RANDOM MATRIX-VALUED MULTIPLICATIVE FUNCTIONS AND LINEAR RECURRENCES IN HILBERT-SCHMIDT NORMS OF RANDOM MATRICES

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Abstract. We introduce the notion of a random matrix-valued multiplicative function, generalizing Rademacher random multiplicative functions to matrices. We provide an asymptotic for the second moment based on a linear recurrence property for Hilbert-Schmidt norms of successive products of random matrices. Moreover, we provide upper bounds for the higher even moments related to the generalized joint spectral radius.

1. Introduction

A Rademacher random multiplicative function is a family \((f(n))_{n \in \mathbb{N}}\) (with the convention \(0 \notin \mathbb{N}\)) of random variables taking values in \(\{\pm 1, 0\}\) such that

- \(n \mapsto f(n)\) is supported on squarefree integers,
- \((f(p))_{p \text{ prime}}\) are independent, each taking the values \(\pm 1\) with probability \(\frac{1}{2}\) and
- when \(n = p_1 \cdots p_r\) is squarefree then we have \(f(n) = f(p_1) \cdots f(p_r)\).

Moments of these functions have been studied in a great amount of detail. It is a classical fact that

\[
\mathbb{E}\left[\left(\sum_{n \leq x} f(n)\right)^2\right] = \frac{6}{\pi^2} x + O(\sqrt{x})
\]

and it was proven by Harper, Nikeghbali and Radziwiłł in [5, Theorem 4] and independently by Heap and Lindqvist [6, Theorem 4] in the even case that for all integers \(k \geq 3\) there exists a constant \(C_k > 0\) such that

\[
\mathbb{E}\left[\left(\sum_{n \leq x} f(n)\right)^k\right] \sim C_k x^{k/2}(\log x)^{(k-1)/2} - k.
\]

In this work we will consider the following matrix-valued generalisation of Rademacher multiplicative functions.

**Definition 1.** Let \(d \geq 1\) be an integer. A random matrix-valued multiplicative function is a family \((f(n))_{n \in \mathbb{N}}\) of random variables taking values in \(\mathbb{C}^{d \times d}\) such that

- \(n \mapsto f(n)\) is supported on squarefree integers,
- \((f(p))_{p \text{ prime}}\) are independent identically distributed (i.i.d.) and
- when \(n = p_1 \cdots p_r\) is squarefree with \(p_1 < \cdots < p_r\) then we have \(f(n) = f(p_1) \cdots f(p_r)\).
Our goal is to obtain estimates for the even moments

\[ \mathbb{E}\left[ \left\| \sum_{n \leq x} f(n) \right\|_{HS}^{2k} \right] , \]

where \( \| \cdot \|_{HS} \) denotes the Hilbert-Schmidt norm defined by \( \| A \|_{HS}^2 = \text{Tr}(A^*A) \) for \( A \in \mathbb{C}^{d \times d} \).

In section 3 we will prove the following estimate for the second moment based on a linear recurrence property of the Hilbert-Schmidt norm, which will be the subject of section 2.

**Theorem 1.** Let \( d \geq 1 \) be an integer, let \( X \) be a \( \mathbb{C}^{d \times d} \)-valued random variable and let \( f \) be the associated matrix-valued multiplicative function. Suppose that \( \mathbb{E}X = 0 \) and

\[ \mathbb{E}\left[ \| X \|_{HS}^2 \mathbb{1}(\| X \|_{HS}^2 > R) \right] \xrightarrow{R \to \infty} 0, \]

where \( \mathbb{1}(E) \) denotes the characteristic function of an event \( E \). Define

\[ T : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d} , \]

\[ A \mapsto \mathbb{E}[X^*AX], \]

let \( l := d^2 \) and assume that \( T \) is diagonalizable with eigenvalues \( \lambda_1, \ldots, \lambda_l \) arranged in descending order according to their real parts. Then for any \( N \in \mathbb{N} \) there are constants \( C_{i,m}, m = 1, \ldots, N, i = 1, \ldots, l \) such that

\[ \mathbb{E}\left[ \left\| \sum_{n \leq x} f(n) \right\|_{HS}^{2k} \right] = x \sum_{m=1}^{N} \sum_{i=1}^{l} C_{i,m}(\log x)^{\lambda_i - m} + O \left( x(\log x)^{\lambda_1 - N - 1} \right) \]

holds for all \( x \geq 2 \).

Our argument also extends to the case when \( T \) is not diagonalizable, even though our estimate becomes less precise in this case. The exact statement without the assumption of diagonalizability will be given and proven in section 3.

Section 4 will be devoted to proving an upper bound for higher moments that will be related to what is known as the generalized joint spectral radius.

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## 2. A Linear Recurrence for Hilbert-Schmidt Norms

The goal of this section is to prove the following result which may be of independent interest.

**Theorem 2.** Let \( d, k \geq 1 \) be fixed integers. Suppose that \( X, X_1, X_2, \ldots \) is a sequence of i.i.d. \( \mathbb{C}^{d \times d} \)-valued random variables such that

\[ \mathbb{E}\left[ \| X \|_{HS}^{2k} \mathbb{1}(\| X \|_{HS}^{2k} > R) \right] \to 0 \]

as \( R \to \infty \), and define

\[ a_n := a_n^{(2k)} := \mathbb{E}\left[ \| X_1 \cdots X_n \|_{HS}^{2k} \right] . \]
Then the sequence \((a_n)_n\) satisfies a linear recurrence of length
\[
l^C := \left( k + \frac{d^2 - 1}{k} \right).
\]
If the random variables are in fact \(\mathbb{R}^{d \times d}\)-valued, then the sequence \((a_n)_n\) satisfies a linear recurrence of length
\[
l^R := \left( k + \frac{d+1}{2} - 1 \right).
\]

Before proving this Theorem, we first recall the following standard fact about linear recurrences.

