Abstract

Five essential spectra of linear relations are defined in terms of semi-Fredholm properties and the index. Basic properties of these sets are established and the perturbation theory for semi-Fredholm relations is then applied to verify a generalisation of Weyl’s theorem for single-valued operators. We conclude with a Möbius transform spectral mapping theorem.

1 Introduction

While the study of the spectrum of bounded linear operators generalises the theory of eigenvalues of matrices, the essential spectra of linear operators characterise the non-invertibility of operators \( \lambda - T \). The latter have been considered in terms of two key related directions of investigation, namely the study of the ascent and descent (as well as the nullity and defect) of \( \lambda - T \) and in terms of semi-Fredholm properties of \( \lambda - T \). Today there are several related definitions of essential spectra and comprehensive reviews may be found in [17], [22], [23], [24], [27] and [34]. In [19], the refinements of the spectrum in terms of ascent and descent were investigated in terms of states of operators, using the terminology of [31] (see also [8] for the states of linear relations). On the other hand, the perturbation theory of semi-Fredholm operators provides a more general context for the early observations of H. Weyl, who showed that limit points of the spectrum (i.e. all points of the spectrum, except isolated eigenvalues of finite multiplicity) of a bounded symmetric transformation on a Hilbert space are invariant under perturbation by compact symmetric operators (cf. Riesz and Sz-Nagy [28]).

In this paper we apply the theory of Fredholm relations to show that theory for essential spectra of linear operators can be extended naturally to linear relations. In particular, we extend preliminary results of Cross [8], where the set \( \sigma_{e1}(\cdot) \) defined below is introduced. The definitions in this paper are based on the classifications given in Edmunds and Evans [9] for single-valued operators.

We commence with a recollection of some preliminary properties required in the sequel.

2 Semi-Fredholm Linear Relations

We first clarify some notation and terminology. Let \( X \) and \( Y \) be normed linear spaces, and let \( B(X, Y) \) and \( L(X, Y) \) denote the classes of bounded and unbounded linear operators, respectively, from \( X \) into \( Y \). A multivalued linear operator \( T : X \to Y \) is a set-valued map such that its graph \( G(T) = \{(x, y) \in X \times Y \mid y \in Tx\} \) is a linear subspace of \( X \times Y \). We use the term linear relation or simply relation, to refer to such a multivalued linear operator denoted \( T \in LR(X, Y) \) (cf. Arens [2] and Lee and Nashed [21]). A relation \( T \in LR(X, Y) \) is said to be closed if its graph \( G(T) \) is a closed subspace. The closure of a linear relation \( T \), denoted \( \overline{T} \) is defined in terms of its corresponding graph: \( G(\overline{T}) := \overline{G(T)} \subset X \times Y \).

The conjugate \( T' \) (cf. [8], III.1.1) of a linear relation \( T \in LR(X, Y) \) is defined by

\[
G(T') := G((-T)^{-1})^\perp \subset Y' \times X'
\]

where \([\langle y, x \rangle, \langle y', x' \rangle] := [x, x'] + [y, y'] = x'x + y'y\). For \( \langle y', x' \rangle \in G(T') \) we have \( y'y = x'x \) whenever \( x \in D(T) \).
Let \( Q_T \), or simply \( Q \), when there is no ambiguity about the relation \( T \), denote the natural quotient map \( Q^Y_{\text{norm}} : Y \to Y/T(0) \) with kernel \( T(0) \). For \( x \in D(T) \) define \( \|Tx\| \) by
\[
||Tx|| := ||QTx||,
\]
and let the quantity \( ||T|| \) be defined
\[
||T|| := ||QT||.
\]
Clearly \( QT \) is a single-valued linear operator. It follows from the definition that \( ||Tx|| = d(y, T(0)) \) for all \( y \in Tx \), and that \( ||T|| = \sup_{x \in B_{D(T)}} ||Tx|| \). The quantity \( ||T|| \) is referred as the norm of \( T \), though we note that it is in fact a pseudonorm since \( ||T|| = 0 \) does not imply \( T = 0 \).

A relation \( T \in LR(X, Y) \) is said to be continuous if for any neighbourhood \( V \subset R(T) \), the inverse image \( T^{-1}(V) := \{ u \in D(T) \mid V \cap Tu \neq \emptyset \} \) is a neighbourhood in \( D(T) \), and \( T \) is said to be open if its inverse \( T^{-1} \) is continuous. It can be shown that \( T \) is continuous if and only if \( ||T|| < \infty \) (cf. [8], II.3.2).

The minimum modulus of \( T \in LR(X, Y) \) is the quantity
\[
\gamma(T) := \sup \{ \lambda \in \mathbb{R} : ||Tx|| \geq \lambda d(x, N(T)) \text{ for } x \in D(T) \},
\]
and \( T \) is open if and only if \( \gamma(T) > 0 \) ([8], II.3.2). The quantity \( \gamma(T) \) is related to the norm quantity by \( \gamma(T) = ||T^{-1}||^{-1} \).

