WOLFF’S PROBLEM OF IDEALS IN THE MULTIPLIER ALGEBRA ON DIRICHLET SPACE

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Abstract. We establish an analogue of Wolff’s theorem on ideals in $H^\infty(\mathbb{D})$ for the multiplier algebra of Dirichlet space.

In 1962 Carleson [C] proved his famous “Corona theorem” characterizing when a finitely generated ideal in $H^\infty(\mathbb{D})$ is actually all of $H^\infty(\mathbb{D})$. Independently, Rosenblum [R], Tolokonnikov [To], and Uchiyama gave an infinite version of Carleson’s work on $H^\infty(\mathbb{D})$. In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(\mathbb{D})$, Wolff [G] proved the following version:

Theorem A (Wolff). If
\[
\{f_j\}_{j=1}^n \subset H^\infty(\mathbb{D}), \quad H \in H^\infty(\mathbb{D}) \quad \text{and}
\]
\[
|H(z)| \leq \left( \sum_{j=1}^n |f_j(z)|^2 \right)^{\frac{1}{2}} \quad \text{for all } z \in \mathbb{D},
\]
then
\[
H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n),
\]
the ideal generated by $\{f_j\}_{j=1}^n$ in $H^\infty(\mathbb{D})$.

It is known that (1) is not, in general, sufficient for $H$ itself to be in $\mathcal{I}(\{f_j\}_{j=1}^n)$, see Rao [G]; or even for $H^2$ to be in $\mathcal{I}(\{f_j\}_{j=1}^n)$, see Treil [T].

Recall that if we consider the radical of the ideal, $\mathcal{I}(\{f_j\}_{j=1}^n)$, i.e.
\[
Rad(\{f_j\}_{j=1}^n) \overset{def}{=} \{ h \in H^\infty(\mathbb{D}) : \exists n \in \mathbb{N} \text{ with } h^n \in \mathcal{I}(\{f_j\}_{j=1}^n) \},
\]
then (1) gives a characterization of radical ideal membership.

2010 Mathematics Subject Classification. Primary: 30H50, 31C25, 46J20.
Key words and phrases. corona theorem, Wolff’s theorem, Dirichlet space.
That is:

**Theorem B (Wolff).** Let \( \{ H, f_j : j = 1, \ldots, n \} \subset H^\infty(\Bbb D) \). Then \( H \in \text{Rad} \left( \{ f_j \}_{j=1}^n \right) \) if and only if there exists \( C_0 < \infty \) and \( m \in \Bbb N \) such that

\[
|H^m(z)| \leq C_0 \sum_{j=1}^n |f_j(z)|^2 \quad \text{for all } z \in \Bbb D.
\]

For the algebra of multipliers on Dirichlet space, the analogue of the corona theorem was established in Tolokonnikov [To] and, for infinitely many generators, this was done in Trent [Tr2]. The purpose of this paper is to establish an analogue of Wolff’s results, Theorems A and B, for the algebra of multipliers on Dirichlet space.

We use \( \mathcal{D} \) to denote the Dirichlet space on the unit disk, \( \Bbb D \). That is,

\[
\mathcal{D} = \{ f : \Bbb D \to \Bbb C \mid f \text{ is analytic on } \Bbb D \text{ and for } f(z) = \sum_{n=0}^{\infty} a_n z^n, \]

\[
\|f\|^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2 < \infty \}.
\]

We will use other equivalent norms for smooth functions in \( \mathcal{D} \) as follows,

\[
\|f\|^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\Bbb D} |f'(z)|^2 d\sigma d\omega \quad \text{and}
\]

\[
\|f\|^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} d\sigma d\theta.
\]

Also, we will consider \( \overset{\infty}{\oplus} \mathcal{D} \) as an \( l^2 \)-valued Dirichlet space. The norms in this case are exactly as above but we will replace the absolute value by \( l^2 \)-norms. Moreover, we use \( \mathcal{H}\mathcal{D} \) to denote the harmonic Dirichlet space (restricted to the boundary of \( \Bbb D \)). The functions in \( \mathcal{D} \) have only vanishing negative Fourier coefficients, whereas the functions in \( \mathcal{H}\mathcal{D} \) may have negative fourier coefficients which do not vanish. Again, if \( f \) is smooth on \( \partial D \), the boundary of the unit disk \( D \), then

\[
\|f\|^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^2} d\sigma d\theta.
\]

We use \( \mathcal{M}(\mathcal{D}) \) to denote the multiplier algebra of Dirichlet space, defined as: \( \mathcal{M}(\mathcal{D}) = \{ \phi \in \mathcal{D} : \phi f \in \mathcal{D} \text{ for all } f \in \mathcal{D} \} \), and we will denote the multiplier algebra of harmonic Dirichlet space by \( \mathcal{M}(\mathcal{H}\mathcal{D}) \), defined similarly (but only on \( \partial D \)).
Given \( \{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}) \), we consider \( F(z) = (f_1(z), f_2(z), \ldots) \) for \( z \in \mathcal{D} \). We define the row operator \( M^R_F : \bigoplus_1^\infty \mathcal{D} \to \mathcal{D} \) by
\[
M^R_F \left( \{h_j\}_{j=1}^{\infty} \right) = \sum_{j=1}^{\infty} f_j h_j \text{ for } \{h_j\}_{j=1}^{\infty} \in \bigoplus_1^\infty \mathcal{D}.
\]
Similarly, define the column operator \( M^C_F : \mathcal{D} \to \bigoplus_1^\infty \mathcal{D} \) by
\[
M^C_F (h) = \{f_j h\}_{j=1}^{\infty} \text{ for } h \in \mathcal{D}.
\]

