RATES OF CONVERGENCE FOR GIBBS SAMPLING IN THE ANALYSIS OF ALMOST EXCHANGEABLE DATA

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Abstract. Motivated by de Finetti’s representation theorem for partially exchangeable arrays, we want to sample \( p \in [0,1]^d \) from a distribution with density proportional to \( \exp(-A^2 \sum c_{ij} (p_i - p_j)^2) \). We are particularly interested in the case of an almost exchangeable array (\( A \) large).

We analyze the rate of convergence of a coordinate Gibbs sampler used to simulate from these measures. We show that for every fixed matrix \( C = (c_{ij}) \), and large enough \( A \), mixing happens in \( \Theta(A^2) \) steps in a suitable Wasserstein distance. The upper and lower bounds are explicit and depend on the matrix \( C \) through few relevant spectral parameters.

1. Introduction

As part of his monumental attempt \([9]\) to define a subjective notion of inference based on probability theory, Bruno de Finetti introduced various notions of symmetries among data to justify Bayesian inference. On the mathematical ground, the justification is provided by a class of representation theorems, in the same spirit as the celebrated de Finetti’s theorem for exchangeable sequences \([6]\).

1.1. Background on exchangeability and its variations. In its original formulation, de Finetti’s theorem represents every exchangeable sequence on \( \{0,1\}^\infty \) as a mixture of coin-tossing processes. More precisely, let \( S_n \) be the symmetric group on \( n \) elements. A sequence \( X_1,\ldots,X_n \) is said to be completely exchangeable (or exchangeable) if for every \( n \in \mathbb{N} \), \( \sigma \in S_n \), and every sequence \( (\epsilon_i)_{i<n} \in \{0,1\}^n \), one has

\[
P(X_1 = \epsilon_1,\ldots,X_n = \epsilon_n) = P(X_{\sigma(1)} = \epsilon_1,\ldots,X_{\sigma(n)} = \epsilon_n).
\]

For any exchangeable sequence, de Finetti’s theorem provides a unique measure \( \pi \) on \( [0,1] = M_1(\{0,1\}) \), the space of probability measure on \( \{0,1\} \), such that

\[
P(X_1 = \epsilon_1,\ldots,X_n = \epsilon_n) = \int_{[0,1]} p^s (1 - p)^{n-s} \pi(dp), \quad s := \sum_{i \in [n]} \epsilon_i.
\]

When \( \pi \) is a delta measure at a point \( p \in [0,1] \), the \( X_i \)’s are independent tosses of a \( p \)-coin. More generally, an exchangeable sequence can be generated according to a two step procedure: first select \( p \) according to \( \pi \), and then flip independent \( p \)-coins.

Since its original formulation, de Finetti’s theorem has been generalized in various ways. A comprehensive summary of results on the probabilistic side is \([20]\) and references therein. More recently, representation theorems a la de Finetti played a major role in understanding Bose-Einstein condensation for a gas with mean-field interactions \([24]\) and mean-field spin glasses \([24]\).

One natural generalization is that of partial exchangeability, introduced in \([7]\): a
sequence of random variables \((X_{i,j})_{i\in\mathbb{N}, j\in[d]}\) is said to be partially exchangeable if, for every \(n_1, \ldots, n_d \in \mathbb{N}^d\), every \((\sigma_1, \ldots, \sigma_d) \in \mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_d}\) and every sequence \((\epsilon_{i,j})_{i\in[n_1], j\in[d]} \in [0,1]^{n_1 \times \cdots \times n_d}\), one has
\[
P(X_{i,j} = \epsilon_{i,j}, i \in [n_j], j \in [d]) = \mathbb{P}(X_{\sigma(i), j} = \epsilon_{i,j}, i \in [n_j], j \in [d]).
\]
In analogy to (1.1), partial exchangeable arrays admit a unique representation in terms of a measure \(\pi\) on \([0,1]^d = M_1([0,1]^d)\). Using the notation \(\mathbf{p} := (p_1, \ldots, p_d)\), one has
\[
P(X_{1,1} = \epsilon_{1,1}, \ldots, X_{n_d,d} = \epsilon_{n_d,d}) = \int_{[0,1]^d} \prod_{j=1}^d p_j^{s_j} (1-p_j)^{n_j-s_j} \pi(d\mathbf{p}), \quad s_j := \sum_{i\in[n_j]} \epsilon_{i,j}.
\]
Formula (1.2) reduces to (1.1) if \(\pi\) is supported on the main diagonal \(\mathcal{D} := \{\mathbf{p}1 : p \in [0,1]\} \subset [0,1]^d\), \(\mathbf{1} := (1, \ldots, 1)\).

Therefore, in order to understand whether a partially exchangeable array is close to be completely exchangeable, one can determine instead whether its directing measure \(\pi\) is approximately concentrated on \(\mathcal{D}\). More generally, questions about extra symmetries for a partially exchangeable arrays can be rephrased in conditions on the support of \(\pi\), that should approximately coincide with some manifold [8].

In the context of Bayesian statistics, one starts with some prior \(\pi\). Once data are collected, a posterior distribution is determined via the maximum likelihood principle. \(\pi\) might thus represent the initial belief on exchangeability. In absence of additional information about the \(X_{i,j}\)’s, de Finetti [8] suggests to start with priors \((\pi_{A,C}(\mathbf{p})\) of the form
\[
\pi_{A,C}(\mathbf{p}) = \frac{1}{Z} \exp \left\{ -A^2 \sum_{i<j} c_{ij} (p_i - p_j)^2 \right\} d\mathbf{p}, \quad A > 0, \quad C = (c_{ij})_{i,j\in[d]},
\]
where \(d\mathbf{p}\) denotes Lebesgue measure on \([0,1]^d\) and \(Z = Z(A,C)\) is a normalization constant. Here, \(C := (c_{ij})_{i,j\in[d]}\) is a symmetric matrix with non-negative entries and zeros on the diagonal, measuring the discrepancies among different categories. The parameter \(A\) interpolates between complete exchangeability \((A = \infty)\) and complete lack of information \((A = 0)\).

This measure closely resembles a Discrete Gaussian Free Field (see e.g. [23]), if one forgets about the constraint \(\mathbf{p} \in [0,1]^d\). Because of this condition, however, even the computation of the normalizing constant of \(\pi_{A,C}(\mathbf{p})\) can be challenging [15]. In [1], the authors raise the question of how to sample efficiently from these measures, particularly when dealing with almost exchangeable priors \((A\) large).

1.2. Turning to Gibbs sampling. From now on, we restrict our attention to the prior measure defined in (1.3). The main goal of the paper is to analyze a Gibbs sampling procedure from this distribution. Given a measure \(\pi\) on a product space, the Gibbs sampler – also known as Glauber dynamics – is a Markov chain whose single step consists in updating a randomly chosen coordinate according to the conditional distribution given all the others (a more precise definition will be given in Section 2).

Introduced in [15], the Gibbs sampler is now used in virtually all fields of science (see e.g. [13]). Despite its widespread use, sharp bounds on its rate of convergence have been obtained only in few cases. Results in discrete state spaces are known
for the Ising model \cite{2,13,22}, graph colouring \cite{28,14}, and some integrable models \cite{11,21}, via a combinations of spectral and coupling techniques. In continuous state spaces, there are fewer results available, with few exceptions such as the Gibbs sampler for the uniform distribution on the unit simplex \cite{27}. The mainstream approach via drift and minorization \cite{29,19} tends instead to underestimate possible diffusive behaviors in the chain.

In the case of the measure \( \pi_{A,C} \) of (1.3), we will show that the Gibbs sampler consists in replacing a random coordinate in the hypercube with a weighted average of the others, and then perturbed it with a (truncated) normal noise. A drift and minorization conditions leads to an upper bound exponentially bad in the parameter \( A \). In \cite{17}, the first author was able to show that, in \( d = 2 \), \( \Theta(A^2) \) steps are necessary and sufficient for this chain to mix in total variation. The chain has a diffusive behavior without exhibiting cutoff in the sense of \cite{10}, and the projection of the walk onto the main diagonal of the square has a Brownian-like behavior.

This work aims to generalize the result of \cite{17} to an arbitrary \( d \) and a general matrix \( C \), showing how the mixing time, for \( A \) large, is still governed by the meandering along the diagonal. While our results are obtained in Wasserstein distance, we conjecture similar bounds in total variation to hold as well.

1.3. Structure. The rest of the paper is structured as follows: in Section 2 we specify the model and state our main result. Section 3 deals with technical ingredients of the proof: various moment bounds to deal with the non-homogeneous nature of the Gibbs sampler, a monotone and contracting grand-coupling, and a control on large deviations of the walk away from the diagonal. In Section 4 we verify the core block for the result, an approximate diffusive behavior of a suitable projection of the walk onto the main diagonal. We then collect all elements to complete the proof of Theorem 2.1 in Section 5 and point out open questions and future directions in Section 6.

