EVOLUTION DYNAMICS OF CONFORMAL MAPS WITH QUASICONFORMAL EXTENSIONS

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ABSTRACT. We study one-parameter curves on the universal Teichmüller space \( T \) and on the homogeneous space \( M = \text{Diff} S^1/\text{Rot} S^1 \) embedded into \( T \). As a result, we deduce evolution equations for conformal maps that admit quasiconformal extensions and, in particular, such that the associated quasidisks are bounded by smooth Jordan curves. Some applications to Hele-Shaw flows of viscous fluids are given.

1. INTRODUCTION

Let \( U \) denote the unit disk in the Riemann sphere \( \hat{\mathbb{C}} \) and \( U^* = \hat{\mathbb{C}} \setminus \hat{U} \), where \( \hat{U} \) is the closure of \( U \). By \( S \) we denote the class of all holomorphic univalent functions in \( U \) normalized by \( f(\zeta) = \zeta + a_2 \zeta^2 + \ldots, \zeta \in U \), and by \( \Sigma \), the class of all univalent meromorphic functions in \( U^* \) normalized by \( f(\zeta) = \zeta + c_0 + \frac{c_1}{\zeta} + \ldots, \zeta \in U^* \), \( \Sigma_0 \) stands for all functions from \( \Sigma \) with \( c_0 = 0 \). These classes have been one of the principal subjects of research in Complex Analysis for a long time. The most inquisitive problem for the class \( S \) posed by L. Bieberbach in 1916 [7] finally has been solved in 1984 by L. de Branges [8] who proved that \( |a_n| \leq n \) for any \( f \in S \) and the equality is attained only for the Koebe function \( k(z) = z(1 - z e^{i\theta})^{-2}, \theta \in [0, 2\pi) \). The main tool of the proof turned out to be a parametric representation of a function from \( S \) by the Löwner homotopic deformation of the identity map given by the \textit{Löwner differential equation}. The parametric method emerged almost 80 years ago in the celebrated paper by K. Löwner [32] who studied a one-parameter semigroup of conformal one-slit maps of \( U \) coming then at an evolution equation called after him. His main achievement was an infinitesimal description of a semi-flow of such maps by the Schwarz kernel that led him to the Löwner equation. This crucial result was generalized, then, in several ways. Attempts have been made to derive an equation that allowed to describe a representation of the whole class \( S \). Nowadays, it

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is rather difficult to follow the correct history line of the development of the parametric method because in the middle of the 20-th century a number of works dedicated to this general equation appeared independently. In particular, P. P. Kufarev [28] studied a one-parameter family of domains \( \Omega(t) \), and regular functions \( f(z,t) \) defined in \( \Omega(t) \). He proved differentiability of \( f(z,t) \) with respect to \( t \) for \( z \) from the Carathéodory kernel \( \Omega(t_0) \) of \( \Omega(t) \), and derived a generalization of the Löwner equation. Ch. Pommerenke [38] proposed to consider subordination chains of domains that led him to a general evolution equation. We mention here also papers by V. Gutlyanskiĭ [19] and V. Goryainov [15] in this direction. One can learn more about this method in monographs [1, 10, 39] (see also the references therein). Let us draw reader’s attention to Goryainov’s approach [15] who suggested to use a method of semigroups to derive several other parametric representations of classes of analytic maps and to apply them to study dynamics of stochastic branching processes. This approach is based on the study of one-parameter semi-flows on semigroups of conformal maps, their infinitesimal descriptions, and evolution equations (see also [43]).

In 1959 Shah Dao-Shing [42] suggested a parametric method for quasiconformal automorphisms of \( U \). In another form this method appeared in the paper by F. Gehring and E. Reich [13], and then, in [29]. Later, Cheng Qi He [22] obtained an analogous equation for classes of quasiconformally extendable univalent functions (to be more precise, in terms of inverse functions). Unlike the parametric method for conformal maps, its analogue for quasiconformal maps did not receive so much attention.

Several attempts have been launched to specialize the Löwner-Kufarev equation to obtain conformal maps that admit quasiconformal extensions (see [2, 3, 4, 20]).

Surprisingly, an analogous equation appeared in Fluid Dynamics in the study of plane free boundary problems, where the time dependence of the phase domain \( \Omega(t) \) in a Hele-Shaw cell was described by a one-parameter chain of univalent maps satisfying an equation that now is known as the Polubarinova-Galin equation. It appeared in the pioneering works by P. Ya. Polubarinova-Kochina [36, 37] and L. A. Galin [11] (see surveys [24], [45]). In contrast to the classical Löwner-Kufarev equation the latter is a non-linear (even non-quasilinear) integro-differential equation and many elegant properties of the Löwner-Kufarev equation are less clear for Polubarinova-Galin’s one. A typical feature of the Hele-Shaw flow is that starting with a simply connected phase domain \( \Omega(0) \) with a smooth boundary possible cusps may be developed during time evolution. They are caused by vanishing boundary derivatives as well as by topology change.

The principal goal of our paper is to study evolution equations for conformal maps with quasiconformal extensions. In particular, we are interested in maps smoothly extendable onto the unit circle. Our approach is based on the study of evolutions on the universal Teichmüller space \( T \) and on the manifold Diff \( S^1/\text{Rot} \ S^1 \) embedded into \( T \).
question we are interested in is how a Hele-Shaw flow is seen on the universal Teichmüller space as on a general parametric space.

2. The Löwner-Kufarev and Polubarinova-Galin equations

Let us consider a subordination chain of simply connected hyperbolic domains \( \Omega(t) \) in the Riemann sphere \( \hat{\mathbb{C}} \), which is defined for \( 0 \leq t < t_0 \). This means that \( \Omega(t) \subset \Omega(s) \) whenever \( t < s \). We suppose that all \( \Omega(t) \) are unbounded and \( \infty \in \Omega(t) \) for all \( t \). By the Riemann Mapping Theorem we construct a subordination chain of mappings \( f(\zeta,t) \), \( \zeta \in U^* \), where each function \( f(\zeta,t) = \alpha(t)\zeta + a_0(t) + \frac{a_1(t)}{\zeta} + \ldots \) is a meromorphic univalent map of \( U^* \) onto \( \Omega(t) \) for every fixed \( t \). Ch. Pommerenke \[38, 39\] first introduced such chains in order to generalize Löwner’s equation. His result says that given a subordination chain of domains \( \Omega(t) \) defined for \( t \in [0, t_0) \) with a differentiable real-valued coefficient \( \alpha(t) \) (in particular, \( e^{-t} \)), there exists an analytic regular function \( p(\zeta,t) = p_0(t) + \frac{p_1(t)}{\zeta} + \frac{p_2(t)}{\zeta^2} + \ldots, \zeta \in U^* \), such that \( \Re p(\zeta,t) > 0 \) in \( \zeta \in U^* \) and

\[
\frac{\partial f(\zeta,t)}{\partial t} = -\zeta \frac{\partial f(\zeta,t)}{\partial \zeta} p(\zeta,t),
\]

for almost all \( t \in [0, t_0) \). The coefficient \( \alpha(t) = \alpha(0) \exp(-\int_0^t p_0(\tau)d\tau) \) is the conformal radius of \( \Omega(t) \). This equation now-a-days is known as the Löwner-Kufarev equation due to the contribution by K. Löwner \[32\] and P. P. Kufarev \[28\].

We consider two main questions:

- If \( \partial \Omega(t) \) is a quasicircle, what is \( p(\zeta,t) \)?
- If \( \partial \Omega(t) \) is a smooth Jordan curve, what is \( p(\zeta,t) \)?

We draw reader’s attention to the case of smooth boundaries and their connection to free boundary problems of fluid dynamics. In 1898 H. S. Hele-Shaw \[23\] proposed his famous cell that was a device for investigating a flow of viscous fluid in a narrow gap between two parallel plates.

The dimensionless model of a moving viscous incompressible fluid in the Hele-Shaw cell is described by a potential flow with the velocity field \( \mathbf{V} = (V_1, V_2) \). The pressure \( p \) is the potential for the fluid velocity

\[
\mathbf{V} = -\frac{h^2}{12\mu} \nabla p,
\]

where \( h \) is the cell gap and \( \mu \) is the viscosity of the fluid (see, e.g. \[35, 41\]). Through the similarity in the governing equations, Hele-Shaw flows can be used to study models
of saturated flows in porous media governed by Darcy’s law. Over the years various particular cases of such a flow have been considered. Different driving mechanisms were employed, such as surface tension or external forces (suction, injection). We mention here a 600-paper bibliography of free and moving boundary problems for Hele-Shaw and Stokes flows since 1898 up to 1998 collected by K. A. Gillow and S. D. Howison [14].

Since the work by Hele-Shaw several principal steps have been made. Among them we distinguish the papers by P. Ya. Polubarinova-Kochina [36, 37] and L. A. Galin [11] who suggested in 1945 a complex variable approach that now is one of the basic tools for investigating the Hele-Shaw evolution.

Let us consider the flow of a viscous fluid in a plane Hele-Shaw cell under injection through a unique well which is placed at infinity. Suppose that at the initial moment the phase domain \( \Omega_0 \) occupied by the fluid is simply connected and bounded by a smooth analytic curve \( \Gamma_0 \). This model can be thought of as a receding air bubble in a viscous flow. The evolution of the phase domains \( \Omega(t) \) is described by an auxiliary conformal mapping \( f(\zeta, t) \) of \( U^* \) onto \( \Omega(t) \), \( \Omega(0) = \Omega_0 \), normalized by \( f(\zeta, t) = \alpha(t)\zeta + a_0(t) + a_1(t)\zeta + \ldots, \alpha(t) > 0 \). Here we denote the derivatives by \( f' = \partial f/\partial \zeta \), \( \dot{f} = \partial f/\partial t \), and \( t \) is the time parameter. This mapping satisfies the equation

\[
\Re \left[ \dot{f}(\zeta, t)\overline{f'(\zeta, t)} \right] = -1, \quad \zeta = e^{i\theta},
\]

under suitable rescaling. L. A. Galin [11], P. Ya. Polubarinova-Kochina [36, 37] first derived the equation (2) and stimulated deep investigation in complex variable approach to free boundary problems (see, e.g., [24, 45] and the references therein).

From (2) one can derive a Löwner-Kufarev type equation by the Schwarz-Poisson formula:

\[
\dot{f} = -\zeta f' \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} \frac{\zeta + e^{i\theta}}{\zeta - e^{i\theta}} d\theta,
\]

where \( \zeta \in U^* \).

The equation (2) is equivalent to the kinematic condition on the free boundary and, in particular, implies that the phase domains \( \Omega(t) \) form a subordination chain. In contrary to the classical Löwner-Kufarev equation [11] the equation (3) even is not quasilinear and the problem of the short-time existence and uniqueness of the solution is much more difficult. First it was proved by Yu. P. Vinogradov, P. P. Kufarev [46] in 1948 and later in 1993 by M. Reissig, L. Von Wolfersdorf [40]. In fact, starting with a smooth domain \( \Omega_0 \) the solution to (2) exists and unique locally in time. It is known that the domains \( \Omega(t) \) remain to have smooth (even analytic) boundaries up to the time \( t_0 \) when possible cusps
are developed or the domain fails to be simply connected. This means, in particular, that \( \Omega(t) \) fails to be a quasidisk as \( t \to t_0 \).

We ask the following question: given an initial smooth phase domain \( \Omega_0 \), and the Hele-Shaw evolution \( \Omega(t) \), what kind of evolution it produces on the universal Teichmüller space as a natural general parametric space?

