Abstract

We generalize the Stirling numbers of the first kind $s(a, k)$ to the case where $a$ may be an arbitrary real number. In particular, we study the case in which $a$ is an integer. There, we discover new combinatorial properties held by the classical Stirling numbers, and analogous properties held by the Stirling numbers $s(n, k)$ with $n$ a negative integer.

Généralisation des nombres de Stirling

On généralise ici les nombres de Stirling du premier ordre $s(a, k)$ au cas où $a$ est un réel quelconque. On s’intéresse en particulier au cas où $a$ est entier. Ceci permet de mettre en évidence de nouvelles propriétés combinatoires auxquelles obéissent les nombres de Stirling usuels et des propriétés analogues auxquelles obéissent les nombres de Stirling $s(n, k)$ où $n$ est un entier négatif.

Dedicated to
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1 Introduction

The Stirling numbers are some of the most important combinatorial constants known. It is the hope of this paper, to generalize the Stirling numbers of the first kind $s(n, k)$ to the case where $n$ need not be a nonnegative integer. This leads to equation (8) which points out the link between the two types of Stirling numbers.

When $n$ is an integer, we discover many new combinatorial properties held by the classical Stirling numbers, and analogous properties held by the Stirling numbers $s(n, k)$ with $n$ a negative integer. However, we defer some of their interesting combinatorial properties to another paper [6].

Finally, one should refer to [4] for one of the most important applications of these new constants: the calculation of the harmonic logarithms which form a basis for the Iterated Logarithmic Algebra.
1.1 The Lower Factorial

Recall that for \( n \) a nonnegative integer, one defined the lower factorial function \((x)_n\) to be the product
\[
(x)_n = x(x-1) \cdots (x-n+1).
\]
Similarly, for \( n \) a negative integer, we can define
\[
(x)_n = \prod_{i=k}^{-1} (y-i)^{-1} = 1/(y+1)(y+2) \cdots (y-k)
\]
Equivalently, \((y)_k\) can be defined recursively by the requirements:
\[
(y)_0 = 1 \quad \text{for all } k.
\]
We make a brief digression on notation in order to prevent any possible confusion. We use the symbol \((x)_n\) to denote the “falling powers” \( y^n = y(y-1) \cdots (y-n+1) \); however, many researchers—for example, Askey and Henrici—reserve this notation for the “rising powers” \( y^\overleftarrow{n} = y(y+1) \cdots (y+n-1) \). Actually, as Knuth has pointed out, Pochhammer who devised this notation did not intend either of these definitions; he used \((y)_n\) to denote \( y(y-1) \cdots (y-n+1)/n! \).

End of Digression.

For \( a \) a real number, we could define \((y)_a\) in terms of the Gamma function.

DEFINITION 1.1 (Lower Factorial) Let \( a \) be a real number, and let \( y \) be a formal variable. The define the lower factorial by
\[
(y)_a = \frac{\Gamma(y+1)}{\Gamma(y-a+1)}
\]
where \( \Gamma(y) \) denotes the formal power series expansion in the variable \( y \) of the Gamma function.

Note that for all real numbers \( a \),
\[
(y)_a = (y-a+1)(y)_{a-1}. \tag{1}
\]
We digress to indicate how one could proceed more formally. Require \( [1] \) for all real numbers \( a \) that
\[
\sum_{i \in \mathbb{Z}} \frac{(y)_{a+i}}{[a+i]!} t^{a+i} = \sum_{i \in \mathbb{Z}} \frac{y^{a+i}}{[a+i]!} \log(1+t)^{a+i}
\]
where $y$ and $t$ are variables. It then follows (see [1] or [12]) that

$$(y)_{n+i} = y^{n+i}(1 + A)$$

where

$$A = \sum_{k \geq 1} \sum_{j=1}^{k} (-1)^{j} \binom{a + i + j - 1}{j} \binom{a + i + k}{k - j} S(j + k, j)$$

and the $S(j + k, j)$ are the Stirling numbers of the second kind. Note that $A$ is not a Laurent series so the inner summation does not give the Stirling numbers of the first kind. \textit{End of Digression.}

1.2 The Stirling Numbers of the First Kind

We generalize the classical definition of the Stirling numbers of the first kind, and derive some remarkable identities satisfied by them.