The nullity and deficiency of a linear relation \( T \in LR(X, Y) \) are defined respectively as follows:
\[
\alpha(T) := \dim N(T), \quad \text{and} \quad \beta(T) := \codim R(T) := \dim Y/R(T).
\]

If either \( \alpha(T) < \infty \) or \( \beta(T) < \infty \), then the index of \( T \) is defined as follows:
\[
\kappa(T) := \alpha(T) - \beta(T),
\]
where the value of the difference is computed as \( \kappa(T) := \infty \) if \( \alpha(T) \) is infinite and \( \beta(T) < \infty \) and \( \kappa(T) := -\infty \) if \( \beta(T) \) is infinite and \( \alpha(T) < \infty \).

If \( X \) and \( Y \) are Banach spaces and \( T : X \to Y \) is a closed single-valued operator, then \( T \) is said to be a Fredholm operator, usually denoted \( T \in \Phi(X, Y) \), if \( R(T) \) is closed and both \( \alpha(T) < \infty \) and \( \beta(T) < \infty \); \( T \) is said to be upper semi-Fredholm, denoted \( T \in \Phi_+(X, Y) \), if \( R(T) \) is closed and \( \alpha(T) < \infty \); and \( T \) is said to be lower semi-Fredholm, denoted \( T \in \Phi_-(X, Y) \), if \( R(T) \) is closed and \( \beta(T) < \infty \).

**Definitions 2.1.** The essential resolvent sets, \( \rho_{ei}(T) \) for \( i = 1, 2, 3, 4, 5 \), of \( T \in LR(X) \) are defined as follows:
\[
\rho_{e1}(T) := \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_+ \cup \Phi_- \}
\]
\[
\rho_{e2}(T) := \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_+ \}
\]
\[
\rho_{e3}(T) := \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi_\}
\]
\[
\rho_{e4}(T) := \{ \lambda \in \mathbb{C} \mid (\lambda - T) \in \Phi \text{ and } \kappa(\lambda - T) = 0 \}
\]
\[
\rho_{e5}(T) := \bigcup_n \rho_{e1}^{(n)}(T) \text{ where } \rho_{e1}^{(n)}(T) \text{ is a component of } \rho_{e1}(T)
\]
\[
\text{and } \rho_{e1}^{(n)}(T) \cap \rho(T) \neq \emptyset
\]

The essential spectra, \( \sigma_{ei}(T) \), \( i = 1, 2, 3, 4, 5 \), of \( T \in LR(X) \) are the respective complements of the essential resolvents:
\[
\sigma_{ei}(T) := \mathbb{C} \setminus \rho_{ei}(T), \quad i = 1, 2, 3, 4, 5.
\]
Let Proposition 2.4.\( (iii) \) There exists a non-precompact bounded subset \( W \).

Let Proposition 2.3.

Clearly we have that \( \rho_{e_1}(T) \supset \rho_{e_2}(T) \) for \( i < j < 4 \), and, thus, \( \sigma_{e_1}(T) \subset \sigma_{e_3}(T) \) for \( i < j < 4 \). We will see later that \( \rho_{e_4}(T) \supset \rho_{e_5}(T) \).

For the rest of this section we recall a selection of results from Cross \[8\] which are used in the sequel.

Proposition 2.2. If \( T \in LR(X, Y) \) is continuous with finite dimensional range, then \( T \) is compact.

Proposition 2.3. The following are equivalent:
\( (i) \) \( T \notin \Phi_+ \).
\( (ii) \) There exists a non-precompact bounded subset \( W \) of \( D(T) \).
\( (iii) \) \( T \) has a singular sequence.

Proposition 2.4. Let \( T \in LR(X, Y) \) with \( \gamma(T) > 0 \). Suppose \( S \in LR(X, Y) \) satisfies \( D(S) \supset D(T) \), \( S(0) \subset \overline{T(0)} \) and \( ||S|| < \gamma(T) \). Then \( \alpha(T + S) \leq \alpha(T) \) and \( \beta(T + S) \leq \beta(T) \).

The next result is a general version of the so-called small perturbation theorem for linear relations.

Proposition 2.5. Let \( S, T \in LR(X, Y) \). If \( S(0) \subset \overline{T(0)} \) then \( \Delta(S) < \Gamma(T) \Rightarrow T + S \in \Phi_+ \), where
\[
\Gamma(T) := \inf_{M \in I(D(T))} ||T_M||, \quad \Delta(S) := \sup_{M \in I(D(S))} \Gamma(S_M),
\]
and \( I(X) \) denotes the collection of infinite dimensional subsets of \( X \).

Proposition 2.6. Let \( T \in \Phi(X, Y) \) and suppose \( S \in LR(X, Y) \) satisfies \( D(S) \supset D(T) \), \( S(0) \subset \overline{T(0)} \) and \( ||S|| < \gamma(T) \), then \( \kappa(T + S) = \kappa(T) \).

Proposition 2.7. Let \( S, T \in LR(X, Y) \), \( D(S) \supset D(T) \) and let \( T \in \Phi_- \).
\( (a) \) If \( \dim R(S) < \infty \), then \( T + S \in \Phi_- \).
\( (b) \) If \( S \) is precompact, then \( T + S \in \Phi_- \).
\( (c) \) If \( ||S|| < \gamma(T) \), then \( T + S \in \Phi_- \).

