Numerical analysis of plasma column dynamics in two-fluid EMHD

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Abstract. We propose a new numerical method for the investigation of plasma column dynamics described by the equations of electromagnetic hydrodynamics (EMHD). Special attention is given to the statement of the boundary conditions, as well as testing the algorithm. Our difference scheme for finding solutions of the EMHD equations in Lagrangian coordinates takes into account the motion of the plasma-vacuum boundary and involves discretization of the electrical equation for the external circuit. The main difference from the MHD theory is due to the complex structure of the generalized Ohms law, which is expressed by a boundary value problem for the electrical field on an interval. This algorithm can be effectively used for finding both the electric and the magnetic fields in the column plasma.

1. Introduction

Some important problems in the dynamics of rarefied plasma, as well as plasma with strongly inhomogeneous density, require the use of certain generalizations of MHD equations that take into account electron inertia. Lately, the corresponding systems of equations have been referred to as the extended MHD equations and the inertial MHD equations [1]. An advanced version of the EMHD system, called the equations of electromagnetic hydrodynamics (EMHD), was proposed in 1988 [2]. In the present paper, the EMHD equations are used for the investigation of plasma column compression in devices of z-pinch type employed in the early stages of the fusion program. For devices like z-pinch or plasma focus, the dynamics of a plasma column (from its appearance to its destruction) has not been fully described so far, since its investigation encounters serious problems. A numerical approach has been effective only with regard to some stages of the column evolution. Initially, numerical solutions were obtained in [3] for one-dimensional MHD equations (taking into account finite plasma conductivity) by the finite-difference method in the Lagrangian coordinates associated with radial vibrations of the plasma column. In [4], a difference scheme was proposed for studying plasma column evolution in the case of cylindrical geometry and the MHD approximation taking into account the dissipative effects (the magnetic and hydrodynamic viscosities, as well as the heat conductivity of plasma). A completely conservative difference scheme for cylindrically symmetric EMHD equations is considered in [5]. A numerical study of two-dimensional axially symmetric evolution of the z-pinch was conducted in [6] by the so-called free points method, which later gave rise to the SPH algorithms (Smooth Particle Hydrodynamics). The results of [3-7] made it possible to investigate
the dynamics of a plasma column up to the instant of its maximal compression, when sausage-
type instabilities appear in the column. Investigation of subsequent stages of column evolution
required the introduction of non-hydrodynamic (hybrid) models of plasma [8], in which the
ion component is described by a kinetic equation, while the hydrodynamic approximation is
kept for the electron component. A numerical realization of non-hydrodynamic models can
be found in [9]. It turns out that: (i) the MHD approach is incapable of fully describing the
dynamics of a plasma column and the mechanism of its destruction on the stage of sausage-type
instabilities, and one has to take into account electron inertia; (ii) it is important to examine
vacuum plasma [10]. It appears that further modeling of plasma column dynamics should be
based on the EMHD theory. In the present paper, a simple new difference scheme is proposed
for solving EMHD equations in the Lagrangian coordinates associated with radial vibrations of
the plasma column. This scheme is tested on certain nontrivial analytical solutions (the so called
so-called analytical solutions (the so-called analytical solutions) whose dynamics is described
in [11]). In the case of one-dimensional problems, the main difference between the MHD and the EMHD approaches is that the latter involves a nontrivial version
of the generalized Ohms law, which is expressed as a boundary value problem for the electric
field on an interval. Therefore, special attention is given to the statement and the numerical
realization of boundary conditions for the electric field. It is assumed that there is a precise
geometric plasma-vacuum boundary and the influence of the external electric circuit of the device
on compression processes is taken into account.

