Neumann algebra generated by \(\{M_s, M_p\}\) is \(L^\infty(b\Gamma)\), which is a maximal abelian von Neumann algebra. This completes the proof.

By Theorem 4, the above theorem can be rephrased in the bidisc set up.

Corollary 10. Let \(M_{z_1+z_2}\) and \(M_{z_1z_2}\) denote the multiplication operators on \(L^2_{\text{anti}}(\mathbb{T}^2)\). Then any bounded operator \(M\) on \(L^2_{\text{anti}}(\mathbb{T}^2)\) that commutes with both \(M_{z_1+z_2}\) and \(M_{z_1z_2}\) is of the form \(M_\varphi\), for some function \(\varphi \in L^\infty(\mathbb{T}^2)\).

The above characterization of Laurent operators, apart from its key role in characterizing Toeplitz operators, will have significant implications in the last section where we deal with dual Toeplitz operators.

If \(f\) is a function holomorphic in a neighbourhood of \(\Gamma\), then \(\|f\|_{\infty, \Gamma}\) denotes the supremum norm of \(f\) over the compact set \(\Gamma\). We note a simple fact about this co-ordinate multiplication pair, viz., if \(f\) is any polynomial in two variables, then

\[
\|f(M_s, M_p)\| = \|MF\| = \|f\|_{\infty, \Gamma}.
\]

So far, we have discussed function theory on the symmetrized bidisc and on its distinguished boundary. We have also made a connection between a function in \(L^\infty(b\Gamma)\) and an operator on \(L^2(b\Gamma)\). However, no general operator theory on the symmetrized bidisc has been discussed yet. According to the following definition, the pair \((M_s, M_p)\) is a \(\Gamma\)-unitary.

Definition 11. Agler and Young introduced the following classes of operator pairs.

1. A commuting pair \((R, U)\) is called a \(\Gamma\)-unitary if \(R\) and \(U\) are normal operators and the joint spectrum \(\sigma(R, U)\) of \((R, U)\) is contained in the distinguished boundary of \(\Gamma\).

2. A commuting pair \((T, V)\) acting on a Hilbert space \(\mathcal{K}\) is called a \(\Gamma\)-isometry if there exist a Hilbert space \(\mathcal{N}\) containing \(\mathcal{K}\) and a \(\Gamma\)-unitary \((R, U)\) on \(\mathcal{N}\) such that \(\mathcal{K}\) is left invariant by both \(R\) and \(\mathcal{P}\), and

\[
T = R|_{\mathcal{K}} \quad \text{and} \quad V = U|_{\mathcal{K}}.
\]
In other words, $(R, U)$ is a $\Gamma$-unitary extension of $(T, V)$. In block operator matrix form,

\[ R = \begin{pmatrix} T & * \\ 0 & * \end{pmatrix} \text{ and } U = \begin{pmatrix} V & * \\ 0 & * \end{pmatrix}. \]

Thus, it is clear from the above block matrices that if $f$ is a polynomial in two variables, then

\[ f(R, U) = \begin{pmatrix} f(T, V) & * \\ 0 & * \end{pmatrix}. \]

Consequently,

\[ \| f(T, V) \| \leq \| f(R, U) \| \]

(2.1)

This von Neumann type inequality will also remain true for another class of operator pairs $(S, P)$. Suppose $\mathcal{H}$ is such a subspace of $\mathcal{K}$ that is invariant under $T^*$ and $V^*$. On $\mathcal{H}$, we consider the operators $S$ and $P$ which are defined by

\[ S^* = T^*|_{\mathcal{H}} \text{ and } P^* = V^*|_{\mathcal{H}}. \]

So, $S$ and $P$ are compressions of $T$ and $V$ to a co-invariant subspace. If $f(s, p) = \sum a_{ij}s^i p^j$, consider the polynomial $\tilde{f}(s, p) = \sum a_{ij}s^i p^j$. Then

\[ \| f(S, P) \| = \| f(S, P)^* \| = \| \tilde{f}(S^*, P^*) \| \leq \| \tilde{f}(T^*, V^*) \| = \| f(T, V)^* \| \leq \| f \|_{\infty, \Gamma}. \]

It is a remarkable fact that any pair $(S, P)$ satisfying the inequality (2.3) has to be of the form (2.2), see [3] and [8]. Such a pair $(S, P)$ is called a $\Gamma$-contraction. The two following theorems are from [3] and [8] and characterize $\Gamma$-unitaries and $\Gamma$-isometries.

**Theorem 12.** Let $(R, U)$ be a pair of commuting operators defined on a Hilbert space $\mathcal{H}$. Then the following are equivalent:

1. $(R, U)$ is a $\Gamma$-unitary;
2. there exist commuting unitary operators $U_1$ and $U_2$ on $\mathcal{H}$ such that
   \[ R = U_1 + U_2, \quad U = U_1 U_2; \]
3. $U$ is unitary, $R = R^* U$, and $r(R) \leq 2$, where $r(R)$ is the spectral radius of $R$;
4. $(R, U)$ is a $\Gamma$-contraction and $U$ is a unitary;
5. $U$ is a unitary and $R = W + W^* U$ for some unitary $W$ commuting with $U$.

**Theorem 13.** Let $T, V$ be commuting operators on a Hilbert space $\mathcal{H}$. The following statements are all equivalent:

1. $(T, V)$ is a $\Gamma$-isometry;
2. $(T, V)$ is a $\Gamma$-contraction and $V$ is isometry;
3. $V$ is an isometry, $T = T^* V$ and $r(T) \leq 2$. 

A $\Gamma$-isometry $(T, V)$ is said to be a pure $\Gamma$-isometry if $V$ is a pure isometry, i.e., there is no non trivial subspace of $\mathcal{H}$ on which $V$ acts as a unitary operator.

**Lemma 14.** The pair $(T_s, T_p)$ is a pure $\Gamma$-isometry with $(M_s, M_p)$ as its minimal $\Gamma$-unitary extension and $\sigma(T_s, T_p) = \Gamma$.

**Proof.** The pair $(T_s, T_p)$ is a $\Gamma$-isometry because it is the restriction of the $\Gamma$-unitary $(M_s, M_p)$ to the invariant subspace $H^2(\mathbb{G})$. The operator $T_p$ is pure because by Corollary 13, $T_p$ is unitarily equivalent to $M_{z_1 z_2} | H^2_{\text{anti}}(\mathbb{D}^2)$, which is pure. The extension $(M_s, M_p)$ is minimal because $M_{z_1 z_2}$ is the minimal unitary extension of $M_{z_1 z_2} | H^2_{\text{anti}}(\mathbb{D}^2)$.

It remains to prove that $\sigma(T_s, T_p) = \Gamma$. This is easily accomplished by noting that $H^2(\mathbb{G})$ is a reproducing kernel Hilbert space. Its kernel is

$$k_\mathbb{G}(s_1, p_1), (s_2, p_2) = \frac{1}{(1 - p_1 \bar{p}_2)^2 - (s_1 - \bar{s}_2 p_1)(\bar{s}_2 - s_1 \bar{p}_2)}.$$  

This is called the Szegö kernel of the symmetrized bidisc. If $(s, p)$ is a point of $\mathbb{G}$, then $(\bar{s}, \bar{p})$ is a joint eigenvalue of $(T^*_s, T^*_p)$ with the eigenvector $k(\cdot, (s, p))$. Since $(s, p)$ is in $\mathbb{G}$ if and only if $(\bar{s}, \bar{p})$ is in $\mathbb{G}$, we now have entire $\mathbb{G}$ in the joint point spectrum of $(T^*_s, T^*_p)$. Since the spectrum is a closed set, $\sigma(T_s, T_p) = \sigma(T^*_s, T^*_p) = \Gamma$.  

We progress with basic properties of Toeplitz operators. Although, a Toeplitz operator is defined in terms of an $L^\infty$ function, it is a difficult question of how to recognize a given bounded operator $T$ on the relevant Hilbert space as a Toeplitz operator. This question was answered for the Hardy space of the unit disc by Brown and Halmos in Theorem 6 of [13] where they showed that $T$ has to be invariant under conjugation by the unilateral shift.

We show that in our context one needs both $T_s$ and $T_p$ to give such a characterization.

**Definition 15.** Let $T$ be a bounded operator on $H^2(\mathbb{G})$. We say that $T$ satisfies the Brown-Halmos relations with respect to the $\Gamma$-isometry $(T_s, T_p)$ if

$$(2.4) \quad T_s^* T T_p = T T_s^* \text{ and } T_p^* T T_p = T.$$  

It is a natural question whether any of the two Brown-Halmos relations implies the other. We give here an example of an operator $Y$ which satisfies the second one, but not the first.

