NONCOMMUTATIVE SYMPLECTIC GEOMETRY
OF THE ENDOMORPHISM ALGEBRA
OF A VECTOR BUNDLE

ZAKARIA GIUNASHVILI

Abstract. We study noncommutative generalizations of such notions of the classical symplectic geometry as degenerate Poisson structure, Poisson submanifold and quotient manifold, symplectic foliation and symplectic leaf for associative Poisson algebras. We consider these structures for the case of the endomorphism algebra of a vector bundle, and give the full description of the family of Poisson structures for this algebra.

1. Introduction

In this work we describe a noncommutative (algebraic) approach to such geometrical objects as degenerate Poisson structures, Symplectic foliations for Poisson structures, symplectic submanifolds and quotient manifolds. We generalize these notions for the case of noncommutative differential calculus for an associative algebra. Our constructions are based on the definitions of noncommutative submanifold and quotient manifold given in the work [6] of Thierry Masson.

In Section 2 we give a brief overview of the definitions and some facts from the theory of noncommutative submanifolds and quotient manifolds. Also, we introduce a notion of locally proper submanifold algebra which we need further as a noncommutative analogue of a submanifold with codim > 0.

Section 3 is devoted to the demonstration and investigation of noncommutative submanifolds and quotient manifolds for the endomorphism algebra of a finite-dimensional complex vector bundle. There are several works devoted to the noncommutative geometry of this algebra (e.g., [7], [4]). Our purpose, in this section, is to give some description of submanifold ideals for this algebra.

Section 4 is devoted to degenerate Poisson structures on general associative algebra \( A \). We introduce the definition of nondegenerate/degenerate Poisson structure and the notion of locally proper Poisson ideal in \( A \). Such ideal is a noncommutative analogue of Poisson submanifold and when it is maximal it can be considered as a noncommutative symplectic leaf. The existence of a locally proper Poisson ideal is some kind of indicator of degeneracy of a Poisson structure. We study the relation between Poisson structures and symplectic structures which is rather different from the classical case.

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In Section 5 we apply the investigation and results of the previous sections to the endomorphism algebra of a vector bundle: \( \text{End}(E) \). The main result of this section is the full description of the family of Poisson structures on the endomorphism algebra of a vector bundle, with \( \dim(\text{fiber}) \geq 2 \); in this case any Poisson structure on \( \text{End}(E) \) is of the form \( \{ \Phi, \Psi \} = \lambda \cdot [\Phi, \Psi] \), where \( \lambda \) is a smooth function on the base manifold of the vector bundle. The proof is mostly based on the fact that any Poisson structure on the matrix algebra is of the form \( \{ a, b \} = k \cdot [a, b] \) for some constant number \( k \) (the proof of this statement is also given in this section), and the fact that the multiplicative commutator \([\text{End}(E), \text{End}(E)]\) generates the entire algebra \( \text{End}(E) \). It is worth to notice the general proposition presented in this section which states that for any such algebra \( A \), the commutator of which generates the entire algebra \( A \) and any derivation vanishing on the center is inner, a Hamiltonian map is always a \( Z(A) \)-module homomorphism from \( A \) to \( A/Z(A) \), where \( Z(A) \) denotes the center of \( A \). It seems that the center of an associative algebra plays very important role for Poisson structures on this algebra and somehow “fixes” a Poisson structure. In this section we give also the description of the symplectic leaves (in the noncommutative sense) for the algebra \( \text{End}(E) \).

2. Submanifolds and Quotient Manifolds in Noncommutative Differential Calculus: a Brief Review

In this section we give a brief overview of definitions and facts from the theory of noncommutative submanifolds and quotient manifolds in the derivation based noncommutative differential calculus, which is based on the work of Thierry Masson (see [6]).

Let \( A \) be an associative unital complex or real algebra. Let \( I \) be a two-side ideal in \( A \). We denote by \( S \) the quotient algebra \( A/I \) and by \( p \) the quotient map from \( A \) to \( S \). We have the following short exact sequence of algebra homomorphisms

\[
0 \rightarrow I \rightarrow A \xrightarrow{p} S \rightarrow 0
\]

Let \( \text{Der}(A) \) be the Lie algebra of derivations of \( A \). Introduce the following two Lie subalgebras of \( \text{Der}(A) \)

\[
\text{Der}(A, I) = \left\{ X \in \text{Der}(A) \mid X(I) \subset I \right\}
\]

and its subalgebra

\[
\text{Der}(A, I)_0 = \left\{ X \in \text{Der}(A) \mid X(A) \subset I \right\}
\]

The short exact sequence \( 1 \) induces the following exact sequence of Lie algebra homomorphisms

\[
0 \rightarrow \text{Der}(A, I)_0 \rightarrow \text{Der}(A, I) \xrightarrow{\pi} \text{Der}(S) \rightarrow 0
\]

This exact sequence, in general, is not necessarily short.

**Definition 1** (see [9]). The quotient algebra \( S = A/I \) is said to be a submanifold algebra for \( A \) if the homomorphism \( \pi : \text{Der}(A, I) \rightarrow \text{Der}(S) \) is an epimorphism.

In other words, in this case we have the short exact sequence

\[
0 \rightarrow \text{Der}(A, I)_0 \rightarrow \text{Der}(A, I) \xrightarrow{\pi} \text{Der}(S) \rightarrow 0
\]
The ideal $I$ is called the constraint ideal for the submanifold algebra $S$. Sometimes, for brevity, we shall call $I$ a submanifold ideal.

The classical geometric meaning of the Lie algebra $\text{Der}(A, I)$ is the following: it is the set of such vector fields on a smooth manifold $M$ which are tangent to a submanifold $N \subset M$. It is clear that if the submanifold $N$ is such that for each point $x \in N$ we have $T_x(N) = T_x(M)$, then $\text{Der}(A, I) = \text{Der}(A)$. This is the case when $\text{codim}(N) = 0$. For example, if $N$ is an open subset in $M$. Translating this situation on the algebraic language, we introduce the following

**Definition 2.** We call a submanifold algebra locally proper if $\text{Der}(A, I)$ is a proper subalgebra of $\text{Der}(A)$, i.e., $\text{Der}(A, I) \neq \text{Der}(A)$.

Now, let us recall the noncommutative generalization of the notion of quotient manifold. Let $B$ be a subalgebra of $A$. Define the following two Lie subalgebras of $\text{Der}(A)$

$$T(A, B) = \left\{ X \in \text{Der}(A) \mid X(B) \subset B \right\}$$

and its subalgebra

$$V(A, B) = \left\{ X \in \text{Der}(A) \mid X(B) = \{0\} \right\}$$

$V(A, B)$ is a Lie algebra ideal in $T(A, B)$. Moreover, $V(A, B)$ is the kernel of the Lie algebra homomorphism $\rho : T(A, B) \rightarrow \text{Der}(B)$, which is just the restriction map.

