On the location of zeros of the Laplacian matching polynomials of graphs

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Abstract
The Laplacian matching polynomial of a graph $G$, denoted by $\mathcal{LM}(G, x)$, is a new graph polynomial whose all zeros are nonnegative real numbers. In this paper, we investigate the location of zeros of the Laplacian matching polynomials. Let $G$ be a connected graph. We show that $0$ is a zero of $\mathcal{LM}(G, x)$ if and only if $G$ is a tree. We prove that the number of distinct positive zeros of $\mathcal{LM}(G, x)$ is at least equal to the length of the longest path in $G$. It is also established that the zeros of $\mathcal{LM}(G, x)$ and $\mathcal{LM}(G - e, x)$ interlace for each edge $e$ of $G$. Using the path tree of $G$, we present a linear algebraic approach to investigate the largest zero of $\mathcal{LM}(G, x)$ and particularly to give tight upper and lower bounds on it.

Keywords  Graph polynomial · Matching · Subdivision of graphs · Zeros of polynomials

Mathematics Subject Classification Primary: 05C31 · 05C70; Secondary: 05C05 · 05C50 · 12D10

1 Introduction

The graph polynomials, such as the characteristic polynomial, the chromatic polynomial, the independence polynomial, the matching polynomial, and many others, are widely studied and play important roles in applications of graphs in several diverse...
fields. The location of zeros of graph polynomials is a main topic in algebraic combinatorics and can be used to describe some structures and parameters of graphs. In this paper, we focus on the location of zeros of the Laplacian matching polynomials of graphs. For more results on the location of zeros of graph polynomials, we refer to [9].

Throughout this paper, all graphs are assumed to be finite, undirected, and without loops or multiple edges. Let $G$ be a graph. We denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. Let $M$ be a subset of $E(G)$. We denote by $V(M)$ the set of vertices of $G$ each of which is an endpoint of one of the edges in $M$. If no two distinct edges in $M$ share a common endpoint, then $M$ is called a matching of $G$. The set of matchings of $G$ is denoted by $\mathcal{M}(G)$. A matching $M \in \mathcal{M}(G)$ is said to be perfect if $V(M) = V(G)$. The matching polynomial of $G$ is

$$
\mathcal{M}(G, x) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} x^{|V(G)\setminus V(M)|}
$$

which was formally defined by Heilmann and Lieb [7] in studying statistical physics, although it has appeared independently in several different contexts.

The matching polynomial is a fascinating mathematical object and attracts considerable attention of researchers. For instance, by studying the multiplicity of zeros of the matching polynomials, Chen and Ku [8] gave a generalization of the Gallai–Edmonds theorem, which is a structure theorem in classical graph theory. For another instance, using a well-known upper bound on zeros of the matching polynomials, Marcus, Spielman, and Srivastava [10] established that infinitely many bipartite Ramanujan graphs exist. Some earlier facts on the matching polynomials can be found in [4].

We want to summarize here some basic features of the zeros of the matching polynomial. For this, let us first introduce some more notations and terminology which we need. For a vertex $v$ of a graph $G$, we denote by $N_G(v)$ the set of all vertices of $G$ adjacent to $v$. The degree of $v$ is defined as $|N_G(v)|$ and is denoted by $d_G(v)$. The maximum degree and the minimum degree of the vertices of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a subset $W$ of $V(G)$, we shall use $G[W]$ to denote the induced subgraph of $G$ induced by $W$, and we simply use $G - W$ instead of $G[V(G) \setminus W]$. Also, for a vertex $v$ of $G$, we simply write $G - v$ for $G - \{v\}$. For an edge $e$ of $G$, we denote by $G - e$ the subgraph of $G$ obtained by deleting the edge $e$.

Let $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_m$ be, respectively, the zeros of two real rooted polynomials $f$ and $g$ with $\deg f = n$ and $\deg g = m$. We say that the zeros of $f$ and $g$ interlace if either

$$
\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots
$$

or

$$
\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots
$$

in which case one clearly must have $|n - m| \leq 1$. We adopt the convention that the zeros of any polynomial of degree 0 interlace the zeros of any other polynomial.
For any connected graph $G$, the assertions given in (1.1)–(1.3) are known.

All the zeros of $\mathcal{M}(G, x)$ are real. Moreover, if $\Delta(G) \geq 2$, then the zeros of $\mathcal{M}(G, x)$ lie in the interval $(-2\sqrt{\Delta(G) - 1}, 2\sqrt{\Delta(G) - 1})$ \[7\]. (1.1)

The number of distinct zeros of $\mathcal{M}(G, x)$ is at least equal to $\ell(G) + 1$, where $\ell(G)$ is the length of the longest path in $G$ \[5\]. (1.2)

For each vertex $v \in V(G)$, the zeros of $\mathcal{M}(G - v, x)$ interlace the zeros of $\mathcal{M}(G, x)$. In addition, the largest zero of $\mathcal{M}(G, x)$ has the multiplicity 1 and is greater than the largest zero of $\mathcal{M}(G - v, x)$ \[6\]. (1.3)

Recently, Mohammadian \[11\] introduced a new graph polynomial that is called the Laplacian matching polynomial and is defined for a graph $G$ as

$$L\mathcal{M}(G, x) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \left( \prod_{v \in V(G) \setminus V(M)} (x - d_G(v)) \right). \quad (1.4)$$

Mohammadian proved that all zeros of $L\mathcal{M}(G, x)$ are real and nonnegative, and moreover, if $\Delta(G) \geq 2$, then the zeros of $L\mathcal{M}(G, x)$ lie in the interval $[0, \Delta(G) + 2\sqrt{\Delta(G) - 1})$. By observing this interval, it is natural to ask: What is the sufficient and necessary condition for 0 is a zero of $L\mathcal{M}(G, x)$? More generally, as a new real rooted graph polynomial, it is natural to investigate the properties of zeros such as the interlacing of zeros, the upper and lower bounds of the largest zero, the maximum multiplicity of zeros, and the number of distinct zeros. In this paper, we mainly prove that the assertions given in (1.5)–(1.7) hold for any connected graph $G$, letting $\ell(G)$ be the length of the longest path in $G$.

