Toward Thermalization in Heavy Ion Collisions at Strong Coupling

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\text{ABSTRACT:} We find the trapped surface for a collision of two sourceless shock waves in AdS\textsubscript{5} and conclude that such collisions always lead to a creation of a black hole in the bulk. Due to holographic correspondence, in the boundary gauge theory this result proves that a thermalized medium (quark-gluon plasma) is produced in heavy ion collisions at strong coupling (albeit in $\mathcal{N} = 4$ super-Yang-Mills theory). We present new evidence supporting the analytic estimate for the time of thermalization that exists in the literature and find that thermalization time is parametrically much shorter than the time of shock wave stopping, indicating that our result may be relevant for description of heavy ion collision experiments.

\text{KEYWORDS:} AdS/CFT Correspondence, Heavy Ion Collisions, Shock Waves, Trapped Surface
1. Introduction

The problem of understanding the physics behind thermalization of the medium produced in ultrarelativistic heavy ion collisions is one of the main open questions in heavy ion theory. It has become especially important in recent years after hydrodynamic simulations indicated that a very short thermalization time of the order of 1 fm/c is required to describe RHIC data [1,2]. Lately the problem of thermalization has been studied in the strong coupling framework of the Anti-de Sitter space/conformal field theory (AdS/CFT) correspondence [3–5] with the goal of learning about the dynamics of the strongly-coupled QCD medium by studying the strongly coupled medium in $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory [6–14].

The 5-dimensional gravity dual of a shock wave (ultrarelativistic nucleus) in our 4-dimensional space-time was first constructed in [6]. The AdS$_5$ shock wave metrics are shown below in Eqs. (2.1) and (2.2). These metrics are solutions of Einstein equations in the AdS$_5$ bulk without sources. In four dimensions they correspond to nuclei of infinite transverse extent with a uniform distribution of matter in the transverse plane. Collisions of shock waves in Eqs. (2.1) and (2.2) have been studied in [7,9,10,15] with the goal of explicitly constructing the metric after the collision. While the shock wave collisions in AdS$_3$ allowed for an exact solution of the problem [7], it turned out to be significantly harder in AdS$_5$, allowing only for a perturbative solution of Einstein equations in graviton
exchanges [9, 10]. While all-order graviton exchanges with one of the shock waves (corresponding to proton-nucleus collisions) were resummed exactly in [15], the full problem of nucleus-nucleus collisions involving all-order graviton exchanges with both nuclei still remains unsolved in AdS$_5$.

In [8] an alternative to the exact solution of Einstein equations was proposed: the authors of [8] constructed a trapped surface for a head-on collision of two shock waves with sources in the AdS bulk following [16, 17]. Sources in the bulk lead to the nuclei in the boundary theory having some transverse coordinate dependence in their matter distributions. Formation of a trapped surface before the collision indicates that a black hole will be formed in the future, after the collision. Thus the authors of [8] have proven that black hole is formed in a collision of two shock waves with point sources in the bulk. Generalizations of [8] to the case of nuclear collisions with non-zero impact parameter were presented in [11, 12]. Also a trapped surface was found in [11] for an important case of collision of two shock waves with extended (not point-like) bulk sources.

However, the exact implications of a source in the bulk for the boundary theory are still not entirely clear. The same energy-momentum tensor of the boundary theory can be given by metrics with extended sources at different bulk locations. It is possible that the sources would manifest themselves in fluctuations of the metric, but more research is needed to understand which bulk source gives the “right” fluctuations most accurately describing real-life heavy ion collisions. In [11] it was suggested that the position of the source in the bulk is related to the saturation scale of the shock wave. Initial steps on determination of saturation scale in shock waves were done in [18–20]. It appears more work is needed to clarify the complete impact of the bulk source on the boundary gauge theory.

Interestingly the trapped surfaces found in [8, 11, 12] are always formed around the source in the bulk. One may therefore wonder whether the source is required for the trapped surface to form. No trapped surface analysis has been performed to date for the sourceless shock waves of Eqs. (2.1) and (2.2) to answer this question.

Here we perform a trapped surface analysis for a collision of two sourceless shock waves from Eqs. (2.1) and (2.2). We first consider the trapped surface obtained in [11] for a collision of two shock waves with extended sources in the bulk, and then take the limit in which the sources are moved to the deep infrared (IR) while keeping the energies of the shock waves in the boundary gauge theory fixed. Interestingly enough, the trapped surface does not disappear in this source-free limit, its lower boundary remains at finite value of the 5th dimension coordinate $z$ with its finite area giving a finite expression for the produced entropy. We argue that collisions of two shock waves with sources in the deep IR (at $z = \infty$) are indistinguishable from collisions of two shock waves without bulk sources by performing a perturbative solution of Einstein equations for the shock wave with sources in the bulk and taking the sources to $z = \infty$. We also note that the trapped surface which remains after we send the sources to $z = \infty$ does not depend on how the limit was taken and on which sources were sent to infinity: the remaining trapped surface is the same for extended and point-like sources sent to the IR.
We therefore conclude that a collisions of two sourceless shock waves in AdS$_5$ leads to creation of a black hole in the bulk. The absence of bulk sources leaves no uncertainty in the interpretation of the physics and makes application of AdS/CFT correspondence better justified. For the boundary theory this result proves that thermalized quark-gluon plasma is produced in heavy ion collisions at strong coupling.

The paper is structured as follows. In Sect. 2 we present the problem at hand and describe how the limit of sending the sources to the IR should be taken without changing the bulk physics. In Sect. 3 we present a lowest-order perturbative solution of Einstein equations for a collision of two shock waves with sources along the lines of a similar calculation for the sourceless shock waves in [10]. We take the limit of the shock waves sources going to the IR and show that our solution exactly maps onto the metric produced in a collision of two sourceless shock waves found in [10]. This provides a strong argument that the shock waves with sources at $z = \infty$ collide in the same way as the shocks without any sources. In Sect. 4 we perform the trapped surface analysis and demonstrate that the trapped surface does not disappear when the sources are send to the deep IR. Thus we obtain the trapped surface for the collision of two sourceless shock waves. In Sect. 5 we conclude by presenting a guess for the thermalization time inspired by our analysis (see also [9]).

We argue that thermalization proper time is likely to be parametrically shorter than the light-cone stopping time for shock waves found in [10,15], which indicates that our conclusions may be applied to real-life heavy ion collisions at least at the qualitative level. We note however that the numbers generated by our approximate thermalization time estimate are too short to describe RHIC physics.

2. The Problem

High energy heavy ion collision can be realistically modeled by a collision of two ultrarelativistic shock waves. In [6], using the holographic correspondence [21], the geometry in AdS$_5$ dual to each one of the nuclei in the boundary theory is given by the following metric

$$ds^2 = \frac{L^2}{z^2} \left\{ -2 dx^+ dx^- + t_1(x^-) z^4 dx^- + dx^2 + dz^2 \right\}$$

(2.1)

for nucleus 1 and by

$$ds^2 = \frac{L^2}{z^2} \left\{ -2 dx^+ dx^- + t_2(x^+) z^4 dx^+ + dx^2 + dz^2 \right\}$$

(2.2)

for nucleus 2. Here $dx^2 = (dx^1)^2 + (dx^2)^2$ is the transverse metric and $x^\pm = (x^0 \pm x^3)/\sqrt{2}$ where $x^3$ is the collision axis. $L$ is the radius of S$_5$ and $z$ is the coordinate describing the 5th dimension with the boundary of AdS$_5$ at $z = 0$. We have also defined

$$t_1(x^-) \equiv \frac{2}{N_c^2} \langle T_{1--}(x^-) \rangle, \quad t_2(x^+) \equiv \frac{2}{N_c^2} \langle T_{2++}(x^+) \rangle$$

(2.3)
in accordance with the prescription of holographic renormalization [21]. Here \( \langle T_{1-}(x^-) \rangle \) and \( \langle T_{2+}(x^+) \rangle \) are the energy-momentum tensors of the two shock waves in the gauge theory. We assume that the nuclei are so large and homogeneous that one can neglect transverse coordinate dependence in \( \langle T_{1-}(x^-) \rangle \) and \( \langle T_{2+}(x^+) \rangle \). Following [6] we take

\[
\langle T_{1-}(x^-) \rangle = \mu_1 \delta(x^-), \quad \langle T_{2+}(x^+) \rangle = \mu_2 \delta(x^+). \tag{2.4}
\]

For simplicity we also put \( \mu_1 = \mu_2 = \mu \).

