Margenau-Hill operator valued measures and joint measurability

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Abstract

We employ the Margenau-Hill (MH) correspondence rule for associating classical functions with quantum operators to construct quasi-probability mass functions. Using this we obtain the fuzzy one parameter quasi measurement operator (QMO) characterizing the incompatibility of non-commuting spin observables of qubits, qutrits and 2-qubit systems. Positivity of the fuzzy MH-QMO places upper bounds on the associated unsharpness parameter. This serves as a sufficient condition for measurement incompatibility of spin observables. We assess the amount of unsharpness required for joint measurability (compatibility) of the non-commuting qubit, qutrit and 2-qubit observables. We show that the degree of compatibility of a pair of orthogonal qubit observables agrees perfectly with the necessary and sufficient conditions for joint measurability. Furthermore, we obtain analytical upper bounds on the unsharpness parameter specifying the range of joint measurability of spin components of qutrits and pairs of orthogonal spin observables of a 2-qubit system. Our results indicate that the measurement incompatibility of spin observables increases with Hilbert space dimension.

I. INTRODUCTION

In the classical framework, physical observables are all compatible and they can be measured jointly. In contrast, measurement of observables, which do not commute, are declared to be incompatible in the quantum scenario. The notion of compatibility of measurements is limited to that of commutativity of the observables only when one restricts to projective valued (PV) measurements. An extended notion of incompatibility emerges in the generalized framework of fuzzy measurements using positive operator valued measures (POVMs). Quantum measurement incompatibility offers a conceptual way to separate quantum and classical features.

In a different direction, Wigner’s quasi-phase-space distribution function approach brings out differences between quantum and classical phase-space descriptions. In particular, the quantum phase-space distribution differs from its classical counterpart in that it can assume negative values or it is more singular than the Dirac delta function distribution.

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Hence, it cannot be treated as a genuine probability distribution function. For this reason, Wigner’s quantum phase-space distribution function is referred to as quasi-probability distribution.

Based on Wigner’s approach, it is possible to reproduce expectation values of quantum observables as phase-space averages of corresponding classical functions. Besides Wigner distribution function, which makes use of Weyl’s operator correspondence rule \[ q^k p^l \leftrightarrow \frac{1}{2^k} \sum_{r=0}^{k} \frac{k!}{r! (k-r)!} \hat{q}^r \hat{p}^l \hat{q}^{k-r}, \quad k, r = 1, 2, \ldots \]
to associate classical functions \( f(q, p) \) of canonical position and momentum variables \( q, p \) with quantum operators \( \hat{f}(\hat{q}, \hat{p}) \), various other distribution functions have also been developed by designing suitable operator correspondence rules \[7–13\]. The phase-space distribution formalism, initiated by Wigner for canonical position and momentum observables, has also been extended to develop discrete probability mass functions for spin observables \[9–13\].

In this work we employ the Margenau-Hill (MH) operator correspondence rule \[7–13\] for associating classical functions \( f(x_1, x_2) \) with quantum operator functions \( \hat{f}(\hat{X}_1, \hat{X}_2) \) of non-commuting observables \( \hat{X}_1, \hat{X}_2 \) in a \( d \)-dimensional Hilbert space and construct quasi-probability mass functions \( P_{\text{MH}}(x_1, x_2) \) such that

\[
\sum_{x_1, x_2=1}^d P_{\text{MH}}(x_1, x_2) f(x_1, x_2) \equiv \text{Tr} \left[ \hat{\rho} \hat{f}(\hat{X}_1, \hat{X}_2) \right] = \left\langle \hat{f}(\hat{X}_1, \hat{X}_2) \right\rangle
\]

where \( \hat{\rho} \) denotes the density operator. Based on this approach we explicitly construct the fuzzy one parameter MH quasi measurement operator (QMO) \( \{ \hat{G}_Q(x_1, x_2; \eta), \quad 0 \leq \eta \leq 1 \} \) corresponding to the quasi mass functions \( P_{\text{MH}}(x_1, x_2) \), where the parameter \( \eta \) denotes fuzziness of measurement. The observables \( \hat{X}_1, \hat{X}_2 \) are compatible or jointly measurable if the MH-QMO turns out to be a legitimate parent POVM \[1–3\]. In Sec. 2 we give a brief outline on measurement incompatibility of a set of finite outcome quantum observables. The terms compatibility and joint measurability are used interchangeably throughout our discussion.
II. JOINT MEASURABILITY OF NON-COMMUTING OBSERVABLES

We start with a brief description of generalized measurements and the notion of joint measurability:

- A POVM is a set \( \{ \hat{E}(x) \} \) of positive semidefinite operators acting on a Hilbert space of dimension \( d \), satisfying the normalization condition
  \[
  \sum_{x=1}^{d} \hat{E}(x) = \hat{I}_d,
  \]
  where \( \hat{I}_d \) denotes the identity operator. The element \( \hat{E}(x) \) of the POVM is associated with the measurement outcome \( x = 1, 2, \ldots, d \).

- Incompatibility of measurements of a set of POVMs associated with the measurement of non-commuting observables is defined as the impossibility of a single parent measurement POVM from which the results of individual measurements can be registered [1-3].

Let \( \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n \) denote a set of non-commuting observables on a \( d \)-dimensional Hilbert space with finite measurement outcomes denoted by \( x_1, x_2, \ldots, x_n \) respectively. Let \( \hat{E}_k \equiv \{ \hat{E}_\eta(x_k) \}, k = 1, 2, \ldots, n \) denotes the POVM associated with the generalized measurements of the observable \( \hat{X}_k \), with the real parameter \( 0 \leq \eta \leq 1 \) characterizing unsharpness or fuzziness [15].

In principle, the POVMs \( \hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n \) are jointly measurable if there exists a parent POVM \( \{ \hat{G}(x_1, x_2, \ldots, x_n; \eta); 0 \leq \eta \leq 1 \} \) acting on the \( d \)-dimensional Hilbert space, which obeys the following properties [1-3, 14]:

(i) \( 0 \leq \hat{G}(x_1, x_2, \ldots, x_n; \eta) \leq \hat{I}_d \).

