Progress on Ultraviolet Finiteness of Supergravity

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Abstract

In this lecture we summarize recent calculations pointing to the possible ultraviolet finiteness of N = 8 supergravity in four dimensions. We outline the modern unitarity method, which enables multiloop calculations in this theory and allows us to exploit a remarkable relation between tree-level gravity and gauge-theory amplitudes. We also describe a link between observed cancellations at loop level and improved behavior of tree-level amplitudes under large complex deformations of momenta.

1 Introduction

For over 25 years the prevailing wisdom has been that it is impossible to construct a perturbatively ultraviolet finite point-like quantum field theory of gravity in four dimensions (see e.g. refs. [1]). In this lecture we describe recent concrete calculations that call into question this belief.

Of all unitary quantum gravity field theories, maximally supersymmetric N = 8 supergravity [2] is the most promising one to investigate for possible ultraviolet finiteness. Its high degree of supersymmetry suggests that it has the best ultraviolet properties of any gravity field theory with two derivative couplings. Moreover, with the modern unitarity method [2,4,5,6,8,9], the high degree of supersymmetry can be exploited to greatly simplify calculations. In fact, the striking simplicity of the theory led to the recent suggestion that N = 8 supergravity may in a sense be the simplest quantum field theory [10]. The unitarity method allows us to exploit a remarkable relation between gravity and gauge-theory tree amplitudes [11,12,13], allowing us to map gravity calculations into algebraically simpler gauge-theory calculations.

In a classic paper 't Hooft and Veneziano showed that gravity coupled to matter generically diverges at one loop in four dimensions [14,15]. Due to the dimensionful nature of the coupling, the divergences cannot be absorbed by a redefinition of the original parameters of the Lagrangian, rendering the theory non-renormalizable. Pure Einstein gravity does not possess a viable counterterm at one loop, delaying the divergence to two loops [14,16,17]. The two-loop divergence of pure Einstein gravity was established by Goroo and Sagnotti and by van de Ven, through direct computation [13,19].

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Figure 1: Gauge theories have three- and four-point vertices in a Feynman diagrammatic description.

Figure 2: Gravity theories have an infinite number of higher-point contact interactions in a Feynman diagrammatic description.

Over the years supersymmetry has been studied extensively as a mechanism for delaying the onset of divergences in gravity theories. No supergravity theory can diverge until at least three loops [20, 21, 1]. With additional assumptions this can be delayed further [22, 23, 24, 25, 26], perhaps even to nine loops. However, all studies to date based on supersymmetry considerations alone point to protection against divergences from supersymmetry failing at some loop order. An additional mechanism is needed to prevent divergences.

Here we review direct evidence at three loops [27, 28] that such a mechanism does indeed exist in gravity theories. It is striking that all explicit complete calculations of $N = 8$ supergravity scattering amplitudes [6, 29, 30, 31, 27, 32, 33, 34, 10, 28] find a power counting identical to that of $N = 4$ super-Yang-Mills theory, which is known to be finite in four dimensions [35, 36, 37]. Interestingly, M theory has also been used to argue for the finiteness of $N = 8$ supergravity [38], though issues with decoupling towers of massive states [39] may alter these conclusions.

Here we focus only on the possible order-by-order finiteness of the perturbative series of $N = 8$ supergravity, leaving aside non-perturbative issues or how one might go about constructing a realistic ultraviolet finite theory. We are interested in perturbative ultraviolet finiteness of gravity because the existence of cancellations sufficient to render the theory finite would imply a new symmetry or dynamical mechanism to enforce these. We can expect that a proper understanding of the mechanism behind such cancellations will have a profound impact on our understanding of gravity.

2 Quantum gravity

2.1 Feynman diagram approach

Before turning to the on-shell methods used in modern calculations, it is helpful to first survey gravity Feynman diagrams. We compare the Feynman diagrams of gravity to
those of gauge theory. Consider the Einstein-Hilbert and Yang-Mills Lagrangians,

\[ L_{YM} = \frac{1}{4} F^a F^a ; \quad L_{EH} = \frac{2}{3} \varepsilon^{abc} \mathcal{R} : \]

Following standard Feynman diagrammatic methods we gauge \( x \) and then expand the Lagrangians in a set of vertices. As illustrated in figs. 1 and 2, with standard gauge choices for gauge theory there are three- and four-point interactions, while for gravity there are an infinite number of contact interactions. (As we shall discuss below, all interactions beyond three points are, in fact, unnecessary when using non-shellless methods.) Perhaps more striking than the infinite number of interactions is the complexity of these interactions. Consider, for example, the three-graviton interaction. In standard de Donder gauge, the three-graviton vertex is of the form,

\[ G_3 ; ; (k_1 ; k_2 ; k_3) = \frac{i}{2} \text{Sym} \left( \frac{1}{2} P_3 (k_1 \ k) \right) \left( \frac{1}{2} P_6 (k_1 \ k_1 ) \right) \left( \frac{1}{2} P_3 (k_1 \ k) \right) \]

