GENERALIZED RAYCHAUDHURI EQUATIONS FOR STRINGS AND MEMBRANES

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Abstract

A recent generalisation of the Raychaudhuri equations for timelike geodesic congruences to families of $D$ dimensional extremal, timelike, Nambu–Goto surfaces embedded in an $N$ dimensional Lorentzian background is reviewed. Specialising to $D = 2$ (i.e the case of string worldsheets) we reduce the equation for the generalised expansion $\theta_a, (a = \sigma, \tau)$ to a second order, linear, hyperbolic partial differential equation which resembles a variable–mass wave equation in $1 + 1$ dimensions. Consequences, such as a generalisation of geodesic focussing to families of worldsheets as well as exactly solvable cases are explored and analysed in some detail. Several possible directions of future research are also pointed out.
I. INTRODUCTION

It is a well established fact today that the proof of the existence of spacetime singularities in the general theory of relativity (GR) largely relies on the consequences obtained from the Raychaudhuri equations for null/timelike geodesic congruences [1,2,3]. Even though the applications of the Raychaudhuri equations are mostly confined to the domain of GR, it is important to note that these equations contain some basic statements about the nature of geodesics in a Riemannian/pseudo–Riemannian geometry. GR comes into the picture when one assumes the Einstein field equation and thereby reduces one crucial term containing information about geometry into an object related to matter stress–energy. Subsequently, if one imposes an Energy condition (such as the Weak Energy condition which implies that the energy density of matter is always positive in all frames of reference) it is possible to derive the fact that geodesic congruences necessarily converge within a finite value of the affine parameter. This is known as the focussing theorem, which, along with other assumptions about causality, essentially imply the existence of spacetime singularities.

In string and membrane theories, the notion of the point particle (and its associated world–line) which is basic to GR as well as other relativistic field theories, gets replaced by the string/membrane (and its corresponding world–surface). This is a radically new concept and has paid rich dividends in recent times. For instance, it is claimed that quantum gravity as well as a unification of forces comes out naturally from quantizing string theories.

If one accepts the string/membrane viewpoint then it should, in principle, be possible to derive the corresponding generalized Raychaudhuri equations for timelike/null worldsheet congruences and arrive at similar focussing and singularity theorems in Classical String theory. Very recently, Capovilla and Guven [4] have written down the generalized Raychaudhuri equations for timelike worldsheet congruences. In this talk, we shall first give a brief review of these equations. Thereafter, we construct explicit examples of these rather complicated set of equations by specializing to certain simple extremal families of surfaces. Our principal aim is to extract some information regarding focussing of families of surfaces in a way
similar to the results for geodesic congruences in GR.

Finally, we shall summarize some recent work in progress and point out several future directions of research.

II. REVIEW

A. What are Raychaudhuri Equations?

An useful way of visualising the content of the Raychaudhuri equations is to look at an analogy with fluid flow. The flow lines of a fluid (which are the integral curves of the velocity field) form a congruence of curves. If we focus our attention on the cross-sectional area enclosing a certain number of these curves we find that it is different at different points. The gradient of the velocity field contains crucial information about the behaviour of this area. What can happen to this area as we move along the family of curves? The obvious answer is–it can expand (i.e. a smaller circle may become a larger circle which is concentric to the former), it can become sheared (i.e. a circle can become an ellipse) or it can twist or rotate. The information about each of these objects (i.e. the expansion, shear and the rotation) is encoded in the gradient of the velocity field). Recall that this gradient is a second rank tensor. Such a quantity can be split into its symmetric traceless, antisymmetric and trace parts – these are respectively the shear, rotation and expansion of the congruence/flow lines. For geodesic congruences in Riemannian/pseudo-Riemannian geometry we have to replace the flow lines with geodesics and the velocity field by the tangent vectors to the geodesic curves.

