Conditional Moments of Anticipative $\alpha$-Stable Markov Processes

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Abstract

The anticipative $\alpha$-stable autoregression of order 1 (AR(1)) is a stationary Markov process undergoing explosive episodes akin to bubbles in financial time series data. Although featuring infinite variance, integer conditional moments up to order four may exist. The conditional expectation, variance, skewness and kurtosis are provided at any forecast horizon under any admissible parameterisation. During bubble episodes, these moments become equivalent to that of a Bernoulli distribution charging complementary probabilities to two polarly-opposite outcomes: pursued explosion or collapse. Parallel results are obtained for the continuous time anticipative $\alpha$-stable Ornstein-Uhlenbeck process. The proofs build heavily on and extend properties of arbitrary, not necessarily symmetric $\alpha$-stable bivariate random vectors. Other processes are considered such as the anticipative AR(2) and the aggregation of anticipative AR(1).

Keywords: Anticipative/Noncausal processes, Stable processes, Explosive bubbles, Conditional moments.

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1 Introduction

Dynamic models often admit solution processes for which the current value of the variable is a function of future values of an independent error process. Such solutions, called anticipative, have attracted increasing attention in the financial and econometric literatures. In particular, anticipative processes have been shown to be convenient for modelling speculative bubbles [8, 17, 19, 20, 23, 24, 25, 26] (see also [1, 9, 33, 34]). However, lack of knowledge about the predictive distribution of anticipative processes is impeding the ability to forecast them, thus limiting their use in practical applications. Partial results have been obtained in [20] for the anticipative stable AR(1), defined as the stationary solution of

\[ X_t = \rho X_{t+1} + \varepsilon_t, \quad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, 0), \quad (1.1) \]

where \(0 < |\rho| < 1\) and \(\mathcal{S}(\alpha, \beta, \sigma, 0)\) denotes the univariate \(\alpha\)-stable distribution with tail parameter \(\alpha \in (0, 2)\), asymmetry \(\beta \in [-1, 1]\) and scale \(\sigma > 0\). Figure 1 depicts a typical simulated path of an anticipative stable AR(1) featuring multiple bubbles.

![Figure 1: Sample path of the solution of (1.1) with \(\varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(1.7, 0.8, 0.1, 0)\) and \(\rho = 0.95\).](image)

This paper proposes a complete characterisation of the conditional moments at any horizon, when existing, for the stable AR(1) process and two related models: the anticipative Ornstein-Uhlenbeck (OU) process and the aggregated stable process, defined as a linear combination of \(\alpha\)-stable anticipative processes of the form (1.1). Explicit expressions of the conditional moments generally have a complex form. However, we will show that the conditional distributions of \(X_{t+h}\), say, given \(X_t = x\) displays dramatic simplifications when \(x \to \pm \infty\), which provides illuminating
interpretations on the behaviour of the process during bubble episodes.

Section 2 starts by recalling characterisations and properties of multivariate stable distributions and then provides our results on the anticipative stable AR(1) and OU processes. Section 3 analyses the aggregation of AR(1). Section 4 finds a new upper bound for the existence of conditional moments of anticipative AR(2) processes. Complementary results on bivariate stable vectors are stated in Appendix A and B. Proofs are collected in Appendix C and in a Supplementary file.

2 Anticipative $\alpha$-stable Markov processes

Before analysing the anticipative $\alpha$-stable AR(1) and OU processes, we begin by recalling some characterisations of multivariate stable distributions which will be the cornerstone of our proofs.

2.1 Characterisation of $\alpha$-stable random vectors

Stable random vectors are defined in a similar way as when considering stable variables on the real line. Denote by $\ll d \gg$ the equality in distribution between two random variables.

Definition 2.1 A random vector $X = (X_1, \ldots, X_d)$ is said to be a stable random vector in $\mathbb{R}^d$ if for any positive numbers $A$ and $B$ there is a positive number $C$ and a non-random vector $D \in \mathbb{R}^d$ such that

$$AX^{(1)} + BX^{(2)} \ll d \gg CX + D,$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of $X$. Moreover, if $X$ is stable, then there exists a constant $\alpha \in (0, 2]$ such that the above holds with $C = (A^\alpha + B^\alpha)^{1/\alpha}$, and $X$ is then called $\alpha$-stable.

We exclude the intensively-studied Gaussian case ($\alpha = 2$) from our analysis. Let $S_d$ be the unit sphere of $\mathbb{R}^d$ equipped with the Euclidean norm $\| \cdot \|$ induced by the canonical scalar product, denoted $\langle \cdot, \cdot \rangle$. The distribution of stable random vectors are characterised (see Theorem 2.3.1 in [40]) by a unique pair $(\Gamma, \mu^0)$, where $\Gamma$ is a finite measure on the unit sphere $S_d$ and a vector $\mu^0 \in \mathbb{R}^d$. Let $0 < \alpha < 2$, then $X = (X_1, \ldots, X_d)$ is an $\alpha$-stable random vector if and only if there exists a unique pair $(\Gamma, \mu^0)$ such that, for any $u \in \mathbb{R}^d$, the characteristic function of $X$ writes

$$\varphi_X(u) := \mathbb{E}[e^{i\langle u, X \rangle}] = \exp \left\{ - \int_{S_d} |\langle u, s \rangle|^\alpha \left( 1 - i \text{sign}(\langle u, s \rangle) w(\alpha, \langle u, s \rangle) \right) \Gamma(ds) + i \langle u, \mu^0 \rangle \right\},$$

(2.1)

where $w(\alpha, s) = \text{tg} \left( \frac{\pi \alpha}{2} \right)$, if $\alpha \neq 1$, and $w(1, s) = -\frac{2}{\pi} \ln |s|$ otherwise, for $s \in \mathbb{R}$. The finite measure $\Gamma$ is called the spectral measure of $X$ and captures the information about the scale, asymmetry
and dependence between its components. The non-random vector $\mu^0$ is a location parameter and is called the shift vector. The pair $(\Gamma, \mu^0)$ is said to be the spectral representation of the random vector $X$. In the univariate case, (2.1) boils down to

$$
\varphi_X(u) = \exp \left\{ -\sigma^\alpha |u|^\alpha \left( 1 - i\beta \text{sign}(u) w(\alpha, u) \right) + iu\mu \right\},
$$

for some $\sigma > 0$, $\beta \in [-1, 1]$ and $\mu \in \mathbb{R}$.

Stable distributions are known to have very little moments. However, the distribution of one component conditionally on the others can have more moments according to the degree of dependence between them. A sufficient condition for the existence of conditional moments of bivariate $\alpha$-stable vectors $(X_1, X_2)$ is given in the following Proposition.

**Proposition 2.1 (Samorodnitsky and Taqqu (Theorem 5.1.3, 1994))** Let $X = (X_1, X_2)$ be an $\alpha$-stable random vector with spectral measure $\Gamma$, satisfying

$$
\int_{S_2} |s_1|^{-\nu} \Gamma(ds) < +\infty, \text{ for some } \nu \geq 0.
$$

Then, $E[|X_2|^\gamma |X_1 = x] < +\infty$ for almost every $x$ if

$$
0 \leq \gamma < \min(\alpha + \nu, 2\alpha + 1).
$$

The less concentrated around the points $(0,1)$ and $(0,-1)$ of the unit circle is the spectral measure, the higher the moments of $X_2|X_1$.

### 2.2 Discrete time: the anticipative $\alpha$-stable AR(1)

Operating the arsenal of properties of multivariate $\alpha$-stable distributions we provide in the previous section and Appendix A, we analyse in detail the predictive distribution of the anticipative $\alpha$-stable AR(1) solution of (1.1), $X_t = \sum_{k \geq 0} \rho^k \varepsilon_{t+k}$. The following result shows that when the noise sequence $(\varepsilon_t)$ in (1.1) is $\alpha$-stable distributed, then $(X_t, X_{t+h})$ is itself an $\alpha$-stable random vector with a very specific spectral representation.

**Proposition 2.2** Let $(X_t)$ be the anticipative AR(1) solution of (1.1) with $0 < \alpha < 2$, $\beta \in [-1, 1]$ and $|\rho| < 1$. Then, for any $h \geq 1$, $(X_t, X_{t+h})$ is $\alpha$-stable and its spectral representation, denoted $(\Gamma_h, \mu^0)$ with $\mu^0 = (\mu^0_1, \mu^0_2)$, is such that

$$
\Gamma_h = \frac{\bar{\sigma}^\alpha}{2} \sum_{\vartheta \in S_0} \left[ (1 - |\rho|^h + (1 - (\rho^{\alpha_h})^h) \vartheta \bar{\beta}) \delta_{(\vartheta, 0)} + (1 + |\rho|^{2h})^{\alpha/2} (1 + \vartheta \bar{\beta}) \delta_{(\vartheta h)} \right],
$$

where $h$ is an integer.
where $S_0 = \{-1, +1\}$, $\delta_{\{x\}}$ is the Dirac measure at point $x \in \mathbb{R}$, $\bar{\sigma}^\alpha = \frac{\sigma^\alpha}{1 - |\rho|^\alpha}$, $\bar{\beta} = \beta \frac{1 - |\rho|^\alpha}{1 - \rho^{<\alpha>}}$, $y^{<\rho>} = \text{sign}(y)|y|^\rho$ for any $y, r \in \mathbb{R}$ and $s_h = \frac{(\rho^h, 1)}{\sqrt{1 + |\rho|^{2h}}} \in S_2$. Moreover, if $\alpha \neq 1$, then $\mu^0 = (0, 0)$, and if $\alpha = 1$ then,
\[
\mu_1^0 = \bar{\mu} - \frac{2}{\pi \bar{\sigma} \bar{\beta} \ln |\rho|} (1 - \rho), \hspace{1cm} \mu_2^0 = \rho^{-\beta} \bar{\mu} - \frac{2}{\pi \bar{\sigma} \bar{\beta} \ln |\rho|} \left(h + \frac{\rho}{1 - \rho}\right).
\]
with $\bar{\mu} = -\frac{\bar{\sigma} \bar{\beta}}{\pi} \rho^h \ln \left(1 + \rho^{-2h}\right)$.

It can be noticed from the previous Proposition that the spectral measure of $(X_t, X_{t+h})$ is discrete and concentrated on at most four points of the unit circle: $(\pm 1, 0)$ and $\pm (\rho^h, 1)/\sqrt{1 + |\rho|^{2h}}$. It collapses on exactly two points when $\rho > 0$ and $\beta = 1$ (resp. $\beta = -1$), that is, when the marginal distribution of $X_t$ is totally skewed to the right (resp. to the left). In particular, for any fixed $h \geq 1$, $\Gamma_h$ is always charging zero mass to sufficiently small neighbourhoods around the points $(0, \pm 1)$, which leads to the following result and the existence of conditional moments.

**Lemma 2.1** Let $(X_t)$ be the anticipative AR(1) solution of (1.1) with $0 < \alpha < 2$, $\beta \in [-1, 1]$ and $0 < |\rho| < 1$. Then, for any $h \geq 1$, the spectral measure of $(X_t, X_{t+h})$ is such that
\[
\int_{S_2} |s_1|^{-\nu} \Gamma_h(ds) < +\infty, \text{ for any } \nu \geq 0. \tag{2.4}
\]

**Proof.**

Let $\nu \geq 0$ and $h \geq 1$. Decompose the integral of (2.4) into two parts
\[
\int_{S_2} |s_1|^{-\nu} \Gamma_h(ds) = \int_{S_2 \cap \{s \in S_2 : |s_1| \leq |\rho|^h / 2\sqrt{1 + |\rho|^{2h}}\}} |s_1|^{-\nu} \Gamma_h(ds) + \int_{S_2 \cap \{s \in S_2 : |s_1| > |\rho|^h / 2\sqrt{1 + |\rho|^{2h}}\}} |s_1|^{-\nu} \Gamma_h(ds).
\]
In view of (2.3), the second term on the right-hand side is finite while the first one is zero.

**Corollary 2.1** Let $(X_t)$ be the anticipative AR(1) solution of (1.1) with $0 < \alpha < 2$, $\beta \in [-1, 1]$ and $0 < |\rho| < 1$. Then, for any $h \geq 1$,
\[
\mathbb{E} \left[|X_{t+h}|^\gamma |X_t, X_{t-1}, \ldots\right] < +\infty, \text{ a.s. for any } 0 < \gamma < 2\alpha + 1.
\]

When $\rho > 0$ and $\beta = 1$ (resp. $\beta = -1$), we have from Gouriéroux and Zakoïan (2017) that the marginal distribution of $X_t$ is univariate $\alpha$-stable with asymmetry parameter $\beta_1 = \beta \frac{1 - |\rho|^\alpha}{1 - \rho^{<\alpha>}} = 1$ (resp. $\beta_1 = -1$). Zolotarev (1986) call such distributions **totally skewed to the right** (resp. to the left).
Proof. As the anticipative AR(1) is a Markov process (see Proposition 2 in [20]), we have
\[ \mathbb{E}[X_{t+h}^\gamma | X_t, X_{t-1}, \ldots] = \mathbb{E}[X_{t+h}^\gamma | X_t] \] for any \( h \) and \( \gamma \). The existence of conditional moments up to order \( 2\alpha + 1 \) is now a direct consequence of Lemma 2.1 and Proposition 2.1. \( \square \)

Analytical formulae were so far available only for the first and second conditional moments of the anticipative stable AR(1) processes, moreover only in the symmetric (\( \beta = 0 \)) and Cauchy (\( \alpha = 1 \) and \( \beta = 0 \)) cases [20]. Thus we extend the formulae to any admissible parameterisations \( (\alpha, \beta) \in (0, 2) \times [-1, 1] \) and also provide the forms of the third and fourth conditional moments in the next Theorem. For expository purposes, the more intricate case \( \alpha = 1 \) has been singled out in Appendix B. Recall that the anticipative AR(1) is a Markov process and that integer conditional moments may exist only up to order four under the most favourable dispositions of Corollary 2.1.

\textbf{Theorem 2.1} Let \( (X_t) \) be the anticipative \( \alpha \)-stable AR(1) solution of (1.1) with \( \beta \in [-1, 1] \) and \( 0 < |\rho| < 1 \). Let \( h > 0 \).

For \( \alpha \in (0, 2) \), \( \alpha \neq 1 \),
\[ \mathbb{E}[X_{t+h} | X_t = x] = \kappa_1 x + \frac{a(\lambda_1 - \beta_1 \kappa_1)}{1 + a^2 \beta_1^2} \left[ a \beta_1 x + \frac{1 - x \mu(x)}{\pi f_X(x)} \right]. \] (2.5)

For \( \alpha \in (1/2, 2) \), \( \alpha \neq 1 \),
\[ \mathbb{E}[X_{t+h}^2 | X_t = x] = \kappa_2 x^2 + \frac{ax(\lambda_2 - \beta_1 \kappa_2)}{1 + (a \beta_1)^2} \left[ a \beta_1 x + \frac{1 - x \mu(x)}{\pi f_X(x)} \right] \] (2.6)
\[ - \frac{\alpha^2 \sigma_1^{2\alpha}}{2\pi f_X(x)} H(2, \theta_1; x). \]

For \( \alpha \in (1, 2) \),
\[ \mathbb{E}[X_{t+h}^3 | X_t = x] = \kappa_3 x^3 + \frac{ax^2(\lambda_3 - \beta_1 \kappa_3)}{1 + (a \beta_1)^2} \left[ a \beta_1 x + \frac{1 - x \mu(x)}{\pi f_X(x)} \right] \] (2.7)
\[ - \frac{\alpha^2 \sigma_1^{2\alpha}}{2\pi f_X(x)} \left[ x H(2, \theta_2; x) + \alpha \sigma_1^\alpha H(3, \theta_3; x) \right]. \]

For \( \alpha \in (3/2, 2) \),
\[ \mathbb{E}[X_{t+h}^4 | X_t = x] = \kappa_4 x^4 + \frac{ax^3(\lambda_4 - \beta_1 \kappa_4)}{1 + (a \beta_1)^2} \left[ a \beta_1 x + \frac{1 - x \mu(x)}{\pi f_X(x)} \right] \] (2.8)
\[ - \frac{\alpha^2 \sigma_1^{2\alpha}}{2\pi f_X(x)} \left[ x^2 \frac{H(2, \theta_4; x)}{2} + \frac{ax \sigma_1^\alpha}{3} H(3, \theta_5; x) + \frac{\alpha^2 \sigma_1^{2\alpha}}{3} H(4, \theta_6; x) \right]. \]

\(^2\)Higher conditional moments may however exist in some boundary cases, such as when \( X_t \) is totally skewed either to the right or left.
Here, \( a = \tan(\pi\alpha/2) \), and

\[
\sigma_1^2 = \frac{\sigma^2}{1 - |\rho|^2}, \quad \beta_1 = \beta \frac{1 - |\rho|^2}{1 - \rho^{<\alpha>}}, \quad \kappa_p = |\rho|^p \rho^{-hp}, \quad \lambda_p = \beta_1 (\rho^{<\alpha>})^h \rho^{-hp},
\]

for \( p \in \{1, 2, 3, 4\} \). Furthermore, for any \( n \in \mathbb{N} \), \( \theta_i = (\theta_{i1}, \theta_{i2}) \in \mathbb{R}^2 \), \( x \in \mathbb{R} \), \( \mathcal{H} \) is defined by

\[
\mathcal{H}(n, \theta_i; x) = \int_0^{+\infty} e^{-\sigma_1^2 u^n n^{-1}} \left( \theta_{i1} \cos(ux - a\beta_1 \sigma_1^2 u^n) + \theta_{i2} \sin(ux - a\beta_1 \sigma_1^2 u^n) \right) du,
\]

and we denote \( \mathcal{H}(\cdot) := \mathcal{H}(0, (0, 1); \cdot) \), and \( f_X := \frac{1}{\pi} \mathcal{H}(0, (1, 0); \cdot) \) \(^3\)

Finally, \( \theta_1 = (\theta_{11}, \theta_{12}) \) in (2.6) is given by

\[
\theta_{11} = \kappa_1^2 - a^2 \lambda_1^2 + a^2 \beta_1 \lambda_2 - \kappa_2, \quad \theta_{12} = a(\lambda_2 + \beta_1 \kappa_2) - 2a\lambda_1 \kappa_1,
\]

and the remaining \( \theta_i \)'s in (2.7)–(2.8), which depend only on \( \alpha, \beta_1 \), and the \( \kappa_p \)'s and \( \lambda_p \)'s above, are given in (C.18)–(C.27) in Appendix C. If \( \alpha < 1 \) and \( \beta_1 = 1 \) (resp. \( \beta_1 = -1 \)), Relations (2.5) and (2.6) are well defined only for \( x \geq 0 \) (resp. \( x \leq 0 \)).

**Remark 2.1** In the special cases considered in [20], the conditional expectation and variance are respectively linear and quadratic functions of the past. This does not appear to be the case in general. For instance, a necessary and sufficient condition for the linearity of the conditional expectation (A.3) requires that \( \lambda_1 = \beta_1 \kappa_1 \). In the case of the anticipative \( \text{AR}(1) \), this holds if and only if at least one of the following is true: \( i \) \( \beta = 0 \), \( ii \) \( \rho > 0 \), or \( iii \) \( h \) is even.

**Remark 2.2** From a computational perspective, note that the above moments can be inexpensively calculated for various horizons \( h \) and conditioning values \( x \). Notice indeed that the functions \( \mathcal{H}(n, \theta_i; x) \), \( n = 2, 3, 4 \), can be decomposed into \( a_n u_n(x) + b_n v_n(x) \), where \( a_n \) and \( b_n \) are constants depending only on \( h \) -and fixed parameters of the process-, and \( u_n(x) = \mathcal{H}(n, (0, 1); x) \) and \( v_n = \mathcal{H}(n, (1, 0); x) \) are simple integrals which depend only on \( x \). The constants \( a_n \) and \( b_n \) can be inexpensively computed for any horizons while the simple integrals \( u_n(x) \) and \( v_n(x) \) need only to be computed once for a given conditioning value.

Let us denote by \( \mu(x, h) \), \( \sigma^2(x, h) \), \( \gamma_1(x, h) \) and \( \gamma_2(x, h) \) the conditional expectation, variance, skewness and excess kurtosis respectively. When they are well defined, we denote for \( x \in \mathbb{R} \) and

\(^3\)Notice that \( f_X \) is the density of \( X_t \sim \mathcal{S}(\alpha, \beta_1, \sigma_1, 0) \) when \( \alpha \neq 1 \).
\( h > 0, \)

\[
\begin{align*}
\mu(x, h) & := \mathbb{E}[X_{t+h}|X_t = x], \\
\sigma^2(x, h) & := \mathbb{E}\left[ \left( X_{t+h} - \mu(x,h) \right)^2 | X_t = x \right], \\
\gamma_1(x, h) & := \mathbb{E}\left[ \left( \frac{X_{t+h} - \mu(x,h)}{\sigma(x,h)} \right)^3 | X_t = x \right], \\
\gamma_2(x, h) & := \mathbb{E}\left[ \left( \frac{X_{t+h} - \mu(x,h)}{\sigma(x,h)} \right)^4 | X_t = x \right] - 3.
\end{align*}
\] (2.11) (2.12) (2.13) (2.14)

To illustrate the results of Theorem 2.1, the conditional moments of the anticipative 1.7-stable AR(1) with \( \rho = 0.95, \beta = 0.8 \) and \( \sigma = 0.1 \) are depicted on Figure 2 as functions of the past observation \( X_t = x \) and the horizon \( h \). Notice in particular that the conditional volatility \( \sigma(\cdot, h) \) appears naturally smile-shaped, which reproduces a well-known stylised fact of implied volatilities and news impact curves on financial markets.

Although \( X_t \) is marginally stable-distributed, the conditional distribution of \( X_{t+h} \) given \( X_t = x \) is typically non-stable. For \( \rho > 0 \), a clear interpretation of the distribution \( X_{t+h}|X_t = x \) appears during explosive/bubble episodes, that is, as \( x \) becomes large relative to the central values of process \( (X_t) \).

**Corollary 2.2** Let \((X_t)\) be the anticipative strictly stationary solution of (1.1) with \( \rho > 0 \) and \( \beta \in [-1, 1] \). If \( |\beta_1| = 1 \), let \( \beta_1 x \to +\infty \), and if \( |\beta_1| \neq 1 \), let \( x \to \pm \infty \).\footnote{See Remark A.3 for details regarding the different behaviours when \( |\beta_1| \neq 1 \) and \( |\beta_1| = 1 \).} Also, let \( s = 1 \) if \( x \to +\infty \) and \( s = -1 \) if \( x \to -\infty \). Then, for any \( h \geq 1 \), as \( x \to \pm \infty \),

\[
\begin{align*}
\mu(x, h) & \sim (\rho^{-h}x)^{\rho^\alpha h}, & \text{if } \alpha \in (0, 2), \\
\sigma^2(x, h) & \sim (\rho^{-h}x)^2(1 - \rho^\alpha h), & \text{if } \alpha \in (1/2, 2), \\
\gamma_1(x, h) & \to s \frac{1 - 2\rho^\alpha h}{\sqrt{\rho^\alpha h(1 - \rho^\alpha h)}}, & \text{if } \alpha \in (1, 2), \\
\gamma_2(x, h) & \to \frac{1}{\rho^\alpha h} + \frac{1}{1 - \rho^\alpha h} - 6, & \text{if } \alpha \in (3/2, 2).
\end{align*}
\]

**Remark 2.3** The strikingly simplistic forms of the conditional moments during explosive/bubble episodes yielded by Corollary 2.2 are characteristic of a weighted Bernoulli distribution charging probability \( \rho^\alpha h \) to the value \( \rho^{-h}x \) and probability \( 1 - \rho^\alpha h \) to 0. It is thus natural to interpret \( \rho^\alpha h \) as the probability that the bubble survives at least \( h \) more time steps, conditionally on reaching
Figure 2: Conditional moments $\mu(\cdot)$, $\sigma(\cdot)$, $\gamma_1(\cdot)$, $\gamma_2(\cdot)$ given by (2.11)-(2.14) of the stable anticipative AR(1) solution of (1.1) with $\varepsilon_t \overset{i.i.d.}{\sim} S(1.7, 0.8, 0.1, 0)$ and $\rho = 0.95$, for horizons $h = 1, \ldots, 30$ and conditioning values $X_t = x \in (-10, 10)$. Lower is darker, higher is whiter.

The interpretation surprisingly implies that the survival probability does not depend on the past longevity of the bubble neither on its current height. The bubbles generated by the stable anticipative AR(1) appear to display a memory-less property.

**Remark 2.4** Corollary 2.2 also echoes the bubble model that was initially proposed in [3] and further studied recently in [30]. The approach therein consists in modelling $X_t$ as

$$X_t = s_t \rho^h X_{t-1} + \eta_t, \quad \text{for} \ t \geq 1, \quad (2.15)$$

The interpretation of $\rho^h$ as a survival probability of bubbles can also be reached using point processes under the more general assumption that the errors of (1.1) belong to the domain of attraction of an $\alpha$-stable distribution (see the Supplementary file).
given initial values $X_0 = \eta_0$, $s_0 = 0$, and where $\rho^* > 1$, $(\eta_t)$ is a finite variance i.i.d. sequence and $s_t$ is a $0-1$ Bernoulli taking value 1 with probability $p \in (0, 1)$. The stable anticipative AR(1) \cite{[11]} is reminiscent of \cite{[2.15]} in two aspects. On the one hand, it is the unique solution of the linear recursive equation $X_t = \rho^* X_{t-1} + \varepsilon_t^*$ with explosive AR coefficient $\rho^* = 1/\rho > 1$. On the other hand, Corollary \cite{[2.2]} shows that the anticipative AR(1) also behaves as a two-point conditional distribution during bubble episodes.

2.3 Continuous time: the anticipative $\alpha$-stable Ornstein-Uhlenbeck

Financial applications are often inclined towards continuous-time representations and efforts are deployed to advance discrete- and continuous-time techniques side-by-side, including when it comes to bubble modelling \cite{[10]}. A continuous time analogue of the AR(1) is the well-known Ornstein-Uhlenbeck process. When it is driven by a Brownian motion ($\alpha$-stable Lévy process with $\alpha = 2$), it is the only continuous in probability stationary Markov Gaussian process. However, when driven by an $\alpha$-stable Lévy process with $0 < \alpha < 2$, at least two distinct processes arise that are continuous in probability, stationary and Markov: the direct time OU and its reverse time counterpart (see Chapter 3 Section 6 in \cite{[40]}).

Let us first introduce the objects upon which continuous time $\alpha$-stable moving averages are defined. We borrow from the very concise introduction in \cite{[28]}. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and $L_0(\Omega)$ be the set of all real random variables defined on it. Let also $(E, \mathcal{E}, m)$ be an arbitrary measurable space, $\beta : E \to [-1, 1]$ be a measurable function and define the set $\mathcal{E}_0 = \{ A \in \mathcal{E} : m(A) < +\infty \}$.

**Definition 2.2** An independently scattered $\sigma$-additive set function $M : \mathcal{E}_0 \to L_0(\Omega)$ such that for each $A \in \mathcal{E}_0$

$$M(A) \sim \mathcal{S}\left(\alpha, \frac{\int A \beta(x)m(dx)}{m(A)}, (m(A))^{1/\alpha}, 0\right)$$

is called $\alpha$-stable random measure on $(E, \mathcal{E})$ with control measure $m$ and skewness intensity $\beta$.

Independent scatteredness means that for any disjoints sets $A_1, A_2, \ldots, A_n \in \mathcal{E}_0$, $n \in \mathbb{N}$, the random variables $M(A_1), \ldots, M(A_n)$ are independent. One can consider random processes of the form

$$X_t = \int_E f(x-t)M(dx), \quad t \in \mathbb{R}, \quad (2.16)$$

\footnote{Two-sided OU processes are also mentioned in \cite{[10]} \cite{[22]}, where it is noticed they admit higher conditional than marginal moments. Anticipative stable OU are also alluded to in \cite{[27]}.}
where \( f : E \rightarrow \mathbb{R} \) is a measurable function such that \( \int_E |f(x)|^\alpha m(dx) < +\infty \) and in the case \( \alpha = 1 \), additionally, \( \int_E |f(x)\beta(x)| \ln |f(x)| m(dx) < +\infty \). As underlined in [28], the integral in (2.16) is constructed in the natural way by approximating the function \( f \) by simple functions in Chapter 3 Section 4 in [40].

We will focus on random processes for which \( E = \mathbb{R} \) and \( m \) is the Lebesgue measure.

Definition 2.3 Let \( \lambda > 0 \) and \( M \) be an \( \alpha \)-stable random measure with Lebesgue control measure and constant skewness intensity \( \beta \in [-1, 1] \). The non-anticipative and anticipative \( \alpha \)-stable Ornstein-Uhlenbeck, denoted \( X_{na} \) and \( X_a \) respectively, are defined as

\[
X_{na}(t) = \int_{-\infty}^t e^{-\lambda(t-x)} M(dx), \quad t \in \mathbb{R}, \tag{2.17}
\]

\[
X_a(t) = \int_t^{+\infty} e^{-\lambda(x-t)} M(dx), \quad t \in \mathbb{R}. \tag{2.18}
\]

Remark 2.5 The non-anticipative and anticipative \( \alpha \)-stable OU are Markov processes. Indeed, for \( s < t \), \( X_{na}(t) - e^{-\lambda(t-s)} X_{na}(s) = \int_s^t e^{-\lambda(t-x)} M(dx) \) and \( X_a(s) - e^{-\lambda(t-s)} X_a(t) = \int_s^t e^{-\lambda(x-s)} M(dx) \).

