Existence or not of many body localization in interacting quasiperiodic systems

Yi-Ting Tu, DinhDuy Vu, and S. Das Sarma
Condensed Matter Theory Center and Joint Quantum Institute,
Department of Physics, University of Maryland, College Park, Maryland 20742, USA

Coupling a 1D quasiperiodic interacting system to a Markovian bath, we study the avalanche instability of the many body localized phase numerically, finding that many body localization (MBL) likely exists in pseudorandom quasiperiodic systems in the thermodynamic limit for a disorder strength \( W > 8 \) (to be compared with \( W > 18 \) in the corresponding randomly disordered case). We support our conclusion by additionally developing real space RG arguments, and provide a detailed comparison between quasiperiodic and random MBL from the avalanche instability perspective, concluding that the two belong to different universality classes.

Introduction - Many-body localization (MBL) [1, 2] is a manifestation of the generic failure of the eigenstate thermalization hypothesis (ETH) [3, 4] and as such violates the basic premise of quantum statistical mechanics in that an isolated generic interacting quantum system does not thermalize. This is a lofty claim, and the subject has remained controversial since the evidence for MBL has been mostly through small system numerical simulations, and an intricate mathematical proof for arbitrarily large disorder [5]. Does MBL really exist as a thermodynamic phenomenon or is it just like a glass with very slow relaxation with no eventual violation of ergodicity?

An important conceptual advance was recently made in the subject by Morningstar et al. [6], see also [7], where the issue was inverted asking the question which mechanism(s) could conceivably destabilize MBL (and how), assuming its existence at very large disorder in the thermodynamic limit. The issue can be studied by coupling the system weakly to a Markovian bath, and then carefully following the system dynamics to discern whether the bath induced thermal inclusions grow (“avalanche”) or stabilize with system size and disorder. The conclusion of Refs. [6, 7] for interacting randomly disordered 1D problem is that thermodynamic MBL is definitely destabilized for disorder strength \( W < 18 \), by contrast, all existing studies of MBL find a critical disorder \( \sim 5 \) for the MBL transition, showing that the putative MBL so far studied in the literature is only a finite-size MBL or at best a crossover phenomenon. In the existing MBL simulations, the relaxation is slow, but the system would thermalize eventually with increasing system size, and at best, a finite-size-MBL is being observed.

Since the studies of Refs. [6, 7] are restricted only to the interacting Anderson model (i.e. random disorder), a natural question arises about the nature of MBL in the interacting incommensurate quasiperiodic system (e.g. Aubry-André model [8]) where the disorder is strictly deterministic (and not random at all). This is important conceptually since the nature of single-particle localization is qualitatively different in 1D random and quasiperiodic models (e.g. the random model is always localized whereas the quasiperiodic model allows for extended or localized single-particle states depending on the disorder strength), and also from an experimental perspective, since all substantive experimental studies of 1D MBL have focused on the quasiperiodic systems (and not on random systems) making the understanding of MBL in quasiperiodic systems of considerable empirical significance [9–14].

In the current work, we study the MBL avalanche instability in the interacting Aubry-André 1D chain model, finding that MBL is relatively (compared with the interacting Anderson model) stable in the quasiperiodic model, with the thermodynamic MBL likely happening at a relatively modest disorder strength of \( W > 8 \), contrasting with most existing finite-size simulations (and experiments of the interacting quasiperiodic systems) reporting MBL at a critical disorder of \( \sim 5 \). Thus, the reported finite-size MBL in the quasiperiodic interacting model is much closer to the thermodynamic MBL than the corresponding random interacting models where the reported MBL transition for a critical disorder of \( \sim 5 \) has little to do with any thermodynamic MBL. Our work also provides an explanation for why it has been so much easier to experimentally observe MBL in interacting quasiperiodic systems than in random systems although the reverse is true for single particle localization where there are hundreds of reported experimental studies of 1D Anderson localization (and very few for the Aubry-André localization). Obviously, interactions make the Anderson disordered system much more ergodic than the corresponding quasiperiodic system where the MBL appears to be much more stable (although the single-particle localization is less stable since the Aubry-André model allows for extended states at weak disorder). We emphasize that bath induced thermal inclusions, leading to quantum avalanches (as studied in our work), have recently been directly observed in interacting quasiperiodic 1D atomic systems [15], providing a direct motivation for our work.

