Asymptotic results for linear combinations of spacings generated by i.i.d. exponential random variables

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September 7, 2021

Abstract

We prove large (and moderate) deviations for a class of linear combinations of spacings generated by i.i.d. exponentially distributed random variables. We allow a wide class of coefficients which can be expressed in terms of continuous functions defined on $[0, 1]$ which satisfy some suitable conditions. In this way we generalize some recent results by Giuliano et al. (2015) which concern the empirical cumulative entropies defined in Di Crescenzo and Longobardi (2009a).

Keywords: large deviations, moderate deviations, cumulative entropy, $L$-statistics.

2000 Mathematical Subject Classification: 60F10, 62G30, 94A17.

1 Introduction

Empirical processes and their applications to statistics are widely studied (see e.g. Shorack and Wellner (1986) as a monograph on this topic). An important part of the results on this topic concerns linear combinations of order statistics (called $L$-statistics) and, more in particular, linear combinations of spacings (a spacing is a difference between two consecutive order statistics). Among the references with results on large deviations for $L$-statistics here we recall Aleshkyavichene (1991), Bentkus and Zikitis (1990), Groeneboom et al. (1979) and Groeneboom and Shorack (1981). In some cases the large deviation results are formulated in terms of the concept of large deviation principle (see e.g. Dembo and Zeitouni (1998)) and, among the references with this kind of results, here we recall Boistard (2007) and Duffy et al. (2011).

The aim of this paper is to generalize the results in Giuliano et al. (2015) concerning a particular sequence of linear combinations of spacings $\{C_n : n \geq 1\}$ generated by a sequence of independent and identically distributed (i.i.d. for short) random variables $\{X_n : n \geq 1\}$. We recall that the random variables $\{C_n : n \geq 1\}$ are the empirical cumulative entropies defined in Di Crescenzo and Longobardi (2009a) for a sequence of i.i.d. positive random variables $\{X_n : n \geq 1\}$ with a (common) absolutely continuous distribution function. Moreover the results in Giuliano et al. (2015) concern the case of exponentially distributed random variables $\{X_n : n \geq 1\}$ and, in such a case, the joint distribution of the spacings has some nice properties. In this paper the random variables $\{X_n : n \geq 1\}$ are again exponentially distributed, and we allow a wide class

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of sequences of linear combinations of spacings \( \{C_n(w) : n \geq 1\} \), where \( w \) is a continuous function on \([0, 1]\) which satisfies some suitable conditions.

We conclude with the outline of the paper. Section 2 is devoted to some preliminaries; in particular we also illustrate the connections with some references as \cite{di_crescenzo-longobardi1}, \cite{di_crescenzo-longobardi2} and \cite{gao-zhao1}. In Section 3 we generalize the results in \cite{giuliano-etal1}. The connections between our moderate deviation result and the moderate deviation result for \( L \)-statistics in \cite{gao-zhao1} is discussed in Section 4. Finally, in Section 5 we discuss some possible choices of the function \( w \) based on some empirical entropies in the literature.

2 Preliminaries

We start with some preliminaries on large deviations. We also present the sequence studied in this paper, and some connection with the literature.

2.1 Preliminaries on large deviations

Here we briefly recall some basic preliminaries on large deviations (see e.g. \cite{dembo Zeitouni}, pages 4-5). Let \( \mathcal{X} \) be a topological space equipped with its completed Borel \( \sigma \)-field. A sequence of \( \mathcal{X} \)-valued random variables \( \{Z_n : n \geq 1\} \) satisfies the large deviation principle (LDP for short) with speed function \( v_n \) and rate function \( I \) if: \( \lim_{n \to \infty} v_n = \infty \); the function \( I : \mathcal{X} \to [0, \infty] \) is lower semi-continuous; we have the upper bound

\[
\limsup_{n \to \infty} \frac{1}{v_n} \log P(Z_n \in C) \leq - \inf_{x \in C} I(x) \text{ for all closed sets } C,
\]

and the lower bound

\[
\liminf_{n \to \infty} \frac{1}{v_n} \log P(Z_n \in O) \geq - \inf_{x \in O} I(x) \text{ for all open sets } O.
\]

A rate function \( I \) is said to be good if its level sets \( \{x \in \mathcal{X} : I(x) \leq \eta \} : \eta \geq 0 \} \) are compact. In the LDPs presented in this paper we always have \( \mathcal{X} = \mathbb{R} \). In some cases we apply the Gārtn er Ellis Theorem (see e.g. Theorem 2.3.6 in \cite{dembo Zeitouni}) with the speed function \( v_n \) and rate function \( I \) if: \( \lim_{n \to \infty} v_n = \infty \); the function \( I : \mathcal{X} \to [0, \infty] \) is lower semi-continuous; we have the upper bound

\[
\limsup_{n \to \infty} \frac{1}{v_n} \log \mathbb{E}[e^{v_n \theta Z_n}] \text{ for all } \theta \in \mathbb{R},
\]

and the lower bound

\[
\liminf_{n \to \infty} \frac{1}{v_n} \log \mathbb{E}[e^{v_n \theta Z_n}] \text{ for all } \theta \in \mathbb{R},
\]

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\liminf_{n \to \infty} \frac{1}{v_n} \log \mathbb{E}[e^{v_n \theta Z_n}] \text{ for all } \theta \in \mathbb{R},
\]

the origin belongs to the interior of

\[
\mathcal{D}(\Lambda) := \{\theta \in \mathbb{R} : \Lambda(\theta) < \infty\},
\]

and the function \( \Lambda \) is essentially smooth (see e.g. Definition 2.3.5 in \cite{dembo Zeitouni}) and lower semi-continuous, then \( \{Z_n : n \geq 1\} \) satisfies the LDP with speed function \( v_n \) and good rate function \( \Lambda^* \) defined by \( \Lambda^*(z) := \sup_{\theta \in \mathbb{R}} \{\theta z - \Lambda(\theta)\} \). For the sake of completeness we recall that the function \( \Lambda \) is essentially smooth if the interior of \( \mathcal{D}(\Lambda) \) is non-empty, it is differentiable throughout the interior of \( \mathcal{D}(\Lambda) \), and \( |\Lambda'(\theta_n)| \to \infty \) whenever \( \{\theta_n\} \) is a sequence of points in the interior of \( \mathcal{D}(\Lambda) \) which converges to a boundary point of \( \mathcal{D}(\Lambda) \).
2.2 Preliminaries on the sequence \( \{C_n(w) : n \geq 1\} \)

