Upper tail bounds for cycles

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Abstract

This paper examines bounds on upper tails for cycle counts in $G_{n,p}$. For a fixed graph $H$ define $\xi_H = \xi_H^{n,p}$ to be the number of copies of $H$ in $G_{n,p}$. It is a much studied and surprisingly difficult problem to understand the upper tail of the distribution of $\xi_H$, for example, to estimate

$$\mathbb{P}(\xi_H > 2E\xi_H).$$

The best known result for general $H$ and $p$ is due to Janson, Oleszkiewicz, and Ruciński, who, in 2004, proved

$$\exp\left[ -O_{H,\epsilon}(\mathcal{M}_H(n,p) \ln(1/p)) \right] < \mathbb{P}(\xi_H > (1 + \epsilon)E\xi_H)$$

$$< \exp\left[ -\Omega_{H,\epsilon}(\mathcal{M}_H(n,p)) \right].$$

Thus they determined the upper tail up to a factor of $\ln(1/p)$ in the exponent. There has since been substantial work to improve these bounds for particular $H$ and $p$. We close the $\ln(1/p)$ gap for cycles, up to a constant in the exponent. Here the lower bound in (1) is the truth for $l$-cycles when $p > \ln(1/(l-2)n)$.

1 Introduction

Let $G = G(m,p)$ be the usual (Erdős-Rényi) random graph. A copy of $H$ in $G$ is a subgraph of $G$ isomorphic to $H$. It is a much studied question to estimate, for $\eta > 0$ and $\xi_H = \xi_H^{m,p}$ the number of copies of $H$ in $G_{m,p}$,

$$\mathbb{P}(\xi_H > (1 + \eta)E\xi_H).$$

(2)

To avoid irrelevancies we will always assume $p \geq m^{-1/m_H}$, where (see [20, pg. 56])

$$m_H = \max\{e_K/v_K : K \subseteq H, v_K > 0\}.$$

(So in the case of cycles we assume $p \geq m^{-1}$.) Then $m^{-1/m_H}$ is a threshold for "$G \supseteq H$" (see [20, Theorem 3.4]). For smaller $p$ (and bounded $\eta$) the quantity in (2) is $\Theta(\min\{m^\epsilon_K p^\epsilon_K : K \subseteq H, \epsilon_K > 0\})$ (see [20, Theorem 3.9] for a start).
Investigation the distribution of $\xi_H$ began in 1960 with Erdős and Rényi \cite{10}. In the case of triangles it is easy to see that the upper tail is lower bounded by $\exp[-O(n^2p^2\ln(1/p))]$ (since this is the probability that $G_{n,p}$ contains a complete graph on, say, $2np$ vertices). This is, usually, much bigger than the naive guess, $\exp[-\Omega(n^3p^3)]$, a first indication that the problem is hard. In fact, not much was known about the upper tail until 2000 when Vu proved the first exponential tail bound in \cite{22}. More information on what was known prior to 2002 can be found in \cite{13}. A breakthrough occurred in 2004 when, in \cite{15}, Kim and Vu showed, using the “polynomial concentration method” of \cite{14}, that when $H$ is a triangle and $p > \log_m m$, 

$$\mathbb{P}(\xi_H > (1 + \eta)\mathbb{E}\xi_H) < \exp[-\Omega(\eta(m^2p^2))]$$

Thus they determined the probability in (2) up to a factor of $O(\ln(1/p))$ in the exponent for constant $\eta > 0$. This remains the best result for general $H$ and $p$. The first progress towards closing the $\ln(1/p)$ gap was made by Chatterjee in \cite{5} and DeMarco and Kahn in \cite{9} who independently closed it for triangles, showing that, for $p > \log m/m$, the lower bound is the truth (up to the constant in the exponent). DeMarco and Kahn also gave the order of the exponent for smaller $p > 1/m$ where the lower bound in (3) (namely $\exp[-\Omega(n^2p^2\ln n)]$) is no longer the answer. Later, in \cite{8}, DeMarco and Kahn closed the gap for $l$-cliques, showing that (for $p \geq m^{-2/(l-1)}$, $\eta > 0$, and $l > 1$) 

$$\mathbb{P}(\xi_{K_l} > (1 + \eta)\mathbb{E}\xi_{K_l}) < \exp[-\Omega_{l,\eta}(\min\{m^2p^{l-1}\log(1/p), m^lp^{(l)}\})].$$

When $H$ is a “strictly balanced” graph and $p$ is small ($p \leq m^{-v/e}\log C_h m$), Warnke, in \cite{23}, used a combinatorial sparsification idea based on the BK inequality \cite{3, 18} to close the $\ln(1/p)$ gap, improving on work in \cite{22, 21}. There was a breakthrough in 2016 when Chatterjee and Dembo introduced a “nonlinear large deviation” framework \cite{6}. This has been used to close the gap for
general $H$ and large $p$ (i.e. $p > m^{-\alpha_H}$) \cite{6,16}. Recently this technique was used, in \cite{7}, by Cook and Dembo to close the gap — including determining the correct constant in the exponent — for cycles when $p \gg m^{-1/2}$ (among other results). Additionally, outside of the large deviation framework, Warnke and Šileikis, in \cite{17}, recently determined the correct upper tail bound for stars (including in the case where $\eta \geq n^{-\alpha}$ rather than a constant).

Here we settle the question for cycles (i.e. the order of magnitude of the exponent), where, with the $l$-cycle denoted $C_l$,

$$M_{C_l}(m, p) = m^2 p^2.$$  

Formally, letting $\xi_l = \xi_l(G)$ be the number of copies of $C_l$ in $G$ we prove:

**Theorem 1.2.** For any fixed $l$, $\eta > 0$, and $p \in [0, 1]$,

$$P(\xi_l > (1 + \eta)E[\xi_l]) < \exp[-\Omega_{\eta,l}(\min\{m^2 p^2 \ln(1/p), m^l p^l\})].$$

We are most interested in the range where $m^2 p^2 \ln(1/p) < m^l p^l$, so essentially when $p > \frac{\ln^{-2(l-2)}}{m}$. As in \cite{9}, it is convenient to work with an $l$-partite version of the random graph. Let $\mathbb{H}$ be the random $l$-partite graph on $ln$ vertices where the vertex set is the disjoint union of $l$ $n$-sets, say $V = V(\mathbb{H}) = V_1 \cup \cdots \cup V_l$, and $P(xy \in E(\mathbb{H})) = p$ whenever $x \in V_i$ and $y \in V_{i+1}$ for some $i$ (all subscripts mod $l$), these choices made independently. There are no edges between other pairs $(V_i, V_j)$ or within a $V_i$. We always take $v_i$ to be a vertex of $V_i$. A copy of $C_l$ in $\mathbb{H}$ is any subgraph, with vertices $v_1, v_2, \ldots, v_l$ isomorphic to $C_l$. Note these are not all of the subgraphs of $H$ isomorphic to $C_l$ since we demand each vertex of the cycle is in a different $V_i$. We denote the number of copies of $C_l$ in $\mathbb{H}$ by $\xi'_l$. A copy of the $l-1$ path (denoted $P_{l-1}$) is any path $v_1, v_2, \ldots, v_l$ isomorphic to $P_{l-1}$ (i.e. $v_i \sim v_{i+1}$ for $1 \leq i < l$). We use $(v_1, \ldots, v_l)$ to denote both copies of $C_l$ and copies of $P_{l-1}$, since it will always be clear which interpretation is intended. We show the following bound.

**Theorem 1.3.** For any fixed $l$, $\delta > 0$, and $p \in [0, 1]$,

$$P(\xi'_l > (1 + \delta)n^l p^l) < \exp[-\Omega_{\delta,l}(\min\{n^2 p^2 \ln(1/p), n^l p^l\})].$$

(4)

That Theorem 1.3 implies Theorem 1.2 is likely well known and an easy generalization from the $l = 3$ case which can be found in \cite{9}. However, for completeness we will still give the general argument.

**Proposition 1.4.** Theorem 1.3 implies Theorem 1.2.

This is proved in Section 2. The rest of the paper is organized as follows. Section 3 gives notation and states the two main assertions that give Theorem 1.3. These are proved in Sections 5-7, with Section 4 devoted to preliminaries.

## 2 Reduction

For completeness we give the proof of Proposition 1.4 following \cite{9}. 

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Proof of Proposition 1.4. We first claim that it is enough to prove Proposition 1.4 for \( m = \ln \). Assuming we know Proposition 1.4 for \( m = \ln \) we show it still holds when \( m = -k \mod l \). Given \( \eta \) and \( l \), we may assume \( m \) is large (formally \( m > m_{\eta,l} \)). So, for example,

\[
(1 + \eta) \left( \frac{m}{l} \right) > (1 + \eta/2) \left( \frac{m + k}{l} \right).
\]

Therefore,

\[
\mathbb{P} \left( \xi_l > (1 + \eta) \left( \frac{m}{l} \right) p^l \right) \leq \mathbb{P} \left( \xi_l > (1 + \eta/2) \left( \frac{m + k}{l} \right) p^l \right)
\]

\[
< \exp \left[ -\Omega_{\eta/2,l} \{ \min \{ (m + k)^2 p^2 \ln(1/p), (m + k)^l p^l \} \} \right]
\]

\[
= \exp \left[ -\Omega_{\eta,l} \{ \min \{ m^2 p^2 \ln(1/p), m^l p^l \} \} \right].
\]

Note the second inequality holds since \( m + k \) is a multiple of \( l \).

