Research Article

Ulam–Hyers Stability for a Class of Caputo-Type Fractional Stochastic System with Delays

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In this study, we explore the Caputo-type fractional stochastic differential equations (FSDEs) with delays. Firstly, the existence and uniqueness of the solutions for the considered system are obtained under the non-Lipschitz condition by using the Carathéodory approximation method with the aid of Fatous Lemma, Itô isometry, Cauchy–Schwarz inequality, and Gronwall’s inequality. Secondly, we study the Ulam–Hyers (U–H) stability of the addressed equations with the aid of the Cauchy–Schwarz inequality and Gronwall’s inequality. Finally, an example with numerical simulation is presented to support the validity of our theory.

1. Introduction

In 1695, the German mathematician Leibniz and the French mathematician L’Hospital first discussed the one-half derivative in their correspondence. Since then, the fractional derivative has appeared in the sight of many scholars. With the research of Bernoulli, Euler, Lagrange, Riemann, Liouville, Lacroix, Abel, Caputo, and other scholars, the definition of fractional calculus is more and more perfect, and its properties, applications are gradually discovered. So far, the fractional calculus has experienced more than three hundred years of development. The study of fractional calculus was focused on pure mathematics at first and then gradually developed in the field of practical application. Nowadays, we can see the application of fractional calculus to fractional physics, viscous elasticity, automatic control theory, chemical physics, anomalous diffusion, and chain unlinking of polymer materials. For more details on fractional calculus, please refer to [1–8]. Authors of [1–7] introduced, in detail, the origin and definition of various types of fractional calculus and elaborated the boundary value problem of fractional differential equations (FDEs) and its research methods and introduced the application fields and application models of fractional calculus. Agarwal et al. [8] introduced a new extension of the Caputo fractional derivative operator involving the generalized hypergeometric-type function \( F_p(a, b; c; z; k) \).

Due to the development of FDEs in various fields, some experts found that the traditional differential equations cannot accurately describe the long tail phenomenon and power law phenomenon, as well as polymer materials, all kinds of anomalous diffusion in the field of porous media percolation mechanics phenomenon, but FSDEs were considered better able to describe these phenomena. So, FSDEs were proposed to solve those problems that were affected by random factors in real life. Therefore, FSDEs are considered to be very useful, and more and more scholars have carried out research on FSDEs. Refer to [9–11] for some results related to FSDEs. Mishura and Zili [9] were devoted to general information from the theory of Gaussian processes, fractional and subfractional Brownian motions, and mixed fractional and mixed subfractional Brownian motions. Mishura [10] was devoted to the Wiener integration with respect to fractional Brownian motion, stochastic integration with respect to fractional Brownian motion, and other aspects of stochastic calculus of fBm and filtering problems in the mixed fractional models. Meerschaert and Sikorskii [11] covered basic limit theorems for
random variables and random vectors with heavy tails and the basic ideas of fractional calculus and anomalous diffusion were introduced in the context of probability theory. In order to facilitate the further development and application of FSDEs, it is necessary to study some important properties of FSDEs. On the one hand, we should explore the existence and uniqueness of solutions. Many scholars had made a lot of important contributions in this area. In [12], the authors employed two methods to prove the existence and uniqueness of fractional stochastic neutral differential equations, in which the contraction mapping principle was used under the Lipschitz condition at first, and then, the Picard approximation was applied under the non-Lipschitz condition. In [13], the author proved existence and uniqueness of \( \psi \)-Caputo fractional stochastic evolution equations with varying-time delay driven by fBm using the Schauder fixed-point theorem [14] and the Banach contraction principle, separately. In [15], Picard iterations were used to explore the existence and uniqueness of multi-dimensional FSDEs with variable order under non-Lipschitz condition. Although [13, 15] both used Picard approximation, the mild solution in [13] was solved by Laplace transform and its inverse, while the mild solution in [15] was obtained by integrating both sides of the differential equations. In addition, Yang [13] investigated the constant fractional order, while Moualkia [15] explored variable fractional order. In [16], the authors studied the existence of fractional impulsive neutral stochastic differential equations with infinite delay by using fractional calculus, operator semigroup, and fixed-point theorem. For more types of FDEs and the methods used to study the existence and uniqueness of solutions, one can refer to [17–21].

On the contrary, it is necessary to study the stability of solution to FSDEs, and U–H stability is an important content. In 1940, Ulam [22] first proposed the stability of functional equations in a speech at Wisconsin University. In 1941, Hyers [23] gave the answer to this question, and then, U–H stability came into being. The study of U–H stability for FDEs can not only provide an important theoretical basis for the existence and uniqueness of solutions but also provide a reliable theoretical basis for approximate solution of FSDEs. For the system with U–H stability, we can substitute an approximate solution for an exact one when it is relatively difficult to obtain the exact solution and the U–H stability can guarantee the reliability of the approximated solution to a certain extent. At present, many scholars had studied the U–H stability of various types of FDEs. For example, in [24], the researchers used the technique of Picard operator and Gronwall’s integral inequalities to explore U–H stability for conformable FDEs with delay. The authors applied the tools of fractional calculus and Hölder’s inequality of integration to derive the U–H stability of nonlinear differential equations with fractional integrable impulses in [25]. For further contributions on this topic, see [26–30] and its references. The stability of stochastic differential equations has also been studied in [31–34], but there are relatively few papers on the U–H stability of FSDEs. Among them, Ahmadova et al. [35, 36] have made important contributions for U–H stability of FSDEs, and they studied the U–H stability of Caputo-type fractional stochastic neutral differential equations with weighted maximum norm and Itoh isometry in [36].

