Random matrices and Lyapunov coefficients regularity

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Abstract: Analyticity and other properties of the largest or smallest Lyapunov exponent of a product of real matrices with a “cone property” are studied as functions of the matrices entries, as long as they vary without destroying the cone property. The result is applied to stability directions, Lyapunov coefficients and Lyapunov exponents of a class of products of random matrices and of dynamical systems. The method is based on the classical theory of the Mayer series in Statistical Mechanics of rarefied gases.

1 Introduction and paradigm

Regularity of the Lyapunov exponents of products of random matrices has been studied thoroughly in [11]. The study dealt with various aspects and consequences of the following cone property (as it will be called here):

Definition 1: A sequence \( \{T_j\}_{j=-\infty}^{\infty} \) of \( d \times d \) real invertible matrices has the \((\Gamma, \Gamma')\)-cone property if there are proper, closed convex cones \( \Gamma, \Gamma' \subset \mathbb{R}^d \), with apex at the origin \( O \) and \( \Gamma' \subset \Gamma \), such that \( T_j \Gamma \subset \Gamma' \) and \( \Gamma'/O \) is contained in the interior \( \Gamma^0 \) of \( \Gamma \).

The proofs in [11] are based on the implicit function theorem and on the results in [2]. Here the aim is to obtain most of the results in [11] relative to the finite dimensional case, with a different and self contained technique. The “cone property” importance for studies beyond the Lyapunov exponents, like decay of correlations in smooth and non smooth dynamical systems, has been developed in [7]: the new techniques of the latter work (and in the many stemming out of it) are also quite different from the ones presented here.

To present the main idea, pursued here, imagine the \( T_n \) diagonalizable; but the results will cover the general case. Then \( T_n \) will be written:

\[
T_n = \sum_{\sigma=0}^{d-1} \lambda_{n,\sigma} |\sigma, n\rangle \langle \sigma, n|
\]  (1.1)

where \( \lambda_{n,\sigma} \) are the eigenvalues of \( T_n \) and the vectors \( |n, \sigma\rangle, \langle n, \sigma| \) are the corresponding right and left eigenvectors which will be supposed normalized.
to \( \langle n, \sigma | n, \sigma' \rangle \equiv \delta_{\sigma\sigma'} \). The eigenvalues will be labeled by decreasing modulus \(|\lambda_{n,0}| \geq |\lambda_{n,1}| \geq \ldots \geq |\lambda_{n,d-1}| \) and \( \lambda_{n,\sigma} \neq 0 \) (invertibility condition).

Remark that \( H_v(n,N) \overset{\text{def}}{=} T_n \cdots T_N v, \ v \in \Gamma' \), can be written as

\[
H_v(n,N) = \sum_{\sigma_n,\ldots,\sigma_N} |n,\sigma_n\rangle \cdot \left( \prod_{j=n}^N \lambda_{j,\sigma_j} \right) \prod_{j=n}^N \langle j,\sigma_j|j+1,\sigma_{j+1}\rangle \tag{1.2}
\]

with \(|N+1,\sigma_{N+1}\rangle \equiv v\) (here \(\sigma_{N+1}\) is just a label and not an index of summation, being used only for uniformity of notation).

The representation in Eq.(1.2) suggests an alternative approach to the analysis of such products of matrices, directly inspired by the methods of 1-dimensional statistical mechanics of spin systems.

To bring it to a more familiar form: consider intervals of integers \(J = [h,h'] = (h,h+1,\ldots,h')\) with \(1 \leq h \leq h' \leq N\) and, on them, spin configurations \(\sigma_J = (\sigma_h,\ldots,\sigma_{h'})\) with \(\sigma_j = 1,\ldots,d-1\) (i.e. \(\sigma_j \neq 0\), see Eq.(1.1)).

Call \(Y = (J,\sigma_J)\) a “polymer” with base \(J\) and structure \(\sigma_J\) (if \(|J| = \ell\); there are \(\ell^{d-1}\) polymers with base \(J\)): then Eq.(1.2) can be interpreted as an expectation value evaluated in an ensemble of polymers as follows.

A “configuration” of polymers in \([n,N]\) will be \(Y = (J_1,\sigma_{J_1},\ldots,J_s,\sigma_{J_s})\) with \(J_1 < J_2 < \ldots < J_s \subset [n,N]\), \(s > 0\): i.e. \(Y\) is a configuration of non overlapping polymers (“hard core polymers”). With each polymer \((J,\sigma_J)\) associate an “activity” \(I(J,\sigma_J)\) (\(I\) might even be complex).

A formal probability distribution (“ensemble”) on the \(Y\)’s is obtained by attributing a weight \(\zeta(Y) \overset{\text{def}}{=} \prod_{i=1}^s I(J_i,\sigma_{J_i})\) equal to the product of the activities; the empty configuration is given weight 1. The ensemble thus defined is formal as \(I(J,\sigma_J) \geq 0\) is not required.

After some meditation, it is recognized that Eq.(1.2), can be rewritten imagining the sites with \(\sigma_i = 0\) as “empty sites” in polymer configurations, and it can be cast in the form

\[
H_v(n,N) = K(n,N) \sum_{\sigma_n,\ldots,\sigma_N} \sum_{s \geq 0} \sum_{J_1 < \ldots < J_s} |n,\sigma_n\rangle \frac{\prod_{i=1}^s I(J_i,\sigma_{J_i})}{\Omega(n,N)} \tag{1.3}
\]
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where $\Lambda(n, N), \Omega(n, N)$ are normalization factors and

\[
I(J, \sigma) \overset{\text{def}}{=} \left( \prod_{j=h}^{h' - 1} \frac{\lambda_{j, \sigma_j}}{\lambda_{j, 0}} \right) \left( \prod_{j=h}^{h' - 1} \frac{\langle j, \sigma_j | j + 1, \sigma_{j+1} \rangle}{\langle j, 0 | j + 1, 0 \rangle} \right) \cdot \left( \frac{\langle h - 1, 0 | h, \sigma_h \rangle}{\langle h - 1, 0 | h, 0 \rangle} \right) \delta_{h > n} \left( \frac{\langle h', \sigma_h | h' + 1, 0 \rangle}{\langle h', 0 | h' + 1, 0 \rangle} \right)
\]

\[
\Omega_{vN}(n, N) \overset{\text{def}}{=} \sum_{s \geq 0} \sum_{J_1 < \ldots < J_s} \prod_{i=1}^{s} I(J_i, \sigma_{J_i})
\]

\[
\Lambda_{vN}(n, N) \overset{\text{def}}{=} \Omega_{vN}(n, N) \prod_{j=n}^{N} \left( \lambda_{j, 0} \langle j, 0 | j + 1, 0 \rangle \right)
\]

where $|N + 1, \sigma\rangle$ has to be interpreted, instead, as $v$, see Eq. (1.2), and, for $s > 0$, $J_1 < \ldots < J_s$ are consecutive intervals in $[n, N]$ not empty and disjoint; $s = 0$ contributes 1 to $\Omega(n, N)$.

Eq. (1.3) maps the problem of studying $H_v(n, N)$ into studying $\Lambda(n, N)$ times a formal average in what is known, in statistical mechanics, as a Fisher model, [1]. The latter is well known, since [1], as a machine for examples and counterexamples in statistical mechanics and in dynamical systems (for some applications see [3], [5, Def. D7.3.1]).

Here Eq. (1.3) will be the starting point to obtain, with an alternative method, the following theorems, special cases of results already in [11]:

**Theorem 1:** (1) Let $T_n$ be a sequence as in definition 1 (hence with the $(\Gamma, \Gamma')$-cone property) with $||T_n|| < B_0$ for some $B_0 > 0$. Then given any sequence $\{v_j\}_{j \geq 1}$ of unit vectors in $\Gamma'$, the limits

\[
h(n) = \lim_{N \to \infty} \frac{T_n T_{n+1} \cdots T_N v_N}{\Lambda_{vN}(n, N)} = \lim_{N \to \infty} \frac{H_{vN}(n, N)}{\Lambda_{vN}(n, N)}
\]

\[
\Lambda(n, p) = \lim_{N \to \infty} \frac{\Lambda_{vN}(p, N)}{\Lambda_{vN}(n, N)} > 0,
\]

exist $\forall 1 \leq n \leq p$ and are independent of the sequence $\{v_j\}_{j \geq 1}$.

