Solvable model of quantum phase transitions and the symbolic-manipulation-based study of its multiply degenerate exceptional points and of their unfolding

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Abstract
It is known that the practical use of non-Hermitian (i.e., typically, $\mathcal{PT}$-symmetric) phenomenological quantum Hamiltonians $H \neq H^\dagger$ requires an efficient reconstruction of an ad hoc Hilbert-space metric $\Theta = \Theta(H)$ which would render the time-evolution unitary. Once one considers just the $N$-dimensional matrix toy models $H = H^{(N)}$, the matrix elements of $\Theta(H)$ may be defined via a coupled set of $N^2$ polynomial equations. Their solution is a typical task for computer-assisted symbolic manipulations. The feasibility of such a model-completion construction is illustrated here via a discrete square well model $H = p^2 + V$ endowed with a $k$-parametric close-to-the-boundary interaction $V$. The model is shown to possess (possibly, multiply degenerate) exceptional points marking the phase transitions which are attributable, due to the exact solvability of the model at any $N < \infty$, to the loss of the regularity of the metric. In the parameter-dependence of the energy spectrum near these singularities one encounters a broad variety of alternative, topologically non-equivalent scenarios.

1 Introduction
In the two most recent collections [1] of papers on the applicability of non-Hermitian operators in quantum physics one can find multiple samples of the advantages which are provided by the use of manifestly non-Hermitian effective quantum Hamiltonians $H \neq H^\dagger$ in several areas of phenomenology. Among these advantages one of the key roles is played by the capability of these sufficiently general phenomenological Hamiltonians of mimicking the quantum phase transitions and/or an onset of quantum chaos in many-body systems, etc. The growth of popularity of this area of research motivated also our present study in which we intend to pay particular attention to the role of computer-assisted symbolic manipulations.

From the point of view of mathematics an explanation of the deepest essence of at least some of the above-mentioned phenomena is not too difficult since many of them are simply caused by
the so called spontaneous breakdown of certain symmetries. For the most elementary illustration let us recall, e.g., paper [2] where we explained that and how the spontaneous breakdown of the combined parity and time-reversal symmetry (conveniently abbreviated as \( \mathcal{PT} \)-symmetry in the physics literature [3]) plays the role of a trigger of transition between the observability and non-observability of the energy in an elementary toy model of quantum dynamics. The message delivered by this and similar elementary examples is nontrivial and unexpectedly deep showing, e.g., that the possibility of transitions between different dynamical regimes is closely connected to the presence of branch-point singularities, say, on the Riemann energy surface \( \mathbb{E}(g) \) in the complex plane of coupling (or of any other tunable parameter) \( g \).

In the closely related Kato’s monograph [4] on the mathematics of perturbation theory the latter singularities were systematically studied via finite-dimensional matrix models and they were also given the well-chosen name of “exceptional points” (EP). The same finite-dimensional-matrix methodical strategy will be also accepted in what follows. In Introduction let us also mention that in a broader mathematical setting of the geometric singularity theory one can find the same (or at least very similar) concepts in the Thom’s classical theory of catastrophes [5] (with multisided applications [6]) as well as in its multiple newer descendants: \textit{pars pro toto} let us mention our recent proposals [7, 8] of the simplest possible quantum analogues of such a classical singularity classification pattern.

In the mathematically narrower square-root-branch EP context the studies of the Riemann-surface singularities found particularly numerous explicit applications in quantum physics. In the context of perturbative quantum field theory and in a way enhancing our understanding of quantum anharmonic oscillations in potentials \( V(x) = x^2 + gx^2+δ \) the singularities of this class became widely known under a nickname of “Bender-Wu singularities” [9]. In optics, the alternative theoretical identification of EPs with the points of non-Hermitian degeneracies [10] encountered an enormous experimental popularity recently [11]. This success was supported not only by the availability of innovated metamaterials possessing anomalous refraction indices but also by the underlying analogies between the Maxwell and Schrödinger equations in the dynamical regime of phase transitions [12].

After a return to the standard quantum mechanics of stable systems or to the atomic, molecular or nuclear phenomenology [13], the studies of concrete models reveal the existence of EP hypersurfaces \( \partial \mathcal{D} \) playing the role of certain natural horizons of observability of quantum systems (i.e., of certain separation boundaries between different phases), with numerous important physical as well as mathematical consequences [14, 15]. In our present paper we shall be inspired by this particular problem. We shall describe some of its aspects in detail, emphasizing that their clarification finds a very natural methodical support in the symbolic as well as advanced numerical manipulations mediated, typically, by MAPLE [16] and/or by similar, mostly commercially available software.
The concept of hidden Hermiticity of Hamiltonians

During practically all of the history of quantum theory it has been overlooked that its applicability is restricted by the use of concrete representations of the physical Hilbert space $\mathcal{H}^{(P)}$ made in parallel with a concrete self-adjoint representation of observables (say, generators $\mathfrak{h}^{(P)}$ of the unitary evolution with time). A criticism of such a paradigm emerged, e.g., in Refs. [3, 17].

