Observation of oscillation coupling ratios and the meridional flow

A Schad\textsuperscript{1,2}, M Roth\textsuperscript{2} and J Timmer\textsuperscript{1,3}

\textsuperscript{1} Freiburg Center for Data Analysis and Modeling, University of Freiburg, 79104 Freiburg, Germany
\textsuperscript{2} Kiepenheuer-Institut für Sonnenphysik, 79104 Freiburg, Germany
\textsuperscript{3} Freiburg Institute for Advanced Studies, University of Freiburg, 79104 Freiburg, Germany
E-mail: ariane.schad@fdm.uni-freiburg.de

Abstract. Measurements from local helioseismology indicate the existence of a meridional flow in the Sun with a strength up to 15 m/s near the solar surface. The flow profile at depths below \( \approx 15 \text{ Mm} \) are not accessible. We propose a method using global helioseismic measurements with the prospect to infer the meridional flow profile throughout the solar convection zone and show its performance on simulated data.

1. Introduction

From perturbation analysis it is known that large-scale flows in the solar interior, like convection, giant cells, and differential rotation, lead to couplings between unperturbed eigenfunctions of acoustic modes [1, 2, 3, 4]. These couplings manifest in shifts of the mode eigenfrequencies and distortions of the eigenfunctions. Here we focus on the perturbation of the eigenfunctions due to a meridional flow. In a simulation we show how to use the perturbation signature to infer the meridional flow from global helioseismic observations.

2. The meridional flow and perturbation of solar eigenfunctions

The meridional flow \( \mathbf{u} \) is a zonal poloidal flow. It is assumed that the flow is confined between the surface and the bottom of the solar convection zone. In spherical geometry \((\theta, \phi)\) it can be decomposed into radial and horizontal components of different degree \( s \), with corresponding radial flow strengths \( u_s(r) \) and \( v_s(r) \),

\[
\mathbf{u}(r, \theta) = \sum_s \left[ u_s(r) Y^0_s(\theta, \phi) \hat{e}_r + v_s(r) \partial_\theta Y^0_s(\theta, \phi) \hat{e}_\theta \right].
\] (1)

In case of a stationary meridional flow the horizontal flow \( v_s \) can be expressed in terms of \( u_s \) by means of the mass conservation law.

The coupling of global p-modes due to a meridional flow can be investigated using quasi-degenerate perturbation theory. In algebraic terms the coupling is represented by the supermatrix \( \mathbf{Z} \) with respect to the unperturbed mode eigenfunctions \( \xi_0^i \) [1].
We regard a single mode $i = (n, l, m)$ and a subset $M_i = \{j_1, ..., j_{N_i}\}$ of modes with quasi-degenerate frequencies coupling with mode $i$. The corresponding block-matrix $Z_i$ of $Z$ is of the form

$$Z_i = \begin{pmatrix}
a_i & H_{ij_1} & H_{ij_2} & \cdots & H_{ij_{N_i}} \\
H_{ij_1} & a_{j_1} & 0 & \cdots & 0 \\
H_{ij_2} & 0 & a_{j_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
H_{ij_{N_i}} & 0 & 0 & \cdots & a_{j_{N_i}}
\end{pmatrix}.$$  

(2)

The diagonal elements are determined by the squared unperturbed mode frequency $\omega_j^2$. The off-diagonal matrix elements result from small perturbations due to the flow. From perturbation theory applied to $Z_i$ the first eigenvector of $Z_i$ corresponding to mode $i$ can be approximated up to first order by

$$\xi_i = C_{ii} \xi_i^0 + \sum_{j \in M_i} C_{ij} \xi_j^0,$$

(3)

with coefficients

$$C_{ii} = 1 \quad \text{and} \quad C_{ij} = \frac{H_{ij}}{\omega_i^2 - \omega_j^2} \quad \text{for} \ j \in M_i, \ j \neq i.$$

(4)

3. Coupling ratios and $b$-coefficients

The contribution of a perturbed mode $i$ with oscillation amplitude $a_i(t)$ to the observable Doppler velocity field $v$ is given by

$$v_i = a_i(t) \left[ C_{ii} \xi_i^0 + \sum_{j \in M_i} C_{ij} \xi_j^0 \right].$$

(5)

Therefore, a projection of the Doppler field onto the spherical harmonic function $Y_{l}^{m}$ leads to additional contributions of a mode $i$ to the global solar oscillations of modes in $M_i$. In the frequency domain we find that the ratios of mode amplitudes $A(\omega)$ at the frequency $\omega_i$ between mode $i$ and coupling modes in $M_i$ reflect the coefficients $C_{ij}$ by the relation

$$\frac{A_j(\omega_i)}{A_i(\omega_i)} \propto \frac{C_{ij}}{C_{ii}} \quad \text{for} \ j \in M_i,$$

(6)

and we define the ratios $C_{ij}/C_{ii}$ as coupling ratios.