If $\Delta(G) \geq 2$, then the zeros of $L\mathcal{M}(G, x)$ are contained in the interval $[0, \Delta(G) + 2\sqrt{\Delta(G) - 1}) \cos \frac{\pi}{2\ell(G)+1}$, and in addition, the upper bound of (1.5) the interval is a zero of $L\mathcal{M}(G, x)$ if and only if $G$ is a cycle.

The number of distinct positive zeros of $L\mathcal{M}(G, x)$ is at least equal to $\ell(G)$.

Also, if $\delta(G) \geq 2$, then $L\mathcal{M}(G, x)$ has at least $\ell(G) + 1$ distinct positive zeros.

For each edge $e \in E(G)$, the zeros of $L\mathcal{M}(G, x)$ and $L\mathcal{M}(G - e, x)$ interlace in the sense that, if $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_n$ are, respectively, the zeros of $L\mathcal{M}(G, x)$ and $L\mathcal{M}(G - e, x)$ in which $n = |V(G)|$, then $\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_n \leq \alpha_n$. Further, the largest zero of $L\mathcal{M}(G, x)$ has the multiplicity 1 and is strictly greater than the largest zero of $L\mathcal{M}(H, x)$ for any proper subgraph $H$ of $G$.\[123\]
It should be mentioned that the Laplacian matching polynomial is recently studied under a different name and expression by Chen and Zhang [17].

For a graph \( G \), the subdivision of \( G \), denoted by \( S(G) \), is the graph derived from \( G \) by replacing every edge \( e = \{a, b\} \) of \( G \) with two edges \( \{a, v_e\} \) and \( \{v_e, b\} \) along with the new vertex \( v_e \) corresponding to the edge \( e \). We know from a result of Yan and Yeh [16] that

\[
\mathcal{M}(S(G), x) = x^{|E(G)|-|V(G)|} \mathcal{L}\mathcal{M}(G, x^2) \tag{1.8}
\]

for any graph \( G \), which is also proved by Chen and Zhang [17] by different methods. The equality (1.8) shows that the problem of the location of zeros of the Laplacian matching polynomial of a graph \( G \) can be transformed into the problem that deals with the location of zeros of the matching polynomial of \( S(G) \). For an instance, using (1.8) and the first statement in (1.1), it immediately follows that the zeros of \( \mathcal{L}\mathcal{M}(G, x) \) are nonnegative real numbers. Assertion (1.6) is proved in Sect. 2 by the subdivision of graphs.

One of the most important tools in the theory of the matching polynomial is the concept of ‘path tree’, which is introduced by Godsil [5]. Given a graph \( G \) and a vertex \( u \in V(G) \), the path tree \( T(G, u) \) is the tree which has as vertices the paths in \( G \) which start at \( u \) where two such paths are adjacent if one is a maximal proper subpath of the other. In Sect. 3, we show that the path tree is also applicable for the Laplacian matching polynomial by making some appropriate adjustments. Using this, we prove (1.5), which is a slight improvement of the second statement of Theorem 2.6 of [11]. Assertion (1.7) is proved in Sect. 3 by linear algebra arguments.

Let us introduce more notations and definitions before moving on to the next section. We use \( \lambda(f(x)) \) to denote the largest zero of a real rooted polynomial \( f(x) \). For a square matrix \( M \), we shall use \( \phi(M, x) \) to denote the characteristic polynomial of \( M \) in the indeterminate \( x \). If all the zeros of \( \phi(M, x) \) are real, then its largest zero is denoted by \( \lambda(M) \). For a graph \( G \), the adjacency matrix of \( G \), denoted by \( A(G) \), is a matrix whose rows and columns are indexed by \( V(G) \) and the \((u, v)\)-entry is 1 if \( u \) and \( v \) are adjacent and 0 otherwise. Let \( D(G) \) be the diagonal matrix whose rows and columns are indexed as the rows and the columns of \( A(G) \) with \( d_G(v) \) in the \( v \)th diagonal position. The matrices \( L(G) = D(G) - A(G) \) and \( Q(G) = D(G) + A(G) \) are, respectively, said to be the Laplacian matrix and the signless Laplacian matrix of \( G \). It is known that \( \mathcal{M}(G, x) = \phi(A(G), x) \) if and only if \( G \) is a forest [14]. In addition, it is proved that \( \mathcal{L}\mathcal{M}(G, x) = \phi(L(G), x) \) if and only if \( G \) is a forest [11]. Among other results, we present a generalization of these results in Sect. 2.

### 2 Subdivision of graphs and the Laplacian matching polynomial

In this section, we examine the location of zeros of the Laplacian matching polynomial by establishing a relation between the Laplacian matching polynomial of a graph and the matching polynomial of the subdivision of that graph. Then, by analysing the structures of the subdivision of graphs, we will prove (1.6). To begin with, we recall the multivariate matching polynomial that covers both the matching polynomial and the
Laplacian matching polynomial. This multivariate graph polynomial was introduced by Heilmann and Lieb [7].

Let $G$ be a graph and associate the vector $x_G = (x_v)_{v \in V(G)}$ with $G$ in which $x_v$ is an indeterminate corresponding to the vertex $v \in V(G)$. Notice that, for a subgraph $H$ of $G$, $x_H$ is the vector that has the same coordinate as $x_G$ in the positions corresponding to the vertices in $V(H)$. The multivariate matching polynomial of $G$ is defined as

$$M(G, x) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \left( \prod_{v \in V(G) \setminus V(M)} x_v \right).$$ (2.1)

Let $1_G$ be the all one vector of length $|V(G)|$. Also, for a subgraph $H$ of $G$, let $d_G, H = (d_G(v))_{v \in V(H)}$. For simplicity, we write $d_G$ instead of $d_{G,G}$. We sometimes drop the subscript of the vector symbols if there is no possible confusion. It is easy to see that

$$M(G, x1_G) = M(G, x)$$ (2.2)

and

$$M(G, x1_G - d_G) = LM(G, x).$$ (2.3)

Note that

$$M(G_1 \cup G_2, (x_{G_1}, x_{G_2})) = M(G_1, x_{G_1})M(G_2, x_{G_2}),$$

where $G_1 \cup G_2$ denotes the disjoint union of two graphs $G_1$ and $G_2$. So, in what follows, we often restrict our attention on connected graphs.

We need the following useful lemma in the sequel.

