Abstract

A key aspect where extreme values methods differ from standard statistical models is through having asymptotic theory to provide a theoretical justification for the nature of the models used for extrapolation. In multivariate extremes many different asymptotic theories have been proposed, partly as a consequence of the lack of ordering property with vector random variables. One class of multivariate models, based on conditional limit theory as one variable becomes extreme, developed by Heffernan and Tawn (2004), has developed wide practical usage. The underpinning value of this approach has been supported by further theoretical characterisations of the limiting relationships by Heffernan and Resnick (2007) and Resnick and Zeber (2014). However Drees and Janßen (2017) provided a number of counterexamples of their results, which potentially undermine the trust in these statistical methods. Here we show that in the Heffernan and Tawn (2004) framework, which involves marginal standardisation to a common exponentially decaying tailed marginal distribution, the problems in these examples are removed.

Keywords: Conditional multivariate extreme value theory; Copulas; Laplace marginal distribution.

1 Introduction

Multivariate extreme value problems are important across a range of subject domains, such as sea levels (Coles and Tawn, 1994), air pollution (Heffernan and Tawn, 2004), rainfall (Davison et al., 2012) and river flow (Engelke and Hitz, 2020) which all feature in influential discussion papers. The typical formulation is to have \( n \) independent and identically distributed replicate observations \( (x_1, \ldots, x_n) \), from a \( d \)-dimensional vector random variable \( \mathbf{X} \) with unknown joint distribution \( F_{\mathbf{X}} \). Here the aim is to estimate \( \Pr(\mathbf{X} \in A) \) where \( A \subset \mathbb{R}^d \), such that all the elements in \( A \) are in the upper tail of at least one of the marginal distributions of \( \mathbf{X} \), with the formulation of \( A \) depending on the characteristics of the problem of interest. The typical approach to make such inference is to estimate both the marginal distributions and dependence structure (copula) with a focus on their behaviour in their upper extremes. Univariate extreme value methods are
well established (Coles, 2001; Davison and Smith, 1990), with multivariate dependence modelling being the key challenge.

In the bivariate case, that we will focus on for variables \((X, Y)\), there are two distinct types of extremal dependence, which are easiest explained via the coefficient of asymptotic dependence \(\chi\), given by

\[
\chi = \lim_{p \to 1} \Pr [F_Y(Y) > p \mid F_X(X) > p],
\]

where \(F_X\) and \(F_Y\) are the marginal distributions of \(X\) and \(Y\) respectively. Having \(\chi > 0\) coincides with asymptotic dependence between \(X\) and \(Y\), a situation in which both variables can take their largest values simultaneously; while when \(\chi = 0\), termed asymptotic independence, such limiting dependence is impossible. Measures of sub-asymptotic dependence exist for the asymptotically independent case, with the dependence measure \(\overline{\chi}\), covering a form of extremal positive and negative dependence and independence for the values \(0 < \overline{\chi} < 1\), \(-1 < \overline{\chi} < 0\) and \(\overline{\chi} = 0\) respectively, whilst \(\overline{\chi} = 1\) corresponds to asymptotic dependence; see Coles et al. (1999). Since many models for bivariate extremes are only suitable in one of these situations, distinguishing between them, or having a model that incorporates both cases in a flexible way, can play a crucial role in model selection.

For example, multivariate max-stable distributions (Gudendorf and Segers, 2012) and multivariate generalised Pareto distributions (Kiriliouk et al., 2019) only allow \(\chi > 0\) or have \(\chi = \overline{\chi} = 0\), while the Gaussian copula with correlation parameter \(-1 < \rho < 1\) gives \(\chi = 0\) and \(\overline{\chi} = \rho\) (Ledford and Tawn, 1996).

One class of multivariate models, based on conditional limit theory as one variable becomes extreme, developed by Heffernan and Tawn (2004), has developed wide practical usage, with applications linked to widespread river flooding (Keef et al., 2013b), time series dependence in heatwaves (Winter and Tawn, 2017), spatial air temperature extremes (Wadsworth and Tawn, 2022), spatio-temporal sea-surface temperatures (Simpson and Wadsworth, 2021), offshore metocean environmental design contours (Ewans and Jonathan, 2014), coastal flooding (Gouldby et al., 2017), food chemicals (Paulo et al., 2006), and laboratory trials (Southworth and Heffernan, 2012).

This Heffernan and Tawn (2004) class of models has considerable flexibility as it covers both asymptotic dependence and asymptotic independence classes. Furthermore, in the multivariate case it allows for different extremal dependence classes between separate subsets of the variables, unlike models such as Wadsworth et al. (2017) and Huser and Wadsworth (2019). Since its initial presentation, the model proposed by Heffernan and Tawn (2004) has been extended by Keef et al. (2013a) to its current most widely adopted form. Specifically, for \((X, Y)\) marginally transformed to have Laplace marginals, denoted \((X_L, Y_L)\), it is assumed that there exists values \((\alpha_{|X}, \beta_{|X}) \in [-1, 1] \times (-\infty, 1)\) such that
for $x > 0$ and $z \in \mathbb{R}$

$$
\Pr \left\{ \frac{Y_L - \alpha_{|X_L} X_L}{X_L^{\beta_{|X_L}}} \leq z, X_L - t > x \mid X_L > t \right\} \to G_{|X}(z) \exp(-x) \quad \text{as } t \to \infty,
$$

where $G_{|X}$ is the distribution function of a non-degenerate random variable, subject to the condition that $\lim_{z \to \infty} G_{|X}(z) = 1$ to ensure that $\alpha$ is uniquely identifiable. This relation gives that the normalised $Y_L$ is conditionally independent of $X_L$ in the limit. Here the conditioning event $\{X_L > t\}$ differs from the condition event $\{X_L = t\}$ of Heffernan and Tawn (2004), with the latter corresponding to the former when joint densities exist. In statistical applications, the limit (2) is taken to hold for finite $t$, and using realisations of $(X_L,Y_L)$, such that $X_L > t$, the parameters $\alpha_{|X}$ and $\beta_{|X}$ are estimated using regression methods (for $Y_L$ given $X_L$) whilst non-parametric methods are used in inference for $G_{|X}$, based on the standardised residuals of this regression. To characterise the full joint tail of $(X_L,Y_L)$ in addition to limit (2) we also need the equivalent relationship for the reverse conditional distribution of $X_L$ given $Y_L$ is large.

Despite the strong applied value of the conditional modelling framework, some concerns about the broader theoretical restrictions of the limiting assumptions exist. Attempts to formalise the method and weaken some of these assumptions include Heffernan and Resnick (2007) and Resnick and Zeber (2014). However, Drees and Janßen (2017) provided a number of counterexamples of their results. A side effect of this has been to undermine the potential wider practical adoption of the Heffernan and Tawn (2004) conditional multivariate extremes framework.

This paper explores the counterexamples of Drees and Janßen (2017) to see if they undermine any of the asymptotic justification for the statistical methods stemming from the Heffernan and Tawn (2004) framework. There is a critical difference between the framework studied in Heffernan and Resnick (2007), Resnick and Zeber (2014), Drees and Janßen (2017) and the Heffernan and Tawn (2004) framework, specifically, that latter requires an initial marginal standardisation, so that after transformation of $(X,Y)$ they are assumed to have identical marginal distributions before studying the conditional extremes behaviour. This transformation was taken to be Gumbel in Heffernan and Tawn (2004) and Laplace (as above) in Keef et al. (2013a). Such standardisation of variables to common margins is quite usual in the study of dependence structure, e.g., Nelsen (1999) and Beirlant et al. (2004), as this makes relationships more easy to model through linearity, with exponential margins being particularly desirable for this, as shown by Papastathopoulos et al. (2017).

