Dilaton, Antisymmetric Tensor and Gauge Fields in String Effective Theories at the One–loop Level*

P. Mayr and S. Stieberger

Physik Department
Institut für Theoretische Physik
Technische Universität München
D–8046 Garching, W–Germany

and

Max–Planck–Institut für Physik
—Werner–Heisenberg–Institut—
P.O. Box 401212
D–8000 München, W–Germany

ABSTRACT

We investigate the dependence of the gauge couplings on the dilaton field in string effective theories at the one–loop level. First we resolve the discrepancies between statements based on symmetry considerations and explicit calculations in string effective theories on this subject. A calculation of the relevant one–loop scattering amplitudes in string theory gives us further information and allows us to derive the exact form of the corresponding effective Lagrangian. In particular there is no dilaton dependent one–loop correction to the holomorphic $f$–function arising from massive string modes in the loop. In addition we address the coupling of the antisymmetric tensor field to the gauge bosons at one–loop. While the string $S$–matrix elements are not reproduced using the usual supersymmetric Lagrangian with the chiral superfield representation for the dilaton field, the analogue Lagrangian with the dilaton in a linear multiplet naturally gives the correct answer.

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1 Introduction

String theory is the only known theory consistent with quantized gravity. Whether it is also consistent with the world where we all live is a different and very difficult question. To tackle this problem one tries to derive an effective Lagrangian, which describes the low–energy limit of the string theory in a field–theoretical language. One question of current interest are modifications of the one–loop effective field theory through massive string modes running in the loops. While the contributions of the massless modes are generated by the field theory itself, those of the massive ones have to be plugged in by hand. In general these stringy effects have to be determined by explicit one–loop calculations [1, 2, 3, 4, 5, 6] directly in four–dimensional string theory.

In lucky cases there is yet another way to derive the string loop corrections: if there are severe stringy symmetries and in addition they are known. These symmetries can give significant restrictions on the possible structure of the Lagrangian terms which can not be understood from the point of field theory. This is because the field theory as the low–energy limit has to respect the symmetries of its stringy roots. This method was partially successful in the case of a restricted class of orbifold models for the moduli and gauge group dependent part of the gauge couplings. The discrete reparametrization symmetry SL(2,\(\mathbb{Z}\)) of the moduli space was sufficient to infer the exact form of the one–loop string corrections from the requirement of anomaly freedom of the effective theory with respect to SL(2,\(\mathbb{Z}\)) [7, 8]. However already in this favourable case it was not possible to determine the Lagrangian uniquely from symmetry considerations [7]: to fix the coefficients of the two independent counterterms, a Green–Schwarz term and a holomorphic contribution from massive modes, again string calculations are necessary [3]. Furthermore it was shown in ref. [9] that for the majority of orbifold models the known symmetry group is smaller than SL(2,\(\mathbb{Z}\)) and therefore this method fails due to the lack of reliable information.

Another scalar field with a perturbatively flat potential is the dilaton field, whose vev determines the gauge coupling at tree–level [10]. Therefore it seems natural to ask for the dependence of the one–loop corrections to the gauge coupling on this field. Although there exist no appropriate quantum symmetries comparable to the discrete reparametrization symmetries of the moduli space, there are classical symmetries [11] which have quite strong implications. Earlier investigations based on this symmetry have led to the statement, that the one–loop piece of the gauge coupling is independent of the dilaton vev [12]. We will find in sect. 2 that this kind of statement is too rigorous as has been already indicated by the results of explicit calculations in effective theories, including dilaton dependent gauge couplings [3, 8, 13]. Nevertheless the above mentioned symmetries are sufficient to determine the exact dependence of the one–loop gauge couplings on the dilaton vev.

To confirm the symmetry arguments as well as to gain information also about the couplings of the quantum fields, a string loop calculation is desirable. Since string
theory is an S–matrix theory there are problems with on–shell singularities in the computation of scattering amplitudes. We will use Minahan’s off–shell prescription \[15\] for the momentum variables to handle this difficulty and present in the appendix the derivation using the background–field method \[1, 5\] as an alternative concept. Another issue is related to the fact that string theory calculations provide always S-matrix elements: To obtain the 1PI vertices generating effective Lagrangian one has to assign a string correction to corrections to 1PI vertices in an appropriate way. This step plays an important role for the translation of string loop calculations into a field–theoretical language.

The outline of this paper is the following. In sect. 2 we discuss the special role of the dilaton in string theory and its effective theories. In sect. 3 we compute a three–point scattering amplitude of two gauge bosons and a physical boson of the supergravity multiplet on the torus. This can be a graviton, an antisymmetric tensor field or the dilaton. In sect. 4 we discuss the obtained results in terms of effective supersymmetric Lagrangians and give our conclusions.

2 The dilaton as the loop expansion parameter

In this section we want to analyze earlier investigations on the dilaton dependence of the gauge coupling at one–loop in string theory based on symmetry considerations. The well–known statement arising from this kind of reasoning is that the one–loop correction to the gauge coupling is independent of the dilaton vev. This has to be contrasted with the results of explicit calculations in string effective field theories, which include a one–loop coupling of the dilaton to the gauge bosons \[13, 8, 14\]. A closer examination of the underlying assumptions present in the symmetry arguments shows that there are no discrepancies to worry about. Furthermore we will derive in this way the exact dilaton dependence of the one–loop contribution to the gauge coupling in string effective theories.

First let us consider the situation directly in string theory \[16\]. The two–dimensional action of the superstring contains the expression \[1\]

$$
\frac{1}{4\pi\alpha'} \int d^2 z \sqrt{h} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) + \frac{1}{4\pi} \int d^2 z \sqrt{h} R^{(2)} \phi(X) ,
$$

where \(h\) and \(R^{(2)}\) are the two–dimensional metric and curvature, respectively, and \(\alpha'\) is the slope parameter. \(g_{\mu\nu}(X)\) corresponds to the background metric, while \(\phi(X)\) is a scalar field. The two–dimensional integration over the constant mode \(\phi_0\) of \(\phi\) yields \(1\)

$$
\frac{1}{4\pi} \int d^2 z \sqrt{h} R^{(2)} \phi_0 = \chi = 2(1 - l)\phi_0 ,
$$

where \(\chi\) is the Euler characteristic and \(l\) is the genus of the world–sheet manifold. For this reason the contribution of higher genus will be suppressed by a factor of \(e^{-2l\phi_0}\) for each loop. This is the same functional dependence on the genus as that
of the weight of a Riemannian surface in the path integral, $g^{2l}$, where $g$ is a free parameter, the string coupling constant \[17\]. Therefore a change in the coupling constant can be compensated by a redefinition of the field $\phi$ and in this way is determined by its vev. To relate the field $\phi$ (or parts of it) to the dilaton field one uses the following form of the dilaton vertex operator in the case of vanishing momentum \[16\]:

$$V_0^D(k \to 0) = \frac{a}{4\pi \alpha'} \int d^2z \sqrt{h} \ h^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu \nu} + \frac{b}{4\pi} \int d^2z \sqrt{h} R^{(2)} . \quad (3)$$