Inspired by the above papers, in this study, we consider the following system:

\[
^C D_0^λ a(s) = Ba(s) + K(s, a(s), a(s − τ)) + N(s, a(s), a(s − τ)) \frac{dw(s)}{ds}, \ s ∈ [0, U],
\]

\[
a(s) = b(s), \ s ∈ [−τ, 0],
\]

where \(^C D_0^λ\) is the Caputo fractional derivative, \(1/2 < λ < 1\), \(B \in \mathbb{R}^{m×n}, K: [0, U] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(N: [0, U] \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{m×n}\) are measurable functions, \(w(s)\) is an \(m\)-dimensional Brownian motion on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), \(b: [−τ, 0] \rightarrow \mathbb{R}^n\) is a continuous function, \(a(0) = b(0) = b_0\), and \(\mathbb{E}\|a(0)\|^2 < \infty\). In this study, we use the Carathéodory approximation method to study the existence and uniqueness and use the direct method to prove the U–H stability of the solution to system (1).

The main innovations of this study at least include the following aspects:

(1) The system in which we study the existence and uniqueness of solutions by the Carathéodory approximation method is more generalized. In [37], the system is FSDEs, and in [38, 39], the system is first-order stochastic differential equations with delay while the system in this study is FSDEs with delay.

(2) The U–H stability and the existence and uniqueness of solutions are proved by using weaker non-Lipschitz condition. This is a contribution to the study of FSDEs.

(3) We give a numerical simulation to verify the U–H stability of the system, and this is the deficiency in [24–28, 36].

The structure of this study is as follows. In Section 2, we give some definitions, lemmas, and assumptions to promote the progress of the following paper. In Section 3, we prove the existence and uniqueness of system (1) under non-Lipschitz conditions by using Carathéodory approximation. The main purpose of section 4 is to explore the U–H stability of FSDEs. In Section 5, we test the effectiveness of our theory through an example with numerical simulation.

2. Preliminaries

In the section, we give some definitions, lemmas, and assumptions which will be used throughout the study. Let \(\| \cdot \|\) be the standard Euclidean norm of a vector \(x \in \mathbb{R}^n\) or a matrix \(A \in \mathbb{R}^{m×n}\) be defined by

\[
\|x\| = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}, \quad i = 1, 2, \ldots, n, \quad \|A\| = \sqrt{\lambda},
\]
where $\lambda$ is the maximum eigenvalue of $A^T A$.

**Definition 1** (see [1]). For $\omega \in ([0, U], \mathbb{R}^n)$, the Caputo derivative of fractional order $0 < \lambda < 1$ is defined as

$$D_0^\lambda \omega(\eta) = \frac{1}{\Gamma(n - \lambda)} \int_0^\eta (\eta - \sigma)^{n-\lambda-1} \omega^{(i)}(\sigma) d\sigma,$$

(3)

where $i - 1 < \lambda < i, i \in \mathbb{N}$.

**Definition 2** (see [37]). For $\omega : [0, U] \rightarrow \mathbb{R}^n$, the Riemann–Liouville integral operator of fractional order $0 < \lambda < 1$ is defined as

$$I_0^\lambda \omega(\eta) = \frac{1}{\Gamma(\lambda)} \int_0^\eta (\eta - \sigma)^{\lambda-1} \omega(\sigma) d\sigma,$$

(4)

where $\eta > 0$ and $\Gamma : (0, \infty) \rightarrow \mathbb{R}$ is the well-known Euler's Gamma function. In particular, for $\lambda \in (0, 1)$, we have

$$I_0^1 D_0^1 \omega(\eta) = \omega(\eta) - \omega(0).$$

(5)

**Lemma 1** (see [40]). Suppose that $r(x), t(x), f(x) \in C([0, U], \mathbb{R}^n)$, $s(x) \in C([0, U], \mathbb{R}^n)$, and $r(x)$ and $s(x)$ are nondecreasing and $r(0) = s(0)$. If $\mu(x) \in C([-\tau, U], \mathbb{R}^n)$ and

$$\begin{cases} 
\mu(x) \leq r(x) + \int_0^\tau (\mu(\nu) f(\nu) + f(\nu) \mu(\nu - \tau) ) d\nu, & x \in [0, v], \\
\mu(x) \leq s(x), & x \in [-\tau, 0], 
\end{cases}$$

(6)

then

$$\mu(x) \leq r(x) \exp \left( \int_0^x (t(\nu) f(\nu) ) d\nu \right), \quad x \in [0, v].$$

(7)

**Lemma 2** (see [12]). For each $U > 0$, $2\lambda > 1$, if $g(\cdot)$ is an $\mathbb{R}^n$-value stochastic process and $\int_0^U \| g(\eta) \|^2 d\eta < \infty$, then the following inequality holds:

$$\sup_{0 \leq \eta \leq U} \mathbb{E} \left( \int_0^\eta (\eta - \nu)^{1-2\lambda} g(\nu) d\nu \left\| g(\nu) \right\|^2 \right) \leq \frac{U^{2\lambda-1}}{2\lambda - 1} \mathbb{E} \left( \int_0^U \| g(\nu) \|^2 d\nu \right).$$

(8)

**Lemma 3** Let us consider the solution for the following system:

$$\begin{align*}
D_0^\lambda a(s) &= Ba(s) + K(s), & s \in [0, U], \\
a(s) &= b(s), & s \in [-\tau, 0],
\end{align*}$$

(9)

where $b(0) = b_0$.

When $s \in [0, U]$, system (9) is equivalent to the integral equation as follows:

$$a(s) = b_0 + \frac{1}{\Gamma(\lambda)} \int_0^s (s - \nu)^{\lambda-1} Ba(\nu) d\nu + \frac{1}{\Gamma(\lambda)} \int_0^s (s - \nu)^{\lambda-1} K(\nu) d\nu.$$

(10)

Applying Laplace transformation, it is easy to obtain

$$\tilde{a}(u) = \frac{b_0}{u} + \frac{B}{u^\lambda} \tilde{a}(u) + \frac{1}{u^\lambda} \tilde{K}(u).$$

(11)

Then,

$$\tilde{a}(u) = \frac{b_0 u^{\lambda-1}}{u^\lambda - B} + \frac{\tilde{K}(u)}{u^\lambda - B},$$

(12)

$$\{ s^{\lambda-1} M_\lambda(s^\lambda B); u \} + \tilde{K}(u) L,$$

Utilizing inverse Laplace transformation, we get the solution to system (9):

$$a(s) = b_0 M_\lambda(s^\lambda B) + \int_0^s (s - \nu)^{\lambda-1} M_\lambda(s^\lambda B)K(\nu) d\nu, \quad s \in [0, U],$$

