Heat Transfer Analysis of Unsteady Natural Convection Flow of Oldroyd-B Model in the Presence of Newtonian Heating and Radiation Heat Flux

TALHA ANWAR, POOM KUMAM, ILYAS KHAN, ASIFA, AND PHATIPHAT THOUNTHONG

1Department of Mathematics, Faculty of science, King Mongkut’s University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand
2KMUTT Fixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Department of Mathematics, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand
3Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Science Laboratory Building, Faculty of Science, King Mongkut’s University of Technology Thonburi (KMUTT), Bangkok 10140, Thailand
4Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan
5Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia
6Department of Mathematics, COMSATS University Islamabad, Islamabad 44000, Pakistan
7Renewable Energy Research Centre, Department of Teacher Training in Electrical Engineering, Faculty of Technical Education, King Mongkut’s University of Technology North Bangkok, Bangkok 10800, Thailand

Corresponding author: Poom Kumam (poom.kumam@mail.kmutt.ac.th)

This work received funding from the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT and Petchra Pra Jom Kla Ph.D. Research Scholarship.

ABSTRACT
The main focus of this theoretical inspection is to explore the control of Newtonian heating on heat transfer for an unsteady natural convection flow of Oldroyd-B fluid confined to an infinitely long, vertically static plate. Partial differential equations are constructed effectively to describe the fluid flow and heat transfer. Some appropriate dimensionless quantities and Laplace transformation are employed as basic tools to evaluate the solutions of these differential equations. However, due to the complex nature of velocity field, solution is approximated by using Durbin’s numerical Laplace inverse algorithm. This solution is further validated by obtaining the velocity solution through algorithms proposed by Stehfest and Zakian. The temperature and velocity gradient are also determined to anticipate the heat transfer rate and skin friction at wall. Some well known results in literature are also deduced from the considered model. Conclusively, to have a deep understanding of the physical mechanism of considered model, and influence of implanted parameters, some outcomes are elucidated with the assistance of tables and graphs. As a result, it is found that under the effect of Newtonian heating, freely convective viscous fluid has greater velocity than Oldroyd-B fluid, Maxwell fluid and second grade fluid.

INDEX TERMS
Free convection, Laplace transform, Newtonian heating, Oldroyd-B model, thermal radiation.

NOMENCLATURE
$\rho$ Fluid density ($kgm^{-3}$)
$g$ Acceleration due to gravity ($ms^{-2}$)
$\beta$ Coefficient of thermal expansion ($K^{-1}$)
$T$ Fluid temperature ($K$)
$x, y, z$ Spatial variables ($m$)
$u, v, w$ Velocity components ($ms^{-1}$)
$\mu$ Dynamic viscosity ($kgms^{-1}$)
$\lambda$ Relaxation time ($s$)
$\lambda_r$ Retardation time ($s$)
$S$ Shear stress ($Nm^{-2}$)
$\nu$ Kinematic viscosity ($m^2s^{-1}$)
$k$ Thermal conductivity ($Wm^{-1}K^{-1}$)
$c_p$ Specific heat ($Jkg^{-1}K^{-1}$)
$\sigma_1$ Stefan-Boltzmann coefficient ($Wm^{-2}K^{-4}$)
$K_1$ Rosseland absorption constant ($m^{-1}$)
$T_\infty$ Ambient temperature ($K$)

The associate editor coordinating the review of this manuscript and approving it for publication was Hamid Mohammad-Sedighi.
model and drawn a comparison with zero-slip condition at
studied the role of slip condition in flow of Oldroyd-B
namic (MHD) motion of Oldroyd-B model. Shakeel et al.
analytically probed the role of porosity in magnetohydrody-
92480
VOLUME 8, 2020
neous movement of a surface inside a channel. Khan et al.
time-dependent motion of Oldroyd-B fluid due to sponta-
tions and melts, clay, and oil. A capable and simple model,
viscous and elastic behavior such as toothpaste, polymer solu-
one of fluid. A wide range of fluids reveal a combination 
non-linear term results in precise anticipation of behavior 
of fluid. A wide range of fluids reveal a combination of 
and elastic behavior such as toothpaste, polymer solu-
ations and melts, clay, and oil. A capable and simple model, 
which records the flow history and provides an adequate 
approximation of viscoelastic nature of fluids is recognized 
as Oldroyd-B model. Since James G. Oldroyd provided 
this model to forecast the elasticity and memory effects, 
therefore it is remembered as Oldroyd-B model [1]. The 
current model is capable of retaining rheological effects 
case of unidirectional flows, while it contains a non-physical 
singularity when extensional flows are under consideration. 
Furthermore, in shear flow, Oldroyd-B fluid provides an 
excellent approximation of viscoelastic fluids. The model can 
also be presented split into viscoelastic part separately from 
the solvent part. Extensively, if solvent has zero viscosity, 
this model reduces to Upper Convected Maxwell model. In 
this regard, Upper Convected model can be viewed as a 
special case of considered model and moreover conventional 
Maxwell and viscous fluids can be deduced from it by making 
simple substitutions [2].

According to authors’s knowledge, first exact solutions 
of Oldroyd-B model were evaluated by Tanner [3]. Later, 
Waters and King [4], [5] employed Laplace transformation 
to calculate the analytic results of these viscoelastic fluids. 
Fetecau et al. [6] provided a note on flow of Oldroyd-B model 
induced as a result of an accelerating surface. A systematic 
study was conducted by Fetecau et al. [7] to investigate the 
time-dependent motion of Oldroyd-B fluid due to sponta-
aneous movement of a surface inside a channel. Khan et al. [8] 
analytically probed the role of porosity in magnetohydrody-
amic (MHD) motion of Oldroyd-B model. Shakeel et al. [9] 
studied the role of slip condition in flow of Oldroyd-B model 
and drawn a comparison with zero-slip condition at