The change of the paradigm has been encouraged by the practical needs of applications of quantum theory in nuclear physics. Besides the often cited Dyson-inspired non-Hermitian variational approach to the so called interacting boson models of heavy nuclei [14] one might also recall another manifestly non-Hermitian variational method based on the judicious, Hilbert-space-metric-employing coupling of clusters [18], etc. One of the main obstacles of the necessary conceptual separation of the simultaneous choices of the Hilbert spaces and Hamiltonian operators was the difficulty of its implementation in calculations. Only too often, people prefer the choice of the simplest Hilbert-space representations (with, say, $\mathcal{H}^{(P)} \equiv L^2(\mathbb{R}^d)$) and of the simplest dynamics (cf. also the critical commentary in [19] in this context).

On abstract level, the amended quantum theory admitting a broader class of quantum dynamics may be found summarized in [20]. In the spirit of Ref. [14] one finds the simultaneous introduction of space $\mathcal{H}^{(P)}$ and self-adjoint observable $\mathfrak{h}^{(P)}$ overrestrictive. The information about dynamics is separated into the choice of Hilbert space $\mathcal{H}^{(F)}$ (which remains friendly, cf. the superscript) and a given Hamiltonian operator $H$ (which need not necessarily remain self-adjoint in the same space, $H \neq H^\dagger$). The emerging apparent puzzle (“does one violate the requirements of unitarity and Stone’s theorem?”) has an elementary explanation [3, 14]: The initial Hilbert space (say, $\mathcal{H}^{(F)} \equiv L^2(\mathbb{R})$) is reclassified as auxiliary and unphysical. In parallel, the apparently non-Hermitian Hamiltonian (take, for illustration, just the most popular Bessis’ imaginary cubic oscillator $H^{(ICO)} = p^2 + ix^3$ [21] with real spectrum [22]) is also reclassified. As “crypto-Hermitian”, i.e., by definition [20], as self-adjoint in another, “standard” Hilbert space $\mathcal{H}^{(S)}$.

One makes the Hamiltonian $H$ self-adjoint in $\mathcal{H}^{(S)}$ by using just an ad hoc redefinition of the inner product in $\mathcal{H}^{(F)}$. Once we have the two different operators $H$ and $H^\dagger \neq H$ acting on the ket vectors $|\phi\rangle$ in $\mathcal{H}^{(F)}$, we simply change

$$\langle \psi|\phi \rangle^{(F)} \rightarrow \langle \psi|\phi \rangle^{(S)} := \langle \psi|\Theta|\phi \rangle^{(F)}.$$  \hspace{1cm} (1)

The self-adjoint and positive definite operator $\Theta = \Theta^\dagger > 0$ may be perceived as playing the role of the Hilbert space metric (the mathematical conditions are listed, say, in [14]). Thus, the usual Hermiticity of observables $\Lambda$ is now required in $\mathcal{H}^{(S)}$,

$$\Lambda^\dagger := \Theta^{-1} \Lambda^\dagger \Theta = \Lambda.$$ \hspace{1cm} (2)

For the Hamiltonian $\Lambda_0 = H$ (with real spectrum), in particular, the hidden Hermiticity property

$$H^\dagger \Theta = \Theta H$$  \hspace{1cm} (3)

is often re-read as an implicit, ambiguous [14] definition of a suitable metric $\Theta = \Theta(H)$. 

3
3 An update of the concept of solvability

The concept of solvability is often restricted to the availability of the closed-form eigenstates of ordinary differential Hamiltonians \[23\]. The phenomenologically oriented search for the measurable aspects of quantum systems forced many physicists to search for various extensions of the concept. The question reemerged, recently, in connection with the new wave of interest in phase transitions described in terms of the spontaneous breakdown of antilinear symmetries. Typically, the same \(PT\)–symmetric Hamiltonian \(H = H(\lambda)\) is considered before and after the phase transition. During the phase transition itself such a Hamiltonian becomes “anomalous” (i.e., some of its eigenvalues get complex). In other words, one leaves the physical domain of parameters \(D\).

During the application of such an idea to the above-mentioned imaginary cubic oscillator \(H^{(ICO)}\) people encountered serious mathematical (i.e., technical \[24\] as well as much more serious conceptual \[25\]) difficulties. A partial escape out of such a trap has been found in the exceptional differential-operator and boundary-delta-function model \(H^{(Robin)}\) of Refs. \[26\] which proved sufficiently representative though still completely solvable \[27\].