We notice that \( \mathcal{D} \) is a reproducing kernel (r.k.) Hilbert space with r.k.
\[
k_w(z) = \frac{1}{z\overline{w}} \log \left( \frac{1}{1 - z\overline{w}} \right) \text{ for } z, w \in \mathbb{D}
\]
and it is well known (see [AM]) that
\[
\frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} c_n (z\overline{w})^n, \ c_n > 0, \text{ for all } n.
\]
Hence, Dirichlet space has a reproducing kernel with “one positive square” or a “complete Nevanlinna-Pick” kernel. This property will be used to complete the first part of our proof.

An important relationship between the multipliers and reproducing kernels is that for \( \phi \in \mathcal{M}(\mathcal{D}) \) and \( z \in \mathbb{D} \),
\[
M^*_\phi k_z = \overline{\phi(z)} k_z.
\]

This automatically implies that \( \|\phi\|_\infty \leq \|M\phi\| \), so \( \mathcal{M}(\mathcal{D}) \subseteq H^\infty(\mathbb{D}) \).

Similarly, if \( \phi_{ij} \in \mathcal{M}(\mathcal{D}) \) and \( M_{\{\phi_{ij}\}} \in B(\bigoplus_1^\infty \mathcal{D}) \), then for \( x \in l^2 \) and \( z \in \mathbb{D} \), we have
\[
M^*_{\phi_{ij}} (x k_z) = [\phi_{ij}(z)]^* (x k_z).
\]
Again, it follows that
\[
sup_{z \in \mathcal{D}} \| [\phi_{ij}(z)] \|_{B(l^2)} \leq \|M_{\phi_{ij}}\|_{B(\bigoplus_1^\infty \mathcal{D})}
\]
and so
\[
\mathcal{M}(\bigoplus_1^\infty \mathcal{D}) \subseteq H^\infty_B(l^2)(\mathbb{D}).
\]

It is clear that \( \mathcal{M}(H^2(\mathbb{D})) = H^\infty(\mathbb{D}) \) but \( \mathcal{M}(\mathcal{D}) \nsubseteq H^\infty(\mathbb{D}) \) (e.g., \( \sum_{n=1}^{\infty} \frac{n^3}{n^2} \) is in \( H^\infty(\mathbb{D}) \) but is not in \( \mathcal{D} \) and so not in \( \mathcal{M}(\mathcal{D}) \)). Hence, \( \mathcal{M}(\mathcal{D}) \nsubseteq H^\infty(\mathbb{D}) \cap \mathcal{D} \).
Also, it is worthwhile to note that the pointwise hypothesis that
\( F(z)^* F(z) \leq 1 \) for \( z \in \mathbb{D} \), implies that the analytic Toeplitz operators
\( T^R_F \) and \( T^C_F \) defined on \( \oplus_1^\infty H^2(\mathbb{D}) \) and \( H^2(\mathbb{D}) \) in analogy to that of \( M^R_F \) and \( M^C_F \) are bounded and
\[
\|T^R_F\| = \|T^C_F\| = \sup_{z \in \mathbb{D}} \left( \sum_{j=1}^\infty |f_j(z)|^2 \right)^{\frac{1}{2}} \leq 1.
\]

But, since \( M^C(\mathbb{D}) \subsetneq H^\infty(\mathbb{D}) \), the pointwise upperbound hypothesis will not be sufficient to conclude that \( M^R_F \) and \( M^C_F \) are bounded on Dirichlet space. However, \( \|M^R_F\| \leq \sqrt{18} \|M^C_F\| \) from [Tr2]. Thus, we will replace the natural normalization that \( F(z)^* F(z) \leq 1 \) for all \( z \in \mathbb{D} \), by the stronger condition that \( \|M^C_F\| \leq 1 \).

Then we have the following theorem:

**Theorem 1.** Let \( H, \{f_j\}_{j=1}^\infty \subset \mathcal{M}(\mathbb{D}) \). Assume that

(a) \( \|M^C_G\| < \infty \)

and (b) \( |H(z)| \leq \sqrt{\sum_{j=1}^\infty |f_j(z)|^2} \) for all \( z \in \mathbb{D} \).

Then there exist \( \{g_j\}_{j=1}^\infty \subset \mathcal{M}(\mathbb{D}) \) with

\( \|M^G_G\| < \infty \)

and \( F G^T = H^3 \).

Of course, it should be noted that for only a finite number of multipliers, \( \{f_j\} \), condition (a) of Theorem 1 can always be assumed, so we have the exact analogue of Wolff’s theorem in the finite case.

