One New Blowup Criterion for the 2D Full Compressible Navier-Stokes System

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Abstract

We establish a blowup criterion for the two-dimensional (2D) full compressible Navier-Stokes system. The criterion is given in terms of the divergence of the velocity field only, and is independent of the temperature. The criterion tells that once the strong solution blows up, the $L^\infty$-norm for the divergence of velocity blows up.

Keywords: Full compressible Navier-Stokes system, blowup criterion, vacuum.

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1 Introduction

The motion of compressible viscous, heat-conductive, ideal polytropic fluid is governed by the full compressible Navier-Stokes system. Suppose that the domain occupied by the fluid is $\Omega$. The whole system on $(0, T) \times \Omega$ consists of the following equations

$$
\begin{cases}
\rho_t + \text{div} (\rho u) = 0, \\
(\rho u)_t + \text{div} (\rho u \otimes u) - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \nabla P = 0, \\
c_v [(\rho \theta)_t + \text{div} (\rho u \theta)] - \kappa \Delta \theta + P \text{div} u = 2\mu |\mathcal{D}(u)|^2 + \lambda (\text{div} u)^2.
\end{cases}
$$

(1.1)

together with the initial-boundary conditions

$$
\begin{align*}
(\rho, u, \theta)|_{t=0} &= (\rho_0, u_0, \theta_0), \\
u &= 0, \quad \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{align*}
$$

(1.2)

In this paper, we consider the two-dimensional case, i.e., $\Omega$ is a bounded smooth domain in $\mathbb{R}^2$. $\rho, u = (u^1, u^2)^{tr}, \theta$ and $P = R \rho \theta (R > 0)$ represent respectively the fluid density, velocity, absolute temperature and pressure. In addition, $\mathcal{D}(u)$ is the deformation tensor:

$$
\mathcal{D}(u) = \frac{1}{2}(\nabla u + (\nabla u)^{tr}).
$$

The constant viscosity coefficients $\mu$ and $\lambda$ satisfy the physical restrictions:

$$
\mu > 0, \quad \mu + \lambda \geq 0.
$$

(1.4)
Positive constants $c_v$ and $\kappa$ are respectively the heat capacity and the ratio of the heat conductivity coefficient over the heat capacity.

There is a huge amount of literature on the existence and large time behavior of solutions to (1.1). For the case that the initial density is far away from vacuum, local existence and uniqueness of classical solutions were proved in [21, 26, 28]. Matsumura-Nishida [25] first obtained the global classical solution when the initial data is close to a non-vacuum equilibrium in some Sobolev space $H^s$. Later, Hoff [8] constructed a global weak solution for discontinuous initial data, with the assumption that the initial density is strictly positive.

Normally the presence of vacuum state makes the problem more complicated. We recall some results about the weak solution for this case first. For the global weak solution to the barotropic case, the major breakthrough is due to Lions [24] and subsequently improved by Jiang-Zhang [20] and Feireisl [7]. They succeeded in constructing a global weak solution with finite energy when the pressure $P = \rho^\gamma$, $\gamma > \frac{N}{2}$, where $N$ is the dimension. Recently, Huang-Li [12] obtained the global weak solution to the full Navier-Stokes system (1.1) provided the initial energy is suitably small. Strong solutions have also been under investigation. The first local existence result was derived by Cho-Kim [4]. For a global classical solution with small energy, refer to [12, 16]. On the other hand, Xin [33] first contributed a remarkable blow-up result. He proved that if the initial density has compact support, any smooth solution to the Cauchy problem of the full Navier-Stokes equations without heat conduction blows up in finite time. In this direction, for more recent progress, see [3, 27] and references therein.

Taking into consideration both the local existence results and the blowup results, then it is important to study the mechanism of possible blowup and structure of possible singularity. For the blowup criterion for the compressible flow, there have been many results, [3,5,6,10,11,14,15,18,19,30,32]. It should be mentioned here that Huang-Li-Xin [15] first established a Beale-Kato-Majda type blowup criterion for the barotropic case. In fact they proved that if $T^*$ is the maximal time of existence for local strong solution, then

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty} \, dt = \infty,$$  

under the assumption $7\mu > \lambda$ when $\Omega$ is a three-dimensional domain. Jiang-Ou [19] extended this criterion to the full Navier-Stokes system (1.1) over a periodic domain or unit square domain of $\mathbb{R}^2$ and proved that

$$\lim_{T \to T^*} \int_0^T \|\nabla u\|_{L^\infty} \, dt = \infty.$$  

Recently, Huang-Li-Wang [13] obtained a Serrin type blow up criterion for (1.1) in $\mathbb{R}^N$. Here is the criterion,

$$\lim_{T \to T^*} \int_0^T \left( \|\text{div} u\|_{L^\infty} + \|u\|_{L^r}^s \right) \, dt = \infty, \quad \frac{2}{s} + \frac{N}{r} = 1, \quad N < r \leq \infty.$$  

which is analogue to the Serrin criterion for the 3D incompressible Navier-Stokes equations. In particular, for $N = 2$, if one can bound a priorily $\|u\|_{L^2(0,T;L^\infty)}$-norm or $\|u\|_{L^4(0,T;L^4)}$-norm, then (1.7) can be replaced by

$$\lim_{T \to T^*} \int_0^T \|\text{div} u\|_{L^\infty} \, dt = \infty.$$  

(1.8)
If (1.8) is proved, it is an improvement of the work by Jiang-Ou [19] and it reveals some connection between the compressible and incompressible Navier-Stokes equations, since global strong solution with vacuum was obtained for 2D incompressible case [17]. The question is we can not get the uniform bound of \( \| u \|_{L^2(0,T;L^4)} \) or \( \| u \|_{L^4(0,T;L^4)} \) from the a priori energy estimate. The aim of our paper is to verify (1.8) and the key trick is the use of Lemma 2.3 below, one critical Sobolev embedding inequality.

The results such as (1.5) or (1.6) or (1.7), notice that they are all in terms of velocity field only. There is another big class of results which are in terms of density \( \rho \) and temperature \( \theta \). For example, Sun-Wang-Zhang [30] obtained the following criterion in 3D,

\[
\lim_{T \to T^*} \sup_{0 \leq t \leq T} \{ \| \rho \|_{L^\infty} + \| \rho^{-1} \|_{L^\infty} + \| \theta \|_{L^\infty} \} = \infty. \tag{1.9}
\]

Fang-Zi-Zhang [6] extended the result to the 2D problem with a refiner form,

\[
\lim_{T \to T^*} \sup_{0 \leq t \leq T} \{ \| \rho \|_{L^\infty} + \| \theta \|_{L^\infty} \} = \infty. \tag{1.10}
\]

Before stating our main result, we first explain the notations and conventions used throughout this paper. We denote

\[
\int f dx = \int \Omega f dx.
\]

For \( 1 \leq p \leq \infty \) and integer \( k \geq 0 \), the standard homogeneous and inhomogeneous Sobolev spaces are denoted by:

\[
\begin{align*}
L^p &= L^p(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = W^{k,2}, \\
W_0^{1,p} &= \{ u \in W^{1,p} \mid u = 0 \text{ on } \partial \Omega \}, \quad H_0^1 = W_0^{1,2}.
\end{align*}
\]

Let

\[
\dot{f} := f_t + u \cdot \nabla f
\]

denote the material derivative of \( f \).

