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3D-shortest paths for a hypersonic glider in a heterogeneous environment

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Abstract: Shortest paths in 3-dimensional space of a hypersonic glider in a heterogeneous environment are considered in this paper. The environment is heterogeneous in the sense that the maximum curvature of the vehicle path varies and depends on the position of the vehicle. Path generation is based on the Dubins-like model. It assumes that initial and final states are sufficiently far from each other so that the CSC (Curve-Straight line-Curve) path is the shortest path between both states. Paths are calculated based on the optimal control theory and a geometrical approach. This method is computationally fast and easy to implement in a real time system. Moreover, paths found by this method are more realistic than existing Dubins’ paths.

Keywords: Aerospace trajectories, optimal trajectories, autonomous vehicle, trajectory planning, aerospace control.

1. INTRODUCTION

The purpose of this paper is to generate the shortest path in 3-dimensional space for an aerial vehicle flying in a heterogeneous environment where the maximum curvature of the path varies in the environment. In case of aerial vehicles, the maximum curvature of the path that the vehicle can perform depends on the vehicle position. Unmanned aerial vehicles are mostly subjected to aerodynamic forces to maneuver. Moreover, the aerodynamic forces, i.e. lift \( f_L \) and drag \( f_D \), depend on aerodynamic coefficients \( C_L \) and \( C_D \), surface of reference \( S \), vehicle speed \( v \) and air density \( \rho(z) \).

\[
\begin{align*}
   f_L &= \frac{1}{2} \rho(z) SC_L v^2 \\
   f_D &= \frac{1}{2} \rho(z) SC_D v^2
\end{align*}
\]

Thus, the maneuverability, i.e. maximum curvature, of the aerial vehicles depends on the air density which decreases exponentially with altitude \( z \) (see Section 2.1).

The shortest path between two vehicle states is a key element in many planning algorithms. In a 2-dimensional plane, the shortest path of Dubins’ vehicle is usually used to determine the distance between two states. The study was initiated and solved geometrically in Dubins (1957) with a vehicle only moving forward. In his study, a vehicle with constant turning radius was used. Dubins stated that the shortest path between initial and final states was a combination of straight lines (S) and arcs of circle (C), i.e. CSC paths, CCC paths or degenerated forms of these. Dubins’ work inspired a lot of researchers later on. Reeds and Shepp (1990) extended the study with a vehicle moving forward and backward. Pontryagin’s Minimum Principle was used to solve the Dubins’ problem in Boissonnat et al. (1991). The study was extended to the unmanned aerial vehicle whose dynamics was the same as the Dubins’ vehicle in McGee et al. (2005) by analyzing the effect of constant wind. In Dolinskiaya and Maggiar (2012), a vehicle moving in an anisotropic environment, which meant that the minimum turning radius depended on the orientation of the vehicle, was considered. Later, some generalizations of the Dubins’ vehicle in a heterogeneous environment were studied. In Sanfelice and Frazzoli (2008), the environment having two different property planes where the vehicle could maneuver with the same turning rate was considered. Then, in Hérissé and Pepy (2013), shortest paths in heterogeneous environments were considered. The environments are heterogeneous in the sense that the maximum curvature of the vehicle path varies and depends on the position of the vehicle.

The study of Dubins’ vehicle has also taken a different path into a 3-dimensional space problem. In Sussmann (1995), it was also demonstrated that, for sufficiently close distance between two states, the helicoidal arc could be shorter than the CSC path. Then, it was shown that the shortest path in a 3-dimensional plane was a helicoidal arc, a CSC path, a CCC path or a degenerated form of these Dubins’ paths. Later, in Shanmugavel (2007), Dubins’ path in 2-dimensional plane was extended to 3-dimensional plane for multiple UAVs path planning. Suboptimal paths of CCSC type were used.

Recently, the shortest path of Dubins’ vehicle in 3-dimensional space was studied for a vehicle with a constant turning radius in Hota and Ghose (2010). The shortest Dubins’ path was calculated by using a geometrical approach between initial and final states that were sufficiently far
The environment is considered heterogeneous because of variation of air density $\rho(z)$, decreasing exponentially with altitude $z$. The simplified environment model can be expressed as:

$$\rho(z) = \rho_0 e^{-z/z_r},$$

where $\rho_0$ is the air density at standard atmosphere at sea level and $z_r$ is a reference altitude.

