The impulsive effect can be appeared in several biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated system etc. See the cited papers [1–3, 6, 7] for more detail of this topic.

Motivated by [1–3], in this paper we consider a class of fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

\[ c^\alpha D^\alpha x(t) = Ax(t) + J^{1-\alpha} f(t,x(t),G_1(x(t)),G_2(x(t))), \]

\[ t \in (s_i, t_{i+1}), i = 0, 1, 2, \ldots, m \]

\[ x(t) = g_i(t,x(t)), \quad t \in (t_i, s_i), i = 1, 2, \ldots, m \]

\[ x(0) = x_0, \]

where \( c^\alpha D^\alpha \) is the Caputo fractional derivative of order \( 0 < \alpha \leq 1 \), \( J^{1-\alpha} \) is Riemann-Liouville fractional integral operator and \( f \) is operational interval. The map \( A : D(A) \subset X \rightarrow X \) is a closed linear sectorial operator defined on a Banach space \( (X, \| \cdot \|) \), \( x_0 \in X, 0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \cdots < t_m \leq s_m < t_{m+1} = T \) are fixed numbers, \( g_i \in C((t_i, s_i) \times X, X), f \in [0, T] \times X^3 \rightarrow X \) is a nonlinear function and the functions \( G_1 \) and \( G_2 \) are defined by

\[ G_1(x(t)) = \int_0^t h(t, s, x(s)) ds \quad \text{and} \quad G_2(x(t)) = \int_0^T \tilde{h}(t, s, x(s)) ds, \]

\[ h, \tilde{h} : \Delta \times X \rightarrow X, \] where \( \Delta = \{ (x, s) : 0 \leq s \leq x \leq \tau \} \) are given functions which satisfies assumptions to be specified later on.
The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence results of problem (1.1). An example is given in Section 4 to illustrate the results.

2. Preliminaries

Let us set \( J = [0, T], J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, t_{m+1}] \) and introduce the space \( PC(J, X) := \{ u : J \rightarrow X \mid u \in C(J_0, X), k = 0, 1, 2, \ldots, m, \text{ and there exist } u(t_{k-1}^+) \text{ and } u(t_{k+1}^-), k = 1, 2, \ldots, m, \text{ with } u(t_{k+1}^-) = u(t_k) \} \). It is clear that \( PC(J, X) \) is a Banach space with the norm \( \| u \|_{PC} = \sup \{ \| u(t) \| : t \in J \} \).

Let us recall the following definition of mild solutions for fractional evolution equations involving the Caputo fractional derivative.

Definition 2.1. [4] Caputo’s derivative of order \( \alpha > 0 \) with lower limit \( a \), for a function \( f : [a, \infty) \rightarrow \mathbb{R} \) is defined as

\[
\frac{C^\alpha}{a} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds = aJ^{n-\alpha}f^n(t)
\]

where \( a \geq 0, n \in \mathbb{N} \). The Laplace transform of the Caputo derivative of order \( \alpha > 0 \) is given as

\[
L\left\{ \frac{C^\alpha}{a} f(t); \lambda \right\} = \lambda^\alpha \hat{f}(\lambda) - \sum_{k=0}^{n-1} \frac{\lambda^{a-k-1} f^{(k)}(0)}{\Gamma(k+1)}; \quad n-1 < \alpha \leq n.
\]

Definition 2.2. [2] A closed and linear operator \( A \) is said to be sectorial if there are constants \( \omega \in \mathbb{R}, \theta \in [\frac{\pi}{2}, \pi], M_\theta > 0, \) such that the following two conditions are satisfied:

1. \( \Sigma(\theta, \omega) := \{ \lambda \in C : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta \} \subset \rho(A) \)
2. \( \| R(\lambda, A) \|_{L(X)} \leq \frac{M_\theta}{|\lambda - \omega|}; \lambda \in \Sigma(\theta, \omega) \).

where \( X \) is the complex Banach space with norm denoted \( \| \cdot \|_X \).