(13)

where $M_\lambda$ and $M_{\lambda, \lambda}$ are matrix Mittag–Leffler functions and $M_\lambda, M_{\lambda, \lambda} : \mathbb{R}^{\text{non}} \rightarrow \mathbb{R}^{\text{non}}$ are defined as [36]

$$M_\lambda(s^\lambda B) = \sum_{j=0}^{\infty} B^j \frac{s^j}{\Gamma(j\lambda + 1)}, \quad M_{\lambda, \lambda}(s^\lambda B) = \sum_{j=0}^{\infty} B^j \frac{s^j}{\Gamma(j\lambda + \lambda)}.$$  

(14)

Since $M_\lambda(s^\lambda B)$ and $M_{\lambda, \lambda}(s^\lambda B)$ are bounded on $s \in [0, U]$, for convenience, we assume

$$\sigma_1 = \sup_{0 \leq s \leq U} \left\| M_\lambda(s^\lambda B) \right\|^2, \quad \sigma_2 = \sup_{0 \leq s \leq U} \left\| M_{\lambda, \lambda}(s^\lambda B) \right\|^2.$$  

(15)

**Definition 3** (see [37]). If an $\mathbb{R}^n$-value stochastic process $\{a(s)\}_{s \in \mathbb{R}^{\text{non}}}^{\text{non}}$ is satisfied,

1. $\{a(s)\}$ is $\mathcal{F}_s$-adapted and continuous.
2. $\{K(s, a(s), a(s - \tau))\} \in L([0, U] \times \mathbb{R}^n \times \mathbb{R}^n)$ and $\{N(s, a(s), a(s - \tau))\} \in L^2([0, U] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{\text{non}})$.  
3. For $s \in [-\tau, 0], a(s) = b(s)$ and for $s \in [0, U]$,
Assumption 1. (non-Lipschitz condition). Suppose that there exists a function \( H: [0, +\infty) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that
\[
\mathbb{E} \left( \int_{-\tau}^{U} \|a(s)\|^2 \, ds \right) < \infty.
\]

(4) For each other solution \( \bar{a}(s) \), we obtain \( P \{ a(s) = \bar{a}(s), \ -\tau \leq s \leq U \} = 1 \); then, \( \{ a(s) \}_{-\tau \leq s \leq U} \) is a solution to (1), where \( \{ K(s, a(s), a(s - \tau)) \} \in L([0, U] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \) represents the Lebesgue integrability of the function \( K \).

Assumption 2. Let \( K(\theta, 0, 0) = 0 \) and \( \forall \theta \in [0, U] \); it follows that
\[
\|K(\theta, 0, 0)\|^2 \leq \sigma_3,
\]
where \( \sigma_3 \) is a constant.

Lemma 4 (see \[37\]). Given that the function \( H(v, u): [0, U] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is concave, there exist \( c_1(v), c_2(v): [0, U] \rightarrow \mathbb{R}^+ \) such that
\[
H(v, u) \leq c_1(v) + c_2(v)u,
\]
where
\[
\int_{0}^{U} c_1(v) \, dv < \infty, \quad \int_{0}^{U} c_2(v) \, dv < \infty.
\]

Assumption 3. In equation (1), for any \( s \in [-\tau, 0] \), there exists a nondecreasing function \( T(\cdot) \in C([-\tau, 0], \mathbb{R}^+) \) and
\[
T(0) = 3b_0\sigma_1 + \frac{12\sigma_2U^{2\lambda - 1}}{2\lambda - 1} \left[ U \left( \sigma_3 + \sup_{0 \leq s \leq U} c_1(v) \right) + U \sup_{0 \leq s \leq U} \sup_{-\tau \leq \tau \leq 0} \left( \frac{b^2(\tau) + 2b^2(\tau)}{2} \right) \right],
\]

such that
\[
\sup_{-\tau \leq \tau \leq s} b^2(\tau) \leq T(s), \quad s \in [-\tau, 0].
\]

3. Existence and Uniqueness

This section explores the existence and uniqueness of the addressed system. Let us consider Carathéodory successive approximations as follows: for each integer \( m \geq 1 \), define \( a_m \) on \([-1 + \tau, U]\) by

\[
a_m(s) = b(\tau), \quad s \in [-1 + \tau, -\tau],
\]
\[
a_m(s) = b(s), \quad s \in [-\tau, 0],
\]
\[ a_m(s) = b_0 M_0(s^2 B) + \int_0^s (s-v)^{1-1} M_0, ((s-v)^4 B) K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \]

\[ \text{Theorem 1. Suppose that assumptions 1-3 hold; then, equation (I) has a unique solution } a(t), t \in [-\tau, U]. \]

**Proof 1.** For \(-\tau \leq s \leq 0\), obviously \([a_m(s), m \geq 1]\) is bounded, equicontinuous, and a Cauchy sequence. For \(0 \leq s \leq U\), the proof will be divided into the following steps.

**Step 1:** \([a_m(s), m \geq 1]\) is bounded.

By (26), we have

**Theorem 1.** Suppose that assumptions 1-3 hold; then, equation (I) has a unique solution \(a(t), t \in [-\tau, U]\).

Proof 1. For \(-\tau \leq s \leq 0\), obviously \([a_m(s), m \geq 1]\) is bounded, equicontinuous, and a Cauchy sequence. For \(0 \leq s \leq U\), the proof will be divided into the following steps.

