On Generalized Axion Reductions

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ABSTRACT

Recently interest in using generalized reductions to construct massive supergravity theories has been revived in the context of M-theory and superstring theory. These compactifications produce mass parameters by introducing a linear dependence on internal coordinates in various axionic fields. Here we point out that by extending the form of this simple ansatz, it is always possible to introduce the various mass parameters simultaneously. This suggests that the various “distinct” massive supergravities in the literature should all be a part of a single massive theory.
Dimensional reduction provides an important window on the duality relations amongst the various superstring theories, as well as eleven-dimensional supergravity. Recently generalized Scherk-Schwarz reductions\cite{1} have received a renewed interest\cite{2, 3, 4, 5, 6}. This activity began with the remarkable discovery\cite{2} that the massive IIa supergravity of Romans\cite{7} is related by T-duality to a Scherk-Schwarz compactification of the massless IIb theory. This result then provides a massive extension of the standard T-duality between type IIa and IIb superstring theories compactified on $S^1\cite{8}$. Further, renewed interest stems from the recent investigations of extended objects in string theory. Massive supergravities are particularly relevant in the case of domain walls\cite{3, 4}. Some earlier investigations of Scherk-Schwarz reductions in string theory were made both at the level of the low-energy supergravity action\cite{9}, and at the level of the world-sheet conformal field theory\cite{10}.

The key to the generalized Scherk-Schwarz reductions\cite{1, 6} is that, using global symmetries arising in a compactification, the fields may be given a (specific) dependence on the internal coordinates. However, the resulting theory is still independent of all of the internal coordinates. The recent discussions\cite{2, 3, 4, 5} in the context of low-energy string or M-theory focus on toroidal compactifications and various axionic symmetries, \textit{i.e.}, constant shifts of certain scalar fields. In the simplest cases then, the axions appear in the action covered by derivatives, \textit{i.e.}, the scalar field $\chi$ appears everywhere in the action only as $\partial_\mu \chi$, or in form notation as $d\chi$. If upon compactification such axions are given a linear dependence on the internal coordinates, only the slope of this dependence appears in the reduced action\cite{3}, \textit{i.e.},

$$\chi(x, z) = \chi(x) + m z \quad \longrightarrow \quad d\chi(x, z) = d\chi(x) + m dz$$  \hspace{1cm} (1)

The slope parameters then play the role of masses in the compactified theory.

A fundamental axion scalar appears in the ten-dimensional IIb supergravity and plays the central role in the T-duality to the massive IIa theory\cite{2}. In general, however, the axions of interest arise in a partially reduced theory as internal components of gauge fields, form-fields or the metric. The translation symmetry of these scalars is then a residue of a local gauge invariance in the uncompactified theory. Introducing the linear ansatz can also then be regarded as giving an expectation to certain field strengths on the internal space, or introducing a twist or curvature in the internal geometry. Further, these axions may have nonderivative couplings through descendants of ‘Chern-Simons’ interactions in the unreduced theory. Introducing the linear ansatz\cite{1} then requires certain field redefinitions to cover the appropriate scalars with derivatives\cite{3}. As a result, however, a conflict may arise in simultaneously introducing the linear ansatz for several different axions. Below, we show that this conflict can be resolved by the introduction of a slightly generalized ansatz, which is quadratic (or higher order) in the internal coordinates. Field redefinitions may be found to reduce the internal dependence to a linear one, although still not of the simple form given in eq. (1).
In this letter, we make the discussion explicit by referring to a specific example considered in ref. [3]. Cowdall et al. [3] applied Scherk-Schwarz reductions to eleven-dimensional supergravity to produce a variety of maximally-supersymmetric massive supergravities in $D \leq 8$. They discussed the case of simultaneously applying the linear ansatz (1) to several axions, but were limited by the problem discussed above. The present discussion provides an explicit extension of their results, and generalizing this approach to other cases is a straightforward exercise.

In the toroidal compactification of eleven-dimensional supergravity to $D = 8$, three axions $A_0^{(ij)}$ (with $i,j = 1,2,3$ and $i < j$) appear in the off-diagonal components of the internal metric. The appropriate dreibein on the internal torus may be written (using the notation of [11]):

$$
e^A_M = \begin{pmatrix}
e^{-\phi_1} & e^{-\phi_1}A_0^{(12)} & e^{-\phi_1}A_0^{(13)} \\
0 & e^{-\phi_2} & e^{-\phi_2}A_0^{(23)} \\
0 & 0 & e^{-\phi_3}
\end{pmatrix}$$

(2)

where $A$ and $M$ denote the tangent-space and holonomic indices, respectively. The kinetic terms of the axions are governed by the “field strengths”

$$F_1^{(12)} = dA_0^{(12)} \quad F_1^{(13)} = dA_0^{(13)} - A_0^{(23)} dA_0^{(12)} \quad F_1^{(23)} = dA_0^{(23)}$$

