New mass limit for white dwarfs: super-Chandrasekhar type Ia supernova as a new standard candle

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Type Ia supernovae, sparked off by exploding white dwarfs of mass close to Chandrasekhar limit, play the key role to understand the expansion rate of universe. However, recent observations of several peculiar type Ia supernovae argue for its progenitor mass to be significantly super-Chandrasekhar. We show that strongly magnetized white dwarfs not only can violate the Chandrasekhar mass limit significantly, but exhibit a different mass limit. We establish from foundational level that the generic mass limit of white dwarfs is 2.58 solar mass. This explains the origin of over-luminous peculiar type Ia supernovae. Our finding further argues for a possible second standard candle, which has many far reaching implications, including a possible reconsideration of the expansion history of the universe.

PACS: 97.60.Bw, 97.20.Rp, 98.80.Es, 97.10.Ld, 71.70.Di

Introduction. — Recently, some peculiar type Ia supernovae: e.g. SN 2006gz, SN 2007if, SN 2009dc, SN 2003fg, have been observed with exceptionally higher luminosities but lower kinetic energies. The kinetic energy is proportional to the difference between the obtained nuclear energy which arises from the synthesis of elements in the explosion through fusion and the binding energy of the white dwarf. Most of the light-curves of above mentioned peculiar supernovae appear over-luminous, slow-rising, which do not allow them to be calibrated as standard candles. This questions the use of all type Ia supernovae in measuring distances of far away regions and hence unraveling the expansion history of the universe. However, assuming the progenitor to be a highly super-Chandrasekhar mass white dwarf reproduces the low kinetic energies and thus velocities seen in the above supernovae. This is because a larger mass implies a larger binding energy of the star and hence a smaller velocity for the same and/or higher luminosity (due to nuclear fusions) than that observed in a standard type Ia supernova. The progenitor masses required to explain the above supernovae lie in the range 2.1 – 2.8\(M_\odot\), when \(M_\odot\) being the mass of sun, subject to the model chosen to estimate the nickel mass. Now naturally the following vital questions arise. Is there any fundamental basis behind the formation of a highly super-Chandrasekhar white dwarf? How to address the significant violation of Chandrasekhar mass limit? What is the ultimate mass limit of a white dwarf? Here we plan to address all the above issues by exploiting the effects of magnetic field in compact objects. This will lead, as we will show, to a natural explanation of the so called “peculiar” type Ia supernovae. This might eventually lead these supernovae to be considered as altogether new standard candles. This has many far reaching implications, including a possible reconsideration of the expansion history of the universe.

Before proceeding further, let us recall the physics of a white dwarf and its link to the type Ia supernova. When a star exhausts its nuclear fuel, it converts to either of the three compact objects: white dwarf, neutron star or black hole, depending on the initial mass of the evolving star. It is generally believed that the fate is a white dwarf when the mass of the initial star in its main sequence is \(M_\odot \leq 5M_\odot\). Such a main sequence star undergoing collapse leading to a small volume consists of a lot of electrons. Being in a small volume many such electrons tend to occupy same energy states, hence making them to degenerate electrons, as the energy of a particle depends on its momentum which is determined by the total volume of the system. However, being a fermion an electron obeys Pauli’s exclusion principle which says that no two fermions can occupy the same quantum state. Hence, once up to the Fermi level, which is the maximum allowed energy of a fermion, is filled by the electrons, there is no available space for the remaining electrons in a small volume of a collapsing star, which expels the electrons to move out leading to an outward pressure. In a white dwarf, the inward gravitational force is balanced by the force due to outward pressure created by to degenerate electrons. Chandrasekhar in one of his celebrated papers showed that the mass of a white dwarf cannot be more than 1.44\(M_\odot\) which sets the famous Chandrasekhar mass limit of a white dwarf.