\textbf{Definition 1.2} (Stirling Numbers of the First Kind) For all real numbers $a$ and for all nonnegative integers $k$, we define the Stirling number of the first kind $s(a, k)$ of degree $a$ and order $k$ to be the coefficient $[y^k](y)_a$ in the Taylor expansion of the lower factorial. Thus,

$$(y)_a = \sum_{k \geq 0} s(a, k) y^k.$$ 

Note that for $a$ a positive integer, this corresponds to the usual definition of Stirling numbers of the first kind.

\textbf{Example 1.1} 

1. For $n$ a nonnegative integer, $s(n, k)$ is the usual Stirling number of the first kind. That is, $(-1)^{n+k} s(n, k)$ is the number of permutations of $n$ elements which is the product of $k$ disjoint cycles.

2. For all nonnegative integers $k$, $s(0, k) = \delta_{0,k}$.

3. $s(a, 0) = 1$ except when $a$ is a positive integer in which case $s(a, 0) = 0$.

4. See Table [7]

\textbf{Theorem 1.3} For all $a$ and for all positive integers $k$, 

$$s(a + 1, k) = s(a, k - 1) - as(a, k).$$

\textit{Proof:} Equation (1).
Table 1: Stirling Numbers of the First Kind, \( s(n, k) \)

| \( k \backslash n \) | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---------------|----|----|----|----|----|---|---|---|---|---|---|---|
| 0             | 1  | 1  | 1  | 1  | 1  | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1             | 1  | -2 | -3 | -4 | -5 | 1 | 0 | -1 | 2 | -6 | 24 | -120 |
| 2             | 1  | -2 | -2 | -2 | -1 | 0 | 0 | 1 | -3 | 11 | -50 | 274 |
| 3             | 1  | -2 | -5 | -7 | -9 | 0 | 0 | 0 | 1 | -6 | 35 | -225 |
| 4             | 1  | -2 | -7 | -10| -13| 0 | 0 | 0 | 0 | 1 | -10| 85  |
| 5             | 1  | -2 | -10| -14| -18| 0 | 0 | 0 | 0 | 0 | 1 | -15 |
| 6             | 1  | -2 | -13| -20| -26| 0 | 0 | 0 | 0 | 0 | 0 | 1  |

2 Properties of the Stirling Numbers of the First Kind

Nonpositive Degree

In this section, we derive several identities which hold for Stirling numbers of the first kind with degrees which are nonpositive integers.

Recall that a linear partition \( \rho \) is a nonincreasing infinite sequence, \((\rho_i)_{i \geq 1}\), of nonnegative integers which is eventually zero. For example, \( \rho = (17, 2, 2, 1, 0, \ldots) \) is a linear partition. Each nonzero \( \rho_i \) is called a part \( \rho \).

A linear partition \( \rho \) is said to be a partition of \( n \) if the sum of its parts is \( n \), and we write \( \rho \vdash n \). The product of the parts of \( \rho \) is denoted \( \pi(\rho) \).

The set of all linear partitions is denoted by \( \mathcal{P} \). A linear partition is said to have distinct parts if its multiset of parts is, in fact, a set. The set of all linear partitions with distinct parts is denoted by \( \mathcal{P}^* \).

The zero sequence is a linear partition. It has no parts. Thus, it is a partition of zero with distinct parts, and the product of its parts is one.

As promised, Proposition 2.1 and its corollaries give enumerative interpretations of the Stirling numbers of nonnegative degree.

