Analytic formula for the Geometric Phase of an Asymmetric Top

Nicholas A. Mecholsky
Vitreous State Laboratory
The Catholic University of America
Washington, DC 20064
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The motion of a handle spinning in space has an odd behavior. It seems to unexpectedly flip back and forth in a periodic manner as seen in a popular YouTube video. As an asymmetrical top, its motion is completely described by the Euler equations and the equations of motion have been known for more than a century. However, recent concepts of the geometric phase have allowed a new perspective on this classical problem. Here we explicitly use the equations of motion to find a closed form expression for the geometric phase of the asymmetric force-free top and explore some consequences of this formula with the particular example of the spinning handle for demonstration purposes. As one of the simplest dynamical systems, the asymmetric top should be a canonical example to explore the classical analogy of the Berry phase.

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I. INTRODUCTION

The motion of spinning objects is ubiquitous. From a seemingly mundane and classical motion, new mysteries are still being debated and surprising features are revealed. One important observation is the identification of a Berry phase of rotating classical objects. Geometric phases have become popular recently due to their applications to material properties (see Ref. (7) for a recent review) and other theoretical and calculational advantages. Examples have relied on Berry’s original paper describing the quantum mechanical effect and relegating the classical version to the term ‘classical analog’. The canonical classical example of the Hannay angle usually involves Foucault’s pendulum. However, even in the force-free spinning of a top, the Hannay angle is apparent, yet not typically discussed in elementary treatments of the subject. Evidently Jacobi was the first to write down the exact analytic expressions for the motion of the asymmetric top in the 19th century. In Landau and Lifshitz’s Mechanics, it was already appreciated that the motion of the top was not perfectly periodic in time. This was identified by Landau and Lifshitz when they stated “this incommensurability has the result that the top does not at any time return to its original position.” (see Ref. (13), page 120) However, this observation was not investigated further and the connection to the Hannay angle was not included in their work which would have predated the original work by Hannay, but perhaps not the original concept evidently introduced by Pancharatnam in the 1950s.

In 1991, Richard Montgomery, and around the same time, Mark Levi, published papers that derived a formula for computing the Berry phase of rigid bodies

\[ \Delta \alpha = \frac{2ET}{M} - \Omega, \] (1)

where \( T \) is the period of the angular velocity vector in space (here we compute it from Eq. (14)) and \( \Omega \) is the signed solid area swept out by the angular momentum vector. The dynamic part \( \left( \frac{2ET}{M} \right) \) is the integrated angle of the angular velocity projected onto the angular momentum vector. The geometric part is a fraction of a unit sphere swept out by the path of the angular velocity vector.

Even though much work has been done in this area in the past few decades (see for example Ref. 16), the connection and explicit formula to an asymmetric top was never worked out. Part of the goal of this paper is to take the last step and cast the exact expressions for the motion of the asymmetric top in the framework of the geometric phase. A closed form expression for the geometric phase will be presented and explored. We use the example of the T-shaped handle to demonstrate some of these expressions, however the formulas are for a general asymmetric top. In some ways, this is the most elementary example of geometric phase available in classical physics. Even in this simple system, a wide variety of questions and lines of investigation are possible. To that end, an exploration and some observations of the formula are in order.

The paper begins by reviewing the dynamics of the asymmetric top. The angular velocity and Euler angles are determined as a function of time. From these expressions, the geometric phase is determined directly and compared to Montgomery’s formula. Finally, some observations about the geometric phase are made and some conclusions are discussed.