boundary. Khan et al. [10] examined the porosity effects 
on Oldroyd-B model by calculating the exact solutions. 
Gul et al. [11] analyzed transient MHD thin motion of Oldroyd-B model near an inclined oscillating surface. 
Riaz et al. [12] explored the fractional flow of Oldroyd-B model in a circular duct by deriving the analytic results. 
Khan et al. [13] discussed the hydromagnetic rotatory motion 
of Oldroyd-B model in a permeable material. Recently, some 
new global results for incompressible Oldroyd-B fluid were 
discussed by Wan [14]. Application of Oldroyd-B model to 
hemodynamics and its numerical simulation was presented 
by Elhanafy et al. [15]. Hullender [16] studied pretransient 
turbulent motion in circular lines by employing Oldroyd-B 
transient model. Tiwana et al. [17] evaluated the influence of 
ramped temperature and ramped boundary motion on tran-
sient MHD convection flow of Oldroyd-B model. Keeping in 
mind the above literature, it is found that effect of Newtonian 
heating on Oldroyd-B fluid has not been investigated yet. 
This study is an attempt to fulfill this gap by analyzing 
the role of Newtonian heating and nonlinear heat flux in 
unsteady naturally convective motion of Oldroyd-B fluid past 
a vertically static plate.

Idea of Newtonian heating was initiated by Merkin [18], 
as he noted that by applying Newtonian heating from surface, 
convective flows can be set up. Such kind of flows are called 
conjugate convective flows. He provided a numerical solu-
tion comprised of analytic solution near the leading edge 
and numerical solution far downstream. The mechanism of 
Newtonian heating occurs in numerous engineering processes 
such as heat exchanger, petroleum industry, solar radiation, 
and conjugate heat transfer about fins. Salleh et al. [19] 
studied the impact of Newtonian heating on energy and fluid 
motion near a stretching surface. Haq et al. [20] explored the 
influence of convective boundary conditions and magnetic 
field on two dimensional flow of Casson nanofluid near a 
stretching surface. Nadeem et al. [21] further extended this 
study to numerically probe the same flow in three dimensions. 
Consequences of imposed Newtonian heating, MHD, and slip 
condition on Casson fluid flow over a nonlinearly stretch-
ing surface embedded in porous material were explored by 
Ullah et al. [22]. Imran et al. [23] examined control of New-
tonian heating and slip effect on MHD boundary layer flow 
of generalized Maxwell model. Hayat et al. [24] analyzed 
power law nanofluid’s steady motion near stretching sheet 
with Newtonian heating. Ramzan [25] reported the influence 
of Joule heating, Newtonian heating, and viscous dissipation 
on three dimensional MHD flow of nanofluid.

The three major modes of convection are known as nat-
ural, mixed and forced convection [26]. From these three 
mechanisms, natural convection is considered in this study. 
In natural convection, temperature gradient in turn causes 
the heat transfer and buoyancy force induces the flow. The 
flows based on natural convection are particularly significant 
in electric machinery, electronic power supplies, designing of 
spaceships, drying of porous substances in textile industries, 
and nuclear reactors. Furthermore applications of natural
convection in engineering and sciences include solar ponds, oceanic and atmospheric circulation, and formation of micro-structure [27], [28]. Initially, Siegel [29] provided the foundation to study natural convection by investigating the time dependent natural convective flow over a semi infinite vertical plate. Ahmed [30] analyzed the effects of heat source, Hall current, MHD, thermal diffusion, and porosity of medium on transient natural convective motion over an upright plate. Exact analysis of unsteady naturally convective motion of a radiative gas under imposed magnetic field past an infinite inclined plate was reported by Narahari [31]. Impacts of ramped temperature and slip condition at wall for transient free convective flow of viscous fluid were examined by Haq et al. [32]. Shen et al. [33] studied unsteady natural convection flow of second grade nanofluid with a new definition of time-space fractional applied to heat conduction.

Thermal radiation has a vital role when it comes to space technology and many other environmental process which take place at high temperature. For instance, hypersonic flights, rocket combustion chambers, gas-cooled nuclear reactors, missile reentry, and power plants for interplanetary flights. All these practical applications have urged the researchers to put their special attention on this significant energy transferring mechanism and interpret the outcomes in an improved and understandable fashion. Das et al. [34] examined the influence of radiative flux and Newtonian heating on transient free convection motion over a vertical wall. Closed form solutions for freely convective nanofluid motion near a moving plate influenced by heat radiation and magnetic field were reported by Das and Jana [35]. Izadi et al. [36] scrutinized thermogravitational flow of a micro-polar nanoliquid to evaluate the impression of radiative heat flux and MHD in porous channel.

In the light of above investigations, this study is an attempt to analyze the profiles of unsteady free convective flow of Oldroyd-B fluid over a static vertical plate and radiative heat transfer with Newtonian heating at boundary. Semi analytic solutions of flow and heat governing equations are obtained by applying Laplace transformation. Durbin’s numerical algorithm [37] is executed to approximate the velocity solution in real time domain and this approximation is further validated by using Stehfest’s [38] and Zakian’s algorithm [39]. The temperature gradient and Skin friction are precisely evaluated at wall in pursuance to their significant applications in mechanics and engineering. Finally, current velocity, temperature and velocity of some limiting fluids’ cases are graphically portrayed to get a clear insight of physical features of considered model.

II. MATHEMATICAL MODELING OF PHYSICAL PROCESS

The principal equations to express the laminar, unsteady freely convective flow of Oldroyd-B fluid together with applied Boussinesq’s approximation for buoyancy force are described as [40]–[42]

\[ \nabla \cdot \mathbf{V} = 0, \]

(1)

\[ \rho \left( \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right) = \mathbf{g} \rho \beta T - \mathbf{g} \rho \beta T_\infty + \nabla \cdot \mathbf{T}, \]

(2)

where \( g \) is standard gravitational pull, \( t \) is time, \( \beta \) is coefficient of thermal volume expansion, \( \rho \) is density, \( T \) is temperature, \( T_\infty \) is ambient temperature and \( \nabla \) is gradient operator. Moreover, Cauchy tensor of stress \( \mathbf{T} \) and one dimensional velocity of laminar flow \( \mathbf{V} \) are respectively described as

\[ \mathbf{T} = -p \mathbf{I} + \mathbf{S}, \]