**Proposition 2.1 (Harmonic Relation)** For any nonnegative integers \( n \) and \( k \), the Stirling number of degree \( n \) and order \( k \) is given by the sum

\[
s(-n, k) = \frac{(-1)^k}{n!} \sum_{\substack{\rho \in \mathcal{P} \\ \ell(\rho) = k \\ \rho_i \leq n}} \pi(\rho)^{-1}.
\]

over all linear partitions \( \rho \) with \( k \) parts and with no part greater than \( n \).
We offer two proofs. First,

**Proof 1:** Let \( n \) and \( k \) be as above, and define

\[
d^{(k)}_n = \sum_{\rho \in P \atop \ell(\rho) = k \atop \rho_1 \leq n} \pi(\rho)^{-1}.
\]

By Theorem 1.3, it suffices to verify the recursion

\[
nd^{(k)}_n - d^{(k-1)}_n = nd^{(k)}_{n-1}
\]  
(3)

for \( n \) and \( k \) positive, since the boundary conditions are easy to verify.

Consider the following series of equalities.

\[
nd^{(k)}_n - d^{(k-1)}_n = \left(n \sum_{\mu \in P \atop \ell(\mu) = k \atop \mu_1 \leq n} \pi(\mu)^{-1}\right) - \left(n \sum_{\mu \in P \atop \ell(\mu) = k-1 \atop \mu_1 \leq n} \pi(\mu)^{-1}\right)
\]

\[
= \left(n \sum_{\mu \in P \atop \ell(\mu) = k \atop \mu_1 \leq n} \pi(\mu)^{-1}\right) - \left(n \sum_{\mu \in P \atop \ell(\mu) = k \atop \mu_1 = n} \pi(\mu)^{-1}\right)
\]

\[
= n \sum_{\mu \in P \atop \ell(\mu) = k \atop \mu_1 < n} \pi(\mu)^{-1}
\]

Thus, equation (3) holds.

Alternately, we could adopt the following proof due to Y. C. Chen.

**Proof 2:** For \( n \) positive,

\[
S(-n, k) = [y^k](y)_n = \frac{1}{n!} \frac{y^k}{(1 + y)(1 + y/2) \cdots (1 + y/n)}
\]
Properties of the Stirling Numbers of the First Kind

\[
\begin{align*}
\left(\frac{-1}{n!}\right)^k &\left[ y^k \right] \left( \sum_{\rho_1 \geq 0} y^{\rho_1} \right) \left( \sum_{\rho_2 \geq 0} (y/2)^{\rho_2} \right) \cdots \left( \sum_{\rho_n \geq 0} (y/n)^{\rho_n} \right) \\
= &\left(\frac{-1}{n!}\right)^k \sum_{\rho \in P} \pi(\rho)^{-1}. \quad \square
\end{align*}
\]

From the second proof, we have the following Porism involving the complete symmetric function \( h_n(x_1, x_2, \ldots) \). It is defined explicitly by the sum

\[
h_n(x_1, x_2, \ldots) = \sum_{1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n} \prod_{k=1}^{n} x_{\alpha_k}, \quad (4)
\]

and it is defined implicitly by the generating function

\[
\prod_{n \geq 1} (1 - x_n y)^{-1} = \sum_{n \geq 0} h_n(x_1, x_2, \ldots) y^n.
\]

**Porism 2.2** Let \( n \) and \( k \) be nonnegative integers. Then

\[ s(-n, k) = h_k(-1, -1/2, \ldots, -1/n)/n!. \quad \square \]

Thus, we see from the Harmonic Relation (Proposition 2.1) that the Stirling numbers are simply related to the partial sums of the harmonic series. For \( k = 1 \), Proposition 2.1 yields a sum over partitions of length one with no part greater than \( n \). There are \( n \) such partitions; they are the integers from 1 to \( n \). Thus, the sum is the sum of the reciprocals of the first \( n \) integers, so that

\[
n! s(-n, 1) = -\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) \quad (5)
\]

For the Stirling numbers of order 2, we obtain similarly:

\[
n! s(-n, 2) = 1 + \frac{1}{2} \left(1 + \frac{1}{2}\right) + \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right), \quad (6)
\]

and for order 3, we obtain

\[
-n! s(-n, 2) = 1 + \frac{1}{2} \left[1 + \frac{1}{2} \left(1 + \frac{1}{2}\right)\right] + \frac{1}{3} \left[1 + \frac{1}{2} \left(1 + \frac{1}{2}\right) + \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3}\right)\right] + \cdots + \frac{1}{n} \left[1 + \frac{1}{2} \left(1 + \frac{1}{2}\right) + \frac{1}{3} \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right].
\]
Proposition 2.3 turns out to be very useful in the calculation of Stirling numbers of the first kind.

