FUJITA’S CONJECTURE FOR QUASI-ELLIPTIC SURFACES

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ABSTRACT. We show that Fujita’s conjecture is true for quasi-elliptic surfaces. Explicitly, for any quasi-elliptic surface $X$ and an ample line bundle $A$ on $X$, we have $K_X + tA$ is base point free for $t \geq 3$ and is very ample for $t \geq 4$.

1. Introduction

Let $X$ be a smooth projective variety and $A$ be an ample line bundle. One of the crucial classical problems is understanding under what conditions the adjoint linear system $K_X + A$ is base point free or very ample. Thanks to Serre’s theorem, we know that $K_X + tA$ is very ample for $t$ sufficiently large, and there is great interest in understanding the smallest value of $t$ for which this holds. The following conjecture is due to Fujita in [Fuj88].

Conjecture 1.1 (Fujita). Let $X$ be a smooth projective variety of dimension $n$ and $A$ be an ample line bundle. Then $K_X + tA$ is base point free (resp. very ample) whenever $t \geq n + 1$ (resp. $t \geq n + 2$).

This conjecture for curves follows from the Riemann-Roch theorem. In characteristic zero, the conjecture is entirely proved for surfaces by Reider’s theorem [Rei88]. For the base point freeness part of this conjecture in characteristic zero, it has been proved up to dimension five in [EL93], [Hel97], [Kaw97], and [YZ20]. In positive characteristic, Shepherd-Barron showed in [SB91] that the conjecture is true for surfaces that are neither quasi-elliptic (see Definition 2.2) nor of general type. Recently, Gu, Zhang, and Zhang claimed in [GZZ20] that there are counterexamples to the conjecture for surfaces. Those examples are of general type.

In this paper, we show that Fujita’s conjecture is true for quasi-elliptic surfaces.

Theorem 1.2 (=Theorem 3.2). Fujita’s conjecture is true for quasi-elliptic surfaces $X$. (See Definition 2.2.) That is, given a quasi-elliptic surface $X$ and any ample line bundle $A$ on $X$, we have

1. $K_X + tA$ is base point free for $t \geq 3$; and
2. $K_X + tA$ is very ample for $t \geq 4$.

To prove this result, we follow the ideas of [DCF15] and make a careful case-by-case study. Note that, in [DCF15], it is proved that, when $p = 3$, $K_X + tA$ is base point free for $t \geq 4$ and it is very ample for $t \geq 8$; and when $p = 2$, $K_X + tA$ is base point free for $t \geq 5$ and it is very ample for $t \geq 19$.

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2. Preliminaries

In this section, we recall some definitions and results which will be used later. We will always assume the base field $k$ is algebraically closed and of positive characteristic $p$.

Lemma 2.1. Let $X$ be a smooth projective surface over $k$ and $N$ a nef divisor on $X$. Then for any divisor $D$ on $X$, we have

$$N^2D^2 \leq (N.D)^2.$$ Moreover, if $N$ is ample, then the equality holds only when $D$ is numerically proportional to $N$.

Proof. Since we can approximate nef divisors by ample $\mathbb{Q}$-divisors and the desired inequality is homogeneous, we can reduce to the case when $N$ is ample.

Now we consider $E = (N.D)N - N^2D$. Notice that $E.N = 0$. Then, by the Hodge index theorem, we have $E^2 \leq 0$, and we get the desired inequality. Moreover, the equality holds only when $E \equiv 0$, that is, $D$ is numerically proportional to $N$. □

Definition 2.2. A smooth projective surface $X$ over $k$ is said to be quasi-elliptic if there is a fibration $f : X \to C$ where $C$ is a smooth curve such that $f_*\mathcal{O}_X = \mathcal{O}_C$ and such that the general fibers of $f$ are rational curves with one (ordinary) cusp. Such $f : X \to C$ is called a quasi-elliptic fibration.

Remark 2.3. The general fibers of $f$ have arithmetic genus 1. Moreover, by a result of Tate in [Tat52], quasi-elliptic surfaces exist only when $p = 2$ or 3.

