EXISTENCE AND NONEXISTENCE OF SUBSOLUTIONS FOR AUGMENTED HESSIAN EQUATIONS

LIMEI DAI
School of Mathematics and Information Science
Weifang University
Weifang 261061, China

(Communicated by Isabeau Birindelli)

Abstract. In this paper, we consider the augmented Hessian equations
\[ S^k \left[D^2 u + \sigma(x) I \right] = f(u) \text{ in } \mathbb{R}^n \text{ or } \mathbb{R}^n_+ \]
We first give the necessary and sufficient condition of the existence of classical subsolutions to the equations in \( \mathbb{R}^n \) for \( \sigma(x) = \alpha \), which is an extended Keller-Osserman condition. Then we obtain the nonexistence of positive viscosity subsolutions of the equations in \( \mathbb{R}^n \text{ or } \mathbb{R}^n_+ \) for \( f(u) = u^p \) with \( p > 1 \).

1. Introduction
The augmented Hessian equation
\[ S_k^k [D^2 u + A(x, u, Du)] = B(x, u, Du) \text{ in } \Omega \]  
(1)
is a class of fully nonlinear elliptic equation which has attracted many interests and has many applications, where \( D^2 u \) is the second derivative of \( u \), \( \Omega \) is an open set in \( \mathbb{R}^n \), \( A : \Omega \times \mathbb{R}^n \rightarrow S^{n \times n} \), \( S^{n \times n} \) denotes the set of \( n \times n \) real symmetric matrices, \( B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a positive function and \( S_k \) is the \( k \)-Hessian operator defined by
\[ S_k [W] = S_k (\lambda) \]
with \( \lambda \) being the eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \) of the \( n \times n \) symmetric matrix \( W \) and \( S_k (\lambda) \) being the \( k \)-th elementary symmetric function given by
\[ S_k (\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \ 1 \leq k \leq n. \]
If \( A \equiv 0 \), (1) becomes the standard Hessian equation. In this case, when \( k = n \) it is the standard Monge-Ampère equation \( \det(D^2 u) = B(x, u, Du) \). For these standard Hessian equations and Monge-Ampère equations, they are very well known and investigated extensively, see [2, 3, 7, 27, 28], etc. If \( A \neq 0 \), in this case, for \( k = n \), (1) corresponds to a class of Monge-Ampère type equations which has been studied in [15, 18, 23, 25], etc. If \( A \neq 0 \), in this case, for \( k = n \), (1) corresponds to a class of Monge-Ampère type equations which has been studied in [15, 18, 23, 25], etc. If \( A \neq 0 \), in this case, for \( k = n \), (1) corresponds to a class of Monge-Ampère type equations which has been studied in [15, 18, 23, 25], etc. If \( A \neq 0 \), in this case, for \( k = n \), (1) corresponds to a class of Monge-Ampère type equations which has been studied in [15, 18, 23, 25], etc. If \( A \neq 0 \), in this case, for \( k = n \), (1) corresponds to a class of Monge-Ampère type equations which has been studied in [15, 18, 23, 25], etc. If \( A \neq 0 \), in this case, for \( k = n \), (1) corresponds to a class of Monge-Ampère type equations which has been studied in [15, 18, 23, 25], etc.
estimates and the existence of the solutions to the Dirichlet problem in bounded domains are given in [17, 19].

In this paper we consider the subsolutions for the augmented Hessian equations

\[ S_k^+ [D^2 u + \sigma(x) I] = f(u) \]  

(2)

in the whole space \( \mathbb{R}^n \) or in the half space \( \mathbb{R}^n_+ := \{ x = (x', x_n) \in \mathbb{R}^n | x_n > 0 \} \), where \( \sigma(x) \) is a positive continuous function in \( \mathbb{R}^n \), \( I \) is the identity matrix and \( f \) is a positive continuous function on \( \mathbb{R} \). To work in the realm of elliptic equations, we let

\[ \Gamma_k = \{ \lambda \in \mathbb{R}^n | S_j(\lambda) > 0, j = 1, \ldots, k \} \]

\( \Gamma_k \) is a symmetric cone, that is, any permutation of \( \lambda \) is in \( \Gamma_k \) if \( \lambda \in \Gamma_k \). We always assume that \( \lambda(D^2 u + \sigma(x) I) \in \Gamma_k \) for \( x \in \mathbb{R}^n \). A function \( u \in C^2(\mathbb{R}^n) \) is called a classical subsolution of (2), if \( u \) satisfies

\[ S_k^+ [D^2 u + \sigma(x) I] \geq f(u) \]  

in \( \mathbb{R}^n \).

We first study the necessary and sufficient condition for the existence of sub-\( S_k^+ \)olutions to the augmented Hessian equations

\[ S_k^+(D^2 u + \alpha I) = f(u) \]  

(3)

where \( 1 \leq k < n \), \( \alpha \geq 0 \) is a constant. Osserman [26] proved that the equation

\[ \Delta u = f(u) \]  

(4)

has a subsolution if and only if

\[ \int_0^\infty \left( \int_0^\tau f(t)dt \right)^{-\frac{1}{2}} d\tau = \infty, \]  

(5)

where we omit the lower limit to admit any positive constant. For \( f(u) = u^p, p > 1 \), Keller [21] proved that (4) has no positive solution. Then the growth condition (5) is the well known Keller-Osserman condition. Ji and Bao [16] proved that if \( f \) is nonnegative, nondecreasing and continuous on \( \mathbb{R} \), the Hessian equation

\[ S_k^+(D^2 u) = f(u) \]  

in \( \mathbb{R}^n \)

has a positive subsolution \( u \in C^2(\mathbb{R}^n) \) if and only if

\[ \int_0^\infty \left( \int_0^\tau f^k(t)dt \right)^{-\frac{1}{k+1}} d\tau = \infty. \]  

(6)

Covei [8] studied the Keller-Osserman condition on the boundary blow up solution of Hessian equations. The Keller-Osserman condition on the existence of entire solutions is similar to that on the existence of boundary blow up solutions, we can also refer to [11]. For the Keller-Osserman condition on \( k- \)Yamabe type equations, we can see [1]. Analogous results for fully nonlinear equations have been obtained by Dolcetta, Leoni and Vitolo [4, 5], Cutr`ı and Leoni[10], Felmer and Quaas [12], Lu and Zhu [24] and the references therein.

Let the positive constant \( \alpha_0 \) satisfy

\[ \alpha_0 > \frac{n}{k} \left( \frac{n!}{C_{n-1}^k} \right)^\frac{1}{k} \alpha = \frac{n}{k} \left( C_n^k \right)^\frac{1}{k} \alpha, \]  

(7)

where \( C_n^m = \frac{n!}{m!(n-m)!}, 0 \leq m \leq n. \)
Theorem 1.1. Let \(1 \leq k < n\) and \(f(t)\) be a continuous and nondecreasing function on \(\mathbb{R}\) and \(f(t) \geq \alpha_0\). Then there exists a subsolution \(u \in C^2(\mathbb{R}^n)\) of (3) if and only if (6) holds.

In the proof of necessity of Theorem 1.1, due to the impact of \(\alpha I\), different from the standard Hessian equations, we cannot have a good estimate as [16]. By giving the condition \(f \geq \alpha_0\), through Lemma 2.2 and the convexity inequality, we overcome the impact of \(\alpha I\). We need not only \(\varphi'(r) > 0\), but also
\[
\varphi'(r) \geq \left((C_k^k)^{-1/k} \alpha_0 - \alpha\right)r,
\]
see Lemma 2.2.

Remark 1. If the monotonicity of \(f\) fails, we are not sure about the existence of solutions.