By Theorem 3.5.3 in [40], we have the independence between \( X_{na}(t) - e^{-\lambda(t-s)} X_{na}(s) \) and the \( \sigma \)-algebra generated by \( \{ X_{na}(u), u \leq s \} \) on the one hand, and between \( X_a(s) - e^{-\lambda(t-s)} X_a(t) \) and the \( \sigma \)-algebra generated by \( \{ X_a(u), u \leq s \} \) on the other hand.

Close to this framework, generalised OU processes driven by Lévy processes (not necessarily stable) are also defined in integral forms and studied in [29]. In [36], these Lévy-driven OU are pointed out to be solutions to stochastic differential equations (SDE) of the form \( dV_t = V_t - dU_t + dL_t \), where \((U, L)\) is a bivariate Lévy process. It was moreover shown in [2] that the latter SDE may admit anticipative solutions.

The two definitions of the OU processes in (2.17) and (2.18) are very practical in our context as they can be readily embedded in the bivariate \( \alpha \)-stable vector framework. Similarly to the discrete time case, we will consider for any \( t \in \mathbb{R} \) and \( h > 0 \) the vectors \((X_i(t), X_i(t+h))\), for \( i = a, na \).

Just as for the \( \alpha \)-stable non-anticipative AR(1), the non-anticipative OU does not feature more moments than the marginal distribution, namely \( \mathbb{E}[|X_{na}(t+h)|^p|X_{na}(t)] = +\infty \) whenever \( p \geq \alpha \). It displays infinite variance, and the expectation is also ill-defined when \( 0 < \alpha \leq 1 \). On the contrary, the anticipative OU features conditional moments up to \( 2\alpha + 1 \). From now on, we shall focus solely on the anticipative OU, hence we drop the subscript \( \ll a \) and simply denote the process satisfying Equation (2.18) as \( X_t \), for \( t \in \mathbb{R} \). The next Lemma shows that, just as for the discrete time counterpart of the anticipative OU, the spectral measure of \((X_t, X_{t+h})\) is concentrated on either two or four points of the unit circle.
Proposition 2.3 Let \( \{X_t, t \in \mathbb{R}\} \) be the anticipative \( \alpha \)-stable OU process defined by (2.18) with \( \lambda > 0 \) and \( M \) an \( \alpha \)-stable random measure with Lebesgue control measure and constant skewness intensity \( \beta \in [-1,1] \). Then, for any \( h \in \mathbb{R}_+^\star \), \( (X_t, X_{t+h}) \) is \( \alpha \)-stable and its spectral representation, denoted \( (\Gamma_h, \mu_0) \), with \( \mu_0 = (\mu_0^1, \mu_0^2) \), is such that

\[
\Gamma_h = \frac{1}{\alpha \lambda} \sum_{\vartheta \in \mathbb{S}_0} \frac{1 + \vartheta \beta}{2} \left[ \left( 1 - e^{-\alpha \lambda h} \right) \delta_{\{\vartheta,0\}} + \left( 1 + e^{-2\lambda h} \right)^{\alpha/2} \delta_{\{\vartheta,s_h\}} \right],
\]

with \( s_h = \sqrt{\frac{(1 - e^{-\lambda h}, 1)}{1 + e^{-2\lambda h}}} \). Moreover, if \( \alpha \neq 1 \), then \( \mu_0 = (0,0) \), and if \( \alpha = 1 \) then,

\[
\mu_0^1 = \mu + \frac{2}{\lambda \pi} \beta, \quad \mu_0^2 = e^{\lambda h} \mu + \frac{2}{\lambda \pi} \beta(1 + \lambda h),
\]

where \( \mu = -\frac{\beta}{\lambda \pi} e^{-\lambda h} \ln(1 + e^{2\lambda h}) \).

The following Theorem summarises the previous considerations and gives the expressions of the conditional moments in the case \( \alpha \neq 1 \). The case \( \alpha = 1 \) has been singled out in Appendix, Proposition B.2 for expository purposes.

Theorem 2.2 Let \( \{X_t, t \in \mathbb{R}\} \) be the anticipative \( \alpha \)-stable OU process, \( \alpha \neq 1 \), defined by (2.18) with \( \lambda > 0 \) and \( M \) an \( \alpha \)-stable random measure with Lebesgue control measure and constant skewness intensity \( \beta \in [-1,1] \). Then, for any \( h \in \mathbb{R}_+^\star \), the following hold

i) The anticipative \( \alpha \)-stable Ornstein-Uhlenbeck is a Markov process.

ii) If \( 0 \leq \gamma < 2\alpha + 1 \), then, \( \mathbb{E}[|X_{t+h}|^\gamma |X_t] < +\infty \).

iii) The first four moments of \( X_{t+h} |X_t \), when they exist, are given by Theorem 2.1 with

\[
\sigma_1^2 = \frac{1}{\alpha \lambda}, \quad \beta_1 = \beta, \quad \kappa_p = e^{\lambda h(\alpha - p)}, \quad \lambda_p = \beta \kappa_p, \quad \text{for } p \in \{1,2,3,4\}.
\]

The expressions of the conditional moments simplify during explosive/bubble events.

Corollary 2.3 Let \( \{X_t, t \in \mathbb{R}\} \) be the anticipative \( \alpha \)-stable OU as defined in Theorem 2.2. If \( |\beta_1| = 1 \), let \( \beta_1 x \to +\infty \), and if \( |\beta_1| \neq 1 \), let \( x \to \pm\infty \).\(^7\) Also, let \( s = 1 \) if \( x \to +\infty \) and \( s = -1 \) if \( x \to -\infty \). Then, for any \( h \in \mathbb{R}_+^\star \),

\[
\begin{align*}
\mu(x,h) &\sim (e^{\lambda h}x)e^{-\alpha \lambda h}, \quad \text{if } \alpha \in (0,2), \\
\sigma^2(x,h) &\sim (e^{\lambda h}x)^2 e^{-\alpha \lambda h}(1 - e^{-\alpha \lambda h}), \quad \text{if } \alpha \in (1/2,2), \\
\gamma_1(x,h) &\to s \frac{1 - 2e^{-\alpha \lambda h}}{\sqrt{e^{-\alpha \lambda h}(1 - e^{-\alpha \lambda h})}}, \quad \text{if } \alpha \in (1,2), \\
\gamma_2(x,h) &\to \frac{1}{\alpha e^{-\alpha \lambda h}} + \frac{1}{1 - e^{-\alpha \lambda h}} - 6, \quad \text{if } \alpha \in (3/2,2),
\end{align*}
\]

\(^7\)See Remark A.3 for details regarding the different behaviours when \( |\beta_1| \neq 1 \) and \( |\beta_1| = 1 \).
Remark 2.6 Echoing Remark 2.3, the anticipative OU behaves as its discrete time counterpart in that $X_{t+h}|X_t = x$, as $x$ becomes large, can be interpreted as a distribution charging probability $e^{-\alpha \lambda h}$ to the value $e^{\lambda h} x$ and probability $1 - e^{-\alpha \lambda h}$ to 0. Focusing on the limiting behaviour of the conditional kurtosis, it can be easily seen that the function $h \mapsto \frac{1}{e^{-\alpha \lambda h}} + \frac{1}{1 - e^{-\alpha \lambda h}} - 6$ is strictly convex and diverges to infinity as $h \to +\infty$, but also as $h \to 0$, illustrating that the paths of the anticipative OU are continuous only in probability. It reaches its global minimum at $h_0$ such that $e^{-\alpha \lambda h_0} = \frac{1}{2}$, yielding $h_0 = \frac{\ln 2}{\alpha \lambda}$, and takes value $-2$ corresponding to the lowest achievable excess kurtosis amongst all probability distributions. Last, the horizon $h_0$ achieving the minimum is further away in the future for heavier-tailed and more persistent processes.

3 Aggregated anticipative AR(1)

Heavy-tailed anticipative AR processes generate trajectories that feature locally explosive phenomena such as financial bubbles. The higher the order of the AR process, the more complex patterns it is able to mimic (see [17] for some examples). However, a given AR($p$) process is constrained by the fact that it is specific to one particular explosive pattern which occurs recurrently through time. It is proposed in [20] to consider processes resulting from the aggregation of multiple AR(1) with different autoregressive coefficients. More formally, a process from this family can be defined by

$$X_t = c \sum_{j=1}^{J} \pi_j X_{j,t}, \quad X_{j,t} = \rho_j X_{j,t+1} + \varepsilon_{j,t}, \quad 0 < |\rho_j| < 1, \quad j = 1, \ldots, m \quad (3.1)$$

where $c > 0$, $\pi_j \in (0,1)$ for any $j$, $\sum_{j=1}^{J} \pi_j = 1$ and $(\varepsilon_{j,t})_{t \in \mathbb{Z}} \overset{i.i.d.}{\sim} S(\alpha, \beta_j, \sigma_j, 0)$ are mutually independent sequences of i.i.d. noise. Process $(X_t)$ will generate explosive/bubble episodes with rates of increase $1/\rho_j$. Unlike the latent $X_{j,t}$’s however, it is not a Markov process, and nothing is known about the predictive distribution of $X_{t+h}$ given its past. We now give results regarding the conditional distribution of $X_{t+h}$ given $X_t$, by first noticing that $(X_t, X_{t+h})$ can also be embedded in the multivariate $\alpha$-stable framework. For $j = 1, \ldots, J$, denote $(\Gamma_{j,h}, \mu^0_j, \mu^1_j, \mu^2_j)$ the spectral representation of $(X_{j,t}, X_{j,t+h})$ given by Proposition 2.2. For each $j = 1, \ldots, J$, denote also $\sigma_{1,j}$, $\beta_{1,j}$, $\kappa_{p,j}$ and $\lambda_{p,j}$ the quantities defined at Theorem 2.1 where $\rho$, $\sigma$ and $\beta$ are replaced by $\rho_j$, $\sigma_j$ and $\beta_j$. 

where the left-hand side quantities are defined in (2.11)-(2.14).
Lemma 3.1 Let \((X_t)\) be defined according to (3.1) with \(0 < \alpha < 2\). Then, for any \(h \geq 1\), \((X_t, X_{t+h})\) is a bivariate \(\alpha\)-stable vector and its spectral representation, denoted \((\Gamma_h, \mu^0)\) with \(\mu^0 = (\mu^0_1, \mu^0_2)\), is such that

\[
\Gamma_h = c^\alpha \sum_{j=1}^J \pi_j^\alpha \Gamma_j,h,
\]

and,

\[
\mu^0_1 = c \sum_{j=1}^J \pi_j \left( \mu^0_{1,j} - 1 \right) \frac{2}{\pi} \sigma_1 j \beta_1,j \ln |c\pi_j|, \quad \mu^0_2 = c \sum_{j=1}^J \pi_j \left( \mu^0_{2,j} - 1 \right) \frac{2}{\pi} \sigma_1 j \lambda_1,j \ln |c\pi_j|.
\]

The techniques used in the previous sections are therefore available here as well and we are able to characterise the moments of \(X_{t+h}\) given \(X_t\). As previously, we provide here the moments for \(\alpha \neq 1\), the remaining case being given in Proposition \(B.3\) in Appendix.

Proposition 3.1 Let \((X_t)\) be defined according to (3.1) with \(0 < \alpha < 2\). Let \(h \geq 1\).

i) If \(\gamma < 2\alpha + 1\), then \(E\left[|X_{t+h}|^\gamma \big| X_t = x\right] < +\infty\).

\(\nu\) The first four moments of \(X_{t+h}|X_t\), when they exist, are given by Theorem 2.1 with

\[
\sigma_1^\alpha = c^\alpha \sum_{j=1}^J \pi_j^\alpha \sigma_1^\alpha, \quad \beta_1 = \mathbb{E}(B), \quad \kappa_p = \mathbb{E}(K_p), \quad \lambda_p = \mathbb{E}(L_p), \quad \text{for } p \in \{1, 2, 3, 4\}
\]

where \(B, K_p\) and \(L_p\) are discrete random variables such that \(\mathbb{P}\left((B, K_p, L_p) = (\beta_{1,j}, \kappa_{p,j}, \lambda_{p,j})\right) = w_j\) and \(w_j = \frac{\pi_j^\alpha \sigma_1^\alpha}{\sum_{i=1}^J \pi_i^\alpha \sigma_1^\alpha}\) for \(j = 1, \ldots, J\).

Proof. i) From Lemma 3.1 we know that the spectral measure of \((X_t, X_{t+h})\) writes \(\Gamma_h = c^\alpha \sum_{j=1}^J \pi_j^\alpha \Gamma_j,h\), for \(0 < \alpha < 2\), where the \(\Gamma_j,h\)'s are the spectral measures of \((X_{j,t}, X_{j,t+h})\), with the \(X_{j,t}\)'s being simple AR(1) processes. We know by Lemma 2.1 that for any \(j\), any \(h\) and any \(\nu \geq 0\),

\[
\int_{S_2} |s_1|^{-\nu} \Gamma_j,h(ds) < +\infty.
\]

Hence, for any \(\nu \geq 0\),

\[
\int_{S_2} |s_1|^{-\nu} \Gamma_h(ds) = c^\alpha \sum_{j=1}^J \pi_j^\alpha \int_{S_2} |s_1|^{-\nu} \Gamma_j,h(ds) < +\infty.
\]

The existence of conditional moments follows from Proposition 2.1.

\(\nu\) The form of the conditional moments follow from Theorems A.1, A.3, A.5 and A.6. The parameters of the \(X_j\)'s are obtained by first noticing that,

\[
\sigma_1^\alpha = \int_{S_2} (s_2/s_1)^\nu |s_1|^{-\nu} \Gamma_h(ds) = c^\alpha \sum_{j=1}^J \pi_j^\alpha \int_{S_2} (s_2/s_1)^\nu |s_1|^{-\nu} \Gamma_j,h(ds) = c^\alpha \sum_{j=1}^J \pi_j^\alpha \sigma_1^\alpha.
\]
And thus, for instance,

$$\kappa_p = \frac{1}{\sigma_1^\alpha} \int_{S_2} (s_2/s_1)^p |s_1|^\alpha \Gamma_h(d\mathbf{s}) = \frac{c_\alpha}{\sigma_1^\alpha} \sum_{j=1}^J \pi_j^\alpha \int_{S_2} (s_2/s_1)^p |s_1|^\alpha \Gamma_{j,h}(d\mathbf{s}) = \sum_{j=1}^J \frac{\pi_j^\alpha \sigma_j^\alpha}{\sum_{i=1}^J \pi_i^\alpha \sigma_i^\alpha} \kappa_{p,j}.$$

**Remark 3.1** For the non-aggregated anticipative AR(1) considered at Section 2.2, linearity of the conditional expectation occurs when $\rho > 0$. However, assuming $\rho_j > 0$ for $j = 1, \ldots, J$ for the aggregated process $X_t$ does not guarantee linearity in general. Indeed in Proposition A.1, linearity is achieved if and only if $\lambda_1 - \beta_1 \kappa_1 = 0$, which is equivalent to

$$\text{Cov} (B, K_1) + \mathbb{E} \left[ B(|K_1| - K_1) \right] = 0,$$

since $L_1 = B|K_1|$. Hence, if $\rho_j > 0$ for $j = 1, \ldots, J$, then $K_1 > 0$ a.s. and this condition becomes

$$\text{Cov} (B, K_1) = 0.$$

**Remark 3.2** It is easy to construct examples for which $\mathbb{E} \left[ X_{j,t+h} | X_{j,t} = x \right]$ are all linear in $x$ for any $j$ and $h$, and yet such that $y \mapsto \mathbb{E} \left[ X_{t+h} | X_t = y \right]$ is a non-linear function of $y$. In view of the previous Remark, this can be achieved by taking for instance $J = 2$, $\rho_1 = \beta_1 = 0.1$ and $\rho_2 = \beta_2 = 0.9$ in (3.1).

### 4 A higher bound for the moments of $X_3 | X_2, X_1$

To the best of our knowledge, Proposition 2.1 is the only result quantifying up to which order the conditional moments of a stable random vector may exist. It is however restricted to the bivariate framework and whether this bound holds for higher dimension of the conditioning space is unknown. In this section, we take advantage both of the Markov property of anticipative AR(2) processes as shown in [17] and the result of Proposition 2.1 to show that a higher sufficient bound may hold when the dimension of conditioning is at least 2. Let $(X_t)$ be the strictly stationary solution of

$$X_t = \psi_1 X_{t+1} + \psi_2 X_{t+2} + \varepsilon_t, \quad \varepsilon_t \overset{i.i.d.}{\sim} \mathcal{S}(\alpha, \beta, \sigma, 0),$$

(4.1)

where $\psi(z) := 1 - \psi_1 z - \psi_2 z^2 = (1 - a_1 z)(1 - a_2 z)$ for some real numbers $a_1, a_2$ such that $0 < |a_i| < 1$ for $i = 1, 2$. We exclude the uninteresting case $\psi_1 = 0$ since it implies that $\{X_{2t}, t \in \mathbb{Z}\}$

---

8 Sufficient conditions for the finiteness of the conditional variance are also known in higher dimensions (see [16] for instance) but do not tell anything about higher, possibly fractional, orders.
Proposition 4.1 Let $X_t$ be the anticipative strictly stationary solution of (4.1) with $0 < \alpha < 2$. Then,

$$
E\left[|X_t|^\gamma |X_{t-1}, X_{t-2}\right] < +\infty, \text{ a.s. for any } 0 \leq \gamma < 3\alpha + 2. \quad (4.2)
$$

Remark 4.1 Proposition 4.1 in particular demonstrates that for some $\alpha$-stable random vectors $(X_1, X_2, X_3)$, the moments of $X_3 | X_2, X_1$ may exist up to order $3\alpha + 2 \in (2, 8)$. Obtaining bounds such as the latter and the one of Proposition 2.1 for general $\alpha$-stable random vectors $(X_1, X_2, X_3)$ is particularly delicate. Attempting a proof as in [11, 14] would require the sixth derivative of the characteristic function of $X_3 | X_2, X_1$, knowing that in the bivariate case, the fourth derivative is already a sum of more than 20 terms requiring a two-page classification.

Proof. For any $(x_0, x_1, x_2) \in \mathbb{R}^3$, 

$$
\hat{f}_{X_t}(x_{t+1}, x_{t+2}) = (x_1, x_2)(x_0) = \hat{f}_t(x_0 - \psi_1 x_1 - \psi_2 x_2),
$$

because $\varepsilon_t$ is independent from $X_{t+1}, X_{t+2}$. By the Bayes formula,

$$
\hat{f}_{X_t}(x_{t+1}, x_{t+2}) = (x_1, x_2)(x_0) = \frac{\hat{f}_{X_{t+2}}(x_0, x_1)(x_2)}{\hat{f}_{X_{t+1}, X_{t+2}}(x_1, x_2)} \hat{f}_{X_{t+1}}(x_0, x_1).
$$

Thus,

$$
\hat{f}_{X_{t+2}}(x_0, x_1)(x_2) = \frac{\hat{f}_t(x_0 - \psi_1 x_1 - \psi_2 x_2)\hat{f}_{X_{t+2}|X_{t+1}=x_1}(x_2)\hat{f}_{X_{t+1}}(x_1)}{\hat{f}_{X_{t+1}, X_{t+2}}(x_0, x_1)}.
$$

On the one hand, when $|x_2| \to +\infty$,

$$
\hat{f}_t(x_0 - \psi_1 x_1 - \psi_2 x_2) = O(|x_2|^{-\alpha - 1}),
$$

thus, for any $\gamma > 0$,

$$
|x_2|^\gamma \hat{f}_{X_{t+2}|X_{t+1}=x_1}(x_2) \mid_{x_2 \to +\infty} = O\left(|x_2|^\gamma - \alpha - 1 f_{X_{t+2}|X_{t+1}=x_1}(x_2)\right). \quad (4.3)
$$

On the other hand, we will show that $x \longmapsto |x|^r f_{X_t | X_{t+1}=x_1}(x)$ is integrable on $\mathbb{R}$ for any $r < 2\alpha + 1$, from which the conclusion will follow.
The integrability of the later function is equivalent to the finiteness of $E \left[ |X_t|^r \right]_{X_{t-1} = x_1}$ which we will show using Proposition 2.1. From Lemma C.1 $(X_t, X_{t+1})$ is $\alpha$-stable and

$$\int_{S_2} |s_1|^{-\nu} \Gamma(ds) = \sigma^\alpha \left( 1 + \sum_{k=1}^{+\infty} |d_k|^{-\nu} (d_k^2 + d_{k-1}^2)^{\alpha+\nu/2} \right).$$

Given the form of the coefficients $d_k$’s for $X_t$ satisfying (4.1), we have for $k$ large enough say $|d_k| \sim C(k) |a|^k$ where $|a| \in (0, 1)$ and $C$ is a polynomial with degree 0 or 1. It is easy to see that $|d_{k-1}/d_k| \to \ell$, for some $\ell \geq 0$. Hence,

$$|d_k|^{-\nu} (d_k^2 + d_{k-1}^2)^{\alpha+\nu/2} = |d_k|^{\alpha} (1 + (d_{k-1}/d_k)^2)^{\alpha+\nu/2} \sim C(k)^\alpha |a|^\alpha k (1 + \ell^2)^{(\alpha+\nu)/2},$$

which is the term of an absolutely convergent series for any $\nu \geq 0$. Thus, $\int_{S_2} |s_1|^{-\nu} \Gamma(ds) < +\infty$ for all $\nu \geq 0$ and we conclude invoking Proposition 2.1.

5 Concluding remarks

Our results constitute a first step towards a quantification of the odds of crashes of bubbles, which could be valuable for risk/portfolio managers and regulators. Specifically in a portfolio allocation context, where managers would decide both the composition of their portfolios and when to pull out from speculative assets for instance, the functional forms per se of the higher order moments could be valuable [22, 27]. Our results also open the possibility for alternative point predictors for the stable anticipative AR(1) and OU processes that exploit higher order conditional moments, as opposed to other predictors that were proposed to circumvent the infinite variance of $\alpha$-stable processes, such as as minimum $L^\alpha$-dispersion or maximum covariation ([28] and the references therein).

APPENDIX

This Appendix is composed of three sections. The first provides the form of the conditional moments for arbitrary bivariate stable vectors $(X_1, X_2)$. The second section completes Theorem 2.1 and Propositions 2.3, 4.1 in the case $\alpha = 1$. The third gathers the main proofs. Complementary results and proofs are collected in a Supplementary file.

A Conditional moments of bivariate $\alpha$-stable random vectors

The conditional moments stated in Theorem 2.1 for the particular AR(1) case originate from the broader bivariate $\alpha$-stable framework that was much studied in a series of papers in the 90s.
Theorem A.1 (Samorodnitsky and Taqqu (Theorem 5.2.2, 1994)) Let \((X_1, X_2)\) be \(\alpha\)-stable, \(\alpha \in (0, 2) \setminus \{1\}\), with spectral representation \((\Gamma, 0)\). If \(0 < \alpha < 1\), let \(\Gamma\) satisfy \([2.2]\) for some \(\nu > 1 - \alpha\). Then, for almost every \(x\),

\[
E[X_2 | X_1 = x] = \kappa_1 x + \frac{a(\lambda_1 - \beta_1 \kappa_1)}{1 + a^2 \beta_1^2} \left[ a \beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)} \right], \tag{A.3}
\]

where \(a = \tan \left(\frac{\pi \alpha}{2}\right)\) and \(\sigma_1, \beta_1, \kappa_\nu's and the \(\lambda_\nu's as in \([A.1]\) and \([A.2]\).

If \(\alpha < 1\) and \(\beta_1 = 1\), Relation \([A.3]\) is well defined only for \(x \geq 0\), and if \(\alpha < 1\) and \(\beta_1 = -1\), it is well defined only for \(x \leq 0\).

The conditional expectation in the case \(\alpha = 1\) has also been considered in the literature and is more intricate.
Let the case form of the conditional variance for an arbitrary, skewed bivariate distribution of \( X \) be as in (2.2) with \( \nu > 0 \). Then, for almost every \( x \),

\[
\mathbb{E}[X_2|X_1 = x] = -\frac{2\sigma_1}{\pi}q_0 + \kappa_1(x - \mu_1) + \frac{\lambda_1 - \beta_1\kappa_1}{\beta_1}[(x - \mu_1) - \frac{\sigma_1}{\pi f_{X_1}(x)} U(x)],
\]

(A.4)

if \( \beta_1 \neq 0 \), and

\[
\mathbb{E}[X_2|X_1 = x] = -\frac{2\sigma_1}{\pi}q_0 + \kappa_1(x - \mu_1) - \frac{2\sigma_1}{\pi} \lambda_1 \frac{V(x)}{\pi f_{X_1}(x)},
\]

(A.5)

if \( \beta_1 = 0 \). Here \( a = 2/\pi, \sigma_1, \beta_1, \) the \( \kappa_p \)'s and the \( \lambda_p \)'s are as in (A.1) and (A.2), and

\[
U(x) = \int_0^{+\infty} e^{-\sigma_1 t} \sin \left( t(x - \mu_1) + a\sigma_1\beta_1 t \ln t \right) dt,
\]

\[
V(x) = \int_0^{+\infty} e^{-\sigma_1 t} (1 + \ln t) \cos \left( t(x - \mu_1) + a\sigma_1\beta_1 t \ln t \right) dt,
\]

\[
q_0 = \frac{1}{\sigma_1} \int_{S_2} s_2 \ln |s_1| \Gamma(ds), \quad \mu_1 = -a \int_{S_2} s_1 \ln |s_1| \Gamma(ds).
\]

If \( \alpha < 1 \) and \( \beta_1 = 1 \) (resp. \( \beta_1 = -1 \)), Relation (A.3) is well defined only for \( x \geq 0 \) (resp. \( x \leq 0 \)).

Regarding the conditional variance, studies have focused most exclusively on the \( S_\alpha S \) case (see \([5, 16, 42]\)). One notable exception is Theorem 3.1 in \([12]\) which states without proof the functional form of the conditional variance for an arbitrary, skewed bivariate \( \alpha \)-stable vector for \( \alpha \neq 1 \). We therefore provide a proof for the second moment as well and fill the gap for \( \alpha = 1 \). We start with the case \( \alpha \neq 1 \).

**Theorem A.3** Let \( (X_1, X_2) \) be \( \alpha \)-stable, \( \alpha \in (1/2, 2) \setminus \{1\} \), with spectral representation \((\Gamma, 0)\), where \( \Gamma \) satisfies (2.2) with \( \nu > 2 - \alpha \). Then, for almost every \( x \),

\[
\mathbb{E}[X_2^2|X_1 = x] = \kappa_2 x^2 + \frac{ax(\lambda_2 - \beta_1 \kappa_2)}{1 + (a\beta_1)^2} \left[ a\beta_1 x + \frac{1 - xH(x)}{\pi f_{X_1}(x)} \right] - \frac{\alpha^2 \sigma_1^2}{\pi f_{X_1}(x)} H(2, \theta_1; x),
\]

(A.6)

where \( a = \tan \left( \frac{\pi \alpha}{2} \right) \), \( \theta_1 = (\theta_{11}, \theta_{12}) \) with

\[
\theta_{11} = \kappa_1^2 - a^2 \lambda_1^2 + a^2 \beta_1 \lambda_2 - \kappa_2, \quad \theta_{12} = a(\lambda_2 + \beta_1 \kappa_2) - 2a\lambda_1 \kappa_1,
\]

and \( \sigma_1, \beta_1, \) the \( \kappa_p \)'s and the \( \lambda_p \)'s are as in (A.1) and (A.2).

If \( \alpha < 1 \) and \( \beta_1 = 1 \) (resp. \( \beta_1 = -1 \)), Relation (A.6) is well defined only for \( x \geq 0 \) (resp. \( x \leq 0 \)).

We now give the formulae for the second conditional moment when \( \alpha = 1 \). As for the conditional expectation when \( (X_1, X_2) \) is not \( S_1 S, \) two different results hold according to whether the marginal distribution of \( X_1 \) is skewed or symmetric.
Theorem A.4 Let \((X_1, X_2)\) be \(\alpha\)-stable, with \(\alpha = 1\) and spectral representation \((\Gamma, 0)\), where \(\Gamma\) satisfies (2.2) with \(\nu > 1\). Then, for almost every \(x\),
\[
\mathbb{E}
\left[
X_2^2 \mid X_1 = x
\right]
= \sigma_1^2 (a^2 q_0^2 - \kappa_1^2) + \frac{2\sigma_1 \lambda_1}{\beta_1} \left(\sigma_1 \kappa_1 - a q_0 (x - \mu_1)\right) + \frac{\lambda_1^2}{\beta_1} \left((x - \mu_1)^2 - \sigma_1^2\right)
+ \left(a \sigma_1 q_0 (\lambda_1 - \beta_1 \kappa_1) + (\kappa_1 \lambda_1 - \lambda_2) (x - \mu_1)\right) \frac{2 \sigma_1 U(x)}{\beta_1 f_{X_1}(x)}
+ \left(\lambda_2 + \beta_1 \kappa_2 - 2 \kappa_1 \lambda_1 + a^2 \sigma_1 \beta_1 (\lambda_1^2 - \beta_1 \lambda_2) W(x)\right) \frac{\sigma_1}{\beta_1 f_{X_1}(x)},
\]
if \(\beta_1 \neq 0\), and
\[
\mathbb{E}
\left[
X_2^2 \mid X_1 = x
\right]
= \sigma_1^2 (\kappa_2 + a^2 q_0^2 - \kappa_1^2) - 2 a \sigma_1 \kappa_1 q_0 (x - \mu_1) + \kappa_2 (x - \mu_1)^2
+ a \sigma_1 (\lambda_2 - 2 \lambda_1 \kappa_1) \frac{F_{X_1}(x) - 1/2}{f_{X_1}(x)} + \frac{a \sigma_1 \lambda_1}{\pi f_{X_1}(x)} \left[2 (a \sigma_1 q_0 - \kappa_1 (x - \mu_1)) V(x) + a \sigma_1 \lambda_1 W(x)\right],
\]
if \(\beta_1 = 0\). Here, \(a = 2/\pi\), \(\sigma_1\), \(\beta_1\), the \(\kappa_p\)'s and the \(\lambda_p\)'s are as in (A.1) and (A.2), \(U\), \(V\), \(q_0\) and \(\mu_1\) are as in Theorem A.2 and
\[
W(x) = \int_0^{+\infty} e^{-\sigma_1 t} (1 + t \ln t)^2 \cos \left(t (x - \mu_1) + a \sigma_1 \beta_1 t \ln t\right) dt.
\]

Remark A.1 Note that when \(\alpha = 1\), \(\pi f_{X_1}(x) = \int_0^{+\infty} e^{-\sigma_1 t} \cos \left(t (x - \mu_1) + a \sigma_1 \beta_1 t \ln t\right) dt\). If in addition \(\beta_1 = 0\), then \(X_1\) is marginally Cauchy distributed and its density and cumulative distribution function are known explicitly.

