LAGRANGIAN MEAN CURVATURE FLOW IN PSEUDO-EUCLIDEAN SPACE

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ABSTRACT. We establish the longtime existence and convergence results of the mean curvature flow of entire Lagrangian graphs in Pseudo-Euclidean space which is related to Logarithmic gradient flow.

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1. INTRODUCTION

The mean curvature flow in high codimension has been studied extensively in the last few years (cf. [1], [2], [3], [4], [5], [6], [7]). In this paper we consider the Lagrangian mean curvature flow in Pseudo-Euclidean space.

Let $\mathbb{R}^{2n}$ be an $2n$-dimensional Pseudo-Euclidean space with the index $n$. The indefinite flat metric on $\mathbb{R}^{2n}$ (cf. [8]) is defined by

$$ds^2 = \sum_{i=1}^{n} (dx^i)^2 - \sum_{\alpha=n+1}^{2n} (dx^\alpha)^2$$

Consider the logarithmic gradient flow (cf. [9]) on $\mathbb{R}^n$:

$$\partial u \partial t - \frac{1}{n} \ln \det D^2 u = 0, \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$u = u_0(x), \quad t = 0, \quad x \in \mathbb{R}^n.$$

By Proposition 2.1 there exist a family of diffeomorphisms

$$r_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

such that

$$F(x, t) = (r_t, Du(r_t, t)) \subset \mathbb{R}^{2n}_n$$

is a solution to the mean curvature flow of a complete space-like submanifold in pseudo-Euclidean space

$$\left\{ \begin{array}{l} \frac{dF}{dt} = \vec{H}, \\ F(x, 0) = F_0(x), \end{array} \right.$$ 

where $\vec{H}$ is the mean curvature vector of the submanifold $F(x, t) \subset \mathbb{R}^{2n}_n$ at $F(x, t)$ with

$$F_0(x) = (x, Du_0(x)).$$
Definition 1.1. Assume function $u_0(x) \in C^2(\mathbb{R}^n)$. We call $u_0(x)$ satisfying

(i) Condition A if

$$u_0(x) = \frac{u_0(Rx)}{R^2}, \ \forall R > 0.$$

(ii) Condition B if

$$\Lambda I \geq D^2u_0(x) \geq \lambda I, \quad x \in \mathbb{R}^n,$$

where $\Lambda, \lambda$ are positive constants and $I$ is the identity matrix.

We now state the main theorems of this paper.

Theorem 1.2. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function which satisfies Condition B. Then there exists a unique strictly convex solution of (1.1) such that

$$u(x, t) \in C^\infty(\mathbb{R}^n \times (0, +\infty)) \cap C(\mathbb{R}^n \times [0, +\infty)),$$

where $u(\cdot, t)$ satisfies Condition B. More generally, for $l = \{3, 4, 5, \ldots \}$ and $\varepsilon_0 > 0$,

$$\|D^lu(\cdot, t)\|_{C(\mathbb{R}^n)}^2 \leq C, \quad \forall t \in (\varepsilon_0, +\infty),$$

where $C$ depends only on $n, \lambda, \Lambda, \frac{1}{\varepsilon_0}$.

Consider the following Monge-Ampère type equation

$$\det D^2u = \exp\{n(u - \frac{1}{2} \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i})\}.$$

According to the definition in [10], we can show that an entire solution to (1.5) is a self-expanding solutions to Lagrangian mean curvature flow in Pseudo-Euclidean space.

Theorem 1.3. There exists a one-to-one correspondence between smooth self-expanding solutions satisfying condition B and functions satisfying Condition A and Condition B.

The following theorem shows that we can obtain the self-expanding solution by the Logarithmic gradient flow.

Theorem 1.4. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function which satisfies Condition B. Assume that

$$\lim_{\lambda \to +\infty} \tau^{-2}u_0(\tau x) = U_0(x)$$

for some $U_0(x) \in C^2(\mathbb{R}^n)$. Let $u(x, t)$ and $U(x, t)$ be solutions to (1.1) with initial data $u_0(x)$ and $U_0(x)$ respectively. Then,

$$\lim_{t \to +\infty} t^{-1}u(\sqrt{t}x, t) = U(x, 1).$$

Here the convergence is uniform and smooth in any compact subset of $\mathbb{R}^n$, and $U(x, 1)$ is a self-expanding solution of (1.2).

To describe the asymptotic behavior of the Lagrangian mean curvature flow (1.2), we will prove
Theorem 1.5. Suppose that $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function which satisfies Condition B and $\sup_{x \in \mathbb{R}^n} |D u_0(x)|^2 < +\infty$. Then the evolution equations of mean curvature flow (1.2) has a longtime smooth solution and the graph $(x, Du(x,t))$ converges to a plane in $\mathbb{R}^n$ as $t$ goes to infinity. If we assume in addition that $|D u_0(x)| \to 0$ as $|x| \to \infty$, then the graph $(x, Du(x,t))$ converges smoothly on compact sets to the coordinate plane $(x,0)$ in $\mathbb{R}^n$.