Definition 2.4 ([SB91], [DCF15]). A rank-two vector bundle $\mathcal{E}$ on $X$ is unstable if it fits into a short exact sequence, which will be called a de-stabilizing sequence for $\mathcal{E}$,

$$
0 \longrightarrow \mathcal{O}_X(D_1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) \longrightarrow 0
$$

where $D_1$ and $D_2$ are Cartier divisors, $\mathcal{I}_Z$ is the ideal sheaf of a finite subscheme $Z$ of $X$, and $D_1 - D_2 \in C_{++}(X)$, the positive cone of $\text{NS}(X)$. Notice that $Z$ could be empty, and by convention, $\mathcal{I}_Z = \mathcal{O}_X$ when $Z$ is empty. We also recall that

$$C_{++}(X) = \{x \in \text{NS}(X)|x^2 > 0 \text{ and } x.H > 0 \text{ for some ample divisor } H\} = \{x \in \text{NS}(X)|x^2 > 0 \text{ and } x \text{ is big}\}.$$

Definition 2.5 ([DCF15]). A big divisor $D$ on a smooth surface $X$ with $D^2 > 0$ is $m$-unstable for a positive integer $m$ if $h^1(X, \mathcal{O}_X(-D)) \neq 0$ and there exists a nonzero effective divisor $E$ such that $mD - 2E$ is big and $(mD - E).E \leq 0$.

In [Bog78], Bogomolov showed that, in characteristic zero, every rank-two vector bundle $\mathcal{E}$ on a smooth surface with $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$ is unstable. Also, in positive characteristic, there is a result related to the unstability of vector bundles.

Theorem 2.6 ([SB91] Theorem 1]). Let $\mathcal{E}$ be a rank-two vector bundle on a smooth projective surface $X$ over an algebraically closed field $k$ of positive characteristic $p > 0$ such that $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$. Then there exists an integral surface $Y$ contained in the ruled threefold $\mathbb{P}(\mathcal{E})$ such that

(1) the composition $\rho : Y \to X$ is purely inseparable of degree $p^e$ for some $e > 0$; and
(2) \((F^e)^*\mathcal{E}\) is unstable where \(F : X \to X\) is the absolute Frobenius morphism.

Moreover, we have

\[ K_Y \equiv \rho^* \left( K_X - \frac{p^e - 1}{p^e}(D_1 - D_2) \right) \]

where

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_X(D_1) & \to & F^e*\mathcal{E} & \to & \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) & \to & 0
\end{array}
\]

is a de-stablizing sequence for \((F^e)^*\mathcal{E}\).

We recall the construction of \(Y\) in Theorem 2.6. Assume we have shown that \((F^n)^*\mathcal{E}\) is unstable for some positive integer \(n\). Let \(e\) be the smallest one such that \(\tilde{E} := (F^e)^*\mathcal{E}\) is unstable. We have the following cartesian diagram

\[
\begin{array}{ccc}
\mathbb{P}(\tilde{E}) & \xrightarrow{\psi} & \mathbb{P}(E) \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
X & \xrightarrow{F^n} & X.
\end{array}
\]

From a de-stablizing sequence for \((F^e)^*\mathcal{E}\), we have a surjection \((F^e)^*\mathcal{E} \to \mathcal{I}_Z \cdot \mathcal{O}_X(D_2)\), which gives a quasi-section \(Y' \subset \mathbb{P}(\tilde{E})\). Then \(Y\) is the schematic image of \(Y'\) in \(\mathbb{P}(E)\).

**Lemma 2.7.** If \(D\) is big with \(D^2 > 0\) and \(h^1(X, \mathcal{O}_X(-D)) \neq 0\), then \(D\) is \(p^e\)-unstable for some \(e > 0\).

**Proof.** Indeed, this is contained in [SB91, Lemma 16]. The reader can also see [DCF15, Remark 2.10]. For the reader’s convenience, we include the proof.