Now to prove Proposition 1.4 when \( m = \ln \) let \( \eta \) be as in Theorem 1.2, and set \( \delta = \eta/2 + \eta \). We can choose \( H \) by first choosing \( G \) on \( V = \lceil \ln \rceil \) and then selecting a uniform equipartition \( V_1 \cup \cdots \cup V_l \), and setting

\[
E(H) = \{ xy \in E(G) : x, y \text{ belong to consecutive } V_i \s's \}.
\]

Note that, for any possible value \( G \) of \( G \)

\[
\mathbb{E}[\xi'|G = G] = \rho \xi(G), \tag{5}
\]

where \( \rho = n^l/\binom{l}{l} \). On the other hand, letting

\[
\alpha(G) = \mathbb{P}(\xi' < (1 - \delta)\rho \xi(G)|G = G),
\]

we have

\[
\mathbb{E}[\xi'|G = G] \leq \alpha(G)(1 - \delta)\rho \xi(G) + (1 - \alpha(G))\xi(G). \tag{6}
\]

Combining (5) and (6) gives \( \alpha(G) \leq 1 - \frac{\delta \rho}{1 - \rho + \delta \rho} := 1 - \beta \). We also have, by Theorem 1.3

\[
\exp \left[ -\Omega_{\delta,l} \{ \min \{ n^2 p^2 \ln(1/p), n^l p^l \} \} \right] > \mathbb{P}(\xi'_l > (1 + \delta)n^l p^l).
\]

Additionally, we know

\[
\mathbb{P}(\xi'_l > (1 + \delta)n^l p^l)
\]

\[
\geq \mathbb{P} \left( \xi'_l > (1 + \delta)n^l p^l|\xi_l > 1 + \delta \left( \frac{\ln}{l} \right) p^l \right) \mathbb{P} \left( \xi_l > 1 + \delta \left( \frac{\ln}{l} \right) p^l \right)
\]

\[
\geq \beta \mathbb{P} \left( \xi_l > 1 + \delta \left( \frac{\ln}{l} \right) p^l \right).
\]

Here the final inequality holds since \( (1 - \delta)\rho \frac{\ln}{l} p^l = (1 + \delta)n^l p^l \) and, as we showed, \( \alpha(G) \) is always at most \( (1 - \beta) \). Since \( \frac{\ln}{l} = 1 + \eta \), Theorem 1.2 follows.
3 Main Lemmas

Recall that we always take \(v_i\) to be a vertex in \(V_i\); indices are always written mod \(l\); and copy of \(C_i\), copy of \(P_{l-1}\) were defined just before the statement of Theorem 1.3. We use \(C\) to denote the set of copies of \(C_i\) in \(H\). Additionally, we abusively use just cycle for “copy of \(C_i\)” and full path for “copy of \(P_{l-1}\”). As usual \(N_V(x) = \{y \in Y : xy \in E(H)\}\), \(d_Y(x) = |N_Y(x)|\), \(d(x,y) = |N_V(x) \cap N_V(y)|\), and \(\Delta\) is the maximum degree in \(H\) (we also use \(N(x) = N_V(x)\) and \(d(v) = d_V(x)\)). Let

\[
\hat{d}(v_i) = \max\{d_{V_{i-1}}(v_i), d_{V_{i+1}}(v_i)\}.
\]

We will abusively refer to \(\hat{d}(v)\) as the degree of \(v\). For disjoint \(X, Y \subseteq V\) we use \(\nabla(X)\) (resp. \(\nabla(X, Y)\)) for the set of edges with one end in \(X\) (resp. one end in each of \(X, Y\)).

Much of the set-up that follows is borrowed from or inspired by [9]. Set \(t = \ln(1/p)\) and \(s = \min\{t, n^{l-2}p^{l-2}\}\) (so the exponent in (4) is \(-\Omega_\delta,l(n^{2/p^2}s))\). For simplicity set \(\gamma = \frac{1}{5\ell^2}\) and

\[
\epsilon = \frac{\delta}{(27\ell)^{l+1}}.
\]

(7)

Note that for a fixed \(\nu\) and \(p > \nu\), Theorem 1.2 is covered by Theorem 1.1. For us it is convenient to pick \(\nu = e^{-4/\gamma} = e^{-20l^2}\). Of course, the partite version (Theorem 1.3) was not considered in [19], but it is not too hard to get this from Theorem 1.1.

Proposition 3.1. For \(p > e^{-20l^2}\) Theorem 1.3 follows from Theorem 1.1.

This will be proved at the end of the section.

In view of Proposition 3.1 we may assume for the proof of Theorem 1.3 that

\[
p \leq e^{-4/\gamma} = e^{-20l^2}.
\]

(8)

We may also assume: \(\delta\) — so also \(\epsilon\) — is (fixed but) small (since [14] becomes weaker as \(\delta\) grows); given \(\delta\) and \(l\), \(n\) is large (formally, \(n > n_{\delta,l}\)); and, say,

\[
p > \epsilon^{4n^{-1}}
\]

(9)

(since for smaller \(p\), Theorem 1.3 is trivial for an appropriate \(\Omega_{\delta,l}\)). We say that an event occurs with large probability (w.l.p.) if its probability is at least

\[1 - \exp[-T \epsilon^4 n^2 p^3 t]\]

for some fixed \(T > 0\) and small enough \(\epsilon\). We write “\(\alpha <^* \beta\)” for “w.l.p. \(\alpha < \beta\)” if \(\alpha <^* \beta\) holds w.l.p. We write “\(\alpha < \beta\)” if \(\alpha < \beta\) holds w.l.p. also holds w.l.p.

Let \(V'_i = \{v \in V_i : \hat{d}(v) < np^{l-1}\}\) and let \(f(v_1, v_l)\) be the number of full paths with endpoints \(v_1\) and \(v_l\) in which each vertex is in the appropriate \(V'_i\). The next two assertions imply Theorem 1.3.

\[
\text{w.l.p. } |\{(v_1, \ldots, v_l) \in C : \exists i (v_i \notin V'_i)\}| < (\delta/2)n^l p^l;
\]

(10)
\[ \mathbb{P}(\{ (v_1, \ldots, v_l) \in C : \forall i (v_i \in V_i') \} ) > (1 + \delta/2)n^l p^l \] < \exp[-\Omega_{\delta,l}(n^2 p^2 s)]. \] (11)

We prove (10) in Section 5 and (11) in Section 7. In Section 6 we prove that
\[ \sum_{v_1, v_l} f(v_1, v_l) \prec^* (1 + \delta/8)n^l p^{l-1}, \] (12)
which will be used in the proof of (11).

We now give the proof of Proposition 3.1. To do so we require the following tail bound due to Janson ([12]; see also [20, Theorem 2.14]).

**Lemma 3.2.** Let \( \Gamma \) be a set of size \( N \) and \( \Gamma_p \) the random subset of \( \Gamma \) in which each element is included with probability \( p \) (independent of the other choices). Assume \( S \) is a family of non-empty subsets of \( \Gamma \), and for each \( A \in S \) let \( I_A = 1[ A \subseteq \Gamma_p ] \). Additionally, let \( X = \sum_{A \in S} I_A \). Define
\[ \bar{\Delta} = \sum_{A \cap B \neq \emptyset} \mathbb{E}(I_A I_B). \]

Then for \( 0 \leq t \leq \mathbb{E}X \),
\[ \mathbb{P}(X \leq \mu - t) \leq \exp\left[ -\frac{t^2}{2 \bar{\Delta}} \right]. \]

**Proof of Proposition 3.1.** Let \( H \) be as in Theorem 1.3 and regard \( H \) as a subgraph of \( G = G_{ln,p} \). Set \( \xi = \xi_l(G) \), \( \xi' = \xi'_l(H) \), and \( \xi'' = \xi - \xi' \); thus \( \xi'' \) is the number of cycles in \( G \) that are not of the form \((v_1, \ldots, v_l)\). Then
\[ \mathbb{E}[\xi''] = \left( \frac{(ln)^3}{(ln-1)!} - n^l \right) p^l. \]

We first use Lemma 3.2 to show
\[ \mathbb{P}(\xi'' < (1 - \epsilon)\mathbb{E}\xi'') \leq \exp[-\Omega_{l,\epsilon}(n^2)]. \] (13)

To apply Lemma 3.2 we take \( S \) to be the set cycles in \( G \) not of the form \((v_1, \ldots, v_l)\) (so each \( A \in S \) is the edge set of a particular cycle). Note that when \( |A \cap B| = k \) we have \( \mathbb{E}[I_A I_B] = p^{2l-k} \). Furthermore, the number of pairs of cycles sharing exactly \( k \geq 1 \) edges is at most \( c k n 2^{l-(k+1)} \) (for some constants \( c \)). Thus we have
\[ \bar{\Delta} \leq \sum_{k} c k n 2^{l-(k+1)} p^{2l-k} = c l n 2^{-2}, \]
since \( p = \Omega(1) \). Lemma 3.2, with \( t = \epsilon \mathbb{E}\xi'' \), gives
\[ \mathbb{P}(\xi'' < (1 - \epsilon)\mathbb{E}\xi'') \leq \exp[-\Omega_{l,\epsilon}(n^2)]. \] (13)

Furthermore, we claim that for any \( \delta' > 0 \)
\[ \mathbb{P}(\xi' > (1 + \delta')\mathbb{E}\xi') \leq \mathbb{P}(\xi'' < (1 - \delta'')\mathbb{E}\xi'') + \mathbb{P}(\xi > (1 + \delta)\mathbb{E}\xi), \] (14)
provided $\delta$ and $\delta''$ are such that $\delta \mathbb{E}\xi + \delta'' \mathbb{E}\xi'' < \delta' \mathbb{E}\zeta'$. This is because occurrence of the event on the l.h.s. implies occurrence of one of the events on the r.h.s.; namely, if
\[ \xi'' \geq (1 - \delta'') \mathbb{E}\xi'' \quad \text{and} \quad \xi \leq (1 + \delta) \mathbb{E}\xi, \]
then
\[ \xi' = \xi - \xi'' \leq (1 + \delta) \mathbb{E}\xi - (1 - \delta'') \mathbb{E}\xi'' = \mathbb{E}\xi' + \delta \mathbb{E}\xi + \delta'' \mathbb{E}\xi'' < (1 + \delta') \mathbb{E}\xi'. \]

Therefore, for any $\eta > 0$ we can select $\delta$ and $\delta''$ such that
\[ \mathbb{P}(\xi' > (1 + \eta) \mathbb{E}\xi') \leq \mathbb{P}(\xi'' < (1 - \delta'') \mathbb{E}\xi'') + \mathbb{P}(\xi > (1 + \delta) \mathbb{E}\xi) \]
\[ < \exp[-\Omega_{\delta'',l}(n^2)] + \mathbb{P}(\xi > (1 + \delta) \mathbb{E}\xi), \]
where the second inequality holds by (13) and the third by Theorem 1.1.

4 Preliminaries

To prove (10) and (11) we need the following preliminaries, where $B(m, \alpha)$ is used for a random variable with the binomial distribution $\text{Bin}(m, \alpha)$. The first two of these are standard large deviation bounds; see e.g. [1, Theorem A.1.12], [20, Theorem 2.1(a)] and [2, Lemma 8.2]. The others are applications of Lemma 4.1 that we will use repeatedly.

