About an alternative distribution function for fractional exclusion statistics

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Abstract

We show that it is possible to replace the actual implicit distribution function of the fractional exclusion statistics by an explicit one whose form does not change with the parameter $\alpha$. This alternative simpler distribution function given by a generalization of Pauli exclusion principle from the level of the maximal occupation number is not completely equivalent to the distributions obtained from the level of state number counting of the fractional exclusion particles. Our result shows that the two distributions are equivalent for weakly bosonized fermions ($\alpha >> 0$) at not very high temperatures.

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The principle of the fractional exclusion statistics (FES) was for the first time proposed about 60 years ago by Gentile[1] who suggested an intermediate maximum occupation number changing from 1 (for fermions) to $\infty$ (for bosons). This idea was later recognized and developed in the study of anyons and quasi-particle excitations for some low dimensional systems relevant to fractional quantum Hall effect and to superconductivity[2].
Years ago, the study in this direction gets a new dimension by the consideration of the influence of interactions on the number $d$ of one particle state\textsuperscript{[3, 4]}. The starting point is the assumption\textsuperscript{[3]} that the variation of the number of states $d_i$ of a single particle energy level $i$ is proportional to the variation of the number of particles $N_i$ occupying the level, i.e., $\Delta d_i = -\alpha \Delta N_i$ in a simple case without mutual statistics\textsuperscript{[4]}. Then by a state counting procedure\textsuperscript{[4]} on the basis of the assumption $d_i = G_i - \alpha(N_i - 1)$ for “bosons” or $d_i = G_i - (1 - \alpha)(N_i - 1)$ for “fermions”, one finds the number of quantum states $W_i$ interpolating between $W_b$ ($\alpha = 0$) and $W_f$ ($\alpha = 1$). Here $G_i$ can be considered as the number of one particle states for ideal case. The total number of a system of $N$ particles and $\omega$ levels is then $W = \prod_{i=1}^{\omega} W_i$.

These generalized bosons or fermions can then be treated as ideal boson gas as usual\textsuperscript{[4]} under the additivity conditions of $E = \sum_i N_i e_i$ and $N = \sum_i N_i$ where $E$ is the total energy of the system of $N$ particles and $e_i$ is the energy of one particle level $i$. The most probable distribution is given by the average occupation number $n_i = N_i/G_i$, i.e.

$$n_i = \frac{1}{f(e^{-\beta(e_i-\mu)}) + \alpha}$$  \hspace{1cm} (1)

where $f(x)$ is a function satisfying

$$f^\alpha(x)(1 + f(x))^{1-\alpha} = x$$  \hspace{1cm} (2)

and $\alpha$ varies between $0 \leq \alpha \leq 1$. The reader can find in Figure 1 this distribution (full lines) for different $\alpha$ values. The constant $\alpha$ turns out to be the inverse maximum occupation number $(1/\alpha)$ of these fermionized bosons or bosonized fermions. The exclusion principle of Pauli ($\alpha = 0$ or 1) has been generalized here.

The functional form of Eq.(1) depends on the value of $\alpha$. Eq.(1) and Eq.(2) do not necessarily have explicit solutions for any $\alpha$. In a previous paper\textsuperscript{[5]}, by avoiding the state counting procedure and using an usual method to calculate the grand partition function, we proposed a simpler, explicit fractional distribution which seemed identical to Eq.(1) but whose functional form did not change with $\alpha$.

The purpose of the present letter is to describe the method giving the alternative fractional distribution and to provide a detailed analysis of the
relationship between the two fractional exclusion distributions. The reader will find that this approach does not need the two different generalizations $d_i$ for “bosons” and “fermions”. It is only supposed that the quasi-particles obey the fractional exclusion principle, i.e. the maximal number of particles $n_m$ at a state may be different from 1 (fermions) or $\infty$ (bosons). In average, $n_m < 1$ may also make sense, so we propose $0 < n_m < \infty$.

According to the hypothesis of the fractional ideal gas[4], the $N$ quasi-particles should be described by Boltzmann-Gibbs statistics. So that the grand partition function can be given by[6] (let $n_i = N_i$ for the moment):

$$Z = \sum_{N=0}^{\infty} \sum_{\{n_i\}_N} e^{-\sum_i n_i (e_i - \mu)} = \sum_{N=0}^{\infty} \prod_i \left[ e^{-\beta (e_i - \mu)} \right]^{n_i} \tag{3}$$

where $\{n_i\}_N$ signifies all the possible sets $\{n_i\}$ obeying $N = \sum_i n_i$ for a given $N$. The summations in Eq.(3) are equivalent to[6]

$$Z = \prod_i \sum_{n_i=0}^{n_m} \left[ e^{-\beta (e_i - \mu)} \right]^{n_i} \tag{4}$$

The summation over $n_i$ is a geometric progression whose result is given by

$$Z = \prod_i \frac{1 - e^{-(1+n_m)\beta (e_i - \mu)}}{1 - e^{-\beta (e_i - \mu)}} \tag{5}$$

Then as usual, the average occupation number $n$ of an one-particle state of energy $e$ is calculated via the grand potential $\Omega = -kT \ln Z$ as follows

$$n = \frac{\partial \Omega}{\partial e} = -kT \frac{\partial (\ln Z)}{\partial e} = \frac{1}{e^\beta(e-\mu) - 1} - \frac{1+\alpha}{e^{1+\alpha\beta(e-\mu)} - 1} \tag{6}$$

which recovers the standard boson distribution if $\alpha = 0$ and the fermion distribution if $\alpha = 1$. Note that here we have put $n_m = 1/\alpha$ according to the result of Eq.(1) that $1/\alpha$ is the maximal occupation number.