Our intuition is that having marginal variables on completely different marginal tail behaviours (explicitly different shape parameters/tail indices) imposes a major restriction on a conditional approach using affine transformations, such as in the norming of $Y_L$ in limit (2). The results presented in this paper show that working with standard-
ised marginal distributions overcomes all of the counterexamples. We believe that these findings further illustrate the versatility of the Heffernan and Tawn (2004) conditional multivariate extremes framework.

The paper is structured as follows: In Section 2 we present the background theory of the different conditional representations. In Section 3 we cover each of the counterexamples given by Drees and Janßen (2017), with simulations to help interpretation, and state which features of Das and Resnick (2011) and Resnick and Zeber (2014) they show are not appropriate. In each case we illustrate how the problems are overcome through an initial standardisation of the marginal distributions. Some technical details of the calculations for the examples are given in the Appendix.

2 Background Theory

2.1 Multivariate and Conditional Extremes

For notational simplicity, we focus on the bivariate case with $(X,Y)$, where $X$ and $Y$ are continuous random variables. Classical multivariate extreme value models assume that the marginal distributions $F_X$ and $F_Y$ of $(X,Y)$ belong to the domain of attraction of some extreme value distribution: $F_X$ is in the domain of attraction of an extreme value distribution if there exist functions $p_X: \mathbb{R} \to \mathbb{R}^+$ and $q_X: \mathbb{R} \to \mathbb{R}$ such that

$$F_X \left\{ p_X(t)x + q_X(t) \right\} \to \exp \left\{ - (1 + \gamma_X x)^{-1/\gamma_X} \right\} \quad \text{as} \quad t \to \infty \quad (3)$$

for some $\gamma_X \in \mathbb{R}$ and all $x \in E^{(\gamma_X)} := \{ x \in \mathbb{R} \mid 1 + \gamma_X x > 0 \}$. Multivariate extreme value distributions then arise as the limiting joint distribution of the componentwise maxima of independent and identically distributed random variables $(X_i,Y_i)$, for $i = 1, \ldots, t$, with joint distribution function $F_{X,Y}$ and marginal distribution functions $X_i \sim F_X$ and $Y_i \sim F_Y$. Specifically, it as assumed that there exist functions $p_X, q_X$ as in limit (3), and similarly $p_Y, q_Y$, such that

$$\Pr \left( \max_{i=1,\ldots,d} \frac{X_i - q_X(t)}{p_X(t)} \leq x, \max_{i=1,\ldots,d} \frac{Y_i - q_Y(t)}{p_Y(t)} \leq y \right)$$

$$= [F_{x,y}(p_X(t)x + q_X(t), p_Y(t)y + q_Y(t))]^t \to H(x,y) \quad \text{as} \quad t \to \infty,$$

where $H$ is a bivariate distribution function with non-degenerate marginal distributions, given by limit form (3), with tail indices of $\gamma_X$ and $\gamma_Y$ respectively, and with a copula possessing a specific max-stable property which, amongst other features, excludes the possibility of negative dependence, see Coles et al. (1999) and Beirlant et al. (2004).

Heffernan and Tawn (2004) propose examining the dependence in the tail of $(X,Y)$ by first standardising the marginals via the probability integral transformation to have Gum-
bel distributions, denoted \((X_G, Y_G)\), with \(\Pr(X_G \leq x) = \Pr(Y_G \leq x) = \exp\{-\exp(-x)\} (x \in \mathbb{R})\), and considering the conditional \(Y_G \mid (X_G = t)\) as \(t \to \infty\). The assumption underlying their approach is that there exist normalising functions \(\tilde{a}_{|X}(y) : \mathbb{R}_+ \to \mathbb{R}\) and \(\tilde{b}_{|X}(y) : \mathbb{R}_+ \to \mathbb{R}_+\) such that

\[
\Pr \left\{ \frac{Y_G - \tilde{a}_{|X}(X_G)}{\tilde{b}_{|X}(X_G)} \leq z \mid X_G = t \right\} \to \tilde{G}_{|X}(z) \quad \text{as } t \to \infty, \tag{4}
\]

where the limit distribution \(\tilde{G}_{|X}\) is non-degenerate. To ensure that \(\tilde{a}_{|X}, \tilde{b}_{|X}\) and \(\tilde{G}_{|X}\) are well-defined, we require \(\lim_{z \to \infty} \tilde{G}_{|X}(z) = 1\), i.e., \(\tilde{G}_{|X}\) has no mass at \(+\infty\), and \(\tilde{b}_{|X}(x)/x \to 0\) as \(x \to \infty\) (Keef et al., 2013a). Heffernan and Tawn (2004) find that (up to type) the functions \(\tilde{a}_{|X}\) and \(\tilde{b}_{|X}\) in (4) have a common parametric form for all copulas described by Joe (1997) and Nelsen (1999).

Heffernan and Resnick (2007) modify and extend the original framework by Heffernan and Tawn (2004). Their main modification is to replace the condition \(X = t\) in (4) by \(X > t\), i.e., they analyse

\[
\Pr \left\{ \frac{Y - a_{|X}(X)}{b_{|X}(X)} \leq z \mid X > t \right\} \to G_{|X}(z) \quad \text{as } t \to \infty. \tag{5}
\]

This is the most widely applied and considered conditional extreme value model framework and we will use it in the remainder of the paper. Heffernan and Resnick (2007) further drop the assumption that \(X\) and \(Y\) have Gumbel margins and provide theoretical results subject to \(F_X\) lying in the domain of attraction of some extreme value distribution.

Keef et al. (2013a) focus on \(X\) and \(Y\) having standard Laplace margins, i.e.,

\[
\Pr(X < x) = \begin{cases} 
\frac{1}{2} \exp(x) & \text{if } x \leq 0, \\
1 - \frac{1}{2} \exp(-x) & \text{if } x > 0.
\end{cases}
\]

Under these conditions, the functions in (5) are of the form \(a_{|X}(x) = \alpha x\) and \(b_{|X}(x) = x^\beta\), with \((\alpha, \beta) \in [-1, 1] \times (-\infty, 1)\), in all of the standard copulas studied by Heffernan and Tawn (2004). When \(X\) and \(Y\) are positively associated, standardisation of \(X\) and \(Y\) to Laplace margins gives the same limiting behaviour as when the variables were transformed to Gumbel margins. However, the limiting behaviours differ when \(X\) and \(Y\) are negatively associated, with the symmetry of the Laplace margins giving a simpler form. Estimation of the parameters \((\alpha, \beta)\) and the distribution function \(G_{|X}\) for Laplace margins are considered in Keef et al. (2013a,b).
2.2 Linking Multivariate and Conditional Extremes Models

One early question asked about the conditional extremes model by Heffernan and Tawn (2004) concerned its link to established multivariate extreme value models, e.g., Hüsler and Reiss (1989) or Tawn (1990). This motivated Heffernan and Resnick (2007) to define the class of conditional extreme value models (CEVM) which does not require the margins to be standardised to common margins. The limit distribution of \( Y \mid (X > t) \) as \( t \to \infty \) lies in the CEVM class if

1. The distribution function \( F_X \) of \( X \) is in a domain of attraction of an extreme value distribution with parameter \( \gamma_X \in \mathbb{R} \).