Further, for any algebra $A$, we denote by $Z(A)$ the center of $A$.

**Definition 3** (see [R]). The subalgebra $B$ in $A$ is said to be a quotient manifold algebra if the following three conditions are satisfied

q1. $Z(B) = B \cap Z(A)$
q2. the restriction map $\rho : T(A, B) \rightarrow \text{Der}(B)$ is an epimorphism
q3. $B = \left\{ b \in A \mid X(b) = 0, \forall X \in V(A, B) \right\}$

### 3. Submanifolds and Quotient Manifolds for the Endomorphism Algebra of a Vector Bundle

In this section we investigate the endomorphism algebra of a finite-dimensional vector bundle in the framework of the definitions and notions of the previous section.

Let $\pi : E \rightarrow M$ be a finite-dimensional complex (or real) vector bundle over a smooth manifold $M$. We denote the algebra of endomorphisms of this bundle by $\text{End}(E)$. Any element $\Phi \in \text{End}(E)$ can be considered as a section of the bundle of endomorphisms. For any point $x \in M$ we denote by $\Phi_x$ the value of this section at this point: $\Phi_x \in \text{End}(E_x)$, where $E_x = \pi^{-1}(x)$. If $\Phi$ is an element of the center of the algebra $\text{End}(E)$, then for any $x \in M$, the operator $\Phi_x$ is in the center of the algebra $\text{End}(E_x)$. But the algebra $\text{End}(E_x)$ is isomorphic to the algebra of $n \times n$ matrices, where $n = \text{dim}(E_x)$. The center of the matrix algebra is $\mathbb{C} \cdot 1$, where $1$ denotes the identity matrix. Hence we have that the center of the algebra $\text{End}(E)$ is the set

$$\left\{ f \cdot 1 \mid f \in C^\infty(M) \right\} \equiv Z(\text{End}(E))$$
For any associative algebra $A$ its center is invariant under the action of any derivation of this algebra

$$\forall X \in \text{Der}(A), \forall z \in Z(A), \forall a \in A : \ za = az \Rightarrow X(za) = X(az) \Rightarrow \ X(z)a + zX(a) = X(a)z + aX(z) \Rightarrow X(z)a = aX(z) \Rightarrow X(z) \in Z(A)$$

Therefore we have the Lie algebra homomorphism

$$\rho : \text{Der}(A) \rightarrow \text{Der}(Z(A))$$

The Lie algebra of inner derivations of $A$ is defined as

$$\text{Int}(A) = \left\{ \text{ad}(a) = [a, \cdot] : A \rightarrow A \mid a \in A \right\}$$

It is clear that $\text{Int}(A) \subset \ker \rho$. Therefore, in the case when $\rho$ is an epimorphism and $\ker \rho = \text{Int}(A)$, we have that $Z(A)$ is a quotient manifold algebra for $A$ (see [6]).

Consider the above construction for the case when $A = \text{End}(E)$. In this case $\text{Der}(Z(\text{End}(A))) = \text{Der}(C^\infty(M)) = \mathfrak{g}(M)$, where $\mathfrak{g}(M)$ denotes the Lie algebra of vector fields on the manifold $M$. Hence, we have a homomorphism

$$\rho : \text{Der}(\text{End}(E)) \rightarrow \mathfrak{g}(M)$$

**Lemma 1.** The kernel of the homomorphism $\rho : \text{Der}(\text{End}(E)) \rightarrow \mathfrak{g}(M)$ is the Lie algebra of inner derivations of $\text{End}(E)$.

**Proof.** Let $X \in \ker \rho$, then for any $f \in C^\infty(M)$ and $\Phi \in \text{End}(E)$, we have

$$X(f \cdot \Phi) = f \cdot X(\Phi)$$

which implies that $X : \text{End}(E) \rightarrow \text{End}(E)$ is a homomorphism of $C^\infty(M)$-modules. Therefore, $X$ corresponds to some homomorphism of the endomorphism bundle. Hence, its action is pointwise

$$X_m : \text{End}(E_m) \rightarrow \text{End}(E_m), \quad \forall m \in M$$

But as $\text{End}(E_m)$ is isomorphic to the matrix algebra, it has only inner derivations. And so $X$ is an inner derivation. \[\square\]

We denote by $\Gamma(E)$ the space of smooth sections of the vector bundle $(E, M, \pi)$. Let

$$\nabla : \Gamma(E) \rightarrow \Gamma(T^*(M)) \otimes \Gamma(E)$$

be a connection (covariant derivation) on the vector bundle $E$. For any $X \in \mathfrak{g}(M)$ define an operator $D_X : \text{End}(E) \rightarrow \text{End}(E)$ as:

$$(3) \quad D_X(\Phi)(s) = \nabla_X(\Phi(s)) - \Phi(\nabla_X(s)), \quad \forall \Phi \in \text{End}(E) \text{ and } \forall s \in \Gamma(E)$$

**Lemma 2.** For any $X \in \mathfrak{g}(M)$ the operator $D_X$ is a derivation of the algebra $\text{End}(E)$ and $\rho(D_X) = X$. 

Proof. For any pair of endomorphisms $\Phi, \Psi \in \text{End}(E)$ and $s \in \Gamma(E)$ we have the following
\[ D_X(\Phi \Psi)(s) = \nabla_X(\Phi(\Psi(s))) - (\Phi \Psi)(\nabla_X(s)) = \]
\[ = D_X(\Phi)(\Psi(s)) + \Phi(\nabla_X(\Psi(s))) - (\Phi \Psi)(\nabla_X(s)) = \]
\[ = D_X(\Phi)(\Psi(s)) + \Phi(D_X(\Psi)(s) + \Psi(\nabla_X(s))) - (\Phi \Psi)(\nabla_X(s)) = \]
\[ = (D_X(\Phi) \circ \Psi + \Phi \circ D_X(\Psi))(s) \]
which implies that for any vector field $X \in \mathfrak{g}(M)$, the operator $D_X$ on $\text{End}(E)$ is a derivation.

By definition of $\rho$, we have that for any $f \in C^\infty(M)$
\[ (\rho(D_X))(f \cdot 1)s = \nabla_X(fs) - f\nabla_X(s) = X(f) \cdot s \]
which implies that $\rho(D_X) = X, \forall X \in \mathfrak{g}(M)$. \qed

In fact, $D$ is the associated connection to $\nabla$ on the fiber bundle $\text{End}(E)$ (see $\S$). It follows from the above lemma that $\rho$ is an epimorphism and this fact together with the previous lemma implies that the center of the endomorphism algebra $\text{End}(E)$, which is $C^\infty(M) \cdot 1$, is a quotient manifold algebra for $\text{End}(E)$.