Let \( \{X_n : n \geq 1\} \) be a sequence of i.i.d. positive random variables and let \( X_{1:n} \leq \cdots \leq X_{n:n} \) be the ascending order statistics of \( X_1, \ldots, X_n \) (for all \( n \geq 1 \)); moreover we set \( X_{0:n} = 0 \). Then we consider the sequence \( \{C_n(w) : n \geq 1\} \) defined by

\[
C_n(w) := \sum_{k=0}^{n-1} w(k/n)(X_{k+1:n} - X_{k:n}),
\]

for some function \( w : [0, 1] \rightarrow \mathbb{R} \). So we have

\[
C_n(w) = \sum_{k=0}^{n-1} w(k/n)X_{k+1:n} - \sum_{k=0}^{n-1} w(k/n)X_{k:n} = \sum_{k=1}^{n} w((k - 1)/n)X_{k:n} - \sum_{k=0}^{n-1} w(k/n)X_{k:n}
\]

and, by taking into account \( X_{0:n} = 0 \), we get

\[
C_n(w) = \sum_{k=1}^{n-1} (w((k - 1)/n) - w(k/n))X_{k:n} + w((n - 1)/n)X_{n:n}.
\]

Actually in this paper we assume that the common distribution of the random variables \( \{X_n : n \geq 1\} \) is \( \mathcal{E}(\lambda) \) for some \( \lambda > 0 \), i.e. their (common) distribution function is

\[
F(t) := 1 - e^{-\lambda t} \quad \text{for all } t \geq 0.
\]

Then, in such a case, it is known (see e.g. Subsection 2.3 in [Pyke (1965)]) that the spacings

\[
\{X_{1:n} - X_{0:n}, X_{2:n} - X_{1:n}, \ldots, X_{n:n} - X_{n-1:n}\}
\]

are independent and, for all \( k \in \{0, \ldots, n-1\} \), the distribution of \( X_{k+1:n} - X_{k:n} \) is \( \mathcal{E}(\lambda(n-k)) \). This result yields some explicit formulas for moment generating function, mean and variance of \( C_n(w) \). Firstly, for all \( \theta \in \mathbb{R} \), we have

\[
\mathbb{E}\left[e^{\theta C_n(w)}\right] = \prod_{k=0}^{n-1} \mathbb{E}\left[e^{\theta w(k/n)(X_{k+1:n} - X_{k:n})}\right],
\]

and therefore

\[
\mathbb{E}\left[e^{\theta C_n(w)}\right] = \begin{cases} 
\prod_{k=0}^{n-1} \frac{\lambda(n-k)}{\lambda(n-k)-\theta w(k/n)} & \text{if } \theta w(k/n) < \lambda(n-k) \text{ for all } k \in \{0, \ldots, n-1\} \\
\infty & \text{otherwise.}
\end{cases}
\]

Moreover

\[
\mathbb{E}[C_n(w)] = \frac{1}{\lambda} \sum_{k=0}^{n-1} \frac{w(k/n)}{n-k} \quad \text{and} \quad \text{Var}[C_n(w)] = \frac{1}{\lambda^2} \sum_{k=0}^{n-1} \frac{w^2(k/n)}{(n-k)^2}.
\]

Now we discuss the almost sure convergence and the asymptotic normality following the lines of some proofs in [Di Crescenzo and Longobardi (2009a)] and [Di Crescenzo and Longobardi (2009b)].

We introduce the following condition.

**Condition 1.** The function \( w : [0, 1] \rightarrow \mathbb{R} \) is continuous and there exist \( x_0 \in (0, 1) \), \( \beta \in (0, 1] \) and \( c > 0 \) such that \( |w(x)| \leq c(1-x)^\beta \) for all \( x \in [1-x_0, 1] \).

We start with a generalization of Proposition 2 in [Di Crescenzo and Longobardi (2009b)]. In view of what follows we recall that Condition 1 yields \( w(1) = 0 \), and this condition is needed to have the finiteness of the almost sure limit \( \int_0^\infty w(F(z))dz \) (see (3) below).
Proposition 2.1. Assume that Condition I holds. Let \( \{X_n : n \geq 1\} \) be a sequence of i.i.d. positive random variables in \( L^p \) for some \( p \) such that \( \beta p > 1 \), with (common) distribution function \( F \) possibly different from the one in (3). Then

\[
C_n(w) \to \int_0^\infty w(F(z))dz \text{ a.s. (as } n \to \infty). \tag{6}
\]

Proof. We follow the lines of the proof of Proposition 2 in Di Crescenzo and Longobardi (2009b) (see also the proof of Theorem 9 in Rao et al. (2004)). Obviously we have

\[
C_n(w) = \int_0^\infty w(\hat{F}_n(z))dz \text{ (for all } n \geq 1),
\]

where \( \hat{F}_n(x) := \frac{1}{n} \sum_{k=1}^n 1\{X_k \leq x\} \) is the empirical distribution function. We take \( a_0 > 0 \) such that \( F(a_0) \geq 1 - \frac{x_0}{2} \) and, by the Glivenko Cantelli Theorem, for \( n \) large enough we have

\[
F(a_0) + \frac{x_0}{2} \geq \hat{F}_n(a_0) \geq F(a_0) - \frac{x_0}{2}.
\]

Thus for all \( z \geq a_0 \) we have

\[
\hat{F}_n(z) \geq \hat{F}_n(a_0) \geq 1 - x_0,
\]

which yields

\[
|w(\hat{F}_n(z))| \leq c(1 - \hat{F}_n(z))^\beta
\]

by Condition I. We also remark that

\[
1 - \hat{F}_n(z) = 1 - \frac{1}{n} \sum_{k=1}^n 1\{X_k \leq z\} \leq \frac{1}{n} \sum_{k=1}^n 1\{X_k > z\} \leq \frac{1}{n} \sum_{k=1}^n X_k^p \frac{X_k^p}{z^p} \leq \frac{1}{n} \sum_{k=1}^n X_k^p \frac{X_k^p}{z^p} \leq \frac{\alpha}{z^p},
\]

where \( \alpha := \sup_{n \geq 1} \frac{1}{n} \sum_{k=1}^n X_k^p < \infty \) a.s. (in fact, since the random variables \( \{X_n : n \geq 1\} \) are in \( L^p \), \( \alpha \) is the supremum of a sequence that converges a.s.); thus

\[
|w(\hat{F}_n(z))| \leq c \frac{\alpha^\beta}{z^{\beta p}}.
\]

So, by the Glivenko Cantelli Theorem, we can apply the dominated convergence theorem (noting that \( \int_0^\infty \frac{dz}{z^{\beta p}} < \infty \) because \( \beta p > 1 \)) and we have

\[
\int_{a_0}^\infty w(\hat{F}_n(z))dz \to \int_{a_0}^\infty w(F(z))dz \text{ a.s. (as } n \to \infty).
\]