**Proposition 2.3 (Knuth)** Let $n$ and $k$ be nonnegative integers (not both zero). Then $s(n,k)$ is given by the following finite sum:

$$s(-n, k) = \frac{(-1)^{k+1}}{n!} \sum_{m=1}^{n} \binom{n}{m} (-1)^m m^{-k}.\quad (7)$$

**Proof:** By consideration of the examples above, the Proposition holds for $n = 0$ and $k = 0$. Now, by induction we need only show that equation (3) holds for the summation on the right side of equation (7).

$$\sum_{m \geq 1} (-1)^{m-1} \binom{n}{m} (-1)^m m^{-k} + \frac{1}{n} \sum_{m \geq 1} \binom{n}{m} (-1)^m m^{1-k}$$

$$= \sum_{m \geq 1} (-1)^m m^{-k} \left( \binom{n-1}{m} + m \binom{n}{m} \right)$$

$$= \sum_{m \geq 1} (-1)^m m^{-k} \left( \binom{n-1}{m} + \binom{n-1}{m-1} \right)$$

$$= \sum_{m \geq 1} (-1)^m m^{-k} \binom{n-1}{m}.$$

We would like to generalize to the case where $n$ need not be a nonnegative integer; however, this is impossible since in this case the sum is not only infinite but divergent.

Note that Proposition 2.3 is the analog of the following classical result involving Stirling numbers of the second kind $S(k, n)$.

$$S(k, n) = \frac{(-1)^n}{n!} \sum_{m=1}^{k} \binom{n}{m} (-1)^m m^k.$$

Thus, in some sense we can say that

$$S(k, n) = (-1)^n s(-n, -k). \quad (8)$$

Proposition 2.3 has several more corollaries:

**Corollary 2.4** For nonnegative integers $n$ and $k$, the Stirling number of degree $n$ and order $k$ is given by the sum

$$s(-n, k) = \frac{(-1)^{kn}}{(n-1)!} \sum_{\rho \in \mathcal{P}} \pi(\rho)^{-1}$$

over all linear partitions $\rho$ with $k + 1$ parts and whose largest part is $n$. 
Proof: Add the part $n$ to each partition $\rho$ being summed over in Corollary 2.1.

**Corollary 2.5** Let $n$ and $k$ be nonnegative integers. Then the Stirling number of the first kind of degree $-n$ and order $k$ is given by the sums

$$s(-n, k) = \frac{(-1)^k}{n!} \sum_{M \subseteq \{1, 2, \ldots, n\}} \left( \prod_{m \in M} \frac{1}{m^{-1}} \right)$$

over multisets $M$ where all of the products are computed with the proper multiplicities.

Proof: Every linear partition is associated with a unique multiset of positive numbers called its parts. In the identities from Proposition 2.1 and Corollary 2.4, sum over these multisets instead of the partitions themselves.

**Corollary 2.6** Let $n$ and $k$ be nonnegative integers. Then the Stirling number of the first kind of degree $-n$ and order $k$ is given by the sum

$$s(-n, k) = \frac{(-1)^k}{n!} \sum \left( \prod_{i=1}^{n} i^{-m_i} \right)$$

over all sequences $(m_i)_{i=1}^{n}$ of $n$ nonnegative integers which sum to $k$.

Proof: Every linear partition is determined by the number $m_i$ of times each integer $i$ occurs as a part. Hence, we can sum over sequences of nonnegative integers $m_i$.

We note that

$$\lim_{k \to +\infty} s(-n, k) = \frac{(-1)^k}{(n - 1)!}$$

for all nonnegative $n$. 
Positive Degree

Now, we develop the classical Stirling numbers (those of positive degree) in a similar vein.