**Example 16.** This example produces an operator that commutes with $T_p$ so that the second of the Brown-Halmos relations is obviously satisfied. Nevertheless, the operator does not satisfy the first one. We shall use the unitary equivalences we established in Section 1 for this example. We define an operator $X$ on $H^2_{\text{anti}}(\mathbb{D}^2)$. It is enough to define it on the basis elements, and on the basis elements its action is given by

$$(2.5) \quad X(z_1 z_2)(z_1^{j+1} - z_2^{j+1}) = (z_1 z_2)(z_1^{j+1} - z_2^{j+1}) \quad \text{for } i = 0, 1, \ldots \text{ and } j = 1, 2, \ldots.$$  

Then $X$ commutes with $M_{z_1 z_2} | H^2_{\text{anti}}(\mathbb{D}^2)$ clearly. By applying the unitary $U$, we see that $U^* X U$ commutes with $T_p$. Let $Y = U^* X U$. We shall show that the first of the Brown-Halmos relations is not satisfied although the second one is satisfied by commutativity of $Y$ with $T_p$. To that end, we note that

$$T_s^* Y T_p = T_s^* T_p Y = T_s Y.$$
so that the question boils down to whether $Y$ commutes with $T_s$ or not which is easy to resolve because

$$YT_s(1) = U^*XUT_s(1) = \frac{1}{\|J\|}U^*X(z_1^2 - z_2^2) = \frac{1}{\|J\|}U^*(z_1^3 - z_2^3) = s^2 - p$$

and

$$T_sY(1) = T_sU^*XU(1) = \frac{1}{\|J\|}T_sU^*X(z_1 - z_2) = \frac{1}{\|J\|}T_sU^*(z_1 - z_2) = T_s s = s^2.$$

**Theorem 17.** A Toeplitz operator satisfies the Brown-Halmos relations and vice versa.

**Proof.** We first prove that the condition is necessary. Let $T$ be a Toeplitz operator with symbol $\varphi$. Then for $f, g \in H^2(\mathbb{D})$,

$$\langle T_pT_\varphi T_pf, T_pg \rangle = \langle PrM_\varphi T_pf, T_pg \rangle = \langle M_\varphi M_pf, M_pg \rangle = \langle M_\varphi f, g \rangle = \langle T_\varphi f, g \rangle.$$  

Also,

$$\langle T_sT_\varphi T_pf, T_pg \rangle_{H^2} = \langle PrM_\varphi T_pf, T_sg \rangle_{H^2} = \langle M_\varphi M_pf, M_sg \rangle_{L^2} = \langle M^*_\varphi M_\varphi f, g \rangle_{L^2} = \langle M_\varphi M_f, g \rangle_{L^2} = \langle PrM_\varphi M_sg, f \rangle_{H^2} = \langle T_\varphi T_sf, g \rangle_{H^2}.$$  

In the above computation, we have used the equality $M_s = M^*_s M_p$.

Now we prove that the condition is sufficient. To this end we work on $H^2_{\text{anti}}(\mathbb{D}^2)$ and invoke Corollary 5 to draw the conclusion. So let $T$ be a bounded operator on $H^2_{\text{anti}}(\mathbb{D}^2)$ satisfying $T_{z_1 z_2}TT_{z_1 z_2} = TT_{z_1 z_2}$ and $T_{z_1 z_2}TT_{z_1 z_2} = T$. For two different integers $i$ and $j$, let $e_{i,j} := z_i^* z_j^2 - z_i^* z_j^2$. Note that for $n \geq 0$, $M^n_{z_1 z_2} e_{i,j} = e_{i+n,j+n}$. Therefore for every different integers $i$ and $j$, there exists a sufficiently large $n$ such that $M^n_{z_1 z_2} e_{i,j} \in H^2_{\text{anti}}(\mathbb{D}^2)$. For each $n \geq 0$, define an operator $T_n$ on $L^2_{\text{anti}}(\mathbb{T}^2)$ by

$$T_n := M^n_{z_1 z_2} TP_n M^n_{z_1 z_2},$$

where $P_n$ is the orthogonal projection of $L^2_{\text{anti}}(\mathbb{T}^2)$ onto $H^2_{\text{anti}}(\mathbb{D}^2)$. Let $i, j, k$ and $l$ be integers such that $i \neq j$ and $k \neq l$, then for sufficiently large $n$, we have

$$\langle T_n e_{i,j}, e_{k,l} \rangle = \langle TM^n_{z_1 z_2} e_{i,j}, M^n_{z_1 z_2} e_{k,l} \rangle = \langle Te_{i+n,j+n}, e_{k+n,l+n} \rangle. \tag{2.6}$$

Since $T^n_{z_1 z_2} TT^n_{z_1 z_2} = T$, we have for every $n \geq 0$, $T^n_{z_1 z_2} TT^n_{z_1 z_2} = T$. Let $i, j, k$ and $l$ be non-negative integers such that $i \neq j$ and $k \neq l$, then for every $n \geq 0$,

$$\langle Te_{i,j}, e_{k,l} \rangle = \langle TT^n_{z_1 z_2} e_{i,j}, T^n_{z_1 z_2} e_{k,l} \rangle = \langle Te_{i+n,j+n}, e_{k+n,l+n} \rangle. \tag{2.7}$$

Since $\{e_{i,j} : i \neq j \in \mathbb{Z}\}$ is an orthogonal basis for $L^2_{\text{anti}}(\mathbb{T}^2)$ and the sequence of operators $T_n$ on $L^2_{\text{anti}}(\mathbb{T}^2)$ is uniformly bounded by $\|T\|$, by (2.6) and (2.7) the sequence $T_n$ converges weakly to some operator $T_\infty$ (say) acting on $L^2_{\text{anti}}(\mathbb{T}^2)$.

We now use Corollary 10 to conclude that $T_\infty = M_\varphi$, for some $\varphi \in L^\infty_{\text{sym}}(\mathbb{T}^2)$. Therefore we have to show that $T_\infty$ commutes with both $M_{z_1 z_2}$ and $M_{z_1 z_2}$. That $T_\infty$ commutes
with $M_{z_1 z_2}$ is clear from the definition of $T\infty$. The following computation shows that $T\infty$ commutes with $M_{z_1 z_2}$ also. Let $i, j, k$ and $l$ be integers such that $i \neq j$ and $k \neq l$. Then

$$
\langle M_{z_1 + z_2}^* T_{\infty}^* e_{i, j}, e_{k, l} \rangle \\
= \lim_{n} \langle M_{z_1 + z_2}^* M_{z_1 z_2}^{n} T_{\infty}^* P_{a} M_{z_1 z_2}^{n} e_{i, j}, e_{k, l} \rangle \\
= \lim_{n} \langle T_{z_1 + z_2}^* M_{z_1 z_2}^{n} e_{i, j}, M_{z_1 z_2}^{n} e_{k, l} \rangle \quad \text{(for sufficiently large $n$)} \\
= \lim_{n} \langle T_{z_1 z_2}^* M_{z_1 z_2}^{n} e_{i, j}, M_{z_1 z_2}^{n} e_{k, l} \rangle \quad \text{(applying (2.4))} \\
= \lim_{n} \langle M_{z_1 z_2}^{n+1} P_{a} M_{z_1 z_2}^{n+1} * M_{z_1 z_2}^{*} e_{i, j}, e_{k, l} \rangle \\
= \lim_{n} \langle M_{z_1 z_2}^{n+1} P_{a} M_{z_1 z_2}^{n+1} * M_{z_1 z_2}^{*} e_{i, j}, e_{k, l} \rangle \quad \text{(since $M_{z_1 z_2} = M_{z_1 z_2}^*$)} \\
= \langle T_{\infty}^* M_{z_1 z_2}^* e_{i, j}, e_{k, l} \rangle.
$$

Therefore there exists a $\varphi \in L_{\text{sym}}^\infty(T^2)$ such that $T\infty = M_{\varphi}$. Now for $f$ and $g$ in $H^2_{\text{anti}}(\mathbb{D}^2)$, we have

$$
\langle P_{a} M_{\varphi} f, g \rangle = \langle M_{\varphi} f, g \rangle = \langle T_{\infty} f, g \rangle = \lim_{n} \langle T_{n} f, g \rangle = \lim_{n} \langle T T_{z_1 z_2}^{n} f, T_{z_1 z_2}^{n} g \rangle = \langle T f, g \rangle.
$$

Hence $T$ is the Toeplitz operator with symbol $\varphi$. \hfill \square

The following is a straightforward consequence of the characterization of Toeplitz operators obtained above.

**Corollary 18.** If $T$ is a bounded operator on $H^2(\mathbb{G})$ that commutes with both $T_{s}$ and $T_{p}$, then $T$ satisfies the Brown-Halmos relations and hence is a Toeplitz operator.

