Three-point Functions in $\mathcal{N} = 4$ SYM at Finite $N_c$ and Background Independence

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Abstract

We compute non-extremal three-point functions of scalar operators in $\mathcal{N} = 4$ super Yang-Mills at tree-level in $g_{YM}$ and at finite $N_c$, using the operator basis of the restricted Schur characters. We make use of the diagrammatic methods called quiver calculus to simplify the three-point functions. The results involve an invariant product of the generalized Racah-Wigner tensors ($6j$ symbols). Assuming that the invariant product is written by the Littlewood-Richardson coefficients, we show that the non-extremal three-point functions satisfy the large $N_c$ background independence; correspondence between the string excitations on $\text{AdS}_5 \times S^5$ and those in the LLM geometry.
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1 Introduction

Recently we have seen remarkable progress in the computation of the correlation functions of $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in the hope of establishing the AdS/CFT correspondence [1]. There are two complementary approaches to this problem.

The first approach is based on the integrability of $\mathcal{N} = 4$ SYM in the planar limit. The planar three-point functions of single-trace operators are regarded as a pair of hexagons glued together, where each hexagon form-factor is severely constrained by the centrally-extended $\mathfrak{su}(2|2)$ symmetry [2]. The $n$-point functions of BPS operators can be studied by *hexagonization*. The gluing of four hexagons give us the planar four-point functions [3–5], and the gluing of $2n - 4 + 4g$ hexagons should give the $g$-th non-planar corrections [6–8]. Furthermore, certain four-point functions in the large charge limit decompose into a pair of octagons [9,10], which can be resumed [11,12].

The integrability approach tells us how single-trace correlation functions depend on the ’t Hooft coupling $\lambda = N_c g_{\text{YM}}^2$. However, only the non-extremal correlation functions have been studied, because the non-extremality is related to the so-called bridge length (the number of Wick contractions between a pair of operators), which suppresses the complicated wrapping corrections to the asymptotic formula [13–17].

The second approach is based on the finite-group theory. In this approach, one obtains the results valid for any values of $N_c$, though most results are limited to tree-level or a few orders of small $\lambda$ expansion. In the finite-group approach, extremal correlation functions are often studied, because they are roughly equal to the two-point functions at tree level.

Quite recently the author studied the $n$-point functions of multi-trace scalar operators at tree-level of $\mathcal{N} = 4$ SYM with $U(N_c)$ gauge group, based on the finite group methods [18]. Those results are written in terms of permutations, meaning that they are valid to any orders of $1/N_c$ expansions, but not at any values of $N_c$ because the finite-$N_c$ constraints are not taken into consideration. The primary purpose of this paper is to generalize the permutation-based results to finite $N_c$, by taking a Fourier transform of symmetric groups.

Two types of operator bases of $\mathcal{N} = 4$ SYM are well-known, which carry a set of Young diagrams as the operator label, diagonalize tree-level two-point functions at finite $N_c$, generalizing the pioneering work of [19]. The covariant basis (also called BHR basis) introduced in [20,21] respects the global (or flavor) symmetry of the operator. As such, one can construct $O(N_f)$ singlets for general $N_f$ [22]. The restricted Schur basis was introduced in a series of papers [23–25] and related to multi-matrix models in [26,27]. The restricted Schur basis respects the permutation symmetry of the operator, and suitable for explicit calculation. In other words, one has to specify a state inside the irreducible representation of the global (or flavor) symmetry, like the highest weight state. Here is a brief comparison of the two representation bases [28]:

\footnote{Note that the restricted Schur basis can compute the observables of a multi-matrix model, which are not the function of the multi-matrix eigenvalues only.}


| Operator basis     | Symmetry respected | Analogy                  |
|--------------------|--------------------|--------------------------|
| Covariant          | Global symmetry    | Spherical coordinates    |
| Restricted Schur   | Permutation of constituents | Cartesian coordinates |

In this paper, we consider general non-extremal three-point functions of the scalar operators in the restricted Schur basis. There are several important ideas in this computation. The first idea is the Schur-Weyl duality between $U(N_c)$ and $S_L$, which converts powers of $N_c$ into the irreducible characters of the symmetric group $S_L$. The second idea is the *quiver calculus* initiated by [29]. This is a set of diagrammatic rules which enormously simplify the manipulation of representation-theoretical objects. The third idea is the generalized Racah-Wigner tensor. Since the three-point function is non-extremal, we need to compute a non-trivial overlap between the states under different subgroup decompositions of $S_L$. The invariant products we encounter are more general than Wigner’s $6j$ symbols\(^2\).

Let us summarize the main results. Our notation is explained in Appendix A. We are particularly interested in two types of the non-extremal three-point functions (or equivalently non-extremal OPE coefficients). The first type is the super-protected three-point functions [32] in the restricted Schur basis, given by (3.70).

\[
\text{Fourier transform of } \langle \langle \text{tr}_{L_1}(\alpha_1 Z^{\otimes L_1}) \text{ tr}_{L_2}(\alpha_2 \tilde{Z}^{L_2}) \text{ tr}_{L_3}(\alpha_3 \overline{Z}^{L_3}) \rangle \rangle = \left( \prod_{i=1}^{3} \frac{L_i!}{\hat{L}_i!} \right) \sum_{R+L} \sum_{d_{R_1}d_{R_2}d_{R_3}} \left( \prod_{i=1}^{3} d_{Q_i} \right) G_{123}. \tag{1.1} \]

The second type is the three-point functions of the scalar operators made of three pairs of complex scalars in $\mathcal{N} = 4$ SYM, given by (3.90).

\[
\text{Fourier transform of } \langle \langle \text{tr}_{L_1}(\alpha_1 X^{(\ell_3)} \bar{Y}^{h_3} Z^{(\ell_2+h_2)}) \rangle \rangle \times \left( \prod_{i=1}^{3} \frac{L_i!}{\hat{L}_i!} \right) \sum_{R+L} \sum_{d_{R_1}d_{R_2}d_{R_3}} \left( \prod_{i=1}^{3} d_{Q_i} \right) \delta_{\nu_1-\nu_2} \delta_{\nu_2-\nu_3} \delta_{\nu_3-\nu_1} G'_{123}. \tag{1.2} \]

The objects $G_{123}$ and $G'_{123}$ are related to the invariant products of the generalized Racah-Wigner tensors.

Mathematically, the branching coefficient of $R = \oplus (r \otimes s)$ is the building block of the restricted Schur character and the generalized Racah-Wigner tensor. In the literature, the orthonormal basis of $r \otimes s$ is called the split basis [33], and the branching coefficients are called fractional parentage coefficients [34], subduction coefficients [35, 36] or the split-standard transformation coefficients [33, 37, 38]. In general, explicit computation of the branching coefficients is a hard problem. See \(^2\)The $6j$ symbol is also called Racah’s $W$ coefficient or recoupling coefficient. The $6j$ symbols of symmetrical groups are called $6j$ symbols in [30], and they are related to the $6j$ symbols of unitary groups by the through the duality factor [31].
for the recent results on the branching coefficients, and on the construction of the restricted Schur basis [42].

Likewise, it is difficult to compute $G_{123}$, $G'_{123}$ explicitly. We conjecture that they can be written by the Littlewood-Richardson coefficients, based on the fact that they satisfy certain sum rules.

From (1.1) and (1.2), it is straightforward to show the large $N_c$ background independence in $\mathcal{N} = 4$ SYM [43]. The background independence is a conjectured correspondence between the operators with $\mathcal{O}(N_c^0)$ canonical dimensions and those with $\mathcal{O}(N_c^2)$ canonical dimensions, where the latter is constructed from the former by “attaching” a large number of background boxes. By AdS/CFT, this conjecture implies that the stringy excitations in AdS$_5 \times S^5$ and those in the (centralized configuration of) LLM geometry [44].

On the gauge theory side, the large $N_c$ background independence has been checked for the case of two-point functions and extremal $n$-point functions. On the gravity side, some string spectrum in the $\text{SL}(2)$ sector has been studied in [45]. We find that the non-extremal OPE coefficients in the LLM background are essentially given by the rescaling of $N_c$ in (1.1), (1.2). Our results provide strong support that the large $N_c$ background independence can be found also in the string interactions.

## 2 Two-point functions in the representation basis

We review the construction of the restricted Schur basis, and introduce the diagrammatic computation methods called quiver calculus.

### 2.1 Set-up

We consider $\mathcal{N} = 4$ SYM of $U(N_c)$ gauge group at tree-level. This theory has three complex scalars $(X, Y, Z)$, which satisfy the $U(N_c)$ Wick rule,

$$X^b_a(x) X^d_c(0) = Y^b_a(x) Y^d_c(0) = Z^b_a(x) Z^d_c(0) = |x|^{-2} \delta^d_a \delta^b_c. \quad (2.1)$$

With $\alpha \in S_{l+m+n}$, we define a multi-trace operator in the permutation basis

$$\mathcal{O}_{\alpha}^{(l,m,n)} = \text{tr}_{m+n} (\alpha X^{\otimes l} Y^{\otimes m} Z^{\otimes n}) \equiv \sum_{i_1, i_2, \ldots, i_{l+m+n} = 1}^{N_c} X_{i_{\alpha(1)}}^{i_1} \cdots X_{i_{\alpha(l)}}^{i_l} \ Y_{i_{\alpha(l+1)}}^{i_{l+1}} \cdots Y_{i_{\alpha(l+m)}}^{i_{l+m}} \ Z_{i_{\alpha(l+m+1)}}^{i_{l+m+1}} \cdots Z_{i_{\alpha(l+m+n)}}^{i_{l+m+n}}. \quad (2.2)$$

The usual single-trace operators can be expressed in the permutation basis as

$$\text{tr} (X^l Y^m Z^n) \rightarrow \text{tr}_L (\alpha X^{\otimes l} Y^{\otimes m} Z^{\otimes n}), \quad (\alpha_l \in Z_{l+m+n}). \quad (2.3)$$

The correspondence between a multi-trace operator and $\alpha \in S_L$ is not one-to-one, because $\alpha$ is defined modulo conjugation,

$$\mathcal{O}_{\alpha}^{(l,m,n)} = \mathcal{O}_{\gamma \alpha \gamma^{-1}}, \quad \gamma \in S_l \otimes S_m \otimes S_n \quad (2.4)$$
which we call the flavor symmetry (or global symmetry). For example,
\[
\text{tr } (XXZZ) = \text{tr}_{L=4}((1234) X^{\otimes 2} Z^{\otimes 2}) = \text{tr}_{L=4}((2143) X^{\otimes 2} Z^{\otimes 2}) = \ldots
\]
\[
\text{tr } (XZXX) = \text{tr}_{L=4}((1234) X^{\otimes 2} Z^{\otimes 2}) = \text{tr}_{L=4}((3142) X^{\otimes 2} Z^{\otimes 2}) = \ldots
\]
where \ldots represents the other permutations generated by the flavor symmetry $[2.4]$. We define the complex conjugate operator by
\[
\bar{O}_{\alpha}^{(l,m,n)} = \text{tr}_{m+n} \left( \alpha X^{\otimes l} Y^{\otimes m} Y^{\otimes n} \right)
\]
The two-point function between $O_{\alpha_1}^{(l,m,n)}$ and $O_{\alpha_2}^{(l,m,n)}$ at tree-level is given by
\[
\langle O_{\alpha_1}^{(l,m,n)}(x) O_{\alpha_2}^{(l,m,n)}(0) \rangle = |x|^{-2(l+m+n)} \sum_{\gamma \in S_l \otimes S_m \otimes S_n} N_c^C(\alpha_1 \gamma \alpha_2 \gamma^{-1})
\]
where $C(\omega)$ counts the number of cycles in $\omega \in S_{l+m+n}$. We write $\langle O_1 \bar{O}_2 \rangle \equiv \langle O_1(1) \bar{O}_2(0) \rangle$.

### 2.2 Diagonalizing the tree-level two-point

Following [29], we show how to “derive” the representation basis of operators starting from the two-point functions on the permutation basis [2.7]. The resulting tree-level two-point functions are diagonal at any $N_c$. The readers familiar with the restricted Schur basis can skip this subsection. The basic formulae are summarized in Appendix A.3.

First, we rewrite the equation (2.7) by using (A.40) as
\[
\langle O_{\alpha_1}^{(l,m,n)} O_{\alpha_2}^{(l,m,n)} \rangle = \sum_{R \vdash (l+m+n)} \sum_{\gamma \in S_l \otimes S_m \otimes S_n} \dim_{N_c}(R) \chi^R(\alpha_1 \gamma \alpha_2 \gamma^{-1})
\]
\[
= \sum_{R \vdash (l+m+n)} \dim_{N_c}(R) \sum_{\gamma \in S_l \otimes S_m \otimes S_n} \alpha_1 \gamma^{-1} \alpha_2
\]
where we used the quiver calculus notation of Appendix A in the second line. We introduce $\gamma = \gamma_1 \circ \gamma_2 \circ \gamma_3 \in S_l \otimes S_m \otimes S_n$ and the branching coefficients for $S_{l+m+n} \downarrow (S_l \otimes S_m \otimes S_n)$ to make use of the identity (A.20). The equation (2.8) becomes
\[
\langle O_{\alpha_1}^{(l,m,n)} O_{\alpha_2}^{(l,m,n)} \rangle = \sum_{R \vdash (l+m+n)} \dim_{N_c}(R) \sum_{\gamma_1 \in S_l} \sum_{\gamma_2 \in S_m} \sum_{\gamma_3 \in S_n} \chi^{r_1 \gamma_1} \chi^{r_2 \gamma_2} \chi^{r_3 \gamma_3} \chi^{\nu_1 \nu_2} \chi^{\nu_3 \nu_4}
\]

\[
\langle O_{\alpha_1}^{(l,m,n)} O_{\alpha_2}^{(l,m,n)} \rangle = \sum_{R \vdash (l+m+n)} \dim_{N_c}(R) \sum_{\gamma_1 \in S_l} \sum_{\gamma_2 \in S_m} \sum_{\gamma_3 \in S_n} \chi^{r_1 \gamma_1} \chi^{r_2 \gamma_2} \chi^{r_3 \gamma_3} \chi^{\nu_1 \nu_2} \chi^{\nu_3 \nu_4}
\]
\[
\langle O_{\alpha_1}^{(l,m,n)} O_{\alpha_2}^{(l,m,n)} \rangle = \sum_{R \vdash (l+m+n)} \dim_{N_c}(R) \sum_{\gamma_1 \in S_l} \sum_{\gamma_2 \in S_m} \sum_{\gamma_3 \in S_n} \chi^{r_1 \gamma_1} \chi^{r_2 \gamma_2} \chi^{r_3 \gamma_3} \chi^{\nu_1 \nu_2} \chi^{\nu_3 \nu_4}
\]
We apply the grand orthogonality \( [B.4] \) to the matrix elements of \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) to obtain

\[
\langle O^{(l,m,n)}_{\alpha_1} | O^{(l,m,n)}_{\alpha_2} \rangle = \sum_{R:(l+m+n)} \text{Dim}_{\nu}(R) \sum_{r_1,r_2,r_3,\nu_-,\nu_+} \frac{l! m! n!}{d_{r_1} d_{r_2} d_{r_3}} \chi^{R,(r_1,r_2,r_3),(\nu_-,\nu_+)}(\alpha_1) \chi^{R,(r_1,r_2,r_3),(\nu_+,\nu_-)}(\alpha_2)
\]

where \( \chi^{R,(r_1,r_2,r_3),(\nu_-,\nu_+)}(\alpha) \) is the restricted characters defined through branching coefficients,

\[
\chi^{R,(r_1,r_2,r_3),(\nu_+,\nu_-)}(\sigma) = \sum_{l,j} \sum_{i,j} B_{l \rightarrow (i,j)}^{R,(r_1,r_2,r_3)\nu_+} (B_{l \rightarrow (i,j)}^{T})_{j \rightarrow (i,j)}^{R,(r_1,r_2,r_3)\nu_-} D_{ij}(\sigma).
\]

The restricted characters satisfy the orthogonality relations \( \langle A.51 \rangle \). It is straightforward to find a linear combination of operators which diagonalizes the two-point function;

\[
O^{S,(s_1,s_2,s_3),\mu_+,\mu_-}_{(t_1,t_2,t_3),\eta_+\eta_-}(x) = \frac{1}{l! m! n!} \sum_{\alpha \in S_{l+m+n}} \chi^{S,(s_1,s_2,s_3),\mu_+,\mu_-}(\alpha) O^{(l,m,n)}_{\alpha}(x)
\]

\[
\mathcal{O}^{S,(t_1,t_2,t_3),\eta_+\eta_-}_{(t_1,t_2,t_3),\eta_+\eta_-}(y) = \frac{1}{l! m! n!} \sum_{\alpha \in S_{l+m+n}} \chi^{T,(t_1,t_2,t_3),\eta_+\eta_-}(\alpha) \mathcal{O}^{(l,m,n)}_{\alpha}(y).
\]

It follows that

\[
\langle O^{S,(s_1,s_2,s_3),\mu_+,\mu_-}_{(t_1,t_2,t_3),\eta_+\eta_-} | \mathcal{O}^{S,(t_1,t_2,t_3),\eta_+\eta_-} \rangle = \left( \frac{1}{l! m! n!} \right)^2 \sum_{R,(r_1,r_2,r_3,\nu_-,\nu_+)} \text{Dim}_{\nu}(R) \frac{l! m! n!}{d_{r_1} d_{r_2} d_{r_3}} \times \sum_{\alpha_1,\alpha_2 \in S_{l+m+n}} \chi^{S,(s_1,s_2,s_3),\mu_+,\mu_-}(\alpha_1) \chi^{T,(t_1,t_2,t_3),\eta_+\eta_-}(\alpha_2) \chi^{R,(r_1,r_2,r_3),(\nu_-,\nu_+)}(\alpha_1) \chi^{R,(r_1,r_2,r_3),(\nu_+,\nu_-)}(\alpha_2)
\]

\[
= \text{Dim}_{\nu}(S) \frac{(l + m + n)!^2}{l! m! n!} \frac{d_{s_1} d_{s_2} d_{s_3}}{d_s^3} \delta^{ST} \delta^{s_1 t_1} \delta^{s_2 t_2} \delta^{s_3 t_3} \delta^{\mu_+ \eta_-} \delta^{\mu_- \eta_-}
\]

\[
= \text{Wt}_{\nu}(S) \frac{\text{hook}_{s_1} \text{hook}_{s_2} \text{hook}_{s_3}}{\text{hook}_{s}} \delta^{ST} \delta^{s_1 t_1} \delta^{s_2 t_2} \delta^{s_3 t_3} \delta^{\mu_+ \eta_-} \delta^{\mu_- \eta_-}
\]

\[
\text{where we used } [A.5].
\]

Recall that \( O^{(l,m,n)}_{\alpha} \) in \( [2.22] \) becomes half-BPS when \( l = m = 0 \), and the restricted character \( [2.10] \) reduces to the usual irreducible characters of \( S_n \). The two-point function \( [2.12] \) becomes

\[
\langle O^S \mathcal{O}^T \rangle = \text{Wt}_{\nu}(S) \delta^{ST}
\]

which gives the same normalization of half-BPS operators as in \( [19] \).
\section{Three-point functions in the representation basis}

In \cite{ref}, tree-level formulae of the \( n \)-point functions of general scalar operators in the permutation basis have been derived. We consider three-point functions of scalar operators in the restricted Schur basis below. The three-point functions of \( \mathcal{N} = 4 \) SYM are related to the OPE coefficients by

\begin{equation}
\langle O_1(x_1)O_2(x_2)O_3(x_3) \rangle = \frac{C_{123}}{|x_1 - x_2|^\Delta_1 + \Delta_2 + \Delta_3 \ |x_2 - x_3|^\Delta_2 + \Delta_3 - \Delta_1 \ |x_3 - x_1|^\Delta_3 + \Delta_1 - \Delta_2} \tag{3.1}
\end{equation}

thanks to the conformal symmetry. By abuse of notation, we write (3.1) as

\begin{equation}
\langle O_1O_2O_3 \rangle = C_{123}. \tag{3.2}
\end{equation}

\subsection{Set-up}

Let us recall the tree-level permutation formula for three-point functions in \cite{ref}. That formula has been derived based on the following idea. Consider a non-extremal three-point function of the operators labeled by \( \alpha_i \in S_L \) for \( i = 1, 2, 3 \). We expect that the tree-level Wick contractions give the quantity like \( N_c^{C(\alpha_1\alpha_2\alpha_3)} \). However, we cannot define the multiplication of elements in \( S_L \) if \( L_1 \neq L_2 \). This problem can be solved by extending \( \alpha_i \) to \( \hat{\alpha}_i \in S_L \) for some \( L \), which makes the quantity \( N_c^{C(\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3)} \) well-defined.

