Low-dimensional linear representations of mapping class groups

Mustafa Korkmaz$^{1,2}$

$^1$Department of Mathematics, Middle East Technical University, Ankara, Turkey
$^2$Max-Planck Institut für Mathematik, Bonn, Germany

Abstract

Let $S$ be a compact orientable surface of genus $g$ with marked points in the interior. Franks–Handel (Proc. Amer. Math. Soc. 141 (2013) 2951–2962) proved that if $n < 2g$ then the image of a homomorphism from the mapping class group Mod($S$) of $S$ to $\text{GL}(n, \mathbb{C})$ is trivial if $g \geq 3$ and is finite cyclic if $g = 2$. The first result is our own proof of this fact. Our second main result shows that for $g \geq 3$ up to conjugation there are only two homomorphisms from Mod($S$) to $\text{GL}(2g, \mathbb{C})$: the trivial homomorphism and the standard symplectic representation. Our last main result shows that the mapping class group has no faithful linear representation in dimensions less than or equal to $3g - 3$. We provide many applications of our results, including the finiteness of homomorphisms from mapping class groups of nonorientable surfaces to $\text{GL}(n, \mathbb{C})$, the triviality of homomorphisms from the mapping class groups to $\text{Aut}(F_n)$ or to $\text{Out}(F_n)$, and homomorphisms between mapping class groups. We also show that if the surface $S$ has $r$ marked point but no boundary components, then Mod($S$) is generated by involutions if and only if $g \geq 3$ and $r \leq 2g - 2$.

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1 | INTRODUCTION AND THE RESULTS

1.1 | Statements of the main results

Let $S$ be a compact connected oriented surface of genus $g \geq 1$ with $p \geq 0$ boundary components and with $r \geq 0$ marked points in the interior. Let $\text{Mod}(S)$ denote the mapping class group of $S$, the group of isotopy classes of orientation-preserving diffeomorphisms $S \to S$. We assume that diffeomorphisms and isotopies of $S$ are the identity on the marked points and near the boundary of $S$.

Gluing a disk along each boundary component of $S$ and forgetting the marked points give rise to a closed surface $\tilde{S}$ and a natural surjective homomorphism $\text{Mod}(S) \to \text{Mod}(\tilde{S})$. After fixing a basis for the first homology $H_1(\tilde{S};\mathbb{Z})$ of $\tilde{S}$, the action of $\text{Mod}(\tilde{S})$ on $H_1(\tilde{S};\mathbb{Z})$ gives the classical representation $\text{Mod}(\tilde{S}) \to \text{Sp}(2g,\mathbb{Z})$ onto the symplectic group. Precomposing this map...
with \( \text{Mod}(S) \rightarrow \text{Mod}(\hat{S}) \) and postcomposing with the inclusion \( \text{Sp}(2g, \mathbb{Z}) \hookrightarrow \text{GL}(2g, \mathbb{C}) \) give a map \( P : \text{Mod}(S) \rightarrow \text{GL}(2g, \mathbb{C}) \).

We prove the following.

**Theorem 1.** Let \( g \geq 1, n \leq 2g - 1, p \geq 0, r \geq 0 \) and let \( \phi : \text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C}) \) be a homomorphism. Then the image of \( \phi \) is abelian, so that it is

(i) a quotient of the cyclic group \( \mathbb{Z}_{12} \) if \((g, p) = (1, 0)\) and of the free abelian group \( \mathbb{Z}^p \) if \( g = 1 \) and \( p \geq 1 \),

(ii) a quotient of the cyclic group \( \mathbb{Z}_{10} \) of order 10 if \( g = 2 \), and

(iii) trivial if \( g \geq 3 \).

**Theorem 2.** Let \( g \geq 3 \) and let \( \phi : \text{Mod}(S) \rightarrow \text{GL}(2g, \mathbb{C}) \) be a group homomorphism. Then \( \phi \) is either trivial or conjugate to the homomorphism \( P : \text{Mod}(S) \rightarrow \text{GL}(2g, \mathbb{C}) \).

In the first version of their paper [10], Franks–Handel asked whether every homomorphism \( \text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C}) \) is trivial for \( n \leq 2g - 1 \). They proved that, in fact, this is the case for \( g \geq 3 \) and \( n \leq 2g - 4 \), improving a result of Funar [11]. About at the same time of the first version of this paper written in 2011, they improved their result to \( n \leq 2g - 1 \), getting Theorem 1. The statements of Theorem 1 and of [10, Theorem 1.1.] are almost the same. We present our own proof, starting the induction from the cases \( g = 2 \) and \( n \leq 3 \) using the information that the commutator subgroup of \( \text{Mod}(S) \) is perfect.

I was informed by Bridson (Private Communication) that he conjectured Theorem 1 (see [5, p. 2]).

Theorem 2 says that up to conjugation the map \( P : \text{Mod}(S) \rightarrow \text{GL}(2g, \mathbb{C}) \) is the only nontrivial complex representation of \( \text{Mod}(S) \) in this dimension. We recall that two homomorphisms \( \phi_1 \) and \( \phi_2 \) from a group \( G \) to a group \( H \) are conjugate if there exists an element \( y \in H \) such that \( \phi_2(x) = y\phi_1(x)y^{-1} \) for all \( x \in G \).

One of the outstanding unsolved problems in the theory of mapping class groups is the existence of a faithful linear representation \( \text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C}) \) for some \( n \) (cf. [4]). Our third theorem shows that in dimensions \( n \leq 3g - 3 \), there is no such representation of the mapping class group.

**Theorem 3.** Let \( g \geq 3 \) and let \( n \leq 3g - 3 \). Then there is no injective homomorphism \( \text{Mod}(S) \rightarrow \text{GL}(n, \mathbb{C}) \).

The linearity problem for the braid group, the mapping class group of a disk with marked points, was solved by Bigelow [2] and Krammer [26, 27]. Using the linearity of the braid group, the author [22] observed that the mapping class group of a sphere with marked points and the hyperelliptic mapping class groups, in particular the mapping class group of the closed surface of genus 2, are linear, which were also obtained by Bigelow–Budney [3].

On the other hand, Farb–Lubotzky–Minsky [8] proved that no homomorphism from a subgroup of finite index of the mapping class group into \( \text{GL}(n, \mathbb{C}) \) is injective for \( n < 2 \sqrt{g - 1} \). This result was improved by Funar [11] who showed that every map from the mapping class group into \( \text{SL}(n, \mathbb{C}) \) has finite image for \( n \leq \sqrt{g + 1} \).

We also prove that if \( S \) is closed, that is, \( p = 0 \), then the group \( \text{Mod}(S) \) can be generated by involutions if and only if \( g \geq 3 \) and \( r \leq 2g - 2 \). This result is stated below as Theorem 2.7 and is needed for the applications.
1.2 | Applications

We give a number of applications of our theorems above.

Nonorientable surfaces
Let $N$ be a closed nonorientable surface of genus $g$ with $r \geq 0$ marked points. The genus of a nonorientable surface is the number of projective planes in a connected sum decomposition. Equivalently, it is the maximal number of disjoint simple closed curves on $N$ whose complement is connected. (This definition applies to the orientable case too.) The mapping class group $\text{Mod}(N)$ of $N$ is defined to be the group of isotopy classes of all diffeomorphisms $N \to N$ preserving the set of marked points (isotopies are assumed to fix the marked points).

The action of $\text{Mod}(N)$ on the first homology of the closed surface $\overline{N}$ obtained from $N$ by forgetting the marked points gives rise to an automorphism of $H_1(\overline{N}; \mathbb{Z})$ preserving the associated $\mathbb{Z}_2$-valued intersection form. It was proved by McCarthy and Pinkall [30], and also by Gadgil and Pancholi [12], that, in fact, all automorphisms of $H_1(\overline{N}; \mathbb{Z})$ preserving the $\mathbb{Z}_2$-valued intersection form are induced by diffeomorphisms $\overline{N} \to \overline{N}$. By fixing a basis for $H_1(\overline{N}; \mathbb{Z})$ and by dividing out the torsion subgroup of $H_1(\overline{N}; \mathbb{Z})$, we get a representation $\text{Mod}(N) \to \text{GL}(g - 1, \mathbb{C})$. It is now natural to ask the triviality of the lower dimensional representations of $\text{Mod}(N)$. As the mapping class group $\text{Mod}(N)$ has nontrivial (finite) first integral homology, one cannot expect that every such homomorphism is trivial. Instead, one may ask the following question.

**Question 1.1.** Let $g \geq 3$, and let $n \leq g - 2$. Is the image of every homomorphism $\text{Mod}(N) \to \text{GL}(n, \mathbb{C})$ finite?

As an application of Theorem 1, we answer this question leaving only one case open†.

**Theorem 4.** Let $g \geq 3$, and let $n \leq g - 2$ if $g$ is odd and $n \leq g - 3$ if $g$ is even. Then the image of any homomorphism $\text{Mod}(N) \to \text{GL}(n, \mathbb{C})$ is finite.

The mapping class group of a nonorientable surface with boundary components may also be considered, but we restrict ourself to surfaces with marked points only in order to make the proof simpler.

Automorphisms of free groups
As another application of Theorem 1, we prove the following result on the homomorphisms from the mapping class group of an orientable surface to $\text{Aut}(F_n)$ and to $\text{Out}(F_n)$. (Compare Theorem 5 with [6, Question 16].)

**Theorem 5.** Let $g \geq 2$ and let $S$ be a closed orientable surface of genus $g$ with $r \leq 2g - 2$ marked points. Let $n$ be a positive integer with $n \leq 2g - 1$ and let $H \in \{\text{Aut}(F_n), \text{Out}(F_n)\}$. Then the image of any homomorphism $\text{Mod}(S) \to H$ is

(i) trivial if $g \geq 3$, and
(ii) a quotient of $\mathbb{Z}_{10}$ if $g = 2$.

† Szepietowski settled this case in [32].
Suppose that $S$ is closed and has two marked points, say $x_0$ and $x_1$. Let $R$ denote the surface $S$ with the only marked point $x_1$. The action of the mapping class group $\text{Mod}(S)$ on the fundamental group $\pi_1(R, x_0)$ gives us an injective homomorphism $\text{Mod}(S) \to \text{Aut}(F_{2g})$, so that the number $2g - 1$ in Theorem 5 is the best bound for the rank of the free group.

**Homomorphisms between mapping class groups**

We now give another proof of a theorem of Harvey and the author [15], by generalizing it in the following way.

**Theorem 6.** Let $g > h \geq 1$. Let $S$ be a closed connected orientable surface of genus $g$ with at most $2g - 2$ marked points, and let $R$ be an compact connected oriented surface of genus $h$ with finitely many marked points. Then the image of any homomorphism $\text{Mod}(S) \to \text{Mod}(R)$ is

(i) trivial if $g \geq 3$, and
(ii) a quotient of $\mathbb{Z}_2$ if $g = 2$.

This theorem was proved in [15] for closed surfaces by investigating the normal closures of various torsion elements. Here, the proof follows from Theorem 1, using the fact that the commutator subgroup of $\text{Mod}(S)$ is generated by torsion elements (cf. Theorem 2.6) and that the Torelli group is torsion-free.

**Quotients of mapping class groups**

The followings are immediate corollaries to our results, some of which might be known to the experts.

**Corollary 7.** Let $g \geq 2$, $n \leq 2g - 1$ and let $Q$ be a quotient of $\text{Mod}(S)$. The image of any homomorphism $Q \to \text{GL}(n, \mathbb{C})$ is trivial if $g \geq 3$, and is isomorphic to a quotient of $\mathbb{Z}_{10}$ if $g = 2$.

Note that the groups $\text{Sp}(2g, \mathbb{Z})$, $\text{Sp}(2g, \mathbb{Z}_m)$, $\text{PSp}(2g, \mathbb{Z})$, and $\text{PSp}(2g, \mathbb{Z}_m)$ are quotients of $\text{Mod}(S)$. As $\text{Mod}(S)$ is residually finite [14, 17], there are many more finite quotients.

**Corollary 8.** If $g \geq 3$, then up to the conjugation by an element of $\text{GL}(2g, \mathbb{C})$ there are exactly two homomorphisms $\text{Sp}(2g, \mathbb{Z}) \to \text{GL}(2g, \mathbb{C})$, the trivial homomorphism and the injection given by the inclusion.

**Corollary 9.** If $g \geq 3$, then every homomorphism $\text{PSp}(2g, \mathbb{Z}) \to \text{GL}(2g, \mathbb{C})$ is trivial, where $\text{PSp}(2g, \mathbb{Z})$ is the group $\text{Sp}(2g, \mathbb{Z})$ divided out by its center.

**Corollary 10.** If $g \geq 3$ and if $Q$ is a finite quotient of $\text{Mod}(S)$, then every homomorphism $Q \to \text{GL}(2g, \mathbb{C})$ is trivial.

To prove Corollaries 7–10, consider the composition of the given homomorphism $H \to \text{GL}(n, \mathbb{C})$ with the surjective map $\text{Mod}(S) \to H$ and apply the theorems above.

**Other mapping class groups**

In the definition of the mapping class group, if we allow the diffeomorphisms of $S$ to permute the marked points on $S$, then we get a group $\mathcal{M}(S)$, which contains $\text{Mod}(S)$ as a subgroup of index $r!$. 
Corollary 11. Let \( g \geq 2 \) and let \( n \leq 2g - 1 \). The image of any homomorphism \( \mathcal{M}(S) \to GL(n, \mathbb{C}) \) is finite.

Suppose that \( S \) has no boundary components. Let \( \mathcal{M}^\ast(S) \) denote the extended mapping class group of \( S \), the group of isotopy classes of all diffeomorphisms of \( S \). Then the group \( \mathcal{M}(S) \) in the above corollary can be replaced by \( \mathcal{M}^\ast(S) \), because \( \mathcal{M}^\ast(S) \) contains \( \mathcal{M}(S) \) as a subgroup of index two.

### 1.3 Outline of the paper

Here is an outline of the paper. Section 2 gives the relevant background from the theory of mapping class groups and surface topology. A new result in this section is that \( \text{Mod}(S) \) is generated by involutions if and only if \( p = 0 \) and \( r \leq 2g - 2 \). Note that if \( p > 0 \) then \( \text{Mod}(S) \) is torsion-free.

In Section 3, we give preliminary information on matrices. Section 4 investigates the properties of eigenvalues and eigenspaces of the images in \( GL(m, \mathbb{C}) \) of the Dehn twists about nonseparating simple closed curves. In Section 5, we prove Theorem 1. The proof is induction on the genus of the surface. We prove Theorems 4, 5, and 6 in Section 6.

Section 7 starts with a criteria for the triviality of a representation of the mapping class group into \( GL(m, \mathbb{C}) \). A technical lemma, we name Main Lemma, needed in the proofs of Theorems 2 and 3 is proved in the section. Section 8 is devoted to the proof of Theorem 2 for \( g \geq 4 \), and an outline of the proof for \( g = 3 \). The proof uses the assumption \( g \geq 3 \) in an essential way, and it seems that it does not work for \( g = 2 \). Our third main theorem, Theorem 3, is proved in Section 9, by induction. The initial case of the induction follows from Theorem 2. In fact, we prove a slightly more general result.

## 2 BACKGROUND RESULTS ON MAPPING CLASS GROUPS

Let \( S \) be a compact connected oriented surface of genus \( g \) with a finite number of marked points in the interior. In this section we state the necessary results from the theory of mapping class groups needed in this paper. For further information on mapping class groups, the reader is referred to \([9],[7],[18]\) or \([19]\).