If a white dwarf having mass close to the Chandrasekhar limit gains more mass (e.g. by accretion, when the mass is supplied by a companion star of the white dwarf), then its mass exceeds the Chandrasekhar limit, which leads to a gravitational force stronger than the outward force that arises due to the degenerate electrons. Hence, this leads to the contraction of the white dwarf and a subsequent increase of its temperature, which is favorable for the initiation of fusion reactions again. If the white dwarf mostly consists of carbon and oxygen, namely the carbon-oxygen white dwarf (which is commonly the case), then nuclear fusion of carbon (and oxygen) takes place. Subsequently, within a few seconds, a substantial fraction of the white dwarf matter undergoes a runaway reaction which releases huge
energy $\sim 10^{51}$ erg to unbind it in an explosion, namely type Ia supernova explosion. This eventually leads to a complete gravitational collapse of the star without leaving any remnant.

As all the commonly observed type Ia supernovae are produced by (almost) the same mechanism, namely the mass of the progenitor white dwarf exceeding Chandrasekhar limit and subsequent processes, the underlying variations of luminosity as functions of time, namely light-curves, appear alike for all the explosions. All of these supernovae exhibit consistent peak luminosity, the relation between the peak luminosity and width of the light-curve, due to the uniform mass of white dwarfs (Chandrasekhar limit) which finally explode, e.g., because of the accretion process. Very importantly, the apparent stability of this value helps the underlying supernovae to be used as standard candles in order to measure the distances to their host galaxies. Since these supernovae are exceptionally bright, they can be observed across huge cosmic distances. The variation of their brightness with distance (or redshift) is an extremely important tool for measuring various cosmological parameters, which in turn shed light on the expansion history of the Universe. Their enormous importance is self evident and was brought into the prime focus by the awarding of the Nobel Prize in Physics in 2011, for the discovery (made possible by the observation of distant type Ia supernovae) that the universe is undergoing an accelerated expansion.

Now we move on to our goal of establishing a new mass limit for super-Chandrasekhar white dwarfs, of whose formation there is no understanding from foundational level — a caveat behind the hypothesis of super-Chandrasekhar progenitor for the peculiar type Ia supernovae, raised by the earlier authors. In fact they emphasized on the pursuit of theoretical studies in order to assess the hypothesis. Although, based on a numerical code for stellar binary evolution, the rotating white dwarfs are suggested to hold mass up to $2.7M_\odot$, a foundational level calculation is missing and there is no estimate of mass limit of such a star either.

In this letter, we show that (highly) magnetized white dwarfs not only can have mass $\sim 2.6M_\odot$, but also exhibit its ultimate limit of mass. Hence, we propose a fundamentally new mass limit for white dwarfs, which eventually helps in explaining light-curves of peculiar type Ia supernovae. This may further lead to establishing these supernovae as modified standard candles for distance measurement.

The motivation behind our approach lies in the discovery of several isolated magnetized white dwarfs through the Sloan Digital Sky Survey (SDSS) with surface fields $10^6 - 10^7G$. Hence their expected central fields could be $2-3$ orders of magnitude higher. Moreover, about 25% of accreting white dwarfs, namely cataclysmic variables (CVs), are found to have high magnetic fields $10^7 - 10^8G$.

Equation of state.— As the starting choice is the magnetized white dwarf, we first recall degenerate electrons under the influence of magnetic field which are known to be Landau quantized. Larger the magnetic field, smaller is the number of Landau levels occupied (see supplement). Recent works establish that Landau quantization due to strong magnetic field modifies the equation of state (EoS) of the electron degenerate gas, which results in a significant modification of the the mass-radius relation of the underlying white dwarf. Interestingly, these white dwarfs are found to have super-Chandrasekhar masses. The main aim of this letter is to obtain the maximum possible mass of such a white dwarf (which is magnetized), and therefore a (new) mass limit. Hence we look for the regime of high density of electron degenerate gas and the corresponding EoS, which further corresponds to the high Fermi energy ($E_F$) of the system. This is because high density corresponds to high momentum, which implies high energy (see supplement). Note that the maximum Fermi energy ($E_{F,max}$) corresponds to the maximum central density of the star. Consequently, conservation of magnetic flux (technically speaking flux freezing theorem which is generally applicable for a compact star) argues for the maximum possible field of the system, which implies that only the ground Landau level will be occupied by the electrons. For the expressions of density, pressure and the EoS for such a highly magnetized system, see supplement. Hence, in the limit of $E_F >> m_e c^2$, when $m_e$ is the mass of the electrons and $c$ the speed of light, for a given magnetic field exhibiting the system of one Landau level, the EoS is