Remark A.2 The conditional variance when \((X_1, X_2)\) is S1S (derived in [12]) is encompassed by the second statement of the Theorem. Indeed, when \((X_1, X_2)\) is S1S, its spectral measure satisfies \(\Gamma(\mathcal{A}) = \Gamma(\mathcal{A})\) for any \(A \in \mathcal{S}_2\) and it can be shown that \(\beta_1 = \mu_1 = q_0 = \lambda_i = 0\), for \(i = 1, 2\). This then yields
\[
\mathbb{V}
\left[
X_2 \mid X_1 = x
\right]
= (\kappa_2 - \kappa_1^2) \left(x^2 + \sigma_1^2\right).
\]

We now provide the analytical form for the third conditional moment.

Theorem A.5 Let \((X_1, X_2)\) be \(\alpha\)-stable, \(\alpha \in (1, 2)\), with spectral representation \((\Gamma, 0)\), where \(\Gamma\) satisfies (2.2) with \(\nu > \alpha - 3\). Then, for almost every \(x\),
\[
\mathbb{E}
\left[
X_2^3 \mid X_1 = x
\right]
= \kappa_3 x^3 + \frac{a \sigma_1 \kappa_3}{1 + (a \beta_1)^2} \left[a \beta_1 x + \frac{1 - x H(x)}{\pi f_{X_1}(x)}\right]
- \frac{a^2 \sigma_1^2}{2 \pi f_{X_1}(x)} \left[x \mathcal{H}(2, \theta_2; x) + a \sigma_1^2 \mathcal{H}(3, \theta_3; x)\right],
\]
where the \(\theta_i\)'s are given in (C.24)-(C.27) in Appendix C with \(\sigma_1\), \(\beta_1\), the \(\kappa_p\)'s and the \(\lambda_p\)'s as in (A.1) and (A.2).
Finally, under the most favourable dispositions of Proposition 2.1, the fourth conditional moment exists and its analytical form is given in the following Proposition.

**Theorem A.6** Let \((X_1, X_2)\) be \(\alpha\)-stable, \(\alpha \in (3/2, 2)\), with spectral representation \((\Gamma, 0)\), where \(\Gamma\) satisfies (2.2) with \(\nu > \alpha - 4\). Then, for almost every \(x\),

\[
\mathbb{E}\left[ X_4^2 \mid X_1 = x \right] = \kappa_4 x^4 + \frac{\alpha x^3 (\lambda_4 - \beta_1 \kappa_4)}{1 + (a_{\beta_1})^2} \left[ a_{\beta_1} x + \frac{1 - x H(x)}{\pi f_{X_1}(x)} \right] \\
- \frac{\alpha^2 \sigma_1^2 \alpha}{\pi f_{X_1}(x)} \left[ \frac{x^2}{2} H(2, \theta_4; x) + \frac{\alpha \sigma_1^2 \alpha}{6} \mathcal{H}(3, \theta_5; x) + \frac{\alpha^2 \sigma_2^2 \alpha}{3} \mathcal{H}(4, \theta_6; x) \right],
\]

where the \(\theta_i\)’s are given in (C.18)-(C.23) in Appendix C with \(\sigma_1, \beta_1, \kappa_\rho\)’s and the \(\lambda_\rho\)’s as in (A.1) and (A.2).

The previous expressions of the conditional moments simplify when one considers the asymptotics with respect to the conditioning variable, as \(X_1 = x\) becomes large.

**Proposition A.1** Let \(p \in \{1, 2, 3, 4\}\) and let \((X_1, X_2)\) be \(\alpha\)-stable with \(\alpha \in (0, 2)\), and spectral representation \((\Gamma, 0)\) such that the conditional moment of order \(p\) exists. If \(|\beta_1| \neq 1\), then

\[
\begin{align*}
\frac{x^{-p} \mathbb{E}\left[ X_2^p \mid X_1 = x \right]}{x^{-p} \mathbb{E}\left[ X_2^p \mid X_1 = x \right]} & \xrightarrow{x \to +\infty} \frac{\kappa_\rho + \lambda_\rho}{1 + \beta_1}, \\
\frac{x^{-p} \mathbb{E}\left[ X_2^p \mid X_1 = x \right]}{x^{-p} \mathbb{E}\left[ X_2^p \mid X_1 = x \right]} & \xrightarrow{x \to -\infty} \frac{\kappa_\rho - \lambda_\rho}{1 - \beta_1},
\end{align*}
\]

and if \(|\beta_1| = 1\) and \(\beta_1 x \to +\infty\), then,

\[
\frac{x^{-p} \mathbb{E}\left[ X_2^p \mid X_1 = x \right]}{x^{-p} \mathbb{E}\left[ X_2^p \mid X_1 = x \right]} \xrightarrow{\nu_\rho} \kappa_\rho.
\]

**Remark A.3** The difference between the cases \(|\beta_1| = 1\) and \(|\beta_1| \neq 1\) can be seen as a consequence of the different tail behaviours that prevail. When \(|\beta_1| \neq 1\), both the left and right tail of the density of \(X_1\) display power law decay as \(O(|x|^{-\alpha - 1})\). However, when \(\beta_1 = -1\) for instance, the distribution of \(X_1\) is said to be totally skewed to the left. The left tail still decays as \(O(|x|^{-\alpha - 1})\), but the right tail decays much faster and another asymptotics holds.\(^9\)

\(^9\)If \(X_1 \sim S(\alpha, -1, 1, 0)\), and \(x \to +\infty\), then by Theorem 5.2.2 in [13],

\[
\begin{align*}
f_{X_1}(x) & \sim \frac{(x/\alpha)^{(\alpha-2)/2(\alpha-1)}}{\sqrt{2\pi\sigma(1 - \alpha)}} \exp\left\{ -\frac{1}{2\sigma} e^{(x/\alpha)\alpha/(\alpha-1)} \right\}, \quad \text{if } \alpha > 1, \\
f_{X_1}(x) & \sim \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x - 1}{2} - e^{x-1} \right\}, \quad \text{if } \alpha = 1.
\end{align*}
\]

If \(\alpha < 1\), the support of \(f_{X_1}\) is \(\mathbb{R}_-\) and conditioning by \(x > 0\) makes no sense. Note however that when \(x \to 0\), a formula similar to the case \(\alpha > 1, x \to +\infty\) holds.
B  Complementary results for $\alpha = 1$

B.1 Conditional moments of the AR(1), OU and aggregated AR(1) when $\alpha = 1$

Theorems [2.1, 2.2] and Proposition [3.1] give the conditional moments of the three considered processes in the case $\alpha \neq 1$. We provide here the remaining more intricate case $\alpha = 1$. As can be seen in Propositions [2.2, 2.3] and Lemma [3.1], the bivariate vectors $(X_t, X_{t+h})$ of each process are $\alpha$-stable and their spectral representations display a none-zero shift parameter $\mu^0$. For the sake of simplicity, we cancel this shift by considering the vector $(\hat{X}_t, \hat{X}_{t+h}) := (X_t, X_{t+h}) - \mu^0$. We start with the anticipative AR(1).

**Theorem B.1** Let $(X_t)$ be the anticipative $\alpha$-stable AR(1) solution of [(1.1)] with $\alpha = 1$, $\beta \in [-1,1]$ and $0 < |\rho| < 1$. Let $h \geq 1$. Then, $E[\hat{X}_{t+h}|\hat{X}_t = x]$ and $E[\hat{X}_{t+h}^2|\hat{X}_t = x]$ are given respectively by Theorem [A.2] and [A.4] with,

$$
\begin{align*}
\sigma_1 &= \frac{\sigma}{1 - |\rho|}, & \beta_1 &= \beta \frac{1 - |\rho|}{1 - \rho}, & \mu_1 &= \frac{1}{\pi} \sigma_1 \beta_1 \rho^h \ln(1 + \rho^{-2h}), & q_0 &= -\frac{1}{2} \beta_1 \ln(1 + \rho^{-2h}) \\
\kappa_p &= |\rho|^{h-p} \rho^{-hp}, & \lambda_p &= \beta_1 \rho^{h(1-p)}, & & & \text{for } p \in \{1,2\}.
\end{align*}
$$

**Theorem B.2** Let $\{X_t, t \in \mathbb{R}\}$ be the anticipative $\alpha$-stable OU process, with $\alpha = 1$, defined by [(2.18)] with $\lambda > 0$ and $M$ an $\alpha$-stable random measure with Lebesgue control measure and constant skewness intensity $\beta \in [-1,1]$. Let $h \in \mathbb{R}^*_+$. Then, $E[\hat{X}_{t+h}|\hat{X}_t = x]$ and $E[\hat{X}_{t+h}^2|\hat{X}_t = x]$ are given respectively by Theorem [A.2] and [A.4] with,

$$
\begin{align*}
\sigma_1 &= \frac{1}{\lambda}, & \beta_1 &= \beta, & \mu_1 &= \frac{\beta}{\lambda \pi} e^{-\lambda h} \ln(1 + e^{2\lambda h}), & q_0 &= -\frac{1}{2} \beta \ln(1 + e^{2\lambda h}) \\
\kappa_p &= e^{-\lambda h(1-p)}, & \lambda_p &= \beta \kappa_p, & & & \text{for } p \in \{1,2\}.
\end{align*}
$$

In addition to $\sigma_{1,j}$, $\beta_{1,j}$, $\kappa_{p,j}$ and $\lambda_{p,j}$, denote for each $j = 1, \ldots, J$, the quantities $q_{0,j}$ defined at Theorem B.1 where $\rho, \sigma$ and $\beta$ are replaced by $\rho_j$, $\sigma_j$ and $\beta_j$.

**Theorem B.3** Let $(X_t)$ be the aggregated anticipative AR(1) defined according to [(3.1)] with $\alpha = 1$. Let $h \geq 1$. Then, $E[\hat{X}_{t+h}|\hat{X}_t = x]$ and $E[\hat{X}_{t+h}^2|\hat{X}_t = x]$ are given respectively by Theorem [A.2] and [A.4] with,

$$
\begin{align*}
\sigma_1 &= c \sum_{j=1}^{J} \pi_j \sigma_{1,j}, & \beta_1 &= \mathbb{E}(B), & \mu_1 &= c \sum_{j=1}^{J} \pi_j \mu_{1,j}, & q_0 &= \mathbb{E}(Q_0) \\
\kappa_p &= \mathbb{E}(K_p), & \lambda_p &= \mathbb{E}(L_p), & & & \text{for } p \in \{1,2\}.
\end{align*}
$$
for $p \in \{1, 2\}$, where $B$, $K_p$, $L_p$ and $K_0$ are discrete random variables such that $\mathbb{P}((B, K_p, L_p, Q_0) = (\beta_{1,j}, \kappa_{p,j}, \lambda_{p,j}, q_{0,j})) = w_j$ and $w_j = \frac{\pi_j \sigma_{1,j}}{\sum_{i=1}^{J} \pi_i \sigma_{1,i}}$ for $j = 1, \ldots, J$. Stable random noise.

C Proofs

C.1 Proof of Proposition 2.2

To prove Proposition 2.2, we begin with a Lemma that gives the forms of the spectral measure and shift vector for more general, discrete time vectors of linear moving averages driven by $\alpha$-stable noise. Let $\varepsilon_t \overset{i.i.d.}{\sim} S(\alpha, \beta, \sigma, \mu)$, $m$ an integer such that $m \geq 2$ and let $\{d_{k,i}, k \in \mathbb{Z}, i = 1, \ldots, m\}$ be a real deterministic sequence verifying

\[
\text{for any } i = 1, \ldots, m, \quad \sum_{k \in \mathbb{Z}} |d_{k,i}|^s < +\infty, \quad \text{for some } s < \alpha, \quad s \leq 1. \quad (C.1)
\]

Consider the vector

\[
X_t = (X_{1,t}, \ldots, X_{m,t}), \quad \text{with} \quad X_{i,t} = \sum_{k \in \mathbb{Z}} d_{k,i} \varepsilon_{t+k}, \quad \text{for } i = 1, \ldots, m. \quad (C.2)
\]

It follows from Proposition 13.3.1 in [4] that the infinite series converge almost surely and $X_t$ is well defined. Denote $d_k = (d_{k,1}, \ldots, d_{k,m})$ for any $k \in \mathbb{Z}$ and $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$.

Lemma C.1 Let $0 < \alpha < 2$ and let $X_t$ satisfy (C.1) and (C.2). Then, $X_t$ is an $\alpha$-stable random vector in $\mathbb{R}^m$, with spectral measure $\Gamma$ on the unit sphere $S_m$ and location vector $\mu^0 \in \mathbb{R}^m$ such that

\[
\Gamma = \sigma^\alpha \frac{1 + \beta}{2} \sum_{k \in \mathbb{Z}} \|d_k\|^\alpha \delta_{\{d_k\over \|d_k\|}} + \sigma^\alpha \frac{1 - \beta}{2} \sum_{k \in \mathbb{Z}} \|d_k\|^\alpha \delta_{\{-d_k\over \|d_k\|}}, \quad (C.3)
\]

\[
\mu^0 = \sum_{k \in \mathbb{Z}} d_k \mu - \prod_{\{\alpha = 1\}} \frac{2}{\pi} \sigma^\beta \sum_{k \in \mathbb{Z}} d_k \ln \|d_k\|
\]

where $\delta_{\{x\}}$ is the dirac measure at point $x \in \mathbb{R}$ and by convention, if for some $k \in \mathbb{Z}$, $d_k = 0$, i.e. $\|d_k\| = 0$, then the kth term vanishes from the sums.

Proof. The characteristic function of $X_t$ reads, for any $u \in \mathbb{R}^m$:

\[
\varphi_{X_t}(u) = \mathbb{E} \left( \exp \left\{ i \sum_{j=1}^{m} u_j X_{j,t} \right\} \right) = \prod_{k \in \mathbb{Z}} \mathbb{E} \left[ i \left( \sum_{j=1}^{m} u_j d_{k,j} \right) \varepsilon_{t+k} \right]
\]

We obtain that for $\alpha \neq 1$,

\[
\varphi_{X_t}(u) = \exp \left\{ - \sum_{k \in \mathbb{Z}} \sigma^\alpha | \sum_{j=1}^{m} u_j d_{k,j}|^\alpha \left( 1 - i \beta \text{sign} \left( \sum_{j=1}^{m} u_j d_{k,j} \right) \tan \frac{\pi \alpha}{2} \right) + i \sum_{j=1}^{m} u_j \sum_{k \in \mathbb{Z}} d_{k,j} \mu \right\}. \quad (C.4)
\]

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If $\alpha = 1$,

$$
\varphi_{X_t}(u) = \exp \left\{ -\sum_{k \in \mathbb{Z}} \sigma \left| \sum_{j=1}^{m} u_j d_{k,j} \right| \left( 1 + \frac{2i\beta}{\pi} \text{sign} \left( \sum_{j=1}^{m} u_j d_{k,j} \right) \ln \left| \sum_{j=1}^{m} u_j d_{k,j} \right| \right) + i \sum_{j=1}^{m} u_j \sum_{k \in \mathbb{Z}} d_{k,j} \mu \right\}. 
$$

(C.5)

Replacing (C.3) in (2.1) allows to retrieve the two above formulae.

Let us now prove Proposition 2.2. From Lemma C.1, taking $X_t = (X_t, X_{t+h})$, we have for the anticipative AR(1), $d_k = (\rho^k \mathds{1}_{k \geq 0}, \rho^{k-h} \mathds{1}_{k \geq h})$ for any $h \in \mathbb{Z}$ and $h \geq 1$, and

$$
\Gamma_h = \sigma^2 \sum_{\varrho \in S_0} \frac{1 + \varrho^2}{2} \left[ \sum_{k=0}^{h-1} |\rho|^k \varrho \delta \left\{ \varrho \text{sign}(\rho)^k(1,0) \right\} + \sum_{k \geq h} |\rho|^{2k} + \rho^{2(k-h)} \right]^{\alpha/2} \left\{ \varrho \sqrt{2\rho^{2(k-h)}} \right\} 
$$

$$
= \sigma^2 \sum_{\varrho \in S_0} \frac{1 + \varrho^2}{2} \left[ \sum_{k=0}^{h-1} |\rho|^k \varrho \delta \left\{ \varrho \text{sign}(\rho)^k(1,0) \right\} + \left( 1 + |\rho|^{-2h} \right)^\alpha \sum_{k \geq h} |\rho|^{2k} + \rho^{2(k-h)} \right]^{\alpha/2} \left\{ \varrho \sqrt{2\rho^{2(k-h)}} \right\} 
$$

$$
= \sigma^2 \sum_{\varrho \in S_0} \left[ \left( 1 - |\rho|^h \right) + \left( 1 - \rho^{<\alpha>h} \right) \varrho \delta \left\{ \varrho \text{sign}(\rho)^h(0,0) \right\} + \left( 1 + |\rho|^{-2h} \right)^\alpha \left( 1 - |\rho|^h \right) \delta \left\{ \varrho \right\} \right] 
$$

$$
= \frac{\sigma^2}{2} \sum_{\varrho \in S_0} \left[ \left( 1 - |\rho|^h \right) + \left( 1 - \rho^{<\alpha>h} \right) \varrho \delta \left\{ \varrho \text{sign}(\rho)^h(0,0) \right\} + \left( 1 + |\rho|^{-2h} \right)^\alpha \left( 1 - |\rho|^h \right) \delta \left\{ \varrho \right\} \right] 
$$

From Proposition C.1 we also have $\mu^0 = \mathbf{0}$ for $\alpha \neq 1$ (since $\mu = 0$ in (1.1)). For $\alpha = 1$, we have

$$
\mu_1^0 = -\frac{2}{\pi} \sigma \beta A_1, \quad \mu_2^0 = -\frac{2}{\pi} \sigma \beta A_2,
$$

with

$$
A_1 = \ln |\rho| \sum_{k=0}^{+\infty} k \rho^k + \frac{1}{2} \ln \left( 1 + |\rho|^{-2h} \right) \sum_{k=0}^{+\infty} \rho^k, \quad A_2 = \rho^{-h} \left[ \ln |\rho| \sum_{k=0}^{+\infty} k \rho^k + \frac{1}{2} \ln \left( 1 + |\rho|^{-2h} \right) \sum_{k=0}^{+\infty} \rho^k \right].
$$

It is easily shown that $\sum_{k=0}^{+\infty} k \rho^k = \frac{\rho h^h}{1 - \rho} + \frac{\rho^h}{(1 - \rho)^2}$ for $h \geq 0$. Substituting in $A_1$ and $A_2$ yields the conclusion.
C.2 Proof of Theorem 2.1

From Proposition 2.2, we know that \((X_t, X_{t+h})\) is an \(\alpha\)-stable vector with spectral representation denoted \((\Gamma_h, 0)\). Corollary 2.1 gives a sufficient condition for the existence of conditionals moments and Theorems A.1, A.3, A.5 and A.6 give their analytical forms in terms of the spectral measure. Given \(\Gamma_h\) as in (2.3), the constants \(\sigma_1, \beta_1, \kappa_p\)'s and \(\lambda_p\)'s simplify. For instance:

\[
\sigma_1^\alpha = \int_{S_2} |s_1|^{\alpha} \Gamma_h(ds) \\
= \frac{\sigma^\alpha}{2} \sum_{\vartheta \in S_0} \left( (1 - |\rho|^h + (1 - (\rho^{-\alpha}_r)^h) \vartheta \beta_1^\alpha + (1 + |\rho|^{2h})^{\alpha/2} (1 + \vartheta \beta_1^2) \frac{\vartheta \rho^h}{1 + |\rho|^{2h}} \right)^\alpha = \bar{\sigma}_1^\alpha.
\]

C.3 Proof of Corollary 2.2

We will give the proof for the excess kurtosis. The other limits and equivalents are obtained in a similar manner. Letting \(\alpha \in (3/2, 2)\) ensures the existence of the fourth order moment.

Since we assume \(\rho > 0\), it follows that \(\lambda_p = \beta_1 \kappa_p\) for \(p = 1, 2, 3, 4\). Using Proposition A.1, it is straightforward to show that as \(x\) tends to infinity

\[
\gamma_2(x, h) \rightarrow \frac{\kappa_4 - 4 \kappa_1 \kappa_3 + 6 \kappa_1^2 \kappa_2 - 3 \kappa_1^4}{(\kappa_2 - \kappa_1^2)^2} - 3.
\]

Substituting the \(\kappa_p\)'s by the expressions in Theorem 2.1 and rearranging terms yields the conclusion.

C.4 Preliminary elements for the proofs of the main results

Notations for the proofs of Theorems A.3, A.5 and Proposition A.1

Let \(X = (X_1, X_2)\) be an \(\alpha\)-stable vector, with \(0 < \alpha < 2, \alpha \neq 1\), and spectral representation \((\Gamma, 0)\). Its characteristic function will be denoted \(\varphi_X(t, r)\) for any \((t, r) \in \mathbb{R}^2\), and reads

\[
\varphi_X(t, r) = \exp\left\{-\int_{S_2} g_1(ts_1 + rs_2) \Gamma(ds)\right\},
\]

where \(g_1(z) = |z|^{\alpha} - iaz^{\alpha}\) for \(z \in \mathbb{R}\), and \(a = \tan(\pi \alpha/2)\). As we assume \(\sigma_1 > 0\) so that \(X_1\) is not degenerate, the conditional characteristic function of \(X_2\) given \(X_1 = x\), denoted \(\phi_{X_2|x}(r)\) for \(r \in \mathbb{R}\), equals

\[
\phi_{X_2|x}(r) := 1 + \frac{1}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-i \tau x} \left( \varphi_X(t, r) - \varphi_X(t, 0) \right) dt.
\]

where \(f_{X_1}\) denotes the density of \(X_1 \sim S(\alpha, \beta_1, \sigma_1, 0)\). The following notation of the \(\mathcal{H}\) family function will be more handy than that in (2.9): for any \(y > -1\) and \(\theta = (\theta_1, \theta_2) \in \mathbb{R}^2\), define the
Lemma C.2 Let $\{X_1, X_2\}$ be an $\alpha$-stable vector, $0 < \alpha < 2, \alpha \neq 1$, with conditional characteristic function $\phi_{X_2|x}$ as given in (C.7). Let $r \in \mathbb{R}$. If $1 < \alpha < 2$, or if $0 < \alpha < 1$ and (2.2) holds with $\nu > 1 - \alpha$, the first derivative of $\phi_{X_2|x}$ is given by

$$
\phi_{X_2|x}^{(1)}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \Lambda \left( \int_{S_2} g_{2s_2} \right).
$$

If $1/2 < \alpha < 2$ and (2.2) holds with $\nu > 2 - \alpha$, the second derivative is given by

$$
\phi_{X_2|x}^{(2)}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \left[ ix \Lambda \left( \int_{S_2} g_{2s_2}^2 s_2^{-1} \right) + \alpha \left( \Lambda \left( \int_{S_2} g_{2s_2}^2 s_2^{-1} \right) \left( \int_{S_2} g_{2s_2} \right) - \Lambda \left( \int_{S_2} g_{2s_2}^2 \right) \right) \right],
$$

and if $1 < \alpha < 2$ and (2.2) holds with $\nu > 3 - \alpha$, the third derivative is given by

$$
\phi_{X_2|x}^{(3)}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \left( ix (1 - \alpha)(I_1 - \alpha I_2) + \alpha^2 (I_3 - I_4) + \alpha (\alpha - 1)(I_5 + I_6 - 2I_7) \right),
$$

with

$$
\begin{align*}
I_1 &= \Lambda \left( \int_{S_2} g_{3s_2}^3 s_2^{-1} \right), \\
I_2 &= \Lambda \left( \int_{S_2} g_{2s_2} \right) \left( \int_{S_2} g_{2s_2}^2 s_2^{-1} \right), \\
I_3 &= \Lambda \left( \int_{S_2} g_{2s_2}^3 \right), \\
I_4 &= \Lambda \left( \int_{S_2} g_{2s_2} \right) \left( \int_{S_2} g_{2s_2}^2 s_2^{-1} \right), \\
I_5 &= \Lambda \left( \int_{S_2} g_{2s_2}^2 s_2^{-1} \right) \left( \int_{S_2} g_{3s_2 s_1} \right), \\
I_6 &= \Lambda \left( \int_{S_2} g_{2s_2} \right) \left( \int_{S_2} g_{3s_2} s_2^{-1} \right), \\
I_7 &= \Lambda \left( \int_{S_2} g_{2s_2} \right) \left( \int_{S_2} g_{3s_2}^2 \right).
\end{align*}
$$
If \(3/2 < \alpha < 2\) and \((\ref{2.2})\) holds with \(\nu > 4 - \alpha\), the fourth derivative is given by

\[
\phi^{(4)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \left[ i\alpha x \left( \alpha (3J_1 - 2J_2) + (\alpha - 1) \left( 2J_3 - 3J_4 + J_5 \right) \right) + \alpha x^2 J_6 - (\alpha - 1)x^2 J_7 
+ \alpha^2 (\alpha - 1) \left( J_8 + J_9 + J_{10} - 3(2J_{11} + J_{12} - J_{13}) \right) 
+ \alpha (\alpha - 1)^2 \left( 4J_{14} - 3J_{15} - J_{16} \right) 
+ \alpha^3 \left( 3J_{17} - J_{18} - J_{19} \right) \right],
\]

with

\[
J_1 = \Lambda \left( \int_{S_2} g_{2s_2^2 s_1^2} \right) \left( \int_{S_2} g_{2s_2^2} \right)^2, \quad J_{11} = \Lambda \left( \int_{S_2} g_{2s_2^2 s_1^2} \right) \left( \int_{S_2} g_{3s_2 s_1} \right) \left( \int_{S_2} g_{2s_2} \right),
\]

\[
J_2 = \Lambda \left( \int_{S_2} g_{2s_2^3 s_1^2} \right) \left( \int_{S_2} g_{2s_1} \right) \left( \int_{S_2} g_{2s_2} \right), \quad J_{12} = \Lambda \left( \int_{S_2} g_{2s_2^2 s_1^2} \right) \left( \int_{S_2} g_{2s_1} \right) \left( \int_{S_2} g_{2s_2} \right),
\]

\[
J_3 = \Lambda \left( \int_{S_2} g_{3s_2^2 s_1^2} \right) \left( \int_{S_2} g_{2s_1} \right), \quad J_{13} = \Lambda \left( \int_{S_2} g_{3s_2^2} \right) \left( \int_{S_2} g_{2s_2} \right)^2,
\]

\[
J_4 = \Lambda \left( \int_{S_2} g_{3s_2^3 s_1^2} \right) \left( \int_{S_2} g_{2s_2} \right), \quad J_{14} = \Lambda \left( \int_{S_2} g_{3s_2^2 s_1^2} \right) \left( \int_{S_2} g_{3s_2 s_1} \right),
\]

\[
J_5 = \Lambda \left( \int_{S_2} g_{2s_2^3 s_1^2} \right) \left( \int_{S_2} g_{3s_2 s_1} \right), \quad J_{15} = \Lambda \left( \int_{S_2} g_{3s_2^2} \right)^2,
\]

\[
J_6 = \Lambda \left( \int_{S_2} g_{2s_2^3 s_1^2} \right) \left( \int_{S_2} g_{2s_2} \right), \quad J_{16} = \Lambda \left( \int_{S_2} g_{3s_2^4 s_1^2} \right) \left( \int_{S_2} g_{3s_1^2} \right),
\]

\[
J_7 = \Lambda \left( \int_{S_2} g_{3s_2^4 s_1^2} \right), \quad J_{17} = \Lambda \left( \int_{S_2} g_{2s_2^2 s_1^2} \right) \left( \int_{S_2} g_{2s_1} \right) \left( \int_{S_2} g_{2s_2} \right)^2,
\]

\[
J_8 = \Lambda \left( \int_{S_2} g_{2s_2^3 s_1^2} \right) \left( \int_{S_2} g_{3s_1^2} \right) \left( \int_{S_2} g_{2s_2} \right), \quad J_{18} = \Lambda \left( \int_{S_2} g_{2s_2^2} \right)^4,
\]

\[
J_9 = \Lambda \left( \int_{S_2} g_{2s_2^3 s_1^2} \right) \left( \int_{S_2} g_{3s_2 s_1} \right) \left( \int_{S_2} g_{2s_1} \right), \quad J_{19} = \Lambda \left( \int_{S_2} g_{2s_2^3 s_1^2} \right) \left( \int_{S_2} g_{2s_1} \right)^2 \left( \int_{S_2} g_{2s_2} \right),
\]

\[
J_{10} = \Lambda \left( \int_{S_2} g_{3s_2^4 s_1^2} \right) \left( \int_{S_2} g_{2s_1} \right)^2.
\]