II. DYNAMICS AND GEOMETRIC PHASE

As an example for visualization, and without loss of generality, we may use a T-shaped handle to explore our dynamics. There is a handle for every physical moment of inertia. Consider a handle of uniform density, an example
of which can be seen in Fig. 1(b).

\[ I_{xx} = \frac{6l_1 l_2 w^2 + 2l_1^2 w^2 + l_2^2 w (6l_1 + w) + l_1^2 + 4l_1 l_2^3}{12 (l_1 + l_2)^2} \]  
\[ I_{yy} = \frac{(l_1 + 2l_2) w^2 + l_2^2}{12 (l_1 + l_2)} \]  
\[ I_{zz} = \frac{l_1 l_2 (l_1^2 + 5w^2 + 4l_2^2) + l_1^2 (l_1^2 + w^2) + l_2^2 w (6l_1 + w) + l_1^2}{12 (l_1 + l_2)^2} \]  

where the coordinates have been chosen to have a diagonal moment of inertia tensor. An ellipsoid (with semi-major axes lengths \(a, b,\) and \(c\)) with the same moment of inertia is given by

\[ a = \sqrt{\frac{5}{3}} \sqrt{l_2 w^2 + l_1^2} \] \[ b = \sqrt{\frac{5}{3}} \sqrt{2l_1 l_2 (3l_2 w + 2l_2^2 + 2w^2) + l_1^2 w^2 + l_2^2} \] \[ c = \frac{1}{\sqrt{5}} \sqrt{3} w. \]  

Here we restrict \(l_1 > l_2.\)

It will turn out that the only important parameters are those that determine the direction that the moment of inertia points in the moment of inertia space (the polar and azimuthal angles relative to the \(I_x, I_y,\) and \(I_z\) axes). Thus the absolute magnitude of the vector of the eigenvalues of the moment of inertia is not important, only the direction. We will find it useful to use a moment of inertia that is parameterized by two variables, \(b\) and \(c,\) the semi-axes of a solid ellipsoid. The third semi-axis of this ellipsoid is determined such that the moment of inertia vector (composed from the eigenvalues) has unit magnitude. The moment of inertia for this particular choice is given by

\[ I_1 = b^2 + c^2 \]  
\[ I_2 = \frac{1}{2} \left( b^2 + \sqrt{2 - 3b^4 - 2b^2c^2 - 3c^4 + c^2} \right) \]  
\[ I_3 = \frac{1}{2} \left( b^2 + \sqrt{2 - 3b^4 - 2b^2c^2 - 3c^4 - c^2} \right) \]  

with the third semi-axis of the ellipsoid given by

\[ a = \frac{\sqrt{2 - 3b^4 - 2b^2c^2 - 3c^4 - b^2 - c^2}}{\sqrt{2}} \]  

This particular choice of semi-axes of the ellipsoid is always chosen to be labeled so that \(I_1 < I_2 < I_3.\) However, as long as \(a > b > c\) in the ellipsoid, this will always be the case.

In the next section we take a small digression to identify the full space of moments of inertia.
A. Space of Possible Moments of Inertia

All triplets \((I_1, I_2, I_3)\) of positive numbers are not valid moments of inertia. To have a valid (physical) moment of inertia we must satisfy the following relations,

\[
I_1 + I_2 \geq I_3 \tag{6}
\]
\[
I_2 + I_3 \geq I_1 \tag{7}
\]
\[
I_3 + I_1 \geq I_2. \tag{8}
\]

these may be referred to as the inertial inequalities.

Without loss of generality (we may relabel axes in the object if necessary) we may further restrict ourselves to the region, \(I_1 < I_2 < I_3\). For labeling purposes, consider only the moments of inertia where \(I_1^2 + I_2^2 + I_3^2 = 1\). In this case (we will see that these are the only cases we need consider) we can show the region of possible physical moments of inertia as those moments of inertia inside the red boundary in the \(I_1 - I_2\) plane in Fig. 1. The curves that bound this region are the curve of all prolate spheroids \(\left(I_2 = \sqrt{1 - I_1^2} \right)\), the curve of all oblate spheroids \((I_2 = I_1)\), and a curve of degenerate ellipsoids where \(c = 0\), \(I_2 = \frac{1}{2} \left(\sqrt{2 - 3I_1^2} - I_1\right)\). Thus picking \(0 < c < b < 1\) fixes a particular asymmetric top. For the rest of the paper, we assume a rigid object with moments of inertia fixed.

