A KAM Theorem for finitely differentiable Hamiltonian systems

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Abstract
Given \( l > 2\nu > 2d \geq 4 \), we prove the persistence of a Cantor–family of KAM tori of measure \( O(\varepsilon^{1/2-\nu/l}) \) for any non–degenerate nearly integrable Hamiltonian system of class \( C^l(\mathcal{D} \times \mathbb{T}^d) \), where \( \nu - 1 \) is the Diophantine power of the frequencies of the persistent KAM tori and \( \mathcal{D} \subset \mathbb{R}^d \) is a bounded domain, provided that the size \( \varepsilon \) of the perturbation is sufficiently small. This extends a result by D. Salamon in [1] according to which we do have the persistence of a single KAM torus in the same framework. Moreover, it is well–known that, for the persistence of a single torus, the regularity assumption can not be improved.

Keywords : Nearly integrable Hamiltonian systems; KAM Theory; Smooth KAM Tori; Arnold’s scheme; Cantor–like set; Smoothing techniques.

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1 Introduction

KAM Theory asserts that, for sufficiently regular non–degenerate nearly integrable Hamiltonian systems, a Cantor–like family of KAM tori of the unperturbed part survive any perturbation, being only slightly deformed, provided the perturbation is small enough. Moreover, the family of KAM tori of the perturbed system is of positive Lebesgue measure and tends to fill up the phase space as the perturbation tends to zero. A natural question is:

Question 1 In a fixed degrees of freedom $d$, how regular has to be the integrable Hamiltonian and the perturbation in order to get KAM tori?

It was Arnold [2], inspired by the breakthrough of Kolmogorov [3], who first proved the persistence of positive measure set of KAM tori of a real-analytic integrable Hamiltonian under a real-analytic perturbation, provided the latter is small enough. In 1962, J. Moser[4, 5] proved in the framework of area–preserving twist mappings of an annulus, the persistence of invariant curves of integrable analytic systems under $C^k$ perturbation, but for $k$ very high ($k = 333$); which, later on, was brought down by H. Rüssmann [6] to 5 and finally to the optimal value 3 by M.R. Herman [11].
Moser [8] proved the continuation of a single torus of an integrable real–analytic Hamiltonian under a perturbation of class \( C^{l+2} \), with \( l > 2d \). Then Pöschel [9, 10], following an idea due to Moser, showed that a Cantor–like family of KAM tori, of positive measure, of a non-degenerate integrable real–analytic Hamiltonian survive any sufficiently small perturbation of class \( C^k \), provided \( k > 3d - 1 \), and also showed that, for the persistence of a single torus of an integrable real–analytic Hamiltonian, it is sufficient to require the perturbation to be of class \( C^l \), provided \( l > 2d \). Later, refining this idea of Moser, D. Salamon [1] showed that, for the persistence of single torus, it sufficient that both of the integrable and perturbed part are of class \( C^l \), with \( l > 2d \). And, regarding the continuation of a single torus of the integrable system, the regularity assumption \( l > 2d \) turns out to be also sharp (see e.g. [11, 12]). Indeed, M. Herman [11] gave a counterexample of an exact area–preserving twist mappings of an annulus for which a given invariant curve can be destroyed by arbitrarily small perturbation of class \( C^{3-\iota} \). It corresponds in the Hamiltonian context to \( d = 2 \) and \( l = 4 - \iota \). In [11], he also provided a counterexample of an exact area–preserving twist mappings of an annulus of class \( C^{2-\iota} \) without any invariant curve, corresponding in our context to \( d = 2 \) and \( l = 3 - \iota \). Then, it has been widespread that

Conjecture 2  In \( d \)–degrees of freedom, a small perturbation of class \( C^l \) of a non–degenerate integrable Hamiltonian which is also of class \( C^l \), exhibits a positive measure set of KAM tori iff \( l > 2d \).

Albrecht has proven in [13] the persistence of KAM tori of a non–degenerate real–analytic integrable system under small enough perturbations of class \( C^{2d} \), provided that the moduli of continuity of the \( 2d \)–th partial derivatives of the perturbation satisfy some integral condition, weaker that the Hölder continuity condition. Yet, the KAM tori of the perturbed system form a zero measure set.

Given \( \alpha > 0 \), \( \tau > 0 \), a vector \( \omega \in \mathbb{R}^d \) is called \((\alpha, \tau)\)–Diophantine if

\[
|\omega \cdot k| \geq \frac{\alpha}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}.
\]

In this paper, we prove the “if” part of the Conjecture 2 i.e. roughly speaking:

Theorem 3  Consider a Hamiltonian of the form \( H(y, x) = K(y) + P(y, x) \) where \( K, P \in C^l(\mathcal{D} \times \mathbb{T}^d) \) and \( \mathcal{D} \subset \mathbb{R}^d \) is a non–empty and bounded domain.\(^2\) If \( K \) is non–degenerate and \( l > 2\nu > 2d \) then, all the KAM tori of the integrable system \( K \) whose frequency are \((\alpha, \tau)\)–Diophantine, with \( \alpha \simeq \varepsilon^{1/2-\nu/l} \) and \( \tau := \nu - 1 \), do survive, being only slightly

\(^1\)The positive constant \( \tau \) (resp. \( \alpha \)) is then called a Diophantine power (resp. Diophantine constant) of \( \omega \).

\(^2\)A domain is an open and connected set.
deformed, where $\varepsilon$ is the $C^l$–norm of the perturbation $P$. Moreover, letting $\mathcal{K}$ be the corresponding family of KAM tori of $H$, we have $\text{meas}(\mathcal{D} \times \mathbb{T}^d \setminus \mathcal{K}) = O(\varepsilon^{1/2-\nu/l})$.

To our best knowledge, the best result in this direction is due to A. Bounemoura and consigned in his nice paper [14], where he proved the persistence of positive measure set filled by the KAM tori of $H$ under the assumptions $K \in C^{l+2}$ and $P \in C^l$, with $l > 2d$. Bounemoura also proved that the region free of KAM tori is of measure $O(\sqrt{\varepsilon})$ which turns out to be sharp (see e.g. [15, 16]).

In the present paper, under the sharper assumption $K, P \in C^l$, we show that the measure estimate of the region free of KAM tori is of $O(\varepsilon^{1/2-\nu/l})$, which, netherless, yields in the limit $l \to \infty$ the optimal bound in the real–analytic case i.e. $O(\sqrt{\varepsilon})$ (see e.g. [15, 16]).

The proof shares two main features with [14]. Firstly, our proof uses also a quantitative approximation method of smooth functions by analytic functions introduced by Moser; however, here we have to approximate not only the pertubed part, but also the integrable part at each step of the KAM scheme as, unlike [14], we do not linearize the integrable part. Secondly, we also use the refined approximation given in [17, Theorem 7.2, page 134] instead of truncating the Fourier expansion of the perturbation at each step of the KAM scheme. But, unlike [14], in this paper we use a KAM scheme à la Arnold.\textsuperscript{3}

The strategy is to prove a general quantitative KAM Step for real–analytic perturbation of non–degenerate real–analytic integrable Hamiltonian systems (see Lemma 6). Then, one approximates, in a quantitative manner, both the integrable and perturbed part by a sequence of real–analytic functions on complex strips of widths decreasing to zero (see Lemma 7), yielding a suitable real–analytic approximation of the perturbed Hamiltonian, to each of which we apply the KAM Step. Then, one proves that indeed the procedure converges.

2 Notation

- For $d \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $x, y \in \mathbb{C}^d$, we let $x \cdot y := x_1\bar{y}_1 + \cdots + x_d\bar{y}_d$ be the standard inner product; $|x|_1 := \sum_{j=1}^d |x_j|$ be the 1–norm, and $|x| := \max_{1 \leq j \leq n} |x_j|$ be the sup–norm.
- $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$ is the $d$–dimensional (flat) torus.

\textsuperscript{3}Usually, in the literature, Moser’s idea is combined with his own KAM scheme (like [9, 10, 14] ) or with Kolmogorov scheme (like [1, 18]).
For $\alpha > 0$, $\tau \geq d - 1 \geq 1$,

$$\Delta^\alpha_\tau := \left\{ \omega \in \mathbb{R}^d : |\omega \cdot k| \geq \frac{\alpha}{|k|^\tau}, \quad \forall \ 0 \neq k \in \mathbb{Z}^d \right\},$$

(1)

is the set of $(\alpha, \tau)$–Diophantine numbers in $\mathbb{R}^d$.

- We denote by $\text{meas}$, the Lebesgue (outer) measure on $\mathbb{R}^d$;

- Given $l \in \mathbb{R}$, we shall denote its integer part by $\lfloor l \rfloor$ and its fractional part by $\{l\}$;

- For $l > 0$, $A$ any subset of $\mathbb{R}^d$ or of $\mathbb{R}^d \times \mathbb{T}^d$, we denote by $C^l(A)$ the set of continuously differentiable functions $f$ on $A$ up to the order $\lfloor l \rfloor$ such that $f^{[l]}$ is Hölder–continuous with exponent $\{l\}$ and with finite $C^l$–norm define by:

$$\|f\|_{C^l(A)} := \max \{\|f\|_{C^0(A)}, \|f^l\|_{C^0(A)}\},$$

$$\|f\|_{C^0(l)(A)} := \max_{k \in \mathbb{N}^d} \sup_{0 \leq |k| \leq \lfloor l \rfloor} |\partial^k_y f|,$$

$$\|f^l\|_{C^0(l)(A)} := \max_{k \in \mathbb{N}^d} \sup_{y_1, y_2 \in A, y_1 - y_2 \neq 0} \frac{|\partial^k_y f(y_1) - \partial^k_y f(y_2)|}{|y_1 - y_2|^{\lfloor l \rfloor}}.$$

When $A = \mathbb{R}^d$ or $A = \mathbb{R}^d \times \mathbb{T}^d$, we will simply write $\|f\|_{C^l}$ for $\|f\|_{C^l(A)}$.

- For $l > 0$, $A$ any subset of $\mathbb{R}^d$, we denote by $C^l_\mathcal{W}(A)$, the set of functions of class $C^l$ on $A$ in the sense of Whitney.