Then we easily conclude the proof noting that we also have

\[
\int_0^{a_0} w(\hat{F}_n(z))dz \to \int_0^{a_0} w(F(z))dz \text{ a.s. (as } n \to \infty)
\]

again by the Glivenko Cantelli Theorem and the dominated convergence theorem (noting that \( w \) is continuous and therefore bounded, and the integral is over a bounded interval). \( \square \)

In particular, if \( F \) is the distribution function in (3), it is easy to check that the limit value is

\[
\int_0^\infty w(F(z))dz = \int_0^\infty w(1 - e^{-\lambda z})dz = \frac{1}{\lambda} \int_0^1 \frac{w(x)}{1-x}dx =: \mu_w, \tag{7}
\]

which is finite by Condition I moreover, if we take the mean value in (5), we have

\[
\lim_{n \to \infty} \mathbb{E}[C_n(w)] = \mu_w. \tag{8}
\]
We conclude with a brief comment on the asymptotic Normality of the empirical estimators, i.e. the weak convergence of $\frac{C_n(w) - E[C_n(w)]}{\sqrt{\text{Var}[C_n(w)]}}$ to the standard Normal distribution. We can follow the lines of the proof of Theorem 7.1 in Di Crescenzo and Longobardi (2009a) and, in particular, the Lyapunov condition for the sequence $\{C_n(w) : n \geq 1\}$ is

$$\lim_{n \to \infty} \frac{1}{(\lambda n)^3} \sum_{k=0}^{n-1} \frac{|w(k/n)|^3}{(1-k/n)^3} = 0.$$  

(9)

**Remark 2.1.** By taking into account Condition 1, it is easy to check that (9) holds if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{|w(k/n)|^3}{(1-k/n)^3} = \int_0^1 \frac{|w(x)|^3}{(1-x)^3} dx < \infty;$$  

(10)

this yields $3(1-\beta) < 1$, and therefore $\beta > \frac{2}{3}$.

**Remark 2.2.** We have

$$\lim_{n \to \infty} n \text{Var}[C_n(w)] = \frac{1}{\lambda^2} \int_0^1 \frac{w^2(x)}{(1-x)^2} dx = \sigma_w^2,$$

(11)

thus the above weak convergence of $\frac{C_n(w) - E[C_n(w)]}{\sqrt{\text{Var}[C_n(w)]}}$ to the standard Normal distribution is equivalent to the weak convergence of $\sqrt{n}(C_n(w) - E[C_n(w)])$ to the centered Normal distribution with variance $\sigma_w^2$.

Some examples for the function $w$ with $\beta = 1$ will be presented just after Condition 2 (see (13)). An example with $\beta \in (0,1)$ is $w(x) = (1-x)^\beta$; then, by (7) and (11), we have

$$\mu_w = \frac{1}{\lambda} \int_0^1 (1-x)^{\beta-1} dx = \frac{1}{\lambda \beta}$$

and, if $\beta > 1/2$,

$$\sigma_w^2 = \frac{1}{\lambda^2} \int_0^1 (1-x)^{2\beta-2} dx = \frac{1}{\lambda^2 (2\beta - 1)}.$$

2.3 Connections with some literature

We note that the sequence $\{C_n(w) : n \geq 1\}$ defined by (11) (see also (2)) coincides with the sequence $\{L_n : n \geq 1\}$ of $L$-statistics in Gao and Zhao (2011) (Section 4.6) if we take $w(\cdot) = w(J; \cdot)$, where $w(J; \cdot)$ is defined by

$$w(J; x) := \int_x^1 J(u) du$$  

(12)

for some function $J$ called score function (a sequence of estimators of this kind appears in several references; here we recall Jones and Zitikis (2003), eqs. (18) and (19), for the estimation of risk measures and related quantities). In Gao and Zhao (2011) it is not required that the i.i.d. random variables $\{X_n : n \geq 1\}$ are exponentially distributed.

Moreover, if we consider the score function

$$\bar{J}(u) := \log u + 1,$$

we get

$$w(\bar{J}; x) := \int_x^1 \log u + 1 du = [u \log u]_{u=x}^1 = -x \log x;$$
In this section we generalize the results for the sequence \( \{C_n(w)\} \) which yields its essential smoothness required in the statement of Proposition 3.1.

Let \( \lambda \) have to check the steepness of \( \lambda \).

So \( \{C_n(w)\} \) coincides with:

- \( \{CE(\tilde{F}_n) : n \geq 1\} \) in Di Crescenzo and Longobardi (2009a) (Section 7), when \( \{X_n : n \geq 1\} \) are i.i.d. and positive random variables;
- \( \{C_n : n \geq 1\} \) in Giuliano et al. (2015) (Section 4), when \( \{X_n : n \geq 1\} \) are i.i.d. \( \mathcal{E} \mathcal{X} \mathcal{P}(\lambda) \) distributed random variables.

## 3 Results

In this section we generalize the results for the sequence \( \{C_n : n \geq 1\} \) in Giuliano et al. (2015) (Section 4). In view of what follows we introduce the following condition.

**Condition 2.** Let \( w : [0, 1] \to \mathbb{R} \) be a function as in Condition 1 with \( \beta = 1 \), and set \( h_w(x) := \frac{w(x)}{1-x} \) for \( x \in [0, 1) \). Moreover let \( \Lambda_w : \mathbb{R} \to \mathbb{R} \cup \{\infty\} \) be the function defined by

\[
\Lambda_w(\theta) := \begin{cases} 
\int_0^1 \log \left( \frac{\lambda}{\lambda-\theta h_w(x)} \right) dx & \text{if } \sup_{x \in (0,1)} \{\theta h_w(x)\} \leq \lambda \\
\infty & \text{otherwise},
\end{cases}
\]

and assume that \( \Lambda_w \) is finite in a neighbourhood of the origin \( \theta = 0 \).

We remark that the function \( \Lambda_w \) would not be finite in a neighbourhood of the origin \( \theta = 0 \) if we have Condition 1 with \( \beta \in (0, 1) \).

### Some examples for the function \( w \).

We consider the following functions:

\[
w_1(x) := 1 - x; \quad w_2(x) := (1 - x)^2; \quad w_3(x) := (1 - x)(1 - \sqrt{x}).
\]

For all these cases Condition 1 holds with \( \beta = 1 \); moreover: \( \sup_{x \in (0,1)} \{\theta h_w(x)\} \leq \lambda \) if and only if \( \theta \leq \lambda, \Lambda_w \) is lower semicontinuous, and there exists \( \Lambda'_w(\theta) \) for \( \theta < \lambda \). Thus, for each function, we have to check the steepness of \( \Lambda_w \), i.e.

\[
\lim_{\theta \to \lambda^-} \Lambda'_w(\theta) = \infty,
\]

which yields its essential smoothness required in the statement of Proposition 3.1.