Remark 2. One could observe that, if \(u\) is a subsolution of (3) for a given \(\alpha\), then \(u\) is an entire subsolution for any bigger \(\alpha\). In fact, remaining in the cone \(\Gamma_k\) of admissible functions, \(S_k\) is elliptic and therefore, if \(\alpha_1 > \alpha\), then
\[
S^\frac{1}{k}(D^2u + \alpha_1 I) \geq S^\frac{1}{k}(D^2u + \alpha I) = f(u).
\]
So assuming \(f(t) \geq \alpha_0\): if the Keller-Osserman type condition (6) holds, there exist entire subsolutions for all \(\alpha > 0\); vice versa, if there exists an entire subsolution with \(\alpha\) small enough as in condition (7), then the Keller-Osserman type condition (6) must hold.

Corollary 1. Let \(p > 0\) and
\[
f(t) = \begin{cases} 
  t^p + \alpha_0, & t > 0, \\
  \alpha_0, & t \leq 0.
\end{cases}
\]
Then for \(\alpha \geq 0\) there exists a subsolution of (3) if and only if \(p \leq 1\).

Corollary 2. Let \(l \geq 0\) and
\[
f(t) = \begin{cases} 
  e^{lt}, & t > 0, \\
  1, & t \leq 0.
\end{cases}
\]
If \(0 \leq \alpha < 1\) then for \(l = 0\), there exists a subsolution of (3), and for \(l > 0\), there exists no subsolution of (3).

Then we consider the viscosity subsolutions (the definition of viscosity subsolutions will be given in section 4) of the equation
\[
S_k^\frac{1}{k}[D^2u + \sigma(x)I] = u^p
\]
in the whole space \(\mathbb{R}^n\) or in the half space \(\mathbb{R}^n_+\), where \(\sigma(x) \in C(\mathbb{R}^n)\) is a positive function, \(\sigma(x) \leq \sigma_0\) for some positive constant \(\sigma_0\) and \(p\) is a positive constant. Jin, Li and Xu [20] investigated the viscosity subsolutions for the equation
\[
S_k^\frac{1}{k}(D^2u) = u^p
\]
in the whole space \(\mathbb{R}^n\) or in the half space \(\mathbb{R}^n_+\). In this paper, we extend the results in [20] to the equation (8).

Theorem 1.2. If \(p > 1\), then (8) has no positive continuous viscosity subsolution in \(\mathbb{R}^n\).

Let \(g(x, t)\) be defined on \(\partial\mathbb{R}^n_+ \times [0, \infty)\) such that \(g(x, t) > 0\) for any \(x \in \partial\mathbb{R}^n_+\) and \(t > 0\).
If \( p > 1 \), then the problem
\[
\begin{align*}
\frac{\partial^2}{\partial x^2} [D^2 u + \sigma(x) I] &= u^p & \text{in } \mathbb{R}^n_+, \\
\frac{\partial u}{\partial x_n} &= g(x, u) & \text{on } \partial \mathbb{R}^n_+
\end{align*}
\] (9)
has no positive continuous viscosity subsolution.

**Question 1.** If \( 0 < p \leq 1 \), how about the solutions of (8) and (9)?

This paper is arranged as follows. In section 2, we give some basic results of radial functions. Theorem 1.1 will be proved in section 3 and Theorems 1.2 and 1.3 will be proved in section 4.

2. Some basic results of radial functions. In this section, some results for the radial functions will be given. Let \( r = |x| = \sqrt{x_1^2 + \cdots + x_n^2} \).

**Lemma 2.1.** Suppose that \( \varphi : [0, \infty) \rightarrow \mathbb{R} \) is a \( C^2 \) radial function satisfying \( \varphi'(0) = 0 \). Let \( u(x) = \varphi(r) \), then \( u(x) \in C^2(\mathbb{R}^n) \) and the eigenvalues of \( D^2 u + \alpha I \) are
\[
\lambda(D^2 u + \alpha I) = \begin{cases} 
(\varphi''(r) + \alpha, \frac{\varphi'(r)}{r} + \alpha, \cdots, \frac{\varphi'(r)}{r} + \alpha), & r \in (0, \infty), \\
(\varphi''(0) + \alpha, \varphi''(0) + \alpha, \cdots, \varphi''(0) + \alpha), & r = 0.
\end{cases}
\]
Hence
\[
S_k(D^2 u + \alpha I) = \begin{cases} 
C_{k-1}(\varphi''(r) + \alpha) \frac{(\varphi'(r) + \alpha)^{k-1}}{r^{k-1}} + C_k \frac{(\varphi'(r) + \alpha)^k}{r^k}, & r \in (0, \infty), \\
C_k(\varphi''(0) + \alpha)^k, & r = 0.
\end{cases}
\]

We can prove Lemma 2.1 similarly as Lemma 2.1 in [16]. Here we omit its proof. By Lemma 2.1, we conclude that \( u(x) = \varphi(r) \) is a \( C^2 \) radial solution of (3) if and only if \( \varphi(r) \) satisfies
\[
C_{k-1}(\varphi''(r) + \alpha) \frac{(\varphi'(r) + \alpha)^{k-1}}{r^{k-1}} + C_k \frac{(\varphi'(r) + \alpha)^k}{r^k} = f^k(\varphi(r)), & r \in (0, +\infty).
\] (10)

**Lemma 2.2.** Assume that \( f(t) \) is a continuous function on \( \mathbb{R} \), \( f(t) \geq \alpha_0 \) and \( \varphi(r) \in C^2[0, \infty) \) satisfies (10) with \( \varphi'(0) = 0 \), then
\[
\varphi'(r) \geq ((C_k)^{-1/k} \alpha_0 - \alpha)r > 0, r \in (0, +\infty).
\]

In addition, if \( f \) is nondecreasing, then \( \varphi''(r) + \alpha > 0, r \in (0, +\infty) \).

**Proof.** From (10), we have
\[
C_{k-1}(\varphi''(r) + \alpha) \frac{(\varphi'(r) + \alpha)^{k-1}}{r^{k-1}} = kr^{n-1}f^k(\varphi(r)).
\]
Noticing that \( \varphi'(0) = 0 \) and integrating on \( r \), we know that
\[
\varphi'(r) = r^{\frac{k-n}{k}} \left( \int_0^r \frac{k}{C_{k-1}} s^{n-1} f^k(\varphi(s)) ds \right)^{\frac{1}{k}} - \alpha r.
\] (11)
Since \( f \geq \alpha_0 \), by (7), then
\[
\varphi'(r) \geq r^{\frac{k-n}{k}} \left( \int_0^r \frac{k}{C_{k-1}} \alpha_0 s^{n-1} ds \right)^{\frac{1}{k}} - \alpha r
\]
\[
= ((C_k)^{-1/k} \alpha_0 - \alpha)r > 0.
\] (12)
Lemma 2.3. Assume that \( f(t) \) is a continuous function on \( \mathbb{R} \). Let \( \varphi(r) \in C^0[0, R] \cap C^1(0, R) \) satisfy (11). Then \( \varphi(r) \in C^2[0, R] \) and \( \varphi \) satisfies (10) with \( \varphi'(0) = 0 \).

Proof. On one hand, we have

\[
\lim_{r \to 0} \varphi'(r) = \lim_{r \to 0} r^{-\frac{k-n}{k}} \left( \int_0^r \frac{k}{C_{n-1}^{k-1}} f_k(s) s^{n-1} ds \right)^{\frac{1}{k}} - \alpha r
= 0 = \varphi'(0).
\]

So \( \varphi(r) \in C^1[0, R] \).