2. Initial equations
The EMHD equations have the form

\[
\begin{align*}
(a) \quad & \frac{\partial \rho}{\partial t} + \text{div}\rho \mathbf{U} = 0, \quad (b) \quad \frac{\partial \rho \mathbf{U}}{\partial t} + \text{Div}\Pi = 0, \\
(c) \quad & \frac{\partial p_\pm}{\partial t} + \mathbf{U} \cdot \nabla p_\pm + \gamma p_\pm \text{div}\mathbf{U} \pm \lambda \rho \gamma^{-1} \mathbf{j} \cdot \nabla \left( \frac{p_\pm}{\rho^\gamma} \right) = (\gamma - 1) \frac{m_+ m_-}{m_\Sigma} \frac{j^2}{\sigma}, \\
(d) \quad & c^{-1} \frac{\partial \mathbf{H}}{\partial t} + \text{rot}\mathbf{E} = 0, \quad (e) \quad \text{div}\mathbf{H} = 0, \\
(f) \quad & E + \frac{c^2 \lambda_+ \lambda_-}{4\pi \rho} \text{rot}\text{rot}\mathbf{E} = -c^{-1}[\mathbf{U}, \mathbf{H}] + \rho^{-1} \text{Div}\mathbf{W} + \frac{\mathbf{j}}{\sigma}, \\
g) \quad & \text{rot}\mathbf{H} = \frac{4\pi}{c} \mathbf{j},
\end{align*}
\]

where subscripts \( \pm \) refer to the ion (+) and the electron (-) components of plasma, \( \lambda_\pm = m_\pm/e_\pm \),
\( \lambda = \lambda_+ + \lambda_- \), \( m_\Sigma = m_+ + m_- \); \( \mathbf{U} \) is the mass velocity, \( \rho \) is the sum of densities of electrons and ions, \( \sigma \) is plasma conductivity; the electrons and ions are assumed to be ideal polytropic gases
with a common adiabatic index \( \gamma > 1 \); \( \Pi \) is the internal stress tensor and \( \mathbf{W} \) is the electrodynamic
state tensor:

\[
\Pi = \Pi^{(h)} + \Pi^{(p)} + \Pi^{(c)},
\]

\[
\mathbf{W} = (\lambda_- - \lambda_+)(\Pi^{(p)} + \Pi^{(c)}) + (\lambda_+ p_+ - \lambda_- p_-)I_3 + \lambda_+ \lambda_- (\mathbf{jU} + \mathbf{Uj}),
\]

\[
\Pi^{(h)} = \rho \mathbf{UU} + p_\Sigma I_3, \quad \Pi^{(p)} = \frac{H^2}{8\pi} I_3 - \frac{\mathbf{HH}}{4\pi}, \quad \Pi^{(c)} = \lambda_+ \lambda_- \frac{\mathbf{jj}}{\rho},
\]

where \( I_3 \) is the three-dimensional identity tensor, \( p = p_+ + p_- \).

3. Lagrangian coordinates and boundary conditions
Consider a cylindrically symmetric plasma column of radius \( R(t) \) whose dynamics is described
by system (1), (2). We restrict ourselves to the case of \( U_\varphi = 0, U_z = 0, H_\varphi = 0, H_z = 0, E_\varphi = 0 \)
and set \( U = U_r, H = H_\varphi, E = E_z \). It is assumed that there is vacuum outside the column up
to the lateral walls of the device chamber. Let \( I(t) \) be the total current through the column at
time $t$. The function $I(t)$ is either given or is to be determined from the electrical equation for the external circuit (see section 4). Introducing the mass Lagrangian coordinates $(\tau, m)$:

$$\tau = t, \quad m(t, r) = \int_0^r \rho(t, r) dr,$$

we seek dimensionless cylindrically symmetric solutions of system (1), (2) from the following equations:

$$\frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) = \frac{\partial U}{\partial m}, \quad \frac{\partial U}{\partial t} + r \frac{\partial}{\partial m} \left( \frac{p}{M} + \frac{\kappa^2}{2} H^2 \right) + \frac{\kappa^2}{r \rho} H^2 = 0,$$

$$\frac{\partial}{\partial t} \left( \frac{p}{\rho} \right) + (\gamma - 1) p \frac{\partial r U}{\partial m} = (\gamma - 1) \nu_m M \kappa^2 \frac{m}{m^2} \rho \left( \frac{\partial H}{\partial m} \right)^2,$$