Since we are going to work with the blowup criterion of the strong solutions, we’d like to recall the result for the existence of the local strong solution. The solution to the 3D full Navier-Stokes system with vacuum was obtained by Cho-Kim [4]. The method there can be applied to the case in this paper, i.e. the case that \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \). And the corresponding result can be stated as follows (or refer to [6]):

**Theorem 1.1** Let \( q \in (2, \infty) \) be a fixed constant. Assume that the initial data satisfy

\[
\rho_0 \geq 0, \quad \rho_0 \in W^{1,q}, \quad u_0 \in H_0^1 \cap H^2, \quad \theta_0 \in H^2,
\]

with the compatibility conditions

\[
\begin{align*}
\mu \Delta u_0 + (\lambda + \mu) \nabla \text{div} u_0 - R \nabla (\rho_0 \theta_0) &= \frac{1}{\rho_0^2} g_1, \quad (1.11) \\
\kappa \Delta \theta_0 + \frac{\mu}{2} \nabla u_0 + (\nabla u_0)^{tr} \nabla \theta_0 + \lambda (\text{div} u_0)^2 &= \frac{1}{\rho_0^2} g_2, \quad (1.12)
\end{align*}
\]

for some \( g_1, g_2 \in L^2 \). Then there exist a positive constant \( T_0 \) and a unique strong solution \((\rho, u, \theta)\) to the system (1.7)-(1.3) such that

\[
\begin{align*}
\rho &\geq 0, \quad \rho \in C([0,T_0]; W^{1,q}), \quad \theta \in C([0,T_0]; H^2), \quad (1.13) \\
u &\in C([0,T_0]; H_0^1 \cap H^2), \quad (u, \theta) \in L^2([0,T_0]; W^{2,q}), \quad (1.14) \\
(u_t, \theta_t) &\in L^2([0,T_0]; H^1), \quad (\sqrt{\rho} u_t, \sqrt{\rho} \theta_t) \in L^\infty([0,T_0]; L^2). \quad (1.15)
\end{align*}
\]
Regarding the blowup criterion for the local strong solution, here is our main theorem.

**Theorem 1.2** Suppose the assumptions in Theorem 1.1 are satisfied and \((\rho, u, \theta)\) is the strong solution. Let \(T^*\) be the maximal time of existence for that strong solution. If \(T^* < \infty\), then

\[
\lim_{T \to T^*} \| \text{div} u \|_{L^1(0,T;L^\infty)} = \infty.
\]

A few remarks are in order,

**Remark 1.1** It is worth noting that the conclusion in Theorem 1.2 is somewhat surprising since the criterion \((1.16)\) is independent of the temperature and is the same as that of barotropic case (\([14]\)). In fact, it seems that the nonlinearity of the highly nonlinear terms \(|\mathcal{D}(u)|^2\) and \((\text{div} u)^2\) in the temperature equation is stronger than that of \(\text{div}(\rho u \otimes u)\) in the momentum equations (\([31]\)), however, \((1.16)\) shows that the nonlinear term \(|\nabla u|^2\) can be controlled provided one can control \(\text{div}(\rho u \otimes u)\).

**Remark 1.2** It is well known that the 2D incompressible homogeneous Navier-Stokes system has a unique global strong solution if the initial velocity belongs to \(L^2\) or some more regular space, and recently it is proved in \([17]\) that the 2D incompressible non-homogenous Navier-Stokes system also has a unique global strong solution under some compatibility conditions, so the result in our paper is reasonable from this point. The blowup criterion here shows that \(\text{div} u\) plays an important role in the fluid dynamics.

**Remark 1.3** The techniques in this paper can be easily adapted to the two dimensional periodic case. And the same criterion will be derived.

The rest of the paper is organized as follows: In Section 2, we collect some elementary facts and inequalities. The main result, Theorem 1.2, will be proved in Section 3.

### 2 Preliminaries

In this section, we recall some known facts and elementary inequalities that will be used later.

The first proposition is for the Lamé system, which comes from the momentum equation \((1.1)_2\). Assume that \(\Omega \subset \mathbb{R}^2\) is a bounded smooth domain. Suppose \(U \in H^1_0\) is a weak solution to the Lamé system,

\[
\begin{aligned}
\mu \Delta U + (\mu + \lambda) \nabla \text{div} U &= F, \quad \text{in} \ \Omega, \\
U(x) &= 0, \quad \text{on} \ \partial \Omega.
\end{aligned}
\]

In what follows, we denote \(LU = \mu \Delta U + (\mu + \lambda) \nabla \text{div} U\). Owing to the uniqueness of solution, we denote \(U = L^{-1} F\).

The system is an elliptic system under the assumption \((1.4)\), hence some regularity estimates can be derived. For a proof, refer to \([30]\).

**Proposition 2.1** Let \(q \in (1, \infty)\). Then there exists some constant \(C\) depending only on \(\lambda, \mu, p\) and \(\Omega\) such that
• if $F \in L^p$, then
  \[ \|U\|_{W^{2,p}} \leq C \|F\|_{L^p}; \tag{2.2} \]

• if $F \in W^{-1,p}$ (i.e., $F = \text{div} f$ with $f = (f_{ij})_{2 \times 2}$, $f_{ij} \in L^p$), then
  \[ \|U\|_{W^{1,p}} \leq C \|f\|_{L^p}. \tag{2.3} \]

Moreover, for the endpoint case, if $f_{ij} \in L^\infty \cap L^2$, then $\nabla U \in BMO(\Omega)$ and there exists some constant $C$ depending only on $\lambda, \mu, \Omega$ such that
  \[ \|\nabla U\|_{BMO(\Omega)} \leq C \left( \|f\|_{L^\infty} + \|f\|_{L^2} \right). \tag{2.4} \]

Here $\|g\|_{BMO(\Omega)} := \|g\|_{L^2} + [g]_{BMO(\Omega)}$, with
  \[ [g]_{BMO(\Omega)} := \sup_{x \in \Omega, r \in (0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |g(y) - g_{\Omega_r(x)}| \, dy, \]
  \[ g_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} g(y) \, dy, \]
where $\Omega_r(x) = B_r(x) \cap \Omega$ and $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

Two logarithmic Sobolev inequalities will be presented, which originate from the work owing to Brezis-Gallouet [1] and Brezis-Wainger [2]. The first one, together with Proposition 2.1, will give the estimate of $\nabla \rho$. For its proof, see also [30].

**Lemma 2.2** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^2$ and $f \in W^{1,p}$ with $p \in (2, \infty)$, there exists a constant $C$ depending only on $p$ such that
  \[ \|f\|_{L^\infty} \leq C \left( 1 + \|f\|_{BMO(\Omega)} \ln(e + \|f\|_{W^{1,p}}) \right). \tag{2.5} \]

The second inequality is in terms of both space integral and time integral. The proof can be found in [17] or refer to [23] for a similar proof. It plays an important role in the proof of Lemma 3.3.

**Lemma 2.3** Let $\Omega$ be a smooth domain in $\mathbb{R}^2$, and $f \in L^2(0, T; H_0^1 \cap W^{1,p})$, with $p > 2$. Then there exists a constant $C$ depending only on $p$ such that
  \[ \|f\|_{L^2(0,T;L^\infty)}^2 \leq C \left( 1 + \|f\|_{L^2(0,T;H^1)}^2 \ln(e + \|f\|_{L^2(0,T;W^{1,p})}) \right). \tag{2.6} \]

### 3 Proof of Theorem 1.2.

Let $(\rho, u, \theta)$ be a strong solution described in Theorem 1.2. Suppose that (1.16) is false, i.e.,
  \[ \lim_{T \to T^*} \|\text{div}u\|_{L^1(0,T;L^\infty)} \leq M_0 < +\infty, \tag{3.1} \]
which together with (1.11) yields immediately the following upper bound of the density (see [14, Lemma 3.4]).
Lemma 3.1 Assume that (3.1) holds. Then it is true that for $0 \leq T < T^*$,
\[
\sup_{0 \leq t \leq T} \| \rho \|_{L^\infty} \leq C, \tag{3.2}
\]
where and in what follows, $C, C_1, C_2$ and $C_3$ denote generic constants depending only on $M_0, \mu, \lambda, R, \kappa, c_v, T^*$, and the initial data.

