Intrinsic Stochastic Differential Equation on Manifolds using Regular Lagrangian.

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Abstract

A general way of representing Intrinsic Stochastic Differential Equations (SDEs) on manifolds is based on Schwartz morphism. Hence, construction of Schwartz morphism is of importance. In this manuscript we show that it is possible to construct Schwartz morphism from \( \mathbb{R}^{p+1} \) to \( M \) using a special map that we call as diffusion generator. We show that one of the ways of constructing the diffusion generator is by using regular Lagrangian.

Keywords: Stochastic Differential Geometry, Stochastic Differential Equations on Manifolds, Ito Stochastic Differential Equations on Manifolds, Schwartz Stochastic Differential Equations, Schwartz second order geometry.

1 Introduction

Stochastic Differential Equation (SDE) evolving on linear spaces is a well studied subject. Some of the popular books on this subject are [2, 9]. On manifolds, however, the subject of SDEs is an active research area. Ever since Kiyosi Ito first described the coordinate transformation rules on manifolds, the subject has evolved and taken a form of what is now broadly known as Stochastic Differential Geometry. In linear spaces, Stratonovich SDE representation and Ito SDE representation are two popular ways of representing semi-martingale in form of SDEs. It is natural that there will be equivalent ways of describing SDEs on manifold. In case of Stratonovich SDEs, it is enough to consider sections of tangent bundle (vector fields) to describe the drift and the noise coefficients. However, similar statement cannot be made for Ito type SDE on manifolds due to the additional drift correction term. To address this problem Laurent Schwartz, in [11], introduced the idea of the second order tangent bundle. It is because of this special construction that the study of Stochastic Differential Equations on manifolds gets a special name of Stochastic Differential Geometry. A complete account of Schwartz’s second order geometry can be found in [5]. One of the central ideas in Schwartz’s Stochastic Differential Geometry is that the stochastic differential is considered as an infinitesimal element of Schwartz’s second order tangent space or what we will call as diffusion space. These stochastic differentials are also called Intrinsic differentials or Schwartz differentials.

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∗In order to avoid confusing Schwartz’s second order tangent bundle with the usual second tangent bundle \( TT M \), we will call Schwartz’s second order tangent bundle as diffusion bundle.
On the other hand, Ito Stochastic Differential on manifolds can be developed independently using the idea of Ito-Bundle presented in Yuri Gliklikh’s book [3]. As per this approach, if a manifold is equipped with a connection, then it is possible to describe an Ito SDE on the manifold. The highlight of the book is the description of Ito Equation on manifolds using in Belopolskya-Daletskii form (section 7.3 of [6]), which can be exploited for numerical computations. Yet another approach is that of Stochastic development and anti-development, mainly due to David Elworthy et.al., that can be found in chapter 2 of [7] or in [4]. However, we do not consider any of these approach here.

We consider the Schwartz’s approach or the so called Intrinsic SDEs. The description of Intrinsic SDEs on manifolds depend on Schwartz morphism that can morph semi-martingales from source manifold to a semi-martingale on the target manifold. If we consider the source manifold as \( \mathbb{R}^p \) with \( (Y_t) \) as a semi-martingale on \( \mathbb{R}^p \), then the Schwartz morphism will morph a semi-martingale \( Y_t \in \mathbb{R}^p \) into a semi-martingale on some target manifold, \( M \). However, the problem remains in the construction of the Schwartz morphism. Schwartz addressed this problem by constructing the Schwartz morphism using a map \( F: \mathbb{R}^p \to M \) that gives the semi-martingale \( F(Y_t) \). An approach based on 2-jets is presented in presented in [1], in which the authors have constructed the Schwartz morphism using functions of type \( F: \mathbb{R}^p \times M \to M \). Although, the authors have shown that their approach is useful in numerical computations, we feel that there is a scope of having yet another approach of constructing Schwartz morphism that not only helps in numerical computation but also make coordinate invariant way of analysis easier. In this manuscript we present a way of constructing a Schwartz morphism that morphs the process \( (t, W_t) \in \mathbb{R}^{p+1} \) into a semi-martingale on \( M \). In particular, our focus is on construction of Schwartz morphism using regular Lagrangians.

In section 3 we consider a special class of fiber preserving maps from the tangent bundle to the diffusion bundle that can be used to construct a Schwartz morphism. We call this special fiber preserving map as diffusion generator. Our main result, in section 4 is that of construction of Schwartz morphism using regular Lagrangian. In section 5 we explore the possibility of converting an Intrinsic SDE into an equivalent Belopolskya-Daletskii form so as to exploit it for numerical computations. Some of the key elements of Schwartz’s stochastic differential geometry are explained in section 2.

2  Prerequisites

This section presents some elementary definitions and notations that will be used through out the article. Some of the well known ideas and results in the theory of Stochastic Differential Equations on Manifolds are also presented. Readers are encouraged to refer [3] for complete account of Schwartz’s second order geometry and its application to stochastic analysis. We recommend that the readers who are familiar with Schwartz’s second order geometry skip this section after going through the basic notations and definition in the next section.

2.1  Basic definitions and notations.

We will denote the set of all sections of any fiber bundle \( F \) by \( \Gamma(F) \). The set of all smooth vector fields will be denoted by \( \mathfrak{X}(M) \) and the set of all smooth function by \( \mathcal{F}(M) \). A second order tangent space at a point \( x \) on an n-manifold \( M \) is defined as a vector space of all differential operators of upto order 2. We will denote it as \( \mathcal{D}_x M \). Locally, every second order differential operator is symmetric and is represented as \( \partial_i^2 \). Therefore, every differential operator upto second order is locally of the form \( a^i \partial_i + b^{ij} \partial_i \partial_j \). Symmetry of the second order differential operators means that the dimension of the second order tangent space is \( n + (1/2)n(n+1) \). We will call the elements of \( \mathcal{D}_x M \) as diffusors at point \( x \in M \) and \( \mathcal{D}_x M \) as a diffusion space. With this definitions, it is clear that a tangent vector is also a diffusor i.e. \( T_x M \subset \mathcal{D}_x M \forall x \in M \).