2. Logarithmic gradient flow related to Lagrangian mean curvature flow

Let $(x^1, \cdots, x^n; y^1, \cdots, y^n)$ be null coordinates in $\mathbb{R}^{2n}$. Then the indefinite metric (cf. [8]) is defined by

$$ds^2 = \frac{1}{2} \sum_{i=1}^n dx^i dy^i.$$  

Suppose $u$ be a smooth convex function. We consider the graph $M$ of $\nabla u$, defined by

$$(x^1, \cdots, x^n; \frac{\partial u}{\partial x^1}, \cdots, \frac{\partial u}{\partial x^n}).$$

The induce Riemannian metric on $M$ is defined by

$$ds^2 = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i dx^j.$$ 

Choose a tangent frame field $\{e_1, \cdots, e_n\}$ along $M$, where

$$e_i = \frac{\partial}{\partial x^i} + \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{\partial}{\partial y^j}.$$ 

We use $\langle , \rangle$ to denote the inner product induced from (2.1). Then

$$\langle e_i, e_j \rangle = \frac{\partial^2 u}{\partial x^i \partial x^j}.$$ 

Let $\{\eta_1, \cdots, \eta_n\}$ be the normal frame field of $M$ in $\mathbb{R}^{2n}$ defined by

$$\eta_i = \frac{\partial}{\partial x^i} - \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{\partial}{\partial y^j}$$

with

$$\langle \eta_i, \eta_j \rangle = - \frac{\partial^2 u}{\partial x^i \partial x^j}.$$ 

The mean curvature vector of $M$ is given by

$$\vec{H} = - \frac{1}{2ng} \frac{\partial g}{\partial x^i} g^{jk} \eta_k,$$

where $g = \text{det} D^2 u$.

If $u(x,t) \in C^{3,\frac{3}{2}+}$, $u$ is strictly convex function in $\mathbb{R}^n$ and

$$F(x(t),t) = (x^1, \cdots, x^n; \frac{\partial u}{\partial x^1}, \cdots, \frac{\partial u}{\partial x^n})$$

satisfies (1.2). Then

$$\frac{dx^i}{dt} = - \frac{1}{2ng} \frac{\partial g}{\partial x^i} g^{ji}, \quad \frac{du_j}{dt} = \frac{1}{2ng} \frac{\partial g}{\partial x^j} g^{lk} \frac{\partial^2 u}{\partial x^k \partial x^l}, \quad i,j = 1, \cdots, n.$$
where \( u_j = \frac{\partial u}{\partial x_j} \), \([g_{ij}] = D^2u\), \([g^{ij}] = [g_{ij}]^{-1}\). However,

\[
\frac{du_j}{dt} = \frac{\partial u_j}{\partial t} + \frac{\partial u_j}{\partial x_k} \frac{dx_k}{dt}, \quad j = 1, 2, \cdots, n.
\]

So that

\[
\frac{\partial u_j}{\partial t} = \frac{1}{2n} \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} g^i_j \frac{\partial^2 u}{\partial x_k \partial x_j} + \frac{1}{2ng} \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} g^i_k \frac{\partial^2 u}{\partial x_k \partial x_j}
\]

\[
= \frac{1}{n} \frac{\partial}{\partial x_j} \ln g, \quad j = 1, 2, \cdots, n.
\]

Then \( u(x, t) \) satisfies (1.1).

Conversely, if \( u(x, t) \in C^{2,1} \) and \( u \) is strictly convex function in \( \mathbb{R}^n \). Then we define in the obvious way

\[
\tilde{F}(x, t) = (x^1, \cdots, x^n; \frac{\partial u}{\partial x^1}, \cdots, \frac{\partial u}{\partial x^n}).
\]

Let \( r : \mathbb{R}^n \times [0, T) \to \mathbb{R}^n \) be the solution of the following system of ordinary differential equations:

\[
\begin{align*}
\frac{dx_i}{dt} &= -\frac{1}{2ng} \frac{\partial g}{\partial x_i} g^i_k, & i = 1, 2, \cdots, n, \\
x^i(0) &= x^i, & i = 1, 2, \cdots, n.
\end{align*}
\]

Then \( r_t \) be a family of diffeomorphisms \( \mathbb{R}^n \to \mathbb{R}^n \) and \( F(x, t) = \tilde{F}(r(x, t), t) \) be the solution of (1.2).

