The Mixed-Isospin Vector Current Correlator in Chiral Perturbation Theory and QCD Sum Rules

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Abstract

The mixed-isospin vector current correlator, \( \langle 0 | T (V_\rho^\mu V_\omega^\nu) | 0 \rangle \) is evaluated using both QCD sum rules and Chiral Perturbation Theory (ChPT) to one-loop order. The sum rule treatment is a modification of previous analyses necessitated by the observation that those analyses produce forms of the correlator that fail to be dominated, near \( q^2 = 0 \), by the most nearby singularities. Inclusion of contributions associated with the \( \phi \) meson rectify this problem. The resulting sum rule fit provides evidence for a significant direct \( \omega \rightarrow \pi \pi \) coupling contribution in \( e^+ e^- \rightarrow \pi^+ \pi^- \). It is also pointed out that results for the \( q^2 \)-dependence of the correlator cannot be used to provide information about the (off-shell) \( q^2 \)-dependence of the off-diagonal element of the vector meson propagator unless a very specific choice of interpolating fields for the vector mesons is made. The results for the value of the correlator near \( q^2 = 0 \) in ChPT are shown to be more than an order of magnitude smaller than those extracted from the sum rule analysis and the reasons why this suggests slow
convergence of the chiral series for the correlator given.

11.55.Hx, 12.39.Fe, 14.40.Cs, 24.85.+p
I. INTRODUCTION

Non-electromagnetic isospin breaking is well-established in many strongly interacting systems (e.g., splittings in the hadron spectrum, binding energy differences in mirror nuclei, asymmetries in polarized np scattering, binding energies and level splittings of light Λ hypernuclei [1]). In few-body systems, an important source of this breaking has been thought to be the mixing of isoscalar and isovector mesons appearing in meson exchange diagrams. In particular, the bulk of the non-Coulombic contributions to the charge symmetry breaking $nn$-$pp$ scattering length difference and to the $A=3$ binding energy difference, and of the np asymmetry at 183 MeV, can be explained [2,3] using the value of $\rho - \omega$ mixing extracted from an analysis of $e^+e^- \to \pi^+\pi^-$ in the $\rho - \omega$ interference region [4,5]. The plausibility of this explanation (which employs the observed mixing, measured at $q^2 = m_\omega^2$, unchanged in the spacelike region $q^2 < 0$) has, however, recently been called into question by Goldman, Henderson and Thomas [6] who pointed out that, in the context of a particular model, the relevant $\rho - \omega$ mixing matrix element has significant $q^2$-dependence. Subsequently, various authors, employing various computational and/or model framewords, have showed that the presence of such $q^2$-dependence appears to be a common feature of isospin-breaking in both meson-propagator- and current-correlator matrix elements [7–16].

In the present paper we will concentrate on the isospin-breaking vector current correlator

$$\Pi_{\mu\nu}(q^2) = i \int d^4x \ e^{iq.x} \langle 0 | T(\Gamma^{\rho}_\mu(x) \Gamma^{\omega}_\nu(0)) | 0 \rangle , \quad (1.1)$$

where

$$\Gamma^{\rho}_\mu = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/2 , \quad \Gamma^{\omega}_\mu = (\bar{u}\gamma_\mu u + \bar{d}\gamma_\mu d)/6 . \quad (1.2)$$

This correlator was first analyzed using QCD sum rules in Ref. [17], and the analysis updated by the authors of Ref. [12] who, in particular, stressed the $q^2$-dependence of the correlator implicit in the results of this analysis. As will be shown below, a worrisome feature of the resulting fit is that the phenomenological representation of the correlator near $q^2 = 0$
is not dominated by the most nearby singularities, suggesting that some ingredient may be missing from the form chosen for this representation. This missing ingredient is identified below and it is shown that a reanalysis of the correlator, which includes it, rectifies the problem. The resulting correlator still displays a very strong \( q^2 \)-dependence, and, in addition, provides evidence for the presence of significant direct \( \omega \to \pi \pi \) coupling in \( e^+e^- \to \pi^+\pi^- \). The behavior of the resulting correlator near \( q^2 = 0 \) is then compared with that obtained from ChPT to one-loop. The latter is found to be more than an order of magnitude smaller than the former, the reason why this suggests the likelihood of a slow convergence of the chiral series for the correlator explained.

The paper is organized as follows. In Section II, those features of the behavior of quantum field theories under field redefinitions relevant to attempts in the literature to relate meson propagators and current correlators are discussed, and it is explained why the freedom of field redefinition implies that (1) one cannot obtain off-shell information about the off-diagonal element of the vector meson propagator from the off-diagonal element of the vector current correlator without making specific choices for the vector meson interpolating fields, and (2) if one writes the off-diagonal element of the vector meson propagator as

\[
\Delta_{\mu\nu}^{\rho\omega}(q^2) \equiv -(g_{\mu\nu} - q_\mu q_\nu/q^2)(q^2/m_\rho^2)(q^2/m_\omega^2)
\]  

(1.3)

\( \theta^{\rho\omega}(q^2) \) cannot, in general, be \( q^2 \)-independent. In Section III we return to the QCD sum rule analysis of the vector current correlator, first explaining why certain features of the existing analyses suggest the need for a modified analysis, and then performing this analysis. The results both correct the apparently unphysical features of the previous analyses and provide evidence for non-negligible direct \( \omega \to \pi^+\pi^- \) contributions to \( e^+e^- \to \pi^+\pi^- \) in the \( \rho - \omega \) interference region. In Section IV, the correlator is computed to one-loop in ChPT, and the results compared to those obtained from the sum rule analysis. Implications for the discrepancy between the results of the two approaches are discussed there. Finally, in Section V, a brief summary of the main results of the paper is given.
II. CONSEQUENCES OF THE FREEDOM OF FIELD REDEFINITION

Let us begin by clarifying the relation (or lack thereof) between the vector-meson-propagator and vector-current-correlator matrices. The former is an, in general, off-shell Green function, which we may think of as being associated with some low-energy effective Lagrangian, $L_{\text{eff}}$, in which the vector meson degrees of freedom have been made explicit. As is well-known \[18–20\], the form of such a Lagrangian is not unique: if $\phi$ and $\chi$ are two possible field choices describing a given particle, related by $\phi = \chi F(\chi)$, with $F(0) = 1$, then $L_{\text{eff}}[\phi]$ and $L'_{\text{eff}}[\chi] \equiv L_{\text{eff}}[\chi F(\chi)]$ produce exactly the same experimental observables \[18\]. However, while the S-matrix elements of the two theories are identical, this is not true of the general off-shell Green functions. One is free to make field redefinitions of the form above (as is done, e.g., in order to obtain the canonical form of the effective Lagrangian for ChPT \[19–21\]) without changing the physical consequences of the theory; the Green functions, however, are not in general invariant under such field redefinitions. Useful pedagogical illustrations of this general principle, for pion Compton scattering and the linear $\sigma$-model, are given in Ref. \[22\] and Chapter IV of Ref. \[23\], respectively. In the case of interest to us, what this means is that, when we make a redefinition of the $\rho$, $\omega$ fields in $L_{\text{eff}}[\rho, \omega]$, we generate a new effective Lagrangian, $L'_{\text{eff}}[\rho', \omega']$, the Green functions of which are, in general, different from those of $L_{\text{eff}}$ (though when we piece such Green functions together to form S-matrix elements, these differences produce no net effect). The off-shell behavior of the vector meson propagator is thus dependent on the particular choice of fields used to represent the vector mesons (the choice of “interpolating field”). It is not a physical observable. In contrast, the vector current correlators $\Pi_{\mu\nu}^{ab}(q^2) = i \int d^4x \ e^{iq.x} \langle 0 | T(V_{\mu}^{a}(x)V_{\nu}^{b}(0)) | 0 \rangle$ are, in fact, physical objects, independent of interpolating field choice. The spectral functions for $\Pi_{\mu\nu}^{33}$ and $\Pi_{\mu\nu}^{88}$ are, for example, accessible from a combination of $\tau^- \to \nu_{\tau}\pi^-\pi^0$ and $e^+e^- \to \pi\pi$, $\pi\pi\pi\pi$ data, and that for $\Pi_{\mu\nu}^{38}$ could in principle be obtained from a careful analysis of the deviation of the ratio of the differential decay rates for $\tau^- \to \nu_{\tau}\pi^-\pi^0$ and $e^+e^- \to \pi^+\pi^-$ from that predicted by isospin symmetry. As such there can be no general (i.e. valid for all choices of
interpolating field) relation between the correlator and propagator matrices. This point is the source of some confusion in Ref. [12] where an attempt is made to obtain the off-shell propagator based on an analysis of the correlator.