**Lemma 2.1** *(Amini [1])* Let $G$ be a graph. For any vertex $v \in V(G)$,

$$M(G, x) = x_v M(G - v, x) - \sum_{w \in N_G(v)} M(G - v - w, x_{G-v-w}).$$

By combining Lemma 2.1 and (2.2), we get

$$M(G, x) = xM(G - v, x) - \sum_{w \in N_G(v)} M(G - v - w, x),$$ (2.4)

which is a well-known recursive formula for the matching polynomial.

The following theorem, which is a generalization of (1.8), plays a crucial role in our proofs in Sect. 3.

**Theorem 2.2** Let $G$ be a graph. For any subset $W$ of $V(G)$,

$$M(S(G) - W, x) = x^{|E(G)| - |V(G)| + |W|} M(G - W, x^2 1_{G-W} - d_{G,G-W}).$$
Proof. For simplicity, let $k = |V(G) \setminus W|$ and $m = |E(G)|$. We prove the assertion by induction on $k$. If $W = V(G)$, then $S(G) - W$ consists of $|E(G)|$ isolated vertices and the claimed equality holds by adopting a convention. If $V(G) \setminus W = \{u\}$ for some vertex $u \in V(G)$, then $S(G) - W$ consists of a star on $d_G(u) + 1$ vertices and $|E(G)| - d_G(u)$ isolated vertices. Therefore,

$$\mathcal{M}(S(G) - W, x) = x^{m+1} - d_G(u)x^{m-1}$$

and

$$\mathcal{M}(G - W, x^2 - d) = x^2 - d_G(u).$$

So, the claimed equality holds for $k = 1$. Assume that $k \geq 2$. Choose a vertex $u \in V(G) \setminus W$ and let $H = S(G) - W - u$. By Lemma 2.1, the induction hypothesis, and (2.4), we have

$$x^{m-k+2}\mathcal{M}(G - W, x^2 - d) = x(x^2 - d_G(u))x^{m-k+1}\mathcal{M}(G - W - u, x^2 - d)$$

$$- \sum_{v \in N_{G-W}(u)} x^{m-k+2}\mathcal{M}(G - W - u - v, x^2 - d)$$

$$= x(x^2 - d_G(u))\mathcal{M}(H, x) - \sum_{v \in N_{G-W}(u)} \mathcal{M}(H - v, x)$$

$$= x^2\mathcal{M}(S(G) - W, x) + x^2 \sum_{v \in N_{S(G)-W}(u)} \mathcal{M}(H - v, x)$$

$$- d_G(u)x\mathcal{M}(H, x) - \sum_{v \in N_{G-W}(u)} \mathcal{M}(H - v, x).$$

Hence, in order to complete the induction step, it suffices to prove that

$$d_G(u)x\mathcal{M}(H, x) = x^2 \sum_{v \in N_{S(G)-W}(u)} \mathcal{M}(H - v, x) - \sum_{v \in N_{G-W}(u)} \mathcal{M}(H - v, x).$$

(2.5)

To establish (2.5), let $N_G(u) \cap W = \{a_1, \ldots, a_s\}$ and $N_G(u) \setminus W = \{b_1, \ldots, b_t\}$. Also, for $i = 1, \ldots, s$, let $a'_i$ be the vertex of $S(G)$ corresponding to the edge $\{u, a_i\}$ of $G$ and, for $j = 1, \ldots, t$, let $b'_j$ be the vertex of $S(G)$ corresponding to the edge $\{u, b_j\}$ of $G$. Notice that, if one of $N_G(u) \cap W$ and $N_G(u) \setminus W$ is empty, then we may derive (2.5) by the same discussion as below. We have $d_G(u) = s + t$ and $N_{S(G)-W}(u) = N_{S(G)}(u) = \{a'_1, \ldots, a'_s, b'_1, \ldots, b'_t\}$. The structure of $H$ is illustrated in Fig. 1.

We have $d_H(a'_i) = 0$ for $i = 1, \ldots, s$ and $d_H(b'_j) = 1$ for $j = 1, \ldots, t$. By applying (2.4) for $a'_i$ and $b'_j$, we find that

$$\mathcal{M}(H, x) = x\mathcal{M}(H - a'_i, x)$$

$\square$ Springer
\[ E(G[W]) \] isolated vertices

\[ |E(G[W])| \] isolated vertices

\[ S(G - W - u) \]

Fig. 1 The structure of \( H \)

and

\[
x \mathcal{M}(H, x) = x^2 \mathcal{M}(H - b'_j, x) - x \mathcal{M}(H - b_j - b'_j, x)
\]

\[
= x^2 \mathcal{M}(H - b'_j, x) - \mathcal{M}(H - b_j, x).
\]

Therefore,

\[
d_G(u)x \mathcal{M}(H, x) = sx \mathcal{M}(H, x) + tx \mathcal{M}(H, x)
\]

\[
= x^2 \sum_{i=1}^{s} \mathcal{M}(H - a'_i, x) + x^2 \sum_{j=1}^{t} \mathcal{M}(H - b'_j, x)
\]

\[
- \sum_{j=1}^{t} \mathcal{M}(H - b_j, x)
\]

\[
= x^2 \sum_{v \in N_{S(G)} - W(u)} \mathcal{M}(H - v, x) - \sum_{v \in N_{G - W(u)}} \mathcal{M}(H - v, x),
\]

which is exactly (2.5). This completes the proof. \( \square \)

In what follows, we prove some results about the Laplacian matching polynomial by analysing the structures of the subdivision of graphs. The following consequence immediately follows from Theorem 2.2 and the first statement in (1.1). It worth mentioning that the following result is proved in [17] for a different expression of the Laplacian matching polynomial.
Corollary 2.3 Let $G$ be a graph. Then,

$$\mathcal{M}(S(G), x) = x^{|E(G)| - |V(G)|} \mathcal{L}(G, x^2).$$

In particular, the zeros of $\mathcal{L}(G, x)$ are nonnegative real numbers.

For a graph $G$, it is proved that $\mathcal{L}(G, x) = \varphi(L(G), x)$ if and only if $G$ is a forest [11]. Since 0 is an eigenvalue of $L(G)$, we deduce that $\mathcal{L}(G, 0) = 0$ if $G$ is a forest. From (1.4), we get the combinatorial identity

$$\sum_{M \in \mathcal{M}(F)} (-1)^{|M|} \prod_{v \in V(F) \setminus V(M)} d_F(v) = 0$$

for any forest $F$. The following theorem, which is proved in [17], gives a necessary and sufficient condition for 0 to be a zero of the Laplacian matching polynomial. We present here a different proof for it.