Since \(h^1(X, \mathcal{O}_X(-D)) \neq 0\), there exists a non-split short exact sequence

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_X & \to & \mathcal{E} & \to & \mathcal{O}_X(D) & \to & 0
\end{array}
\]

given by a nonzero element of \(\text{Ext}^1(\mathcal{O}_X(D), \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X(-D))\), where \(\mathcal{E}\) is a vector bundle of rank two. Note that \(c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) = D^2 > 0\). By Theorem 2.6, we have the following diagram.

\[
\begin{array}{ccccccccccc}
0 & \to & \mathcal{O}_X(D_1) & \xrightarrow{f_1} & (F^e)^*\mathcal{E} & \xrightarrow{f_2} & \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) & \to & 0 \\
\downarrow g_1 & & \downarrow \tau & & \downarrow g_2 & & \downarrow & & \downarrow 0 \\
0 & & \mathcal{O}_X(p^e D) & & & & & & 0
\end{array}
\]

We claim that \(\tau = g_2 \circ f_1\) is not zero. Indeed, if \(\tau = 0\), then \(f_1 = g_1 \circ \tau'\) where \(\tau'\) is a nonzero map from \(\mathcal{O}_X(D_1)\) to \(\mathcal{O}_X\). That means \(-D_1\) is linearly equivalent to an effective...
divisor. Now notice that \( D_1 + D_2 \equiv c_1((F^e)^*\mathcal{E}) \equiv p^eD \) is big and, for any ample divisor \( H \), we have
\[
0 < p^eD.H = (D_1 + D_2).H = -(D_1 - D_2).H - (2D_2).H < 0, \text{ which is impossible.}
\]

Hence, we have \( \tau \neq 0 \) and so \( D_2 \equiv c_1((F^e)^*\mathcal{E}) - D_1 \equiv p^eD - D_1 \) is effective. So \( p^eD - 2D_2 \equiv D_1 - D_2 \) is big and
\[
(p^eD - D_2).D_2 = D_1.D_2 = c_2((F^e)^*\mathcal{E}) - \deg Z = -\deg Z \leq 0.
\]

Also \( D_2 \neq 0 \) since otherwise the vertical exact sequence
\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(D) \longrightarrow 0
\]
splits, which is a contradiction.

To sum up, \( D \) is \( p^e \)-unstable. \( \square \)

**Proposition 2.8.** Let \( \pi : Y \to X \) be a birational morphism between two smooth surfaces and let \( \widetilde{D} \) be a big Cartier divisor on \( Y \) such that \( \widetilde{D}^2 > 0 \). Assume there is a non-zero effective divisor \( \widetilde{E} \) such that \( \widetilde{D} - 2\widetilde{E} \) is big and \( (\widetilde{D} - \widetilde{E}).\widetilde{E} \leq 0 \).

Set \( D = \pi_*\widetilde{D} \), \( E = \pi_*\widetilde{E} \) and \( \alpha = D^2 - \widetilde{D}^2 \). If \( D \) is nef and \( E \) is a non-zero effective divisor, then \( 0 \leq D.E < \alpha/2 \) and \( D.E - \alpha/4 \leq E^2 \leq (D.E)^2/D^2 \).

**Proof.** For a reference, see [Sak90, Proposition 2].

**Corollary 2.9.** Let \( \pi : Y \to X \) be a birational morphism between two smooth surfaces and let \( \widetilde{D} \) be a big Cartier divisor on \( Y \) such that \( \widetilde{D}^2 > 0 \). Assume that \( h^1(X, \mathcal{O}_X(-\widetilde{D})) \neq 0 \) and \( \widetilde{D} \) is \( m \)-unstable for some \( m > 0 \). That is, there exists a non-zero effective divisor \( \widetilde{E} \) such that \( m\widetilde{D} - 2\widetilde{E} \) is big and \( (m\widetilde{D} - \widetilde{E}).\widetilde{E} \leq 0 \).

Set \( D = \pi_*\widetilde{D} \), \( E = \pi_*\widetilde{E} \) and \( \alpha = D^2 - \widetilde{D}^2 \). If \( D \) is nef and \( E \) is a non-zero effective divisor, then \( 0 \leq D.E < m\alpha/2 \) and \( mD.E - m\alpha^2/4 \leq E^2 \leq (D.E)^2/D^2 \).

**Proof.** Write \( \widetilde{B} = m\widetilde{D} \). Since \( \widetilde{D} \) is \( m \)-unstable, \( \widetilde{B} \) is \( 1 \)-unstable. Thus, we can use Proposition 2.8 above. Note that \( \alpha_B = B^2 - \widetilde{B}^2 = m(D^2 - \widetilde{D}^2) = m^2\alpha_D \). \( \square \)

3. Fujita’s Conjecture for Quasi-Elliptic Surfaces

From now on, \( X \) and \( Y \) are quasi-elliptic surfaces, and \( A \) is an ample divisor on \( X \). We first improve [DCF15, Proposition 4.3].