\[ \sup_{0 \leq y \leq s} \mathbb{E}[a_m(y)]^2 \]

\[ \leq 3 \sup_{0 \leq y \leq s} \mathbb{E}[b_0 M_0(y^4 B)]^2 + 3 \sup_{0 \leq y \leq s} \mathbb{E}\left[ \int_0^y (y-v)^{1-1} M_0, ((y-v)^4 B) K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \right] \]

\[ \leq 3 \sup_{0 \leq y \leq s} \mathbb{E}\left[ \int_0^y (y-v)^{1-1} M_0, ((y-v)^4 B) N(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \right] \]

where we use the following inequality:

\[ \left\| \sum_{i=1}^n x_i \right\|^2 \leq n \sum_{i=1}^n \left\| x_i \right\|^2. \quad \text{(28)} \]

\[ \sup_{0 \leq y \leq s} \mathbb{E}[a_m(y)]^2 \]

\[ \leq 3b_0 \sigma_1 + 3\sigma_2 \sup_{0 \leq y \leq s} \mathbb{E}\left[ \int_0^y (y-v)^{2}\frac{1}{m} - \frac{1}{m} \right] dv, \]

\[ \left\| K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) \right\|^2 dv \]

\[ + 3 \sigma_2 \frac{1}{2} - \frac{1}{2} \mathbb{E}\left[ \int_0^y \left\| N(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) \right\|^2 dv \right] \]

\[ \leq 3b_0 \sigma_1 + 3\sigma_2 \frac{1}{2}\mathbb{E}\left[ \int_0^y \left\| K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m}) - K(v, 0, 0) + K(v, 0, 0) \right\|^2 dv \]

\[ + 3 \sigma_2 \frac{1}{2}\mathbb{E}\left[ \int_0^y \left\| N(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) - N(v, 0, 0) + N(v, 0, 0)^2 dv \right\| \right] \]

\[ \leq 3b_0 \sigma_1 + 12\sigma_2^2 \frac{1}{2}\mathbb{E}\left[ \int_0^y K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m}) - k(v, 0, 0) \right\|^2 dv \]

Applying Cauchy–Schwarz inequality, Lemma 2, inequality (28), and assumption 1 (ii), we attain
From Lemma 4, it follows that

\[
\sup_{0 \leq y \leq x} \mathbb{E} \|a_m(y)\|^2 \\
\leq 3b_0 \sigma_1 + 12 \sigma_1 \sigma_2 \frac{U^{2\lambda}}{2\lambda - 1} \\
+ 12 \sigma_1 \frac{U^{2\lambda}}{2\lambda - 1} \left( \int_0^s (C_1(v) + (C_2(v)) \|a_m(v - \frac{1}{m})\|^2 + C_2(v) \|a_m(v - \frac{1}{m})\|^2 \right) dv \\
+ 12 \sigma_1 \frac{U^{2\lambda}}{2\lambda - 1} \sup_{0 \leq y \leq v} C_2(v) \int_0^s \sup_{0 \leq y \leq v} \mathbb{E} \left\| a_m(v - \frac{1}{m}) \right\|^2 dv \\
+ 12 \sigma_1 \frac{U^{2\lambda}}{2\lambda - 1} \sup_{0 \leq y \leq v} C_2(v) \int_0^s \sup_{0 \leq y \leq v} \mathbb{E} \left\| a_m(v - \frac{1}{m}) \right\|^2 dv \\
\leq f_1 + f_2 \int_0^s \left( \sup_{0 \leq y \leq \frac{1}{m}} \mathbb{E} \left\| a_m(v - \frac{1}{m}) \right\|^2 + \sup_{1/m \leq y \leq v} \mathbb{E} \left\| a_m(v - \frac{1}{m}) \right\|^2 \right) dv \\
+ f_2 \int_0^s \left( \sup_{0 \leq y \leq \frac{1}{m}} \mathbb{E} \left\| a_m(v - \tau - \frac{1}{m}) \right\|^2 + \sup_{1/m \leq y \leq v} \mathbb{E} \left\| a_m(v - \tau - \frac{1}{m}) \right\|^2 \right) dv \\
\leq f_1 + f_2 \int_0^s \left( b^2(\tau) + \sup_{-\tau \leq y \leq 0} \mathbb{E} \|a_m(v)\|^2 + \sup_{0 \leq y \leq v} \mathbb{E} \|a_m(v)\|^2 \right) dv \\
+ f_2 \int_0^s \left( b^2(\tau) + \sup_{-\tau \leq y \leq 0} \mathbb{E} \|a_m(v)\|^2 \right) dv \\
\leq f_1 + f_2 \int_0^s \left( b^2(\tau) + \sup_{-\tau \leq y \leq 0} b^2(v) + \sup_{0 \leq y \leq v} \mathbb{E} \|a_m(v)\|^2 \right) dv \\
+ f_2 Ub^2(\tau) + f_2 \int_0^s \sup_{0 \leq y \leq 0} \mathbb{E} \|a_m(v - \tau)\|^2 dv \\
\leq f_1 + f_2 U^2 \sup_{-\tau \leq y \leq 0} \mathbb{E} \|a_m(v)\|^2 + 2 f_2 U b^2(\tau)
where \( f_1 = 3b_0 \sigma_1 + 12\sigma_2 \sigma_3 U_0^{21}/2\lambda - 1 + 12\sigma_3 U_0^{22}/2\lambda - 1 \sup_{0 \leq y \leq c_1(v)} \) and \( f_2 = 12\sigma_2 U_0^{21}/2\lambda - 1 \sup_{0 \leq y \leq c_2(v)} \).

Then, Lemma 1 and assumption 3 read

\[
\sup_{0 \leq y \leq 1} E \|a_m(y)\|^2 \leq (0) \exp\left(\int_0^1 (f_1 + f_2) \, dv\right)
\leq T(0) e^{2fU}.
\]