(3)

It is clear here that upon compactifying to $D = 7$ one can straightforwardly introduce the linear ansatz (1) for $A_0^{(12)}$ and $A_0^{(13)}$. To apply this ansatz to $A_0^{(23)}$, one must redefine the fields [3] as $\tilde{A}_0^{(13)} = A_0^{(13)} - A_0^{(23)} A_0^{(12)}$, such that

$$F_1^{(13)} = d\tilde{A}_0^{(13)} + A_0^{(12)} dA_0^{(23)}.$$  \hspace{1cm} (4)

Now in reducing to $D = 7$, one can apply the ansatz

$$A_0^{(23)}(x, z) = A_0^{(23)}(x) + m^{(23)} z.$$ \hspace{1cm} (5)

However, from eq. (3), one sees that this ansatz may no longer be applied to $A_0^{(12)}$.

As an alternative to making the above field redefinition, one could extend the compactification ansatz slightly as follows:

$$A_0^{(12)}(x, z) = A_0^{(12)}(x)$$

$$A_0^{(23)}(x, z) = A_0^{(23)}(x) + m^{(23)} z$$

$$A_0^{(13)}(x, z) = A_0^{(13)}(x) + m^{(23)} z A_0^{(12)}(x)$$

(6)

The additional term added to $A_0^{(13)}$ is a reflection of the fact that the axion shift symmetry of $A_0^{(23)}$ in the original theory is accompanied by a compensating shift of

\footnote{We have simplified this notation with respect to the scalars $\phi_i$, which do not play an important role in the following.}
\( \mathcal{A}_0^{(13)} \) so as to leave \( F_1^{(13)} \) invariant. We see by replacing (3) into the original expression for the field strengths (3) that all of the explicit \( z \) dependence cancels. (Alternatively, we note that this ansatz is identical to the original one in which implicitly we have reduced the new axion as \( \tilde{A}_0^{(13)}(x, z) = A_0^{(13)}(x) \).) While the extended ansatz (6) does not resolve the problem of simultaneously introducing two mass parameters, \( m^{(23)} \) and \( m^{(12)} \), it does show that explicitly covering the axions with derivatives is not essential to introducing the mass parameters. This was anticipated in ref. [6], where it was noted that the Scherk-Schwarz construction [1] applies for general global symmetries. Thus one might believe that a modified ansatz would allow for the simultaneous inclusion of both parameters. While we originally constructed such an extended ansatz by trial and error, in fact, it appears quite naturally using the full formalism originally developed by Scherk and Schwarz [1].

Within the Scherk-Schwarz formalism, one begins by identifying the relevant global symmetries. Here, they are a part of the \( SL(3, R) \) symmetry acting on the internal three-torus, which acts on the dreibein (2) as \( e^{A_M} \rightarrow e^{A_N} T^N_M \). The translations of the axions can be identified as the three transformations with generators

\[
M^{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M^{(13)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M^{(23)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\]

For example, \( e^{A_M} \rightarrow e^{A_N} \exp[\lambda M^{(12)}]^N_M \) accomplishes a shift \( A_0^{(12)} \rightarrow A_0^{(12)} + \lambda \). Note that \( M^{(23)} \) produces \( A_0^{(13)} \rightarrow A_0^{(13)} + \lambda A_0^{(12)} \). A distinguishing property of these three generators (7) is that they are nilpotent.

Now in the Scherk-Schwarz reduction [1] to \( D = 7 \), one introduces the following specific dependence on the new internal coordinate \( z \) into the dreibein (2):

\[
e^{A_N}(x, z) = e^{A_N}(x) U(z)^N_M = e^{A_N}(x) \exp[M z]^N_M
\]

where \( M = \sum m^{(ij)} M^{(ij)} \). If we consider only a single nonvanishing mass parameter at a time, it is clear that this ansatz reproduces the usual linear ansatz discussed above because the individual generators are nilpotent, i.e., the exponential reduces to \( 1 + M z \). However, in the case that \( m^{(12)} \) and \( m^{(23)} \) are simultaneously chosen to be nonvanishing, \( (M)^2 = m^{(12)} m^{(23)} M^{(13)} \neq 0 \) while \( (M)^3 = 0 \). Thus in this situation the linear ansatz is naturally extended to one quadratic in the internal coordinate \( z \).

