Large and small group homology

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Abstract
For several instances of metric largeness like enlargeability or having hyperspherical universal covers, we construct non-large vector subspaces in the rational homology of finitely generated groups. The functorial properties of this construction imply that the corresponding largeness properties of closed manifolds depend only on the image of their fundamental classes under the classifying map. This is applied to construct examples of essential manifolds whose universal covers are not hyperspherical, thus answering a question of Gromov (1986), and, more generally, essential manifolds which are not enlargeable.

1. Introduction
Gromov and others [11, 14, 15] introduced notions of largeness for Riemannian manifolds. These include enlargeability and having hypereuclidean or hyperspherical universal covers, and universal covers with infinite filling radii. While precise definitions are given later in this paper, we point out that in spite of their reference to Riemannian metrics these properties are independent of the chosen metrics in the case of closed manifolds.

In this paper we will elaborate on the topological–homological nature of several largeness properties of closed manifolds and more generally of homology classes of simplicial complexes with finitely generated fundamental groups.

To explain some of our results in greater detail, let us recall here the definition of enlargeability (cf. [15]). In this paper all manifolds are assumed to be smooth and connected unless otherwise stated.

Definition 1.1. Let $M$ be a closed orientable manifold of dimension $n$ and let $g$ be a Riemannian metric on $M$. Then $(M,g)$ is called enlargeable, if, for every $\varepsilon > 0$, there is a Riemannian cover $\tilde{M}_\varepsilon \rightarrow M$ and an $\varepsilon$-contracting (that is, $\varepsilon$-Lipschitz) map $\tilde{M}_\varepsilon \rightarrow S^n$ to the $n$-dimensional unit sphere which is constant outside a compact set and of non-zero degree.

On closed manifolds all Riemannian metrics are in bi-Lipschitz correspondence and hence the described property does not depend on the particular choice of $g$. However, it is important that $g$ remains fixed for the different choices of $\varepsilon$. Examples of enlargeable manifolds include tori and more generally manifolds admitting Riemannian metrics of non-positive sectional curvature.

Received 5 February 2009; revised 7 December 2009.
2000 Mathematics Subject Classification 53C23 (primary), 20J06 (secondary).

The authors gratefully acknowledge the financial support from the Deutsche Forschungsgemeinschaft.

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Definition 1.2 [9]. Let $M$ be a closed oriented manifold with fundamental class $[M]$ and let $\Phi : M \to B\pi_1(M)$ be the classifying map of the universal cover. We call $M$ essential if $\Phi_*[M] \neq 0 \in H_*(B\pi_1(M); \mathbb{Q})$.

In [19] it was shown by index theoretic methods that enlargeable manifolds are essential, if the cover $\overline{M} \to M$ can always be assumed to be finite. Relying on ideas from coarse geometry, [18, Corollary 1.3] states the essentialness for manifolds with hyperspherical universal covers (these are manifolds where $\overline{M}$ may always be chosen as the universal cover). Elaborating on the methods from [19], the essentialness of all enlargeable manifolds was proved in [20].

Extending the results of [18–20] we will show that manifolds which are either enlargeable or have universal covers which are (coarsely) hyperspherical, (coarsely) hypereuclidean or macroscopically large (these notions will be defined in Section 2) are essential. Our approach is independent of index theory or coarse geometry (unless the largeness condition under consideration refers to coarse notions).

We will moreover prove that each of these largeness properties depends only on the image $\Phi_*[M] \in H_*(B\pi_1(M); \mathbb{Q})$ of the fundamental class under the classifying map. This property may be called homological invariance of largeness.

Both conclusions are implied by the following theorem which will be proved in Section 3 of our paper.

Theorem 1.3. Let $\Gamma$ be a finitely generated discrete group and let $n$ be a natural number. Let $P$ denote one of the properties of being enlargeable or having a universal cover which is (coarsely) hyperspherical, (coarsely) hypereuclidean or macroscopically large. Then there is a rational vector subspace $H_{n}^{\text{sm}(P)}(B\Gamma; \mathbb{Q}) \subset H_{n}(B\Gamma; \mathbb{Q})$ of ‘non-large’ (that is, small) classes with respect to $P$. In the case when $\Gamma$ is finitely presentable, the following holds. If $M$ is a closed oriented $n$-dimensional manifold with fundamental group $\Gamma$ and classifying map $\Phi : M \to B\Gamma$, then $\Phi_*[M] \in H_{n}^{\text{sm}(P)}(B\Gamma; \mathbb{Q})$ if and only if $M$ is not large in the respective sense.

In Definition 3.1 we introduce largeness properties in general for homology classes of connected simplicial complexes with finitely generated fundamental groups by adapting the classical definitions accordingly. The hard part of this approach lies in the verification that the required properties depend on the given homology class only. In this respect homological invariance of largeness is built into the definition right away. Once this has been achieved it will be easy to show that the classes which are ‘small’, that is, not large in the respective sense, enjoy the nice algebraic property of forming a vector subspace; see Theorem 3.6.

If the simplicial complex in question is the classifying space of a finitely generated group, this approach emphasizes our point of view that the largeness properties in Theorem 1.3 should be regarded as metric properties of finitely generated groups (respectively, their rational homology) much like the quasi-isometry type of the word metric itself.

Homological invariance of largeness for the classical case of closed manifolds is a simple consequence of the functorial properties for large homology classes proved in Proposition 3.4. Together with the fact that the non-large classes form a vector subspace (and hence contain the class zero in each degree), this indeed shows that enlargeable manifolds and more generally manifolds with (coarsely) hyperspherical, (coarsely) hypereuclidean or macroscopically large universal covers are essential.

In Section 4 we will illustrate by examples that the subspaces $H_{n}^{\text{sm}(P)}$ are, in general, different from zero and may even depend on the specific largeness property $P$. In particular we will prove the following consequence of Theorem 1.3.
Theorem 1.4. For all \( n \geq 4 \) there are enlargeable (hence essential) manifolds of dimension \( n \) whose universal covers are neither (coarsely) hyperspherical, (coarsely) hypereuclidean nor macroscopically large.

Gromov asked in [11, p. 113] whether universal covers of essential manifolds would always be hyperspherical. Theorem 1.4 gives a negative answer. It also shows the remarkable fact that enlargeable manifolds do not always have hyperspherical universal covers. This provides a late justification for the organization of an argument in [18]: in that paper the proof that enlargeable manifolds are Baum–Connes essential was much easier for manifolds with hyperspherical universal covers (these were called universally enlargeable in [18]) than for general enlargeable manifolds. Now we see that the general case cannot be reduced to the case of manifolds with hyperspherical universal covers.

By a refinement of our methods we also get the following result, showing that even the more flexible property of enlargeability is not always implied by essentialness.

Theorem 1.5. For all \( n \geq 4 \), there are \( n \)-dimensional closed manifolds that are essential, but not enlargeable.

Since arbitrary covers of these manifolds need to be controlled, the proof of this result is technically much harder than that of Theorem 1.4. Our argument makes essential use of the Higman 4-group [21].

These examples are interesting because enlargeability is the most flexible of the largeness properties considered in this paper in terms of which coverings of the given manifold can be used. In this respect they point to a principal limitation of the use of largeness of closed manifolds for proving the strong Novikov conjecture in full generality; cf. [18, 19].

We remark, however, that none of our examples appearing in Theorem 1.4 or 1.5 is aspherical. In the case of aspherical manifolds there is apparently room to use metric largeness properties in order to prove general theorems on the non-existence of positive scalar curvature metrics and related properties, notably if the fundamental group has finite asymptotic dimension [6, 7].

The characterization of metric largeness by certain subspaces of group homology has been remarked before in the context of vanishing simplicial volume [9, Section 3.1] and (in some more restricted setting) in the context of positive scalar curvature metrics on high-dimensional manifolds with non-spin universal covers [24]. Recently in [2] the first author of the present paper showed that the systolic constant, the minimal volume entropy and the spherical volume of closed manifolds only depend on the image of their fundamental classes in the integral homology of their fundamental groups under the classifying maps of their universal covers.

In the definition of enlargeability the maps \( \overline{M}_\varepsilon \to S^n \) are assumed to contract distances. If they are only required to contract volumes of \( k \)-dimensional submanifolds, then \( M \) is called \( k \)-enlargeable. In the case \( k = 2 \), this property is also called area-enlargeability [15].

Relying on index theory it was shown in [20] that area-enlargeable manifolds are essential. In Section 5, we will prove the following more general statement by methods similar to those employed in Section 3.

Theorem 1.6. Let \( M \) be a closed oriented \( n \)-dimensional manifold. If \( M \) is \( k \)-enlargeable and satisfies

\[
\pi_i(M) = 0 \quad \text{for} \ 2 \leq i \leq k - 1,
\]

then \( M \) is essential. In particular, area-enlargeable manifolds are essential.
For $k \leq 2$ the condition on the homotopy groups is to be understood as empty. Note that, for $k > 2$, the condition is in fact necessary: let $M$ be an enlargeable manifold. Then the product $M \times S^2$ is 3-enlargeable, but the classifying map $M \times S^2 \to B\pi_1(M)$ sends the fundamental class to zero, that is, $M \times S^2$ is not essential. This is in accordance with Theorem 1.6 as $\pi_2(M \times S^2) \neq 0$.

For $k \geq n + 1$ the $k$-dimensional volume of any subset of $S^n$ is zero, of course. Therefore, in this case the assumption of $k$-enlargeability in Theorem 1.6 can be dropped and the remaining non-trivial requirement is $\pi_i(M) = 0$ for $2 \leq i \leq k - 1$. The inequality $k \geq n + 1$ and the Hurewicz theorem then imply that all homotopy groups of the universal cover of $M$ vanish. In other words, Theorem 1.6 includes the well-known statement that aspherical manifolds are essential.

Hence the conditions in Theorem 1.6 interpolate between two extreme cases: enlargeable and area-enlargeable manifolds on the one side and aspherical ones on the other. It was shown in [15] that area-enlargeable spin manifolds do not carry metrics of positive scalar curvature and it has been conjectured that the conclusion is valid for all aspherical manifolds. In this respect it seems reasonable to conjecture that the conditions in Theorem 1.6 are also obstructions to the existence of positive scalar curvature metrics. In fact the strong Novikov conjecture implies that essential spin manifolds do not admit positive scalar curvature metrics, cf. [23].

Concerning homological invariance, $k$-enlargeability for $k \geq 2$ seems to behave less favourably than the other largeness properties considered in our work. However, we will not pursue this question further in this paper. Theorem 1.6 is related in spirit to [4, Theorem 2.5], which deals with functorial properties of hyperbolic cohomology classes.

This paper is based in part on a chapter of the first author’s thesis [3]. In particular, homological invariance of largeness properties (which is part of Theorem 1.3 of the present paper) and Theorem 1.6 were first proved there. The most important novelties of the present paper are a more systematic treatment of large homology classes in Section 3 and, based on that, the construction of interesting examples of enlargeable manifolds without hyperspherical universal covers (Theorem 1.4) and of essential manifolds that are not enlargeable (Theorem 1.5).

