It is well known that by applying the first law of thermodynamics to the apparent horizon of a Friedmann-Robertson-Walker universe, one can derive the corresponding Friedmann equations in Einstein, Gauss-Bonnet, and more general Lovelock gravity. Is this a generic feature of any gravitational theory? Is the prescription applicable to other gravities? In this paper we would like to address the above questions by examining the same procedure for Horava-Lifshitz gravity. We find that in Horava-Lifshitz gravity, this approach does not work and we fail to reproduce a corresponding Friedmann equation in this theory by applying the first law of thermodynamics on the apparent horizon, together with the appropriate expression for the entropy in Horava-Lifshitz gravity. The reason for this failure seems to be due to the fact that Horava-Lifshitz gravity is not diffeomorphism invariant, and thus, the corresponding field equation cannot be derived from the first law around horizon in the spacetime. Without this, it implies that the specific gravitational theory is not consistent, which shows an additional problematic feature of Horava-Lifshitz gravity. Nevertheless, if we still take the area formula of geometric entropy and regard Horava-Lifshitz sector in the Friedmann equation as an effective dark radiation, we are able to extract the corresponding Friedmann equation from the first law of thermodynamics.

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I. INTRODUCTION

Inspired by Lifshitz theory in solid state physics, recently, Horava proposed a field theory model for a UV complete theory of gravity \[1\]. This theory is a nonrelativistic renormalizable theory of gravity and reduces to Einstein’s general relativity at large scales. The theory is usually referred to as the Horava-Lifshitz (HL) theory. It also has manifest three-dimensional spatial general covariance and time reparametrization invariance. Various aspects of HL gravity have been investigated in the literature. A specific direction of the research is the investigation of the thermodynamic properties of HL gravity. In particular, black hole solutions in this gravity theory have been attracted much attention. Here we review some thermodynamical properties of HL gravity investigated previously, though our list is not complete. Thermodynamics and stability of black holes in HL gravity have been studied in \[2\–6\]. It was shown that HL black holes are thermodynamically stable in some parameter space and an unstable phase also exists in other parameter spaces \[6\]. In \[7\–8\], the validity of the generalized second law of thermodynamics has been explored in a universe governed by HL gravity. In \[9\] the relationship between the first law of thermodynamics and the gravitational field equation of a static, spherically symmetric black hole in HL gravity has been explored. It was shown that, gravitational field equations of static, spherically symmetric black holes in HL theory can be written as the first law of thermodynamics on the black hole horizon \[9\]. It was argued that this approach can lead to extracting an expressions for the entropy and mass of HL black holes which are consistent with those obtained from other approaches \[9\]. In the cosmological setups, some attempts have been done to disclose the connection between thermodynamics and gravitational field equations of the Friedmann-Robertson-Walker (FRW) universe in HL theory. For example, following the entropic interpretation of gravity proposed by Verlinde \[10\], a modified Friedmann equation in HL gravity was obtained in \[11\]. Following \[11\], the connection between the Debye model for the entropic force scenario and the modified Friedmann equations in HL gravity was also studied \[12\], although the results obtained in \[11,12\] seem to be incorrect. The reason is that the Friedman-like equation obtained in these papers from the entropic force scenario is not the same as the one directly derived from the field equations of HL gravity \[2,7\]. This indicates that one cannot derive the Friedmann equation in HL cosmology from the entropic gravity perspective. Other studies on HL gravity have been carried out in \[13\].

On the other side, it was shown that the gravitational field equation of a static spherically symmetric spacetime in Einstein, Gauss-Bonnet, and more general Lovelock gravity can be transformed as the first law of thermodynamics \[14\]. The studies were also extended to other gravity theories such as \( f(R) \) gravity \[15\] and scalar-tensor gravity \[16\]. In the cosmological setup, it was shown that the differential form of the Friedmann equation of FRW universe can be transformed to the first law of thermodynamics on the apparent horizon \[17\–18\]. The extension of this connection has also been carried out in the braneworld cosmology \[19\–21\]. The deep connection between the gravitational equation describing the gravity in the bulk and the first law of thermodynamics on the apparent horizon reflects some deep ideas of holography.