For any manifolds \( M \) and \( N \), consider \( L \in \mathcal{D}_x M \); if \( \phi: M \to N \), then the push forward of \( L \) by \( \phi \) at
a specific point \( x \in M \) is written as \( \mathcal{D}_x \phi (L) \) such that \( \mathcal{D}_x \phi : \mathcal{D}_x M \to \mathcal{D}_{\phi(x)} N \). Moreover, \( \forall f \in \mathcal{F}(N), \ \mathcal{D}_x \phi (L)[f] = L[f(\phi)] = L[\phi^* f] \). This pushforward map is linear. The vector bundle over the manifold \( M \), with diffusion space \( \mathcal{D}_x M \) as the fibers, will be called **diffusion bundle**, \( \mathcal{D} M \). Dually, if the fiber at point \( x \) is the codiffusion space \( \mathcal{D}^*_x M \), then the resulting vector bundle will be called **codiffusion bundle**, \( \mathcal{D}^* M \). A smooth **diffusor field** \( \zeta \) is defined as a smooth section of the diffusion bundle \( \mathcal{D} M \). Following our usual symbol for section of a fiber bundle, the set of all smooth diffusor fields on a manifold \( M \) will be denoted by \( \Gamma(\mathcal{D} M) \). Similarly, a smooth **codiffusor field** \( \chi \) is defined as a smooth section of \( \mathcal{D}^* M \) and the set of all smooth codiffusor fields on a manifold \( M \) will be denoted by \( \Gamma(\mathcal{D}^* M) \). For \( \phi : M \to N \), we will call the fiber preserving map over \( \phi \), \( \mathcal{D}\phi : \mathcal{D} M \to \mathcal{D} N \) as **diffusion map**. Locally in charts \((U, \psi)\) on \( M \) and \((V, \chi)\) on \( N \), for all \( L \in \mathcal{D} M \) such that \( L|_U = a^i \partial_i + b^i \partial^i \),

\[
\mathcal{D}\phi (L)|_V = [a^i \partial_i \phi^k + b^i \partial^i \partial_\phi \phi^k] \partial_k + [b^i \partial_i \phi^k \partial_j \phi^l \partial^2]_{ij} .
\]

(1)

Given \( L \in \mathcal{D}_x M \), consider a symmetric contravariant tensor \( \hat{L} \in T^2_M \) such that

\[
\hat{L}(df, dg) = \frac{1}{2} (L[fg] - fL[g] - gL[f]).
\]

(2)

The fact that \( \hat{L} \) is indeed symmetric can be verified locally by considering \( L = a^i \partial_i + b^i \partial^i \). So, locally

\[
\hat{L}(df, dg) = b^i \partial_i f \partial_j g.
\]

(3)

In other words, \( \hat{L} \) can be interpreted as the symmetric part of the diffusor \( L \).

### 2.2 Schwartz’s second order stochastic differential geometry

A stochastic process \( X_t \) on a manifold \( M \) is said to be a **semimartingale** if \( f(X_t) \) is a semi-martingale \( \forall f \in \mathcal{F}(M) \). Let \( X_t \) be a continuous semi-martingale on manifold \( M \). If \( X^i_t \) are the local components of \( X_t \) in some chart, then the local Ito differentials \( dX^i_t \) and \( \frac{1}{2} d[ X^i_t, X^j_t ] \) can be taken as coefficients to construct an infinitesimal diffusor

\[
dX_t = (dX^i_t) \partial_i + \left( \frac{1}{2} d[ X^i_t, X^j_t ] \right) \partial^2_{ij}.
\]

(4)

The diffusor \( dX_t \) is known as the **Schwartz differential** of \( X_t \).

Consider two manifolds \( M \) and \( N \) with \( x \in M \) and \( y \in N \). If there exists a linear map \( J(x,y) : \mathcal{D}_x M \to \mathcal{D}_y N \) such that \( \text{Img}(J|_{T_x M}) \subset T_y N \) and \( J\hat{L} = (J|_{T_x M} \otimes J|_{T_x M}) \hat{L} \), then such a map \( J \) is called a **Schwartz morphism**. If there exists a linear map \( S_{(x,y)} : T_x M \to T_y N \), then such a map is called a **Stratonovich operator**.

The following two theorems are due to M. Emery and the proofs can be found in his book [5].

**Theorem 2.1.** If \( \phi : M \to N \) is a smooth map, then the diffusion map \( \mathcal{D}_x \phi : \mathcal{D}_x M \to \mathcal{D}_{\phi(x)} N \) is a Schwartz operator from point \( x \to \phi(x) \). Moreover, if \( U_t \) is a semi-martingale on \( M \), then the semi-martingale \( \phi(U_t) \) on \( N \) is given by the solution of the Schwartz Stochastic Differential Equation (SDE),

\[
dX_t = \mathcal{D}U_t \phi(\mathcal{D}U_t).
\]

(5)

In other words, the Schwartz differential \( d(\phi(U_t)) \) is obtained by the push forward of the Schwartz differential \( dU_t \) by \( \phi \); i.e. \( d(\phi(U_t)) = \mathcal{D}U_t \phi(\mathcal{D}U_t) \).
Theorem 2.2. For every Stratonovich operator \( S(x, y) \), there exists a unique Schwartz operator \( J(x, y) \), such that the Stratonovich SDE
\[
\begin{align*}
  dX_t &= S(U_t, X_t) \circ dU_t \\
\end{align*}
\]
has the same solution as that of the Schwartz SDE
\[
\begin{align*}
  dX_t &= J(U_t, X_t) dU_t;
\end{align*}
\]
such that, for smooth curves \((x(t), y(t)) \in M \times N\), if \( \dot{y}(t) = S(x(t), y(t)) \dot{x}(t) \), then \( \dot{y}(t) = J(x(t), y(t)) \dot{x}(t) \).