Lemma 4.1. For any $\beta \in (0, 1), K \geq 1 + \beta, m,$ and $\alpha$ we have,
\[ \mathbb{P}(B(m, \alpha) \geq K \alpha m) \leq \begin{cases} \exp[-\beta^2\alpha m / 4] & \text{if } K \leq 4, \\
(\epsilon / K)^{Kn\alpha} & \text{if } K > 4. \end{cases} \]  
(15)

When $m = n$ and $\alpha = p$ (which is what we have when our binomial random variable is $d_{V_{e-1}}(v_i)$ or $d_{V_{e+1}}(v_i)$) and $K \geq 1 + \epsilon$ we use $q_K$ for the right hand side of (15); that is,
\[ q_K := \begin{cases} \exp[-\epsilon^2np / 4] & \text{if } K \leq 4, \\
(\epsilon / K)^{Kn\alpha} & \text{if } K > 4. \end{cases} \]  
(16)

First note that for any $K \geq 1 + \epsilon$ we have,
\[ q_K \leq \exp[-\epsilon^2Kn\alpha / 16]. \]  
(17)

Of course this is unnecessarily weak when $K$ is not close to 1 (as was the first bound in (15)), but is often enough for our purposes and will be used
repeatedly below. It will also be useful to have the following upper bound on $q_K$ when $K \geq p^{-\gamma/2}$ (recall $\gamma$ was defined before (7)):

$$q_K \leq \exp[-\gamma Knpt/4] < n^{-2}. \quad (18)$$

To show the first inequality holds note that $K \geq p^{-\gamma/2}$ and $p \leq e^{-4/\gamma}$ (see (8)) imply $K \geq e^2$ and

$$(e/K)^{Knpt} \leq \exp\left[Knp\left(1 - \frac{\gamma}{2}t\right)\right].$$

Again $p \leq e^{-4/\gamma}$ implies $t \geq 4/\gamma$ giving the first inequality in (18):

$$q_K = (e/K)^{Knpt} \leq \exp[-\gamma Knpt/4].$$

The second inequality in (18) follows easily from the combination of $t \geq 4/\gamma$ and the fact that $p$ is not extremely small (see (9)).

Lemma 4.2. Suppose $w_1, \ldots, w_m \in [0, z]$. Let $\zeta_1, \ldots, \zeta_m$ be independent Bernoullis, $\zeta = \sum \zeta_i w_i$, and $E\zeta = \mu$. Then for any $\nu > 0$ and $\lambda > \nu \mu$,

$$P(\zeta > \mu + \lambda) < \exp[-\Omega(\lambda/z)].$$

The last two lemmas are the basis for much of what follows. Lemma 4.4 in particular may be regarded as perhaps the main idea for sections 5 and 6; it allows us to bound sums of atypically large degrees, which we then use to bound the number of cycles that include vertices of “large” degree (in Section 5) and the number of full paths without vertices of “large” degree (in Section 6).

Lemma 4.3. For $K \geq 1 + \epsilon$ and any $i$,

$$|\{v_i \in V_i : d(v_i) \geq Knpt\}| <^* r_K := \begin{cases} 6eK^{-1}n & \text{if } q_K > n^{-2}, \\ \frac{e^{2npt}}{K \ln K} & \text{otherwise.} \end{cases} \quad (19)$$

The first, ad hoc value is for use in Section 6 while the second will be used throughout. Convenient bounds for the second expression in (19) are

$$\frac{e^{2npt}}{K \ln K} < \begin{cases} 2enpt/K & \text{if } K > 1 + \epsilon, \\ enp/K & \text{if } K > p^{-\epsilon}. \end{cases} \quad (20)$$

Proof of Lemma 4.3. Let $q = q_K$ and $r = \min\{r_K, 1\}$. We let $r = \min\{r_K, 1\}$ because later it will be helpful to have $n/r \leq n$. We can enforce this lower bound on $r$ because if $r_K < 1$ then

$$P(|\{v_i \in V_i : d(v_i) \geq Knpt\}| \geq r) = P(|\{v_i \in V_i : d(v_i) \geq Knpt\}| \geq 1).$$

Without loss of generality, let $i = 1$. We show

$$|\{v_1 \in V_1 : d(v_1) \geq Knpt\}| <^* r/2. \quad (21)$$
Write $N$ for the left hand side of (21). We first assume $q \leq n^{-2}$. Since the $d_{v_i}(v_1)$'s ($v_1 \in V_1$) are independent copies of $B(n, p)$, two applications of Lemma 4.1 give

$$P(N \geq r) < P(B(n, q) \geq \lceil r/2 \rceil) \leq (2\epsilon q/r)^{r/2} < (2e\sqrt{q})^{r/2} < \exp[-\Omega(e^4n^2p^2 t)].$$

The third inequality holds since $q \leq n^{-2}$, so $n/r \leq n \leq q^{-1/2}$.

Now assume $q > n^{-2}$. Recall from (17) that we always have

$$q \leq \exp[-\epsilon^2Kn/p].$$

So,

$$n^{-2} < q \leq \exp[-\epsilon^2Kn/p]$$

implies

$$Kn/p < 32\epsilon^{-2}\log n,$$  \hfill (22)

On the other hand (9) gives

$$q < \exp[-\epsilon^2Kn/p] < \exp[-\epsilon^{-2}K/16] < \epsilon K^{-l}.$$  

The last inequality uses the fact that $\exp[-\epsilon^{-2}K/16]K^{-l}$ is minimized at $K = 16\epsilon^{-2}$ and $\epsilon < (\frac{\epsilon}{16})^{1/(2l-1)}$ (as we may assume). Hence

$$P(N \geq r/2) < P(B(n, q) \geq r/2) < \exp[-\Omega(\epsilon nK^{-l})] < \exp[-\Omega(n^2p^2 t)],$$

where the second inequality uses $r/2 > 3nq$ (and Lemma 4.1) and the (very crude) third inequality uses $K^{-l-2} < n/\log^3 n$ which follows from (22) and (9).

**Lemma 4.4.** For $p > \frac{64\epsilon^{-2}\ln n}{n}$ and any $i$,

$$\sum \left\{ \hat{d}(v_i) : \hat{d}(v_i) > (1 + \epsilon)np \right\} <^* \epsilon^2 n^2p^2 t, \hfill (23)$$

and

$$\sum \left\{ \hat{d}(v_i) : \hat{d}(v_i) > np^{1-\gamma/2} \right\} <^* \epsilon n^2 p^2. \hfill (24)$$

There is nothing special about $\gamma/2$ here; it is simply a value that will work for our purposes. The reason for the particular — and not very important — lower bound on $p$ will appear following (26).

**Proof.** First we show (23). To slightly lighten the notation we fix $i$ and set

$$W = \{ v_i : \hat{d}(v_i) > (1 + \epsilon)np \}.$$
We partition $W = \bigcup_{j=0}^{J} W^j$ (where $J := \log_2((p(1+\epsilon))^{-1}) - 1 < 2t$), with

$$W^j = \{ v_i : 2^j(1 + \epsilon)np < \hat{d}(v_i) \leq 2^{j+1}(1 + \epsilon)np \}.$$

It suffices to show

$$\sum_{j=0}^{J} |W^j|2^{j+1}(1 + \epsilon)np < \epsilon^2 n^2 p^2 t. \quad (25)$$

Lemma 4.1 (using just (17)) gives

$$\mathbb{P}(v_i \in W^j) \leq \mathbb{P}(\hat{d}(v_i) > 2^j(1 + \epsilon)np)$$

$$\leq 2 \exp[-\epsilon^22^{j-4}np] < \exp[-\epsilon^22^{j-5}np].$$

Thus, for any $(a_0, \ldots, a_J)$,

$$\mathbb{P}(|W^0| = a_0, \ldots, |W^J| = a_J) < \exp\left[ \sum_{j=0}^{J} -a_j \epsilon^2 2^{j-5}np \right] \prod_{j=0}^{J} \binom{n}{a_j}$$

$$< \exp\left[ \sum_{j} a_j (\ln n - \epsilon^2 2^{j-5}np) \right]$$

$$\leq \exp\left[ \sum_{j} -a_j \epsilon^2 2^{j-6}np \right]. \quad (26)$$

For (26) we note that $p > \frac{54-2}{n} \ln n$, so $\epsilon^2 2^{j-5}np \geq 2 \ln n$.

On the other hand, for (25) it is enough to show

$$\sum_{(a_0, \ldots, a_J)} \mathbb{P}(|W^0| = a_0, \ldots, |W^J| = a_J) < \exp[-T \epsilon^4 n^2 p^2 t] \quad (27)$$

for some constant $T > 0$ (not depending on $\epsilon$), where we sum over $(a_0, \ldots, a_J)$ satisfying

$$\sum_{j} a_j 2^{j+1}(1 + \epsilon)np > \epsilon^2 n^2 p^2 t. \quad (28)$$

Here we can just bound the number of terms in (27) by the trivial

$$n^J < \exp[2t \log n],$$

while (in view of (28), (26) bounds the individual summands in (27) by

$$\exp[-\Omega(\epsilon^4 n^2 p^2 t)].$$

Moreover, the lemma’s lower bound on $p$ (or the weaker $p \gg \frac{\log^{1/2} n}{n}$) implies $n^2 p^2 t \gg t \log n$. So the left hand side of (27) is at most

$$\exp[2t \log n - \Omega(\epsilon^4 n^2 p^2 t)] = \exp[-\Omega(\epsilon^4 n^2 p^2 t)],$$

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To show (24) we now let \( W = \{ v_i : \hat{d}(v_i) > np^{1-\gamma/2} \} \). As before, we partition \( W = \bigcup_{j=0}^{J} W^j \) (where \( J := \log_2(p^{-1+\gamma/2}) - 1 < 2t \)) with
\[
W^j = \{ v_i : 2^j np^{1-\gamma/2} < d(v_i) \leq 2^{j+1} np^{1-\gamma/2} \}.
\]
It suffices to show
\[
\sum_{j=0}^{J} |W^j|2^{j+1} np^{1-\gamma/2} < \epsilon n^2 p^2. \tag{29}
\]
Lemma 4.1 and (18) give
\[
P(v_i \in W^j) \leq P(\hat{d}(v_i) > 2^j np^{1-\gamma/2}) \leq 2 \exp[-\gamma 2^{j-2} np^{1-\gamma/2} t] < \exp[-\gamma 2^{j-3} np^{1-\gamma/2} t].
\]
Thus, for any \((a_0, \ldots, a_J)\),
\[
P(|W^0| = a_0, \ldots, |W^J| = a_J) < \exp \left[ \sum_{j=0}^{J} -a_j \gamma 2^{j-3} np^{1-\gamma/2} t \right] \prod_{j=0}^{J} \binom{n}{a_j} \]
\[
< \exp \left[ \sum_{j} a_j (\ln n - \gamma 2^{j-3} np^{1-\gamma/2} t) \right] \]
\[
< \exp \left[ \sum_{j} -a_j \gamma 2^{j-4} np^{1-\gamma/2} t \right]. \tag{30}
\]
(30) follows from \( \gamma 2^{j-3} np^{1-\gamma/2} t \gg \ln n \), in this case a very weak consequence of our assumed lower bound on \( p \).