Here $a$ and $b$ are some real numbers which can be determined by demanding unitarity and conformal invariance or alternatively from the limiting field theory \[18\], \[19\]. The term proportional to $R^{(2)}$ has been discussed extensively in the literature \[20\]. However this issue will be of no importance for our discussion. A small shift $\delta D$ of the dilaton vev can be taken into account by the insertion of (3) into each correlator:

$$\langle V_1...V_N \rangle \longrightarrow \langle V_1...V_N \rangle + \delta D \langle V_1...V_N V_0^D(k \to 0) \rangle . \quad (4)$$

In turn this can be absorbed to leading order in the associated constant field $\delta D$ by modifying the background fields in (1) as

$$\phi(X) \longrightarrow \phi(X) + b \delta D ,$$
$$g_{\mu \nu}(X) \longrightarrow g_{\mu \nu}(X) (1 + a \delta D) . \quad (5)$$

It follows that parts of the field $\phi$ can be identified with the dilaton field and that also the normalization of the space–time metric will be influenced by the vev of the dilaton field. A second possibility would be to make the replacements

$$\phi(X) \longrightarrow \phi(X) + b \delta D ,$$
$$\alpha' \longrightarrow (1 - a \delta D) \alpha' . \quad (6)$$

Note that (5) corresponds to a fixed string scale $M_S \sim \alpha'^{-1/2}$, while (6) will leave unchanged the Planck mass $M_{pl}$ if $a$ and $b$ are chosen correctly. Here the Planck mass is defined as the coefficient of the Einstein term in the low–energy Lagrangian or, equivalently, by the strength of the three graviton interaction in string theory.

Since there is the relation $2\kappa = (2\alpha')^{d-2}g$ connecting the three parameters $\alpha', g$ and $\kappa \sim M_{pl}^{-2}$ of the effective field theory in $d$ dimensions \[21\], \[22\], there is the possibility to form a dimensionless parameter proportional to $\alpha'^{(d-2)/4} \kappa^{-1}$. The above argumentation recovers the well–known result that such a parameter will be determined dynamically rather than representing a free input parameter \[23\].

What are the implications for the dilaton dependence of a genuine correlator? From eqs. (1) and (6) one can see that the statement that a correlator is independent of the dilaton vev, if the second term on the r.h.s. of eq. (4) vanishes relies heavily on the field normalizations. In this context it is helpful to note that the background field $g_{\mu \nu}$ appearing in eqs. (1) and (6) is not the canonical normalized
gravitational field as it is used in the original determination of the effective
Lagrangian calculating string amplitudes [21, 22]. This point can be easily seen also in
the field–theoretical formulation. Eqs. (5) and (6) correspond to the following two
Lagrangians, respectively:

\[
\mathcal{L} = \frac{\Phi^{4+d^2}}{\kappa^2} \left[ -\frac{1}{2} R - \frac{1}{4} F^2 - \frac{3}{4} H^2 + \ldots \right], \tag{7}
\]

\[
\mathcal{L} = -\frac{1}{2\kappa^2} R - \frac{1}{4 \Phi^{-1/2}} F^2 - \frac{3\kappa^2}{4 g_d^2} \Phi^{-1} H^2 + \ldots . \tag{8}
\]

We use the standard notations for the fields [21] except for \(\Phi\) which is defined to
be \(\Phi = e^{\frac{d}{\sqrt{\phi^2 + D}}\rangle}\), where \(D\) is the \(d\)–dimensional dilaton field. In eq. (7) the space–
time metric has been rescaled with respect to (8) as \(g_{\mu\nu} \rightarrow \Phi^{-1/2} g_{\mu\nu}\) and the string
scale \(M_S \equiv g_d/\kappa\) has been set to 1. Apparently the factor in front of (7) can be
identified with the loop expansion parameter [24]. Therefore in this special field
basis the one–loop gauge coupling should be independent of the dilaton VEV in any
dimension. This Lagrangian corresponds to the case (3) with a dilaton independent
normalization of the background metric \(g_{\mu\nu}\). This choice causes a dilaton dependent
coefficient of the Einstein term in the effective Lagrangian. In the more physical
formulation of (8) where \(M_{pl}\) is fixed, the terms of the Lagrangian have an individual
dependence on \(\Phi\) and the loop expansion parameter is in general different from the
coefficient of the \(F^2\) term. Therefore one can no longer infer a dilaton independence
of the gauge coupling at one–loop and we see again that the usual reasoning based
on symmetry considerations is only valid for a certain choice of field normalizations
corresponding to (7).

It is important that the dependence of the gauge coupling on \(\Phi\) in (8) is different
for any number of dimensions. This reflects in some sense the singular behavior of
a Yang–Mills theory in \(d\) dimensions. In \(d=4\) the special situation arises that the \(\Phi\)
dependence of the coefficients of the \(F^2\) terms in (7) and (8) agree. Nevertheless a
simple cancellation of couplings will not be sufficient to ensure the dilaton indepen-
dence of the one–loop contribution to the gauge kinetic term: the Lagrangian (8)
describes a field theory with a dilaton dependent physical cutoff which should show
up in an one–loop effective gauge coupling. In summary we can state the following
result: The dilaton dependence of the gauge coupling at one–loop in the physical
field basis with \(M_{pl} \neq M_{pl} \langle \langle D \rangle \rangle\) is entirely due to the dilaton dependence of the
cutoff \(\sim M_S\) (in four dimensions).

It is useful to notice the precise assertion of the above argumentation. First note
that it is valid for all one–loop corrections to the gauge coupling, regardless whether
they arise as a one–loop term of the holomorphic \(f\)–function of supergravity [13] or
from non–holomorphic terms [2]. On the other hand it is restricted to the constant
part of the dilaton field, that is, its vev.

A similar discussion can be carried on for the symmetry arguments of ref. [12],
valid for the four–dimensional effective theory in the limit of infinitely small comp-
actification radius. In this paper the (at least) classical symmetries of the effective
field theory arising from the degeneration of the string vacuum were used to narrow down the possible form of the effective Lagrangian. Two kinds of symmetries were considered there: First two classical scale invariances of the theory representing the flat directions in the effective potential of the dilaton and the universal modulus field related to the overall radius of the compactification manifold. Secondly two axial $U(1)$ symmetries given by constant shifts of the supersymmetric pseudoscalar partners of the above mentioned fields. This axial symmetries are valid to all orders of perturbation theory and can be traced back to the invariance of the field strength of an antisymmetric tensor field.

Two important results were derived in ref. [12] from the presence of these symmetries: using the scale invariances together with the input that the loop expansion parameter is given by $S/T$, it was shown that the one–loop kinetic term of the gauge fields should be independent of the $S$ field. Here $S$ ($T$) is the chiral superfield containing the four–dimensional dilaton (overall modulus) field. In fact this result was stated only for the holomorphic $f$–function, but the assumption of holomorphicity was not yet used at this point. In addition the axial $U(1)$ symmetries together with holomorphicity were used to prove that the gauge coupling receives no contributions from higher than one–loop. This remarkable conclusion is not transferable to the now well–known non–holomorphic one–loop corrections, but is only valid for the gauge coupling of the effective action defined in the spirit of ref. [25]. It is clear that the field basis with dilaton dependent $M_{pl}$ was used throughout ref. [12].

