Star exponential functions as two-valued elements

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Abstract
We propose a relatively new notion of two-valued elements, which arises naturally in constructing the star exponential functions of the quadratics in the Weyl algebra over the complex number field. This notion enables us to describe the group like objects of the set of star exponential functions of quadratics in the Weyl algebra.

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1 Introduction

Geometries are described within a framework of manifolds which are set up among the topological spaces. The question then may arise as to whether there are possibilities to employ other notions rather than manifolds. In this paper, we attempt to propose a notion of two-valued elements, which seems to renew a geometric concept.

A nontrivial example of objects we propose in this paper is given by the Hopf-fibering $S^3 \rightarrow S^2$. Viewing $S^3 = \bigsqcup_{q \in S^2} S^1_q$ (disjoint union), we consider the double covering $\tilde{S}^1_q$ of each fiber $S^1_q$, which is denoted by $\tilde{S}^3$. When $\tilde{S}^3$ is considered as a point set, we are able to define local trivializations of $\tilde{S}^3|_{V_i} \cong V_i \times S^1$ naturally through the trivializations $S^3|_{V_i}$ given on a simple open covering $\{V_i\}_{i \in \Gamma}$ of $S^2$. This structure permit us to treat $\tilde{S}^3$ as a local Lie group, and hence it looks like a topological space. On the other hand, we have a projection $\pi : \tilde{S}^3 = \bigsqcup_{q \in S^2} S^1_q \rightarrow S^3 = \bigsqcup_{q \in S^2} S^1_q$ as the union of fiberwise projections, as if it were a non-trivial double covering. $\tilde{S}^3$ cannot be a manifold, since $S^3$ is simply connected. This might suggest us to make the notion of points vague. In particular, the “points” of $\tilde{S}^3$ should be regarded as two-valued elements with ± ambiguity.

We now consider a 1-parameter subgroup $S^1$ of $S^3$ and the inverse image $\pi^{-1}(S^1)$. Since all points of $\tilde{S}^3$ are “two-valued”, this simply looks like a combined object of $S^1 \times \mathbb{Z}_2$ and the double covering group, i.e. in some restricted region, this object can be regarded as a point set by several ways. In such a region, the ambiguity only arises in the case two pictures of point sets are mixed up. Similar phenomena appear in constructing star exponential functions of quadratic forms in the suitably extended Weyl system, which leads us to open a new concept of geometry as a noncommutative (quantum) aspect.

In the paper [9], we have shown strange phenomena which break associativity for the Weyl algebra over $\mathbb{C}$ generated by two generators $u$ and $v$. Furthermore, we have shown that the Lie algebra over $\mathbb{C}$ of quadratic forms can be exponentiated to the “group” which looks as if it were a double covering group of $SL_C(2)$ which is simply connected, or the complexification of the metaplectic group $Mp(2, \mathbb{R})$.

As a sequel to this work, we develop to the case of Weyl algebra with $2m$-generators $u_1, \cdots, u_m, v_1, \cdots, v_m$, and show that similar phenomena occur also in the case of $2m$-generators. We show that star exponential functions can be viewed as two-valued elements. We note that an approach using the notion of gerbes will be a possibility to describe such phenomena (cf. [3], [4], [9]), which will also give rise to a new geometrical formulation.
2 Weyl algebra and orderings

2.1 Weyl algebra

The Weyl algebra $W_\hbar$ is the algebra over $\mathbb{C}$ generated by $u_1, \cdots, u_m, v_1, \cdots, v_m$ with the following commutation relations:

$$[u_i, v_j] = -i\hbar \delta_{ij},$$  

(1)

where $\hbar$ is a positive constant and $[a, b] = a \ast b - b \ast a$. Here, the product on $W_\hbar$ is denoted by $\ast$. For abbreviation, we set $u = (u_1, \cdots, u_m)$, and $z = (u, v) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$. Let $\text{Sym}(2m, \mathbb{C})$ be the set of complex symmetric matrix $A = (A_{ij})$. For $A \in \text{Sym}(2m, \mathbb{C})$, we define a quadratic form by

$$A \ast (z) = \sum_{i,j=1}^{2m} A_{ij} \frac{1}{2} (z_{i} \ast z_{j} + z_{j} \ast z_{i}).$$  