Recall that the trivial partition has no parts, and therefore the product of its parts is one; however, there are no partitions with $-1$ parts.

**Proposition 2.7 (Harmonic Relation)** Let $k$ be a nonnegative integer, and let $n$ be a positive integer. Then the Stirling number of the first kind of degree $n$ and order $k$ is given by the sums

$$s(n, k) = (-1)^{n+k}(n-1)! \sum_{\mu \in \mathcal{P}^* \atop \ell(\mu) = k-1 \atop \mu_1 < n} \pi(\mu)^{-1}$$

$$= (-1)^{n+k} \sum_{\mu \in \mathcal{P}^* \atop \ell(\mu) = n-k \atop \mu_1 < n} \pi(\mu),$$

over all linear partitions $\mu$ with $k-1$ parts all of which are distinct and less than $n$.

**Proof:** Note that

$$s(n, k) = [x^k](x)_n$$

$$= [x^k] \prod_{i=0}^{n-1} (x-i)$$

$$= [x^k] \sum_{\mu \in \mathcal{P}^* \atop \mu_1 < n} (-1)^{\ell(\mu)} x^{n-\ell(\mu)} \pi(\mu)$$

$$= [x^k] \sum_{\mu \in \mathcal{P}^* \atop \mu_1 < n} \frac{(n-\ell(\mu))!}{(n-\ell(\mu)-k)!} (-1)^{\ell(\mu)} x^{n-\ell(\mu)-k} \pi(\mu)$$

$$= \sum_{\mu \in \mathcal{P}^* \atop \mu_1 < n \atop \ell(\mu) = n-k} k! (-1)^{n-k} \pi(\mu)$$

$$= (-1)^{n+k}(n-1)! \sum_{\nu \in \mathcal{P}^* \atop \nu_1 < n \atop \ell(\nu) = k-1} \pi(\nu)^{-1}.$$

Note also that $s(n, k) = 0$ if $k > n > 0$ or if $k = 0$ and $n > 0$, since there is no partition with $k$ distinct parts all less than $k$, or with $-1$ parts.
By way of example, let us consider the extreme cases. If \( k = n > 0 \), then we must have \( k - 1 \) distinct parts less than \( k \). There is only one way to do this; we must use the partition consisting of the integers from 1 through \( k - 1 \). Thus, \( s(k, k) = 1 \).

Conversely, for \( k = 1 \) and \( n > 0 \), we sum over partitions with no parts. The trivial partition is the only such partition, so \( s(n, 1) = (-1)^{n-1}(n-1)! \).

The Stirling numbers of positive degree and order 2 are related to the partial sums of the harmonic series

\[
s(n, 2) = (-1)^n(n-1)! \left( \frac{1}{2} + \cdots + \frac{1}{n-1} \right).
\]

(10)

Again, as happened for Stirling numbers of nonnegative degree, higher orders correspond to generalizations of the harmonic series.

Some useful equivalent formulations of Proposition 2.7 follow:

**Corollary 2.8** Let \( k \) be a nonnegative integer, and let \( n \) be a positive integer. Then

\[
s(n, k) = (-1)^{n+k}n! - n \sum_{\mu \in P^*, \ell(\mu) = k, \mu_1 = n} \pi(\mu)^{-1}.
\]

*Proof:* Add the part \( n \) to each partition \( \mu \) being summed over in Proposition 2.7.

**Corollary 2.9** Let \( k \) be a nonnegative integer, and let \( n \) be a positive integer. Then

\[
s(n, k) = (-1)^{n+k}(n-1)! \sum_{S \subseteq \{1,2,\ldots,n-1\}, |S| = k-1} \left( \prod_{s \in S} s^{-1} \right) = \sum_{S \subseteq \{1,2,\ldots,n-1\}, |S| = n-k} \left( \prod_{s \in S} (-s) \right)
\]

where the sums range over sets \( S \).