- For $r, s > 0$, $y_0 \in \mathbb{C}^d$, $\emptyset \neq \mathcal{D} \subseteq \mathbb{C}^d$, we denote:

$$\mathbb{T}_s^d := \left\{ x \in \mathbb{C}^d : |\text{Im} x| < s \right\} / 2\pi \mathbb{Z}^d,$$

$$B_r(y_0) := \left\{ y \in \mathbb{R}^d : |y - y_0| < r \right\}, \quad (y_0 \in \mathbb{R}^d),$$

$$D_r(y_0) := \left\{ y \in \mathbb{C}^d : |y - y_0| < r \right\},$$

$$D_{r,s}(y_0) := D_r(y_0) \times \mathbb{T}_s^d,$$

$$D_{r,s} : \emptyset := \bigcup_{y_0 \in \mathcal{D}} D_{r,s}(y_0).$$

- If $\mathbb{I}_d := \text{diag}(1)$ is the unit $(d \times d)$ matrix, we denote the standard symplectic matrix by

$$\mathbb{J} := \begin{pmatrix} 0 & -\mathbb{I}_d \\ \mathbb{I}_d & 0 \end{pmatrix}.$$

\footnote{We refer the reader for instance to [15, Appendix E, page 207] for details.}
For $D \subset \mathbb{C}^d$, $A_{r,s}(D)$ denotes the Banach space of real-analytic functions with bounded holomorphic extensions to $D_{r,s}(D)$, with norm

$$\| \cdot \|_{r,s,D} := \sup_{D_{r,s}(D)} | \cdot |.$$

We equip $\mathbb{C}^d \times \mathbb{C}^d$ with the canonical symplectic form

$$\varpi := dy \wedge dx = dy_1 \wedge dx_1 + \cdots + dy_d \wedge dx_d,$$

and denote by $\phi_H^t$ the associated Hamiltonian flow governed by the Hamiltonian $H(y, x)$, $y, x \in \mathbb{C}^d$.

$\pi_1: \mathbb{C}^d \times \mathbb{C}^d \ni (y, x) \mapsto y$ is the projection on the first $d$–components and, $\pi_2: \mathbb{C}^d \times \mathbb{C}^d \ni (y, x) \mapsto x$ is the projection on the last $d$–components.

Given a linear operator $\mathcal{L}$ from the normed space $(V_1, \| \cdot \|_1)$ into the normed space $(V_2, \| \cdot \|_2)$, its “operator–norm” is given by

$$\| \mathcal{L} \| := \sup_{x \in V_1 \setminus \{0\}} \frac{\| \mathcal{L} x \|_2}{\| x \|_1},$$

so that $\| \mathcal{L} x \|_2 \leq \| \mathcal{L} \| \| x \|_1$ for any $x \in V_1$.

Given $\omega \in \mathbb{R}^d$, the directional derivative of a $C^1$ function $f$ with respect to $\omega$ is given by

$$D_\omega f := \omega \cdot f_x = \sum_{j=1}^d \omega_j f_{x_j}.$$

If $f$ is a (smooth or analytic) function on $\mathbb{T}^d$, its Fourier expansion is given by

$$f = \sum_{k \in \mathbb{Z}^d} f_k e^{ik \cdot x}, \quad f_k := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) e^{-ik \cdot x} \, dx,$$

(where, as usual, $e := \exp(1)$ denotes the Neper number and $i$ the imaginary unit). We also set:

$$\langle f \rangle := f_0 = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(x) \, dx.$$
3 Assumptions

- Let \( l > 2\nu := 2(\tau + 1) > 2d \geq 4 \) and \( \mathcal{D} \subset \mathbb{R}^d \) be a non-empty, bounded domain.

- On the phase space \( \mathcal{D} \times \mathbb{T}^d \), consider the Hamiltonian
  \[
  H(y, x) := K(y) + P(y, x),
  \]
  where \( K, P \in C^l(\mathcal{D} \times \mathbb{T}^d) \) are given functions with finite \( l \)-norms \( \|K\|_{C^l(\mathcal{D})} \) and \( \varepsilon := \|P\|_{C^l(\mathcal{D} \times \mathbb{T}^d)} \).

- Assume that \( K_y \) is locally-uniformly invertible; namely that \( \det K_{yy} \neq 0 \) for all \( y \in \mathcal{D} \) and
  \[
  T := \|T\|_{C^0(\mathcal{D})} < \infty, \quad T(y) := K_{yy}(y)^{-1}.
  \]

  Set\(^5\)
  \[
  K := \max\{1, \|K\|_{C^l(\mathcal{D})}\}, \quad \theta := TK \geq 1.
  \]

- Let \( \alpha \in (0, 1) \) and set
  \[
  \alpha_* := \alpha^{\frac{1}{2-2\nu}}, \quad \mathcal{D} := \{y \in \mathcal{D} : B_{\alpha_*}(y) \subseteq \mathcal{D}\}
  \]
  and
  \[
  \mathcal{D}_\alpha := \{y \in \mathcal{D} : K_y(y) \in \Delta^\tau\}.
  \]

- Finally, set
  \[
  \sigma := \left(\frac{\varepsilon^{3/2}}{\theta^{2/(\nu \alpha \sqrt{K})}}\right)^{1/(l+\nu)}, \quad \rho := \frac{2C_1K \varepsilon}{\alpha^{2\sigma}} \varepsilon, \quad \beta_0 := \min\left\{\frac{l}{2\nu} - 1 + \frac{1}{\nu} , 2\right\},
  \]
  for some suitable constant \( C_1 = C_1(d, l) > 1 \).\(^6\)

4 Theorem

Under the notations and assumptions of § 2 and 3, the following Theorem holds.

**Theorem 4**

**Part I:** There exists a positive constant \( c = c(d, \tau, l) < 1 \) such that, if

\[
\alpha \leq c K, \quad \text{and} \quad \varepsilon \leq c K^{\frac{l+2\nu}{l-2\nu} \theta^{-a} \alpha^{\frac{2l}{l-2\nu}}},
\]

\(^{5}\)Indeed, \( \theta \geq \|T(y_0)\| \|K_{yy}(y_0)\| = \|T(y_0)\| \|T^{-1}(y_0)\| \geq 1 \), for any \( y_0 \in \mathcal{D} \).

\(^6\)\( C_1 \) is actually the constant appearing in Lemma 7.
where \( a := (l-2\nu)^{-1} \max \{ (6 + 2l^{-1})(l+\nu) - 2l(l-\nu), 2l(l+3\nu)^{-1} \} \), then, the following holds. There exist a Cantor–like set \( \mathcal{D} \subset \mathbb{D} \), an embedding \( \phi_*(u, v) : \mathcal{D} \times \mathbb{T}^d \to \mathcal{K} := \phi_*(\mathcal{D} \times \mathbb{T}^d) \subset \mathcal{D} \times \mathbb{T}^d \) of class \( C^1_{\mathcal{W}}(\mathcal{D} \times \mathbb{T}^d) \) such that the map \( \xi \mapsto \phi_*(y, x) \) is of class \( C^\beta(\mathbb{T}^d) \), for any given \( y \in \mathcal{D} \) (for any \( \nu^{-1} < \beta < \beta_0 \)), a function \( K_\nu \in C^2_{\mathcal{W}}(\mathcal{D}, \mathbb{R}) \), satisfying

\[
\mathbf{H} \circ \phi_*(y, x) = K_\nu(y, x), \quad \forall (y, x) \in \mathcal{D} \times \mathbb{T}^d.
\]

Moreover, the map \( G^* := (\partial_y K_* )^{-1} \circ \partial_y K : \mathcal{D} \to \mathcal{D} \) is well-defined and is a lypeomorphism onto \( \mathcal{D} \), \( B_{\alpha/2}(\mathcal{D}) \subset \mathcal{D} \), and \( \mathcal{K} \) is foliated by KAM tori of \( H \), each of which is a graph of a map of class \( C^\nu(\mathbb{T}^d) \). Furthermore,

\[
\|G^* - \text{id}\|_{L^\alpha} \leq \varepsilon \frac{\beta}{\tau^{\nu+\epsilon}} \mathbf{K}^{\frac{\nu}{2(\nu+\epsilon)}}, \quad \|G^* - \text{id}\|_{L^\alpha} < 1/2, \quad (5)
\]

\[
\sup_{\mathcal{D} \times \mathbb{T}^d} \max \left\{ \|W(\phi_* - \text{id})\|, \|\pi_2(\partial_{\nu} \phi_* - \text{id})\| \right\} \leq \theta^2 \log(\rho^{-1})^{-2\nu} < 1, \quad (6)
\]

where \( W := \text{diag}(K(\alpha\sigma^\nu)^{-1} \mathbb{I}_d, \sigma^{-1} \mathbb{I}_d) \).

**Part II:** Assume furthermore that the boundary \( \partial \mathcal{D} \) of \( \mathcal{D} \) is a smooth hypersurface of \( \mathbb{R}^d \) and

\[
0 < \alpha \leq \min \left\{ \frac{R(\mathcal{D})}{6}, \frac{1}{2} \text{minfoc}(\partial \mathcal{D}) \right\}, \quad (7)
\]

where \( \text{minfoc}(\partial \mathcal{D}) \) denotes the minimal focal distance of \( \partial \mathcal{D} \) and

\[
R(\mathcal{D}) := \sup \{ R > 0 : B_R(y) \subset \mathcal{D}, \text{ for some } y \in \mathcal{D} \}.
\]

Then, the following measure estimate holds:

\[
\text{meas}(\mathcal{D} \times \mathbb{T}^d \setminus \mathcal{K}) \leq (3\pi)^d \left( 2H^{d-1}(\partial \mathcal{D}) \varepsilon + C \varepsilon^2 + \text{meas}(\mathcal{D} \setminus \mathcal{D}_\alpha) \right), \quad (8)
\]

where \( R^{\partial \mathcal{D}} \) denotes the curvature tensor of \( \partial \mathcal{D} \), \( k_{2j}(R^{\partial \mathcal{D}}) \), the \( (2j) \)-th integrated mean curvature of \( \partial \mathcal{D} \) in \( \mathbb{R}^d \),

\[
\varepsilon := \max \left\{ \varepsilon \frac{\beta}{\tau^{\nu+\epsilon}} K^{\frac{\nu}{2(\nu+\epsilon)}}, \alpha \right\},
\]

and

\[
C = C(d, \tau, l, \varepsilon, \alpha, T, K, R^{\partial \mathcal{D}}) := 2 \sum_{j=1}^{d-1} \frac{\varepsilon^{2j-1} k_{2j}(R^{\partial \mathcal{D}})}{1 \cdot 3 \cdots (2j + 1)}.
\]

