Reduction of path integrals for interacting systems: The case of using dependent coordinates in the description of reduced motion on the orbit space

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Abstract

We consider a reduction procedure in Wiener-type path integral for a finite-dimensional mechanical system with a symmetry representing the motion of two interacting scalar particles on a manifold that is the product of the total space of the principal fiber bundle and a vector space. By analogy with what is done in gauge theories, the local description of the reduced motion on the space of orbits is carried out using dependent coordinates. The factorization of the measure in the path integral, which is necessary for the reduction, is based on the application of the stochastic differential equation of the optimal nonlinear filtering from the theory of stochastic processes. The non-invariance of the measure in the path integral under the reduction is shown. The integrand of the path integral reduction Jacobian is generated by the projection of the mean curvature vector field of the orbit onto the submanifold, which is used to determine the adapted coordinates in the principal fiber bundle associated with the problem under study.

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1 Introduction

Dynamical systems with symmetry are characterized by a deep relationship with geometry. These include, for example, dynamical systems with gauge symmetries, especially those studied in gauge field theories, where the Yang-Mills fields are represented by connections on the principal fiber bundles.

In gauge theories, the main problem is to describe quantum evolution in terms of gauge invariant variables. This is the same as describing the evolution given on the space of orbits of the group action. A satisfactory solution has not yet been found for this problem. At present, in the path integral quantization of the gauge systems, an approach based on the Faddeev-Popov mechanism [1] is used. In this approach, the additional restrictions are imposed on the gauge fields, and therefore we are forced to describe the quantum dynamics on the orbit space in terms of constrained (or dependent) variables.

However, the transition to constrained gauge fields, carried out in path integrals, looks like a heuristic method of the path integral transformations. To clarify the question of the correctness of such a transition, we studied how this happens in path integrals for finite-dimensional dynamical systems with symmetry [2,3]. This was done because for finite-dimensional Wiener-type path integrals, unlike of path integrals used in gauge theories, the integration measure can be rigorously determined.

The model system we studied was a mechanical system describing the motion of a scalar particle on a finite-dimensional manifold, on which the free isometric action of a compact semisimple Lie group is given. The dynamics of this system is largely due to the presence of the principal fiber bundle associated with the considered dynamical system. The original configuration space of the system can be viewed as a total space of this bundle. Therefore, as coordinates in the original configuration space, one can use the coordinates that are usually given in the principal fiber bundle. The possibility of using a coordinate system including dependent coordinates follows from the existence of a local isomorphism between the original principal bundle and the trivial principal bundle whose base space is a local submanifold (‘gauge surface’) [4].

However, in this approach, the problem of the global description of quantum evolution remains unsolved. From a geometric point of view, this is due to the fact that, in the general case, the principal bundles of gauge theories are nontrivial. Therefore, there are no global sections in the principal bundle, and we can only deal with local gauge fixing surfaces. It also follows from this that a global description of evolution with the help of dependent variables defined on these surfaces is possible only when considering trivial principal bundles.

In our paper, the path integral reduction procedure consists in trans-
formation of the original path integral, whose measure is generated by a stochastic process, to the path integral that determines the diffusion (or “quantum evolution”) of the reduced dynamical system given on the orbit space of the principal bundle. In this transformation, the main role is played by the factorization of the measure in the original path integral into the ‘group’ measure (given in the space of paths on a group) and a measure defined in path space on a special surface of the principal bundle known as a ‘gauge fixing surface’ (or ‘gauge surface’).

In our papers [2, 5, 6] we used the definition of path integrals proposed by Daletskii and Belopolskaya in [7, 8]. According to their approach, the measure of path integrals is determined by those stochastic processes that are associated with local evolution semigroups. The global path integral is defined as the limit of these semigroups upon refining the subdivision of the time interval. Transformations of path integrals (local evolution semigroups) can be obtained from transformations of local stochastic differential equations whose solutions are stochastic processes that generate measures in these path integrals. In our works, the factorization of the measure in the path integrals was carried out using the stochastic differential equation of the optimal nonlinear filtering from the stochastic process theory. Note that a similar method for studying reduction in path integrals was used in [9].

The path integral reduction leads to the integral relation between the original path integral and the path integral used to describe the “quantum evolution” on the orbit space of the principal fiber bundle. We also note that the Hamilton operator of the dynamical system obtained as a result of the reduction of the path integral has an additional potential term, which is the integrand of the path integral reduction Jacobian.

In our articles [10, 11], the path integral reduction method was applied to a simple model system describing the interaction of two scalar particles that are given on a manifold represented by the product of two manifolds: a smooth compact Riemannian manifold and a vector space. The product manifold was endowed with a free proper isometric action of a semisimple compact unimodular Lie group. Such a group action gives rise to a principal fiber bundle, so that the original configuration space, the product manifold, can be regarded as the total space of this bundle. In the papers cited, we considered the case where the gauge surface (used to obtain a local section of the principal fiber bundle) has a parametric representation and, therefore, it is possible to define invariant variables that can serve as coordinates in the orbit space of this bundle.

The purpose of the present paper is to consider the case of the path integral reduction when, as in gauge theories, dependent variables are used to describe the motion on the orbit space. It should be noted that the geometric
properties of the considered mechanical system are similar to the properties of an infinite-dimensional dynamical system describing the interaction of the Yang-Mills gauge field with a scalar field. This makes it possible to use the mechanical system under study as a finite-dimensional model for such field theories.

The paper is organized as follows. In Section 2 we give basic definitions that are in fact the same as those from our previous articles. In Section 3 we briefly describe the geometry of the problem and discuss the introduction of the adapted coordinates. Some of the information presented in this section was also borrowed from our previous work. In Sections 4 we define the drift coefficients of our stochastic differential equations. Section 5 deals with the factorization of the measure in the path integral and the calculation of the reduction Jacobian for the case of the reduction onto a zero-momentum level. In Conclusion, the details of the obtained result are discussed. Some relationships between Christoffel symbols used in the article are given in Appendix.

2 Definitions

In this article, we consider the path integrals that are used to represent the solution of the backward Kolmogorov equation which is given on a smooth compact Riemannian manifold $\tilde{P} = \mathcal{P} \times \mathcal{V}$:

$$\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t_a} + \frac{1}{2} \mu^2 \kappa \left[ \Delta_{\mathcal{P}}(p_a) + \Delta_{\mathcal{V}}(v_a) \right] + \frac{1}{\mu^2 \kappa m} V(p_a, v_a) \right) \psi_{t_a}(p_a, t_a) = 0, \\
\psi_{t_b}(p_b, v_b, t_b) = \phi_0(p_b, v_b), \\
(t_b > t_a),
\end{array} \right. \quad (1)$$

where $\mu^2 = \frac{\hbar}{m}$, $\kappa$ is a real positive parameter, $V(p, f)$ is the group-invariant potential term: $V(pg, g^{-1}v) = V(p, v)$, $g \in G$, $\Delta_{\mathcal{P}}$ is the Laplace–Beltrami operator on the Riemannian manifold $\mathcal{P}$. The scalar Laplacian $\Delta_{\mathcal{V}}$ acts on the space of functions defined on the finite-dimensional vector space $\mathcal{V}$. In a chart $(U_{\mathcal{P}} \times U_{\mathcal{V}}, \varphi^P)$, $\varphi^P = (\varphi^A, \varphi^a)$, from the atlas of the manifold $\tilde{P}$, where the point $(p, v)$ has the coordinates $(Q^A, f^a) = \varphi^P(p, v)$, $\Delta_{\mathcal{P}}$ is given by the following expression\footnote{In our formulas we assume that there is sum over the repeated indices. The indices denoted by the capital letters run from 1 to $n_{\mathcal{P}} = \dim \mathcal{P}$, and the indices of small Latin letters, except $i, j, k, l$, run from 1 to $n_{\mathcal{V}} = \dim \mathcal{V}$.}

$$\Delta_{\mathcal{P}}(Q) = G^{-1/2}(Q) \frac{\partial}{\partial Q^A} G^{AB}(Q) G^{1/2}(Q) \frac{\partial}{\partial Q^B}, \quad (2)$$
where $G = \text{det}(G_{AB})$ and $G_{AB}(Q)$ represents the Riemannian metric on $\mathcal{P}$: $G_{AB}(Q) = G(\frac{\partial}{\partial Q^A}, \frac{\partial}{\partial Q^B})$.

The coordinate expression of the operator $\Delta_V$ is

$$\Delta_V(f) = G^{ab} \frac{\partial}{\partial f^a} \frac{\partial}{\partial f^b},$$

where $G^{ab}$ is an inverse matrix to the matrix $G_{ab}$ representing the metric on $\mathcal{V}$ in the coordinate basis $\{\frac{\partial}{\partial f^a}\}$. We assume that the matrix $G_{ab}$ consists of fixed constant elements.

To obtain the Schrödinger equation, one should perform the transition from the equation (1) to the forward Kolmogorov equation and then set $\kappa = i$.

With the definition of the path integral from [7], the solution of the equation (1) can be written as

$$\psi_t(p_a, v_a, t_a) = \mathbb{E}\left[ \phi_0(\eta_1(t_b), \eta_2(t_b)) \exp\left\{ -\frac{1}{\mu^2 \kappa m} \int_{t_a}^{t_b} V(\eta_1(u), \eta_2(u)) du \right\} \right],$$

where $\eta(t) = (\eta_1(t), \eta_2(t))$ is a global stochastic process on a manifold $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$, the measure $\mu^n$ in the space of paths $\Omega_n = \{ \omega(t) = (\omega^1(t), \omega^2(t)) : \omega^1, \omega^2(t_a) = 0, \eta_1(t) = p_a + \omega^1(t), \eta_2(t) = v_a + \omega^2(t) \}$ is determined by the probability distribution of the stochastic process $\eta(t)$.

Note that (3) is a symbolical notation of the global semigroup, defined according to [7,8] as a limit (under refinement of the subdivision of the time interval) of the superposition of the local semigroups:

$$\psi_t(p_a, v_a, t_a) = \lim_{q \to 0} \mathbb{E}\left[ \tilde{U}_\eta(t_a, t_1) \cdots \tilde{U}_\eta(t_{n-1}, t_b) \phi_0 \right](p_a, v_a),$$

where each local evolution semigroup $\tilde{U}_\eta$ acting in the space of functions on the manifold $\tilde{\mathcal{P}}$ is given by

$$\tilde{U}_\eta(s, t) \phi(p, v) = \mathbb{E}_{s,p,v}[\phi(\eta_1(t), \eta_2(t))], \quad s < t, \quad \eta_1(s) = p, \quad \eta_2(s) = v,$$

in which $\tilde{U}_\eta(s, t) \phi(p, v)$ should be understand as $[\tilde{U}_\eta(s, t) \phi](p, v)$.

These local semigroups are also expressed in terms of path integrals with the integration measures determined by the local representatives $\varphi^\mathcal{P}(\eta_t) = \eta^{1^\mathcal{P}}(t) \equiv \{ \eta_1^A(t), \eta_2^A(t) \} \in R^{n \mathcal{P}}$ of the global stochastic process $\eta(t)$.

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2In what follows, the potential term of the Hamilton operator, as being insignificant for our transformations of path integrals, will be temporarily omitted in the path integrals. In the final formulae, the potential term will be recovered.
The local processes \( \{ \eta_A^1(t), \eta_a^2(t) \} \) are solutions of the stochastic differential equations

\[
\begin{align*}
d\eta_A^1(t) &= \frac{1}{2} \mu^2 \kappa G^{-1/2} \frac{\partial}{\partial Q} (G^{1/2} G^{AB}) dt + \mu \sqrt{\kappa} \mathcal{K} \bar{M} (\eta_1(t)) dw^{\bar{M}}(t), \\
d\eta_a^2(t) &= \mu \sqrt{\kappa} \mathcal{K} \bar{b} dw^{\bar{b}}(t),
\end{align*}
\]

in which the stochastic differentials are taken in the sense of Itô and where \( \mathcal{X}_A^\alpha \) and \( \mathcal{X}_b^c \) are defined by the local equalities \( \sum_{\alpha=1}^{n^A} \mathcal{X}_A^\alpha \mathcal{X}_B^\beta = G^{AB} \) and \( \sum_{c=1}^{n^c} \mathcal{X}_b^a \mathcal{X}_b^c = G^{ac} \), \( dw^{\bar{M}}(t) \) and \( dw^{\bar{b}}(t) \) are the independent Wiener processes. (Here and further we denote the Euclidean indices by over-barred indices.) Also note that equations (6) and (7) are the Stratonovich equations.

Note that with the local processes \( \{ \eta_A^1(t), \eta_a^2(t) \} \) one can rewrite (5) as

\[
\begin{align*}
\tilde{U}_\eta(s, t) \phi(p, v) &= E_{s, \phi \tilde{P}(p, v)} [\phi(\eta_\phi \tilde{P}(s))], \quad \eta_\phi \tilde{P}(s) = \phi \tilde{P}(p, v). \quad (8)
\end{align*}
\]

The fundamental solution, also known as the Green’s function \( \tilde{G}_\tilde{P} \), is the kernel of the semigroup, with the help of which the left-hand side of (3) is determined as follows:

\[
\psi_{t_b}(p_a, v_a, t_a) = \int \tilde{G}_\tilde{P}(p_b, v_b; p_a, v_a, t_a) \phi_0(p_b, v_b) dv_\tilde{P}(p_b, v_b),
\]

where \( dv_\tilde{P}(p, v) \) is a volume element on the manifold \( \tilde{P} \). Note that in this formula the integration is carried out using a partition of unity subordinate to the local finite covering of the manifold \( \tilde{P} \).

The probabilistic representation for the kernel \( \tilde{G}_\tilde{P} \) can be formally obtained by replacing \( \phi_0 \) in (3) with the delta-function. This means that the measure of the path integral for the Green’s function \( \tilde{G}_\tilde{P} \) will be defined in the space of paths with both ends fixed. The same can be done more correctly by considering functions approximating the delta function (instead of \( \phi_0 \)) and then taking the appropriate limit.

The fundamental solution to the equation (1) also satisfies the forward Kolmogorov equation. As for the existence of a fundamental solution to the equation (1), this depends on the smoothness constraints imposed on the coefficients of the equation, and in this article we assume that in our equation the coefficients are chosen in accordance with these requirements.
3 Adapted coordinates in the principal fiber bundle

Our main assumption in the article is that on the Riemannian manifold $\tilde{\mathcal{P}} = \mathcal{P} \times \mathcal{V}$, which is the configuration space of our original mechanical system, there is a smooth isometric free and proper action of the compact semisimple Lie group $\mathcal{G}$. We consider the right action of the group by which the point $(p, v)$ is mapped to the point $(\tilde{p}, \tilde{v})$ so that $(\tilde{p}, \tilde{v}) = (p, v)g = (pg, g^{-1}v)$. If the point $(p, v)$ has the local coordinates $(Q^A, f^a)$ (in the coordinate basis $(\frac{\partial}{\partial Q^A}, \frac{\partial}{\partial f^a})$), then this action in coordinates can be written as

$$\tilde{Q}^A = F^A(Q, g), \quad \tilde{f}^b = \tilde{D}^b_a(g)f^a,$$

where $\tilde{D}^b_a(g) \equiv D^b_a(g^{-1})$, and by $D^b_a(g)$ we denote the matrix of the finite-dimensional representation of the group $\mathcal{G}$ acting in the vector space $\mathcal{V}$.

For the right action, we have the compatibility relation

$$F(F(Q, g_1), g_2) = F(Q, \hat{\Phi}(g_1, g_2)),$$

where the function $\hat{\Phi}$ determines the group multiplication law in the space of the group parameters.

The manifold $\tilde{\mathcal{P}}$ is equipped with the following Riemannian metric:

$$ds^2 = G_{AB}(Q)dQ^AdQ^B + G_{ab}df^a df^b. \quad (9)$$

We note, that the components of the metric are not arbitrary, but must satisfy certain relations due to the isometric action of the group $\mathcal{G}$ on $\tilde{\mathcal{P}}$.