The next estimate is similar to an energy estimate.

Lemma 3.2 Under the assumption (3.1), it holds that for $0 \leq T < T^*$,
\[
\sup_{0 \leq t \leq T} \int (\rho \theta + \rho |u|^2) \, dx + \int_0^T \| \nabla u \|_{L^2}^2 \, dt \leq C. \tag{3.3}
\]

Proof. Applying standard maximum principle to (1.1) together with $\theta_0 \geq 0$ (c.f. [5, 7]) shows that
\[
\inf_{\mathbb{R}^3 \times [0,T]} \theta(x,t) \geq 0. \tag{3.4}
\]
It follows from (1.1) that the specific energy $E \equiv c_v \theta + \frac{1}{2} |u|^2$ satisfies
\[
(\rho E)_t + \text{div}(\rho E u + P u) = \Delta (\kappa \theta + \frac{1}{2} \mu |u|^2) + \mu \text{div}(u \cdot \nabla u) + \lambda \text{div}(u \text{div} u). \tag{3.5}
\]
Integrating (3.5) over $\Omega \times [0,T]$ directly yields
\[
\sup_{0 \leq t \leq T} \int (\rho \theta + \rho |u|^2) \, dx \leq C. \tag{3.6}
\]
Multiplying (1.1) by $u$ and integrating the resulting equation over $\Omega$, we obtain after using (3.4) and (3.6) that
\[
\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 \, dx + \mu \| \nabla u \|_{L^2}^2 + (\mu + \lambda) \| \text{div} u \|_{L^2}^2 \\
\leq C \| \text{div} u \|_{L^\infty} \int \rho \theta \, dx \\
\leq C \| \text{div} u \|_{L^\infty},
\]
which together with (3.1) and (3.6) gives (3.3). It completes the proof of Lemma 3.2.

For a slightly higher order estimate, we derive the bound for the $L^\infty(0,T;L^2)$-norm of $\nabla u$, which will play a key role in obtaining more high order estimates.

Lemma 3.3 Under the condition (3.1), it holds that for $0 \leq T < T^*$,
\[
\sup_{0 \leq t \leq T} \int (\rho \theta^2 + |\nabla u|^2) \, dx + \int_0^T \left( |\nabla \theta|^2 + \theta |\nabla u|^2 + \rho |\dot{u}|^2 \right) \, dx \, dt \leq C. \tag{3.7}
\]

Before the proof of Lemma 3.3 we present an auxiliary lemma, which controls $L^p$-norm of $\theta$ by $\| \nabla \theta \|_{L^2}$. And it will help the proof of Lemma 3.3.

Lemma 3.4 Under the condition (3.1), it holds on $[0,T^*)$ that for every $p \in [1, \infty)$,
\[
\| \theta \|_{L^p} \leq C + C(p) \| \nabla \theta \|_{L^2}. \tag{3.8}
\]
Proof. Denote by \( \bar{\theta} = \frac{1}{|\Omega|} \int \theta \, dx \), the average of \( \theta \),
\[
|\bar{\theta}| \int \rho \, dx \leq \left| \int \rho \theta \, dx \right| + \left| \int \rho (\theta - \bar{\theta}) \, dx \right| \leq C + C\|\nabla \theta\|_{L^2},
\]
which together with Poincaré’s inequality implies
\[
\|\theta\|_{L^2} \leq C + C\|\nabla \theta\|_{L^2}.
\]
And consequently, (3.8) holds.

Proof of Lemma 3.3. First, multiplying (1.13) by \( \theta \) and integrating the resulting equation over \( \Omega \) lead to
\[
\frac{d}{dt} \int \rho \theta^2 \, dx + 2\kappa\|\nabla \theta\|_{L^2}^2 \leq C\|\text{div}u\|_{L^\infty} \int \rho \theta \, dx + C \int |\nabla u|^2 \, dx.
\]
To make the estimate close, one needs to bound the term \( \int \theta |\nabla u|^2 \, dx \) in (3.11). To achieve that, we borrowed the idea from [13], multiplying (1.12) by \( u \theta \) and integrating the resulting equation over \( \Omega \). Then
\[
\mu \int |\nabla u|^2 \theta \, dx + (\mu + \lambda) \int |\text{div} u|^2 \theta \, dx = - \int \rho \dot{u} \cdot u \theta \, dx - \mu \int u \cdot \nabla u \cdot \nabla \theta \, dx - (\mu + \lambda) \int \text{div}(u \cdot \nabla \theta) \, dx - \int \nabla P \cdot u \theta \, dx.
\]
We estimate the terms on the righthand of (3.12). By the Young’s inequality,
\[
\left| \int \rho \dot{u} \cdot u \theta \, dx \right| \leq \eta \int |\dot{u}|^2 \, dx + C_\eta \int \rho \theta^2 |u|^2 \, dx,
\]
and
\[
\mu \int u \cdot \nabla u \cdot \nabla \theta \, dx + (\mu + \lambda) \int \text{div}(u \cdot \nabla \theta) \, dx \leq \frac{\epsilon}{4}\|\nabla \theta\|_{L^2}^2 + C_\epsilon \int |u|^2 |\nabla u|^2 \, dx,
\]
where \( \eta, \epsilon \) are small positive constants to be determined later. Using integration by parts,
\[
\left| \int \nabla P u \theta \, dx \right| = \left| \int P \theta \text{div} u \, dx + \int P u \cdot \nabla \theta \, dx \right| \leq \frac{\epsilon}{4}\|\nabla \theta\|_{L^2}^2 + C\|\text{div} u\|_{L^\infty} \int \rho \theta^2 \, dx + C_\epsilon \int \rho \theta |\nabla u|^2 \, dx.
\]
Combining the estimates (3.11)-(3.15), we obtain after choosing \( \epsilon \) suitably small that
\[
\frac{d}{dt} \int \rho \theta^2 \, dx + \int (\theta |\nabla u|^2 + \kappa |\nabla \theta|^2) \, dx \leq C\eta \int |\dot{u}|^2 + C\|\text{div} u\|_{L^\infty} \int \rho \theta^2 \, dx + C_\eta \int (\rho \theta^2 |u|^2 + |u|^2 |\nabla u|^2) \, dx.
\]
Note that there are some terms such as $\int \rho |\dot{u}|^2 \, dx$ whose estimates are not clear. These terms look less frightening than $\int \theta |\nabla u|^2 \, dx$, if one is familiar with the techniques for regularity estimates of compressible Navier-Stokes equation. One standard way is to multiply (1.1) by $u_t$ and integrate the resulting equation over $\Omega$. Then by the Young’s inequality, we obtain that

\[
\frac{1}{2} \frac{d}{dt} \int [\mu |\nabla u|^2 + (\mu + \lambda)(\text{div}u)^2] \, dx + \int \rho |\dot{u}|^2 \, dx = \int \rho \dot{u} \cdot (u \cdot \nabla)u \, dx + \int \text{Pdiv}u_t \, dx \leq \frac{1}{4} \int \rho |\dot{u}|^2 \, dx + C \int |u|^2 |\nabla u|^2 \, dx + \frac{d}{dt} \int \text{Pdiv}u \, dx - \int \text{Pdiv}u \, dx. \tag{3.17}
\]