A more systematic realization of the project (i.e., in the language of mathematics, of an exhaustive solution of the operator Eq. (3)) has been revealed and described, in Refs. \[28, 29\], as based on the use of suitable finite-dimensional toy models. Their solvability opened new perspectives in a choice of the Hilbert-space metric in Eq. (2). The key point was that one did not need to start from a metric which would be given \textit{a priori}. Even in a generalized (e.g., \(PT\)–symmetric) setting one was suddenly able to reconstruct the metric from the given Hamiltonian \(H \neq H^\dagger\) non-numerically.

One of the key difficulties now emerged in connection with the ambiguity of the general solution \(\Theta = \Theta_\kappa(H)\) of Eq. (3). Here the subscripted (multi)index \(\kappa\) numbers the alternatives (see \[14\] for an exhaustive discussion of this point). The new problem only remains reasonably tractable in the finite-dimensional Hilbert spaces with \(\dim \mathcal{H}^{(F,S)} = N < \infty\). In these cases it appears sufficient \[30\] to solve the conjugate Schrödinger equation

\[
H^\dagger |\Xi_n\rangle = E_n^\dagger |\Xi_n\rangle
\]

(note that by assumption the spectrum is real and discrete here). One then defines the general metric by the formula

\[
\Theta_\kappa(H) = \sum_{n=1}^{N} |\Xi_n\rangle \kappa_n \langle \Xi_n|.
\]

An arbitrary optional \(N\)–plet of coefficients \(\kappa_n > 0\) is admitted. In other words, the solvability of the model proves crucial under the \(N < \infty\) assumption.

3.1 Oscillator-type solvable models

Even if one decides to work with dimensions \(N < \infty\), serious technical difficulties with the analysis of spectra of \(H\) and/or with the explicit construction of the metrics already emerge at dimensions
as low as \( N = 4 \) [31]. It is recommended to work with the matrices of specific forms as sampled by the well-motivated non-Hermitian and PT-symmetric version of the popular Bose-Hubbard complex Hamiltonian \( H^{(BH)} \) [32] and by some other realistic physical models [33].

Alas, strictly speaking, many of the realistic choices of these computationally tractable (i.e., complex and, typically, tridiagonal or pentadiagonal) Hamiltonians \( H \) cease to be solvable, in spite of their frequent merit of being well described by perturbation theory [32]. In this sense the first decisive step toward the exactly solvable family has only been made in Ref. [34] where we picked up and studied the anharmonic-oscillator-related real and tridiagonal anharmonic-like matrices

\[
H^{(N)}_{(ATM)} = \begin{bmatrix}
1 - N & g_1 & 0 & 0 & \cdots & 0 \\
-g_1 & 3 - N & g_2 & 0 & \cdots & 0 \\
0 & -g_2 & 5 - N & \ddots & \ddots & \vdots \\
0 & 0 & -g_3 & \ddots & g_2 & 0 \\
\vdots & \vdots & \ddots & \ddots & N - 3 & g_1 \\
0 & 0 & \cdots & 0 & -g_1 & N - 1
\end{bmatrix}
\] (6)

which appeared particularly construction-friendly. With the purpose of their further necessary simplification at the larger \( N \) we imposed an additional requirement \( g_{N-1} = g_1, g_{N-2} = g_2, \ldots \), having enhanced the symmetry of the underlying multiparametric coupling pattern perceivably. We were rewarded by the discovery of the exact solvability of the resulting model at all \( N < \infty \) [34].

This discovery proved heavily dependent on the availability of the symbolic manipulations with polynomials in MAPLE. Pars pro toto let us recall that for the very localization of the degenerate EP value of the very first coupling \( g_{N-1} = g_1 := \sqrt{D} \) (i.e., of the first coordinate of the vertex of the boundary manifold \( \partial\mathcal{D} \)) at the not too large sample dimension \( N = 8 \) we had to determine this particular EP value (equal, incidentally, exactly to \( \sqrt{7} \)) as a unique (sic!) root of the sixteenth-degree secular-like polynomial equation

\[
314432 \, D^{17} - 5932158016 \, D^{16} + 4574211144896 \, D^{15} + 3133529909492864 \, D^{14} + \\
+917318495163561932 \, D^{13} + 167556261648918275684 \, D^{12} + \\
+14670346929744822064505 \, D^{11} + 720991093724510065469933 \, D^{10} + \\
+6249137451114251409236415 \, D^9 + 676326278232758784369966787 \, D^8 + \\
+40525434802944282153115803370 \, D^7 + 2361976444746440513605248930610 \, D^6 - \\
-14575983663685012145070948315366 \, D^5 + \\
+8129925258122948689157916436170874 \, D^4 + \\
-68875673245487669398850290405642067 \, D^3 + \\
+235326754101824439936800228806905073 \, D^2 - \\
-453762279414621179815552897029039797 \, D + 
\]
This and similar polynomials were generated by means of the Groebner-basis technique as implemented in MAPLE. Needless to add, even the very proof of the uniqueness of this root (which we never published, due to its length) required an even more extensive use of the MAPLE software. Let us also add that the similar computer-assisted constructions of the EP boundaries had to be performed just at a few not too large dimensions $N$. Due to the immanent friendliness of our highly symmetric toy models $H^{(N)}_{(TAM)}$ we were able to extrapolate the resulting closed formulae to all $N$, we clarified their structure and we pointed out their relevance in Ref. [35]. The readers may find more details therein.

