Global Exponential Stability for Complex-Valued Recurrent Neural Networks With Asynchronous Time Delays

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Abstract—In this paper, we investigate the global exponential stability for complex-valued recurrent neural networks with asynchronous time delays by decomposing complex-valued networks to real and imaginary parts and construct an equivalent real-valued system. The network model is described by a continuous-time equation. There are two main differences of this paper with previous works: (1) time delays can be asynchronous, i.e., delays between different nodes are different, which makes our model more general; (2) we prove the exponential convergence directly, while the existence and uniqueness of the equilibrium point is just a direct consequence of the exponential convergence. By using three generalized norms, we present some sufficient conditions for the uniqueness and global exponential stability of the equilibrium point for delayed complex-valued neural networks. These conditions in our results are less restrictive because of our consideration of the excitatory and inhibitory effects between neurons, so previous works of other researchers can be extended. Finally, some numerical simulations are given to demonstrate the correctness of our obtained results.

Index Terms—Asynchronous, complex-valued, global exponential stability, recurrent neural networks, time delays.

I. INTRODUCTION

Recurrently connected neural networks, including Hopfield neural networks[1], Cohen-Grossberg neural networks[2], and cellular neural networks [3]-[4], have been extensively studied in past decades and found many applications in different areas, such as signal and image processing, pattern recognition, optimization problems, associative memories, and so on. Until now, many criteria about the stability of equilibrium are obtained in the literature, see [5]-[20] and references therein. CVNN can be regarded as an extension of real-valued recurrent neural networks, which has complex-valued state, output, connection weight, and activation functions. For example, they are suited to deal with complex state composed of amplitude and phase. This is one of the core concepts in physical systems dealing with electromagnetic, light, ultrasonic, quantum waves, and so on. Moreover, many applications heavily depend on the dynamical behaviors of networks. Therefore, analysis of these dynamical behaviors is a necessary step toward practical design of these neural networks. In [35], a CVNN model on time scales is studied based on delta differential operator. In [36]-[39], discrete-time CVNNs are also discussed. Stability of complex-valued impulsive system is investigated by [40]. Until now, there have been various methods to study the stability of CVNNs, such as the Lyapunov functional method [41], the synthesis method [42], and so on.

In particular, in hardware implementation, time delays inevitably occur due to the finite switching speed of the amplifiers and communication time. What’s more, to process moving images, one must introduce time delays in the signals transmitted among the cells. Furthermore, time delay is frequently a source of oscillation and instability in neural networks. Therefore, neural networks with time delays have much more complicated dynamics due to the incorporation of delays, and the stability of delayed neural networks has become a hot topic of great theoretical and practical importance, and a great deal of significant results have been reported in the literature. For example, [43] investigates the stability and synchronization for discrete-time CVNNs with time-varying delays; [44] studies the stability of complex-valued impulsive system with delay. The global exponential and asymptotical stability of CVNNs with time-delays is studied by [45] with two assumptions of activation functions, while [46] and [47] point out the mistakes in the proof of [45] and give some new conditions and criteria to ensure the existence, uniqueness, and globally asymptotical stability of the equilibrium point of CVNNs.

In practice, the interconnections are generally asynchronous, that is to say, the inevitable time delays between different
nodes are generally different. For example, in order to model vehicular traffic flow \[43\] \[49\], the reaction delays of drivers should be considered, and for different drivers, the reaction delays are different depending on physical conditions, drivers’ cognitive and physiological states, etc. Moreover, in the load balancing problem \[50\], for a computing network consisting of \(n\) computers (also called nodes), except for the different communication delays, the task-overflow delays \(\tau_{jk}\) also should be considered, which depends on the number of tasks to be transferred from node \(k\) to node \(j\). More related examples can be found in \[51\] and references therein. Hence, based on above discussions, it is necessary to study the dynamical behavior of neural networks with asynchronous time (varying) delays. To our best knowledge, there have been few works to report the stability of CVNNs with asynchronous time delays, see \[52\], \[53\]. For example, \[52\] focuses on the existence, uniqueness and global robust stability of equilibrium point for CVNNs with multiple time-delays and under parameter uncertainties with respect to two activation functions; while \[53\] investigates the dynamical behaviors of CVNNs with mixed time delays. However, all these works \([45\] - \[47\], \[52\], \[53\]) apply the homeomorphism mapping approach proposed by \[7\] to prove the existence of equilibrium; step 2, prove its stability. In \[9\] and \[10\], a direct approach to analyze global and local stability of networks was first revealed. It was revealed that the finiteness of trajectory \(x(t)\) under some norms, i.e., \(\int_{0}^{\infty} \|\dot{x}(t)\| \, dt < \infty\), is a sufficient condition for the existence, and global stability of the equilibrium point. This idea was also used in \[13\]. In this paper, we will adopt this approach. Moreover, we give several criteria based on three generalized \(L_\infty\) norm, \(L_1\) norm, \(L_2\) norm, respectively. In particular, based on \(L_\infty\)-norm, we can discuss the networks with time-varying delays.

This paper is organized as follows. In Section \[II\] we give the model description, decompose the complex-valued differential equations to real part and imaginary part, and then recast it into an equivalent real-valued differential system, whose dimension is double that of the original complex-valued system. Some definitions, lemmas and notations used in the paper are also given. In Section \[III\] we present some criteria for the uniqueness and global exponential stability of the equilibrium point for recurrent neural networks models with asynchronous time delays by using the generalized \(\infty\)-norm, \(1\)-norm, and \(2\)-norm, respectively. Some comparisons with previous M-matrix results are also presented. In Section \[IV\] some numerical simulations under constant and time-varying-delays are given to demonstrate the effectiveness of our obtained results. Finally, conclusion is given and some discussions about our future investigation of CVNNs are presented in Section \[V\].

II. Preliminaries

In this section, we give some definitions, lemmas and notations, which will be used throughout the paper.

At first, let us give a definition of asynchronous time delays.

**Definition 1:** (Synchronous and asynchronous time delays) For any node \(j\) in a coupled neural network, the synchronous time delay means that at time \(t\), node \(j\) receives the information from other nodes at the same time \(t - \tau_j(t)\); while the asynchronous time delays mean that at time \(t\), node \(j\) receives the information from other nodes at different times \(t - \tau_{jk}(t)\), i.e., for nodes \(k_1 \neq k_2\), \(\tau_{jk_1}(t)\) and \(\tau_{jk_2}(t)\) can be different.

Obviously, the network models of asynchronous time delays have a larger scope than that of synchronous time delays.

In this paper, we will investigate the CVNN with asynchronous time delays as follows:

\[
\dot{z_j}(t) = -d_j z_j(t) + \sum_{k=1}^{n} a_{j,k} f_k(z_k(t)) + \sum_{k=1}^{n} b_{j,k} g_k(z_k(t - \tau_{jk})) + u_j, \quad j = 1, \ldots, n
\]

where \(z_j \in \mathbb{C}\) is the state of \(j\)-th neuron, \(\mathbb{C}\) is the set of complex numbers; \(d_j > 0\) represents the positive rate with which the \(j\)-th unit will reset its potential to the resting state in isolation when disconnected from the network; \(f_j(\cdot) : \mathbb{C} \to \mathbb{C}\) and \(g_j(\cdot) : \mathbb{C} \to \mathbb{C}\) are complex-valued activation functions; matrices \(A = (a_{jk})\) and \(B = (b_{jk})\) are complex-valued connection weight matrices without and with time delays; \(\tau_{jk}\) are asynchronous constant time delays; \(u_j \in \mathbb{C}\) is the \(j\)-th external input.

**Remark 1:** When \(\tau_{jk} = \tau\), system (1) becomes the model investigated in \[45\]; when activation functions \(f_j\) and \(g_j\) are real functions, system (1) becomes the model investigated by \[10\]. Therefore, this model has a larger scope than previous works, and all the obtained results in the next section can be applied to these special cases.

For any complex number \(z\), we use \(z^R\) and \(z^I\) to denote its real and imaginary part respectively, so \(z = z^R + i \cdot z^I\), where \(i\) denotes the imaginary unit, that is \(i = \sqrt{-1}\).