\footnote{See (i) in Remark 5 below.}

\footnote{Observe that the condition \( \alpha \leq R(\mathcal{D})/6 \) ensures that the interior of \( \mathcal{D} \) is non–empty.}

\footnote{We refer the reader to \([16, 15]\) for more details.}
Remark 5  (i) By definition,
\[ \partial y_\ast K_\ast \circ G^* = \partial y K \quad \text{on} \quad D_\alpha. \]  

Now, from (9) and (4), one deduces that the embedded $d$–tori
\[ \mathcal{T}_{\omega_*;\varepsilon} := \phi_* \left( y_*; \mathbb{T}^d \right), \quad y_* \in D_\ast, \quad \omega_* := \partial y_* K_\ast (y_\ast) \in \Delta_{\alpha}^\ast, \]  

are non–degenerate, invariant, Lagrangian Kronecker tori of class $C_\text{W}^{\beta_\nu}$ (for any $\nu^{-1} < \beta < \beta_0$) for $H$, i.e. KAM tori, with Diophantine frequency $\omega_\ast$ i.e.
\[ \phi^t_H \circ \phi_*(y_\ast, x) = \phi_*(y_\ast, x + \omega_* t), \quad \forall \, x \in \mathbb{T}^d. \]  

Indeed, as each $\phi^t_j$ is symplectic, we have
\[ \phi^t_{H_{j-1}} \circ \phi^t_j = \phi^t_j \circ \phi^t_{H_{j-1} \circ \phi^t_j}. \]  

Now, pick $y_* \in D_\ast$ and $y_j \in D_j$ converging to $y_*$. Letting $\omega_* := \partial y_* K_\ast (y_\ast)$, we have
\[ \phi^t_{H_{j-1} \circ \phi^t_j}(y_j, x) \overset{(28)}{=} (y_j, x + t \omega_\ast) + O(r_j^{-1} \| P_j \| r_j, s_j, \varphi_j + \| y_j - y_\ast \|), \quad \lim_{j \to \infty} r_j^{-1} \| P_j \| r_j, s_j, \varphi_j \overset{(31)}{=} 0. \]  

Then, recalling that $H_j$ converges uniformly to $H$ on $\mathbb{R}^d \times \mathbb{T}^d$, we have, for any $x \in \mathbb{T}^d$,
\[ \phi^t_H \circ \phi_*(y_\ast, x) = \lim_{j \to \infty} \phi^t_{H_{j-1} \circ \phi^t_j}(y_j, x) \overset{(12)}{=} \lim_{j \to \infty} \phi^t_j \circ \phi^t_{H_{j-1} \circ \phi^t_j}(y_j, x) \overset{(13)}{=} \phi_*(y_\ast, x + t \omega_\ast), \]

and (11) is proven. In particular, each torus $\mathcal{T}_{\omega_*;\varepsilon}$ is Lagrangian by Lemma C.1.10

(ii) Choosing $\alpha \simeq \varepsilon^{1/2 - \nu/l}$ in (3), we get $\hat{\varepsilon} = O(\varepsilon^{1/2 - 1/l})$ and therefore, plugging them into (8), we obtain
\[ \text{meas}(D \setminus \mathcal{K}) = O(\varepsilon^{3/4 - \delta}), \]  

which agrees for $l \to \infty$ with the sharp measure of KAM tori for smooth Hamiltonian systems i.e. $O(\sqrt{\varepsilon})$. Moreover, (5) yields the sharp bound $O(\varepsilon^{3/4 - \delta})$ on the displacement of each persistent invariant torus from the corresponding unperturbed one i.e.
\[ \| G^* - \text{id} \|_{D_\ast} = O(\varepsilon^{3/4 - \delta}). \]

10 An alternative proof is as follows. Let $v_j := \pi_1 \phi_j$ and $u_j := \pi_2 \phi_j$. Then, as $\phi_j$ is symplectic, we have $(\partial x u_j)^T \partial x v_j = (\partial x v_j)^T \partial x u_j$, and letting $j \to \infty$, we get $(\partial x u_\ast)^T \partial x v_\ast = (\partial x v_\ast)^T \partial x u_\ast$ i.e. the tori $\mathcal{T}_{\omega_*;\varepsilon}$ are Lagrangian, where $A^T$ denotes the transpose of the matrix $A$. 

9
The relation in (6) yields oscillations of $O(\sqrt{\varepsilon})$ for each perturbed torus:
\[
\sup_{y \in \mathcal{D}_y, x, x' \in T^d} \sup |v_{y*}(y, x) - v_{y*}(y, x')| = O(\sqrt{\varepsilon}),
\]
which is sharp (see e.g. [19]). It is worth mentioning that, in order to get (14), the smoothness assumption on the boundary of the domain can be removed using a different argument. The argument consists in slicing the domain into small pieces, then construct in each of those pieces a family of KAM tori and estimate their respective relative measures, and finally some them all up (see [16, 15] for more details).

5 Proof of Theorem 4

5.1 General step of the KAM scheme

Lemma 6 Let $r > 0$, $0 < \sigma \leq 1$, $0 < 2\sigma < s \leq 1$, $\mathcal{D}_2 \subset \mathbb{R}^d$ be a non-empty, bounded domain. Consider the Hamiltonian
\[
H(y, x) := K(y) + P(y, x),
\]
where $K, P \in \mathcal{A}_{r, s}(\mathcal{D}_2)$. Assume that
\[
\begin{align*}
det K_{yy}(y) &\neq 0, \\
\|K_{yy}\|_{r, \mathcal{D}_2} &\leq K, \\
\|P\|_{r, s, \mathcal{D}_2} &\leq \varepsilon, \\
T(y) &:= K_{yy}(y)^{-1}, \quad \forall y \in \mathcal{D}_2, \\
\|T\|_{\mathcal{D}_2} &\leq T, \\
K_y(\mathcal{D}_2) &\subset \Delta^\alpha.
\end{align*}
\]
Assume that
\[
\sigma^{-\nu} \frac{\varepsilon}{\alpha r} \leq \rho \leq \frac{1}{4} \quad \text{and} \quad r \leq \frac{\alpha}{K^\sigma \rho}. \tag{16}
\]
Let
\[
\begin{align*}
\theta &:= TK, \\
\lambda &:= \log \rho^{-1}, \\
\kappa &:= 6\sigma^{-1} \lambda, \\
\check{r} &:= \frac{r}{32dTK}, \\
\bar{r} &\leq \min \left\{ \frac{\alpha}{2dK^{\kappa}}, \check{r} \right\}, \\
\tilde{r} &:= \frac{\check{r}}{16dTK}, \\
\bar{s} &:= s - \frac{2}{3} \sigma, \\
\bar{s}' &:= s - \sigma, \\
L &:= C_0 \frac{\theta T \varepsilon}{\tilde{r} \bar{r}}. \tag{17}
\end{align*}
\]
Assume:
\[
L \leq \frac{\bar{s}}{3}. \tag{18}
\]
Then, there exists a diffeomorphism $G : D_{\tilde{r}}(\mathcal{D}_2) \to G(D_{\tilde{r}}(\mathcal{D}_2))$, a symplectic change of coordinates
\[
\phi' = \text{id} + \tilde{\phi} : D_{\bar{r} + \bar{s}' / 3}(\mathcal{D}_2') \to D_{\bar{r} + \bar{r} / 3}(\mathcal{D}_2'), \tag{19}
\]
such that
\[
\begin{cases}
H \circ \phi' =: H' =: K' + P', \\
\partial_y K' \circ G = \partial_y K, \quad \det \partial_y^2 K' \circ G \neq 0 \quad \text{on } D,
\end{cases}
\]
with \(K'(y') := K(y') + \tilde{K}(y') := K(y') + \langle P(y'), \cdot \rangle\). Indeed, \(G = (\partial_y K')^{-1} \circ K_y\). Moreover, letting \((\partial_y^2 K'(y'))^{-1} =: T(y') + \tilde{T}(y')\), \(y' \in G(D)\), the following estimates hold.

\[
\|\partial_y^2 \tilde{K}\|_{r/2, \mathscr{A}_2} \leq KL, \quad \|G - \text{id}\|_{\mathcal{F}, \mathscr{A}_2} \leq \tau L, \quad \|\tilde{T}\|_{\mathscr{A}_2} \leq TL,
\]

\[
\max\{\|W_\tilde{\phi}\|_{r/2,s', \mathscr{A}} , \|\pi_2 \partial_{s'} \tilde{\phi}\|_{r/2,s', \mathscr{A}}\} \leq C_1 \frac{\varepsilon}{\alpha r \sigma^p}, \quad \|P'\|_{r/2,s', \mathscr{A}} \leq C_1 \rho \varepsilon,
\]

where
\[
D := G(D), \quad \left(\partial_y^2 K'(y')\right)^{-1} =: T \circ G^{-1}(y') + \tilde{T}(y'), \quad \forall y' \in D,
\]
\[W := \text{diag}(r^{-1} \mathbb{1}_d, \sigma^{-1} \mathbb{1}_d)\).

**Proof** The proof follows essentially the same lines as the one of the KAM Step in [15] (see also [16]) modulo two changes:

(i) To construct the generating function, as in [14], we use the approximation given in [17, Theorem 7.2, page 134] instead of truncating the Fourier expansion of \(P\).

(ii) We use systematically the estimate in 2. of Lemma A.2 to estimate the generating function as well as its derivatives.

Those two modifications improve a lot the KAM Step; in particular it yields the optimal power of the lost of regularity \(\sigma\), which is crucial in the KAM Theory for finitely differentiable Hamiltonian systems, at least from the Moser’s “analyticing” idea point of view. We refer the reader to Appendix C for an outline of the proof. \(\blacksquare\)

### 5.2 Characterization of smooth functions by mean of real–analytic functions

The following two Lemmata, which will be needed from Lemma 9 on and may be found in [20, 1].

