Fractional Brownian motion in presence of two fixed adsorbing boundaries

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Abstract. We study the long-time asymptotics of the probability $P_t$ that the Riemann-Liouville fractional Brownian motion with Hurst index $H$ does not escape from a fixed interval $[-L, L]$ up to time $t$. We show that for any $H \in ]0, 1]$, for both subdiffusion and superdiffusion regimes, this probability obeys

$$\ln(P_t) \sim -t^{2H}/L^2,$$

i.e. may decay slower than exponential (subdiffusion) or faster than exponential (superdiffusion). This implies that survival probability $S_t$ of particles undergoing fractional Brownian motion in a one-dimensional system with randomly placed traps follows

$$\ln(S_t) \sim -n^{2/3}t^{2/3H},$$

as $t \to \infty$, where $n$ is the mean density of traps.

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1. Introduction

Consider stochastic process $X_t^H$ defined as

$$X_t^H = \frac{1}{\Gamma(H + 1/2)} \int_0^t \frac{d\tau \zeta(\tau)}{(t - \tau)^{1/2 - H}},$$

(1)

where $\zeta(\tau)$ is Gaussian, delta-correlated noise and $H$ - the Hurst index, is a real number in $]0, 1]$.

The process in Eq. (1) is called the Riemann-Liouville fractional Brownian motion (fBm), since $X_t^H$ is the solution of the Langevin equation in which random force is the fractional Riemann-Liouville derivative of Gaussian noise. The fBm was first formulated in Kolmogorov’s paper [1] and later in the papers of Lévy [2], but first systematic analysis belongs to Mandelbrot and Van Ness [3].

The fBm $X_t^H$ is a continuous-time Gaussian process starting at zero, with mean zero, having the correlation function:

$$E[X_t^H X_s^H] = \frac{(H + 1/2)t^{H - 1/2}s^{H + 1/2}}{\Gamma^2(H + 3/2)} 2F_1 \left( \frac{1}{2} - H, 1; H + \frac{3}{2}; \frac{s}{t} \right),$$

(2)

where $s < t$ and $F$ denotes the Gauss hypergeometric function, and the variance:

$$E \left[ (X_t^H)^2 \right] = \frac{t^{2H}}{2H \Gamma^2(H + 1/2)}.$$

(3)
Equations (2) and (3) signify that $X^H_t$ has self-similar but not independent increments: for $H > 1/2$ the increments of the process are positively correlated and the fBm shows a superdiffusive behavior, while for $H < 1/2$ the increments of the process are negatively correlated and one deals with subdiffusion. In case $H = 1/2$ one recovers standard Brownian motion with diffusion coefficient $D = 1/2$.

Here we study a simple first-passage problem for the fBm, whose general understanding is a basic aspect of stochastic processes [4, 5]. Namely, we analyze the asymptotic long-time behavior of the probability $P_t$ that the fBm $X^H_t$ in Eq. (1) does not escape from the interval $[-L, L]$ up to time $t$. We note that contrary to a single boundary case, for which several rigorous results are available [6, 7], understanding of the two boundaries case is still rather controversial:

(a) One source of confusion stems from a recent tendency of describing all wealth of naturally occurring anomalous diffusive processes in terms of the so-called *fractional* diffusion equation, fractional Fokker-Planck and other fractional differential equations, regardless of the origin, intrinsic correlations and physics underlying these processes. If non-Markovian fBm-type processes are indeed described by fractional diffusion equations [8], one expects that $P_t$ will have an *algebraic* tail. This would imply that the distribution of the adsorption or first exit time from a fixed interval will not have all moments. This is, of course, a rather counterintuitive conclusion.

(b) Survival of a tagged bead of an infinitely long Rouse polymer chain, (whose dynamics is an fBm-type process with $H = 1/4$), in presence of two adsorbing boundaries has been discussed in Ref. [9]. Using a path-integral formulation with an exact measure of trajectories of such a bead [10], and adapting a classic method of images, it was shown that the bead’s survival probability obeys

$$-\ln(P_t) \sim \frac{t^{1/2}}{L^2},$$

i.e., is described by a stretched-exponential function of time.