First, let’s outline the method of our proof. Assume that \( F \in \mathcal{M}_{l^2}(\mathbb{D}) \) and \( H \in \mathcal{M}(\mathbb{D}) \) satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant \( K < \infty \), so that
\[
M_{H^3} M^*_F H^3 \leq K^2 M^R_F M^*_R F.
\]  \( (2) \)

Given (2), a commutant lifting theorem argument as it appears in, for example, Trent [Tr2], completes the proof by providing a \( G \in \mathcal{M}_{l^2}(\mathbb{D}) \), so that \( \|M^G_G\| \leq K \) and \( F G^T = H^3 \).
But (2) is equivalent to the following: there exists a constant $K < \infty$ so that, for any $h \in \mathcal{D}$, there exists $u_h \in \oplus_1^\infty \mathcal{D}$ such that

$$
\begin{align*}
(i) \quad & M^R_F(u_h) = H^3h \quad \text{and} \\
(ii) \quad & \|u_h\|_\mathcal{D} \leq K \|h\|_\mathcal{D}.
\end{align*}
$$

Hence, our goal is to show that (3) follows from (a) and (b). For this we need a series of lemmas.

**Lemma 1.** Let $\{c_j\}_{j=1}^\infty \in l^2$ and $C = (c_1, c_2, \ldots) \in B({l^2, \mathbb{C}})$. Then there exists $Q$ such that the entries of $Q$ are either 0 or $\pm c_j$ for some $j$ and $CC^*I - C^*C = QQ^*$. Also, range of $Q = \text{kernel of } C$.

We will apply this lemma in our case with $C = F(z)$ for each $z \in \mathbb{D}$, when $F(z) \neq 0$. A proof of a more general version can be found in Trent [Tr2].

Given condition (b) of Theorem 1 for all $z \in \mathbb{D}$, $F \in \mathcal{M}_{l^2}(\mathcal{D})$ and $H \in \mathcal{M}(\mathcal{D})$ with $H$ being not identically zero, we lose no generality assuming that $H(0) \neq 0$. If $H(0) = 0$, but $H(a) \neq 0$, let $\beta(z) = \frac{a-z}{1-\overline{a}z}$ for $z \in \mathbb{D}$. Then since (b) holds for all $z \in \mathbb{D}$, it holds for $\beta(z)$. So we may replace $H$ and $F$ by $H\beta F$ and $\beta F$, respectively. If we prove our theorem for $H\beta F$ and $\beta F$, then there exists $G \in \mathcal{M}_{l^2}(\mathcal{D})$ so that $(\beta F)G = H\beta F$ and hence $F(G(\beta)^{-1}) = H$ and $G(\beta)^{-1} \in \mathcal{M}_{l^2}(\mathcal{D})$, so we were done. Thus, we may assume that $H(0) \neq 0$ in (b), so $\|F(0)\|_2 \neq 0$. This normalization will let us apply some relevant lemmas from [Tr1].

It suffices to establish (i) and (ii) for any dense set of functions in $\mathcal{D}$, so we will use polynomials. First, we will assume $F$ and $H$ are analytic on $\mathbb{D}_{1+\epsilon}(0)$. In this case, we write the most general solution of the pointwise problem on $\overline{\mathbb{D}}$ and find an analytic solution with uniform bounds. Then we remove the smoothness hypotheses on $F$ and $H$.

For a polynomial, $h$, we take

$$
\begin{equation}
\begin{aligned}
\mathcal{U}_h(z) &= F(z)^* (F(z)F(z)^*)^{-1} H^3 h - Q(z)k(z),
\end{aligned}
\end{equation}
$$

where $k(z) \in l^2$ for $z \in \overline{\mathbb{D}}$.

We have to find $k(z)$ so that $\mathcal{U}_h \in \oplus_1^\infty \mathcal{D}$. Thus we want $\bar{\partial}_z \mathcal{U}_h = 0$ in $\mathbb{D}$.

Therefore, we will try

$$
\begin{equation}
\begin{aligned}
\mathcal{U}_h &= \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q \cdot FF^* H^3 h}{(FF^*)^2} \right).
\end{aligned}
\end{equation}
$$
where \( \hat{k} \) is the Cauchy transform of \( k \) on \( \overline{D} \). Note that for \( k \) smooth on \( \overline{D} \) and \( z \in D \),

\[
\hat{k}(z) = -\frac{1}{\pi} \int_D \frac{k(w)}{w - z} \, dA(w) \quad \text{and} \quad \overline{\partial} \hat{k}(z) = k(z) \quad \text{for} \quad z \in \mathbb{D}.
\]

See [A] for background on the Cauchy transform.

Then it’s clear that \( M_R^R(u_h) = H^3 h \) and \( u_h \) is analytic. Hence, we will be done in the smooth case if we are able to find \( K < \infty \), independent of the polynomial, \( h \), and \( \epsilon > 0 \), such that

\[
\|u_h\|_D \leq K \|h\|_D \quad \text{(4)}
\]

Lemma 2. Let \( w \) be a harmonic function on \( \overline{D} \), then

\[
\int_D \|Q'w\|_I^2 \, dA \leq 8 \|w\|_{\text{HD}}^2.
\]

Proof. Let \( w \) be a vector-valued harmonic function on \( \overline{D} \). Write \( w = \underline{x} + \bar{y} \), where \( \underline{x} \) and \( \bar{y} \) are respectively the analytic and co-analytic parts of \( w \).