Hence, we have a short exact sequence of Lie algebra homomorphisms
\[ 0 \rightarrow \text{Int}(\text{End}(E)) \rightarrow \text{Der}(\text{End}(E)) \overset{\rho}{\rightarrow} \text{Der}(C^\infty(M)) \cong \mathfrak{g}(M) \rightarrow 0 \]
and the mapping $\mathfrak{g}(M) \ni X \mapsto D_X \in \text{Der}(\text{End}(E))$ gives a splitting of this short exact sequence, but it is a splitting of the short exact sequence of $C^\infty(M)$-module homomorphisms, because $X \mapsto D_X$ is a Lie algebra homomorphism only when the connection $\nabla$ is flat.

Further, in this section, we study the submanifold algebras for the algebra $\text{End}(E)$.

If $B \subset A$ is a quotient manifold algebra for an associative algebra $A$, and $I \subset B$ is a submanifold ideal in $B$, then a natural candidate for a submanifold ideal in $A$ is the two-side ideal in $A$ generated by $I$. In general, such ideal is not always submanifold ideal, but this method gives a positive result in some “good” cases. Let $N (\partial N = \emptyset)$ be a compact submanifold of $M$ and $I_N$ be the ideal in $C^\infty(M)$ consisting of the functions vanishing on $N$. The ideal in $\text{End}(E)$ generated by $I_N$ coincides with the set of sections of the endomorphisms bundle vanishing on $N$. The quotient algebra $\text{End}(E)/(I_N \cdot \text{End}(E))$ is canonically isomorphic to $\text{End}(E_N)$, where $E_N$ denotes the restriction bundle of the vector bundle $E$ to the submanifold $N$: $\pi_N = \pi|_N : E_N = \pi^{-1}(N) \rightarrow N$. The quotient map
\[ p : \text{End}(E) \rightarrow \text{End}(V_n) = \text{End}(E)/(I_N \cdot \text{End}(E)) \]
is just the restriction map, which maps any endomorphism of $\Phi : E \rightarrow E$ to its restriction $\Phi_N : E_N \rightarrow E_N$.

Let $\nabla$ be a connection on $E$. If $X \in \mathfrak{g}(M)$ is such that $X(I_N) \subset I_N$, then the ideal $I_N \cdot \text{End}(E)$ is invariant for the operator $D_X$, because:
\[ D_X(f \Phi) = X(f)\Phi + f \cdot D_X(\Phi) \in I_N \cdot \text{End}(E), \quad \forall f \in I_N \text{ and } \forall \Phi \in \text{End}(E) \]
Therefore, the connection $\nabla$ defines a splitting of the exact sequence
\[ 0 \rightarrow \text{Int}(\text{End}(E_N)) \rightarrow \text{Der}(\text{End}(E_N)) \overset{\pi_N}{\rightarrow} \mathfrak{g}(N) \rightarrow 0 \]
This implies that any derivation $U_N \in \text{Der}(\text{End}(E_N))$ can be represented as

$$U_N = ad(\Phi_N) + D_{x_N}$$

where $\Phi_N \in \text{End}(E_N)$ and $x_N \in \mathfrak{F}(N)$. Let $X \in \mathfrak{F}(M)$ be an extension of $X_N$ and $\Phi \in \text{End}(E)$ be an extension of $\Phi_N$, then we obtain that the derivation $U = ad(\Phi) + D_{x_E} \in \text{Der}(\text{End}(E))$ is an extension of the derivation $U_N$. Hence the derivations of $\text{End}(E_N) = \text{End}(E)/(I_N \cdot \text{End}(E))$ are “covered” by the derivations of $\text{End}(E)$, which implies that $\text{End}(E_N)$ is a submanifold algebra for $\text{End}(E)$, and the ideal $I_N \cdot \text{End}(E)$ is the corresponding constraint ideal. Moreover, we have the following

**Proposition 3.** Let $I$ be an ideal in the algebra $\text{End}(E)$ and $I'$ be $I \cap (C^\infty(M) \cdot 1)$. If $I'$ is a submanifold ideal in the algebra $C^\infty(M) \cdot 1$, corresponding to some compact closed submanifold $N$, then $I$ is a submanifold ideal in the algebra $\text{End}(E)$

**Proof.** By assumption $I' = I_N = \left\{ f \in C^\infty(M) \mid f(N) = \{0\} \right\}$. Consider the ideal in $\text{End}(E)$ generated by $I_N$. As it was mentioned above, this ideal coincides with the set of such endomorphisms $\Phi \in \text{End}(E)$ that $\Phi|_N \equiv 0$. As $I_N \cdot 1 \subset I$, we have that $I_N \cdot \text{End}(E) \subset I$. For any $x \in N$, consider the evaluation map

$$\delta_x : \text{End}(E) \rightarrow \text{End}(E_x), \quad \delta_x(\Phi) = \Phi(x)$$

As $\delta_x$ is an epimorphism, the image of the ideal $I$ by $\delta_x$ is an ideal in $\text{End}(E_x)$. But any ideal in $\text{End}(E_x)$ is the trivial one – $\{0\}$, or the entire $\text{End}(E_x)$. If $\delta_x(I) = \text{End}(E_x)$, then there is an endomorphism $\Phi \in I$, such that $\Phi_x = 1$. This implies that, there exists such neighborhood $U$ of the point $x$ in $M$, that $\Phi_u$ is invertible for each $u \in U$. Consider two smaller neighborhoods of $x$: $W \subset V \subset U$, and a function $f \in C^\infty(M)$, such that $f(W) = \{1\}$ and $f(M \setminus V) = \{0\}$. Construct the endomorphism

$$\Psi = \begin{cases} f \cdot \Phi^{-1} & \text{on } V \\ 0 & \text{on } M \setminus V \end{cases}$$

We have that $\Psi \Phi = f \cdot 1$. But as $\Phi \in I$, we obtain that $f \cdot 1 \in I$ which contradicts to the assumption that $(C^\infty(M) \cdot 1) \cap I = I_N$, because $f(x) = 1 \Rightarrow f \notin I_N$. Hence, the assumption that $\delta(I) = \text{End}(E_x)$ is false. Therefore for any $\Phi \in I$ and any $x \in N$, the value $\Phi_x$ is 0. This, itself, implies that $I \subset I_N \cdot \text{End}(E)$. This together with $I_N \cdot \text{End}(E) \subset I$, gives the equality $I = I_N \cdot \text{End}(E)$. We obtain that any such ideal $I \subset \text{End}(E)$ that $I \cap (C^\infty(M) \cdot 1)$ is a submanifold ideal in $C^\infty(M) \cdot 1$, for some compact, closed submanifold $N$, is of the form $I = I_N \cdot \text{End}(E)$. But as we have shown, such ideals in $\text{End}(E)$ are submanifold ideals. \hfill $\square$

**Problem:** Is any submanifold ideal in the algebra $\text{End}(E)$ of the form $I \cdot \text{End}(E)$ where $I$ is a submanifold ideal in $C^\infty(M)$?