The mapping class group \( \text{Mod}(S) \) of \( S \) is defined to be the group of isotopy classes of orientation-preserving self-diffeomorphisms of \( S \), where diffeomorphisms and isotopies are assumed to fix the marked points and the points near the boundary. For the operation of the group \( \text{Mod}(S) \), we use the functional notation: If \( f \) and \( h \) are two diffeomorphisms \( S \to S \), the composition \( fh \) means that \( h \) is applied first. In this paper, all diffeomorphisms and all curves are considered up to isotopy.

We first state the next theorem on the topology of curves on \( S \).

**Theorem 2.1** [25, Theorem 1.2]. Let \( g \geq 1 \) and let \( a \) and \( b \) be two nonseparating simple closed curves on \( S \). Then there is a sequence

\[
a = a_0, a_1, a_2, \ldots, a_k = b
\]

of nonseparating simple closed curves such that \( a_{i-1} \) intersects \( a_i \) transversely at only one point.
2.1 | Dehn twists

For a simple closed curve $a$ on $S$, we denote by $t_a$ the (isotopy class of the) right Dehn twist about $a$. We start with the following well-known relations among Dehn twists.

Let $a, b, c_1, c_2, c_3$ and $c_4$ be simple closed curves on $S$.

(i) If $f \in \text{Mod}(S)$, then $f t_a f^{-1} = t_{f(a)}$.

(ii) (Commutativity) If $a$ and $b$ are disjoint, then $t_a$ and $t_b$ commute.

(iii) (Braid relation) If $a$ intersects $b$ transversely at one point, then the Dehn twists about them satisfy the braid relation

$$t_a t_b t_a = t_b t_a t_b.$$

(iv) (One–holed and two–holed torus relations) If $c_1$ intersects $c_2$ transversely at one point, then a regular neighborhood of $c_1 \cup c_2$ is a torus with one boundary component, say $a$. The corresponding Dehn twists satisfy the one-holed torus relation

$$\left(t_{c_1} t_{c_2}\right)^6 = t_a.$$

If $c_2$ intersects each of the curves $c_1$ and $c_3$ transversely at one point and if $c_1$ is disjoint from $c_3$, then a regular neighborhood of $c_1 \cup c_2 \cup c_3$ is a torus with two boundary components, say $a$ and $b$. We then have the two-holed torus relation

$$\left(t_{c_1} t_{c_2} t_{c_3}\right)^4 = t_a t_b.$$

(v) (Lantern relation) If $c_1, c_2, c_3, c_4$ are disjoint and cobound a subsurface $\Sigma$ of $S$ homeomorphic to a sphere with four boundary components, then there are three more simple closed curves $x, y, z$ on $\Sigma$, as illustrated in Figure 1, such that the Dehn twists about these seven curves satisfy the lantern relation

$$t_{c_1} t_{c_2} t_{c_3} t_{c_4} = t_x t_y t_z.$$
We write \( \nu = p + r \).

It is known that for \( g \geq 2 \), the group \( \text{Mod}(S) \) is generated by the Dehn twists about (finitely many) nonseparating simple closed curves. If the surface is closed this is due to Dehn \cite{Dehn1910} and Lickorish \cite{Lickorish1962}:

**Theorem 2.2** \cite{Korkmaz2002}. If \( g \geq 2 \), then the mapping class group \( \text{Mod}(S) \) is generated by the Dehn twists about nonseparating simple closed curves

\[ \{a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g, c_1, c_2, \ldots, c_{g-1}, e_1, e_2, \ldots, e_\nu\} \]

illustrated in Figure 2.

We note that the above theorem is not true for \( g = 1 \). More precisely, if \( S \) is a torus with at least two boundary components, then the Dehn twists about nonseparating simple closed curves are not sufficient to generate \( \text{Mod}(S) \); one also needs the Dehn twists about the curves parallel to the boundary components \cite{Korkmaz2005, Korkmaz2006}. We also note that in fact this generating set is excessive.

### 2.2 Commutator subgroup and first homology

Recall that, for a group \( G \), the commutator subgroup \( G' \), also denoted by \( [G, G] \), is the (normal) subgroup of \( G \) generated by all commutators \([x, y] = xyx^{-1}y^{-1}\). The first homology group \( H_1(G; \mathbb{Z}) \) of \( G \) is isomorphic to the abelianization \( G/G' \).

**Theorem 2.3** \cite[Theorem 5.1]{Korkmaz2005}. Let \( S \) be a compact orientable surface of genus \( g \geq 1 \) with \( p \) boundary components and \( r \) marked points in the interior. The first homology group \( H_1(\text{Mod}(S); \mathbb{Z}) \) is

(i) trivial if \( g \geq 3 \),
(ii) isomorphic to the cyclic group \( \mathbb{Z}_{10} \) if \( g = 2 \),
(iii) isomorphic to the cyclic group \( \mathbb{Z}_{12} \) if \( (g, p) = (1, 0) \), and
(iv) isomorphic to the free abelian group \( \mathbb{Z}^p \) if \( g = 1 \) and \( p \geq 1 \).

In other words, the group \( \text{Mod}(S) \) is perfect if \( g \geq 3 \). In the case \( g = 2 \), \( \text{Mod}(S) \) is not perfect, but its commutator subgroup is perfect.

**Theorem 2.4** \cite[Theorem 4.2]{Korkmaz2006}. If \( g \geq 2 \) then the commutator subgroup of \( \text{Mod}(S) \) is perfect.
Theorem 2.5 [25, Theorem 2.7]. Let $g \geq 2$. Let $a$ and $b$ be two nonseparating simple closed curves on $S$ intersecting transversely at one point. The commutator subgroup of Mod($S$) is generated normally by $t_at_b^{-1}$.

Theorem 2.5 should be interpreted as follows: If a normal subgroup of Mod($S$) contains $t_at_b^{-1}$, then it contains the commutator subgroup Mod($S$)'.

Note that, in [25], the group Mod($S$) of this paper is denoted by $\mathbb{PM}_S$. In that paper, Theorems 2.4 and 2.5 above are proved for surfaces with marked points (=punctures), but the same proofs apply to surfaces with boundary components as well.

2.3 Generating Mod($S$) by involutions

An involution in a group is an element of order two. If the surface $S$ has boundary components, that is, $p > 0$, then Mod($S$) is torsion-free [9]. Assume that $S$ is closed and has $r \geq 0$ marked points.

If $g \leq 2$, then the group Mod($S$) cannot be generated by involutions, because in this case $H_1$ (Mod($S$); $\mathbb{Z}$) is not a direct sum of cyclic groups of order two. McCarthy–Papadopoulos [29] showed that if $g \geq 3$ and $r = 0$, then Mod($S$) is generated by involutions. We extend this result to the surfaces with marked points.

Theorem 2.6. Let $g \geq 2$ and let $S$ be a closed oriented surface of genus $g$ with $r \leq 2g - 2$ marked points. The commutator subgroup Mod($S$)' of the mapping class group Mod($S$) is generated by involutions.

Proof. Let us embed the closed surface $S$ in $\mathbb{R}^3$ in such a way that $S$ intersects the $y$-axis at $2g - 2$ points and that it is invariant under the rotation by $\pi$, say $\rho$, about the $y$-axis. We assume that the marked points are on the $y$-axis.

It is easy to find a pair of disjoint nonseparating simple closed curves on $S$ such that the complement of $c_1 \cup c_2$ is connected. We may assume up to a diffeomorphism of $S$ that $\rho$ interchanges $c_1$ and $c_2$. We then have that $t_{c_1}t_{c_2}^{-1} = (\rho t_{c_2} \rho)^{-1}$ is the product of the involutions $\rho$ and $t_{c_2}$. Let $a$ and $b$ be two simple closed curves on $S$ intersecting transversely at one point. As $g \geq 2$, there is a nonseparating curve $c$ on $S$ disjoint from both $a$ and $b$. Note that the complements of $a \cup c$ and $b \cup c$ must be connected. Now we may write $t_at_b^{-1} = (t_{c_1} t_{c_2})^{-1} t_{c_2}$ as a product of four involutions. As Mod($S$)' is generated normally by $t_at_b^{-1}$, the theorem follows. \hfill \Box

Theorem 2.7. Let $S$ be a closed oriented surface of genus $g$ with $r$ marked points. The group Mod($S$) is generated by involutions if and only if $g \geq 3$ and $r \leq 2g - 2$.

Proof. If $g \geq 3$ and $r \leq 2g - 2$, the claim follows from Theorem 2.6 because Mod($S$) is perfect.

On the other hand, if $g < 3$, as $H_1$ (Mod($S$); $\mathbb{Z}$) is not generated by involutions, the group Mod($S$) is not generated by involutions either. Suppose that $g \geq 3$ and $r > 2g - 2$. Let $\sigma$ be an involution of the surface $S$ and let $\text{Fix}(\sigma)$ denote the set of fixed points of $\sigma$, including the marked points. Denote the cardinality of $\text{Fix}(\sigma)$ by $r'$. Note that $r' \geq r$.

We have a two-fold covering map from $S - \text{Fix}(\sigma)$ to the quotient of $S - \text{Fix}(\sigma)$ with $\sigma$, a surface of genus $h$ with $r'$ punctures. By comparing the Euler characteristics of the total and base spaces, we get $2g - 2 + r' = 2(2h - 2 + r')$. It follows that $h = 0$ and $r' = 2g + 2$. It is known that an involution with $2g + 2$ fixed points is a hyperelliptic involution. Therefore, $\sigma$ acts as the minus
identity on the first homology of $\tilde{S}$, the closed surface obtained from $S$ by forgetting the punctures. We conclude that $\text{Mod}(S)$ cannot be generated by involutions. □

2.4 | Homomorphisms to abelian groups

If $a$ and $b$ are two nonseparating simple closed curves on $S$, then there is a diffeomorphism mapping $a$ to $b$. It follows that the Dehn twists $t_a$ and $t_b$ are conjugate in $\text{Mod}(S)$, so that their classes in $H_1(\text{Mod}(S); \mathbb{Z})$ are equal. In particular, we conclude the next lemma.

Lemma 2.8. Let $g \geq 1$, and let $a$ and $b$ be two nonseparating simple closed curves on $S$. If $H$ is an abelian group and if $\varphi : \text{Mod}(S) \to H$ is a homomorphism, then $\varphi(t_a) = \varphi(t_b)$.

2.5 | Kernel of $\text{Mod}(S) \to \text{Sp}(2g, \mathbb{Z})$

Let $S$ be a closed connected oriented surface of genus $g$ with $r \geq 0$ marked points, $z_1, z_2, \ldots, z_r$.

For each $0 \leq k \leq r$, let $S_k$ denote the closed surface of genus $g$ with the marked points $z_1, z_2, \ldots, z_k$ obtained from $S = S_r$ by forgetting the marked points $z_{k+1}, z_{k+2}, \ldots, z_r$. If $k \geq 1$ and if the Euler characteristic of $S_{k-1}$ is negative, by forgetting the marked point $z_k$, we get the Birman short exact sequence

$$1 \longrightarrow \pi_1(S_{k-1}, z_k) \xrightarrow{\partial_k} \text{Mod}(S_k) \xrightarrow{\delta_k} \text{Mod}(S_{k-1}) \longrightarrow 1,$$

(1)

where the map $\partial_k(\alpha)$ is the diffeomorphism obtained by pushing the base point $z_k$ once along the loop $\alpha$ (cf. [9, Theorem 4.6]). On the surface $S_{k-1}$, we think of the marked points as punctures.

If a loop $\alpha$ in $S_{k-1}$ based at $z_k$ is simple and if $c_1$ and $c_2$ are the boundary components of a regular neighborhood of $\alpha$ in $S_k$, then

$$\partial_k(\alpha) = t_{c_1} t_{c_2}^{-1}$$

for appropriate labeling of $c_1$ and $c_2$. We also note that the homomorphism $\partial_k$ is $\text{Mod}(S_k)$-equivariant: If $f \in \text{Mod}(S_k)$ and $\alpha \in \pi_1(S_{k-1}, z_k)$, then

$$\partial_k(f(\alpha)) = f \partial_k(\alpha) f^{-1}.$$

Let $\tilde{S} = S_0$ be the closed surface obtained from $S$ by forgetting all marked points. Let $\Delta : \text{Mod}(S) \to \text{Mod}(\tilde{S})$ be the resulting homomorphism, so that

$$\Delta = \delta_1 \delta_2 \cdots \delta_{r-1} \delta_r.$$

Proposition 2.9. Let $S$ be a compact oriented surface of genus $g \geq 1$ with $p$ boundary components and with $r$ marked points, and let $P : \text{Mod}(S) \to \text{Sp}(2g, \mathbb{Z})$ be the homomorphism obtained from the action of $\text{Mod}(S)$ on $H_1(\tilde{S}; \mathbb{Z})$. Then the kernel of $P$ is torsion-free.

Proof. If $p > 0$, then the group $\text{Mod}(S)$ is torsion-free. Thus, we may assume that $p = 0$. 

It is easy to see that if
\[ 1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1 \]
is a short exact sequence of groups with $H$ and $K$ torsion-free, then so is $G$.

Let us denote the marked points of $S$ by $z_1, z_2, \ldots, z_r$, and let $S_k$ denote the surface $S$ with the first $k$ marked points, so that $S_r = S$ and $S_0 = \tilde{S}$. Let $\varphi_k : \text{Mod}(S_k) \to \text{Mod}(\tilde{S})$ be the homomorphism obtained by forgetting all marked points. Note that $\varphi_k = \varphi_{k-1} \delta_k$.

Let $I_g$ denote the Torelli subgroup of $\text{Mod}(S_0)$, the subgroup consisting of mapping classes acting trivially on the first integral homology of $S_0$. It is known that $I_g$ is torsion-free (cf. [9, 17]).

Suppose first that $g \geq 2$. From the Birman exact sequence (1), we get the short exact sequence
\[ 1 \longrightarrow \pi_1(S_{k-1}) \longrightarrow \varphi_{k-1}^{-1}(I_g) \delta_k \longrightarrow \varphi_{k-1}^{-1}(I_g) \longrightarrow 1 \]
for all $k \geq 1$. The fundamental group of $S_{k-1}$ is torsion-free. As $\varphi_0^{-1}(I_g) = I_g$ is torsion-free, $\varphi_1^{-1}(I_g)$ is torsion-free. By repeated application of the exact sequence (2), we conclude that $\varphi_r^{-1}(I_g) = \ker(P)$ is torsion-free.

If $g = 1$, then $I_1$ and $\varphi_1^{-1}(I_1)$ are trivial. If $k \geq 2$, then $\pi_1(S_k)$ is free, hence torsion-free, and we still have a short exact sequence (2). It follows that $\varphi_r^{-1}(I_1) = \ker(P)$ is torsion-free. $\square$

3 | LINEAR ALGEBRAIC PRELIMINARIES

Throughout the paper, let $\mathbb{C}$ denote the field of complex numbers, $\text{GL}(n, \mathbb{C})$ denote the general linear group, and let $U$, $\tilde{U}$ and $V_x$ denote the $2 \times 2$ matrices defined by
\[ U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad V_x = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}, \]
where $x$ is a nonzero complex number.