Theorem 2.4 (Chen, Zhang [17]) Let $G$ be a connected graph. Then, 0 is a zero of $\mathcal{L}(G, x)$ if and only if $G$ is a tree.

Proof If $G$ is a tree, then $|E(G)| = |V(G)| - 1$ and so $\mathcal{L}(G, x) = x \mathcal{M}(S(G), x)$ by Corollary 2.3, implying that 0 is a zero of $\mathcal{L}(G, x)$. We prove that 0 is not a zero of $\mathcal{L}(G, x)$ if $G$ is not a tree. For this, assume that $|E(G)| \geq |V(G)|$. One may easily consider $S(G)$ as a bipartite graph with the bipartition $\{V(G), E(G)\}$ after identifying each new vertex $v_e$ of $S(G)$ with its corresponding edge $e$ of $G$.

We claim that $S(G)$ has a matching that saturates the part $V(G)$. If $G$ contains a vertex $u$ with degree 1 and $e$ is the edge incident to $u$ in $G$, then it suffices to prove that $S(G - u)$ has a matching that saturates the part $V(G - u)$, since the union of such matching and the edge $\{u, v_e\}$ forms a matching of $S(G)$ that saturates the part $V(G)$. Thus, we may assume that $d_G(v) \geq 2$ for all vertices $v \in V(G)$. We are going to establish that $S(G)$ satisfies Hall’s condition [2, Theorem 16.4]. For a subset $W$ of $V(G)$, we shall use $N_G(W)$ to denote the set of vertices of $G$ each of which is adjacent to a vertex in $W$ and $\partial_G(W)$ to denote the set of edges of $G$ each of which has exactly one endpoint in $W$. For any subset $U$ of the part $V(G)$, since $d_G(v) \geq 2$ for all vertices $v \in V(G)$,

$$|\partial_{S(G)}(U)| \geq 2|U|. \quad (2.6)$$

On the other hand, $d_{S(G)}(v_e) = 2$ for each $e \in E(G)$, so

$$|\partial_{S(G)}(N_{S(G)}(U))| = 2|N_{S(G)}(U)|. \quad (2.7)$$

Clearly, $|\partial_{S(G)}(N_{S(G)}(U))| \geq |\delta_{S(G)}(U)|$ which implies that $|N_{S(G)}(U)| \geq |U|$ using (2.6) and (2.7). This means that $S(G)$ satisfies Hall’s condition, as required.

We proved that $S(G)$ has a matching that saturates the part $V(G)$. This means that the smallest power of $x$ in $\mathcal{M}(S(G), x)$ is $|E(G)| - |V(G)|$ by (2.1) and (2.2). In
view of Corollary 2.3, $\mathcal{M}(S(G), x) = x^{[E(G)]-|V(G)|} \mathcal{L}(G, x^2)$ which shows that the constant term in $\mathcal{L}(G, x)$ is nonzero. So, 0 is not a zero of $\mathcal{L}(G, x)$. This completes the proof.

\[
\square
\]

In the next theorem, we give a lower bound on the number of distinct zeros of the Laplacian matching polynomial.

**Theorem 2.5** Let $G$ be a connected graph and let $\ell(G)$ be the length of the longest path in $G$. Then, the number of distinct positive zeros of $\mathcal{L}(G, x)$ is at least equal to $\ell(G)$. Also, if $\delta(G) \geq 2$, then $\mathcal{L}(G, x)$ has at least $\ell(G) + 1$ distinct positive zeros.

**Proof** For convenience, let $\ell = \ell(G)$. Denote by $\ell'$ the length of the longest path in $S(G)$. From (1.2), $\mathcal{M}(S(G), x)$ has at least $\ell' + 1$ distinct zeros. By Corollary 2.3, $\mathcal{M}(S(G), x) = x^{[E(G)]-|V(G)|} \mathcal{L}(G, x^2)$ which shows that $\mathcal{L}(G, x^2)$ has at least $\ell'$ distinct nonzero zeros. Since all zeros of $\mathcal{L}(G, x)$ are real and nonnegative by Corollary 2.3, it follows that $\mathcal{L}(G, x)$ has at least $\lceil \ell'/2 \rceil$ distinct positive zeros.

For each edge $e \in E(G)$, denote by $v_e$ the vertex of $S(G)$ corresponding to $e$. Let $v_0, v_1, \ldots, v_\ell$ be a path in $G$. Then, $v_0, v_{e_1}, v_1, \ldots, v_{e_\ell}, v_\ell$ is a path in $S(G)$ of length $2\ell$, where $e_i = \{w_{i-1}, w_i\} \in E(G)$ for $i = 1, \ldots, \ell$. Thus, $\ell' \geq 2\ell$ and so $\mathcal{L}(G, x)$ has at least $\ell$ distinct positive zeros.

Now, assume that $\delta(G) \geq 2$. This assumption allows us to consider a vertex $w' \in N_G(v_0) \setminus \{v_1\}$. Then, $S(G)$ contains the path $v_{e'}, v_0, v_{e_1}, v_1, \ldots, v_{e_\ell}, v_\ell$ of length $2\ell + 1$, where $e' = \{w', w_0\} \in E(G)$. Therefore, $\ell' \geq 2\ell + 1$ and so $\mathcal{L}(G, x)$ has at least $\lceil \ell'/2 \rceil \geq \ell + 1$ distinct positive zeros. This completes the proof.

\[
\square
\]

**Remark 2.6** The second statement in Theorem 2.5 implies that, if $G$ is a graph with a Hamilton cycle, then the zeros of $\mathcal{L}(G, x)$ are all distinct.

Given a graph $G$, it is known that $\mathcal{M}(G, x) = \varphi(A(G), x)$ if and only if $G$ is a forest [14]. Also, as we mentioned before, it is established that $\mathcal{L}(G, x) = \varphi(L(G), x)$ if and only if $G$ is a forest [11]. Below, we present a general result which shows that the multivariate matching polynomial of a forest has a determinantal representation in terms of its adjacency matrix, which will be used in the next section.