**Definition 4.** For any ideal $I \subset \text{End}(E)$, we call the set of points

$$M_I = \left\{ x \in M \mid \Phi_x = 0, \forall \Phi \in I \right\}$$

the 0-set of the ideal $I$.

**Proposition 4.** If an ideal $I \subset \text{End}(E)$ is such that $M_I = \emptyset$ and the manifold $M$ is locally compact, then $I = \text{End}(E)$.
Proof. For any point \( x \in M \), consider the evaluation map

\[ \delta_x : \text{End}(E) \rightarrow \text{End}(E_x), \quad \delta_x(\Phi) = \Phi_x \]

This map is an epimorphism of associative algebras. Therefore \( \delta_x(I) \) is an ideal in \( \text{End}(E_x) \), which can be only 0 or the entire \( \text{End}(E_x) \). By assumption, the 0-set of the ideal \( I \) is empty. Therefore \( \delta_x(I) = \text{End}(E_x) \). Let \( \{ \Phi^1, \ldots, \Phi^m \} \) be any basis of the complex vector space \( \text{End}(E_x) \), and \( \{ \Phi^1, \ldots, \Phi^m \} \) be such elements from \( I \), that \( \delta_x(\Phi^i) = \Phi^i_x \), \( i = 1, \ldots, m \). As the system \( \{ \Phi^1, \ldots, \Phi^m \} \) is linearly independent at the point \( x \), there exists such neighborhood \( U \) of \( x \) in \( M \), that the system \( \{ \Phi^1|_U, \ldots, \Phi^m|_U \} \) is also linearly independent over \( C^\infty(U) \) and therefore forms a basis for \( \text{End}(E_U) \). Let \( \bigcup U_k = M \) be such locally finite covering of \( M \) that for any \( U_k \) we have a system \( \{ \Phi^1_k, \ldots, \Phi^m_k \} \subset I \) such that \( \{ \Phi^1_k|_{U_k}, \ldots, \Phi^m_k|_{U_k} \} \) is a local basis of \( \text{End}(E_{U_k}) \). For any \( \Psi \in \text{End}(E) \), we have: \( \Psi_{U_k} = \sum_i f^k_i \Phi^i_k|_{U_k} \), where \( \{ f^1_k, \ldots, f^m_k \} \subset C^\infty(U_k) \). Let \( \{ \varphi_k \} \) be a partition of unity corresponding to the covering \( \{ U_k \} \). Consider the endomorphisms

\[ \bar{\Psi}_k = \sum_i f^k_i \Phi^i_k, \quad \bar{\Psi}_k = \begin{cases} \varphi_k f^k_i & \text{on } U_k \\ 0 & \text{on } M \setminus U_k \end{cases} \]

As \( \Phi^i_k \in I \), we have that \( \bar{\Psi}_k \in I \), but on the other hand: \( \sum_k \bar{\Psi}_k = \Psi \). Hence, we obtain that \( \Psi \in I \).

It follows from the definition of the 0-set, that if \( I_1 \subset I_2 \) then \( M_{I_2} \subset M_{I_1} \). Therefore for any subset \( S \subset M \), the ideal \( I_S \cdot \text{End}(E) \), where

\[ I_S = \left\{ f \in C^\infty(M) \bigg| f|_S \equiv 0 \right\} \]

is the greatest ideal with 0-set equal to \( S \). From this follows that maximal proper ideals in \( \text{End}(E) \) are the ideals of the form \( I_x \cdot \text{End}(E) \approx \left\{ \Phi \in \text{End}(E) \bigg| \Phi_x = 0 \right\} \), where \( x \) is a point in \( M \). As it was mentioned, such ideal is a submanifold ideal and the corresponding submanifold algebra is \( \text{End}(E)/(I_x \cdot \text{End}(E)) \equiv \text{End}(E_x) \) (see 3).

4. Degenerate Poisson Structures in Noncommutative Geometry

As before, let \( A \) be a unital associative complex or real algebra. A Poisson structure on \( A \) is defined as a Lie bracket \( \{ , \} : A \wedge A \rightarrow A \), which is a biderivation (see 2, 3):

\[ \{ a, bc \} = b\{ a, c \} + \{ a, b \}c, \quad \forall \{ a, b, c \} \subset A \]

The pair \( (A, \{ , \}) \) is called a Poisson algebra. For any element \( a \in A \), the derivation \( A \ni x \mapsto \{ a, x \} \in A \) is called the Hamiltonian derivation (or just Hamiltonian) corresponding to the element \( a \) and we denote this derivation by \( \text{ham}(a) \).

One method of defining a Poisson structure, which has its well-known classical analogue, is the method which uses a symplectic form (see 2, 3). Let \( \omega \) be a 2-form in the derivation-based differential calculus for \( A \); i.e., \( \omega \) is a \( Z(A) \)-bilinear antisymmetric mapping:

\[ \omega : \text{Der}(A) \times \text{Der}(A) \rightarrow A \]

Such form is said to be nondegenerate if for any element \( a \in A \), there exists a derivation \( X_a \in \text{Der}(A) \) such that \( i_{X_a} \omega = -da \). The 1-forms \( i_{X_a} \omega \) and \( da : \text{Der}(A) \rightarrow A \) are defined as \( (da)(X) = X(a) \) and \( (i_{X_a} \omega)(X) = \omega(X_a, X), \forall X \in \text{Der}(A) \). It is easy to verify
that if the form \( \omega \) is nondegenerate then the mapping \( X \mapsto i_X \omega \) from \( \text{Der}(A) \) to the space of 1-forms is a monomorphism.