(3)

\[ \mathbf{V} = [u(y, t), 0, 0], \]

(4)

where \( y \) is space variable, \( u \) is velocity component in \( x \)-direction and \(-p \mathbf{I}\) is the indeterminate stress tensor. Furthermore, relation for extra stress tensor \( \mathbf{S} \) is given as

\[ (1 + \lambda \dot{\mathbf{S}}) = (1 + \lambda_r \dot{\mathbf{A}}) \mu, \]

(5)

where \( \mu \) is dynamic viscosity, \( \lambda \) is relaxation time and \( \lambda_r \) is retardation time. Additionally, Rivlin-Ericksen tensor \( \mathbf{A} \) and material time derivative represented by superposed dot are given as

\[ \mathbf{A} = (\nabla \mathbf{V}) + (\nabla \mathbf{V})^T = \begin{pmatrix} 0 & u_t \\ u_y & 0 \end{pmatrix}, \]

(6)

\[ \frac{D \mathbf{S}}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}. \]

(7)

Operating equations (3) – (7) in momentum equation (2), we acquire

\[ \rho \left( 1 + \lambda \frac{\partial}{\partial t} \right) \frac{\partial u}{\partial t} = \mu \left( 1 + \lambda_r \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial y^2} + \rho g \beta \left( 1 + \lambda \frac{\partial}{\partial t} \right) (T - T_\infty). \]

(8)

The considered model is geometrically presented in Fig. 1.

**FIGURE 1.** Geometrical description of current flow model.

Application of Rosseland approximation [43], and assumptions of small temperature difference and small thermal Reynolds number formulate the following equations for velocity, shear stress, and energy of Oldroyd-B fluid

\[ \frac{\partial u}{\partial t} + \lambda \frac{\partial^2 u}{\partial t^2} = \nu \frac{\partial^2 u}{\partial y^2} + \lambda_r \nu \frac{\partial^3 u}{\partial y^3 \partial t} + \beta g \left( 1 + \lambda \frac{\partial}{\partial t} \right) (T - T_\infty), \]

(9)
\[ S + \lambda \frac{\partial S}{\partial \tau} = \mu \frac{\partial u}{\partial y} + \lambda \mu \frac{\partial^2 u}{\partial y^2}, \]
\[ \frac{\partial T}{\partial \tau} = \frac{k}{\rho c_p} \left( 1 + \frac{16\sigma_1 T_\infty}{3K_1} \right) \frac{\partial^2 T}{\partial y^2}, \]

where \( \nu \) is kinematic viscosity, \( S \) is nontrivial shear stress, \( k \) is thermal conductivity, \( c_p \) is specific heat at constant pressure, \( \sigma_1 \) is Stefan-Boltzmann coefficient and \( K_1 \) is Rosseland absorption coefficient.

The appropriate initial and boundary conditions corresponding to governing equations are
\[ u(y, 0) = 0, \quad T(y, 0) = T_\infty, \quad \left. \frac{\partial u(y, t)}{\partial y} \right|_{y=0} = 0, \]
\[ u(0, \tau) = 0, \quad \left. \frac{\partial T(y, \tau)}{\partial y} \right|_{y=0} = -\frac{h}{k} T(0, \tau), \]
\[ u(y, \tau) \to 0, \quad T(y, \tau) \to T_\infty \quad \text{as} \quad y \to \infty. \]

The purpose of reducing the number of variables in current model is achieved by introducing following dimensionless terms
\[ \eta = \left( \frac{h}{k} \right) y, \quad u_1 = \left( \frac{k}{h \nu} \right) u, \quad \tau = \left( \frac{h}{k} \right)^2 t, \]
\[ \lambda_1 = \nu \left( \frac{h}{k} \right)^2 \lambda, \quad \lambda_2 = \nu \left( \frac{h}{k} \right)^2 \lambda, \]
\[ F = \frac{\rho}{\mu^2} \left( \frac{k}{h} \right)^2 S, \quad \theta = \frac{T - T_\infty}{T_\infty}. \]

Using above quantities in equations (9)-(11), we get
\[
\left( 1 + \lambda_1 \frac{\partial}{\partial \tau} \right) \frac{\partial u_1}{\partial \tau} = \left( 1 + \lambda_2 \frac{\partial}{\partial \tau} \right) \frac{\partial^2 u_1}{\partial \eta^2} + Gr \left( 1 + \lambda_1 \frac{\partial}{\partial \tau} \right) \theta, \\
\left( 1 + \lambda_1 \frac{\partial}{\partial \tau} \right) F = \left( 1 + \lambda_2 \frac{\partial}{\partial \tau} \right) \frac{\partial u_1}{\partial \eta}, \]
\[ \frac{\partial^2 \theta}{\partial \eta^2} = \frac{\partial^2 \theta}{\partial \eta^2}, \]
where non-dimensional numbers are given as
\[ Gr = \frac{g \beta T_\infty}{\nu^3} \left( \frac{k}{h} \right)^3, \quad Pr = \frac{\mu c_p}{k}, \]
\[ Nr = \frac{16\sigma_1 T_\infty^3}{3kK_1}, \quad a = \frac{Pr}{1 + Nr}. \]

The initial and boundary conditions turn out as
\[ u_1(\eta, 0) = 0, \quad \theta(\eta, 0) = 0, \quad \left. \frac{\partial u_1(\eta, \tau)}{\partial \eta} \right|_{\eta=0} = 0, \]
\[ u_1(0, \tau) = 0, \quad \left. \frac{\partial \theta(\eta, \tau)}{\partial \eta} \right|_{\eta=0} = -\left[ \theta(0, \tau) + 1 \right], \]
\[ u_1(\eta, \tau) \to 0, \quad \theta(\eta, \tau) \to 0 \quad \text{as} \quad \eta \to \infty. \]

### III. SOLUTION OF PROBLEM

Laplace transformation \[ \mathcal{L} \] is an efficient technique to derive the analytical solutions of current problem. Its formulation in integral form is provided as
\[ \mathcal{L}[M](\eta, \tau) = \int_0^\infty M(\eta, \tau)e^{-\tau} d\tau = \tilde{M}(\eta, s). \]