\[ + P_6 (k_1 \ k_1 ) + 2P_3 (k_1 \ k_1 ) P_3 (k_1 \ k_2 ) \]

\[ + P_3 (k_1 \ k_2 ) + P_6 (k_1 \ k_1 ) + 2P_3 (k_1 \ k_2 ) \]

\[ + 2P_3 (k_1 \ k_2 ) P_3 (k_1 \ k) ; \quad (2) \]

where \( \text{Sym} \) signifies a symmetrization in \( , , \) and \( , \), and \( P_3 \) and \( P_6 \) signify a symmetrization over the three external legs, generating three or six terms respectively. Here the coupling \( \eta \) is related to Newton's constant by \( \eta = 32 \mu \). In total the vertex has on the order of 100 terms. The \( k_i \) are the momenta of the three gravitons and the \( g \)-metric. The precise form of the vertex depends on the gauge, but in general the three vertex is rather complicated. We may contrast this to the relatively simple three-gluon vertex in Feynman gauge,

\[ V_{3}^{abc} (k_1 ; k_2 ; k_3) = i g f^{abc} (k_1 \ k_2) + \text{cyclic} ; \quad (3) \]

Obviously, based on these considerations gravity would seem much more complicated than gauge theory.

One can do better with special gauge choices and appropriate notations \[13, 14\], considerably simplifying the Feynman rules. Still, loop calculations in gravity based directly on Feynman diagrams are extremely difficult, if not impossible, even with the power of modern computers. Because of this difficulty, few explicit determinations of counterterms have been carried out using Feynman diagrams, with some notable exceptions \[14, 15, 18, 19\].

### 2.2 Power counting in gravity theories

In four dimensions Newton's constant is dimensionful, implying that gravity theories in four dimensions are non-renormalizable by power counting. The standard argument that gravity theories are badly behaved in the ultraviolet follows simply from loop-level Feynman diagrams. Consider a Feynman diagram such as the three-loop one in fig. 3. If
the diagram is for gluon scattering of Yang-Mills theory, the diagram yields a Feynman integral of the form,

$$Z \prod_{j=1}^{\nu} \frac{d^D p_j}{(2\pi)^D} \frac{Q^{(g f)}_{m}}{(\frac{p_j^2}{m} + i)}; \quad (4)$$

where the numerator factor of the form $(g f_{abc} p_j)$ signifies a vertex factor, given by a coupling $g$, color factor $f_{abc}$ and a momentum $p_j$. The denominators are the Feynman propagators of the diagram, carrying momentum depending on the independent loop and external momenta $p_j$ and $k_i$. This may be contrasted with the corresponding gravity diagram, whose form is similar except the vertices have two powers of momenta in the numerator for each vertex,

$$Z \prod_{j=1}^{\nu} \frac{d^D p_j}{(2\pi)^D} \frac{Q^{(p p)}_{m}}{(\frac{p_j^2}{m} + i)}; \quad (5)$$

Generically, the momenta appearing in the vertices will be loop momenta. Because of the larger number of loop momenta in the numerators we may expect each gravity Feynman integral to be worse behaved in the ultraviolet, compared to the corresponding gauge theory diagram, unless there are non-trivial cancellations. Based on these simple power-counting considerations, as the number of loops increases we may expect an ever-worsening behavior compared to gauge theory.

We can easily determine the structure of a counterterm given a divergence at a given loop order. Since every loop gains an extra power of $G_N \sim M_{\text{Planck}}^2$ every additional loop must gain two powers of mass dimension to compensate. For graviton amplitudes this corresponds to an additional power of the Riemann tensor, $R_{ijkl}$, or to a covariant derivative $D^2$. At one loop, pure-gravity counterterms in the Lagrangian involve two powers of the Riemann tensor, but by an accident of four dimensions, the Gauss-Bonnet theorem eliminates the potential on-shell counterterm. However, if matter is added, then generically a one-loop divergence appears \([14,15]\). At two loops the potential counterterm is of the form $R_{ijkl} R_{ijkl}$. In pure gravity, thanks to calculations by Gor'ko and Sagnotti and by van de Ven \([18,19]\), we know for a fact that this theory diverges and the coefficient of this counterterm is nonvanishing.

The first divergence in any four-dimensional supergravity theory can be no earlier than three loops, since the potential one- and two-loop counterterms cannot be made consistent with supersymmetry \([20]\). In this case, the potential counterterm consistent with
supersymmetry is an $R^4$ term [1] with indices appropriately contracted, corresponding to the square of the Bel-Robinson tensor. With additional assumptions we can raise the loop order where a divergence is first expected in $N = 8$ supergravity. For example, if an o-shell superspace with $N = 6$ supersymmetries manifest were to exist, potential divergences in four dimensions would be delayed to at least five loops [22]. This is supported by recent supersymmetry arguments which do not rely on the existence of such a superspace [26]. One can go beyond this by assuming the existence of a superspace with $N = 7$ supersymmetries manifest which would delay the first potential divergence to at least six loops [22]. Continuing in this way, if one were to assume the existence of a fully covariant o-shell superspace with $N = 8$ supersymmetries manifest, then the first potential divergence would be pushed to seven loops [22]. It is important to note that since no o-shell superspace beyond $N = 4$ has been constructed for supergravity theories, bounds based on assuming their existence are not firm. A full superspace invariant, which could act as a potential counterterm, has been constructed at eight loops, suggesting that a divergence might appear at this loop order, if it does not appear earlier [41]. The appearance of the first potential divergence can even be pushed to nine loops with an additional speculative assumption that all fields respect ten-dimensional general coordinate invariance [28]. This bound coincides with the one argued [26] from the type II string theory non-renormalization theorem of Berkovits [24]. Beyond this order, no purely supersymmetric mechanism has been suggested for preventing the onset of divergences. Indeed, for a supergravity theory to be ultraviolet finite at all orders of the perturbative expansion, novel cancellations beyond the known supersymmetric ones must exist.