3 The two gauge boson – dilaton amplitude in string theory

In the previous section we derived the precise dilaton dependence of the one–loop corrections to the gauge coupling using symmetry consideration. To confirm our results as well as to get information also about the coupling of the non–constant part of the dilaton field to the gauge bosons at one–loop, we calculate the CP even part of the two gauge boson – dilaton coupling at one–loop in string theory.

We will consider in general a three–point function on the torus involving two gauge bosons and a physical boson from the supergravity multiplet. This provides a useful check of the correct normalizations relative to existing calculations. Furthermore we want to get information about the coupling of the antisymmetric tensor field to the gauge bosons. Our conventions and normalizations are given in appendix A. We will compute a scattering amplitude and compare the result with that of a background–field calculation in appendix B.

The CP even part of the three–point function at one–loop is given by

$$\mathcal{A} = \frac{1}{2} \sum_{\text{even } s} (-)^{(s_1 + s_2)}$$

\[\text{See also ref. [26].}\]
where \( s_1, s_2 \) label the three CP even spin structures \( (s_1, s_2) = (0,0), (0,1), (1,0) \). A zero (one) corresponds to a NS (R) boundary condition. \( \tau = \tau_1 + i \tau_2 \) is the single Teichmüller parameter of the worldsheet torus and \( \Gamma \) its fundamental domain \( \Gamma = \{ \tau \mid |\tau_1| < 1/2 \ ; \ |\tau_2| > 1 \} \). The partition function for the heterotic string is \( Z(\tau, \bar{\tau}, s) \equiv Tr_{s_1}[(-)^{s_2}F q^{H-1/2} \bar{q}^{\bar{H}-1}] \) with \( q = e^{2\pi i \tau} \) and \( V_0^A, V_0^G \) are the zero–picture vertex operators for the gauge boson and the supergravity tensor, respectively:

\[
V_0^A(z_1, \bar{z}_1) = \frac{4g}{\pi} \epsilon_{1\mu} J_1^\mu (\partial X_1^\mu + i\bar{\psi}_1^\mu k_1 \cdot \psi_1)e^{ik_1z_1} , \\
V_0^G(z_1, \bar{z}_1) = \frac{8\kappa}{\pi} \epsilon^{(1)}_{\mu\nu} (\partial X_1^\mu (\partial X_1^\nu + i\bar{\psi}_1^\nu k_1 \cdot \psi_1)e^{ik_1z_1} .
\]

Here we use the notation \( f(z_i) \equiv f_i \), \( f(\bar{z}_i) \equiv f_i \bar{f}_i \), \( g \) is the dimensionless string coupling parameter and \( \kappa \) the gravitational coupling. We will set \( \alpha' = \frac{1}{2} \) in most of the following unless an explicit representation is useful to distinguish terms of different order in \( \alpha' \). The full dependence can be recovered by simple dimensional considerations.

Because of supersymmetry, summing over the spin structures will give a zero result unless there are right–handed fermions in the correlation function. However, since the fermions come in normal–ordered pairs, there must actually be four fermions to get a non–vanishing result. Therefore, performing all possible contractions which fulfill this condition yields the following expression for the amplitude:

\[
\mathcal{A} = (2\alpha') \frac{4\kappa g^2}{\pi^3} \sum_{\text{even } s} (-)^{(s_1+s_2)} \int_{\tau \in \Gamma} \frac{d^2 \tau}{\Im \tau} Z(\tau, \bar{\tau}, s) \int \prod_{i=1}^3 d^2 z_i \prod_{i<j} |\chi_{ij}|^{\epsilon_i k_j \alpha'} \langle J_2 J_3 \rangle \\
\times \left\{ \left( \sum_{i<j} \epsilon_i \cdot k_j \epsilon_j \cdot k_i - \epsilon_i \cdot \epsilon_j k_i \cdot k_j \right) G_{ij}^2(s) \right\} \left( \sum_{p \neq i} \epsilon_p \cdot k_q \partial_p \ln|\chi_{pq}|^2 \right) \\
-4t(1,2,3) G_{12}(s) G_{13}(s) G_{23}(s) \right\} \left\{ \epsilon_1^* \cdot k_2 \partial_2 \ln|\chi_{12}|^2 + \epsilon_3^* \cdot k_3 \partial_3 \ln|\chi_{13}|^2 \right\} \\
- \frac{16}{2\alpha'} V \left( \sum_{i<j} (\epsilon_i \cdot k_j \epsilon_j \cdot k_i - \epsilon_i \cdot \epsilon_j k_i \cdot k_j) G_{ij}^2(s) \right) \left( \sum_{p \neq i,j} \epsilon_p \right) .
\]

\( G(s) \) is the spin structure dependent fermionic two–point function and \( t(1,2,3) \) is the kinematical factor of \( \mathcal{A}_{\mathcal{S}} \) (for further details see appendix A). In \( (\mathbb{II}) \) the polarization tensor \( \epsilon^{(1)}_{\mu\nu} \) has been replaced by \( \epsilon_1^* \mu \epsilon_1^* \nu \) for calculational reasons. Eq. \( (\mathbb{II}) \) differs from that of ref. \( (\mathbb{III}) \) by an additional term proportional to \( V \equiv -\frac{\pi}{4 \Im \tau} \) in \( (\mathbb{II}) \) which stems from the contraction of the fields \( \partial X_1 \) and \( \bar{\partial} X_1 \) in the dilaton vertex operator. In general a contraction of the fields \( \partial X \) and \( \bar{\partial} X \) yields

\[
\langle \partial X_1 \bar{\partial} X_2 \rangle = \partial_1 \bar{\partial}_2 \left( -\frac{1}{4} \ln|\chi_{12}|^2 \right) = \frac{\pi}{4} \delta^2(z_{12}) - \frac{\pi}{4 \Im \tau} ,
\]

\(^2\)We will denote the dependence \( f(z_i, \bar{z}_i) \) again with \( f_i \) if no confusion is possible.
where the second term is the contribution coming from the zero modes. While it is obvious that the \( \delta \)-function disappears if normal–ordered fields are considered, more involved arguments are necessary to show that this part of the two–point function does not contribute even in the more general case of unordered fields. Actually there are several different ways to derive this result \([13, 27, 28]\).

The next step is to extract from (11) the terms of lowest order in \( \alpha' \). Apart from the obvious terms proportional to \( V \) there arise additional ones from the integrations over the \( z \)'s in regions where some of the vertex operators are close together on the world–sheet. The resulting singularities can be treated along the line of Minahan’s off–shell prescription for the kinematical variables, which respects both, modular and conformal invariance \([15]\). The crucial point is that it is by no means fixed a priori, whether the momentum integral is performed before or after the \( z \)–integrations. This allows one to choose the kinematical variables which respect only the constraints from conformal and modular invariance but not from momentum conservation during parts of the calculation. The correctness of this procedure was confirmed in ref. \([29]\), where the authors show that the same result is obtained by factorising a four–point amplitude.