(2) The limits are (real) analytic functions of the matrices entries, as long as their variations are small enough.

(3) The vectors $h(n)$ are “eigenvectors” for the product of the inverse matrices $T_j^{-1}$, i.e. there are “Lyapunov coefficients”, $\Lambda(n, p)$ such that:

\[
T_{p-1}^{-1} \cdots T_n^{-1} h(n) = \Lambda(n, p) h(p),
\]
(4) There is $B$ such that $B^{-1} \leq ||h(n)|| \leq B$ and the upper and lower limits of $\frac{1}{p} \log |\Lambda(n,p)|$, as $p \to \infty$, are $n$ independent.

Remarks: (a) The arbitrariness of $v_N$ implies uniqueness of $h(n)$ and that $h(n)$ is the unique eigenvector (i.e. “covariant vector”).
(b) $\frac{h(n)}{||h(n)||}$ is called covariant unit (or direction) vector at site $n$.
(c) Imagine the matrices $(T_j)_{j=-\infty}^{\infty}$ be a sequence of random variables and that their entries are distributed with a distribution $\rho$ which is invariant with respect to the (left) translations, and with samples restricted to keep the cone property, i.e. with the cones $\Gamma, \Gamma'$ which do not depend on the choice of the matrices. Then the upper and lower limits of $\frac{1}{p} \log |\Lambda(n,p)|$ as $p \to \infty$ will be constant under translation, i.e. $n$-independent as a consequence of the $n$-independence in item (4). They will be shown below to exist almost everywhere, as a consequence of the ergodic theorem, hence if $\rho$ is ergodic they will be equal and constant with $\rho$ probability 1.

Theorem 1 is a paradigm: it is the basis for similar theorems for dynamical systems: see the next section, before proceeding to the proofs.

2 Dynamical systems

Let $\mathcal{F}$ be a smooth compact manifold and $\tau$ a smooth, smoothly invertible, map on $\mathcal{F}$ (take smooth to mean $C^\infty$, for simplicity). At each point $x \in \mathcal{F}$ there are proper closed convex cones $\Gamma(x) \supset \Gamma'(x)$, with apex at $x$ in a linear space $E(x)$ of dimension $d$ (and call its adjoint $E(x)^*$).

Definition 2: The minimum angle between vectors on the boundary of $\Gamma(x)$ and on that of $\Gamma'(x)$ will be called inclination $\varepsilon(x)$; while the maximum angle between vectors in $\Gamma(x)$ will be called $\vartheta(x)$, likewise define $\vartheta'(x)$. Let $T(x), x \in \mathcal{F}$, be an invertible mapping of $E(x)$ onto $E(\tau x)$, and $T(x)^{-1}$ maps $\Gamma(x)$ into $\Gamma'(x) \subset \Gamma(x)$ with $\Gamma'(x)/\{x\} \subset \Gamma(x)^0$ and $T(x), \Gamma(x), \varepsilon(x), \vartheta(x), \vartheta'(x)$ are continuous, $\varepsilon(x) > 0$.

If $T^{-1}(x)$ is diagonalizable write it, with $|x, \sigma \rangle \in E(x), \langle x, \sigma | \in E(\tau x)^*$:

$$T(x)^{-1} = \sum_{\sigma=0}^{d} \lambda_\sigma(x) |x, \sigma \rangle \langle x, \sigma | \quad (2.1)$$

Imitating the previous section, given $x \in \mathcal{F}$ and a function $v(x) \in \Gamma'(x)$, $||v(x)|| \equiv 1$, let $x, \tau x, \tau^2 x, \ldots$ be the future trajectory of $x$: then the
vector \( H_v(x, N) \) can be expressed, as in Sec 1 if \( J = [h, h'], 0 \leq h \leq h' \leq N \), in terms of

\[
I(x, J, \sigma, J) \overset{\text{def}}{=} \left( \prod_{j=h}^{h'} \lambda(\sigma x) \langle \tau^j x, \sigma \rangle \langle \tau^{j+1} x, \sigma \rangle \right)
\]

\[
\cdot \left( \left( \langle \tau^{h-1} x, 0 | \tau^h x, \sigma_h \rangle \right)^{\delta_{\sigma=1}} \langle \tau^{h'} x, \sigma_{h'} | \tau^{h+1} x, 0 \rangle \right)
\]

\[
\Omega(x, N) \overset{\text{def}}{=} \sum_{s \geq 0} \sum_{j_1 < \ldots < j_s} \prod_{i=1}^{s} I(x, J_i, \sigma, J_i)
\]

\[
\Lambda(x, N) \overset{\text{def}}{=} \Omega(x, N) \prod_{j=1}^{N} \left( \lambda(\tau^j x) \langle \tau^j x, 0 | \tau^{j+1} x, 0 \rangle \right)
\]

where \( |\tau^{N+1} x, \sigma_{N+1}| \) means \( v(\tau^{N+1} x) \), as in Sec 1.

Here \( x \) plays the role of \( n \) in Eq. 2.4. Then \( H(x, N) \) is written as:

\[
H_v(x, N) \overset{\text{def}}{=} \Lambda_v(x, N) \sum_{s \geq 0} \sum_{\sigma_{j_1} \ldots \sigma_{j_s}} |x, \sigma_0\rangle \prod_{i=1}^{s} I(x, J_i, \sigma, J_i) \Omega_v(x, N)
\]

**Definition 3:** Let \( T(x) \) be as in definition 2 and suppose that \( T(x) \) depend on a variable \( z \), with a \( x \)-uniformly convergent and bounded power series for \( |z| < \nu \), and \( T^{-1}(x) \Gamma(x) \subset \Gamma'(x) \) for \( z \) real, \( |z| < \nu \), then \( T \) is said “\( z \)-analytic with cone property” and with radius \( \nu \).

Assuming \( T(x) \) to satisfy the properties in Definitions 2,3 above (hence not necessarily diagonalizability, used only to illustrate in detail a simple case via Eq. 2.2), an analogue of theorem 1 holds:

**Theorem 2:** Let \( T(x) \) be as in definitions 2,3, hence \( T(x)^{-1} \Gamma(x) \subset \Gamma'(x) \). Given a measurable function \( x \to v(x) \in \Gamma'(x), ||v(x)|| \equiv 1 \) (not necessarily continuous), it is

1. There are continuous functions \( x \to b(x) \in \Gamma(x) \), and \( x \to \Lambda(x, p), p = 0, 1, \ldots \), such that, for all \( p > 0, x \in F, v \), exist the limits

\[
b(x) = \lim_{N \to \infty} \frac{T^{-1}(x) \cdots T^{-1}(\tau^{N-1} x) v(\tau^N x)}{\Lambda_v(x, N)}
\]

\[
\Lambda(x, p) = \lim_{N \to \infty} \frac{\Lambda_v(\tau^p x, N)}{\Lambda_v(x, N)} > 0,
\]
(2) The vectors \( b(x) \) are eigenvectors for the products of \( T(\tau^j x) \) in the sense
\[
b(\tau^p x) = \Lambda(x, p) T(\tau^{p-1} x) \ldots T(x) b(x)
\] (2.5)

(3) \( b(x), \Lambda(x, p) \) are \( \nu \)-independent and continuous in \( x \in \mathcal{F} \) and \( \exists B \) such that \( B^{-1} < \|b(x)\| < B \); the unit vector \( \frac{b(x)}{\|b(x)\|} \) will be called the stable unit vector, or stable direction, at \( x \).

(4) \( T(x) \) \( z \)-analytic with cone property implies \( \Lambda(x, p) \) analytic in \( z, |z| < \nu' \leq \nu \) for some \( \nu' > 0 \).

(5) The upper and lower limit values of \( \frac{1}{p} \log |\Lambda(x, p)| \) as \( p \to \infty \) are constant along trajectories, i.e. \( k \)-independent if evaluated at \( \tau^k x \).

If \( x \) is given and \( T_n^{\text{def}} = T(\tau^n x)^{-1} \) the result is reduced to Theorem 1, because the extra continuity statements can be checked (by checking that the limits above are attained uniformly in \( x \), see below).