Of course, no power counting arguments can prove the existence of a divergence, only that divergences cannot appear prior to a certain loop order. If the theory were to possess a hidden symmetry, not accounted for in the power counting bound, the bound may suggest the appearance of a divergence when, in fact, there is none. The possibility of hidden symmetries is real: the three-loop divergence thought to appear in the three-loop four-point amplitude in the studies from 1980’s [1] is now known to not be present [27,28,26]. As already mentioned in the introduction, there are a number of good reasons to question the conventional wisdom and to reexamine the ultraviolet behavior of $N = 8$ supergravity using modern on-shell methods.

3 On-shell methods

The computations we summarize in the next section were obtained using on-shell methods. A key feature of these methods is that the elementary building blocks for obtaining new amplitudes are previously obtained on-shell amplitudes. These methods fall into two basic categories: on-shell recursion [42,43] and the modern unitarity method [25]. On-shell recursion is suitable for obtaining complete tree-level amplitudes while the unitarity method is suitable at loop level. At one loop a hybrid bootstrap approach making use of both unitarity and on-shell recursion has also been developed [44]. The use of these methods have become much more widespread in recent years because of their computational efficiency, especially at loop level. These methods have been applied to a wide variety of problems, including QCD (see refs. [45] for examples of recent applications).
AdS/CFT studies of $N = 4$ super-Yang-Mills amplitudes [46,47], and to quantum gravity which we discuss here. Here we only briefly review on-shell methods, focusing on the salient features for gravity, and refer the reader to various reviews [48] for more extensive expositions.

3.1 On-shell vertices

As an explicit example of the inherent simplification of on-shell methods, consider an on-shell version of the three-graviton vertex in eq. (2), where we dot the three legs with physical polarizations tensors, satisfying the on-shell conditions, $k_i^2 = "_1 k_i = "_1 k_i = " = 0$. This gives the simplified vertex,

\[ G_3(k_1;k_2;k_3) = \frac{\hbar}{i} h_{123} (k_1) + \text{cyclic}; \]

(6)

which is not much more complicated than the corresponding on-shell Yang-Mills vertex,

\[ V_3^{abc}(k_1;k_2;k_3) = \frac{\hbar}{i} 2g^{abc} (k_1) + \text{cyclic}; \]

(7)

where the polarization vector satisfies $\kappa_1 = 0$. As we shall discuss below, the structure of the on-shell graviton vertex as a product of the kinematic part of gauge-theory vertices is not accidental, but reflects a profound and important property of gravity.

If we proceed naively with an on-shell formalism, we encounter a difficulty: the on-shell three vertex for gravitons or gluons actually vanishes, because the process is kinematically forbidden. The solution is to use complex momenta [13,49,50], which allows us to satisfy the on-shell conditions and momentum conservation, yet define a nonvanishing vertex.

In general, in four dimensions it is best to express gauge and gravity amplitudes in terms of spinors. We define the two helicity configurations gluons as

\[ "(k_1;q_1) = \frac{\hbar k_1^i j k_i}{2\hbar k_1^i j k_i}; \]

(8)

where \( k_1 \) and \( q_1 \) are Weyl spinors and the \( q_1 \) are arbitrary null reference momenta. The graviton polarization tensors are simply products of these,

\[ "(k_1;q_1) = "(k_1;q_1)"(k_1;q_1): \]

(9)

For later reference we define spinor inner products, as

\[ h_{ji} = h_{kj} j k_j^i; \quad [ji] = h_{kj}^j k_i^k. \]

(10)

These products are antisymmetric, \( h_{ji} = -h_{ji}, [ji] = -[ji] \). For massless momenta, spinor inner products are complex square roots of Lorentz inner products, satisfying \( h_{ji}[ji] = 2k_i^j k_j \). For further details, see refs. [50].