On the other hand,

\[
\varphi''(0) = \lim_{r \to 0} \frac{\varphi'(r) - \varphi'(0)}{r - 0}
= \lim_{r \to 0} r^{\frac{k-n}{k}} \left( \int_0^r \frac{k}{C_{n-1}^{k-1}} f_k(s) s^{n-1} ds \right)^{\frac{1}{k}} - \alpha r
= \left( \frac{k}{nC_{n-1}^{k-1}} \right)^{\frac{1}{k}} f(\varphi(0)) - \alpha.
\]

It is clear that \( \varphi(r) \in C^2(0, R) \). For \( r \in (0, R) \) by (13),

\[
\lim_{r \to 0} \varphi''(r)
= \lim_{r \to 0} \frac{k-n}{k} r^{-\frac{k-n}{k}} \left( \int_0^r \frac{k}{C_{n-1}^{k-1}} f_k(s) s^{n-1} ds \right)^{\frac{1}{k}}
+ \lim_{r \to 0} r^{\frac{k-n}{k}} \left( \int_0^r \frac{k}{C_{n-1}^{k-1}} f_k(s) s^{n-1} ds \right)^{\frac{1}{k}-1} \frac{1}{C_{n-1}^{k-1}} f_k(s) s^{n-1} ds \right)^{\frac{1}{k}-1} - \alpha
= \left( \frac{k}{nC_{n-1}^{k-1}} \right)^{\frac{1}{k}} f(\varphi(0)) - \alpha
= \varphi''(0).
\]

Consequently, \( \varphi(r) \in C^2[0, R] \).

Finally, by (11) and (13), we know that \( \varphi \) satisfies (10). \( \square \)
By Lemma 2.2 and Lemma 2.3, we have that $\varphi(r) \in C^2[0, R]$ and $\varphi$ satisfies (10) with $\varphi'(0) = 0$ if and only if $\varphi(r) \in C^0[0, R] \cap C^1(0, R)$ and $\varphi$ satisfies (11).

Next we prove an existence result for (11).

**Lemma 2.4.** Let $f(t) \geq \alpha_0$. Suppose that $f(t)$ is continuous and nondecreasing on $\mathbb{R}$. Then for any constant $a$, there exists a positive number $R$ such that (11) with the initial value

$$\varphi(0) = a, \quad \varphi'(0) = 0$$

has a solution $\varphi \in C^2(0, R) \cap C^0[0, R]$.

**Proof.** As the proof of Lemma 2.3 in [16]. We define a functional $F(\cdot, \cdot)$ on $\mathbb{R} = [0, l] \times \{ \varphi \in C^2[0, l] : a - h < \varphi < a + h \}$ as

$$F(r, \varphi) := r^{k-n} \left( \int_0^r \frac{k}{C_{n-1}^k} s^{n-1} f^k(\varphi(s)) ds \right) - \alpha r,$$

where $l$ and $h$ are positive constants small enough. Then (11) can be rewritten as

$$\varphi'(r) = F(r, \varphi).$$

(15)

By Lemma 2.2, $F > 0$ for $r > 0$.

Define an Euler's break line on $[0, l]$ as

$$\begin{cases} 
  \psi(0) = a, \\
  \psi(r) = \psi(r_{i-1}) + F(r_{i-1}, \psi(r_{i-1}))(r - r_{i-1}), \quad r_{i-1} < r \leq r_i,
\end{cases}$$

(16)

where $0 = r_0 < r_1 < \cdots < r_m = l$. Then $\psi \in C^2[0, l]$. We claim that $(r, \psi) \in \mathcal{R}$.

Indeed, for any $(r, \psi) \in \mathcal{R}$, we have

$$F(r, \psi(r)) \leq r^{k-n} f(a + h) \left( \int_0^r \frac{k}{C_{n-1}^k} s^{n-1} ds \right)^{\frac{1}{2}} \leq \left( \frac{k}{nC_{n-1}^{k-1}} \right)^{\frac{1}{2}} l f(a + h).$$

(17)

Then for the break line $\psi$, we have

$$a - h \leq \psi(r) \leq a + \sum_i F(r_{i-1}, \psi(r_{i-1}))(r - r_{i-1}) \leq a + Cl^2 f(a + h), \quad r \in [0, l].$$

Thus if $h$ is fixed, we can choose $l$ sufficiently small such that

$$a - h \leq \psi(r) \leq a + h.$$

Next, we prove that the Euler's break line $\psi$ is an $\varepsilon-$approximation solution of (11). For this, we only need to prove that for any $\varepsilon > 0$, we can choose points $\{r_i\}_{i=1, \ldots, m}$ such that the break line satisfies

$$\left| \frac{d\psi(r)}{dr} - F(r, \psi(r)) \right| < \varepsilon, \quad r \in [0, l].$$

(18)
Indeed, we see from the second line in (17) that
\[
\lim_{r \to 0} F(r, \psi(r)) \leq \lim_{r \to 0} f(a + h) \left( \int_0^r \frac{k}{C_{n-1}^{k-1}} s^{n-1} ds \right)^{\frac{1}{q}} = 0.
\]
Then for any \( \psi \in C^2[0, l] \), \( a - h \leq \psi \leq a + h \),
\[
\lim_{r \to 0} F(r, \psi) = 0.
\]
So for any \( \varepsilon > 0 \), there exists \( \bar{r} \in (0, l) \) such that for \( 0 \leq r < \bar{r} \), we have
\[
F(r, \psi) < \frac{\varepsilon}{2}.
\]
Then
\[
\left| \frac{d\psi(r)}{dr} - F(r, \psi(r)) \right| = |F(r_{i-1}, \psi(r_{i-1})) - F(r, \psi(r))| < \varepsilon.
\]
If \( \bar{r} \leq r \leq l \), we have
\[
\frac{d\psi(r)}{dr} - F(r, \psi(r)) = F(r_{i-1}, \psi(r_{i-1})) - F(r, \psi(r))
\]
\[
= \left| \int_0^{r_{i-1}} \frac{k}{C_{n-1}^{k-1}} s^{n-1} f^k(\psi(s)) ds \right| - \alpha r_{i-1}
\]
\[
\left| \frac{r_{i-1}^{k-1}}{C_{n-1}^{k-1}} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds \right| + \alpha |r - r_{i-1}|
\]
\[
\leq \left( \frac{k}{C_{n-1}^{k-1}(r_{i-1})^{n-k}} \right)^{\frac{1}{q}} \left| r_{i-1}^{k-1} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds \right| + \alpha |r - r_{i-1}|.
\]
On the other hand, for \( \tau \leq r \leq l \), we have
\[
\frac{r_{i-1}^{k-1}}{C_{n-1}^{k-1}(r_{i-1})^{n-k}} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds - r_{i-1}^{k-1} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds
\]
\[
\leq \left( \left| r_{i-1}^{k-1} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds - r_{i-1}^{k-1} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds \right| \right)^{\frac{1}{q}}
\]
\[
= \left( \left| r_{i-1}^{k-1} \int_{r_{i-1}}^{r} s^{n-k} f^k(\psi(s)) ds \right| + \left| r_{i-1}^{k-1} - r_{i-1}^{n-k} \int_0^{r_{i-1}} s^{n-k} f^k(\psi(s)) ds \right| \right)^{\frac{1}{q}}
\]
\[
\leq r^{-\frac{n-k}{2}} \int_{r_{i-1}}^{r} s^{n-1} f^k(\psi(s)) ds \left| \frac{d}{dr} \right| + r^{n-k} - r_{i-1}^{n-k} \left| \frac{d}{dr} \right| \left( \int_{0}^{r} s^{n-1} f^k(\psi(s)) ds \right) \left| \frac{d}{dr} \right|
\leq n^{-\frac{1}{2}} l^{\frac{n-k}{2}} f(a + h) \left| r^n - r_{i-1}^n \right|^{\frac{1}{2}} + n^{-\frac{1}{2}} l^{\frac{n-k}{2}} f(a + h) \left| r^n - r_{i-1}^n \right|^{\frac{1}{2}}.
\]

Then if \( r \leq l \),
\[
\frac{d\psi(r)}{dr} - F(r, \psi(r)) \leq \left( \frac{k}{c_{n-1}^{k-1/2}(n-k)} \right)^{\frac{1}{2}} n^{-\frac{1}{2}} l^{\frac{n-k}{2}} f(a + h) \left| r^n - r_{i-1}^n \right|^{\frac{1}{2}} + \left( \frac{k}{c_{n-1}^{k-1/2}(n-k)} \right)^{\frac{1}{2}} n^{-\frac{1}{2}} l^{\frac{n-k}{2}} f(a + h) \left| r^n - r_{i-1}^n \right|^{\frac{1}{2}} + \alpha \left| r - r_{i-1} \right|.
\]

Since \( r^n \) is Lipschitz continuous on \([\bar{r}, l]\), for the above \( \varepsilon \), there exists \( 0 < \delta(\varepsilon) < \varepsilon \) such that for \( r', r'' \in [\bar{r}, l] \) and \( |r' - r''| < \delta(\varepsilon) \),
\[
|r^n - r'^n| < \left( \frac{k}{c_{n-1}^{k-1/2}(n-k)} \right)^{\frac{1}{2}} n^{-\frac{1}{2}} l^{\frac{n-k}{2}} f(a + h) \varepsilon^k,
\]
and
\[
|r^{n-k} - r'^{n-k}| < \frac{\varepsilon^k}{c_{n-1}^{k-1/2}(n-k)n^k f(a + h)}.
\]

Let \( r_1 = \bar{r} \) and
\[
\max_{2 \leq i \leq m} |r_{i-1} - r_i| < \min \{ \bar{r}, \delta(\varepsilon) \},
\]
then we have (18).