For (29) it is enough to show
\[
\sum_{(a_0, \ldots, a_J)} P(|W^0| = a_0, \ldots, |W^J| = a_J) < \exp[-T\epsilon^4 n^2 p^2 t] \tag{31}
\]
for some constant \( T > 0 \) (not depending on \( \epsilon \)) where we sum over \((a_0, \ldots, a_J)\) satisfying
\[
\sum_{j} a_j 2^{j+1} np^{1-\gamma/2} > \epsilon n^2 p^2. \tag{32}
\]
Again we can just bound the number of terms in (31) by the trivial
\[
n^J < \exp[2t \log n],
\]
while (in view of (32)), (30) bounds the individual summands by
\[
\exp[-\Omega(\epsilon n^2 p^2 t)].
\]
Again since the lemma’s lower bound on $p$ (or the weaker $p \gg \log^{1/2} n$) implies $n^2 p^2 t \gg t \log n$, the left hand side of (31) is at most
\[
\exp[2t \log n - \Omega(en^2 p^2 t)] = \exp[-\Omega(en^2 p^2 t)],
\]
as desired. \hfill \Box

We will also make use of the fact that for any $\beta > 0$, $k$, and $p$,
\[
p^\beta \ln k (1/p) \leq (ke^{\beta/k})^k.
\] (33)
To see this let $f(p) = p^\beta \ln k (1/p)$, and notice that
\[
f'(p) = -kp^{\beta-1} \ln k^{-1}(1/p) + \beta p^{\beta-1} \ln k (1/p)
= p^{\beta-1} \ln k^{-1}(1/p)(-k + \beta \ln (1/p)).
\]
Thus $f(p)$ is maximized at $p = e^{-k/\beta}$, where it equals the r.h.s. of (33).

5 Proof of (10)

We first rule out very small $p$, showing that when
\[
p < n^{-\gamma/4},
\]
w.l.p. $\Delta < np^{1-\gamma}$, (34)
so that (10) is vacuously true. For (34), with $K = (1/2)p^{-\gamma}$ (and $x$ any vertex), Lemma 4.1 (and the union bound) give
\[
\Pr(\Delta \geq np^{1-\gamma}) \leq ln \cdot \Pr(d(x) \geq 2Kn) \leq ln \cdot \exp[-2Kn \ln (K/e)] = ln \cdot \exp[-np^{1-\gamma}(\gamma t - \ln(2))].
\] (35)
But for $p < n^{-\gamma/4}$ (which is the same as $np^{1-\gamma} > n^2 p^2$), the r.h.s. of (35) is $\exp[-\Omega_\delta,i(n^2 p^2 t)]$ (note that (8) implies $\gamma t \geq 4$ and the initial $\ln$ disappears because (9) makes $\gamma n^2 p^2 t$ a large multiple of $\log n$). Therefore for the remainder of the proof of (10) we may assume that
\[
p \geq n^{-\gamma/4}.
\] (36)
We say $v$ has large degree if $\hat{d}(v) > np^{1-\gamma/2}$ and intermediate degree if $np^{1-\gamma/2} \geq \hat{d}(v) > 2np$. We classify the cycles appearing in (10) according to the positions of their large and intermediate vertices. For disjoint $M, N \subset [l]$, say $v_i$ is of type $(M, N)$ if
\[
\hat{d}(v_i) \begin{cases} np^{1-\gamma/2} & \text{if } i \in M, \\
\in (2np, np^{1-\gamma/2}) & \text{if } i \in N, \\
\leq 2np & \text{otherwise,}
\end{cases}
\]
and say a set of vertices is of type $(M, N)$ if each of its members is. We consider various possibilities for $(M, N)$, always requiring that all vertices under discussion are of the given type. To begin note that since we are in (10) we have $M \neq \emptyset$.

A little preview may be helpful. In each case we are trying to show that the size of the set of cycles $(v_1, \ldots, v_l)$ in question is small relative to $mlp^l$, so would like the number of possibilities for $v_i$ to be, in geometric average, somewhat less than $mp$. For example, for $i \in M$ we do much better than this using Lemma 4.3, which, recall, bounds the number of $v_i$’s of such large degree by $mp^{1+\gamma/2}$ (or $emp^{1+\gamma/2}$ but here the $e$ is minor). On the other hand, for $i \notin M \cup N$ we have only the naive bound $m$, which is clearly unaffordable. To control the number of such $v_i$ we rely on first selecting some $v_{i-1}$ (or $v_{i+1}$) and then bounding the number of choices for $v_i$ by $\hat{d}(v_{i-1})$ (or $\hat{d}(v_{i+1})$). If $i-1, i \notin M \cup N$ then given $v_{i-1}$ we simply use $\hat{d}(v_{i-1}) \leq 2mp$ as a bound on the number of choices for $v_i$. However if, for example, $i-1 \in M \cup N$ and $i \notin M \cup N$ we require Lemma 4.4 to bound the choices for $(v_{i-1}, v_i)$ (with $v_{i-1} \sim v_i$).

We now consider cycles of type $(M, \emptyset)$. Here the absence of intermediate vertices will allow us to relax our assumption that there is at least one vertex of degree at least $np^{1-\gamma}$; we will only need to assume that there is at least one vertex of degree at least $np^{1-\gamma/2}$. Let

$$M^* = \{i \in M : i + 1 \notin M\},$$

with subscripts interpreted mod $l$. Note that $M \neq \emptyset$ implies $M^* = \emptyset$ only when $M = [l]$. Here and in the future we will tend to somewhat abusively omit “w.l.p.” in situations where this is clearly what is meant. We will bound:

(i) for $i \in M \setminus M^*$, the number of possibilities for $v_i$;

(ii) for $i \in M^*$, the number of possibilities for $(v_i, v_{i+1})$;

(iii) given the choices in (ii), the number of possibilities for vertices of the cycle not chosen in (i) and (ii).

Note that the number of vertices chosen in (iii) is $l - |M| - |M^*|$. The reason for treating $i \in M^*$ in (ii) rather than (i) is (roughly) that it is through these vertices that we control the number of choices for the vertices that follow them (the $v_{i+1}$’s of (ii)). For (i) we just recall that Lemma 4.3 bounds the number of choices for $v_i$ (of large degree) by $emp^{1+\gamma/2}$; so the total number of possibilities in (i) is at most

$$(emp^{1+\gamma/2})^{l - |M| - |M^*|}.$$ 

For $i$ as in (ii) the number of possibilities for $(v_i, v_{i+1})$ is at most

$$\sum (\hat{d}(v_i) : \hat{d}(v_i) > np^{1-\gamma/2}) <^* en^2p^2,$$

with the inequality given by Lemma 4.4. Thus the total number of possibilities in (ii) is at most

$$(en^2p^2)^{|M^*|}.$$
Finally, we may choose the \( v_i \)'s in (iii) in an order for which each \( v_{i-1} \) is chosen before \( v_i \) (either because \( v_{i-1} \) is chosen in (ii) or because \( i-1 \) precedes \( i \) in our order; e.g. we can use any cyclic order that begins with an \( i \) for which \( i-1 \in M^* \) — if \( M^* = \emptyset \) then \( M = [l] \), so all vertices were chosen in (i)). But since \( N = \emptyset \), the number of choices for \( v_i \) given \( v_{i-1} \) is at most \( 2np \).

Combining the above bounds we find that, for a given \( M \), the number of cycles of type \((M, \emptyset)\) is at most

\[
\left( \epsilon n^2 p^2 \right)^{|M'|} (\epsilon n^{1+\gamma/2} |M| - |M^*|) (2np)^{|M| - |M^*|} < \epsilon^2 n^l p^l < \frac{\delta}{2^{\gamma/2}} n^l p^l,
\]

(using (7) for the last inequality). So, since there are fewer than \( 2^l \) possibilities for \( M \),

the number of cycles of any type \((M, \emptyset)\) is at most \( \frac{\delta}{4} n^3 p^3 \).

Next we consider cycles of type \((M, N)\) with \( N \neq \emptyset \). We may assume (at the cost of a negligible factor of \( l \) in our eventual bound) that \( 1 \in N \), and that \( k \) is an index for which \( \hat{d}(v_k) > np^{1-\gamma} \) (which exists since we are in (10); again, we will pay a factor of \( l-1 \) for the choice of \( k \)). We further define

\[
N_1 = (N \cup M) \cap \{2, \ldots, k-1\},
\]

\[
N_2 = (N \cup M) \cap \{k+1, \ldots, l\},
\]

\[
N_1^* = \{ i \in N_1 \setminus \{k-1\} : i+1 \notin N_1 \}, \text{ and}
\]

\[
N_2^* = \{ i \in N_2 \setminus \{k+1\} : i-1 \notin N_2 \}.
\]

We split into cases based on whether \( 2 \in N_1 \cup \{k\} \) and/or \( l \in N_2 \cup \{k\} \). First assume \( 2 \notin N_1 \cup \{k\} \) and \( l \notin N_2 \cup \{k\} \). We will bound:

(i) the number of possibilities for \( v_k \);

(ii) the number of possibilities for \( (v_2, v_1, v_l) \);

(iii) for \( i \in (N_1 \cup N_2) \setminus (N_1^* \cup N_2^*) \) the number of possibilities for \( v_i \);

(iv) for \( i \in N_1^* \), the number of possibilities for \( (v_i, v_{i+1}) \);

(v) for \( i \in N_2^* \), the number of possibilities for \( (v_i, v_{i-1}) \);

(vi) given the choices in (ii), (iv), and (v), the number of possibilities for vertices of the cycle not chosen in (i)-(v).

For (i) we just recall that Lemma 4.3 bounds the number of choices for \( v_k \) by

\[\epsilon np^{1+\gamma}.\]

For (ii) the number of possibilities for \( (v_2, v_1, v_l) \) is bounded by

\[
\sum \left\{ \hat{d}(v_1)^2 : np^{1-\gamma/2} \geq \hat{d}(v_1) > 2np \right\} \leq \left( np^{1-\gamma/2} \right) \sum \left\{ \hat{d}(v_1) : \hat{d}(v_1) > 2np \right\} < \epsilon^2 n^3 p^{3-\gamma/2} t,
\]

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where the second inequality is given by Lemma 4.4.

For (iii) Lemma 4.3 bounds the number of choices for \( v_i \) (of intermediate or large degree) by \( cnpt \); so the number of possibilities in (iii) is at most

\[
(\varepsilon npt|N_1| + |N_2| - |N_1^*| - |N_2^*|).
\]

For \( i \) as in (iv) the number of possibilities for \((v_i, v_{i+1})\) is at most

\[
\sum \{\hat{d}(v_i) : \hat{d}(v_i) > 2np\} < \varepsilon n^2 p^2 t,
\]

with the inequality given by Lemma 4.4. Thus the number of possibilities in (iv) is at most

\[
(\varepsilon n^2 p^2 t)^{|N_1^*|}.
\]

Similarly, the total number of possibilities in (v) is at most

\[
(\varepsilon n^2 p^2 t)^{|N_2^*|}.
\]

Finally, for (vi) we choose the remaining \( v_i \)'s with \( i < k \) in increasing order (of their indices) and those with \( i > k \) in decreasing order. In the first case, when we come to \( v_i \) the number of possibilities is at most \( \hat{d}(v_i-1) \leq 2np \) (since \( v_{i-1} \notin N_1 \)), and similarly in the second case this number is at most \( \hat{d}(v_{i+1}) \leq 2np \) since \( v_{i+1} \notin N_2 \). Thus, the number of possibilities in (vi) is at most

\[
(2np)^{|N_1| - |N_2| - |N_1^*| - |N_2^*| - 4}.
\]

Combining the above bounds we find that, for a given \( M \) and \( N \), the number of cycles of type \((M, N)\) is at most

\[
\varepsilon^3 n^l p^{l+\gamma/2} 2^l < \varepsilon^3 n^l p^{l+10l^3} < \frac{\delta n^l p^l}{4L^23^l},
\]

where the second inequality uses (33).