**Lemma 7 (Jackson, Moser, Zehnder)** Given \(l > 0\), there exists \(C_1 = C_1(d, l) > 0\) such that for any \(f \in C^l(\mathbb{R}^d)\) and for any \(s > 0\), there exists a real–analytic function \(f_s: O_s := \{(y, x) \in \mathbb{C}^d \times \mathbb{C}^d : |\text{Im} \ (y, x)| < s\} \to \mathbb{C}\) satisfying the following:

\[
\sup_{O_s} |f_s| \leq C_1 \|f\|_{C^0}, \quad \sup_{\alpha \in \mathbb{N}^d} \sup_{O_s} |\partial^\alpha f_s - \partial^\alpha f_s'| \leq C_1 \|f\|_{C^l} s^{l-l'}, \quad \|f - f_s\|_{C^l} \leq C_1 \|f\|_{C^l} s^{l-l'},
\]

\[
(22)
\]
for any $0 < s' < s$ and any $0 \leq l' \leq l$ with $l' \in \mathbb{N}$. If, in addition, $f$ is periodic in some component $y_j$ or $x_j$, then so is $f_z$ in that component.

Lemma 8 (Bernstein, Moser) Assume that \{f_j\}_{j \geq 0} is a sequence of real-analytic functions defined respectively on $O_j := \{(y, x) \in \mathbb{C}^d \times \mathbb{C}^d : \text{Im}(y, x) < s_j\}$ such that

$$\sup_{O_j} |f_j - f_{j-1}| \leq \gamma s_j^{j-1}, \quad \forall \, j \geq 1,$$

where $l \in \mathbb{R} \setminus \mathbb{Z}$, $\gamma > 0$ and $s_j := s_0 \xi_j$, with $s_0 > 0$ and $0 < \xi < 1$. Then, $f_j$ converges uniformly on $\mathbb{R}^d$ to a function $f \in C^d(\mathbb{R}^d \times \mathbb{T}^d)$. Moreover, if all the $f_j$ are periodic in some component $y_i$ or $x_i$, then so is $f$ in that component.

5.3 Iteration of the KAM step and convergence

Let $K, T, \theta, \varepsilon, \sigma, \rho, \alpha_*$, $\beta_0$ be as in §3 and 4. Let $0 < m < l/2 - \nu$, $0 < \widehat{m} < \min\{(m + 1)/\nu, 2\}$, $l' := \max\{(6 + 2l/\nu)(l + \nu)/(l - 2\nu) - 2l(l - \nu)/(l - 2\nu), 2l(l + 3\nu)/(\nu(l - 2\nu))\}$ and for $j \geq 0$, let

\[
\begin{align*}
\sigma_0 &:= C_2^{-1} \sigma, \quad s_0 := 4\sigma_0, \quad r_0 := \alpha \sigma_0^\nu/(2K), \quad \lambda := \log \rho^{-1}, \\
\xi &:= (C_2 \theta^{1/\nu} \lambda)^{-1}, \quad \sigma_j := \sigma_0 \xi_j, \quad s_j := 4\sigma_j = 4\sigma_0 \xi_j, \quad \sigma_j := \xi_j, \quad \kappa_j := 6\sigma_j^{-1} \lambda, \\
r_j &:= r_0 \xi_j, \quad \tilde{r}_{j+1} := \frac{r_0}{64d\theta} \xi_j, \quad \tilde{r}_{j+1} := \frac{r_0}{2^{11} d^2 \theta} \xi_{j+1}, \quad \xi_0 := s_0, \quad \xi_{j+1} := \sigma_j, \\
S_j &:= \{y \in \mathbb{C}^d : \text{Im}(y) < \xi_j\}, \quad O_j := \{(y, x) \in \mathbb{C}^d \times (\mathbb{C}^d / \mathbb{T}^d) : \text{Im}(y, x) < \xi_j\}, \\
\|\cdot\|_{\xi_j} &:= \sup_{O_j} |\cdot|.
\end{align*}
\]

First of all, we extend $K$ and $P$ to the whole phase space $\mathbb{R}^d \times \mathbb{T}^d$.

5.3.1 Extension of $K$ and $P$ to the whole space

First of all, there exist $C_0 = C_0(d, l) > 0$ and a Cut-off $\chi \in C(\mathbb{C}^d) \cap C^\infty(\mathbb{R}^d)$ with $0 \leq \chi \leq 1$, supp $\chi \subseteq D_{\alpha_*/2}(\mathscr{O})$, $\chi \equiv 1$ on $D_{\alpha_*/2}(\mathscr{O})$ and for any $k \in \mathbb{N}$ with $|k|_1 \leq l$,

$$\|\partial_y^k \chi\|_{\mathbb{R}^d} \leq C_0 \alpha_*^{-|k|_1}.$$

By the Fàa Di Bruno’s Formula [21], there exists $C_1 = C_1(d, l) > 0$ such that for any $f \in C^l(\mathbb{R}^d \times \mathbb{T}^d)$, we have

$$\|\chi \circ \pi_1 \cdot f\|_{C^l} \leq C_1 \alpha_*^{-l} \|f\|_{C^l}. \quad (23)$$

\[\footnote{see for instance [15, Lemma 2.2.1]}\]
Let $\hat{K} \in C^l(\mathbb{R}^d)$ such that\(^{12}\) $|T|_\mathcal{D} \| \hat{K} - K \|_{C^l(\mathcal{D})} \leq C_1^{-1} \alpha_s^l / 4$. Thus, $\hat{K}_{yy} = K_{yy}(1_d + T(\hat{K}_{yy} - K_{yy}))$ is invertible on $\mathcal{D}$ and $\| (\hat{K}_{yy})^{-1} \|_\mathcal{D} \leq 2 |T|_\mathcal{D}$. Then, $K := \hat{K} + \chi \cdot (K - \hat{K}) \in C^l(\mathbb{R}^d \times \mathbb{T}^d)$, $K \equiv K$ on $D_{\alpha_s/2}(\mathcal{D}')$ and

$$\| K \|_{C^l} \leq \| K \|_{C^l} + C_1 \alpha_s^{-l} \| \hat{K} - K \|_{C^l} \leq \| K \|_{C^l} + |T|_{C^l}^{-1} / 4 < 2 \| K \|_{C^l}$$

and

$$\| (\hat{K}_{yy})^{-1} \|_{C^l} \leq \| (\hat{K}_{yy})^{-1} \|_{C^l} \cdot C_1 \alpha_s^{-l} \| \hat{K} - K \|_{C^l} \leq 1 / 2.$$

Therefore, $K_{yy}$ is in particular invertible and $\| (K_{yy})^{-1} \|_{C^l} \leq 2 \| (\hat{K}_{yy})^{-1} \|_{C^l} \leq 4 |T|_{C^l}$. Similarly, one extends $P$ to a function $P \in C^l(\mathbb{R}^d \times \mathbb{T}^d)$ such that $K \equiv K$ on $D_{\alpha_s/2}(\mathcal{D}')$ and $\| P \|_{C^l} \leq 2 \| P \|_{C^l}$. Now, letting $H := K + P$, we have $H_{|D_{\alpha_s/2}(\mathcal{D}')} = H$. Hence, it does not make any difference for us replacing $H$ by $H$ since the invariant tori of $H$ we shall construct live precisely in $D_{\alpha_s/2}(\mathcal{D}')$ as $r_0 < \alpha_s / 2$.

Let $\mathcal{K}_j$ (resp. $\mathcal{P}_j$) be the real–analytic approximation $K_{\xi_j}$ (resp. $P_{\xi_j}$) of $K$ (resp. $P$) defined on $\mathcal{O}_j$ given by Lemma 7. Then, the following holds.

### 5.3.2 Iteration of the KAM Step

There exist constants $C_j = C_j(d, \tau, l) > 1$ ($j = 1, \ldots, 6$) such that the following holds.

**Lemma 9** Set $\mathcal{D}_0 := \{ y \in \mathbb{R}^d : \partial_y \mathcal{K}_0(y) \in \partial_y K(\mathcal{D}_0) \}$. Assume that

$$\alpha \leq C_3^{-1} K \quad \text{and} \quad C_3 \in K^{\frac{l+2\nu}{2\nu} \theta'} \alpha^{-\frac{2l}{2\nu}} \leq 1 . \tag{24}$$

Then, the following assertions ($\mathcal{P}_j$), $j \geq 1$, hold. There exist a sequence of sets $\mathcal{D}_j$, a sequence of diffeomorphisms $G_j : D_{r_j}(\mathcal{D}_{j-1}) \rightarrow G_j(D_{r_j}(\mathcal{D}_{j-1}))$, a sequence of real–analytic symplectic transformations

$$\phi_j = (v_j, u_j) : D_{r_j, s_j}(\mathcal{D}_j) \rightarrow D_{r_{j-1}, s_{j-1}}(\mathcal{D}_{j-1}) , \tag{25}$$

such that, setting $\mathcal{H}_{j-1} := \mathcal{K}_{j-1} + \mathcal{P}_{j-1}$, we have

$$G_j(\mathcal{D}_{j-1}) = \mathcal{D}_j \subset \mathcal{D}_j , \quad G_j = (\partial_y K_j)^{-1} \circ \partial_y K_{j-1} , \tag{26}$$

$$\det \partial_y^2 K_j(y) \neq 0 , \quad T_j(y) := \partial_y^2 K_j(y)^{-1} , \quad \forall y \in \mathcal{D}_j , \tag{27}$$

$$H_j := H_{j-1} \circ \phi_j = : K_j + P_j \quad \text{on} \quad D_{r_j, s_j}(\mathcal{D}_j) , \tag{28}$$

where $\phi_j := \phi_1 \circ \phi_2 \circ \cdots \circ \phi_j$ and $K_0 := \mathcal{K}_0$.

\(^{12}\)Observe that $C_3^{-1} \alpha_s^l / 4 < 1 / 4$. 

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Moreover,
\[
\|G_j - \text{id}\|_\mathcal{V}_{j-1} \leq \bar{r}_j \xi^{2^v} \xi^{m(j-1)}, \quad \|\partial_y G_j - \text{id}\|_{\mathcal{V}_{j-1}} \leq \xi^{2^v} \xi^{2m(j-1)}, \tag{29}
\]
\[
\|\partial^2_y K_j\|_{\mathcal{V}_j} < 2K, \quad \|T_j\|_{\mathcal{V}_j} < 2T, \quad T_j := \left(\partial^2_y K_j\right)^{-1}, \tag{30}
\]
\[
\|P_j\|_{\mathcal{V}_{j-1}} \leq C_1 K \xi^{l_j}, \tag{31}
\]
\[
\max \left\{ \|W_j(\phi_j - \text{id})\|_{\mathcal{V}_j}, \|\pi_2 \partial_x (\phi_j - \text{id})\|_{\mathcal{V}_j} \right\} \leq \xi^{2^v} \xi^{m(j-1)}, \tag{32}
\]
where \(W_j := \text{diag} \left( r_{j-1}^{-1} \mathbb{1}_d, \sigma_{j-1}^{-1} \mathbb{1}_d \right) \).