$$P = K_m \rho^2,$$

when $P$ and $\rho$ are respectively the pressure and density of the gas and the constant $K_m$ is given by

$$K_m = \frac{m_e c^2 \pi^2 \lambda^3}{(\mu_e m_H)^2 B_D},$$

where $\lambda = h/m_e c$, the Compton wavelength of electron, $h$ the Planck’s constant divided by $2\pi$, $\mu_e$ the mean molecular weight per electron, $m_H$ the mass of hydrogen atom and $B_D$ the magnetic field in the units of $4.14 \times 10^{13}$G.

Mass limit of white dwarfs.— Now following the Lane-Emden formalism, we obtain the mass of the magnetized white dwarf

$$M = 4\pi^2 \rho_c \left(\frac{K_m}{2\pi G}\right)^{3/2}.$$
and the corresponding radius

\[ R = \sqrt{\frac{\pi K_m}{2G}}, \quad (4) \]

when \( \rho_c \) is the central density of the white dwarf supplied as a boundary condition in addition to the condition that \( \frac{dp}{dr} = 0 \) at \( r = 0 \), when \( r \) is the radial distance from the center of the star such that at the surface \( r = R \), and \( G \) is the Newton’s gravitation constant. See supplementary information [18] for detailed calculations.

Now the expression of \( \rho_c \) for a one Landau level system in the limit \( E_F = E_{F_{\text{max}}} \gg m_e c^2 \) is given by (see supplement [18])

\[ \rho_c = \frac{\mu_e m_H}{\sqrt{2\pi^2} \lambda_c^3} B_D^{3/2}. \quad (5) \]

Substituting \( \rho_c \) from equation (3) in equation (4), we obtain the mass of the white dwarf, independent of \( \rho_c \) and \( B_D \), given by

\[ M = \left( \frac{\hbar c}{2G} \right)^{3/2} \left( \frac{1}{(\mu_e m_H)^2} \right) \approx 10.312 \frac{\mu_c^3}{\mu_e^2} M_\odot, \quad (6) \]

and from equations (2), (4) and (5) we obtain the radius

\[ R = \left( \frac{\pi^{2/3} \hbar c}{2^{7/3} (\mu_e m_H)^{4/3} G} \right)^{1/2} \rho_c^{-1/3} \rightarrow 0 \text{ as } \rho_c \rightarrow \infty, \quad (7) \]

which set the new limits for mass and radius. Note that the Chandrasekhar limit also corresponds to \( \rho \rightarrow \infty \) and \( R \rightarrow 0 \). For \( \mu_e = 2 \) which is the case of a carbon-oxygen white dwarf

\[ M \approx 2.58 M_\odot. \quad (8) \]

Interestingly, while high magnetic field introduces anisotropic effects into the star tending it to be an oblate spheroid, this does not affect the limiting mass as the corresponding radius tends to zero. However, for lighter white dwarfs with finite radii, super-Chandrasekhar mass would have been achieved at a lower field, if the star is appropriately set to be a spheroid rather than a sphere, as justified earlier [3, 20].

