Collapsing Plane Symmetric Source with Heat Flux and Conformal Flatness

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Abstract

This paper deals with the study of collapsing plane symmetric source in the presence of heat flux. For this purpose, we have calculated the Einstein field equations as well as Weyl tensor components. The conditions for the conformal flatness have been determined. The interior source has been matched smoothly with the exterior geometry in single null coordinate. It has been found the pressure is balanced with the outgoing heat flux and the continuity of the masses in two regions has been noted. A simple new model of collapse has been proposed which satisfies flatness condition, also we have discussed the physical properties of the model. For our model, we have calculated the temperature profile by using the approximation scheme.

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1 Introduction

The concept of gravity in terms of spacetime curvature was introduced for the first time by Einstein. According to him, curvature in the spacetime is produced due to the presence of matter in which it exists. This curvature is taken as a source of gravitational field. He formulated a set of equations known as Einstein field equations. The solution of these equations have spacetime singularities. These singularities are formed by gravitational collapse. Gravitational collapse is one of the important problems in general relativity.

In astronomy a gravitational collapse is a phenomenon in which a massive star dozens of times larger than the solar mass contracts under the influence of its own gravity. It occurs when the internal nuclear fuel fails to supply sufficiently high pressure to counterbalance the gravity. There are two types of spacetime singularities, one is a black hole and the other is a naked. The singularity covered by an event horizon is called black hole and an uncovered singularity is called naked. Event horizon is the boundary of the black hole.

Many researchers investigated the solutions of Einstein field equations with heat flux (Schafer and Goenner 2000, Chan et al.2003, Herrera and Santos 2003). Ghosh and Deshker (2007) investigated exact non spherical radiating collapse using junction conditions. Ahmad et al. (2013) investigated the gravitational collapse with heat flux and gravitational waves.

Initially Bronnikov and Kovalchuk (1983) studied cylindrical symmetry and found some exact solutions. The collapse of dust spheroid was studied numerically by Shapiro and Teukolsky (1991) using cylindrical spacetime. They concluded that the end product of collapse was either a black hole or a naked depending on the spheroid compactness. Barrabes et al.(1991) investigated the exact solution for collapsing convex shell. They concluded that in some cases they found the absence of apparent horizons. Sharif and Ahmad (2007) studied high-speed cylindrical collapse of two perfect fluids.

Gutti et al.(2012) investigated the gravitational collapse of an infinite cylindrical distribution of timelike dust, using the matching conditions. The part of anisotropy, radial heat flux and electric charge over the dynamics of non adiabatic cylindrical collapse by using the coupled equation was studied by Sharif and Abbas (2011). Abbas and his collaborators (abbas 2014a, Abbas 2014b Mahmood et al.2015) investigated the expanding and collapsing solutions for charged plane and cylindrical stars. The shearfree cylindrical collapse was studied by Di Prisco et al.(2009). Herrera et al. (2004) studied spherical shearfree radiating collapse and conformal flatness they developed
a relation between dissipation and density inhomogeneity. Also, Herrera and 
Santos (1997) explored the properties of anisotropic self-gravitating spheres
and discussed their stability using the perturbation method. Herrera and his 
collaborators (Herrera et al. 2008a, Herrera et al. 2008b, Herrera et al. 1989,
Herrera et al. 2009a, Herrera et al. 2010, Herrera et al. 2012) have discussed 
the stability and applications of anisotropic solutions to stellar collapse.

In this paper, the work done by Herrera et al. (2004) is extended by taking 
the plane symmetric gravitational with heat flux. The plan of the paper is 
as follows: The matter source and field equations are presented in section 2. 
In section 3, we provide the formulation of solutions by taking into account 
conformal flatness. In section 4, we construct a simple dissipative model 
and studied the temperature profile of the proposed model. We conclude the 
results in the last section.