**Remark 10** Observe that
\[
s_j + \sigma_{j-1}/3 = (12\xi + 1)\sigma_{j-1}/3 < 2s_{j-1}/3 < s_{j-1}/2, \tag{33}
\]
\[
2r_j + r_{j-1}\sigma_{j-1}/3 < r_{j-1}/4 + r_{j-1}/6 < r_{j-1}/2, \tag{34}
\]
\[
2r_j + r_{j-1}\sigma_{j-1}/3 < \sigma_0^j \xi^j + \sigma_{j-1}/3 = \sigma_0^j \sigma_j + \sigma_{j-1}/3 < \sigma_{j-1}, \tag{35}
\]
which combined with (19) imply
\[
\phi_j(Dr_{j}, s_j(\mathcal{V}_j)) \subset D_{\sigma_{j-1}, \sigma_{j-1}}(\mathcal{V}_{j-1}) \cap Dr_{j-1/2, \sigma_{j-1}/2}(\mathcal{V}_{j-1}), \tag{36}
\]
and, in particular, (25). Also,
\[
2r_{j+1} \leq \frac{1}{4} \min \left\{ \frac{\alpha}{2d(2K)\kappa_j^p}, \bar{r}_{j+1} \right\}, \tag{37}
\]
which, together with the definitions of the sequences of the various parameters, implies that (17) is fulfilled for any \(j \geq 1\).

**Proof**
**Step 1:** We check (\(\mathcal{P}_1\)). We claim that we can apply Lemma 6 to \(\mathcal{H}_0\). Indeed, we have
\[
\|\mathcal{P}_0\|_{\mathcal{V}_0} \leq \|\mathcal{P}_0\|_{\mathcal{V}_0} \leq C_1 \|P\|_{\mathcal{V}_0} \leq 2C_1 \varepsilon. \tag{38}
\]
From
\[
\partial^2_y \mathcal{K}_0 = \partial^2_y K(\mathbb{1}_d + T\partial^2_y (\mathcal{K}_0 - \mathcal{K}))
\]
and
\[
\|T\partial^2_y (\mathcal{K}_0 - \mathcal{K})\|_{\mathcal{V}_0} \leq \sup_{\mathcal{S}_0} \|T\partial^2_y (\mathcal{K}_0 - \mathcal{K})\| \leq C_1 \theta \xi^{l-2} \leq \frac{1}{2},
\]

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it follows that $\partial_y^2 K_0$ is invertible, $\|\partial_y^2 K_0\|_{r_0, \varphi_0} \leq 2K$ and

$$\|(\partial_y^2 K_0)^{-1} - T\|_{\varphi_0} \leq 2TC_1 \theta s_0^{-2} < T, \quad \|(\partial_y^2 K_0)^{-1}\|_{\varphi_0} < 2T.$$  \hspace{1cm} (39)

Hence, $H_0$ satisfies the assumptions in (15) with $\varepsilon \sim C_1 \varepsilon$, $r \sim r_0$, $s \sim s_0$, $\sigma \sim \sigma_0$, $K \sim 2K$. Now, observe that

$$\rho = \frac{2C_1 \varepsilon}{\alpha r_0 \sigma_0^\nu} \leq \frac{1}{4} \quad \text{and} \quad r_0 = \frac{\alpha}{2K} \sigma_0^\nu,$$

and, therefore, (16) is verified. Moreover, by the definitions and (37), (17) holds trivially with $T \sim 2T$, $\theta \sim 4TK \kappa \sim \kappa_0$, $\bar{r} \sim \bar{r}_1$, $\bar{r} \sim 4r_1$, $\bar{r} \sim \bar{r}_1$, $\bar{s} \sim s_0 - 2\sigma_0/3$, $s' \sim s_0 - \sigma_0$, $\bar{\sigma} \sim \sigma_0$, $L \sim 32C_0 C_1 K T^2 \bar{\varepsilon}/(r_0\bar{r}_1) \leq \sigma_0/3$. Consequently, we can apply Lemma 6 to $H_0$ to obtain the change of coordinates $\phi^1 = \phi_1$. In particular, (19) yields $(25)_{j=1}$, (20) yields $(26)_{j=1} \div (28)_{j=1}$ and, (21) yields $(29)_{j=1} \div (32)_{j=1}$. Therefore, $(\mathcal{P}_1)$ is proven.

**Step 2:** We assume $(\mathcal{P}_j)$ holds for some $j \geq 1$ and check $(\mathcal{P}_{j+1})$. Write

$$H_j := K_j + P_j = H_{j-1} + (K_j - K_{j-1}) + (P_j - P_{j-1}).$$

By the inductive assumption and (36), we have

$$\begin{align*}
H_j \circ \phi^j &= H_{j-1} \circ \phi^j + (K_j - K_{j-1}) \circ \phi^j + (P_j - P_{j-1}) \circ \phi^j \\
&= K_j + P_j + (K_j - K_{j-1}) \circ \phi^j + (P_j - P_{j-1}) \circ \phi^j \\
&= \mathcal{K}^j + \mathcal{P}^j \quad \text{on} \quad D_{r_j, s_j}(\mathcal{O}_j),
\end{align*}$$

where $\mathcal{K}^j := K_j$ and $\mathcal{P}^j := P_j + (K_j - K_{j-1}) \circ \phi^j + (P_j - P_{j-1}) \circ \phi^j$, with

$$\|\partial_y^2 \mathcal{K}^j\|_{r_j, \varphi_j} < 2K, \quad \|(\partial_y^2 \mathcal{K}^j)^{-1}\|_{\varphi_j} < 2T,$$

by the inductive assumption, provided $\phi^j$ maps $D_{r_j, s_j}(\mathcal{O}_j)$ into $\mathcal{O}_j = \{(y, x) \in \mathbb{C}^d \times \mathbb{C}^d : |\text{Im} (y, x)| < \xi_j\}$ i.e.

$$\sup_{D_{r_j, s_j}(\mathcal{O}_j)} |\text{Im} \phi^j| \leq \frac{\xi_j}{2},$$

(41)

which we now prove. Observe that, for any $1 \leq i \leq j$,

$$\|W_i W_{i+1}^{-1}\| = \xi \quad \text{and} \quad \|W_i (D\phi_i - 1_{2d}) W_{i-1}^{-1}\|_{r_{i,s_i}, \varphi_i} \leq 2 \xi^\nu \xi^{m(i-1)}.$$  \hspace{1cm} (42)

\textsuperscript{13}“$a \sim b$” stands for “$a$ replaced by $b$.”

\textsuperscript{14}\|W_i (D\phi_i - 1_{2d}) W_{i-1}^{-1}\|_{r_{i,s_i}, \varphi_i} = \max\{\|\partial_y v_j - \text{Id}\|_{r_{i,s_i}, \varphi_i}, \frac{s_j^{-1}}{r_{i-1}} \|\partial_x v_j\|_{r_{i,s_i}, \varphi_i}, \frac{r_{i-1}}{s_j^{-1}} \|\partial_y u_j\|_{r_{i,s_i}, \varphi_i}, \|\partial_x u_j - \text{Id}\|_{r_{i,s_i}, \varphi_i}\}.
Thus, writing $W_1D\phi^jW_j^{-1} = (W_1D\phi^1W_1^{-1})(W_1W_2^{-1})\cdots(W_jD\phi^jW_j^{-1})$, we then get from (42):

$$\|W_1D\phi^jW_j^{-1}\|_{r_j,s_j,\mathcal{D}_j} \leq \xi^j \prod_{i=1}^{j}(1 + 2\zeta^{\nu}\zeta^{m(i-1)})$$

$$\leq \xi^j \exp(4\zeta^{\nu}) \leq 2\xi^{j-1}. \hspace{1cm} (43)$$

Now, writing $(y, x) = X + iY$, where $X = X(y, x) := \text{Re}(y, x) \in \mathbb{R}^d \times \mathbb{R}^d$ and $Y = Y(y, x) := \text{Im}(y, x) \in \mathbb{R}^d \times \mathbb{R}^d$, we obtain

$$\phi^j(y, x) = \phi^j(X) + i\mathcal{J}(y, x),$$

where

$$\mathcal{J}(y, x) := W_1^{-1} \int_0^{1} W_1D\phi^j(X + itY)W_j^{-1}dt \cdot W_jY,$$

so that, as $\phi^j$ is real on reals, we have $\phi^j(X) \in \mathbb{R}^d$ and, therefore, $\text{Im}(\phi^j(y, x)) = \text{Re}(\mathcal{J}(y, x))$. We have, for any $y_j \in \mathcal{D}_j \subset \mathbb{R}^d$ and $y \in D_{r_j}(y_j),$

$$\frac{|\text{Im}(y)|}{r_j^{-1}} \leq \frac{1}{r_j^{-1}} (|\text{Im}(y - y_j)| + |\text{Im}(y_j)|) \leq \frac{1}{r_j^{-1}} |y - y_j| \leq \frac{1}{r_j^{-1}} r_j = \zeta^\nu, \hspace{1cm} (44)$$

so that

$$\sup_{D_{r_j},s_j(\mathcal{D}_j)} |W_jY| = \sup_{(y, x) \in D_{r_j},s_j(\mathcal{D}_j)} \max \left\{ \frac{|\text{Im}(y)|}{r_j^{-1}}, \frac{|\text{Im}(x)|}{s_j^{-1}} \right\} \leq \max \{\zeta^\nu, 4\zeta\} \leq 4\zeta. \hspace{1cm} (45)$$