B. Exact Dynamics of Asymmetric Top

Suppose we decide to spin this asymmetric top around some axis with some initial angular velocity without any external torques. Let us say that in the body frame of reference (determined by the principle moments of inertia \(I_1 < I_2 < I_3\)), the initial angular velocity is given by

\[
\omega_0 = (\omega_{01}, \omega_{02}, \omega_{03}). \tag{9}
\]

We will show below that the geometric phase will not depend on the magnitude of \(\omega_0\). Thus we shall take \(|\omega_0| = 1\).

The time dynamics of the angular velocity is then determined by these initial conditions up to a choice of initial orientation angles. In torque-free motion, there are two constants of motion for the asymmetric top. One is the total angular momentum of the top, given by

\[
M = |I \cdot \omega_0| = \sqrt{I_1^2 \omega_{01}^2 + I_2^2 \omega_{02}^2 + I_3^2 \omega_{03}^2}, \tag{10}
\]

and the total energy of rotation (assume the center of mass velocity of the asymmetric top is 0), given by

\[
E = \frac{1}{2} \omega_0 \cdot I \cdot \omega_0 = \frac{1}{2} (I_1^2 \omega_{01}^2 + I_2^2 \omega_{02}^2 + I_3^2 \omega_{03}^2). \tag{11}
\]

The motion of the angular momentum vector in space coordinates is, of course, fixed since no torque is applied to change the angular momentum. In the body coordinates, the angular momentum vector executes periodic motion, where the origin is fixed and the terminus sweeps out a curve in the body frame. The curve that the angular momentum sweeps out in space is constrained by these two constants of motion and is called the polhode. This intersection can be seen in Fig. 1(a).

Equations 10 and 11 represent a sphere of constant magnitude (the angular momentum magnitude is fixed), and an intersecting ellipsoid that is the surface of constant energy in angular momentum coordinates (a space where the coordinates are the projections of the angular momentum along the body’s principle moments of inertia) called the Binet ellipsoid.

The two constant surfaces must intersect for a physical object and this means that the constant angular momentum sphere is smaller than largest Binet ellipsoid axis and bigger than smallest Binet ellipsoid axis. This gives the condition

\[
2EI_1 \leq M^2 \leq 2EI_3 \tag{12}
\]

This will always happen since the alternative is for no

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Full parameter space of asymmetric rigid bodies of fixed moment of inertia magnitude equal to 1. \(I_1\) and \(I_2\) are two of the principal moments of inertia labeled so that \(I_1 < I_2 < I_3\), and the magnitude of the moment of inertia vector is unity. The red curves are degenerate asymmetric rigid bodies: Prolate spheroids (rod-like shapes), spheres, oblate spheroids (coin-like shapes), and degenerate ellipsoids (lines and disks). The equations for these curves are given in the text. The contours are curves of constant minimum values for the geometric phase. Given a shape (parametrized by two of the three values of the moment of inertia), any initial condition will lead to at least a geometric phase of the given value indicated by the contour values.}
\end{figure}
intersection of the two constant surfaces, and thus no physical dynamics. There are two possible types of curves on the angular momentum sphere and the inertial ellipsoid (not including degenerate cases). To avoid degenerate cases (discussed in detail elsewhere\textsuperscript{13,17}) we shall always assume the strict inequalities. We shall also take $M^2 > 2E\omega_0^2$ for the equations below, but for the case $M^2 < 2E\omega_0^2$, the formulas require $I_1 \leftrightarrow I_3$.

The dynamics of the angular velocity in the body frame evolve according to the Euler equations

\begin{align}
I_1\omega_1'(t) &= (I_2 - I_3)\omega_2(t)\omega_3(t), \tag{13a} \\
I_2\omega_2'(t) &= (I_3 - I_1)\omega_3(t)\omega_1(t), \tag{13b} \\
I_3\omega_3'(t) &= (I_1 - I_2)\omega_1(t)\omega_2(t). \tag{13c}
\end{align}