2 Interior Matter Distribution and Field Equations

Assume that a given four dimensional plane symmetric spacetime is divided 
by a three dimensional hypersuface Σ into two regions interior and exterior 
denoted by $V^-$ and $V^+$ respectively. We consider a particular plane symmet-
ric spacetime in interior region given by

$$ds_-^2 = -A^2(t, z)dt^2 + B^2(t, z)[dx^2 + dy^2 + dz^2].$$

(1)

Here the coordinates are labeled as follows: $x^0 = t, x^1 = x, x^2 = y$ and 
$x^3 = z$. We assume that the energy momentum tensor in the interior region 
is given by

$$T_{\alpha\beta}^- = (\mu + P) w_\alpha w_\beta + P g_{\alpha\beta} + w_\alpha q_\beta + q_\alpha w_\beta,$$

(2)

where $\mu$ is the energy density, $P$ the pressure, $w^\alpha$ the four velocity of fluid 
and $q^\alpha$ the heat flux satisfying $q_\alpha w^\alpha = 0$. For interior spacetime $1$ in 
comoving coordinates, we have

$$w^\alpha = \frac{1}{A} \delta^\alpha_0,$$

(3)

$$q^\alpha = q \delta^\alpha_3,$$

(4)

where $q$ depend on $t$ and $z$. 

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In this case the rate the rate of expansion scalar $\Theta = w^\alpha_{;\alpha}$ is given by
\[ \Theta = 3 \frac{B_t}{AB}, \tag{5} \]

Weyl tensor $C_{\alpha\beta\gamma\delta}$ has the following non-zero components for metric (1)
\[ C_{1212} = \frac{B^2}{3} \left[ 2 \frac{B_z}{B} \left( \frac{A_z}{A} - \frac{B_z}{B} \right) - \left( \frac{A_{zz}}{A} - \frac{B_{zz}}{B} \right) \right], \tag{6} \]
\[ C_{1212} = 2 \left( \frac{B}{A} \right)^2 C_{0101} = - \left( \frac{B}{A} \right)^2 C_{0303} = -2C_{1313}. \tag{7} \]

For the interior metric (II) the non vanishing components of Einstein field equations are given by
\[ G^{-}_{00} = \left( \frac{A}{B} \right)^2 \left( -2 \frac{B_{zz}}{B} + \left( \frac{B_z}{B} \right)^2 \right) + 3 \left( \frac{B_t}{B} \right)^2 = \kappa \mu A^2, \tag{8} \]
\[ G^{-}_{11} = - \left( \frac{B}{A} \right)^2 \left( 2 \frac{B_{tt}}{B} - 2 \frac{A_tB_t}{AB} + \left( \frac{B_t}{B} \right)^2 \right) + \frac{A_{zz}}{A} + \frac{B_{zz}}{B} - \left( \frac{B_z}{B} \right)^2 = \kappa PB^2, \tag{9} \]
\[ G^{-}_{22} = G^{-}_{11}, \]
\[ G^{-}_{33} = - \left( \frac{B}{A} \right)^2 \left( 2 \frac{B_{tt}}{B} - 2 \frac{A_tB_t}{AB} + \left( \frac{B_t}{B} \right)^2 \right) + 2 \frac{A_zB_z}{AB} + \left( \frac{B_z}{B} \right)^2 = \kappa PB^2, \tag{10} \]
\[ G^{-}_{03} = - \frac{2 \left( \frac{B_t}{AB} \right) z}{A} = -\kappa qAB^2. \tag{11} \]

Using Eqs. (11) and (5), we get
\[ \kappa qB^2 = \frac{2}{3} \Theta_z, \tag{12} \]

This indicates that for $q > 0$ heat flux is directed outward, implying that $\Theta_z > 0$. For $q = 0$, Eq. (12) implies that $\Theta_z = 0$ which shows that collapse is homogeneous (Herrera and Santos 2003). The Taub’s mass function (Zannias 1990) for the plane symmetric spacetime is given by
\[ m(t, z) = \frac{(g_{11})^{\frac{3}{2}}}{2} R_{12}^{12} = \frac{B}{2} \left[ \frac{B_t^2}{A^2} - \frac{B_{zz}^2}{B^2} \right], \tag{13} \]
Differentiation of $m(t, z)$ with respect to $z$ and $t$ along with field equations (8)-(11) gives

$$m'(t, z) = \frac{\kappa}{2} \left[ B^2 B_z \mu + q B^4 \frac{B_t}{A} \right],$$

(14)

$$\dot{m}(t, z) = -\frac{\kappa}{2} \left[ P B^2 B_t + q B^2 B_z A \right].$$

(15)

The above equations show that the heat flux may effect the gradient and time derivative of $m(z, t)$, and the dissipation reduces the amount of matter, hence the rate of collapse slows down probably.