Thus, for 0 \leq s_1 < s_2 \leq U and each integer \( m \geq 1 \), from (26), we obtain

\[
= a_m(s_2) - a_m(s_1)
\]

\[
b_0 M_\lambda(s_1^2 B) - b_0 M_\lambda(s_1^2 B) + \int_{s_1}^{s_2} \left[(s_2^2 - v)^{k-1} M_{\lambda,1}(s_2^2 - v)^{k} B) - (s_1^2 - v)^{k-1} M_{\lambda,1}(s_1^2 - v)^{k} B)\right] K(v, a_m(v - \frac{1}{m}), a_m(v - \tau - \frac{1}{m})) \, dv
\]

\[
+ \int_{s_1}^{s_2} (s_2^2 - v)^{k-1} M_{\lambda,1}(s_2^2 - v)^{k} B) K(v, a_m(v - \frac{1}{m}), a_m(v - \tau - \frac{1}{m})) \, dv
\]

\[
+ \int_0^{s_1} \left[(s_2^2 - v)^{k-1} M_{\lambda,1}(s_2^2 - v)^{k} B) - (s_1^2 - v)^{k-1} M_{\lambda,1}(s_1^2 - v)^{k} B)\right] N(v, a_m(v - \frac{1}{m}), a_m(v - \tau - \frac{1}{m})) \, dv(v)
\]

\[
+ \int_{s_1}^{s_2} (s_2^2 - v)^{k-1} M_{\lambda,1}(s_2^2 - v)^{k} B) N(v, a_m(v - \frac{1}{m}), a_m(v - \tau - \frac{1}{m})) \, dv(v),
\]

in which, with the aid of (28), Cauchy–Schwarz inequality and Itô isometry yield
\[ \begin{aligned}
&+ 5 \int_0^{s_1} \left\| \frac{(s_2 - v)^{k-1} M_{1,1}\left((s_2 - v)^{4} B\right) - (s_1 - v)^{k-1} M_{1,1}\left((s_1 - v)^{4} B\right) }{v, a_m\left(v - \frac{1}{m}\right), a_m\left(v - \tau - \frac{1}{m}\right)} \right\|^2 dv \\
&+ 5 \int_{s_1}^{s_2} \left\| (s_2 - v)^{k-1} M_{1,1}\left((s_2 - v)^{4} B\right), N \left(v, a_m\left(v - \frac{1}{m}\right), a_m\left(v - \tau - \frac{1}{m}\right)\right) \right\|^2 dv, 
\end{aligned} \] 

where

\[ E \left\| K\left(v, a_m\left(v - \frac{1}{m}\right), a_m\left(v - \tau - \frac{1}{m}\right)\right) \right\|^2 \]

\[ = E \left\| k\left(v, a_m\left(v - \frac{1}{m}\right), a_m\left(v - \tau - \frac{1}{m}\right)\right) - k\left(v, 0, 0\right) + k\left(v, 0, 0\right) \right\|^2 \]

\[ \leq 2E \left\| k\left(v, a_m\left(v - \frac{1}{m}\right), a_m\left(v - \tau - \frac{1}{m}\right)\right) - k\left(v, 0, 0\right) \right\|^2 + 2\| k\left(v, 0, 0\right) \|^2 \]

\[ \leq 2E \left[ H\left(v\left( a_m\left(v - \frac{1}{m}\right) + a_m\left(v - \tau - \frac{1}{m}\right)\right) \right) \right] + 2\sigma_3 \]

\[ \leq 2 \sup_{0 \leq v \leq U} c_1(v) + 2 \sup_{0 \leq v \leq U} c_2(v) \left( \sup_{-1 \leq v_1 \leq 0} E\left\| a_m(v_1) \right\|^2 + \sup_{0 \leq v_1 \leq v} E\left\| a_m(v_1) \right\|^2 \right) \]

\[ + 2 \sup_{0 \leq v \leq U} c_2(v) \left( \sup_{-1 \leq v_2 \leq v} E\left\| a_m(v_2) \right\|^2 + \sup_{0 \leq v_2 \leq v} E\left\| a_m(v_2) \right\|^2 \right) + 2\sigma_3 \]

\[ \leq 2 \sup_{0 \leq v \leq U} c_1(v) + 2 \sup_{0 \leq v \leq U} c_2(v) \left( b^2(\tau) + \sup_{-1 \leq v_1 \leq 0} b^2(v_1) + c \right) \]

\[ + 2 \sup_{0 \leq v \leq U} c_2(v) \left( b^2(\tau) + \sup_{-1 \leq v_2 \leq 0} b^2(v_2) + c \right) + 2\sigma_3 \]

\[ = \mathcal{P}_1. \]

Thus,

\[ E\left\| a_m\left(s_2\right) - a_m\left(s_1\right) \right\|^2 \]

\[ \leq 5 \left\| b_0 M_{1,1}\left(s_2^4 B\right) - b_0 M_{1,1}\left(s_1^4 B\right) \right\|^2 \]

\[ + 5q_1 U \int_0^{s_1} (s_2 - v)^{k-1} M_{1,1}\left((s_2 - v)^{4} B\right) - (s_1 - v)^{k-1} M_{1,1}\left((s_1 - v)^{4} B\right) dv + \right\|^2 dv \\
+ 5q_1 \int_{s_1}^{s_2} \left\| (s_2 - v)^{k-1} M_{1,1}\left((s_2 - v)^{4} B\right) - (s_1 - v)^{k-1} M_{1,1}\left((s_1 - v)^{4} B\right) \right\|^2 dv \\
+ 5q_1 \int_0^{s_1} \left\| (s_2 - v)^{k-1} M_{1,1}\left((s_2 - v)^{4} B\right) - (s_1 - v)^{k-1} M_{1,1}\left((s_1 - v)^{4} B\right) \right\|^2 dv. \]
Let $s_1 \to s_2$; we obtain $E\|a_m(s_2) - a_m(s_1)\| \to 0$ which implies that $\{a_m(s), m \geq 1\}$ is equicontinuous.