For each $i = 1, 2, \ldots, g$, we define the $2g \times 2g$ block diagonal matrices
\[ A_i = \text{Diag}(I_2, \ldots, I_2, U, I_2, \ldots, I_2), \]
\[ B_i = \text{Diag}(I_2, \ldots, I_2, \tilde{U}, I_2, \ldots, I_2). \]
Here, $U$ and $\tilde{U}$ are in the $i^{th}$ block on the diagonal, and $I_n$ is the $n \times n$ identity matrix. In the case $g \geq 2$, for a nonzero $x \in \mathbb{C}$, we also define
\[ C_{k,x} = \begin{bmatrix} I_{2k-2} & 0 & 0 \\ 0 & U & V_x \\ 0 & V_{1/x} & U \\ 0 & 0 & I \end{bmatrix} \]
for each $k = 1, 2, \ldots, g - 1$. The identity matrix in the bottom right corner is of size $2g - 2k - 2$.

We note that $A_i$ (resp., $B_i$ and $C_{k,-1}$) is the (symplectic) matrix of the action of the Dehn twist $t_{a_i}$ (resp., $t_{b_i}$ and $t_{c_k}$) on the first homology group $H_1(\tilde{S}; \mathbb{Z})$ of the closed surface $\tilde{S}$ with respect to the standard ordered basis $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$, where $a_i$ and $b_i$ are the (oriented) simple
closed curves given in Figure 2. Here, we let that the curves $a_i$ and $b_i$ stand in for their homology classes. Recall that when one considers a Dehn twist about a curve on an oriented surface, the orientation of the curve is irrelevant.

**Lemma 3.1.** Let $C = (c_{ij})$ and $D = (d_{ij})$ be two upper triangular matrices in $\text{GL}(n, \mathbb{C})$ with $c_{ii} = c$ and $d_{ii} = d$ for all $i$. Then $C$ and $D$ satisfy the braid relation $CDC = DCD$ if and only if $C = D$.

**Proof.** The cases $n \leq 2$ are elementary. Assume that $n \geq 3$ and $CDC = DCD$. If we write $C$ and $D$ as

$$
C = \begin{bmatrix} C' & \ast \\ 0 & c \end{bmatrix}, \quad D = \begin{bmatrix} D' & \ast \\ 0 & d \end{bmatrix} = \begin{bmatrix} d & \ast \\ 0 & D'' \end{bmatrix},
$$

where $C', C'', D', D'' \in \text{GL}(n-1, \mathbb{C})$, then we see that $C'D'C' = D'C'D'$, $C''D''C'' = D''C''D''$ and $c = d$. By induction, we have the equalities $C' = D'$ and $C'' = D''$.

Therefore, the matrices $C$ and $D$ are of the form

$$
C = \begin{bmatrix} c & M & x \\ 0 & N & Q \\ 0 & 0 & c \end{bmatrix} \text{ and } D = \begin{bmatrix} c & M & y \\ 0 & N & Q \\ 0 & 0 & c \end{bmatrix},
$$

where $x, y \in \mathbb{C}$. From the equality $CDC = DCD$ once again, we obtain $x = y$, so that $C = D$. □

### 3.1 Properties of $A_i$ and $B_i$

We now prove a number of lemmas on the properties of the matrices $U, \hat{U}, A_i$ and $B_i$, culminating in Lemma 3.6 and Remark 3.7.

**Lemma 3.2.** Let $X$, $Y$ and $Z$ be matrices with entries in $\mathbb{C}$ of sizes $2 \times k$, $k \times 2$ and $2 \times 2$, respectively.

(i) $AX = \hat{A}X = X$ if and only if $X = 0$.

(ii) $YU = Y\hat{U} = Y$ if and only if $Y = 0$.

(iii) $ZU = UZ$ and $Z\hat{U} = \hat{U}Z$ if and only if $Z = \alpha I_2$ for some $\alpha \in \mathbb{C}$.

**Proof.** The proof is straightforward. □

**Lemma 3.3.** Let $X$, $Y$ and $Z$ be matrices with entries in $\mathbb{C}$ such that the given multiplications are possible.

(i) $A_iX = B_iX = X$ for all $i$ if and only if $X = 0$.

(ii) $YA_i = YB_i = Y$ for all $i$ if and only if $Y = 0$.

(iii) $ZA_i = A_iZ$ and $ZB_i = B_iZ$ for all $i$ if and only if $Z$ is equal to a diagonal matrix $\text{Diag}(\alpha_1I_2, \alpha_2I_2, ..., \alpha_gI_2)$ for some $\alpha_i \in \mathbb{C}$.

**Proof.** This lemma is a slight generalization of Lemma 3.2, and may easily be proved. □
We remind that the centralizer of a subset $K$ of a group $G$ is defined as the subgroup

$$C_G(K) = \{ x \in G : xk = kx \text{ for all } k \in K \}.$$

**Corollary 3.4.** The centralizer of the set $\{A_1, B_1, A_2, B_2, \ldots, A_g, B_g\}$ in $\text{GL}(2g, \mathbb{C})$ is the subgroup

$$\{\text{Diag}(\alpha_1I_2, \alpha_2I_2, \ldots, \alpha_gI_2) : \alpha_1\alpha_2 \cdots \alpha_g \neq 0, \alpha_i \in \mathbb{C} \}.$$

**Lemma 3.5.** Let $X \in \text{GL}(2, \mathbb{C})$. Then

(i) $XU = UX$ and $X\hat{U}X = \hat{U}X\hat{U}$ if and only if $X = U$, and
(ii) $X\hat{U} = \hat{U}X$ and $UXU = XUX$ if and only if $X = \hat{U}$.

**Proof.** If $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the equality $XU = UX$ yields $c = 0$ and $d = a$. Now the equation $X\hat{U}X = \hat{U}X\hat{U}$ gives the system

$$a(a - b) = a - b,$$

$$a^2 = 2a - b,$$

$$b(2a - b) = b,$$

whose only solution is $a = b = 1$. Thus, $X = U$.

The proof of (ii) is similar. □

**Lemma 3.6.** Let $X \in \text{GL}(2g, \mathbb{C})$. Suppose that

(i) $\lambda = 1$ is the only eigenvalue of $X$,
(ii) $XA_i = A_iX$ for all $i = 1, 2, \ldots, g$,
(iii) $XB_j = B_jX$ for all $j = 2, 3, \ldots, g$, and
(iv) $XB_1X = B_1X^2$.

Then $X = A_1$.

**Proof.** The case $g = 1$ follows from Lemma 3.5. So, suppose that $g \geq 2$.

Let us write $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_1$ and $X_4$ are, respectively, $2 \times 2$ and $(2g - 2) \times (2g - 2)$ matrices. For each $i \geq 2$, we write

$$A_i = \begin{bmatrix} I_2 & 0 \\ 0 & \hat{A}_{i-1} \end{bmatrix}, \quad B_i = \begin{bmatrix} I_2 & 0 \\ 0 & \hat{B}_{i-1} \end{bmatrix}.$$

For $i \geq 2$, the equations $XA_i = A_iX$ and $XB_i = B_iX$ yield that

- $X_2\hat{A}_{i-1} = X_2\hat{B}_{i-1} = X_2$,
- $\hat{A}_{i-1}X_3 = \hat{B}_{i-1}X_3 = X_3$, and
- $X_4\hat{A}_{i-1} = \hat{A}_{i-1}X_4$ and $X_4\hat{B}_{i-1} = \hat{B}_{i-1}X_4$.

It now follows from Lemma 3.3 and the assumption (i) that $X_2 = 0, X_3 = 0$ and $X_4 = I_{2g-2}$. 

Moreover, from the equalities $XA_1 = A_1X$ and $XB_1X = B_1XB_1$, we obtain

- $X_1U = UX_1$, and
- $X_1\tilde{U}X_1 = \tilde{U}X_1\tilde{U}$.

Now these two equalities and Lemma 3.5 give us $X_1 = U$, so that $X = A_1$, the desired result. □

**Remark 3.7.** The above proof may easily be modified to prove the following version of Lemma 3.6:

Suppose that

(i) $\lambda = 1$ is the only eigenvalue of $X$,
(ii) $XA_i = A_iX$ for all $i = 1, 2, \ldots, g$,
(iii) $XB_j = B_jX$ for all $1 \leq j \leq g$ with $j \neq k$, and
(iv) $XB_kX = B_kXB_k$.

Then $X = A_k$. The lemma is also true if the roles of $A_i$ and $B_i$ are interchanged.

**Lemma 3.8.** Let $g \geq 2$, let $1 \leq k \leq g - 1$ and let $X \in \text{GL}(2g, \mathbb{C})$. Suppose that

(i) $\lambda = 1$ is the only eigenvalue of $X$,
(ii) $XA_i = A_iX$ for all $i \in \{1, 2, \ldots, g\}$,
(iii) $XB_j = B_jX$ for all $j \in \{1, 2, \ldots, g\}\setminus\{k, k+1\}$, and
(iv) $XB_jX = B_jXB_j$ for $j \in \{k, k+1\}$.

Then $X = C_{k,x}$ for some $x \neq 0$.

**Proof.** Suppose first that $g = 2$. Write

$$X = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix},$$

where each $Y_i$ is a $2 \times 2$ matrix. The equalities $XA_i = A_iX$ give $UY_1 = Y_1U$, $UY_4 = Y_4U$, $UY_2 = Y_2U = Y_2$, $UY_3 = Y_3U = Y_3$. As the only eigenvalue of $X$ is 1, we conclude that

$$Y_1 = \begin{bmatrix} 1 & y_1 \\ 0 & 1 \end{bmatrix}, \quad Y_2 = V_{y_2}, \quad Y_3 = V_{y_3} \quad \text{and} \quad Y_4 = \begin{bmatrix} 1 & y_4 \\ 0 & 1 \end{bmatrix}$$

for some $y_1, y_2, y_3, y_4$. The relations $XB_jX = B_jXB_j$ for $j = 1, 2$ then yield $y_1 = y_4 = 1$ and $y_2y_3 = 1$, giving the desired result.

Suppose now $g \geq 3$. Assume that $k = 1$. Let us write $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$, where $X_1$ and $X_4$ are, respectively, $4 \times 4$ and $(2g - 4) \times (2g - 4)$ matrices.

For each $i \geq 3$, let us write

$$A_i = \begin{bmatrix} I_4 & 0 \\ 0 & \tilde{A}_{i-2} \end{bmatrix}, \quad B_i = \begin{bmatrix} I_4 & 0 \\ 0 & \tilde{B}_{i-2} \end{bmatrix}.$$

The equalities $XA_i = A_iX$ and $XB_i = B_iX$ imply that
\[ X_2 \tilde{A}_{i-2} = X_2 \tilde{B}_{i-2} = X_2, \]
\[ \tilde{A}_{i-2} X_3 = \tilde{B}_{i-2} X_3 = X_3 \] and
\[ X_4 \tilde{A}_{i-2} = \tilde{A}_{i-2} X_4, \]
\[ X_4 \tilde{B}_{i-2} = \tilde{B}_{i-2} X_4. \]

By Lemma 3.3, Corollary 3.4, and the assumption (i), we get \( X_2 = 0, X_3 = 0 \) and \( X_4 = I_{2g-4}. \)

For \( i = 1, 2 \), let us now write
\[ A_i = \begin{bmatrix} A'_i & 0 \\ 0 & I_{2g-4} \end{bmatrix}, B_i = \begin{bmatrix} B'_i & 0 \\ 0 & I_{2g-4} \end{bmatrix}. \]

The equalities \( X A_i = A_i X \) and \( X B_i X = B_i X B_i \) yield
\[ \cdot X_1 A'_i = A'_i X_1 \] and
\[ \cdot X_1 B'_i X = B'_i X_1 B'_i. \]

It follows from the case \( g = 2 \) that \( X_1 = \begin{bmatrix} U & V \\ V_{1/x} & U \end{bmatrix} \) for some nonzero \( x \), so that \( X = C_{1,x}. \)

The general case may be reduced to the case \( k = 1 \) by conjugating everything by a change-of-basis matrix. \( \square \)

### 3.2 Solvable groups

For a group \( G \), let \( G^{(0)} = G \). For each integer \( k \geq 1 \), we inductively define the \( k \)th derived subgroup \( G^{(k)} \) of \( G \) by
\[ G^{(k)} = [G^{(k-1)}, G^{(k-1)}]. \]

Recall that a group \( G \) is called solvable if \( G^{(k)} = 1 \) for some \( k \).

**Lemma 3.9.** The subgroup of \( GL(n, \mathbb{C}) \) consisting of upper triangular matrices is solvable.

**Lemma 3.10.** Let \( r \) and \( s \) be two positive integers. Then the subgroup of \( GL(r + s, \mathbb{C}) \) consisting of all matrices of the form \( \begin{bmatrix} I_r & * \\ 0 & I_s \end{bmatrix} \) is abelian.

**Lemma 3.11** [10, Lemma 2.2]. Let \( G \) be a perfect group, and \( H \) a solvable group. Then any homomorphism \( G \to H \) is trivial.

### 4 Eigenvalues and Eigenspaces of \( \phi(t_a) \)

Let \( S \) be a compact connected oriented surface of genus \( g \), perhaps with a finite number of marked points in the interior and let \( \phi : Mod(S) \to GL(n, \mathbb{C}) \) be a homomorphism. For a simple closed curve \( a \) on \( S \), following [10], we write
\[ L_a = \phi(t_a) \]

for the image of the Dehn twist \( t_a \). If \( \lambda \) is an eigenvalue of a linear operator \( L : \mathbb{C}^n \to \mathbb{C}^n \), we denote the corresponding eigenspace by \( E^a_{\lambda}(L) \). We simply write \( E^a_{\lambda} \) for \( E^a_{\lambda}(L_a) \).
If \( a \) and \( b \) are two nonseparating simple closed curves on \( S \), there is a diffeomorphism of \( S \) mapping \( a \) to \( b \), so that \( t_a \) is conjugate to \( t_b \). It follows that \( L_a = \phi(t_a) \) is similar to \( L_b = \phi(t_b) \). In particular, \( L_a \) and \( L_b \) have the same eigenvalues.

For an eigenvalue \( \lambda \) of a matrix \( M \), let \( \lambda_\#(M) \) denote the multiplicity of \( \lambda \). As the matrix \( M \) is going to be clear from the context, we omit it from the notation and simply write \( \lambda_\# \).

### 4.1 Eigenspaces under the action of \( \text{Mod}(S) \)

The next lemma is elementary and well-known, but we state it anyway for the convenience of the reader.

**Lemma 4.1.** Let \( L \) and \( M \) be two linear automorphisms of \( \mathbb{C}^n \) with \( LM = ML \). If \( \lambda \) is an eigenvalue of \( L \), then \( \ker(L - \lambda I)^k \) is \( M \)-invariant for all \( k \geq 1 \). In particular, the eigenspace \( E_\lambda(L) = \ker(L - \lambda I) \) is \( M \)-invariant.

**Proof.** Let \( F = \phi(f) \). The assumptions \( f(x) = a \) and \( f(y) = b \) imply that \( ft_x f^{-1} = t_a \) and \( ft_y f^{-1} = t_b \), and hence \( FL_x F^{-1} = L_a \) and \( FL_y F^{-1} = L_b \). Therefore,

\[
E_\lambda^a = E_\lambda(L_a) = E_\lambda(FL_x F^{-1}) = F(E_\lambda^x)
\]

and

\[
E_\lambda^b = E_\lambda(L_b) = E_\lambda(FL_y F^{-1}) = F(E_\lambda^y).
\]

The lemma now follows from these two equalities.