**Lemma 3.** Let \((a_n)_n\) be a sequence of complex numbers, and let
\[
p(x) = x^l + c_1 x^{l-1} + \cdots + c_l = (x - \lambda_1)^{m_1} \cdots (x - \lambda_t)^{m_t}
\]
be a polynomial such that
\[
a_{n+l} + c_1 a_{n+l-1} + \cdots + c_l a_n = 0
\]
holds for all \(n\), where \(\lambda_1, \ldots, \lambda_t\) are distinct complex numbers. Then there exist unique polynomials \(g_i\) of degrees < \(m_i\) for \(i = 1, \ldots, t\) such that
\[
a_n = g_1(n)\lambda_1^n + \cdots + g_t(n)\lambda_t^n.
\]

**Proof of Theorem.** Let \(\mu\) be the law of \(X\). Let \(S_d\) denote the space of complex-symmetric \(d \times d\) matrices and let
\[
V^C := \text{Sym}^k(C^{d \times d}) \quad \text{and} \quad V^R := \text{Sym}^k(S_d).
\]
Note that \(V^C\) resp. \(V^R\) is a complex vector space of dimension \(l^C\) resp. \(l^R\).

Further, define
\[
T^C : V^C \rightarrow V^C,
\]
\[
v \mapsto \mathbb{E}[(X^*)^\otimes k v X^\otimes k]
\]
and let \(T^R\) be its restriction to \(V^R\) whenever \(X\) is real-valued.

In the following, we will shorten notation by writing \(V, T\) and \(l\) in place of the corresponding real and complex objects whenever a statement holds in both cases. We will adopt this convention for objects defined later on. Moreover, we will write \(\mathbb{K}\) as a placeholder for \(\mathbb{R}\) and \(\mathbb{C}\).

Finally, we denote by
\[
p_T(x) = x^l + c_1 x^{l-1} + \cdots + c_l
\]
the characteristic polynomial of \(T\).

**Part 1: \(\mu\) has finite support.**
Inductively applying the mixed-product identity of the Kronecker product
\[
(A_1 \otimes A_2)(A_3 \otimes A_4) = A_1 A_3 \otimes A_2 A_4
\]
for $A_1, A_2, A_3, A_4 \in \mathbb{C}^{d \times d}$ implies that
\[
    a_n = \mathbb{E} \left[ \text{Tr}(X_n^* \cdots X_1^* X_1 \cdots X_n)^k \right] = \mathbb{E} \left[ \text{Tr} \left( (X_n^* \cdots X_1^* X_1 \cdots X_n)^\otimes k \right) \right] \\
    = \text{Tr} \left( \mathbb{E} \left[ (X_n^* \otimes k \cdots (X_1^* \otimes k X_1^\otimes k \cdots X_n^\otimes k) \right] \right).
\]

We claim that the sequence $(a_n)_n$ satisfies the recurrence defined by the characteristic polynomial of $T$, i.e. for all $n \in \mathbb{N}$ we have
\[
a_{n+l} + c_1 a_{n+l-1} + \cdots + c_l a_n = 0.
\]

In order to see this, assume first that $T$ is diagonalizable. Then we can write the identity $I := I_d^\otimes k \in V$ as a linear combination of eigenvectors of $T$, i.e. there are $\lambda_1, \ldots, \lambda_l, \alpha_1, \ldots, \alpha_l \in \mathbb{C}$ and non-zero $v_1, \ldots, v_l \in V$ such that
\[
    T v_i = \lambda_i v_i \quad \text{and} \quad I = \sum_{i=1}^l \alpha_i v_i.
\]
This implies that
\[
a_n = \text{Tr} \left( \mathbb{E} \left[ (X_n^* \otimes k \cdots (X_1^* \otimes k \sum_{i=1}^l \alpha_i v_i X_1^\otimes k \cdots X_n^\otimes k \right] \right) \\
    = \sum_{i=1}^l \alpha_i \lambda_i \text{Tr} \left( \mathbb{E} \left[ (X_n^* \otimes k \cdots (X_2^* \otimes k v_i X_2^\otimes k \cdots X_n^\otimes k \right] \right) \right). \\
\]
Inductively, we obtain
\[
a_n = \sum_{i=1}^l \alpha_i \text{Tr}(v_i) \lambda_i^n,
\]
so that the sequence $(a_n)_n$ indeed satisfies the characteristic polynomial of $T$ under the assumption that this operator is diagonalizable.

Now fix $m$ and weights $p_1, \ldots, p_m > 0$ with $\sum p_i = 1$. Given $B_1, \ldots, B_m \in \mathbb{K}^{d \times d}$, define the (finitely supported) probability measure
\[
    \mu := \sum_{i=1}^m p_i \delta_{B_i},
\]
where $\delta_B$ denotes the Dirac measure at $B \in \mathbb{K}^{d \times d}$. Set $\beta = 1$ resp. $2$ when $X$ is real- resp. complex-valued. Endowing $(\mathbb{K}^{d \times d})^m = \mathbb{R}^{\beta md^2}$ with the Zariski topology, we claim that the set
\[
    M := \{(B_1, \ldots, B_m) \in (\mathbb{K}^{d \times d})^m : T \text{ diagonalizable} \} \subseteq \mathbb{R}^{\beta md^2}
\]
is dense. \footnote{We do not claim that this is a dense condition in $\mathbb{C}^{md^2}$ in the complex case, but only in $\mathbb{R}^{2md^2}$.} First, note that this set is non-empty: Choose $B_1 = \cdots = B_m = I$ all to be the identity matrix. Then $T$ is the identity on $V$, hence diagonalizable.
The next step is to prove that $M$ is Zariski-open. But this follows from the fact that the map

$$\tau : \mathbb{R}^{\beta md^2} \to \text{End}(V),$$

$$(B_1, \ldots, B_m) \mapsto T$$

is polynomial in the entries of the $B_i$ and diagonalizability of $T$ is an open condition on the right-hand side. Hence, $M$ is indeed a dense set. Note that $\tau$ is not polynomial on $\mathbb{C}^{dm^2}$ in the complex case.

Lastly, consider for fixed $n$ the composition of maps

$$\mathbb{R}^{\beta md^2} \to \text{End}(V) \times \mathbb{R}^{l+1} \to \mathbb{C}^l \times \mathbb{R}^{l+1} \to \mathbb{C}$$

given by

$$(B_1, \ldots, B_m) \mapsto (T, (a_{n+l}, \ldots, a_n)),$$

$$(T, (b_1, \ldots, b_0)) \mapsto (p_T, (b_1, \ldots, b_0)),$$

$$((c_1, \ldots, c_l), (b_1, \ldots, b_0)) \mapsto b_l + c_1b_{l-1} + \cdots + c_lb_0,$$

where in the second map we send an operator to its characteristic polynomial viewed as a vector in its coefficients. It is clear that each of these maps is continuous, and we know that their composition

$$(B_1, \ldots, B_m) \mapsto a_{n+l} + a_{n+l-1}c_1 + \cdots + a_nc_l$$

vanishes on the dense set $M$, hence everywhere, which settles Part 1.