Figure 1 presents a comparison of Eq.(6) with Eq.(1). Surprisingly, at low temperature, Eq.(6) (symbols) perfectly reproduces the distribution (full lines) given by Eq.(1). On the other hand, a small difference can be seen at higher temperatures. This difference will be analyzed with the Fermi energies of the two distributions.

With Eq.(6), it is straightforward to show that, if $T = 0$, $n = 1/\alpha$, $n = 1/2\alpha$ and $n = 0$ for $e < \mu$, $e = \mu$ and $e > \mu$, respectively. So the
Fermi energy \( (e^0_f) \) at \( T = 0 \) can be identified to \( \mu \). For a 2-dimensional gas, \( e^0_f = \alpha \frac{\hbar^2 N}{2mV} \) where \( m \) is the mass of the particle and \( V \) the volume of the system. When \( T \neq 0 \), \( e_f \) is determined by

\[
\frac{(e^{1/\alpha} e^0_f - 1)}{(e^{1/\alpha} e^0_f - 1)} = e^{e^0_f}
\]

which is plotted (symbols) versus temperature in Figure 2 in comparison with the Fermi energy (full lines) given by Eq.(1) [4]

\[
e^{e^0_f} = e^{e^0_f} - e^{1/\alpha e^0_f}
\]

Note that in Figure 2 the particle density \( N/V \) is chosen to give \( e^0_f = 1 \) eV when \( \alpha = 1 \) (usual ideal fermions).

At low temperatures, there is no significant difference between the two Fermi-energies. But at very high temperatures up to several K, important difference is observed for \( \alpha \) very different from 0 and unity. This difference is studied in Figure 3 through the relative difference \( (R) \) between the two Fermi energies, where \( R = \frac{|e^{e^0_f} - e^0_f|}{\sqrt{|e^{e^0_f} e^0_f|}} \) and \( e^{e^0_f} \) are the Fermi energy given by Eq.(3) (lines) and that given by the present work in Eq.(7) (symbols), respectively. For sufficiently large \( \alpha \), \( R \) is very small up to high temperature. On the other hand, for small \( \alpha \), \( R \) may diverge at temperatures at which \( e^{e^0_f} \) or \( e^0_f \) tends to zero.

In summery, a major result of this work is to provide an analysis of the relationship between two possible distribution functions for fractional exclusion particles in the Haldane-Wu’s sense [3, 4]. The first was obtained by Wu [4] through state counting and maximum entropy procedure on the basis of Haldane hypothesis \( \Delta d = -\alpha N \). The second via the grand partition function calculated by supposing \( n_{max} = 1/\alpha \) for the maximal occupation number. From the point of view of the fractional Pauli exclusion principle, one would expect that these two approaches should be equivalent. In fact, at temperatures up to ambiant ones, there is no significant difference between these two distributions. However, for very small \( \alpha \) (weakly fermionized bosons) or very high temperatures, one should be careful because the relative difference between the two Fermi energies may diverge at some temperatures. We hope that the explicit FES distribution function of present work may be helpful for some calculations and applications.

It is worth noticing that the difference between the distributions Eq.(1) and Eq.(6) means that the generalization of Pauli exclusion principle at the
level of state number counting as described briefly at the beginning of this letter may be different from that at the level of maximal occupation number à la Gentile\[1\] as described in this paper. This is a major conclusion of the present work.

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Figure 1: Comparison of the FES distribution given in [4] by Wu [Eq.(1)] (lines) with that given by the present work [Eq.(6)] (symbols) for different α values and temperatures (300 K and 2000 K). We see that the two distributions are rather equivalent at low temperature. A small difference takes place between these two distributions at high temperature (see the lines for 2000 K).
Figure 2: Temperature dependence of 2-Dimensional Fermi energies given in [4] by Wu and by present work for different values of $\alpha$. There are important differences at very high temperatures for $\alpha$ values very different from 0 and unity. For low temperatures, the two Fermi energies are sufficiently close to each other. But for very small $\alpha$, the relative difference may be important at low temperature as shown in Figure 3.
Figure 3: Temperature dependence of the relative differences ($R$) between the two Fermi energies plotted in Figure 2. Here $R = \frac{|e_f^W - e_f^P|}{\sqrt{|e_f^W|} \sqrt{|e_f^P|}}$, where $e_f^W$ and $e_f^P$ are the Fermi energy given by Wu in [4] and that given by the present work, respectively. For sufficiently large $\alpha$, $R$ is very small up to high temperature. On the other hand, for small $\alpha$, $R$ may diverge at temperatures at which $e_f^W$ or $e_f^P$ tends to zero (the peaks).