After this, we can state that if \( \omega \) is nondegenerate then for any \( a \in A \), the derivation \( X_a \) is unique. For a nondegenerate \( \omega \) define an antisymmetric bracket on \( A \): \( \{a, b\} = \omega(X_a, X_b) \). It can be verified by direct calculations that this bracket is a biderivation and for any \( a, b, c \in A \) we have (see [2]):

\[
\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = (d\omega)(X_a, X_b, X_c)
\]

Therefore if the form \( \omega \) is closed, then the Jacobi identity for this bracket is true and so the pair \( (A, \{\ , \\}) \) is a Poisson algebra. In this case the derivation \( X_a \) is the same as the Hamiltonian corresponding to \( a \). When the \( Z(A) \)-module generated by the set \( \{\text{ham}(a) \mid a \in A\} \) coincides with the entire space \( \text{Der}(A) \), the two conditions: Jacobi identity for \( \{\ , \\} \) and \( d\omega = 0 \), are equivalent. A nondegenerate and closed 2-form \( \omega \) is called a symplectic form and the pair \( (A, \omega) \) is called the corresponding symplectic structure, or symplectic algebra. Let us formulate a noncommutative analogue of the notion of degenerate/nondegenerate Poisson structure. The classical situation, when \( A \) is an algebra of \( C^\infty \)-class functions on some \( C^\infty \)-class manifold, dictates the following

**Definition 5.** A poisson bracket on an associative algebra \( A \) is said to be nondegenerate if and only if the \( Z(A) \)-submodule generated by the Hamiltonian derivations in \( \text{Der}(A) \) coincides with the entire \( \text{Der}(A) \). Otherwise, the Poisson structure is said to be degenerate.

A noncommutative generalization of a symplectic leaf for a degenerate Poisson structure can be obtained by using of the notion of Poisson ideal.

**Definition 6.** A two-side ideal \( I \) in \( A \) is said to be a Poisson ideal if \( \{I, A\} \subseteq I \), i.e., \( I \) is also a Lie algebra ideal. A Poisson ideal is said to be **locally proper** if

\[
\text{Der}(A, I) = \left\{ X \in \text{Der}(A) \mid X(I) \subseteq I \right\} \neq \text{Der}(A).
\]

If \( I \) is also a submanifold ideal in \( A \) (i.e., the mapping \( \pi : \text{Der}(A, I) \to \text{Der}(A/I) \) is an epimorphism) then the submanifold algebra \( A/I \) can be regarded as a noncommutative analogue of a symplectic submanifold of a Poisson manifold.

Using the above definition we formulate the following criteria for the degeneracy of a Poisson structure on \( A \).

**Proposition 5.** If a Poisson algebra \( A \) contains at least one locally proper Poisson ideal then the Poisson structure on \( A \) is degenerate.

**Proof.** The \( Z(A) \)-module generated by \( \text{ham}(A) \) is the set of the elements of the form \( \sum_{i=1}^k z_i \cdot \text{ham}(a_i) \) for some \( z_i \in Z(A), a_i \in A, \ 1 \leq i \leq k \). It follows from the definition of locally proper Poisson ideal, that there exists at least one \( X \in \text{Der}(A) \), such that \( X(I) \not\subseteq I \). If \( X \in Z(A) \cdot \text{ham}(A) \) then \( X = \sum z_i \cdot \text{ham}(a_i) \) and for any \( u \in I \) we have \( X(u) = \sum_{i \in I} z_i \cdot (a_i, u) \in I \). This contradicts to the condition \( X(I) \not\subseteq I \).

Therefore the module \( Z(A) \cdot \text{ham}(A) \) does not coincide with the entire \( \text{Der}(A) \) and hence, the Poisson bracket \( \{\ , \\} \) is degenerate in the sense of the definition. \( \square \)

In the classical case, when \( A = C^\infty(M) \), the above criteria is a necessary and sufficient condition, because if a Poisson structure on a smooth manifold \( M \) is
Proof. Let \( A \) algebra \( M \) forms on \( M \). In the classical differential calculus there is a one-to-one correspondence, between the family of nondegenerate Poisson structures on \( M \) and the family of symplectic forms on \( M \). In the noncommutative case we have the following

**Proposition 6.** Let \( \{ \cdot , \cdot \} \) be a nondegenerate Poisson bracket on an associative algebra \( A \) (i.e., \( Z(A) \cdot \text{ham}(A) = \text{Der}(A) \)). Then there exists a symplectic form \( \omega \) such that the Poisson bracket on \( A \) defined by \( \omega \) coincides with \( \{ \cdot , \cdot \} \).

*Proof.* Let \( X, Y \in \text{Der}(A) \). As the Poisson bracket \( \{ \cdot , \cdot \} \) is nondegenerate, by the definition we have \( Z(A) \cdot \text{ham}(A) = \text{Der}(A) \). Therefore we have \( X = \sum_{i=1}^{m} u_i \cdot \text{ham}(x_i) \) and \( Y = \sum_{j=1}^{n} v_j \cdot \text{ham}(y_j) \), for some \( \{ u_1, \ldots, u_m, v_1, \ldots, v_n \} \subset Z(A) \) and \( \{ x_1, \ldots, x_m, y_1, \ldots, y_n \} \subset A \). Define the value of a form \( \omega \) on the pair of vectors \((X, Y)\) as \( \omega(X, Y) = \sum_{i,j} u_i v_j \{ x_i, y_j \} \). Let us verify that the right side of the equality is independent of the choice of the representations of \( X \) and \( Y \). Let \( X = \sum_{i=1}^{k} u'_i \cdot \text{ham}(x'_i) \) be another representation of \( X \). Then for any \( 1 \leq j \leq n \) we have

\[
\sum_{i} u'_i v_j \{ x'_i, y_j \} = \sum_{i} v_j u'_i \{ x'_i, y_j \} = v_j X(y_j) = \sum_{p} v_j u_p \{ x_p, y_j \}
\]

which implies the independence of the expression \( \sum_{i,j} u_i v_j \{ x_i, y_j \} \) from the representation of \( X \). As the form \( \omega \) is antisymmetric, from this follows the independence from the representation of \( Y \).

The identity of the bracket defined by the form \( \omega \) and the original one is a tautological result of the definition.

The form is closed because of the equality

\[
(d\omega)(\text{ham}(a), \text{ham}(b)\text{ham}(c)) = \{ \{ a, b \}, c \} + \{ \{ b, c \}, a \} + \{ \{ c, a \}, b \}
\]

and the assumption that \( Z(A) \cdot \text{ham}(A) = \text{Der}(A) \). \( \square \)

When \( A = C^\infty(M) \) for some smooth manifold \( M \), the converse is also true: a Poisson structure defined by some symplectic form is nondegenerate in the sense that \( C^\infty(M) \cdot \text{ham}(M) = \mathfrak{g}(M) \), where \( \text{ham}(M) \) denotes the space of Hamiltonian vector fields on \( M \).

5. **Poisson Structures for the Endomorphism Algebra of a Vector Bundle**

In this section we describe the family of all Poisson structures and the noncommutative analogue of the corresponding symplectic foliations for the endomorphism algebra of a finite-dimensional vector bundle. We start from the description of Poisson structures on the matrix algebra.

