NON-LOCAL PROPERTIES IN
EUCLIDEAN QUANTUM GRAVITY

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Abstract. In the one-loop approximation for Euclidean quantum gravity, the boundary conditions which are completely invariant under gauge transformations of metric perturbations involve both normal and tangential derivatives of the metric perturbations $h_{00}$ and $h_{0i}$, while the $h_{ij}$ perturbations and the whole ghost one-form are set to zero at the boundary. The corresponding one-loop divergency for pure gravity has been recently evaluated by means of analytic techniques. It now remains to compute the contribution of all perturbative modes of gauge fields and gravitation to the one-loop effective action for problems with boundaries. The functional determinant has a non-local nature, independently of boundary conditions. Moreover, the analysis of one-loop divergences for supergravity with non-local boundary conditions has not yet been completed and is still under active investigation.

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1. Introduction

This paper is devoted to local and non-local properties which are relevant for the analysis of Euclidean quantum gravity in the presence of boundaries. Before presenting the technical details, it is necessary to describe why boundaries are so important in quantum gravity. As far as we can see, there are at least two main motivations:

(i) The propagator of quantum gravity may be expressed formally as a path integral over all Riemannian four-geometries matching the boundary data on two (compact) Riemannian three-geometries \((\Sigma_1, h_1)\) and \((\Sigma_2, h_2)\), where \(h_i\) is the metric induced on the surface \(\Sigma_i\), with \(i = 1, 2\). It is then necessary to understand how to fix the boundary data on \((\Sigma_1, h_1)\) and \((\Sigma_2, h_2)\). In quantum cosmology, this analysis leads to a prescription for the quantum state of the universe, i.e. a functional of the three-geometry which solves the Wheeler-DeWitt equation and represents the probability amplitude of having data on a compact Riemannian three-geometry \((\Sigma, h)\) [1,2]. These data consist of the metric configuration on \((\Sigma, h)\), and of matter field configurations (e.g. fermionic fields or bosonic gauge fields).

(ii) The effective action remains the main tool of perturbative quantum field theory [3-5]. Its one-loop approximation contains relevant information about trace anomalies and one-loop divergences [6,7], and is at the heart of symmetry-breaking phenomena [8-13]. The general form of volume terms in the corresponding asymptotic heat kernel has been obtained, after many years of dedicated work, by DeWitt, Gilkey, Avramidi [14-16]. In the presence of boundaries, surface terms occur which have a rich geometric structure and are necessary to obtain the correct values of the trace anomalies and to investigate the non-local nature of the one-loop effective action. For spinor fields, gauge fields and gravitation, the correct values of these one-loop divergences have been obtained for the first time only very recently, by using analytic or geometric techniques [6,17-25]. There is agreement, by now, between the analytic (mode-by-mode) and geometric (space-time covariant) calculations of the trace anomalies for gauge fields and one-loop divergences for gravitation subject
to local boundary conditions, when the Faddeev-Popov formalism is used with manifestly covariant gauges [22-25]. However, the presence of boundaries leads to severe technical complications, and the geometric form of such divergences with non-covariant gauges and other families of boundary conditions is not yet completely understood.

In section 2 we derive in detail a set of mixed boundary conditions in Euclidean quantum gravity. In section 3 we discuss the open problems in this branch of perturbative quantum gravity.

2. Mixed Boundary Conditions for Euclidean Quantum Gravity

For gauge fields and gravitation, the boundary conditions are mixed, in that some components of the field (more precisely, a one-form or a two-form) obey a set of boundary conditions, and the remaining part of the field obeys another set of boundary conditions. Moreover, the boundary conditions are invariant under local gauge transformations providing suitable boundary conditions are imposed on the corresponding ghost zero-form or one-form.

We are here interested in the derivation of mixed boundary conditions for Euclidean quantum gravity. The knowledge of the classical variational problem, and the principle of gauge invariance, are enough to lead to a highly non-trivial quantum boundary-value problem. Indeed, it is by now well known that, if one fixes the three-metric at the boundary in general relativity, the corresponding variational problem is well posed and leads to the Einstein equations, providing the Einstein-Hilbert action is supplemented by a boundary term whose integrand is proportional to the trace of the second fundamental form [6,26-27]. In the corresponding quantum boundary-value problem, which is relevant for the one-loop approximation in quantum gravity [6], the perturbations $h_{ij}$ of the induced three-metric are set to zero at the boundary. Moreover, the whole set of metric perturbations $h_{\mu\nu}$ are subject to the so-called gauge transformations [25]

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} + \nabla_{(\mu} \varphi_{\nu)} , \quad (2.1)$$
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where $\nabla$ is the Levi-Civita connection of the background four-geometry with metric $g$, and $\varphi_\nu \, dx^\nu$ is the ghost one-form [25]. In geometric language, the difference between $\hat{h}_{\mu\nu}$ and $h_{\mu\nu}$ is given by the Lie derivative along $\varphi$ of the four-metric $g$.