The metrics in Eqs. (2.1) and (2.2) solve Einstein equations in the empty AdS\(_5\) space:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{6}{L^2} g_{\mu\nu} = 0. \tag{2.5}
\]

However, as we will see below, it is hard to perform the trapped surface analysis with the sourceless shock waves. To this end, as we have mentioned above, it will be more convenient to represent sourceless shock waves as limiting cases of the shock waves with sources, when the sources are sent to \( z = \infty \) while keeping energy-momentum tensor of the nuclei in the boundary theory intact.

We therefore need to construct shock waves with sources in the bulk, which we will do following [8, 11]. We need to satisfy Einstein equations in AdS\(_5\) with sources in the bulk

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{6}{L^2} g_{\mu\nu} = 8 \pi G_5 J_{\mu\nu} \tag{2.6}
\]

where \( J_{\mu\nu} \) is the energy momentum tensor for bulk sources. We will not specify what fields contribute to create non-zero \( J_{\mu\nu} \) in the bulk: as for us the source will serve as an IR regulator we do not need to know the origin of \( J_{\mu\nu} \) in detail. The 5-dimensional Newton constant is

\[
G_5 = \frac{\pi L^3}{2 N_c^2}. \tag{2.7}
\]

Eq. (2.6) can be rewritten as

\[
R_{\mu\nu} + \frac{4}{L^2} g_{\mu\nu} = 8 \pi G_5 \left( J_{\mu\nu} - \frac{1}{3} g_{\mu\nu} J \right) \tag{2.8}
\]

with

\[
J = J_{\mu} = J_{\mu\nu} g^{\nu}. \tag{2.9}
\]

Following [11] for one shock wave we will consider a source without any transverse \((x^1, x^2)\) coordinate dependence, with the only non-zero component of the energy-momentum tensor

\[
J_{\alpha}^{(0)} = \frac{E}{z_0 L} \delta(x^-) \delta(z - z_0). \tag{2.10}
\]
The source is located at $z = z_0$ and $x^- = 0$, spans the transverse directions and moves along the $x^+$ axis. $E$ is a yet unspecified parameter with dimension of energy. To find the metric of the shock wave satisfying Eq. (2.8) with the source (2.10) we look for it in the following form generalizing Eq. (2.1)

$$ds^2 = \frac{L^2}{z^2} \{ -2 dx^+ dx^- + \phi(z) \delta(x^-) dx^2 + dx_\perp^2 + dz^2 \}.$$  \hspace{1cm} (2.11)

Plugging Eqs. (2.11) and (2.10) into Eq. (2.8) we get the following equation for the “$-$” component of Einstein equations [11]

$$\frac{3}{2 z} \phi'(z) - \frac{1}{2} \phi''(z) = 8 \pi G_5 \frac{E}{z_0 L} \delta(z - z_0).$$  \hspace{1cm} (2.12)

When solving this equation we require that $\phi(z) \rightarrow 0$ as $z \rightarrow 0$ and that $\phi(z)$ is regular as $z \rightarrow +\infty$. The latter condition is needed to avoid the singular behavior of metrics (2.1) and (2.2) in the IR. While the singularity of metrics (2.1) and (2.2) does not affect curvature invariants and is thus not unphysical, it is easier to perform trapped surface analysis which we intend to do below on a metric with is regular in the IR.

Solving Eq. (2.12) with the boundary condition that $\phi(z) \rightarrow 0$ as $z \rightarrow 0$ and $\phi(z)$ is regular at $z \rightarrow +\infty$ yields [11]

$$\phi(z) = \frac{4 \pi G_5 E}{L} \begin{cases} \frac{z^4}{z_0^4}, & z \leq z_0 \\ 1, & z > z_0. \end{cases}$$  \hspace{1cm} (2.13)

Eqs. (2.13) and (2.11) give us the metric of a single shock wave with the bulk source (2.10).

Using holographic renormalization [21] (see e.g. Eq. (2.3) above) we conclude that the energy-momentum tensor corresponding to the metric (2.11) has only one non-zero component [11]

$$\langle T_{--} \rangle = \frac{L^3}{4 \pi G_5} \delta(x^-) \lim_{z \rightarrow 0} \frac{\phi(z)}{z^4} = \frac{E L^2}{z_0^4} \delta(x^-).$$  \hspace{1cm} (2.14)

It is clear that this energy-momentum tensor would be the same as for the sourceless shock wave (2.1) given by Eq. (2.4) if we identify

$$\mu = \frac{E L^2}{z_0^4}$$  \hspace{1cm} (2.15)

obtaining

$$\langle T_{--} \rangle = \mu \delta(x^-).$$  \hspace{1cm} (2.16)

The difference between the metrics for the shock wave with source in Eqs. (2.13) and (2.11) and the sourceless shock wave in Eq. (2.1) is that the source regulates the metric in the IR. It is
important to note that if we take \( z_0 \to \infty \) limit of the metric in Eqs. (2.11) and (2.13) keeping \( E/z_0^4 \) and therefore \( \mu \) in Eq. (2.13) fixed we would recover the metric in Eq. (2.1) without modifying the energy-momentum tensor of the gauge theory given by (2.10). At any finite \( z \) the metric of Eqs. (2.11) and (2.13) becomes equivalent to (2.1) in this limit, which sends the source at \( z_0 \) to the IR infinity. The question arises whether the metric (2.1) is equivalent to the \( z_0 \to \infty, E/z_0^4 = \text{const} \) limit of the metric in Eqs. (2.11) and (2.13). In other words, is having the sources at infinity identical to having no sources at all?

We are interested in the answer to this question in the context of collisions of two shock waves. The question then becomes whether colliding shock waves from Eqs. (2.1) and (2.2) are identical to colliding the shock wave in Eqs. (2.11) and (2.13) with its counterpart with \( x^+ \leftrightarrow x^- \) in the limit \( z_0 \to \infty, E/z_0^4 = \text{const} \) of the resulting post-collision metric?

The intuitive answer to the this question is “yes”. Indeed it is highly unlikely that sources at \( z_0 = \infty \) would affect any physics at finite \( z \). Even in empty AdS\(_5\) space light propagates with velocity 1 along the \( z \)-direction. It would take light an infinite time to travel to any finite \( z \) from \( z_0 = \infty \) after the collision. The metric modification in the collision is only likely to lower the light velocity in the \( z \)-direction: in the “extreme” case when a black hole is created no signal from \( z = \infty \) would be able to propagate outside of the horizon. Even more minor modifications of the metric are likely to only change the speed of light in \( z \)-direction leaving it finite and not changing the above arguments. Hence any modification of sources at \( z_0 = \infty \) in the collision is not going to affect the physics at finite \( z \). Hence the collision of two shock waves with sources at \( z_0 = \infty \) should be indistinguishable from the collision of two sourceless shock waves in Eqs. (2.1) and (2.2).