Proposition 2.8. \( (a) \) Suppose \( T \in \Phi_+(X, Y) \) and \( S \in LR(X, Y) \) is strictly singular. If \( ||S|| < \infty \), \( D(S) \supset D(T) \), \( S(0) \subset \overline{T(0)} \), then \( \kappa(T + S) = \kappa(T) \).
\( (b) \) Suppose \( T \in \Phi_-(X, Y) \) and \( S \in LR(X, Y) \) is such that \( S' \) is strictly singular. If \( ||S'|| < \infty \), \( D(S) \supset D(T) \), \( S(0) \subset \overline{T(0)} \), then \( \kappa(T + S) = \kappa(T) \).

3 Properties of the Essential Spectra

We begin this section by showing that the various essential spectra are closed, and then illustrate some characteristic properties. In the single-valued case, the set \( \bigcap_{p \in K_T} \sigma(T + K) \) is referred to as the Weyl essential spectrum. Proposition \[6.4\] shows that \( \sigma_{e_4}(T) \) can be characterised in terms of the Weyl essential spectrum in the multivalued case as well (cf. Edmunds and Evans \[9\]). We conclude this section by giving properties of the quantities \( \alpha(\lambda - T) \), \( \beta(\lambda - T) \) and \( \kappa(\lambda - T) \) for \( \lambda \) in the essential spectra, and deduce in Proposition \[6.9\] the inclusions
\[
\sigma_{e_1}(T) \subset \sigma_{e_2}(T) \subset \sigma_{e_3}(T) \subset \sigma_{e_4}(T) \subset \sigma_{e_5}(T) \subset \sigma(T).
\]

Proposition \[6.5\] is included here for application in Proposition \[6.9\] and is based on the single-valued analogue given in Goldberg \[13\].
Proposition 3.1. For \( i = 1, 2, 3, 4, 5 \), \( \sigma_{e_i}(T) \) is closed.

**PROOF**

Suppose \( \lambda \in \rho_{e_i}(T) \), \( i = 1, 2, 3, 4, 5 \). Since \( R(\lambda - T) \) is closed, it follows from the Open Mapping Theorem ([8], III.4.2), that \( \gamma(\lambda - T) > 0 \). If \( \lambda - T \in F_+ \) and \( |\mu| < \gamma(\lambda - T) \), then by Theorem 2.8 \( \mu + \lambda - T \in F_+ \). Similarly, if \( \lambda - T \in F_- \) and \( |\mu| < \gamma(\lambda - T') \), then by Theorem 2.7 \( \mu + \lambda - T \in F_- \). Thus, \( \rho_{e_1}(T) \), \( \rho_{e_2}(T) \) and \( \rho_{e_3}(T) \) are open. Furthermore, by Theorem 2.6 \( \kappa(\mu + \lambda - T) = \kappa(\lambda - T) \), i.e. \( \rho_{e_4}(T) \) is open. Since each component of \( \rho_{e_5}(T) \) is open, so is \( \rho_{e_5}(T) \).

Proposition 3.2. Let \( T \in LR(X) \). Then

(a) \( \sigma_{e_1}(T') = \sigma_{e_1}(T) \) for \( i = 1, 3, 4, 5 \)

(b) \( \sigma_{e_2}(T') = \sigma'_{e_2}(T) \)

**PROOF**

(a) Suppose \( \lambda \in \rho_{e_1}(T), \ i = 1, 3, 4 \). By [8], III.7.2, \( \alpha(\lambda - T') = \beta(\lambda - T) \) since \( R(\lambda - T) \) is closed. By the Closed Range Theorem ([8], III.4.4), \( R(\lambda - T') \) if and only if \( R(\lambda - T) \) is closed and, since \( \lambda - T \) is open, \( \beta(\lambda - T') = \alpha(\lambda - T) \). Thus, the result holds for \( i = 1, 3 \) and 4. Since \( \rho_{e_1}(T) = \rho_{e_1}(T') \) and \( \rho(T) = \rho(T') \), it follows that \( \rho_{e_1}^{(n)}(T') = \rho_{e_1}^{(n)}(T) \), i.e. the result holds for \( i = 5 \).

(b) follows from the reasons given in (a).

Proposition 3.3. \( \lambda \in \sigma_{e_2}(T) \) if and only if \( \lambda - T \) has a singular sequence.

**PROOF**

Since \( \lambda \in \sigma_{e_2}(T) \) if and only if \( \lambda - T \notin F_+ \), the result follows from Theorem 2.8.

Proposition 3.4.

\[
\sigma_{e_4}(T) = \bigcap_{K \in K_T} \sigma(T + K),
\]

where \( K_T := \{ K \in LR(X) \mid K \ is \ compact \ and \ K(0) \subset \overline{T(0)} \} \).