For \( n \in \mathbb{N} \), let \( M_n(\mathbb{C}) \) be the algebra of \( n \times n \) complex matrices. As on any associative algebra, there is a natural Poisson structure on \( M_n(\mathbb{C}) \) defined by the multiplicative commutator: \( \{ a, b \} = [a, b] = ab - ba \). This Poisson structure is
nondegenerate because the Hamiltonian corresponding to an element \( a \in M_n(\mathbb{C}) \) is the inner derivation: \( \text{ham}(a) = ad(a) = [a, \cdot] \), and all the derivations of the algebra \( M_n(\mathbb{C}) \) are inner: \( \text{Der}(M_n(\mathbb{C})) = \text{Int}(M_n(\mathbb{C})) \).

We have a family of Poisson structures on \( M_n(\mathbb{C}) \) parametrized by \( \mathbb{C} \):
\[
\{a, b\}_k = k \cdot [a, b], \quad k \in \mathbb{C}
\]
All of these brackets are nondegenerate except of the case when \( k = 0 \).

**Proposition 7.** Any Poisson bracket on the algebra \( M_n(\mathbb{C}) \) is of the form
\[
\{a, b\}_k = k \cdot [a, b], \quad \forall a, b \in M_n(\mathbb{C})
\]
for some \( k \in \mathbb{C} \).

**Proof.** Let \( p \in M_n(\mathbb{C}) \) be a projector: \( p^2 = p \). There is a one-to-one correspondence between the subset of all projectors in \( M_n(\mathbb{C}) \) and the set of decompositions of the vector space \( \mathbb{C}^n \) into a pair of complementar subspaces: \( \mathbb{C}^n = X \oplus Y \). For such decomposition the corresponding projector \( p \) is the projection operator on \( X \) along \( Y \) and the operator \( 1 - p \) is the projection operator on \( Y \) along \( X \). As the Poisson structure is a biderivation, for any \( a \in M_n(\mathbb{C}) \) we have the following
\[
\{a, p\} = \{a, p^2\} = p\{a, p\} + \{a, p\}p \Rightarrow \{a, p\}(1 - p) = p\{a, p\}
\]
Consider the decomposition of \( \mathbb{C}^n \), corresponding to the projector \( p \):
\[
\mathbb{C}^n = \text{Im}(p) \oplus \text{Im}(1 - p)
\]
If \( x \in \text{Im}(p) \), then we have
\[
p\{a, p\}x = \{a, p\}(1 - p)x = 0 \Rightarrow \{a, p\}x \in \text{Im}(1 - p)
\]
If \( y \in \text{Im}(1 - p) \), then
\[
\{a, p\}(1 - p)y = p\{a, p\}y \Rightarrow p\{a, p\}y = \{a, p\}y \Rightarrow \{a, p\}y \in \text{Im}(p)
\]
From these we obtain that the operator \( \{a, p\} \) maps the subspace \( \text{Im}(p) \) into the subspace \( \text{Im}(1 - p) \) and the subspace \( \text{Im}(1 - p) \) into \( \text{Im}(p) \). This implies that for any \( a \in M_n(\mathbb{C}) \), the decomposition of the operator \( \{p, a\} \), corresponding to the decomposition \( \mathbb{C}^n = \text{Im}(p) \oplus \text{Im}(1 - p) \) is of the form
\[
\{p, a\} = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}
\]
for some \( b_1 : \text{Im}(1 - p) \rightarrow \text{Im}(p) \) and \( b_2 : \text{Im}(p) \rightarrow \text{Im}(1 - p) \). By definition of a Hamiltonian derivation and because all derivations of \( M_n(\mathbb{C}) \) are inner, we have
\[
\{p, a\} = \text{ham}(p)a = [\text{ham}(p), a], \quad \forall a \in M_n(\mathbb{C})
\]
Let \( \text{ham}(p) = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \) for the decomposition \( \mathbb{C}^n = \text{Im}(p) \oplus \text{Im}(1 - p) \). Then, for \( a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) we obtain
\[
\{p, a\} = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} -p_2 & p_1 - q_2 \\ 0 & p_2 \end{pmatrix} \Rightarrow p_2 = 0
Similarly, for $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ we obtain
\[
\begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ q_2 - p_1 & -q_1 \end{pmatrix} \Rightarrow q_1 = 0
\]
So, we have $\text{ham}(p) = \begin{pmatrix} p_1 & 0 \\ 0 & q_2 \end{pmatrix}$. Then, for any $a = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ we obtain
\[
\{p, a\} = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix} = \begin{pmatrix} [p_1, x] & 0 \\ 0 & [q_2, y] \end{pmatrix} \Rightarrow [p_1, x] = [q_2, y] = 0
\]
which implies that $p_1$ and $q_2$ are just scalars, and $\text{ham}(p)$ is of the form
\[
\text{ham}(p) = \begin{pmatrix} k_1 \cdot 1 & 0 \\ 0 & k_2 \cdot 1 \end{pmatrix}, \quad k_1, k_2 \in \mathbb{C}
\]
Rewrite the latter expression for $\text{ham}(p)$ in the form
\[
\text{ham}(p) = \left( (k_1 - k_2) \cdot 1 \right) 0 + \begin{pmatrix} k_2 \cdot 1 & 0 \\ 0 & k_2 \cdot 1 \end{pmatrix}
\]
As the matrix $\begin{pmatrix} k_2 \cdot 1 & 0 \\ 0 & k_2 \cdot 1 \end{pmatrix}$ is just a scalar: $k_2 \cdot 1$, it has no effect in the commutator. Hence we can conclude that the hamiltonian mapping on the subset of projectors is of the form
\[
\text{ham}(p) = \left( f(p) \cdot 1 \right) 0 = f(p) \cdot p
\]
where $f$ is a complex-valued function on the subset of projectors in $M_n(\mathbb{C})$. From this easily follows that if two projectors $p$ and $q$ does not commute then the equality $\{p, q\} = [\text{ham}(p), q] = -[\text{ham}(q), p]$ implies that $f(p) = f(q)$. Now consider the set of elementary matrices: $E_{ij} \in M_n(\mathbb{C}), \{i, j\} \subset \{1, \ldots, n\}$, which forms a basis for $M_n(\mathbb{C})$. Construct the following set of projectors: $\forall \{i, j\} \subset \{1, \ldots, n\}$
\[
\text{P}_{ij} = \begin{cases} 
E_{ij} + E_{jj} & \text{if } i \neq j, \\
E_{ii} & \text{if } i = j.
\end{cases}
\]
It is clear that the set $\{\text{P}_{ij}\}$ is also a basis of the vector space $M_n(\mathbb{C})$. For any $i \neq j$ we have the following commutation relations: $[\text{P}_{ij}, \text{P}_{jj}] = E_{ij} \neq 0$ and $[\text{P}_{ij}, \text{P}_{ii}] = -E_{ij} \neq 0$. Therefore for any two $P_{mk}$ and $P_{kl}$, where $m \neq k$ and $k \neq l$, we have $P_{kk}$, which does not commute with both of them. This implies that $f(P_{mk}) = f(P_{kk}) = f(P_{kj})$. By using a series of such equalities, we can obtain that $f(P_{ij}) = \text{const} \equiv k \in \mathbb{C}, \{i, j\} \subset \{1, \ldots, n\}$. As $\{P_{ij}\}$ forms a basis of $M_n(\mathbb{C})$ and $\text{ham}$ is a linear map, we obtain that $\text{ham}(T) = k \cdot T$, $\forall T \in M_n(\mathbb{C})$.