The additional terms arise in the following way: the correlation functions behave for \( z_{ij} \to 0 \) as

\[
G_{ij} \to \frac{1}{4z_{ij}} , \quad \partial_{ij} \ln |\chi_{ij}|^2 \to \frac{1}{z_{ij}} , \quad |\chi_{ij}|^2 \to |z_{ij}|^2 . \tag{13}
\]

Integration over those terms in (11) which provide a pole in \( z_{ij} \) yields, by analytic continuation

\[
\int_{|z_{ij}|<\epsilon} d^2z_{ij} \frac{1}{|z_{ij}|^2} |z_{ij}|^{\alpha' k_i \cdot k_j} = \frac{2\pi}{\alpha' k_i \cdot k_j} . \tag{14}
\]

Therefore this mechanism provides a source for new \( O(\alpha'^0) \) terms\(^3\) with poles in \( k_i \cdot k_j \), which cancel similar factors in the kinematical factors of (11). The resulting expression of \( O(\alpha'^0) \) reads

\[
\mathcal{A} = \frac{16\kappa g^2}{\pi^2} \sum_{\text{even} s} \frac{z_{23}^s}{s_{1} + s_{2}} \int_{\tau \in \Gamma} \frac{d^2\tau}{\text{Im} \, \tau} Z(\tau, \bar{\tau}, s) \int d^2z_{23} \langle \bar{J}_{2} J_{3} \rangle \tag{15}
\]

\[
\times \left\{ \text{Im} \, \tau \, \text{tr} (\epsilon^{(1)}_{as} \epsilon_{2} \cdot k_{3} \epsilon_{3} \cdot k_{2} G_{23}^2(s)) + 2(k_{3} \cdot \epsilon^{(1)}_{as} \cdot \epsilon_{2} \cdot k_{1} + k_{2} \cdot \epsilon^{(1)}_{as} \cdot \epsilon_{3} \cdot k_{1}) \int d^2z G^2(z, s) \right\} ,
\]

where

\[
\epsilon^{(1)}_{as} \equiv \frac{1}{2} (\epsilon_{1} - \epsilon_{1}^T) . \tag{16}
\]

Note that there is no contribution arising from \( z_{23} \to 0 \). In addition also the region where all vertex operators come close together does not contribute because of supersymmetry.

\(^3\)Of course there is an additional dependence on \( \alpha' \) implicit in \( \kappa \).
The two terms inside the curly brackets of (15) form separately gauge invariant objects. In fact two of the long distance contributions proportional to $V$ together with the pole terms are gauge invariant as well as the single term involving $\text{tr}(\epsilon^{(1)})$. This is not surprising because exactly this structure asserts gauge invariance for the three different types of polarization tensors corresponding to a graviton, an antisymmetric tensor or a dilaton.

In (15) the only spin structure dependent factors are the fermionic correlation functions. Therefore we can rewrite them using (A.5). After this replacement we perform the remaining $z$–integrations and splitting the partition function in space–time and internal part we get

$$A = \mathcal{K} \frac{k g^2}{8\pi^2} \int_{\tau \in \Gamma} \frac{d^2 \tau}{\text{Im} \, \tau} B_a \ , \quad (17)$$

where

$$\mathcal{K} = -\frac{1}{2} \text{tr}(\epsilon^{(1)}) \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 + k_3 \cdot \epsilon^{(1)}_{as} \cdot \epsilon_3 \epsilon_2 \cdot k_1 + k_2 \cdot \epsilon^{(1)}_{as} \cdot \epsilon_2 \epsilon_3 \cdot k_1 \ , \quad (18)$$

$$B_a = \sum_{\text{even } s} (-)^{(s_1 + s_2)} 2 |q(\tau)| -4 \frac{1}{2\pi i} d\tau Z_\psi(s) \times \text{Tr} \left[ \left( Q_a^2 + \frac{k_a}{4\pi \text{Im} \, \tau} \right)(-)^{s_2 F} q^{H-3/8} q^{B-11/12} \right]_{(c,\bar{c})=(9,22)} \ . \quad (19)$$

From (18) it is clear that there is a zero result for the case where the polarization tensor is that of a graviton. In the case of the antisymmetric tensor the result agrees with that of ref. [6] after the identification

$$k_3 \cdot \epsilon^{(1)}_{as} \cdot \epsilon_3 \epsilon_2 \cdot k_1 + k_2 \cdot \epsilon^{(1)}_{as} \cdot \epsilon_2 \epsilon_3 \cdot k_1 \longrightarrow \frac{1}{2} A_\mu \partial_\nu A_\rho (\partial^{\mu} B^{\nu \rho} + \partial^\nu B^{\rho \mu} + \partial^\rho B^{\mu \nu}) \ , \quad (20)$$

in spite of the fact that the derivation as well as the interpretation therein seems to be incorrect\footnote{The distinction in pole and long distance terms is only convention: by partial integration one can change this 'classification'.}. In fact there is also a non–vanishing result if $\epsilon^{(1)}$ is chosen to describe a dilaton. In this case the polarization tensor is given by

$$\epsilon^{(1)}_{\mu \nu} = \frac{1}{\sqrt{d-2}} (\eta_{\mu \nu} - k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu) \ , \quad (21)$$

where we have defined a vector $\bar{k}$ such that $\bar{k}^2 = 0$ and $k \cdot \bar{k} = 1$. Thus

$$\text{tr}(\epsilon^{(1)}) = \sqrt{d-2} = \sqrt{2} \ , \quad (22)$$

and (17) becomes

$$A = -\sqrt{2}(\epsilon_2 \cdot k_3 \epsilon_3 \cdot k_2 - \epsilon_2 \cdot \epsilon_3 k_2 \cdot k_3) \frac{k g^2}{16\pi^2} \int_{\tau \in \Gamma} \frac{d^2 \tau}{\text{Im} \, \tau} B_a \ , \quad (23)$$

\footnote{Compare with the discussion of eq.(12).}
where we have reinstated a kinematical term which was dropped during the calculations. Eq. (23) gives the following term to an one–loop S-matrix element:

\[
\frac{1}{4\Delta_a} F_{\mu\nu}^a F_{\mu\nu}^a D ,
\]  

(24)

where

\[
\tilde{\Delta}_a^D = \sqrt{2\kappa} \Delta_a^g = \sqrt{2\kappa} \left( \frac{1}{16\pi^2} \int_{\tau \in \Gamma} \frac{d^2 \tau}{\Im \tau} B_a \right) .
\]  

(25)

The tilde on \( \tilde{\Delta}_a^D \) denotes that it represents a correction to an S-matrix element rather than to an 1PI vertex. Furthermore \( \Delta_a^g \) is the one–loop correction to the gauge coupling derived in [1] and the gauge fields have been rescaled as \( A \to gA \).