In summary by the regularity theory of parabolic equation we have the following result:

**Proposition 2.1.** Let \( u_0 : \mathbb{R}^n \to \mathbb{R} \) be a strictly convex \( C^2 \) function. Then (1.1) has a strictly convex smooth solution on \( \mathbb{R}^n \times (0, T) \) with initial condition \( u(x, 0) = u_0(x) \) if and only if (1.2) has a smooth solution \( F(x, t) \) on \( \mathbb{R}^n \times (0, T) \) with strictly convex potential and with initial condition \( F(x, 0) = (x, \nabla u_0(x)) \). In particular, there exists a smooth family of diffeomorphisms \( r(x, t) : \mathbb{R}^n \to \mathbb{R}^n \) for \( t \in [0, T) \) such that \( F(x, t) = (r(x, t), \nabla u(r(x, t), t)) \) solves (1.2) on \( \mathbb{R}^n \times [0, T) \).

A solution \( F(\cdot, t) \) of (1.2) is called self-expanding if it has the form

\[
M_t = \sqrt{t} M_1 \quad \text{for all } t > 0,
\]

where \( M_t = F(\cdot, t) \).

Assume that \( F(x, t) \) is a self-expanding solution of (1.2). Following Proposition 2.1, \( u(x, t) \) satisfies

\[
\frac{\partial u}{\partial t} - \frac{1}{n} \ln \det D^2u = 0, \quad t > 0, \quad x \in \mathbb{R}^n.
\]

Hence,

\[
D(u(x, t) - tu(x \sqrt{t}^{-1}, 1)) = 0,
\]

i.e.

\[
u(x, t) = tu(x \sqrt{t}^{-1}, 1), \quad t > 0.
\]
Thus combining (2.6), (2.4) and letting $t = 1$, we can verify that $u(x, 1)$ satisfies (1.5).

We want to use the continue methods to prove the solvability of (1.1).

**Definition 2.1.** Given $T > 0$. Let $\tau \in [0, 1]$. We say $u \in C^{\frac{5}{2}}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T))$ is a solution of ($\star_{\tau}$) if $u$ satisfies

\[
\begin{cases}
\frac{\partial u}{\partial t} - \frac{\tau}{n} \ln \det D^2 u - (1 - \tau)\Delta u = 0, & t > 0, \ x \in \mathbb{R}^n, \\
u = u_0(x), & t = 0, \ x \in \mathbb{R}^n.
\end{cases}
\]

Set

\[
u_0(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u_0(y) \exp\left[-\frac{|x-y|^2}{4t}\right] dy.
\]

Clearly $\nu_0(x, t)$ is a solution of (2.5) with $\tau = 0$. Let

\[I = \{\tau \in [0, 1] : (\star_{\tau}) \text{ has a solution}\}.
\]

The long time existence of the flow (1.2) holds if we can show that $I$ is both closed and open. To prove that the classical solution of (1.1) must be strictly convex we need the following conclusion which is proved by Pierre-Louis Lions, Marek Musiela (cf. Theorem 3.1 in [11]).

**Lemma 2.2.** Let $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ be a solution of a fully nonlinear equations of the form

\[
\frac{\partial u}{\partial t} = F(D^2 u)
\]

where $F$ is a $C^2$ function defined on the cone $\Gamma_+$ of definite symmetric matrices, which is monotone increasing (that is, $F(A) \leq F(A + B)$ whenever $B$ is a positive definite matrix), and such that the function

\[F^*(A) = -F(A^{-1})
\]

is concave on $\Gamma_+$. If $D^2 u \geq 0$ everywhere on $\mathbb{R}^n$ for $t = 0$. Then $D^2 u \geq 0$ everywhere on $\mathbb{R}^n$ for $0 \leq t \leq T$.

From Lemma 2.2 we obtain

**Corollary 2.3.** Suppose that $u : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ be a solution of a fully nonlinear equations of the form

\[
\frac{\partial u}{\partial t} = F(D^2 u)
\]

where $F$ satisfies the conditions in Lemma 2.2 and $F$ is concave on the cone $\Gamma_+$. If $\lambda I \leq D^2 u \leq \Lambda I$ (for some $0 < \lambda < \Lambda$) everywhere on $\mathbb{R}^n$ for $t = 0$. Then $\lambda I \leq D^2 u \leq \Lambda I$ everywhere on $\mathbb{R}^n$ for $0 \leq t \leq T$.

**Proof.** Step 1. We will prove that $D^2 u \geq \lambda I$ everywhere on $\mathbb{R}^n$ for $0 \leq t \leq T$.

Set $\bar{u} = u - \frac{\lambda}{2} |x|^2$. Then $\bar{u}$ satisfies

\[
\frac{\partial \bar{u}}{\partial t} = F(D^2 \bar{u} + \lambda I)
\]

with $D^2 \bar{u}|_{t=0} \geq 0$. Define

\[\bar{F}(D^2 \bar{u}) = F(D^2 \bar{u} + \lambda I),
\]
Lemma 2.4.