One may also think of a source at \( z_0 \) as providing an (externally imposed) infrared cutoff \( 1/z_0 \) on the transverse momenta \( k_T \) of the partons inside the shock wave in the boundary gauge theory (see [11]). With this interpretation the limit of \( z_0 \to \infty, E/z_0^4 = \text{const} \) can be interpreted in the boundary theory as removing the IR cutoff on the transverse momenta of the partons while keeping the energy of the shock wave fixed. The shock waves without sources would then correspond to nuclei without an ad hoc IR cutoff on the transverse momenta of their partons in the boundary theory. Hence, from the standpoint of the boundary theory, the \( z_0 \to \infty \) limit imposed on the four-dimensional shock waves dual to the shocks with sources in the bulk would simply remove the IR cutoff on partons’ \( k_T \). This would make the boundary theory shock waves identical to those dual to the sourceless shock waves in the bulk. Therefore, with the IR \( k_T \)-cutoff interpretation of \( 1/z_0 \) [11] the \( z_0 \to \infty \) limit also appears to be a justified way of obtaining duals of sourceless bulk shock waves in the boundary theory.

To verify the above arguments we will perform a perturbative solution of Einstein equations for a collision of two shock waves with sources in the next Section. We will explicitly demonstrate that taking the \( z_0 \to \infty, E/z_0^4 = \text{const} \) limit of the obtained metric produced in the collision would simply reduce it to the metric produced in the collision of two sourceless shock waves found previously in [9,10], thus substantiating our intuitive argument above.
3. Perturbative Solution of Einstein Equations for Colliding Shock Waves with Bulk Sources

Consider a collision of two shock waves with sources like the one given in Eq. (2.10). The general metric for such a collision could be written as

\[ ds^2 = \frac{L^2}{z^2} \left\{ -\left[2 + g(x^+, x^-, z)\right] \, dx^+ \, dx^- + \left[\phi(z) \, \delta(x^-) + f(x^+, x^-, z)\right] \, dx^-^2 \right. \\
+ \left. \left[\phi(z) \, \delta(x^+) + \tilde{f}(x^+, x^-, z)\right] \, dx^+^2 + \left[1 + h(x^+, x^-, z)\right] \, dx^2_\perp + dz^2\right\}. \]  (3.1)

The functions \( f, \tilde{f}, g, \) and \( h \) are non-zero only for \( x^+ \geq 0, x^- \geq 0 \). Before the collision (for \( x^- < 0 \) and \( x^+ < 0 \)) the superposition of the metrics of colliding shocks (the terms with \( \phi \)'s above) solves Einstein equations (2.8) exactly.

We will follow [9,10,15] and find the functions \( f, \tilde{f}, g, \) and \( h \) perturbatively at the lowest order treating the shock waves as perturbations of the empty AdS_5 space. As \( \phi(z) \sim \mu \) one can argue that \( f, \tilde{f}, g, \) and \( h \) start at order \( \mu^2 \) [10,15]. Our strategy is to expand Einstein equations to the order linear in \( f, \tilde{f}, g, \) and \( h \) and quadratic in \( \phi \). This is the same procedure as used in [10,15] for a collision of two sourceless shock waves.

The main difference in the case at hand is that the shock waves now have sources. The energy-momentum tensors of the sources, given by the following non-vanishing components before the collision (order \( \mu \), see Eqs. (2.15 and (2.10))

\[ J^{(0)}_{-} = \mu \frac{z_0^3}{L^3} \delta(x^-) \delta(z - z_0), \quad J^{(0)}_{++} = \mu \frac{z_0^3}{L^3} \delta(x^+) \delta(z - z_0), \]  (3.2)

get modified in the collision. In principle to understand modifications of the bulk source one needs to know the field content of the source and the corresponding equations of motion for the fields. However, it turns out that this is not really necessary. Following a similar procedure for perturbative construction of classical Yang-Mills fields in nuclear collisions [22] we note that Einstein equations (2.6) imply

\[ \nabla_\mu J^{\mu\nu} = 0 \]  (3.3)

where \( \nabla_\mu \) is the covariant derivative. Imposing causality and using Eq. (3.3) along with Einstein equations one can perturbatively construct the bulk energy-momentum tensor order-by-order in \( \mu \). Using the symmetries of the problem one can argue that it is unlikely that colliding sources would recoil in the transverse or \( z \) directions. This limits the non-zero contributions to the bulk energy-momentum tensor to \( J_{++}, J_{--} \) and \( J_{+-} = J_{-+} \). Note that to find \( J^{\mu\nu} \) at order \( \mu^2 \) one only need the metric (3.1) at order \( \mu \). This means one does not yet need to know the functions \( f, \tilde{f}, g, \)
and \( h \). It is then not too hard to infer the sources up to order \( \mu^2 \): the non-vanishing components of the bulk energy-momentum tensor are

\[
J_{++} = \mu \frac{z^3}{L^3} \delta(z - z_0) \left[ \delta(x^+) + \frac{1}{2} \theta(x^-) \delta'(x^-) [z \phi'(z) - \phi(z)] + \ldots \right]
\]

\[
J_{--} = \mu \frac{z^3}{L^3} \delta(z - z_0) \left[ \delta(x^-) + \frac{1}{2} \theta(x^+) \delta'(x^+) [z \phi'(z) - \phi(z)] + \ldots \right]
\]

\[
J_{+-} = J_{-+} = -\mu \frac{z^3}{L^3} \delta(z - z_0) \delta(x^+) \delta(x^-) \left[ \phi(z) + \frac{1}{2} z \phi'(z) \right] + \ldots.
\]

Plugging Eqs. (3.1) and (3.4) into (2.8) and expanding the result in powers of \( \mu \) we obtain at order \( \mu^2 \) the following expressions for the \( \perp \perp \) and the \( \perp \perp \) components of Einstein equations

\[
(\perp \perp) \quad g_z + 5 h_z - z h_{zz} + 2 z h_{x^x} = 2 \delta(x^+) \delta(x^-) \phi(z) \phi'(z)
\]

\[- \frac{16 \pi G_5}{3} \frac{z_0^5}{L^3} \mu \delta(x^+) \delta(x^-) (z - z_0) \phi'(z) \]

\[
(\perp \perp) \quad g_z + 2 h_z - z g_{zz} - 2 z h_z = - \delta(x^+) \delta(x^-) [2 \phi(z) \phi'(z) + (\phi'(z))^2 + 2 z \phi(z) \phi''(z)]
\]

\[- \frac{16 \pi G_5}{3} \frac{z_0^5}{L^3} \mu \delta(x^+) \delta(x^-) (z - z_0) \phi'(z). \]

Here the subscripts indicate partial derivatives. Solving Eq. (3.5a) for \( g_z \) and substituting the result into Eq. (3.5b) yields

\[
-3 h_z + 3 z h_{zz} - z^2 h_{zzz} + 2 z^2 h_{x^x} = \delta(x^+) \delta(x^-)
\]

\[
\times \left[ z [\phi'(z)]^2 - \frac{16 \pi G_5}{3} \frac{z_0^5}{L^3} \mu z [\delta'(z - z_0) \phi'(z) + \delta(z - z_0) \phi''(z)] \right]. \quad (3.6)
\]

Eq. (3.6) can be rewritten as

\[
z^2 \frac{\partial_z}{z} \left[ \frac{3}{z} h_z - h_{zz} + 2 h_{x^x} \right] = \delta(x^+) \delta(x^-)
\]

\[
\times \left[ z [\phi'(z)]^2 - \frac{16 \pi G_5}{3} \frac{z_0^5}{L^3} \mu z [\delta'(z - z_0) \phi'(z) + \delta(z - z_0) \phi''(z)] \right]. \quad (3.7)
\]