Before proceeding to the reanalysis of the correlator, let us be more precise about the problems with the interpretation of the results of Ref. [12], in the light of the above comments. The authors begin by writing a general form for the spectral function of the correlator:

$$\text{Im } \Pi_{\mu\nu}(q^2) = A_0 \text{Im } \Pi_{\mu\nu}^{\rho\omega}(q^2) + A_1 \text{Im } \Pi_{\mu\nu}^{\rho'\omega'} + \cdots \quad (2.1)$$

where the superscripts on the RHS should, for the moment, be taken only as labelling the region of spectral strength, and where $+ \cdots$ refers to all other contributions (we return to this below). Eqn. (2.1) is, of course, completely general. The authors of Ref. [12], however, then identify $A_0$ with $m_\rho^2 m_\omega^2 / g_\rho g_\omega$, where $g_{\rho,\omega}$ are the vector meson decay constants, defined by

$$\langle 0 | V_{\mu}^{\rho,\omega} | \rho, \omega \rangle \equiv \frac{m_\rho^2 m_\omega^2}{g_{\rho,\omega}} \epsilon_\mu \quad (2.2)$$

and $\Pi_{\mu\nu}^{\rho\omega}$ with the off-diagonal element of the vector meson propagator. This amounts to assuming that the isospin-unmixed $I = 1 \rho$ state, $\rho(0)$, couples only to $V_{\mu}^{\rho}$, and the isospin-unmixed $I = 0 \omega$ state, $\omega(0)$, only to $V_{\mu}^{\omega}$, the isospin-breaking contribution to $\Pi_{\mu\nu}$ of Eqn. (2.2) from the $\rho, \omega$ region then resulting solely from the $\rho(0)-\omega(0)$ mixing in the meson propagator. In this interpretation, fixing the imaginary part of the correlator in the $\rho, \omega$ region (via the sum rule analysis) allows one to obtain the isospin-breaking parameters of the imaginary part of the vector meson propagator, and, via a dispersion relation, the behavior of the off-diagonal element of the propagator off-shell. However, as explained above, such a possibility is excluded on general grounds. The problem with going from $A_0$ and $\Pi_{\mu\nu}^{\rho\omega}$ of Eqn. (2.2) to the interpretation of these quantities in Ref. [12] is that, not one, but three sources of isospin breaking exist in the contributions to $\Pi_{\mu\nu}$ from the $\rho, \omega$ region: that due to $\rho(0)-\omega(0)$ mixing (discussed above), that due to the direct coupling of $V_{\mu}^{\rho}$ to $\omega(0)$, and that
due to the direct coupling of $V_\omega$ to $\rho^{(0)}$. The same $\Delta I = 1$ strong operator which gives rise to non-zero $\rho^{(0)}$-$\omega^{(0)}$ mixing will also necessarily give rise to the latter two couplings. These couplings would be described by new isospin breaking parameters, $\phi^{(\rho)\omega}$ and $\phi^{(\omega)\rho}$,

$$
\langle 0 | V_\mu^{\omega} | \rho^{(0)} \rangle \equiv \frac{m_\omega^2}{g_\omega} \phi^{(\omega)\rho} \epsilon_\mu
$$

$$
\langle 0 | V_\mu^{\rho} | \omega \rangle \equiv \frac{m_\rho^2}{g_\rho} \phi^{(\rho)\omega} \epsilon_\mu
$$

where $\phi^{(\omega)\rho}$, $\phi^{(\rho)\omega}$ are, in general, $q^2$-dependent, and also interpolating-field-dependent off-shell. Thus, off-shell, the $\rho$-$\omega$ region contribution to $\Pi_{\mu\nu}$ depends not only on the (interpolating-field-choice-dependent) isospin-breaking parameters of the off-diagonal element of the vector meson propagator, but also on the (interpolating-field-choice-dependent) isospin-breaking parameters $\phi^{(\omega)\rho}$, $\phi^{(\rho)\omega}$. The total contribution is independent of the interpolating field choice, but the individual contributions are not. One is, of course, free to choose a convenient set of $\rho$, $\omega$ interpolating fields and work with these, provided one calculates contributions to $S$-matrix elements. Since, to $O(m_d - m_u)$ Eqn. (2.2) remains valid when we replace $\rho$ and $\omega$ with $\rho^{(0)}$ and $\omega^{(0)}$, the fields

$$
\rho_{\mu}^{(0)c} \equiv \frac{g_\rho}{m_\rho^2} V_\mu^{\rho},
$$

$$
\omega_{\mu}^{(0)c} \equiv \frac{g_\omega}{m_\omega^2} V_\mu^{\omega},
$$

satisfy $\langle 0 | \rho_{\mu}^{(0)c} | \rho^{(0)} \rangle = \epsilon_\mu$ and $\langle 0 | \omega_{\mu}^{(0)c} | \omega^{(0)} \rangle = \epsilon_\mu$, and hence serve as possible choices of interpolating fields for $\rho^{(0)}$ and $\omega^{(0)}$. With this choice of interpolating fields (and not with others) one obtains

$$
\Pi_{\mu\nu}(q^2) = \frac{m_\rho^2 m_\omega^2}{g_\rho g_\omega} \Delta_{\mu\nu}^{(c)\rho\omega}(q^2)
$$

where $\Delta_{\mu\nu}^{(c)\rho\omega}(q^2)$ is the off-diagonal element of the vector meson propagator for the interpolating field choice above. If one simultaneously evaluates, e.g. $NN\rho$, $NN\omega$ vertex form factors using the same interpolating fields, one could, of course, piece the resulting vertices and propagators together to obtain contributions to $NN$ scattering $S$-matrix elements which are independent of field choice. For a general choice of interpolating field, however, neither
nor $\Pi_{\mu\nu}$ is proportional to $\Delta_{\mu\nu}^{\rho\omega}$. Given the existence of QCD sum rules and ChPT methods, which are rather efficient at handling current-current correlators and vector current vertex functions, the choice (2.4) for vector meson interpolating fields would appear to be a convenient and sensible one. With this choice, Eqn. (2.1) provides the basis of a spectral representation of $\Delta^{(c)\rho\omega}$, but for other choices of the vector meson interpolating fields this is not the case.