2. Model and result

2.1. The role of the graph structure. One can think of \( C \) as a network, its entries being the weights on a graph with \( d \geq 2 \) vertices. Throughout the paper, we will assume

**Assumption 1.** The network \( C \) is connected with unit total weight.

The normalizing assumption is not restrictive because of the presence of the scaling parameter \( A \). Together, they imply

\[
\begin{align*}
c_i := \sum_{j \neq i} c_{ij} > 0, \quad \sum_i c_i = 1.
\end{align*}
\]

Here, \( c = (c_i)_{i \in [d]} \) is nothing but the stationary distribution of the simple random walk on the network. Equivalently, the diagonal matrix \( D \) defined by \( D_{i,i} := c_i \) is non-singular with \( \text{Tr}(D) = 1 \).

The main goal of this paper is to describe the mixing property of the Gibbs sampler in terms of few spectral parameters of the network \( C \). To this aim, for \( p \in \mathbb{R}^d \), we introduce the quantities

\[
\hat{p} := D^{-1} C p, \quad \Delta p := (I - D^{-1} C) p = p - \hat{p}.
\]
The vector \( \hat{p} \) represents the single update distribution of a simple random walker on
the network \( C \) who starts with distribution \( p \). \( \Delta \) is the Laplacian of the network,
which is a symmetric operator with respect to the inner product induced by the
stationary distribution \( c \), i.e.,
\[
\langle \Delta p, q \rangle = \langle p, \Delta q \rangle, \quad \langle p, q \rangle := p^T D q = \sum_{i \in [d]} c_i p_i q_i.
\] (2.3)

Here, we dropped the dependence on \( D \) in the scalar product for convenience.
Under Assumption 1, the Perron-Frobenius theorem is in force for
\( C \). As a consequence, the Laplacian has \( d \) real eigenvalues that satisfy
\[
0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_d \leq 2,
\] with the one dimensional (right) eigenspace corresponding to \( \lambda_1 = 0 \) being spanned
by the normalized eigenvector \( 1 \). The connection between the Laplacian and mea-
sure (1.3) comes from the identity
\[
\sum_{i<j} c_{ij} (p_i - p_j)^2 = \langle p, \Delta p \rangle.
\] (2.4)

It is convenient to introduce the quantities
\[
\lambda := \lambda_2, \quad \gamma := \max(|1 - \lambda_2|, |1 - \lambda_d|), \quad \beta := \min_{i \in [d]} c_i.
\] (2.5)

For a heuristic understanding, recall that \( \lambda \) is intimately related to the
connectivity of the network, \( \gamma \) to the mixing properties of the associated random walk, and \( \beta \) to
the accessibility of the stationary distribution. In particular, since these parameters
are can be estimates with various probabilistic and geometric tools, and are stable under
small perturbations of the weights \( c_{ij} \), our bounds are very robust.

2.2. The Gibbs sampler. In our context, the Gibbs sampler with random updates
is a Markov chain on \([0, 1]^d\), with stationary distribution \( \pi_{A,C} \). Given the current
state \( p = (p_1, \ldots, p_d) \), it updates according to the following rules:

1. sample \( i \in [d] \) uniformly at random;
2. update the \( i \)-th coordinate of \( p \) according to the conditional distribution
   \( \pi_{A,C}(p_1, \ldots, p_{i-1}, \cdot, p_{i+1}, \ldots, p_d) \).

Conditional on the choice of index \( i \) in the first step, the update in the second step
can be easily derived from the identity (2.4), and gives
\[
p_i \to \tilde{p}_i + \epsilon(\sigma_i^2, \tilde{p}_i), \quad \sigma_i^2 = \sigma_i^2(A,C) := \frac{1}{2A^2 c_i}.
\] (2.6)

Here, \( \epsilon(\sigma^2, p) \) denotes a normal random variable with mean 0, variance \( \sigma^2 \) and
conditioned to lie in \([-p, 1 - p]\). Sampling from a truncated normal random variable
can be done efficiently (see e.g. [5]), and the usefulness of the Gibbs sampler is
then related to its rate of convergence to stationarity.

Formula (2.6) shows that the walk consists of a non-local move (replacing a random
coordinate with a weighted average of all the others), and a non-homogeneous
diffusive move (adding a truncated normal noise). The non-locality creates some
difficulties in approximating the walk with a diffusion, and the lack of homogeneity
requires to take care of the effect of the boundary of the hypercube. Intuitively,
for large \( A \) one expects the behavior of the chain at the two corners \( 0 \) and \( 1 \) –
where coordinates are clustered together and in the vicinity of the boundary of the
hypercube – to represent the main challenge.

We now write (2.6) in a more compact form. Let \( \Pi_i \) be the (random) projection
onto the \( i \)-th basis vector, where \( i \) is chosen uniformly at random, and for
of one step of the Gibbs sampler as conditioned to lie in \( \mathcal{P} \). Owing to (2.2) and (2.6), we can write the update of one step of the Gibbs sampler as

\[
p \rightarrow p + \Pi_i \left[ -\Delta p + \epsilon(\Sigma, \hat{p}) \right], \quad \Sigma = \Sigma(A, C) := \frac{D^{-1}}{2A^2}.
\]

We will denote by \( K^{\ast k}_{A,C}(p) \) the law of a random variable obtained after performing \( k \) steps of the Gibbs sampler, starting from the point \( p \). By \( K_{A,C}(p) \), we will mean the single step update.

2.3. Main theorem and examples. Our result is expressed in term of \( \infty \)-Wasserstein distance on \( M_1([0, 1]^d) \), defined as

\[
d_{\infty}(\mu, \nu) := \inf_{p \sim \mu, q \sim \nu} \mathbb{E}\left[ \|p - q\|_{\infty} \right], \quad \mu, \nu \in M_1([0, 1]^d).
\]

Here, the infimum is taken over all couplings of \( p \) and \( q \) with marginals \( \mu \) and \( \nu \), respectively, and \( \|p - q\|_{\infty} = \sup_{i \in [d]} |p_i - q_i| \). We can now state the main Theorem.

**Theorem 2.1.** Let \( C \) be a network on \( d \) vertices satisfying Assumption 7 and let \( \lambda, \gamma, \beta \) be defined as in (2.5). Then, for any \( \delta > 0 \), there exists \( m = m(\delta), M = M(\delta) \) such that, for all \( \delta > 0 \), one has:

- if \( k = m\frac{d}{2} A^2 \), then
  \[
  \sup_{p \in [0, 1]^d} d_{\infty}(K^{\ast k}_{A,C}(p), \pi_{A,C}) \geq \frac{1}{4} - \delta;
  \]
- if \( k = M d A^2 \), then
  \[
  \sup_{p \in [0, 1]^d} d_{\infty}(K^{\ast k}_{A,C}(p), \pi_{A,C}) \leq \delta;
  \]

**Remark 2.2.** Our result shows that for a fixed network \( C \), similarly to the two dimensional case analyzed in [17], \( \Theta(A^2) \) steps are necessary and sufficient to mix, with no cutoff.

**Remark 2.3.** When the underlying network is the complete graph with uniform weights, the Laplacian is maximally symmetric on the orthogonal complement of the vector \( 1 \). In particular,

\[
\lambda = \frac{d}{d - 1}, \quad \gamma = \frac{1}{d - 1}, \quad \frac{\lambda}{\gamma} = d.
\]

Therefore, \( \Theta(d A^2) \) are necessary and sufficient to mix. This is well illustrated by Figures 1, 2, and 3, that show the mixing time of some relevant statistics.

**Remark 2.4.** Looking at the proof of the lower bound, one see that \( \frac{1}{2} = d_{\infty}(\delta_1, \delta_D) \), where \( \delta_D \) is the uniform measure on the main diagonal. We conjecture that it is possible to replace it with \( \frac{1}{2} = d_{\infty}(\delta_0, \delta_D) \).

**Remark 2.5.** Given a network, quantities like \( \lambda \) and \( \gamma \) can be also estimated by geometric-analytic methods, such as Cheeger inequalities [4] or Poincaré/Nash-Sobolev inequalities [12]. Our proof gives explicit bounds on the size of \( A^2 \) in terms of these quantities.

**Remark 2.6.** If the network is disconnected, the stationary measure becomes a product measure over different connected components. Moreover, it is easy to see that the Gibbs sampler updates in different connected components, conditioned on the choices of the indices, are also independent. Therefore, if the connectivity in Assumption 1 is dropped, we can still show that \( \Theta(A^2) \) steps are necessary and
Lemma 3.2. The following facts hold:

\[ \epsilon \]

by means of uniform bounds on the moments of \( \epsilon \), we denote by a standard normal random variable, which will be denoted by \( Z \). Where the last bound follows from the spectral theorem for \( D \).