**Part 2: $\mu$ has compact support.**

Our goal is to show that the equation

$$a_{n+l} + c_1a_{n+l-1} + \cdots + c_la_n = 0$$

still holds for all $n$. Let $K = \text{supp } \mu$ and let $(\mu_m)_m$ be a sequence of probability measures with finite support contained in $K$ such that $\mu_m \to \mu$ weakly, i.e. for all continuous bounded functions $f : \mathbb{K}^{d \times d} \to \mathbb{R}$ we have

$$\int_{\mathbb{K}^{d \times d}} f d\mu_m \to \int_{\mathbb{K}^{d \times d}} f d\mu$$

as $m \to \infty$. Let $(X_{n,m})_n$ be i.i.d. sequences of random variables distributed according to $\mu_m$, let

$$a_{n,m} = \mathbb{E}[(\|X_{1,m} \cdots X_{n,m}\|_H)^{2k}]$$

and similarly define $T_m$ and $c_{i,m}$ w.r.t. $\mu_m$. Since the measures $\mu_m$ have finite support, we know that

$$a_{n+l,m} + c_{1,m}a_{n+l-1,m} + \cdots + c_{l,m}a_{n,m} = 0$$

holds for all $n,m$. It thus suffices to show that for any fixed $n$ and $i$ we have $a_{n,m} \to a_n$ and $c_{i,m} \to c_i$ as $m \to \infty$. 
It is a standard fact that the weak convergence of $(\mu_m)_m$ implies the weak convergence of the product measures $\mu_m^{\otimes n} \to \mu^{\otimes n}$. Moreover, we have

\[
a_n = \int_{[\mathbb{R}^{d \times d}]^n} \|A_1 \cdots A_n\|^{2k}_{HS} d\mu^{\otimes n}(A_1, \ldots, A_n),
\]

\[
a_{n,m} = \int_{[\mathbb{R}^{d \times d}]^n} \|A_1 \cdots A_n\|^{2k}_{HS} d\mu_m^{\otimes n}(A_1, \ldots, A_n).
\]

Letting $f : ([\mathbb{K}^{d \times d}]^n \to \mathbb{R}$ be a bounded continuous function which coincides with $(A_1, \ldots, A_n) \mapsto \|A_1 \cdots A_n\|^{2k}_{HS}$ on $K^{\times n}$ implies the convergence $a_{n,m} \to a_n$ as $m \to \infty$ for any fixed $n$.

To show the convergence of $c_{i,m}$, note that it suffices to show that each entry of $T_m$ in some fixed basis converges to the corresponding entry of $T$. But for this, in turn, it suffices in both the real and the complex case to show the same property for the extended operator

\[
\hat{T} : ([\mathbb{C}^{d \times d}]^\otimes k \to ([\mathbb{C}^{d \times d}]^\otimes k
\]

and the corresponding operators $\hat{T}_m$, since $T$ and $T_m$ are just restrictions of these operators to a common invariant subspace. Let us take the standard basis given by $e_{i_1j_1} \otimes \cdots \otimes e_{i_kj_k}$ with $i_1, j_1, \ldots, i_k, j_k \in \{1, \ldots, d\}$, where $e_{ij} \in \mathbb{C}^{d \times d}$ is the matrix with entry $ij$ being $1$ and the rest $0$. One verifies that

\[
(\hat{T}(e_{i_1j_1} \otimes \cdots \otimes e_{i_kj_k}))e_{i_1'j_1'} \otimes \cdots \otimes e_{i_k'j_k'} = \mathbb{E}[\sum_{i_1 \neq i_1'} X_{i_1i_1'} X_{j_1j_1'} \cdots \sum_{i_k \neq i_k'} X_{i_ki_k'} X_{j_kj_k'}]
\]

\[
= \int_{[\mathbb{R}^{d \times d}]^n} \sum_{i_1 \neq i_1'} A_{i_1i_1'} A_{j_1j_1'} \cdots \sum_{i_k \neq i_k'} A_{i_ki_k'} A_{j_kj_k'} d\mu(A),
\]

where in the complex case the integral is taken over real and imaginary part separately; analogous statements hold for $\hat{T}_m$. Again taking bounded continuous functions $f : [\mathbb{K}^{d \times d}] \to \mathbb{R}$ which coincide with real resp. imaginary part of $A \mapsto \sum_{i_1 \neq i_1'} A_{i_1i_1'} A_{j_1j_1'} \cdots \sum_{i_k \neq i_k'} A_{i_ki_k'} A_{j_kj_k'}$ on $K$ implies the claim.

**Part 3: The general case.**

Let $R > 0$ be sufficiently large so that $\mu(B_R(0)) > 0$. Define the (conditional) probability measure

\[
\mu_c^R(M) := \frac{\mu(M \cap B_R(0))}{\mu(B_R(0))}.
\]

Denoting by $(X_n^R)_n$ a family of i.i.d. random variables corresponding to $\mu_c^R$, we can set

\[
a_n^R := \mathbb{E}[\|X_1^R \cdots X_n^R\|^{2k}_{HS}]
\]

and similarly $T^R$ and $c_i^R$. Since $\mu_c^R$ has compact support, we know that

\[
a_n^R + c_1^R a_{n-1}^R + \cdots + c_i^R a_n^R = 0
\]

holds for all $n$. It thus suffices to show that for any fixed $n$ and $i$ we have $a_n^R \to a_n$ and $c_i^R \to c_i$ as $R \to \infty$. 
We have
\[ a_n^R = \int_{\mathbb{R}^{d \times d}} \|A_1 \cdots A_n\|_{HS}^{2k} \, d(\mu^R_{c} \times \cdots \times n)(A_1, \ldots, A_n) \]
\[ = \frac{1}{\mu(B_R(0)^n)} \int_{\mathbb{R}^{d \times d}} \|A_1 \cdots A_n\|_{HS}^{2k} \mathbb{1}(\|A_1\|_{HS}^{2k} \leq R) \cdots \mathbb{1}(\|A_n\|_{HS}^{2k} \leq R) \, d\mu_{\circ}^n(A_1, \ldots, A_n) \]