Hence we can conclude that the family of Poisson structures on the algebra of matrices $M_n(\mathbb{C})$ is parametrized by $\mathbb{C}$ and each Poisson bracket is of the form
\[
\{S, T\}_k = k \cdot [S, T], \quad k \in \mathbb{C}
\]

As in the section 3, let $\text{End}(E)$ be the endomorphism algebra of a finitedimensional complex or real vector bundle $\pi : E \to M$. As in the case of the matrix algebra, we denote by $\text{ad}(\Phi), \Phi \in \text{End}(E)$ the Hamiltonian derivations for the Poisson bracket defined by the multiplicative commutator. In this case the $C^\infty(M)$-module generated by the Hamiltonian derivations is not the entire $\text{Der(End}(E))$, because a derivation of the type $D_X, X \in \mathfrak{g}(M)$ (see the formula 4), defined by using of
some connection on $E$, is not Hamiltonian. This follows from the fact that for any hamiltonian derivation $\text{ham}(\Phi) = [\Phi, \cdot]$ and an endomorphism of $f \cdot 1$, $f \in C^\infty(M)$, we have the following

$$\text{ham}(\Phi)(f \cdot 1) = 0 \text{ and } D_X(f \cdot 1) = X(f) \cdot 1$$

Moreover, there are proper Poisson ideals in the Poisson algebra $(\text{End}(E), [\cdot, \cdot])$. Such ideal can be constructed by a proper subset $S \subset M$, for example when $S$ consists of only one point.

**Lemma 8.** For any point $x_0 \in M$, the ideal $I_{x_0}(\text{End}(E)) = I_{x_0} \cdot \text{End}(E)$ in the algebra $\text{End}(E)$, generated by $I_{x_0} = \{ f \in C^\infty(M) \mid f(x_0) = 0 \}$ is a locally proper Poisson ideal in $\text{End}(E)$.

**Proof.** The inclusion $\{ I_{x_0}(\text{End}(E)), \text{End}(E) \} \subset I_{x_0}(\text{End}(E))$ easily follows from the definition of the ideal $I_{x_0}(\text{End}(E))$. Now, let us verify that there exists such derivation $U \in \text{Der}(\text{End}(E))$ that $U(I_{x_0}(\text{End}(E))) \not\subset I_{x_0}(\text{End}(E))$. Let $f \in C^\infty(M)$ is such that $f(x_0) = 0$ and for some tangent vector $v \in T_{x_0}(M)$: $f'(x_0)v \neq 0$. Choose such $\Phi \in \text{End}(E)$ that $\Phi(x_0) \neq 0$. It is clear that $f \cdot \Phi \in I_{x_0}(\text{End}(E))$. Let $X$ be such vector field on $M$ that $X(x_0) = v$. For some connection on $E$ consider the derivation $D_X : \text{End}(E) \rightarrow \text{End}(E)$. For these data we have the following

$$D_X(f \cdot \Phi) = X(f) \cdot \Phi + f \cdot D_X(\Phi)$$

which implies that $D_X(f \cdot \Phi)_{x_0} \neq 0$ and therefore $D_X(f \cdot \Phi) \notin I_{x_0}(\text{End}(E))$. $\square$

In the case when we have a compact, closed submanifold $N \subset M$, the ideal $I_N(\text{End}(E)) = I_N \cdot \text{End}(E)$ is a submanifold ideal (see 3). At the same time, the quotient $\text{End}(E)/(I_N \cdot \text{End}(E)) \cong \text{End}(E_N)$ is a Poisson algebra under the bracket induced from $\text{End}(E)$. This situation can be considered as a noncommutative Poisson submanifold in a Poisson manifold. When $N = x_0$ is a Point, the corresponding ideal $I_{x_0}(\text{End}(E))$ is maximal and the quotient $\text{End}(E)/(I_{x_0} \cdot \text{End}(E)) \cong \text{End}(E_{x_0})$ is a nondegenerate Poisson algebra with the bracket induced from $\text{End}(E)$, and can be regarded as a noncommutative analogue of a symplectic leaf of a Poisson structure.

Before we start the description of the family of Poisson structures on the endomorphism algebra let us concern some general facts about Poisson algebras.

Let $(A, \{ \cdot, \cdot \})$ be a Poisson algebra. As the center of the algebra $A$ is invariant for any derivation, and the bracket $\{ \cdot, \cdot \}$ is a biderivation, we have that the center of $A$ is a Lie algebra ideal for the Poisson bracket: $\{ Z(A), A \} \subset Z(A)$.

**Lemma 9.** Let $\text{Comm}(A)$ be the minimal subalgebra of the associative algebra $A$ containing the commutator $[A, A] = \{ [a, b] = ab - ba \mid a, b \in A \}$ (in fact $\text{Comm}(A)$ is the set of finite sums of the elements of the type $\prod_{i=1}^{k} [a_i, b_i]$). For any Poisson structure on $A$, the Poisson bracket of the elements of the center of $A$ with the elements of $\text{Comm}(A)$ is equal to $0$: $\{ Z(A), \text{Comm}(A) \} = \{ 0 \}$.

**Proof.** Because of the Leibniz rule it is sufficient to verify the statement of the lemma for the elements of the type $[a, b]$, $a, b \in A$. For any $z \in Z(A)$ we have the
following
\[ \{z, [a, b]\} = z\{a, b\} - \{z, ba\} = a\{z, b\} + \{z, a\}b - b\{z, a\} - \{z, b\}a = 0 \]

Corollary 1. If \(\text{Comm}(A) = A\) then the multiplicative center \(Z(A)\) is also in the center of the Lie algebra corresponding to the Poisson bracket.