2. There exist normalising functions \( c, f : \mathbb{R} \to \mathbb{R} \) and \( d, g : \mathbb{R} \to \mathbb{R}_+ \), such that

\[
\Pr \left\{ \frac{X - c(t)}{d(t)} > x, \frac{Y - f(t)}{g(t)} \leq y \right\} \to \mu_{Y \mid X>}(\{x, \infty\} \times [-\infty, y]) \quad \text{as} \quad t \to \infty, \quad (6)
\]

where \( \mu_{Y \mid X>}(\{x, \infty\} \times [-\infty, y]) \) is a non-degenerate distribution function in \( x \), and \( \mu_{Y \mid X>}(\{x, \infty\} \times [-\infty, y]) < \infty \).

Resnick and Zeber (2014) add the condition \( \mu_{Y \mid X>}(\{x, \infty\} \times \{\infty\}) = 0 \) to ensure uniqueness of the limit measure. However, Example 2.3 in Drees and Janßen (2017) show that this condition has to be strengthened to

\[
\lim_{y \to \infty} \mu_{Y \mid X>}(\{x, \infty\} \times [y, \infty)) = \lim_{y \to \infty} \mu_{Y \mid X>}(\{x, \infty\} \times (-\infty, -y]) = 0,
\]

i.e., the limit measure cannot put any mass for \( Y \) at \( \{-\infty\} \) or \( \{\infty\} \).

Returning to the link between the CEVM framework and multivariate extreme value models assume that \( X \) and \( Y \) belong to the domain of attraction of some extreme value distribution, with parameters \( \gamma_X \) and \( \gamma_Y \) respectively. Theorem 2.1 in Das and Resnick (2011) claims that \( (X,Y) \) lies in the domain of attraction of a multivariate extreme value distribution if \( Y \mid (X > t) \) and \( X \mid (Y > t) \) both lie in the CEVM class by Heffernan and Resnick (2007). Example 4.4 in Drees and Janßen (2017) illustrates, however, that this result is not true, unless the normalisations of \( Y \mid (X > t) \) and \( X \mid (Y > t) \) in (6) are identical.

Now consider the case that \( (X,Y) \) lies in the domain of attraction of a multivariate extreme value distribution. Suppose that (i) \( X \) and \( Y \) are asymptotically dependent and (ii) \( \gamma_X, \gamma_Y \leq 0 \). Drees and Janßen (2017) show that these conditions are sufficient for the limits of \( Y \mid (X > t) \) and \( X \mid (Y > t) \) to lie in the class of conditional extreme value models. The restriction \( \gamma_X, \gamma_Y \leq 0 \) is required, as demonstrated by Example 4.2 in Drees and Janßen (2017). Note, the conditions (i) and (ii) are not necessary conditions for \( Y \mid (X > t) \) and \( X \mid (Y > t) \) to lie in the CEVM class; see Section 5 in Heffernan and Resnick (2007) for the case of \( X \) and \( Y \) being asymptotically independent.
2.3 Standardisation of Marginals in the CEVM Class

Heffernan and Resnick (2007) examined how the standardisation of \( X \) to a standard Pareto distributed random variable \( X_P \), but leaving \( Y \) unchanged, effected the limiting measure \( \mu_{Y|X>} \) in (6). They show that the limiting behaviour of \((X_P, Y)\) satisfies

\[
\lim_{t \to \infty} t \Pr \left\{ X_P > x, \frac{Y - f(t)}{g(t)} \leq y \right\} = \begin{cases} 
\mu_{Y|X>} \left( \left[ \frac{x^{\gamma_X} - 1}{\gamma_X}, \infty \right] \times [-\infty, y] \right) & \text{if } \gamma_X \neq 0, \\
\mu_{Y|X>} \left( [\log x, \infty] \times [-\infty, y] \right) & \text{if } \gamma_X = 0,
\end{cases}
\]

where the measure \( \mu_{Y|X>} \) corresponds to that in (6), and where \( c(t) = 0 \) and \( d(t) = t \) due to the standardisation to \( X_P \).

The standardisation of \( Y \) is more challenging than that of \( X \), because the CEVM (6) does not require \( F_Y \) to be in a domain of attraction of an extreme value distribution, unlike the Heffernan and Tawn (2004) and Keef et al. (2013a) formulations of limit (4). Heffernan and Resnick (2007) and Das and Resnick (2011) consider the task of finding a monotone and unbounded function \( h : \mathbb{R} \to \mathbb{R}_+ \) such that

\[
\lim_{t \to \infty} t \Pr \left\{ \frac{X_P}{t} > x, \frac{h(Y)}{t} \leq y \right\} \to \tilde{\mu}_{Y|X>} \left( [x, \infty] \times [-\infty, y] \right) \quad \text{as } t \to \infty,
\]

where \( \tilde{\mu}_{Y|X>} \) is finite and non-degenerate, \( \tilde{\mu}_{Y|X>}([x, \infty] \times \{\infty\}) = 0 \) and \( \tilde{\mu}_{Y|X>}({\{\infty\}} \times [-\infty, y]) = 0 \) for all \( x, y \). Das and Resnick (2011) argue that such a function \( h \) exists if, and only if, \( \mu_{Y|X>} \) is not a product measure. However, examples in Drees and Janßen (2017) Section 3 illustrate that neither implication is true, and the limit measures \( \mu_{Y|X>} \) and \( \tilde{\mu}_{Y|X>} \) may convey different information if they exist. Drees and Janßen (2017) further provide two sufficient sets of conditions on the functions \( f \) and \( g \) in expression (6) for such a function \( h \) to exist.

3 Investigating Drees and Janßen (2017) Examples

3.1 Strategy

Most examples in Drees and Janßen (2017) work with the joint distribution of \((X_P, Y)\), where \( \Pr(X_P > x) = 1 - x^{-1} \) (\( x > 1 \)), i.e., the conditioning variable has standard Pareto distribution, and the distribution of \( Y \) is given indirectly through the distributions of \( X_P \) and \( Y | X_P \). In the following, we consider the examples by Drees and Janßen (2017) highlighted in Section 2 and examine the obtained limiting behaviour, in each case using their numbering of the examples. We work within the framework of Keef et al. (2013a), therefore, we consider the limiting behaviour after the variables \( X_P \) and \( Y \) have been transformed to Laplace margins, denoted by \((X_L, Y_L)\).
The standardisation of $X_P$ to Laplace margins yields the variable $X_L$ and the link between the values $x_L$ of $X_L$ and $x$ of $X_P$ is given by

$$
\frac{1}{x} = \begin{cases} 
1 - \frac{1}{2} \exp(x_L) & \text{if } x_L \leq 0, \\
\frac{1}{2} \exp(-x_L) & \text{if } x_L > 0.
\end{cases}
$$

(8)

When transforming $Y$ to $Y_L$, we first derive the distribution function $F_Y$ and then derive the expression for the transformed value $y_L$ of $Y_L$ as $y_L = F_{Y_L}^{-1}[F_Y(y)]$, where $F_{Y_L}^{-1}$ is the inverse distribution function of a Laplace random variable.