Now, we introduce some classes of activation functions.

**Definition 2:** Assume \(f_j(z)\) can be decomposed to its real and imaginary part as \(f_j(z) = f_j^R(z^R, z^I) + i f_j^I(z^R, z^I)\) where \(z = z^R + i z^I\), \(f_j^R(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}\) and \(f_j^I(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}\). Suppose the partial derivatives of \(f_j(\cdot, \cdot)\) with respect to \(z^R, z^I\) : \(\partial f_j^R / \partial z^R, \partial f_j^I / \partial z^R, \partial f_j^R / \partial z^I, \partial f_j^I / \partial z^I\) exist. If these partial derivatives are continuous, positive and bounded, i.e., there exist positive constant numbers \(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{LR}, \lambda_j^{LI}\), such that

\[
0 < \partial f_j^R / \partial z^R \leq \lambda_j^{RR}, \quad 0 < \partial f_j^R / \partial z^I \leq \lambda_j^{RI},
\]

\[
0 < \partial f_j^I / \partial z^R \leq \lambda_j^{LR}, \quad 0 < \partial f_j^I / \partial z^I \leq \lambda_j^{LI},
\]

then \(f_j(z)\) is said to belong to class \(H_1(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{LR}, \lambda_j^{LI})\).

**Remark 2:** If \(f_j^R\) and \(f_j^I\) are absolutely continuous, then their partial derivatives exist almost everywhere.

**Definition 3:** Assume \(g_j(z)\) can be decomposed to its real and imaginary part as \(g_j(z) = g_j^R(z^R, z^I) + ig_j^I(z^R, z^I)\), where \(z = z^R + i z^I\), \(g_j^R(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}\) and \(g_j^I(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}\). Suppose the partial derivatives of \(g_j(\cdot, \cdot)\) with respect to \(z^R, z^I\) : \(\partial g_j^R / \partial z^R, \partial g_j^I / \partial z^R, \partial g_j^R / \partial z^I, \partial g_j^I / \partial z^I\) exist. If these partial derivatives are continuous and bounded, i.e., there exist positive constant numbers \(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{LR}, \mu_j^{LI}\), such that

\[
|\partial g_j^R / \partial z^R| \leq \mu_j^{RR}, \quad |\partial g_j^R / \partial z^I| \leq \mu_j^{RI},
\]

\[
|\partial g_j^I / \partial z^R| \leq \mu_j^{LR}, \quad |\partial g_j^I / \partial z^I| \leq \mu_j^{LI},
\]
\[ \left| \frac{\partial g_j^T}{\partial z^R} \right| \leq \mu_j^{JR}, \quad \left| \frac{\partial g_j^T}{\partial z^I} \right| \leq \mu_j^{JI}, \] (3)

then \( g_j(z) \) is said to belong to class \( H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}) \).

Remark 3: Definition 2 is the usual assumption for activation functions in the literature of CNNs, which can be found in [45, 52, 53] and references therein. However, the activation functions defined in Definition 2 is more restrictive, which will be useful when considering the signs of entries in connection weights, i.e., there is a trade-off between the assumption on activation functions and obtained final criteria.

Therefore, by decomposing CNN II to real and imaginary parts, we can get two equivalent real-valued systems:

\[ \dot{z}_j^R(t) = -d_j z_j^R(t) \]
\begin{align*}
&+ \sum_{k=1}^{n} a_{jk} R f_k^R \left( z_k^R(t), z_k^I(t) \right) - \sum_{k=1}^{n} a_{jk} I f_k^I \left( z_k^R(t), z_k^I(t) \right) \\
&+ \sum_{k=1}^{m} b_{jk} R g_k \left( z_k^R(t - \tau_{jk}), z_k^I(t - \tau_{jk}) \right) \\
&- \sum_{k=1}^{m} b_{jk} I g_k \left( z_k^R(t - \tau_{jk}), z_k^I(t - \tau_{jk}) \right) + u_j^R, \quad (4)
\end{align*}

and

\[ \dot{z}_j^I(t) = -d_j z_j^I(t) \]
\begin{align*}
&+ \sum_{k=1}^{n} a_{jk} R f_k^R \left( z_k^R(t), z_k^I(t) \right) + \sum_{k=1}^{n} a_{jk} I f_k^I \left( z_k^R(t), z_k^I(t) \right) \\
&+ \sum_{k=1}^{m} b_{jk} R g_k \left( z_k^R(t - \tau_{jk}), z_k^I(t - \tau_{jk}) \right) \\
&+ \sum_{k=1}^{m} b_{jk} I g_k \left( z_k^R(t - \tau_{jk}), z_k^I(t - \tau_{jk}) \right) + u_j^I. \quad (5)
\end{align*}

Remark 4: The method of decomposing the CNNs into two real-valued networks makes the network dimension grow two times, which may cause more calculations. However, this expansion of dimension can also bring some benefits. For example, the number (or dimension) of equilibria can be doubled, which enlarges the capacity of neural networks. It is a trade-off.

The following three generalized norms are used throughout the paper.

Definition 4: (See [10]) For any vector \( v(t) \in R^{m \times 1} \),
1. \( \{ \xi, \infty \} \)-norm. \( \| v(t) \|_{\{ \xi, \infty \}} = \max_j | \xi_j |^1 v_j(t) \), where \( \xi_j > 0, j = 1, \ldots, m \).
2. \( \{ \xi, 1 \} \)-norm. \( \| v(t) \|_{\{ \xi, 1 \}} = \sum_j | \xi_j | v_j(t) \), where \( \xi_j > 0, j = 1, \ldots, m \).
3. \( \{ \xi, 2 \} \)-norm. \( \| v(t) \|_{\{ \xi, 2 \}} = \left( \sum_j | \xi_j | v_j(t)^2 \right)^{1/2} \), where \( \xi_j > 0, j = 1, \ldots, m \).

Lemma 1: (See [10]) Let \( C = (c_{jk}) \in R^{m \times m} \) be a nonsingular matrix with \( c_{jk} \leq 0, j, k = 1, \ldots, m, j \neq k \). Then all the following statements are equivalent.
1. \( C \) is an M-matrix, i.e., all the successive principal minors of \( C \) are equivalent.
2. \( C^T \) is an M-matrix, where \( C^T \) is the transpose of \( C \).
3. The real part of all eigenvalues are positive.
4. There exists a vector \( \xi = (\xi_1, \ldots, \xi_m)^T \) with all \( \xi_j > 0, j = 1, \ldots, m \) such that \( \xi^T C > 0 \), or \( C \xi > 0 \).

Notation 1: For any real scalar \( a \), denote \( a^+ = \max\{0, a\} \).

In the following, we denote \( n \times n \) matrices \( A^R = (a_{jk}^R), A^I = (a_{jk}^I), B^R = (b_{jk}^R), B^I = (b_{jk}^I), \) and \( F^{RR} = \text{diag} \{ \lambda_1^{RR}, \ldots, \lambda_n^{RR} \}, F^{RI} = \text{diag} \{ \lambda_1^{RI}, \ldots, \lambda_n^{RI} \}, F^{IR} = \text{diag} \{ \lambda_1^{IR}, \ldots, \lambda_n^{IR} \}, F^{II} = \text{diag} \{ \lambda_1^{II}, \ldots, \lambda_n^{II} \}, G^{RR} = \text{diag} \{ \mu_1^{RR}, \ldots, \mu_n^{RR} \}, G^{RI} = \text{diag} \{ \mu_1^{RI}, \ldots, \mu_n^{RI} \}, G^{IR} = \text{diag} \{ \mu_1^{IR}, \ldots, \mu_n^{IR} \}, G^{II} = \text{diag} \{ \mu_1^{II}, \ldots, \mu_n^{II} \} \).