To deal with the term $\int \text{Pdiv}u \, dx$, we employ some technique which is a combination of those from [30] and [32]. First, we split $u$ into two parts, $v$ and $w$. Let $v = L^{-1} \nabla P$ and $w = u - v$. (In what follows, $v$ and $w$ always denote $L^{-1} \nabla P$ and $u - v$. ) As noted in [30], $\text{div}w$ acts as the effective viscous flux for the bounded domain case. Now

\[
\int \text{Pdiv}u \, dx = \int \text{Pdiv}v \, dx + \int \text{Pdiv}w \, dx.
\]

Herein

\[
\int \text{Pdiv}v \, dx = -\int \nabla \text{Pdiv} \, dx = -\int (\mathcal{L}v)_t \, dx \tag{3.18}
\]

and according to [35],

\[
\int \text{Pdiv}w \, dx = R \left\{ \int (c_v + R) \rho \theta u - \kappa \nabla \theta - \mu \nabla u \cdot u - \mu u \cdot \nabla u - \lambda u \text{div}u \right\} \cdot \nabla \text{div}w \, dx + \frac{1}{2} \int \text{div}(\rho u)|u|^2 \text{div}w \, dx - \int \rho u \cdot \text{div}w \, dx
\]

By virtue of Proposition 2.1 we have

\[
\|\nabla v\|_{L^2} \leq C \|\rho \theta\|_{L^2}, \tag{3.20}
\]
By Proposition 2.1 and Lemma 3.4, we get that
\[
\|\nabla^2 w\|_{L^2} \leq C \|\rho \dot{u}\|_{L^2}.
\] (3.21)

Making use of these two inequalities (3.20) and (3.21), we obtain that
\[
- \int P \text{div} w \, dx \leq C \left( \|\sqrt{\rho} \|_{L^2} \|u\|_{L^\infty} + \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2} \|u\|_{L^\infty} \right) \|\nabla \text{div} w\|_{L^2}
\]
\[
+ C \|\sqrt{\rho} \|_{L^2} \|u\|_{L^\infty} \|\text{div} w\|_{L^2}
\]
\[
\leq C \delta \|\sqrt{\rho} \|_{L^2} \|u\|_{L^\infty}^2 + C \delta \|\nabla \theta\|_{L^2}^2 + C \delta \|\nabla u\|_{L^2}^2 \|u\|_{L^\infty}^2 + \delta \|\sqrt{\rho} \|_{L^2}^2.
\] (3.22)

Substituting (3.18) and (3.22) into (3.17), we obtain after choosing \(\delta\) suitably small that
\[
\frac{d}{dt} \int \left( \mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2 \right) \, dx + \int \rho |\dot{u}|^2 \, dx
\]
\[
\leq C \|\sqrt{\rho} \|_{L^2} \|u\|_{L^\infty}^2 + C \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|u\|_{L^\infty}^2.
\] (3.23)

Choose some constant \(C_2 \geq C_1 + 1\) suitably large such that
\[
\mu |\nabla u|^2 - 2 \rho \text{div} u + C_2 \rho \theta^2 \geq \frac{3}{4} |\nabla u|^2 + \rho \theta^2.
\]

Adding (3.16) multiplied by \(C_2\) to (3.23), we have after choosing \(\eta\) suitably small that
\[
\frac{d}{dt} \int \left( \mu |\nabla u|^2 + (\mu + \lambda) (\text{div} u)^2 - 2 \rho \text{div} u + C_2 \rho \theta^2 + \|(-L)^{1/2} v\|^2 \right) \, dx
\]
\[
+ \frac{1}{2} \int \rho |\dot{u}|^2 \, dx + \frac{1}{2} \int \rho |\nabla u|^2 \, dx + \int \rho \theta^2 \, dx + \int \kappa |\nabla \theta|^2 \, dx
\]
\[
\leq C \|\text{div} u\|_{L^\infty} \int \rho \theta^2 \, dx + C \|u\|_{L^\infty}^2 \left( \int \rho \theta^2 \, dx + \|\nabla u\|_{L^2}^2 \right).
\] (3.24)

Let
\[
\Psi(t) = e + \sup_{\tau \in [0,t]} \left( \|\nabla u(\tau)\|_{L^2}^2 + \int \rho \theta^2(\tau) \, d\tau \right) + \int_0^t \int (\rho |\dot{u}|^2 + \theta |\nabla u|^2 + |\nabla \theta|^2) \, dx \, d\tau.
\]

By virtue of Gownwall’s inequality, for every \(0 \leq s \leq T < T^*\),
\[
\Psi(T) \leq C \Psi(s) \exp \left\{ C \int_s^T \|u(\tau)\|_{L^\infty}^2 \, d\tau \right\}.
\] (3.25)

Now it is time to get a good control of \(\|u\|_{L^2(s,T;L^\infty)}\). Making use of Lemma 2.3, we can get that
\[
\|u\|_{L^2(s,T;L^\infty)}^2 \leq C \left( 1 + \|u\|_{L^2(s,T;H^1)}^2 \right) \ln \left( e + \|u\|_{L^2(s,T;W^{1,3})} \right).
\] (3.26)

By Proposition 2.1 and Lemma 3.4,
\[
\|u\|_{W^{1,3}} \leq C \|w\|_{W^{2,6}} + \|v\|_{W^{1,3}}
\]
\[
\leq C \|\rho \dot{u}\|_{L^6} + C \|P\|_{L^3} + C \|u\|_{L^2}
\]
\[
\leq C \|\rho \dot{u}\|_{L^2} + C \|\nabla \theta\|_{L^2} + C \|\nabla u\|_{L^2} + C,
\]
which implies that

\[ \|u\|_{L^2(s,T;W^{1,3})} \leq C \|\rho \dot{u}\|_{L^2(s,T;L^2)} + C \|\nabla \theta\|_{L^2(s,T;L^2)} + C \|\nabla u\|_{L^2(s,T;L^2)} + C \]

\[ \leq C \Psi(T). \quad (3.27) \]

Substituting (3.27) to (3.26),

\[ \|u\|_{L^2(s,T;L^\infty)} \leq C \left( 1 + \|u\|_{L^2(s,T;H^1)}^2 \ln (C \Psi(T)) \right). \quad (3.28) \]

Taking this inequality (3.28) back to (3.25), then we get

\[ \Psi(T) \leq C \Psi(s) C \Psi(T)^{3 \|u\|_{L^2(s,T;H^1)}^2} . \]

Recalling the energy like estimate (3.3), we choose some \( s \) which is close enough to \( T^* \) such that

\[ \lim_{T \to T^-} C_3 \|u\|_{L^2(s,T;H^1)}^2 \leq \frac{1}{2}, \]

then

\[ \Psi(T) \leq C \Psi(s)^2 < \infty, \quad (3.29) \]

which completes the proof for Lemma 3.3.

Lemma 3.3 tells that

\[ \lim_{T \to T^-} \|\nabla u\|_{L^\infty(0,T;L^2)} < \infty, \]

which implies that

\[ \lim_{T \to T^-} \|u\|_{L^4(0,T;L^4)} < \infty. \]

According Huang-Li-Wang [13]’s criterion (1.7), we can claim here that the strong solution can be extended. For readers’ convenience, we give the complete proof. The remaining proof consists of higher order estimates of the solutions which are needed to guarantee the extension of local strong solution to be a global one under the conditions (3.1) and (3.7). Compared to [13], there are some slight changes, since we consider the bounded case, instead of the whole space one.

