Decoherence in quantum cosmology at the onset of inflation

A. O. Barvinsky¹, A. Yu. Kamenshchik²³, C. Kiefer⁴ and I. V. Mishakov⁵

¹Theory Department, Lebedev Institute and Lebedev Research Center in Physics, Leninsky Prospect 53, Moscow 117924, Russia
²L. D. Landau Institute for Theoretical Physics, Russian Academy of Sciences, Kosygin Street 2, Moscow, 117334, Russia
³Landau Network-Centro Volta, Villa Olmo, Via Cantoni 1, 22100 Como, Italy
⁴Fakultät fur Physik, Universität Freiburg, Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany
⁵“Polyprom M”, Komsomolsky Prospect 13, Moscow, Russia

Abstract

We calculate the reduced density matrix for the inflaton field in a model of chaotic inflation by tracing out degrees of freedom corresponding to various bosonic fields. We find a qualitatively new contribution to the density matrix given by the Euclidean effective action of quantum fields. We regularise the ultraviolet divergences in the decoherence factor. Dimensional regularisation is shown to violate the consistency conditions for a density matrix as a bounded operator. A physically motivated conformal redefinition of the environmental fields leads to well-defined expressions. They show that due to bosonic fields the Universe acquires classical properties near the onset of inflation.

PACS: 98.80.Hw, 98.80.Bp, 04.60.-m
Keywords: Quantum cosmology; inflation; quantum-to-classical transition
1. Introduction

The combination of general relativity with particle physics can yield viable models for the early Universe. This leads, in particular, to the idea that the Universe underwent a period of accelerated expansion ("inflation") at an early stage. Inflation not only loses the dependence on initial conditions, it also provides a quantitative scenario for structure formation. The interesting question is of course why and how inflation arises in the first place. It is generally assumed that a satisfactory answer can only be found in the realm of quantum cosmology. In this respect, ideas about the quantum nucleation of the Universe from "nothing" could yield a quantitative scenario for the emergence of classical inflation. The problem of classical initial conditions is then replaced by the question for an appropriate choice of the quantum cosmological wave function. Well-known choices are the no-boundary proposal [1] and the tunneling proposal [2], which are in fact a whole class of proposals. There exist other interesting proposals such as the SIC put forward in [3], that is in particular well equipped to deal with the case of a classically recollapsing universe.

Inflation is assumed to take place near the GUT mass scale that is thought to lie about five orders of magnitude below the Planck scale. For this reason it is not necessary to rely on the most fundamental level of quantum gravity (such as superstring theory), and effective theories such as canonical quantum gravity should give an excellent approximation. Using either the Euclidean path integral or the Wheeler-DeWitt equation, the "transition" from the classically forbidden region to the classical inflationary regime has been investigated in great detail. Using normalised wave functions, a probability peak for the mass scale of inflation has been calculated in various models [4, 5]. Such a probability peak has been interpreted – assuming that we inhabit a generic Universe – as providing a criterium to select amongst the members of an "ensemble" of classical universes.

However, quantum theory does not yield a classical ensemble. Since interference effects can play a crucial role, it would be inconsistent to interpret these results directly as giving probabilities for inflationary universes with different Hubble parameters. A complete analysis would thus have to include a quantitative discussion of the quantum-to-classical transition.

It is now generally accepted that the classical properties for a subsystem arise from the irreversible interaction of this quantum system with its natural environment. Starting with the pioneering work of Zeh in the seventies [6], this concept of decoherence has been developed extensively, see [7] and [8] for reviews. Quite recently, this continuous loss of coherence was observed in quantum-optical experiments [9].

Quantum-to-classical transition through decoherence has also proven fruitful in quantum cosmology. The first application of this concept to gravitational systems was performed in
a Newtonian framework where the (formally quantised) gravitational field acquired classical properties through interaction with masses [10]. In quantum cosmology, it was suggested to consider background degrees of freedom (such as the scale factor) as the relevant system and higher-order perturbations (such as gravitational waves or density fluctuations) as an environment [11]. A quantitative discussion of this idea was first done in [12], where it was demonstrated how the scale factor and the inflaton field can acquire classical properties. This idea was further pursued in many papers [7]. The quantum-to-classical transition through decoherence plays also a crucial role for the primordial fluctuations that eventually serve as classical seeds for galaxies and clusters of galaxies [13].

One big problem that remained in most of the above papers was the issue of regularisation. Since there are infinitely many environmental modes involved, there arise formal divergences that have to be dealt with. This was done in most cases only heuristically through the choice of a more or less appropriate cutoff in the number of modes. It was argued in [14] that dimensional regularisation is only applicable for the phase part of the decoherence factor, not its absolute value. This observation will be confirmed in detail in our paper.

Motivated by the finiteness of the decoherence factor in QED, a redefinition of the fields was performed in a particular quantum cosmological model in [15] to find a finite result. (A similar proposal was discussed recently in [16].) In the present paper we shall find in a wide class of models that there exists a distinguished redefinition that renders the decoherence factor finite.

A redefinition of environmental degrees of freedom changes, however, the reduced density matrix for the system (in particular, the system momentum is changed). This was discussed in [17], where it was suggested to use a redefinition that eliminates that part of the decoherence effect that solely arises from the change in three-volume, so that only decoherence due to “particle creation” remains. It is thus clear that a physical principle has to be invoked to select the appropriate choice of environmental variables.

The purpose of our paper is to explore the possibilities to get a physically reasonable, finite, decoherence factor that can sensibly be calculated to understand the onset of inflation. For this purpose, the non-diagonal elements in the reduced density matrix have to be discussed. Using the criterium of normalisability for the wave function in one-loop quantum cosmology, the diagonal part was already discussed in the earlier papers [18, 1, 19, 3, 20] where the focus was on the derivation of probabilities. The same criterium will be employed here for the study of decoherence. We shall apply our discussion to a variety of models. The case of fermions, however, is discussed in a separate paper [21], since a new formalism has to be applied and novel features appear.

Our paper is organised as follows. In section 2 we briefly review the form of the quantum
cosmological wave function when the higher-order modes are in their vacuum state. We
then give a derivation of the correct normalisation for this state, which is different from
a (wrong) normalisation that is sometimes used in the literature. In section 3 we discuss
the diagonal part of the reduced density matrix for the inflaton field and, in particular,
its dependence on the one-loop effective action. Sections 4–6 comprise the main part of
our paper. In section 4 we discuss general properties of the non-diagonal elements of the
reduced density matrix. In section 5 we demonstrate that the application of dimensional
regularisation violates basic properties of the density matrix. Section 6 then shows that there
exists a distinguished redefinition of the environmental degrees of freedom that renders the
decoherence factor finite. Section 7 is devoted to a physical interpretation of the results
obtained and an outlook on future work.

2. Cosmological wave function

We consider the quantum-cosmological wave function in the context of chaotic inflation.
This includes all cases of boundary conditions mentioned in the Introduction [1, 2, 3]. The
background variables are the scale factor $a$ of the inflationary universe and the inflaton field
$\varphi$. In addition we assume the presence of spatially inhomogeneous fields $f(x)$ that may be
the higher modes of the inflaton or some other field. This fields play the role of microscopic
(environmental) modes and are treated perturbatively. They will thus be integrated out
below to study their decohering influence on the background variables.

We shall consider the cosmological quantum state in the framework of the reduced phase
space quantisation, that is, in the representation of physical variables. This has the advantage
that a well-defined inner product is available for the interpretation of the density matrix. We
must emphasise, though, that this reduced quantisation is limited to semiclassical branches
of the wave function, but this is the framework where the present investigation takes place.

In this representation the role of $f(x)$ is played by physical polarisations of linear fields
of all possible spins, and there is one physical degree of freedom in the homogeneous sector
of $(a, \varphi)$, which without loss of generality can be identified with $\varphi$. Thus, the full set of
physical variables is

$$\phi(x) = [\varphi(t), f(t, x)], \quad x = (t, x).$$

With the decomposition of $f(t, x)$ into a discrete series of spatial orthonormal harmonics
$Q_n(x)$ on a section of a closed three-space,

$$f(t, x) = \sum_n f_n(t)Q_n(x),$$

(2.1)
\( \phi(x) \) can equally be represented by the countable set of \((\varphi(t), f_n(t))\).