3.2 Behavior of amplitudes under large complex shifts

On-shell recursion [42,43,51,52] gives us a simple means for systematically constructing new tree-level amplitudes in gravity and in gauge theory using previously constructed
lower-point amplitudes. The proof of the recursion relations employs a complex shift of two momenta,

\[
\begin{align*}
k_j & \to k_j(z) = k_j + \frac{z}{2}j + l; \\
k_i & \to k_i(z) = k_i + \frac{z}{2}j + l;
\end{align*}
\]

so that they remain massless, \( k_j(z) = k_j(z) = 0 \), and overall momentum conservation is maintained. An on-shell amplitude containing the momenta \( k_j \) and \( k_i \) then depends on the parameter \( z \),

\[
A(z) = A(k_1; \ldots ; k_j(z); k_{j+1}; \ldots ; k_i(z); k_{i+1}; \ldots ; k_n);
\]

where the physical amplitude is recovered by setting \( z = 0 \).

A crucial step in the proof of on-shell recursion relations is the requirement that \( A(z) \) vanishes as \( z \to 1 \). This property has been demonstrated for a wide class of shifts in gauge [43, 53] and gravity theories [51, 52]. This property is especially surprising for gravity, where the larger number of powers of loop momentum in the vertices would tend to make the diagram ill-behaved as \( z \to 1 \). In general, individual Feynman diagrams are not well-behaved as \( z \to 1 \); only the complete amplitudes vanish in this limit because of a strong cancellation between diagram s. However, with a space-like gauge similar to light-cone gauge [54] it is possible to arrange the diagrams in both gauge theory [55] and gravity [52] such that each diagram is well-behaved at large \( z \). The improved large \( z \) behavior appears connected to an enhanced Lorentz symmetry appearing in this limit [52].

An important consequence of the on-shell recursion relations is that we can recursively construct tree-level scattering amplitudes starting from on-shell three-point vertices. This can even be carried out for pure gravity in \( D \) dimensions [52]. Using the unitarity method these tree amplitudes are all that are needed to systematically construct all higher-loop amplitudes. This construction does not use four- and higher-point vertices in any step for either gauge or gravity theories. Thus, rather surprisingly the four- and higher-point vertices displayed in figs.1 and 2 are irrelevant, as far as scattering amplitudes to any loop order are concerned.

Note the behavior of the amplitudes as \( z \to 1 \) is connected to the high-energy behavior of the theory, albeit in a complex direction. As we shall discuss below, the vanishing of \( A(z) \) as \( z \to 1 \) is connected directly to loop-level ultraviolet cancellations [52].

3.3 Kawai-Lewellen-Tye tree-level relations

From the perspective of Lagrangians or on-shell Feynman rules it is very difficult to discern any simple relations between gravity and gauge theory amplitudes. Nevertheless, tree-level gravity amplitudes can be rewritten in terms of gauge-theory amplitudes, a fact first uncovered in string theory by Kawai, Lewellen and Tye (KLT) [11, 12, 13]. These relations also hold in the theory, as the low-energy limit of string theory. In this limit, the KLT relations for four- and ve-point amplitudes are

\[
\begin{align*}
M_{4}^{\text{tree}}(1;2;3;4) &= i s_{12} a_{4}^{\text{tree}}(1;2;3;4) \bar{A}_{4}^{\text{tree}}(1;2;4;3); \\
M_{5}^{\text{tree}}(1;2;3;4;5) &= i s_{12}s_{34} a_{5}^{\text{tree}}(1;2;3;4;5) \bar{A}_{5}^{\text{tree}}(2;1;4;3;5) \\
&\quad + i s_{13}s_{24} a_{5}^{\text{tree}}(1;3;2;4;5) \bar{A}_{5}^{\text{tree}}(3;1;4;2;5);
\end{align*}
\]

(13) (14)
Here the $M_n$'s are amplitudes in a gravity theory stripped of couplings, the $A_n$'s and $\tilde{A}_n$'s are two distinct color-ordered gauge theory amplitudes and the Mandelstam invariants are $s_{ij} \equiv (k_i + k_j)^2$, with $k_i$ being the outgoing momentum of leg $i$. The color-ordered gauge-theory amplitudes correspond to the coefficient of color traces with a given ordering of fm atrices. The gravity states are direct products of gauge-theory states for each external leg. Explicit formulæ for $n$-point amplitudes may be found in refs. [29,56].

As a simple example, consider the four-graviton amplitude $M_4^{\text{tree}}(1;2;3;4^+)$, where the labels refer to the helicity of the gravitons in an all outgoing convention. Using the four-gluon color-ordered amplitudes [50],

$$A_4^{\text{tree}}(1;2;3;4^+) = \frac{i \hbar^2}{2} \frac{2^{12}}{s_{12}} \frac{2^{12}}{h_{23i} h_{34i} h_{41i} \hbar} ;$$

$$A_4^{\text{tree}}(1;2;4^+;3^+) = \frac{i \hbar^2}{2} \frac{2^{12}}{s_{12}} \frac{2^{12}}{h_{24i} h_{34i} h_{31i} \hbar} ;$$

from the KLT relation [13], we obtain immediately,

$$M_4^{\text{tree}}(1;2;3;4^+) = \frac{2^{12}}{s_{12}} \frac{2^{12}}{h_{23i} h_{34i} h_{41i} \hbar} ;$$

where we have restored the gravity coupling constant. We invite the reader to evaluate this four-graviton amplitude using Feynman rules starting from the Einstein action; the result will match eq. (16). These relations hold not only for any theory corresponding to the low energy limit of a string theory, but appear to hold for a much broader range of gravity theories [13].