3 Intrinsic representation of Stochastic Differential Equations on Manifolds

As per Schwartz’s second order geometry, we know that the Intrinsic type SDE on manifolds are equalities of two infinitesimal Schwartz differentials. Therefore, if we have infinitesimal stochastic diffusor \( \delta \alpha \in \mathfrak{D}M \) that depends on the process \( X_t \), then the Intrinsic SDE will be
\[
\begin{align*}
  dX_t &= \delta \alpha(\omega, X_t).
\end{align*}
\]
The problem, however, is that not every equalities of this form have a solution. Moreover, we want an equality that gives the usual form of Ito SDE in the local chart. We will focus on Ito SDEs that are driven by Weiner processes. Locally if we consider Ito SDE,
\[
\begin{align*}
  dX^i_t &= V^i dt + \frac{1}{2} a^i dt + \sum_{l=1}^{p} \sigma^i_l dW^l_t,
\end{align*}
\]
then
\[
\begin{align*}
  d[X^i, X^j]_t &= \sum_{l=1}^{p} \sigma^i_l \sigma^j_l dt.
\end{align*}
\]
As we know that locally the Schwartz differential \( dX^i_t|_U = dX^i_t \partial_i + \frac{1}{2} d[X^i, X^j]_t \partial^2_{ij} \), we get
\[
\begin{align*}
  dX^i_t|_U &= V^i dt + \frac{1}{2} \left( a^i \partial_i + \sum_{l=1}^{p} \sigma^i_l \sigma^j_l \partial^2_{ij} \right) dt + \sum_{l=1}^{p} (\sigma^i_l \partial_i) dW^l_t.
\end{align*}
\]
So we see that the drift term (the coefficient of \( dt \)) is a diffusor. If we want this equality to be globally valid, then we must assume that the drift term is the local representation of a diffusor. In other words, \( dX^i_t|_U \) must obey coordinate transformation via diffusion map of the transition function. If we assume that there exists a diffusor \( \alpha \in \mathfrak{D}M \) such that locally in chart \((U, \chi)\),
\[
\begin{align*}
  \alpha|_U &= \left( a^i \partial_i + \sum_{l=1}^{p} \sigma^i_l \sigma^j_l \partial^2_{ij} \right),
\end{align*}
\]
then it allows us to have a global representation of equation 11 as
\[
\begin{align*}
  dX_t &= Vdt + \frac{1}{2} \alpha dt + \sum_{l=1}^{p} \sigma_l dW^l_t.
\end{align*}
\]
Alternatively, if we have a Schwartz morphism \( \beta \) from \( \mathbb{R}^{p+1} \) to \( M \), then we know that
\[
\begin{align*}
  dX_t &= \beta((t, W_t), X_t)dt, W_t)
\end{align*}
\]
the diffusor \( f \) of the ways is by assuming

\[
\alpha
\]

Clearly, there is no unique way of constructing the Schwartz morph ism given by

\[
\beta
\]

processes. In chart \((U, \chi)\), we know that \( \hat{\beta} \) is given by

\[
\beta((t, W_t), X_t) d(t, W_t)|_U = \left[ f^i_0 \partial_i \begin{bmatrix} f^j_0 \partial_j \end{bmatrix} \right] \begin{bmatrix} \frac{dt}{dW_t^i} \end{bmatrix},
\]

we get

\[
\beta((t, W_t), X_t) d(t, W_t)|_U = f^i_0 dt + \frac{1}{2} \left( \sum_{i=1}^{p} f^i_0 \partial_i \right) dt + \sum_{i=1}^{p} (\sigma^i_0 \partial_i) dW_t^i.
\]

Comparing this with equation (14) we find that \( f^i_0 = V^i \), \( f^j_0 = \sigma^j_0 \), and \( f^i_0 \partial_i = a^i \) i.e. trace \( \sigma^i_0 \partial_i = a^i \). Clearly, there is no unique way of constructing the Schwartz morphism \( \beta \) due to the trace term (one of the ways is by assuming \( f^i_0 = \frac{1}{p} a^i_0 \partial_i \)). Nonetheless we feel that SDEs in form of equation (13) is a better way of describing Intrinsic SDEs. In order to facilitate the description of SDEs in form of equation (13) we will consider the following definition.

**Definition 3.1.** We will call a fiber preserving map \( G : TM \rightarrow \mathcal{D}M \) a **diffusion generator** if \( \forall Y \in TM \),

\[
\hat{G}(Y) = Y \otimes Y.
\]

We will denote the set of all diffusion generators on \( M \) as \( \mathcal{G}(M) \).

We would like to point out that diffusion generator is not the same as the *generator* of a stochastic processes.

**Proposition 3.1.** If there exists a diffusion generator \( G \in \mathcal{G}(M) \), then the equality

\[
dX_t = \left[ V + \frac{1}{2} \sum_{i=1}^{p} G(\sigma^i) \right] dt + \sum_{i=1}^{p} \sigma^i dW_t^i
\]

is coordinate invariant. Moreover, there exists a unique semi-martingale \( X_t \in M \) that satisfies the equation locally in time, for any initial condition \( X_0 \in M \).

**Proof.** Suppose for a vector field \( \sigma \in \mathfrak{X}(M) \), locally in chart \((U, \chi)\) with coordinates \((x^1, x^2, ..., x^n)\), the diffusor \( \alpha = G(\sigma) \) is given as \( \hat{\alpha} = G(\sigma)|_U = a^i \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial}{\partial x^i \partial x^j} \). Suppose in other chart \((U, \Upsilon)\) with coordinates \((y^1, y^2, ..., y^n)\), \( \hat{\alpha} = G(\sigma)|_U = \hat{\alpha}^i \frac{\partial}{\partial y^i} + \hat{\sigma}^i \hat{\sigma}^j \frac{\partial}{\partial y^i \partial y^j} \). Using the change of coordinates formula, we know that \( \hat{\alpha} = \hat{\partial} Y \hat{\alpha} \). Therefore, it can be concluded that

\[
\hat{\alpha}^i = \frac{\partial Y^i}{\partial x^j} a^j + \sigma^m \sigma^m \frac{\partial Y^i}{\partial x^m} \frac{\partial Y^m}{\partial x^m}.
\]

In chart \((U, \chi)\), the left hand side of equation (18) is given by

\[
dX_t|_U = dX^i_t \frac{\partial}{\partial x^i} + \frac{1}{2} d[X^i_t, X^j_t] \frac{\partial^2}{\partial x^i \partial x^j},
\]

where \( X^i_t = \chi^i(X_t) \). Therefore, in chart \((U, \chi)\), we get the Ito SDEs,

\[
dX^i_t = (V^i + \frac{1}{2} a^i) dt + \sigma^i dW_t^i
\]
and
\[ d[X^i_t, X^j_t] = \sigma^j_i(X_t)\sigma^j_i(X_t)dt. \] (22)

Therefore, the Schwartz differential generated by the above Ito differentials matches the right hand side of equation [18] on the manifold in the given chart.