### 3.2 Example 2.3

Let $B$ be a discrete random variable that is uniformly distributed on $\{0, 1\}$ that is independent of $X_P$ and define $Y = B + (1 - B)(2 - 1/X_P)$. The variable $Y$ can take any value in the interval $[1, 2)$, with the highest values occurring when $B = 0$ and $X_P$ large, and its marginal distribution is given by $\Pr(Y = 1) = 1/2$ and $\Pr(Y < y) = y/2$ for $1 < y \leq 2$. This example in Drees and Janßen (2017) showed that the condition $\mu_{Y|X}([1, x, \infty] \times \{\infty\}) = 0$ in Resnick and Zeber (2014) is not sufficient to ensure uniqueness of the limit measure in expression (6), and that the stronger condition $\mu_{Y|X}([1, x, \infty] \times \{-\infty, \infty\}) = 0$ is required.

As outlined in Section 3.1, we are interested in the limiting behaviour of the transformed variable $Y_L$ given that the transformed variable $X_L$ is large. Transformation of $Y$ to Laplace margins gives $Y = 2 - \exp(-Y_L)$ for $B = 0$, while $Y = 1$ when $B = 1$; this second case implies $Y_L = 0$ irrespective of $X$ for $B = 1$ and, thus, the lower tail of $Y_L$ is not Laplace distributed. Figure 1 left panel shows that realised values of $Y$ are close to...
$y = 1$ or $y = 2$ for large values of $X_P$, while the transformed variable $Y_L$, shown in the right panel, has no upper bound. When $B = 0$, substituting the realisations $x$ and $y$ by their transformed values $x_L$ and $y_L$, gives for $x_L > 0$

$$y = 2 - \frac{1}{x} \iff 2 - \exp(-y_L) = 2 - \frac{1}{2}\exp(-x_L) \iff y_L = \log(2) + x_L.$$  

This linear relationship between the values, for $x_L > 0$ and $B = 0$, is also visible in Figure 1 right panel, while, for $B = 1$, we have $y_L = 0$ for all possible values $x_L$.

From the calculations above, we conclude that the functions $a|_X(x) = x$ and $b|_X(x) = 1$ in expression (5) give the limiting behaviour as

$$\Pr(Y_L - X_L \leq z \mid X_L > x_L) \to \frac{1}{2}(1 + \mathbf{1}\{\log 2 \leq z\}) = G|_X(z) \quad \text{as } x_L \to \infty,$$

where $\mathbf{1}$ denotes the indicator function, and $G|_X$ is a non-degenerate distribution function. The result $\lim_{z \to -\infty} G|_X(z) = 0.5$ is due to the case $B = 1$ which occurs with probability 0.5. Other choices for $a|_X$ and $b|_X$ lead to a degenerate limiting distribution $G|_X$, contradicting the Heffernan and Tawn (2004) assumption, or yield $\mu_{Y|X>}(x, \infty) \times \{-\infty\} = 0$, violating the constraint $\lim_{z \to \infty} G|_X(z) = 1$ by Keef et al. (2013a).

In terms of the measure $\mu_{Y|X>}$ in (6), we have for $x_L > 0$

$$\mu_{Y|X>}(x_L, \infty) \times [-\infty, y_L]) = \frac{1}{2}(1 + \mathbf{1}\{\log 2 \leq y_L\}) \times \frac{1}{2}\exp(-x_L).$$

While this result is similar to the first limit found by Drees and Janßen (2017), we do not require the additional constraint $\mu_{Y|X>}(x, \infty) \times \{-\infty\} = 0$, introduced by Drees and Janßen (2017), to ensure a unique limiting behaviour, because we transformed the variables to common Laplace margins.

### 3.3 Example 3.1

Let $B$ be a discrete random variable that is uniformly distributed on $\{-1, 1\}$ and independent of the Pareto distributed random variable $X_P$. The variable $Y$ is defined as $Y = 2 - B/X_P$. For large $X_P$, the values of $Y$ are concentrated around 2 (see Figure 2 left panel). The marginal distribution of $Y$ is $Y \sim \text{Uniform}(1,3)$. Drees and Janßen (2017) present this and the following Example 3.2, to illustrate that the result by Das and Resnick (2011) linked to the standardisation (7) of $Y$ does not hold in general.

We again start by transforming the random variables $X_P$ and $Y$ to Laplace margins. Substitution of the values $y$ by their transformed values $y_L$ gives

$$y = \begin{cases} 1 + \exp(y_L) & \text{if } y_L \leq 0, \\ 3 - \exp(-y_L) & \text{if } y_L > 0. \end{cases}$$
and the transformation of \( X_P \) to Laplace margins is given in (8). For the case \( B = 1 \), \( Y \) takes values smaller than 2, while only values greater than 2 are observed for \( Y \) when \( B = -1 \). Therefore, we have to consider the transformation with \( y_L \leq 0 \) for \( B = 1 \), and \( y_L > 0 \) for \( B = -1 \). For the case \( B = 1 \), we find that

\[
y = 2 - \frac{1}{x} \quad \Leftrightarrow \quad 1 + \exp(y_L) = 2 - \frac{1}{2} \exp(-x_L) \quad \Leftrightarrow \quad y_L = \log \left\{ \frac{1 - \exp(-x_L)}{2} \right\}.
\]

The final equation implies that we can approximate \( y_L \) by \(-\frac{1}{2} \exp(-x_L)\) as \( x_L \to \infty \). Similar calculations for the case \( B = -1 \) give that as \( x_L \to \infty \)

\[
y = 2 + \frac{1}{x} \quad \Leftrightarrow \quad y_L = -\log \left\{ \frac{1 - \exp(-x_L)}{2} \right\} \sim \frac{1}{2} \exp(-x_L).
\]

Without norming, \( Y_L \mid (X_L > x) \to^P 0 \) as \( x \to \infty \). To avoid this degeneracy, we need to take the functions in (5) to be \( a_{X}(x) = 0 \) and \( b_{X}(x) = \exp(-x) \), the limiting distribution \( G_{|X} \) then assigns probability 1/2 to each of the values \( z = -0.5 \) and \( z = 0.5 \).

The expression for \( b_{X}(x) \) is not of the form for Laplace margins found by Keef et al. (2013a), with \( b_{X}(x) \) tending to zero very rapidly. This form is needed given the speed of convergence of \( Y_L \mid (X_L > x) \) towards zero as \( x \to \infty \), as seen in Figure 2 right panel. This is not too surprising as it is known that the simple parametric forms of Keef et al. (2013a) for the norming functions do not always hold, with Papastathopoulos and Tawn (2016) already identifying that it is possible to have \( a_{X}(x) = x \mathcal{L}_{a}(x) \) and \( b_{X}(x) = x^{\beta} \mathcal{L}_{b}(x) \), with \( \mathcal{L}_{a}(x) \) and \( \mathcal{L}_{b}(x) \) being slowly varying functions and \( \beta \in (-\infty, 1) \). Here we have an example that is outside that class with \( \beta = 0 \) and \(-\log\{\mathcal{L}_{b}(x)\}\) being...
regularly varying. With our norming, the limiting measure $\mu_{Y\mid X_L}$, as defined in (6), is

$$
\mu_{Y\mid X_L}((x_L, \infty] \times [-\infty, y_L]) = \frac{1}{2} (I \{-0.5 \leq y_L\} + I \{0.5 \leq y_L\}) \times \frac{1}{2} \exp(-x_L).
$$

Thus, although Drees and Janßen (2017) obtain a non-product limiting measure in this example, the result above shows that standardisation of marginals to a common form leads to a simpler product limit measure; thus, providing further evidence that standardisation helps in extremal dependence modelling.