This may be solved exactly in terms of Jacobian elliptic functions.\textsuperscript{13} The motion of the angular velocity vector is perfectly periodic in time with period

\begin{equation}
T(E, M, I_1, I_2, I_3) = \frac{4K(k(E, M, I_1, I_2, I_3)^2)}{\sqrt{(I_3-I_2)(M^2-2EI_1)}} \quad \text{with}
\end{equation}

\begin{equation}
k(E, M, I_1, I_2, I_3) = \frac{(I_2-I_1)(2EI_3-M^2)}{(I_3-I_2)(M^2-2EI_1)}, \tag{14a}
\end{equation}

where $K(m)$ is the Complete Elliptic Integral of the First Kind, namely,

\begin{equation}
K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}. \tag{15}
\end{equation}

In some notations, the argument of the complete elliptic integral $K$ is $k$ and not $k^2$. We choose the definition in Eq. (15). Note that when $M^2 < 2EI_2$, replace $I_1 \leftrightarrow I_3$ in the quantities above.

Given the initial angular velocity vector $\omega_0$ in the body frame, the initial angular momentum ($M_0 = I_1\omega_0$) is not colinear with the angular velocity (since $I_1 \neq I_2 \neq I_3$). The initial angular momentum is fixed in space and we may define the z axis of the space coordinates to be this angular momentum vector.

Since the angular velocity vector is explicitly determined in time (See Ref. \textsuperscript{13} Eq. (37.10)), we may further express the angular velocity in terms of the Euler angles. Choosing the Euler angle convention of Ref. \textsuperscript{17}, two of the Euler angles ($\theta$ and $\psi$) are given algebraically in terms of the angular velocity functions, and the third ($\phi$) is given as a first-order non-linear differential equation in terms of the angular velocity components. This may be integrated to give $\phi$ in terms of theta functions.\textsuperscript{13,18}

\section{Formula for Geometric Phase}

Even though the equation for the angular velocity is exactly periodic in time, the orientation and angles after a period are not periodic and will not in general return to the same orientation. This angle mismatch after a period of the angular velocity is called the geometric phase or Hannay angle.\textsuperscript{4,5} Sometimes the term geometric phase is used to refer to just the geometric part of the Hannay angle. Here we refer to the geometric phase as the full angular mismatch (or the same quantity mod 2$\pi$). We reserve Berry phase as the quantum mechanical version where a wave function is explicitly transported in parameter space. In contrast, we are investigating the natural (so to speak) dynamical phase associated with the force-free evolution.

In computing the geometric phase, we could use Montgomery’s formula,\textsuperscript{5} Eq. (17)

\begin{equation}
\Delta \alpha = \frac{2ET}{M} - \Omega, \tag{16}
\end{equation}

where $T$ is the period from Eq. (14) and $\Omega$ is the signed solid area swept out by the angular momentum vector.

The dynamic part $\left(\frac{2ET}{M}\right)$ is the integrated angle of the angular velocity projected onto the angular momentum vector over one period. The geometric part is a fraction of a unit sphere swept out by the angular momentum vector over one period.

However considering that we have exact expressions for the Euler angles, we already have a formula for the geometric phase. The angular momentum vector after one period will be identical to its starting value.\textsuperscript{5} Thus the angular difference around this axis, $\phi$, is the geometric phase. If we assume that the Euler angle $\phi$ at time $t = 0$ is 0, then we have the change in angle, $\Delta \alpha$, over one period is

\begin{equation}
\Delta \alpha = \phi(T). \tag{17}
\end{equation}

Hence the geometric phase is the Euler angle $\phi$ evaluated after one period, $T$. It reduces to

\begin{equation}
\Delta \alpha_{\phi} = \frac{MT}{I_3} + \frac{M(I_3 - I_1)}{I_3} \mathcal{P}\left(\frac{(I_3-I_2-I_1)}{I_1(I_2-I_3)}; \quad \text{am}(4K \frac{(k^2)}{k^2}) \quad \text{am}(4K \frac{(k^2)}{k^2}) \right) \quad \text{am}(4K \frac{(k^2)}{k^2}) \quad \text{am}(4K \frac{(k^2)}{k^2}), \quad \text{am}(4K \frac{(k^2)}{k^2}), \tag{18}
\end{equation}
where $\Pi(n; \phi|m)$ is the Incomplete Elliptic Integral of the Third Kind defined by

$$\Pi(n; \phi|m) = \int_0^\phi \frac{d\theta}{(1 - n \sin^2(\theta))(1 - m \sin^2(\theta))^{1/2}},$$

