Equation for spin decoherence rate in an all-electric ring

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Abstract

There is a quantitative error in the derivation for the spin decoherence rate by Talman in IPAC2012. The crucial point is a subtle confusion between the concept of ‘longitudinal’ as in ‘along the reference orbit’ and ‘parallel to the particle velocity.’ They are not the same direction, and the distinction is significant for high-precision experiments to search for a possible nonzero electric dipole moment (EDM) of a charged particle in a storage ring.

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I. INTRODUCTION

Richard and John Talman published a paper in IPAC2012 [1] giving some details of the orbital and spin motion in an all-electric storage ring. John Talman is Richard’s son, but the sins of the father will not be visited upon the son, at least for the moment; here and below, ‘Talman’ will mean exclusively Richard Talman.

In particular, Talman derived an equation for the spin decoherence rate, specifically something called $d\alpha/dt$ (eq. (10) in [1], see below). There is a quantitative error in Talman’s derivation. The core of the matter is a subtle but significant detail: the concept of ‘longitudinal.’ There is a confusion between ‘longitudinal’ as in ‘along the reference orbit’ and ‘longitudinal’ as in ‘parallel to the particle velocity.’ They are not the same direction, and the distinction is significant for high-precision experiments to search for a possible nonzero electric dipole moment (EDM) of a charged particle in a storage ring.

II. BASIC NOTATION

I shall treat a particle of mass $m$, charge $e$, with velocity $v = \beta c$ and Lorentz factor $\gamma = 1/\sqrt{1-\beta^2}$. (Talman [1] denotes the particle mass by $m_p$, and I shall also use this notation below.) The particle spin $s$ is treated as a unit vector and $g$ denotes the particle $g$-factor. There is no magnetic field in the model, the ring is all-electric. For simplicity of the exposition, I shall treat a smooth focusing model below. I employ cylindrical polar coordinates $(r, \theta, z)$. Following Talman [1], I shall treat orbital and spin motion in the horizontal plane only. In the horizontal plane, the electric field points radially.

III. HELICITY: ANGLE $\alpha$

First, I need to define the important angle $\alpha$, which is the angle between the spin unit vector $s$ and the unit vector in the direction of the velocity be $\hat{v}$. (Both vectors are assumed to lie in the horizontal plane, as stated above.) Following Talman [1], we go counterclockwise...
from \( \hat{\beta} \) to \( s \). Then

\[
s \cdot \hat{\beta} = \cos \alpha, \quad s \times \hat{\beta} = -\sin \alpha \hat{z}.
\]  

(1)

Also define \( x \) via \( r = r_0 + x \), where \( r_0 \) is a reference radius. Talman defines the electric field to point radially inward, so \( \mathbf{E} = -E(x) \hat{r} \). Then Talman writes (eq. (10) in [1])

\[
\frac{d\alpha}{dt} = \frac{eE(x)}{m_pc} \left( \frac{g\beta(x)}{2} - \frac{1}{\beta(x)} \right).
\]

(2)

Talman [1] cites Jackson [2] for the above equation. (Note that all references to Jackson’s textbook in this note are to the second edition.) However Jackson’s textbook does not contain the above equation. It has been derived from an equation in Jackson’s text. We must therefore begin our analysis from the actual equation written by Jackson.

**IV. EQUATION FROM JACKSON**

Jackson writes the following equation for the evolution of the longitudinal spin component (eq. (11.171) in [2])

\[
\frac{d}{dt}(\hat{\beta} \cdot s) = -\frac{e}{mc} s_\perp \cdot \left[ \left( \frac{g}{2} - 1 \right) \hat{\beta} \times \mathbf{B} + \left( \frac{g\beta}{2} - \frac{1}{\beta} \right) \mathbf{E} \right].
\]

(3)