### C.5 Proof of Lemma \((\ref{C.2})\)

For each of the derivatives, the proof involves two main steps: 1) computation of the derivative 2) justifying inversion of integral and derivation signs. Regarding computation, we detail only the case of the second derivative, whereas for the justification, we detail only the case of the third. Those cases are representative of the main techniques employed for the others.
C.5.1 Computation: second derivative

Note that if \( f(x) = |x|^b \), for \( x, b \in \mathbb{R}, b \neq 0 \), then for \( x \neq 0 \), \( f'(x) = b x^{b-1} \) and if \( f : x \rightarrow x^{b} \), then \( f'(x) = b|x|^{b-1} \). This can be shown by distinguishing the cases \( x > 0 \) and \( x < 0 \). Formal computation of the second derivative yields divergent terms when \( 1/2 < \alpha < 1 \) and a special manipulation called «appropriate integration by parts» in (p.106) is needed.

\[
\phi^{(2)}_{X_2|x}(r) = \frac{\partial}{\partial r} \phi^{(1)}_{X_2|x}(r)
= -\frac{\alpha}{2\pi f_X(x)} \lim_{h \to 0} \frac{1}{h} \left[ \int_{\mathbb{R}} \int_{S_2} e^{-itx} \varphi_X(t, r + h) g_2((t + h)s_2) s_2 \Gamma(ds)dt
- \int_{\mathbb{R}} \int_{S_2} e^{-itx} \varphi_X(t, r) g_2((t + r)s_2) s_2 \Gamma(ds)dt \right]
= -\frac{\alpha}{2\pi f_X(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} e^{-itx} \left[ \varphi_X(t, r + h) - \varphi_X(t, r) \right] g_2((t + h)s_2) s_2 \Gamma(ds)dt
+ \frac{\alpha}{2\pi f_X(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} e^{-itx} \varphi_X(t, r) \left[ g_2((t + h)s_2) - g_2((t + r)s_2) \right] s_2 \Gamma(ds)dt
:= A_1 + A_2.
\]

The first limit can be straightforwardly obtained:

\[
A_1 = \frac{\alpha^2}{2\pi f_X(x)} \int_{\mathbb{R}} e^{-itx} \varphi_X(t, r) \left( \int_{S_2} g_2((t + r)s_2) s_2 \Gamma(ds) \right)^2 dt
= \frac{\alpha^2}{2\pi f_X(x)} \Lambda \left( \int_{S_2} g_2 s_2 \right)^2.
\]

The second one requires «appropriate integration by parts». With the change of variable \( t' = t + \frac{hs_2}{s_1} \),

\[
A_2 = -\frac{\alpha}{2\pi f_X(x)} \lim_{h \to 0} \frac{1}{h} \left[ \int_{S_2} \int_{\mathbb{R}} e^{-itx} \varphi_X(t, r) g_2((t + h)s_2) s_2 dt \Gamma(ds)
- \int_{S_2} \int_{\mathbb{R}} e^{-itx} \varphi_X(t, r) g_2((t + r)s_2) s_2 dt \Gamma(ds) \right]
= \frac{\alpha}{2\pi f_X(x)} \int_{S_2} \int_{\mathbb{R}} e^{-itx} \varphi_X(t, r) \left[ g_2((t + h)s_2) - g_2((t + r)s_2) \right] s_2 dt \Gamma(ds)
= \frac{\alpha}{2\pi f_X(x)} \int_{S_2} \int_{\mathbb{R}} s_2 s_1^{-1} g_2((t + r)s_2) \lim_{h \to 0} \frac{1}{hs_2/s_1} \left[ e^{-i(t - \frac{hs_2}{s_1})} \varphi_X(t - \frac{hs_2}{s_1}, r) - e^{-itx} \varphi_X(t, r) \right] dt \Gamma(ds)
\]
\[\begin{align*}
\frac{A_2}{2\pi f_{X_1}(x)}&= -\frac{i\alpha}{2\pi f_{X_1}(x)} \left( \int_{S_2} g_2 s_2^2 s_1^{-1} \right) \Lambda \left( \int_{S_2} g_2 s_2^2 s_1^{-1} \right) \left( \int_{S_2} g_2 s_2^2 s_1^{-1} \right)
\end{align*}\]

Combining the expressions obtained for \(A_1\) and \(A_2\) yields the second derivative.

### C.5.2 Justifying inversion of integral and derivation signs: third derivative

Let \(\alpha \in (1, 2)\) and let \([2.2]\) hold with \(\nu > 3 - \alpha\). Starting from the second derivative of \(\phi_{X_2|x}^{(2)}(r)\) given at \([C.13]\), with obvious notations

\[\phi_{X_2|x}^{(2)}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \left[ ixI_1(r) + \alpha(I_3(r) - I_2(r)) \right]\]

On the one hand, it can be shown that the dominated convergence theorem applies to \(I_1\) using the usual arguments the fact that \([2.2]\) holds with \(\nu > 3 - \alpha\). On the other hand, after some elementary manipulations, we get that

\[I_3 - I_2 = \int_{\mathbb{R}} e^{-itx+ia} \int_{S_2} \left( ts_1 + rs_2 \right)^{\alpha-1} \bar{\Gamma}(ds) \int_{S_2} \left( ts_1 + rs_2 \right)^{\alpha-1} \Gamma(ds) \int_{S_2} \left( ts_1 + rs_2 \right)^{\alpha-1} \Gamma(ds) dt\]

\[\times \int_{S_2} \int_{S_2} \left\{ \left( ts_1 + rs_2 \right)^{\alpha-1} \left( ts_1' + rs_2' \right)^{\alpha-1} - a^2 |ts_1 + rs_2|^{\alpha-1} |ts_1' + rs_2'|^{\alpha-1}
\]

\[\times \left( \left( ts_1 + rs_2 \right)^{\alpha-1} \left( ts_1' + rs_2' \right)^{\alpha-1} + \left( ts_1 + rs_2 \right)^{\alpha-1} \left( ts_1' + rs_2' \right)^{\alpha-1} \right) \right\}
\]

\[\times \left[ s_2^2 s_1^{-1} s_1' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') dt\]

The previous expression can be decomposed into terms of the form

\[\int_{\mathbb{R}} \int_{S_2} \int_{S_2} \text{trig} \left( \frac{tx + a}{S_2} \left( ts_1 + rs_2 \right)^{\alpha-1} \bar{\Gamma}(ds) \right) \times e^{-itx+ia} \int_{S_2} \left( ts_1 + rs_2 \right)^{\alpha-1} \bar{\Gamma}(ds) \times \left( ts_1 + rs_2 \right)^{\alpha-1} \times \left( ts_1' + rs_2' \right)^{\alpha-1} \times \left[ s_2^2 s_1^{-1} s_1' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') dt,\]

where «trig» is to be replaced by a sine or cosine function. Each of these terms can be treated in
a similar way to show that the dominated convergence theorem applies. We will consider

\[
J(r) = \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)}
\times |ts_1 + rs_2|^{\alpha-1} (ts_1' + rs_2')^{<\alpha-1>} \left[ s_2^{-1} - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') dt.
\]

We have

\[
J'(r) = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \left[ \cos \left( tx - a \int_{S_2} (ts_1 + (r+h)s_2)^{<\alpha>} \Gamma(ds) \right) - \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) \right]
\times e^{-\int_{S_2} |ts_1 + (r+h)s_2|^{\alpha} \Gamma(ds)}
\times \left[ s_2^{-1} s' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') dt
\]

\[
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)}
\times \left[ ts_1 + (r+h)s_2 \right]^{<\alpha-1} - \left| ts_1 + rs_2 \right|^{<\alpha-1}
\times (ts_1' + (r+h)s_2')^{<\alpha-1>} \left[ s_2^{-1} s' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') dt
\]

\[
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)}
\times \left[ (ts_1' + (r+h)s_2')^{<\alpha-1>} - (ts_1' + rs_2')^{<\alpha-1>} \right]
\times \left| ts_1 + rs_2 \right|^{<\alpha-1>} \left[ s_2^{-1} s' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') dt
\]

\[
:= K_1 + K_2 + K_3 + K_4.
\]

We will show that we can apply the dominated convergence theorem to the \( K_i \)’s. Let us begin with \( K_1 \). Its integrand converges to

\[
aa \int_{S_2 \times S_2 \times S_2} \sin \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)}
\times |ts_1 + rs_2|^{<\alpha-1>} (ts_1' + rs_2')^{<\alpha-1>} \left[ s_2^{-1} s' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') \Gamma(ds'').
\]

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For any $h$, $|h| < |r|$, the integrand of $K_1$ can be bounded using the mean value theorem on the cosine and Lemma [C.6] by

$$\frac{|a|}{|h|} \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>} - (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \left| e^{2\alpha|\sigma_2^0} e^{-2^{1-\alpha} \sigma_1^0 |t|^\alpha} \right| \int_{S_2} \int_{S_2} |ts_1 + (r + h)s_2|^{\alpha-1}(ts_1' + (r + h)s_2')^{<\alpha-1>} \left[ s_2^2 s_1^{-1} s_1' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') \right| . \quad (C.16)$$

Hence, by inequality [H.7] and given that $0 < \alpha - 1 < 1$, the quantity (C.16) can be bounded by

$$\alpha |a| \Gamma(S_2) e^{2\alpha |\sigma_2^0| |t|^\alpha} e^{-2^{1-\alpha} \sigma_1^0 |t|^\alpha} (|t|^{\alpha-1} + 2|r|^{\alpha-1}) \times \left| \int_{S_2} \int_{S_2} |ts_1 + (r + h)s_2|^{\alpha-1}(ts_1' + (r + h)s_2')^{<\alpha-1>} \left[ s_2^2 s_1^{-1} s_1' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') \right| \leq \alpha |a| \Gamma(S_2) e^{2\alpha |\sigma_2^0| |t|^\alpha} e^{-2^{1-\alpha} \sigma_1^0 |t|^\alpha} (|t|^{\alpha-1} + 2|r|^{\alpha-1})^3 \left( \Gamma(S_2) + \int_{S_2} |s_1|^{-1} \Gamma(ds) \right) \leq \text{const} \ e^{-2^{1-\alpha} \sigma_1^0 |t|^\alpha} (|t|^{\alpha-1} + 2|r|^{\alpha-1})^3,$$

where const is a finite nonnegative constant because of (2.2) with $\nu > 3 - \alpha > 1$ and the fact that $\Gamma$ is a finite measure. This last bound, independent of $h$, is integrable with respect to $t$ on $\mathbb{R}$. The dominated convergence theorem applies to $K_1$. Consider now $K_2$. Its integrand converges to

$$\alpha \int_{S_2} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)} \times \left| ts_1 + rs_2 \right|^{\alpha-1}(ts_1' + rs_2')^{<\alpha-1>} s_2^2 s_1^{-1} s_1' - s_2 s_2' \right| \Gamma(ds) \Gamma(ds') \right| . \quad (C.17)$$

By (H.3), the integrand of $K_2$ can be bounded by

$$\Gamma(S_2) e^{2\alpha |\sigma_2^0| |t|^\alpha} e^{-2^{1-\alpha} \sigma_1^0 |t|^\alpha} \alpha (|t|^{\alpha-1} + 2|r|^{\alpha-1}) \times \left| \int_{S_2} \int_{S_2} |ts_1 + (r + h)s_2|^{\alpha-1}(ts_1' + (r + h)s_2')^{<\alpha-1>} \left[ s_2^2 s_1^{-1} s_1' - s_2 s_2' \right] \Gamma(ds) \Gamma(ds') \right|$$

Which can be further bounded by an integrable function of $t$ in a similar way as for the integrand of $K_1$. The dominated convergence theorem applies to $K_2$. Consider now $K_3$. Its integrand converges to

$$(\alpha - 1) \int_{S_2} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)} \times (ts_1 + rs_2)^{<\alpha-2>} (ts_1' + (r + h)s_2')^{<\alpha-1>} s_2^2 s_1^{-1} s_1' - s_2 s_2' \right| \Gamma(ds) \Gamma(ds')$$

Using Lemmas [C.6] [C.5] (\alpha) and the triangle inequality, the integrand of $K_3$ can be bounded by

$$\frac{1}{|h|} e^{\alpha |\sigma_2^0| |t|^\alpha} \int_{S_2} \int_{S_2} |hs_2||ts_1 + rs_2|^{\alpha-2}|ts_1' + (r + h)s_2'|^{\alpha-1}\left| s_2^2 s_1^{-1} s_1' - s_2 s_2' \right| \Gamma(ds) \Gamma(ds') \leq e^{\alpha |\sigma_2^0| |t|^\alpha} \Gamma(S_2) \int_{S_2} e^{-2^{1-\alpha} \sigma_1^0 |t|^\alpha} |ts_1 + rs_2|^{\alpha-2}(|t|^{\alpha-1} + 2|r|^{\alpha-1}) \left| 1 + |s_1|^{-1} \Gamma(ds) \right|$$

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To show the integrability with respect to $t$ of the last bound we make use of Lemma C.7 with $\eta = \alpha - 2$, $b = 0, \alpha - 1$ and $p = 0$ and the fact that with $1 < \alpha < 2$, $\int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^3_t |t|^\alpha} |t|^{\alpha-2} dt < +\infty$ and $\int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^3_t |t|^\alpha} |t|^{2\alpha-3} dt < +\infty$

$$e^{r|\sigma^2_t \Gamma(S_2)} \int_{S_2} \left| 1 + |s_1|^{-1} \right| \int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^2_t |t|^\alpha} |s_1|^{\alpha-2} \left[ t + r \frac{s_2}{s_1} \right]^{\alpha-2} \left( |t|^{\alpha-1} + 2 |r|^{\alpha-1} \right) dt \Gamma(ds)$$

$$\leq e^{r|\sigma^2_t \Gamma(S_2)} \int_{S_2} \left| 1 + |s_1|^{-1} \right| \int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^2_t |t|^\alpha} |s_1|^{\alpha-2} \left[ t + r \frac{s_2}{s_1} \right]^{\alpha-2} \left| t + r \frac{s_2}{s_1} \right|^{\alpha-2} - |t|^{\alpha-2} + |t|^{\alpha-2} |t|^{\alpha-1} dt$$

$$+ 2 |r|^{\alpha-1} \int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^2_t |t|^\alpha} |t + r \frac{s_2}{s_1}|^{\alpha-2} - |t|^{\alpha-2} |t|^{\alpha-1} dt$$

$$+ \int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^2_t |t|^\alpha} |t|^{2\alpha-3} dt$$

$$+ 2 |r|^{\alpha-1} \int_{\mathbb{R}} e^{-2^{1-\alpha} \sigma^2_t |t|^\alpha} |t|^{\alpha-2} dt \right] \Gamma(ds)$$

$$\leq \text{const} \int_{S_2} \left| 1 + |s_1|^{-1} \right| |s_1|^{\alpha-2} \Gamma(ds)$$

$$\leq \text{const} \left( \int_{S_2} |s_1|^{\alpha-2} \Gamma(ds) + \int_{S_2} |s_1|^{\alpha-3} \Gamma(ds) \right),$$

which is finite because of (2.2) with $\nu > 3 - \alpha$. Hence, the dominated convergence theorem applies to $K_3$. The case of $K_4$ is similar, using Lemma C.5 (ii) instead of (i) to bound the term $\left| (ts'_1 + (r+h)s'_2)^{\alpha-2} - (ts'_1 + rs'_2)^{\alpha-2} \right|$. The dominated convergence theorem applies to all the $K_4$'s and we can invert the integration and derivation signs in $J'$.

### C.6 Proof of Theorem A.6

The conditional moments are obtained by evaluating the derivatives of the conditional characteristic function at $r = 0$. We provide here the proof for the fourth order, which yields the expressions of the vectors $\theta_4, \theta_5$ and $\theta_6$ appearing in Theorems 2.1 and A.6. The fourth order derivative of the characteristic function of $X_2|X_1 = x$ is given by (C.15) in Lemma C.2. It can be shown that the $J'$'s evaluated at $r = 0$ write
$iJ_1 = 2\sigma_1^{3}\mathcal{H}(3(\alpha - 1), \theta_1^J; x)$,
$iJ_2 = 2\sigma_1^{3}\mathcal{H}(3(\alpha - 1), \theta_2^J; x)$,
$iJ_3 = 2\sigma_1^{3}\mathcal{H}(2\alpha - 3, \theta_3^J; x)$,
i
$iJ_4 = iJ_5 = 2\sigma_1^{2}\mathcal{H}(2\alpha - 3, \theta_4^J; x)$,
$J_6 = 2\sigma_1^{2}\mathcal{H}(2(\alpha - 1), \theta_6^J; x)$,
$J_7 = 2\sigma_1^{3}\mathcal{H}(\alpha - 2, \theta_7^J; x)$,
$J_8 = J_9 = J_{12} = 2\sigma_1^{3}\mathcal{H}(3\alpha - 4, \theta_8^J; x)$,
$J_10 = 2\sigma_1^{3}\mathcal{H}(3\alpha - 4, \theta_10^J; x)$,

where $\theta_I^J = (\theta_{i1}^J, \theta_{i2}^J)$, for $i = 1, \ldots, 19$,

$\theta_{11}^J = a\left(\lambda_2(\kappa_1^2 - a^2\lambda_1^2) + 2\kappa_1\kappa_2\lambda_1\right)$,
$\theta_{21}^J = a\left(K + \beta_1L\right)$,
$\theta_{31}^J = a\left(\beta_1\kappa_4 + \lambda_4\right)$,
$\theta_{41}^J = aK$,
$\theta_{61}^J = L$,
$\theta_{71}^J = \kappa_4$,
$\theta_{81}^J = L - a^2\beta_1K$,
$\theta_{101}^J = \kappa_4(1 - a^2\beta_1^2) - 2a^2\beta_1\lambda_4$,
$\theta_{111}^J = \theta_{12}^J$,
$\theta_{141}^J = L$,
$\theta_{151}^J = \kappa_2^2 - a^2\lambda_2^2$,
$\theta_{161}^J = \kappa_4 - a^2\beta_1\lambda_4$,
$\theta_{171}^J = \theta_{12}^J - a\beta_1\theta_{11}^J$,
$\theta_{181}^J = \kappa_4^2 - 6a^2\kappa_1^2\lambda_2^2 + a^4\lambda_4^2$,
$\theta_{191}^J = L(1 - a^2\beta_1^2) - 2a^2\beta_1K$,

$J_{11} = J_{13} = 2\sigma_1^{3}\mathcal{H}(3\alpha - 4, \theta_{11}^J; x)$,
$J_{14} = 2\sigma_1^{2}\mathcal{H}(2\alpha - 4, \theta_{14}^J; x)$,
$J_{15} = 2\sigma_1^{2}\mathcal{H}(2\alpha - 4, \theta_{15}^J; x)$,
$J_{16} = 2\sigma_1^{2}\mathcal{H}(2\alpha - 4, \theta_{16}^J; x)$,
$J_{17} = 2\sigma_1^{4}\mathcal{H}(4(\alpha - 1), \theta_{17}^J; x)$,
$J_{18} = 2\sigma_1^{4}\mathcal{H}(4(\alpha - 1), \theta_{18}^J; x)$,
$J_{19} = 2\sigma_1^{4}\mathcal{H}(4(\alpha - 1), \theta_{19}^J; x)$,
and $K = \kappa_1 \lambda_3 + \lambda_1 \kappa_3$, $L = \kappa_1 \kappa_3 - a^2 \lambda_1 \lambda_3$. Hence,

$$
\mathbb{E}[X_2^4|X_1 = x] = \varphi^{(4)}_{X_2|x}(0)
= \frac{-\alpha}{\pi f_{X_1}(x)} \left[ \alpha x (\alpha K_1 + (\alpha - 1) K_2) + \alpha x^2 K_6 - (\alpha - 1) K_7 + \alpha^2 (\alpha - 1) K_3 + \alpha (\alpha - 1)^2 K_4 + \alpha^3 \right],
$$

where

$$
K_1 = \sigma_1^{20} \mathcal{H} \left( 3(\alpha - 1), \theta^K_1 ; x \right), \quad \text{with } \theta^K_1 = 3 \theta^J_1 - 2 \theta^J_2,
K_2 = \sigma_1^{20} \mathcal{H} \left( 2 \alpha - 3, \theta^K_2 ; x \right), \quad \text{with } \theta^K_2 = 2(\theta^J_3 - \theta^J_4),
K_3 = \sigma_1^{20} \mathcal{H} \left( 3 \alpha - 4, \theta^K_3 ; x \right), \quad \text{with } \theta^K_3 = \theta^J_{10} - 3 \theta^J_{11} - \theta^J_8,
K_4 = \sigma_1^{20} \mathcal{H} \left( 2 \alpha - 4, \theta^K_4 ; x \right), \quad \text{with } \theta^K_4 = 4 \theta^J_{14} - 3 \theta^J_{15} - \theta^J_6,
K_5 = \sigma_1^{20} \mathcal{H} \left( 4 - \alpha, \theta^K_5 ; x \right), \quad \text{with } \theta^K_5 = 3 \theta^J_{17} - \theta^J_{18} - \theta^J_9,
K_6 = \sigma_1^{20} \mathcal{H} \left( 2(\alpha - 1), \theta^K_6 ; x \right), \quad \text{with } \theta^K_6 = \theta^J_0,
K_7 = \sigma_1^2 \mathcal{H} \left( \alpha - 2, \theta^K_7 ; x \right), \quad \text{with } \theta^K_7 = \theta^J_7.
$$

Invoking Lemmas C.10 ($\mu$) for $n = 1, 2, 3$ and C.11 we get

$$
\mathbb{E}[X_2^4|X_1 = x] = \frac{-\alpha}{\pi f_{X_1}(x)} \left[ x^3 \sigma_1^0 \left( \theta^K_{12} C_1(x) - \theta^K_{11} S_1(x) \right) + \frac{\alpha x^2 \sigma_1^{20}}{2} C_2(x) \left( - \theta^K_{22} + 2 \theta^K_{61} - 2 \left( \theta^K_{12} + \alpha \theta^K_{11} + \alpha \theta^K_{10} \right) - \frac{\alpha - 1}{2 \alpha - 3} \theta^K_{41} \right) + \frac{\alpha x^2 \sigma_1^{20}}{2} S_2(x) \left( \theta^K_{21} + 2 \theta^K_{62} - 2 \left( \theta^K_{22} - \alpha \theta^K_{12} \right) - \frac{\alpha - 1}{2 \alpha - 3} \theta^K_{42} \right) \right] + \frac{\alpha^2 x \sigma_1^{30}}{6} C_3(x) \left( 6 \theta^K_{11} + 3 \left( \theta^K_{21} + \alpha \theta^K_{22} \right) - 2 \theta^K_{31} + 5 \frac{\alpha - 1}{2 \alpha - 3} \left( \alpha \theta^K_{11} - \theta^K_{41} \right) \right) + \frac{\alpha^2 x \sigma_1^{30}}{6} S_3(x) \left( 6 \theta^K_{21} + 3 \left( \theta^K_{22} - \alpha \theta^K_{21} \right) + 2 \theta^K_{31} + 5 \frac{\alpha - 1}{2 \alpha - 3} \left( \theta^K_{11} + \alpha \theta^K_{41} \right) \right) + \frac{\alpha^3 \sigma_1^{40}}{3} C_4(x) \left( \theta^K_{31} + \alpha \theta^K_{32} + \frac{\alpha - 1}{2 \alpha - 3} \left( \theta^K_{41} \left( 1 - a^2 \beta^2_1 \right) + 2 \alpha \theta^K_{11} \right) + 3 \theta^K_{51} \right) + \frac{\alpha^3 \sigma_1^{40}}{3} S_4(x) \left( \theta^K_{32} - \alpha \theta^K_{31} + \frac{\alpha - 1}{2 \alpha - 3} \left( \theta^K_{42} \left( 1 - a^2 \beta^2_1 \right) - 2 \alpha \theta^K_{11} \right) + 3 \theta^K_{52} \right) \right].
$$

Using Lemma C.10 ($\mu$) yields the conclusion. The coefficients $\theta$'s in the expression of Proposition A.6 are deduced from the $\theta^K$'s and $\theta^J$'s as follows:
\[ \theta_{41} = -\theta_{22}^K + 2\theta_{61}^K - 2\left(\theta_{71}^K + \alpha \beta_1 \theta_{42}^K\right) - \frac{\alpha - 1}{2\alpha - 3} \theta_{41}^K, \quad (C.18) \]
\[ \theta_{42} = \theta_{21}^K + 2\theta_{62}^K - 2\left(\theta_{72}^K - \alpha \beta_1 \theta_{41}^K\right) - \frac{\alpha - 1}{2\alpha - 3} \theta_{42}^K, \quad (C.19) \]
\[ \theta_{51} = 6\theta_{11}^K + 3\left(\theta_{21}^K + \alpha \beta_1 \theta_{31}^K\right) - 2\theta_{32}^K + 5\frac{\alpha - 1}{2\alpha - 3} \left(\theta_{41}^K + \alpha \beta_1 \theta_{42}^K\right), \quad (C.20) \]
\[ \theta_{52} = 6\theta_{21}^K + 3\left(\theta_{22}^K - \alpha \beta_1 \theta_{32}^K\right) + 2\theta_{31}^K + 5\frac{\alpha - 1}{2\alpha - 3} \left(\theta_{41}^K + \alpha \beta_1 \theta_{42}^K\right), \quad (C.21) \]
\[ \theta_{61} = \theta_{31}^K + \alpha \beta_1 \theta_{32}^K + \frac{\alpha - 1}{2\alpha - 3} \left(\theta_{41}^K(1 - a^2\beta_1^2) + 2a\beta_1 \theta_{42}^K\right) + 3\theta_{51}^K, \quad (C.22) \]
\[ \theta_{62} = \theta_{32}^K - \alpha \beta_1 \theta_{31}^K + \frac{\alpha - 1}{2\alpha - 3} \left(\theta_{42}^K(1 - a^2\beta_1^2) - 2a\beta_1 \theta_{41}^K\right) + 3\theta_{52}^K, \quad (C.23) \]

\section*{C.7 Vectors $\theta_2$ and $\theta_3$ of Theorems 2.1 and A.5}

We provide here the expressions of $\theta_2 = (\theta_{21}, \theta_{22})$, $\theta_3 = (\theta_{31}, \theta_{32})$, which intervene in the form of the third conditional moments:

\[ \theta_{21} = 3(L + a^2 \beta_1 \lambda_3 - \kappa_3), \quad (C.24) \]
\[ \theta_{22} = 3a(\lambda_3 + \beta_1 \kappa_3 - K), \quad (C.25) \]
\[ \theta_{31} = a \left( \lambda_3(1 - a^2 \beta_1^2) + 2\beta_1 \kappa_3 + 2\lambda_1(3\kappa_1^2 - a^2 \lambda_1^2) - 3(K + \beta_1 L) \right), \quad (C.26) \]
\[ \theta_{32} = \kappa_3(1 - a^2 \beta_1^2) - 2a^2 \beta_1 \lambda_3 + 2(\kappa_3^3 - 3a^2 \kappa_1 \lambda_1^2(3) + 3(a^2 \beta_1 K - L), \quad (C.27) \]

with $K = \kappa_1 \lambda_2 + \kappa_2 \lambda_1$ and $L = \kappa_1 \kappa_2 - a^2 \lambda_1 \lambda_2$.