We have

\[
\int_D \|Q'w\|_I^2 \, dA = \int_D \|Q'\underline{x} + Q'\bar{y}\|_I^2 \, dA \\
\leq 2 \int_D \|Q'\underline{x}\|_I^2 \, dA + 2 \int_D \|Q'\bar{y}\|_I^2 \, dA.
\]

Now

\[
\int_D \|Q'\underline{x}\|_I^2 \, dA = \int_D < Q'^* Q' \underline{x}, \underline{x} >_I^2 \, dA \\
\leq \int_D < F'^* F' \underline{x}, \underline{x} >_I^2 \, dA \\
\leq \int_D \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\bar{f}_j x_k'|^2 \, dA \\
\leq 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_D |(f_j x_k)'|^2 \, dA + 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_D |f_j x_k'|^2 \, dA \\
\leq 2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \|M_f x_k\|_D^2 + 2 \sum_{k=1}^{\infty} \|x_k\|_D^2 \, dA \\
\leq 2 \sum_{k=1}^{\infty} \|M_{f_j}^\circ x_k\|^2_D + 2 \sum_{k=1}^{\infty} \|x_k\|_D^2 \, dA \\
= 4 \|\underline{x}\|_I^2_D.
\]
Similarly, we can show that \( \int_D \|Q'y\|_D^2 \, dA \leq 4 \|y\|_D^2 \).

Thus,

\[
\begin{align*}
\int_D \|Q'w\|_D^2 \, dA & \leq 8 \|x\|_D^2 + 8 \|y\|_D^2 \\
& = 8 \|x + y\|_{\mathcal{H}^D}^2 \\
& = 8 \|w\|_{\mathcal{H}^D}^2.
\end{align*}
\]

\(\square\)

**Lemma 3.** Let the operator \(T\) be defined on \(L^2(\mathbb{D}, dA)\) by

\[
(Tf)(\lambda) = \int_D \frac{f(z)}{(z - \lambda)(1 - z \bar{\lambda})} \, dA(z),
\]

for \(\lambda \in \mathbb{D}\) and \(f \in L^2(\mathbb{D}, dA)\). Then

\[
\|Tf\|_A^2 \leq 100 \pi^2 \|f\|_A^2.
\]

**Proof.** To show that the singular integral operator, \(T\), is bounded on \(L^2(\mathbb{D}, dA)\), we apply Zygmund’s method of rotations [Z] and apply Schur’s lemma an infinite number of times.

Let \(f(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}z^j \bar{z}^k\), where \(a_{ij} = 0\) except for a finite number of terms. For \(z = re^{i\theta}\), we relabel, so that

\[
f(re^{i\theta}) = \sum_{l=-\infty}^{\infty} f_l(r) e^{il\theta}, \quad \text{where} \quad f_l(r) = \sum_{k=0}^{\infty} a_{l+k,k} r^{l+2k}.
\]

Then

\[
\|f\|_A^2 = \sum_{l=-\infty}^{\infty} \|f_l(r)\|_{L^2[0,1]},
\]

where the measure on \(L^2[0,1]\) is “\(rdr\)”. 

Now

\[(Tf)(\lambda) = \int_D \frac{f(z)}{(z - \lambda)(1 - z\lambda)} dA(z)\]

\[= \sum_{n=0}^{\infty} \lambda^n \int_D \left[ \frac{1}{z - \lambda} \right] z^n f(z) dA(z)\]

\[= \sum_{n=0}^{\infty} \lambda^n \left[ \int_{|z|<|\lambda|} \frac{1}{z - \lambda} + \int_{|z|>|\lambda|} \frac{1}{z - \lambda} \right] z^n f(z) dA(z)\]

\[= \sum_{n=0}^{\infty} \lambda^n \left[ \frac{1}{-\lambda} \int_{|z|<|\lambda|} \sum_{p=0}^{\infty} \frac{z^p}{\lambda^p} + \int_{|z|>|\lambda|} \frac{1}{z} \sum_{p=0}^{\infty} \frac{\lambda^p}{z^p} \right] z^n f(z) dA(z)\]

\[= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1) \lambda^n \frac{1}{\lambda} \int_{|z|<|\lambda|} \frac{z^{n+p}}{\lambda^p} \left( \sum_{l=-\infty}^{\infty} f_i(r) e^{il\theta} \right) dA(z)\]

\[+ \sum_{n=0}^{\infty} \lambda^n \sum_{p=0}^{\infty} \int_{|z|>|\lambda|} \frac{\lambda^p}{zp+1} z^n \left( \sum_{l=-\infty}^{\infty} f_i(r) e^{il\theta} \right) dA(z).\]

Therefore,

\[(Tf)(se^{it}) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1) s^n e^{-int} \frac{e^{-it}}{s} \int_{-\pi}^{\pi} \int_0^s \frac{r^{n+p} e^{i(n+p+l)\theta}}{s^p e^{ipt}} f_i(r) r dr d\theta\]

\[+ \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} s^n e^{-int} \sum_{p=0}^{\infty} \int_{-\pi}^{\pi} \int_s^1 \frac{s^p e^{ipt}}{r^{p+1} e^{i(p+1)\theta}} r^n e^{in\theta} e^{it\theta} f_i(r) r dr d\theta. \quad (*)\]