Since \( \mu(B_R(0)) \to 1 \) as \( R \to \infty \), it suffices to show that this integral converges to \( a_n \) as \( R \to \infty \). But

\[ \left| a_n - \int_{\mathbb{R}^{d \times d}} \|A_1 \cdots A_n\|_{HS}^{2k} \mathbb{1}(\|A_1\|_{HS}^{2k} \leq R) \cdots \mathbb{1}(\|A_n\|_{HS}^{2k} \leq R) \, d\mu_{\circ}^n(A_1, \ldots, A_n) \right| \]
\[ \leq n \int_{\mathbb{R}^{d \times d}} \|A_1 \cdots A_n\|_{HS}^{2k} \mathbb{1}(\|A_1\|_{HS}^{2k} > R) \, d\mu_{\circ}^n(A_1, \ldots, A_n) \]
\[ \leq n \int_{\mathbb{R}^{d \times d}} \|A_1\|_{HS}^{2k} \mathbb{1}(\|A_1\|_{HS}^{2k} > R) \, d\mu(A_1) \int_{\mathbb{R}^{d \times d}} \|A_2\|_{HS}^{2k} \, d\mu(A_2) \cdots \int_{\mathbb{R}^{d \times d}} \|A_n\|_{HS}^{2k} \, d\mu(A_n) \xrightarrow{R \to \infty} 0 \]

by assumption.

For the convergence of \( c_i^R \) to \( c_i \), it again suffices to show that every entry of \( T^R \) converges to the corresponding entry of \( T \) in some fixed basis. Again, it suffices to show this for the extended operators \( \tilde{T} \) and \( \tilde{T}^R \) defined in the obvious way. But we have

\[ \left| \tilde{T} - \frac{1}{\mu(B_R(0))^{2k}} \tilde{T}^R(e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k})e_{i_1' j_1'} \otimes \cdots \otimes e_{i_k' j_k'} \right| \]
\[ = \left| \int_{\mathbb{R}^{d \times d}} A_{i_1 j_1} \cdots A_{i_k j_k} A_{i_1' j_1'} \cdots A_{i_k' j_k'} \mathbb{1}(\|A\|_{HS}^{2k} > R) \, d\mu(A) \right| \leq \int_{\mathbb{R}^{d \times d}} \|A\|_{HS}^{2k} \mathbb{1}(\|A\|_{HS}^{2k} > R) \, d\mu(A) \to 0 \]
as \( R \to \infty \), hence the claim.

\( \square \)

**Remark.** The idea of reducing matrix dimensions by looking at symmetric algebras in a similar context of Theorem 2 has been considered in [1,8] related to Kronecker and semidefinite lifting.

One might be interested in the optimality of \( l^R \). In the real case, we can in fact prove that \( l^R \) is optimal in the sense that for all \( d, k \geq 1 \) there exists \( X \) such that the sequence \( (a_n) \) does not satisfy a linear recurrence of any shorter length.

In this case, it in fact suffices to take \( X \) to be a deterministic distribution supported in a single point \( A \). Let \( \lambda_1, \ldots, \lambda_d \) be the eigenvalues of \( A \) and assume for simplicity that they are algebraically independent. One verifies that the eigenvalues of

\[ S_d \to S_d \]
\[ B \mapsto B^T AB \]

are given by \( \lambda_i \lambda_j \) for \( 1 \leq i \leq j \leq d \), which we will denote by \( \mu_1, \ldots, \mu_{d'} \) with \( d' = \binom{d+1}{2} \). Moreover, it is elementary to see that the eigenvalues of \( T \) are then given by \( \mu_i \cdots \mu_k \) for \( 1 \leq i_1 \leq \cdots \leq i_k \leq d' \), and that they are pairwise distinct. A generic choice of \( A \) will satisfy \( \alpha_i \text{Tr}(v_i) \neq 0 \) for all \( i \), which then implies the claim.
In the complex case, taking a deterministic $X$ and doing the same construction as in the real case gives an operator $T$ which can have at most

$$\tilde{l}_C = \left(\frac{k + d - 1}{k}\right)^2$$

distinct eigenvalues. The above argument does prove that there are $X$ such that $(a_n)_n$ satisfies a linear recurrence of no shorter length than $\tilde{l}_C$, but it does not give optimality of $\tilde{l}_C$. For this, one would need to take a more complicated $X$, for which it is significantly more difficult to explicitly compute the eigenvalues of $T$. Nonetheless, numerical evidence in this case does suggest that $\tilde{l}_C$ might still be optimal.

The eigenvalue of $T$ of largest real part is essentially the generalized joint spectral radius of $X$. More details on this can be found in section 4.

For further reference, we would like to record the following

**Corollary 4.** Let $X, X_1, X_2, \ldots$ be i.i.d. $\mathbb{C}^{d \times d}$-valued random variables with

$$\mathbb{E} \left[ \|X\|_{HS} \mathbb{1}(\|X\|_{HS} > R) \right] \xrightarrow{R \to \infty} 0$$

and define

$$a_n := \mathbb{E} \left[ \|X_1 \cdots X_n\|_{HS}^2 \right].$$

Let

$$T : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d},$$

$$A \mapsto \mathbb{E} [X^* AX]$$

and denote by

$$p_T(x) = x^l + c_1 x^{l-1} + \cdots + c_l$$

the characteristic polynomial of $T$, where $l := d^2$ is the dimension of $\mathbb{C}^{d \times d}$. Then for any $n \in \mathbb{N}$, we have

$$a_{n+1} + c_1 a_{n+l-1} + \cdots + c_l a_n = 0.$$

3. **Second-Moment Estimate for Random Matrix-Valued Multiplicative Functions**

We will fix the following notation: We set

$$P(s, z) := \prod_p \left(1 + \frac{z}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^z$$

and

$$F(s, z) := \frac{P(s, z)}{\Gamma(z)},$$

as well as $P(z) := P(1, z)$ and $F(z) := F(1, z)$. 
3.1. The Diagonalizable Case. Using Theorem 2 we are now in a position to prove Theorem 1.