Similarly, in chart \((U, \Upsilon)\), the Ito SDE is given by
\[ d\hat{X}^i_t = (\hat{V}^i + \frac{1}{2} \hat{\sigma}^i_j)dt + \hat{\sigma}^i_l dW^l_t, \] (23)
where \(\hat{X}^i_t = \Upsilon^i(X_t)\). Let the transition map from chart \((U, \chi)\) to \((U, \Upsilon)\) be given by \(\Psi = \Upsilon \circ \chi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n\). Let the coordinates in the codomain of the chart \((U, \chi)\) be given by \((\hat{x}^1, \hat{x}^2, ..., \hat{x}^n)\) and let \(\hat{X}_t = (X^1_t, X^2_t, ..., X^n_t)\) i.e., \(\hat{X}_t = \chi(X_t)\). By It\^{o}'s lemma,
\[ d(\Psi^i(\hat{X}_t)) = \frac{\partial \Psi^i}{\partial \hat{x}_j} (V^i + \frac{1}{2} (a^i_j(X_t))dt + \sigma^j_i(X_t)dW^j_t) + \frac{1}{2} \frac{\partial^2 \Psi^i}{\partial \hat{x}_j \partial \hat{x}_k} \sigma^j_l(X_t)\sigma^k_l(X_t)dt. \] (24)

As \(\Psi^i(\hat{X}_t) = \Upsilon^i(X_t)\), if \(\hat{X}^i_t = \Upsilon^i(X_t)\), we can conclude that
\[ d\hat{X}^i_t = \frac{\partial \Psi^i}{\partial \hat{x}_j} (V^i + \frac{1}{2} (a^i_j(X_t))dt + \sigma^j_i(X_t)dW^j_t) + \frac{1}{2} \frac{\partial^2 \Psi^i}{\partial \hat{x}_j \partial \hat{x}_k} \sigma^j_l(X_t)\sigma^k_l(X_t)dt. \] (25)

But we know that in chart \((U, \Upsilon)\), the Ito SDE representation for \(\hat{X}^i_t = \Upsilon^i(X_t)\) is given by
\[ d\hat{X}^i_t = (\hat{V}^i + \frac{1}{2} \hat{\sigma}^i_j)dt + \hat{\sigma}^i_l dW^l_t, \] (26)
using equation [19]
\[ d\hat{X}^i_t = \left(\frac{\partial \Upsilon^i}{\partial x^j} V^j + \frac{1}{2} \frac{\partial^2 \Upsilon^i}{\partial x^j \partial x^j} \sigma^j_l + \frac{1}{2} \sigma^j_l \sigma^j_l \right) dt + \hat{\sigma}^i_l dW^l_t \] (27a)
\[ = \left(\frac{\partial \Upsilon^i}{\partial x^j} V^j + \frac{1}{2} \frac{\partial^2 \Upsilon^i}{\partial x^j \partial x^j} \sigma^j_l + \frac{1}{2} \sigma^j_l \sigma^j_l \right) dt + \hat{\sigma}^i_l dW^l_t. \] (27b)

As it is known that \(\frac{\partial \Upsilon^i}{\partial x^j} = \frac{\partial \Psi^i}{\partial \hat{x}_j}\) and \(\frac{\partial^2 \Upsilon^i}{\partial x^j \partial x^j} = \frac{\partial^2 \Psi^i}{\partial \hat{x}_j \partial \hat{x}_j}\), equation [25] and equation [26] are equivalent. Therefore, the equality in form of equation [18] is coordinate invariant. Hence, for existence of unique solution, it is enough to prove the existence of unique solution in one of the charts. But this is already known to be true in the chart. Hence, for any point \(X_0 \in M\) as the initial condition, there exists a unique semi-martingale \(X_t \in M\) that satisfies equation [18] locally in time. \(\square\)

This proposition allows the following definition.

**Definition 3.2.** We define **Intrinsic Stochastic Differential Equation** on a manifold \(M\) as a 3-tuple \((V, \{\sigma_i\}, G)\), where \(V \in \mathfrak{X}(M)\), \(\sigma_i \in \mathfrak{X}(M)\) for \(i \in \{1, 2, ..., p\}\), and \(G \in \mathcal{G}(M)\). A **solution** of the SDE \((V, \{\sigma_i\}, G)\) is a stochastic process \(X_t \in M\) that satisfies equation [18] in all the charts.

As discussed earlier, for every SDE in form of equation [18] there exists a Schwartz morphism \(\beta\) from \(\mathbb{R}^{p+1}\) to \(M\) such that
\[ dX_t = \beta((t, W_t), X_t)d(t, W_t) = Vdt + \frac{1}{2}\alpha dt + \sum_{i=1}^{p} \sigma_i(x)dW^i_t, \] (28)
where \( W_t = (W^1_t, W^2_t, ..., W^p_t) \) is a Weiner process in \( \mathbb{R}^p \). Therefore, as the intrinsic SDE for \((V, \{\sigma_i\}, G)\) is given by equation (18), which also happen to be in form of equation (13), we have a Schwartz morphism from \( \mathbb{R}^{p+1} \) to \( M \) such that

\[
dX_t = \beta((t, W_t), X_t)dt + \frac{1}{2} \sum_{i=1}^{p} G(\sigma_i)dt + \sum_{i=1}^{p} \sigma_i(x) dW^i_t. \tag{29}
\]

However, as mentioned earlier, there is no unique way of obtaining the Schwartz morphism due to the presence of the trace term.

### 3.1 Construction of diffusion generators using flow of a vector field on manifold \( M \)

We have demonstrated that we can construct a Schwartz morphism if we have a diffusion generator. Therefore, construction of Schwartz morphism boils down to construction of diffusion generator. In this section we will demonstrate that it is possible to construct the diffusion generator using the flow of a vector field on the given manifold \( M \).

We know that for any smooth curve \( c(t) \) in chart \( (U, \chi) \),

\[
\frac{dc}{dt} \mid_U = \dot{c} \sigma_i \frac{\partial}{\partial x^i} + \sigma_i \sigma_j \frac{\partial^2}{\partial x^i \partial x^j}.
\]

As \( \frac{dc}{dt} = \sigma \frac{\partial}{\partial t} \), the function mapping \( \sigma \rightarrow \frac{dc}{dt} \) will be a diffusion generator.

**Lemma 3.1.** For every vector field \( \sigma \in \mathfrak{X}(M) \) there exists a unique diffusor field \( \alpha \in \Gamma(\mathfrak{D}M) \) such that locally, in chart \( (U, \chi) \) with coordinates \( (x^1, x^2, ..., x^n) \),

\[
\hat{\alpha} = \alpha \mid_U = d\sigma^i \cdot \sigma \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}, \tag{31}
\]

where \( \sigma^i = d\chi^i \cdot \sigma \).