Let us explain how this idea works. First, we extend the operator \( O_i \) by adding identity fields,

\begin{equation}
\hat{O}_i \equiv O_{\alpha_i} \times \text{tr} (1)^{\mathcal{T}_i} \equiv \prod_{p=1}^{L} \left( \Phi^{(i)}_{\hat{A}_p} \right)^{\alpha_{\hat{\alpha}_i}(p)}, \quad \hat{\alpha}_i = \alpha_i \circ 1_{\mathcal{T}_i} \in S_{L_1} \times S_{L_2} \subset S_L \tag{3.3}
\end{equation}

where

\begin{equation}
L = \frac{L_1 + L_2 + L_3}{2}, \quad \mathcal{T}_i = L - L_i. \tag{3.4}
\end{equation}

The permutation \( \hat{\alpha}_i \) acts as the identity at the position \( p \) at which \( \Phi^{(i)}_{\hat{A}_p} = 1 \). The (edge-type) permutation formula reads

\begin{equation}
C_{123} = \frac{1}{\prod_{i=1}^{2} L_i !} \frac{1}{L!} \sum_{\{\mathcal{U}_i\} \in S_L^{3 \beta}} \left( \prod_{p=1}^{L} h^{\hat{A}_i (i) A_p (2) A_p (3)} \right) N_c^{C(\hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3)} \tag{3.5}
\end{equation}

where \( \hat{A}_p^{(i)} \equiv \hat{A}_{U_i(p)}; \hat{\alpha}_i \equiv U_i^{-1} \hat{\alpha}_i U_i \) and

\begin{equation}
\begin{aligned}
h^{ABC} &= h^{AB} \delta_1^C + h^{BC} \delta_1^A + h^{CA} \delta_1^B, \\
h^{AB} &= \begin{cases} g^{AB} \equiv \langle \Phi^A(1) \Phi^B(0) \rangle & (A \neq 1, B \neq 1) \\
0 & (\text{otherwise}). \end{cases} \tag{3.6}
\end{aligned}
\end{equation}

We call \( h^{ABC} \) a triple Wick contraction.

We will consider two types of three-point functions. The first type is the three-point functions of half-BPS multi-trace operators,

\begin{equation}
C_{\alpha\alpha\alpha} = \left( \text{tr}_{L_1}(\alpha_1 Z^{L_1}) \text{tr}_{L_2}(\alpha_2 \tilde{Z}^{L_2}) \text{tr}_{L_3}(\alpha_3 \tilde{Z}^{L_3}) \right), \quad \tilde{Z} = (Z + \overline{Z} + Y - \overline{Y}). \tag{3.7}
\end{equation}
The field $\tilde{Z}$ belongs to the one-parameter family of operators used in \cite{2,32},

$$3_i(a) = (Z + a_i (Y - \overline{Y}) + a_i^2 \overline{Z})(x_i), \quad x_i = (0, a_i, 0, 0).$$

(3.8)

The second type is general three-point functions of the scalar multi-trace operator \cite{2,22},

$$C^{XYZ}_h \equiv \left\langle \text{tr}_{L_1} \left( \alpha_1 X^{\otimes (\ell_1 - h_2)} Y^{\otimes h_2} Z^{\otimes (\ell_2 - h_3 + h_2)} \right) \times \text{tr}_{L_2} \left( \alpha_2 X_{n_1}^{\otimes h_1} Y^{\otimes (\ell_3 - h_1 + h_3)} Z^{\otimes (\ell_2 - h_3)} \right) \times \text{tr}_{L_3} \left( \alpha_3 X^{\otimes (\ell_1 - h_2 + h_1)} Y^{\otimes (\ell_3 - h_1)} Z^{\otimes h_2} \right) \right\rangle$$

(3.9)

where $\ell_{ij}$ is the number of tree-level Wick contractions between $O_i$ and $O_j$ (called the bridge length), given by

$$\ell_{12} = \frac{L_1 + L_2 - L_3}{2}, \quad \ell_{23} = \frac{L_2 + L_3 - L_1}{2}, \quad \ell_{31} = \frac{L_3 + L_1 - L_2}{2}$$

(3.10)

and $h_i$ is an integer inside the range

$$0 \leq h_1 \leq \ell_{23}, \quad 0 \leq h_2 \leq \ell_{31}, \quad 0 \leq h_3 \leq \ell_{12}.$$  

(3.11)

### 3.2 Partial Fourier transform

We construct the three-point functions in the restricted Schur basis by taking the Fourier transform of $C^{XYZ}_h$ in \cite{2,7} and $C^{XYZ}_h$ \cite{2,9}. Recall that the usual Fourier transform of the delta function is a constant. In the Fourier transform over a finite group, the Fourier transform of the identity permutation should be a sum over all representations. In other words, if we write

$$R_i \vdash L_i \leftrightarrow \text{FT of } \alpha_i \in S_{L_i}, \quad t_i \vdash T_i \leftrightarrow \text{FT of } 1_{T_i} \in S_{T_i}$$

(3.12)

then we should sum $t_i$ over all possible partitions of $T_i$. In fact, $t_i$ is an unphysical parameter, and we can perform a calculation without using $t_i$. Thus we call the procedure (3.12) a partial Fourier transform.

In order to treat $C^{XY}_{\alpha}$ and $C^{XYZ}_h$ simultaneously, we extend the multi-trace operator \cite{2,22} as in \cite{2,33},

$$O_{\alpha_i}^{(l_i, m_i, n_i, T_i)}[X, Y, Z, 1] = \text{tr}_{L_i} \left( \alpha_i X^{\otimes l_i} Y^{\otimes m_i} Z^{\otimes n_i} \right) \times \text{tr} \left( 1_{T_i} \right)$$

(3.13)

$$l_i + m_i + n_i = L_i, \quad L_i + L_i = L, \quad \alpha_i = \alpha_i \circ 1_{T_i} \in S_L$$

and define the partial Fourier transform by

$$O^{R_i(T_i)}[X, Y, Z, 1] = \frac{1}{l_i! m_i! n_i!} \sum_{\alpha_i \in S_{L_i}} \chi^{R_i(\alpha_i)} O_{\alpha_i}^{(l_i, m_i, n_i, T_i)}[X, Y, Z, 1]$$

(3.14)

$$R_i = \{ R_i, (q_i, r_i, s_i), \nu_i, - \nu_i \}, \quad (R_i \vdash L_i, q_i \vdash l_i, r_i \vdash m_i, s_i \vdash n_i).$$

The partial Fourier transform can be rewritten as a linear combination of the complete Fourier transform. To see this, we recall \cite{3,33} and

$$\chi^{R_i \otimes l_i}(\alpha_i \circ 1_{T_i}) = \chi^{R_i(\alpha_i)} d_i, \quad \sum_{t_i \vdash T_i} d_i^2 = T_i$$

(3.15)
giving us a dummy representation $t_i$ to be summed over the partitions of $\bar{L}_i$. It follows that
\[
\hat{\Omega}_i^{R_i(T_i)}[X, Y, Z, 1] = \frac{1}{l_i! \cdot m_i! \cdot n_i! \cdot |\bar{L}_i|!} \sum_{t_i \vdash \bar{L}_i} \sum_{\hat{\alpha}_i \in S_{L_i} \times \mathbf{1}_{T_i}} d_{t_i} \chi^{R_i \otimes t_i}(\hat{\alpha}_i) \bar{\Omega}^{(l_i, m_i, n_i, T_i)}_i[X, Y, Z, 1]. \tag{3.16}
\]

As for $C_{000}$, we introduce the Fourier transform of the half-BPS operators as
\[
\tilde{\Omega}_1 = \hat{\Omega}_1^{R_1(T_1)}[Z, 1], \quad \tilde{\Omega}_2 = \hat{\Omega}_2^{R_2(T_2)}[\tilde{Z}, 1], \quad \tilde{\Omega}_3 = \hat{\Omega}_3^{R_3(T_3)}[\bar{Z}, 1], \quad R_i = R_i \vdash L_i \tag{3.17}
\]
and define
\[
\tilde{C}_{000} = \left\langle \hat{\Omega}_1^{R_1(T_1)}[Z, 1] \hat{\Omega}_2^{R_2(T_2)}[\tilde{Z}, 1] \hat{\Omega}_3^{R_3(T_3)}[\bar{Z}, 1] \right\rangle. \tag{3.18}
\]
As for $C_{XYZ}^h$, we take the Fourier transform of the operators in (3.9) as
\[
\tilde{\Omega}_1 = \hat{\Omega}_1^{R_1(T_1)}[X, Y, Z, 1] \quad \left(l_1, m_1, n_1 = (\ell_{31} - h_2, h_3, \ell_{12} - h_3 + h_2)\right)
\]
\[
\tilde{\Omega}_2 = \hat{\Omega}_2^{R_2(T_2)}[X, Y, \bar{Z}, 1] \quad \left(l_2, m_2, n_2 = (h_1, \ell_{23} - h_1 + h_3, \ell_{12} - h_3)\right)
\]
\[
\tilde{\Omega}_3 = \hat{\Omega}_3^{R_3(T_3)}[X, Y, Z, 1] \quad \left(l_3, m_3, n_3 = (\ell_{31} - h_2 + h_1, \ell_{23} - h_1, h_2)\right) \tag{3.19}
\]
and define
\[
\tilde{C}_{h}^{XYZ} = \left\langle \hat{\Omega}_1^{R_1(T_1)}[X, Y, Z, 1] \hat{\Omega}_2^{R_2(T_2)}[X, Y, \bar{Z}, 1] \hat{\Omega}_3^{R_3(T_3)}[X, Y, Z, 1] \right\rangle. \tag{3.20}
\]
We collectively denote the three-point functions of the operators in the representation basis by
\[
\tilde{C}_{123} \equiv \langle \tilde{\Omega}_1 \tilde{\Omega}_2 \tilde{\Omega}_3 \rangle. \tag{3.21}
\]
From (3.5) we get
\[
\tilde{C}_{123} = \frac{1}{\prod_{i=1}^3 l_i! \cdot m_i! \cdot n_i! \cdot |\bar{L}_i|!} \sum_{\{U_i\} \in S_{L_i} \otimes \mathbf{1}_{T_i}} \left( \prod_{p=1}^3 h^{3}_{U_1^{(1)} U_2^{(2)} U_3^{(3)}} \right) \sum_{\{t_i \vdash \bar{L}_i\}} \left( \prod_{i=1}^3 d_{t_i} \right) \times
\]
\[
\sum_{\{\hat{\alpha}_i \in S_{L_i} \times \mathbf{1}_{T_i}\}} \left( \prod_{i=1}^3 \chi^{R_i \otimes t_i}(\hat{\alpha}_i) \right) N_c^{C(U_1^{-1} \hat{\alpha}_1 U_1 U_2^{-1} \hat{\alpha}_2 U_2 U_3^{-1} \hat{\alpha}_3 U_3)}. \tag{3.22}
\]

Consider the second line of (3.22). We use the identity (A.40) and (A.9) to obtain
\[
\sum_{\{\hat{\alpha}_i \in S_{L_i} \times \mathbf{1}_{T_i}\}} \left( \prod_{i=1}^3 \chi^{R_i \otimes t_i}(\hat{\alpha}_i) \right) N_c^{C(U_1^{-1} \hat{\alpha}_1 U_1 U_2^{-1} \hat{\alpha}_2 U_2 U_3^{-1} \hat{\alpha}_3 U_3)} \tag{3.23}
\]
\[
= \sum_{\{\hat{\alpha}_i \in S_{L_i} \times \mathbf{1}_{T_i}\}} \sum_{R \vdash L} \text{Dim}_{N_c}(\tilde{R}) \left( \prod_{i=1}^3 \chi^{R_i \otimes t_i}(\hat{\alpha}_i) \tilde{D}_{\hat{R}} \right) \tilde{D}_{\hat{R}} J_{1,2,3}(U_1 U_2^{-1}) \tilde{D}_{\hat{R}} J_{2,3,1}(U_2 U_3^{-1}) \tilde{D}_{\hat{R}} J_{3,1,2}(U_3 U_1^{-1}).
\]
We simplify the sum over $\{\hat{\alpha}_i\}$ in the last line. The character is given by (3.15). We decompose the matrix elements $D_{\hat{R}}^{\hat{R}}(\hat{\alpha}_i)$ according to the restriction
\[
S_L \downarrow (S_{L_i} \otimes S_{T_i}), \quad \hat{R} = \bigoplus_{R_i \vdash L_i} \bigoplus_{T_i \vdash \bar{L}_i} \bigoplus_{\mu_i=1} g(R_i, t_i; \tilde{R}) (R_i \otimes T_i)_{\mu_i}. \tag{3.24}
\]
When $\hat{C}_{123} = \hat{C}_{000}$, we have $R_i = R_i$. From (3.24) we get

$$\sum_{\{\lambda_i\}} \chi_{R_i^*}^{R_i^*} (\hat{\lambda}_i) D_{I_iJ_i}^R (\hat{\lambda}_i) = \sum_{\alpha_i \in S_L} \sum_{R_i^*} \sum_{I_i} \sum_{J_i} \sum_{\mu_i = 1} g(R_i^*, I_i, J_i) \chi_{R_i}^{R_i} (\alpha_i) (B_{I_iJ_i}^R (\alpha_i)) d_{I_iJ_i}^R (\alpha_i)$$

$$= \sum_{R_i^*} \sum_{I_i} \sum_{J_i} \sum_{\mu_i = 1} L_i d_{d_{I_iJ_i}}^R (\alpha_i) = \sum_{\mu_i = 1} g(R_i^*, I_i, J_i) L_i d_{I_iJ_i}^R (\alpha_i) \quad (3.25)$$

where we used (3.15), (A.20), (A.29) and (A.40). When $\hat{C}_{123} = \hat{C}^{XYZ}$, by using the definition of the restricted character (A.24) we find

$$\sum_{\{\lambda_i\} \in S_L \times 1_T} \chi_{R_i^*}^{R_i^*} (\hat{\lambda}_i) D_{I_iJ_i}^R (\hat{\lambda}_i)$$

$$= \sum_{R_i^*} \sum_{I_i} \sum_{J_i} \sum_{\mu_i = 1} L_i d_{I_iJ_i}^R (\alpha_i) \quad (3.26)$$

where we introduced the double projector

$$\mathcal{P}_{I_iJ_i}^{R_i \rightarrow RT_{i-},\mu} = \sum_{j,k,l,c} B_{I_iJ_i}^{R_i \rightarrow RT_{i-},\mu} (B_{I_iJ_i}^R) \mathcal{P}_{I_iJ_i}^{R_i \rightarrow RT_{i+}}$$

$$\mathcal{P}_{I_iJ_i}^{R_i \rightarrow RT_{i+}} = \sum_{l} d_{R_i} B_{I_iJ_i}^{R_i \rightarrow RT_{i+},\mu}$$

which come from the double restriction $S_L \downarrow (S_L \otimes S_T) \downarrow (S_i \otimes S_m \otimes S_n \otimes S_L)$. Here we should keep in mind that the restriction to the subgroup of $S_L$ is different for each $i = 1, 2, 3$. We will revisit this issue in Section 3.4.

Now the equation (3.23) is simplified as

$$\sum_{\{\lambda_i\} \in S_L \times 1_T} \left( \prod_{i=1}^{3} (\chi_{R_i^*}^{R_i^*} (\hat{\lambda}_i)) \right) N_{c} C_{U_i} (U_i U_j U_k) \quad (3.29)$$

$$= \sum_{\{\mu_i\} \in S_L} \left( \prod_{i=1}^{3} \mathcal{P}_{I_iJ_i}^{R_i \rightarrow sub} \right) D_{I_iJ_i}^{R_i} (U_i U_j U_k) \quad (3.30)$$
where the projector $\mathcal{P}^{R \to \text{sub}}_{i,j}$ is given by

$$\mathcal{P}^{R \to \text{sub}}_{i,j} = \begin{cases} 
\mathcal{P}^{R \to \text{sub}}_{(I,T),\mu,\nu} = B^{R \to (R_t,T_t),\mu} (B^T)^{R \to (R_t,T_t),\mu} (J)_{j \to (I,T)} & \text{(for } \tilde{C}_{\text{XXX}}) \\
\mathcal{P}^{R \to RT_{i,-,+}}_{i,j} = B^{R \to RT_{i,-,+}} (B^T)^{R \to RT_{i,+}} (J)_{j \to (j,k,l,c)} & \text{(for } \tilde{C}_{\text{XYZ}}) 
\end{cases}$$

(3.31)

The three-point function (3.22) becomes

$$\tilde{C}_{123} = \left( \prod_{i=1}^{3} \frac{L_i!}{l_i!} \right) \frac{1}{L!} \sum_{\{U_i\} \in S^3_L} \left( \prod_{p=1}^{L} \mathcal{H}_{(U_{12}^{(1)} U_{23}^{(2)} U_{31}^{(3)})} \right) \sum_{R+L} \text{Dim}_N(R) \times \sum_{\{T_i,\mu_i\}} \left( \prod_{i=1}^{3} \mathcal{P}^{R \to \text{sub}}_{i,j} \right) D^{R}_{j_1 l_2} (U_1 U_2^{-1}) D^{R}_{j_2 l_3} (U_2 U_3^{-1}) D^{R}_{j_3 l_1} (U_3 U_1^{-1})$$

(3.32)

where (3.15) is used to sum over $t_i$.

### 3.3 Sum over Wick contractions

We simplify the sum over the Wick contractions, denoted by $\{U_i\} \in S^3_L$ in (3.32).

#### 3.3.1 Symmetry of the permutation formula

To begin with, let us review the symmetry in the permutation formula (3.3) for a fixed $\{U_i\}$,

$$C_{123}(\{U_i\}) = \frac{1}{L_1! L_2! L_3! L!} \left( \prod_{p=1}^{L} \mathcal{H}_{(U_{12}^{(1)} U_{23}^{(2)} U_{31}^{(3)})} \right) N e^{C(U_1^{-1}\alpha_1 U_1^{-1} \alpha_2 U_2^{-1} \alpha_3 U_3^{-1})}.$$  

(3.33)

Since $\tilde{C}_{123}$ is a linear combination of $C_{123}$, the equation (3.32) should inherit the same symmetry.

First, $C_{123}(\{U_i\})$ is invariant under the simultaneous transformation

$$(U_1, U_2, U_3) \rightarrow (U_1 V_0, U_2 V_0, U_3 V_0), \quad \forall V_0 \in S_L$$

(3.34)

which corresponds to the relabeling $p \rightarrow V_0(p)$ in (3.33). Second, $C_{123}(\{U_i\})$ is invariant under the permutation of identity fields

$$(U_1, U_2, U_3) \rightarrow (V_i U_1, V_2 U_2, V_3 U_3)$$

$$(V_i, V_2, V_3) \in (1_{L_1} \otimes S_{T_1}, 1_{L_2} \otimes S_{T_2}, 1_{L_3} \otimes S_{T_3}) \subset S^3_L$$

(3.35)

which follows from the definition $\alpha_i = \alpha_i \circ 1_{T_i}$. Third, $C_{123}(\{U_i\})$ is invariant under the flavor symmetry (2.1),

$$(U_1, U_2, U_3) \rightarrow (W_1 U_1, W_2 U_2, W_3 U_3),$$

$$(W_1, W_2, W_3) \in (S_{l_1} \otimes S_{m_1} \otimes S_{n_1} \otimes 1_{T_1}, S_{l_2} \otimes S_{m_2} \otimes S_{n_2} \otimes 1_{T_2}, S_{l_3} \otimes S_{m_3} \otimes S_{n_3} \otimes 1_{T_3})$$

(3.36)

The redundancy (3.34) and (3.35) are unphysical, which should be canceled by the numerical factors $L!$ and $\prod_i L_i!$ in (3.33). The last operation (3.36) is the symmetry of the external operators, and interchanges different Wick contractions.