If the internal part of the theory is given by an orbifold compactification [30] it is possible to make more concrete statements. In this case the moduli dependence of the one–loop correction to the gauge coupling \( \Delta_a^g \) has been discussed first in ref. [2]. In particular it has been argued there that only moduli corresponding to a plane which is left unchanged by a subgroup of the orbifold group can enter the corrections. The orbifold sectors which are generated by elements of this subgroup leave intact two space–time supersymmetries and therefore build up an N=2 sector of the theory. For the restricted class of orbifold compactifications where the six–dimensional torus splits into a direct sum, \( T_6 = T_2 \oplus T_4 \), and the fixed plane lies in \( T_2 \), explicit expressions for \( \Delta_a^g \) have been obtained with the help of an auxiliary model involving a two–dimensional torus compactification. The functional dependence of \( \Delta_a^g \) on a generic modulus \( T \) has been found to be an automorphic function of the duality group \( SL(2, \mathbb{Z}) \):

\[
\Delta_a^g \sim \ln[|\eta(T)|^4(T + \bar{T})] ,
\]  

(26)

where \( \eta(T) \) is the Dedekind function. Recently the more general case where the internal torus is generated by a genuine six–dimensional lattice, was presented in ref. [9]. It was shown there that the so–called threshold function \( \Delta_a^g \) is no longer an automorphic function of the modular group \( SL(2, \mathbb{Z}) \) but only of a certain subgroup of \( SL(2, \mathbb{Z}) \).

4 Connection to field theory

We turn now to the discussion of the previous string calculation in terms of an effective Lagrangian. In the field theory two different types of diagrams provide corrections to the S–matrix element \( \tilde{\Delta}_a^D \) of eq. (25): wave function renormalizations \( \delta \Pi_a \) and \( \delta \Pi_D \) in an external gauge and dilaton leg, respectively, as shown in fig. 1 as well as genuine 1PI corrections to the three–particle vertex shown in fig. 2:

\[
\tilde{\Delta}_a^D = (p_1^\nu p_2^\mu - g^{\mu\nu} p_1 \cdot p_2) \left[ \{g_{a,D}^{-2}\}_\text{1–loop} + (2\delta \Pi_a + \delta \Pi_D) \{g_{a,D}^{-2}\}_\text{tree} \right] ,
\]  

(27)
where \( \{g_{a,D}^{-2}\}^{1\text{-loop}} \) and \( \{g_{a,D}^{-2}\}^{\text{tree}} \) denote the effective linear coupling of the dilaton to gauge bosons at one–loop and tree–level, respectively. In contrast to the analogous relation for the moduli dependent correction to the gauge coupling \[2\], where the second term is absent since \( \{g_{a,T}^{-2}\}^{\text{tree}} = 0 \) (\( T \) denotes the moduli field), the loops in the external legs now contribute because the dilaton couples to the gauge bosons at tree–level:

\[
\{g_{a}^{-2}\}^{\text{tree}} = e^{-\sqrt{2}kD}.
\]

To assign the string correction \( \tilde{\Delta}_{a,D} \) to the two terms in eq. \[27\] let us focus on the gauge group dependent part \( \sim Q_{a}^{2} \) in eq. \[19\]. If we consider only differences between the corrections to the gauge couplings of different gauge groups, as it was introduced in ref. \[2\] to get rid of the modular non–invariant regulator terms of the correlator of the Kac–Moody currents, the term \( \sim \delta\Pi_{D} \) drops. The expression for the wave–function renormalization factor \( \delta\Pi_{a} \) can be extracted from ref. \[1\] and multiplying it by the tree–level coupling \( \{g_{a,D}^{-2}\}^{\text{tree}} \) of eq. \[28\] we reproduce exactly the string correction \( \tilde{\Delta}_{a,D} \). In conclusion there is no one–loop correction to the effective dilaton–gauge boson coupling \( \{g_{a,D}^{-2}\}^{1\text{-loop}} \) at all.

On the other side we have to consider the corrections calculated in field theory as well as the presence of the dilaton dependent cutoff \( \sim M_{S} \). From the discussion of section 2 we expect that the complete effective Lagrangian should depend on the dilaton through \( M_{S} \). Obviously this will be a subject involving only massless string modes. Because of the lack of information about the explicit form of the effective theory in general four–dimensional string models, we have to restrict our investigation from now on to the case of orbifold compactifications. The usual representation of the dilaton field in the supergravity theory is to use a chiral superfield \( S \) with a lowest scalar component given by \( \text{Re } s = e^{-\sqrt{2}kD} \). There are two sources of a coupling of the superfield \( S \) to the gauge superfields at one–loop. The first is due to the tree–level coupling contained in the supersymmetric completion of a Lagrangian term corresponding to eq.\[28\]. The second one is the presence of \( S \) in the Kähler connection, which in turn appears in the covariant derivatives of the fermionic fields \[7, 13\]. The Kähler connection is related to the U(1) rotations on the fermions which are necessary to make the supersymmetric Lagrangian invariant under Kähler transformations. If this U(1) symmetry is gauged at the superfield level, the arising theory is called Kähler superspace \[34\]. The coupling of the Kähler connection and on–shell gauge bosons generated by triangle graphs with fermions running in the loop can be represented by a non–local contribution to the effective Lagrangian \[32, 34, 4, 5\]:

\[
L_{nl} = \frac{1}{4} \frac{1}{(4\pi)^2} \int d^{4}\theta \sum_{a}(W^{a}W^{a})\frac{D^{2}}{\Box} \frac{1}{2} \left( c^{a}_{\text{adj}} - \sum_{R} c^{a}_{R} \right) \kappa^{2} K + \text{h.c.}
\]

\[
= -\frac{1}{4} \frac{1}{(4\pi)^2} \int d^{4}\theta \sum_{a}(W^{a}W^{a})\frac{D^{2}}{\Box} \frac{1}{2} \left( c^{a}_{\text{adj}} - \sum_{R} c^{a}_{R} \right) \kappa^{2} K \tag{29}
\]
where we have only indicated the pure dilaton and moduli dependent part of the Kähler potential. $W^a$ is the chiral superfield containing the Yang–Mills field strength $F_{\mu\nu}^a$ of the gauge group labeled by $a$, $c_{adj}^a$ ($c_R^a$) is the quadratic Casimir operator in the adjoint (relevant matter) representation and the sum runs over the matter representations $R$. In general it is also possible for a chiral superfield to couple to the fermions through the covariantization of the derivatives with respect to sigma–model general coordinate transformations \[7, 13\]. Since this coupling is absent for the superfield $S$ in string inspired supergravity theories because there are simply no charged fields with a dilaton dependent metric at the tree–level, it is irrelevant for this discussion. Nevertheless we have yet to take into account the above mentioned contribution due to the tree–level coupling of $S$ to the gauge bosons. In ref. \[13\] it has been shown that this term gives an expression similar to that of eq. (29) with a group theoretical factor exactly of the size to complete the coefficient in eq. (29) to that of the full $\beta$–function. In this way one gets the following algebraic equation for the effective gauge couplings, valid to all orders and written in general as a function of a chiral superfield $\Phi$:

\[
  g_a^{-2}(\Phi, \bar{\Phi}, p) = \Re f_a(\Phi) + \frac{b_a}{8\pi^2} \ln \frac{M_{pl}}{p} - \frac{\kappa^2}{16\pi^2} \left( c_{adj}^a - \sum_R c_R^a \right) K(\Phi, \bar{\Phi}) + \frac{c_{adj}^a}{8\pi^2} \ln g_a^{-2}(\Phi, \bar{\Phi}, p) - \frac{1}{8\pi^2} \sum_R c_R^a \ln \det Z_R(\Phi, \bar{\Phi}, p),
\]