Explicitly the axions are chosen as:

\[
\begin{align*}
\mathcal{A}_0^{(12)}(x, z) &= \mathcal{A}_0^{(12)}(x) + m^{(12)} z \\
\mathcal{A}_0^{(23)}(x, z) &= \mathcal{A}_0^{(23)}(x) + m^{(23)} z \\
\mathcal{A}_0^{(13)}(x, z) &= \mathcal{A}_0^{(13)}(x) + m^{(13)} z + m^{(23)} z \mathcal{A}_0^{(12)}(x) + \frac{1}{2} m^{(12)} m^{(23)} z^2
\end{align*}
\]
One can verify that there is no explicit $z$ dependence in $F_{1}^{(13)}$ with this ansatz. Thus within the full Scherk-Schwarz framework[14], one finds that there is no obstacle to turning on all of the mass parameters simultaneously.

These axions also couple to other fields in the $D = 8$ supergravity, and one must also choose a consistent compactification ansatz to ensure that the corresponding field strengths do not introduce a $z$ dependence in the compactified theory. The Scherk-Schwarz formalism provides a precise prescription to accomplish this result. Essentially any of the fields carrying internal holonomic indices are also contracted with the same matrix $U$ appearing in eq. (8). In the present case of the compactification of eleven-dimensional supergravity, one must consider the components of the three-form potential, e.g., $A_{mN_{1}N_{2}}(x) U(z)^{N_{1}M_{1}} U(z)^{N_{2}M_{2}}$ and $A_{m_{1}m_{2}N}(x) U(z)^{N_{M}}$. Following the notation of [11], these correspond to the one-forms $A_{1}^{(ij)}$ and two-forms $A_{2}^{(i)}$. In the end, one arrives at the following reduction ansatz for the one-forms:

$$
A_{1}^{(12)}(x, z) = \left( \begin{array}{c} A_{1}^{(12)}(x) \\ A_{0}^{(12)}(x) \end{array} \right) \quad \quad A_{1}^{(13)}(x, z) = \left( \begin{array}{c} A_{1}^{(13)}(x) + m_{(23)} z A_{1}^{(12)}(x) \\ A_{0}^{(13)}(x) + m_{(23)} z A_{0}^{(12)}(x) \end{array} \right)
$$

$$
A_{1}^{(23)}(x, z) = \left( \begin{array}{c} A_{1}^{(23)}(x) + m_{(12)} z A_{1}^{(13)}(x) - \left( m^{(13)} z - \frac{1}{2} m_{(12)} m_{(23)} z^{2} \right) A_{1}^{(12)}(x) \\ A_{0}^{(23)}(x) + m_{(12)} z A_{0}^{(13)}(x) - \left( m^{(13)} z - \frac{1}{2} m_{(12)} m_{(23)} z^{2} \right) A_{0}^{(12)}(x) \end{array} \right)
$$

and for the two-forms:

$$
A_{2}^{(1)}(x, z) = \left( \begin{array}{c} A_{2}^{(1)}(x) \\ A_{1}^{(14)}(x) \end{array} \right) \quad \quad A_{2}^{(2)}(x, z) = \left( \begin{array}{c} A_{2}^{(2)}(x) + m_{(12)} z A_{2}^{(1)}(x) \\ A_{1}^{(24)}(x) + m_{(12)} z A_{1}^{(14)}(x) \end{array} \right)
$$

$$
A_{2}^{(3)}(x, z) = \left( \begin{array}{c} A_{2}^{(3)}(x) + m_{(23)} z A_{2}^{(1)}(x) + \left( m^{(13)} z + \frac{1}{2} m_{(12)} m_{(23)} z^{2} \right) A_{2}^{(1)}(x) \\ A_{1}^{(34)}(x) + m_{(23)} z A_{1}^{(24)}(x) + \left( m^{(13)} z + \frac{1}{2} m_{(12)} m_{(23)} z^{2} \right) A_{1}^{(14)}(x) \end{array} \right)
$$

Here, we see that the quadratic terms make their presence felt in $A_{1}^{(23)}$ and $A_{2}^{(3)}$. Again one may explicitly verify that with this ansatz no dependence on $z$ appears in the corresponding field strengths[15]. One must also consider the axion $A_{0}^{(123)}$ which corresponds to the three-form potential component with three internal indices. Following the Scherk-Schwarz prescription, the compactification ansatz is

$$
A_{N_{1}N_{2}N_{3}}(x) U(z)^{N_{1}M_{1}} U(z)^{N_{2}M_{2}} U(z)^{N_{3}M_{3}} = A_{M_{1}M_{2}M_{3}}(x) \det U .
$$

However, $\det U = 1$, so this scalar is unaffected by the above Scherk-Schwarz ansatz. In more general settings, one could not expect such a cancellation to occur. Further, one might also consider the spacetime vectors arising from the off-diagonal components of the eleven-dimensional metric. However, with the present notation of ref. [14], one does not introduce any $z$ dependence for these vectors — note that the notations of refs. [11] and [11] differ for these fields.