Taking \(l = 0\) in \((*)\), we get that

\[(Tf_0)(se^{it}) = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} (-1) s^n e^{-int} \frac{e^{-it}}{s} \int_{-\pi}^{\pi} \int_0^s \frac{r^{n+p} e^{i(n+p)\theta}}{s^p e^{ipt}} f_0(r) r dr d\theta\]

\[+ \sum_{n=0}^{\infty} s^n e^{-int} \sum_{p=0}^{\infty} \int_{-\pi}^{\pi} \int_s^1 \frac{s^p e^{ipt}}{r^{p+1} e^{i(p+1)\theta}} r^n e^{in\theta} f_i(r) r dr d\theta.\]

Simplifying the above,

\[(Tf_0)(se^{it}) = -2\pi \int_0^1 \chi_{(0,s)}(r) \frac{f_0(r) e^{-it}}{s} r dr \]

\[+ 2\pi s e^{-it} \sum_{n=0}^{\infty} s^{2n} \int_0^1 \chi_{(s,1)}(r) f_0(r) r dr.\]
So
\[(T f_0) (se^{it}) = 2 \pi e^{-it} (T_0 f_0) (s),\]
where we define \( T_0 \) on \( L^2([0, 1], rdr) \) by
\[(T_0 f_0) (s) = - \int_0^1 \chi_{(0,s)} (r) \left( \frac{r}{s} \right) f_0 (r) \, dr + \frac{s}{1 - s^2} \int_0^1 \chi_{(s,1)} (r) f_0 (r) \, rdr.\]

A similar calculation shows that when \( l \geq 1 \), then
\[(T f_l (r \cdot e^{i\theta}) (se^{it}) = 2 \pi e^{i(l-1)t} (T_l f_l) (s),\]
where
\[(T_l f_l) (s) = \frac{1}{1 - s^2} \int_0^1 \chi_{(s,1)} (r) \left( \frac{s}{r} \right)^{l-1} f_l (r) \, rdr.\]
Similarly, when \( l < 0 \),
\[(T f_l (r \cdot e^{i\theta}) (se^{it}) = 2 \pi e^{i(l-1)t} (T_l f_l) (s),\]
where
\[(T_l f_l) (s) = - \left( \sum_{n=0}^{l-1} s^{2n} \right) \int_0^1 \chi_{(0,s)} (r) \left( \frac{r}{s} \right)^{l-1} f_l (r) \, dr + \frac{1}{1 - s^2} \int_0^1 \chi_{(s,1)} (r) (rs)^{l-1} f_l (r) \, dr.\]
Hence,
\[(T f) (se^{it}) = 2 \pi \sum_{l=-\infty}^{\infty} e^{i(l-1)t} (T_l f_l) (s),\]
for \((T_l f_l) (s) = \begin{cases} 
- \left( \sum_{n=0}^{l-1} s^{2n} \right) \int_0^1 \chi_{(0,s)} (r) \left( \frac{r}{s} \right)^{l-1} f_l (r) \, dr \\
+ \frac{1}{1 - s^2} \int_0^1 \chi_{(s,1)} (r) (rs)^{l-1} f_l (r) \, dr & \text{ for } l \leq 0 \\
\frac{1}{1 - s^2} \int_0^1 \chi_{(s,1)} (r) \left( \frac{s}{r} \right)^{l-1} f_0 (r) \, rdr & \text{ for } l > 0.
\end{cases}\]

By our construction,
\[\|T f\|^2_A = 4 \pi^2 \sum_{l=-\infty}^{\infty} \|T_l f_l\|^2_{L^2[0,1]},\]
where the measure on \( L^2 [0, 1] \) is “rdr”. Thus to prove our lemma, it suffices to prove that
\[\sup_{l} \|T_l\|_{B(L^2[0,1], L^2[0,1])} \leq 5 < \infty. \quad (**)\]
Once we prove (**) we can conclude that

\[ \| T f \|_A^2 \leq 100 \pi^2 \sum_{l=-\infty}^{\infty} \| f_l \|_{L^2[0,1]}^2 = 100 \pi^2 \| f \|_A^2. \]

For the case \( l = 0 \), we get that

\[
\int_0^1 |(T_0 f_0)(se^{it})|^2 s ds \\
\leq 2 \int_0^1 \left| - \int_0^1 \chi_{(0,s)}(r) \frac{f_0(r)}{s} e^{-it} r dr \right|^2 s ds \\
+ 2 \int_0^1 \left| \frac{s}{1 - s^2} (e^{-it}) \int_0^1 \chi_{(s,1)}(r) f_0(r) r dr \right|^2 s ds \\
\leq 2 \int_0^1 \frac{1}{s^2} \left[ \int_0^1 \chi_{(0,s)}(u) |f_0(u)| u du \int_0^1 \chi_{(0,s)}(v) |f_0(v)| v dv \right] s ds \\
+ 2 \int_0^1 \frac{s^2}{(1 - s^2)^2} \left[ \int_0^1 \chi_{(s,1)}(x) |f_0(x)| x dx \int_0^1 \chi_{(s,1)}(y) |f_0(y)| y dy \right] s ds.
\]