Note that the above discussion also clarifies one ongoing point of debate in the literature, namely that concerning the $q^2$-dependence of the quantity $\theta^{\rho\omega}(q^2)$ appearing in Eqn. (1.3). Defining $\hat{\Pi}(q^2)$ by

$$\Pi_{\mu\nu}(q^2) \equiv (g_{\mu\nu} - q_{\mu}q_{\nu}/q^2) \hat{\Pi}(q^2),$$

the absence of massless singularities implies that $\hat{\Pi}(0) = 0$. This in turn implies, with

$$\Delta_{\mu\nu}^{(c)\rho\omega}(q^2) \equiv -(g_{\mu\nu} - q_{\mu}q_{\nu}/q^2) \Delta^{(c)\rho\omega}(q^2),$$

$\Delta^{(c)\rho\omega}(q^2) = 0$, and hence $\theta^{\rho\omega}(0) = 0$. Since this is true for one choice of the vector meson interpolating fields, it is incumbent upon those advocating $\theta^{\rho\omega}(q^2) = \theta^{\rho\omega}(m_\omega^2)$ (2.8) to explicitly demonstrate the existence of an interpolating field choice for the vector mesons for which Eqn. (2.8) is valid; the relation cannot be true in general.

### III. THE QCD SUM RULE ANALYSIS OF $\Pi_{\mu\nu}(Q^2)$ REVISITED

With the above discussion in mind, let us turn to the sum rule analysis of the vector correlator, first briefly reviewing the treatment and results of Refs. [12, 17]. The sum rule approach consists of writing an operator product expansion (OPE) representation for the correlator, valid in the region of validity of perturbative QCD, and a second, phenomenological, representation in terms of hadronic parameters, and then Borel transforming both. The Borel transform serves to extend the ranges of validity of both representations and, in
addition, to (1) emphasize the operators of lowest dimension in the OPE representation and (2) give higher weight to the parameters of the lowest lying resonances in the phenomenological representation. One then matches the transformed representations in order to make predictions for the relevant hadronic parameters.

The OPE for the correlator of interest was performed long ago [17]. Truncating the expansion at operators of dimension six, one finds that, defining \( \Pi(q^2) \) by

\[
\Pi_{\mu\nu} \equiv (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2),
\]

one has

\[
\Pi^{OPE}(Q^2) = \frac{1}{12} \left[ -c_0 \log(Q^2) + \frac{c_1}{Q^2} + \frac{c_2}{Q^4} + \frac{2c_3}{Q^6} \right]
\]

where \( Q^2 = -q^2 \) and

\[
\begin{align*}
c_0 &= \frac{\alpha_{EM}}{16\pi^3} \\
c_1 &= \frac{3}{2\pi^2} (m_d^2 - m_u^2) \\
c_2 &= \frac{(m_d - m_u)}{(m_d + m_u)} 2f_\pi^2 m_\pi^2 \left[ 1 + \left( \frac{\gamma}{2 + \gamma} \right) \left( \frac{m_d + m_u}{m_d - m_u} \right) \right] \\
c_3 &= -\frac{224}{81} \pi [\alpha_s \langle \bar{q} q \rangle_0]^2 \left[ \frac{\alpha_{EM}}{8\alpha_s(\mu^2)} - \gamma \right]
\end{align*}
\]

with \( \gamma \equiv \langle \bar{d}d \rangle_0 / \langle \bar{u}u \rangle_0 - 1 \). Taking for the phenomenological representation (in the narrow resonance approximation)

\[
\text{Im} \Pi^{\text{phen}}(q^2) = \frac{\pi}{12} \left[ f_\rho \delta(q^2 - m_\rho^2) - f_\omega \delta(q^2 - m_\omega^2) + f_{\rho'} \delta(q^2 - m_{\rho'}^2) - f_{\omega'} \delta(q^2 - m_{\omega'}^2) \right] + \frac{\alpha_{EM}}{192\pi^2},
\]

(where \( f_\rho, f_\omega, f_{\rho'}, f_{\omega'} \) may be thought of as the parameters to be determined from the sum rule analysis) one finds, upon Borel transformation and matching,

\[
\frac{1}{M^2} \left[ f_\rho \exp(-m_\rho^2/M^2) - f_\omega \exp(-m_\omega^2/M^2) + f_{\rho'} \exp(-m_{\rho'}^2/M^2) - f_{\omega'} \exp(-m_{\omega'}^2/M^2) \right] + \frac{\alpha_{EM}}{16\pi^3} \exp(-s_0/M^2) = c_0 + \frac{c_1}{M^2} + \frac{c_2}{M^4} + \frac{c_3}{M^6},
\]
where $M$ is the Borel mass. As pointed out in Ref. [12], to $O(\delta m^2, \delta m'^2)$, where $\delta m^2 = m_\omega^2 - m_\rho^2$, $\delta m'^2 = m_{\omega'}^2 - m_{\rho'}^2$, Eqn. (3.5) can be rewritten in terms of the parameters $\xi$, $\beta$, $\xi'$ and $\beta'$, where

$$\begin{align*}
\xi &= \frac{\delta m^2}{m^4} \left( \frac{f_\rho + f_\omega}{2} \right), \\
\beta &= \frac{(f_\omega - f_\rho)}{m^2 \xi}, \\
\xi' &= \frac{\delta m'^2}{m'^4} \left( \frac{f_{\rho'} + f_{\omega'}}{2} \right), \\
\beta' &= \frac{(f_{\omega'} - f_{\rho'})}{m'^2 \xi'}. 
\end{align*}$$

(3.6)

with $m^2 \equiv (m_\rho^2 + m_\omega^2)/2$ and $m'^2 \equiv (m_{\rho'}^2 + m_{\omega'}^2)/2$, as

$$\begin{align*}
\xi \frac{m^2}{M^2} \left( \frac{m^2}{M^2} - \beta \right) \exp(-m^2/M^2) + \xi' \frac{m'^2}{M^2} \left( m'^2 - \beta' \right) \exp(-m'^2/M^2) \\
+ \frac{\alpha_{EM}}{16\pi^3} \exp(-s_0/M^2) = c_0 + \frac{c_1}{M^2} + \frac{c_2}{M^4} + \frac{c_3}{M^6}.
\end{align*}$$

(3.7)