**Lemma 4.2.** Suppose that \( a, b, x, y \) are four simple closed curves on \( S \) such that there is an orientation-preserving diffeomorphism \( f : S \to S \) with \( f(x) = a \) and \( f(y) = b \). For an eigenvalue \( \lambda \) of \( L_a = \phi(t_a) \), \( E_\lambda^a = E_\lambda^b \) if and only if \( E_\lambda^x = E_\lambda^y \).

**Proof.** Let \( X = \phi(f) \). The assumptions \( f(x) = a \) and \( f(y) = b \) imply that \( ft_x f^{-1} = t_a \) and \( ft_y f^{-1} = t_b \), and hence \( FL_x F^{-1} = L_a \) and \( FL_y F^{-1} = L_b \). Therefore,

\[
E_\lambda^a = E_\lambda(L_a) = E_\lambda(FL_x F^{-1}) = F(E_\lambda^x)
\]

and

\[
E_\lambda^b = E_\lambda(L_b) = E_\lambda(FL_y F^{-1}) = F(E_\lambda^y).
\]

The lemma now follows from these two equalities.

**Lemma 4.3.** Let \( g \geq 2 \), let \( a, b \) be two nonseparating simple closed curves on \( S \) intersecting transversely at one point and let \( \lambda \) be an eigenvalue of \( L_a \) and \( L_b \). Suppose that \( E_\lambda^a = E_\lambda^b \). Then \( E_\lambda^a \) is \( \text{Mod}(S) \)-invariant, that is, \( \phi(f)(E_\lambda^a) = E_\lambda^a \) for all \( f \in \text{Mod}(S) \).

**Proof.** Let \( x \) be a nonseparating simple closed curve on \( S \). By Theorem 2.1, there is a sequence \( a = a_0, a_1, a_2, \ldots, a_k = x \) of nonseparating simple closed curves such that \( a_{i-1} \) intersects \( a_i \) at one point for all \( 1 \leq i \leq k \). As there exists a diffeomorphism \( f_i \) of \( S \) mapping \( (a, b) \) to \( (a_{i-1}, a_i) \), we have \( E^a_{\lambda_{i-1}} = E^a_{\lambda_i} \) by Lemma 4.2. It follows that \( E^x_\lambda = E^a_\lambda \) for all nonseparating simple closed curves \( x \). In particular, \( E^a_\lambda \) is invariant under \( L_x = \phi(t_x) \). As \( \text{Mod}(S) \) is generated by the Dehn twists about nonseparating curves, we conclude that the subspace \( E^a_\lambda \) is \( \text{Mod}(S) \)-invariant.

## 5 REPRESENTATIONS INTO \( \text{GL}(n, \mathbb{C}) \) FOR \( n \leq 2g - 1 \)

In this section, we prove Theorem 1. The proof is by induction on the genus \( g \).
Proposition 5.1. Let $g = 2$ and $n \leq 3$, and let $\phi : \text{Mod}(S) \to \text{GL}(n, \mathbb{C})$ be a homomorphism. Then the image of $\phi$ is a quotient of the cyclic group $\mathbb{Z}_{10}$.

Proof. As $H_1(\text{Mod}(S); \mathbb{Z})$ is isomorphic to the cyclic group $\mathbb{Z}_{10}$, it suffices to prove that $\phi(\text{Mod}(S))$ is abelian, or equivalently that $\phi(\text{Mod}(S)')$ is trivial. As $\text{Mod}(S)'$ is generated normally by a single element $t_xt_y^{-1}$ for any two simple closed curves $x$ and $y$ intersecting transversely at one point (cf. Theorem 2.5), it is also sufficient to find two such curves with $L_x = L_y$.

Let $a$, $b_1$, and $b_2$ be three nonseparating simple closed curves on $S$ such that $a$ is disjoint from $b_1 \cup b_2$, and that $b_1$ intersects $b_2$ transversely at one point. Then $t_{b_1}$ and $t_{b_2}$ satisfy the braid relation

$$t_{b_1}t_{b_2}t_{b_1} = t_{b_2}t_{b_1}t_{b_2},$$

and hence we have the braid relation

$$L_{b_1}L_{b_2}L_{b_1} = L_{b_2}L_{b_1}L_{b_2} \quad (3)$$

in $\text{GL}(n, \mathbb{C})$. If $L_{b_1}$ and $L_{b_2}$ commute, then we conclude from (3) that $L_{b_1} = L_{b_2}$. Therefore, it suffices to prove that $L_{b_1}L_{b_2} = L_{b_2}L_{b_1}$.

Case 1. If $n = 1$, then $\text{GL}(n, \mathbb{C}) = \mathbb{C}^*$ is abelian. Hence, the proof follows.

Case 2. Suppose that $n = 2$. The Jordan form of $L_a$ is one of the following three matrices, where $\lambda_1 \neq \lambda_2$:

(i) $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, (ii) $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, (iii) $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Choose an ordered basis of $\mathbb{C}^2$ so that $L_a$ in this basis is in its Jordan form. Consider $L_{b_i}$ as written in this basis.

(i). As each eigenspace of $L_a$ is invariant under $L_{b_1}$ and $L_{b_2}$, they are diagonal too. Hence, they commute.

(ii). As $L_x$ is conjugate to $L_a$ for every nonseparating simple closed curve $x$ on $S$, we have $L_x = \lambda I$. In particular, $L_{b_1} = \lambda I = L_{b_2}$.

(iii). As $L_{b_1}$ and $L_{b_2}$ preserve the eigenspace of $L_a$, they are upper triangular with $\lambda$ along the diagonal. Thus, they commute.

Case 3. Suppose finally that $n = 3$. In this case, the Jordan form of $L_a$ is one of the following six matrices:

(i) $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, (ii) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$, (iii) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{bmatrix}$,

(iv) $\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, (v) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$, (vi) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{bmatrix}$.

Here, different notations represent distinct eigenvalues. In each case, we fix an ordered basis of $\mathbb{C}^3$ with respect to which the matrix $L_a$ is in its Jordan form, and we assume that the matrices of linear operators are written in this basis. We now analyze each case.

(i). As each eigenspace of $L_a$ is invariant under $L_{b_1}$, the matrices $L_{b_1}$ and $L_{b_2}$ are diagonal. Hence, they commute.
(ii). From the proof of Case 2 (ii) above, we get that $L_{b_1} = L_{b_2}$.

(iii). As the matrices $L_{b_1}$ and $L_{b_2}$ preserve eigenspaces $E_\lambda^a$ and $E_\mu^a$, and also $\ker(L_a - \mu I)^2$, they are of the form

$$L_{b_i} = \begin{bmatrix}
x_i & 0 & 0 \\
0 & y_i & u_i \\
0 & 0 & z_i \\
\end{bmatrix}.$$  

The braid relation (3) implies that $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $z_1 = z_2 = z$. The equality $L_{b_1} L_a = L_a L_{b_1}$ gives $y = z$. Hence, $x = \lambda$ and $y = z = \mu$, so that

$$L_{b_i} = \begin{bmatrix}
\lambda & 0 & 0 \\
0 & \mu & u_i \\
0 & 0 & \mu \\
\end{bmatrix}.$$  

But we then have $L_{b_1} L_{b_2} = L_{b_2} L_{b_1}$.

(iv). In this case, $\ker(L_a - \lambda I) = E_\lambda^a$ is 1-dimensional and $\ker(L_a - \lambda I)^2$ is 2-dimensional. As they are both $L_{b_i}$-invariant for $i = 1, 2$ and are conjugate to $L_a$, it follows that $L_{b_i}$ is upper triangular with $\lambda$ along the diagonal. Now the braid relation (3) and Lemma 3.1 imply that $L_{b_1} = L_{b_2}$.

(v). The eigenspace $E_\lambda^a$ is 2-dimensional in this case. If $E_\lambda^a = E_\mu^c$ for some (hence all) curve $c$ intersecting $a$ at one point, then by Lemma 4.3 the eigenspace $E_\lambda^a$ is $\text{Mod}(S)$-invariant, so that $\phi$ induces a homomorphism $\bar{\phi} : \text{Mod}(S) \rightarrow \text{GL}(E_\lambda^a) = \text{GL}(2, \mathbb{C})$. By Case 2, $\bar{\phi}(\text{Mod}(S))$ is cyclic, and hence $\bar{\phi}(f) = I_2$ all $f \in \text{Mod}(S)'$, so that the matrix of $\phi(f)$ is of the form

$$\begin{bmatrix}1 & 0 & * \\
0 & 1 & * \\
0 & 0 & *\end{bmatrix}.$$  

As the subgroup of $\text{GL}(3, \mathbb{C})$ consisting of upper triangular matrices is solvable and as $\text{Mod}(S)'$ is perfect, $\bar{\phi}(\text{Mod}(S)')$ is trivial.

If $E_\lambda^a \neq E_\mu^c$ for some (hence all) simple closed curve $c$ intersecting $a$ transversely at one point, we choose such a curve $c$ so that it is disjoint from $b_1$ and $b_2$. Then $E_\lambda^a \cap E_\mu^c$ and $E_\lambda^a$ are $L_{b_i}$-invariant subspaces. Let us take a (ordered) basis of $\mathbb{C}^3$ whose first element is in $E_\lambda^a \cap E_\mu^c$ and the second element in $E_\lambda^a$. In this basis, $L_{b_i}$ is an upper triangular matrix with $\lambda$ along the diagonal. The relation (3) and Lemma 3.1 give $L_{b_1} = L_{b_2}$.

(vi). In this last case, the eigenspace $E_\lambda^a$ is, again, 2-dimensional. If $E_\lambda^a = E_\lambda^c$ for some simple closed curve $c$ intersecting $a$ transversely at one point, then as in the case (v) we conclude that $\bar{\phi}(\text{Mod}(S)')$ is trivial.

Suppose finally that, in the case (vi), $E_\lambda^a \neq E_\lambda^c$ for some simple closed curve $c$ intersecting $a$ transversely at one point. We choose such a curve $c$ so that it is disjoint from $b_1$ and $b_2$. By Lemma 4.2, $E_\lambda^a \neq E_\lambda^y$ for all simple closed curves $x$ and $y$ intersecting once. Set $a = b_4$ and $c = b_5$, and choose a simple closed curve $b_3$ intersecting $b_2$ and $b_4$ transversely once and is disjoint from $b_1$ and $b_5$. Thus, we have that
• $b_i$ intersects $b_j$ at one point if $|i - j| = 1$, and
• $b_i$ is disjoint from $b_j$ if $|i - j| \geq 2$.

Let $v_1 \in E^a_{\lambda} \cap E^c_{\lambda}$, $v_2 \in E^a_{\lambda}$ and $v_3 \in E^a_{\mu}$ so that $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{C}^3$. In this basis, the matrix of $L_a$ is its Jordan form. As $E^a_{\lambda} \cap E^c_{\lambda}$, $E^a_{\lambda}$ and $E^a_{\mu}$ are $L_{b_i}$-invariant for $i = 1, 2$, we have

$$L_{b_i} = \begin{bmatrix}
x_i & w_i & 0 \\
0 & y_i & 0 \\
0 & 0 & z_i
\end{bmatrix},$$

with $\{x_i, y_i, z_i\} = \{\lambda, \mu\}$. Then the relation (3) gives $x_1 = x_2 = x$, $y_1 = y_2 = y$ and $z_1 = z_2 = z$.

If $z = \mu$ then $x = y = \lambda$, and hence $L_{b_1}L_{b_2} = L_{b_2}L_{b_1}$.

Suppose that $z = \lambda$. We show that this case is not possible. If $x = \lambda$ then $y = \mu$. But then we have $E^a_{\lambda} = E^a_{\mu}$, a contradiction. If $x = \mu$ then $y = \lambda$. As $E^a_{\mu}$ is $L_{b_3}$-invariant, we have

$$L_{b_3} = \begin{bmatrix}
u & \star & \star \\
0 & \star & \star \\
0 & \star & \star
\end{bmatrix}.$$

Now the braid relation $L_aL_{b_3}L_a = L_{b_3}L_aL_{b_3}$ gives $u = \lambda$, while the braid relation $L_{b_3}L_{b_2}L_{b_3} = L_{b_2}L_{b_3}L_{b_2}$ gives $u = \mu$. As $\lambda \neq \mu$, we get a contradiction again.

This completes the proof of the proposition. $\square$

Finally, we are ready to prove Theorem 1.

**Proof of Theorem 1.** If we show that the image of $\phi$ is abelian, the other claims follows from Theorem 2.3.

If $g = 1$, then $\phi : \text{Mod}(S) \to \text{GL}(1, \mathbb{C})$ and the group $\text{GL}(1, \mathbb{C}) = \mathbb{C}^*$ is abelian. The case $g = 2$ is proved in Proposition 5.1. Assume that $g \geq 3$ and that the theorem is true for all surfaces of genus at most $g - 1$. As $\text{GL}(n - 1, \mathbb{C})$ is isomorphic to a subgroup of $\text{GL}(n, \mathbb{C})$, it suffices to prove the theorem for $n = 2g - 1$. We set $n = 2g - 1$ and let $\phi : \text{Mod}(S) \to \text{GL}(n, \mathbb{C})$.

Let $a$ and $b$ be two simple closed curves on $S$ intersecting transversely at one point and let us fix a subsurface $R$ in $S$ of genus $g - 1$ with one boundary component in the complement of $a \cup b$. We embed $\text{Mod}(R)$ into $\text{Mod}(S)$ by extending diffeomorphisms $R \to R$ to $S \to S$ by the identity and identify $\text{Mod}(R)$ with its image, so that it is a subgroup of $\text{Mod}(S)$.

We claim that if there exists a $\text{Mod}(R)$-invariant subspace $V$ of dimension $m$ with $2 \leq m \leq n - 2$, then $\phi$ is trivial. Suppose that there exists such a subspace. Then $\phi$ induces two homomorphisms

$$\phi_1 : \text{Mod}(R) \to \text{GL}(V) = \text{GL}(m, \mathbb{C})$$

and

$$\phi_2 : \text{Mod}(R) \to \text{GL}(\mathbb{C}^n/V) = \text{GL}(n - m, \mathbb{C}).$$

Note that $m \leq 2(g - 1) - 1$ and $n - m \leq 2(g - 1) - 1$. By assumption, the image of each $\phi_i$ is cyclic. In particular, if $c$ and $d$ are two simple closed curves on $R$ intersecting transversely at
one point, then $\phi_i(t_c) = \phi_i(t_d)$ by Lemma 2.8, that is, $\phi_i(t_c t_d^{-1}) = I$. As the commutator subgroup $\text{Mod}(R)'$ of $\text{Mod}(R)$ is generated normally in $\text{Mod}(R)$ by $t_c t_d^{-1}$ we get that $\phi_i(f) = I$ for all $f \in \text{Mod}(R)'$. It follows that, with respect to some basis of $\mathbb{C}^n$,

$$
\phi(f) = \begin{bmatrix} I_m & F \\ 0 & I_{n-m} \end{bmatrix}
$$

for all $f \in \text{Mod}(R)'$. As the subgroup of $\text{GL}(n, \mathbb{C})$ consisting of such matrices is abelian and as $\text{Mod}(R)'$ is perfect, we conclude that $\phi(\text{Mod}(R)')$ is trivial. In particular, we have $\phi(t_c t_d^{-1}) = I$. As $\text{Mod}(S)' = \text{Mod}(S)$ is generated normally by $t_c t_d^{-1}$, we conclude that $\phi(\text{Mod}(S))$ is trivial.