Lemma 2.3. [2] Let \( f \) satisfies the uniform Holder condition with exponent \( \beta \in (0, 1) \) and \( A \) is a sectorial operator. Consider the fractional equations of order \( 0 < \alpha < 1 \)

\[
C_a^\alpha x(t) = Ax(t) + J^{1-\alpha} f(t),
\]

\( t \in J = [a, t], \quad a \geq 0, \quad x(a) = x_0 \). Then a function \( x(t) \in C([a, t], X) \) is the solution of the equation (2.1) if it satisfies the following integral equation

\[
x(t) = Q_\alpha(t-a)x_0 + \int_a^t Q_\alpha(t-s) f(s)ds
\]

where \( Q_\alpha(t) \) is solution operator generated by \( A \) defined as

\[
Q_\alpha(t) = \frac{1}{2\pi i} \int_G e^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha I - A)^{-1} d\lambda
\]

\( \Gamma \) is a suitable path lying on \( \Sigma_{\theta, \omega} \).

Remark 2.4. If \( A \in \mathcal{A}(\theta, \omega) \), then strongly continuous

\[
\| Q_\alpha(t) \| \leq M\epsilon^\alpha. \quad \text{Let } M_\theta := \sup_{0 \leq \tau \leq T} \| Q_\alpha(t) \|_{L(X)}.
\]

So we have \( \| Q_\alpha(t) \|_{L(X)} \leq M_\theta \).

Now, we recall the following important Lemma which is very useful to prove our main result.

Lemma 2.5. [5] Let \( 0 < \rho < 1, \gamma > 0 \),

\[
S = \rho^n + D_1^\rho n^{-1} \gamma + \frac{D_2^\rho n^{-2} \gamma^2}{2!} + \cdots + \frac{\gamma^n}{n!}; \quad n \in \mathbb{N}.
\]

Then, for all constant \( 0 < \xi < 1 \) and all real number \( s > 1 \), we get

\[
S \leq O\left( \frac{\xi^n}{\sqrt{n}} \right) + O\left( \frac{1}{n^s} \right) = O\left( \frac{1}{n^s} \right), \quad n \to +\infty.
\]

Definition 2.6. A function \( x \in PC(J, X) \) is said to be a PC-mild solution of problem (1.1) if it satisfies the following relation:

\[
x(t) = \begin{cases}
Q_\alpha(t)x_0 + \int_0^t Q_\alpha(t-s) f(x(s), G_1 x(s), G_2 x(s))ds, & t \in [0, t_1] \\
G_1(t, x(t)), & t \in (t_1, s_1] \\
Q_\alpha(t-s_1) G_1(s_1, x(s_1)) + \int_{s_1}^t Q_\alpha(t-s) f(x(s), G_1 x(s), G_2 x(s))ds, & t \in (s_1, t_{i+1}] 
\end{cases}
\]

for all \( i = 1, 2, \ldots, m \).

3. Existence Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under general Banach contraction principle fixed point theorem.

From Definition 2.3, we define an operator \( \Upsilon : PC(J, X) \rightarrow PC(J, X) \) as \( \Upsilon x)(t) = (\Upsilon_1 x)(t) + (\Upsilon_2 x)(t) \), where

\[
(\Upsilon_1 x)(t) = \begin{cases}
Q_\alpha(t)x_0, & t \in [0, t_1] \\
G_1(t, x(t)), & t \in (t_1, s_1] \\
Q_\alpha(t-s_1) G_1(s_1, x(s_1)), & t \in (s_1, t_{i+1}] 
\end{cases}
\]

and

\[
(\Upsilon_2 x)(t) = \begin{cases}
\int_0^t Q_\alpha(t-s) f(s, x(s), G_1 x(s), G_2 x(s))ds, & t \in [0, t_1] \\
0, & t \in (t_1, s_1] \\
\int_{s_1}^t Q_\alpha(t-s) f(s, x(s), G_1 x(s), G_2 x(s))ds, & t \in (s_1, t_{i+1}] 
\end{cases}
\]

To prove our first existence result we introduce the following assumptions:
Theorem 3.1. If hypotheses H(f), H(\tilde{h}) and H(g) hold and 0 \leq \Lambda < 1 (\Lambda = \max \left\{ L_{g_k}, M_0 L_{\tilde{R}_k} \right\}) , then problem (1.1) has a unique PC-mild solution x' \in PC(J, X).