We can now substitute \( \phi(z) \) from Eq. (2.13) into Eq. (3.7). There is a small subtlety: the derivative of \( \phi(z) \) is discontinuous at \( z = z_0 \). It is therefore not clear which value of the derivative to choose, the one at \( z - z_0 \rightarrow 0^+ \) or the one at \( z - z_0 \rightarrow 0^- \). As for \( z > z_0 \) all derivatives of \( \phi(z) \) are zero, plugging the derivatives at \( z - z_0 \rightarrow 0^+ \) into Eq. (3.7) would simply eliminate all bulk source effects. It therefore seems more physical to use the derivatives at \( z - z_0 \rightarrow 0^- \). This gives

\[
\frac{3}{z} h_z - h_{zz} + 2 h_{x^x} = \frac{1}{3} \left( \frac{16 \pi G_5 \mu}{L^3} \right)^2 \delta(x^+) \delta(x^-) \left[ \frac{z_0^6}{2} \theta(z - z_0) - \frac{z_0^6}{2} \theta(z - z_0) - \frac{z_0^7}{2} \delta(z - z_0) \right]. \quad (3.8)
\]
Eq. (3.8) is easy to solve as the Green function for the operator on its left hand side was found in [15, 23]. Defining the Green function by

\[
\left[ \frac{3}{z} \partial_z - \partial_z^2 + 2 \partial_+ \partial_- \right] G(x^+, x^-, z; x'^+, x'^-, z') = \delta(x^+ - x'^+) \delta(x^- - x'^-) \delta(z - z')
\]  

(3.9)

one can find an integral expression [15, 23]

\[
G(x^+, x^-, z; x'^+, x'^-, z') = \frac{1}{2} \theta(x^+ - x'^+) \theta(x^- - x'^-) \frac{z^2}{z'^2} \int_0^\infty dm
\times m J_0 \left( m \sqrt{2(x^+ - x'^+)(x^- - x'^-)} \right) J_2(m z) J_2(m z')
\]  

(3.10)

which can be integrated to give

\[
G(x^+, x^-, z; x'^+, x'^-, z') = \frac{1}{2 \pi} \theta(x^+ - x'^+) \theta(x^- - x'^-) \theta(s) \theta(2 - s) \frac{z}{z'^2} \frac{1 + 2 s (s - 2)}{\sqrt{s (2 - s)}}
\]  

(3.11)

with

\[
s \equiv \frac{2 (x^+ - x'^+) (x^- - x'^-) - (z - z')^2}{2 z z'}.
\]  

(3.12)

With the help of Eq. (3.10) we solve Eq. (3.8) and write

\[
h(x^+, x^-, z) = \int_{-\infty}^{x^+} dx'^+ \int_{-\infty}^{x^-} dx'^- \int_0^\infty dz' \frac{z'^2}{2 z'} \int_0^\infty dm m J_0 \left( m \sqrt{2(x^+ - x'^+)(x^- - x'^-)} \right) J_2(m z) J_2(m z')
\times \frac{1}{3} \left( \frac{16 \pi G_5 \mu}{L^3} \right)^2 \delta(x'^+) \delta(x^-) \left[ \frac{z'^6}{2} \theta(z_0 - z') - \frac{z_0^6}{2} \theta(z' - z_0) - z_0^7 \delta(z' - z_0) \right].
\]  

(3.13)

Integrating over \( x'^+ \) and \( x'^- \) trivially yields

\[
h(x^+, x^-, z) = \frac{1}{3} \left( \frac{16 \pi G_5 \mu}{L^3} \right)^2 \theta(x^+) \theta(x^-) \int_0^\infty dz' \frac{z'^2}{2 z'} \int_0^\infty dm m J_0 (m \tau) J_2(m z) J_2(m z')
\times \left[ \frac{z'^6}{2} \theta(z_0 - z') - \frac{z_0^6}{2} \theta(z' - z_0) - z_0^7 \delta(z' - z_0) \right]
\]  

(3.14)

where we defined the proper time

\[
\tau = \sqrt{2 x^+ x^-}.
\]  

(3.15)
Let us evaluate the three terms in the brackets in Eq. (3.14) separately. Start with the last term: it is proportional to
\[
\int_0^\infty \frac{dz'}{2z'} \int_0^\infty dm \, m \, J_0(m \tau) \, J_2(m \, z') \, z_0^2 \, \delta(z' - z_0) = \frac{z^2}{2z_0^2} \int_0^\infty dm \, m \, J_0(m \tau) \, J_2(m \, z) \, J_2(m \, z_0)
\]
which is
\[
= \frac{z^2}{2 \pi} \, z_0^6 \, \theta(s_0) \, \theta(2 - s_0) \, \frac{1 + 2 \, s_0 \,(s_0 - 2)}{\sqrt{s_0 \,(2 - s_0)}} \tag{3.16}
\]
with
\[
s_0 = \frac{\tau^2 - (z - z_0)^2}{2 \, z \, z_0}. \tag{3.17}
\]
We see that taking \(z_0 \to \infty\) and keeping \(\mu\) fixed gives \(s_0 \approx -z_0/(2 \, z)\) such that the expression in Eq. (3.16) becomes zero due to \(\theta(s_0)\). Hence the last term in the brackets of Eq. (3.14) does not contribute in the \(z_0 \to \infty\) limit.

The second term in the brackets of Eq. (3.14) is proportional to
\[
\int_{z_0}^\infty dz' \frac{1}{z'} \int_0^\infty dm \, m \, J_0(m \tau) \, J_2(m \, z) \, J_2(m \, z') = \frac{1}{z_0} \int_0^\infty dm \, m \, J_0(m \tau) \, J_2(m \, z) \, J_1(m \, z_0) = 0 \tag{3.18}
\]
with the last step being valid for \(z_0 > z + \tau\), i.e., for the large \(z_0\) we are interested in.

We are left with the first term in the brackets of Eq. (3.14). Hence at large \(z_0\) we have
\[
h(x^+, x^-, z) = \frac{1}{3} \left( \frac{8 \, \pi \, G_5 \, \mu}{L^3} \right)^2 \theta(x^+) \, \theta(x^-) \, z^2 \int_{z_0}^0 dz' z'^5 \int_0^\infty dm \, m \, J_0(m \tau) \, J_2(m \, z) \, J_2(m \, z')
\]
which is
\[
= \frac{1}{3} \left( \frac{8 \, \pi \, G_5 \, \mu}{L^3} \right)^2 \theta(x^+) \, \theta(x^-) \, z^4 \int_0^\infty \frac{dm}{m} \, J_0(m \tau) \, J_2(m \, z) \, [6 \, J_4(m \, z_0) - m \, J_5(m \, z_0)]
\]
\[
= \left( \frac{8 \, \pi \, G_5 \, \mu}{L^3} \right)^2 \theta(x^+) \, \theta(x^-) \, z^4 \left[ \tau^2 + \frac{1}{3} \, z^2 \right]. \tag{3.19}
\]
This is exactly the solution found for sourceless shock waves in [10]! Using \(h\) from Eq. (3.19) in Eq. (3.5a) one would obtain function \(g\), which, for \(z_0 \to \infty\) would also be \(z_0\)-independent and would also correspond to that found for sourceless shock waves in [10]. Similarly one can show that \(f\) and \(\tilde{f}\) would also reduce to the ones from [10] in the \(z_0 \to \infty\) limit. We conclude that, at least at this lowest non-trivial order in \(\mu\), colliding shock waves with sources gives a metric which in the limit of \(z_0 \to \infty\) (keeping \(\mu\) fixed) reduces to that produced in the collision of two shock waves without sources. This presents a strong argument supporting our earlier assertion that collisions of the shock waves with sources at \(z_0 = \infty\) are equivalent to collisions of the shock waves without the sources.
4. Trapped Surface Analysis

Below we will present trapped surface analysis for a collision of two shock waves without bulk sources. We will begin by outlining general concepts of the trapped surface analysis and will present a naive attempt to find the trapped surface for a collision of shock waves from Eqs. (2.1) and (2.2). We will then obtain the trapped surface for a collision of two shock waves with bulk sources and take the limit of \( z_0 \to \infty \), deriving the trapped surface for a collision of sourceless shock waves. We will solidify our above conclusion of the equivalence between the sourceless shock wave and the one with sources at \( z = \infty \) by taking the limit of sources going to the IR for a collision of two different shock waves with extended sources at \( z_1 \) and \( z_2 \) and showing that the limiting trapped surface is the same as obtained before.