In particularly,

$$G_{AB}(Q) = G_{DC}(F(Q, g))F^D_A(Q, g)F^C_B(Q, g), \quad (10)$$

with $F^B_A(Q, g) \equiv \partial F^B(Q, g)/\partial Q^A$, and

$$G_{pq} = G_{ab}\tilde{D}^a_p(g)\tilde{D}^b_q(g). \quad (11)$$

The last relation can be derived from the linear isometrical action of the group $\mathcal{G}$ in the vector space $\mathcal{V}$.

The Killing vector fields $K_\alpha$, $\alpha = 1, \ldots, n_\mathcal{G}, n_\mathcal{G} = \dim \mathcal{G}$, of the manifold $\tilde{\mathcal{P}}$ with the metric (9) are expressed in terms of local coordinates $(Q^A, f^b)$ as follows:

$$K_\alpha = K^A_\alpha(Q)\partial/\partial Q^A + K^b_\alpha(f)\partial/\partial f^b,$$

where $K^A_\alpha(Q) = \partial \tilde{Q}^A/\partial a^\alpha|_{a=e}$ and

$$K^b_\alpha(f) = \partial \tilde{f}^b/\partial a^\alpha|_{a=e} = \partial \tilde{D}^b_c(a)/\partial a^\alpha|_{a=e}f^c \equiv (\tilde{J}_\alpha)c_f^c.$$
The generators $\bar{J}_\alpha$ of the representation $\bar{D}^\xi (a)$ satisfy the commutation relation $[\bar{J}_\alpha, \bar{J}_\beta] = \bar{c}^\gamma_{\alpha\beta} \bar{J}_\gamma$, where the structure constants $\bar{c}^\gamma_{\alpha\beta} = -\bar{c}^\gamma_{\alpha\beta}$.

As it follows from the general theory (see, for example, [12]) the action of the group $\mathcal{G}$ on the manifold $\tilde{P}$ leads to the principal fiber bundle $\pi': \mathcal{P} \times \mathcal{V} \rightarrow \mathcal{P} \times_\mathcal{G} \mathcal{V}$. We denote this principal bundle as $P(\tilde{\mathcal{M}}, \mathcal{G})$, where the orbit space $\tilde{\mathcal{M}} = \mathcal{P} \times_\mathcal{G} \mathcal{V}$ is the base space of the bundle.

The fact that the principal bundle is related to our problem means that the original manifold $\tilde{P}$ can be considered as the total space of this bundle. It follows that we can express the local coordinates $(Q^A, f^a)$ of the point $(p, v) \in \mathcal{P}$ in terms of the coordinates defined in the principal fiber bundle.

As the coordinates in the principal bundle, we will use the adapted coordinates. This choice is due to the fact that the quantization of gauge fields is mainly carried out using such coordinates [13–15].

Adapted coordinates are defined with the help of `gauge fixing surfaces', which are local submanifolds $\tilde{\Sigma}_i$ in the total space of the principal bundle. There is a certain correspondence between these local submanifolds and the open neighborhoods $\tilde{U}_i$ from the atlas of the manifold defined on $\tilde{\mathcal{M}}$. In addition, $\tilde{\Sigma}_i$ must have a transversal intersection with the group orbits.

In our case, to determine $\tilde{\Sigma}_i$ in the principal fiber bundle $P(\tilde{\mathcal{M}}, \mathcal{G})$, we use local surfaces $\Sigma_i$ in the total space $\mathcal{P}$ of the principal bundle $P(\mathcal{M}, \mathcal{G})$ with the base space $\mathcal{M} = \mathcal{P} / \mathcal{G}$. In fact, the local surfaces $\Sigma_i$ serve to define local sections of this bundle. On the other hand, if the sections are given, then the images of these sections are the local surfaces $\Sigma_i$.

We assume that local surfaces $\Sigma_i$ form a global surface $\Sigma$ in the original manifold $\mathcal{P}$ or, which is the same, that there exists a global section in the principal fiber bundle $P(\mathcal{M}, \mathcal{G})$. This implies that our principal fiber bundle is trivial and hence $\mathcal{P} \sim \mathcal{M} \times \mathcal{G}$. Such a case corresponds to what we have in gauge theories, where in practice one has to deal with trivial principal bundles. Note also that triviality of $P(\mathcal{M}, \mathcal{G})$ is interrelated with the triviality of the principal fiber bundle $P(\mathcal{M}, \mathcal{G})$.

Let’s consider how we can apply the general scheme of introducing adapted coordinates in our case. The submanifold $\Sigma$ of $\mathcal{P}$ in this approach is given by the system of equations $\chi^\alpha = 0$, $\alpha = 1, ..., n_G$. And the points belonging to $\Sigma$ have coordinates $Q^{*A}$ for which $\chi^\alpha(Q^{*A}) = 0$. For this reason, the coordinates $Q^{*A}$ are called dependent coordinates.

The group coordinates $a^\alpha(Q)$ of a point $p \in \mathcal{P}$ are defined by the solution of the following equation:

$$\chi^\alpha(F^A(Q, a^{-1}(Q))) = 0.$$
This means that

\[ Q^* A = F^A(Q, a^{-1}(Q)). \]

That is, the group element \( g^{-1}(p) \) with the coordinates \( a^{-1}(Q) \) carries the point \( p \) to the submanifold \( \Sigma \).

Therefore, if we are given the (global) gauge fixing surface \( \Sigma \) in the principal fiber bundle \( \pi: P \to M \), then the section of the bundle can be defined as follows. Restricting the projection \( \pi \) to \( \Sigma \), we have \( \pi|_{\Sigma}: U_{\Sigma} \to U_M \) for local neighborhoods on \( \Sigma \) and \( M \). The section \( \sigma_{\Sigma} \) is defined as a map \( \sigma_{\Sigma}: U_M \to U_{\Sigma} \) such that \( \pi|_{\Sigma} \circ \sigma_{\Sigma} = \text{id}_{\Sigma} \). It follows from the above coordinate transformations that \( \sigma_{\Sigma}(p) = pg^{-1}(p) \).

Using \( \sigma_{\Sigma} \) one can define the section \( \sigma_{\tilde{\Sigma}} \) of the principal fiber bundle \( P(\tilde{\mathcal{M}}, G) \) as the map \( \sigma_{\tilde{\Sigma}}: P \times V \to \tilde{\Pi} \times V \). This section is defined as

\[ \sigma_{\tilde{\Sigma}}(p, \tilde{v}) = (\pi_{\tilde{\Sigma}}(p), g(p)\tilde{v}) = (p, g(p)\tilde{v})g^{-1}(p). \]

Having obtained \( \tilde{\Sigma} \), we can define the principal bundle \( \pi'_{(\tilde{\Sigma})}: P \times V \to \tilde{\Sigma} \) with \( \pi'_{(\tilde{\Sigma})} = \sigma_{\tilde{\Sigma}} \circ \pi' \). This principal bundle isomorphic to \( P(\tilde{\mathcal{M}}, G) \).

The group element \( g^{-1}(p) \) obtained above (participating in the transition (along the orbit) from the point \( p \in P \) to the corresponding point given on the gauge fixing surface \( \Sigma \) in the total space of the principal bundle \( P(\mathcal{M}, G) \)) is also used to determine adapted coordinates on the manifold \( \tilde{\mathcal{P}} \).

The coordinate functions \( \tilde{\varphi}^{-1} \) that make the transition to adapted coordinates are as follows:

\[ \tilde{\varphi}^{-1}: (Q^A, f^b) \to (Q^A(Q), \tilde{f}^b(Q), a^a(Q)), \]

where

\[ \tilde{f}^b(Q) = D^b_{\alpha}(a(Q)) f^c, \]

\((\tilde{D}^b_{\alpha}(a^{-1}) \equiv D^b_{\alpha}(a))\) and \( a(Q) \) are the coordinates of the group element \( g(p) \).

The inverse transformation is given by the coordinate functions \( \tilde{\varphi} \):

\[ \tilde{\varphi}: (Q^* B, \tilde{f}^b, a^a) \to (F^A(Q^*, a), \tilde{D}^b_{\alpha}(a)\tilde{f}^b). \]

These functions, \( \tilde{\varphi} \) and \( \tilde{\varphi}^{-1} \), represent the bundle maps \( \tilde{\varphi} : \tilde{\Sigma} \times G \to P \times V \) and \( \tilde{\varphi}^{-1} : P \times V \to \tilde{\Sigma} \times G \) for the isomorphic trivial principal bundles \( \tilde{\Sigma} \times G \to \tilde{\Sigma} \) and \( P(\mathcal{M}, G) \).

Note also that in the general case when \( P(\tilde{\mathcal{M}}, G) \) is nontrivial, the trivial principal bundles \( \Sigma_{\tilde{i}} \times G \to \Sigma_{\tilde{i}} \) are only locally isomorphic to the principal bundle \( P(\tilde{\mathcal{M}}, G) \) \[14,15,16\]. In this case the adapted coordinates can only be used to describe the local motion on the orbit space \( \tilde{\mathcal{M}} \).
Thus, we have defined the special bundle coordinates in the total space of the principal fiber bundle $\pi' : \mathcal{P} \times \mathcal{V} \to \mathcal{P} \times \mathcal{G} \mathcal{V}$. They are such that $\tilde{\phi}^B(p, v) = (Q^A, \tilde{f}^b, a^\alpha)$ where $\tilde{\phi}^B := \tilde{\phi}^{-1} \circ \phi^B$.

Note that adapted coordinates can be obtained in the same way using a finite set of fixing surfaces $\tilde{\Sigma}_i$. But in this case, it is also required that adapted coordinates must be consistent to each other on the overlapping of the charts.

Replacing coordinates $\left( Q^A, f^a \right)$ of a point $(p, v) \in \mathcal{P} \times \mathcal{V}$ with new coordinates $\left( Q^A, f^a \right)$, $Q^A = F^A(Q^*B, a^\alpha)$, $f^a = D^a_b(a)\tilde{f}^b$,

leads to the following transformations of the local coordinate vector fields:

$$\frac{\partial}{\partial f^a} = D^b_a(a) \frac{\partial}{\partial \tilde{f}^b},$$

$$\frac{\partial}{\partial Q^B} = \frac{\partial Q^*A}{\partial Q^B} \frac{\partial}{\partial Q^A} + \frac{\partial a^\alpha}{\partial Q^B} \frac{\partial}{\partial a^\alpha} + \frac{\partial \tilde{f}^b}{\partial Q^B} \frac{\partial}{\partial \tilde{f}^b},$$

$$= \tilde{F}^C_B \left( N^A_C(Q^*) \frac{\partial}{\partial Q^A} + \chi^\mu_C(\Phi^{-1})^\beta_\mu a^\alpha(\tilde{\phi}) \frac{\partial}{\partial a^\alpha} - \chi^\mu_C(\Phi^{-1})^\beta_\mu (\tilde{J}_\nu)^b_\mu (\tilde{J}_\nu)^p \frac{\partial}{\partial \tilde{f}^p} \right)$$

(13)

Here $\tilde{F}^C_B \equiv F^C_B(F(Q^*, a), a^{-1})$ is an inverse matrix to the matrix $F^A_B(Q^*, a)$,

$\chi^\mu_C \equiv \frac{\partial \chi^\mu_C(Q)}{\partial Q^A} \bigg|_{Q=Q^*}$, $(\Phi^{-1})^\beta_\mu \equiv (\Phi^{-1})^\beta_\mu(Q^*)$ – the matrix which is inverse to the Faddeev–Popov matrix

$$(\Phi_\mu)^\beta(Q) = K^A_\mu(Q) \frac{\partial \chi^\beta(Q)}{\partial Q^A},$$

the matrix $\tilde{v}^\beta_\alpha(a)$ is inverse of the matrix $\tilde{u}^\beta_\alpha(a)$.

The operator $N^A_C$, defined as

$$N^A_C(Q) = \delta^A_C - K^A_\alpha(Q)(\Phi^{-1})^\alpha_\mu(Q)\chi^\mu_C(Q),$$

is the projection operator $(N^A_B N^B_C = N^A_C)$ onto planes perpendicular to the gauge orbits in $\mathcal{P}(\mathcal{M}, \mathcal{G})$, since $N^A_C K^C_\alpha = 0$ [13]. $N^A_C(Q^*)$ is the restriction of $N^A_C(Q)$ to the submanifold $\Sigma$:

$$N^A_C(Q^*) \equiv N^A_C(F(Q^*, e)), \quad N^A_C(Q^*) = F^B_C(Q^*, a)N^M_B(F(Q^*, a))\tilde{F}^A_M(Q^*, a),$$

$^4$det $a^\alpha_\beta(a)$ is the density of the right-invariant measure given on the group $\mathcal{G}$.
where \( e \) is the unity element of the group.

As an operator, the vector field \( \frac{\partial}{\partial Q^\alpha} \) is determined by means of the following rule:

\[
\frac{\partial}{\partial Q^\alpha} \varphi(Q^*) = (P_\perp)^A_B(Q^*) \frac{\partial \varphi(Q^*_{\perp})}{\partial Q^B} \bigg|_{Q^*=Q^*_{\perp}},
\]

where the projection operator \( (P_\perp)^A_B \) onto the tangent plane to the submanifold \( \Sigma \) is given by

\[
(P_\perp)^A_B = \delta^A_B - \chi_B^\alpha (\chi\chi^\top)^{-1}\beta (\chi^\top)^A_{\beta}.
\]

In this formula, \( (\chi^\top)^A_{\beta} \) is a transposed matrix to the matrix \( \chi_B^\alpha \):

\[
(\chi^\top)^A_{\mu} = G^{AB} \gamma_{\mu \nu} \chi_B^\nu,
\]

\[
\gamma_{\mu \nu} = K^A_{\mu} G_{AB} K^B_{\nu}.
\]

Using the above explicit expressions for the projection operators, it is easy to get the following properties:

\[
(P_\perp)^A_B N_A^C = (P_\perp)^C_B, \quad N_A^A (P_\perp)_A^C = N_C^C.
\]

In the new coordinate basis \( (\partial/\partial Q^\alpha, \partial/\partial \tilde{\bar{T}}^\alpha, \partial/\partial a^\alpha) \) the metric \( (\tilde{\bar{G}}) \) of the original manifold \( \mathcal{P} \times \mathcal{V} \) is written as follows:

\[
\tilde{\bar{G}}_{AB}(Q^*, \tilde{\bar{T}}, a) = \begin{pmatrix}
G_{CD}(P_\perp)_A^C (P_\perp)_B^D & 0 & G_{CD}(P_\perp)_A^C K^D_{\nu} \tilde{\bar{u}}^\nu_{\alpha} \\
0 & G_{ab} & G_{ap} K^p_{\nu} \tilde{\bar{u}}^\nu_{\alpha} \\
G_{BC} K^C_{\mu} \tilde{\bar{u}}^\mu_{\beta} & G_{ip} K^p_{\nu} \tilde{\bar{u}}^\nu_{\beta} & d_{\mu \nu} \tilde{\bar{u}}^\mu_{\alpha} \tilde{\bar{u}}^\nu_{\beta}
\end{pmatrix},
\]

where \( G_{CD}(Q^*) \equiv G_{CD}(F(Q^*, c)) \):

\[
G_{CD}(Q^*) = F_C^M(Q^*, a) F_D^N(Q^*, a) G_{MN}(F(Q^*, a)),
\]

the projection operator \( P_\perp \) and the components \( K^A_{\mu} \) of the Killing vector fields depend on \( Q^*, \tilde{\bar{u}}^\mu_{\beta} = \tilde{\bar{u}}^\mu_{\beta}(a), \ K^p_{\nu} = K^p_{\nu}(\tilde{\bar{T})} \), \( d_{\mu \nu}(Q^*, \tilde{\bar{T})} \) is the tensor used to determine the metric given on the orbits of the group action. Its components are given by the relation

\[
d_{\mu \nu}(Q^*, \tilde{\bar{T})} = K^A_{\mu}(Q^*) G_{AB}(Q^*) K^B_{\nu}(Q^*) + K^a_{\nu}(\tilde{\bar{T})} G_{ab} K^b_{\mu}(\tilde{\bar{T})}
\]

\[
\equiv \gamma_{\mu \nu}(Q^*) + \gamma^\prime_{\mu \nu}(\tilde{\bar{T})}.
\]

Also we note that when obtaining \( (14) \), the following transformations were used:

\[
df^a = D^a_p(a) df^p + \frac{\partial D^a_p(a)}{\partial a^\mu} df^p da^\mu.
\]
and 
\[ \frac{\partial \tilde{D}_\rho^a(a)}{\partial \alpha^\mu} \tilde{f}^\rho = (\tilde{J}_\rho)_\rho^\nu \tilde{D}_c^a(a) \tilde{u}_\mu^\rho(a) \tilde{f}^\mu = K^c_\beta(\tilde{f}) \tilde{D}_c^a(a) \tilde{u}_\mu^\rho(a). \]

The last equality is due to the identity 
\[ D_\rho^a(a)(\tilde{J}_\rho)_\rho^\nu D_\rho^p(a) = \rho_\alpha^a(a)(\tilde{J}_\beta)_\rho^c, \]
in which \( \rho_\alpha^a(a) = \tilde{u}_\alpha^a(a) v_\alpha^a(a) \) is the matrix of the adjoint representation of the group \( \mathbb{G} \).