### 2.2 Vehicle model

In this paper, a simplified model of an aerial vehicle is used. It is modeled as a rigid body maneuvering in a 3-dimensional plane. Three frames (Fig. 1) are introduced to describe the motion of the vehicle: an Earth-Centred Earth-Fixed (ECEF) reference frame $\mathcal{I}$ centered at point $O$ and associated with the basis vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$; a body-fixed frame $\mathcal{B}$ attached to the vehicle at its center of mass $C$ with the vector basis $(\mathbf{e}_1^b, \mathbf{e}_2^b, \mathbf{e}_3^b)$; and a velocity frame $V$ attached to the vehicle at $C$ with the vector basis $(\mathbf{e}_1^v, \mathbf{e}_2^v, \mathbf{e}_3^v)$ where the translational velocity of the vehicle is denoted $\mathbf{v} = v\mathbf{e}_1^v$ and $v$ is the speed of the vehicle. Position and velocity defined in $\mathcal{I}$ are denoted $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ and $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})^T \in \mathbb{R}^3$. Denote $\gamma$ and $\chi$ the orientation of the velocity. The flight angle is denoted by $\gamma$ and the azimuth angle is denoted by $\chi$.

Since it is a simplified model, to eliminate all the external factor to the problem, a zero wind assumption is applied. Then, the translational velocity $\mathbf{v}$ is assumed to coincide with the apparent velocity. Besides, an unpowered hypersonic aerial vehicle such as an interceptor missile during midcourse phase is studied in this paper. Thus, the gravity can be neglected which is a strong hypothesis that is only valid for missile-like aircraft flying in a short distance. Moreover, the drag can be ignored since we are interested in the shortest path between two states, i.e. the path of minimum length. Thus, the dynamics of velocity does not need to be considered.

Therefore, the dynamics of a hypersonic aerial vehicle can be written as

$$
\begin{align*}
\dot{x} &= v \cos \gamma \cos \chi, \\
\dot{y} &= v \cos \gamma \sin \chi, \\
\dot{z} &= v \sin \gamma, \\
\dot{\gamma} &= v \frac{1}{2m} \rho(z) SC_{\text{max}} \mu = v c(z) \mu, \\
\dot{\chi} &= v \frac{1}{2m} \rho(z) SC_{\text{max}} \frac{\eta}{\cos \gamma} = v c(z) \frac{\eta}{\cos \gamma},
\end{align*}
$$

where $\mu, \eta$ are the normalized control inputs bounded by condition $\sqrt{\mu^2 + \eta^2} \leq 1$, $m$ is the mass of the vehicle, $\gamma \in [-\pi/2, \pi/2]$, $\chi \in [-\pi, \pi]$, and $c(z) \in \mathbb{R}_+$ is the curvature of the vehicle path. The curvature depends on the attitude of the vehicle $z$ whose maximum value can be written as

$$c(z) = c_0 e^{-z/z_r}.$$

The optimal control problem consists in minimizing the cost function

$$s_f = \int_0^{t_f} v \, dt$$

where $s_f$ is the final path length and $t_f$ is the final time.

Since we are interested in the minimum length path, a change of variables from time $t$ to curvilinear abscissa $s(t) = \int_0^t v(u) \, du$ is made. Then, the dynamics can be rewritten as:

$$
\begin{align*}
\frac{dx}{ds} &= \cos \gamma \cos \chi, \\
\frac{dy}{ds} &= \cos \gamma \sin \chi, \\
\frac{dz}{ds} &= \sin \gamma, \\
\frac{d\gamma}{ds} &= c(z) \mu, \\
\frac{d\chi}{ds} &= c(z) \frac{\eta}{\cos \gamma}.
\end{align*}
$$

Thus, the dynamics of the forward velocity does not need to be specified in this study.
3. 3D DUBINS’ PATHS IN HETEROGENEOUS ENVIRONMENT