Therefore, expressed in mathematical language we have:

\[ v_{\mu,\nu} = \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{3} \theta h_{\mu\nu} \]  

(1)

The evolution equation for each of these quantities – expansion, shear, rotation along the geodesic congruence are what are known as the Raychaudhuri equations.
For example, the equation for the expansion $\theta$ for a timelike geodesic congruence turns out to be:

$$\frac{d\theta}{d\lambda} + \frac{1}{3} \theta^2 + \sigma^2 - \omega^2 = -R_{\mu\nu} \xi^\mu \xi^\nu$$  \hspace{1cm} (2)$$

An analysis of the nature of the solutions of this equation leads us to the concept of geodesic focusing. First, let us convert this equation into a second order linear equation by redefining $\theta = 3\frac{F'}{F}$. This results in the following:

$$\frac{d^2 F}{d\lambda^2} + \frac{1}{3} \left( \sigma^2 - \omega^2 + R_{\mu\nu} \xi^\mu \xi^\nu \right) F = 0$$  \hspace{1cm} (3)$$

Assuming zero rotation, one can prove that if $\theta$ is negative somewhere it has to go to $-\infty$ within a finite value of the affine parameter if the coefficient of the second term in the above equation is greater than or equal to zero. This largely follows from the theorems on the existence of zeros of a general class of such second order, linear equations proofs of which can be found in the paper by Tipler [6].

What happens when $\theta \to -\infty$? If $A_1$ is the area at $\lambda_1$ and $A_2$ at $\lambda_2$ we can write $\theta$ as $\theta = \frac{A_2 - A_1}{A_2}$. Therefore $\theta$ can go to $-\infty$ if $A_2 \to 0$. Thus, the family of geodesics must focus at a point (i.e. at $\lambda = \lambda_2$).

A word about the constraints on the geometry which are necessary for focusing. Notice that if the rotation is zero we must have $R_{\mu\nu} \xi^\mu \xi^\nu \geq 0$. Using the Einstein field equations one can write this as $T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \geq 0$. This is what is known as the Strong Energy Condition. An Energy Condition, in general, defines a certain class of matter which is physically possible. For instance, the Weak Energy Condition states that $T_{\mu\nu} \xi^\mu \xi^\nu \geq 0$ which physically implies the positivity of energy density in all frames of reference.

We now ask the following questions:

Suppose we replace extremal curves (geodesics) by extremal $D$ dimensional timelike surfaces embedded in an $N$ dimensional background.

- Are there generalisations of the Raychaudhuri equations?
• Is there an analog of geodesic focusing?

The remaining part of this article is devoted to answering these two questions.

**B. Geometry of Embedded Surfaces**

We begin with a brief account of the differential geometry of embedded surfaces.

A $D$-dimensional surface in an $N$-dimensional background is defined through the embedding $x^\mu = X^\mu(\xi^a)$ where $\xi^a$ are the coordinates on the surface and $x^\mu$ are the ones in the background. Furthermore, we construct an orthonormal basis $(E_\mu^a, n_i^\mu)$ consisting of $D$ tangents and $N - D$ normals at each point on the surface. $E_\mu^a, n_i^\mu$ satisfy the following properties.

$$g(E_a, E_b) = \eta_{ab} ; \quad g(E_a, n_i) = 0 ; \quad g(n_i, n_j) = \delta_{ij} \quad (4)$$

We can write down the Gauss–Weingarten equations using the usual definitions of extrinsic curvature, twist potential and the worldsheet Ricci rotation coefficients.

$$D_a E_b = \gamma^c_{ab} E_c - K^i_{ab} n_i \quad (5)$$

$$D_a n_i = K^i_{ab} E^b + \omega^{ij}_a n_j \quad (6)$$

where $D_a \equiv E_\mu^a D_\mu$ ( $D_\mu$ being the usual spacetime covariant derivative). The quantities $K^i_{ab}$ (extrinsic curvature), $\omega^{ij}_a$ and $\gamma^c_{ab}$ are defined as :

$$K^i_{ab} = -g(D_a E_b, n^i) = K^i_{ba} \quad (7)$$

$$\omega^{ij}_a = g(D_a n^i, n^j) \quad (8)$$

$$\gamma_{abc} = g(D_a E_b, E_c) = -\gamma_{acb} \quad (9)$$

In order to analyse deformations normal to the worldsheet we need to consider the normal gradients of the spacetime basis set. The corresponding analogs of the Gauss–Weingarten equations are :
\[ D_i E_a = J_{aij} n^j + S_{abi} E^b \] (10)

\[ D_i n_j = -J_{aij} E^a + \gamma^k_{ij} n_k \] (11)

where \( D_i \equiv n^\mu_i D_\mu \). The quantities \( J_{aij}^v \), \( S_{abi} \) and \( \gamma^k_{ij} \) are defined as:

\[ S_{ab}^i = g(D^i E_a, E_b) = -S_{ba}^i \] (12)

\[ \gamma_{ijk} = g(D_i n_j, n_k) = -\gamma_{ikj} \] (13)

\[ J_{aij}^a = g(D^i E_a, n^j) \] (14)