$$E^* - \xi^2 \frac{\partial}{\partial m} \left[ \frac{r^2 \rho \partial E^*}{\rho} \right] = \zeta \nu_m \rho \frac{\partial r H}{\partial m} - \zeta \nu^2 \frac{\partial}{\partial m} \left[ r^2 \rho H \frac{U}{r} \right],$$

$$0 \leq m \leq M = \int_0^{R(t)} \rho(t, r) dr, \quad t \geq 0,$$

where $\xi, \zeta, \kappa, M, \nu_m$ are similarity numbers,

$$\xi = \frac{cm_{m}^{1/2}}{2\pi^{1/2}e^{[n]^{1/2}|L|Z^{1/2}}, \quad \zeta = \frac{[U][H]}{c[E]}, \quad \kappa = \frac{v_A}{U}, \quad M = \frac{[U]^2}{[p], \quad \nu_m = \frac{c^2}{4\pi \sigma [L][U]}},$$

Here and in what follows, square brackets are used to indicate the scale of the corresponding quantity (length $[L]$, density $[\rho]$, particle concentration $[n]$, speed $[U]$, etc.); $c = c_-, Z = e_+/e_-, v_A = [H]/4\pi(p)=1/2$ is the characteristic Alfvén velocity, $M$ is the Mach number multiplied by $\gamma$, $\nu_m$ is the dimensionless magnetic viscosity. When passing to dimensionless quantities, we have assumed that $[\rho] = m_+ [n]$, $[t] = [L]/[U]$, $j = c[H]/4\pi[L]$, $[T] = [p]/(k_B[n])$, $k_B$ is the Boltzmann constant. Plasma conductivity $\sigma = \sigma_\pm$ has been taken of the form $[12]

$$\sigma = \frac{e_+ e_-}{\eta}, \quad \eta = 0.5129 \frac{m_e}{\tau_e n}, \quad \tau_e = \frac{3m_e}{(k_B T_e)^{3/2}} \frac{\Lambda e^4 Z^2 n}{4(2\pi)^{1/2} \Lambda e^4 Z^2 n},$$

where $m_e = m_-$, $T_e = T_-$ is the temperature of electrons measured in Kelvins and $\Lambda = 15$ is the Coulomb logarithm. In view of the above expressions, we have

$$\nu_m = \nu_0 / T_e^{3/2}, \quad \nu_0 = 12.825 \cdot 10^{-12} [n]^{3/2} [U]^{-1} [L]^{-1} [p]^{-3/2},$$

where $T_e = T_-$ is the dimensionless value of the electron temperature related to the dimensionless values of $p$ and $\rho$ as follows: $T_- = (p_-/\rho)Z^{-1}(1 + \lambda_-/\lambda_+)$. Finally, in (3) we have also used the dimensionless value of the electric field $E^* = E + H U / c$ in a reference frame fixed to the moving plasma. The MHD-limit is obtained as $\xi \rightarrow 0$, and for $\xi = 0$, system (3) describes radial vibrations of the plasma column in the MHD-theory. Thus, electron inertia affects only
the generalized Ohm’s law (1f), from which the electric field \(E^*\) is sought. In order to find the electric field \(E^*\) for \(\xi > 0\), one has to impose the boundary conditions

\[
m = 0 : \quad E^*(0) - 2\xi^2 \frac{\partial E^*}{\partial m}(0) = 2\nu m \zeta \frac{H}{r} \bigg|_{m=0},
\]

\[
m = M_* : \quad \xi^{-1} \frac{\partial E^*}{\partial m} \bigg|_{m=M_*} = \left[ \frac{I(t)}{pr^2} + \frac{I(t)}{r} \left( \frac{\partial U}{\partial m} - \frac{U}{pr^2} \right) \right] \bigg|_{m=M_*},
\]

where the dot on top of a variable indicates its differentiation in \(t\). Boundary conditions (5) are obtained from the Faraday law combined with the relations \(H(t, R(t)) = 2I(t)/(cR(t))\), \(\dot{R}(t) = U(t, R(t))\) (kinematic condition) on the boundary \(r = R(t)\) of the plasma column, from which it follows that

\[
\frac{\partial E}{\partial r} \bigg|_{r=0} = 0, \quad \frac{\partial E}{\partial r} \bigg|_{r=R(t)} = \left( \frac{2I(t)}{c^2r} - \frac{U}{c} \left( \frac{\partial H}{\partial r} + \frac{2I(t)}{cr^2} \right) \right) \bigg|_{r=R(t)}.
\]