- For \( w = w_1 \) we have

\[
\Lambda_{w_1}(\theta) = \begin{cases} \log \left( \frac{\lambda}{\lambda-\theta} \right) & \text{if } \theta < \lambda \\
\infty & \text{otherwise.}
\end{cases}
\]

So we have \( \Lambda_{w_1}(\lambda) = \infty \), and therefore \( 14 \) holds; indeed we have

\[
\lim_{\theta \to \lambda^-} \Lambda'_{w_1}(\theta) = \lim_{\theta \to \lambda^-} \frac{1}{\lambda - \theta} = \infty.
\]

- For \( w = w_2 \) we have

\[
\Lambda_{w_2}(\lambda) = -\int_0^1 \log(1 - h_{w_2}(x))dx = 1;
\]

however, even if \( \Lambda_{w_2}(\lambda) < \infty \), \( 14 \) holds because

\[
\lim_{\theta \to \lambda^-} \Lambda'_{w_2}(\theta) = \lim_{\theta \to \lambda^-} \int_0^1 \frac{h_{w_2}(x)}{\lambda - \theta h_{w_2}(x)} dx = \frac{1}{\lambda} \int_0^1 \frac{h_{w_2}(x)}{1 - h_{w_2}(x)} dx = \frac{1}{\lambda} \left( \int_0^1 \frac{1}{x} dx - 1 \right) = \infty.
\]
• For $w = w_3$ we have
  \[ \Lambda_{w_3}(\lambda) = -\int_0^1 \log(1 - h_{w_3}(x))dx = \frac{1}{2} \]
  and
  \[ \lim_{\theta \to 0} \Lambda'_{w_3}(\theta) = \lim_{\theta \to 0} \int_0^1 \frac{h_{w_3}(x)}{\lambda - \theta h_{w_3}(x)}dx = \frac{1}{\lambda} \int_0^1 \frac{h_{w_3}(x)}{1 - h_{w_3}(x)}dx = \frac{1}{\lambda} \left( \int_0^1 \frac{1}{\sqrt{x}}dx - 1 \right) = \frac{1}{\lambda}; \]
  thus (11) fails.

We start with the first result, which is the analogue of Proposition 4.1 in [Giuliano et al. 2015].

**Proposition 3.1.** Assume that \(\{X_n : n \geq 1\}\) are i.i.d. and \(\mathcal{E}\mathcal{X}\mathcal{P}(\lambda)\) distributed, Condition [2] holds, and \(\Lambda_w\) is essentially smooth and lower semi-continuous. Then the sequence \(\{C_n(w) : n \geq 1\}\) defined by (1) satisfies the LDP with speed function \(v_n = n\) and good rate function \(\Lambda'_w\) defined by
  \[ \Lambda'_w(y) := \sup_{\theta \in \mathbb{R}} \{\theta y - \Lambda_w(\theta)\}. \]

**Proof.** We want to apply Gärtner Ellis Theorem; thus we have to check that
  \[ \lim_{n \to \infty} \frac{1}{n} \log \mathbb{E}[e^{n\theta C_n(w)}] = \Lambda_w(\theta) \quad (\text{for all } \theta \in \mathbb{R}). \tag{16} \]

We remark that, by (11), we have
  \[ \frac{1}{n} \log \mathbb{E}[e^{n\theta C_n(w)}] = \frac{1}{n} \sum_{k=0}^{n-1} \log \left( \frac{\lambda(n - k)}{\lambda(n - k) - n\theta w(k/n)} \right) = \frac{1}{n} \sum_{k=0}^{n-1} \log \left( \frac{\lambda (1 - \frac{k}{n})}{\lambda (1 - \frac{k}{n}) - \theta w(k/n)} \right) \]
  for all \(\theta \in \mathbb{R}\) such that
  \[ \theta w(k/n) < \lambda \left( 1 - \frac{k}{n} \right) \quad \text{for all } j \in \{0, \ldots, n - 1\} \tag{17} \]
  (and \(\frac{1}{n} \log \mathbb{E}[e^{n\theta C_n(w)}]\) equal to infinity otherwise). Moreover condition (17) holds (for any fixed \(n \geq 1\)) if and only if
  \[ \theta h_w(k/n) < \lambda \quad \text{for all } k \in \{0, \ldots, n - 1\}. \]
  Thus the limit in (16) trivially holds if \(\sup_{x \in [0,1]} \{\theta h_w(x)\} > \lambda\) while, if \(\sup_{x \in [0,1]} \{\theta h_w(x)\} \leq \lambda\), the limit (16) can be checked noting that we have a limit of an integral sum (possibly equal to infinity). In conclusion the desired LDP holds as a straightforward application of the Gärtner Ellis Theorem.

**Remark 3.1.** It is well-known that \(\Lambda'_w(y) = 0\) if and only if \(y = \Lambda_w(0)\). Then, since we can differentiate under the integral sign by Condition [2] we get
  \[ \Lambda'_w(0) = \frac{1}{\lambda} \int_0^1 h_w(x)dx = \frac{1}{\lambda} \int_0^1 \frac{w(x)}{1 - x}dx, \]
  i.e. \(\Lambda'_w(0)\) coincides with \(\mu_w\) in (7).

**Remark 3.2.** If the random variables in Proposition 3.1 are not exponentially distributed, then we cannot rely on some properties of the spacings cited above (independence and exponential distributions with different parameters); so we have some difficulties to apply the Gärtner Ellis Theorem. A possible way to overcome this problem is to try to apply Theorem 2.2 in [Najim 2002]. Some technical conditions should be checked and this could be done in a successive work.
Remark 3.3. Here we consider Proposition 3.1 with \( w = w_1 \), where \( w_1 \) is the function in (13). Thus \( \Lambda_\nu \) coincides with the function \( \Lambda_{w_1} \) in (15); moreover we can check (after some easy computations) that \( \Lambda^*_w \) coincides with

\[
\Lambda^*_w(y) = \begin{cases} 
\lambda y - 1 - \log(\lambda y) & \text{if } y > 0 \\
\infty & \text{otherwise.}
\end{cases}
\]

Then we have the rate function provided by the Cramér Theorem (see e.g. Theorem 2.2.3 in Dembo and Zeitouni (1998)) for the sequence of empirical means \( \left\{ \frac{X_1 + \ldots + X_n}{n} : n \geq 1 \right\} \) when (as happens in Proposition 3.1) \( \{X_n : n \geq 1\} \) is a sequence of i.i.d. and \( \mathcal{E}\mathcal{X}\mathcal{P}(\lambda) \) distributed random variables. In fact it is easy to check that

\[
C_n(w_1) = \frac{1}{n} \sum_{k=0}^{n-1} (n-k)(X_{k+1:n} - X_{k:n}) \quad (\text{for all } n \geq 1)
\]

by (11) and the definition of \( w_1 \) in (13), and therefore \( \{C_n(w_1) : n \geq 1\} \) and \( \left\{ \frac{X_1 + \ldots + X_n}{n} : n \geq 1 \right\} \) are equally distributed by taking into account the independence and the distributions of the spacings (indeed, for each \( n \geq 1 \), the law of \( C_n(w_1) \) and \( \frac{X_1 + \ldots + X_n}{n} \) is the Gamma distribution with probability density function \( g(z) = \frac{\lambda^n}{(n-1)!} z^{n-1} e^{-\lambda z} 1_{(0,\infty)}(z) \)).