Note that this equation is coordinate-free, and is completely general, and is not restricted to motion in a plane. Here \( \mathbf{E} \) and \( \mathbf{B} \) are the external electric and magnetic fields, respectively, and \( s_\perp \) is the spin component orthogonal to the direction of the velocity \( \hat{\beta} \), viz.

\[
s_\parallel = s \cdot \hat{\beta} \hat{\beta}, \quad s_\perp = s - s_\parallel = s - \hat{\beta} \hat{\beta} = \hat{\beta} \times (s \times \hat{\beta}).
\]

(4)

To make contact with eq. (2), we restrict the motion to the horizontal plane. As stated previously, there is no magnetic field in our model and the electric field in the horizontal plane is radial \( \mathbf{E} = -E(x)\hat{r} \). Then

\[
\frac{d}{dt}(\hat{\beta} \cdot s) = \frac{eE(x)}{m_pc} \left( \frac{g\beta}{2} - \frac{1}{\beta} \right) (\hat{\beta} \times (s \times \hat{\beta})) \cdot \hat{r}
\]

(5)

\[
\sin \alpha \frac{d\alpha}{dt} = \frac{eE(x)}{m_pc} \left( \frac{g\beta}{2} - \frac{1}{\beta} \right) (\hat{\beta} \times \hat{z} \sin \alpha) \cdot \hat{r}
\]

(6)

\[
\frac{d\alpha}{dt} = \frac{eE(x)}{m_pc} \left( \frac{g\beta}{2} - \frac{1}{\beta} \right) \hat{\beta} \cdot \hat{\theta}.
\]

(7)
The final factor of $\hat{\beta} \cdot \hat{\theta}$ is absent from Talman’s IPAC2012 paper [1] (see eq. (2) above). This is the subtlety of the concept of ‘longitudinal,’ viz. the direction along the reference orbit is not the same as the direction of the particle velocity. This has consequences which I shall spell out below.

V. EQUATION FOR $d\alpha/d\theta$

Starting from the equation for $d\alpha/dt$, Talman [1] derives an equation for $d\alpha/d\theta$ (eq. (12) in [1]). An equation for $d\alpha/d\theta$, essentially the rate of change of $\alpha$ per turn around the ring, is more useful for accelerator physics work. I derive the equation for $d\alpha/d\theta$ as follows. Note that

$$\hat{\beta} \cdot \hat{\theta} = \frac{v_\theta}{c} = \frac{r}{\beta c} \frac{d\theta}{dt}.$$ (8)

Hence from eq. (7),

$$\frac{d\alpha}{dt} = eE(x) m_p c \left( g\beta^2 - 1 \right) \frac{r}{\beta c} \frac{d\theta}{dt}.$$ (9)

$$\frac{d\alpha}{d\theta} = eE(x)r \left( \frac{g\beta^2 - 1}{\beta} \right).$$ (10)

Talman’s derivation to obtain an equation for $d\alpha/d\theta$ proceeds as follows. Talman notes that the (vertical) angular momentum $L$ is conserved, since the force (electric field) is radial, hence (eq. (11) in [1])

$$\frac{d\theta}{dt} = \frac{L}{\gamma m_p r^2}.$$ (11)

Hence from eq. (2)

$$\frac{d\alpha}{d\theta} = \frac{eE(x)r^2 \gamma}{Lc} \left( \frac{g\beta^2 - 1}{\beta} \right)$$

$$= \frac{eE(x)(r_0 + x)^2}{Lc} \left( \frac{g}{2} \beta^2 \gamma - \gamma \right)$$

$$= \frac{eE(x)(r_0 + x)^2}{Lc} \left( \left( \frac{g}{2} - 1 \right) \gamma - \frac{g/2}{\gamma} \right).$$ (12)

This is eq. (12) in [1].
VI. DIFFERENCES/CONSEQUENCES

Comparing the two expressions, I have $E(x) r/\beta$ (eq. (10)) whereas Talman has $E(x) r^{2\gamma}$ (eq. (12)). There are of course also factors of $L$, $m_p$ and $c$ to balance the dimensions, but they are constants. The different dependence on the orbit is sufficient to yield noticeable quantitative differences in a high-precision analysis. The distinction between the direction along the reference orbit and the direction of the particle velocity is subtle but significant.