(ii) \( \sum_{x_1, x_2, \ldots, x_n} \hat{G}(x_1, x_2, \ldots, x_n; \eta) = \hat{I}_d \).

(iii) \( \hat{E}_\eta(x_k) = \sum_{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n} \hat{G}(x_1, x_2, \ldots, x_n; \eta) \).

Several measures to quantify incompatibility of measurements in terms of a single real parameter \( \eta \in [0, 1] \) have been proposed in the literature [1-3, 16].
Heinosaari et al. [3] consider degree of compatibility of the POVMs $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n$ as the supremum value (least upper bound) of a single real parameter $\eta \in [0, 1]$ which ensures that the fuzzy POVMs $\hat{E}_k = \{\hat{E}_\eta(x_k)\}, k = 1, 2, \ldots, n$ are all jointly measurable.

A real parameter $\eta \in [0, 1]$ is used as an index to quantify minimum amount of noise added so that the set of observables are compatible (see Ref. [16] and references therein). The supremum $\eta^*$ of this real parameter is called the incompatibility robustness. This interpretation comes from the convex-geometric structure of POVMs [3, 16].

The region $0 \leq \eta \leq \eta^*$ in which a bonafide parent POVM $\{\hat{G}(x_1, x_2, \ldots, x_n; \eta)\}$ is guaranteed has been termed as the compatibility region [3] of the POVMs $\{\hat{E}_\eta(x_k)\}, k = 1, 2, \ldots, n$. Larger the compatibility region more compatible are the observables. It has been realized that the region of compatibility decreases with the increase of Hilbert space dimension [3, 16].

For a detailed overview on robustness based measures describing how incompatible a set of quantum observables are to specific noise models see Ref. citenjp19 Also see Ref. [3] for a review on measurement incompatibility in more general operational theory or probability theory framework.

The purpose of this work is to construct quasi-measurement operators associated with non-commuting observables by employing the Margenau-Hill correspondence rule [7–13] and explore the compatibility region of quantum spin observables of qubits, qutrits and 2-qubit systems [17].
III. MARGENAU-HILL CHARACTERISTIC FUNCTION FOR NON-COMMUTING OBSERVABLES

Margenau-Hill (MH) characteristic function for non-commuting observables \( \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n \) is defined by \[9, 10\]

\[
\phi_{\text{MH}}(u_1, u_2, \ldots, u_n) = \left\langle \left\{ e^{i \hat{X}_1 u_1} e^{i \hat{X}_2 u_2} \cdots e^{i \hat{X}_n u_n} \right\}_{\text{Sym}} \rightangle = \frac{1}{n!} \sum_{P} \left\langle \left\{ e^{i \hat{X}_1 u_1} e^{i \hat{X}_2 u_2} \cdots e^{i \hat{X}_n u_n} \right\} \rightangle = \frac{1}{n!} \sum_{P} \text{Tr} \left( \hat{\rho} \left\{ e^{i \hat{X}_1 u_1} e^{i \hat{X}_2 u_2} \cdots e^{i \hat{X}_n u_n} \right\} \right) = \sum_{x_1, x_2, \ldots, x_n} P_{\text{MH}}(x_1, x_2, \ldots, x_n) e^{i \sum_{k} x_k u_k}. \tag{1}
\]

where \( \{ \cdot \}_{\text{Sym}} \) denotes symmetrization and \( \sum_{P} \) stands for summation over \( n! \) permutations of the factors \( e^{i \hat{X}_1 u_1}, e^{i \hat{X}_2 u_2}, \ldots, e^{i \hat{X}_n u_n} \).

1. The set of real numbers \( \{x_1, x_2, \ldots, x_n\} \) denote eigenvalues i.e., measurement outcomes of the observables \( \hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n \).

2. The map \( \hat{\rho} \mapsto P_{\text{MH}}(x_1, x_2, \ldots, x_n) \) assigns MH quasi-probability mass function \( P_{\text{MH}}(x_1, x_2, \ldots, x_n) \) with any arbitrary quantum state \( \hat{\rho} \) and is obtained \[9, 11\] by taking discrete Fourier transform of the characteristic function \( \phi_{\text{MH}}(u_1, u_2, \ldots, u_n) \).

3. The MH probability mass function is a real, normalized function satisfying the relations \( P_{\text{MH}}(x_1, x_2, \ldots, x_n) = P_{\text{MH}}^*(x_1, x_2, \ldots, x_n) \) and \( \sum_{x_1, x_2, \ldots, x_n} P_{\text{MH}}(x_1, x_2, \ldots, x_n) = 1 \).

4. The marginal probability mass functions say, \( P_{\text{MH}}(x_k) \), is obtained by summing \( P_{\text{MH}}(x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) \) over all the outcomes except \( x_k \) i.e.,

\[
P_{\text{MH}}(x_k) = \sum_{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, x_n} P_{\text{MH}}(x_1, x_2, \ldots, x_n).
\]

5. The operator correspondence rule

\[
x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \mapsto \{ \hat{X}_1^{r_1} \hat{X}_2^{r_2} \cdots \hat{X}_n^{r_n} \}_{\text{Sym}},
\]

\[
= \frac{1}{n!} \sum_{P} \{ \hat{X}_1^{r_1} \hat{X}_2^{r_2} \cdots \hat{X}_n^{r_n} \}, \quad r_1, r_2, \ldots, r_n = 0, 1, 2 \ldots \tag{2}
\]
leads to the evaluation of expectation values of any quantum observables $\hat{f}(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n)$ in terms of averages of corresponding classical functions $f(x_1, x_2, \ldots, x_n)$ i.e.,

$$\text{Tr} \left[ \hat{\rho} \hat{f}(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n) \right] = \sum_{x_1, x_2, \ldots, x_n} P_{\text{MH}}(x_1, x_2, \ldots, x_n) f(x_1, x_2, \ldots, x_n).$$ (3)

6. The MH probability mass function $P_{\text{MH}}(x_1, x_2, \ldots, x_n)$ is not necessarily positive and hence is referred to as MH quasi-probability mass function.