(2)

Denote by $A_\hbar$ the set of $A \ast (z)$, where $A \in \text{Sym}(2m, \mathbb{C})$. It is easily seen that $A_\hbar$ forms a complex Lie algebra isomorphic to $\text{sp}_{\mathbb{C}}(m)$.

2.2 Orderings

Orderings are treated in the physical literature (cf. [1]) in quantum mechanics as the rule of association from $c$-number functions to $q$-number functions. There are typical orderings, called the standard ordering, the antistandard ordering and the Weyl ordering, and in case of complex variables $\zeta_k = u_k + iv_k$, $\zeta_k^* = u_k - iv_k$, the normal ordering, the antinormal ordering and the Weyl ordering.

However, from the mathematical viewpoint, it is better to go back to the original understanding of Weyl which says that the ordering is the problem of realization of the Weyl algebra $W_\hbar$. Since the Weyl algebra is the universal enveloping algebra of Heisenberg Lie algebra, the Poincaré-Birkhoff-Witt theorem shows that this algebra can be viewed as an algebra defined on the space of polynomials.

For precise formulations of ordering prescriptions in formal deformation quantization, one can refer to the article [2], but the theory using a formal deformation parameter gives only a probe for genuine quantum theory. We emphasize here that the deformation parameter $\hbar$ in this note is not a formal parameter, but a parameter moving among positive reals.

Thus, we generalize orderings as follows. Let $J$ be a $2m \times 2m$ matrix defined by $J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$. For every symmetric complex $2m \times 2m$ matrix $K = (K^{ij})$, we set the product

$$f(z) \ast_K g(z) = f \exp \{ i\hbar \frac{1}{2} (\sum_{i,j=1}^{2m} \bar{\partial}_{z_i} \Gamma^{ij} \partial_{z_j}) \} g,$$  

(3)

where $\Gamma = (\Gamma^{ij}) = (K^{ij} + J^{ij})$. The product formula (3) is well-defined for all $f, g \in \mathbb{C}[z]$, where $\mathbb{C}[z] = \mathbb{C}[z_1, \cdots, z_{2m}]$, and this satisfies

$$z_i \ast_K z_j - z_j \ast_K z_i = i\hbar J^{ij},$$  

(4)
which give the same commutation relations \(i\) as the Weyl algebra \(W_h\).

**Proposition 2.1** For every complex symmetric \(2m \times 2m\) matrix \(K\), \((\mathbb{C}[z], \ast_K)\) forms an associative algebra isomorphic to \(W_h\).

Proposition 2.1 gives a realization of the Weyl algebra \(W_h\), and at the same time, it also gives the way of the expressions of elements of the Weyl algebra \(W_h\). For instance, computing \(u_i \ast u_j \ast u_k\) by using (3) gives the expression of \(u_i \ast u_j \ast u_k\) as a polynomial. Thus, the product formula (3) will be referred to as \(K\)-ordering, i.e. giving an ordering is nothing but a product formula which gives the Weyl algebra \(W_h\) where generators are fixed. Note that according to the choice of \(K\):

\[
\begin{bmatrix}
0 & I_m \\
I_m & 0
\end{bmatrix},
\begin{bmatrix}
0 & -I_m \\
-I_m & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]

the product formulas (3) gives the standard ordering, the antistandard ordering and the Weyl ordering respectively.