Suppose that the point \( x \) is chosen randomly in \( \mathcal{F} \) with respect to a \( \tau \)-invariant distribution \( \rho(dx) = \rho(\tau^{-1} dx) \); then the limits
\[
\limsup_{p \to \infty} \frac{1}{p} \log |\Lambda(x, p)|, \quad \liminf_{p \to \infty} \frac{1}{p} \log |\Lambda(x, p)|
\] (2.6)
are \( \rho \)-measurable constants of motion on the trajectories \( x \to \tau^k x, k = 0, 1, \ldots \). This is particularly relevant because of the following theorem:

**Theorem 3:** Let \( \rho \) be a \( \tau \)-invariant distribution on \( \mathcal{F} \) for the map \( \tau \). Let \( T(x) \) be as in theorem 2. Then:

(1) the limits in Eq.(2.6) are \( \rho \)-almost everywhere equal;
(2) if \( \rho \) is ergodic they are also \( \rho \)-almost everywhere \( x \)-independent.

**Remark:** (1) If \( \rho \) is ergodic and the matrices \( T \) are \( z \)-analytic with cone property, see definition 3 \( \rho \)-almost everywhere equal, hence constant, value of the limits in Eq.(2.6), whose logarithm can be identified with the least Lyapunov exponent (of the family of random matrices), is analytic in \( z \). This follows by integrating the analytic (in \( z \)) function \( \frac{1}{p} \log \Lambda(x, p) \) over \( x \); the uniform bounds that will be derived on it imply that the limit as \( p \to \infty \) is analytic. Hence the minimum Lyapunov exponent would correspondingly be analytic.

(2) Theorems, symmetric to the above theorems 2,3, hold if the cone property is \( T(x)\Gamma(x) \subset \Gamma^{\prime}(\tau x) \) and the sequence \( T_n = T(\tau^{-n} x) \) is considered: in this case the maximum Lyapunov exponent would be analytic (as above) and the vector \( \frac{b(x)}{\|b(x)\|} \) would determine the unstable direction at \( \tau x \).
The analyticity properties in theorem 1, and theorems 2, 3 follow because the various quantities in theorem 1, 2, 3 are expressed via an absolutely and uniformly convergent expansion in terms of quantities, the $I(J, \sigma_J)$, which are analytic, as it is seen from the estimates in Sec. 3 and Appendix B; hence also the derivatives can be expressed via convergent series of analytic functions.

The situation in theorems 2, 3 may arise in the theory of hyperbolic systems $(F, \tau)$, [7, 11]; more generally, the considerations in in [11, Sec. 4] can be largely adapted to the present analysis.

3 Paradigmatic case (theorem 1)

The essence of the proof lies in understanding the case in which the matrices $T_j$ are diagonalizable with real eigenvectors and eigenvalues uniformly (in $j$) pairwise separated, and possess a $(\Gamma, \Gamma')$-cone property; furthermore
\[ \max_{n, \sigma=1, \ldots, d-1} \left| \frac{\lambda_{\sigma,n}}{\lambda_0} \right| < \epsilon_0 \]
with $\epsilon_0$ small enough.

The more general case contemplated in theorem 1 (i.e. just the cone property and a uniform bound on $\|T_n\|$ is supposed) will be eventually reduced to the latter one via the following algebraic lemma, see proof in Appendix A:

**Lemma 1:** Let $T$ be a (single) $d \times d$ matrix with the $(\Gamma, \Gamma')$-cone property. Then:
(a) $\frac{T^n v}{\|T^n v\|} \overset{n \to +\infty}{\longrightarrow} b$ exponentially fast and $\frac{T^n b}{\|T^n b\|} = b$ for all $v \in \Gamma$. A corresponding property holds for the transposed $T^*$ with $b$ replaced by a $b^*$.
(b) The eigenvalue $\lambda_0$ of $T$ with maximum modulus is simple and positive, hence there is $\gamma < 1$ such that $\max_{\sigma > 0} \left| \frac{\lambda_{\sigma}}{\lambda_0} \right| \leq \gamma < 1$.

**Definition 4:** Let $\lambda$ the modulus of the eigenvalue of a matrix with largest modulus and $\lambda'$ the maximum modulus of the other eigenvalues; call, here, $\frac{\lambda'}{\lambda}$ the matrix “spectral gap”.

The proof of lemma 1 leads to, see Appendix A:

**Lemma 2:** Suppose that the sequence $T_1, T_2, \ldots$ satisfies the cone property with respect to the pair of cones $\Gamma \supset \Gamma'$ and let $T' \text{def} = T_1 \cdot T_2 \cdots T_p$. Then there are constants $c, \alpha > 0$ with $\alpha < 1$ such that the spectral gap of $T'$ is $\leq \alpha^p$ for $p$ large enough. Furthermore the matrix elements of $T'/\|T'\|$ on the basis formed by $b$ (see lemma 1) and by $d-1$ unit vectors in the plane orthogonal
to $b^*$ are all bounded by $c_\alpha p$ with the exception of the entry $T_{0,0}^{[p]}/||T^{[p]}|| \geq \frac{1}{c_\alpha}$.

The $c, \alpha$ depend on the inclination $\varepsilon$ and on the openings $\vartheta, \vartheta'$ of the cones.

This implies that, if $p$ is large enough, the sequence $T'_n = T_{np+1}T_{np+2}\cdots T_{(n+1)p}$, $n = 0, 1, \ldots$, satisfies the cone property with respect to the same pair of cones $\Gamma \supset \Gamma'$ and the spectral gap of the matrices $T'_n$ can be made as small as wished by taking $p$ large enough.

Hence it is sufficient to prove theorem 1 for matrices $T'_j$ with the above defined spectral gap $\lambda' = \gamma$ as small as needed. Once the sequence $b'_n$ for $T'_n$ is obtained the sequence $b_n$ that has to be found will be:

$$b_{np+k} = T_{np+k}\cdots T_{n(p+1)}b'_{n(p+1)}, \quad k = n, n-1, \ldots 0 \quad (3.1)$$

and all the statements in theorem 1 will, as well, follow for the sequence $T_n$.

**proof of theorem 1:** Suppose at first that the matrices $T_n$ have real eigenvalues with reciprocal distance, as $n$ varies, greater than a positive lower bound. The $\Omega_{\nu_{\eta}}(n, N)$ in Eq.(1.4) can be interpreted as the partition function of a gas of polymers represented by (lattice) intervals $J = [h, h'] \subset [1, N]$ with “base” $J$, “structure” $\sigma_J = (\sigma_h, \ldots, \sigma_{h'})$ and “activity” $I(J, \sigma_J)$ defined in Eq.(1.4). Of course this is only an analogy as $\sum_{\sigma_J} |I(J, \sigma_J)|$, although real numbers (under the restrictive temporary assumption) might be $< 0$.

A simple bound can be set on $|I(J, \sigma_J)|$ by remarking that the cone property implies a $n$-independent lower bound $\frac{1}{\eta'} > 0$ on the scalar products appearing in the denominators in the definition Eq.(1.4): actually $\frac{1}{\eta'} > 0$ can be a lower bound for the absolute value of the product of any pair or unit vectors in $\Gamma$. If $\gamma$ is an upper bound to the matrices spectral gaps, see definition 4, Eq.(1.4) implies, also defining $\eta$:

$$\sum_{\sigma_J} |I(J, \sigma_J)| \leq ((d-1)\delta^{-1}\gamma)^{h'-h+1} \equiv \eta^{h'-h+1} \quad (3.2)$$

Remarks: (1) As shown by the last bound, forgetting that the activities may be negative and treating $I(J, \sigma_J)$ as weight (or activity) of the polymer $(J, \sigma_J)$ in the formal probability distribution of the polymers appearing in Eq.(2.3), the number of $J$'s should be very small if $\eta$ (i.e. $\gamma$) is small: the “preferred state” is no $J$ at all; but the $I(J, \sigma_J)$ may be negative and cannot be regarded as probability weights.

(2) Hence it will be necessary to evaluate the “averages” (like Eq.(1.3)) with respect to the “distribution” in which a configuration of polymers with bases $J_1 < \ldots < J_s$ and spin structures $\sigma_{J_1}, \ldots, \sigma_{J_s}$ has weight $\prod_{i=1}^s I(J_i, \sigma_{J_i})$:
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this must be done algebraically, i.e. without profiting of positivity properties. It is natural to have recourse to the cluster expansion: which is a method that was designed, in statistical mechanics, precisely for such tasks.

