ON THE EXISTENCE OF UNIVERSAL FAMILIES OF MARKED IRREDUCIBLE HOLOMORPHIC SYMPLECTIC MANIFOLDS

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Abstract. We prove the existence of a global family with natural universal properties over every component of the moduli space of marked irreducible holomorphic symplectic manifolds. The analogous result follows for the Teichmüller spaces.

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1. Introduction

An irreducible holomorphic symplectic manifold is a simply connected compact Kähler manifold $X$, such that $H^0(X, \Omega^2_X)$ is one dimensional spanned by an everywhere non-degenerate holomorphic 2-form [Be1, Hu1]. The second integral cohomology of $X$ is endowed with a non-degenerate integral symmetric bilinear pairing of signature $(3, b_2 - 3)$, where $b_2$ is the second Betti number of $X$. The pairing is known as the Beauville-Bogomolov-Fujiki pairing. A marking for $X$ is an isometry $\eta : H^2(X, \mathbb{Z}) \to \Lambda$ with a fixed lattice $\Lambda$. An isomorphism of two marked pairs $(X_i, \eta_i), i = 1, 2,$ consists of an isomorphism $f : X_1 \to X_2$, such that $\eta_1 \circ f^* = \eta_2$.

Given an analytic space $B$ and a discrete group $H$, denote by $H_B$ the trivial local system over $B$ with fiber $H$. A family of marked irreducible holomorphic symplectic manifolds over an analytic base $B$ consists of a family $\pi : \mathcal{X} \to B$ of such manifolds, together with an isometric trivialization $\eta : R^2\pi_*\mathbb{Z} \to \Lambda_B$. We will call the pair $(\pi, \eta)$ a $\Lambda$-marked family for short. Two $\Lambda$-marked families $(\pi : \mathcal{X} \to B, \eta)$ and $(\tilde{\pi} : \tilde{\mathcal{X}} \to B, \tilde{\eta})$ are isomorphic, if there exists an isomorphism $f : \mathcal{X} \to \tilde{\mathcal{X}}$, such that $\tilde{\pi}f = \pi$ and $\tilde{\eta} = \eta \circ f^*$, where $f^* : R^2\tilde{\pi}_*\mathbb{Z} \to R^2\pi_*\mathbb{Z}$ is the isomorphism induced by $f$. Given a marked family $(\pi : \mathcal{X} \to B, \eta)$ and a morphism $\kappa : \tilde{B} \to B$, we get the pulled back family $\kappa^*(\pi) : \mathcal{X} \times_B \tilde{B} \to \tilde{B}$ with the marking $\kappa^*(\eta)$. Let $\mathcal{F}_\Lambda$ be the functor, from the category of analytic spaces to the category of sets, which associates to an analytic space $B$ the set of isomorphism classes of $\Lambda$-marked families $(\pi, \eta)$ over $B$. There exists a non-Hausdorff (disconnected) complex manifold $\mathcal{M}_\Lambda$ of dimension $\text{rank}(\Lambda) - 2$, which coarsely represents $\mathcal{F}_\Lambda$ [Hu1].

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Let $X$ be an irreducible holomorphic symplectic manifold. Denote by $\text{Aut}_0(X)$ the subgroup of the automorphism group $\text{Aut}(X)$ of $X$, consisting of elements which act trivially on $H^2(X, \mathbb{Z})$. Huybrechts proved that $\text{Aut}_0(X)$ is a finite group \cite[Prop. 9.1(v)]{Huybrechts}. Associated to every family $\pi : X \to B$ of irreducible holomorphic symplectic manifolds we have a local system $\text{Aut}_0(\pi)$ over $B$, whose fiber over $b \in B$ is the group $\text{Aut}_0(X_b)$ of the fiber $X_b$ of $\pi$ over $b$, by \cite[Theorem 2.1]{Huybrechts}.

Given an $\text{Aut}_0(\pi)$ torsor $\mathcal{P}$ over $B$, we get the family $\tilde{X} := X \times \text{Aut}_0(\pi) \mathcal{P}$. Denote by $\tilde{\pi} : \tilde{X} \to B$ the natural projection. Note that the local systems $\mathbb{R}^2 \pi_* \mathbb{Z}$ and $\mathbb{R}^2 \tilde{\pi}_* \mathbb{Z}$ are naturally isomorphic, and so a marking $\eta$ for the former induces a marking $\tilde{\eta}$ for the latter. We denote this new marked family by $(\tilde{\pi}, \tilde{\eta}) := (\pi, \eta) \times \mathcal{P}$. Two $\Lambda$-marked families $(\pi : X \to B, \eta)$ and $(\tilde{\pi} : \tilde{X} \to B, \tilde{\eta})$ are said to be equivalent, if there exists an $\text{Aut}_0(\pi)$ torsor $\mathcal{P}$ over $B$, such that $(\tilde{\pi}, \tilde{\eta}) = (\pi, \eta) \times \mathcal{P}$. The map $\mathcal{P} \mapsto (\pi, \eta) \times \mathcal{P}$ is a bijection between the set $\text{H}^1(B, \text{Aut}_0(\pi))$, of isomorphism classes of $\text{Aut}_0(\pi)$ torsors, and the set of isomorphism classes of $\Lambda$-marked families equivalent to $(\pi, \eta)$.

Its inverse sends $(\tilde{\pi}, \tilde{\eta})$ to the isomorphism class of the $\text{Aut}_0(\pi)$ torsor $\text{Isom}((\pi, \eta), (\tilde{\pi}, \tilde{\eta}))$ of local isomorphisms of the two families compatible with the markings. We elaborate on this bijection in Remark \ref{Remark 1.2}.

Let $\Lambda$ be the trivial local system over $\mathcal{M}_\Lambda$ with fiber $\Lambda$. The main result of this paper is the following statement.

**Theorem 1.1.** There exists a family $\pi : X \to \mathcal{M}_\Lambda$ of irreducible holomorphic symplectic manifolds and a marking $\eta : \mathbb{R}^2 \pi_* \mathbb{Z} \to \Lambda$ satisfying the following universal property. Given a family of $\Lambda$-marked irreducible holomorphic symplectic manifolds $(\tilde{\pi} : \tilde{X} \to B, \tilde{\eta})$ over an analytic space $B$, the pullback $(\kappa^*(\pi), \kappa^*(\eta))$ via the classifying morphism $\kappa : B \to \mathcal{M}_\Lambda$ is equivalent to $(\tilde{\pi}, \tilde{\eta})$. The marked family $(\pi, \eta)$ satisfying this property is unique, up to isomorphism. $\text{Aut}_0(\pi)$ restricts to each connected component of $\mathcal{M}_\Lambda$ as a trivial local system.

The Theorem is proved in section \ref{section 4}. The Teichmüller space of an irreducible holomorphic symplectic manifold $X$ maps to the moduli space of marked pairs \cite[Cor. 4.31]{Verbitsky}, and so the universal family over the latter pulls back to one over the Teichmüller space.

Consider the moduli space $\mathcal{M}_{\Lambda,G}$ of isomorphism classes of triples $(X, \eta, \psi)$, where $(X, \eta)$ is a marked irreducible holomorphic symplectic manifold, and $\psi : \text{Aut}_0(X) \to G$ is an isomorphism with a fixed finite group $G$. The automorphism group $\text{Aut}(X, \eta)$ of the marked pair $(X, \eta)$ is $\text{Aut}_0(X)$, and the automorphism group $\text{Aut}(X, \eta, \psi)$ of the triple is the center of $\text{Aut}_0(X)$.

**Remark 1.2.** If the center of $G$ is trivial, then the moduli space $\mathcal{M}_{\Lambda,G}$ represents the functor from the category of analytic spaces to the category of sets, which associates to an analytic space $B$ the set of isomorphism classes of triples $(\pi, \eta, \psi)$, consisting of a family $\pi : X \to B$ of irreducible holomorphic symplectic manifolds, an isometric trivialization $\eta : \mathbb{R}^2 \pi_* \mathbb{Z} \to \Lambda|_B$, and a trivialization $\psi : \text{Aut}_0(\pi) \to G|_B$. The local universal families glue in this case uniquely to a global universal family.