**PROOF**

We show first that \( \sigma_{e_4}(T) \subset \bigcap_{K \in K_T} \sigma(T + K) \). Suppose \( \lambda \notin \bigcap_{K \in K_T} \sigma(T + K) \). Then there exists \( K \in K_T \) such that \( \lambda \notin \rho(T + K) \). Thus \( \lambda \in \rho_{e_4}(T + K) \). By Propositions 2.6 and 2.7 \( \lambda - T = \lambda - T - K + K \in \Phi \), and by Theorem 2.8

\[
\kappa(\lambda - T) = \kappa(\lambda - T - K + K) = \kappa(\lambda - T - K).
\]

Thus, \( \lambda \in \rho_{e_4}(T) \), i.e. \( \lambda \notin \sigma_{e_4}(T) \).

Conversely, suppose \( \lambda \in \rho_{e_4}(T) \). Then \( R(\lambda - T) \) is closed, and \( \alpha(\lambda - T) = \beta(\lambda - T) = n \), say. Let \( \{x_1, \ldots, x_n\} \) and \( \{y'_1, \ldots, y'_n\} \) be bases for \( N(\lambda - T) \) and \( R(\lambda - T)^\perp = N(\lambda - T') \), respectively. Choose \( x'_j \in X' \) and \( y_j \in X, \ j = 1, \ldots, n \) such that

\[
x'_j x_k = \delta_{jk}, \quad \text{and} \quad y'_j y_k = \delta_{jk},
\]

where \( \delta_{jk} = 0 \) if \( j \neq k \) and \( \delta_{jk} = 1 \) if \( j = k \), and define \( K \in LR(X) \) as follows:

\[
K x := \sum_{k=1}^n (x'_k x) y_k, \quad x \in X
\]

Then \( \dim R(K) \leq \infty \) and

\[
||K|| \leq \left( \sum_{k=1}^n ||x'_k|| \right) ||x||.
\]
By Proposition 2.2 it follows that $K$ is a compact operator. By Propositions 2.5 and 2.7 it follows that
\[ \lambda - (T + K) \in \Phi \quad \text{and by Theorem 2.8} \quad \kappa(\lambda - (T + K)) = \kappa(\lambda - T). \]

Without loss of generality, assume $\lambda = 0$. Now if $x \in N(T)$, then $x = \sum_{k=1}^{n} a_k x_k$ and $x'_k(x) = a_j$, $1 \leq j \leq n$. On the other hand, if $x \in N(K)$, then $x'_k(x) = 0$. Thus $N(T) \cap N(K) = 0$.

Similarly, if $y \in R(K)$, then $y = \sum_{k=1}^{n} a_k y_k$ and $y'_j(y) = a_j$, $1 \leq j \leq n$, and if $y \in R(T)$, then $y'_j(y) = 0$. Thus $R(K) \cap R(T) = 0$.

Next, suppose $x \in N(T + K)$. Then $Tx = -Kx + T(0)$. It follows from the argument above, that $Tx = T(0)$, i.e. $x \in N(T)$. Thus, $x = \sum_{k=1}^{n} a_k x_k$ and $x'_k(x) = a_k$, $1 \leq k \leq n$. Since $Kx = \sum_{k=1}^{n} (x'_k x)y_k = 0$, it follows that $x'_k(x) = 0$, $1 \leq k \leq n$, and hence $x = 0$. Thus, $\alpha(T + K) = 0 = \beta(T + K)$, i.e. $0 \in \rho_{-\lambda}(T + K)$.

**Proposition 3.5.** Suppose $T \in \Phi_+ \cup \Phi_-$ and $S \in LR(X, Y)$ satisfies $D(S) \supset D(T)$, $S(0) = \overline{S(0)} \subset \overline{T(0)}$, and $||S|| < \gamma(T)$. Then $\exists \nu > 0$ such that $\alpha(T + \lambda S)$ and $\beta(T + \lambda S)$ are constant in the annulus $0 < |\lambda| < \nu$.

**PROOF**

We first assume $\alpha(T) < \infty$. Let $\lambda \neq 0$ and let $x \in N(T + \lambda S)$. Then
\[ Tx \supset -\lambda S x, \]
whence
\[ Sx \subset R(T) =: R_1, \quad \text{and} \quad x \in S^{-1} R_1 =: D_1. \]

Thus
\[ -\lambda S x \subset Tx \subset TD_1 =: R_2, \quad \text{and} \quad x \in S^{-1} R_2 =: D_2. \]

Proceeding in this way, we obtain
\[ R_{k+1} := TD_k, \quad \text{where} \quad D_k := S^{-1} R_k. \]

Clearly
\[ R_1 \supset R_2 \supset \ldots \quad \text{and} \quad D_1 \supset D_2 \supset \ldots \]

It follows from the construction of these sequences of subspaces that
\[ N(T + \lambda S) \subset \bigcap_{k=1}^{\infty} D_k. \quad (1) \]

By induction, we have that $R_n$ are closed subspaces of $Y$, and $D_n$ are relatively closed subspaces of $D(S)$: from the hypothesis, $R_1$ is closed, and, hence, since $S$ is continuous, and $S(0)$ is closed, $D_1$ is relatively closed in $D(S)$; if $R_k$ and $D_k$ are closed and relatively closed, respectively, then, since $T|D_k \in \Phi_+ \cup \Phi_-$, it follows that $R_{k+1} = TD_k$ is closed, and, since $S$ is continuous, and $S(0)$ is closed, $D_{k+1} = S^{-1} R_{k+1}$ is relatively closed in $D(S)$.