The KLT relations have recently been clarified, giving a more transparent form of the relation [50]. Consider the diagrammatic expansion of color-dressed gauge-theory amplitudes,

$$A_n^{\text{tree}}(1;2;3;\ldots;n) = g^n \frac{2^{X}}{\prod_{i} \left( \frac{n_i c_i}{j p_i^j} \right)} ;$$

$$\tilde{A}_n^{\text{tree}}(1;2;3;\ldots;n) = g^n \frac{2^{X}}{\prod_{i} \left( \frac{n_i c_i}{j p_i^j} \right)} ;$$

where the $A_n^{\text{tree}}$ and $\tilde{A}_n^{\text{tree}}$ are two distinct gauge-theory amplitudes including color factors, $c_i$, and $g$ is the coupling constant. The sum runs over all diagrams with only cubic vertices, so each diagram has exactly $n = 3$ propagators. The numerator ($\prod_{i} \left( \frac{n_i c_i}{j p_i^j} \right)$) represents the Feynman propagators of the $i$th diagram. As discussed in ref. [50], the $n_i$ and $c_i$ num erators can be constrained to satisfy a kinematic numerator identity similar to the Jacobi identity satisfied by the $q$ color factors. If the numerators satisfy this constraint, the gravity amplitudes, including factors of the coupling, are then given by,

$$M_n^{\text{tree}}(1;2;3;\ldots;n) = g^n \frac{2^{X}}{\prod_{i} \left( \frac{n_i n_i}{j p_i^j} \right)} ;$$

The sum runs over the same set of diagrams as for the gauge theory case in eq. (17). This equation has been explicitly checked to be equivalent to the KLT relations through eight points [56].
Figure 4: Examples of generalized cuts for determining a three-loop four-point amplitude. Each blob represents an on-shell tree amplitude, and the displayed intermediate lines are all on shell.

Although the KLT relations are at present used merely as a technical trick to efficiently evaluate gravity amplitudes, they point towards a non-trivial \(\text{ELD}\)-theory unification of gravity and gauge theory. As is well known, string theory automatically encodes this unification.

3.4 Modern unitarity method for obtaining loop amplitudes

The modern unitarity method \([3]\) gives us a systematic means for constructing loop amplitudes using on-shell tree amplitudes as input. To obtain an amplitude one first constructs an initial ansatz in terms of integral functions that reproduces one cut. Then, subsequent cuts of the amplitude are compared against the corresponding cuts of the ansatz. If any discrepancy is found in a later cut, additional terms that vanish when all the earlier cut conditions are imposed are added to the ansatz. Once a complete set of cuts have been checked, this procedure gives an integral representation of the loop amplitude with the correct cuts in all channels. The result is the complete amplitude, equivalent to what would have been obtained with Feynman diagrams.

At one loop, massless amplitudes in supersymmetric gauge theories are determined completely by their four-dimensional cuts \([3]\). Unfortunately, this has not been done at higher loops. Even if the theory is ultraviolet finite, infrared singularities are present in four dimensions requiring use of a version of dimensional regularization compatible with supersymmetry \([57]\). To guarantee that no terms are dropped in the construction, the unitarity cuts must be evaluated in four dimensions. Unfortunately, evaluating the cuts in \(D\) dimensions \([4]\) makes the calculation significantly more difficult, because powerful four-dimensional spinor methods \([50]\) can no longer be used. For \(N = 4\) super-Yang-Mills theory, some of this additional complexity can be sidestepped by performing internal-state sums in terms of the gauge supermultiplet of \(D = 10; N = 1\) super-Yang-Mills theory instead of the \(D = 4; N = 4\) multiplet; this counts the same states but simplifies the bookkeeping.

In practical calculations it is useful to first construct an ansatz for the amplitude based on using four-dimensional momenta in the generalized cuts, since we can use powerful spinor methods. Recently, there has been considerable progress in bookkeeping the superpartners crossing unitarity cuts in four dimensions \([59,10]\), based on extensions of Nair's on-shell superspace \([60]\) to general amplitudes. After an ansatz for the complete

\(^3\)An interesting paper with a six-dimension helicity-like formalism recently appeared \([58]\).
amplitudes is constructed based on four-dimensional cuts, to guarantee that no terms are dropped, it must be check against the D-dimensional cuts. However, for four-point amplitudes in $N = 4$ super-Yang-Mills theory the weight of evidence points to all terms being detectable through four loops by four-dimensional cuts [61,6,62,6]. Beyond this we do not expect cuts with four-dimensional kinematics to be sufficient. Indeed, for two-loop six-point amplitudes, terms appear that vanish in four dimensions [47].