Let

\[ \tilde{F}^*(A) = -F(A^{-1} + \lambda I), \]

\[ \tilde{F}^*(\lambda_1, \lambda_2, \ldots, \lambda_n) = -F(\lambda_1^{-1} + \lambda, \lambda_2^{-1} + \lambda, \ldots, \lambda_n^{-1} + \lambda), \]

\[ \Sigma = \{\lambda_1 > 0, \lambda_1 > 0, \ldots, \lambda_1 > 0\}. \]

It follows from [12] that \( \tilde{F}^*(A) \) is concave on \( \Gamma_+ \) if and only if \( \tilde{F}^*(\lambda_1, \lambda_2, \ldots, \lambda_n) \) is concave on \( \Sigma \). Note that for all \( \xi \in \mathbb{R}^n \),

\[ \frac{\partial^2 \tilde{F}^*}{\partial \lambda_i \partial \lambda_i} \xi_i \xi_j = -\tilde{F}_{ij} \bar{\xi}_i \bar{\xi}_j - 2\tilde{F}_i \lambda_i \xi_i^2 \]

where \( \bar{\xi}_i = \frac{\xi_i}{\lambda_i} \). Since \( \tilde{F}^*(A) = -F(A^{-1}) \) is concave on \( \Gamma_+ \), we have

\[ -\tilde{F}_{ij} \bar{\xi}_i \bar{\xi}_j|_{\lambda=0} - 2\tilde{F}_i \lambda_i \xi_i^2|_{\lambda=0} \leq 0. \]

So that

\[ -\tilde{F}_{ij} \bar{\xi}_i \bar{\xi}_j \leq 2\tilde{F}_i \frac{\lambda_i}{1 + \lambda \lambda_i} \xi_i^2, \]

Clearly,

\[ \frac{\partial^2 \tilde{F}^*}{\partial \lambda_i \partial \lambda_i} \xi_i \xi_j = -\tilde{F}_{ij} \bar{\xi}_i \bar{\xi}_j - 2\tilde{F}_i \lambda_i \xi_i^2 \leq 2\tilde{F}_i \frac{\lambda_i}{1 + \lambda \lambda_i} \xi_i^2 - 2\tilde{F}_i \lambda_i \xi_i^2 \leq 0. \]

Such that one can apply Lemma [27] that \( D^2 \bar{u} \geq 0 \) for for \( 0 \leq t \leq T \).

Step 2. We will prove that \( D^2 \bar{u} \leq \Lambda \bar{I} \) everywhere on \( \mathbb{R}^n \) for \( 0 \leq t \leq T \).

Introduce the Legendre transformation of \( u \)

\[ \tau = t, \quad y^i = \frac{\partial u}{\partial x^i}, \quad i = 1, 2, \ldots, n, \quad u^*(y^1, \ldots, y^n) := \sum_{i=1}^n x^i \frac{\partial u}{\partial x^i} - u(x). \]

In terms of \( \tau, y^1, \ldots, y^n, u^*(y^1, \ldots, y^n, \tau) \), one can easily check that

\[ \frac{\partial u^*}{\partial \tau} = -\frac{\partial u}{\partial t}, \quad \frac{\partial^2 u^*}{\partial y^i \partial y^j} = \left[ \frac{\partial^2 u}{\partial x^i \partial x^j} \right]^{-1}. \]

Then \( u^* \) is a solution of the form

\[ \frac{\partial u^*}{\partial \tau} = F^*(D^2 u^*). \]

Since \( F^{**} = F \) is concave on the cone \( \Gamma_+ \), using the conclusions of step 1 we obtain \( D^2 u^* \geq \frac{1}{\bar{\lambda}} \bar{I} \) for \( 0 \leq t \leq T \) and this yields our desired results. \( \square \)

For the problem [2.5] we have

Lemma 2.4. \( I \) is closed.

Proof. Suppose that \( u \) is a solution of \((\ast_\tau)\). For \( A \in \Gamma_+ \), set

\[ F(A) = \frac{T}{n} \ln \text{det} A + (1 - \tau) \text{Tr} A. \]

Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( A \). Define

\[ f(\lambda_1, \lambda_2, \ldots, \lambda_n) = F(A) = \frac{T}{n} \ln \lambda_1 \lambda_2 \cdots \lambda_n + (1 - \tau)(\lambda_1 + \lambda_2 + \cdots + \lambda_n) \]

and

\[ f^*(\lambda_1, \lambda_2, \ldots, \lambda_n) = F^*(A) = \frac{T}{n} \ln \lambda_1 \lambda_2 \cdots \lambda_n - (1 - \tau)(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n}). \]
Lemma 2.6. Define the Banach spaces $B$ and $\Gamma$ of definite symmetric matrix matrices, which is monotone increasing.