Define
\[ X_1 := \bigcap_{k=1}^{\infty} D_k, \quad \text{and} \quad Y_1 := \bigcap_{k=1}^{\infty} R_k. \]

Then, by the definitions of $R_k$ and $D_k$, it follows that
\[ TX_1 \subset Y_1 \quad \text{and} \quad SX_1 \subset Y_1. \]

Now define \( T_1 \) and \( S_1 \) by:
\[
T_1 := T|_{D(T) \cap X_1}, \quad \text{and} \quad S_1 := S|_{D(T) \cap X_1}.
\]

Then \( R(T_1) \subset Y_1 \) and \( R(S_1) \subset Y_1 \), and since \( T \) is closed and \( X_1 \) is relatively closed in \( D(S) \) and hence also in \( D(T) \), \( T_1 \) is a closed relation. To see that \( T_1 \) is surjective, let \( y \in Y_1 = \bigcap_{n=1}^{\infty} TD_n \). Then for each \( n \geq 1 \), there exists \( x_n \in D_n \) such that \( y \in Tx_n \). Since \( \alpha(T) < \infty \) and \( D_n \supset D_{n+1} \), there exists \( k_0 \) such that for \( k \geq k_0 \),
\[
N(T) \cap D_{k_0} = N(T) \cap D_k,
\]
and for \( x_k \in D_k \), and \( x_{k_0} \in D_{k_0} \),
\[
x_k - x_{k_0} \in N(T) \cap D_{k_0} = N(T) \cap D_k \subset D_k.
\]
From this it follows that
\[
x_{k_0} \in \bigcap_{k \geq k_0} D_k = X_1, \quad \text{and} \quad y \in Tx_{k_0}.
\]

i.e. \( T_1 \) is surjective. By the Open Mapping Theorem (\textit{[8]}, III.4.2), \( T_1 \) is open.

By Theorem 2.4, Propositions 2.5 and 2.7, and by Theorem 2.6, Proposition 3.6.

Since \( \lambda \) is the smallest non-negative integer attained by \( \alpha(T) \) on \( \rho_{\nu_1}^{(n)}(T) \). Suppose \( \alpha(T') \neq n_1 \) for some \( T' \). Since \( \rho_{\nu_1}^{(n)}(T) \) is connected, there exists an arc \( \Lambda \) in \( \rho_{\nu_1}^{(n)}(T) \) with endpoints \( \lambda_0 \) and \( T' \). Since \( \lambda - T' \in \Phi_+ \cup \Phi_- \), it follows from Proposition 3.5 that for each \( \mu \in \Lambda \) there exists an open ball \( B_\mu \) contained in \( \rho_{\nu_1}^{(n)}(T) \) such that \( \alpha(\lambda) \) is constant on \( B_\mu \setminus \{ \mu \} \). Since \( \Lambda \) is compact, there exists a finite set of points \( \lambda_1, \lambda_2, \ldots, \lambda_n = T' \) such that \( B_{\lambda_0}, B_{\lambda_1}, \ldots, B_{\lambda_n} \) cover \( \Lambda \), and, for \( 0 \leq i \leq n - 1 \),
\[
B_{\lambda_i} \cap B_{\lambda_{i+1}} \neq \emptyset. \quad (6)
\]
It follows from Theorem 2.4 that \( \alpha(\lambda) \leq \alpha(\lambda_0) \) for \( \lambda \) sufficiently close to \( \lambda_0 \). Thus, since \( \alpha(\lambda_0) \) is the minimum value attained by \( \alpha(T) \) on \( \rho_{\nu_1}^{(n)}(T) \), it follows that \( \alpha(\lambda) = \alpha(\lambda_0) \) for \( \lambda \) sufficiently close to \( \lambda_0 \). Since \( \alpha(T) \) is constant for all \( \lambda \neq \lambda_0 \) in \( B_{\lambda_0} \), this constant must be \( \alpha(\lambda_0) \). Similarly \( \alpha(T) \) is constant on \( B_{\lambda_i} \setminus \{ \lambda_i \} \) for \( 1 \leq i \leq n \). Thus, by (6) that \( \alpha(\lambda) = \alpha(\lambda_0) \) for all \( \lambda \in B_{\lambda'} \setminus \{ \lambda' \} \) and \( \alpha(\lambda') > n_1 \).

To see that the result holds for \( \beta(T - \lambda) \), we pass to the conjugate of \( T \) and apply the above, and the equality
\(\alpha(\lambda - T') = \beta(\lambda - T)\).

The proofs for \(\rho_{e_2}^{(n)}(T)\) and \(\rho_{e_3}^{(n)}(T)\) are similar.

