Probing singularities in quantum cosmology with curvature scalars

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We provide further evidence that the canonical quantization of cosmological models eliminates the classical Big Bang singularity, using the DeBroglie-Bohm interpretation of quantum mechanics. The usual criterion for absence of the Big Bang singularity in Friedmann-Robertson-Walker (FRW) quantum cosmological models is the non-vanishing of the expectation value of the scale factor. We compute the ‘local expectation value’ of the Ricci and Kretschmann scalars, for some quantum FRW models. We show that they are finite for all time. Since these scalars are elements of general scalar polynomials in the metric and the Riemann tensor, this result indicates that, for the quantum models treated here, the ‘local expectation value’ of these general scalar polynomials should be finite everywhere. Therefore, according to the classification introduced in Refs. [9, 10], we have further evidence that the quantization of the models treated here eliminates the classical Big Bang singularity.

Keywords: Quantum cosmology, Big Bang singularity, DeBroglie-Bohm interpretation of quantum mechanics.

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The presence of singularities in cosmological models in general relativity is an old issue [11]. In a series of articles, Hawking, Penrose and Geroch showed that, if certain very general conditions are satisfied, singularities are always present in cosmological models based on general relativity [12]. One of such singularities is the Big Bang singularity, which is believed to represent the very beginning of the Universe. Here, one has a fundamental problem because, if the beginning of the Universe is a singular event of general relativity, that theory cannot describe it. In order to overcome that fundamental problem, many authors proposed the quantization of gravity. Quantum cosmology was the first of such attempts and, since the first model, has showed good signs toward the solution of the above mentioned problem [1]. Since then, many important works have been done by computing the wave function of the universe (Ψ) in different minisuperspace models. Some authors find Ψ by solving the Wheeler-DeWitt equation [5, 6, 13–18, 24] and others by using the path integral approach [19–21]. As a common result, in most of them the problem of the initial singularity was claimed to be solved. The main argument used to support those claims depends on the quantum mechanical interpretation used in each particular work.

The two interpretations most frequently used in quantum cosmology are the Many Worlds one [22] and the DeBroglie-Bohm one [2, 3]. As in the usual Copenhagen interpretation of quantum mechanics, in the Many Worlds interpretation one cannot talk about trajectories of the canonical variables, but only about mean values of those variables. On the other hand, in the DeBroglie-Bohm interpretation the trajectories of the canonical variables are meaningful and can, in principle, be computed by solving a system of differential equations involving derivatives of the wave function phase. In the minisuperspace models treated using the Many Worlds interpretation, the common argument used to justify the absence of a Big Bang singularity is the fact that the mean value of the scale factor (a), as a function of a chosen time, never vanishes [3, 6, 14, 18, 24]. In the models investigated according to the DeBroglie-Bohm interpretation, the argument was that the scale factor Bohmian trajectories a(t) as a function of a chosen time never go through a = 0 [2, 3]. That result is supported by the fact that the quantum potential present in the dynamical equation of a, for those models, is repulsive for a near to zero [5, 8]. In the present work, we shall restrict our attention to the DeBroglie-Bohm interpretation of quantum mechanics.

Quantum cosmology was the first attempt in order to remove the Big Bang singularity by quantizing the gravita-

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tional theory \cite{1}. The DeBroglie-Bohm interpretation of quantum mechanics \cite{2, 3}, is frequently used in quantum cosmology. In the minisuperspace models treated using the DeBroglie-Bohm interpretation the common argument used to justify the absence of a Big Bang singularity is the fact that the scale factor Bohmian trajectories \(a(t)\), as a function of a chosen time, never go through \(a = 0\) \cite{5–8}.

