Anderson’s Orthogonality Catastrophe for One-dimensional Systems

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Abstract. We derive rigorously the leading asymptotics of the so-called Anderson integral in the thermodynamic limit for one-dimensional, non-relativistic, spin-less Fermi systems. The coefficient, $\gamma$, of the leading term is computed in terms of the S-matrix. This implies a lower and an upper bound on the exponent in Anderson’s orthogonality catastrophe, $\tilde{C}N^{-\gamma} \leq D_N \leq CN^{-\gamma}$ pertaining to the overlap, $D_N$, of ground states of non-interacting fermions.

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1. Introduction

In 1967, P.W. Anderson \cite{Anderson67} studied the transition probability between the ground state of $N$ free fermions and the ground state of $N$ fermions subject to an exterior (radially symmetric) potential in $\mathbb{R}^3$. Interestingly, he found that this probability decays like $N^{-\gamma}$ with some explicit $\gamma > 0$ (in terms of phase shifts of the potential) as $N \to \infty$. Here, we give a rigorous analysis of this so-called orthogonality catastrophe for one-dimensional systems.

To begin with, let us briefly sketch the many-particle problem underlying our considerations. The state space of $N$ fermions is the $N$-fold antisymmetric tensor product

$$\mathcal{H}^N := \mathcal{H} \wedge \ldots \wedge \mathcal{H} \quad \text{N-times}$$

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of some one-particle space $\mathcal{H}$ (e.g. $\mathcal{H} = L^2(\Omega) \otimes \mathbb{C}^s$, $\Omega \subset \mathbb{R}^d$, $s,d \in \mathbb{N}$) where a one-particle Hamilton operator $H : D(H) \to \mathcal{H}$ is defined. Since we assume our particles to not interact the corresponding operator $H^N$ on $\mathcal{H}^N$ is simply a sum

$$H^N := H \wedge 1 \wedge \ldots \wedge 1 + \cdots + 1 \wedge \ldots \wedge 1 \wedge H.$$ 

If $H$ has a discrete spectrum consisting of (simple) eigenvalues $\lambda_1 < \lambda_2 < \cdots$ with corresponding eigenvectors $\varphi_1, \varphi_2, \ldots$ one can easily construct the analogous $N$-particle quantities. In particular, the ground state $\varphi^N$ is a Slater determinant and the eigenvalue $\lambda^N$ a sum, i.e.

$$\varphi^N = \varphi_1 \wedge \ldots \wedge \varphi_N, \lambda^N = \lambda_1 + \cdots + \lambda_N.$$ 

Note that the definition of the wedge product contains the factor $(N!)^{-1/2}$ whereby the product of normalized vectors automatically becomes normalized. Let $H^N_V := H^N + V$ be a second operator on $\mathcal{H}^N$ with (simple) eigenvalues $\mu_1 < \mu_2 < \cdots$ and eigenvectors $\psi_1, \psi_2, \ldots$. The operator $H^N_V$ is defined analogously to $H^N$ and thus the new ground state and its energy are

$$\psi^N = \psi_1 \wedge \ldots \wedge \psi_N, \mu^N = \mu_1 + \cdots + \mu_N.$$ 

The transition probability, $\mathcal{D}_N$, studied by Anderson is given through the scalar product

$$\mathcal{D}_N := |(\varphi^N, \psi^N)|^2 = |\det((\varphi_j, \psi_k))_{j,k=1,\ldots,N}|^2. \quad (1.1)$$ 

It can be estimated (see 5.21) as

$$\mathcal{D}_N \leq e^{-I^N}, \quad I_N := \sum_{j=1}^{N} \sum_{k=N+1}^{\infty} |(\varphi_j, \psi_k)|^2. \quad (1.2)$$ 

Here, $I_N$ is the so-called 'Anderson integral' which is the object of our main interest. The asymptotics we wish to analyze involves a second parameter $L$ reflecting the system length so that $\mathcal{H} = \mathcal{H}_L = L^2([0, L]^d)$ is the Hilbert space of (spin-less) fermions confined to the box $[0, L]^d$. Therefore, we work with a sequence of Hilbert spaces $\mathcal{H}_L$ and ground states $\varphi^N = \varphi^N_L, \psi^N = \psi^N_L$ with $L > 0$. In the thermodynamic limit we let $N, L \to \infty$ with the particle density $\rho = N/L^d$ being kept fixed. The main result (Theorem 5.3) is an asymptotic formula for the Anderson integral

$$I_{N,L} = \gamma \ln N + O(1), \quad N, L \to \infty,$$

in dimension $d = 1$ and with a slightly different convention for the box size (namely $2L$ instead of $L$) and the density $\rho = (N + \frac{1}{2})/(2L)$. The coefficient can be computed explicitly, Corollary 5.4

$$\gamma(\nu) = \frac{1}{\pi^2} (1 - \text{Re} t(\sqrt{\nu})), \quad \nu := \pi^2 \rho^2,$$

where $t(\sqrt{\nu})$ is the transmission coefficient at energy $\nu$ (cf. [3]). Scattering theory tells us (see [3], [10]) that usually $\gamma(\nu) > 0$ in which case the transition probability behaves precisely as (Corollary 5.6)

$$\tilde{C} N^{-\gamma(\nu)} \leq \mathcal{D}_{N,L} \leq C N^{-\gamma(\nu)}, \quad N, L \to \infty.$$
Here, $\tilde{\gamma}(\nu) > 0$ can be derived from $\gamma(\nu)$.

The main ingredient of the proof is an integral formula for $I_{N,L}$ (Proposition 2.1), which holds true under rather general conditions. It rests essentially upon the Riesz integral formula for spectral projections and Krein’s resolvent formula. In order to adapt it to Schrödinger operators we derive a resolvent formula involving abstract differentiation and multiplication operators (Proposition 2.2). Via this formula, a sequence of scalar functions comes into play which tends at least informally to a Dirac delta function. This is made precise in Sections 3.3 and 4, hence the name delta-term and delta-estimate. The singularity represented by the delta sequence reflects in a way the singular transition from a discrete spectrum to a continuous spectrum as $L \to \infty$.

Our method requires a rather detailed and precise knowledge of the free Dirichlet problem, in particular of the resolvent. Almost everything one needs to know about the perturbed problem, however, can be read off from the so-called T-operator. The perturbed eigenvalues do not enter in the actual asymptotic analysis. We only need to make sure that the number of perturbed eigenvalues below some fixed (Fermi) energy is asymptotically the same for large $N$ as for the free problem (see Proposition 3.10). This is related to the spectral shift function (see [9] for potentials with compact support). Interestingly, a lot of work has been done to derive asymptotic formulae for the perturbed eigenvalues at large energies. Except for [11], we are not aware of studies that include also the dependence on $L$ as well.

Anderson’s orthogonality catastrophe has attracted a lot of interest in solid state physics since its discovery. There are early attempts to determine the exact asymptotics of the determinant $D_N$ itself. Rivier and Simanek [17] used the adiabatic theorem to express $D_N$ through the solution of a Wiener-Hopf equation. However, they could not deal satisfactorily with certain limit procedures underlying the method. This was improved upon by Hamann [7] who, likewise, could treat the thermodynamic limit only informally. A clarification of that method can be found in [14]. Recent numerical investigations have been carried out by Weichselbaum, Münfer, and von Delft [19] who also present some physical background and refer to further reading.