Notation 2: For any two non-negative functions \( f(t), g(t) : (-\infty, +\infty) \to [0, +\infty) \), \( f(t) = O(g(t)) \) means that for all \( t \in R \), there is a positive constant scalar \( c \) such that \( f(t) \leq c \cdot g(t) \). For any symmetric matrix \( A \), \( \lambda_{\max}(A) \) means its largest eigenvalue. A \( n \)-dimensional vector \( p = (p_1, \ldots, p_n)^T \) is called a positive vector, if its all elements are positive, i.e., \( p_i > 0, i = 1, \ldots, n \).

III. MAIN RESULTS

In this section, we prove some criteria for the uniqueness and global exponential stability of the equilibrium.

A. Criteria with \( \{ \xi, \infty \} \)-norm

Theorem 1: For dynamical systems [4] and [5], suppose the activation function \( f_j(z) \) belongs to class \( H_2(\lambda_j^{RR}, \lambda_j^{RI}, \lambda_j^{IR}, \lambda_j^{II}) \) and \( g_j(z) \) belongs to class \( H_2(\mu_j^{RR}, \mu_j^{RI}, \mu_j^{IR}, \mu_j^{II}) \), \( j = 1, \ldots, n \). If there exists a positive vector \( \xi = (\xi_1, \ldots, \xi_n, \phi_1, \ldots, \phi_n)^T \) \( > 0 \) from \( e > 0 \) such that, for \( j = 1, \ldots, n \),

\[ T1(j) = \xi_j \left( -d_j + \epsilon + \{ a_{jj}^R + \lambda_j^{RR} \} + \{ -a_{jj}^I + \lambda_j^{RI} \} \right) + \sum_{k=1}^{n} \xi_k a_{jk}^R \lambda_j^{RR} + \sum_{k=1}^{n} \phi_k a_{jk}^R \mu_j^{RR} + \sum_{k=1}^{n} \xi_k a_{jk}^I \lambda_j^{RI} + \sum_{k=1}^{n} \phi_k a_{jk}^I \mu_j^{RI} \]

\[ + \sum_{k=1}^{n} \xi_k b_{jk}^R \lambda_j^{RR} + \sum_{k=1}^{n} \phi_k b_{jk}^R \mu_j^{RR} + \sum_{k=1}^{n} \xi_k b_{jk}^I \lambda_j^{RI} + \sum_{k=1}^{n} \phi_k b_{jk}^I \mu_j^{RI} \]

\[ e^{\tau_{jk}} \leq 0, \quad (6) \]

and

\[ T2(j) = \phi_j \left( -d_j + \epsilon + \{ a_{jj}^R + \lambda_j^{RR} \} + \{ -a_{jj}^I + \lambda_j^{RI} \} \right) + \sum_{k=1}^{n} \xi_k a_{jk}^R \lambda_j^{RR} + \sum_{k=1}^{n} \phi_k a_{jk}^R \mu_j^{RR} + \sum_{k=1}^{n} \xi_k a_{jk}^I \lambda_j^{RI} + \sum_{k=1}^{n} \phi_k a_{jk}^I \mu_j^{RI} \]

\[ + \sum_{k=1}^{n} \xi_k b_{jk}^R \lambda_j^{RR} + \sum_{k=1}^{n} \phi_k b_{jk}^R \mu_j^{RR} + \sum_{k=1}^{n} \xi_k b_{jk}^I \lambda_j^{RI} + \sum_{k=1}^{n} \phi_k b_{jk}^I \mu_j^{RI} \]

\[ e^{\tau_{jk}} \leq 0, \quad (7) \]

then dynamical systems [4] and [5] have a unique equilibrium \( z^R = (z_1^R, \ldots, z_n^R)^T \) and \( z^I = (z_1^I, \ldots, z_n^I)^T \), respectively. Moreover, for any solution

\[ Z(t) = (z_1^R(t), \ldots, z_n^R(t), z_1^I(t), \ldots, z_n^I(t))^T, \quad (8) \]

there hold

\[ \| \dot{Z}(t) \|_{\{ \xi, \infty \}} = O(e^{-\epsilon t}), \quad (9) \]
\[ Z(t) = (Z^{RT}, Z^{IT})_{\| \xi \|_{\infty}} = O(e^{-\epsilon t}). \] (10)

Its proof can be found in Appendix A.

**Corollary 1:** For dynamical systems (4) and (5), suppose the activation function \( f_j(z) \) belongs to class \( H_1(\lambda_1^{RR}, \lambda_1^{RI}, \lambda_1^{IR}, \lambda_1^{II}) \) and \( g_j(z) \) belongs to class \( H_2(\mu_2^{RR}, \mu_2^{RI}, \mu_2^{IR}, \mu_2^{II}), \) \( j = 1, \cdots, n. \) If there exists a positive vector \( \xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0 \), such that, for \( j = 1, \cdots, n, \)

\[ T3(j) = \xi_j \left( -d_j + \{a_{j+}^{R} \} + \lambda_{j}^{RR} + \{ -a_{j+}^{I} \} + \lambda_{j}^{IR} \right) \]
\[ + \sum_{k=1}^{n} \xi_k |a_{jk}^{R}| \lambda_{k}^{RR} + \sum_{k=1}^{n} \phi_k |a_{jk}^{R}| \lambda_{k}^{RI} + \sum_{k=1}^{n} \xi_k |a_{jk}^{I}| \lambda_{k}^{RI} \]
\[ + \sum_{k=1}^{n} \phi_k |a_{jk}^{I}| \lambda_{k}^{II} + \sum_{k=1,j\neq j}^{n} \xi_k |b_{jk}^{R}| \mu_{k}^{RR} + \sum_{k=1,j\neq j}^{n} \phi_k |b_{jk}^{R}| \mu_{k}^{RI} + \sum_{k=1,j\neq j}^{n} \xi_k |b_{jk}^{I}| \mu_{k}^{RI} \]
\[ + \sum_{k=1,j\neq j}^{n} \phi_k |b_{jk}^{I}| \mu_{k}^{II} < 0, \] (11)

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

If conditions (11) and (12) hold, then we can find a sufficient small constant \( \epsilon > 0 \), such that inequalities (6) and (7) hold. Therefore, this corollary is a direct consequence of Thm. 1.

**Corollary 2:** For dynamical systems (4) and (5), suppose the activation function \( f_j(z) \) belongs to class \( H_2(\lambda_2^{RR}, \lambda_2^{RI}, \lambda_2^{IR}, \lambda_2^{II}) \) and \( g_j(z) \) belongs to class \( H_3(\mu_3^{RR}, \mu_3^{RI}, \mu_3^{IR}, \mu_3^{II}), \) \( j = 1, \cdots, n. \) If there exists a positive vector \( \xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0 \), such that, for \( j = 1, \cdots, n, \)

\[ T4(j) = \phi_j \left( -d_j + \{ a_{j+}^{R} \} + \chi_{j}^{RI} + \{ a_{j+}^{I} \} + \lambda_{j}^{RI} \right) \]
\[ + \sum_{k=1}^{n} \xi_k |a_{jk}^{R}| \lambda_{k}^{RR} + \sum_{k=1}^{n} \phi_k |a_{jk}^{R}| \lambda_{k}^{RI} + \sum_{k=1}^{n} \xi_k |a_{jk}^{I}| \lambda_{k}^{RI} \]
\[ + \sum_{k=1}^{n} \phi_k |a_{jk}^{I}| \lambda_{k}^{II} + \sum_{k=1,j\neq j}^{n} \xi_k |b_{jk}^{R}| \mu_{k}^{RR} + \sum_{k=1,j\neq j}^{n} \phi_k |b_{jk}^{R}| \mu_{k}^{RI} + \sum_{k=1,j\neq j}^{n} \xi_k |b_{jk}^{I}| \mu_{k}^{RI} \]
\[ + \sum_{k=1,j\neq j}^{n} \phi_k |b_{jk}^{I}| \mu_{k}^{II} < 0, \] (12)

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

**Remark 5:** Theorem 1 can be generalized to the system with time-varying delays

\[ \dot{z}_j(t) = -d_j z_j(t) \]
\[ + \sum_{k=1}^{n} a_{jk} f_k(z_k(t)) + \sum_{j=1}^{n} b_{jk} g_k(z_k(t - t_j,k(t))) + u_j, \]
\[ j = 1, \cdots, n \] (15)

where \( t_j,k(t) \) can be bounded or unbounded. In fact, by Theorem 1, system (15) has an equilibrium, which is globally \( \mu \) stable (for the concept of \( \mu \) stability first proposed in [11] and details, readers can refer to [11, 12]).