**Theorem 2.7** Let $F$ be a forest. Then $\mathcal{M}(F, x_F) = \det(X_F - A(F))$, where $X_F$ is a diagonal matrix whose rows and columns are indexed by $V(F)$ and the $(v, v)$-entry is $x_v$ for any vertex $v \in V(F)$. In particular, $\mathcal{M}(F, x) = \varphi(A(F), x)$ and $\mathcal{L}(F, x) = \varphi(L(F), x)$.

**Proof** We prove that $\mathcal{M}(F, x_F) = \det(X_F - A(F))$ by induction on $|E(F)|$. The equality is trivially valid if $|E(F)| = 0$. So, assume that $|E(F)| \geq 1$. As $F$ is a forest, we may consider two vertices $u, v \in V(F)$ with $N_F(u) = \{v\}$. Without loss of generality, we may assume that the first row and column of $A(F)$ are corresponding to $u$ and the second row and column of $A(F)$ are corresponding to $v$. Expanding the determinant of $X_F - A(F)$ along its first row, we obtain by the induction hypothesis and Lemma 2.1 that

\[
\det(X_F - A(F)) = x_u \det(X_{F-u} - A(F-u)) - \det(X_{F-u-v} - A(F-u-v))
\]
as desired. The ‘in particular’ statement immediately follows from (2.2) and (2.3). □

**Corollary 2.8** For a tree \( T \), the multiplicity of 0 as a zero of \( \mathcal{L}(T, x) \) is 1.

**Proof** It is well known that the number of connected components of a graph \( \Gamma \) is equal to the multiplicity of 0 as a zero of \( \varphi(L(\Gamma), x) \) [3, Proposition 1.3.7]. So, the result follows from \( \mathcal{L}(T, x) = \varphi(L(T), x) \) which is given in Theorem 2.7. □

### 3 The largest zero of the Laplacian matching polynomial

The purpose of this section is to investigate the location of the largest zero of the Laplacian matching polynomial. We give a linear algebraic approach to study the largest zero of the Laplacian matching polynomial and present sharp upper and lower bounds on it. Assertions (1.5) and (1.7) are also proved in this section based on the linear algebraic approach.

Let \( G \) be a connected graph and \( u \in V(G) \). Let \( T(G, u) \) be the path tree of \( G \) respect to the vertex \( u \) which is introduced in Sect. 1. Consider two vectors \( x_G = (x_v)_{v \in V(G)} \) and \( x_{T(G,u)} = (x_P)_{P \in V(T(G,u))} \) of indeterminates associated with \( G \) and \( T(G, u) \), respectively. For every vertex \( P \in V(T(G, u)) \), we may identify \( x_P \) with \( x_{u(P)} \) in which \( u(P) \) is the terminal vertex of the path \( P \) in \( G \). In such way, \( G \) and \( T(G, u) \) will be equipped with two vectors consisting of the same indeterminates, which are simply denoted by \( x \) when there is no ambiguity. In what follows, for every subgraph \( H \) of \( G \) and vertex \( u \in V(H) \), we denote by \( D_G(T(H, u)) \) the diagonal matrix whose rows and columns are indexed by \( V(T(H, u)) \) and the \((P, P)\)-entry is \( d_G(u(P)) \).

The univariate version of the following theorem, which is proved by Godsil [5], has a key role in the theory of the matching polynomial. Notice that, for a graph \( G \) and a vertex \( u \in V(G) \), \( u \) is a path in \( G \) and the corresponding vertex in \( T(G, u) \) will also be referred to as \( u \).

**Theorem 3.1** (Amini [1]) Let \( G \) be a connected graph and let \( u \in V(G) \). Then

\[
\frac{\mathcal{M}(G - u, x)}{\mathcal{M}(G, x)} = \frac{\mathcal{M}(T(G, u) - u, x)}{\mathcal{M}(T(G, u), x)},
\]

and moreover, \( \mathcal{M}(G, x) \) divides \( \mathcal{M}(T(G, u), x) \).

For a connected graph \( G \) and a vertex \( u \in V(G) \), Theorem 3.1 and Theorem 2.7 yield that \( \mathcal{M}(G, x) \) divides \( \varphi(A(T(G, u)), x) \). Since all zeros of the characteristic polynomial of a symmetric matrix are real, the first statement in (1.1) is obtained as an application of Theorem 3.1. For the Laplacian matching polynomial, we get the following result.

**Corollary 3.2** Let \( G \) be a connected graph, \( H \) be a subgraph of \( G \), and \( u \in V(H) \). If \( H \) is connected, then \( \mathcal{M}(H, x1_H - d_{G,H}) \) divides \( \varphi(D_G(T(H, u)) + A(T(H, u)), x) \).
In particular, \( \varphi(D_G(T(G,u)) + A(T(G,u)), x) \) is divisible by \( L.M(G,x) \) for every vertex \( u \in V(G) \).

\textbf{Proof} By Theorem 3.1, we find that \( M(H,x1_H - d_{G,H}) \) divides \( M(T(H,u), x1_H - d_{G,H}) \). It follows from Theorem 2.7 that

\[
\begin{align*}
M(T(H,u), x1_H - d_{G,H}) &= \det \left( xI - D_G(T(H,u)) - A(T(H,u)) \right) \\
&= \varphi\left( D_G(T(H,u)) + A(T(H,u)), x \right),
\end{align*}
\]

which establishes what we require. Since \( M(G,x1_G - d_G) = L.M(G,x) \) using (2.3), the ‘in particular’ statement immediately follows. \( \square \)

\textbf{Remark 3.3} The matrix \( D_G(T(G,u)) + A(T(G,u)) \), which appeared in Corollary 3.2, is a symmetric diagonally dominant matrix with nonnegative diagonal entries, so all of its eigenvalues are nonnegative real numbers. Hence, Corollary 3.2 gives us another proof for the fact that all zeros of the Laplacian matching polynomial are real and nonnegative which is also proved in Corollary 2.3.

It is well known that the largest zero of the matching polynomial of a graph is equal to the largest eigenvalue of the adjacency matrix of a path tree of that graph. This fact is obtained by combining the Perron–Frobenius theorem [3, Theorem 2.2.1] and Theorems 2.7 and 3.1. The following theorem can be considered as an analogue of the fact. Indeed, the following theorem presents a linear algebra technique to treat with the largest zero of the Laplacian matching polynomial.