(30)

where $p$ is the renormalization scale, $b_a$ is the group theoretical coefficient of the $\beta$–function $\beta_a = b_a g^3_a/16\pi$ and $Z_R$ is the field dependent Kähler metric for the matter fields in the representation $R$. If we insert the tree–level Kähler potential for $S$ as well as the tree–level gauge coupling and furthermore replace Re $S$ by its vev, this term has the effect to shift the scale at which the couplings begin to run from $M_{pl}$ to $M_S$:

\[
  b_a \ln \frac{M_{pl}^2}{\mu^2} - b_a \ln \langle \text{Re } S \rangle = b_a \ln \frac{M_S^2}{\mu^2},
\]

(31)

due to the relation $M_{pl}/M_S = \langle \text{Re } s \rangle^{1/2}$. Here $s$ denotes the lowest component of the superfield $S$. This is the result expected from the discussion of the previous sections. The fact that the parity even piece of the non–local Lagrangian (29) can be interpreted as a supersymmetric cutoff was first pointed out in ref. \[13\].

There are two further points worth mentioning: first we emphasize that in general there could be yet a holomorphic piece in the one–loop correction to the gauge coupling, which contains the field $S$. This would correspond to a one–loop piece of the $f$–function. From the previous results it is clear that this term is absent in the string effective theory, $\Re f_a^{-\text{loop}}(S) = 0$. Secondly it is nice to record the consistent normalization of the $T$ field. At first sight it looks conspicuous that the
T field in the field theory has to be normalized in units of $M_{pl}$, while that appearing in the string calculations is apparently normalized in units of $M_S$. But it is easy to convince oneself that in the string theory an expansion of the field–dependent gauge coupling in powers of $T$ includes a factor $g$ for each field $T$. Therefore the precise expression for the $T$ dependent functions appearing in the string calculations of refs. [3] is obtained by the replacements $T[M_S] \rightarrow gT[M_S] = T[M_{pl}]$ in the expressions given therein, in agreement with the field–theoretical description. Here we have used $g = (\text{Re } s)^{-1/2}$.

Now let us take a look at the coupling of the antisymmetric tensor field $B_{\mu \nu}$ to the gauge bosons at one–loop, which is related to that of the dilaton field by supersymmetry. From eq. (17) we find $\Delta_a^B = -\Delta_a^g$. Again there is a tree–level coupling due to the presence of the Chern–Simons term in the definition of the field strength $H$:

$$H_{\mu \nu \rho} = \partial_{[\mu}B_{\nu \rho]} - \frac{\kappa}{4} \text{tr} \left( A_{[\mu}F_{\nu \rho]} - \frac{g}{3} A_{[\mu}A_{\nu A_{\rho]} } \right).$$

(32)

Subtracting from $\Delta_a^B$ the contribution of the loops in the external legs, as we did before for the case of the dilaton field, we find again a zero result. That is, the one–loop S–matrix element in string theory is entirely reproduced by the diagram shown in fig. 3.

In an unnoticed way we have run into trouble. On one hand we have shown that there are no one–loop corrections to the three particle vertices involving two gauge bosons and a dilaton or an antisymmetric tensor field in the effective string theory. On the other hand there is the expression in eq. (29) which shifts nicely the scale $M_{pl}$ to $M_S$ after it is supplemented with the contribution related to the tree–level coupling of $S$ to the gauge bosons and if the field $(S + \bar{S})$ is replaced by its vev. However, if we make an expansion around this vev, the same term induces linear couplings of two gauge bosons and a dilaton or an antisymmetric tensor field:

$$\mathcal{L}_{nl} = -\frac{1}{4} \frac{1}{(4\pi)^2} \int d^2 \theta \sum_a (W^a W^a) \tilde{D}^2 D^2 \frac{1}{2} b_a \times \left[ \langle \ln (S + \bar{S}) \rangle + \frac{1}{\langle S + \bar{S} \rangle} (\tilde{S} + \bar{\tilde{S}}) + \ldots \right] + \text{h.c.}$$

(33)

where the tilde on $\tilde{S}$ identifies the quantum field. This is in apparent contradiction with the situation we have found in string theory. We will argue that this problem appears due to an inconsistent formulation of the effective theory in terms of the chiral superfield $S$.

In fact it is known that the four–dimensional N=1 supersymmetric effective target space superstring theories correspond to so–called new–minimal supergravities
with a linear multiplet containing the dilaton field \[34, 35\]. The general coupling of a linear multiplet to the supergravity–matter system and Yang–Mills fields was derived in \[36\] and recently the complete supersymmetric action was constructed \[37\].

It is well–known that a supergravity theory including a linear multiplet can be related to a theory with only chiral matter fields via a duality transformation \[36\]. Such a transformation will connect an effective theory in the linear multiplet formalism to another one in the chiral multiplet formalism in a classical sense. Therefore one should not start to compute quantum corrections of the two theories related in this way and compare the results with each other\footnote{We thank J.P. Derendinger for an explanation on this issue.}. This is exactly what happens in the above mentioned problem: we have compared the quantum corrections of a theory in the chiral multiplet formalism obtained from a tree–level duality transformation to that of the one–loop corrections in a theory formulated with a linear multiplet. Indeed if we replace the Kähler potential of the dilaton superfield \(S\), \(K(S, \bar{S}) = -\kappa^{-2} \ln(S + \bar{S})\) by that of the linear multiplet containing the dilaton, \(K(L) = \kappa^{-2} \ln L\) we get

\[
\mathcal{L}_{nl} = \frac{1}{4 (4\pi)^2} \int d^2 \theta \sum_a (W^a W^a) \frac{\bar{D}^2 D^2}{\Box} \frac{1}{2} b_a \\
\times \left[ \langle \ln L \rangle + \frac{1}{\langle L \rangle} L + \ldots \right] + \text{h.c.} \\
= -\frac{1}{8 (4\pi)^2} \sum_a b_a \left[ F^a F^a \left( \langle \ln L \rangle + \ldots \right) - \bar{F}^a \bar{F}^a \left( 0 + \ldots \right) \right].
\]  

Here we have used \(D^2 L = \bar{D}^2 L = 0 + \ldots\), where the dots represent terms involving field strengths arising from the modified linearity conditions

\[
(D_{\alpha} D^{\alpha} - 8R) L = 2k \text{tr} (W^a W^a), \\
(D^a D_a - 8 R^a) L = 2k \text{tr} (\bar{W}^a \bar{W}^a),
\]  

where \(R\) is one of the torsion superfields of supergravity. We have written only the coupling to the gauge field strength on the r.h.s. of (35) which corresponds to adding the Yang–Mills Chern–Simons form to the field strength of the antisymmetric tensor. In general also Chern–Simons terms for the Lorentz and U(1) group are necessary to get an anomaly free theory. From eq. (34) we can see that there are no linear couplings of dilaton or antisymmetric tensor fields to the gauge bosons in the linear multiplet formalism.