The scalar of the Weyl tensor in terms of Kretchsmann scalar $\tilde{R}^2 = R^\alpha_\beta\gamma_\delta R_{\alpha\beta\gamma\delta}$, the Ricci tensor $R_{\alpha\beta}$ and Ricci scalar $R$ is defined as

$$C^2 = \tilde{R} - 2R^\alpha_\beta R_{\alpha\beta} + \frac{1}{3} R^2.$$ 

(16)

Using Eqs.(6), (8) and (13) in Eq. (16), we obtain Weyl tensor in terms of pure gravitational mass given by

$$C^2 = 48 \left( \frac{m}{B^3} - \frac{\kappa \mu}{6} \right)^2 = 48 \frac{m_c^2}{B^6},$$

(17)

where $m_c = m - \frac{\kappa}{6} \mu B^3$.

3 Conformally flat solution

In this section, we impose conformal flatness condition to metric (1), this requirement implies that all the components of the Weyl tensor must be equal to zero. Hence, from Eqs.(6) and (7), we see that if $C_{1212} = 0$, this condition is justified and we have

$$\left( \frac{A_{zz}}{A} - \frac{B_{zz}}{B} \right) - 2 \frac{B_z}{B} \left( \frac{A_z}{A} - \frac{B_z}{B} \right) = 0.$$ 

(18)

Integrating above equation and after reparametrizing $t$, we get

$$A = [C_1(t)z + 1] B.$$ 

(19)

where $C_1$ is arbitrary function of $t$. Also, from the isotropy of pressure $G_{11} = G_{33}$ with Eq.(19), gives

$$\frac{B_{zz}}{B_z} - 2 \frac{B_z}{B} = 0,$$

(20)
integration of Eq. (20) yields
\[ B = \frac{1}{C_2(t)z + C_3(t)}, \tag{21} \]
where \( C_2 \) and \( C_3 \) are the constant of integration depends on \( t \). Stephani (1967) and Kramer et al. (1981) investigated all conformally flat solutions with \( q = 0 \).

Conformal flatness imposes \( C = 0 \) and put \( m_c = 0 \) in Eq. (17), we get
\[ m = \frac{\kappa}{6} \mu B^3. \tag{22} \]
Using Eqs. (14) and (22) we get
\[ \mu' = qB^2 \Theta, \tag{23} \]
which describes that for \( q > 0 \) and \( \Theta < 0 \) then \( \mu' < 0 \) implying that the density reduces with increasing \( z \), while from (5) and (12) we get
\[ \kappa \mu' = \frac{1}{3}(\Theta^2)_{,z}, \tag{24} \]
integrating (24) we get
\[ \kappa \mu = \frac{\Theta^2}{3} + g(t), \tag{25} \]
where \( g \) depend on \( t \) only.

Putting solution (19) and (21) into (8), (9) and (11) it follows
\[ \kappa \mu = 3 \left( \frac{\dot{C}_2 z + \dot{C}_3}{C_1 z + 1} \right)^2 - 4C_2^2, \tag{26} \]
\[ \kappa P = \frac{1}{C_1 z + 1} \left[ 2(\dot{C}_2 z + \dot{C}_3)(C_2 z + C_3) - 3(\dot{C}_2 z + \dot{C}_3)^2 \right. \]
\[ - \frac{2\dot{C}_1}{C_1 z + 1}(\dot{C}_2 z + \dot{C}_3)(C_2 z + C_3)z \]
\[ + \frac{1}{C_1 z + 1} [C_1 C_2^2 z + 3C_2^2 - 2C_1 C_2 C_3], \tag{27} \]
\[ \kappa q = 2(C_1 \dot{C}_3 - \dot{C}_2) \left( \frac{C_2 z + C_3}{C_1 z + 1} \right)^2. \tag{28} \]
The expansion of the fluid given by (3) along with Eqs. (19), (21) and (28), yields
\[ \Theta = -3 \frac{\dot{C}_2 z + \dot{C}_3}{C_1 z + 1} = -3 \left[ \dot{C}_3 - \frac{\kappa q z}{2 (C_2 z + C_3)^2} \right]. \] (29)