**Proposition 3.7.** \(\lambda \in \rho_{e_5}(T)\) if and only if \(\lambda \in \rho_{e_4}(T)\) and a deleted neighbourhood of \(\lambda\) lies in \(\rho(T)\).

**PROOF**
Suppose \(\lambda \in \rho_{e_5}(T)\). Then, by definition, \(\lambda\) lies in a component \(\rho_{e_1}^{(n)}(T)\) of \(\rho_{e_1}(T)\) which intersects \(\rho(T)\). Let \(C\) be such a component. Clearly \(C \cap \rho(T)\) is open.

Since \(\mu \in C \cap \rho(T)\) implies \(\alpha(\mu - T) = \beta(\mu - T) = \kappa(\mu - T) = 0\), it follows from Theorem 2.6 that \(\kappa(\lambda - T) = 0\) for \(\lambda \in C\) when \(\lambda\) is sufficiently close to \(\mu\), and, hence for all \(\lambda \in C\). Applying Proposition 3.6, we see that \(\alpha(\lambda - T) = \beta(\lambda - T) = 0\) for all except some isolated points, say \(\lambda_j\) where \(\alpha(\lambda_j - T) > 0\) and \(\beta(\lambda_j - T) > 0\). Thus if \(\lambda \in \rho_{e_5}(T)\), then either \(\lambda \in \rho(T)\) or \(\lambda\) is one of these isolated points in \(\rho_{e_4}(T)\).

Clearly the converse is true.

**Corollary 3.8.** If \(\rho_{e_4}(T)\) is connected and \(\rho(T) \neq \emptyset\), then \(\rho_{e_5}(T) = \rho_{e_4}(T)\).

**PROOF**
Since \(\rho(T) \subset \rho_{e_4}(T)\), it follows from the hypothesis and Proposition 3.6 that \(\alpha(\lambda - T) = \beta(\lambda - T) = 0\) for all \(\lambda \in \rho_{e_4}(T)\) except perhaps at isolated points, i.e., a deleted neighbourhood of \(\lambda\) lies in \(\rho(T)\). The result follows from Proposition 3.7.

**Proposition 3.9.**

\[\sigma_{e_1}(T) \subset \sigma_{e_2}(T) \subset \sigma_{e_3}(T) \subset \sigma_{e_4}(T) \subset \sigma_{e_5}(T) \subset \sigma(T)\]

**PROOF**
Clearly

\[\rho_{e_1}(T) \supset \rho_{e_2}(T) \supset \rho_{e_3}(T) \supset \rho_{e_4}(T)\]

The remaining inclusions follow from Proposition 3.7.

**Proposition 3.10.** The index is constant in each connected component \(\rho_{e_k}^{(n)}(T)\) of \(\rho_{e_k}(T)\), \(k = 1, 2, 3, 4, 5\).

**PROOF**
Clearly the result holds for \(\rho_{e_4}^{(n)}(T)\), and it follows from Proposition 3.7 that the result hold for \(\rho_{e_5}^{(n)}(T)\).

Let \(\lambda\) and \(\lambda'\) be distinct points in \(\rho_{e_k}^{(n)}(T)\), \(k = 1, 2, 3\). Let \(\Lambda\) be an arc in \(\rho_{e_k}^{(n)}(T)\) with endpoints \(\lambda\) and \(\lambda'\). By Theorem 2.6 there exists \(\varepsilon > 0\) such that \(\kappa(\mu - T) = \kappa(\lambda - T)\) for any \(\mu\) such that \(|\mu - \lambda| < \varepsilon\). Clearly the open balls \(B(\lambda), \lambda \in \Lambda\) cover \(\Lambda\). Since \(\Lambda\) is compact, a finite number of these balls suffices to cover \(\Lambda\). Since each of these balls overlap, it follows that \(\kappa(\lambda - T) = \kappa(\lambda' - T)\).

## 4 Perturbation of the Essential Spectra

We now apply perturbation theorems for semi-Fredholm relations to verify the stability properties of the essential spectra under small and compact perturbation. In particular we arrive at generalisations of Weyl’s theorem for linear operators to a relatively compact case \(4\). First we recall Propositions 4.1 to 4.3 which are proved in Cross 8.

**Proposition 4.1.** Let \(T \in LR(X,Y)\) and let \(G = G_T\) denote the graph operator of \(T\), i.e., \(G_T\) is the identity injection of \(X_T\) into \(X\) (\(G_T x = x\)) and \(X_T\) is the vector space \(D(T)\) endowed with the norm \(||x||_T := ||x|| + ||Tx||\) for \(x \in D(T)\). Then \(TG\) is open if and only if \(T\) is open and

\[\gamma(TG) = \frac{\gamma(T)}{1 + \gamma(T)},\quad \text{provided } T \neq 0,\]

with the cases \(\infty \cdot 1 = 1\) and \(\gamma(TG) := \infty\) if \(T = 0\).
Proposition 4.2. The norms $\|\cdot\|_T$ and $\|\cdot\|_{\lambda-T}$ are equivalent.