2. Large Riemannian manifolds

In this section we will recall classical notions of metric largeness for Riemannian manifolds, most of which were first formulated by Gromov; for example, see [11, 12, 15] and also [5, 16, 17]. They include the following properties:

(i) being $k$-hypereuclidean and $k$-hyperspherical (see Definition 2.2);
(ii) being $k$-enlargeable (see Definition 2.3);
(iii) having infinite filling radius (see Definition 2.5);
(iv) being macroscopically large (see Definition 2.7).

In Proposition 2.10 we characterize hypersphericity in terms of the existence of a Lipschitz map of non-zero degree to the balloon space, which was introduced in [18]. Upon passing from Lipschitz to large-scale Lipschitz maps, this allows us to define coarsely hyperspherical manifolds, a notion similar to coarsely hypereuclidean manifolds, which by definition admit coarse maps to euclidean space of non-zero degree. Furthermore, we will discuss several implications between these largeness properties. In particular, in Proposition 2.8 we will show that the classes of macroscopically large manifolds and of manifolds with infinite filling radii coincide. This is remarkable because it relates a coarse (that is, quasi-isometric) property to a bi-Lipschitz one.

Let $f : (M, g_M) \to (N, g_N)$ be a smooth map between (not necessarily compact) Riemannian manifolds, and let $k$ be a positive integer.
**Definition 2.1.** The \( k \)-dilation of \( f \) is defined as
\[
\dil_k(f) := \sup_{p \in M} \|\Lambda^k D_pf\| \in \mathbb{R} \cup \{ \infty \},
\]
the supremum of the norms of the \( k \)-fold exterior product of the differential \( Df \).

In other words, the \( k \)-dilation is the smallest number \( \varepsilon \) such that, for any \( k \)-dimensional submanifold \( A \subset M \), the \( k \)-dimensional volume \( \text{Vol}_k(f(A)) \) of the image \( f(A) \subset N \) is bounded by \( \varepsilon \cdot \text{Vol}_k(A) \). The 1-dilation is the smallest Lipschitz constant for \( f \).

Let \( p \in M \) be a point and let \( n \) be the dimension of \( M \). Denote by \( \lambda_1 \geq \ldots \geq \lambda_n > 0 \) the eigenvalues of the Gram matrix of the pull-back \( (D_pf)^*(g_N)_{f(p)} \) with respect to \((g_M)_p\). Then
\[
\|\Lambda^k D_pf\|^2 = \lambda_1 \cdot \ldots \cdot \lambda_k.
\]
Therefore, the inequality
\[
dil_1(f)^{1/l} \leq \dil_k(f)^{1/k}
\]
holds for all \( l \geq k \).

Let \((V,g)\) be a complete orientable Riemannian manifold of dimension \( n \). A choice of orientation for \( V \) defines a fundamental class \([V] \in H_{lf}^n(V;\mathbb{Z})\) in locally finite homology. In this context the mapping degree is well defined for proper maps to oriented manifolds and for maps to closed oriented manifolds \( Z \) that are almost proper, that is, constant outside a compact set. This can be made rigorous by adding an infinite whisker to \( Z \), extending the given almost proper map to a proper map with the target \( Z \cup \text{whisker} \), and observing that \( H_{lf}^n(Z \cup \text{whisker}) = \tilde{H}_*(Z) \).

In the following, we equip euclidean spaces and unit spheres with their standard metrics.

**Definition 2.2.** We call \((V,g)\) \( k \)-hypereuclidean if there is a proper map
\[
f : V \longrightarrow \mathbb{R}^n
\]
of non-zero degree and of finite \( k \)-dilation. It is called \( k \)-hyperspherical if, for every \( \varepsilon > 0 \), there is an almost proper map
\[
f_\varepsilon : V \longrightarrow S^n
\]
of non-zero degree such that \( \dil_k(f_\varepsilon) \leq \varepsilon \). For \( k = 1 \) we omit the number, and for \( k = 2 \) we speak of area-hypereuclidean and area-hyperspherical manifolds.

By the above inequality, every \( k \)-hypereuclidean or \( k \)-hyperspherical manifold is also, respectively, \( l \)-hypereuclidean or \( l \)-hyperspherical for any \( l \geq k \). Since \( \mathbb{R}^n \) is obviously hyperspherical, any \( k \)-hypereuclidean manifold is also \( k \)-hyperspherical. Note also that both notions depend only on the bi-Lipschitz type of the metric \( g \).

Closely related is the notion of enlargeability. It was introduced by Gromov and Lawson in [14] and in the following more general form in [15].

**Definition 2.3.** An orientable \( n \)-dimensional manifold \( V \) is called \( k \)-enlargeable if, for every complete Riemannian metric \( g \) on \( V \) and every \( \varepsilon > 0 \), there is a Riemannian cover \( \tilde{V}_\varepsilon \) of \( V \) and an almost proper map
\[
f_\varepsilon : \tilde{V}_\varepsilon \longrightarrow S^n
\]
of non-zero degree such that \( \dil_k(f_\varepsilon) \leq \varepsilon \). As before, we omit the number \( k \) in the case \( k = 1 \) and speak of area-enlargeable manifolds in the case \( k = 2 \).
If $V$ is closed, then all Riemannian metrics on $V$ are bi-Lipschitz to each other and it is enough that $V$ satisfies the above condition with respect to one Riemannian metric.

The significance of the notion of enlargeability is demonstrated by the following theorem; see [15, Theorem 6.1].

**Theorem 2.4.** If $V$ is area-enlargeable and the covers $\bar{V}_\varepsilon$ in Definition 2.3 may be chosen spin, then $V$ does not carry a complete Riemannian metric of uniformly positive scalar curvature.

Next, we revisit the notion of filling radius. Recall that every Riemannian metric $g$ on $V$ induces a path metric $d_g$ on $V$. Denote by $L^\infty(V)$ the vector space of all functions $V \to \mathbb{R}$ with the uniform ‘norm’ $\| - \|_\infty$. This is not a norm proper since it may take infinite values. Therefore the induced ‘metric’ is not an actual metric. Nevertheless, the Kuratowski embedding $\iota_g : (V, d_g) \hookrightarrow L^\infty(V)$, $v \mapsto d_g(v, -)$ is an isometric embedding by the triangle inequality.

One could replace $L^\infty(V)$ by its affine subspace $L^\infty(V)_b$ that is parallel to the Banach space of all bounded functions on $V$ and contains the distance function $d_g(v, -)$ for some $v \in V$. Then the image of the Kuratowski embedding is contained in $L^\infty(V)_b$, and the ‘norm’ $\| - \|_\infty$ induces an actual metric on $L^\infty(V)_b$. Since all points of $L^\infty(V)$ outside of this affine subspace are already infinitely far away from it, this would not change the following definition.

**Definition 2.5.** The filling radius of $(V, g)$ is defined as

$$\text{FillRad}(V, g) := \inf \{ r > 0 | \iota_g^*[V] = 0 \in H^H_n(U_r(\iota_g V); \mathbb{Q}) \},$$

where $U_r(\iota_g V) \subset L^\infty(V)$ denotes the open $r$-neighbourhood of the image $\iota_g V \subset L^\infty(V)$. If the set on the right-hand side is empty, we say that $(V, g)$ has infinite filling radius.

Note that, for closed manifolds, $L^\infty(V)_b$ is the vector space of all bounded functions on $V$ and the above definition of the filling radius coincides with the usual definition (see [10, Section 1]). For non-compact manifolds the filling radius need not be finite. For instance, the filling radius of the euclidean space is infinite.

It follows from the definition that the property of having infinite filling radius depends only on the bi-Lipschitz type of the metric $g$.

We recall the following implication that was shown in [11] (see also [5]).

**Proposition 2.6.** If $(V, g)$ is hyperspherical, then its filling radius is infinite.
Let \( X \) be a metric space. A cover \( \mathcal{U} \) of \( X \) is called uniform if the diameters of its members are uniformly bounded and if every bounded set in \( X \) meets only finitely many members of \( \mathcal{U} \). A family \( (\mathcal{U}_i)_{i \in I} \) of uniform covers is called an anti-Čech system if, for every \( r > 0 \), there exists a cover \( \mathcal{U}_r \) with Lebesgue number at least \( r \).

The nerve of a cover \( \mathcal{U} \) is denoted by \( |\mathcal{U}| \). It is the simplicial complex whose simplices are finite subsets of \( \mathcal{U} \) with a non-empty intersection in \( X \). In particular, the set of vertices is equal to \( \mathcal{U} \). The nerve of a uniform cover is locally finite.

If \( \mathcal{U} \) and \( \mathcal{V} \) are two uniform covers such that the Lebesgue number of \( \mathcal{V} \) is bigger than the uniform bound on the diameters of the sets of \( \mathcal{U} \), then we write \( \mathcal{U} \leq \mathcal{V} \). In this way the set of uniform covers of \( X \) becomes directed. If \( \mathcal{U} \leq \mathcal{V} \), then there is a proper simplicial map \( |\mathcal{U}| \to |\mathcal{V}| \) mapping each vertex \( U \in \mathcal{U} \) to some vertex \( V \in \mathcal{V} \) that contains \( U \). The proper homotopy class of this map is uniquely determined.

If \( X \) is proper (that is, bounded subsets are precompact), then anti-Čech systems always exist. Given an anti-Čech system, one defines the coarse homology of \( X \) as

\[
HX_k(X; \mathbb{Q}) := \lim_{\to} H^\lf_k(|\mathcal{U}|; \mathbb{Q}).
\]

This is independent of the choice of the anti-Čech system.

For proper \( X \) and for any uniform cover \( \mathcal{U} \) there is a proper map \( X \to |\mathcal{U}| \) that sends each point \( x \in X \) to a point in the simplex spanned by those \( U \in \mathcal{U} \) that contain \( x \). Moreover, the proper homotopy class of such a map is uniquely determined. Therefore, one gets an induced homomorphism

\[
c : H^\lf_k(X; \mathbb{Q}) \to HX_k(X; \mathbb{Q}),
\]

which will be called the character homomorphism of \( X \).

**Definition 2.7** \cite{8}. A complete oriented \( n \)-dimensional Riemannian manifold \( V \) is called macroscopically large, if

\[
c[V] \neq 0 \in HX_n(V; \mathbb{Q}).
\]

Note that this property depends only on the quasi-isometry class of the metric. We will show that macroscopic largeness is equivalent to having infinite filling radius. This proves that the property of having infinite filling radius depends only on the quasi-isometry class of the Riemannian metric. Note that the quasi-isometry class strictly includes the bi-Lipschitz class. It is not known whether hypereuclideaness or hypersphericity are also invariant under quasi-isometries.

**Proposition 2.8.** Let \( V \) be a complete orientable Riemannian manifold. Then \( V \) is macroscopically large if and only if its filling radius is infinite.

For the proof we need the notion of a coarse map. The general definition is a bit involved. However recall from \cite[Section 1.3]{22} that a map \( f : X \to Y \) from a path metric space to a metric space is coarse if and only if it is large-scale Lipschitz and (metrically) proper.

**Proof of Proposition 2.8.** Identify \( V \) with its image under the Kuratowski embedding, and let \( n \) be the dimension of \( V \).

