Killing horizons, throats and bottlenecks in the ergoregion of the Kerr spacetime

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The properties of Kerr black holes (BHs) and naked singularities (NSs) are investigated by using stationary observers and their limiting frequencies. We introduce the concept of NS Killing throats and bottlenecks for slowly spinning NSs to describe the frequency of stationary observers. In particular, we show the frequency on the horizon can be used to point out a connection between BHs and NSs and to interpret the horizon in terms of frequencies. The analysis is performed on the equatorial plane of the ergoregion.

Keywords: Black holes; Naked singularities; Killing horizons

1. Introduction

We study the orbital angular frequencies of stationary observers in the Kerr spacetime. We introduce the concept of Killing throats that arise in the spacetime of NSs and can be interpreted as the “opening” and disappearance of Killing horizons. Killing bottlenecks are identified as “restrictions” of Killing throats that appear in the case of weak naked singularities (WNSs) for which the spin-mass ratio is close to the value of the extreme BH. To explore these NS effects, and considering the dynamics of the zero angular momentum observers (ZAMOs), we introduce the concept of “metric bundles” and “extended planes”. In this work, we limit the analysis of the equatorial plane of the Kerr spacetime. The generalization to the case of the Reissner-Nordström and Kerr-Newmann spacetimes is presented elsewhere.

We first analyze the behavior of the frequency of a stationary observer in terms of the radial distance and the spin parameter of the source. In this way, we find in the NS region a particular set of curves that we identify as the Killing throat. In the case of WNSs, for which , the Killing throats show “restrictions” identified as Killing bottlenecks. To explore the properties of the bottlenecks, we introduce the concept of extended plane which is a graph relating a particular characteristic of a spacetime in terms of the parameters entering the corresponding spacetime metric. Any curve on the extended plane represents, therefore, a metric bundle, i.e., a family of spacetimes defined by a characteristic photon orbital frequency and characterized by a particular relation between the metrics parameters. In the case of the Kerr spacetime, the extended plane turns out to establish a relation

The concept of strong and weak NSs, defined through the values of the spin parameter, has been explored in several works. However, they can also be defined as strong curvature singularities.
between BHs and NSs. As a consequence, WNSs turn out to be related to the appearance of (a portion of) the inner horizon, whereas strong naked singularities (SNSs) with \( a > 2M \) are related to the outer horizon.

This work is organized as follows. In Sec. (2), we discuss the concept of stationary observers, introducing the concepts of Killing throats and Killing bottlenecks. A discussion of the significance of the Killing bottlenecks and their possible origin is presented in Sec. (2.0.1). In Sec. (3), we introduce a possible generalization of the Killing horizon definition in the extended plane. Finally, we discuss a possible connection between black holes and naked singularities, revisiting the definition of horizons and the role played by NSs in horizon construction. Final remarks follow in Sec. (4).

2. Stationary observers and light surfaces

Consider the Kerr spacetime in Boyer-Lindquist (BL) coordinates, \((t, r, \phi, \theta)\), with \( M \geq 0 \) as the mass parameter and \( a \equiv J/M \geq 0 \) as the specific angular momentum or spin, where \( J \) is the total angular momentum of the gravitational source. The horizons and ergospheres radii are given by

\[
\begin{align*}
    r_{\pm} &= M \pm \sqrt{M^2 - a^2} \\
    r_{\pm}^\epsilon &= M \pm \sqrt{M^2 - a^2 \cos^2 \theta}
\end{align*}
\]

Stationary observers are characterized by a four-velocity of the form

\[
u^\alpha = \gamma (\xi^\alpha_t + \omega \xi^\alpha_\phi), \quad \gamma^{-2} = -\kappa (\omega^2 g_{\phi\phi} + 2 \omega g_{t\phi} + g_{tt}),
\]

where \( \gamma \) is a normalization factor with

\[
\kappa = -g_{\alpha\beta} u^\alpha u^\beta, \quad \xi_\phi \text{ is the rotational Killing field, } \xi_t \text{ is the time-translational Killing field, and } \omega \text{ is a uniform angular velocity (dimensionless quantity).}
\]

In BL coordinates, in which the metric tensor depends on \((r, \theta)\) only; this means that \( r \) and \( \theta \) are constants along the worldline of each stationary observer (the observer does not see the spacetime changing along the trajectory). The spacetime causal structure of the Kerr spacetime can be also studied by considering stationary observers.