3.2 Classical-orthogonal-polynomials-related solvable models

The above family of solvable physics-motivated quantum models $H^{(N)}_{(TAM)}$ has been complemented by the other, mathematically motivated and classical-orthogonal-polynomials-related models of Refs. [36]. Subsequently [37] we added more details of the symbolic-manipulation-assisted constructions of the necessary Hilbert-space metrics $\Theta$ for the underlying specific Hamiltonians. For illustration we choose there the Gegenbauer-related family of Hamiltonians

$$
\begin{pmatrix}
0 & 1/2 \ a^{-1} & 0 & 0 & \ldots & 0 \\
2 \frac{a}{2a+2} & 0 & (2a+2)^{-1} & 0 & \ldots & \vdots \\
0 & 2a+1 & 0 & (2a+4)^{-1} & \ddots & 0 \\
0 & 0 & 2a+2 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & 0 & \frac{2a+N-1}{2a+2N-2} & 0
\end{pmatrix}
\tag{7}
$$

This enabled us to explain that besides the above-mentioned key contribution of computer facilities to the feasibility of the symbolic-manipulation constructions of the eligible metrics $\Theta$, an equally important role appeared to be played by the MAPLE-supported numerical software which enables one to control the numerical precision needed in the, in general, very ill-conditioned task of the localization of the eigenvalues of $\Theta$. Yielding the guarantee of the necessary strict positivity of all of these eigenvalues and of their inverse values. The resulting explicit knowledge of the boundaries of the domain at which the eigenvalues of the metric $\Theta = \Theta_{\kappa}(H)$ were losing their positivity appeared to be of a particular relevance in the quantum analogue of the classical theory of catastrophes as described in Ref. [8].

3.3 Quantum-graph solvable models

Just for completeness let us also mention our quantum-graph proposals [38] in which the third family of the solvable quantum models emerged after a suitable discretization of coordinates, $x \in \mathbb{R}$
$\rightarrow x_j, j \in \mathbb{Z}$. This trick helped us to make the approximate models tractable by the standard tools of linear algebra.

The simplest dynamically nontrivial though still topologically trivial model of the latter discrete-coordinate family dates already back to Ref. [28]. In this paper the most elementary special case was characterized by the following next-to-trivial tridiagonal matrix Hamiltonian

$$H^{(N)}(\lambda) = \begin{pmatrix}
2 & -1 - \lambda & 0 & \ldots & 0 & 0 \\
-1 + \lambda & 2 & -1 & 0 & \ldots & 0 \\
0 & -1 & 2 & \ddots & \ddots & \vdots \\
\vdots & 0 & \ddots & \ddots & -1 & 0 \\
0 & \vdots & \ddots & -1 & 2 & -1 + \lambda \\
0 & 0 & \ldots & 0 & -1 - \lambda & 2
\end{pmatrix}.$$  \hspace{1cm} (8)

In our present paper we intend to generalize such a one-parametric boundary-interaction family, keeping in mind, i.a., the not yet explored possibility of connecting this and similar $N < \infty$ quantum Hamiltonians and bound-state spectra with their respective analogues as defined and derived in continuous limits [39].

4 New, $k$–parametric boundary-interaction toy model

The highly restricted flexibility of the exactly solvable one-parametric discrete square well model (8) of Ref. [28] is disappointing. This disappointment is only partially compensated by the immanent merit of the possible connection of the model to its continuous $N \rightarrow \infty$ limit and analogue as proposed and described in Refs. [26]. On the other hand, one of the key shortcomings of the one-parametric and discrete $N < \infty$ models (8) is that they do not allow us to perform any EP-degeneracy fine-tuning, found and accessible in several multiparametric $N < \infty$ toy models, say, of Refs. [32, 34]. In our recent paper [40] we turned attention, therefore, to the two- and three-parametric extensions of the above-mentioned model (8). We revealed that the extended models remain solvable. In our present further extension of the latter paper we shall introduce and study, therefore, the entirely general family of the $k$–parametric and $N$ by $N$ dimensional matrix quantum Hamiltonians