For considered problem, \( M \in \{ \theta, u_1, F \} \) and the condition \( Re(s) > a_0 \) ensures the convergence of above integral. Here \( s = a + ib \) with \( a \) and \( b \) stand for some real constants and \( i \) is standard imaginary unit. The inverse Laplace transformation to obtain solutions in \( \tau \) domain is accomplished by employing following relation
\[ \mathcal{L}^{-1}\tilde{M}(\eta, s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \tilde{M}(\eta, s)e^{\tau \tau} d\tau = M(\eta, \tau). \]

### A. TEMPERATURE DISTRIBUTION

Operating Laplace transform (23) to the equation (18) and using initial condition (20) yields
\[ \frac{\partial^2 \bar{\theta}}{\partial \eta^2} - sa\bar{\theta} = 0, \]
where \( \bar{\theta} \) satisfies the following boundary conditions
\[ \left. \frac{\partial \bar{\theta}(\eta, s)}{\partial \eta} \right|_{\eta=0} = -\left[ \bar{\theta}(0, s) + 1 \right], \]
\[ \bar{\theta}(\eta, s) \to 0 \quad \text{as} \quad \eta \to \infty. \]

The solution of equation (25) under conditions in equation (26) is evaluated as
\[ \bar{\theta}(\eta, s) = \frac{1}{\sqrt{a} \sqrt{s - 1}} e^{-\sqrt{a} \eta}. \]

Implementation of inverse Laplace transform (24) provides following result
\[ \theta(\eta, \tau) = e^{(\frac{s}{2} - \frac{\tau}{2})} \text{erfc} \left( \frac{\eta}{2\sqrt{\tau} - \sqrt{a}} \right) - \text{erfc} \left( \frac{\eta}{2\sqrt{\tau} - \sqrt{a}} \right). \]

To anticipate the rate of heat transfer from plate to fluid, Nusselt number is evaluated by plugging equation (28) into the following expression
\[ Nu = -\left. \frac{\partial \theta(\eta, \tau)}{\partial \eta} \right|_{\eta=0} = -\left[ -\theta(0, \tau) + 1 \right] \]
\[ = e^{(\frac{s}{2})} \left[ 2 - \text{erfc} \left( \frac{\sqrt{\tau}}{\sqrt{a}} \right) \right]. \]

### B. VELOCITY DISTRIBUTION

Applying Laplace transform on equations (16), and using initial condition (20)1 emits
\[ (1 + \lambda_1 s) \tilde{u}_1 = (1 + \lambda_2 s) \frac{\partial^2 \tilde{u}_1}{\partial \eta^2} + Gr (1 + \lambda_1 s) \bar{\theta}, \]
\[ \left. \frac{\partial u_1(\eta, \tau)}{\partial \eta} \right|_{\eta=0} = \left. \frac{\partial \theta(\eta, \tau)}{\partial \eta} \right|_{\eta=0} = -[\theta(0, \tau) + 1], \]
\[ u_1(0, \tau) = 0, \quad \theta(\eta, \tau) \to 0 \quad \text{as} \quad \eta \to \infty. \]
where $\tilde{u}_1$ follows the conditions mentioned below

$$\tilde{u}_1(0, s) = 0 \quad \text{and} \quad \tilde{u}_1(\eta, s) \to 0 \quad \text{when} \quad \eta \to \infty. \quad (31)$$

Introducing equation (27) into equation (30), we obtain

$$(1 + \lambda_2 s) x \tilde{u}_1 - (1 + \lambda_1 s) s \tilde{u}_1 = -\frac{Gr (1 + \lambda_1 s)}{s a (s - \frac{1}{\sqrt{a}})} e^{-\sqrt{a} \eta}. \quad (32)$$

The solution of above equation is derived as

$$\tilde{u}_1(\eta, s) = \frac{(1 + \lambda_1 s)Gr}{bs(\sqrt{s} - \frac{1}{\sqrt{a}})((s + m)^2 - m^2)} \times e^{-\sqrt{a} \eta}, \quad (33)$$

where

$$b = \sqrt{a}(\lambda_2 a - \lambda_1), \quad m = \frac{a - 1}{2(\lambda_2 a - \lambda_1)}. \quad (34)$$

Since $\tilde{u}_1(\eta, s)$ is a complex function containing multi-valued combinations of Laplace parameter “s”, therefore numerical inverse Laplace transform is applied to approximate solution in real time domain. Particularly, we operated Durbin’s method, which is based on Fourier series expansion such as

$$u_1(\eta, \tau) = \frac{e^{\alpha_1 \tau}}{T} \left( -\frac{1}{2} \text{Re} \left\{ \tilde{u}_1(\eta, a) \right\} \right. \right.$$  
$$\left. + \sum_{p=0}^{\infty} \text{Re} \left\{ \tilde{u}_1 \left( \eta, a + \frac{p \pi}{T} i \right) \right\} \cos \left( \frac{p \pi \tau}{T} \right) \right.$$  
$$\left. - \sum_{p=0}^{\infty} \text{Im} \left\{ \tilde{u}_1 \left( \eta, a + \frac{p \pi}{T} i \right) \right\} \sin \left( \frac{p \pi \tau}{T} \right) \right). \quad (35)$$

In order to validate our numerical Laplace inversion, we obtained approximation of $\tilde{u}_1(\eta, s)$ with the help of Stehfest’s algorithm and Zakian’s algorithm respectively in following manner.

$$u_1(\eta, \tau) = \frac{\ln(2)}{\tau} \sum_{k=1}^{2n} d_k \tilde{u}_1 \left( \eta, \frac{k \ln(2)}{\tau} \right), \quad \text{with}$$

$$d_k = (-1)^{k+n} \sum_{j=\left[\frac{k}{2}+1\right]}^{\min(k,a)} \frac{j^m(2j)!}{(n-j)!(j-1)!(2j-k)!}, \quad (36)$$

$$u_1(\eta, \tau) = \frac{2}{\tau} \sum_{i=1}^{s} \text{Re} \left\{ K_0 \tilde{u}_1 \left( \eta, \frac{\alpha_i}{\tau} \right) \right\}, \quad (37)$$

where $n$ is a positive integer and $\alpha_i$ and $K_i$ are fixed complex values.