**Case 1.** Suppose that a direct sum of eigenspaces of $L_a$ is a subspace $V$ of $\mathbb{C}^n$ with $2 \leq \dim(V) \leq n - 2$. In this case, $V$ is $\text{Mod}(R)$-invariant, and hence, $\phi(\text{Mod}(S))$ is trivial by the above claim. Notice that if $L_a$ has at least three distinct eigenvalues, then there exists such a subspace.

**Case 2.** Suppose that there is no subspace $V$ as in Case 1. In particular, $L_a$ has at most two eigenvalues and each eigenspace of $L_a$ is either 1-dimensional, or $(n - 1)$-dimensional, or $n$-dimensional. Thus, the Jordan form of $L_a$ is one of the following four matrices:

(i) $\lambda I_n$, (ii) $\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$, (iii) $\begin{bmatrix} \lambda I_{n-2} & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$,

(iv) $\begin{bmatrix} \lambda I_{n-1} & 0 \\ 0 & \mu \end{bmatrix}$.

We fix a basis so that the matrix $L_a$ in this basis is equal to its Jordan form.

In the case (i), if $x$ is a nonseparating simple closed curve on $S$, then $L_x = \lambda I$ because it is conjugate to $L_a$. As $\text{Mod}(S)$ is generated by the Dehn twists about nonseparating simple closed curves, $\phi(\text{Mod}(S))$ is cyclic, and hence trivial.

In the case (ii), the subspace $\ker(L_a - \lambda I)^2$ is a $\text{Mod}(R)$-invariant subspace of dimension 2.

In the cases (iii) and (iv), the eigenspaces $E^a_{\lambda}$ and $E^b_{\lambda}$ are of dimension $n - 1$. If $E^a_{\lambda} \neq E^b_{\lambda}$, then $E^a_{\lambda} \cap E^b_{\lambda}$ is a $\text{Mod}(R)$-invariant subspace of dimension $n - 2$.

Suppose finally that $E^a_{\lambda} = E^b_{\lambda}$. It follows from Lemma 4.2 that $E^x_{\lambda} = E^y_{\lambda}$ for any two simple closed curves $x$ and $y$ on $S$ intersecting transversely at one point. We then conclude from Theorem 2.1 that $E^a_{\lambda} = E^x_{\lambda}$, and hence $L_x$ is of the form

$$
\begin{bmatrix} \lambda I_{n-1} & * \\ 0 & * \end{bmatrix}
$$

for every nonseparating curve $x$. As $\text{Mod}(S)$ is generated by the Dehn twists about nonseparating simple closed curves, $\phi(\text{Mod}(S))$ is contained in the subgroup of $\text{GL}(n, \mathbb{C})$ consisting of upper triangular matrices. As this subgroup is solvable and $\text{Mod}(S)$ is perfect, we conclude from Lemma 3.11 that $\phi(\text{Mod}(S))$ is trivial.

This completes the proof of Theorem 1.
6 | APPLICATIONS OF THEOREM 1

In this section, we prove Theorems 4–6 as applications of Theorem 1.

6.1 | Nonorientable surfaces

Let $\phi : \text{Mod}(N) \to \text{GL}(n, \mathbb{C})$ be a homomorphism, where $N$ is a closed connected nonorientable surface of genus $g \geq 3$ with $r \geq 0$ marked points, and $n \leq g - 2$ for odd $g$ and $n \leq g - 3$ for even $g$.

**Proof of Theorem 4.** If $g = 3$ or $g = 4$ then $n = 1$, and $\text{GL}(1, \mathbb{C}) = \mathbb{C}^*$ is abelian, so that $\phi$ factors through the first homology $H_1(\text{Mod}(N); \mathbb{Z})$. As $H_1(\text{Mod}(N); \mathbb{Z})$ is finite (cf. [21, 24]), the result follows. So, we assume that $g \geq 5$.

Let $T$ denote the subgroup of the mapping class group $\text{Mod}(N)$ generated by the Dehn twists about two-sided simple closed curves on $N$. Write $g = 2h + 1$ if $g$ is odd and $g = 2h + 2$ if $g$ is even. Hence, we have $h \geq 2$ and $n \leq 2h - 1$.

Let $S$ be a compact orientable surface of genus $h$ with one boundary component embedded in $N$, so that the boundary of $S$ bounds a Möbius band (resp., Klein bottle with one boundary) on $N$ with $r$ marked points if $g$ is odd (resp., even). By extending the diffeomorphisms $S \to S$ to $N \to N$ by the identity, we get a homomorphism $\eta : \text{Mod}(S) \to \text{Mod}(N)$. The composition $\phi \eta$ is a homomorphism from $\text{Mod}(S)$ to $\text{GL}(n, \mathbb{C})$:

$$
\begin{array}{ccc}
\text{Mod}(N) & \xrightarrow{\phi} & \text{GL}(n, \mathbb{C}) \\
\downarrow{\eta} & & \\
\text{Mod}(S) & \xrightarrow{\phi \eta} & \\
\end{array}
$$

If $h \geq 3$ (i.e., $g \geq 7$) then $\phi \eta$ is trivial by Theorem 1. It follows that $\phi(t_a) = I_n$ for any Dehn twist $t_a$ supported in $S$. If $b$ is a two-sided nonseparating simple closed curve on $N$ such that the complement of it is nonorientable, it follows from [24, Theorems 3.1 and 5.3] that a Dehn twist $t_b$ is conjugate to a Dehn twist supported in $S$. Hence, $\phi(t_b)$ is trivial for all such $b$. As $T$ is generated by such Dehn twists (cf. proof of [24, Theorem 5.12]), we get that $\phi(T)$ is trivial. We also know that the index of $T$ in $\text{Mod}(N)$ is $r! \cdot 2^{r+1}$ [24, Corollary 6.2]. The conclusion of the theorem now follows.

If $h = 2$ ($g = 5$ or $g = 6$) then the image of $\phi \eta$ is cyclic by Theorem 1. It follows that $\phi(t_at_b^{-1}) = I$ for any two nonseparating simple closed curves on $S$. Let $x$ and $y$ be two two-sided nonseparating simple closed curves on $N$ intersecting at one point such that the complement of each is nonorientable. It can easily be shown that there exists a diffeomorphism $f : N \to N$ such that $f(x \cup y) \subset S$, so that $t_x t_y^{-1}$ can be conjugated to $t_at_b^{-1}$, where $a$ and $b$ are on $S$. Hence, $\phi(t_xt_y^{-1}) = I$, or $\phi(t_x) = \phi(t_y)$. We now apply [24, Theorem 3.1] to conclude that $\phi(t_x) = \phi(t_y)$ for all two-sided nonseparating simple closed curves whose complements are nonorientable. (Such simple closed curves are called essential in [24].) As $T$ is generated by such Dehn twists, we get that $\phi(T)$ is cyclic, so that $\phi(T') = \{I\}$, where $T'$ is the commutator subgroup. Stukow proved that the index of $T'$ in $T$ is $2$ (cf. [31, Theorem 8.1]). We conclude that $\phi(\text{Mod}(N))$ is a finite group of order at most $r! \cdot 2^{r+2}$.

This finishes the proof of Theorem 4. □
Remark 6.1. If \( g \geq 7 \) and if \( N \) is closed, then the above proof, together with the fact that \( H_1(\text{Mod}(N); \mathbb{Z}) = \mathbb{Z}_2 \), shows that the image of \( \phi \) is either trivial or is isomorphic to \( \mathbb{Z}_2 \). In fact, it is easy to find a homomorphism whose image is \( \mathbb{Z}_2 \); send all Dehn twists to 0 and all crosscap slides (also called \( Y \)-homeomorphisms) to 1.

6.2 Homomorphisms to \( \text{Aut}(F_n) \) and \( \text{Out}(F_n) \)

Next, we prove Theorem 5. Let \( S \) be a closed connected oriented surface of genus \( g \geq 2 \) with at most \( 2g - 2 \) marked points. Let \( n \leq 2g - 1 \).

Let \( F_n \) denote the free group of rank \( n \). The action of the automorphism group \( \text{Aut}(F_n) \) of \( F_n \) on \( H_1(F_n; \mathbb{Z}) = \mathbb{Z}^n \) gives rise to a surjective homomorphism \( \eta : \text{Aut}(F_n) \to \text{GL}(n, \mathbb{Z}) \), the kernel of which is denoted by \( IA_n \). Let \( \text{Out}(F_n) \) denote the group of outer automorphisms of \( F_n \), so that it is the quotient of \( \text{Aut}(F_n) \) with the (normal) subgroup \( \text{Inn}(F_n) \) consisting of inner automorphisms. As inner automorphisms of a group \( G \) act trivially on the abelianization of \( G \), \( \eta \) induces a surjective homomorphism \( \text{Out}(F_n) \to \text{GL}(n, \mathbb{Z}) \).

Proof of Theorem 5. The proof in the case \( n = 1 \) is trivial, because \( F_1 = \mathbb{Z} \) and \( \text{Aut}(\mathbb{Z}) = \text{Out}(\mathbb{Z}) = \mathbb{Z}_2 \). So, we assume that \( 2 \leq n \leq 2g - 1 \).

Let \( \varphi : \text{Mod}(S) \to \text{Aut}(F_n) \) be a homomorphism and let \( \phi \) be the composition of \( \varphi \) with \( \eta \), so that we have a commutative diagram

\[
\begin{array}{ccc}
\text{Mod}(S) & \xrightarrow{\varphi} & \text{Aut}(F_n) \\
\downarrow \varphi & & \downarrow \eta \\
1 & \longrightarrow & IA_n \longrightarrow \text{GL}(n, \mathbb{Z}) \longrightarrow 1.
\end{array}
\]

It suffices to prove that \( \phi \) factors through the abelianization map \( \text{Mod}(S) \to H_1(\text{Mod}(S); \mathbb{Z}) \).

By Theorem 1, the image of \( \phi \) is abelian, implying that \( \phi(\text{Mod}(S)') \) is trivial. Thus, \( \varphi(\text{Mod}(S)') \) is contained in \( IA_n \). As \( IA_n \) is torsion-free by a result of Baumslag–Taylor [1] and as \( \text{Mod}(S)' \) is generated by involutions by Theorem 2.6, we conclude that \( \varphi(\text{Mod}(S)') \) is trivial.

This completes the proof of Theorem 5 for \( \text{Aut}(F_n) \).

The case \( \text{Out}(F_n) \) is completely similar and uses the fact that the subgroup \( IA_n/\text{Inn}(F_n) \) is torsion-free (cf. [1]). \( \square \)

6.3 Homomorphisms between mapping class groups

Let \( g > h \) be positive integers, and let \( S \) and \( R \) be compact connected oriented surfaces of genera \( g \) and \( h \), respectively. Suppose that \( S \) has no boundary components and has at most \( 2g - 2 \) marked points, and that \( R \) has finitely many marked points.

Proof of Theorem 6. Let \( \varphi : \text{Mod}(S) \to \text{Mod}(R) \) be a homomorphism. Let \( q : \text{Mod}(R) \to \text{Sp}(2h, \mathbb{Z}) \) be the homomorphism obtained from the action of \( \text{Mod}(R) \) on \( H_1(\tilde{R}; \mathbb{Z}) \), where \( \tilde{R} \) is the closed surface of obtained from \( R \) by gluing a disk along each boundary component and by forgetting...
the marked points. Consider the diagram

\[
\begin{array}{c}
\text{Mod}(S) \\
\downarrow \varphi \\
\ker(q) \\
\downarrow q \\
\text{Mod}(R) \\
\downarrow q \\
\text{Sp}(2h, \mathbb{Z}) \\
\end{array}
\]

The image of the map \( q\varphi \) is abelian by Theorem 1. Hence, the restriction of \( q\varphi \) to \( \text{Mod}(S)' \) is trivial, so that \( \varphi(\text{Mod}(S)') \) is contained in the kernel of \( q \). As the kernel of \( q \) is torsion-free by Proposition 2.9 and as the group \( \text{Mod}(S)' \) is generated by involutions by Theorem 2.6, \( \varphi(\text{Mod}(S)') \) is trivial. Consequently, the image of \( \varphi \) is abelian.

\[\square\]

7 | MORE ON EIGENVALUES OF \( L_u \)

7.1 | A criterion for the triviality of a representation

For a compact connected oriented surface \( S \) of genus \( g \geq 2 \) perhaps with marked points, we give a criterion for the triviality of a representation of the mapping class group into the general linear group \( \text{GL}(m, \mathbb{C}) \), which is inspired by the first version of [10]. So, let \( \phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C}) \) be a homomorphism.

**Lemma 7.1.** If there is a flag

\[0 = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_k = \mathbb{C}^m\]

of \( \text{Mod}(S) \)-invariant subspaces such that \( \dim(W_i/W_{i-1}) \leq 2g - 1 \) for each \( i = 1, 2, \ldots, k \), then the image of \( \phi \) is abelian.

**Proof.** For each \( i = 1, 2, \ldots, k \), choose an ordered basis \( \beta_i \) of \( W_i \) satisfying \( \beta_1 \subset \beta_2 \subset \cdots \subset \beta_k = \beta \).

As each \( W_i \) is \( \text{Mod}(S) \)-invariant, for \( f \in \text{Mod}(S) \), the matrix \( \phi(f) \) in the basis \( \beta \) is of the form

\[
\phi(f) = \begin{bmatrix}
F_1 & * & * & \cdots & * \\
0 & F_2 & * & \cdots & * \\
0 & 0 & F_3 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & F_k
\end{bmatrix}
\]

Then, for each \( i \), the correspondence \( f \mapsto F_i \) is a homomorphism

\[\phi_i : \text{Mod}(S) \to \text{GL}(W_i/W_{i-1}).\]

As \( \dim(W_i/W_{i-1}) \leq 2g - 1 \) and \( g \geq 2 \), the image of \( \phi_i \) is cyclic (trivial if \( g \geq 3 \)) by Theorem 1.

As the image of \( \phi_i \) is abelian, we get \( \phi_i(f) = I \) for all \( f \in \text{Mod}(S)' \) and for all \( i \). Hence, \( \phi(f) \) is upper triangular with 1 along the diagonal for all \( f \in \text{Mod}(S)' \). The subgroup of \( \text{GL}(m, \mathbb{C}) \)
Corollary 7.2. Let \( R \) be a compact connected oriented surface of genus \( g - 1 \geq 3 \) and let \( \phi : \text{Mod}(R) \to \text{GL}(2g, \mathbb{C}) \) be a homomorphism. If there is a \( \text{Mod}(R) \)-invariant subspace \( W \) with \( 3 \leq \dim(W) \leq 2g - 3 \), then \( \phi \) is trivial.

Proof. The flag \( 0 \subset W \subset \mathbb{C}^{2g} \) is a \( \text{Mod}(R) \)-invariant subspaces with \( \dim(W) \leq 2(g - 1) - 1 \) and \( \dim(\mathbb{C}^{2g}/W) \leq 2(g - 1) - 1 \). Now Lemma 7.1 applies and \( \text{Mod}(R) \) is perfect. \( \square \)

7.2 Eigenvalues of \( L_a \) and their multiplicities

Let \( \phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C}) \) be a homomorphism. Recall that we write \( L_a \) for \( \phi(t_a) \) and \( E^a_{\lambda} \) for the eigenspace of \( L_a \) corresponding to an eigenvalue \( \lambda \).