(c) Numerical simulations of a tagged particle dynamics in a one-dimensional hard-core lattice gas - another fBm-type process with $H = 1/4$, - in presence of two adsorbing boundaries [11], and more recent simulations of dynamics of a tagged bead of a finite Rouse chain between two traps [12], suggested both a faster decay of the survival probability:

$$-\ln(P_t) \sim \frac{t}{L^4},$$

We set out to show here that, for $0 < H \leq 1$, i.e., both in the subdiffusive and superdiffusive regimes, the survival probability $P_t$ obeys

$$-\ln(P_t) \sim \text{const} \frac{t^{2H}}{L^2},$$

which expression can be rewritten, taking advantage of Eq. (3), as

$$-\ln(P_t) \sim \text{const} \frac{E [(X^H_t)^2]}{L^2}.$$  

Note that our result in Eq. (6) confirms Eq. (4) and contradicts (a) and (c), Eq. (5).
2. Basic equations

Since we are concerned with the large-\(t\) behavior, it will not matter much how we define \(\zeta(\tau)\) - as a continuous in time function or as a discrete process, provided that we keep all essential features of noise. We thus divide, at a fixed \(t\), the interval \([0, t]\) into \(N\) (\(N \gg 1\)) small subintervals \(\Delta\), (such that \(\Delta N \equiv t\)), and assume that within the \(k\)-th subinterval, \(k = 0, 1, \ldots, N - 1\), the noise \(\zeta(\tau)\) is constant and equal to \(\zeta_k/\sqrt{\Delta}\), where \(\{\zeta_k\}\) are independent random variables with normal distribution \(N[0, 1]\).

Then, the Riemann-Liouville fBm in Eq.\((1)\) can be written down as a "weighted" sum of independent random variables:

\[
X^H_t = \frac{1}{\Gamma(H + 1/2)} \sum_{k=0}^{N-1} \frac{\zeta_k}{\sqrt{\Delta}} \int_{k\Delta}^{(k+1)\Delta} \frac{d\tau}{(t - \tau)^{1/2 - H}} = \sum_{l=1}^{N} \sigma_l \zeta_{N-l},
\]

(8)

where \(\sigma_l\) is a non-random function:

\[
\sigma_l = \frac{\Delta^H}{\Gamma(H + 3/2)} \left[H + 1/2 - (l - 1)^H + 1/2\right].
\]

(9)

Note that \(\sigma_l\) is a \textit{monotonically decreasing} function of \(l\) for \(H < 1/2\), a constant for \(H = 1/2\), and a \textit{monotonically increasing} function of \(l\) for \(H > 1/2\).

Now, let rect\(_L\)(\(X\)) denote the rectangular function:

\[
\text{rect}_L(X) = \begin{cases} 
1, & |X| < L, \\
1/2, & X = \pm L, \\
0, & |X| > L
\end{cases}
\]

(10)

Representing \(\text{rect}_L(X)\) via its Fourier transform:

\[
\text{rect}_L(x) = \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{\sin(Lk)}{k} \exp[ikX],
\]

(11)

we may now write down the indicator function \(I\left(\max|X^H_t| \leq L\right)\) of the event that an \(N\)-step trajectory \(X^H_t\) did not leave the interval \([-L, L]\) as the following \(N\)-fold integral:

\[
I\left(\max|X^H_t| \leq L\right) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{l=1}^{N} \frac{dk_l}{\pi} \frac{\sin(Lk_l)}{k_l} \exp\left[ i \sum_{j=1}^{N-l+1} k_j \right].
\]

(12)

Averaging the latter equation, we have then that the probability \(P_N\) of this event is given by

\[
P_N = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{l=1}^{N} \frac{dk_l}{\pi} \frac{\sin(Lk_l)}{k_l} \exp\left[ -\frac{\sigma_l^2}{2} \left( \sum_{j=1}^{N-l+1} k_j \right)^2 \right].
\]

(13)
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Next, changing next the integration variables:

\[ Y_1 = k_1 + k_2 + \ldots + k_N, \]
\[ Y_2 = k_1 + k_2 + \ldots + k_{N-1}, \]
\[ Y_3 = k_1 + k_2 + \ldots + k_{N-2}, \]
\[ \ldots \]
\[ Y_N = k_1, \]

we obtain

\[ P_N = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{l=1}^{N} \frac{dY_l}{\pi} \frac{\sin (L (Y_l - Y_{l+1}))}{Y_l - Y_{l+1}} \exp \left[ -\frac{1}{2} \sum_{l=1}^{N} \frac{\sigma_l^2 Y_l^2}{2} \right], \quad Y_{N+1} \equiv 0. \]  