Therefore, we obtain

\[ \langle \Delta \Delta p, \Delta p \rangle = \langle \Delta \Delta p, \Delta \Delta \hat{p} \rangle \geq \lambda \langle p, \Delta p \rangle, \quad p \in \mathbb{R}^d. \]  

The following bound will also be useful.

Lemma 3.1. If \( C \) satisfies Assumption 2, then

\[ \max_{i \in [d]} c_i \leq \gamma. \]

Proof. Without loss of generality, assume \( \max_{i \in [d]} c_i = c_1 =: c \). Consider the vector

\[ p = \left( \sqrt{\frac{1 - c}{c}}, -\sqrt{\frac{c}{1 - c}}, \ldots, -\sqrt{\frac{c}{1 - c}} \right). \]

An easy computation shows \( \langle p, p \rangle = 1, \langle p, 1 \rangle = 0 \), as well as

\[ \langle p, \hat{p} \rangle = -\frac{c}{1 - c}. \]

Therefore, we obtain

\[ \max_{i \in [d]} c_i = c \leq \left| -\frac{c}{1 - c} \right| = |\langle p, \hat{p} \rangle| \leq \gamma, \]

where the last bound follows from the spectral theorem for \( D^{-1}C \) and (5.1).

3.1. Truncated normal distributions. Recall that, for \( \sigma > 0 \) and \( p \in [0, 1] \), we denote by \( \epsilon(\sigma^2, p) \) a normal random variable with zero mean, variance \( \sigma^2 \) and conditioned to lie in \([-p, 1 - p] \). When \( \sigma \) is small, the truncation has little effect as long as \( p \) is in the inner part of \([0, 1]\). We quantify this effect, and its failure, by means of uniform bounds on the moments of \( \epsilon(\sigma^2, p) \). Here and after, we will always denote by \( f(x), F(x) \) the density and distribution function, respectively, of a standard normal random variable, which will be denoted by \( Z \).

Lemma 3.2. The following facts hold:

1. For all \( p \in [0, 1] \) and \( \sigma > 0 \), one has \( \epsilon(\sigma^2, p) \mathcal{N}(\sigma^2, 1 - \sigma^2) \);
2. For any \( \sigma > 0 \) and \( p \in [0, \frac{1}{2}] \),

\[ 0 \leq E(\epsilon(\sigma^2, p)) \leq 2\sigma e^{-\frac{\sigma^2}{2\sigma^2}}; \]
3. For any \( \sigma > 0 \) and \( p \in [\frac{1}{2}, 1] \),

\[ -2\sigma e^{-\frac{\sigma^2}{2\sigma^2}} \leq E(\epsilon(\sigma^2, p)) \leq 0; \]
4. For any \( \sigma > 0 \), \( p \in [0, 1] \),

\[ \text{Var}(\epsilon(\sigma^2, p)) \leq \sigma^2, \quad E[(\epsilon(\sigma^2, p))^2] \leq 5\sigma^2. \]
Proof. Part (1) follows from $Z \overset{d}{=} -Z$. Standard computations (see e.g. [3]) show
\[
E(\epsilon(\sigma, p)) = \sigma \frac{f\left(\frac{x}{\sigma}\right) - f\left(\frac{1-x}{\sigma}\right)}{F\left(\frac{1-x}{\sigma}\right) + F\left(\frac{x}{\sigma}\right) - 1},
\]
which gives immediately the lower bound in (3.3). As for the upper bound, if $\sigma \geq \frac{10}{29}$, $p \in [0, \frac{1}{2}]$ there is nothing to prove since the right side in (3.3) – which is monotone in $\sigma$ – is greater than one while $|\epsilon(\sigma^2, p)| \leq 1$. If $\sigma \leq \frac{10}{29}$ we use
\[
E(\epsilon(\sigma, p)) \leq \sigma \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{2\sigma^2}}}{F\left(\frac{1}{2\sigma}\right) - \frac{1}{2}} \leq 2\sigma e^{-\frac{x^2}{2\sigma^2}},
\]
where in the last step we bound
\[
\sqrt{2\pi} [F\left(\frac{1}{2\sigma}\right) - \frac{1}{2}] \geq \int_{-\infty}^{\frac{1}{\sigma}} e^{-\frac{x^2}{2}} dx \geq \frac{1}{2}.
\]
This proves part (2). Combining part (1) and part (2) we get part (3). Another standard computation (see again [3]) gives
\[
\frac{\text{Var}(\epsilon(\sigma^2, p))}{\sigma^2} = 1 - \frac{1}{\sigma} \frac{p f\left(\frac{1}{\sigma}\right) + (1-p) f\left(\frac{1-x}{\sigma}\right)}{F\left(\frac{1}{\sigma}\right) + F\left(\frac{x}{\sigma}\right) - 1} - \left( \frac{f\left(\frac{1}{\sigma}\right) - f\left(\frac{1-x}{\sigma}\right)}{F\left(\frac{1}{\sigma}\right) + F\left(\frac{x}{\sigma}\right) - 1} \right)^2.
\]
(3.6)
The first bound in (4) follows immediately for any $\sigma > 0$, while the second bound is obtained combining (2) with the upper bound on the variance. As for (5), owing to part (1) we can assume $p \in [0, \frac{1}{2}]$ without loss of generality. Since, for fixed $p \neq 0$ and $\sigma \to 0$, the right side in (3.6) converges to one, we only need to show that it is bounded away from 0 as $\sigma \to 0$ and for $p$, say, in $[0, \frac{1}{10}]$. In this case, we can rewrite (3.6) as
\[
\frac{\text{Var}(\epsilon(\sigma^2, p))}{\sigma^2} \sim 1 - x f(x) \frac{F(x)}{F(x)} - \left( \frac{f(x)}{F(x)} \right)^2, \quad x := \frac{p}{\sigma}.
\]
Since, for all $x > 0$, we have the bounds $\frac{f(x)}{F(x)} \leq \sqrt{\frac{2}{\pi}}$ and $F(x) \geq \frac{1}{4}$, we obtain
\[
x \frac{f(x)}{F(x)} + \left( \frac{f(x)}{F(x)} \right)^2 \leq \sqrt{\frac{2}{\pi}} x + \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}} < 1,
\]
where the last bound follows by computing the explicit maximum. \hfill \Box

Remark 3.3. The upper bound in (3.3) shows that the drift for the truncated normal noise induced by the boundaries decays rapidly when $p$ is at distance $O\left(\frac{1}{\sigma}\right)$ from 0 or 1.

Remark 3.4. The lower bound (3.5) can be extended to the case where $\sigma \in (0, K)$, obtaining a corresponding lower bound $\rho_K$. However, $\rho_K \to 0$ as $K \to +\infty$, since the left side is bounded by one. On the other hand, as $K \to 0$, one still has $\lim_{K \to 0} \rho_K < 1$.

Remark 3.5. Combining (2.6) and (3.5), at every step of the Gibbs sampler we get a uniform lower bound on the variance of the truncated normal step given by
\[
\text{Var}(\epsilon(\sigma, p)) \geq \rho \sigma^2,
\]
as long as $A^2 \geq \frac{1}{2\pi}$. This is one of the reason for why $A$ has to be large enough in our result.

3.2. A monotone and contractive coupling. Consider two nearby points $p \leq q$—say in $[0, \frac{1}{2}]$—and perturb them with a truncated normal noise. Because of [3.3], we may hope to couple them in such a way that they will get even closer, since the one to the left will encounter more drift. Owing to the log-concavity of the normal distribution, we can do better, and maintain their order as well.

**Lemma 3.6.** Fix $\delta, \sigma > 0$. Let $p, q \in [0, 1]$ such that $0 \leq p - q \leq \delta$. Then, it is possible to couple two random variables

$$X_p \overset{d}{=} p + \epsilon(\sigma^2, p), \quad X_q \overset{d}{=} q + \epsilon(\sigma^2, q)$$

in such a way that $0 \leq X_p - X_q \leq \delta$.