First assume that \( \text{FillRad}(V, g) < r \) for some finite \( r \). Then there is a locally finite complex \( X \subset U_r(V) \) containing \( V \) such that \([V] = 0 \in H^\lf_n(X; \mathbb{Q})\). Moreover, the inclusion \( V \to X \) is a coarse equivalence since the coarse map that assigns to a point \( x \in X \) a point \( v \in V \) with
$d(x, v) \leq r$ being an inverse. The commutative diagram

$$
\begin{array}{ccc}
H_n^H(V; \mathbb{Q}) & \longrightarrow & H_n^H(X; \mathbb{Q}) \\
\downarrow & & \downarrow \\
HX_n(V; \mathbb{Q}) & \longrightarrow & HX_n(X; \mathbb{Q})
\end{array}
$$

shows that $c[V] = 0 \in HX_n(V; \mathbb{Q})$, that is, $(V, g)$ is not macroscopically large. (Note that $U_r(V)$ is also coarsely equivalent to $V$ but it is not proper. Therefore, it is not clear whether it admits a character homomorphism.)

To prove the converse implication, assume that $(V, g)$ is not macroscopically large. By the definition of the direct limit there is a uniform cover $\mathcal{U}$ of $V$ such that $\phi_*[V] = 0 \in H_n^H(|\mathcal{U}|; \mathbb{Q})$, where $\phi : V \to |\mathcal{U}|$ is a proper map that sends each point $v \in V$ to a point in the simplex spanned by those $U \in \mathcal{U}$ that contain $v$. Let $r > 0$ be an upper bound on the diameters of the sets of $\mathcal{U}$.

Define a map $\psi : |\mathcal{U}| \to L^\infty(V)$ by sending each vertex $U \in \mathcal{U}$ to some point $\psi(U) \in U \subset V$ and by extending this linearly over each simplex of the nerve.

Let $p$ be a point in $|\mathcal{U}|$. It may be written as $p = \sum \lambda_i U_i$ with $\sum \lambda_i = 1$, $\lambda_i > 0$ and $U_i \in \mathcal{U}$ such that $\bigcap U_i \neq \emptyset$. Then $\psi(p) = \sum \lambda_i \psi(U_i)$ and

$$d(\psi(p), \psi(U_1)) = \left\| \sum \lambda_i \psi(U_i) - \psi(U_1) \right\|_{\infty} \leq \sum \lambda_i \left\| \psi(U_i) - \psi(U_1) \right\|_{\infty} \leq 2r,$$

since $U_i \cap U_1 \neq \emptyset$ for all $i$. This shows that the image of $\psi$ lies in the $2r$-neighbourhood of $V$ in $L^\infty(V)$. Hence

$$(\psi \circ \phi)_* [V] = 0 \in H_n^H(U_{2r}(V); \mathbb{Q}).$$

Let $v \in V$ be a point which is contained in the sets $U_1, \ldots, U_m \in \mathcal{U}$ and in no other set of $\mathcal{U}$. Then $\phi(v) = \sum_{i=1}^m \lambda_i U_i$ for some $\lambda_i \geq 0$ with $\sum \lambda_i = 1$. Therefore

$$d(\psi(\phi(v)), v) = \left\| \sum \lambda_i \psi(U_i) - v \right\|_{\infty} \leq \sum \lambda_i \left\| \psi(U_i) - v \right\|_{\infty} \leq r,$$

since $v \in U_i$ for all $i = 1, \ldots, m$. Thus the linear homotopy from the inclusion $V \hookrightarrow L^\infty(V)$ to $\psi \circ \phi$ is proper and lies entirely in $U_r(V)$. We conclude

$$[V] = (\psi \circ \phi)_* [V] \in H_n^H(U_{2r}(V); \mathbb{Q}),$$

and consequently $[V] = 0 \in H_n^H(U_{2r}(V); \mathbb{Q})$, and hence $\text{FillRad}(V, g) \leq 2r < \infty$.

Propositions 2.6 and 2.8 show that complete hyperspherical manifolds are macroscopically large. If the given hyperspherical manifold is the universal cover of a closed manifold, then this is proved directly in [18, Proposition 3.1] using the balloon space $B^n$. This path metric space is defined as a real half-line $[0, \infty)$ with an $n$-dimensional round sphere $S^n_i$ of radius $i$ attached at each positive integer $i \in [0, \infty)$. 

\[ \Box \]
**Proposition 2.9** [18, Proposition 2.2]. The $n$-dimensional coarse homology of the balloon space is given by

$$HX_n(B^n; \mathbb{Q}) = \left( \prod_{i=1}^{\infty} \mathbb{Q} \right) \bigg/ \left( \bigoplus_{i=1}^{\infty} \mathbb{Q} \right).$$

Moreover, for the locally finite homology we have $H^lf_n(B^n; \mathbb{Q}) = \prod_{i=1}^{\infty} \mathbb{Q}$, and the character homomorphism $c : H^lf_n(B^n; \mathbb{Q}) \to HX_n(B^n; \mathbb{Q})$ is the canonical projection.

Using this computation we obtain the following characterization of hyperspherical manifolds.

**Proposition 2.10.** A complete oriented Riemannian manifold $V$ of dimension $n$ is hyperspherical, if and only if there exists a proper Lipschitz map $f : V \to B^n$ such that $f_*(\mathcal{V}) \neq 0 \in HX_n(B^n; \mathbb{Q})$.

**Proof.** First, assume that $V$ is hyperspherical. We shall construct a sequence of closed balls

$$\emptyset = B_0 \subset B_1 \subset B_2 \subset \ldots \subset V$$

that exhausts $V$ and a sequence of 1-Lipschitz maps $f_i : B_i \setminus \bar{B}_{i-1} \to S^n_i \setminus [i, i+1] \subset B^n$ such that $f_i(\partial B_{i-1}) = i$, $f_i(\partial B_i) = i + 1$, and such that $f_i$ is of non-zero degree as a map to $S^n_i$.

Assume that the balls and maps have been constructed up to index $i - 1$. Let $S^n_R$ be the round sphere of radius $R$ with a large $R$ which will be specified later. Choose a 1-Lipschitz map $f'_i : V \to S^n_R$ that is constant outside a compact set $K_i$ and that is of non-zero degree. Without loss of generality, we may assume that $B_{i-1} \subset K_i$ and that $f'_i(B_{i-1})$ and the point $f'_i(V \setminus K_i)$ avoid a ball of radius $\pi i$ in $S^n_R$ (for instance, choose $R \geq 2i + r/\pi$, where $r$ is the radius of $B_{i-1}$). Let $g_i : S^n_R \to S^n_i$ be a non-expanding map that contracts everything outside this ball of radius $\pi i$ to the base point of $S^n_i$.

Choose a ball $B_i \subset V$ such that $K_i \subset B_i$ and such that $d(\partial B_i, K_i) \geq 1$. Define $f_i$ as follows:

$$f_i(v) :=
\begin{cases}
  g_i \circ f'_i(v) & \text{for } v \in B_i \setminus \bar{B}_{i-1} \text{ and } d(v, \partial B_i) \geq 1,
  i + 1 - d(v, \partial B_i) & \text{for } v \in B_i, d(v, \partial B_i) \leq 1.
\end{cases}$$

Then $f_i$ has the asserted properties.

The collection of the maps $f_i$ defines a proper 1-Lipschitz map $f : V \to B^n$ such that every entry of $f_*[\mathcal{V}] \in H^lf_n(B^n; \mathbb{Q}) \cong \prod_{i=1}^{\infty} \mathbb{Q}$ is non-zero. In particular $f_*[\mathcal{V}] \neq 0 \in HX_n(B^n; \mathbb{Q})$ as required.

For the converse, let $f : V \to B^n$ be a proper Lipschitz map such that $f_*[\mathcal{V}] \neq 0 \in HX_n(B^n; \mathbb{Q})$. Let $\varepsilon > 0$, and choose an integer $i \geq \text{dil}_1(f)/\varepsilon$ such that the $i$th entry of $f_*[\mathcal{V}] \in H^lf_n(B^n; \mathbb{Q}) \cong \prod_{i=1}^{\infty} \mathbb{Q}$ is not zero. This is possible since by assumption there are infinitely many non-vanishing entries.

Let $f_\varepsilon$ be the composition of $f$ with the canonical quotient map from $B^n$ to the $i$th sphere $S^n_i$ and the dilation from this sphere of radius $i$ to the unit sphere. Then $f_\varepsilon$ is constant outside a compact set, has non-zero degree, and its dilation is given by $\text{dil}_1(f)/i \leq \varepsilon$. This proves that $V$ is indeed hyperspherical. \qed

In [18, Proposition 3.1] it is shown that hyperspherical universal covers of closed Riemannian manifolds admit proper Lipschitz maps to the balloon space, sending the locally finite fundamental class of the universal covers to non-zero classes. The corresponding implication in our Proposition 2.10 is slightly more general in that it is not assumed that the metric on $V$ is invariant under a cocompact group action.
Definition 2.11. A complete oriented Riemannian manifold \( V \) of dimension \( n \) is called \textit{coarsely hypereuclidean}, if there is a coarse map
\[
f : V \longrightarrow \mathbb{R}^n
\]
such that \( f_*[V] \neq 0 \in HX_n(\mathbb{R}^n; \mathbb{Q}) \cong \mathbb{Q} \). It is called \textit{coarsely hyperspherical} if there is a coarse map
\[
f : V \longrightarrow B^n
\]
to the balloon space such that \( f_*[V] \neq 0 \in HX_n(B^n; \mathbb{Q}) \).

These two notions depend only on the quasi-isometry class of the metric on \( V \).

The following diagram summarizes the known implications between some of the largeness properties on a complete Riemannian manifold discussed in this section.

\[
\begin{array}{c}
\text{hypereuclidean} \implies \text{coarsely hypereuclidean} \\
\downarrow \\
\text{hyperspherical} \implies \text{coarsely hyperspherical} \\
\downarrow \\
\text{infinite filling radius} \iff \text{macroscopically large}
\end{array}
\]

As \( H_n^\text{lf}(\mathbb{R}^n; \mathbb{Q}) \cong HX_n(\mathbb{R}^n; \mathbb{Q}) \cong \mathbb{Q} \), hypereuclidean manifolds are coarsely hypereuclidean, and Proposition 2.10 implies that hyperspherical manifolds are coarsely hyperspherical. This explains the two upper horizontal arrows. Moreover, the proof of Proposition 2.10 shows the existence of a coarse map \( \mathbb{R}^n \rightarrow B^n \) sending the coarse fundamental class of \( \mathbb{R}^n \) to a non-zero class in \( HX_n(B^n; \mathbb{Q}) \). This implies the upper vertical arrow on the right. The lower right vertical implication follows by the very definition of macroscopic largeness. Apart from the lower horizontal arrow it is not known if any of the implications is an equivalence.

We also remark that the properties on the left-hand side are invariants of the bi-Lipschitz class of the given metric, and those on the right-hand side of its quasi-isometry class.

3. Largeness in homology

In this section we shall formulate the concept of largeness for rational homology classes in simplicial complexes with finitely generated fundamental groups. We will then prove Theorem 1.3. In this section, the term \textit{large} is a placeholder for one of the properties of being enlargeable or having a universal cover which is (coarsely) hyperspherical, (coarsely) hypereuclidean or macroscopically large.

The method of extending differential–geometric concepts from smooth manifolds to more general spaces like simplicial complexes has occurred at other places in the literature in similar contexts; for example, see \[2\].