Is the inverse procedure also always possible? (That is extracting the general field equations from the first law of thermodynamics?) Jacobson \[22\] was the first who disclosed that the Einstein field equation of general relativity can be derived from the relation between the horizon area and entropy, together with the Clausius relation \(dQ = TdS\). For so called \( f(R) \) gravity, Eling et al. \[23\] argued that the corresponding field equation describing gravity can be derived from thermodynamics by using the procedure in \[22\], but a treatment with nonequilibrium thermodynamics of spacetime is needed. By using the entropy expression associated with the horizon of the static spherically symmetric black hole solutions in Einstein gravity, and replacing the horizon radius \( r_h \) with the apparent horizon radius, \( \tilde{r}_A \), and taking the ansatz for the temperature of the apparent horizon, it was shown that the Friedmann equations of the FRW universe can be derived by applying the first law of thermodynamics to the apparent horizon for a FRW universe with any spatial curvature \[24\]. Employing the entropy relation of black holes to apparent horizon in Gauss-Bonnet gravity and in the more general Lovelock gravity, one also can get the corresponding Friedmann equations in these theories \[24\]. Also, starting from the first law of thermodynamics, \( dE = T_h dS_h + W dV \), at the apparent horizon of a FRW universe, and assuming that the associated entropy with the apparent horizon has a quantum corrected relation, \( S = \frac{A}{4} - \alpha \ln \frac{A}{4} + \beta \frac{A}{4} \), one is able to derive the modified Friedmann equations describing the dynamics of the universe with any spatial curvature \[25\]. These results indicate the holographic properties of the gravitational field equations in a wide range of gravity theories. For a recent review on the thermodynamical aspects of gravity see \[26\].

In the present work, we would like to address the question on the connection between thermodynamics and HL gravity, by applying the first law of thermodynamics on the apparent horizon of a FRW universe and examine whether we can extract the corresponding Friedmann equation in this gravity theory or not. Our strategy here is to pick up the entropy expression associated with the black hole horizon in HL gravity, assuming that the entropy formula also holds for the apparent horizon of a FRW universe in HL gravity. In other words, the apparent horizon has the same expression for entropy but we replace the black hole horizon radius \( r_h \) by the apparent horizon radius \( \tilde{r}_A \). We find out that the resulting Friedmann equation from the first law of thermodynamics differs from one obtained directly by
varying the action of HL gravity with respect to the FRW metric. This shows an inconsistency in HL gravity that originates from the fact that this theory is not diffeomorphism invariant, and thus, the corresponding field equation cannot derive from the first law around the horizon [27].

This paper is structured as follows. In the next section we review the IR modified HL theory and derive directly the corresponding Friedman equation by varying the action. In Sec. III, we apply the first law of thermodynamics on the apparent horizon, together with the appropriate expressions for the entropy and temperature, and extract the Friedmann-like equation of the modified HL cosmology. Only in the IR limit does the result of this section coincide on the apparent horizon. We finish our paper with a summary and discussion in Sec. V.

II. FRIEDMAN EQUATION IN IR MODIFIED HL COSMOLOGY

In this section we first review the cosmological model which is governed by HL gravity. The dynamical variables are the lapse and shift functions, \(N\) and \(N_i\), respectively, and the spatial metric \(g_{ij}\). Using the Arnowitt-Deser-Misner (ADM) formalism, the metric is parametrized as

\[
ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt). \tag{1}
\]

The Einstein-Hilbert action can be expressed as

\[
I_{EH} = \frac{1}{16\pi G} \int d^4x\sqrt{|g|}N \left[ (K_{ij}K^{ij} - K^2) + R - 2\Lambda \right], \tag{2}
\]

where \(K_{ij}\) is the extrinsic curvature which takes the form

\[
K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i), \tag{3}
\]

and a dot denotes a derivative with respect to \(t\) and covariant derivatives defined with respect to the spatial metric \(g_{ij}\). The action of HL gravity is given by

\[
I_{S H} = \int dt d^3x(\mathcal{L}_0 + \dot{\mathcal{L}}_1 + \mathcal{L}_m), \tag{4}
\]

\[
\mathcal{L}_0 = \sqrt{|g|}N \left\{ \frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1 - 3\lambda)} \right\}, \tag{5}
\]