**Lemma 3.5** Under the condition (3.1), it holds that for \( 0 \leq T < T^* \),

\[ \sup_{0 \leq t \leq T} \int \left( (\nabla \theta)^2 + \rho |\dot{u}|^2 \right) dx + \int_0^T \int \left( \rho \dot{\theta}^2 + |\nabla \dot{u}|^2 \right) dx dt \leq C. \quad (3.30) \]

**Proof.** First, applying \( \dot{u}^j \left[ \partial_t + \text{div}(u^j) \right] \) to (3.1) and integrating the resulting equation over \( \Omega \), we obtain after integration by parts that

\[ \frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 \, dx = - \int \dot{u}^j \left[ \partial_j P_t + \text{div}(u \partial_j P) \right] \, dx + \mu \int \dot{u}^j \left[ \Delta u^j_t + \text{div}(u \Delta u^j) \right] \, dx \]

\[ + (\mu + \lambda) \int \dot{u}^j \left[ \partial_j \text{div} u_t + \text{div}(u \partial_j \text{div} u) \right] \, dx \]

\[ = \sum_{i=1}^3 N_i. \quad (3.31) \]
We get after integration by parts and using the equation (1.1) that

\[ N_1 = - \int \dot{u}^i [\partial_j P_t + \text{div}(\partial_j Pu)] \, dx \]
\[ = R \int \partial_j \dot{u}^j \left( \rho \dot{\theta} - \rho u \cdot \nabla \theta - \theta u \cdot \nabla \rho - \theta \rho \text{div} u \right) \, dx + \int \partial_k \dot{u}^j \partial_j P u^k \, dx \]
\[ = R \int \partial_j \dot{u}^j \cdot \rho \theta \, dx - \int P \partial_k \dot{u}^j \partial_j u^k \, dx \]
\[ \leq \frac{\mu}{8} |\nabla \dot{u}|^2_{L^2} + C \| \rho \dot{\theta} \|^2_{L^2} + C \int \rho^2 \theta^2 |\nabla u|^2 \, dx \]
\[ \leq \frac{\mu}{8} |\nabla \dot{u}|^2_{L^2} + C \| \rho \dot{\theta} \|^2_{L^2} + C \| \theta \|_{L^4}^4 + C \| \nabla u \|_{L^4}^4. \] (3.32)

Integration by parts leads to

\[ N_2 = \mu \int \dot{u}^i [\Delta u^i + \text{div}(u \Delta u^i)] \, dx \]
\[ = -\mu \int \left( \partial_i \dot{u}^j \partial_j u^i + \Delta u^i u \cdot \nabla \dot{u}^j \right) \, dx \]
\[ = -\mu \int \left( |\nabla \dot{u}|^2 - \partial_i \dot{u}^j \partial_j \partial_i u^i - \partial_i \dot{u}^j \partial_j \partial_k u^i + \Delta u^i u \cdot \nabla \dot{u}^j \right) \, dx \] (3.33)
\[ = -\mu \int \left( |\nabla \dot{u}|^2 + \partial_i \dot{u}^j \partial_j \text{div} u - \partial_i \dot{u}^j \partial_j \partial_k u^i - \partial_i \partial_j \partial_k \partial_i \dot{u}^j \right) \, dx \]
\[ \leq -\frac{7\mu}{8} \int |\nabla \dot{u}|^2 \, dx + C \int |\nabla u|^4 \, dx. \]

Similarly, we have

\[ N_3 \leq -\frac{7}{8} (\mu + \lambda) \| \text{div} \dot{u} \|_{L^2}^2 + C \int |\nabla u|^4 \, dx. \] (3.34)

Substituting (3.32)-(3.34) into (3.31) implies

\[ \frac{d}{dt} \int \rho |\dot{u}|^2 \, dx + \mu \| \nabla \dot{u} \|_{L^2}^2 \leq C \int \rho \dot{\theta}^2 \, dx + C \| \theta \|_{L^4}^4 + C \int |\nabla u|^4 \, dx \]
\[ \leq C \int \rho \dot{\theta}^2 \, dx + C \| \nabla \theta \|_{L^2}^4 + C \| \sqrt{\rho \dot{u}} \|_{L^2}^4 + C, \] (3.35)

where for the last inequality we have used the fact,

\[ \| \nabla u \|_{L^4} \leq \| \nabla v \|_{L^4} + \| \nabla w \|_{L^4} \leq C \| \rho \theta \|_{L^4} + C \| \rho \dot{u} \|_{L^4/3} \]
\[ \leq C \| \nabla \theta \|_{L^2} + C \| \sqrt{\rho \dot{u}} \|_{L^2} + C, \] (3.36)

owing to Proposition 2.1 and Lemma 3.3.

Next, multiplying (1.1) by \( \dot{\theta} \) and integrating the resulting equation over \( \Omega \) yield that

\[ c_v \int \rho |\dot{\theta}|^2 \, dx + \frac{\kappa}{2} \frac{d}{dt} \int |\nabla \theta|^2 \, dx = \kappa \int \Delta \theta \cdot (u \cdot \nabla \theta) \, dx + \lambda \int (\text{div} u)^2 \dot{\theta} \, dx \]
\[ + 2\mu \int |\mathcal{D}(u)|^2 \dot{\theta} \, dx - R \int \rho \text{div} u \dot{\theta} \, dx \]
\[ \triangleq \sum_{i=1}^4 I_i. \] (3.37)
We estimate each $I_i (i = 1, \cdots, 4)$ as follows:

First, it follows from Sobolev embedding theory that for any $\epsilon \in (0, 1]$, we have

$$\int \theta^2 |\nabla u|^2 \, dx \leq C \|\theta\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \leq \epsilon \|\nabla^2 \theta\|_{L^2}^2 + C_\epsilon \|\nabla \theta\|_{L^2}^2 + C_\epsilon,$$  \hspace{1cm} (3.38)

which together with the standard $W^{2,2}$-estimate of (1.13) gives

$$\|\theta\|_{H^2}^2 \leq C \int \rho \theta^2 \, dx + C \int \rho^2 \theta^2 |\nabla u|^2 \, dx + C \int |\nabla u|^4 \, dx + C \|\theta\|_{L^2}^2 \leq C \int \rho \theta^2 \, dx + C \|\nabla \theta\|_{L^2}^2 + C_\delta \|\nabla \theta\|_{L^2}^2 + C \int |\nabla u|^4 \, dx + C_\delta.$$

Hence, choosing some $\epsilon$ small enough, we have

$$\|\theta\|_{H^2}^2 \leq C \int \rho \theta^2 \, dx + C \|\nabla \theta\|_{L^2}^2 + C \int |\nabla u|^4 \, dx + C.$$ \hspace{1cm} (3.39)

Consequently, by Gagliardo-Nirenberg inequality,

$$|I_1| \leq C \int |\nabla \theta| |\nabla^2 \theta| |u| \, dx \leq C \|\nabla^2 \theta\|_{L^2} \|\nabla \theta\|_{L^4} \|u\|_{L^4} \leq C \|\nabla^2 \theta\|_{L^2} \left(\|\nabla^2 \theta\|_{L^2} + \|\nabla \theta\|_{L^2}\right)^{1/2} \|\nabla \theta\|_{L^2}^{1/2} \leq \delta \|\nabla^2 \theta\|_{L^2}^2 + C_\delta \|\nabla \theta\|_{L^2}^2 \leq C \delta \int \rho \theta^2 \, dx + C_\delta \|\nabla \theta\|_{L^2}^2 + C_\delta \int |\nabla u|^4 \, dx + C_\delta.$$ \hspace{1cm} (3.40)