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3.3.2 Fixing redundancy

Let us rewrite the flavor factor $\prod_p h^{ABC}$ in (3.33) as

$$\tilde{\mathcal{H}} \left[ \hat{A}^{(i)}_{U_i(p)} \right] = \prod_{p=1}^{L} h^{(1)}_{U_1(p)} \hat{A}^{(2)}_{U_2(p)} \hat{A}^{(3)}_{U_3(p)}$$

(3.37)

where $\left[ \hat{A}^{(i)}_{U_i(p)} \right]$ is the $3 \times L$ Wick-contraction matrix\(^3\)

$$\left[ \hat{A}^{(i)}_{U_i(p)} \right] = \left[ \begin{array}{ccc} \hat{A}^{(1)}_{U_1(1)} & \cdots & \hat{A}^{(1)}_{U_1(L)} \\ \hat{A}^{(2)}_{U_2(1)} & \cdots & \hat{A}^{(2)}_{U_2(L)} \\ \hat{A}^{(3)}_{U_3(1)} & \cdots & \hat{A}^{(3)}_{U_3(L)} \end{array} \right].$$

(3.38)

Note that the position of each column is unimportant for computing the flavor factor (3.37),

$$\left[ \hat{A}^{(i)}_{U_i(p)} \right] \simeq \left[ \hat{A}^{(i)}_{U_i(\sigma(p))} \right], \quad \forall \sigma \in S_L.$$

(3.39)

We fix the redundancy of $V_0$ in (3.34) as follows. Let us choose the position of the identity fields for each operator as

$$\Phi \hat{A}^{(1)}_{U_1(p)} = 1_p, \quad (p = 1, 2, \ldots, \ell_1)$$

$$\Phi \hat{A}^{(2)}_{U_2(p)} = 1_p, \quad (p = \ell_1 + 1, \ell_1 + 2, \ldots, \ell_1 + \ell_2)$$

$$\Phi \hat{A}^{(3)}_{U_3(p)} = 1_p, \quad (p = \ell_1 + \ell_2 + 1, \ell_1 + \ell_2 + 2, \ldots, L).$$

(3.40)

Here the subscript of $1$ is a dummy index, which will disappear after the identification (3.39). The Wick-contraction matrix becomes

$$\left[ \hat{A}^{(i)}_{U_i(p)} \right] = \left[ \begin{array}{cccc} 1 & \cdots & 1_{\ell_1} & \hat{A}^{(1)}_{U_1(\ell_1+1)} & \cdots & \hat{A}^{(1)}_{U_1(L)} \\ \hat{A}^{(2)}_{U_2(1)} & \cdots & \hat{A}^{(2)}_{U_2(\ell_2+1)} & 1_{\ell_2+1} & \cdots & 1_L \\ \hat{A}^{(3)}_{U_3(1)} & \cdots & \hat{A}^{(3)}_{U_3(\ell_3+1)} & \hat{A}^{(3)}_{U_3(L)} & 1_{L+1} & \cdots & 1_L \end{array} \right].$$

(3.41)

The residual redundancy of $V_0$ is now $V_0' \in S_{\ell_1} \otimes S_{\ell_2} \otimes S_{\ell_3}$.

After the partial gauge fixing (3.40), $\{U_i\}$ permute the non-identity fields only,

$$U_1 \in S_{\ell_1} \otimes 1_{\ell_2}, \quad U_2 \in S_{\ell_2} \otimes 1_{\ell_3}, \quad U_3 \in S_{\ell_3} \otimes 1_{\ell_2}.$$ 

(3.42)

There is still residual redundancy generated by a combination of $V_0'$ and $V_i$ in (3.35),

$$\tilde{V} : \{U_i\} \mapsto \{U_i'\}, \quad \hat{A}_{U_i(p)}^{(i)} = \begin{cases} 1_p & \text{(if } \hat{A}_{U_i(p)}^{(i)} = 1_p) \\ \hat{A}^{(i)}_{U_i(p)} \hat{A}_V^{-1} & \text{(if } \hat{A}_{U_i(p)}^{(i)} \neq 1_p) \end{cases}$$

(3.43)

for any $\tilde{V} \in S_{\ell_1} \otimes S_{\ell_2} \otimes S_{\ell_3}$. This map does not permute identity fields, but permutes the non-identity fields sitting in the same column.

\(^3\)Each element of this matrix represents the flavor data. Note that this notation is slightly different from [18], where the Wick-contraction matrix is defined by the color data.
3.3.3 Counting inequivalent Wick contractions

We pick up one set of partially gauge-fixed permutations \( \{ U_i^\bullet \} \) such that \( \prod_{p=1}^L h_{\hat{A}(1)_p;\hat{A}(2)_p;\hat{A}(3)_p} \neq 0 \). We generate other \( \{ U_i \} \) by applying the flavor symmetry, \( U_i^\bullet \rightarrow W_i U_i^\bullet \) in (3.36).

This procedure generates all non-vanishing Wick pairings. To show this, consider two sets of permutations \( \{ U_i^\bullet \} \) and \( \{ U_i^\circ \} \), both of which are subject to the partial gauge fixing (3.42) and giving the non-vanishing flavor factor (3.37). Define

\[
U_i^\bullet \equiv W_i^\bullet U_i^\circ, \quad W_i^\bullet \in S_{L_i} \otimes 1_{\mathcal{T}_i}.
\]

(3.44)

Since any permutation consists of a product of transpositions, we may assume \((W_1^\bullet, W_2^\bullet, W_3^\bullet) = ((ab), 1, 1) \in S_{L_1} \otimes S_{L_2} \otimes S_{L_3}\) without loss of generality. Let us represent the Wick contractions of \( \{ U_i^\bullet \} \) by

\[
\left\langle \text{tr} (\Phi_{A(1)} \hat{A}_{(3)} \ldots \text{tr} (\Phi_{A(1)} \hat{A}_{(3)} \ldots \text{tr} (\Phi_{A(1)} \hat{A}_{(3)} \ldots ) \rightangle = \left\langle \Phi_{A(1)} \Phi_{A(2)} \Phi_{A(3)} \ldots \right\rangle \neq 0.
\]

(3.45)

Then, the Wick contractions of \( \{ U_i^\circ \} \) are written as

\[
\left\langle \text{tr} (\Phi_{A(1)} \hat{A}_{(3)} \ldots \text{tr} (\Phi_{A(1)} \hat{A}_{(3)} \ldots \text{tr} (\Phi_{A(1)} \hat{A}_{(3)} \ldots ) \rightangle = \left\langle \Phi_{A(1)} \Phi_{A(2)} \Phi_{A(3)} \ldots \right\rangle \neq 0.
\]

(3.46)

Since both (3.45) and (3.46) are non-zero, and since \( \Phi = (X, Y, Z) \) have orthogonal inner products, we should have \( \Phi_{A(1)} = \Phi_{A(1)}^\circ \). This implies that \( W_i^\circ \in S_{l_i} \otimes S_{m_i} \otimes S_{n_i} \otimes 1_{\mathcal{T}_i} \), which is part of the flavor symmetry (3.36).

The range of \( \{ U_i \} \) in (3.42) now becomes

\[
U_1 \in S_{l_1} \otimes S_{m_1} \otimes S_{n_1} \otimes 1_{\mathcal{T}_1} \equiv S_1 \\
U_2 \in S_{l_2} \otimes S_{m_2} \otimes S_{n_2} \otimes 1_{\mathcal{T}_2} \equiv S_2 \\
U_3 \in S_{l_3} \otimes S_{m_3} \otimes S_{n_3} \otimes 1_{\mathcal{T}_3} \equiv S_3
\]

(3.47)

The sum over \( (S_1, S_2, S_3) \) counts each inequivalent Wick pairing more than once. The multiplicity comes from the residual redundancy (3.43),

\[
|S_{L_1} \otimes S_{L_2} \otimes S_{L_3}| = L_1! L_2! L_3!.
\]

(3.48)

The number of inequivalent Wick contractions is given by

\[
|\text{Wick}| \equiv \left| \frac{S_{L_1} \otimes S_{L_2} \otimes S_{L_3}}{S_{L_1} \otimes S_{L_2} \otimes S_{L_3}} \right| = \prod_{i=1}^3 \frac{l_i! m_i! n_i!}{L_i}
\]

(3.49)

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3.3.4 The OPE coefficients simplified

We collected all non-vanishing Wick contractions by restricting the sum \( \{U_i\} \) over the ranges (3.31). The OPE coefficient (3.32) becomes

\[
\tilde{C}_{123} = \left( \prod_{i=1}^{3} \frac{L_i!}{l_i! m_i! n_i!} \right) \sum_{R \vdash L} \frac{\dim_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \times \sum_{\{T_i, \mu_i\}} \sum_{u_1 \in S_1} \sum_{u_2 \in S_2} \sum_{u_3 \in S_3} \sum_{l_1, l_2} (U_1 U_2^{-1}) D_{j_1 l_2}^R (U_2 U_3^{-1}) D_{j_2 l_3}^R (U_3 U_1^{-1}).
\] (3.50)

Recall that the projector is equal to the product of branching coefficients, \( \mathcal{P} = \mathcal{B} \mathcal{B}^T \) as in (3.31). We can simplify the second line by using the identity of branching coefficients (A.21)

\[
\sum_j D_{j f}^R (u \circ v \circ w) B_{j \to (j, k, l)}^{R \to (q, r, s) \nu} = \sum_{a, b, c} D_{a j}^q (u) D_{b k}^r (v) D_{c l}^s (w) B_{I \to (a, b, c)}^{R \to (q, r, s) \nu}.
\] (3.51)

If we bring \( U_k = u_k \otimes v_k \otimes w_k \) and \( U_k^{-1} = u_k^{-1} \otimes v_k^{-1} \otimes w_k^{-1} \) across the double branching coefficients \( \mathcal{B} \) or \( \mathcal{B}^T \), they annihilate each other; see (3.54).

Let us define a triple-projector product

\[
\mathcal{I}_{123}^{R \to \text{sub}} \equiv \mathcal{P}_{I_1 I_2}^{R \to \text{sub}} \mathcal{P}_{I_2 I_3}^{R \to \text{sub}} \mathcal{P}_{I_3 I_1}^{R \to \text{sub}}
\] (3.52)

where we used the symbols \( \mathcal{P} \) and \( \mathcal{P}^{\dagger} \) to keep in mind that the branching coefficients come from different restrictions of \( S_L \). Then

\[
\tilde{C}_{123} = \left( \prod_{i=1}^{3} \frac{L_i!}{l_i! m_i! n_i!} \right) |\text{Wick}| \sum_{R \vdash L} \frac{\dim_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{\{T_i, \mu_i\}} \mathcal{I}_{123}^{R \to \text{sub}} \]
\[
= \left( \prod_{i=1}^{3} \frac{L_i!}{l_i!} \right) \sum_{R \vdash L} \frac{\dim_{N_c}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{\{T_i, \mu_i\}} \mathcal{I}_{123}^{R \to \text{sub}}
\] (3.53)

where we used (3.49).
In the notation of the quiver calculus in Appendix B, we can express the above calculation as

\[ \tilde{C}_{123} \sim \sum_{\hat{R} \vdash L} \text{Dim}_{Nc}(\hat{R}) \sum_{\{U_i \in S_i\}} \hat{R}_{1} \hat{R}_{2} \hat{R}_{3} \sum_{\nu_1, \nu_2, \nu_3} U_{1}^{\nu_1} U_{2}^{\nu_2} U_{3}^{\nu_3} \]

\[ \sim \sum_{\hat{R} \vdash L} \text{Dim}_{Nc}(\hat{R}) \| \text{Wick} \| \]

(3.54)

From this diagram, we see that \( \mathcal{I}_{123}^{R \to \text{sub}} \) in (3.52) is also a triple product of the transformation matrices (A.16).

### 3.4 Sum over the triple-projector products

We compute the OPE coefficients by evaluating a sum over the triple-projector products,

\[ \sum_{\{T_i, \mu_i\}} \mathcal{I}_{123}^{\hat{R} \to \text{sub}} = \sum_{T_1 + T_2 + T_3} \sum_{\mu_1 + \mu_2 + \mu_3} \mathcal{R}^{\hat{R} \to \text{sub}}_{I_1 I_2} \mathcal{R}^{\hat{R} \to \text{sub}}_{I_2 I_3} \mathcal{R}^{\hat{R} \to \text{sub}}_{I_3 I_1} \]

(3.55)

where the projector is given by (3.31). The main idea is to decompose each projector further into a sum of sub-projectors, so that we can make use of the orthogonality of the sub-projectors on the fully-split space, \( V_{FS} \).

Below we discuss the two cases \( \tilde{C}_{ooo} \) in (3.18) and \( \tilde{C}_{XYZ}^{XYZ} \) in (3.20) separately.

#### 3.4.1 Case of \( \tilde{C}_{ooo} \)

Recall that \( \tilde{C}_{ooo} \) is a linear combination of \( C_{ooo} \) given in (3.7). The Wick-contraction matrix of \( C_{ooo} \) after a partial gauge-fixing (3.41) is given by

\[
\left[ \hat{A}_{U,(p)}^{(i)} \right] = \begin{bmatrix}
1_1 & \cdots & 1_{L_1} & Z_{t}(L_3) & Z_{U_1(L_3+1)} & \cdots & Z_{U_1(L)} \\
\tilde{Z}_{u_1} & \cdots & \tilde{Z}_{u_1(L_3)} & 1_{L_3} & \tilde{Z}_{U_2(L_3+1)} & \cdots & \tilde{Z}_{U_2(L)} \\
\tilde{Z}_{u_3} & \cdots & \tilde{Z}_{u_3(L_3)} & \tilde{Z}_{U_3(L_3+1)} & 1_{L_3+1} & \cdots & 1_L \\
\end{bmatrix}
\]

(3.56)
which shows that \( S_i = S_L \otimes S_{\tilde{L}_i} \) in place of \( (3.47) \). We represent \( (3.56) \) as in the following figure,

\[
\begin{array}{c|c|c|c}
\hat{O}_1 & 1 & Z \\
& & U_1 U_2^{-1} \\
\hat{O}_2 & \bar{Z} & 1 \\
& & U_2 U_3^{-1} \\
\hat{O}_3 & Z & \bar{Z} \\
& & U_3 U_4^{-1} \\
\end{array}
\]

(3.57)

Let us choose the fully-split space as

\[
V_{FS} = V_{T_1} \otimes V_{T_2} \otimes V_{T_3}
\]

(3.58)

which induces the restriction \( S_L \downarrow S_{FS} \), where

\[
S_{FS} = S_{T_1} \otimes S_{T_2} \otimes S_{T_3}.
\]

(3.59)

On the space \( V_{FS} \), the states decompose as

\[
\left| \hat{R}_I \right| = \left| R_i T_i \atop I_i \ c_i \right| \left( B^T \right)_{I \rightarrow (I_i, c_i)} = \left| Q_i Q'_i T_i \atop b_i \ b'_i \ c_i \right| \left( B^T \right)_{I \rightarrow (I_i, c_i)} \left( B^T \right)_{I \rightarrow (b_i, b'_i)}
\]

(3.60)

where we used \( (A.13) \). We introduce the fully-split branching coefficients by

\[
\mathfrak{B} \hat{R} \rightarrow (R_i T_i)_{\mu_i \rightarrow (Q_i, Q'_i, T_i), \rho_i, \rho_i} = \sum_{I_i=1}^{d_{R_i}} B_{I \rightarrow (I_i, c_i)} B_{I \rightarrow (b_i, b'_i)}
\]

(3.61)

and the corresponding sub-projector by

\[
\mathfrak{P} \hat{R} \rightarrow (R_i T_i)_{\mu_i \rightarrow (Q_i, Q'_i, T_i), \rho_i, \rho_i}
\]

\[
= \sum_{b_i, b'_i, c_i} \mathfrak{B} \hat{R} \rightarrow (R_i T_i)_{\mu_i \rightarrow (Q_i, Q'_i, T_i), \rho_i, \rho_i} \left( B^T \right)_{I \rightarrow (b_i, b'_i)}
\]

(3.62)

We rewrite the original projectors in \( (3.31) \) as a sum over sub-projectors on \( V_{FS} \) as

\[
\mathfrak{G} \hat{R} \rightarrow (R_1 T_1)_{\mu_1, \rho_1} = \sum_{Q_1, Q'_1, \rho_1} \mathfrak{P} \hat{R} \rightarrow (R_1 T_1)_{\mu_1 \rightarrow (Q_1, Q'_1, T_1), \rho_1, \rho_1}
\]

\[
\mathfrak{G} \hat{R} \rightarrow (R_2 T_2)_{\mu_2, \rho_2} = \sum_{Q_2, Q'_2, \rho_2} \mathfrak{P} \hat{R} \rightarrow (R_2 T_2)_{\mu_2 \rightarrow (Q_2, Q'_2, T_2), \rho_2, \rho_2}
\]

\[
\mathfrak{G} \hat{R} \rightarrow (R_3 T_3)_{\mu_3, \rho_3} = \sum_{Q_3, Q'_3, \rho_3} \mathfrak{P} \hat{R} \rightarrow (R_3 T_3)_{\mu_3 \rightarrow (Q_3, Q'_3, T_3), \rho_3, \rho_3}
\]

(3.63)
By construction, all sub-projectors follow from the same restriction

$$S_L \downarrow S_{FS}, \quad \hat{R} = \bigoplus_{Q,Q',T} g(Q,Q',T;\hat{R}) \bigoplus_{\gamma=1} (Q \otimes Q' \otimes T)_{\eta}$$  \hspace{1cm} (3.64)$$

and all sub-representations should be synchronized when evaluating $\mathcal{I}^{\hat{R} \rightarrow \text{sub}}_{123}$ in \((3.53)\). The states can also be decomposed as

$$\left| \begin{array}{c} \hat{R} \\ \hat{I} \end{array} \right> = |Q Q' T \rangle (b b' c \eta)_{\gamma}$$  \hspace{1cm} (3.65)$$

in addition to \((3.60)\). The consistency of the two decompositions suggests that the multiplicity labels can be rewritten as

$$\xi_i \equiv \{ \mu_i, \rho_i \}, \quad 1 \leq \xi_i \leq g(Q_i, Q'_i; R_i) g(R_i, T_i; \hat{R}).$$  \hspace{1cm} (3.66)$$

In \((3.63)\), the representations $T_i$ come from the Fourier transform of identity fields $1$, and $Q_i, Q'_i$ come from the non-identity fields, $Z, \tilde{Z}, \tilde{Z}$. Since the OPE coefficient $C_{\text{oo}o}$ has the Wick-contraction structure given in \((3.57)\), we should identify the representations $\{ Q_i, Q'_i, T_i \}$ with those acting on the constituent of $V_{FS}$ as

$$T_1 = Q'_2 = Q_3 \in \text{Hom}(V_{\tilde{Z}_1})$$

$$Q_1 = T_2 = Q'_3 \in \text{Hom}(V_{\tilde{Z}_2})$$

$$Q'_1 = Q_2 = T_3 \in \text{Hom}(V_{\tilde{Z}_3}).$$  \hspace{1cm} (3.67)$$

We can show \((3.67)\) from another argument. The triple-projector product is equal to the product of generalized Racah-Wigner tensors in Appendix [C]

$$\text{tr} \left( \Psi_{I,J}^{\hat{R} \rightarrow \cdots \rightarrow (Q_1, Q'_1, T_1), \xi_1} \Psi_{I,J}^{\hat{R} \rightarrow \cdots \rightarrow (Q_2, Q'_2, T_2), \xi_2} \Psi_{I,J}^{\hat{R} \rightarrow \cdots \rightarrow (Q_3, Q'_3, T_3), \xi_3} \right) = \text{tr} (U_R \tilde{U}_R \tilde{U}_R)$$  \hspace{1cm} (3.68)$$

which we conjecture as \([C,19]\),

$$\text{tr} (U_R \tilde{U}_R \tilde{U}_R) = \delta^{T_1 Q_2} \delta^{Q'_2 Q_3} \delta^{Q_1 T_2} \delta^{T_2 Q_3} \delta^{Q'_1 Q_2} \delta^{Q_2 T_3} \left( \prod_{i=1}^{3} d_{Q_i} \right) \mathcal{G}_{123}$$  \hspace{1cm} (3.69)$$

$$\mathcal{G}_{123} = \frac{g(Q_1, Q_2; R_1) g(R_1, Q_3; \hat{R}) g(Q_2, Q_3; R_2) g(R_2, Q_1; \hat{R}) g(Q_3, Q_1; R_3) g(R_3, Q_2; \hat{R})}{g(Q_1, Q_2, Q_3; \hat{R})^2}.$$ 

The three-point function \((3.53)\) becomes

$$\tilde{C}_{\text{ooo}} = \left( \prod_{i=1}^{3} \frac{L_i!}{L_i!} \right) \sum_{\hat{R} \rightarrow \mathcal{I}} \frac{\text{Dim}_{\mathcal{I}}(\hat{R})}{d_{R_1} d_{R_2} d_{R_3}} \sum_{Q_1 \rightarrow \mathcal{I}} \sum_{Q_2 \rightarrow \mathcal{I}} \sum_{Q_3 \rightarrow \mathcal{I}} \left( \prod_{i=1}^{3} d_{Q_i} \right) \mathcal{G}_{123}.$$  \hspace{1cm} (3.70)$$

Here, the Littlewood-Richardson coefficients in $\mathcal{G}_{123}$ put constraints on the sum over $\{Q_i\}$. In other words, we should find all $\{Q_i\} = \{Q'_i\}$ such that

$$R_1 = Q'_1 \otimes Q'_2, \quad R_2 = Q'_2 \otimes Q'_3, \quad R_3 = Q'_3 \otimes Q'_1, \quad \hat{R} = Q'_1 \otimes Q'_2 \otimes Q'_3.$$  \hspace{1cm} (3.71)$$
The conditions (3.71) can be summarized as

\[ \tilde{O}_1 \quad \tilde{O}_2 \quad \tilde{O}_3 \]

\[ \begin{array}{ccc}
Q_3^* & R_1 & \\
Q_1^* & R_2 & \\
R_3 & Q_2^* & \tilde{R}
\end{array} \]

(3.72)

**Extremal case.** As a check, consider the situation \( L_1 + L_2 = L_3 = L \). From (3.72), this corresponds to

\[ Q_2 = \emptyset, \quad R_1 = Q_1, \quad R_2 = Q_3, \quad \tilde{R} = R_3. \]

(3.73)

We get

\[ G_{123} = \frac{g(R_1, Q_3; \tilde{R}) g(R_2, Q_1; \tilde{R}) g(Q_3, Q_1; R_3)}{g(Q_1, Q_3; R)^2} = g(R_1, R_2; R_3) \]

(3.74)

and therefore

\[ \tilde{C}_{ooo} = L_3! \frac{\text{Dim}_{Nc}(R_3)}{d_{R_3}} g(R_1, R_2; R_3). \]

(3.75)

This result agrees with the literature [19] including the normalization of the two-point function given in (2.13).