\[
\dot{\mathcal{L}}_1 = \sqrt{|g|}N \left\{ \frac{\kappa^2\mu^2(1 - 4\lambda)}{32(1 - 3\lambda)}R^2 - \frac{\kappa^2}{2\omega^2} \left( C_{ij} - \frac{\mu\omega^2}{2}R_{ij} \right) \left( C^{ij} - \frac{\mu\omega^2}{2}R^{ij} \right) \right\}, \tag{6}
\]

where \(\kappa^2, \lambda, \) and \(\omega\) are dimensionless constant parameters, while \(\mu\) and \(\Lambda_W\) are constant parameters with mass dimensions. Here \(\mathcal{L}_m\) stands for the Lagrangian of the matter field, \(R\) and \(R_{ij}\) are a three-dimensional spatial Ricci scalar and Ricci tensor, and \(C_{ij}\) is the Cotton tensor defined as

\[
C^{ij} = \epsilon^{ikl}\nabla_k \left( R^j_l - \frac{1}{4}R\delta^j_l \right) = \epsilon^{ikl}\nabla_k R^j_l - \frac{1}{4}\epsilon^{ikl}\partial_k R, \tag{7}\]

where \(\epsilon^{ikl}\) is the totally antisymmetric unit tensor. It is worth mentioning that the IR vacuum of this theory is anti-de-Sitter (AdS) spacetimes. Hence, it is interesting to take a limit of the theory, which may lead to a Minkowskian spacetime in the IR sector. For this purpose, one may modify the theory by introducing \(\mu^4 R\) and then, taking the \(\Lambda_W \rightarrow 0\) limit [2]. This does not alter the UV properties of the theory, but it changes the IR properties. That is, there exists a Minkowski vacuum, instead of an AdS vacuum. We will now consider the limit of this theory such that \(\Lambda_W \rightarrow 0\). The deformed action of the nonrelativistic renormalizable gravitational theory is hence given by

\[
I_{S H} = \int dt d^3x\sqrt{|g|}N \left\{ \frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2\omega^2}C_{ij}C^{ij} - \frac{\mu^2\kappa^2}{8}R_{ij}R^{ij} + \frac{\kappa^2\mu\omega^2}{2}R_{kl}R^{kl} + \frac{\kappa^2\mu^2(1 - 4\lambda)}{32(1 - 3\lambda)}R^2 + \mu^4 R \right\}. \tag{8}
\]

In the IR limit, action [8] can be written as the standard Einstein-Hilbert action in the ADM formalism given in Eq. [2], provided [2]

\[
\lambda = 1, \quad \epsilon^2 = \frac{\kappa^2\mu^4}{2}, \quad G = \frac{\kappa^2}{32\pi c}. \tag{9}
\]
The constant $\omega$ is given by

$$\omega = \frac{8\mu^2(3\lambda - 1)}{\kappa^2}. \tag{10}$$

Besides, for $\omega \to \infty$ (equivalently, $\kappa^2 \to 0$), action (8) reduces to the action of Einstein gravity and hence we expect all its solutions also recover to their respective ones in general relativity.

We now consider a homogeneous and isotropic cosmological solution to the theory with the standard FRW geometry

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\Omega^2)\right]. \tag{11}$$

As usual, $k = -1, 0, +1$ corresponds to an open, flat, or closed universe, respectively. Suppose that the energy-momentum tensor of the total matter and energy in the universe has the form of a perfect fluid $T_{\mu\nu} = pg_{\mu\nu} + (\rho+p)U_{\mu}U_{\nu}$ where $U^\nu$ denotes the four-velocity of the fluid and $\rho$ and $p$ are the total energy density and pressure, respectively.

The Friedmann equation, resulting from a variation of action (8) with respect to the FRW metric, turns out to be

$$H^2 = \frac{\kappa^2}{6(3\lambda - 1)}\left(\rho - \frac{6k\mu^4}{a^2} - \frac{3k^2\mu^2}{8(3\lambda - 1)a^2}\right), \tag{12}$$

where $H = \dot{a}/a$ is the Hubble parameter and we have imposed so called projectability condition and set $N = 1$ and $N^i = 0$. The term proportional to $a^{-4}$ in (12) is the usual “dark radiation”. The energy conservation law $\nabla_\mu T^{\mu\nu} = 0$ leads to the continuity equation in the form