Frank, Lewin, Lieb, and Seiringer [5, Eq. (11)] considered the related problem of proving a lower bound to the energy difference – in our notation below $\text{tr}(H_V \Pi - HP)$ – directly in the thermodynamic limit in terms of semi-classical quantities.

Gebert, Kütter, and Müller [6] using different methods have recently established a rigorous lower bound

$$I_{N,L} \geq \gamma' \ln N, \quad N, L \to \infty,$$

in any dimension (even with a periodic background potential but with positive and compactly supported exterior potential $V$). Remarkably, their value $\gamma'$ agrees with Anderson’s prediction. In our framework, the expression for $\gamma$ that at first came out from Theorem 5.3 is rather implicit. Only after some computation could we confirm that $\gamma = \gamma'$, Corollary 5.4. Thus, one can
reasonably conjecture that $\gamma' \ln N$ is indeed the exact leading asymptotics in any dimension.

2. Representation of the Anderson Integral

Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot, \cdot)$, which is anti-linear in the first component and linear in the second component and let $\| \cdot \|$ be the corresponding vector norm. The induced operator norm will be denoted with the same symbol $\| \cdot \|$. We consider a self-adjoint operator $H : D(H) \to \mathcal{H}$, $D(H) \subset \mathcal{H}$, and a bounded operator $V : \mathcal{H} \to \mathcal{H}$. Then, $H_V = H + V$ is self-adjoint as well with $D(H_V) = D(H)$. We denote by $\sigma(H)$ and $\sigma(H_V)$ the spectrum of $H$ and $H_V$, respectively, and by

$$R(z) := (z 1 - H)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(H), \quad R_V(z) := (z 1 - H_V)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(H_V)$$

(2.1)

their resolvents. From spectral theory we know

$$\|R(z)\| = \frac{1}{\text{dist}(z, \sigma(H))}, \quad \|R_V(z)\| = \frac{1}{\text{dist}(z, \sigma(H_V))}. \quad (2.2)$$

We borrow some notation from scattering theory (see e.g. [18, 3.6]). Note, that for $z \notin \sigma(H)$ the operator $(1 - VR(z))^{-1}$ exists and is bounded if and only if $z \notin \sigma(H_V)$. The same holds true for $(1 - R(z)V)^{-1}$. Hence, the so-called transition operator or T-operator

$$T(z) := (1 - VR(z))^{-1}V = V(1 - R(z)V)^{-1} \quad (2.3)$$

exists for $z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_V))$ with

$$\|T(z)\| \leq \frac{\text{dist}(z, \sigma(H))}{\text{dist}(z, \sigma(H_V))}\|V\| \quad (2.4)$$

and is analytic there as a function of $z$. Krein’s resolvent formula

$$R_V(z) - R(z) = R(z)T(z)R(z) \quad (2.5)$$

relates the resolvents $R(z)$ and $R_V(z)$ with each other whenever the T-operator exists. The operator $V$ plays an important role via its modified polar decomposition

$$V = \sqrt{|V|}J\sqrt{|V|}, \quad J^* = J, \quad J^2 = 1, \quad \|J\| = 1, \quad (2.6)$$

which is obvious for the multiplication operators used below. Like in scattering theory it is advantageous to look at operators relative to $V$. More precisely, we will use (cf. [18, 3.6.1, 1])

$$\sqrt{|V|R(z)\sqrt{|V|}}, \quad \Omega(z) := (1 - \sqrt{|V|R(z)\sqrt{|V|}}J)^{-1} \quad (2.7)$$

with the sandwiched resolvent being called Birman-Schwinger operator. Note the relation

$$T(z) = \sqrt{|V|J\Omega(z)\sqrt{|V|}} \quad (2.8)$$

Obviously, the Birman-Schwinger operator exists and is bounded for $z \in \mathbb{C} \setminus \sigma(H)$. For $\Omega(z)$ to exist as a bounded operator it is required that $z \notin \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_V))$. The converse is true, too. That is to say, if $z \notin \sigma(H)$
and \( \Omega(z) \) exists and is bounded then \( z \notin \sigma(H_V) \). In order to see this one first shows that \( 1 - R(z)V \) is injective and has dense range and, in a second step, that the range is closed.

### 2.1. Operators with a common spectral gap

Riesz’s integral formula yields a handy expression for the Anderson integral when the operators \( H \) and \( H_V \) have a common spectral gap. That is to say their spectra can be written as

\[
\sigma(H) = \sigma_1(H) \cup \sigma_2(H), \quad \sigma(H_V) = \sigma_1(H_V) \cup \sigma_2(H_V)
\]  

such that there is a closed contour \( \Gamma \subset \mathbb{C} \) with each \( \sigma_1 \) being inside and each \( \sigma_2 \) outside of \( \Gamma \). Let \( P \) be the spectral projection of \( H \) belonging to \( \sigma_1(H) \) and let \( \Pi \) be defined likewise for \( H_V \). The Anderson integral in question is

\[
\mathcal{I} := \text{tr} \left[ P(1 - \Pi) \right].
\]  

In our application \( P \) is trace class and hence \( 0 \leq \mathcal{I} < \infty \). The Riesz formula reads

\[
P = \frac{1}{2\pi i} \int_{\Gamma} R(z) \, dz, \quad \Pi = \frac{1}{2\pi i} \int_{\Gamma} R_V(z) \, dz.
\]  

Note that both integrals have the same \( \Gamma \) from above. For our purposes, an infinite contour is more appropriate. In particular, due to the special form of the free Green function (see (3.12)) a parabola will do best.

**Proposition 2.1.** Let \( P \) be trace class. We assume the sets \( \sigma_{1,2} \) in (2.9) to satisfy

\[
\sup \sigma_1(H) < \nu < \inf \sigma_2(H), \quad \sup \sigma_1(H_V) < \nu < \inf \sigma_2(H_V)
\]  

with some \( \nu \in \mathbb{R} \) and define the parabola \( \Gamma_\nu := \{ z = (\sqrt{\nu} + is)^2 \mid s \in \mathbb{R} \} \). Then, the difference of the spectral projections has the representation

\[
\Pi - P = \frac{1}{2\pi i} \int_{\Gamma_\nu} R(z) T(z) R(z) \, dz
\]  

and the Anderson integral (2.10) can be written as

\[
\mathcal{I} = \frac{1}{2\pi i} \int_{\Gamma_\nu} \text{tr} \left[ PR(z) T(z) R(z)^2 T(z) \right] \, dz.
\]  