**Scaling behaviors of mass and radius of the white dwarf with its central density.** Now we provide general scaling laws for the mass and radius of the white dwarf describing their variations with its central density (as is known for non-magnetized white dwarfs proposed by Chandrasekhar) and magnetic field strength. By Lane-Emden formalism

\[ M \propto K^{3/2} \rho_c^{\frac{2-n}{2}}, \quad (9) \]

where \( K \) depends on \( B_D \) and \( \rho_c \) for a magnetized white dwarf (see supplement [18]). For the extremely high density regime of the white dwarf, the EoS reduces to a polytropic form \( P = K \rho^n \) (can be verified from supplement [18]), when \( \Gamma = 1 + 1/n \), is the polytropic index. In this regime, \( \Gamma = 2 \), consequently \( n = 1 \) and \( K = K_m \propto B_D^{-1} \propto \rho_c^{-2/3} \), as shown by equations (5), (2), (5). This finally renders \( M \) in equation (9) to be independent of \( \rho_c \), as already shown by equation (6). Similarly, the scaling of radius is obtained as

\[ R \propto K^{1/2} \rho_c^{\frac{1-n}{2}}, \quad (10) \]

which reveals \( R \propto \rho_c^{-1/3} \) for \( n = 1 \), as already shown by equation (7).

**Justification of high magnetic field in white dwarfs.** So far we have employed Landau quantization in search of a new mass limit giving rise to super-Chandrasekhar white dwarfs. However, the effect of Landau quantization becomes significant only at a high field \( \sim B_D \times 10^{13} G \) = \( B_{\text{cr}} \). How can we justify such a high field in a white dwarf?

Let us consider the commonly observed phenomenon of a magnetized white dwarf accreting mass from its companion. Now the surface field of an accreting white dwarf, as observed, could be \( \sim 10^9 G \ll B_{\text{cr}} \). Its central field however can be several orders of magnitude higher \( \sim 10^{12} G \), which is also less than \( B_{\text{cr}} \). Naturally, such a magnetized CV, commonly known as a polar, still lies on the mass radius relation obtained by Chandrasekhar. However, in contrast with Chandrasekhar’s work (which did not include magnetic field in the calculations), we will see that a non-zero initial field in the white dwarf, however ineffective for rendering Landau quantization effects, will prove to be crucial in supporting the additional mass accumulated due to accretion. As the magnetized white dwarf accretes mass, its total
mass increases which in turn increases the gravitational power and hence the white dwarf contracts in size due to the increased gravitational pull. However, the total magnetic flux is conserved in such a process and, hence, as a result of the above decrease in size of the star, the central (as well as surface) magnetic field also increases. Here we are interested in the evolution of the central field, since it is this field which is primarily responsible for rendering super-Chandrasekhar mass to the white dwarf, as justified in [3]. Since accretion is a continuous process, the deposition of matter on the surface of the white dwarf, followed by its contraction and subsequent increase of magnetic field, continues in a cycle. In such a process, eventually the central magnetic field could exceed $B_{\text{cr}}$. As a result, the EoS of the electron degenerate matter gets modified as shown in Figure 1. Hence, the inward gravitational force is balanced by the outward force due to this modified pressure and a quasi-equilibrium state is attained. In this way, a very high magnetic field is generated, which in turn prevents the white dwarf from collapsing, thus making it more massive. Subsequently, with the continuation of accretion the white dwarf approaches the new mass limit $\sim 2.58M_\odot$, as obtained above, which sparks off a violent thermonuclear reaction with further accretion, thus exploding it and giving rise to a super-Chandrasekhar type Ia supernova. The evolution of the mass-radius relation of a polar into that of a super-Chandrasekhar white dwarf of the maximum possible mass is shown in Figure 2, along with a few typical mass-radius relations for different magnetic field strengths describing possible stars in intermediate equilibrium states. The ultimate white dwarf, corresponding to the maximum mass $\sim 2.58M_\odot$, lies on the mass-radius relation for a one Landau level system, but the intermediate white dwarfs having weaker magnetic fields correspond to multilevel systems. The one Landau level system corresponds to the central magnetic field $8.8 \times 10^{17}$ G. The intermediate systems of 200-level and 50124-level correspond to central magnetic fields $4.4 \times 10^{15}$ G and $1.7 \times 10^{13}$ G respectively (see supplement [18] for relevant formula), when $E_{F_{\text{max}}} = 200m_ec^2$. This value of $E_{F_{\text{max}}}$ is found to produce the theoretical mass limit with good numerical accuracy. Note that one can in principle go up to higher $E_{F_{\text{max}}}$, which will lead to a further decrease in the radius of the white dwarf keeping the mass practically same. In order to construct this evolutionary track, we consider the values of $\rho_c$ corresponding to the density at the ground-to-first Landau level transition of the respective EoSs, since only this choice leads to the maximum possible mass. We emphasize here that the range of masses for the super-Chandrasekhar progenitors obtained from observations is not very strict. Hence, if the new mass limit obtained by us is taken into account, one could possibly do away with the mass distribution altogether. However, if the distribution is indeed real, then it could be attributed to the difference in accretion rates found in different CVs.