Timelike stationary particles have orbital frequencies in the range

\[
\omega \in ]\omega_-, \omega_+[, \quad \omega_\pm \equiv \omega_Z \pm \sqrt{\omega_Z^2 - \omega_s^2}, \quad \omega_s \equiv \frac{g_{tt}}{g_{\phi\phi}}, \quad \omega_Z \equiv -\frac{g_{\phi t}}{g_{\phi\phi}}
\]

where \( \omega_Z \) is the orbital angular frequency of the zero angular momentum observers (ZAMOS) and \( \omega_\pm \) are the limiting frequencies of photons orbits, which are solutions of the equation \( g_{\alpha\beta} L^\alpha_+ L^\beta_+ = 0 \) with \( L^\alpha_\pm \equiv \xi^\alpha_t + \omega_\pm \xi^\alpha_\phi \). Killing vectors \( L^\alpha_\pm \) are generators of Killing horizons. The Killing vector \( \xi^\alpha_t + \omega_\phi \xi^\alpha_\phi \) becomes null at \( r = r_+ \), defining the frequency \( \omega_+(r_+) = \omega_h \).

On the equatorial plane \( \theta = \pi/2 \) of the Kerr spacetime we have that

\[
\omega_\pm \equiv \frac{2aM^2 \pm M \sqrt{r^2 \Delta}}{r^3 + a^2(2M + r)}, \quad \Delta \equiv r^2 - 2Mr + a^2
\]

\( ^b \)The particular case \( \omega = 0 \) defines static observers; these observers cannot exist in the ergoregion.
with the asymptotic behavior
\[
\lim_{r\to\infty} \omega_\pm = 0, \quad \lim_{a\to\infty} \omega_\pm = 0, \quad \lim_{r\to0} \omega_\pm \equiv \omega_0 = \frac{M}{a}
\]
and the particular values
\[
\omega_h \equiv \omega_\pm(r_+) = \omega_Z(r_+), \quad \omega_e \equiv \omega_\pm(r_+^\tau) = \frac{aM}{2M^2 + a^2}.
\]
We can see that $\omega_- < 0$ for $r > r_+^\tau$, and $\omega_- > 0$ inside the ergoregion, while $\omega_+ > 0$ everywhere. Moreover, since $\omega_+ = \omega_-$ on the horizon, stationary observers cannot exist inside this surface. Therefore, $\omega_\pm$ are limiting angular velocities for physical observers. The behavior of the frequencies $\omega_\pm$ is depicted in Fig. 1.

Fig. 1. Left panel: Plots of the frequency surfaces $\omega_\pm$ as functions of the radial distance $r$ in Cartesian coordinates $(x,y)$ for different spin values $a$, including BHs (upper plots) and NSs (bottom plots). Timelike stationary observers are defined in the region bounded by these planes. In the BH case, the horizons are clearly identified; as the spin increases, the horizons merge and, in the NS region, a rippled configuration (bottleneck) appears specially in the case of WNSs with $(a \in [M,2M])$. Right panel: Plot of the frequency interval $\Delta \omega_\pm = \omega_+ - \omega_-$ as a function of the radius $r/M$ and the spin $a/M$. The extrema $r_{\pm}^{\tau}$ and $r_{\pm}^{\omega}$ are solutions of the equations $\partial_r \Delta \omega_\pm = 0$ and $\partial_a \Delta \omega_\pm = 0$, respectively. Bottlenecks are shown explicitly.

Instead of the spin parameter $a$, we could use the frequency $\omega_0$ or $\omega_h$ to parameterize the Kerr spacetime by using the relations
\[
a = \frac{M}{\omega_0} \quad \text{or} \quad a = \frac{4M\omega_h}{1 + 4\omega_h^2},
\]
respectively, which can be prove to be $1 - 1$ relations. This simple observation has very important consequences. In fact, if we could measure all the values of $\omega_0 \in [0,\infty[$ or $\omega_h \in [0,1/2]$, the entire Kerr family of spacetimes would be described by these frequencies. From a practical point of view, however, we can expect that the frequency on the (outer) horizon $\omega_h$ is a more suitable candidate for being
measured. This, again, is an interesting fact that seems to be related to the well-known property that all the physical degrees of freedom of a black hole are encoded on the horizon.

The angular frequency $\omega_0$ (“frequency of the singularity”) emerges as a relevant quantity in this analysis and, as it is clear from Fig. it can be used to differentiate between low spin $a \in [0, M]$ sources, where the frequency gap $\Delta \omega_\pm = \omega_+ - \omega_-$ is larger, and WNSs and SNSs, where stationary observers are not defined (asymptotically, for large $a/M$, $\omega_0 \approx 0$ and $\Delta \omega_\pm \approx 0$ for a fixed orbit $r$).