\textbf{C. Sketch of Derivation}

The full set of equations governing the evolution of deformations can now be obtained by taking the worldsheet gradient of \( J_{aij}^v \). This turns out to be (for details see Appendix of Ref. [4]).

\[ \tilde{\nabla}_b J_{aij}^v = -\tilde{\nabla}^i K^j_{ab} - J^i_{bk} J^k_{aj} - K^i_{bc} K^c_{aj} - g(R(E_b, n^i) E_a, n^j) \] (15)

where the extrinsic curvature tensor components are \( K^i_{ab} = -g_{\mu\nu} E_a^\alpha (D^\alpha E^\mu_b) n^{\nu i} \)

On tracing over worldsheet indices we get

\[ \tilde{\nabla}_a J_{aij}^v = -J^i_{ak} J^k_{aj} - K^i_{ac} K^c_{aj} - g(R(E_a, n^i) E_a, n^j) \] (16)

where we have used the equation for extremal membranes (i.e. \( K^i = 0 \))

The antisymmetric part of (12) is given as:

\[ \tilde{\nabla}_b J_{aij}^v - \tilde{\nabla}_a J_{aij}^v = G_{ab}^{ij} \] (17)

where \( g(R(Y_1, Y_2) Y_3, Y_4) = R_{\alpha\beta\mu\nu} Y_1^\alpha Y_2^\beta Y_3^\mu Y_4^\nu \) and

\[ G_{ab}^{ij} = -J^i_{bk} J^k_{aj} - K^i_{bc} K^c_{aj} - g(R(E_b, n^i) E_a, n^j) - (a \rightarrow b) \] (18)

One can further split \( J_{aij} \) into its symmetric traceless, trace and antisymmetric parts

\( \bar{J}_{ai} = \Sigma_{ai} + \Lambda_{ai} + \frac{1}{N} \delta_{ai} \theta_a \) and obtain the evolution equations for each of these quantities.

The one we shall be concerned with mostly is given as
\[ \Delta \gamma + \frac{1}{2} \partial_a \gamma \partial^a \gamma + (M^2)^i_i = 0 \]  \hspace{1cm} (19)

with the quantity \((M^2)^{ij}\) given as:

\[ (M^2)^{ij} = K^i_{ab}K^{ij} + R_{\mu
u\rho\sigma}E^\mu_a n^\nu_i E^\rho_a n^\sigma_j \]  \hspace{1cm} (20)

\(\nabla_a\) is the worldsheet covariant derivative \((\Delta = \nabla^a \nabla_a)\) and \(\partial_a \gamma = \theta_a\). Notice that we have set \(\Sigma_{ij}^a\) and \(\Lambda_{ij}^a\) equal to zero. This is possible only if the symmetric traceless part of \((M^2)^{ij}\) is zero. One can check this by looking at the full set of generalized Raychaudhuri equations involving \(\Sigma_{ij}^a\), \(\Lambda_{ij}^a\) and \(\theta_a\) [4]. For geodesic curves the usual Raychaudhuri equations can be obtained by noting that \(K_{00}^i = 0\), the \(J_{aij}\) are related to their spacetime counterparts \(J_{\mu\nu a}\) through the relation \(J_{\mu\nu a} = n^i_\mu n^j_\nu J_{aij}\), and the \(\theta\) is defined by contracting with the projection tensor \(h_{\mu\nu}\).

The \(\theta_a\) or \(\gamma\) basically tell us how the spacetime basis vectors change along the normal directions as we move along the surface. If \(\theta_a\) diverges somewhere, it induces a divergence in \(J_{aij}\), which, in turn means that the gradients of the spacetime basis along the normals have a discontinuity. Thus the family of worldsheets meet along a curve and a cusp/kink is formed. This, we claim, is a focussing effect for extremal surfaces analogous to geodesic focussing in GR where families of geodesics focus at a point if certain specific conditions on the matter stress energy are obeyed.