**B. Criteria with \( (\xi, 1)-\text{norm} \)**

**Theorem 2:** For dynamical systems (4) and (5), suppose the activation function \( f_j(z) \) belongs to class \( H_1(\lambda_1^{RR}, \lambda_1^{RI}, \lambda_1^{IR}, \lambda_1^{II}) \) and \( g_j(z) \) belongs to class \( H_2(\mu_2^{RR}, \mu_2^{RI}, \mu_2^{IR}, \mu_2^{II}), \) \( j = 1, \cdots, n. \) If there exists a positive vector \( \xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0 \) and \( \epsilon > 0 \), such that, for \( k = 1, \cdots, n, \)

\[ T7(k) = \xi_k (-d_k + \epsilon) \]
\[ + \left[ \xi_k a_{k,k}^{R} + \sum_{j=1,j \neq k}^{n} \xi_j |a_{jk}^{R}| + \sum_{j=1}^{n} \phi_j |a_{jk}^{R}| \right]^{+} \lambda_{k}^{RR} \]
\[ + \left[ - \xi_k a_{k,k}^{R} + \sum_{j=1,j \neq k}^{n} \xi_j |a_{jk}^{R}| + \sum_{j=1}^{n} \phi_j |a_{jk}^{R}| \right]^{+} \lambda_{k}^{RI} \]
\[ + \sum_{j=1}^{n} \left( \sum_{j=1,j \neq k}^{n} \xi_j |b_{jk}^{R}| \mu_{k}^{RR} + |b_{jk}^{R}| \mu_{k}^{RI} \right) e^{\epsilon t_{jk}} \leq 0, \]

\[ T8(k) = \phi_k (-d_k + \epsilon) \]
\[ + \left[ \phi_k a_{k,k}^{R} + \sum_{j=1,j \neq k}^{n} \phi_j |a_{jk}^{R}| + \sum_{j=1}^{n} \xi_j |a_{jk}^{R}| \right]^{+} \lambda_{k}^{II} \]
\[ + \left[ \phi_k a_{k,k}^{R} + \sum_{j=1,j \neq k}^{n} \phi_j |a_{jk}^{R}| + \sum_{j=1}^{n} \xi_j |a_{jk}^{R}| \right]^{+} \lambda_{k}^{RI} \leq 0, \] (14)

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.
then dynamical systems \(4\) and \(5\) have a unique equilibrium \(Z^R = (\tau_1^R, \cdots, \tau_n^RT)\) and \(Z^I = (\tau_1^I, \cdots, \tau_n^IT)\) respectively. Moreover, for any solution \(Z(t)\) defined by \(3\), equations \(4\) and \(10\) hold, while the norm is \(\{\xi,1\}\)-norm.

Its proof can be found in Appendix B.

Corollary 3: For dynamical systems \(4\) and \(5\), suppose the activation function \(f_j(z)\) belongs to class \(H_1(\lambda^{RR}_j, \lambda^{RI}_j, \lambda^{IR}_j, \lambda^{II}_j)\) and \(g_j(z)\) belongs to class \(H_2(\mu^{RR}_j, \mu^{RI}_j, \mu^{IR}_j, \mu^{II}_j), j = 1, \cdots, n.\) If there exists a positive vector \(\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0,\) such that, for \(k = 1, \cdots, n,\)

\[
T9(k) = -\xi_k d_k + \left[ \xi_k R_k \right] + \sum_{j=1}^{n} \xi_j a_{jk}^{R} \xi_j a_{jk}^{I} + \sum_{j=1}^{n} \xi_j a_{jk}^{I} \right] \lambda_k^{RR} + \sum_{j=1}^{n} \phi_j (|b_{jk}^R| + |b_{jk}^I|) + \phi_j (|b_{jk}^R| + |b_{jk}^I|) \lambda_k^{II} < 0,
\]

\[
T10(k) = -\phi_k d_k + \left[ \phi_k a_k^R \right] + \sum_{j=1}^{n} \xi_j a_{jk}^{R} \xi_j a_{jk}^{I} + \sum_{j=1}^{n} \xi_j a_{jk}^{I} \right] \lambda_k^{II} + \sum_{j=1}^{n} \xi_j (|b_{jk}^R| + |b_{jk}^I|) + \phi_j (|b_{jk}^R| + |b_{jk}^I|) \lambda_k^{II} < 0,
\]

then any solution of systems \(4\) and \(5\) respectively converges to a unique equilibrium exponentially.

Corollary 4: For dynamical systems \(4\) and \(5\), suppose the activation function \(f_j(z)\) belongs to class \(H_2(\lambda^{RR}_j, \lambda^{RI}_j, \lambda^{IR}_j, \lambda^{II}_j)\) and \(g_j(z)\) belongs to class \(H_2(\mu^{RR}_j, \mu^{RI}_j, \mu^{IR}_j, \mu^{II}_j), j = 1, \cdots, n.\) Denote

\[
\Delta = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix},
\]

\[
\Delta = \begin{pmatrix} |A| & |A| \\ |A| & |A| \end{pmatrix}, \quad \Delta = \begin{pmatrix} (F^{RR}) & (F^{RI}) \\ (F^{IR}) & (F^{II}) \end{pmatrix},
\]

\[
\Delta = \begin{pmatrix} |B| & |B| \\ |B| & |B| \end{pmatrix}, \quad \Delta = \begin{pmatrix} (G^{RR}) & (G^{RI}) \\ (G^{IR}) & (G^{II}) \end{pmatrix}.
\]

\[
\text{If } \Delta - \Delta F - \Delta G \text{ is a nonsingular M-matrix, then any solution of systems } 4 \text{ and } 5 \text{ respectively converges to a unique equilibrium exponentially.}
\]

\[
\text{Proof: If } \Delta - \Delta F - \Delta G \text{ is a nonsingular M-matrix, according to Lemma 1 there exists vector } \xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0, \text{ such that } (\Delta - \Delta F - \Delta G) \xi > 0, \text{ that is, inequalities } 13 \text{ and } 14 \text{ hold. Therefore, the conclusion is a direct consequence of Corollary 2.}
\]

On the other hand, if \(\Delta - \Delta F - \Delta G\) is a nonsingular M-matrix, according to Lemma 1 there exists vector \(\xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0, \text{ such that } \xi^T (\Delta - \Delta F - \Delta G) \xi > 0, \text{ that is, inequalities } 16 \text{ and } 17 \text{ hold. Therefore, the conclusion is also a direct consequence of Corollary 4.}
\]

Remark 6: Criterion based on M-matrix was also reported in [45]. However, it neglects the signs of entries in the connection matrices \(A\) and \(B\), and thus, the difference between excitatory and inhibitory effects might be ignored. Comparatively, the criteria given in Theorem 1, Theorem 2 Corollary 1 Corollary 3 are more powerful.

In the following, we give a comparison between Corollary 1 and Theorem 3 by using the matrix theory. Denote matrices

\[
P_1 = \text{diag}(\{a_{11}^R, \cdots, a_{1n}^R, \cdots, a_{nn}^R\});
\]

\[
P_2 = \text{diag}(\{a_{11}^I, \cdots, a_{1n}^I, \cdots, a_{nn}^I\});
\]

\[
P_3 = \text{diag}(\{a_{11}, \cdots, a_{1n}, \cdots, a_{nn}\}).
\]

Obviously, these matrices are all non-negative definite. Define

\[
\Delta = \begin{pmatrix} P_1 F^{RR} + P_2 F^{RI} \\ 0 \\ P_1 F^{II} + P_3 F^{IR} \end{pmatrix},
\]

(19)
so it is also non-negative definite. Using this notation, and from Corollary 1 the sufficient condition for global stability is that

$$\mathcal{D} - \mathcal{A}F - BG + \Sigma$$

(20)

should be a nonsingular M-matrix. Obviously, if \(\mathcal{D} - \mathcal{A}F - BG\) is a nonsingular M-matrix, the above matrix (20) is also a nonsingular M-matrix; instead, if matrix (20) is a nonsingular M-matrix, \(\mathcal{D} - \mathcal{A}F - BG\) may be not.