The constant $\omega_h$ is also important in the context of BH thermodynamics. Indeed, from the fundamental Bekenstein-Hawking entropy equation, $S = \frac{1}{4} A_h$, where $A_h = A_h(r_+)$ is the area of the outer horizon, one can derive all the thermodynamic properties of a BH, including its temperature (surface gravity) and angular velocity at the horizon. The frequency $\omega_h$ also determines the uniform (rigid) angular velocity on the horizon, representing the fact that the BH rotates rigidly. This quantity enters directly into the definition of the BH surface gravity and, consequently, into the formulation of the rigidity theorem. BH thermodynamics is established through the quantities $(r_+, \omega_h)$ and their variation (in $r > r_+$); consequently, from the point of view of the geometric laws of the BH thermodynamics considered here, the universe in the outer region, $r > r_+$, is regulated exclusively by the outer horizon $r_+$ (the region “inside” the black hole, i.e., $r < r_+$, does not directly enter in the outer region thermodynamics). This aspect will have a correspondence in the concept of inner horizon confinement (in the extended plane) that we will show in Sec. (3). The BH surface gravity is defined from the variation of the Killing field $L$, norm on the outer BH horizon. In this analysis we study this norm and its variation on the equatorial plane of the Kerr geometries (i.e. $r > 0$ and $\theta = \pi/2$). We focus particularly on the ergoregion as the outer boundary of the ergosurface ($r^+_e = 2M$) is a spin-independent quantity (in the static limiting metric, where $r^+_e = 2M$, there is an event horizon associated to the Killing $\xi_t$–apparent, trapped, Killing and events horizons in stationary and static spacetimes coincide.). The relevance of this special feature of the outer ergosurface on the equatorial plane consists in the variation of the norm of the Killing field $L$ with the dimensionless spacetime spin $a/M$. The off-equatorial case is instead considered in[7].

Since $S$ depends only on the radius of the outer horizon, which in turn depends only on $M$ and $a$, we can replace the spin parameter $a$ by the frequency parameter $\omega_h$, by using the $1 - 1$ relation [6]. This means that $\omega_h$ can play the role of a thermodynamic variable for a BH. It would be interesting to investigate how the use of the frequency $\omega_h$, instead of the angular momentum $J = aM$, as a thermodynamic variable would affect the thermodynamics of the corresponding BHs.

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4In fact, one can write the Hawking temperature as $T_H = \hbar c / 2\pi k_B$, where $k_B$ is the Boltzmann constant and $\kappa$ is the surface gravity. Temperature $T = \kappa / (2\pi)$; entropy $S = A/(4\hbar G)$, where $A =$ area of the horizon $A = 8\pi mr_+$; pressure $p = -\omega_h$; volume $V = GM/c^3$ ($J = amc^2/G$); internal energy $U = GM (M = c^2m/G=$ mass) and $m$ is the mass.
Killing vectors and Killing horizons

It is convenient here to review some well-known facts about Killing horizons and Killing vectors. A Killing horizon is a null surface, \( S_0 \), whose null generators coincide with the orbits of an one-parameter group of isometries (i.e., there is a Killing field \( \mathcal{L} \) which is normal to \( S_0 \)). Therefore, it is a lightlike hypersurface (generated by the flow of a Killing vector) on which the norm of a Killing vector goes to zero. In static BH spacetimes, the event, apparent, and Killing horizons with respect to the Killing field \( \xi_t \) coincide. In the Schwarzschild spacetime, therefore, \( r = 2M \) is the Killing horizon with respect to the Killing vector \( \partial_t \). The event horizons of a spinning BH are Killing horizons with respect to the Killing field \( \mathcal{L}_h = \partial_t + \omega_h \partial_\phi \), where \( \omega_h \) is defined as the angular velocity of the horizon (the event horizon of a stationary black hole must be a Killing horizon). In \( 7 \), we also consider the case of the Reissner–Nordström and Kerr-Newman spacetimes.

Killing vectors and BH surface gravity

The surface gravity of a BH may be defined as the rate at which the norm of the Killing vector vanishes from outside. The surface gravity for the Kerr BH metric, \( SG_{\text{Kerr}} = \frac{(r_+ - r_-)}{2(r_+^2 + a^2)} \), is a conformal invariant of the metric, but it re-scales with the conformal Killing vector. Therefore, it is not the same on all generators (but obviously it is constant along one specific generator because of the symmetries).

Surface gravity and frequencies

The Kerr BH surface gravity can be decomposed as \( \kappa = \kappa_s - \gamma_a \), where \( \kappa_s \equiv 1/4M \) is the Schwarzschild BH surface gravity, while \( \gamma_a = M \omega_h^2 \) is the contribution due to the additional component of the BH intrinsic spin; \( \omega_h \) is, therefore, the angular velocity (in units of \( 1/M \)) on the event horizon. The (strong) rigidity theorem connects then the event horizon with a Killing horizon, stating that, under suitable conditions, the event horizon of a stationary (asymptotically flat) solution satisfying suitable hyperbolic equations is a Killing horizon. The surface area of the BH event horizon is non-decreasing with time (which is the content of the second law of black hole thermodynamics). The BH event horizon of the stationary solution is a Killing horizon with constant surface gravity (zeroth law \( \Lambda \)). Thus \( \Lambda = \mathcal{L}^\alpha \mathcal{L}_\alpha \) is constant on the horizon. The surface gravity is then defined as the constant \( \kappa : \nabla^\alpha \Lambda = -2\kappa \mathcal{L}^\alpha \) (on the outer horizon \( r_+ \)). Alternatively, it holds that \( \mathcal{L}^\beta \nabla_\alpha \mathcal{L}_\beta = -\kappa \mathcal{L}_\alpha \) and \( L_\mathcal{L} \kappa = 0 \), where \( L_\mathcal{L} \) is the Lie derivative (a non affine geodesic equation). In other words, \( \kappa \) is constant on the orbits of \( \mathcal{L} \).