Proposition 10. If \(\text{Comm}(A) = A\) and every derivation \(X \in \text{Der}(A)\) such that \(X(Z(A)) = 0\), is inner derivation, then any Poisson structure on \(A\) is defined by a linear mapping \(f : A \to A/Z(A)\) via the equality \(\{a, b\} = [f(a), b], \quad a, b \in A\); and the mapping \(f\) is \(A/Z(A)\)-linear.

Proof. As it follows from the previous corollary, if \(\text{Comm}(A) = A\), we have that for any \(a \in A\)
\[ \{a, Z(a)\} = \{0\} \Rightarrow \text{ham}(a)Z(A) = \{0\} \Rightarrow \text{ham}(a) \in \text{Int}(A) \cong A/Z(A) \]
In fact, in this case the map \(f : A \to A/Z(A)\) is the Hamiltonian map.

For \(z \in Z(A)\) and \(a \in A\), we have
\[ [f(za), x] = [za, f(x)] = za, f(x) = z[f(a), x] = [zf(a), x] \Rightarrow \]
\[ \Rightarrow [f(za) - zf(a), x] = 0, \quad \forall x \in A \Rightarrow f(za) - zf(a) \in Z(A) \Rightarrow \]
\[ \Rightarrow \quad [f(za)] = [zf(a)] = z[f(a)] \quad \text{in the quotient} A/Z(A) \]
which implies that \(f\) is \(A/Z(A)\)-linear. \qed

The antisymmetric property and the Jacoby identity for \(\{, \}\) gives the following properties of the mapping \(f : \forall \{x, y, z\} \subset A\)

1. \( [f(x), y] = [x, f(y)] \)
2. \( [f(x), [f(y), z]] + [f(y), [f(z), x]] + [f(z), [f(x), y]] = 0 \)
   or equivalently:
   \[ [[f(x), f(y)], z] + [[f(y), f(z)], x] + [[f(z), f(x)], y] = 0 \]
   or using the previous property:
   \[ [[f^2(x), y], z] + [[f^2(y), z], x] + [[f^2(z), x], y] = 0 \]

Let us summarize by the following

Proposition 11. Let \(A\) be an associative algebra such that \(\text{Comm}(A) = A\) and any derivation of \(A\) which vanishes on the center of \(A\) is inner. Then there is a one-to-one correspondence between the family of Poisson brackets on \(A\) and the family of such mappings (Hamiltonians) \(f : A \to A/Z(A)\) which satisfy the conditions

H1. \(f\) is a homomorphism of \(Z(A)\)-modules
H2. \([f(a), b] = [a, f(b)], \quad \forall a, b \in A\)
H3. \([f^2(a), b], c] + [[f^2(b), c], a] + [[f^2(c), a], b] = 0, \quad \forall a, b, c \in A\)

It is clear that any mapping of the type \(f(x) = c \cdot x\) for fixed \(c \in Z(A)\) satisfies the above conditions.

Let us apply the above result to the case when \(A\) is the endomorphism algebra of a finite-dimensional complex (or real) vector bundle \(\pi : E \to M\). First,
recall that any derivation of the algebra $\text{End}(E)$, which vanishes on the center $Z(\text{End}(E)) \cong C^\infty(M) \cdot 1$, is inner. Then, for the matrix algebra $M_n(\mathbb{C})$, when $n \geq 2$, we have that $\text{Comm}(M_n(\mathbb{C})) = M_n(\mathbb{C})$. This easily follows from the fact that any elementary matrix $E_{ij}$ is a commutator, when $i \neq j$: $E_{ij} = [E_{ij}, E_{jj}]$, and is a product of two commutators when $i = j$: $E_{ii} = E_{ik} \cdot E_{ki} = [E_{ik}, E_{kk}] \cdot [E_{ki}, E_{ii}]$ for some $k \neq i$. Therefore the same is true for the endomorphism algebra:

$$\text{Comm}(\text{End}(E)) = \text{End}(E), \text{ when } \dim(\pi^{-1}(x)) \geq 2$$

After these, from the Proposition 11 follows that any Poisson bracket on $\text{End}(E)$ is of the form

$$\{\Phi, \Psi\} = [f(\Phi), \Psi], \text{ where } f : \text{End}(E) \longrightarrow \text{End}(E)/C^\infty(M)$$

is a homomorphism of $C^\infty(M)$-modules. The module $\text{End}(E)/C^\infty(M)$ is canonically isomorphic to the module of sections of the subbundle of $\text{End}(E)$ consisting of the traceless endomorphisms. As $f$ is a $C^\infty(M)$-module homomorphism it is induced by by some homomorphism of vector bundles. Hence, $f$ is pointwise:

$$f(\Phi)_x = f_x(\Phi)_x, \forall x \in M.$$ 

Therefore, it induces a Poisson structure on each fiber of the endomorphism bundle: $\{\Phi_x, \Psi_x\} = \{\Phi, \Psi\}_x = [f_x(\Phi)_x, \Psi_x]$. As it was shown, any Poisson bracket on the matrix algebra is of the form $\{a, b\} = k \cdot [a, b], k \in \mathbb{C},$ and any Hamiltonian is of the form $f(a) = k \cdot a$. Therefore, the mapping

$$f : \text{End}(E) \longrightarrow \text{End}(E)/C^\infty(M)$$

is of the form $f(\Phi) = \lambda \cdot [\Phi]$, where $\lambda \in C^\infty(M)$.

To summarize, we formulate the following (the full description of the family of Poisson structures on the endomorphism algebra of a vector bundle with $\dim(\text{fiber}) \geq 2$)

**Proposition 12.** Any Poisson bracket on the endomorphism algebra of a finite-dimensional complex (or real) vector bundle, with fiber the dimension of which is $\geq 2$, is of the form

$$\{\Phi, \Psi\}_\lambda = \lambda \cdot [\Phi, \Psi], \quad \forall \Phi, \Psi \in \text{End}(E)$$

where $\lambda \in C^\infty(M)$.

Hence, the family of Poisson structures on the endomorphism algebra $\text{End}(E)$ is parametrized by $C^\infty(M)$.

**Corollary 2.** Any Poisson structure on the endomorphism algebra of a finite-dimensional complex (or real) vector bundle with $\dim(\text{fiber}) \geq 2$ is degenerate.

If the fiber of a vector bundle is 1-dimensional, then the endomorphism algebra coincides with the commutative algebra of smooth functions on the base manifold, and the Poisson structures on this algebra are given by involutive bivector fields on the base manifold (see [5]).

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\textbf{Department of Theoretical Physics, Institute of Mathematics, Georgian Academy of Sciences, Tbilisi, Georgia}  
\textit{E-mail address: zaqro@gtu.edu.ge}