**D. Is the evolution of \(J_{aij}\) constrained?**

For each pair \((ij)\) there are \(D\) quantities \(J_{aij}^i\). To analyse whether the evolution of \(J_{aij}^i\) is constraint–free or not we split the antisymmetric equations into two sets.

\[ \tilde{\nabla}_A J_{ij}^i - \tilde{\nabla}_A J^i_0 = G_{ab}^{ij} \]  \hspace{1cm} (21)

\[ \tilde{\nabla}_B J_{ij}^i - \tilde{\nabla}_A J_{ij}^j = G_{ab}^{ij} \]  \hspace{1cm} (22)

Thus first of these contains a total of \(D\) equations for each \(ij\). Eqn (12) on the other hand contains one equation. Thus the total number of equations contained in these is \(D\).
which is the number necessary to specify the evolution of the quantity $J^{ij}_a$. Therefore the second set above is actually a set of constraints on the evolution.

However, note that the number of equations in this second set is equal to $\frac{1}{2}(D-1)(D-2)$. Therefore for $D = 1$ (curves) and $D = 2$ (string worldsheets) these equations are vacuous and the evolution of $J^{ij}_a$ is entirely constraint-free! Even more surprising is the fact that for any general $D > 2$ also the second set of equations are identities (for a proof see [4]). Therefore, if the constraints and equations of motion are satisfied at any initial time they continue to be so for all future values.

III. FOCUSING OF STRING WORLDSHEETS [5]

Two dimensional timelike surfaces embedded in a four dimensional background are the objects of discussion in this section. We begin by writing down the generalised Raychaudhuri equation for the case in which $\Sigma^{ij}_a$ and $\Lambda^{ij}_a$ are set to zero (i.e implicitly assuming that $(M^2)^{ij}$ does not have a nonzero symmetric traceless part). Thus we have

$$\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma^2} + \Omega^2(\sigma, \tau)(M^2)^{i}_{i}(\sigma, \tau)F = 0$$  \hspace{1cm} (23)

where $\Omega^2$ is the conformal factor of the induced metric written in isothermal coordinates. Notice that the above equation is a second–order, linear, hyperbolic partial differential equation. On the contrary, the Raychaudhuri equation for curves is a linear, second order, ordinary differential equation. The easiest way to analyse the solutions of this equation is to assume separability of the quantity $\Omega^2(M^2)^{i}_{i}$. Then, we have

$$\Omega^2(M^2)^{i}_{i} = M^2_1(\tau) + M^2_2(\sigma)$$  \hspace{1cm} (24)

$$F(\tau, \sigma) = F_1(\tau) \times F_2(\sigma)$$  \hspace{1cm} (25)

With these we can now split the partial differential equation into two ordinary differential equations given by

$$\frac{d^2 F_1}{d\tau^2} + (\omega^2 - M^2_1(\tau))F_1 = 0$$  \hspace{1cm} (26)
\[
\frac{d^2 F_2}{d\sigma^2} + (\omega^2 + M_2^2(\sigma)) F_2 = 0
\]  

(27)

Since the expansions along the \(\tau\) and \(\sigma\) directions can be written as \(\theta_\tau = \frac{F_1}{F_2}\) and \(\theta_\sigma = \frac{F_1'}{F_2}\), we can analyse focussing effects by locating the zeros of \(F_1\) and \(F_2\) much in the same way as for geodesic curves [6]. The well–known theorems on the existence of zeros of ordinary differential equations as discussed in [6] make our job much simpler. The theorems essentially state that the solutions of equations of the type \(\frac{d^2 A}{dx^2} + H(x) A = 0\) will have at least one zero iff \(H(x)\) is positive definite. Thus for our case here, we can conclude that, focussing along the \(\tau\) and \(\sigma\) directions will take place only if

\[
\omega^2 \geq \text{max}[M_1^2(\tau)] \quad ; \quad \omega^2 \geq \text{max}[-M_2^2(\sigma)]
\]  

(28)

For stationary strings, one notes that \((M^2)^{ij}\) will not have any dependence on \(\tau\). Thus we can set \(M_1^2\) equal to zero. Thus, focussing will entirely depend on the sign of the quantity \(M_2^2\). We can write \(M_2^2\) alternatively as follows. Consider the Gauss–Codazzi integrability condition:

\[
R_{\mu\nu\alpha\beta} E^\mu_a E^\nu_b E^\alpha_c E^\beta_d = R_{abcd} - K_{aci} K^{aci} + K_{bdi} K^{bdi}
\]  

(29)

Trace the above expression on both sides with \(\eta^{ac}\eta^{bd}\) and rearrange terms to obtain :

\[
K_{abi} K^{abi} = -2 R + K^i K_i + R_{\mu\nu\alpha\beta} E^\mu_a E^\nu_b E^\alpha_c E^\beta_d
\]  