The pseudoinverse matrix \( \tilde{G}^{AB}(Q^*, \tilde{f}, a) \) to the matrix \( (14) \) is given by the following expression:

\[
\begin{pmatrix}
G^E F N^A E N^B F & -G^E F N^A E \Lambda^\nu E K^a F \\
-G^E F N^A E \Lambda^\nu E K^b F & G^{ba} + G^E F N^A E \Lambda^\mu E K^b F K^a F \\
G^E F N^A E \nu E K^b F & -G^E F N^A E \Lambda^\nu E K^b F \nu E K^b F \\
G^E F \Lambda^\nu E K^b F \nu E K^b F & G^E F \Lambda^\nu E \nu E K^b F \nu E K^b F
\end{pmatrix}.
\]

Here \( \Lambda^\nu E \equiv (\Phi^{-1})_\rho^\nu(Q^*) \chi^E_\rho(Q^*) \).

The pseudoinversion of \( \tilde{G}_{AB} \) means that

\[ \tilde{G}^{\tilde{A} \tilde{D}} \tilde{G}_{\tilde{D} \tilde{B}} = \begin{pmatrix} (P_{\perp})_{A}^{B} & 0 & 0 \\ 0 & \delta^a_b & 0 \\ 0 & 0 & \delta^\alpha_\beta \end{pmatrix}. \]

The determinant of the matrix \( (14) \) is defined as

\[ \det \tilde{G}_{AB} = (\det d_{\mu}) (\det \tilde{u}_\mu^a)^2 H, \]

with

\[ H = \det \begin{pmatrix} (P_{\perp})_{A}^{B} \tilde{G}^{H}_{AB} \tilde{G}_{AB}^{H} (P_{\perp})_{B}^{B} \\ (P_{\perp})_{B}^{A} \tilde{G}^{H}_{BA} \tilde{G}_{BA}^{H} (P_{\perp})_{A}^{A} \tilde{G}^{H}_{AB} \end{pmatrix}, \]

where \( \tilde{G}^{H}_{AB} = G_{AB} - G_{AC} K^{\mu}_{a} d^{\nu a} K^{\nu} G_{DB}, \quad \tilde{G}^{H}_{AB} = -G_{AB} K^{b}_{\mu} d^{\nu b} K^{b} G_{ba}, \)

\[ \tilde{G}^{H}_{ba} = G_{ba} - G_{bc} K^{b}_{c} d^{\nu b} K^{\nu} G_{pa}. \]

Note that \( \det \tilde{G}_{AB} \) does not vanish only on the surface \( \tilde{\Sigma} \). On this surface \( \det(P_{\perp})_{B}^{A} \) is equal to unity.

It is also worth noting that the matrix \( (14) \), representing the metric on \( \mathcal{P} \times \mathcal{V} \), can also be written in terms of the components of the mechanical connection existing in the principal fiber bundle \( \mathcal{P}(\hat{\mathcal{M}}, \mathbb{G}) \). This connection is given by a Lie-algebra valued one-form, which in local coordinates is represented as

\[ \omega^\alpha = \omega^\alpha_D dQ^D + \omega^\alpha_p d\tilde{f}^p + u^\alpha_d a^u, \]

where

\[ \omega^\alpha_D(Q^*, \tilde{f}) = d^\alpha_\beta K^{\beta}_{\gamma} G_{DC}, \quad \omega^\alpha_p(Q^*, \tilde{f}) = d^\alpha_\beta K^{\beta}_{\gamma} G_{ap}. \]

\(^5\) In what follows, \( \det d_{\mu} \) will be denoted by \( d \).
In this case, \( \tilde{G}_{A\dot{B}} \) is given by the following expression:

\[
\begin{pmatrix}
(P_{\perp})_{A}^{t}(P_{\perp})_{B} \left( \tilde{G}_{AB}^{H} + d_{\mu\nu} \phi_{\mu}^{\alpha} \phi_{\nu}^{\beta} \right) & 0 \\
0 & G_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
(P_{\perp})_{A}^{t} \phi_{\alpha}^{\mu} d_{\mu\nu} \tilde{u}_{\nu}^{\alpha} \\
\phi_{\beta}^{\mu} d_{\mu\nu} \tilde{u}_{\nu}^{\beta}
\end{pmatrix}. \quad (17)
\]

The pseudoinverse matrix to the matrix (17) is

\[
\begin{pmatrix}
G^{AB} N_{A}^{\alpha} N_{B}^{\beta} & G^{AB} N_{A}^{\alpha} N_{B}^{\beta} \phi_{C}^{\mu} K_{\mu}^{a} - G^{AB} N_{A}^{\alpha} N_{B}^{\beta} \phi_{C}^{\beta} v_{\beta}^{a} \\
G^{AB} N_{A}^{\alpha} N_{B}^{\beta} \phi_{C}^{\mu} K_{\mu}^{a} & G^{AB} N_{A}^{\alpha} N_{B}^{\beta} + G^{ab}
\end{pmatrix}, \quad (18)
\]

where \( \phi_{C}^{\mu} = \gamma_{\nu}^{\mu} K_{\nu}^{B} G_{BC} \), \( N_{A}^{\dot{B}} = -K_{\mu}^{a} \Lambda_{B}^{\mu} \) is the component of the projection operator \( N_{A}^{\dot{B}} = (N_{A}^{1}, 0, N_{B}^{2}, B_{b}) \), (in our notation, the index \( \tilde{A} = (A, a) \)). Since \( N_{A}^{\dot{B}} K_{\alpha}^{B} = 0 \), \( N_{A}^{\dot{B}} \) projects onto planes perpendicular to the gauge orbits in \( P(\mathcal{M}, \mathcal{G}) \).

Note that the elements of the matrix (18) can also be rewritten somewhat differently, using the following identities:

\[
G^{BB'} \Lambda_{A}^{a} A_{B'}^{\beta} = \left( \gamma_{a}^{\beta} + G^{AB'} N_{A}^{B} \phi_{A}^{a} \phi_{B}^{\beta} \right),
\]

\[
G^{AB} N_{A}^{a} N_{B}^{b} + G^{ab} = \left( \gamma_{a}^{b} + G^{AB'} N_{A}^{B} \phi_{A}^{a} \phi_{B}^{b} \right) K_{a}^{B} K_{\beta}^{b} + G^{ab}.
\]

4 Stochastic differential equations in coordinates of the principal fiber bundle

Transition from the original coordinates \( (Q^{A}, f^{a}) \), given on the local chart of the manifold \( \tilde{P} \), to the special coordinates \( (Q^{*A}, \tilde{f}^{b}, a^{a}) \) leads to the need for a corresponding replacement of the original stochastic process \( \eta \) generating the measure in the path integral (3). The local components \( (Q_{t}^{*A}, \tilde{f}_{t}^{b}, a^{a}) \) of a new process must be solutions of those local stochastic differential equations that can be obtained from the equations (6) and (7) as a result of the change of variables performed in stochastic processes. These equations can be derived using the Itô’s differentiation formula.

Consider, for example, how to obtain a stochastic differential equation for the component \( Q^{*A}(t) = Q_{t}^{*A} \) of a new stochastic process. We know that the local coordinate \( Q^{*A} \) is defined by the following equation \( Q^{*A} = \)
$F^A(Q, a^{-1}(Q))$. The same equation must hold for the local stochastic variable $Q^*_t$ and the stochastic variable $\eta^A(t)$: $\eta^A(t) = F^A(Q^*_A(t), a^a(t))$. Then, applying the Itô’s differentiation formula to the stochastic variable $Q^*_t$, we obtain

$$dQ^*_r = \left( \frac{\partial Q^*_A}{\partial Q^B} \right) d\eta^B_r(t) + \frac{1}{2} \left( \frac{\partial^2 Q^*_A}{\partial Q^C \partial Q^D} \right) < d\eta^C_r(t), d\eta^D_r(t) >.$$  

Using the stochastic differential equation (6) for $\eta^A(t)$ in this equation and re-expressing $\eta^A(t)$ through $Q^*_A(t)$ and $a^a(t)$ in all terms depending on $\eta^A(t)$ on the right-hand side of the above equation, we obtain expressions for the drift and diffusion coefficients of the stochastic differential equation for $Q^*_t$. A similar approach can be used to write down the stochastic differential equations for the processes $\tilde{f}^b_t$ and $a^a_t$.

Another way to derive stochastic differential equations for transformed variables is based on the use of the Laplace-Beltrami operator which is obtained from the original differential operator of the equation (1) as a result of introducing new coordinates on the manifold $\hat{\mathcal{P}}$. Namely, the drift coefficients of these equations will be represented by the coefficients at the first derivatives of thus obtained Laplace-Beltrami operator, and the diffusion coefficients can be found from the coefficients at the second partial derivatives of this operator. Using this method, we get the following stochastic differential equations:

$$dQ^*_t = \frac{1}{2} \left( \mu^2 \kappa \right) \left[ \frac{1}{\sqrt{dH}} \frac{\partial}{\partial Q^A} \left( \sqrt{dH} G^{A'B'} N^A_A N^B_B \right) \right] dt + \mu \sqrt{\kappa} N^B_B \mathcal{X}^C_M dw^M_t,$$

(19)

where $\mathcal{X}^A_M(Q^*_t)$ is determined from the local equality $\sum_{M=1}^{np} \mathcal{X}^A_M \mathcal{X}^B_M = G^{AB}$. 

$$d\tilde{f}^b_t = \frac{1}{2} \left( \mu^2 \kappa \right) \left[ \frac{1}{\sqrt{dH}} \frac{\partial}{\partial Q^A} \left( \sqrt{dH} G^{A'B'} N^A_A N^B_B \mathcal{X}^c_B K^b_c \right) \right] dt + \mu \sqrt{\kappa} (N^b_B \mathcal{X}^C_M dw^M_t + \mathcal{X}^b_c dw^c_t),$$

(20)

and where $\mathcal{X}^a_c$ is defined by the local equality $\sum_{c=1}^{nv} \mathcal{X}^a_c \mathcal{X}^b_c = G^{ab}$. 

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\[ \begin{align*}
  da^a_t &= \frac{1}{2}(\mu^2 \kappa) \left[ \frac{1}{\sqrt{dH}} \frac{\partial}{\partial Q^A} \left( -\sqrt{dH} G^{EF} N^E F^C \mathcal{A}^\beta \right) \bar{v}^a_\beta \right] \\
  &+ \frac{1}{\sqrt{dH}} \frac{\partial}{\partial f^b} \left( -\sqrt{dH} G^{EC} \Lambda^\beta_C \Lambda^a_C K^b_\mu \bar{v}^a_\beta \right) \\
  &+ G^{BC} \Lambda^\alpha_C \Lambda^\beta_C \bar{v}^a_\beta \frac{\partial}{\partial a} \bar{v}^a_\alpha dt + \mu \sqrt{\kappa} \Lambda^\beta_C \bar{v}^a_\beta \mathcal{F}_C^a dw^M_t. \tag{21}
\end{align*} \]

A short notation of the above stochastic differential equations is as follows:

\[ \begin{align*}
  dQ^B_t &= (\mu^2 \kappa) b^B_t dt + \mu \sqrt{\kappa} N^B_C \mathcal{F}_M^C dw^M_t, \\
  d\tilde{f}^b_t &= (\mu^2 \kappa) b^b_t dt + \mu \sqrt{\kappa} (N^a_C \mathcal{F}_M^C dw^M_t + \mathcal{F}_e^a dw^e_t), \\
  da^a_t &= (\mu^2 \kappa) a^a_t dt + \mu \sqrt{\kappa} \Lambda^\alpha_C \bar{v}^a_\beta \mathcal{F}_C^a dw^M_t.
\end{align*} \]

The drift coefficients of the first two equations can be rewritten to obtain the stochastic differential equations, which are used in stochastic theory to describe diffusion on a manifold embedded in some ambient manifold (cf. ref. [20]).

It can be shown that these drift coefficients are equal to the coefficients standing at the first partial derivatives with respect to \(Q^A\) and \(\tilde{f}^b\) in the Laplacian obtained by regrouping the terms of the original Laplace-Beltrami operator in the following form:

\[ \frac{1}{2} \hat{G}^{\hat{C}\hat{B}} \nabla_{\hat{C}} \nabla_{\hat{B}} = \frac{1}{2} \left( \hat{\Pi}^D_C \hat{\Pi}^D_B - K^A_\alpha d^\rho\beta (\nabla_A K^B_\beta) \nabla_B + K^A_\alpha d^\rho\beta \nabla_A K^B_\beta \nabla_B \right), \tag{22} \]

where it is also necessary to replace the local coordinates \((Q^A, f^b)\) with the coordinates \((Q^A, \tilde{f}^b, a^a)\).

In the first term on the right side of this equation, the horizontal projection operator

\[ \hat{\Pi} = \begin{pmatrix}
  \hat{\Pi}^D_C \\
  \hat{\Pi}^b_C \\
  \hat{\Pi}^a_C
\end{pmatrix} \]

such that \(\hat{\Pi}^D_C K^C_\alpha = 0\), with the components

\[ \begin{align*}
  \hat{\Pi}^D_C &= \delta^D_C - K^D_\alpha d^\rho\beta K^B_\beta G_{BC}, \\
  \hat{\Pi}^D_a &= -K^D_a d^\rho\beta K^b_\beta G_{ba}, \\
  \hat{\Pi}^b_C &= -K^b_\alpha d^\rho\beta K^B_\beta G_{BC}, \\
  \hat{\Pi}^b_a &= \delta^b_a - K^b_\alpha d^\rho\beta K^b_\beta G_{ba},
\end{align*} \]

\(^{15}\)Here and in what follows, we adopt the notation in which subscripts (and superscripts), indicated by capital letters with a tilde, denote multi-indices consisting of two indices, for example \(\hat{B} = (B, b)\). It is also assumed that summation occurs over these repeated indices.
has the following properties:

\[ \tilde{\Pi}^A_L N^L_C = \Pi_C^A, \quad \tilde{\Pi}^E_B N^E_B = N^B_B, \quad \tilde{\Pi}^E_B N^B_E = 0. \]

Note also that in our case

\[ (\tilde{G}^{CB} \nabla_C \nabla_B)(Q, f) = (G^{AB} \nabla_A \nabla_B)(Q) + G^{pq} (\nabla_p \nabla_q)(f). \]

The symbol \( \nabla_A \) denotes the covariant derivative defined using the Christoffel symbols of the manifold \( \mathcal{P} \). However, due to the special choice of the initial metric on the manifold \( \mathcal{P} \), in our case we have \( \nabla_q(f) = \frac{\partial}{\partial f^q} \).