\textbf{Theorem 3.4} Let \( G \) be a connected graph, \( H \) be a subgraph of \( G \), and \( u \in V(H) \). If \( H \) is connected, then

\[
\lambda(M(H,x1_H - d_{G,H})) = \lambda\left( D_G(T(H,u)) + A(T(H,u)) \right). \tag{3.1}
\]

In particular, \( \lambda(L.M(G,x)) = \lambda(D_G(T(G,u)) + A(T(G,u))) \). Also, the largest zero of \( L.M(G,x) \) has the multiplicity 1.

\textbf{Proof} We prove (3.1) by induction on \( |V(H)| \). Clearly, (3.1) is valid for \( |V(H)| = 1 \). Assume that \( |V(H)| \geq 2 \). We first show that

\[
\lambda(M(H-u,x1_{H-u} - d_{G,H-u})) < \lambda(M(H,x1_H - d_{G,H})). \tag{3.2}
\]

To see (3.2), we apply Theorem 2.2 and (1.3) to get that

\[
\begin{align*}
\lambda(M(H-u,x2_{H-u} - d_{G,H-u})) &= \lambda\left( M(S(G) - W - u, x) \right) \\
&< \lambda\left( M(S(G) - W, x) \right) \\
&= \lambda(M(H,x2_{H} - d_{G,H})),
\end{align*}
\]
where \( W = V(G) \setminus V(H) \). This clearly proves (3.2). Now, let \( N_H(u) = \{u_1, \ldots, u_k\} \) and let \( T(H_i, u_i) \) be the connected component of \( T(H, u) - u \) containing the vertex which corresponds to path \( u, u_i \) in \( H \) for \( i = 1, \ldots, k \). By the induction hypothesis,

\[
\lambda(\mathcal{M}(H_i, x1_{H_i} - d_{G,H_i})) = \lambda(D_G(T(H_i, u_i)) + A(T(H_i, u_i)))
\]

(3.3)

for \( i = 1, \ldots, k \). It is not hard to see the \( k \times k \) block diagonal matrix whose \( i \)th block diagonal entry is \( D_G(T(H_i, u_i)) + A(T(H_i, u_i)) \), say \( R \), is a principal submatrix of \( D_G(T(H, u)) + A(T(H, u)) \) with size \( |T(H, u)| - 1 \). Hence, by the interlacing theorem [3, Corollary 2.5.2], it follows that \( \lambda(R) \) is greater than or equal to the second largest eigenvalue of \( D_G(T(H, u)) + A(T(H, u)) \). Further, it follows from (3.3) and (3.2) that

\[
\lambda(R) = \max \left\{ \lambda(\mathcal{M}(H_i, x1_{H_i} - d_{G,H_i})) \mid 1 \leq i \leq k \right\} = \lambda(\mathcal{M}(H - u, x1_{H - u} - d_{G,H - u})) < \lambda(\mathcal{M}(H, x1_H - d_{G,H})).
\]

Thus, \( \lambda(\mathcal{M}(H, x1_H - d_{G,H})) \) is strictly greater than the second largest eigenvalue of \( D_G(T(H, u)) + A(T(H, u)) \). On the other hand, Corollary 3.2 implies that \( \lambda(\mathcal{M}(H, x1_H - d_{G,H})) \) is a zero of \( \varphi(D_G(T(H, u)) + A(T(H, u)), x) \). So, we conclude that \( \lambda(\mathcal{M}(H, x1_H - d_{G,H})) \) is the largest eigenvalue of \( D_G(T(H, u)) + A(T(H, u)) \). This completes the induction step and demonstrates that (3.1) holds.

For the ‘in particular’ statement, note that (3.1) and (2.3) yield that

\[
\lambda\left(D_G(T(G, u)) + A(T(G, u))\right) = \lambda(\mathcal{M}(G, x1_G - d_G)) = \lambda(L(\mathcal{M}(G), x)),
\]

and further, the connectedness of \( G \) implies that \( D_G(T(G, u)) + A(T(G, u)) \) is an irreducible matrix with nonnegative entries, and consequently, its largest eigenvalue has the multiplicity 1 by the Perron–Frobenius theorem [3, Theorem 2.2.1].

\[\square\]

**Corollary 3.5** Let \( G \) be a connected graph and \( u \in V(G) \). Then

\[
\lambda(\mathcal{M}(G, x)) \geq \lambda(L(T(G, u)))
\]

(3.4)

with the equality holds if and only if \( G \) is a tree.

**Proof** We first recall the fact that a graph \( \Gamma \) is bipartite if and only if \( \varphi(L(\Gamma), x) = \varphi(Q(\Gamma), x) \) [3, Proposition 1.3.10]. For each \( P \in V(T(G, u)) \), we have \( d_{T(G,u)}(P) \leq d_G(\nu(P)) \), where \( \nu(P) \) is the terminal vertex of the path \( P \) in \( G \). Therefore, \( R = D_G(T(G, u)) + A(T(G, u)) - Q(T(G, u)) \) has nonnegative entries, and thus, Theorem 3.4, the Perron–Frobenius theorem [3, Theorem 2.2.1], and the above-mentioned fact yield that

\[
\lambda(\mathcal{M}(G, x)) = \lambda\left(D_G(T(G, u)) + A(T(G, u))\right)
\]
Let $G$ be a connected graph. Then

$$= \lambda \left( R + Q(T(G, u)) \right)$$

$$\geq \lambda \left( Q(T(G, u)) \right)$$

$$= \lambda \left( L(T(G, u)) \right), \quad (3.5)$$

proving (3.4). If $G$ is a tree, then $G$ is isomorphic to $T(G, u)$ and since $\mathcal{LM}(G, x) = \varphi(L(G), x)$ by Theorem 2.7, the equality in (3.4) is attained. Conversely, assume that the equality in (3.4) holds. Consequently, the equality in (3.5) occurs, and hence, the Perron–Frobenius theorem [3, Theorem 2.2.1] implies that $R = 0$. This means that $d_{T(G, u)}(P) = d_G(\upsilon(P))$ for each $P \in V(T(G, u))$. We assert that $G$ is a tree. Towards a contradiction, suppose that there is a cycle $C$ with the equality holds if and only if $G$ is a star.