The KLT relations give us a rather efficient means of evaluating gravity generalized cuts that reduce the amplitudes to products of tree amplitudes summed over intermediate states. These relations allow us to re-express cuts of gravity amplitudes to sums of products of cuts of gauge-theory amplitudes. The gauge-theory cuts are generally much simpler to evaluate. An important feature of this construction is that once the superpartner sum is performed for $N = 4$ super-Yang-Mills generalized cuts, the corresponding superpartner sum in $N = 8$ supergravity follows directly from the KLT relations. This holds in D dimensions as well. More generally, any simplifications performed on the gauge-theory generalized cuts carry over immediately to gravity cuts.

4 Ultraviolet properties of $N = 8$ supergravity

4.1 Early higher-loop computations

In refs. [61,6], the two-loop four-point amplitudes of $N = 4$ super-Yang-Mills theory and $N = 8$ supergravity were determined in terms of scalar integrals via the unitarity method. A class of unitarity cuts, the "iterated two-particle cuts," were evaluated to all loop orders, yielding a partial reconstruction of the four-point amplitudes at any loop order. The iterated cuts are fairly simple to evaluate because the same sewing algebra appears at each loop order [61]. For the case of $N = 4$ super-Yang-Mills theory, examining the powers of loop momenta in the numerator of the generic iterated two-particle cut contributions suggests the finiteness bound [6],

$$D < \frac{6}{L} + 4 \quad (L > 1);$$

where $D$ is the dimension of space-time and $L$ the loop order. (The case of one loop, $L = 1$, is special, with the amplitudes finite for $D < 8$, not $D < 10$.) The bound [6] differs somewhat from earlier superspace power counting [67], although all bounds concern the ultraviolet finiteness of $N = 4$ super-Yang-Mills theory in $D = 4$. This all-loop-order bound [13] has since been confirmed [22] using $N = 3$ harmonic superspace [63]. Explicit computations demonstrate this bound is saturated through at least four loops [61,6,64].

In ref. [6], the iterated two-particle cuts of $N = 8$ supergravity amplitudes were also analyzed, leading to the proposal that the four-point $N = 8$ supergravity amplitude should be ultraviolet finite for

$$D < \frac{10}{L} + 2 \quad (L > 1);$$

(Again the one-loop case is special with the $N = 8$ supergravity amplitudes being finite for $D < 8$ [34].) This bound implies that in $D = 4$ the first potential divergence
Figure 5: Any one-loop n-point amplitude can be expressed as a linear combination of scalar box, triangle and bubble integrals. In $N = 4$ super-Yang-Mills and $N = 8$ supergravity, after reducing all integrals to this basis, the coefficients of all triangle and bubble integrals vanish.

in $N = 8$ supergravity may appear at five loops. The formula is also consistent with bounds obtained by Howe and Stelle [22], assuming the existence of an $N = 6$ harmonic superspace [63]. It is also consistent with a newer analysis [26] predicting divergences for $L = 4; D = 5$ and $L = 5; D = 4$, but which does not rely on the existence of such a superspace. Since bound (20), proposed in ref. [6] is based on a partial calculation, if there are additional hidden cancellations with uncalculated terms then the true finiteness bound would be improved compared to eq. (20).

4.2 Novel cancellations at one loop

A key clue for the existence of novel cancellations at higher loops [64] comes from one loop. A one-loop theorem [65] states that near four dimensions any n-point massless amplitude can be expressed as

$$A_n = \sum_{i} d_i I^i_4 + \sum_{i} c_i I^i_3 + \sum_{i} b_i I^i_2;$$

(21)

where the scalar integrals $I^{\alpha_1\alpha_2\alpha_3}$ are respectively bubbles, triangles, and boxes and $d_i; c_i; b_i$ are (possibly dimension dependent) rational coefficients. These basis scalar integrals are displayed in fig. 5 and are the same ones that would appear in '3' theory. Some time ago, for the special case of maximally helicity violating one-loop amplitudes of $N = 8$ supergravity, a curious cancellation was observed, setting the coefficients of all triangle and bubble integrals to zero [29]. More recently, it became clear that the vanishing of the triangle and bubble integral coefficients was a general property of all one-loop $N = 8$ supergravity amplitudes [30, 31, 33], with proofs given in refs. [34, 52]. This has been called the 'no-triangle property', although it also implies lack of bubble integrals. For $N = 8$ supergravity this one-loop behavior is rather surprising and suggests the existence of hidden cancellations not apparent in naive power counting. We can use this to limit the number of powers of loop momenta which can appear in the numerators of one-loop amplitudes in $N = 8$ supergravity. Under the Brown-Feynman or Passarino-Veltman reduction [65] each power of $l \cdot k_j$ where $l$ is a loop momentum and $k_j$ an external momentum will give a linear combination of integrals with either no or one canceled propagator. For example, an amplitude with a hexagon integral with a numerator of the form $(l \cdot k)^3$, after integral reductions will contain box and triangle integrals. With an additional power of $l \cdot k$, bubble integrals will also appear.