4.1 Generalities

Let us start with outlining some generalities of trapped surface. Consider the collision of two shock waves given by the following metric before the collision:

\[
\begin{align*}
\frac{ds^2}{z^2} &= \frac{L^2}{z^2} \{-2 \, dx^+ dx^- + dx_+^2 + dz^2\} + \frac{L}{z} \Phi_1(x_\perp, z) \delta(x^+) \, dx^+ \, dz + \frac{L}{z} \Phi_2(x_\perp, z) \delta(x^-) \, dx^- \, dz \\
&= \frac{L^2}{z^2} \{-2 \, dx^+ dx^- + dx_+^2 + dz^2\} + \frac{L}{z} \Phi_1(x_\perp, z) \delta(x^+) \, dx^+ \, dz + \frac{L}{z} \Phi_2(x_\perp, z) \delta(x^-) \, dx^- \, dz
\end{align*}
\]  

\( (4.1) \)

where (cf. Eq. (3.1))

\[
\Phi_i(x_\perp, z) = \frac{L}{z} \phi_i(x_\perp, z), \quad i = 1, 2. 
\]  

\( (4.2) \)

The marginally trapped surface is found from the condition of vanishing of expansion \( \theta \) [24]. The trapped surface is made up of two pieces: \( S = S_1 \cup S_2 \). \( S_1(S_2) \) is associated with shock wave at \( x^+ = 0 \) (\( x^- = 0 \)) before the collision. An additional condition is imposed requiring that the outer null normal to \( S_1 \) and \( S_2 \) must be continuous at the intersection \( C = S_1 \cap S_2 \) point \( x^+ = x^- = 0 \) to avoid delta function in the expansion.

To calculate the trapped surface associated with shock wave at \( x^+ = 0 \), we use the following coordinate transformation [8, 17]:

\[
x^- \to x^- + \frac{\phi_1(x_\perp, z)}{2} \theta(x^+) 
\]  

\( (4.3) \)

to eliminate the delta-function discontinuity at \( x^+ = 0 \).\(^1\) The trapped surface \( S_1 \) can then be parametrized by [17]

\[
x^+ = 0, \quad x^- = -\frac{\psi_1(x_\perp, z)}{2}. 
\]  

\( (4.4) \)

\(^1\)Note a different definition for the light-cone coordinates used in [8, 17].
The condition of marginally trapped surface is the vanishing of expansion \( \theta \equiv h^{\mu\nu} \nabla_\mu l_\nu \), with \( h^{\mu\nu} \) the induced metric and \( l_\nu \) the outer null normal to the trapped surface. Similarly to \([8, 11, 12]\), the condition gives rise to

\[
(\Box - \frac{3}{L^2}) [\Psi_1(x_\perp, z) - \Phi_1(x_\perp, z)] = 0
\]

with \( \Psi_1(x_\perp, z) = \frac{L}{z} \psi_1(x_\perp, z) \) and the Laplacian is defined with respect to Euclidean AdS\(_3\) space

\[
ds^2 = \frac{L^2}{z^2} \{dx_\perp^2 + dz^2\}.\]

By analogy, we have the condition defining the trapped surface \( S_2 \):

\[
(\Box - \frac{3}{L^2}) [\Psi_2(x_\perp, z) - \Phi_2(x_\perp, z)] = 0.
\]

The continuity of trapped surface \( S_1 \) and \( S_2 \) and their outer null normal on the cusp of the light-cone \( x^+ = x^- = 0 \) reduce to the boundary conditions

\[
\Psi_1(x_\perp, z)|_C = \Psi_2(x_\perp, z)|_C = 0 \quad (4.8a)
\]
\[
\nabla \Psi_1(x_\perp, z) \cdot \nabla \Psi_2(x_\perp, z)|_C = 8 \quad (4.8b)
\]

where the boundary \( C \) is to be determined from Eq. \((4.8)\). The covariant derivative \( \nabla \) is again defined with respect to Eq. \((4.6)\).

Having the equations for the trapped surface with arbitrary shock wave \((4.3), (4.7)\) and \((4.8)\) at hand, we are ready to apply them to the collision of source-free shock waves \((2.1)\) and \((2.2)\). With the symmetry \( \phi_1(z) = \phi_2(z) \equiv \phi(z) \) (and thus \( \psi_1(z) = \psi_2(z) \equiv \psi(z) \)), they take a particularly simple form

\[
z^2 \Psi''(z) - z \Psi'(z) - 3 \Psi(z) = 0\]

\[
\Psi(z_a) = \Psi(z_b) = 0\]

\[
\frac{z_a^2}{L^2} \Psi'(z_a)^2 = \frac{z_b^2}{L^2} \Psi'(z_b)^2 = 8.\]

The boundary \( C \) in this case is given by \( z_a < z < z_b \), as there is no dependence on transverse coordinates. Eq. \((4.9)\) is easily solved by

\[
\Psi(z) = C_1 z^3 + \frac{C_2}{z}\]

with \( C_1 \) and \( C_2 \) arbitrary constants.

Obviously we cannot have \( C_1 = C_2 = 0 \) because of Eq. \((4.9c)\). It is easy to see then Eq. \((1.9)\) would immediately require \( z_a = z_b \). Similar phenomenon of no trapped surface was observed in \([17]\)
for collisions of gravitational shock waves in asymptotically Minkowskian 4-dimensional space-time. One may be tempted to conclude that trapped surface formation is not possible in collisions of source-free shock waves. Before accepting such conclusion, let us point out that the reason we choose $C$ to be bounded by $z_a < z < z_b$ from both sides in the bulk is because the trapped surface has to be closed. However, AdS$_5$ is different from asymptotically Minkowskian spaces: it appears not quite clear whether the requirement of a closed trapped surface necessarily implies finite $z_b$.² If one searches for the trapped surface with $z_b = \infty$, i.e., with $z > z_a$ constraint only, such that conditions in Eqs. (4.13) are imposed only at $z_a$, one gets

$$
\Psi(z) = \frac{L}{\sqrt{2}} \left[ \frac{z^3}{z^3_a} - \frac{z^3}{z} \right] (4.11)
$$

giving

$$
\psi(z) = \frac{1}{z^3} \left[ z^4 - z^4_a \right]. (4.12)
$$

Unfortunately the conditions in Eqs. (4.13) are insufficient to fix $z_a$ uniquely.

However, $z_a$ in Eq. (4.11) can be fixed if we choose to study a closely relevant situation. Let us consider the trapped surface formation in the collision of two sourced shock waves

$$
ds^2 = L^2 z^2 \left\{ -2 dx^+ dx^- + \phi_1(z) \delta(x^-) dx^- + dx_\perp^2 + dz^2 \right\} (4.13a)
$$

$$
ds^2 = L^2 z^2 \left\{ -2 dx^+ dx^- + \phi_2(z) \delta(x^-) dx^- + dx_\perp^2 + dz^2 \right\} (4.13b)
$$

with the sources $J_{++} = \frac{E_1}{z_1 L} \delta(x^+) \delta(z - z_1)$ and $J_{--} = \frac{E_2}{z_2 L} \delta(x^-) \delta(z - z_2)$ corresponding to each of the shock waves. As discussed in the previous sections, we keep $\frac{E_1 L^2}{z_1^4} = \frac{E_2 L^2}{z_2^4} = \mu$ such that the nuclei on the boundary have the same energy density.