Therefore, Corollary 1 presents a better criterion than that by previous works, like [43], because it considers the signs of entries in the connection matrix \(A\), whose positive effect is described by the above nonnegative matrix \(\Sigma\) defined in (19). Moreover, from this result, we can also find that in order to make the CVNNs have the stable equilibrium, \(P_1, P_2, P_3\) should be as large as possible, so one way is to make all \(a_{jj}, j = 1, \ldots, n\) be negative numbers.

Remark 7: The function \(M(t) = \max_i \max_{t=1, \ldots, m} |u_i(t)|\) proposed in [10] is a powerful tool in dealing with delayed systems. In particular, for the time-varying delays.

Remark 8: It can be seen that in computing the integral \(\int_0^\infty ||Z(t)||dt\), the estimation of \(\frac{\partial}{\partial t} ||Z(t)||\) plays an important role.

Let \(A(t) = (a_{ij})^N_{i,j=1}, \xi_i > 0, i = 1, \ldots, N\) and

$$\frac{dw}{dt} = Aw(t)$$

(21)

It has been shown that (see [9, 10])

$$\max \frac{d}{dt} ||w(t)||_{\{\xi_i\}, 1} = \max_j \{a_{jj} + \sum_{i \neq j} \xi_i [a_{ij}]\},$$

$$\max \frac{d}{dt} ||w(t)||_{\{\xi_i\}, \infty} = \max_i \{\xi_i [a_{ii}]\},$$

$$\max \frac{d}{dt} ||w(t)||_{\{\xi_i\}, 2} = \lambda_{\max} (\Xi A + A^T \Xi), \Xi = \text{diag}(\xi).$$

$$\frac{d}{dt} ||w(t)||_{\{\xi_i\}, 1} = \sum_{i=1}^n \frac{d}{dt} (w_i(t)) \xi_i \sum_{j=1}^{n} a_{ij} w_j(t)$$

$$= \sum_{j=1}^n \sum_{i \neq j} \text{sign}(w_i(t)) \xi_i a_{ij} w_j(t)$$

$$\leq \sum_{j=1}^n a_{jj} + \sum_{i \neq j} \xi_i [a_{ij}] ||w(t)||_{\{\xi_i\}, 1}$$

$$\leq \max_j \{a_{jj} + \sum_{i \neq j} \xi_i [a_{ij}]\} ||w(t)||_{\{\xi_i\}, 1}$$

Therefore,

$$\max \frac{d}{dt} ||w(t)||_{\{\xi_i\}, 1} = \max_j \{a_{jj} + \sum_{i \neq j} \xi_i [a_{ij}]\}.$$

Similarly, we can prove the other two equalities.

These three equalities play very important role in discussing stability of the neural networks or other dynamical systems. For example, if \(\max_j \{a_{jj} + \sum_{i \neq j} \xi_i [a_{ij}]\} \leq -\alpha < 0\), then \(\frac{d}{dt} ||w(t)||_{\{\xi_i\}, 1} \leq -\alpha ||w(t)||_{\{\xi_i\}, 1}\), which implies \(||w(t)||_{\{\xi_i\}, 1} = O(e^{-\alpha t})\).

It happens that these three equalities are closely relating to the matrix measure of \(A\) with respect to three norms.

D. Criteria with \((\xi, 2)\)-norm

Theorem 4: For dynamical systems (4) and (5), suppose the activation function \(f_j(z)\) belongs to class \(H_1(\lambda^{RR}, \lambda^{RI}, \lambda^{IR}, \lambda^{II}, \lambda^{II})\) and \(g_j(z)\) belongs to class \(H_2(\mu^{RR}, \mu^{RI}, \mu^{IR}, \mu^{II}), j = 1, \ldots, n\). If there exists a positive vector \(\xi = (\xi_1, \ldots, \xi_n, \phi_1, \ldots, \phi_n)^T > 0\) and \(\epsilon > 0\), such that, for \(j = 1, \ldots, n\),

$$T13(j) = 2\xi_j (-d_j + \epsilon + \{a^{RR}_{jj}\} + \lambda^{RR}_j + \{-a^{II}_{jj}\} + \lambda^{II}_j)$$

$$+ \sum_{k=1, k \neq j}^n \xi_k (|a^{RR}_{jk}| \lambda^{RR}_k + |a^{IR}_{jk}| \lambda^{IR}_k) \pi 1_{jk}$$

$$+ \sum_{k=1, k \neq j}^n \xi_k (|a^{RR}_{jk}| \lambda^{RR}_k + |a^{IR}_{jk}| \lambda^{IR}_k) \pi 1_{jk}^{-1}$$

$$+ \sum_{k=1}^n \xi_j (|a^{RR}_{jk}| \lambda^{RR}_k + |a^{IR}_{jk}| \lambda^{IR}_k) \pi 2_{jk}$$

$$+ \sum_{k=1}^n \xi_j (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \pi 4_{jk}$$

$$+ \sum_{k=1}^n \xi_j (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \pi 3_{jk}^{-1}$$

$$+ \sum_{k=1}^n \phi_k (|a^{RR}_{jk}| \mu^{RR}_k + |a^{IR}_{jk}| \mu^{IR}_k) \omega 1_{jk}$$

$$+ \sum_{k=1}^n \phi_k (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \omega 3_{jk}^{-1} e^{2\tau_{jk}} \leq 0,$$

$$T14(j) = 2\phi_j (-d_j + \epsilon + \{a^{RR}_{jj}\} + \mu^{RR}_j + \{a^{II}_{jj}\} + \mu^{II}_j)$$

$$+ \sum_{k=1, k \neq j}^n \phi_k (|a^{RR}_{jk}| \mu^{RR}_k + |a^{IR}_{jk}| \mu^{IR}_k) \omega 1_{jk}$$

$$+ \sum_{k=1, k \neq j}^n \phi_k (|a^{RR}_{jk}| \mu^{RR}_k + |a^{IR}_{jk}| \mu^{IR}_k) \omega 2_{jk}$$

$$+ \sum_{k=1, k \neq j}^n \phi_k (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \omega 4_{jk}$$

$$+ \sum_{k=1}^n \phi_j (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \omega 3_{jk}^{-1}$$

$$+ \sum_{k=1}^n \phi_j (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \omega 2_{jk}^{-1}$$

$$+ \sum_{k=1}^n \phi_j (|b^{RR}_{jk}| \mu^{RR}_k + |b^{IR}_{jk}| \mu^{IR}_k) \omega 4_{jk}^{-1}.$$
+ \sum_{k=1}^{n} \phi_k([b_k^R|\mu_j^R] + [b_k^I|\mu_j^I])\omega_4 k_j^3) e^{2\pi k_j} \leq 0,

where \( \pi_{1jk}, \pi_{2jk}, \pi_{3jk}, \pi_{4jk}, \omega_{1jk}, \omega_{2jk}, \omega_{3jk}, \omega_{4jk} \) are positive numbers. Then dynamical systems (4) and (5) have a unique equilibrium \( Z^R \) and \( Z^I \) respectively. Moreover, for any solution \( Z(t) \) defined by (4), equations (2) and (10) hold, where the norm is \( \{2 \} \)-norm.

Its proof can be found in Appendix C.