The above equation show that for \( q = 0 \) the contraction is homogeneous, however for \( q \neq 0 \), collapse is inhomogeneous, which has already been concluded in (12). From Eqs. (26) and (25), we see that \( g(t) = -4C_2^2 \). Here, we matched the interior spacetime with the exterior spacetime that is described by plane symmetric spacetime in single null coordinate given by
\[ ds^2 = \frac{2m(\nu)}{z} d\nu^2 - 2d\nu dz + z^2 (dy^2 + dz^2), \] (30)
from Eqs. (11) and (30) on hypersurface \( \Sigma \) along with field equations (8)-(11) and the mass function (13), we get
\[ B = z_\Sigma, \] (31)
\[ P = (qB)\Sigma, \] (32)
\[ m(\nu) = \frac{B}{2} \left[ \frac{B_i^2}{A^2} - \frac{B_i^2}{B^2} \right] \] (33)

Using Eqs. (27)-(28) and (32) we obtain
\[ \left[ \frac{\dot{C}_2 z + \dot{C}_3}{2 (C_2 z + C_3)} - \frac{(C_1 z + 1)^2}{2(C_2 z + C_3)} \right] = 0. \] (34)

4 A simple model

A simple approximation solution satisfying the junction condition (34) for the functions \( C_1(t), C_2(t) \) and \( C_3(t) \) is
\[ C_1 = \varepsilon c_1(t), \ C_2(t) = 0, \ C_3(t) = \frac{a}{t^2}, \] (35)
where $0 < \epsilon << 1$, $0 \leq t \leq -\infty$ and $a > 0$ a constant proportional to the total mass in the plane. Using (35) and (34) we get upto order of $\epsilon$,

\[
\dot{c}_1 + \frac{c_1}{\Sigma} \approx 0,
\]

(36)

integrating (36) we get,

\[
c_1(t) \approx c_1(0) \exp\left[-\frac{t}{\Sigma}\right].
\]

(37)

Putting the solution (35) into (26)-(28), we get

\[
\kappa_\mu \approx \frac{12a^2}{t^6}(1 - 2\epsilon c_1 z),
\]

(38)

\[
\kappa P \approx -\frac{4\epsilon c_1 a^2}{t^5} \frac{z}{\Sigma},
\]

(39)

\[
\kappa q \approx -\frac{4\epsilon c_1 a^3}{t^7},
\]

(40)

above conditions hold for a realistic process in stellar evaluation. We would like to mention that for $\epsilon \neq 0$ the value of $t$ is constrained by some physical situation. Thus, if one requires that the central pressure remain less than the value of the central energy density, then there exists the inequality

\[
\frac{3}{t_2} > \epsilon c_1.
\]

(41)

From Eq.(38) and Eq.(39), we see that the energy density and pressure decrease in the outer region due to dissipation. It can be noted from (40) that there is increase in the heat flow outwardly.

From (13) with (19) and (21) we get

\[
m(t, z) \approx \frac{2}{a}(1 - 2\epsilon c_1 z),
\]

(42)

which indicates that mass of the gravitating system is being decreased due to heat flux. Now the expansion scalar (35) becomes

\[
\Theta \approx \frac{6a}{t^3}(1 - \epsilon c_1 z).
\]

(43)
The above equation shows that dissipation causes to decrease the rate of collapse. The adiabatic index in the presence of heat flux is defined as follows

\[ \Gamma_{\text{eff}} = \frac{d\ln P}{d\ln \mu} = \left( \frac{\dot{P}}{P} \right) \left( \frac{\mu}{\dot{\mu}} \right), \] (44)

It gives the dynamical instability at a particular time, now computing (44) at \( z = 0 \) and \( z = z_\Sigma \) (using Eq.(35) and (37)) upto order of \( \epsilon \), we get

\[ \Gamma_{z=0} \approx \frac{t}{6z_\Sigma} + \frac{5}{6}, \] (45)

\[ \Gamma_{z=z_\Sigma} \approx \frac{t}{6z_\Sigma} + \frac{5}{6}. \] (46)

The above equations imply that the collapse is homogeneous at the center and on the hypersurface \( \Sigma \).