In order to derive the physical content of any operator \(\hat{A}(x, p_x)\), in the DeBroglie-Bohm interpretation of quantum mechanics, one must compute the so-called ‘local expectation value’ of that operator defined as \cite{3}

\[
A(x, t) = \text{Re} \left( \frac{\Psi^*(x, t) (\hat{A}\Psi)(x, t)}{\Psi^*(x, t) \Psi(x, t)} \right),
\]

where \((\hat{A}\Psi)(x, t) = \int \hat{A}(x, x')\Psi(x', t) d^3x'\). If we apply this definition in quantum cosmology, it is easy to see that the ‘local expectation value’ of the scale factor operator is the real, time dependent, scale factor function. On the other hand, the ‘local expectation value’ of more complicated operators constructed out of the scale factor and its canonically conjugate momentum will be much more difficult to compute and in general will require a specific factor ordering prescription.

The Big Bang singularities that occur in the classical Friedmann-Robertson-Walker (FRW) models are said to be ‘scalar polynomial singularities’ \cite{9}. A ‘scalar polynomial singularity’ is the end point of at least one curve on which a scalar polynomial in the metric and the Riemann tensor becomes infinite \cite{10}. In the present work, we wish to give further evidence, besides the usual one, that the canonical quantization of FRW models removes the initial Big Bang singularities of those models. We shall compute, for all FRW models considered, the ‘local expectation value’ of the Ricci \((\equiv g_{\alpha\beta}g_{\gamma\delta}R^{\delta\alpha\gamma\beta})\) and the Kretschmann scalars \((\equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})\). As we shall see, they are finite for all time. Therefore, since these scalars are components of general scalar polynomials in the metric and the Riemann tensor, that result indicates that the ‘local expectation value’ of a general scalar polynomial in the metric and the Riemann tensor should be free from singularities in those models.

In the present work, we shall consider FRW cosmological models coupled to a radiative perfect fluid, treated by means of the variational formalism developed by Schutz \cite{23}. Our main motivation to choose the matter content of the model as radiation is because we would like to describe the very early Universe, the so called ‘radiation dominated era’. At that time, the quantum effects were more important. The models are described by the Hamiltonian \cite{6}

\[
\mathcal{H} = \frac{p_a^2}{24} + 6ka^2 - p_T,
\]

where \(p_a\) and \(p_T\) are, respectively, the momenta canonically conjugate to the scale factor \((a)\) and the radiation variable \((T)\). The parameter \(k\) is related to the spatial curvature of the model and may assume the values +1 (positive curvature), −1 (negative curvature) and zero (no curvature). We employ the natural system of units, defined by \(\hbar = c = 16\pi G = 1\).

The quantization of those models follows the Dirac formalism for quantizing constrained systems. The application of this formalism for the present models result in the following Wheeler-DeWitt equation for the wave function \(\Psi(a, T)\) \cite{6}.

First we introduce a wave function which is a function of the canonical variables \(a\) and \(T\),

\[
\Psi = \Psi(a, T).
\]

Then, we impose the appropriate commutators between the operators \(a\) and \(T\) and their conjugate momenta \(p_a\) and \(p_T\). Working in the Schrödinger picture, the operators \(a\) and \(T\) are simply multiplication operators, while their conjugate momenta are represented by the differential operators

\[
p_a \to -i \frac{\partial}{\partial a}, \quad p_T \to -i \frac{\partial}{\partial T}.
\]

Finally, we demand that the operator corresponding to \(\mathcal{H}\) annihilate the wave function \(\Psi\), which leads to the Wheeler-DeWitt equation. From Eq. \(\mathcal{H}\), this formalism leads to

\[
\frac{\partial^2 \Psi}{\partial a^2} - 144ka^2 \Psi + 24i \frac{\partial \Psi}{\partial \tau} = 0,
\]

where \(T = -\tau\). Several solutions to equation \(\mathcal{H}\) are known.
Consider first the case \( k = 0 \). Then the solution to Eq. (5) is given by \[24\],
\[
\Psi(a, \tau) = \left( \frac{4\sigma}{\pi} \right)^{1/4} \sqrt{\frac{1}{(2\sigma \tau - i)(2\tau + i)}} \times \exp \left\{ \frac{6i}{\tau} \left[ 1 + \frac{i}{2\sigma \tau - i} \right] a^2 \right\}.
\] (6)