**Proof.** To prove that \( \alpha \in \Gamma(\mathfrak{D}M) \) is a diffusor field, we need to prove that it is coordinate invariant. This can be achieved by considering another chart \( (U', \chi') \) with coordinates \( (z^1, z^2, ..., z^n) \). In chart \( (U', \chi') \),

\[
\hat{\alpha} = \alpha \mid_U = d\hat{\sigma}^i \cdot \sigma \frac{\partial}{\partial z^i} + \hat{\sigma}^i \hat{\sigma}^j \frac{\partial^2}{\partial z^i \partial z^j}, \tag{32}
\]

where \( \hat{\sigma}^i = d\chi'^i \cdot \sigma \). The smoothness of the diffusor field in the chart follows from the smoothness of the vector fields.

\[
\Rightarrow \hat{\alpha} = d(d\hat{\sigma}^i \cdot \sigma) \frac{\partial}{\partial z^i} + (d\hat{\sigma}^i \cdot \sigma)(d\hat{\sigma}^j \cdot \sigma) \frac{\partial^2}{\partial z^i \partial z^j} \tag{33a}
\]

\[
= \left( \frac{\partial}{\partial x^i} (\hat{\sigma}^i \sigma^j) \right) \frac{\partial}{\partial z^i} + \left( \frac{\partial}{\partial x^i} \sigma^j \frac{\partial}{\partial x^k} \sigma^k \right) \frac{\partial^2}{\partial z^i \partial z^j} \tag{33b}
\]

\[
= \left( \frac{\partial^2 \hat{\sigma}^i}{\partial x^i \partial x^j} \sigma^j \right) \frac{\partial}{\partial z^i} + \left( \frac{\partial \hat{\sigma}^i}{\partial x^j} \frac{\partial}{\partial x^j} \sigma^j \right) \frac{\partial}{\partial z^i} + \left( \frac{\partial \hat{\sigma}^i}{\partial x^j} \frac{\partial \hat{\sigma}^j}{\partial x^k} \sigma^k \right) \frac{\partial^2}{\partial z^i \partial z^j} \tag{33c}
\]

\[
= \mathfrak{D} \hat{\sigma} \hat{\alpha}. \tag{33d}
\]

Therefore, there exists a diffusor \( \alpha \in \Gamma(\mathfrak{D}M) \) such that locally, in chart \( (U, \chi) \) with coordinates \( (x^1, x^2, ..., x^n) \),

\[
\alpha \mid_U = d\sigma^i \cdot \sigma \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}. \tag{34}
\]
Lemma 3.2. There exists a unique diffusion generator $G_S \in \mathcal{G}(M)$ on the manifold $M$ such that the solution of the ODE $\dot{x} = \sigma(x)$ is also the solution of the Schwartz differential equation

$$\frac{dx}{dt} = G_S(\sigma(x)),$$

where $\sigma \in \mathcal{X}(M)$.

Proof. If there exists a diffusion generator $G_S \in \mathcal{G}(M)$, then in chart $(U, \chi)$,

$$G_S(\sigma)|_U = a^i \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}.$$  

(36)

If $x(t)$ is the solution for the ODE $\dot{x} = \sigma(x)$, then locally

$$\frac{dx}{dt}|_U = \frac{d^2}{dt^2}((\chi^i \circ x) \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}).$$

(37)

If the equation $\frac{dx}{dt} = G_S(\sigma)$ is satisfied by $x(t)$, then

$$a^i = \frac{d^2}{dt^2}(\chi^i \circ x) = \frac{d}{dt}(d\chi^i \cdot \sigma) = \frac{d\sigma^i}{dt} = d\sigma^i \cdot \sigma.$$  

(38)

Therefore,

$$G_S(\sigma)|_U = d\sigma^i \cdot \sigma \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}.$$  

(39)

From lemma 3.1 we know that the $G_S(\sigma)$ is a diffusor and the above equation is its local representation.

Conversely, if we consider the diffusor $G_S(\sigma)$ such that $G_S(\sigma)|_U = d\sigma^i \cdot \sigma \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}$, then the solution of the ODE $\dot{x} = \sigma(x)$ is the same as the solution of the Schwartz ODE $\frac{dx}{dt} = G_S(\sigma(x))$. The uniqueness follows due to the fact that $G_S(\sigma) = \sigma \otimes \sigma$.

Lemma 3.2 is just a special case of a more general result presented in theorem 7.22 of Emery’s book [5] (or theorem 2.2). Infact, this SDE is an Intrinsic representation of the Stratonovich SDE. This is because if we consider the SDE

$$dX = \left[V + \frac{1}{2} \sum_{l=1}^{p} G_S(\sigma_l)\right]dt + \sum_{l=1}^{p} \sigma_l dW^l_t,$$

(40)

then we see that the local Ito SDE

$$dX^i_I = \left[V^i + \frac{1}{2} \sum_{l=1}^{p} \frac{\partial \sigma^i_l}{\partial x^j} \sigma^j_l\right]dt + \sum_{l=1}^{p} \sigma^i_l dW^l_t,$$

(41)

is the same as the Stratonovich SDE,

$$dX^i = V^i dt + \sum_{l=1}^{p} \sigma^i_l \circ dW^l_t.$$  

(42)

Therefore, we use the subscript $S$ to indicate the special diffusion generator $G_S$, which can convert the Stratonovich SDE $(V, \{\sigma_1, ..., \sigma_p\})$ into Intrinsic SDE $(V, \{\sigma_1, ..., \sigma_p\}, G_S)$.
Definition 3.3. The unique diffusion generator \( G_S \in \mathcal{G}(M) \) on the manifold \( M \) such that the solution of the ODE \( \dot{x} = \sigma(x) \) is also the solution of the Schwartz differential equation

\[
\frac{dx}{dt} = G_S(\sigma(x)),
\]  

(43)

where \( \sigma \in \mathcal{X}(M) \), will be called \textbf{Stratonovich diffusion generator}.

If we have an arbitrary diffusion generator, then the conversion of Intrinsic SDE into Stratonovich SDE is a little different. We would like to point out that this conversion from Stratonovich representation to Intrinsic representation has been considered earlier by other authors e.g. in chapter 1 of [10].