### 3.4 Example 3.2

Let $B$ be a discrete random variable that is uniformly distributed on $\{-1, 1\}$, $U \sim$ Uniform$(0, 1)$, and $X_P$, $B$ and $U$ are all independent. Define $Y = B(1 - U/X_P)$, with the random variable $Y$ taking negative and positive values for $B = -1$ and $B = 1$ respectively. Figure 3 left panel shows that the values of $Y$ are close to $y = -1$ and $y = 1$ for large values of $X_P$. For $-1 < y < 0$, we calculate the marginal distribution of $Y$ as $\Pr(Y < y) = (y + 1)(1 - \log(y + 1))/2$; see Section A.1 for details. Using similar calculations, we find $\Pr(Y < y) = (1 + y)/2 + (1 - y) \log(1 - y)/2$ for $0 \leq y < 1$.

To transform $Y$ to Laplace margins for $y < 0$, which corresponds to $y_L < 0$, we use the relationship $(1/2)\{(y + 1)(1 - \log(y + 1))\} = (1/2)\exp(y_L)$. Since we cannot find an analytical close form for $y$ in terms of $y_L$, we consider approximations in order derive the link between $y_L$ and $y$ in the limit as $y \to -1$. The calculations in Appendix A.2 give that

$$
y + 1 \sim -\frac{\exp(y_L)}{y_L}
$$
for $y \downarrow -1$. Using similar approximations, we find $1 - y \sim \exp(-y_L)/y_L$ for $y \uparrow 1$. For $B = 1$, the limiting behaviour of $Y$, as $X_P$ becomes large, is thus described by

$$y = -1 + \frac{u}{x} \iff -\frac{\exp(y_L)}{y_L} - 1 = -1 + \frac{u}{2} \exp(-x_L) \iff y_L - \log(-y_L) = \log \left( \frac{u}{2} \right) - x_L,$$

where $u$ denotes the realisation of the random variable $U$. Considering $x_L \to \infty$, we obtain that for $B = -1$

$$y_L = -x_L + \log x_L + o_P(\log x_L),$$

where the stochasticity is due to $U$. So, $y_L \xrightarrow{p} -\infty$ as $x_L \to \infty$; this can also be seen in Figure 3 right panel. Similar calculations give $-y_L - \log(y_L) = \log (u/2) - x_L$ when $B = 1$. Hence, for $B = 1$, $y_L = x_L - \log x_L + o_P(\log x_L)$ with $y_L \xrightarrow{p} \infty$ as $x_L \to \infty$.

At first sight these results appear to correspond to there being non-unique choices for $a_{|X}$ and $b_{|X}$ in (4) that yield a non-degenerate limiting distribution $G_{|X}$. However, there is only one such choice (up to type) with $a_{|X}(x) = x$ and $b_{|X}(x) = \log x \ (x > 1)$ giving $G_{|X}$ placing mass of $1/2$ at $\{-\infty\}$ and $\{-1\}$, i.e., $G_{|X}(x) = 0.5$ for $-\infty < z < -1$ and $G_{|X}(z) = 1$ for $-1 \leq z < \infty$. As in Example 3.1, the derived norming function $b_{|X}(x)$ is not of the simple power parametric form of Keef et al. (2013a). Another possible norming has $b_{|X}(x) = x$ with $a_{|X}(x) = o(b_{|X}(x))$ as $x \to \infty$, giving $G_{|X}$ with mass at of $1/2$ at $\{-1\}$ and $\{1\}$, but this type of norming is not permitted as $b_{|X}(x)$ cannot grow as fast as $x$ (Keef et al., 2013a).

### 3.5 Example 4.2

Let $B$ be a discrete random variable that is uniformly distributed on $\{0, 1\}$. Define the function $g(x) := x(2 + \sin \log x)$ for $x \geq 1$ and we consider $Y = BX_P + (1 - B) \{-g^{-1}(2X_P)\}$, where $g^{-1}$ is the inverse of $g$. Figure 4 indicates that $Y$ tends to $-\infty$ and $+\infty$ as $X_P$ becomes large. The purpose of this example in Drees and Janßen (2017) is to illustrate that $(X, Y)$ being multivariate extreme value distributed is not a sufficient condition for $Y \mid (X_P > t)$, as $t \to \infty$, to lie in the class of CEVMs of the form (6).

We derive the marginal distribution of $Y$ as

$$\Pr(Y < y) = \begin{cases} 
1/g(-y) & \text{if } y < -1, \\
1/2 & \text{if } -1 \leq y \leq 1 \\
1 - 1/2y & \text{if } y > 1,
\end{cases}$$

and the calculations are provided in Appendix A.3. Transformation of $Y$ to Laplace margins gives $y = -g^{-1}(2/\exp(y_L))$ for $y \leq -1$, and $y = \exp(y_L)$ when $y \geq 1$. The limiting behaviour of the transformed variable $Y_L$ as $X_L$ becomes large, and for $B = 1$, is given by $\exp(y_L) \sim 2 \exp(x_L)$, which is equivalent to $y_L = \log 2 + x_L + o(1)$ as $x_L \to \infty$. 

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For the case $B = 0$, we find $y_L = -\log 2 - x_L + o(1)$ as $x_L \to \infty$. This symmetry in the limiting behaviour on Laplace marginal distributions is also visible in Figure 4 right panel. Consequently, we have that as $x_L \to \infty$,

$$Y_L = \begin{cases} 
\log 2 + X_L + o(1) & \text{if } B = 1, \\
-\log 2 - X_L + o(1) & \text{if } B = 0. 
\end{cases}$$

Defining $a_{|X}(x) = x$ and $b_{|X}(x) = 1$ yields a non-degenerate limiting distribution $G_{|X}$ with $G_{|X}(z) = 0.5$ for $-\infty < z < \log 2$ and $G_{|X}(z) = 1$ for $\log 2 \leq z < \infty$. While the normalising functions $a_{|X}(x) = -x$ and $b_{|X}(x) = 1$ also yield a non-degenerate limiting distribution $G_{|X}$, this choice is not permissible because $G_{|X}$ would have mass at $+\infty$ (Keef et al., 2013b). Consequently, the normalising functions $a_{|X}$ and $b_{|X}$ are well-defined (up to type). Furthermore, this example shows that a transformation to Laplace margins can result in the distribution of $Y_L \mid (X_L > t)$ as $t \to \infty$ being in the class of conditional extreme models by Keef et al. (2013a), although $Y \mid (X_P > t)$ as $t \to \infty$ does not lie in the class of CEVMs introduced by Heffernan and Resnick (2007). When we consider the distributions of $X_L \mid (Y_L = y_L)$, where $y_L > 0$, there is deterministic relationship between $X_L$ and $Y_L$, and thus there cannot exist a non-degenerate limiting distribution $G_{|Y}$, but the behaviour is trivial $X_L = Y_L - \log 2 \mid Y_L > y_L$ for any $y_L > 0$.