In the cases \( k = \pm 1 \), the solution of Eq. (5) may be written in the following form \[6\]:
\[
\Psi(a, \tau) = \left( \frac{4\sigma}{\pi} \right)^{1/4} \sqrt{\frac{k}{\cos^2(\sqrt{k}\tau) \left[ 2\sigma \tan(\sqrt{k}\tau) - i\sqrt{k} \right] \left[ 2\tan(\sqrt{k}\tau) + i\sqrt{k} \right]}} \times \exp \left\{ \frac{6i\sqrt{k}}{\tan(\sqrt{k}\tau)} \left[ 1 + \frac{i\sqrt{k}}{\cos^2(\sqrt{k}\tau) \left[ 2\sigma \tan(\sqrt{k}\tau) - i\sqrt{k} \right] \left[ 2\tan(\sqrt{k}\tau) + i\sqrt{k} \right] \right] a^2 \right\}.
\] (7)

In order to use the DeBroglie-Bohm interpretation we must rewrite \( \Psi \) in the polar form
\[
\Psi(a, \tau) = \left( \frac{4\sigma}{\pi} \right)^{1/4} \Theta \exp(iS)
\] (8)

Consider first the case \( k = 0 \). Then the solution to Eq. (5) is given in polar form by \[24\],
\[
\Theta = g_0(\tau) \times \exp\left( \frac{-12\sigma a^2}{1 + 4\sigma^2 \tau^2} \right)
\] and
\[
S = f_0(\tau) + \frac{24\sigma^2 \tau}{1 + 4\sigma^2 \tau^2} a^2.
\] (9) (10)

For the cases \( k = \pm 1 \) one has \[6\]
\[
\Theta = g_k(\tau) \times \exp\left( \frac{-12\sigma k a^2}{k \cos^2(\sqrt{k}\tau) + 4\sigma^2 \sin^2(\sqrt{k}\tau)} \right)
\] and
\[
S = f_k(\tau) + \frac{12\sqrt{k}}{2\tan(\sqrt{k}\tau)} \left[ 1 - \frac{k}{k \cos^2(\sqrt{k}\tau) + 4\sigma^2 \sin^2(\sqrt{k}\tau)} \right] a^2.
\] (11) (12)

Following the DeBroglie-Bohm interpretation and the fact that \( p_a = 12\dot{a} \), where the dot means differentiation with respect to \( \tau \), we may compute the scale factor Bohmian trajectory from \[7\]
\[
p_a = \frac{\partial S}{\partial a}.
\] (13)

For the case \( k = 0 \), from Eqs. (9) and (10) we have \[7\]
\[
a(\tau) = a_0(1 + 4\sigma^2 \tau^2)^{1/2},
\] (14)

where \( a_0 \) is an integration constant and
\[
Q = \frac{\sigma}{1 + 4\sigma^2 \tau^2} - \frac{24\sigma^2 a^2}{(1 + 4\sigma^2 \tau^2)^2}.
\] (15)

For the cases \( k = \pm 1 \), Eqs. (11) and (12) yield \[7\]
\[
a(\tau) = a_k(k \cos^2(\sqrt{k}\tau) + 4\sigma^2 \sin^2(\sqrt{k}\tau))^{1/2},
\] (16)
where $a_k$ is an arbitrary integration constant, associated to a given $k$, and

$$Q = \frac{\sigma k}{k \cos^2 (\sqrt{k \tau}) + 4 \sigma^2 \sin^2 (\sqrt{k \tau})} - \frac{24 \sigma^2 k^2 a^2}{(k \cos^2 (\sqrt{k \tau}) + 4 \sigma^2 \sin^2 (\sqrt{k \tau}))^2}. \quad (17)$$

In all these cases it is clear that the scale factor Bohmian trajectories never reach $a = 0$. Therefore, one may conclude that these models are free from the Big Bang singularity.