By the above formulation of orderings, intertwiners between \(K\)-orderings are explicitly given as follows:

**Proposition 2.2** For every pair of complex symmetric \(2m \times 2m\) matrices \(K, K'\), we have the intertwiner \(T_h : (\mathbb{C}[z], \ast_K) \rightarrow (\mathbb{C}[z], \ast_{K'})\) defined as

\[
T_h(f) = \exp \left( \frac{\hbar}{2i} \sum_{i,j} (K^{ij} - K'^{ij}) \partial_{z_i} \partial_{z_j} \right) f.
\]  

(5)

Namely the following identity

\[
T_h(f \ast_K g) = T_h(f) \ast_{K'} T_h(g),
\]  

(6)

holds for any \(f, g \in \mathbb{C}[z]\).

Although the precise statement will be given in the forthcoming paper, the intertwiner can be extended for a certain class of functions. However, as it has been shown in [9] the intertwiner behaves only a 2-to-2 mappings in the space of exponential functions of quadratic forms, since the square root appears in the amplitude part of intertwined functions.

We think this is a basic phenomena which breaks the associativity of \(\ast\)-product in the space of closed linear hull of the exponential functions of quadratic forms. Such strange phenomena occurs only in the the case that the deformation parameter is a non-formal parameter. In spite of this, it is important that one can consider one parameter subgroups via the theory of ordinary differential equations.
3 Star exponential functions

3.1 Star exponential functions

We give the explicit formula for the star exponential function $e^{tA \ast(z)}$ via $K$-ordering. For $A \in \text{Sym}(2m, \mathbb{C})$, we denote by $A[z]$ the symmetric quadratic function defined by

$$A[z] = \sum_{i,j=1}^{2m} A_{ij} z_i z_j. \quad (7)$$

Set $\mathbb{C}^\times = \mathbb{C} - \{0\}$, and we denote by $\mathcal{F}$ the set defined by

$$\mathcal{F} = \{ F = g \exp Q \mid g \in \mathbb{C}^\times, Q \in \text{Sym}(2m, \mathbb{C}) \}. \quad (8)$$

For $A \in \text{Sym}(2m, \mathbb{C})$, we set as

$$A_{*K}(z) = \sum_{i,j=1}^{2m} A_{ij} \frac{1}{2} (z_i \ast K z_j + z_j \ast K z_i). \quad (9)$$

The product formula (8) gives

$$A_{*K}(z) = A[z] + i\hbar \text{Tr}K. \quad (10)$$

We realize the star exponential functions of $A_{*}(z)$, for $A \in \text{Sym}(2m, \mathbb{C})$ with the help of $K$-ordering. Namely, in order to get the formula $e^{tA_{*K}(z)}$, we set $F_K(t) = e^{A_{*K}(z) \ast K}$, and consider the following equation:

$$\left\{ \begin{array}{l} \partial_t F_K(t) = A_{*K}(z) \ast K F_K(t), \\ F_K(0) = 1. \end{array} \right. \quad (11)$$

By the product formulas (8) and (9), the evolution equation (11) can be expressed as a differential equation. Thus the uniqueness of the real analytic solution holds, if it exists.

By setting

$$F_K(t) = g_K(t) \exp Q_K(t)[z], \quad \text{where } Q_K(t)[z] \in \text{Sym}(2m, \mathbb{C}), \quad (12)$$

the evolution equation (11) is reduced to a system of ordinary differential equations on $g_K(t)$ and $Q_K(t)[z]$. By a direct computation, although it is rather complicated, we have

**Theorem 3.1** The evolution equation (11) has the unique analytic solution $F_K(t) \in \mathcal{F}$ explicitly given by

$$F_K(t) = g_K(t) \exp Q_K(t)[z], \quad (13)$$

where

$$Q_K(t) = -\frac{J}{\hbar} (\tan \hbar t J A) \cdot (I - iK \tan \hbar t J A)^{-1} \quad (14)$$

$$g_K(t) = \left( \det (\cos \hbar t J A - iK \sin \hbar t J A) \right)^{-1/2}. \quad (15)$$


Remark Millard [5] also obtained this product formula by solving the successive power series of a Riccati-type equation. Remark also that for every $t \in \mathbb{C}$ there is $K$-ordering such that $F_K(t)$ is well-defined.