Proof. For any x, y \in PC(J, X), by (3.1) we sustain

\| (Y_1x)(t) - (Y_1y)(t) \| \leq \Lambda^i \|x - y\|_{PC},

which means

\| (Y_1x)(t) - (Y_1y)(t) \| \leq \Lambda \|x - y\|_{PC},

where t \in [0, t_1] \cup \{t_i, s_i\} \cup \{s_i, t_{i+1}\}, i = 1, 2, \ldots, m. Then we obtain

\| (Y_1^2x)(t) - (Y_1^2y)(t) \| \leq \Lambda^2 \|x - y\|_{PC},

which means

\| (Y_1^2x)(t) - (Y_1^2y)(t) \| \leq \Lambda^2 \|x - y\|_{PC},

where t \in [0, t_1] \cup \{t_i, s_i\} \cup \{s_i, t_{i+1}\}, i = 1, 2, \ldots, m. It is clear that, we have

\| (Y_1^n x)(t) - (Y_1^n y)(t) \| \leq \Lambda^n \|x - y\|_{PC},

where t \in [0, t_1] \cup \{t_i, s_i\} \cup \{s_i, t_{i+1}\}, i = 1, 2, \ldots, m.

For any real number 0 < \varepsilon < 1, there exists a continuous function \phi(s) such that \int_0^T | \phi(s) | ds < \varepsilon, where

\ell(s) = M_0 L_{f_1}(s) + L_{f_2}(s) L_R + L_{f_3}(s) L_{\tilde{R}} \] is a Lebesgue integrable function. For any t \in [0, t_1], x, y \in PC(J, X) and by (3.2), we obtain

\| (Y_2x)(t) - (Y_2y)(t) \|

\leq \int_0^T \|Q_a(t - s)\| \|f(x(s), G_1x(s), G_2y(s)) - f(x(s), G_1y(s), G_2y(s))\| ds

\leq M_0 \int_0^T \|L_{f_1}(s) + L_{f_2}(s) L_R + L_{f_3}(s) L_{\tilde{R}}\| (x(s) - y(s)) \| ds

\leq \int_0^T \ell(s) \|x - y\|_{PC}

\leq (\varepsilon + \lambda t) \|x - y\|_{PC}

\leq \left( D_k^0 \varepsilon^k + D_k^1 \left( \frac{\lambda t}{1!} \right)^k + \cdots + D_k^k \left( \frac{\lambda t}{k!} \right)^k \right) \|x - y\|_{PC},

where max_{i \in J} | \phi(t) | = \lambda.

Assume that, for any natural number k, we get

\| (Y_2^k x)(t) - (Y_2^k y)(t) \|

\leq \left( D_k^0 \varepsilon^k + D_k^1 \left( \frac{\lambda t}{1!} \right)^k + \cdots + D_k^k \left( \frac{\lambda t}{k!} \right)^k \right) \|x - y\|_{PC},

From the above inequality and the formula D_k^{n+1} = D_k^n + D_k^{n-1}, we obtain

\| (Y_2^{n+1} x)(t) - (Y_2^{n+1} y)(t) \|

\leq \left( D_k^n \varepsilon^k + D_k^1 \left( \frac{\lambda t}{1!} \right)^k + \cdots + D_k^k \left( \frac{\lambda t}{k!} \right)^k \right) \|x - y\|_{PC},

By mathematical methods of induction, for any natural number n, we get

\| Y_2^n x - Y_2^n y \|_{PC}

\leq \left( D_k^n \varepsilon^k + D_k^1 \left( \frac{\lambda t}{1!} \right)^k + \cdots + D_k^n \varepsilon^n \right) \|x - y\|_{PC},

\| Y_2^n x - Y_2^n y \|_{PC} = \left( D_k^n \varepsilon^k + D_k^1 \left( \frac{\lambda t}{1!} \right)^k + \cdots + D_k^n \varepsilon^n \right) \|x - y\|_{PC},
where \( \zeta = \lambda T \). By Lemma 2.2, we have
\[
\|\mathcal{Y}^n x - \mathcal{Y}^n y\|_{PC} \leq \left( A^n + O \left( \frac{1}{n^\mu} \right) \right) \|x - y\|_{PC}
\]
(3.5)
where \( 0 < \eta < 1, \mu > 1 \). It is easy to see that the above equation (3.5) holds for \( t \in (s_i, t_{i+1}), i = 1, 2, \ldots, m \). By (3.4) and (3.5), we obtain
\[
\|\mathcal{Y}^n x - \mathcal{Y}^n y\|_{PC} \leq \left( A^n + O \left( \frac{1}{n^\mu} \right) \right) \|x - y\|_{PC}, \quad \forall n > n_0.
\]
Thus, for any fixed constant \( \mu > 1 \), we can find a positive integer \( n_0 \) such that, for any \( n > n_0 \), we get \( 0 < A^n + \frac{1}{n^\mu} < 1 \). Therefore, for any \( x, y \in PC(J, X) \), we have
\[
\|\mathcal{Y}^n x - \mathcal{Y}^n y\|_{PC} \leq \left( A^n + \frac{1}{n^\mu} \right) \|x - y\|_{PC} \leq \|x - y\|_{PC}, \quad \forall n > n_0.
\]
By the general Banach contraction mapping principle, we get that the operator \( \mathcal{Y} \) has a unique fixed point \( x^* \in PC(J, X) \), which means that problem (1.1) has a unique \( PC \)-mild solution.