**Proof.** It is given that $T T_{p} = T_{p} T$. Multiplying both sides from the left by $T_{p}^{*}$, we get that $T_{p}^{*} T T_{p} = T$ because $T_{p}$ is an isometry. The following simple computation shows that $T$ also satisfies the other relation.

$$
T_{s} T T_{p} = T_{s} T_{p} T = T_{s} T = T T_{s},
$$

where we used the fact that $(T_{s}, T_{p})$ is a $\Gamma$-isometry and hence $T_{s} = T_{s} T_{p}$. \hfill \square

**Lemma 19.** For $\varphi \in L_{\text{sym}}^\infty(\partial \mathbb{G})$ if $T_{\varphi}$ is the zero operator, then $\varphi = 0$, a.e. In other words, the map $\varphi \mapsto T_{\varphi}$ from $L_{\text{sym}}^\infty(\partial \mathbb{G})$ into the set of all Toeplitz operators on the symmetrized bidisc, is injective.

**Proof.** Let $\varphi \circ \pi(z_{1}, z_{2}) = \sum_{i, j \in \mathbb{Z}} \alpha_{i, j} z_{1}^{i} z_{2}^{j} \in L_{\text{sym}}^\infty(T^2)$. Then $T_{\varphi \circ \pi}$ on $H^2_{\text{anti}}(\mathbb{D}^2)$ is the zero operator. Now we have for every $m, k \geq 0$ and $n, l \geq 1$,

$$
0 = \langle T_{\varphi \circ \pi}(z_{1} z_{2})^{m} (z_{1}^{n} - z_{2}^{n}), (z_{1} z_{2})^{l} (z_{1}^{l} - z_{2}^{l}) \rangle \\
= \langle \sum_{i, j \in \mathbb{Z}} \alpha_{i, j} z_{1}^{i+m+n} z_{2}^{l+j+m+n}, (z_{1} z_{2})^{l} (z_{1}^{l} - z_{2}^{l}) \rangle \\
= \alpha_{k+l-m-n, k-m} + \alpha_{k-m, k+l-m} - \alpha_{k+l-m, k-m} - \alpha_{k-m, k+l-m} \\
= 2(\alpha_{k+l-m-n, k-m} - \alpha_{k-m, k+l-m}).
$$
To obtain the last equality we have used the fact that $\alpha_{i,j} = \alpha_{j,i}$ for all $i, j \in \mathbb{Z}$. Since the sequence $\{\alpha_{i,j}\}$ is square summable, the above computation says that for every $m, k \geq 0$ and $n, l \geq 1$,

$$\alpha_{k-m-n+l, k-m} = \alpha_{k-m, n, k-m+l} = 0.$$ 

Note that $\{k-m : m \geq 0\} = \mathbb{Z}$ and for fixed $k, m \geq 0$, $\{(k-m)-(n-l) : n, l \geq 1\} = \mathbb{Z}$. Hence $\alpha_{i,j} = 0$ for all $i, j \in \mathbb{Z}$. This completes the proof. \qed

It is easy to see that the space $H^\infty(G)$ consisting of all bounded analytic functions on $G$ is contained in $H^2(G)$. We identify $H^\infty(G)$ with its boundary functions. In other words, $H^\infty(G) = \{\varphi \in L^\infty(b\Gamma) : \varphi \circ \pi \text{ has no negative Fourier coefficients}\}$.

**Definition 20.** A Toeplitz operator with symbol $\varphi$ is called an analytic Toeplitz operator if $\varphi$ is in $H^\infty(G)$. A Toeplitz operator with symbol $\varphi$ is called a co-analytic Toeplitz operator if $T^*_\varphi$ is an analytic Toeplitz operator.

Our next goal is to characterize analytic Toeplitz operators. But to be able to do that we need to define the following notion and the proposition following it.

**Definition 21.** Let $\varphi \in L^\infty(b\Gamma)$. The operator $H_\varphi : H^2(G) \rightarrow L^2(b\Gamma) \ominus H^2(G)$ defined by

$$H_\varphi f = (I - Pr)M_\varphi f$$

for all $f \in H^2(G)$, is called a Hankel operator.

We write down few observations about Toeplitz operators some of which will be used in the theorem following it. The proofs are not written because they go along the same line as the one dimensional case.

**Proposition 22.** Let $\varphi \in L^\infty(b\Gamma)$. Then

1. $T^*_\varphi = T^*_\varphi$.
2. If $\psi \in L^\infty(b\Gamma)$ is another function, then the product $T^*_\varphi T^*_\psi$ is another Toeplitz operator if $\varphi$ or $\psi$ is analytic. In each case, $T^*_\varphi T^*_\psi = T^*_\varphi \psi$.
3. If $\psi \in L^\infty(b\Gamma)$, then $T^*_\varphi T^*_\psi - T^*_\varphi \psi = -H^*_\varphi H^*_\psi$.
4. For an operator $T$, let $\Pi(T)$ be the approximate point spectrum of $T$. Then the essential range of $\varphi = \Pi(M_\varphi) = \sigma(M_\varphi) \subseteq \Pi(T) \subseteq \sigma(T^*_\varphi)$.

Hence

(a) $\|\varphi\|_{\infty} = \|M_\varphi\| = \|T^*_\varphi\| = r(T^*_\varphi)$ and

(b) $\|T^*_\varphi - K\| \geq \|T^*_\varphi\|$, for every compact operator $K$ on $H^2(G)$.

Now we are ready to characterize Toeplitz operators with analytic symbol.

**Theorem 23.** Let $T_\varphi$ be a Toeplitz operator. Then the following are equivalent:

(i) $T_\varphi$ is an analytic Toeplitz operator;

(ii) $T_\varphi$ commutes with $T_p$;

(iii) $T_\varphi(RanT_p) \subseteq RanT_p$;

(iv) $T_p T_\varphi$ is a Toeplitz operator;
(v) $T_\varphi$ commutes with $T_s$;
(vi) $T_sT_\varphi$ is a Toeplitz operator.

Proof. (i) $\iff$ (ii): That (i) $\implies$ (ii) is easy. To prove the other direction, we use part (3) of Proposition 22 to get that $H^*_\varphi H_\varphi = 0$. This shows that the corresponding product of Hankel operators on $\mathbb{H}^2_\text{anti}(\mathbb{D}^2)$ is also zero, that is $H^*_\text{symm} H_\text{anti} = 0$. Let the power series expansion of $\varphi \circ \pi \in L^\infty_{\text{symm}}(\mathbb{T}^2)$ be

$$
\varphi \circ \pi(z_1, z_2) = \sum_{m, n \in \mathbb{Z}} \alpha_{m,n} z_1^m z_2^n \text{ for all } z_1, z_2 \in \mathbb{T}.
$$

Since $\varphi \circ \pi$ is symmetric we have $\alpha_{m,n} = \alpha_{n,m}$, for every $m, n \in \mathbb{Z}$. For $k, r \geq 0$ and $l \geq 1$, we have

$$
0 = \langle H_\text{anti}(z_1, z_2)^r(z_1^l - z_2^l), H^*_\text{symm}(z_1^{k+1} - z_2^{k+1}) \rangle_{L^2(\mathbb{T}^2)}

= \sum_{m, n \in \mathbb{Z}} \alpha_{m,n} z_1^m z_2^n (z_1^l - z_2^l)(z_1^{k+1} - z_2^{k+1}) \rangle_{L^2(\mathbb{T}^2)}

= \sum_{m, n \in \mathbb{Z}} \alpha_{m,n} (z_1^{m+r+l} z_2^r - z_1^r z_2^{m+l}) (z_1^{k+1} - z_2^{k+1}) \rangle_{L^2(\mathbb{T}^2)}

= 2(\alpha_{k-r-l-1, r-1} - \alpha_{k-r-1, r-1-k})
$$

where to obtain the last equality we have used $\alpha_{m,n} = \alpha_{n,m}$ for every $m, n \in \mathbb{Z}$. Now since the sequence $\{\alpha_{m,n}\}$ is square summable, we conclude that for every $k, r \geq 0$ and $l \geq 1$

$$
\alpha_{r-1, (k-l)} = \alpha_{(r+l)-1, k} = 0.
$$

From these equalities we claim that $\alpha_{m,n} = 0$, unless both of $m$ and $n$ are non-negative, which would imply that $\varphi$ is analytic. First we show that if $m \geq 0$ and $n \geq 1$, then $\alpha_{-n,m} = \alpha_{m,-n} = 0$. For that we choose $r = n - 1$ and $k, l$ such that $k - l = m + n - 1$. For this choice of $k, r$ and $l$ we have $0 = \alpha_{r-1, (k-l)-r} = \alpha_{m,n}$. Now we show that if $m \geq 1$ and $n \geq 0$, then $\alpha_{m,n} = 0$. To this end, we choose $r = m - 1$ and $k, l$ such that $k - l = m - n - 1$. For this choice of $k, r$ and $l$ we have $0 = \alpha_{r-1, (k-l)-r} = \alpha_{m,n}$. 