A general scheme of the proposed study is shown in Figure 1.

3. **Infinitesimal structures of the universal Teichmüller space \( T \)**

Let us consider the family \( \mathcal{F} \) of all quasiconformal automorphisms of \( U \). Every such map \( f \) satisfies the Beltrami equation \( f_\zeta = \mu_f(\zeta) f_\zeta \) in \( U \) in the distributional sense, where \( \mu_f \) is a measurable essentially bounded function (\( L^\infty(U) \)) in \( U \), \( \|\mu_f\| = \text{ess sup}_U |\mu_f(\zeta)|_\infty < 1 \).

Conversely, for each measurable Beltrami coefficient \( \mu \) essentially bounded as above, there exists a quasiconformal automorphism of \( U \), that satisfies the Beltrami equation, which is unique if provided with some conformal normalization, e.g., three point normalization \( f(\pm1) = \pm1, f(i) = i \). Two normalized maps \( f_1 \) and \( f_2 \) are said to be equivalent, \( f_1 \sim f_2 \), if being extended onto the unit circle \( S^1 \), the superposition \( f_1 \circ f_2^{-1} \) restricted to \( S^1 \) is the identity map. The quotient set \( \mathcal{F}/\sim \) is called the universal Teichmüller space \( T \). It is a covering space for all Teichmüller spaces of analytically finite Riemann surfaces. By definition we have two realizations of \( T \): as a set of equivalence classes of quasiconformal maps and, due to the relation between \( \mathcal{F}/\sim \) and the unit ball \( B \subset L^\infty(U) \), as a set of equivalence classes of corresponding Beltrami coefficients.
The normalized maps from $\mathcal{F}$ form a group $\mathcal{F}_0$ with respect to superposition and the maps that act identically on $S^1$ form a normal subgroup $\mathcal{I}$. Thus, $T$ is the quotient of $T = \mathcal{F}_0/\mathcal{I}$.

If $g \in \mathcal{F}$, $f \in \mathcal{F}_0$, then there exists a Möbius transformation $h$, such that $h \circ f \circ g^{-1} \in \mathcal{F}_0$. Let us denote by $[f] \in T$ the equivalence class represented by $f \in \mathcal{F}_0$. Then, one defines the universal modular group $\mathcal{M}$, $\omega \in \mathcal{M}$, $\omega : T \rightarrow T$, by the formula $\omega([f]) = [h \circ f \circ g^{-1}]$. Its subgroup $\mathcal{M}_0$ of right translations on $T$ is defined by $\omega_0([f]) = [f \circ g^{-1}]$, where $f, g \in \mathcal{F}_0$.

An important fact (see [31, Chapter III, Theorem 1.1]) is that there are real analytic mappings in any equivalence class $[f] \in T$.

Given a Beltrami coefficient $\mu \in B \subset L^\infty(U)$ let us extend it by zero into $U^*$. We normalize the corresponding quasiconformal map $f$, which is conformal in $U^*$, by $f(\zeta) = \zeta + a_1/\zeta + \ldots$ about infinity. Then, two Beltrami coefficients $\mu$ and $\nu$ are equivalent if and only if the corresponding normalized mappings $f^\mu$ and $f^\nu$ map $U^*$ onto one and the same domain in $\overline{\mathbb{C}}$. Thus, the universal Teichmüller space can be thought of as the family of all normalized conformal maps of $U^*$ admitting quasiconformal extension. Moreover, any compact subset of $T$ consists of conformal maps $f$ of $U^*$ that admit quasiconformal extension to $U$ with $\|\mu_f\|_{\infty} \leq k < 1$ for some $k$.

As we mentioned above, a normalized conformal map $f \in [f] \in T$ defined in $U^*$ can have a quasiconformal extension to $U$ which is real analytic in $U$, but on the unit circle $f$ may behave quite irregularly. For example, the resulting quasicircle $f(S^1)$ can have the Hausdorff dimension greater than 1.

**Remark.** Given a bounded $K$-quasicircle $\Gamma$, $K = (1 + k)/(1 - k)$, in the plane let $N(\varepsilon, \Gamma)$ denote the minimal number of disks of radius $\varepsilon > 0$ that are needed to cover $\Gamma$. Let

$$\beta(K) = \sup_\Gamma \limsup_{\varepsilon \to 0} \log N(\varepsilon, \Gamma)/\log(1/\varepsilon)$$

denote the supremum of the Minkowski dimension of curves $\Gamma$ where $\Gamma$ ranges over all bounded $K$-quasicircles. The Hausdorff dimension of $\Gamma$ is bounded from above by $\beta(K)$ (see [31]). In [31] it was also established several explicit estimates for $\beta(K)$, e.g., $\beta(K) \leq 2 - cK^{-3.41}$.

Let us denote by $\Sigma_{0}^{qc} \subset \Sigma_0$ the class of those univalent conformal maps $f$ defined in $U^*$ which admit a quasiconformal extension to $U$, normalized by $f(\zeta) = \zeta + a_1/\zeta + \ldots$. Let $x, y \in T$ and $f, g \in \Sigma_{0}^{qc}$ be such that $\mu_f \in x$ and $\mu_g \in y$. Then, the Teichmüller distance $\tau(x, y)$ on $T$ is defined as

$$\tau(x, y) = \inf_{\mu_f \in x, \mu_g \in y} \frac{1}{2} \log \frac{1 + \|\mu_g \circ f^{-1}\|_{\infty}}{1 - \|\mu_g \circ f^{-1}\|_{\infty}}.$$
For a given $x \in T$ we consider an extremal Beltrami coefficient $\mu^*$ such that $\|\mu^*\|_\infty = \inf_{\nu \in \mathcal{X}} \|\nu\|_\infty$. Let us remark that $\mu^*$ need not be unique. A geodesic on $T$ can be described in terms of the extremal coefficient $\mu^*$ as a continuous homomorphism $x_t : [0, 1] \mapsto T$ such that $\tau(0, x_t) = t\tau(0, x_1)$. Due to the above remark the geodesic need not be unique as well.

We consider the Banach space $B(U)$ of all functions holomorphic in $U$ equipped with the norm

$$\|\varphi\|_{B(U)} = \sup_{\zeta \in U} |\varphi(\zeta)|(1 - |\zeta|^2)^2.$$  

For a function $f$ in $\Sigma$ the Schwarzian derivative

$$S_f(\zeta) = \frac{\partial}{\partial \zeta} \left( \frac{f''(\zeta)}{f'(\zeta)} \right) - \frac{1}{2} \left( \frac{f''(\zeta)}{f'(\zeta)} \right)^2$$

is defined and Nehari’s [31] estimate $\|S_f(1/\zeta)\|_{B(U)} \leq 6$ holds. Given $x \in T$, $\mu \in x$ we construct the mapping $f^\mu \in \Sigma_{qc}^0$ and have the homeomorphic embedding $T \mapsto B(U)$ by the Schwarzian derivative.

The universal Teichmüller space $T$ is an analytic infinite dimensional Banach manifold modelled on $B(U)$. The Banach space $B(U)$ is an infinite dimensional vector space that can be thought of as the cotangent space to $T$ at the initial point (represented by $\mu \equiv 0$). More rigorously, let the map $f^\mu$ be a quasiconformal homeomorphism of the unit disk $U$. It has a Fréchet derivative with respect to $\mu$ in a direction $\nu$. Let us construct the variation of $f^{\tau \nu} \in \Sigma_{qc}^0$, $\mu = \tau \nu$, with respect to a small parameter $\tau$:

$$f^{\tau \nu}(\zeta) = \zeta + \tau V(\zeta) + o(\tau), \quad \zeta \in U^*.$$  

Taking the Schwarzian derivative in $U^*$ we get

$$S_{f^{\tau \nu}} = \tau V''(\zeta) + o(\tau), \quad \zeta \in U^*,$$

locally uniformly in $U^*$. Taking into account the normalization of the class $\Sigma_{qc}^0$ we have (see, e.g., [31])

$$V(\zeta) = -\frac{1}{\pi} \int_0^\infty \frac{\nu(w) \nu(w)}{w - \zeta}, \quad V''(\zeta) = -\frac{6}{\pi} \int_0^\infty \frac{\nu(w) \nu(w)}{(w - \zeta)^4}.$$  

The integral formula implies $V''(A(\zeta))A'(\zeta)^2 = V''(\zeta)$ (subject to the relation for the Beltrami coefficient $\nu(A(\zeta))A'(\zeta) = \nu(\zeta)A'(\zeta)$) for any Möbius transform $A$. Now let us change variables $\zeta \rightarrow 1/\bar{\zeta}$ and reduce the first variation to a holomorphic function in the unit disk by changing $f^{\tau \nu}(\zeta)$ to $g^{\tau \nu}(\zeta) \equiv f^{\tau \nu}(1/\bar{\zeta})$. Setting $\Lambda(\nu)(\zeta) = S_{g^{\tau \nu}}(\zeta)$ and
\[ \hat{\Lambda}_\nu(\zeta) = \frac{1}{\zeta} \nabla^m(1/\zeta) \] we have (see, e.g., [12, Section 6.5, Theorem 5]) that
\[ \Lambda_\nu(\zeta) - \tau \dot{\Lambda}_\nu(\zeta) = \frac{o(\tau)}{(1 - |\zeta|^2)^2}. \]

So the operator \( \dot{\Lambda}_\nu \) is the derivative of \( \Lambda_\nu \) at the initial point of the universal Teichmüller space with respect to the norm of the Banach space \( B(U) \). The reproducing property of the Bergman integral gives
\[ \varphi(\zeta) = 3 \int_\mathbb{D} \frac{\varphi(w)(1-|w|^2)^2 d\sigma_w}{(1-\bar{w}\zeta)^4}, \quad \varphi \in B(U). \]

The latter integral leads us to the so-called harmonic (Bers’) Beltrami differential
\[ \nu(\zeta) = \Lambda^*(\varphi) = -\frac{1}{2} \overline{\varphi(\zeta)} (1-|\zeta|^2)^2, \quad \zeta \in U. \]

Let us denote by \( A(U) \) the Banach space of analytic functions with the finite \( L^1 \) norm in the unit disk. We have that \( A(U) \hookrightarrow B(U) \) is a continuous inclusion (see, e.g., [33, Section 1.4.2]). On \( L^\infty(U) \times A(U) \) one can define a coupling
\[ \langle \mu, \varphi \rangle := \int_\mathbb{D} \mu(\zeta) \varphi(\zeta) d\sigma_\zeta. \]

Denote by \( N \) the space of locally trivial Beltrami coefficients, which is the subspace of \( L^\infty(U) \) that annihilates the operator \( \langle \cdot, \varphi \rangle \) for all \( \varphi \in A(U) \). Then, one can identify the tangent space to \( T \) at the initial point with the space \( H := L^\infty(U)/N \). It is natural to relate it to a subspace of \( L^\infty(U) \). The superposition \( \dot{\Lambda}_\nu \circ \Lambda^* \) acts identically on \( A(U) \) due to (4). The space \( N \) is also the kernel of the operator \( \dot{\Lambda}_\nu \). Thus, the operator \( \Lambda^* \) splits the following exact sequence
\[ 0 \rightarrow N \hookrightarrow L^\infty(U) \xrightarrow{\dot{\Lambda}_\nu} A(U) \rightarrow 0. \]

Then, \( H = \Lambda^*(A(U)) \cong L^\infty(U)/N \). The coupling \( \langle \mu, \varphi \rangle \) defines \( A(U) \) as a cotangent space. Let \( A^2(U) \) denote the Banach space of analytic functions \( \varphi \) with the finite norm
\[ ||\varphi||_{A^2(U)} = \int_\mathbb{D} |\varphi(\zeta)|^2 (1-|\zeta|^2)^2 d\sigma_\zeta. \]

Then \( A(U) \hookrightarrow A^2(U) \) and Petersson’s Hermitian product [44] is defined on \( A^2(U) \) as
\[ (\varphi_1, \varphi_2) = \int_\mathbb{D} \varphi_1(\zeta) \overline{\varphi_2(\zeta)} (1-|\zeta|^2)^2 d\sigma_\zeta. \]

The Kählerian Weil-Petersson metric \( \{ \nu_1, \nu_2 \} = \langle \nu_1, \dot{\Lambda}_{\nu_2} \rangle \) can be defined on the tangent space to \( T \) and gives a Kählerian manifold structure to \( T \).
The universal Teichmüller space is a smooth manifold on which a Lie group \( \text{Diff} \, T \) of real sense preserving diffeomorphisms is defined. The tangent bundle is defined on \( T \) and is represented by the harmonic differentials from \( H \) translated to all points of \( T \). We will consider tangent vectors from \( H \) at the initial point of \( T \) represented by the map \( f(\zeta) \equiv \zeta \).