Let’s take the first term, which is

\[
\int_0^1 \int_0^1 |f_0(u)| |f_0(v)| \left( \int_0^1 \chi_{(0,s)}(u) \chi_{(0,s)}(v) \frac{ds}{s} \right) u du v dv \\
= \int_0^1 \int_0^1 |f_0(u)| |f_0(v)| \ln \left( \frac{1}{\max \{u, v\}} \right) u du v dv.
\]

By Schur’s Test we know that

\[
\int_0^1 \int_0^1 |f_0(u)| |f_0(v)| \ln \left( \frac{1}{\max \{u, v\}} \right) u du v dv \leq C_1^2 \int_0^1 |f(u)|^2 u du
\]

if and only if there exist \( p \geq 0 \) a.e. and \( C_1 < \infty \), satisfying

\[
\int_0^1 \ln \left( \frac{1}{\max \{u, v\}} \right) p(u) u du \leq C_1 p(v)
\]

for a.e. \( v \in [0, 1] \).

We take \( p(u) = 1 \). Now

\[
\int_0^v \ln \left( \frac{1}{u} \right) u du = \frac{1}{4} \ln \left( \frac{1}{u^2} \right) \leq \frac{1}{4}
\]

and

\[
\int_v^1 \ln \left( \frac{1}{u} \right) u du \leq 1 - v \leq 1.
\]
Therefore, 
\[ C_1 = \frac{1}{4} + 1 = \frac{5}{4}. \]
Again, we will repeat the same argument for the second term,
\[ \int_0^1 \int_0^1 |f_0(x)| |f_0(y)| \left[ \int_0^{\min\{x,y\}} \frac{s^2}{(1-s^2)^2} s \,ds \right] xdx \, ydy. \]
Changing variables and computing, we get that
\[ \int_0^1 \int_0^1 |f_0(x)||f_0(y)| \left[ \int_0^{\min\{x,y\}} \frac{s^2}{(1-s^2)^2} s \,ds \right] xdx \, ydy \leq \int_0^1 \int_0^1 |f_0(x)||f_0(y)| \left[ \frac{1}{2} \frac{1}{(1-(\min\{x,y\})^2)} \right] xdx \, ydy. \]
For this second term, we take \( p(x) = \frac{1}{\sqrt{1-x^2}}. \) Therefore,
\[ \int_0^y \frac{1}{2} \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}} xdx \leq \frac{1}{2} \frac{1}{\sqrt{1-y^2}}. \]
Also,
\[ \int_y^1 \frac{1}{2} \frac{1}{\sqrt{1-y^2}} \frac{1}{\sqrt{1-x^2}} xdx = \frac{1}{2} \frac{1}{\sqrt{1-y^2}}. \]
So we get \( C_2 \leq 1. \) Hence
\[ \int_0^1 |(T_0 f_0)(s)|^2 \, ds \leq \frac{9}{2} \int_0^1 |f_0(u)|^2 \, du. \]
Applying Schur’s Test for \( l \geq 1 \) with \( p(u) = \frac{1}{\sqrt{1-u^2}}, \) we get the estimate \( C_l \leq \frac{3}{2}, \) independent of \( l. \) Similarly, for \( l < 0 \) with \( p(u) = 1 \) and \( p(u) = \frac{1}{\sqrt{1-u^2}} \) for each of the two terms, respectively, we get the estimate \( C_l \leq 5, \) independent of \( l. \) Thus we conclude that
\[ \sup_l \|T_l\|_{B(L^2[0,1], L^2[0,1])} \leq 5. \]
This finishes the proof of Lemma 3.

Lemma 4. If \( Q \) is a multiplier of \( \mathcal{D}, \) then
\[ (1 - |z|^2) |Q'(z)| \leq \|M_Q\|_{B(\mathcal{D})} \text{ for all } z \in \mathbb{D}. \]

Proof. Define \( \varphi : D \to D \) as \( \varphi(z) = \frac{Q(z)}{|M_Q|} \) for all \( z \in \mathbb{D}. \) Now use the Schwarz lemma and the fact that \( \|\varphi\|_{\infty, \mathbb{D}} \leq \|M_\varphi\| \) to complete the proof. \( \square \)
We are now ready to prove Theorem 1.

Proof. First, we will prove the theorem for smooth functions on $\mathbb{D}$ and get a uniform bound. Then we will remove the smoothness hypothesis.

Assume that (a) and (b) of Theorem 1 hold for $F$ and $H$ and that $F$ and $H$ are analytic on $D_{1+\epsilon}(0)$.

Then our main goal is to show that there exists a constant $K < \infty$, independent of $\epsilon$, so that for any polynomial, $h$, there exists $u_h \in \oplus_i D$ such that $M_{\mathbb{D}}^R(u_h) = H^3 h$ and $\|u_h\|_{\mathbb{D}}^2 \leq K \|h\|_{\mathbb{D}}^2$.

We take $u_h = \frac{F^* H^3 h}{F F^*} - Q\left(\frac{Q^* F^* H^3 h}{(F F^*)^2}\right)$. Then $u_h$ is analytic and $M_{\mathbb{D}}^R(u_h) = H^3 h$.

