On the Hawking radiation as tunneling for a class of dynamical black holes

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Abstract

The instability against emission of massless particles by the trapping horizon of an evolving black hole is analyzed with the use of the Hamilton-Jacobi method. The method automatically selects one special expression for the surface gravity of a changing horizon. Indeed, the strength of the horizon singularity turns out to be governed by the surface gravity as was defined a decade ago by Hayward using Kodama’s theory of spherically symmetric gravitational fields. The theory also applies to point masses embedded in an expanding universe, were the surface gravity is still related to Kodama-Hayward theory. As a bonus of the tunneling method, we gain the insight that the surface gravity still defines a temperature parameter as long as the evolution is sufficiently slow that the black hole pass through a sequence of quasi-equilibrium states.

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1 Introduction

With the exception of few lower dimensional models where dynamical computations can be done, the semi-classical theory of black hole radiation and evaporation has perhaps reached a satisfactory state of development only for the case of stationary black holes. It includes now a vast range of topics, ranging from the various derivations of the quantum Hawking’s effect to the efforts aiming to a statistical interpretation of the area law and the associated thermodynamical description. One of the surprising aspects of these results is that the radiation caused by the changing metric of the collapsing star approaches a steady outgoing flow for large times, implying a drastic violation of energy conservation if one neglects the back reaction of the quantum radiation on the causal structure of spacetime. But the back reaction problem has not been solved yet in a satisfactory way. As pointed out by Fredenhagen and Haag long
ago [1], if the back reaction is taken into account by letting the mass of the black hole to change with time, then the radiation will originate from the surface of the black hole at all times after its formation and “it will no longer be precisely calculable from a scaling limit” [1] (the term originates from the link between the Hawking’s radiation and the short distance behaviour on the horizon of the two-point function of the quantum field). Thus one is concerned to show, in the first place, that at least some sort of instability really occurs near the horizon of the changing black hole. This question is non trivial since a changing horizon is not even a null hypersurface, although it is still one of infinite red shift. In this vein we shall analyze this question for a class of dynamical black hole solutions that was inspired by problems not directly related to black hole physics, although these were subsequently reconsidered in the light of the black hole back reaction problem in the early Eighties. The metrics we shall consider are the Vaidya radiating metric [2], as revisited by J. Bardeen [3] and J. York [4], together with what really is a fake dynamical black holes, the McVittie solution representing, in author’s mind, a point mass in cosmology [5]. And the strategy we shall use is a variant of a by now well known method due to F. Wilczek and M. Parikh [6] according to which the process of Hawking radiation is akin to a tunneling effect through the horizon. The method was refined and extended to more general cases in [7] and others papers as well [8]. For criticism and counter criticism see also [9]. It must be kept in mind that these solutions do not correspond precisely to the standard notion of a black hole, but they do have horizons (apparent and/or trapping) to which the familiar black hole theorems seems to apply [10].

The variant referred to above is the Hamilton-Jacobi method introduced in [11], so called after the comparison analysis with the Parikh-Wilczek method done by B. Kerner and R. Mann [12]. The tunneling method provides not only new physical insight to an understanding of the black hole radiation, but is also a powerful way to compute the surface gravity for a vast range of solutions. As a matter of facts, in the past decade some different definitions of the surface gravity of an evolving black hole were proposed [10], [13], until one which met with all requirements was introduced in [14] using a key ingredient invented by H. Kodama [15] (more on this later). The HJ method can also be applied to the more elaborate theory of isolated horizons of Ashtekhar and co-workers (see [16] and references therein. The literature is quite extensive). Results in this direction appeared recently in [17], where the Parikh-Wilczek and HJ methods are compared and showed to agree. An early study of the evolution of evaporating black holes in inflationary cosmology is in [18].