The rest of the proof is similar to the proof of Lemma 2.3 in [16].

Then according to Lemma 2.3, we know that the solution \( \varphi(r) \) of (11) with the initial value \( \varphi(0) = a, \varphi'(0) = 0 \) is in \( C^2[0, R] \).

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1. The proof of Theorem 1.1 includes the following three Lemmas.

**Lemma 3.1.** Let \( f(t) \) be continuous, nondecreasing and positive on \( \mathbb{R} \). Suppose that (10) has a solution \( \varphi(r) \in C^2[0, R] \) satisfying \( \varphi'(0) = 0 \), and \( \varphi(r) \to +\infty \) as \( r \to R \). Then if \( u \) is a subsolution of (3), we have \( u(x) \leq \varphi(|x|) \) in \( B_R = \{ x : |x| \leq R \} \).

**Proof.** Let \( v(x) = \varphi(r) = \varphi(|x|) \), then \( v(x) \) satisfies
\[
S_k(D^2 v + \alpha I) = f^k(v), \ |x| < R.
\]

We only need to prove that \( u(x) \leq v(x) \) in \( B_R \). Suppose on the contrary that \( u > v \) at some point. Then there exists some positive constant \( \tau \) such that \( u - \tau \) touches \( v \) from below at some interior point \( \tau \in B_R \), i.e., \( u(\tau) - \tau - v(\tau) = 0 \), and \( u - \tau - v \leq 0 \) in \( B_R \). Due to the fact that \( v(x) \to +\infty \) as \( |x| \to R \), we can choose some \( |\tau| < R' < R \) such that \( u - \tau - v \leq 0 \) in \( B_{R'} \) and \( \sup_{\partial B_{R'}} (u - \tau - v) < 0 \).
In addition, we have in \( B_R' \),

\[
S_k[D^2(u - \tau) + \alpha I] - f^k(u - \tau) = S_k(D^2u + \alpha I) - f^k(u - \tau)
\geq f^k(u) - f^k(u - \tau)
\geq 0 = S_k(D^2v + \alpha I) - f^k(v).
\]

By linearizing the operator \( S_k \) and using the maximum principle of linear elliptic equations \([6, 13]\), we know that in \( B_R' \),

\[
0 = \sup_{B_R'}(u - \tau - v) = \sup_{\partial B_R'}(u - \tau - v) < 0.
\]

This is a contradiction. Then we complete the proof of Lemma 3.1. \(\square\)

**Lemma 3.2.** Suppose that \( f(t) \) is a continuous and nondecreasing function and \( f(t) \geq \alpha_0 \) on \( \mathbb{R} \), then there exists a subsolution \( u \) of \( (3) \) if and only if \( (11) \) has a solution \( \varphi \in C^2[0, +\infty) \) satisfying \( \varphi'(0) = 0 \) and \( \varphi(0) = a \) for any constant \( a \).

**Proof.** **Sufficiency.** Assume that \( (11) \) has a solution \( \varphi \), then clearly \( v(x) = \varphi(x) = \varphi(|x|) \) is the desired solution of \( (3) \).

**Necessity.** Conversely, suppose that \( (11) \) has no such function \( \varphi(r) \) existing globally. From Lemma 2.4 and Lemma 2.3, \( (11) \) has a \( C^2 \) solution \( \varphi \) on some interval with \( \varphi(0) = a, \varphi'(0) = 0 \) which cannot be a global solution. Then there is a maximal interval \( [0, R] \) in which the solution exists. From Lemma 2.2, we have \( \varphi'(r) > 0 \) for \( r \in (0, R) \), then \( \varphi(r) \to \infty \) as \( r \to R \). In fact, if \( \varphi \) is bounded above, then by (15) \( \varphi' \) is bounded above as well, and this contradicts the maximality of \( R \) if \( R < +\infty \). So from Lemma 3.1, any subsolution of \( (3) \) satisfies \( u(x) \leq \varphi(|x|) \) for \( |x| < R \). In particular, we have \( u(0) \leq \varphi(0) = a \). However since \( a \) is arbitrary, letting \( a = u(0)/2 \), we know that it is a contradiction. Lemma 3.2 is proved. \(\square\)

**Lemma 3.3.** If \( f(t) \) is continuous and nondecreasing and \( f(t) \geq \alpha_0 \) on \( \mathbb{R} \), then \( (11) \) has a solution \( \varphi \in C^2[0, +\infty) \) satisfying \( \varphi'(0) = 0 \) if and only if \( (6) \) holds.

**Proof.** **Sufficiency.** Suppose that there is no such solution of \( (11) \). Just as the proof of Lemma 3.2, \( (11) \) has a \( C^2 \) solution \( \varphi(r) \) on the maximal interval \( [0, R] \) satisfying \( \varphi'(0) = 0, \varphi(0) = 0 \) and \( \varphi(r) \to \infty \) as \( r \to R \). Hence \( \varphi \) satisfies \( (10) \). Since by Lemma 2.2, we have \( \varphi'(r) > 0, \varphi''(r) + \alpha > 0 \), then we know that

\[
f^k(\varphi(r)) = (\varphi''(r) + \alpha)C_{n-1}^k(\varphi'(r) + \alpha r)^{k-1} + C_{n-1}^kC_{n-1}^k(\varphi'(r) + \alpha r)^k \geq C_{n-1}^kC_{n-1}^k(\varphi'(r) + \alpha r)^{k-1} \geq C_{n-1}^kC_{n-1}^k(\varphi'(r))^{k-1} \frac{\varphi'(r)}{r^{k-1}}.
\]

Therefore,

\[
C_{n-1}^kC_{n-1}^k(\varphi'(r))^{k-1} \leq f^k(\varphi(r)r^{k-1}).
\]

For \( 0 < r < R \), we have that

\[
C_{n-1}^kC_{n-1}^k(\varphi'(r))^{k-1} \leq f^k(\varphi(r))r^{k-1}.
\]

Multiplying \( \varphi'(r) \) on both sides, we have

\[
\varphi''(r)(\varphi'(r))^k \leq \frac{1}{C_{n-1}^k}R^{k-1}f^k(\varphi(r))\varphi'(r),
\]

that is,

\[
((\varphi'(r))^{k+1})' \leq \frac{k+1}{C_{n-1}^k}R^{k-1}f^k(\varphi(r))\varphi'(r).
\]
Integrating on $r$ from 0 to $r$, we have
\[(\varphi'(r))^{k+1} \leq \frac{k+1}{C_{n-1}^k} R^{k-1} \int_{\varphi(0)}^{\varphi(r)} f^k(t) dt.\]

Thus
\[\varphi'(r) \leq \left(\frac{k+1}{C_{n-1}^k} R^{k-1}\right)^{\frac{1}{k+1}} \left(\int_{\varphi(0)}^{\varphi(r)} f^k(t) dt\right)^{\frac{1}{k+1}}.\]

Consequently,
\[
\left(\int_{\varphi(0)}^{\varphi(r)} f^k(t) dt\right)^{\frac{1}{k+1}} d\varphi \leq \left(\frac{k+1}{C_{n-1}^k} R^{k-1}\right)^{\frac{1}{k+1}} R^{\frac{2k}{k+1}} < \infty.
\]

Noting that $\varphi(0) = 0$ and $\varphi(R) = \infty$ and integrating on $r$ from 0 to $R$, we know
\[
\int_0^\infty \left(\int_0^\tau f^k(t) dt\right)^{\frac{1}{k+1}} d\tau < \infty,
\]
which contradicts with (6).