We here consider \( \Lambda \) as a function of \( (r, a) \) (as we are restricting the analysis to the equatorial plane). To analyze the dependence on \( a \), we introduce the concept of

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4 Progenitors, as stars or galaxies, have generally spin \( a = J/(Mc) \) usually bigger than their mass \( m = GM/c^2 \). During the gravitational collapse, the body should lose mass and angular momentum, generating the Killing horizons, ending up as a black hole. Penrose Cosmic Censorship Hypothesis constrains the gravitational collapse from good (physically realistic) initial conditions for the progenitors to end up in a BH; this result, however, is still an hypothesis, strictly depending on the “physical” initial conditions.
the extended plane $\pi^+$ as the set of points $(a/M, Q)$, where $Q$ is any quantity that characterizes the spacetime and depends on $a$. In general, the extended plane is an $(n + 1)$-dimensional surface, where $n$ is the number of independent parameters that enter $Q$. An example of an extended plane is plotted in Fig. 1, which corresponds to the 2D set of points $(a/M, \omega \pm)$. The frequency $\omega \pm$ shows “ripples”, which emerge when the spin is within the interval $a \in [M, 2M]$. The surfaces become connected and the two horizons disappear, leaving as “remnants” in the planes the “Killing bottlenecks”. Those ripples appeared also in different analysis of the stationary observes.

2.0.1. Light surfaces

Light surfaces are determined by the solutions of the equation $g_{\alpha \beta}L_\alpha L_\beta = 0$, which in the case under consideration can be expressed as

$$
\frac{r_s^+}{M} = \frac{2\beta_1 \sin \left(\frac{1}{3} \arcsin \beta_0\right)}{\sqrt{3}}, \quad \frac{r_s^-}{M} = \frac{2\beta_1 \cos \left(\frac{1}{3} \arccos(-\beta_0)\right)}{\sqrt{3}} \quad (7)
$$

where $\beta_1 \equiv \sqrt{1 - \frac{1}{\omega^2} - \frac{1}{\omega_0^2}}$, $\beta_0 \equiv \frac{3\sqrt{3}\beta_1 \omega^2}{(\frac{\omega}{\omega_0} + 1)^2}$. \quad (8)

The surfaces $r_s = r_s^+ \cup r_s^-$ are represented for BHs and NSs in Fig. 2. In the BH case, we note the disconnection due to the presence of the horizons, which corresponds to the disconnection in the frequency planes of Fig. 1 where we limit the plots to the region $r_s < r_s^+$. Physical (timelike) observers are located between the inner cone and the outer cylinder surfaces of Fig. 2. As the spin increases into the NS regime, the surfaces merge at $a \geq M$, giving rise to a connected surface $r_s$ ($r_s^+ \cup r_s^-$), the Killing throat (or tunnel). For spins $a \in [0, 2M]$, the Killing bottleneck appears as restriction of the surface $r_s = r_s^+ \cup r_s^-$. We focus the analysis on the Killing bottleneck considering a 2D representation of the Killing throat versus the frequency $\omega$ at different spins, close to values where the bottleneck appears. Fig. 4

![Fig. 2. Plots of the surfaces $r_s^+$ (in units of mass) versus the frequency $\omega$ for different spin values $a/M$, including BH and NS geometries. The Killing throat and bottleneck are plotted.](image-url)
shows the behavior for BHs (gray region) and NSs. In the BH case, we do not consider the region inside the inner horizon (i.e. \( r < r_- \)). The outer horizon is the tangent point \( r_+ \) to the curve \( r_s \) and the minimum (regular) point of the surface \( r_s(\omega) \). For the extreme Kerr spacetime the horizon is a (non regular) cusp point in this representation. Each Killing throat is associated to one spacetime geometry. For example, the Killing throat defined for \( a = 2M \) has a characteristic singularity frequency \( \omega_0 = M/a = 0.5 \), which is the limiting frequency at the singularity (note that the restriction of the throat approaching the singularity corresponds to a null frequency gap \( \Delta \omega_{\pm} \), meaning that there are no physical stationary observers close to the singularity). At a fixed value \( r_s(\omega) \), one orbit on a throat \( a = 2M \), corresponds to two (positive) photon orbital frequencies \( \omega_+ \) and \( \omega_- \) (a part on the singularity \( r_s = 0 \)); viceversa, there can be two orbits \( r_1(\omega) < r_2(\omega) \) on the surface \( r_s \) (vertical lines in Fig. 2), where photons have equal orbital frequency (note that the photon orbital frequency \( \omega_{\pm} \) are limiting frequencies for timelike particles; therefore, they determine also the range of possible values of \( \omega \) for the physical stationary observers).