4 Construction of diffusion generator using regular Lagrangian.

We have already seen that the diffusion generator obtained by the first order vector field results in Intrinsic representation of Stratonovich SDE. Now, we will try to construct the diffusion generator using second order differential equations. In general second order differential equations are vector fields on \( TM \) such that if \( c(t) \) is the solution curve of the vector field, then \( Tc \cdot \dot{c}(t) = c(t) \). In terms of the covariant derivative \( \nabla \), the second order equations are given as \( \nabla \dot{x} = V(x) \), for some \( V \in \mathcal{X}(M) \). In order to define a covariant derivative, we need a connection on the manifold. With this we know that for \( \nabla \dot{x} = V(x) \), \( \ddot{x} = V_i(x) - \Gamma^i_{jk} \dot{x}^j \dot{x}^k \). Therefore, in local coordinates \((U, \chi)\) the diffusion generator is given as,

\[
G(\dot{x})|_U = V^i(x) - \Gamma^i_{jk} \dot{x}^j \dot{x}^k \partial_i + \dot{x}^j \partial^2_{ij}.
\]  

(44)

\( V = 0 \) is a special case, in which the solution curve is a geodesic. When we consider \( V(x) = 0 \), then from [6] and [5] we know that the resulting SDE is the Intrinsic representation of the Ito SDE on manifold with connection. Conventionally, these equation are simply written as

\[
dX_t = V dt + \sum_{l=1}^{p} \sigma_l dW^l_t.
\]  

(45)

However, in order to ensure that the coordinate invariant nature of the equality is preserved, we will always represent Ito SDEs in its Intrinsic form

\[
dX_t = V dt + \frac{1}{2} \sum_{l=1}^{p} G_I(\sigma_l) dt + \sum_{l=1}^{p} \sigma_l dW^l_t.
\]  

(46)

The subscript \( I \) is used for the diffusion generator to indicate that the diffusion generator is obtained via the geodesic equation and that it corresponds to the Ito SDEs on manifolds with connection. We will call these equations as \textbf{standard Ito SDEs} and the diffusion generator \( G_I \) as \textbf{Ito diffusion generator}.

The reason for calling

Another way of obtaining second order differential equation is by using Lagrangians. In this section we will focus on construction of diffusion generator using regular Lagrangian.

Proposition 4.1. For every regular Lagrangian \( L \in \mathcal{F}(TM) \) there exists a diffusion generator \( G_L \in \mathcal{G}(M) \) such that locally in chart \((U, \chi)\), for all \( \sigma \in T_x M \),

\[
G_L(\sigma_x)|_U = \left[ \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \right]_{(x, \sigma)}^{-1} \left( \frac{\partial L}{\partial x^j} \right|_{(x, \sigma)} - \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \frac{\partial}{\partial x^j} \sigma^k \right) \frac{\partial}{\partial \dot{x}^i} \sigma^j + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}.
\]  

(47)
Moreover, if \( z(t) \) is the solution of the Hamiltonian dynamics \( \dot{z} = \omega_t^L dE \) (where \( \omega_t^L = FL^*\omega_0 \), and \( E \in \mathcal{F}(TM) \) such that \( E(v) = FL(v) \cdot v - L(v) \)), then the solution of the Schwartz equation

\[
\frac{dx}{dt} = G_L(T_M(\omega_t^L dE(x(t)))),
\]

is given by \( x(t) = \tau_M(z(t)) \) provided \( z(0) = (x(0), \dot{x}(0)) \).

**Proof.** From basic mechanics we know that in local coordinates, the equation \( \dot{x} = \frac{\partial L}{\partial \dot{x}} \) with initial condition \( x(0) = x_0 \) is equivalent to the Euler-Lagrange equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial^2 L}{\partial x \partial \dot{x}} \frac{\partial \dot{x}}{\partial x}.
\]

For initial condition \( z(0) = (x_0, v_0) \) it is equivalent to the Euler-Lagrange equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial^2 L}{\partial x \partial \dot{x}} \frac{\partial \dot{x}}{\partial x}.
\]

As the Lagrangian is regular the inverse of \( \frac{\partial \dot{x}}{\partial x} \) exists. Therefore, for initial condition \( x(0) = x \) and \( \dot{x}(0) = \sigma \), the solution \( x(t) \) is locally given by the Euler-Lagrange equation and at \( t = 0 \) the acceleration is given by

\[
\ddot{x}(0) = \left( \frac{\partial^2 L}{\partial x^2} \right)^{-1} \left( \frac{\partial L}{\partial x} - \frac{\partial^2 L}{\partial x \partial \dot{x}} \frac{\partial \dot{x}}{\partial x} \right) \sigma^k.
\]

This gives a fiber preserving map \( G_L : TM \to \mathcal{D}M \) such that locally in chart \((U, \chi)\), for all \( \sigma \in T_xM \),

\[
G_L(\sigma_x)|_U = \left[ \frac{\partial^2 L}{\partial x^i \partial \dot{x}^j} \right]^{-1} \left( \frac{\partial L}{\partial x^i} \right) \sigma^j + \sigma^j \sigma^k \frac{\partial^2 L}{\partial x^i \partial \dot{x}^j}. \]

In general, we will say that an SDE is generated by a Lagrangian \( L \), if the SDE is in the form

\[
dX = \left[ V + \frac{1}{2} \sum_{l=1}^{p} G_L(\sigma_l) \right] dt + \sum_{l=1}^{p} \sigma_l dW_t^l.
\]

**4.1 Manifold \( M \) with a non-degenerate \( T^0_2M \) tensor-field \( \alpha. \)**

As \( \alpha \in T^0_2M \) is non-degenerate, if \( L \in (TM) \) such that

\[
L(v) = \frac{1}{2} \alpha(v, v),
\]

for all \( v \in TM \), then from proposition [4.1] we know that

\[
G_L(\sigma_x)|_U = \left[ \frac{\partial^2 L}{\partial x^i \partial \dot{x}^j} \right]^{-1} \left( \frac{\partial L}{\partial x^i} \right) \sigma^j + \sigma^j \sigma^k \frac{\partial^2 L}{\partial x^i \partial \dot{x}^j}.
\]

Therefore,

\[
G_L(\sigma_x)|_U = \alpha^{ij} \left( \frac{1}{2} \frac{\partial^2 L}{\partial x^k \partial \dot{x}^l} \sigma^k \sigma^l \right) \frac{\partial}{\partial x^i} + \sigma^j \sigma^k \frac{\partial^2 L}{\partial x^i \partial \dot{x}^l}.
\]

**4.2 Riemannian manifold, \((M, g)\), with Kinetic energy as the Lagrangian.**

A special case of proposition [4.1] is a regular Lagrangian \( L \in \mathcal{F}(TM) \) such that

\[
L(v) = \frac{1}{2} g^b v \cdot v,
\]
where $g$ is the Riemannian metric on the manifold $M$. In Mechanics, such a Lagrangian is called the Kinetic Energy on $TM$. From, basic mechanics we know that if the initial condition is $v \in TM$ and the solution is given by $z(t)$, then $x(t) = \tau_M(z(t))$ is a geodesic in the direction of $v \in TM$ i.e. $x(t) = exp_{\tau_M(v)}(vt) = exp_v(x(0)t)$.