Theory - We consider the Heisenberg model with onsite potential, which is an open 1D spin-\( \frac{1}{2} \) chain with the Hamiltonian

\[
H = \frac{1}{4} \sum_{j=1}^{L-1} \vec{\sigma}_j \cdot \vec{\sigma}_{j+1} + \frac{1}{2} \sum_{j=1}^{L} h_j Z_j, \tag{1}
\]

where \( \vec{\sigma}_j = (X_j, Y_j, Z_j) \) are the Pauli matrices on site \( j \). We consider two models for the onsite potential: (i) quasiperiodic potential \( h_j = W \cos(2\pi \varphi j + \phi) \) with the golden ratio \( \varphi = (1 + \sqrt{5})/2 \), and (ii) random potential where \( h_j \) is a random variable drawn from a uniform distribution \([-W,W]\). For the former, we average over an ensemble of...
random $\phi$; while for the latter, the average is performed over random configurations. The quasi-periodic potential case is our new result compared to Refs. [6, 7], which only study the random model. In the following, we demonstrate and then explain that the difference between these two models is not simply quantitative but is actually qualitative, and reflects distinguishable physics.

**Open system decay rate** - In this part, we identify the avalanche landmark for the quasi-periodic model, following the approach in [6, 7]. (We also reproduce the avalanche landmark for the random model, which is shown in [16].) We couple the first spin perturbatively to a thermal bath, so that the spin chain of small size $L$ can be used to simulate the induced thermalization behavior near a thermal inclusion in a large MBL system. The dynamics of this small spin chain is described by the Lindblad equation

$$\mathcal{L}[\rho] = -i[H, \rho] + \gamma \sum_{\mu} \left( L_{\mu} \rho L_{\mu}^\dagger - \frac{1}{2} \{ L_{\mu}^\dagger L_{\mu}, \rho \} \right),$$

where $\gamma \ll 1$ is the perturbation parameter characterizing the weak coupling and $L_{\mu} = (X_{\mu}, Y_{\mu}, Z_{\mu})$. We then denote $\lambda_1$ as the eigenvalue of $\mathcal{L}$ with the second largest real part as zero is always the eigenvalue with the largest real part, corresponding to the maximally mixed steady state. $\tau = -1/\text{Re}(\lambda_1)$ is the life time of the slowest decay mode and can be considered as the lower bound of the time scale in which the thermal bath thermalizes a spin at distance $L$ away. We consider the first-order perturbation in $\gamma$, in which $\lambda_1$ equals the second largest eigenvalue of the projected Lindbladian

$$\mathcal{L}_{nm} = \langle m | \mathcal{L} | n \rangle (|n| |n\rangle |m\rangle)$$

where $|n\rangle$’s are the eigenstates of $H$. The resulting decay rate $1/\tau$ is proportional to $1/\gamma$ and the matrix size is reduced to $2L \times 2L$ (dense) compared with $4L \times 4L$ (sparse) for the exact diagonalization. We follow Refs. [6, 7] and use 80th percentile as the “typical” decay rate value and estimate the errors based on the 68 percent bootstrap confidence interval. Data points with $L \leq 9$ are calculated from 6000 random choices of the initial phase $\phi$. For larger $L$ the number of random choices is adaptive and varies around 100 to 1000. In the context of avalanche instability, this small chain thermalization describes the expansion of a thermal bubble in an MBL system. In particular, if the rate $1/\tau$ grows faster than $4^L$, the bubble can expand to fill the entire system, causing thermalization and destroying the finite-size MBL. Otherwise, this thermal region stops growing at some finite size, and since thermal seeds are rare to begin with in the MBL phase, the system remains localized. Therefore, we display our decay rate in the scale of $4^{-L}\gamma$. This is the same strategy as in Refs. [6, 7], now adapted for a quasiperiodic “disorder”.