Lemma 7.3. Let \( g \geq 3 \) and let \( a \) be a nonseparating simple closed curve on \( S \). If \( L_a \) has only one eigenvalue \( \lambda \) with \( \dim(E^a_{\lambda}) = m - 1 \), then \( \lambda = 1 \).

Proof. Suppose that \( L_a \) has only one eigenvalue \( \lambda \) and \( \dim(E^a_{\lambda}) = m - 1 \). As \( g \geq 3 \), we can choose six nonseparating simple closed curves \( c_1, c_2, c_3, c_4, c_5, c_6 \) on \( S \) disjoint from \( a \) such that \( a, c_1, c_2, c_3 \) bound a sphere with four boundary components and that the Dehn twists about them satisfy lantern relation

\[
t_a t_{c_1} t_{c_2} t_{c_3} = t_{c_4} t_{c_5} t_{c_6}.
\]

Let \( \beta \) be an ordered basis of \( \mathbb{C}^m \) whose first \( m - 1 \) elements are in \( E^a_{\lambda} \), so that

\[
L_a = \begin{bmatrix}
\lambda I_{m-1} & * \\
0 & \lambda
\end{bmatrix}
\]

in the basis \( \beta \). As each \( c_i \) is nonseparating, \( L_{c_i} \) is conjugate to \( L_a \) and, hence, \( \lambda \) is the only eigenvalue of \( L_{c_i} \). As \( L_{c_i} \) commutes with \( L_a \), we have \( L_{c_i}(E^a_{\lambda}) = E^a_{\lambda} \). Thus, the matrix of \( L_{c_i} \) in the basis \( \beta \) is of the form

\[
L_{c_i} = \begin{bmatrix}
M_i & * \\
0 & \lambda
\end{bmatrix}.
\]

Now the lantern relation (4) gives

\[
L_a L_{c_1} L_{c_2} L_{c_3} = L_{c_4} L_{c_5} L_{c_6}
\]

which implies that \( \lambda^4 = \lambda^3 \). As the matrix \( L_a \) is invertible, \( \lambda \neq 0 \), so that \( \lambda = 1 \). \( \square \)

Lemma 7.4. Let \( g \geq 3 \) and let \( a \) be a nonseparating simple closed curve on \( S \). If \( \lambda \) is an eigenvalue of \( L_a \) with the multiplicity \( \lambda_\# \leq 2g - 3 \), then \( \lambda = 1 \) and the dimension of the eigenspace \( E^a_{\lambda} \) is \( \lambda_\# \).
Proof. If \( m \leq 2g - 1 \), then \( \phi \) is trivial by Theorem 1. In particular, \( L_\alpha \) has only one eigenvalue \( \lambda = 1 \) and \( E_1^\alpha = \mathbb{C}^m \). Thus, we assume that \( m \geq 2g \).

Let \( \lambda, \lambda_1, \lambda_2, \ldots, \lambda_s \) be all distinct eigenvalues of \( L_\alpha \). Set

\[
K = \ker(L_\alpha - \lambda I)^m \quad \text{and} \quad K' = \bigoplus_{i=1}^s \ker(L_\alpha - \lambda_i I)^m
\]

so that \( \dim(K) = \lambda_# \), and that \( \mathbb{C}^m = K \oplus K' \). Let \( \beta_1 \) be a basis \( K \) and \( \beta_2 \) be a basis of \( K' \) such that with respect to the basis \( \beta = \beta_1 \cup \beta_2 \) of \( \mathbb{C}^m \), the matrix of \( L_\alpha \)

\[
L_\alpha = \begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}
\]

is in the Jordan form, where \( A \in \text{GL}(\lambda_#, \mathbb{C}) \).

Let \( R \) be the complement of a regular neighborhood \( N \) of the curve \( a \). Extending self-diffeomorphisms of \( R \) to \( S \) by the identity on \( N \) gives a homomorphism \( q : \text{Mod}(R) \to \text{Mod}(S) \). Let \( \psi = \phi q \) and let \( a' \) be a simple closed curve on \( R \) isotopic to \( a \) on \( S \). For \( f \in \text{Mod}(R) \), \( \psi(f) \) commutes with \( L_\alpha \) so that it preserves the subspaces \( K \) and \( K' \). Hence, the matrix of \( \psi(f) \) in the basis \( \beta \) is of the form

\[
\psi(f) = \begin{bmatrix} F & 0 \\ 0 & F' \end{bmatrix}
\]

where \( F \in \text{GL}(\lambda_#, \mathbb{C}) \).

Now, the correspondence \( f \mapsto F \) defines a homomorphism \( \psi' : \text{Mod}(R) \to \text{GL}(K) = \text{GL}(\lambda_#, \mathbb{C}) \). As \( \lambda_# \leq 2g - 3 = 2(g - 1) - 1 \) and the genus of \( R \) is \( g - 1 \geq 2 \), the image of the map \( \psi' \) is cyclic by Theorem 1. It is easy to find six simple closed curves \( b, c, d, x, y, z \) that are nonseparating on \( R \) such that there is the lantern relation \( t_a t_b t_c t_d = t_x t_y t_z \). It can be seen easily that each of \( t_x t_b^{-1}, t_y t_c^{-1} \) and \( t_z t_d^{-1} \) is a commutator in \( \text{Mod}(R) \), so that

\[
t_{a'} = (t_x t_b^{-1})(t_y t_c^{-1})(t_z t_d^{-1})
\]

is contained in the commutator subgroup of \( \text{Mod}(R) \). In particular, \( \psi'(t_{a'}) = I \). As a result of this, we have

\[
L_\alpha = L_{a'} = \begin{bmatrix} \psi'(t_{a'}) & 0 \\ 0 & A' \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A' \end{bmatrix}.
\]

Hence, \( \lambda = 1 \) and \( \dim(E_\lambda^\alpha) = \lambda_# \). \( \square \)

**Corollary 7.5.** Let \( g \) and \( a \) be as in Lemma 7.4. If \( m \leq 4g - 4 \) then \( L_\alpha \) has at most two distinct eigenvalues.
7.3 The main lemma

A major step in the proof of Theorem 2 is the next lemma, which is also used in the proof of Theorem 3. Consider the curves on $S$ given in Figure 2.

**Lemma 7.6.** Let $g \geq 1$, $m \geq 2g$ and let $\phi : \text{Mod}(S) \to \text{GL}(m, \mathbb{C})$ be a homomorphism. Suppose that $a$ is a nonseparating simple closed curve on $S$ such that $L_a$ has only one eigenvalue $\lambda = 1$ and that the dimension of the eigenspace $E^a_1$ is $m - 1$. Suppose also that there exists a simple closed curve $b$ intersecting $a$ transversely at one point such that $E^a_1 \neq E^b_1$. Then there is a basis of $\mathbb{C}^m$ with respect to which

$$
L_{a_i} = \begin{bmatrix} A_i & 0 \\ 0 & I \end{bmatrix} \text{ and } L_{b_i} = \begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix},
$$

where $I$ is the identity matrix of size $(m - 2g) \times (m - 2g)$.

**Proof.** We prove the lemma in five steps.

**Step 1:** For $i = 1, 2, \ldots, g$, we set

$$
\widetilde{A}_i = \begin{bmatrix} A_i & 0 \\ 0 & I \end{bmatrix}, \quad \widetilde{B}_i = \begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix},
$$

$L_{2i-1} = L_{a_i}$, $L_{2i} = L_{b_i}$, $E^{2i-1} = E^a_1$ and $E^{2i} = E^b_1$. As $E^a_1 \neq E^b_1$ and because there exists a diffeomorphism mapping $(a, b)$ to $(a_i, b_i)$, we have $E^{2i-1} \neq E^{2i}$ by Lemma 4.2. We also have the braid relation

$$
L_{2i-1}L_{2i}L_{2i-1} = L_{2i}L_{2i-1}L_{2i}
$$

for $1 \leq i \leq g$, and $\dim(E^j) = m - 1$ for $1 \leq j \leq 2g$.

**Step 2:** We first prove that we may choose $L_1 = \widetilde{A}_1$ and $L_2 = \widetilde{B}_1$ in some basis. As $E^1 \neq E^2$, the subspace $W_1 = E^1 \cap E^2$ is of dimension $m - 2$. Let $\{w_3, w_4, \ldots, w_m\}$ be a basis of $W_1, w_1 \in E^1 \setminus E^2$ and $w_2 \in E^2 \setminus E^1$. With respect to the ordered basis

$$
\beta_0 = \{w_1, w_2, \ldots, w_m\}
$$

of $\mathbb{C}^m$, we have

$$
L_1 = \begin{bmatrix} 1 & x_1 \\ 0 & 1 \\ 0 & X \end{bmatrix} \begin{bmatrix} 0 \\ I_{m-2} \end{bmatrix}, \quad L_2 = \begin{bmatrix} 1 & 0 \\ 0 & Y \\ Y' & 0 \end{bmatrix} \begin{bmatrix} 0 \\ I_{m-2} \end{bmatrix},
$$

where $X = [x_3, x_4, \ldots, x_m]^t$.

If $x_1 = 0$ we conclude from the braid relation (5) that $y'_2 = 0$. But then $L_1$ and $L_2$ commute, and hence $L_1 = L_2$, contradicting to $E^1 \neq E^2$. Therefore, $x_1$ is nonzero.
Let
\[ v_1 = x_1 w_1 + x_3 w_3 + x_4 w_4 + \cdots + x_m w_m. \]
In the basis
\[ \{v_1, w_2, w_3, w_4, \ldots, w_{n-1}, w_m\}, \]
we have
\[
L_1 = \begin{bmatrix} U & 0 \\ 0 & I_{m-2} \end{bmatrix} = \tilde{A}_1, \quad L_2 = \begin{bmatrix} 1 & 0 \\ y_2 & 1 \\ Y & 0 \end{bmatrix} I_{m-2},
\]
where \( Y = [y_3 \ y_4 \ \cdots \ y_m]^t \). The braid relation (5) then gives \( y_2 = -1 \). If we let
\[ v_2 = w_2 - (y_3 w_3 + y_4 w_4 + \cdots + y_m w_m), \]
in the basis
\[ \beta_1 = \{ v_1, v_2, w_3, w_4, \ldots, w_{m-1}, w_m \} \]
we have \( L_1 = \tilde{A}_1 \) and \( L_2 = \tilde{B}_1 \).

**Step 3:** Suppose that \( k < g \) and that there is a basis
\[ \beta_k = \{ v_1, v_2, \ldots, v_{m-1}, v_m \} \]
with respect to which
\[ L_{2i-1} = \tilde{A}_i \text{ and } L_{2i} = \tilde{B}_i \]
for all \( i = 1, 2, \ldots, k \). Note that in this case the subset
\[ \alpha = \{ v_{2k+1}, v_{2k+2}, \ldots, v_{m-1}, v_m \} \]
of \( \beta_k \) is contained in \( W_k = \bigcap_{i=1}^{2k} E_i \). It can easily be shown that, indeed, \( \alpha \) is a basis for \( W_k \), so that \( \dim(W_k) = m - 2k \).

**Step 4:** Next, we consider \( L_{2k+1} \) and \( L_{2k+2} \). Let \( s \in \{2k + 1, 2k + 2\} \). As the subspace \( W_k \) is \( L_s \)-invariant, with respect to the basis \( \beta_k \),
\[
L_s = \begin{bmatrix} Z_s & 0 \\ Y_s & X_s \end{bmatrix},
\]
where \( Z_s \in \text{GL}(2k, \mathbb{C}) \). As \( L_s \) commutes with each one of
\[
L_{2i-1} = \begin{bmatrix} A_i & 0 \\ 0 & I \end{bmatrix} \text{ and } L_{2i} = \begin{bmatrix} B_i & 0 \\ 0 & I \end{bmatrix},
\]
for \( i = 1, 2, \ldots, k \), where the matrix \( \tilde{A}_i \) is the \( 2k \times 2k \) block diagonal matrix \( \text{Diag}(I_2, \ldots, I_2, U, I_2, \ldots, I_2) \) whose \( i^{th} \) block is \( U \) and \( \tilde{B}_i \) is obtained from \( \tilde{A}_i \) by replacing \( U \) with \( \hat{U} \), we get that

- \( Z_s \tilde{A}_i = \tilde{A}_i Z_s, Y_s \tilde{A}_i = Y_s \)
- \( Z_s \tilde{B}_i = \tilde{B}_i Z_s, Y_s \tilde{B}_i = Y_s \)

for each \( i \). As 1 is the only eigenvalue of \( L_s \), we conclude from Lemma 3.3 that \( Z_s = I_{2k} \) and \( Y_s = 0 \), so that

\[
L_s = \begin{bmatrix}
I_{2k} & 0 \\
0 & X_s
\end{bmatrix}.
\]

In particular, \( v_1, v_2, \ldots, v_{2k} \) are eigenvectors of \( L_s \).

If \( W_k \) were a subspace of \( E^s \), then we would have \( L_s = I_m \). By this contradiction, both of the subspaces \( W_k \cap E^{2k+1} \) and \( W_k \cap E^{2k+2} \) are of dimension \( m - 2k - 1 \). If, furthermore, we had \( W_k \cap E^{2k+1} = W_k \cap E^{2k+2} \), then we would conclude that \( E^{2k+1} = E^{2k+2} \), again arriving at a contradiction. Hence, the subspaces \( W_k \cap E^{2k+1} \) and \( W_k \cap E^{2k+2} \) are different, so that

\[
W_{k+1} = W_k \cap E^{2k+1} \cap E^{2k+2}
\]
is of dimension \( m - 2k - 2 \).

Let \( \{w_{2k+3}, w_{2k+4}, \ldots, w_m\} \) be a basis of \( W_{k+1} \) and choose two vectors \( w_{2k+1} \) and \( w_{2k+2} \) with

- \( w_{2k+1} \in W_k \cap E^{2k+1}, w_{2k+1} \notin W_{2k+1} \),
- \( w_{2k+2} \in W_k \cap E^{2k+2}, w_{2k+2} \notin W_{2k+1} \).