**Necessity.** Suppose on the contrary, $\int_0^\infty \left(\int_0^\tau f^k(t) dt\right)^{\frac{1}{k+1}} d\tau < \infty$.

We first prove that
\[
\frac{f(t)}{t} \to \infty, \text{ as } t \to \infty. \tag{19}
\]

Indeed, let
\[g(\tau) = \left(\int_0^\tau f^k(t) dt\right)^{\frac{1}{k+1}}.\]

Then $\int_0^\infty g(\tau) d\tau < \infty$. Thus we know that $\int_0^\infty g(\tau) d\tau \to 0$ as $s \to \infty$. Moreover, we note that $g(\tau)$ is nonincreasing in $(0, \infty)$. Hence
\[0 \leq \tau g(\tau) \leq 2 \int_0^\tau g(s) ds < 2 \int_\tau^\infty g(s) ds \to 0, \text{ as } \tau \to \infty.
\]

So
\[\tau^{-(k+1)} \int_0^\tau f^k(t) dt = (\tau g(\tau))^{-(k+1)} \to \infty, \text{ as } \tau \to \infty.
\]

Since $f$ is nondecreasing, then we have
\[\tau^{-(k+1)} \int_0^\tau f^k(t) dt \leq \tau^{-(k+1)} \tau f^k(\tau) = \tau^{-k} f^k(\tau) = \left(\frac{f(\tau)}{\tau}\right)^k.
\]

Thus as $\tau \to \infty$, we have $f(\tau)/\tau \to \infty$, and (19) holds.

Then by (19), there exists $t_1 > 0$ such that $f(t) \geq t - \varphi(0)$ for $t > t_1$. Integrating (12) on both sides, we know that
\[
\varphi(r) \geq \frac{1}{2} \left(\left(\frac{k}{nC_{n-1}^k}\right)^{1/k} \alpha_0 - \alpha\right) r^2 + \varphi(0).
\]
Thus for the above $t_1$, there is $r_1 > 0$ such that $\varphi(r) > t_1$ for $r > r_1$. As a result,
$$f(\varphi(r)) \geq \varphi(r) - \varphi(0), r > r_1. \quad (20)$$

Furthermore, by (10), we have
$$C_{n-1}^{k-1}(\varphi''(r) + \alpha) \left( \frac{\varphi'(r) + \alpha r}{r} \right)^{k-1} \leq f^k(\varphi(r)).$$

Multiplying the above inequality by $\varphi'(r) + \alpha r$ on both sides, then we know that
$$C_{n-1}^{k-1}((\varphi'(r) + \alpha r)^k) \leq (k + 1)r^{k-1}f^k(\varphi(r))\varphi'(r) + \alpha(k + 1)r^k f^k(\varphi(r)).$$

Since $f$ and $\varphi$ are nondecreasing, we have by integrating the above inequality from 0 to $r$,
$$C_{n-1}^{k-1}(\varphi'(r) + \alpha r)^k + 1 \leq (k + 1)r^{k-1}f^k(\varphi(r))\varphi'(r) + \alpha(k + 1)r^k f^k(\varphi(r)).$$

So by (20), we know that for $r > r_1$,
$$C_{n-1}^{k-1}(\varphi'(r) + \alpha r)^k \leq (k + 1)r^{k-1}f^k(\varphi(r))\varphi'(r) + \alpha(k + 1)r^k f^k(\varphi(r)).$$

Therefore
$$\left( \frac{\varphi'(r) + \alpha r}{r} \right)^k \leq f^k(\varphi(r)) \left[ \frac{k + 1}{\alpha} \left( \frac{1}{r^2} + \frac{\alpha}{f(\varphi(r))} \right) \right].$$

and then
$$C_{n-1}^{k-1} \left( \frac{\varphi'(r) + \alpha r}{r} \right)^k \leq f^k(\varphi(r)) \left[ \frac{(C_{n-1}^{k-1})^{k^2} + 1}{r^2} + \frac{(C_{n-1}^{k-1})^{k^2}}{f(\varphi(r))} \right].$$

Since $f \geq \alpha_0$, by (7), we know that
$$\left( \frac{C_{n-1}^{k+1}}{C_{n-1}^{k+1}} \frac{\alpha}{f(\varphi(r))} \right)^{\frac{k}{k+1}} < 1 - \frac{k}{n}.$$
Multiplying $\varphi'(r) + \alpha r$ on both sides, we have for $r > r_2$
\[
(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^k \geq \frac{k}{n}(C_{n-1}^{k-1} - 1) - 1 r^k(\varphi(r)) + \frac{k}{n} \alpha(C_{n-1}^{k-1} - 1 r^k f^k(\varphi(r))).
\]
By (7) and the fact that $f \geq \alpha_0$,
\[
(\varphi''(r) + \alpha)(\varphi'(r) + \alpha r)^k \geq \frac{k}{n}(C_{n-1}^{k-1} - 1) - 1 r^k(\varphi(r)) + \alpha k^r r^k.
\]
Hence for $r > r_2 > 1$, we know that
\[
((\varphi'(r) + \alpha r)^{k+1})' > \frac{k(k + 1)}{n}(C_{n-1}^{k-1} - 1) r^k(\varphi(r)) \varphi'(r) + (k + 1) \alpha k^r r^k.
\]
Integrating the above inequality from $r_2$ to $r$, we have that
\[
(\varphi'(r) + \alpha r)^{k+1} - (\varphi'(r_2) + \alpha r_2)^{k+1} > \frac{k(k + 1)}{n}(C_{n-1}^{k-1} - 1) \int_{\varphi(r_2)}^{\varphi(r)} f^k(t) dt + \alpha k^r (r^{k+1} - r_2^{k+1}).
\]
Thus
\[
(\varphi'(r) + \alpha r)^{k+1} \geq \frac{k(k + 1)}{nC_{n-1}^{k-1} - 1} \int_{\varphi(r_2)}^{\varphi(r)} f^k(t) dt + \alpha k^r r^{k+1}.
\]
Due to Lemma 2.2, we know that
\[
r \leq \frac{1}{(C_{n-1}^{k-1} - 1/k \alpha_0 - \alpha \varphi'(r)}.
\]
So by the convexity inequality,
\[
(\varphi'(r) + \alpha r)^{k+1} \leq \left( \frac{1}{2} (2 \varphi'(r)) + \frac{1}{2} (2 \alpha r) \right)^{k+1} \\
\leq 2^k ((\varphi'(r))^{k+1} + (\alpha r)^{k+1}) \\
= 2^k (\varphi'(r))^{k+1} + (2 - 1)(\alpha r)^{k+1} + (\alpha r)^{k+1} \\
\leq C_0 (\varphi'(r))^{k+1} + (\alpha r)^{k+1},
\]
where $C_0$ is some positive constant. Noting that $nC_{n-1}^{k-1}/k = C_n^k$, thus from (21), we have
\[
C_0(\varphi'(r))^{k+1} + (\alpha r)^{k+1} \geq \frac{k + 1}{C_{n-1}^{k-1} C_0} \int_{\varphi(r_2)}^{\varphi(r)} f^k(t) dt + \alpha k^r r^{k+1},
\]
so,
\[
(\varphi'(r))^{k+1} \geq \frac{k + 1}{C_{n-1}^{k-1} C_0} \int_{\varphi(r_2)}^{\varphi(r)} f^k(t) dt.
\]
Hence
\[
\left( \int_{\varphi(r_2)}^{\varphi(r)} f^k(t) dt \right)^{- \frac{1}{k+1}} \cdot d\varphi \geq \left( \frac{k + 1}{C_{n-1}^{k-1} C_0} \right)^{\frac{1}{k+1}} \cdot dr = C_1 dr,
\]
where $C_1 = \left( \frac{k + 1}{C_{n-1}^{k-1} C_0} \right)^{\frac{1}{k+1}}$. Integrating on $r$ from $r_2$ to $r$, we know
\[
C_1 (r - r_2) \leq \int_{\varphi(r_2)}^{\varphi(r)} \left( \int_{\varphi(r_2)}^{\varphi(r)} f^k(t) dt \right)^{- \frac{1}{k+1}} \cdot dt \\
\leq \int_{\varphi(r_2)}^{\infty} \left( \int_{\varphi(r_2)}^{r} f^k(t) dt \right)^{- \frac{1}{k+1}} \cdot dr.
\]
(22)
Since \( f \) is nondecreasing, then for \( \tau \geq 2\varphi(r_2) \), we know
\[
\int_0^{\varphi(r_2)} f^k(t) dt \leq \varphi(r_2)f^k(\varphi(r_2)) \leq \frac{\tau}{2} f^k\left(\frac{\tau}{2}\right) \leq \int_0^{\tau} f^k(t) dt.
\]
Thus
\[
\int_0^{\tau} f^k(t) dt = \int_0^{\varphi(r_2)} f^k(t) dt + \int_{\varphi(r_2)}^{\tau} f^k(t) dt - \int_0^{\tau} f^k(t) dt \leq \int_{\varphi(r_2)}^{\tau} f^k(t) dt.
\]
Therefore, by (22),
\[
C_1(r - r_2) \leq \int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau f^k(t) dt\right)^{-\frac{1}{1+p}} d\tau = 2 \int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau f^k(t) dt\right)^{-\frac{1}{1+p}} d\tau.
\]
Letting \( r \to \infty \), we get a contradiction.  \( \square \)