Proposition 4.3. Let $T \in LR(X, Y)$ and suppose $S \in LR(X)$ satisfies $D(S) \supset D(T)$ and $S(0) \subset T(0)$, and is $T$-bounded with $a, b > 0$, $b < 1$ such that for $x \in D(T)$, $\|Sx\| \leq a\|x\| + b\|Tx\|$.
(a) The norms $\|\cdot\|_T$ and $\|\cdot\|_{T+S}$ are equivalent.
(b) If $X$ and $Y$ are complete and $T$ is closed, then $T + S$ is closed.

Theorem 4.4. Let $T \in LR(X)$ be closed and suppose $S \in LR(X)$ is $T - compact$ with $T -$ bound $b < 1$, and $D(S) \supset D(T)$ and $S(0) \subset T(0)$. Then for $i = 1, 2, 3, 4$

$$\sigma_{ei}(T + S) = \sigma_{ei}(T).$$

If additionally $\rho_{e4}$ is connected and neither $\rho(T)$ nor $\rho(T + S)$ are empty, then

$$\sigma_{e5}(T + S) = \sigma_{e5}(T).$$

PROOF
By Corollary 4.2 the norms $\|\cdot\|_T$ and $\|\cdot\|_{\lambda-T}$ are equivalent and hence, $S$ is $(\lambda-T)$-compact. Let $G_{\lambda-T}$ denote the graph operator from space $X_{\lambda-T} := (X, ||x||_{\lambda-T})$ into $X$. Suppose $\lambda-T \in \Phi_{\pm}$. Clearly $R(TG_{\lambda-T}) = R(T)$, and as subsets of the set $X$, we have $N(TG_{\lambda-T}) = N(T)$. By Proposition 4.1 $(\lambda-T)G_{\lambda-T}$ is open, and hence $(\lambda-T)G_{\lambda-T} \in \Phi_{\pm}$.

Thus, by Propositions 2.5 and 2.7 it follows that $(\lambda-T) - S = \lambda-(T + S) \in \Phi_{\pm}$ and by Theorem 2.8 $\kappa(\lambda-(T + S)) = \kappa(\lambda-T)$.

On the other hand, suppose $\lambda-(T+S) \in \Phi_{\pm}$. By the equivalence of the norms $\|\cdot\|_T$ and $\|\cdot\|_{\lambda-(T+S)}$ (Proposition 4.3 and Corollary 4.2), it follows that $S$ is $(\lambda-(T+S))$-compact. Arguing as before, it follows that $\lambda-T \in \Phi_{\pm}$ and $\kappa(\lambda-T) = \kappa(\lambda-(T+S))$.

Thus, $\rho_{ei}(T + S) = \rho_{ei}(T)$ for $i = 1, 2, 3, 4$. It follows from the additional hypotheses, Corollary 3.8 and what has just been proved that

$$\rho_{e5}(T) = \rho_{e4}(T) = \rho_{e4}(T + S) = \rho_{e5}(T + S).$$

5 Functions of the Essential Spectra

The Möbius transform, $\eta(\lambda) = (\mu - \lambda)^{-1}$, is a topological homeomorphism from $\mathbb{C} \cup \{\infty\}$, endowed with the usual topology, onto itself. Theorem 5.2 below is analogous to the Theorem on the Möbius transform of the spectrum in Cross [3]. For its proof, we first recall the following index theorem:

Proposition 5.1. Let $T \in LR(X, Y)$ and $S \in LR(Y, Z)$. Suppose $D(S) = Y$ and that $T$ and $S$ have finite indices. Then

$$\kappa(ST) = \kappa(T) + \kappa(S) - \dim(T(0) \cap N(S)).$$

Theorem 5.2. Let $T \in LR(X)$ be closed. Suppose $\mu \in \rho(T)$. Then for $i = 1, 2, 3, 4, 5$

$$\lambda \in \sigma_{ei}(T) \iff (\mu - \lambda)^{-1} \in \sigma_{ei}(T_{\mu}).$$

PROOF
Let $S := (\mu - \lambda)((\mu - \lambda)^{-1} - T_{\mu})$. It can be shown that $\lambda - T = S(\mu - T)$ ([5], IV.4.2). Since $T$ is closed, so is $\lambda - T$, and since $R(\mu - T) = X$ it follows that

$$R(\lambda - T) = R(S).$$

(7)

Since $T_{\mu}$ is single valued,

$$\alpha(\lambda - T) = \dim T_{\mu}S^{-1}(0) \leq \dim S^{-1}(0) = \alpha(S).$$
Thus, \( S \in \Phi_\pm \) implies that \( \lambda - T \in \Phi_\pm \), i.e. \( (\mu - \lambda)^{-1} \in \rho_{ei}(T_\mu) \) implies that \( \lambda \in \rho_{ei}(T) \) for \( i = 1, 2, 3 \).