Then \( \{w_{2k+1}, w_{2k+2}, w_{2k+3}, w_{2k+4}, \ldots, w_{m-1}, w_m\} \) is a basis of \( W_k \). Now consider the basis

\[
\tilde{\beta}_k = \{v_1, v_2, \ldots, v_{2k}, w_{2k+1}, w_{2k+2}, \ldots, w_{m-1}, w_m\}
\]
of \( C^m \). With respect to this basis,

\[
L_{2i-1} = \tilde{A}_i, \ L_{2i} = \tilde{B}_i \text{ for } i = 1, 2, \ldots, k,
\]

and

\[
L_{2k+1} = \begin{bmatrix}
I_{2k} & 0 & 0 \\
0 & X_1 & 0 \\
0 & X_2 & I
\end{bmatrix} \text{ and } L_{2k+2} = \begin{bmatrix}
I_{2k} & 0 & 0 \\
0 & Y_1 & 0 \\
0 & Y_2 & I
\end{bmatrix},
\]

where

\[
X_1 = \begin{bmatrix}
1 & x \\
0 & 1
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
0 & 0 & \ldots & 0 \\
x_{2k+3} & x_{2k+4} & \ldots & x_m
\end{bmatrix}^t, \quad Y_1 = \begin{bmatrix}
y' \\
0
\end{bmatrix}, \quad \text{ and } \quad Y_2 = \begin{bmatrix}
y'_{2k+3} & y'_{2k+4} & \ldots & y'_m
\end{bmatrix}^t.
\]

The rest of the proof proceeds as above: if \( x = 0 \) then the braid relation (5) gives \( y' = 0 \), so that \( L_{2k+1} \) and \( L_{2k+2} \) commute. Hence, \( L_{2k+1} = L_{2k+2} \), a contradiction. Thus, \( x \neq 0 \).
Define
\[ w'_{2k+1} = xw_{2k+1} + (x_2w_{2k+3} + x_2w_{2k+4} + \cdots + x_m w_m) \]
and let \( \tilde{\beta}'_k \) be the basis obtained from \( \tilde{\beta}_k \) by replacing \( w_{2k+1} \) with \( w'_{2k+1} \). With respect to \( \tilde{\beta}'_k \), the matrices of \( L_1, \ldots, L_{2k} \) are the same, and \( L_{2k+1} = \tilde{A}_{k+1} \). The matrix of \( L_{2k+2} \) turns into a new matrix of the form (6), where \( y' \) is replaced by some \( y \), and \( y'_j \) is replaced by some \( y_j \). The braid relation (5) then implies that \( y = -1 \). If we let
\[ w'_{2k+2} = w_{2k+2} - (y_{2k+3}w_{2k+3} + y_{2k+4}w_{2k+4} + \cdots + y_m w_m) \]
and
\[ \tilde{\beta}_{k+1} = \{ v_1, v_2, \ldots, v_{2k}, w'_{2k+1}, w'_{2k+2}, w_{2k+3}, w_{2k-4}, \ldots, w_{m-1}, w_m \}, \]
we have \( L_{2i-1} = \tilde{A}_i \) and \( L_{2i} = \tilde{B}_i \) for all \( 1 \leq i \leq k+1 \) with respect to \( \tilde{\beta}_{k+1} \).

**Step 5:** By repeating the argument in Step 4 for \( k = 1, 2, 3, \ldots, g-1 \), we see that, with respect to some basis \( \beta'_g \) of \( \mathbb{C}^m \), \( L_{2i-1} = \tilde{A}_i \) and \( L_{2i} = \tilde{B}_i \) for all \( 1 \leq i \leq g \).

This finishes the proof of the lemma. \( \square \)

### 8 | REPRESENTATIONS INTO \( \text{GL}(2g, \mathbb{C}) \)

Let \( S \) be a compact connected oriented surface of genus \( g \geq 3 \) with \( b \geq 0 \) boundary components and with \( r \geq 0 \) marked points in the interior, and let
\[ \phi : \text{Mod}(S) \to \text{GL}(2g, \mathbb{C}) \]
be a homomorphism.

We fix the basis \( \{a_1, b_1, \ldots, a_g, b_g\} \) for \( H_1(S; \mathbb{Z}) \) (cf. Figure 2). Note that if \( P : \text{Mod}(S) \to \text{GL}(2g, \mathbb{C}) \) is the composition of the maps
\[ \text{Mod}(S) \to \text{Mod}(\tilde{S}) \to \text{Sp}(2g, \mathbb{Z}) \hookrightarrow \text{GL}(2g, \mathbb{C}), \]
then \( P(t_{a_i}) = A_i, P(t_{b_i}) = B_i, P(t_{c_j}) = C_{j-1} \) and \( P(t_{e_k}) = A_1 \) for \( 1 \leq i \leq g, 1 \leq j \leq g-1 \) and \( 0 \leq k \leq b + r \).

We prove in this section that either \( \phi \) is trivial or \( \phi(t_x) = P(t_x) \) for \( x \in \{a_i, b_i, c_j, e_k\} \) with respect to a suitable basis of \( \mathbb{C}^{2g} \). As Mod(S) is generated by these Dehn twists, it follows that \( \phi \) is conjugate to \( P \).

Recall that, for a simple closed curve \( a \), \( E^a_\lambda \) denotes the eigenspace corresponding to an eigenvalue \( \lambda \) of \( L_a = \phi(t_a) \) and \( \lambda_\# \) denotes the multiplicity of \( \lambda \).

#### 8.1 | Three lemmas

**Lemma 8.1.** Let \( g \geq 4 \), let \( a \) be a nonseparating simple closed curve on \( S \) and let \( \lambda \) be an eigenvalue of \( L_a \). If \( \lambda_\# \geq 3 \) then \( \dim(E^a_\lambda) \geq 2g - 1 \). In particular, \( \lambda_\# \geq 2g - 1 \).
Proof. Let $b$ be a simple closed curve on $S$ intersecting $a$ transversely at one point, and let $R$ denote the complement of a regular neighborhood of $a \cup b$, so that it is a subsurface of genus $g - 1$. Then $\text{Mod}(R)$ injects into $\text{Mod}(S)$. By identifying $\text{Mod}(R)$ with its image, we assume that $\text{Mod}(R)$ is a subgroup of $\text{Mod}(S)$.

Suppose first that $\dim(E^a_{\lambda}) \leq 2g - 3$. Define a subspace $W$ by

$$W = \begin{cases} 
\ker(L_a - \lambda I)^4, & \text{if } \lambda \not\in (2g - 3), \\
\ker(L_a - \lambda I)^3, & \text{if } \lambda \not\in (2g - 2) \text{ and } \dim(E^a_{\lambda}) = 1, \\
\ker(L_a - \lambda I)^2, & \text{if } \lambda \not\in (2g - 2) \text{ and } \dim(E^a_{\lambda}) = 2, \\
E^a_{\lambda}, & \text{if } \lambda \not\in (2g - 2) \text{ and } 3 \leq \dim(E^a_{\lambda}) \leq 2g - 3.
\end{cases}$$

Note that the dimension of $W$ satisfies $3 \leq \dim(W) \leq 2(g - 1) - 1$. The elements of $\text{Mod}(R)$ commute with the Dehn twist $t_a$, so that $W$ is $\text{Mod}(R)$-invariant. As the genus of $R$ is $g - 1 \geq 3$, $\phi(\text{Mod}(R))$ is trivial by Corollary 7.2. It follows that $L_a = I$. This says, in particular, that $\dim(E^a_{\lambda}) = 2g$, contradicting to the assumption.

Suppose now that $\dim(E^a_{\lambda}) = 2g - 2$. If $E^a_{\lambda} \neq E^b_{\lambda}$, then $E^a_{\lambda} \cap E^b_{\lambda}$ is a $\text{Mod}(R)$-invariant subspace of dimension $2g - 3$ or $2g - 4$. Hence, $\phi$ is trivial on $\text{Mod}(R)$ by Corollary 7.2. We conclude, again, that $L_a = I$, arriving at a contradiction. If $E^a_{\lambda} = E^b_{\lambda}$ then by Lemma 4.3 the eigenspace $E^a_{\lambda}$ is a $\text{Mod}(S)$-invariant subspace of dimension $2g - 2$, so that $0 \subset E^a_{\lambda} \subset \mathbb{C}^{2g}$ is a $\text{Mod}(S)$-invariant flag. Now Lemma 7.1 applies to conclude that $\phi$ is trivial, arriving at a contradiction again.

Therefore, the dimension of the eigenspace $E^a_{\lambda}$ is at least $2g - 1$, finishing the proof of the lemma.

\begin{lemma}
Let $g \geq 4$, let $a$ be a nonseparating simple closed curve on $S$ and let $\lambda$ be an eigenvalue of $L_a$. If $\dim(E^a_{\lambda}) \geq 2g - 1$ then $\lambda = 1$.
\end{lemma}

\begin{proof}
As the genus of $S$ is at least 4, we can choose seven nonseparating simple closed curves $a = x_1, x_2, x_3, x_4, x_5, x_6, x_7$ on $S$ such that the Dehn twists about which satisfy the lantern relation

$$t_{x_1}t_{x_2}t_{x_3}t_{x_4} = t_{x_5}t_{x_6}t_{x_7},$$

so that

$$L_{x_1}L_{x_2}L_{x_3}L_{x_4} = L_{x_5}L_{x_6}L_{x_7}. \tag{7}$$

Choose a nonzero vector $v$ in the nontrivial subspace $\bigcap_{i=1}^{7} E^a_{x_i}$. Evaluating both sides of (7) at $v$ gives $\lambda^4 v = \lambda^3 v$, giving the desired result $\lambda = 1$.
\end{proof}

\begin{lemma}
If $g = 3$ and if $a$ is a nonseparating simple closed curve on $S$, then $L_a$ has only one eigenvalue $\lambda = 1$ and $\dim(E^a_{\lambda})$ is equal to 5 or 6.
\end{lemma}

\begin{proof}
We may assume that $\phi$ is nontrivial. Let $b$ be a simple closed curve on $S$ intersecting $a$ transversely at one point and let $R$ be the complement of a regular neighborhood of $a \cup b$. Choose two simple closed curves $x_1$ and $x_2$ on $R$ intersecting each other transversely at one point.

\end{proof}
The element $L_a \in \text{GL}(6, \mathbb{C})$ has at most two distinct eigenvalues by Corollary 7.5. Suppose first that $L_a$ has two distinct eigenvalues $\lambda$ and $\mu$ with $\lambda \mu \geq \mu^2$. By Lemma 7.4, we have $\lambda \mu \geq 4, \mu^2 \leq 2$, $\mu = 1$ and $\dim(E_{\lambda}^a) = \mu^2$. Let $k = \dim(E_{\lambda}^a)$. Consider the following $\text{Mod}(R)$-invariant flags:

- $0 \subset \ker(L_a - \lambda I)^3 \subset \mathbb{C}^6$ if $k = 1$,
- $0 \subset \ker(L_a - \lambda I) \subset \ker(L_a - \lambda I)^2 \subset \mathbb{C}^6$ if $k = 2$,
- $0 \subset E^a_{\lambda} \subset \mathbb{C}^6$ if $k = 3$.

Thus, in the cases $k \leq 3$, Lemma 7.1 gives that the commutator subgroup of $\text{Mod}(R)$ is contained in the kernel of $\phi$. In particular, $\phi(t \cdot x)^{-1} = I$. As the group $\text{Mod}(S)$ is generated normally by $t \cdot x$, the map $\phi$ is trivial, giving a contradiction.

In the case $k = 4$, the flag $0 \subset E^a_{\lambda} \cap E^b_{\lambda} \subset E^a_{\lambda} \subset \mathbb{C}^6$ is $\text{Mod}(R)$-invariant if $E^a_{\lambda} \neq E^b_{\lambda}$, and the flag $0 \subset E^a_{\lambda} \subset \mathbb{C}^6$ is $\text{Mod}(S)$-invariant if $E^a_{\lambda} = E^b_{\lambda}$, both implying that $\phi$ is trivial, a contradiction again.

Suppose now that $k = 5$. Fix a basis of $\mathbb{C}^6$ such that

$$L_a = \begin{bmatrix} \lambda I_5 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

As $H_1(\text{Mod}(S); \mathbb{Z})$ is trivial, by composing $\phi$ with the determinant function $\text{GL}(6, \mathbb{C}) \to \mathbb{C}$ we see that $\lambda^5 = 1$. For any simple closed curve $x$ disjoint from $a$, the matrix of $\phi(t \cdot x) = L_x$ is of the form

$$L_x = \begin{bmatrix} \ast & 0 \\ 0 & \gamma_x \end{bmatrix}$$

for some nonzero complex number $\gamma_x$. If $x$ is nonseparating, then $\gamma_x$ is either 1 or $\lambda$, as it is an eigenvalue of $L_x$.

It is easy to see that there is a nonseparating simple closed curve $x_3$ on $S$ with the following properties: the curve $x_3$ is disjoint from $a$ and $x_1$, it intersects $x_2$ and $b$ transversely once, and a regular neighborhood of $x_1 \cup x_2 \cup x_3$ is a torus with two boundary components $a$ and another curve, say $a'$ (cf. Figure 3). We then have the two-holed torus relation $(t \cdot x_1 t \cdot x_2 t \cdot x_3)^4 = t \cdot a t \cdot a'$.

From the braid relations, we get $\gamma_{x_1} = \gamma_{x_2} = \gamma_{x_3}$. If $\gamma_{x_1} = 1$ then $E^a_{\lambda} = E^b_{\lambda} = E^a_{\lambda}$. It follows that $E^a_{\lambda} = E^a_{\lambda}$ for all nonseparating curve $x$ on $S$, so that $E^a_{\lambda}$ is $\text{Mod}(S)$-invariant. One may deduce from this that $\phi$ is trivial, a contradiction. If $\gamma_{x_1} = \lambda$, then the relation $(t \cdot x_1 t \cdot x_2 t \cdot x_3)^4 = t \cdot a t \cdot a'$ and $\lambda^5 = 1$ imply that $\gamma_{a'} = \lambda^2$, which is an eigenvalue of $L_{a'}$. On the other hand, as the eigenvalues of $L_{a'}$ are 1 and $\lambda$, we conclude that $\lambda = 1$, a contradiction again.

From these contradictions, we get that $L_a$ must have only one eigenvalue, say $\lambda$.

If $\dim(E^a_{\lambda}) \leq 4$ then one arrives at a contradiction by deducing that $\phi$ must be trivial, as in the proof of Lemma 8.1. If $\dim(E^a_{\lambda}) = 6$ then $\phi$ is trivial. If $\dim(E^a_{\lambda}) = 5$, then Lemma 7.3 gives $\lambda = 1$. □
8.2 Proof of Theorem 2

We give the proof in several steps. Recall that, for a simple closed curve $x$ on $S$, we denote $\phi(t_x)$ by $L_x$. Let $a$ and $b$ be two simple closed curve on $S$ intersecting transversely at one point.

**Step 1:** We claim that $L_a$ has a unique eigenvalue $\lambda = 1$ and that $\dim(E_1^a)$ is either $2g - 1$ or $2g$.

The case $g = 3$ is proved in Lemma 8.3. So, we assume that $g \geq 4$.

First of all, $L_a$ has at most two eigenvalues by Corollary 7.5. Suppose that it has two distinct eigenvalues $\lambda$ and $\mu$ with multiplicities $\lambda_\# \geq \mu_\#$, so that $\mu_\# \leq g \leq \lambda_\#$. We now get from Lemma 7.4 that $\mu = 1$, hence $\lambda \neq 1$. On the other hand, by Lemma 8.1, $\lambda_\# = 2g - 1$ and the dimension of the eigenspace $E_1^a$ is $2g - 1$. Now Lemma 8.2 implies that $\lambda = 1$, giving the desired contradiction. Hence, $L_a$ must have only one eigenvalue, say $\lambda$. Therefore, $L_x$ has a unique eigenvalue $\lambda$ for every nonseparating curve $x$.

We now have that $\dim(E_1^a)$ is equal to either $2g - 1$ or $2g$ by Lemma 8.1, and that $\lambda = 1$ by Lemma 8.2.

**Step 2:** Suppose first that $\dim(E_1^a) = 2g$, so that $\phi(t_a) = I_{2g}$. As the group Mod$(S)$ is generated by the Dehn twists about nonseparating simple closed curves, which are conjugate to $t_a$, $\phi$ is trivial.

Suppose now that $\dim(E_1^a) = 2g - 1$.

**Step 3:** We claim that $E_1^a \neq E_1^b$. If $E_1^a = E_1^b$, then by Lemma 4.3 the subspace $E_1^a$ is Mod$(S)$-invariant. It follows from Lemma 7.1 and from the fact that Mod$(S)$ is perfect, the image of $\phi$ is trivial, a contradiction.