Other variables (including lapse and shift functions, unphysical components of linear vector and tensor fields) are parametrised in terms of physical ones within a particular choice of gauge conditions fixing the local gauge and coordinate symmetries. In the sector of \((a, \varphi)\), for example, in the cosmic-time gauge this parametrisation can have a very simple form in the slow-roll approximation, corresponding to slowly varying (practically constant) inflaton field. It can be taken to coincide with an approximate classical solution with the cosmic time \(t\),

\[
\begin{align*}
\varphi(t) &= \varphi, \\
a(t) &= \frac{1}{H} \cosh Ht, \quad H = H(\varphi),
\end{align*}
\]

(2.3)

(2.4)

where \(H(\varphi) = 8\pi V(\varphi)/3m_p^2\) is the Hubble constant generated by the inflaton potential \(V(\varphi)\). We emphasise that the time parameter that appears in the reduced formalism is equivalent to the standard WKB-time parameters that appears in the semiclassical approximation to the Wheeler-DeWitt equation [22].

In the no-boundary or tunneling prescription of the cosmological wave function, the classical background (2.4) arises as an analytic continuation of the solution of Euclidean Einstein equations,

\[
\begin{align*}
\varphi(\tau) &= \varphi, \\
a(\tau) &= \frac{1}{H} \sin H\tau, \quad H = H(\varphi),
\end{align*}
\]

(2.5)

into the complex plane of Euclidean time \(\tau\),

\[
\tau = \frac{\pi}{2H} + it.
\]

(2.6)

This is usually interpreted as a “quantum nucleation” of the Lorentzian DeSitter spacetime from the gravitational instanton – the Euclidean four-dimensional hemisphere with radius \(R = 1/H(\varphi)\). In more standard language, the Euclidean section of this instanton just corresponds to a classically forbidden region, and the analytic continuation into the Lorentzian regime corresponds to the emergence of time from timeless quantum gravity [23]. The cosmological wave function in Lorentzian time can also be understood as an analytic continuation of the Euclidean wave function, which in the semiclassical one-loop approximation has the form

\[
\Psi_E(\tau|\varphi, f) = \frac{1}{\sqrt{u_{\varphi}(\tau)}} e^{-I(\tau, \varphi)} \prod_n \psi_n^E(\tau, \varphi|f_n).
\]

(2.7)

5
Here $I(\tau, \varphi)$ denotes the Euclidean Hamilton-Jacobi function for the solution (2.3), and

$$
\psi_n^E(\tau, \varphi | f_n) = \frac{1}{\sqrt{u_n(\tau)}} \exp \left( -\frac{1}{2} a^3(\tau) \frac{\dot{u}_n(\tau)}{u_n(\tau)} f_n^2 \right), \tag{2.8}
$$

where $u_n(\tau)$ is the set of basis functions of the Euclidean linearised equations of motion for the Euclidean action $I[\varphi, f]$ of physical variables; it satisfies the regularity condition on the Euclidean hemisphere containing the pole $\tau = 0$,

$$
F_n(d/d\tau)u_n(\tau) = 0, \quad u_n(\tau) = \text{reg}, \quad F_n(d/d\tau)\delta(\tau - \tau') = \frac{\delta^2 I[\varphi, f]}{\delta f_n(\tau) \delta f_n(\tau')}. \tag{2.9}
$$

Analogously, $u_\varphi(\tau)$ in (2.7) denotes the basis function of a spatially homogeneous inflaton mode. This expression was obtained both by solving the Wheeler-DeWitt equation and by calculating the Euclidean path integral [19, 24, 25, 26]. It is important that in the above expressions the basis functions and Gaussian states of quantum variables $f_n$ are all calculated on the background of the classical solution parametrised by the inflaton variable $\varphi$ which, as an argument of the wave function, is also an essential quantum degree of freedom.

The analytic continuation to the Lorentzian wave function

$$
\Psi(t|\varphi, f) = \Psi_E(\pi/2H + it|\varphi, f) \tag{2.11}
$$

takes place by introducing the time-independent Euclidean action $I(\varphi)$ of the DeSitter instanton and the Lorentzian Hamilton-Jacobi function $S(t, \varphi)$,

$$
I(\pi/2H + it, \varphi) = \frac{1}{2} I(\varphi) - iS(t, \varphi), \tag{2.12}
$$

$$
I(\varphi) = 2I(\pi/2H, \varphi) \simeq -\frac{3m_P^4}{8V(\varphi)}, \tag{2.13}
$$

and the basis functions of the Lorentzian linearised field equations,

$$
v_n(t) = [u_n(\pi/2H + it)]^*, \quad v_n^*(t) = u_n(\pi/2H + it)], \tag{2.14}
$$

$$
F_n^L(d/dt)v_n(t) = 0, \quad F_n^L(d/dt)\delta(t - t') = \frac{\delta^2 S}{\delta f_n(t) \delta f_n(t')} \tag{2.15}
$$

The latter turn out to be the basis functions of the DeSitter-invariant Bunch-Davies vacuum [23, 30]. The cosmological wave function thus takes the form of the product

$$
\Psi(t|\varphi, f) = \frac{1}{\sqrt{v_\varphi^*(t)}} e^{-I(\varphi)/2 + iS(t, \varphi)} \prod_n \psi_n(t, \varphi | f_n) \tag{2.16}
$$

\footnote{Here, for simplicity, we consider the case of a scalar field with $F_n(d/d\tau) = -(d/d\tau)a^3(d/d\tau) + ...$, while for the generic case [23] $a^3\dot{u}_n$ in the exponential should be replaced by the corresponding Wronskian operator linear in time derivative acting on $u_n$.}
of the background wave function and the infinite set of Gaussian states of harmonic oscillators representing this vacuum,

$$\psi_n(t, \varphi|f_n) = \frac{1}{\sqrt{v_n^*(t)}} \exp \left(-\frac{1}{2} \Omega_n(t) f_n^2 \right),$$  \hspace{1cm} (2.17)$$

$$\Omega_n(t) = -ia^3(t) \frac{\dot{v}_n^*(t)}{v_n^*(t)}. \hspace{1cm} (2.18)$$

An answer similar to (2.16) holds also for the case of the tunneling wave function – the only difference is the opposite sign of the Euclidean action in the tree-level part. In what follows we shall mainly consider the no-boundary wavefunction, but the main conclusions and algorithms for the decoherence aspects of the system will be equally applicable in the tunneling case as well.

An important property of these vacuum states is that their norm is conserved along any semiclassical solution (2.4),

$$\langle \psi_n, \psi_n \rangle \equiv \int df_n |\psi_n(f_n)|^2 = \sqrt{2\pi} [\Delta_n(\varphi)]^{-1/2},$$  \hspace{1cm} (2.19)$$

$$\Delta_n(\varphi) \equiv ia^3(v_n^* \dot{v}_n - \dot{v}_n^* v_n) = \text{constant}. \hspace{1cm} (2.20)$$

Note that $\Delta_n(\varphi)$ is just the (constant) Wronskian corresponding to (2.15). We want to emphasise, however, that $\Delta_n$ is a nontrivial function of the background variable $\varphi$, since it is defined on full configuration space and not only along a semiclassical trajectory. It can thus not be factored out by a deliberate normalisation of these states to unity – a procedure that is sometimes assumed in the literature [16]. The replacement of $\psi_n(t, \varphi|f_n)$ with the normalised states

$$\phi_n(t, \varphi|f_n) = \left(\frac{\Delta_n}{2\pi}\right)^{1/4} \psi_n(t, \varphi|f_n) = \left(\frac{\text{Re} \Omega_n}{\pi}\right)^{1/4} \left(\frac{v_n}{v_n^*}\right)^{1/2} \exp \left(-\frac{1}{2} \Omega_n f_n^2 \right),$$  \hspace{1cm} (2.21)$$

$$\langle \phi_n, \phi_n \rangle = 1$$  \hspace{1cm} (2.22)$$

would be inconsistent from the viewpoint of the full Wheeler-DeWitt equation or the path-integral representation of its solution, because the multiplication of this solution by a configuration space-dependent object would violate the Wheeler-DeWitt equation. We shall

\footnote{In the case of the tunneling wave function, the Euclidean path integral derivation does not work, and it can be obtained as an alternative solution of the Wheeler-DeWitt equation linearly independent from the no-boundary one. Strictly speaking, the behaviours of both of these wave functions in the Euclidean and Lorentzian domains does not reduce to a simple analytic continuation presented above. For a no-boundary state one underbarrier branch matches with two complex-conjugated branches in the Lorentzian domain, while for the tunneling state the situation is reversed. For our presentation here these subtleties are immaterial. We shall briefly get back to this issue in the concluding section when discussing the decoherence between the branches.}
give in the rest of this section the justification of the use of the unnormalised vacuum wave functions in \(2.17\).