The general theory of the cluster expansion for polymers, see [3, Ch.7] (it is recalled from scratch, for completeness, in Appendix B), yields a formal expression for \( \Omega_{vN}(n, N) \) as

\[
\Omega_{vN}(n, N) = \exp \sum_{Y} \varphi^T(Y) \zeta(Y), \quad \zeta(Y) = \prod_{J \in Y} I(J, \sigma_J)
\]

where the summation runs over all polymer configurations \( Y \) consisting of \( Y_1, \ldots, Y_s \), with \( Y_i = (J_i, \sigma_{J_i}) \), in which the constraint of no overlap on the polymers base intervals \( J_i \) (implied by the \( * \) in Eq. (1.4)) is dropped and \( \varphi^T(Y) \) are suitable (real) combinatorial coefficients, see Appendix B, Eq. (B.8).

The coefficients \( \varphi^T(Y) \) have the important property of being translation invariant, under a simultaneous translation of the polymers in \( Y \), and of vanishing unless the intervals \( J \) which are bases of the polymers in \( Y \) overlap, in the sense that \( \cup_{J \in Y} J \) is a connected interval, called a cluster (imagine to draw the \( J_i \)'s as continuous segments joining their extremes).

Abridge \( \varphi^T(Y) \zeta(Y) \) into \( \hat{\varphi}(Y) \): the sum \( \sum_{Y, J_i \subset [n, N]} \hat{\varphi}(Y) \) over the clusters \( Y \) will be an absolutely convergent series, summing to \( \log \Omega_{vN}(n, N) \), if it will be shown that, for a suitable choice of \( r(J) \), it is

\[
\mu \overset{\text{def}}{=} \sup_{J, \sigma_J} \frac{|\zeta(J, \sigma_J)|}{r(J)} \exp \sum_{S, \sigma_S} r(S) < 1 \quad (3.4)
\]

where the sum is over the polymers \( (S, \sigma_S) \) overlapping with \( J \), i.e. \( S \cap J \neq \emptyset \), as recalled in a self contained proof, in Appendix B.

This is a non trivial property: certainly the overlap condition strongly reduces the number of addends in the series for \( \Omega_{vN}(n, N) \) and helps together with the Eq. (3.4), which implies that \( \zeta(Y) \) is exponentially small with the size of the interval covered by the clusters of the polymers bases in \( Y \). However the help is not sufficient and important combinatorial cancellations have to be taken into account: they are exhibited through the remark that the coefficients \( \varphi^T(Y) \zeta(Y) \) satisfy an identity, see Eq. (B.12), reducible to an identity known in Physics as Kirkwood-Salsburg equations, [4].

Take \( r(S) \overset{\text{def}}{=} \eta S |S| \), see Eq. (3.2); if \( \eta = (d-1)\delta^{-1}\gamma \) is small enough:

\[
\mu \leq \sup_{|J|=j \geq 1} \frac{\eta^j}{\eta^{2j}} \exp \left( \sum_{\ell=0}^{\infty} (\ell + 1) \frac{\eta^{2(\ell+1)}}{\eta^{4(\ell+1)}} \right) = \eta^{\frac{1}{2}} \exp \frac{\eta^{\frac{1}{2}}}{(1 - \eta^{\frac{1}{2}})^2} < 1 \quad (3.5)
\]
Setting, see Eq.(1.3),

\[ h_{vN}(n, N) \stackrel{d.f.}{=} \sum_{s \geq 0} \sum_{J_1 < \cdots < J_s \sigma_{J_1}, \ldots, \sigma_{J_s}} |n, \sigma_n) \prod_{i=1}^s I(J_i, \sigma_{J_i}) \Omega_{vN}(n, N) \] (3.6)

this, as remarked in Sec.1, can be interpreted as

\[ P_{n,\sigma,N} \stackrel{d.f.}{=} \Omega_{vN}(n, N) \sum_{s \geq 0} \sum_{J_1 < \cdots < J_s \sigma_{J_1}, \ldots, \sigma_{J_s}} \prod_{i=1}^s I(J_i, \sigma_{J_i}) \Omega_{vN}(n, N) \] (3.7)

where the \( \ast \sigma \) indicates that the sum is restricted to polymers configurations with \( \sigma_n = \sigma \): i.e. with \( n \in J_1 \) and \( \sigma_n = \sigma \) if \( \sigma \geq 1 \) or with \( n \notin J_1 \) if \( \sigma_n = 0 \). The \( v_N \) dependence of \( P_{n,\sigma,N} \) is not explicitly marked.

Hence if the activities \( I(J, \sigma_J) \) were non negative \( P_{n,\sigma,N} \) would be the probability of finding a configuration with spin \( \sigma \) at site \( n \), and \( P_{n,0,N} + \sum_{\sigma=1}^{d-1} P_{n,\sigma,N} \equiv 1 \). This relation is a purely algebraic property and is identically satisfied (as long as the addends are well defined).

For the \( P_{n,\sigma,N} \) with \( \sigma \geq 1 \) the cluster expansion (see Eq.(B.15)) yields

\[ P_{n,\sigma,N} = \frac{1}{\Omega_{vN}(n, N)} \sum_{J \ni n, J_1, \sigma_{J_1} = \sigma} \langle D_{J,\sigma_J} \hat{\varphi}^T \rangle, \quad |P_{n,\sigma,N}| \leq \frac{\eta^2}{(1 - \mu)(1 - \eta^2^2)} \] (3.8)

where \( \langle D_{J,\sigma_J} \hat{\varphi}^T \rangle \stackrel{d.f.}{=} \sum_Y \hat{\varphi}^T((J, \sigma_J) \cup Y) \); the bound is obtained by using the mentioned overlap property that the \( \hat{\varphi}^T(Y) \) vanish unless the bases of the polymers in \( Y \) form a connected interval, see below.

The overlap property shows that the sum \( \langle D_{J,\sigma_J} \hat{\varphi}^T \rangle \) is restricted to polymers that contain the site \( n \): hence the union of the bases of the polymers contributing terms that depend on the boundary vector \( v \) must cover the whole \([n, N]\) and therefore contribute a quantity that is exponentially small as \( N \rightarrow \infty \), as mentioned above.

Explicit bounds, see Eq.(B.13), (B.14), if Eq.(3.4) holds, give \( |P_{n,\sigma,N}| \leq \sum_{j=1}^{\infty} \sum_{m=0}^{\infty} \eta^{\frac{1}{2}} \mu^m \) where \( j \) is the length of the polymer \( J \) containing \( n \). Therefore \( P_{n,\sigma,N} \) depends on the boundary condition vector \( v \): but the dependence disappears in the limit \( N \rightarrow \infty \). Hence the limits \( P_{n,\sigma} \) of \( P_{n,\sigma,N} \)

\(^1\)The site \( n \) must be contained since \( J \) must contain \( n \) because \( \sigma_n \geq 1 \); and the site \( N \) must be contained as otherwise the \( \varphi^T \) does not depend on \( v \).
as \( N \to \infty \) exist and

\[
|P(n, \sigma)| \leq \begin{cases} 
\frac{\eta^{\frac{1}{2}}}{(1-\mu)(1-\eta^{\frac{1}{2}})} & \text{if } \sigma \geq 1 \\
1 + \frac{\eta^{\frac{1}{2}}}{(1-\mu)(1+\eta^{\frac{1}{2}})}(d-1) & \text{if } \sigma = 0 
\end{cases} \tag{3.9}
\]

where the bound for \( \sigma = 0 \) reflects the algebraic identity (due to the probabilistic interpretation) \( P(n, 0, N) + \sum_{\sigma=1}^{d-1} P(n, \sigma, N) \equiv 1 \) (which holds whether or not the activities \( I(J, \sigma_j) \) are \( \geq 0 \), provided convergence holds).