The shear stress on the wall $F_w$ is approximated by introducing the derivative of equation (33) into following relation

$$F_w = \frac{(1 + \lambda_2 \frac{\partial}{\eta})}{(1 + \lambda_1 \frac{\partial}{\pi})} \frac{\partial}{\eta} \left|_{\eta=0} \right.$$  
$$\frac{\sqrt{a}}{s a (s - \frac{1}{\sqrt{a}})[(s + m)^2 - m^2]} \times \left( (1 + \lambda_2 s)Gr \right)$$  
$$\left. - \sqrt{a} s \bar{u} \left( \eta, \frac{1 + \lambda_2 s}{1 + \lambda_2 s} \right) \frac{\partial}{\eta} \right|_{s (s + m)^2 - m^2}. \quad (38)$$

IV. LIMITING CASES

This section is comprised of special cases deduced from the current work.

A. MAXWELL FLUID WITH NEWTONIAN HEATING

Making $\lambda_2 \to 0$ in equation (33), we derive the following solution

$$\tilde{u}_1(\eta, \tau) = \frac{(1 + \lambda_1 s)Gr}{\lambda_1 s(1 - s \sqrt{a})} \left( \frac{(s + \frac{a-1}{2a})^2 - \left( \frac{a-1}{2a} \right)^2}{s + \frac{a-1}{2a}} \right). \quad (39)$$

B. SECOND GRADE FLUID WITH NEWTONIAN HEATING

Making $\lambda_1 \to 0$ in equation (33), we get the following result

$$\tilde{u}_1(\eta, \tau) = \frac{(1 + \lambda_1 s)Gr}{\lambda_2 s(a-1)\left( \frac{1}{s} - \frac{1}{s} \right)} \left( \frac{(s + \frac{a-1}{2a})^2 - \left( \frac{a-1}{2a} \right)^2}{s + \frac{a-1}{2a}} \right). \quad (40)$$

C. NEWTONIAN FLUID WITH NEWTONIAN HEATING

Making $\lambda_1 \to 0$ and $\lambda_2 \to 0$ in equation (33), we derive the following solution

$$\tilde{u}_1(\eta, \tau) = \frac{Gr}{\sqrt{a^6(a-1)} \left( s - \frac{1}{\sqrt{a}} \right)} \left( s - \frac{1}{\sqrt{a}} \right). \quad (41)$$

V. RESULTS AND DISCUSSION

To deeply analyze the practical applicability of considered problem, we have studied the physical significance of Grashof number $Gr$, time $\tau$, relaxation time $\lambda_1$, radiation parameter $Nr$, retardation time $\lambda_2$ and Prandtl number $Pr$ in momentum and energy equations and attained conclusions are interpreted through graphs. The effect of same parameters on rate of heat transfer and wall shear stress is observed and numerical computations are presented in tabular form. The cases $Gr = 0$ and $Nr = 0$ correspond to absence of buoyancy force and thermal radiation respectively. The default values of relevant parameters are mentioned inside the respective figures.

Fig. 2 presents the relationship between radiation coefficient $Nr$ and dimensionless temperature of fluid $\theta(\eta, \tau)$. It is
found that temperature profile gets elevation with increase in value of \( \text{Nr} \). Since increase in \( \text{Nr} \) at fixed values of \( T_\infty \) and \( k \), decreases the value of \( K_1 \), therefore gradient of radiative thermal flux \( \partial q_r/\partial y \) increases which leads to increase the radiative heat transfer rate and eventually temperature of fluid rises. It means that thickness of energy boundary layer reduces and temperature is distributed more uniformly.

The variation of fluid temperature due to Prandtl number \( \text{Pr} \) is described in Fig. 3. It is witnessed that temperature of fluid goes through a decay due to increasing variation of \( \text{Pr} \). It is justified by the fact that a high \( \text{Pr} \) value is associated to low thermal conductivity, which reduces both conduction and thermal boundary layer thickness. Ultimately, fluid faces more thermal resistance and temperature of fluid decreases. Fig. 4 accounts the transient nature of flow and reveals that temperature rises with the extension of time duration \( \tau \). The fluid temperature is high adjacent to the wall and asymptotically drops down to zero value, as fluid creeps away from the wall. Table 1 exhibits an enhancement in Nusselt number due to increasing variation of \( \text{Nr} \), while an inverse profile is observed as a response of higher \( \text{Pr} \) value. This is supported by the fact that increase in \( \text{Nr} \) means temperature gradient has strong dominance which results in higher rate of heat transfer. Contrarily, when fluid gains higher \( \text{Pr} \) value, its thermal conductivity reduces and therefore, its capacity of heat conduction vanishes. Hence thickness of thermal boundary layer decreases and eventually rate of heat transfer reduces. Additionally, respective table indicates that rate of heat transfer increases as time \( \tau \) evolves.

In order to verify our velocity approximation calculated by Durbin’s method, we have also computed the solution by Stehfest’s method and Zakian’s method. In Fig. 5, velocity solution approximated by these three approaches is presented at time steps \( \tau = 3.0 \) and \( \tau = 5.0 \), and an excellent agreement between all the solutions is noticed at both time steps. This comparison validates the reliability of solution.

Fig. 6 demonstrates the velocity distribution for different values of relaxation time \( \lambda_1 \). It is observed that fluid’s velocity
increases with an increase in $\lambda_1$. This is realized by the fact that $\lambda_1$ reduces the boundary layer thickness and corresponding to this decrease in thickness, velocity illustrates significant behavior in main stream region and later fluid attains zero velocity. The significance of Grashof number $Gr$ in fluid flow is presented in Fig. 7. It is clearly sighted that fluid gets accelerated when $Gr$ increases. This is due to the fact that $Gr$ value is responsible for relative contribution of buoyancy force and viscous force to the fluid flow. Increase in $Gr$ means, buoyancy force dominates the viscous force which leads to more rapid flow of fluid. Therefore, as value of $Gr$ increases, velocity profile of fluid rises. An interesting observation is made that when $Gr$ has zero value fluid has absolutely no motion. This observation justifies the free convection nature of flow and indicates that fluid flow is purely generated by buoyancy force in this case.