IV. MARGENAU-HILL QUASI MEASUREMENT OPERATOR (MH-QMO)

Expressing the quasi MH probability mass function $P_{\text{MH}}(x_1, x_2, \ldots, x_n)$ as

$$P_{\text{MH}}(x_1, x_2, \ldots, x_n) = \text{Tr}[\hat{\rho} \hat{G}_Q(x_1, x_2, \ldots, x_n)]$$ (4)

it is possible to identify a set of operators $\{\hat{G}_Q(x_1, x_2, \ldots, x_n)\}$ which we refer to as MH quasi measurement operator (QMO) associated with the observables $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$. By construction (see (4)) the QMO $\{\hat{G}_Q(x_1, x_2, \ldots, x_n)\}$ obey

\begin{align*}
(1) \quad & \hat{G}_Q^\dagger(x_1, x_2, \ldots, x_n) = \hat{G}_Q(x_1, x_2, \ldots, x_n) \\
(2) \quad & \sum_{x_1, x_2, \ldots, x_n} \hat{G}_Q(x_1, x_2, \ldots, x_n) = \hat{I}_d, \quad \text{(Identity operator)} \\
(3) \quad & \sum_{x_n} \hat{G}_Q(x_1, x_2, \ldots, x_n) = \hat{G}_Q(x_1, x_2, \ldots, x_{n-1}), \\
& \quad \vdots \quad \vdots \quad \vdots \\
(5) \quad & \sum_{x_1, x_2, \ldots, x_n} \hat{G}_Q(x_1, x_2, \ldots, x_n) = \hat{G}_Q(x_1, x_2) \\
& \sum_{x_2, \ldots, x_n} \hat{G}_Q(x_1, x_2, \ldots, x_n) = \hat{E}(x_1). 
\end{align*}

As the MH joint mass function $P_{\text{MH}}(x_1, x_2, \ldots, x_n)$ is not necessarily positive the operators $\hat{G}_Q(x_1, x_2, \ldots, x_n)$ are not positive in general. We refer to the set of operators $\{\hat{G}_Q(x_1, x_2, \ldots, x_n)\}$ as Margenau-Hill Quasi Measurement Operator (MH-QMO) associated with the observables $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$. The observables $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ are jointly measurable if $\{\hat{G}_Q(x_1, x_2, \ldots, x_n)\}$ happens to be a legitimate parent POVM i.e.,

$$\hat{G}_Q(x_1, x_2, \ldots, x_n) \geq 0$$ (6)
in addition to the defining properties (5) of MH-QMO. The criterion given in (6) serves as a sufficient condition for joint measurability of the observables $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$.

It is pertinent to point out here that Heinosaari et. al. had alluded to the construction of an observable (see Sec.3.2 of Ref. [3])

\[ J_n(\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n) = \frac{1}{n!} \sum_{p} \{\hat{E}_1, \hat{E}_2 \ldots \hat{E}_n\} \]  

(7)

associated with a set $\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n$ of POVMs, by generalizing the notion of the Jordan product. (In (7) ‘P’ stands for the set of all permutations). The Jordan product (7) can be employed as a joint observable whenever $J_n(\hat{E}_1, \hat{E}_2, \ldots, \hat{E}_n) \geq 0$. While the construction (7) resembles the MH operator correspondence rule given in (2) (see right hand side of (2) with $r_1, r_2, \ldots, r_n = 1$), we emphasize that the conceptual origin of MH operator correspondence rule (2) is different from that of the Jordan product.

V. MARGENAU-HILL QUASI MEASUREMENT OPERATORS (MH-QMOS) FOR A PAIR OF ORTHOGONAL SPIN OBSERVABLES

For a pair of non-commuting observables $\hat{X}$ and $\hat{Z}$ with discrete eigenvalues $x, z$, the MH-QMO $\{\hat{G}_Q(x, z)\}$ associated with the Margenau-Hill probability distribution mass function $P_{\text{MH}}(x, z)$ is obtained by using (4):

\[ P_{\text{MH}}(x, z) = \text{Tr} [\hat{\rho} \hat{G}_Q(x, z)] = \langle \hat{G}_Q(x, z) \rangle. \]  

(8)

It may be noted that the relations (see (1), (4))

\[ \phi_{\text{MH}}(u, v) = \frac{1}{2} \left\langle e^{iXu}e^{iZv} + e^{iZv}e^{iXu} \right\rangle \]

\[ = \sum_{x, z} \langle \hat{G}_Q(x, z; \eta) \rangle e^{i(xu+zu)} \]

lead to an explicit evaluation of $\hat{G}_Q(x, z; \eta)$ from the coefficient of $e^{i(xu+zu)}$ in the characteristic function $\phi_{\text{MH}}(u, v)$.

We now proceed to discuss specific examples of MH-QMO for the orthogonal spin observables of qubits, qutrits and 2-qubits. We choose the orthogonal spin observables corresponding to measurements along $x$ and $z$ directions.
A. Joint measurability of orthogonal qubit observables

Consider the orthogonal qubit observables

\[ \hat{X} = \sigma_x, \quad \hat{Z} = \sigma_z, \quad [\sigma_x, \sigma_z] = -i \sigma_y \]

\[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (9)\]

The MH characteristic function associated with the above pair of qubit observables is given by

\[ \phi_{\text{MH}}(u, v) = \frac{1}{2!} \langle e^{i \sigma_x u} e^{i \sigma_z v} + e^{i \sigma_z v} e^{i \sigma_x u} \rangle = \sum_{x, z} P_{\text{MH}}(x, z) e^{i (x u + z v)} = \sum_{x, z} \langle G_Q(x, z) \rangle e^{i (x u + z v)}. \quad (10)\]

On simplification we obtain

\[ \phi_{\text{MH}}(u, v) = \langle I_2 \rangle \cos u \cos v + i \langle \sigma_x \rangle \sin u \cos v + i \langle \sigma_z \rangle \cos u \sin v. \quad (11)\]

The qubit MH-QMO \( \hat{G}_Q^{\text{qubit}}(x, z) \) is then identified using the coefficient of \( e^{i (x u + z v)} \) i.e.,

\[ \hat{G}_Q^{\text{qubit}}(x, z) = \frac{1}{4} (I_2 + x \sigma_x + z \sigma_z), \quad x, z = \pm 1. \quad (12)\]