Next, integration by parts yields that, for any $\eta, \delta \in (0, 1]$, we have

$$I_2 = \lambda \int (\text{div} u)^2 \theta \, dx + \lambda \int (\text{div} u)^2 u \cdot \nabla \theta \, dx$$

$$= \lambda \left( \int (\text{div} u)^2 \theta \, dx \right)_t - 2\lambda \int \theta \text{div} \text{div}(\dot{u} - u \cdot \nabla u) \, dx$$

$$+ \lambda \int (\text{div} u)^2 u \cdot \nabla \theta \, dx$$

$$= \lambda \left( \int (\text{div} u)^2 \theta \, dx \right)_t - 2\lambda \int \theta \text{div} \text{div} \dot{u} \, dx$$

$$+ 2\lambda \int \theta \text{div} \text{div}(u \cdot \nabla u) \, dx + \lambda \int (\text{div} u)^2 u \cdot \nabla \theta \, dx$$

$$= \lambda \left( \int (\text{div} u)^2 \theta \, dx \right)_t - 2\lambda \int \theta \text{div} \text{div} \dot{u} \, dx$$

$$+ 2\lambda \int \theta \text{div} \delta_t u^j \partial_j u^i \, dx + \lambda \int u \cdot \nabla (\theta (\text{div} u)^2) \, dx$$

$$\leq \lambda \left( \int (\text{div} u)^2 \theta \, dx \right)_t + C \|\theta\| \|\nabla u\|_{L^2} \left(\|\nabla u\|_{L^2} + \|\nabla u\|_{L^4}^2\right)$$

$$\leq \lambda \left( \int (\text{div} u)^2 \theta \, dx \right)_t + \eta \|\nabla \dot{u}\|_{L^2}^2 + C_\delta \int \rho \theta^2 \, dx + C_\delta \|\nabla \theta\|_{L^2}^2 + C_\delta \|\nabla u\|_{L^4}^2 + C_\delta \eta,$$ \hspace{1cm} (3.41)
where in the last inequality, we have used (3.38) and (3.39).

Then, similar to (3.41), we have that, for any \( \eta, \delta \in (0, 1) \),

\[
I_3 \leq 2\mu \left( \int |\mathcal{D}(u)|^2 \theta \, dx \right)_t + \eta \|\nabla \hat{u}\|_{L^2}^2 + C\delta \int \rho \hat{\theta}^2 \, dx \\
+ C_{\delta, \eta} \|\nabla \theta\|_{L^2}^2 + C_{\delta} \|\nabla u\|_{L^4}^4 + C_{\delta, \eta}.
\]  

Finally, it follows from (3.38) and (3.39) that

\[
|I_4| \leq \delta \int \rho \hat{\theta}^2 \, dx + C_{\delta} \int \theta^2 |\nabla u|^2 \, dx \\
\leq C\delta \int \rho \hat{\theta}^2 \, dx + C_{\delta} \|\nabla \theta\|_{L^2}^2 + C_{\delta}.
\]  

(3.42)

Substituting (3.40)-(3.43) into (3.37), we obtain after choosing \( \delta \) suitably small that, for any \( \eta \in (0, 1) \),

\[
\frac{d}{dt} \int \left( \frac{K}{2} |\nabla \theta|^2 - \theta \left[ \lambda (\text{div} u)^2 + 2\mu |\mathcal{D}(u)|^2 \right] \right) \, dx + \frac{C_v}{2} \int \rho \hat{\theta}^2 \, dx \\
\leq C\eta \|\nabla \hat{u}\|_{L^2}^2 + C_{\eta} \|\nabla u\|_{L^4}^4 + C_{\eta} \|\nabla \theta\|_{L^2}^4 + C_{\eta} \\
\leq C\eta \|\nabla \hat{u}\|_{L^2}^2 + C_{\eta} \|\sqrt{\rho} \hat{u}\|_{L^2}^4 + C_{\eta} \|
\]  

(3.43)

where the last inequality is owing to (3.36).

Noticing that

\[
\int \theta \left[ \lambda (\text{div} u)^2 + 2\mu |D(u)|^2 \right] \, dx \\
\leq C \|\theta\|_{L^5} \|\nabla u\|_{L^{12/5}}^2 \\
\leq C (\|\nabla \theta\|_{L^2} + 1) \cdot \|\nabla u\|_{L^2}^{4/3} \|\nabla u\|_{L^4}^{2/3} \\
\leq C (\|\nabla \theta\|_{L^2} + 1) \left( \|\sqrt{\rho} \hat{u}\|_{L^2}^{2/3} + \|\rho \theta\|_{L^2}^{2/3} + 1 \right) \\
\leq \frac{K}{4} \|\nabla \theta\|_{L^2}^2 + \eta^{1/2} \|\sqrt{\rho} \hat{u}\|_{L^2}^2 + C_{\eta},
\]  

(3.45)

so adding (3.35) multiplied by \( 2\eta^{1/2} \) to (3.41), we obtain (3.30) after choosing \( \eta \) suitably small and using Gronwall’s inequality. Thus we complete the proof of Lemma 3.5.

As a corollary, we can bound \( \|\theta\|_{L^4} \) and \( \|\nabla u\|_{L^4} \).

**Corollary 3.6** Under the condition (3.1), it holds that for \( 0 \leq T < T^* \),

\[
\sup_{0 \leq t \leq T} (\|\theta\|_{L^4} + \|\nabla u\|_{L^4}) \leq C.
\]  

(3.46)

**Proof.** By virtue of Lemma 3.4 and Lemma 3.5,

\[
\|\theta\|_{L^4} \leq C \|\nabla \theta\|_{L^2} + C \leq C.
\]

Consequently, according to (3.36) and Lemma 3.5,

\[
\|\nabla u\|_{L^4} \leq C \|\rho \hat{u}\|_{L^2} + C \|\rho \theta\|_{L^4} + C \|\nabla u\|_{L^2} + C \leq C.
\]

Next, we will derive the desired estimates for \( \hat{\theta} \). In fact, we have
Lemma 3.7  Under the condition (3.1), it holds that for $0 \leq T < T^*$,
\[
\sup_{0 \leq t \leq T} \int \rho \dot{\theta}^2 \, dx + \int_0^T \| \nabla \dot{\theta} \|^2_{L^2} \, dt \leq C. \tag{3.47}
\]

Proof. Applying the operator $\partial_t + \text{div}(u \cdot \cdot \cdot)$ to (3.4), and using (3.1), we get
\[
c_v \rho \left( \partial_t \dot{\theta} + u \cdot \nabla \dot{\theta} \right) = \kappa \Delta \dot{\theta} + \kappa \left( \text{div} \Delta \theta - \partial_t \left( \partial_t u \cdot \nabla \theta \right) - \partial_t \dot{\theta} \nabla \theta \right) - R \rho \dot{\theta} \text{div} u - R \rho \text{div} \dot{\theta} u + \left( \lambda (\text{div} u)^2 + 2 \mu |\nabla u|^2 \right) \text{div} u + 2 \lambda \left( \text{div} \dot{\theta} u \partial_t u^k \right) \text{div} u + \mu \left( \partial_t u^j + \partial_t \dot{\theta}^i \right) \left( \partial_t \dot{\theta}^j + \partial_t u^i - \partial_t u^k \partial_t \dot{\theta}^j - \partial_t \dot{\theta}^k \partial_t \dot{\theta}^i \right). \tag{3.48}
\]

Multiplying (3.48) by $\dot{\theta}$, we obtain after integration by parts and Corollary 3.6 that
\[
\frac{c_v}{2} \left( \int \rho \dot{\theta}_t^2 \, dx \right) + \kappa \| \nabla \dot{\theta} \|^2_{L^2} \leq C \int |\nabla u| \left( |\nabla^2 \theta| |\dot{\theta}| + |\nabla \dot{\theta}| |\nabla \dot{\theta}| \right) \, dx + C \int |\nabla u|^2 |\dot{\theta}| |\nabla \dot{\theta}| \, dx + \int \rho \dot{\theta}_t^2 |\nabla u| \, dx + C \int \rho \dot{\theta}_t |\nabla \dot{\theta}| \, dx + C \int |\nabla \dot{\theta}| |\partial_t \dot{\theta}| \, dx \leq C \||\nabla u\|_{L^4} \||\nabla^2 \theta\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} + C \|
\]