Fig. 8 exhibits the impact of radiation parameter $Nr$ on velocity of fluid. It is spotted that thickness of momentum boundary layer increases with higher values of $Nr$. Physically, rate of energy transfer justifies this increment. As $Nr$ enlarges, rate of energy transfer to the fluid increases which in turn results to loose the bonds between fluid particles. As a result, these loosely connected particles collectively offer a much weaker viscosity to fluid motion and eventually fluid gets accelerated. The effect of retardation time $\lambda_2$ on velocity distribution is revealed in Fig. 9. It is realized that flow is retarded due to higher values of $\lambda_2$. This is supported by the fact that increase in $\lambda_2$ reduces the momentum boundary layer.
thickness and fluid shows significant behavior in main stream region but away from wall it gradually calms down. Fig. 10 describes the contribution of Pr in velocity distribution. It is shown that fluid is decelerated with an increase in value of Pr. The physical phenomenon justifying this profile is dominance of viscous forces. For larger values of Pr, viscosity of fluid increases and it offers strong resistance to fluid flow. Eventually, velocity of fluid faces a decay.

Fig. 11 illustrates that progress of time $\tau$ accelerates the fluid flow. A comparison between velocity profile of Oldroyd-B fluid, Maxwell fluid, second grade fluid and Newtonian fluid is shown in Fig. 12.

### Table 2. Numerical Laplace inversion of velocity by Durbin’s, Stehfest’s and Zakian’s algorithm.

| $\eta$ | $u_1(\eta, \tau)$ [Durbin’s] | $u_1(\eta, \tau)$ [Stehfest’s] | $u_1(\eta, \tau)$ [Zakian’s] |
|--------|-------------------------------|---------------------------------|-------------------------------|
| 0.0    | 0.00000000                   | 0.00000000                     | 0.00000000                    |
| 0.2    | 0.4526357                    | 0.4526354                     | 0.4526175                     |
| 0.4    | 0.7108129                    | 0.7108137                     | 0.7108309                     |
| 0.6    | 0.8332767                    | 0.8332768                     | 0.8332910                     |
| 0.8    | 0.8644130                    | 0.8644115                     | 0.8644085                     |
| 1.0    | 0.8871126                    | 0.8371105                     | 0.8371040                     |
| 1.2    | 0.7751806                    | 0.7751785                     | 0.7751767                     |
| 1.4    | 0.6953549                    | 0.6953520                     | 0.6953556                     |
| 1.6    | 0.608934                     | 0.6089792                     | 0.6089837                     |
| 0.18   | 0.5233977                    | 0.5233929                     | 0.5233994                     |
| 0.20   | 0.4430178                    | 0.4430126                     | 0.4430183                     |

### Table 3. Variation of shear stress for implanted factors.

| $\tau$ | $\lambda_1$ | $\lambda_2$ | Shear stress [Durbin’s] | Shear stress [Stehfest’s] |
|--------|-------------|-------------|-------------------------|--------------------------|
| 3.0    | 1.0842424   | 1.0842482   | 1.6360516               | 1.6360953                |
| 4.0    | 1.6360516   | 1.6360953   | 2.2942936               | 2.2963461                |
| 5.0    | 2.2942936   | 2.2963461   | 3.0714890               | 3.0739958                |
| 6.0    | 3.0714890   | 3.0739958   | 1.7121803               | 1.7122523                |
| 7.0    | 1.7121803   | 1.7122523   | 1.6360516               | 1.6360953                |
| 8.0    | 1.6360516   | 1.6360953   | 2.5704830               | 2.570581                |
| 9.0    | 2.5704830   | 2.570581    | 1.5145571               | 1.5145793                |
| 10.0   | 1.5145571   | 1.5145793   | 0.0000000               | 0.0000000               |
| 11.0   | 0.0000000   | 0.0000000   | 0.8180308               | 0.8180476               |
| 12.0   | 0.8180308   | 0.8180476   | 2.0  | 1.6360516 | 1.6360953   | 2.4540924   | 2.4541416   |
| 13.0   | 2.4540924   | 2.4541416   | 3.0  | 1.5951662 | 1.5951666   | 1.6728645   | 1.7338491   |
| 14.0   | 1.5951662   | 1.5951666   | 2.0  | 1.7806314 | 1.7806743   | 1.7338491   | 1.7806743   |
Newtonian fluid is drawn in Fig. 12. It is found that under the effect of Newtonian heating, Newtonian fluid attains highest velocity and on the other hand, second grade fluid exhibits slowest motion profile. Table 2 shows the computations of velocity by Durbin’s, Stehfest’s and Zakian’s algorithms at different spatial steps to observe the authenticity of solution upto desired accuracy. Table 3 predicts that shear stress at wall is controlled by gradually raising the value of \( \lambda \), while an inverse behavior is observed for higher values of \( \lambda \). However, velocity decreases for larger values of \( \lambda \) and \( \tau \). Approximated shear stress computations are also compared for the purpose of verification. Usually, one is attracted to calculate the skin friction (shear stress) for the technical purposes. Moreover enhanced skin friction is considered to be a limitation in engineering exercises. Table 4 is presented to numerically compare the velocity of various fluids at different spatial steps.

### VI. CONCLUSION

The purpose of this investigation is to analyze the effect of Newtonian heating on rate of heat transfer for an unsteady free convection flow of Oldroyd-B model over a static infinite vertical plate. Some appropriate dimensionless quantities are introduced in principal flow and heat governing equations, and boundary conditions to obtain the unit-less form of current model. Solutions of this unit-less model are calculated by applying Laplace transform. Graphical and tabular illustrations of solutions are provided to observe the physical importance of emerging parameters. Temperature and velocity gradient at wall are evaluated to analyze the Nusselt number and shear stress respectively. A comparison between velocity profile of different fluid models deduced from current model is also drawn to clearly examine the difference between flows.