Passing to the field \(E^*\) and the Lagrangian variables in the above relations, we obtain (5). For the remaining functions, the boundary conditions are obvious:

\[
U(0) = 0, \quad H(0) = 0, \quad r(0) = 0, \quad p(M_*) = 0, \quad H(M_*) = I(t)/R(t), \quad r(t, M_*) = R(t).
\]

We consider two alternatives when solving system (3) – (6) on the interval \(0 \leq m \leq M_*, \ t \geq 0\):

(i) the total current \(I(t)\) is given; (ii) \(I(t)\) is sought (together with plasma parameters in the column) from the electrical equation for the external circuit.

4. Electrical equation for the external circuit

The total current \(I(t)\) flowing through the plasma column forms a closed contour with an external circuit, which initially contains an electric charge whose discharge generates the plasma column in the gas chamber. The discharge dynamics depends on the structure and the parameters of the external circuit. Consider the simplest circuit shown in figure 1, where \(L_*\) is the inductance, \(C_*\) is the capacitance, \(R_*\) is the resistance of the electric circuit and \(Z\) is the discharge chamber of the z-pinch. This chamber is a gas-filled cylinder of radius \(R_{ex}\) and height \(z_0\). Voltage is applied to the copper bases of the cylinder and its lateral surface is formed by a dielectric [13].

![Figure 1. External circuit structure.](image)

The electrical equation for the external circuit follows from the law of conservation of energy: the Joule heat generated due to the circuit resistance is equal to the decrease (per time unit) of the total energy, which is a sum of the energies of plasma in the column, the electromagnetic field in the chamber outside plasma, the capacitor and the inductance coil:

\[
- \frac{d}{dt} \left\{ \int_{V_p} \varepsilon dx + \int_{V_0\setminus V_p} \frac{H^2}{8\pi} dx + \frac{L_* I^2}{2c^2} + \frac{Q^2}{2C_*} \right\} = R_* I^2,
\]

(7)
where \( Q(t) \) is the charge on the capacitor plates at time \( t \), \( V_p \) is the part of the chamber occupied by plasma, \( V_0 \) is the whole chamber region, and

\[
\varepsilon = \rho \left( \frac{U^2}{2} + \frac{p}{(\gamma - 1)\rho} + \frac{\lambda_+ \lambda_- f^2}{2\rho^2} \right) + \frac{H^2}{8\pi}.
\]

In cylindrical coordinates, we have

\[
V_p = \{0 \leq z \leq z_0, 0 \leq r \leq R(t), 0 \leq \varphi \leq 2\pi\}, \quad V_0 = \{0 \leq z \leq z_0, 0 \leq r \leq R_{ex}, 0 \leq \varphi \leq 2\pi\}.
\]

For the electromagnetic field in vacuum, we have

\[
H_\varphi(t,r) = \frac{2\dot{I}(t)}{cr}, \quad E_z(t,r) = \frac{2\dot{I}(t)}{c^2} \ln \frac{r}{R(t)} + E_z(t), \quad R(t) \leq r \leq R_{ex},
\]

where \( E_z(t) \) is the electric field on the moving boundary \( r = R(t) \) of the column. Using these expressions and (7), we obtain the electrical equation for the external circuit

\[
\dot{I} \left[ \frac{2z_0}{c^2} \ln \frac{R_{ex}}{R(t)} + \frac{L_s}{c^2} \right] + \frac{Q}{C_\ast} + R_\ast I + z_0 E_z(t) = 0, \quad \dot{Q} = I.
\]