### 3.4.2 Case of \( \tilde{C}_h^{XYZ} \)

Our discussion is quite parallel to Section 3.4.1. Recall that \( \tilde{C}_h^{XYZ} \) is a linear combination of \( C_h^{XYZ} \) given in (3.9). We represent the Wick-contraction matrix by

\[ \hat{O}_1 \quad \hat{O}_2 \quad \hat{O}_3 \]

\[ \begin{array}{ccc}
\ell_{31} - h_2 & 1 & \ell_{23} - h_3 \quad 1 & Z_{\ell_{12} - h_3 + h_2} \\
U_1U_2^{-1} & & & \\
1 & \ell_{23} - h_1 & 1 & \ell_{12} - h_3 \quad 1 \\
U_2U_3^{-1} & & & \\
X_{\ell_{31} - h_2 + h_1} & 1 & \ell_{23} - h_1 & 1 & \ell_{12} \quad \ell_3 \quad U_3U_1^{-1} & \\
1, 2, \ldots & \ldots, L & & & & &
\end{array} \]

(3.76)

where \( h_i \) are constrained by (3.11),

\[ 0 \leq h_1 \leq \ell_{23} = \ell_1, \quad 0 \leq h_2 \leq \ell_{31} = \ell_2, \quad 0 \leq h_3 \leq \ell_{12} = \ell_3. \]

(3.77)
We choose the fully-split space as
\[
V_{FS} = V_{\ell_31-h_2} \otimes V_{h_1} \otimes V_{h_3} \otimes V_{\ell_23-h_1} \otimes V_{\ell_12-h_3} \otimes V_{h_2}
\] (3.78)
and decompose the original projectors (3.31). From (3.76), one finds that the new branch coefficients are needed for
\[
\begin{align*}
S_{\ell_23-h_3+h_2} & \downarrow (S_{\ell_23-h_3} \otimes S_{h_2}) \quad \text{and} \quad S_{\ell_31} \downarrow (S_{h_1} \otimes S_{\ell_23-h_1}) \quad \text{for } O_1 \\
S_{\ell_23-h_3+h_1} & \downarrow (S_{\ell_23-h_3} \otimes S_{h_3}) \quad \text{and} \quad S_{\ell_31} \downarrow (S_{\ell_31-h_2} \otimes S_{h_2}) \quad \text{for } O_2 \\
S_{\ell_23-h_3+h_1} & \downarrow (S_{\ell_31-h_2} \otimes S_{h_1}) \quad \text{and} \quad S_{\ell_12} \downarrow (S_{h_3} \otimes S_{\ell_12-h_3}) \quad \text{for } O_3.
\end{align*}
\] (3.79)
For example, we rewrite the states for $O_1$ on the space $V_{FS}$ as
\[
\left[\hat{R}_{\hat{I}}\right] = \left[ R_1 T_1 \right]_{I_1 \left( c_1 \right)} \mu_1 \left( B^T \right)_{I \rightarrow (I_1, c_1)} R^{-1}(R_1, T_1, \mu_1)_{I \rightarrow (I_1, c_1)}
\]
\[
= \left[ q_1 r_1 s_1 T_1 \right]_{I_1 \left( c_1 \right)} \mu_1 \left( B^T \right)_{I \rightarrow (I_1, c_1)} R^{-1}(R_1, T_1, \mu_1)_{I \rightarrow (I_1, c_1)} (B^T)_{I \rightarrow (I_1, c_1)} R_1 \rightarrow (q_1, r_1, s_1, \nu_1)_{I_1 \rightarrow (I_1, c_1)} \\
= \left[ q_1 r_1 s_1'' t''_1 \right]_{I_1 \left( c_1 \right)} \mu_1 \left( B^T \right)_{I \rightarrow (I_1, c_1)} R^{-1}(R_1, T_1, \mu_1)_{I \rightarrow (I_1, c_1)} (B^T)_{I \rightarrow (I_1, c_1)} R_1 \rightarrow (q_1, r_1, s_1'' \nu_1)_{I_1 \rightarrow (I_1, c_1)} (B^T)_{I \rightarrow (I_1, c_1)} R_1 \rightarrow (t''_1, \nu_1)_{I_1 \rightarrow (I_1, c_1)} \times (B^T)_{I \rightarrow (I_1, c_1)} R_1 \rightarrow (t''_1, \nu_1)_{I_1 \rightarrow (I_1, c_1)}
\]
and introduce the fully-split branching coefficients by
\[
\mathcal{B}_{I \rightarrow (I_1, k, l, t''_1, \nu_1)}^{R \rightarrow (R_1, T_1, \mu_1)} = B_{I \rightarrow (I_1, c_1)}^{R \rightarrow (R_1, T_1, \mu_1)} B_{I \rightarrow (I_1, c_1)}^{R_1 \rightarrow (q_1, r_1, s_1'' \nu_1)} B_{I \rightarrow (I_1, c_1)}^{R_1 \rightarrow (t''_1, \nu_1)} B_{I \rightarrow (I_1, c_1)}^{R_1 \rightarrow (t''_1, \nu_1)}
\] (3.81)
The original projector (3.31) becomes a sum over the sub-projectors $\mathcal{P} = \mathcal{B} \mathcal{B}^T$,
\[
\mathcal{B}_{I \rightarrow (I_1, k, l, t''_1, \nu_1)}^{R \rightarrow (R_1, T_1, \mu_1)} = \sum_{s'_1, s''_1, t'_1, t''_1, \rho_1, \xi_1} \mathcal{P}_{I \rightarrow (I_1, k, l, t''_1, \nu_1)}^{R \rightarrow (R_1, T_1, \mu_1)}
\] (3.82)
and similarly
\[
\mathcal{B}_{I \rightarrow (I_1, k, l, t''_1, \nu_1)}^{R \rightarrow (R_1, T_1, \mu_1)} = \sum_{s'_1, s''_1, t'_1, t''_1, \rho_1, \xi_1} \mathcal{P}_{I \rightarrow (I_1, k, l, t''_1, \nu_1)}^{R \rightarrow (R_1, T_1, \mu_1)}
\] (3.83)
When summing over $\{t'_1, t''_1\}$ we can forget the constraint $t'_1 \otimes t''_1 \simeq T_1$, because the OPE coefficient (3.53) contains sums over $\{T_1\}$. All sub-projectors come from the irreducible decompositions of $\hat{R}$ under the restriction $S_L \downarrow S_{FS}$ ,
\[
\hat{R} = \bigoplus_{g(q, q', r, r', s, s') \hat{R}} \bigoplus_{\eta=1} (q' \otimes q'' \otimes r' \otimes r'' \otimes s' \otimes s'')_{\eta}
\] (3.84)
Since the OPE coefficient $C_{h_1}^{XYZ}$ has the Wick contraction structure of (3.76), we should identify the representations as
\[
\begin{align*}
q_1 = t'_3 & \in \text{Hom}(V_{\ell_31-h_2}) \\
r_1 = r'_3 & \in \text{Hom}(V_{h_1}) \\
s'_1 = s_2 & \in \text{Hom}(V_{\ell_31-h_2}) \\
q_2 = t''_3 & \in \text{Hom}(V_{h_1}) \\
r_2 = r''_3 & \in \text{Hom}(V_{h_1}) \\
s''_1 & \in \text{Hom}(V_{\ell_31-h_2})
\end{align*}
\] (3.85)
and replace the multiplicity labels by
\[ \xi_{i+} = \{ \mu_i, \nu_{i+}, \rho_i, \xi_i \}. \] (3.86)

Again, the trace over the product of sub-projectors is given by the generalized Racah-Wigner tensors \( \langle C \rangle \),
\[
\text{tr}_R \left( \Psi_{I_1 I_2} \rightarrow (q_1, r_1, s_1', t_1', t_1'') \xi_1 \rightarrow \xi_{i+} \right) \Psi_{I_1 I_3} \rightarrow (q_2, r_2, s_2, t_2, t_2') \xi_2 \rightarrow \xi_{i+}
\]
\[
\text{tr}_R \left( \Psi_{I_1 I_2} \rightarrow (q_3, r_3, s_3, t_3, t_3'') \xi_3 \rightarrow \xi_{i+} \right) \]

The three-point function \( (3.87) \)
\[
\text{tr} (W_R \tilde{W}_R \tilde{W}_R) = \left( D_{123} d_{q_1} d_{q_2} d_{q_3} d_{r_1} d_{r_2} d_{r_3} d_{s_2} d_{s_3} \right) \delta^{\xi_1 - \xi_2 +} \bar{\delta}^{\xi_2 - \xi_3 +} \bar{\delta}^{\xi_3 - \xi_1 +}
\]
\[
D_{123} = \delta^{q_1 t_2} \delta^{q_2 t_3} \delta^{q_3 t_2} \delta^{q_1 r_2} \delta^{q_2 r_3} \delta^{q_3 r_2} \delta^{q_1 s_2} \delta^{q_2 s_3} \delta^{q_3 s_1} \delta^{t_1 t_2} \delta^{t_2 t_3} \delta^{t_3 t_1} \delta^{s_1 s_2} \delta^{s_2 s_3} \delta^{s_3 s_1}.
\]

We need to sum over the representations and multiplicity labels. We conjecture that the result is given by \( (3.88) \),
\[
\sum_{\xi_1 \xi_2 \xi_3} \text{tr} (W_R \tilde{W}_R \tilde{W}_R) = \left( D_{123} d_{q_1} d_{q_2} d_{q_3} d_{r_1} d_{r_2} d_{r_3} d_{s_2} d_{s_3} \right) \bar{\delta}^{\nu_1 - \nu_2 +} \bar{\delta}^{\nu_2 - \nu_3 +} \bar{\delta}^{\nu_3 - \nu_1 +} G'_{123}
\]
\[
G'_{123} = \frac{M_{R_1,s_1,\nu_1-} M_{R_1,s_1,\nu_1+} M_{R_2,r_2,\nu_2-} M_{R_2,r_2,\nu_2+} M_{R_3,t_3,\nu_3-} M_{R_3,t_3,\nu_3+}}{M_{\text{tot}}^3}
\]

where \( M_{R,r,v} \) is the slice of the total multiplicity space constrained by \( (R, r, v) \).

The three-point function \( (3.53) \) becomes
\[
\tilde{C}_{h}^{XYZ} = \left( \prod_{i=1}^{3} \frac{L_i!}{L_i!} \right) \sum_{R \vdash L} \frac{\text{Dim}_N(R)}{d_{R_1} d_{R_2} d_{R_3}} (d_{q_1} d_{q_2} d_{q_3} d_{r_1} d_{r_1} d_{r_3} d_{s_1} d_{s_2} d_{s_3}) \bar{\delta}^{\nu_1 - \nu_2 +} \bar{\delta}^{\nu_2 - \nu_3 +} \bar{\delta}^{\nu_3 - \nu_1 +} G'_{123}. \] (3.90)

Here \( \{ q_i, r_i, s_i \} \) must be consistent with \( R_3 \) in \( (3.14) \). This condition is implicitly included in the definition of \( \bar{\delta} \) in \( (C.36) \). In other words, the OPE coefficients are non-zero only if \( (q_1, q_2, r_1, r_3, s_2, s_3) \)
\[
q_1 \otimes q_2 = q_3, \quad r_1 \otimes r_3 = r_2, \quad s_2 \otimes s_3 = s_1, \quad q_1 \otimes q_2 \otimes r_1 \otimes r_3 \otimes s_2 \otimes s_3 = R
\]
\[
(R_1)_{\nu_{i+}} = q_1 \otimes r_1 \otimes (s_2 \otimes s_3), \quad (R_2)_{\nu_{i+}} = q_2 \otimes (r_1 \otimes r_3) \otimes s_2, \quad (R_3)_{\nu_{i+}} = (q_1 \otimes q_2) \otimes r_3 \otimes s_3
\]
which can be represented by

\[
\begin{array}{c|ccc|c}
\text{O}_1 & q_1 & q_2 & r_1 & r_3 & s_1 \\
\text{O}_2 & q_1 & q_2 & r_2 & r_3 & s_2 & s_3 \\
\text{O}_3 & q_3 & r_1 & r_3 & s_2 & s_3 \\
\end{array}
\]

\[
\tilde{R}
\]

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We find some difference from the case of \( \tilde{C}_{\text{occ}} \) in \[(3.70)\]. First, we do not have a sum over \((q^1, q^2_1, r^1_3, s^2_2, s^3_3)\). This is because \( \tilde{C}_{\text{XYZ}}^\rightarrow \) has the same structure of the Wick contractions as the extremal correlators for each flavor \( X, Y, Z \) \[4\] Thus, the first line of \[(3.91)\] is trivial. Second, there is no sum over \( \{\nu_{\pm}\} \), because \( \{\nu_{\pm}\} \) are part of the operator data \( R_i = \{R_i, (q_i, r_i, s_i), \nu_{\pm}, \nu_{\mp}\} \). We should pick up the right combination of multiplicities consistent with \( R_i \).

**Extremal case.** Consider the situation where the operators consist of \( Z \) or \( \overline{Z} \) only. This means

\[
0 = h_1 = \ell_{31} - h_3, \quad \ell_{23} = 0, \quad V_{FS} = V_{t_{12}} \otimes V_{t_{31}}
\]

\[
q_i = r_i = \emptyset, \quad R_i = s_i, \quad \tilde{R} = R_1.
\]

(3.93)

In particular, we do not need to specify \( \nu_{\pm} \).

The quantity \( G'_{123} \) becomes

\[
G'_{123} = \frac{|M_{R_1}|^2 |M_{R_2}|^2 |M_{R_3}|^2}{|M_{\text{tot}}|^3} = g(R_2, R_3; R_1)
\]

(3.94)

where we used

\[
|M_{R_1}| = 1, \quad |M_{R_2}| = |M_{R_3}| = |M_{\text{tot}}| = g(R_2, R_3; R_1).
\]

(3.95)

The three-point function \[(3.90)\] becomes

\[
\tilde{C}_{h_{\text{XYZ}}}^\rightarrow = L_1! \frac{\text{Dim}_{N_c}(R_1)}{d_{R_1}} g(R_2, R_3; R_1)
\]

(3.96)

which agrees with \[(3.75)\] after relabeling.

In Appendix C.3 we consider the restricted Littlewood-Richardson coefficients, which are related to the extremal three-point functions of different type.

## 4 Background independence at large \( N_c \)

We study the tree-level three-point functions in the representation basis, and check the background independence conjectured in \[43\]. Our proof is based on the conjectured relations for the generalized Racah-Wigner tensor in Appendix C.

### 4.1 The LLM operators

Let us review the argument on the large-\( N_c \) background independence \[43\]. They mapped the \( \mathcal{N} = 4 \) SYM operators with the \( \mathcal{O}(N_c^0) \) canonical dimensions to those with the \( \mathcal{O}(N_c^2) \) canonical dimensions by attaching a large number of background boxes. We call the latter LLM operators, because they correspond to stringy excitations on the LLM geometry. Recall that the LLM geometries are the half-BPS solutions of IIB supergravity. This implies that the addition of \( \mathcal{O}(N_c^2) \) boxes should consist of a single holomorphic scalar like \( \sim Z^{N_c^2} \).

\[4\] Recall that \( \langle ZZ \rangle = 0 \) whereas any of \( \langle \overline{Z} \overline{Z} \rangle, \langle Z \overline{Z} \rangle, \langle \overline{Z} Z \rangle \) are non-zero.
For simplicity, we consider the operator mixing in the $\mathfrak{su}(2)$ sector, at one-loop in $\lambda$ at any $N_c$. We expand the dilatation eigenstates in terms of the restricted Schur basis as

$$\mathcal{D}_1 \mathcal{O}_\Delta = \Delta_1 \mathcal{O}_\Delta, \quad \mathcal{O}_\Delta = \sum_{R,r,s,\nu_\pm} c_{R,(r,s),\nu_-,\nu_+} \mathcal{O}^{R,(r,s),\nu_-,\nu_+}. \quad (4.1)$$

We denote the action of the one-loop dilatation on the restricted Schur basis by

$$\mathcal{D}_1 \mathcal{O}^{R,(r,s),\nu_-,\nu_+} = \sum_{T,t,u,\mu_-,\mu_+} N_{T,(t,u),\mu_-,\mu_+} \mathcal{O}^{R,(t,u),\nu_-,\nu_+} \quad (4.2)$$

and define the LLM operator by

$$\mathcal{O}_\Delta \to \mathcal{O}_\Delta^{\text{LLM}} = \sum_{R,(r,s),\nu_\pm} c_{R,(r,s),\nu_-,\nu_+} \mathcal{O}^{R,(r,s),\nu_-,\nu_+}. \quad (4.3)$$

The operation $r \to (\mathbf{+} r)$ can be exemplified as

$$r = \boxed{\phantom{c}} \to \ (\mathbf{+} r) = \boxed{\phantom{c}} \quad (4.4)$$

Here there are $\mathcal{O}(1)$ white boxes, and $\mathcal{O}(N_c^2)$ gray boxes in total. Each edge of the gray block has the length of $\mathcal{O}(N_c)$. The general form of the background Young diagram $\mathcal{B}$ is shown in Figure 1.