(19)

$K(m)$ is defined in Eq. (15), and $am(u|m)$ is the Jacobi Amplitude. If $u = F(\phi|m)$ then $\phi = am(u|m)$, where $F$ is the Incomplete Elliptic Integral of the First Kind,

$$F(\phi|m) = \int_0^\phi \frac{d\theta}{(1 - m \sin^2(\theta))^{1/2}},$$

(20)

for $-\pi/2 < \phi < \pi/2$.

It is worthwhile to note that the geometric phase formula in Eq. (18) seems to depend on $E$, $M$, $I_1$, $I_2$, and $I_3$. However it can be shown that it does not depend on the magnitude of the moment of inertia ($I_1^2 + I_2^2 + I_3^2$) nor on the magnitude of the angular velocity.

The fact that the initial direction of the axis of the angular velocity only enters the geometric phase formula is somewhat surprising. However, considering that the initial angular momentum (hence angular velocity) follows closed curves on the Binet ellipsoid means that every point along that path has the same geometric phase since every point has the same quantities of $E$, $M$, $I_1$, $I_2$, and $I_3$.

Thus curves of the same geometric phase foliate the Binet ellipsoid. An example of this can be seen in Fig. 3 for the case of $(I_1, I_2, I_3) = (0.286827 Js^2, 0.533256 Js^2, 0.795844 Js^2)$. Here, the color of the curve corresponds to the geometric phase modulo $2\pi$.

In the next section, we check Eq. (18) by evaluating both Eq. (18) and by calculating Eq. (1) directly.

D. Demonstration of formula

For illustrative purposes, we choose a handle shape where $l_1 = 8$, $I_2 = 4$, and $l_1 = 1$ as shown in Fig. 1(b). We further choose the total mass (from a uniform density) to be such that the moment of inertia has unit magnitude. The initial angular velocity has unit magnitude, and the direction is given by a polar angle (measured from the space $z$ axis), $\theta_o = \pi/2 - 0.15$, and an azimuthal angle, $\phi_o = \pi/2 - 0.2$, measured from the $x$ axis. This corresponds to a large initial velocity about the middle moment of inertia, $I_2$. This sets the total angular momentum and total rotational energy. Those, along with the moments of inertia are: $E = 0.264805 J$, $M = 0.533252 Js$, $I_1 = 0.286827 Js^2$, $I_2 = 0.533256 Js^2$, and $I_3 = 0.795844 Js^2$. The dynamics of $\omega$ in the body frame are shown in Fig. 1(c). In this case, the period calculated from Eq. (14) is 23.8181 s. We can see from the shape of the curves that this corresponds visually to the period of Fig. 1(c). One aspect to note from this plot is that the dashed green curve shows the body $y$-component of the angular velocity (the axis of the intermediate moment of inertia), suddenly reverses twice, describing the flip known as the Dzhanibekov effect or the tennis racket theorem. Interestingly, most classic text in dynamics describe this behavior as ‘unstable,’ however the motion is completely determined and is periodic, eventually coming back to the ‘unstable’ point.

We may now compute some of the quantities that have been developed. The dynamic part of the Berry phase from Eq. (18) is $\frac{2ET}{M} = 23.6555$. If we integrate the projected angular velocity along the angular momentum axis for one period, we get

$$\int_0^T \frac{\omega \cdot M}{|M|} dt = 23.6555,$$

(21)

which matches the dynamic part. If we calculated the area of intersection between the sphere of angular momentum and the inertia ellipsoid (seen in Fig. 4(b), we get the signed area $\Omega = -4.1579$.

This gives a calculated geometric phase of 27.8135. The formula from Eq. (18) gives an identical value.