4. Application

In this section, we are going to present an example to validate the results established in Theorem 3.1. Consider the fractional partial integro-differential equation of the form
\[
c^{\alpha}\frac{D^\alpha x(t, z)}{Dz} = \frac{\partial^2 x}{\partial z^2} + J^{1-\alpha} P \left( t, x(t, z), \int_0^t h(t, s, x(s, z))ds, \int_0^T \tilde{h}(t, s, x(s, z))ds \right),
\]
a.e. \( (t, z) \in \cup_{i=1}^m (s_i, t_{i+1}) \times [0, \pi] \) (4.1)
\[
x(t, 0) = x(t, \pi) = 0, \quad t \in [0, T] \quad (4.2)
\]
\[
x(0, z) = x_0(z), \quad z \in [0, \pi] \quad (4.3)
\]
\[
x(t, z) = G_i(t, x(t, z)), \quad t \in (t_i, s_i), \quad z \in [0, \pi],
\]
\[
i = 1, 2, \ldots, N. \quad (4.4)
\]
where \( 0 = t_0 = s_0 < t_1 \leq s_1 \leq \ldots \leq s_m < t_{m+1} = T, P \in C(\{0, T\} \times \mathbb{R}^3, \mathbb{R}) \) and \( G_i \in C((t_i, s_i] \times \mathbb{R}, \mathbb{R}) \) for all \( i = 1, 2, \ldots, m \).

Let \( X = L^2([0, \pi]) \). Define an operator \( A : D(A) \subseteq X \rightarrow X \) by \( Ax = x'' \) with \( D(A) = \{ x \in X : x'' \in X, x(0) = x(\pi) = 0 \} \).
It is well known that \( A \) is the infinitesimal generator of an analytic semigroup \( \{ T(t) : t \geq 0 \} \) in \( X \). Moreover, the subordination principle of solution operator implies that \( A \) is the infinitesimal generator of a solution operator \( \{ Q_A(t) \}_{t \geq 0} \) such that \( \| Q_A(t) \|_{L(X)} \leq \bar{Q}_S \) for \( t \in [0, 1] \).

Set
\[
x(t, z) = x(t, z), J^{1-\alpha} f(t, x(t), G_1(x(t), G_2(x(t))))
\]
\[
= J^{1-\alpha} P \left( t, x(t, z), \int_0^T h(t, s, x(s, z))ds, \int_0^T \tilde{h}(t, s, x(s, z))ds \right)
\]
\[
g_i(t, x(t)) = G_i(t, x(t, z)).
\]
Thus with this set-up, equations (4.1)-(4.4) can be written in the abstract form for the system (1.1).

References

[1] X. Fu, X. Liu and B. Lu, On a new class of impulsive fractional evolution equations, Advances in Difference Equations, (2015) 2015:227.
[2] Ganga Ram Gautam and Jeydev Dabas, Mild solution for nonlocal fractional functional differential equation with non-instantaneous impulse, International Journal of Nonlinear Science, 21(3)(2016), 151–160.
[3] E. Hernández and D. O’Regan, On a new class of abstract impulsive differential equations, Proc.Amer. Math. Soc., 141 (2013), 1641–1649.
[4] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, 2006.
[5] L.S. Liu, C.X. Wu and F. Guo, Existence theorems of global solutions of initial value problems for nonlinear integro-differential equations of mixed type in Banach spaces and applications, Comput. Math. Appl., 47(2004), 13–22.
[6] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal., Real World Appl., 11(2010), 4465–4475.
[7] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, Comput. Math. Appl., 59(2010), 1063–1077.

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