Proof of Theorem 1. We have
\[ E\left[\left\| \sum_{n \leq x} f(n) \right\|_{HS}^2 \right] = \sum_{n_1, n_2 \leq x} E[\text{Tr}(f(n_1)^*f(n_2))] = \sum_{n \leq x} \text{Tr}(E[f(n)^*f(n)]). \]

By definition of \( f \), the contribution of squarefree \( n \) to this sum depends only on \( \omega(n) \) and is given by
\[ a_{\omega(n)} = \text{Tr}(E[X_{\omega(n)}^*X_{\omega(n)}]) \]
for i.i.d. random variables \( X, X_1, \ldots, X_{\omega(n)} \), where \( a_{\omega(n)} \) is defined as in Corollary 4. But this implies
\[ E\left[\left\| \sum_{n \leq x} f(n) \right\|_{HS}^2 \right] = \sum_{n \leq x} \mu^2(n) a_{\omega(n)} = \sum_{n \leq x} \mu^2(n) \lambda^2_{\omega(n)}. \]

It thus remains to prove the following

Proposition 5. For any \( N \in \mathbb{N} \) and \( z \in C \) there are explicit constants \( C_1, \ldots, C_N \) (depending on \( z \)) such that
\[ \sum_{n \leq x} \mu^2(n) z^{\omega(n)} = x \sum_{m=1}^{N} C_i (\log x)^{z-m} + O(x (\log x)^{z-N-1}). \]

For example, we have
\[ \sum_{n \leq x} \mu^2(n) z^{\omega(n)} = P(z) x (\log x)^{z-1} + \frac{(\gamma z - 1) P(z) + P_s(z)}{\Gamma(z - 1)} x (\log x)^{z-2} + O(x (\log x)^{z-3}), \]
where \( P_s(z) \) denotes the derivative of \( P(s, z) \) w.r.t. \( s \) evaluated at \( s = 1 \), and where \( \gamma \) is the Euler-Mascheroni constant. In fact, the error terms are uniform over \( |z| < A \).

This follows from [4, Theorem, p. 188] by setting
\[ a_z(n) := \mu^2(n) z^{\omega(n)} \]
in the notation there, so that
\[ f(s, z) := \sum_{n \geq 1} \frac{a_z(n)}{n^s} = \sum_{n \geq 1} \frac{\mu^2(n) z^{\omega(n)}}{n^s}, \]
which gives
\[ g(s, z) := (s - 1)^s f(s, z) = [(s - 1)\zeta(s)]^z \zeta(s)^{-z} f(s, z) = [(s - 1)\zeta(s)]^z P(s, z). \]

Taylor expansion of \((s - 1)\zeta(s)\) around \( s = 1 \) and application of the Binomial Theorem quickly yields, for example,
\[ [(s - 1)\zeta(s)]^z = 1 + \gamma z(s - 1) + O((s - 1)^2). \]
This quickly gives the second part of the assertion.

Since we can compute an arbitrary number of terms in this Taylor series and also the one for \( P \) around \( s = 1 \), this gives the first claim by [4, Theorem, p. 188]. This concludes the proof of Theorem 1.
Remark. The proof shows that the constants $C_{i,m}$ in Theorem 1 are explicit. Let $v_1, \ldots, v_l \in V$ be the eigenvectors of $T$ associated to $\lambda_1, \ldots, \lambda_l$, and let $\alpha_1, \ldots, \alpha_l \in \mathbb{C}$ be such that $I = \sum \alpha_i v_i$. Then, for example, we have

$$C_{i,1} = \alpha_i \text{Tr}(v_i) F(\lambda_i)$$

and

$$C_{i,2} = \alpha_i \text{Tr}(v_i) \left( \frac{\gamma \lambda_i - 1}{\Gamma(\lambda_i - 1)} \right).$$

By the methods outlined in the proof of Proposition 5 one can compute arbitrarily many such constants.

We also remark that if $X$ is real-valued then by Theorem 2 we can restrict $T$ to $S_d$ and set $l = \left( \begin{array}{c} d+1 \\ 2 \end{array} \right)$.

Example 6. Let $X$ be the uniform distribution on the set

$$S = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$$

and let $f$ be the associated matrix-valued multiplicative function. Then we have $\text{Im} f = \text{SL}_2(\mathbb{Z}) \cup \{0\}$ almost surely. If

$$T : S_d \to S_d,$$

$$A \mapsto \mathbb{E}[X^T A X]$$

then it is verified by evaluating at $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ that $T$ can be represented by the matrix

$$T = \frac{1}{4} \begin{pmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

with eigenvalues

$$\lambda_1 = \frac{3 + \sqrt{3}}{4}, \quad \lambda_2 = \frac{3 - \sqrt{3}}{4} \quad \text{and} \quad \lambda_3 = \frac{1}{2}$$

and eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Moreover, we can write the identity matrix as $I_2 = \sum_{i=1}^3 \alpha_i v_i$ with

$$\alpha_1 = \frac{3 + \sqrt{3}}{6}, \quad \alpha_2 = -\frac{3 + \sqrt{3}}{6} \quad \text{and} \quad \alpha_3 = 0.$$ 

We obtain

$$C_{1,1} = \left( 1 + \frac{2}{\sqrt{3}} \right) F(\lambda_1) = 1.256 \ldots, \quad C_{2,1} = \left( 1 - \frac{1}{\sqrt{3}} \right) F(\lambda_2) = -0.048 \ldots \quad \text{and} \quad C_{3,1} = 0$$

as well as

$$C_{1,2} = 0.251 \ldots, \quad C_{2,2} = -0.017 \ldots \quad \text{and} \quad C_{3,2} = 0.$$
We infer
\[ E \left[ \| \sum_{n \leq x} f(n) \|_{HS}^2 \right] = x \left( C_{1,1}(\log x)^{\lambda_1-1} + C_{2,1}(\log x)^{\lambda_2-1} + C_{1,2}(\log x)^{\lambda_1-2} + C_{2,2}(\log x)^{\lambda_2-2} \right) \\
+ O \left( x(\log x)^{\lambda_1-3} \right). \]

3.2. The Non-Diagonalizable Case. If \( T \) is not diagonalizable, it turns out that we need to find an estimate for a more difficult quantity, and we are only able to prove an ineffective asymptotic. More precisely, we need the following