Killing bottleneck appears as restriction of the Killing throat in the light surfaces analysis and as ripples in the frequency planes of Fig. 1. We considered the possibility that these “horizons remnants” in WNSs were caused by failure of the BL coordinates “close” to the \( a = M \), in the region near the horizon \( r_+ = r_- = M \); nevertheless, Killing bottlenecks appear in the whole spin range \( a \in [M, 2M] \). To understand if the Killing bottlenecks are due to some known geometrical properties of the singularities, we focus our analysis on the spins \( a \in [M, 2M] \). Considering ZAMOs dynamical properties, we highlight some particular spin values of this range.

![Fig. 3. Curves \( r_-^\pm = \text{constant} \) and \( r_s^\pm = \text{constant} \) (inside panel) in the plane \((\omega, a/M)\). The numbers denote the constant radii \( r_s^\pm /M \) (light cylinders). The angular momentum and the velocity \((a, \omega)\) for \( r_s^\pm (a, \omega) = 0 \) are related by \( \omega = M/a \).](image-url)
Fig. 4. The radii $r^\pm$ versus the frequency $\omega$ for different values of the spin $a/M$ (numbers close to the curves). The gray region is the only region allowed for the case of BH spacetimes. The surfaces $\hat{r}^\pm$ at $a = M$ (extreme-BH-case) are shown in black-thick (from 4).

**Zero Angular Momentum Observers**

Some properties of spacetimes with bottlenecks shown in Fig. 2 can be related to the dynamics of ZAMOs, which are defined by the condition

$$L \equiv u^\alpha \xi_\alpha = g_{\alpha\beta} \xi^\alpha \xi^\beta = g_{t\phi} \xi_t + g_{\phi\phi} \xi_\phi = 0 \quad (d\phi/dt = -g_{\phi t}/g_{\phi\phi} \equiv \omega_Z = (\omega_+ + \omega_-)/2).$$

(9)

The variation of the orbital frequencies of the ZAMOs with the spin, for different orbits, shows the existence of extreme orbits as shown in Fig. (6), $\partial_a \omega_Z|_{\pi/2} = 0$, $\omega_e \equiv \omega_Z(r_e)$, where

$$r_e \equiv \frac{\sqrt{3}a^2 + \Upsilon^2}{3\sqrt{3} \Upsilon}, \quad \Upsilon \equiv \sqrt{9Ma^2 + \sqrt{3}a^4(27M^2 - a^2)}.$$  

(10)

The relation between ZAMOs orbital frequency and light frequencies $\omega_{\pm}$ can be inferred from Eq. (2). In Figs. 5 and 6 some properties of the ZAMOs dynamics are shown, which follow from the analysis of the Killing bottleneck and the light surfaces, Fig. 4.

These spins are significant for the orbital properties of the ZAMOs which exist exclusively inside the ergoregion of WNS with $a \leq 1.31M$. A more detailed analysis is performed in Figs. 5 and 6. See also $^4$ From the analysis of Fig. 5 it

$^4$The constant $\mathcal{L}$ and $\mathcal{E}$ shown in Figs 5 and Figs 6 are constant of motion associated respectively to $\xi_t$ and $\xi_\phi$ and compose the rotational version of the Killing fields i.e. the canonical vector fields $\check{V} \equiv (r^2 + a^2)d\theta + ad\phi$, and $\check{W} \equiv \partial_\phi + a\sigma^2\partial_t$, $\sigma = \sin \theta$. The contraction of the geodesic four-velocity with $\check{W}$ leads to the non-conserved quantity $\mathcal{L} = \mathcal{E}a\sigma^2$, which on the equatorial plane reduces to the constant $\mathcal{L} = \mathcal{E}a$. Considering the principal null congruence $\gamma_\pm \equiv \pm \partial_\tau + \Delta^{-1}\check{V}$, there is the angular momentum $\mathcal{L} = a\sigma^2$, that is $\bar{\ell} = 1$ (and $\mathcal{E} = +1$, in proper units), every principal null geodesic is then characterized by $\bar{\ell} = 1$, with $\mathcal{L} = \mathcal{E} = 0$ on the horizon. In this analysis, the dimensionless radius $R \equiv r/a$ is relevant. For more discussions on the role of this ratio as $\mathcal{L} = \mathcal{L}/a$ and $\ell = \ell/\mathcal{E}a$, see $^6$. From the analysis of Fig. 5 it
Fig. 5. The radius \( r(a) \), solution of \( \partial_a \Delta \omega_{\pm} = 0 \), i.e., it represents the critical points of the separation parameter \( \Delta \omega_{\pm} \equiv (\omega_+ - \omega_-)|_{\pi/2} \) on the equatorial plane \( \theta = \pi/2 \). The radius \( r^\pm_\omega \), where the orbital energy \( \mathcal{E} = 0 \), and the orbits \( \hat{r}^\pm_\omega \), for which \( \mathcal{L} = 0 \), are also plotted. Dashed lines represent the spins \( a_\sigma = 1.064306 M \), \( a_\mu = 4 \sqrt{2}/3 \approx 1.08866 M \), \( a_\Delta = 1.16905 M \) and \( a_3 = 3 \sqrt{3}/4 M \). The black region corresponds to \( r < r^+_\omega \). The radii \( r^\pm_\sigma : \partial_\alpha \Delta \omega_{\pm} = 0 \) are plotted as functions of \( a/M \)–from (3).