*Proof:* These identities can be obtained from Proposition 2.7 and Corollary 2.8 by summing over the set of parts of \( \mu \) instead of \( \mu \) itself.
**Corollary 2.10** For all nonnegative integers $k$ and all positive integers $n$, the Stirling number of the first kind of degree $n$ and order $k$ is given by the sum

$$s(n, k) = (-1)^{n+k} n! \sum_{\mu \vdash n, \ell(\mu) = k} (\pi(\mu)^{-1}) \left( \prod_{j=1}^{n} m_i(\mu)! \right)^{-1}$$

over partitions $\rho$ of the number $n$ into exactly $k$ parts, and $m_i(\rho)$ denote the number of times $i$ occurs as part of $\rho$.

**Proof**: $|s(n, k)|$ is the number of permutations of $n$ letters with $k$ cycles. The number of permutations of cycle type $\rho$ is $n! \left( \prod_{i \geq 1} i^{m_i} m_i! \right)^{-1}$ where $m_i$ is the number of parts of $\rho$ equal to $i$. □

In contrast to equation (9),

**Proposition 2.11** For $n$ positive,

$$\sum_{k=1}^{n} (-1)^{k} s(n, k) = (-1)^{n+1} / (n - 1)!.$$  \hspace{1cm} (11)

Furthermore,

$$\sum_{k \geq 0} (-1)^{k} s(n, k) = (-1)^{n+1} / (n - 1)!.$$  \hspace{1cm} (12)

**Proof 1**: It suffices to demonstrate equation (12), since $s(n, k) = 0$ if $k = 0$ or if $k > n$. Let us expand the left hand side of the equation (12) as follows:

$$\sum_{k \geq 0} (-1)^{k} s(n, k) = (-1)^{n} \sum_{k \geq 0} |s(-n, k)|$$

$$= (-1)^{n} \sum_{k \geq 0} |\{ \pi \in S_{-n} : \pi \text{ has } k \text{ cycles}\}|$$

$$= (-1)^{n} n!.$$ □

Alternately, we have the following proof.

**Proof 2**: As mentioned above, it suffices to verify equation (12).

$$\sum_{k \geq 0} s(n, k)x^k = (y)_n$$

$$\sum_{k \geq 0} s(n, k)(-1)^k = (-1)(-1 - 1) \cdots (-1 - n + 1)$$

$$= (-1)^{n} n!.$$ □
Recall the definition of the elementary symmetric function \( e_n(x_1, x_2, \ldots) \). It is defined explicitly by the sum

\[
e_n(x_1, x_2, \ldots) = \sum_{0<\alpha_1<\alpha_2<\cdots<\alpha_n} \prod_{k=1}^{n} x_{\alpha_k}
\]

all linear partitions \( \mu \) with distinct parts, and it is defined implicitly by the generating function

\[
\prod_{n \geq 1} (1 + x_n y) = \sum_{n \geq 0} e_n(x_1, x_2, \ldots) y^n.
\]

**Proposition 2.12** For \( k \) nonnegative and \( n \) positive,

\[
s(n, k) = (-1)^n(n-1)!(e_{k-1}(-1, -1/2, \ldots, -1/n) = e_{n-k}(-1, -2, -3, \ldots, -n+1).
\]

We digress to discuss the implications of Porism 2.2 and Proposition 2.12. The Stirling numbers of the first type with nonnegative degree are merely examples of the complete elementary symmetric function, and those with negative degree are merely examples of the elementary elementary symmetric function. This duality is not too surprising in light of [10] which interprets the elementary symmetric function as an extension of the complete symmetric function to a “negative” number of variables. In fact, if we adopt the notation of [6], then we deduce

\[
s(n, k) = \lim_{\epsilon \to 0} \frac{h_k\left(-\frac{1}{1+\epsilon}, -\frac{1}{2+\epsilon}, \ldots, -\frac{1}{n+\epsilon}\right)}{\Gamma(n+1+\epsilon)}
\]

for all integers \( n \) and nonnegative integers \( k \). End of Digression.
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