4.1 Calculation of Temperature

This section deals with the calculation of temperature profile \( T(z, t) \) for our model. The heat transport equation in Maxwell-Cattaneo theory (Herrera et al.2004) is

\[ \tau h^{\alpha\beta} w^\gamma q_{\beta\gamma} + q^\alpha = -Kh^{\alpha\beta}(T_{,\beta} + a_\beta T), \] (47)

where the quantities \( \tau, K \) and \( h^{\alpha\beta} = g^{\alpha\beta} + w^{\alpha}w^{\beta} \) are the time relaxation, thermal conductivity and the projection tensor respectively. Considering Eqs. (1)-(4) then (47) becomes

\[ \tau B_B(Bq),_t + qAB^2 = -K(TA),_z, \] (48)

Applying Eqs.(19), (21) and (28) into (48) and taking \( C_2 = 0 \), we obtain upto \( O(\epsilon) \)

\[ \tau(C_1C_3\dot{C}_3),_t + C_1\dot{C}_3 = -\frac{\kappa K}{2}[T(c_1z + 1)],_z. \] (49)

Now if there is no dissipation we have \( (C_1 = \epsilon c_1 = 0) \) from (42) it follows that \( T = T_0(t) \) this shows that for non-dissipative systems, the temperature is homogenous throughout the system. Thus for dissipative case \( C_1 \neq 0 \), we have the following form of temperature profile

\[ T = T_0(t) + \epsilon T_1(z, t). \] (50)
substituting (50) into (49) we getupto $O(\epsilon)$

$$T \approx T_c(t) + \epsilon c_1 \left( \frac{4a}{\kappa K t^3} - T_0(t) \right) z - \frac{4\tau \epsilon c_1 a^2}{\kappa K t^5} \left( \frac{5}{t} + \frac{1}{z_\Sigma} \right) z.$$  \hspace{1cm} (51)

Here, it is assumed that $K > 0$ and $T_c$ is the central temperature. The dissipation can decrease the temperature at the loss of energy, as it is evident from second term of term of above equation while last term predicts the support of relaxational effects.

5 Conclusion

It is well known that gravitational collapse of a stellar object is extremely dissipative process. Therefore, it becomes important to study the effects of dissipation during the collapse of a radiating star. During the dynamical process of gravitational collapse of a non-adiabatic star, the enough amount of energy losses in the form of outward heat flow and radiations. Herrera et al. (2009a) derived the dynamical equations by including the dissipation term in the source in the form of heat flow, radiation, shear and bulk viscosity and then coupled the field equations with the causal transport equation. The inertia due to heat flux and its effect during the dynamics of dissipative collapse with outgoing radiations was studied by Herrera (2006).

In this paper, we investigate the plane gravitational collapse of radiating fluid with conformal flatness condition. For this purpose, we have explored some the effects of dissipation on the dynamics of self-gravitating body. By determining the conformal flatness condition of the collapsing star, we have found that the system of equations is exactly solvable and there appears three arbitrary integrating functions which depend on $t$. This implies that conformal flatness conditions to leads to the homogeneous solutions. Further, the interior solution has been matched smoothly with the plane symmetric metric which is defined in single null coordinate. We would like to mention that exterior geometry emits the radiations in the form of outgoing radiations. We constructed a simple model which satisfies the matching conditions.

We have calculated the adiabatic index which shows that gravitating system is instable at particular instant of time. The perturbation analysis of the adiabatic index implies that the collapse is homogeneous at the center as well as on the hypersurface $\Sigma$. Using the Maxwell-Cattaneo heat transport equation, we have calculated the temperature profile of the radiating star.
The importance of the proposed model lies in the fact that it presents the influence of relaxational effects on the temperature profile in a sophisticated way and thereby on the evolution of the system. It is concluded that the inhomogeneities in the energy density are directly related to dissipation, even though the underlying spacetime is conformally flat.

6 Conflict of Interest

The authors declare that they have no conflict of interest.

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