Proposition 7. For fixed \( z \in \mathbb{C} \setminus \mathbb{Z}^- \) and \( r \in \mathbb{N}_0 \), we have
\[ \sum_{n \leq x} \mu^2(n)\omega(n)^rz^{\omega(n)} \sim z^r F(z)x(\log x)^{\lambda_1-2}(\log 2x)^{\lambda_1-3}. \]

Proof. The idea is to find an asymptotic as \( s \to 1 \) for the associated Dirichlet series and then to apply Delange’s Theorem [3, Théorème IV], compare also [11, Theorem 7.28]. A crucial point in proving this is that
\[ \sum_{n \geq 1} \mu^2(n)\omega(n)(\omega(n) - 1) \cdots (\omega(n) - r + 1)z^{\omega(n)} = z^r \frac{d^r}{dz^r} \left( \sum_{n \geq 1} \mu^2(n)z^{\omega(n)} \right). \]
Expanding the falling factorials using the Stirling numbers of the second kind, denoted by curly brackets, we obtain
\[ \sum_{n \geq 1} \mu^2(n)\omega(n)^rz^{\omega(n)}n^s = \sum_{k=0}^r \left\{ r \right\} k^k \frac{d^k}{dz^k} \left( \sum_{n \geq 1} \mu^2(n)z^{\omega(n)} \right). \]
But we have
\[ \frac{d}{dz} \left( \sum_{n \geq 1} \frac{\mu^2(n)z^{\omega(n)}}{n^s} \right) = \frac{d}{dz} (\zeta(s)^2 F(s, z)) = (\log \zeta(s))^2 F(s, z) + \zeta(s)^2 F_z(s, z). \]
Here, \( F_z \), denotes the derivative of \( F \) in the second component. Inductively we obtain expansions of the form
\[ \frac{d^k}{dz^k} \left( \sum_{n \geq 1} \frac{\mu^2(n)z^{\omega(n)}}{n^s} \right) = \zeta(s)^2 \left[ (\log \zeta(s))^k F(s, z) + \ldots \right], \]
where the other terms involve lower powers of \( \log \zeta(s) \) as well as derivatives of \( F \). We thus obtain an expansion of the form
\[ \sum_{n \geq 1} \frac{\mu^2(n)\omega(n)^rz^{\omega(n)}}{n^s} = z^r \zeta(s)^2 \left[ (\log \zeta(s))^r F(s, z) + \ldots \right], \]
where again the other terms involve lower (non-negative, integral) powers of \( \log \zeta(s) \) multiplied by functions of \( s \) and \( z \) which are holomorphic around \( s = 1 \) for any \( z \). We are thus in a position to
apply Delange’s Theorem, which indeed implies that

$$\sum_{n \leq x} \mu^2(n)\omega(n) r^{\omega(n)} \sim z^r F(z)x(\log x)^{z-1}(\log_2 x)^r$$

when $z \in \mathbb{C} \setminus \mathbb{Z}^-$, as claimed. \qed

This Proposition allows us to prove the following

**Theorem 8.** Let $d \geq 1$ be an integer, let $X$ be a $\mathbb{C}^{d \times d}$-valued random variable and let $f$ be the associated matrix-valued multiplicative function. Suppose that $\mathbb{E}X = 0$ and

$$\mathbb{E} \left[ \|X\|_{HS}^2 \mathbf{1}(\|X\|_{HS} > R) \right] \xrightarrow{R \to \infty} 0.$$

Define

$$T : \mathbb{C}^{d \times d} \to \mathbb{C}^{d \times d},$$

$$A \mapsto \mathbb{E}[X^*AX],$$

and let $\lambda_1, \ldots, \lambda_t$ be the (distinct) eigenvalues of $T$ arranged in descending order according to their real parts. Let $p_T$ be the characteristic polynomial of $T$ and let $c_1, \ldots, c_l$ and $m_1, \ldots, m_t$ be such that

$$p_T(x) = x^l + c_1x^{l-1} + \cdots + c_l = (x - \lambda_1)^{m_1} \cdots (x - \lambda_t)^{m_t}.$$  

Further, define

$$a_n := \mathbb{E} \left[ \|X_1 \cdots X_n\|_{HS}^{2k} \right],$$

where $X_1, X_2, \ldots$ are i.i.d. copies of $X$, and let $g_1, \ldots, g_t$ be the polynomials satisfying $d_i := \deg g_i < m_i$ and

$$a_n = g_1(n)\lambda_1^n + \cdots + g_t(n)\lambda_t^n$$

(see Theorem 2 and Lemma 3). Let $R$ be the maximal real part among those $\lambda_i$ with $d_i > 0$. Define $L_1, L_2$ and $L_3$ to be the collection of $i$ such that $\Re \lambda_i > R$, $\Re \lambda_i = R$ and $\Re \lambda_i < R$, respectively. Lastly, let $d_{\text{max}} = \max_{i \in L_2} d_i$ and $L_2' = \{i \in L_2 : d_i = d_{\text{max}}\}$. Then for any $N \in \mathbb{N}$ there are explicit constants $C_{i,m}$ for $i \in L_1$, $m = 1, \ldots, N$ and $C_j$ for $j \in L_2'$ such that

$$\mathbb{E} \left[ \left\| \sum_{n \leq x} f(n) \right\|_{HS}^2 \right] = x \sum_{m=1}^N \sum_{i \in L_1} C_{i,m}(\log x)^{\lambda_i - m} + (1 + o(1)) \sum_{j \in L_2'} C_j x(\log x)^{\lambda_j - 1}(\log_2 x)^{d_{\text{max}}}$$

$$+ O \left( x(\log x)^{\lambda_1 - N - 1} \right).$$
Proof. Writing $b_i$ for the leading coefficient of $g_i$, the same argument as in Theorem 1 implies

$$
\mathbb{E}\left[ \left\| \sum_{n \leq x} f(n) \right\|_{HS}^2 \right] = \sum_{n \leq x} \mu^2(n) \sum_{i=1}^{t} g_i(\omega(n)) \lambda_i^{\omega(n)}
$$

$$
= \sum_{i \in L_1} b_i \sum_{n \leq x} \mu^2(n) \lambda_i^{\omega(n)} + \sum_{i \in L_2} b_i \sum_{n \leq x} \mu^2(n) \omega(n) d_{max} \lambda_i^{\omega(n)}
$$

$$
+ O \left( \sum_{i \in L_3} \sum_{n \leq x} \mu^2(n) \omega(n) d_{max} - 1 \lambda_i^{\omega(n)} \right) + O_\varepsilon \left( \sum_{i \in L_3} \sum_{n \leq x} \mu^2(n) (\lambda_i + \varepsilon)^{\omega(n)} \right).
$$