**Proof of Theorem 1.1.** By Lemma 3.2 and Lemma 3.3, we know that Theorem 1.1 is true.  \( \square \)

**Proof of Corollary 1.** For \( 0 < p \leq 1 \),
\[
\int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau f^k(t) dt\right)^{-\frac{1}{1+p}} d\tau = \int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau (t^p + \alpha_0)^k dt\right)^{-\frac{1}{1+p}} d\tau = \infty.
\]
For \( p > 1 \),
\[
\int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau f^k(t) dt\right)^{-\frac{1}{1+p}} d\tau < \infty.
\]
Therefore by Theorem 1.1, we can prove that the corollary is true.  \( \square \)

**Proof of Corollary 2.** If \( l = 0 \), then \( f(t) = 1 \), thus
\[
\int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau f^k(t) dt\right)^{-\frac{1}{1+p}} d\tau = \int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau dt\right)^{-\frac{1}{1+p}} d\tau = \infty.
\]
From Theorem 1.1, (3) has a subsolution.

If \( l > 0 \), then
\[
\int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau f^k(t) dt\right)^{-\frac{1}{1+p}} d\tau = \int_{\varphi(r_2)}^{\tau} \left(\int_0^\tau e^{kt} dt\right)^{-\frac{1}{1+p}} d\tau < \infty.
\]
By Theorem 1.1, (3) has no positive subsolution.  Corollary 2 is proved.  \( \square \)

4. **Proofs of Theorem 1.2 and Theorem 1.3.** First, we give the definition of viscosity solutions to (8).  One can also refer to [9, 20].

**Definition 4.1.** We say a nonnegative function \( u \in C(\mathbb{R}^n) \) is a viscosity subsolution of (8) in \( \mathbb{R}^n \), if for any \( x_0 \in \mathbb{R}^n \) there exists an \( \epsilon > 0 \) such that for any \( \phi \in C^2(B_\epsilon(x_0)) \) satisfying \( \phi(x_0) = u(x_0) \),
\[
\phi \geq u \text{ and } \lambda(D^2\phi + \sigma(x)I) \in \Gamma_k \text{ in } B_\epsilon(x_0),
\]
there holds
\[
S^\frac{1}{k}_k [D^2\phi(x_0) + \sigma(x_0)I] \geq \phi^p(x_0).
\]
Similarly, we say a nonnegative function \( u \in C(\mathbb{R}^n) \) is a viscosity supersolution of (8) in \( \mathbb{R}^n \), if for any \( x_0 \in \mathbb{R}^n \) there exists an \( \epsilon > 0 \) such that for any \( \phi \in C^2(B_{\epsilon}(x_0)) \) satisfying \( \phi(x_0) = u(x_0) \),
\[
\phi \leq u \text{ and } \lambda(D^2\phi + \sigma(x)I) \in \Gamma_k \text{ in } B_{\epsilon}(x_0),
\]
there holds
\[
S_k^+ [D^2\phi(x_0) + \sigma(x_0)I] \leq \phi^{\rho}(x_0).
\]

We say a nonnegative function \( u \in C(\mathbb{R}^n) \) is a viscosity solution of (8) in \( \mathbb{R}^n \), if it is both a viscosity subsolution and a viscosity supersolution of (8).

Next we give a comparison principle in [20]. Let \( A(\cdot, \cdot, \cdot): \Omega \times (0, \infty) \times \mathbb{R}^n \to S^{n \times n} \) and \( h(x, t) \) be a positive function defined on \( \Omega \times (0, \infty) \).

\textbf{Lemma 4.2.} ([20]) Suppose that \( \Omega \subset \mathbb{R}^n \) is an open bounded set, and that \( \gamma \to \gamma^{-1}h(x, \gamma) \) is strictly increasing on \( (0, \infty) \) for each \( x \in \Omega \). Let \( u \in C(\overline{\Omega}) \) be a positive viscosity subsolution of
\[
S_k^+ [D^2u + A(x, u, Du)] = h(x, u) \text{ in } \Omega. \tag{23}
\]
Assume that \( v \in C^2(\Omega) \cap C(\overline{\Omega}) \) is a positive classical supersolution of (23) with \( \lambda(D^2v + \gamma^{-1}A(x, \gamma v, \gamma Dv)) \in \Gamma_k \) for each \( \gamma \geq 1 \). Assume also that for \( x \in \Omega \) and \( \xi, p \in \mathbb{R}^n \) the function
\[
\gamma \to \gamma^{-1}\langle A(x, \gamma, \gamma p)\xi, \xi \rangle
\]
is nonincreasing on \( (0, \infty) \). If \( u \leq v \) on \( \partial \Omega \), then \( u \leq v \) on \( \overline{\Omega} \).

\textbf{Proof of Theorem 1.2.} Let \( j = 1, 2, \ldots \) be the positive integers. Suppose that \( B_j \) is the ball with radius \( j \) and center at the origin. Then
\[
\bigcup_{j=1}^{\infty} B_j = \mathbb{R}^n.
\]
In the following we will construct a sequence of positive functions \( \{v_j\} \) with \( v_j \in C^2(B_j) \) such that
\[
\lambda(D^2v_j + \sigma(x)I) \in \Gamma_k \text{ in } B_j, \tag{24}
\]
\[
S_k^+ [D^2v_j + \sigma(x)I] \leq v_j^{\rho} \text{ in } B_j, \tag{25}
\]
\[
v_j(x) \to +\infty \text{ uniformly as } d(x, \partial B_j) \to 0 \tag{26}
\]
and
\[
v_j(x) \to 0 \text{ as } j \to \infty \text{ for any fixed } x \in \mathbb{R}^n. \tag{27}
\]
Assume that such functions \( v_j \) exist and \( u \in C(\mathbb{R}^n) \) is a positive viscosity subsolution of (8) in \( \mathbb{R}^n \), by Lemma 4.2, we can get that
\[
u(x) \leq v_j(x), x \in B_j \text{ for any } j.
\]
Letting \( j \to \infty \), by (27) we have \( u(x) \equiv 0 \), which is a contradiction.