We know that

$$\|u_h\|_{\mathbb{D}}^2 = \int_{-\pi}^{\pi} \|u_h(e^{it})\|^2 d\sigma(t) + \int_D (u_h(z))' \|dA(z)\|.$$

Condition (b) implies that

$$\int_{-\pi}^{\pi} \left\| \frac{F^* H^3 h}{F F^*} - Q\left(\frac{Q^* F^* H^3 h}{(F F^*)^2}\right) \right\|^2 d\sigma(t) \leq C_0^2 \|h\|_{\sigma}^2,$$

where $C_0$ can be chosen to be 15 (See [Tr1]). Hence, we only need to show that

$$\int_D \left\| \left( \frac{F^* H^3 h}{F F^*} - Q\left(\frac{Q^* F^* H^3 h}{(F F^*)^2}\right) \right) \right\|^2 dA(z) \leq C^2 \|h\|_{\mathbb{D}}^2$$

for some $C < \infty$. 
Now
\[
\int_D \left\| \left( \frac{F^* H^3 h}{F F^*} - Q \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right) \right)' \right\|^2 dA(z)
\]
\[
\leq 2 \int_D \left\| \left( \frac{F^* H^3 h}{F F^*} \right)' \right\|^2 dA(z) + 2 \int_D \left\| \left( Q \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right) \right)' \right\|^2 dA(z)
\]
\[
\leq 4 \int_D \left\| \frac{F^* 3 H^2 H' h}{F F^*} \right\|^2 dA(z) + 8 \int_D \left\| \frac{F^* H^3 H'}{F F^*} \right\|^2 dA(z)
\]
\[
+ 8 \int_D \left\| \frac{F^* H^3 h' F F^*}{(F F^*)^2} \right\|^2 dA(z) + 4 \int_D \left\| Q' \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right) \right\|^2 dA(z)
\]
\[
+ 4 \int_D \left\| Q \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right)' \right\|^2 dA(z).
\]

Then
\[
(a') = \int_D \left\| \frac{F^* 3 H^2 H' h}{F F^*} \right\|^2 dA(z) = 9 \int_D \left\| \frac{F^*}{\sqrt{F F^*}} \right\| H H' h \right\|^2 dA(z)
\]
\[
\leq 9 \int_D \left\| H H' \right\|^2 dA(z)
\]
\[
\leq 18 \left( \| M_H \|^2 + \| H \|^2 \right) \| h \|^2_D
\]
\[
\leq 36 \| M_H \|^2 \| h \|^2_D.
\]

(b') = \int_D \left\| \frac{F^* H^3 H'}{F F^*} \right\|^2 dA(z) \leq \int_D \left\| h' \right\|^2 dA(z) \leq \| h \|^2_D.

(c') = \int_D \left\| \frac{F^* H^3 h F F^*}{(F F^*)^2} \right\|^2 dA(z) = \int_D \left\| \frac{F^* F' F^*}{\sqrt{F F^*}} \right\|^2 dA(z)
\]
\[
\leq \int_D \left\| \frac{F^* F' F^*}{\sqrt{F F^*}} \right\|^2 dA(z)
\]
\[
\leq \int_D \left\| F^* F' \right\|^2 dA(z)
\]
\[
\leq 4 \| h \|^2_D.
\]

We use condition (a) of the theorem and the boundedness of the Beurling transform on \( L^2(\mathbb{D}, dA) \) (with bound 14) to conclude that
\[
(e') \leq 56(14)^2 \int_D \left\| F^* h \right\|^2 dA \leq 224(14)^2 \| h \|^2_D.
\]
So we only need estimate \((d')\). For this, we have

\[
\int_D \|Q' \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \|^2 dA(z) = \int_D \|Q' \hat{w} \|^2 dA(z),
\]

where \(\hat{w} = \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right)\) is a smooth function on \(\overline{\mathbb{D}}\).

Therefore,

\[
\int_D \|Q' \hat{w} \|^2 dA(z) \leq 2 \int_D \|Q' \hat{w} - Q' \tilde{\hat{w}} \|^2 dA(z) + 2 \int_D \|Q' \tilde{\hat{w}} \|^2 dA(z),
\]

where \(\tilde{\hat{w}}(z) = \int_{-\pi}^{\pi} \frac{1-|z|^2}{1-\bar{e}^{-it}z} \hat{w}(e^{it}) d\sigma(t)\) is the harmonic extension of \(\hat{w}\) from \(\partial \mathbb{D}\) to \(\mathbb{D}\).

Lemma (2) tells us that

\[
\int_D \|Q' \tilde{\hat{w}} \|^2 dA(z) \leq 8 \|\tilde{\hat{w}}\|^2_{\mathcal{H}D}.
\]

Also, a lemma of [Tr2] implies that

\[
\|\tilde{\hat{w}}\|^2_{\mathcal{H}D} \leq \|w\|^2_A + \|\hat{w}\|^2_{\mathcal{H}D}.
\]

But, as we showed above

\[
\|w\|^2_A = \int_D \|Q^* F'^* H^3 h \|^2 (FF^*)^2 dA(z) \leq \int_D \|F'^* h \|^2 dA(z) \leq 4 \|h\|^2_D
\]

and

\[
\|\hat{w}\|^2_{\mathcal{H}D} = \int_{-\pi}^{\pi} \left( \frac{Q^* F'^* H^3 h}{(FF^*)^2} \right) \|^2 d\sigma(t) \leq 15 \|h\|^2_{\mathcal{H}D}
\]

from [Tr2].