Among the formula (13) of the star exponential functions of $A_*(z)$, we particularly have for the standard ordering and the Weyl ordering by plugging $K = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$, and $K = 0$, respectively.

In particular, we have

**Corollary 3.2** For any $A \in \text{Sym}(2m, \mathbb{C})$, the star exponential function $e^{tA_(z)}$ is expressed as

$$e^{tA_(z)} = \det(\cos \hbar tJ A)^{-1/2} \cdot \exp \left( -\frac{J}{\hbar} \tan \hbar tJ A \right) [z]$$

(16)

via the Weyl ordering.

### 3.2 Star exponential functions of rank one quadratics

We examine the product formula (11) by restricting the quadratics to the rank one.

For $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m) \in \mathbb{C}^m$, we set $\langle x, y \rangle = \sum_{i=1}^m x_i y_i$. For $a, b \in \mathbb{C}^m$, we consider $\langle a, u \rangle = \sum_{i=1}^m a_i u_i$ and $\langle b, v \rangle = \sum_{i=1}^m b_i v_i$ as elements of $W_\hbar$. It is easy to see

$$[\langle a, u \rangle, \langle b, v \rangle]_* = -i\hbar \langle a, b \rangle. \quad (17)$$

Hence, if $\langle a, a \rangle = 1$, then $\langle a, u \rangle$ and $\langle a, v \rangle$ form a canonical conjugate pair. Let $S^{m-1}_\mathbb{C} = \{ a \in \mathbb{C}^m \mid \langle a, a \rangle = 1 \}$. For every $a \in S^{m-1}_\mathbb{C}$, and $\alpha, \beta, \gamma \in \mathbb{C}$, we consider a quadratic form

$$B_*(\alpha, \beta, \gamma) = \alpha \langle a, u \rangle * \langle a, u \rangle + \beta \langle a, v \rangle * \langle a, v \rangle + \gamma (\langle a, u \rangle * \langle a, v \rangle + \langle a, v \rangle * \langle a, u \rangle), \quad (18)$$

which is called a quadratic form of rank one.

In the following, we assume that the discriminant $D = \gamma^2 - \alpha \beta = 1$. We now write down the star exponential for the quadratic form $B_*(\alpha, \beta, \gamma)$ of rank one. We denote by $F_M(\alpha, \beta, \gamma)$ and $F_N(\alpha, \beta, \gamma)$ the solution of (11) for $A_*(z) = B_*(\alpha, \beta, \gamma)$ with respect to $K = 0$ and $\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$ respectively. Then, we have (see also [9]):

**Corollary 3.3** Assume that $D = \gamma^2 - \alpha \beta = 1$. Then, for $a \in \mathbb{C}^m$ such that $\langle a, a \rangle = 1$, we have

$$F_M(t, \alpha, \beta, \gamma) = g_M(t, \alpha, \beta, \gamma) \cdot \exp Q_M(t, \alpha, \beta, \gamma), \quad (19)$$

where

$$g_M(t, \alpha, \beta, \gamma) = (\cos \hbar t)^{-1}. \quad (20)$$
\[ Q_M(t, \alpha, \beta, \gamma) = \frac{1}{\hbar} \tan(\hbar t) \cdot (\alpha \langle \mathbf{a}, \mathbf{u} \rangle^2 + \beta \langle \mathbf{a}, \mathbf{v} \rangle^2 + 2\gamma \langle \mathbf{a}, \mathbf{u} \rangle \langle \mathbf{a}, \mathbf{v} \rangle) \]  

(21)

Similarly, we have

\[ F_N(t, \alpha, \beta, \gamma) = g_N(t, \alpha, \beta, \gamma) \cdot \exp Q_N(t, \alpha, \beta, \gamma). \]  

(22)