We thank A.R. Rao of TIFR for insightful suggestions. B.M. acknowledges partial support through research Grant No. ISRO/RES/2/367/10-11. U.D. thanks CSIR, India for financial support.

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FIG. 2: Mass-Radius relations — the pure solid line represents Chandrasekhar’s result and the one marked with filled circles represents the evolutionary track of the white dwarf with the increase of magnetic field. The dot-dashed, dotted and dashed lines represent the white dwarfs with 50124-level, 200-level and 1-level systems of Landau quantization respectively (corresponding to increasing magnetic fields). $E_{F_{\text{max}}} = 200m_e c^2$.

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Supplementary information for the letter

I. BASIC EQUATIONS FOR ONE LANDAU LEVEL SYSTEM OF DEGENERATE ELECTRONS

The Fermi energy level \( E_F \) of a Landau quantized system is given by
\[
E_F^2 = p_F(\nu)^2 c^2 + m_e^2 c^4 (1 + 2\nu B_D),
\]
where \( p_F(\nu) \) is the Fermi momentum of \( \nu \)th Landau level (\( \nu = 0, 1, 2, \ldots \)), \( m_e \) the mass of electrons, \( c \) the speed of light, \( B_D \) the magnetic field in units of \( 4.414 \times 10^{13} \)G. From the condition that \( p_F(\nu)^2 \geq 0 \), the maximum number of occupied Landau levels is given by
\[
\nu_m = \left( \frac{E_{F_{\text{max}}}}{m_e c^2} \right)^2 - 1.
\]

For one Landau level system, when only ground Landau level (\( \nu = 0 \)) is occupied, \( \nu_m = 1 \). Similarly, for two level system, when ground (\( \nu = 0 \)) and first (\( \nu = 1 \)) levels are occupied, \( \nu_m = 2 \), and so on. Hence, for one Landau level system, density \( \rho \) and pressure \( P \) of the electron degenerate gas are given by
\[
\rho = \frac{\mu_e m_H B_D}{2\pi^2 \lambda_e^3} \sqrt{\left( \frac{E_F}{m_e c^2} \right)^2 - 1} \frac{\mu_e m_H}{2\pi^2 m_e c^2 \lambda_e^3} B_D E_F \quad \text{for} \quad E_F >> m_e c^2, \tag{I.3}
\]
\[
P = \frac{B_D m_e c^2}{4\pi^2 \lambda_e^3} \left( \frac{E_F \sqrt{E_F^2 - m_e^2 c^4}}{(m_e c^2)^2} \ln \left( \frac{E_F + \sqrt{E_F^2 - m_e^2 c^4}}{m_e c^2} \right) \right), \tag{I.4}
\]
when \( \mu_e \) is the mean molecular weight, \( m_H \) the mass of proton, \( \lambda_e \) the Compton wavelength of electron. Now eliminating \( E_F \) from equations (I.3) and (I.4) for an arbitrary \( E_F \), we arrive at the equation of state (EoS) for a one level system given by
\[
P = \frac{m_e c^2}{2Q\mu_e m_H} \left( \rho \sqrt{Q^2 + \rho^2} - Q^2 \ln \left( \frac{\rho + \sqrt{Q^2 + \rho^2}}{Q} \right) \right), \tag{I.5}
\]
when \( Q = \mu_e m_H B_D/2\pi^2 \lambda_e^3 \).