We begin in Sec. II to present the relevant solutions to which the method will be applied. This will actually be done in Sec. III, where we shall also display the “generalized first law” and the conclusions to be drawn from it.
We consider first spherically symmetric spacetimes which outside the horizon (if there is one) are described by a metric of the form

$$ds^2 = -e^{2\Psi(r,v)}A(r,v)dv^2 + 2e^{\Psi(r,v)}dvdr + r^2dS_{D-2}^2.$$  \hspace{1cm} (2.1)

where the coordinate $r$ is the areal radius commonly used in relation to spherical symmetry and $v$ is intended to be an advanced null coordinate. In an asymptotically flat context one can always write (we use geometrized units in which the Newton constant $G = 1$)

$$A(r,v) = 1 - 2m(r,v)/r^{D-3}.$$  

This metric was first proposed by Vaidya [2] in $D = 4$ dimensions, and studied in an interesting paper during the classical era of black hole physics by Lindquist et al [19]. It has been generalized to Einstein-Maxwell systems and de Sitter space by Bonnor-Vaidya and Mallet, respectively [20]. In the special case $\Phi(r,v) = \Psi(r,v)$ it was then extensively used by Bardeen [3] and York [4] in their semi-classical analysis of the back reaction problem, the former to establish the stability of the event horizon in the geometry modified by Hawking radiation back reaction (contrary to a claim of F. Tipler), the latter in an attempt to explain dynamically the origin of the entropy of the black holes. We shall call it the Vaidya-Bardeen metric. If one wishes the metric can also be written in double-null form. In the $(v,r)$-plane one can introduce null coordinates $x^\pm$ such that the dynamical Vaidya-Bardeen BHs may be written as

$$ds^2 = -2f(x^+,x^-)dx^+dx^- + r^2(x^+,x^-)dS_{D-2}^2,$$  \hspace{1cm} (2.2)

for some differentiable function $f$. The remaining angular coordinates contained in $dS_{D-2}^2$ do not play any essential role. In the following we shall use both forms of the metric, depending on computational convenience. The field equations of the Vaidya-Bardeen metric in $D = 4$ dimensions are of interest. They read

$$\frac{\partial m}{\partial v} = 4\pi r^2 T^v_r, \hspace{0.5cm} \frac{\partial m}{\partial r} = -4\pi r^2 T^r_v, \hspace{0.5cm} \frac{\partial \Psi}{\partial r} = 4\pi r e^\Psi T^v_r.$$  \hspace{1cm} (2.3)

The second example we are interested in is the McVittie solution [5] for a point mass in a Friedmann-Robertson-Walker flat cosmology. In D-dimensional spacetime in isotropic spatial coordinates it is given by [21]

$$ds^2 = -A(\rho,t)dt^2 + B(\rho,t)\left(d\rho^2 + \rho^2 dS^2_{D-2}\right)$$  \hspace{1cm} (2.4)

with

$$A(\rho,t) = \left[1 - \left(\frac{m}{a(t}\rho\right)^{D-3}\right]^2, \hspace{1cm} B(\rho,t) = a(t)^2\left[1 - \left(\frac{m}{a(t}\rho\right)^{D-3}\right]^{2/(D-3)}.$$  \hspace{1cm} (2.5)

When the mass parameter $m = 0$, it reduces to a spatially flat FRW solution with scale factor $a(t)$; when $a(t) = 1$ it reduces to the Schwarzschild metric with mass $m$. In four dimensions this solution has had a
strong impact on the general problem of matching the Schwarzschild solution with cosmology, a problem faced also by Einstein and Dirac. Besides McVittie, it has been extensively studied by Nolan in a series of papers [22]. To put the metric in the general form of Kodama theory, we use what may be called the Nolan gauge, in which the metric reads

\[ ds^2 = -\left(A_s - H^2(t)r^2 \right) dt^2 + A_s^{-1} dr^2 - 2A_s^{-1/2}H(t)r \, dr \, dt + r^2 dS^2_{D-2} \]  