**Proof.** Let $\Gamma(p, \cdot)$ the inverse distribution function of the random variable $X_p$. If $U$ is uniformly distributed on $[0, 1]$, then $\Gamma(p, U) = X_p$. Using the same noise $U$ for constructing $X_q$, we have

$$X_p - X_q = \Gamma(p, U) - \Gamma(q, U).$$

In particular, owing to the mean value theorem, it suffices to show

$$\frac{\partial \Gamma}{\partial p}(p, u) \in [0, 1], \quad p, u \in [0, 1]. \quad (3.7)$$

Let $\Phi(p, \cdot)$ the distribution function of $X_p$, so that

$$\Phi(p, u) = \frac{\int_0^u f\left(\frac{x - p}{\sigma}\right)dx}{F\left(\frac{1-u}{\sigma}\right) + F\left(\frac{u}{\sigma}\right) - 1}$$

Consider the identity

$$\Phi(p, \Gamma(p, u)) = u, \quad p, u \in [0, 1].$$

Differentiating with respect to $p$ and rearranging, we obtain

$$\frac{\partial \Gamma}{\partial p}(p, u) = 1 - \frac{(1-u)f\left(\frac{u}{\sigma}\right) + uf\left(\frac{1-u}{\sigma}\right)}{f\left(\frac{p - \Gamma(p, u)}{\sigma}\right)}.$$

The upper bound in [3.7] follows immediately. As for the lower bound, we need to show

$$G(p, u) := f\left(\frac{p - \Gamma(p, u)}{\sigma}\right) - (1-u)f\left(\frac{p}{\sigma}\right) - uf\left(\frac{1-u}{\sigma}\right) \geq 0, \quad p, u \in [0, 1].$$

Since $G(p, 0) = G(p, 1) = 0$ for all $p \in [0, 1]$, it is enough to show that $\frac{\partial G}{\partial u}(p, u)$ is decreasing in $u$ for all $p, u \in [0, 1]$. The claim then follows as $G(p, u)$ is concave in $u$ for each $p$. Using the definition of $\Gamma(p, u)$ and inverse differentiation rule we get

$$\frac{\partial G}{\partial u}(p, u) = \frac{p - \Gamma(p, u)}{\sigma} f\left(\frac{p - \Gamma(p, u)}{\sigma}\right) - \frac{f(p)}{\sigma} - \frac{f\left(\frac{1-u}{\sigma}\right)}{\sigma}.$$

Since the only dependence on $u$ is through $\Gamma(p, u)$, which is increasing in $u$, we conclude.

**Remark 3.7.** Fix $\Phi$ log-concave on $\mathbb{R}$. Lemma 3.6 remains valid in the same way once $\epsilon(\sigma^2, p)$ is replaced by $\epsilon(p)$ whose density on $[0, 1]$ is proportional to $\Phi(\cdot - p)$. 
We can equip $\mathbb{R}^d$ with the coordinate-wise partial order, so that $p \leq q$ if and only if $p_i \leq q_i$ for all $i \in [d]$. Owing to Lemma 3.6, we can couple two walkers starting from different points in a monotone and contractive fashion (with respect to the $\| \cdot \|_1$ norm). By using the same uniform random variable $U$ given in the proof of Lemma 3.6 to update multiple walkers, we can actually do better and construct a monotone and contractive grand-coupling for an arbitrary number of initial conditions.

**Lemma 3.8.** Let $C$ be a network on $d$ vertices satisfying Assumption 2 and let $A > 0$. Fix $\delta > 0$, $s \in \mathbb{N} \cup \{\infty\}$, and consider a partially ordered system of $p(1), \ldots, p(s) \in [0,1]^d$. Then, we can construct random variables $p'(1), \ldots, p'(s) \in [0,1]^d$ on a common probability space, with $p'(i) \sim K_{A,C}(p(i))$, so that whenever for some $i, j \in [s]$ we have

$$0 \leq p(i) - p(j) \leq \delta 1,$$

then for the resulting random variables we also have

$$0 \leq p'(i) - p'(j) \leq \delta 1.$$

**Proof.** Choose the same uniformly random coordinate $k \in [d]$ for updating all the $p(i)$’s. Because of (3.8) and convexity, for a pair $i, j \in [s]$ of interest above we obtain

$$0 \leq \hat{p}_k(i) - \hat{p}_k(j) \leq \delta.$$

Finally, we conclude owing to Lemma 3.6 since we can use the same noise to generate all the truncated normal updates at once in a monotone and contractive fashion.

\[ \square \]

3.3. **Clustering on the main diagonal.** While we will show that the mixing time is dictated by the diffusion on the diagonal, it is still important to keep control of how spread out the coordinates are. There are two forces in competition with each other: while the Laplacian tends to attract coordinates together, the normal noise will play the opposite role. A measure of the clustering of the coordinates is given by $\langle p, \Delta p \rangle$ since, under Assumption 1, vanishes only when $p \in D$. The following Lemma describes its evolution when $p$ evolves according to the Gibbs sampler.

**Lemma 3.9.** Let $C$ be a network on $d$ vertices satisfying Assumption 2 and let $A > 0$. For any $k \in \mathbb{N}$ and $p_0 \in [0,1]^d$, let $p(k) \sim K_{A,C}^{d,k}(p_0)$. Then

$$\mathbb{E}\left( \langle p(k), \Delta p(k) \rangle \right) \leq \left( 1 - \frac{\lambda}{d} \right)^k \mathbb{E}\left( \langle p_0, \Delta p_0 \rangle \right) + \frac{5d}{2\lambda A^2}. $$

In particular, if $p_0 \in D$, then

$$\mathbb{E}\left( \langle p(k), \Delta p(k) \rangle \right) \leq \frac{5d}{2\lambda A^2}. \tag{3.9}$$

**Proof.** Write $q' := p(k)$ and $q := p(k - 1)$. According to (2.7), we can write

$$\langle q', \Delta q' \rangle - \langle q, \Delta q \rangle = \langle \Pi_i \left( -\Delta q + \epsilon(\Sigma, \hat{q}) \right), \Delta q + \epsilon(\Sigma, \hat{q}) \rangle.$$

Taking expected values and using $\sum_{i \in [d]} \Pi_i = I$,

$$\mathbb{E}\left( \langle q', \Delta q' \rangle \right) - \mathbb{E}\left( \langle q, \Delta q \rangle \right) = \frac{1}{d} \mathbb{E}\left[ - \langle \Delta q, \Delta q \rangle + \langle \epsilon(\Sigma, \hat{q}), \epsilon(\Sigma, \hat{q}) \rangle \right].$$

Using (3.2), (2.6), and (3.4), we obtain

$$\mathbb{E}\left( \langle q', \Delta q' \rangle \right) \leq \left( 1 - \frac{\lambda}{d} \right) \mathbb{E}\left( \langle q, \Delta q \rangle \right) + \frac{1}{d} \sum_{i \in [d]} \frac{5c_i}{2\lambda_i A^2} = \left( 1 - \frac{\lambda}{d} \right) \mathbb{E}\left( \langle q, \Delta q \rangle \right) + \frac{5}{2A^2}. $$
The result then follows by iterating in \( k \) the inequality above and bounding
\[
\sum_{j=0}^{k} \left(1 - \frac{\lambda}{d}\right)^j \leq \frac{d}{\lambda}.
\]

\[ \square \]

In order to prove the upper bound in Theorem \((2.1)\), we will need a more refined control on the probability of excursions of the Gibbs sampler away from the main diagonal.

**Lemma 3.10.** Let \( C \) be a network on \( d \) vertices satisfying Assumption \([2]\) and \( A^2 \geq \frac{1}{d^2} \). For any \( \delta > 0 \), let \( \eta > 0 \) be such that
\[
\frac{\beta \delta^2}{d} - 2dA^2\eta^2 \geq \eta. \tag{3.10}
\]
Then, we have the bound
\[
\mathbb{P}\left(\left|p_i(t) - \hat{p}_i(t)\right| \geq \delta \text{ for some } i \in [d], t \in [k]\right) \leq 13ke^{-A^2\eta}. \tag{3.11}
\]

**Proof.** Because of \((2.3), (2.5)\), and the bound \( \langle \Delta p, \Delta p \rangle \leq 2\langle p, \Delta p \rangle \), we have the inclusion of events
\[
\left\{\left|p_i(t) - \hat{p}_i(t)\right| \geq \delta \text{ for some } i \in [d], t \in [k]\right\} \subset \left\{\max_{t \in [k]}\langle \Delta p(t), p(t) \rangle \geq \frac{\beta \delta^2}{d}\right\} =: B_k
\]
In what follows, \( I = I_t \) denotes the coordinate selected by the Gibbs sampler at step \( t \). For \( \eta > 0 \) as in \((3.10)\) and \( m := dA^2\eta \), define the events
\[
B_k(\eta) := \left\{\langle \Delta p(t), p(t) \rangle - \langle \Delta p(t-1), p(t-1) \rangle \geq \eta \text{ for some } t \in [k]\right\},
\]
\[
B_k(m) := \left\{\exists T = \{h, \ldots, h + m - 1\} \subset [k], \left|I_{t-1} \neq \arg \max_{i \in [d]} c_i|\Delta p_i(t-1)|^2, t \in T\right\},
\]
\[
B_k(m, \eta) := B_k(\eta) \cup B_k(m) \cdot
\]
We use the notation \( \mathbf{q} = p(t - 1) \), \( \mathbf{q}' = p(t) \). As we have shown in Lemma \(3.9\)
\[
\langle \Delta \mathbf{q}', \mathbf{q}' \rangle - \langle \Delta \mathbf{q}, \mathbf{q} \rangle = -c_1|\Delta \mathbf{q}|^2 + c_1\epsilon_1(\sigma \tilde{q}_t, \tilde{q}_t)|^2. \tag{3.12}
\]
Therefore, if \( Z \) denotes a standard normal random variable, we obtain
\[
\mathbb{P}(B_k(\eta)) \leq k \frac{\mathbb{P}(Z^2 \geq 2A^2\eta)}{\mathbb{P}(Z \in [-2c_1A^2\tilde{q}_t, 2c_1A^2(1 - \tilde{q}_t)])} \leq 6ke^{-A^2\eta},
\]
where we used a union bound, \( A^2c_1 \geq A^2\beta \geq 1 \), \( \mathbb{P}(Z \in [0, 1]) \geq \frac{1}{2} \) and the Chernoff bound for tail probabilities of a standard normal random variable.