2.1. Justification of wave function prefactors: reduction method for functional determinants

One type of justification comes from the general semiclassical theory of constrained dynamics \([27]\) including the Wheeler-DeWitt equation as a particular case. As shown in \([27]\), there exists a unitary map between the solutions of the quantum Dirac constraints and physical wave functions in reduced phase-space quantisation. These wave functions, like \((2.8)\), have the form of a semiclassical packet with a prefactor given by the Jacobian of the transformation from the initial coordinates of the flow of classical trajectories to their final coordinates. The matrix of this Jacobian is the solution of linearised equations of motion, just like the basis functions in the prefactors of \((2.8)\), and these basis functions are not normalised to unity in the sense of a Wronskian inner product \((2.20)\).

A more useful justification of this fact comes from the Euclidean path-integral representation of the no-boundary wave function, which goes as follows. The Hartle-Hawking path integral in the one-loop approximation reads

\[
\Psi_{E}^{1\text{-loop}}(\tau|\varphi, f) = \int D\phi(x)e^{-I[g(x)]}_{\text{one-loop}} = e^{-I(\tau|\varphi, f)} \left[ \text{Det} F \right]^{-1/2}.
\] (2.23)

It contains the tree-level classical action which when expanded up to quadratic order in microscopic variables,

\[
I(\tau|\varphi, f) = I(\tau, \varphi) + \frac{1}{2} a^{3}(\tau) \frac{\dot{u}_{n}(\tau)}{u_{n}(\tau)} f_{n}^{2},
\] (2.24)

gives rise to exponentials of Gaussian oscillator functions; it also contains the one-loop functional determinant of the operator

\[
F = \frac{\delta^{2}I[\phi]}{\delta\phi(x)\delta\phi(y)}
\] (2.25)

subject to Dirichlet conditions at the boundary of the Euclidean spacetime ball of “radius” \(\tau\). These boundary conditions follow from the fact that the integration in the path integral requires fixed values of fields at the boundary as prescribed by the arguments of the wave function. The definition of the functional determinant can then be given in terms of the eigenvalues of the corresponding eigenvalue problem

\[
F\phi_{\alpha}(\tau') = \lambda_{\alpha}\phi_{\alpha}(\tau'), \quad 0 \leq \tau' \leq \tau, \quad \phi_{\alpha}(\tau) = 0, \quad \lambda_{\alpha} > 0,
\] (2.26)

\[
\ln \text{Det} F = \text{Tr} \ln F = \sum_{\alpha} \ln \lambda_{\alpha}.
\] (2.27)
Here $\alpha$ is a collective index of arbitrary nature enumerating these eigenvalues.

Suppose now that we can write down an equation which determines these eigenvalues in the form

$$E(\lambda) = 0. \tag{2.28}$$

Its left-hand side $E(\lambda)$ can be used to represent the sum over the eigenvalues as an integral in the complex plane of $\lambda$ over the contour $C_+$ surrounding the spectrum of the operator – so that the residues in the roots of (2.28) reproduce the contributions of different eigenvalues

$$\sum_\alpha \ln \lambda_\alpha = \frac{1}{2\pi i} \int_{C_+} d\lambda \ln \lambda \frac{d}{d\lambda} \ln E(\lambda). \tag{2.29}$$

In what follows we assume that this spectrum is positive and lies on the positive real axis. Omitting some technical details [28] we integrate in this integral by parts and rotate the contour of integration into the contour $C_-$ so that it begins surrounding the origin and running over the upper and lower shores of the negative real axis. In view of the absence of other roots in the complex plane of $\lambda$, the only arising residue of the resulting integral gives the final answer in terms of the left-hand side of the eigenvalue equation

$$\sum_\alpha \ln \lambda_\alpha = -\frac{1}{2\pi i} \int_{C_-} d\lambda \frac{\lambda}{\ln E(\lambda)} = \ln E(0). \tag{2.30}$$

In the basis of orthonormal spatial harmonics the operator has a diagonal form, so that the total equation on its eigenvalues decomposes into an infinite product of partial equations

$$F = \text{diag}\{F_n(d/d\tau)\}, \quad E(\lambda) = \prod_n E_n(\lambda), \tag{2.31}$$

and the final answer reads

$$\sum_\alpha \ln \lambda_\alpha = \sum_n \ln E_n(0) \equiv \sum_{n=0}^\infty \text{dim}(n) \ln E_n(0), \tag{2.32}$$

where the summation over the collective index of these harmonics is rewritten as a sum involving the degeneracies of the corresponding quantum numbers.

Let us now get back to the calculation of a particular functional determinant in the no-boundary wave function. The role of eigenfunctions in this case is played by the basis functions of the operator modified by the extra (mass) term, satisfying the condition of regularity at $\tau = 0$,

$$(F_n(d/d\tau') - \lambda)u_{n,\lambda}(\tau') = 0, \quad 0 < \tau' < \tau. \tag{2.33}$$
Therefore the left-hand side of the eigenvalue equation takes the form

$$E_n(\lambda) = u_{n,\lambda}(\tau), \quad (2.34)$$

and its value at zero argument coincides with the original regular basis function $E_n(0) = u_{n,0}(\tau) = u_n(\tau)$ taken at $\tau$ - the “radial” coordinate of the boundary. The use of the equation (2.32) then immediately gives

$$[\text{Det} F]^{-1/2} = \exp \left( -\frac{1}{2} \sum \alpha \ln \lambda_\alpha \right) = \frac{1}{\sqrt{u_0(\tau)}} \prod_n \frac{1}{\sqrt{u_n(\tau)}}, \quad (2.35)$$

where we have also included the contribution of the inflaton sector of the total diagonal operator $F = \text{diag}\{F_\varphi(d/d\tau), F_n(d/d\tau)\}$.

The above derivation is presented in a simplified form, because it disregards special measures to provide the vanishing of the arc integrals at the infinity of the complex plane in the transition from (2.29) to (2.30), regularisation of ultraviolet divergences and the precautions against spurious roots of (2.32) caused by multiplication of $E_n(\lambda)$ by functions of $\lambda$. These problems are considered in much detail in the authors’ paper [28], where it is shown, in particular, that the normalization of the Euclidean basis functions in (2.34) and (2.35) should be chosen in such a way that the coefficient of their power-like asymptotic behaviour at $\tau = 0$,

$$u_n(\tau) \simeq U_n \tau^{\mu_n}, \quad \tau \to 0, \quad (2.36)$$

is a trivial field-independent constant. Such a normalisation is very different from the normalisation to unity with respect to the Wronskian inner product.

3. Wave function norm, Euclidean effective action, and the diagonal of the density matrix

The nontrivial normalisation of vacuum states results in a nontrivial quantum distribution of cosmological parameters in inflaton space, yielding a probability distribution for the inflaton. This mechanism was used in [18, 23] to analyse the normalisability of the cosmological wave function and to find initial conditions for inflation [3, 4, 19]. We briefly repeat these results here, for they will play a substantial role also in the decoherence properties of the density matrix.