Here, we would like to give further evidence, besides the non-vanishing of the scale factor, that those models are free from the Big Bang singularity at the quantum level. In order to do that, we shall compute the ‘local expectation value’ of the Ricci and the Kretschmann scalars. As it will be seen, they remain finite for all time. These scalars are components of general scalar polynomials in the metric and the Riemann tensor. Therefore, we have an additional indication that the ‘local expectation value’ of these general scalar polynomials should be finite everywhere.

For the present models, the expressions of the Ricci ($R$) and the Kretschmann ($K$) scalars are given, respectively, by

$$R = g^{ac}g^{bd}R_{abcd} = \frac{\dot{p}_a}{2a^3} + \frac{6k}{a^2} \quad \text{and}$$

$$K = R^{abcd}R_{abcd} = \frac{p_a^2}{12a^6} - \frac{\dot{p}_a p_a^2}{72a^7} + \frac{p_a^4}{864a^8} + \frac{kp_a^2}{6a^6} + \frac{12k^2}{a^4}, \quad (19)$$

where we used $p_a = 12\dot{a}$ in order to write $R$ and $K$ in terms of $p_a$, the momentum canonically conjugate to $a$.

It is important to notice, before we proceed, that $R$ and $K$ given by Eqs. (18) and (19) are to be promoted to quantum operators. Since, at the quantum level, $p_a$ and $a$ do not commute, we shall have to introduce a specific factor ordering in order to correctly describe the terms involving products of powers of $a$ and $p_a$. Here, we shall use a symmetrization procedure known as the Weyl ordering [25]. In order to obtain the Weyl-ordered expression of a product $(a^n p_a^m)$, one first randomly orders the $a$'s and $p_a$'s, with each different ordering counted once, then divides the result by the number of terms present in the final expression [25].

In Eqs. (18) and (19) we notice the presence of the time rate of change of the momentum $p_a$. Quantum mechanically, it is an operator and in the DeBroglie-Bohm interpretation it has the following value [3],

$$\dot{p}_a = -\nabla (V + Q) \quad (20)$$

Where $V$ is the classical potential present in the Hamiltonian [2] and $Q$ is the quantum potential. From Eq. (2) $V$ is given by

$$V = 6ka^2. \quad (21)$$

Using expressions (15) and (17) for $Q$, and Eq. (21) for $V$, we may compute the time rate of change of the momentum $p_a$.

For $k = 0$ we obtain

$$\dot{p}_a = \frac{48 \sigma^2 a^4}{a^3}, \quad (22)$$

while for $k = \pm 1$ we find

$$\dot{p}_a = -12ka + \frac{48 \sigma^2 a^4}{a^3}. \quad (23)$$

Introducing the above values of $\dot{p}_a$ in the expressions for the Ricci and Kretschmann scalars we obtain new forms which depend only on the operators $a$ and $p_a$.

For $k = 0$, introducing $\dot{p}_a$ given by Eq. (22) in Eqs. (18) and (19), we get

$$R = \frac{24 \sigma^2}{a_0^7} \frac{1}{(1 + 4 \sigma^2 \tau^2)^{3/2}}, \quad (24)$$
expression of \( R \) in the polar form (8). Then, we obtain the expressions for the 'local expectation values' of each operator in Eq. (30) of expressions, such that where

\[
\sigma \mathbf{R} = 0. \quad \text{Thus, } R \text{ is regular for all values of } k \text{ at the beginning moment of the universes described by the corresponding models. In order to compare the behavior of the } R \text{'local expectation value', as a function of } \tau, \text{ with the classical expression of } R, \text{ we produced Figs. (1), (3) and (5), one for each value of } k. \text{ The classical scalar factor was derived with initial conditions compatible with those of the scale factor Bohmian trajectories. One may easily see from those figures that, for the cases where the spatial sections are open, both quantities coincide for large } \tau. \text{ In order to derive the physical content of the operator } K, \text{ in the DeBroglie-Bohm interpretation of quantum mechanics, we must compute its 'local expectation value' Eq. (1). Due to the presence of the terms}

\[
\frac{p_a^4}{a^8}, \quad \frac{p_a^2}{a^{10}}, \quad \frac{p_a^2}{a^8}
\]

in the expressions of \( K \), Eqs. (23) and (27), we choose to use the Weyl ordering (25).