**Corollary 5:** For dynamical systems (4) and (5), suppose the activation function \( f_j(z) \) belongs to class \( H_1(\lambda_j^R, \lambda_j^I, \lambda_j^I, \lambda_j^I) \) and \( g_j(z) \) belongs to class \( H_2(\mu_j^R, \mu_j^R, \mu_j^R, \mu_j^R), \ j = 1, \cdots, n. \) If there exists a positive vector \( \xi = (\xi_1, \cdots, \xi_n, \phi_1, \cdots, \phi_n)^T > 0, \) such that, for \( j = 1, \cdots, n, \)

\[
T15(j) = 2\xi_j(-d_j + [a^R_{jk}] + \lambda_j^R + (-a^I_{jk} + \lambda_j^I) + \sum_{k=1, k\neq j}^{n} \xi_j([a^R_{jk}] + [a^I_{jk}]) \pi_{1jk} + \sum_{k=1}^{n} \xi_j([a^R_{jk}] + [a^I_{jk}]) \pi_{2jk} + \sum_{k=1, k\neq j}^{n} \xi_j([a^R_{jk}] + [a^I_{jk}]) \pi_{3jk} + \sum_{k=1}^{n} \xi_j([a^R_{jk}] + [a^I_{jk}]) \pi_{4jk} + \sum_{k=1}^{n} \phi_k([a^R_{jk}] + [a^I_{jk}]) \omega_1 k_j^3) - \sum_{k=1}^{n} \xi_j([a^R_{jk}] + [a^I_{jk}]) \psi_3 k_j^3 < 0,
\]

\[
T16(j) = 2\phi_j(-d_j + [a^R_{jk}] + \mu_j^R + [a^I_{jk} + \mu_j^I] + \sum_{k=1, k\neq j}^{n} \phi_j([a^R_{jk}] + [a^I_{jk}]) \omega_1 k_j + \sum_{k=1}^{n} \phi_j([a^R_{jk}] + [a^I_{jk}]) \omega_2 k_j + \sum_{k=1, k\neq j}^{n} \phi_j([a^R_{jk}] + [a^I_{jk}]) \omega_3 k_j^3) + \sum_{k=1}^{n} \xi_k([b^R_{kj}] + [b^I_{kj}]) \psi_2 k_j^3 < 0,
\]

then any solution of systems (4) and (5) respectively converges to a unique equilibrium exponentially.

**Remark 9:** As for how to use the norms \( \| \cdot \|_{\{1 \}} \) and \( \| \cdot \|_{\{2 \}} \) to discuss the time-varying delayed networks, readers can refer to the papers [15, 16].

**IV. Numerical Example**

In this section, some numerical simulations are presented to show the effectiveness of our obtained results.

Consider a two-neuron complex-valued recurrent neural network described as follows:

\[
\begin{align*}
\dot{z}_1(t) &= -d_1 z_1(t) + a_{11} f_1(z_1(t)) + a_{12} f_2(z_2(t)) + b_{11} g_1(z_1(t - 1)) + b_{12} g_2(z_2(t - 2)) + u_1 \\
\dot{z}_2(t) &= -d_2 z_2(t) + a_{21} f_1(z_1(t)) + a_{22} f_2(z_2(t)) + b_{21} g_1(z_1(t - 3)) + b_{22} g_2(z_2(t - 4)) + u_2
\end{align*}
\]

(22)
where \( z_k = z_k^R + iz_k^I, k = 1, 2, D = \text{diag}(d, d) = 19I_2, \) and

\[
A = (a_{jk})_{2 \times 2} = \begin{pmatrix}
-2 - 3i & 3 - i \\
4 - 2i & -1 + 2i
\end{pmatrix},
\]

\[
B = (b_{jk})_{2 \times 2} = \begin{pmatrix}
-1 + 2i & 2 + i \\
3 - 4i & -3 + 2i
\end{pmatrix},
\]

\[
u = (u_1, u_2)^T = (-3 + i, 2 + 4i)^T,
\]

\[
f_k(z_k) = \frac{1 - \exp(-2z_k^R - z_k^I)}{1 + \exp(-2z_k^R - z_k^I)} + i \frac{1}{1 + \exp(-z_k^R - 2z_k^I)},
\]

\[
g_k(z_k) = \frac{1}{1 + \exp(-z_k^R - 2z_k^I)} + i \frac{1 - \exp(-2z_k^R - z_k^I)}{1 + \exp(-2z_k^R - z_k^I)}.
\]

From simple calculations, we have, for \( j = 1, 2, \)

\[0 < \frac{\partial f_j^R}{\partial z_j^R} \leq 1 = \lambda_j^{RR}, \quad 0 < \frac{\partial f_j^R}{\partial z_j^I} \leq 0.5 = \lambda_j^{RI};\]

\[0 < \frac{\partial f_j^I}{\partial z_j^R} \leq 0.25 = \lambda_j^{IR}, \quad 0 < \frac{\partial f_j^I}{\partial z_j^I} \leq 0.5 = \lambda_j^{II};\]

therefore, \( f_j(z) \) belongs to class \( H_1(1, 0.5, 0.25, 0.5), j = 1, 2. \) Similarly, we can prove that \( g_j(z) \) belongs to class \( H_2(0.25, 0.5, 1, 0.5), j = 1, 2. \)

From the notations defined in [18], we have \( \overline{D} = 19I_4, \)

\[
\overline{A} = \begin{pmatrix}
2 & 3 & 3 & 1 \\
4 & 1 & 2 & 2 \\
3 & 1 & 2 & 3 \\
2 & 2 & 4 & 1
\end{pmatrix}, \quad \overline{T} = \begin{pmatrix}
1 & 0 & 0.5 & 0 \\
0 & 1 & 0 & 0.5 \\
0.25 & 0 & 0.5 & 0 \\
0 & 0.25 & 0 & 0.5
\end{pmatrix},
\]

\[
\overline{B} = \begin{pmatrix}
1 & 2 & 2 & 1 \\
3 & 3 & 4 & 2 \\
2 & 1 & 1 & 2 \\
4 & 2 & 3 & 3
\end{pmatrix}, \quad \overline{G} = \begin{pmatrix}
0.25 & 0 & 0.5 & 0 \\
0 & 0.25 & 0 & 0.5 \\
1 & 0 & 0.5 & 0 \\
0 & 1 & 0 & 0.5
\end{pmatrix}.
\]

Calculations show that eigenvalues of \( \overline{D} - \overline{A} \overline{F} - \overline{B} \overline{G} \) are: \(-0.7655, 18.6670, 20.9701, 19.8784,\) so it is not an M-matrix, which means that Theorem [13] is not satisfied. However, according to Corollary [1] and Remark [6] we have \( P_1 = \text{diag}\{2, 1\}, P_2 = \text{diag}\{0, 2\}, P_3 = \text{diag}\{3, 0\}, \) and

\[
\overline{A} = \text{diag}\{P_1 + 0.25P_2, 0.5(P_1 + P_3)\},
\]

then eigenvalues of \( \overline{D} - \overline{A} \overline{F} - \overline{B} \overline{G} + \overline{A} \) are \(-0.4884, 20.0717, 22.7947, 21.5348,\) therefore Corollary [1] holds, so the above system can achieve its equilibrium exponentially.

The following simulations present the correctness of our claim. We choose five cases for initial values. Case 1: \( z_1(t) = -4 + 3i, z_2(t) = -5 - i, t \in [-4, 0]; \) Case 2: \( z_1(t) = 2 + i, z_2(t) = -3 + 2.5i, t \in [-4, 0]; \) Case 3: \( z_1(t) = 3 - 5i, z_2(t) = 6 + 3i, t \in [-4, 0]; \) Case 4: \( z_1(t) = -2 - 4i, z_2(t) = -7 + 4i, t \in [-4, 0]; \) Case 5: \( z_1(t) = 1 + 4i, z_2(t) = -5 - 1.5i, t \in [-4, 0]. \) Figures [14] depict the trajectories of \( z_1^R(t), z_1^I(t), z_2^R(t), z_2^I(t) \) respectively. For different initial values, they converge to the same equilibrium \((-0.0351, 0.1423, 0.0912, 0.2239)^T,\) i.e., the unique equilibrium has the global exponential stability property.

Moreover, if we choose the initial values as Case 1, and only the external control \( \nu \) are different, i.e., different external controls \( (u_1, u_2)^T = (-3 + i, 2 + 4i)^T \) and \( (u_1', u_2')^T = (3 + \)

Fig. 1. Trajectories of \( z_1^R(t) \) for different initial values, which show the global exponential stability of equilibrium.