\section*{C.8 Proof of Proposition A.1 in the case $\alpha \neq 1$}

First assume that $|\beta_1| \neq 1$. We will focus on the case $x \to +\infty$. The case $x \to -\infty$ can be obtained by considering the vector $(X_1, X_2)$, whose parameter are $\beta_1^* = -\beta_1$, $\kappa_1^* = -\kappa_1$ and $\lambda_1^* = \lambda_1$ and noticing that $\mathbb{E}\left[X_2^p | X_1 = x\right] = \mathbb{E}\left[X_2^p | X_1 = -x\right]$. For $p = 1$, the result is already known (see [21]). For $p = 2, 3, 4$, we have from the proofs of Propositions A.6, A.7 and A.8 that

\[ \mathbb{E}\left[X_2^p | X_1 = x\right] = \frac{\alpha \sigma_\tau^p}{\pi f_{X_1}(x)} \left[ x^{p-1} \mathcal{H}\left(\alpha - 1, (a\lambda_p, \kappa_p); x\right) + \sum_{i=2}^{p} b_{i,p} x^{p-i} \mathcal{H}\left(\alpha - 1, \nu_i; x\right) \right], \]

for some coefficients $b$'s. From the proof of Corollary 3.2 in [21], we deduce the following limit:

\[ x^\alpha \mathcal{H}\left(\alpha - 1, (a\lambda_p, \kappa_p); x\right) \xrightarrow{x \to +\infty} \left(\kappa_p + \lambda_p\right) \sin\left(\frac{\pi \alpha}{2}\right) \Gamma(\alpha). \]

We also have

\[ x^{\alpha+1} f_{X_1}(x) \xrightarrow{x \to +\infty} \frac{1}{\pi} \sigma_\tau^p (1 + \beta_1) \sin\left(\frac{\pi \alpha}{2}\right) \Gamma(1 + \alpha). \quad (C.28) \]
Hence,

\[ x^{-p} \frac{\alpha \sigma_1^\alpha x^{p-1}}{\pi f_{X_1}(x)} H(\alpha - 1, (a, \lambda_p, \kappa_p); x) \rightarrow \frac{\kappa_p + \lambda_p}{1 + \beta_1}, \]

as \( x \rightarrow +\infty \). It remains to be shown that \( \sum_{i=2}^{p} b_i x^{p-i} H(i(\alpha - 1), \nu_i; x) \frac{x^{p-1} H(\alpha - 1, (a, \lambda_p, \kappa_p); x)}{x^{p-1} H(\alpha - 1, (a, \lambda_p, \kappa_p); x)} \rightarrow 0 \). By Theorem 127 in [41], for \( i = 2, 3, 4 \),

\[ H(i(\alpha - 1), \nu_i; x) \rightarrow x^{(\alpha - 1)(i - 1)} O(x^{-i}). \]

Hence,

\[ \left| \frac{x^{p-i} H(i(\alpha - 1), \nu_i; x)}{x^{p-1} H(\alpha - 1, (a, \lambda_p, \kappa_p); x)} \right| \rightarrow x^{(\alpha - 1)(i - 1)} \frac{x^{-i}}{x^{-i}} O(x^{\alpha(1-i)}) \rightarrow 0. \]

Now assume that \( |\beta_1| = 1 \). For instance if \( \beta_1 = 1 \), the distribution of \( X_1 \) is **totally skewed to the right**. On the one hand, we have \( \lambda_p = \beta_1 \kappa_p \). On the other hand, the right tail of \( f_{X_1} \) still decays as [C.28], yielding the conclusion.

The following elementary Lemmas, stated without proof, are used to establish Theorems A.3-A.6.

**Lemma C.3** For \( x, y \in \mathbb{R} \),

\[ |e^{-x} - e^{-y}| \leq e^{-\min(x,y)} |x - y|, \quad (C.29) \]
\[ |e^{-x} - e^{-y}| \leq e^{y} e^{-x} |x - y|. \quad (C.30) \]

**Lemma C.4** For \( \alpha > 1 \) and \( x, y \in \mathbb{R} \),

\[ \max \left( 2^{1-\alpha} |x|^\alpha - |y|^\alpha, 2^{1-\alpha} |y|^\alpha - |x|^\alpha \right) \leq |x + y|^\alpha \leq 2^{\alpha-1} \left( |x|^\alpha + |y|^\alpha \right). \]

**Lemma C.5** For \( z \in \mathbb{R} \) and \( 0 < b \leq 1 \),

\[ \begin{align*}
\text{(i)} & \quad \left| 1 + z^b - 1 \right| \leq |z|, \\
\text{(ii)} & \quad \left| 1 + z^{<b>} - 1 \right| \leq 2|z|.
\end{align*} \]

**Lemma C.6** (Lemma 3.3, Cioszek-Georges and Taqqu (1998)) For \( \alpha > 1 \) and \( t, r \in \mathbb{R} \),

\[ \exp \left\{ - \int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(ds) \right\} \leq \exp \{|r|\sigma_2^\alpha \} \exp \{-2^{1-\alpha} \sigma_1^\alpha |t|^\alpha \}. \]
Lemma C.7 (Lemma 3.1, Cioszek-Georges and Taqqu (1998)) The following inequality holds for $c > 0$, $0 < \alpha < 2$, $-1 < \eta < 0$ and $-1 - \eta < b$:

$$\int_{\mathbb{R}} \exp(-c|t|^\alpha) |t + z|^\eta - |t|^\eta |t|^b dt \leq \text{const.} |z|^p$$

with

$$0 \leq p < b + \eta + 1 \quad \text{for} \quad -1 - \eta < b < 0,$$

and

$$0 \leq p < \eta + 1 \quad \text{or} \quad b \leq p < b + \eta + 1, p \leq 1 \quad \text{for} \quad 0 \leq b.$$

const. depends only on $c$, $\alpha$, $\eta$, $b$ and $p$.

Lemma C.8 (Corollary 3.1, Cioszek-Georges and Taqqu (1998)) The following inequality holds for $c > 0$, $0 < \alpha < 2$, $-1/2 < \eta < 0$ and $0 \leq p < 2\eta + 1$:

$$\int_{\mathbb{R}} \exp(-c|t|^\alpha) |t + z_1|^\eta |t + z_3|^\eta - |t + z_2|^\eta |t + z_4|^\eta dt \leq \text{const.} (|z_1 - z_2|^p + |z_3 - z_4|^p),$$

where const depends only on $c$, $\alpha$, $\eta$ and $p$.

Lemma C.9 (Lemma 3.12, Cioszek-Georges and Taqqu (1998)) The following inequality holds for $c > 0$, $0 < \alpha < 2$, $-1 < \eta < 0$, $b \geq 0$ and $0 \leq p < \eta + 1$:

$$\int_{\mathbb{R}} \exp(-c|t|^\alpha) |t + z_1|^\eta - |t + z_2|^\eta |t|^b dt \leq \text{const.} |z_1 - z_2|^p,$$

where const depends only on $c$, $\alpha$, $\eta$, $b$ and $p$.

Lemma C.10 Let $\alpha \in (1, 2)$, $b > 0$, $c \in \mathbb{R}$. Define for $n \geq 1$ and $x \in \mathbb{R}$

$$C_n(x) = \int_{0}^{+\infty} e^{-bt\alpha} t^{n(\alpha-1)} \cos(tx - ct\alpha) dt, \quad F_n(x) = \int_{0}^{+\infty} e^{-bt\alpha} t^{n(\alpha-1)-1} \cos(tx - ct\alpha) dt,$$

$$S_n(x) = \int_{0}^{+\infty} e^{-bt\alpha} t^{n(\alpha-1)} \sin(tx - ct\alpha) dt, \quad G_n(x) = \int_{0}^{+\infty} e^{-bt\alpha} t^{n(\alpha-1)-1} \sin(tx - ct\alpha) dt.$$

i) Then the following hold for any $n \geq 1$ and $x \in \mathbb{R}$

$$F_n(x) = \frac{1}{n(\alpha - 1)} \left[ \alpha \left(bC_{n+1}(x) - cS_{n+1}(x)\right) + xS_n(x) \right],$$

$$G_n(x) = \frac{1}{n(\alpha - 1)} \left[ \alpha \left(cC_{n+1}(x) + bS_{n+1}(x)\right) - xC_n(x) \right].$$

ii) For any $n \geq 1$, $\theta_1, \theta_2 \in \mathbb{R}$ and $x \in \mathbb{R}$:

$$\theta_1 F_n(x) + \theta_2 G_n(x) = \frac{\alpha}{n(\alpha - 1)} \left[ C_{n+1}(x) \left( b\theta_1 + c\theta_2 \right) + S_{n+1}(x) \left( b\theta_2 - c\theta_1 \right) \right]$$

$$+ \frac{x}{n(\alpha - 1)} \left[ - \theta_2 C_n(x) + \theta_1 S_n(x) \right].$$
We have for $x \in \mathbb{R}$, $b = \sigma_1^\alpha$ and $c = a\beta_1\sigma_1^\alpha$:

$$C_1(x) = \frac{1}{\alpha\sigma_1^\alpha(1 + (a\beta_1)^2)} \left[ a\beta_1 x \pi f_{X_1}(x) + 1 - x H(x) \right],$$

$$S_1(x) = \frac{1}{\alpha\sigma_1^\alpha(1 + (a\beta_1)^2)} \left[ x \pi f_{X_1}(x) - a\beta_1(1 - x H(x)) \right].$$

Lemma C.11 Let $\alpha \in (3/2, 2)$, $b > 0$, $c \in \mathbb{R}$. Define for $x \in \mathbb{R}$

$$h_c(x) = \int_0^{+\infty} e^{-bt^\alpha} t^{2\alpha-4} \cos(tx - ct^\alpha) dt, \quad h_s(x) = \int_0^{+\infty} e^{-bt^\alpha} t^{2\alpha-4} \sin(tx - ct^\alpha) dt.$$

Then for any $\theta_1, \theta_2 \in \mathbb{R}$ and $x \in \mathbb{R}$,

$$\theta_1 h_c(x) + \theta_2 h_s(x) = \frac{\alpha^2}{3(2\alpha - 3)(\alpha - 1)} \left[ C_4(x) \left( \theta_1(b^2 - c^2) + 2bc\theta_2 \right) + S_4(x) \left( \theta_2(b^2 - c^2) - 2bc\theta_1 \right) \right]$$

$$+ \frac{5\alpha x}{6(2\alpha - 3)(\alpha - 1)} \left[ C_3(x) \left( c\theta_1 - b\theta_2 \right) + S_3(x) \left( b\theta_1 + c\theta_2 \right) \right]$$

$$- \frac{x^2}{2(2\alpha - 3)(\alpha - 1)} \left[ \theta_1 C_2(x) + \theta_2 S_2(x) \right].$$

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D Interpreting $\rho^{\alpha h}$ using point processes

The quantity $\rho^{\alpha h}$ appearing in Corollary 2.2 has the intuitive interpretation of a survival probability at horizon $h$ of a bubble generated by (1.1). This conclusion can also be reached using point processes under the less restrictive assumption that the errors of (1.1) belong to the domain of attraction of an $\alpha$-stable distribution. Consider $n$ observations $X_1, \ldots, X_n$ of (1.1) where now $(\varepsilon_t)$ is an i.i.d. sequence of random variables such that:

$$
P(|\varepsilon_0| > x) = x^{-\alpha}L(x), \quad \text{and} \quad \lim_{x \to \infty} \frac{P(\varepsilon_0 > x)}{P(|\varepsilon_0| > x)} \to c \in [0, 1],$$

with $L$ a slowly varying function at infinity. Let $a_n = \inf\{u : P(|\varepsilon_0| > u) \leq n^{-1}\}$. Then, adapting Section 3.D in [1], we can study the time indexes $k \in \{1, \ldots, n\}$ for which $a_n^{-1}X_k$ falls outside the interval $(-x, x)$, for $x > 0$, that is, the time indexes for which $(X_t)$ undergoes extreme events. The corresponding point process converges as the number of observations $n$ grows to infinity:

$$
\sum_{k=1}^{n} \delta_{(k/n, a_n^{-1}X_k)} \left( \cdot \cap B_x \right) \overset{d}{\to} \sum_{k=1}^{+\infty} \xi_k \delta_{Y_k},
$$

where $\delta$ is the Dirac measure, $B_x = (0, +\infty) \times ((-\infty, -x) \cup (x, +\infty))$, $\{Y_k, k \geq 1\}$ are the points of a homogeneous Poisson Random Measure (PRM) on $(0, +\infty)$ with rate $x^{-\alpha}$ and $\xi_k = \text{Card}\{i \in \mathbb{Z} : J_k(\rho^i) > 1\}$ where $\{J_k, k \geq 1\}$ are i.i.d. on $(1, +\infty)$, independent of $\{Y_k\}$, with common density:

$$
f(z) = \alpha z^{-\alpha-1}1_{(1, +\infty)}(z). \tag{D.1}
$$

See [15]: $\{Y_k, k \geq 1\}$ are the points of a homogeneous PRM on $(0, +\infty)$ with rate $x^{-\alpha}$ if and only if, for any $\ell \geq 1$, nonnegative integers $a_1, \ldots, a_\ell$ and $b_1, \ldots, b_\ell$ such that $a_i < b_i \leq a_{i+1}$, $i = 1, \ldots, \ell$, and any nonnegative integers $n_1, \ldots, n_\ell$:

$$
P(N(a_i, b_i) = n_i, i = 1, \ldots, \ell) = \prod_{i=1}^{\ell} \frac{[x^{-\alpha}(b_i - a_i)]^{n_i}}{n_i!} \exp\left\{-x^{-\alpha}(b_i - a_i)\right\},
$$

where $N(a_i, b_i)$ denotes the number of terms of $\{Y_k, k \geq 1\}$ falling in the half-open interval $(a_i, b_i]$, $i = 1, \ldots, \ell$. 

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The sequences \( \{Y_k\} \) and \( \{\xi_k\} \) are interpreted (see [35]) as describing respectively the occurrence dates of clusters of extreme events and the size of these clusters (i.e. the number of co-occurring extreme events, which here corresponds to the duration of bubble episodes). Since \( \xi_k = \text{Card}\{i \in \mathbb{Z} : J_k|\rho^i| > 1\} = \max_{i \geq 1}\{J_k > |\rho|^{-i}\} \), we can obtain explicitly the distribution of the bubble duration using (D.1). For any \( h \geq 1 \),

\[
P(\xi_k \geq h) = P(\xi_k \geq h) = |\rho|^{ah},
\]

which as announced, is precisely the probability parameter of the Bernoulli variable intervening in the suggested interpretation of Corollary 2.2.

**E   Proof of Proposition 2.3**

The \( \alpha \)-stable random vector \((X_t, X_{t+h})\) admits the integral representation

\[
(X_t, X_{t+h}) = \left( \int_{\mathbb{R}} f_1(x-t)M(dx), \int_{\mathbb{R}} f_2(x-t)M(dx) \right),
\]

(E.1)
where $f_1(x) = e^{-\lambda x}1_{\{x \geq 0\}}$ and $f_2(x) = f_1(x-h)$, for $x \in \mathbb{R}$. Let $u = (u_1, u_2) \in \mathbb{R}^2$. For $\alpha \neq 1$, by Proposition 3.4.1(i) in [40], its characteristic function reads

\[
E \left[ \exp \left\{ i \sum_{j=1}^{2} u_j \int_{\mathbb{R}} f_j M(dx) \right\} \right] \\
\quad = \exp \left\{ \int_{\mathbb{R}} \left[ - \left| \sum_{j=1}^{2} u_j f_j \right|^\alpha + i\alpha \beta \left( \sum_{j=1}^{2} u_j f_j \right)^{<\alpha>} \right] dx \right\} \\
\quad = \exp \left\{ \int_{0}^{h} \left[ - \left| u_1 e^{-\lambda x} \right|^\alpha + i\alpha \beta \left( u_1 e^{-\lambda x} \right)^{<\alpha>} \right] dx \right. \\
\quad \left. + \int_{h}^{+\infty} \left[ - \left| u_1 e^{-\lambda x} + u_2 e^{-\lambda(x-h)} \right|^\alpha + i\alpha \beta \left( u_1 e^{-\lambda x} + u_2 e^{-\lambda(x-h)} \right)^{<\alpha>} \right] dx \right\} \\
\quad = \exp \left\{ \left( -|u_1|^\alpha + i\alpha \beta u_1^{|<\alpha>} \right) \frac{1 - e^{-\alpha \lambda h}}{\alpha \lambda} \right. \\
\quad \left. + \left( |u_1 + u_2 e^{\lambda h}|^\alpha + i\alpha (u_1 + u_2 e^{\lambda h})^{<\alpha>} \right) \frac{e^{-\alpha \lambda h}}{\alpha \lambda} \right\} \\
\quad = \exp \left\{ \frac{1}{\alpha \lambda} \sum_{\vartheta \in S_0} \frac{1 + \vartheta \beta}{2} \left[ \left( -|\vartheta u_1|^\alpha + i\alpha (\vartheta u_1)^{<\alpha>} \right) (1 - e^{-\alpha \lambda h}) \right. \\
\quad \left. + \left( - u_1 \frac{\vartheta e^{-\lambda h}}{\sqrt{1 + e^{-2\lambda h}}} + u_2 \frac{\vartheta}{\sqrt{1 + e^{-2\lambda h}}} \right)^{<\alpha>} \right) (1 + e^{-2\lambda h})^{\alpha/2} \right\} \\
\quad = \exp \left\{ - \int_{S_2} |\langle u, s \rangle|^\alpha - i\alpha (\langle u, s \rangle)^{<\alpha>} \Gamma_h(ds) \right\},
\]
Hence, with $a = \tan(\pi\alpha/2)$ and $\Gamma_h$ as in the Proposition. If $\alpha = 1$, then with $a = 2/\pi$, we have by Proposition 3.4.1(ii) in [40]

\[
E\left[ \exp\left\{ i \sum_{j=1}^{2} u_j \int_{\mathbb{R}} f_j M(dx) \right\} \right] \\
= \exp\left\{ - \int_{\mathbb{R}} \left[ \sum_{j=1}^{2} u_j f_j \right] dx \right\} \\
= \exp\left\{ - \int_{0}^{h} \left[ u_1 e^{-\lambda x} + i a \beta (u_1 e^{-\lambda x}) \ln |u_1 e^{-\lambda x}| \right] dx \right\} \\
- \int_{h}^{\infty} \left[ u_1 e^{-\lambda x} + u_2 e^{-\lambda (x-h)} \right] dx \\
+ i a \beta (u_1 e^{-\lambda x} + u_2 e^{-\lambda (x-h)}) \ln |u_1 e^{-\lambda x} + u_2 e^{-\lambda (x-h)}| dx \right\}
\]

\[
= \exp\left\{ - \left( |u_1| + i a \beta (u_1) \ln |u_1| \right) \int_{0}^{h} e^{-\lambda x} dx + i a \lambda \beta u_1 \int_{0}^{h} x e^{-\lambda x} dx \right\} \\
- \left( |u_1 + u_2 e^{\lambda h}| + i a \beta (u_1 + u_2 e^{\lambda h}) \ln |u_1 + u_2 e^{\lambda h}| \right) \int_{h}^{\infty} e^{-\lambda x} dx \\
+ i a \lambda \beta (u_1 + u_2 e^{\lambda h}) \int_{h}^{\infty} x e^{-\lambda x} dx \right\}
\]

\[
= \exp\left\{ - \frac{1}{\lambda} \sum_{\vartheta \in S_0} \frac{1 + \vartheta \beta}{2} \left[ (|\vartheta u_1| + i a (\vartheta u_1) \ln |\vartheta u_1|) (1 - e^{-\lambda h}) \right. \\
+ \left. \left( \left| u_1 \frac{\vartheta e^{-\lambda h}}{\sqrt{1 + e^{-2\lambda h}}} + u_2 \frac{\vartheta}{\sqrt{1 + e^{-2\lambda h}}} \right| \\
+ i a \left( u_1 \frac{\vartheta e^{-\lambda h}}{\sqrt{1 + e^{-2\lambda h}}} + u_2 \frac{\vartheta}{\sqrt{1 + e^{-2\lambda h}}} \right) \ln |u_1 \frac{\vartheta e^{-\lambda h}}{\sqrt{1 + e^{-2\lambda h}}} + u_2 \frac{\vartheta}{\sqrt{1 + e^{-2\lambda h}}} | \right) \right. \\
\left. \left. \left( 1 + e^{-2\lambda h} \right) \right. \\
- \left. i a \beta (u_1 e^{-\lambda h} + u_2) (h + \frac{\ln(1 + e^{-2\lambda h})}{2\lambda}) + i a \lambda \beta (u_1 \int_{0}^{h} x e^{-\lambda dx} + u_2 e^{\lambda h} \int_{h}^{\infty} x e^{-\lambda dx}) \right\},
\right.
\]

and

\[
\lambda \left( u_1 \int_{0}^{\infty} x e^{-\lambda dx} + u_2 e^{\lambda h} \int_{h}^{\infty} x e^{-\lambda dx} \right) = u_1 \lambda^{-1} + u_2 (h + \lambda^{-1}),
\]

\[
h + \frac{\ln(1 + e^{-2\lambda h})}{2\lambda} = \ln(1 + e^{2\lambda h}).
\]

Hence,

\[
E\left[ \exp\left\{ i \left( u_1 X_t + u_2 X_{t+h} \right) \right\} \right] = \exp\left\{ - \int_{S_2} |\langle u, s \rangle| + i a (\langle u, s \rangle) \ln |\langle u, s \rangle| \Gamma_h(d s) + i \langle u, \mu^0 \rangle \right\},
\]

with $\Gamma_h$ and $\mu^0$ as claimed for $\alpha = 1$. 

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F Proof of Theorem 2.2

Let \( \{X_t, t \in \mathbb{R}\} \) be the anticipative OU defined in (2.18) with \( \lambda > 0 \) and let \( h > 0 \). Since, the Markov property of the anticipative OU has already been discussed in the relevant section, let us focus on the points two last points, namely the existence and the form of the conditional moments. The proof could be done similarly to those of the discrete time AR(1) using the expression of the spectral measure obtained at Proposition 2.3. However, we propose another proof which has the advantage of illustrating how \( \alpha \)-stable vectors with different integrand functions \( f_1 \) and \( f_2 \) in (E.1) could be considered. Formulae are given by Proposition 3.1 in [39] for expressing the constants like the ones we introduced in Equations (A.1) and (A.2) in terms of these integrand functions. The condition (2.2) can be translated in terms of \( f_1 \) and \( f_2 \) as

\[
\int_{S_2} |s_1|^{-\nu} \Gamma_h(ds) < +\infty \iff \int_{\mathbb{R}^+} \frac{|f_2(x)|^{\alpha+\nu}}{|f_1(x)|^\nu} dx = e^{\lambda h(\alpha+\nu)} \int_h^{+\infty} e^{-\alpha \lambda x} dx = \frac{e^{\lambda h \nu}}{\alpha \lambda} < +\infty,
\]

which is satisfied for any \( \nu \geq 0 \), hence (iii). Let us turn to point (iii). The conditional moments are given by Theorems A.1, A.3, A.5 and A.6 for an arbitrary spectral measure \( \Gamma \). From the proof of Proposition 3.1 in [39], we know how we can rewrite the constants \( \sigma_1^\alpha, \beta_1, \kappa_p \) and \( \lambda_p \), for \( p \in \{1, 2, 3, 4\} \) which are expressed in terms of an integral of \( s_1, s_2 \) and \( \Gamma \) into expressions involving \( f_1, f_2 \) and the Lebesgue measure. It can be shown that

\[
\sigma_1^\alpha = \int_{\mathbb{R}} |f_1(x)|^\alpha dx = \int_{\mathbb{R}}^{+\infty} e^{-\alpha \lambda x} dx = \frac{1}{\alpha \lambda},
\]

\[
\beta_1 = \frac{\int_{\mathbb{R}} f_1(x)^{<\alpha>} \beta(x) dx}{\sigma_1^\alpha} = \frac{\beta}{\sigma_1^\alpha} \int_{\mathbb{R}} f_1(x)^\alpha dx = \beta,
\]

\[
\kappa_p = \frac{\int_{\mathbb{R}^+} (f_2(x)/f_1(x))^p |f_1(x)|^{\alpha} dx}{\sigma_1^\alpha} = \frac{1}{\sigma_1^\alpha} \int_{\mathbb{R}}^{+\infty} \left( \frac{e^{-\lambda(x-h)}}{e^{-\lambda x}} \right)^p e^{-\alpha \lambda x} dx = e^{\lambda h (\alpha-p)},
\]

\[
\lambda_p = \frac{\int_{\mathbb{R}^+} (f_2(x)/f_1(x))^p |f_1(x)|^{<\alpha>} \beta(x) dx}{\sigma_1^\alpha} = \beta \frac{\int_{\mathbb{R}^+} (f_2(x)/f_1(x))^p |f_1(x)|^{\alpha} dx}{\sigma_1^\alpha} = \beta \kappa_p,
\]

\[
\varphi_0 = \frac{\int_{\mathbb{R}^+} f_2(x) \beta(x) ln \left| \frac{f_1(x)}{\sqrt{f_2^2(x)+f_2^2(x)}} \right| dx}{\sigma_1} = -\frac{1}{2} \beta \int_{\mathbb{R}}^{+\infty} e^{-\lambda(x-h)} ln(1+e^{2h}) dx = \frac{1}{2} \beta ln(1+e^{2h}).
\]
G Proof of Lemma 3.1

Using the independence between the $X_{j,t}$’s and denoting $X_j = (X_{j,t}, X_{j,t+h})$,

$$
E[e^{iuX_{t}+ivX_{t+h}}] = E\left[ \exp \left\{ iuc \sum_{j=1}^{J} \pi_j X_{j,t} + ivc \sum_{j=1}^{J} \pi_j X_{j,t+h} \right\} \right]
$$

$$
= \prod_{j=1}^{J} E \left[ \exp \left\{ i\langle uc\pi_j, X_j \rangle \right\} \right]
$$

$$
= \prod_{j=1}^{J} \exp \left\{ - \int_{S_2} |\langle uc\pi_j, s \rangle|^\alpha \left( 1 - i \text{sign}(\langle uc\pi_j, s \rangle)w(\alpha, \langle uc\pi_j, s \rangle) \right) \Gamma_{j,h}(ds) \right. \\
\left. \left. + i \langle uc\pi_j, \mu_0 \rangle \right\}, \right.
$$

When $\alpha \neq 1$, then $w(\alpha, \cdot) = \tan(\pi \alpha / 2)$ and

$$
E[e^{iuX_{t}+ivX_{t+h}}] = \exp \left\{ - c^\alpha \sum_{j=1}^{J} \pi_j^\alpha \int_{S_2} |\langle u, s \rangle|^\alpha \left( 1 - i \text{sign}(\langle u, s \rangle)w(\alpha, \langle u, s \rangle) \right) \Gamma_{j,h}(ds) \right. \\
\left. \left. + i \langle uc\pi_j, \mu_0 \rangle \right\}, \right.
$$

When $\alpha = 1$, with $a = 2/\pi$,

$$
E[e^{iuX_{t}+ivX_{t+h}}] = \prod_{j=1}^{J} \exp \left\{ - c \int_{S_2} |\langle u, s \rangle| + ia \langle u, s \rangle \ln |\langle u, s \rangle| \sum_{j=1}^{J} \pi_j \Gamma_{j,h}(ds) \right. \\
\left. \left. + i \sum_{j=1}^{J} \left( \langle u, c\pi_j \mu_0 \rangle - ac\pi_j \ln |c\pi_j| \int_{S_2} \langle u, s \rangle \Gamma_{j,h}(ds) \right) \right\}, \right.
$$

and

$$
i \sum_{j=1}^{J} \left( \langle u, c\pi_j \mu_0 \rangle - ac\pi_j \ln |c\pi_j| \int_{S_2} \langle u, s \rangle \Gamma_{j,h}(ds) \right) = i\langle u, c \sum_{j=1}^{J} \pi_j (\mu_0 - a \ln |c\pi_j| \int_{S_2} s \Gamma_{j,h}(ds)) \rangle \\
= i\langle u, c \sum_{j=1}^{J} \pi_j (\mu_0 - a\sigma_{1,j} \ln |c\pi_j| \beta_{1,j}) \rangle \left( \frac{\lambda_{1,j}}{\lambda_{1,j}} \right). \right\)
H Proof of Lemma [C.2]

H.1 A special manipulation to obtain the fourth derivative

Fourth derivative
Before derivating \( \phi^{(3)}_{X_2|x} \), we follow the advice stated in [14] (p.48) and integrate by parts the terms containing \( \int_{S_2} g_3(t_1 + r s_2)^3 s_1^{-1} \Gamma(ds) \) and \( \int_{S_2} g_3(t_1 + r s_2)^2 s_1^{-2} \Gamma(ds) \), namely \( I_1, I_6 \) and \( I_7 \). This is done in order to guarantee the validity of the representation of the fourth derivative when \( \nu > 4 - \alpha \). If we did not do this step first, the obtained fourth derivative would be valid only when \( \nu > 5 - \alpha \). We obtain

\[
\phi^{(3)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \left[ i \alpha x \left( I_{11} - I_2 + I_{62} - 2I_{72} \right) - x^2 I_{12} + \alpha^2 \left( I_3 - I_4 - 2I_{71} + I_{61} \right) + \alpha(\alpha - 1) \left( I_5 - I_{63} + 2I_{73} \right) \right],
\]

where, in addition to \( I_2, I_3, I_4 \) and \( I_5 \) defined in the Lemma,

\[
I_{11} = \Lambda \left( \int_{S_2} g_2 s_3^3 s_1^{-2} \right) \left( \int_{S_2} g_2 s_1 \right), \quad I_{12} = \Lambda \left( \int_{S_2} g_2 s_3^3 s_1^{-2} \right),
\]

\[
I_{61} = \Lambda \left( \int_{S_2} g_2 s_3^3 s_1^{-2} \right) \left( \int_{S_2} g_2 s_1 \right)^2, \quad I_{71} = \Lambda \left( \int_{S_2} g_2 s_3^2 s_1^{-1} \right) \left( \int_{S_2} g_2 s_1 \right) \left( \int_{S_2} g_2 s_2 \right),
\]

\[
I_{62} = \Lambda \left( \int_{S_2} g_2 s_3^3 s_1^{-2} \right) \left( \int_{S_2} g_2 s_1 \right), \quad I_{72} = \Lambda \left( \int_{S_2} g_2 s_3^2 s_1^{-1} \right) \left( \int_{S_2} g_2 s_2 \right),
\]

\[
I_{63} = \Lambda \left( \int_{S_2} g_2 s_3^3 s_1^{-2} \right) \left( \int_{S_2} g_3 s_1^2 \right), \quad I_{73} = \Lambda \left( \int_{S_2} g_2 s_3^2 s_1^{-1} \right) \left( \int_{S_2} g_3 s_2 s_1 \right).
\]

The fourth derivative is obtained from this representation by techniques similar to those used to get the first and second derivatives.