We equip the \( n \)-dimensional simplex \( \Delta^n \) with the metric induced by the standard embedding into \( \mathbb{R}^{n+1} \). Recall from \[13\], Chapter 1 that each connected simplicial complex comes with a canonical path metric restricting to this standard metric on each simplex. Furthermore, if \( p : X \rightarrow Y \) is a covering map of path-connected topological spaces and \( Y \) is equipped with a path metric, then there is a unique path metric on \( X \) so that \( p \) is a local isometry. If \( Y \) is a simplicial complex with the canonical path metric and \( X \) carries the induced simplicial structure, then this is the canonical path metric on \( X \).
A connected subcomplex \( S \) of a connected simplicial complex \( X \) is called \( \pi_1 \)-surjective, if the inclusion induces a surjection of fundamental groups, and we say that \( S \) carries a homology class \( c \in H_\ast(X; \mathbb{Q}) \), if \( c \) is in the image of the map in homology induced by the inclusion.

If \( p : \hat{X} \to X \) is a (not necessarily connected) cover of a simplicial complex \( X \) and \( c \in H_n(X; \mathbb{Q}) \) is a simplicial homology class, then the transfer \( p^!(c) \in H_n^\text{lf}(\hat{X}; \mathbb{Q}) \) is represented by the formal sum of all preimages of simplices in a chain representative of \( c \), with appropriate coefficients.

**Definition 3.1.** Let \( X \) be a connected simplicial complex with finitely generated fundamental group and let \( c \in H_n(X; \mathbb{Q}) \) be a (simplicial) homology class. Choose a finite connected \( \pi_1 \)-surjective subcomplex \( S \subset X \) carrying \( c \). (This subcomplex exists because \( \pi_1(X) \) is finitely generated.)

The class \( c \in H_n(X; \mathbb{Q}) \) is called enlargeable if the following holds. Let \( \varepsilon > 0 \). Then there is a connected cover \( p : \hat{X} \to X \) and an almost proper \( \varepsilon \)-contracting map \( f_\varepsilon : \hat{S} \to S^n \) which sends the class \( p^!(c) \) to a non-zero class in \( H_n(S^n; \mathbb{Q}) \). Here \( \hat{S} := p^{-1}(S) \) (which is connected as \( S \) is \( \pi_1 \)-surjective) is equipped with the canonical path metric.

The class \( c \) is called, respectively, (coarsely) hypereuclidean, (coarsely) hyperspherical or macroscopically large if the complex \( \hat{S} = p^{-1}(S) \) associated to the universal cover \( p : \hat{X} \to X \) together with the transfer class \( p^!(c) \) enjoys the according property.

It is important to work with \( \pi_1 \)-surjective subcomplexes \( S \), because then the covers \( \hat{S} \) are connected and hence equipped with canonical path metrics. We could have replaced the conditions in Definition 3.1 by first representing the homology class in question as the image of the fundamental class under a map \( \phi : M \to X \) from a closed oriented \( n \)-dimensional manifold \( M \) to \( X \) and requiring that \( M \) or appropriate covers thereof share the corresponding largeness property. This indeed works if \( \phi \) induces a surjection in \( \pi_1 \). We preferred the above definition because it applies as well to homology classes that are not representable by closed manifolds (which is relevant in low dimensions), works for fundamental groups which are not necessarily finitely presented and last but not least shows that metric largeness properties are not linked to differential–topological properties of manifolds, but only to metric properties of simplicial complexes. Of course this will not prevent us from applying the results of the present section mainly to the classical case of closed manifolds later on.

We remark that a map defined on a connected simplicial complex \( X \) (with the path metric) to a metric space is \( \varepsilon \)-contracting if and only if this holds for the restriction of the map to each simplex in \( X \).

We need to show that Definition 3.1 does not depend on the choice of \( S \). This is in fact the main technical argument in this section and we shall explain it first in detail for enlargeable classes. We start with the following basic extension lemma.

**Lemma 3.2.** Let \( k, n \geq 1 \) be natural numbers, and equip the disc \( D^k \) and its boundary \( \partial D^k \) with fixed but arbitrary piecewise smooth Riemannian metrics (which need not be related). Then there are positive constants \( \delta \) and \( C \), which depend only on the chosen metrics on \( D^k \) and \( \partial D^k \) and on \( n \), so that the following holds. Let \( 0 < \varepsilon < \delta \) and let \( f : \partial D^k \to S^n \) be a piecewise smooth \( \varepsilon \)-contracting map. Then \( f \) can be extended to a piecewise smooth \((C \cdot \varepsilon)\)-contracting map \( D^k \to S^n \).

**Proof.** Since any two piecewise smooth metrics on \( D^k \) are in bi-Lipschitz correspondence, it is enough to treat the case when the metric on \( D^k \) is given in polar coordinates by \( dr^2 + r^2 \cdot g \), where \( g \) is the given piecewise smooth Riemannian metric on \( \partial D^k \).
We can choose a $\delta > 0$ so that, for any $0 < \varepsilon < \delta$ and any $\varepsilon$-contracting piecewise smooth map $f : \partial D^k \to S^n$, the image of $f$ is contained in a closed hemisphere $D \subset S^n$. Using polar coordinates, we identify the hemisphere with the unit ball $D^n \subset \mathbb{R}^n$ by a diffeomorphism $\omega : D^n \to D$. This diffeomorphism is bi-Lipschitz with Lipschitz constants independent of $f$. In particular there is a constant $C$, independent of $\varepsilon$ and $f$, so that $\omega^{-1} \circ f : \partial D^k \to D^n$ is $(C \cdot \varepsilon)$-contracting. This implies that the diameter of the image of $\omega^{-1} \circ f$ is bounded above by $\text{diam}(\partial D^k) \cdot C \cdot \varepsilon$.

Now map the midpoint $P$ of $D^k$ to some point contained in the image of $\omega^{-1} \circ f$ and extend $\omega^{-1} \circ f$ to $D^k$ linearly along the radial lines joining $P$. This extended map is piecewise smooth and using polar coordinates on $D^k$ for computing the lengths of piecewise smooth curves in $D^k$ (and the fact that the metric on $D^k$ has the special form described above), this extended map is $C \cdot \varepsilon$-contracting with a constant $C$ which is independent of $\varepsilon$ and $f$.

Upon composing this map with the Lipschitz map $\omega$, the claim of the lemma follows. \qed

Returning to Definition 3.1, let $S' \subset S$ be a smaller finite connected $\pi_1$-surjective subcomplex which carries $c$. If one of the properties described in Definition 3.1 holds for $S$, then it holds as well for $S'$ by the naturality of $p^T$ and because the lifted inclusion $S' \to \tilde{S}$ is 1-Lipschitz for any connected cover $\tilde{X} \to X$. Now let $S, S' \subset X$ be two finite connected $\pi_1$-surjective subcomplexes carrying $c$. Then there is a third finite connected $\pi_1$-surjective subcomplex $T \subset X$ carrying $c$ and containing $S$ and $S'$. Hence, it remains to show that in Definition 3.1 we may pass from $S$ to a larger finite connected $\pi_1$-surjective subcomplex $T$ of $X$.

Let $\varepsilon > 0$. By assumption there is a connected cover $p : \tilde{X} \to X$ and an almost proper $\varepsilon$-contracting map $f_\varepsilon : \tilde{S} \to S^n$ satisfying $(f_\varepsilon)_*(p^T(c)) \neq 0$. We will show that if $\varepsilon$ is small enough, then $f_\varepsilon$ can be extended to a $(C \cdot \varepsilon)$-contracting almost proper map $\tilde{T} \to S^n$, where $C > 0$ is a constant that depends only on $S$ and $T$, but not on $\varepsilon$.

The proof is by induction on the $k$-skeletal $T^{(k)}$ of $T$, where $0 \leq k \leq \text{dim} T$. However, the start of the induction is a bit involved, because we need to treat the cases $k = 0$ and $k = 1$ simultaneously.

Let us first assume that $T \setminus S$ contains just one vertex $v$. Let $V \subset \tilde{T}$ be the set of lifts of $v$. For each $\tilde{v} \in \tilde{V}$, let $F(\tilde{v}) \subset \tilde{S}$ be the set of all vertices having a common edge with $\tilde{v}$. Note that because $\tilde{T}$ is connected and locally finite, the set $F(\tilde{v})$ is non-empty and finite. Furthermore, $\text{diam} F(\tilde{v})$ (measured with respect to the path metric on $\tilde{S}$) is independent of $\tilde{v} \in \tilde{V}$. Let $F(\tilde{v}) \subset \tilde{S}$ be the subset defined in an analogous fashion as $F(\tilde{v})$ but with $\tilde{S}$ replaced by the universal cover $\tilde{S} \to S$ (and $\tilde{v}$ by a point $\tilde{v} \in \tilde{S}$ over $v$) and set

$$d := \text{diam} F(\tilde{v})$$

measured with respect to the path metric on $\tilde{S}$. Then $d$ is independent of the choice of $\tilde{v}$ and $\varepsilon$, and furthermore,

$$\text{diam} F(\tilde{v}) \leq d.$$

Let $e \subset T$ be a fixed edge connecting $v$ with a vertex in $S$. For $\tilde{v} \in \tilde{V}$, we set $f_\varepsilon(\tilde{v}) := f_\varepsilon(v_1(\tilde{v}(\tilde{v}))),$ where $\tilde{e}(\tilde{v})$ is the unique lift of $e$ containing $\tilde{v}$ and $\tilde{v}_1(\tilde{e}(\tilde{v}))$ is the vertex of this lift different from $\tilde{v}$. The extension $f_\varepsilon : \tilde{S} \cup \{v\} \to S^n$ defined in this way satisfies $d(f_\varepsilon(v_0), f_\varepsilon(v_1)) \leq \max\{d, 1\} \cdot \varepsilon$, if $v_0$ and $v_1$ are the vertices in some 1-simplex of $\tilde{S} \cup T^{(1)} = p^{-1}(S \cup T^{(1)})$. Next, assuming that $\max\{d, 1\} \cdot \varepsilon < \delta_1$, we extend $f_\varepsilon$ to a $(\max\{d, 1\} \cdot C_1 \cdot \varepsilon)$-contracting map $\tilde{S} \cup T^{(1)} \to S^n$ using Lemma 3.2. Here, $\delta_1$ and $C_1$ are given by Lemma 3.2 and depend only on $S$ and $T$.

If $T \setminus S$ contains more than one vertex, we apply this process inductively, where in each induction step, we pick a vertex in $T$ which has a common edge with some vertex in the subcomplex where $f_\varepsilon$ has already been defined (note that in each step, this subcomplex is connected). In this way, we get (for small enough $\varepsilon$) a $(C'_1 \cdot \varepsilon)$-contracting, almost proper
extension $\overline{S \cup T^{(1)}} \to S^n$ of $f_\varepsilon$ where $C'_1 > 0$ is an appropriate constant that just depends on $S$ and $T$.

Now, if, for $k \geq 3$ or $k = 2$, we have, respectively, a $(C'_1 \cdot C_2 \cdot \ldots \cdot C_{k-1} \cdot \varepsilon)$-contracting or $(C'_1 \cdot \varepsilon)$-contracting almost proper extension $\overline{S \cup T^{(k-1)}} \to S^n$ of $f_\varepsilon$, then we extend this map to a $(C'_1 \cdot \ldots \cdot C_k \cdot \varepsilon)$-contracting almost proper map $\overline{S \cup T^{(k)}} \to S^n$. This is possible by Lemma 3.2, if $\varepsilon$ is small enough (where the smallness just depends on $S$ and $T$).