Now, it is expedient to use the following integral identity for the sinc-function:

\[ \frac{\sin (L (Y_l - Y_{l+1}))}{Y_l - Y_{l+1}} = \frac{1}{2} \int_{-L}^{L} dX \exp \left[ iX (Y_l - Y_{l+1}) \right]. \]  

Plugging Eq. (16) into Eq. (15), and performing integrations over \( \{Y_l\} \), we finally arrive at the following meaningful representation of the survival probability:

\[ P_N = \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1}^{N} \frac{dX_l}{\sqrt{2\pi} \sigma_l} \exp \left[ -\frac{1}{2} \sum_{l=1}^{N} \frac{(X_l - X_{l-1})^2}{2\sigma_l^2} \right] = \]

\[ = \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1}^{N} dX_l \exp \left[ -\sum_{l=1}^{N} \frac{(X_l - X_{l-1})^2}{2\sigma_l^2} \right] / \]

\[ / \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{l=1}^{N} dX_l \exp \left[ -\sum_{l=1}^{N} \frac{(X_l - X_{l-1})^2}{2\sigma_l^2} \right], \quad X_0 \equiv 0. \]  

Note that the integrand in Eq. (17) is an analog of the Wiener measure for the Riemann-Liouville fractional Brownian motion.

3. Hurst index \( 1/4 < H \leq 1/2 \): fBm as a Brownian motion in an expanding cage.

Change the integration variables \( X_l = \sigma_l x_l \). Then, Eq. (17) reads

\[ P_N = \int_{-L/\sigma_1}^{L/\sigma_1} \ldots \int_{-L/\sigma_N}^{L/\sigma_N} \prod_{l=1}^{N} \frac{dX_l}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \sum_{l=1}^{N} \left( \frac{x_l - \sigma_{l-1} x_{l-1}}{\sigma_l} \right)^2 \right], \]  

where \( x_0 \equiv 0 \). Next, we represent

\[ \sum_{l=1}^{N} \left( \frac{x_l - \sigma_{l-1} x_{l-1}}{\sigma_l} \right)^2 = \sum_{l=1}^{N} (x_l - x_{l-1})^2 + F(\{x_l\}), \]  

where
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\[ F\left(\{x_l\}\right) = 2 \sum_{l=1}^{N} \left(1 - \frac{\sigma_{l-1}}{\sigma_l}\right) x_{l-1} (x_l - x_{l-1}) + \sum_{l=1}^{N} \left(1 - \frac{\sigma_{l-1}}{\sigma_l}\right)^2 x_{l-1}^2. \]  

(20)

Now, note that it follows from Eq. (20) that \( \sigma_1 \sim l^{H-1/2} \) and \( (1 - \sigma_{l-1}/\sigma_l) \sim 1/l \) as \( l \to \infty \). Hence, for any \( H > 0 \) and any \( N \), the second sum on the right-hand-side of Eq. (20) is bounded:

\[
\max \left| \sum_{l=1}^{N} \left(1 - \frac{\sigma_{l-1}}{\sigma_l}\right)^2 x_{l-1}^2 \right| \leq L^2 C, \quad C \equiv \sum_{l=1}^{\infty} \sigma_{l-1}^2 \left(1 - \frac{\sigma_{l-1}}{\sigma_l}\right)^2.
\]  

(21)

On the other hand, an elementary analysis shows that for \( 0 < H < 1/2 \) the maximal absolute value of the first sum on the right-hand-side of Eq. (20) grows with \( N \):

\[
\max \left| \sum_{l=1}^{N} \left(1 - \frac{\sigma_{l-1}}{\sigma_l}\right) x_{l-1} (x_l - x_{l-1}) \right| \sim L^2 C_1 N^{1-2H},
\]  

(22)

where \( C_1 \) is \( N \)-independent constant.

Consequently, the survival probability \( P_N \) obeys the following double-sided inequality:

\[
\exp \left[-L^2 \left( C_1 N^{1-2H} + C \right)\right] \Psi_N \leq P_N \leq \exp \left[L^2 \left( C_1 N^{1-2H} + C \right)\right] \Psi_N,
\]  

(23)

in which \( \Psi_N \) is given explicitly by

\[
\Psi_N = \int_{-L/\sigma_1}^{L/\sigma_1} \cdots \int_{-L/\sigma_N}^{L/\sigma_N} \prod_{l=1}^{N} \frac{dx_l}{\sqrt{2\pi}} \exp \left[-\sum_{l=1}^{N} \frac{(x_l - x_{l-1})^2}{2}\right],
\]  

(24)

with \( x_0 = 0 \).