Fix $w \in N_G(\upsilon(P_1)) \cap V(C)$ and let $P_2$ be the path on $C$ between $\upsilon(P_1)$ and $w$ whose length is more than 1. If $P$ is the path between $u$ and $w$ formed by $P_1$ and $P_2$, then it is clear that $d_{T(G, u)}(P) < d_G(\upsilon(P))$. This contradiction completes the proof. \qed

In the following consequence, we give some lower bounds on the largest zero of the Laplacian matching polynomial.

**Corollary 3.6** Let $G$ be a connected graph. Then

$$\lambda(\mathcal{LM}(G, x)) \geq \max \left\{ \Delta(G) + 1, \delta(G) + \sqrt{\Delta(G)} \right\}$$

with the equality holds if and only if $G$ is a star.

**Proof** Let $u \in V(G)$ be of degree $\Delta(G)$. Indeed, $d_{T(G, u)}(u) = d_G(u)$ and therefore $\Delta(T(G, u)) = \Delta(G)$. For each connected graph $\Gamma$, Proposition 3.9.3 of [3] states that $\lambda(L(\Gamma)) \geq \Delta(\Gamma) + 1$ with the equality holds if and only if $\Delta(\Gamma) = |V(\Gamma)| - 1$. By this fact and Corollary 3.5, we obtain that $\lambda(\mathcal{LM}(G, x)) \geq \lambda(L(T(G, u))) \geq \Delta(T(G, u)) + 1 = \Delta(G) + 1$, and moreover, the equality $\lambda(\mathcal{LM}(G, x)) = \Delta(G) + 1$ holds if and only if $G$ is a star.

For each connected graph $\Gamma$, the Perron–Frobenius theorem [3, Theorem 2.2.1] implies that $\lambda(A(\Gamma)) \geq \sqrt{\Delta(\Gamma)}$ with the equality holds if and only if $\Gamma$ is a star. Using this fact, Theorem 3.4, and the Weyl inequality [3, Theorem 2.8.1], we derive

$$\lambda(\mathcal{LM}(G, x)) = \lambda \left( D_G(T(G, u)) + A(T(G, u)) \right)$$

$$\geq \delta(G) + \lambda \left( A(T(G, u)) \right)$$

$$\geq \delta(G) + \sqrt{\Delta(T(G, u))}$$

$$= \delta(G) + \sqrt{\Delta(G)}. \quad (3.6)$$

Suppose that the equality $\lambda(\mathcal{LM}(G, x)) = \delta(G) + \sqrt{\Delta(G)}$ holds. So, the equality in (3.6) is attained, and thus, $T(G, u)$ is a star. This implies that $G$ is a star, and
then, $\lambda(\mathcal{LM}(G, x)) = \delta(G) + \sqrt{\Delta(G)}$ forces that $|V(G)| \leq 2$. Since the equality $\lambda(\mathcal{LM}(G, x)) = \delta(G) + \sqrt{\Delta(G)}$ is valid for the stars $G$ on at most 2 vertices, the proof is complete. \hfill \Box

In the following theorem, we establish (1.5) which slightly improves the second statement of Theorem 2.6 of [11].

**Theorem 3.7** Let $G$ be a connected graph with $\Delta(G) \geq 2$ and let $\ell(G)$ be the length of the longest path in $G$. Then,

$$
\lambda(\mathcal{LM}(G, x)) \leq \Delta(G) + 2\sqrt{\Delta(G)} - 1 \cos \frac{\pi}{2\ell(G) + 2}
$$

(3.7)

with the equality holds if and only if $G$ is a cycle.

**Proof** For simplicity, let $\Delta = \Delta(G)$ and $\ell = \ell(G)$. For every positive integers $d$ and $k \geq 2$, the Bethe tree $B_{d,k}$ is a rooted tree with $k$ levels in which the root vertex is of degree $d$, the vertices on levels $2, \ldots, k - 1$ are of degree $d + 1$, and the vertices on level $k$ are of degree 1. By Theorem 7 of [13],

$$
\lambda(A(B_{d,k})) = 2\sqrt{d} \cos \frac{\pi}{k + 1}.
$$

(3.8)

Let $u \in V(G)$. It is not hard to check that $T(G, u)$ is isomorphic to a subgraph of $B_{\Delta-1, 2\ell+1}$. For this, it is enough to correspond $u \in V(T(G, u))$ to an arbitrary vertex on level $\ell+1$ in $B_{\Delta-1, 2\ell+1}$. By applying Theorem 3.4, the Weyl inequality [3, Theorem 2.8.1], the interlacing theorem [3, Corollary 2.5.2], and (3.8), we derive

$$
\lambda(\mathcal{LM}(G, x)) = \lambda(D_G(T(G, u)) + A(T(G, u))) \\
\leq \lambda(D_G(T(G, u))) + \lambda(A(T(G, u))) \\
\leq \Delta + \lambda(A(B_{\Delta-1, 2\ell+1})) \\
= \Delta + 2\sqrt{\Delta} - 1 \cos \frac{\pi}{2\ell + 2},
$$

(3.9)

proving (3.7). Now, assume that the equality in (3.7) is achieved. Therefore, the equality in (3.9) occurs, and thus, the Perron–Frobenius theorem [3, Theorem 2.2.1] implies that $T(G, u)$ is isomorphic to $B_{\Delta-1, 2\ell+1}$. Since $\Delta \geq 2$, one can easily obtain that $G$ is a cycle. Conversely, if $G$ is a cycle, then $T(G, u)$ is a path on $2\ell + 1$ vertices. By Theorem 3.4 and (3.8), we get

$$
\lambda(\mathcal{LM}(G, x)) = 2 + 2 \cos \frac{\pi}{2\ell + 2}.
$$

This completes the proof. \hfill \Box
Stevanović [15] proved that the eigenvalues of the adjacency matrix of a tree \( T \) are less than \( 2\sqrt{\Delta(T)} - 1 \). The corollary below gives an improvement of this upper bound for the subdivision of trees.

**Corollary 3.8** Let \( G \) be a graph with \( \Delta(G) \geq 2 \). Then

\[
\lambda(M(S(G), x)) < 1 + \sqrt{\Delta(G) - 1}.
\]  

(3.10)

In particular, if \( F \) is a forest with \( \Delta(F) \geq 2 \), then \( \lambda(A(S(F))) < 1 + \sqrt{\Delta(F) - 1} \).