As a result of the calculations, it follows that in each of the equations (19) and (20), the drift coefficients are represented by the sum of two terms, \( b_l \) and \( b_{II} \), coming only from the first and the second terms on the right-hand side of the equation (22). The third term does not contribute to the drift coefficients of stochastic differential equations on the \( \tilde{\Sigma} \).

4.1 The drift coefficient \( b_{II} \) as a projection of the mean curvature vector field of the orbit onto the submanifold \( \tilde{\Sigma} \)

In this subsection we consider the geometry of the drift term \( b_{II} \). Our calculation show that this term is given by the projection on the submanifold \( \tilde{\Sigma} \) of the mean curvature vector of the orbit.

In our case, the Killing vector fields which in the coordinates \((Q^A, f^a)\) are given by the formula

\[ \tilde{K}_\alpha = K^A_{\alpha}(Q) \frac{\partial}{\partial Q^A} + K^a_{\alpha}(f) \frac{\partial}{\partial f^a}, \]

are tangent to the group orbit.

The mean curvature vector field (mean curvature normal) of the orbit is defined as follows:

\[ \bar{\mu} = \frac{1}{2} \delta^{\rho\beta} \left[ \left( \tilde{\Pi}_C^D (\nabla_{K^\alpha} K^\beta)^C + \tilde{\Pi}_a^D (\nabla_{K^\alpha} K^\beta)^a \right) \frac{\partial}{\partial Q^D} \right. \]

\[ + \left. \left( \tilde{\Pi}_C^b (\nabla_{K^\alpha} K^\beta)^C + \tilde{\Pi}_a^b (\nabla_{K^\alpha} K^\beta)^a \right) \frac{\partial}{\partial f^b} \right]. \tag{23} \]

By \( (\nabla_{K^\alpha} K^\beta)^C(Q) \), used in \( \bar{\mu} \), we denote

\[ K^A_{\alpha}(Q) \frac{\partial}{\partial Q^A} K^C_{\beta}(Q) + K^A_{\alpha}(Q) K^B_{\beta}(Q) \Gamma^C_{AB}(Q), \]
where
\[ \Gamma_{AB}(Q) = \frac{1}{2} G^{CE}(Q) \left( \frac{\partial}{\partial Q^A} G_{EB}(Q) + \frac{\partial}{\partial Q^B} G_{EA}(Q) - \frac{\partial}{\partial Q^E} G_{AB}(Q) \right) . \]

To obtain the projection of \( \bar{\mu} \) onto the submanifold \( \tilde{\Sigma} \), one must first find how the terms on the right-hand side of the equation (23) are expressed in terms of the coordinates \((Q^*, \tilde{f}^a, a^a)\). It can be shown that \((\nabla_{K^*} K_\beta)^b(f)\) is transformed as
\[ (\nabla_{K^*} K_\beta)^b(f) = \rho^a_\beta(a) \rho^b_\beta(a) K^a_\beta(\tilde{f}) (\bar{J}_\mu)^b_\beta \bar{D}_b(a). \]

In derivation of this representation it was used the formula
\[ (\nabla K^a_\alpha K^b_\beta)^A_{\alpha\beta} = \rho^A_\alpha(a) \rho^b_\beta(a) (\bar{J}_\mu)^b_\beta \bar{D}_b(a). \]

For \((\nabla_{K^*} K_\beta)^A(Q)\) we have the following expression:
\[ (\nabla_{K^*} K_\beta)^A(Q) = F^A_D K^D_\mu(Q^*) c^D_\alpha\beta \rho^\mu_\alpha(a) + N^D_B \bar{D}_b(a). \]

As a result of the performed transformations, the mean curvature vector field \( \bar{\mu}(Q^*, \tilde{f}, a) \) is represented as
\[ \bar{\mu} = \frac{1}{2} d^{a\beta} (\nabla_{K^*} K_\beta)^b(Q^*) \left( N^B_B \frac{\partial}{\partial Q^*} + \Pi^C_B \bar{D}_b(a) \frac{\partial}{\partial a^\nu} + N^b_B \frac{\partial}{\partial f^b} \right) \]
\[ + \frac{1}{2} d^{a\beta} (\nabla_{K^*} K_\beta)^b(\tilde{f}) \left( \Pi^E_B \bar{D}_b(a) \frac{\partial}{\partial a^\nu} + \frac{\partial}{\partial f^b} \right) . \]

(We recall that \( N^b_B = -\Lambda^b_B K^b_\mu \).)

The projection of the vector field \( \bar{\mu}(Q^*, \tilde{f}, a) \) on the submanifold \( \tilde{\Sigma} \in \mathcal{P} \times \mathcal{V} \) is defined by the following formula:
\[ \tilde{G}_L^S \tilde{G} \left( \tilde{\mu}, \frac{\partial}{\partial Q^*} \right) - \tilde{G}_{L}^S \tilde{G} \left( \tilde{\mu}, \frac{\partial}{\partial f} \right) \]
\[ + \tilde{G}_{Q^*}^a \tilde{G} \left( \tilde{\mu}, \frac{\partial}{\partial f} \right) + \tilde{G}_{Q^*}^L \tilde{G} \left( \tilde{\mu}, \frac{\partial}{\partial f} \right) \]

Applying this formula, we get the following representation for \( \tilde{b}_{II}(Q^*, \tilde{f}) \):
\[ \tilde{b}_{II} = -\frac{1}{2} G^{CC'} N^C_B N^C_B G_{BB'} \frac{\partial}{\partial Q^*} (\nabla_{K^*} K_\beta)^B + \frac{\partial}{\partial f^b} \]
\[ - \frac{1}{2} d^{a\beta} \left( N^B_B (\nabla_{K^*} K_\beta)^B + (\nabla_{K^*} K_\beta)^a \right) \frac{\partial}{\partial f^b} . \]
Since the first line of (25) can be also rewritten as
\[ \frac{1}{2} d^{\alpha \beta} N^L_B (\nabla_{K_\alpha} K_\beta)^B \frac{\partial}{\partial Q^*_L}, \]
then, using the shorthand notation, \( \tilde{b}_{II} \) can be written in the following form:
\[ \tilde{b}_{II} = -\frac{1}{2} d^{\alpha \beta} N^L_B (\nabla_{K_\alpha} K_\beta) \tilde{B} \frac{\partial}{\partial \tilde{Q}^L}, \]
(that is, here \( \tilde{L} = (L, a) \) and \( \tilde{Q}^L \equiv (Q^*_L, \tilde{f}^a) \)).

4.2 The drift coefficient \( b^A_I \)

In this subsection, we show that the drift coefficient \( b^A_I \) can be obtained using the corresponding stochastic differential equation given on the orbit space \( \tilde{M} \) (which is locally isomorphic to \( \tilde{\Sigma} \)). The differential generator of the semigroup associated with the stochastic process described by this stochastic differential equation arises from the first term on the right-hand side of the equation (22) after its restriction to the manifold \( \tilde{M} \). The differential generator thus obtained is the Laplace-Beltrami operator on \( \tilde{M} \).

Note that in our case, the local coordinates on the orbit space manifold \( \tilde{M} \) are given by the invariant coordinates \((x^i, \tilde{f}^a)\), where \( x^i, i = 1, \ldots, n_M \), are coordinates on \( M \). They are defined under the condition that the local submanifold \( \Sigma \) can also be represented in the parametric form: \( Q^A = Q^*_A(x^i) \). This implies that the coordinates of points on \( \Sigma \) must satisfy the equation \( \chi^a(Q^*_A(x^i)) = 0 \).

The metric tensor of the Riemannian manifold \( \tilde{M} \) has the following components in the coordinate basis \( \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial \tilde{f}^a} \right) \):
\[\begin{pmatrix}
\tilde{h}_{ij} & \tilde{G}^H_{Aa} Q^*_i B \\
\tilde{G}^H_{Aa} Q^*_i & \tilde{G}^H_{ba}
\end{pmatrix}, \tag{26}\]
where \( \tilde{h}_{ij} = Q^*_A \tilde{G}^H_{AB} (Q^*_i(x)) Q^*_j B \) with \( Q^*_i \equiv \frac{\partial}{\partial x^i} Q^*_i(x) \). This means that the Riemannian manifold \( \tilde{M} \) can be considered as a submanifold in the (Riemannian) manifold \( (\tilde{P}, \tilde{G}^H_{AB}) \) with degenerate metric \( \tilde{G}^H_{AB} \).

The elements of the inverse matrix to matrix (26) are given by
\[\tilde{h}^{ij} = h^{ij} := h^{SD} T^i_D T^j_S = G^{EF} N^S_E N^P_D T^i_D T^j_S, \quad \tilde{h}^{ib} := h^{BP} T^i_P = G^{EF} N^b_P N^P_E T^i_P, \quad \tilde{h}^{ab} := h^{ab} = G^{ab} + G^{EF} N^a_E N^b_F, \]

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where the operator
\[ T_i^D = (P_\perp)_D^B(Q^*(x))G_{BL}^H(Q^*(x))Q_m^L(x)h^{mi}(x), \]
with the properties \( T_i^D Q_j^D = \delta_i^j \) and \( T_i^D Q_i^B = (P_\perp)_D^B \), is defined by using the projection operator \( P_\perp \) on the tangent plane to the submanifold \( \Sigma \) of the manifold \( P \).

The diffusion on \( \tilde{M} \) is given by the stochastic process locally represented by the processes \((x^i_t, \tilde{f}_a^i)\). But our goal is to obtain a description of diffusion on \( \tilde{M} \) in terms of the local stochastic differential equations for stochastic variables associated with the ambient manifold, that is in our case for the stochastic variable \( Q^*_A \), which is associated with the dependent variable \( Q^*_A \), and for \( \tilde{f}_a^i \).

Since on \( M \) the variable \( Q^*_A \) is now the function of \( x^i \), the stochastic differential equation for \( Q^*_A(x(t)) \) can be obtained by means of the replacement of the variables in the stochastic differential equations:
\[
dQ^*_A(x(t)) = Q^*_A(x(t))dx^i_t + \frac{1}{2}Q^*_A(x(t))<dx^i_t, dx^j_t>. \tag{27}
\]
In turn, the drift coefficient \( b^i_I(t) \) of the stochastic differential equation for \( x_t \)
\[
dx^i_t = \mu^2 \kappa b^i_I(t)dt + \sqrt{\kappa} X^i_t(t)dw^i_t
\]
can be easily found using the Laplace-Beltrami operator of the manifold \( \tilde{M} \):
\[
b^i_I = \frac{1}{2} \left( \frac{1}{H^{1/2}} \frac{\partial}{\partial x^j} (H^{1/2} h_{ij}) + \frac{1}{H^{1/2}} \frac{\partial}{\partial f^a} (H^{1/2} \tilde{h}_{ai}) \right). \tag{28}
\]
On the other hand,
\[
b^i_I = -\frac{1}{2} \left( h_{kn} \Gamma^i_{kn} + \tilde{h}_{kn} \Gamma^i_{kn} + h_{am} \Gamma^i_{am} + \tilde{h}_{ab} \Gamma^i_{ab} \right), \tag{28}
\]
where \( \Gamma \) are Christoffel symbols of \( \tilde{M} \) with the metric tensor \[26\].

The Christoffel symbol \( \Gamma^i_{kn} \) is defined as
\[
\Gamma^i_{kn} = \tilde{h}^{im} \Gamma_{kmn} + \tilde{h}^{ia} \Gamma_{kna}, \tag{29}
\]
where
\[
\Gamma_{kmn} = \frac{1}{2} (\tilde{h}_{km,n} + \tilde{h}_{nm,k} - \tilde{h}_{kn,m}),
\]
\[
\Gamma_{kna} = \frac{1}{2} (\tilde{h}_{ka,n} + \tilde{h}_{na,k} - \tilde{h}_{kn,a}).
\]
In the same way one can get that following formula:

\[ \Gamma_{knm} = \hat{\Gamma}_{BMT}^{\alpha} \hat{Q}^{\beta}_{k_n} \hat{Q}^{\gamma}_{m} + \hat{G}_{AB}^{H} \hat{Q}^{\alpha}_{k} \hat{Q}^{\beta}_{m}, \]
\[ \Gamma_{kna} = \hat{\Gamma}_{BMa}^{\alpha} \hat{Q}^{\beta}_{k_n} + \hat{G}_{Aa}^{H} \hat{Q}^{\alpha}_{mn}, \]

where

\[ \hat{\Gamma}_{BMD}^{i} = \frac{1}{2} (G_{BD,M}^{H} + \hat{G}_{MD,B}^{H} - \hat{G}_{BM,D}^{H}), \]
\[ \hat{\Gamma}_{BMa}^{i} = \frac{1}{2} (G_{Ba,M}^{H} + \hat{G}_{MD,B}^{H} - \hat{G}_{BM,a}^{H}). \]

To determine the Christoffel symbol \( \hat{\Gamma}_{BM}^{R} \) we use the following equality:

\[ \hat{\Gamma}_{BM}^{R} = \hat{\Gamma}_{RD}^{H} \hat{\Gamma}_{BM}^{R}. \]

Note, however, that this defines \( \hat{\Gamma}_{BM}^{R} \) only modulo such terms \( T_{BC}^{M} \) that satisfy \( \hat{G}_{AM}^{H} T_{BC}^{M} = 0 \). In addition, note that by our notation [31] also means that

\[ \hat{\Gamma}_{BM}^{i} = \hat{\Gamma}_{R}^{H} \hat{\Gamma}_{BM}^{R}. \]

Substituting [31] and [32] into [30], and then the result in [29], we obtain, after appropriate transformations, the following representation for \( \Gamma_{kn}^{i} \):

\[ \Gamma_{kn}^{i} = T_{S \alpha}^{i} N_{R}^{S} (\hat{\Gamma}_{BM}^{R} \hat{Q}^{\alpha}_{k} \hat{Q}^{\beta}_{n} + \hat{Q}^{\beta}_{kn}). \]

Such a representation is obtained using the following identities:

\[ N_{E}^{D} \hat{G}_{RF}^{H} = \hat{G}_{RF}^{H}, G^{EF} \hat{G}_{RF}^{H} = \hat{\Pi}_{R}^{E}, N_{E}^{A} \hat{\Pi}_{R}^{E} = N_{A}, N_{E}^{A} \hat{\Pi}_{R}^{E} = 0. \]

In the same way one can get that

\[ \Gamma_{kh}^{i} = T_{S \alpha}^{i} N_{R}^{S} \hat{\Gamma}_{Ab}^{R} Q^{A}_{k} \eta_{i}^{n}, \quad \Gamma_{km}^{i} = T_{S \alpha}^{i} N_{R}^{S} \hat{\Gamma}_{ab}^{R} Q^{B}_{m} \eta_{i}^{n}, \quad \Gamma_{ab}^{i} = T_{S \alpha}^{i} N_{R}^{S} \hat{\Gamma}_{ab}^{R} Q^{B}_{m}. \]

Using the obtained Christoffel symbols in [28], it can be shown that the expression for the drift coefficient \( b_{j}^{A} \) of the equation [27] is given by the following formula:

\[ b_{j}^{A} = \frac{1}{2} N_{R}^{A} (G^{EF} N_{E}^{B} \hat{G}_{BM}^{H} \hat{\Gamma}^{R}_{BM} + G^{EF} N_{E}^{B} \hat{G}_{Bb}^{R} + G^{EF} N_{E}^{B} \hat{G}_{F}^{H} \hat{\Gamma}^{R}_{ab} + (G^{ab} + G^{EF} N_{E}^{B}) \hat{\Gamma}^{R}_{ab} + h^{kn} Q^{A}_{kn}) + \frac{1}{2} h^{ij} Q^{A}_{ij}, \]

where the terms depend on \( Q^{A}(x) \) and \( \tilde{f} \) and, therefore, in [27] on \( Q^{A}(x_{t}) \) and \( \tilde{f}_{t} \). Also note that, using a shorthand notation, \( b_{j}^{A} \) can be rewritten as

\[ b_{j}^{A} = \frac{1}{2} N_{R}^{A} (h^{BM} \hat{\Gamma}^{R}_{BM} + h^{kn} Q^{A}_{kn}) + \frac{1}{2} h^{ij} Q^{A}_{ij}. \]
4.3 The drift coefficient \( b^a_I \)

The drift coefficient \( b^a_I \) of the stochastic differential equation for the stochastic variable \( \tilde{f}_t \) can also be found using the Laplace-Beltrami operator of the manifold \( \tilde{M} \). This term is given by the following expression:

\[
b^a_I = \frac{1}{2} \left( \frac{1}{H^{1/2}} \frac{\partial}{\partial x^i} (H^{1/2} \tilde{h}^i_{ab}) + \frac{1}{H^{1/2}} \frac{\partial}{\partial f^b} (H^{1/2} \tilde{h}^a_{ab}) \right).
\]

This expression can be rewritten using Christoffel symbols as follows:

\[
b^a_I = -\frac{1}{2} \left( \tilde{h}^{kn} \Gamma^a_{kn} + \tilde{h}^{kb} \Gamma^a_{kb} + \tilde{h}^{bn} \Gamma^a_{bm} + \tilde{h}^{bc} \Gamma^a_{bc} \right).
\]

We omit here the steps with the necessary transformations to obtain the representation of \( b^a_I \), since they are similar to what we did when obtaining the drift coefficient \( b^A_I \). The relationships between Christoffel symbols required for these transformations are given for reference in Appendix.