At first sight this seems to be a strange result because the Kähler connection in the linear multiplet formalism contains a term \[37\]

\[
V^L_\mu = \ldots - \frac{i}{8 \lambda} \epsilon_{\mu \nu \rho \lambda} \partial^\nu B^\rho \lambda. \tag{36}
\]
Therefore $B_{\mu\nu}$ indeed couples to the fermions analogue to the coupling of the corresponding pseudoscalar $a = \text{Im} s$ in the chiral multiplet formalism:

$$V^C = ... - \frac{i}{2} \frac{A^a}{S + \bar{S}} \partial_\mu a.$$  \hspace{1cm} (37)

We have seen in eq. (33) that this coupling causes a non–vanishing one–loop correction to the two gauge boson – pseudoscalar vertex.

To analyze the different outcome of the calculation of the triangle diagrams in the two formalisms we write down the action of the operator $\Box^{-1} \bar{D}^2 D^2$ on $S$ and $L$, respectively:

$$\Box^{-1} \bar{D}^2 D^2 S \big|_{\theta = \bar{\theta} = 0} = \Box^{-1} 16 \Box (\text{Re} s + ia),$$

$$\Box^{-1} \bar{D}^2 D^2 L \big|_{\theta = \bar{\theta} = 0} = \Box^{-1} 4i \partial_\mu A^\mu,$$  \hspace{1cm} (38)

where $A_\mu = \epsilon_{\mu\nu\rho\lambda} \partial^\nu B^{\rho\lambda}$. In the first line of eq. (38) the operator $\Box^{-1}$ is ill–defined on–shell, but since it is off–shell everywhere equal to one, it is one also on–shell, by analyticity. The contrary applies to the second line of eq. (38). First note that the dilaton dependent term has already dropped out because of an algebraic identity. Furthermore the expression $\epsilon_{\mu\nu\rho\lambda} p^\mu p^\nu B^{\rho\lambda}$ vanishes for any (complex) momentum $p_\mu$. Therefore, by analyticity, $\Box^{-1} \partial_\mu A^\mu$ has to be zero also on–shell. If we replace $A_\mu$ by the duality transformed expression $\partial_\mu a$, the algebraic zero reflecting the gauge invariance of the antisymmetric tensor field strength turns into a weaker on–shell zero of the Laplacian acting on the pseudoscalar field $a$. This fact is in the heart of the difference between the non–local Lagrangians of eqs. (33) and (34).

Of course this effect can be seen also at the component level and is not restricted to a supersymmetric model. The explicit expression for the triangle graph with two vector gauge bosons $A_1^\sigma$, $A_2^\rho$ and one axial gauge boson $A_3^\mu$ was given long ago in ref. [38]:

$$R_{\sigma\rho\mu}(k_1, k_2) = A_3 \left( k_1 \cdot k_2 k_1^\sigma \epsilon_{\tau\sigma\rho\mu} - k_1 \cdot k_2 k_2^\tau \epsilon_{\tau\sigma\rho\mu} + k_1^\rho k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho\mu} - k_2^\rho k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho\mu} \right)$$

$$+ A_4 \left( k_2^\rho k_1^\tau \epsilon_{\tau\sigma\rho\mu} - k_1^\rho k_2^\tau \epsilon_{\tau\sigma\rho\mu} + k_2^\rho k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho\mu} - k_1^\rho k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho\mu} \right),$$  \hspace{1cm} (39)

where $k_1$ and $k_2$ are the momenta of the vector gauge bosons and $A_3$ and $A_4$ are some integrals over Feynman parameters. The ambiguity of the linearly divergent diagram has been fixed by the requirement of gauge invariance in the vector channel. With the handicap of massless fermions in the loop it is straightforward to show the following relations for on–shell vector bosons $A_1^\sigma$ and $A_2^\rho$:

$$R_{\sigma\rho\mu} A^\mu = 0$$

$$R_{\sigma\rho\mu} A^\mu = 8 \pi^2 k_1^\xi k_2^\tau \epsilon_{\xi\tau\sigma\rho\mu}$$  \hspace{1cm} for $A^\mu = \epsilon^{\mu\nu\lambda\kappa} \partial_\nu B^{\lambda\kappa}$.

In agreement with the previous result in the supersymmetric formulation there is only a non–vanishing coupling if the axial vector boson is written in terms of a pseudoscalar.
In conclusion we have shown that the one–loop gauge coupling in string effective theories depends on the dilaton vev in a way determined by the dilaton dependence of the cutoff scale $M_S$. This result can be derived by symmetry considerations and applies to the holomorphic as well as to the non–holomorphic piece of the one–loop correction. The identification of the dilaton dependence as a field–theoretical effect shows that it is entirely due to massless fields and, in particular, that there is no contribution from massive modes providing a dilaton dependent one–loop correction to the holomorphic $f$–function. Calculating a three point string amplitude involving two gauge bosons and one dilaton we have shown that there is no one–loop correction to the three particle vertex in the 1PI vertices generating effective Lagrangian. The same statement applies to the coupling of the antisymmetric tensor field to the gauge bosons. This result is not reproduced by the usual non–local term in the effective Lagrangian in the formalism, where the dilaton is in a chiral multiplet. On the other hand the analogue effective theory written in terms of a linear multiplet is in full agreement with string theory.

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Appendix

A Conventions and normalizations

The overall normalizations in the operator formalism can be fixed by comparison with results from a path integral calculation. In our conventions the correlation function for the bosonic fields (with $\alpha' = \frac{1}{2}$) reads

$$\langle X^\mu(z_1, \bar{z}_1) X^\nu(z_2, \bar{z}_2) \rangle = -\frac{1}{4} \eta^{\mu\nu} \ln |\chi(z_{12}, \tau)|^2 , \quad (A.1)$$

where $z_{12} = z_1 - z_2$ and

$$\chi_{ij} \equiv \chi(z_{ij}) = 2\pi \exp \left[ -\frac{\pi (\text{Im} z_{ij})^2}{\text{Im} \tau} \right] \left| \frac{\vartheta_1(z_{ij}|\tau)}{\vartheta_1'(0|\tau)} \right| . \quad (A.2)$$