Regarding the first summand, Proposition 5 directly tells us that for any $N \in \mathbb{N}$ there are explicit constants $C_{i,m}$ such that

$$
\sum_{i \in L_1} b_i \sum_{n \leq x} \mu^2(n) \lambda_i^{\omega(n)} = x \sum_{m=1}^{N} C_{i,m} (\log x)^{\lambda_i - m} + O \left( x(\log x)^{\max \Re \lambda_i - N - 1} \right).
$$

Using Proposition 7 on the second summand directly implies

$$
\sum_{j \in L_2} b_j \sum_{n \leq x} \mu^2(n) \omega(n) d_{max} \lambda_j^{\omega(n)} = (1 + o(1)) \sum_{j \in L_2^\prime} C_j^\prime (\log x)^{\lambda_j - 1}(\log_2 x)^{d_{max}}
$$

for some explicit constants $C_j^\prime$.

Proposition 7 furthermore implies

$$
\sum_{j \in L_2^\prime} \sum_{n \leq x} \mu^2(n) \omega(n) d_{max} - 1 \lambda_j^{\omega(n)} = O \left( x(\log x)^{R - 1}(\log_2 x)^{d_{max} - 1} \right) = o \left( x(\log x)^{R - 1}(\log_2 x)^{d_{max}} \right).
$$

For the last error term, fix $\varepsilon > 0$ such that $\lambda_i + \varepsilon < R$ for all $i \in L_3$. Then

$$
\sum_{i \in L_3} \sum_{n \leq x} \mu^2(n) (\lambda_i + \varepsilon)^{\omega(n)} = o \left( x(\log x)^{R - 1} \right)
$$

and the claim follows. \qed

4. An Upper Bound for Higher Even Moments

Let $s \geq 1$, and let $X, X_1, X_2, \ldots$ be i.i.d. $\mathbb{C}^{d \times d}$-valued random variables with

$$
\mathbb{E}[\|X\|_{HS}^s] < \infty.
$$

Then

$$
\rho_s := \rho_s(X) := \lim_{n \to \infty} \mathbb{E} \left[ \|X_1 \cdots X_n\|_{HS}^s \right]^{1/n}
$$

will be called the spectral $s$-radius of $X$. If $S \subset \mathbb{C}^{d \times d}$ is bounded then

$$
\rho_\infty(S) := \lim_{k \to \infty} \sup \{ \|A_1 \cdots A_k\|_{HS}^{1/k} : A_i \in S \}
$$

is called the joint spectral radius of $S$. Note that all these quantities are in fact independent of the chosen norm, since all norms on $\mathbb{C}^{d \times d}$ are equivalent.
The joint spectral radius has been studied in great detail in contexts such as dynamical systems, wavelets, optimization and control. We refer the interested reader to [7]. For the generalized joint spectral radius, its geometric interpretation and relation to Kronecker products, see e.g. [9,10].

We note at this point that by Hölder’s inequality, \( \rho_s \) is monotonically increasing and if \( X \) is the uniform distribution on a bounded set \( S \) then we have \( \rho_s \uparrow \rho_\infty \) as \( s \to \infty \). Also, note that \( \rho_{2k} = \lambda_1^{1/2k} \), where \( \lambda_1 \geq 0 \) is as in Theorem [8].

The goal of this section is to prove the following

**Theorem 9.** Let \( k \geq 2 \) be an integer, and let \( X \) be a symmetric \( \mathbb{C}^{d \times d} \)-valued random variable satisfying \( \mathbb{E} X = 0 \) and

\[
\mathbb{E} \left[ \|X\|_{HS}^{2k} \mathbbm{1}(\|X\|_{HS}^2 > R) \right] \to 0
\]

as \( R \to \infty \). Let \( f \) be the random matrix-valued multiplicative function associated to \( X \). Then we have

\[
\mathbb{E} \left[ \left\| \sum_{n \leq x} f(n) \right\|_{HS}^{2k} \right] \ll x^k (\log x) [\rho_{2k}^2 + 1]^2 - 2k,
\]

where \( \lfloor \cdot \rfloor \) denotes the integral part.

**Proof.** Denoting by \( \square \) a generic square, we have

\[
\mathbb{E} \left[ \left\| \sum_{n \leq x} f(n) \right\|_{HS}^{2k} \right] = \mathbb{E} \left[ \text{Tr} \left( \sum_{n_1, n_2 \leq x} f(n_1)^* f(n_2) \right)^k \right]
\]

\[
= \sum_{n_1, \ldots, n_{2k} \leq x} \text{Tr} \left( \mathbb{E} \left[ f(n_1)^* f(n_2) \otimes \cdots \otimes f(n_{2k-1})^* f(n_{2k}) \right] \right)
\]

\[
= \sum_{n_1, \ldots, n_{2k} \leq x} \text{Tr} \left( \mathbb{E} \left[ f(n_1) \otimes f(n_3) \otimes \cdots \otimes f(n_{2k-1})^* f(n_2) \otimes \cdots f(n_{2k}) \right] \right)
\]

\[
\leq \sum_{n_1, \ldots, n_{2k} \leq x} \mathbb{E} \left[ \left\| f(n_1) \otimes \cdots \otimes f(n_{2k-1}) \right\|_{HS} \left\| f(n_2) \otimes \cdots \otimes f(n_{2k}) \right\|_{HS} \right]
\]

\[
\leq \sum_{n_1, \ldots, n_{2k} \leq x} \mathbb{E} \left[ \left\| f(n_1) \right\|_{HS}^{2k} \right]^{1/2k} \cdots \mathbb{E} \left[ \left\| f(n_{2k}) \right\|_{HS}^{2k} \right]^{1/2k}.
\]