The second result, which is the analogue of Proposition 4.2 in Giuliano et al. (2015), provides an upper bound of the rate function \( \Lambda^*_w \) in Proposition 3.1 when \( h_w(x) > 0 \) almost everywhere with respect to \( x \). This upper bound can be expressed in terms of the relative entropy (see e.g. Kullback and Leibler (1951)) of an exponential distribution with respect to another one. We recall that, given two absolutely continuous real valued random variables \( X_1 \) and \( X_2 \) with densities \( f_1 \) and \( f_2 \), the relative entropy of \( X_1 \) with respect to \( X_2 \) is defined by

\[
H(X_1|X_2) := \int_{\mathbb{R}} f_1(x) \log \frac{f_1(x)}{f_2(x)} dx;
\]

thus \( H(X_1|X_2) \) actually depends on the laws of the random variables \( X_1 \) and \( X_2 \). Then the relative entropy of the distribution \( \mathcal{E}\mathcal{X}\mathcal{P}(\lambda_1) \) with respect to the distribution \( \mathcal{E}\mathcal{X}\mathcal{P}(\lambda_2) \) is

\[
H(\mathcal{E}\mathcal{X}\mathcal{P}(\lambda_1)|\mathcal{E}\mathcal{X}\mathcal{P}(\lambda_2)) = \frac{\lambda_2}{\lambda_1} - 1 - \log \frac{\lambda_2}{\lambda_1}.
\]

Proposition 3.2. Let \( h_w \) be as in Condition 2 and assume that \( h_w(x) > 0 \) almost everywhere with respect to \( x \). Moreover set \( M_w(y) := \int_0^1 H(\mathcal{E}\mathcal{X}\mathcal{P}(1/y)|\mathcal{E}\mathcal{X}\mathcal{P}(\lambda h_w^{-1}(x)))\, dx \). Then: (i) \( \Lambda^*_w(y) \leq M_w(y) \) for all \( y \in (0,\infty) \); (ii) \( \Lambda^*_w(y) = \infty \) for all \( y \in (-\infty,0) \); (iii) the infimum of \( M_w(y) \) is attained at \( y = \bar{y}_w \), where \( \bar{y}_w := (\lambda \int_0^1 h_w^{-1}(x)\, dx)^{-1} \).

Proof. We start with the proof of (i). We remark that, for \( y > 0 \), we have

\[
\sup_{\theta < \eta} \left\{ \theta y - \log \left( \frac{\eta}{\eta - \theta} \right) \right\} = H(\mathcal{E}\mathcal{X}\mathcal{P}(1/y)|\mathcal{E}\mathcal{X}\mathcal{P}(\eta))
\]

for \( \eta > 0 \); then we get

\[
\Lambda^*_w(y) = \sup_{\theta \sup_{z \in (0,1)} h_w(z) \leq \lambda} \left\{ \theta y - \int_0^1 \log \left( \frac{\lambda}{\lambda - \theta h_w(x)} \right) \, dx \right\}
\]

\[
= \sup_{\theta \sup_{z \in (0,1)} h_w(z) \leq \lambda} \left\{ \theta y - \int_0^1 \log \left( \frac{\lambda h_w^{-1}(x)}{\lambda h_w^{-1}(x) - \theta} \right) \, dx \right\}
\]

\[
\leq \int_0^1 \sup_{\theta \sup_{z \in (0,1)} h_w(z) \leq \lambda} \left\{ \theta y - \log \left( \frac{\lambda h_w^{-1}(x)}{\lambda h_w^{-1}(x) - \theta} \right) \right\} \, dx
\]

\[
\leq \int_0^1 \sup_{\theta < \lambda h_w^{-1}(x)} \left\{ \theta y - \log \left( \frac{\lambda h_w^{-1}(x)}{\lambda h_w^{-1}(x) - \theta} \right) \right\} \, dx = \int_0^1 H(\mathcal{E}\mathcal{X}\mathcal{P}(1/y)|\mathcal{E}\mathcal{X}\mathcal{P}(\lambda h_w^{-1}(x)))\, dx.
\]
Now the proof of (ii): for $y < 0$ we have

$$
\Lambda^*_w(y) \geq \sup_{\theta \leq 0} \left\{ \theta y - \int_0^1 \log \left( \frac{\lambda h^{-1}_w(x)}{\lambda h^{-1}_w(x) - \theta} \right) dx \right\} \geq \sup_{\theta \leq 0} \{ \theta y \} = \infty;
$$

for $y = 0$ (this case was forgotten in the proof of Proposition 4.2 in Giuliano et al. (2015)) we have

$$
\Lambda^*_w(0) \geq \sup_{\theta \leq 0} \left\{ - \int_0^1 \log \left( \frac{\lambda h^{-1}_w(x)}{\lambda h^{-1}_w(x) - \theta} \right) dx \right\} = \lim_{\theta \to -\infty} - \int_0^1 \log \left( \frac{\lambda h^{-1}_w(x)}{\lambda h^{-1}_w(x) - \theta} \right) dx = \infty.
$$

Finally the proof of (iii). One can check that

$$
M_w(y) = \lambda \int_0^1 h^{-1}_w(x)dx \cdot y - 1 - \log \lambda - \int_0^1 \log(h^{-1}_w(x))dx - \log y
$$

and its derivative is

$$
M'_w(y) = \lambda \int_0^1 h^{-1}_w(x)dx - \frac{1}{y}.
$$

So we have $M'_w(y) = 0$ if and only if $y = \bar{y}_w$, and $y = \bar{y}_w$ is a minimizer by the convexity of $M_w$. \qed

The third result, which is the analogue of Proposition 4.3 in Giuliano et al. (2015), concerns moderate deviations. In view of its proof we remark that

$$
\text{there exists } \delta > 0 \text{ such that } \log(1 + x) \leq x - \frac{x^2}{2} + \frac{x^3}{3} \text{ for all } |x| < \delta \tag{18}
$$

(which can be proved by checking that the function $g$ defined by $g(x) := \log(1 + x) - (x - \frac{x^2}{2} + \frac{x^3}{3})$ has a local maximum at $x = 0$) and

$$
\text{for all } v > \frac{1}{2}, \text{ there exists } \delta > 0 \text{ such that } \log(1 + x) \geq x - vx^2 \text{ for all } |x| < \delta \tag{19}
$$

(which can be proved by checking that the function $g$ defined by $g(x) := \log(1 + x) - (x - vx^2)$ has a local minimum at $x = 0$).