follows that the radii connected with the the frequency interval \( \Delta \omega_{\pm} \equiv \omega_+ - \omega_- \) are relevant to Killing bottlenecks:

\[
r^+_\omega \equiv \eta \cos \left[ \frac{1}{3} \arccos \left( -\frac{8a^2}{\eta^2} \right) \right], \quad r^-_\omega \equiv \eta \sin \left[ \frac{1}{3} \arcsin \left( \frac{8a^2}{\eta^2} \right) \right], \quad \eta \equiv \frac{2 \sqrt{8M^2 - a^2}}{\sqrt{3}},
\]

for \( r \in [0, 2\sqrt{2}M] \) where \( r^\pm_\sigma : \partial_\alpha \Delta \omega_{\pm} \big|_{\sigma} = 0 \) (maximum points) (11)

The plots of Fig.5 show also the radii related to the frequency gap \( \Delta \omega_{\pm} \). These radii are related to properties generally associated with so called “repulsive effects” of NSs. In this sense, the presence of ripples in the frequency sheets seem to be a

Fig. 6. Left panel: The ratio \( \mathcal{E}_-^{\pm} / \mathcal{L}_-^{\pm} \) and the angular momentum of the ZAMOs \( \omega_\epsilon^{\pm} \) as a function of \( a/M \) in the static limit \( r = r_\epsilon^{\pm} \). The angular momentum \( \omega_\epsilon^{\pm} \equiv \omega_\epsilon(r_\epsilon^{\pm}) \) which is a boundary frequency for the stationary observer (outer light surface) is plotted (gray curve). The radius \( r_\epsilon^{\pm} \) is defined by the condition \( \omega_\epsilon - (r_\epsilon^{\pm}) = 0 \), \( \omega_\epsilon \) is the ZAMOs angular velocity on \( r = r_\epsilon \), i.e. \( \omega_\epsilon(r_\epsilon^{\pm}) = \omega_\epsilon \). The maxima, related to the orbits \( r_\epsilon(a) \), are denoted by points. A zoom of this plot in the BH region is in the right panel.
related effect. It is also to be noted that the ripples are defined as a gap restriction in the frequency sheets; more precisely, they could be interpreted as due to the existence of two null orbits $r_1 \leq r_2$, where $\Delta \omega_{\pm}(r_1,r_2)$ is a minimum. That is, there is a pair of points $(r_1, r_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, on the light surface of a selected spacetime, where the photon orbital frequencies interval (range of possible timelike orbital frequency) is minimized. Clearly, the limiting case occurs for $r_1 = r_2 = r_\ast$: $\Delta \omega_{\pm}(r_\ast) = 0$, i.e., in $r_\ast \in \{0, r_\pm\}$. In this sense, the horizons and singularities can be interpreted as the limiting cases of the Killing bottlenecks (“horizons remnants”). For more details see [2].

3. Horizon extension: Unveiling BH–NS connections

Figure 5 shows a bottleneck configuration for spins $a \in [M, 1.31M]$. We now introduce the concept of metric bundle $g_{\omega}$ as the collection of Kerr metrics within the parameter range $a \in [a_0, a_g]$ with

$$a_{\pm}(r, \omega; M) \equiv \frac{2M^2\omega \pm \sqrt{r^2\omega^2 - M^2}}{(r + 2M)\omega^2}$$

and a constant frequency $\omega$, which characterizes the bundle $g_{\omega}$. Any geometry of the bundle possesses two distinct lightlike orbits, $r_1 \leq r_2$, whose frequencies coincide with the characteristic frequency of the bundle, i.e., $\omega(r_1) = \omega(r_2) = \omega$, constrained within the limiting geometries with $a_0$ and $a_g$. Moreover, the orbital distance $(r_2 - r_1)$ reaches a maximum in the bundle and is null on the borders $a_0$ and $a_g$. All and only the geometries of that bundle share this property. From

![Image](image.png)

Fig. 7. The metric bundle $g_{\omega}$ for the spin $\omega_{\pm} = \omega_h = 1/2$ and bundle origin $a_0 = 2M$. The horizon curve $(a_+(r) \equiv \sqrt{r(2M - r)})$ and $a_-$, solution of $r = r_\gamma$ where $r_\gamma \in \Sigma^+$ is the photon orbit in the ergoregion of the Kerr BH, are also plotted. The right panel shows a confinement of the metric bundle in the region bounded by the inner horizon and the bundle at $\omega = 0.5$.