(ii) $\iff$ (iii): The part (ii) $\implies$ (iii) is easy. Conversely, suppose that $\text{Ran} T_p$ is invariant under $T_\varphi$. Since $\text{Ran} T_p$ is closed, we have for every $f \in H^2(\mathbb{G})$,

$$
T_\varphi T_pf = T_pg_f \text{ for some } g_f \in H^2(\mathbb{G}),
\implies T_\varphi T_\varphi T_pf = g_f \implies T_\varphi f = g_f \text{ (by Theorem 17)}.
$$

Hence $T_\varphi T_p = T_p T_\varphi$.

(iii) $\iff$ (iv): If $T_\varphi$ commutes with $T_p$, then $T_p T_\varphi$ is same as $T_\varphi T_p$, which is a Toeplitz operator by Proposition 22. Conversely, if $T_p T_\varphi$ is a Toeplitz operator, then it satisfies Brown-Halmos relations, the second one of which implies that $T_\varphi$ commutes with $T_p$.

(i) $\iff$ (v): For an analytic symbol $\varphi$, $T_\varphi$ obviously commutes with $T_s$. The proof of the converse direction is done by the same technique as in the proof of (ii) $\implies$ (i). If $T_\varphi$ commutes with $T_s$, then by part (3) of Proposition 22 we have $H^*_\varphi H_\varphi = 0$. Suppose
$\varphi \circ \pi \in L^\infty_{\text{symm}}(T^2)$ has the following power series expansion
$$\varphi \circ \pi(z_1, z_2) = \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} z_1^m z_2^n$$
for all $z_1, z_2 \in \mathbb{T}$.

For every $k, l \geq 1$ and $r \geq 0$, we have
$$0 = \langle H_{\varphi \circ \pi}(z_1 z_2)^x (z_1^l - z_2^l), H_{\bar{z}_1 + \bar{z}_2}(z_1^k - z_2^k) \rangle_{L^2(T^2)}$$
$$= \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} z_1^m z_2^n (z_1 z_2)^x (z_1^l - z_2^l), (z_1^k - z_2^k) \rangle_{L^2(T^2)}$$
$$= \sum_{m,n \in \mathbb{Z}} \alpha_{m,n} (z_1^{m+r+l} z_2^{n+r} - z_1^{m+r} z_2^{n+r+l}), (z_1^k - z_2^k) \rangle_{L^2(T^2)}$$
$$= 2(\alpha_{-r-1, (k-l)-r} - \alpha_{-(r+l)-1, k-r}).$$

Similar argument as in the proof of $(ii) \Rightarrow (i)$ reveals that $\alpha_{m,n} = 0$, if either of $m$ and $n$ is negative, in other words, $\varphi$ is analytic.

$(v) \iff (vi)$: The implication $(v) \Rightarrow (vi)$ follows from Proposition 22. Conversely suppose that $T_s T_\varphi$ is a Toeplitz operator. Therefore applying Theorem 17 and the relation $T_s = T_s^* T_p$, we get $T_s T_\varphi = T_s^* T_p T_p T_\varphi T_s = T_s T_\varphi$. \hfill $\square$

The following is a direct consequence of the preceding theorem.

**Corollary 24.** Let $T_\psi$ be a Toeplitz operator. Then the following are equivalent:

(i) $T_\psi$ is a co-analytic Toeplitz operator;
(ii) $T_\psi$ commutes with $T_p^*$;
(iii) $T_\psi \circ (\text{Ran} T_p) \subseteq \text{Ran} T_p$;
(iv) $T_\psi^* T_\psi^*$ is a Toeplitz operator;
(v) $T_\psi^*$ commutes with $T_s$;
(vi) $T_\psi^* T_\psi$ is a Toeplitz operator.

We end this section with two facts about Toeplitz operators on the symmetrized bidisc - one is similar to the unit disc and the other is dissimilar.

**Theorem 25.** The only compact Toeplitz operator on the symmetrized bidisc is zero.

**Proof.** The proof is similar to that case of the unit disc. Let $T_\varphi$ be a compact Toeplitz operator. For every $m > n \geq 0$, let $e_{m,n} = z_1^n \bar{z}_2^n - z_1^m \bar{z}_2^m$ be the orthogonal basis of $H^2_{\text{symm}}(T^2)$. Since $T_\varphi$ is compact, $\|T_{\varphi \circ \pi} e_{m,n}\| \to 0$ as $m, n \to \infty$. Also $T_{z_1 z_2}^* T_{\varphi \circ \pi} T_{z_1 z_2} = T_{\varphi \circ \pi}$, so we have for every $r \geq 0$,
$$|\langle T_{\varphi \circ \pi} e_{m,n}, e_{k,l} \rangle| = |\langle T_{\varphi \circ \pi} e_{m+r,n+r}, e_{k+r,l+r} \rangle| \leq \sqrt{2} \|T_{\varphi \circ \pi} e_{m+r,n+r}\| \to 0 \text{ as } r \to \infty,$$
which shows that $T_{\varphi \circ \pi}$ is zero, since $m > n \geq 0$ and $k > l \geq 0$ are arbitrary. \hfill $\square$

It has been observed over the last decade that the symmetrized bidisc enjoys some one dimensional phenomena, e.g., the automorphism group of $\mathbb{G}$ is isomorphic to the automorphism group of the unit disc (Theorem 4.1 of [1]), the minimal normal boundary dilation of this domain acts on the minimal normal boundary dilation of the unit disc [1].
However, the following example shows that the Coburn Alternative, which has several useful consequences in the study of Toeplitz operators on the unit disc, fails to hold true in the symmetrized bidisc.

**Theorem 26 (The Coburn Alternative).** For a non-zero function \( \varphi \) in \( L^\infty(T) \), either \( T_\varphi \) or \( T_\varphi^* \) is injective.

See Theorem 3.3.10 of the book [22] for a proof of this theorem. To show that it fails in the case of the symmetrized bidisc, we choose the symbol to be \( \varphi(z_1, z_2) = z_1^2 \overline{z_2} + \overline{z_1}^2 z_2^2 \). Note that \( \varphi \) is in \( L^\infty_{sym}(T^2) \) and \( T_\varphi(z_1 - z_2) = 0 = T_\varphi^*(z_1 - z_2) \).

### 3. Asymptotic Toeplitz Operators and Compactness

We know that if \( T \) is a bounded operator on \( H^2(\mathbb{D}) \) such that \( T_s^n T T_p^n \) converges weakly to some operator \( B \) on \( H^2(\mathbb{D}) \), then \( B \) is a Toeplitz operator. Although the multiplication by the second component \( T_p \) of \( H^2(\mathbb{G}) \) is unitarily equivalent to \( T_z \) on a vector-valued Hardy space on the unit disc, we have seen an example which shows that an operator need not be a Toeplitz operator even if it commutes with \( T_p \).

Therefore convergence of \( T_p^n T T_p^n \) to some operator \( B \) weakly does not imply that \( B \) is a Toeplitz operator. The following lemma gives a necessary and sufficient condition when \( B \) can be Toeplitz.

**Lemma 27.** Let \( T \) and \( B \) be bounded operators on \( H^2(\mathbb{G}) \) such that \( T_p^n T T_p^n \to B \) weakly. Then \( B \) is Toeplitz operator if and only if

\[
T_p^n [T, T_s] T_p^n \to 0 \text{ weakly},
\]

where \([T, T_s]\) denotes the commutator of \( T \) and \( T_s \).