Step 3: $\{a_m(s), m \geq 1\}$ is a Cauchy sequence. For integer $j > m \geq 1$, based on (28), Cauchy–Schwarz inequality and Lemma 2, we can derive

\[
\sup_{0 \leq s \leq t} E\|a_j(y) - a_m(y)\|^2 \\
\leq 2 \sup_{0 \leq s \leq t} E\left( \int_0^s (y - v)^{k-1} M_{\lambda_1} \left[ K(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}), a_m(v - \frac{1}{m})) - K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) \right] dv \right)^2 \\
+ 2 \sup_{0 \leq s \leq t} E\left( \int_0^s (y - v)^{k-1} M_{\lambda_1} \left[ N(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}), a_m(v - \frac{1}{m})) - N(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m})) \right] dv \right)^2 \\
\leq 2\sigma \frac{U^{k-1}}{2\lambda - 1} E \left( \int_0^s K(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \right)^2 \\
+ 2\sigma \frac{U^{k-1}}{2\lambda - 1} E \left( \int_0^s N(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \right)^2 \\
= 2\sigma \frac{U^{k-1}}{2\lambda - 1} E \left( \int_0^s K(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \right)^2 \\
+ 2\sigma \frac{U^{k-1}}{2\lambda - 1} E \left( \int_0^s N(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}), a_m(v - \frac{1}{m})) dv \right)^2 \\
\leq 4\sigma \frac{U^{k-1}}{2\lambda - 1} \int_0^s E K(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m})) dv \\
- K(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}))^2 dv \\
+ 4\sigma \frac{U^{k-1}}{2\lambda - 1} \int_0^s E K(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m})) dv \\
- K(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m}))^2 dv \\
+ 4\sigma \frac{U^{k-1}}{2\lambda - 1} \int_0^s E N(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m})) dv \\
- N(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m}))^2 dv \\
+ 4\sigma \frac{U^{k-1}}{2\lambda - 1} \int_0^s E N(v, a_j(v - \frac{1}{j}), a_j(v - \frac{1}{m})) dv \\
- N(v, a_m(v - \frac{1}{m}), a_m(v - \frac{1}{m}))^2 dv.
\]
\[ \mathbb{E}\left\| K\left( v, a_j\left( v - \frac{1}{m}\right), a_j\left( v - \tau - \frac{1}{m}\right)\right) - K\left( v, a_m\left( v - \frac{1}{m}\right), a_m\left( v - \tau - \frac{1}{m}\right)\right) \right\|^2 \leq \mathbb{E}H\left(v, \left\| a_j\left( v - \frac{1}{m}\right) - a_m\left( v - \frac{1}{m}\right)\right\|^2 + \left\| a_j\left( v - \tau - \frac{1}{m}\right) - a_m\left( v - \tau - \frac{1}{m}\right)\right\|^2 \right) \]
\[ \leq H\left(v, \mathbb{E}\left\| a_j\left( v - \frac{1}{m}\right) - a_m\left( v - \frac{1}{m}\right)\right\|^2 \right) + \mathbb{E}\left\| a_j\left( v - \tau - \frac{1}{m}\right) - a_m\left( v - \tau - \frac{1}{m}\right)\right\|^2 \right) \]
\[ \leq H\left(v, \left\| a_j\left( v - \frac{1}{m}\right) - a_m\left( v - \frac{1}{m}\right)\right\|^2 \right) \]
\[ \leq H\left(v, \left( \sup_{0 \leq y_1 \leq v} \mathbb{E}\left\| a_j(v_1) - a_m(v_1)\right\|^2 + \sup_{0 \leq y_2 \leq v} \mathbb{E}\left\| a_j(v_2) - a_m(v_2)\right\|^2 \right) \right) \]
\[ = H\left(v, 2 \sup_{0 \leq y_1 \leq v} \mathbb{E}\left\| a_j(v_1) - a_m(v_1)\right\|^2 \right). \]  

Then, we have
\[ \sup_{0 \leq y \leq s} \mathbb{E}\left\| a_j(y) - a_m(y)\right\|^2 \]
\[ \leq 8\sigma_2^2 \frac{U^{2l-1}}{2\lambda - 1} \int_0^s \mathbb{E}H\left(v, \left\| a_j\left( v - \frac{1}{j}\right) - a_j\left( v - \frac{1}{m}\right)\right\|^2 \right) \, dv + 8\sigma_2^2 \frac{U^{2l-1}}{2\lambda - 1} \int_0^s H\left(v, 2 \sup_{0 \leq y_1 \leq v} \mathbb{E}\left\| a_j(v_1) - a_m(v_1)\right\|^2 \right) \, dv. \]  

Noticing the Fatous lemma and continuity of \( H(\theta, \cdot) \), we obtain
\[ \lim_{j,m \to \infty} \sup_{0 \leq y \leq s} \mathbb{E}\left\| a_j(y) - a_m(y)\right\|^2 \]
\[ \leq 16\sigma_2^2 \frac{U^{2l-1}}{2\lambda - 1} \int_0^s \mathbb{E}H\left(v, \lim_{j,m \to \infty} \sup\left\| a_j\left( v - \frac{1}{j}\right) - a_j\left( v - \frac{1}{m}\right)\right\|^2 \right) \, dv \]
\[ + 16\sigma_2^2 \frac{U^{2l-1}}{2\lambda - 1} \int_0^s \mathbb{E}H\left(v, \lim_{j,m \to \infty} \sup\left( 2 \sup_{0 \leq y_1 \leq v} \mathbb{E}\left\| a_j(v_1) - a_m(v_1)\right\|^2 \right) \, dv. \]
As \( j, m \to \infty \), we get \( v - 1/j \to v - 1/m \) and \( v - \tau - 1/j \to v - \tau - 1/m \). Then, by assumption 1 (i) and Step 2, we obtain

\[
H(v, \lim_{j,m \to \infty} \sup \left( \left\| a_j(v - 1/j) - a_j(v - 1/m) \right\|^2 + \left\| a_j(v - \tau - 1/j) - a_j(v - \tau - 1/m) \right\|^2 \right)) = H(v,0) = 0.
\]

Let

\[
X(s) = \lim_{j,m \to \infty} \sup_{j,m \geq s} \left( \sup_{0 \leq y \leq s} E\left\| a_j(y) - a_m(y) \right\|^2 \right).
\]

We derive

\[
X(s) \leq 16\sigma^2 \frac{U^{2\lambda - 1}}{2\lambda - 1} \int_0^s H(v, X(v)) dv. \tag{42}
\]