As for \( B_k(m) \), a union bound, \( \ln(1 + x) \leq x \ (x \in \mathbb{R}) \) and the definition of \( m \) gives
\[
\mathbb{P}(B_k(m)) \leq k \left(1 - \frac{1}{d}\right)^m \leq ke^{-\frac{m}{d}} \leq ke^{-\eta A^2}
\]
In the case of \( B_k(m, \eta) \), consider the time \( t^* \) at which \( \langle \Delta p(t^*), p(t^*) \rangle \) reaches the target \( \beta \delta^2/\lambda \). Going backwards in time for \( m \) steps, we cannot be further than \( m\eta \) from the target. Also, the coordinate maximizing \( c_i|\Delta p_i(t)|^2 \) was selected at least once in the time interval \([t^* - m, t^* - 1]\). More precisely, the occurrence of
\( B_k(m, \eta) \) implies that there is \( t \in [t^* - m, t^* - 1] \) where, using again the notation \( q = p(t - 1), q' = p(t) \),

\[
\langle \Delta q, q' \rangle \in \left( \frac{\beta \delta^2}{\lambda} - m \eta, \frac{\beta \delta^2}{\lambda} \right), \quad I = \arg \max_{i \in [d]} c_i |\Delta q_i|^2, \quad \langle \Delta q', q' \rangle \geq \frac{\beta \delta^2}{\lambda} - m \eta.
\]

Combining with (3.12) and Lemma 3.1, the occurrence of \( B_k(m, \eta) \) implies

\[
c_I \epsilon_I (\sigma_I^2, \hat{q}_I)^2 \geq -m \eta + c_I |\Delta q_I|^2
\]

\[
\geq -m \eta + \frac{1}{d} \langle \Delta q, \Delta q \rangle
\]

\[
\geq -m \eta + \frac{\lambda}{d} \langle \Delta q, q \rangle
\]

\[
\geq \frac{\beta \delta^2}{d} - \left( \frac{\lambda}{d} + 1 \right) d A^2 \eta^2
\]

\[
\geq \eta
\]

where we also used \( \frac{\lambda}{d} < 1 \), the definition of \( m \) and (3.10). Therefore, using a union bound and \( A^2 \beta \geq 1 \),

\[
\mathbb{P}(B_k') \leq 6k e^{-A^2 \eta}.
\]

Collecting all the terms and using a union bound over all coordinates, we obtain the first claim. As for the second, notice that for \( A \geq A^* (\delta, \beta) \), the choice of

\[
\eta = \frac{\sqrt{\beta \delta}}{\sqrt{2dA}},
\]

satisfies (3.10). Here we used that, since \( \beta \leq \frac{1}{A^2} \) for any network \( C \), we can define \( A^* (\delta, \beta, d) = A^* (\delta, \beta) \).

4. Diffusive behavior

This section is devoted to an in-depth understanding of the long-term movement of the walk both inside the hypercube and also when approaching the boundary, which will serve as the core of the proof of Theorem 2.1. The relevant quantity to follow is the projection onto the diagonal of the hypercube. For a point \( p \in [0, 1]^d \), we define its barycenter by

\[
\overline{p} := \langle p, 1 \rangle.
\]

Let \( p_0 \in [0, 1]^d \), and \( p(k) \sim K_{A,C}(p_0) \). The main intuition behind the \( A^2 \) scaling appearing in Theorem 2.1 is twofold: the evolution of \( \overline{p}(k) \), the barycenter of \( p(k) \), resembles that of a simple random walk, and also once \( \overline{p}(k) \) gets close to stationarity, all coordinates of \( p(k) \) will be clustered together. The first claim is misleading – the evolution of \( \overline{p}(k) \) involves jumps and the knowledge of the whole chain – but we will show how a certain diffusive-like behavior emerges when \( A \) is sufficiently large. As for the second claim, since we work in the Wasserstein metric (2.8), it is enough to control how clustered are the coordinates by the time \( \overline{p}(k) \) has diffused, that we have already covered in Lemma 3.10.

4.1. Anticoncentration vs. concentration. Before analyzing the walker’s barycenter, we need to properly understand the stationary distribution \( \pi_{A,C} \), for \( C \) fixed and \( A \) large enough. Under \( \pi_{A,C} \) we expect the barycenter to be roughly uniformly distributed on the unit interval. We are able to obtain the following quantitative statement, which gives sharp anti-concentration bounds. In the following, \( \pi_{A,C} \) will denote the distribution of \( \overline{p} \), where \( p \sim \pi_{A,C} \).
Lemma 4.1. Let $C$ be a network on $d$ vertices satisfying Assumption 1. Let $\delta > 0$ and let $A$ be such that
\begin{equation}
A^2 \geq \frac{e}{2\lambda \delta^2}.
\end{equation}
Then, for every $s \in (\delta, 1-\delta)$, one has
\begin{equation}
\pi_{A,C}([s, 1-s]) \leq \frac{1 - 2s}{(1 - 2\delta)^2}.
\end{equation}
In particular, for any $\delta > 0$ and all $A \geq A^e(\delta, \beta, \lambda)$, if $\overline{\rho} \sim \pi_{A,C}$ then
\begin{equation}
E\left(\left|\overline{\rho} - \frac{1}{2}\right|\right) \geq \frac{1}{4} - \delta.
\end{equation}

Proof. With a bit of abuse of notation, let $\pi_{A,C}(t)$ denote also the density of $\pi_{A,C}$ at $t \in [0, 1]$. Write $p = p_1 + q$ with $q \in D_\perp$. By means of (2.4) we can write
\begin{equation}
\pi_{A,C}(t) = \int_{q \in [-t(1-t)]^1 \cap D_\perp} e^{-A^2(q, \Delta q)} dq.
\end{equation}
For any $t \in [0, 1]$, the right side in (4.5) is upper bounded by
\begin{align*}
\int_{D_\perp} e^{-A^2(q, \Delta q)} dq &= \frac{Z(C)}{A^{d-1}}, \quad Z(C) := \int_{D_\perp} e^{-\langle q, \Delta q \rangle} dq < +\infty,
\end{align*}
where the $(d-1)$-scaling appears since we are integrating on a $(d-1)$ subspace. On the other hand, for $t \in (\delta, 1-\delta)$ we have
\begin{align*}
\frac{Z(C)}{A^{d-1}} \pi_{A,C}(t) &\leq \frac{1}{A^{d-1}} \int_{D_\perp \cap [-A\delta, A\delta]^c} e^{-\langle q, \Delta q \rangle} dq \\
&\leq \frac{1}{A^{d-1}} \frac{1}{\lambda \beta d A^2 \delta^2} \int_{D_\perp} e^{-\langle q, \Delta q \rangle} dq \\
&\leq \frac{Z(C)}{A^{d-1}} \frac{e}{\lambda \beta A^2 \delta^2}.
\end{align*}
Here, in the second last inequality we used that, owing to (2.3) and (3.1),
\begin{equation*}
\langle q, \Delta q \rangle \geq \lambda \langle q, q \rangle \geq \lambda \beta \sum_{i=1}^d q_i^2 \geq \lambda \beta d A^2 \delta^2, \quad q \in [-A\delta, A\delta]^c,
\end{equation*}
while the last inequality follows from $x \leq e^x$ ($x \in \mathbb{R}$), a change of variables and
\begin{equation*}
\left(1 - \frac{1}{d}\right)^{d-1} \geq \frac{1}{e}, \quad d > 1.
\end{equation*}
All together, for $A$ as in (4.2), we obtain
\begin{equation*}
\inf_{t, t' \in [\delta, 1-\delta]} \frac{\pi_{A,C}(t)}{\pi_{A,C}(t')} \geq 1 - 2\delta.
\end{equation*}
Therefore, for $s \in [\delta, 1-\delta]$,
\begin{equation*}
\pi_{A,C}([s, 1-s]) \leq \frac{\pi_{A,C}([s, 1-s])}{\pi_{A,C}([\delta, 1-\delta])} \leq \frac{1 - 2s}{(1 - 2\delta)^2},
\end{equation*}
which concludes the proof of (4.3). As for (4.4), fix $\delta > 0$, and apply (4.3) with $\delta' = \delta'(\delta)$ such that
\begin{equation*}
1 - \frac{1}{(1 - 2\delta')^2} \leq \frac{\delta}{2}, \quad \delta' \leq \frac{\delta}{2}.
\end{equation*}
and choose $A$ large accordingly. Then