We specify a corner of the background Young diagram $\mathcal{B}$, and consider a set of all Young diagrams attached to that corner. This set of states has many interesting properties. First, from the Littlewood-Richardson rule, we find

$$g(r,s; R) \simeq g(\mathbf{+} r, s; \mathbf{+} R), \quad (N_c \gg 1). \quad (4.5)$$

This allows us to use the same multiplicity labels $\nu_\pm$ before and after the $\mathbf{+}$ operation. Note that the tensor product $(\mathbf{+} r) \otimes s$ contains representations in which boxes are attached to multiple corners of $\mathcal{B}$. However, the overlap between such states and $(\mathbf{+} r)$ is suppressed by $1/N_c$. Second, the hook length of $(\mathbf{+} r)$ factorizes as

$$\frac{\text{hook}_{\mathbf{+} r}}{\text{hook}_r \cdot \text{hook}_{\mathcal{B}}} \simeq (\eta_{\mathcal{B}})^{|r|} \quad (N_c \gg 1) \quad (4.6)$$

where $\eta_{\mathcal{B}}$ is the factor which depends only on $\mathcal{B}$,

$$\eta_{\mathcal{B}} \equiv \prod_{k=1}^{C} \frac{L(k,C)}{L(k,C) - N_k} \prod_{l=C+1}^{D} \frac{L(C+1,l)}{L(C+1,l) - M_l}, \quad L(a,b) = \sum_{k=a}^{b} (M_k + N_k) \quad (4.7)$$

assuming that the small diagram $r$ is put at the $C$-th corner of $\mathcal{B}$ in Figure 1. It follows that

$$\frac{|\mathcal{B}| + |r|)!}{|\mathcal{B}|!} \simeq |\mathcal{B}|^{|r|}, \quad \frac{d_{\mathbf{+} r}}{d_r} \simeq \frac{1}{|r|!} \left( \frac{|\mathcal{B}|}{\eta_{\mathcal{B}}} \right)^{|r|} \quad (N_c \gg 1). \quad (4.8)$$

Since position of the $C$-th corner is $(i,j) = (1 + \sum_{l=C+1}^{D} M_l, 1 + \sum_{k=1}^{C} N_k)$, from (A.5) we get

$$\frac{\text{Dim}_{N_c}(\mathbf{+} R)}{\text{Dim}_{N_c}(\mathcal{B})} \simeq \text{Dim}_{N_c'}(R), \quad N_c' = N_c + \sum_{l=C+1}^{D} M_l - \sum_{k=1}^{C} N_k. \quad (4.9)$$
Figure 1: The general background Young diagram $\mathcal{B}$ having a staircase shape, which corresponds to the LLM geometry of concentric shapes by AdS/CFT. All $M_i$ and $N_i$ are $\mathcal{O}(N_c)$, and $\sum_i N_i = N_c$. The gray and black boxes represent localized string excitations. To define the operation $\dagger$ we should choose one gray box.
In [43] they found that the operator mixing coefficients satisfy the identity
\[ N_{T_1(T_2)T_3}^+ \nu_+ \mu_+ \simeq N_{T_1(T_2)T_3}^\nu \mu_+ \] (4.10)
showing that
\[ \mathcal{D}_1 \mathcal{O}_\Delta^{LLM} \simeq \Delta_1 \mathcal{O}_\Delta^{LLM} \] (4.11)

### 4.2 Tree-level OPE coefficients

We revisit two types of OPE coefficients in Section 3. We will show that the OPE coefficients of non-extremal three-point functions in \( \mathcal{N} = 4 \) SYM are essentially same as those of the LLM operators, after redefinition of \( N_c \).

#### 4.2.1 Adding a background tableau to \( \tilde{C}_{\infty} \)

Recall that \( \tilde{C}_{\infty} \) is given by (3.70),
\[ \tilde{C}_{\infty} = \left\langle \hat{O}_1^{R_1(T_1)}[Z,1] \hat{O}_2^{R_2(T_2)}[\tilde{Z},1] \hat{O}_3^{R_3(T_3)}[\bar{Z},1] \right\rangle = \left( \prod_{i=1}^3 \frac{L_i^!}{L_i!} \right) \sum_{R_1+L_2} \text{Dim}_{N_c}(\hat{R}) \sum_{Q_1+L_2} \sum_{Q_2+L_3} \sum_{Q_3+L_1} \left( \prod_{i=1}^3 \frac{d_{Q_i}}{d_{R_i}} \right) \mathcal{G}_{123}. \] (4.12)

We obtain the OPE coefficients of the LLM operators by the substitution \( Q_1 \rightarrow (+Q_1) \), while leaving \( Q_2, Q_3 \) as before. From (3.71) it follows that
\[ (+R_1) = (+Q_1) \otimes Q_2, \quad R_2 = Q_2 \otimes Q_3^*, \quad (+R_3) = Q_3 \otimes (+Q_1) \]
and thus
\[ \tilde{C}_{\infty}^{LLM} \equiv \left( \hat{O}_1^{+R_1(T_1)}[Z,1] \hat{O}_2^{R_2(T_2)}[\tilde{Z},1] \hat{O}_3^{R_3(T_3)}[\bar{Z},1] \right) = \left( \prod_{i=1}^3 \frac{L_i^!}{L_i!} \right) \sum_{R_1+L_2} \text{Dim}_{N_c}(\hat{R}) \sum_{Q_1+L_2} \sum_{Q_2+L_3} \sum_{Q_3+L_1} (d_{+Q_1}d_{Q_2}d_{Q_3}) \mathcal{G}_{123}^{LLM}. \] (4.14)

By using the identities in Section [4.1] we find
\[ \tilde{C}_{\infty}^{LLM} \simeq (\eta_{\mathscr{B}})^L \text{Wt}_{N_c}(\mathscr{B}) \frac{L_1^!L_2^!L_3^!}{L_1^!L_2^!L_3^!} \sum_{R_1+L_2} \text{Dim}_{N_c}(\hat{R}) \sum_{Q_1+L_2} \sum_{Q_2+L_3} \sum_{Q_3+L_1} (d_{Q_1}d_{Q_2}d_{Q_3}) \mathcal{G}_{123}. \] (4.15)

If we remove the \( \mathscr{B} \)-dependent prefactor \( (\eta_{\mathscr{B}})^L \text{Wt}_{N_c}(\mathscr{B}) \), the OPE coefficient \( \tilde{C}_{\infty}^{LLM} \) agrees with \( \tilde{C}_{\infty} \) up to the redefinition of \( N_c \rightarrow N_c' \) in (4.9).
4.2.2 Adding a background tableau to $\tilde{C}^{XYZ}_h$

Recall that $\tilde{C}^{XYZ}_h$ is given by (3.90),

$$\tilde{C}^{XYZ}_h = \left\{ \hat{O}_1^{R_1(T_1)}|X, Y, Z, 1| \hat{O}_2^{R_2(T_2)}|X, Y, Z, 1| \hat{O}_3^{R_3(T_3)}|X, Y, Z, 1| \right\}$$

$$= \left( \prod_{i=1}^{\frac{3}{N}} \frac{L_i!}{L_i^i} \right) \sum_{\overline{R}+L} \frac{\text{Dim}_{\overline{R}}(\overline{R})}{d_{\overline{R}}!} \frac{(d_{q_1} d_{q_2} d_{r_1} d_{r_2} d_{s_2} d_{s_3}) \delta^{\nu_1 \nu_2+} \delta^{\nu_2 \nu_3+} \delta^{\nu_3 \nu_1+} G'_{123}}{G_{123}}$$

where $R_i$ is defined in (3.14) as

$$R_i = \{ R_i, \{q_i, r_i, s_i\}, \nu_i-\nu_i+ \} \text{, \quad } (R_i \models L_i).$$

We obtain the OPE coefficients in the LLM background by the substitution $(s_1, s_2, s_3) \rightarrow (+s_1, +s_2, s_3)$, while $q_i, r_i$ are the same as before. From (3.91) we find

$$q_1 \otimes q_2 = q_3, \quad r_1 \otimes r_3 = r_2, \quad (+s_2) \otimes s_3 = (+s_1), \quad q_1 \otimes q_2 \otimes r_1 \otimes r_3 \otimes (+s_2) \otimes s_3 = \hat{R}$$

$$(+R_1)_{\nu_1+} = q_1 \otimes r_1 \otimes (+s_2) \otimes s_3$$

$$(+R_2)_{\nu_2+} = q_2 \otimes r_1 \otimes r_3 \otimes (+s_2)$$

$$(R_3)_{\nu_3+} = q_1 \otimes q_2 \otimes r_1 \otimes r_3 \otimes s_3.$$ (4.18)

It follows that

$$(\tilde{C}^{XYZ}_{h})_{\text{LLM}} = \left( \frac{(+L_1)!(+L_2)!L_3!}{L_1!L_2!(+L_3)!} \right) \sum_{\overline{R}+L} \frac{\text{Dim}_{\overline{R}}(\overline{R})}{d_{\overline{R}}!} \frac{(d_{q_1} d_{q_2} d_{r_1} d_{r_2} d_{s_2} d_{s_3}) \times \delta^{\nu_1 \nu_2+} \delta^{\nu_2 \nu_3+} \delta^{\nu_3 \nu_1+} G'_{123}}{G_{123}}.$$ (4.19)

At large $N_c$, we can simplify this result following our discussion in Section 4.1 as

$$(\tilde{C}^{XYZ}_{h})_{\text{LLM}} = \frac{L_3!}{(L_3 - |r_1|)!} \left( \frac{\eta_{\overline{R}}}{|\overline{R}|} \right)^{|r_1|} \eta_{\overline{R}} \text{Wt}_{N_c}(\overline{R}) \times$$

$$\frac{L_1!L_2!L_3!}{L_1!L_2!L_3!} \sum_{\overline{R}+L} \frac{\text{Dim}_{\overline{R}}(\overline{R})}{d_{\overline{R}}!} \frac{(d_{q_1} d_{q_2} d_{r_1} d_{r_2} d_{s_2} d_{s_3}) \times \delta^{\nu_1 \nu_2+} \delta^{\nu_2 \nu_3+} \delta^{\nu_3 \nu_1+} G'_{123}}{G_{123}}.$$ (4.20)

The first line is a numerical prefactor, and the second line agrees with $(\tilde{C}^{XYZ}_{h})$ by the redefinition of $N_c \rightarrow N'_c$ in (1.9).

5 Conclusion and Outlook

In this paper, we have studied general non-extremal three-point functions of scalar multi-trace operators at tree level valid for any values of $N_c$ in gauge theory including $\mathcal{N} = 4$ SYM, by using the representation theory of symmetric groups.

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We made full use of various new mathematical techniques. The quiver calculus of [29] gives a collection of diagrammatic method which simplifies various objects in the representation theory. The generalized Racah-Wigner tensor is introduced as an extension of the $6j$ symbols. We conjectured formulae about the invariant products of the generalized Racah-Wigner tensors, written in terms of the Littlewood-Richardson coefficients.

With these formulae, we provide strong evidence on the large $N_c$ background independence, a correspondence between small ($O(N_c^0)$) and huge ($O(N_c^2)$) operators of $\mathcal{N} = 4$ SYM. The background independence has been checked for two-point functions as well as extremal three-point functions. Our argument demonstrates that it extends to non-extremal three-point functions. These results will clarify the properties of stringy excitations on the LLM backgrounds, particularly how they differ from the usual strings on AdS$_5 \times S^5$.

Let us comment on some important future directions.

The first direction is to find a connection with the integrability results of the planar $\mathcal{N} = 4$ SYM. Clearly, the operators in the representation basis are not the eigenstates of the dilatation operator of $\mathcal{N} = 4$ SYM. One should think of the representation basis as a tool for the finite $N_c$ computation. The two-point functions of single-trace operators in the $\mathfrak{su}(2)$ sector have been computed in this way [27, 46], generalizing the old results of the complex matrix model [47, 48]. A particularly interesting question is to determine the so-called octagon frame, namely the tree-level part of the “simplest” four-point functions of $\mathcal{N} = 4$ SYM in the large charge limit [11]. The finite group methods developed in this paper can be used for the exact finite-$N_c$ computation, because it is a generalization of the character expansion methods familiar in the matrix models [49–51].

The second direction is to refine our computation. The conjectured formula for the invariant products of generalized Racah-Wigner tensor should be proven. The computation of the $n$-point functions in the representation basis is also important. It is interesting to ask whether one can bootstrap four-point functions out of two- and three-point data.

The third direction is to investigate a possible relation between quiver calculus and knot theory. The $6j$ symbol of the unitary group has been extensively studied in the context of knot theory and integrable systems [52]. Since the $6j$ symbols of symmetrical groups are related to those of unitary groups, the quiver calculus could give a new insight into the study of knot polynomials. For example, some non-trivial conjectures about the $6j$ symbols have been made [53–55], though most of them discuss the multiplicity-free cases only. Since the new invariants $G_{123}$ and $G'_{123}$ discussed in this paper are closely related to the multiplicity structure, studying similar quantity in the case of unitary groups is a fascinating problem.

Finally, we hope to find a clear understanding of the AdS/CFT correspondence of the operators with huge anomalous dimensions, including giant gravitons [56, 57] and the fluctuation in the LLM geometry [43, 58, 59]. Some correlation functions have been studied such as three giants [60, 62], two giants and one single-trace [63, 70].

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A Survey of finite-group representation theory

We explain our notation and formulae used in the main text, while providing a brief survey of the representation theory of finite groups. Our notation is similar to the one used in [22]. For more details on finite groups, see textbooks like [71,72].

A.1 Basic notation

The symmetric group permuting $L$ elements is denoted by $S_L$. We denote the conjugacy class of $S_L$ by

$$
C_\alpha = \frac{1}{|S_L|} \sum_{\gamma \in S_L} \gamma \alpha \gamma^{-1},
$$

(A.1)

The $\delta$-function over $S_L$ (or $\mathbb{C}[S_L]$) is defined by

$$
\delta(\beta) = \begin{cases} 
1 & (\beta = 1 \in S_L) \\
0 & \text{(otherwise)}.
\end{cases}
$$

(A.2)

A permutation cycle is denoted by $(12\ldots L) \in \mathbb{Z}_L$. Any element of $S_L$ consists of permutation cycles. The number of length-$k$ cycles in $\sigma \in S_L$ is denoted by $\text{Cyc}_k(\sigma)$. The number of cycles in $\sigma$ is

$$
C(\sigma) = \sum_k \text{Cyc}_k(\sigma)
$$

(A.3)

so that $C(\text{id}) = C((1)(2)\ldots(L)) = L$.

A partition of $L$, or equivalently a Young diagram with $L$ boxes, is denoted by $R \vdash L$. Define

$$
d_R = \frac{L!}{\text{hook}_R}, \quad \text{hook}_R = \prod_{(i,j) \in R} \left(\text{hook length at } (i,j)\right)
$$

(A.4)

$$
\text{Dim}_N(R) = \frac{d_R}{L!} \text{Wt}_N(R), \quad \text{Wt}_N(R) = \prod_{(i,j) \in R} (N + i - j)
$$

(A.5)

where $d_R$ is the dimension of $R$ as the representation of $S_L$, and $\text{Dim}_N(R)$ is the dimension of $R$ as the representation of $U(N)$[5]. For example, $\text{hook}_R$ and $\text{Wt}_N(R)$ of the Young tableau $R = \begin{array}{llll}
\end{array}$

\[5\text{Wt}_N(R)\] is also denoted by $f_R$ in the literature, e.g. [23].
are given by
\[ \begin{array}{cccc}
5 & 4 & 2 & 1 \\
2 & 1 & & \\
\hline
N & N+1 & N+2 & N+3 \\
N-1 & N & & 
\end{array} \]  \Rightarrow  \text{hook} = 5 \times 4 \times 2 \times 2 \times 1 = 5 \times 4 \times 2 \times 2 \times 1 \times 1 \tag{A.6}

\[ \begin{array}{cccc}
\hline
N & N+1 & N+2 & N+3 \\
N-1 & N & & 
\end{array} \]  \Rightarrow  \text{Wt}_N = (N-1)N^2(N+1)(N+2)(N+3). \]

We assume that all representations are real and orthogonal.\[6\] Denote the \(I\)-th component of the irreducible representation \(R\) of \(S_L\) by \(|\sigma^{(I)}\rangle\), with \(I = 1, 2, \ldots, d_R\). Introduce the dual basis by
\[ \langle \sigma^{(I)} | \sigma^{(J)} \rangle = \delta_{IJ}. \tag{A.7} \]

Let \(D_{I,J}^R(\sigma)\) be the representation matrix of \(\sigma \in S_{m+n}\) of the representation \(R \vdash L\),
\[ D_{I,J}^R(\sigma) = \langle \sigma | \sigma^{(I)} \rangle = D_{J,I}^R(\sigma^{-1}). \tag{A.8} \]

The character of the representation \(R\) for the group element \(\sigma\) is denoted by\[7\]
\[ \chi^R(\sigma) = \sum_{I=1}^{d_R} D_{I,I}^R(\sigma). \tag{A.9} \]

By restricting \(\sigma \in S_L = S_{m+n}\) to \(S_m \otimes S_n\), we obtain the irreducible decomposition\[8\]
\[ R = \bigoplus_{r^+ S^+ m \prec \kappa s^+ S^+ n} \bigoplus_{\nu=1}^{\nu=1} g(r, s; R) \bigoplus_{r^+ S^+ m \prec \kappa s^+ S^+ n} (r \otimes s)_\nu. \tag{A.10} \]

where \(g(r, s; R)\) is the Littlewood-Richardson coefficient. It counts the number of \(r \otimes s\) appearing in the irreducible decomposition of \(R\). The subscript \(\nu\) is called the multiplicity label. With an appropriate change of basis, we can transform the representation matrix into a block-diagonal form,
\[ D_{I,J}^R(\sigma) = B \begin{pmatrix}
D_{1,1}^{(1) \otimes (1)}(\sigma) \\
D_{2,2}^{(2) \otimes (2)}(\sigma) \\
D_{3,3}^{(3) \otimes (3)}(\sigma) \\
\vdots
\end{pmatrix} B^T \quad (\sigma \in S_m \otimes S_n) \tag{A.11} \]
such that it matches\[A.10\]. By definition of the irreducible decomposition, there are no off-block-diagonal elements including the multiplicity labels. For general \(\sigma \in S_{m+n}\), the matrix\[A.11\] has off-block-diagonal elements\[9\].

\[6\]The orthogonal form of the Young-Yamanouchi basis satisfies these conditions.
\[7\]Often we sum over the repeated indices of matrices. The symbol \(\sum\) is written explicitly in Appendix A.
\[8\]The restriction to a subgroup is also called subduction in the literature.
\[9\]The restricted Schur basis should have off-block-diagonal elements with respect to the multiplicity labels, which can be checked by counting the dimensions\[40].
Let \( \left| \begin{array}{c} r, s \\ i, j \end{array} \right| \) be an orthonormal basis of \( r \otimes s \) at the \( \nu \)-th multiplicity, satisfying
\[
\left\langle \begin{array}{c} r, s \\ i, j \end{array} \left| \begin{array}{c} r, s \\ i, j \end{array} \right| = \delta_{i, j} \right\rangle \]  
(A.12)
for \( \nu_k = 1, 2, \ldots, g(r_k, s_k; R) \). The rotation matrix is called the branching coefficients, defined by
\[
B_{I \rightarrow (i, j)}^{R \rightarrow (r, s), \nu} = \left( R \left| \begin{array}{c} r, s \\ i, j \end{array} \right| I \right), \quad (B^T)_{I \rightarrow (i, j)}^{R \rightarrow (r, s), \nu} = \left( R \left| \begin{array}{c} r, s \\ i, j \end{array} \right| I \right). \]  
(A.13)

### A.2 Branching coefficients

We find from (A.11) that the branching coefficients satisfy the completeness relations
\[
\sum_{r, s, \nu} \sum_{i, j} B_{I \rightarrow (i, j)}^{R \rightarrow (r, s), \nu} (B^T)_{I \rightarrow (i, j)}^{R \rightarrow (r, s), \nu} = \delta_{i, j} \]  
(A.14)
\[
\sum_{i, j} (B^T)_{I \rightarrow (i, j)}^{R \rightarrow (r_1, r_2), \nu} B_{I \rightarrow (i, j), \nu} = \delta_{r_1, s_1} \delta_{r_2, s_2} \delta_{\nu, \mu} \delta_{i, j} \delta_{i, j}. \]  
(A.15)

In (A.15), we assume that two product representations \( r_1 \otimes r_2 \) and \( s_1 \otimes s_2 \) descend from the same restriction \( S_{m+n} \downarrow (S_m \otimes S_n) \). If they descend from different restrictions, then the two branching coefficients \( B \) and \( \tilde{B} \) are unrelated, and we obtain another orthogonal matrix
\[
\sum_{i, j} (B^T)_{I \rightarrow (i, j)}^{R \rightarrow (r_1, r_2), \nu} \tilde{B}_{I \rightarrow (j, i), \mu} = \left\langle \begin{array}{c} r_1 r_2 \\ i_1 i_2 \end{array} \left| \begin{array}{c} s_1 s_2 \\ j_1 j_2 \end{array} \right| \right\rangle. \]  
(A.16)

For example, given two irreducible decompositions
\[
S_6 \downarrow (S_4 \otimes S_2), \quad S_6 \downarrow (S_3 \otimes S_3),
\]  
any pairs \( r_1 \otimes r_2 \) and \( s_1 \otimes s_2 \) from different restrictions can have non-vanishing overlap, e.g.
\[
\left\langle \begin{array}{c} i_1, i_2 \end{array} \left| \begin{array}{c} j_1, j_2 \end{array} \right| \right\rangle \neq 0. \]  
(A.18)

Sometimes we take the coordinates explicitly in order to distinguish \( S_{m+n} \downarrow (S_m \otimes S_n) \) and \( S_{m+n} \downarrow (S_n \otimes S_m) \). For example, the following two restrictions
\[
S_{m+n} \downarrow (S_m \otimes S_n) \sim \text{Permute} \left( \{1, 2, \ldots, m\} \times \text{Permute} \left( \{m+1, \ldots, m+n\} \right) \right)
S_{m+n} \downarrow (S_n \otimes S_m) \sim \text{Permute} \left( \{1, 2, \ldots, n\} \times \text{Permute} \left( \{n+1, \ldots, n+m\} \right) \right)
\]  
(A.19)
define different branching coefficients, \( B_{I \rightarrow (i, i), \nu}^{R \rightarrow (r_1, r_2), \nu} \) and \( \tilde{B}_{I \rightarrow (j, j), \mu}^{R \rightarrow (s_1, s_2), \mu} \).