Let us sketch the proof of Theorem \ref{Theorem 1.1}. We have the forgetful morphism $\phi : \mathcal{M}_{\Lambda,G} \to \mathcal{M}_\Lambda$. We first show that $\phi$ restricts to each connected component $\mathcal{M}^0_{\Lambda,G}$ of $\mathcal{M}_{\Lambda,G}$ as an isomorphism onto the corresponding connected component $\mathcal{M}^0_\Lambda$ of $\mathcal{M}_\Lambda$ (Lemma \ref{Lemma 2.2}).

Fix a connected component $\mathcal{M}^0_{\Lambda,G}$ of $\mathcal{M}_{\Lambda,G}$. Let $\Lambda$ be the trivial local system over $\mathcal{M}^0_{\Lambda,G}$ with fiber $\Lambda$ and define $G$ similarly. Consider the stack $G$ over $\mathcal{M}^0_{\Lambda,G}$, which associates to each subset $U$ of $\mathcal{M}^0_{\Lambda,G}$, open in the classical topology, the following category $G(U)$. Objects of $G(U)$ are triples $(\pi, \eta, \psi)$, consisting of a family $\pi : X \to U$ of irreducible holomorphic symplectic manifolds $X$.


manifolds, an isometric isomorphism of local systems \( \eta : R^2 \pi_* Z \to A_U \), and an isomorphism of local systems \( \psi : Aut_0(\pi) \to G_U \), such that the triple \((X_t, \eta_t, \psi_t)\) over a point \( t \) of \( U \) represents the isomorphism class parametrized by \( t \) as a point of the coarse moduli space \( M^{0}_{A,G} \). The morphisms of \( G(U) \) are isomorphisms of families, compatible with the trivializations of the two local systems.

We observe next that \( G \) is a gerbe over \( M^{0}_{A,G} \) with band the trivial sheaf of groups \( Z \) with fiber the center \( Z \) of \( G \) (Lemma 3.6). See section 3 for the definitions. Equivalence classes of gerbes with band \( Z \), satisfying a technical property shared by \( G \), are parametrized by \( \tilde{H}^2(M^{0}_{A,G}, Z) \) [Gi, Theorem 5.2.8], [Moe, Theorem 3.1], and Lemma 3.7 below. The existence of a universal family over \( M^{0}_{A,G} \) is equivalent to the triviality of the class \([G] \) in \( \tilde{H}^2(M^{0}_{A,G}, Z) \). Denote by \( \phi_*[G] \) the image of \([G] \) in \( \tilde{H}^2(M^{0}_{A,G}, Z) \) via the isomorphism \( \phi : M^{0}_{A,G} \to M_A^{0} \).

Associated to a Kähler class \( \omega \) on \( X \) and a marking \( \eta \) is a twistor line \( T_{\omega,\eta} \) in \( M_A^{0} \) through the marked pair \((X, \eta)\) [Hu1, 1.13]. \( T_{\omega,\eta} \) is isomorphic to a projective line \( \mathbb{P}^1 \). Finally, we show that the restriction homomorphism \( \tilde{H}^2(M^{0}_{A,G}, Z) \to \tilde{H}^2(T_{\omega,\eta}, Z) \) is an isomorphism (Lemma 3.2). The image of \( \phi_*[G] \) in \( \tilde{H}^2(T_{\omega,\eta}, Z) \) vanishes, by the existence of the twistor family over \( T_{\omega,\eta} \). Hence, the class \([G] \) is trivial and Theorem 1.1 follows.

Irreducible holomorphic symplectic manifolds of \( K3^{[n]} \)-type are those, which are deformation equivalent to the Hilbert scheme of length \( n \) subschemes of a \( K3 \) surface. If \( X \) is of \( K3^{[n]} \)-type, then any automorphism of \( X \), which acts trivially on \( H^2(X, Z) \), is the identity. This follows for Hilbert schemes by a result of Beauville [Be2], and consequently also for their deformations, see [HT, Sec. 2]. The automorphism group of every marked pair \((X, \eta)\), with \( X \) of \( K3^{[n]} \)-type, is thus trivial, and Theorem 1.1 is known in this case.

**Example 1.3.** Fix an integer \( n \geq 2 \). Let \( T \) be a two-dimensional compact complex torus, \( T^{[n+1]} \) its Douady space of length \( n + 1 \) subschemes, \( T^{(n+1)} \) its \((n+1)\) symmetric product, and consider the fiber \( K^{[n]}(T) \) over \( 0 \in T \) of the composition \( T^{[n+1]} \to T^{(n+1)} \to T \), where the left arrow is the Hilbert Chow morphism, and the right is summation. The fiber \( K^{[n]}(T) \) is a \( 2n \)-dimensional irreducible holomorphic symplectic manifold known as the generalized Kummer manifold associated to \( T \) [Be1]. Translation by points of \( T \) of order \( n+1 \) induce automorphisms of \( T^{[n+1]} \) leaving \( K^{[n]}(T) \) invariant, as does multiplication by \(-1\). These automorphisms generate \( Aut_0(K^{[n]}(T)) \), by [BNS, Cor. 5]. When \( X \) is deformation equivalent to \( K^{[n]}(T) \), \( Aut_0(X) \) is thus isomorphic to the semidirect product \( \mathbb{Z}/(n+1)\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \), where the non-trivial element of \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z}/(n+1)\mathbb{Z} \) via multiplication by \(-1\) (see [BNS, Theorem 3 and Corollary 5]). The center of \( Aut_0(X) \) is trivial, if \( n \) is even, and it is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \), if \( n \) is odd.

The group \( Aut_0(X) \) acts faithfully on the total cohomology ring \( H^*(X, Z) \), when \( X \) is deformation equivalent to a generalized Kummer manifold, by a result of Oguiso [Og, Theorem 1.3]. An alternative proof of Theorem 1.1 for this deformation type follows from Oguiso’s result via the argument used in the proof of Lemma 2.2 below.

The classification of irreducible holomorphic symplectic manifolds is an open problem. Two additional deformation types are known at present, one of six dimensional manifolds [O'G2] and one of ten dimensional manifolds [O'G1].

This work was motivated by the talk of Zhiyuan Li at the workshop “Hyper-Kähler Manifolds, Hodge Theory, and Chow Groups” at the Tsinghua Sanya International Mathematics Forum in December 2016. In his talk Li surveyed consequences of the existence of universal families over
Teichmüller spaces to the study of cycles on moduli spaces of polarized irreducible holomorphic symplectic manifolds, generalizing previous work in the $K3$ surface case [BLMM].

2. Moduli spaces of marked pairs and triples

Fix a lattice $\Lambda$ isometric to the Beauville-Bogomolov-Fujiki lattice of some irreducible holomorphic symplectic manifold. Set $\Lambda_C := \Lambda \otimes \mathbb{Z} \otimes \mathbb{C}$. Let $\Omega_\Lambda$ be the period domain of irreducible holomorphic symplectic manifolds with Beauville-Bogomolov-Fujiki lattice $\Lambda$

\[(2.1) \quad \Omega_\Lambda := \{ \ell \in \mathbb{P}(\Lambda_C) : (\ell, \ell) = 0, \text{ and } (\ell, \bar{\ell}) > 0 \} .\]

Fix a connected component $\mathfrak{M}_\Lambda^0$ of the moduli space of marked irreducible holomorphic symplectic manifolds. The period map $P : \mathfrak{M}_\Lambda^0 \to \Omega_\Lambda$ sends the isomorphism class of a marked pair $(X, \eta)$ to $\eta(H^{2,0}(X))$. Given a point $\ell \in \Omega_\Lambda$, denote by $\Lambda^{1,1}(\ell)$ the sublattice of $\Lambda$ orthogonal to $\ell$. The period map is a surjective local homeomorphism $[\text{Hu1}]. \quad \mathcal{L}$ is generically injective and any two points in the same fiber of $\mathcal{P}$ are inseparable, by Verbitsky’s Global Torelli Theorem $[\text{Ver}, \text{Hu2}].$ A point $(X, \eta)$ in $\mathfrak{M}_\Lambda^0$ is a separated point, if and only if the Kähler cone of $X$ is equal to its positive cone, where the latter is the connected component of $\{ \alpha \in H^{1,1}(X, \mathbb{R}) : (\alpha, \alpha) > 0 \}$ containing the Kähler cone $[\text{Ver}, \text{Hu2}]$ (see also $[\text{Ma}, \text{Theorem 2.2 (4)}]).$ Consequently, if $\Lambda^{1,1}(\ell)$ is trivial, or cyclic spanned by a class $\lambda$ with $(\lambda, \lambda) \geq 0$, then the fiber $P^{-1}(\ell)$ consists of a single separated point of $\mathfrak{M}_\Lambda^0 [\text{Hu1}, \text{Corollaries 5.7 and 7.2}]$ (see also $[\text{Ma}, \text{Theorem 2.2 (5)}]).$

Lemma 2.1. Every local system over $\mathfrak{M}_\Lambda^0$ is trivial.