(2.6)

where \( H(t) = \dot{a}/a \) is the Hubble parameter and, for example, in the charged 4-dimensional case, \( A_s = 1 - 2m/r + q^2/r^2 \) and in \( D \) dimension \( A_s = 1 - 2M/r^{D-3} + Q^2/r^{2D-6} \). In passing to the Nolan gauge a choice of sign in the cross term \( dr \, dt \) has been done, corresponding to an expanding universe; the transformation \( H(t) \rightarrow -H(t) \) changes this into a contracting one. In the following we shall consider \( D = 4 \) and \( q = 0 \); then the Einstein-Friedmann equations read

\[ 3H^2 = 8\pi \rho, \quad 2A_s^{-1/2} \dot{H}(t) + 3H^2 = -8\pi p. \]  

(2.7)

It follows that \( A_s = 0 \), or \( r = 2m \), is a curvature singularity. In fact, it plays the role that \( r = 0 \) has in FRW models, namely it is a big bang singularity. When \( H = 0 \) one has the Schwarzschild solution. Note how the term \( H^2 r^2 \) in the metric strongly resembles a varying cosmological constant; in fact for \( H \) a constant, it reduces to the Schwarzschild-de Sitter solution in Painlevé coordinates. As we will see, the McVittie solution possesses in general both apparent and trapping horizons, and the spacetime is dynamical. However, it is really not a dynamical black hole in the sense we used it above, since the mass parameter is strictly constant: for this reason we called it a fake dynamical BH. This observation prompts one immediately for an obvious extension of the solution: to replace the mass parameter by a function of time and radius, but this will not be pursued here.

The solutions being given, one may ask whether they have anything to do with black holes. Unfortunately it is not entirely clear what should be considered a black hole in a dynamical regime. However, some kind of horizon must be present, and moreover, the evolution should be sufficiently slow to permit a comparison with the more familiar stationary case. In order for a horizon to exist some metric components must have a zero somewhere. To clarify the issue we have to introduce few definitions, so at this point we have to refer the reader to references [10], [14], [23], [24] and [25] for more details. The expansions \( \theta_{\pm} \) of the null geodesic congruences orthogonal to a closed surface \( S \) with codimension two and measure \( \mu(S) = \Omega_{D-2} r^{D-2} \), are defined by \( \theta_{\pm} = \mu(S)^{-1} \partial_{\pm} \mu(S) = (D-2)r^{-1} \partial_{\pm} r \). A marginal surface \( S \) is a \((D-2)\)-dimensional spacelike surface with vanishing expansion, and a dynamical horizon is a hypersurface which is foliated by marginal surfaces. It is possible to show that the area of such marginal surfaces is non decreasing if the energy inflow is non negative. A future dynamical horizon, say \( H^+ \), is the hypersurface implicitly determined by the condition \( \theta_+ = 0 \). It will be called a trapping horizon if

\[ \text{There are dual definitions involving } \theta_- \text{ and some interchanging of “future” with “past”, see the references for details.} \]
\( \theta_+ \) is strictly decreasing on crossing the horizon from the outside (so that \( \partial_- \theta_+ < 0 \) on \( H^+ \)), a condition which insure the non vanishing of the surface gravity (indeed it is violated for extremal black holes and naked singularities). We just mentioned the surface gravity; a geometrical definition of this quantity for a trapping horizon is [14], \( \kappa = (D - 2)^{-\frac{1}{2}} g^{+\pm} (D - 2) \partial_\pm \theta_+ + \theta_+ \partial_- |_{\theta_+ = 0} \). We stress that this quantity is not the same surface gravity that was defined few years before this new proposal, although it fitted equally well to a generalized first law. Later on we will show that \( \kappa \) fixes the expansion of the metric near the horizon along a future null direction.