One notices now that Eq. (24) describes the probability that an \( N \)-step Brownian motion trajectory \( x_l \), starting at the origin, does not escape from the interval whose boundaries move deterministically away from the origin as \( \pm L/\sigma_l \), \( l = 1, 2, \ldots, N \). This classical problem has been extensively studied in the probability theory (see Ref. [13] and references therein). A lucid derivation of main results and description of different approaches can be also found in Ref. [14].

At sufficiently large \( N \), \( \Psi_N \) obeys [13] (note that we appropriately change the notations):

\[
\Psi_N \sim \exp \left[ -\frac{\pi^2}{8L^2} \sum_{l=1}^{N} \sigma_{l}^2 \right] \sim \exp \left[ -\frac{\pi^2}{16HT^2(H+1/2)} \frac{(\Delta N)^{2H}}{L^2} \right] = \exp \left[ -\frac{\pi^2}{8} E \left[ \left( X_H^l \right)^2 \right] \right] \]  

(25)

Note now that for \( 1/4 < H < 1/2 \), \( \Psi_N \) in Eq. (25) decays faster than \( \exp[-N^{1-2H}] \), and hence, in virtue of the inequality in Eq. (23), \( \Psi_N \) determines the decay of the survival probability \( P_N \), which yields the result in Eq. (6).
4. Hurst index $1/2 < H \leq 1$: fBm as a Brownian motion in a shrinking cage.

Consider next the superdiffusive case when $1/2 < H < 1$. One notices that here \( \sigma_l \sim l^{H-1/2} \to \infty \) as \( l \to \infty \), while \( (1 - \sigma_{l-1}/\sigma_l) \sim 1/l \), and hence, \( F(\{x_l\}) \), Eq.(20), is bounded by a constant for any \( N \), i.e., \[
\max|F(\{x_l\})| \leq 2L^2C_2, \quad C_2 \equiv \sum_{l=1}^{\infty} \left( 1 - \frac{\sigma_{l-1}}{\sigma_l} \right) \sigma_{l-1}^{-1} \left( \frac{1}{\sigma_l} + \frac{1}{\sigma_{l-1}} \right) + \frac{1}{2} \sum_{l=1}^{\infty} \left( 1 - \frac{\sigma_{l-1}}{\sigma_l} \right)^2 \sigma_{l-1}^2, \tag{26} \]
Consequently, for $1/2 < H \leq 1$, the survival probability \( P_N \) obeys:
\[
\exp[-L^2C_2] \Psi_N \leq P_N \leq \exp[L^2C_2] \Psi_N, \tag{27} \]
where \( \Psi_N \) is defined by Eq.(24).

Contrary to the situation discussed in the previous subsection, here, i.e. for $1/2 < H \leq 1$, \( \Psi_N \) describes the probability that an \( N \)-step Brownian motion trajectory \( x_l \), commencing at the origin, does not escape from the interval whose boundaries move deterministically towards the origin, i.e. that it survives in a shrinking cage.

In this subsection we estimate the long-time asymptotical behavior of \( \Psi_N \) using an adiabatic approximation described in Ref.[14]. To ascertain the accuracy of this approach, in the next section we will present the results of a more rigorous analysis.

To define an asymptotic behavior of \( \Psi_N \), consider the solution of a diffusion equation
\[
\frac{\partial P(X,t)}{\partial t} = \frac{1}{2\Delta} \frac{\partial^2 P(X,t)}{\partial X^2}, \quad P(X,t=0) = \delta(X), \tag{28} \]
subject to the boundary conditions
\[
P(X = \pm L(t), t) = 0, \tag{29} \]
where \( L(t) = L/\sigma_t \) and \( \sigma_t \sim \sqrt{\Delta} t^{H-1/2}/\Gamma(H + 1/2) \), Eq.(23). The basic idea behind the adiabatic approximation is that, if the cage expands or shrinks sufficiently slowly, the density distribution approaches the same form as in the fixed-cage case, except that the parameters in this probability distribution acquire time dependence to satisfy the moving boundary condition [14]. Within this approximation, one takes
\[
P(X,t) \sim f(t) \cos \left( \frac{\pi X}{2L(t)} \right) = f(t) \cos \left( \frac{\pi X}{2L} \frac{\sqrt{\Delta}}{\Gamma(H + 1/2)} t^{H-1/2} \right), \tag{30} \]
where the amplitude \( f(t) \) is to be determined. Substituting Eq.(30) into Eq.(28), one finds
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\[
\frac{df(t)}{dt} = \left( \frac{\pi^2}{8\Delta L^2(t)} \right) f(t) - \left( \frac{\pi X}{2L^2(t)} \right) \tanh \left( \frac{\pi X}{2L(t)} \right) \frac{dL(t)}{dt} f(t). \tag{31}
\]