Moreover, since $\|W_1^{-1}\| \leq \sigma_0$, we have, for any $(y, x) \in D_{r_j,s_j}(\mathcal{D}_j),$

$$|\text{Im}(\phi^j(y, x))| = |\text{Re}(\mathcal{J}(y, x))| \leq \|W_1^{-1}\| \left\| \int_0^{1} W_1D\phi^j(X + itY)W_j^{-1}dt \right\||W_jY|$$

$$\leq \sigma_0 \cdot 2\xi^{j-1} \cdot 4\zeta = 8\sigma \xi < \xi_j, \hspace{1cm} (43)+(45)$$
which completes the proof of (41). Hence,\(^\text{15}\)
\[\|\mathcal{P}^j\|_{r_j,s_j,\mathcal{D}_j} \leq \|P_j\|_{r_j,s_j,\mathcal{D}_j} + \|(K_j - K_{j-1}) \circ \phi^j\|_{r_j,s_j,\mathcal{D}_j} + \|(P_j - P_{j-1}) \circ \phi^j\|_{r_j,s_j,\mathcal{D}_j} \]
\[\leq C_1 K_{\xi_j} + \|K_j - K_{j-1}\|_{\xi_j} + \|P_j - P_{j-1}\|_{\xi_j} \]
\[\leq C_1 K_{\xi_j} + \|K_j - K_{j-1}\|_{\xi_j} + C_1 \in \xi_{j-1} \]
\[\leq 3 C_1 K_{\xi_{j-1}} \]
(46)

Thus, thanks to (40), \(H_j \circ \phi^j = K^j + \mathcal{P}^j\) satisfies the assumptions in (15) with \(\varepsilon \sim \|\mathcal{P}^j\|_{r_j,s_j,\mathcal{D}_j}, r \sim r_j, s \sim s_j, \sigma \sim \sigma_j, \kappa \sim 2K\) as

\[\partial_y^2 K^j(\mathcal{D}_j) \overset{\text{def}}{=} \partial_y^2 K_j(G_j(\mathcal{D}_j)) = \partial_y^2 K_{j-1}(\mathcal{D}_j) = \cdots = \partial_y^2 K_0(\mathcal{D}_0) \subset \Delta_\alpha.\]

Hence, in order to apply Lemma 6 to \(H_j \circ \phi^j = K^j + \mathcal{P}^j\), we need only to check (16), (17) and (18). But, by the definitions and (37), (17) holds trivially with \(T \sim 2T, \theta \sim 4TK\)

\[\kappa \sim \kappa_j, r \sim r_{j+1}, \bar{r} \sim 4r_{j+1}, \bar{\sigma} \sim \bar{\sigma}_j, \bar{\bar{r}} \sim 4r_{j+1}, s \sim s_j - 2\sigma_j/3, s' \sim s_{j+1} < s_j - \sigma_j,\]

\[L \sim 32C_0K^2\|\mathcal{P}^j\|_{r_j,s_j,\mathcal{D}_j}/(r_j \bar{r}_{j+1}).\]

Moreover, we have,

\[r_j = r_0C^\nu j \leq \frac{\alpha}{2K}\sigma_j^{(40)}/\alpha r_1 \leq C_2\sigma_0 K \leq \rho^{-1} \leq 1,\]

\[3C_0\frac{\theta T\|\mathcal{P}^j\|_{r_1,s_1,\mathcal{D}_1}}{r_1 \bar{r}_2 \bar{\sigma}_1} \leq C_2\sigma_0^{-2\nu} \frac{\theta^{m + 2\nu} K^2}{\alpha^2} \lambda^{2(\nu + m)} \leq \xi^{2\nu},\]

and, for \(j \geq 2,\)

\[\sigma_j^{-\nu} \frac{\|\mathcal{P}^j\|_{r_j,s_j,\mathcal{D}_j}}{\alpha r_j} \leq C_2\sigma_0 \frac{K}{\varepsilon} \leq C_2\sigma_0 \xi^{-2l} \leq 1,\]

\[3C_0\frac{\theta T\|\mathcal{P}^j\|_{r_j,s_j,\mathcal{D}_j}}{r_j \bar{r}_{j+1} \bar{\sigma}_j} \leq C_2\sigma_0^{-2\nu} \frac{\theta^{m + 2\nu} K^2}{\alpha^2} \lambda^{2l} \leq \xi^{2\nu},\]

\(^{15}\)It may seem artificial treating \((K_j - K_{j-1}) \circ \pi_1 \circ \phi^j\) as a reminder, as the latter is of order 1 compared to \((P_j - P_{j-1}) \circ \phi^j\). Indeed, to get (46), we have bounded \(\varepsilon\) by \(K\), and the latter is of order 1. The point is that the norm of \(\phi^j - \text{id}\) is in fact large, it is basically of order \(\|\phi^j - \text{id}\|_{r_1,s_1,\mathcal{D}_1}\). Hence, extracting the integrable part of \((K_j - K_{j-1}) \circ \pi_1 \circ \phi^j\) does not yield significant improvement.
which concludes the verification of (16) and (18). Therefore, Lemma 6 applies to \( \mathcal{H}_j \) and yields the desired symplectic change of coordinates \( \phi_{j+1} \). In particular, (19) yields (25) \( j+1 \), (20) yields (26) \( j+1 \), (28) \( j+1 \) and, (21) yields (29) \( j+1 \), as

\[
C_1 \rho \cdot 3C_1 K_j^j = C_1 K_j^j \cdot C_1 \rho (\theta^{1/\nu} \log \rho^{-1})^l \leq C_1 K_j^j \cdot C_1 \theta^{l/\nu} \rho^{1/2} \leq C_1 K_j^j.
\]

This ends the proof of (\( \mathcal{P}_{j+1} \)), and, consequently, of the Lemma.

5.3.3 Convergence of the procedure

Now, we are in position to prove the convergence of the KAM scheme.

Lemma 11 Under the assumptions and notation in Lemma 9, the following holds.

(i) the sequence \( G^j := G_j \circ G_{j-1} \circ \cdots \circ G_2 \circ G_1 \) converges uniformly on \( \mathcal{D}_0 \) to a lipeo-
morphism \( G_*: \mathcal{D}_0 \to \mathcal{D}_* := G_*(\mathcal{D}_0) \subset \mathcal{D} \) and \( G_* \in C^1 (\mathcal{D}_0) \).

(ii) \( P_j \) converges uniformly to 0 on \( \mathcal{D}_* \times T^d_{**} \) in the \( C^2_W \) topology;

(iii) \( \phi^j \) converges uniformly on \( \mathcal{D}_* \times T^d \) to a symplectic transformation

\[
\phi_*: \mathcal{D}_* \times T^d \mapsto \mathcal{D} \times T^d;
\]

with \( \phi_* \in C^m (\mathcal{D}_* \times T^d) \) and \( \phi_*(y_*, \cdot) \in C^{m} (T^d) \), for any given \( y_* \in \mathcal{D}_* \).

(iv) \( K_j \) converges uniformly on \( \mathcal{D}_* \) to a function \( K_* \in C^2 (\mathcal{D}_*) \), with

\[
\partial_{y_*} K_* \circ G_* = \partial_{y} K_0 \quad \text{on} \quad \mathcal{D}_0, \quad (48)
\]

\[
H \circ \phi_*(y_*, x) = K_*(y_*), \quad \forall (y_*, x) \in \mathcal{D}_* \times T^d. \quad (49)
\]

Proof The proof is essentially the same as for [15, Lemma 6.3.3, page. 167], which, in turn, is based on [15, Lemma E.2, page. 207]. For the reader’s convenience, we give the proof for \( \phi^j \); the proofs for \( G^j \) and \( P_j \) are similar. Writing \( \phi^j - \phi^{j-1} = \phi^{j-1} \circ \phi_j - \phi^{j-1} \), it follows, for any \( j \geq 2, \)

\[
\| W_1 (\phi^j - \phi^{j-1}) \|_{r_j, s_j, \mathcal{D}_j} \leq \| W_1 D \phi^{j-1} W_{j-1} \|_{r_j-1/2, s_j-1/2, \mathcal{D}_j} \| W_{j-1} W_j^{-1} \| \| W_j (\phi_j - \id) \|_{r_j, s_j, \mathcal{D}_j} \leq 4 \xi^{2 \nu} \xi (m+1) (j-1),
\]

(32)+(43)+(42)
so that
\[ \sum_{j \geq 2} r_j^{n} \| W_1(\phi^j - \phi^{j-1}) \|_{r_j,s_j,\mathcal{D}_j} \leq 4\xi^{2\nu} r_1^{n} \sum_{j \geq 2} \xi^{(m+1)(j-1)} < \infty, \]  
from which we conclude that \( \phi_* \in C^{\tilde{m}}_W(\mathcal{D}_* \times \mathbb{T}^d) \), and, in particular,  
\[ \sup_{\mathcal{D}_* \times \mathbb{T}^d} \max \{ |W_1(\phi_* - \text{id})|, \| \partial_x(u_* - \text{id}) \| \} \leq 8\xi^{2\nu}. \]  
Moreover,  
\[ \sup_{y_* \in \mathcal{D}_*, \nu \geq 2} \sum_{T_{x_j}^d} s_j^{n} \sup_{T_{x_j}^d} |W_1(\phi^j(y_*, \cdot) - \phi^{j-1}(y_*, \cdot))| \leq \infty, \]  
which implies that \( \phi_* \) is \( C^{\tilde{m} \nu} \) in the angle variable \( i.e. \) for any given \( y_* \in \mathcal{D}_* \), the map \( \phi_*(y_*, \cdot) : x \mapsto \phi_* (y, x) \) is \( C^{\tilde{m} \nu} (\mathbb{T}^d) \subset C^1 (\mathbb{T}^d) \).

Now, by Lemma 8, it follows that the sequence \( H_j \) converges in the \( C^t \)-topology uniformly to \( H \) on \( \mathbb{R}^d \times \mathbb{T}^d \). Thus, letting \( j \to \infty \) in \( K_j = H_j \circ \phi^j - P_j \) yields (48) and (49).]