H.2 Justifying inversion of integral and derivation signs: First derivative

Case \( \alpha \in (0, 1) \)
Assume $\alpha \in (0, 1)$. We begin with the first derivative of the imaginary part of $\phi_{X_2|x}$.

\[
\frac{d}{dr}(\text{Im}\phi_{X_2|x}(r)) = \frac{-1}{2\pi f_{X_1}(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ e^{-\int_{S_2} |t_{s_1+(r+h)s_2}|^\alpha \Gamma(ds)} \sin \left( tx-a \int_{S_2} (ts_1+r+h)s_2)^{<\alpha>\Gamma} (ds) \right] \\
- e^{-\int_{S_2} |t_{s_1+r}s_2|^\alpha \Gamma(ds)} \sin \left( tx-a \int_{S_2} (ts_1+r)s_2)^{<\alpha>\Gamma} (ds) \right] dt
\]

\[
= \frac{-1}{2\pi f_{X_1}(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ \sin \left( tx-a \int_{S_2} (ts_1+(r+h)s_2)^{<\alpha>\Gamma} (ds) \right) \\
- \sin \left( tx-a \int_{S_2} (ts_1+r)s_2)^{<\alpha>\Gamma} (ds) \right) \right] \\
\times \exp \left\{ - \int_{S_2} |ts_1+r+s_2|^\alpha \Gamma(ds) \right\} dt
\]

\[
= \frac{1}{2\pi f_{X_1}(x)} \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ \exp \left\{ - \int_{S_2} |ts_1+(r+h)s_2|^\alpha \Gamma(ds) \right\} \\
- \exp \left\{ - \int_{S_2} |ts_1+r+s_2|^\alpha \Gamma(ds) \right\} \right] \\
\times \sin \left( tx-a \int_{S_2} (ts_1+(r+h)s_2)^{<\alpha>\Gamma} (ds) \right) dt
\]

\[
:= I_1 + I_2. \quad \text{(H.2)}
\]

The integrand of $I_1$ converges to

\[
-\alpha \cos \left( tx-a \int_{S_2} (ts_1+r+s_2)^{<\alpha>\Gamma} (ds) \right) \times \int_{S_2} |ts_1+r+s_2|^{\alpha-1}s_2 \Gamma(ds) \times \exp \left\{ - \int_{S_2} |ts_1+r+s_2|^\alpha \Gamma(ds) \right\}
\]

Using the mean value theorem, the triangle inequality and the inequality $-|x+y|^\alpha \leq -|x|^\alpha + |y|^\alpha$ when $0 < \alpha < 1$, the integrand of $I_1$ can be bounded for any $h$, $|h| < |r|$, by

\[
|\cos(y)| \left| \frac{a}{h} \int_{S_2} |(ts_1+(r+h)s_2)^{<\alpha>\Gamma} -(ts_1+r+s_2)^{<\alpha>\Gamma} | \Gamma(ds) \right| \exp \left\{ \int_{S_2} -|ts_1|^\alpha + |r+s_2|^\alpha \Gamma(ds) \right\}
\leq 2|a|e^{r^\alpha \sigma_2} e^{-\sigma_1^\alpha |r|^\alpha} \int_{S_2} |ts_1+r+s_2|^{\alpha-1}s_2 \Gamma(ds), \quad \text{(H.3)}
\]

where $\sigma_2 = \left( \int_{S_2} s_2 \Gamma(ds) \right)^{1/\alpha}$, $y \in \mathbb{R}$, and we used the bound

\[
\frac{|(ts_1+(r+h)s_2)^{<\alpha>\Gamma} -(ts_1+r+s_2)^{<\alpha>\Gamma}|}{h} \leq 2|ts_1+r+s_2|^{\alpha-1}s_2, \quad \text{(H.4)}
\]

for $ts_1+r+s_2 \neq 0$, which is a consequence of $||1+z|^{<\alpha>\Gamma} - 1| \leq 2|z|$, for $z \in \mathbb{R}$ (see Lemma C.5 (\(\mu\)) below). Bound (H.3) does not depend on $h$ and is integrable with respect to $t$. Indeed, invoking
Lemma C.7 with \( \eta = \alpha - 1 \), \( b = p = 0 \), and (2.2) with \( \nu > 2 - \alpha > 1 - \alpha \)

\[
\left| \int_{\mathbb{R}} e^{-\sigma_1^2 |t|^\alpha} \int_{S_2} |t + \frac{s_2}{s_1}|^{\alpha-1} s_1^{\alpha-1} \Gamma(ds) dt - \int_{\mathbb{R}} \int_{S_2} e^{-\sigma_1^2 |t|^\alpha} |t|^{\alpha-1} s_1^{\alpha-1} \Gamma(ds) dt \right|
\]

\[
\leq \int_{S_2} |s_1|^{\alpha-1} \int_{\mathbb{R}} e^{-\sigma_1^2 |t|^\alpha} \left| t + \frac{s_2}{s_1} \right|^{\alpha-1} - |t|^{\alpha-1} dt \Gamma(ds)
\]

\[
\leq \text{const} \int_{S_2} |s_1|^{\alpha-1+\nu} |s_1|^{-\nu} \Gamma(ds)
\]

\[
\leq \text{const} \int_{S_2} |s_1|^{-\nu} \Gamma(ds)
\]

\[
< +\infty,
\]  

(H.5)

and the integrability with respect to \( t \) follows from the fact that \( \int_{\mathbb{R}} e^{-\sigma_1^2 |t|^\alpha} |t|^{\alpha-1} dt < +\infty \). Hence the Lebesgue dominated convergence theorem applies to \( I_1 \) and we can invert integration and derivation. Focusing on \( I_2 \), its integrand tends to

\[
-\alpha \int_{S_2} (t_1 + r s_2)^{\alpha-1} s_2^{\alpha} \Gamma(ds) \exp \left\{ - \int_{S_2} |t_1 + r s_2|^{\alpha} \Gamma(ds) \right\} \sin \left( t x - a \int_{S_2} |t_1 + r s_2|^{<\alpha>} \Gamma(ds) \right).
\]

Using the inequality

\[
\left| \frac{(t_1 + (r + h) s_2)^\alpha - (t_1 + r s_2)^\alpha}{h} \right| \leq |t_1 + r s_2|^{\alpha-1} |s_2|,
\]

for \( t_1 + r s_2 \neq 0 \), which is a consequence of \(|1 + z|^{\alpha} - 1 |z|, \) for \( z \in \mathbb{R} \) (Lemma C.5(u) below) and the inequality \(|e^{-x} - e^{-y}| \leq e^{-y} |x - y|\), for \( x, y \in \mathbb{R} \), we can bound the integrand of \( I_2 \) for any \(|h| < |r|\) by

\[
\exp \left\{ - \int_{S_2} |t_1 + r s_2|^{\alpha} \Gamma(ds) \right\} \exp \left\{ \left| \int_{S_2} |t_1 + (r + h) s_2|^{\alpha} - |t_1 + r s_2|^{\alpha} \Gamma(ds) \right| \right\}
\]

\[
\times \left| \frac{1}{h} \int_{S_2} |t_1 + (r + h) s_2|^{\alpha} - |t_1 + r s_2|^{\alpha} \Gamma(ds) \right|
\]

\[
\leq e^{2|\alpha| \sigma_2^2} e^{-\sigma_1^2 |t|^\alpha} \int_{S_2} |t + r \frac{s_2}{s_1}|^{\alpha-1} s_1^{\alpha-1} \Gamma(ds).
\]

The integrability with respect to \( t \) is deduced as for (H.5) using Lemma C.7 with \( \eta = \alpha - 1 \), \( b = p = 0 \). Thus, the Lebesgue-dominated convergence theorem applies to \( I_2 \) and we can invert integration and derivation. The real part of \( \phi_{X_2|x}(r) \) can be treated in a similar way, allowing us to derive the under the integral.

**Case \( \alpha \in (1, 2) \)**

Assume \( \alpha \in (1, 2) \). Just as for the case \( \alpha \in (0, 1) \), the imaginary part of \( \phi_{X_2|x} \) is given by (H.2)

\[
\frac{d}{dr} \left( \text{Im} \phi_{X_2|x}(r) \right) = I_1 + I_2.
\]
The integrands of $I_1$ and $I_2$ still converge to the same limits, however a different argument is needed to bound them. For $|h| < |r|$, the mean value theorem, the triangle inequality and the inequality of Lemma C.6 yield the following bound for the integrand of $I_1$

$$\left(\frac{a}{h}\right) \int_{S_2} |(ts_1 + (r + h)s_2)^{\alpha} - (ts_1 + rs_2)^{\alpha}| \Gamma(ds) e^{\alpha\sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |r|^\alpha}, \quad (H.6)$$

where $y \in \mathbb{R}$. By the triangle inequality and the mean value theorem, we have for some $u \in \left(\min (ts_1 + (r + h)s_2, ts_1 + rs_2), \max (ts_1 + (r + h)s_2, ts_1 + rs_2)\right)$

$$\left|\int_{S_2} (ts_1 + (r + h)s_2)^{\alpha} - (ts_1 + rs_2)^{\alpha}\Gamma(ds)\right| = \int_{S_2} \alpha hs_2 |u|^{\alpha-1}\Gamma(ds) \leq \alpha |h| \int_{S_2} |t|^{\alpha-1} + 2|r|^{\alpha-1}\Gamma(ds) \leq \alpha |h| \Gamma(S_2)(|t|^{\alpha-1} + 2|r|^{\alpha-1}) \quad (H.7)$$

Thus, (H.6) can be bounded by

$$\alpha |a| \Gamma(S_2) e^{\alpha\sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |r|^\alpha} (|t|^{\alpha-1} + 2|r|^{\alpha-1}),$$

which is certainly integrable with respect to $t$ on $\mathbb{R}$ for $\alpha > 1$. Let us now turn to $I_2$. We have again by the mean value theorem,

$$\left|\frac{|ts_1 + (r + h)s_2|^{\alpha} - |ts_1 + rs_2|^{\alpha}}{h}\right| \leq \alpha (|t|^{\alpha-1} + 2|r|^{\alpha-1}),$$

if $|h| < |r|$, and thus

$$\left|\frac{e^{-\int_{S_2} |ts_1 + (r + h)s_2|^\alpha \Gamma(ds)} - e^{-\int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(ds)}}{h}\right| \leq \max \left(\frac{e^{-\int_{S_2} |ts_1 + (r + h)s_2|^\alpha \Gamma(ds)} - e^{-\int_{S_2} |ts_1 + rs_2|^\alpha \Gamma(ds)}}{h}\right) \times \int_{S_2} \left|\frac{|ts_1 + (r + h)s_2|^{\alpha} - |ts_1 + rs_2|^{\alpha}}{h}\Gamma(ds)\right| \leq \Gamma(S_2) e^{2\alpha\sigma_2^\alpha} e^{-2^{1-\alpha}\sigma_1^\alpha |r|^\alpha} \alpha (|t|^{\alpha-1} + 2|r|^{\alpha-1}), \quad (H.8)$$

by Lemma C.3 (C.29) and Lemma C.6. The latter bound is again integrable with respect to $t$ on $\mathbb{R}$. Hence the dominated convergence theorem applies to $I_1$, $I_2$ and therefore to $\frac{d}{dr}(\text{Im} \tilde{\phi}_X(r))$ and we can invert the integration and derivation signs. Similar arguments show the dominated convergence theorem applies to the real part of the conditional characteristic function as well.
H.3 Justifying inversion of integral and derivation signs: Second derivative

Case \( \alpha \in (1/2, 1) \)

In an expanded fashion, \( \phi^{(1)}_{X_2|x}(r) \) can be written,

\[
\phi^{(1)}_{X_2|x}(r) = \frac{-\alpha}{2\pi f_{X_1}(x)} \left[ J_1 - aJ_2 - i(J_3 + aJ_4) \right],
\]

with,

\[
J_1(r) = \int_{\mathbb{R}} e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha\gamma} \Gamma(ds) \right) \int_{S_2} (ts_1 + rs_2)^{<\alpha-1>} s_2 \Gamma(ds) dt,
\]

\[
J_2(r) = \int_{\mathbb{R}} e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)} \sin \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha\gamma} \Gamma(ds) \right) \int_{S_2} |ts_1 + rs_2|^{<\alpha-1>} s_2 \Gamma(ds) dt,
\]

\[
J_3(r) = \int_{\mathbb{R}} e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)} \sin \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha\gamma} \Gamma(ds) \right) \int_{S_2} (ts_1 + rs_2)^{<\alpha-1>} s_2 \Gamma(ds) dt,
\]

\[
J_4(r) = \int_{\mathbb{R}} e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds)} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha\gamma} \Gamma(ds) \right) \int_{S_2} |ts_1 + rs_2|^{<\alpha-1>} s_2 \Gamma(ds) dt.
\]

To obtain \( \phi^{(2)}_{X_2|x}(r) \), we will show that the dominated convergence theorem applies to \( J'_1 \). Let us
consider,
\[
J'_1(r) = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ \exp \left\{ - \int_{S_2} |t s_1 + (r + h) s_2|^\alpha \Gamma(ds) \right\} \cos \left( t x - a \int_{S_2} (t s_1 + (r + h) s_2)^{<\alpha>} \Gamma(ds) \right) \right.
\]
\[
\times \int_{S_2} (t s_1 + (r + h) s_2)^{<\alpha-1>} s_2 \Gamma(ds) \right. \\
- \exp \left\{ - \int_{S_2} |t s_1 + r s_2|^\alpha \Gamma(ds) \right\} \cos \left( t x - a \int_{S_2} (t s_1 + r s_2)^{<\alpha>} \Gamma(ds) \right) \right.
\]
\[
\times \int_{S_2} (t s_1 + r s_2)^{<\alpha-1>} s_2 \Gamma(ds) \right] \\ dt
\]
\[
= \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ \exp \left\{ - \int_{S_2} |t s_1 + (r + h) s_2|^\alpha \Gamma(ds) \right\} - \exp \left\{ - \int_{S_2} |t s_1 + r s_2|^\alpha \Gamma(ds) \right\} \right]
\]
\[
\times \cos \left( t x - a \int_{S_2} (t s_1 + (r + h) s_2)^{<\alpha>} \Gamma(ds) \right) \int_{S_2} (t s_1 + r s_2)^{<\alpha-1>} s_2 \Gamma(ds) \\ dt
\]
\[
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \exp \left\{ - \int_{S_2} |t s_1 + (r + h) s_2|^\alpha \Gamma(ds) \right\} \cos \left( t x - a \int_{S_2} (t s_1 + (r + h) s_2)^{<\alpha>} \Gamma(ds) \right) \\
\times \left[ \int_{S_2} (t s_1 + (r + h) s_2)^{<\alpha-1>} s_2 \Gamma(ds) - \int_{S_2} (t s_1 + r s_2)^{<\alpha-1>} s_2 \Gamma(ds) \right] \\ dt
\]
\[
:= K_1 + K_2 + K_3. \tag{H.10}
\]

It can be shown that the dominated convergence theorem applies to \( K_1 \) following the proof in [11] (p.105) for \( I_1 \). Consider \( K_2 \). The integrand converges to
\[
\alpha a \left( \int_{S_2} |t s_1 + r s_2|^{\alpha-1} s_2 \Gamma(ds) \right) \left( \int_{S_2} (t s_1 + r s_2)^{<\alpha-1>} s_2 \Gamma(ds) \right)
\]
\[
\times \sin \left( t x - a \int_{S_2} (t s_1 + r s_2)^{<\alpha>} \Gamma(ds) \right) \exp \left\{ - \int_{S_2} |t s_1 + r s_2|^\alpha \Gamma(ds) \right\}.
\]

Using the mean value theorem, \( (H.4) \) and the triangle inequality, we can bound the integrand for
any $|h| < |r|$ by
\[
\frac{1}{h} \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>} - (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \\
\times |\sin(y)|e^{2|\nu|^\alpha} e^{-|\nu|^\alpha} \int_{S_2} |t + r \frac{s_2}{s_1}|^{\alpha - 1} |s_2||s_1|^{\alpha - 1} \Gamma(ds)
\leq 2e^{2|\nu|^\alpha} \left( \int_{S_2} |t + r \frac{s_2}{s_1}|^{\alpha - 1} \Gamma(ds) \right)^2 e^{-|\nu|^\alpha} 
\] (H.12)

where $y \in \mathbb{R}$. The bound (H.12) does not depend on $h$ and is integrable with respect to $t$: invoking (2.9) Lemma 2.2 in [11],
\[
\left| \int_{\mathbb{R}} \int_{S_2} e^{-\sigma \alpha |t|^\alpha} |t + r \frac{s_2}{s_1}|^{\alpha - 1} \Gamma(ds) \Gamma(ds') \right|
\leq \left( \int_{S_2} \int_{S_2} (s_1)^{\alpha - 1} \int_{\mathbb{R}} e^{-\sigma \alpha |t|^\alpha} \left[ |t + r \frac{s_2}{s_1}|^{\alpha - 1} - |t + r \sigma \frac{s_2}{s_1}|^{\alpha - 1} \right] dt \Gamma(ds) \Gamma(ds') \right)^2
\leq \text{const} \left( \int_{S_2} (s_1)^{\alpha - 1} \Gamma(ds) \right)^2
< +\infty,
\] (H.14)

where const is a constant depending only on $\alpha$ and $\sigma \alpha$. The integrability of (H.12) follows from (H.14), the fact that $\int_{\mathbb{R}} e^{-\sigma \alpha |t|^\alpha} |t|^{2\alpha - 2} dt < +\infty$ and (2.2) with $\nu > 2 - \alpha > 1 - \alpha$. Hence the dominated convergence theorem applies to $K_2$. Let us now turn to $K_3$: «this [a] case when appropriate
"integration by part" is needed (11). With the change of variable \( t' = t + \frac{hs_2}{s_1} \),

\[
K_3 = \lim_{h \to 0} \frac{1}{h} \left[ \int_{\mathbb{R}} \exp \left\{ - \int_{S_2} |ts_1 + (r + h)s_2|^\alpha \Gamma(ds) \right\} \cos \left( tx - a \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>} \Gamma(ds) \right) \times \int_{S_2} \left( t + \frac{hs_2}{s_1} + \frac{rs_2}{s_1} \right)^{<\alpha-1>} s_2 s_1^{<\alpha-1>} \Gamma(ds')dt \right.
\]

\[
- \int_{\mathbb{R}} \exp \left\{ - \int_{S_2} |ts_1 + (r + h)s_2|^\alpha \Gamma(ds) \right\} \cos \left( tx - a \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>} \Gamma(ds) \right) \times \int_{S_2} \left( t + \frac{rs_2}{s_1} \right)^{<\alpha-1>} s_2 s_1^{<\alpha-1>} \Gamma(ds')dt \right]
\]

\[
= \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \frac{1}{h s_2^{s_1}} \left[ \cos \left( t + \frac{hs_2}{s_1} \right) x - a \int_{S_2} \left( t + \frac{hs_2}{s_1} \right) s_1 + (r + h)s_2 \right]^{<\alpha>} \Gamma(ds) \right)
\]

\[
- \cos \left( tx - a \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>} \Gamma(ds) \right) \times \exp \left\{ - \int_{S_2} |ts_1 + (r + h)s_2|^\alpha \Gamma(ds) \right\} \left( t + \frac{rs_2}{s_1} \right)^{<\alpha-1>} s_2 s_1^{<\alpha-2>} \Gamma(ds')dt \right]
\]

\[
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \frac{1}{h s_2^{s_1}} \exp \left\{ - \int_{S_2} \left( t + \frac{hs_2}{s_1} \right) s_1 + (r + h)s_2 \right]^{<\alpha>} \Gamma(ds) \right\} - \exp \left\{ - \int_{S_2} |ts_1 + (r + h)s_2|^\alpha \Gamma(ds) \right\} \times \cos \left( t + \frac{rs_2}{s_1} \right) x - a \int_{S_2} \left( t + \frac{rs_2}{s_1} \right) s_1 + (r + h)s_2 \right]^{<\alpha>} \Gamma(ds) \right)
\]

\[
\times \left( t + \frac{rs_2}{s_1} \right)^{<\alpha-1>} s_2 s_1^{<\alpha-2>} \Gamma(ds')dt \right]
\]

\[
= K_{31} + K_{32}.
\]

The case of \( K_{32} \) is similar to that of \( J_{22} \) in (11) (p.106-108), the dominated convergence theorem.
applies. We focus on $K_{31}$. Its integrand converges to

$$\sin \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<\alpha>} \Gamma(ds) \right) \exp \left\{ - \int_{S_2} |ts_1 + rs_2|^{\alpha} \Gamma(ds) \right\}$$

$$\times \left( x - aa \int_{S_2} |ts_1 + rs_2|^{\alpha-1} s_1 \Gamma(ds) \right) \left( \int_{S_2} (ts_1' + rs_2')^{<\alpha-1>} s_2' s_1'^{-1} \Gamma(ds') \right).$$

Using the mean value theorem and Lemma C.5 (ν), we can bound the integrand of $K_{31}$ for any $|h| < |r|$ by

$$|\sin(y)|e^{2|x|\sigma_2^2} e^{-|t|\sigma_1} \int_{S_2} \left| t + \frac{r^2}{s_2} s_1'^{-1} \right|^\alpha \left( |x| + 2a \left( t + (r + h)s_2 \right) \right)^\alpha \Gamma(ds')$$

$$\leq e^{2|x|\sigma_2^2} e^{-|t|\sigma_1} \int_{S_2} \left| t + \frac{r^2}{s_2} s_1'^{-1} \right|^\alpha \left( |x| + 2a \left( t + (r + h)s_2 \right) \right)^\alpha \Gamma(ds')$$

$$\leq |x|e^{2|x|^2} e^{-|t|^2} \int_{S_2} \left| t + \frac{r^2}{s_2} s_1'^{-1} \right|^\alpha \left( |x| + 2a \left( t + (r + h)s_2 \right) \right)^\alpha \Gamma(ds').$$

The integrability with respect to $t$ of the first (resp. second) term is obtained in the same way as for (H.5) (resp. (H.14)) and concluding using (2.2) with $\nu > 2 - \alpha$. Thus, the dominated convergence theorem applies to $K_{31}$, which finally shows that the dominated convergence theorem applies to $J_1'$. The other $J$’s can be treated in a similar fashion.

**Case $\alpha \in (1, 2)$**

After derivation, $\phi^{(1)}_{X_2|x}(r)$ is given by (H.9) with functions $J$’s of the form

$$\int_{\mathbb{R}} e^{-\int_{S_2} |ts_1 + rs_2|^{<\alpha>} \Gamma(ds)} \sin \left( tx - a \int_{S_2} |ts_1 + rs_2|^{<\alpha>} \Gamma(ds) \right) \int_{S_2} (ts_1 + rs_2)^{<\alpha-1>} \alpha \Gamma(ds) dt,$$

which are similar to deal with. Consider for instance $J_1(r)$. It’s derivative can be written as in (H.11)

$$J_1'(r) = K_1 + K_2 + K_3.$$

For the integrand of $K_1$, we can use (H.8) and the triangle inequality to bound it by

$$\Gamma(S_2)e^{2|x|^2} e^{-2|x|^2} \alpha (|t|^{\alpha-1} + 2|r|^{\alpha-1}) \int_{S_2} |ts_1 + rs_2|^{\alpha-1} s_2 \Gamma(ds).$$
Since 0 < α − 1 < 1, we can further bound it by
\[ \Gamma(S_2)e^{2r|σ|^2}e^{-2}\left[\frac{1}{|σ|^2}\right]t^2α(|t|α-1 + 2|t|α-1)^2, \]
which is integrable with respect to t. The same bound can be obtained for the integrand of \( K_2 \) using the mean value theorem, \([H.7]\) and Lemma \([C.6]\). As for \( K_3 \), there is no need to perform "appropriate integration by parts" since 0 < α − 1 < 1. Its integrand converges to
\[ (α - 1) \exp \left\{ - \int_{S_2} |ts_1 + rs_2|^α \Gamma(ds) \right\} \cos \left( tx - a \int_{S_2} (ts_1 + rs_2)^{<α>\Gamma(ds)} \right) \int_{S_2} |ts_1 + rs_2|^{α-2}s_2^2\Gamma(ds). \]
Using Lemmas \([C.6]\) and \([C.5]\), it can be bounded for any |h| < |r| by
\[ \frac{2}{|h|}\Gamma(S_2)e^{2r|σ|^2}e^{-2}\left[\frac{1}{|σ|^2}\right]t^2 \int_{S_2} |ts_1 + rs_2|^{α-2}|hs_2|\Gamma(ds), \]
\[ \leq \Gamma(S_2)e^{2r|σ|^2}e^{-2}\left[\frac{1}{|σ|^2}\right]t^2 \int_{S_2} \left| t + \frac{rs_2}{s_1} \right|^{α-2}|s_1|^{α-2}\Gamma(ds). \]
We can show that this bound is integrable with respect to t using Lemma \([C.7]\) with \( η = α - 2, b = 0 \) and \( p = 0 \), the fact that \( \int_{\mathbb{R}} e^{-2}\left[\frac{1}{|σ|^2}\right]t^α|t|^{α-2}dt < +∞ \) for \( α ∈ (1, 2) \) and \([2.2]\) with \( ν > 2 - α \). The dominated convergence theorem thus applies and we get
\[ \phi^{(2)}_{X_2|z}(r) = \frac{-α}{2\pi f_{X_1}(x)} - \alpha \int_{\mathbb{R}} e^{-itx}\varphi(t,r) \left( \int_{S_2} g_2(ts_1 + rs_2)s_2\Gamma(ds) \right)^2 dt \]
\[ + (α - 1) \int_{\mathbb{R}} e^{-itx}\varphi(t,r) \left( \int_{S_2} g_3(ts_1 + rs_2)s_2^2\Gamma(ds) \right) dt. \]
with \( g_3(z) = |z|^{α-2} - iαz^{<α-2>} \) for \( z ∈ \mathbb{R} \). Integrating by parts the terms \( |ts_1 + rs_2|^{α-2} \) or \( α-2 \) involved in the expression \( \int_{\mathbb{R}} e^{-itx}\varphi(t,r) \left( \int_{S_2} g_3(ts_1 + rs_2)s_2^2\Gamma(ds) \right) dt \) yields the expression \([C.13]\) obtained in the case \( α ∈ (1/2, 1) \). Hence, the same representation for the second order conditional moment of Proposition \([A.3]\) holds when \( α > 1 \).

### H.4 Justifying inversion of integral and derivation signs: Fourth derivative

Showing that the dominated convergence theorem holds when differentiating \([H.1]\) is the most delicate for the terms: \( I_5, I_{63} \) and \( I_{73} \) - the terms involving the function \( g_3 \), that is, \( |ts_1 + rs_2| \) to the power \( α - 2 \). Arguments and bounds that have already been encountered can be used for the others.

Let us show the dominated convergence theorem applies to \( I_5 \). The cases of \( I_{63} \) and \( I_{73} \) are similar. We decompose \( I_5 \) into terms of the form
\[ \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \text{trig} \left( -tx + a \int_{S_2} (ts_1 + rs_2)^{<α>\Gamma(ds)} \right) e^{-\int_{S_2} |ts_1 + rs_2|^{α}\Gamma(ds)} \]
\[ \times |ts_1 + rs_2|^{α-1} \text{ or } |ts_1 + rs_2|^{α-1} \text{ or } <α-2>|ts_1 + rs_2|^{α-2} \text{ or } <α-2> s_2s_1^{-1}s_2^2\Gamma(ds)\Gamma(ds')dt. \]
Consider for instance

\[ J(r) := \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \cos\left(-tx + a \int_{S_2} (ts_1 + rs_2)^{<\alpha>\Gamma(ds)}\right) e^{-\int_{S_2} [ts_1 + rs_2]^{\alpha}\Gamma(ds)} \times |ts_1 + rs_2|^{-1}|ts_1' + rs_2'|^{-2}s_2^{-1}s_1^{-1}s_2's_1\Gamma(ds)\Gamma(ds')dt. \]

We have

\[
J'(r) = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \int_{S_2} \left[ ts_1' + (r + h)s_2'|^{\alpha-2} - |ts_1' + rs_2'|^{\alpha-2} \right] \times \cos\left(-tx + a \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>\Gamma(ds)}\right) e^{-\int_{S_2} |ts_1 + (r + h)s_2|^{\alpha}\Gamma(ds)} \times |ts_1 + rs_2|^{-1}s_2's_1\Gamma(ds)\Gamma(ds')dt \\
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \int_{S_2} |ts_1' + rs_2'|^{\alpha-2} \times \cos\left(-tx + a \int_{S_2} (ts_1 + (r + h)s_2)^{<\alpha>\Gamma(ds)}\right) e^{-\int_{S_2} |ts_1 + (r + h)s_2|^{\alpha}\Gamma(ds)} \times |ts_1 + rs_2|^{-1}s_2's_1\Gamma(ds)\Gamma(ds')dt \\
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2} \int_{S_2} |ts_1' + rs_2'|^{\alpha-2} \times \cos\left(-tx + a \int_{S_2} (ts_1 + rs_2)^{<\alpha>\Gamma(ds)}\right) e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha}\Gamma(ds)} \times \left[ e^{-\int_{S_2} |ts_1 + (r + h)s_2|^{\alpha}\Gamma(ds)} - e^{-\int_{S_2} |ts_1 + rs_2|^{\alpha}\Gamma(ds)} \right]^{-1}s_2's_1\Gamma(ds)\Gamma(ds')dt \\
:= K_1 + K_2 + K_3 + K_4
\]

The integrand of \( K_1 \) can be bounded using inequality (C.17), (H.8) and invoking Lemma C.7 and (2.2) with \( \nu > 4 - \alpha \). The integrand of \( K_3 \) can be bounded using (H.7), Lemma C.6 and concluding with Lemma C.7 and (2.2) with \( \nu > 4 - \alpha \). Focus now on \( K_2 \). Using Lemmas C.6 and C.5 (i), its integrand can be bounded by

\[
e^{2\nu}e^{-\eta}e^{-2^{\alpha-2-\alpha^2}t^{\alpha-2}s_1^{-1}s_2^{-1}s_1^{-1}}t^{\alpha-2}s_2^{-1}s_1^{-3}s_1^{-1}s_2^{-1}|s_1'|s_2'|.
\]

The later bound does not depend on \( h \) and can be shown to be integrable with respect to \( t \) using (2.2) with \( \nu > 4 - \alpha \), Lemma C.8 with \( \eta = \alpha - 2, z_2 = z_4 = 0, p = 0 \) and the fact that
\[ \int_{\mathbb{R}} e^{-c|t|^\alpha} |t|^{2(\alpha-2)} < +\infty \text{ for } \alpha \in (3/2, 2). \] Let us now turn to the term \( K_1 \) which is more intricate.