In order to reduce the Weyl-ordered expressions of each one of the above products of \( a' \)'s and \( p_a' \)'s, we use the commutation relation between the operators \( a \) and \( p_a \). With its aid, we write all terms, in each of the Weyl-ordered expressions, such that \( p_a \) or a power of \( p_a \) must appear to the right of \( a \) or a power of \( a \). This procedure simplifies very much the Weyl-ordered expressions because most of the terms combine with each other. With the aid of the following commutators,

\[
[a^n, p_a^2] = 2ina^{n-1}p_a + n(n-1)a^{n-2}
\]

\[
[a^n, p_a^4] = 4ina^{n-1}p_a^3 + 6n(n-1)a^{n-2}p_a^2 - 4in(n-1)(n-2)a^{n-3}p_a
- n(n-1)(n-2)(n-3)a^{n-4},
\]

where \( n \) is a positive or negative integer, we obtain the following Weyl-ordered expressions for each one of the products of \( a' \)'s and \( p_a' \)'s in Eq. (23):

\[
\left( \frac{p_a^4}{a^8} \right)_W = \frac{1}{a^8}p_a^4 + 16i\frac{1}{a^9}p_a^3 - 116\frac{1}{a^{10}}p_a^2 - 440i\frac{1}{a^{11}}p_a + 10831\frac{1}{15a^{12}},
\]

\[
\left( \frac{p_a^2}{a^{10}} \right)_W = \frac{1}{a^{10}}p_a^2 + 10i\frac{1}{a^{11}}p_a - 175\frac{1}{6a^{12}},
\]

\[
\left( \frac{p_a^2}{a^8} \right)_W = \frac{1}{a^8}p_a^2 + 6i\frac{1}{a^9}p_a - 23\frac{1}{2a^{10}}.
\]

Now, we substitute \( p_a \) given by \(-i\partial/\partial a\) into these Weyl-ordered expressions and compute their 'local expectation value' Eq. (1), using the wave function \( S \). It is important to remember that the wave function \( \Psi \) must be written in the polar form \( \ref{8} \). Then, we obtain the expressions for the 'local expectation values' of each operator in Eq. (30) as functions of \( \Theta(a, \tau) \) and derivatives of \( \Theta(a, \tau) \) and \( S(a, \tau) \) with respect to \( a \).

\[
\left< \frac{p_a^4}{a^8} \right>_L = \frac{(\partial S(a, \tau)/\partial a)^4}{a^8} - \frac{3(\partial^2 S(a, \tau)/\partial a^2)^2}{a^8} - \frac{116(\partial S(a, \tau)/\partial a)^2}{a^{10}}
\]

For \( k = \pm 1 \), inserting \( p_a \) from Eq. (23) into Eqs. (18) and (19), we obtain

\[
R = \frac{24\sigma^2}{a_k^2} \frac{1}{(k \cos^2 (\sqrt{k} \tau) + 4\sigma^2 \sin^2 (\sqrt{k} \tau))^3}.
\]

\[
K = \frac{1}{864 a^8} \frac{p_a^4}{a^8} - \frac{2a^4\sigma^2}{3} \frac{p_a^2}{a^{10}} + \frac{k p_a^2}{3 a^6} + 192\sigma^4 a_k^8 \frac{1}{a^{12}} - 96\sigma^2 k^3 a_k^4 \frac{1}{a^8} + 24 \frac{1}{a^4}.
\]
Finally, we compute the \( K \) ‘local expectation value’ for both cases of \( k = 0 \) Eq. (24) and \( k = \pm 1 \) Eq. (27). Then, in order to compute the \( K \) ‘local expectation value’, After that, we introduce the values of \( \Theta(a, \tau) \) and \( S(a, \tau) \) for each \( k \), Eqs. (31)(12) in the ‘local expectation values’ of (\( p_a^2/a^2 \))\(_W\), (\( p_a^2/a^{10} \))\(_W\) and (\( p_a^2/a^6 \))\(_W\).