If $c_{0-3}$ were precisely known, Eqn. (3.5) or Eqn. (3.7) could, in principle, be used to determine the parameters $\xi$, $\beta$, $\xi'$, $\beta'$. There are, however, some uncertainties in the values of the $c_i$, associated with the imprecision in our knowledge of the values of the four-quark condensates and of the isospin-breaking ratio of the $\langle \bar{u}u \rangle_0$ and $\langle \bar{d}d \rangle_0$ condensates. The authors of Ref. [12] (which updates Ref. [17]) consider a range of possibilities for these quantities, and also take for $r \equiv (m_d - m_u)/(m_d + m_u)$ the value $r = 0.28$, obtained from an analysis of pseudoscalar isomultiplet splittings [24] employing Dashen’s theorem [25] for the electromagnetic contributions to these splittings. The last ingredient of the analysis of Ref. [12] is the imposition of an external constraint on the hadronic parameter $\xi$, based on the observed interference in the $\rho$-$\omega$ interference region in $e^+e^- \rightarrow \pi^+\pi^-$. This constrained value, $\xi = 1.13 \times 10^{-3}$, is based on (1) the assumed connection between the correlator and the propagator (presumably valid for the essentially on-shell value of the mixing, though not elsewhere) and (2) the assumption that direct $\omega^{(0)} \rightarrow \pi\pi$ contributions to $e^+e^- \rightarrow \pi^+\pi^-$ can be neglected (see Ref. [26] for a discussion of these issues). There appears to be no particularly good reason for the latter assumption, and, indeed, it would seem appropriate to allow
\(\xi\) to be fit by the sum rule analysis as a test of this assumption (as will be done below), but let us follow the analysis of Ref. [12] for the moment. Using the sum rule, Eqn. (3.7), and imposing the constraint \(\xi = 1.13 \times 10^{-3}\), as discussed above, the authors of Ref. [12] solve for \(\beta, \xi'\) and \(\beta'\) for four different input sets \(\{c_i\}\). Using the expression (3.4) for \(\text{Im } \Pi_{\text{phen}}(q^2)\) and the fact that \(\Pi(q^2)\) satisfies an unsubtracted dispersion relation, one may show that, to first order in \(\delta m^2\) and \(\delta m'^2\),

\[
\text{Re } \Pi(0) = \frac{1}{12} [\xi (1 - \beta) + \xi'(1 - \beta')] .
\]

(3.8)

Using the values of the parameters obtained in Ref. [12], one finds that the ratios of the contributions to \(\text{Re } \Pi(0)\) from the \(\rho' - \omega'\) region to those from the \(\rho - \omega\) region are 1.8, 0.8, 0.3 and 0.8 for input sets I, II, III, IV, respectively. The failure of the results to be dominated by the nearby \((\rho, \omega)\) singularities suggests that the phenomenological form employed for the spectral function may well be incomplete, either in missing low-lying contributions or in failing to include the effect of even more distant singularities. If we consider Eqns. (3.4) and (3.8) for a moment an interesting possibility becomes evident. If one had all isospin-breaking effects generated solely by \(\rho(0) - \omega(0)\) mixing, and if the physical vector mesons were a simple rotation of the isospin-pure basis (not in general true when the wavefunction renormalization matrix of the system is non-diagonal), we would have \(f_{\rho} = f_{\omega}\) for \(f_{\rho}, f_{\omega}\) as written in Eqn. (3.4). While the assumptions required to arrive at this conclusion are certainly not satisfied in general, this nonetheless indicates that there should be significant cancellation between the \(\rho\) and \(\omega\) contributions to the correlator. Thus, a single isolated resonance, even with a coupling much smaller than that of the \(\rho\) or \(\omega\), could in fact contribute significantly to \(\Pi_{\mu\nu}\). This suggests that the \(\phi\) contribution to \(\text{Im } \Pi_{\mu\nu}\), neglected in Ref. [12], may well be non-negligible. In fact we can make a rough estimate of the expected size of \(f_{\phi}\) (where \(f_{\phi}\) is defined by adding a contribution \(\frac{f_{\phi}}{12} f_{\phi} \delta(q^2 - m_{\phi}^2)\) to \(\text{Im } \Pi_{\text{phen}}(q^2)\) in Eqn. (3.4)) as follows. \(\phi\) is known to be not quite pure \(\bar{s}s\). If, e.g., we take the Particle Data Group (PDG) [27] value for the octet-singlet mixing angle, \(\theta = 39^0\) (quadratic fit), \(\phi \simeq \phi(0) - \delta \omega(0)\), where \(\phi(0)\) is the pure \(\bar{s}s\) state and \(\delta = .065\) rad is the deviation of \(\theta\) from ideal mixing. The
contribution of the $\phi$ pole term to $\Pi_{\mu\nu}$ due to mixing in the propagator should then be of order $-\delta$ times that associated with the $\omega$ pole, i.e. $\simeq 0.065 f_\omega \simeq 0.065 f_\rho$. There will, of course, also, in general, be isospin-breaking contributions from direct couplings to the current vertices, not just from mixing in the propagator, but the above discussion shows that $f_\phi \simeq (0.05 - 0.10) f_{\rho,\omega}$ should be a reasonable expectation. As we will see below, this (rather crude) estimate is indeed borne out by the sum rule analysis.

Let us, therefore, add a term $\frac{\pi}{12} f_\phi \delta(q^2 - m_\phi^2)$ to $\text{Im} \Pi_{\text{phen}}(q^2)$ on the RHS of Eqn. (3.4), and perform a reanalysis of that equation. We will follow Ref. [12] in choosing the range of input values for the $\{c_i\}$, with, however, the following modifications. First, the small $c_1$ term dropped in Ref. [12] will be retained, though, as pointed out there, it in fact has little effect on the final results. The numerical value is obtained by using $(m_d + m_u)(1 \text{GeV}) = 12.5\pm 2.5 \text{MeV}$ from Ref. [28] and the updated value of $r$ discussed below. The main modification to the input is in the parameter $r$. There is now considerable evidence that Dashen’s theorem is significantly violated [29–31], Refs. [30,31] in particular suggesting that

$$(m_{K^+}^2 - m_{K^0}^2)_{EM} \simeq 1.9 (m_{\pi^+}^2 - m_{\pi^0}^2)_{\text{exp}} \quad (3.9)$$

(where the factor 1.9 on the RHS of Eqn. (3.9) is absent in Dashen’s theorem). Using Eqn. (3.9) in place of Dashen’s theorem for the electromagnetic contribution to the kaon mass splitting produces a rescaling of $r$ by $1.22$. The resulting change in the $c_i$ is essentially to rescale the values of $c_2$ in Ref. [12] by this same factor. In assessing the effect of the uncertainties in the values of the $\{c_i\}$ for a given input set, the input errors on $c_2$ have also been rescaled by this factor of 1.22. Finally, since the masses of all the resonances appearing above, including the $\rho'$ and $\omega'$, are known, we may take these as input and use the sum rule to extract the isospin-breaking parameters, $\{f_k\}$, where $i = 1 \cdots 5$ correspond to $\rho$, $\omega$, $\rho'$, $\omega'$ and $\phi$, respectively. Note that, in taking this approach, we are abandoning the constraint on $\xi$ employed in Ref. [12]. If the direct $\omega^{(0)} \to \pi^+\pi^-$ coupling is, indeed, negligible in $e^+e^- \to \pi^+\pi^-$, this will manifest itself by the value of $\xi$ resulting from the sum rule analysis being near $1.13 \times 10^{-3}$. 