**Proposition 3.3.** Assume that $\{X_n : n \geq 1\}$ are i.i.d. and $\mathcal{E}XP(\lambda)$ distributed, and Condition $\mathbb{2}$ holds. Then, for any positive sequence $\{a_n : n \geq 1\}$ such that

$$
a_n \to 0 \quad \text{and} \quad na_n \to \infty \text{ (as } n \to \infty) \tag{20},
$$

the sequence $\{\sqrt{n}a_n(C_n(w) - \mathbb{E}[C_n(w)]) : n \geq 1\}$ satisfies the LDP with speed function $v_n = 1/a_n$ and good rate function $\hat{\Lambda}^*_w(y)$ defined by $\hat{\Lambda}^*_w(y) := \frac{y^2}{2\sigma^2_w}$, where $\sigma^2_w$ is the expression in (III).

**Proof.** We want to apply the Gärtner Ellis Theorem with speed function $1/a_n$; thus we have to check that

$$
\liminf_{n \to \infty} a_n \log \mathbb{E} \left[ \exp \left( \theta \sqrt{\frac{\mathbb{E}}{a_n}(C_n(w) - \mathbb{E}(C_n(w)))} \right) \right] \geq \sigma^2_w \theta^2 \tag{21}
$$

and

$$
\limsup_{n \to \infty} a_n \log \mathbb{E} \left[ \exp \left( \theta \sqrt{\frac{\mathbb{E}}{a_n}(C_n(w) - \mathbb{E}(C_n(w)))} \right) \right] \leq \sigma^2_w \theta^2 \tag{22}
$$

for all $\theta \in \mathbb{R}$. 

9
It is useful to remark that, by (11) and the mean value in (15) (together with some computations), we have

\[
\log E \left[ \exp \left( \theta \sqrt{\frac{n}{a_n}} (C_n(w) - \mathbb{E}[C_n(w)]) \right) \right] = \log E \left[ \exp \left( \theta \sqrt{\frac{n}{a_n}} C_n(w) \right) \right] - \log \sqrt{\frac{n}{a_n}} \mathbb{E}[C_n(w)]
\]

\[
= \sum_{k=0}^{n-1} \log \frac{\lambda(n-k)}{\lambda(n-k) - \theta \sqrt{\frac{2}{a_n} w(k/n)}} - \frac{\theta}{\lambda} \sqrt{\frac{n}{a_n}} \sum_{k=0}^{n-1} w(k/n) / (n-k)
\]

\[
= - \sum_{k=0}^{n-1} \left( \log \left( 1 - \frac{\theta w(k/n)}{\lambda \sqrt{na_n} (1-k/n)} \right) + \frac{\theta w(k/n)}{\lambda \sqrt{na_n} (1-k/n)} \right)
\]

for all \( \theta \in \mathbb{R} \) such that

\[
\frac{\theta w(k/n)}{\lambda \sqrt{na_n} (1-k/n)} < 1 \text{ for all } k \in \{0, \ldots, n-1 \}
\]

(and \( \log E \left[ \exp \left( \theta \sqrt{\frac{n}{a_n}} (C_n(w) - \mathbb{E}[C_n(w)]) \right) \right] \) equal to infinity otherwise). Then, by Condition (2) and by \( na_n \to \infty \), for all \( \delta > 0 \) there exists \( \bar{n} \) such that

\[
\left| \frac{\theta w(k/n)}{\lambda \sqrt{na_n} (1-k/n)} \right| < \delta \text{ for all } k \in \{0, \ldots, n-1 \}
\]

for all \( n > \bar{n} \) (in fact \( \left| \frac{\theta w(k/n)}{\lambda \sqrt{na_n} (1-k/n)} \right| \leq \frac{\theta (c \lambda \sqrt{ma_n} - 0 \to 0 \text{ as } n \to \infty) \right) \).

Now we are ready for the proof of (21) and (22); this will be done by using (18) and (19) for \( \delta > 0 \) chosen above and for suitable choices of \( x \) which depend on \( n > \bar{n} \). We start with the proof of (21). If we combine the above computations in this proof and (18) (with \( x = -\frac{\theta w(k/n)}{\lambda \sqrt{na_n} (1-k/n)} \)), we have

\[
a_n \log E \left[ \exp \left( \theta \sqrt{\frac{n}{a_n}} (C_n(w) - \mathbb{E}[C_n(w)]) \right) \right] \geq a_n \sum_{k=0}^{n-1} \left( \frac{\theta^2}{2 \lambda^2 n a_n (1-k/n)^2} + \frac{\theta^3}{3 \lambda^3 (na_n)^{3/2} (1-k/n)^3} \right)
\]

hence, by taking into account the limit for the variance in (11) and

\[
\lim_{n \to \infty} \frac{1}{n \sqrt{na_n}} \sum_{k=0}^{n-1} \frac{w^3(k/n)}{(1-k/n)^3} = 0
\]

(because \( na_n \to \infty \) by (20) and, as explained in Remark 2.1, \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{w^3(k/n)}{(1-k/n)^3} = \int_0^1 \frac{w^3(x)}{(1-x)^3} dx \) is finite because \( \beta > \frac{1}{2} \)), we obtain

\[
\liminf_{n \to \infty} a_n \log E \left[ \exp \left( \theta \sqrt{\frac{n}{a_n}} (C_n(w) - \mathbb{E}[C_n(w)]) \right) \right] \geq \sigma_w^2 \frac{\theta^2}{2}
\]

and (21) is proved. The proof of (22) is similar. We have to consider (19) instead of (18) and, again, we take into account the limit of the variance (11); then we obtain

\[
\limsup_{n \to \infty} a_n \log E \left[ \exp \left( \theta \sqrt{\frac{n}{a_n}} (C_n(w) - \mathbb{E}[C_n(w)]) \right) \right] \leq \limsup_{n \to \infty} a_n \sum_{k=0}^{n-1} \frac{w^2(k/n)}{na_n (1-k/n)^2} = \sigma_w^2 v \theta^2,
\]

and we get (22) by letting \( v \) go to \( \frac{1}{2} \).

In the following remark we recall some typical features on moderate deviations.
Remark 3.4. The class of LDPs in Proposition 3.3 fill the gap between two asymptotic regimes.

1. The almost sure convergence of $C_n(w)$ to $\mu_w$, which is equivalent (by (8)) to the almost sure convergence of $C_n(w) - E[C_n(w)]$ to zero.