Finally, we combine them following Eq. (24) for \( k = 0 \) and Eq. (27) for \( k = \pm 1 \).

For \( k = 0 \) we find

\[
\langle K \rangle_L = \frac{C^0 a_0^8 + C^6 a_6^6 + C^4 a_4^4 + C^2 a_2^2 + C_0^0}{a_0^{12} \left(1 + 4\sigma^2 \tau^2\right)^6},
\]

where

\[
C^0_8 = 6144\sigma^8 \tau^4 - 10752\sigma^6 \tau^2 + 960\sigma^4,
\]
\[
C^0_6 = -1920\sigma^5 \tau^2 + 304\sigma^3,
\]
\[
C^0_4 = -56\sigma^4 \tau^2 + \frac{601}{9} \sigma^2,
\]
\[
C^2_2 = 9\sigma
\]
\[
C^0_0 = 10831
\]
\[
\text{For } k = 1 \text{ we get}
\]

\[
\langle K \rangle_L = \frac{C^1 a_1^8 + C^1 a_1^6 + C^1 a_1^4 + C^1 a_1^2 + C^1_0}{a_1^{12} \left(\cos^2 \tau + 4\sigma^2 \sin^2 \tau\right)^6},
\]

where

\[
C^1_8 = 24(256\sigma^8 + 256\sigma^6 - 160\sigma^4 + 16\sigma^2 + 1) \cos^4 \tau
\]
\[
+ 96\sigma^2(-128\sigma^6 - 16\sigma^4 + 40\sigma^2 - 7) \cos^2 \tau
\]
\[
+ 192\sigma^4(32\sigma^4 - 24\sigma^2 + 5),
\]
\[
C^1_6 = 80\sigma(16\sigma^4 - 8\sigma^2 + 1) \cos^4 \tau + 40\sigma(-16\sigma^4 + 16\sigma^2
\]
\[
- 3) \cos^2 \tau - 16\sigma^3(40\sigma^2 + 19),
\]
\[
C^1_4 = (17/15)(-4\sigma^4 + 2\sigma^2 - (1/4)) \cos^4 \tau + (1/5)\sigma(988/3)\sigma^4
\]
\[
- (34/3)\sigma^2 - (71/4) \cos^2 \tau + (2/3)\sigma^2(-92\sigma^2 + 757/15),
\]
\[
C^1_2 = (27/10)\sigma,
\]
\[
C^1_0 = (10831/12960).\]
Finally, for \( k = -1 \) we obtain

\[
\langle K \rangle_L = \frac{C_8^{-1}a_8^{-1} + C_6^{-1}a_6^{-1} + C_4^{-1}a_4^{-1} + C_2^{-1}a_2^{-1} + C_0^{-1}}{a_1^{12} (\cosh^2 \tau + 4\sigma^2 \sinh^2 \tau)^6},
\]

where

\[
C_8^{-1} = 24(256\sigma^8 - 256\sigma^6 - 160\sigma^4 - 16\sigma^2 + 1) \cosh^4 \tau \\
+ 96\sigma^2(-128\sigma^6 + 16\sigma^4 + 40\sigma^2 + 7) \cosh^2 \tau \\
+ 192\sigma^4(32\sigma^4 + 24\sigma^2 + 5), \\
C_6^{-1} = 80\sigma(-16\sigma^4 - 8\sigma^2 - 1) \cosh^4 \tau + 40\sigma(16\sigma^4 + 16\sigma^2 \\
+ 3) \cosh^2 \tau + 16\sigma^3(40\sigma^2 + 19), \\
C_4^{-1} = (17/15)(4\sigma^4 + 2\sigma^2 + 1/4) \cosh^4 \tau + (1/5)(-988/3)\sigma^4 \\
- (34/3)\sigma^2 + (71/4)) \cosh^2 \tau + (2/3)\sigma^2(92\sigma^2 + (757/15)), \\
C_2^{-1} = (27/10)\sigma, \\
C_0^{-1} = (10831/12960).
\]