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The analysis of the modified version of the sum rule, (3.5), proceeds as follows. First, from the terms of $\mathcal{O}(M^0)$, $c_0 = \alpha_{EM}/16\pi^3$. One may check that, as in Ref. [12], the analysis is very insensitive to the value of the EM threshold parameter, $s_0$. We will, therefore, quote all results below for the value, $s_0 = 1.8$ GeV, employed in a number of the results quoted in Ref. [12]. Second, again as in Ref. [12], we impose the local duality relation

$$
\int_0^\infty ds \, \text{Im} \Pi^{\text{phen}}(s) = \mathcal{O}(\alpha_{EM}, m_2^q) \quad (3.10)
$$

(which is equivalent to matching the coefficients of the $\mathcal{O}(1/M^2)$ terms in Eqn. (3.5)). With the index $k = 1, \cdots, 5$ labelling $\rho, \omega, \rho', \omega'$ and $\phi$, respectively, as above, this relation is

$$
\sum_k (-1)^{k+1} f_k = c_0 s_0 + c_1 \quad (3.11)
$$

(Note that the $c_i$ tabulated in Ref. [12] have had the appropriate factors of $m^2$ required to leave the remaining coefficient dimensionless factored out of them. Thus, e.g., $c_1$ in Eqn. (3.11) is $m^2$ times that tabulated in Ref. [12].) The remaining four relations required to obtain a solution for the five unknowns, $\{f_k\}$, are obtained by acting on Eqn. (3.5) with $(-1)^n \frac{\partial^n}{\partial(1/M^2)^n}$ for $n = 1, \cdots, 4$. One may check that the results are not sensitive to using precisely the PDG values for the $\rho'$ and $\omega'$ masses. Indeed, shifting either mass by 50 MeV induces changes of $< 4\%$ in $\xi$, $< 2.5\%$ in $\beta$, $< 5\%$ in $\beta'$ and $< 20\%$ in $\xi'$. The resulting changes in the correlator itself are even smaller: e.g. $\Pi(0)$ and $\frac{d\Pi}{dq^2}(0)$ are changed by $< 2\%$ by the above mass shifts.

In Table 1, the results of the modified sum rule analysis are displayed for the input sets I, III, IV of Ref. [12], modified as described above. The errors shown in the table correspond to the uncertainties in the input parameters, $c_2$ and $c_3$, (those quoted in Ref. [12] in the case of $c_3$ and the rescaled version thereof in the case of $c_2$). The stability of the analysis is illustrated, for input set IV, in Figs. 1-5, which display the parameters $\xi$, $\beta$, $\xi'$, $\beta'$, $f_\phi$ as a function of the Borel mass, $M$, in the range 1-10 GeV (the choice of the first four parameters, rather than corresponding $f_k$ values, is made in order to facilitate comparison with Ref. [12]). Set I generates results of comparable stability, while the results of set III
are even more stable than those of set IV. In all three cases a wide stability window exists in the Borel mass for all five output parameters. This stability window, moreover, occurs without the necessity of using unphysical values for the the average of the $\rho'$ and $\omega'$ masses.

As noted previously in Ref. [12], results for input set II are considerably less stable than for the other sets: in fact, no stability window exists anywhere in the range $M = 1$ and $M = 10$ GeV, apart from for the very lower edge of the error band for the magnitude of $c_3$, for which values input set II is very close to the upper end of the corresponding error band for input set I. The instability of the analysis for input set II is illustrated (for the central values of $c_2$ and $c_3$) in Fig. 6, where the parameter, $f_\phi$, is plotted as a function of the Borel mass, $M$. As a result of this instability, results corresponding to input set II are not quoted in the table; for most of the input range (i.e. for larger values of the magnitude of $c_3$) the input set appears, from the sum rule analysis, to be unphysical.

A number of features are evident from the results of the above analysis. First, from Table 1, we see that the magnitude of $\xi$ differs significantly from that which would be expected from the analysis of $e^+e^- \to \pi^+\pi^-$, neglecting $\omega(0) \to \pi^+\pi^-$ contributions, suggesting that the latter are, indeed, not negligible. It should be stressed that the errors quoted in the table correspond to varying $c_2$ and $c_3$ separately within the range of quoted errors, and taking the maximum variation of the resulting output. One can obtain even lower values of $\xi$, i.e. closer to that expected if one can indeed neglect $\omega(0) \to \pi^+\pi^-$ contributions to $e^+e^- \to \pi^+\pi^-$, by letting $c_2$ lie at the bottom of its error band and, simultaneously, the magnitude of $c_3$ lie at the top of its error band in set I. However, such a combination (which produces $\xi = 1.43 \times 10^{-3}$) is quite unstable, the values of $\xi'$, e.g., varying by more than 20% between $M = 3$ and 5 GeV. A similar result, $\xi = 1.48 \times 10^{-3}$, can be obtained from set II for the central value of $c_2$ and the lower edge of the error band for the magnitude of $c_3$, with comparable ($\approx 20\%$ over the range $M = 3$ to 5 GeV) instability. All other portions of the set II error band are even more unstable. Thus it appears very clear that the value $\xi = 1.13 \times 10^{-3}$ is excluded by the sum rule analysis. The second observation is that the inclusion of the $\phi$ pole term in the phenomenological representation of the correlator
rectifies the problem of the strength of the distant singularities. This can be seen from
the relative size of $\xi$ and $\xi'$ in Table 1, but is more evident in Table 2, where the output
values for the parameters $\{f_k\}$ are tabulated, for the central values of the input parameters
$\{c_i\}$, for input sets I, III, IV. The ratios of $f_\phi$ to $f_\omega$ are 0.062, 0.068 and 0.066 for sets I,
III and IV, respectively. This is in (better than should be expected) agreement with the
rough estimate given above, confirming the physical plausibility of the solutions obtained.
Moreover, $f_\rho'$ and $f_\omega'$ are now a factor of 40-60 smaller than $f_\rho$ and $f_\omega$. The structure of
the resulting contributions to the correlator near $q^2 = 0$ is shown in Table 3, where the $\rho$,
$\omega$, and also the $\rho'$, $\omega'$ contributions have been combined. Note that the individual $\rho$ and
$\omega$ contributions are a factor of $\approx 13$ larger than the $\phi$ contribution, but the cancellation
between them is such that the $\phi$ contribution is approximately twice as large as their sum.
The $\rho'$-$\omega'$ region contribution is then less than 10% of the $\phi$ contribution. The more distant
singularities, thus, have only a small effect, justifying, a posteriori, the neglect of yet more
distant singularities in the phenomenological side of the sum rule analysis. Given that the
results satisfy all the above tests for being physically sensible and stable, it appears that the
resulting values for the correlator and its slope with respect to $q^2$ at $q^2 = 0$ should be taken
as good estimates, within the uncertainties resulting from those in the input parameters.
The fact that, due to cancellation between the otherwise dominant $\rho$ and $\omega$ contributions,
the $\phi$ contribution is actually dominant, no doubt accounts for the unphysical behavior of
the spectral distribution of the correlator obtained in the absence of the $\phi$ term. Note that,
despite the significant changes in the fit, as compared to Ref. [12], the slope of the correlator
remains large in the present results.