For problems with boundaries, Eq. (2.1) implies that

$$\hat{h}_{ij} = h_{ij} + \varphi_{(ij)} + K_{ij} \varphi_0,$$

(2.2)

where the stroke denotes, as usual, three-dimensional covariant differentiation tangentially with respect to the intrinsic Levi-Civita connection of the boundary, while $K_{ij}$ is the extrinsic-curvature tensor of the boundary. Of course, $\varphi_0$ and $\varphi_i$ are the normal and tangential components of the ghost one-form, respectively. Note that boundaries make it necessary to perform a 3+1 split of space-time geometry and physical fields. As such, they introduce non-covariant elements in the analysis of problems relevant for quantum gravity. This seems to be an unavoidable feature, although the boundary conditions may be written in a covariant way [28].

In the light of (2.2), the boundary conditions

$$\left[ h_{ij} \right]_{\partial M} = 0$$

(2.3a)

are gauge invariant, i.e.

$$\left[ \hat{h}_{ij} \right]_{\partial M} = 0,$$

(2.3b)

if and only if the whole ghost one-form obeys homogeneous Dirichlet conditions, so that

$$\left[ \varphi_0 \right]_{\partial M} = 0,$$

(2.4)

$$\left[ \varphi_i \right]_{\partial M} = 0.$$  

(2.5)

To prove necessity and sufficiency of the conditions (2.4)-(2.5), one has to bear in mind the independent expansions in harmonics of $\varphi_0$ and $\varphi_i$. These obey a factorization property, and hence the three-dimensional covariant derivatives only act on the spatial harmonics, so that $\varphi_{(ij)}$ vanishes at the boundary if and only if (2.5) holds [25].
The problem now arises to impose boundary conditions on the remaining set of metric perturbations. The key point is to make sure that the invariance of such boundary conditions under the transformations (2.1) is again guaranteed by (2.4)-(2.5), since otherwise one would obtain incompatible sets of boundary conditions on the ghost one-form. Indeed, on using the Faddeev-Popov formalism for the amplitudes of quantum gravity, it is necessary to use a gauge-averaging term in the Euclidean action, of the form [24]

\[ I_{g.a.} \equiv \frac{1}{32\pi G\alpha} \int_M \Phi_\nu \Phi^\nu \sqrt{\det g} d^4x , \tag{2.6} \]

where \( \Phi_\nu \) is any relativistic gauge-averaging functional which leads to self-adjoint elliptic operators on metric and ghost perturbations. In particular, if the de Donder gauge is chosen [24],

\[ \Phi_{dD}^{dD}(h) \equiv \nabla^\mu \left( h_{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} h_{\rho\sigma} \right) , \tag{2.7} \]

one finds that [25]

\[ \Phi_{dD}^{dD}(h) - \Phi_{dD}^{dD}(\hat{h}) = -\frac{1}{2} \left( g_{\mu\nu} \square + R_{\mu\nu} \right) \phi^\mu = \frac{\lambda}{2} \phi^\nu , \tag{2.8} \]

where \( \square \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu , \) \( R_{\mu\nu} \) is the Ricci tensor of the background, and \( \lambda \) denotes the eigenvalues of the elliptic operator \( -\left( g_{\mu\nu} \square + R_{\mu\nu} \right) . \) Indeed, our notation in the second equality of (2.8) is loose, and it is enough to emphasize the elliptic nature of the operator acting on the ghost one-form. Thus, if one imposes the boundary conditions

\[ \left[ \Phi_{dD}^{dD}(h) \right]_{\partial M} = 0 , \tag{2.9a} \]
\[ \left[ \Phi_{dD}^{dD}(\hat{h}) \right]_{\partial M} = 0 , \tag{2.10a} \]

their invariance under (2.1) is guaranteed when (2.4)-(2.5) hold, by virtue of (2.8). Hence one also has

\[ \left[ \Phi_{dD}^{dD}(\hat{h}) \right]_{\partial M} = 0 , \tag{2.9b} \]
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\[
\left[ \Phi_i^{dD}(\hat{h}) \right]_{\partial M} = 0. \tag{2.10b}
\]

Note that the boundary conditions on the ghost one-form become redundant if one also imposes the conditions (2.3b), (2.9b) and (2.10b). Nevertheless, we shall always write them explicitly, since the ghost one-form plays a key role in quantum gravity.

The most general scheme does not depend on the choice of the de Donder term, so that it relies on (2.3a)-(2.3b), (2.4)-(2.5), jointly with [7,25]

\[
\left[ \Phi_i(h) \right]_{\partial M} = 0, \tag{2.11a}
\]

\[
\left[ \Phi_0(h) \right]_{\partial M} = 0, \tag{2.11b}
\]

\[
\left[ \Phi_i(\hat{h}) \right]_{\partial M} = 0, \tag{2.12a}
\]

\[
\left[ \Phi_i(\hat{h}) \right]_{\partial M} = 0. \tag{2.12b}
\]

Again, it is enough to write (2.3a), (2.11a), (2.12a), (2.4)-(2.5), or (2.3a)-(2.3b) jointly with (2.11a)-(2.11b) and (2.12a)-(2.12b).