**Proof.** By Riesz’s and Krein’s formulae, (2.11) and (2.5),

\[
\Pi - P = \frac{1}{2\pi i} \int_{\Gamma} (R_V(z) - R(z)) \, dz = \frac{1}{2\pi i} \int_{\Gamma} R(z) T(z) R(z) \, dz
\]  

with the closed contour \( \Gamma \) used in (2.9). For the Anderson integral note \( P(\Pi - 1) = P(\Pi - P) \) which allows us to use (2.15). Since \( P \) is trace class and the other operators are bounded we may take the trace. Using the cyclic commutativity we obtain

\[
\mathcal{I} = -\frac{1}{2\pi i} \int_{\Gamma} \text{tr} \left[ PR(z) T(z) R(z) \right] \, dz = -\frac{1}{2\pi i} \int_{\Gamma} \text{tr} \left[ PR(z)^2 T(z) \right] \, dz
\]
since \( P \) commutes with \( R(z) \). Recall that \( R(z) \) is differentiable in \( z \in \mathbb{C} \setminus \sigma(H) \) with \( R'(z) = -R(z)^2 \). Since all functions involved are analytic for \( z \in \mathbb{C} \setminus (\sigma(H) \cup \sigma(H_V)) \) we may integrate by parts, \[
abla = -\frac{1}{2\pi i} \int_\Gamma \text{tr} [PR(z)T'(z)] \, dz = \frac{1}{2\pi i} \int_\Gamma \text{tr} [PR(z)T(z)R(z)^2T(z)] \, dz. \tag{2.16}
\]
By the estimates (2.2) and (2.4) the integrands in (2.15) and (2.16) decay fast enough at infinity so that we may bend the closed contour \( \Gamma \) into the parabola \( \Gamma_\nu \) to obtain (2.13) and (2.14), respectively. \( \square \)

The integral formula (2.14) for the Anderson integral was intentionally made more complicated via integration by parts. For, in the application of the delta-estimate to (2.14) it will be important to have the smooth cut-off factor \( PR(z) \) instead of just \( P \).

2.2. Schrödinger-type operators

A typical Schrödinger operator is built from differentiation and multiplication operators. Let us introduce two operators \( \nabla \) and \( X \) satisfying
\[
[\nabla, X] = 1. \tag{2.17}
\]
We assume \( \nabla : D(\nabla) \to \mathcal{H}, D(\nabla) \subset \mathcal{H}, \) to be densely defined on \( \mathcal{H} \) and \( X : \mathcal{H} \to \mathcal{H} \) to be bounded such that \( XD(\nabla) \subset D(\nabla) \). Thus, (2.17) is meant to hold true on \( D(\nabla) \). Self-adjointness of Schrödinger operators often results from boundary conditions which usually lessen the domain of definition. Let \( -\nabla^2 \) have a self-adjoint restriction \( H : D(H) \to \mathcal{H}, \) i.e. \( D(H) \subset D(\nabla^2) \) and
\[
H = -\nabla^2 \text{ on } D(H). \tag{2.18}
\]
The resolvent of \( H \),
\[
R(z) = (z1 - H)^{-1}, \quad z \in \mathbb{C} \setminus \sigma(H)
\]
is a well-defined and bounded operator with \( R(z) : \mathcal{H} \to D(H) \). The latter implies
\[
(z1 + \nabla^2)R(z) = (z1 - H)R(z) = 1. \tag{2.19}
\]
In general, this equality fails to hold true when the order of terms is switched as can be seen in Proposition 3.6. This is the reason why the following resolvent formula gives non-trivial results.

**Proposition 2.2.** For operators \( \nabla \) and \( X \) as in (2.17) let us assume in addition \( XD(H) \subset D(H) \). Then, the decomposition
\[
R(z)^2 = \frac{1}{z}R(z) - \frac{1}{2z}[X\nabla, R(z)] + \frac{1}{z}D(z) = \frac{1}{2z}(R(z) - C(z)) + \frac{1}{z}D(z) \tag{2.20}
\]
holds true on \( D(\nabla^2) \) and for \( z \in \mathbb{C} \setminus \sigma(H) \). Here,
\[
D(z) := \left( \frac{1}{2}X - R(z)\nabla \right)[\nabla, R(z)], \quad C(z) := X\nabla R(z) - R(z)\nabla X. \tag{2.21}
\]
The operator \( D(z) \) is the so-called 'delta-term' and satisfies
\[
(z1 + \nabla^2)D(z) = 0 \tag{2.22}
\]
on $D(\nabla^2)$ and for $z \in \mathbb{C} \setminus \sigma(H)$.

Proof. We start off from the elementary formula

$$R(z)^2 = \frac{1}{z} R(z) + \frac{1}{z} R(z) HR(z) \quad (2.23)$$

and rewrite the last term. By the product rule the commutator in (2.20) becomes

$$[X \nabla, R(z)] = X[\nabla, R(z)] + [X, R(z)] \nabla. \quad (2.24)$$

Formula (2.17) implies $[\nabla^2, X] = 2 \nabla$. By noting $XD(H) \subset D(H)$ and $R(z)(z1 - H) = 1$ on $D(H)$ we obtain

$$[X, R(z)] = R(z)[z1 - H, X]R(z) = R(z)[\nabla^2, X]R(z) = 2R(z)\nabla R(z).$$

Thus,

$$\frac{1}{2} [X, R(z)] \nabla = R(z)\nabla R(z) \nabla = R(z)\nabla^2 R(z) + R(z)\nabla [R(z), \nabla]$$

Recalling (2.18) we solve for $R(z)HR(z)$, insert this into (2.23), and use (2.24). Then,

$$R(z)^2 = \frac{1}{z} R(z) - \frac{1}{2z} [X, R(z)] \nabla - \frac{1}{z} R(z) \nabla [\nabla, R(z)]$$

$$= \frac{1}{z} R(z) - \frac{1}{2z} [X \nabla, R(z)] + \frac{1}{2z} X[\nabla, R(z)] - \frac{1}{z} R(z) \nabla [\nabla, R(z)].$$

With the definition (2.21) of $D(z)$ this is the first equality in (2.20). The second one follows by means of the commutation relation $X \nabla = -1 + \nabla X$. Finally, by (2.17)

$$(z1 + \nabla^2)(\frac{1}{2} X - R(z) \nabla) = \frac{1}{2} X(z1 + \nabla^2).$$

Then,

$$(z1 + \nabla^2)[\nabla, R(z)] = \nabla(z1 + \nabla^2)R(z) - (z1 + \nabla^2)R(z) \nabla = 0$$

shows (2.22).