### 3.6 Example 4.4

Define the function $g_c(u) = u(1 + c \sin \log u)$, where $0 < u \leq 1$ and $|c| < 1/\sqrt{2}$, and $\psi_c(z) = g_c^{-1}(1/z)$ with $z \geq 1$. Let $Z_P$ be standard Pareto distributed, $\Pr(Z_P > z) = 1 - z^{-1}$ ($z > 1$), and $B$ be discrete and uniformly distributed on $\{1, 2, 3, 4\}$ and also
independent of $Z_P$. The random variables $X$ and $Y$ are then defined as

$$ (X, Y) := \begin{cases} 
(2 - \psi_{1/2}(Z_P), 2 - 1/Z_P) & \text{if } B = 1, \\
(2 - \psi_{-1/2}(Z_P), 2 - 1/\sqrt{Z_P}) & \text{if } B = 2, \\
(1 - 1/Z_P, 2 - 1/Z_P) & \text{if } B = 3, \\
(2 - 1/Z_P, 1 - 1/Z_P) & \text{if } B = 4.
\end{cases} \tag{9} $$

The purpose of this example by Drees and Janßen (2017) is to show that $(X, Y)$ does not lie in the class of multivariate extreme value models despite $Y \mid (X > t)$ and $X \mid (Y > t)$ belonging to the CEVM class by Heffernan and Resnick (2007) as $t \to 2$. This inconsistency of the CEVM class with $(X, Y)$ being in the domain of attractions of a bivariate extreme value distribution indicates that these conditional distributions fall outside the framework of the standard assumptions for bivariate extreme values.

We now investigate this inconsistency for the bivariate distribution (9) after marginal standardisation. We start by calculating the marginal distributions of $X$ and $Y$ to see if they are individually in the domain of attraction of the univariate extreme value distribution (3). Figure 5 left panel shows that the random variable $X$ ($Y$) respectively can take values between 0 and 1 when $B = 3$ ($B = 4$), while $B \neq 3$ ($B \neq 4$) leads to the values of $X$ ($Y$) lying between 1 and 2. The cumulative distribution function of $X$ is

$$ \Pr(X \leq x) = \begin{cases} 
x/4 & \text{if } 0 \leq x \leq 1, \\
3x/4 - 1/2 & \text{if } 1 \leq x \leq 2,
\end{cases} $$

and for $Y$ we have

$$ \Pr(Y \leq y) = \begin{cases} 
y/4 & \text{if } 0 \leq y \leq 1, \\
y/2 - (2 - y)^2/4 & \text{if } 1 \leq y \leq 2.
\end{cases} $$

Detailed calculations for $\Pr(X \leq x)$ and $\Pr(Y \leq y)$ are provided in Appendix A.4. For these two marginals it is straightforward to show that they are each in the domain of attraction of the univariate extreme value distribution with parameters $\gamma_X = \gamma_Y = -1$.

Transformation of $X$ and $Y$ to Laplace margins gives

$$ x = \begin{cases} 
2 \exp(x_L) & \text{if } x_L \leq - \log 2, \\
2/3 + (2/3) \exp(x_L) & \text{if } - \log 2 < x_L \leq 0, \\
2 - (2/3) \exp(-x_L) & \text{if } x_L > 0,
\end{cases} $$
and

\[
y = \begin{cases} 
2 \exp(y_L) & \text{if } y_L \leq -\log 2, \\
3 - \sqrt{5 - 2 \exp(y_L)} & \text{if } -\log 2 < y_L \leq 0, \\
3 - \sqrt{1 + 2 \exp(-y_L)} & \text{if } y_L > 0.
\end{cases}
\]

Applying these transformations does not change the property that the marginal distributions are in the domain of attractions of univariate extremes value distribution, only now \( \gamma_{X_L} = \gamma_{Y_L} = 0 \), and we still have that \((X_L, Y_L)\) is not in domain of attraction of a bivariate extreme value distribution. The following calculations show that, with standardisation to Laplace margins, the conditional limiting distribution of \( Y_L \mid (X_L > x_L) \) fails to meet the conditions of Heffernan and Tawn (2004) as \( x_L \to \infty \), unlike the conditional \( Y \mid (X > t) \), as \( t \to 2 \), that falls in the CEVM class of Heffernan and Resnick (2007).

We explore the conditional distributions by looking at the relations between \((X_L, Y_L)\) for each value of \( B \). For \( B = 1 \), the expressions \( X = 2 - \psi_{1/2}(Z) \) and \( Y = 2 - 1/Z \) give \( Y = 2 - g_{1/2}(2 - X) \). To study the limiting behaviour, we again replace \( X \) and \( Y \) by their Laplace distributed transformed expressions. Figure 5 right panel shows that, for \( B = 1 \), large values of \( X_L \) lead to large values of \( Y_L \) and we thus consider the equality

\[
3 - \sqrt{1 + 2 \exp(-y_L)} = 2 - g_{1/2}\{(2/3) \exp(-x_L)\} \\
= 2 - (1/3) \exp(-x_L)[2 + \sin\{\log(2/3) - x_L\}].
\]

For notational brevity, we define \( h_1(x_L) = (1/3) \{2 + \sin\{\log(2/3) - x_L\}\} \) and we note...
(1/3) ≤ h_1(x_L) ≤ 1. By simplifying the terms and taking squares on both sides, we get

\[ 1 + 2 \exp(-y_L) = 1 + 2 \exp(-x_L)h_1(x_L) + \exp(-2x_L) [h_1(x_L)]^2 . \]

Further simplifying the terms and taking logs, we end up with

\[ y_L = x_L - \log h_1(x_L) + \log [1 + (1/2) \exp(-x_L)h_1(x_L)] , \tag{10} \]

which we can write as

\[ y_L = x_L - \log h_1(x_L) + (1/2) \exp(-x_L)h_1(x_L) + O(\exp(-2x_L)) . \]

For B = 2, the expressions X = 2 − ψ_{−1/2}(Z_P) and Y = 2 − 1/Z_P give that Y = 2 − √g_{−1/2}(2 − X). On Laplace scale, we then have

\[ 3 - \sqrt{1 + 2 \exp(-y_L)} = 2 - \sqrt{g_{-1/2} \{ (2/3) \exp(-x_L) \}} = 2 - \exp(-x_L/2)h_2(x_L), \]

where h_2(x_L) = √(2/3) − (1/3) \sin \{ \log(2/3) − x_L \}. By taking squares on both sides,

\[ 1 + 2 \exp(-y_L) = 1 + 2 \exp(-x_L/2)h_2(x_L) + \exp(-x_L) [h_2(x_L)]^2 . \]

Following the same steps as for B = 1 yields

\[ y_L = x_L/2 - \log h_2(x_L) + (1/2) \exp(-x_L/2)h_2(x_L) + O(\exp(-x_L)) . \]

The case B = 3 leads to x_L < −log 2, i.e., x_L is not becoming large, and thus this mixture component can be ignored when studying Y_L | (X_L > x_L) as x_L → ∞. Finally, B = 4 gives Y = X − 1 with 2 \exp(Y_L) = 2 − (2/3) \exp(-X_L) − 1. Consequently,

\[ y_L = \log [1/2 - (1/3) \exp(-x_L)] \sim -\log(2) \quad \text{as } x_L \to \infty. \]