In Fig. 1, we show the 80th percentile value of the decay rate in the quasi-periodic spin chain. Starting from $L = 8$, a common intersection at $W_c \sim 7 - 8$ emerges. As a result, for $W < W_c$, the decay rate varies with $L$ faster than $4^{-L}$, indicating an avalanche instability. The quite small sizes of our simulation naturally raise a question of whether these intersections really converge to a finite $W_c$ or they drift to infinity, ruling out any true MBL. To determine how the avalanche landmark evolves in large systems, we plot in Fig. 2 the decay rate with respect to $L$. Here, a decreasing (increasing) function indicates MBL (ETH) while the minimum marks the finite-size avalanche criticality. For $W > 8$, within numerical fluctuations, the scaled decay rates decrease monotonically for all values of $L$, implying that $W \sim 7 - 8$ is the upper bound for avalanche instability in the thermodynamic limit; while for $W = 7.69$, a broad minimum emerges starting at $L = 10$, indicating that the avalanche landmark most likely saturates for $L \geq 10$. In Fig. 3, we plot the intersecting $W$ also with the system length $L$ and compare with the random model. Data points are calculated by fitting each $L$ and $L + 2$ curves near their intersection with linear or quadratic functions in the log of $4^L P_{90}(1/\tau)$-log $W$ plane, with the intersection point as the shared parameter. For the random model, the intersection drifts continuously with $L$ but appears to slows down near $L = 12, W = 17$. From this fact, Ref. [6] postulates a possible thermodynamic avalanche landmark. For our quasi-periodic model, this indicator is much sharper as one can see that the convergence of the intersections starts at smaller $L$ and lower value of disorder strength $W$ than the random spin chain. Another prominent distinction is that the avalanche landmark is much closer to the finite-size MBL transition obtained through the level statistics data for the quasi-random spin chain (dashed lines in Fig. 3). This is in contrast to the random case where the avalanche instability can happen deep in the finite-size MBL phase [6], thus raising some questions about the thermodynamic MBL. Our bath coupling results, on the other hand, suggest that for the quasi-periodic model, the finite-size MBL transition is much more likely to survive in the thermodynamic limit.
FIG. 2. Scaled decay rates versus the length $L$ of the quasiperiodic spin chain for a given disorder strength $W$. The value of $W$ increases logarithmically from the top to the bottom lines, and we mark several values of $W$ for reference. Dashed line indicates $\gamma^{-1} P_{\text{res}}(1/\tau) = 100 \epsilon$ with $\epsilon$ being the machine epsilon of the double-precision floating-point numbers. We drop data points below this line.

FIG. 3. The value of $W$ at which the spin chain of length $L$ and $L+2$ have the same $4^L/\tau$. Data of the random model with $W \geq 14$ are taken from Figure 5 of Ref. [6]. The dashed lines denote the finite-size ($L = 10 - 16$) MBL transition of the respective model (color-wise) from the level statistics [16].

Real-space renormalization - One might think that the difference between the two models is simply quantitative and due to a specific choice of the quasi-periodic potential instead of a more general one, i.e. $h(x) = h(x + \varphi^{-1})$. In the following discussion, we establish a stronger scenario and demonstrate that they belong to different universality classes. As such, MBL in quasi-periodic systems is more stable (in the thermodynamic limit) than in the random systems.

To show this, we use the strong randomness real-space renormalization [17–19] where the initial state of the 1D spin chain is composed of alternating thermal (T) and insulating (I) blocks. Here, the T segments are only characterized by their physical length $L^T$ while the I blocks, beside the physical length $L^I$, have extra index $d^I = L^I/\epsilon - L^I$, called the primary length, that indicates how close the block is to avalanche instability. $\xi$ is normalized so that $\xi < 1$ is associated with localization, so $d^I$ is always positive. The following rules of block decimation apply. If $L^T_i < d^I_{i-1}$, $d^I_{i+1}$, the three-block segment merges into a new insulating block with $L^T_i = L^T_{i-1} + L^T_{i+1} + L^I_{i+1}$ and $d^I_i = d^I_{i-1} - L^T_{i-1} + d^I_{i+1}$. Otherwise, if the smallest block (in primary length) is insulating, i.e. $d^I_i < L^T_{i-1}, L^T_{i+1}$, the composite block becomes thermal with $L^T_i = L^T_{i-1} + L^T_{i+1} + L^I_{i+1}$. The details for the RG flow in random and quasi-periodic models are described in Refs. [19–21] but the initial block configuration is too simple for our microscopic models. On the other hand, there are RG schemes for generic spin chain models, but the procedure is much more intricate [5, 22, 23].