Thus,

\[
\int_D \|Q' \tilde{\hat{w}} \|^2 dA(z) \leq 8 \left[ 4 \|h\|^2_D + 15 \|h\|^2_{\mathcal{H}D} \right].
\]

Now we are just left with estimating \((\alpha)\). We will use Lemmas 3 and 4. We have

\[
(\alpha) = \int_D \|Q' \tilde{\hat{w}} - Q' \tilde{\hat{w}} \|^2 dA(z) \]

\[
= \int_D \|Q' \left[ -\frac{1}{\pi} \int_D \frac{w(u)}{u - z} dA(u) - \int_{-\pi}^{\pi} \frac{1-|z|^2}{1-\bar{e}^{-it}z} \hat{w}(e^{it}) d\sigma(t) \right] \|^2 dA(z)
\]
\[
\begin{align*}
\frac{1}{\pi^2} \int_D \| Q' \int_D w(u) \left[ -\frac{1}{u-z} + \frac{\bar{z}}{1-u\bar{z}} \right] dA(u) \|^2 dA(z) & \\
\quad + \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|1 - e^{-it}z|} e^{-it} \frac{1}{1-ue^{-it}} d\sigma(t) dA(u) \|^2 dA(z) & \\
\quad + \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|1 - e^{-it}z|} e^{-it} \frac{1}{1-ue^{-it}} d\sigma(t) dA(z) & \\
\quad \qquad \quad \| M_Q \|^2 \quad \| T(w) \|^2 dA(z) & \\
\quad \leq \| M_Q \|^2 \quad \| T(w) \|^2 dA(z) & \\
\quad \leq 100 \pi^2 \| M_Q \|^2 \| w \|^2 & \\
\quad \leq 100 \pi^2 \| M_Q \|^2 \| h \|^2 & \leq 1800 \| h \|^2.
\end{align*}
\]

Combining all these pieces, we see that in the smooth case

\[
\| u_h \|^2_D \leq K \| h \|^2_D
\]

for some constant \( K < \infty \), which is independent of \( h \) and \( \epsilon > 0 \).

By the proof of Theorem 1 in the smooth case, we have

\[
M_{F^*_r}(M_{F^*_r})^* \leq K^2 M_{H^3_r}, \quad \text{for } 0 \leq r < 1.
\]

Using a commutant lifting argument, there exists \( G_r \in \mathcal{M}(\mathcal{D}, \mathcal{D}) \)
so that \( M_{F^*_r} M_{G^*_r} = M_{H^3_r} \) and \( \| M_{F^*_r} \| \leq K \). Then \( M_{F^*_r} \rightarrow M_{F^*_r} \)
and \( M_{H^3_r} \rightarrow M_{H^3} \) as \( r \uparrow 1 \) in the \( \ast \)-strong topology.

By compactness, we may choose a net with \( G_{r_\alpha}^* \rightarrow G^* \) as \( r_\alpha \rightarrow 1^- \).
Since the multiplier algebra (as operators) is WOT closed, \( G \in \mathcal{M}(\mathcal{D}, \mathcal{D}) \).
Also, since \( F_{r_\alpha}^* \xrightarrow{\delta} F^* \), we get \( M_{H^3_r} = M_{G^*_r} M_{F^*_r} \xrightarrow{\text{WOT}} M_{G}^C M_{F}^R \) and so \( M_{F}^R M_{G}^C = M_{H^3} \) with entries of \( G \) in \( \mathcal{M}(\mathcal{D}) \) and \( \| M_{G}^C \| \leq K \).

It might be of some interest to note that the norm of the operator, \( \| M_{G}^C \| \), doesn’t exceed \( \sqrt{144 \| M_{H^3} \|^2 + 73, 104} \).

This ends our proof. \( \square \)
Just as Wolff gets Theorem B for free, we get

**Theorem 2.** Let \( \{H, f_j : j = 1, \ldots, n\} \subset \mathcal{M}(\mathcal{D}) \). Then \( H \in \text{Rad}(\{f_j\}_{j=1}^n) \) if and only if there exist \( C_0 < \infty \) and \( m \in \mathbb{N} \) such that

\[
|H^m(z)| \leq C_0 \sum_{j=1}^n |f_j(z)|^2 \quad \text{for all } z \in \mathbb{D}.
\]

This paper discusses when \( H^3 \) belongs to \( \mathcal{I}(\{f_j\}_{j=1}^n) \), the ideal generated by \( \{f_j\}_{j=1}^n \) in \( \mathcal{M}(\mathcal{D}) \) and characterizes membership in the radical of the ideal, \( \mathcal{I}(\{f_j\}_{j=1}^n) \). The question of strong sufficient conditions for \( H \) itself to belong to \( \mathcal{I}(\{f_j\}_{j=1}^n) \) is more subtle. The first author has obtained some interesting results in this direction.

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