IV. THE CORRELATOR TO ONE-LOOP ORDER IN CHPT

The starting point for the computation of the mixed-isospin correlator, $\Pi_{\mu\nu}(q^2)$, is the
effective chiral Lagrangian of Ref. [21],

$$
L_{\text{eff}} = \frac{1}{4}f^2\text{Tr}(D_\mu \Sigma D^\mu \Sigma^\dagger) + \frac{1}{2}f^2\text{Tr}[B_0 M(\Sigma + \Sigma^\dagger)] + L_1[\text{Tr}(D_\mu \Sigma D^\mu \Sigma^\dagger)]^2
$$
\[ + L_2 \text{Tr}(D_\mu \Sigma D_\nu \Sigma^\dagger) \text{Tr}(D^\mu \Sigma D^\nu \Sigma^\dagger) + L_3 \text{Tr}(D_\mu \Sigma^\dagger D_\nu \Sigma \Sigma^{\dagger} D^\nu \Sigma) + L_4 \text{Tr}(D_\mu \Sigma \Sigma^{\dagger} D_\nu \Sigma \Sigma^{\dagger} D^\nu \Sigma) \]

\[ + L_5 \text{Tr}(D_\mu \Sigma \Sigma^{\dagger} D_\nu \Sigma \Sigma^{\dagger} D^\nu \Sigma) + L_6 [\text{Tr}(2B_0 M (\Sigma + \Sigma^\dagger))]^2 + L_7 [\text{Tr}(2B_0 M (\Sigma - \Sigma^\dagger))]^2 \]

\[ + L_8 \text{Tr}(4B^2_0 M \Sigma \Sigma^\dagger M \Sigma^\dagger M \Sigma) + L_9 \text{Tr}(F^{\mu \nu}_R F^{\mu \nu}_L + F^{\mu \nu}_L F^{\mu \nu}_L) + H_1 \text{Tr}(F^{\mu \nu}_R F^{\mu \nu}_L + F^{\mu \nu}_L F^{\mu \nu}_L) + H_2 \text{Tr}(4B^2_0 M^2) \] \tag{4.1}

In Eqn. (4.1), \(B_0\) is a mass scale related to the value of the quark condensate in the chiral limit, \(\Sigma = \exp(i \vec{\lambda} \cdot \vec{\pi} / f)\) (with \(\vec{\lambda}\) the usual \(SU(3)\) Gell-Mann matrices and \(\vec{\pi}\) the octet of pseudoscalar (pseudo-) Goldstone boson fields), \(f\) is a dimensionful constant, equal to \(f_\pi\) in leading order, \(M\) is the current quark mass matrix, and \(D_\mu\) is the covariant derivative

\[ D_\mu \Sigma = \partial_\mu \Sigma - i (v_\mu + a_\mu) \Sigma + i \Sigma (v_\mu - a_\mu). \] \tag{4.2}

In Eqn. (4.1) the external pseudoscalar sources (which occur in the most general form of \(\mathcal{L}_{\text{eff}}\)) have already been set to zero, and the external scalar source to \(2B_0\) times the current quark mass matrix, since we are interested here only in the vector current correlator and, therefore, require only the external vector sources. For this same reason we may drop the external axial vector sources from the expression for the covariant derivative in Eqn. (4.2).

The left and right external source field strength tensors, \(F^{\mu \nu}_L, R\), then both reduce to \(F^{\mu \nu}_R = \partial_\mu v_\nu - \partial_\nu v_\mu - i [v_\mu, v_\nu]\), where \(v_\mu = \frac{\lambda^a}{2} v^{a}_\mu\), with \(v^{a}_\mu\) the octet of external \(SU(3)\) vector fields. In principle, Eqn. (4.1) should be supplemented by terms involving \(\text{Tr}(F^{\mu \nu})\) in order to treat the case at hand, since the current \(V^{\omega}_\nu\) contains both octet and singlet pieces. However, to one-loop order, the additional terms do not contribute to the isospin-mixed correlator (the correlator is identical to \(\Pi^{38}_{\mu \nu}(q^2) / 3 \sqrt{3}\), with \(\Pi^{38}_{\mu \nu}(q^2) = i \int d^4x \langle 0 | T(V^{3}_\mu(x)V^{8}_8(0)) | 0 \rangle\), to this order), so we will not explicitly display these terms. The unrenormalized higher order coefficients \(L_1, \ldots, L_{10}\) and \(H_1, H_2\) appearing in Eqn. (4.1) contain divergent pieces which cancel those of the one-loop graphs involving vertices from the first two terms in \(\mathcal{L}_{\text{eff}}\) above, and also finite, renormalization-scale-dependent pieces, \(L_i^r\). Expressions for the divergent pieces of the \(L_i, H_i\), relevant to one-loop calculations, may be found in Ref. [21].
Contributions to the correlator resulting from Eqn. (4.1) are of two types, corresponding to the two types of contribution to the low-energy representation of the product of currents, $V_\mu V_\nu$: (1) those terms arising from the product of the low-energy representations of the individual currents, $V_\mu$ and $V_\nu$ (obtained from the terms in $\mathcal{L}_{\text{eff}}$ linear in the $v_a^\alpha$, $a = 0, \cdots, 8$), and (2) contact terms (generated by the terms in $\mathcal{L}_{\text{eff}}$ quadratic in the $v_a^\alpha$). To leading order (i.e. keeping only the first two terms in $\mathcal{L}_{\text{eff}}$) the correlator vanishes. This is because it is isospin-breaking and the only isospin-breaking at leading order lies in the term involving the quark mass matrix, which does not contain the external vector sources, and hence does not contribute to the correlator in zero-loop graphs. The leading contributions to the correlator are, therefore, next-to-leading order in the usual chiral counting. As such, the contributions consist of one-loop contact and non-contact graphs (where the current vertices are obtained from the first term in the effective Lagrangian above) and meson-field-independent contact terms from the remainder of $\mathcal{L}_{\text{eff}}$. These latter contributions, which would in general produce terms involving the $L_i^r$, may be easily shown to vanish for the case at hand. Thus only the contact and non-contact graphs mentioned above contribute. It is straightforward to demonstrate then that, to one-loop order, the $\mathcal{O}(m_d - m_u)$ expression for the correlator is

$$\Pi(q^2) = \frac{1}{12} \left[ \frac{\log(m_{K^0}^2/m_{K^+}^2)}{48\pi^2} + \left( \frac{4m_{K^0}^2}{3q^2} - \frac{1}{3} \right) J_{K^0}(q^2) - \left( \frac{4m_{K^+}^2}{3q^2} - \frac{1}{3} \right) J_{K^+}(q^2) \right]$$

(4.3)

where

$$J_P(q^2) = -\frac{1}{16\pi^2} \int_0^1 dx \, \log \left[ 1 - x(1-x)q^2/m_P^2 \right].$$

(4.4)

For our purposes we will not need the general expression for $J$ (which is quoted in Appendix A of Ref. [21]), but only the behavior near $q^2 = 0$, which is given by

$$J_P(q^2) = \frac{1}{96\pi^2} \frac{q^2}{m_P^2} + \frac{1}{960\pi^2} \frac{q^4}{m_P^4} + \cdots.$$  

(4.5)

In Eqn. (4.3), $m_{K^0,K^+}^2$ are the leading-order expressions for the kaon squared-masses, $m_{K^0}^2 = B_0(m_s + m_d)$ and $m_{K^+}^2 = B_0(m_s + m_u)$ and terms have been kept only to $\mathcal{O}(m_d - m_u)$. As
such, we must also expand all terms occurring there to the same order. Doing so, and making the expansion of Eqn. (4.5) for the loop integrals \( \bar{J}_{K^0,K^+} \), we obtain, for the behavior of the correlator in the vicinity of \( q^2 = 0 \),

\[
\Pi(q^2) = \frac{1}{12} \left( \frac{(m_{K^0}^2 - m_{K^+}^2)}{48\pi^2\bar{m}_K^2} \right) \left( 1 + \frac{q^2}{10\bar{m}_K^2} + \cdots \right),
\]