From riemannian geometry it is known that, locally in chart $(U, \chi)$,

$$\frac{d}{dt}\bigg|_{t=0}(exp_{\tau_M(v)}(vt)) = v$$

and

$$\frac{d^2}{dt^2}\bigg|_{t=0}(exp_{\tau_M(v)}(vt)) = \langle v, \nabla_v g^k d\chi^k \rangle = Hess(\chi^k)(v, v);$$

where $exp^k = \chi^k \circ exp$, and $Hess$ is called the Hessian. From the proof of lemma \ref{lemma} we can conclude that there exists a function $G \in \mathcal{G}(M)$ such that locally

$$G(v)|_U = Hess(\chi^k)(v, v)\frac{\partial}{\partial x^i} + v^i v^j \frac{\partial^2}{\partial x^i \partial x^j}. \tag{58}$$

From the earlier discussion, we know that this is just the Intrinsic representation of the standard Ito SDE $(V, \{\sigma_1, \sigma_2, ..., \sigma_p\})$ on the Riemannian manifold $(M, g)$.

### 4.3 Riemannian manifold, $(M, g)$, with Kinetic energy + Potential Energy as the Lagrangian.

Let $\Phi : M \to \mathbb{R}$ be the potential energy. Therefore, the Lagrangian is given by $L \in \mathcal{F}(TM)$ such that

$$L(v) = \frac{1}{2}g^p v \cdot v - \Phi(\tau_M(v)). \tag{59}$$

Using proposition \ref{proposition} we get

$$G_L(\sigma_x)|_U = \left\{ \frac{\partial^2 L}{\partial x^i \partial x^j} \right\}_{(x, \sigma)}^{-1} \left( \frac{\partial L}{\partial x^i} \right)_{(x, \sigma)} - \frac{\partial^2 L}{\partial x^i \partial x^j} \left( \begin{array}{c} \sigma^k \\ \sigma^k \\ \sigma^k \end{array} \right) \frac{\partial}{\partial x^k} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}. \tag{60}$$

Therefore,

$$G_L(\sigma_x)|_U = g^{ij}(x) \left( \frac{\sigma^i \partial g_{lm}(x) \sigma^m - \partial \Phi}{2 \partial x^j} - \frac{\partial \Phi}{\partial x^j}(x) - \frac{\partial g_{lm}(x) \sigma^m}{\partial x^j} \right) \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}. \tag{61}$$

In other words,

$$G_L(\sigma_x)|_U = \left( Hess(\chi^i)(\sigma, \sigma) - g^{ij}(x) \frac{\partial \Phi}{\partial x^j}(x) \right) \frac{\partial}{\partial x^i} + \sigma^i \sigma^j \frac{\partial^2}{\partial x^i \partial x^j}. \tag{62}$$

#### 4.3.1 Example

Let us consider the space to be $\mathbb{R}^2$ and coordinates $(x, y)$.

A Lagrangian $L$ is defined as $L(x, y, v_1, v_2) = v_1^2 + v_2^2 + v_1 + v_2 + v_3^2 + v_4^2 - x^2 - y^2$. From equation \[47\] we know that as the Lagrangian is regular

$$G_L = \begin{pmatrix} -x \\ -y \\ 6v_1^2 + 1 \\ -6v_2^2 + 1 \end{pmatrix}, \begin{pmatrix} v_1v_1 & v_1v_2 \\ v_2v_1 & v_2v_2 \end{pmatrix}. \tag{63}$$
We will take drift to be
\[
V(x, y) = \frac{1}{\sin(5\pi x)}
\]
and the noise vectors as
\[
\sigma_1(x, y) = \begin{bmatrix} y \\ 0 \end{bmatrix},
\]
and
\[
\sigma_2(x, y) = \begin{bmatrix} 0 \\ y \end{bmatrix}.
\]
Therefore, the Intrinsic SDE is given by
\[
d \begin{bmatrix} x \\ y \end{bmatrix} = \left[ \frac{1}{\sin(5\pi x)} + \frac{1}{2} G_L(\sigma_1) + \frac{1}{2} G_L(\sigma_2) \right] dt + \begin{bmatrix} y \\ 0 \end{bmatrix} dW^1_t + \begin{bmatrix} 0 \\ y \end{bmatrix} dW^2_t.
\]
As \(dX_t = \left( dX_t, \frac{1}{2} d[X_t, X_t] \right)\), we can say that the underlying Ito SDE for the current example is given as
\[
d \begin{bmatrix} x \\ y \end{bmatrix} = \left[ \frac{1}{\sin(5\pi x)} + \frac{1}{2} \left( \frac{-x}{6y^2 + 1} \right) + \frac{1}{2} \left( \frac{-x}{6y^2 + 1} \right) \right] dt + \begin{bmatrix} y \\ 0 \end{bmatrix} dW^1_t + \begin{bmatrix} 0 \\ y \end{bmatrix} dW^2_t.
\]
On the other hand, the standard Ito SDE representation will depend on the metric on \(\mathbb{R}^2\). Moreover, as we will see in the next section, the standard Ito SDE (that is defined on manifolds with a connection) is not the same as the Ito SDE that is only defined on linear spaces.

5 Converting Intrinsic SDEs into Belopolskya-Daletskii form for numerical computations

A naive approach of numerical computations in local chart is always available. Alternatively, one can convert an Intrinsic SDE into a Stratonovich SDE and use numerical methods from \cite{8}. In literature one finds many numerical methods for Stratonovich SDEs on manifolds, e.g. in \cite{8, 3}. Readers will find plenty of references on other numerical methods for Stratonovich SDEs in \cite{3}. In \cite{1}, 2-jet approach has been considered to construct Intrinsic SDEs so that it assists in numerical computations. In this section we show that we can convert the Intrinsic SDE into an equivalent Belopolskya-Daletskii type SDE to exploit the underlying flow of second order differential equation for numerical computations. In order to get the Belopolskya-Daletskii form for the given Intrinsic SDE, we first convert the given Intrinsic SDE into standard Ito SDE and then consider the Belopolskya-Daletskii form for the resulting standard Ito SDE. The idea of converting Intrinsic SDEs into standard Ito SDE/Stratonovich SDE and vice-versa is not a new one and has been discussed in chapter 1 of \cite{10}. What we consider here are equivalent representations of Intrinsic SDEs obtained using the diffusion generator approach.