The Weil-Petersson metric defines a Lie algebra of vector fields on \( T \) by the Poisson-Lie bracket \([\nu_1, \nu_2] = \{\nu_1, \nu_2\} - \{\nu_2, \nu_1\}\), where \( \nu_1, \nu_2 \in H \). One can define the Poisson-Lie bracket at all other points of \( T \) by left translations from \( \text{Diff} \, T \). To each element \([x]\) from \( \text{Diff} \, T \) an element \( x \) from \( T \) is associated as an image of the initial point. Therefore, a curve in \( \text{Diff} \, T \) generates a traced curve in \( T \) that can be realized by a one-parameter family of quasiconformal maps from \( \Sigma \text{qc}_0 \).

For each tangent vector \( \nu \in H \) there is a one-parameter semi-flow in \( \text{Diff} \, T \) and a corresponding flow \( x^\tau \in T \) with the velocity vector \( \nu \). To make an explicit representation we use the variational formula for the subclass \( \Sigma \text{qc}_0 \) of \( \Sigma_0 \) of functions with quasiconformal extension (see, e.g., [31]) to \( \overline{\mathbb{C}} \). If \( f^\mu \in \Sigma \text{qc}_0 \), \( \nu \in H \) and

\[
\mu_f(\zeta, \tau) = \begin{cases} 
\tau \nu(\zeta) + o(\tau) & \text{if } \zeta \in U, \\
0 & \text{if } \zeta \in U^*,
\end{cases}
\]

then the map

\[
f^\mu(\zeta) = \zeta - \tau \pi \int_U \frac{\nu(w)d\sigma_w}{w - \zeta} + o(\tau)
\]

locally describes the semi-flow \( x^\tau \) on \( T \).

4. Diff \( S^1/\text{Rot} \, S^1 \) EMBEDDED INTO \( T \)

In this section we study a diffeomorphic embedding of the homogeneous manifold \( \text{Diff} \, S^1/\text{Rot} \, S^1 \) into the universal Teichmüller space \( T \).

4.1. Homogeneous manifold Diff \( S^1/\text{Rot} \, S^1 \). We denote the Lie group of \( C^\infty \) sense preserving diffeomorphisms of the unit circle \( S^1 \) by \( \text{Diff} \, S^1 \). Each element of \( \text{Diff} \, S^1 \) is represented as \( z = e^{i\phi(\theta)} \) with a monotone increasing, \( C^\infty \) real-valued function \( \phi(\theta) \), such that \( \phi(\theta + 2\pi) = \phi(\theta) + 2\pi \). The Lie algebra for \( \text{Diff} \, S^1 \) is identified with the Lie algebra \( \text{Vect} \, S^1 \) of smooth \( (C^\infty) \) tangent vector fields to \( S^1 \) with the Poisson - Lie bracket given by

\[
[\phi_1, \phi_2] = \phi_1 \phi'_2 - \phi_2 \phi'_1.
\]
Fixing the trigonometric basis in \( \text{Vect} \, S^1 \) the commutator relations take the form

\[
\begin{align*}
\left[ \cos n\theta, \cos m\theta \right] &= \frac{n-m}{2} \sin (n+m)\theta + \frac{n+m}{2} \sin (n-m)\theta, \\
\left[ \sin n\theta, \sin m\theta \right] &= \frac{m-n}{2} \sin (n+m)\theta + \frac{n+m}{2} \sin (n-m)\theta, \\
\left[ \sin n\theta, \cos m\theta \right] &= \frac{m-n}{2} \cos (n+m)\theta - \frac{n+m}{2} \cos (n-m)\theta.
\end{align*}
\]

There is no general theory of infinite dimensional Lie groups, example of which is under consideration. The interest to this particular case comes first of all from the string theory where the Virasoro algebra appears as the central extension of \( \text{Vect} \, S^1 \). Entire necessary background for the construction of the theory of unitary representations of \( \text{Diff} \, S^1 \) is found in the study of Kirillov’s homogeneous Kählerian manifold \( M = \text{Diff} \, S^1 / \text{Rot} \, S^1 \), where \( \text{Rot} \, S^1 \) denotes the group of rotations of \( S^1 \). The group \( \text{Diff} \, S^1 \) acts as a group of translations on the manifold \( M \) with \( \text{Rot} \, S^1 \) as a stabilizer. The Kählerian geometry of \( M \) has been described by Kirillov and Yuriev in \[25\]. The manifold \( M \) admits several representations, in particular, in the space of smooth probability measures, symplectic realization in the space of quadratic differentials. We will use its analytic representation that is based on the class \( \tilde{\Sigma}_0 \) of functions from \( \Sigma_0 \) which being extended onto the closure \( \overline{U^*} \) of \( U^* \) are supposed to be smooth on \( S^1 \). The class \( \tilde{\Sigma}_0 \) is dense in \( \Sigma_0 \) in the local uniform topology of \( U^* \).

Let \( \tilde{S} \) denote the class of all univalent holomorphic maps in the unit disk \( g(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \ldots \) which are smooth on \( S^1 \). Then, for each \( f \in \tilde{\Sigma}_0 \) we have \( \infty \in f(U^*) \) and there is an adjoint map \( g \in \tilde{S} \) such that \( \overline{\mathbb{C}} \setminus f(U^*) = g(U) \). The superposition \( g^{-1} \circ f \) restricted to \( S^1 \) is in \( M \) (see Figure 2). Reciprocally, for each element of \( M \) there exist such \( f \) and \( g \). A piece-wise smooth closed Jordan curve is a quasicircle if and only if it has no cusps. So any function \( f \) from \( \tilde{\Sigma}_0 \) has a quasiconformal extension to \( U \). By this realization the manifold \( M \) is naturally embedded into the universal Teichmüller space.
T. Moreover, the Kählerian structure on $M$ corresponds to the Kählerian structure on $T$ given by the Weil-Petersson metric.

The Goluzin-Schiffer variational formulae lift the actions from the Lie algebra $\text{Vect} \ S^1$ onto $\tilde{\Sigma}_0$. Let $f \in \tilde{\Sigma}_0$ and let $d(e^{i\theta})$ be a $C^\infty$ real-valued function in $\theta \in (0, 2\pi]$ from $\text{Vect} \ S^1$ making an infinitesimal action as $\theta \mapsto \theta + \tau d(e^{i\theta})$. Let us consider a variation of $f$ given by

$$\delta_d f(\zeta) = -\frac{1}{2\pi i} \int_{S^1} \left( \frac{w f'(w)}{f(w)} \right)^2 \frac{wd(w)dw}{f(w) - f(\zeta)}.$$ 

Kirillov and Yuriev [25], [26] have established that the variations $\delta_d f(\zeta)$ are closed with respect to the commutator and the induced Lie algebra is the same as $\text{Vect} \ S^1$. Moreover, Kirillov’s result [27] states that there is the exponential map $\text{Vect} \ S^1 \to \text{Diff} \ S^1$ such that the subgroup Rot $S^1$ coincides with the stabilizer of the map $f(\zeta) \equiv \zeta$ from $\tilde{\Sigma}_0$.

4.2. Douady-Earle extension. Let $\varphi: S^1 \to S^1$ be a circle quasisymmetric homeomorphism, i.e., a homeomorphism that possesses a quasiconformal extension into $U$ (for a precise definition see, e.g., [31]). Then $\varphi$ has infinitely many quasiconformal extensions into $U$, one of the most remarkable of which is the Beurling-Ahlfors extension [6]. In 1986 Douady and Earle [9] defined for any such $\varphi: S^1 \to S^1$ a conformally natural extension $h: \overline{U} \to \overline{U}$ from $F$. The map $h$ is a homeomorphism which is real analytic in the interior. The idea was to introduce the concept of a conformal barycenter of a measure on $S^1 = \partial U$. Douady and Earle proved that $w = h(\zeta) \in F$ satisfies the functional equation

$$F(\zeta, w) \equiv \frac{1}{2\pi i} \int_{S^1} \left( \frac{\varphi(z) - w}{1 - \overline{w}\varphi(z)} \right) \frac{1 - |\zeta|^2}{|\zeta - z|^2} |dz| = 0.$$ 

An advantage of this extension is that if $\sigma, \tau \in \text{Möb}(U)$, then the extension of $\sigma \circ \varphi \circ \tau$ is given by $\sigma \circ h \circ \tau$, what is not true for the Beurling-Ahlfors extension. The three-point boundary normalization of $F_0$ can be always attained, and thus, the Douady-Earle extension is compatible with the definition of the universal Teichmüller space. Later, in 1988, another proof of Douady-Earle’s result has appeared in [30] where the authors worked with the inverse function. The functional equation (6), in particular, implies that a $C^\infty$ mapping $\varphi$ representing an element from the manifold $M$ has a real analytic extension $h \in F$ which is $C^\infty$ on $S^1$.

Let $f \in \tilde{\Sigma}_0$ represent an element from $\varphi \in M$. Let $g \in \tilde{S}$ be the adjoint map, $g^{-1} \circ f |_{S^1} = \varphi$. If $h$ is the Douady-Earle extension of $\varphi$, then $g \circ h |_{S^1} \equiv f |_{S^1}$ and $g \circ h$ is a quasiconformal extension of $f \in \tilde{\Sigma}_0$. Given $\varphi \in M$ we construct the mapping $f^\mu$ that
satisfies the normalization of the class $\tilde{\Sigma}_0$ and whose Beltrami coefficient is

$$
\mu_f(\zeta) = \frac{F_\zeta F_w - F_{\bar{\zeta}} F_{\bar{w}}}{F_{\bar{\zeta}} F_{\bar{w}} - F_\zeta F_w}, \quad w = h(\zeta), \quad \zeta \in U,
$$

with $\mu_f(\zeta) = 0$ for $\zeta \in U^*$. The equivalence class $[f^\mu]$ is a point of the universal Teichmüller space $T$. So the Douady-Earle extension defines an explicit embedding of $M$ into $T$.

4.3. **Semi-flows on $T$ and $M$.** As it was mentioned, the Weil-Petersson metric defines a Lie algebra of vector fields on $T$ by the Poisson bracket $[\nu_1, \nu_2] = \{\nu_2, \nu_1\} - \{\nu_1, \nu_2\}$, where $\nu_1, \nu_2 \in H$. One can define the Poisson bracket at all other points of $T$ by left translations of the universal modular group.