As a result, we get the following representation for \( b^a_I \):

\[
b^a_I = -\frac{1}{2} \left( \tilde{h}^{kn} \Gamma^a_{kn} + \tilde{h}^{kb} \Gamma^a_{kb} + \tilde{h}^{bn} \Gamma^a_{bm} + \tilde{h}^{bc} \Gamma^a_{bc} \right).
\]

The same expression can be rewritten as

\[
b^a_I = -\frac{1}{2} \left( \tilde{h}^{kn} \Gamma^a_{kn} + \tilde{h}^{kb} \Gamma^a_{kb} + \tilde{h}^{bn} \Gamma^a_{bm} + \tilde{h}^{bc} \Gamma^a_{bc} \right).
\]

From (33) and (35) we see that the expressions for \( b^A_I \) and \( b^a_I \) include still “untransformed” terms with second partial derivatives of \( Q^A(x) \) with respect to \( x^k \) and \( x^n \). It turns out that the solution to this problem can be found using the mean curvature normal of the orbit space \( \tilde{M} \).

The mean curvature normal is defined as the trace of the second fundamental form. In our case, the local basis (the frame) on the tangent plane to the manifold of the orbit space (which is the submanifold in the ambient
space with the degenerated metric) is given by the coordinate vector fields \((\frac{\partial}{\partial Q_i}, \frac{\partial}{\partial f^a})\) \(\equiv (e_i, e_a)\). To calculate the mean curvature normal we use the following representation for these coordinate vector fields: \((Q^*_i(x) \frac{\partial}{\partial Q^*_i}, \frac{\partial}{\partial f^a}) = (Q^*_i e_A, e_a)\), so the mean curvature normal \(\vec{j}_I\) must be given as

\[
\vec{j}_I = j^A \frac{\partial}{\partial Q^*_A} + j^a \frac{\partial}{\partial f^a}.
\]

The components of the mean curvature normal \(\vec{j}_I\) are defined as follows:

\[
j^A = \frac{1}{2}(\delta^A_B - N^A_B)(\ldots)^B,
\]

(36)

where

\[
(\ldots)^B = \left(h^{ij} [\nabla_{e_i} e_j] + \tilde{h}^{ia} [\nabla_{e_i} e_a] + \tilde{h}^{ab} [\nabla_{e_a} e_b] \right)^B.
\]

To calculate the mean curvature normal \(\vec{j}_I\), we use the formula (36), which states that \(\vec{j}_I\) is defined as the projection of the vector field, which in our case is represented by the above expression given in parentheses \((\ldots)\), on the normal bundle to the orbit space [21]. This projection is performed by an operator whose components are specified using \((\delta^A_B - N^A_B)\), where \(N^A_B = (N^A_B, N^a_B, 0, N^a_B = \delta^a_B)\).

Note that from (36) it follows that

\[
j^A = \frac{1}{2}(\delta^A_B - N^A_B)(\ldots)^B \quad \text{and} \quad j^c = -\frac{1}{2}N^c_B(\ldots)^B.
\]

In our local basis given by the tangent vector fields, we have

\[
\nabla_{e_i} e_j = \nabla_{Q^*_i(x)e_A} Q^*_j(x)e_B
\]

\[
= h^{ij}(Q^*_i Q^*_j \hat{\Gamma}_{AB} + Q^*_i Q^*_j \hat{\Gamma}_{AB}) \frac{\partial}{\partial Q^*_C} + h^{ij} Q^*_i Q^*_j \hat{\Gamma}_{AB} \frac{\partial}{\partial f^a}
\]

\[
= (G_{EF} N^A_E N^B_F \hat{\Gamma}_{AB} + h^{ij} Q^*_i Q^*_j) \frac{\partial}{\partial Q^*_C} + G_{EF} N^A_E N^B_F \hat{\Gamma}_{AB} \frac{\partial}{\partial f^a},
\]

where \(G_{EF}, N^A_E, N^B_F\) depend on \(Q^*(x)\), \(\hat{\Gamma}\) depend on \(Q^*(x)\) and \(\hat{f}\). The other components of \((\ldots)^B\) are found similarly.

As a result, we get that

\[
\tilde{j}^a_i = -\frac{1}{2} N^a_C \left( G_{EF} N^A_E N^B_F \hat{\Gamma}_{AB} + G_{EF} N^B_E N^b_F \hat{\Gamma}_{Bb} + G_{EF} N^b_E N^B_F \hat{\Gamma}_{bB} 
\]

\[
+ (G^{db} + G_{EF} N^d_E N^b_F) \hat{\Gamma}_{dB} + h^{ij} Q^*_i Q^*_j \right) \frac{\partial}{\partial \hat{f}^C} E \hat{f}^C + h^{ij} Q^*_i Q^*_j \right)
\]

\[
= -\frac{1}{2} N^a_C \left( h^{ef} \hat{\Gamma}_{EF} + h^{ij} Q^*_i Q^*_j \right)
\]

(37)
Note that in \( j^n \) (the last line follows from differentiating \( \mathbf{C} \)),

Comparing (33) and (38), we conclude that

\[
j^A_j = \frac{1}{2} (\delta^A_C - N^A_C) \left( G^{EF} N^A_E N^B_F H_{A'B'} + G^{EF} N^B_F N^b_E H_{Bb} \right. \\
&+ G^{EF} N^b_E N^b_F H_{BB} + (G^{db} + G^{EF} N^d_E N^b_F) H_{db} + h^{ij} Q^C_{ij} \right) \\
= \frac{1}{2} (\delta^A_C - N^A_C) (h^{EF} H_{EF} C^A_{EF} + h^{ij} Q^C_{ij}).
\] (38)

Comparing (33) and (38), we conclude that

\[
b^A_j = -\frac{1}{2} h^{BM} \hat{H}_{BM} + j^A_j. \quad (39)
\]

Similarly, from (35) and (37) it follows that

\[
b^B_j = -\frac{1}{2} h^{BM} \hat{H}_{BM} + j^B_j. \quad (40)
\]

Since the mean curvature normal can be also defined by the Weingarten map, i.e. without using explicit coordinate expressions, \( j^i \) is the function given on a submanifold \( \mathcal{M} \), so \( j^i_j \equiv j^A_j(Q^*(x), \tilde{f}) \) and \( j^i_j \equiv j^B_j(Q^*(x), \tilde{f}). \)

In our case, this can be shown by using in (38) the identity

\[
N^A_{B,D} Q^B_i Q^D_j + N^A_B Q^B_i j^B_j = Q^A_{ij},
\]

which is derived from the identity \( N^A_B Q^B_i = Q^A_i \), together with the relation \( h^{ij} Q^A_i Q^B_j = G^{EF} N^A_E N^B_F \). As a result, we get the following representation for \( j^A_j \):

\[
j^A_j = \frac{1}{2} h^{BM} N^A_{BM} + \frac{1}{2} h^{BM} \left( H_{BM} - N^A_C H_{BM} \right). \quad (41)
\]

Note that in \( j^A_j \) not all \( N^A_{B,M} \) components are nonzero, because in our case \( N^A_6 = 0 \) and then its derivatives are also equal to zero.

It also follows from \( N^a_R Q^a R_{ij} = N^a_R N^R_{B,D} Q^B_i Q^D_j \) that

\[
j^a_j = -\frac{1}{2} N^a_C \left( h^{BM} N^C_{B,M} + h^{BM} \hat{H}_{BM} \right) \\
= \frac{1}{2} h^{CM} N^a_{C,M} - \frac{1}{2} N^a_C h^{BM} \hat{H}_{BM} \quad (42)
\]

(the last line follows from differentiating \( N^a_C N^C_B = 0 \) and taking into account \( h^{BM} N^C_B = h^{CM} \)). Then we can conclude that the resulting \( b^A_j(Q^*(x), \tilde{f}) \) and \( b^a_j(Q^*(x), \tilde{f}) \) should be used to determine the drift coefficients of the stochastic differential equation for the stochastic process on \( \mathcal{M} \), which is locally represented by the processes \( (Q^A(x(t)), \tilde{f}(t)) \).
4.4 Diffusion coefficients and stochastic differential equations

The diffusion coefficients of the equation (27) can be obtained if we first assume that the equality

\[ Q^\ast_i X^t_i dw^m_t = \hat{\mathcal{F}}^A_i dw^M_t \]

holds for some matrix \( \hat{\mathcal{F}}^A_i \). Taking the ‘square’ of the equality and using the main property of the Wiener process, according to which

\[ < dw^m_i, dw^N_t > = \delta^{mn} dt \]

and also

\[ < dw^M_i, dw^N_t > = \delta^{MN} dt \]

we find that the matrix \( \hat{\mathcal{F}}^A_i \) satisfies the local relation

\[ \sum_M \hat{\mathcal{F}}^A_M \hat{\mathcal{F}}^B_M = G^{CD} N^A_C N^B_D. \]

This can be verified by using a special representation for the projection operator

\[ N^A_B(Q^\ast(x)) = C_H^{BD} (Q^\ast(x)) Q^D_i h^{ij} Q^*_j. \]

In turn, the resulting relation allows us to define matrix \( \hat{\mathcal{F}}^A_M \) so that

\[ \hat{\mathcal{F}}^A_M = N^A_B \hat{\mathcal{F}}^C_B, \]

and hence \( \sum_M \mathcal{F}^A_M \mathcal{F}^B_M = G^{AB} \).

Note that the relations that we have used to derive the diffusion coefficients are possible only because these coefficients in stochastic differential equations with Wiener processes are determined up to orthogonal transformations.

To obtain a stochastic differential equation describing diffusion on a submanifold \( \hat{\mathcal{M}} \) in terms of the variable given on the external manifold (i.e., in our case, using dependent coordinates on charts of the external manifold), it is necessary to redefine the coordinates \( Q^\ast(x(t)) \) of local stochastic processes \( (Q^A(x(t)), \tilde{f}^a(t)) \) for new coordinates \( Q^\ast(t) \). In addition, we require that the new process, represented by the local processes \( (Q^A(t), \tilde{f}^a(t)) \), at the initial moment of time also be given on the submanifold \( \mathcal{M} \).

Thus, taking into account the new representations obtained for the drift and diffusion coefficients, our local stochastic differential equations (19) and (20) can be rewritten as follows:

\[ dQ^A(t) = \mu^2 \kappa \left( -\frac{1}{2} h^{BM} \hat{H}_B^A \hat{F}^A_{BM} + j^A_I + j^A_{II} \right) dt + \mu \sqrt{\kappa} N^A_C \mathcal{F}^C_M dw^M_t, \]

\[ d\tilde{f}^a(t) = \mu^2 \kappa \left( -\frac{1}{2} h^{BM} \hat{H}_B^a \hat{F}^a_{BM} + j^a_I + j^a_{II} \right) dt + \mu \sqrt{\kappa} \left( N^a_C \mathcal{F}^C_M dw^M_t + \mathcal{F}^a_B dw^B_t \right). \]

The terms on the right-hand sides of the above equations now depend on \( Q^\ast(t) \) and \( \tilde{f}(t) \). Also note, that in these equations we have introduced a new notation for the drift coefficients \( b_{II} \) defined in (23). Hereinafter they will be denoted as \( j_{II} \).

\(^7\)We denote new stochastic variable by the same letter.
Solutions of the local equations (13), (44), and (21) are the coordinate representatives of the local stochastic process $\zeta^{\tilde{\varphi}}(t) = (Q_t^A, \tilde{f}_t^a, a_t^0)$ given on a chart of the principal fiber bundle. The set of solutions of such equations on local charts determines the stochastic evolution family of mappings of the manifold $\tilde{\mathcal{P}}$, considered as the total space of the principal fiber bundle $\pi'$. As in [7, 8], these local evolution families of mappings generate a global stochastic evolution family, which by definition of the cited papers, is a global stochastic process $\zeta$ in the principal fiber bundle $\mathcal{P}(\mathcal{M}, \mathcal{G})$. The local stochastic process $\zeta^{\tilde{\varphi}}(t) = \tilde{\varphi}(\zeta_t)$ is a local representative of the global stochastic process $\zeta$ on the chart of the principal fiber bundle with the coordinate homeomorphism $\tilde{\varphi} = \tilde{\varphi}^{-1} \circ \varphi^{\tilde{\varphi}}$.

We also note that transition from the local stochastic processes $(Q_t^A, f_t^b)$ to the local processes $(Q_t^A, \tilde{f}_t^a, a_t^0)$, performed using the mapping $\tilde{\varphi}^{-1}$, is the phase-space transformation of the stochastic processes. It is known that such a transformation does not change the probabilities and also the transition probabilities. This allows us to rewrite the right-hand side of (8) as the expectation which is taken over the distribution of the local process $\zeta^{\tilde{\varphi}}(t)$:

$$E_{s,\zeta^{\tilde{\varphi}}(p,v)}[\phi((\varphi^{\tilde{\varphi}})^{-1}(\tilde{\varphi}(\zeta^{\tilde{\varphi}}(t))))] = E_{s,\zeta^{\tilde{\varphi}}(p,v)}[\tilde{\phi}(\zeta^{\tilde{\varphi}}(t))].$$

When deriving such a representation, it was taken into account that $\eta^{\varphi^{\tilde{\varphi}}} = \tilde{\varphi}(\zeta_t)$ and $(\varphi^{\tilde{\varphi}})^{-1} \circ \tilde{\varphi} = (\tilde{\varphi}^{-1} \circ \varphi^{\tilde{\varphi}})^{-1} = (\tilde{\varphi})^{-1}$. Moreover, we have replaced the function $\tilde{\phi}$ under the expectation sign with the function $\tilde{\phi} = \phi \circ (\varphi^{\tilde{\varphi}})^{-1}$.

The global evolution semigroup for the process $\zeta(t)$ is given by the limit of the superposition of the local semigroups associated with local stochastic processes $\zeta^{\tilde{\varphi}}(t)$:

$$\psi_{t_a}(p_a, v_a, t_a) = \lim_{t \to t_a} \left[ \tilde{U}_{\zeta^{\tilde{\varphi}}}(t_a, t_1) \cdots \tilde{U}_{\zeta^{\tilde{\varphi}}}(t_{n-1}, t_b) \tilde{\phi}_0 \right] (Q_{t_a}^*, \tilde{f}_a, \theta_a), \quad (45)$$

where the boundary values of $\zeta^{\tilde{\varphi}}(t_a) \equiv (Q_{t_a}^*, \tilde{f}_a, \theta_a)$ in the right-hand side of the equation (15) should be expressed in terms of $(p_a, v_a)$ with the help of inverse transformation $(\varphi^{\tilde{\varphi}})^{-1}$.