It is clear from Eqs. (34), (36) and (38) that the \( K \) ‘local expectation value’, for each model, is regular for all \( \tau \). Including the limit \( \tau \to 0 \), that is, at the beginning moments of the corresponding classical universes. In order to compare the behavior of the \( K \) ‘local expectation value’, as a function of \( \tau \), with the classical expression of \( K \), we produced Figs. 2, 4 and 6, one for each value of \( k \). The classical scalar factor was derived with initial conditions compatible with those of the scale factor Bohmian trajectories. One may easily see from those figures that, for the cases where the spatial sections are open, both quantities coincide for large \( \tau \). It is important to notice, that this result is independent of the factor ordering used here. In fact, observing Eqs. (34), (36) and (38), we conclude that they are regular mainly because the scale factor Bohmian trajectories \( a(t) \) never go through \( a = 0 \). Therefore, from the operatorial expression of \( K \) (27), it is not difficult to see that whatever factor ordering we decide to use the denominator of the \( K \) ‘local expectation value’ will be a polynomial in the scale factor. Then, if we take in account that it does not vanish for any \( \tau \), the \( K \) ‘local expectation value’ will always be regular. Now, since, \( R \) and \( K \) are elements of general scalar polynomials in the metric and the Riemann tensor, the above results indicate that, for the quantum models treated here, the ‘local expectation value’ of these general scalar polynomials should be free of singularities. Therefore, according to the classification introduced in Refs. [9, 10], we have further evidence that the quantization of the models treated here eliminates the classical Big Bang singularity.

We believe that the above result may be extended to FRW models with matter contents described by other types of perfect fluids. This is the case because, as we have mentioned above, the main reason for the regularity of the ‘local expectation values’ of \( R \) and \( K \) is that the scale factor Bohmian trajectories \( a(t) \), as a function of a chosen time, never go through \( a = 0 \). Therefore, if we consider FRW models with matter contents described by other types of perfect fluids and we obtain Bohmian trajectories \( a(t) \) that never go through \( a = 0 \), very likely the ‘local expectation values’ of \( R \) and \( K \) of those models will always be regular. In Reference [3], the authors calculated the scale factor Bohmian trajectory, as a function of the proper time, for a flat \( (k = 0) \) FRW model with matter content described by a generic perfect fluid. A generic perfect fluid is described by the equation of state \( p = w\rho \), where \( w \) is a constant. In the radiation case \( w = 1/3 \). There, they found that the scale factor never go through \( a = 0 \) whatever value \( w \) assumes. Therefore, for that case we are very confident to say that the ‘local expectation values’ of \( R \) and \( K \) should always be regular. Unfortunately, we do not have such general result for models with \( k = \pm 1 \). On the other hand, if one believes that the quantization of those models will solve the singularity problem, they will also have scale factor Bohmian trajectories that never go through \( a = 0 \). In this way, the ‘local expectation values’ of \( R \) and \( K \) of those models should, also, always be regular.