[1] R. Talman and J. Talman, Proceedings of IPAC2012, New Orleans, 3203–3207 (2012).
[2] J. D. Jackson, Classical Electrodynamics 2nd ed., Wiley, New York (1975).
Appendix A: Averages over the orbit: errors of algebra by Talman

This Appendix is for accelerator specialists only. Talman [1] published the following formula for the spin decoherence rate for orbital and spin motion in the horizontal plane in an all-electric ring (eq. (17) in [1])

\[-\left\langle \frac{d\alpha}{d\theta} \right\rangle \approx \frac{E_0 r_0 \gamma_0}{(p_0 c/e) \beta_0} \left( \left\langle \frac{\gamma}{\gamma_0} - 1 \right\rangle + m \left\langle \frac{x}{r_0} \right\rangle - \frac{m^2 - m}{2} \frac{x^2}{r_0^2} \right) \tag{A1}\]

Talman employs the notation \( m \) for the field index (see below) and uses \( m_p \) for the particle mass. There is an error of algebra in the above formula, in addition to other errors I have pointed out above. Talman eq. (13) states

\[-\left\langle \frac{d\alpha}{d\theta} \right\rangle \approx \left\langle \frac{eE_0(r_0 + x)^2}{(L c \beta(x))} \right\rangle \left( \left( \frac{g}{2} - 1 \right) \left\langle \gamma \right\rangle - \frac{g}{2} \left\langle \frac{1}{\gamma} \right\rangle \right) \tag{A2}\]

Next Talman employs the relativistic virial theorem to deduce (Talman eq. (16))

\[\left\langle \frac{1}{\gamma} \right\rangle = \left\langle \gamma \right\rangle - \frac{E_0 r_0}{m_p c^2/e} \left\langle \frac{r_0^m}{r_{m}} \right\rangle \tag{A3}\]

Use this in eq. (A2). Also we operate at the magic gamma, so \( a = 1/(\beta_0^2 \gamma_0^2) \) and \( g/2 = 1 + a = 1/\beta_0^2 \). Then

\[-\left\langle \frac{d\alpha}{d\theta} \right\rangle \approx \frac{eE_0 r_0^2}{(p_0 c/e) \beta_0} \left( \left\langle \gamma \right\rangle - \frac{1}{\beta_0^2} \frac{m_p c^2 \gamma_0 \beta_0^2}{m_p c^2} \left\langle \frac{r_0^m}{r_{m}} \right\rangle \right) \]

\[\leq \frac{E_0 r_0 \gamma_0}{(p_0 c/e) \beta_0} \left( \left\langle \frac{\gamma}{\gamma_0} \right\rangle - \left\langle \frac{r_0^m}{r_{m}} \right\rangle \right) \tag{A4}\]

\[\leq \frac{E_0 r_0 \gamma_0}{(p_0 c/e) \beta_0} \left( \left\langle \frac{\gamma}{\gamma_0} \right\rangle - \left( 1 - m \frac{x}{r_0} + \frac{m(1+m)}{2} \frac{x^2}{r_0^2} \right) \right) \]

\[= \frac{E_0 r_0 \gamma_0}{(p_0 c/e) \beta_0} \left( \left\langle \frac{\gamma}{\gamma_0} - 1 \right\rangle + m \left\langle \frac{x}{r_0} \right\rangle - \frac{m^2 + m}{2} \frac{x^2}{r_0^2 \gamma_0} \right) \]

Hence there is an error of algebra in the last term of Talman eq. (17) (see eq. (A1)); the coefficient should be \( m^2 + m \) not \( m^2 - m \).