In Fig. 1 we show an example of simulated coupling ratios for $l = 120, n = 5$ in dependence on azimuthal order $m$ for a simulated meridional flow which is described below. The dependence of the matrix elements $H_{ij}$ on azimuthal order $m$ can be expressed in terms of orthogonal polynomials $P_{l}^{s}$ via the equation

$$H_{nlm}^{n'}(m) = \sum_{s} b_{n,l,n',m}^{s} P_{l}^{s}(m),$$

(7)

where we identified the subscript $i$ by its radial order and degree $(n, l)$ and $j$ by $(n', l')$. The polynomials are given by Wigner-3j-symbols analogously to the definitions in [5]. The $b$-coefficients are here defined by

$$b_{n,l,n',m}^{s} = \int_{0}^{R} K_{s}^{nlm}(r) u_{s}(r) dr.$$

(8)

They relate the matrix elements to the radial meridional flow strength of corresponding degree $s$. The integral kernel $K_{s}^{nlm}$ describes the coupling between modes $(n, l)$ and $(n', l')$ due to a meridional flow component of degree $s$ and is given in [4].
4. Inference of the meridional flow - a simulation
In a simulation we generated a meridional flow as defined in [4]. The flow consists of components with degree \( s = 1, \ldots, 5 \) and a return flow at \( r = 0.713 R \). The horizontal flow strength at the surface varies between 0.5 – 56 m/s. We used Solar Model S [6] to compute the meridional flow kernels and the coupling ratios. We estimated the \( b \)-coefficients by fitting polynomials as defined before. The observation error was assumed to be Gaussian with unit variance and independent for all coupling ratios. From application of the angular momentum selection rules and inclusion of restrictions for the observable frequency range we obtain 8048 couplings between p-modes with frequencies of 1.1 – 3.7 mHz and degree \( l = 1 – 299 \).

The meridional flow and the \( b \)-coefficients are formally connected by the integral equation of the first kind in Eq. (8). For the problem of inference of the meridional flow we adopted the Subtractive Optimally Localized Averaging (SOLA) technique which is successfully applied for the inversion of the differential rotation [7]. For the SOLA technique target kernels sensitive to different depths \( r \) are defined with a Gaussian shape. Using meridional flow kernels we determine averaging kernels which optimally approximate the target kernels (Fig. 2). This finally yields estimates of the radial flow profiles as shown in Fig. 3. Vertical errorbars for \( u_s \) are derived from the error magnification of the SOLA technique. Horizontal errorbars correspond to the radial resolution of the averaging kernels. The horizontal flow profiles are derived from the estimated radial flow profiles using the mass conservation law. The inversion results show a good estimation of the true underlying flow profiles. The vertical errorbars for \( s = 3 – 5 \) are very small due to the very well agreement of averaging kernels and target kernels. A bias of the estimated flow is observed at \( r = 0.7 R \) where the simulated flow profile has an imposed artificial discontinuity at the transition region of the convection zone and radiation zone.

5. Discussion
We have derived a method to infer the meridional flow based on the perturbation of the eigenfunctions of global p-modes. In a simulation we have shown that from a determination of the coupling ratios it is possible to infer the meridional flow by means of the SOLA inversion
**Figure 3.** Simulated and estimated profiles of the radial $u_s$ and horizontal $v_s$ flow components in dependence on depth $r/R$ and degree $s$. Black lines mark the simulated flow profiles. Gray asterisks show the estimated profiles. The vertical errorbars for $s = 3, ..., 5$ are of small size. For reasons of visibility we magnified them by a factor of ten.

Technique. In further studies we will investigate the influence of leakage due to incomplete observations of the Doppler velocity field on the determination of coupling ratios from global oscillations. A more extensive treatment of the observation error and its propagation from $u_s$ to $v_s$ throughout the inversion procedure is in work. An application of the method to observed global oscillations will be performed.

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