These definitions look somewhat artificial, but in fact they are very natural and connected directly with what is known for the stationary black holes. To see this one notes, following Kodama [15], that any metric like (2.1) or (2.2), admits a unique (up to normalization) vector field \( K^a \) such that \( K^a G_{ab} \) is divergence free, where \( G_{ab} \) is the Einstein tensor; for instance, using the double-null form, one finds \( K = -g^{\pm}(\partial_+ r \partial_- - \partial_- r \partial_+) \), while using (2.1) one has \( K = e^{-\Phi} \partial_v \). In any case the defining property of \( K \) show that it is a natural generalization of the time translation Killing field of a static black hole. Moreover, by Einstein equations \( K_a T^{ab} \) will be conserved so for such metrics there exists a natural (Kodama wrote “preferable”) localizable energy flux and its conservation law. Now consider the expression \( E = \text{div} \left( e^{-\phi} \partial_v \right) \), which for a Killing vector field \( \nabla_b K_a \) is anti-symmetric so the definition reduces to the usual one. The study of black holes requires also a notion of energy; the natural choice would be to use the charge associated to Kodama conservation law, but this turns out to be the Misner-Sharp energy, which for a sphere with areal radius \( r \) is the same as the Hawking mass [26] (here \( D = 4 \)), \( E = 2^{-\frac{1}{2}} r (1 - 2^{-\frac{1}{2}} r^2 g^{\pm-} \Theta_+ \Theta_-) \). If this definition is found not particularly illuminating, then one can use the metric (2.1) to obtain the equivalent expression \( g^{\mu \nu} \partial_\mu r \partial_\nu r = 1 - 2E/r \), where \( g_{\mu \nu} \) is the reduced metric in the plane normal to the sphere of symmetry (the plane \( (v, r) \)). In this form it is clearly a generalization of the Schwarzschild mass. As we said, \( E \) is just the charge associated to Kodama conservation law; as showed by Hayward [23], in vacuo \( E \) is also the Schwarzschild energy, at null infinity it is the Bondi-Sachs energy and at spatial infinity it reduces to the ADM mass.

Let us apply this general theory to the two classes of dynamical BH we have considered. Using Eq. (2.1), we have \( \theta_+ = (D - 2)e^{2\Psi_+ - \Phi} A(r, v)/2r \). The condition \( \theta_+ = 0 \) leads to \( A(r_H, v) = 0 \), which defines a curve \( r_H = r_H(v) \) giving the location of the apparent horizon; writing the solution in the Vaidya-Bardeen form, that is with \( A(r, v) = 1 - 2m(r, v)/r \), the Misner-Sharp energy of the black hole is \( E = m(r_H(v), v) \), and the horizon will be trapping if \( m'(r_H, v) < 1/2 \), a prime denoting the radial derivative. The surface gravity associated with the Vaidya-Bardeen dynamical horizon is

\[
\kappa(v) = \frac{A'(r, v)}{2} \bigg|_{r = r_H} = \frac{m'(r_H, v)}{r_H^2} - \frac{m'(r_H, v)}{r_H} = \frac{1}{2r_H} - \frac{m'(r_H, v)}{r_H}. \tag{2.8}
\]

We see the meaning of the trapping condition: it ensures the positivity of the surface gravity. In the case
of McVittie BHs, we obtain \( \theta_{\pm} = \pm(D - 2)(\sqrt{A_s} \mp H r)/2r f_{\pm} \), where the functions \( f_{\pm} \) determine null coordinates \( x^{\pm} \) such that \( dx^{\pm} = f_{\pm} \left[ (\sqrt{A_s} \pm H r) \, dt \pm A_s^{1/2} \, dr \right] \). One may compute from this the dual derivative fields \( \partial_{\pm} \) such that

\[
dx^{\pm} = f_{\pm} \left[ (\sqrt{A_s} \pm H r) \, dt \pm A_s^{-1/2} \, dr \right].
\]