$\vartheta_1(z|\tau)$ is one of the Riemann theta–functions. Correlation functions of the fields $\partial X, \bar{\partial} X$ can be obtained by simply taking derivatives of (A.1) except for $\langle \partial X \bar{\partial} X \rangle$ as discussed in the text. They obey the identities

$$\int d^2 z \, \partial \ln |\chi(z)| = 0 , \quad \int d^2 z \, \partial_z^2 \ln |\chi(z)| = 0 . \quad (A.3)$$
The fermionic correlation function is
\[ G_{12}(s) \equiv \langle \psi_1 \psi_2 \rangle_s = \frac{1}{4} \frac{\vartheta_\alpha(z_{12}|\tau)\vartheta'_\alpha(0|\tau)}{\vartheta_\alpha(z_{12}|\tau)\vartheta_\alpha(0|\tau)} , \tag{A.4} \]
where \( \alpha = 2,3,4 \) for \( (s_1,s_2) = (1,0), (0,0), (0,1) \), respectively. Furthermore we have defined \( \vartheta'_\alpha(0|\tau) = \partial_z \vartheta_\alpha(z|\tau)|_{z=0} \). If there is no additional spin structure dependence but that of \( (G_{12}(s))^2 \) only the spin structure dependent terms inside this correlation function will survive and we can replace
\[ (4G_{12}(s))^2 = -2\partial^2_{12} \ln \chi_{12} - \hat{G}_2(\tau) - e_{\alpha-1}(\tau) \longrightarrow 4\pi i \frac{d}{d\tau} \ln \frac{\vartheta_\alpha(0|\tau)}{\eta(\tau)} , \tag{A.5} \]
where \( \hat{G}_2(\tau) \) is the Eisenstein function of modular weight two and the \( e_{\alpha-1}(\tau) \) are defined in terms of the Weierstrass function.

The correlation function for the Kac–Moody currents is given by
\[ \langle J^a A^b \rangle = \frac{1}{16} [ -k_a \delta^{ab} \partial^2_{12} \ln \Theta_1(\bar{z}_{12} | \bar{\tau}) - 4\pi^2 Q_a Q_b ] , \tag{A.6} \]
where \( k_a \) is the level of the Kac–Moody algebra and the \( Q_a \)'s are the charges of the propagating states.

The four-dimensional free plus ghost part of the partition function is given by
\[ Z_{\psi}(\tau, \bar{\tau}, s) = \frac{\vartheta_\alpha(0|\tau)}{\eta(\tau)} , \quad Z_B(\tau, \bar{\tau}) = \frac{1}{|\eta(\tau)|^4} \frac{1}{(2\pi \mathrm{Im} \tau)^2} , \tag{A.7} \]
for the fermions and bosons, respectively. In addition there is a contribution \( Z_{\text{int}} \) from the internal \( (c,\bar{c}) = (9,22) \) superconformal theory.

The contraction of six fermions inside a correlator yields the kinematical factor
\[ t(1,2,3) = \epsilon_1 \cdot k_3 \epsilon_2 \cdot k_1 \epsilon_3 \cdot k_2 - \epsilon_1 \cdot k_2 \epsilon_2 \cdot k_3 \epsilon_3 \cdot k_1 + \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_1 k_2 \cdot k_3 - \epsilon_3 \cdot k_2 k_1 \cdot k_3 + \epsilon_2 \cdot \epsilon_3 \epsilon_1 \cdot k_3 k_1 \cdot k_2 - \epsilon_1 \cdot k_1 k_2 \cdot k_3 + \epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot k_1 k_2 \cdot k_3 , \tag{A.8} \]

**B Background-field calculation**

The vertex operators for the background fields can be obtained from (10) by the replacements
\[ \epsilon_\mu e^{ikX} \rightarrow A_\mu(X) , \quad \epsilon_\mu^\nu e^{ikX} \rightarrow G_{\mu\nu}(X) . \tag{B.1} \]

\[ ^7 \text{For the usual conventions see [39].} \]
The fields $A_{\mu}(X)$ and $G_{\mu\nu}(X)$ are solutions of the classical equations of motions. Note that the $X$–dependence of the background fields has to be taken into account when the contractions on the world–sheet are performed. Keeping only the first terms in an expansion gives

$$V_0^{A}(z_1, \bar{z}_1) = \frac{4g}{\pi} J_1^\mu (A_{1\mu} \partial X_1^\mu + \partial_{\nu} A_{1\mu} X_1^\nu \partial X_1^\mu + \psi_1^\mu \psi_1^\nu \partial_{\nu} A_{1\mu}) ,$$

$$V_0^{G}(z_1, \bar{z}_1) = \frac{8\kappa}{\pi} \bar{\partial} X_1^\mu (\partial X_1^\nu G_{1\mu\nu}^{(1)} + \partial X_1^\nu X^\lambda G_{1\mu\nu}^{(1)} + \psi_1^\nu \psi_1^\lambda \partial_{\lambda} G_{1\mu\nu}^{(1)}) .$$ (B.2)

Only three terms in the correlator $\langle V_0^{G} V_0^{A} V_0^{A} \rangle$ yield non–vanishing contributions of $O(\alpha'^0)$. They are

$$\langle \bar{\partial} X_1^\mu \partial X_1^\nu \rangle \langle \psi_2^\gamma \psi_2^\beta \psi_3^\alpha \psi_3^\beta \rangle \langle J_2 J_3 \rangle G_{1\mu\nu}^{(1)} \partial_{\gamma} A_{2\rho} \partial_{\alpha} A_{2\beta} ,$$

$$\langle \bar{\partial} X_1^\mu \partial X_3^\rho \rangle \langle \psi_1^\lambda \psi_2^\alpha \psi_2^\beta \psi_3^\gamma \psi_3^\beta \rangle \langle J_2 J_3 \rangle A_{3\alpha} \partial_{\lambda} G_{1\mu\nu}^{(1)} \partial_{\gamma} A_{2\rho} ,$$

$$\langle \bar{\partial} X_1^\mu \partial X_2^\rho \rangle \langle \psi_1^\lambda \psi_2^\gamma \psi_3^\beta \psi_3^\beta \rangle \langle J_2 J_3 \rangle A_{2\rho} \partial_{\lambda} G_{1\mu\nu}^{(1)} \partial_{\alpha} A_{3\beta} .$$ (B.3)

The further calculation is quite similar to the case involving the usual vertex operators. The second and third term in (B.3) supply the amplitude where $G_{\mu\nu}$ refers to the antisymmetric tensor. If we restrict our attention to the dilaton field, (B.3) results in

$$+ \frac{\kappa}{4} \text{tr} G_{1\mu\nu}^{(1)} F_{\alpha\beta}^{a} F_{\alpha\beta}^{a} \left( \frac{g^2}{16\pi^2} \int_{\tau \in \Gamma} \frac{d^2\tau}{\text{Im} \, \tau} B_{a} \right) ,$$ (B.4)

in agreement with (24).
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Figure Captions

**Figure 1:** One–loop contributions to the three–particle S–matrix element with one dilaton and two gauge bosons arising from loops in the external legs. Wavy lines denote gauge bosons while the dashed line represents the dilaton.

**Figure 2:** One–loop correction to the same S–matrix element associated to the 1PI three–particle vertex. The wavy–solid line denotes any massless field.

**Figure 3:** Contributions to the one–loop S–matrix element with one antisymmetric tensor field and two gauge bosons. The double line represents the antisymmetric tensor field.