**Proof.** Note that if \( T \) and \( B \) are bounded operators on \( H^2(\mathbb{G}) \) such that \( T_p^n T T_p^n \to B \) weakly, then \( T_p^n B T_p \to B \). Suppose \( T_p^n [T, T_s] T_p^n \to 0 \) weakly. To prove that \( B \) is Toeplitz, it remains to show that \( B \) satisfies the first Brown-Halmos relation with respect to the \( \Gamma \)-isometry \((T_s, T_p)\).

\[
T_p^n B T_p = \text{w-lim} \ T_s (T_p^n T T_p^n) T_p = \text{w-lim} \ T_p^n (T_s T T_p^n) T_p = \text{w-lim} \ T_p^{n+1} T_s^n T T_p^{n+1} = \text{w-lim} \ T_p^{n+1} T T_p^{n+1} T_s = B T_s.
\]

Conversely, suppose that the weak limit \( B \) of \( T_p^n T T_p^n \) is a Toeplitz operator and hence satisfies the Brown-Halmos relations. Thus,

\[
w-\lim T_p^n (T T_s - T_s T) T_p^n = w-\lim (T_p^n T T_p^n T_s - T_s T_p^n T T_p^n) = w-\lim (T_p^n T T_p^n T_s - T_s T_p^{n-1} T T_p^{n-1} T_p) = B T_s - T_s B T_p = 0.
\]

\( \square \)

We now proceed towards characterizing the compact operators on \( H^2(\mathbb{G}) \). Let us start by recalling the analogous result for the polydisc, recently discovered in [21]. The one dimensional case was proved by Feintuch [19].
Theorem 28. A bounded operator $T$ on $H^2(\mathbb{D}^n)$ is compact if and only if $T_{zi}^{*m}T_{zi}^{m} → 0$ in norm for all $1 ≤ i, j ≤ n$, where $T_{zi}$ is the multiplication by the $i$-th co-ordinate $z_i$ on $H^2(\mathbb{D}^n)$.

Let us write this theorem from [21] in an equivalent form. For every $m = 1, 2, \ldots$, define a completely positive map $\eta_m : \mathcal{B}(H^2(\mathbb{D}^n)) \to \mathcal{B}(\oplus_{i=1}^n H^2(\mathbb{D}^n))$ by

$$\eta_m(T) = \begin{pmatrix} T_{z_1}^{*m} & 0 & \cdots & 0 \\
0 & T_{z_2}^{*m} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
n0 & 0 & \cdots & T_{z_n}^{*m}\end{pmatrix} \begin{pmatrix} T_{z_1}^{m} & T_{z_2}^{m} & \cdots & T_{z_n}^{m} \end{pmatrix}.$$ 

Then the criterion for compactness of $\eta_m$ is that $\eta_m(T) → 0$ in norm as $m → \infty$. This shows the importance of the forward shifts in characterizing the compact operators.

In case of the space $H^2_{anti}(\mathbb{D}^2)$, a typical basis element is of the form

$$e_{k,l} = (z_1 z_2)^k (z_1^* - z_2^*)$$

where $k ≥ 0$ and $l ≥ 1$.

Note that the operator $T_{z_1 z_2}$ has the property $T_{z_1 z_2} e_{k,l} = e_{k+1,l}$ for every $k ≥ 0$ and $l ≥ 1$.

Let us analyze the operator $X$ defined in [25] a bit more. Let

$$X_0 = P_\mathcal{E} X,$$

where $\mathcal{E} = \text{span} \{z_1^j - z_2^j : 1 ≤ j < \infty\} \subset H^2_{anti}(\mathbb{D}^2)$ is the coefficient space of the symmetrized bidisc, as defined in Lemma 7. Then $X_0 T_{z_1 z_2} = 0 = T_{z_1 z_2}^* X_0$ and

$$X = \sum_{n=0}^\infty T_{z_1 z_2}^n X_0 T_{z_1 z_2}^* n.$$

It is easy to see from the definition (25) of $X$ that $X$ is an isometry. We show below that $X$ is pure, i.e., $X^n f → 0$ as $n → \infty$ for every $f ∈ H^2_{anti}(\mathbb{D}^2)$. Indeed, if $f(z_1, z_2) = \sum_{k,l=1}^\infty a_{k,l} (z_1 z_2)^k (z_1^* - z_2^*)$ is in $H^2_{anti}(\mathbb{D}^2)$, then for every $\epsilon > 0$ there exist $M$ and $N$ large enough so that $\sum_{k,l ≥ M, l ≥ N} |a_{k,l}|^2 ≤ \epsilon$. Now since $X_0 T_{z_1 z_2} = 0 = T_{z_1 z_2}^* X_0$, for every $n ≥ 1$, we have

$$(X^n)^n = (X_0^n + T_{z_1 z_2} X_0^n T_{z_1 z_2} + T_{z_1 z_2}^2 X_0^n T_{z_1 z_2}^* + \cdots ) f$$

$$= (X_0^n + T_{z_1 z_2} X_0^n T_{z_1 z_2} + T_{z_1 z_2}^2 X_0^n T_{z_1 z_2}^* + \cdots + T_{z_1 z_2}^{M-1} X_0^n T_{z_1 z_2}^{M-1} ) f$$

$$+ (T_{z_1 z_2}^M X_0^n T_{z_1 z_2}^M + \cdots ) f.$$ 

Clearly, if $n ≥ N$, then the second summand in the above equation is less than $\epsilon$. Since there are only finitely many terms in the first summand and $X_0^n → 0$, we can make the first summand less than $\epsilon$ by choosing sufficiently large $n$. Hence $(X^n)^n → 0$ strongly. We have the following characterization of compact operators on $H^2(\mathbb{G})$ using the two pure isometries $Y$ and $T_p$. 


Theorem 29. Define a sequence of completely positive maps $\eta_n : \mathcal{B}(H^2(\mathbb{G})) \to \mathcal{B}(H^2(\mathbb{G}) \oplus H^2(\mathbb{G}))$ by

$$\eta_n(T) = \left( \begin{array}{c} Y^{*n} \\ T_p^{*n} \end{array} \right) T \left( \begin{array}{c} Y^n \\ T_p^n \end{array} \right), \text{ for } n = 1, 2, \ldots$$

where $Y = U^*XU$. A bounded operator $T$ on $H^2(\mathbb{G})$ is compact if and only if $\eta_n(T) \to 0$ in norm.

Proof. The necessity follows from a straightforward application of Lemma 3.1 of [21].

Conversely, suppose that $T$ is a bounded operator on $H^2(\mathbb{G})$ satisfying the convergence conditions in the statement of the theorem. We shall now find an approximation of $T$ by finite rank operators. Denote $Y_0 = U^*X_0U$. Then $Y$ has the following expression

$$Y = \sum_{n=0}^{\infty} T_p^n Y_0 T^{*n}_p.$$ 

Note that $U^*P_zU = U^*(I - T_{z1z2}T_{z1z2})U = I - T_pT^*_p$. Consider the finite rank operator

$$F_n = U^*(P_z - X_0^n X_0^n) + T_{z1z2}(P_z - X_0^n X_0^n)T_{z1z2} + \cdots + T_{z1z2}^{n-1}(P_z - X_0^n X_0^n)T_{z1z2}^{n-1}) U$$

$$= (I - T_p T^*_p - Y_0^n Y^{*n}) + T_p(I - T_p T^*_p - Y_0^n Y^{*n})T_p + \cdots + T_p^{n-1}(I - T_p T^*_p - Y_0^n Y^{*n})T_p^{n-1}$$

$$= I - T_p T^{*n} - (Y_0^n Y^{*n} + T_p Y_0^n Y^{*n} T_p + \cdots + T_p^{n-1} Y_0^n Y^{*n} T_p^{n-1}).$$

Then the operator $\tilde{F}_n = TF_n + F_nT - F_nTF_n$ is also a finite rank operator and note that $T - \tilde{F}_n = (I - F_n)T(I - F_n)$. Let $P_n$ be the projection of $H^2(\mathbb{G})$ onto the subspace generated by

$$\{T^j_p f : f \in \text{Ran}(I - T_p T^*_p), 0 \leq j \leq n - 1\}.$$ 

Note that $Y_0^{*n} + T_p Y_0^{*n} T^*_p + T_p^2 Y_0^{*n} T^*_p + \cdots + T_p^{n-1} Y_0^{*n} T_p^{n-1} = Y^{*n} P_n = P_n Y^{*n}$. Therefore

$$I - F_n = Y_0^n Y^{*n} + T_p Y_0^n Y^{*n} T_p + \cdots + T_p^{n-1} Y_0^n Y^{*n} T_p^{n-1} + T_p^n T^{*n}$$

$$= Y_0^n Y^{*n} + T_p Y_0^n T_p^* Y_0^n T_p + \cdots + T_p^{n-1} Y_0^n T_p^{n-1} Y_0^n T_p^{n-1} + T_p^n T^{*n}$$

$$= P_n Y^n Y^{*n} P_n + T_p^n T^{*n}.$$ 

The following computation shows that $\tilde{F}_n$ converges to $T$ in norm.

$$\|T - \tilde{F}_n\| = \|(I - F_n)T(I - F_n)\|$$

$$\leq \|(P_n Y^n Y^{*n} P_n + T_p^n T^{*n}) T (P_n Y^n Y^{*n} P_n + T_p^n T^{*n})\|$$

$$\leq \|Y^n T Y^{*n}\| + \|Y^n T^n T^{*n}\| + \|T^n T Y^{*n}\| + \|T^n T^n T^{*n}\|$$

$$\to 0,$$ because of the hypothesis.