Thus, the future dynamical horizon defined by \( \theta_{+} = 0 \), has a radius which is a root of the equation \( \sqrt{A_s} = H r \), which in turn implies \( A_s = H^2 r_H^2 \). Hence the horizon radius is a function of time. The Misner-Sharp mass and the related surface gravity are

\[
E = m + \frac{1}{2} H(t)^2 r_H^3, \quad \kappa(t) = \frac{1}{2} \left[ A_s'(r_H) \right] - H^2 r_H - \frac{\dot{H}}{2H} = \frac{m}{r_H} - H^2 r_H - \frac{\dot{H}}{2H}.
\]

Note that \( E = r_H/2 \). In the static cases everything agrees with the standard results. The surface gravity has an interesting expression in terms of the sources of Einstein equations and the Misner-Sharp mass. Let \( T_{(2D)} \) be the reduced trace of the stress tensor in the space normal to the sphere of symmetry, evaluated on the horizon \( H^+ \). For the Vaidya-Bardeen metric it is, by Einstein’s equations (2.3),

\[
T_{(2D)} = T_v^v + T_r^r = -\frac{1}{2\pi r_H} \frac{\partial m}{\partial r} \bigg|_{r=r_H}
\]

For the McVittie’s solution, this time by Fredmann’s equations (2.7) one has

\[
T_{(2D)} = -\rho + p = -\frac{1}{4\pi} \left( 3H^2 + \frac{\dot{H}}{H r_H} \right)
\]

We have then the formula, \( \kappa = r_H^{-2} E + 2\pi r_H T_{(2D)} \). It is worth mentioning the pure FRW case, i.e. \( A_s = 1 \), for which \( \kappa(t) = -\left( H(t) + \dot{H}/2H \right) \). We feel that these expressions for the surface gravity are non trivial and display deep connections with the emission process. Indeed it is the non vanishing of \( \kappa \) that is connected with the imaginary part of the action of a massless particle, as we are going to show in the next section.

3 Tunneling within the Hamilton-Jacobi method, and the conclusions

The essential property of the tunneling method is that the action \( I \) of an outgoing massless particle emitted from the horizon has an imaginary part which for stationary black holes is \( \text{Im} \, I = \pi \kappa^{-1} E \), where \( E \) is the Killing energy and \( \kappa \) the horizon surface gravity. The imaginary part is obtained by means of Feynman \( i\epsilon \)-prescription, as explained in [6, 11]. As a result the particle production rate reads

\[
\Gamma = \exp(-2\text{Im} \, I) = \exp(-2\pi \kappa^{-1} E) .
\]

One then recognizes the Boltzmann factor, from which one deduces the well-known temperature \( T_H = \kappa/2\pi \). But more than this, an explicit expression for \( \kappa \) is actually obtained in terms of radial derivatives of the metric on the horizon. Let us consider now the case of a dynamical black hole in the double-null form. We have for a massless particle along a radial geodesic the Hamilton-Jacobi equation \( \partial_+ I \partial_- I = 0 \). Since the particle is outgoing \( \partial_- I \) is not vanishing, and we
arrive at the simpler condition $\partial_s I = 0$. First, let us apply this condition to the Vaidya-Bardeen BH. One has then
\begin{equation}
2e^{-\Psi(r,v)}\partial_r I + A(r,v)\partial_v I = 0.
\end{equation}

Since the particle will move along a future null geodesic, to pick the imaginary part we expand the metric along a future null direction starting from an arbitrary event $(r_H(v_0), v_0)$ on the horizon, i.e. $A(r_H(v_0), v_0) = 0$. Thus, shortening $r_H(v_0) = r_0$, we have $A(r,v) = \partial_r A(r_0, v_0)\Delta r + \partial_v A(r_0, v_0)\Delta v \ldots = 2\kappa(v_0)(r - r_0) + \ldots$, since along a null direction at the horizon $\Delta v = 0$, according to the metric \ref{eq:2.1}; here $\kappa(v_0)$ is the surface gravity, Eq. \ref{eq:2.8}. From \ref{eq:3.1} and the expansion, $\partial_r I$ has a simple pole at the event $(r_0, v_0)$; as a consequence
\begin{equation}
\text{Im} I = \text{Im} \int \partial_r I dr = -\text{Im} \int dr \frac{2e^{-\Psi(r,v)}\partial_r I}{A(r_0, v_0)(r - r_0 - i0)} = \frac{\pi\omega(v_0)}{\kappa(v_0)}.
\end{equation}