Hence the limit \( h(n) \overset{\text{def}}{=} \lim_{N \to \infty} h_v(n, N) \) exists and by Eq.\( \text{(1.3)} \) is

\[
h(n) = \lim_{N \to \infty} \frac{T_n \cdots T_N v}{\Lambda_v(n, N)} = \sum_{\sigma=0}^{d-1} P_{n,\sigma}|n, \sigma|, \quad \text{with}
\]

\[
\Lambda_v(n, N) \overset{\text{def}}{=} \Omega_v(n, N) \prod_{j=n}^{N} \left( \lambda_j,0 \langle j,0|j+1,0 \rangle \right)
\]

\[
B^{-1} \leq ||h(n)|| \leq B,
\]

with \( B = (1 + 2d\frac{\eta^{\frac{1}{2}}}{(1-\mu)(1-\eta^{\frac{1}{2}})}) \), if \( \eta \) is small enough so that \( \mu, \eta < 1 \) (together with Eq.\( \text{(3.5)} \) this is the condition on the spectral gap that determines the parameter \( \gamma \)). Notice that the convergence of the cluster expansion also implies \( \Omega_v(n, N) > 0 \) and upper and lower bounds on \( \Omega_v(n, N) \):

\[
| \log \Omega_v(n, N) | \leq \left| \sum_Y \hat{\phi}^T(Y) \right| \leq \sum_{\gamma \subset [n,N]} \sum_Y |\hat{\phi}^T(\gamma \cup Y)|
\]

\[
\leq \sum_{\gamma \subset [n,N]} r(\gamma) \sum_{m=1}^{\infty} I_m \leq \sum_{k=1}^{\infty} \frac{(N-n+1)}{1-\mu} \eta^{\frac{1}{2}k} \leq \frac{(N-n+1)\sqrt{\eta}}{(1-\mu)(1-\sqrt{\eta})}, \tag{3.11}
\]

of course not uniform in \( N \). Thus exhibiting the important cancellation that occurs in the ratios \( \frac{\Omega_v(n, N)}{\Omega_v(p, N)} \) above, thus estimated uniformly in \( N \).

Drop, for simplicity, the \( v \) labels form the \( \Omega_v \) and define

\[
\Lambda(n, p) \overset{\text{def}}{=} \lim_{N \to \infty} \frac{\Omega(p, N) \prod_{j=p}^{N} \left( \lambda_j,0 \langle j,0|j+1,0 \rangle \right) \prod_{j=n}^{N} \left( \lambda_j,0 \langle j,0|j+1,0 \rangle \right)}{\Omega(n, N) \prod_{j=n}^{N} \left( \lambda_j,0 \langle j,0|j+1,0 \rangle \right)}
\]

\[
= \left( \prod_{j=n}^{p-1} \lambda_j,0 \langle j,0|j+1,0 \rangle \right)^{-1} \lim_{N \to \infty} \frac{\Omega(p, N)}{\Omega(n, N)} \tag{3.12}
\]

\( \Lambda \) can be evaluated again by the cluster expansion, which also allows us to see the cancellation that shows the \( v \)-independence of the last limit, as:
\[
\lim_{N \to \infty} \frac{\Omega(p, N)}{\Omega(n, N)} = \exp \sum_{\mathbf{Y}} \hat{\varphi}^T(\mathbf{Y}) = \exp \sum_{q=n}^{p-1} \Phi(q)
\] (3.13)

where \(Y = (J_1, \sigma_1, \ldots, J_s, \sigma_s)\) and the * means that the cluster \((\bigcup J_i)\) overlaps with \([n, p]\) and \(\Phi(q)\) is the sum \(\sum_{Y, J_1 \geq q, J_1 \ni q} \hat{\varphi}^T(\mathbf{Y})\) over all polymer configurations which are to the right of \(q\) and \(q\) is the first point of \(J_1\): in Eq. (3.13) numerator and denominator are exponentials of sums of many terms which are common (see Eq. (3.3)), hence cancel, except those relative to \(Y\)’s with bases touching \([n, p]\). Such polymers are independent of \(v\) unless their bases touch also \(N\): hence their contributions to \(\Phi(q)\) tend to 0 as \(N \to \infty\) at fixed \(n, p\).

Remark: (1) It is important to stress that \(\Phi(q)\) depends only on the matrices \(T_j\) with \(j \geq q\): \(\Phi(q) = F(T_q, T_{q+1}, \ldots)\) and each \(\Phi(q)\) is given by an absolutely convergent series because of the bounds on \(\hat{\varphi}^T(\mathbf{Y})\) in Eq. (3.13), (3.14). Furthermore the function \(F\) is independent of \(q\) (i.e. it is translation invariant as a function of the sequence of matrices), because of the translation invariance of \(\varphi^T(\mathbf{Y})\).

(2) The uniform convergence of the series defining \(\Phi(q)\), Eq. (3.13), implies that all limits points as \(p \to \infty\) of \(\Omega(n, p)^{1/p}\) are \(n\)-independent.

(3) Uniformity of the limits defining \(P_{n, \sigma}\) also holds if the matrix elements of the matrices \(T_j\) are varied keeping the cone property independent of the variations and their norms bounded by \(B_0\): this is due to the uniformity of the bounds on \(I(J, \sigma_J)\) only depending on the inclination \(\varepsilon\) of \(\Gamma'\) in \(\Gamma\), on the opening angles \(\vartheta, \vartheta'\) between the cones and on the bounds on the matrices norms.

(4) The activities \(I(J, \sigma_J)\), which so far have been real valued, are holomorphic functions of small complex variations of the matrices entries (since it is temporarily being supposed that the eigenvalues are real and keep pairwise a positive minimal distance) still satisfying the same bounds possibly with slightly different constants: hence \(h(n), \Lambda(n, p)\) are analytic in the entries.

(5) The \(h(n)\) form, essentially by definition once convergence has been established, a covariant family of vectors in the sense of theorem 1, Eq. (1.6), as implied by Eq. (3.10). In fact \(h(n)\) and likewise \(\Lambda(n, p), \Omega(n, N)\) are given by convergent expansions (see the series in Eq. (3.5), for instance) in the activities \(I(J, \sigma_J)\): the latter are given by simple algebraic expressions, see Eq. (1.4), which are analytic in the matrix elements of the \(T(x)\).

(6) If the matrix elements of the \(T_j\) are chosen randomly with respect to a distribution \(\rho\) on \(\prod_{j=1}^{\infty} R^d \times R^d\), which is invariant under the (right) transla-
tions and has samples satisfying the \((\Gamma, \Gamma')\)-cone property the limit

\[
\lim_{p \to \infty} \frac{1}{p} \log \Lambda(n, p) = \lim_{p \to \infty} \frac{1}{p} \left( \sum_{j=n}^{p} \log(\lambda_{j,0} \langle j, 0 | j+1, 0 \rangle + \sum_{j=n}^{p-1} \Phi(j)) \right) \tag{3.14}
\]

exists, consequence of the ergodic theorem, because the addends in the sums are translates to the right of the same functions of the random matrices (i.e. in the language of ergodic theory they are “Birkhoff averages”). By the same argument the limit \(\lim_{n \to -\infty} \frac{1}{n} \log \Lambda(n, p)\) exists almost everywhere, and it is \(= - \lim_{p \to \infty} \frac{1}{p} \log \Lambda(n, p)\) almost everywhere.

(7) Furthermore \(n\)-independence of the limit points, remarked in Eq. (3.14), implies that the limits in Eq. (3.14) are constants of motion under the right translations. Hence if \(\rho\) is also ergodic the limits are constant almost everywhere and the maximum Lyapunov exponent \(\lambda_+\) is the integral with respect to \(\rho\) of \(\log(\lambda_{1,0} \langle 1, 0 | 2, 0 \rangle + \Phi(1))\); hence it is analytic in the parameters on which the matrices may depend.

Finally the assumption that \(T_j\) have spectrum consisting of pairwise uniformly separated eigenvalues has to be removed: lemma 2 shows that the matrices \(T_j = T_{j+p+1}T_{j+p+2} \cdots T_{(j+1)p}\) to which we want to apply the analysis, see remark after Lemma 2, have a spectral representation as \(T_j = \lambda_{0,j} (|j, 0 \rangle \langle j, 0 | + \sum_{\sigma,\sigma' \geq 1} (\Theta_j)_{\sigma,\sigma'} |j, \sigma \rangle \langle j, \sigma' |)\), where the bases \(|j, \sigma \rangle, \langle j, \sigma' |\) consist of the vectors \(|j, 0 \rangle\) and \(|j, 0 \rangle\) and correspondingly \(d-1\) other orthogonal vectors, respectively to \(|j, 0 \rangle\) and \(|j, 0 \rangle\), arbitrarily chosen. The matrix elements are bounded above by \(c \alpha^p, \alpha < 1\).

Since the basis vectors \(|j, \sigma \rangle, \langle j, \sigma |\) for \(\sigma > 0\) need not be eigenvectors of \(T_j\) the basis can be taken real: hence the matrix \(\Theta_j\) can be taken real.