- We construct a one-parameter family of generalized unsharp measurements \( \{ \hat{G}_Q^{\text{qubit}}(x, z; \eta) \} \) by replacing \( \hat{X} = \sigma_x, \hat{Z} = \sigma_z \) by \( \eta \hat{X} = \eta \sigma_x \) and \( \hat{Z} = \sigma_z \) by \( \eta \hat{Z} = \eta \sigma_z \), where \( 0 \leq \eta \leq 1 \), to obtain (see (12))

\[ \hat{G}_Q^{\text{qubit}}(x, z; \eta) = \frac{1}{4} (I_2 + \eta x \sigma_x + \eta z \sigma_z) = \frac{1}{4} \begin{pmatrix} 1 + \eta z & \eta x \\ \eta x & 1 - \eta z \end{pmatrix}. \quad (13)\]

- Note that the elements of fuzzy POVM associated with the individual measurements of the orthogonal qubit observables \( \sigma_x, \sigma_z \) obey the conditions

\[ \sum_{x=\pm 1} \hat{E}_x^{\text{qubit}}(x) = I_2, \quad \hat{E}_x^{\text{qubit}}(x) \geq 0 \]

\[ \sum_{z=\pm 1} \hat{F}_z^{\text{qubit}}(z) = I_2, \quad \hat{F}_z^{\text{qubit}}(z) \geq 0. \quad (14)\]
• The eigenvalues of $\hat{G}_Q^{\text{qubit}}(x, \eta)$ are readily found to be

$$\lambda_{\pm} = \frac{1}{4}(1 \pm \eta \sqrt{2})$$  \hspace{1cm} (15)

and it is seen that the least eigenvalue $\lambda_- < 0$ when

$$\eta > \frac{1}{\sqrt{2}} \approx 0.707.$$  \hspace{1cm} (16)

Thus $\{\hat{G}_Q^{\text{qubit}}(x = \pm 1, z = \pm 1; \eta)\}$ is positive in the range $0 \leq \eta \leq \frac{1}{\sqrt{2}}$. In other words the MH-QMO $\{\hat{G}_Q^{\text{qubit}}(x, z; \eta)\}$ with $\eta \in (0, \frac{1}{\sqrt{2}} \approx 0.707)$ turns out to be a valid parent POVM for joint measurements of the orthogonal qubit observables $\sigma_x, \sigma_z$. It may be seen that the degree of compatibility $\eta_{\text{qubit}} = \frac{1}{\sqrt{2}}$ matches perfectly with the results for joint measurability of a pair of orthogonal qubit observables [15, 19–21].

B. Joint measurability of orthogonal qutrit observables

Consider the qutrit spin observables

$$\hat{X} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{Z} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \hspace{1cm} (17)$$

which are $x, z$ components of spin-1 quantum system respectively.

In order to evaluate the MH characteristic function

$$\phi_{\text{MH}}(u, v) = \frac{1}{2} \langle \{e^{iXu} e^{iZv}\} \rangle = \frac{1}{2} \langle e^{iXu} e^{iZv} + e^{iZv} e^{iXu} \rangle$$

and to identify MH-QMO $\{\hat{G}_Q(x, z; \eta)\}$ associated with the pair of qutrit spin observables $(\hat{X}, \hat{Z})$ we prescribe the following steps.

• We employ the 2-qubit basis $\{|0\_1,0\_2\rangle, |0\_1,1\_2\rangle, |1\_1,0\_2\rangle, |1\_1,1\_2\rangle\}$, instead of the qutrit basis $\{|jm\rangle, j = 1, 0, m = -j \rightarrow j\}$ to express $\hat{X}, \hat{Z}$ as

$$\hat{U}_{\text{CG}}^\dagger(\hat{X} \oplus 0) \hat{U}_{\text{CG}} = \frac{1}{2}(I_2 \otimes \sigma_x + \sigma_z \otimes I_2), \quad \hat{U}_{\text{CG}}^\dagger(\hat{Z} \oplus 0) \hat{U}_{\text{CG}} = \frac{1}{2}(I_2 \otimes \sigma_z + \sigma_x \otimes I_2) \hspace{1cm} (18)$$

Here the symbol $\oplus$ denotes direct sum and the unitary matrix $\hat{U}_{\text{CG}}$ given by

$$\hat{U}_{\text{CG}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \hspace{1cm} (19)$$
The MH characteristic function \( \phi_{\text{MH}}(u, v) = \frac{1}{2} \langle e^{i X u} e^{i Z v} + e^{i Z v} e^{i X u} \rangle \) assumes the form

\[
\phi_{\text{MH}}(u, v) = \frac{1}{4} (I_2 \otimes I_2 (1 + \cos u + \cos v + \cos u \cos v) + i(\sigma_x \otimes I_2 + I_2 \otimes \sigma_x) (\sin u + \sin u \cos v) + i(\sigma_x \otimes I_2 + I_2 \otimes \sigma_x) (\sin v + \sin v \cos u) - \sigma_x \otimes \sigma_x (1 - \cos u + \cos v - \cos u \cos v) - \sigma_y \otimes \sigma_y (1 - \cos u - \cos v + \cos u \cos v) - \sigma_z \otimes \sigma_z (1 + \cos u - \cos v - \cos u \cos v))
\]

\[
= \sum_{x, z = \pm 1, 0} \langle \hat{G}_Q(x, z) \rangle e^{i(x u + z v)},
\]

(20)