This, in dimensionless form, with the field \( E_z^* = E_z + UH/c \), yields

\[
\alpha_0 \frac{dI}{dt} + d \left( I \ln \frac{R_{ex}}{R(t)} \right) + \eta_0 Q + \eta_1 I + \eta_2 E_z^*(t) = 0, \quad \dot{Q} = I,
\]

where \( \alpha_0 = L_\ast(2z_0)^{-1} \), \( \eta_0 = c^2[t][Q](2z_0 C_\ast[I])^{-1} \), \( \eta_1 = (c^2 R_\ast[t])(2z_0)^{-1} \), \( \eta_2 = c^2[E][t](2[I])^{-1} \), and it is assumed that \( [Q] = [t][I], [I] = cR_0[H]/2 \). Therefore, unless the current \( I(t) \) is given, system (3) – (6) should be supplemented with equation (8) and the initial conditions \( I(0) = I_0, \ Q(0) = Q_0 \) (the initial current and the initial charge on the capacitor plates).

5. A numerical method for solving system (3) – (6) with a given \( I(t) \)

On the interval \([0, T]\), consider integer and half-integer nodes with step \( h = M_s/N \). At the integer nodes \( k h \), \( 0 \leq k \leq N \), we approximate the functions \( U, r, E^* \), and at the half-integer nodes \((k + 1/2)h \), \( 0 \leq k < N \), we approximate the functions \( p_\pm, p = p_+ + p_- \), \( \rho, H, B = H/r \).

For the grid functions approximating the plasma parameters \( f = U, r, \rho, H, E^*, p_\pm \), the transition \( f^n \to f^{n+1} \) from the lower \((t = t_n)\) to the upper \((t = t_n + \tau)\) time layer is defined by the following formulas (for simplicity, the layer number \( n \) is dropped):

### Hydrodynamic equations

1) \[
\frac{U_k^1 - U_k^0}{\tau} + \frac{r_k^0}{h} \left[ \left( \frac{p + \omega}{M} + \frac{\kappa^2}{2} H^2 \right)_k^{k+1/2} - \left( \frac{p + \omega}{M} + \frac{\kappa^2}{2} H^2 \right)_k^{k-1/2} \right] + \frac{H_k^{k+1/2}}{2} \left[ \left( \frac{H^2}{\rho} \right)_k^{k+1/2} - \left( \frac{H^2}{\rho} \right)_k^{k-1/2} \right] = 0, \quad 0 < k \leq N, \quad U_0^1 = 0,
\]

\[
\omega_{k+1/2} = \mu_0 \rho_{k+1/2} (U_k + U_k^0) \cdot 0.5[(U_k + U_k^0) - |U_k + U_k^0|], \quad 0 \leq k < N,
\]

\[
\omega_{N+1/2} = -\omega_{N-1/2}, \quad f_{N+1/2} = 2f_{N-1/2} - f_{N-3/2}, \quad f = p^{-1}, \rho, H^2,
\]

2) \[
\frac{r_k^1 - r_k^0}{\tau} = U_k^0, \quad 0 \leq k \leq N,
\]
3) \[ \frac{[(1/\rho)^{1/2}_{k+1/2} - (1/\rho)_0^{1/2}_{k+1/2}]}{h} = \frac{(r^{1/2} U^1_{k+1} - (r^{1/2} U^1)_k}{h}, 0 \leq k < N, \quad r^{1/2}_k = \frac{r^0_k + r^1_k}{2} \]

Electrodynamic equations

4) \[ \frac{[(B/\rho)^{1/2}_{k+1/2} - (B/\rho)_0^{1/2}_{k+1/2}]}{h} = \zeta^{-1} \left[ \frac{E^*_k - E^*_k}{r} \right], 0 \leq k < N, \]