$$H^{(N)}(\lambda, -\mu, \ldots) = \begin{pmatrix}
2 & -1 - \lambda & 0 & \ldots & \ldots & 0 \\
-1 + \lambda & 2 & -1 + \mu & 0 & \ldots & \vdots \\
0 & -1 - \mu & 2 & -1 - \nu & 0 & \ldots \\
\vdots & 0 & -1 + \nu & 2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & -1 - \mu & 0 \\
0 & \ldots & \ldots & 0 & -1 - \lambda & 2
\end{pmatrix}.$$  \hspace{1cm} (9)
They contain an antisymmetrized and sign-changing sequence of the couplings \( \lambda, \mu, \ldots \) entering the elements
\[
-1 - \lambda, -1 + \mu, -1 - \nu, \ldots, -1 + \nu, -1 - \mu, -1 + \lambda
\]
of the upper diagonal and, mutatis mutandis, also the elements of the lower diagonal.

### 4.1 The construction of the simplest Hermitizing metric

Our first result may be now formulated as the statement that models (9) remain solvable at any number of parameters \( k \). In order to illustrate the contents of such a result, let us now take the sufficiently representative \( N = 11 \) sample of Hamiltonian (9) containing, say, the four non-vanishing couplings in the matrix \( H^{(11)}(\lambda, -\mu, \nu, -\rho) \) of the bidiagonal form with vanishing main diagonal and with the upper diagonal such that
\[
\{ 1 + H_{j+1,j}^{(11)}(\lambda, -\mu, \nu, -\rho), \ j = 1, 2, \ldots, N-1 \} = \{-\lambda, -\mu, -\nu, \rho, 0, 0, -\rho, \nu, -\mu, \lambda\}
\]
and with the lower diagonal such that
\[
\{ 1 + H_{j,j+1}^{(11)}(\lambda, -\mu, \nu, -\rho), \ j = 1, 2, \ldots, N-1 \} = \{\lambda, -\mu, -\nu, 0, 0, \rho, -\nu, -\mu, -\lambda\}.
\]
We may feel inspired by papers [28, 41] and use the diagonal ansatz for the metric,
\[
\Theta_{j,j}^{(\text{diag})}, \ j = 1, 2, \ldots, N\}
\]
Its insertion converts the crypto-Hermiticity condition (3) into a set of coupled nonlinear algebraic equations. Their more or less routine solution (using symbolic manipulations) leads to the unambiguous step-by-step elimination and specification of the unknown metric-matrix elements,
\[
z_4 = \frac{1 + \rho}{1 - \rho} \equiv f(-\rho), \quad z_3 = \frac{-1 + \nu - \rho + \nu \rho}{-1 + \rho - \nu + \nu \rho} = f(-\rho) f(\nu),
\]
\[
z_2 = -\frac{1 - \mu + \nu + \nu \mu - \rho - \mu \rho + \nu \rho + \nu \rho \mu}{1 - \rho + \nu - \nu \rho - \mu - \rho \mu - \nu \mu + \nu \rho \mu} = f(-\rho) f(\nu) f(-\mu)
\]
plus, finally, \( z_1 = \)
\[
= \frac{1 - \lambda + \mu - \mu \lambda - \nu + \nu \lambda - \nu \mu + \nu \mu \lambda + \rho - \rho \lambda + \rho \mu - \mu \lambda - \mu \rho + \nu \rho \mu + \nu \rho \mu \lambda}{1 - \rho + \nu - \nu \rho - \mu + \rho \mu - \nu \mu + \nu \rho \mu + \nu \rho \mu \lambda} = f(-\rho) f(\nu) f(-\mu) f(\lambda).
\]
The extrapolation pattern is now obvious, yielding the proof of the following, entirely general

**Proposition 1.** Every \( k \)-parametric Hamiltonian \( H^{(N)} = H^{(N)}(\lambda_1, \lambda_2, \ldots, \lambda_k) \) of Eq. (9) with the real parameters \( \lambda_1 = +\lambda, \lambda_2 = -\mu, \lambda_3 = +\nu \) etc which are all smaller than one in absolute value is self-adjoint in Hilbert space \( \mathcal{H}^{(S)} \sim \mathbb{R}^N \) using the diagonal non-Dirac Hilbert space metric \( \Theta^{(\text{diag})} \neq I \) which differs from the unit matrix just by its \( 2k \) outermost diagonal matrix elements
\[
\Theta_{kk}^{(\text{diag})} = \Theta_{N+1-k,N+1-k}^{(\text{diag})} = f(\lambda_k),
\]
\[ \Theta_{k-1k-1}^{(diag)} = \Theta_{N+2-k,N+2-k}^{(diag)} = f(\lambda_k)f(\lambda_{k-1}), \]

\[ \ldots, \]

\[ \Theta_{11}^{(diag)} = \Theta_{N,N}^{(diag)} = f(\lambda_k)f(\lambda_{k-1}) \ldots f(\lambda_1) \]

where \( f(x) := (1 - x)/(1 + x) \).