It follows from (3.49) and Lemma 3.5 that
\[
\| \nabla^2 \theta \|_{L^2} \leq C \||\sqrt{\rho} \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} \||\nabla \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} + C \||\nabla \dot{\theta}\|_{L^2} \||\partial_t \dot{\theta}\|_{L^2} + C \|
\]

For the estimate for $\| \dot{\theta} \|_{L^4}$, we will follow the method used in Lemma 3.4. Let $\bar{\theta} = \frac{1}{\| \theta \|} \int \theta \, dx$,
\[
\bar{\theta} \int \rho \, dx \leq \int \rho \left( \dot{\theta} - \bar{\theta} \right) \, dx + \int \rho \bar{\theta} \, dx \leq C \| \nabla \dot{\theta} \|_{L^2} + C \| \sqrt{\rho} \dot{\theta} \|_{L^2},
\]

which together with Poincaré’s inequality leads to
\[
\| \dot{\theta} \|_{L^4} \leq C \| \nabla \dot{\theta} \|_{L^2} + C \| \bar{\theta} \| \leq C \| \nabla \dot{\theta} \|_{L^2} + C \| \sqrt{\rho} \dot{\theta} \|_{L^2}. \tag{3.52}
\]

And $\| \theta \|_{L^\infty}$ can be estimated as follows,
\[
\| \theta \|_{L^\infty} \leq C \| \nabla^2 \theta \|_{L^2} + C \| \theta \|_{L^2} \leq C \| \sqrt{\rho} \dot{\theta} \|_{L^2} + C. \tag{3.53}
\]
Substituting (3.50)-(3.53) to (3.49), we arrive at
\[
c_v \left( \int \rho |\dot{\theta}|^2 \, dx \right)_t + \kappa \| \nabla \dot{\theta} \|^2_{L^2}
\leq C \int \rho |\dot{\theta}|^2 \, dx + C\| \sqrt{\rho} \dot{\theta} \|_{L^2} \| \nabla \dot{\theta} \|_{L^2} + C\| \nabla \dot{\theta} \|^2_{L^2}
+ C\| \nabla \dot{u} \|_{L^2} \int \rho |\dot{\theta}|^2 \, dx + C\| \nabla \dot{u} \|_{L^2} \| \sqrt{\rho} \dot{\theta} \|_{L^2} + C\| \nabla \dot{u} \|^2_{L^2} + \frac{1}{4} \| \nabla \dot{\theta} \|^2_{L^2} + C,
\]
which together with the Gronwall’s inequality completes the proof for Lemma 3.7.

As a corollary, the bounds for \( \| \theta \|_{H^2} \) and \( \| \theta \|_{L^\infty} \) can be derived.

**Corollary 3.8** Under the condition (3.1), it holds that for \( 0 \leq T < T^* \),
\[
\sup_{0 \leq t \leq T} (\| \theta \|_{H^2} + \| \theta \|_{L^\infty}) \leq C. \tag{3.55}
\]

**Proof.** First, it follows from (3.39), Lemma 3.5, Corollary 3.6 and Lemma 3.7 that
\[
\| \nabla^2 \theta \|_{L^2} \leq C. \tag{3.56}
\]
Hence,
\[
\| \theta \|_{L^\infty} \leq C \| \theta \|_{H^2} \leq C. \tag{3.57}
\]

Up to now, we have get the bounds for \( \| \rho \|_{L^\infty} \) and \( \| \theta \|_{L^\infty} \), which imply other necessary high order estimates for the extension of the strong solution, according to the theorem proved in [6]. We sketch the proof for completeness.

**Corollary 3.9** Under the condition (3.1), it holds that for \( 0 \leq T < T^* \),
\[
\sup_{0 \leq t \leq T} \| w \|_{H^2} + \int_0^T \left( \| \nabla^2 w \|_{L^p}^2 + \| \nabla w \|_{L^\infty}^2 \right) \, dt \leq C, \quad p \in (2, \infty). \tag{3.58}
\]

**Proof.** By virtue of Proposition 2.1 and Lemma 3.5
\[
\| w \|_{H^2} \leq C \| \rho \dot{u} \|_{L^2} \leq C,
\]
and by Sobolev embedding inequality,
\[
\| \nabla w \|_{L^\infty} \leq C \| \nabla w \|_{W^{1,p}} \leq C \| \rho \dot{u} \|_{L^p} \leq C \| \nabla \dot{u} \|_{L^2},
\]
which implies (3.58).

The next lemma is used to bound the density gradient and \( \| u \|_{H^2} \).

**Lemma 3.10** Under the condition (3.1), it holds that for \( 0 \leq T < T^* \),
\[
\sup_{0 \leq t \leq T} (\| \rho \|_{W^{1,q}} + \| u \|_{H^2}) \leq C. \tag{3.59}
\]
Hence,

\[
\begin{aligned}
&\text{Substituting (3.62) into (3.60), we get that}
\end{aligned}
\]

\[
\begin{aligned}
\frac{d}{dt} \|\nabla \rho\|_{L^p} &\leq C(1 + \|\nabla u\|_{L^8})\|\nabla \rho\|_{L^p} + C\|\nabla^2 u\|_{L^p} \\
&\leq C(1 + \|\nabla v\|_{L^8} + \|\nabla w\|_{L^8})\|\nabla \rho\|_{L^p} + C(\|\nabla^2 v\|_{L^p} + \|\nabla^2 w\|_{L^p}) \\
&\leq C(1 + \|\nabla v\|_{L^8} + \|\nabla w\|_{L^8})\|\nabla \rho\|_{L^p} + C\|\nabla^2 w\|_{L^p} + C,
\end{aligned}
\]

where for the last inequality we used the fact

\[
\|\nabla^2 v\|_{L^p} \leq C\|\nabla (p\theta)\|_{L^p} \\
\leq C\|\nabla \rho\|_{L^p} \|\theta\|_{L^8} + C\|\nabla \theta\|_{L^p} \|\rho\|_{L^\infty} \\
\leq C\|\nabla \rho\|_{L^p} + C.
\]

To bound \(\|\nabla v\|_{L^\infty}\), we make use of the endpoint case of Proposition 2.21 Lemma 2.2 and (3.61),

\[
\|\nabla v\|_{L^\infty} \leq C(1 + \|\nabla v\|_{BMO(\Omega)} \ln(1 + \|\nabla v\|_{W^{1,p}})) \\
\leq C(1 + (\|P\|_{L^\infty} + \|P\|_{L^2}) \ln(1 + \|\nabla v\|_{W^{1,p}})) \\
\leq C(1 + \ln(e + \|\nabla \rho\|_{L^p})).
\]

Substituting (3.62) into (3.60), we get that

\[
\begin{aligned}
&\frac{d}{dt} (e + \|\nabla \rho\|_{L^p}) \\
&\leq C(1 + \|\nabla w\|_{L^8})\|\nabla \rho\|_{L^p} + C\ln(e + \|\nabla \rho\|_{L^p})\|\nabla \rho\|_{L^p} + C\|\nabla^2 w\|_{L^p} + C,
\end{aligned}
\]

which together with Gronwall’s inequality and Corollary 3.9 gives that

\[
\sup_{0 < T < T^*} \|\nabla \rho\|_{L^p} \leq C.
\]

Let \(p = q\), then we get the bound of \(\|\rho\|_{W^{1,q}}\).