Appropriate «integration by parts» is required. With the change of variable \( t = t + \frac{h s'_1}{s_1} \),

\[
K_1 = \lim_{h \to 0} \frac{1}{h} \int_{S_2} \int_{S_2} \int_{\mathbb{R}} \left[ e - \int_{S_2} \left( (t - \frac{h s'_1}{s_1}) s_1 + (r + h)s_2 \right)^{\alpha} \Gamma(ds) \right]
\times \cos \left( (t - \frac{h s'_1}{s_1}) x - a \int_{S_2} \left( (t - \frac{h s'_1}{s_1}) s_1 + (r + h)s_2 \right)^{<\alpha>} \Gamma(ds) \right)
\times \left| (t - \frac{h s'_1}{s_1}) s_1 + (r + h)s_2 \right|^{\alpha-1} - \left| ts'_1 + r s'_2 \right|^{\alpha-1}
\times \left| ts'_1 + r s'_2 \right|^2 s_2 s_1 dt \Gamma(ds) \Gamma(ds')
\]

\[
+ \lim_{h \to 0} \frac{1}{h} \int_{S_2} \int_{S_2} \int_{\mathbb{R}} e - \int_{S_2} \left| ts'_1 + r s'_2 \right|^2 s_2 s_1 dt \Gamma(ds) \Gamma(ds')
\times \left[ \cos \left( (t - \frac{h s'_1}{s_1}) x - a \int_{S_2} \left( (t - \frac{h s'_1}{s_1}) s_1 + (r + h)s_2 \right)^{<\alpha>} \Gamma(ds) \right)
\right.
\left. - \cos \left( ts'_1 + r s'_2 \right)^{<\alpha>} \Gamma(ds) \right]
\times \left| ts'_1 + r s'_2 \right|^{\alpha-1} - \left| ts'_1 + r s'_2 \right|^{\alpha-1}
\times \left| ts'_1 + r s'_2 \right|^2 s_2 s_1 dt \Gamma(ds) \Gamma(ds')
\]

\[ := K_{11} + K_{12} + K_{13}. \]

It can be shown that the generalised Lebesgue convergence theorem applies to the terms \( K_{11} \) and \( K_{12} \) following the proof in [14] (p.50-52). Regarding the integrand of \( K_{13} \), using the mean value
theorem on the cosine, Lemma C.6 and (H.7), we get for $|h| < |r|$

$$
\frac{1}{|h_{s_2}'|} e^{2r|\sigma_2^a e^{-2^{1-\alpha}g}|\alpha|} |ts_1 + (r + h)s_2|^{\alpha-1} |ts_1' + rs_2'|^{\alpha-2} s_2^{-1} |s_2'|^2 \\
\times \left( \frac{h s_2'}{s_1'} x + a \int_{S_2} \left( (t - \frac{h s_2'}{s_1'}) s_1 + (r + h) s_2 \right)^{<\alpha>} - \left( ts_1 + (r + h) s_2 \right)^{<\alpha>} \right) \Gamma(ds) \\
\leq \frac{1}{|h_{s_2}'|} e^{2r|\sigma_2^a e^{-2^{1-\alpha}g}|\alpha|} |ts_1 + (r + h)s_2|^{\alpha-1} |ts_1' + rs_2'|^{\alpha-2} s_2^{-1} |s_2'|^2 \\
\times \left[ \left| \frac{h s_2'}{s_1'} x \right| + \left| \frac{a s_2'}{s_1'} \right| \int_{S_2} |s_1||ts_1 + (r + h) s_2|^{\alpha-1} \Gamma(ds) \right] \\
\leq e^{2r|\sigma_2^a e^{-2^{1-\alpha}g}|\alpha|} |t + \frac{rs_2'}{s_1'} s_2^{-1} s_1'^{\alpha-2} |s_2|^{\alpha-2} \\
\times \left( |t|^{\alpha-1} + |2r|^{\alpha-1} \right) \left[ |x| + |a| \Gamma(S_2)(|t|^{\alpha-1} + |2r|^{\alpha-1}) \right].
$$

The last bound can be shown to be integrable with respect to $t$ using Lemma C.9 with $\eta = \alpha - 2$, $b = 0$, $\alpha - 1$, $2(\alpha - 1)$, $p = 0$ and (2.2) with $\nu > 4 - \alpha$. We established that we can invert the derivation and integration signs in all the $K_i$'s, hence in $J'$. 
I Proof of Theorem A.3

The second order derivative of the characteristic function of $X_2 | X_1 = x$ is given by (C.13) in Lemma C.2. Evaluating it at $r = 0$ yields

$$\mathbb{E}[X_2^2 | X_1 = x] = -\phi^{(2)}_{X_2|x}(0) = -\frac{\alpha}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx + i\alpha \sigma_1 \beta_1 t^{<\alpha>}} e^{-\sigma_1^2 t^2}$$

$$= \frac{\alpha \sigma_1^2}{2\pi f_{X_1}(x)} \int_{\mathbb{R}} e^{-itx + i\alpha \sigma_1 \beta_1 t^{<\alpha>}} e^{-\sigma_1^2 t^2}$$

$$\times \left[ i\alpha \sigma_1 (\kappa_2 t^{<\alpha>} - i\alpha \lambda_2 t^{(\alpha-1)}) - \alpha \sigma_1^2 (\kappa_1 t^{<\alpha>} - i\alpha \lambda_1 t^{(\alpha-1)})^2 
+ \alpha \sigma_1^2 (\kappa_2 t^{<\alpha>} - i\alpha \lambda_2 t^{(\alpha-1)}) (t^{<\alpha> - i\alpha \beta_1 t^{(\alpha-1)}}) \right] dt$$

$$= \frac{\alpha \sigma_1^2}{\pi f_{X_1}(x)} \left[ ax \lambda_2 C_1(x) + \kappa_2 x S_1(x) - \alpha \sigma_1^2 (\kappa_1^2 - a^2 \lambda_1^2 + \alpha \lambda_2 t^{(\alpha-1)}) C_2(x) - \alpha \sigma_1^2 (a(\lambda_2 + \beta_1) - 2a\lambda_1 \kappa_1) S_2(x) \right],$$

where the $\kappa_i$'s and $\lambda_i$'s are given at (A.2). Invoking Lemma C.10 (ii) yields

$$\mathbb{E}[X_2^2 | X_1 = x] = \frac{x}{1 + (a \beta_1)^2} \left[ (a^2 \lambda_2 \beta_1 + \kappa_2)x + a(\lambda_2 - \kappa_2 \beta_1) \frac{1 - x H(x)}{\pi f_{X_1}(x)} \right]$$

$$- \frac{\alpha^2 \sigma_1^2}{\pi f_{X_1}(x)} H(2(\alpha - 1), \theta_1; x)$$

$$= \kappa_2 x^2 + \frac{ax (\lambda_2 - \beta_1 \kappa_2)}{1 + (a \beta_1)^2} \left[ a \beta_1 x + \frac{1 - x H(x)}{\pi f_{X_1}(x)} \right] - \frac{\alpha^2 \sigma_1^2}{\pi f_{X_1}(x)} H(2(\alpha - 1), \theta_1; x),$$

where $H$ is given in (C.8) with

$$\theta_{11} = \kappa_1^2 - a^2 \lambda_1^2 + a^2 \beta_1 \lambda_2 - \kappa_2, \quad \theta_{12} = a(\lambda_2 + \beta_1 \kappa_2) - 2a \lambda_1 \kappa_1.$$
Proof of Theorem A.5

The third order derivative of the characteristic function of $X_2|X_1 = x$ is given by \((\text{C.14})\) in Lemma C.2. It can be shown that the $I$'s evaluated at $r = 0$ write

\[
I_1 = 2\sigma_1^3\mathcal{H}(\alpha - 2, \theta_1^I; x), \quad \theta_1^I = \left(\kappa_3, -a\lambda_3\right),
\]

\[
I_2 = 2\sigma_1^{2\alpha}\mathcal{H}(2(\alpha - 1), \theta_2^I; x), \quad \theta_2^I = \left(L, -aK\right),
\]

\[
i I_3 = 2\sigma_1^{3\alpha}\mathcal{H}(3(\alpha - 1), \theta_3^I; x), \quad \theta_3^I = \left(a\lambda_1(3\kappa_1^2 - a^2\lambda_1^2), \kappa_3^3 - 3a^2\kappa_1\lambda_1^2\right),
\]

\[
i I_4 = 2\sigma_1^{3\alpha}\mathcal{H}(3(\alpha - 1), \theta_4^I; x), \quad \theta_4^I = \left(a\left(K + \beta_1L\right), L - a^2\beta_1K\right),
\]

\[
i I_5 = iI_7 = 2\sigma_1^{2\alpha}\mathcal{H}(2\alpha - 3, \theta_5^I; x), \quad \theta_5^I = \left(aK, L\right),
\]

\[
i I_6 = 2\sigma_1^{2\alpha}\mathcal{H}(2\alpha - 3, \theta_6^I; x), \quad \theta_6^I = \left(a(\lambda_3 + \beta_1\kappa_3), \kappa_3 - a^2\beta_1\lambda_3\right),
\]

with $K = \kappa_1\lambda_2 + \lambda_1\kappa_2$ and $L = \kappa_1\kappa_2 - a^2\kappa_1\lambda_2$. Hence,

\[
\mathbb{E}[X_2^3|X_1 = x] = -i\phi_{X_2|X_1}(0) = \frac{\alpha}{\pi f_{X_1}(x)} \left[-x \left((\alpha - 1)K_1 - aK_2\right) + \alpha^2K_3 + \alpha(\alpha - 1)K_4\right],
\]

with

\[
K_1 = \sigma_1^\alpha\mathcal{H}(\alpha - 2, \theta_1^K; x), \quad \text{with } \theta_1^K = \theta_1^I,
\]

\[
K_2 = \sigma_1^{2\alpha}\mathcal{H}(2(\alpha - 1), \theta_2^K; x), \quad \text{with } \theta_2^K = \theta_2^I,
\]

\[
K_3 = \sigma_1^{3\alpha}\mathcal{H}(3(\alpha - 1), \theta_3^K; x), \quad \text{with } \theta_3^K = \theta_3^I - \theta_4^I.
\]

\[
K_4 = \sigma_1^{2\alpha}\mathcal{H}(2\alpha - 3, \theta_4^K; x), \quad \text{with } \theta_4^K = \theta_6^I - \theta_5^I.
\]

Invoking Lemma \((\text{C.10}(\alpha))\) for $n = 1, 2$ and regrouping the terms, we get

\[
\mathbb{E}[X_2^3|X_1 = x] = \frac{\alpha x^2\sigma_1^\alpha}{\pi f_{X_1}(x)} \left(\theta_{12}C_1(x) - \theta_{11}S_1(x)\right)
\]

\[
+ \frac{\alpha}{\pi f_{X_1}(x)} \left[\frac{\alpha x\sigma_1^{2\alpha}}{2}C_2(x)\left(-2\left(\theta_{11}K_{11} + a\beta_1\theta_{12}K_{12}\right) + 2\theta_{21}K_{21} - \theta_{42}K_{42}\right)\right.
\]

\[
+ \frac{\alpha x\sigma_1^{2\alpha}}{2}S_2(x)\left(-2\left(\theta_{12}K_{12} - a\beta_1\theta_{11}K_{11}\right) + 2\theta_{22}K_{22} + \theta_{41}K_{41}\right)
\]

\[
+ \frac{\alpha^2\sigma_1^{3\alpha}}{2}C_3(x)\left(2\theta_{31}K_{31} + \theta_{41}K_{41} + a\beta_1\theta_{42}K_{42}\right)
\]

\[
+ \frac{\alpha^2\sigma_1^{3\alpha}}{2}S_3(x)\left(2\theta_{32}K_{32} + \theta_{42}K_{42} - a\beta_1\theta_{41}K_{41}\right)\right].
\]
Using Lemma C.10 \((iii)\) yields the conclusion with \(\theta_2 = (\theta_{21}, \theta_{22}), \theta_3 = (\theta_{31}, \theta_{32})\) such that
\[
\theta_{21} = 3(L + a^2\beta_1\lambda_3 - \kappa_3), \\
\theta_{22} = 3a(\lambda_3 + \beta_1\kappa_3 - K), \\
\theta_{31} = a\left(\lambda_3(1 - a^2\beta_1^2) + 2\beta_1\kappa_3 + 2\lambda_1(3\kappa_1^2 - a^2\lambda_1^2) - 3(K + \beta_1L)\right), \\
\theta_{32} = \kappa_3(1 - a^2\beta_1^2) - 2a^2\beta_1\lambda_3 + 2(\kappa_1^3 - 3a^2\kappa_1\lambda_1^2) + 3(a^2\beta_1K - L),
\]
with \(K = \kappa_1\lambda_2 + \kappa_2\lambda_1, \ L = \kappa_1\kappa_2 - a^2\lambda_1\lambda_2.\)

**K Proof Theorem A.4**

Let \(X = (X_1, X_2)\) be an \(\alpha\)-stable vector with \(\alpha = 1\) and spectral representation \((\Gamma, 0)\). Its characteristic function, denoted \(\varphi_X(t, r)\) for any \((t, r) \in \mathbb{R}^2\), reads
\[
\varphi_X(t, r) = \exp\left\{ -\int_{S_2} |ts_1 + rs_2| + ia(ts_1 + rs_2) \ln|ts_1 + rs_2| \Gamma(ds) \right\}, \tag{K.1}
\]
with \(a = 2/\pi\). The conditional characteristic function of \(X_2\) given \(X_1 = x\), denoted \(\phi_{X_2|x}(r)\) for \(r \in \mathbb{R}\), is still given by (C.7).

**Lemma K.1** Let \((X_1, X_2)\) be an \(\alpha\)-stable random vector with \(\alpha = 1\) and spectral representation \((\Gamma, 0)\). If (2.2) holds with \(\nu > 0\), the first derivative of \(\phi_{X_2|x}\) is given by
\[
\phi_{X_2|x}(r) = \frac{-1}{2\pi f_{X_1}(x)} \left( A_1 + iaA_2 \right),
\]
with
\[
A_1 = \int_{\mathbb{R}} e^{-itx} \varphi_X(t, r) \left( \int_{S_2} s_2(ts_1 + rs_2)^{<0>} \Gamma(ds) \right) dt, \tag{K.2}
\]
\[
A_2 = \int_{\mathbb{R}} e^{-itx} \varphi_X(t, r) \left( \int_{S_2} s_2(1 + \ln|ts_1 + rs_2|) \Gamma(ds) \right) dt \tag{K.3}
\]
If (2.2) holds with \(\nu > 1\), the second derivative of \(\phi_{X_2|x}\) is given by
\[
\phi_{X_2|x}^{(2)}(r) = \frac{-1}{2\pi f_{X_1}(x)} \left( -B_1 + ixB_2 + B_3 \right), \tag{K.4}
\]

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where,

\[ B_1 = \int_{\mathbb{R}} e^{-itx} \varphi X(t, r) \left( \int_{S_2} s_2(t_{s_1} + r_{s_2})^{<0>} + ias_2(1 + \ln |ts_1 + rs_2|\Gamma(ds) \right)^2 dt, \]

\[ B_2 = \int_{\mathbb{R}} e^{-itx} \varphi X(t, r) \left( \int_{S_2} (t_{s_1} + r_{s_2})^{<0>} + i(1 + \ln |ts_1 + rs_2|) s_2^2 s_1^{-1} \Gamma(ds) \right) dt, \]

\[ B_3 = \int_{\mathbb{R}} e^{-itx} \varphi X(t, r) \left( \int_{S_2} s_1(t_{s_1} + r_{s_2})^{<0>} + ias_1(1 + \ln |ts_1 + rs_2|\Gamma(ds) \right) \times \left( \int_{S_2} (t_{s_1} + r_{s_2})^{<0>} + i(1 + \ln |ts_1 + rs_2|) s_2^2 s_1^{-1} \Gamma(ds) \right) dt. \]

**K.1 Justifying inversion of integral and derivative signs**

**First derivative**

The terms depending on \( r \) in the right-hand side of (K.1) are of the form (omitting the factor \( 1/2\pi f_{X_1}(x) \))

\[ \int_{\mathbb{R}} e^{-\int_{S_2} |t_{s_1} + r_{s_2}|\Gamma(ds)} \text{trig} \left( -tx - a \int_{S_2} (t_{s_1} + r_{s_2}) \ln |ts_1 + rs_2|\Gamma(ds) \right) dt. \]

Consider for instance the term obtained by replacing \( \text{trig} \) by the cosine function, denoted \( I_1 \).

\[ I'_1(r) = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ e^{-\int_{S_2} |t_{s_1} + (r+h)_{s_2}|\Gamma(ds)} - e^{-\int_{S_2} |t_{s_1} + r_{s_2}|\Gamma(ds)} \right] \times \cos \left( tx + a \int_{S_2} (t_{s_1} + (r+h)_{s_2}) \ln |ts_1 + (r+h)_{s_2}|\Gamma(ds) \right) dt 
\]

\[ + \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} e^{-\int_{S_2} |t_{s_1} + r_{s_2}|\Gamma(ds)} \left[ \cos \left( tx + a \int_{S_2} (t_{s_1} + (r+h)_{s_2}) \ln |ts_1 + (r+h)_{s_2}|\Gamma(ds) \right) 
\]

\[ - \cos \left( tx + a \int_{S_2} (t_{s_1} + r_{s_2}) \ln |ts_1 + r_{s_2}|\Gamma(ds) \right) \right] dt \]

\[ := I_{11} + I_{12} \]

The integrand of \( I_{11} \) converges to

\[ -e^{-\int_{S_2} |t_{s_1} + r_{s_2}|\Gamma(ds)} \cos \left( tx + a \int_{S_2} (t_{s_1} + r_{s_2}) \ln |ts_1 + r_{s_2}|\Gamma(ds) \right) \int_{S_2} s_2(t_{s_1} + r_{s_2})^{<0>} \Gamma(ds). \]

Using (C.30) we can bound the integrand of \( I_{11} \) by

\[ \frac{1}{|h|} \left| \int_{S_2} |t_{s_1} + (r+h)_{s_2}| - |t_{s_1} + r_{s_2}|\Gamma(ds) \right| e^{-\int_{S_2} |t_{s_1} + r_{s_2}|\Gamma(ds)} e^{\int_{S_2} |t_{s_1} + (r+h)_{s_2}| - |t_{s_1} + r_{s_2}|\Gamma(ds)}. \]

By Lemma C.5 (i) and the triangle inequality, we can further bound it for \( |h| < |r| \) by

\[ \sigma_2 e^{\sigma_2 (1 + |r|) - \sigma_1 |t|}, \]

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which does not depend on \( h \) and is integrable with respect to \( t \) on \( \mathbb{R} \). The dominated convergence theorem applies to \( I_{11} \). Turning to \( I_{12} \), its integrand converges to

\[
-ae^{-\int_{S_2}|ts_1+rs_2|\Gamma(ds)} \sin \left( tx + a \int_{S_2} (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right) \int_{S_2} s_2 (1 + \ln |ts_1 + rs_2|) \Gamma(ds).
\]

Using the mean value theorem on the cosine, its integrand can be bounded by

\[
\frac{a}{|h|} \int_{S_2} |ts_1+rs_2| \Gamma(ds) \left| (ts_1 + (r + h)s_2) \ln |ts_1 + (r + h)s_2| - (ts_1 + rs_2) \ln |ts_1 + rs_2| \right| \Gamma(ds)
\]

\[
\leq ae^{\sigma_2|v| - \sigma_1|t|} \frac{1}{|h|} \int_{S_2} |ts_1 + (r + h)s_2) \ln |ts_1 + (r + h)s_2| - (ts_1 + rs_2) \ln |ts_1 + rs_2| \right| \Gamma(ds)
\]

\[
:= ae^{\sigma_2|v| - \sigma_1|t|} \left( Q_1 + Q_2 \right),
\]

where the two terms \( Q_1 \) and \( Q_2 \) involve integrals over \( S_2 \cap \{ s : |ts_1 + rs_2| \geq 2|h| \} \) and \( S_2 \cap \{ s : |ts_1 + rs_2| < 2|h| \} \). Focus on \( Q_2 \). Introduce the function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) defined for any \( z \geq 0 \) by

\[
f(z) = z|\ln z|.
\]

It is such that \( f(0) = 0 \) and for \( z \) small enough \((0 < z < e^{-1})\), \( f \) is monotone increasing. Since \(|ts_1 + rs_2| < 2|h|\), we also have \(|ts_1 + (r + h)s_2| < 3|h|\). Thus, for \( 0 < |h| < (3e)^{-1} \), the integrand of \( Q_2 \) can be bounded by

\[
|h|^{-1} \left( |f(3|h|)| + |f(2|h|)| \right) \leq 2|h|^{-1} |f(3|h|)| \leq 6 |\ln 3h|
\]

Using Lemma \([L.1] \) we can bound the latter quantity for any \( v > 0 \) by

\[
6v^{-1} \left( 2 + |3h|^v + |3h|^{-v} \right).
\]

From \(|ts_1 + rs_2|/2 < |h| < (3e)^{-1}\), we deduce that \(|3h|^{-v} < \left( 3|ts_1 + rs_2|/2 \right)^{-v}\) and

\[
6v^{-1} \left( 2 + |3h|^v + |3h|^{-v} \right) \leq 6v^{-1} \left( 2 + e^{-v} + \left( 3|ts_1 + rs_2|/2 \right)^{-v} \right) \leq \text{const}_1 + \text{const}_2 |ts_1 + rs_2|^{-v},
\]

for some nonnegative constants \( \text{const}_1 \) and \( \text{const}_2 \). Hence, the term involving \( Q_2 \) in \([K.5] \) can be further bounded for any \( v > 0 \) by

\[
ae^{\sigma_2|v| - \sigma_1|t|} \left( \text{const}_1 + \text{const}_2 \int_{S_2} \left| t + \frac{rs_2}{s_1} \right|^{-v} |s_1|^{-v} \Gamma(ds) \right).
\]

The term with \( \text{const}_1 \) is clearly integrable with respect to \( t \) on \( \mathbb{R} \). Letting \([2.2] \) hold with \( \nu > 0 \), choose some \( v \in (0, \min(\nu, 1)) \). We show that the second term is bounded by an integrable function of \( t \) as we did in Equation \([H.5] \) using Lemma \([C.7] \) with \( \eta = v, b = 0, p = 0 \), the fact that \( \int_{\mathbb{R}} e^{-\sigma_1|t|} |t|^{-\nu} dt < +\infty \) and \([2.2] \) with \( \nu > v > 0 \). There remains to be bounded the part involving
$Q_1$ in (K.5). For this term, we apply the mean value theorem to the function $z \mapsto z \ln |z|$ and get that

$$|h|^{-1} \left| (ts_1 + (r + h)s_2) \ln |ts_1 + (r + h)s_2| - (ts_1 + rs_2) \ln |ts_1 + rs_2| \right|$$

$$\leq |h|^{-1} |h s_2| 1 + \ln |u|$$

$$\leq 1 + |\ln |u||,$$

for some $u \in [ts_1 + (r + h)s_2 \wedge ts_1 + rs_2, ts_1 + (r + h)s_2 \vee ts_1 + rs_2]$. Since $Q_1$ is an integral over $S_2 \cap \{s : |ts_1 + rs_2| \geq 2|h|\}$, we have $|u| \in \left[ \frac{|ts_1 + rs_2|}{2}, 2|ts_1 + rs_2| \right]$, and because of the quasi-convexity of the function $z \mapsto |\ln |z||$, we can bound the above term by

$$1 + \left| \ln \frac{ts_1 + rs_2}{2} \right| + \ln |2(ts_1 + rs_2)| \leq \text{const} + 2 \ln |ts_1 + rs_2|.$$ 

Using Lemma L.1, we can bound this term for any $v > 0$ by

$$\text{const} + 2v^{-1} \left( 2 + |ts_1 + rs_2|^v + |ts_1 + rs_2|^{-v} \right) \leq \text{const}_1 + \text{const}_2 |t|^v + \text{const}_3 t + \frac{rs_2}{s_1} |s_1|^{-v}$$

Hence, the term in (K.5) involving $Q_1$ can be bounded for any $v > 0$ by

$$ae^{\sigma_2 v - \sigma_1 |t|} \left( \text{const}_1 + \text{const}_2 |t|^v + \text{const}_3 \int_{S_2} t + \frac{rs_2}{s_1} |s_1|^{-v} \Gamma(ds) \right).$$

(K.7)

which can be shown to be integrable with respect to $t$ on $\mathbb{R}$ as we did above for the term with $Q_2$. The dominated convergence theorem applies to $I_{12}$ and thus to $I_1$. We can derivate $\phi_{X_{2|t}}$ under the integral sign.