**Proposition 3.1.** Let \( X \) be a quasi-elliptic surface with a quasi-elliptic fibration \( f : X \to C \) and \( D \) be a big divisor on \( X \) with \( D^2 > 0 \) and \( h^1(X, \mathcal{O}_X(-D)) \neq 0 \). Then \( D \) is \( p \)-unstable.

Moreover, let \( F \) be a general fiber of the fibration \( f \) and \( E \) be a non-zero effective divisor whose existence is guaranteed by \( p \)-unstability of \( D \) (see Definition 2.8). Then we have \( (3D - 2E).F = 1 \) when \( p = 3 \) and \( (D - E).F = 1 \) when \( p = 2 \).

**Proof.** Since \( h^1(X, \mathcal{O}_X(-D)) \neq 0 \), there exists a non-split short exact sequence
\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(D) \longrightarrow 0
\]
given by a non-zero element of \( \text{Ext}^1(\mathcal{O}_X(D), \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X(-D)) \), where \( \mathcal{E} \) is a vector bundle of rank two. Note that \( c_2(\mathcal{E}) = 4c_2(\mathcal{E}) = D^2 > 0 \). By Theorem 2.6 we have
(\(F^e\))∗\(E\) is unstable for some \(e > 0\) and \(\rho : Y \to X\) is a purely inseparable morphism of degree \(p^e\). Let

\[
0 \to \mathcal{O}_X(D_1) \to (\mathcal{F}^e)^*\mathcal{E} \to \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) \to 0
\]

be a de-stabilizing sequence for \((\mathcal{F}^e)^*\mathcal{E}\).

By Lemma 2.7 and its proof, \(D\) is \(p^e\)-unstable. Let \(E\) be a non-zero effective divisor whose existence is guaranteed by \(p^e\)-unstability of \(D\), \(F\) be a general fiber of \(f : X \to B\), and \(C = \rho^*F\). Note that

\[
-K_Y.C = \rho^*\left(\frac{p^e-1}{p^e}(D_1 - D_2) - K_X\right).C
\]

\[
= \rho^*\left(\frac{p^e-1}{p^e}(p^eD - 2E) - K_X\right).C
\]

\[
= p^e\left(\frac{p^e-1}{p^e}(p^eD - 2E) - K_X\right).F
\]

\[
= (p^e - 1)(p^eD - 2E).F > 0
\]

where

1. the first equality follows from Theorem 2.6
2. the second equality follows from

\[
D_1 - D_2 = (D_1 + D_2) - 2D_2 \equiv c_1(\mathcal{F}^e\mathcal{E}) - 2E \equiv p^eD - 2E,
\]

3. the third equality follows from projective formula,
4. the fourth equality follows since \(F\) has arithmetic genus one, and
5. the last inequality follows since \(p^eD - 2E\) is big and \(F\) is a general fiber of \(f\).

Notice that \(Y\) is a local complete intersection since \(Y\) is a divisor in a smooth variety.

Then by [DCF15, Corollary 2.14], we have \(-K_Y.C \leq 3\). This gives

\[
3 \geq -K_Y.C = (p^e - 1)(p^eD - 2E).F.
\]

When \(p = 3\), we have \((p^e - 1)(p^eD - 2E).F \geq 3^e - 1 \geq 8\) if \(e \geq 2\), which is impossible.

When \(p = 2\), we have \((p^e - 1)(p^eD - 2E).F = 2(2^e - 1)(2^{e-1}D - E).F \geq 2(2^e - 1) \geq 6\) if \(e \geq 2\), which is impossible.

Thus, \(e\) must be 1 and \(D\) is \(p\)-unstable.

Moreover, when \(p = 3\), we have

\[
(p^e - 1)(p^eD - 2E).F = 2(3D - 2E).F
\]

which is a positive even integer less than 3. So we have \((3D - 2E).F = 1\). When \(p = 2\), we have

\[
(p^e - 1)(p^eD - 2E).F = 2(D - E).F
\]

which is a positive even integer less than 3. So we have \((D - E).F = 1\).