From (A.11), we obtain the following identities for the matrix elements of \( \gamma = \gamma_1 \circ \gamma_2 \in S_m \otimes S_n \)
\[
D_{I J}^{R \rightarrow (r_1, r_2), \nu} (\gamma_1 \circ \gamma_2) = \sum_{r_1, r_2, \nu_{i, j, k, l}} D_{I i}^{r_1} (\gamma_1) D_{J j}^{r_2} (\gamma_2) B_{I \rightarrow (i, j, k, l)}^{R \rightarrow (r_1, r_2), \nu} (B^T)_{J \rightarrow (k, l)}^{R \rightarrow (r_1, r_2), \nu} \]  
(A.20)
By multiplying $B_{J→I(k,i)}^{R→(r_1,r_2)_\mu}$ to (A.21) and summing over $J$, we find
\[
\sum_J D_{I,j}^R(\gamma_1 \circ \gamma_2) B_{J→I(k,i)}^{R→(r_1,r_2)_\mu} = \sum_{i,j} D_{i,k}^R(\gamma_1) D_{j,l}^R(\gamma_2) B_{I→(i,j)}^{R→(r_1,r_2)_\mu}. \tag{A.21}
\]
Again, by multiplying $(B^T_{I→(i',j')})^{R→(r_1,r_2)_\mu}$ to (A.21) and summing over $J$, we find
\[
\sum_{I,j} D_{i,j}^R(\gamma_1 \circ \gamma_2) (B^T_{I→(i,j)})^{R→(r_1,r_2)_\mu} B_{J→(k,i)}^{R→(r_1,r_2)_\mu} = \delta_{\mu\nu} D_{i,k}^R(\gamma_1) D_{j,l}^R(\gamma_2). \tag{A.22}
\]
In the RHS, the matrix elements of $\gamma_1 \circ \gamma_2$ in the split basis are independent of the multiplicity labels $\mu, \nu$. This can be understood also from the construction of the Young-Yamanouchi basis.

The branching coefficients (A.13) for general restriction $S_L \downarrow (S_{m_1} \otimes S_{m_2} \otimes \cdots \otimes S_{m_e})$ are given by
\[
B_{I→(i_1,i_2,\ldots,i_\ell)}^{R→(r_1,r_2,\ldots,r_\ell)} = \left\langle R \left| I \begin{array}{cccc} r_1 & r_2 & \cdots & r_\ell \\ i_1 & i_2 & \cdots & i_\ell \end{array} \right| \right\rangle, \quad (B^T_{I→(i_1,i_2,\ldots,i_\ell)})^{R→(r_1,r_2,\ldots,r_\ell)} = \left\langle r_1 \cdots r_\ell \left| I \begin{array}{cccc} \ell \cdots \ell \\ i_1 & i_2 & \cdots & i_\ell \end{array} \right| R \right\rangle \tag{A.23}
\]
for $\nu = 1, 2, \ldots, g(r_1, r_2, \ldots, r_\ell; R)$. The restricted characters (A.24) are generalized accordingly.

### A.3 Restricted Schur basis

Consider the restriction $S_M \downarrow (S_{m_1} \otimes S_{m_2} \otimes S_{m_3})$ with $M = m_1 + m_2 + m_3$, which corresponds to the multi-trace operators with three complex scalars in (2.2).

Define the restricted Schur characters by using the branching coefficients [29].
\[
\chi^{R,(r_1,r_2,r_3),\nu_+\nu_-}(\sigma) \equiv \sum_{I,J} B_{I→(i,j,k)}^{R→(r_1,r_2,r_3)} (B^T_{I→(i,j,k)})^{R→(r_1,r_2,r_3)} \cdot D_{J,j}^R(\sigma), \quad (\sigma \in S_M). \tag{A.24}
\]

Define the operator in the restricted Schur basis by
\[
\mathcal{O}^{R,(r_1,r_2,r_3),\nu_+\nu_-}[X,Y,Z] = \frac{1}{m_1!m_2!m_3!} \sum_{\alpha \in S_M} \chi^{R,(r_1,r_2,r_3),\nu_+\nu_-}(\alpha) \text{tr}_M \left( \alpha X^{\otimes m_1} Y^{\otimes m_2} Z^{\otimes m_3} \right). \tag{A.25}
\]
The inverse transformation from the restricted Schur basis to the permutation basis is
\[
\text{tr}_M \left( \alpha X^{\otimes m_1} Y^{\otimes m_2} Z^{\otimes m_3} \right) = m_1!m_2!m_3! M! \sum_{R,r_1,r_2,r_3,\mu_+,\mu_-} d_R d_{r_1} d_{r_2} d_{r_3} \chi^{R,(r_1,r_2,r_3),\mu_+\mu_-}(\alpha) \mathcal{O}^{R,(r_1,r_2,r_3),\mu_+\mu_-} \tag{A.26}
\]
which can be checked by the row orthogonality of the restricted characters [A.51],
\[
\frac{1}{M!} \sum_{\sigma \in S_M} \chi^{R,(r_1,r_2,r_3),\nu_+\nu_-}(\sigma) \chi^{S,(s_1,s_2,s_3),\mu_+\mu_-}(\sigma) = \frac{d_{r_1} d_{r_2} d_{r_3}}{d_R} \delta_{R S} \delta_{r_1 s_1} \delta_{r_2 s_2} \delta_{r_3 s_3} \delta^{\nu_+\mu_-} \delta^{\nu_-\mu_+}. \tag{A.27}
\]
As discussed in Section 2.2, the tree-level two-point function is
\[
\langle \mathcal{O}^{R,(r_1,r_2,r_3),(\nu_+\nu_-)}[X,Y,Z]|x\rangle \mathcal{O}^{S,(s_1,s_2,s_3),(\mu_+\mu_-)}[X',Y',Z']|0\rangle = \frac{\text{Wt}_N(R)}{|x|^M} \frac{\text{hook}_R}{\text{hook}_{r_1} \text{hook}_{r_2} \text{hook}_{r_3}} \delta_{RS} \delta_{r_1 s_1} \delta_{r_2 s_2} \delta_{r_3 s_3} \delta^{\nu_+\mu_-} \delta^{\nu_-\mu_+}. \tag{A.28}
\]
A.4 Formulae

The formulae for the irreducible characters and the restricted characters will be summarized below. For simplicity, we mostly consider the restriction \( S_{m+n} \downarrow (S_m \otimes S_n) \). Generalization to \( S_M \downarrow (\otimes_k S_{m_k}) \) is straightforward.

**Character Orthogonality.** Let \( R, S \) be the irreducible representations of \( S_L \). The representation matrices satisfy the grand orthogonality relation

\[
\sum_{\sigma \in S_L} D^R_{ij}(\sigma) D^S_{kl}(\sigma^{-1}) = \frac{L!}{d_R} \delta_{il} \delta_{jk}.
\]

(A.29)

By taking the trace, we obtain the row (or first) orthogonality relation of irreducible characters,

\[
\sum_{\sigma \in S_L} \chi^R(\sigma) \chi^S(\sigma^{-1}) = L! \delta^{RS}.
\]

(A.30)

The irreducible characters also satisfy the column (or second) orthogonality relation,

\[
\sum_{R \vdash L} \chi^R(\sigma) \chi^R(\tau) = \sum_{\gamma \in S_L} \delta(\sigma \gamma \gamma^{-1}) = \begin{cases} |C_{\sigma}| & (C_{\sigma} = C_{\tau}) \\ 0 & \text{(otherwise)} \end{cases}
\]

(A.31)

where \( |C_{\sigma}| \) is the number of elements in a given conjugacy class \( [A.1] \). This relation follows from the fact that any class function can be expanded by irreducible characters

\[
f(\sigma) = f(\gamma \sigma \gamma^{-1}), \quad (\forall \gamma \in S_L) \quad \Leftrightarrow \quad f(\sigma) = \sum_{R \vdash L} \hat{f}_R \chi^R(\sigma).
\]

(A.32)

As a corollary, the \( \delta \)-function can be written as

\[
\delta(\beta) = \frac{1}{L!} \sum_{R \vdash L} d_R \chi^R(\beta).
\]

(A.33)

**Multiplicity label.** There are several ways to understand Littlewood-Richardson coefficients.

The first way is by restriction \( S_{m+n} \downarrow (S_m \otimes S_n) \) as in \([A.10]\)

\[
R = \bigoplus_{r \vdash m, s \vdash n} g(r, s; R) (r \otimes s).
\]

(A.34)

The second way is by induction,

\[
r \otimes s = \bigoplus_{R} g(r, s; R) R
\]

(A.35)

Frobenius reciprocity guarantees the consistency between \([A.35]\) and \([A.34]\). Finally, the Littlewood-Richardson coefficient can be computed by

\[
g(r, s; R) = \frac{1}{|S_m \otimes S_n|} \sum_{\alpha \in S_m} \sum_{\beta \in S_n} \chi^r(\alpha) \chi^s(\beta) \chi^R(\alpha \circ \beta)
\]

(A.36)
where $\alpha \circ \beta \in S_m \otimes S_n \subset S_{m+n}$.

The generalized Littlewood-Richardson coefficient for $\otimes_{k=1}^l S_{m_k}$ is given by

$$g(r_1, r_2, \ldots, r_l; R) = \frac{1}{\left| \otimes_{k=1}^l S_{m_k} \right|} \sum_{\{\sigma_k \in S_{m_k}\}} \left( \prod_{k=1}^l \chi^{r_k}(\sigma_k) \right) \chi^R(\sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_l). \quad (A.37)$$

They satisfy a recursion relation

$$\sum_{R \vdash M} g(r_1, r_2, \ldots, r_l; R) g(R, r_{l+1}; S) = g(r_1, r_2, \ldots, r_{l+1}; S), \quad \left( M = \sum_{k=1}^l m_k \right) \quad (A.38)$$

which can be shown from (A.31). The equation (A.38) implies an important identity for multiple branching coefficients

$$B^{S \rightarrow (r_1, r_2, \ldots, r_{l+1}); \eta}_{I \rightarrow (a_1, a_2, \ldots, a_{l+1})} = \sum_{R} \sum_{A=1}^{d_R} B^{S \rightarrow (R, r_{l+1}); \mu}_{I \rightarrow (A, a_{l+1})} B^{R \rightarrow (r_1, r_2, \ldots, r_l); \rho}_{A \rightarrow (a_1, a_2, \ldots, a_l)} \quad (A.39)$$

$\eta = 1, 2, \ldots, g(r_1, r_2, \ldots, r_{l+1}; S), \quad \mu = 1, 2, \ldots, g(R, r_{l+1}; S), \quad \rho = 1, 2, \ldots, g(r_1, r_2, \ldots, r_l; R)$.

**Schur-Weyl duality.** The quantity $N^C(\sigma)$ is a class function. We obtain its irreducible decomposition (A.32) by using the Schur-Weyl duality [19] as

$$N^C(\sigma) = \sum_{R \vdash \lambda} \text{Dim}_N(R) \chi^R(\sigma). \quad (A.40)$$

Note that $\text{Dim}_N(R) = 0$ if the height of the Young diagram $R$ is larger than $N$, as can be seen from (A.3). By applying the grand orthogonality relation (A.29), we find

$$\sum_{\sigma \in S_L} D_{I,J}^S(\sigma) N^C(\sigma) = \delta_{IJ} \text{Dim}_N(S) \text{hook}_S = \delta_{IJ} \text{Wt}_N(S). \quad (A.41)$$

By multiplying the branching coefficients as in (A.43), we obtain another formula [23]

$$\sum_{\sigma \in S_{m+n}} \chi^{R,(r,s); \nu^+, \nu^-}(\sigma) N^C(\sigma) = \delta^{\nu^+, \nu^-}_s d_s \text{Wt}_N(R). \quad (A.42)$$

**Restricted projector.** We define the restricted projector

$$\mathcal{P}^{R,(r_1, r_2); \nu^+, \nu^-}_s = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi^{R,(r_1, r_2); \nu^+, \nu^-}(\sigma) \sigma \in \mathbb{C}[S_{m+n}] \quad (A.43)$$

so that [46]

$$\chi^{R,(r_1, r_2); \nu^+, \nu^-}(\sigma) = \chi^R(\mathcal{P}^{R,(r_1, r_2); \nu^+, \nu^-}_s \sigma) \quad (A.44)$$

$$\mathcal{P}^{R,(r_1, r_2); \nu^+, \nu^-}_s \mathcal{P}^{S,(s_1, s_2); \mu^+, \mu^-}_t = \delta^{RS} \delta_{r_1 s_1} \delta_{r_2 s_2} \delta^{\nu^+, \mu^-} \mathcal{P}^{R,(r_1, r_2); \nu^+, \mu^-}_t. \quad (A.45)$$

By comparing (A.44) and (A.24), one finds

$$\mathcal{P}^{R,(r_1, r_2); \nu^+, \nu^-}_s \equiv D_{I,J}^R \left( \mathcal{P}^{R,(r_1, r_2); \nu^+, \nu^-}_s \right) = \sum_{i,j} B_{I \rightarrow (i,j)}^{R \rightarrow (r,s); \nu^+} (B^T)_{J \rightarrow (i,j)}^{R \rightarrow (r,s); \nu^-}. \quad (A.46)$$

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It follows that
\[
\chi^R (\otimes R, (r_1, r_2), \nu_+, \nu_-) = \sum_i \sum_{i,j} B_{I \to (i,j)}^{R \to (r_s), \nu_+} \left( (B^T)_{I \to (i,j)}^{R \to (r_s), \nu_-} \right) = \delta^{\nu_+ \nu_-} d_{r_1} d_{r_2}. \tag{A.47}
\]

The restricted projector is useful for fixing the normalization. These formulae as well as the following identities can be proven by using the quiver calculus in Appendix B.

**Restricted Character Orthogonality.** The restricted characters \([A.24]\) satisfy the identities
\[
\chi^{R, (r_s), \nu_+, \nu_-} (\sigma) = \chi^{R, (r_s), \nu_-, \nu_+} (\sigma^{-1}) \tag{A.48}
\]
\[
\chi^{R, (r_s), \nu_+, \nu_-} (\gamma \sigma^{-1}) = \chi^{R, (r_s), \nu_+, \nu_-} (\sigma) \quad (\forall \gamma \in S_m \otimes S_n) \tag{A.49}
\]
\[
\chi^{R, (r_s), \nu_+, \nu_-} (\sigma_1 \circ \sigma_2) = \delta^{\nu_+ \nu_-} \chi^R (\sigma_1) \chi^R (\sigma_2) \quad (\forall \sigma_1 \circ \sigma_2 \in S_m \otimes S_n) \tag{A.50}
\]
where the last relation is consistent with \([A.22]\). The row and column orthogonality relations \([A.31]\) are generalized as
\[
\frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} \chi^{R, (r_1, r_2), \nu_+, \nu_-} (\sigma) \chi^{S, (s_1, s_2), \mu_+, \mu_-} (\sigma) = \frac{d_{r_1} d_{r_2}}{d_R} \delta^{RS} \delta^{r_1 s_1} \delta^{r_2 s_2} \delta^{\nu_+ \mu_+} \delta^{\nu_- \mu_-} \tag{A.51}
\]
\[
\sum_{R, r_1, r_2, \nu_+, \nu_-} \frac{d_R}{d_{r_1} d_{r_2}} \chi^{R, (r_1, r_2), \nu_+, \nu_-} (\sigma) \chi^{R, (r_1, r_2), \nu_+, \nu_-} (\tau) = \frac{(m+n)!}{m! n!} \sum_{\gamma \in S_m \otimes S_n} \delta (\gamma \sigma^{-1} \tau^{-1}) \tag{A.52}
\]

One can generalize the grand orthogonality relation \([A.29]\) with the branching coefficients in two ways. First, let \(R\) and \(S\) be the irreducible representations of \(S_{m+n}\). A sum over \(S_{m+n}\) gives
\[
\frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} D^R_{IJ}(\sigma) B^{R \to (r_1, r_2), \nu_+} G^{J \to (k,l)}^{(r_1, r_2), \nu_-} D^S_{MN}(\sigma) B^{S \to (s_1, s_2), \mu_+} G^{M \to (m,n)}^{(s_1, s_2), \mu_-} D^S_{MN}(\sigma) B^{S \to (s_1, s_2), \mu_+} G^{M \to (m,n)}^{(s_1, s_2), \mu_-} = \frac{d_R}{d_{r_1} d_{r_2}} \delta^{\nu_+ \mu_+} \delta^{\nu_- \mu_-} \delta^{r_1 s_1} \delta^{r_2 s_2} \delta_{i,m} \delta_{j,n} \delta_{k,p} \delta_{l,q} \tag{A.53}
\]
which reduces to \([A.51]\) by taking the trace over \(r_1 \otimes r_2 = s_1 \otimes s_2\). Second, let \((r_1, r_2)\) and \((s_1, s_2)\) be the irreducible representations of \(S_m \otimes S_n\). A sum over \(S_m \otimes S_n\) gives
\[
\frac{1}{m! n!} \sum_{\sigma \in S_m \otimes S_n} D^R_{IJ}(\sigma) B^{R \to (r_1, r_2), \nu_+} G^{J \to (k,l)}^{(r_1, r_2), \nu_-} D^S_{MN}(\sigma) B^{S \to (s_1, s_2), \mu_+} G^{M \to (m,n)}^{(s_1, s_2), \mu_-} D^S_{MN}(\sigma) B^{S \to (s_1, s_2), \mu_+} G^{M \to (m,n)}^{(s_1, s_2), \mu_-} = \frac{d_R}{d_{r_1} d_{r_2}} \delta^{\nu_+ \mu_+} \delta^{\nu_- \mu_-} \delta_{i,m} \delta_{j,n} \delta_{k,p} \delta_{l,q} \tag{A.54}
\]
where we used \([A.22]\).

**B Quiver calculus**

Let us introduce a graphical notation of various representation-theoretical objects following \([29]\). We denote the indices of \(R \vdash L = (m+n)\) by a double line, and those of \(r_1 \vdash m\) or \(r_2 \vdash n\) by a single line. We use different lines to distinguish two set of representations \(\{R, (r_1, r_2)\}\) and \(\{S, (s_1, s_2)\}\).
The matrix representation of a permutation group element is represented by

\[ D_{IJ}^R(\sigma) = \begin{pmatrix} I & J \\ \sigma & \sigma^{-1} \end{pmatrix}, \quad D_{IJ}^R = D_{IJ}^{\sigma^{-1}} \]  

by using (A.8). Note that the matrix transposition is represented as flipping all the arrow directions.