$$
E\left(\left|\frac{1}{2} - \frac{1}{2}\right|\right) = \int_0^1 \left(1 - \pi_{A,C}([s, 1 - s])\right) ds
$$

$$
\geq \int_0^{\frac{1}{2}-\delta} (1 - (1 - 2s)) ds - \frac{\delta}{2}
$$

$$
\geq \int_0^{\frac{1}{2}} 2s ds - \delta
$$

$$
= \frac{1}{4} - \delta.
$$

Consider now the situation where $p_0 = \frac{1}{2}1$. After few steps of the Gibbs sampler, coordinates will not have enough time to travel away from the middle. We use the following submartingale approach to quantify the concentration around the mean.

**Lemma 4.2.** Let $C$ be a network on $d$ vertices satisfying Assumption[1] and $A > 0$. Let $p_0 = \frac{1}{2}1$, $p(k) \sim K_{A,C}(p_0)$ with barycenter $\bar{p}(k)$. Then, for every $\delta > 0$,

$$
E\left[\left|\bar{p}(k) - \frac{1}{2}\right|^2\right] \leq \frac{27k\gamma}{\lambda A^2}.
$$

In particular, for $m = m(\delta)$ sufficiently small and $k = m\frac{1}{2}A^2$,

$$
E\left[\left|\bar{p}(k) - \frac{1}{2}\right|\right] \leq \delta.
$$

**Proof.** Since the walk is invariant under reflection of the hypercube with respect to $p_0$, we get immediately $E(\bar{p}(k)) = \frac{1}{2}$ for all $k$. For the sake of convenience, in the following we write $q := p(k - 1), \bar{q} := p(k)$ (and, correspondingly, $q, q'$ for their barycenters), and drop the dependence on $k$ for the moment. From (2.7) and (4.1), we get

$$
\left(q' - \frac{1}{2}\right)^2 - \left(q - \frac{1}{2}\right)^2 = 2\left(q - \frac{1}{2}\right)\langle -\Delta(q) + \epsilon(\Sigma, q), \Pi, 1\rangle
$$

$$
+ \left(\langle -\Delta q + \epsilon(\Sigma, \bar{q}), \Pi, 1\rangle\right)^2.
$$

Notice the identity

$$
\sum_{i \in [d]} \Pi_i v = v, \quad \sum_{i \in [d]} |\langle v, \Pi_i 1\rangle|^2 = \langle Dv, v\rangle, \quad v \in \mathbb{R}^d,
$$

and the fact $\Delta q \in \mathcal{D}_\perp$. We now take expectation on both sides and use the tower property to exploit the above (first conditioning with respect to $q$, then with respect to $\epsilon$). The right side then becomes

$$
\frac{2}{d^2}E\left(\left(q - \frac{1}{2}\right)\langle \epsilon(\Sigma, \bar{q}) | q\rangle, 1\right) + \frac{1}{d}E\left(\langle D(\Delta q + \epsilon(\Sigma, \bar{q})), (\Delta q + \epsilon(\Sigma, \bar{q}))\rangle\right).
$$

Since $\Delta q \in \mathcal{D}_\perp$, $q - \frac{1}{2} = \langle q - \frac{1}{2}1, 1\rangle = \langle q - \frac{1}{2}1, 1\rangle$. Therefore, if we expand the inner products in the first summand and an easy application on Cauchy-Schwartz...
on the second one,
\[ E\left( \left( \bar{q} - \frac{1}{2} \right)^2 - \left( \bar{q} - \frac{1}{2} \right)^2 \right) \leq \frac{2}{d} \sum_{i,j \in [d]} c_i c_j E\left( (\bar{q}_i - \frac{1}{2}) E(\epsilon_j(\sigma_j, \hat{q}_j)|q) \right) \]
\[ + \frac{2}{d} E(\langle D \Delta q, \Delta q \rangle) \]
\[ + \frac{2}{d} E(\langle D \epsilon(\Sigma, \hat{q}), \epsilon(\Sigma, \check{q}) \rangle) \]
\[ =: E_1 + E_2 + E_3. \]

We start bounding \( E_1 \). Owing to (3.3), positive contribution only comes from terms where \( \hat{q}_i \) and \( \hat{q}_j \) have opposite signs, in which case \( |\hat{q}_i - \frac{1}{2}| \leq |\hat{q}_i - \hat{q}_j| \). Thanks to this observation,
\[ E_1 \leq \frac{2}{d} \sum_{i,j \in [d]} c_i c_j E\left( (|\hat{q}_i - \hat{q}_j|) E(\epsilon_j(\sigma_j, \hat{q}_j)|q) \right) \]
\[ \leq \frac{2\sqrt{2}}{dA} \sum_{i,j \in [d]} c_i \sqrt{c_j} \left| \hat{q}_i - \hat{q}_j \right| \]
\[ \leq \frac{2\sqrt{2}}{dA} \left( \sum_{i,j \in [d]} c_i c_j \right) \frac{1}{2} \left( \sum_{i,j \in [d]} c_i \right)^{-\frac{1}{2}} \]
\[ = \frac{4}{Ad} \left( \sqrt{\langle \check{q} - \check{q}_1, \hat{q} - \check{q}_1 \rangle} \right)^{\frac{1}{2}} \]
where we used (3.3), the Cauchy-Schwartz inequality and Pythagorean theorem. Finally, using (3.2), (3.1) and (3.9) we obtain
\[ E_1 \leq \frac{4\gamma}{A\sqrt{d}} \left( \sqrt{\langle \check{q} - \check{q}_1, \hat{q} - \check{q}_1 \rangle} \right)^{\frac{1}{2}} \]
\[ \leq \frac{4\gamma}{A\sqrt{d} \sqrt{A}} \left( \sqrt{\langle \check{q}, \Delta q \rangle} \right)^{\frac{1}{2}} \]
\[ \leq \frac{2\sqrt{10}\gamma}{\lambda A^2}. \]

As for \( E_2 \), by means of (3.1), (3.9) and Lemma 3.1 we can bound
\[ E_2 \leq \frac{4}{d} \max_{e \in [d]} c_e E(\langle \Delta q, q \rangle) \]
\[ \leq \frac{10\gamma}{\lambda A^2}. \]

Finally, for \( E_3 \) we can use (3.4), Lemma 3.1 and \( \lambda \leq 2 \) to obtain
\[ E_3 \leq \frac{2}{d} \max_{e \in [d]} c_e \sum_{i,j \in [d]} \frac{5}{2} \frac{c_i A^2}{2c_i A^2} \]
\[ \leq \frac{10\gamma}{\lambda A^2}. \]

Combining the bounds for \( E_1, E_2, E_3 \) and using a telescoping sum, we obtain
\[ E\left( \left( \hat{p}(k) - \frac{1}{2} \right)^2 \right) \leq \frac{27k\gamma}{\lambda A^2}. \]
The last statement then follows at once from Jensen inequality. \( \square \)

The claim of Lemma 4.2 is in line with simulation results as shown below in Figure 1 for the complete graph with uniform weights. Observe that the value stabilizes around \( \approx \frac{1}{12} \) when \( A \) large, corresponding to the variance of the uniform
distribution on $[0, 1]$ (a good approximation of $\pi_{A,C}$ for $A$ large because of Lemma 4.1). The scaling of the variance (quadratic in $A$, linear in $d$ because of Remark 2.3) agrees with our bound.

![Sample paths](image)

**Figure 1.** A few sample paths of the evolution of the squared distance of the barycenter from $\frac{1}{2}$ together with an empirical mean of 1000 runs for each, for different values of $d$ and $A$, considering the complete graph with uniform weights.