• Collecting the coefficients of \( e^{i(xu+zw)} \), \( x, z = \pm 1, 0 \), we obtain the elements of MH-QMO:

\[
\begin{align*}
\hat{G}_Q(0, 0) &= \frac{1}{4} [I_2 \otimes I_2 - \sigma_x \otimes \sigma_x - \sigma_z \otimes \sigma_z - \sigma_y \otimes \sigma_y] \\
\hat{G}_Q(x, 0) &= \frac{1}{8} [I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y - \sigma_z \otimes \sigma_z + x(I \otimes \sigma_x + \sigma_x \otimes I)] \\
\hat{G}_Q(0, z) &= \frac{1}{8} [I_2 \otimes I_2 - \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z + \sigma_y \otimes \sigma_y + z(I \otimes \sigma_z + \sigma_z \otimes I)] \\
\hat{G}_Q(x, z) &= \frac{1}{16} [I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z - \sigma_y \otimes \sigma_y + z(I \otimes \sigma_z + \sigma_z \otimes I) + x(I \otimes \sigma_x + \sigma_x \otimes I) + x z(\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x)].
\end{align*}
\]

(21)

Then we replace \( \sigma_x \rightarrow \eta \sigma_x \) and \( \sigma_z \rightarrow \eta \sigma_z \), \( 0 \leq \eta \leq 1 \) in (21) to obtain elements of the one-parameter fuzzy MH-QMO [23]

\[
\begin{align*}
\hat{G}_Q(0, 0; \eta) &= \frac{1}{4} [I_2 \otimes I_2 - \eta^2 \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z - \eta^2 \sigma_y \otimes \sigma_y] \\
\hat{G}_Q(x, 0; \eta) &= \frac{1}{8} [I_2 \otimes I_2 + \eta^2 \sigma_x \otimes \sigma_x + \eta^2 \sigma_y \otimes \sigma_y + \eta^2 \sigma_z \otimes \sigma_z + x(I \otimes \sigma_x + \sigma_x \otimes I)] \\
\hat{G}_Q(0, z; \eta) &= \frac{1}{8} [I_2 \otimes I_2 - \eta^2 \sigma_x \otimes \sigma_x - \sigma_z \otimes \sigma_z - \eta^2 \sigma_y \otimes \sigma_y + z(I \otimes \sigma_z + \sigma_z \otimes I) + \eta x(I \otimes \sigma_x + \sigma_x \otimes I) + \eta z(\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x)].
\end{align*}
\]

(22)

We then transform the \( 4 \times 4 \) matrices \( \hat{G}_Q \)'s of (22) back to the qutrit basis using the unitary transformation (19) i.e.,

\[
\hat{U}_{\text{CG}} \hat{G}_Q(x, z; \eta) \hat{U}_{\text{CG}}^\dagger = \begin{pmatrix}
\hat{G}^{\text{qutrit}}_{Q}(x, z; \eta) & 0_{3 \times 1} \\
0_{1 \times 3} & \hat{G}^{\text{singlet}}_{Q}(x, z; \eta)
\end{pmatrix}
\]

(23)
where

\[ \hat{G}_{Q_{qutrit}}(0, 0; \eta) = \frac{1}{4} \begin{pmatrix}
1 - \eta^2 & 0 & \eta^2(\eta^2 - 1) \\
0 & 1 - \eta^4 & 0 \\
\eta^2(\eta^2 - 1) & 0 & 1 - \eta^2
\end{pmatrix} \]

\[ \hat{G}_{Q_{qutrit}}(x = \pm1, 0; \eta) = \frac{1}{8} \begin{pmatrix}
1 - \eta^2 & \sqrt{2}x\eta & \eta^2(1 - \eta^2) \\
\sqrt{2}x\eta & (1 + \eta^2)^2 & \sqrt{2}x\eta \\
\eta^2(1 - \eta^2) & \sqrt{2}x\eta & 1 - \eta^2
\end{pmatrix} \]  

(24)

\[ \hat{G}_{Q_{qutrit}}(0, z = \pm1; \eta) = \frac{1}{8} \begin{pmatrix}
1 + 2z\eta + \eta^2 & 0 & -\eta^2(1 + \eta^2) \\
0 & (\eta^2 - 1)^2 & 0 \\
-\eta^2(1 + \eta^2) & 0 & 1 - 2z\eta + \eta^2
\end{pmatrix} \]

\[ \hat{G}_{Q_{qutrit}}(x = \pm1, z = \pm1; \eta) = \frac{1}{16} \begin{pmatrix}
1 + 2z\eta + \eta^2 & \sqrt{2}x\eta(1 + z\eta) & \eta^2(1 + \eta^2) \\
\sqrt{2}x\eta(1 + z\eta) & 1 - \eta^4 & x\sqrt{2}\eta(1 - z\eta) \\
\eta^2(1 + \eta^2) & \sqrt{2}x\eta(1 - z\eta) & 1 - 2z\eta + \eta^2
\end{pmatrix} \]

are the elements of qutrit MH-QMO.

- The qutrit POVM \( \{ \hat{E}_{qutrit}(x), x = \pm1, 0 \} \) associated with fuzzy measurement of the observable \( \hat{X} \) is then obtained using the qutrit MH-QMO (see [24]) as follows:

\[ \hat{E}_{qutrit}(x = \pm1) = \sum_{z=\pm1, 0} \hat{G}_{Q_{qutrit}}(x, z; \eta) \]

\[ = \frac{1}{4} \begin{pmatrix}
1 & \sqrt{2}x\eta & \eta^2 \\
\sqrt{2}x\eta & 1 + \eta^2 & \sqrt{2}x\eta \\
\eta^2 & \sqrt{2}x\eta & 1
\end{pmatrix} \]

\[ \hat{E}_{qutrit}(x = 0) = \sum_{z=\pm1, 0} \hat{G}_{Q_{qutrit}}(0, z; \eta) \]

\[ = \frac{1}{2} \begin{pmatrix}
1 & 0 & -\eta^2 \\
0 & 1 - \eta^2 & 0 \\
-\eta^2 & 0 & 1
\end{pmatrix} \]  

(25)
Similarly the qutrit POVMs for the measurement of $\hat{Z}$ take the form

$$ \hat{F}_q^{\text{qutrit}}(z = \pm 1) = \sum_{x=\pm 1, 0} \hat{G}_q^{\text{qutrit}}(x, z; \eta) $$

$$ = \frac{1}{4} \begin{pmatrix} 1 + 2z\eta + \eta^2 & 0 & 0 \\ 0 & 1 - \eta^2 & 0 \\ 0 & 0 & 1 - 2z\eta + \eta^2 \end{pmatrix} $$

$$ \hat{F}_q^{\text{qutrit}}(z = 0) = \sum_{x=\pm 1, 0} \hat{G}_q^{\text{qutrit}}(x, 0; \eta) $$

$$ = \frac{1}{2} \begin{pmatrix} 1 - \eta^2 & 0 & 0 \\ 0 & 1 + \eta^2 & 0 \\ 0 & 0 & 1 - \eta^2 \end{pmatrix} $$  \hspace{1cm} (26)