5) \[ E^*_k - \frac{\zeta}{h} \left[ (r^2 \rho)^0_{k+1/2} \frac{E^*_{k+1} - E^*_k}{h} - (r^2 \rho)^0_{k-1/2} \frac{E^*_k - E^*_{k-1}}{h} \right] = \frac{\nu_0 \zeta (r^2 B)^0_{k+1/2} - (r^2 B)^0_{k-1/2}}{h} + \zeta \frac{W^0_{k+1/2} B^0_{k+1/2} - W^0_{k-1/2} B^0_{k-1/2}}{h}, 0 < k < N, \]

\[ W_{k+1/2} = 2(U r)_{k+1/2} - (r^2 \rho)_{k+1/2} \frac{(r^2 B)^0_{k+1/2} - (r^2 B)^0_k}{h}, 0 \leq k < N, \]

\[ f_k = \frac{f_{k+1/2} + f_{k-1/2}}{2}, f = T, \rho, 0 < k < N, \quad f_{k+1/2} = \frac{f_k + f_{k+1}}{2}, f = r^2, U r, 0 \leq k < N. \]

Equations for pressures

6) \[ \frac{[(p_+ / \rho)^{1/2}_{k+1/2} - (p_+ / \rho)_0^{1/2}_{k+1/2}]}{h} = \frac{(\gamma - 1) (p_+ + \omega_{\pm})^0_{k+1/2} (r^{1/2} U^1)^{k+1} - (r^{1/2} U^1)_k}{h} = \frac{m_{\pm} \nu_0 M_{k}^2}{\rho^{1/2}_{k+1/2} (T^0_{k+1/2}^{3/2})} - \frac{\rho^{1/2}_{k+1/2}}{h} \left[ \frac{(r^2 B)^0_{k+1/2} - (r^2 B)^0_k}{h} \right]^2, 0 \leq k < N, \]

\[ (r^2 B)^0_k = 0, \quad (r^2 B)^0_{k+1/2} = I(t_n), \quad B_k = \frac{B_{k+1/2} + B_{k-1/2}}{2}, 0 < k < N, \]

\[ \omega_{\pm} = \alpha_+ \alpha_-, \quad \alpha_+ + \alpha_- = 1, \quad \alpha_{\pm} \geq 0, \quad (\alpha_+ = \alpha_- = 1/2). \]

Approximation of boundary conditions for \( E^* \)

7) \[ E^*_0 - 2 \zeta \frac{E^*_1 - E^*_0}{h} = \frac{2 \nu_0 \zeta}{(T^0_{1/2}^{3/2})} B_0, \]

\[ \zeta^{-1} \frac{E^*_{N-1} - E^*_N}{h} = \frac{I(t_n)}{(\rho^2_N)^0} + \frac{I(t_n)}{(r^2_N)^0} \left[ \frac{U^0_N - U^0_{N-1}}{h} - \frac{U^0_N}{(\rho^2_N)^0} \right], \]

\[ B_0 = \frac{3}{2} B_{1/2} - \frac{1}{2} B_{3/2}, \quad (T_e)_0 = \frac{3}{2} (T_e)_{1/2} - \frac{1}{2} (T_e)_{3/2}, \quad (\rho^{-1})_N = \frac{3}{2} (\rho^{-1})_{N-1/2} - \frac{1}{2} (\rho^{-1})_{N-3/2}. \]

Stability condition. The step \( \tau \) for each time level is chosen from the Courant – Friedrichs – Lewy condition

8) \[ \tau = k_0 h / U_{\text{eff}}, \quad 0 < k_0 < 1, \]

\[ U_{\text{eff}} = \max_{0 \leq k < N} \left\{ \left( r^2 \rho \right)^0_{k+1/2} \left[ \left( U^0_{k+1/2} + \left[ M^{-1} (\gamma p / \rho)^0_{k+1/2} + \kappa^2 (H^2 / \rho)^0_{k+1/2} \right]^{1/2} \right) \right] \right\}, \]

\[ U_{k+1/2} = \frac{U_k + U_{k+1}}{2}, \quad r_{k+1/2} = \frac{r_k + r_{k+1}}{2}. \]

The coefficient \( k_0 \) and the value \( \mu_0 \) in the expression for artificial viscosity \( \omega_{k+1/2} \) are chosen experimentally. Typically, one has \( k_0 = 0.1 \div 0.8, \mu_0 = 2 \div 4 \). The computation procedure is obvious. From equations 1) we find the grid function \( U^1 \), then from 2) we determine \( r^1 \), and
then seek $\rho^1$ from equations 3). Further, from 5) and 7), solving the system of linear equations by the sweep method, we calculate $E^*$ and then find $B^1$ from 4). Finally, we seek $p^1_\pm$ from 6) and calculate $p^1 = p^1_+ + p^1_-$. After that, using 8), we find the next time step $\tau$ and the computation process is repeated.