Now we assume \( 2 \in N_1 \cup \{k\} \), but \( l \notin N_2 \cup \{k\} \). In this case (i) (iii) (iv) and (v) and their respective bounds all remain the same. However, now we replace (ii) with

(ii') the number of possibilities for \((v_1, v_l)\).

This is because \( v_2 \) will be selected in either (i) (iii) or (iv). Our new (ii') is bounded by

\[
\sum \{\hat{d}(v_1) : \hat{d}(v_1) > 2np\} < \varepsilon^2 n^2 p^2 t,
\]

where the inequality comes from Lemma 4.4. Additionally, in (vi) there are now \( l - |N_1| - |N_2| - |N_1^*| - |N_2^*| - 3 \) vertices left to choose. Thus our bound for (vi) becomes

\[
(2np)^{|N_1| - |N_2| - |N_1^*| - |N_2^*| - 3}.
\]
Combining these bounds with our previous bounds for (i) and (iii)-(v) we find that, for a given $M$ and $N$, the number of cycles of type $(M, N)$ is at most

$$e^3 n^l p^{l+\gamma l^2} 2^l < e^3 n^l p^l (4l^3)^l < \frac{\delta n^l p^l}{4l^2 3^l},$$

where the second bound is again given by (33).

The argument for $2 \notin N_1 \cup \{k\}$, $l \in N_2 \cup \{k\}$ is essentially identical to the preceding one, so we will not discuss it further.

It remains to consider the case when we have both $2 \in N_1 \cup \{k\}$ and $l \in N_2 \cup \{k\}$. Again, there is no change in (i) and (iii)-(v) and we replace (ii) in this case, by

(ii)” the number of possibilities for $v_1$

(since $v_2$ and $v_l$ will be among the vertices chosen in (i) and (iii)-(v)). By Lemma 4.3 the number of possibilities here (i.e. for $v_1$) is at most

$$e^2 np^l.$$

Additionally, in (vi) we are now selecting $l-|N_1|-|N_2|-|N_1^*|-|N_2^*|-2$ vertices; so, our bound becomes

$$(2np)^{l-|N_1|-|N_2|-|N_1^*|-|N_2^*|-2}.$$ 

Again, combining bounds, we find that the number of cycles of type $(M, N)$ is at most

$$e^3 n^l p^{l+\gamma l^2} 2^l < e^3 n^l p^l (4l^3)^l < \frac{\delta n^l p^l}{4l^2 3^l}.$$ 

So to recap, we have shown that, for any given $M$, $N \neq \emptyset$ (where we assume $\hat{d}(v_k) > np^{1-\gamma}$ and $np^{1-\gamma/2} \geq \hat{d}(v_1) > 2np$) there are at most

$$\frac{\delta n^l p^l}{4l^2 3^l}$$

cycles of type $(M, N)$.

Since there are fewer than $3^l$ choices for $(M, N)$ and the assumptions on 1 and $k$ only cost a factor of $l^2$, there are at most

$$\frac{\delta n^l p^l}{4}$$

cycles of all types $(M, N)$ with $N \neq \emptyset$; recalling (see (37)) that we showed the same bound for the number of cycles of types $(M, \emptyset)$ (with $M \neq \emptyset$), we have the desired bound, $(\delta/2) n^l p^l$, on the l.h.s. of (10).

\[\square\]
6 Proof of \((12)\)

For the rest of our discussion we may ignore bad vertices, meaning those of degree at least \(np^{1-\gamma}\), since cycles involving such vertices are excluded from \((12)\). (Recall we are calling \(\tilde{d}(v)\) the degree of \(v\).)

What’s really going on here is as follows. We think of choosing \(\nabla(V_1, V_l)\) after all other edges have been specified. The number of cycles (again, avoiding bad vertices) is then

\[
\sum_{v_1 \sim v_l} f(v_1, v_l)
\]

(recall \(f(v_1, v_l)\) is the number of full paths with endpoints \(v_1\) and \(v_l\) in which there are no bad vertices). Given \(G \setminus \nabla(V_1, V_l)\), this is a weighted sum of independent binomials with expectation

\[
p \sum_{v_1 \sim v_l} f(v_1, v_l),
\]

to which we may hope to apply the large deviation bound in Lemma 4.2. In this section we give a good (w.l.p.) bound on the sum in (39) (namely \((12)\)). Once we have this, the only difficulty is that some of the “weights” \(f(v_1, v_l)\) may be too large to support finishing via the lemma. We will handle this difficulty in Section 7.

To prove \((12)\) we first consider full paths \((v_1, \ldots, v_l)\) in which each of \(v_1, \ldots, v_{l-1}\) has degree at most \((1 + \epsilon)np\). There are at most

\[
(1 + \epsilon)^l n^l p^{l-1} < (1 + \delta/16)n^l p^{l-1}
\]

such paths.

Now all the paths \((v_1, \ldots, v_l)\) left to consider must have some \(v_i\) (where \(i \in [l-1]\)) such that \(\tilde{d}(v_i) > (1 + \epsilon)np\). To count the number of such paths we split the argument based on \(p\). First assume

\[
p > \frac{\ln^2 n}{n},
\]

(This is not a tight bound for either argument, but it is a convenient cut-off.) Given \((11)\) we know

\[
q_K \leq \exp \left[\frac{-\epsilon^2 K np}{16}\right] < \exp \left[\frac{-\epsilon^2 \ln^2 n}{16}\right] < n^{-2}
\]

for all \(K \geq 1 + \epsilon\) (see \((16)\) for the definition of \(q_K\), so in applications of Lemma 4.3 we are always using the second value of \(r_K\) (namely, \(r_K = \frac{\epsilon^2 np}{K \ln K}\)). Additionally since \(p > \frac{\ln^2 n}{n}\) Lemma 4.4 applies. As in Section 5 we classify paths according to the positions of vertices with \(\tilde{d}(v_i) > (1 + \epsilon)np\). For \(M \subseteq [l-1]\), say \(v_i\) is of type \(M\) if

\[
\tilde{d}(v_i) \begin{cases} 
> (1 + \epsilon)np, & \text{if } i \in M, \\
\leq (1 + \epsilon)np & \text{otherwise},
\end{cases}
\]
and say a set of vertices is of type $M$ if each of its members is either of type $M$ or in $V_l$. Note we have already shown that there are at most

$$(1 + \delta/16)n^l p^{l-1}$$

full paths of type $\emptyset$, so we now assume $M \neq \emptyset$. Let $m$ be the smallest element of $M$ and let

$$M^* = \{i \in M : i + 1 \notin M\}.$$

We will bound:

(i) for $i \in M \setminus M^*$, the number of possibilities for $v_i$;

(ii) for $i \in M^*$, the number of possibilities for $(v_i, v_{i+1})$;

(iii) given the choices in (ii), the number of possibilities for vertices of the path not chosen in (i) and (ii).

For i as in (i) we recall that by Lemma 4.3 the number of $v_i$’s of degree at least $(1 + \epsilon)np$ is at most $\epsilon n^t p^2$. So, the total number of possibilities in (i) is at most

$$(\epsilon n^t p^2)^{|M| - |M^*|}.$$  \hfill (42)

For i as in (ii) the number of possibilities for $(v_i, v_{i+1})$ is at most

$$\sum \left\{ \hat{d}(v_i) : \hat{d}(v_i) > (1 + \epsilon)np \right\} < \epsilon^2 n^2 p^2 t,$$

with the inequality given by Lemma 4.4. Thus the total number of possibilities in (ii) is at most

$$(\epsilon^2 n^2 p^2 t)^{|M^*|}.$$  \hfill (43)

Finally for (iii) we choose the remaining $v_i$’s with $i > m$ in increasing order (of the indices). When we come to $v_i$ we know $i - 1 \notin M$, so given $v_{i-1}$ there are at most $(1 + \epsilon)np$ choices for $v_i$. If $m = 1$ then we have selected all the vertices in the path. If not, then we next select $v_{m-1}$. Since we are ignoring vertices of degree at least $np^{l-\gamma}$ we know that given $v_m$, there are at most $np^{l-\gamma}$ ways to select $v_{m-1}$. If $m = 2$ then we are done, and if not then we select the $v_i$’s with $i < m - 1$ in decreasing order (of the indices). Since $i + 1 \notin M$, given $v_{i+1}$ there are at most $(1 + \epsilon)np$ choices for $v_i$. Thus, the number of possibilities in (iii) is at most

$$\begin{cases} (1 + \epsilon)np^{l-|M^*|} & \text{if } m > 1, \\ (1 + \epsilon)np^{l-1} & \text{if } m = 1. \end{cases}$$  \hfill (44)

Combining (42), (43), and the appropriate bound from (i) we find that, for a given $M$, there are at most

$$\epsilon(1 + \epsilon)^l n^l p^{l-\gamma} t^l < \epsilon(2l)! n^l p^{l-1} < \frac{\delta n^l p^{l-1}}{2^{l+3}}.$$
full paths of type $M$ (where the first inequality uses (53)). Since there are less than $2^{l-1}$ possibilities for $M \neq \emptyset$ there are at most

$$\frac{\delta n^l p^l}{16}$$

full paths of type other than $\emptyset$. Together with our earlier bound on the number of full paths of type $\emptyset$ this bounds the total number of full paths (without vertices of degree at least $np^{1-\gamma}$) by

$$(1 + \delta/8) n^l p^l, \tag{46}$$

as desired.

When

$$p \leq \frac{\ln^2 n}{n} \tag{45}$$

we first note that we have a better bound on $\Delta$ (the maximum degree) than $np^{-\gamma}$. For (45) Lemma 4.1 with $K = (\ln^3 n)/2$ (and $x$ any vertex) gives

$$P(\Delta > \ln^3 n(np)) \leq \ln P(d(x) > \ln^3 n(np))$$

$$< \ln \exp[-np(n^3 n)(\ln \ln n)]$$

$$< \exp[-\Omega_{\delta,l}(n^2 p^2 t)],$$

using $npt < \ln^3 n$ and absorbing the initial $\ln$ into the exponent (since (9) gives $np(\ln^3 n) > \epsilon^{-2}(\ln^3 n)$). Thus, $\Delta < \ln^3 n(np) \leq \ln^5 n$.