Following previous work [2], we now approximate the above EoS and EoSs for any other \( \nu_m \) by a polytropic relation \( P = K\rho^\Gamma \) such that the polytropic index \( \Gamma = 1 + 1/n \) is piecewise constant in different density ranges and \( K \) being a dimensional constant. This will prove to be useful in order to understand scaling behaviors of mass and radius of the white dwarf with its central density.

Now in terms of \( Q \), equation (I.3) can be rewritten as
\[
\rho = Q \frac{E_F}{m_e c^2} \tag{I.6}
\]
In the limit \( E_F >> m_e c^2 \), we note that, \( \rho >> Q \). Thus in this limit, equation (I.5) reduces to
\[
P = \frac{m_e c^2}{2Q\mu_e m_H} \left( \rho^2 - Q^2 \ln \left( \frac{2\rho}{Q} \right) \right). \tag{I.7}
\]
The logarithmic term is much smaller than the first term in the above equation and hence by neglecting it we obtain
\[
P = \frac{m_e c^2}{2Q\mu_e m_H} \rho^2, \tag{I.8}
\]
which corresponds to the polytropic EoS with \( \Gamma = 2 \).
II. LANE-EMDEN EQUATION AND EXPRESSIONS FOR MASS AND RADIUS

The underlying white dwarf following above EoS obeys the magnetostatic equilibrium condition

\[
\frac{1}{\rho} \frac{d}{dr} \left( P + \frac{B^2}{8\pi} \right) = F_g + \frac{\vec{B} \cdot \nabla \vec{B}}{4\pi \rho},
\]

(II.1)

when \( r \) is the radial distance from the center of white dwarf, \( \vec{B} \) the magnetic field in \( G \), \( B^2 = \vec{B} \cdot \vec{B} \), \( F_g \) the gravitational force. This equation is supplemented by the estimate of mass \( (M) \) within any \( r \) given by

\[
\frac{dM}{dr} = 4\pi r^2 \rho
\]

(II.2)

approximating the star to be spherical. The white dwarf in the present context can be considered in the Newtonian framework and the magnetic field therein is nearly constant in the regime of smaller radii (as justified previously [3]). Moreover, at a very large density, as in the limiting case to be considered here, the star becomes so compact as if the magnetic field remains constant throughout. Hence, taking above facts into consideration and combining equations (II.1) and (II.2), the equilibrium condition may be read at any \( r \) as

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dP}{dr} \rho \right) = -4\pi G \rho,
\]

(II.3)

where \( G \) is the Newton’s gravitation constant. We now briefly recall the Lane-Emden formalism (see, e.g., [4]). Let us define

\[
\rho = \rho_c \theta^n,
\]

(II.4)

where \( \rho_c \) is the central density of the white dwarf and \( \theta \) is a dimensionless variable and

\[
r = a \xi,
\]

(II.5)

where \( \xi \) is another dimensionless variable and constant \( a \) carries the dimension of length defined as

\[
a = \left[ \frac{(n+1)K \rho_c^{\frac{n+1}{n}}}{4\pi G} \right]^{1/2}.
\]

(II.6)

Thus using equations (II.4), (II.5) and (II.6) along with the polytropic form of EoS, equation (II.3) reduces to

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n,
\]

(II.7)

which is the famous Lane-Emden equation. Equation (II.7) can be solved for a given \( n \), along with the boundary conditions

\[
\theta(\xi = 0) = 1
\]

(II.8)

and

\[
\left( \frac{d\theta}{d\xi} \right)_{\xi=0} = 0.
\]

(II.9)

Note that for \( n < 5 \), \( \theta \) becomes zero for a finite value of \( \xi \), say \( \xi_1 \), which basically corresponds to the surface of the white dwarf such that its radius

\[
R = a \xi_1.
\]

(II.10)

Also by combining equations (II.2), (II.4), (II.5) and (II.7) we obtain the mass of the white dwarf

\[
M = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n \, d\xi.
\]

(II.11)

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