Hence, $T$ is compact. \hfill \square

Definition 30. A bounded operator $T$ on $H^2(\mathbb{G})$ is called an asymptotic Toeplitz operator if $T^{*n}_p[T, T^*_p]T_p \to 0$, $T_p^{*n} T^n T^{*n}_p \to B$ and $\eta_n(T - B) \to 0$.

Theorem 31. A bounded operator $T$ on $H^2(\mathbb{G})$ is an asymptotic Toeplitz operator if and only if $T$ is the sum of a compact operator and a Toeplitz operator.
Proof. If \( T \) is an asymptotic Toeplitz operator and \( T^*nTT^n_p \) converges to \( B \), then it follows from Lemma 27 that \( B \) is a Toeplitz operator because \( T^*n[T, T_p]T^n_p \rightarrow 0 \). Also, since \( \eta_n(T - B) \rightarrow 0 \), by Theorem 29 \( T - B \) is a compact operator. Hence \( T \) is the sum of a compact operator and a Toeplitz operator.

Conversely, let \( T = K + T_\varphi \), where \( K \) is some compact operator. Then by Theorem 29 \( T^*nTT^n_p \rightarrow T_\varphi \). Since \( T_\varphi \) is Toeplitz, by Lemma 27 \( T^*n[T, T_p]T^n_p \rightarrow 0 \). And finally, since \( K \) is compact, by Theorem 29 \( \eta_n(T - T_\varphi) \rightarrow 0 \). Hence \( T \) is asymptotic Toeplitz operator. \( \square \)

Remark 32. If \( T \) is an operator such that both \( T^*nTT^n_p \) and \( Y^*nTY^n \) converge to \( T \), even then it is not necessary that \( T \) is a Toeplitz operator. For example, choose \( T = Y \). Because \( Y \) is an isometry and it commutes with \( T_p \), for every \( n \geq 0 \), \( Y^*nYY^n = Y \) and \( T^*nYT^n_p = Y \). But we have noticed that \( Y \) is not a Toeplitz operator.

4. COMMUTANT LIFTING AND DUAL TOEPLITZ OPERATORS

It is a natural generalization of the concept of Toeplitz operators to replace the multiplication by the co-ordinate multiplier by a more general isometry (in the classical case of Brown and Halmos). Moreover, depending on the domain, one can introduce a tuple of operators with a suitable property. Prunaru did it for the Euclidean ball \( \mathbb{B}_d \). The natural operator tuple to consider there is a spherical isometry, i.e., a commuting tuple \( T = (T_1, T_2, \ldots, T_d) \) of bounded operators with the property \( T_1^*T_1 + T_2^*T_2 + \cdots + T_d^*T_d = I \), its prototypical example being the tuple of co-ordinate multiplications \( T_\alpha = (T_{\alpha 1}, T_{\alpha 2}, \ldots, T_{\alpha d}) \) on the Hardy space of the Euclidean ball. Prunaru called an operator \( X \) a Toeplitz operator with respect to a given spherical isometry \( T \) if \( T_1^*XT_1 + T_2^*XT_2 + \cdots + T_d^*XT_d = X \).

Definition 33. Given a Hilbert space \( \mathcal{H} \), a \( \Gamma \)-isometry \((S, P)\) on \( \mathcal{H} \) and a bounded operator \( T \) on \( \mathcal{H} \), we say that \( T \) satisfies the Brown-Halmos relation with respect to the \( \Gamma \)-isometry \((S, P)\) (or just satisfies the Brown-Halmos relation when the pair \((S, P)\) is clear from the context) if

\[
S^*TP = TS \quad \text{and} \quad P^*TP = T.
\]

Definition 34. We say that a family \( \mathcal{F} = \{(S_\alpha, P_\alpha) : \alpha \in \Lambda\} \) of \( \Gamma \)-isometries on a Hilbert space \( \mathcal{H} \) is commuting if the union \( \cup_{\alpha \in \Lambda}\{S_\alpha, P_\alpha\} \) is a commutative set of operators.

For a commuting family \( \mathcal{F} \) of \( \Gamma \)-isometries on a Hilbert space \( \mathcal{H} \), let \( \mathcal{T}(\mathcal{F}) \) be the set of all operators \( X \in \mathcal{B}(\mathcal{H}) \) such that

\[
S_\alpha^*XP_\alpha = XS_\alpha \quad \text{and} \quad P_\alpha^*XP_\alpha = X, \quad \text{for all} \ \alpha \in \Lambda.
\]

In other words, an element of \( \mathcal{T}(\mathcal{F}) \) satisfies the Brown-Halmos condition for each \( \alpha \).

Given a family \( \mathcal{F} = \{(S_\alpha, P_\alpha) : \alpha \in \Lambda\} \) of \( \Gamma \)-isometries on a Hilbert space \( \mathcal{H} \), we say that a commuting family \( \mathcal{G} = \{(R_\alpha, U_\alpha) : \alpha \in \Lambda\} \) of \( \Gamma \)-unitaries on a Hilbert space \( \mathcal{K} \) containing \( \mathcal{H} \) extends \( \mathcal{F} \), if each pair \((R_\alpha, U_\alpha)\) is an extension of \((S_\alpha, P_\alpha)\). Moreover, \( \mathcal{G} \) is called the minimal extension of \( \mathcal{F} \), if \( \mathcal{K} \) is the smallest reducing subspace of each \( R_\alpha \) and \( U_\alpha \) containing \( \mathcal{H} \).

Remark 35. Two remarks are in order.
(1) Since the family $\mathcal{F}$ is commuting, $\mathcal{T}(\mathcal{F})$ contains $\mathcal{F}$. 

(2) Consider an $X \in \mathcal{B}(\mathcal{H})$ which commutes with each $P_\alpha$. Then $X$ is in $\mathcal{T}(\mathcal{F})$ if and only if $X$ commutes with each $S_\alpha$. Thus $\mathcal{T}(\mathcal{F})$ contains the commutant of $\mathcal{F}$.

One of the main results of this section is the following. It is similar in spirit to Theorem 1.2 of Prunaru [26] whose roots can be traced back to Section 3 of Beltita and Prunaru [7]. The difference in our theorem lies in the $S_\alpha$. We shall apply Beltita and Prunaru’s ideas to obtain simultaneous dilation of the $P_\alpha$ and then note how the representation acts on $S_\alpha$. It will be clear in course of the proof that the dilation space is no bigger than that of the simultaneous dilation of $P_\alpha$.

**Theorem 36.** Let $\mathcal{F} = \{(S_\alpha, P_\alpha) : \alpha \in \Lambda\}$ be a commuting family of $\Gamma$-isometries on a Hilbert space $\mathcal{H}$. Then

1. There exists a commuting family $\mathcal{G} = \{(R_\alpha, U_\alpha) : \alpha \in \Lambda\}$ of $\Gamma$-unitaries acting on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that $\mathcal{G}$ is the minimal extension of $\mathcal{F}$. In fact, $\mathcal{K} = \{U_{n_1}^{m_1}U_{n_2}^{m_2} \cdots U_{n_k}^{m_k} : h \in \mathcal{H}, n \in \mathbb{N} \text{ and for } 1 \leq j \leq k, n_j \in \Lambda \text{ and } m_j \in \mathbb{Z}\}$.

Moreover, any operator $X$ acting on $\mathcal{H}$ commuting with $\mathcal{F}$ if and only if $X$ has a unique norm preserving extension $Y$ acting on $\mathcal{K}$ commuting with $\mathcal{G}$.

2. An operator $X$ is in $\mathcal{T}(\mathcal{F})$ if and only if there exists an operator $Y$ in the commutant of the von-Neumann algebra generated by $\{R_\alpha, U_\alpha : \alpha \in \Lambda\}$ such that $X = P_\mathcal{H}Y|_\mathcal{H}$.

3. Let $\mathcal{C}^*(\mathcal{F})$ and $\mathcal{C}^*(\mathcal{G})$ denote the unital $\mathcal{C}^*$-algebras generated by $\{S_\alpha, P_\alpha : \alpha \in \Lambda\}$ and $\{R_\alpha, U_\alpha : \alpha \in \Lambda\}$, respectively and $\mathcal{I}(\mathcal{F})$ denote the closed ideal of $\mathcal{C}^*(\mathcal{F})$ generated by all the commutators $XY - YX$ for $X,Y \in \mathcal{C}^*(\mathcal{F}) \cap \mathcal{I}(\mathcal{F})$. Then there exists a short exact sequence

$$0 \rightarrow \mathcal{I}(\mathcal{F}) \hookrightarrow \mathcal{C}^*(\mathcal{F}) \xrightarrow{\pi_0} \mathcal{C}^*(\mathcal{G}) \rightarrow 0$$

with a completely isometric cross section, where $\pi_0 : \mathcal{C}^*(\mathcal{F}) \rightarrow \mathcal{C}^*(\mathcal{G})$ is the canonical unital $*$-homomorphism which sends the generating set $\mathcal{F}$ to the corresponding generating set $\mathcal{G}$ that is, $\pi_0(P_\alpha) = U_\alpha$ and $\pi_0(S_\alpha) = R_\alpha$ for all $\alpha \in \Lambda$.