Our motivation behind the resolvent formula (2.20) in Proposition 2.2 is that it splits the integrand $\text{tr}[PR(z)T(z)R(z)T(z)]$ in the integral representation of the Anderson integral, Proposition 2.1, into a sum of two terms. The first term, $\text{tr}[PR(z)T(z)(R(z) - C(z))T(z)]$, will be subdominant, i.e. $O(1)$, as shown in Section 5.1 whereas the second term $\text{tr}[PR(z)T(z)D(z)T(z)]$ is of the leading order $\ln N$, see Section 5.2. The operator $D(z)$ quantifies the difference between the resolvent of the Laplace operator with and without Dirichlet boundary conditions.
3. One-dimensional Schrödinger Operators

We look into the special case of Schrödinger operators with Dirichlet boundary conditions on the finite interval \([-L, L]\). Our Hilbert space then becomes \(H = L^2[-L, L]\). Actually, it ought to bear an index \(L\) as well as all operators defined on it and related quantities. However, since this dependence is ubiquitous we tacitly suppress it. In our concrete case, \(\nabla = \frac{d}{dx}\), \((X\varphi)(x) = x\varphi(x)\). (3.1)
The domain \(D(\nabla)\) as well as \(D(\nabla^2)\) can be described with the aid of Sobolev spaces which we do not need in detail herein. One can show that \(X(\nabla) \subset D(\nabla)\). The operator \(H\) becomes
\[H = -\nabla^2 = -\frac{d^2}{dx^2}\text{ on } D(H),\]
where \(D(H)\) is restricted by Dirichlet boundary conditions. Because of that we have \(XD(H) \subset D(H)\). The corresponding eigenvalue problem reads
\[\varphi'' = \lambda \varphi, \quad \varphi(-L) = 0 = \varphi(L).\] (3.2)
The eigenvalues \(\lambda_j\) and normalized eigenfunctions \(\varphi_j, j \in \mathbb{N}\), are
\[\lambda_j = \left(\frac{\pi j}{2L}\right)^2, \quad \varphi_j(x) = \begin{cases} \frac{1}{\sqrt{L}} \sin\left(\frac{\pi j}{2L}x\right) & \text{for } j \text{ even}, \\ \frac{1}{\sqrt{L}} \cos\left(\frac{\pi j}{2L}x\right) & \text{for } j \text{ odd}. \end{cases}\] (3.3)
We translate the integral formula in Proposition 2.1 and the resolvent formula (2.20) into the framework of Schrödinger operators. For the \(\nu \in \mathbb{R}\) in Proposition 2.1 separating the two parts of the spectrum we choose the so-called Fermi energy
\[\nu_N := \left[\frac{\pi}{2L}(N + \frac{1}{2})\right]^2.\] (3.4)
Thereby, the spectrum of \(H\) decomposes into \(\sigma(H) = \sigma_1(H) \cup \sigma_2(H)\),
\[\sigma_1(H) := \{\lambda_j \mid 1 \leq j \leq N\}, \quad \sigma_2(H) := \{\lambda_j \mid j \geq N + 1\}\]
and the parabola \(\Gamma_{\nu_N}\) becomes what we call Fermi parabola
\[\Gamma_N := \{z = (\sqrt{\nu_N} + is)^2 \mid -\infty < s < \infty\}, \quad dz = 2i(\sqrt{\nu_N} + is)ds.\] (3.5)
The distance of the Fermi parabola from the spectrum is
\[|z - \lambda_j| = |\sqrt{\nu_N} + is + \sqrt{\lambda_j}||\sqrt{\nu_N} + is - \sqrt{\lambda_j}| \geq (\nu_N + s^2)^{\frac{1}{2}}((\sqrt{\nu_N} - \sqrt{\lambda_j})^2 + s^2)^{\frac{1}{2}},\] (3.6)
which will be used at various points in particular with \(s = 0\). The spectral projection \(P\) in the Anderson integral (2.10) becomes
\[P_N := \sum_{j=1}^{N} (\varphi_j, \cdot)\varphi_j.\] (3.7)
The perturbed operator \(H_V\) is given by
\[H_V = H + V,\]
where \( V \) is the operator of multiplication by a real-valued function \( V \), the potential, denoted by the same symbol for the sake of simplicity. Some results further below will be uniform in \( L \). In order to formulate this conveniently we assume that the potential \( V \) is already defined on the whole of \( \mathbb{R} \) and not only on the interval \([-L, L]\). Thus, we denote by \( \|V\|_r, 1 \leq r \leq \infty \) the \( L^r(\mathbb{R}) \) norms of the function \( V \). If \( V \in L^\infty(\mathbb{R}) \) then the operator \( V \) is bounded regardless of \( L \), which is in line with Section 2. In particular, \( D(H_V) = D(H) \). Furthermore, since the free eigenfunctions are obviously delocalized, \( V \in L^1(\mathbb{R}) \) implies

\[
\|\sqrt{|V|}\varphi_j\| \leq \frac{1}{\sqrt{L}} \|V\|_1, \quad (3.8)
\]

which will be used throughout. The spectrum of \( H_V \) is given through the corresponding Dirichlet problem

\[
-\psi'' + V\psi = \mu\psi, \quad \psi(-L) = 0 = \psi(L). \quad (3.9)
\]

It consists solely of simple eigenvalues, which follows easily via uniqueness results for ordinary differential equations. We denote them by \( \mu_k, k \in \mathbb{N} \) with the usual ordering \( \mu_1 < \mu_2 < \cdots \). The decomposition (2.9) of \( \sigma(H_V) \) will be studied in Section 3.4. The normalized eigenfunctions of \( H_V \) are \( \psi_k, k \in \mathbb{N} \) and the spectral projection \( \Pi \) in (2.10) reads

\[
\Pi_M := \sum_{k=1}^M (\psi_k, \cdot)\psi_k. \quad (3.10)
\]

Note that in general \( M \neq N \) (see Section 3.5).

### 3.1. Free resolvent

The spectral representation of the free resolvent (2.1) with (3.3) reads

\[
R(z) = \sum_{j=1}^\infty \frac{1}{z - \lambda_j} (\varphi_j, \cdot)\varphi_j. \quad (3.11)
\]

The corresponding kernel or Green function is given by

\[
R(z; x, y) = \frac{1}{W(z)} \begin{cases}
\sin(\sqrt{z}(x - L)) \sin(\sqrt{z}(y + L)) & -L \leq y \leq x \leq L \\
\sin(\sqrt{z}(x + L)) \sin(\sqrt{z}(y - L)) & -L \leq x \leq y \leq L
\end{cases} \quad (3.12)
\]

with the Wronski determinant

\[
W(z) = 2\sqrt{z} \sin(L\sqrt{z}) \cos(L\sqrt{z}) = \sqrt{z} \sin(2L\sqrt{z}). \quad (3.13)
\]

By rewriting the Green function one can cast the resolvent into a form where the \( L \) dependence is more tangible

\[
R(z) = \frac{1}{2\sqrt{z}} \left[ \frac{\cos(L\sqrt{z})}{\sin(L\sqrt{z})} P_s(z) - \frac{\sin(L\sqrt{z})}{\cos(L\sqrt{z})} P_c(z) + G(z) \right]. \quad (3.14)
\]
The operators $P_s(z)$, $P_c(z)$, and $G(z)$ have the kernels
\[ P_s(z; x, y) := \sin(\sqrt{z}x) \sin(\sqrt{z}y), \quad P_c(z; x, y) := \cos(\sqrt{z}x) \cos(\sqrt{z}y), \]
\[ G(z; x, y) := \sin(\sqrt{z}|x - y|). \quad (3.15) \]