**Remark 1.** It is more than appropriate to add here that once we managed to construct the positive and invertible diagonal metric \( \Theta^{(diag)} \), we need not bother about the proof of the reality of the spectrum of the related crypto-Hermitian matrix \( H^{(N)} = H^{(N)}(\lambda_1, \lambda_2, \ldots, \lambda_k) \) anymore. Indeed, the latter matrix is, by construction, self-adjoint in the “new auxiliary” Hilbert space \( \mathcal{H}^{(NA)} \sim \mathbb{R}^N \) which is assumed endowed with the \( (S) \)–superscripted inner product \( \mathfrak{I} \) where \( \Theta = \Theta^{(diag)} \neq I \).

Following paper [41] one should add that the latter space need not coincide with the ultimate physical Hilbert space \( \mathcal{H}^{(S)} \) of Ref. [20]. Still, the (hidden) Hermiticity \( H = H^\dagger \) in \( \mathcal{H}^{(NA)} \) implies the reality of the energy spectrum of course.

### 4.2 Non-equivalent tridiagonal Hermitizing metrics

In the spirit of paper [28] let us now recall the highly ambiguous nature of the general, \( N \)–parametric, spectral-expansion-resembling formula (5) for the metric and let us make use of this great flexibility in assuming that there might exist some still sufficiently elementary next-to-diagonal metric of the form

\[ \Theta^{(tridiag)} = \Theta^{(diag)} + v\mathcal{P} \quad (10) \]

in which the \( N \)–dimensional and real diagonal metric \( \Theta^{(diag)} = \Theta^{(diag)}(\lambda_1, \ldots, \lambda_k) \) of Proposition 1 is “perturbed” by a suitable bidiagonal pseudometric. For the sake of clarity let us first set \( N = 11 \) and insert the ansatz

\[
\mathcal{P} = \begin{bmatrix}
0 & t_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_1 & 0 & t_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_2 & 0 & t_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t_3 & 0 & t_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t_4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & t_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_4 & 0 & t_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_3 & 0 & t_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_2 & 0 & t_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_1 & 0
\end{bmatrix}
\]
in Eq. (3) again. In a more or less routine manner we may again solve the resulting set of the $N^2 = 121$ coupled algebraic equations yielding the following unique result,

$$t_4 = 1 + \rho \equiv (1 - \rho) z_4, \quad t_3 = \frac{-1 + \nu - \rho + \nu \rho}{-1 + \rho} \equiv (1 + \nu) z_3,$$

$$t_2 = \frac{-1 - \mu + \nu + \nu \mu - \rho - \rho \mu + \nu \rho + \nu \rho \mu}{-1 + \rho - \nu + \nu \rho} \equiv (1 - \mu) z_2$$

plus, similarly and finally, $t_1 = (1 + \lambda) z_1$. It is rather easy to generalize now this construction and to reformulate it into a detailed proof of the following

**Proposition 2.** Every $k-$parametric Hamiltonian $H^{(N)} = H^{(N)}(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is also self-adjoint in another Hilbert space $H^{(S)} \sim \mathbb{R}^N$ where a tridiagonal non-Dirac Hilbert space metric $\Theta^{(tridiag)} \neq I$ is used in the form of a positive definite linear combination \((10)\) of the diagonal metric of preceding Proposition with the bidiagonal pseudometric $\mathcal{P}$. In the latter matrix the main diagonal vanishes while its non-vanishing upper and lower diagonals have the same form filled with units, with the exception of the $2k$ outermost matrix elements

$$\mathcal{P}_{kk+1} = \mathcal{P}_{N-k,N+1-k} = (1 + \lambda_k) f(\lambda_k),$$

$$\mathcal{P}_{k-1k} = \mathcal{P}_{N+1-k,N+2-k} = (1 + \lambda_{k-1}) f(\lambda_k) f(\lambda_{k-1}),$$

$$\ldots,$$

$$\mathcal{P}_{12} = \mathcal{P}_{N-1,N} = (1 + \lambda_1) f(\lambda_k) f(\lambda_{k-1}) \ldots f(\lambda_1)$$

where the function $f(x) := (1 - x)/(1 + x)$ is the same as above.

**Remark 2.** Whenever the real parameter $v$ in formula \((10)\) remains sufficiently small, the resulting tridiagonal metric $\Theta^{(tridiag)}$ remains “acceptable”, i.e., safely positive and invertible. For the larger values of $v$ the analysis is more difficult. One must proceed in full methodical parallel with the analogous problem as studied in Ref. [37] and mentioned also in paragraph 5.2 above.