We proceed restricting ourselves to $M$ embedded into $T$. The complex form of Green’s formula implies that (5) for $f(\zeta) \equiv \zeta$ is equivalent to

$$
\delta_d \zeta = -\frac{1}{\pi} \iint_U \frac{\partial_w (wd(w))d\sigma_w}{w - \zeta},
$$

where the distributional derivative $\partial_w d(w)$ is given in the unit disk $U$, $d(w)$ is a continuous extension of the $C^\infty$ function $d(e^{i\theta}) \in \text{Vect} \ S^1$ into $U$ that has $L^s(U)$ distributional derivatives in $U$, $s > 2$, and $d\sigma_w$ is the area element in $U$. Thus, one can extract the elements from $H$ that are of the form $\nu(\zeta) = \zeta \partial_\zeta d(\zeta)$, where $\partial_\zeta$ means $\partial/\partial \zeta$.

We are going to deduce an exact form of $\nu$ using the Douady-Earle extension. For this we start with the variation of the element

$$
\varphi(e^{i\theta}, \tau) = e^{i\theta}(1 + \tau d(e^{i\theta})) + o(\tau), \quad \varphi \in M, \quad d \in \text{Vect} \ S^1,
$$

and $\tau$ is small. The Beltrami coefficient of the extended quasiconformal map $h$ has its variation as $\mu_h(\zeta) = \tau \nu(\zeta) + o(\tau)$, where

$$
\nu(\zeta) = \frac{\partial}{\partial \tau} \left( \frac{F_\zeta F_w - F_{\bar{\zeta}} F_{\bar{w}}}{F_{\bar{\zeta}} F_{\bar{w}} - F_\zeta F_w} \right) \Bigg|_{\tau=0, w=\zeta}, \quad \zeta \in U,
$$

where

$$
F^\tau(\zeta, w) = \frac{1}{2\pi} \int_{S^1} \left( \frac{\varphi(z, \tau) - w}{1 - \bar{w}\varphi(z, \tau)} \right) \frac{1 - |\zeta|^2}{|\zeta - z|^2} |dz| = 0.
$$

Thus, $\nu(\zeta)$ depends only on $d(e^{i\theta})$. We will give explicit formulae in the next section. They can be obtained substituting $\varphi(e^{i\theta}, 0) = e^{i\theta}$, and taking into account that

$$
\left. \frac{F_\zeta F_w - F_{\bar{\zeta}} F_{\bar{w}}}{F_{\bar{\zeta}} F_{\bar{w}} - F_\zeta F_w} \right|_{\tau=0, w=\zeta} = 0.
$$
The Lie algebra Vect $S^1$ is embedded into the Lie algebra of $H$ by (9), (10). Hence, a flow given on $M$ corresponding to a vector $d \in \text{Vect} S^1$ is represented as a flow on the universal Teichmüller space $T$ corresponding to the vector $\nu \in H$ given by (9).

5. Infinitesimal descriptions of semi-flows

First of all we give an explicit formula that connects the vectors $d(e^{i\theta})$ from Vect $S^1$ with corresponding tangent vectors $\nu(\zeta) \in H$ to the universal Teichmüller space $T$ making use of the Douady-Earle extension. These vectors give the infinitesimal description of semi-flows on $M$ and $T$ respectively.

**Theorem 1.** Let $d(e^{i\theta}) \in \text{Vect} S^1$ be the infinitesimal description of a flow $\varphi$ in $M$. Then, the corresponding infinitesimal description $\nu(\zeta) \in H$ of this flow embedded into $T$ is given by the function

\[(11) \quad \nu(\zeta) = \frac{3}{2\pi} \int_0^{2\pi} \left( \frac{1 - |\zeta|^2}{1 - e^{i\theta}\zeta} \right)^2 e^{2i\theta} d(e^{i\theta}) d\theta.\]

**Proof.** Let $\varphi(\zeta, \tau) = e^{i(\theta + \tau d(e^{i\theta}))}$ and $h(\zeta, \tau)$ be the Douady-Earle extension of $\varphi$ into the unit disk $U$, $\zeta \in U$ by means of (10). If $\tau = 0$, then $h(\zeta, 0) \equiv \zeta$. We calculate

\[
\begin{align*}
\partial_\zeta F^\tau(\zeta, w) &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\varphi(e^{i\theta}, \tau) - w}{1 - \overline{w}\varphi(e^{i\theta}, \tau)} \right) e^{i\theta} \frac{e^{i\theta}(\bar{\zeta} - e^{-i\theta})^2}{|\zeta - e^{i\theta}|^2} d\theta, \\
\partial_\bar{\zeta} F^\tau(\zeta, w) &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\varphi(e^{i\theta}, \tau) - w}{1 - \overline{w}\varphi(e^{i\theta}, \tau)} \right) e^{-i\theta} \frac{e^{-i\theta}(\bar{\zeta} - e^{i\theta})^2}{|\zeta - e^{i\theta}|^2} d\theta, \\
\partial_w F^\tau(\zeta, w) &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{-1}{1 - \overline{w}\varphi(e^{i\theta}, \tau)} \right) e^{i\theta} \frac{1 - |\zeta|^2}{|\zeta - e^{i\theta}|^2} d\theta, \\
\partial_{\bar{w}} F^\tau(\zeta, w) &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{\varphi(e^{i\theta}, \tau)(\varphi(e^{i\theta}, \tau) - w)}{(1 - \overline{w}\varphi(e^{i\theta}, \tau))^2} \right) e^{i\theta} \frac{1 - |\zeta|^2}{|\zeta - e^{i\theta}|^2} d\theta.
\end{align*}
\]
Substituting \( \tau = 0 \) and \( w = \zeta \) we have
\[
\begin{align*}
\partial_\zeta F^\tau(\zeta, w) \bigg|_{\tau=0, w=\zeta} &= \frac{1}{1 - |\zeta|^2}, \\
\partial_\zeta F^\tau(\zeta, w) \bigg|_{\tau=0, w=\zeta} &= 0, \\
\partial_w F^\tau(\zeta, w) \bigg|_{\tau=0, w=\zeta} &= \frac{-1}{1 - |\zeta|^2}, \\
\partial_{\bar{w}} F^\tau(\zeta, w) \bigg|_{\tau=0, w=\zeta} &= 0.
\end{align*}
\]

We will use the properties of the Douady-Earle extension. Let us fix a point \( \zeta_0 \in U \) and choose two Möbius transformations \( \sigma, \delta \) of \( U \) such that \( \delta(0) = \zeta_0 \) and \( \sigma(0) = h(\zeta_0, \tau) \). We set \( g = \sigma^{-1} \circ h \circ \delta \). Then, \( g(0, \tau) = 0 \), \( \dot{g}(0, \tau) = 0 \) and
\[
\begin{align*}
\partial_\zeta g(0, \tau) &= \partial_\zeta h(\zeta_0, \tau) \frac{\delta'(0)}{\sigma'(0)}, \\
\partial_\zeta \dot{g}(0, \tau) &= \partial_\zeta h(\zeta_0, \tau) \frac{\delta''(0)}{\sigma''(0)}.
\end{align*}
\]
So we see that
\[
\frac{\partial_\zeta h(\zeta_0, \tau)}{\partial_\zeta h(\zeta_0, \tau)} \frac{\delta'(0)}{\partial_\zeta g(0, \tau)} \frac{\delta'(0)}{\partial_\zeta h(\zeta_0, \tau)} = \frac{\partial_\zeta h(\zeta_0, \tau)}{\partial_\zeta g(0, \tau)} \frac{\delta''(0)}{\delta'(0)}.
\]
By the property of the Douady-Earle extension we have that the function \( g(\zeta, \tau), \zeta \in U \) is the extension of \( g(e^{i\theta}, \tau) \) by means of (10). If \( \tau = 0 \), then \( g(\zeta, 0) \equiv \zeta \). Now we put \( \psi(e^{i\theta}, \tau) = g(e^{i\theta}, \tau) \) in (10) and calculate variations in \( \tau \)
\[
\begin{align*}
\frac{\partial}{\partial \tau} \partial_\zeta F^\tau(\zeta, w) \bigg|_{\tau=0, w=\zeta=0} &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{e^{i\theta} - \zeta}{(1 - \zeta e^{i\theta})^3} \right) \frac{1 - |\zeta|^2}{|\zeta - e^{i\theta}|^2} \left. \partial_\tau \psi(e^{i\theta}, \tau) \right|_{\tau=0, \zeta=0} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta} \dot{\psi}(e^{i\theta}, 0) d\theta, \\
\frac{\partial}{\partial \tau} \partial_w F^\tau(\zeta, w) \bigg|_{\tau=0, w=\zeta=0} &= \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{2e^{i\theta} - \zeta - |\zeta|^2 e^{i\theta}}{(1 - \zeta e^{i\theta})^3} \right) \frac{1 - |\zeta|^2}{|\zeta - e^{i\theta}|^2} \left. \partial_\tau \psi(e^{i\theta}, \tau) \right|_{\tau=0, \zeta=0} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} 2e^{i\theta} \dot{\psi}(e^{i\theta}, 0) d\theta.
\end{align*}
\]
Then, we can obtain the explicit form of the variation of the Beltrami coefficient by (9) as

\[
\frac{\partial}{\partial \tau} \frac{\partial \zeta g(0, \tau)}{\partial \bar{\zeta}} \bigg|_{\tau=0} = -\frac{3}{2\pi} \int_0^{2\pi} e^{i\theta} \psi(e^{i\theta}, 0) d\theta.
\]

The Möbius transformation $\delta$ does not depend on $\tau$ whereas $\sigma$ does. Explicitly, we put

\[
\sigma^{-1} \circ h \circ \delta = \frac{h(\delta(\zeta), \tau) - h(\zeta_0, \tau)}{1 - h(\delta(\zeta), \tau) h(\zeta_0, \tau)},
\]

where $\delta(\zeta) = (\zeta + \zeta_0)(1 + \bar{\zeta}_0)^{-1}$. We denote $e^{i\alpha} = \delta(e^{i\theta})$. Therefore, denoting by $e^{i\alpha} = \delta(e^{i\theta}) = e^{i\theta} + \zeta_0 \bar{\zeta}_0$, we have

\[
\dot{g}(e^{i\theta}, 0) = \frac{\dot{h}(e^{i\alpha}, 0)(1 - |\zeta_0|^2) - \dot{h}(\zeta_0, 0)(1 - \bar{\zeta}_0 e^{i\alpha}) + \overline{h(\zeta_0, 0)} e^{i\alpha} (e^{i\alpha} - \zeta_0)}{(1 - \bar{\zeta}_0 e^{i\alpha})^2}.
\]

Then,

\[
e^{i\alpha} d\theta = \frac{1 - |\zeta_0|^2}{(1 - e^{i\alpha} \bar{\zeta}_0)^2} e^{i\alpha} d\alpha,
\]

and changing variables in (12), we obtain

\[
\frac{\partial}{\partial \tau} \frac{\partial \zeta g(0, \tau)}{\partial \bar{\zeta}} \bigg|_{\tau=0} = -\frac{3}{2\pi} \int_0^{2\pi} \left( 1 - |\zeta_0|^2 \right)^2 e^{2i\alpha} d(e^{i\alpha}) d\alpha.
\]

Taking into account that $\delta'(0) = 1$ we come to the statement of the theorem. □

**Corollary 1.** If $q = \max_{\theta \in [0, 2\pi]} |d(e^{i\theta})|$, then

\[
|\nu(\zeta)| \leq 3 \frac{1 + |\zeta|^2}{1 - |\zeta|^2} q.
\]

**Proof.** The formula given in the preceding theorem implies

\[
\nu(\zeta) = \frac{3}{2\pi} \int_0^{2\pi} \frac{1 - |\zeta|^2}{(1 - e^{i\alpha} \bar{\zeta})^2} e^{i\alpha} d(\delta(e^{i\theta})) e^{i\theta} d\theta.
\]

Changing variables $\alpha \to \theta$ we obtain

\[
\nu(\zeta) = \frac{3}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + \zeta (1 + e^{i\theta} \bar{\zeta})^2}{1 + e^{i\theta} \bar{\zeta}} \frac{1 - |\zeta|^2}{1 - |\zeta|^2} e^{i\theta} d(\delta(e^{i\theta})) d\theta.
\]
Next, we obviously estimate $|\nu|$ as in the statement of the corollary.