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{13}$$

For $k = 0$, there is no contribution from the higher order derivative terms in the action. However, for $k \neq 0$, the higher derivative terms are significant for small volumes i.e., for small $a$, and become insignificant for large $a$, where they agree with general relativity. The standard Friedmann equation is recovered, in units where $c = 1$, provided we define

$$G_{\text{cosm}} = \frac{\kappa^2}{16\pi(3\lambda - 1)}, \tag{14}$$

and

$$\frac{\kappa^2\mu^4}{3\lambda - 1} = 1, \tag{15}$$

where condition (15) also agrees for $\lambda = 1 = c$ with the second relation in (9). Here $G_{\text{cosm}}$ is the “cosmological” Newton’s constant. It is worth mentioning that in theories such as HL, where the Lorentz invariance is broken, the “gravitational” Newton’s constant $G$ (that is, the one that is present in the gravitational action) does not coincide with the cosmological Newton’s constant $G_{\text{cosm}}$ (that is, the one that is present in Friedmann equations), unless Lorentz invariance is restored [7]. In the IR limit where $\lambda = 1$, the Lorentz invariance is restored, and hence $G_{\text{cosm}} = G$. Using the above identifications, as well as definition (10), the Friedmann equation (12) can be rewritten as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{\text{cosm}}}{3}\rho + \frac{k^2}{2\omega a^4}. \tag{16}$$

One can easily see that in the limit $\omega \to \infty$, the dark radiation term vanishes and the standard Friedmann equation is restored for $\lambda = 1$ ($G_{\text{cosm}} = G$), as expected. On the other hand, one may also further rewrite the Friedmann equation (16) in the form

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{\text{cosm}}}{3}(\rho + \rho_{\text{rad}}), \tag{17}$$

where in general $\rho = \rho_m + \rho_D$ is the total energy density of matter and dark energy and we have defined

$$\rho_{\text{rad}} = \frac{3k^2}{16\pi\omega a^4 G_{\text{cosm}}}. \tag{18}$$

Therefore, by regarding the dark radiation term which incorporates the effect of HL gravity in the cosmological equation, we see that the Friedmann equation of the modified HL cosmology can be written completely in the form of standard cosmology. In such a case, one does not take into account the richness of the HL gravity. Note that in the modified HL cosmology, $\rho_{\text{rad}}$ cannot be interpreted as the effective dark energy fluid, since at the late time ($a \to \infty$) where the dark energy should be dominated, it goes to zero. Besides, for a universe with spatial curvature, it also has no contribution, which is not reasonable. This is in contrast to HL cosmology in the presence of a cosmological constant, where one can interpret the contribution from $\Lambda$ and $\rho_{\text{rad}}$ as an effective dark energy [7, 8].
III. FRIEDMAN EQUATION IN HL COSMOLOGY FROM THE FIRST LAW

In order to extract the cosmological equation governing the evolution of the FRW universe in HL gravity, we need to have the entropy expression of static spherically symmetric black holes in this theory. It was argued that there is a deep connection between the entropy expression and gravitational field equations \[24, 22, 29\]. Having the entropy expression at hand, one can derive the corresponding cosmological equations in a wide range of gravity theories including Einstein, Gauss Bonnet and Lovelock gravity \[24\]. The entropy expression depends on the gravity theory and takes different forms for different gravity theories. In the deformed HL gravity, the entropy associated with the event horizon of the static spherically symmetric black holes has the form \[5\]

\[
S_h = \frac{A}{4G} + \frac{\pi}{\omega} \ln \frac{A}{G},
\]

(19)

where \(A = 4\pi r_h^2\) is the area of the black hole horizon. Throughout this paper, we set \(k_B = c = h = 1\) for simplicity. As one can see the entropy formula has a logarithmic term, which is a characteristic of HL gravity theory \[5\]. The parameter \(\omega\) can be regarded as a characteristic parameter in the deformed HL gravity and the entropy relation will recover to the well-known area law as \(\omega \to \infty\). In general, the entropy expression for HL black holes may have an additional constant term \(S_0\). However, as the HL parameter \(\omega \to \infty\), the HL gravity should be reduced to the Einstein gravity; thus, we fix the constant term in the entropy expression equal to zero.