The key findings of this analysis are outlined as:

- Temperature has higher profile for increasing values of \( \text{Nr} \) and \( \tau \), while it has lower profile for higher \( \text{Pr} \) values.
- Rate of heat transfer (Nusselt number) is decreasing function of \( \text{Pr} \) and it increases for an increment in \( \text{Nr} \).
- Fluid has no motion for \( Gr = 0 \), which shows that flow is induced due to buoyancy force (free convection).
- Fluid gets accelerated with an increase in \( Gr, \lambda, \text{Nr} \) and \( \tau \). However, velocity decreases for larger values of \( \lambda \) and \( \text{Pr} \).
- Skin friction reduces as value of relaxation time \( \lambda \) increases and has an inverse behavior for similar variation of retardation time \( \lambda \).
- In Comparison, Newtonian fluid has highest velocity profile and on the other hand second grade fluid has the slowest flow.
- Our approximated velocity solutions through numerical Laplace inversion methods named as Durbin’s algorithm, Stehfest’s algorithm and Zakian’s algorithm are equal.

### CONFLICTS OF INTEREST

The author declares that there is no competing interest.

### REFERENCES

[1] J. G. Oldroyd, “On the formulation of rheological equations of state,” Proc. R. Soc. Lond. A, vol. 200, no. 1063, pp. 523–541, 1950.
[2] J. C. Maxwell, “Iv. on the dynamical theory of gases,” Philos. Trans. Roy. Soc., vol. 200, no. 157, pp. 49–88, 1867.
[3] R. I. Tanner, “Note on the Rayleigh problem for a visco-elastic fluid,” Zeitschrift für Angew. Math. Phys., vol. 13, no. 6, pp. 573–580, Nov. 1962.
[4] N. D. Waters and M. J. King, “Unsteady flow of an elastico-viscous liquid,” Rheologica Acta, vol. 9, no. 4, p. 614, Nov. 1970.
[5] N. D. Waters and M. J. King, “The unsteady flow of an elastico-viscous liquid in a straight pipe of circular cross section,” J. Phys. D, Appl. Phys., vol. 4, no. 2, pp. 204–211, Feb. 1971.
[6] C. Fetecau, S. C. Prasad, and K. R. Rajagopal, “A note on the flow induced by a constantly accelerating plate in an Oldroyd-B fluid,” Appl. Math. Model., vol. 31, no. 4, pp. 647–654, Apr. 2007.
[7] C. Fetecau, T. Hayat, M. Khan, and C. Fetecau, “Erratum to: Unsteady flow of an Oldroyd-B fluid induced by the impulsive motion of a plate between two side walls perpendicular to the plate,” Acta Mechanica, vol. 216, nos. 1–4, pp. 359–361, Jan. 2011.
[8] M. Khan, T. Hayat, and S. Asghar, “Exact solution for MHD flow of a generalized Oldroyd-B fluid with modified Darcy’s law,” Int. J. Eng. Sci., vol. 44, nos. 5–6, pp. 333–339, Mar. 2006.
[9] A. Shakeel, S. Ahmad, H. Khan, N. A. Shah, and S. U. Haq, “Flows with slip of Oldroyd-B fluids over a moving plate,” Adv. Math. Phys., vol. 2016, May 2016, Art. no. 8619634.
[10] I. Khan, M. Imran, and K. Fakhar, “New exact solutions for an Oldroyd-B fluid in a porous medium,” Int. J. Math. Math. Sci., vol. 2011, Jun. 2011, Art. no. 408132.
M. B. Riaz, M. A. Imran, and K. Shabbir, “Unsteady MHD thin film flow of an Oldroyd-B fluid over an oscillating inclined belt,” *PLoS ONE*, vol. 10, no. 7, 2015, Art. no. e0126698.

M. B. Riaz, M. A. Imran, and K. Shabbir, “Analytic solutions of Oldroyd-B fluid with fractional derivatives in a circular duct that applies a constant couple,” *Alexandria Eng. J.*, vol. 55, no. 4, pp. 3267–3275, Dec. 2016.

I. Khan, K. Fakhar, and M. Anwar, “Hydromagnetic rotating flows of an Oldroyd-B fluid in a porous medium,” *Special Topics Rev. Porous Media Int. J.*, vol. 3, no. 1, pp. 89–95, 2012.

R. Wan, “Some new global results to the incompressible Oldroyd-B model,” *Zeitschrift für Angew. Math. und Physik*, vol. 70, no. 1, p. 28, Feb. 2019.

A. Elhanafy, A. Gualiy, and A. Elsaid, “Numerical simulation of Oldroyd-B fluid with application to hemodynamics,” *Adv. Mech. Eng.*, vol. 11, no. 5, May 2019, Art. no. 168781401985284.

A. D. Hullender, “Analytical non-newtonian Oldroyd-B transient model for pretransient turbulent flow in smooth circular lines,” *J. Fluids Eng.*, vol. 141, no. 2, Feb. 2019, Art. no. 021303.

M. H. Tiwana, A. B. Mann, M. Rizwan, K. Maqbool, S. Javeed, S. Raza, and M. S. Khan, “Unsteady magnetohydrodynamic convective fluid flow of Oldroyd-B model considering ramped wall temperature and ramped wall velocity,” *Mathematics*, vol. 7, no. 8, p. 676, 2019.

J. H. Merkin, “Natural-convective boundary-layer flow on a vertical surface with Newtonian heating,” *Int. J. Heat and Fluid Flow*, vol. 15, no. 5, pp. 392–398, Oct. 1994.

M. Z. Salleh, R. Nazar, and I. Pop, “Boundary layer flow and heat transfer over a stretching sheet with Newtonian heating,” *J. Taiwan Inst. Chem. Eng.*, vol. 41, no. 6, pp. 651–655, 2010.

R. Haq, S. Nadeem, Z. Khan, and T. Okedayo, “Connective heat transfer and MHD effects on casson nanofluid flow over a shrinking sheet,” *Open Phys.*, vol. 12, no. 12, pp. 862–871, Jan. 2014.