where $\omega(v_0) = e^{-\Psi(r_0,v_0)}\partial_r I$, is to be identified with the energy of the particle at the time $v_0$. Note that the Vaidya-Bardeen metric has a sort of gauge invariance due to conformal reparametrizations of the null coordinate $v$: the map $v \rightarrow \tilde{v}(v)$, $\Psi(v,r) \rightarrow \tilde{\Psi}(\tilde{v},r) + \ln(\partial \tilde{v}/\partial v)$ leaves the metric invariant, and the energy is gauge invariant too. Thus we see that it is the Hayward-Kodama surface gravity that is relevant to the process of particles emission. The emission probability, $\Gamma = \exp(-2\pi\omega(v)/\kappa(v))$, has the form of a Boltzmann factor, suggesting a locally thermal spectrum. For the McVittie BH, the situation is similar. In fact, the condition $\partial_s I = 0$ becomes $\partial_r I = -F(r,t)^{-1}\partial_t I$, where $F(r,t) = \sqrt{A_s(r)(\sqrt{A_s(r)} - rH(t))}$. As before, we pick the imaginary part by expanding this function at the horizon along a future null direction, using the fact that for two neighbouring events on a null direction in the metric \ref{eq:2.0}, one has $t - t_0 = (2H_0^{-1}/r_0)^{-1}(r - r_0)$, where $H_0 = H(t_0)$. We find the result
\begin{equation}
F(r,t) = \left(\frac{1}{2}A_s'(r_0) - r_0 H_0^2 + \frac{H_0}{2H_0}\right)(r - r_0) = \kappa(t_0)(r - r_0) \ldots
\end{equation}

where this time $r_0 = r_H(t_0)$. From this equation we see that $\partial_r I$ has a simple pole at the horizon; hence, making use again of Feynman $ie$-prescription, one finds $\text{Im} I = \pi\kappa(t_0)^{-1}\omega(t_0)$, where $\omega(t) = \partial_t I$ is again the energy at time $t$, in complete agreement with the geometric evaluation of the previous section.

Obviously, if $\kappa$ vanishes on the horizon there is no simple pole and the black hole should be stable.\footnote{However, charged extremal black holes can radiate \cite{27}.} The kind of instability producing the Hawking flux for stationary black holes evidently persists in the dynamical arena, and so long as the evolution is sufficiently slow the black hole seems “to evaporate thermally” (paraphrasing \cite{3}). Note that the imaginary part, that is the instability, is attached to the horizon all the time, confirming the Fredenhagen-Haag suggestion quoted in the introduction. It is worth mentioning the role of $\kappa$ in the analogue of the first law for dynamical black holes (contributions to this problem for Vaidya black holes were given in \cite{28}). Using the formulas of the projected stress tensor
\( T_{(2D)} \) given above, and the expression of the Misner-Sharp energy, one obtains the differential law

\[
dE = \frac{\kappa dA_H}{8\pi} - \frac{T_{(2D)}}{2} dV_H
\]

(3.4)

provided all quantities were computed on the horizon. Here \( A_H = 4\pi r_H^2 \) is the horizon area and \( V_H = 4\pi r_H^3/3 \) is a formal horizon volume. If one interprets the “d” operator as a derivative along the future null direction one gets Hayward’s form of the first law. But one can also interpret the differential operation more abstractly, as referring to an ensemble. Indeed, to obtain Eq. (3.4) it is not necessary to specify the meaning of the “d”. It is to be noted that the same law can be proved with other, inequivalent definitions of the surface gravity, even maintaining the same meaning of the energy. Thus it was not a trivial problem to identify the correct one: the tunneling method has made the choice.

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