Noticing next that the second term on the right-hand-site of Eq. (31) decays faster than the first one, we find

\[
f(t) \sim \exp \left[ -\frac{\pi^2}{8\Delta} \int_0^t \frac{H(\pi^2(H + 1/2)/L^2)}{16H^2(1/2)} d\tau \right] \sim \exp \left[ -\frac{\pi^2}{16H^2(H + 1/2)} t^{2H} \right], \tag{32}
\]

and hence, \( \Psi_t \) follows

\[
\Psi_t \sim \int_{-L(t)}^{L(t)} dX P(X, t) = \frac{4}{\pi} L(t) \exp \left[ -\frac{\pi^2}{16H^2(H + 1/2)} \frac{t^{2H}}{L^2} \right] \sim \exp \left[ -\frac{\pi^2}{8} \frac{E(\frac{(X^H)^2}{L^2})}{L^2} \right]. \tag{33}
\]

As in the case \( 1/4 < H < 1/2 \), \( \Psi_t \) determines the decay of the survival probability \( P_t \) and hence, we obtain the result in Eq. (6). Note also that the exponential term in Eq. (33), describing leading behavior in the superdiffusive regime with \( 1/2 < H \leq 1 \), is exactly the same as the one in Eq. (25), describing subdiffusion, but here it defines a faster than exponential decay of the survival probability.

5. Arbitrary Hurst index, \( 0 < H \leq 1 \): Upper and lower bounds.

In this section, our analysis will be based on the following two keystones:

(i) The probability \( P \left( \max |X_t^{H=1/2}| \leq M \right) \) that Brownian motion \( X_t^{H=1/2} \), starting at the origin, does not leave an interval \([-M, M]\) up to time \( t \), \( t \) sufficiently large, is given by (see, e.g., Ref. [15]):

\[
P \left( \max |X_t^{H=1/2}| \leq M \right) \sim \exp \left[ -\frac{\pi^2}{8M^2 t} \right]. \tag{34}
\]

(ii) A fundamental property of \( P_N \), Eq. (17): \( P_N = P_N(\sigma_1, \sigma_2, \ldots, \sigma_N) \) is a monotonically decreasing function of any variable \( \sigma_k \), i.e.

\[
\frac{\partial P_N}{\partial \sigma_k} \leq 0, \quad 0 \leq \sigma_k < \infty \tag{35}
\]

Equation (35) signifies that replacing any or all \( \sigma_k \) by \( \Sigma(k) \), such that \( \sigma_k \leq \Sigma(k) \) for any \( k \), we will decrease the survival probability and arrive at the lower bound on \( P_N \); if, on contrary, we will replace one or all \( \sigma_k \) by \( \Sigma(k) \), such that \( \sigma_k \geq \Sigma(k) \) for any \( k \), we will increase the survival probability and obtain an upper bound on \( P_N \).

We are unaware of any statement similar to (ii) made in the literature, apart of a less general "Pascal Principle" [16]. It might be thus instructive to demonstrate first its validity.

Let us first single out terms dependent on \( X_k \) and \( X_{k-1} \) in Eq. (17), writing down \( P_N \) formally as
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\[ P_N = \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1, l \neq k-1, k}^{N} \frac{dX_l}{\sqrt{2\pi\sigma_l}} B(\{X_l\}) \times \]
\[ \times \left( \int_{-L}^{L} \frac{dX_k}{\sqrt{2\pi\sigma_k}} \int_{-L}^{L} \frac{dX_{k-1}}{\sqrt{2\pi\sigma_{k-1}}} B'(X_{k-1}, X_k, X_{k+1}) \right), \]  

where

\[ B(\{X_l\}) = \exp \left[ -\sum_{l=1, l \neq k-1, k+1}^{N} \frac{(X_l - X_{l-1})^2}{2\sigma_l^2} \right], \quad X_0 \equiv 0, \]  

and

\[ B'(X_{k-1}, X_k, X_{k+1}) = \exp \left[ -\sum_{l=k-1}^{k+1} \frac{(X_l - X_{l-1})^2}{2\sigma_l^2} \right]. \]  