The equations (4.13) for the trapped surface now take the following form:

$$
z^2 \Psi''_i(z) - z \Psi'_i(z) - 3 \Psi_i = -16 \pi G_5 E_i \delta(z - z_i) (4.14a)
$$

$$
\Psi_i(z_a) = \Psi_i(z_b) = 0 (4.14b)
$$

$$
\frac{z^2}{L^2} \Psi'_i(z_a) \Psi'_i(z_a) = \frac{z^2}{L^2} \Psi'_i(z_b) \Psi'_i(z_b) = 8 (4.14c)
$$

where the boundary $C$ is again $z_a < z < z_b$ and $i = 1, 2$. Eq. (4.14) is solved by

$$
\Psi_i = \begin{cases} 
C_i \left( \frac{z^3}{z^3_i} - \frac{z^3}{z} \right), & z < z_i \\
D_i \left( \frac{z^3}{z^3_i} - \frac{z^3}{z} \right), & z > z_i
\end{cases} (4.15)
$$

²Requirement that the trapped surface has to be closed appears to stem from the cosmic censorship conjecture, which we assume to be true in AdS$_5 \times S^5$: the issue of whether trapped surfaces in AdS$_5$ necessarily have to be closed may require further investigation.
with the constants

\[
\begin{aligned}
C_i &= -\frac{4\pi G_5 E_i}{z_i^4} \frac{(\frac{z_i^4}{z_j^4}-1) z_j}{z_i^4 z_j^4} \\
D_i &= -\frac{4\pi G_5 E_i}{z_i^4} \frac{(\frac{z_i^4}{z_j^4}-1) z_j}{z_i^4 z_j^4}.
\end{aligned}
\]  

(4.16)

The third equation in (4.14) gives the following simple relations:

\[
C_1 C_2 = D_1 D_2 = \frac{L^2}{2}.
\]  

(4.17)

4.2 Shock Waves with Identical Sources

It is instructive to first consider a collision of identical shock waves in AdS$_5$. Putting $z_1 = z_2 = z_0$ and $E_1 = E_2 = E$ in Eqs. (4.15) and (4.16) above we obtain from Eq. (4.17) [11]

\[
\begin{aligned}
z_a + z_b &= \frac{4\sqrt{2}\pi G_5}{L} E \\
\frac{(z_a+z_b)^2-3z_a z_b}{z_a^4 z_b^4} &= \frac{1}{z_0^4}.
\end{aligned}
\]  

(4.18)

We want to take $z_0 \to \infty$ limit while keeping the energy of the shock wave in the boundary theory fixed. That is we want to hold

\[
\mu = \frac{E L^2}{z_0^4}
\]  

(4.19)

fixed. We rewrite Eq. (4.18) in terms of $\mu$ as

\[
\begin{aligned}
z_a + z_b &= \frac{2\sqrt{2}\pi^2}{N_c^2} \mu z_0^4 \\
\frac{(z_a+z_b)^2-3z_a z_b}{z_a^4 z_b^4} &= \frac{1}{z_0^4}
\end{aligned}
\]  

(4.20)

where we have replaced $G_5 = \pi L^3/2 N_c^2$. Now, taking $z_0 \to \infty$ keeping $\mu$ fixed we can easily infer the asymptotics of $z_a$ and $z_b$. First one can consider the case that in this limit $z_a$ and $z_b$ are of the same order, $z_a \sim z_b$. In such case the first equation in (4.20) gives $z_a \sim z_b \sim z_0^4$, which can not satisfy the second equation in (4.20). As $z_a < z_b$ by definition, we are left to consider the case when, in the $z_0 \to \infty$ limit one has $z_a \ll z_b$. Then the first equation in (4.20) yields

\[
z_b \approx \frac{2\sqrt{2}\pi^2}{N_c^2} \mu z_0^4
\]  

(4.21)
which, when plugged into the second equation in (4.20) along with the assumption that \( z_a \ll z_b \) gives

\[
\begin{aligned}
    z_a &\approx \frac{1}{\left( \frac{2\sqrt{2}\pi^2}{N_c^2\mu} \right)^{1/4}} \equiv z^*_a. \\
        (4.22)
\end{aligned}
\]

The values of \( z_a \) and \( z_b \) given by Eqs. (4.22) and (4.21) satisfy \( z_a \ll z_b \) condition when \( z_0 \) is large, which confirms that they give the correct asymptotics. In the strict \( z_0 \to \infty \) limit we see that \( z_b \to \infty \), but \( z_a \) remains finite given by Eq. (4.22). Indeed Eqs. (4.22) and (4.21) can also be obtained by solving Eqs. (4.20) explicitly and taking the \( z_0 \to \infty \) limit: the exact solution of Eq. (4.20) giving real \( z_a \) and \( z_b \) is

\[
\begin{aligned}
    z_a &= \frac{\tilde{\mu} z_0^4}{2} - \frac{1}{4\xi} \sqrt{2^{11/3}\xi^3 - 2^{1/3}z_0^4\xi + 4 z_0^8 \tilde{\mu}^2 \xi^2} \\
    z_b &= \frac{\tilde{\mu} z_0^4}{2} + \frac{1}{4\xi} \sqrt{2^{11/3}\xi^3 - 2^{1/3}z_0^4\xi + 4 z_0^8 \tilde{\mu}^2 \xi^2} \\
        (4.23a) & & (4.23b)
\end{aligned}
\]

where

\[
\tilde{\mu} = 2 \sqrt{2} \frac{\pi^2}{N_c^2} \mu
\]

and

\[
\xi = \left( z_0^6 \sqrt{4 + \frac{z_0^{12}}{\tilde{\mu}^4} - \frac{z_0^{12}}{\tilde{\mu}^2}} \right)^{1/3}. \\
(4.25)
\]

One can readily check that the \( z_0 \to \infty \) asymptotics of Eqs. (4.23a) and (4.23b) is given by Eqs. (4.22) and (4.21).³

Taking the \( z_0 \to \infty \) limit in Eq. (4.15) one can see that the trapped surface is described by

\[
\psi(z) = \frac{2\pi^2}{N_c^2} \mu \left[ z^4 - z_a^* z^4 \right] = \frac{\tilde{\mu}}{\sqrt{2}} \left[ z^4 - \tilde{\mu}^{-4/3} \right],
\]

which is exactly Eq. (4.12) with \( z_a \) now fixed by Eq. (4.22). We see that introducing bulk source as a regulator of the metric in the IR and then taking \( z_0 \to \infty \) limit allows one to fix \( z_a \) and hence determines the trapped surface uniquely.

Now let us verify that the obtained value of \( z_a \) in Eq. (4.22) is independent of the way we take the limit of sending the bulk sources to infinite IR. Let us show that the same trapped surface arises in a more general case when the two shock waves are different from each other.

³Note that for \( z_0 < (2/\tilde{\mu})^{1/3} \) both \( z_a \) and \( z_b \) from Eqs. (4.23a) and (4.23b) become complex and trapped surface ceases to exist: however this small-\( z_0 \) limit is the exact opposite of the \( z_0 \to \infty \) case we would like to consider here.
4.3 Shock Waves with Sources at Different Bulk Locations

We consider a collision of shock waves with sources at different locations and with different \( E_i \)'s: now we have \( z_1 \neq z_2 \) and \( E_1 \neq E_2 \) but with \( \frac{E_1 L^2}{z_1^2} = \frac{E_2 L^2}{z_2^2} = \mu \). It proves useful to set \( \frac{z_i^4}{z_a z_b} = \lambda_i \), \( \frac{z_i^2}{z_a z_b} = \lambda_2 \) and rewrite Eq. (4.17) as

\[
\begin{align*}
\frac{z_i^2}{z_a z_b} + \frac{z_i^2}{z_a z_b} + 1 &= \frac{\lambda_i + \lambda_2 + 1}{\lambda_1 \lambda_2} \\
(\frac{z_i z_a}{z_a z_b})^3 \frac{z_i z_a}{z_a z_b} &= \left( \frac{N_c^2}{2 \pi^2 \mu} \right)^2 \frac{1}{2(1 - \lambda_1 \lambda_2)}
\end{align*}
\] (4.27)

eliminating \( z_1 \) and \( z_2 \). Finding solution for \( z_a \) and \( z_b \) seems to be a hard task. We instead first solve the first equation in (4.27) for \( z_a/z_b \) and use the obtained ratio in the second equation in (4.27) to find \( z_a z_b \). Using the product \( z_a z_b \) in \( \frac{z_i^4}{z_a z_b} = \lambda_1, \frac{z_i^2}{z_a z_b} = \lambda_2 \) we can write \( z_1 \) and \( z_2 \) as \( (i = 1, 2) \)

\[
z_i^4 = \lambda_i \left( \frac{N_c^2}{2 \pi^2 \mu} \right)^{4/3} \left[ \frac{\lambda_1 + \lambda_2 + 1 - 3 \lambda_1 \lambda_2}{2 (1 - \lambda_1 \lambda_2)} \right]^{2/3} \frac{[\lambda_1 + 1] (\lambda_2 + 1)^{1/3}}{\lambda_1 \lambda_2}.
\] (4.28)