Earlier, in section 4 we have observed that the standard Ito SDE \((V, \{\sigma_1, ..., \sigma_p\})\), is the same as the Intrinsic SDE \((V, \{\sigma_1, ..., \sigma_p\}, G_I)\). However, we do not know if Intrinsic SDEs with arbitrary diffusion generator \(G\) can have a standard Ito representation. Therefore, it seems reasonable that the Intrinsic SDE \((V, \{\sigma_1, ..., \sigma_p\}, G)\) is the same as the standard Ito SDE \(\left( V + \frac{1}{2} \sum_{i=1}^p (G(\sigma_i) - G_I(\sigma_i)), \{\sigma_1, ..., \sigma_p\} \right)\).

However, we need to prove that \(G(\sigma_i) - G_I(\sigma_i)\) is indeed a tangent vector.

**Lemma 5.1.** For every two diffusion generators \(G, G_\alpha \in \mathcal{G}(M)\), there exists a tangent bundle valued 1-form \(\nabla_\alpha^G \in \Omega^1(M; TM)\) such that \(\nabla_\alpha^G(X) = G(X) - G_\alpha(X) \quad \forall \ X \in TM\).
Proof. As per the definition of diffusion generator, for any \( G \in \mathcal{G}(M), \) \( G(X) = X \otimes X, \) \( \forall X \in TM. \) Therefore, \( G(X) - G_{\alpha}(X) = 0 \) i.e., \( G(X) - G_{\alpha}(X) \in TM \forall X \in TM. \) The existence of \( \nabla_{\alpha}^G \in \Omega^1(M; TM) \) follows because we are given that \( G - G_{\alpha} \) is \( \nabla_{\alpha}^G. \)

Lemma 5.2. \((V, \{\sigma_1, ..., \sigma_p\}, G)\) is equivalent to \( \left( V + \frac{1}{2} \sum_{l=1}^{p} \nabla_{\alpha}^G(\sigma_l), \{\sigma_1, ..., \sigma_p\}, G_{\alpha}\right). \)

Proof.

\[
\text{d}X_t = V dt + \frac{1}{2} \sum_{l=1}^{p} G(\sigma_l) dt + \sum_{l=1}^{p} \sigma_l(x) dW^l_t \tag{69}
\]

\[
= V dt + \frac{1}{2} \sum_{l=1}^{p} \left( \nabla_{\alpha}^G(\sigma_l) + G_{\alpha}(\sigma_l) \right) dt + \sum_{l=1}^{p} \sigma_l(x) dW^l_t \tag{70}
\]

From lemma 5.1, we know that \( \nabla_{\alpha}^G(\sigma_l) \) is a vector. Hence, \( \text{d}X_t \) can be considered as the SDE \( \left( V + \frac{1}{2} \sum_{l=1}^{p} \nabla_{\alpha}^G(\sigma_l), \{\sigma_1, ..., \sigma_p\}, G_{\alpha}\right). \)

Due to this lemma, if the manifold is equipped with a connection, then the Intrinsic SDE \((V, \{\sigma_1, ..., \sigma_p\}, G)\) has the standard Ito representation

\[
\left( V + \frac{1}{2} \sum_{l=1}^{p} \nabla_{\alpha}^G(\sigma_l), \{\sigma_1, ..., \sigma_p\}, G_{\alpha}\right). \tag{72}
\]

Similarly, the Intrinsic SDE \((V, \{\sigma_1, ..., \sigma_p\}, G)\) has the Stratonovich representation

\[
\left( V + \frac{1}{2} \sum_{l=1}^{p} \nabla_{\alpha}^S(\sigma_l), \{\sigma_1, ..., \sigma_p\}\right). \tag{73}
\]

From [6], we know that the Belopolskya-Daletskii form for the standard Ito SDE \((V, \{\sigma_1, ..., \sigma_p\})\) is given by

\[
\text{d}X_t = \exp_{X_t} \left( V(X_t) dt + \sum_{l=1}^{p} \sigma_l(X_t) dW^l_t \right), \tag{74}
\]

where the exponential map is due to the connection. Therefore, we get the following statement.

Lemma 5.3. The Intrinsic SDE \((V, \{\sigma_1, ..., \sigma_p\}, G)\) has an equivalent Belopolskya-Daletskii form that is given by

\[
\text{d}X_t = \exp_{X_t} \left( V(X_t) dt + \frac{1}{2} \sum_{l=1}^{p} \nabla_{\alpha}^G(\sigma_l) dt + \sum_{l=1}^{p} \sigma_l(X_t) dW^l_t \right). \tag{75}
\]

This lemma allows us to take advantage of the underlying exponential map for numerical computations. Let,

\[
X_{t+\delta t} = \exp_{X_t}(Y_{t+\delta t} - Y_t), \tag{76}
\]
where $Y_t$ is a stochastic process in the tangent space $T_{X_t}M$ such that,

$$
\text{d}Y_s = \left[ V(X_t) + \frac{1}{2} \sum_{l=1}^p \nabla^l \sigma_l(X_t) \right] \text{d}s + \sum_{l=1}^p \sigma_l(X_t) \text{d}W^l_s.
$$

(77)

The numerical solution can be given by the following two steps:

1. Solve equation (77) in the tangent space $T_{X_t}M$ with standard numerical techniques for SDEs in finite linear spaces. Moreover, we assume that the drift and the noise coefficients are constant in the tangent space $T_{X_t}M$.

2. Use time stepping iteration in form of equation (76) to compute $X_{t+1}$, where $t_{i+1} = t_i + \delta t$. The exponential map can be computed using standard numerical integration methods on manifolds.

In this approach, other than the numerical errors in evaluating (76) and equation (77), an additional error is introduced when we assume that the drift and the noise coefficients are constant in the tangent space $T_{X_t}M$. Nonetheless, now we have a numerical method for Intrinsic SDEs.

6 Conclusion

We have presented a way of constructing Schwartz morphism from $\mathbb{R}^{p+1} \to M$ such that it morphs the process $(t, W_t)$ into a process on $M$. This approach is based on construction of diffusion generator, which is a special type of fiber preserving map from the tangent bundle to the diffusion bundle. Therefore, we have demonstrated that the construction of Schwartz morphism boils down to the construction of the diffusion generator. We have constructed the diffusion generator using flow of first order differential equations and second order differential equations. We find that the SDE obtained by diffusion generator of first order differential equation is nothing but the Intrinsic representation of the Stratonovich SDE. On the other hand, for the second order differential equation, we consider Hamiltonian dynamics on the tangent bundle $T_M$ using a regular Lagrangian on $T_M$ to construct the diffusion generator. We have illustrated this with examples of diffusion generator constructed using different regular Lagrangians. Finally we have obtained Belopolsky-Daletskii type representation for the Intrinsic SDE on a manifold equipped with a connection and have shown that this can be used for numerical computations.

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