It follows from Corollary 2.3 that if $u_0(x)$ satisfies Condition B then $u(x,t)$ does so. For $s > 0$, $\Omega \subset \mathbb{R}^n$ define

$$\Omega_T = \Omega \times [0,T), \quad \Omega_{T,s} = \Omega \times [s,T).$$

Furthermore by the regularity theory of parabolic equation (cf. [13]) we have

$$\|u\|_{C^{2,1}(\Omega_T)} \leq C_1, \quad \|u\|_{C^{2+\alpha, 2+\alpha}(\Omega_{T,s})} \leq C_2,$$

where $0 < \alpha < 1$, $C_1$ is a positive constant depending only on $u_0, \Omega, T$, and $C_2$ relies on $\lambda, \Lambda, \Omega, T, \frac{1}{s}$. By (2.6), a diagonal sequence argument and the regularity theory of parabolic equation shows that $I$ is closed. \hfill \Box

To prove that $I$ is open we need the following lemma (cf. Theorem 17.6 in [14]).

**Lemma 2.5.** Let $B_1, B_2$ and $X$ be Banach spaces and $G$ is a mapping from an open subset of $B_1 \times X$ into $B_2$. Let $(u_0, \tau_0)$ be a point in $B_1 \times X$ satisfying:

(i) $G[u_0, \tau_0] = 0$;
(ii) $G$ is continuously differentiable at $(u_0, \tau_0)$;
(iii) the partial Fréchet derivative $L = G^1_{(u_0, \tau_0)}$ is invertible.

Then there exists a neighbourhood $\mathcal{N}$ of $\tau_0$ in $X$ such that the equation $G[u, \tau] = 0$, is solvable for each $\tau \in \mathcal{N}$, with solution $u = u_\tau \in B_1$.

Based on the implicit function theorem we have the following conclusions.

**Lemma 2.6.** $I$ is open.

**Proof.** Define the Banach spaces

$$B_1 = C^{3, \frac{3}{2}}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T)), \quad X = \mathbb{R},$$

$$B_2 = C^{3, \frac{3}{2}}(\mathbb{R}^n \times (0, T)) \times C(\mathbb{R}^n),$$

and a continuously differentiable map from $B_1 \times X$ into $B_2$,

$$G: (u, \tau) \rightarrow [\frac{\partial u}{\partial t} - \frac{\tau}{n} \ln \det D^2u - (1 - \tau)\Delta u, u - u_0].$$

Take an open set of $B_1 \times X$:

$$\Theta = \{u|\frac{\lambda}{2}I < D^2u(x,t) < \frac{3\lambda}{2}I, \quad u \in C^{5, \frac{5}{2}}(\mathbb{R}^n \times (0, T)) \cap C(\mathbb{R}^n \times [0, T)) \} \times (0, 1).$$

Suppose that $(u_0, \tau_0) \in \Theta$. Then the partial Fréchet derivative $L = G^1_{(u_0, \tau_0)}$ is invertible if and only if the following cauchy problem is solvable

$$\left\{ \begin{array}{l}
\frac{\partial w}{\partial t} - \frac{\tau_0}{n} u_{ij} \frac{\partial^2 w}{\partial x^i \partial x^j} - (1 - \tau_0)\Delta w = f, \quad t > 0, \quad x \in \mathbb{R}^n, \\
\tau = g, \quad t = 0, \quad x \in \mathbb{R}^n,
\end{array} \right.$$
Thereby applying Lemma 2.5 we prove that I is open.

Given \( x_0 \in \mathbb{R}^n, \kappa > 0 \), define
\[
Q_{1, x_0} = \{ x \mid |x - x_0| \leq 1 \} \times [\kappa, \kappa + 1), \quad Q_{\frac{1}{2}, x_0} = \{ x \mid |x - x_0| \leq \frac{1}{2} \} \times [\kappa + \frac{1}{4}, \kappa + \frac{1}{2}),
\]
\[
Q_{\frac{1}{3}, x_0} = \{ x \mid |x - x_0| \leq \frac{1}{3} \} \times [\kappa + \frac{1}{3}, \kappa + \frac{5}{12}), \quad B_{1, x_0} = \{|x - x_0| \leq 1\}.
\]

The following two lemmas which will be mentioned below may be used repeatedly (cf. [13]).

**Lemma 2.7.** (Theorem 14.7 in [13]). Let \( u : \mathbb{R}^n \times [0, T) \rightarrow \mathbb{R} \) be a classical solution of a fully nonlinear equation of the form
\[
\begin{aligned}
\frac{\partial u}{\partial t} - F(D^2 u) &= 0, \quad t > 0, \ x \in \mathbb{R}^n, \\
u &= u_0(x), \quad t = 0, \ x \in \mathbb{R}^n,
\end{aligned}
\]
where \( F \) is a \( C^2 \) concave function defined on the cone \( \Gamma_+ \) of definite symmetric matrices, which is monotone increasing with
\[
\lambda I \leq \frac{\partial F}{\partial r_{ij}} \leq \Lambda I.
\]
Then there exists a strictly convex solution of (1.1) satisfying (1.3) and \( u(\cdot, t) \) satisfies Condition B.