## 5 The descriptive and spectral properties of the model

### 5.1 The EP degeneracy phenomenon from $H^{(N)}(\lambda, -\mu, \ldots)$ using $N = 11$

Near the EP boundary $\partial \mathcal{D}$ some of the elements of the diagonal metric $\Theta^{(diag)}$ of Proposition 1 will, for the consistency reasons, almost vanish or almost diverge. We may expect, therefore, that the most elementary parametric path of couplings $\lambda = \mu = \ldots = t$ will certainly cross the EP boundary at $t = \pm 1$. This expectation is confirmed by Fig. 1 in which the $t-$dependence of the real spectrum as well as the phase-transition-marking loss of its reality at $t = \pm 1$ are displayed, for illustration, at $N = 11$ and $k = 4$.

The most interesting features illustrated by the latter picture seem to be the $t-$independence of the exceptional level $E = 0$, the presence of the two outer “spectator-like” levels and, last
but not least, the nine-tuple EP degeneracy which occurs at \( t = \pm 1 \). In a complementary step of analysis one can easily switch to symbolic manipulations and derive the corresponding exact secular equation

\[
E^{11} - (10 - 8 t^2) E^9 + (36 - 58 t^2 + 22 t^4) E^7 - (56 - 136 t^2 + 104 t^4 - 24 t^6) E^5 + \\
+ (35 - 114 t^2 + 132 t^4 - 62 t^6 + 9 t^8) E^3 - (6 - 24 t^2 + 36 t^4 - 24 t^6 + 6 t^8) E = 0.
\]

On this ground we may confirm the above-mentioned graphical result rigorously. Indeed, this secular equation degenerates to the trivial relation \( E^{11} - 2 E^9 = 0 \) at the two EP-marking parameters \( t = \pm 1 \), etc.

### 5.2 A typology of the unfoldings of the EP degeneracies

In an attempt of exploring the small vicinity of the maximal EP degeneracy at \( \lambda^{(MEP)} = \mu^{(MEP)} = \nu^{(MEP)} = \rho^{(MEP)} = 1 \) let us now fix one of the individual parameters near the EP boundary \( \partial D \) and let us keep the selected parameters, one by one, \( t \)-independent. This change will define the four new phase-transition parametrizing paths of the couplings. One may expect that the nine-fold degeneracy of the energies of Fig. 1 at \( t = \pm 1 \) will unfold in different ways forming the alternative phase-transition patterns.

In Fig. 2 we see the first \( t \)-dependent spectrum in which we fixed the innermost coupling \( \rho = 9/10 \) and in which we kept the other three couplings in the same form as above, \( \lambda = \mu = \nu = t \). With the two outer, “spectator” real levels left out of the picture we see that the eight innermost levels remain degenerate at \( t = -1 \) while the degeneracy of the remaining pair gets shifted rather far to the left.

In the subsequent picture provided by Fig. 3 we see the more thoroughly modified \( t \)-dependence of the spectrum which is caused by the move to the next scenario in which we choose the constant

Figure 1: Real eigenvalues of the \( k = 4 \) Hamiltonian \( H^{(11)}(t, -t, t, -t) \) and the graphical localization of the degenerate phase-transition EP points at \( t = \pm 1 \).
Figure 2: The nine central real eigenvalues of the \( k = 4 \) Hamiltonian \( H^{(11)}(t, -t, t, -0.9) \) and the “weakest” unfolding of the degeneracy near the phase-transition point \( t = -1 \).

\( \nu = 9/10 \) while keeping the remaining couplings variable as above, \( \lambda = \mu = \rho = t \). Ignoring now still the two outermost spectator levels as less relevant, we observe that another outer pair of the real levels has got separated at \( t = -1 \) and that it only becomes degenerate and complexified more to the left. Marginally, it is worth noticing that at the values of \( t \ll -1 \) which already lie out of the picture (i.e., very far to the left) the previously shifted second outer pair gets merged with the respective upper or lower “spectators” so that merely the single, exceptional constant energy \( E = 0 \) remains real in the \( |t| \gg 1 \) asymptotic region.

A return to the asymptotic reality of the triplet of the energies (including again two outermost spectators, i.e., not visible in Fig. 4) is the phenomenon which characterizes, a bit unexpectedly, the next choice of \( \mu = 9/10 \) together with \( \lambda = \nu = \rho = t \). A graphical explanation is provided by the parallel change of behaviour of the levels near the MEP degeneracy. In Fig. 4 we see that the sequential unfolding of this degeneracy further continues in a way which is a bit more subtle. In fact, the “expectable” complexification of the further two nontrivial innermost energy trajectories gets replaced by their mere crossing, followed by the two separate subsequent complexifications which only occur again a bit later, i.e., further to the left.