We further assume the entropy expression (19) is also valid for the apparent horizon of the FRW universe in HL gravity. Replacing the horizon radius \(r_h\) with the apparent horizon radius \(\tilde{r}_A\), we have \(A = 4\pi \tilde{r}_A^2\) for the apparent horizon area in Eq. (19). Now we take the differential of the entropy (19),

\[
dS = \frac{\partial S}{\partial A} dA = \frac{2\pi \tilde{r}_A}{G} \left(1 + \frac{G}{\omega \tilde{r}_A^2}\right) d\tilde{r}_A.
\]

(20)

The line element of the FRW universe can be written as

\[
ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(21)

where \(x^0 = t, x^1 = r, \tilde{r} = a(t)r\), and \(h_{ab} = \text{diag} (-1, a^2/(1 - kr^2))\) represents the two-dimensional metric. Here \(k\) specifies the curvature of the spatial part of the metric. Solving equation \(h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0\), we obtain the dynamical apparent horizon radius of the FRW universe as

\[
\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}.
\]

(22)

The temperature associated with the apparent horizon is defined as \(T_h = \kappa/2\pi\), where \(\kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b \tilde{r})\) is the surface gravity. It is easy to show that the surface gravity at the apparent horizon of FRW universe can be written as \[20\]

\[
\kappa = -\frac{1}{\tilde{r}_A} \left(1 - \frac{\tilde{r}_A}{2H\tilde{r}_A}\right).
\]

(23)

Since for \(\tilde{r}_A < 2H\tilde{r}_A\), we have \(\kappa < 0\), which leads to the negative temperature, one may, in general, define the temperature on the apparent horizon as \(T_h = |\kappa|/2\pi\). In addition, since we associate a temperature with the apparent horizon, one may expect that the apparent horizon has a kind of Hawking radiation just like a black hole event horizon. This issue was previously addressed \[30\] by showing the connection between temperature on the apparent horizon and the Hawking radiation. This study gives more solid physical implication of the temperature associated with the apparent horizon.

The next quantity we need to have is the work density. In our case it can be calculated as \[31\]

\[
W = -\frac{1}{2} T^{\mu\nu} h_{\mu\nu} = \frac{1}{2}(\rho - p).
\]

(24)

Then, we suppose the first law of thermodynamics on the apparent horizon of the universe in HL gravity holds and has the form

\[
dE = T_h dS_h + W dV,
\]

(25)
where $S_h$ is the entropy associated with the apparent horizon in HL cosmology which has the form Eq. (19). The term $WdV$ in the first law comes from the fact that we have a volume change for the total system enveloped by the apparent horizon. As a result, the work term should be considered in the first law. For a pure de Sitter space, $\rho = -p$, and the work term reduces to the standard $-pdV$; thus, we obtain exactly the standard first law of thermodynamics, $dE = TdS - pdV$.

Assume the total energy content of the universe inside a three-sphere of radius $\tilde{r}_A$ is $E = \rho V$, where $V = \frac{4\pi}{3}\tilde{r}_A^3$ is the volume enveloped by three-dimensional sphere with the area of apparent horizon $A = 4\pi\tilde{r}_A^2$. Then we have

$$dE = 4\pi\tilde{r}_A^2\rho d\tilde{r}_A + \frac{4\pi}{3}\tilde{r}_A^3\dot{\rho} dt$$
$$= 4\pi\tilde{r}_A^2\rho d\tilde{r}_A - 4\pi H\tilde{r}_A^3(\rho + p) dt.$$  
(26)

where we have used the continuity equation (13) in the last step. Substituting Eqs. (20), (24) and (26) in the first law (25) and using the definition of the temperature associated with the apparent horizon, we can get the differential form of the Friedmann-like equation

$$\frac{1}{4\pi G} \frac{d\tilde{r}_A}{\tilde{r}_A^3} \left(1 + \frac{G}{\omega\tilde{r}_A^2}\right) = H(\rho + p) dt.$$  
(27)

After using the continuity equation (13), we reach

$$-2\frac{d\tilde{r}_A}{\tilde{r}_A^3} \left(1 + \frac{G}{\omega\tilde{r}_A^2}\right) = \frac{8\pi G}{3} d\rho.$$  
(28)

Integrating (28) yields

$$\frac{1}{\tilde{r}_A^2} + \frac{G}{2\omega\tilde{r}_A^2} = \frac{8\pi G}{3} \rho,$$  
(29)

where an integration constant has been absorbed into the energy density $\rho$. Substituting $\tilde{r}_A$ from Eq. (22) we obtain a Friedmann-like equation of a FRW universe in deformed HL gravity,

$$H^2 + \frac{k}{a^2} + \frac{G}{2\omega} \left(H^2 + \frac{k}{a^2}\right)^2 = \frac{8\pi G}{3} \rho.$$  
(30)