Note that $P_s(z)$ and $P_c(z)$ are rank-one operators which makes the resolvent differ from the operator $G(z)$ by a rank-two perturbation. We would like to apply the delta-estimate from Section 4 directly to $R(z)$ and $\Omega(z)$ (cf. (2.7)). However, the prefactors of $P_s(z)$ and $P_c(z)$ in (3.14) behave too singularly at $z = \nu_N$ to do that. In a first step we therefore replace $z$ in the benevolent operators $P_{s,c}(z)$ and $G(z)$ by $\nu_N$ and retain the malevolent dependence in the function $\tau$. This motivates the definition of the operators $R_{\pm}\left(\nu_N, Ls\right)$ and $\Omega_{\pm}\left(\nu_N, Ls\right)$ in (3.20) and (3.44), respectively. In (3.32), we estimate the difference between $R(z)$ and $R_{\pm}\left(\nu_N, Ls\right)$. Later, in our main Theorem 5.3, we use these operators to compute the coefficient of the leading asymptotic $N$-behaviour of the Anderson integral. To begin with, we have a closer look at (3.14). At the Fermi energy (3.31)
\[ \sin(L\sqrt{\nu_N}) = \frac{1}{\sqrt{2}}(-1)^{\lfloor \frac{N}{2} \rfloor}, \quad \cos(L\sqrt{\nu_N}) = \frac{1}{\sqrt{2}}(-1)^{\lfloor \frac{N}{2} \rfloor}, \quad (3.16) \]

which implies on the Fermi parabola (3.35)
\[ \sin(L(\sqrt{\nu_N} + is)) = \frac{1}{\sqrt{2}}(-1)^{\lfloor \frac{N}{2} \rfloor}(\cosh(Ls) + i(-1)^N \sinh(Ls)), \]
\[ \cos(L(\sqrt{\nu_N} + is)) = \frac{1}{\sqrt{2}}(-1)^{\lfloor \frac{N}{2} \rfloor}((-1)^N \cosh(Ls) - i \sinh(Ls)). \quad (3.17) \]

Furthermore, we have
\[ \frac{\cos(L(\sqrt{\nu_N} + is))}{\sin(L(\sqrt{\nu_N} + is))} = (-1)^N \tau((-1)^N Ls) \quad (3.18) \]
where
\[ \tau(s) := \frac{\cosh s - i \sinh s}{\cosh s + i \sinh s}, \quad \tau(-s) = \bar{\tau}(s), \quad |\tau(s)| = 1, \quad s \in \mathbb{R}, \quad \lim_{s \to \infty} \tau(s) = -i. \quad (3.19) \]

Now, we keep the $s$-dependence only in the scalar function $\tau$ but not in the operators $P_{s,c}(z)$ and $G(z)$ and introduce
\[ R_{\pm}\left(\nu_N, Ls\right) := \frac{1}{2\sqrt{\nu_N}} \left[ \pm \tau(\pm Ls)P_s(\nu_N) \mp \tau(\mp Ls)P_c(\nu_N) + G(\nu_N) \right]. \quad (3.20) \]

This can be seen, in a way, as the limit of the resolvent as $L \to \infty$ (cf. (3.32)). Note that $R_{\pm}\left(\nu_N, s\right)$ differs from $G(\nu_N)$ by a rank-two perturbation.

The operator $C(z)$ in (2.21) has the kernel $(x, y \in [-L, L])$
\[ C(z; x, y) = \frac{\sqrt{z}}{2W(z)} \begin{cases} 
  x \cos(\sqrt{z}(x - L)) \sin(\sqrt{z}(y + L)) & y \leq x, \\
  +y \sin(\sqrt{z}(x - L)) \cos(\sqrt{z}(y + L)) & x \leq y.
\end{cases} \quad (3.21) \]
The Green function and related quantities are to be evaluated on the Fermi
parabola $\Gamma_N$.

**Lemma 3.1.** For all $s \in \mathbb{R}$, $L > 0$, $N \in \mathbb{N}$, and $\nu_N$ as in (3.4) we have

$$\frac{1}{\sin(L(\sqrt{\nu_N} + is))} \leq 4e^{-2L|s|}, \quad \frac{1}{\cos(L(\sqrt{\nu_N} + is))} \leq 4e^{-2L|s|}. \quad (3.22)$$

Moreover, for $z \in \Gamma_N$ (see (3.15))

$$|R(z; x, y)| \leq \frac{2}{(\nu_N + s^2)^{\frac{1}{2}}} e^{-|s||x-y|}, \quad |C(z; x, y)| \leq 2(|x| + |y|)e^{-|s||x-y|}. \quad (3.23)$$

Let $z = (a + is)^2$, $a, s \in \mathbb{R}$. Then, the kernels of the operators $P_{s,c}(z)$ and $G(z)$ from (3.13) satisfy

$$|P_{s,c}(z; x, y) - P_{s,c}(a^2; x, y)| \leq |s|(|x| + |y|)e^{|s|(|x|+|y|)},$$

$$|G(z; x, y) - G(a^2; x, y)| \leq |s||x-y|e^{|s||x-y|}.$$  

**Proof.** (a) From (3.17) we deduce

$$|\sin(L(\sqrt{\nu_N} + is))|^2 = |\cos(L(\sqrt{\nu_N} + is))|^2 = \frac{1}{4}(e^{2Ls} + e^{-2Ls}) \geq \frac{1}{4}e^{2L|s|}$$

which proves (3.22).

(b) For $L \geq x \geq y \geq -L$

$$|\sin((\sqrt{\nu_N} + is)(x - L))\sin((\sqrt{\nu_N} + is)(y + L))| \leq e^{2L-|x-y|},$$

where we estimated the sine by the exponential function. For $x \leq y$ the bound looks the same. Using (3.22) we obtain

$$|R(z; x, y)| \leq \frac{1}{(\nu_N + s^2)^{\frac{1}{2}}} \frac{e^{2L-|x-y|}}{2|\sin(L(\sqrt{\nu_N} + is))||\cos(L(\sqrt{\nu_N} + is))|}$$

$$\leq \frac{2}{(\nu_N + s^2)^{\frac{1}{2}}} e^{-|s||x-y|},$$

which proves the first estimate in (3.23). The estimate for $C(z; x, y)$ in (3.23) follows likewise.

(c) We write the difference as an integral

$$P_s(z; x, y) - P_s(a^2; x, y)$$

$$= \int_0^s \frac{d}{dt} (\sin((a + it)x)\sin((a + it)y)) \, dt$$

$$= i \int_0^s (x\cos((a + it)x)\sin((a + it)y) + y\sin((a + it)x)\cos((a + it)y)) \, dt,$$

and estimate

$$|P_s(z; x, y) - P_s(a^2; x, y)| \leq \int_0^{|s|} (|x| + |y|)e^{t(|x|+|y|)} \, dt \leq |s|(|x| + |y|)e^{|s|(|x|+|y|)}.$$  

The estimates for $P_c(z)$ and $G(z)$ follow in like manner. \qed
Similar to the Birman-Schwinger operator (2.7) we need to study operators of the form \( \sqrt{|V|} P_{s,c}(z) \sqrt{|V|} \). To this end, we introduce the functions \( \omega_s(z), \omega_c(z) \),

\[
\omega_s(z;x) := \sqrt{|V(x)|} \sin(\sqrt{z}x), \quad \omega_c(z;x) := \sqrt{|V(x)|} \cos(\sqrt{z}x),
\]

for \( z \in \mathbb{C}, \; x \in \mathbb{R} \), so that the kernels read (cf. (3.15))

\[
\sqrt{|V|} P_{s,c}(z) \sqrt{|V|}(x,y) = \omega_{s,c}(z;x) \omega_{s,c}(z;y).
\]