Moreover, Let \(p = 2\) in (3.64), then by Corollary 3.9 and (3.61),

\[
\|\nabla^2 u\|_{L^2} \leq \|\nabla^2 w\|_{L^2} + \|\nabla^2 v\|_{L^2} \leq C + C\|\nabla \rho\|_{L^2} \leq C.
\]

It completes the proof of Lemma 3.10.

In view of Lemmas 3.1, 3.10 it is enough to extend the strong solution \((\rho, u, \theta)\) beyond \(t = T^*\). In fact, note that the generic constants \(C\) in Lemmas 3.1, 3.10 remains uniformly bounded for all \(T < T^*\), so the functions \((\rho, u, \theta)(x, T^*) \equiv \lim_{t \to T^*} (\rho, u, \theta)(x, t)\) satisfy the conditions imposed on the initial data at the time \(t = T^*\). Furthermore, standard arguments yield that \(\rho \dot{u}, \rho \dot{\theta} \in C([0, T^*]; L^2)\), which implies

\[
(\rho \dot{u}, \rho \dot{\theta})(x, T^*) = \lim_{t \to T^*} (\rho \dot{u}, \rho \dot{\theta})(x, t) \in L^2.
\]

Hence,

\[
\begin{aligned}
\mu \Delta u + (\mu + \lambda)\nabla \div u - R\nabla (\rho \dot{\theta})|_{t = T^*} = \sqrt{\rho}(x, T^*)g_1(x), \\
\kappa \Delta \theta + \frac{\mu}{2} \nabla u + |\nabla u|^2 + \lambda \div u^2 |_{t = T^*} = \sqrt{\rho}(x, T^*)g_2(x),
\end{aligned}
\]

\[
\begin{aligned}
\end{aligned}
\]

\[
\begin{aligned}
\end{aligned}
\]
with
\[ g_1(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)(\rho\dot{u})(x, T^*), & x \in \{x \mid \rho(x, T^*) > 0\}, \\ 0, & x \in \{x \mid \rho(x, T^*) = 0\}, \end{cases} \]
and
\[ g_2(x) \triangleq \begin{cases} \rho^{-1/2}(x, T^*)[\rho\dot{\theta} + R\rho\theta\text{div}\nu](x, T^*), & x \in \{x \mid \rho(x, T^*) > 0\}, \\ 0, & x \in \{x \mid \rho(x, T^*) = 0\}, \end{cases} \]
satisfying \( g_1, g_2 \in L^2 \) due to Lemma 3.10 and Corollary 3.8. Thus \( (\rho, u, \theta)(x, T^*) \) satisfies (1.11) and (1.12) also. Therefore, one can take \( (\rho, u, \theta)(x, T^*) \) as the initial data and apply Theorem 1.1 to extend the local strong solution beyond \( T^* \). This contradicts the assumption on \( T^* \). We thus finish the proof of Theorem 1.2.

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**References**

[1] H. Brezis and T. Gallouet, Nonlinear Schrödinger evolution equations, *Nonlinear Anal., T. M. A.* 4 (1980), 677-681.

[2] H. Brezis and S. Wainger, A note on limiting cases of Sobolev embeddings and convolution inequalities, *Comm. Partial Differential Equations* 5 (1980), 773-789.

[3] Y. Cho, B. J. Jin, Blow-up of viscous heat-conducting compressible flows, *J. Math. Anal. Appl.* 320 (2006), 819-826.

[4] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum, *J. Differential Equations* 228 (2006), 377-411.

[5] J. S. Fan, S. Jiang, Y. B. Ou: A blow-up criterion for compressible viscous heat-conductive flows, *Annales de l’Institut Henri Poincare (C) Analyse non lineaire* 27 (2010), 337-350.

[6] D. Y. Fang, R. Z. Zi., T. Zhang: A blow-up criterion for two dimensional compressible viscous heat-conductive flows, *Nonlinear Anal.* 75 (2012), 3130-3141.

[7] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford Science Publication, Oxford, 2004.

[8] D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential. Equations.* 120 (1995), 215-254.

[9] D. Hoff, Discontinuous solutions of the Navier-Stokes equation for multidimensional flows of heat-conducting fluids. *Arch. Rational Mech. Anal.* 139 (1997), 303–354.

[10] X. D. Huang, *Some results on blowup of solutions to the compressible Navier-Stokes equations*. PhD Thesis. The Chinese University of Hong Kong, 2009.
11] X. D. Huang, J. Li, On breakdown of solutions to the full compressible Navier-Stokes equations, *Methods Appl. Anal.* 16 (2009), no. 4, 479-490.

12] X. D. Huang, J. Li, Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations, [http://arxiv.org/abs/1107.4655](http://arxiv.org/abs/1107.4655).

13] X. D. Huang, J. Li, Y. Wang, Serrin-type blowup criterion for the full compressible Navier-Stokes system, submitted.

14] X. D. Huang, J. Li, Z. P. Xin, Serrin type criterion for the three-dimensional viscous compressible flows, *SIAM J. Math. Anal.* 43, (2011), 1872–1886.

15] X. D. Huang, J. Li, Z. P. Xin, Blowup criterion for viscous barotropic flows with vacuum states, *Comm. Math. Phys.* 301 (2011), 23-35.

16] X. D. Huang, J. Li, Z. P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, *Comm. Pure Appl. Math.* 65 (2012), no.4, 549-585.

17] X. D. Huang, Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system, *J. Differential Equations* 254 (2013), 511-527.

18] X. D. Huang, Z. P. Xin, A blow-up criterion for classical solutions to the compressible Navier-Stokes equations, *Sci. China Math.*, 53 (2010), no. 3, 671-686 (2010).

19] S. Jiang, Y. B. Ou: A blow-up criterion for compressible viscous heat-conductive flows, *Acta Math. Sci. Ser. B Engl. Ed.* 30 (2010), no. 6, 1851-1864.

20] S. Jiang, P. Zhang: On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations, *Comm. Math. Phys.* 215 (2001), no. 3, 559-581.

21] A. V. Kazhikhov, Cauchy problem for viscous gas equations, *Siberian Math. J.* 23 (1982), 44-49.

22] A. V. Kazhikhov, V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *J. Appl. Math. Mech.* 41 (1977), 273–282.

23] H. Kozono, T. Ogawa and Y. Taniuchi, The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations, *Math. Z.* 242 (2002), 251-278.

24] P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 2. Compressible models*, Oxford University Press, New York, 1998.

25] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases, *J. Math. Kyoto Univ.* 20 (1980), 67-104.

26] J. Nash, *Le problème de Cauchy pour les équations différentielles d’un fluide général*, *Bull. Soc. Math. France.* 90 (1962), 487-497.

27] O. Rozanova, Blow up of smooth solutions to the compressible Navier–Stokes equations with the data highly decreasing at infinity, *J. Differential Equations* 245 (2008), 1762-1774.
[28] J. Serrin, On the uniqueness of compressible fluid motion, *Arch. Rational. Mech. Anal.* 3 (1959), 271–288.

[29] J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.* 9 (1962) 187-195.

[30] Y. Z. Sun, C. Wang, Z. F. Zhang, A Beale-Kato-Majda Blow-up criterion for the 3-D compressible Navier-Stokes equations, *J. Math. Pures Appl.* 95 (2011) 36-47.

[31] Y. Z. Sun, C. Wang, Z. F. Zhang, A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows, *Arch. Rational Mech. Anal.* 201 (2011), 727-742.

[32] H. Y. Wen and C. J. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier-Stokes equations with vacuum, [http://arxiv.org/pdf/1111.2657.pdf](http://arxiv.org/pdf/1111.2657.pdf)

[33] Z. P. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure Appl. Math.* 51 (1998), 229-240.