A similar argument works for hypereuclidean and hyperspherical classes.

If we are dealing with coarse largeness properties, then the preceding argument can be replaced by the simple observation that the inclusion $\tilde{S} \to \tilde{T}$ is a coarse equivalence, where $\tilde{S} = p^{-1}(S)$ is the preimage of $S$ under the universal covering map $p : \tilde{X} \to X$ and similarly for $T$.

As a further consequence of our argument we note the following slightly different characterization of large classes which will be important for one of our constructions in the next section.

**Proposition 3.3.** Let $X$ be a connected simplicial complex with finitely generated fundamental group and let $c \in H_n(X; \mathbb{Q})$ be a homology class. Then $c$ is enlargeable, if and only if the following holds. Choose a finite connected $\pi_1$-surjective subcomplex $S \subset X$ carrying $c$. Let $C \subset S$ be a finite subcomplex carrying $c$ which is not necessarily connected or $\pi_1$-surjective. Then, for any $\varepsilon > 0$, there is a connected cover $p : \tilde{X} \to X$ and an $\varepsilon$-contracting almost proper map $\tilde{C} \to S^n$ mapping $p'(c)$ to a non-zero class. Here, $\tilde{C} := p^{-1}(C) \subset \tilde{S} = p^{-1}(S)$ is equipped with the restriction of the canonical path metric on $\tilde{S}$.

**Proof.** It is easy to see that the given property is necessary for the enlargeability of $c$. Conversely, if this condition is satisfied by $C$, then we apply the preceding argument in order to show that it is also satisfied by the larger subcomplex $S$. The only difference is that in the beginning of the induction (that is, for $k = 0, 1$) we work with the restrictions of the path metrics from $\tilde{S}$ to $\tilde{C}$ and from $\tilde{S}$ to $\tilde{C}$ (for the definition of $d$). Here $\tilde{C} := \psi^{-1}(C)$, where $\psi : \tilde{X} \to X$ is the universal cover.

We remark that analogues of Proposition 3.3 hold for (coarsely) hypereuclidean, (coarsely) hyperspherical or macroscopically large classes.

Next we study functorial properties of large homology classes.

**Proposition 3.4.** Let $X$ and $Y$ be connected simplicial complexes with finitely generated fundamental groups and let $\phi : X \to Y$ be a continuous (not necessarily simplicial) map. Then the following implications hold.

(i) If $\phi$ induces a surjection of fundamental groups and $\phi_*(c)$ is enlargeable, then $c$ is enlargeable.

(ii) If $\phi$ induces an isomorphism of fundamental groups and $c$ is enlargeable, then also $\phi_*(c)$ is enlargeable.

If we are dealing with (coarsely) hyperspherical, (coarsely) hypereuclidean or macroscopically large classes and if we assume that $\phi$ induces an isomorphism of fundamental groups, then $c$ is large in the respective sense if and only if $\phi_*(c)$ is.
Proof. First, assume that $\phi_*(c)$ is enlargeable and $\phi$ is surjective on $\pi_1$. Let $S \subset X$ be a finite connected $\pi_1$-surjective subcomplex carrying $c$. Then $\phi(S)$ is contained in a finite connected $\pi_1$-surjective subcomplex $T \subset Y$ which carries $\phi_*(c)$. Since $S$ and $T$ are compact, the map $\phi : S \to T$ is Lipschitz with a Lipschitz constant $\lambda$, say.

Let $\varepsilon > 0$ and choose a connected cover $p_Y : \bar{Y} \to Y$ together with an almost proper $\varepsilon$-contracting map $T \to S^n$ mapping $(p_Y)^1(c)$ to a non-zero class. Let $p_X : \bar{X} \to X$ be the pull-back of this cover under $\phi$. Since $\phi$ is surjective on $\pi_1$, it follows that $\bar{X}$ is connected and we get a map of covering spaces

$$
\begin{array}{ccc}
\bar{S} & \xrightarrow{\tilde{\phi}} & \bar{T} \\
\downarrow_{p_X} & & \downarrow_{p_Y} \\
S & \xrightarrow{\phi} & T
\end{array}
$$

which restricts to a bijection on each fibre. In particular, the map $\tilde{\phi}$ is proper and $\lambda$-Lipschitz (as usual, $\bar{S}$ is equipped with the canonical path metric). Hence, if $f_\varepsilon : T \to S^n$ is an almost proper $\varepsilon$-contracting map, then $f_\varepsilon \circ \phi$ is almost proper, $\lambda \cdot \varepsilon$ contracting and maps $(p_X)^1(c)$ to a non-zero class.

Now assume that $c$ is large and $\phi$ induces an isomorphism of fundamental groups. By the first part of this proof, we can replace $Y$ by a homotopy equivalent complex and hence we may assume that $\phi$ is a simplicial inclusion. Let $S \subset X$ be a finite connected subcomplex carrying $c$. Then $S$ is also a subcomplex of $Y$ and it carries $\phi_*(c)$. Since $\phi$ induces an isomorphism on $\pi_1$, each connected cover of $X$ can be written as the restriction of a connected cover of $Y$. This shows that $\phi_*(c)$ is also enlargeable.

Again, the other largeness properties are treated in a similar manner.

Proposition 3.4 implies the important fact that the large classes form a well-defined subset of $H_\ast(B\Gamma; \mathbb{Q}) = H_\ast(B\pi_1(M); \mathbb{Q})$ for each finitely generated group $\Gamma$ in the following sense: for each simplicial model $X$ of $B\Gamma$, the large classes form a subset of $H_\ast(X; \mathbb{Q})$ and if $X$ and $Y$ are two models of $B\Gamma$ and $X \to Y$ is the (up to homotopy unique) homotopy equivalence inducing the identity on $\pi_1 = \Gamma$, then the induced map in homology identifies the large classes in $H_\ast(X; \mathbb{Q})$ and $H_\ast(Y; \mathbb{Q})$.

The next corollary states homological invariance of classical largeness properties.

**Corollary 3.5.** Let $M$ be a closed oriented manifold of dimension $n$. Then $M$ is large if and only if $\phi_*[M] \in H_n(B\pi_1(M); \mathbb{Q})$ is large.

**Proof.** This is implied by Proposition 3.4, because $M$ is large if and only if $[M] \in H_n(M; \mathbb{Q})$ is large.

The next theorem indicates that the subset of non-large classes should actually be our main concern.

**Theorem 3.6.** Let $X$ be a connected simplicial complex with finitely generated fundamental group. Then the non-large homology classes in $H_\ast(X; \mathbb{Q})$ form a rational vector subspace.
Proof. The class $0 \in H_n(X; \mathbb{Q})$ is not large: we take any connected finite $\pi_1$-surjective subcomplex $S \subset X$. Clearly, $S$ carries $0$. By a direct application of Definition 3.1 it follows that $0$ is not large.

It is obvious that if $c \in H_n(X; \mathbb{Q})$ is not large, then no rational multiple of $c$ is large.

In order to show that the subset of non-large classes is closed under addition, we need to show the following. Let $c, d \in H_n(X; \mathbb{Q})$ and assume that $c + d$ is large. Then one of $c$ and $d$ must also be large. For a proof, let $S \subset X$ be a connected finite $\pi_1$-surjective subcomplex carrying $c$ and $d$. Then $S$ also carries $c + d$. Assume that $c + d$ is enlargeable (the other largeness properties are easier and left to the reader). Let $\varepsilon := 1/k$ for a natural number $k \geq 1$. Since $c + d$ is enlargeable, there is a connected cover $p : \tilde{X} \rightarrow X$ and an almost proper $\varepsilon$-contracting map $f_\varepsilon : \tilde{S} \rightarrow S^n$ mapping $p^!(c + d)$ to a non-zero class in $\tilde{H}_n(S^n; \mathbb{Q})$. This can hold only if either $p^!(c)$ or $p^!(d)$ is mapped to a non-zero class. Hence, for infinitely many $k$, either $p^!(c)$ or $p^!(d)$ is mapped to a non-zero class (for appropriate covers $p : \tilde{X} \rightarrow X$) and consequently either $c$ or $d$ is enlargeable.

Definition 3.7. If one largeness property $P$ is chosen and $X$ is a connected simplicial complex with finitely generated fundamental group, then we denote the rational vector subspace of $H_\ast(X; \mathbb{Q})$ consisting of classes that are not large with respect to $P$ by $H_{\text{sm}}^{\ast}(P)(X; \mathbb{Q})$. This is the small homology of $X$ with respect to $P$ and depends a priori on the given largeness property $P$.

Theorem 3.6 implies that $0 \in H_{\text{sm}}^{n}(P)(B\Gamma; \mathbb{Q})$ for each finitely generated group and each largeness property $P$. Together with Corollary 3.5 this shows that large manifolds are essential.

The results of this section leave it as a central problem to determine $H_{\text{sm}}^{n}(P)(B\Gamma; \mathbb{Q})$ for different largeness properties and finitely generated groups $\Gamma$. This will be the topic of the next section.

4. Examples and applications

The following theorem illustrates the usefulness of our systematic approach to large homology in Section 3.

Theorem 4.1. The small homology of $B\mathbb{Z}^k = T^k$, with $k \geq 1$, is calculated as follows.

(i) If $P$ denotes enlargeability, then we have $H_{\text{sm}}^{n}(P)(T^k; \mathbb{Q}) = 0$ for all $n \geq 1$.

(ii) If $P$ denotes (coarse) hypereuclideaness, (coarse) hypersphericity or macroscopic largeness, then we have

$$H_{\text{sm}}^{n}(P)(T^k; \mathbb{Q}) = \begin{cases} 0 & \text{for } n = k, \\ H_n(T^k; \mathbb{Q}) & \text{for } 1 \leq n < k. \end{cases}$$

Proof. We equip $T^k = \mathbb{R}^k/\mathbb{Z}^k$ with the metric induced from $\mathbb{R}^k$. Let $0 \neq c \in H_n(T^k; \mathbb{Q})$. We write

$$c = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq k} \alpha_{i_1, \ldots, i_n} t_{i_1, \ldots, i_n}.$$
where \( t_{i_1, \ldots, i_n} \in H_n(T^k; \mathbb{Z}) \) is represented by the embedding
\[
T^n \to T^k,
\]
\[(\xi_1, \ldots, \xi_n) \mapsto (1, \ldots, 1, \xi_1, 1, \ldots, 1, \xi_n, 1, \ldots, 1)\]
of the \( n \)-torus into \( T^k = (S^1)^k \), where \( \xi_i \) is put at the \( i \)-th entry, and each \( \alpha_{i_1, \ldots, i_n} \in \mathbb{Q} \). Without loss of generality we may assume that \( \alpha_{i_1, \ldots, i_n} \neq 0 \).