In order to describe how \( \omega_{s,c}(z) \) and derived quantities behave in the complex plane we associate to any \( V \in L^1(\mathbb{R}) \) the transformed function \( V_L \in C^\infty(\mathbb{R}) \),

\[
V_L(s) := \int_{-L}^{L} |V(x)| e^{s|x|} \, dx, \quad s \in \mathbb{R}.
\]

Its derivatives satisfy

\[
0 \leq V_L^{(p)}(0) \leq V_L^{(p)}(s), \quad V_L^{(p+q)}(0) \leq L^p \| X^q V \|_1,
\]

\[
V_L^{(p+q)}(s) \leq L^p \| X^q V \|_\infty \int_{-L}^{L} e^{s|x|} \, dx
\]

for \( p, q \in \mathbb{N}_0 \) provided that \( X^q V \in L^1(\mathbb{R}) \) and \( X^q V \in L^\infty(\mathbb{R}) \), respectively.

**Lemma 3.2.** Let \( V \in L^1(\mathbb{R}) \) and \( z = (a + is)^2 \) with \( a, s \in \mathbb{R} \). Then,

\[
\| \omega_{s,c}(z) \| \leq V_L(2|s|)^{1/2},
\]

\[
\| \omega_{s,c}(z) - \omega_{s,c}(a^2) \| \leq |s| V_L^{(2)}(2|s|)^{1/2}.
\]

**Proof.** In order to prove (3.27) we estimate

\[
\| \omega_s(z) \|^2 = \int_{-L}^{L} |V(x)| \sin((a + is)x)^2 \, dx \leq \int_{-L}^{L} |V(x)| e^{2|x|} \, dx.
\]

For (3.28) we compute

\[
\| \omega_s(z) - \omega_s(a^2) \|^2 = \int_{-L}^{L} |V(x)| \cdot |\sin((a + is)x) - \sin(ax)|^2 \, dx
\]

and use the estimate

\[
|\sin((a + is)x) - \sin(ax)| = |ix \int_0^s \cos((a + it)x) \, dt| \leq |x||s| e^{sx},
\]

which yields (3.28). The estimates for \( \omega_c(z) \) follow in like manner. \( \square \)

One could use Lemma 3.1 to study the norms of \( R(z) \) or \( G(z) \). However, the applications we have in mind require that to be done for the Birman-Schwinger (see (2.7)) and suchlike operators (with \( \sqrt{|V|} \) multiplied from left and right).

**Lemma 3.3.** Let \( V \in L^1(\mathbb{R}) \) and \( z \in \Gamma_N \). Then, the Birman-Schwinger operator satisfies

\[
\| \sqrt{|V|} R(z) \sqrt{|V|} \| \leq \frac{4}{\sqrt{\nu_N} + s} \| V \|_1.
\]
If $X^2V \in L^1(\mathbb{R})$ with $X$ as in (3.1) the operator $C(z)$ from (3.21) satisfies
\[ \| \sqrt{|V|} C(z) \sqrt{|V|} \| \leq 8 \| X^2V \|_1^{\frac{1}{2}} \| V \|_1^{\frac{1}{2}}. \] (3.30)
Furthermore, for the operators $G(\nu_N)$ from (3.15) and $R^\pm_\infty(\nu_N, s)$ from (3.20) we have
\[ \| \sqrt{|V|} G(\nu_N) \sqrt{|V|} \| \leq \| V \|_1, \quad \| \sqrt{|V|} R^\pm_\infty(\nu_N, s) \sqrt{|V|} \| \leq \frac{3}{2\sqrt{\nu_N}} \| V \|_1. \] (3.31)
Finally,
\[ \| \sqrt{|V|}(R_\infty(\nu_N, Ls) - R((\sqrt{\nu_N} + is)^2)) \sqrt{|V|} \| \leq \frac{3}{2\nu_N} V_L(0)|s| + \frac{3}{\sqrt{\nu_N}} |s| V_L(2)|s|^\frac{1}{2} V_L(2)|s|^\frac{1}{2} \] (3.32)
where $R_\infty$ stands for $R^+_\infty$, $R^-_\infty$ depending on whether $N$ in $\nu_N$ is even or odd.

**Proof.** Let the kernel $W(x, y)$ of the integral operator $W$ be bounded by
\[ |W(x, y)| \leq |W_1(x)| f(x, y) W_2(x) |, \]
where $W_1, W_2 \in L^2(\mathbb{R})$ and $f \in L^\infty(\mathbb{R}^2)$. By the Cauchy-Schwarz inequality
\[ \| W\varphi \|^2 \leq \int_{-L}^{L} |W_1(x)|^2 \int_{-L}^{L} |f(x, y) W_2(y)|^2 \, dy \, dx \, \| \varphi \|^2 \]
\[ \leq \| f \|_\infty^2 \| W_1 \|_2^2 \| W_2 \|_2^2 \| \varphi \|^2 \]
for $\varphi \in L^2[-L, L]$. The norms of $W_{1,2}$ and $f$ pertain to $\mathbb{R}$ and $\mathbb{R}^2$, respectively. Hence
\[ \| W \| \leq \| f \|_\infty \| W_1 \|_2 \| W_2 \|_2. \]
In order to prove (3.29) we can take $W_{1,2} = \sqrt{|V|}$ and $f \equiv 1$ because of (3.23). By the same estimate we can prove (3.30) by using $W_1 = X \sqrt{|V|}$,
$W_2 = \sqrt{|V|}$, $f \equiv 1$ and $W_1 = \sqrt{|V|}$, $W_2 = X \sqrt{|V|}$, $f \equiv 1$.
Because of the obvious bound $|G(\nu_N; x, y)| \leq 1$ (cf. (3.15)) we obtain
\[ \| \sqrt{|V|} G(\nu_N) \sqrt{|V|} \| \leq V_L(0), \]
which proves (3.31) for $G(\nu_N)$ via (3.26). Using this and (3.27) along with $|\tau(Ls)| = 1$ we obtain for all $s \in \mathbb{R}$
\[ 2\sqrt{\nu_N} \| \sqrt{|V|} R^\pm_\infty(\nu_N, s) \sqrt{|V|} \| \]
\[ \leq \| \sqrt{|V|} P_\nu(\nu_N) \sqrt{|V|} \| + \| \sqrt{|V|} P_\nu(\nu_N) \sqrt{|V|} \| + \| \sqrt{|V|} G(\nu_N) \sqrt{|V|} \| \]
\[ \leq 3V_L(0), \]
which gives (3.31) for $R^\pm_\infty(\nu_N, s)$ via (3.26).