The composition of permutations is

\[ D_{IJ}^R(\sigma \tau) = \sum_{K=1}^{d_R} D_{IK}^R(\sigma) D_{KJ}^R(\tau) = \begin{pmatrix} I & J \\ \sigma \tau & \tau \end{pmatrix}, \quad D_{IJ}^{\sigma \tau} = D_{IJ}^{\tau \sigma^{-1}} \]  

The grand orthogonality relation (A.29) is

\[ \frac{1}{L!} \sum_{\sigma \in S_L} \begin{pmatrix} I & K \\ \sigma & \sigma^{-1} \end{pmatrix} = \frac{\delta^{RS}}{d_R} \begin{pmatrix} I & K \\ J & L \end{pmatrix} \]

or equivalently

\[ \frac{1}{L!} \sum_{\sigma \in S_L} \begin{pmatrix} I & K \\ \sigma & \sigma^{-1} \end{pmatrix} = \frac{\delta^{RS}}{d_R} \delta_{IL} \delta_{JK}. \]

The branching coefficients (A.13) are represented as

\[ B_{I \rightarrow (i,j)}^{R \rightarrow (r_1,r_2)} = \begin{pmatrix} I & J \\ \nu & \nu \end{pmatrix}, \quad (B^T)_{I \rightarrow (i,j)}^{R \rightarrow (r_1,r_2)} = \begin{pmatrix} I & J \\ \nu & \nu \end{pmatrix} \]

We use double lines for the indices of $S_{m+n}$, wavy lines for $S_m$ and straight lines for $S_n$. The
completeness relations of the branching coefficients (A.14), (A.15) are

\[ \sum_{r_1,r_2,\nu} = \delta^{\nu\mu} \delta^{r_1 s_1} \delta^{r_2 s_2} \] (B.6)

where we assumed that \( r_1 \otimes r_2 \) and \( s_1 \otimes s_2 \) follow from the same restriction of \( R \). If the two product representations descend from different restrictions, we get the orthogonal matrix (A.16)

\[ \nu \mu \] (B.7)

The relation (A.21) is expressed as

\[ \gamma_1 \circ \gamma_2 \] (B.8)

The identity for multiple branching coefficients (A.39) is

\[ \sum_{S} = \sum_{S} \] (B.9)
The character and the restricted characters are

\[
\chi^R(\sigma) = \chi^R(\sigma^{-1}) = \sigma \quad \text{and} \quad \chi^{R(r_1,r_2)(\nu_+\nu_-)}(\sigma) = \sigma^{-1} \quad \text{(B.10)}
\]

We can show the row orthogonality of the restricted character as

\[
\frac{1}{L!} \sum_{\sigma \in S_L} \delta R S \sum_{\gamma \in S_L} D_{ij}^R(\gamma) D_{kl}^R(\gamma^{-1}), \quad (i, j, k, l = 1, 2, \ldots, d_R).
\]

To show the column orthogonality, we insert the resolution of identity on the irreducible representation \( R \) by (A.29),

\[
\delta_{il} \delta_{jk} = \frac{d_R}{L} \sum_{\gamma \in S_L} D_{ij}^R(\gamma) D_{kl}^R(\gamma^{-1}), \quad (i, j, k, l = 1, 2, \ldots, d_R).
\]

We obtain

\[
\sum_{R+L} \delta R S \sum_{\gamma \in S_L} \delta(\sigma \gamma \tau^{-1} \gamma^{-1}) \quad \text{(B.13)}
\]

where we used (A.33). Note that

\[
\sum_{\gamma \in S_L} \delta(\sigma \gamma \tau^{-1} \gamma^{-1}) = \sum_{\omega \in S_L} \delta(\sigma \omega \tau \omega^{-1}), \quad (\omega \tau = \gamma \in S_L).
\]

(B.14)
Similarly, we can derive the column orthogonality for the restricted characters \((A.52)\). By using

\[
\delta_{il} \delta_{jk} = \frac{d_{r_1}}{m!} \sum_{\gamma \in S_m} D_{ij}^r(\gamma_1) D_{kl}^r(\gamma_1^{-1}), \quad (i, j, k, l = 1, 2, \ldots, d_{r_1})
\]

\[
\delta_{mq} \delta_{np} = \frac{d_{r_2}}{n!} \sum_{\gamma \in S_n} D_{mn}^r(\gamma_2) D_{pq}^r(\gamma_2^{-1}), \quad (i, j, k, l = 1, 2, \ldots, d_{r_2})
\]

we find

\[
\sum_{R, r_1, r_2, \nu_+, \nu_-} \frac{d_R}{d_{r_1} d_{r_2}} = \sum_{R, r_1, r_2, \nu_+, \nu_-} \frac{d_R}{m! n!} \sum_{\gamma_1 \in S_m} \sum_{\gamma_2 \in S_n} \delta_{\sigma \nu_+ \gamma_1} \delta_{\tau^{-1} \nu_+ \gamma_2} \delta_{\tau^{-1} \nu_- \gamma_1^{-1}} \delta_{\nu_- \gamma_2^{-1}}.
\]

\[
= \sum_{R+L} \frac{d_R}{m! n!} \sum_{\gamma \in S_m \otimes S_n} \delta(\sigma \gamma^{-1} \tau^{-1} \gamma).
\]

In the last line, we cannot use \((B.14)\), because \(\gamma \in S_m \otimes S_n \subseteq S_{m+n}\).
We can show the restricted grand orthogonality (A.53) by

\[
\frac{1}{L} \sum_{\sigma \in S_L} \sum_{\nu^+, \nu^-} = \frac{\delta^{RS}}{d_R} \sum_{\nu^+, \nu^-} \delta^{\nu_+ \mu_+} \delta^{\nu_- \mu_-} \delta^{i_1, s_1} \delta^{i_2, s_2} \delta^{i, m} \delta^{\mu, n} \delta^{k, p} \delta^{q, q}.
\]

**(B.17)**

**Restricted projector.** The restricted projector (A.43) can be represented as

\[
\mathcal{G}_{R,(r_1, r_2), \nu^+, \nu^-} = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \sigma \cdot
\]

which is an element of \(\mathbb{C}[S_{m+n}]\) and not a number. Its matrix elements are given by the branching coefficients (A.46), which can be shown by

\[
\mathcal{G}_{R, (r_1, r_2), \nu^+, \nu^-} = \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \sigma \cdot
\]

\[
= \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \sigma^{-1}
\]

**(B.19)**
The identity (A.45) follows from the calculation

\[
\frac{d_R d_S}{(m+n)!^2} \sum_{\sigma, \tau \in S_{m+n}} \sigma \tau 
\]

\[
= \frac{d_R d_S}{(m+n)!^2} \sum_{\sigma, \rho \in S_{m+n}} \rho 
\]

\[
= \delta^{RS} d_R \sum_{\rho \in S_{m+n}} \rho 
\]

\[
= \delta^{RS} \delta^{r_1 s_1} \delta^{r_2 s_2} \delta^{\nu_+ - \mu_+} \frac{d_R}{(m+n)!} \sum_{\sigma \in S_{m+n}} \rho 
\]

(C.20)

\[
\begin{align*}
\{ j_1 & \ j_2 & j_{1+2} \\
\ j_3 & \ J & j_{2+3} \}
\colon \text{Hom}(j_1 \otimes j_2 \otimes j_3, J) \to \text{Hom}(j_1 \otimes (j_2 \otimes j_3), J).
\end{align*}
\]

(C.1)

The problem of computing 6j symbol is called the Racah-Wigner calculus.

We construct a slightly general object from the branching coefficients. The generalized 6j symbol is covariant under the action of symmetric groups, and contains four multiplicity labels.

C Generalized Racah-Wigner tensor
C.1 Case of $\tilde{C}_{ooo}$

Consider two ways of the double restriction

$$S_L \downarrow (S_{L_1 + L_2} \otimes S_{L_3}) \downarrow (S_{L_1} \otimes S_{L_2} \otimes S_{L_3}), \quad S_L \downarrow (S_{L_1} \otimes S_{L_2 + L_3}) \downarrow (S_{L_1} \otimes S_{L_2} \otimes S_{L_3})$$

(C.2)

with $L = L_1 + L_2 + L_3$, which corresponds to the calculation of $\tilde{C}_{ooo}$ in Section 3.4.1. They induce the irreducible decompositions

$$\hat{R} = \bigoplus_{R_{12},q_3} g(R_{12}, q_3; \hat{R}) R_{12} \otimes q_3 \equiv \bigoplus_{q_1,q_2,q_3} g(q_1, q_2; R_{12}) g(R_{12}, q_3; \hat{R}) q_1 \otimes q_2 \otimes q_3$$

$$\hat{R} = \bigoplus_{R_{23},q_1} g(R_{23}, q_1; \hat{R}) q_1 \otimes R_{23} \equiv \bigoplus_{q_2',q_3'} g(q_2', q_3'; R_{23}) g(R_{23}, q_1; \hat{R}) q_1 \otimes q_2' \otimes q_3'$$

(C.3)

The corresponding branching coefficients are

$$\hat{R} \bigg|_{\hat{I}} = \bigoplus_{R_{12},q_3} \bigg| R_{12} \bigg| \bigg|_{R_{12},q_3} \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\mu \bigg) \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\rho \bigg) = \bigg| q_1 q_2 q_3 \bigg|_{I \rightarrow (I,c)} \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\mu \bigg) \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\rho \bigg)$$

$$\hat{R} \bigg|_{\hat{I}} = \bigoplus_{q_1,q_2,q_3} \bigg| q_1 q_2 q_3 \bigg|_{I \rightarrow (I,c)} \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\mu \bigg) \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\rho \bigg)$$

(C.4)

The multiplicity labels $(\mu, \rho)$ and $(\mu', \rho')$ run over the spaces

$$\xi \equiv (\mu, \rho) \in \mathcal{M}_{12}, \quad |\mathcal{M}_{12}| = g(q_1, q_2; R_{12}) g(R_{12}, q_3; \hat{R})$$

$$\xi' \equiv (\mu', \rho') \in \mathcal{M}_{23}, \quad |\mathcal{M}_{23}| = g(q_2, q_3; R_{23}) g(R_{23}, q_1; \hat{R})$$

(C.5)

which are subsets of the total multiplicity space induced by the irreducible decomposition

$$\hat{R} = \bigoplus_{q_1,q_2,q_3} \bigoplus_{\eta \in \mathcal{M}_{12,3}} (q_1 \otimes q_2 \otimes q_3)_{\eta}, \quad \bigg| \hat{R} \bigg|_{\hat{I}} = \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\eta \bigg) \bigg| \bigg|_{I \rightarrow (I,c)} \bigg( B^{T}_{I \rightarrow (I,c)} \bigg| R_{12},q_3,\eta \bigg)$$

$$\eta \in \mathcal{M}_{\text{tot}}, \quad |\mathcal{M}_{\text{tot}}| = g(q_1, q_2, q_3; \hat{R}).$$

(C.6)

From the identity (A.39), we obtain the following relation between the branching coefficients in (C.4) and (C.6),

$$\left< \tilde{q}_1 \tilde{q}_2 \tilde{q}_3 \left| a \ b \ c \ \tilde{\mu} \ \tilde{\rho} \right| q_1 q_2 q_3 \right> = \sum_{R_{12}} \left< \tilde{q}_1 \tilde{q}_2 \tilde{q}_3 \left| a \ b \ c \ \tilde{\mu} \ \tilde{\rho} \right| q_1 q_2 q_3 \right>$$

$$= \sum_{R_{12}} \sum_{\eta} \sum_{\mu,\rho} \sum_{a,b,c} \sum_{\mu',\rho'} \delta_{\mu' a} \delta_{\rho' b} \delta_{\rho a} \delta_{\mu b} \delta_{\mu' c} \delta_{\rho' c}$$

(C.7)

where the RHS depends on $R_{12}$ through the multiplicity space of $(\mu, \rho)$ in (C.5).

We define the orthogonal matrix (A.16) between the two states by

$$U_{R} \left( \begin{array}{ccc} q_1 & q_2 & q_3 \\ q_1' & q_2' & q_3' \end{array} \right) = \sum_{I=1}^{d_{R}} \sum_{I=1}^{d_{R}} \sum_{I=1}^{d_{R}} \left( B^{T} \right|_{I \rightarrow (I,c)} \left( R_{12},q_3,\mu \right) \left( B^{T} \right|_{I \rightarrow (I,c)} \left( R_{12},q_3,\rho \right) \tilde{B}^{R_{12},q_3,\mu} \tilde{B}^{R_{12},q_3,\rho}$$

$$\equiv \left< q_1 q_2 q_3 \left| a \ b \ c \ \mu \ \rho \right| q_1 q_2 q_3 \right> \sum_{a,b,c} \sum_{\mu,\rho} \sum_{a',b',c'} \sum_{\mu',\rho'} \delta_{\mu' a} \delta_{\rho' b} \delta_{\rho a} \delta_{\mu b} \delta_{\mu' c} \delta_{\rho' c}$$

(C.8)
and call it the generalized Racah-Wigner tensor. Our notation is slightly redundant because the generalized Racah-Wigner tensor is proportional to \( \prod_{i=1}^{3} \delta^{q_i q'_i} \). The usual 6j symbol for a symmetric group is given by

\[ \text{tr} (U_R) \equiv \sum_{a,b,c} U_R \left( \begin{array}{ccc} q_1 & q_2 & q_3 \\ q_1' & q_2' & q_3' \end{array} \right) \left( \begin{array}{ccc} R_{12} & \mu & \rho \\ R_{23} & \mu' & \rho' \end{array} \right)_{abc,abc}. \] (C.9)

The generalized Racah-Wigner tensor can be depicted as

\[ U_R \left( \begin{array}{ccc} q_1 & q_2 & q_3 \\ q_1' & q_2' & q_3' \end{array} \right) \left( \begin{array}{ccc} R_{12} & \mu & \rho \\ R_{23} & \mu' & \rho' \end{array} \right)_{abc,a'b'c'}. \] (C.10)

We want to compute the products of generalized Racah-Wigner tensors

\[ \text{tr} (U_R \tilde{U}_R) \equiv \sum_{a,b,c} U_R \left( \begin{array}{ccc} q_1 & q_2 & q_3 \\ q_1' & q_2' & q_3' \end{array} \right) \left( \begin{array}{ccc} R_{12} & \mu & \rho \\ R_{23} & \mu' & \rho' \end{array} \right)_{abc,a'b'c'} U_R \left( \begin{array}{ccc} q'_1 & q'_2 & q'_3 \\ q'_1 & q'_2 & q'_3 \end{array} \right) \left( \begin{array}{ccc} R_{12} & \mu' & \rho' \\ R_{23} & \mu & \rho' \end{array} \right)_{abc,a'b'c'} \] (C.11)

which are rewriting of the product of projectors \([A.45]\),

\[ \text{tr} (U_R \tilde{U}_R) = \text{tr}_R \left( \mathbf{R} \rightarrow \cdots \rightarrow (q_1, q_2, q_3), \mu, \mu', \mu'' \right) \] (C.12)

By using \( \xi, \xi', \xi'' \) in \([C.5]\), we depict these products as

\[ \text{tr} (U_R \tilde{U}_R) = \] (C.13)

By grouping pairs of nodes with the same color, we obtain the projector representation \([C.12]\). From the identity of the projectors \([A.45]\), we get

\[ \text{tr} (U_R \tilde{U}_R) = \left( \prod_{i=1}^{3} \delta^{q_i q'_i} d_{q_i} \right) \delta_{\xi_1 \xi_2} \delta_{\xi_2 \xi_1} \] (C.14)
where we sum over the repeated indices ($\xi_i$’s).

The product $\text{tr} (\hat{U}_R \tilde{U}_R)$ satisfies the following sum rules,

$\sum_{R_{23}} \text{tr} (\hat{U}_R \tilde{U}_R) = \left( \prod_{i=1}^{3} \delta^{q_i q'_i} d_{q_i} \right) g(q_1, q_2; R_{12}) g(R_{12}, q_3; \hat{R})$ \hfill (C.15)

$\sum_{R_{12}} \text{tr} (\hat{U}_R \tilde{U}_R) = \left( \prod_{i=1}^{3} \delta^{q_i q'_i} d_{q_i} \right) g(q_2, q_3; R_{23}) g(R_{23}, q_1; \hat{R})$.

We can derive these sum rules by using the identities (A.39), (A.15) and (C.7), as

\begin{align*}
\sum_{R_{23}} \sum_{\mu, \rho, \mu', \rho'} & \delta^{q_1} \delta^{q_2} \delta^{q_3} \sum_{\eta, \eta'} g(R_{12}, q_3; \hat{R}) g(q_1, q_2; R_{12}) \\
&= \sum_{\mu, \rho, \eta'} \delta^{q_1} \delta^{q_2} \delta^{q_3} \sum_{\eta, \eta'} g(R_{12}, q_3; \hat{R}) g(q_1, q_2; R_{12}) .
\end{align*}

A solution to the equations (C.15) is

$$\text{tr} (\hat{U}_R \tilde{U}_R) = \left( \prod_{i=1}^{3} \delta^{q_i q'_i} d_{q_i} \right) \frac{g(q_1, q_2; R_{12}) g(R_{12}, q_3; \hat{R}) g(q_2, q_3; R_{23}) g(R_{23}, q_1; \hat{R})}{g(q_1, q_2, q_3; \hat{R})} . \hfill (C.17)$$
We conjecture that both sides are equal, and continue the discussion below. Similarly, we find

\[
\sum_{R_{12}} \text{tr} \left( U_{\tilde{R}} \tilde{U}_{\tilde{R}} \tilde{U}_{\tilde{R}} \right) = \left( \prod_{i=1}^{3} \delta_{q_i} \delta_{\tilde{q}_i} \delta_{\tilde{q}_i} \right) \sum_{\mu, \rho, \mu', \rho'} \left( \begin{array}{c} U_{\tilde{R}} \left( \begin{array}{cccc} q_1 & q_2 & q_3 & R_{12} \\ q_1' & q_2' & q_3' & R_{12} \end{array} \right) \mu & \rho \\ \mu' & \rho' \end{array} \right)_{abc,a'b'c'} \times \]

\[
\left( \begin{array}{c} U_{\tilde{R}} \left( \begin{array}{cccc} q_1' & q_2' & q_3' & R_{12} \\ q_1 & q_2 & q_3 & R_{12} \end{array} \right) \mu & \rho \\ \mu' & \rho' \end{array} \right)_{a'b'c',abc} \right. 
\]

\[
\sum_{R_{23}} \text{tr} \left( U_{\tilde{R}} \tilde{U}_{\tilde{R}} \tilde{U}_{\tilde{R}} \right) = \left( \prod_{i=1}^{3} \delta_{q_i} \delta_{\tilde{q}_i} \delta_{\tilde{q}_i} \right) \sum_{\mu, \rho, \mu', \rho''} \left( \begin{array}{c} U_{\tilde{R}} \left( \begin{array}{cccc} q_1 & q_2 & q_3 & R_{12} \\ q_1' & q_2' & q_3' & R_{31} \end{array} \right) \mu & \rho \\ \mu'' & \rho'' \end{array} \right)_{abc,a'b'c''} \times \]

\[
\left( \begin{array}{c} U_{\tilde{R}} \left( \begin{array}{cccc} q_1 & q_2 & q_3 & R_{12} \\ q_1' & q_2' & q_3' & R_{31} \end{array} \right) \mu & \rho \\ \mu'' & \rho'' \end{array} \right)_{a'b'c',a'b'c''} \right. 
\]

A solution to these equations is

\[
\text{tr} \left( U_{\tilde{R}} \tilde{U}_{\tilde{R}} \tilde{U}_{\tilde{R}} \right) = \left( \prod_{i=1}^{3} \delta_{q_i} \delta_{\tilde{q}_i} \delta_{\tilde{q}_i} \right) d_{q_i} \times \frac{g(q_1, q_2; R_{12}) g(R_{12}, q_3; \tilde{R}) g(q_2, q_3; R_{23}) g(R_{23}, q_1; \tilde{R}) g(q_3, q_1; R_{31}) g(R_{31}, q_2; \tilde{R})}{g(q_1, q_2, q_3; \tilde{R})^2}. \tag{C.19} \]

In view of (C.14), our conjecture is summarized as

\[
\sum_{\xi_1 \in M_{12}} \sum_{\xi_2 \in M_{23}} \delta_{\xi_1} \delta_{\xi_2} \delta_{\xi_2} = \frac{|M_{12}| |M_{23}|}{|M_{\text{tot}}|} \tag{C.20} \]

\[
\sum_{\xi_1 \in M_{12}} \sum_{\xi_2 \in M_{23}} \sum_{\xi_3 \in M_{31}} \delta_{\xi_1} \delta_{\xi_2} \delta_{\xi_3} \delta_{\xi_1} = \frac{|M_{12}| |M_{23}| |M_{31}|}{|M_{\text{tot}}|^2}. \tag{C.20} \]