After the last possible choice of \( \lambda = 9/10 \) with \( \mu = \nu = \rho = t \) all the nontrivial levels get again complex at the sufficiently large \( |t| \gg 1 \). The whole real spectrum is shown in Fig. 5 where we see that the complexification now involves the sextuplet of outer energy levels. The extreme phase transition pattern of Fig. 1 is now changed most thoroughly. The original degeneracy of the spectrum survives just in the weakest form at \( t = \pm 1 \). This and similar observations may be also deduced from the \( t = \pm 1 \) form of the secular equation,

\[
E^{11} - \frac{119}{50} E^9 + \frac{7961}{10000} E^7 - \frac{361}{5000} E^5 = 0.
\]

Due to the unexpected exact solvability of this \( t = \pm 1 \) algebraic equation one can confirm the expected presence of the five degenerate inner roots \( E = 0 \) and also the less expectable double
Figure 3: The central real eigenvalues of the $k = 4$ Hamiltonian $H^{(11)}(t, -t, 0.9, -t)$ and the second form of the unfolding of the degeneracy near the phase-transition point $t = -1$.

degeneracy of the two other, non-vanishing roots $E = \pm \sqrt{19}/10$ which moved away from zero. The last though, possibly, just marginal surprise is that the last two non-degenerate, outer energy roots $E = \pm \sqrt{2}$ did not move after the change of path at all.

6 Summary

In an overall applied-mathematics context and, more explicitly, within the framework of the use of the crypto-Hermitian representations of observables in quantum mechanics [20] our present paper and main model-building message may be read as based on the following three methodical assumptions, viz.,

- {1} the requirement of the feasibility of constructive considerations
- {2} a correspondence-principle connection
- {3} an offer of insight.

We interpreted point {1} as our convenient restriction of the Hilbert spaces in question to the finite-dimensional ones. In section 4 we realized the second assumption {2} via a “derivation” of our main difference-operator $N < \infty$ toy-model example from its differential-operator $N = \infty$ predecessor of Ref. [26] (cf. also Ref. [28] for more details). Thirdly, in connection with item {3} it is worth emphasizing that our present results just reconfirmed that the symbolic manipulations and the related advanced software (like MAPLE, etc) became, with time, an inseparable condition sine qua non of all the similar, manifestly constructive (i.e., basically, applied-linear-algebra) projects.

Our present concrete results were aimed, basically, at a deeper understanding of the spectral features of Hamiltonians sampled by the new model $H^{(N)}(\lambda, -\mu, \ldots)$ of section 4. The motivation
of our study was firmly rooted in the underlying physics. Briefly, it was aimed at a constructive analysis of the parametric domain $D$ of the unitarity of the underlying hypothetical physical quantum system.

Such a global purpose and aim have been achieved in several directions. Firstly, in contrast to the most common and plainly Hermitian toy models (where, typically, $D \equiv \mathbb{R}^d$ has no accessible boundary) we (re-)emphasized that the domains $D$ of our present interest were, typically, compact, possessing a nontrivial EP boundary alias horizon $\partial D \neq \emptyset$. Secondly, we demonstrated that the constructive study of many features of these horizons may be rendered feasible via an interactive use of a suitable graphical software, of a sufficiently advanced computer arithmetics and, first of all, of certain extensive symbolic manipulation package.

Speaking in technical terms we used MAPLE and profited from its Gröbner basis facilities, etc. Thus, although our original motivation came from the physical background (concerning, typically, the questions of the stability of quantum systems), our concrete main tasks (typically, the manipulations with secular polynomials) and results (typically, the recurrent construction of non-Dirac metrics $\Theta \neq I$, etc) were of a more mathematical nature.

Within this framework, a subsequent return to physics might be inspired, in the nearest future, by the “next-step” transition to the complex-matrix model-building. Such a next-step enrichment of the tunable dynamics would already lead us very close to experimental setups. Typically, they might be explored, using coupled waveguides, in a way sampled, more concretely, in Ref. [42].

In conclusion let us mention that in the future study of the model the emphasis may be expected to get shifted, formally speaking, beyond the horizons $\partial D$ and out of the quantum-stability domains $D$. The emergent complex spectra and unstable physical mechanisms seem to open new challenging questions. Curiously enough, we are witnessing an enormous increase of interest in similar phenomena not only in the theoretical studies of quantum catastrophes [8] and in many innovative and non-standard practical quantum-model analyses [13, 32, 43, 44] but even
Figure 5: Real eigenvalues of the $k = 4$ Hamiltonian $H^{(11)}(0.9, -t, t, -t)$ and the strongest form of the unfolding of the MEP degeneracy at the phase-transition points $t = \pm 1$.

far beyond the quantum physics itself and, in particular, as we already mentioned, in classical optics [11, 45].

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