Fig. 2. Trajectories of \( z_1^I(t) \) for different initial values, which show the global exponential stability of equilibrium.

Fig. 3. Trajectories of \( z_2^R(t) \) for different initial values, which show the global exponential stability of equilibrium.
In the final simulation, we will consider the time-varying delays, thus we choose the equation as

\[
\begin{aligned}
\dot{z}_1(t) &= -d_1 z_1(t) + a_{11} f_1(z_1(t)) + a_{12} f_2(z_2(t)) + b_{11} g_1(z_1(t - 1 - \sin(t))) + b_{12} g_2(z_2(t - 2 - \cos(t))) + u_1 \\
\dot{z}_2(t) &= -d_2 z_2(t) + a_{21} f_1(z_1(t)) + a_{22} f_2(z_2(t)) + b_{21} g_1(z_1(t - 3 + \sin(t))) + b_{22} g_2(z_2(t - 4 + \cos(t))) + u_2
\end{aligned}
\]  

(23)

All the parameters, including the external control, are the same as defined in the above simulations. Similarly, according to Corollary 1, Remark 5 and Remark 6, this system can achieve its equilibrium exponentially. Figures 6-9 depict the trajectories of \(z_1^0(t), z_1^1(t), z_2^0(t), z_2^1(t)\) respectively. Moreover, the equilibrium is also \((-0.0351, 0.1423, 0.0912, 0.2239)^T\), i.e., the equilibriums are the same for system (22) and system (23) even though they have different time delays.

V. CONCLUSION AND DISCUSSIONS

In this paper, we first propose a complex-valued recurrent neural network model with asynchronous time delays. This feature is the first difference of this paper with previous works. Then under the assumptions of activation functions, we prove the exponential convergence directly by using the \(\infty\)-norm, 1-norm and 2-norm respectively, the existence and uniqueness of the equilibrium point is a direct consequence of the exponential convergence; while previous works in the literature
always use two proving steps: step 1, prove the existence of equilibrium; step 2, prove the stability. This is also a novelty of this paper for investigating the equilibrium of CVNNs. Moreover, considering the signs of coupling matrix, some sufficient conditions for the uniqueness and global exponential stability of the equilibrium point are presented, which are more general and less restrictive than previous works, i.e., the $M$-matrix property of $\mathcal{D} = AF - BG$ is just a special case of criteria for exponential stability. These are our main theoretical results. Finally, three numerical examples are given to show the correctness of our obtained results.

In the end, we give some discussions about future directions of the complex-valued neural networks:

1) This paper deals with complex-valued neural network by decomposing it to real and imaginary parts and constructing an equivalent real-valued system. To ensure this decomposition, we assume the partial derivatives of activation functions exist and bounded, see Definition

2) How to find an efficient way to analyze complex-valued system using the complex nature of system and consider its properties on complex planes will be our future direction.

2) In this paper, we consider the asynchronous time delays, which can be regarded as discrete delays. However, a distribution of propagation delays can exist for neural networks due to the multitude of parallel pathways with a variety of axon sizes and lengths. Therefore, continuously distributed delays can be a good choice, so investigation of stability under distributed delays will also be our future direction.

3) As for the dynamical behaviors of complex-valued neural networks, the global existence and exponential stability is just an aspect, there are also many other interesting dynamical behaviors for future research, for example, the multistability, the robustness of uncertain neural networks, the existence and stability of periodic (or almost periodic) solutions, chaotic behaviors for (delayed) complex-valued neural networks, etc.

APPENDIX A: THE PROOF OF THEOREM

**Proof:** Define

$$x_j(t) = e^{\epsilon t} z_j^R(t), \quad y_j(t) = e^{\epsilon t} z_j^I(t), \quad j = 1, 2, \ldots, n. \quad (24)$$

Then we have

$$\dot{x}_j(t) = (-d_j + \epsilon) x_j(t)$$

$$+ \sum_{k=1}^n a_{jk}^R \left( \frac{\partial f_k^R}{\partial z_k} x_k + \frac{\partial f_k^R}{\partial z_k} y_k \right) - \sum_{k=1}^n a_{jk}^I \left( \frac{\partial f_k^I}{\partial z_k} x_k + \frac{\partial f_k^I}{\partial z_k} y_k \right)$$

$$+ \sum_{k=1}^n b_{jk}^R e^{\epsilon \tau_{jk}} \left( \frac{\partial g_k^R}{\partial z_k} x_k(\tau_{jk}) + \frac{\partial g_k^R}{\partial z_k} y_k(\tau_{jk}) \right)$$

$$- \sum_{k=1}^n b_{jk}^I e^{\epsilon \tau_{jk}} \left( \frac{\partial g_k^I}{\partial z_k} x_k(\tau_{jk}) + \frac{\partial g_k^I}{\partial z_k} y_k(\tau_{jk}) \right), \quad (25)$$

and

$$\dot{y}_j(t) = (-d_j + \epsilon) y_j(t)$$

$$+ \sum_{k=1}^n a_{jk}^R \left( \frac{\partial f_k^R}{\partial z_k} x_k + \frac{\partial f_k^R}{\partial z_k} y_k \right) + \sum_{k=1}^n a_{jk}^I \left( \frac{\partial f_k^I}{\partial z_k} x_k + \frac{\partial f_k^I}{\partial z_k} y_k \right)$$

$$+ \sum_{k=1}^n b_{jk}^R e^{\epsilon \tau_{jk}} \left( \frac{\partial g_k^R}{\partial z_k} x_k(\tau_{jk}) + \frac{\partial g_k^R}{\partial z_k} y_k(\tau_{jk}) \right)$$

$$+ \sum_{k=1}^n b_{jk}^I e^{\epsilon \tau_{jk}} \left( \frac{\partial g_k^I}{\partial z_k} x_k(\tau_{jk}) + \frac{\partial g_k^I}{\partial z_k} y_k(\tau_{jk}) \right), \quad (26)$$

where $\frac{\partial f_k^R}{\partial z_k}, \frac{\partial f_k^I}{\partial z_k}$ denotes $\frac{\partial f_k^R(z_k^R(t), z_k^I(t))/\partial z_k}{\partial z_k}$, $a, b = R, I$; $x_k(\tau_{jk}) = x_k(t - \tau_{jk})$, and $y_k(\tau_{jk}) = y_k(t - \tau_{jk})$; while $\frac{\partial f_k^R}{\partial z_k}, \frac{\partial f_k^I}{\partial z_k}$ denotes $\frac{\partial f_k^R(z_k^R(t - \tau_{jk}), z_k^I(t - \tau_{jk}))/\partial z_k}{\partial z_k}$, $a, b = R, I$.

Let

$$X(t) = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_n(t))^T \in \mathbb{R}^{2n \times 1},$$

Fig. 8. Trajectories of $z_j^R(t)$ for different initial values, which show the global exponential stability of equilibrium

Fig. 9. Trajectories of $z_j^I(t)$ for different initial values, which show the global exponential stability of equilibrium
so \( X(t) = e^{\tau \dot{Z}(t)} \), and \( \|X(t)\| \in \{\xi, \infty\} = \max\{\max_i \{\| x_j(t) \|\}, \max_j \{\| y_j(t) \|\}\}. 