\(\square\)

Now we are ready to prove

**Theorem 3.2.** Fujita’s conjecture is true for quasi-elliptic surfaces \(X\). That is, given a quasi-elliptic surface \(X\) and any ample line bundle \(A\) on \(X\), we have

1. \(K_X + tA\) is base point free for \(t \geq 3\); and
2. \(K_X + tA\) is very ample for \(t \geq 4\).
Proof. We divide the proof into several steps. We first prove that $K_X + tA$ is base point free for $t \geq 3$.

(Step 1) (Preparation.) Let $D = tA$ and assume that $|K_X + D|$ has a base point at $x \in X$. Let $\pi : Y \to X$ be the blow-up at $x$. Since $x$ is a base point, we have that

$$h^1(Y, \mathcal{O}_Y(K_Y + \pi^* D - 2E_x)) = h^1(X, \mathcal{O}_X(K_X + D) \otimes \mathfrak{m}_x) \neq 0$$

where $E_x$ is the exceptional divisor of $\pi$. Let $\widetilde{D} = \pi^* D - 2E_x$.

In order to apply Proposition 3.1 we need to check that $\widetilde{D}$ is big and $\widetilde{D}^2 > 0$. Note that

$$h^0(Y, \mathcal{O}_Y(t\widetilde{D})) = h^0(Y, \mathcal{O}_Y(\ell(\pi^* D - 2E_x)))$$

$$= h^0(X, \mathcal{O}_X(\ell D) \otimes \mathfrak{m}_x^{2\ell})$$

$$\geq \frac{D^2}{2} \ell^2 + O(\ell) - \left(\frac{2\ell + 1}{2}\right)$$

$$= \frac{t^2A^2 - 4}{2} \ell^2 + O(\ell).$$

So $\widetilde{D}$ is big whenever $t \geq 3$. Also note that $\widetilde{D}^2 = D^2 - 4 = t^2A^2 - 4 \geq 5$ when $t \geq 3$.

Applying Proposition 3.1 on $Y$ and $\widetilde{D}$, we have that $\widetilde{D}$ is $p$-unstable. So there is a nonzero effective divisor $\tilde{E}$ such that $p\tilde{D} - 2\tilde{E}$ is big and $(p\tilde{D} - \tilde{E}).\tilde{E} \leq 0$. Let $E = \pi_*\tilde{E}$. Note that $E$ is an effective divisor. If $E = 0$, then $\tilde{E} = bE_x$ for $b > 0$. Thus, $(p\tilde{D} - \tilde{E}).\tilde{E} = b(b + 2p) > 0$, which is a contradiction. Therefore, $E$ is non-zero.

Also $\pi_*\tilde{D} = D = tA$ is ample and $\alpha = D^2 - \tilde{D}^2 = 4$. Hence, by Corollary 2.9 we have

$$0 < tA.E < 2p \leq 6 \text{ and } ptA.E - p^2 \leq E^2 \leq (A.E)^2/A^2.$$  

So we have $0 < A.E < \frac{6}{t} \leq 2$ and thus, $A.E = 1$ and $E$ is an irreducible curve. The second inequality in (1) becomes

$$pt - p^2 \leq E^2 \leq 1/A^2 \leq 1.$$  

(Step 2) If $p = 2$, then $2 \leq 2t - 4 \leq E^2 \leq 1$, which is impossible.

(Step 3) If $p = 3$, then $3t - 9 \leq E^2 \leq 1$. This happens only when $t = 3$ and $E^2 = 0$ or 1.

Now, by Proposition 3.1 we have

$$1 = (3\tilde{D} - 2\tilde{E}).\pi^*F = (9A - 2E).F.$$  

Since $F$ is nef and $A$ is ample, we get $9A.F \geq 9$ and so, by equality (3), we have

$$E.F \geq 4.$$  

Note that $E + F$ is nef since $(E + F).E \geq 0 + F,E \geq 4$ and $(E + F).F = E.F \geq 4$.

(Step 4) If $E^2 = 1$, then $A^2 = 1$ by inequality (2) and $A$ is numerically equivalent to $E$ by Hodge inequality. Thus, by equality (3), we have $7A.F = 1$, which is impossible.