### C.2 Case of $\hat{C}^{XYZ}_{\tilde{R}}$

Consider another set of restrictions

\[
S_L \downarrow \left( (S_L \otimes S_{L_6}) \otimes S_{L_4} \otimes S_{L_3} \otimes (S_L \otimes S_{L_1}) \right) \tag{C.21} \]

\[
S_L \downarrow \left( (S_{L_3} \otimes S_{L_4}) \otimes S_{L_2} \otimes S_{L_5} \otimes (S_L \otimes S_{L_1}) \right) \tag{C.21} \]

\[
S_L \downarrow \left( (S_L \otimes S_{L_2}) \otimes S_{L_4} \otimes S_{L_6} \otimes (S_{L_3} \otimes S_{L_5}) \right) \tag{C.21} \]
with \( L = \sum_{i=1}^{6} L_i \), which correspond to the case of \( \tilde{C}_{Ri}^{XYZ} \) in Section 3.4.2. They induce the irreducible decomposition

\[
\tilde{R} = \bigoplus_{R,T \in \{q_i\}} \bigoplus_{q_i} \left\{ g(q_5, q_6; Q) g(q_1, q_3; R) g(q_2, q_4; T) g(R, T; \tilde{R}) \bigotimes_{i=1}^{6} q_i \right\}
\]

\[
\tilde{R} = \bigoplus_{Q', R', T' \in \{q_i'\}} \bigoplus_{q_i'} \left\{ g(q'_5, q'_6; Q') g(q'_1, q'_3; R') g(q'_2, q'_4; T') g(R', T'; \tilde{R}) \bigotimes_{i=1}^{6} q'_i \right\} \quad (C.22)
\]

\[
\tilde{R} = \bigoplus_{Q'', R'', T'' \in \{q_i''\}} \left\{ g(q''_5, q''_6; Q'') g(q''_1, q''_3; R'') g(q''_2, q''_4; T'') g(R'', T''; \tilde{R}) \bigotimes_{i=1}^{6} q''_i \right\}.
\]

We fix the representations \((R, Q), (R', Q'), (R'', Q'')\) and the multiplicity labels \(\nu, \nu', \nu''\) according to the external operators. The space of multiplicities run over the spaces

\[
\xi \in \mathcal{M}_{R,Q,\nu}, \quad \xi' \in \mathcal{M}_{R',Q',\nu'}, \quad \xi'' \in \mathcal{M}_{R'',Q'',\nu''}
\]

where

\[
|\mathcal{M}_{R,Q,\nu}| = g(q_5, q_6; Q) g(q_2, q_4; T) g(R, T; \tilde{R})
\]

\[
|\mathcal{M}_{R',Q',\nu'}| = g(q'_5, q'_6; Q') g(q'_2, q'_4; T') g(R', T'; \tilde{R}) \quad (C.24)
\]

\[
|\mathcal{M}_{R'',Q'',\nu''}| = g(q''_5, q''_6; Q'') g(q''_2, q''_4; T'') g(R'', T''; \tilde{R})
\]

They are subsets of the total multiplicity space

\[
|\mathcal{M}_{\text{tot}}| \equiv g(q_1, q_2, q_3, q_4, q_5, q_6; \tilde{R}),
\]

\[
|\mathcal{M}_{\text{tot}}| = \sum_{R,Q} \sum_{\nu = 1}^{6} |\mathcal{M}_{R,Q,\nu}| = \sum_{R',Q'} \sum_{\nu' = 1}^{6} |\mathcal{M}_{R',Q',\nu'}| = \sum_{R'',Q''} \sum_{\nu'' = 1}^{6} |\mathcal{M}_{R'',Q'',\nu''}|
\]

Since the restricted Schur characters have two multiplicity labels \(A.24\), we introduce

\[
\xi_{\pm} \in \mathcal{M}_{R_{\pm},Q_{\pm},\nu_{\pm}}, \quad \xi'_{\pm} \in \mathcal{M}_{R'_{\pm},Q'_{\pm},\nu'_{\pm}}, \quad \xi''_{\pm} \in \mathcal{M}_{R''_{\pm},Q''_{\pm},\nu''_{\pm}}
\]

where the \(\pm\) signs are correlated.\(^{10}\)

Let us define the generalized Racah-Wigner tensor by

\[
W_{R_+} \left( q_1, q_2, \ldots, q_6 \bigg| \begin{array}{c} R_- \xi_- \\ \xi'_+ \end{array} \right)_{ab \ldots f} \equiv \left\langle q_1 q_2 \ldots q_6 \bigg| \begin{array}{c} q'_1 q'_2 \ldots q'_6 \\ \xi_- \end{array} \bigg| \begin{array}{c} \xi'_+ \\ a' b' \ldots f' \end{array} \right\rangle
\]

\(^{10}\)Note that \((R_-, R'_-, R''_-) = (R_+, R'_+, R''_+)\) in the main text. We removed these constraints for convenience.
which is again proportional to $\prod_{i=1}^{6} \delta^{q_i q_i'}$. We want to compute their products

$$\text{tr} \left( W_R \tilde{W}_R \right) = \sum_{\xi_+ \xi'_+} \sum_{\xi_+ \xi'_+} W_R \left( \begin{array}{c|c} q_1 & q_2 & \ldots & q_6 \\ \hline q'_1 & q'_2 & \ldots & q'_6 \\ \end{array} \right) \left( \begin{array}{c} R_- \xi_+ \end{array} \right)_{a b \ldots f, a \prime b \prime \ldots f'} \times W_R \left( \begin{array}{c|c} q'_1 & q'_2 & \ldots & q'_6 \\ \hline q_1 & q_2 & \ldots & q_6 \\ \end{array} \right) \left( \begin{array}{c} R'_- \xi'_+ \end{array} \right)_{a \prime b \prime \ldots f', a b \ldots f}. \tag{C.28}$$

$$\text{tr} \left( W_R \tilde{W}_R \tilde{W}_R \right) = \sum_{\xi_+ \xi'_+} \sum_{\xi_+ \xi'_+} W_R \left( \begin{array}{c|c} q_1 & q_2 & \ldots & q_6 \\ \hline q'_1 & q'_2 & \ldots & q'_6 \\ \end{array} \right) \left( \begin{array}{c} R'_- \xi'_+ \end{array} \right)_{a \prime b \prime \ldots f', a b \ldots f'} \times W_R \left( \begin{array}{c|c} q'_1 & q'_2 & \ldots & q'_6 \\ \hline q_1 & q_2 & \ldots & q_6 \\ \end{array} \right) \left( \begin{array}{c} R'_- \xi'_+ \end{array} \right)_{a \prime b \prime \ldots f', a b \ldots f}. \tag{C.29}$$

They are identical to the product of projectors (3.55),

$$\text{tr} \left( W_R \tilde{W}_R \right) = \text{tr}_R \left( \left( \begin{array}{c} q_1 \ldots q_6, q'_1 \ldots q'_6 \end{array} \right) \xi_- \xi_+ \right) \text{tr}_I \left( \left( \begin{array}{c} q_1 \ldots q_6, q'_1 \ldots q'_6 \end{array} \right) \xi_- \xi'_+ \right) \quad \text{tr} \left( W_R \tilde{W}_R \tilde{W}_R \right) = \text{tr}_R \left( \left( \begin{array}{c} q_1 \ldots q_6, q'_1 \ldots q'_6 \end{array} \right) \xi_- \xi_+ \right) \text{tr}_I \left( \left( \begin{array}{c} q_1 \ldots q_6, q'_1 \ldots q'_6 \end{array} \right) \xi_- \xi'_+ \right) \text{tr}_I \left( \left( \begin{array}{c} q_1 \ldots q_6, q'_1 \ldots q'_6 \end{array} \right) \xi_- \xi''_+ \right). \tag{C.30}$$

These products are depicted as

$$\text{tr} \left( W_R \tilde{W}_R \right) = \quad \text{tr} \left( W_R \tilde{W}_R \tilde{W}_R \right) = \quad \tag{C.31}$$

The identity of the projectors (A.45) suggests that

$$\text{tr} \left( W_R \tilde{W}_R \right) = \left( \prod_{i=1}^{6} \delta^{q_i q_i'} d_{q_i} \right) \delta^{\xi_- \xi_+} \delta^{\xi_- \xi'_+} \quad \text{tr} \left( W_R \tilde{W}_R \tilde{W}_R \right) = \left( \prod_{i=1}^{6} \delta^{q_i q_i'} \delta^{q_i q_i''} d_{q_i} \right) \delta^{\xi_- \xi_+} \delta^{\xi_- \xi'_+} \delta^{\xi_- \xi''_+}. \tag{C.32}$$

By summing $\{\xi_+, \xi'_+, \xi''_+\}$ over the ranges $\{\mathcal{M}_{R_+,Q_+,\nu_-}, \mathcal{M}_{R'_+,Q'_+,\nu'_+}, \mathcal{M}_{R''_+,Q''_+,\nu''_+}\}$, we discover the overlap

$$\sum_{\xi_- \in \mathcal{M}_{R_+,Q_-,\nu_-}} \sum_{\xi'_+ \in \mathcal{M}_{R'_+,Q'_+,\nu'_+}} \delta^{\xi_- \xi'_+} = \left| \mathcal{M}_{R_+,Q_-,\nu_-} \cap \mathcal{M}_{R'_+,Q'_+,\nu'_+} \right|. \tag{C.33}$$
The overlap satisfies the sum rules
\[
\sum_{R_{-},Q_{-},\nu_{-}} \sum_{R'_{+},Q'_{+},\nu'_{+}} \left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \cap \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right| = \left| \mathcal{M}_{\text{tot}} \right|
\]
\[
\sum_{R_{-},Q_{-},\nu_{-}} \left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \cap \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right| = \left| \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right|
\] (C.34)
\[
\sum_{R'_{+},Q',\nu'_{+}} \left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \cap \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right| = \left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \right|
\]

As a solution to the sum rules, we conjecture that
\[
\left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \cap \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right| = \delta_{\nu_{-},\nu'_{-}} \frac{\left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \right| \left| \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right|}{\left| \mathcal{M}_{\text{tot}} \right|}
\] (C.35)

where \( \delta_{\nu_{-},\nu'_{-}} \) should be understood as the intersection inside \( \mathcal{M}_{\text{tot}} \)
\[
\delta_{\nu_{-},\nu'_{-}} = \begin{cases} 1 & \left( \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \cap \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \neq \emptyset \right) \\ 0 & \text{(otherwise)} \end{cases}
\] (C.36)

It follows that
\[
\sum_{\xi_{+},\xi'_{+}} \text{tr} (W_{R} \hat{W}_{R}) = \left( \prod_{i=1}^{6} \delta^{q_{i}' q_{i}} d_{q_{i}} \right) \delta_{\nu_{-},\nu'_{-}} \delta_{\nu'_{+},\nu_{+}} \times
\]
\[
\frac{\left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \right| \left| \mathcal{M}_{R_{+},Q_{+},\nu_{+}} \right| \left| \mathcal{M}_{R'_{-},Q'_{-},\nu'_{-}} \right| \left| \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right|}{\left| \mathcal{M}_{\text{tot}} \right|^2}
\] (C.37)
\[
\sum_{\xi_{+},\xi'_{+},\xi''_{+}} \text{tr} (W_{R} \hat{W}_{R} \hat{W}_{R}) = \left( \prod_{i=1}^{6} \delta^{q_{i}' q_{i}} \delta^{q_{i}' q''_{i}} d_{q_{i}} \right) \delta_{\nu_{-},\nu'_{-}} \delta_{\nu'_{+},\nu''_{+}} \delta_{\nu''_{+},\nu_{+}} \times
\]
\[
\frac{\left| \mathcal{M}_{R_{-},Q_{-},\nu_{-}} \right| \left| \mathcal{M}_{R_{+},Q_{+},\nu_{+}} \right| \left| \mathcal{M}_{R'_{-},Q'_{-},\nu'_{-}} \right| \left| \mathcal{M}_{R'_{+},Q'_{+},\nu'_{+}} \right| \left| \mathcal{M}_{R''_{+},Q''_{+},\nu''_{+}} \right|}{\left| \mathcal{M}_{\text{tot}} \right|^3}
\] (C.38)

C.3 Restricted Littlewood-Richardson coefficients

Let us compute the restricted Littlewood-Richardson coefficients in [27] in our method. We will find the perfect agreement. However, they considered multiplicity-free cases only. Thus, this agreement does not provide non-trivial checks of our conjectured formula.

We define the restricted Littlewood-Richardson coefficients by
\[
F_{\{1\} \{2\}}^{(3)} = \frac{1}{L_{1}! L_{2}!} \sum_{\sigma_{1} \in S_{L_{1}}} \sum_{\sigma_{2} \in S_{L_{2}}} \chi^{R_{1}}(\sigma_{1}) \chi^{R_{2}}(\sigma_{2}) \chi^{R_{3}}(\sigma_{1} \circ \sigma_{2})
\]
\[
L_{i} = m_{i} + n_{i} , \quad R_{i} = \{ R_{i}, (r_{i}, s_{i}), (\nu_{i-}, \nu_{i+}) \}
\] (C.39)
The definition used in [27] is
\[
f_{\{1\}(2)}^{\{3\}} = \frac{1}{m_1!n_1!m_2!n_2!} \frac{m_3!n_3!}{L_3!} \frac{d_{R_3} d_{s_3}}{d_{r_3} d_{s_2}} \sum_{\sigma_1 \in S_{L_1}} \sum_{\sigma_2 \in S_{L_2}} \chi_{R_1}^{R_1}(\sigma_1) \chi_{R_2}^{R_2}(\sigma_2) \chi_{R_3}^{R_3}(\sigma_1 \circ \sigma_2). \tag{C.40}
\]

The two definitions are related by
\[
F_{\{1\}(2)}^{\{3\}} = \frac{m_1!n_1!m_2!n_2!}{m_3!n_3!} \frac{L_3!}{L_1!L_2!} \frac{d_{r_3} d_{s_3}}{d_{r_1} d_{s_2}} f_{\{1\}(2)}^{\{3\}}. \tag{C.41}
\]

The restricted Littlewood-Richardson coefficients \(F_{\{1\}(2)}^{\{3\}}\) can be computed as follows. First, consider the restriction \(S_{L_3} \downarrow (S_{L_1} \otimes S_{L_2})\), which gives
\[
R_3 = \bigoplus_{T_1, T_2} g(T_1, T_2; R_3) (T_1 \otimes T_2). \tag{C.42}
\]

The restricted character in (C.39) becomes
\[
\chi_{R_3}^{R_3}(\sigma_1 \circ \sigma_2) = \sum_{T_1, T_2, \mu} g(T_1, T_2; R_3) D_{h_1 h_1'}^{T_1}(\sigma_1) D_{h_2 h_2'}^{T_2}(\sigma_2) \tilde{B}_{I \rightarrow (h_1 h_2)}^{R_3 \rightarrow (T_1, T_2) \mu} (B^{T}_{I \rightarrow (h_1 h_2)^\prime})^{R_3 \rightarrow (T_1, T_2) \mu} \times B_{I \rightarrow (i, j)}^{R_3 \rightarrow (r_3, s_3), \mu \rightarrow (r_3, s_3), \mu \rightarrow (r_3, s_3) + \nu_3}. \tag{C.43}
\]

In the quiver notation, we can depict this equation as
\[
\chi_{R_3(r_3, s_3), (\nu_3- \nu_3+)}^{R_3}(\sigma_1 \circ \sigma_2) = \sum_{T_1, T_2, \mu} \chi_{R_3(r_3, s_3), (\nu_3- \nu_3+)}^{R_3}(\sigma_1 \circ \sigma_2) = \sum_{T_1, T_2, \mu} \gamma_{(\sigma_1 \circ \sigma_2)}^{T_1 T_2}. \tag{C.44}
\]

By summing over \(\sigma_1\) and \(\sigma_2\) in (C.39), we get \(\delta_{T_1, R_1} \delta_{T_2, R_2}\) and another sets of branching coefficients in place of \(\sigma_1, \sigma_2\) in (C.44), giving us
\[
\chi_{R_3(r_3, s_3), (\nu_3- \nu_3+)}^{R_3}(\sigma_1 \circ \sigma_2) = \sum_{T_1, T_2, \mu} \gamma_{(\sigma_1 \circ \sigma_2)}^{T_1 T_2} = \text{tr} (\mathcal{P} \mathcal{H}). \tag{C.45}
\]
The restricted Littlewood-Richardson coefficient (C.39) becomes
\[
F^{(3)}_{\{1\}} = \frac{1}{d_{R_1}d_{R_2}} \sum_{\mu} \text{tr} \left( \mathcal{D}^{R_3 \rightarrow (r_3,s_3),(\nu_3,v_3)} \mathcal{D}^{R_1 \rightarrow (R_1,R_2)} \mathcal{D}^{R_2 \rightarrow (\mu_{1},s_1,r_1,s_1)} (\mu_{1},(v_1,v_2),((\nu_1,\nu_2))) \right). \tag{C.46}
\]

To evaluate the projectors, we introduce the permutations on the fully-split space
\[
S_{FS} = S_{n_1} \otimes S_{n_2} \otimes S_{n_1} \otimes S_{n_2} \tag{C.47}
\]
and consider sub-projectors. The total multiplicity space for the restriction \(S_{L_3} \downarrow S_{FS}\) is
\[
|\mathcal{M}_{\text{tot}}| = g(r_1, r_2, s_1, s_2; R_3). \tag{C.48}
\]

The multiplicity space for the first projector \(\mathcal{D}^{R_3 \rightarrow (r_3,s_3),(\nu_3,v_3)}\) is
\[
\sum_{r_3,s_3} \sum_{\nu_3,v_3} |\mathcal{M}_{r_3,s_3,\nu_3,v_3}| = \sum_{r_3,s_3} \sum_{\nu_3,v_3} |\mathcal{M}_{r_3,s_3,\nu_3}| = |\mathcal{M}_{\text{tot}}|. \tag{C.49}
\]
The multiplicity space for the second projector \(\mathcal{D}^{R_3 \rightarrow \cdots \rightarrow (r_1,s_1,r_2,s_2),(\nu,\nu_1,\nu_2)}\) is
\[
\sum_{R_1,R_2} \sum_{\nu_1,v_1} \sum_{\nu_2,v_2} |\mathcal{M}_{R_1,R_2,\nu_1,v_1,\nu_2,v_2}| = \sum_{R_1,R_2} \sum_{\nu_1,v_1} \sum_{\nu_2,v_2} |\mathcal{M}_{R_1,R_2,\nu_1,v_2}| = |\mathcal{M}_{\text{tot}}|. \tag{C.50}
\]

From the identity of the projector (A.45), we obtain
\[
\text{tr} (\mathcal{D} \mathcal{D}) = \delta^{\nu_3\nu_3} (v_1,v_2) \delta^{(v_1,v_2)} v_3 - d_{r_1} d_{r_2} d_{s_1} d_{s_2} \mathcal{G}_{LR} \tag{C.51}
\]
where we grouped \((\nu_1,\nu_2)\) so that they can be compared with \(\nu_3\). Just like before, we conjecture that
\[
\mathcal{G}_{LR} = \frac{|\mathcal{M}_{r_3,s_3,\nu_3,v_3}| |\mathcal{M}_{r_3,s_3,\nu_3}| |\mathcal{M}_{R_1,R_2,\nu_1,v_1,\nu_2,v_2}| |\mathcal{M}_{R_1,R_2,\nu_1,v_2}|}{|\mathcal{M}_{\text{tot}}|^2} \tag{C.52}
\]
In summary, we get
\[
F^{(3)}_{\{1\}} = \frac{d_{r_1} d_{r_2} d_{s_1} d_{s_2}}{d_{R_1} d_{R_2}} \left( \frac{g(R_1,R_2; R_3) g(r_1,r_2,s_1,s_2)}{g(r_1,r_2,s_1,s_2;R_3)} \right)^2. \tag{C.53}
\]

Three cases have been considered in [27]. The first case is the antisymmetric representations,
\[
(R_i, r_i, s_i) = \left( [m_i+n_i], [1^n], [1^m] \right) \tag{C.54}
\]
and the second case is the symmetric representations,
\[
(R_i, r_i, s_i) = \left( [m_i+n_i], [m_i], [n_i] \right). \tag{C.55}
\]
In both cases, all representations are one-dimensional and multiplicity-free. Therefore \( F^{(3)}_{\{1\}\{2\}} = 1 \), which means
\[
f^{(3)}_{\{1\}\{2\}} = \frac{m_3! n_3! L_1! L_2!}{m_1! n_1! m_2! n_2! L_3!}.
\]  
\( \text{(C.56)} \)

The last case is \( r_2 = s_1 = \emptyset \), implying that
\[
R_1 = r_1 = r_3, \quad R_2 = s_2 = s_3, \quad F^{(3)}_{\{1\}\{2\}} = 1
\]  
\( \text{(C.57)} \)

and hence
\[
f^{(3)}_{\{1\}\{2\}} = \delta_{R_1, r_3} \delta_{R_2, s_3} \frac{L_1! L_2!}{L_3!} \frac{d_{R_3}}{d_{r_3} d_{s_3}}.
\]  
\( \text{(C.58)} \)

All the results agree with \[27\].

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