Last through inequality (42) and assumption 1 (iii), it is easy to verify

\[
X(s) = \lim_{j,m \to \infty} \sup_{0 \leq y \leq s} E\left\| a_j(y) - a_m(y) \right\|^2 = 0, \tag{43}
\]

which implies that \( \{a_m(s), m \geq 1\} \) is a Cauchy sequence. Then, we know from the Borel–Cantelli lemma that \( a_m(s) \to a(s) (n \to \infty) \) holds uniformly for \( s \in [-\tau, U] \). So, if we take the limits of both sides of equations (24), (25), and (26), we get that \( a(s) \) is a solution to (1) and

\[
\sup_{0 \leq y \leq s} E\left\| a(y) \right\|^2 \leq \infty, \quad 0 \leq s \leq U. \tag{44}
\]

Hence, the existence has been proved. The uniqueseness can be proved by the same method as Step 3, and then, the proof is completed. \( \square \)

### 4. Ulam–Hyers Stability Analysis of FSDEs

**Definition 4** (see [41]). For system (1), if there exists a constant \( \delta > 0 \) such that \( \forall \varepsilon > 0 \) and for each solution \( r(\cdot) \in C([0,U], \mathbb{R}^n) \) of the following inequality,

\[
\sup_{0 \leq s \leq U} E\left\| D_{0+}^\alpha r(s) - Br(s) - K(s,r(s),r(s-\tau)) \right\| \leq \varepsilon.
\]

\[ -N(s,r(s),r(s-\tau)) \frac{dw(s)}{ds} \leq \varepsilon. \tag{45} \]

There exists a solution \( a(\cdot) \in C([0,U], \mathbb{R}^n) \) of (1) with

\[
\sup_{0 \leq s \leq U} E\left\| r(s) - a(s) \right\| \leq \varepsilon \delta. \tag{46}
\]

Then, system (1) is called U-H stable.

**Remark 1** (see [41]). A function \( r(\cdot) \in ([0,U], \mathbb{R}^n) \) is a solution of (45) if there is a function \( z(\cdot) \in ([0,U], \mathbb{R}^n) \), such that

\[
\begin{align*}
(1) & \quad \sup_{0 \leq s \leq U} E\left\| z(s) \right\|^2 \leq \varepsilon \\
(2) & \quad D_{0+}^\alpha r(s) = Br(s) + k(s,r(s),r(s-\tau)) + N(s,r(s),r(s-\tau)) + z(s)
\end{align*}
\]

**Theorem 2.** If assumption 1 (i) (ii) hold and there exists a constant \( \beta > 0 \) such that \( \sup_{0 \leq s \leq U} c_1(\cdot) \leq \beta \), then the FSDEs (1) is U-H stable on \([0,U]\).

**Proof 2.** Taking Lemma 3 and Remark 1 into account, we find

\[
r(s) = b_0 M_0(s^\beta B) \\
+ \int_0^s (s-v)^{\lambda-1} M_{\lambda,1}(s-v)^B K(v,r(v),r(v-\tau)) dv \\
+ \int_0^s (s-v)^{\lambda-1} M_{\lambda,1}(s-v)^B N(v,r(v),r(v-\tau)) dw(v) \\
+ \int_0^s (s-v)^{\lambda-1} M_{\lambda,1}(s-v)^B z(v) dv. \tag{47}
\]
In view of Definition 3 and (47), we obtain

\[ r(s) - a(s) \]
\[ = \int_0^s (s - v)^{\lambda - 1} M_{\lambda, \lambda}((s - v)^3 B)[K(v, r(v), r(v - \tau)) - K(v, a(v), a(v - \tau))] dv \]
\[ + \int_0^s (s - v)^{\lambda - 1} M_{\lambda, \lambda}((s - v)^3 B)[N(v, r(v), r(v - \tau)) - N(v, a(v), a(v - \tau))] dw(v) \]
\[ + \int_0^s (s - v)^{\lambda - 1} M_{\lambda, \lambda}((s - v)^3 B)z(v) dv. \] \hspace{1cm} (48)

Then, using (28) Cauchy–Schwarz inequality and Lemma 2, we have

\[ \sup_{0 \leq y \leq s} \mathbb{E}[r(y) - a(y)] \leq 3 \sup_{0 \leq y \leq s} \mathbb{E}\| (y - v)^{\lambda - 1} M_{\lambda, \lambda}((y - v)^3 B)K(v, r(v), r(v - \tau)) - K(v, a(v), a(v - \tau)) \| dv \]
\[ + 3 \sup_{0 \leq y \leq s} \mathbb{E}\| (y - v)^{\lambda - 1} M_{\lambda, \lambda}((y - v)^3 B)N(v, r(v), r(v - \tau)) - N(v, a(v), a(v - \tau)) \| dw(v) \]
\[ + 3 \sup_{0 \leq y \leq s} \mathbb{E}\| (y - v)^{\lambda - 1} M_{\lambda, \lambda}((y - v)^3 B)z(v) \|^2 \] \hspace{1cm} (49)
\[ \leq 3d_1 \frac{U^{2\lambda - 1}}{\lambda - 1} \int_0^s \mathbb{E}\|K(v, r(v), r(v - \tau)) - K(v, a(v), a(v - \tau))\|^2 dv \]
\[ + 3d_1 \frac{U^{2\lambda - 1}}{\lambda - 1} \int_0^s \mathbb{E}\|N(v, r(v), r(v - \tau)) - N(v, a(v), a(v - \tau))\|^2 dv \]
\[ + 3d_2 \frac{U^{2\lambda}}{2\lambda - 1} \varepsilon. \]

By assumption 1 (ii) and Lemma 4, we obtain

\[ \sup_{0 \leq y \leq s} \mathbb{E}[r(y) - a(y)]^2 \]
\[ \leq 6d_1 \frac{U^{2\lambda - 1}}{2\lambda - 1} \int_0^s \mathbb{E}H(v,(||r(v) - a(v)||^2 + ||r(v - \tau) - a(v - \tau)||^2)) dv \]
\[ + 3d_2 \frac{U^{2\lambda}}{2\lambda - 1} \varepsilon \]
\[ \leq 6d_2 \frac{U^{2\lambda - 1}}{2\lambda - 1} \int_0^s \mathbb{E}[c_1(v) + c_2(v)||r(v) - a(v)||^2 + c_2(v)||r(v - \tau) - a(v - \tau)||^2] dv \]
where

\[
\begin{align*}
\sup_0 & \quad \bigg[ c_1(v) + 2c_2(v) \sup_{0 \leq v \leq T} \| r(v_1) - a(v_1) \|^2 + c_2(v) \sup_{-r \leq v \leq 0} \| r(v_2) - a(v_2) \|^2 
\end{align*}
\]