**Proof.** A commuting family of $\Gamma$-isometries is a commuting family of isometries $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ along with the $S_\alpha$ which satisfy $S_\alpha = S_\alpha^*P_\alpha$ and $r(S_\alpha) \leq 2$. This is because of the characterization of $\Gamma$-isometries delineated in Theorem 13. The extension to a commuting family of $\Gamma$-unitaries will be achieved by an application of Stinespring’s dilation theorem of an appropriate completely positive map. To produce this map, we take recourse to the following result of Beltita and Prunaru which can be found in [26].

**Lemma 37** (Lemma 2.3 of [26]). Let $\{\Phi_\alpha : \alpha \in \Lambda\}$ be a set of commuting completely positive unital and normal mappings acting on $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then there exists a completely positive mapping $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ whose range is precisely the set

$$\{X \in \mathcal{B}(\mathcal{H}) : \Phi_\alpha(X) = X, \alpha \in \Lambda\}$$

and such that $\Phi \circ \Phi = \Phi$. 

In our context, for each $\alpha \in \Lambda$, define $\Phi_\alpha : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by
\[
\Phi_\alpha(X) = P_\alpha^*X P_\alpha.
\]
Then the family $\{\Phi_\alpha\}_{\alpha \in \Lambda}$ satisfies the hypotheses of the lemma above. Therefore there exists a complete positive mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\Phi \circ \Phi = \Phi$,
\[
\text{Ran} \Phi = \{X \in \mathcal{B}(\mathcal{H}) : \Phi_\alpha(X) = P_\alpha^*X P_\alpha = X, \text{ for all } \alpha \in \Lambda\};
\]
and in particular, $\Phi(X) = X$ for all $X \in \mathcal{T}(\mathcal{P})$ where
\[
\mathcal{T}(\mathcal{P}) = \{X : P_\alpha^*X P_\alpha = X, \text{ for all } \alpha \in \Lambda\}.
\]
Also since $\Phi$ is an idempotent unital completely positive map it follows from a well-known result of \cite{14} that
\[
(4.2) \quad \Phi(\Phi(X)Y) = \Phi(X\Phi(Y)) = \Phi(\Phi(X)\Phi(Y))
\]
for all $X, Y \in \mathcal{B}(\mathcal{H})$. Let $C^*(\mathcal{T}(\mathcal{P}))$ denote the $C^*$-algebra generated by $\mathcal{T}(\mathcal{P})$ and $\Phi_0$ denote the restriction of $\Phi$ to $C^*(\mathcal{T}(\mathcal{P}))$. Let $\pi : C^*(\mathcal{T}(\mathcal{P})) \to \mathcal{B}(\mathcal{K})$ denote the minimal Stinespring dilation of $\Phi_0$ so that $\Phi_0(X) = V^*\pi(X)V$ for some isometry $V : \mathcal{H} \to \mathcal{K}$ and for all $X \in \mathcal{B}(\mathcal{H})$. It follows from \cite{12} that $\text{Ker} \Phi_0$ is an ideal of $C^*(\mathcal{T}(\mathcal{P}))$ and therefore $\text{Ker} \Phi_0 = \text{Ker} \pi$ and the mapping $\rho : \pi(C^*(\mathcal{T}(\mathcal{P}))) \to \mathcal{B}(\mathcal{H})$ defined by $\rho(\pi(X)) = V^*\pi(X)V$ for $X \in C^*(\mathcal{T}(\mathcal{P}))$ is a complete isometry such that $\pi \circ \rho = \text{id}_{\pi(C^*(\mathcal{T}(\mathcal{P})))}$ and $\text{Ran} \rho = \text{Ran} \Phi$. We denote $\pi(P_\alpha)$ by $U_\alpha$. Below we list few properties of the representation $\pi$ which can be obtained from the proof of Theorem 1.2 of Prunaru \cite{26} applied to $\mathcal{P}$:

\textbf{(P1)} The commuting family of unitaries $\mathcal{U} = \{U_\alpha = \pi(P_\alpha) : \alpha \in \Lambda\}$ is a minimal unitary extension of the family of isometries $\mathcal{P}$, i.e.,
\[
P_\alpha = V^*U_\alpha V \text{ and } U_\alpha(V\mathcal{H}) \subseteq V\mathcal{H}
\]
for all $\alpha \in \Lambda$ and $\mathcal{K}$ is the minimal reducing subspace containing $V\mathcal{H}$ for the family $\mathcal{U}$.

\textbf{(P2)} Any operator $X \in \mathcal{B}(\mathcal{H})$ belongs to the commutant of $\mathcal{P}$ if and only if there exists a unique norm preserving extension of $X$ in the commutant of $\mathcal{U}$ that is, there exists a unique operator $\widehat{X}$ in the commutant of $\mathcal{U}$ which leaves $V\mathcal{H}$ invariant, $X = V^*\widehat{X}V$ and $\|X\| = \|\widehat{X}\|$. In fact, it turns out that $\widehat{X} = \pi(X)$.

For the rest of the proof we identify $\mathcal{H}$ with $V\mathcal{H}$ and view $\mathcal{H}$ as a subspace of $\mathcal{K}$. We now turn our attention to $S_\alpha$ and denote $R_\alpha = \pi(S_\alpha)$ for all $\alpha \in \Lambda$. Since $S_\alpha$ belongs to the commutant of $\mathcal{P}$, by property (P2) above, we get $R_\alpha$ to be an extension of $S_\alpha$ and $\|R_\alpha\| = \|S_\alpha\|$ for all $\alpha \in \Lambda$. Since $S_\alpha = S_\alpha^*P_\alpha$, we have $R_\alpha = R_\alpha^*U_\alpha$ for each $\alpha \in \Lambda$. Hence by part (3) of Theorem \cite{12} we know $(R_\alpha, U_\alpha)$ to be a $\Gamma$-unitary for each $\alpha$. It is now clear from property (P1) that the commuting family of $\Gamma$-unitaries $\mathcal{G} = \{(R_\alpha, U_\alpha) : \alpha \in \Lambda\}$ is a minimal normal extension of the commuting family of $\Gamma$-isometries $\mathcal{F}$. For the last part of part (1), note that if $X$ commutes with $\mathcal{F}$, then $X$ belongs to the commutant of $\mathcal{P}$. Therefore again by property (P2), $\pi(X)$ is the unique norm preserving extension of $X$ in the commutant of $\mathcal{U}$. Moreover, $\pi(X)$ belongs to the commutant of $\mathcal{G}$ as $X$ commutes with $S_\alpha$ for all $\alpha \in \Lambda$. This proves part (1) of the theorem.
To prove part (2), let us first suppose that \( X \) is in \( \mathcal{T}(\mathcal{F}) \), which means that \( X \) satisfies \( S_\alpha X P_\alpha = X S_\alpha \) and \( P_\alpha X P_\alpha = X \), for each \( \alpha \in \Lambda \). Let \( Y = \pi(X) \). Then it follows that \( X \) is the compression of \( Y \) to \( \mathcal{H} \). Now applying \( \pi \) on the above equations we get \( R_\alpha^* Y U_\alpha = Y R_\alpha \) and \( U_\alpha^* Y U_\alpha = Y \) for all \( \alpha \in \Lambda \). This implies that \( Y \) commutes with both \( R_\alpha \) and \( U_\alpha \), for each \( \alpha \in \Lambda \). This proves one direction of part (2). For the converse part, note that for each \( \alpha \in \Lambda \), \( R_\alpha \), \( U_\alpha \) and \( Y \) have the following matrix representation with respect to the decomposition \( \mathcal{H} \oplus (\mathcal{K} \oplus \mathcal{H}) \)

\[
\begin{pmatrix}
S_\alpha & * \\
0 & *
\end{pmatrix}
\begin{pmatrix}
P_\alpha & * \\
0 & *
\end{pmatrix}
\begin{pmatrix}
0 & * \\
X & *
\end{pmatrix},
\]

respectively and they satisfy \( R_\alpha^* Y U_\alpha = R_\alpha Y \) and \( U_\alpha^* Y U_\alpha = Y \). Now it follows from a simple block matrix computation that \( X \), the compression of \( Y \) to \( \mathcal{H} \), is in \( \mathcal{T}(\mathcal{F}) \).