The difference approximations of all spatial derivatives in the equations have the order $O(h^2)$, while the time derivatives are of the order $O(\tau)$. The approximations of values beyond the boundary in equations 1) and in the boundary conditions 7) are less precise, namely, $O(h)$. It is not difficult to refine these approximations, but this, however, does not improve the calculation results.

At the initial time instant, the grid functions $U, r, \rho, H, p_\pm$ are given. The grid functions $B = H/r$ and $H$ are expressed through one another by the interpolation formulas $B_{k+1/2} = 2H_{k+1/2}/(r_k + r_{k+1})$, $0 \leq k < N$.

6. A numerical method for solving system (3) – (8)

In this case, the scheme proposed in section 4 should be modified in its part pertaining to the calculation of $E_k^*$, $0 \leq k \leq N$, since the boundary condition 5) at the right end-point is changed (at the left end-point, it is the same). Indeed, eliminating $\dot{I}(t)$ from equation (8) and substituting the resulting expression into the second relation in (5), we obtain the following modified boundary condition for $E^*$ at the right end:

$$\frac{\eta_2}{\rho r^2} \left( \alpha_0 + \ln \frac{R_{ex}}{r} \right)^{-1} E^* + \zeta^{-1} \frac{\partial E^*}{\partial m} = AI(t) + BQ(t),$$

where $A = \frac{\partial}{\partial m} \left( \frac{U}{r} \right) + \left( \frac{U}{r} - \eta_1 \right) \frac{1}{\rho r^2} \left( \alpha_0 + \ln \frac{R_{ex}}{r} \right)^{-1}$, $B = -\frac{\eta_0}{\rho r^2} \left( \alpha_0 + \ln \frac{R_{ex}}{r} \right)^{-1}$.

Here, $A, B$ and all other coefficients in (9) are calculated for $m = M_*$ and $\zeta$ is the same as in section 3.

Now, consider the difference scheme for the electrical equation (8) for the external circuit,

$$\frac{I^{n+1} - I^n}{\tau} + \frac{I^{n+1} \ln(R_{ex}/R^{n+1}) - I^n \ln(R_{ex}/R^n)}{2} + \eta_1 \frac{I^n + I^{n+1}}{2} + \eta_2 E_{z_{mn}}^n = 0,$$

where $Q^0, I^0$ are given and the step $\tau$ is the same as in (1) – (8). Taking into account that $R^{n+1} = r_{N}^1$, $R^n = r_{N}^0$, $E_{z_{mn}} = E_{N}^*$ (see section 4), we come to the following modification of the algorithm of section 4. The changes pertain merely to the calculation of the field $E^*$. First, using the sweep method and taking into account the modified boundary condition (9), we find the function $E^*$ from 5), in particular, we obtain $E_{N}^*$ (in the difference version of (9), the values of $Q(t)$ and $I(t)$ are taken from the $n$-th time layer and are equal to $Q^n$ and $I^n$, respectively).

Now, solving system (10), we obtain $I^{n+1}, Q^{n+1}$ in explicit form:

$$I^{n+1} = \left[ \left( \alpha_0 + \ln \frac{R_{ex}}{R^n} - \frac{\tau \eta_1}{2} - \frac{\tau^2 \eta_0}{4} \right) I^n - \tau \eta_0 Q^n - \tau \eta_2 E_{N}^* \right] D^{-1},$$

$$Q^{n+1} = \left[ \left( \alpha_0 + \ln \frac{R_{ex}}{R^n} + \frac{\tau \eta_1}{2} + \frac{\tau^2 \eta_0}{4} \right) Q^n + \frac{\tau}{2} \left( \alpha_0 + \ln \frac{R_{ex}^2}{R^n R^{n+1}} \right) I^n - \frac{\tau^2}{2} \eta_2 E_{N}^* \right] D^{-1},$$

where $D = \alpha_0 + \ln(R_{ex}/R^{n+1}) + \tau \eta_1/2 + \tau^2 \eta_0/4$. 