3.1 Problem formulation

Let $x(s) = (x(s), y(s), z(s), \gamma(s), \chi(s))^T$ denote a state vector and $u(s) = (\eta(s), \mu(s))^T$ denote a control input. The boundary conditions are $x(0) = x_0$ and $x(s_f) = x_f$ where $s_f$ is the distance from initial state $x_0$ to goal state $x_f$. The optimal control problem is to minimize the length of the path by minimizing

$$J(x_0, x_f, u) = \int_0^{s_f} ds$$

In a 2-dimensional plane, it was shown in Dubins (1957) and Boissonnat et al. (1991) that the shortest path between two fixed states of Dubins’ vehicle with a constant turning radius is composed of straight line (S) and arc of circle of minimum turning radius (C), i.e., CSC or CCC type. In Shkel and Lumelsky (2001), it was proven that CSC or CCC type is the shortest path if two states are sufficiently far from each other.

As in 3-dimensional problems, it was proven in Sussmann (1995) that, unlike in a 2-dimensional plane, there can exist a helicoidal arc which is shorter than the CSC path. Thus, the shortest path in 3-dimensional space was a helicoidal arc, a CSC path, a CCC path or a degenerated form of these Dubins’ paths. However, in path planning for hypersonic aerial vehicle such as interceptor missile, the target or the mission is known a priori using high performance technologies. Therefore, path planning is usually executed between states that are sufficiently far from each other. As a consequence, in this paper, only the CSC path in a heterogeneous environment using geometric approach is demonstrated with the hypothesis that $x_0$ and $x_f$ are sufficiently far from each other.

In this paper, a 3-dimensional length-optimal path between two given states on a heterogeneous environment model, i.e., equation (1), on the plane $P$ is rewritten as

$$\rho(z_p) = \rho_0 e^{-z/z_r} = \rho_0 e^{-z_p \cos \phi/z_r}.$$  

Moreover, curvature equation (3) can be written as

$$c(z_p) = c_0 e^{-z/z_r} = c_0 e^{-z_p \cos \phi/z_r}.$$  

In order to derive the optimal solution with curve of maximum curvature, the magnitude of the control input $u_p$ in system (7) is set to 1. By differentiating $\theta_p'$ with respect to $s$, we obtain

$$\theta_p'' = -\frac{\sin \phi}{z_r} \theta_p' \sin \theta_p.$$  

Define $\zeta = \tan \left(\frac{\theta_p'}{2}\right)$. After some straightforward trigonometry, we have

$$\cos^2 \theta_p = \frac{1 - \zeta^2}{1 + \zeta^2}$$  

By integrating equation (10) and applying some trigonometric techniques, we have

$$\theta_p' = \cos \phi \frac{z_r}{\cos \phi} \left(\theta_p'' - \cos \theta_{p_0} + 1 - 2 \cos^2 \left(\frac{\theta_p}{2}\right)\right)$$  

With equations (11), (12) and (13), we obtain

$$\zeta' = A + B \zeta^2,$$

$$A = \frac{\cos \phi}{2 z_r} \left(\frac{z_r}{\cos \phi} \theta_p'' - \cos \theta_{p_0} + 1\right),$$

$$B = A - \frac{\cos \phi}{z_r}.$$
According to system (14), there are four types of curves depending on the values of $A$ and $B$:

1. **$C_1$ curve if $AB > 0$,**
   \[
   \zeta_1(s) = \sqrt{\frac{A}{B}} \tan \left[ A \sqrt{\frac{B}{A}} s + \arctan \left( \sqrt{\frac{B}{A}} \theta_0 \right) \right]
   \] (15)

   The $C_1$ curve is illustrated in Fig. 3.

2. **$C_2$ curve if $AB < 0$,**
   \[
   \zeta_2(s) = \sqrt{\frac{A}{B}} \tanh \left[ A \sqrt{\frac{B}{A}} s + \text{arctanh} \left( \sqrt{\frac{B}{A}} \theta_0 \right) \right]
   \] (16)

   The $C_2$ curve is also illustrated in Fig. 3. This curve has oblique asymptotes, i.e., $\zeta_2 \in [-\sqrt{\frac{A}{B}}, \sqrt{\frac{A}{B}}]$. This condition must be verified for both $\zeta_0$ and $\zeta_2(s)$. If one or both variables do not respect this condition, there is no solution.