(30)

Thereafter, use this expression and the fact that \(\eta^{\mu\nu} n^i_\nu = g^{\mu\nu} - E^{\mu a} E^a_\nu\) in the original expression for \((M^2)^i_j\) (see Eqn.(11)) to get

\[
M_2^2 = -2 R + R_{\mu\nu} E^{\mu a} E^a_\nu
\]  

(31)

One can notice the following features from the above expression:

(i) If the background spacetime is a vacuum solution of the Einstein equations then the positivity of \(M_2^2\) is guaranteed iff \(2R \leq 0\). Thus all string configurations in vacuum spacetimes which have negative Ricci curvature everywhere will necessarily imply focussing. This includes the well known string solutions in Schwarzschild and Kerr backgrounds.
(ii) If the background spacetime is a solution of the Einstein equations then we can replace the second and third terms in the expressions for $M_2^2$ by the corresponding ones involving the Energy Momentum tensor $T_{\mu\nu}$ and its trace. Thus we have

$$M_2^2 = \left(-\frac{1}{2}g_{\mu\nu}R + T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)E^{\mu\nu}E^\nu_a$$

Notice that if we split the quantity $E^{\mu\nu}E^\nu_a$ into two terms such as $E^\mu_\tau E^\nu_\tau$ and $E^\mu_\sigma E^\nu_\sigma$ then we have :

$$M_2^2 = -2R + \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)E^{\mu\nu}E^\nu_\tau + \left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)E^{\mu\sigma}E^\nu_\sigma$$

The second term in the above equation is the L. H. S. of the Strong Energy Condition (SEC). Apart from this we have two other terms which are entirely dependent on the fact that we are dealing with extended objects. The positivity of the whole quantity can therefore be thought of as an Energy Condition for the case of strings. Thus even if the background spacetime satisfies the SEC, focussing of string world–sheets is not guaranteed–worldsheet curvature and the projection of the combination $T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$ along the $\sigma$ direction have an important role to play in deciding focussing/defocussing.

Let us now try to understand the consequences of the above equations for certain specific flat and curved backgrounds for which the string solutions are known.

**A. Rindler Spacetime**

The metric for four dimensional Rindler spacetime is given as

$$ds^2 = -a^2 x^2 dt^2 + dx^2 + dy^2 + dz^2$$

We recall from [7] the a string solution in a Rindler spacetime:

$$t = \tau ; \quad x = ba \cosh a\sigma_c$$

$$y = ba^2 \sigma_c ; \quad z = z_0 \quad (constant)$$
where \( d\sigma_c = \frac{ds}{a^2x^2} \) and \( b \) is an integration constant. The orthonormal set of tangents and normals to the worldsheet can be chosen to be as follows:

\[
E^\mu_\tau \equiv \left( \frac{1}{ax}, 0, 0, 0 \right) ; \quad E^\mu_\sigma \equiv (0, \tanh a\sigma_c, \text{sech} a\sigma_c, 0) \quad (36)
\]

\[
n^\mu_1 \equiv (0, 0, 0, 1) \quad ; \quad n^\nu_2 \equiv (0, \text{sech} a\sigma_c, -\tanh a\sigma_c, 0) \quad (37)
\]

In the worldsheet coordinates \( \tau, \sigma_c \) the induced metric is flat and the components of the extrinsic curvature tensor turn out to be

\[
K^1_{ab} = 0 \quad ; \quad K^2_{\tau\tau} = -a^2K^2_{a\sigma_c} = \frac{1}{ba \cosh^2 a\sigma_c} \quad ; \quad K^2_?= 0 \quad (38)
\]

The quantity \( (M^2)_i^i \) which is dependent only on the extrinsic curvature of the worldsheet (the background spacetime being flat) turns out to be

\[
(M^2)_i^i = \frac{2}{b^2 a^2 \cosh^4 a\sigma_c} \quad (39)
\]

Therefore the generalized Raychaudhuri equation turns out to be

\[
-\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma_c^2} + \frac{2a^2}{\cosh^2 a\sigma_c} F = 0 \quad (40)
\]

Separating variables \( F = T(\tau)\Sigma(\sigma) \) one gets the harmonic oscillator equation for \( T \) and the Poschl-Teller equation for positive eigenvalues for \( \Sigma \) which is given as:

\[
\frac{d^2\Sigma}{d\sigma^2} + \left( \omega^2 + \frac{2a^2}{\cosh^2 a\sigma_c} \right) \Sigma = 0 \quad (41)
\]

From the results of Tipler [6] on the zeros of differential equations one can conclude that focussing will occur \( (H(\sigma) > 0 \text{ always}) \).