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[15] These field strengths are explicitly listed in ref. [15]. Note that it is important to explicitly retain certain higher order terms, i.e., $F_{3}^{(3)} = dA_{2}^{(3)} - (A_{0}^{(13)} - A_{0}^{(12)} A_{0}^{(23)}) dA_{2}^{(1)} - A_{0}^{(12)} A_{0}^{(2)} + \ldots$. 
It should be clear at this point that if we were to extend this discussion to generalized Scherk-Schwarz compactifications to lower dimensions, the linear axion ansatz would again be extended to include cubic and higher order terms in the internal coordinate. We also note, however, that there does remain the possibility of using field redefinitions to simplify the ansatz to one with only linear dependence on the internal coordinates. In the present example, redefining \( \tilde{A}_0^{(13)} = A_0^{(13)} - \frac{1}{2} A_0^{(12)} A_0^{(23)} \) eliminates the quadratic terms in the compactification ansatz (9) leaving
\[
\tilde{A}_0^{(13)}(x, z) = \tilde{A}_0^{(13)}(x) + m^{(13)} z + \frac{1}{2} m^{(23)} z A_0^{(12)}(x) - \frac{1}{2} m^{(12)} z A_0^{(23)}(x).
\]
Similarly redefining
\[
\tilde{A}_1^{(23)} = A_1^{(23)} - \frac{1}{2} A_0^{(12)} A_0^{(23)} A_1^{(12)},
\]
\[
\tilde{A}_2^{(3)} = A_2^{(3)} - \frac{1}{2} A_0^{(12)} A_0^{(23)} A_2^{(1)}
\]
removes the quadratic terms from eqs. (10) and (11). Although linear in \( z \), this reduction ansatz still does not take the original simple form of eq. (1).

In summary, one finds that there is no obstacle to simultaneously applying a Scherk-Schwarz reduction for all four of the eight-dimensional axions, \( A_0^{(12)}, A_0^{(13)}, A_0^{(23)} \) and \( A_0^{(123)} \) — we have not considered the latter above, but there is no conflict in introducing \( m^{(123)} \) along with any of the other mass parameters \( m \). The conclusion also applies to other compactifications. Hence, the various massive supergravities presented in ref. [3] as distinct theories should actually be regarded as belonging to a single family of theories. After suitable field redefinitions one finds in the present example that generically there is a three-parameter scalar potential involving \( m^{(123)} \), \( m^{(12)} \) and \( m^{(23)} \), while \( m^{(13)} \) can be completely removed from the action. The latter is essentially accomplished by absorbing \( m^{(13)} \) in the expectation value of the axion \( A_0^{(23)} \) (as long as \( m^{(12)} \) is nonvanishing) [3]. Given the Scherk-Schwarz framework, one should be able to extend this theory further by beginning with the massive type IIA theory in ten dimensions and compactifying down to seven dimensions. This would introduce a fourth mass parameter. On the IIb side, this would correspond to a compactification of the ten-dimensional theory on \( T^3 \) which simultaneously introduces a twist in the RR axion along with a twist in the torus geometry, as well as constant internal expectation values of the NS-NS and RR three-form field strengths. This is likely to be the most general massive seven-dimensional supergravity which can be produced using the axionic translation symmetries.

Many aspects of these results apply universally for generalized axionic reductions. Individually, the axionic symmetries will correspond to nilpotent generators of the global symmetry group. Hence the Scherk-Schwarz reduction will coincide with the simple linear ansatz (1) when an individual mass parameter is introduced. However, when several masses are simultaneously turned on, the reduction ansatz may involve
quadratic and higher order terms as in eq. (9). These terms result from the failure of the various nilpotent generators to commute with each other.

It would be interesting to investigate the interplay of U-duality with these Scherk-Schwarz reductions — some aspects of this issue have been addressed recently, in [12]. Introducing the mass parameters generically breaks some part of the global symmetry group which would otherwise appear in the compactified theory. However, it should be possible to write the massive theory in a U-duality invariant form, as long as the symmetry breaking parameters, i.e., the masses, are endowed with the appropriate transformation properties [13]. Thus, as is standard in spontaneous symmetry breaking, a broken symmetry will act as a transformation between distinct massive theories, or distinct “vacua” of the higher dimensional theory. In the present case, the full supergravity duality group in seven-dimensions is $SL(5,\mathbb{R})$. While we have argued that the mass parameters should form a representation of this group, we have only identified four such parameters for the seven-dimensional theory. Thus, the full massive theory must contain new masses beyond those considered here. In the context of the Scherk-Schwarz framework, it may be that the latter are associated with symmetries other than the axionic ones identified here, e.g., eq. (7). Thus one probably has to extend the reduction ansatz to include more general global symmetries [1, 5] to produce a U-duality invariant form. Another aspect of these constructions which would be interesting to study in the context of U-duality is the non-Abelian gauge symmetries which arise in the Scherk-Schwarz reductions [1, 5].

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