Interestingly, the observed novel cancellations appear to be generic in gravity theories, as suggested by the one-loop study of ref. [32]. In fact, these type of cancellations
Figure 6: An L-loop contribution as proposed in ref. [6] based on a partial analysis of unitarity cuts. In $N = 4$ super-Yang-Mills theory the numerator factor is $((1 + k_4)^2)^{L/2}$, after dividing out factors which are independent of loop momenta. In the $N = 8$ supergravity case the proposed corresponding factor was $((1 + k_4)^2)^{2L/2}$. This proposal violates the no-triangle property of the one-loop amplitude isolated by the cut in Fig. 7(a), implying additional cancellations with other contributions.

Figure 7: The no-triangle property ensures that all one-loop subamplitudes appearing in a multiloop $N = 8$ supergravity amplitude have the same degree of divergence as that of $N = 4$ super-Yang-Mills theory. The cut (a) is an L-particle cut of an L-loop amplitude. Cut (b) makes use of generalized unitarity to isolate a one-loop subamplitude. These cuts may be used to demonstrate the existence of novel cancellations to all loop orders.

may be generic to any theory where color factors do not prevent cancellations between contributions with a differing ordering of legs [56]. The novel one-loop cancellations were also shown to follow from the remarkably good high-energy behavior of gravity tree amplitudes under the complex deformations used to prove on-shell recursion relations in gravity [43, 51, 10].

4.3 Novel cancellations to all loop orders

By itself, the cancellation of one-loop triangle and bubble integrals at n-points does not directly say anything about the ultraviolet properties of the theory. However, as pointed out in ref. [64], for a certain class of terms, the unitarity method gives us a powerful means to non-trivially constrain the ultraviolet behavior of multiloop amplitudes to all loop orders [64], starting from the one-loop no-triangle cancellations. In particular, consider the diagrammatic contributions to the L-loop four-point amplitude of the form in Fig. 6. Based on evaluating the iterated two-particle cuts, ref. [6] suggested that for $N = 4$ super-Yang-Mills theory the numerator of the integral corresponding to this diagram be dressed by a factor of

$$((1 + k_4)^2)^{L/2};$$ (22)
after removing factors which are independent of the loop momentum $l$. For $N = 8$ supergravity, the proposed corresponding factor was,\[ (l^4 + k_4^4)^{\frac{q}{2}}: \quad (23) \]

which is much worse behaved as $L$ increases compared to the super-Yang-Mills numerator $([22])$. Power counting the integral in $g^4$ with this numerator gives the $N = 8$ finiteness bound in eq. (22).

However, additional cancellations must exist because the too high a power of loop momentum in the proposed numerator $[23]$ would lead to a violation of the no-triangle-integral property in the one-loop subamplitude isolated by the unitarity cut shown in $g^4(a)$. This violation would start at three loops. To prevent this, non-trivial cancellations with other integrals would need to start at three loops, effectively reducing the number of powers of loop momentum in the numerators. Besides constraints on the power counting from the cut in $g^4(a)$, non-trivial additional constraints follow from the set of all generalized cuts isolating a one-loop subamplitude, schematically depicted in $g^4(b)$. After the cancellations take place, at three loops only two powers of loop momentum should remain, matching the number appearing in the $N = 4$ super-Yang-Mills case $[22]$. This has been confirmed by explicit calculation $[27,28]$.

It is interesting that there does not appear to be a purely supersymmetric explanation of these cancellations $[22,26]$. If it is not entirely supersymmetric then what might be behind the cancellations? The unitarity method implies that cancellations must have their origin in tree-level cancellations. An obvious guess is that the loop-level cancellations should be related to the tree-level ones under large complex deformations of the momenta discussed in section 3.2. Indeed this is the case $[31,32]$. From an analysis of one-loop cancellations using an integration formalism due to Forde $[67]$, the following picture emerges $[32]$: Most of the one-loop cancellations observed in $N = 8$ supergravity leading to the no-triangle property are already present in non-supersymmetric gravity and are directly tied to the tree-level cancellations. Schematically, in $N = 8$ supergravity we find that a one-loop integral with the 2n loop momentum numbers of powers of loop momentum undergoes a cancellations of $n + 4$ powers, leaving $n = 4$ powers. Of the canceled powers of loop momentum, 8 are due to supersymmetry while $n = 4$ are generic and cancel also in pure Einstein gravity. The generic cancellations have recently been attributed to the unordered nature of the graviton amplitudes $[34,66]$.

Although the above arguments demonstrate the existence of novel ultraviolet cancellations at all loop orders, they do not demonstrate that these cancellations exist for all contribution. In particular, contributions not detectable by isolating a one-loop subamplitude will not be constrained. We therefore need to inspect such terms. The three-loop four-point amplitude is the logical place to do so because it is the simplest example expected to exhibit the novel cancellations.

4.4 Super finiteness at three loops

In refs. $[27,28]$ the full three-loop four-point amplitude of $N = 8$ supergravity was obtained using the unitarity method. These computations establish the existence of novel cancellations in terms not sensitive to cuts isolating a one-loop subamplitude.
Figure 8: The different parent diagrams in terms of which four-point three-loop amplitudes may be expressed.