Note that $\{\hat{E}_q^{\text{qutrit}}(x)\}, \{\hat{F}_q^{\text{qutrit}}(z)\}$ obey the properties

$$ \sum_{x=\pm 1, 0} \hat{E}_q^{\text{qutrit}}(x) = I_3, \hspace{0.2cm} \hat{E}_q^{\text{qutrit}}(x) \geq 0 $$

$$ \sum_{z=\pm 1, 0} \hat{F}_q^{\text{qutrit}}(z) = I_3, \hspace{0.2cm} \hat{F}_q^{\text{qutrit}}(z) \geq 0 $$  \hspace{1cm} (27)

and they correspond to one-parameter fuzzy measurements of the observables $\hat{X}, \hat{Z}$ respectively.

Now we proceed to evaluate the eigenvalues of the joint MH-QMO $\{\hat{G}_q^{\text{qutrit}}(x, z; \eta); x, z = \pm 1, 0\}$ and find that the lowest eigenvalue of both $\hat{G}_q^{\text{qutrit}}(x = \pm 1, 0; \eta)$ and $\hat{G}_q^{\text{qutrit}}(0, z = \pm 1; \eta)$ is given by

$$ \lambda(\eta) = \frac{1 + \eta^2 - \eta\sqrt{4 + \eta^2(1 + \eta^2)^2}}{8} $$  \hspace{1cm} (28)

which assumes negative values (see Fig.1) when $\sqrt{\sqrt{2} - 1} < \eta \leq 1$. Moreover the eigenvalues of $\hat{G}_q^{\text{qutrit}}(x = \pm 1, z = \pm 1; \eta)$ given by

$$ \lambda_1(\eta) = \frac{1 - 2\eta^2 - \eta^4}{16}, \hspace{0.2cm} \lambda_2(\eta) = \frac{1 + 2\eta^2 - \eta\sqrt{\eta^6 + 8}}{16} $$  \hspace{1cm} (29)

assume negative values when $\eta \geq \sqrt{\sqrt{2} - 1}$.

In other words, the qutrit MH-QMO $\{\hat{G}_q^{\text{qutrit}}(x, z; \eta); x, z = \pm 1, 0\}$ happens be a legitimate parent POVM enabling joint measurements of the orthogonal qutrit observables $\hat{X}, \hat{Z}$ (see [17]) when $0 \leq \eta \leq \sqrt{\sqrt{2} - 1} \approx 0.64359$.  

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FIG. 1. A plot of the lowest eigenvalue $\lambda(\eta) = \frac{1+\eta^2-\eta\sqrt{1+\eta^4(1+\eta^2)^2}}{8}$ of the elements $\hat{G}_Q^{\text{qutrit}}(0, z; \eta)$, for $z = \pm 1$ and $\hat{G}_Q^{\text{qutrit}}(x, 0; \eta)$ for $x = \pm 1$ of the MH-QMO as a function of the unsharpness parameter $\eta$. It is seen that $\lambda(\eta) < 0$ when $\eta \geq \sqrt{\sqrt{2} - 1} \approx 0.64359$.

FIG. 2. A plot of the eigenvalues $\lambda_1(\eta) = \frac{1-2\eta^2-\eta^4}{16}$ (dot-dashed line), $\lambda_2 = \frac{1+2\eta^2-\eta\sqrt{\eta^4+8}}{16}$ (undashed line) of the elements $\hat{G}_Q^{\text{qutrit}}(x = \pm 1, z = \pm 1; \eta)$ of the MH-QMO as a function of $\eta$. It is seen that $\lambda_1(\eta), \lambda_2(\eta) < 0$ when $\eta \geq \sqrt{\sqrt{2} - 1} \approx 0.64359$.

Note that the region of compatibility of the qutrit observables $\hat{X}, \hat{Z}$ i.e.,

$$0 \leq \eta \leq 0.64359$$  \hspace{1cm} (30)

is smaller than that of a qubit (see (16)). Thus the incompatibility of the pair of spin observables $\hat{X}, \hat{Z}$ is seen to increase with the Hilbert space dimension. This result is
consistent with the observation of Ref. [3] that the finite dimensional position and momentum observables become more incompatible with increasing dimension.

C. Joint measurability of 2-qubit observables

In this section, we consider joint measurements of the set \{\hat{X}_1, \hat{Z}_1; \hat{X}_2, \hat{Z}_2\} of 2-qubit observables:

\[
\hat{X}_1 = \sigma_x \otimes I_2, \quad \hat{Z}_1 = \sigma_z \otimes I_2 \\
\hat{X}_2 = I_2 \otimes \sigma_x, \quad \hat{Z}_2 = I_2 \otimes \sigma_z.
\]

The observable pairs (\hat{X}_k, \hat{Z}_k), k = 1, 2 represent x, z components of qubit spin observables (expressed in the 2-qubit computational basis \{0_10_2, 0_11_2, 1_10_2, 1_11_2\}).

Note that

- the x,z components of qubit spin observables are shown to be jointly measurable in the range \(0 \leq \eta \leq 0.707\) (see [16]) of the fuzziness parameter;

- the observable pair (\hat{X}_1, \hat{Z}_1) of qubit 1 commutes with the set (\hat{X}_2, \hat{Z}_2) of qubit 2.

So, it appears natural to anticipate that the set \{\hat{X}_1, \hat{Z}_1; \hat{X}_2, \hat{Z}_2\} of 2-qubit observables (see [31]) is compatible when \(0 \leq \eta \leq 0.707\). With the help of an explicit construction of MH-QMO for the 2-qubit observables [31] we show that even though pairwise joint measurability of the observable pairs (\hat{X}_1, \hat{Z}_1) and (\hat{X}_2, \hat{Z}_2) is ensured when \(0 \leq \eta \leq 0.707\), the set of all four 2-qubit observables \{\hat{X}_1, \hat{Z}_1; \hat{X}_2, \hat{Z}_2\} is incompatible.