Given $p$, let $K$ be minimal with $q_K \leq n^{-2}$. We first bound the number of cycles containing at least one $v$ with $d(v) > Knp$. Lemma 4.3 says there are at most $\frac{\ln^2 npt}{Kn}$ such vertices (in all of $V$). Once such a vertex $v$ has been specified there are at most

$$\Delta^{l-1} < \ln^{5(l-1)} n$$

ways to select the remaining vertices in a full path containing $v$. So, w.l.p. we have at most

$$\frac{\ln^2 n pt \ln^{5(l-1)} n}{K \ln K} = o(n^l p^{l-1}) \tag{46}$$

full paths containing at least one $v$ as above. (The quite weak $o(n^l p^{l-1})$ follows from the lower and upper bounds on $p$ in (9) and (15), respectively.)

Now we count paths in which every vertex has degree at most $Knp$ and at least one vertex has degree at least $(1 + \epsilon)np$ (recalling that we have already treated those violating either condition). Say $v$ is of type $i$ if

$$(1 + \epsilon)2^i np < d(v) \leq (1 + \epsilon)2^{i+1} np,$$

and let $U_i = \{\text{vertices of type } i\}$. We say the type of a path $P$ is the largest $i$ for which $P$ contains a vertex of type $i$. Lemma 4.3 gives

$$|U_i| < 6\epsilon 2^{-d} n.$$

Note we have already bounded the number of full paths of type $i$ where $i > \log_2 K - 1$. For smaller $i$ we think of specifying a path $P$ of type $i$ by choosing
(i) some \( v \) of type \( i \), and then
(ii) the remaining vertices of the path.

Here the bounds are easy: the number of possibilities in (i) is at most
\[
|U_i| < 6\ell 2^{-i}n, \tag{47}
\]
and the number of possibilities in (ii) is at most
\[
((1 + \epsilon)2^{i+1}np)^{l-i},
\]
since, given the choice in (i), we may order the remaining choices so that each new vertex is drawn from the at most \((1 + \epsilon)2^{i+1}np\) neighbors of some vertex chosen earlier. Thus the number of full paths of type \( i \) is bounded by
\[
6\ell(1 + \epsilon)^{l-i}2^{l-i}n^p < \ell 2^{l-i}n^p.
\]
Summing over \( i \) we find that w.l.p. there are at most
\[
\log_2 K - 1 \sum_{i=0}^{\log_2 K - 1} \ell 2^{2l-i}n^p < \frac{\delta}{17} n^p.
\]  

(48)

full paths of all types up to \( \log_2 K - 1 \) (where the inequality follows easily from our choice of \( \epsilon \) — see (12)). Adding (48) to the numbers of full paths with all degrees at most \((1 + \epsilon)np\) and those of type \( i \) for \( i > \log_2 K - 1 \) ((40) and (46)) we find that w.l.p. there are at most
\[
(1 + \delta/8)n^p
\]
full paths (with all vertices of degree at most \( np^{1-\gamma} \)). So, regardless of \( p \), we have
\[
\sum f(v_1, v_l) < (1 + \delta/8)n^p
\]
as desired.

\[ \square \]

7 Proof of (11)

As explained at the start of Section 6 we want to use (12) and finish via Lemma 4.2, but some \( f(v_1, v_l) \)'s may be too large to support this. To handle this difficulty we introduce the notion of a “heavy path” below. We then set
\[
C' = \{ (v_1, \ldots, v_l) \in C : (\forall i)v_i \in V'_i \text{ and } (v_1, \ldots, v_l) \text{ is not heavy} \},
\]
and show
\[
\mathbb{P}(|C'| > (1 + \delta/4)n^p) < \exp[-\Omega_{\delta,i}(n^2p^2 s)], \tag{49}
\]
w.l.p. \([|\{(v_1, \ldots, v_l) \in C : \forall i(v_i \in V'_i), (v_1, \ldots, v_l) \text{ heavy}\}| < (\delta/4)n^p]. \tag{50}
\]

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It will turn out that we need different definitions of “heavy path”, depending on $p$. Either of these will say that the number of non-heavy paths, say $g(v_1, v_l)$, joining any $v_i, v_l$ satisfies

$$g(v_1, v_l) \leq \frac{4^l n^{l-2} p^{l-2}}{s}. \quad (51)$$

(Recall $s = \min\{t, n^{l-2} p^{l-2}\}$.) We will return to the definitions of heavy path and the proof of (50) in Subsections 7.1 and 7.2; here we assume (51) and give the easy proof of (49).

As suggested above this is a straightforward application of Lemma 4.2. Let $V_1 = \{x_1, \ldots, x_n\}$ and $V_l = \{y_1, \ldots, y_n\}$. Then with

$$w_{i,j} = g(x_i, y_j) \leq \frac{4^l n^{l-2} p^{l-2}}{s} =: z$$

and $\zeta_{i,j}$ the indicator of the event $\{x_iy_j \in \mathbb{H}\}$ we have

$$|C'| = \zeta := \sum \zeta_{i,j}w_{i,j}.$$ 

In addition, recalling (12), we have

$$\mathbb{E}\zeta = p \sum w_{i,j} \leq p \sum f(v_1, v_l) <^* (1 + \delta/8)n^l p^l.$$ 

Hence Lemma 4.2 with $\lambda = (\delta/8)n^l p^l$ gives

$$\mathbb{P}(|C'| > (1 + \delta/4)n^l p^l) < \exp[-\Omega_{\delta,l}(n^2 p^2 s)],$$

as desired.

### 7.1 Proof of (51) when $p > n^{\frac{5l}{5l+1}}$

For $p > n^{\frac{5l}{5l+1}}$ we say $(v_1, v_l)$ is heavy if

$$f(v_1, v_l) > \frac{4^l n^{l-2} p^{l-2}}{s},$$

and $(v_1, \ldots, v_l)$ is a heavy path if $(v_1, v_l)$ is heavy. (Note that here we have $s = t(= \ln(1/p)).$) So, in this case the notion of heavy depends only on the endpoints of the path. Note that this definition trivially implies (51).

A brief indication of why we need two definitions of a heavy path may be helpful. In the present case (i.e. $p > n^{\frac{5l}{5l+1}}$) we bound the number of cycles $(v_1, \ldots, v_l)$ for which $(v_1, \ldots, v_l)$ is a heavy path by first bounding the number of $v_1$’s (and similarly $v_l$’s) that are in heavy paths. To do this we show that for $v_1$ to be in a heavy path there must be some $v_3$ for which $d(v_1, v_3) := |N(v_1) \cap N(v_3)|$ is “large”, and we use this necessary condition to bound the number of $v_1$’s in heavy paths.
Let

\[ V_1^* = \{ v_1 \in V_1' : \exists v_i \in V_1 \text{ with } (v_1, v_i) \text{ heavy} \}, \]
\[ V_l^* = \{ v_l \in V_l' : \exists v_1 \in V_1 \text{ with } (v_1, v_l) \text{ heavy} \}. \]

Thus every cycle, \((v_1, \ldots, v_l)\), considered in this section must have \(v_1 \in V_1^*\) and \(v_l \in V_l^*\). We first bound \(|V_1^*|\) and \(|V_l^*|\), and then use this to bound \(|\nabla(V_1^*, V_l^*)|\).

A necessary condition for \(v_1 \in V_1^*\) is

there exists \(v_3\) such that \(d(v_1, v_3) \geq np^{1+\gamma(l-1)}\). (52)

To see this, fix \(v_1\) and recall that \(\hat{d}(v) < np^{1-\gamma}\) for every vertex under discussion in \([41]\). Thus, we know that for any \(v_l\) there are at most \((np^{1-\gamma})^{l-3}\) paths \((v_1, \ldots, v_3)\). To pick \(v_2\) to complete such a path with \(v_1\) we require \(v_2 \in N(v_1) \cap N(v_3)\). Thus if \(d(v_1, v_3) < np^{1+\gamma(l-1)}\) for all \(v_3\) then for any \(v_l\),

\[ f(v_1, v_l) < n^l - 2p^{l-2} < \frac{5l^2n^l - 2p^{l-2}}{2e} < 4tn^{l-2} / s. \]

(Here the middle inequality comes from \((33)\) with \(\beta = 2\gamma\) and \(k = 1\).) So in order to bound \(|V_l^*|\) it suffices to bound the number of \(v_1's\) satisfying \((52)\).

Since \(\hat{d}(v_3) < np^{1-\gamma}\), Lemma \([4, 1]\) (with \(m = np^{1-\gamma}\), \(\alpha = p\), and \(K = p^{-1+\gamma l}\))

gives

\[ \mathbb{P}(v_1 \in V_1^*) \leq n\mathbb{P}(B(m, p) > Kmp) \]
\[ < n \exp \left[ np^{1+\gamma(l-1)}(1 - (1 - \gamma) t) \right]. \]

Note that \(p \leq e^{-4/\gamma}\) (see \((5)\)) implies \(t \geq 4/\gamma\), so

\[ \exp(np^{1+\gamma(l-1)}(1 - (1 - \gamma) t)) < \exp[-np^{1+\gamma(l-1)}t/2]. \]

Thus,

\[ \mathbb{P}(v_1 \in V_1^*) < n \exp[-np^{1+\gamma(l-1)}t/2] \]
\[ < \exp[-np^{1+\gamma(l-1)}t/3]. \]

The initial \(n\) disappears since \(p > n^{\frac{5l}{5l^2+5}}\) implies \(np^{1+\gamma(l-1)} > n^{1/(5l^2+5)}\).

Next we show that w.l.p. \(|V_1^*|\) and \(|V_l^*|\) are at most \(\exp[-np^{1+\gamma(l-1)}t/3]t/3\) then \(|U| < \epsilon np^{1+\gamma(l-1-\epsilon)}\).

**Lemma 7.1.** If \(c \in [1, 3]\) and \(U\) is a random subset of \(V_i\) in which each \(v_i\) is included independently with probability at most \(\exp[-np^{1+\gamma(l-1-\epsilon)}t/3]\) then \(|U| < \epsilon np^{1+\gamma(l-1-\epsilon)}\).

**Proof.** Here we apply Lemma \([4, 1]\) with \(m = n\), \(\alpha = \exp[-np^{1+\gamma(l-1-\epsilon)}t/3]\), and \(K = \epsilon p^{1-\gamma(l-\epsilon)} \alpha^{-1}\). Note that since \(p > n^{\frac{5l}{5l^2+5}}\) we know, say, \(K/e > \alpha^{-1/2}\); so
Lemma 4.1 gives
\[
P \left( |U| > np^{1-\gamma(l-c)} \right) < (e/K)\epsilon np^{1-\gamma(l-c)}/2
\]
\[= \exp[-\epsilon^2 n^2 p^2 t/6].
\]

Hence $|V_i^*|, |V_{i+1}^*| < \epsilon np^{1-\gamma(l-1)}$.