2. The weak convergence of $\sqrt{n}(C_n(w) - E[C_n(w)])$ to the centered Normal distribution with variance $\sigma_w^2$ (see Remark 3.1), and

Then we recover these two cases by taking the sequence of random variables in Proposition 3.3 with $a_n = \frac{1}{n}$ and $a_n = 1$, respectively; in both cases one condition in (20) holds, and the other one fails.

Moreover we know that the LDP in Proposition 3.1 which concerns the almost sure convergence of $C_n(w)$ to $\mu_w$, is governed by the rate function $\Lambda_w''(y)$ which uniquely vanishes at $y = \Lambda_w'(0)$ (see Remark 2.2), and $(\Lambda_w'^n(0)) = (\Lambda_w''(0))^{-1}$. So, since we can differentiate (twice) under the integral sign by Condition 2, we get (see also (11))

$$
\Lambda_w''(0) = \frac{1}{\lambda} \int_0^1 h_w(x)dx = \frac{1}{\lambda x^2} \int_0^1 \frac{w^2(x)}{(1-x)^2}dx = \sigma_w^2,
$$

i.e. the variance of the weak limit law of $\frac{C_n(w) - E[C_n(w)]}{\sqrt{n}}$.

In some sense we can say that we have an asymptotic normality result as a consequence of an LDP; an interesting discussion on this issue can be found in Budy (1993).

Finally we show how to obtain a lower bound for the asymptotic variance $\sigma_w^2$ in Remark 3.3 (and in Remark 2.2).

Remark 3.5. Here we assume that $\gamma_w := \int_0^1 \frac{w(x)}{1-x}dx \neq 0$. Then, by (11) and an easy application of the Jensen’s inequality, we have

$$
\sigma_w^2 \geq \frac{1}{\lambda x^2} \left( \int_0^1 \frac{w(x)}{1-x}dx \right)^2 = \frac{\gamma_w^2}{\lambda x^2}.
$$

So, if we consider the function $w_1$ in (13), the inequality turns into an equality if and only if

$$
w(x) = \gamma_w w_1(x) = \gamma_w (1-x).
$$

From now on we set $\ell_w(x) := \gamma_w (1-x)$; moreover we take $\gamma_w > 0$ and we follow the same lines of some parts of Remark 3.4. Firstly we have $\Lambda_{\ell_w}''(\theta) = \Lambda_{w_1}''(\theta \gamma_w)$ for all $\theta \in \mathbb{R}$ and

$$
\Lambda_{\ell_w}''(y) = \sup_{\theta \in \mathbb{R}} \{ y \theta - \Lambda_{\ell_w}(\theta) \} = \sup_{\theta \in \mathbb{R}} \{ y \theta - \Lambda_{w_1}(\theta \gamma_w) \}
$$

$$
= \Lambda_{w_1}''(y \gamma_w^{-1}) = \left\{ \begin{array}{ll}
\lambda \gamma_w^{-1} y - 1 - \log(\lambda \gamma_w^{-1} y) & \text{if } y > 0 \\
\infty & \text{otherwise}.
\end{array} \right.
$$

Moreover $\Lambda_{\ell_w}''$ is the rate function provided by the Cramér Theorem for a sequence of empirical means of i.i.d. and $\mathcal{E}XP(\lambda \gamma_w^{-1})$ distributed random variables; indeed we have

$$
C_n(\ell_w) = \frac{\gamma_w}{n} \sum_{k=0}^{n-1} (n-k)(X_{k+1:n} - X_{k:n}) \ (\text{for all } n \geq 1)
$$

by (11) and the definition of $\ell_w$, and therefore $\{C_n(\ell_w) : n \geq 1\}$ is a sequence of such empirical means by the independence and the distributions of the spacings.
4 Some analogies with a moderate deviation result for $L$-statistics

In this section we discuss some connections between Theorem 4.8 in [Gao and Zhao (2011)] with $F$ as the exponential distribution function in [3], and Proposition 3.3 in this paper with $w(\cdot) = w(J; \cdot)$ as in [12].

Firstly, since $F$ is the exponential distribution function in [3], we can give the following formulas for $m(J, F)$ and $\sigma^2(J, F)$ in Theorem 4.8 in [Gao and Zhao (2011)]:

$$m(J, F) := \int_0^\infty xJ(1 - e^{-\lambda x})\lambda e^{-\lambda x}dx;$$

$$\sigma^2(J, F) := \int_0^\infty \int_0^\infty J(1 - e^{-\lambda x})J(1 - e^{-\lambda y})\{1 - e^{-\lambda(x+y)} - (1 - e^{-\lambda x})(1 - e^{-\lambda y})\}dxdy.$$  

Then, under suitable hypotheses (some of them concern the score function $\sigma$ as in (12)), we get

$$\int_0^\infty J(1 - e^{-\lambda x})J(1 - e^{-\lambda y})\{1 - e^{-\lambda(x+y)} - (1 - e^{-\lambda x})(1 - e^{-\lambda y})\}dxdy = 0.$$ 

Thus (a(n))^{-2} in [23] plays the role of $a_n$ in [20]; moreover, as typically happens for the results on moderate deviations, both rate functions $\Lambda_w^r$ in Proposition 3.3 and $I^L$ are quadratic functions that uniquely vanish at the origin $y = 0$.

We remark that, if we compare the rate functions $\Lambda_w^r$ in Proposition 3.3 and $I^L$ are quadratic functions with an integration by parts, we have

$$m(J, F) = \int_0^1 \frac{\log(1 - r)}{\lambda} J(r)\lambda(1 - r) \frac{dr}{\lambda(1 - r)} = \frac{1}{\lambda} \int_0^1 \log(1 - r)J(r)dr$$

$$= \frac{1}{\lambda} \left\{ [-w(J; r) \log(1 - r)]_{r=0}^{r=1} - \int_0^1 \frac{w(J; r)}{1 - r} dr \right\} = \frac{1}{\lambda} \int_0^1 \frac{w(J; r)}{1 - r} dr = \mu_w(J;),$$

indeed $[-w(J; r) \log(1 - r)]_{r=0}^{r=1} = 0$ because the score function $J$ is bounded, continuous and trimmed (i.e. it is equal to zero near $r = 0$ and $r = 1$).