As we see, the given estimate is good enough when $|\zeta|$ is not close to 1. Let us now give an asymptotic estimate for $|\nu(\zeta)|$ in the case $|\zeta| \sim 1$.

**Corollary 2.** There exists a constant $M$ independent of $\zeta$ such that

$$|\nu(\zeta)| \leq M \frac{1 - |\zeta|^2}{|\zeta|^2}.$$ 

In particular, $|\nu(\zeta)| = O(1 - |\zeta|^2)$ as $|\zeta| \sim 1$.

**Proof.** We integrate by parts the right-hand side in the formula (11) twice and come to the following expression

$$\nu(\zeta) = -\frac{(1 - |\zeta|^2)^2}{4\pi \zeta^2} \int_0^{2\pi} \frac{1 - |\zeta|^2}{(1 - e^{i\theta} \zeta)^2} \left[ \partial [e^{i\theta} d(e^{i\theta})] + \partial^2 [e^{i\theta} d(e^{i\theta})] \right] d\theta.$$ 

The absolute value of the above integral is bounded because of the Poisson kernel in it and due to the smoothness of the function $d$. □

6. **Parametric representation of univalent maps with quasiconformal extensions**

6.1. **Semigroups of conformal maps.** The basic ideas that we use in this section come from Goryainov’s works [15, 16] and the monograph by Shoikhet [43].

We consider the semigroup $\mathcal{G}$ of conformal univalent maps from $U^*$ into itself with composition as the semigroup operation. This makes $\mathcal{G}$ a topological semigroup with respect to the topology of local uniform convergence on $U^*$. We impose the natural normalization for such conformal maps: $\Phi(\zeta) = \beta \zeta + b_0 + \frac{b_1}{\zeta} + \ldots$, $\zeta \in U^*$, $\beta > 0$. The unit of the semigroup is the identity. Let us construct on $\mathcal{G}$ a one-parameter semi-flow $\Phi^\tau$, that is, a continuous homomorphism from $\mathbb{R}^+$ into $\mathcal{G}$, with the parameter $\tau \geq 0$. For any fixed $\tau \geq 0$ the element $\Phi^\tau$ is from $\mathcal{G}$ and is represented by a conformal map $\Phi(\zeta, \tau) = \beta(\tau) \zeta + b_0(\tau) + \frac{b_1(\tau)}{\zeta} + \ldots$ from $U^*$ onto the domain $\Phi(U^*, \tau) \subset U^*$. The element $\Phi^\tau$ satisfies the following properties:

- $\Phi^0 = id$;
- $\Phi^{\tau+s} = \Phi(\Phi(\zeta, \tau), s)$, for $\tau, s \geq 0$;
- $\Phi(\zeta, \tau) \to \zeta$ locally uniformly in $U^*$ as $\tau \to 0$.

In particular, $\beta(0) = 1$. This semi-flow is generated by a vector field $v(\zeta)$ if for each $\zeta \in U^*$ the function $w = \Phi(\zeta, \tau)$, $\tau \geq 0$ is a solution of an autonomous differential equation $dw/d\tau = v(w)$ with the initial condition $w|_{\tau=0} = \zeta$. The semi-flow can be
extended to a symmetric interval \((-t, t)\) by putting \(\Phi^{-\tau} = \Phi^{-1}(\zeta, \tau)\). Certainly, the latter function is defined on the set \(\Phi(U^*, \tau)\). Admitting this restriction for negative \(\tau\) we define a one-parameter family \(\Phi^\tau\) for \(\tau \in (-t, t)\).

For a semi-flow \(\Phi^\tau\) on \(G\) there is an infinitesimal generator at \(\tau = 0\) constructed by the following procedure. Any element \(\Phi^\tau\) is represented by a conformal map \(\Phi(\zeta, \tau)\) that satisfies the Schwarz Lemma for the maps \(U^* \to U^*\), and hence,

\[
\Re \frac{\zeta}{\Phi(\zeta, \tau)} \leq \left| \frac{\zeta}{\Phi(\zeta, \tau)} \right| \leq 1, \quad \zeta \in U^*,
\]

where the equality sign is attained only for \(\Phi^0 = id \simeq \Phi(\zeta, 0) \equiv \zeta\). Therefore, the following limit exists (see, e.g., \([15]\), \([16]\), \([43]\))

\[
\lim_{\tau \to 0} \Re \frac{\zeta - \Phi(\zeta, \tau)}{\tau \Phi(\zeta, \tau)} = -\Re \left. \frac{\partial \Phi(\zeta, \tau)}{\partial \tau} \right|_{\tau = 0} \leq 0,
\]

and the representation

\[
\left. \frac{\partial \Phi(\zeta, \tau)}{\partial \tau} \right|_{\tau = 0} = \zeta p(\zeta)
\]

holds, where \(p(\zeta) = p_0 + p_1/\zeta + \ldots\) is an analytic function in \(U^*\) with positive real part, and

\[
(15) \quad \left. \frac{\partial \beta(\tau)}{\partial \tau} \right|_{\tau = 0} = p_0.
\]

In \([17]\) it was shown that \(\Phi^\tau\) is even \(C^\infty\) with respect to \(\tau\). The function \(\zeta p(\zeta)\) is an infinitesimal generator for \(\Phi^\tau\) at \(\tau = 0\), and the following variational formula holds

\[
(16) \quad \Phi(\zeta, \tau) = \zeta + \tau \zeta p(\zeta) + o(\tau), \quad \beta(\tau) = 1 + \tau p_0 + o(\tau).
\]

The convergence is thought of as local uniform. We rewrite (16) as

\[
(17) \quad \Phi(\zeta, \tau) = (1 + \tau p_0)\zeta + \tau \zeta (p(\zeta) - p_0) + o(\tau) = \beta(\tau)\zeta + \tau \zeta (p(\zeta) - p_0) + o(\tau).
\]

Now let us proceed with the semigroup \(G^{qc} \subset G\) of quasiconformal automorphisms of \(\overline{C}\). A quasiconformal map \(\Phi\) representing an element of \(G^{qc}\) satisfies the Beltrami equation in \(\overline{C}\)

\[
\Phi_\zeta = \mu_\Phi(\zeta) \Phi_\zeta,
\]

with the distributional derivatives \(\Phi_\zeta\) and \(\Phi_\zeta\), where \(\mu_\Phi(\zeta)\) is a measurable function vanishing in \(U^*\) and essentially bounded in \(U\) by

\[
||\mu_\Phi|| = \text{ess sup}_U |\mu_\Phi(\zeta)| \leq k < 1,
\]
for some \( k \). If \( k \) is sufficiently small, then the function \( \Phi - b_0 \beta \) satisfies the variational formula (see, e.g., [31])

\[
\frac{\Phi(\zeta) - b_0}{\beta} = \zeta - \frac{1}{\pi} \int_U \frac{\mu(\omega) d\sigma_w}{w - \zeta} + o(k),
\]

where \( d\sigma_w \) stands for the area element in the \( w \)-plane.

Now for each \( \tau \) small and \( \Phi^\tau \in G^{qc} \) the mapping \( h(\zeta, \tau) = \frac{\Phi(\zeta, \tau) - b_0(\tau)}{\beta(\tau)} \) is from \( \Sigma_0^{qc} \) and represents an equivalence class \([h^\tau] \in T\). Consider the one-parameter curve \( x^\tau \in T \) that corresponds to \([h^\tau] \) and a velocity vector \( \nu(\zeta) \in H \) (that is not trivial), such that

\[
\mu_\nu(\zeta, \tau) = \mu_\Phi(\zeta, \tau) = \tau \nu(\zeta) + o(\tau).
\]

We take into account that \( \Phi(\zeta, 0) \equiv \zeta \) in \( U^* \) and is extended up to the identity map of \( \mathbb{C} \).

The formula (18) can be rewritten for \( \Phi(\zeta, \tau) \) as

\[
\frac{\Phi(\zeta, \tau) - b_0(\tau)}{\beta(\tau)} = \zeta - \frac{\tau}{\pi} \int_U \frac{\nu(w) d\sigma_w}{w - \zeta} + o(\tau).
\]

Comparing with (17) we come to the conclusion about \( \Phi \):

\[
\Phi(\zeta, \tau) = \beta(\tau) \zeta + \tau p_1 - \frac{\tau}{\pi} \int_U \frac{\nu(w) d\sigma_w}{w - \zeta} + o(\tau).
\]

The relations (16, 17, 20) imply that

\[
p(z) = \frac{p_1}{\zeta} - \frac{1}{\pi} \int_U \frac{\nu(w) d\sigma_w}{\zeta(w - \zeta)}.
\]

The constants \( p_0, p_1 \) and the function \( \nu \) must be such that \( \text{Re} p(z) > 0 \) for all \( z \in U^* \).

We summarize these observations in the following theorem.

**Theorem 2.** Let \( \Phi^\tau \) be a semi-flow in \( G^{qc} \). Then it is generated by the vector field \( \nu(\zeta) = \zeta p(\zeta) \),

\[
p(z) = p_0 + \frac{p_1}{\zeta} - \frac{1}{\pi} \int_U \frac{\nu(w) d\sigma_w}{\zeta(w - \zeta)},
\]

where \( \nu(\zeta) \in H \) is a harmonic Beltrami differential and the holomorphic function \( p(\zeta) \) has positive real part in \( U^* \).

This theorem implies that at any point \( \tau \geq 0 \) we have

\[
\frac{\partial \Phi(\zeta, \tau)}{\partial \tau} = \Phi(\zeta, \tau) p(\Phi(\zeta, \tau)).
\]
6.2. Evolution families and differential equations. A subset $\Phi^{t,s}$ of $\mathcal{G}$, $0 \leq s \leq t$ is called an *evolution family* in $\mathcal{G}$ if

- $\Phi^{t,t} = id$;
- $\Phi^{t,s} = \Phi^{t,r} \circ \Phi^{r,s}$, for $0 \leq s \leq r \leq t$;
- $\Phi^{t,s} \rightarrow id$ locally uniformly in $U^*$ as $t, s \rightarrow \tau$.

In particular, if $\Phi^{\tau}$ is a one-parameter semi-flow, then $\Phi^{t-s}$ is an evolution family. We consider a subordination chain of mappings $f(\zeta, t)$, $\zeta \in U^*$, $t \in [0, t_0)$, where the function $f(\zeta, t) = \alpha(t)z + a_0(t) + a_1(t)/\zeta + \ldots$ is a meromorphic univalent map $U^* \rightarrow \overline{\mathbb{C}}$ for each fixed $t$ and $f(U^*, s) \subset f(U^*, t)$ for $s < t$. Let us assume that this subordination chain exists for $t$ in an interval $[0, t_0)$.