To prove part (3), we first note that the representation \( \pi_0 \) in the statement of the theorem is actually the restriction of \( \pi \) to \( \mathcal{C}^*(\mathcal{F}) \) as the representation \( \pi \) also maps the generating set \( \mathcal{F} \) of \( \mathcal{C}^*(\mathcal{F}) \) to the generating set \( \mathcal{G} \) of \( \mathcal{C}^*(\mathcal{G}) \). Since \( \pi_0(\mathcal{F}) = \mathcal{G} \), range of \( \pi_0 \) is \( \mathcal{C}^*(\mathcal{G}) \). Therefore to prove that the following sequence

\[
0 \to \mathcal{I}(\mathcal{F}) \leftarrow \mathcal{C}^*(\mathcal{F}) \xrightarrow{\pi_0} \mathcal{C}^*(\mathcal{G}) \to 0
\]

is a short exact sequence, all we need to show is that \( \ker \pi_0 = \mathcal{I}(\mathcal{F}) \). Since \( \pi_0(\mathcal{C}^*(\mathcal{F})) \) is commutative, we have \( XY - YX \) in the kernel of \( \pi_0 \), for any \( X, Y \in \mathcal{C}^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F}) \). Hence \( \mathcal{I}(\mathcal{F}) \subseteq \ker \pi_0 \). To prove the other inclusion, let us agree to denote by \( \mathcal{F}^* \), for a family \( \mathcal{F} \) of operators, the adjoints of members of \( \mathcal{F} \). Let \( Z_1 \) be a finite product of members of \( \mathcal{F}^* \) and \( Z_2 \) be a finite product of members of \( \mathcal{F} \) and call \( Z = Z_1Z_2 \). Then by the commutativity of the family \( \mathcal{F} \), we have for each \( \alpha \in \Lambda \), \( \Phi_\alpha(Z) = Z \) and hence \( \Phi_0(Z) = Z \), where \( \Phi_0 \) and \( \Phi_\alpha \)'s are as in the proof of part (1). Note that \( \Phi_0(Z) = P_{\mathcal{H}} \pi_0(Z)|_{\mathcal{H}} \), for every \( Z \in \mathcal{C}^*(\mathcal{F}) \). Now let \( Z \) be any arbitrary finite product of members from \( \mathcal{F} \) and \( \mathcal{F}^* \). Since \( \pi_0(\mathcal{F}) = \mathcal{G} \), which is a family of normal operators, we obtain, by Fuglede-Putnam’s theorem that, action of \( \Phi_0 \) on \( Z \) has all the members from \( \mathcal{F}^* \) at the left and all the members from \( \mathcal{F} \) at the right. It follows from \( \ker \pi = \ker \Phi \) and \( \Phi \) is idempotent that \( \ker \pi_0 = \{ X - \Phi_0(X) : X \in \mathcal{C}^*(\mathcal{F}) \} \). Also, because of the above description of \( \Phi_0(X) \), if \( X \) is a finite product of elements from \( \mathcal{F} \) and \( \mathcal{F}^* \) then \( X - \Phi_0(X) \) belongs to the ideal generated by all the commutators \( XY - YX \), where \( X, Y \in \mathcal{C}^*(\mathcal{F}) \cap \mathcal{T}(\mathcal{F}) \). This shows that \( \ker \pi_0 = \mathcal{I}(\mathcal{F}) \). In order to find a completely isometric cross section, recall the completely isometric map \( \rho : \pi(\mathcal{C}^*(\mathcal{T}(\mathcal{P}))) \to \mathcal{B}(\mathcal{H}) \) such that \( \pi \circ \rho = id_{\pi(\mathcal{C}^*(\mathcal{T}(\mathcal{P})))} \). Set \( \rho_0 := \rho|_{\pi(\mathcal{C}^*(\mathcal{F}))} \). Then by the definition of \( \rho \) it follows that \( \text{Ran} \rho_0 \subseteq \mathcal{C}^*(\mathcal{F}) \) and therefore is a completely isometric cross section. This completes the proof of the theorem.

We separate out a corollary which has the flavour of a commutant lifting theorem. This is a special case of part (2) above. The reason for writing this rather simple special case separately is that it will play a significant role in the study of dual Toeplitz operators below.

**Corollary 38.** Let \((S, P)\) on \( \mathcal{H} \) be a \( \Gamma \)-isometry and \((R, U)\) on \( \mathcal{K} \) be its minimal \( \Gamma \)-unitary extension. An operator \( X \) satisfies the Brown-Halmos relations with respect to \((S, P)\) if and
only if there exists an operator $Y$ in the commutant of the von-Neumann algebra generated by \{R, U\} such that $X = P_d Y |_{\mathcal{H}}$.

A remark on the matrix representation of the operator $Y$ in Corollary 38 is in order.

**Remark 39.** It should be noted that the operator $Y$ in Corollary 38 need neither be an extension nor a co-extension of the operator $X$, in general. For example, choose the $\Gamma$-isometry to be $(T_s, T_p)$. Then by Theorem 17, any operator that satisfies the Brown-Halmos relations with respect to this $\Gamma$-isometry is a Toeplitz operator with some symbol $\varphi \in L^\infty(b\Gamma)$ and $Y$, by Theorem 9, would be $M_{\varphi}$, which has the matrix representation as in (4.3).

Dual Toeplitz operators have been studied for a while on the Bergman space of the unit disc $D$ in [27] and on the Hardy space of the Euclidean ball $B_d$ in [17]. In our setting, consider the space $H^2(G)^\perp = L^2(b\Gamma) \ominus H^2(G)$. Let $(I - Pr)$ be the projection of $L^2(b\Gamma)$ onto $H^2(G)^\perp$. If $\varphi \in L^\infty(b\Gamma)$, define the dual Toeplitz operator on $H^2(G)^\perp$ by $DT_{\varphi} = (I - Pr)M_{\varphi}|_{H^2(G)^\perp}$. With respect to the decomposition above,

\begin{equation}
M_{\varphi} = \begin{pmatrix}
T_{\varphi} & H_{\varphi}^* \\
H_{\varphi} & DT_{\varphi}
\end{pmatrix}.
\end{equation}

**Lemma 40.** The special pair $D = (DT_s, DT_p)$ is a $\Gamma$-isometry with $(M_s, M_p)$ as its minimal $\Gamma$-unitary extension.

**Proof.** It is a $\Gamma$-isometry because it is the restriction of the $\Gamma$-unitary $(\bar{M}_s, \bar{M}_p)$ to the space $H^2(G)^\perp$. And this extension is minimal because $M_p$ is the minimal unitary extension of $DT_p$. \hfill $\square$

**Theorem 41.** A bounded operator $T$ on $H^2(G)^\perp$ is a dual Toeplitz operator if and only if it satisfies the Brown-Halmos relations with respect to $D$.

**Proof.** The easier part is showing that every dual Toeplitz operator on $H^2(G)^\perp$ satisfies the Brown-Halmos relations with respect to $(DT_s, DT_p)$. It follows from the following identities

$M_s^* M_{\varphi} M_p = M_{\varphi} M_s$ and $M_p^* M_{\varphi} M_p = M_{\varphi}$ for every $\varphi \in L^\infty(b\Gamma)$

and from the $2 \times 2$ matrix representations of the operators in concern. For the converse, let $T$ on $H^2(G)^\perp$ satisfy the Brown-Halmos relations with respect to the $\Gamma$-isometry $(DT_s, DT_p)$. By Corollary 38 and Theorem 9 there is a $\varphi \in L^\infty(b\Gamma)$ such that $T$ is the compression of $M_{\varphi}$ to $H^2(G)^\perp$. \hfill $\square$

5. **Epilogue**

The **fundamental operator** $F$ of a $\Gamma$-contraction $(S, P)$ is the unique bounded operator on $D_F$ that satisfies the equation

$S - S^* P = D_F F D_F$. 

Its existence was discovered in [8]. Apart from the elegance it brings in characterizing Γ-contractions (Theorem 4.4 in [8]), it has proved to be immensely useful, see for example Theorem 4.4 in [9] and the Abstract of [25]. The fundamental operator appears in this paper too, establishing that it is an essential part of the study of operator theory in the symmetrized bidisc. The operator $Y$ that appeared in Example 16 and also in Theorem 29 while characterizing compact operators on $H^2(\mathbb{G})$ is the adjoint of the fundamental operator of the Γ-coisometry ($T_\ast^s, T_\ast^p$). This follows from [11] and was discussed in detail in Section 5 of [11].

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