Proof. The period domain $\Omega_\Lambda$ is simply-connected, by $[\text{Hu3}, \text{Cor. 3}].$ Hence, it suffices to prove that every local system $\mathcal{L}$ over $\mathfrak{M}_\Lambda^0$ is the pullback of a local system over $\Omega_\Lambda$ via the period map $P$. Choose a covering $\mathcal{U} := \{ U_i : i \in I \}$ of $\mathfrak{M}_\Lambda^0$ by simply connected open subsets $U_i$, such that $P$ restricts to each $U_i$ as a homeomorphism. Set $V_i := P(U_i).$ We get the open covering $\mathcal{V} := \{ V_i : i \in I \}$ of $\Omega_\Lambda$, by the surjectivity of the period map. Given an $n$-tuple $i := (i_1, i_2, \ldots, i_n) \in I^n$, denote by $J_i$ the set of connected components of $U_{i_1} \cap \cdots \cap U_{i_n}$. Set $V_i := \cap_{k=1}^{n} V_{i_k}$. The period map restricts to an open embedding of $U_i$ into $V_i$. The complement of $P(U_i)$ in $V_i$ is contained in the intersection of $V_i$ with the countable union of closed complex analytic hyperplanes $\Omega_\Lambda \cap \lambda^{-1}$, as $\lambda$ varies in the set of primitive classes in $\Lambda$ with $(\lambda, \lambda) < 0$, by the Global Torelli Theorem. Consequently, each connected component of $V_i$ contains the image of a unique connected component of $U_i$, by $[\text{Ver}, \text{Lemma 4.10}]$. We get a one-to-one correspondence between the set of connected components of $U_i$ and $V_i$. The restriction $\mathcal{L}_{|U_i}$ of $\mathcal{L}$ to $U_i$ is a trivial local system, for all $i \in I$, as $U_i$ is simply connected. Set $\Gamma(U_i, \mathcal{L}) := H^0(U_i, \mathcal{L}_{|U_i}).$ The evaluation homomorphism $ev_i^t : \Gamma(U_i, \mathcal{L}) \to \mathcal{L}_i$ is an isomorphism, for every fiber $\mathcal{L}_i$ of $\mathcal{L}$ over a point $t$ of $U_i$. Given $i \in I^2$ and $c \in J_i$, denote by $U_c$ the corresponding connected component of $U_i$. Given points $t_i$ and $t'_i$ in $U_i$, parallel transport along any path from $t_i$ to $t'_i$ in $U_i$ is given by $ev_{t'_i}^i(\rho_i^c)^{-1}$. The restriction homomorphism $\rho_i^k : \Gamma(U_{ik}, \mathcal{L}) \to \Gamma(U_{ik}, \mathcal{L})$ is an isomorphism, for $k = 1, 2,$ and for every $c \in J_i$. Hence, the gluing of $\mathcal{L}_{|U_{i_1}}$ and $\mathcal{L}_{|U_{i_2}}$ along $U_c$ is determined by the isomorphism

\[ f_{c,i_1,i_2} := (\rho_{i_1}^c)^{-1} \circ \rho_{i_2}^c : \Gamma(U_{i_2}, \mathcal{L}) \to \Gamma(U_{i_1}, \mathcal{L}) \]
Hence, connected components of \( \text{Out}(\text{Aut}_0) \) denote both orbit spaces by \( \text{Out}(\text{Aut}_0) \) fiber over the point corresponding to the isomorphism class \((X,\eta)\). We get a natural identification \( \text{Isom}(\text{Aut}_0) \) group isomorphisms from \( \text{Aut}_0 \) automorphisms of \( G \).

**Proof.**

Given a point \( t \in U_1 \cap U_{i_2} \), we have \( g_{i_1,i_2}(t) := f_{c,i_1,i_2} : \Gamma(U_{i_2}, \mathcal{L}) \to \Gamma(U_{i_1}, \mathcal{L}) \), where \( U_c \) is the connected component containing \( t \). The transformation \( g_{i_1,i_2} \) glues the trivializations \( \mathcal{L}|_{U_{i_1}} \cong \Gamma(U_{i_1}, \mathcal{L})_{U_{i_1}} \) and \( \mathcal{L}|_{U_{i_2}} \cong \Gamma(U_{i_2}, \mathcal{L})_{U_{i_2}} \) along \( U_{i_1,i_2} \).

Let \( P_i : U_i \to V_i \) be the restriction of \( P \) to \( U_i \). Denote by \( P_i^* : \Gamma(V_i, P_{i,*}\mathcal{L}|_{U_i}) \to \Gamma(U_i, \mathcal{L}|_{U_i}) \) the natural isomorphism and let \( P_{i,*} \) be its inverse. We get the isomorphisms

\[
\bar{f}_{c,i_1,i_2} := P_{i,*} \circ f_{c,i_1,i_2} \circ P_{i,*} : \Gamma(V_{i_2}, P_{i,*}\mathcal{L}|_{U_{i_2}}) \to \Gamma(V_{i_1}, P_{i,*}\mathcal{L}|_{U_{i_1}}).
\]

The latter determines a gluing of \( P_{i,*}\mathcal{L}|_{U_{i_1}} \) and \( P_{i,*}\mathcal{L}|_{U_{i_2}} \) along the connected component \( V_c \) of \( V_{i_1,i_2} \), for every \( c \in J_\zeta \), hence along all the connected components of \( V_{i_1,i_2} \). Denote by \( \bar{g}_{i_1,i_2} \) the gluing of the trivializations \( P_{ik,*}\mathcal{L}|_{U_{ik}} \cong \Gamma(V_{ik}, P_{ik,*}\mathcal{L}|_{U_{ik}}) \), \( k = 1, 2 \), along \( V_{i_1,i_2} \). These gluing transformations satisfy the co-cycle condition \( \bar{g}_{i,k} = \bar{g}_{i,j}\bar{g}_{j,k} \), since each \( \bar{g}_{i,j} \) pulls back to \( g_{i,j} \) and the \( g_{i,j} \)'s satisfy the co-cycle condition. Let \( \mathcal{L} \) be the local system over \( \Omega(X) \) determined by the covering \( \mathcal{V} \) and the gluing transformations \( \bar{g}_{i_1,i_2} \). Then the restriction of \( P^*\mathcal{L} \) to \( U_i \) is naturally identified with that of \( \mathcal{L} \) and each gluing transformation \( P^*\bar{g}_{i_1,i_2} \) restricts to \( g_{i_1,i_2} \). Hence, \( \mathcal{L} \) is isomorphic to \( P^*\mathcal{L} \).

We keep the notation of the introduction. Fix a group \( G \) isomorphic to the group \( \text{Aut}_0(X) \) of some irreducible holomorphic symplectic manifold \( X \). The moduli space \( \mathfrak{M}_{A,G} \) is constructed by gluing all Kuranishi families \( \pi : X \to U \), each endowed with a choice of an isometric trivialization \( \eta : R^2\pi_*\mathcal{Z} \to \Delta_U \) and a trivialization \( \psi : \text{Aut}_0(\pi) \to \mathcal{G} \). The construction is completely analogous to that of \( \mathfrak{M}_A \) in [Hu2, Prop. 4.3]. The following is an immediate corollary of Lemma 2.1.