Therefore it appears that this case simply leads to more complicated formulae in which at each site \(j\) are now associated two spins \((\sigma_j, \sigma'_j)\), instead of just 1, and the products like \(\prod_{j=h}^{j' = b} \frac{\lambda_{j,\sigma} \langle j, \sigma | j+1, \sigma' \rangle}{\lambda_{0,j} \langle j, 0 | j+1, 0 \rangle}\), appearing in Eq. (3.14) must be replaced by the product \(\prod_{j=h}^{j' = b} (\Theta_j)_{\sigma_j, \sigma'_j} \frac{\langle j, \sigma | j+1, \sigma' \rangle}{\langle j, 0 | j+1, 0 \rangle}\).

This amounts at more values of the spins associated with each site: from \(d-1\) to \((d-1)^2 + 1\). There is no need to perform a full spectral decompostion and therefore to worry about degeneracies, complex eigenvalues and eigenvectors,\(^2\) eigenvalues crossing and the like. The only property needed

\(^2\)Only the largest eigenvalue, which is separated from the rest of the spectrum by an \(x\)-independent factor \(\alpha < 1\), and the relative eigenvector are needed.
is that the $\Theta_j$ be as small as necessary for the cluster expansion and to be analytic in the matrices entries: this is achieved, as mentioned, by taking $p$ large enough.

This concludes a proof of Theorem 1. Also theorems 2 and 3 are proved by applying theorem 1 to the sequences $T_n = T(\tau^n x)^{-1}$ (or $T_n = T(\tau^{-n} x)$ depending on the assumed cone property): the claim on the continuity in $x$ simply follows from the uniformity in $x$ of the limits considered, due to the uniformity of the convergence of the cluster expansion implied by the uniformity of the bounds on the activities, Eq.(3.2), (3.4), see also the last remark in Appendix A: details are skipped.

4 Comments

Neither the theorem in [2], nor its extensions in [5], have been used, the ergodic theorem being sufficient in the simple cases considered. The general and deep result in [2] does not give the analyticity: for analyticity a more restricted class of matrices has to be considered like the class considered in [11] or here.

A simple application of theorem 1 is to matrices $(T_n)_{\sigma,\sigma'} > 0$ with $1 \leq \frac{\max_{\sigma,\sigma'}(T_n)_{\sigma,\sigma'}}{\min_{\sigma,\sigma'}(T_n)_{\sigma,\sigma'}} \leq C < \infty$: they have the cone property with $\Gamma$ the cone of the vectors with components $\geq 0$ and some, $n$-independent, cone $\Gamma'$, [2, Sec.3]. This can be immediately applied to obtain free energy analyticity for a 1D spin glass with short range interaction as remarked in [11, p.69]: indeed the positivity of $(T_n)_{\sigma,\sigma'} > 0$ turns the problem into one in Statistical Mechanics with interaction $J_{\sigma,\sigma'} = \log(T_n)_{\sigma,\sigma'}$, [10, p.121].

The analysis is fully constructive for what concerns the contents of Theorem 1. In fact lemma 1 can be replaced by the solution of finitely many eigenvalue problems, like the determination of the largest eigenvalue of the matrices $T_1, \ldots, T_{N_0}$ or their products with $N_0$ that can be computed a priori, if the approximation needed is given. $N_0$ is directly related to the maximum size of the polymers necessary to achieve a desired approximation: it is a priori determinable through the value of $\alpha$ appearing in lemma 2, the estimate Eq.(3.4) and the cluster expansion estimates.

Constructivity is only lost, as usual, in the application of the ergodic theorem, as there is no control on which is the set of the $x$ for which the limits like Eq.(2.6), (3.14), exist.

Nevertheless if $x$ is random with respect to a $\tau$-ergodic measure the determination of the maximal Lyapunov exponent (or minimal, depending
on the cone property holding)) can be again expressed constructively, as the integral of the function appearing inside the sum in Eq. (3.14) setting \( j = 1 \), in which the first term is explicitly known while the second, i.e. \( \Phi(1) \), can be expressed to any prefixed accuracy by the cluster expansion.

The theorems 1, 2, 3 are not optimal: for instance the invertibility condition of the matrices \( T_j \) or \( T(x)^{-1} \), absent in [11, 8], is used only in item (3) of theorem 1 or item (2) of theorem 2: for the remaining statements it is not needed.

The point of this work has been to show that the cluster expansion technique can be of great help in problems that can be cast into a statistical mechanics context: after all it has been among the major achievements in equilibrium statistical mechanics of the XX century. Its use is limited to special problems but when it can be applied (as here) it gives a complete and constructive solution.

**Appendices**

**A  Algebraic properties of cones**

*proof of Lemma 1:* (following [11]) Let \( T \) be a \( d \times d \) matrix and let \( \Gamma, \Gamma' \) be proper, convex, closed cones (with apex at the origin \( O \)) in \( \mathbb{R}^d \). Suppose that \( T \Gamma \subset \Gamma' \) and a relative inclination (see definition in Sec. 2) \( \varepsilon > 0 \) of \( \Gamma \) to \( \Gamma' \). Let \( \Gamma^* = \{ w|\langle w|v \rangle \geq 0, \forall v \in \Gamma \} \). Then, fixed \( 0 \neq v_0 \in \Gamma \) and \( 0 \neq w_0 \in \Gamma^* \) the maps

\[
\begin{align*}
v & \rightarrow \frac{T v}{\langle T^* w_0 |v \rangle}, \\
 w & \rightarrow \frac{T^* w}{\langle w |T v_0 \rangle}
\end{align*}
\]  

(A.1)

map continuously the convex compact sets \( \{ v|\langle w_0|v \rangle = 1 \} \) and, respectively, \( \{ w|\langle w|v_0 \rangle = 1 \} \) strictly into themselves. Hence \( \exists a \in \Gamma \) and \( a^* \in \Gamma^* \) which are fixed points of the maps, respectively; hence, if \( b = a \frac{a}{||a||} \) and \( b^* = a^* \frac{a^*}{||a^*||} \)

\[
\begin{align*}
\frac{T b}{||T b||} &= b, \\
\frac{T^* b^*}{||T^* b^*||} &= b^*
\end{align*}
\]  

(A.2)

Let \( T \xi = \frac{T \xi}{||T \xi||} \) and let

\[
K = \{ \xi|\langle b^*|\xi \rangle = 0, \text{ and } b + \xi \in \Gamma \}
\]  

(A.3)

Since \( \frac{T b}{||T b||} = b \) and \( \frac{T^* b^*}{||T^* b^*||} = b^* \) the set \( K \) is mapped into itself by \( T \) (e.g. \( \langle b^*|T \xi \rangle = 0 \) and \( b + \frac{T \xi}{||T \xi||} = \frac{T(b + \xi)}{||T(b + \xi)||} \in \Gamma \)) and since the cone \( \Gamma \) is shrunk
by $T$ the set $K$ is mapped into $TK \subset \alpha K$ with $\alpha < 1$ (determined by the inclination and opening angles, see Sec[2]).

Hence $T^n(b + \xi) = b + T^n\xi$ and $||T^n\xi|| \leq \alpha^n$. For any $v \in \Gamma, v \neq 0$, there is a $\nu \neq 0$ such that $v = \nu b + \xi, \xi \in K$, so that

$$\frac{T^n(\nu b + \xi)}{||T^n(\nu b + \xi)||} = \frac{T^n(\nu b + \xi)}{||T^n(\nu b + \xi)||} = \frac{O(\alpha^n)}{n \rightarrow +\infty} \lambda b$$

(A.4)

because $T$ and $T'$ are proportional: notice that the above analysis implies that the largest eigenvalue $\lambda_0$ of $T$ is positive and that it is the unique eigenvalue of $T$ with maximum modulus.

proof of Lemma 2: (following [11]) Let $T^d \equiv T_1T_2 \cdots T_p$ and let $\vartheta, \vartheta^*$ be the, respective, normalized eigenvectors with maximum modulus eigenvalue $\lambda' > 0$ for $T'$ and $(T')^*$ (existing by lemma 1). Define

$$\vartheta_p \equiv \vartheta, \vartheta_{p-1} = \frac{T_p\vartheta_p}{||T_p\vartheta_p||}, \ldots, \vartheta_0 = \frac{T_1\vartheta_1}{||T_1\vartheta_1||} = \vartheta$$

$$\vartheta^*_0 \equiv \vartheta^*, \vartheta^*_1 = \frac{T_1^*\vartheta^*_0}{||T_1^*\vartheta^*_0||}, \ldots, \vartheta^*_p = \frac{T_p^*\vartheta^*_{p-1}}{||T_p^*\vartheta^*_{p-1}||} = \vartheta^*$$

(A.5)

By the argument in the proof of lemma 1 the action of $T_j$ on the plane orthogonal to $\vartheta^*_j$ maps it on the plane $\vartheta^*_{j+1}$ and contracts by at least $\alpha < 1$. Therefore $T_1T_2 \cdots T_p$ contracts by at least $\alpha^p$ in the space orthogonal to $\vartheta^*$, proving Lemma 2.