Note now that, trivially,

\[ \exp \left[ -\frac{(X_{k-1} - X_{k-2})^2}{2\sigma_{k-1}^2} - \frac{(X_{k+1} - X_k)^2}{2\sigma_{k+1}^2} \right] \leq 1, \]  

and hence, \( P_N \) is bounded from above by

\[ P_N \leq \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1, l \neq k-1, k}^{N} \frac{dX_l}{\sqrt{2\pi\sigma_l}} B(\{X_l\}) \times \]
\[ \times \left( \int_{-L}^{L} \frac{dX_k}{\sqrt{2\pi\sigma_k}} \int_{-L}^{L} \frac{dX_{k-1}}{\sqrt{2\pi\sigma_{k-1}}} \exp \left[ -\frac{(X_k - X_{k-1})^2}{2\sigma_k^2} \right] \right) \]  

Performing integration over \( X_k \) and \( X_{k-1} \), and differentiating both sides of the inequality in Eq.(40) with respect to \( \sigma_k \), we get

\[ \frac{\partial P_N}{\partial \sigma_k} \leq -\frac{1}{\pi\sigma_{k-1}} \left( 1 - \exp \left[ -\frac{2L^2}{\sigma_k^2} \right] \right) \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1, l \neq k-1, k}^{N} \frac{dX_l}{\sqrt{2\pi\sigma_l}} B(\{X_l\}) \leq 0 \]  

This proves the inequality in Eq.(35).

5.1. Bounds: fBm as a Brownian motion with variance \( \sigma_N^2 \).

Suppose that we set in Eq.(17) all \( \sigma_l \) equal to \( \sigma_N \) such that the survival probability in Eq.(17) becomes

\[ P'_N = \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1}^{N} \frac{dX_l}{\sqrt{2\pi\sigma_N}} \exp \left[ -\sum_{l=1}^{N} \frac{(X_l - X_{l-1})^2}{2\sigma_N^2} \right], \quad X_0 \equiv 0. \]  

In virtue of the fundamental property (ii) of \( P_N \), for subdiffusion, i.e., for \( H \) such that \( 0 < H < 1/2 \), we have the following upper bound:
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\[ P_N \leq P'_N, \quad (43) \]

since we have replaced \( \sigma_l \), (which is for subdiffusion a \textit{monotonically decreasing} function of \( l \)), by its lowest possible value \( \sigma_N \). On contrary, for superdiffusion we have an inverse inequality, i.e., a \textit{lower} bound,

\[ P_N \geq P'_N, \quad (44) \]

since in the superdiffusion case \( \sigma_l \) is a \textit{monotonically increasing} function of \( l \) and thus we have replaced it by its highest possible value.

Now, one notices that \( P'_N \) describes the probability \( P\left(\max|X^H_{1/2}| \leq M\right) \) that an \( N \)-step Brownian trajectory, starting at the origin, does not leave an interval \([-M,M]\), where \( M = L/\sigma_N \). Hence, in virtue of Eq.(34), we have

\[ P'_N \sim \exp\left[-\frac{\pi^2 \sigma^2_N}{8L^2} \right] \sim \exp\left[-\frac{\pi^2}{8\Gamma^2(H+1/2)} \frac{(\Delta N)^{2H}}{L^2} \right] = \exp\left[-2H \frac{\pi^2}{8} E \left[ \left( X^H_t \right)^2 \right] \right] \]

The result in Eq.(45) defines a \textit{lower} bound on the survival probability \( P_N \) in the superdiffusive case, and simultaneously, represents an \textit{upper} bound on \( P_N \) for subdiffusion, \( 0 < H < 1/2 \).

5.2. Bounds: fBm as a Brownian motion in time \( t^{2H} \).

Consider finally a lower and an upper bounds on \( P_N \) which still rely on the fundamental property (ii) of the survival probability \( P_N \) but also involve a little bit different type of arguments.