### 5.4 Completion of the Proof of Theorem 4

Choose \( \beta := \tilde{m} \). Then, one checks easily that (3) implies (24) and, therefore, Lemmata 9 and 11 hold. Thus, the map \( G_0 := (\partial_y K_0)_{|B_{\tilde{r}_0} / (\mathcal{D}_0)}^{-1} \circ \partial_y K \) is well-defined on \( B_{\tilde{r}_0} / (\mathcal{D}_0) \) and satisfies
\[ G_0 (B_{\tilde{r}_0} / (\mathcal{D}_0)) \subset B_{\tilde{r}_1 / 2} (\mathcal{D}_0), \quad \max \{ \| G_0 - \text{id} \|_{\tilde{r}_0, \mathcal{D}_0}, \xi_0 \| \partial_y G_0 - 1 \|_{\tilde{r}_0, \mathcal{D}_0} \} \leq 2C_1 \theta \xi_0^{-1}, \]  
where \( K_0 := K_0 \) and \( \tilde{r}_0 := \tilde{r}_1 / (16d\theta) \). Indeed, fix \( y_0 \in \mathcal{D}_0 \) and consider the auxiliary function \( f : B_{\tilde{r}_1 / 4}(y_0) \times B_{\tilde{r}_0}(y_0) \ni (y, z) \mapsto \partial_y K_0(y) - \partial_y K(z) \). Then, for any \( (y, z) \in B_{\tilde{r}_1 / 4}(y_0) \times B_{\tilde{r}_0}(y_0) \)
\[ \| 1 - T(y_0)f_y(y, z) \| \leq \| T(y_0) \| \| \partial_y^2 (K_0 - K)(y_0) + (\partial_y^2 K_0)(y) - (\partial_y^2 K_0)(y_0) \| \leq \| T \| (C_1 K \xi_0^{-2} + 4dC_1 K \xi_0^{-1} \tilde{r}_1 / 4) \leq 2^{\mu - 7} C_1 \xi_0^{-2} \left( \frac{\alpha}{K} \sigma_0^\tau \right) \leq \frac{1}{2}, \]  
and
\[ 2 \| T(y_0) \| | f(y_0, z) | \leq 2T | \partial_y (K_0 - K)(y_0) + (\partial_y K)(y_0) - \partial_y K(z) | \leq 2T (C_1 K \xi_0^{-1} + dK \tilde{r}_0) < 4d\theta \tilde{r}_0 = \tilde{r}_1 / 4. \]
Thus, by Lemma A.2, \( G_0 = (\partial_y K_0)^{-1} \circ \partial_y K \) is well-defined\(^{16}\) on \( B_{\bar{r}_0}(\mathcal{D}_\alpha) \) and the first part of (52) holds and we now prove its second part. In fact, for any \( y \in B_{\bar{r}_0}(y_0) \),

\[
|G_0(y) - y| = |(\partial_y K_0)^{-1}(K_y(y)) - (\partial_y K_0)^{-1}(K_y(y) + \partial_y (K_0 - K)(y))| \\
\leq \| (\partial_y^2 K_0)^{-1} \|_{\bar{r}_1/4, \mathcal{D}_\alpha} \| \partial_y (K_0 - K) \|_{C^1} \overset{(22)}{\leq} 2TC_1K\xi_{\underline{t}}^{\xi_{\underline{t}}-1}.
\]

Moreover,

\[
\| \partial_y G_0 - I_d \|_{\bar{r}_0, \mathcal{D}_\alpha} = \| (1_d + T\partial_y^2 (K_0 - K))^{-1} - 1_d \|_{\bar{r}_0, \mathcal{D}_\alpha} \leq 2T\| \partial_y^2 (K_0 - K) \|_{\bar{r}_0, \mathcal{D}_\alpha} \overset{(22)}{\leq} 2C_1\theta\xi_{\underline{t}}^{\xi_{\underline{t}}-2},
\]

which completes the proof of (52).

Now, observe that \( G^* = G_0 \circ G_0 \) is well-defined by (52). Thus, \( \mathcal{D}_0 \cap B_{\bar{r}_1/4}(\mathcal{D}_\alpha) = G_0(\mathcal{D}_\alpha) \) and, therefore, denoting \( G^*(\mathcal{D}_\alpha) \) again by \( \mathcal{D}_* \), the relations (9) and (4) then follows. Observe that (51) implies (6).

Next, we prove (5). Set \( G^0 := G_0, G^{-1} := I_d \) and \( \mathcal{D}_{-1} := \mathcal{D}_\alpha \). Then, for any \( j \geq 0 \),

\[
\| G^j - I_d \|_{\mathcal{D}_\alpha} = \sum_{i \geq 0} \| G^{i+1} - G^i \|_{\mathcal{D}_\alpha} \leq \sum_{i \geq 0} \| G_{i+1} \circ G^i - G^i \|_{\mathcal{D}_\alpha} \]

\[
= \sum_{i \geq 0} \| G_{i+1} - I_d \|_{\mathcal{D}_\alpha} \leq \sum_{i \geq 0} \| G_{i+1} - I_d \|_{\bar{r}_1/2, \mathcal{D}_\alpha} \overset{(29) + (52)}{\leq} 2^{2l-1}C_1\theta\sigma_{0}^{l-1} + 2\bar{r}_1\xi_{\underline{t}}^{2l} \overset{(3)}{\leq} \alpha\sigma_{0}^{l}/\theta^2,
\]

then, letting \( j \to \infty \) yields the first part of (5).

Next, we show that \( \| G^* - I_d \|_{L, \mathcal{D}_\alpha} < 1 \), which will imply that\(^{17}\) \( G^*: \mathcal{D}_\alpha \overset{onto}{\rightarrow} \mathcal{D}_* \) is a lipeomorphism. Indeed, for any \( j \geq 0 \), we have

\[
\| G^j - I_d \|_{L, \mathcal{D}_\alpha} + 1 = \| (G_j - I_d) \circ G^j - (G^{j-1} - I_d) \|_{L, \mathcal{D}_\alpha} + 1 \\
\leq \| G_j - I_d \|_{L, G^{j-1}(\mathcal{D}_\alpha)} \| G^j \|_{L, \mathcal{D}_\alpha} + \| G^{j-1} - I_d \|_{L, \mathcal{D}_\alpha} + 1 \\
\leq \| G_j - I_d \|_{L, G^{j-1}(\mathcal{D}_\alpha)} (\| G^{j-1} - I_d \|_{L, \mathcal{D}_\alpha} + 1) + \| G^{j-1} - I_d \|_{L, \mathcal{D}_\alpha} + 1 \\
\leq (\| \partial \xi G_j - I_d \|_{\bar{r}_j/2, \mathcal{D}_\alpha} + 1)(\| G^{j-1} - I_d \|_{L, \mathcal{D}_\alpha} + 1)
\]

\(^{16}\)In fact, the graph of \( G_0 \) is precisely the set of solutions of the equation \( f(y, z) = 0 \).

\(^{17}\)See [22, Proposition II.2].
Thus, letting $j \to \infty$, we get that $G^\ast$ is Lipschitz continuous and (5) is proven. Observe that (51) implies (6). Next, observe that, thanks to [15, Theorem 6.2.2, page 148] (see also [16]), (5) yields (8).

Finally, we show that each KAM torus, as a graph, is of class $C^\nu(T^d)$. Set $\varphi_j(y, x) := v_j(y, u_j^{-1}(y, x)), \varphi^1 := \varphi_1$ and $\varphi^{j+1}(y, x) := \varphi^j(\varphi_j(y, x), x), j \geq 1$. By definition, $\varphi_j(y, x) := y + \partial_x g_j(y, x)$ so that, for any $j \geq 2$,

$$
\max \left\{ \| \varphi_j - \pi_1 \|_{r_j, s_j, \vartheta_j}, r_j, j \| \partial_y \varphi_j - \pi_1 \|_{r_j, s_j, \vartheta_j} \right\} \leq (1 + 2\rho) C_0 \frac{\| P_j \|_{r_j-1, s_j-1, \vartheta_j-1}}{\alpha \sigma_j^{\nu}}
$$

and

$$
\leq C_2 \rho \sigma_j^{\nu} \xi^{(l-2\nu)j-2l} \gamma_j^{\nu j-1}.
$$

(53)

Now, observe that $\partial_y \varphi^j = \partial_y \varphi_1 \cdot \partial_y \varphi_2 \cdots \partial_y \varphi_j$. Thus, by the usual telescoping argument, (53) yields

$$
\zeta_2 := \sup_{j \geq 2} \| \partial_y \varphi_j \|_{r_j, s_j, \vartheta_j} < \infty.
$$

Therefore,

$$
\sup_{y \in D_{r_j}(\vartheta_j)} \sum_{s_j+1}^{s_j+\nu} \sup_{T^d_j} \| \varphi^{j+1}(y, \cdot) - \varphi^j(y, \cdot) \| (29) \leq \sum_{j \geq 2} s_j^{\nu} \| \varphi^{j+1} - \varphi_j \|_{r_j+1, s_j+1, \vartheta_j+1}
\leq \zeta_2 \sum_{j \geq 2} s_j^{\nu} \| \varphi_j + 1 - \pi_1 \|_{r_j+1, s_j+1, \vartheta_j+1}
\leq \zeta_1 \zeta_2 s_1^{\nu} \sum_{j \geq 2} \xi^{(l-2\nu)j} < \infty.
$$

\footnote{Recall that $e^x - 1 \leq x e^x, \forall x \geq 0$.}

\footnote{By (32), for any given $y \in D_{2r_j}(\vartheta_j)$, the map $x \mapsto u_j(y, x)$ is a real–analytic diffeomorphism from $T^d_{s_j}$ onto its image, and we denote by $u_j^{-1}(y, x)$ the value of its inverse at $(y, x)$.}
Hence, $\varphi^j$ converges uniformly on $\mathcal{D}_* \times \mathbb{T}^d$ to the the map $\varphi_*$ defined by $\varphi_*(y_*, x) := v_*(u_*^{-1}(y_*, x), x)$. In particular, $\varphi_* \in C^0(\mathcal{D}_* \times \mathbb{T}^d)$ and, for any given $y_* \in \mathcal{D}_*$, the map $x \mapsto \varphi_*(y_*, x)$ is $C^\nu(\mathbb{T}^d)$ and its graph is precisely the KAM torus $\phi_*(y_*, \mathbb{T}^d)$. ■

Appendix

A Reminders

A.1 Classical estimates (Cauchy, Fourier, Cohomological Equation)

Lemma A.1 ([23]) 1. Let $p \in \mathbb{N}$, $r, s > 0, y_0 \in \mathbb{C}^d$ and $f$ a real–analytic function $D_{r,s}(y_0)$ with

$$\|f\|_{r,s} := \sup_{D_{r,s}(y_0)} |f|.$$  

Then,

(i) For any multi–index $(l, k) \in \mathbb{N}^d \times \mathbb{N}^d$ with $|l|_1 + |k|_1 \leq p$ and for any $0 < r' < r$, $0 < s' < s$, 20

$$\|\partial_y^l \partial_x^k f\|_{r', s'} \leq p! \|f\|_{r,s}(r - r')^{|l|_1}(s - s')^{|k|_1}.$$  

(ii) For any $k \in \mathbb{Z}^d$ and any $y \in D_r(y_0)$

$$|f_k(y)| \leq e^{-|k|_1 s} \|f\|_{r,s}.$$  

2. Let $p \in \mathbb{N}$, $\omega \in \Delta^r_\alpha$ and $f \in A_{r,s}$ and $\langle f \rangle = 0$. Then, for any $0 < \sigma < s$, the system

$$D_\omega g = f, \quad \langle g \rangle = 0$$

has a unique solution in $A_{r,s-\sigma}$ such that for any multi–index $k \in \mathbb{N}^d$ with $|k|_1 = l$

$$\|\partial^k_x g\|_{r,s-\sigma} \leq C_1 \|f\|_s \alpha^{-(\sigma + l)},$$

where $C_1 := 2^{\sigma+1-(\sigma+l)} \sqrt{\Gamma(2(\sigma + l) + 1)}$ (see [24, 23]).