As one can see the obtained Friedmann equation from the first law of thermodynamics differs from the one obtained in (16) from varying the action of the modified HL theory. This shows that at least for the HL cosmology, the general prescription for deriving the field equation from the first law of thermodynamics does not work. This indicates a problematic feature of HL gravity. In the IR limit of the theory where $\omega \to \infty$, one recovers the standard Friedmann equation

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho.$$  
(31)

Also for the late time cosmology where the scale factor $a$ becomes large, Eqs. (16) and (30) coincide. This is also consistent with the fact that the HL theory of gravity reduces to Einstein’s general relativity at large scales.

### IV. AN EFFECTIVE APPROACH TO HL THERMODYNAMICS

In this section, we are going to use the effective approach and try to reproduce the effective Friedmann equation of the deformed HL cosmology given in (16), by applying the first law of thermodynamics on the apparent horizon. In the effective approach all extra information of HL gravity can be absorbed in an effective energy density and so, we consider that the universe contains matter (and possible dark energy), plus this dark radiation. Thus, the total energy density in the effective approach can be written as $\rho_{\text{tot}} = \rho + \rho_{\text{rad}}$, where $\rho = \rho_m + \rho_D$. The total energy density as well as the dark radiation sector also satisfy the continuity equations

$$\dot{\rho}_{\text{tot}} + 3H(\rho_{\text{tot}} + p_{\text{tot}}) = 0,$$  
(32)

$$\dot{\rho}_{\text{rad}} + 3H(\rho_{\text{rad}} + p_{\text{rad}}) = 0.$$  
(33)
Since the effective gravitational sector is now just the standard general relativity, the entropy associated with the apparent horizon obeys the well-known area formula,

$$S_h = \frac{A}{4G_{\text{cosm}}}$$  \hspace{1cm} (34)

where $A$ is the horizon area and $G_{\text{cosm}}$ is the cosmological Newton’s constant given in [14]. Taking the differential form of the above entropy, one finds

$$dS_h = \frac{2\pi \tilde{r}_A}{G_{\text{cosm}}} d\tilde{r}_A.$$  \hspace{1cm} (35)

Now we assume that the total effective energy content of the universe is $E_{\text{eff}} = \rho_{\text{tot}} V$. Differentiating we get

$$dE_{\text{eff}} = 4\pi \tilde{r}_A^2 \rho_{\text{tot}} d\tilde{r}_A + \frac{4\pi}{3} \tilde{r}_A^3 \rho_{\text{tot}} dt$$

$$= 4\pi \tilde{r}_A^2 \rho_{\text{tot}} d\tilde{r}_A - 4\pi H \tilde{r}_A^3 (\rho_{\text{tot}} + p_{\text{tot}}) dt.$$  \hspace{1cm} (36)

where we have used the continuity equation (32) in the last step. Inserting Eqs. (24), (35) and (36) in the first law of the form $dE_{\text{eff}} = T_h dS_h + W dV$, and using the definition of the temperature associated with the apparent horizon, we find the following equation:

$$\frac{1}{4\pi G_{\text{cosm}}} \frac{d\tilde{r}_A}{\tilde{r}_A^3} = H (\rho_{\text{tot}} + p_{\text{tot}}) dt.$$  \hspace{1cm} (37)

After using the continuity equation (32), we reach

$$-\frac{2d\tilde{r}_A}{\tilde{r}_A^3} = \frac{8\pi G_{\text{cosm}}}{3} d\rho_{\text{tot}}.$$  \hspace{1cm} (38)

Integrating and then substituting $\tilde{r}_A$ from Eq. (22), we obtain the effective Friedmann equation of the FRW universe in deformed HL gravity,

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{\text{cosm}}}{3} (\rho + \rho_{\text{rad}}).$$  \hspace{1cm} (39)

If we use the definition for $\rho_{\text{rad}}$ given in Eq. (18), we can further rewrite the above equation as

$$H^2 + \frac{k}{a^2} = \frac{8\pi G_{\text{cosm}}}{3} \rho + \frac{k^2}{2\omega a^4}.$$  \hspace{1cm} (40)

which is exactly the result obtained in Eq. (16). This is an expected result, since in the effective approach, all extra information of HL gravity has been absorbed in the energy sector and the gravity sector is just standard Einstein’s gravity. However, in this case one does not take into account the richness of the effects of gravity. The fact that the results of two approaches coincide in the IR limit ($\omega \to \infty$) can be easily understood, since in this case the correction term in the entropy expression vanishes.