**Second derivative**

Let us start with $A_2$, which is the most delicate. It is composed of terms of the form

$$\int_{\mathbb{R}} e^{-\int_{S_2} |ts_1 + rs_2| \Gamma(ds)} \text{trig} \left( -tx - a \int_{S_2} (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right)$$

$$\times \left( \int_{S_2} s_2(1 + \ln |ts_1 + rs_2|) \Gamma(ds) \right) dt,$$
where «trig» stands for sine or cosine. Denoting the one with cosine as $K_2$, we have

$$K_2 = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \left[ e^{-\int_{s_2} |ts_1+(r+h)s_2|\Gamma(ds)} - e^{-\int_{s_2} |ts_1+rs_2|\Gamma(ds)} \right]$$

$$\times \cos \left( tx + a \int_{s_2} (ts_1 + (r + h)s_2) \ln |ts_1 + (r + h)s_2|\Gamma(ds) \right)$$

$$\times \left( \int_{s_2} s_2(1 + \ln |ts_1 + (r + h)s_2|)\Gamma(ds) \right) dt$$

$$+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} e^{-\int_{s_2} |ts_1+rs_2|\Gamma(ds)} \left[ \cos \left( tx + a \int_{s_2} (ts_1 + (r + h)s_2) \ln |ts_1 + (r + h)s_2|\Gamma(ds) \right) \right.$$}

$$\left. - \cos \left( tx + a \int_{s_2} (ts_1 + rs_2) \ln |ts_1 + rs_2|\Gamma(ds) \right) \right]$$

$$\times \left( \int_{s_2} s_2(1 + \ln |ts_1 + rs_2|)\Gamma(ds) \right) dt$$

$$+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} e^{-\int_{s_2} |ts_1+rs_2|\Gamma(ds)} \cos \left( tx + a \int_{s_2} (ts_1 + rs_2) \ln |ts_1 + rs_2|\Gamma(ds) \right)$$

$$\times \left[ \int_{s_2} s_2 \ln |ts_1 + (r + h)s_2| - s_2 \ln |ts_1 + rs_2|\Gamma(ds) \right] dt$$

$$:= K_{21} + K_{22} + K_{23}.$$ 

The integrand of $K_{21}$ converges to

$$- e^{-\int_{s_2} |ts_1+rs_2|\Gamma(ds)} \cos \left( tx + a \int_{s_2} (ts_1 + rs_2) \ln |ts_1 + rs_2|\Gamma(ds) \right)$$

$$\times \left( \int_{s_2} s_2(t_1 + rs_2)^{0>\Gamma(ds)} \right) \left( \int_{s_2} s_2(1 + \ln |ts_1 + rs_2|)\Gamma(ds) \right).$$

Using (C.30), the triangle inequality and (C.6), it can be bounded by

$$\sigma_2 e^{\sigma_2(1+|r|)-\sigma_1|t|} \int_{s_2} |s_2|^{1 + \ln |ts_1 + (r + h)s_2|\Gamma(ds)}.$$  (K.8)

The integrand of the above expression can be bounded using Lemma [L.1] for any $v > 0$ by

$$1 + v^{-1} \left( 2 + |ts_1 + (r + h)s_2|^v + |ts_1 + (r + h)s_2|^{-v} \right)$$

$$\leq \text{const}_1 + \text{const}_2|t|^v + \text{const}_3 \left| t + \frac{(r + h)s_2}{s_1} \right|^v |s_1|^{-v}.$$ 

hence, (K.8) is bounded by

$$\sigma_2 e^{\sigma_2(1+|r|)-\sigma_1|t|} \left( \text{const}_1 + \text{const}_2|t|^v + \text{const}_3 \int_{s_2} \left| t + \frac{(r + h)s_2}{s_1} \right|^v |s_1|^{-v} \Gamma(ds) \right).$$

The terms involving const$_1$ and const$_2$ are clearly integrable with respect to $t$. The last term is more intricate as it still depends on $h$. We will show that the generalised Lebesgue dominated
convergence theorem (Theorem 19, p.89 in [38]) applies. Denoting
\[ T(h) = e^{-\sigma_1|t| \left| t + \frac{(r + h)s_2}{s_1} \right|^v |s_1|^{-v}}, \]
it can be shown that \( T(0) \) is integrable with respect to \( t \) on \( \mathbb{R} \) and \( \Gamma \) on \( S_2 \) invoking the usual arguments. Also, choosing some \( v \in (0, 1) \), with have by Lemma C.9 with \( \eta = -v \), \( b = 0 \) and \( 0 < p < 1 - v \),
\[
\left| \int T(h) - T(0) \right| \leq \int_{S_2} |s_1|^{-v} \left| t + \frac{(r + h)s_2}{s_1} \right|^v \left| t + \frac{r s_2}{s_1} \right|^{-v} dt \Gamma(ds)
\]
\[
\leq \text{const} \int_{S_2} |s_1|^{-v} \left| \frac{h s_2}{s_1} \right|^p \Gamma(ds)
\]
\[
\leq \text{const} \left| h \right|^p \int_{S_2} |s_1|^{-v - p} \Gamma(ds) \xrightarrow{h \to 0} 0,
\]
because (2.2) holds with \( \nu > 1 \) and \( v + p < v + 1 - v < 1 \). Since \( T(0) \) is integrable and \( \lim_{h \to 0} \int T(h) = \int T(0) \), the generalised dominated convergence theorem applies to \( K_{21} \). We turn to \( K_{22} \). Its integrand converges to
\[
-ae^{-\int_{S_2} |ts_1 + rs_2| \Gamma(ds)} \sin \left( tx + a \int_{S_2} (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right)
\]
\[
\times \left( \int_{S_2} s_2 (1 + \ln |ts_1 + rs_2|) \Gamma(ds) \right)^2.
\]
With the usual inequalities and Lemma L.1, it can be bounded for any \( v > 0 \) by
\[
\frac{a}{|h|} e^{\sigma_2|r| - \sigma_1|t|} \left| \int_{S_2} (ts_1 + (r + h)s_2) \ln |ts_1 + (r + h)s_2| - (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right|
\]
\[
\times \left| \int_{S_2} s_2 (1 + \ln |ts_1 + (r + h)s_2|) \Gamma(ds) \right|
\]
\[
\leq ae^{\sigma_2|r| - \sigma_1|t|} (Q_1 + Q_2) \left( \sigma_2 + \int_{S_2} \ln |ts_1 + (r + h)s_2| \Gamma(ds) \right)
\]
\[
\leq ae^{\sigma_2|r| - \sigma_1|t|} (Q_1 + Q_2) \left( \text{const}_1 + \text{const}_2 |t|^v + \text{const}_3 \int_{S_2} \left| t + \frac{(r + h)s_2}{s_1} \right|^{-v} \left| s_1 \right|^{-v} \Gamma(ds) \right),
\]
where, similarly to (K.5), the two terms \( Q_1 \) and \( Q_2 \) involve integrals over \( S_2 \cap \{ s : |ts_1 + rs_2| \geq 2|h| \} \) and \( S_2 \cap \{ s : |ts_1 + rs_2| < 2|h| \} \). After expansion, the terms with \( \text{const}_1 \) and \( \text{const}_2 \) are readily dealt with by following the method developed for (K.5). Focus on the remaining term
\[
a \int_{S_2} e^{\sigma_2|r| - \sigma_1|t|} (Q_1 + Q_2) \left| t + \frac{(r + h)s_2}{s_1} \right|^{-v} \left| s_1 \right|^{-v} \Gamma(ds).
\]
In view of the bounds (K.6) and (K.7), the integrand can be bounded (up to a multiplicative constant) by
\[
U(h) = e^{-\sigma_1|t|} \left| t + \frac{rs_2}{s_1} \right|^{-v} \left| t + \frac{(r + h)s_2}{s_1} \right|^{-v} \left| s_1 \right|^{-v} \left| s_1 \right|^{-v}.
\]
Choosing some \( v \in (0, 1/2) \), we can invoke Lemma (C.8) with \( \eta = -v \), \( p = 0 \) and the fact that 
\[
\int_{\mathbb{R}} e^{-\sigma_1|t|} |t|^{-2v} dt < +\infty
\]
to show that \( U(0) \) is integrable on the one hand. On the other hand we can again invoke Lemma (C.8), this time with \( \eta = -v \), \( 0 < p < 1 - 2v \), and the fact that \( (2.2) \) holds with \( \nu > 1 > v + 1 - 2v > v + p \) to show that \( \int U(h) \rightarrow \int U(0) \). The generalised dominated convergence theorem applies to \( K_{12} \).
We turn to \( K_{23} \) for which «appropriate integration by parts» is required. After obvious manipulations,

\[
K_{23} = \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2^2} s_2' \ln |ts_1' + r s_2'| \left[ e^{-\int_{S_2^2} \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2} \Gamma(ds) - e^{-\int_{S_2^2} |ts_1 + r s_2| \Gamma(ds)} \right] \\
\times \cos \left( t - \frac{hs_2'}{s_1'} \right) x + a \int_{S_2^2} \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2 \ln \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2 \Gamma(ds) \right) \Gamma(ds')
\]

\[
+ \lim_{h \to 0} \frac{1}{h} \int_{\mathbb{R}} \int_{S_2^2} s_2' \ln |ts_1' + r s_2'| e^{-\int_{S_2^2} |ts_1 + r s_2| \Gamma(ds)} \\
\times \left[ \cos \left( t - \frac{hs_2'}{s_1'} \right) x + a \int_{S_2^2} \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2 \ln \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2 \Gamma(ds) \right] \Gamma(ds')
\]

\[
:= L_1 + L_2.
\]

Starting with \( L_1 \), its integrand converges to

\[
e^{-\int_{S_2^2} |ts_1 + r s_2| \Gamma(ds)} \cos \left( t x + a \int_{S_2^2} (ts_1 + r s_2) \ln |ts_1 + r s_2| \Gamma(ds) \right)
\]

\[
\times \left( \int_{S_2^2} s_1 (ts_1 + r s_2)^{<0>} \Gamma(ds) \right) \left( \int_{S_2^2} \ln |ts_1 + r s_2| s_2^{-1} \Gamma(ds) \right)
\]

It can be bounded using (C.29) and Lemma C.5 \((v)\) by

\[
\left| s_2' \ln |ts_1' + r s_2'| \right| \exp \left\{ - \min \left( \int_{S_2^2} \left| \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2 \right| \Gamma(ds), \int_{S_2^2} |ts_1 + r s_2| \Gamma(ds) \right) \right\}
\]

\[
\times \left| \int_{S_2^2} \left( t - \frac{hs_2'}{s_1'} \right)s_1 + r s_2 \Gamma(ds) \right| - |ts_1 + r s_2| \Gamma(ds)
\]

\[
\leq e^{\sigma_2|v|} \exp \left\{ - \sigma_1 \min \left( \left| t - \frac{hs_2'}{s_1'} \right| |t| \right) \right\} \left| s_2' \ln |ts_1' + r s_2'| \right| \left| \frac{1}{h} \int_{S_2^2} \frac{hs_2'}{s_1'} s_1 \Gamma(ds) \right|
\]

\[
\leq \sigma_1 e^{\sigma_2|v|} \exp \left\{ - \sigma_1 \min \left( \left| t - \frac{hs_2'}{s_1'} \right| |t| \right) \right\} \left| \ln |ts_1' + r s_2'| \right| \left| s_2' \right| s_1^{-1}
\]

\[
:= V(h).
\]
We follow a similar procedure as the one used in [14] (p.51) to deal with the min inside the exponential. Focus on the case $h s_2 / s_1 > 0$ (the converse case is similar). We have

$$\min\left(\left|t - \frac{h s_2}{s_1}\right|, |t|\right) = \begin{cases} \left|t - \frac{h s_2}{s_1}\right|, & \text{if } t \geq h s_2 / 2 s'_1, \\ |t|, & \text{if } t < h s_2 / 2 s'_1. \end{cases}$$

Thus, up to a multiplicative constant,

$$\int_{\mathbb{R}} V(h) dt = \int_{\mathbb{R}} e^{-\sigma_1 |t-h s_2 / s_1|} |\ln| t_1 + r s_2 | || s_2^2 | s_1 |^{-1} dt + \int_{-\infty}^{+\infty} e^{-\sigma_1 |t-h s_2 / s_1|} |\ln| t_1 + r s_2 | || s_2^2 | s_1 |^{-1} dt$$

$$= \int_{-\infty}^{+\infty} e^{-\sigma_1 |t|} |\ln| t_1 + r s_2 + \frac{h s_2}{s_1} | || s_2^2 | s_1 |^{-1} dt + \int_{-\infty}^{+\infty} e^{-\sigma_1 |t|} |\ln| t_1 + r s_2 | || s_2^2 | s_1 |^{-1} dt$$

$$= \int_{\mathbb{R}} e^{-\sigma_1 |t|} \left[ |\ln| t_1 + (r + h) s_2 | || s_2^2 | s_1 |^{-1} dt + |\ln| t_1 + r s_2 | || s_2^2 | s_1 |^{-1} dt. \right]$$

Thus, using Lemma A.1, we can bound the integrand for any $v > 0$ and $|h| < |r|$ by

$$e^{-\sigma_1 |t|} \left[ |\ln| t_1 + (r + h) s_2 | || s_2^2 | s_1 |^{-1} \right] \leq v^{-1} e^{-\sigma_1 |t|} \left[ \text{const}_1 + \text{const}_2 |t|^v \right.$$

$$+ \text{const}_3 |t + \frac{r s_2}{s_1}|^{-v} |s_1|^{-v} + \text{const}_4 |t + \frac{(r + h) s_2}{s_1}|^{-v} |s_1|^{-v} \left] |s_2^2 | s_1 |^{-1}. \right.$$

Clearly, the terms involving $\text{const}_1$ and $\text{const}_2$ are integrable with respect to $t$ and $\Gamma$. Denoting the last term as $V_4(h) := e^{-\sigma_1 |t|} |t + \frac{(r + h) s_2}{s_1}|^{-v} |s_2^2 | s_1 |^{-1-v}$, we show that the generalised dominated convergence theorem applies. As (2.2) holds for some $\nu > 1$, choose $v = \nu - \frac{1}{2} > 0$ if $\nu < 2$, and some $v \in (0, 1)$ if $\nu \geq 2$. The integrability of $V_4(0)$ (and at the same time, of the term involving $\text{const}_3$) is obtained from Lemma C.7 with $\eta = -v$, $b = 0$, $p = 0$ and the fact that $\int_{\mathbb{R}} e^{-\sigma_1 |t|} |t|^{-v} dt < \infty$. Doing so indeed yields

$$\left| \int_{S_2} |s_2^2 | s_1 |^{-1-v} \int_{\mathbb{R}} e^{-\sigma_1 |t|} |t + \frac{r s_2}{s_1}|^{-v} - |t|^{-v} |s_2^2 | s_1 |^{-1-v} dt \right| \Gamma(ds)$$

$$\leq \int_{S_2} \int_{\mathbb{R}} e^{-\sigma_1 |t|} \left|t + \frac{r s_2}{s_1}|^{-v} - |t|^{-v} \right| dt \Gamma(ds)$$

$$\leq \text{const} \int_{S_2} |s_1|^{-v} |s_1|^{\nu-1-v} \Gamma(ds)$$

$$\leq \text{const} \int_{S_2} |s_1|^{-\nu} \Gamma(ds)$$

$$< +\infty,$$
since \( \nu - 1 - v = \frac{\nu - 1}{2} > 0 \) if \( \nu \in (1, 2) \) and \( \nu - 1 - v > \nu - 2 > 0 \) if \( \nu \geq 2 \). The convergence 
\[ \int V_4(h) \to \int V_4(0) \] can be obtained from Lemma C.9 with \( \eta = -v, b = 0 \) and \( 0 < p < v \). The generalised dominated convergence hence applies to \( L_1 \).

We turn to \( L_2 \). Its integrand converges to

\[
e^{-\int_{s_2}^{s_2 + rs_2} \frac{\Gamma(ds)}{\Gamma(ds)}} \sin \left( tx + a \int_{s_2}^{s_2 + rs_2} (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right)
\]

\[
\times \left( x + a \int_{s_2}^{s_2 + rs_2} s_1(1 + \ln |ts_1 + rs_2|) \Gamma(ds) \right) \ln |ts_1 + rs_2|^{2s_2^2 s_1^{-1}}.
\]

Applying the mean value theorem to the cosine function and the usual bounds, we can bound it by

\[
e^{\sigma_2 |r| - \sigma_1 |t|} \left| \frac{Q_2 s_1^{t-1} \ln |ts_1 + rs_2|}{s_2^{2s_2^2 s_1^{-1}}} \right|
\]

\[
\left| \frac{1}{h_{s_1}^2} x + a \int_{s_2}^{s_2 + rs_2} \left( \frac{t - h_{s_1}^2}{s_1} \right) s_1 + rs_2 \ln \left| \frac{t - h_{s_1}^2}{s_1} \right| s_1 + rs_2 - (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right|
\]

\[
\leq e^{\sigma_2 |r| - \sigma_1 |t|} \left| \frac{Q_2 s_1^{t-1} \ln |ts_1 + rs_2|}{s_2^{2s_2^2 s_1^{-1}}} \right|
\]

\[
\left( x + a \int_{s_2}^{s_2 + rs_2} \left( \frac{t - h_{s_1}^2}{s_1} \right) s_1 + rs_2 \ln \left| \frac{t - h_{s_1}^2}{s_1} \right| s_1 + rs_2 - (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right).
\]

(K.9)

The term involving \( |x| \) can be treated using the usual arguments. The one with the integral is of course the most delicate. Let us split this integral into two parts as:

\[
\int_{s_2}^{s_2 + rs_2} \frac{1}{h_{s_1}^2} \left| \left( \frac{t - h_{s_1}^2}{s_1} \right) s_1 + rs_2 \ln \left| \frac{t - h_{s_1}^2}{s_1} \right| s_1 + rs_2 - (ts_1 + rs_2) \ln |ts_1 + rs_2| \Gamma(ds) \right|
\]

\[
:= Q_1 + Q_2,
\]

where \( Q_1 \) and \( Q_2 \) involve integrals over \( S_2 \cap \{ s : |ts_1 + rs_2| \geq 2|h_{s_1}^2/s_1^2| \} \) and \( S_2 \cap \{ s : |ts_1 + rs_2| < 2|h_{s_1}^2/s_1^2| \} \) respectively. We will first majorise \( Q_1 \) and \( Q_2 \), and then use these bounds in inequality (K.9). Consider \( Q_2 \) and define the function \( g \) such that for any \( z > 0 \)

\[
g(z) = \begin{cases} 
  f(z) = z|\ln z|, & \text{if } 0 < z < e^{-1}, \\
  z(2 + \ln z), & \text{if } z \geq e^{-1}.
\end{cases}
\]

It is easily checked that \( g \) is continuous, strictly increasing and such that for any \( z > 0 \), \( 0 \leq f(z) \leq
\( g(z) \). The integrand of \( Q_2 \) can be bounded as

\[
\frac{1}{h_{s_2'}^2} \left( f \left( t - \frac{h s_2'}{s_1'} s_1 + r s_2 \right) + \left| f(t s_1 + r s_2) \right| \right) \leq \frac{1}{h_{s_2'}^2} \left( \left| g \left( t - \frac{h s_2'}{s_1'} s_1 + r s_2 \right) + g(t s_1 + r s_2) \right| \right) \\
\leq \frac{1}{h_{s_2'}^2} \left| g \left( \frac{3 h s_2'}{s_1'} \right) \right| + \left| g \left( \frac{2 h s_2'}{s_1'} \right) \right| \\
\leq \frac{2}{h_{s_2'}^2} g \left( \frac{3 h s_2'}{s_1'} \right).
\]

By Lemma (L.1), with bound further the right-hand side for any \( v > 0 \) by

\[
\frac{2}{h_{s_2'}^2} g \left( \frac{3 h s_2'}{s_1'} \right) \leq \text{const}_1 + \text{const}_2 \left| \frac{3 h s_2'}{s_1'} \right|^v + \text{const}_3 \left| \frac{3 h s_2'}{s_1'} \right|^{-v}.
\]

On the one hand if \( \left| \frac{3 h s_2'}{s_1'} \right| < e^{-1} \), given that \( (3|t s_1 + r s_2|/2)^{-v} > (3 h s_2'/s_1')^{-v}, \)

\[
\text{const}_1 + \text{const}_2 \left| \frac{3 h s_2'}{s_1'} \right|^v + \text{const}_3 \left| \frac{3 h s_2'}{s_1'} \right|^{-v} \leq \text{const}_1 + \text{const}_2 \left| t + \frac{r s_2}{s_1} \right|^{-v} \left| s_1 \right|^{-v}.
\]

On the other hand if \( \left| \frac{3 h s_2'}{s_1'} \right| \geq e^{-1} \), then for \( |h| < |v|, \)

\[
\text{const}_1 + \text{const}_2 \left| \frac{3 h s_2'}{s_1'} \right|^v + \text{const}_3 \left| \frac{3 h s_2'}{s_1'} \right|^{-v} \leq \text{const}_1 + \text{const}_2 \left| s_1^\prime \right|^{-v}. \quad \text{(K.10)}
\]

Focusing now on \( Q_1 \), we can use the mean value theorem to bound its integrand by

\[
\left| s_1 \right| \left( 1 + \ln |u| \right),
\]

for some \( u \in \left[ t s_1 + r s_2 - h s_{2^\prime} / s_{1^\prime} \land t s_1 + r s_2, t s_1 + r s_2 - h s_{2^\prime} / s_{1^\prime} \lor t s_1 + r s_2 \right] \). Given that \( |t s_1 + r s_2| \geq 2|h s_{2^\prime} / s_{1^\prime}|, \) we have \( |u| \in \left[ \frac{|t s_1 + r s_2|}{2}, 2|t s_1 + r s_2| \right] \) and thus, we further bound the above inequality using Lemma (L.1) for any \( v > 0 \) by

\[
\left| s_1 \right| \left( \text{const}_1 + \text{const}_2 \left| t s_1 + r s_2 \right|^v + \text{const}_3 \left| t s_1 + r s_2 \right|^{-v} \right) \leq \text{const}_1 + \text{const}_2 \left| t \right|^v + \text{const}_3 \left| t + \frac{r s_2}{s_1} \right|^{-v} \left| s_1 \right|^{1-v}. \quad \text{(K.11)}
\]

Hence, using (K.10) and (K.11) in (K.9), and making use again of Lemma (L.1) to bound \( |\ln |t s_1^\prime + r s_2^\prime| | \), we can bound integrand of \( L_2 \) for any \( v > 0 \) by

\[
e^{-\sigma_1} |t| \left( \text{const}_1 + \text{const}_2 \left| t \right|^v + \text{const}_3 \left| t + \frac{r s_2^\prime}{s_1^\prime} \right|^{-v} \right) \left| s_1^\prime \right|^{-1-v} \\
\times \left( \left| x \right| + \text{const}_4 + \text{const}_5 \left| t \right|^v + \text{const}_6 \left| s_1^\prime \right|^{-v} + \text{const}_7 \left| t + \frac{r s_2}{s_1} \right|^{-v} \left| s_1 \right|^{1-v} \right)
\]

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It can be shown that all the terms obtained after expansion can be bounded by functions integrable with respect to $t$ and $\Gamma$ using the usual combinations of either Lemma C.7 or Lemma C.8 with $\eta = -v$, $b = 0$, $p = 0$, the fact that $\int_{\mathbb{R}} e^{-\sigma_1|t|} |t|^{-v} < +\infty$, $\int_{\mathbb{R}} e^{-\sigma_1|t|} |t|^{-2v} < +\infty$ for appropriately chosen values $v > 0$, and (2.2) with $\nu > 1$. The detail we have to pay attention to is precisely to chose an appropriate exponent $v > 0$ so that it satisfies the constraint (2.2) and ensures the finiteness of the two integrals in $t$. The later imposes us to have $v \in (0, 1/2)$. Regarding the former, we identify that the most negative power of which $|s_1|$ appears in the above bound after expansion is $-1 - 2v$. We need $\nu - 1 - 2v > 0$. Choosing $v = (\nu - 1)/4$ if $1 < \nu < 3$ and any $v \in (0, 1/2)$ if $\nu \geq 3$ enables to satisfy both constraints, validating the use of the dominated convergence theorem for $L_2$, and finally, for $B_2$ in (K.3).

The proof is essentially similar, somewhat easier, for $B_1$ in (K.2) for which the only difficulty is to perform the «appropriate integration by parts» when it comes to differentiating the term involving $(ts_1 + rs_2)^{-\nu}$.

**K.2 Evaluating at $r = 0$**

Since $E[X_2^2|X_1 = x] = -\phi^{(2)}_{X_2|x}(0)$, we evaluate (K.4) at $r = 0$ and get

$$
\varphi_X(t, 0) = \exp\{-\sigma_1|t| - ia\sigma_1 \beta_1 t \ln |t| + it\mu_1\},
$$

$$
A_{1/2} = \sigma_1^2 \left( (\kappa_1^2 - a^2 \eta_0^2)H_c(0) + 2a\kappa_1f_0H_s(0) \right)
+ 2a\sigma_1^2 \left( -a\sigma_1 H_c(1) + \kappa_1 H_s(1) \right) - a^2\sigma_1^2 \sigma_2^2 H_c(2),
$$

$$
iA_{2/2} = \sigma_1 \left( -a^2 \kappa_1 H_c(0) + \kappa_2 H_s(0) \right) - a\lambda_2 \kappa_1 H_c(1),
$$

$$
A_{3/2} = \sigma_1 \left( (\sigma_1 \kappa_2 + a\kappa_1 \kappa_1)H_c(0) + (\sigma_1 \kappa_1 - \mu_1 \kappa_2)H_s(0) \right)
+ a\sigma_1 \left( (\lambda_2 \mu_1 - a\sigma_1 \beta_1 \kappa_1)H_c(1) + \sigma_1 (\lambda_2 + \beta_1 \kappa_2)H_s(1) \right) - a^2\sigma_1^2 \beta_1 \lambda_2 H_c(2),
$$

where $k_1 = \sigma_1^{-1} \int_{s_2} (s_2^2/s_1^2)^2 s_1 \ln |s_1| \Gamma(ds)$, and the $H_c$’s and $H_s$’s are defined at Lemma L.2. Using the result of the same Lemma under $\beta_1 \neq 0$ and $\beta_1 = 0$, and regrouping the terms allows to retrieve the two formulae of Theorem A.4.

**L Proof of Proposition A.1 in the case $\alpha = 1$**

**Case $\beta_1 \neq 0$** The conditional second order moment when $\alpha = 1$ has a particular form. We only consider the case $|\beta_1| \neq 1$ and $x \to +\infty$. Since $|x| \to +\infty$, we have $x - \mu_1 \sim x$ and we may
assume that $\mu_1 = 0$. From [21], we know that $U(x) \sim x^{-1}$. Notice that

$$W(x) = \int_0^{+\infty} e^{-\sigma_1 t}(1 + \ln t)^2 \cos(\alpha \sigma_1 t \ln t) \cos(tx)dt$$

$$- \int_0^{+\infty} e^{-\sigma_1 t}(1 + \ln t)^2 \sin(\alpha \sigma_1 t \ln t) \sin(tx)dt.$$  

Because the factors of $\cos(tx)$ and $\sin(tx)$ are integrable, we have by the Riemann-Lebesgue Lemma that $W(x) \xrightarrow{x \to +\infty} 0$. Having also

$$f_{X_1}(x) \sim \frac{\sigma_1(1 + \beta_1)}{\pi} x^{-2},$$

we deduce the following limits

$$\left(2a\sigma_1 q_0(\lambda_1 - \beta_1 \kappa_1) + 2(\kappa_1 \lambda_1 - \lambda_2)x\right) \frac{\sigma_1 U(x)}{\beta_1 \pi f_{X_1}(x)} x^{-2} \xrightarrow{x \to +\infty} \frac{2(\kappa_1 \lambda_1 - \lambda_2)}{(1 + \beta_1)\beta_1},$$

$$\left(\lambda_2 + \beta_1 \kappa_2 - 2\kappa_1 \lambda_1 + a^2 \sigma_1 \beta_1 (\lambda_1^2 - \beta_1 \lambda_2) W(x)\right) \frac{\sigma_1 x^{-2}}{\pi f_{X_1}(x)} \xrightarrow{x \to +\infty} \frac{\lambda_2 + \beta_1 \kappa_2 - 2\kappa_1 \lambda_1}{(1 + \beta_1)\beta_1}.$$  

Hence,

$$x^{-2}\mathbb{E}\left[X_2^2 \mid X_1 = x\right] \xrightarrow{x \to +\infty} \frac{\lambda_2}{\beta_1} + \frac{2(\kappa_1 \lambda_1 - \lambda_2)}{(1 + \beta_1)\beta_1} + \frac{\lambda_2 + \beta_1 \kappa_2 - 2\kappa_1 \lambda_1}{(1 + \beta_1)\beta_1} = \kappa_2 + \lambda_2$$

**Case $\beta_1 = 0$** From [21],

$$V(x) \xrightarrow{-\frac{\pi}{2x}},$$

hence,

$$2a\sigma_1 \lambda_1 \left(a\sigma_1 q_0 - \kappa_1 (x - \mu_1)\right) \frac{V(x)}{\pi f_{X_1}(x)} x^{-2} \xrightarrow{-a\pi \lambda_1 \kappa_1}.$$

Moreover,

$$a\sigma_1 \frac{F_{X_1}(x) - 1/2}{f_{X_1}(x)} x^{-2} \xrightarrow{-\frac{1}{2}a\pi (\lambda_2 - 2\kappa_1 \lambda_1)}.$$

It can be shown that $W(x) \xrightarrow{} 0$. Therefore,

$$x^{-2}\mathbb{E}\left[X_2^2 \mid X_1 = x\right] \xrightarrow{x \to +\infty} \kappa_2 + \frac{1}{2}a\pi (\lambda_2 - 2\kappa_1 \lambda_1) + a\pi \kappa_1 \lambda_1 = \kappa_2 + \lambda_2$$

\square

**Lemma L.1** For any $x > 0$ and $v > 0$

$$|\ln x| \leq \frac{1}{v} \left(2 + x^v + x^{-v}\right).$$

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We provide here two Lemmas which are used in the proof of Theorem A.4.

**Lemma L.2** Let for any \( n \geq 0 \),
\[
H_c(n) = \int_0^{+\infty} e^{-\sigma_1 t}(1 + \ln t)^n \cos \left( t(x - \mu_1) + a\sigma_1 \beta_1 t \ln t \right) dt,
\]
\[
H_s(n) = \int_0^{+\infty} e^{-\sigma_1 t}(1 + \ln t)^n \sin \left( t(x - \mu_1) + a\sigma_1 \beta_1 t \ln t \right) dt.
\]

Then, if \( \beta_1 \neq 0 \),
\[
H_c(1) = \frac{1}{a\sigma_1 \beta_1} \left( \sigma_1 H_s(0) - (x - \mu_1)H_c(0) \right), \quad H_s(1) = \frac{1}{a\sigma_1 \beta_1} \left( 1 - \sigma_1 H_c(0) - (x - \mu_1)H_s(0) \right).
\]

If \( \beta_1 = 0 \),
\[
H_c(0) = \pi f_{X_1}(x), \quad H_s(0) = \frac{x - \mu_1}{\sigma_1} \pi f_{X_1}(x), \quad H_s(1) - \frac{x - \mu_1}{\sigma_1} H_c(1) = \frac{\pi F_{X_1}(x)}{\sigma_1}.
\]

**Proof.** The equalities of Lemmas C.10\textsuperscript{L.2} can be obtained by integrating by parts. We provide details for the last equality of Lemma L.2 when \( \beta_1 = 0 \). Integrating the exponential by parts, we obtain
\[
H_s(1) = \frac{1}{\sigma_1} \int_0^{+\infty} e^{-\sigma_1 t} t^{-1} \sin \left( t(x - \mu_1) \right) dt + \frac{x - \mu_1}{\sigma_1} H_c(1)
\]

Denote \( A(x) = \int_0^{+\infty} e^{-\sigma_1 t} t^{-1} \sin \left( t(x - \mu_1) \right) dt \) for \( x \in \mathbb{R} \) (\( A \) is well defined since \( e^{-\sigma_1 t} t^{-1} \sin \left( t(x - \mu_1) \right) \to x - \mu_1 \) as \( t \to 0 \)). It can be shown that we can derivate \( A \) under the integral sign and get
\[
A'(x) = \int_0^{+\infty} e^{-\sigma_1 t} \cos \left( t(x - \mu_1) \right) dt = \pi f_{X_1}(x),
\]

Since \( X_1 \) is Cauchy distributed when \( \alpha = 1 \) and \( \beta_1 = 0 \),
\[
A(x) = \pi F_{X_1}(x) + \text{const} = \arctg \left( \frac{x - \mu_1}{\sigma_1} \right) + \frac{\pi}{2} + \text{const},
\]
and evaluating the integral form of \( A \) at \( \mu_1 \), we deduce that \( \text{const} = -\pi/2 \). Thus, \( A(x) = \pi \left( F_{X_1}(x) - 1/2 \right) \).
Additional References

[1] Davis, R. and S., Resnick. 1985. Limit Theory for Moving Averages of Random Variables with Regularly Varying Tail Probabilities. *Annals of Probability*, 13, 179-195.