Consider the quantum distribution function of the inflaton field which is just the diagonal element of the reduced density matrix obtained by tracing out the microscopic variables,

$$\rho(t|\varphi) \equiv \rho(t|\varphi, \varphi) = \int df |\Psi(t|\varphi, f)|^2. \quad (3.1)$$
It involves the infinite product of norms (2.19)-(2.20) that can be reinterpreted by using the reduction technique for functional determinants of the previous section. For this, let us introduce two sets of Euclidean basis functions inhabiting the total spherical manifold of the DeSitter-instanton (the full sphere with the scale factor (2.15) extended to the full range of the latitude angle $0 < \tau/H < \pi$),

$$F^E (d/d\tau) u^\pm (\tau) = 0, \quad u^-(\tau) = u(\tau), \quad u^+(\tau) = u(\pi/H - \tau).$$  \hspace{1cm} (3.2)

They are related by mirror map with respect to the equatorial section of the instanton $\tau = \pi/2H$ at which the nucleation into the Lorentzian domain takes place. Generically, $u_-(\tau)$ is regular at $\tau = 0$ but singular at the antipodal pole of the sphere, $\tau = \pi/H$, and vice versa for $u_+(\tau)$. In the Lorentzian domain they give rise to negative and positive frequency basis functions

$$u^-(\pi/2H + it) = v^*(t), \quad u^+(\pi/2H + it) = v(t),$$  \hspace{1cm} (3.3)

the Wronskian norm of which can thus be rewritten as a Wronskian product of two different Euclidean basis functions calculated at $\tau = \pi/2H$,

$$\Delta_n(\varphi) \equiv ia^3 (v_n^* \dot{v}_n^* - \dot{v}_n^* v_n^*)_{t=0} = a^3 (u_+^+ \frac{du^-}{d\tau} - \frac{du^+}{d\tau} u^-)_{\tau = \pi/2H}. \hspace{1cm} (3.4)$$

Let us now modify these functions by the $\lambda$-term in analogy with the previous section, $u^\pm_\lambda (\tau)$,

$$(F_n (d/d\tau) - \lambda) u^\pm_\lambda (\tau) = 0, \quad 0 < \tau < \pi/H,$$  \hspace{1cm} (3.5)

still demanding the regularity of $u^-_\lambda (\tau)$ at $\tau = 0$ and of $u^+_\lambda (\tau)$ at $\tau = \pi/H$, and define the function

$$E_n(\lambda) = a^3 \left( u_\lambda^+ \frac{d u^-}{d\tau} - \frac{d u^+}{d\tau} u^-_\lambda \right)_n. \hspace{1cm} (3.6)$$

Equation (2.34) with this function will be just the requirement of the functional linear dependence of $u^-_\lambda (\tau)$ and $u^+_\lambda (\tau)$ (in view of the second-order equation of motion the equality of the functions and their derivatives at one point implies their equality at any $\tau$). This linear dependence means that there are solutions of (3.3) that are regular on the whole instanton including both its poles at particular values of $\lambda$ satisfying this equation. These values represent the spectrum of the operator on the whole DeSitter instanton, and according to (2.32) the product of norms arising in (3.1) reads, up to a numerical constant, in terms of this spectrum,

$$\prod_n \langle \psi_n, \psi_n \rangle = C\sqrt{\Delta_\varphi} \exp \left( -\frac{1}{2} \sum_\alpha \ln \lambda_\alpha \right) = C\sqrt{\Delta_\varphi} \exp \left( -\frac{1}{2} \text{Tr} \ln F \right).$$  \hspace{1cm} (3.7)
It can thus be expressed as the one-loop Euclidean effective action of the theory calculated on the DeSitter-instanton of the radius \(1/H(\varphi)\) \[18, 19, 26, 29\],

\[
\Gamma_{\text{1-loop}}(\varphi) = \frac{1}{2} \text{Tr} \ln F .
\] (3.8)

(The total one-loop action contains the contribution of the inflaton mode missing in this equation; this explains the origin of an additional \(\Delta\varphi\) factor in (3.7).)

Thus, the distribution function takes the form

\[
\rho(t|\varphi, \varphi') = C \sqrt{\Delta \varphi} \exp(-I(\varphi) - \Gamma_{\text{1-loop}}(\varphi)) .
\] (3.9)

The derivation of this result was presented here for the case of the no-boundary wave function. It is valid, however, also in the tunneling case of the model with microscopic oscillator modes \[27\]. The basic difference in this case is the opposite sign of the tree-level Euclidean action \(I(\varphi)\) in the exponential.

The advantage of this representation is that there exist powerful methods of calculation and covariant renormalisation of \(\Gamma_{\text{1-loop}}(\varphi)\). These methods, in particular, lead to the asymptotic scaling behaviour

\[
\Gamma_{\text{1-loop}}(\varphi) = Z \ln \frac{H(\varphi)}{\mu} , \quad H(\varphi) \to \infty ,
\] (3.10)

with an easily calculable coefficient \(Z\) \[4, 5\]. For big positive \(Z\) the one-loop contribution guarantees the normalisability of the cosmological wave function at large \(\varphi\). In the model with large negative nonminimal coupling of the inflaton it leads to a sharp probability peak with the energy-scale parameters belonging to the GUT domain rather than the Planck domain \[4, 5\] and serves as a “source” for inflation.

4. The density matrix

We now proceed to discuss the off-diagonal elements of the reduced density matrix that are needed for the discussion of decoherence. We shall restrict ourselves to one semiclassical branch, i.e. to a wave function of the form \(2.16\). We shall comment on decoherence between branches in section 7.

The off-diagonal elements of the reduced density matrix read

\[
\rho(t|\varphi, \varphi') = \int df \Psi(t|\varphi, f) \Psi(t|\varphi', f) ,
\] (4.1)
where $\Psi$ is given by (2.17). After the Gaussian integration one obtains
\[
\rho(t|\varphi,\varphi') = C \frac{\Delta_\varphi \Delta'_\varphi}{\sqrt{v_\varphi(t)v'_\varphi(t)}} \exp \left[ -\frac{1}{2} I - \frac{1}{2} I' + i(S - S') \right] \prod_n \left[ v_n^* v'_n (\Omega_n + \Omega'_n) \right]^{-1/2}. \tag{4.2}
\]

It is useful to extract from this density matrix the factor $\sqrt{\rho(t|\varphi)\rho(t|\varphi',\varphi')}$, cf. (3.10), the remaining factor describing the dynamical entanglement during the Lorentzian evolution. By using the relation $\Delta_n = 2|v_n|^2 \text{Re} \Omega_n$ and the expression for the one-loop effective action obtained above, we get
\[
\rho(t|\varphi,\varphi') = C \frac{\Delta_\varphi \Delta'_\varphi}{\sqrt{v_\varphi(t)v'_\varphi(t)}} \exp \left( -\frac{1}{2} \Gamma - \frac{1}{2} \Gamma' + i(S - S') \right) D(t|\varphi,\varphi'), \tag{4.3}
\]
where
\[
\Gamma = I(\varphi) + \Gamma_{1\text{-loop}}(\varphi) \tag{4.4}
\]
is the full Euclidean effective action including the classical part, and
\[
D(t|\varphi,\varphi') = \prod_n \left( \frac{4 \text{Re} \Omega_n \text{Re} \Omega'_n}{(\Omega_n + \Omega'_n)^2} \right)^{1/4} \left( \frac{v_n^* v'_n}{v_n v'_n} \right)^{1/4} \tag{4.5}
\]
is the conventional decoherence factor. The form (4.3) of the density matrix corresponds to an “ideal measurement” by the environmental variables, since the diagonal part (the probabilities) remains unchanged; only the coherence is affected.

It should be emphasised here that, although the decoherence factor coincides with the one known in the literature \cite{14, 15, 16}, the total density matrix (4.3) is different, because it contains the effective-action factors related to both arguments of $\rho(t|\varphi,\varphi')$. This is the effect of a correct normalisation of the vacuum Gaussian states for microscopic variables. The factors are entirely defined on the DeSitter-instanton and, therefore, independent of the Lorentzian time. Therefore they play an essential role only at the onset of inflation, provided the decoherence effects rapidly grow during the Lorentzian evolution. As we shall see, this will indeed be the case for all fields except for the case of massless conformally invariant fields.

The amplitude of the decoherence factor can be rewritten in the form
\[
|D(t|\varphi,\varphi')| = \exp \frac{1}{4} \sum_n \ln \frac{4 \text{Re} \Omega_n \text{Re} \Omega'_n}{(\Omega_n + \Omega'_n)^2}, \tag{4.6}
\]
and the convergence of this series is far from being guaranteed \cite{14, 13, 16}. Moreover, the divergences might be not renormalisable by local counterterms in the bare quantised action. We shall now analyse this question of coherence for all types of bosonic fields; the case of fermions will be dealt with in a separate paper \cite{21}.