Now we construct such functions \( v_j \). Let
\[
v(x) = (1 - |x|^2)^{-\beta}, x \in B_1,
\]
where \( \beta \) is a positive constant to be determined. Let \( r = |x| \), then \( v(x) = \varphi(r) = (1 - r^2)^{-\beta} \). By straightforward calculation, we know that
\[
\frac{\partial^2 v}{\partial x_k \partial x_l} = \varphi''(r) \frac{r^2}{r^3} x_k x_l - \varphi'(r) \frac{r^3}{r^4} x_k x_l + \frac{\varphi'(r)}{r} \delta_{kl},
\]
we have

\[ v_{kl}(x) + \sigma(x)\delta_{kl} = \frac{\varphi''(r)}{r^2} x_k x_l - \frac{\varphi'(r)}{r^3} x_k x_l + \left( \frac{\varphi'(r)}{r} + \sigma(x) \right) \delta_{kl}. \]

Set

\[ v_j(x) = c_j^\beta (j^2 - |x|^2)^{-\beta} \text{ in } B_j, \]

where \( c \) is a positive constant. Denote \( x/j = X \) and \( v(x/j) = v(X) \), then

\[ v_j(x) = c_j^{-\beta} v(X), \]

\[
(v_j)_{kl} + \sigma(x)\delta_{kl} = c_j^{-\beta - 2} \frac{\varphi''(r)}{r^2} \frac{x_k x_l}{j^2} - c_j^{-\beta - 2} \frac{\varphi'(r)}{r^3} \frac{x_k x_l}{j^3} + \left( c_j^{-\beta - 2} \frac{\varphi'(r)}{r} + \sigma(x) \right) \delta_{kl}, \quad \bar{r} = \frac{|x|}{j}.
\]

Therefore we have that the eigenvalues of the matrix \((v_j)_{kl} + \sigma(x)\delta_{kl}\) are

\[ \lambda_1 = c_j^{-\beta - 2} \frac{\varphi''(r)}{r^2} \left( \frac{|x|}{j} \right) + \sigma(x), \]

\[ \lambda_2 = \cdots = \lambda_n = c_j^{-\beta - 2} \frac{j}{|x|} \frac{\varphi'(r)}{r} \left( \frac{|x|}{j} \right) + \sigma(x). \]

Furthermore we know that

\[
\varphi'(r) = 2\beta r[\varphi(r)]^{\frac{\beta + 1}{\beta}}, \\
\varphi''(r) = 2\beta[\varphi(r)]^{\frac{\beta + 2}{\beta}} [1 + (2\beta + 1)r^2].
\]

Thus,

\[ \lambda_1 = 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left[ 1 + (2\beta + 1) \frac{|x|^2}{j^2} \right] + \sigma(x), \]

\[ \lambda_2 = \cdots = \lambda_n = 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left( 1 - \frac{|x|^2}{j^2} \right) + \sigma(x), \]

\[
S_k [D^2 v_j + \sigma(x)I] = \left\{ 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left( 1 + (2\beta + 1) \frac{|x|^2}{j^2} \right) + \sigma(x) \right\}
\]

\[ - C_{n-1}^k \left\{ 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left( 1 - \frac{|x|^2}{j^2} \right) + \sigma(x) \right\}^{k-1}
\]

\[ + C_{n-1}^k \left\{ 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left( 1 - \frac{|x|^2}{j^2} \right) + \sigma(x) \right\}^k. \]

By the fact that

\[ 1 - \frac{|x|^2}{j^2} \leq 1 + (2\beta + 1) \frac{|x|^2}{j^2}, \quad x \in B_j, \]

we have

\[ 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left( 1 - \frac{|x|^2}{j^2} \right) \]

\[ \leq 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right] \frac{\beta + 2}{\beta} \left( 1 + (2\beta + 1) \frac{|x|^2}{j^2} \right). \]
Therefore
\[ S_k[D^2v_j + \sigma(x)I] \leq C_n^k \left\{ 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right]^{\frac{\beta + 2}{\beta}} \left( 1 + (2\beta + 1)\frac{|x|^2}{j^2} \right) + \sigma(x) \right\}^k. \]

On the other hand,
\[ v^p_j(x) = c^p j^{-\beta p} \left[ v \left( \frac{x}{j} \right) \right]^p. \]

Let \( \beta = \frac{2}{p-1} \), then \( p = \frac{\beta + 2}{\beta} \), so
\[
S_k^\frac{\beta}{\beta} [D^2v_j + \sigma(x)I] - v^p_j \leq (C_n^k)^{\frac{\beta}{\beta}} 2c\beta j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right]^{\frac{\beta + 2}{\beta}} \left( 1 + (2\beta + 1)\frac{|x|^2}{j^2} \right) - c^p j^{-\beta p} \left[ v \left( \frac{x}{j} \right) \right]^p + (C_n^k)^{\frac{\beta}{\beta}} \sigma(x)
\]
\[ \leq c[(C_n^k)^{\frac{\beta}{\beta}} 4\beta (\beta + 1) - c^{p-1}]j^{-\beta - 2} \left[ v \left( \frac{x}{j} \right) \right]^{\frac{\beta + 2}{\beta}} + (C_n^k)^{\frac{\beta}{\beta}} \sigma(x), \ x \in B_j.
\]

If \( x \in B_j \), we have \( v(x/j) \geq 1 \). Then for \( p > 1 \), we can choose \( c \) large enough such that
\[ S_k^\frac{1}{\beta} [D^2v_j + \sigma(x)I] \leq v^p_j, \ x \in B_j.
\]

Therefore \( v_j \) satisfies (24), (25), (26) and (27) and then we complete the proof. \( \square \)

Suppose that \( \Omega \) is an open set in \( \mathbb{R}^n \), the boundary \( \partial \Omega \neq \emptyset \) is smooth and \( \nu \) is the unit inner normal to \( \partial \Omega \). Let \( \Sigma \subset \partial \Omega \) be an open subset of \( \partial \Omega \). Assume that \( h(x,t) \) is a positive function defined on \( \Omega \times (0, \infty) \) and \( g(x,t) \) is any function defined on \( \Sigma \times [0, \infty) \). Consider the problem
\[
S_k^\frac{1}{\beta} [D^2u + \sigma(x)I] = h(x,u) \quad \text{in} \ \Omega,
\]
\[
\frac{\partial u}{\partial \nu} = g(x,u) \quad \text{on} \ \Sigma.
\]

If a positive function \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) is a classical subsolution of \(29\) in \( \Omega \) and \( \partial u/\partial \nu \geq g(x,u) \) on \( \Sigma \), then we call \( u \) is a classical subsolution of \( 29 \) and \( 30 \). The definition of viscosity subsolution of \(29 \) and \( 30 \) is the following which can be referred to \( [20] \) and can also be referred to \( [9] \).