Note that the local semigroups $\tilde{U}_{\zeta^{\tilde{\varphi}}} \tilde{\phi}$ are now defined as

$$\tilde{U}_{\zeta^{\tilde{\varphi}}}(s, t) \tilde{\phi}(Q_t^*, \tilde{f}_t, \theta_t) = E_{s,(Q_{t_a}^*, \tilde{f}_a, \theta_a)}[\tilde{\phi}(Q^*_s, \tilde{f}(s), a(s))]$$

$$s < t, \quad Q^*_s = Q_{t_a}^*, \quad \tilde{f}(s) = \tilde{f}_a, \quad a(s) = \theta_a.$$ \quad (46)

We may consider the global stochastic process $\zeta(t)$ as consisting of two components: $\zeta(t) = \{\xi_{\tilde{\xi}}(t), a_{\tilde{a}}(t)\}$. The first process describes a special
stochastic evolution (due to a certain form of stochastic differential equations and their initial conditions) on the submanifold $\tilde{\Sigma}$ (on the gauge surface), and the second one describes the evolution on the orbits of the principal fiber bundle. Then, taking into account the potential term, the global semigroup associated with the stochastic process $\zeta(t)$ can be symbolically written in the following form:

$$
\psi_{t_0}(p_a, v_a, t_a) = E\left[ \tilde{\phi}_0(\xi_{\tilde{\Sigma}}(t_b), a_G(t_b)) \exp\left\{ \frac{1}{\mu^2 \kappa m} \int_{t_a}^{t_b} \tilde{V}(\xi_{\tilde{\Sigma}}(u)) du \right\} \right],
$$

where $\xi_{\tilde{\Sigma}}(t_a)$ is such a point on $\tilde{\Sigma}$ that has local coordinates $(Q^*_a, \tilde{f}_a)$, and the point $a_G(t_a) \in G$ has the local coordinates $\theta_a$. Also, $\tilde{\phi}(p_a, v_a) = (Q^*_a, \tilde{f}_a, \theta_a)$.

### 5 Factorization of the path integral measure

To perform the reduction procedure in the path integral, i.e. to transform the original path integral into the path integral describing the “quantum evolution” on the orbit space of the principal bundle, we must first factorize the path integral measure in the original path integral. For dynamical systems with symmetry this can be done by the method developed in our works [2,5,6,10,11], and as well as in [9]. This method is based on using the optimal nonlinear filtering stochastic differential equation from the stochastic process theory [16,17].

The equation deals with the evolution of the conditional mathematical expectation of a function that depends on both an unknown signal process (the process $a_G(t)$ in our case) and the observation process (the stochastic process $\xi_{\tilde{\Sigma}}(t)$) relative to the sub-$\sigma$-algebra generated by the observation process.

When deriving such an equation in the stochastic theory, it is assumed that the signal process $Z_t$ and the observation process $Y_t$ satisfy the following stochastic differential equations:

\[
\begin{align*}
\frac{dZ_t}{dt} &= \varphi(Y, Z, t) dt + X(Y, Z, t) \, dw_1(t) + X'(Y, Z, t) \, dw_2(t), \\
\frac{dY_t}{dt} &= \varphi_1(Y, Z, t) dt + X_1(Y, t) \, dw_2(t),
\end{align*}
\]

where $w_1(t)$ and $w_2(t)$ are independent Wiener processes.

These equations satisfy two main requirements. The first is that the diffusion coefficient $X_1$ must not depend on the process $Z_t$. And the second requirement is that in the equation for the observation process $Y_t$ there should not be a term with $dw_1(t)$. Note that in our stochastic differential
equations, the second requirement will be satisfied due to the presence of the corresponding projection operators in the diffusion coefficients.

The nonlinear filtering stochastic differential equation for the conditional mathematical expectation \( \tilde{f} = E[f(Y_t, Z_t, t)|\mathcal{Y}^t_{t_0}] \), \( \mathcal{Y}^t_{t_0} \) is the sub-\( \sigma \)-algebra generated by the observation process \( Y_t \), has the following form:

\[
d\tilde{f}(t) = E[f_t + f_z \varphi + \frac{1}{2} f_{zz}(XX^T)|\mathcal{Y}^t_{t_0}]dt + E\left[f(\varphi_1 - \hat{\varphi}_1) + f_z(XX^T)|\mathcal{Y}^t_{t_0}\right] \left(X_1X_1^T\right)^{-1}(dY_t - \hat{\varphi}_1 dt), \quad (47)
\]

where \( \hat{\varphi}_1 \equiv E[\varphi_1(Y_t, Z_t, t)|\mathcal{Y}^t_{t_0}] \).

By the properties of the conditional mathematical expectation of Markov processes, each local semigroup \([16]\) of the global semigroup \([45]\) can be represented as follows:

\[
\hat{U}_{\zeta^t_i}(s, t)\hat{\phi}(Q^t_{\zeta}, \tilde{f}(t), \theta_0) = E\left[E\left[\hat{\phi}(Q^t_{\zeta}(t), \tilde{f}(t), a(t)) \mid (\mathcal{F}_{(Q^s_{\zeta}, f)})_s\right]\right]. \quad (48)
\]

We are interested in the optimal nonlinear filtering equation for the conditional expectation of \( \hat{\phi} \) with respect to the sub-\( \sigma \)-algebra \( (\mathcal{F}_{(Q^s_{\zeta}, f)})_s \):

\[
\hat{\varphi}(Q^t_{\zeta}(t), \tilde{f}(t)) \equiv E\left[\hat{\phi}(Q^t_{\zeta}(t), \tilde{f}(t), a(t)) \mid (\mathcal{F}_{(Q^s_{\zeta}, f)})_s\right].
\]

Such an equation can be obtained from \([47]\) provided that the process \( \xi_{\tilde{\zeta}} \), which is locally determined by the processes \( Q^{sA}_{\zeta}(t) \) and \( \tilde{f}^a(t) \), should be considered as an observation process corresponding to the stochastic process \( Y_t \) in the above stochastic differential equations. Then, in our case, the local stochastic differential equation for the observation process can be written in the following form:

\[
\begin{pmatrix}
\frac{dQ^{sA}_{i(t)}}{d\tilde{f}^a_{i(t)}}
\end{pmatrix} = (\mu^2 \kappa) \begin{pmatrix} b^A \\ b^a \end{pmatrix} dt + \mu \sqrt{\kappa} \begin{pmatrix} N^A_C \mathcal{D}^C_M & 0 \\ N^a_C \mathcal{D}^C_M & \mathcal{D}^a_b \end{pmatrix} \begin{pmatrix} dw^M_i \\ dw^b_i \end{pmatrix}.
\quad (49)
\]

Note that the local stochastic process \( (Q^{sA}_{\zeta}(t), \tilde{f}^a(t)) \) is given on \( \tilde{\zeta} \) and is used to describe the stochastic evolution on the base space \( \mathcal{M} \) of the principal fiber bundle \( \mathcal{P}(\mathcal{M}, \mathcal{G}) \).

Also note that the stochastic process \( a^a(t) \) satisfying the equation

\[
da^a_i = (\mu^2 \kappa)b^a dt + \mu \sqrt{\kappa} \Lambda^a_C \tilde{v}^a_\beta \mathcal{D}^C_M dw^M_i
\]
corresponds to the signal process \( Z_t \). Therefore, in our case, instead of the coefficient \( X \) of the equation \([47]\), we should use \( \Lambda^a_C \tilde{v}^a_\beta \mathcal{D}^C_M \), and \( X_1 \) must be given by the diffusion matrix of the equation \([49]\):

\[
X_1 = \begin{pmatrix} N^A_C \mathcal{D}^C_M & 0 \\ N^a_C \mathcal{D}^C_M & \mathcal{D}^a_b \end{pmatrix}.
\]
The remaining coefficients of the equation [17], which are necessary for the derivation of the nonlinear filtering stochastic differential equation of our problem, are easy to find, and they are given by the following expressions:

\[ X \cdot X^T = G^{CD} \Lambda_\alpha^{\lambda} \Lambda_\beta^{\gamma} \tilde{v}_\alpha^{\gamma}, \]

\[ X \cdot X_1^T = \tilde{v}_\beta^{\alpha} \cdot \left( \begin{array}{cc} N_B^D \Lambda_\beta^{\gamma} G^{CC'} & N_B^D \Lambda_\beta^{\gamma} G^{CD} \\ 0 & 0 \end{array} \right), \]

\[ X_1 \cdot X_1^T = \left( \begin{array}{cc} G^{CE} N_C^B N_D^B & G^{CD} N_C^B N_D^B \\ G^{EB} N_E^D N_B & G^{AB} N_A^B + G^{ab} \end{array} \right), \]

\[ (X_1 \cdot X_1^T)^{-1} = \left( \begin{array}{cc} (P_+)_{A'} G^{H_{A'B'}} (P_+)_{B'} P_+ A' G^{H_{A'a}} \end{array} \right). \]

As a result, we get the following stochastic differential equation of the optimal nonlinear filtering:

\[
\dot{\tilde{\phi}}(Q^*(t), \tilde{f}(t)) = \mu^2 \kappa \left\{ -\frac{1}{2} \left[ d^{-1/2} H^{-1/2} \frac{\partial}{\partial Q^*} \left( d^{1/2} H^{1/2} G^{EF} N_E^A N_F^C \phi \right) \right] \\
+ \frac{1}{2} \left( G^{BC} \Lambda_\beta^{\gamma} \Lambda_\mu^{\lambda} d^{-1/2} H^{-1/2} \frac{\partial}{\partial f_b^t} \left( d^{1/2} H^{1/2} K_{ib}^b \right) \right) E [ \tilde{L}_\beta \tilde{\phi}(Q^*, \tilde{f}, a_t) (F_{Q^*}, j) ] \\
+ \frac{1}{2} \left( G^{BC} \Lambda_\beta^{\gamma} \Lambda_\mu^{\lambda} E [ \tilde{L}_\beta \tilde{\phi}(Q^*, \tilde{f}, a_t) (F_{Q^*}, j) ] \right) dt \\
+ \frac{1}{2} \left( \mu \frac{\sqrt{\bar{\kappa}}} {\Lambda_\alpha^{\lambda} \Lambda_\beta^{\gamma} E [ \tilde{L}_\beta \tilde{\phi}(Q^*, \tilde{f}, a_t) (F_{Q^*}, j) ]} \right) dw_t \\
+ \frac{1}{2} \left( \mu \frac{\sqrt{\bar{\kappa}}} {\Lambda_\alpha^{\lambda} \Lambda_\beta^{\gamma} \mu E [ \tilde{L}_\beta \tilde{\phi}(Q^*, \tilde{f}, a_t) (F_{Q^*}, j) ]} \right) dw_t^b. \]

To simplify this equation, we apply the Peter-Weyl theorem to the function \( \tilde{\phi} \), considered as a function given on a group \( G \).

According to this theorem, the function \( \tilde{\phi} \) can be represented as

\[ \tilde{\phi}(Q^*, \tilde{f}, a) = \sum_{\lambda,p,q} c_{pq}^\lambda (Q^*, \tilde{f}) D_{pq}^\lambda (a), \]

where \( D_{pq}^\lambda (a) \) are the matrix elements of an irreducible representation \( T^\lambda \) of a group \( G \): \( \sum_q D_{pq}^\lambda (a) D_{qa}^\mu (b) = D_{pa}^\mu (ab) \).

It follows from the properties of conditional mathematical expectations that

\[ E [ \tilde{\phi}(Q^*(t), \tilde{f}(t), a(t)) (F_{Q^*}, j) ] = \sum_{\lambda,p,q} c_{pq}^\lambda (Q^*(t), \tilde{f}(t)) E [ D_{pq}^\lambda (a(t)) (F_{Q^*}, j) ] \]

\[ \equiv \sum_{\lambda,p,q} c_{pq}^\lambda (Q^*(t), \tilde{f}(t)) \tilde{D}_{pq}^\lambda (Q^*(t), \tilde{f}(t)), \]

\footnote{We have now introduced a different notation for the matrix elements of an irreducible representation to distinguish them from those used previously.}
where
\[ c_{pq}^\Lambda (Q^*(t), \tilde{f}(t)) = d^\Lambda \int_G \tilde{\phi}(Q^*(t), \tilde{f}(t), \theta) \tilde{D}_{pq}^\Lambda (\theta) d\mu(\theta), \]
d^\Lambda is a dimension of an irreducible representation and \( d\mu(\theta) \) is a normalized \( (\int_G d\mu(\theta) = 1) \) invariant Haar measure on a group \( G \).

In this equation, \( D_{pq}^\Lambda \) is the multiplicative stochastic integral. This integral is defined as a limit of the sequence of time–ordered multipliers that have been obtained as a result of the piecewise breaking of the time interval \([s, t]\). They are defined as \( D_{pq}^\Lambda (a) \). We do not write out explicit expressions for the coefficients \( \Gamma_\beta \), since they can be easily obtained using the equation (50).

We do not write out explicit expressions for the coefficients \( \Gamma_1^\beta \) and \( \Gamma_2^\alpha \), since they can be easily obtained using the equation (50).

Also note that the conditional expectations \( \tilde{D}_{pq}^\Lambda (Q^*(t), \tilde{f}(t)) \) also depend on the initial points \( Q_0^\Lambda = Q^*(s), \tilde{f}_0^a = \tilde{f}^a(s) \) and \( \theta_0^a = a^a(s) \). To avoid awkward notation, we have omitted this dependence in \( \tilde{D}_{pq}^\Lambda \).

The solution of the linear matrix stochastic differential equation (51) can be written \([18,19]\) as
\[ \tilde{D}_{pq}^\Lambda (Q^*(t), \tilde{f}(t)) = (\tilde{\exp})_{pq}^\Lambda (Q^*(t), \tilde{f}(t), t, s) E[D_{pq}^\Lambda (a(s)) | (F_{Q^*})^t_{s}], \]
where
\[ (\tilde{\exp})_{pq}^\Lambda (Q^*(t), \tilde{f}(t), t, s) = \tilde{\exp} \int_s^t \left\{ \mu^2 \kappa \left[ \frac{1}{2} \frac{\partial^\alpha \theta(Q^*(u), \tilde{f}(u))(J_\alpha)_{pq}^\Lambda (J_\nu)_{rn}^\Lambda}{\partial Q^* A} \right] \right. \]
\[ \left. - \frac{1}{2} \sqrt{dH} G^{EF}_E b^{EF}_E \left( \sqrt{dH} G^{EF}_E b^{EF}_E (Q^*(u)) \right) (J_\alpha)_{pq}^\Lambda \right. \]
\[ - \frac{1}{2} \left( G^{EC}_E b^{EC}_E \right) \frac{\partial}{\partial Q^* A} \left( \sqrt{dH} K^{\Lambda^\mu}_{pq} \right) (J_\nu)_{pq}^\Lambda \]
\[ \left. + \mu \sqrt{R} \Lambda^\beta C (Q^*(u))(J_\beta)_{pq}^\Lambda \left[ \tilde{\Pi}_E^{\text{aC}} F_E^{\text{aC}} (Q^*(u)) dw^E (u) + \tilde{\Pi}_a^{\text{aC}} F_a^{\text{aC}} dw^a (u) \right] \right\} \]
\[ (H, d, \tilde{\Pi}_E^{\text{aC}}, \tilde{\Pi}_a^{\text{aC}} \) depend on \( Q^*(u) \) and \( \tilde{f}(u) \).

is the multiplicative stochastic integral. This integral is defined as a limit of the sequence of time–ordered multipliers that have been obtained as a result of the piecewise breaking of the time interval \([s, t] \), \( s = t_0 \leq t_1 \ldots \leq t_n = t \).
In (53), the time order of these multipliers is indicated by the arrow directed to the multipliers given at greater times.