13
5. Inconsistencies in the dimensionally regularised density matrix

In this section we shall show that dimensional regularisation leads to density matrices that violate crucial properties and that are therefore inconsistent. We shall start with the case of a massive minimally coupled scalar field. There the equation for the Lorentzian basis functions reads

\[
\frac{d}{dt} \left( a^3 \frac{dv_n}{dt} \right) + \left( \frac{n^2}{a^2} + m^2 \right) v_n = 0.
\] (5.1)

In what follows we shall use dimensional regularisation to regulate the divergencies. In this regularisation the equation retains its form except for the power of \(a\) in the first term, which instead of 3 becomes \(d - 1\) where \(d \to 4\) is the regularisation parameter of the spacetime dimension (and \(n\) is being changed to \(n + (d - 4)/2\)). The corresponding solutions for the DeSitter background, with the Hubble constant \(H\), that are regular on the Euclidean hemisphere of the gravitational instanton (at \(\tau = \pi/2H + it = 0\)) read in this dimension \[28\]

\[
v_n(t) = (cosh Ht)^{(d-2)/(d+1)} P_{(n+4-d/2)}^{-1} \sqrt{\frac{m^2}{H^2}} (i \sin Ht).
\] (5.2)

The behaviour of the series (4.6) depends on the behaviour of \(\Omega_n\) with respect to \(n\). Since we shall be strongly interested in the limit of large masses, it is worth presenting the above function in the form of an \(1/m\) asymptotic expansion uniform in \(n\) \[28, 31\]. It reads

\[
P_{-1/2+i\lambda}(z) = \frac{1}{|\Gamma(n + i\lambda + 1/2)|} \sqrt{\lambda} \times \exp \left( \frac{\mu}{2} \ln \left( \frac{1 - v}{1 + v} - \lambda \arctan \frac{v}{\alpha} \right) \sum_{k=0}^{\infty} \frac{T_k(v, \alpha)}{\lambda^k}, \lambda \to \infty, \right.
\] (5.3)

where

\[
\alpha = \frac{\mu}{\lambda}, \beta^2 = 1 + \alpha^2, v = \frac{\alpha z}{(\beta^2 - z^2)^{1/2}},
\] (5.4)

and the coefficients \(T_k(v, \alpha)\) beginning with \(T_0(v, \alpha) = 1\) are given in \[28\] (they are not needed below). They are uniformly bounded for all values of \(\lambda\) and \(\mu\). In view of this boundedness, the exponential part of this function represents for large \(n\) the conventional high-frequency adiabatic expansion and contributes the dominant part to \(\Omega_n\),

\[
\Omega_n = a^2 \left[ \sqrt{n^2 + m^2 a^2} + i \sin Ht \left( 1 + \frac{1}{2n^2 + m^2 a^2} \right) \right] + O(1/m).
\] (5.5)

14
The corresponding leading contribution to the amplitude of the decoherence factor, 

\[ \ln |D(t|\varphi, \varphi')| \simeq \frac{1}{4} \sum_{n=0}^{\infty} n^2 \ln \frac{4a^2a'^2\sqrt{n^2 + m^2a^2} \sqrt{n^2 + m^2a'^2}}{(a^2 \sqrt{n^2 + m^2a^2} + a'^2 \sqrt{n^2 + m^2a'^2})^2}, \]  

(5.6)

obviously contains cubic and linear divergences which cannot be represented as additive functions of \( a \) and \( a' \). This means that no one-argument counterterm to \( \Gamma \) and \( \Gamma' \) in (4.3) can cancel these divergences of the amplitude – an observation made a number of years ago in [14, 15].

Let us forget for the moment this difficulty and just use the regularisation which identically discards all power divergences. Then, provided that the logarithmic divergences are vanishing, one can at least have a finite quantity subject to physical analysis. Another problem can, however, occur in this case – this regularisation can perform an “oversubtraction” in the sense that the resulting regularised density matrix will be inconsistent – it will violate the necessary properties satisfied for a convergent series (4.9). The calculations briefly presented below really show that the dimensionally-regularised density matrix would have off-diagonal terms infinitely growing for \( \varphi - \varphi' \to \infty \) and thus being inconsistent; for example, \( \text{Tr} \hat{\rho}^2 \) would diverge. We emphasise that this problem has not been discussed before, since reduced density matrices are usually not considered in quantum field theory.

Dimensional regularisation of the divergent sum (5.6) implies that it should be reformulated in \( d \)-dimensional spacetime, which boils down to two main modifications – the degeneracy of the \( n \)-th eigenvalue of the spatial Laplacian \( n^2 \), and the eigenvalues \( \lambda_n = -n^2/a^2 \) itself should be replaced by their \( d \)-dimensional versions \( \text{dim}(n,d) \) and \( \lambda_n(d) \). From the theory of irreducible representations of the group \( O(d + 1) \) it is well known that

\[ \text{dim}(n,d) = \frac{(2n + d - 4)\Gamma(n + d - 3)}{\Gamma(n)\Gamma(d - 1)} = n^2 - \epsilon \frac{(2 - \epsilon/n)(1 - \epsilon/n)\Gamma(n - \epsilon) n^\epsilon}{\Gamma(3 - \epsilon)} \sim n^{2-\epsilon}, \quad \epsilon \equiv 4 - d, \]  

(5.7)

while the \( d \)-dimensional \( \lambda_n(d) \) can be obtained from \( \lambda_n \) by shifting the quantum number \( n \), \( n \to n - (4 - d)/2 \). The latter results in the following modification of the frequency,

\[ \Omega_n(d) = a^2 \sqrt{(n - \frac{4 - d}{2})^2 + m^2a^2 + O(m^0)}. \]  

(5.8)

Note that the main effect of regularisation in (5.7) consists in the multiplication of the summation measure \( n^2 \) by the factor \( n^{-\epsilon} \) which for large positive \( \epsilon \equiv 4 - d \) can make the divergent series convergent. The analytic continuation of the result to \( \epsilon = 0 \) allows one to
disentangle the divergent part of the whole answer as a pole in dimensionality and get a finite contribution.

The further calculations are based on the use of the summation Euler-Maclaurin formula, 
\[
\sum_{n=1}^{\infty} f(n) = \int_0^\infty dy f(y) - \frac{1}{2} f(0) + i \int_0^\infty dy \frac{f(iy) - f(-iy)}{e^{2\pi y} - 1},
\]
which allows one to reduce the infinite sum to the combination of integrals along real and imaginary axes in the complex plane of \( n = y \). Because of the exponential damping factor in the integral over imaginary axes, the regularisation can be removed in its integrand from the very beginning. This would mean that its contribution is vanishing (because the integrand is an even function of its argument), except for the possible residues at the singular points of the integrand given by the equation \( n^2 + (ma)^2 \equiv -y^2 + (ma)^2 = 0 \). These residues will, however, contribute the exponentially small terms \( O(e^{-ma}) \) negligible in the approximation of large masses \( ma \gg 1 \). Thus, the decoherence factor reduces in the main to the contribution of the integral over the real axes, which after the change of the integration variable \( y = ma/x \), takes the following form,
\[
\ln |D| = \int_0^\infty dx x^{\epsilon-4} f(x, \epsilon) + O(e^{-ma}).
\]
Its integrand can be obtained from (5.6) by the substitution of \( n = ma/x \) and the dimensional modifications of the above type, and for nonvanishing \( \epsilon \) it looks rather complicated. However, its complexity due to the nontrivial dimensionality can be represented in the following form,
\[
f(x, \epsilon) = \frac{2(ma)^{-\epsilon}}{\Gamma(3-\epsilon)} f(x, 0) \left[ 1 + O(\epsilon/ma) \right],
\]
\[
f(x, 0) = \frac{1}{4} (ma)^3 \ln \frac{4\delta^2 \sqrt{1 + x^2} \sqrt{1 + \delta^2 x^2}}{(\sqrt{1 + x^2} + \delta^2 \sqrt{1 + \delta^2 x^2})^2}, \quad \delta \equiv \frac{a'}{a}.
\]
with the function \( f(x, 0) \) trivially obtained from (5.6) by introducing the new parameter \( \delta \). Due to this, the resulting expression does not look symmetric in \( a \) and \( a' \), but it will allow us easily to analyse the behaviour of far off-diagonal elements corresponding to \( \delta \ll 1 \).