**Lemma 2.2.** The forgetful morphism \( \phi : \mathfrak{M}_{A,G} \to \mathfrak{M}_A \) restricts to each connected component \( \mathfrak{M}_{A,G}^0 \) of \( \mathfrak{M}_{A,G} \) as an isomorphism onto the corresponding connected component \( \mathfrak{M}_A^0 \) of \( \mathfrak{M}_A \). The set of connected components of \( \mathfrak{M}_{A,G} \) over \( \mathfrak{M}_A^0 \) forms a torsor under the group of outer automorphisms of \( G \).

**Proof.** Let \( \text{Ad}_g \in \text{Aut}(G) \) be conjugation by \( g \in G \). \( G \) acts on the set \( \text{Isom}(\text{Aut}_0(X), G) \), of group isomorphisms from \( \text{Aut}_0(X) \) onto \( G \), by the action \( \psi \mapsto \text{Ad}_g \circ \psi \). Similarly, \( f \in \text{Aut}_0(X) \) acts on \( \text{Isom}(\text{Aut}_0(X), G) \) via \( \psi \mapsto \psi \circ \text{Ad}_{f^{-1}} \). The set of orbits for the two actions coincide and we get a natural identification \( \text{Isom}(\text{Aut}_0(X), G)/G = \text{Isom}(\text{Aut}_0(X), G)/\text{Aut}_0(X) \), so we denote both orbit spaces by \( \text{Out}(\text{Aut}_0(X), G) \). Over \( \mathfrak{M}_A \) we have the local system \( \mathcal{O}_{Out} \), whose fiber over the point corresponding to the isomorphism class \((X,\eta)\) is \( \text{Out}(\text{Aut}_0(X), G) \). If \( f \) is an automorphism of a marked pair \((X,\eta)\), then \( f \) belongs to \( \text{Aut}_0(X) \). Hence, the local system \( \mathcal{O}_{Out} \) is well defined.

The moduli space \( \mathfrak{M}_{A,G} \) is simply the total space of the local system \( \mathcal{O}_{Out} \). The local system \( \mathcal{O}_{Out} \) restricts to a trivial local system over each connected component \( \mathfrak{M}_A^0 \), by Lemma 2.1. Hence, connected components of \( \mathfrak{M}_{A,G} \) are simply global sections of \( \mathcal{O}_{Out} \).
Given an abelian group $A$ and a topological space $S$, denote by $\check{H}^i(S, A)$ the $i$-th Čech cohomology of $S$ with coefficients in $A$.

**Lemma 2.3.** The pull back homomorphism $P^* : \check{H}^i(\Omega, A) \to \check{H}^i(\mathbb{M}_\Lambda^0, A)$ is an isomorphism, for every abelian group $A$.

**Proof.** We keep the notation of the proof of Lemma [2.1]. The simply connected open sets $U$ of $\mathbb{M}_\Lambda^0$, such that $P$ restricts to $U$ as a homeomorphism, form a basis for the topology of $\mathbb{M}_\Lambda^0$. Given an open covering $U = \{U_i\}_{i \in I}$, consisting of such open sets, we get an open covering $V := \{V_i := P(U_i)\}_{i \in I}$ of $\Omega$. The covering $U$ is a refinement of the covering $\{P^{-1}(V_i)\}_{i \in I}$, so we get a pullback homomorphism

$$P^* : \check{H}^i(V, A) \to \check{H}^i(U, A).$$

Every open covering of $\Omega$ admits a refinement by an open covering $V$ as above. Hence, it suffices to prove that the pullback homomorphism displayed above is an isomorphism, for such a covering $U$ of $\mathbb{M}_\Lambda^0$ and the induced covering $V$ of $\Omega$. The degree $n$ group $C^n(U, A)$ in the Čech complex is $\oplus_{i \in I_{n+1}} \oplus_{c \in J_i} \Gamma(U_c, A)$ and $\Gamma(U_c, A) = A$. The natural bijection between the connected components of $U_i$ and $V_i$, observed in the proof of Lemma [2.1], implies that the pullback homomorphism $P^*$ induces an isomorphism of Čech complexes. \qed

3. **Gerbes**

**Definition 3.1.** A fibered category (or a presheaf of categories) $\mathcal{F}$ over a topological space $S$ consists of the following.

1. A category $\mathcal{F}(U)$ for each open subset $U$ of $S$.
2. A functor $i^* : \mathcal{F}(U) \to \mathcal{F}(V)$, for each inclusion $i : V \to U$ of open sets.
3. A natural isomorphism $\tau_{ij} : (ij)^* \to j^* i^*$ for each composition $W \xrightarrow{j} V \xrightarrow{i} U$ of inclusions.

The natural isomorphisms are assumed to satisfy a natural associativity property [Gi] [Br] Def. 5.2.1]. See also [Moe] Def. 2.1].

**Definition 3.2.** A fibered category $\mathcal{F}$ over a topological space $S$ is called a prestack if, for any pair of objects $a, b$ of $\mathcal{F}(U)$, the presheaf $\text{Hom}(a, b)$ over $U$, associating to an inclusion $i : V \to U$ of an open subset $V$ the set $\text{Hom}_{\mathcal{F}(V)}(i^* a, i^* b)$, is a sheaf. A prestack $\mathcal{F}$ is called a stack, if it satisfies an additional descent condition for objects [Gi] [Br] (see also [Moe] Def. 2.6]).

**Definition 3.3.** ([Gi] [Br] Def. 5.2.4], or [Moe] Def. 3.1]). A gerbe over a topological space $S$ is a stack $\mathcal{G}$ satisfying the following properties.

1. Each category $\mathcal{G}(U)$ is a groupoid.
2. $S$ admits a covering by open sets $U$, such that $\mathcal{G}(U)$ has an object.
3. Given objects $a, b$ of $\mathcal{G}(U)$, any point $x \in U$ has an open neighborhood $V \subset U$, such that $\text{Hom}_{\mathcal{G}(V)}(i^* a, i^* b)$ is non-empty, where $i : V \to U$ is the inclusion.

Let $\mathcal{G}$ be a gerbe over a topological space $S$, such that for every open set $U$ and every object $a$ of $\mathcal{G}(U)$, the group $\text{Aut}_{\mathcal{G}(U)}(a)$ is abelian. We define next a sheaf $Z$ of abelian groups called the band of $\mathcal{G}$. For a more general definition, for arbitrary gerbes, see [Gi] or [Moe] Def. 3.2]. We can choose an open covering $U := \{U_\alpha\}$ and an object $a_\alpha$ in $\mathcal{G}(U_\alpha)$, yielding a sheaf $\text{Aut}(a_\alpha)$ over each $U_\alpha$, by Axiom (2) of Definition 3.3. These sheaves admit a canonical gluing to a sheaf $Z$ of abelian groups as follows. Choose a covering $\{U_{\alpha\beta}\}$ of each $U_{\alpha\beta}$, such that there exists
an isomorphism $f^\xi_{a\beta}$ from the restriction $(a_\beta)|_{U^\xi_{a\beta}}$ of $a_\beta$ to $U^\xi_{a\beta}$ to the restriction $(a_\alpha)|_{U^\xi_{a\beta}}$ of $a_\alpha$. Such a covering exists, by Axioms (1) and (3) of Definition 3.3. Conjugation by $f^\xi_{a\beta}$ induces a sheaf isomorphism

$$\lambda^\xi_{a\beta} : \text{Aut} \left( (a_\beta)|_{U^\xi_{a\beta}} \right) \to \text{Aut} \left( (a_\alpha)|_{U^\xi_{a\beta}} \right).$$

Now $\lambda^\xi_{a\beta}$ is independent of the choice of $f^\xi_{a\beta}$, since any other choice is of the form $hf^\xi_{a\beta}$, for an element $h$ of the abelian group $\text{Aut}_{|G(U^\xi_{a\beta})}((a_\alpha)|_{U^\xi_{a\beta}})$. In particular, $\lambda^\xi_{a\beta}$ and $\lambda^{\xi'}_{a\beta}$ agree over overlaps $U^\xi_{a\beta} \cap U^{\xi'}_{a\beta}$ and define an isomorphism

$$\lambda_{a\beta} : \text{Aut} \left( (a_\beta)|_{U^\xi_{a\beta}} \right) \to \text{Aut} \left( (a_\alpha)|_{U^\xi_{a\beta}} \right).$$

The isomorphisms $\lambda_{a\beta}$ satisfy the co-cycle condition $\lambda_{a\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha} = 1$ in $\text{Aut} \left( \text{Aut} \left( (a_\alpha)|_{U^\xi_{a\beta}} \right) \right)$, as the left hand side is an inner automorphism of an abelian group. Hence, the sheaves $\text{Aut}(a_\alpha)$ glue to a sheaf Band($G$) of abelian groups, called the band of the gerbe $G$. The isomorphism class of the sheaf Band($G$) is independent of the choice of the covering $\{U_\alpha\}$ and of the objects $\{a_\alpha\}$.