In the original calculation of the three-loop four-point amplitude \([24]\), the result was given in a form where individual diagrams have a worse power counting than the complete answer. An improved form, was subsequently found using the method of maximal cuts \([8]\). In the latter form, the individual contributions are no worse behaved than the contributions of the corresponding super-Yang-Mills amplitude \([28]\).

The three-loop amplitudes can be described in terms of the "parent diagrams" displayed in Figure 8 dressed by appropriate numerator factors. The integrals composing the amplitudes are of the form,

\[
I^{(x)} = i \frac{Z}{4} \prod_{i=1}^{n} \frac{d^D q_i}{(2\pi)^D} N^{(x)}(q_i; k_j) \; ;
\]

where the \(q_i\)'s are three independent loop momentum, the \(k_j\)'s are the momentum carried by the propagators of the diagrams, and the \(N^{(x)}(q_j; k_j)\) are numerator factors.

In the improved form given in ref. \([28]\), no numerator factors of \(N = 8\) supergravity have a worse power counting than the corresponding factors of \(N = 4\) super-Yang-Mills theory. That is, numerators in either theory satisfy,

\[
\theta_{q_i} \theta_{q_j} \theta_{q_k} N^{(x)}(q_i; k_m) = 0 \; ;
\]

for the independent loop momentum with \(i; j; k\) taking on the values 1; 2; 3. Thus, no more than two powers of loop momentum appear and the power counting of the three-loop four-point \(N = 8\) supergravity amplitude is identical to that of the corresponding \(N = 4\) super-Yang-Mills amplitude. Not only does this computation prove that the three-loop

\[\text{\footnote{A related method is the leading singularity method, which has also been used in calculations of various maximally supersymmetric multiloop amplitudes.}}\]
Divergence is not present in four dimensions, the amplitude exhibits "super finiteness" or cancellations beyond those required for finiteness. Instead of the finiteness condition (20), at \( L = 3 \) the correct power count for the four-point amplitude is the one matching the corresponding super-Yang-Mills amplitude (19). Explicit calculations show that the finiteness bound (19) is saturated in both theories, so at three loops both theories do diverge in six dimensions.

5 Conclusions

Maximally supersymmetric \( N = 8 \) supergravity theory is the most promising candidate for a unitary perturbatively ultraviolet-nine point-like quantum field theory of gravity. The modern unitarity method offers a powerful means to probe its multiloop ultraviolet behavior, mitigating the rather rapid increase in complexity of Feynman diagrams at higher-loop orders. We summarized the results of refs. [27,28], where a loop-integral representation of the three-loop four-point amplitude of \( N = 8 \) supergravity was presented, exhibiting cancellations beyond those needed for finiteness. The three-loop power counting matched the one for \( N = 4 \) super-Yang-Mills theory, which is known to be ultraviolet finite.

As we discussed, for a subset of contributions novel all-loop cancellations exist with no known supersymmetry explanation. This follows from an investigation of a class of generalized unitarity cuts [64], making use of the one-loop "no-triangle property" [29, 30, 31, 63, 32, 34, 10]. In ref. [32] one-loop cancellations in generic theories of gravity were linked to unexpectedly soft behavior of tree-level gravity amplitudes under large complex shifts of their momenta [51,52]. This mechanism was also proposed as a source of all-loop cancellations, perhaps sufficiently strong, in conjunction with supersymmetry cancellations, to render \( N = 8 \) supergravity ultraviolet finite. This improved behavior is unaccounted for in superspace power counting.

An important question is whether any of the known symmetries of \( N = 8 \) supergravity can be used to constrain its multiloop ultraviolet behavior. For example, \( N = 8 \) supergravity contains a non-compact \( E_{7(1)} \) duality symmetry [2,69]. Its explicit action on fields and amplitudes has been presented recently [70,10]. Improved ultraviolet properties in \( N = 8 \) supergravity may also be linked to M-theory dualities [38] and to string theory non-renormalization theorems [23,24], though issues with decoupling of massive states [35] remain to be clarified.

A next step in unraveling the UV properties of \( N = 8 \) supergravity will be the calculation of the four-loop four-point amplitude. We have every reason to believe the four-loop amplitude will be finite in four dimensions, but crucially, guiding information will come from identifying the smallest dimension in which the divergence appears. This will allow us to test whether the observed cancellations at three loops might have an explanation solely in supersymmetry, along the lines of refs. [22,28]. In particular, recent supersymmetry arguments predict a four-loop divergence in \( N = 8 \) supergravity analytically continued to six dimensions [26]. On the other hand, generalized unitarity arguments [64] point to \( N = 8 \) supergravity being no more divergent than \( N = 4 \) super-Yang-Mills, which does not have a four-loop six-dimensional divergence. A four-loop four-point calculation currently in progress should be able to resolve this decisively [71].
A key open problem is to develop a physical interpretation of the observed ultraviolet cancellations in gravity theories and whether this can be used to prove the ultraviolet finiteness of $N=8$ supergravity. More generally, it would be important to explore the surprising and profound relations between gravity and gauge theories. In particular, the KLT relations [11,12,13] and their recent clarification [56] hint at a unification between these theories of gravity and gauge theory of the sort implied by string theory.

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