The further transformations of (5.10) consist in the consecutive integrations by parts in the convergence domain of the integral and the analytic continuation to \( \epsilon = 0 \):
\[
\int_0^\infty dx x^{\epsilon-4} f(x, \epsilon) = \frac{f'''(0, 0)}{\epsilon} - \frac{1}{6} \int_0^\infty dx \ln x f^{(iv)}(x, 0)
\]
\[
- \left. \frac{d}{d\epsilon} f'''(0, \epsilon) \right|_{\epsilon=0} - \left. \frac{d}{d\epsilon} f'''(0, \epsilon) \left|_{\epsilon=0} \right. \right). (5.13)
\]
In view of (5.11)-(5.12) one has
\[
f'''(0, 0) = 0, \quad \frac{d}{d\epsilon} f'''(0, \epsilon) \bigg|_{\epsilon=0} = (ma)^3 O(1/m),
\]
\]
16
which means the absence of logarithmic divergences, and the final result reads
\[ \ln |D| = \int_0^\infty dx \ln x \frac{d^4 f(x, 0)}{dx^4} = \frac{\pi}{24}(ma)^2 + O(m^3), \quad a \gg a'. \] (5.15)

This implies the inconsistency of the regularised density matrix in view of its infinitely growing off-diagonal elements.

The behaviour of the regularised density-matrix elements in close vicinity of its diagonal is also unsatisfactory. For small \( |\varphi - \varphi'| \) the dimensionally regularised decoherence factor in the limit of large mass gives the result
\[ \ln |D(t|\varphi, \varphi')| \approx \frac{7}{64} m^3 \bar{a}(a - a')^2, \] (5.16)
also leading to the inconsistency of the density matrix.

The physical inconsistency of the dimensional regularisation is even more transparent in applications to massless conformal invariant fields. On Friedmann backgrounds, such fields decouple from the gravitational variables, so one should expect the complete absence of decoherence effects, \( D(t|\varphi, \varphi') = 1 \). This is, however, not the case. Simple calculations show that for a massless conformally coupled scalar field the basis function \( v_n^*(t) \) and the frequency function have the form:
\[ v_n^*(t) = \frac{1}{\cosh Ht} \left( \frac{1 + i \sinh Ht}{1 - i \sinh Ht} \right)^{\frac{n}{2}}, \] (5.17)
\[ \Omega_n = na^2 + ia^2 \sqrt{H^2 a^2 - 1}. \] (5.18)

The corresponding decoherence factor is then given by the divergent sum,
\[ \ln |D| = -\frac{1}{4} \sum_{n=1}^{\infty} n^2 \ln \left( A + \frac{B}{n^2} \right), \] (5.19)
\[ A = \left( \frac{a^2 + a'^2}{2a a'} \right)^2, \quad B = \left( \frac{a^2 \sqrt{H^2 a^2 - 1} - a'^2 \sqrt{H^2 a'^2 - 1}}{2a a'} \right)^2. \] (5.20)

Its dimensional regularisation gives the exact result
\[ \ln |D(t|\varphi, \varphi')| = \frac{\pi}{12} \left( \frac{B}{A} \right)^{3/2}, \] (5.21)
again demonstrating the wrong behaviour of the off-diagonal elements of the density matrix in view of the obvious growth of the exponential \( (B/A)^{3/2} \to \infty, \quad |\varphi - \varphi'| \to \infty \). For a minimally coupled scalar field the situation is analogous to the conformal case.

Thus, in the massless limit the regularised decoherence matrix turns out to be as inconsistent as for massive fields. Moreover, for conformally invariant fields it leads to an anti-intuitive conclusion of growing quantum correlations between the environment and gravitational background in spite of their actual decoupling.
6. Conformal parametrisation of bosonic fields

In this section we discuss a possible solution to the problems encountered with dimensional regularisation. It was remarked in the Introduction that a redefinition of the environmental fields can change the reduced density matrix. Moreover, as was remarked in [17], such a redefinition can be motivated by physical considerations. In fact, it was already shown for a specific model in [15] that a redefinition can lead both to a finite decoherence factor and to a consistent density matrix. Here we shall demonstrate that such a viable redefinition is not arbitrary and can lead to a unique result.

Let us consider the following redefinition of a bosonic scalar field,

\[ f(t) \rightarrow \tilde{f}(t) = a^\mu(t) f(t) \]  

(6.1)

with an arbitrary numerical parameter \( \mu \). Under this redefinition the kinetic term of the Lagrangian changes as

\[ L(f) = \frac{1}{2} a^3 \left( \frac{df}{dt} \right)^2 + ... = \frac{1}{2} a^{3-2\mu} \left( \frac{d\tilde{f}}{dt} \right)^2 + ... \]  

(6.2)

and the Wronskian operator and basis functions get replaced by

\[ W(\frac{d}{dt}) = a^3 \frac{d}{dt} \rightarrow \tilde{W}(\frac{d}{dt}) = a^{3-2\mu} \frac{d}{dt}, \]

\[ v_n(t) \rightarrow \tilde{v}_n(t) = a^\mu(t) v_n(t). \]  

(6.3)

(6.4)

The new frequency function then reads

\[ \tilde{\Omega}_n = -ia^{3-2\mu} \frac{d}{dt} \ln \tilde{v}_n^* \]

\[ = a^{2-2\mu} \left[ \sqrt{n^2 + m^2a^2} + i \sinh Ht \left( 1 - \mu + \frac{1}{2} \frac{m^2a^2}{n^2 + m^2a^2} \right) \right] + O(1/m). \]  

(6.5)

For a particular choice of the parameter \( \mu = 1 \) two important things happen with this function – the leading \( n \) behaviour of \( \tilde{\Omega}_n \) becomes independent of the macroscopic variable, and its imaginary part decreases with growing \( n \) as \( 1/n^2 \),

\[ \tilde{\Omega}_n = \sqrt{n^2 + m^2a^2} + b_n, \]  

(6.6)

\[ \text{Re} b_n = O(1/n), \]  

(6.7)

\[ \text{Im} b_n = \sinh Ht \frac{1}{2} \frac{m^2a^2}{n^2 + m^2a^2} + O(1/m) = O(1/n^2). \]  

(6.8)

Substituting this behaviour into (4.5), one finds

\[ \ln \tilde{D}(t|\varphi, \varphi') \simeq \frac{1}{4} \sum_{n=0}^{\infty} n^2 \left( i \frac{\text{Im} b_n'}{n} - i \frac{\text{Im} b_n}{n} + O(1/n^4) \right). \]  

(6.9)
This expression has at most logarithmic divergences which are imaginary and, thus, affect only the phase of the density matrix. Moreover, since $b_n$ and $b'_n$ depend on $a$ and $a'$, respectively, these divergences decompose into an additive sum of one-argument functions and, therefore, can be cancelled by counterterms to the classical action $S$ (and $S'$) in (4.2). The real part is simply convergent and generates the finite decoherence amplitude. This result is formally similar to the result for the decoherence factor in QED [15].

Let us show now that no inconsistencies arise with this redefined density matrix. To begin with, let us consider the far off-diagonal behaviour of the decoherence-factor amplitude for the arbitrary parameter of reparametrisation $\mu$ in the large-mass limit. In this limit $\tilde{\Omega}_n \simeq a^2 - 2^{\mu} \sqrt{n^2 + m^2 a^2}$, and the decoherence factor can approximately be represented by the integral (replacing the summation, $n = may$)

$$\ln |\tilde{D}(t|\varphi, \varphi')| \simeq \frac{(ma)^3}{4} \int_0^\infty dy y^2 \ln \frac{4\delta^{2^{\mu}} \sqrt{1 + y^2} \sqrt{y^2 + \delta^2}}{(\sqrt{1 + y^2} + \delta^{2^{\mu}} \sqrt{y^2 + \delta^2})^2}.$$  \hspace{1cm} (6.10)

where $\delta = a'/a$. This integral is convergent only for $\mu = 1$, and for $a \gg a'$ it yields the answer which shows the suppression of far off-diagonal terms,

$$|\tilde{D}(t|\varphi, \varphi')| \simeq \exp \left[ -\frac{(ma)^3}{24} \left( \pi - \frac{8}{3} \right) + O(m^2) \right].$$  \hspace{1cm} (6.11)

For other values of $\mu$ this integral should be regulated to give the final answer. Dimensional regularisation in the main changes the power of $y$ in the integration measure from $y^2$ to $y^{2-\varepsilon}$. Then the integration gives two different results for different ranges of $\mu$

$$|\tilde{D}(t|\varphi, \varphi')| \simeq \exp \left[ -\frac{\pi}{24} (ma)^3 \right], \quad \mu > 1,$$

$$|\tilde{D}(t|\varphi, \varphi')| \simeq \exp \left[ \frac{\pi}{24} (ma)^3 \right], \quad \mu < 1.$$  \hspace{1cm} (6.12, 6.13)

In the latter case this obviously coincides with the behaviour of the inconsistent dimensionally regularised density matrix (5.15) that was naively defined for $\mu = 0$. Thus, although a sensible result can be obtained for $\mu > 1$, the value $\mu = 1$ is distinguished in yielding directly a finite result. In spite of this, the results for $\mu = 1$ and $\mu > 1$ are qualitatively the same.