3. **$C_3$ curve if $A = 0$,**
   \[
   \frac{1}{\zeta_3(s)} - \frac{1}{\zeta_0} = Bs
   \] (17)

4. **$C_4$ curve if $B = 0$,**
   \[
   \zeta_4(s) = \zeta_0 + As
   \] (18)

**Remark 1.** $C_3$ and $C_4$ curves are the extremal cases of the first two types. They are rarely obtained in reality. Thus, no illustration of these curves is presented in this paper.

$
\theta_p, x_p$ and $z_p$ can be derived as function of $k(s)$ as follows:

\[
\begin{align*}
\theta_p(\zeta) &= 2 \arctan \zeta + k(s) \pi \\
x_p(\zeta) &= \frac{z_r}{\cos \phi} (\theta_p(\zeta) - \theta_m) - \frac{z_r}{\cos \phi} (A + B)s \\
z_p(\zeta) &= \frac{z_r}{\cos \phi} \log \left( \frac{1 + \zeta^2}{A + B \zeta^2} \right)
\end{align*}
\] (19)

where $k(s)$ is an integer depending on the distance $s$. In case of $C_1$ path $k(s)$ is calculated as follows

\[
k(s) = \left\lfloor s \sqrt{AB} / \left( u_p \frac{\pi}{2} - \arctan \left( \sqrt{\frac{B}{A}} \theta_0 \right) \right) \right\rfloor, \quad u_p = \pm 1
\] (20)

3.3 **3D paths generation**

In order to find the shortest Dubins’ path in a heterogeneous environment shown in Fig. 4, let $l \in \mathbb{R}^3$ denote a line which lies in both plane $P_1$ and $P_2$, i.e., $l \in P_1$ and $l \in P_2$.

In the following, the cross product of $u$ and $v$ is defined by $u \times v$. In order to find both curves, the following normal vector to each particular plane $P_1$ and $P_2$ must be defined:

- The unit vector perpendicular to the first plane:
  \[
  \mathbf{b}_1 = \frac{1 \times \mathbf{v}_0}{||1 \times \mathbf{v}_0||};
  \] (21)

- The unit vector perpendicular to the second plane:
  \[
  \mathbf{b}_2 = \frac{1 \times \mathbf{v}_1}{||1 \times \mathbf{v}_1||};
  \] (22)

**Remark 2.** In case $1 \times \mathbf{v}_0 = \mathbf{0}$ or $1 \times \mathbf{v}_1 = \mathbf{0}$, it means that there is no curve. Thus, the CSC type degrades to CS, SC, or S type. However, it is pretty rare to reach this condition.

$\theta_{p_1}$ on plane $P_1$ and $\theta_{p_2}$ on plane $P_2$ in equation (19) are defined as follows

\[
\begin{align*}
\theta_{p_1} &= \angle(\mathbf{l}, \mathbf{e}_1^{p_1}) \quad \text{(23)}
\theta_{p_2} &= \angle(\mathbf{l}, \mathbf{e}_1^{p_2}) \quad \text{(24)}
\end{align*}
\]

Then, the position of $\mathbf{e}_1$ on plane $P_1$ and $\mathbf{e}_2$ on plane $P_2$ can be found using the calculation shown in Section 3.2. Then, $\mathbf{e}_1 = (x_1, y_1, z_1)$ and $\mathbf{e}_2 = (x_2, y_2, z_2)$ in $\mathcal{I}$ frame can be found as follows:

\[
\begin{align*}
x_1 &= -z_p \sin \phi_1 \cos \psi_1 - x_p \sin \psi_1 + x_0 \\
y_1 &= -z_p \sin \phi_1 \sin \psi_1 + x_p \cos \psi_1 + y_0 \\
z_1 &= z_p \cos \phi_1 + z_0
\end{align*}
\] (25)

\[
\begin{align*}
x_2 &= -z_p \sin \phi_2 \cos \psi_2 - x_p \sin \psi_2 + x_f \\
y_2 &= -z_p \sin \phi_2 \sin \psi_2 + x_p \cos \psi_2 + y_f \\
z_2 &= z_p \cos \phi_2 + z_f
\end{align*}
\] (26)
Recall that both curves are obtained by considering \((x_p, \bar{x}_p) = (0, 0)\) as a origin and \((\phi_1, \psi_1)\) and \((\phi_2, \psi_2)\) as orientations of \(b_1\) and \(b_2\), respectively.