By Lemma 2.7 we get
\[
[D^2 u]_{C^\alpha(\bar{Q}_{\frac{1}{2}, x_0})} \leq C,
\]
where \( C \) are positive constants depending only on \( n, \lambda, \Lambda \) and \( \frac{1}{\kappa} \).

**Proof of Theorem 1.2**

Using the linear parabolic equations theory (cf. [13]) and combining Lemma 2.4 with Lemma 2.6 we conclude that there exists a unique strictly convex solution of (1.1) satisfying (1.3) and \( u(\cdot, t) \) satisfies Condition B.
For \( m \in \{1, 2, \cdots, n\} \), set \( v = \frac{\partial u}{\partial x_m} \). Then \( v \) satisfies
\[
\begin{cases}
\frac{\partial v}{\partial t} - \frac{1}{n} u^{ij} v_{ij} = 0, & t > 0, \ x \in \mathbb{R}^n, \\
v = v_0(x), & t = 0, \ x \in \mathbb{R}^n.
\end{cases}
\]
Such that by Lemma 2.8 we have
\[
|D^3 u|_{C^0(\bar{Q}_{1/2}, x_0)} \leq C |Du_0|_{C^0(\bar{B}_1, x_0)}.
\]
Let
\[
\tilde{v}(x, t) = v - \frac{\partial u_0}{\partial x_m}(x_0).
\]
It is easy to see that \( \tilde{v} \) satisfies
\[
\begin{cases}
\frac{\partial \tilde{v}}{\partial t} - \frac{1}{n} u^{ij} \tilde{v}_{ij} = 0, & t > 0, \ x \in \mathbb{R}^n, \\
\tilde{v} = \tilde{v}(x, 0), & t = 0, \ x \in \mathbb{R}^n.
\end{cases}
\]
Then by (2.7) and the mean value theorem we arrive at
\[
|D^3 u|_{C^0(\bar{Q}_{1/2}, x_0)} \leq C |Du_0 - Du_0(x_0)|_{C^0(\bar{B}_1, x_0)} \leq C.
\]
Using the similar methods we obtain (1.4) for \( l = \{3, 4, 5 \cdots\} \).

**Proof of Theorem 1.3:**

The main idea comes from [15] and we present here for completeness.

Case 1. If \( u_0 \) satisfies Condition A and B. Then by Theorem 1.2, there exists a unique smooth solution \( u(x, t) \) to (1.1) for all \( t > 0 \) with initial data \( u_0 \). One can verify that
\[
u_R(x, t) := R^{-2} u(Rx, R^2 t)
\]
is a solution to (1.1) with initial data
\[
u_R(x, 0) := R^{-2} u_0(Rx) = u_0(x).
\]
Here we have used that \( u_0 \) satisfies Condition A. Since \( u_R(x, 0) = u_0 \), the uniqueness result in Theorem 1.2 implies
\[
u(x, t) = u_R(x, t)
\]
for any \( R > 0 \). Therefore \( u(x, t) \) satisfies (2.4), and hence \( u(x, 1) \) solves (1.5). In other words, \( u(x, 1) \) is a smooth self-expanding solution.

Case 2. If \( v \) is a smooth solution to (1.5) satisfying Condition B. Define \( u(x, t) \) for \( t > 0 \) by
\[
u(x, t) = tv\left(\frac{x}{\sqrt{t}}\right).
\]
By the definition of \( v \) we conclude that \( u(x, t) \) satisfies the evolution equation (1.1). Now we claim that \( \lim_{t \to 0} u(x, t) \) exists. To see this, note that \( u(0, t) = tv(0) \) for \( t > 0 \), so
\[
\lim_{t \to 0} u(0, t) = 0.
\]
Moreover, it is clear that
\[
Du(0, t) = \sqrt{t} Du(0),
\]
Since
\[
(2.8) \quad D^2 u(x, t) = D^2 u tv\left(\frac{x}{\sqrt{t}}\right) = D^2 v\left(\frac{x}{\sqrt{t}}\right),
\]
we get for any \( t > 0 \),
\[
\lambda I \leq D^2 u(x, t) \leq \Lambda I, \quad x \in \mathbb{R}^n.
\]

Using the medium theorem twice, we have
\[
u(x, t) = u(x, t) - u(0, t) + u(0, t)
= < Du(\xi, t), x > + u(0, t)
= < Du(\xi, t) - Du(0, t), x > + u(0, t)
= \sum_{i,j=1}^n \xi_i u_{ij} x_j + < Du(0, t), x > + u(0, t).
\]

Applying Caffarelli’s regularity theory of Monge-Ampère type equation (cf. [16], [17]) and interior Schauder estimates to the equation (1.5), we may then conclude that, for any sequence \( t_i \to 0 \), there is a subsequence \( t_{k_i} \), such that \( u(x, t_{k_i}) \) converges in \( C^{2,\alpha} \) uniformly in compact subsets of \( \mathbb{R}^n \) for any \( 0 < \alpha < 1 \). This limit is in fact independent of the choice of the subsequence \( \{t_{k_i}\} \). Indeed, let \( u_1 \) and \( u_2 \) be two such limits for subsequences \( \{t_i\} \) and \( \{t_i'\} \) respectively. Since \( u(x, t) \) is a solution to (1.1), \( \frac{\partial u}{\partial t} \) is uniformly bounded for any \( t > 0 \) and \( x \in \mathbb{R}^n \).