(Step 5) So we have $E^2 = 0$. Applying Lemma 2.11 to $9A - 2E$ and $E + F$, we have

$$(9A - 2E)^2(E + F)^2 \leq ((9A - 2E).E + F)^2.$$
Thus, we have
\[(81A^2 - 36)(2F.E) \leq (9A.E + (9A - 2E).F)^2 = 100\]
since \(A.E = 1\) and \((9A - 2E).F = 1\) by equality (3). Thus,
\[5 \leq 9A^2 - 4 \leq \frac{100}{18(F.E)} \leq \frac{100}{18 \times 4} \leq 2,\]
which is impossible.

Hence, we have shown that \(K_X + tA\) is base point free whenever \(t \geq 3\).

Now we prove \(K_X + tA\) is very ample when \(t \geq 4\).

(Step 1) (Preparation.) Let \(D = tA\). It suffices to show \(|K_X + D|\) separates points and tangents. (For a reference, see [Har77, Proposition II.7.3].) Assume that \(|K_X + D|\) does not separate points \(x\) and \(y\) (resp. does not separate tangents at \(x\)). Then we have
\[h^1(Y, \mathcal{O}_Y(K_Y + \pi^*D - 2E_x - 2E_y)) = h^1(X, \mathcal{O}_X(K_X + D) \otimes \mathfrak{m}_x \otimes \mathfrak{m}_y) \neq 0\]
(resp. \(h^1(Y, \mathcal{O}_Y(K_Y + \pi^*D - 3E_x)) = h^1(X, \mathcal{O}_X(K_X + D) \otimes \mathfrak{m}_x^2) \neq 0\))
where \(\pi : Y \rightarrow X\) is the blow-up of \(X\) at \(x, y\) and \(E_x, E_y\) are the exceptional divisor (resp. \(\pi : Y \rightarrow X\) is the blow-up of \(X\) at \(x\) and \(E_x\) is the exceptional divisor.)

Now let \(\widetilde{D} = \pi^*D - 2E_x - 2E_y\) (resp. \(\widetilde{D} = \pi^*D - 3E_x\)). By the similar argument as above, \(\widetilde{D}\) is big and \(\widetilde{D}^2 > 0\) whenever \(t \geq 4\).

Applying Proposition 3.1 to \(Y\) and \(\widetilde{D}\), we have that \(\widetilde{D}\) is \(p\)-unstable. So there is a nonzero effective divisor \(\widetilde{E}\) such that \(p\widetilde{D} - 2\widetilde{E}\) is big and \((p\widetilde{D} - \widetilde{E}).\widetilde{E} \leq 0\). Let \(E = \pi_*\widetilde{E}\). Note that \(E\) is a non-zero effective divisor by the similar argument as above. Also we have \(\pi_*\widetilde{D} = D = tA\) is ample and \(\alpha = D^2 - \widetilde{D}^2 = 8\) (resp. 9).

Hence, by Corollary 2.9 we have
\[0 < tA.E < ptA.E = p^2\alpha/4 \leq E^2 \leq (A.E)^2/A^2.\]

(Step 2) When \(p = 3\), by Proposition 3.1, we have
\[(3\widetilde{D} - 2\widetilde{E}).\pi^*F = 1\]
and thus
\[(3tA - 2E).F = 1.\]
If \(t\) is even, then the left hand side is an even integer, which is impossible.

(Step 3) So \(t\) is odd. From inequalities (3), we have
\[0 < tA.E < \frac{3 \alpha}{2} \leq \frac{27}{2} \text{ and } 3tA.E - \frac{9}{4}\alpha \leq E^2 \leq (A.E)^2/A^2.\]
Then we have \(A.E = 1\) or 2 since \(A.E < \frac{27}{2t} \leq \frac{27}{10} < 3\). If \(A.E = 2\), then we have
\[9 < 6t - \frac{81}{4} \leq 6t - \frac{9}{4}\alpha \leq E^2 \leq 4/A^2 \leq 4,\]
which is impossible.

Thus \(A.E = 1\) and so \(E\) is an irreducible curve. Also, from the second inequality in (7), we have
\[3t - \frac{81}{4} \leq 3t - \frac{9}{4}\alpha \leq E^2 \leq \frac{1}{A^2} \leq 1.\]
Hence $t$ is 5 or 7 and

$$-5 \leq E^2 \leq 1.$$  

Using equality (6), we have $1 + 2E.F = 3tA.F \geq 15$. So

$$E.F \geq 7$$

and $E + F$ is nef since $(E + F).E \geq -5 + 7 = 2$ and $(E + F).F \geq 7$.