20 As usual, $\partial_y^l := \partial_{y_{i_1}}^{l_{i_1}} \cdots \partial_{y_{i_d}}^{l_{i_d}}$, $\forall y \in \mathbb{R}^d$, $l \in \mathbb{Z}^d$.  

22
A.2 Implicit and Inverse function Theorems

Firstly, we recall the classical implicit function Theorem, in a quantitative framework.

Lemma A.2 ([18]) Let \( r, s > 0, n, m \in \mathbb{N}, (y_0, x_0) \in \mathbb{C}^n \times \mathbb{C}^m \) and \(^{21}\)

\[
F: (y, x) \in D^n_r(y_0) \times D^m_s(x_0) \subset \mathbb{C}^{n+m} \rightarrow F(y, x) \in \mathbb{C}^n
\]

be continuous with continuous Jacobian matrix \( F_y \). Assume that \( F_y(y_0, x_0) \) is invertible with inverse \( T := F_y(y_0, x_0)^{-1} \) such that

\[
\sup_{D^n_r(y_0) \times D^m_s(x_0)} \|1_n - TF_y(y, x)\| \leq c < 1 \quad \text{and} \quad \sup_{D^m_s(x_0)} |F(y_0, \cdot)| \leq \frac{(1-c)r}{\|T\|}. \tag{A.1}
\]

Then, there exists a unique continuous function \( g: D^m_s(x_0) \rightarrow D^n_r(y_0) \) such that the following are equivalent

(i) \((y, x) \in D^n_r(y_0) \times D^m_s(x_0)\) and \(F(y, x) = 0\);

(ii) \(x \in D^m_s(x_0)\) and \(y = g(x)\).

Moreover, \( g \) satisfies

\[
\sup_{D^m_s(x_0)} |g - y_0| \leq \frac{\|T\|}{1 - c} \sup_{D^m_s(x_0)} |F(y_0, \cdot)|. \tag{A.2}
\]

B Outline of the proof of Lemma 6

Here, we aim to sketch the proof of the general KAM step. We refer the reader to [16, 15] for more details.

Step 1: Construction of the Arnold’s transformation The symplectomorphism \( \phi' \) is generated by the real–analytic map \( y' \cdot x + g(y', x) \) i.e.

\[
\phi': \begin{cases} 
 y = y' + g_x(y', x) \\
 x = x + g_y(y', x),
\end{cases} \tag{B.1}
\]

in such a way that

\[
\begin{aligned}
H' &:= H \circ \phi' = K' + P' \quad \text{on } D_{r_1,s_1}(\mathcal{D}'_z), \\
\det \partial^2_y K'(y') &\neq 0, \quad \forall y' \in \mathcal{D}'_z, \tag{B.2} \\
\partial^2_y K'(\mathcal{D}'_z) &\equiv \partial_y K(\mathcal{D}_z),
\end{aligned}
\]

\(^{21}\)Here, \( D^n_r(z_0) \) denotes the ball in \( \mathbb{C}^n \) centered at \( z_0 \) and with radius \( r \).
with
\[
\begin{align*}
P'(y', x') := P_+(y', \varphi(y', x')) , & \quad P_+ := P^{(1)} + P^{(2)} + P^{(3)} , \quad P^{(3)} := P - \hat{P} , \\
P^{(1)} := \int_0^1 (1 - t)K_{yy}(tg_x) \cdot g_x \cdot g_x dt , & \quad P^{(2)} := \int_0^1 P_y(y' + tg_x, x) \cdot g_x dt ,
\end{align*}
\]
where \( \varphi(y', \cdot) \) is the inverse of the map \( x' \mapsto x + g_y(y', x) \) and \( \hat{P} \) is the approximation of \( P \) given by [17, Theorem 7.2].\(^{22}\) Moreover,
\[
K_y(y') \cdot n \neq 0 , \quad \forall 0 < |n|_1 \leq \kappa , \quad \forall y' \in D_{\gamma_1}(\mathcal{D}_2^\gamma) \quad (\subset D_{\gamma}(\mathcal{D}_2)) ,
\]
and the generating function \( a \) is given by
\[
g(y', x) := \sum_{0 < |n|_1 \leq \kappa} \frac{-\hat{P}_n(y')}{iK_y(y') \cdot n} e^{-in \cdot x} .
\]

**Step 2** Now, we provide the construction performed in **Step 1** with quantitative estimate. First of all, notice that\(^{23}\)
\[
\| P - \hat{P} \|_{r, \mathcal{S}_2} \leq 2\rho \varepsilon , \quad \| \hat{P} \|_{r, \mathcal{S}} \leq \| P \|_{r, \mathcal{S}_2} + \| P - \hat{P} \|_{r, \mathcal{S}_2} \leq (1 + 2\rho)\varepsilon .
\]
Observe also that for any \( y \in \mathcal{D}_2^\gamma , 0 < |n|_1 \leq \kappa \) and \( y' \in D_{\gamma}(y) ,
\[
|K_y(y') \cdot n| \geq \frac{\alpha}{2|n|_1}.
\]
Now, using Lemma A.1–2. and (B.6), we get
\[
\| g \|_{r, \mathcal{S}_2} \leq C_0 \frac{(1 + 2\rho)\varepsilon}{\alpha} \sigma^{-\tau} , \quad \| g_x \|_{r, \mathcal{S}_2} \leq C_0 \frac{(1 + 2\rho)\varepsilon}{\alpha} \sigma^{-(\tau + 1)} , \quad \max\{ |\hat{\sigma}_y^2 g|_{r, \mathcal{S}_2} , \sigma |\hat{\sigma}_y^3 x g|_{r, \mathcal{S}_2} , \sigma^2 |\hat{\sigma}_y^3 x x g|_{r, \mathcal{S}_2} \} \leq \bar{\Gamma},
\]
where
\[
\bar{\Gamma} := 2C_0 \frac{(1 + 2\rho)\varepsilon}{\alpha r} \sigma^{-\tau} .
\]
We have
\[
\| \hat{\sigma}_y^2 \tilde{K} \|_{r/2, \mathcal{S}_2} \leq \frac{2\varepsilon}{r} , \quad \| \hat{\sigma}_y^2 \tilde{K} \|_{\gamma, \mathcal{S}_2} \leq \frac{4\varepsilon}{r^2} \leq \frac{(18)}{r^2} \leq K\bar{\sigma}_3 .
\]

\(^{22}\)With the choices \( \beta_1 = \cdots = \beta_d = 1/2 , T = \kappa \) and \( \delta = \rho \leq 1/4 .
\(^{23}\)By definition of \( \hat{P} \), see [17, Theorem 7.2].
Next, we construct $D_1$ in (B.2). For, fix $y \in D_1$ and consider

$$F : D_1(y) \times D_1(y) \rightarrow \mathbb{C}^d$$

$$(y, z) \mapsto K_y(y) + \tilde{K}_y(y) - K_y(z).$$

Then, one checks easily that Lemma A.2 applies. Thus, we get that $F^{-1}([0])$ is given by the graph of a real–analytic map $G^y : D_1(y) \rightarrow D_1(y)$. Afterwards, one checks that the pieces of the family $\{G^y\}_{y \in D_1}$ matches, yielding therefore a global map $G$ on $D_1(D_1)$ and that, in fact, $G$ is bi–real–analytic.\textsuperscript{24} Next, one shows that the expression $(K_y + \tilde{K}_y)^{-1} \circ K_y$ defines a map on $D_1(y)$ by means of the Inversion Function Lemma A.2. As a consequence, we get an explicit formula for $G$:

$$G = (K_y + \tilde{K}_y)^{-1} \circ K_y \quad \text{on} \quad D_1(y), \quad \text{(B.9)}$$

and $D_1 = G(D_1)$. The reminder of the proof then goes exactly as in [16] (see also [15]).

\section{Isotropcity Lemma}

\textbf{Lemma C.1 ([7, 25, 26])} \textit{Let $H : \mathcal{M} := \mathbb{R}^d \times \mathbb{T}^d \rightarrow \mathbb{R}$ be a Hamiltonian of class $C^1$ and $\phi : \mathbb{T}^d \ni x \mapsto \phi(x) \in \mathcal{M}$, a $C^1$–mapping. Assume that

$$\phi_t' \circ \phi(x) = \phi(x + t\omega), \quad \forall x \in \mathbb{T}^d,$$

for some rationally independent\textsuperscript{25} $\omega \in \mathbb{R}^d$. Then, the torus $\phi(\mathbb{T}^d)$ is isotropic i.e. $i^*\omega = 0$, where $i : \phi(\mathbb{T}^d) \hookrightarrow \mathcal{M}$ is the inclusion map.}

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\textsuperscript{24} i.e. an invertible real–analytic map whose inverse is real–analytic as well.

\textsuperscript{25} i.e. for any $k \in \mathbb{Z}^d \setminus \{0\}$, $\omega \cdot k \neq 0$. 
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