For our purpose, it is important to treat both random and quasi-periodic models on the same footing, so we aim at performing the strong randomness renormalization starting from the microscopic Hamiltonian in Eq. (1). The parameter $\xi$ implies the connection to the perturbation theory where resonance happens if the off-diagonal terms are larger than diagonal terms. We thus define $\xi_{j,j+1}$ for the respective link as

$$\xi_{j,j+1} = \left( \frac{1}{\Delta h^j_2} \frac{1}{|\Delta h^j_1|} \frac{1}{|\Delta h^j_1|} \right)^{\alpha/4}$$

(4)

with $\Delta h_j = h_{j+1} - h_j$. The parameter $\alpha$ is fixed at 0.1 for both models so that the critical $W$ is close to the values obtained from the open system simulations. Through Eq. (4), we can use the exact microscopic model as the initial configuration for RG [16]. The purpose of this analysis is obviously not predicting the critical disorder strength, but comparing the two models and highlighting the different natures of their seemingly similar ETH-MBL transitions.

For the random disorder model, we sample different random configurations, while for the quasi-periodic model, we randomize the phase. In Fig. 4, we show the probability that the entire system becomes insulating after coarse graining. Note that we choose the system size to be Fibonacci numbers in the quasi-periodic case for better compatibility with the periodic boundary condition. For a generic system size, the situation does not change significantly but the data collapse is slightly worse. The critical disorder strengths in Fig. 4 for the random and quasi-periodic models, i.e. $W_\epsilon = 19$ and 6 respectively, agree with the avalanche critical values obtained from studying the bath-coupled systems. Remarkably, one can contrast the transition of these two MBL models. For the random potential, the severe finite-size effect is visible even for the system as large as 2000 sites, making the ETH-MBL transition manifestly a glassy crossover regime over a wide range of $W$. This means that MBL in random systems would show severe finite-size effects with slow...
FIG. 4. Probability of renormalized system being insulating for the random (a) and quasi-periodic (b) onsite potential. The inset shows the respective collapsed data as a function of $(W - W_c)L^{1/\nu}$ with $W_c = 18.9, \nu = 2.8$ (a) and $W_c = 6.0, \nu = 1.2$ (b).

finite-size MBL is much sharper and thermodynamic MBL is much more likely in the quasiperiodic model than in the random model.

More visible distinctions between the two models are revealed in the details of the RG flow. By decimating blocks, the smallest cluster length progressively increases until it reaches the system size, so we treat this quantity as the flow parameter. In Fig. 5, we show the average number of blocks left as we increase the cutoff length. For the random model, there exists a smooth tail whose extension increases with system size. This shows that the cluster length forms a broad distribution [16], and when the system is large enough, exceptionally large cluster is statistically possible. On the other hand, the cluster decimation in the quasi-periodic model happens abruptly and saturates at a length ($\sim 20$) uncorrelated with and much shorter than the system size. We attribute this fact to the sharpening of the cluster length observed in the quasi-periodic universality class. Essentially, as the RG flows, clusters have their lengths concentrated into narrow bands and thus are decimated almost simultaneously. We numerically demonstrate this fact in [16].

To conclude, we contrast the MBL stability in the two models. For the random model, large thermal regions can emerge and weaken the localization. Due to the statistical nature of these rare regions, as the system size grows, larger thermal regions also occur making the glassy behavior persist to a very large size. For the quasi-periodic model, instead of growing to exceptionally large size (albeit low probability), the sizes of all thermal spots mostly assume only a few values, so they either all become inactive in the MBL side or all expand in the thermal size. As a result, the MBL is more stable and finite-size effect is less severe. Our findings are consistent with the recent state-of-the-arts numerical experiments finding that the power-law decaying imbalance persists in the Anderson model for large system sizes and strong disorders [24]. This behavior does not occur in the quasi-periodic model. Our findings are also consistent with all large system ($L > 100$) experimental observations of MBL being limited only to quasiperiodic atomic chains.

**Conclusion** - We have studied the avalanche instability in an
interacting quasiperiodic chain by coupling to a thermal bath, finding an unexpected stability of MBL for disorder strength $W > 8$, implying the likely existence of thermodynamic MBL in quasiperiodic systems for moderate disorder, in contrast to the randomly disordered chains where thermodynamic MBL does not exist up to a disorder strength of 18. Our work also provides a plausible explanation for why MBL experimental observations in many-atom (>100) 1D systems are only reported for quasiperiodic systems and never for random systems.