V. SUMMARY AND DISCUSSION

In summary, by applying the first law of thermodynamics to the apparent horizon of a FRW universe one can derive the gravitational equations governing the dynamics of the universe in a wide range of gravitational theories including Einstein [22], Gauss-Bonnet, Lovelock [24], and $f(R)$ gravity [23]. Can this prescription be applied to other gravitational theories? In this paper we have shown that the answer is not always positive. Having the entropy expression associated with the horizon of spherically symmetric black holes in infrared modified HL gravity, and applying it to the apparent horizon, we failed to extract the corresponding Friedmann equations in the modified HL gravity. This is the main result we found in this paper, which originates from the fact that HL gravity is not diffeomorphism invariant, and thus the corresponding field equation cannot be derived from the first law around horizon in the spacetime [27]. In order to justify this result, here are some comments as follows.

(i) It was proved [27] that the field equations of any theory of gravity which is diffeomorphism invariant must be expressible as a thermodynamic identity, $TdS = dE$, around any event in the spacetime. Also, in [27] it was shown that if the theory is not diffeomorphism invariant, then the field equation cannot be reexpressed as the first law.
(ii) The action of HL gravity is invariant under a restricted class of diffeomorphism. The fundamental symmetry of HL theory is the invariance under space-independent time reparametrization and time-dependent spatial diffeomorphism \( t \rightarrow t'(t) \), \( \vec{x} \rightarrow \vec{x}'(t, x) \). The time-dependent spatial diffeomorphism allows an arbitrary change of spatial coordinates on each constant time surface. However, the time reparametrization here is not allowed to depend on spatial coordinates. As a result, unlike general relativity, in HL gravity the foliation of spacetime by constant time hypersurfaces is not just a choice of coordinates but is a physical entity.

(iii) Also HL gravity treats space and time in a different way. For the case of isolated black holes, the metric is time independent and hence the spacetime is invariant under infinitesimal diffeomorphism transformations. As a result, according to the argument of \cite{27}, the field equations of the theory can be expressible as a thermodynamic identity, \( TdS = dE \) around the event horizon as it was shown in \cite{9}.

(iv) In the background of a FRW universe, the metric is time dependent and hence the metric is not invariant under the diffeomorphism transformation \( t \rightarrow t'(t) \), \( \vec{x} \rightarrow \vec{x}'(t, x) \). As a result, the field equation cannot be expressed as the thermodynamics identity \( TdS = dE \). Without this, it implies that the specific gravitational theory seems to be inconsistent and shows an additional problematic feature of HL gravity. Our result in this paper confirms the general discussion given in \cite{27}.

It is important to note that if we still take the area formula of geometric entropy and regard the HL sector in the Friedmann equation as an effective dark radiation, we are able to derive the corresponding Friedmann equation from the first law of thermodynamics. Indeed, we have the freedom to choose two approaches: the first is known as the robust approach and the second is called the effective approach. In the robust approach, we consider that the universe contains only matter (and possible dark energy) and the effect of the gravitational sector of HL gravity is incorporated through the modified entropy expression on the horizon. In the effective approach, all extra information of HL gravity can be absorbed in an effective energy density and so we consider that the universe contains matter (and possible dark energy), plus dark radiation. This approach is essentially same as Einstein’s gravity theory. We have shown that in the robust approach, we failed to reproduce the corresponding Friedmann equations in modified HL gravity. Nevertheless, in the effective approach, we successfully derived the effective Friedmann equation of modified HL gravity. Note that the results of the two approaches coincide in the IR limit where \( \omega \rightarrow \infty \). This can be easily understood, since in this limit HL gravity reduces to Einstein’s gravity.

The results obtained in this paper, combined with those in \cite{7,11,12} indicate that HL gravity is not a consistent gravitational theory from a thermodynamic perspective. Is it possible to modify the HL theory in such a way that it can be consistent with thermodynamics? This is quite an interesting question, which deserves further investigation.

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