Applying Proposition elementary algebra for linear relations (\cite{S}, I.4.2) we have
\[
(\mu - T)S = (\mu - T)(\mu - \lambda)((\mu - \lambda)^{-1} - T_\mu)
\]
\[
= (\mu - T) - (\mu - \lambda)(\mu - T)(\mu - T)^{-1}
\]
\[
= (\mu - T) - (\mu - \lambda)(I + (\mu - T)(\mu - T)^{-1} - (\mu - T)(\mu - T)^{-1})
\]
\[
= \lambda - T + (\mu - \lambda)(TT^{-1} - TT^{-1})
\]
\[
= \lambda - T.
\]

Thus, since \( \kappa(\mu - T) \) and \( \kappa(S) \) are finite and \( D(S) = X \), it follows from Proposition 5.1 that
\[
\kappa(\lambda - T) = \kappa(S) + \kappa(\mu - T) - \dim(S(0) \cap N(\mu - T)).
\]

(8)

In particular, if \( (\mu - \lambda)^{-1} \in \rho_{ei}(T_\mu) \) then \( \kappa(S) = 0 \), and, since \( \mu \in \rho(T) \), we have \( \kappa(\mu - T) = 0 = \alpha(\mu - T) \). Thus \( \kappa(\lambda - T) = 0 \), i.e. \( \lambda \in \rho_{ei}(T) \). Applying Proposition 3.7 it follows that the forward implication also holds for \( i = 5 \).

For the reverse implication, it follows from (\cite{S}) that if \( \lambda - T \in \Phi_- \), then \( S \in \Phi_- \), i.e. \( (\mu - \lambda)^{-1} - T_\mu \in \Phi_- \). Now suppose \( \lambda - T \in \Phi_+ \). Then there exists a finite codimensional subset \( M \) of \( D(\lambda - T) \) such that \( (\lambda - T)|_M \) is injective. As in \cite{S} IV.4.2, it follows that \( S|_M \) is injective, and hence \( \alpha(S) < \infty \).

Thus, \( S \in \Phi_+ \), and consequently \( (\mu - \lambda)^{-1} - T_\mu \in \Phi_+ \). We have
\[
\lambda \in \rho_{ei}(T) \Rightarrow (\mu - \lambda)^{-1} \in \rho_{ei}(T_\mu) \text{ for } i = 1, 2, 3.
\]

Now if \( \lambda \in \rho_{ei}(T) \) then \( \kappa(\lambda - T) = 0 \), and since \( \alpha(\mu - T) = \kappa(\mu - T) = 0 \) it follows from (\cite{S}) that \( 0 = \kappa(S) = \kappa((\mu - \lambda)^{-1} - T_\mu) \). Thus \( (\mu - \lambda)^{-1} \in \rho_{ei}(T_\mu) \). Another application of Proposition 3.7 shows that the converse is true for \( i = 5 \).

**Theorem 5.3.** Let \( X \) be complete and let \( T, S \in LR(X) \) be closed.

Suppose \( \mu \in \rho(T) \cap \rho(S) \) and \( T_\mu - S_\mu \) is compact. Then for \( i = 1, 2, 3, 4 \)
\[
\sigma_{ei}(S) = \sigma_{ei}(T).
\]

If additionally \( \rho_{ei}(S) \) is connected then equality holds for \( i = 5 \) as well.

**PROOF**

For \( i = 1, 2, 3, 4 \) it follows from Theorem 5.2 that
\[
\lambda \in \sigma_{ei}(T) \iff (\lambda - \mu)^{-1} \in \sigma_{ei}(T_\mu),
\]
and
\[
\lambda \in \sigma_{ei}(S) \iff (\lambda - \mu)^{-1} \in \sigma_{ei}(S_\mu),
\]
and by Theorem 4.4
\[
\sigma_{ei}(T_\mu - (T_\mu - S_\mu)) = \sigma_{ei}(T_\mu).
\]

Applying Proposition 5.7 shows that the result it true for \( i = 5 \) under the additional hypotheses.

6 Further Notes and Remarks

We note that Proposition 3.1 appeared for case \( \sigma_{e1} \) in \cite{S} (VII.2.3) and that a similar but different generalisation of Weyl’s theorem is proved in a lengthier argument through Theorems VII.2.15 and VII.2.3 of \cite{S}.

Other subsets of the spectrum of a linear operator have also been investigated for stability under perturbation, for example the *Brouwer essential spectrum* defined by:
\[
\sigma_{b}(T) := \bigcup \{ \sigma(T + K) \mid TK = KT \text{ and } K \text{ is compact} \}.\]
It is possible that such investigations may be extended to multivalued linear operators by the methods employed in this work. More recently Sandovici, De Snoo and Winkler [29] have developed results for the ascent, descent, nullity and defect of linear relations.

For simplicity, we have assumed that the spaces on which the relations are defined are complete, and that the operators are closed. Fredholm properties are, however, stable under more general conditions (cf. Cross [8] for the case $\sigma_{e_1}$). Thus, proofs for $\sigma_{e_i}$, $i = 1, 2, 3$ do not necessarily require assumptions of completeness. The index may not be stable under perturbation, though, and hence, generalisations which weaken assumptions of completeness for $\sigma_{e_i}$, $i = 4, 5$ would have to proceed with considerations similar to those applied for the class of Atkinson relations introduced in Wilcox [33] (see also L. Labuschagne [18] and V. Müller-Horrig [25]).

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