The MH characteristic function \(\phi_{\text{MH}}(u_1, v_1; u_2, v_2)\) associated with the 2-qubit observ-
ables \{\hat{X}_1, \hat{Z}_1; \hat{X}_2, \hat{Z}_2\} of (31) is readily evaluated to obtain
\[
\phi_{\text{MH}}(u_1, v_1; u_2, v_2) = \frac{1}{2} \left\langle e^{i \hat{X}_1 u_1} e^{i \hat{X}_2 u_2}, e^{i \hat{Z}_1 v_1} e^{i \hat{Z}_2 v_2} \right\rangle
\]
\[
= \frac{1}{2} \left\langle e^{i (\hat{X}_1 u_1 + \hat{X}_2 u_2)} e^{i (\hat{Z}_1 v_1 + \hat{Z}_2 v_2)} + e^{i (\hat{Z}_1 v_1 + \hat{Z}_2 v_2)} e^{i (\hat{X}_1 u_1 + \hat{X}_2 u_2)} \right\rangle
= I_2 \otimes I_2 \cos u_1 \cos u_2 \cos v_1 \cos v_2 + i \sigma_x \otimes I_2 \sin u_1 \cos u_2 \cos v_1 \cos v_2
+ i I_2 \otimes \sigma_x \cos u_1 \sin u_2 \cos v_2 \cos v_2 + i \sigma_z \otimes I_2 \cos u_1 \cos u_2 \sin v_1 \cos v_2
+ i I_2 \otimes \sigma_z \cos u_1 \cos u_2 \sin v_1 \sin v_2 - \sigma_x \otimes \sigma_x \sin u_1 \sin u_2 \cos v_1 \cos v_2
- \sigma_x \otimes \sigma_z \cos u_1 \cos u_2 \sin v_1 \sin v_2 - \sigma_y \otimes \sigma_y \sin u_1 \sin u_2 \sin v_2 \sin v_2
= \sum_{x_1, z_1; x_2, z_2 = \pm 1} \left\langle G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2) \right\rangle e^{i (x_1 u_1 + z_1 v_1 + x_2 u_2 + z_2 v_2)} \tag{32}\]

We then obtain explicit form of the elements of 2-qubit MH-QMO from the coefficients of \(e^{i (x_1 u_1 + z_1 v_1 + x_2 u_2 + z_2 v_2)}\), \(x_1, z_1, x_2, z_2 = \pm 1\):
\[
G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2) = \frac{1}{16} (I_2 \otimes I_2 + x_1 \sigma_x \otimes I_2 + z_1 \sigma_z \otimes I_2
+ x_2 I_2 \otimes \sigma_x + z_2 I_2 \otimes \sigma_z + x_1 x_2 \sigma_x \otimes \sigma_x + z_1 z_2 \sigma_z \otimes \sigma_z
+ x_1 z_2 \sigma_x \otimes \sigma_z + x_2 z_1 \sigma_z \otimes \sigma_x - x_1 x_2 z_1 z_2 \sigma_y \otimes \sigma_y) \tag{33}\]

In order to bring in measurement fuzziness we replace \(\sigma_x \rightarrow \eta \sigma_x, \sigma_z \rightarrow \eta \sigma_z\) where \(0 \leq \eta \leq 1\). The corresponding 2-qubit fuzzy MH-QMO \(\left\{G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2; \eta)\right\}\) is then given by
\[
G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2; \eta) = \frac{1}{16} \left[ I_2 \otimes I_2 + \eta \left( x_1 \sigma_x \otimes I_2 + z_1 \sigma_z \otimes I_2 \right)
+ \eta \left( x_2 I_2 \otimes \sigma_x + z_2 I_2 \otimes \sigma_z \right)
+ \eta^2 \left( x_1 x_2 \sigma_x \otimes \sigma_x + z_1 z_2 \sigma_z \otimes \sigma_z + x_1 z_2 \sigma_x \otimes \sigma_z
+ x_2 z_1 \sigma_z \otimes \sigma_x \right) - \eta^4 x_1 x_2 z_1 z_2 \sigma_y \otimes \sigma_y \right] \tag{34}\]

It is readily seen that
\[
(1) \quad \sum_{x_1, z_1; x_2, z_2 = \pm 1} G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2; \eta) = I_2 \otimes I_2
\]
\[
(2) \quad \sum_{x_2, z_2 = \pm 1} G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2; \eta) = \frac{1}{4} [ I_2 \otimes I_2 + \eta \left( x_1 \sigma_x \otimes I_2 + z_1 \sigma_z \otimes I_2 \right) ]
= G_Q(x_1, z_1; \eta)
\]
\[
\sum_{x_1, z_1 = \pm 1} G_Q^{2-\text{qubit}}(x_1, z_1, x_2, z_2; \eta) = \frac{1}{4} [ I_2 \otimes I_2 + \eta \left( x_2 I_2 \otimes \sigma_z + z_2 I_2 \otimes \sigma_z \right) ]
= G_Q(x_2, z_2; \eta). \tag{35}\]
• The elements of MH-QMOs

\[
G_Q(x, z; \eta) = \frac{1}{4} [I_x \otimes I_z + \eta (x \sigma_x \otimes I_z + z \sigma_z \otimes I_x)], \quad x, z = \pm 1
\]

of the first and second qubit respectively turn out to be valid parent POVMs when \(0 \leq \eta \leq 0.707\) ensuring that the pairs of observables \((\hat{X}_1, \hat{Z}_1)\) and \((\hat{X}_2, \hat{Z}_2)\) (see (31)) are pairwise measurable (in conformity with the results of Sec. 5.1 on joint measurability of orthogonal qubit observables).