In the particular and relevant case of flat Euclidean four-space bounded by a three-sphere [6,29], the de Donder gauge and the boundary conditions (2.3a), (2.9a), (2.10a) lead to [25]

\[
\left[ \frac{\partial h_{00}}{\partial \tau} + \frac{6}{\tau} h_{00} - \frac{\partial}{\partial \tau} \left( g^{ij} h_{ij} \right) + \frac{2}{\tau^2} h_{0i} \frac{\partial}{\partial x^i} \right]_{\partial M} = 0, \tag{2.13}
\]

\[
\left[ \frac{\partial h_{0i}}{\partial \tau} + \frac{3}{\tau} h_{0i} - \frac{1}{2} \frac{\partial h_{00}}{\partial x^i} \right]_{\partial M} = 0, \tag{2.14}
\]

where \( \tau \) is the Euclidean-time coordinate, which here represents the radius of three-spheres centred at the origin. These boundary conditions have three basic properties:

(i) They involve both normal and tangential derivatives of the \( h_{00} \) and \( h_{0i} \) metric perturbations.

(ii) The ghost boundary conditions cannot be expressed in terms of complementary projection operators on \( \varphi_0 \) and \( \varphi_i \), and are instead Dirichlet on both \( \varphi_0 \) and \( \varphi_i \).
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(iii) They can be combined with non-local boundary conditions in $N = 1$ supergravity, when half of the tangential part of the gravitino potential is set to zero at the boundary. This is a non-local operation, since it makes it necessary to pick out the modes which multiply harmonics having positive eigenvalues for the intrinsic three-dimensional Dirac operator at the boundary. What is non-local is the separation of the spectrum of an elliptic operator into its positive and negative parts, and this leads to one of the two possible sets of mixed boundary conditions for spin-3/2 potentials [6,18].

When a three-sphere bounds flat Euclidean four-space, the symmetries of the problem and the use of zeta-function regularization [30,31] make it possible to evaluate the corresponding one-loop divergency for pure gravity, which is found to be [25]

$$\zeta(0) = -\frac{241}{90}.$$

(2.15)

3. Achievements and Unsolved Problems

Over the past six years, impressive progress has been made in our understanding of boundary terms in the asymptotic heat kernel. Thus, the trace anomalies for massless scalar fields subject to Dirichlet, Neumann or Robin boundary conditions, or massless spin-1/2 fields with local or spectral boundary conditions, or Euclidean Maxwell theory in vacuum with magnetic or electric boundary conditions, or one-loop divergences for linearized gravity with three different sets of mixed boundary conditions, are by now well known [17-25,32,33]. Moreover, when boundaries are present, the $\zeta'(0)$ values for scalar and spin-1/2 fields, and the contributions of physical degrees of freedom to $\zeta'(0)$ for spin-1, spin-3/2 and spin-2 fields have also been obtained [34-37]. Despite this very encouraging progress, where one should also mention the contribution of physical degrees of freedom to the one-loop divergency for gravitino potentials [6,38], many important problems remain unsolved. They are as follows.
(i) What is the geometric form of the one-loop divergency for linearized gravity subject to the boundary conditions of section 2, and of the trace anomaly for massless spin-1/2 fields subject to non-local boundary conditions? Can one find a suitable generalization of the Schwinger-DeWitt ansatz (cf. [39,40])?

(ii) What is the contribution of gauge modes and ghost modes to the one-loop divergency for gravitino potentials with non-local boundary conditions?

(iii) What is the correct form of $\zeta'(0)$ for gauge fields and gravitation, when all perturbative modes are taken into account in the presence of boundaries? [The work in [37] is restricted to the so-called physical degrees of freedom, e.g. the transverse part of the electromagnetic potential, or transverse-traceless perturbations for pure gravity.]

(iv) Can one evaluate the one-loop divergences with arbitrary relativistic gauges? [These gauges [41,42] lead to non-minimal operators, and the presence of boundaries makes it very difficult to perform a mode-by-mode analysis of the quantized field in such a case.]

(v) Can one prove essential self-adjointness of the various elliptic operators occurring in the semi-classical amplitudes?

(vi) Can one pick out a preferred choice of mixed boundary conditions for Euclidean quantum gravity, among the four different sets studied so far in the literature [24,25,33,43-46]?

To make sense of the quantum state of the universe [2], of the path-integral approach to quantum gravity [47] and of the corresponding semi-classical approximation [6], it is essential to understand the various aspects of the problem of boundary conditions, i.e. the form of the boundary, the four-geometries summed over in the path integral and the boundary data chosen in the one-loop calculation. For this purpose, the old geometrodynamical framework [1], jointly with the key principles of modern quantum field theory, e.g. gauge invariance and BRST symmetry, is still the most appropriate for the active investigation of these issues. Hence we hope that, in the near future, the scientific community will come
to appreciate how deep are the issues raised, and possibly solved, by a rigorous analysis of quantum field theories in the presence of boundaries [6,28].

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