Now we consider combining the different mixture components and we set a_iX(x) = x− \exp(-x)h_1(x) and b_iX(x) = −\log h_1(x) + (1/2)h_1(x) \exp(-x) in (5); as −\log h_1(x) ≥ 0 and h_1(x) > 0 this gives b_iX(x) > 0 as required. The limiting behaviour of Y_L | (X_L > x_L) as x_L → ∞, for B = 1, is then

\[ \frac{Y_L - X_L + \exp(-X_L)h_1(X_L)}{-\log h_1(X_L) + (1/2)h_1(X_L) \exp(-X_L)} \sim 1. \]

For the remaining components, B = 2 and B = 4, lim_{x_L \to \infty} (Y_L - a_iX(X_L))/b_iX(X_L) = −∞ for \{X_L > x_L\}. However, there is no limiting distribution G_{iX} as in (5) because Pr(B = 1 | X_L > x_L) oscillates between 1/6 and 1/2 as x_L → ∞, that is, Pr \{Y_L - a_iX(X_L))/b_iX(x_L) ≤ z | X_L > x_L\} does not converge. This oscillating behaviour is found by considering Pr(B = 1 | X > t) for 1 < t < 2. Using similar calculations as in
Appendix A.4, we find
\[
\Pr(B = 1 \mid X > t) = \frac{\Pr(X > t \mid B = 1) \Pr(B = 1)}{\Pr(X > t)} = \frac{1}{3} + \frac{1}{6} \sin \log(2 - t),
\]
which oscillates between 1/6 and 1/2 as \( t \to 2 \), and this implies that \( \Pr(B = 1 \mid X > x_L) \) oscillates between 1/6 and 1/2 as \( x_L \to \infty \). Consequently, \( Y_L \mid (X_L > x_L) \) as \( x_L \to \infty \) does not fall in the class of conditional extreme value models by Keef et al. (2013a), despite \( Y \mid (X > t) \), as \( t \to 2 \), being in the CEVM class by Heffernan and Resnick (2007), see Drees and Janßen (2017).

So far we have focused on the conditional distribution of \( Y_L \mid (X_L > x_L) \) for \( x_L \to \infty \), but there is also interest in the asymptotic behaviour of the reverse conditional \( X_L \mid (Y_L > y_L) \) as \( y_L \to \infty \). Here Figure 5 provides some insight into what happens, with only the mixture terms corresponding to \( B = 1 \) and \( B = 3 \) contributing to the tail of \( Y_L \). Further, expression (9) shows that \( Y_L \mid (B = 1) \) and \( Y_L \mid (B = 3) \) are identical, which gives that the limiting distribution of \( X_L \mid (Y_L > y_L) \) as \( y_L \to \infty \) must be a mixture distribution with weights 1/2 on each component. When \( B = 3 \) we see that \( X_L \) does not grow with \( Y_L \), so with any norming on \( X_L \) that is required to handle the growth of \( X_L \) with \( y_L \) in \( X_L \mid (Y_L > y_L) \) will lead to mass tending to \( -\infty \) when \( B = 3 \). So it remains to consider the \( B = 1 \) case. The deterministic relationship in expression (10) between \( X_L \) and \( Y_L \) determines \( X_L > X_L \) conditional on \( X_L \) being above a sufficiently high threshold, because \(- \log h_1(x) \geq 0 \) and \( \log[1 + (1/2) \exp(-x)h_1(x)] > 0 \) for all \( x \). Furthermore, the relation between \( X_L \) and \( Y_L \) is bijective, because the first derivative in (10) is strictly positive. Consequently, we can invert the relation between \( Y_L \) and \( X_L \) when \( B = 1 \) and this gives for \( y_L \to \infty \) that
\[
x_L = y_L - Q(y_L),
\]
where \( Q(y_L) > 0 \) is an oscillator function that is bounded above. When \( B = 1 \), we thus obtain the limiting behaviour
\[
\frac{x_L - y_L}{Q(y_L)} \sim -1.
\]
Hence we have that for all \( z \in \mathbb{R} \), as \( y_L \to \infty \)
\[
\Pr\left( \frac{X_L - Y_L}{Q(Y_L)} < z \ \mid Y_L > y_L \right) \to 0.5[1 + \mathbf{I}(z > -1)].
\]
Thus, the reverse conditional has a more straight-forward behaviour.

We further note that the transformation to Laplace margins does not lead to \((X_L, Y_L)\) lying in the class of multivariate extreme value models. Drees and Janßen (2017) showed that \((X, Y)\) is not multivariate extreme value distributed either. Consequently, this is an example for which the limiting behaviour \( Y_L \mid (X_L > x_L) \) is not in the class of conditional extremes models by Keef et al. (2013a) and \((X_L, Y_L)\) does not lie in the
domain of attraction of a multivariate extreme value distribution.

4 Conclusions and Discussion

Our calculations show that standardisation to common Laplace margins resolves the problems highlighted by Examples 2.3 to 4.4 in Drees and Janßen (2017). In Example 2.3, this fixed choice of standardisation implied a unique limit measure $G_{|X}$, while the CEVM framework by Heffernan and Resnick (2007) allowed the limit measure $\mu_{Y|X>}$ to vary with the standardisation used. This example also highlighted that it is necessary to allow $G_{|X}$ to have mass at $\{-\infty\}$, because the limit measure might otherwise be degenerate. Consequently, while Drees and Janßen (2017) advocate the condition $\mu_{Y|X>}(\{-\infty, \infty\} \times E_{\gamma X}) = 0$ to ensure uniqueness, it is sufficient to require $\lim_{z \to \infty} G_{X}(z) = 1$ for the conditional extremes model by Keef et al. (2013a).

A non-degenerate limit measure $G_{|X}$ was found in Examples 3.1 and 3.2, however, the functions $a_{|X}$ and $b_{|X}$ in (6) were not of the simple parametric form of Keef et al. (2013a). Example 3.1 further shows that standardisation to Laplace margins can result in a non-degenerate limit measure, despite there not existing a standardisation of the form by Das and Resnick (2011) for the CEVM framework.

Example 4.2 in Drees and Janßen (2017) showed that $(X,Y)$ being multivariate extreme value distributed is not sufficient for the distributions of $Y \mid (X > t)$ and $X \mid (Y > t)$ to be in the CEVM class as $t$ approaches the upper end of $X$ and $Y$ respectively, while Example 4.4 illustrated that $X \mid (Y > t)$ and $Y \mid (X > t)$ being CEVM does not imply that the distribution of $(X,Y)$ is in the domain of attraction of multivariate extreme value distributions. In contrast, after standardisation of $(X,Y)$ to Laplace margins, $(X_{L},Y_{L})$, our calculations show that the link between the conditional extremes models by Keef et al. (2013a) and the class of multivariate extreme value distributions remains an open research question. Specifically, Examples 4.2 and 4.4 are ruled out as being evidence for the limit (5), and the associated result for $X_{L} \mid Y_{L}$, not being equivalent to the domain of attraction condition of a bivariate extreme value distribution.