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Supplemental Materials for “Existence or not of many body localization in interacting quasiperiodic systems”

I. ADDITIONAL SIMULATION DATA

A. Bath-coupled open system

We first verify the use of the perturbation theory for small systems in Fig. 1. With weak bath coupling $\gamma = 0.001$, the perturbation theory produces almost identical results to the exact diagonalization. For strong bath coupling, the agreement obviously degrades but we can still see the intersections manifesting consistently in both methods. As such, we use the perturbation theory in the main text to explore larger system sizes.

![Fig. 1. Scaled decay rate computed from exact diagonalization (solid lines) and perturbation theory (dash lines) in the quasi-periodic model.](image)

In Fig. 2(a), we show the scaled decay rate with respect to $W$ for the random model. Our results reproduce Ref. [1]. Compared with the quasi-periodic model that we show in the main text where the intersections are around $W \sim 7 - 8$, here the intersecting points visibly drift with increasing system size. This observation is made clearer in Supplementary Figs. 2(b-e) where we plot the evolution of the crossing $W$ between $L$ and $L + n$.

B. Level statistics

We provide the level statistics data as a benchmark for finite-size MBL transition in the quasi-periodic spin chain in Fig. 3. The mean level spacing ratio is computed by averaging $r_n = \min(\delta_n, \delta_{n-1})/\max(\delta_n, \delta_{n-1})$ over the ordered spectrum with $\delta_n = E_n - E_{n-1}$ being the spacing between two neighboring energy levels. As we increases the disorder strength, $\langle r \rangle$ changes from 0.5307 for the GOE (ETH) to 0.3863 for the Poisson distribution (MBL) [2]. In the range $L = 10 - 16$, the finite-size MBL transition happens at $W \sim 1.5$ for the quasi-periodic model and $\sim 2.5$ for the random model (data from Ref. [1]). We emphasize that the thermodynamic limit of the avalanche landmark for the two respective modes are $\sim 7 - 8$ and $\sim 16 - 18$.

II. DETAILS OF THE REAL-SPACE RENORMALIZATION

Equation (4) in the main text is inspired from the connection between the off-diagonal coupling and link resonance. In the product state basis $|\eta \rangle = \prod_{j=1}^{L} |\eta_j \rangle$ where $\eta_j \in \mathbb{C}^d$ is one of the basis of the local $d-$ dimensional Hilbert space at site $j$. Assuming only nearest coupling, the Hamiltonian can be written as

$$H = \sum_{j} J_{j,j+1} + U_{j,j+1} + V_{j}$$  \hspace{1cm} (1)
FIG. 2. (a) Scaled decay rate with respect to $W$ for the random model. Data for $W > 14$ are taken from Ref. [1]. (b-e) Intersecting disorder strength between $L$ and $L + n$.

FIG. 3. The mean level spacing ratio for the quasi-periodic model versus disorder strength. The grid lines denote the GOE limit $\langle r \rangle = 0.5307$ and the Poisson limit $\langle r \rangle = 0.3863$. The inset shows the intersection of $\langle r \rangle$ in $W$ between $L$ and $L + 2$ drifting to larger value as the system size increases.

where $J$ and $U$ have support on pairs of nearest sites, and the previously defined product state basis diagonalizes $U$ and $V$. Since only $J$ has off-diagonal matrix elements, it introduces a coupling between product states $|\eta\rangle, |\eta'\rangle$ defined by the ratio

$$G_{j,j+1}(\eta, \eta') = \frac{\langle \eta | V_{j,j+1} | \eta' \rangle}{E_\eta - E_{\eta'}}$$

with $E_\eta = \langle \eta | U + V | \eta \rangle$. For the spin chain model discussed in the main text, we can identify

$$J_{j,j+1} \equiv \frac{1}{2} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y), \quad U_{j,j+1} \equiv \frac{1}{4} Z_j Z_{j+1}, \quad V_j \equiv \frac{1}{2} h_j Z_j.$$ (3)