Using the solution of the matrix stochastic differential equation (51) defined by (52) and (53), the local semigroup (48) can be represented as follows:

\[
\hat{U}_{\xi^p}(s, t) = \sum_{\lambda', p, q, q'} \mathbb{E}\left[ c_{pq}^\lambda(Q^*(t), \tilde{f}(t))(\hat{\exp})_{pq}^\lambda(Q^*(t), \tilde{f}(t), t, s)\right] D^\lambda_{q'q}(\theta_0). \tag{54}
\]

Note that in this representation, the last multiplier on the right is due to

\[
\mathbb{E}[D_{na}^\lambda(a(s)) | (\mathcal{F}_{Q^*})_s] = D_{na}^\lambda(\theta_0).
\]

The global evolution semigroup (45) is now obtained in accordance with \[7,8\] as the limit (under the refinement of the subdivision of the time interval \([t_a, t_b]\)) of the superposition of the local semigroups that are similar to (52).

We will write the global semigroup in the following symbolic form:

\[
\hat{\psi}_{t_b}(p_a, v_a, t_a) = \sum_{\lambda', p, q, q'} \mathbb{E}\left[ c_{pq}^\lambda(\xi_{\Sigma}(t_b))(\hat{\exp})_{pq}^\lambda(\xi_{\Sigma}(t), t_b, t_a)\right] D^\lambda_{q'q}(\theta_a),
\]

\[
(\xi_{\Sigma}(t_a) = \pi'_{\Sigma}(p_a, v_a)), \tag{55}
\]

where the global process \(\xi_{\Sigma}(t) = (\xi_1(t), \xi_2(t))\) is defined on the submanifold \(\tilde{\Sigma}\). The process \(\xi_{\Sigma}(t)\) is described locally by the stochastic equations (49).

Thus, we have obtained that our initial path integral (3) is represented as the sum of the matrix semigroups (the path integrals) given on the submanifold \(\tilde{\Sigma}\). The coordinate representation of the differential generator (the Hamilton operator) of these matrix semigroups is

\[
\frac{1}{2} \left\{ \left[ h^{AB} \frac{\partial^2}{\partial Q^A \partial Q^B} + 2h^{Aa} \frac{\partial^2}{\partial Q^A \partial f^a} + h^{ab} \frac{\partial^2}{\partial f^a \partial f^b} - (h^{BM} \Gamma^A_{BM} - 2(j^A_I + j^A_{II})) \frac{\partial}{\partial Q^A} - (h^{BM} \Gamma^a_{BM} - 2(j^a_I + j^a_{II})) \frac{\partial}{\partial f^a} \right] (I)^\lambda_{pm} \right.
\]

\[
+ 2N^{AB}_C G^{EB} \Lambda^C_j^\beta (J^j_B)_{pm} \frac{\partial}{\partial Q^A} - 2d^{\nu} K^{\beta}_{\nu} (J^j_B)_{pm} \frac{\partial}{\partial f^b} - \left( \frac{1}{\sqrt{dH}} \frac{\partial}{\partial Q^A} \left( \sqrt{dH} h^{AC} \gamma^C_{(\cdot)} \right) + G^{EC} \Lambda^E_j^\alpha (J^j_C) \frac{1}{\sqrt{dH}} \frac{\partial}{\partial f^b} \left( \sqrt{dH} K^{\beta}_{\nu} \right) (J^j_B)_{pm} \right) \left( I^\lambda_{pm} \right).
\]

\[
\right.
\]

\[
+ G^{CD} \Lambda^C_j^\alpha (J^j_B)_{pq} (J^j_B)_{qn} \right\}, \tag{56}
\]

where \((I^\lambda)_{pq}\) is a unity matrix.
The operator (56) acts in the space of the sections \( \Gamma(\tilde{\Sigma}, V^*) \) of the associated covector bundle (we consider the backward Kolmogorov equation). The scalar product in the space of the sections of the associated co-vector bundle is given by
\[
(\psi_n, \psi_m) = \int_{\Sigma} \langle \psi_n, \psi_m \rangle_{V^*} d^{1/2} dv_{\Sigma},
\]
where \( d = \det(d_{\alpha\beta}) \) and \( dv_{\Sigma} \) is the Riemannian volume element on the submanifold \( \tilde{\Sigma} \). In local coordinates, it can be presented as
\[
dv_{\Sigma}(Q^*, \tilde{f}) = H^{1/2}(Q^*, \tilde{f}) dQ^1 ... dQ^n d\tilde{f}^1 ... \tilde{f}^n.
\]
To express the matrix semigroup under the sign of sum in (55) in terms of the original semigroup defined on \( \tilde{\mathcal{P}} \), it is necessary to inverse this equality. In our previous papers [2,6], such an inversion of the analogous equality was made for the kernels of the corresponding local semigroups. To do this, it was assumed that all the necessary restrictions for the existence of semigroup kernels were fulfilled. This method is applicable to the case we are now considering, if we take into account local isomorphism of the principal bundle \( P(\tilde{\Sigma}, G) \) with the trivial bundle \( \tilde{\Sigma} \times G \rightarrow \tilde{\Sigma} \). This isomorphism, as shown in [4], leads to the existence of the relationship between the charts of these principal fiber bundles. As a result, it becomes possible to reverse the relation between local kernels (local Green’s functions) which are defined on charts of the principal bundle \( P(\tilde{\Sigma}, G) \).

If the global semigroup on the left-hand side of (57) can be presented as
\[
\psi_b(p_a, v_a, t_a) = \int G_{\tilde{\mathcal{P}}}(p_b, v_b, t_b; p_a, v_a, t_a) \phi_0(p_b, v_b) dv_{\tilde{\mathcal{P}}}(p_b, v_b),
\]
then comparing the local expression of the right-hand side of (55) (i.e. what is obtained on the charts of the principal fiber bundle \( P(\tilde{\Sigma}, G) \)) with the local expression \( G_{\tilde{\mathcal{P}}} \) given on the corresponding charts from the atlas of the manifold \( \tilde{\mathcal{P}} \) (with account of a local isomorphism of the bundles), one can find the following relation between the local Green’s functions:
\[
\int_{\tilde{\mathcal{P}}} G_{\tilde{\mathcal{P}}}(\alpha_b, F(Q^*_b, \theta_b), \tilde{D}(\theta_b) \tilde{f}_b, t_b; \beta_a, F(Q^*_a, \theta_a), \tilde{D}(\theta_a) \tilde{f}_a, t_a) \\
\times D^\lambda_{pq}(\theta_b) d\mu(\theta_b) = \sum_{q'} G_{q'p}^\lambda(\alpha_b, Q^*_b, \tilde{f}_b, t_b; \beta_a, Q^*_a, \tilde{f}_a, t_a) D^\lambda_{q'q}(\theta_a),
\]
where \( \alpha_b \) and \( \beta_a \) are the labels of the charts. Note that to obtain the above relation, \( \phi_0(p_b, v_b) \) in (58) was also expanded into a series using the Peter-Weyl theorem. This was done after introducing on \( \tilde{\mathcal{P}} \) the bundle coordinates.
The obtained relation between the local Green’s functions can be reversed:

\[
G^\lambda_{mn}(\alpha_b, Q^*_b, \tilde{f}_b, t_b; \beta_a, Q^*_a, \tilde{f}_a, t_a) = \int_{\mathcal{G}} G_{\tilde{p}}(\alpha_b, Q^*_b, \tilde{f}_b, \theta, t_b; \beta_a, Q^*_a, \tilde{f}_a, e, t_a) D^\lambda_{nm}(\theta)d\mu(\theta),
\]

where \(e\) is the unity element of the group \(\mathcal{G}\). Also note that there is the following invariance property:

\[
G_{\tilde{p}}(\alpha_b, Q^*_b, \tilde{f}_b, \theta, t_b; \beta_a, Q^*_a, \tilde{f}_a, \theta, t_a) \equiv G_{\tilde{p}}(\alpha_b, F(Q^*_b, \theta), \tilde{D}(\theta)\tilde{f}_b, t_b; \beta_a, F(Q^*_a, \theta), \tilde{D}(\theta)\tilde{f}_a, t_a).
\]

To extend the resulting equality (59) from local charts to the whole manifold, we need to glue these local Green’s functions. In case of the trivial principal fiber bundle this can be done using the transition coordinate functions from the manifold atlas. As a result, we obtain the global integral relation between the Green’s functions:

\[
G^\lambda_{mn}(\pi'_{(\Sigma)}(p_b, v_b), t_b; \pi'_{(\Sigma)}(p_a, v_a), t_a) = \int_{\mathcal{G}} G_{\tilde{p}}(p_b\theta, v_b\theta, t_b; p_a, v_a, t_a) D^\lambda_{nm}(\theta)d\mu(\theta),
\]

The path integral for the Green’s function \(G^\lambda_{mn}\) can be written symbolically as

\[
G^\lambda_{mn}(\pi'_{(\Sigma)}(p_b, v_b), t_b; \pi'_{(\Sigma)}(p_a, v_a), t_a) = \\
\tilde{E}_{\xi_b(t_b)=\pi'_{(\Sigma)}(p_a, v_a)}\left[\left(\xi_b \exp^{\alpha_b} G^\lambda_{mn}(\xi_b(t), t_b, t_a) \exp\left\{-\frac{1}{\mu^2 km} \int_{t_a}^{t_b} \tilde{V}(\xi_1(u), \xi_2(u))du\right\}\right] \\
= \int_{\xi_b(t_b)=\pi'_{(\Sigma)}(p_a, v_a)} d\xi_b \exp\left\{-\frac{1}{\mu^2 km} \int_{t_a}^{t_b} \tilde{V}(\xi_b(u))du\right\} \\
\times \exp \int_{t_a}^{t_b} \left\{\mu^2 \kappa \left[\frac{1}{2} \partial^\nu (\xi_b(u))(J_\lambda^\nu (J_\nu)^\lambda)_{mn} - \frac{1}{2} \sqrt{d H} \frac{\partial}{\partial Q^A_{\nu}} \left(\sqrt{d H} G^{EF}_{\rho} N^C_F N^C_{(\gamma)} (J_\nu)^\lambda\right) (J_\nu)^\lambda_{mn} - \frac{1}{2} (G^{EC} \kappa E^\nu) \frac{\partial}{\partial f^b} \left(\sqrt{d H} K^b_{\mu} (J_\nu)^\lambda\right) du \\
- \mu \sqrt{\kappa} (J_\beta)^\lambda_{mn} \left[\tilde{\Pi}_b^C \tilde{\mathcal{D}}_M^E (\xi_b(u))dw^M(u) + \tilde{\Pi}_a \mathcal{D}_b^\alpha dw^b(u)\right]\right\}.
\]

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The semigroup with this kernel acts in the space of the equivariant functions given on $\tilde{P}$:
$$\tilde{\psi}_n(pg, vg) = D_{mn}^\lambda(g)\tilde{\psi}_m(p, v).$$

In local coordinates, the isomorphism of these functions with the functions $\psi_n \in \Gamma(\Sigma, V^*)$ is represented as follows:
$$\tilde{\psi}_n(F(Q^*, e), \tilde{D}_c^b(e)\tilde{f}^c) = \psi_n(Q^*, \tilde{f}).$$

The semigroup with the Green’s function $G^\lambda_{mn}$ is used to describe the “quantum evolution” of the reduced dynamical system on the manifold $\tilde{M}$ – the orbit space of the principal fiber bundle. This path integral reduction corresponds to the case of the reduction of a mechanical system with symmetry to a nonzero momentum level [12].

Note that in our case, in order to describe the evolution on the orbit space $\tilde{M}$ we use an additional surface $\tilde{\Sigma}$ (‘gauge surface’) on which diffusion is given locally by the solution of the local stochastic differential equation (49). In fact, this equation is represented by two equations: (43) and (44). We see that in both of these equations, in the drift coefficients, there are terms that we denoted as $j_{I}$ and $j_{II}$. The presence of $j_I$-terms in the drift coefficients of our stochastic equation on $\tilde{\Sigma}$ is essential for the correct description of the stochastic process on the manifold $\tilde{\Sigma}$, considered as a submanifold in the ambient manifold with the horizontal metric. On the other hand, the $j_{II}$-terms in the drift coefficients of stochastic differential equations are not necessary.

Transformation of a measure in a path integral by replacing the stochastic process $\xi_{\tilde{\Sigma}}$, determined by solutions of local stochastic differential equations (49), with the process $\tilde{\xi}_{\tilde{\Sigma}}$ described by the same local stochastic equations, but without the $j_{II}$ terms\footnote{For the components of the local stochastic process representing the global stochastic process $\xi_{\tilde{\Sigma}}(t)$ on charts, we will use the same notation as before, namely $(Q^*, \tilde{f})$.} can be performed using the Girsanov-Cameron-Martin transformation.

In our article, we consider such a transformation for the reduction onto a zero-momentum level. In this case, i.e. when $\lambda = 0$, instead of (60), we get the integral relation between the scalar Green’s functions (semigroup kernels) acting in the spaces of scalar functions given on $\tilde{P}$ and on $\tilde{\Sigma}$.

Note, that the standard Girsanov-Cameron-Martin formula for the path integral measure transformation deals with the case when the matrix representing diffusion coefficient of the stochastic differential equation is nondegenerate. The elements of such a matrix in (49) are defined using projection operators. Therefore, this matrix is degenerate. In spite of this, an analogue
of the Girsanov-Cameron-Martin formula can also be derived in our case. Its derivation (using the Itô's differentiation formula) follows from the assumption that the solution of a parabolic differential equation with the differential operator \( (16) \) (taken at \( \lambda = 0 \)) is unique (modulo those ambiguities that we have in the problem under consideration).

The Radon-Nikodym derivative of the measure \( \mu_{\tilde{\xi}} \) with respect to the measure \( \tilde{\mu}_{\tilde{\xi}} \) is defined as

\[
\frac{d\mu_{\tilde{\xi}}}{d\tilde{\mu}_{\tilde{\xi}}}(\tilde{\xi}(t)) = \exp \int_t^t \left[ \mu^2 \kappa < A^{-1} \tilde{j}_{II}, d\tilde{w}_s \right] \left. - \frac{1}{2} \right| \mu^2 \kappa ||A^{-1} \tilde{j}_{II}||^2 ds \],
\]

where \( A^{-1} \) is the matrix which is (pseudo)inverse to the ‘diffusion’ matrix \( X_1 \) of the local stochastic differential equation \( (19) \), the Wiener process \( \tilde{w}_s \) consists of two independent Wiener processes: \( \tilde{w}_s = (w_s^M, w_s^b) \), and \( \tilde{j}_{II} \) is the projection of the mean curvature vector field of the orbit onto the submanifold \( \tilde{\Sigma} \). It has the following components:

\[
j_{\tilde{A}} = -\frac{1}{2} G^{CC'} N^A_C N^B_C G_{BB'} d_{\alpha\beta} \left( \nabla_{K_{alpha}} K_{beta} \right)^B, \\
j_{\tilde{a}} = -\frac{1}{2} d_{\alpha\beta} \left( N^a_B \left( \nabla_{K_{alpha}} K_{beta} \right)^B + \left( \nabla_{K_{alpha}} K_{beta} \right)^a \right).
\]

To get an explicit representation of \( A^{-1} \tilde{j}_{II} \) in \( (62) \), we first rewrite the terms on the right-hand side of the equations \( (63) \) using the following identities:

\[
d_{\alpha\beta} \left( \nabla_{K_{alpha}} K_{beta} \right)^B = -\frac{1}{2} \left( G^{BC} N^A_C \sigma_A + G^{BC} N^a_C \sigma_a \right), \\
d_{\alpha\beta} \left( \nabla_{K_{alpha}} K_{beta} \right)^a = -\frac{1}{2} G^{a\alpha} \sigma_q,
\]

where \( \sigma_A = d_{\alpha\beta} \frac{\partial}{\partial Q^A} d_{\alpha\beta} \equiv \frac{\partial}{\partial Q^A} \ln d \) and \( \sigma_a = \frac{\partial}{\partial \tilde{f}^a} \ln d, d = \det d_{\alpha\beta} \).