From [7] we derive some general properties of the metric bundles: 1) The vertical axis of the extended planes contains all the origins $a_0 \in [0, \infty]$ of the metric bundles $g_{\omega}$. 2) The curve $a_\pm$ represents the horizon in the extended plane $\pi^+$ and the boundary
spin $a_g \in [0, M]$ is associated to this curve. 3) We can identify a correspondence between the inner horizon $r_-$ and outer horizon $r_+$ on the curve $a_+$, as shown in Fig. 7, we identify the BH-strip in the lower part of the panel and the NS-strip in the upper one as indicated. 4) The metric bundles $g_\omega$ are closed as a consequence of the closing of the $\xi$ orbits. 5) The metric bundles $g_\omega$ are tangent to the horizon.

Translating the bundle origin $a_0$ in all the range $a_0 > 0$ in Fig. 8, some relevant properties of the metric bundles and the Killing horizon appear: i) All the metric bundles are tangent to the horizon. ii) The metric bundles do not penetrate the horizon. iii) The space subtended by the horizon curve in the extended plane is not described by any metric of any bundle (in the sense of our analysis).

From the above properties, we can derive the following consequences: 1) The frequencies $\omega$ of each bundle $g_\omega$ (and on every point $r$ of each bundle) are all and only those of the horizon frequency $\omega_H$. It is, therefore, sufficient to know the horizon frequency $\omega_H$ in the extended plane to fix the photon orbital frequencies (and therefore the physical observer frequency range) in each point of any BH or NS geometry of the Kerr spacetime family. 2) The horizon arises as the envelope surface of all the metric bundles. 3) The part of the horizon curve corresponding
to the inner horizon in $\pi^+$ is built partially by metric bundles all contained in the region inside the inner horizon, i.e., confined in $a_g \in [0, M]$, $r \in [0, M]$ and $a_0 \in [0, M]$. These bundles have origins in $a_0 \in [0, M]$. However, these special bundles are not sufficient to construct, as envelope surface, the whole inner horizon in the extended plane. 4) The bundles necessary for the construction of the other part of the the inner horizon in $\pi^+$ have origin in $a_0 \in [M, 2M]$, i.e., in the WNS geometries, where the Killing bottleneck appear. 5) The Killing bottleneck appears to be related to the properties of these special metric bundles, which are involved in the construction of the inner horizon as envelope surface. 6) The portion of the horizon curve corresponding to the outer Killing horizon on the equatorial plane is constructed by metric bundles with origins $a_0 > 2M$, which we identify as SNS.

We close this section noting that the whole set of photon limiting orbital frequencies $\omega_\pm$ (or alternatively the light surfaces $r_\pm^\pm$) of a single Kerr geometry with spin $\tilde{a}$ (and therefore the range of orbital frequencies for the physical observers) in the extended plane is given by the collection of points of all the metric bundles on the horizontal lines $\tilde{a}$ = constant in Figs. 4 and 5. These frequencies are all and only those of the horizon frequency $\omega_H$. We discuss the interpretation of this result in the next conclusive section. Finally, we mentioned above that the internal bundles with origin in $a_0 \in [0, M]$ are all confined in the region of the inner horizon in $\pi^+$, i.e., in $a_g \in [0, M]$. This can be shown in several ways. In particular, considering again the horizon frequencies $\omega_\pm^\pm$, we introduce the radii $r_\pm^\pm$, defined as

$$r_-^\pm = \frac{1}{2} \left( \sqrt{\frac{32r_-}{a^2} - a^2 + 6\sqrt{1 - a^2} - 22 - r_-} \right) : \omega_-(r_-^\pm) = \omega_-(r_-) = \omega_h^-,$$

$$r_+^\pm = \frac{1}{2} \left( \sqrt{\frac{32r_+}{a^2} - a^2 - 6\sqrt{1 - a^2} - 22 - r_+} \right) : \omega_+(r_+^\pm) = \omega_+(r_+) = \omega_h^+,$$

where $(r_- < r_-^\pm < r_+ < r_+^\pm)$,

where we have used dimensionless units. The radii $(r_+^\pm, r_-^\pm)$ correspond to photon orbits with frequencies $\omega_\pm$ of the BH horizons $\omega_h^\pm$. The orbital frequency of the inner horizon has a replica on an orbit $r_-^\pm \in [0, r_-]$ “inner horizon frequency confinement”. This result is in agreement with the BHs thermodynamic properties discussed in Sec. 2. Also, the relation $(a_g, a_0, r_h)$, where $r_h$ is obtained through $\omega_h$, has been mentioned to be bijective. We can show this through the relation between $(a_g, a_0)$:

$$\forall a_0 > 0, \quad a_g = \frac{4a_0M^2}{a_0^2 + 4M^2}$$

where $a_g \in [0, M]$ and

$$\lim_{a_0 \to 0} a_g = \lim_{a_0 \to \infty} a_g = 0, \quad a_g(a_0 = 2M) = M.$$