Let \( p : \mathbb{R}^n \times T^{k-n} \to T^k \) be the cover associated to the subgroup \( \mathbb{Z}^{k-n} = 0 \times \mathbb{Z}^{k-n} \subset \mathbb{Z}^k \) given by the last \( k-n \) coordinates. For \( \varepsilon > 0 \) let \( f : \mathbb{R}^n \to S^n \) be an \( \varepsilon \)-contracting almost proper map of non-zero degree. Then the composition
\[
f_{\varepsilon} : \mathbb{R}^n \times T^{n-k} \overset{pr_n}{\to} \mathbb{R}^n \overset{f}{\to} S^n
\]
is \( \varepsilon \)-contracting, almost proper and
\[
(f_{\varepsilon})_* (p^i(t_{i_1, \ldots, i_n})) \neq 0 \in H_n(S^n; \mathbb{Q})
\]
if \( (i_1, \ldots, i_n) = (1, 2, \ldots, n) \) and is zero otherwise. Hence, \( (f_{\varepsilon})_* (p^i(c)) \neq 0 \). This shows that \( c \) is enlargeable.

Concerning the other largeness properties it is clear that \( H_k^{sm}(T^k; \mathbb{Q}) = 0 \), because the universal cover \( \mathbb{R}^k \) of \( T^k \) shares all of the mentioned largeness properties. Furthermore, note that the transfer maps each non-zero class in \( H_k(T^k; \mathbb{Q}) \) to a rational multiple of the locally finite fundamental class of \( \mathbb{R}^k \).

Now let \( 1 \leq n < k \) and let \( c \in H_n(T^k; \mathbb{Q}) \). The full space \( T^k \) carries \( c \) and is \( \pi_1 \)-surjective. However, if \( n < k \), then Poincaré duality implies
\[
H_n^H(\mathbb{R}^k; \mathbb{Q}) \cong H^{k-n}(\mathbb{R}^k; \mathbb{Q}) = 0
\]
so that under the universal cover \( p : \mathbb{R}^k \to T^k \), the transfer \( p^i(c) \in H_n^H(\mathbb{R}^k; \mathbb{Q}) \) is equal to zero. This shows that \( c \) cannot be large. \( \square \)

Combined with Corollary 3.5, this calculation has the following interesting consequence.

**Theorem 4.2.** Let \( M \) be a closed orientable manifold of dimension \( n \) and with a fundamental group \( \mathbb{Z}^k \), where \( 1 \leq n < k \). Then \( M \) is enlargeable if and only if it is essential. However, the universal cover of \( M \) is never (coarsely) hypereuclidean, (coarsely) hyperspherical or macroscopically large.

For \( 4 \leq n < k \) we can construct essential \( n \)-dimensional manifolds \( M \) with a fundamental group \( \mathbb{Z}^k \) as follows. We start with the oriented connected sum
\[
C := T^n \sharp (S^k \times S^{n-1})
\]
of an \( n \)-torus with \( k-n \) copies of \( S^1 \times S^{n-1} \). This manifold has the fundamental group \( \pi_1(C) = \mathbb{Z}^n \ast (*^{k-n} \mathbb{Z}) \), the free product of \( \mathbb{Z}^n \) with \( k-n \) copies of \( \mathbb{Z} \). Let \( C \to B\pi_1(C) \) be the classifying map of the universal cover and consider the composition \( \phi : C \to B\pi_1(C) \to B\mathbb{Z}^k \) induced by the abelianization \( \mathbb{Z}^n \ast (*^{k-n} \mathbb{Z}) \to \mathbb{Z}^k \). Then \( \phi \) sends the fundamental class of \( C \) to a non-zero class in \( H_n(T^k; \mathbb{Z}) \). Moreover, it induces a surjective map of fundamental groups \( \pi_1(C) \to \pi_1(B\mathbb{Z}^k) = \mathbb{Z}^k \) whose kernel can be killed by oriented surgeries in \( C \) (here we use the assumption \( n \geq 4 \)). The resulting manifold \( M \) has the stated properties.

Together with Theorem 4.2 this completes the proof of Theorem 1.4.

We now describe a refined construction to obtain essential manifolds that are not enlargeable. As we have to deal with arbitrary covers of the manifolds in question, we need to recall some facts from covering space theory.
Let $X$ be a path-connected space and let $S \subset X$ be a path-connected subspace. We choose a base point $\tilde{x}$ in the universal cover $\tilde{X}$ of $X$, which lies over $S$, and compute all fundamental groups with respect to $\tilde{x}$ and its images in the different covers of $X$. Let $G = \pi_1(X)$, let $H \subset G$ be a subgroup and let $\lambda : \pi_1(S) \to \pi_1(X)$ be the map induced by the inclusion $S \subset X$. We denote the image of this map by $\Sigma \subset \pi_1(X)$. Let $p : \tilde{X} \to X$ be the cover associated to $H$. Recall that $G$ acts on $\tilde{X}$ from the left in a canonical way.

In view of Proposition 3.3 we collect some information on the components of $\tilde{S} := p^{-1}(S)$. The projection $X \to \tilde{X}$ is denoted by $\psi$. If $g_1, g_2 \in G$, then $g_1 \tilde{x}$ and $g_2 \tilde{x}$ are mapped to the same component of $\tilde{S}$ if and only if there are elements $\sigma \in \Sigma$ and $h \in H$ with $g_1 \sigma = h g_2$. Hence the components of $\tilde{S}$ are in one-to-one correspondence with double cosets $H \backslash G / \Sigma$. By $\Sigma$ be the Higman 4-group $\Sigma$. This is a finitely presented group with no proper subgroups of finite index. Assume the contrary and let $\Sigma$ be a subgroup and let $G$ be a central subgroup of $G$ as a normal subgroup of $G$. We define the following:

Let $\text{Hig} := \langle a, b, c, d | a^{-1} b a = b^2, b^{-1} c b = c^2, c^{-1} d c = d^2, d^{-1} a d = a^2 \rangle$

be the Higman 4-group [21]. This is a finitely presented group with no proper subgroups of finite index. By [1] $\text{Hig}$ is integrally acyclic, that is, $H_*(\text{Hig}; \mathbb{Z}) = 0$. Choose a $z \in \text{Hig}$ of infinite order. We define

$$K := \text{Hig} *_{\langle z \rangle} \text{Hig}$$

as the amalgamated free product of $\text{Hig}$ with itself along $z$. We claim that the group $K$ still does not possess any proper subgroups of finite index. Assume the contrary and let $H < K$ be a proper subgroup of finite index. Then the homomorphism from $K$ to the permutation group of $K/H$ induced by the left translation action of $K$ is non-trivial and has finite image; but then the push-out property of the amalgamated free product implies that $\text{Hig}$ also has a non-trivial homomorphism to a finite group, contradicting the fact that $\text{Hig}$ has no proper subgroups of finite index.

A Mayer–Vietoris argument shows that $\tilde{H}_*(K; \mathbb{Z}) = \tilde{H}_*(S^2; \mathbb{Z})$. Let $c \in H^2(K; \mathbb{Z})$ be a generator and define the following:

(i) the group $L$ by the central extension $1 \to \mathbb{Z} \to L \to K \to 1$ classified by $c$;
(ii) the group $N$ by the central extension $1 \to \mathbb{Z} \to N \to \mathbb{Z}^2 \to 1$ classified by a non-trivial generator of $H^2(\mathbb{Z}^2; \mathbb{Z}) \cong \mathbb{Z}$.

Note that $N$ is the fundamental group of the closed oriented 3-manifold arising as the total space of the $S^1$-bundle over $T^2$ with Euler number 1. This can be regarded as the quotient $\text{Nil}^3 / N$ of the corresponding 3-dimensional nilpotent Lie group by the cocompact lattice $N$.

We consider $\mathbb{Z}$ as a central subgroup of $N \times L$ via the diagonal embedding and finally set

$$G := (N \times L) / \mathbb{Z}.$$ 

We regard in the following:

(i) $N$ as a normal subgroup of $G$ via the inclusion $N = N \times 1 \subset N \times L \to G$;
(ii) $\mathbb{Z}$ as a central subgroup of $G$ via the inclusion $\mathbb{Z} \subset N \subset G$.

The following formulates a key property of $G$. 

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**Theorem 4.3.** Let \( H \subset G \) be a subgroup that maps surjectively onto \( K \) under the canonical map

\[
p : G \rightarrow N/\mathbb{Z} \times L/\mathbb{Z} \rightarrow L/\mathbb{Z} = K.
\]

Then the generator \( e \in \mathbb{Z} \subset G \) belongs to \( H \).

**Proof.** Restricting \( p \) to the subgroup \( H \) leads to an extension

\[
1 \rightarrow F \rightarrow H \rightarrow K \rightarrow 1
\]

where \( F = \ker p|_H \subset N \). Let

\[
\phi : G \rightarrow N/\mathbb{Z} \times L/\mathbb{Z} \rightarrow N/\mathbb{Z} = \mathbb{Z}^2
\]

be the canonical map.

First, assume \( \phi(F) = 0 \). This implies that the canonical map

\[
G \rightarrow N/\mathbb{Z} \times L/\mathbb{Z} = \mathbb{Z}^2 \times K
\]

sends \( H \) to a subgroup \( \bar{H} \subset \mathbb{Z}^2 \times K \) which maps isomorphically to \( K \) under the projection onto the second factor. Since any homomorphism of \( K \) to a finitely generated abelian group is trivial (using the fact that \( K \) has no proper subgroups of finite index), we conclude that the projection of \( \bar{H} \) onto \( \mathbb{Z}^2 \) is trivial. In summary this means that \( 1 \rightarrow F \rightarrow H \rightarrow K \rightarrow 1 \) is a subextension of \( 1 \rightarrow \mathbb{Z} \rightarrow L \rightarrow K \rightarrow 1 \). Since \( e \in H^2(K; \mathbb{Z}) \) is indivisible, this implies that \( F = \mathbb{Z} \) and hence \( e \in H \), as desired.

Now we consider the case \( \text{rk} \phi(F) > 0 \). Let

\[
\psi : H \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}
\]

be the composition of \( \phi|_H \) with one of the projections \( \mathbb{Z}^2 \rightarrow \mathbb{Z} \) so that \( \psi(F) \neq 0 \). Then \( \psi(F) \) is a finite index subgroup of \( \psi(H) \). Hence \( H' := \psi^{-1}(\psi(F)) \subset H \) still maps surjectively to \( K \), as \( K \) has no proper subgroups of finite index. Since \( \psi(H') = \psi(F) \) and \( F = \ker p|_H \), this implies that \( \ker \psi \subset H \) also maps surjectively to \( K \). Repeating this argument, if necessary, this shows that we can assume \( \phi(F) = 0 \) and we are in the case treated before.

**Lemma 4.4.** We have \( H_3(N; \mathbb{Z}) \cong \mathbb{Z} \). Moreover, the inclusion \( N \rightarrow G \) induces an isomorphism \( H_3(N; \mathbb{Z}) \cong H_3(G; \mathbb{Z}) \).

**Proof.** The 3-dimensional closed oriented manifold \( \text{Nil}^3/N \) is a model for \( BN \). This implies the first assertion.