Here, \( g_N \) and \( Q_N \) are given by

\[ g_N(t, \alpha, \beta, \gamma) = e^{-i\hbar t\gamma \cdot \left( \cos 2\hbar t - i\gamma \sin 2\hbar t \right)} - \frac{1}{2}, \]  

(23)

\[ Q_N(t, \alpha, \beta, \gamma) = \frac{1}{\hbar} \left( X_N(t) \langle \mathbf{a}, \mathbf{u} \rangle^2 + Y_N(t) \langle \mathbf{a}, \mathbf{v} \rangle^2 + 2Z_N(t) \langle \mathbf{a}, \mathbf{u} \rangle \circ \langle \mathbf{a}, \mathbf{v} \rangle \right), \]  

(24)

where

\[
\begin{align*}
X_N(t) &= \frac{\alpha}{2} \left( \frac{\sin 2\hbar t}{\cos 2\hbar t - i\gamma \sin 2\hbar t} \right), \\
Y_N(t) &= \frac{\beta}{2} \left( \frac{\sin 2\hbar t}{\cos 2\hbar t - i\gamma \sin 2\hbar t} \right), \\
Z_N(t) &= \frac{i}{2} \left( 1 - \frac{1}{\cos 2\hbar t - i\gamma \sin 2\hbar t} \right)
\end{align*}
\]  

(25)

and the \( \circ \) in the product simply means that we use the standard ordering.

4 Polar elements are two-valued elements

4.1 Polar elements

Using the formulas of the star exponential functions (19) and (22), we show how two-valued elements appear.

We give justifications of the star exponential function of quadratic forms \( B^\ast(\alpha, \beta, \gamma) \) as follows. We consider \( B^\ast(\alpha, \beta, \gamma) \) defined by (18), and consider the star exponential functions expressed by the standard ordering and the Weyl ordering. Looking at the formulas in Corollary 3.3 and evaluating for \( t = \frac{\pi}{2\hbar} \), we have that \( F_M(\frac{\pi}{2\hbar}, \alpha, \beta, \gamma) \) diverges, however \( F_N(\frac{\pi}{2\hbar}, \alpha, \beta, \gamma) \) has a meaning.

Thus, we think of \( F_N(\frac{\pi}{2\hbar}, \alpha, \beta, \gamma) \) as a realization of the star exponential function of \( \frac{\pi}{2\hbar} B^\ast(\alpha, \beta, \gamma) \), which is denoted by \( \exp^\ast \left( \frac{\pi}{2\hbar} B^\ast(\alpha, \beta, \gamma) \right) \). However, by the formula (22) in Corollary 3.3 we obtain

**Theorem 4.1** Assume \( \mathbf{a} \in S^{m-1}_C \). For any \( (\alpha, \beta, \gamma) \) with \( \gamma^2 - \alpha \beta = 1 \), we have

\[ \exp^\ast \left( \frac{\pi}{2\hbar} B_s(\alpha, \beta, \gamma) \right) = \sqrt{-1} e^{\frac{\pi \langle \mathbf{a}, \mathbf{u} \rangle \circ \langle \mathbf{a}, \mathbf{v} \rangle}{2}}, \]  

(26)

which is independent of the choice of \( \alpha, \beta, \gamma \).

We will show that the ambiguity of \( \sqrt{-1} \) can not be eliminated for all \( (\alpha, \beta, \gamma) \).