Remark: A simple extension deals with a dynamical system $(\mathcal{F}, \tau)$ where $\mathcal{F}$ is a smooth manifold (of class $C^\infty$, to fix ideas) and $\tau$ an equally smooth invertible map. Suppose that at each point there are cones $\Gamma(x) \supset \Gamma'(x)$, smooth in $x$ and with apex $x$, in the tangent space at $x$ or more generally in a linear space $E(x)$ of dimension $d$. Let $T(x)$ be a $d \times d$ invertible matrix mapping $E(x)$ onto $E(\tau x)$ all depending on $x$ smoothly (of class $C^\infty$, for instance). Suppose, furthermore, $T(x)\Gamma(x) \subset \Gamma'(\tau x)$ with continuous inclination $\varepsilon(x)$ of $\Gamma'$ with respect to $\Gamma$ (see definition [2]).

The previous arguments can be repeated essentially word by word for the matrices $T'(x) \equiv T(x)T(\tau x) \cdots T(\tau^p x)$: but the vectors $b, b^*$ will have to bear a label $x$ so that $b(x) \in E(x), b^*(x) \in E(x)^*$ and $\frac{T(x)b(x)}{||T(x)b(x)||} = b(\tau x)$. The convergence in Eq. (A.4) is exponential at a rate $\alpha$ that can be estimated in terms of $\varepsilon = \min \varepsilon(x)$: hence it is a convergence uniform in $x$ implying continuity of $b(x), b^*(x)$. Analogously uniform convergence and continuity can be obtained if $T(x)^{-1}\Gamma(\tau x) \subset \Gamma'(x)$.
B  Cluster expansion: a rehearsal

This section follows [2, Ch.7] (in turn based on [3, 4]). Cluster expansion is an algorithm to compute the logarithm of a sum

\[ \Xi = \sum_{J} \zeta(J) \equiv \sum_{J} \prod_{i} \zeta(J_i)^{n_i} \]  

(B.1)

where: (1) \( J = (J_1^{n_1}, \ldots, J_N^{n_N}) \) with \( J_i \)'s subsets in a box \( \Lambda \) on a \( d \)-dimensional lattice (here \( d = 1 \)) called polymers and \( n_i \geq 0 \) are integers defining the “multiplicity” of each (or “counting” how many times each set is counted) hence \( N = 2^{|\Lambda|} \). The sets \( J \) could be decorated by associating to each site \( k \in J \) a “spin”, i.e. a variable assuming \( d - 1 \) values. However in the following the decorations will not be mentioned as they would only make the notations heavier. In the applications in Sec.3 the decorations will be necessary and the formulae of this section (which correspond to the case \( d = 2 \), i.e. all spins 1) are directly usable simply by imagining that each \( J \) is in fact a pair \( Y = (J, \sigma_J) \) where \( \sigma_J = (\sigma_j)_{j \in J} \) and \( \sigma_j = 1, \ldots, d - 1 \).

(2) \( \zeta(J) = \prod \zeta(J_i)^{n_i} \) with \( \zeta(J) \) (small) constants called activities, \( \zeta(\emptyset) \equiv 1 \).

(3) the \( * \) means that the sum runs over the \( J \)'s in which no two of the \( J_i \in J \) with multiplicity \( n_i > 0 \) overlap in the sense that they contain pairs of points at distance \( \leq 1 \) on the lattice. If \( \tilde{J} \) denotes the sets in \( J \) which have positive multiplicity then the \( * \) indicates that the sum is restricted to \( J \equiv \tilde{J} \) in which no two of the \( J \)'s intersect.

In applications \( \zeta(J) \neq 0 \) only for a few of the possible subsets of \( \Lambda \). For instance in the present case \( \Lambda \) is the interval \([n, N]\) and the “polymers” are just the subintervals.

The \( \Xi \) can certainly be written as \( \exp(\sum_J \varphi^T(J)\zeta(J)) \) by expanding the \( \log \Xi \) in powers of the \( \zeta(J) \): of course the sum in the exponential will involve \( J \) with \( J \)'s which can overlap or that can be counted many times. The \( \varphi^T(J) \) are suitable combinatorial coefficients.

For instance if \( \Lambda \) is just one point \( \Xi = 1 + z \) can be written as the exponential of \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k \). If \( \Lambda \) consists of two points, say 1 and 2 then the polymers are \( \emptyset, 1, 2, 12 \) and \( \Xi = 1 + z_1 + z_2 + z_{12} \) is the exponential of \( \sum_{k_1+k_2+k_3 \geq 0} \frac{(-1)^{k_1+k_2+k_3+1}(k_1+k_2+k_3-1)!}{k_1!k_2!k_3!} z_1^{k_1} z_2^{k_2} z_{12}^{k_3} \).

The cluster expansion is the general form of the above examples. It is of interest, for instance, if \( \sum_J |\varphi^T(J)||\zeta(J)| < +\infty \) where the \& means that the sum is restricted to \( J \)'s which contain any fixed point \( x \in \Lambda \) (i.e. with \( x \in \cup_{J \in \tilde{J}} J \)). It is therefore necessary to determine conditions that imply the mentioned convergence.
The first step is to define $\mathbf{J} + \mathbf{J}'$ simply as $J_1^{n_1 + n_1'}, \ldots, J_N^{n_N + n_N'}$, i.e. as the family of polymers with multiplicities equal to the sum of the corresponding ones in $\mathbf{J}$ and $\mathbf{J}'$. Let

$$\mathcal{F} = \text{set of functions } F(\mathbf{J})$$

$$\mathcal{F}_0 = \text{set of functions } F(\mathbf{J}) \text{ with } F(\emptyset) = 0$$

$$\mathcal{F}_1 = \text{set of functions } F(\mathbf{J}) \text{ with } F(\emptyset) = 1$$

$$\mathbf{1}(\mathbf{J}) = \begin{cases} 0 & \text{if } \mathbf{J} \neq \emptyset \\ 1 & \text{if } \mathbf{J} = \emptyset \end{cases}$$

$$f \in \mathcal{F}_1 \iff \tilde{f} \equiv f - 1 \in \mathcal{F}_0$$

and remark that $f \in \mathcal{F}_1$ can be written $f = 1 + \tilde{f}$ with $\tilde{f} \in \mathcal{F}_0$.

Then if $f * g(\mathbf{J}) \equiv \sum_{\mathbf{J}_1 + \mathbf{J}_2 = \mathbf{J}} f(\mathbf{J}_1)g(\mathbf{J}_2)$, for $f, g \in \mathcal{F}$ define

$$\text{Exp}(f)(\mathbf{J}) = \sum_{k=0}^{\infty} \frac{f^k(\mathbf{J})}{k!}, \quad f \in \mathcal{F}_0$$

$$\text{Log}(f)(\mathbf{J}) = \sum_{k=1}^{\infty} \frac{(-1)^k \tilde{f}^k(\mathbf{J})}{k}, \quad f = 1 + \tilde{f} \in \mathcal{F}_1$$

$$f^{*-1} = \sum_{k=1}^{\infty} (-1)^k \tilde{f}^k, \quad f = 1 + \tilde{f} \in \mathcal{F}_1$$

$$\langle f \rangle = \sum_{\mathbf{J} \subseteq \Lambda} f(\mathbf{J}), \quad f \in \mathcal{F}$$

here all sums over $k$ are finite sums for $f$ in the corresponding domains.