**Proof** It follows from Theorem 3.7 that \( \lambda(LM(G, x)) < \Delta(G) + 2\sqrt{\Delta(G) - 1} \). Moreover, it follows from Corollary 2.3 that \( \lambda(M(S(G), x)) = \sqrt{\lambda(LM(G, x))} \).

From these, we find that

\[
\lambda(M(S(G), x)) < \sqrt{\Delta(G) + 2\sqrt{\Delta(G) - 1}} = 1 + \sqrt{\Delta(G) - 1},
\]

proving (3.10). As the subdivision of a forest is a forest, the ‘in particular’ statement follows from Theorem 2.7 and (3.10).

**Remark 3.9** Note that \( \Delta(S(G)) = \Delta(G) \) for every graph \( G \) with \( \Delta(G) \leq 2 \). So, for the subdivision of a graph with the maximum degree at least 2, the upper bound which appears in (3.10) is sharper than the upper bound that comes from (1.1).

We demonstrated in Theorem 3.4 that the largest zero of the Laplacian matching polynomial has the multiplicity 1. In the following theorem, we prove the remaining statements of (1.7) as analogues of the results given in (1.3).

**Theorem 3.10** Let \( G \) be a graph and let \( n = |V(G)| \). For each edge \( e \in E(G) \), the zeros of \( LM(G, x) \) and \( LM(G - e, x) \) interlace in the sense that, if \( \alpha_1 \leq \cdots \leq \alpha_n \) and \( \beta_1 \leq \cdots \leq \beta_n \) are respectively, the zeros of \( LM(G, x) \) and \( LM(G - e, x) \), then \( \beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_n \leq \alpha_n \). Also, if \( G \) is connected, then \( \lambda(LM(G, x)) > \lambda(LM(H, x)) \) for any proper subgraph \( H \) of \( G \).

**Proof** Fix an edge \( e \in E(G) \) and denote by \( v_e \) the vertex of \( S(G) \) corresponding to \( e \). Let \( \alpha_1 \leq \cdots \leq \alpha_n \) and \( \beta_1 \leq \cdots \leq \beta_n \) be the zeros of \( LM(G, x) \) and \( LM(G - e, x) \), respectively. Corollary 2.3 yields that \( \sqrt{\alpha_1} \leq \cdots \leq \sqrt{\alpha_n} \) is the end part of the nongenerating sequence which consists of all the zeros of \( LM(G, x) \) and \( \sqrt{\beta_1} \leq \cdots \leq \sqrt{\beta_n} \) is the end part of the nongenerating sequence which consists of all the zeros of \( LM(G - e, x) \). As \( S(G - e) = S(G) - v_e \), it follows from (1.3) that the zeros of \( LM(S(G), x) \) and \( LM(S(G - e), x) \) interlace. So, we find that

\[
\sqrt{\beta_1} \leq \sqrt{\alpha_1} \leq \sqrt{\beta_2} \leq \sqrt{\alpha_2} \leq \cdots \leq \sqrt{\beta_n} \leq \sqrt{\alpha_n}
\]

which means that \( \beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \cdots \leq \beta_n \leq \alpha_n \), as desired.

Now, assume that \( G \) is connected. Let \( H \) be a proper subgraph of \( G \) and let \( u \in V(H) \). As \( T(H, u) \) is a proper subgraph of \( T(G, u) \), if \( R \) denotes the submatrix of \( D_G(T(G, u)) + A(T(G, u)) \) corresponding to the vertices in \( V(T(H, u)) \), then
$R - (D_H(T(H, u)) + A(T(H, u)))$ is a nonzero matrix with nonnegative entries. So, by applying Theorem 3.4 and the Perron–Frobenius theorem [3, Theorem 2.2.1], we get

$$\lambda(\mathcal{LM}(G, x)) = \lambda\left(D_G(T(G, u)) + A(T(G, u))\right) > \lambda(R) \quad > \lambda\left(D_H(T(H, u)) + A(T(H, u))\right) = \lambda(\mathcal{LM}(H, x)).$$

\[\square\]

**Remark 3.11** For every graph $G$ and real number $\alpha$, let $m_G(\alpha)$ denote the multiplicity of $\alpha$ as a zero of $\mathcal{LM}(G, x)$. As a consequence of Theorem 3.10, we have $|m_G(\alpha) - m_{G-e}(\alpha)| \leq 1$ for each edge $e \in E(G)$.

It is known that among all trees with a fixed number of vertices the path has the smallest value of the largest Laplacian eigenvalue [12]. The following result can be considered as an analogue of this fact and is obtained from Theorems 2.7 and 3.10.

**Corollary 3.12** Let $P_n$ and $K_n$ be the path and complete graph on $n$ vertices, respectively. For any connected graph $G$ on $n$ vertices which is not $P_n$ and $K_n$,

$$\lambda(\mathcal{LM}(P_n, x)) < \lambda(\mathcal{LM}(G, x)) < \lambda(\mathcal{LM}(K_n, x)).$$

### 4 Concluding remarks

In this paper, we have discovered some properties of the location of zeros of the Laplacian matching polynomial. Most of our results can be considered as analogues of known results on the matching polynomial. Comparing to the matching polynomial, the Laplacian matching polynomial contains not only the information of the sizes of matchings in the graph but also the vertex degrees of the graph. Hence, it seems to be that more structural properties of graphs can be reflected by the Laplacian matching polynomial rather than the matching polynomial. For an instance, 0 is a zero of $\mathcal{LM}(G, x)$ if and only if $G$ is a forest, in while 0 is a zero of $\mathcal{M}(G, x)$ if and only if $G$ has no perfect matchings.

More interesting facts about the Laplacian matching polynomial can be concerned in further. For example, one may focus on the multiplicities of zeros of the Laplacian matching polynomial as there are many results on the multiplicities of zeros of the matching polynomial. In view of Remark 3.11, for every graph $G$ and real number $\alpha$, one may divide $E(G)$ into three subsets based on how the multiplicity of $\alpha$ changes when an edge of $G$ is removed. The corresponding problem about the matching polynomial is investigated by Chen and Ku [8]. Also, it is a known result that the multiplicity of a zero of the matching polynomial is at most the path partition number of the graph, that is, the minimum number of vertex disjoint paths required to cover
all the vertices of the graph [4, Theorem 6.4.5]. It seems to be an interesting problem to find a sharp upper bound on the multiplicity of a zero of the Laplacian matching polynomial.

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