**Definition 4.2** Assume \( \mathbf{a} \in S^{m-1}_C \).

\[ \varepsilon_{00}(\mathbf{a}) = \exp^\ast \left( \frac{\pi}{2\hbar} B_s(0, 0, 1) \right) \]  

(27)

is called the polar element.
4.2 Two-valued elements

We explain that the polar elements $\varepsilon_{00}(a)$, $a \in S_{C}^{m-1}$ play the same role like the two-valued elements as below. Since $(\alpha, \beta, \gamma) = (0, 0, 1)$ and $(0, 0, -1)$ are arcwise connected in the set $\gamma^2 - \alpha\beta = 1$, and thus, they have to be viewed as a single element. By Theorem 4.1 we have

$$
\varepsilon_{*}^2((a,v)\ast(a,u)+(a,u)\ast(a,v)) = \sqrt{-1}e_{0}^{\frac{2\pi}{\hbar}}(a,u)\circ(a,v) = e_{*}^{-\frac{2\pi}{\hbar}}((a,v)\ast(a,u)+(a,u)\ast(a,v)).
$$

(28)

However, considering the exponential law of the $*$-exponential function

$$
e_{*}^t((a,v)\ast(a,u)+(a,u)\ast(a,v))
$$

for $t \in \mathbb{C} - \{\text{singular set}\}$, we must set

$$
e_{*}^t((a,v)\ast(a,u)+(a,u)\ast(a,v)) = ite_{0}^{\frac{2\pi}{\hbar}}(a,v),
$$

$$
e_{*}^{-t}((a,v)\ast(a,u)+(a,u)\ast(a,v)) = -ite_{0}^{\frac{2\pi}{\hbar}}(a,v).
$$

(29)

If one wants to fix the sign ambiguity, the exponential law and (28) gives

$$
-1 = e_{*}^t((a,v)\ast(a,u)+(a,u)\ast(a,v)) \ast e_{*}^{-t}((a,v)\ast(a,u)+(a,u)\ast(a,v))
$$

$$
= e_{*}^t((a,v)\ast(a,u)+(a,u)\ast(a,v)) \ast e_{*}^{-t}((a,v)\ast(a,u)+(a,u)\ast(a,v)) = 1.
$$

(30)

We choose a continuous path of $(\alpha, \beta, \gamma)$ from $(0, 0, 1)$ to $(0, 0, -1)$ for the case $m = 1$ concretely as follows: Set $(a,u) = u$, $(a,v) = v$ and $\varepsilon_{00}$ stands for $\varepsilon_{00}(a)$. By a careful calculation, we see

$$
Ad(e_{*}^{\frac{2\pi}{\hbar}(u^2+v^2)})e_{*}^{2tuv} = e_{*}^{t\sin 2\theta (u^2-v^2)+\cos 2\theta (2uv)}.
$$

(31)

Since the discriminant of the quadratic form of the right hand side is identically 1, the right hand side is identically $\varepsilon_{00}$ for $t = \frac{2\pi}{\hbar}$. In particular, $Ad(e_{*}^{\frac{2\pi}{\hbar}(u^2+v^2)}e_{2uv} = \varepsilon_{00}$. On the other hand, consider, for each $\theta$, the one parameter subgroup $Ad(e_{*}^{\frac{2\pi}{\hbar}(u^2+v^2)}e_{2uv}$ with respect to $t$, $t \in [0, \frac{2\pi}{\hbar}]$.

We easily see that $Ad(e_{*}^{\frac{2\pi}{\hbar}(u^2+v^2)}e_{2uv} = e_{*}^{-2tuv}$. In particular,

$$
Ad(e_{*}^{\frac{2\pi}{\hbar}(u^2+v^2)})\varepsilon_{00} = -\varepsilon_{00}
$$

by the exponential law. Move $2\theta$ from 0 to $\pi$. Then, we see the desired fact.

Note also that by (31), (29), there is a singularity at $2\theta = \pi$, $t = \frac{2\pi}{\hbar}$.