As a result, we obtain

\[
\left( \begin{array}{c} j_{\tilde{A}} \\ j_{\tilde{a}} \end{array} \right) = \frac{1}{4} \left( \begin{array}{cc} G^{BD} N^A_B N^C_D & G^{BD} N^a_B N^b_D \\ G^{BD} N^b_B N^a_D & G^{ab} \end{array} \right) \left( \begin{array}{c} \sigma_C \\ \sigma_b \end{array} \right),
\]

\[
= \frac{1}{4} \left( X_1 \cdot X_1^\top \right) \left( \begin{array}{c} \sigma_C \\ \sigma_b \end{array} \right).
\]

In turn, this means that

\[ A^{-1} \tilde{j}_{II} = X_1^\top \cdot \tilde{j}_{II} = \frac{1}{4} \left( \begin{array}{cc} N^A_C \tilde{\sigma}_C & N^b_B \tilde{\sigma}_b \\ 0 & \tilde{\sigma}_b \end{array} \right) \left( \begin{array}{c} \sigma_C \\ \sigma_b \end{array} \right). \]
Thus, the terms under the integral in (62) have the following form:

\[
\langle A^{-1} \cdot \vec{j}_{II}, d\vec{w}_s \rangle = \frac{1}{4} [ (N_C^A \mathcal{A}_M^C \sigma_A + N_D^a \mathcal{A}_M^D \sigma_a) dw_s^M + \mathcal{A}_b^a dw_s^b ]
\]

and

\[
||A^{-1} \cdot \vec{j}_{II}||^2
= \frac{1}{16} \left( G^{CD} N_C^A N_D^B \sigma_A \sigma_B + 2 G^{CD} N_C^a N_D^b \sigma_a \sigma_B + (G^{CD} N_C^a N_D^b + G^{ab}) \sigma_a \sigma_b \right)
= \frac{1}{16} \left[ h^{AB} \sigma_A \sigma_B + 2 h^{aB} \sigma_a \sigma_B + h^{ab} \sigma_a \sigma_b \right].
\]

For brevity, the quadratic form in square brackets of the above expression will be further denoted as \( < \partial \sigma, \partial \sigma > _{\tilde{\Sigma}} \).

The argument of the exponential on the right-hand side of the equation (62) is represented by the sum of the stochastic and ordinary integrals. This exponential function can be rewritten to include only ordinary integrals. This can be done using the special identity (known as the Itô’s identity).

In our case, the identity is obtained from the solution of a local stochastic differential equation having \( \exp \sigma(Q^* t_A, \tilde{f}(t)) \) as an unknown function. The equation itself is derived by taking the stochastic Itô differential of this function, provided that now the components \( Q^* t_A \) and \( \tilde{f}(t) \) of the local stochastic process representing the global stochastic process \( \tilde{\xi}_{\tilde{\Sigma}}(t) \) must satisfy the stochastic differential equations (43) and (44) without \( j_{II} \) terms.

Performing calculations, one can find the following identity:

\[
\exp \int_{t_a}^t (\mu \sqrt{\kappa}) \left[ (N_C^A \mathcal{A}_M^C \sigma_A + N_D^a \mathcal{A}_M^D \sigma_a) dw_s^M + \mathcal{A}_b^a dw_s^b \right]
= \left( \frac{\exp(\sigma(Q^*(t), \tilde{f}(t)))}{\exp(\sigma(Q^*(t_a), \tilde{f}(t_a)))} \right) \exp \left\{ \int_{t_a}^t \left( \frac{1}{2} (\mu^2 \kappa) \left[ h^{AB} \sigma_{AB} + 2 h^{aB} \sigma_{Ab} + h^{ab} \sigma_{ab} \right)
- \left( h^{BM} H^{BM} - 2 j_{II}^A \right) \sigma_A \right) \right\} ds \right\}.
\]

From this identity for local representatives of the stochastic process \( \hat{\xi}_{\hat{\Sigma}}(t) \) it follows that (62) can be rewritten as

\[
\frac{d\mu^{\xi_{\Sigma}}}{d\mu^{\xi_{\Sigma}}}(\xi_{\Sigma}(t)) = \left( \frac{\exp(\sigma(\xi_{\Sigma}(t)))}{\exp(\sigma(\xi_{\Sigma}(t_a)))} \right)^{1/4} \times \exp \left\{ \int_{t_a}^t \left( \Delta_{\Sigma} \sigma + \frac{1}{4} < \partial \sigma, \partial \sigma >_{\Sigma} \right) ds \right\}, \quad (65)
\]
where by $\tilde{\Delta}_\Sigma \sigma$ we denote the expression in the square brackets on the right-hand side of the previous formula. We see that in addition to the standard differential expression used to define the Laplace-Beltrami operator, it includes two more terms: $2 j_f^A \sigma_A$ and $2 j_f^a \sigma_a$. It can be shown that the sum of these terms is equal to zero.

Using the representations (41) and (42) for $j_f^A$ and $j_f^a$, we rewrite this sum as follows:

$$h^{BM}(N_{B,M}^A \sigma_A + N_{B,M}^a \sigma_a) + (K^A_\alpha \sigma_A + K^a_\alpha \sigma_a) \Lambda^\alpha_C h^{BM} \tilde{H}^C_{BM}.$$  

To prove that the expression in the second bracket of the above expression is equal to zero, one must first apply the operator $K^A_\alpha \partial_A + K^a_\alpha \partial_a$ to the metric $d_{\mu\nu} = K^C_\mu G_{CD} K^D_\nu + K^p_\mu G_{pq} K^q_\nu$ and then transform the resulting expression using Killing’s relations from Appendix. This leads to

$$K^A_\alpha \partial_A d_{\mu\nu} + K^a_\alpha \partial_a d_{\mu\nu} = d_{\nu\sigma} c^\sigma_{\alpha\mu} + d_{\mu\sigma} c^\sigma_{\alpha\nu}.$$  

By multiplying both sides of this equality by $d_{\mu\nu}$, we can conclude that the right-hand side of the equality is equal to zero for semisimple Lie groups.

The expression in the first bracket of the equality under study is also equal to zero. This can be shown using the following representations:

$$N^A_{B,M} = -K^A_{\alpha,M} \Lambda^\alpha_B - K^A_{\alpha} \Lambda^\alpha_{B,M} \text{ and } N^a_{B,M} = -K^a_{\alpha} \Lambda^\alpha_{B,M},$$

and taking into account that $h^{BM} \Lambda^\alpha_B = 0$ and $K^A_\alpha \sigma_A + K^a_\alpha \sigma_a = 0$.

The global integral relation between the Green’s functions for the reduction onto the zero-momentum level is given by

$$d_b^{-1/4}d_a^{-1/4}G_{\Sigma}(\pi'_{\Sigma}(p_b, v_b), t_b; \pi'_{\Sigma}(p_a, v_a), t_a) = \int_G G_{\tilde{\Sigma}}(p_\theta, v_\theta, t_b; p_a, v_a, t_a) d\mu(\theta),$$

where $d_b$ and $d_a$ are the values of the det($d_{\alpha\beta}$) taken at the points $\pi'_{\Sigma}(p_b, v_b)$ and $\pi'_{\Sigma}(p_a, v_a)$. The Green’s function $G_{\Sigma}$ is presented by the following path integral:

$$G_{\Sigma}(\pi'_{\Sigma}(p_b, v_b), t_b; \pi'_{\Sigma}(p_a, v_a), t_a) = \int \exp \left\{ \frac{1}{\mu^2 \kappa m} \int_{t_a}^{t_b} \left[ \hat{V}(\xi_{\Sigma}(u)) + J(\sigma(\xi_{\Sigma}(u))) \right] du \right\},$$

$$\xi_{\Sigma}(t_a) = \pi'_{\Sigma}(p_a, v_a) \quad \xi_{\Sigma}(t_b) = \pi'_{\Sigma}(p_b, v_b)$$
where \( J \) is the integrand in the reduction Jacobian \(^{65}\):

\[
J = -\frac{1}{8}\mu^2\kappa \left( \Delta_{\Sigma}^H \sigma + \frac{1}{4} < \partial \sigma, \partial \sigma >_{\Sigma} \right).
\]

(66)

In this expression,

\[
\Delta_{\Sigma}^H \sigma = h^{AB} \sigma_{AB} + 2h^{Ab} \sigma_{Ab} - h^{B\tilde{M}} \tilde{H}_{B\tilde{M}} \sigma_A - h^{\tilde{B}\tilde{M}} \tilde{H}_{\tilde{B}\tilde{M}} \sigma_a.
\]

The global semigroup determined by the Green’s function \( G_{\Sigma} \) acts in the Hilbert space of the scalar functions on \( \tilde{\Sigma} \) with the following scalar product

\[
(\psi_1, \psi_2) = \int \psi_1 \psi_2 \, dv_{\tilde{\Sigma}}.
\]

The Green’s function \( G_{\Sigma} \) satisfies the forward Kolmogorov equation with the operator

\[
\hat{H}_\kappa = \frac{\hbar}{2m} \Delta_{\Sigma} - \frac{\hbar \kappa}{8m} \left[ \Delta_{\Sigma}^H \sigma + \frac{1}{4} < \partial \sigma, \partial \sigma >_{\Sigma} \right] + \frac{1}{\hbar \kappa} \tilde{V}.
\]

Note that at \( \kappa = i \) the forward Kolmogorov equation becomes the Schrödinger equation with the Hamilton operator \( \hat{H} = -\frac{i}{\hbar} \hat{H}_\kappa \big|_{\kappa = i} \).

Note also that the geometric properties of the operator \( \tilde{\Delta}_{\Sigma} \),

\[
\tilde{\Delta}_{\Sigma} = \Delta_{\Sigma}^H + 2j^A_I \partial_A + 2j^a_I \partial_a,
\]

as well as the fact that this operator could be transformed into the Laplace-Beltrami operator given on the orbit space \( \tilde{\mathcal{M}} \) with the Riemannian metric \(^{26}\), if in charts of the reduced manifold it were possible to find invariant local coordinates \((x^i, \tilde{f}^a)\), such that \( \chi^a(Q^*(x^i)) \equiv 0 \), lead us to conclusion that the global semigroup with the kernel \( G_{\Sigma} \) can be used to describe the diffusion of interacting scalar particles on the orbit space \( \mathcal{M} \) of the principal fiber bundle \( P(\mathcal{M}, \mathcal{G}) \).

6 Conclusion

In this article we have considered the reduction procedure in the Wiener-type path integral which represents the “quantum evolution” of a special finite-dimensional mechanical system with symmetry, consisting of two interacting scalar particles. The choice of such a system for study is due to its geometric properties, similar to the properties of those gauge theories that are used to describe the interaction of Yang-Mills fields with scalar fields.

Note that in order to apply the standard approach to the quantization of gauge theories, the original gauge fields must first be rewritten in terms of the constrained variables. In accordance with this, in the principal fiber
bundle associated with our problem, a special system of local coordinates (including dependent coordinates) was introduced. The coordinates of this coordinate system are defined using the local surface $\tilde{\Sigma}$ (a submanifold of the total space of the principal fiber bundle $P(\tilde{M}, G)$). And the reduced motion on the orbit space $\tilde{M}$ is locally described in terms of these coordinates.

The reduction procedure performed in the article leads to the integral relation (60) between Green’s functions $G_{mn}^\lambda$ (the kernel of a reduced semigroup acting in the space of sections $\Gamma(\tilde{\Sigma}, V^*)$ for the general case of reduction onto the nonzero momentum level, $\lambda \neq 0$) and $G_{\tilde{P}}$, which is the kernel of the original semigroup. The representation of $G_{mn}^\lambda$ in terms of the path integral is given by the expression (61).

The result obtained essentially depends on what assumption about the submanifold $\tilde{\Sigma}$ was made. Provided that the submanifold $\tilde{\Sigma}$ is a global section of the principal fiber bundle $P(\tilde{M}, G)$ (which is thus a trivial principal bundle isomorphic to the trivial bundle $P(\tilde{\Sigma}, G)$), then in this case it follows that the dependent coordinates $Q^* A$ (and the coordinates $\tilde{f}^a$) can be thought of as global variables defined on $\tilde{\Sigma}$. This case is analogous to what is usually assumed in gauge theories, when the evolution of the reduced system on the orbit space of the gauge group is described in terms of variables with constraints.

However, in the most cases, the principal fiber bundles of gauge theories are nontrivial. Therefore, using this method, we can actually describe the ‘quantum’ dynamics on the orbit space only locally. Note that the problem of transition from a local description of the reduced dynamics given in terms of dependent coordinates to a global description is not solved yet.

The main result of the article is the obtained expression (66) for the integrand $J$ in the path integral reduction Jacobian for the case of reduction onto a zero-momentum level. In particular, it is shown that $J$ is generated by the projection of the mean curvature vector field of the orbit onto the submanifold $\tilde{\Sigma}$.

In addition, we note that the resulting $J$ is a generalization to interacting dynamical systems with symmetry of a similar expression obtained by J. Lott in [22] while studying the quantum potential in pure Yang-Mills theory.

The Girsanov-Cameron-Martin transformation can also be performed in the path integral (61). In this case, the exponential with the integral of $J$, defined by the formula (66), will be the diagonal part of the reduction Jacobian.
Appendix

Relationship between the Christoffel symbols

\[
\Gamma^a_{\, kn} = \tilde{h}^{an} \Gamma_{\, knm} + \tilde{h}^{ab} \Gamma_{\, knb}
\]
\[
\Gamma^a_{\, kb} = \tilde{h}^{an} \Gamma_{\, knb} + \tilde{h}^{ab} \Gamma_{\, kbn}
\]
\[
\Gamma^a_{\, bn} = \tilde{h}^{an} \Gamma_{\, bmn} + \tilde{h}^{ac} \Gamma_{\, bmc}
\]
\[
\Gamma^a_{\, bc} = \tilde{h}^{an} \Gamma_{\, bcn} + \tilde{h}^{ad} \Gamma_{\, bcd}
\]
\[
\Gamma_{\, knb} = H \tilde{\Gamma}_{\, BMb} Q^B_n Q^A_m + \tilde{G}_{\, Ab} Q^A_k Q^B_n
\]
\[
\Gamma_{\, kbn} = H \tilde{\Gamma}_{\, AbB} Q^A_m Q^B_n
\]
\[
\Gamma_{\, bmn} = H \tilde{\Gamma}_{\, bAB} Q^A_m Q^B_n
\]
\[
\Gamma_{\, bmc} = H \tilde{\Gamma}_{\, bAc} Q^A_m
\]
\[
H \tilde{\Gamma}_{\, AbB} = \tilde{G}_{\, Ab} H \tilde{\Gamma}_{\, BMb}
\]
\[
H \tilde{\Gamma}_{\, Ac} = \tilde{G}_{\, Ac} H \tilde{\Gamma}_{\, BAb}
\]
\[
H \tilde{\Gamma}_{\, bAB} = \tilde{G}_{\, bAB} H \tilde{\Gamma}_{\, bAb}
\]

Killing relations

\[
K^A_{\, \alpha} G_{CD, \alpha} = -G_{CR} K^R_{\, \alpha, D} - G_{RD} K^R_{\, \alpha, C},
\]
\[
- G_{q^b} K_{\alpha, a} - G_{ab} K_{\alpha, q} = 0.
\]

Properties of the horizontal metric \( \tilde{G}_{\, AB}^H \)

\[
N^a_E \tilde{G}_{\, RF}^H + N^b_F \tilde{G}_{\, Rb}^H = \tilde{G}_{\, RF}^H,
\]
\[
G^{ab} \tilde{G}_{\, Ab}^H = \tilde{\Pi}^a_A, \quad G^{EF} \tilde{G}_{\, AF}^H = \tilde{\Pi}^E_A, \quad N^a_E \tilde{\Pi}^E_A + \tilde{\Pi}^a_A = N^a_A.
\]
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