The homological spectral sequence for the central extension

\[
1 \rightarrow \mathbb{Z} \rightarrow L \rightarrow K \rightarrow 1
\]

shows that \( H_*(L; \mathbb{Z}) \cong \mathbb{Z} \) in degree 0 and 3, and \( H_*(L; \mathbb{Z}) = 0 \) otherwise. With this information we conclude that the spectral sequence for the normal extension

\[
1 \rightarrow L \rightarrow G \rightarrow N/\mathbb{Z} \rightarrow 1
\]

collapses at the \( E^2 \)-level (recall that \( N/\mathbb{Z} = \mathbb{Z}^2 \)). Since the induced action of \( N/\mathbb{Z} \) on \( L \) is trivial, this implies that \( H_*(G; \mathbb{Z}) \) is free of rank \( 1, 2, 1, 1, 2, 1, 0, \ldots \) in degrees \( 0, 1, 2, 3, 4, 5, 6, \ldots \). With this information we go into the spectral sequence for the normal extension

\[
1 \rightarrow N \rightarrow G \rightarrow L/\mathbb{Z} \rightarrow 1
\]
and conclude (using that the induced action of \( L/Z \) on \( N \) is trivial) that on the \( E^2 \)-level the differential

\[ \partial^2 : E^2_{2,1} \to E^2_{0,2} \]

is an isomorphism (of free abelian groups of rank 2) and all other differentials (on \( E^2 \) or on higher levels) are zero. This implies that \( E^3_{1,3} = H_0(L/Z; H_3(N; \mathbb{Z})) \cong \mathbb{Z} \) cannot be hit by a differential and hence the assertion of Lemma 4.4. \( \square \)

**Theorem 4.5.** The homology group \( H_3(G; \mathbb{Q}) \) (which is different from zero by Lemma 4.4) consists only of non-enlargeable classes.

Before we go into the proof, we deal with the following lemma concerning the manifold \( C := \text{Nil}^3 / N \) considered before. We fix a base point \( c \in C \).

**Lemma 4.6.** Let \( C \) be equipped with some Riemannian metric. Then there is an \( \varepsilon > 0 \) so that the following holds. Let \( P \to C \) be a connected cover that is equipped with the lifted metric and let \( p \in P \) be a point over \( c \). Then an \( \varepsilon \)-contracting almost proper map \( P \to S^3 \) of non-zero degree can only exist if the image of \( \pi_1(P, p) \) in \( \pi_1(C, c) \) does not contain \( e \in \mathbb{Z} \subset N = \pi_1(C, c) \).

**Proof.** Let \( P \to C \) be a connected cover so that \( \pi_1(P, p) \), considered as a subgroup of \( \pi_1(C, c) \), contains the generator \( e \in N \). Then \( P \) can be written as the total space of the pull-back of the bundle \( S^1 \to C \to T^2 \) along a covering map \( \phi : V \to T^2 \).

Let \( D^2 \to D(C) \to T^2 \) be the disc bundle of \( S^1 \to C \to T^2 \) and extend the given Riemannian metric on \( C \) to \( D(C) \). Let \( D^2 \to D(P) \to V \) be the disc bundle of \( S^1 \to P \to V \) equipped with the pull-back metric from \( D(C) \).

There is a relative CW-structure on the pair \( (D(C), C) \) with finitely many cells attached along piecewise smooth maps. Hence the Riemannian manifold \( D(P) \) can be obtained from \( P \) by attaching lifts of the same cells equipped with Riemannian metrics induced from \( D(C) \).

Using Lemma 3.2 inductively over the cells in this relative CW-decomposition of \( (D(P), P) \), we find an \( \varepsilon > 0 \), which is independent of \( \phi \) (that is, of the specific cover \( P \)), so that any almost proper \( \varepsilon \)-contracting map \( P \to S^3 \) can be extended to an almost proper map \( D(P) \to S^3 \). This implies that any \( \varepsilon \)-contracting almost proper map \( P \to S^3 \) must be of zero degree. \( \square \)

**Proof of Theorem 4.5.** We write \( X := BG \). Let \( \text{Nil}^3 / N = BN \to X \) be the map induced by the inclusion \( N \hookrightarrow G \). We assume that \( BN \) is a finite simplicial complex and that this map is an inclusion of simplicial complexes. By Lemma 4.4 it is enough to show that the image \( c \in H_3(X; \mathbb{Q}) \) of the fundamental class \([BN]\) is not enlargeable. Our proof is based on Proposition 3.3. Choose a finite connected \( \pi_1 \)-surjective subcomplex \( S \subset X \) carrying \( c \). We can assume that \( C := BN \) is contained in \( S \). Note that \( C \) is not \( \pi_1 \)-surjective. Since \( G \) is finitely presented, we can and will assume that the inclusion \( S \subset X \) induces an isomorphism on \( \pi_1 \).

Using Lemma 4.6, there is a constant \( \varepsilon > 0 \) with the following property. If \( P \to C \) is a connected cover and \( P \) is equipped with a metric that is dominated by the simplicial path metric, then an \( \varepsilon \)-contracting almost proper map \( P \to S^3 \) of non-zero degree can only exist if the fundamental group of \( P \) does not contain \( e \in \mathbb{Z} \subset N = \pi_1(C) \). We fix an \( \varepsilon \) with this property and (in view of Proposition 3.3) assume that we are given a connected cover \( p : \tilde{X} \to X \) and an almost proper \( \varepsilon \)-contracting map

\[ f_\varepsilon : \tilde{C} \to S^3 \]
so that \((f_\varepsilon)_*(p|_C)^\dagger(c)) \neq 0\), where \(C := p^{-1}(C)\) is equipped with the metric induced from \(\overline{S} := p^{-1}(S)\). We can choose a component \(P \subset \overline{C}\) so that \((f_\varepsilon)_*(p|_P)^\dagger(c)) \neq 0\).

Our aim is to show that \(f_\varepsilon\) has non-zero degree on infinitely many components of \(C\) and therefore cannot be almost proper, which contradicts our assumptions. Roughly speaking, this is implied by the fact that \(\overline{C}\) contains a string of infinitely many components ‘parallel’ to \(P\).

Let \(\hat{x} \in \hat{X}\) be a base point lying over \(P\) and set \(H := p_*\pi_1(\hat{X}) \subset G\), where we choose images of \(\hat{x}\) as the base points of \(\tilde{X}\) and of \(X\). If under the projection \(\phi : G \rightarrow K\) the subgroup \(H\) maps surjectively to \(K\), then it follows from Theorem 4.3 that \(e \in N \cap H = \pi_1(P)\). By the choice of \(\varepsilon\), this is impossible and hence \(H\) cannot map surjectively onto \(K\). Since \(K\) has no proper subgroup of finite index, we conclude \([K : \phi(H)] = \infty\). The set of double cosets \(H \setminus G/N\) is mapped by \(\phi\) to the set of cosets \(\phi(H) \setminus K\) and is hence infinite. Choose a finite set \(\Sigma \subset L \subset G\) mapping to a set of generators of \(K\). Since \([K : \phi(H)] = \infty\), there is a sequence \((\sigma_n)_{n \in \mathbb{N}}\) of elements in \(\Sigma\) so that the set \(\{\sigma_1\sigma_2\ldots\sigma_k \mid k \in \mathbb{N} \setminus \{0\}\}\) maps to an infinite set of cosets in \(\phi(H) \setminus K\).

Now all elements in \(\Sigma\) commute with all elements in \(N\). However, the elements of \(\Sigma\) do not act on \(\overline{S}\), because \(\Sigma\) need not be in the normalizer of \(H = \pi_1(\overline{S})\); but certainly each \(\sigma \in \Sigma\) acts as a deck transformation on the universal cover \(\psi : \hat{S} \rightarrow S\). We set \(\overline{C} := \psi^{-1}(C) \subset \hat{S}\) and denote by \(\hat{P} \subset \overline{C}\) the component containing \(\hat{x}\). Note that \(\hat{P} \rightarrow C\) is a universal covering. Let \(\tau_k : \hat{S} \rightarrow \hat{S}\) be the deck transformation associated to \(\sigma_1\sigma_2\ldots\sigma_k\) and set \(\tau_0 := \text{id}_\hat{S}\). Since \(\Sigma\) is finite and \(G\) acts isometrically on \(\hat{S}\), there is a positive number \(\Delta\) so that

\[
d_S(\tau_k(x), \tau_{k+1}(x)) = d_S(\tau_k(x), \tau_k(\sigma_{k+1}(x))) = d_S(x, \sigma_{k+1}(x)) \leq \Delta
\]

for all \(x \in \hat{P}\) and all \(k \in \mathbb{N}\). For the last inequality we use the facts that the actions of \(N\) and \(L\) commute and that \(\hat{P}\) is invariant under the restricted action of \(N\) with a compact quotient \(C\) (and hence with compact fundamental domain). Note that \(\Delta\) is independent of \(H\) and of \(\varepsilon\). We shall henceforth assume that \(\varepsilon\) has been chosen so small that \(\varepsilon \cdot \Delta < \pi/2\).

Let \(q : S' \rightarrow S\) be the cover associated to the subgroup \(N \cap H \subset H\). Under the covering map \(r : S' \rightarrow \hat{S}\) (recall that the cover \(\hat{S} \rightarrow S\) was associated to \(H \subset G\)), each component of \(C' := q^{-1}(C)\) maps bijectively to a component of \(\overline{C}\) by a map of Lipschitz constant 1 (here we use the metrics from \(S'\) and \(\hat{S}\), respectively). Let \(P' \subset C'\) be the component of the image of \(\hat{x}\). Now each \(\sigma \in \Sigma\) acts on the covering \(S' \rightarrow S\), because \(\Sigma\) is contained in the centralizer of \(N \cap H\). Furthermore, we know that \(d_S(\tau_k(x), \tau_{k+1}(x)) \leq \Delta\) for all \(x \in P'\) and all \(k \in \mathbb{N}\) (this uses the fact that the projection \(\hat{S} \rightarrow S\) has a Lipschitz constant 1).

The compositions

\[
\tau_k(P') \xrightarrow{r} \hat{S} \xrightarrow{f_\varepsilon} S^3
\]

for increasing \(k\) have the same degree, different from zero, because for adjacent \(k \in \mathbb{N}\) the respective maps are \(\pi/2\)-close to each other (after identifying \(\tau_k(P') = P' = \tau_{k+1}(P')\)) and for \(k = 0\) we see the map \(P' \rightarrow P \rightarrow S^3\) which has non-zero degree by assumption. By the choice of the sequence \((\sigma_n)\), this means that \(f_\varepsilon\) is non-constant on infinitely many components of \(\overline{C}\) and therefore not almost proper. 

Before we can prove Theorem 1.5, we need the following extension of Theorem 4.3.

**Proposition 4.7.** Let \(G = (N \times L)/\mathbb{Z}\) be the group considered in Theorem 4.3, let \(k \in \mathbb{N}\) and let

\[
H \subset G \times \mathbb{Z}^k
\]

be a subgroup which maps surjectively onto \(K\) under the canonical map

\[
p : G \times \mathbb{Z}^k \longrightarrow (N/\mathbb{Z} \times L/\mathbb{Z}) \times \mathbb{Z}^k \longrightarrow L/\mathbb{Z} = K.
\]
Then \((e, 0, \ldots, 0) \in H\), where \(e\) is a generator of \(Z \subset N \subset G\) as before.

**Proof.** We adapt the proof of Theorem 4.3 accordingly. We set \(F := \ker p|_H\) and distinguish two cases for \(\phi(F) \subset \mathbb{Z}^2 \times \mathbb{Z}^k\), where

\[
\phi : G \times \mathbb{Z}^k \to (N/\mathbb{Z} \times L/\mathbb{Z}) \times \mathbb{Z}^k \to N/\mathbb{Z} \times \mathbb{Z}^k = \mathbb{Z}^2 \times \mathbb{Z}^k
\]

is the canonical map.