Let us pass to the semigroup $\mathcal{G}^{qc}$. So $\Phi^{t,s}$ now has a quasiconformal extension to $U$ and being restricted to $U^*$ is from $\mathcal{G}$. Moreover, $\Phi^{t,s} \rightarrow id$ locally uniformly in $\mathbb{C}$ as $t, s \rightarrow \tau$.

For each $t$ fixed in $[0, t_0)$ the map $f(\zeta, t)$ has a quasiconformal extension into $U$ (that can be assumed even real analytic). An important presupposition is that $f(\zeta, t)$ generates a *nontrivial path* in the universal Teichmüller space $T$. This means that for any $t_1, t_2 \in [0, t_0)$, $t_1 \neq t_2$, the mapping $f(\zeta, t_2)$, $\zeta \in U^*$, can not be obtained from $f(\zeta, t_1)$ by a Möbius transform, or taking into account the normalization of $f$, by multiplying by a constant. We construct the superposition $f^{-1}(f(\zeta, s), t)$ for $t \in [0, t_0)$, $s \leq t$. Putting $s = t - \tau$ we denote this mapping by $\Phi(\zeta, t, \tau)$.

Now we suppose the following conditions for $f(\zeta, t)$.

(i) The maps $f(\zeta, t)$ form a subordination chain in $U^*$, $t \in [0, t_0)$.
(ii) The map $f(\zeta, t)$ is holomorphic in $U^*$, $f(\zeta, t) = \alpha(t)\zeta + a_0(t) + a_1(t)/\zeta + \ldots$, where $\alpha(t) > 0$ and differentiable with respect to $t$.
(iii) The map $f(\zeta, t)$ is a quasiconformal homeomorphism of $\overline{\mathbb{C}}$.
(iv) The chain of maps $f(\zeta, t)$ is not trivial.
(v) The Beltrami coefficient $\mu_f(\zeta, t)$ of this map is differentiable with respect to $t$ locally uniformly in $U$, vanishes in some neighbourhood of $U^*$ (independently of $t$).

The function $\Phi(\zeta, t, \tau)$ is embedded into an evolution family in $\mathcal{G}$. It is differentiable with regard to $\tau$ and $t$ in $[0, t_0)$, and $\Phi(\zeta, t, 0) = \zeta$. Fix $t$ and let $D_{\tau} = \Phi^{-1}(U^*, t, \tau) \setminus U^*$. Then, there exists $\nu \in H$ such that the Beltrami coefficient $\mu$ is of the form $\mu_f(\zeta, t, \tau) = \tau \nu(\zeta, t) + o(\tau)$ in $U \setminus D_{\tau}$, $\mu_f(\zeta, t, \tau) = \mu_f(\zeta, t - \tau)$ in $D_{\tau}$, and vanishes in $U^*$. We make $\tau$ sufficiently small such that $\mu_f(\zeta, t, \tau)$ vanishes in $D_{\tau}$ too. Therefore, $\zeta = \lim_{\tau \rightarrow 0} \Phi(\zeta, t, \tau)$ locally uniformly in $\mathbb{C}$ and $\Phi(\zeta, t, \tau)$ is embedded now into an evolution family in $\mathcal{G}^{qc}$. The identity map is embedded into a semi-flow $\Phi^\tau \subset \mathcal{G}^{qc}$ (which is smooth) as the initial...
point with the same velocity vector

\[ \frac{\partial \Phi(\zeta, t, \tau)}{\partial \tau} \bigg|_{\tau=0} = \zeta p(\zeta, t), \quad \zeta \in U^*, \]

that leads to equation (11) (the semi-flow \( \Phi^t \) is tangent to the evolution family at the origin). Actually, the differentiable trajectory \( f(\zeta, t) \) generates a pencil of tangent smooth semi-flows with starting tangent vectors \( \zeta p(\zeta, t) \) (that may be only measurable with respect to \( t \)). The projection to the universal Teichmüller space is shown in Figure 3.

**Figure 3.** The pencil of tangent smooth semi-flows

The requirement of non-triviality makes it possible to use the variation (19). Therefore, the conclusion is that the function \( f(\zeta, t) \) satisfies the equation (11) where the function \( p(\zeta, t) \) is given by

\[ p(\zeta, t) = p_0(t) + \frac{p_1(t)}{\zeta} - \frac{1}{\pi} \iint_U \frac{\nu(w, t) d\sigma_w}{\zeta(w - \zeta)}, \]

and has positive real part. The existence of \( p_0(t), p_1(t) \) comes from the existence of the subordination chain. We can assign the normalization to \( f(\zeta, t) \) controlling the change of the conformal radius of the subordination chain by \( e^{-t} \). In this case, changing variables we obtain \( p_0 = 1, p_1 = 0 \).

Summarizing the conclusions about the function \( p(\zeta, t) \) we come to the following result.

**Theorem 3.** Let \( f(\zeta, t) \) be a subordination chain of maps in \( U^* \) that exists for \( t \in [0, t_0) \) and satisfies the conditions (i–v). Then, there are a real valued function \( p_0(t) > 0 \), a complex valued function \( p_1(t) \), and a harmonic Beltrami differential \( \nu(\zeta, t) \), such that \( \text{Re} \, p(\zeta, t) > 0 \) for \( \zeta \in U^* \),

\[ p(\zeta, t) = p_0(t) + \frac{p_1(t)}{\zeta} - \frac{1}{\pi} \iint_U \frac{\nu(w, t) d\sigma_w}{\zeta(w - \zeta)}, \quad \zeta \in U^*, \]
and $f(\zeta,t)$ satisfies the differential equation

$$\frac{\partial f(\zeta,t)}{\partial t} = -\zeta \frac{\partial f(\zeta,t)}{\partial \zeta} p(\zeta,t), \quad \zeta \in U^*,$$

in $t \in [0,t_0]$.

In the above theorem the function $\nu(\zeta,t)$ belongs to the space of harmonic differentials. We ask now about another but equivalent form of $\nu$ as well as whether one can extend the equation (22) onto the whole complex plane.

Writing $w = f(\zeta,t-\tau)$, $\Phi(\zeta,t,\tau) = f^{-1}(w,t)$ we calculate the dilatation of the function $\Phi(\zeta,t,\tau)$ in $U$. Note that $\Phi$ is differentiable by $t, \tau$.

$$\mu_{\Phi} = \frac{\Phi_\zeta}{\Phi_{\bar{\zeta}}} = \frac{f_{w^{-1}} w_\zeta + f_{\bar{w}^{-1}} \bar{w}_\zeta}{w_\zeta + \mu_{f^{-1}} \bar{w}_\zeta} = \frac{w_\bar{w} - \mu_{f^{-1}} f_\zeta}{w_\zeta} \frac{w_\bar{w} - \mu_{f^{-1}} \bar{f}_\zeta}{w_\zeta}.$$

We use that $\mu_{f^{-1}} \circ f = -\mu_{f^{-1}} f_\zeta/\bar{f}_\zeta$. Finally, $\mu_f$, $f_\zeta$, $f_{\bar{\zeta}}$ are differentiable by $t$ almost everywhere in $t \in [0,t_0)$, locally uniformly in $\zeta \in U$, and

$$\nu_0(\zeta,t) = \lim_{\tau \to 0} \frac{\mu_{\Phi}}{\tau} = -\frac{f_\zeta f_\bar{\zeta}}{f_\zeta} \frac{\partial}{\partial t} \left( \frac{\mu_{f^{-1}} f_\zeta}{f_\zeta} \right),$$

where the limit exists a.e. with respect to $t \in [0,t_0)$ locally uniformly in $\zeta \in U$, or in terms of the inverse function

$$\nu_0(\zeta,t) = \left( \frac{f_{w^{-1}}}{f_{\bar{w}^{-1}}} \frac{\partial \mu_{f^{-1}}}{\partial \bar{\zeta}} \right) \circ f(\zeta,t).$$

Sometimes, it is much better to operate just with dilatations, avoiding functions, so we can rewrite the last expression as

$$\nu_0(z,t) = -\mu_f(z,t) \left[ \frac{\partial \log \mu_{f^{-1}}}{\partial t} \circ f(z,t) \right].$$

**Remark.** The function $\nu(\zeta,t)$ in Theorem 3 may be replaced by the function $\nu_0(\zeta,t)$ that belongs to the same equivalence class in $H$.

Let us consider one-parameter families of maps in $U^*$ normalized by $f(\zeta,t) = e^{-t}\zeta + a_1(t) + \ldots$. The inverse result to the Löwner-Kufarev equation states that given a holomorphic function $p(\zeta,t) = 1 + p_1(t)/\zeta + \ldots$ in $\zeta \in U^*$ with positive real part the solution
of the equation (22) presents a subordination chain (see, e.g., [39]). This enable us to give a condition for \( \nu_0 \) that guarantees a normalized one-parameter non-trivial family of maps \( f(\zeta,t) \) to be a subordination chain.

**Theorem 4.** Let \( f(\zeta,t) \) be a normalized one-parameter non-trivial family of maps for \( \zeta \in U^* \) which satisfies the conditions (ii–v) and is defined in an interval \( [0,t_0) \). Let each \( f(\zeta,t) \) be a homeomorphism of \( \mathbb{C} \) which is meromorphic in \( U^* \), is normalized by \( f(\zeta,t) = e^{-t}\zeta + \frac{a_1(t)}{\zeta} + \ldots \), and satisfies (22). Let the quasiconformal extension to \( U \) be given by a Beltrami coefficient \( \mu_f = \mu(\zeta,t) \) which is differentiable with respect to \( t \) almost everywhere in \( t \in [0,t_0) \). If

\[
\|\nu_0\|_\infty < \frac{\pi}{4 \int_0^1 s K(s) ds} \approx 0.706859 \ldots ,
\]

where \( \nu_0(\zeta,t) \) is as above and \( K(\cdot) \) is the complete elliptic integral, then \( f(\zeta,t) \) is a normalized subordination chain.

**Proof.** Let \( |\zeta| = \rho \), \( w = re^{i\theta} \). We calculate

\[
\left| \frac{1}{\pi} \int_U \frac{\nu_0(w,t) d\sigma_w}{\zeta(w-\zeta)} \right| \leq \frac{\|\nu_0\|_\infty}{\rho \pi} \int_U \frac{d\sigma_w}{|w-z|} = \frac{\|\nu_0\|_\infty}{\rho^2 \pi} \int_U \frac{d\sigma_w}{|1-w/z|} \\
= \frac{\|\nu_0\|_\infty}{\pi} \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{\rho^2 |1-re^{i\theta}/z|} \\
= \frac{\|\nu_0\|_\infty}{\pi} \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{\rho^2 |1-re^{i\theta}/\rho|} \\
= \frac{\|\nu_0\|_\infty}{\pi} \int_0^1 \int_0^{2\pi} \frac{r dr d\theta}{\rho^2 \sqrt{1 + \frac{\rho^2}{\rho^2} - 2 \frac{\rho}{\rho} \cos \theta}} \\
= \frac{\|\nu_0\|_\infty}{\pi} \int_0^{2\pi} s ds d\theta \\
\leq \frac{\|\nu_0\|_\infty}{\pi} \int_0^1 \int_0^{\pi} s ds d\theta \\
= 4 \frac{\|\nu_0\|_\infty}{\pi} \int_0^1 s K(s) ds < 1.
\]
Then \( \text{Re } p(z, t) > 0 \) that implies the statement of the theorem. \qed

**Remark.** If \( \|\nu_0(\cdot, t)\|_\infty \leq q \), then

\[
\begin{align*}
1 + |\mu(\zeta, t)| & \leq e^{2tq} \frac{1 + |\mu(\zeta, 0)|}{1 - |\mu(\zeta, 0)|}.
\end{align*}
\]

This obviously follows from the inequality

\[
\frac{\partial |\mu_f|}{\partial t} = \frac{\partial |\mu_{f^{-1}}|}{\partial t} \leq |\mu_{f^{-1}}|.
\]

**Remark.** Let us remark that the function \( \nu_0 \) can be unilateraly discontinuous on \( S^1 \) in \( \overline{U} \), therefore, it is not possible, in general, to use the Borel-Pompeiu formula to reduce the integral in \( p \) to a contour integral.