Another way of checking both formulae are by numerically integrating the Euler equations over one period and solving a system of equations for the rotation matrix after

![FIG. 3: geometric phase (mod 2π) of the body given in Fig. 1(b) for an asymmetric top with (I_1, I_2, I_3) = (0.286827 Js^2, 0.533256 Js^2, 0.795844 Js^2). For each initial angle, a given curve on the Binet ellipsoid is represented. The Binet ellipsoid is the ellipsoid of constant energy in the angular momentum space. Given an initial angle of the angular momentum, the resulting dynamics traces out a closed curve on this surface. The final angular mismatch (mod 2π) is the geophase and corresponding colors relate to corresponding values from 0 to 2π. Note that there are multiple curves where the phase is near 0.](image-url)
one period. The rotation matrix would give the geometric rotation of the initial system to the final orientation after one period. In this case, the rotation around the angular momentum vector is given by 2.68074 which is precisely the value of the geometric phase formula (27.8135), modulo 2π.

III. DISCUSSION OF FORMULA

A. Symmetries of the Geometric phase

As mentioned before, the geometric phase formula has notable symmetries including its independence from |I| = √I_1^2 + I_2^2 + I_3^2 and |ω_0| = √ω_01^2 + ω_02^2 + ω_03^2.

Independence from |I| follows from the observation that |I| is directly proportional to the total mass of a rigid body. Changing |I| amounts to changing the mass scale. If the mass scale were changed, the dynamics would be unchanged, especially the geometric quality of the rotation.

To understand the independence of Eq. (18) from |ω_0|, we see that changing the magnitude of |ω_0| changes the timescale for the dynamics, but geometrically, does not affect the phase. If we had made a video recording of the spinning, and played it back at a slower speed, the geometrical quality of the rotation would also be unchanged.

To further simplify the geometric phase formula, we may use the observation that there are related initial conditions that describe the same geometric phase and these lie on the same closed curve on the Binet ellipsoid. Without loss of generality, we may choose the initial condition to lie on the space x−z plane so that the azimuthal angle of the initial angular velocity, ω_φ is equal to 0. Thus we only need three parameters to specify the geometric phase: two ellipsoidal semi-axes (b and c), and the polar angle of the initial angular velocity in the body frame, θ_ω.

B. Minimum Geometric Phase

Figure 4 shows two plots for the above demonstrated case. The solid red curve is the geometric phase for a given initial condition described by the polar angle of the angular velocity (or angular momentum) in the body frame. The dashed black curve is the corresponding time period, T, for the angular velocity vector. For a given object, there are are two pieces of the curve separated by a critical angle that corresponds to the separatrix for Fig. 3. The critical angle is given by the condition

\[ M^2 = 2EI_2 \]

which reduces to

\[ \theta_{\text{crit}} = \cos^{-1}\left( \sqrt{\frac{I_1(I_1 - I_2)}{I_1^2 - I_1I_2 + I_2I_3 - I_3^2}} \right). \]

This critical angle is displayed as a gray vertical line on Figs. 4 and 5. The critical angle corresponds with a divergence of the period, T, as is typical for motion near a separatrix. Note that the minimum value for the geometric phase occurs at the extremes of 0 and π. This appears to be a generic behavior. Additionally, near the critical angle, the geometric phase diverges.

When the geometric phase modulo 2π is equal to 0, then trajectories that are started on these curves are closed and the Euler angles are periodic with a period whose integer multiple is the period of the angular velocity vector. As the trajectories on the Binet ellipsoid approach the curve \( M^2 = 2EI_2 \), the geometric phase increases without bound. One question that is apparent is what is the minimum geometric phase for a given object? For example are there objects (b and c values) where there are no initial conditions that give rise to a geometric phase of exactly 2π?

In Fig. 5 we see that the minimum geometric phase can be as high as desired. Looking at all objects, the largest minimum geometric phase is unbounded for objects with moments of inertia approaching \( I_1 = I_2 = I_3 = \sqrt{3}/3 \). Note that when \( I_1 = I_2 = I_3 = \sqrt{3}/3 \) exactly, the object is no longer dynamically asymmetric (an example of this is a sphere). On the other hand, the asymmetric top with the smallest minimum possible geometric phase for any initial condition is a geometric phase of 0 for objects with moments of inertia approaching \( I_1 = 0 \) and \( I_2 = I_3 = \sqrt{2}/2 \). An example of this is a thin solid rod.