In order to prove (3.32) we use the kernel estimates in Lemma 3.1 and obtain
\[ \| \sqrt{|V|}(P_{s,c}(z) - P_{s,c}(\nu_N)) \sqrt{|V|} \| \leq 2|s| V_L(2)|s|^{\frac{1}{2}} V_L(2)|s|^{\frac{1}{2}}, \]
\[ \| \sqrt{|V|}(G(z) - G(\nu_N)) \sqrt{|V|} \| \leq 2|s| V_L(2)|s|^{\frac{1}{2}} V_L(2)|s|^{\frac{1}{2}}, \]
which would also be true for other real values than $\nu_N$ but that is not needed here. Using (3.14) and $|\tau(Ls)| = 1$ we obtain
\[
2\| (R_\infty(\nu_N, Ls) - R(z)) \|
\leq \frac{1}{\sqrt{\nu_N} + s^2} \left( \| P_s(\nu_N) - P_s(z) \| + \| P_c(\nu_N) - P_c(z) \| + \| G(\nu_N) - G(z) \| \right)
+ \left| \frac{1}{\sqrt{\nu_N}} - \frac{1}{\sqrt{\nu_N} + is} \right| \left( \| P_s(\nu_N) \| + \| P_c(\nu_N) \| + \| G(\nu_N) \| \right).
\]

Here, $R_\infty$ means $R_\infty^+$ for even $N$ and $R_\infty^-$ for odd $N$. This proves (3.32). □

3.2. Truncated free resolvent

Let $S_N(z) := P_N R(z)$ be the truncated resolvent with the spectral projection from (3.17). We need to control $S_N(z)$ on the entire Fermi parabola $\Gamma_N$ (see (3.5)) and, with more care, at the Fermi energy (3.4).

Lemma 3.4. Let $V \in L^1(\mathbb{R})$ and $z \in \Gamma_N$. Then, $S_N(z) = P_N R(z)$ satisfies
\[
\| \sqrt{|V|} S_N(z) \sqrt{|V|} \| \leq \frac{8}{\pi} \| V \|_1 \frac{1}{\sqrt{\nu_N} + s^2} \ln(N + 1),
\]
\[
\| \sqrt{|V|}(S_N(z) - S_N(\nu_N)) \sqrt{|V|} \| \leq \frac{64}{\pi \nu_N} \| V \|_1 |s|(N + \frac{1}{2}).
\]

Proof. We start off from the spectral representation of $S_N$,
\[
\sqrt{|V|} S_N(z) \sqrt{|V|} = \sum_{j=1}^N \frac{1}{z - \lambda_j} (\sqrt{|V|} \varphi_j, \cdot) \sqrt{|V|} \varphi_j.
\]

Applying the estimate (3.18) and then using (3.6) we obtain
\[
\| \sqrt{|V|} S_N(z) \sqrt{|V|} \| \leq \frac{1}{L} \| V \|_1 \sum_{j=1}^N \frac{1}{|z - \lambda_j|}
\leq \frac{2}{\pi} \| V \|_1 \frac{1}{\sqrt{\nu_N} + s^2} \sum_{j=1}^N \frac{1}{N + \frac{1}{2} - j}.
\]

Likewise, using
\[
\left| \frac{z - \nu_N}{z - \lambda_j} \right|^2 = \frac{4\nu_N + s^2}{(\sqrt{\nu_N} + \sqrt{\lambda_j})^2 + s^2} \leq \frac{s^2}{(\sqrt{\nu_N} - \sqrt{\lambda_j})^2 + s^2}
\]
we find
\[
\| \sqrt{|V|}(S_N(z) - S_N(\nu_N)) \sqrt{|V|} \|
\leq \frac{1}{L} \| V \|_1 \sum_{j=1}^N \frac{|z - \nu_N|}{|z - \lambda_j||\nu_N - \lambda_j|}
\leq \left| s \right| \frac{2}{L} \| V \|_1 \sum_{j=1}^N \frac{1}{(\sqrt{\nu_N} - \sqrt{\lambda_j})(\nu_N - \lambda_j)}
\leq 4 \left| s \right| \frac{\| V \|_1 (2L)^2}{\pi^3} \sum_{j=1}^N \frac{1}{(N + \frac{1}{2} - j)^2}.
\]
Applying (A.1) to the sums in (3.35) and (3.36) we obtain (3.33) and (3.34). □

The asymptotic analysis in Section 5.2 is based upon a formula for the kernel of the truncated resolvent.

**Proposition 3.5.** Let $z > 0$ such that $\sqrt{z} > \frac{\pi}{2L}N$. Then, the kernel $S_N(z; x, y)$ of the operator $S_N(z) = P_NR(z)$ decomposes into

$$S_N(z; x, y) = \kappa_N S_{0,N}(z; x, y) - S_{1,N}(z; x, y) - (-1)^N(\tilde{\kappa}_N \tilde{S}_{0,N}(z; x, y) - \tilde{S}_{1,N}(z; x, y))$$

with the constants

$$\kappa_N := \int_0^\infty e^{-\frac{2L}{\pi} \sqrt{z}v} \frac{\sinh((N + \frac{1}{2})v)}{\sin \frac{v}{2}} dv,$$

$$\tilde{\kappa}_N := \int_0^\infty e^{-\frac{2L}{\pi} \sqrt{z}v} \frac{\cosh((N + \frac{1}{2})v)}{\cosh \frac{v}{2}} dv$$

and the kernel functions

$$S_{0,N}(z; x, y) := \frac{\cos(\sqrt{z}(x - y))}{2\pi \sqrt{z}}, \quad \tilde{S}_{0,N}(z; x, y) := \frac{\cos(\sqrt{z}(x + y))}{2\pi \sqrt{z}},$$

$$S_{1,N}(z; x, y) := \frac{1}{2\pi \sqrt{z}} \int_0^{\pi(x-y)} \sin(\frac{2L}{\pi} \sqrt{z}(u - \frac{\pi(x - y)}{2L})) \frac{\sin((N + \frac{1}{2})u)}{\sin \frac{u}{2}} du,$$

$$\tilde{S}_{1,N}(z; x, y) := \frac{1}{2\pi \sqrt{z}} \int_0^{\pi(x+y)} \sin(\frac{2L}{\pi} \sqrt{z}(u - \frac{\pi(x + y)}{2L})) \frac{\cos((N + \frac{1}{2})u)}{\cos \frac{u}{2}} du.$$

**Proof.** With the eigenfunctions from (3.3) and using the product formulae for sine and cosine we can write

$$S_N(z; x, y) = \sum_{j=1}^N \frac{1}{z - \lambda_j} \varphi_j(x) \bar{\varphi}_j(y)$$

$$= \frac{1}{2L} \left[ \sum_{j=1}^N \frac{1}{z - \lambda_j} \cos(\frac{\pi j(x - y)}{2L}) - \sum_{j=1}^N (-1)^j \frac{1}{z - \lambda_j} \cos(\frac{\pi j(x + y)}{2L}) \right].$$