The equation (22) is just the Löwner-Kufarev equation in partial derivatives with a special function \( p(z, t) \) given in the above theorems.

Now we discuss the possibility of extending the equation (22) to all of \( \mathbb{C} \). We differentiate the function \( \Phi(\zeta, t, \tau) \) with respect to \( \tau \) when \( \zeta \in U \cup U^* \). It follows that

\[
\frac{\partial \Phi(\zeta, t, \tau)}{\partial \tau} \bigg|_{\tau=0} = \frac{-\bar{f}_\zeta}{|f_\zeta|^2 - |f_\zeta|^2} \bar{f} + \frac{\bar{f}_\zeta}{|f_\zeta|^2 - |f_\zeta|^2} f =: G(\zeta, t).
\]

This formula can be rewritten in the following form

\[
\dot{f}(\zeta, t) = -(f_\zeta G(\zeta, t) + f_\zeta \bar{G}(\zeta, t)).
\]

Taking into account the equation (22) in \( U^* \) we have in the whole plane

\[
\dot{f}(\zeta, t) = \begin{cases} 
-(f_\zeta G(\zeta, t) + f_\zeta \bar{G}(\zeta, t)), & \text{for } \zeta \in \overline{U}, \\
-f_\zeta p(\zeta, t), & \text{for } \zeta \in U^*. 
\end{cases}
\]

where \( p(\zeta, t) \) is a holomorphic in \( U^* \) function with the positive real part by Theorem 3.

The variational formula (20) and differentiation of the singular integral imply that \( G(\zeta, t) = \nu_0(\zeta, t), \ z \in U \). Now let us clarify what is \( G \). Let us consider \( \zeta \in U \). The Pompeiu formula leads to

\[
G(\zeta, t) = h(\zeta, t) - \frac{1}{\pi} \int_U \frac{\nu_0(w, t) d\sigma_w}{w - \zeta}, \quad \zeta \in U
\]

where \( h(\zeta, t) \) is a holomorphic function with respect to \( \zeta \). The function \( G \) is continuous in \( \overline{U} \) and by the Cauchy theorem

\[
h(\zeta, t) = \frac{1}{2\pi i} \int_{S^1} \frac{G(w, t)}{w - \zeta} dw.
\]
To obtain the boundary values of the function $G(w, t), |w| = 1$, we will use the second line in (23). Unfortunately, in general, it is not possible to use the same function $f$ in both lines of (23) to obtain boundary values of $G$. Indeed, the mapping $f(\zeta, t)$ is differentiable regarding to $t$ a.e. in $t \in [0, t_0]$ locally uniformly in $\zeta \in \mathbb{C}$, and continuous in $\zeta \in \mathbb{C}$ for almost all $t \in [0, t_0]$. Therefore, the function $-(f_\zeta G(\zeta, t) + \bar{f}_\zeta \bar{G}(\zeta, t))$, $\zeta \in U$ is the extension of $-\zeta f'(\zeta, t)p(\zeta, t)$, $\zeta \in U^*$, whereas $f_\zeta$, $\zeta \in U$ is not necessarily an extension of $f'$, $\zeta \in U^*$.

A simple example of this situation is as follows. Let us consider the function $f(\zeta, t) = \begin{cases} e^{-t} \left( c\zeta + \frac{\bar{\zeta}}{c} \right), & \text{for } \zeta \in \overline{U}, \\ e^{-t} \left( c\zeta + \frac{1}{c\zeta} \right), & \text{for } \zeta \in U^*, \end{cases}$

where $c > 1$. This mapping forms a subordination chain with the dilatation $\mu(\zeta)$ that vanishes in $U^*$ and is the constant $1/c^2$ in $U$. This chain is trivial, but it is not important for our particular goal here because we do not use at this stage the crucial variation. Then,

$$G(\zeta, t) = \begin{cases} \zeta, & \text{for } \zeta \in U, \\ \frac{c^2\zeta^2 + 1}{c^2\zeta^2 - 1}, & \text{for } \zeta \in U^*, \end{cases}$$

and it splits into two parts that can not be glued on $S^1$. The same is for the derivatives $f_\zeta$ in $U$ and $f'$ in $U^*$.

If $\mu(\zeta, t)$ satisfies the condition (v) in a neighbourhood of $S^1$ in $\overline{U}$, then the derivatives $f_\zeta$, $f_{\bar{\zeta}}$, $\zeta \in U$ has a continuation onto $S^1$ and

$$F(\zeta, t) = \frac{\overline{f_{\bar{\zeta}}}f'p(\zeta, t) - f_\zeta f\bar{p}(\zeta, t)}{|f_{\bar{\zeta}}|^2 - |f_\zeta|^2}, \quad \zeta \in S^1,$$

where $\zeta f'p(\zeta, t)$ is thought of as the angular limits that exist a.e. on $S^1$. Moreover, in a neighbourhood of $S^1$ the derivative $f_{\bar{\zeta}}$ vanishes and the function $F(\zeta, t)$ can be written on $S^1$ as $F(\zeta, t) = \zeta p(\zeta, t)$. In turn,

$$h(\zeta, t) = \frac{1}{2\pi i} \int_{S^1} \frac{wp(w, t)}{w - \zeta} dw.$$

This information allows us formulate the following theorem.

**Theorem 5.** Let $f(\zeta, t)$ be a subordination non-trivial chain of maps in $U^*$ that exists for $t \in [0, t_0)$ and satisfies the conditions (i–v).
(i) For \( \zeta \in U^* \) there exists a holomorphic function \( p(\zeta, t) \) given by Theorem 3 such that
\[
\dot{f}(\zeta, t) = -\zeta f'(\zeta, t)p(\zeta, t).
\]

(ii) For \( \zeta \in U \) there exists a continuous in \( \zeta \) function \( F(\zeta, t) \) given by
\[
F(\zeta, t) = \frac{1}{2\pi i} \int_{S^1} \frac{w p(w, t)}{w - \zeta} dw - \frac{1}{\pi} \int_U \nu_0(w, t) d\sigma_w,
\]
\[
\nu_0(\zeta, t) = \frac{f^{-1}_w}{\mu_f^{-1}} \frac{\partial \mu_f^{-1}}{\partial t} \circ f(\zeta, t),
\]
such that
\[
\dot{f}(\zeta, t) = -f_\zeta F(\zeta, t) - f_{\bar{\zeta}} \bar{F}(\zeta, t).
\]

6.3. The Löwner-Kufarev ordinary differential equation. Dually to the Löwner-Kufarev partial derivative equation there is the Löwner-Kufarev ordinary differential equation. A function \( g \in \Sigma_0 \) is represented as a limit
\[
\lim_{t \to \infty} e^{-t} w(\zeta, t),
\]
where the function \( w = g(\zeta, t) \) is a solution of the equation
\[
\frac{dw}{dt} = -wp(w, t),
\]
almost everywhere in \( t \in [0, \infty) \), with the initial condition \( g(\zeta, 0) = \zeta \). The function \( p(\zeta, t) = 1 + p_1(t)/\zeta + \ldots \) is analytic in \( U^* \), measurable with respect to \( t \in [0, \infty) \), and its real part \( \text{Re} \, p(\zeta, t) \) is positive for almost all \( t \in [0, \infty) \). The equation (25) is known as the Löwner-Kufarev ordinary differential equation. The solutions to (25) form a retracting subordination chain \( g(\zeta, t) \), i.e., it satisfies the condition \( g(U^*, t) \subset U^* \), \( g(U^*, s) \subset g(U^*, s) \) for \( t > s \), and \( g(\zeta, 0) = \zeta \).

The connection between (22) and (25) can be thought of as follows. Solving (22) by the method of characteristics and assuming \( s \) as the parameter along the characteristics we have
\[
\frac{dt}{ds} = 1, \quad \frac{d\zeta}{ds} = \zeta p(\zeta, t), \quad \frac{df}{ds} = 0,
\]
with the initial conditions \( t(0) = 0, \zeta(0) = \zeta_0, f(\zeta, 0) = f_0(\zeta) \), where \( \zeta_0 \) is in \( U^* \). We see that the equation (25) is exactly the characteristic equation for (22). Unfortunately, this approach requires the extension of \( f_0(w^{-1}(\zeta, t)) \) into \( U^* \) because the solution of the function \( f(\zeta, t) \) is given as \( f_0(w^{-1}(\zeta, t)) \), where \( \zeta = w(\zeta_0, s) \) is the solution of the initial value problem for the characteristic equation.

Our goal is to deduce a form of the function \( p \) on the case of the subclass \( \Sigma_0^{qc} \). Let a one-parameter family of maps \( w = g(\zeta, t), g \in \Sigma_0^{qc} \), satisfy the following conditions.
The maps \( g(\zeta, t) \) form a retracting subordination chain \( g(U^*, 0) \subset U^* \).

The map \( g(\zeta, t) \) is meromorphic in \( U^* \), 
\[
f(\zeta, t) = \alpha(t)\zeta + a_0(t) + a_1(t)/\zeta + \ldots,
\]
where \( \alpha(t) > 0 \) and differentiable with respect to \( t \).

The map \( g(\zeta, t) \) is a quasiconformal homeomorphism of \( \mathbb{C} \).

The chain of maps \( g(\zeta, t) \) is not trivial.

The Beltrami coefficient \( \mu_g(\zeta, t) \) of this map is differentiable with respect to \( t \) locally uniformly in \( U \).

Note that in this case we need not a strong assumption (v) in Section 6.5.2.

Set
\[
H(\zeta, t, \tau) = g(g(\zeta, t), \tau) = \beta(\tau)w + b_0(\tau) + \frac{b_1(\tau)}{w} + \ldots,
\]
where \( w = g(\zeta, t) \). For each fixed \( t \) the mapping \( g(\zeta, t) \) generates a smooth semi-flow \( H^\tau \) in \( G^c \) which is tangent to the path \( g(\zeta, t + \tau) \) at \( \tau = 0 \). Therefore, we use the velocity vector \( wp(w, t) \) (that may be only measurable regarding to \( t \)) with \( w = g(\zeta, t) \) and obtain
\[
\frac{\partial H(\zeta, t, \tau)}{\partial \tau} \bigg|_{\tau=0} = g(\zeta, t) p(g(\zeta, t), t).
\]

As before, the trajectory \( g(\zeta, t) \) generates a pencil of tangent smooth semi-flows with the tangent vectors \( wp(w, t) \), \( w = g(z, t) \). Since \( g(U^*, t) \in U^* \) for any \( t > 0 \), we can consider the limit
\[
\lim_{\tau \to 0} \frac{H(\zeta, t, \tau) - g(\zeta, t)}{\tau g(\zeta, t)}.
\]

We have that
\[
\frac{\partial H(\zeta, t, \tau)}{\partial \tau} \bigg|_{\tau=0} = \frac{\partial g(\zeta, t)}{\partial t} = g(\zeta, t) p(g(\zeta, t), t),
\]

where \( p(\zeta, t) = p_0(t) + p_1(t)/\zeta + \ldots \) is an analytic function in \( U^* \) that has positive real part for almost all fixed \( t \). The equation defined by (26) is an evolution equation for the path \( g(\zeta, t) \) and the initial condition is given by \( g(\zeta, 0) = \zeta \).