7. Testing the algorithm
The difference schemes of sections 5 and 6 have been tested on some special analytical solutions called homogeneous deformations. As shown in [11], system (3) admits an analytical solution of the form

\[
\rho(t, m) = \frac{\rho_0}{w^2(t)} = \frac{\rho_0 y^2 + 2}{(\rho_0 y^2 + 1)^2}, \quad p(t, m) = \frac{\rho_0}{w^2(t)} \left(1 - \frac{m}{M_*}\right), \quad U(t, m) = \xi \hat{w}(t)y, \\
H(t, m) = \left(\rho_0 + y_*^{-2}\right)^{1/2} \frac{I(t)}{w(t)} y, \quad r(t, m) = \xi w(t)y, \quad p_\pm = \alpha_\pm p, \\
E(t, m) = \frac{\xi \zeta}{\rho_0 y_*} I(t) (\rho_0 y^2 + 1)^{1/2}, \quad 0 \leq m \leq M_*, \quad t \geq 0,
\]

where \( I(t) \in C^1[0, +\infty] \) is an arbitrary function and \( w(t) \) is a solution of the problem

\[
\dot{w} - \frac{K_*}{w^{2\gamma-1}} + \frac{\chi_* I^2(t)}{w} = 0, \quad w(0) = 1, \quad K_* = \frac{p_0}{M M_*}, \quad \chi_* = \frac{\kappa^2 (\rho_0 y_*^2 + 1)}{\rho_0 y_*^2 \zeta^2}.
\]

Here, \( \rho_0 > 0, \rho_0 > 0 \) are arbitrary constants, \( \xi, \zeta \) are the same as in section 3, \( y_* = \xi^{-1} R_0 |L|^{-1}, \) \( M_* = m(y_*), \) where \( m(y) = (\xi^2/2) \left[ \ln(\rho_0 y^2 + 1) + 1 - (\rho_0 y^2 + 1)^{-1} \right], \) \( 0 \leq y \leq y_* \) is a monotonically increasing function from \([0, y_*]\) onto \([0, M_*]\), and \( y = y(m) \) in (11) is the inverse function. In figure 2, we compare the results for the radius obtained numerically by the algorithm of section 5 with those from the analytical solution (11), (12) for \( I(t) = t + 1 \) (under the assumption that \( \dot{w}(0) = 0 \) in (12)). We see that on the time interval of ten vibrations of the column, our difference scheme is practically ideal for modeling both compression and extension of the column (with the error less than \( 10^{-4} \)); subsequently, the phase shift between the numerical and analytical solutions starts to increase, though the amplitudes and the periods of both vibrations are practically identical. A similar picture is observed for other parameters of the flow, as their numerical and analytical values are compared. Tests of this kind can be constructed also for the algorithm of section 6. But now, \( I(t), w(t) \) are sought from system (12), (8), and one should set \( R(t) = \xi y_* w(t), \) \( E_z^2(t) = I(t) \xi \zeta (\rho_0 + y_*^{-2})^{1/2}/p_0 \) in equation (8).

8. Conclusion
We have constructed and tested a difference scheme for solving the EMHD equations in the cylindrically symmetric case. In contrast to the MHD theory, this approach involves a nontrivial version of the generalized Ohms law stated as a boundary value problem for the electric field on an interval. For this problem, we obtain boundary conditions and describe an effective numerical algorithm for finding, jointly, the electric and the magnetic fields. The difference scheme proposed here involves discretization of the electrical equation for the external circuit and takes into account motion of the plasma-vacuum boundary. Our algorithm is tested on certain nontrivial solutions of the EMHD equations, which correspond to homogeneous deformations of
plasma. The numerical method proposed here allows us to examine cylindrically symmetric evolution of a plasma column, taking into account electron inertia, motion of the column boundary, and processes in the external circuit.

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