Finally, we note that the examples of Drees and Janßen (2017) illustrate some statistical limitations of the Heffernan and Tawn (2004) framework even with standardised Laplace marginals of Keef et al. (2013a). Two particular areas relate to handling mixture distributions for $G_{|X}$ and the choice of parametric families for the normalising functions $a_{|X}$ and $b_{|X}$. We discuss these in turn below.

Many of the examples of Drees and Janßen (2017) involved a mixture structure for $(X,Y)$, and hence also for $(X_{L},Y_{L})$. Although it was possible to identify normalising functions to give a non-degenerate $G_{|X}$ in these cases, it was no surprise that $G_{|X}$ was also a mixture distribution. From a statistical perspective the only complication with $G_{|X}$ having a mixture structure is when $G_{|X}$ puts an atom of mass at $\{-\infty\}$; with Example 3.1
being the only example where $\{-\infty\}$ has mass zero. The complication with limiting mass at $\{-\infty\}$ is that at non-asymptotic levels of $x_L$ this mass will be at a finite value with its precise value depending on the associated conditioning value, e.g., $x_L$ in this set up. Statistical methods have recently been developed by Tendijck et al. (2021) which extend the Heffernan–Tawn conditional extreme value model for handling exactly this situation. Keef et al. (2013a) propose parsimonious canonical parametric families for the normalising functions $a_{X}(x) = \alpha x$ and $b_{X}(x) = x^\beta$ which appears suitable for a wide range of published data applications. The examples in Drees and Janßen (2017) add to the list (first noted by Papastathopoulos and Tawn (2016)) of theoretical joint distributions for $(X_L, Y_L)$ with normalising functions that lie outside the canonical class. Clearly, the canonical families cannot be extended to cover all of these theoretical examples in a parsimonious way. So the most natural line of future research is to identify if it is possible to quantify the errors that can arise from the inappropriate usage of the canonical families in these examples, with the error relating to the bias of estimated probabilities of extremes events for finite extrapolations.

A Appendix: Technical details

A.1 Marginal distribution in Example 3.2

For $-1 < y < 0$, we have

$$\Pr(Y < y) = \frac{1}{2} \Pr\left(-1 + \frac{U}{X_P} < y\right)$$

$$= \frac{1}{2} \int_{0}^{1} \Pr\left(X_P > \frac{u}{y + 1}\right) f_U(u) du$$

$$= \frac{1}{2} \int_{0}^{y+1} \Pr\left(X_P > \frac{u}{y + 1}\right) du + \frac{1}{2} \int_{y+1}^{1} \Pr\left(X_P > \frac{u}{y + 1}\right) du$$

$$= \frac{1}{2} \int_{0}^{y+1} 1 du + \frac{1}{2} \int_{y+1}^{1} \frac{y + 1}{u} du$$

$$= \frac{1}{2} (y + 1) \{1 - \log(y + 1)\}.$$  

A.2 Approximation of the limiting behaviour in Example 3.2

Our aim is to find an approximation of the relation between $y$ and the transformed value $y_L$ as $y \to -1$. We first note that $(y + 1) [1 - \log(y + 1)] \approx -(y + 1) \log(y + 1)$ for $y$ close to $-1$. The next step is to define $y + 1 = \epsilon \exp(y_L)$ for some $\epsilon > 0$. If we set $\epsilon = -1/y_L$, we obtain

$$-\frac{1}{y_L} \exp(y_L) \left[-\frac{1}{y_L} - \exp(y_L)\right] = \exp(y_L).$$
Taking logs on both sides, we get $-\log y + y_L + \log (y_L - \log y_L) = y_L$ and the difference of the two sides becomes negligible for $y_L$ large enough. Consequently, we have for $y \approx -1$ that

$$y \approx -\frac{\exp(y_L)}{y_L} - 1.$$

### A.3 Marginal distribution in Example 4.2

From the definition of $Y = BX_P + (1 - B)\{-g^{-1}(2X_P)\}$, it is clear that $Y$ can only take negative values for $B = 0$. Further, $g^{-1} : [2, \infty) \rightarrow [1, \infty)$ gives that $-g^{-1}(2X_P)$ will only take values smaller than -1. Consider $y \leq -1$ and we calculate

$$\Pr(Y \leq y) = \frac{1}{2} \Pr\left(\frac{1}{2}g(-y) \right) = \frac{1}{g(-y)}.$$

Finally, the case $B = 1$ results in $Y$ only taking values greater than 1 with

$$\Pr(Y \leq y) = \frac{1}{2} + \frac{1}{2} \Pr(X_P \leq y) = \frac{1}{2} + \frac{1}{2} \left(1 - \frac{1}{y}\right) = 1 - \frac{1}{2y}$$

as described in Example 4.2.

### A.4 Marginal distributions in Example 4.4

Only $B = 3$ leads to a value of $X$ between 0 and 1, while $X$ lies between 1 and 2 for $B = 1, 2, 4$. Considering $0 < x < 1$,

$$\Pr(X \leq x) = \frac{1}{4} \Pr\left(1 - \frac{1}{Z_P} \leq x\right) = \frac{1}{4} \Pr(Z_P \leq x - 1) = \frac{1}{4} [1 - (x - 1)] = \frac{x}{4}.$$

When $1 < x < 2$, $\Pr(X \leq x)$ is given by

$$\frac{1}{4} [1 + \Pr(X \leq x \mid B = 2) + \Pr(X \leq x \mid B = 3) + \Pr(X \leq x \mid B = 4)]$$

$$= \frac{1}{4} \left[1 + \Pr\left(2 - \psi_{1/2}(Z_P) \leq x\right) + \Pr\left(2 - \psi_{-1/2}(Z_P) \leq x\right) + \Pr\left(2 - \frac{1}{Z_P} \leq x\right)\right]$$

$$= \frac{1}{4} \left[1 + \Pr\left\{2 - x \leq g_{1/2}^{-1}\left(\frac{1}{Z_P}\right)\right\} + \Pr\left\{2 - x \leq g_{-1/2}^{-1}\left(\frac{1}{Z_P}\right)\right\} + \Pr\left(Z_P \leq \frac{1}{2 - x}\right)\right]$$

$$= \frac{1}{4} \left[1 + \{1 - g_{1/2}(2 - x)\} + \{1 - g_{-1/2}(2 - x)\} + 1 - (2 - x)\right]$$

$$= \frac{3x - 2}{4},$$

20
where we use $g_{1/2}(2 - x) + g_{-1/2}(2 - x) = 2(2 - x)$ to obtain the last line.

Considering the variable $Y$, only $B = 4$ gives a value of $Y$ between 0 and 1, while $B = 1, 2, 3$ lead to the realisation of $Y$ to lie in the interval $(1, 2)$. The same calculations as for $X$ give $\Pr(Y \leq y) = y/4$ for $0 < y < 1$. For $1 < y < 2$, 

$$
\Pr(Y \leq y) = \frac{1}{4} \left[ 1 + 2 \Pr \left( 2 - \frac{1}{Z_P} \leq y \right) + \Pr \left( 2 - \frac{1}{\sqrt{Z_P}} \leq y \right) \right]
$$

$$
= \frac{1}{4} \left[ 1 + 2 \{1 - (2 - y)\} + \{1 - (2 - y)^2\} \right]
$$

$$
= \frac{y}{2} - \frac{(2 - y)^2}{4}.
$$

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