We also remark that, if we compare the rate functions $\Lambda_w^r$ and $I^L$, we expect to have $\sigma^2(J, F) = \sigma_w^2(J;).$ In order to check this equality we note that the function inside the integral is symmetric with respect to $(x, y)$; therefore we have the integral over $\{(x, y) : 0 \leq x \leq y\}$ multiplied by 2 and, after some computations, we get

$$\sigma^2(J, F) = 2 \int_0^\infty dy J(1 - e^{-\lambda y})e^{-\lambda y} \int_0^y dx(1 - e^{-\lambda x})J(1 - e^{-\lambda x}).$$

Moreover we consider two further changes of variables: the first one is $r = 1 - e^{-\lambda x}$, and we obtain

$$\sigma^2(J, F) = 2 \int_0^\infty dy e^{-\lambda y} J(1 - e^{-\lambda y}) \int_0^{1 - e^{-\lambda y}} \frac{dr}{\lambda(1 - r)} J(r);$$
the second one is $s = 1 - e^{-\lambda y}$, and we get

$$\sigma^2(J, F) = 2 \int_0^1 \frac{ds}{\lambda(1-s)}(1-s)J(s) \int_0^s \frac{dr}{\lambda(1-r)}rJ(r) = \frac{2}{\lambda^2} \int_0^1 ds J(s) \int_0^s \frac{dr}{1-r}J(r).$$

Finally we conclude with the following computations (again with integration by parts):

$$\sigma^2(J, F) = \frac{2}{\lambda^2} \left\{ \left[ -w(J; s) \int_0^s \frac{r}{1-r}J(r) \right]_{s=0}^{s=1} + \int_0^1 w(J; s)J(s) \frac{s}{1-s} ds \right\}$$

$$= \frac{2}{\lambda^2} \left\{ \left[ -\frac{(w(J; s))^2}{2} \right]_{s=0}^{s=1} + \int_0^1 \frac{(w(J; s))^2}{2} \frac{1}{(1-s)^2} ds \right\}$$

$$= \frac{1}{\lambda^2} \int_0^1 \frac{(w(J; s))^2}{(1-s)^2} ds = \sigma^2_w(J; \cdot);$$

because $\left[ -w(J; s) \int_0^s \frac{r}{1-r}J(r) \right]_{s=0}^{s=1} = 0$ and $\left[ -\frac{(w(J; s))^2}{2} \right]_{s=0}^{s=1} = 0$ by the hypotheses on the score function $J$ recalled above and, for the second equality, by Condition 2 with $\beta = 1$ (which refers to Condition 1 for $w(J; \cdot)$).

**Remark 4.1.** If the random variables in Proposition 3.3 are not exponentially distributed, we have some difficulties to apply the the Gärtner Ellis Theorem (as we said in Remark 3.2 for Proposition 3.1). However one could try to adapt the proof of Theorem 4.8 in Gao and Zhao (2011) (which is proved for $w(\cdot) = w(J; \cdot)$ as in [12] under some suitable hypotheses) for a quite general function $w$. This could be done in a successive work.

## 5 Applications to some empirical entropies

A natural way to estimate a functional $\varphi(F)$ of a distribution function $F$ is to consider $\varphi(\hat{F}_n)$ where, given a sequence $\{X_n : n \geq 1\}$ of i.i.d. random variables with distribution function $F$ (possibly different from that one in $\mathfrak{I}$), $\{\hat{F}_n : n \geq 1\}$ is the sequence of the empirical distribution functions defined by

$$\hat{F}_n(x) := \frac{1}{n} \sum_{k=1}^n 1\{X_k \leq x\}, \quad x \in \mathbb{R}.$$ 

In this section we concentrate our attention on functionals related to the concept of entropy and some other related items.

We recall some preliminaries and we refer to Di Crescenzo and Longobardi (2009a) (see also the references cited therein). The **cumulative entropy** associated to an absolutely continuous distribution function $F$ is defined by

$$\mathcal{CE}(F) = - \int_0^\infty F(z) \log F(z) dz.$$

Then, given a sequence of i.i.d. random variables $\{X_n : n \geq 1\}$ with (common) distribution function $F$, we can consider the sequence of empirical cumulative entropies $\{\mathcal{CE}(\hat{F}_n) : n \geq 1\}$ defined by

$$\mathcal{CE}(\hat{F}_n) := - \int_0^\infty \hat{F}_n(z) \log \hat{F}_n(z) dz.$$ 

It is known that $\mathcal{CE}(\hat{F}_n) \to \mathcal{CE}(F)$ a.s. as $n \to \infty$; see Proposition 2 in Di Crescenzo and Longobardi (2009a). However we can also refer to Proposition 2.1 with $w = w_{(1)}$, where

$$w_{(1)}(x) := -x \log x;$$

13
in fact $C_n(w(1))$ in Proposition 2.1 coincides with $C\mathcal{E}(\hat{F}_n)$. It is easy to check that both Conditions 1 and 2 hold for the function $w(1)$.

We can also consider the fractional generalized cumulative entropy defined by

$$C\mathcal{E}_\alpha(F) := \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty F(z)(-\log F(z))^\alpha dz \ (\text{for all } \alpha > 0)$$

(see eq. (7) in Di Crescenzo et al. (2021); the generalized cumulative entropy with $\alpha$ integer was previously defined in Kayal (2016)); note that we recover $C\mathcal{E}(F)$ defined above for $\alpha = 1$. In this case we have to consider the function

$$w(\alpha)(x) := \frac{1}{\Gamma(\alpha + 1)} x(-\log x)^\alpha;$$

then the function $w = w(\alpha)$ satisfies both Conditions 1 and 2 for $\alpha \geq 1$ while, if $\alpha \in (0, 1)$, Condition 1 holds (with $\beta \leq \alpha$) and Condition 2 fails.

For completeness we also discuss the case of the fractional cumulative residual entropy (see eq. (5) in Xiong et al. (2019))

$$\mathcal{E}_q(F) := \int_0^\infty (1 - F(z))(-\log(1 - F(z)))^q dz \ (\text{for all } q \in [0, 1]);$$

note that we recover the cumulative residual entropy defined in Rao et al. (2004) for $q = 1$. In this case we have to consider the function

$$w_q(x) := (1 - x)(-\log(1 - x))^q;$$

then, even if we also consider $q > 1$, the function $w = w_q$ satisfies Condition 1 with $\beta \in (0, 1)$, and does not satisfy Condition 2 except for $q = 0$.

**Funding**

CC and ML are supported by Indam-GNAMPA and by MIUR-PRIN 2017 Project ”Stochastic Models for Complex Systems” (No. 2017JFFHSH). CM and BP are supported by MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata (CUP E83C18000100006), by University of Rome Tor Vergata (research program ”Beyond Borders”, project ”Asymptotic Methods in Probability” (CUP E89C20000680005)) and by Indam-GNAMPA (research project ”Stime asintotiche: principi di invarianza e grandi deviazioni”).

**Acknowledgements**

We thank Prof. Gao for some discussion on Theorem 4.8 in Gao and Zhao (2011).

**Declaration**

The authors declare that they have no conflict of interest.

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