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[1] B. S. DeWitt, Phys. Rev. D 160 (1967) 1113.
[2] D. Bohm and B. J. Hiley, The undivided universe: an ontological interpretation of quantum theory, Routledge, London, 1993;
[3] P. R. Holland, The quantum theory of motion: an account of the de Broglie-Bohm interpretation of quantum mechanics, Cambridge University Press, Cambridge, 1993.
[4] C. W. Misner, in: J. Klauder (Ed.), Magic without Magic: John Archibald Wheeler, a Collection of Essays in Honor of his 60th Birthday, W. H. Freeman, San Francisco, 1972, p. 441.
[5] F. G. Alvarenga, J. C. Fabris, N. A. Lemos, G. A. Monerat, Gen. Rel. Grav. 34 (2002) 651.
[6] N. A. Lemos, G. A. Monerat, Gen. Rel. Grav. 35 (2003) 423.
[7] J. Acacio de Barros and N. Pinto-Neto, Int. J. Mod. Phys. D 7 (1998) 201;
[8] J. Acacio de Barros, N. Pinto-Neto and M. A. Sagioro-Leal, Phys. Lett. A 241 (1998) 229.
[9] G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Grav. 8 (1977) 915.
[10] F. J. Tipler, C. J. S. Clarke and G. F. R. Ellis, in: A. Held (Ed.), General Relativity and Gravitation - One Hundred Years After the Birth of A. Einstein, Plenum Press, New York, 1980, p. 97.
[11] A. Friedmann, ZS. f. Phys 10 (1922) 377, translated and reproduced in Gen. Rel. Grav. 31 (1999) 1991.
[12] S. W. Hawking, The Large Scale Structure of Space Time, Cambridge University Press, Cambridge, 1973.
[13] V. G. Lapchinskii and V. A. Rubakov, Theor. Math. Phys. 33 (1977) 1076.
[14] M. J. Gotay and J. Demaret, Phys. Rev. D 28 (1983) 2402.
[15] Mariam Bouhmadi-Lopez and Paulo Vargas Moniz, Phys. Rev. D 71 (2005) 063521; I. G. Moss and W. A. Wright, Phys. Rev. D 29 (1984) 1067.
[16] A. Vilenkin, Phys. Rev. D 33 (1986) 3560; A. Vilenkin, Phys. Rev. D 37 (1988) 888.
[17] N. A. Lemos, F. G. Alvarenga, Gen. Relat. Grav. 31 (1999) 1743.
[18] G. A. Monerat, E. V. Corrêa Silva, G. Oliveira-Neto, L. G. Ferreira Filho and N. A. Lemos, Phys. Rev. D 73 (2006) 044022.
[19] J. B. Hartle and S. W. Hawking, Phys. Rev. D 28 (1983) 2960.
[20] M. Anderson, S. Carlip, J.G. Ratcliffe, S. Surya and S. T. Tschantz, Class. Quant. Grav. 21 (2004) 729;
[21] G. Oliveira-Neto, Phys. Rev. D 58 (1998) 10750.
[22] H. Everett, III, Rev. Mod. Phys. 29 (1957) 301.
[23] Schutz, B. F., Phys. Rev. D 2 (1970) 2762;
[24] Schutz, B. F., Phys. Rev. D 4 (1971) 3359.
[25] N. A. Lemos, J. Math. Phys. 37 (1996) 1449.
[26] T. D. Lee, Introduction to Field Theory and Particle Physics, Harwood, New York, 1981, p. 476.

FIG. 1: The $R$ ‘local expectation value’ for $k = 0$, $\sigma = 1$ and $a_0 = 1$. It is always regular and goes to zero for large $\tau$. For this case $R$ evaluated over the classical scale factor is identically zero.

FIG. 2: The $K$ ‘local expectation value’ for $k = 0$, $\sigma = 1$ and $a_0 = 1$. It is always regular and goes to zero for large $\tau$. For this case $K$ evaluated over the classical scale factor is identically zero.
FIG. 3: The lower curve represents the $R$ ‘local expectation value’ for $k = 1$, $\sigma = 1$ and $a_1 = 1$. It is periodic and always regular. The upper curve represents $R$ evaluated over the classical scale factor.

FIG. 4: The lower curve represents the $K$ ‘local expectation value’ for $k = 1$, $\sigma = 1$ and $a_1 = 1$. It is periodic and always regular. The upper curve represents $K$ evaluated over the classical scale factor.

FIG. 5: The upper curve represents the $R$ ‘local expectation value’ for $k = -1$, $\sigma = 1$ and $a_{-1} = 1$. It is always regular. The lower curve represents $R$ evaluated over the classical scale factor. Both curves go to zero for large $\tau$.

FIG. 6: The lower curve represents the $K$ ‘local expectation value’ for $k = -1$, $\sigma = 1$ and $a_{-1} = 1$. It is always regular. The upper curve represents $K$ evaluated over the classical scale factor. Both curves go to zero for large $\tau$. 