• Even though the observable pair \((\hat{X}_1, \hat{Z}_1)\) of first qubit commutes with \((\hat{X}_2, \hat{Z}_2)\) of the second qubit, the total set \((\hat{X}_1, \hat{Z}_1, \hat{X}_2, \hat{Z}_2)\) of 2-qubit observables (see (31)) is not jointly measurable in the range \(0 \leq \eta \leq \frac{1}{\sqrt{2}}\), which is evidenced by the non-positivity of the associated 2-qubit MH-QMO \(\{G_2^{\text{qubit}}(x_1, z_1, x_2, z_2; \eta)\}\).

• From an explicit evaluations we find that the eigenvalues of the 2-qubit MH-QMO \(\{G_2^{\text{qubit}}(x_1, z_1, x_2, z_2; \eta); x_1, z_1, x_2, z_2 = \pm 1\}\) (see (34)) are same as that of the qutrit MH-QMO \(\{G_{\text{qutrit}}(x, z; \eta); x, z = \pm 1\}\) (see (29)). In other words the fuzzy MH-QMO \(\{G_2^{\text{qubit}}(x_1, z_1, x_2, z_2; \eta); x_1, z_1, x_2, z_2 = \pm 1\}\) becomes a legitimate parent POVM of the 2-qubit observables \(\hat{X}_1, \hat{Z}_1, \hat{X}_2, \hat{Z}_2\) (see (31)) in the domain \(\eta \in (0, \sqrt{\frac{1}{2}} - 1)\). So, the compatibility region of the set of 2-qubit observables \(\hat{X}_1, \hat{Z}_1, \hat{X}_2, \hat{Z}_2\) is given by

\[
0 \leq \eta \leq \sqrt{\frac{1}{2}} - 1 \approx 0.64359.
\] (36)

VI. SUMMARY

By employing the Margenau-Hill (MH) correspondence rule for associating classical functions with quantum operators, we have explicitly constructed fuzzy one parameter quasi measurement operators (QMO) to analyze the incompatibility of the non-commuting spin observables of qubits, qutrits and 2-qubit systems. A real parameter \(0 \leq \eta \leq 1\) corresponding to unsharpness of measurement serves as an index of the minimum amount of fuzziness required so that the observables become compatible. We give a summary of our results:
1. MH-QMOs of a pair of orthogonal qubit observables $\hat{X} = \sigma_x$, $\hat{X} = \sigma_z$ are positive in the range $0 \leq \eta_{\text{qubit}} \leq 0.707$ of the unsharpness parameter. This range of values, obtained based on the positivity of MH-QMO are in agreement with the results of Refs. [15, 19–21] for the joint measurability of a pair of orthogonal qubit measurements.

2. Based on the positivity of fuzzy MH-QMOs, which serves as a sufficient condition for joint measurability, we recognize that a pair of qutrit spin observables $\hat{X}, \hat{Z}$ (see (17)) are compatible when $0 \leq \eta \leq 0.64359$ (see (30)). This region of compatibility is smaller than that for the qubit observables i.e., $0 \leq \eta_{\text{qubit}} \leq 0.707$. Thus the pair of qutrit spin observables $\hat{X}, \hat{Z}$ are found to be more incompatible compared to their qubit counterparts. In other words, quantum incompatibility of spin observables appears to increase with the dimension of the Hilbert space. It deserves future attention to recognize the region of incompatibility of spin observables in the limit $d \to \infty$.

3. Designolle et. al. [16] carried out a detailed investigation to determine maximal incompatibility of a pair of observables in any finite dimensional Hilbert space. They list the incompatibility robustness value $\eta^* = 0.6602$ and $\eta^* = 0.6830$ respectively for two specific qutrit observable pairs (See (78), (79) of Ref. [16]). Maximal incompatibility of sets of qudit observables has also been explored in Ref. [24] where numerical computation leads to $\eta^* = 0.6794$ (see table 4 of Ref. [24]) for a maximally incompatible pair of 3-outcome qutrit observables. The qutrit POVMs $\{E_q^{\text{qutrit}}(x), x = \pm 1, 0\}$, $\{F_q^{\text{qutrit}}(z), z = \pm 1, 0\}$ (see (25), (26)) are incompatible when $\eta \geq 0.64359$ and they are found to outperform the ones listed in Refs. [24] and [16].

4. The 2-qubit observable pairs $(\hat{X}_1 = \sigma_x \otimes I_2, \hat{Z}_1 = \sigma_z \otimes I_2)$ of qubit 1 and $(\hat{X}_2 = I_2 \otimes \sigma_x, \hat{Z}_2 = I_2 \otimes \sigma_z)$ of qubit 2 are shown to be compatible in the domain $0 \leq \eta \leq \frac{1}{\sqrt{2}}$ which is in agreement with the region of compatibility of the orthogonal qubit observables $\sigma_x, \sigma_z$. However, eventhough the observable pair $(\hat{X}_1, \hat{Z}_1)$ of first qubit commutes with the set $(\hat{X}_2, \hat{Z}_2)$ of second qubit observables the compatibility region of the entire set $\{\hat{X}_1 = \sigma_x \otimes I_2, \hat{Z}_1 = \sigma_z \otimes I_2; \hat{X}_2 = I_2 \otimes \sigma_x, \hat{Z}_2 = I_2 \otimes \sigma_z\}$ is found to be $0 \leq \eta \leq 0.64359$, highlighting enhanced incompatibility of the set of qubit observables, when they are embedded in a 2-qubit Hilbert space.

We believe that our framework based on MH operator correspondence rule for joint mea-
surability of spin observables provides intuitive insights on quantum measurement incompatibility. Exploring incompatibility based on different operator correspondence rules deserves future attention.

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computation of incompatibility robustness \[3, 16\] to obtain similar results.

[18] Replacing the Hermitian observable \(\hat{X}\) by \(\eta \hat{X}\) indicates that the moments of the observable are fuzzy i.e., \(\langle \hat{X}^k \rangle_{\text{fuzzy}} = \eta^k \langle \hat{X}^k \rangle_{\text{sharp}}\). It also signifies that a POVM, constructed by mixing the sharp PV measurements of the observable \(\hat{X}\) with noise characterized by the parameter \(\eta\) leads to unsharp measurements of the observable \(\hat{X}\).

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