But from the definition of \( \rho_{2k} \), we see that

\[
\mathbb{E} \left[ \left\| f(n) \right\|_{HS}^{2k} \right]^{1/2k} \ll \mu^2(n) (\rho_{2k} + \varepsilon)^\omega(n)
\]

and in particular

\[
\mathbb{E} \left[ \left\| f(n) \right\|_{HS}^{2k} \right]^{1/2k} \ll \mu^2(n) \rho_{2k}^2 + 1]^\omega(n)/2.
\]
We thus obtain
\[
\mathbb{E}\left[\left\| \sum_{n \leq x} f(n) \right\|_{HS}^{2k}\right] \ll \sum_{n_1, \ldots, n_{2k} \leq x} \mu^2(n_1) \cdots \mu^2(n_{2k}) \frac{1}{\log^2(n_1) + \cdots + \log(n_{2k})/2}.
\]

It thus remains to prove that
\[
\sum_{n_1, \ldots, n_{2k} \leq x} \mu^2(n_1) \cdots \mu^2(n_{2k}) m^{\log(k)} \ll x^k \log x \frac{(2k)!}{2^k} \tag{4}
\]
for all \( m \in \mathbb{N} \). We proceed similar to the proof of [5, Theorem 4]. To this end, let \( g \) be the multiplicative function supported on squarefree integers such that \( g(n_1, \ldots, n_{2k}) = m^{\log(n_1) + \cdots + \log(n_{2k})/2} \) when \( n_1 \cdots n_{2k} \) is a square, and 0 otherwise. Then the associated multiple Dirichlet series
\[
G(s) := \sum_{d_1, \ldots, d_{2k} \geq 1} \frac{g(d_1, \ldots, d_{2k})}{d_1^{s_1} \cdots d_{2k}^{s_{2k}}}
\]
has the Euler product representation
\[
G(s) = \prod_p \sum_{0 \leq \alpha_1, \ldots, \alpha_{2k} \leq 1} \frac{m^{\alpha_1 + \cdots + \alpha_{2k}/2}}{p^{\alpha_1 + \cdots + \alpha_{2k} s_{2k}}}.
\]
This factors as
\[
H(s_1, \ldots, s_{2k}) \prod_{1 \leq i < j \leq 2k} \zeta(s_i + s_j)^m
\]
with \( H \) being holomorphic strictly to the left of \( s = \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \). The claim follows from [2, Theorem 2], choosing each of the linear forms \( s_i + s_j \) for \( 1 \leq i < j \leq 2k \) precisely \( m \) times, so that they are \( m \left( \frac{2k}{2} \right) \) and have rank \( 2k \) (this is where we are using that \( k \geq 2 \)).

**Remark.** Note that our argument in fact implies a stronger statement than (4), namely that for fixed \( m \in \mathbb{N} \) we have
\[
\sum_{n_1, \ldots, n_{2k} \leq x} \mu^2(n_1) \cdots \mu^2(n_{2k}) m^{\log(n_1) + \cdots + \log(n_{2k})/2} \sim C_k x^k \log x \frac{(2k)!}{2^k}.
\]
It seems rather natural, also in light of Proposition 5, to conjecture that this asymptotic holds for all fixed \( z > 0 \) (say) in place of \( m \in \mathbb{N} \). However, this is not possible when \( z \) is small: Looking only at the contribution of tuples \( (n_1, \ldots, n_{2k}) = (p_1, \ldots, p_k, p_1, \ldots, p_k) \), we see that for any fixed \( z > 0 \) we have
\[
\sum_{n_1, \ldots, n_{2k} \leq x} \mu^2(n_1) \cdots \mu^2(n_{2k}) z^{\log(n_1) + \cdots + \log(n_{2k})/2} \gg \frac{x^k}{\log x}.\]
When \( z \) is sufficiently small (depending only on \( k \)) then this is clearly a contradiction. It would be very interesting to know what the correct asymptotic for this expression is, or more generally for any multiple Dirichlet series of this type, i.e. to have a generalisation of [2, Theorem 2] to poles...
of non-integral order. Our remark here suggests that this is not as straightforward as one might expect.

Note also that Theorem 2 implies that we can improve (3) to
\[
(\ast) \quad \mathbb{E}[\|f(n)\|_{HS}^{2k}]^{1/2k} \ll \mu^2(n)\omega(n)^{r/2k}\rho_{2k}^{(n)},
\]
where \(0 \leq r < \ell^2\) is the degree of \(g_1\). This leads in a natural way to the even more general question of obtaining an asymptotic (or upper bound) for multiple Dirichlet series with a pole of non-integral order times a logarithmic pole.

In particular, (\ast\) implies that if \(T\) is diagonalizable (or more generally if \(\deg g_1 = 0\)) then we get
\[
\mathbb{E}[\|f(n)\|_{HS}^{2k}]^{1/2k} \ll \mu^2(n)\rho_{2k}^{(n)}.
\]
If in addition \(\rho_{2k}^2\) is an integer, our argument thus gives
\[
\mathbb{E}\left[\left|\sum_{n \leq x} f(n)\right|_{HS}^{2k}\right] \ll x^k (\log x)^{\frac{\rho_{2k}^2(2k) - 2k}{2}}
\]
in place of (\dagger\). In particular, if \(f\) is a Rademacher multiplicative function then \(\rho_{2k} = 1\) for all \(k\) and up to constant we obtain the optimal upper bound. Noting that all our inequalities in the proof are in fact equalities in this case and that (\dagger\) can be improved to an asymptotic, we can recover [5, Theorem 4], but this leads to the identical argument as it is carried out there.

It would be interesting to know if one can obtain a lower bound for the higher moments, for example in terms of the joint spectral subradius.

**Example 10.** We continue with example 6. We were not able to find explicit expressions for \(\rho_{2k}\) when \(k \geq 2\); it seems quite plausible that such expressions don’t exist. However, we can bound \(\rho_{2k}\) from above by the joint spectral radius \(\rho_{\infty}(S)\). Moreover, using the JSR toolbox for Matlab (see [12] for its documentation and instructions for installation), we could compute that \(\rho_{\infty}^2 = 1.8173540 \cdots < 2\). In particular, we see that
\[
\mathbb{E}\left[\left|\sum_{n \leq x} f(n)\right|_{HS}^{2k}\right] \ll x^k (\log x)^{4k(k-1)}
\]
holds for \(k \geq 2\) and \(x \geq 2\).

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