**Definition 3.4.** Let $Z$ be a sheaf of abelian groups. A gerbe with band $Z$ is a gerbe $G$, all of whose objects over all open subsets have abelian automorphism groups, together with an isomorphism $\theta : \text{Band}(G) \to Z$ of sheaves of abelian groups.

There is a notion of equivalence $\phi : F \to G$ between stacks $F$ and $G$ over a topological space $X$, which consists of equivalences of categories $\phi(U) : F(U) \to G(U)$, for every open subset $U$ of $X$, satisfying a natural descent condition [Br Page 199] and [Moe] Def. 2.3, and Rem. (i) page 12. A morphism $\phi : (G, \theta) \to (G', \theta')$ between two gerbes $(G, \theta)$, $(G', \theta')$ with band $Z$ is an equivalence $\phi : G \to G'$ of fibered categories satisfying $\theta'\phi = \theta$. Two gerbes $(G, \theta)$, $(G', \theta')$ with band $Z$ are said to be equivalent if there exists a third gerbe $(F, \rho)$ with band $Z$ and morphisms from $(F, \rho)$ to each of $(G, \theta)$ and $(G', \theta')$ [Moe] Paragraph after Def. 3.2. The set of equivalence classes of gerbes with abelian band $Z$ is denoted by Gerbes($X, Z$). If the pair $(X, Z)$ has the property that every open covering of $X$ admits a refinement $U := \{U_\alpha\}_{\alpha \in I}$, such that $H^1(U_{a\beta}, Z) = 0$, for all $\alpha, \beta \in I$, then there is a bijective correspondence

$$\text{Gerbes}(X, Z) \cong \check{H}^2(X, Z),$$

with the second Čech cohomology group with coefficients in $Z$, by [Gi], [Br] Theorem 5.2.8 and by [Moe] Theorem 3.1. The vanishing $H^1(U_{a\beta}, Z) = 0$ is satisfied for good open coverings (for which all non-empty finite intersections are contractible), and abelian local systems $Z$, and so the above property holds for $(X, Z)$, whenever $X$ is a Hausdorff manifolds and $Z$ is an abelian local system.

Note that the additional vanishing of the cohomologies $H^2(U_{a\beta}, Z(K_\alpha)|_{U_{a\beta}})$ and $H^1(U_{a\beta\gamma}, Z(K_\alpha)|_{U_{a\beta\gamma}})$ with coefficients in the center of a non-abelian band $K$, assumed in [Moe] Theorem 3.1, is not needed as it is only used in the Remark preceding [Moe] Def. 3.2 to lift a cocycle of outer automorphisms to a cocycle of automorphisms. In our case of an abelian band $Z$ such a lift is not needed.
**Definition 3.5.** Denote by \( \text{Gerbes}_0(X, \mathcal{Z}) \) the set of equivalence classes of gerbes \( \mathcal{G} \) with band \( \mathcal{Z} \) such that \( X \) admits an open covering \( \mathcal{U} := \{ U_a \}_{a \in I} \) with objects \( a \in \mathcal{G}(U_a) \), such that \( \text{Hom}(a_a | U_{\alpha \beta}, a_\beta | U_{\alpha \beta}) \) is the trivial \( \text{Aut}(a_a | U_{\alpha \beta}) \)-torsor, for all \( \alpha, \beta \in I \) with non-empty \( U_{\alpha \beta} \).

If \( X \) is a Hausdorff manifold and \( \mathcal{Z} \) is an abelian local system, then \( \text{Gerbes}_0(X, \mathcal{Z}) = \text{Gerbes}(X, \mathcal{Z}) \), by the vanishing \( H^1(U_{\alpha \beta}, \mathcal{Z}) = 0 \) for a good covering. Note that once the property in the above Definition holds for an open covering, it holds also for every refinement of this covering. The above bijection \((\ref{3.2})\) associates to a gerbe \((\mathcal{G}, \theta)\) a Čech cohomology class represented by a 2-cocycle \( g_{\alpha \beta \gamma} := \theta_{\alpha}(f_{\alpha \beta} f_{\beta \gamma} f_{\gamma \alpha}^{-1}) \in \mathcal{Z}(U_{\alpha \beta \gamma}) \) associated to choices of isomorphisms \( f_{\alpha \beta} : a_\beta | U_{\alpha \beta} \to a_\alpha | U_{\alpha \beta} \) \([\text{Moe}, \text{Prop. 3.1}]\). The proof of \([\text{Moe}, \text{Theorem 3.1}]\) constructs, more generally, a bijection \( \text{Gerbes}_0(X, \mathcal{Z}) \cong \tilde{H}^2(X, \mathcal{Z}) \), for any topological space \( X \) and a sheaf of abelian groups \( \mathcal{Z} \).

Fix a lattice \( \Lambda \) and a finite group \( G \). Let \( B \) be an analytic space and \( \kappa : B \to \mathcal{M}_{\Lambda, G} \) a morphism. Consider the fibered category \( \kappa^{-1} \mathcal{G} \) over \( B \), which associates to an open set \( U \subset B \) the following category \( \kappa^{-1} \mathcal{G}(U) \). Objects of \( \kappa^{-1} \mathcal{G}(U) \) are triples \((\pi, \eta, \psi)\) consisting of a family \( \pi : \mathcal{X} \to U \) of irreducible holomorphic symplectic manifolds, an isometric trivialization \( \eta : R^2\pi_* \mathcal{Z} \to \Lambda_U \), and a trivialization \( \psi : \text{Aut}_0(\pi) \to \mathcal{G}_U \), such that for each point \( u \in U \) the triple \((\mathcal{X}_u, \eta_u, \psi_u)\) represents the isomorphism class corresponding to the point \( \kappa(u) \) of \( \mathcal{M}_{\Lambda, G} \). A morphism in \( \text{Hom}_{\kappa^{-1} \mathcal{G}(U)}((\pi, \eta, \psi), (\tilde{\pi}, \tilde{\eta}, \tilde{\psi})) \) is an isomorphism \( f : \mathcal{X} \to \tilde{\mathcal{X}} \), satisfying \( \pi = \tilde{\pi}f \), such that the induced isomorphism of local systems \( f^* : R^2\tilde{\pi}_* \mathcal{Z} \to R^2\pi_* \mathcal{Z} \) satisfies \( \tilde{\eta} = \eta f^* \) and such that \( \tilde{\psi} = \psi Ad_f \), where \( Ad_f : \text{Aut}_0(\tilde{\pi}) \to \text{Aut}_0(\pi) \) is the isomorphism induced by conjugation by \( f \).

**Lemma 3.6.** The fibered category \( \kappa^{-1} \mathcal{G} \) is a gerbe over \( B \) with band \( \mathcal{Z}_B \), where \( \mathcal{Z} \) is the center of \( G \).