**Definition 4.3.** Let \( u \in C(\Omega \cup \Sigma) \) and \( u > 0 \) in \( \Omega \). A function \( u \) is called a viscosity subsolution of \(29\) and \( 30 \), if \( u \) is a viscosity subsolution of \(29 \), and for any \( x_0 \in \Sigma \) there exists a neighborhood \( U \) of \( x_0 \) such that for any function \( \phi \in C^1(U \cap \overline{\Omega}) \)
\[ \phi(x_0) = u(x_0) \quad \text{and} \quad \phi \geq u \ \text{in} \ U \cap \overline{\Omega} \]
we have
\[ \frac{\partial \phi}{\partial \nu}(x_0) \geq g(x_0, \phi(x_0)). \]

**Remark 3.** The viscosity subsolution here is subsolution in the strong viscosity sense of Definition 7.1 in \( [9] \), while the general viscosity sense is given by Definition 7.4 in \( [9] \).
**Proof of Theorem 1.3.** For each \( j \), in \( B_{j} \) we consider the same function \( v_{j} \) in (28). We know that \( v_{j} \) satisfies (24), (25), (26) and (27). Notice that

\[
\frac{\partial v_{j}}{\partial x_{n}} = 0 \text{ on } B_{j} \cap \partial \mathbb{R}_{+}^{n}.
\]

Suppose that the problem (9) has a positive continuous viscosity subsolution \( u \). We will derive that for each \( j \),

\[
u_{j}(x) \leq v_{j}(x), \quad x \in B_{j}^{+},
\]

where \( B_{j}^{+} := B_{j} \cap \mathbb{R}_{+}^{n} \). Letting \( j \to \infty \), then we have \( u \equiv 0 \) in \( \mathbb{R}_{+}^{n} \) by (27), which is a contradiction.

Suppose that (31) is not true, then we can find a constant \( \eta > 1 \) such that \( u \leq \eta v_{j} \) on \( B_{j}^{+} \) and for some \( x_{0} \in B_{j}^{+} \), \( u(x_{0}) = \eta v_{j}(x_{0}) \). Since \( v_{j}(x) \to +\infty \) as \( d(x, \partial B_{j}) \to 0 \), then \( x_{0} \in B_{j} \cap \mathbb{R}_{+}^{n} \). If \( x_{0} \in \partial \mathbb{R}_{+}^{n} \), noting \( g(x, t) > 0 \) in (9), then by Definition 4.3 we know that

\[
0 = \frac{\partial(\eta v_{j})}{\partial x_{n}}(x_{0}) \geq g(x_{0}, \eta v_{j}(x_{0})) > 0.
\]

This is a contradiction. Thus \( x_{0} \) must be in the interior of \( B_{j}^{+} \). Since \( u \) is a viscosity subsolution of the equation in (9), then by Definition 4.1, we know that

\[
\frac{S_{k}^{\frac{1}{p}}}{D^{2}(\eta v_{j})(x_{0})} + \sigma(x_{0}) I \geq (\eta v_{j})^{p}(x_{0}).
\]

Noticing that \( \eta > 1 \) and \( v_{j} \) satisfies (25), then we have

\[
\frac{S_{k}^{\frac{1}{p}}}{D^{2}(\eta v_{j})(x_{0})} + \sigma(x_{0}) I = \eta S_{k}^{\frac{1}{p}}[D^{2}v_{j}(x_{0}) + \eta^{-1}\sigma(x_{0}) I] \leq \eta S_{k}^{\frac{1}{p}}[D^{2}v_{j}(x_{0}) + \sigma(x_{0}) I] \leq \eta v_{j}(x_{0}).
\]

From (32), we have

\[
\eta v_{j}^{p}(x_{0}) \geq (\eta v_{j})^{p}(x_{0}).
\]

So \( \eta^{p-1} < 1 \), which is a contradiction. Therefore (31) is true.

**Acknowledgments.** The author would like to thank Professor Jiguang Bao for offering the opportunity to visit Beijing Normal University, and also thank School of Mathematical Sciences in Beijing Normal University for providing the good research environment. The author is very grateful to the referee for the very helpful comments and suggestions.

**REFERENCES**

[1] J. G. Bao, X. H. Ji and H. G. Li, Existence and nonexistence theorem for entire subsolutions of k-Yamabe type equations, *J. Differential Equations*, 253 (2012), 2140–2160.

[2] L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations, I. Monge-Ampère equation, *Comm. Pure Appl. Math.*, 37 (1984), 369–402.

[3] L. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian, *Acta Math.*, 155 (1985), 261–301.

[4] I. Capuzzo Dolcetta, F. Leoni and A. Vitolo, Entire subsolutions of fully nonlinear degenerate elliptic equations, *Bull. Inst. Math. Acad. Sin. (N.S.)*, 9 (2014), 147–161.

[5] I. Capuzzo Dolcetta, F. Leoni and A. Vitolo, On the inequality \( F(x, D^{2}u) \geq f(u) + g(u)|Du|^{q} \), *Math. Ann.*, 365 (2016), 423–448.

[6] H. Car and R. Pröpper, Removable singularities of m-Hessian equations, *NoDEA Nonlinear Differential Equations Appl.*, 24 (2017), Art. 6, 18 pp.
[7] K. S. Chou and X. J. Wang, A variational theory of the Hessian equation, *Comm. Pure Appl. Math.*, 54 (2001), 1029–1064.
[8] D. P. Covei, The Keller-Osserman problem for the k-Hessian operator, arXiv:1508.04653.
[9] M. G. Crandall, H. Ishii and P. L. Lions, User’s guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc. (N.S.)*, 27 (1992), 1–67.
[10] A. Cutri and F. Leoni, On the Liouville property for fully nonlinear equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17 (2000), 219–245.
[11] S. Dumont, L. Dupaigne, O. Goubet and V. Radulescu, Back to the Keller-Osserman condition for boundary blow-up solutions, *Adv. Nonlinear Stud.*, 7 (2007), 271–298.
[12] P. L. Felmer and A. Quaas, On critical exponents for the Pucci’s extremal operators, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20 (2003), 843–865.
[13] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[14] B. Guan, Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, *Duke Math. J.*, 163 (2014), 1491–1524.
[15] Y. Huang, F. D. Jiang and J. K. Liu, Boundary $C^{2,\alpha}$ estimates for Monge-Ampère type equations, *Adv. Math.*, 281 (2015), 706–733.
[16] X. H. Ji and J. G. Bao, Necessary and sufficient conditions on solvability for Hessian inequalities, *Proc. Amer. Math. Soc.*, 138 (2010), 175–188.
[17] F. D. Jiang and N. S. Trudinger, On the Dirichlet problem for general augmented Hessian equations, arXiv:1903.12410.
[18] F. D. Jiang, N. S. Trudinger and X. P. Yang, On the Dirichlet problem for Monge-Ampère type equations, *Calc. Var. Partial Differential Equations*, 49 (2014), 1223–1236.
[19] F. D. Jiang, N. S. Trudinger and X. P. Yang, On the Dirichlet problem for a class of augmented Hessian equations, *J. Differential Equations*, 258 (2015), 1548–1576.
[20] Q. N. Jin, Y. Y. Li and H. Y. Xu, Nonexistence of positive solutions for some fully nonlinear elliptic equations, *Methods Appl. Anal.*, 12 (2005), 441–449.
[21] J. B. Keller, On solutions of $\Delta u = f(u)$, *Comm. Pure Appl. Math.*, 10 (1957), 503–510.
[22] Y. Y. Li, Some existence results for fully nonlinear elliptic equations of Monge-Ampère type, *Comm. Pure Appl. Math.*, 43 (1990), 233–271.
[23] J. K. Liu, N. S. Trudinger and X. J. Wang, Interior $C^{2,\alpha}$ regularity for potential functions in optimal transportation, *Comm. Partial Differential Equations*, 35 (2010), 165–184.
[24] G. Z. Lu and J. Y. Zhu, The maximum principles and symmetry results for viscosity solutions of fully nonlinear equations, *J. Differential Equations*, 258 (2015), 2054–2079.
[25] X. N. Ma, N. S. Trudinger and X. J. Wang, Regularity of potential functions of the optimal transportation problem, *Arch. Ration. Mech. Anal.*, 177 (2005), 151–183.
[26] R. Osserman, On the inequality $\Delta u \geq f(u)$, *Pacific J. Math.*, 7 (1957), 1641–1647.
[27] N. S. Trudinger, On the Dirichlet problem for Hessian equations, *Acta Math.*, 175 (1995), 151–164.
[28] X. J. Wang, The $k$-Hessian equation, *Geometric analysis and PDEs*, 177–252, Lecture Notes in Math., 1977, Springer, Dordrecht, 2009.

Received for publication April 2019.

E-mail address: limeidai@126.com