Several other examples can be found in [5] and [9].

**IV. FOCUSSING OF HYPERSURFACES [5]**

We now move on to the special case of timelike hypersurfaces. Here we have \( D \) quantities \( J_a \) but only one normal defined at each point on the surface. The Eqn. (8) turns out to be:
\[ \partial_b J_a - \partial_a J_b = 0 \quad (42) \]

Therefore one can write \( J_a = \partial_a \gamma \) and the traced equation (7) becomes,

\[ \Delta \gamma + (\partial_a \gamma)(\partial^a \gamma) + M^2 = 0 \quad (43) \]

with

\[
M^2 = K_{ab} K^{ab} + R_{\nu\sigma} n^\nu n^\sigma \\
= -2R + R_{\mu\nu} E^\mu_a E^\nu_a = -2R + 3R - R_{\mu\nu} n^\mu n^\nu \quad (44)
\]

where we have used \( n^\mu n^\nu = g^{\mu\nu} - E^a_\mu E^a_\nu \) and the Gauss–Codazzi integrability condition.

If we assume that the background spacetime satisfies the Einstein equation then we have:

\[
M^2 = -\left( 2R g_{\mu\nu} + T_{\mu\nu} + T g_{\mu\nu} \right) n^\mu n^\nu \quad (45)
\]

Thus, for stationary two dimensional hypersurfaces (strings in 3D backgrounds) we have the same conclusions as obtained in the previous section. For a two–dimensional hypersurface in three–dimensional flat background the task is even simpler. \( M^2 \) can be shown to be equal to the negative of the Ricci scalar of the membrane’s induced metric and \( 2R \leq 0 \) guarantees focussing.

Let us now turn to a specific case where the equations are exactly solvable.

**A. Hypersurfaces in a 2 + 1 Curved Background**

Our background spacetime here is curved, Lorentzian background and 2 + 1 dimensional. The metric we choose is that of a Lorentzian wormhole in 2 + 1 dimensions given as :

\[
ds^2 = -dt^2 + dl^2 + \left( b_0^2 + l^2 \right) d\theta^2 \quad (46)
\]

A string configuration in this background can be easily found by solving the geodesic equations in the 2D spacelike hypersurface [8]. This turns out to be

\[
t = \tau \quad ; \quad l = \sigma \quad ; \quad \theta = \theta_0 \quad (47)
\]
The tangents and normal vectors are simple enough:

\[ E^\mu_r \equiv (1, 0, 0) \quad ; \quad E^\mu_\sigma \equiv (0, 1, 0) \quad ; \quad n^\mu = \left( 0, 0, \frac{1}{b_0^2 + l^2} \right) \]

(48)

The extrinsic curvature tensor components are all zero as the induced metric is flat. Using the Riemann tensor components (which can be evaluated simply using the standard formula) we can write down the generalised Raychaudhuri equation. This turns out to be:

\[ -\frac{\partial^2 F}{\partial \tau^2} + \frac{\partial^2 F}{\partial \sigma^2} + \left( -\frac{b_0^2}{(b_0^2 + \sigma^2)^2} \right) F = 0 \]

(49)

A separation of variables \( F = T(\tau)\Sigma(\sigma) \) will result in two equations—one of which is the usual Harmonic Oscillator and the other given by:

\[ \frac{d^2 \Sigma}{d\sigma^2} + \left( \omega^2 - \frac{b_0^2}{(b_0^2 + \sigma^2)^2} \right) \Sigma = 0 \]

(50)

The above equation can be recast into the one for Radial Oblate Spheroidal Functions by a simple change of variables \(- \Sigma' = \sqrt{b_0^2 + \sigma^2}\Sigma\).

\[ (1 + \xi^2) \frac{d^2 \Sigma'}{d\xi^2} + 2\xi \frac{d\Sigma'}{d\xi} + \left( \omega^2 b_0^2 (1 + \xi^2) \right) \Sigma' = 0 \]

(51)

where \( \xi = \frac{\sigma}{b_0} \).