(4.6)

where \( \bar{m}_K^2 \) is the average of the \( K^+ \) and \( K^0 \) squared masses. Thus, \( 12\Pi(0) = (m_{K^0}^2 - m_{K^+}^2)/48\pi^2\bar{m}_K^2 \), where the kaon mass difference is that due to the strong isospin-breaking, i.e., with the electromagnetic contribution removed. Using Eqn. (3.9) for the electromagnetic contribution, we find that the RHS of this expression is \( 5.5 \times 10^{-5} \), to be compared with the results of the sum rule analysis, \( \simeq 1 \times 10^{-3} \). The one-loop ChPT result is a factor of \( \simeq 20 \) smaller than the sum rule result.

The discrepancy between the one-loop ChPT and sum rule analyses for the correlator near \( q^2 = 0 \) should actually not come as a complete surprise. Indeed, when the leading-order contribution to a physical quantity (order 2 in the chiral expansion) vanishes, as it does here, one has no obvious scale to use in judging whether or not the next-to-leading-order contribution obtained is abnormally small, i.e., whether or not the resulting one-loop expression is likely to represent a well-converged approximation to the whole chiral expansion. In fact, the structure of the expression, (4.3), above for the correlator, \( \Pi(q^2) \), is such as to suggest that it is unlikely to be well-converged. The reason for this statement is that Eqn. (4.3) is independent of the low-energy constants (LEC’s), \( L_i^r \), and results purely from one-loop graphs involving internal kaon loops. Such loops, for the non-contact graphs, are well-known to be suppressed in size (the coefficient of \( q^2 \) in the leading term of \( \bar{J}_K \) in Eqn. (4.5), e.g., is a factor of \( m_{\pi}^2/m_{K}^2 \) smaller than for the corresponding \( \pi \) loop integral, \( \bar{J}_\pi \) and, moreover, in the case at hand, i.e. the correlator \( \Pi(q^2) \), those terms in which this suppression would be lifted by the presence of the \( m_K^2/q^2 \) factor in the coefficient multiplying \( \bar{J}_K(q^2) \) cancel, since the expression for the correlator involves the difference of the \( K^+ \) and \( K^0 \) loop contributions. The correlator, of course, has a cut beginning at \( q^2 = 4m_{\pi}^2 \), associated with \( \pi\pi \) intermediate states, but such intermediate states do not enter
until two-loop order in the chiral expansion. Since the relevant \( \pi \) loop integral is intrinsically much larger than its kaonic counterpart, it is likely that the two-loop contributions will not be negligible, despite being higher order in the chiral expansion.

Other examples of slow convergence of the chiral series when the leading contribution vanishes and the next-to-leading order contribution results purely from loop graphs (i.e. is independent of the fourth-order LEC’s, \( L_i^4 \)) are, in fact, already known. One is the process \( \gamma \gamma \rightarrow \pi^0\pi^0 \), whose amplitude, to one-loop order, receives contributions only from loop graphs (though in this case, loop graphs with internal \( \pi \) lines). The one-loop expression \( [32,33] \) deviates from the experimental amplitude \( [34] \) even very close to threshold, and one finds that extending the calculation to two-loop order (sixth order in the chiral expansion) produces corrections to the one-loop result of order 30% \( [35] \), which corrections bring the amplitude into agreement with experiment. Even more closely similar to the case at hand is the process \( \eta \rightarrow \pi^0\gamma\gamma \). The one-loop amplitude again has no leading term and no contributions from the fourth order LEC’s, but here, although there are \( \pi \) loop contributions, these contributions are suppressed by a factor \( (m_d - m_u) \). The \( K \) loop contributions are naturally small, as noted above. The result is that the one-loop prediction for the partial rate \( [36] \) is a factor of \( \simeq 170 \) smaller than observed experimentally \( [27] \).

It is worth considering the process \( \eta \rightarrow \pi^0\gamma\gamma \) in somewhat more detail since, not only does it closely parallel the case at hand, but the physical origin of the smallness of the one-loop result for this process is well-understood. The source of the problem lies in the fact that the dominant contribution to the amplitude is known to be due to vector meson exchange \( [37,38] \). As is well-known \( [20,33,40] \), it is possible to make standard field choice for the various meson resonances and write an effective chiral Lagrangian which includes both these resonances and the octet of pseudoscalar (pseudo-) Goldstone bosons. One may then integrate out the (heavy) resonance fields to obtain an effective Lagrangian of the form \( \mathcal{L}_{\text{eff}} \) for the pseudoscalars alone. The resonance contributions to the LEC’s are then determined by the coupling parameters of the original, extended Lagrangian (which are fixed by experimental data). The effect of the resonances (for the initial field choices used)
then lies solely in their contributions to the $L_i^r$ [39,40]. Those $L_i^r$ to which the vector and axial-vector mesons can contribute ($L_i^r$, $i = 1, 2, 3, 9, 10$) are known to be essentially saturated by these contributions [39,40]. Thus, the absence of any $L_i^r$-dependence in the one-loop amplitude implies the absence of the effects of vector meson exchange (for the given initial choice of vector meson interpolating fields) and if, as seems to be the case for $\eta \rightarrow \pi^0\gamma\gamma$, vector meson exchange is the dominant contribution, the one-loop amplitude can be expected to be a poor representation of the full chiral series. The vector meson contributions, in this case, first appear as tree-level contributions arising from the $\mathcal{O}(p^6)$ part of the effective Lagrangian, not included in Eqn. (4.1) above (the general form of the $\mathcal{O}(p^6)$ part of the effective Lagrangian is given in Ref. [11]). Thus, only by including these contributions (which requires, for consistency, a full two-loop-order calculation) can one hope to obtain a well-converged approximation to the full chiral series for the amplitude.

The $\eta \rightarrow \pi^0\gamma\gamma$ discussion above can obviously be transferred directly to the case of the mixed-isospin vector current correlator under consideration here. We expect significant contributions to the correlator from the vector meson resonances and, for a particular choice of vector meson interpolating fields, these contributions are completely absent from the one-loop result. As a consequence, we can expect significant contributions from the tree-level $\mathcal{O}(p^6)$ terms in which such contributions reside. As already discussed, the $\mathcal{O}(p^6)$ loop contributions (arising from two-loop graphs with lowest-order vertices and one-loop graphs with a single $\mathcal{O}(p^4)$ vertex) may also be significant. A two-loop calculation is, therefore, almost certainly required in order to obtain convergence of the chiral series for the mixed-isospin correlator. Similar statements hold for the related correlator, $\Pi_{\mu\nu}^{38}(q^2)$, for which work on the two-loop calculation is in progress [42]. Note that, in the latter case, only a single combination of the $\mathcal{O}(p^6)$ LEC’s enters the two-loop result. This combination, which in the notation of Ref. [13] (where the analogous $\Pi_{\mu\nu}^{33}$ and $\Pi_{\mu\nu}^{88}$ correlators are computed to two-loop order), is written $Q^0(\mu) - 3L_9^{(-1)}(\mu) - 3L_{10}^{(-1)}(\mu)$, with $\mu$ the renormalization scale, is in principle obtainable from experimental data using the chiral sum rules of Ref. [43] (Eqns. (97) and (98) therein). In the case at hand, one further $\mathcal{O}(p^4)$ LEC and one further $\mathcal{O}(p^6)$ LEC will
be present at the two-loop level, but the sum rule analysis above, in combination with a full two-loop evaluation of $\Pi(q^2)$, would provide a useful constraint on these parameters, albeit it with the $\simeq 20 - 30\%$ errors displayed in Table III and associated with the uncertainties in the values of the input parameters $\{c_i\}$ which determine the correlator near $q^2 = 0$. A similar sum rule analysis of the correlator $\Pi^{28}_{\mu\nu}$ would constrain the combination $Q^0(\mu) - 3L_9^{(-1)}(\mu) - 3L_{10}^{(-1)}(\mu)$, mentioned above.