In essence, in this subsection we proceed to show that \( P_N \) can be bounded by

\[ \int_{-L}^{L} P_{lb}(X,t)dX \leq P_t \leq \int_{-L}^{L} P_{ub}(X,t)dX, \quad (46) \]

where \( P_{lb}(X,t) \) and \( P_{ub}(X,t) \) obey:

\[ \frac{\partial P_{lb}(X,t)}{\partial t} = \frac{1}{2T_{lb}(t)} \frac{\partial^2 P_{lb}(X,t)}{\partial X^2}, \quad P_{lb}(X,t = 0) = \delta(X), \quad T_{lb}(t) = \frac{\Gamma^2(H + 1/2)}{t^{2H-1}}, \quad (47) \]

\[ \frac{\partial P_{ub}(X,t)}{\partial t} = \frac{1}{2T_{ub}(t)} \frac{\partial^2 P_{ub}(X,t)}{\partial X^2}, \quad P_{ub}(X,t = 0) = \delta(X), \quad T_{ub}(t) = \frac{\Gamma^2(H + 3/2)}{2H \cdot t^{2H-1}}, \quad (48) \]

which have to be solved subject to the boundary condition:

\[ P_{lb,ub}(X = \pm L, t) \equiv 0 \quad (49) \]

In other words, we will show that survival probability \( P_N \) can be bounded by survival probabilities of Brownian motions evolving in time \( t^{2H} \).
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Note, that equations similar to Eqs.(47) or (48) have been already proposed in the literature \cite{17} as some effective differential equations describing fractional Brownian motion. However, despite the fact that Eq.(47) reproduces correctly the variance, Eq.(3), the Green’s function, and, as we will see, the time-dependence of the survival probability, it is not exact and can not reproduce correctly the correlations in the fBm process, Eq.(2).

Now, we turn back to our result in Eq.(17) and notice that in the general case $0 < H \leq 1$, i.e. for superdiffusion, diffusion and subdiffusion regimes, and for any $l \geq 1$,

$$\sigma_l^2 \leq \Sigma^2(l) = \frac{\Delta^{2H}}{2H \Gamma^2(H + 1/2)} g(l), \quad (50)$$

where

$$g(l) = l^{2H} - (l - 1)^{2H}. \quad (51)$$

In fact, the inequality in Eq.(50) with $g(l)$ defined by Eq.(51) appears to be a very good approximation for $\sigma_l^2$: while $\sigma_l^2$ and $\Sigma(l)$ show the same behavior as functions of $l$ for sufficiently large $l$, $\sigma_\infty/\Sigma(\infty) = 1$, they differ only by a few per cent also for moderate values of $l$.

Now, in virtue of (ii), we have the following lower bound on the survival probability $P_N$:

$$P_N \geq \int_{-L}^{L} \cdots \int_{-L}^{L} \prod_{i=1}^{N} \frac{dX_i}{\sqrt{2\pi \Sigma(l)}} \exp \left[ -\sum_{i=1}^{N} \frac{(X_i - X_{i-1})^2}{2\Sigma^2(l)} \right] = \int_{-M}^{M} \cdots \int_{-M}^{M} \prod_{i=1}^{N} \frac{dZ_i}{\sqrt{2\pi (l^{2H} - (l - 1)^{2H})}} \exp \left[ -\sum_{i=1}^{N} \frac{(Z_i - Z_{i-1})^2}{2(l^{2H} - (l - 1)^{2H})} \right], \quad (52)$$

where $Z_0 \equiv 0$ and $M = \sqrt{2H \Gamma(H + 1/2)L/\Delta^H}$.

One notices next that the expression in the second line in Eq.(52) defines the probability that an $N$-step trajectory of Brownian motion, starting at the origin and evolving in time $T = (\Delta N)^{2H}$ does not escape from the interval $[-M, M]$. Hence, in virtue of Eq.(34), the survival probability $P_N$ is bounded from below by

$$P_N \geq \exp \left[ -\frac{\pi^2}{16H \Gamma^2(H + 1/2)} \frac{(\Delta N)^{2H}}{L^2} \right] = \exp \left[ -\frac{\pi^2 E \left( X_H^2 \right)^2}{8 L^2} \right], \quad (53)$$

We emphasize that this bound holds for any $H \in [0, 1]$ and thus applies to both subdiffusion and superdiffusion regimes. Note also that it coincides with the results in Eqs.(25) and (33).