If \(\phi(F) = 0\), then we conclude similarly as before that the image of \(H\) in \((N/\mathbb{Z} \times L/\mathbb{Z}) \times \mathbb{Z}^k = (\mathbb{Z}^2 \times K) \times \mathbb{Z}^k\) projects onto the trivial subgroup in \(\mathbb{Z}^2 \times \mathbb{Z}^k\). Hence we can again regard 1 \(\to F \to H \to K \to 1\) as a subextension of 1 \(\to \mathbb{Z} \to L \to K \to 1\) and the indivisibility of the class \(c \in H^2(K; \mathbb{Z})\) implies \(F = \mathbb{Z} \times 0 \subset N \times \mathbb{Z}^k\), as desired.

If \(\phi(F) \neq 0\), then we consider one of the compositions

\[
\psi : H \xrightarrow{\phi} \mathbb{Z}^2 \times \mathbb{Z}^k \to \mathbb{Z}
\]

with \(\psi(F) \neq 0\). Arguing as before, the subgroup \(\ker \psi \subset H\) still maps surjectively onto \(K\). Repeating this argument, we are reduced to the case \(\phi(F) = 0\).

**Theorem 4.8.** For each \(k \geq 0\) the class \([\text{Nil}^3/\mathbb{Z}^k \times [T^k] \in H_{3+k}(B(G \times \mathbb{Z}^k); \mathbb{Q})\) is different from zero and non-enlargeable.

**Proof.** The first assertion follows from Lemma 4.4. The proof of the second assertion is analogous to that of Theorem 4.5. The main difference is that \(X\) is replaced by \(B(G \times \mathbb{Z}^k) = BG \times T^k\), the subgroup \(N \subset G\) by \(N \times \mathbb{Z}^k \subset G \times \mathbb{Z}^k\) and \(C\) by the subcomplex \(\text{Nil}^3/\mathbb{Z}^k \times T^k \subset BG \times T^k = X\). In this context, Lemma 4.6, with \(e \in \mathbb{Z} \subset N\) replaced by \((e, 0, \ldots, 0) \in \mathbb{Z} \times \mathbb{Z}^k \subset N \times \mathbb{Z}^k\) and \(S^3\) replaced by \(S^{3+k}\), remains valid so that Proposition 4.7 allows us to adopt the proof of Theorem 4.5.

For the proof of Theorem 1.5, let \(n \geq 4\). We can construct an essential \(n\)-dimensional manifold \(M\) with a fundamental group \(G \times \mathbb{Z}^{n-3}\) by first representing the homology class \([\text{Nil}^3/\mathbb{Z}^k \times [T^{n-3}] \in H_n(B(G \times \mathbb{Z}^{n-3}); \mathbb{Q})\) as \(\phi_*[M]\) with the closed oriented manifold \(M^n = \text{Nil}^3/\mathbb{Z}^k \times T^{n-3}\) and the inclusion \(\phi : M \to B(G \times \mathbb{Z}^{n-3})\) and then improving the map \(\phi\) by surgery so that it induces an isomorphism on \(\pi_1\) (without changing \(\phi_*[M]\)). This is possible by the definition of \(G\) and because \(L\) is finitely presented. Theorem 4.8 says that the class \([\text{Nil}^3/\mathbb{Z}^k \times [T^{n-3}]\) is not enlargeable. Now the claim follows from Corollary 3.5.

5. **Higher enlargeability implies essentialness**

This section contains the proof of Theorem 1.6. We argue by induction, where the inductive argument relies on a bound, shown by Gromov, on the radius of a submanifold by its volume.

The notion of \(k\)-dilation extends in an obvious way to piecewise smooth maps of simplicial complexes with piecewise smooth Riemannian metrics. For triangulated Riemannian manifolds this new definition coincides with Definition 2.1. Moreover, the inequality

\[
\text{dil}_l(f)^{1/l} \leq \text{dil}_k(f)^{1/k}
\]

for \(l \geq k\) remains valid.

**Definition 5.1.** Let \(X\) be a connected finite simplicial complex equipped with smooth Riemannian metrics on each of its simplices. Let \(c \in H_n(X; \mathbb{Q})\) be a homology class. The class
$c$ is called $k$-enlargeable, if for every $\varepsilon > 0$ there exists a connected cover $p : \tilde{X}_\varepsilon \to X$ and a piecewise smooth almost proper map $f_\varepsilon : \tilde{X}_\varepsilon \to S^n$ with $k$-dilation at most $\varepsilon$ and satisfying $(f_\varepsilon)_*(p^!(c)) \neq 0$.

Contrary to Definition 3.1 we allow metrics on the simplices in $X$ that are different from the standard metrics. This flexibility is convenient for the following argument. By the compactness of $X$, Definition 5.1 is independent of the choice of the Riemannian metric on the simplices of $X$.

**Proposition 5.2.** Let $X$ be a connected finite simplicial complex and let $l \leq n$ be positive integers. Let $c \in H_n(X; \mathbb{Q})$ be $l$-enlargeable. Let $X'$ be obtained from $X$ by attaching finitely many $(l+1)$-cells (that is, simplicial $(l+1)$-balls) to the $l$-skeleton $X^{(l)}$ of $X$. Then the image of $c$ in $H_n(X'; \mathbb{Q})$ is $(l+1)$-enlargeable.

Note that in contrast to the extension Lemma 3.2 (which corresponds to the case $l=1$ in Proposition 5.2) and the subsequent argument, we must, at least for $l \geq 2$, pass from the $l$-dilation to the $(l+1)$-dilation upon attaching $(l+1)$-cells.

In the proof we will need the following lemma.

**Lemma 5.3.** There exists a constant $C_n > 0$ depending only on $n$ such that, for any piecewise smooth map $f : N \to S^n$ from an $l$-dimensional manifold $N$ with $l < n$ to the unit $n$-sphere, there is a piecewise smooth map $f' : N \to S^n$ to an $(l-1)$-dimensional subcomplex of $S^n$ such that $d(f(x), f'(x)) \leq C_n \cdot \text{Vol}(f(N))^{1/l}$ for all $x \in N$.

Gromov [10, Proposition 3.1.A] showed a similar statement for the euclidean space instead of the unit sphere. The above lemma may be proved in an analogous manner or can easily be deduced from Gromov’s result.

**Proof of Proposition 5.2.** Let $g$ be a Riemannian metric on $X$. Since the attached cells do not interfere with each other, we may assume that there is only one $(l+1)$-cell to attach. Let $h : S^l \to X$ be the (simplicial) attaching map. Choose a lift $\tilde{h} : S^l \to \tilde{X}$ to the universal covering, and denote the $l$-dimensional volume of the image $\tilde{h}(S^l)$ by $v$.

First assume $l < n$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $C_n \cdot v^{1/l} \cdot \delta^{(l+1)/l} \leq \varepsilon$ and such that $\delta^{(l+1)/l} \leq \varepsilon$. Let $f_\delta : \tilde{X}_\delta \to S^n$ be almost proper with $(f_\delta)_*(p^!(c)) \neq 0$ and $\text{dil}_l(f_\delta) \leq \delta$.

Note that

$$\text{dil}_{l+1}(f_\delta) \leq \text{dil}_l(f_\delta)^{(l+1)/l} \leq \delta^{(l+1)/l} \leq \varepsilon.$$ 

We extend the given Riemannian metric $g$ on $X$ over the attached $(l+1)$-cell as follows. We think of $X'$ as

$$X \bigcup_{h}(S^l \times [-1, 0]) \bigcup_{h}(S^l \times [0, 1]) \bigcup S^l_{+1}$$

and define the metric by taking

$$g, (-th^* g + (1 + t)g_r) + dt^2, g_r + dt^2, g_r$$

on the respective parts. Here $g_r$ denotes the round metric of radius $r$ both on $S^l$ and on the hemisphere $S^l_{+1}$, with $r$ chosen so that $h : (S^l, g_r) \to (X, g)$ is 1-contracting.

Moreover, we extend $f_\delta$ over the attached cells as follows: on $S^l \times [-1, 0]$ we use the projection to $S^l$ and apply the composition $f_\delta \circ \tilde{h}$, where $\tilde{h} : S^l \to \tilde{X}_\delta$ is an appropriate lift of $h$. The
The cylinder lines \( \{ x \} \times [0,1] \) are mapped to minimizing geodesics from \( f_\delta \circ h(x) \) to \( f'(x) \). For small enough \( \delta \) this map is well defined. The remaining cap \( S^l_{x+1} \) may be regarded as the cone over \( S^l \times \{1\} \) and is mapped to some cone over the \((l-1)\)-dimensional subcomplex to which \( S^l \times \{1\} \) is mapped.

By the choice of \( \delta \), this new map has \((l+1)\)-dilation at most \( \varepsilon \) on \( S^l \times [-1,0] \) and the cap \( S^l_{x+1} \), because they are mapped to \( l\)-dimensional subcomplexes, which are zero sets for the \((l+1)\)-dimensional volume, and on \( S^l \times [0,1] \) because the \( l\)-dimensional volume of the first factor is multiplied by a factor of at most \( \delta \) and the second factor is \( C_n(\delta \cdot v)^{1/l} \)-contracted. Hence the image of \( c \) in \( H_n(X';\mathbb{Q}) \) is \((l+1)\)-enlargeable.

Finally assume \( l = n \). For any \( \varepsilon > 0 \) satisfying \( \varepsilon v < \text{Vol}_n(S^n) \) the composition with \( f_\varepsilon \) of any lift \( \bar{h} \) of the attaching map is not surjective. Hence, \( f_\varepsilon \circ \bar{h} \) is null-homotopic and we may extend \( f_\varepsilon \) over \( X_v \) with the new cells attached. Since \( S^n \) is \( n \)-dimensional, every map to it has zero \((n+1)\)-dilation. \( \square \)

Next, we will show Theorem 1.6 by an inductive argument. In this proof, Proposition 5.2 will serve as the induction step.

**Proof of Theorem 1.6.** Let \( M \) be \( k \)-enlargeable and let \( \pi_i(M) \) be trivial for \( 2 \leq i \leq k-1 \). Then it is possible to construct \( B\pi_1(M) \) from \( M \) by attaching only cells of dimension at least \( k+1 \). We may assume that the image of the attaching map of every \( l \)-cell lies in the \((l-1)\)-skeleton.

If \( M \) is not essential, then there is a finite subcomplex \( X \subset B\pi_1(M) \) containing \( M \) such that \([M] = 0 \) in \( H_n(X;\mathbb{Q}) \). We may assume that \( X \) is of dimension \( n+1 \). Then by an induction using Proposition 5.2, the class \([M] \in H_n(X;\mathbb{Q}) \) is \((n+1)\)-enlargeable. However, this contradicts the fact that \([M] \) vanishes in \( H_n(X;\mathbb{Q}) \). Therefore \( M \) has to be essential. \( \square \)

**Acknowledgements.** The first author would like to thank his thesis advisor D. Kotschick for continuous support and encouragement. Both authors gratefully acknowledge the useful comments by D. Kotschick concerning a preliminary version of this paper and numerous helpful remarks by the referee.

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