Case 1: For \( X(t) \), if \( j_0 = j_0(t) \), which depends on \( t \), is such an index that \( \| x_{j_0}(x_j(t)) = \|X(t)\| \in \{\xi, \infty\} \), then 

\[
\xi_{j_0} \frac{d\|X(t)\|}{dt} = \frac{d\|x_{j_0}(t)\|}{dt} = \operatorname{sign}(x_{j_0}(t)) \left\{ \xi_{j_0}(-d_{j_0} + \epsilon)\xi_{j_0}^{-1}x_{j_0}(t) + \sum_{k=1}^{n} \xi_k |a_{jk}^R| |\lambda_k^R| + \sum_{k=1}^{n} |a_{jk}^I| |\lambda_k^I| \right\} 
\]

\[
= \sum_{k=1}^{n} \xi_k |a_{jk}^R| |\lambda_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} \phi_k |a_{jk}^R| |\lambda_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} \phi_k |a_{jk}^I| |\lambda_k^I| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} | \phi_k |a_{jk}^R| |\lambda_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} | \phi_k |a_{jk}^I| |\lambda_k^I| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} \xi_k |a_{jk}^R| |\lambda_k^R| + \sum_{k=1}^{n} |a_{jk}^R| |\lambda_k^R| + \sum_{k=1}^{n} |a_{jk}^I| |\lambda_k^I| \} \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

Furthermore, define 

\[
M(t) = \sup_{t-\tau \leq s \leq t} \|X(s)\| \in \{\xi, \infty\}, 
\]

where \( \tau = \max_{j,k} \tau_{jk} \). Then \( \|X(t)\| \in \{\xi, \infty\} \leq M(t) \), and if \( \|X(t)\| \in \{\xi, \infty\} = M(t) \), we have 

\[
\xi_{j_0} \frac{d\|X(t)\|}{dt} \leq T1(j_0) \cdot M(t) \leq 0. 
\]

Case 2: For \( X(t) \), if \( j_0 = j_0'(t) \), which depends on \( t \), is such an index that \( \| x_{j_0'}(y_{j_0}(t)) = \|X(t)\| \in \{\xi, \infty\} \), then 

\[
\phi_{j_0'} \frac{d\|X(t)\|}{dt} = \frac{d\|y_{j_0}(t)\|}{dt} = \operatorname{sign}(y_{j_0}(t)) \left\{ \phi_{j_0'}(-d_{j_0} + \epsilon)\phi_{j_0'}^{-1}y_{j_0}(t) + \sum_{k=1}^{n} \xi_k |b_{jk}^R| |\mu_k^R| + \sum_{k=1}^{n} |b_{jk}^I| |\mu_k^I| \right\} 
\]

\[
= \sum_{k=1}^{n} \xi_k |b_{jk}^R| |\mu_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} \phi_k |b_{jk}^R| |\mu_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} \phi_k |b_{jk}^I| |\mu_k^I| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} \xi_k |b_{jk}^R| |\mu_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} | \phi_k |b_{jk}^R| |\mu_k^R| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
+ \sum_{k=1}^{n} | \phi_k |b_{jk}^I| |\mu_k^I| \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

\[
= \left\{ \phi_{j_0'}(-d_{j_0} + \epsilon)\phi_{j_0'}^{-1}y_{j_0}(t) + \sum_{k=1}^{n} \xi_k |b_{jk}^R| |\mu_k^R| + \sum_{k=1}^{n} |b_{jk}^I| |\mu_k^I| \right\} \cdot \|X(t)\| \in \{\xi, \infty\} 
\]

From the definition of \( \|X(t)\| \in \{\xi, \infty\} \leq M(t) \), and if \( \|X(t)\| \in \{\xi, \infty\} = M(t) \), 

\[
\phi_{j_0'} \frac{d\|X(t)\|}{dt} \leq T2(j_0') \cdot M(t) \leq 0. 
\]

Therefore, for the above two cases, according to \( \|X(t)\| \in \{\xi, \infty\} \leq M(t) \), and if \( \|X(t)\| \in \{\xi, \infty\} = M(t) \), 

\[
\phi_{j_0'} \frac{d\|X(t)\|}{dt} \leq T2(j_0') \cdot M(t) \leq 0. 
\]

From the definition of \( \|X(t)\| \in \{\xi, \infty\} \leq M(t) \), and if \( \|X(t)\| \in \{\xi, \infty\} = M(t) \), 

\[
\phi_{j_0'} \frac{d\|X(t)\|}{dt} \leq T2(j_0') \cdot M(t) \leq 0. 
\]

From the definition of \( \|X(t)\| \in \{\xi, \infty\} \leq M(t) \), and if \( \|X(t)\| \in \{\xi, \infty\} = M(t) \), 

\[
\phi_{j_0'} \frac{d\|X(t)\|}{dt} \leq T2(j_0') \cdot M(t) \leq 0. 
\]
implies \( \|X(t)\|_{\xi, \infty} = O(1) \) and
\[
\|Z(t)\|_{\xi, \infty} = O(e^{-ct}),
\]
for any equilibrium point of the systems (4) and (5). By the Cauchy convergence principle, we conclude that \( \lim\limits_{t \to +\infty} Z(t) = \bar{z} \), for some \( \bar{z} = (\bar{z}^R, \bar{z}^I)^T \). It is easy to see that the equilibrium point is unique. 

Appendix B: Proof of Theorem 3

Proof: Recall the definition of \( x_j(t) \) and \( y_j(t) \) defined in (24), we can define a Lyapunov function as
\[
L_1(t) = \sum_{j=1}^{n} \xi_j |x_j(t)| + \sum_{j=1}^{n} \phi_j |y_j(t)|
+ \sum_{j,k=1}^{n} \alpha_{jk} e^{\tau_{jk}} \int_{t-\tau_{jk}}^{t} |x_k(s)|ds
+ \sum_{j,k=1}^{n} \beta_{jk} e^{\tau_{jk}} \int_{t-\tau_{jk}}^{t} |y_k(s)|ds,
\]
where
\[
\alpha_{jk} = \xi_j (|b_{jk}^R| \mu_k^R + |b_{jk}^I| \mu_k^I) + \phi_j (|b_{jk}^R| \mu_k^R + |b_{jk}^I| \mu_k^I);
\]
\[
\beta_{jk} = \xi_j (|b_{jk}^R| \mu_k^R + |b_{jk}^I| \mu_k^I) + \phi_j (|b_{jk}^R| \mu_k^R + |b_{jk}^I| \mu_k^I).
\]

Differentiating \( L_1(t) \) along equations (25) and (26), using some calculations (the details are left to interested readers), we have
\[
\dot{L}_1(t) \leq \sum_{k=1}^{n} T7(k) \cdot |x_k(t)| + \sum_{k=1}^{n} T8(k) \cdot |y_k(t)| \leq 0.
\]

By similar arguments used in the proof of Theorem 1 it is easy to see that the equilibrium point is unique.

Appendix C: Proof of Theorem 4

Proof: Recall the definition of \( x_j(t) \) and \( y_j(t) \) defined in (24), we can define a Lyapunov function as
\[
L_2(t) = \sum_{j=1}^{n} \xi_j x_j^2(t) + \sum_{j=1}^{n} \phi_j y_j^2(t)
+ \sum_{j,k=1}^{n} \alpha'_{jk} e^{2\tau_{jk}} \int_{t-\tau_{jk}}^{t} x_k^2(s)ds
+ \sum_{j,k=1}^{n} \beta'_{jk} e^{2\tau_{jk}} \int_{t-\tau_{jk}}^{t} y_k^2(s)ds,
\]
where
\[
\alpha'_{jk} = \xi_j (|b_{jk}^R| \mu_k^RR + |b_{jk}^I| \mu_k^RI) \pi 3^{-1}_{jk} 
+ \sum_{k=1}^{n} \phi_j (|b_{jk}^R| \mu_k^RR + |b_{jk}^I| \mu_k^RI) \omega 3^{-1}_{jk};
\]
\[
\beta'_{jk} = \xi_j (|b_{jk}^R| \mu_k^RI + |b_{jk}^I| \mu_k^RI) \pi 4^{-1}_{jk}
+ \sum_{k=1}^{n} \phi_j (|b_{jk}^R| \mu_k^RI + |b_{jk}^I| \mu_k^RI) \omega 4^{-1}_{jk}.
\]

Differentiating \( L_2(t) \) along equations (25) and (26), using some calculations (the details are left to interested readers), one can get that
\[
\dot{L}_2(t) \leq \sum_{j=1}^{n} T13(j)x_j^2(t) + \sum_{j=1}^{n} T14(j)y_j^2(t) \leq 0.
\]

By similar arguments used in the proof of Theorem 1 it is easy to see that the equilibrium point is unique.

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