4.2. A stopping time for the boundary. In this subsection, $p_0 := 0$, and let $p(k) \sim K_{A,C}(p_0)$. For $\delta \in (0, 1)$, consider the stopping time

$$T_\delta = \min_{k \in \mathbb{N}} \{1 - \hat{p}_i(k) \leq \delta \text{ for some } i \in [d]\}. \quad (4.7)$$

Since the chain is ergodic and $\{1 - \hat{p}_i \leq \delta \text{ for some } i \in [d]\}$ occurs with positive probability according to the stationary distribution, it is easy to show that $T_\delta$ has finite expectation. For $H > 0$, consider the random variable

$$w_H := p(T_\delta)^2 - HT_\delta. \quad (4.8)$$

The reason to consider such an object is the intuition that $p$ is not so different from a random walk with normal steps in the bulk on the interval. Since, by time $T_\delta$, there should not be a significant negative drift, the hope is that for $H = O\left(\frac{1}{\pi^2}\right)$ (with constant depending on all graph parameters) the expected value of $w_H$ will be positive, which can then be turned in an upper bound for $E(T_\delta)$. This is done in the following Lemma.
Lemma 4.3. Let $C$ be a network on $d$ vertices satisfying Assumption \[4.2\]. For $\delta \in (0, 1)$ and $A \geq \frac{1}{2\delta^2}$, let

$$H := \frac{\rho}{2dA^2} - \frac{2\sqrt{2}}{\sqrt{A^2/d}} e^{-\delta^2 \beta A^2} > 0. \quad (4.9)$$

Then, $E(w_H) > 0$. In particular, if $A \geq A^*(\delta, \beta)$, then

$$E(T_\delta) \leq \frac{4dA^2}{\rho} \quad (4.10)$$

Proof. Since $T_\delta$ is finite almost surely, summation by parts gives

$$E(w_H) = \sum_{j=1}^{\infty} E\left(1_{\{T_j = j\}} (q^2(j) - Hj)\right) = \sum_{j=1}^{\infty} E\left(1_{\{T_j = j-1\}} \left(q^2(j) - q^2(j-1) - H\right)\right).$$

Since $\{T_\delta > j-1\}$ is measurable with respect to the $\sigma$-algebra generated by $p(j-1)$, it suffices to show that on its support

$$E(q^2(j) - q^2(j-1) | p(j-1)) \geq H. \quad (4.11)$$

We drop the dependence on $j$, and denote $p(j) = q'$, $p(j-1) = q$. Proceeding as in the proof of Lemma 4.2 using (4.1) and (2.7) we obtain

$$E(q^2 - q^2 | q) = \frac{2\eta}{d} E(\epsilon(\Sigma, q)|q) + \frac{1}{d} E(|D(q + \epsilon(\Sigma, q))|q)$$

$$=: E_1 + E_2.$$

On the event $\{T_\delta > j-1\}$, we have $q \leq (1 - \delta)1$. If we expand the inner product appearing in $E_1$, because of (3.3) we only get negative contributions if $q_i > \frac{1}{2}$. Therefore, we can lower bound

$$E_1 \geq -\frac{2}{d} \sum_{i=1}^{\infty} \epsilon_i \frac{2}{\sqrt{2A^2 \sigma_i^2}} e^{-\delta^2 \sigma_i^2 A^2}$$

$$\geq -\frac{2\sqrt{2}}{A^2 d} e^{-\delta^2 \beta A^2},$$

where we used Cauchy-Schwartz in the last inequality. As for the second term, the definition of variance and (3.5) gives, for every $A \geq \frac{1}{2\delta^2}$,

$$E_2 = \frac{1}{d} \sum_{i=1}^{d} \epsilon_i^2 \left[\left(\epsilon_i (\epsilon_i^2, \hat{q}) - \epsilon_i \hat{q}_i\right)^2 \right]$$

$$\geq \frac{\rho}{d} \sum_{i=1}^{d} \epsilon_i^2 \frac{1}{2\epsilon_i A^2}$$

$$\geq \frac{\rho}{2dA^2}.$$ Therefore, using the definition of $H$ in (4.9) we obtain

$$E(p^2(j) - p^2(j-1) | p(j-1)) \geq E_1 + E_2 \geq H,$$

which proves (4.11). The last statement follows at once from the constraint $p \in [0, 1]^d$. \[\square\]

To get a numerical intuition, see Figure 2 showing the average values of $T_{0.05}$ for a range of $A$ and two distinct value of $d$. The complete graph with uniform weights is used once again for demonstration. The quadratic increase in $A$ and the linear
dependence on \( d \) are visible from the plots. The need for a large enough \( A \) is also apparent.

![Figure 2](image1.png)

**Figure 2.** Mean value of the hitting time of \( T_{0.05} \), with over 1000 runs for each value of \( A \). The stripe with a width of the empirical standard deviation is also shown.

5. **Stitching the elements**

We have now collected all the necessary elements towards our final goal, it remains to put all parts into place.

**Proof of Theorem 2.1.** For the lower bound, let \( p_0 = \frac{1}{2} \textbf{1} \) and choose the optimal coupling \( \langle p(k), q \rangle \) as needed for \( (2.8) \), where \( p(k) \sim K_{A,C}^{*k}(p_0) \) and \( q \sim \pi_{A,C} \).

Notice that \( \| p - q \|_{\infty} \geq \| \pi(k) - \pi \| \) by convexity. For given \( \delta > 0 \), take \( A \) is large enough as in \( (4.4) \) and \( k = m(\delta)\frac{2}{3}A^2 \) as in Lemma 4.2. We can combine the concentration result \( (4.6) \) from Lemma 4.2 together with the anti-concentration result \( (4.4) \) from Lemma 4.1 and obtain

\[
d_x(K_{A,C}^{*k}(p_0), \pi_{A,C}) \geq E(\| p(k) - q \|_{\infty}) \\
\geq E(\| \pi(k) - \pi \|) \\
\geq E\left(\| \pi - \frac{1}{2} \| - E\left(\| \pi(k) - \frac{1}{2} \|\right)\right) \\
\geq \frac{1}{4} - 2\delta,
\]

from which the lower bound follows since \( \delta \) is arbitrary.

We now turn to the upper bound. For a given \( \delta > 0 \), pick \( M = M(\delta) \) to be fixed later, and \( k = MdA^2 \). Given \( p_0 \in [0, 1] \), consider three walkers, one starting at \( p_0 \), one starting at \( 0 \), and one starting at stationarity. If we couple the updates according to Lemma 3.8, once the walker starting at \( 0 \), whose position at step \( k \) will be denoted by \( p(k) \), is within \( \delta \) (in \( \| \cdot \|_{\infty} \)) from \( 1 \), the walker starting from \( p_0 \) will remain within \( \delta \) from the stationary one. Because of that, for \( \delta \in (0, \frac{1}{2}) \) it is convenient to define

\[
T^\prime_\delta := \inf_{k \in \mathbb{N}} \{ 1 - p(k) \leq 2\delta \}.
\]

The considerations above give

\[
\sup_{p_0 \in [0,1]} d_x(K_{A,C}^{*k}(p_0), \pi_{A,C}) \leq 2\delta + \mathbb{P}(T^\prime_\delta > k).
\]
Notice the inclusion of the events
\[ \{ T_\delta > k \} \cup \{ \| p_i(t) - \tilde{p}_i(t) \| \geq \delta \text{ for some } i \in [d], t \in [k] \} . \]

Using a union bound, Markov inequality, the large deviation bound \((3.11)\) from Lemma 3.10 and the bound for \(E(T_\delta)\) in \((4.10)\) from Lemma 4.3, we have that for \(A \geq A^\delta(\delta, \beta)\)
\[
\sup_{p \in [0,1]^d} d_x(K_{A,C}^k(\mathbf{p}), \pi_{A,C}) \leq 2\delta + \frac{4dA^2}{k\rho} + 13ke^{-\frac{\sqrt{d}\beta A}{k}} \\
\leq 2\delta + \frac{4}{\rho M} + 13MdA^2e^{-\frac{\sqrt{d}\beta A}{M}} .
\]
Taking \(M(\delta)\) large enough, upon increasing \(A^\delta\) we obtain
\[
\sup_{p \in [0,1]^d} d_x(K_{A,C}^k(\mathbf{p}), \pi_{A,C}) \leq 4\delta,
\]
from which the upper bound follows since \(\delta\) is arbitrary. \(\square\)

The analogous simulation on \(T_\delta^\prime\) we have seen for \(T_\delta\) in Figure 2 is presented below in Figure 3, for the complete graph with uniform weights, and the same values of \(d\) considered there. The simulations confirm the intuition that for \(A\) large enough, once the maximum is within \(\delta\) of \(1\), all of \(\mathbf{p}\) is within \(2\delta\) thanks to the clustering (with small exceptional probability). The need for a large enough \(A\) is visible again.

\begin{figure}[h]
\centering
\includegraphics[height=0.3\textwidth]{figure3a.png} \hspace{1cm} \includegraphics[height=0.3\textwidth]{figure3b.png}
\caption{Mean value of the hitting time of \(T_{0.05}^\prime\), with over 1000 runs for each value of \(A\). The stripe with a width of the empirical standard deviation is shown.}
\end{figure}

6. Open problems and future directions

6.1. From Wasserstein to total variation. Using the concentration and anti-concentration bounds in Lemma 4.2 and Lemma 4.1, together with Chebyshev inequality, it is easy to upgrade the lower bound in Wasserstein distance to a lower bound \(1 - \delta\) in total variation after \(k = m(\delta)\frac{1}{\delta}A^\delta\) steps.