**Proof.** It suffices to prove that the fibered category \( \mathcal{G} \) over \( \mathcal{M}_{\Lambda, G} \), associated to the identity morphism, is a gerbe with band \( \mathcal{Z}_B \), as the more general \( \kappa^{-1} \mathcal{G} \) described above is simply the inverse image of \( \mathcal{G} \) via \( \kappa \) and is thus a gerbe with band \( \mathcal{Z}_B \), by \([\text{Br}, \text{Prop. 5.2.6}]\). Property \( (\text{1}) \) of Definition \((\ref{3.3})\) holds, by definition of morphisms in \( \mathcal{G} \). The vector space \( H^0(X, TX) \) vanishes for every irreducible holomorphic symplectic manifold \( X \), and the dimension of \( H^1(X, TX) \) is the Hodge number \( h^{1,1}(X) \). Hence, the versal Kuranishi family of \( X \) is universal \([\text{BHPV}, \text{Ch. I, Theorems (10.3) and (10.5)}]\). Every point \( t := (X_0, \eta_0) \) of \( \mathcal{M}_\Lambda \) admits a simply connected open neighborhood \( U \), over which we have a universal family of deformations of \( X_0 \), as \( \mathcal{M}_\Lambda \) is constructed by gluing such families \([\text{Hu2, Prop. 4.3}]\). The same holds for \( \mathcal{M}_{\Lambda, G} \), by Lemma \((\ref{2.2})\). Let \( b \) be a point of \( \mathcal{M}_{\Lambda, G} \) and let \((X_b, \eta_b, \psi_b)\) represent the isomorphism class \( b \). Choose a simply connected open neighborhood \( U \subset \mathcal{M}_{\Lambda, G} \) of \( b \), over which we have a universal family \( \pi : \mathcal{X} \to U \) of deformations of \( X_b \). The local systems \( R^2\pi_* \mathcal{Z} \) and \( \text{Aut}_0(\pi) \) are trivial, since \( U \) is simply connected, and so \( \eta_b \) and \( \psi_b \) extend to an isometric trivialization \( \eta : R^2\pi_* \mathcal{Z} \to \Lambda_U \), and a trivialization \( \psi : \text{Aut}_0(\pi) \to \mathcal{G}_U \). We get the object \((\pi, \eta, \psi)\) in \( \mathcal{G}(U) \). Property \( (\text{2}) \) of Definition \((\ref{3.3})\) follows. Property \( (\text{3}) \) of Definition \((\ref{3.3})\) holds, since the universal family is universal for each of its fibers (see \([\text{BHPV}, \text{Ch. I, Theorems (10.3) and (10.6)}]\) and the Remark following \((\text{10.6)}\)).

Given an open subset \( U \) of \( \mathcal{M}_{\Lambda, G} \) and an object \( a := (\pi, \eta, \psi) \) in \( \mathcal{G}(U) \), the sheaf \( \text{Aut}_{\mathcal{G}(U)}(a) \) is isomorphic to the center of \( \text{Aut}_0(\pi) \) and \( \psi \) restricts to an isomorphism from the center of \( \text{Aut}_0(\pi) \) onto \( \mathcal{Z}_U \). The isomorphisms induced by the \( \psi \)'s are compatible with the gluing transformations \( \lambda_{\alpha \beta}^\xi \) in \((\ref{3.1})\), by definition of morphisms in \( \mathcal{G} \). Hence, the band of \( G \) is isomorphic to \( \mathcal{Z} \). \qed
Lemma 3.7. The gerbe $G$ over $\mathcal{M}_{\Lambda,G}$ satisfies the property in Definition 3.6 and is hence represented by a class $[G]$ in $H^2(\mathcal{M}_{\Lambda,G}, \mathbb{Z})$.

Proof. It suffices to consider the restriction of $G$ to a connected component $\mathcal{M}_{\Lambda,G}^0$ of $\mathcal{M}_{\Lambda,G}$. Set $\tilde{P} := P \circ \phi : \mathcal{M}_{\Lambda,G}^0 \to \Omega_\Lambda$. Let $U := \{U_i\}_{i \in I}$ be an open covering of $\mathcal{M}_{\Lambda,G}^0$, such that there exists over each $U_i$ a universal family $\pi_i : \mathcal{X}_i \to U_i$ with trivializations $\eta_i$ of $\Omega_i$ and $\tilde{P}$ restricts to each $U_i$ as a homeomorphism $\tilde{P} : U_i \to V_{j(i)} := \tilde{P}(U_i)$, where $j : I \to J$ is a function and $\{V_j\}_{j \in J}$ is a good covering of $\Omega_\Lambda$. Such a covering exists, since $\Omega_\Lambda$ is a Hausdorff manifold, and so every open covering of $\Omega_\Lambda$ admits a refinement by a good covering. Then $V_{j(i_1)} \cap V_{j(i_2)}$ is simply connected, for all $i_1, i_2 \in I$, but the intersection $U_{i_1} \cap U_{i_2}$ need not be simply connected.

Denote by $\mathcal{X}_i|\{V_{j(i)}\} \cap \{V_{j(i)}\}$ the restriction to $V_{j(i_1)} \cap V_{j(i_2)}$ of the pullback via $\tilde{P}^{-1} : V_{j(i_1)} \to U_{i_1}$ of $\mathcal{X}_i$. Define $\mathcal{X}_i|\{V_{j(i)}\} \cap \{V_{j(i)}\}$ similarly. The morphisms $\pi_{i_1}$ and $\pi_{i_2}$ are weakly Kähler, by [BL] Lemma 4.14, and hence so is the fiber product

$$\pi_{i_1i_2} : \mathcal{X}_{i_1i_2} \to V_{j(i_1)} \cap V_{j(i_2)}$$

of $\mathcal{X}_i|\{V_{j(i)}\} \cap \{V_{j(i)}\}$ and $\mathcal{X}_i|\{V_{j(i)}\} \cap \{V_{j(i)}\}$ over $V_{j(i_1)} \cap V_{j(i_2)}$, by [F1] (2.2). Let $D \to V_{j(i_1)} \cap V_{j(i_2)}$ be the relative Douady space of $\pi_{i_1i_2}$. We refer to [BL] Def. 2.3 for the definition of weakly Kähler. All we need is that it is the property required to conclude that every irreducible component of $D$ is proper over $V_{j(i_1)} \cap V_{j(i_2)}$, by the main theorem of [F2]. Let $D_0$ be the (finite) union of irreducible components which contain points parametrizing the graph of an isomorphism $f_t : \mathcal{X}_{i_1,t} \to \mathcal{X}_{i_2,t}$, $t \in U_{i_1} \cap U_{i_2}$, compatible with the markings $\eta_{i_k,t}$ and $\psi_{i_k,t}$, $k = 1, 2$. Then $D_0$ intersects the inverse image of $\tilde{P}(U_{i_1} \cap U_{i_2})$ in a $\mathbb{Z}$-torsor over $\tilde{P}(U_{i_1} \cap U_{i_2})$ (where we identify the restriction of $\mathcal{Z}$ to $U_{i_1}$ with the center of $\text{Aut}_0(\pi_{i_1})$ via $\psi_{i_1}$). Equivalently, $D_0$ is a principal $\mathbb{Z}$-bundle over $\tilde{P}(U_{i_1} \cap U_{i_2})$. A priori this $\mathbb{Z}$-bundle extends to a $\mathbb{Z}$-bundle of graphs of isomorphisms away from a closed analytic subset of $V_{j(i_1)} \cap V_{j(i_2)}$, by the properness of $D_0$ over $V_{j(i_1)} \cap V_{j(i_2)}$. This closed analytic subset is $V_{j(i_1)} \cap V_{j(i_2)} \setminus \tilde{P}(U_{i_1} \cap U_{i_2})$, by definition of $\mathcal{M}_{\Lambda,G}$.

Let $C$ be a smooth and connected Riemann surface in $V_{j(i_1)} \cap V_{j(i_2)}$, which intersects $\tilde{P}(U_{i_1} \cap U_{i_2})$. The inverse image $\tilde{C}_0$ of $C_0 := C \cap \tilde{P}(U_{i_1} \cap U_{i_2})$ in $D_0$ extends, abstractly, to a branched cover $\tilde{C}$ of $C$. Note that $C$ intersects $V_{j(i_1)} \cap V_{j(i_2)} \setminus \tilde{P}(U_{i_1} \cap U_{i_2})$ in a union of isolated points, as the intersection is a closed proper analytic subset, by the discussion in the previous paragraph. The triviality of the $\mathbb{Z}$-bundle $D_0$ is equivalent to that of the associated representation of the fundamental group of $U_{i_1} \cap U_{i_2}$ in the abelian group $\mathbb{Z}$. The latter would follow once we show that for every such $C$ the morphism $\tilde{C} \to C$ is in fact unramified. Associated to every point $\tilde{t}$ of $\tilde{C} \setminus \tilde{C}_0$, over $t \in [C \setminus C_0]$, is a limiting cycle $\Gamma_{\tilde{t}} + \sum_k Y_{\tilde{t},k}$ in $\mathcal{X}_{i_1,t} \times \mathcal{X}_{i_2,t}$, such that $\Gamma_{\tilde{t}}$ is the graph of a bimeromorphic map as well as the unique summand dominating the fibers $\mathcal{X}_{i_1,t}$ and $\mathcal{X}_{i_2,t}$, by the proof of [Hu1] Theorem 4.3].