In the vicinity of the diagonal the modified density matrix also acquires a good quasi-Gaussian behaviour. Expansion of the expression given above in powers of small $|a - a'|$ gives for large $m$ the dominant result

$$\ln |\tilde{D}(t|\varphi, \varphi')| = -\frac{m^3 \pi \bar{a} (a - a')^2}{64}.$$  \hspace{1cm} (6.14)
For massless conformally-invariant fields, the modified density matrix justifies the expectations induced by the decoupling of the environment from the gravitational background, cf. [17]. Its decoherence factor turns out to be trivial. This can be seen by noting that, say, in the case of massless conformally-invariant scalar field, the basis and frequency functions have the form:

\[ \tilde{v}_n^*(t) = \left( \frac{1 + i \sinh Ht}{1 - i \sinh Ht} \right)^{\frac{n}{2}}, \]

\[ \tilde{\Omega}_n = -ia \frac{d}{dt} \ln \tilde{v}_n^*(t) = n. \]  

Hence, \( \tilde{D}(t|\varphi, \varphi') \equiv 1 \). The same holds also for the electromagnetic field.

We conclude this section by briefly considering two massless conformally non-invariant fields – a minimally coupled scalar field and gravitons. What they have in common are the basis- and frequency functions in their respective conformal parametrisations:

\[ \tilde{v}_n^*(t) = \left( \frac{1 + i \sinh Ht}{1 - i \sinh Ht} \right)^{\frac{n}{2}} \left( \frac{n - i \sinh Ht}{n + 1} \right), \]

\[ \tilde{\Omega}_n = \frac{n(n^2 - 1)}{n^2 - 1 + H^2 a^2} - i \frac{H^2 a^2 \sqrt{H^2 a^2 - 1}}{n^2 - 1 + H^2 a^2}. \]

They differ only by the range of quantum number \( n \) (\( 2 \leq n \) for inhomogeneous scalar modes and \( 3 \leq n \) for gravitons) and by the degeneracies of the \( n \)-th eigenvalue of the Laplacian,

\[ \dim(n)_{\text{scal}} = n^2, \]

\[ \dim(n)_{\text{grav}} = 2(n^2 - 4). \]

The decoherence factor generated by both of these quantum fields thus has the form of the following, obviously convergent, series:

\[ \ln |\tilde{D}(t|\varphi, \varphi')| = -\frac{1}{4} \sum_{n=2}^{\infty} \dim(n) \ln \left[ 1 + \frac{\left( \sinh^2 Ht - \sinh^2 H't \right)^2}{4(n^2 + \sinh^2 Ht)(n^2 + \sinh^2 H't)} \right. \]

\[ \left. + \frac{\left( \cosh^2 Ht \sinh Ht (n^2 + \sinh^2 H't) - \cosh^2 H't \sinh H't (n^2 + \sinh^2 Ht) \right)^2}{4n^2(n^2 - 1)^2(n^2 + \sinh^2 Ht)(n^2 + \sinh^2 H't)} \right]. \]

The calculation of the corresponding far off-diagonal elements is more complicated here, but qualitatively it reproduces the behaviour (6.11) for a massive scalar field with the mass parameter replaced by the Hubble constant of the gravitational background,

\[ |\tilde{D}(t|\varphi, \varphi')| \sim e^{-C(a')^3}, \quad a \gg a', \quad C > 0. \]
To study the behaviour of matrix elements in the vicinity of diagonal of density matrix it is enough to expand the expression (6.21) up to the second order in the quantity \((H - H')\) and calculate the arising series by using [32]. The exact result even in this order of \((H - H')\) is too long to be fully presented here. Thus we restrict ourselves to the limit of late time \(t\) for this expression,

\[
|\tilde{D}(t|\varphi, \varphi')| \sim \exp\left(-\frac{\pi^2}{32}(H - H')^2t^2e^{4Ht}\right), \quad Ht \gg 1,
\]

which clearly shows a rapid growth of decoherence during the inflationary evolution.

7. Conclusions

Let us recapitulate briefly the main results of the present paper. We have calculated the reduced density matrix for the inflaton field in a model of chaotic inflation by tracing out inhomogeneous degrees of freedom corresponding to various bosonic fields. In the original parametrisation of these fields, ultraviolet divergences arise in the decoherence factor of the density matrix. The dimensional regularisation of these divergences was shown to violate the consistency of a reduced density matrix as a bounded operator, which is apparently related to the fact that these divergences do not have the structure of one-field counterterms usually used for ultraviolet renormalisation. A physically motivated conformal redefinition of the environmental bosonic fields leads, however, to well-defined finite results, which show that the Universe acquires classical properties near the onset of inflation.

The seemingly mysterious finiteness property for the decoherence factor in the conformal parametrisation has a natural explanation. The off-diagonal elements of the density matrix, in the language of Feynman diagrams, are roughly proportional to the series of gravitational-matter vertices with two quantum matter legs and the growing number of the gravitational (background fields) ones. Indeed, when expanded in Taylor series in \(a - a'\) (and \(\varphi - \varphi'\)), the one-loop density matrix can be represented as a quantum loop with the insertion of external legs corresponding to the differentiations with respect to background fields, these insertions forming the vertices in question. Such vertices generally contain two derivatives which make the loop graph strongly divergent. The conformal parametrisation of quantum matter fields on the Friedmann background effectively removes the gravitational variables from the kinetic term of the matter Lagrangian, thus actually removing the derivatives from the corresponding vertex. This makes the relevant one-loop Feynman diagrammes finite.

The obtained results show that the decoherence properties of the system strongly depend on the parametrisation of the quantum fields that are traced out. Although, intuitively,
this sounds unsatisfactory – the observable correlations depend on the choice of variables about which we all the same do not have any information – this situation is very typical in quantum theory. In fact, the density matrix is analogous to the S-matrix taken off shell. As is well known, quantities such as the off-shell S-matrix, the off-shell effective action, strongly depend on the parametrisation of quantum fields, the integration over which is performed when calculating these quantities. A similar situation arises here: In contrast to the diagonal elements given by the on-shell effective action, the off-diagonal elements are ambiguous. For the bosonic fields that we considered here, this situation seems to have a relatively satisfactory resolution – there exists a distinguished parametrisation in which divergences are absent. This parametrisation is distinguished for several reasons. First, it reflects the conformal weight properties of the field in question even in the case when this field is not conformally invariant. Secondly, this parametrisation is attained by an overall multiplication of the field by some power of the scale factor, which does not spoil the locality of the field. As we shall show in a forthcoming paper [21], the situation with fermions is less pleasant. Conformal parametrisation leaves the answers divergent, while the additional reparametrisation that can make them ultraviolet-finite is nonlocal – it has the form of Bogoliubov transformations rotating the different harmonics of the field differently, thus destroying the local nature of the original field.

The results obtained have strong implications for the quantum-to-classical transition within the theory of the quantum origin of the inflationary Universe. They can be especially important in the quantum cosmology of the chaotic inflationary model with big negative nonminimal coupling $\xi$ of the inflaton field to curvature [4, 19, 5, 33]. This model was shown to generate a probability peak in the distribution function of the inflaton field $\phi$ – the diagonal element of the density matrix – at the GUT energy scale for the inflationary Hubble constant $H(\phi) \sim m_p \sqrt{\lambda}/|\xi| \sim 10^{-5} m_p$ ($\lambda$ is a coupling constant in the quartic term of the inflaton potential, and the value of the ratio $\sqrt{\lambda}/|\xi| \sim 10^{-5} \sim \Delta T/T$ follows in this model from the COBE normalisation of the magnitude of the microwave background anisotropy). This probability peak has at the onset of inflation a very narrow quantum width for the Hubble constant, $\Delta H/H \sim \sqrt{\lambda}/|\xi|$. Now we can show that the width of the density matrix in the off-diagonal direction at the initial moment of time $t = 0$ has the same magnitude. This follows from (4.3).