Next, note that for any $l \geq 1$ and any $H \in [0, 1]$, we have

$$\sigma_l^2 \geq \Sigma^2(l) = \frac{\Delta^{2H}}{\Gamma^2(H + 3/2)} g(l), \quad (54)$$

where $g(l)$ is defined by Eq.(51).
Hence, in virtue of (ii), we have the following upper bound on the survival probability $P_N$:

$$P_N \leq \int_{-L}^{L} \ldots \int_{-L}^{L} \prod_{l=1}^{N} \frac{dX_l}{\sqrt{2\pi \Sigma(l)}} \exp \left[ -\sum_{l=1}^{N} \frac{(X_l - X_{l-1})^2}{2\Sigma^2(l)} \right] =$$

$$= \int_{-M}^{M} \ldots \int_{-M}^{M} \prod_{l=1}^{N} \frac{dZ_l}{\sqrt{2\pi(l^{2H} - (l-1)^{2H})}} \exp \left[ -\sum_{l=1}^{N} \frac{(Z_l - Z_{l-1})^2}{2(l^{2H} - (l-1)^{2H})} \right], \quad (55)$$

where $Z_0 \equiv 0$ and $M = \Gamma(H + 3/2)L/\Delta$. Consequently,

$$P_N \leq \exp \left[ -\frac{\pi^2}{8\Gamma^2(H + 3/2)} \frac{(\Delta N)^{2H}}{L^2} \right] = \exp \left[ -\frac{2H}{(H + 1/2)^2} \frac{\pi^2}{8} \frac{E\left[X_t^H\right]}{L^2} \right], \quad (56)$$

This bound also holds for any $H \in [0, 1]$, i.e. for both superdiffusion and subdiffusion.

Note now that bounds in Eqs. (53) and (56) appear to be sharper than those defined by Eqs. (43), (44) and (45). Indeed, for superdiffusion the bound in Eq. (53) is higher than the one defined by Eqs. (44) and (45), since here $2H > 1$. For subdiffusion, the upper bound in Eqs. (43) and (45) is also worse, i.e. higher, than the one defined by Eq. (56) since $2H < 2H/(H + 1/2)^2$ for $H < 1/2$.

Therefore, the main result of the present paper can be represented as the following double-sided inequality on $P_N$:

$$\frac{2H}{(H + 1/2)^2} \leq -\left(\frac{8}{\pi^2} \frac{L^2}{E\left[X_t^H\right]}\right) \ln(P_N) \leq 1, \quad (57)$$

which holds for any $H \in [0, 1]$. Note that the bounds on the right-hand and on the left-hand-side coincide, as they should, for $H = 1/2$.

6. Conclusions

To conclude, we have studied the long-time asymptotical behavior of the probability $P_t$ that the Riemann-Liouville fractional Brownian motion with Hurst index $H$ does not escape from a fixed interval $[-L, L]$ up to time $t$. We have shown that $P_t$ obeys $\ln(P_t) \sim -t^{2H}/L^2$. This result is valid for any $H \in [0, 1]$, for both subdiffusion ($0 < H < 1/2$) and superdiffusion ($1/2 < H \leq 1$) regimes, and consequently, the decay may be slower than exponential (subdiffusion) or faster than exponential (superdiffusion).

This decay law has been obtained by a) showing that for $1/4 < H \leq 1$ the survival probability $P_t$ of the fBm in presence of two fixed adsorbing boundaries is determined by the probability that a Brownian motion does escape from the interval with moving boundaries and b) by elaborating upper and lower bounds on $P_t$ which show the same time dependence. These bounds stem from some fundamental property of the survival probability, Eq. (35), and controllable approximation of the fractional Brownian motion by standard Brownian motion evolving in time $T = t^{2H}$.

The obtained result for the survival probability decay implies, in particular, that the survival probability $S_t$ of particles undergoing fBm in one-dimensional systems with randomly placed traps obeys $\ln(S_t) \sim -n^{2/3}t^{2H/3}$, where $n$ is the mean density
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of traps. This expression generalizes famous Balagurov and Vaks [18] and Donsker and Varadhan [19] result over the case of anomalous diffusion described as fractional Brownian motion. For $H = 1/4$, we find the $\ln(S_t) \sim -t^{1/6}$ law, which was previously obtained in Ref. [9].

Finally, we note that the analysis presented in this paper can be straightforwardly generalized to fBm taking place in higher-dimensional spaces, the case of Weyl fractional Brownian motion, as well as for evaluation of bounds on the distribution function of the range of fBm, and on the survival probability of the fBm in presence of one-sided or two-sided moving boundaries.

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