Again we have replaced \( G_5 = \pi L^3/2 N_c^2 \). We are interested in the limit \( z_1, z_2 \to \infty \) while keeping \( r_{12} = \frac{z_1}{z_2} = \text{finite} \) and \( \mu \) is fixed. It is not difficult to see that the limit can be achieved by taking \( \lambda_1, \lambda_2 \to 0 \). In this limit Eq. (4.28) takes a very simple form:

\[
z_i^4 = \frac{1}{4 \lambda_i} \left( \frac{N_c^2}{2 \pi^2 \mu} \right)^{4/3} \] (4.29a)

\[
z_i^4 = \frac{1}{4 \lambda_i} \left( \frac{N_c^2}{2 \pi^2 \mu} \right)^{4/3} \] (4.29b)

As \( z_b > z_a \), the first equation in (4.27) gives in the \( \lambda_1, \lambda_2 \to 0 \) limit that \( z_b \gg z_a \). Solving the second equation in (4.27) for \( z_b \gg z_a \) one obtains \( z_a \) asymptotics. Using the result in \( \frac{z_i^4}{z_a z_b} = \lambda_1 \) along with the first equation in (4.27) yields

\[
z_a \approx \left( \frac{N_c^2}{2 \sqrt{2} \pi^2 \mu} \right)^{1/3} \equiv z_a^* \] (4.30a)

\[
z_b \approx (z_1 z_2)^2 \left( \frac{N_c^2}{2 \sqrt{2} \pi^2 \mu} \right)^{-1} \to \infty.
\] (4.30b)

These equations are completely analogous to Eqs. (4.22) and (4.21) above. Therefore the trapped surface is independent of the way one send the bulk sources to the IR infinity: the sources do not have to be at the same bulk location to obtain the same answer as we had in the previous Subsection.

To further test the independence of taking the limit of sources going to infinite IR bulk, we have
also taken a similar limit for the point-like sources, first advocated in [8]: the trapped surface found in [8] again reduced to the trapped surface found in this work above.

This completes our analysis of trapped surface in the collision of two shock waves with sources infinitely deep in the bulk. We note unlike source in finite depth [11], no critical value for the energy density is found in the limit. The formation of the trapped surface is always guaranteed. The trapped surface is even independent of the ratio \( r_{12} = \frac{z_1}{z_2} \), i.e. the details of the limit! It is important to stress that the trapped surface does not disappear with the removal of the sources in the bulk, which can be viewed as IR regulators. In all the examples of scattering of shock waves with bulk sources the trapped surface always appears to be more or less centered around the source in the \( x_{\perp}, z \) space. One was tempted to conjecture therefore that the trapped surface is an inherent property of non-zero bulk energy-momentum tensor. Our result proves otherwise, giving an example of the source-free shock waves collision with a well defined trapped surface.

4.4 Limiting Trapped Surface

To summarize our trapped surface analysis let us re-state that the profiles of the trapped surface are given by

\[
\Psi_i(z) = \frac{2\pi^2 L \mu}{N_c^2} z^{*3} \left( \frac{z^{3}}{z^{*3}} - \frac{z^*}{z} \right)
\]

which is exactly Eq. (4.11) with \( z^*_a \) from Eq. (4.22). In the transformed light cone coordinates (see Eq. (4.3)) the trapped surface is then determined by

\[
x^+ = 0, \quad x^- = -\frac{\pi^2}{N_c} \mu \left[ z^4 - z^*_a^4 \right] = -\frac{\bar{\mu}}{2\sqrt{2}} \left[ z^4 - \tilde{\mu}^{-4/3} \right]
\]

(4.32)

with an analogous expression for the other shock wave obtained by interchanging \( x^+ \leftrightarrow x^- \) in Eq. (4.32).

The trapped surface for a collision of source-free shock waves from Eq. (4.32) is illustrated in Fig. 1. One can clearly see that the trapped surface is present at all times before the collision and rises from the deep IR toward finite values of \( z \). Similar behavior was observed for the trapped surface in the numerical model of heavy ion collision involving gravitational perturbations in the 4-dimensional world in [25, 26]. A horizon rising from the deep IR was also deduced in [27] for a model of heavy ion collision involving a rapidity-independent matter distribution after the collision.

It is interesting to point out that the obtained shape of the trapped surface appears to imply that the black hole produced in the collision would have a singularity at \( z = \infty \) with the horizon independent of the transverse coordinates \( x_{\perp} \). This is indeed very similar to the black hole dual to Bjorken hydrodynamics constructed in [6]. The main difference is that in our case the metric (and the energy-momentum tensor in the gauge theory) are rapidity-dependent, as follows from explicit calculations of the metric produced in shock wave collisions [9, 10, 15].
Our estimate for the produced entropy per unit transverse area $A_\perp$ for a collision of two shock waves with the sources at $z_0$ is \cite{11}

$$\frac{S}{A_\perp} = \frac{N_c^2}{2\pi} \left[ \frac{1}{z_a^2} - \frac{1}{z_b^2} \right].$$

(4.33)

Using Eqs. (4.22) and (4.21) we obtain for $z_0 \to \infty$

$$\frac{S}{A_\perp} = [\pi N_c^2 \mu^2]^{1/3}.$$

(4.34)

As $\mu^2 \sim s$ with $s$ the center of mass energy of the collision, we get

$$\frac{S}{A_\perp} \propto s^{1/3}$$

(4.35)

in agreement with the result obtained in \cite{8}.

The entropy from Eq. (4.33) is plotted in Fig. 2 in arbitrary units as a function of the bulk source location $z_0$ (for a collision of two identical shock waves). Fig. 2 demonstrated that produced entropy becomes practically independent of the bulk source position rather fast, approaching its asymptotic value well before $z_0 \mu^{1/3}$ becomes large.