The general equation for Radial Oblate Spheroidal Functions is given as:

\[ (1 + \xi^2) \frac{d^2 V_{mn}}{d\xi^2} + 2\xi \frac{dV_{mn}}{d\xi} + \left( -\lambda_{mn} + k^2 \xi^2 - \frac{m^2}{1 + \xi^2} \right) V_{mn} = 0 \]

(52)

Assuming \( m = 0 \) and \( \lambda_{0n} = -k^2 = -\omega^2 b_0^2 \) we get the equation for our case. The solutions are finite at infinity and behave like simple sine/cosine waves in the variable \( \sigma \). Consulting the tables in [10] we conclude that only for \( n = 0, 1 \) we can have \( \lambda_{0n} \) to be negative. In general, the scattering problem for the Schroedinger–like equation has been analysed numerically in [11].

As regards focusing, one can say from the differential equations and the theorems stated in [5] that the function \( \Sigma' \) will always have zeros if \( \omega^2 \geq \frac{1}{b_0^2} \). Even from the series representations (see [9]) of the Radial Oblate Spheroidal Functions we can exactly locate the zeros and obtain explicitly the focal curves. However, we shall not attempt such a task here.
V. WORK IN PROGRESS AND FUTURE DIRECTIONS

A systematic study of the generalised Raychaudhuri equations has only begun. A large number of open problems therefore exist in this subfield. Here we report briefly on some recently obtained results and list a few of the outstanding issues.

(i) Although we have been able to derive an analog of geodesic focusing for extremal timelike membranes by looking at some special cases a general treatment of the problem is still lacking. For example, recall that we made the crucial assumption of separability in the equation for the case of strings and hypersurfaces. In order to avoid this assumption, a way out could be to define an initial value problem with the expansions in different directions taking on a negative value at specific locations on the surface. The initial value as well as the partial differential equation can be recast into one Volterra integral equation of the second kind. Solving this would then give the necessary condition for focusing. Note that in this approach one does not assume an Energy Condition at the outset because one does not have a choice to do so. Some progress along these directions are to be reported in \[12\].

(ii) What does one need to have an energy condition in this case? Recall that in the Raychaudhuri equation for geodesic congruences the appearance of the term $R_{\mu\nu}\xi^\mu\xi^\nu$ and the Einstein equation relating geometry to matter were the deciding factors. One could translate the purely geometric term into a term containing properties of matter stress energy. Focusing was a consequence of assuming certain physically relevant features of matter. To have such a situation in the case of strings we need to have an Einstein equation in string theory. This may sound outrageous but we will see why it need not be so.

General relativity as a theory has a unique feature in comparison to all other theories that we know of. The motion of test particles can be derived from the Einstein field equations. We must remember that this is a fact which is not true in all other theories. For example, in electrodynamics the Lorentz force law cannot in any way be derived from the Maxwell equations. The basic question therefore reduces to the following:

What is the field equation from which one can arrive at the equation of motion for test

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An answer to this question will help us in analysing worldsheet focussing by assuming an Energy condition much in the same way as one does it for geodesic congruences.

(iii) It is important to note that we have restricted ourselves exclusively to extremal Nambu–Goto type membranes while deriving the generalised Raychaudhuri equations and exploring its consequences. What are the corresponding equations for actions other than Nambu–Goto (rigidity corrections, presence of antisymmetric tensor fields, supersymmetric generalisations etc.)? It has been found [13] that in the presence of antisymmetric tensor fields one has several nontrivialities appearing. Firstly, one cannot set the $\Sigma_{aij}, \Lambda_{aij}$ equal to zero and work with only the equation of the expansions. If one sets $\Sigma_{aij}$ equal to zero then one has to identify the components of $\Lambda_{aij}$ with the projections of $H_{\mu\nu\rho}$. Therefore, we can now attribute a physical meaning to $\Lambda_{aij}$ by associating it with the background antisymmetric tensor field’s projections. Further work is clearly needed to understand the consequences of actions other than Nambu–Goto and is in progress.

(iv) Finally, we indicate a possible application in a totally different area–the theory of biological (amphiphilic) membranes. Deformations of these membranes ($D = 2$ hypersurfaces in a $D = 3$ Euclidean background) can be analysed using the same formalism as presented here and may turn out to be useful in understanding the fluctuations of these objects. The simplest case of the catenoidal membrane is discussed in detail in the appendix to [5].

**ACKNOWLEDGEMENT**

I thank B. R. Iyer and G. Date for inviting me to this meeting and for the excellent hospitality provided at the Institute of Mathematical Sciences, Madras.
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