It is interesting to note that the relation between the sum rule and ChPT results for the mixed-isospin vector correlator is effectively the reverse of what occurs in the mixed-isospin axial correlator case. In the latter case, the ChPT [16] and sum rule [13] results for the value of that piece of the correlator proportional to $q_\mu q_\nu$ at $q^2 = 0$ are comparable, but the ChPT result for the slope of the correlator with $q^2$ is more than an order of magnitude larger than that obtained from a sum rule analysis analogous to that employed above for the vector correlator case [16]. The source of the discrepancy, in the axial correlator case, is that the sum rule result for the slope has the incorrect chiral behavior, being in fact missing its leading contribution in the chiral expansion. This problem with the sum rule treatment is easily exposed using chiral methods, but is completely non-obvious without them. In the present case, since we do not know what portion of the vector meson masses survives in the chiral limit, we cannot make as precise statements about the required chiral behavior of the vector correlator. There is, however, no obvious problem with the form of the sum rule result above. The sum rule result, moreover, provides clear evidence to indicate that the chiral series for the vector correlator is indeed, as suggested by analogy to the known behavior of the $\eta \to \pi^0 \gamma \gamma$ process, slowly converging. The sum rule result, in this case, should also provide useful input for the two-loop analysis in ChPT. The two examples clearly indicate the advantages of applying both methods, within their common range of validity, in any given physical process.
V. SUMMARY OF RESULTS

The basic results of the paper are as follows. We have demonstrated that (1) in making a sum rule analysis of the mixed-isospin vector current correlator, it is necessary to include the $\phi$ pole term in the phenomenological form of the representation of the correlator, and that, when one does so, the spectral structure of the correlator becomes physically sensible; (2) the expression for the correlator away from $q^2 = m_\omega^2$ has no general interpretation as the off-diagonal element of a vector-meson propagator except for a particular vector meson interpolating field choice; (3) the freedom of field redefinition shows that the isospin-breaking factor $\theta^{\omega}(q^2)$, which occurs in the numerator of the expression, (1.3), for the off-diagonal element of the vector meson propagator, cannot, in general, be taken to be independent of $q^2$; (4) the behavior of the correlator near $q^2 = m_\omega^2$ suggests that the direct $\omega(0) \to \pi^+\pi^-$ contribution to $e^+e^- \to \pi^+\pi^-$ is not negligible in the $\rho-$\omega interference region; (5) the discrepancy between the behavior of the correlator near $q^2 = 0$ as obtained from the sum rule analysis and from ChPT to one-loop indicates a slow convergence of the chiral series for the correlator and, in consequence, the necessity of a two-loop calculation of this quantity in ChPT. The sum rule result for the correlator near $q^2 = 0$ can then be used, in such a calculation, to constrain the $\mathcal{O}(p^6)$ LEC’s of ChPT.

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FIGURES

FIG. 1. Dependence of $\xi$ on the Borel mass, $M$, for modified input set IV.

FIG. 2. Dependence of $\beta$ on the Borel mass, $M$, for modified input set IV.
FIG. 3. Dependence of $\xi'$ on the Borel mass, $M$, for modified input set IV.

FIG. 4. Dependence of $\beta'$ on the Borel mass, $M$, for modified input set IV.
FIG. 5. Dependence of $f_\phi$ on the Borel mass, $M$, for modified input set IV.

FIG. 6. Dependence of $f_\phi$ on the Borel mass, $M$, for modified input set II.
TABLE I. Sum rule fit for the parameters $\xi$, $\beta$, $\xi'$, $\beta'$ and $f_{\phi}$.

| Input  | $\xi \times 10^3$ | $\beta$   | $\xi' \times 10^5$ | $\beta'$  | $f_{\phi} \times 10^3$ |
|--------|-------------------|-----------|---------------------|-----------|------------------------|
| Set I  | 2.18±0.39         | 1.49±0.06 | -2.63±0.79         | -5.84±0.12| 2.30±0.52              |
| Set III| 3.10±0.39         | 1.62±0.02 | -4.57±0.69         | -5.72±0.01| 3.57±0.52              |
| Set IV | 2.59±0.39         | 1.55±0.04 | -3.47±0.61         | -5.78±0.04| 2.86±0.45              |

TABLE II. Sum rule fit for the isospin-breaking parameters \{$f_k$\}. Values are quoted for the central values of the input parameters \{$c_i$\}. The units are GeV$^2$.

| Input  | $f_{\rho} \times 10^2$ | $f_{\omega} \times 10^2$ | $f_{\phi} \times 10^3$ | $f_{\rho'} \times 10^4$ | $f_{\omega'} \times 10^4$ |
|--------|------------------------|---------------------------|------------------------|-------------------------|---------------------------|
| Set I  | 3.53                   | 3.73                      | 2.30                   | 5.34                    | 8.45                      |
| Set III| 5.00                   | 5.30                      | 3.57                   | 9.32                    | 14.6                      |
| Set IV | 4.18                   | 4.42                      | 2.86                   | 7.06                    | 11.1                      |

TABLE III. Behavior of the correlator near $q^2 = 0$. Contributions to $12 \Pi(0)$ from the $\rho$-$\omega$, $\phi$ and $\rho'$-$\omega'$ regions are quoted for central values of the input parameters \{$c_i$\} for each input set, while the effect of the uncertainties in these values is displayed explicitly for $12 \Pi(0)$ and $12 \frac{d\Pi}{dq^2}(0)$. All entries are in units of $10^{-3}$, except for $12 \frac{d\Pi}{dq^2}(0)$, which is in units of $10^{-3}$ GeV$^{-2}$.

| Input  | $\rho$-$\omega$ | $\phi$ | $\rho'$-$\omega'$ | $12 \Pi(0)$ | $12 \frac{d\Pi}{dq^2}(0)$ |
|--------|-----------------|-------|-------------------|-------------|---------------------------|
| Set I  | -1.06           | 2.21  | -0.18             | 0.96±0.14   | 3.88±0.61                 |
| Set III| -1.92           | 3.44  | -0.31             | 1.22±0.14   | 5.10±0.60                 |
| Set IV | -1.43           | 2.75  | -0.24             | 1.08±0.14   | 4.43±0.61                 |