We suppose that all \( g(\zeta, t) \) admit real analytic quasiconformal extensions and the family is non-trivial in the above sense. The function \( g(w, \tau) = (H(\zeta, t, \tau) - b_0(\tau))/\beta(\tau) \) can be extended to a function from \( \Sigma^c_0 \) and it represents an equivalence class \([g^\tau] \in T\). There is a one-parameter path \( y^\tau \in T \) that corresponds to a tangent velocity vector \( \nu(w, t) \) such that
\[
\mu_g(w, \tau) = \tau \nu(w, t) + o(\tau), \quad w = g(z, t).
\]

We calculate explicitly the velocity vector making use of the Beltrami coefficient for a superposition:
\[
\nu(w, t) = \lim_{\tau \to 0} \frac{\mu_g(w, \tau) \circ g(\zeta, t)}{\tau} = \lim_{\tau \to 0} \frac{1}{\tau} \frac{1}{1 - \bar{g}(\zeta, t) \mu_{H(\zeta, t, \tau)} g(\zeta, t)}.
\]
or

\[(27) \quad \nu(w, t) = \frac{\partial \mu_{g(\zeta, t)} \overline{g(\zeta, t)}}{\partial \overline{\zeta}} \frac{g(\zeta)}{g_{\zeta} \circ g^{-1}(w, t)}, \quad \zeta \in U.\]

It is natural to implement an intrinsic parametrization using the Teichmüller distance \(\tau_T(0, [g^t]) = t\), and assume the conformal radius to be \(\beta(t) = e^t\) that implies \(p_0 = 1\).

The assumption of non-triviality allows us to use the variational formula \((20)\) to state the following theorem.

**Theorem 6.** Let \(g(\zeta, t)\) be a retracting subordination chain of maps defined in \(t \in [0, t_0)\) and \(\zeta \in U^*.\) Each \(g(\zeta, t)\) is a homeomorphism of \(\overline{\mathbb{C}}\) which is meromorphic in \(U^*\), \(g(\zeta, t) = e^t \zeta + b_1/\zeta + \ldots\), with a \(c^{2t}\)-quasiconformal extension to \(U\) given by a Beltrami coefficient \(\mu(\zeta, t)\) that is differentiable regarding to \(t\) a.e. in \([0, t_0)\). The initial condition is \(g(\zeta, 0) \equiv \zeta.\) Then, there is a function \(p(\zeta, t)\) such that that \(\text{Re} \, p(\zeta, t) > 0\) for \(\zeta \in U^*\), and

\[p(w, t) = 1 - \frac{1}{\pi} \int \int_{g(U,t)} \frac{\nu(u, t) d\sigma_u}{w(u - w)}, \quad w \in g(U^*, t),\]

where \(\nu(u, t)\) is given by the formula \((27)\), \(\|\nu\|_\infty < 1\), and \(w = g(\zeta, t)\) is a solution to the differential equation

\[(28) \quad \frac{dw}{dt} = wp(w, t), \quad w \in g(U^*, t),\]

with the initial condition \(g(\zeta, 0) = \zeta.\)

**Remark.** Taking into account the superposition we have

\[p(g(\zeta, t), t) = 1 - \frac{1}{\pi} \int \int_{U} \frac{\mu g g^2(u, t) d\sigma_u}{g(\zeta, t)(g(u, t) - g(\zeta, t))},\]

where \(u \in U, \zeta \in U^*.\)

**Remark.** The function \(wp(w, t)\) has a continuation into \(g(U, t)\) given by

\[\frac{dw}{dt} = F(w, t),\]

where the function \(F(w, t)\) is a solution to the equation

\[\frac{\partial F}{\partial w} = \frac{g_\zeta^2 \mu_g}{|g_\zeta|^2 - |g_{\overline{\zeta}}|^2} \circ g^{-1}(w, t).\]

In contrary to the Löwner-Kufarev equation in partial derivatives, the function \(F\) is the continuation of \(p\) in \(U\) through \(S^1\). The solution exists by the Pompeiu integral and can
be written as
\[ F(w, t) = h(w, t) - \frac{1}{\pi} \int_{g(U, t)} \frac{g_\xi^2 \mu_g}{|g_\xi|^2 - |g_\xi|^2} \circ g^{-1}(u, t) \frac{d\sigma_u}{u - w}, \]
where \( w \in g(U, t), h(w, t) \) is a holomorphic functions with respect to \( w \), that can be written as
\[ h(w, t) = \frac{1}{2\pi i} \int_{\partial g(U, t)} \frac{up(u, t)}{u - w} du. \]

Reciprocally, given a function \( F(u, t), u \in g(U, t) \), we can write the function \( p(w, t) \) as
\[ p(w, t) = 1 - \frac{1}{\pi} \int_{g(U, t)} \frac{F_u(u, t)d\sigma_u}{w(u - w)}, \]
where \( w \in g(U^*, t) \).

6.4. **Univalent functions smooth on the boundary.** Let us consider the class \( \tilde{\Sigma} \) of functions \( f(\zeta) = \alpha\zeta + a_0 + a_1/\zeta + \ldots, \zeta \in U^* \), such that being extended onto \( S^1 \) they are \( C^\infty \) on \( S^1 \). Repeating considerations of the preceding subsection for the embedding of \( M \) into the Teichmüller space \( T \) we come to the following theorem.

**Theorem 7.** Let \( f(\zeta, t) \) be a non-trivial subordination chain of maps that exists for \( t \in [0, t_0) \) and \( \zeta \in U^* \). Each \( f(\zeta, t) \) is a homeomorphism \( U^* \rightarrow \overline{C} \) and belongs to \( \tilde{\Sigma} \) for every fixed \( t \). All these maps have quasiconformal extensions to \( U \) and there are a real-valued function \( p_0(t) > 0 \), complex-valued functions \( p_1(t) \), real-valued \( C^\infty \) functions \( d(e^{i\theta}, t) \) such that \( \text{Re} p(\zeta, t) > 0 \) for \( \zeta \in U^* \),
\[ p(\zeta, t) = p_0(t) + \frac{p_1(t)}{\zeta} - \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i2\theta}d(e^{i\theta}, t)d\theta}{\zeta(e^{i\theta} - \zeta)}, \quad \zeta \in U^*, \]
and \( f(\zeta, t) \) satisfies the differential equation
\[ \frac{\partial f(\zeta, t)}{\partial t} = -\zeta \frac{\partial f(\zeta, t)}{\partial \zeta} p(\zeta, t), \quad \zeta \in U^*. \]

Theorems 4 and 7 are linked as follows. For a given subordination chain of maps \( f(\zeta, t) \in \tilde{\Sigma}, \) that exists for \( t \in [0, t_0) \) and \( \zeta \in U^* \), there is a \( C^\infty \) function \( d(e^{i\theta}, t) \) by Theorem 7 and we can construct the function \( \nu(\zeta, t) \) by the Douady-Earle extension and
the formula (3). Then, the function $f(\zeta, t)$ satisfies the equation of Theorem 3 with $p(\zeta, t)$ defined by such $\nu(\zeta, t)$.

Let us consider the ordinary Löwner-Kufarev equation for the functions smooth on $S^1$. If the retracting chain $g(\zeta, t)$ is smooth on $S^1$, then we use again the embedding of $M$ into $T$ and reach a similar result.

**Theorem 8.** Let $g(\zeta, t)$ be a retracting non-trivial subordination chain of normalized maps that exists for $t \in [0, t_0]$ and $\zeta \in U^*$. Each $g(\zeta, t)$ is meromorphic in $U^*$, smooth on $S^1$, and $g(\zeta, t) = \beta(t)\zeta + b_0(t) + \frac{b_1(t)}{\zeta} + \ldots$, $\beta(t) > 0$. An additional assumption is that $g : U^* \to U^*$ for each fixed $t$. Then, there are a real-valued function $p_0(t)$, a complex-valued function $p_1(t)$, and a smooth real-valued function $d(e^{i\theta}, t)$, such that $\Re p(\zeta, t) > 0$ for $\zeta \in U^*$,

$$p(\zeta, t) = p_0(t) + \frac{p_1(t)}{\zeta} - \frac{1}{2\pi i} \int_{S^1} \left( \frac{zg'(z, t)}{g(z, t)} \right)^2 \frac{d(z, t)dz}{g(z, t) - \zeta}, \quad \zeta \in U^*,$$

and $w = g(\zeta, t)$ is a solution to the differential equation

$$\frac{dw}{dt} = wp(w, t), \quad w \in g(U^*, t)$$

with the initial condition $g(\zeta, 0) = \zeta$.

**Remark.** If we work with normalized functions

$$g(\zeta, t) = e^t\zeta + \frac{b_1(t)}{\zeta} + \ldots,$$

then $p_0(t) \equiv 1$, $p_1(t) \equiv 0$.

6.5. **An application to Hele-Shaw flows.** Theorem 7 is linked to the Hele-Shaw free boundary problem as follows. Starting with a smooth boundary $\Gamma_0$ the one-parameter family $\Gamma(t)$ consists of smooth curves as long as the solutions exist. Let us consider the equation (3). Under injection we have a subordination chain of domains $\Omega(t)$. The Schwarz kernel can be developed as

$$\frac{\zeta + e^{i\theta}}{\zeta - e^{i\theta}} = 1 + \frac{2e^{i\theta}}{\zeta} + \frac{2e^{2i\theta}}{\zeta(\zeta - e^{i\theta})}.$$

Therefore, in Theorem 7 we can put

$$p_0(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta}, t)|^2} d\theta, \quad p_1(t) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{i\theta}}{|f'(e^{i\theta}, t)|^2} d\theta.$$
and
\[ d(e^{i\theta}, t) = \frac{-2}{|f'(e^{i\theta}, t)|^2}. \]

Apart from the trivial elliptic case there are no self-similar solutions, and therefore the Hele-Shaw dynamics \( f(\zeta, t) \) generates a non-trivial path in \( T \). Thus, given a Hele-Shaw evolution \( \Gamma(t) = f(S^1, t) \) we observe a differentiable non-trivial path on \( T \), such that at any time \( t \) the tangent vector \( \nu \) is a harmonic Beltrami differential given by
\[
\nu(\zeta, t) = \frac{-3}{\pi} \int_0^{2\pi} \frac{(1 - |\zeta|^2)^2}{(1 - e^{i\theta}\zeta)^4} \frac{e^{2i\theta}}{|f'(e^{i\theta}, t)|^2} d\theta.
\]
The corresponding co-tangent vector is
\[
\varphi(\zeta, t) = \frac{6}{\pi} \int_0^{2\pi} \frac{e^{-2i\theta}d\theta}{(1 - e^{-i\theta}\zeta)^4|f'(e^{i\theta}, t)|^2}.
\]

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