We next show that for any $i$
\[
\text{w.l.p. } |\nabla (A, B)| < \epsilon^2 n^2 p^2 \tag{53}
\]
\[
\forall A \subseteq V_i, B \subseteq V_{i+1} \text{ with } |A|, |B| < \epsilon np^{1-\gamma(l-1)}.
\]

We use (53) to bound $|\nabla (V_i^*, V_{i+1}^*)|$ (and again after (59)). To prove (53) we assume $A$ and $B$ are of the appropriate sizes and apply Lemma 4.1 with $m = |A||B|$, $\alpha = p$, and $K = \epsilon^2 n^2 p^2 (mp)^{-1}$. Note that $m < \epsilon^2 n^2 p^2 - 2\gamma(l-1)$, and, generously, $K \geq p^{-1+2/5l} > p^{-1/2}$. Also, since $p \leq e^{-20l^2}$, we have $\ln(K) > t/2 \geq 10l^2$. So for a given $A$ and $B$ of the appropriate size Lemma 4.1 gives
\[
P(|\nabla (A, B)| > \epsilon^2 n^2 p^2) < \exp[-\epsilon^2 n^2 p^2 (\ln(K) - 1)]
\]
\[< \exp[-\epsilon^2 n^2 p^2 t/4]. \tag{54}
\]

Simply taking the union bound with the first sum over all possible $A, B$ and the next two over all $a, b < \epsilon np^{1-\gamma(l-1)}$ we have
\[
\sum_{A, B} \mathbb{P}(|\nabla (A, B)| > \epsilon^2 n^2 p^2) <
\]
\[
\sum_{a, b} l \left( \begin{array}{c} n \\ a \\ \end{array} \right) \left( \begin{array}{c} n \\ b \\ \end{array} \right) \exp[-\epsilon^2 n^2 p^2 t/4] <
\]
\[
\sum_{a, b} l \exp[a \ln(en/a) + b \ln(en/b) - \epsilon^2 n^2 p^2 t/4]. \tag{54}
\]

It is easy to see (using $p > n^{-\epsilon}$ and $\gamma = \frac{1}{5l^2}$) that for $a, b < \epsilon np^{1-\gamma(l-1)}$ we have
\[
n^2 p^2 t \gg \max\{a \ln(en/a) + b \ln(en/b), \ln(n)\}.
\]

So (54) is, for example, at most $\exp[-\epsilon^2 n^2 p^2 t/5]$. Therefore w.l.p.
\[
|\nabla (A, B)| < \epsilon^2 n^2 p^2, \text{ for all } A, B \text{ with } |A|, |B| < np^{1-\gamma(l-1)}, \tag{55}
\]
as desired. Specifically we have (w.l.p.)
\[
|\nabla (V_i^*, V_{i+1}^*)| < \epsilon^2 n^2 p^2. \tag{56}
\]
We next want to bound the number of full paths between $V_1^*$ and $V_l^*$. For $i \in \{2, \ldots, l-1\}$ let

$$V_i^* = \{ v_i : \max_{v \in V_{i-2} \cup V_{i+2}} d(v, v_i) > n p^{1+\gamma(l-3)} \}. $$

We first bound the number of full paths such that at least one vertex $v_i$ in the path is not in the appropriate $V_i^*$. Fixing $v_1$, $v_l$, and an index $i < l-1$ we bound the number of full paths $(v_1, \ldots, v_l)$ with $v_i \notin V_i^*$. Since $\hat{d}(v) < n p^{1-\gamma}$ for all $v$ under consideration, there are at most

$$n^{l-1} p^{(1-\gamma)(l-1)}$$

ways to choose $v_2 \sim \cdots \sim v_i$ with $v_2 \sim v_1$ and

$$n^{l-i-2} p^{(1-\gamma)(l-i-2)}$$

ways to choose $v_{i-1} \sim \cdots \sim v_{i+2}$ with $v_{i-1} \sim v_l$. To complete the path we must have $v_{i+1} \in N(v_i) \cap N(v_{i+2})$. Since we assume $v_i \notin V_i^*$, there are at most $n p^{1+\gamma(l-3)}$ choices for $v_{i+1}$. Thus there are at most

$$(n^{i-1} p^{(1-\gamma)(i-1)})(n^{l-i-2} p^{(1-\gamma)(l-i-2)}) n p^{1+\gamma(l-3)} = n^{l-2} p^{l-2}$$

paths from $v_1$ to $v_l$ with $v_i \notin V_i^*$.

If $i = l-1$ then we instead bound the number of choices for $v_{l-1}$ by

$$\hat{d}(v_l) < n p^{1-\gamma},$$

and the number of ways to choose $v_2 \sim \cdots \sim v_{l-3}$ with $v_2 \sim v_1$ by

$$n^{l-4} p^{(l-4)(1-\gamma)},$$

To complete the path we must have $v_{l-2} \in N(v_{l-3}) \cap N(v_{l-1})$. Again, as we are assuming $v_{l-1} \notin V_{l-1}^*$, there are at most $n p^{1+\gamma(l-3)}$ choices for $v_{l-2}$. So, there are at most

$$(n^{l-4} p^{(1-\gamma)(l-4)})(n p^{1+\gamma(l-3)}) = n^{l-2} p^{l-2}$$

paths from $v_1$ to $v_l$ with $v_{l-1} \notin V_{l-1}^*$.

Now summing over $i$, there are at most $(l-2)n^{l-2} p^{l-2}$ paths using at least one vertex outside of $\bigcup_{i=2}^{l-2} V_i^*$, and combining this with (56) bounds the number of cycles as in (57) (with some vertex outside of $\bigcup_{i=2}^{l-2} V_i^*$) by

$$(l-2)\epsilon n^i p^l < \frac{\delta}{8} n^i p^l.$$ (57)

The only cycles left to count are those with $v_i \in V_i^*$ for all $i$. We first bound $|V_i^*|$. Lemma 2.1 with $m = n p^{1-\gamma}$, $\alpha = p$, and $K = p^{-1+\gamma(l-2)}$ (and the union bound) gives, for any $v \in V_i$,

$$\Pr(v \in V_i^*) < 2 n \Pr(B(m, p) > K mp)$$

$$< 2 n \exp[n p^{1+\gamma(l-3)}(1 - (1 - \gamma(l-2)) t)].$$ (58)
As before, \( t \geq 4/\gamma \) implies the r.h.s. of (58) is at most

\[
2n \exp[-np^{1+\gamma(l-3)}t/2].
\]

Hence,

\[
\Pr(v_i \in V^*_i) < 2n \exp[-np^{1+\gamma(l-3)}t/2] < \exp[-np^{1+\gamma(l-3)}t/3].
\] (59)

Again the initial \( 2n \) disappears since \( p > n^{-\frac{5l}{3(5l^2+l)}} \) implies \( np^{1+\gamma(l-3)} > n^{3/(5l^2+l)} \).

Given (59) Lemma 7.1 gives

\[
|V^*_i| < \epsilon n^{p^{1-\gamma(l-3)}}.
\]

Assuming this, (55) gives

\[
|\nabla(V^*_i, V^*_i+1)| < \epsilon n^2p^2
\]

for all \( i \).

To finish the proof (for \( p \geq n^{-\frac{5l}{3(5l^2+l)}} \)) we use the following lemma due to Shearer [11]. We will use this lemma again when \( p \leq n^{-\frac{5l}{3(5l^2+l)}} \). To state it we require the following definition. (Recall a hypergraph on \( V \) is simply a collection — possibly with repeats — of subsets of \( V \).)

For a hypergraph \( F \) on the vertex set \( V \) and \( H \subseteq V \), the trace of \( F \) on \( V \) is defined to be

\[
\Tr(F, H) = \{ F \cap H : F \in F \}.
\]

Lemma 7.2. Suppose \( F \) is a hypergraph on \( V \) and \( H \) is another hypergraph on \( V \) such that every vertex in \( V \) belongs to at least \( d \) edges of \( H \). Then

\[
|F| \leq \prod_{H \in H} |\Tr(F, H)|^{1/d}.
\] (60)

To apply Lemma 7.2 here, let \( F \) be the hypergraph on \( V = V(\mathbb{H}) \) whose edges are the vertex sets of cycles using only vertices in \( \bigcup_{i=1}^l V^*_i \). So \( |F| \) is the number of cycles using only vertices in \( \bigcup_{i=1}^l V^*_i \). Let \( H \) be the hypergraph on \( V \) with edges \( \{H_i := V_i \cup V_{i+1} \}_{i \in [l]} \). Thus each vertex belongs to exactly two edges of \( H \). Furthermore

\[
|\Tr(F, H)| \leq |\nabla(V^*_i, V^*_i+1)| < \epsilon n^2p^2.
\]

Thus Lemma 7.2 gives

\[
|F| \leq \prod_{H \in H} |\Tr(F, H)|^{1/2} < (\epsilon n^2p^2)^{l/2} < (\delta/8)n^lp^l.
\]

Combining this with (57) gives (59) (for \( p > n^{-\frac{5l}{3(5l^2+l)}} \)).
7.2 Proof of (50) when $p \leq n^{-5/2}$

For $p \leq n^{-5/2}$ we need the following definitions for $j \notin \{1, l\}$ and $i < l - 1$:

\[ N^j(v_l) = \{v_j : \text{there exists a path } (v_j, v_{j+1}, \ldots, v_l)\} \tag{61} \]
\[ V''_i = \{v_i \in V_i : \max_{v_l} d_{N^i(v_l)}(v_l) > 4\}. \tag{62} \]

That is, $v_i \in V''_i$ if, for some $v_l$, $v_i$ has at least 5 neighbors in $V_{i+1}$ that are “directly reachable” from $v_l$. We say a path $(v_1, \ldots, v_i)$ is heavy if $v_i \in V''_i$ for some $i(< l - 1)$. Note (as promised) we still have (51), since

\[ g(v_1, v_l) \leq 4^{l-2} < \frac{4^l n^{-2} p^{l-2}}{s}. \]

(Again recall $s = \min\{t, n^{-2} p^{-l/2}\}$.)

In this section we are bounding the number of cycles $(v_1, \ldots, v_l)$ containing at least one vertex in some $V''_i$. To do this we fix $i$ and bound the number of cycles with $v_i \in V''_i$.