First, apart from the decoherence factor $\tilde{D}(0|\phi, \phi')$ the only strongly suppressing factor in this equation is, cf. (3.10),

$$\exp\left(-\frac{\Gamma_{1\text{-loop}}}{2} - \frac{\Gamma'_{1\text{-loop}}}{2}\right) \sim (HH')^{-Z/2}, \quad (7.1)$$

with a very big parameter of anomalous scaling in this model, $Z \sim |\xi|^2/\lambda$. It gives a power-
like suppression of the far off-diagonal terms, and the quantum width in the off-diagonal direction, in the vicinity of the diagonal,

\[(HH')^{-Z/2} \sim H^{-Z} e^{-(H-H')^2/\Delta H^2},\]  

\[\Delta H \sim H\sqrt{\lambda}/|\xi|,\]

coincides with the width of the wave packet.

As far as the decoherence factor \(\tilde{D}\) is concerned, at the start of inflation it is basically inefficient, \(\tilde{D}(0|\varphi,\varphi') = 1\), for a very simple reason. Large \(|\xi|\) generate in this model by the Higgs mechanism very big values of masses for all quantum fields directly coupled to inflaton. The order of magnitude of their mass parameters is such that the ratio \(m^2/H^2 \sim |\xi|\) is very big and \(\varphi\)-independent. This means that for big masses at \(t = 0\) the dominant argument of the decoherence factor \(ma(0) = m/H\) is \(\varphi\)-independent and \(\tilde{D}(0|\varphi,\varphi') = 1\). Thus, the Universe is essentially quantum at the “start of inflation”. But due to decoherence effects it rapidly becomes classical: the quantum width of the distribution function – the probability – stays basically the same, while the off-diagonal width rapidly decreases. From (6.14) it immediately follows that this width for late time is \(\Delta H \sim \exp(-3Ht/2)/(t|\xi|^{3/4})\).

In our paper we have restricted ourselves to decoherence within one semiclassical branch of the wave function. There is, however, also the possibility that different branches might interfere. For the examples discussed in our paper, this would be the case for the no-boundary state that is a superposition of two complex conjugate semiclassical branches. Decoherence is, however, also effective in suppressing interferences between such branches as long as decoherence within one branch holds [15]. At the technical level, this follows from the expression (4.5) which for the interbranch case has in the denominator \(\Omega_n + \Omega'_n\) rather than \(\Omega_n + \Omega'_n\). This means that the imaginary parts of the frequency functions add up instead of partially cancelling one another. Therefore, the amplitude of the interbranch decoherence factor is smaller than that of one branch. In quantum mechanics, a nice analogy is provided by the case of chiral molecules [7] where the superposition between the left-handed and the right-handed version of the corresponding molecule is being suppressed after the localisation has been taken place by continuous measurement with, for example, light in each version.

We have here not included a discussion of the influence of fermionic fields, since this will need a different formalism and reveal novel aspects. This case will be discussed in a separate paper [21] that, together with the present paper, should present a complete picture of decoherence in one-loop quantum cosmology at the onset of inflation.
Acknowledgements

We are grateful to A.A. Starobinsky for useful discussions. A.K. also thanks V.L. Chernyak for a useful discussion. The work of A.B. was partially supported by RFBR under the grant No 96-02-16287. The work of A.K. was partially supported by RFBR under the grant No 96-02-16220 and under the grant for support of leading scientific schools No 96-15-96458. A.B. and A.K. kindly acknowledge financial support by the DFG grants 436 RUS 113/333/4 during their visit to the University of Freiburg in autumn 1998. The work of A.K. was also supported by the CARIPLO scientific foundation.

References

[1] J.B. Hartle and S.W. Hawking, Phys. Rev. D 28 (1983) 2960; S.W. Hawking, Nucl. Phys. B 239 (1984) 257.

[2] A.D. Linde, JETP 60 (1984) 211, Lett. Nuovo Cim. 39 (1984) 401; V.A. Rubakov, Phys. Lett. B 148 (1984) 280; A. Vilenkin, Phys. Lett. B 117 (1982) 25, Phys. Rev. D 30 (1984) 549; Ya. B. Zeldovich and A.A. Starobinsky, Sov. Astron. Lett. 10 (1984) 135.

[3] H.D. Conradi and H.D. Zeh, Phys. Lett. A 154 (1991) 321; H.D. Conradi, Phys. Rev. D 46 (1992) 612.

[4] A.O. Barvinsky and A.Yu. Kamenshchik, Phys. Lett. B 332 (1994) 270.

[5] A.O. Barvinsky, A.Yu. Kamenshchik and I.V. Mishakov, Nucl. Phys. B 491 (1997) 387.

[6] H.D. Zeh, Found. Phys. 1 (1970) 69; in: Foundations of Quantum Mechanics, ed. by B. d’Espagnat (Academic Press, New York, 1971); Found. Phys. 3 (1973) 109; O. K"uber and H.D. Zeh, Ann. Phys. (N.Y.) 76 (1973) 405.

[7] D. Giulini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu and H.D. Zeh, Decoherence and the Appearance of a Classical World in Quantum Theory (Springer, Berlin, 1996).

[8] C. Kiefer and E. Joos, in: Quantum Future, ed. by P. Blanchard and A. Jadczyk (Springer, Berlin, 1998).

[9] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J.M. Raimond and S. Haroche, Phys. Rev. Lett. 77 (1996) 4887.

[10] E. Joos, Phys. Lett. A 116 (1986) 6.
[11] H.D. Zeh, Phys. Lett. A 116 (1986) 9.

[12] C. Kiefer, Class. Quantum Grav. 4 (1987) 1369.

[13] C. Kiefer, D. Polarski and A.A. Starobinsky, Int. J. Mod. Phys. D 7 (1998) 455; C. Kiefer and D. Polarski, Ann. Phys. (Leipzig) 7 (1998) 137; C. Kiefer, J. Lesgourgues, D. Polarski and A.A. Starobinsky, Class. Quantum Grav. 15 (1998) L67.

[14] J.P. Paz and S. Sinha, Phys. Rev. D 45 (1992) 2823.

[15] C. Kiefer, Phys. Rev. D 46 (1992) 1658.

[16] T. Okamura, Prog. Theor. Phys. 95 (1996) 95.

[17] R. Laflamme and J. Louko, Phys. Rev. D 43 (1991) 3317.

[18] A.O. Barvinsky and A.Yu. Kamenshchik, Class. Quantum Grav. 7 (1990) L181.

[19] A.O. Barvinsky and A.Yu. Kamenshchik, Int. J. Mod. Phys. D 5 (1996) 825.

[20] G. Esposito, A.Yu. Kamenshchik and G. Miele, Phys. Rev. D 56 (1997) 1328.

[21] A.O. Barvinsky, A.Yu. Kamenshchik and C. Kiefer, Effective action and decoherence for fermions in quantum cosmology, in preparation.

[22] C. Kiefer, in: Canonical Gravity: From Classical to Quantum, ed. by J. Ehlers and H. Friedrich (Springer, Berlin, 1994).

[23] C. Kiefer, in: Time, Temporality, Now, ed. by H. Atmanspacher and E. Ruhnau (Springer, Berlin, 1997).

[24] J.J. Halliwell and S.W. Hawking, Phys. Rev. D 31 (1985) 1777; R. Laflamme, Phys. Lett. B 198 (1987) 156.

[25] T. Vachaspati and A. Vilenkin, Phys. Rev. D 37 (1988) 898.

[26] A.O. Barvinsky and A.Yu. Kamenshchik, Phys. Rev. D 50 (1994) 5093.

[27] A.O. Barvinsky and V. Krykhtin, Class. Quantum Grav. 10 (1993) 1957; A.O. Barvinsky, Class. Quantum Grav. 10 (1993) 1985.

[28] A.O. Barvinsky, A.Yu. Kamenshchik and I.P. Karmazin, Ann. Phys. (N.Y.) 219 (1992) 201.
[29] A.O. Barvinsky, Phys. Rep. 230 (1993) 237.

[30] B. Allen, Phys. Rev. D 32 (1985) 3136.

[31] F.W.J. Olver, Introduction to asymptotics and special functions (Academic Press, New York-London, 1974); R.C. Thorne, Phil. Trans. Roy. Soc. London 249 (1957) 597.

[32] I.S. Gradstein and I.M. Ryzhik, Table of integrals, series, and products, (Academic Press, New York, 1980).

[33] A.O. Barvinsky and A.Yu. Kamenshchik, Nucl. Phys. B 532 (1998) 339.