The situation is more complicated for the upper bound. Notice that because of conditions \((1.9)\) and \((3.10)\), our strategy only allows push two walkers within \(\delta \approx A^{-1}\) of each other. On the other hand, in order to couple a even a single coordinate in total variation requires the two walkers to be closer than \(O(A^{-1})\), which is the order of the variance for the single-step update in the Gibbs sampler. We believe
that this is just a limitation of our method (that force the walkers to couple in the vicinity of the corner), but at the present stage we do not know how to overcome the issue.

6.2. Sharper graph-dependent bounds. Upper and lower bound in Theorem 2.1 are close to each other only for small perturbations of complete graphs, thus failing to provide a sharp answer to the following two interesting questions.

A first one comes from the statistical application, and concerns the case $d$ small, but $c_{ij}$ far from being equal. An example, analyzed in [8], concerns the probability of positive feedback from a certain drug in a population split into four categories: smoking men, smoking women, non-smoking men and non-smoking women (we label them 1, 2, 3, 4 respectively). It is then reasonable to start with a prior (the weights are normalizes according to Assumption 1)

$$\frac{1}{4} - 2\epsilon = c_{12} = c_{34} \gg c_{13} = c_{14} = c_{23} = c_{24} = \epsilon.$$ 

If we assume that smoking, rather than sex, is the main factor in the effectiveness of the drug (in their example, $\epsilon \approx 10^{-6}$). Using Theorem 2.1 and a direct computation of the eigenvalues shows that after $k \approx 2A$ steps the chain is not mixed yet in Wasserstein distance. On the other hand, after $k \gg A^2$ the chain is mixed. When $\epsilon$ is small, our result leaves a gap to be filled.

Another question, more mathematically flavored, concerns family of large graphs other than the complete one. For example, for a cycle in $d$ dimensions ($d$ large), our Theorem 2.1 shows that $d^{-2}A^2$, are needed for the mixing, while $dA^2$ suffices. This leaves a $d^2$ factor between the two bounds, raising the question of where does the truth lie in between. It would be interesting to sharpen the results and understand better the relation with the geometry of the graphs.

6.3. The posterior case. While our analysis deals with prior distributions for almost exchangeable experiments, it is interesting future work to analyze the posterior case, once data are selected. As detailed in [11], assume we start with a prior $\pi_{A,C}$ as defined in (1.3), for some network $C$ on $d$ vertices, and data are sampled from some distribution. If $n = (n_i)_{i \in [d]}$ data are collected in each category, and $r = (r_i)_{i \in [d]}$ is the vector of positive outcomes in each category, mild assumptions on the true distributions guarantee – for $n \approx n^2$ large – a posterior of the form

$$\tilde{\pi}_{n,r,A,C}(p) \propto \exp\{-A^2 \sum_{i<j} c_{ij}(p_i - p_j)^2 - nR_{n,r}(p)\},$$

where $R$ is the quadratic polynomial

$$R_{n,r}(p) := \sum_{i \in [d]} \left( \frac{p_i - p_i^*}{\sigma_i} \right)^2, \quad p_i^* := \frac{r_i}{n_i}, \quad \sigma_i^2 := p_i^*(1 - p_i^*) \frac{n_i}{n}.$$

The regime of interest for the measure $\tilde{\pi}_{n,r,A,C}$ lies, as phrased by de Finetti [8], “in the penetration it affords into intermediate situations where the influence of initial opinion and experience balance each other”. In our language, this amounts to set $n = nA^2$ for some fixed $n > 0$, so that the posterior becomes of the form (we drop for convenience some of the dependencies)

$$\tilde{\pi}_{A,Q} \propto \exp\{-A^2Q(p)\}$$

where $Q$ is the quadratic form

$$Q(p) := \sum_{i<j} c_{ij}(p_i - p_j)^2 + nR_{n,r}(p).$$
Unlike the prior distribution, the zero set of $Q$ is not necessarily the main diagonal. Moreover, the situation is complicated by the possible lack of convexity in $Q$. Even simple examples in two dimensions – e.g. data coming from a unit mass at $(1,0)$ or at $(\frac{1}{2}, \frac{1}{2})$ – suggest an interesting behavior and seem worthy of further investigation.

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REFERENCES

[1] S. Bacallado, P. Diaconis, and S. Holmes. de Finetti Priors using Markov chain Monte Carlo computations. *Statistics and Computing*, 25(4):797–808, 2015.
[2] N. Berger, C. Kenyon, E. Mossel, and Y. Peres. Glauber dynamics on trees and hyperbolic graphs. *Probability Theory and Related Fields*, 131(3):311–340, 2005.
[3] J. Burkardt. The truncated normal distribution. 2014.
[4] J. Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Proceedings of the Princeton conference in honor of Professor S. Bochner*, pages 195–199, 1969.
[5] N. Chopin. Fast simulation of truncated gaussian distributions. *Statistics and Computing*, 21(2):275–288, Jan. 2010.
[6] B. de Finetti. Funzione caratteristica di un fenomeno aleatorio. *Atti della R. Accademia Nazionale dei Lincei, Ser. 6. Memorie, Classe di Scienze Fisiche, Matematiche e Naturali*, 4, pages 251–299, 1931.
[7] B. de Finetti. *Sur la condition d’”Egalivalence partelle.”*, 1938.
[8] B. de Finetti. *Probability, Induction and Statistics: The Art of Guessing*. Wiley Series in Probability and Statistics: Probability and Statistics Section Series. J. Wiley, 1972.
[9] B. de Finetti. *Probabilism*. *Erkenntnis*, 31(2-3):169–223, 1989.
[10] P. Diaconis. The cutoff phenomenon in finite Markov chains. *Proceedings of the National Academy of Sciences*, 93(4):1659–1664, 1996.
[11] P. Diaconis, K. Khare, and L. Saloff-Coste. Gibbs Sampling, Exponential Families and Orthogonal Polynomials. *Statistical Science*, 23(2):151–178, 2008.
[12] P. Diaconis and D. Stroock. Geometric bounds for eigenvalues of Markov chains. *The Annals of Applied Probability*, pages 36–61, 1991.
[13] J. Ding, E. Lubetzky, and Y. Peres. The mixing time evolution of glauber dynamics for the mean-field Ising model. *Communications in Mathematical Physics*, 289(2):725–764, 2009.
[14] A. Frieze and E. Vigoda. A survey on the use of Markov chains to randomly sample colourings. 2005.
[15] S. Geman and D. Geman. Stochastic Relaxation, Gibbs Distributions, and the Bayesian Restoration of Images. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-6(6):721–741, 1984.
[16] A. Genz. Numerical Computation of Multivariate Normal Probabilities. *Journal of Computational and Graphical Statistics*, 1(2):141–149, 1992.
[17] B. Gerencsér. Mixing time of an unaligned Gibbs sampler on the square. *Stochastic Processes and their Applications*, 129(9):3570–3584, 2019.
[18] W. Gilks, S. Richardson, and D. Spiegelhalter, editors. *Markov Chain Monte Carlo in Practice*. Chapman and Hall/CRC, 1996.
[19] G. L. Jones and J. P. Hobert. Honest Exploration of Intractable Probability Distributions via Markov Chain Monte Carlo. *Statistical Science*, 16(4):312–334, 2001.
[20] O. Kallenberg. *Probabilistic symmetries and invariance principles*. Springer-Verlag, 2005.
[21] K. Khare and H. Zhou. Rates of convergence of some multivariate Markov chains with polynomial eigenfunctions. *The Annals of Applied Probability*, 19(2):737–777, 2009.
[22] D. A. Levin, M. J. Luczak, and Y. Peres. Glauber dynamics for the mean-field Ising model: cut-off, critical power law, and metastability. *Probability Theory and Related Fields*, 146(1-2):223, 2010.
[23] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, 2016.
[24] D. Panchenko. *The Sherrington-Kirkpatrick model*. Springer Science & Business Media, 2013.
[25] J. S. Rosenthal. Convergence Rates for Markov Chains. *SIAM Review*, 37(3):387–405, 1995.
[26] N. Rougerie. De Finetti theorems, mean-field limits and Bose-Einstein condensation. *arXiv preprint arXiv:1506.05263*, 2015.
[27] A. Smith. A Gibbs sampler on the n-simplex. The Annals of Applied Probability, 24(1):114–130, Feb. 2014.
[28] E. Vigoda. Improved bounds for sampling colorings. Journal of Mathematical Physics, 41(3):1555–1569, 2000.

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