A key remark is

$$\text{Log} \left( \text{Exp}(f) \right) = f \quad \forall f \in \mathcal{F}_0, \quad \text{Exp} \left( \text{Log}(f) \right) = f \quad \forall f \in \mathcal{F}_1$$

$$f^{*-1} * f = 1, \quad \forall f = 1 + \tilde{f} \in \mathcal{F}_1, \quad \langle f * g \rangle = \langle f \rangle \langle g \rangle$$

If $\chi(\mathbf{J}) = \prod \chi(J_i)^{n_i}$ is a multiplicative function $\chi \in \mathcal{F}$ then $\langle f * g \chi \rangle = \langle f \chi \rangle \langle g \chi \rangle$ so that if $\varphi \in \mathcal{F}_1$ and $\overline{\chi}(\mathbf{J}) = \zeta(\mathbf{J})$

$$\langle f \cdot \zeta \rangle = \langle \text{Exp} \left( \text{Log}(f \cdot \zeta) \right) \rangle = \exp \langle \text{Log}(f \cdot \zeta) \rangle$$

(B.5)

Therefore call $\mathbf{J}$ compatible if $n_i = 0, 1$ (i.e. $\mathbf{J} = \tilde{\mathbf{J}}$) and the elements of $\tilde{\mathbf{J}}$ are not connected then if

$$\varphi(\mathbf{J}) = \begin{cases} 0 & \text{if } \mathbf{J} \text{ is not compatible} \\ 1 & \text{otherwise} \end{cases}$$

(B.6)
then $\phi \in \mathcal{F}_1$ and $\phi^T = \log \phi \in \mathcal{F}_0$ makes sense and

$$\Xi = \langle \phi \cdot \zeta \rangle = \exp \langle \phi^T \cdot \zeta \rangle \equiv \exp \sum_J \phi^T(J) \zeta(J) \quad (B.7)$$

which is the exponential of a power series in the $\zeta(J)$ variables.

Calculating $\phi^T(J)$ requires computing the sum of finitely many quantities: if $J$ is represented as a set of “points” or nodes” and if $G$ is the graph obtained by joining all pairs of polymers in $J$ which are “incompatible” (regarding as different, and incompatible with each other, the $n_i$ copies of $J_i$ it is, (e.g. see [6, Eq.(4.21)]),

$$\phi^T(J) = \frac{1}{\prod n_i!} \sum_{C \in G}(-1)^{\# \text{ of edges in } C} \quad (B.8)$$

where the $*$ means that the sum is restricted to the subgraphs of $G$ which visit all polymers in $G$: their number is huge, growing faster than any power in the number of polymers so that convergence occurs because of cancellations due to the relation in Eq.(B.12).

The series in Eq.(B.7) is certainly convergent for $\zeta(J)$’s small enough: however the radius of convergence might be very small and $\Lambda$ dependent.

Define the differentiation operation as

$$(D_\gamma \Psi)(H) \overset{\text{def}}{=} \Psi(\Gamma + H)(\Gamma + H)! \quad (B.9)$$

with $\Gamma! = \prod_{i=1}^s n_i!$. The name is attributed because of the validity of the following rules:

$$D_\gamma (\Psi_1 * \Psi_2) = (D_\gamma \Psi_1) * \Psi_2 + \Psi_1 * (D_\gamma \Psi_2),$$

$$D_\gamma \exp \Psi = D_\gamma \Psi * \exp \Psi,$$  

(B.10)

A direct check of the above relations can be reduced to the case in which $\Gamma = n\gamma$, i.e. to the case in which there is only one polymer species $\gamma$, and the check is left to the reader. The first relation above, *Leibniz rule*, can be seen as a consequence the combinatorial identity $\sum_{p_1+p_2=n} \binom{q_1}{p_1} \binom{q_2}{p_2} = \binom{q_1+q_2}{n}$ for all $n, q_1, q_2$ with $n \leq q_1 + q_2$.

The definitions lead to the derivation of the expression for $\phi^T(\Gamma)$ in (B.8): which not only is quite explicit but also implies immediately that $\phi^T(\Gamma)$ vanishes for nonconnected $\Gamma$’s.

To determine sufficient conditions for the convergence which are independent on the size of $\Lambda$ let $\bar{\varphi}(Y) \overset{\text{def}}{=} \varphi(Y)\zeta(Y)$ and $\Delta J(Y) \overset{\text{def}}{=} \varphi^{s-1} * D_J \varphi(Y)$.
Then if $\gamma$ is a polymer, and $J, Y$ are polymer configurations

$$
\Delta_{\gamma + J}(Y) = \sum_{Y_1 + Y_2 = Y} \tilde{\varphi}^{* - 1} (Y_1) \varphi(\gamma + J + Y_2) \zeta(\gamma + J + Y_2) = \zeta(\gamma) \sum_{Y_1 + Y_2 = Y} \tilde{\varphi}^{* - 1}(Y_1) * \varphi(\gamma + J + Y_2) \zeta(J + Y_2)
$$

(B.11)

Here no factorials appear because $\varphi(J)$ vanishes unless $J = \tilde{J}$.

Remark that $\varphi(\gamma + J + Y_2) = \varphi(J + Y_2) \Pi_{\gamma' \in Y_2} (1 + \chi(\gamma, \gamma'))$ with $\chi(\gamma, \gamma') = 0$ if $\gamma, \gamma'$ do not overlap and $\chi(\gamma, \gamma') = -1$ otherwise, so that $\varphi(\gamma + J + Y_2) = \varphi(J + Y_2) \sum_{S \subseteq Y_2} (-1)^{|S|}$, with $|S|$ = number of polymers in $S = (s_1, s_2, \ldots)$ and $*$ means that the $s_i$ overlap with $\gamma$, for all $i$. Hence setting $Y_2 = S + H$

$$
\Delta_{\gamma + J}(Y) = \zeta(\gamma) \sum_{S \subseteq Y} \sum_{Y_1 + H = Y - S} \tilde{\varphi}^{-1}(Y_1) \varphi(J + S + H) \zeta(J + S + H)
$$

(B.12)

Let $r(\gamma) \geq |\zeta(\gamma)|$ and $r(X) = \Pi_{\gamma \in X} r(\gamma)$; then

$$
I_m \overset{\text{def}}{=} \sup_{1 \leq n \leq m} \sup_{|J| = n, |Y| = m - n} \sum |\Delta_{J}(Y)| \over r(J)
$$

(B.13)

and $I_1$ is then $I_1 = \sup_{\gamma} \frac{|\zeta(\gamma)|}{r(\gamma)}$ and recursively $I_{m+1} \leq \mu^m I_1$ where

$$
\mu \overset{\text{def}}{=} \sup_{\gamma} \frac{|\zeta(\gamma)|}{r(\gamma)} \exp \sum J r(J)
$$

(B.14)

where here $J$ is a single polymer (intersecting $\gamma$): see [5] Eq. 7.1.28 for more details on the algebra. Therefore $I_{m+1} \leq \mu^m I_1$, if $\mu < 1$.

The latter property $\mu < 1$ holds in various applications, notably in the present work, to bound $\Omega(n, N)$ as well as a few more quantities.

The method has several other applications, see [6], [5, Ch.7]. Here the polymers $J$ will be $\sigma_J$ corresponding to intervals $J$ (on the lattice $[1, N]$) with the associated spin structures $\sigma_J$. We shall make use of Eq.(B.7) and, by Eq.(B.7) and the third in (B.10), of

$$
P(J) \overset{\text{def}}{=} \sum_{H \supseteq J} \frac{\zeta(H)}{\Xi} = \frac{\langle D_J \varphi \zeta \rangle}{\langle \varphi \zeta \rangle} = \langle \tilde{\varphi}^{* - 1} * D_J \tilde{\varphi} \rangle
$$

$$
= \langle \tilde{\varphi}^{* - 1} * D_J \exp(\tilde{\varphi}^T) \rangle = \langle D_J \tilde{\varphi}^T \rangle, \quad \tilde{\varphi} \equiv \varphi \zeta
$$

(B.15)
In an ensemble in which the polymer configurations \( J \) in \( \Lambda \) are given a weight proportional to \( \prod_{\gamma \in J} \zeta(\gamma) \) this would be the probability of finding a configuration of polymers containing the polymer \( J \) if \( \zeta(\gamma) \geq 0 \). Hence the complementary sum \( P'(J) \overset{\text{def}}{=} \sum_{H \in J} \zeta(H) \) will be such that \( P(J) + P'(J) = 1 \).

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