As we noted above, for $z_0 \mu^{1/3} < 2^{1/3}$ both $z_a$ and $z_b$ given by Eqs. (4.23a) and (4.23b) become complex and the trapped surface ceases to exist (see also \cite{13} for a similar result). This likely implies that no black hole is formed in collisions of such shock waves. To understand this result
Figure 2: The (lower bound on the) entropy density produced in the collision of two identical shock waves with sources as a function of the source position $z_0$ in units of $\mu^{-1/3}$. The entropy density is in arbitrary units.

from the boundary gauge theory perspective one has to have a rigorous interpretation of what shock wave sources in the bulk are dual to in the gauge theory. Such interpretation is missing at the moment, which inspired our present investigation of collisions of the sourceless shock waves. We may speculate though: following [11] we may assume that the inverse position of the source in the bulk $1/z_0$ provides an IR cutoff on the transverse momenta of the partons in the shock waves’ wave functions in the boundary theory. Reducing $z_0$ would increase the cutoff $1/z_0$ thus decreasing the number of partons: this is likely to lower the number of degrees of freedom produced in the collision, leading to the reduction of the entropy density with decreasing $z_0$ in Fig. 2. Still it is not entirely clear why the trapped surface disappears completely at a finite small $z_0$ forcing the estimate for produced entropy to go to zero. Indeed our delta-function shock waves are described by a single dimensionful parameter $\tilde{\mu}$ (or $\mu$): the largest momentum scale in the problem is therefore $\tilde{\mu}^{1/3}$. If $1/z_0$ is the IR cutoff, then clearly it can not exceed the largest momentum scale: hence $1/z_0 \lesssim \tilde{\mu}^{1/3}$. This, however, can not explain why the trapped surface vanishes entirely at $z_0 = 2^{1/3} \tilde{\mu}^{-1/3}$. Besides nothing pathological seems to happen in the perturbative solution presented in Sect. 3 for small finite $z_0$. In this work we are interested in the large-$z_0$ asymptotics: Fig. 2 demonstrates that the produced entropy density does not seem to change much between having sources at finite large $z_0 > 2^{1/3} \tilde{\mu}^{-1/3}$ and having no sources at all, which seems to agree with the IR cutoff interpretation of the sources and, more importantly, shows that the entropy is “well-behaved” in the $z_0 \to \infty$ limit we are taking. We leave the detailed study of the small-$z_0$ regime for future work.
5. Thermalization Time Estimate and Conclusions

The result fixing $z^*$ in Eq. (4.22) could be predicted if one realizes that in the limit of delta-function shock waves the problem has only one dimensionful parameter $\tilde{\mu}$ which has dimensions of mass cubed. If a non-vanishing trapped surface is created in such collisions it has to be proportional to the only distance scale in the problem: $1/\tilde{\mu}^{1/3}$. Stretching this analogy further one should expect that the proper time of thermalization (the time of black hole formation) is

$$\tau_{\text{therm}} \sim \frac{1}{\tilde{\mu}^{1/3}},$$  \hspace{1cm} (5.1)

as was originally suggested in [9].

An interesting question is the relation between this thermalization time and the time it takes for the shock waves to stop. It was argued in [10,15] that colliding shock waves come to a complete stop shortly after the collision. One can argue that $\mu \sim p^+ \Lambda^2 A^{1/3}$ [10], where $p^+$ is the large longitudinal momentum of a “nucleon” in the shock wave, $\Lambda$ is the typical transverse momentum scale in the shock, and $A$ is the atomic number of the nucleus we model by the shock wave. The characteristic light-cone stopping time for a shock wave moving in the light-cone “plus” direction is given by [10,15]

$$x^+_{\text{stop}} \sim \frac{1}{\Lambda A^{1/3}}.$$  \hspace{1cm} (5.2)

This is of course parametrically much longer than

$$\tau_{\text{therm}} \sim \frac{1}{\tilde{\mu}^{1/3}} \sim \frac{1}{(p^+ \Lambda^2 A^{1/3})^{1/3}}.$$  \hspace{1cm} (5.3)

Hence, if one assumes that thermalization happens at mid-rapidity first, then, as near mid-rapidity $t \approx \tau$, the time of thermalization is $t_{\text{therm}} \approx \tau_{\text{therm}} \ll t_{\text{stop}} = x^+_{\text{stop}}/\sqrt{2}$. It is therefore likely that thermalization happens at times which are parametrically earlier than the stopping time. If our guess of thermalization time is correct, this would imply that the shock waves still move along their light cones when thermalization happens, justifying an assumption commonly used in hydrodynamic simulations of heavy ion collisions. Note also that the thermalization time in Eq. (5.3) is very short, and decreases with center-of-mass energy of the collision. (In fact, as was noticed in [9] this thermalization time is too short: if one plugs in $p^+ = 100$ GeV, $\Lambda = 0.2$ GeV and $A = 196$ into the parametric estimate (5.3) one would obtain $\tau_{\text{therm}} \approx 0.07$ fm/c for RHIC, which is far too short for agreement with hydrodynamic simulations [1,2]. Indeed the thermalization time estimate of Eq. (5.1) is too crude for 0.07 fm/c to be taken literally, and a numerical coefficient in front of the estimate (5.1), if it results from a more exact calculation and from a more realistic model of colliding nuclei, may significantly change this number.)

It is important to stress the difference between the mathematical limit of delta-function shock waves ($a \to 0$ with $a$ being the $x^-$-width of the smeared non-delta-function shock wave [10,15]
moving in the $x^+$ direction or vice versa) and the physical high energy limit of $p^+ \gg \Lambda$ for nuclei. While in the former limit $\tilde{\mu}$ is the only non-vanishing dimensionful parameter in the problem, the latter limit has another non-vanishing dimensionful scale $\tilde{\mu} a$, which in fact gives the stopping time \( x_{\text{stop}}^+ \sim 1/\sqrt{\tilde{\mu} a} \) [10,15]. (As one can easily see $a \sim \Lambda^{1/3}/p^+$ in the center-of-mass frame, such that $\tilde{\mu} a \sim \Lambda^2 A^{2/3}$ is independent of $p^+$ [10,15].) With the presence of two momentum scales in the problem the validity of the thermalization time estimate of [9] shown here in Eq. (5.1) becomes less apparent. Our trapped surface analysis resulting in Eq. (5.2) appears to indicate that it is the momentum scale which depend only on $\tilde{\mu}$ and not on $a$ that matters for thermalization, thus providing new evidence to support the estimate in Eq. (5.1). In other words we show that if one neglects the smaller second momentum scale $\tilde{\mu} a$ and approximates the shock wave profiles by delta-functions, thermalization is achieved in the collisions at the time given in Eq. (5.1). If one treats the problem more carefully and includes the scale $\tilde{\mu} a$ by considering shock waves of finite longitudinal spread [10,15], Eq. (5.1) is likely to get corrections with the relative suppression factor being some positive power of $\tilde{\mu} a/\tilde{\mu}^{2/3}$, which is very small for high energy collisions, thus leaving the estimate in Eq. (5.1) practically unchanged.

One may argue that the strongly-coupled dynamics of the $\mathcal{N} = 4$ SYM medium produced in shock wave collisions may be similar to that of strongly-coupled QCD medium. Then our conclusion of rapid thermalization may be applicable to soft (non-perturbative, $k_T \sim \Lambda_{QCD}$) modes in heavy ion collisions, which would thermalize very quickly. Harder (perturbative) modes may then thermalize through interactions with the soft non-perturbative thermal bath, though more work is needed to justify such thermalization scenario and to modify the thermalization time estimate (5.3) to take into account perturbative dynamics.

Another interesting question would concern understanding the relation between the rather quick thermalization in heavy ion collisions for the theory at strong coupling argued here and the impossibility of thermalization at weak coupling suggested in [28] by one of the authors.\(^4\) While further research is needed to clarify this problem, the solution may have already been suggested in [32,33], where the authors argue that it is possible that there is a critical value $\lambda_c$ of ’t Hooft coupling $\lambda$. For $\lambda > \lambda_c$ black hole formation is likely in high energy collisions. At the same time, for $\lambda < \lambda_c$ the black hole is not formed in high energy collisions [32,33], corresponding to no thermalization in the boundary theory. Indeed in the case of real-life heavy ion collisions, due to the running of the strong coupling constant, the coupling assumes a wide range of values in a single collision. The coupling is always large for soft transverse modes, making thermalization due to large coupling effects likely in light of our above results.

To conclude let us point out once more that we have obtained a trapped surface for a collision of two sourceless shock waves in AdS\(_5\). The shape of the trapped surface is given by Eq. (5.32) and is illustrated in Fig. 1. Existence of this trapped surface proves that a black hole is created in the bulk for a collision of two sourceless shock waves, corresponding to creation of thermalized medium

\(^4\)Note that perturbative thermalization scenarios have been advocated in [29–31].
(quark-gluon plasma) in the boundary gauge theory.

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