![Image](image.png)

FIG. 4: For the demonstrated example of Fig. 1(c), the red curve indicates the geometric phase for different initial conditions with varying polar angle of the angular velocity. A polar angle of 0 is an angular velocity along the space z axis. Here we see that regardless of the initial condition, the geometric phase will be larger than about 5 radians. The dashed curve plots the corresponding period of the angular velocity vector in seconds.
C. Closed Herpolhodes

We may also use this to produce curves of closed herpolhodes. A herpolhode is the curve traced out by the terminus of the angular velocity vector in the space coordinates. In general these do not close. However, if we use initial conditions that have a geometric phase that is an integer multiple of $2\pi$, we find interesting closed curve patterns. Figure 5 shows the geometric phase for all initial polar angle for the angular velocity. The particular object (moments of inertia given by $(I_1, I_2, I_3) = (0.26 \, Js^2, 0.55 \, Js^2, 0.79 \, Js^2)$) has initial conditions with closed herpolhode trajectories. The corresponding angles that result in the smallest closed herpolhodes are (from left to right) 0.9499, 0.9995, 1.006792, 1.008389, 1.009536, 1.01659, 1.05795, and 1.2737. These are exactly the angles where the geometric phase is a multiple of $2\pi$. In this case the multiples are 3, 4, 5, 5, 4, 3, 2, and 1 respectively. The critical angle is 1.00817. Some of the closed herpolhode figures are depicted on the plot.

Some additional patterns are plotted in Fig. 6 for a different set of moments of inertia $(I_1, I_2, I_3) = (0.4 \, Js^2, 0.6 \, Js^2, \sqrt{1-0.4^2-0.6^2} \, Js^2)$. It is unclear what, if any, connection these closed curves and the associated energies have to the Bohr-Sommerfeld quantization in quantum mechanics.

IV. APPLICATIONS AND POSSIBLE CONNECTIONS

The primary purpose of this paper was to use the asymmetric top as an intuitive and natural example of geometric phase. However there are a few other connections that could be made.

Quantization of the asymmetric top leads to quantized energy levels of the quantum mechanical asymmetric top. How this corresponds to the closed orbits of the last section still need to be studied.

When molecules are no longer symmetric, the asymmetry leads to observable spectroscopic modes that allow for analysis and characterization. Understanding the dynamics of the asymmetric top classically gives us an understanding of the connection with the rotational modes of an asymmetric molecule. Making further connections with the Berry phase arising from simple rotation of these molecules could lead to further insight.

Large nuclei such as $^{135}$Pr and $^{163}$Lu, provide examples of asymmetric top systems in nuclear physics. Through X-ray spectroscopy, the so-called ‘wobble’ bands provide a way to measure angular momentum transitions of a spinning nuclei. The connection to closed geometric phase orbits still needs to be made.

V. CONCLUSIONS

In this paper, we have found the geometric phase formula explicitly in terms of special functions. This leads immediately to observations about the geometric phase. The geometric phase of a rigid object is independent of the magnitude of the moment of inertia, independent of the magnitude of the angular velocity, and all initial conditions along the path of the angular momentum vector in space have the same geometric phase. Thus for a given rigid body, the geometric phase only depends on the initial polar angle of the angular velocity vector (for instance). Additionally, in the parameter space of the moments of inertia, each class of asymmetric rigid bodies has a minimum geometric phase; that is, a minimum angle by which an object must rotate after undergoing one period of the angular velocity vector in space. For some objects, the geometric phase can be as high as desired. Closed trajectories are also highlighted and closed herpolhodes are illustrated and discussed briefly.

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FIG. 6: Some examples of closed herpolhode curves of a body with moments of inertia $(I_1, I_2, I_3) = (0.4 Js^2, 0.6 Js^2, \sqrt{1 - 0.4^2 - 0.6^2} Js^2)$. Note that even multiples of $2\pi$ and odd multiples of $2\pi$ have different plane symmetry.