**Step 4:** By Lemma 7.6, there is a basis $\beta = \{v_1, w_1, v_2, w_2, \ldots, v_g, w_g\}$ of $C^{2g}$ with respect to which $L_{a_i} = A_i$ and $L_{b_i} = B_i$ for each $i = 1, 2, \ldots, g$. For each $k = 1, 2, \ldots, g - 1$, as the curve $c_k$ is disjoint from $a_i$ and $b_j$ for $j \neq k, k + 1$ and because it intersects $b_k$ and $b_{k+1}$ transversely at one point, from the relations between the corresponding Dehn twists we get

- $L_{c_k} A_i = A_i L_{c_k}$ for all $i$,
- $L_{c_k} B_j = B_j L_{c_k}$ for $j \neq k, k + 1$, and
- $L_{c_k} B_j L_{c_k} B_j = B_j L_{c_k} B_j$ for $j = k, k + 1$.

We also know that $L_{c_k}$ has only one eigenvalue $\lambda = 1$ because it is conjugate to $L_{a_i}$. By Lemma 3.8, we get that $L_{c_k} = C_{k,x_k}$ for some nonzero $x_k$.

If the basis $\beta$ is changed to the basis

$$\{q_1 v_1, q_1 w_1, q_2 v_2, q_2 w_2, \ldots, q_g v_g, q_g w_g\}$$

the matrix $L_{c_k}$ becomes $C_{k,y_k}$ for each $k$, where $y_k = x_k q_{k+1} / q_k$. In the new basis, we still have $L_{a_i} = A_i$ and $L_{b_i} = B_i$. If we choose

$$q_g = 1 \text{ and } q_k = (-1)^{g-k} x_k x_{k+1} \cdots x_{g-1},$$

we get that $L_{c_k} = C_{k,-1}$ for every $k$.

**Step 5:** Finally, as the curve $e_k$ intersects $b_1$ transversely once and is disjoint from all $a_i$ and $b_j$ for $j \geq 2$, it follows from Lemma 3.6 that $L_{e_k} = A_1$. Hence, $\phi(t_x) = P(t_x)$ for all $x \in \{a_i, b_i, c_j, e_k\}$ in a basis of $C^{2g}$. As Dehn twists about these curves generate the group Mod$(S)$, we conclude that $\phi = P$.

This concludes the proof of Theorem 2.
Our aim in this section is to prove Theorem 3. We do this by proving a slightly stronger result, Theorem 9.1.

For a surface $\Sigma$ of genus $g \geq 2$ with one boundary component, let $K_\Sigma$ denote the (normal) subgroup of $\text{Mod}(\Sigma)$ generated by all Dehn twists about curves on $\Sigma$ bounding a torus with one boundary component. Note that if $g \geq 2$, the subgroup $K_\Sigma$ contains free groups (cf. [16]), so that its derived subgroups $K^{(k)}_\Sigma$ are nontrivial for all nonnegative integers $k$.

If the genus $S$ is at least 4 and if the genus of $\Sigma$ is 3, then we may embed $\Sigma$ into $S$, so that the embedding induces an injection $\text{Mod}(\Sigma) \hookrightarrow \text{Mod}(S)$. We then may regard $\Sigma$ as a subsurface of $S$ and $\text{Mod}(\Sigma)$ as a subgroup of $\text{Mod}(S)$. Therefore, Theorem 3 follows from Theorem 9.1.

**Theorem 9.1.** Let $n \geq 0$ be an integer, let $S$ be a compact connected oriented surface of genus $g \geq n + 3$, perhaps with marked marked points in the interior, and let $\phi : \text{Mod}(S) \to \text{GL}(2g + n, \mathbb{C})$ be a homomorphism. Then the $n$th derived subgroup $K^{(n)}_\Sigma$ of $K_\Sigma$ is contained in the kernel of $\phi$ for any genus-3 subsurface $\Sigma$ with one boundary component.

**Proof.** We prove the theorem by induction on $n$. If $n = 0$ then $g \geq 3$ and Dehn twists about separating curves are in the kernel of $\phi$ by Theorem 2. Hence, the theorem is true in this case.

Let $n \geq 1$, $g \geq n + 3$ and let $\phi : \text{Mod}(S) \to \text{GL}(2g + n, \mathbb{C})$ be a homomorphism. Suppose that the theorem is true for $n - 1$. If $\Sigma$ and $\Sigma'$ are two genus-3 subsurfaces $S$ each with one boundary component, then there is a self-diffeomorphism of $S$ mapping $\Sigma$ to $\Sigma'$. It follows that the subgroups $\text{Mod}(\Sigma)$ and $\text{Mod}(\Sigma')$ of $\text{Mod}(S)$ are conjugate. We conclude that it suffices to prove the theorem for some genus-3 subsurface.

Choose two simple closed curves $a$ and $b$ intersecting each other transversely at one point and let $R$ denote the complement of a regular neighborhood of $a \cup b$, so that it is a compact surface of genus $g - 1$. As usual, by extending diffeomorphisms $R$ to $S$ by the identity, we regard $\text{Mod}(R)$ as a subgroup of $\text{Mod}(S)$. As $2g + n < 4g - 4$, $\phi(t_a) = L_a$ has at most two (distinct) eigenvalues by Corollary 7.5.

**Case 1. $L_a$ has two eigenvalues.** Suppose that $L_a$ has two eigenvalues $\lambda$ and $\mu$, with multiplicities $\lambda_\mu \geq \mu_\#$. As $\lambda_\# + \mu_\# = 2g + n \leq 3g - 3$, we have $\mu_\# < 2g - 3$. We deduce from Lemma 7.4 that $\mu = 1$ and $\dim(E_1^a) = \mu_\#$.

Let $\beta_1$ and $\beta_2$ be ordered bases of $\ker(L_a - \lambda I)^{\lambda_\#}$ and $E_1^a$, respectively, so that $\beta = \beta_1 \cup \beta_2$ is a basis of $\mathbb{C}^{2g + n}$. The matrix of $L_a$ in the basis $\beta$ is of the form

\[
\begin{bmatrix}
* & 0 \\
0 & I_{\mu_\#}
\end{bmatrix}.
\]

We assume that the elements of the image of $\phi$ are represented in this basis.

For $f \in \text{Mod}(R)$, $\phi(f)$ commutes with $L_a$, so that $\ker(L_a - \lambda I)^{\lambda_\#}$ and $E_1^a = \ker(L_a - I)$ are $\phi(f)$-invariant; hence

\[
\phi(f) = \begin{bmatrix}
* & 0 \\
0 & M_f
\end{bmatrix},
\]
where $M_f \in \text{GL}(\mu, \mathbb{C})$. The correspondence $f \mapsto M_f$ is then a homomorphism $\varphi : \text{Mod}(R) \to \text{GL}(\mu, \mathbb{C})$. As the genus of $R$ is $g - 1 \geq 3$, $\varphi$ is trivial by Theorem 1. In particular, $L_x$ is of the form (8) for every nonseparating simple closed curve $x$ on $R$. Hence, the eigenspace of $L_x$ corresponding to the eigenvalue 1 satisfies $E^x_1 = E^a_1$.

It follows that if $c$ and $d$ are two simple closed curves on $R$ intersecting transversely at one point, then $E^c_1 = E^d_1$ ($= E^a_1$). Now apply Lemma 4.3 to deduce that $E^a_1$ is $\text{Mod}(S)$-invariant. As $\dim(E^a_1) < 2g$, the action of $\text{Mod}(S)$ on $E^a_1$ is trivial. In particular, for $f \in \text{Mod}(S)$, the matrix of $\phi(f)$ is of the form

$$
\phi(f) = \begin{bmatrix}
N_f & 0 \\
* & I
\end{bmatrix}.
$$

Now the correspondence $f \mapsto N_f$ defines a homomorphism $\tilde{\phi} : \text{Mod}(S) \to \text{GL}((\lambda, \mathbb{C})$. As $\lambda \leq 2g + n - 1$, we may consider $\text{GL}((\lambda, \mathbb{C})$ as a subgroup of $\text{GL}(2g + n - 1, \mathbb{C})$. By the induction hypothesis $K^{(n-1)}_\Sigma$ is contained in $\ker(\tilde{\phi})$ for any genus-3 subsurface $\Sigma$ of $R$. We conclude from this that $\phi(K^{(n-1)}_\Sigma)$ is abelian, and hence, $\phi(K^{(n)}_\Sigma)$ is trivial.

**Case 2. $L_a$ has only one eigenvalue.** Suppose that $L_a$ has only one eigenvalue, say $\lambda$.

We claim that if there is a $\text{Mod}(R)$-invariant subspace $V$ of dimension $r_1$ with $3 \leq r_1 \leq 2g + n - 3$, then we are done, namely $\phi(K^{(n)}_\Sigma)$ is trivial for some $\Sigma$. For the proof of the claim, suppose that there is such a subspace $V$. Let $r_2 = 2g + n - r_1$ and let $\beta$ be a (ordered) basis of $\mathbb{C}^{2g+n}$ such that first $r_1$ elements is a basis of $V$. With respect to the basis $\beta$, for $f \in \text{Mod}(R)$, $\phi(f)$ is of the form

$$
\phi(f) = \begin{bmatrix}
\phi_1(f) & * \\
0 & \phi_2(f)
\end{bmatrix},
$$

so that we have two homomorphisms

$$
\phi_1 : \text{Mod}(R) \to \text{GL}(V) = \text{GL}(r_1, \mathbb{C})
$$

and

$$
\phi_2 : \text{Mod}(R) \to \text{GL}(\mathbb{C}^{2g+n}/V) = \text{GL}(r_2, \mathbb{C}).
$$

As each $r_i$ satisfies $r_i \leq 2(g - 1) + n - 1$ and because $g - 1 \geq (n - 1) + 3$, by the induction hypothesis, the derived subgroup $K^{(n-1)}_\Sigma$ of $K_\Sigma$ is contained in the kernel of both $\phi_1$ and $\phi_2$ for some genus-3 subsurface $\Sigma$ of $R$. That is to say,

$$
\phi(f) = \begin{bmatrix}
I_{r_1} & * \\
0 & I_{r_2}
\end{bmatrix}
$$

for every $f \in K^{(n-1)}_\Sigma$. It follows from Lemma 3.10 that $\phi(K^{(n-1)}_\Sigma)$ is abelian, so that $\phi(K^{(n)}_\Sigma)$ is trivial, proving the claim.

Let us set $r = \dim(E^a_\lambda)$ for the rest of the proof.
If \( r \leq 2g + n - 3 \), then

\[
V = \begin{cases} 
\ker(L_a - \lambda I)^3, & \text{if } r = 1 \text{ or } 2, \\
E^a_{\lambda}, & \text{if } 3 \leq r \leq 2g + n - 3 
\end{cases}
\]
is a \( \text{Mod}(R) \)-invariant subspace with \( 3 \leq \dim(V) \leq 2g - 3 \).

If \( r = 2g + n \), then \( L_a = \lambda I \). As \( t_x \) is conjugate to \( t_a \) for any nonseparating simple closed curve \( x \) on \( S \), we have \( L_x = \lambda I \). As \( \text{Mod}(S) \) is generated by such Dehn twists, it follows that the image of \( \phi \) is cyclic. As \( \text{Mod}(S) \) is perfect, \( \phi \) is trivial.

Suppose that \( r = 2g + n - 2 \). If \( E^a_{\lambda} \neq E^b_{\lambda} \) then \( E^a_{\lambda} \cap E^b_{\lambda} \) is a \( \text{Mod}(R) \)-invariant subspace of dimension \( 2g + n - 3 \) or \( 2g + n - 4 \). We let \( V = E^a_{\lambda} \cap E^b_{\lambda} \) in this case. If \( E^a_{\lambda} = E^b_{\lambda} \) then it follows from Lemma 4.3 that \( E^a_{\lambda} \) is \( \text{Mod}(S) \)-invariant. Let \( \beta \) be an ordered basis of \( \mathbb{C}^{2g+n} \) whose first \( r \) elements form a basis of \( E^a_{\lambda} \). For \( f \in \text{Mod}(S) \), the matrix of \( \phi(f) \) with respect to \( \beta \) is

\[
\phi(f) = \begin{bmatrix} \phi_1(f) & * \\ 0 & \phi_2(f) \end{bmatrix}.
\]

In this way, we get two homomorphisms \( \phi_1 : \text{Mod}(S) \to \text{GL}(r, \mathbb{C}) \) and \( \phi_2 : \text{Mod}(S) \to \text{GL}(2, \mathbb{C}) \). By Theorem 1, \( \phi_2 \) is trivial, and by the induction hypothesis, \( \phi_1(K_{\Sigma}^{(n-2)}) \) is trivial for some \( \Sigma \). Therefore, for \( f \in K_{\Sigma}^{(n-2)} \), \( \phi(f) \) is of the form

\[
\phi(f) = \begin{bmatrix} I_r & * \\ 0 & I_2 \end{bmatrix}.
\]

It follows that \( \phi(K_{\Sigma}^{(n-2)}) \) is abelian, so that \( \phi(K_{\Sigma}^{(n-1)}) \) is trivial. Hence, \( \phi(K_{\Sigma}^{(n)}) \) is trivial.

Suppose finally that \( r = 2g + n - 1 \). In the view of Lemma 7.3, we have \( \lambda = 1 \). If \( E^a_1 = E^b_1 \) then by Lemma 4.2 we get that \( E^a_1 = E^b_1 \) for every nonseparating simple closed curve \( x \) on \( S \). As the group \( \text{Mod}(S) \) is generated by the Dehn twists about such curves, \( \text{Mod}(S) \) acts trivially on \( E^a_1 \). With respect to a basis of \( \mathbb{C}^{2g+n} \) whose first \( r \) elements belong to \( E^a_1 \), the matrices of \( L_a \) and \( L_b \) are of the form

\[
\begin{bmatrix} I_r & * \\ 0 & 1 \end{bmatrix}.
\]

It follows that \( L_aL_b = L_bL_a \). From the braid relation

\[
L_aL_bL_a = L_bL_aL_b,
\]

we get \( L_a = L_b \), and so \( \phi(t_at_b^{-1}) = I \). As the normal closure of \( t_at_b^{-1} \) in \( \text{Mod}(S) \) is the whole group, we conclude that \( \phi \) is trivial, which is a contradiction to \( \dim(E^a_1) = 2g + n - 1 \).

Therefore, we have \( E^a_1 \neq E^b_1 \). By Lemma 7.6, we have

\[
L_{a_i} = \begin{bmatrix} A_i & 0 \\ 0 & I_n \end{bmatrix} \quad \text{and} \quad L_{b_i} = \begin{bmatrix} B_i & 0 \\ 0 & I_n \end{bmatrix}.
\]
for a suitable basis of $C^{2g+n}$. Let $c$ be the boundary component of a regular neighborhood of $a_1 \cup b_1$. As $t_c = (t_{a_1} t_{b_1})^6$, we have $\phi(t_c) = (L_{a_1} L_{b_1})^6 = I$, so that $K_\Sigma$, and hence $K_\Sigma^{(n)}$, is contained in the kernel of $\phi$ for any genus-3 subsurface $\Sigma$ of $S$.

This completes the proof of the theorem. \hfill \Box

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**ORCID**

*Mustafa Korkmaz* [https://orcid.org/0000-0002-2731-7097](https://orcid.org/0000-0002-2731-7097)

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