The group $\mathbb{Z}$ acts on $\tilde{C}_0$ and this action is free and transitive on all fibers of $\tilde{C} \to C$. Hence, $\mathbb{Z}$ acts transitively on the set of limiting cycles associated to points in the same fiber of $\tilde{C} \to C$. We claim that the latter action is free. Indeed, if $g$ is a bimeromorphic map from $\mathcal{X}_{i_1,t}$ to $\mathcal{X}_{i_2,t}$ and $f_1, f_2$ are elements of $\text{Aut}_0(\mathcal{X}_{i_1,t})$, such that $gf_1 = gf_2$, then $f_1 = f_2$. Hence, $\tilde{C}$ is unramified over $C$. □
Remark 3.8. Jenia Tevelev pointed out to me that gerbes arise naturally from the construction of the rigidification of a stack (see [ACV, Sec. 5.1]). In the above lemma the coarse moduli space \( \mathfrak{M}_{\Lambda G} \) is the rigidification of the stack \( G \). The construction of rigidification of a stack was used by Gorchinskiy and Viviani to reprove the classical result that a universal family exists over the open subset of the coarse moduli space of hyperelliptic curves of genus \( g \) without extra automorphisms apart from the hyperelliptic involution, if and only if \( g \) is odd [GV, Prop. 4.7].

4. Vanishing of the Cohomology Class of a Gerbe

Fix a lattice \( \Lambda \) isometric to the Beauville-Bogomolov-Fujiki lattice of some irreducible holomorphic symplectic manifold. Let \( W \) be a three dimensional positive definite subspace of \( \Lambda_\mathbb{R} \). Let \( Q_W \subset \mathbb{P}(W_\mathbb{C}) \) be the cone of isotropic lines in \( W_\mathbb{C} := W \otimes_\mathbb{R} \mathbb{C} \). Denote by \( \iota : Q_W \to \Omega_\Lambda \) the inclusion into the period domain (2.1).

Lemma 4.1. The inclusion \( \iota \) is a homotopy equivalence.

Proof. Let \( Gr_+(3, \Lambda_\mathbb{R}) \) be the Grassmannian of positive definite three dimensional subspaces of \( \Lambda_\mathbb{R} \). The identity component \( SO_+(\Lambda_\mathbb{R}) \) of the special orthogonal group acts transitively on \( Gr_+(3, \Lambda_\mathbb{R}) \). The stabilizer of \( W \) in \( SO_+(\Lambda_\mathbb{R}) \) is \( SO(W) \times SO(W^\perp) \) realizing \( Gr_+(3, \Lambda_\mathbb{R}) \) as the quotient \( SO_+(\Lambda_\mathbb{R}) / ([SO(W) \times SO(W^\perp)] \). Now \( SO(W) \times SO(W^\perp) \) is a maximal compact subgroup of \( SO_+(\Lambda_\mathbb{R}) \) and so the latter is topologically the product of the former and a Euclidean space, by Cartan’s Theorem [Mos, Theorem 2]. Hence, \( Gr_+(3, \Lambda_\mathbb{R}) \) is contractible.

A point of the period domain \( \Omega_\Lambda \) corresponds to an isotropic line \( \ell \subset \Lambda_\mathbb{C} \), such that \( [\ell + \ell^\perp] \cap \Lambda_\mathbb{R} \) is a positive definite two-dimensional subspace \( V \) of \( \Lambda_\mathbb{R} \). The natural identification of the two dimensional real vector spaces \( \ell \) and \( V \) endows the latter with an orientation. This construction identifies the period domain \( \Omega_\Lambda \) with the Grassmannian of oriented positive definite two-dimensional subspaces of \( \Lambda_\mathbb{R} \). Let \( I \subset \Omega_\Lambda \times Gr_+(3, \Lambda_\mathbb{R}) \) be the incidence correspondence. The fiber of the projection \( q : I \to Gr_+(3, \Lambda_\mathbb{R}) \) over \( W \) is \( Q_W \). The inclusion \( \iota : Q_W \to I \) of the fiber is a homotopy equivalence, since \( q \) is a fibration over a contractible base. The fiber of the projection \( p : I \to \Omega_\Lambda \) over an oriented two dimensional subspace \( V \subset \Lambda_\mathbb{R} \) is the projectivization of the positive cone in the subspace \( V^\perp \) of signature \((1,b_2-3)\), i.e. a hyperbolic space. Hence, \( p \) is a homotopy equivalence, being a fibration with contractible fibers. We conclude that \( \iota = p \circ \iota \) is the composition of two homotopy equivalences and so is such as well. \( \square \)

Let \( X \) be an irreducible holomorphic symplectic manifold and \( \omega \) a Kähler class on \( X \). Set \( V := [H^{2,0}(X) \oplus H^{0,2}(X)] \cap H^2(X, \mathbb{R}) \) and set \( W := V + \mathbb{R} \omega \). Then \( W \) is a positive definite three dimensional subspace of \( H^2(X, \mathbb{R}) \). Let \( I \) be the complex structure of \( X \). There exists a unique Ricci flat hermetian metric \( g \) on \( X \), whose imaginary part is a Kähler form representing the class \( \omega \), by Yau’s proof of the Calabi conjecture [Be1]. Furthermore, there exists two additional complex structures \( J \) and \( K \), covariantly constant with respect to the Riemannian metric which is the real part of the hermetian metric \( g \), such that \( IJ = K \). The identity, \( I \), \( J \), and \( K \) span a subalgebra of endomorphisms of the real tangent bundle, which is isomorphic to the algebra \( H \) of quaternions. The two-sphere

\[ Tw_\omega := \{ aI + bJ + cK : a^2 + b^2 + c^2 = 1 \}, \]

of purely imaginary unit quaternions, consists of integrable complex structures. The Riemannian metric and each of these complex structures \( I_t \in Tw_\omega \) determine a Kähler form \( \omega_t \) on the manifold \( \overline{X} \) underlying \( X \), hence a Hodge structure. Denote by \( X_t \) the complex manifold
The map $\pi : T_w \to \mathbb{P}(H^2(X, \mathbb{C}))$, sending $I_t$ to $H^{2,0}(X_t)$, is a diffeomorphic embedding whose image is $Q_W$. Endow $T_w$ with the complex structure of $Q_W$, which is isomorphic to the complex projective line. We get a complex structure on the smooth manifold $X := X \times T_w$, such that the projection

$$\pi : X \to T_w$$

is holomorphic, and the fiber of $\pi$ over $I_t \in T_w$ is isomorphic to $(X, I_t)$ [HKL, Sec. 3(F)]. The above family is known as the [twistor family] associated to the Kähler form $\omega$.

Choose a marking $\eta_0 : H^2(X, \mathbb{Z}) \to \Lambda$. It extends uniquely to an isometric trivialization $\eta : R^2\pi_*\mathbb{Z} \to \Lambda T_w$, since $T_w$ is simply connected. We get an embedding

$$\kappa_{\pi, \eta} : T_w \to \mathfrak{M}_\Lambda^0$$

into the connected component of $\mathfrak{M}_\Lambda$ containing $(X, \eta_0)$.

**Lemma 4.2.** The pullback homomorphism $\kappa_{\pi, \eta}^* : \bar{H}^*(\mathfrak{M}_\Lambda^0, A) \to \bar{H}^*(T_w, A)$ is an isomorphism, for every abelian group $A$.