Thus for any \( x \in \mathbb{R}^n \) we may have
\[
|u(x, t_i) - u(x, t_i')| \leq C|t_i - t_i'|
\]
for some constant \( C \) independent of \( i \). Letting \( i \to \infty \), we conclude that \( u_1(x) = u_2(x) \). So for different sequences \( \{t_i\} \) converging to 0, the limit is unique. Let
\[
u_0(x) = \lim_{t \to 0} u(x, t).
\]
Then it follows from (2.8) that \( \nu_0 \) satisfies Condition B. Further,
\[
\frac{1}{R^2} \nu_0(Rx) = \frac{1}{R^2} \lim_{t \to 0} tv(Rx, \sqrt{t}) = \lim_{t \to 0} \sqrt{\frac{t}{R}} v(Rx, \sqrt{t}) = \nu_0(x).
\]

Therefore \( \nu_0(x) \) satisfies Condition A.

From the above two cases we see that Theorem 1.3 is established. \( \square \)

We present here the proof of Theorem 1.4 by the methods of [15].

**Proof of Theorem 1.4**

Assume that
\[
u_0(x) = \lim_{R \to +\infty} R^{-2} \nu_0(Rx).
\]
So \( \nu(x, 0) \) satisfies Condition B. As in the proof of Theorem 1.3 we obtain
\[
\nu_0(x) = \lim_{R \to \infty} R^{-2} \nu_0(Rx) = \lim_{R \to \infty} R^{-2} l^{-2} \nu_0(Rlx) = l^{-2} \nu_0(lx).
\]
This implies that \( \nu_0(x) \) satisfies condition A. Thus by Theorem 1.3 we conclude that \( \nu(x, 1) \) is a self-expanding solution.

Define
\[
u_R(x, t) := R^{-2} \nu(Rx, R^2 t).
\]
It is clear that \( \nu_R(x, t) \) is a solution to (1.1) with initial data \( \nu_R(x, 0) = R^{-2} \nu_0(Rx) \) satisfying Condition B. For any sequence \( R_i \to +\infty \), consider the limitation of \( u_{R_i}(x, t) \). For \( t > 0 \), there holds
\[
D^2 u_{R_i}(x, t) = D^2 u(R_i x, R_i^2 t),
\]
Using Theorem 1.2, we have
\[ \lambda I \leq D^2 u_{R_i}(x, t) \leq \Lambda I \]
for all \( x \) and \( t > 0 \). Moreover, according to (1.4) in Theorem 1.2, there holds
\[ \|D^l u_{R_i}(\cdot, t)\|_{C(R^n)}^2 \leq C, \quad \forall t \geq \varepsilon_0, \quad l = \{3, 4, 5 \cdots \}. \]
For any \( m \geq 1, l \geq 0 \), using Schauder estimates there exists constant \( C \) such that
\[ \|\partial^m \partial_t^m D^l u_{R_i}(\cdot, t)\|_{C(R^n)}^2 \leq C, \quad \forall t \geq \varepsilon_0, \quad l = \{3, 4, 5 \cdots \}. \]
We observe that
\[ u_{R_i} = R_i^{-2}u_0 \]
and
\[ Du_{R_i}(0, 0) = R_i^{-1}Du_0(0). \]
are both bounded, thus \( u_{R_i}(0, t) \) and \( Du_{R_i}(0, t) \) are uniformly bounded to \( i \) for any fixed \( t \). By Arzelà-Ascoli theorem, there exists a subsequence \( \{R_{k_i}\} \) such that \( u_{R_{k_i}}(x, t) \) converges uniformly to a solution \( \hat{U}(x, t) \) of (1.6) in compact subsets of \( \mathbb{R}^n \times (0, \infty) \) and \( \hat{U}(x, t) \) satisfies the estimates in Theorem 1.2. Since \( \frac{\partial \hat{U}}{\partial t} \) is uniformly bounded for any \( t > 0 \), \( \hat{U}(x, t) \) converges to some function \( \hat{U}_0(x) \) when \( t \to 0 \). One can verify that
\[ \hat{U}_0(x) = \lim_{t \to 0} \hat{U}(x, t) \]
\[ = \lim_{t \to 0} \lim_{i \to +\infty} R_i^{-2}u(R_i x, R_i^2 t) \]
\[ = \lim_{i \to +\infty} \lim_{t \to 0} R_i^{-2}u(R_i x, R_i^2 t) \]
\[ = \lim_{i \to +\infty} R_i^{-2}u_0(R_i x) \]
\[ = U_0(x). \]
By the uniqueness result, the above limit is independent of the choice of the subsequence \( \{R_i\} \) and
\[ \hat{U}(x, t) = U(x, t). \]
So, letting \( R = \sqrt{t} \), we have \( t^{-1}u(\sqrt{t}x, t) = u_\sqrt{t}(x, 1) \) converging to \( U(x, 1) \) uniformly in compact subsets of \( \mathbb{R}^n \) when \( t \to \infty \). Theorem 1.3 is established. \( \square \)