The orientations \(v_1\) and \(v_2\) can be found by rotating \(v_0\) and \(v_1\) by \(\Delta \theta_1 = \theta_{p1} - \theta_{p0}\) and \(\Delta \theta_2 = \theta_{p2} - \theta_{p1}\) around vectors \(b_1\) and \(b_2\), respectively. We have

\[
\begin{align*}
v_1 &= R_{b_1} v_0 \\
R_{b_1} &= \cos \Delta \theta_1 I_3 + \sin \Delta \theta_1 C_1 + (1 - \cos \Delta \theta_1) D_1 \\
v_2 &= R_{b_2} v_1 \\
R_{b_2} &= \cos \Delta \theta_2 I_3 + \sin \Delta \theta_2 C_2 + (1 - \cos \Delta \theta_2) D_2
\end{align*}
\]

where \(I_3\) is an identity matrix of order 3, \(C_1, D_1, C_2,\) and \(D_2\) are defined as follows:

\[
\begin{align*}
C_1 &= \begin{bmatrix} 0 & -b_{1z} & b_{1y} \\ -b_{1y} & b_{1z} & 0 \\ -b_{1z} & 0 & b_{1x} \end{bmatrix} \\
D_1 &= \begin{bmatrix} b_{1y}^2 & b_{1y} b_{1z} & b_{1y} b_{1x} \\ b_{1z} b_{1y} & b_{1z}^2 & b_{1z} b_{1x} \\ b_{1x} b_{1y} & b_{1x} b_{1z} & b_{1x}^2 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 0 & -b_{2z} & b_{2y} \\ -b_{2y} & b_{2z} & 0 \\ -b_{2z} & 0 & b_{2x} \end{bmatrix} \\
D_2 &= \begin{bmatrix} b_{2y}^2 & b_{2y} b_{2z} & b_{2y} b_{2x} \\ b_{2z} b_{2y} & b_{2z}^2 & b_{2z} b_{2x} \\ b_{2x} b_{2y} & b_{2x} b_{2z} & b_{2x}^2 \end{bmatrix}
\]

Once two curves have been found, a solver is used to find \(l\) by verifying the objective function \(F(l) = 1 - (\xi_2 - \xi_1) = 0\). Thus, a line, which is on both plane \(P_1\) and plane \(P_2\), connecting both curves is found.

**Remark 3.** With this methodology, the conditions \(\mathbf{I} \times \mathbf{v}_1 = \mathbf{0}\) and \(\mathbf{I} \times \mathbf{v}_2 = \mathbf{0}\) are automatically verified.

There can exist four types of CSC paths shown in Fig. 5 where \(u_p = \pm 1\) for both curves. The solutions can be found in the same way as the demonstration. Among these paths, the shortest Dubins’ path is chosen.
Table 1. Boundary conditions and results for simulations

| Case study | \( \mathbf{x}_0 \) (km, km, km, radian, radian) | \( \mathbf{x}_f \) (km, km, km, radian, radian) | 3D Dubins’ path length (km) |
|------------|-----------------------------------------------|-----------------------------------------------|-------------------------------|
| Case 1     | (0, 0, 0.005, 0, \(-\pi/3\))                | (2, 0, 0.5, 0, 2\pi/3)                        | 5.65                          |
| Case 2     | (0, 0, 10, \pi/12, \pi/12)                  | (20, 15, 25, 0, \(-2\pi/3\))                 | 59.84                         |
| Case 3     | (0, 0, 5, \pi/2, \pi/6)                     | (15, 15, 25, \(-\pi/12, 12\pi/3\))           | 37.04                         |

Fig. 7. Case 2: \( \mathbf{x}_0 = (0, 0, 10, \pi/12, \pi/12)^T \) and \( \mathbf{x}_f = (20, 15, 25, 0, 4\pi/3)^T \)

obtained by this method is more realistic than path generated by existing Dubins’ paths.

This method is computationally fast and easy to implement. Moreover, it can give the shortest and more realistic path from the starting to the ending state. Therefore, it can be applied to many applications such as path planning in complex environment in Pharpatara et al. (2013). However, the path generation can be improved by finding a shortest path of helicoidal arc type between two states that are relatively close to each other so that the path generation can cover all cases in 3-dimensional space.

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