+ \sup_0 2 \leq \sup_{0 \leq v \leq T} \| r(v_1) - a(v_1) \|^2 
\end{align*}
\]

Hence, employing Gronwall’s inequality, we may obtain

\[
\begin{align*}
\sup_0 E \| r(y) - a(y) \|^2 
\leq & \quad (2\beta + 1) 3\sigma_2 U^{2\lambda} / 2\lambda - 1 \exp \left( \int_0^T \sup_{0 \leq v \leq T} c_2(v) dv \right) 
\leq & \quad (2\beta + 1) 3\sigma_2 U^{2\lambda} / 2\lambda - 1 \exp \left( \frac{12\sigma_2 U^{2\lambda}}{2\lambda - 1} \sup_{0 \leq v \leq T} c_2(v) \right) 
\leq & \quad \epsilon \delta,
\end{align*}
\]

where \( \delta = (2\beta + 1) 3\sigma_2 U^{2\lambda} / 2\lambda - 1 \exp \left( \frac{12\sigma_2 U^{2\lambda}}{2\lambda - 1} \sup_{0 \leq v \leq T} c_2(v) \right) \) is a positive constant. Therefore, this theorem is proved.

5. Example

Example 1. Consider the U-H stability of the following system:

\[
\begin{align*}
\begin{cases}
\frac{3}{\pi^2} a(s) = 1 + \frac{1}{\pi} a(s) + \frac{1}{7} \cos a(s) + a(s - 1) + \frac{1}{\pi} (a(s) + a(s - 1)) \frac{\text{d}w(s)}{\text{d}s}, & s \in [0, 8], \\
a(s) = 1, & s \in [-1, 0],
\end{cases}
\end{align*}
\]
where \( B = 1/\pi^4 \), and measurable functions are

\[
K(s, a(s), a(s - \tau)) = \frac{1}{7} \cos \sqrt{a(s)} + a(s - 1),
\]

\[
N(s, a(s), a(s - \tau)) = \frac{1}{e^4} (a(s) + a(s - 1))^2.
\]

We have

\[
\|K(s, r(s), r(s - 1)) - K(s, a(s), a(s - 1))\|^2
\]

\[
= \frac{1}{7} \| \cos \sqrt{r(s)} + r(s - 1) - \cos \sqrt{a(s)} + a(s - 1) \|^2
\]

\[
\leq \frac{1}{7} \| \sqrt{r(s)} + r(s - 1) - \sqrt{a(s)} + a(s - 1) \|^2
\]

\[
= \frac{1}{7} \left\| \frac{r(s) - a(s) + r(s - 1) - a(s - 1)}{\sqrt{r(s)} + r(s - 1) + \sqrt{a(s)} + a(s - 1)} \right\|^2
\]

\[
\leq \frac{2}{7} \left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right)
\]

\[
= H\left( s, \left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right) \right)
\]

\[
= 0 + \tilde{k}\left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right)
\]

\[
= c_1(s) + c_2(s) \left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right),
\]

\[
\|N(s, r(s), r(s - 1)) - N(s, a(s), a(s - 1))\|^2
\]

\[
= \frac{1}{e^4} \left( \| r(s) + r(s - 1) \|^2 - (a(s) + a(s - 1))^2 \right)
\]

\[
= \frac{1}{e^4} \left( \| r(s) + r(s - 1) + a(s) + a(s - 1) \|^2 - (r(s) - a(s) + r(s - 1) - a(s - 1))^2 \right)
\]

\[
\leq \frac{2}{e^4} \max_{0 \leq s \leq 8} \left( \| r(s) + r(s - 1) + a(s) + a(s - 1) \|^2 \left( \| r(s) - a(s) \|^2 + \| r(s - 1) - a(s - 1) \|^2 \right) \right)
\]

\[
\leq \tilde{k}\left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right)
\]

\[
= H\left( s, \left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right) \right)
\]

\[
= 0 + \tilde{k}\left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right)
\]

\[
= c_1(s) + c_2(s) \left( \|r(s) - a(s)\|^2 + \|r(s - 1) - a(s - 1)\|^2 \right),
\]
where

\[
\tilde{k} = \max \left\{ \frac{2}{7 \min_{0 \leq s \leq 8} \| r(s) + r(s-1) + \sqrt{a(s) - a(s-1)} \|_2}, \frac{2}{e} \max_{0 \leq s \leq 8} \| r(s) + r(s-1) + a(s) + a(s-1) \|_2 \right\}. \tag{55}
\]

\( c_1(s) = 0 \) and \( c_3(s) = \tilde{k} \). Therefore, \( K \) and \( N \) satisfy assumption 1 (i) and (ii) and we can find constant \( \forall \beta > 0 \) such that \( \sup_{0 \leq s \leq 8} c_1(s) = 0 \leq \beta \varepsilon \), where \( \varepsilon = 0.09 \). According to Theorem 2, equation (52) is stable in U-H sense on \([0, 8]\).

We can see from Figure 1 that the distance between \( a(s) \) and \( r(s) \) is less than a constant, and according to Definition 4, the solution of (52) is U-H stable.

### 6. Conclusion

The original intention of this study is to provide some more generalized results for the study of stability to FSDEs. We mainly prove the U-H stability, existence, and uniqueness of solutions of system (1). To achieve this goal, we utilize the technique of the Caratheodory approximation method, Fatou’s Lemma, Itô’s isometry, Cauchy–Schwarz inequality, and Gronwall’s inequality. Finally, the theoretical results are verified by an example with numerical simulation.

### Data Availability

No data were used to support the findings of the study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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