S. Nadeem, R. U. Haq, and N. S. Akbar, “MHD three-dimensional boundary layer flow of casson nanofluid past a linearly stretching sheet with convective boundary condition,” *IEEE Trans. Nanotechnol.*, vol. 13, no. 1, pp. 109–115, Jan. 2014.

I. Ullah, S. Shafie, and I. Khan, “Effects of slip condition and Newtonian heating on MHD flow of casson fluid over a nonlinearly stretching sheet saturated in a porous medium,” *J. King Saud Univ. Sci.*, vol. 29, no. 2, pp. 250–259, Apr. 2017.

M. A. Imran, M. B. Riaz, N. A. Shah, and A. A. Zafar, “Boundary layer flow of MHD generalized maxwell fluid over an exponentially accelerated infinite vertical surface with slip and Newtonian heating at the boundary,” *Results Phys.*, vol. 8, pp. 1061–1067, Mar. 2018.

T. Hayat, M. Hussain, A. Alsaedi, S. S. Shehzad, K. Abdulaziz, U. Lahore, S. A. K. Abdulaziz, and P. Comsats, “Flow of power-law nanofluid over a stretching surface with Newtonian heating,” *J. Appl. Fluid Mech.*, vol. 8, no. 2, pp. 273–280, Apr. 2015.

M. Ramzan, “Influence of Newtonian heating on three-dimensional MHD flow of couple stress nanofluid with viscous dissipation and Joule heating,” *PloS ONE*, vol. 10, no. 4, 2015, Art. no. e0124699.

M. Ali and S. Sadek, “Free convection heat transfer from different objects,” in *Heat Transfer: Models, Methods and Applications*, London, U.K.: IntechOpen, 2018.

A. Alhashash, “Natural convection of nanoliquid from a cylinder in square porous enclosure using Buongiorno’s two-phase model,” *Sci. Rep.*, vol. 10, no. 1, 2020, Art. no. 143.

G. Yu, Z. Lian, W. Gan, and J. Ji, “Numerical investigation on the effect of harmonic horizontal-axis rotation on laminar natural convection in an air-filled enclosure,” *Int. J. Heat Mass Transf.*, vol. 152, May 2020, Art. no. 115933.

R. Siegel, “Transient free convection from a vertical flat plate,” *Trans. ASME*, vol. 80, no. 2, p. 347, 1958.

N. Ahmed, H. Kalita, and D. Barua, “Unsteady MHD free convective flow past a vertical porous plate immersed in a porous medium with Hall current, thermal diffusion and heat source,” *Int. J. Eng. Sci. Technol.*, vol. 2, no. 6, pp. 89–94, 2010.

N. Marmeni, “An exact solution of unsteady MHD free convection flow of a radiating gas past an infinite inclined isothermal plate,” *Appl. Mech. Mater.*, vols. 110–116, pp. 2228–2233, Oct. 2011.

S. U. Haq, I. Khan, F. Ali, A. Khan, and T. N. A. Abdelhameed, “Influence of slip condition on unsteady free convection flow of viscous fluid with ramped wall temperature,” *Abstract Appl. Anal.*, vol. 2015, Aug. 2015, Art. no. 327975.
POOM KUMAM (Member, IEEE) received the Ph.D. degree in mathematics from Naresuan University, Thailand. He is currently a Full Professor with the Department of Mathematics, King Mongkut’s University of Technology Thonburi (KMUTT), where he is also the Head of KMUTT Fixed Point Theory and Applications Research Group and the Theoretical and Computational Science Center (TaCS-Center) and also the Director of the Computational and Applied Science for Smart Innovation Cluster (CLASSIC Research Cluster). His research targeted fixed-point theory, variational analysis, random operator theory, optimization theory, and approximation theory. Also, fractional differential equations, differential game, entropy and quantum operators, fuzzy soft set, mathematical modeling for fluid dynamics and areas of interest Inverse problems, dynamic games in economics, traffic network equilibria, bandwidth allocation problem, wireless sensor networks, image restoration, signal and image processing, and game theory and cryptology. He has provided and developed many mathematical tools in his fields productively over the past years. He has more than 600 scientific articles and projects either presented or published. Moreover, he is on editorial board journals more than 40 journals and also he delivers many invited talks on different international conferences every year all around the world.

ILYAS KHAN is currently with Majmaah University, Saudi Arabia. He is also a Visiting Professor with the Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, Johar Bahru, Malaysia, and also with the Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh, Vietnam. He has published more than 400 articles in various reputed journals. He has authored several books and book chapters. He is working on both analytical and numerical techniques. His research interests include boundary layer flows, Newtonian and non-Newtonian fluids, heat and mass transfer, renewable energy, and nanofluids.

ASIFA received the B.S. degree (Hons.) from the University of Sargodha, Pakistan, and the M.S. degree in mathematics from COMSATS University Islamabad, Pakistan. She is currently doing the research in heat transfer, mathematical modeling, fractional analysis, nanofluid, MHD, and heat exchangers. She has written a few articles on flow and heat transfer phenomenons of fluids.

PHATIPHAT THOUNTHONG (Senior Member, IEEE) was born in Phatthalung, Thailand, in December 1974. He received the B.S. and M.E. degrees in electrical engineering from the King Mongkut’s Institute of Technology North Bangkok, Bangkok, Thailand, in 1996 and 2001, respectively, and the Ph.D. degree in electrical engineering from the Institute National Polytechnique de Lorraine University’s de Lorraine, Nancy, France, in 2005. Since 2012, he has been a Full Professor with the Department of Teacher Training in Electrical Engineering, King Mongkut’s University of Technology North Bangkok. He has authored 129 scientific articles (including 21 articles in IEEE TRANSACTIONS/Magazines) published in Scopus with citations=3,063 times and h-index=29. His current research interests include power electronics, electric drives, electric vehicles, electrical devices (fuel cells, photovoltaic, wind turbine, batteries, and supercapacitors), nonlinear controls, and observers.