This relation also allows us to formalize the BH-BH correspondence (construction of the inner horizon as an envelope surface in $\pi^+$), the BH-WNS relation (inner horizon construction) and the BH-SNS relation (construction of the outer horizon).
Particularly, the horizon relates in the extended plane BHs with NSs (through the origins of the metrics bundles $a_0$ and $a_g$). In this sense, the NSs can be interpreted as necessary for the construction (as envelope surface) of the inner and outer horizons in the extended plane. Note that the static case of the Schwarzschild geometry, $a = 0$, can be seen as the limiting case in Figs. 7 and 8 where the horizon $a_g = 2M$ is in correspondence with the limiting SNS bundle with origin $a_0 = +\infty$. The plane $\pi^+$ in Figs. (7) and (8) have several symmetries. Note that in Fig. (8) negative values of $a_0$ and $a_g$ are possible in the metric bundles and are related to $\omega_{\pm} < 0$ frequencies, which are possible outside the ergoregion ($r > r^*_+$).

The quantity $A^{+}_{r^*} = \pi/2$ is the area of the region of $\pi^+$ bounded by the horizon (dimensionless quantities); $A$ is the area of the regions in the extended plane $\pi^+$ bounded by the curves $a^{\pm}_\omega$, defining the metric bundles $g_{\omega}$. $A$ is a decreasing function of the frequency $\omega$, shrinking at the origins $a_0 < M$, i.e. $\omega_0 = M/a > M$, where $g_{\omega}^{\pm}$ are all bounded by the inner horizons; vice versa, the region areas grow as the spin-mass ratio increases in the NS geometries. We repeated this analysis in the case of Reissner-Nordström and Kerr-Newman spacetimes and similar results are found.

4. Final remarks

In this work, we investigated the properties of stationary observers on the equatorial plane of the Kerr spacetime. The generalization to the off-equatorial case as well as the Reissner-Nordström and Kerr-Newman spacetimes is presented elsewhere. We focused on the behavior of the frequency of stationary observers. To emphasize its peculiarities, we introduced the concept of Killing throats and bottlenecks. If we consider the frequency as a function of the spin, certain features appear that are better explained by introducing the concepts of extended planes and metric bundles. In the case of the Kerr metric on the equatorial plane, the extended plane is essentially equivalent to the function that relates the frequency with the spin.

Metric bundles and horizons remnants appear related to the concept of pre-horizon regimes. There is a pre-horizon regime in the spacetime when there are mechanical effects allowing circular orbit observers to recognize the close presence of an event horizon. This concept was introduced in and detailed for the Kerr geometry in. The analysis of the pre-horizon structure led to the conclusion that a gyroscope would observe a memory of the static case in the Kerr metric. It is clear that this aspect could have an essential relevance in the investigation of the collapse. In the extended plane of the Kerr metric, the frequency on the horizons determines a set of metrics, a metric bundle, describing in general BHs and NSs, and also the limiting frequencies for stationary observers, which correspond to light-like orbits. This fact can be interpreted as determining a connection between BHs and NSs. In the extended plane, the NS solutions have a clear meaning in relation to the construction as envelope surface of portions of the horizon in $\pi^+$. The inner
BH horizon is connected to the origin of BH bundles and the outer horizon establishes BHs-SNSs correspondence. Moreover, the horizon in the extended plane can be interpreted as the envelope surface of all metric bundles. On the other hand, the metric bundles are all defined by all and only the frequency of the horizon. In this sense, the corresponding inner horizon in $\pi^+$ is partially constructed by BHs metric bundles. The inner horizon is completed by bundles including BHs and WNSs. This property appears related with the Killing bottlenecks appearing in the light surfaces. Interestingly, the outer horizon in $\pi^+$ is generated by SNSs metric bundles. It is then possible to argue that this result could be of interest for the investigation of the gravitational collapse. Indeed, suppose that the collapse is a quasi-stationary process in which each state is described by a Kerr spacetime. Since rotating astrophysical compact objects are characterized by spin parameters, which correspond to NS configurations ($a/M > 1$), the formation of a BH ($a/M \leq 1$) would necessarily imply passing through a series of states with spin parameters in the NS regime. In this case, the extended plane of the Kerr metric as described above could contain the different states which are necessary for the formation of a BH. This fact has the interesting consequence that only horizon frequencies in extended plane determine the frequencies $\omega_{\pm}$ at each point, $r$, on the equatorial plane of a Kerr BH or NS geometry. All the frequencies $\omega_{\pm}(r)$ on the equatorial planes are only those of the horizon in $\pi^+$. Another relevant aspect connected with this fact is the confinement of the horizon in the sense of the frequencies given in Eq. (13). These aspects are currently under investigation.

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