**Proof.** The composition $P \circ \kappa_{\pi, \eta} : T_w \to \Omega_\Lambda$ with the period map $P$ is the embedding of the base $T_w$ of the twistor family as the conic $Q_W$ of isotropic lines in the complexification of the positive three dimensional subspace $\eta(W)$ of $\Lambda_R$. Set $\iota := P \circ \kappa_{\pi, \eta}$. The pullback $\iota^* : \bar{H}^*(\Omega_\Lambda, A) \to \bar{H}^*(T_w, A)$ is an isomorphism, by Lemma 4.1, and the pullback $P^* : \bar{H}^*(\mathfrak{M}_\Lambda^0, A) \to \bar{H}^*(\Omega_\Lambda, A)$ is an isomorphism, by Lemma 2.3. Hence, $\kappa_{\pi, \eta}^* = \iota^* \circ (P^*)^{-1}$ is an isomorphism as well. $\square$

**Proof of Theorem 1.1.** Choose some twistor family (4.1). The local system $\mathcal{A}ut_0(\pi)$ over $T_w$ is trivial, as the latter is simply connected. Choose a trivialization $\psi : \mathcal{A}ut_0(\pi) \to G_T$, let $\kappa_{\pi, \eta, \psi} : T_w \to \mathfrak{M}_G^0$ be the classifying morphism, and let $\mathfrak{M}_G^0$ be the connected component containing its image. Let $Z$ be the center of $G$. Then $\kappa_{\pi, \eta, \psi}^* : \bar{H}^*(\mathfrak{M}_G^0, Z) \to \bar{H}^*(T_w, Z)$ is an isomorphism, by Lemmas 4.2 and 2.2.

Let $\mathcal{G}$ be the gerbe over $\mathfrak{M}_G^0$ associated to the identity morphism from $\mathfrak{M}_G^0$ to itself as in Lemma 3.6. If $\mathcal{F} := \kappa_{\pi, \eta, \psi}^{-1}\mathcal{G}$ be the gerbe over $T_w$ associated to the morphism $\kappa_{\pi, \eta, \psi}$ as in Lemma 3.6. Equivalence classes of gerbes over $T_w$ with band a sheaf $Z$ of abelian groups are in bijection with cohomology classes in $\bar{H}^2(T_w, \mathcal{F})$ [G]. By Theorem 5.2.8. The equivalence class $|\mathcal{F}|$ of $\mathcal{F}$ in $\bar{H}^2(T_w, \mathcal{F})$ vanishes, since we have the object $(\pi, \eta, \psi)$ in $\mathcal{F}(T_w)$. Similarly, the gerbe $\mathcal{G}$ is represented by a class $[\mathcal{G}]$ in $\bar{H}^2(\mathfrak{M}_G^0, Z)$, by Lemma 3.7. The morphism $\kappa_{\pi, \eta, \psi}$ pulls back the class $[\mathcal{G}]$ to $[\mathcal{F}]$. Hence, $\kappa_{\pi, \eta, \psi}^*([\mathcal{G}])$ vanishes. We conclude that the class $[\mathcal{F}]$ vanishes, since $\kappa_{\pi, \eta, \psi}^*$ is an isomorphism. Consequently, $\mathcal{G}(\mathfrak{M}_G^0)$ has an object, by [G] III.2.1.1.2.

The set of isomorphism classes of objects of $\mathcal{G}(\mathfrak{M}_G^0)$ is a torsor for the group $\bar{H}^1(\mathfrak{M}_G^0, Z)$, by [Br] Prop. 5.2.5. The latter is the trivial group, by Lemmas 4.2 and 2.2. Hence, $\mathcal{G}(\mathfrak{M}_G^0)$ has a unique object, up to isomorphism. The forgetful morphism $\phi : \mathfrak{M}_G^0 \to \mathfrak{M}_\Lambda$ (Lemma 2.2), transferring the object of $\mathcal{G}(\mathfrak{M}_G^0)$ to a $\Lambda$-marked family $(\pi : X \to \mathfrak{M}_\Lambda)$.

It remains to prove the universal property of $(\pi, \eta)$. Let $(\tilde{\pi} : \tilde{X} \to B, \tilde{\eta})$ be a $\Lambda$-marked family over a connected analytic space $B$, let $\kappa : B \to \mathfrak{M}_\Lambda$ be the classifying morphism, and let $\mathfrak{M}_\Lambda$ be the connected component containing its image. The family $\pi : X \to \mathfrak{M}_\Lambda$ is locally universal, as it restricts to a universal Kuranishi family over an open neighborhood of each...
point of $\mathcal{M}_0^\Lambda$. Hence, the $\Lambda$-marked families $(\bar{\pi}, \bar{\eta})$ and $(\kappa^*(\pi), \kappa^*(\eta))$ are locally isomorphic. Let $\mathcal{P} := \mathcal{Isom}((\bar{\pi}, \bar{\eta}), (\kappa^*(\pi), \kappa^*(\eta)))$ be the local system of isomorphisms of the two families, which are compatible with the markings. Then $\mathcal{P}$ is an $\text{Aut}(\bar{\pi})$ torsor and $(\kappa^*(\pi), \kappa^*(\eta))$ is isomorphic to $(\bar{\pi}, \bar{\eta}) \times \mathcal{P}$.

**Remark 4.3.** Let $X$ be an irreducible holomorphic symplectic manifold, $\text{Diff}(X)$ its diffeomorphism group, and $\text{Diff}_0(X)$ the subgroup of elements isotopic to the identity. An easy sufficient criterion for the existence of a universal family over the connected component of Teichmüller space is the triviality of the intersection $\text{Aut}(X) \cap \text{Diff}_0(X)$ for every $X$ in this connected component [C, Sec. 1.4]. Such $X$ is called **rigidified** [C Def. 12].

**Remark 4.4.** Let $(\pi : \bar{X} \to B, \bar{\eta})$ be a $\Lambda$-marked family over a connected analytic space $B$, let $\kappa : B \to \mathcal{M}_0^\Lambda$ be the classifying morphism, and let $(\kappa^*(\pi) : \kappa^*\mathcal{X} \to B, \kappa^*(\eta))$ be the pullback of the universal family. We can choose a trivialization $\psi : \text{Aut}_0(\kappa^*(\pi)) \to G_B$, by Lemma [2.2]. Thus, the orbit of the isomorphism class of the principal $G$-bundle $\mathcal{P}((\bar{\pi}, \bar{\eta}) := \mathcal{Isom}((\kappa^*(\pi), \kappa^*(\eta)), (\bar{\pi}, \bar{\eta})))$, under the group $\text{Out}(G)$ of outer automorphisms, is an invariant of the $\Lambda$-marked family $(\bar{\pi}, \bar{\eta})$.

Remark 4.3. Fix a fiber $X$ of $\bar{\pi}$. Let $\text{Diff}'(X)$ be the subgroup of $\text{Diff}(X)$ fixing the connected component of Teichmüller space containing $X$ and acting trivially on $H^2(X, \mathbb{Z})$. $\Gamma := \text{Diff}'(X)/\text{Diff}_0(X)$ is a subgroup of the mapping class group $\text{Diff}(X)/\text{Diff}_0(X)$. The natural homomorphism $h : \text{Aut}_0(X) \to \Gamma$ is an isomorphism. Indeed, $h$ is injective, as $X$ is assumed to be rigidified, and surjective by [Ver] Theorem 4.26(iii) and Cor. 4.31 (the groups $\Gamma$ and $\text{Aut}_0(X)$ are denoted by $G_I$ and $K_I$ in [Ver]). The $\Lambda$-marked family $(\bar{\pi}, \bar{\eta})$ is differentiably locally trivial. Hence, there exists an open covering $\{U_i\}$ of $B$ and trivializations $\bar{X}|_{U_i} \cong X \times U_i$, such that the gluing transformations are given by continuous maps from $U_i \cap U_j$ to $\text{Diff}'(X)$. These gluing transformations yield a principal $\text{Diff}'(X)$-bundle, hence a principal $\Gamma$-bundle, hence a principal $\text{Aut}_0(X)$-bundle, which coincides with the one described above.

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