At one particular link, we can have four off-diagonal coupling $|\uparrow \downarrow \uparrow \uparrow\rangle \rightleftharpoons |\uparrow \downarrow \uparrow \downarrow \rangle$, $|\uparrow \downarrow \uparrow \downarrow\rangle \rightleftharpoons |\uparrow \downarrow \downarrow \uparrow \rangle$, $|\downarrow \uparrow \uparrow \downarrow\rangle \rightleftharpoons |\downarrow \downarrow \uparrow \downarrow \rangle$, and $|\downarrow \downarrow \downarrow \downarrow\rangle \rightleftharpoons |\downarrow \downarrow \downarrow \uparrow \rangle$ with the red spin indicating the link of interest and other spins beyond the two nearest ones are factored.
Following Ref. [3], we take the geometric means of these dimensionless couplings, obtaining

$$\bar{G}_{j,j+1} = \left( \frac{1}{2\Delta h_j} \frac{1}{2\Delta h_j} \frac{1}{2|\Delta h_j + 1|} \frac{1}{2|\Delta h_j - 1|} \right)^{1/4}$$

(4)

Instead of comparing \( \bar{G} \) with unity to define the resonant or perturbative link, we benchmark it against \( 1/2 \) (thus a stronger condition). This \( 1/2 \)–threshold tunes \( P_{\text{insulating}} \) at the critical point not too close to either 0 or 1, allowing better visualization of data collapse. We also introduce a parameter \( \alpha \) to adjust the critical disorder strength so that it agrees with the open system simulations. As a result, the \( \xi \) parameter is given by

$$\xi_{j,j+1} = \left(2\bar{G}_{j,j+1}\right)^{\alpha} = \left( \frac{1}{\Delta h_j^2} \frac{1}{|\Delta h_j + 1|} \frac{1}{|\Delta h_j - 1|} \right)^{\alpha/4}$$

(5)

with \( \xi < (\geq) 1 \) indicating the insulating (thermal) link. For insulating links, we can define the primary length \( d_{j,j+1} = 1/\xi_{j,j+1} - 1 \). After characterizing all links, we group similar-type links into a block whose physical and primary length is given by the sum of component links.

A. Distribution of rare regions in the random potential

In Fig. 4, we show the distribution of thermal and insulating blocks in the random model as we increase the disorder strength passing the critical point. Both type of blocks exhibit continuous broad distributions, with large thermal regions being exponentially rare. As \( W \) increases, the distribution for insulating blocks extends and surpasses that of the thermal blocks for \( W > W_c \).

![FIG. 4. Probability density for thermal block physical length (top panel) and insulating block primary length (bottom panel). The system size is \( L = 6000 \) and blocks are decimated until \( \ln / d_n \geq 50 \).](image-url)
B. Block parameter sharpening in the quasi-periodic model

For the quasi-periodic model, if we instead approximate the irrational $\varphi$ by $F_{k+1}/F_k$ with $F_k$ being the $k$th Fibonacci number, the link parameters repeat itself after $F_k$. After grouping similar-type links into blocks, the block parameter is also periodic after every $\kappa(F_k)$ blocks with $\kappa(F_k)$ being an integer less than $F_k$ because of the link merging. We expect that as $k \to \infty$, $\kappa(F_k) \to \infty$, so the initial block configuration does preserve quasi-periodicity even though we do not know its explicit position dependence. This places our model into the same universality as Ref. [4]. Therefore, we expect the block parameters to sharpen along the RG flow. In Fig. 5, we denote the initial block parameters as step 1 (the x-axis is the block index). We can see there exist insulating blocks both longer and shorter than the thermal blocks. However, these lengths group into approximate bands, in which the lowest band only contains local minimum (block shorter than its two immediate neighbors). Therefore, if we increase the cutoff length beyond this band, the whole band is eliminated simultaneously. As we apply this procedure iteratively, at step 6, all the thermal and insulating blocks already concentrate into two respective bands well separated from each other. It is noticeable that throughout the RG flows, block lengths only assume a few values much shorter than the system size, instead of forming a broad distribution like the random case. This is a direct manifestation of the rather weak finite size effects in the quasiperiodic model, particularly compared with the random model where finite size effects are severe.

![Figure 5](image_url)

**FIG. 5.** Physical/Primary length of thermal (red) and insulating (blue) blocks with respect to the block order index. We simulate a system with size $L = 10946$, $\varphi = 0$, and $W = 5.5$.

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