We also have in the standard ordering for $D = 1$,

$$
e_{*}^\varepsilon((a,u)^2 + \beta(a,v)^2 + \gamma((a,u)\ast(a,v)+(a,v)\ast(a,u))) = \sqrt{-1}e_{0}^{\frac{2\pi}{\hbar}}(a,u)\circ(a,v).
$$
By the exponential law, we see that $\varepsilon_{00}(a)$ satisfies
\[ \varepsilon_{00}(a)^2 = (\varepsilon_{00}^{-1}(a))^2 = -1, \quad \varepsilon_{00}(a) \ast \varepsilon_{00}^{-1}(a) = 1. \tag{32} \]

Therefore, we must conclude that the sign ambiguity cannot be eliminated. One has to set $\varepsilon_{00}(a) = \sqrt{-1}\varepsilon_0^{\pm}(a,u)^{(a,v)}$ with the sign ambiguity. Similar phenomena have been discussed by Olver [6].

By the above observation, the polar element $\varepsilon_{00}(a)$ should be regarded as a two-valued element. Otherwise we do have a contradiction $1 = -1$.

Only this way one can permit the identity $-\varepsilon_{00}(a) = \varepsilon_{00}(a)$. But since such a notion does not exist in the set theory, it is impossible to define $\varepsilon_{00}(a)$ as a point in a point set.

In what follows, we set
\[ \varepsilon_{00}(k) = e^{i\sum_{k}^m (u_k^* v_k + v_k^* u_k)} = \sqrt{-1}\varepsilon_0^{\pm} u_k^* v_k, \quad k = 1, 2, \ldots, m. \]

These are all regarded as two-valued elements. Although it is natural to think $\varepsilon_{00}(k) \ast \varepsilon_{00}(l) = \varepsilon_{00}(l) \ast \varepsilon_{00}(k)$, we also have the equality
\[ \varepsilon_{00}(k) \ast \varepsilon_{00}(l) = -\varepsilon_{00}(l) \ast \varepsilon_{00}(k), \quad (k \neq l) \]
at the same time. Hence we have $\varepsilon_{00}(k) \ast \varepsilon_{00}(k) = \pm 1$, but we see $\varepsilon_{00}(k)^2 = -1$. This is just the same as $\{\pm 1\} \{\pm 1\} = \{\pm 1\}$, but $\{\pm 1\}^2 = 1$.

Hence $\varepsilon_{00}(k)^2$ behaves like an ordinary number in the extended Weyl algebra. In spite of this, $\varepsilon_{00}(k)$ does not behave like an ordinary number $i$, since it is easy to see with the bumping identity (cf. [9]) that
\[ \text{Ad}(\varepsilon_{00}(k)) u_k = \pm u_k, \quad \text{Ad}(\varepsilon_{00}(k)) v_k = \mp v_k. \]

Using this we easily have
\[ \text{Ad}(\varepsilon_{00}(1))(\sum_{i=1}^m b_i u_i) = -b_1 u_1 + \sum_{i=2}^m b_i u_i. \tag{33} \]

Since every $(a, u), a \in S_m^C$ is translated to $u^1$ by a symplectic transformation, we have in general the reflection w.r.t. $a$:
\[ \text{Ad}(\varepsilon_{00}(a))(b, u) = (b - 2(a, b)a, u) \]
\[ \text{Ad}(\varepsilon_{00}(a))(b, v) = (b - 2(a, b)a, v) \tag{34} \]

We introduce a notion called a blurred double covering group, which is a group like object formed by 2-valued elements ([9]).

**Theorem 4.3** $\text{Ad}(\varepsilon_{00}(a) \ast \varepsilon_{00}(b))$ generate $SO(m, \mathbb{C})$, hence $\{\varepsilon_{00}(a) \ast \varepsilon_{00}(b)\}$ generate a blurred double covering group of $SO(m, \mathbb{C})$. However, this blurred double cover has a point set picture of $SO(m, \mathbb{C}) \times \mathbb{C}^\times$. If $a, b$ are restricted in real vectors, then $\text{Ad}(\varepsilon_{00}(a) \ast \varepsilon_{00}(b))$ generate $SO(m)$, hence $\{\varepsilon_{00}(a) \ast \varepsilon_{00}(b)\}$ generate a blurred double covering group of $SO(m)$, which may be viewed as $\text{Spin}(m)$.  

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