At the end of this section, we present the following Bernstein theorem for the equation (1.5).

**Proposition 2.9.** Let
\[ w = u - \frac{1}{2} < x, Du >. \]
If \( u \) is a \( C^2 \) strictly convex solution of (1.5) and \( w \) can take its maximum or minimum at some point \( x \in \mathbb{R}^n \) with \( |x| < +\infty \). Then \( u \) must be a quadratic polynomial.

**Proof.** It follows from Caffarelli’s regularity theory of Monge-Ampère type equation and interior Schauder estimates that \( u \) is a smooth strictly convex solution. From (1.5) \( w \) satisfies
\[ u^{ij}w_{ij} = \frac{1}{2} < x, Dw >. \]
Since \( w \) can take its maximum or minimum at some point \( x \in \mathbb{R}^n \) with \( |x| < +\infty \), for every \( R > 0 \), by strong maximum principle (cf. [18]), we deduce that \( w \) must be constants in \( B_R(x) \). Then \( w \) must be constants in \( \mathbb{R}^n \). Using Pogorelov’s Theorem [19], we show that \( u \) must be a quadratic polynomial. \( \square \)

3. Longtime Existence and Convergence

As in [20], we also can show that a bound on the height of the graphs is preserved along (1.1).

**Lemma 3.1.** If \( u(x, t) \) is a smooth solution to (1.1). Then

\[
(3.1) \quad \sup_{x \in \mathbb{R}^n} |Du(x, t)|^2 \leq \sup_{x \in \mathbb{R}^n} |Du_0(x)|^2
\]

**Proof.** By (1.1) we have

\[
\frac{\partial}{\partial t} |Du(x, t)|^2 - \frac{1}{n} u^{ij} (|Du(x, t)|^2)_{ij} = -2 \frac{n}{n} w^{pq} u_{pi} u_{qi} \leq 0.
\]

Using Lemma 4.2 in [7] and Schauder estimates we obtain the desired results. \( \square \)

To obtain the convergence of the flow (1.2), we introduce the following decay estimates of the high order derivatives according to the solution of (1.1) based on Theorem 1.2 (cf. [21]).

**Proposition 3.2.** Assume that \( u(x, t) \) is a strictly convex solution of (1.1) which satisfies (1.3) and \( u(\cdot, t) \) satisfies Condition A. Then there exist constant \( C \) only depending on \( n, \lambda, \Lambda, \frac{1}{\varepsilon_0} \) such that

\[
(3.2) \quad |D^3 u(\cdot, t)|^2_{C(\mathbb{R}^n)} \leq \frac{C}{t}, \quad \forall t \geq \varepsilon_0.
\]

More generally, for all \( l = \{3, 4, 5 \cdots \} \) there holds

\[
(3.3) \quad \| D^l u(\cdot, t) \|^2_{C(\mathbb{R}^n)} \leq \frac{C}{t^{l-2}}, \quad \forall t \geq \varepsilon_0.
\]

**Proof of Theorem 1.5:**

By Theorem 1.2 and Proposition 2.1 (1.2) has a longtime smooth solution.

Using (3.3) and (3.1), a diagonal sequence argument shows that as \( t \to \infty \), \( Du(x, t) \) converges subsequentially and uniformly on compact subsets of \( \mathbb{R}^n \) to a smooth function \( Du_\infty \) with

\[
\forall y \in \mathbb{R}^n, \quad |D^l_y u_\infty| = 0
\]

for \( l \geq 3 \). So \( Du_\infty \) must be an affine linear function. Hence \( (x, Du_\infty(x)) \) has to be affine linear subspace. It shows that the graph of the mean curvature flow (1.2) converges to a plane in \( \mathbb{R}^n \).

As the proof of Theorem 1.1 in [20], if \( |Du_0(x)| \to 0 \) as \( |x| \to \infty \), then the graph \( (x, Du(x, t)) \) converges smoothly on compact sets to the coordinate plane \( (x, 0) \) in \( \mathbb{R}^n \).

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