Applying Lemma 2.1 to $3tA - 2E$ and $E + F$, we have

$$(3tA - 2E)^2(E + F)^2 \leq ((3tA - 2E).(E + F))^2.$$  

Note that the left hand side

$$(3tA - 2E)^2(E + F)^2 = (9t^2A^2 - 12t + 4E^2)(E^2 + 2E.F) \geq (4E^2 + 141)(E^2 + 2E.F) \geq (4E^2 + 141)(E^2 + 14)$$

where

(a) the first inequality follows from $A^2 \geq 1$, $5 \leq t \leq 7$, and nefness of $E + F$; and

(b) the second inequality follows from inequalities (8) and (9).

Also the right hand side

$$(3tA - 2E).(E + F)^2 = (1 + 3t - 2E^2)^2 \leq (22 - 2E^2)^2$$

where

(a) the equality follows from equality (6); and

(b) the inequality follows from inequality (8) and $t \leq 7$.

To sum up, we have $(4E^2 + 141)(E^2 + 14) \leq (22 - 2E^2)^2$ and thus $E^2 \leq -6$, which contradicts to inequality (8).

(Step 4) Now we deal with $p = 2$. The inequalities (5) becomes

$$0 < tA.E < \alpha \leq 9 \text{ and } 2tA.E - \alpha \leq E^2 \leq (A.E)^2/A^2.$$  

Hence, $A.E = 1$ or 2 since $A.E < \frac{3}{4} \leq \frac{2}{4}$. If $A.E = 2$, then we have $7 \leq 4t - \alpha \leq E^2 \leq 4/A^2 \leq 4$, which is impossible.

(Step 5) Thus, we have $A.E = 1$, and so $E$ is an irreducible curve. Then, from the second inequality in (10), we have

$$2t - 9 \leq 2t - \alpha \leq E^2 \leq 1/A^2 \leq 1.$$  

So $t = 5$ and $E^2 = 1$; or $t = 4$ and $-1 \leq E^2 \leq 1$. By Proposition 3.1 we have $(\widetilde{D} - \widetilde{E}).\pi^*F = 1$ and thus

$$tA - E).F = 1.$$  

So we have

$$E.F = tA.F - 1 \geq 3.$$  

Therefore, $E + F$ is nef since $(E + F).E \geq -1 + 3 = 2$ and $(E + F).F \geq 3$.

(Step 6) If $t = 5$, then we have $E^2 = 1$ and thus $A \equiv E$ by Lemma 2.1. But, from equality (11), we have

$$1 = (5A - E).F = 4A.F,$$  

which is impossible.
(Step 7) Thus \( t = 4 \). Applying Lemma 2.1 to \( 4A - E \) and \( E + F \), we have

\[
(13) \quad (4A - E)^2(E + F)^2 \leq ((4A - E)(E + F))^2 = (1 + 4 - E^2)^2 = (5 - E^2)^2.
\]

(Step 8) When \( E^2 = 1 \), we have \( A \equiv E \) by Lemma 2.1. Thus, from equality (11), we have

\[
1 = (4A - E).F = 3A.F,
\]

which is impossible.

(Step 9) When \( E^2 = 0 \), we have, from inequality (13),

\[
(16A^2 - 8)(2E.F) \leq 25
\]

and thus, by inequality (12),

\[
1 \leq 2A^2 - 1 \leq \frac{25}{16(E.F)} \leq \frac{25}{16 \times 3} < 1,
\]

which is impossible.

(Step 10) When \( E^2 = -1 \), from inequality (13), we have

\[
(16A^2 - 9)(2E.F - 1) \leq 36
\]

and thus, by inequality (12),

\[
7 \leq 16A^2 - 9 \leq \frac{36}{2E.F - 1} \leq \frac{36}{5} < 8.
\]

Therefore, \( A^2 = 1 \) and \( E.F = 3 \). Moreover, by equality (11), we have \( A.F = 1 \). Now applying Lemma 2.1 to \( A \) and \( E + F \), we have

\[
5 = A^2(E + F)^2 \leq (A.(E + F))^2 = 4,
\]

which is impossible.

Hence, we have shown that \( K_X + tA \) is very ample whenever \( t \geq 4 \). 

\[ \square \]

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