We first observe that

\[ \Delta < n^2 p^2 t \tag{63} \]

(where, as usual, $\Delta$ is the maximum degree in $H$.) For (63) Lemma 4.1 with $K = npt/2$ (and $x$ any vertex), together with the union bound, gives

\[ \mathbb{P}(\Delta > n^2 p^2 t) \leq \ln \mathbb{P}(d(x) > 2Kn) \]
\[ < \ln \exp[-2Kn(1 - \ln(K))] \]
\[ < \exp[-n^2 p^2 t]. \]

So we may assume $\Delta < n^2 p^2 t$, whence, for any $j$ and $v_l$,

\[ |N^j(v_l)| \leq \Delta^{l-2} < n^{2l-2} p^{2l-2} t^{l-1} = : m. \tag{64} \]

Note that $m \leq n^{2l+2} \log^{l-1} n$ (since $p \leq n^{-5/2}$).

We next show

\[ |V''_i| < n^2 p^2. \tag{65} \]

Here, for a given $v_l$, we may think of $N^{i+1}(v_l)$ — which does not depend on edges involving $V_i$ — as given. Then for a given $v_l$ we have (using (64))

\[ \mathbb{P}(v_i \in V''_i) < n \mathbb{P}(B(m, p) > 4); \tag{66} \]

so applying Lemma 4.1 with $\alpha = p$ and $K = 4m^{-1} p^{-1} > n^{3/5}$ bounds the r.h.s. of (66) by

\[ n(e/K)^4 < e^4 n^{-7/5} = : q. \]

Another application of Lemma 4.1 with $m = n$, $\alpha = p$, and $K = e^npq^{-1} > n^{2/5}$ now gives (65):

\[ \mathbb{P}(|V''_i| > n^2 p^2) < (e/K)^{n^2 p^2} < \exp[-(e/5)n^2 p^2 t]. \]
We may thus assume from now on that $|V''_i| < cn^2p^2$.

Given $V''_i$ we bound the number of cycles $(v_1, \ldots, v_i, \ldots, v_l)$ with $v_i \in V''_i$. This requires the following definitions (for $i \neq j$):

- $V_{i,j}^0 = \{v_j : \text{there is a path } (v_i, v_{i+1}, \ldots, v_j) \text{ with } v_i \in V''_i\}$,
- $V_{i,j}^1 = \{v_j : \text{there is a path } (v_i, v_{i-1}, \ldots, v_j) \text{ with } v_i \in V''_i\}$,
- $V_{i,j} = V_{i,j}^0 \cap V_{i,j}^1$.

(Note we are reading subscripts mod $l$.)

Thus $v_i \in V_{i,j}$ if and only if some cycle containing $v_j$ meets $V''_i$. We also set $V_{i,i} = V_{i,i}^0 = V_{i,i}^1 = V''_i$. We will show $|\nabla(V_{i,j}, V_{i,j+1})| < \left(\frac{\delta}{4l}\right)^{2l} n^2p^2.$ (67)

To bound the number of cycles involving some $v_i \in V''_i$ we need a bound on $|\nabla(V_{i,j}, V_{i,j+1})|$, but will actually bound the (larger) quantity $|\nabla(V_{i,j}^0, V_{i,j+1}^1)|$.

As elsewhere the point here is to retain some independence; given $V''_i$, $V_{i,j}^0$ and $V_{i,j+1}^1$ do not depend on $\nabla(V_{i,j}^0, V_{i,j+1}^1)$. Thus, having specified $V''_i$ we may think of first exposing the edges of $H$ not involving $\nabla(V_{i,j}^0, V_{i,j+1}^1)$ — thus determining $V_{i,j}^0$ and $V_{i,j+1}^1$ — at which point $\nabla(V_{i,j}^0, V_{i,j+1}^1)$ is just a binomial to which we may apply Lemma 4.1. Note, however, that $\nabla(V_{i,j}^0, V_{i,j+1}^1)$ will not be independent of the choice of $V''_i$, so we will need to take a union bound over possibilities for $V''_i$.

We will show $|\nabla(V_{i,j}^0, V_{i,j+1}^1)| < \left(\frac{\delta}{4l}\right)^{2l} n^2p^2.$ (67)

The eventual punchline here will be an application of Lemma 4.4 (Shearer’s Lemma) similar to the one in Section 7.1. This is the reason for the $\left(\frac{\delta}{4l}\right)^{2l}$ which, in applying the lemma will be raised to the power $l/2$.

Note that for all $i, j$ we have (very crudely in most cases)

$$|V_{i,j}^0||V_{i,j}^1| \leq |V''_i|\Delta^{l-1} < cn^2p^2\Delta^{l-1}.$$  We apply Lemma 4.4 with

$$m = |V_{i,j}^0||V_{i,j+1}^1| < e^2n^4p^4\Delta^{2l-2},$$
- $\alpha = p$, and
- $K = (mp)^{-1}\left(\frac{\delta}{4l}\right)^{2l} n^2p^2$.

A little checking (using $p < n^{5l/11}$) confirms that, for example,

$$K > n^{l/6l}.$$
Thus for specified $i, V''_i$, and $j$ Lemma 4.1 gives

$$\Pr \left( |\nabla(V_{i,j,0}^0, V_{i,j+1}^1)| > \left( \frac{\delta}{4l} \right)^{2/l} n^2 p^2 \right) < \exp \left[ -\frac{(\delta/(4l))^{2/l} n^2 p^2 t}{6l} \right], \quad (68)$$

and summing over possibilities for $i, V''_i$, and $j$ (recalling that we have $|V''_i| < \epsilon n^2 p^2$) gives (67):

$$\Pr \left( \exists i, j \text{ with } |\nabla(V_{i,j,0}^0, V_{i,j+1}^1)| > \left( \frac{\delta}{4l} \right)^{2/l} n^2 p^2 \right)$$

$$< l^2 \sum_{w<\epsilon n^2 p^2} \binom{n}{w} \exp \left[ -\frac{(\delta/(4l))^{2/l} n^2 p^2 t}{6l} \right]$$

$$= \exp[-\Omega(n^2 p^2 t)].$$

Here for the final bound we use that $w \ln(en/w) < \epsilon n^2 p^2 t$ and $\epsilon$ is small enough (see (7)).

To apply Lemma 7.2 here let $F$ be the hypergraph on $V = V(\mathbb{H})$ where each edge is the vertex set of a cycle using only vertices in $\bigcup_{j=1}^l V_{i,j}$. Again let $H$ be the hypergraph on $V$ with edges $\{H_j := V_j \cup V_{j+1}\}_{j \in [l]}$. Thus each vertex belongs to exactly two edges of $H$. Furthermore, (67) says

$$|\text{Tr}(F, H)| \leq |\nabla(V_{i,j,0}^0, V_{i,j+1}^1)| \leq |\nabla(V_{i,j,0}^0, V_{i,j+1}^1)| <^* \left( \frac{\delta}{4l} \right)^{2/l} n^2 p^2.$$

Thus Lemma 7.2 gives

$$|F| \leq \prod_{H \in \mathcal{H}} |\text{Tr}(F, H)|^{1/2} <^* \left( \frac{\delta}{4l} \right)^{2/l} n^2 p^2 \left( \frac{\delta}{4l} \right)^{l/2} n^l p^l,$$

as desired. So, summing over choices for $i$, there are less than $(\delta/4)n^l p^l$ cycles using some $v_i \in V''_i$, as desired.

**References**

[1] N. Alon and J.H Spencer. *The Probabilistic Method*. New York: Wiley, 2015.

[2] J. Beck and W. Chen. *Irregularities of Distribution*. Cambridge: Cambridge Univ. Pr., 1987.

[3] J. Van Den Berg and H. Kesten. “Inequalities with Applications to Percolation and Reliability”. In: *Journal of Applied Probability* 22.3 (Sept. 1985), pp. 556–569.

[4] B. Bollobás. *Modern Graph Theory*. New York: Springer, 1998.
[5] S. Chatterjee. “The missing log in large deviations for triangle counts”. In: *Random Structures & Algorithms* 40 (4 2011), pp. 437–451.

[6] S. Chatterjee and A. Dembo. “Nonlinear large deviations”. In: *Advances in Mathematics* 299 (2016), pp. 396–450.

[7] N. A. Cook and A. Dembo. “Large deviations of subgraph counts for sparse Erdős-Rényi graphs”. In: *ArXiv e-prints* (Sept. 2018). arXiv:1809.11148 [math.PR].

[8] R. DeMarco and J. Kahn. “Tight upper tails bounds for cliques”. In: *Random Structures & Algorithms* 41.4 (2012), pp. 469–487. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/rsa.20440.

[9] R. DeMarco and J. Kahn. “Upper tails for triangles”. In: *ArXiv e-prints* (2010). arXiv:1005.4471 [math.PR].

[10] P. Erdős and A. Rényi. “On the Evolution of Random Graphs”. In: *Publication Of The Mathematical Institute Of The Hungarian Academy Of Sciences*. 1960, pp. 17–61.

[11] F. Chung, P. Frankl, and J.B. Shearer. “Some intersection theorems for ordered sets and graphs”. In: *Journal of Combinatorial Theory, Series A* 43 (1986), pp. 23–37.

[12] S. Janson. “Poisson approximation for large deviations”. In: *Random Structures & Algorithms* (1 1990), pp. 221–230.

[13] S. Janson and A. Ruciński. “The Infamous Upper Tail”. In: *Random Structures & Algorithms* 20.3 (2002), pp. 317–342.

[14] J.H. Kim and V.H. Vu. “Concentration of Multivariate Polynomials and Its Applications”. In: *Combinatorica* 20.3 (2000), pp. 417–434.

[15] J.H. Kim and V.H. Vu. “Divide and conquer martingales and the number of triangles in a random graph”. In: *Random Structures & Algorithms* 24.2 (2004), pp. 166–174.

[16] E. Lubetzky and Y. Zhao. “On the variational problem for upper tails in sparse random graphs”. In: *Random Structures & Algorithms* 50 (2017), pp. 420–436.

[17] M. Šileikis and L. Warnke. “Upper tail bounds for Stars”. In: *arXiv e-prints*, arXiv:1901.10637 (Jan. 2019), arXiv:1901.10637. arXiv:1901.10637 [math.PR].

[18] D. Reimer. “Proof of the Van den Berg–Kesten Conjecture”. In: *Combinatorics, Probability and Computing* 9.1 (2000), pp. 27–32.

[19] S. Janson, K. Oleszkiewicz and A. Ruciński. “Upper tails for subgraph counts in random graphs”. In: *Israel Journal of Mathematics* 142 (2004), pp. 61–92.

[20] S. Janson, T. Łuczak, A. Ruciński. *Random Graphs*. New York: Wiley, 2000.

[21] M. Šileikis. “On the upper tail of counts of strictly balanced subgraphs”. In: *Electronic Journal of Combinatorics* 19.1 (2012).
[22] V.H. Vu. “A Large Deviation Result on the Number of Small Subgraphs of a Random Graph”. In: Combinatorics, Probability and Computing 10.1 (2001), pp. 79–94.

[23] L. Warnke. “On the missing log in upper tail estimates”. In: ArXiv e-prints (Dec. 2016). arXiv:1612.08561 [math.PR]