In order to sum the series we write the fraction as a Laplace transform. It is convenient to put $z = \frac{\pi^2}{4L^2} \tilde{z}^2$. Then,

$$\frac{1}{z - \lambda_j} = \frac{4L^2}{\pi^2} \frac{1}{2\tilde{z}} \left( \frac{1}{\tilde{z} - j} + \frac{1}{\tilde{z} + j} \right) = \frac{4L^2}{\pi^2} \frac{1}{\tilde{z}} \int_0^\infty e^{-\tilde{z}v} \cosh(jv) dv$$

since $\tilde{z} > j$ by assumption. Hence,

$$S_N(z; x, y) = \frac{2L}{\pi^2} \frac{1}{\tilde{z}} \int_0^\infty e^{-\tilde{z}v} \sum_{j=1}^N \cosh(jv) \left[ \cos(\frac{\pi j(x - y)}{2L}) - (-1)^j \cos(\frac{\pi j(x + y)}{2L}) \right] dv.$$
Using \( \cos(\alpha) \cosh(\beta) = \Re(\cos(\alpha + i\beta)) \) for \( \alpha, \beta \in \mathbb{R} \) we obtain

\[
S_N(z; x, y) = \frac{L}{\pi^2 z} (\Re I_s - (-1)^N \Re I_c)
\]

with the integrals

\[
I_s := \int_0^\infty e^{-\tilde{z}v} \frac{\sin(M(a + iv))}{\sin(\frac{1}{2}(a + iv))} \, dv, \quad I_c := \int_0^\infty e^{-\tilde{z}v} \frac{\cos(M(b + iv))}{\cos(\frac{1}{2}(b + iv))} \, dv.
\]

Here we abbreviated

\[
M := N + \frac{1}{2}, \quad a := \frac{(x-y)}{2L}, \quad b := \frac{(x+y)}{2L}.
\]

We evaluate the integral \( I_s \) by changing the integration contour. To this end, put \( w = a + iv \) and \( \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \),

\[
\Gamma_1 = \{a + iv \mid 0 \leq v \leq R\}, \quad \Gamma_2 = \{u + iR \mid 0 \leq u \leq a\},
\]

\[
\Gamma_3 = \{iv \mid 0 \leq v \leq R\}, \quad \Gamma_4 = \{u \mid 0 \leq u \leq a\},
\]

orientated counterclockwise. By Cauchy’s integral theorem

\[
\int_{\Gamma} e^{izw} \frac{\sin(Mw)}{\sin \frac{w}{2}} \, dw = 0
\]

since the integrand has only removable singularities. Because of

\[
e^{izu} \frac{\sin(Mw)}{\sin \frac{w}{2}} = e^{iz(u+iR)} \frac{\sin(M(u+iR))}{\sin(\frac{1}{2}(u+iR))} \sim e^{-(\tilde{z}-M+\frac{1}{2})R}, \quad R \to \infty
\]

and \( \tilde{z} - M + \frac{1}{2} > 0 \) the integral over \( \Gamma_2 \) vanishes as \( R \to \infty \). Hence,

\[
e^{iz_a} I_s = \int_0^\infty e^{-\tilde{z}v} \frac{\sin(Miv)}{\sin(\frac{1}{2}iv)} \, dv + i \int_0^a e^{izu} \frac{\sin(Mu)}{\sin \frac{u}{2}} \, du
\]

and furthermore

\[
\Re I_s = \cos(\tilde{z}a) \int_0^\infty e^{-\tilde{z}v} \frac{\sinh(Mv)}{\sinh \frac{v}{2}} \, dv - \int_0^a \sin(\tilde{z}(u-a)) \frac{\sin(Mu)}{\sin \frac{u}{2}} \, du.
\]

This gives the terms \( S_{0,N} \) and \( S_{1,N} \). The integral \( I_c \) can be treated in like manner and the proof is finished. \( \square \)

Via elementary calculations one can obtain the bounds

\[
|S_{0,N}(z; x, y)| \leq \frac{1}{2\pi \sqrt{z}}, \quad |\tilde{S}_{0,N}(z; x, y)| \leq \frac{1}{2\pi \sqrt{z}} \quad (3.37)
\]

\[
|S_{1,N}(z; x, y)| \leq \frac{1}{\sqrt{z}} \frac{N + \frac{1}{2}}{2L} |x - y|, \quad |\tilde{S}_{1,N}(z; x, y)| \leq \frac{1}{\sqrt{z}} \frac{N + \frac{1}{2}}{2L} |x + y|.
\]

\[
(3.38)
\]

for the above kernel functions. Thereby, Proposition 3.5 helps to separate the \( x, y \) and \( N \) dependence of \( S_N(z; x, y) \) for special real values of \( z \) including the Fermi energy \( \beta \) such that (see Lemma A.1)

\[
S_N(\nu_N; x, y) \sim \frac{\zeta_N}{2\pi \sqrt{\nu_N}} \cos(\sqrt{\nu_N}(x - y)) \text{ with } \zeta_N \sim \ln N, \quad N \to \infty.
\]
By the addition theorem for the cosine this leading term can be written as
\[ S_{0,N}(\nu_N) = \frac{1}{2\pi\sqrt{\nu_N}}(P_s(\nu_N) + P_c(\nu_N)) \] (3.39)
with the rank-one operators from (3.15).

### 3.3. One-dimensional delta-term, \( D(z) \)

The delta-term being non trivial reflects on an abstract level the boundary conditions used in the definition of \( H \), which make up the difference between \( H \) and \(-\nabla^2\).

**Proposition 3.6.** For \( z \in \mathbb{C} \setminus \sigma(H) \), \( \varphi \in D(\nabla^2) \), and the resolvent \( R(z) \) of \( H \) we have
\[
(R(z)(z\mathbb{1} + \nabla^2)\varphi)(x) = \varphi(x) - \frac{\sin(\sqrt{z}(x + L))\varphi(L) - \sin(\sqrt{z}(x - L))\varphi(-L)}{\sin(2\sqrt{z}L)}.
\] (3.40)

Furthermore, the delta-term \( D(z) \) from (2.21) reads
\[
D(z) = \frac{L}{4} \left( \frac{1}{\sin^2(\sqrt{z}L)} P_s(z) + \frac{1}{\cos^2(\sqrt{z}L)} P_c(z) \right)
\] (3.41)
with the rank-one operators \( P_s(z) \) and \( P_c(z) \) from (3.15).

**Proof.** (a) In order to derive (3.40) we integrate by parts two times using the Dirichlet boundary conditions, i.e. \( R(z; x, \pm L) = 0 \), in the first step
\[
(R(z)\nabla^2\varphi)(x) = \int_{-L}^{L} R(z; x, y)\varphi''(y) \, dy
\]
\[
= [R(z; x, y)\varphi'(y)]_{-L}^{L} - \int_{-L}^{L} \frac{\partial R(z; x, y)}{\partial y}\varphi'(y) \, dy
\]
\[
= -\left[ \frac{\partial R(z; x, y)}{\partial y}\varphi(y) \right]_{-L}^{L} + \int_{-L}^{L} \frac{\partial^2 R(z; x, y)}{\partial y^2}\varphi(y) \, dy
\]
\[
= \frac{\partial R(z; x, -L)}{\partial y}\varphi(-L) - \frac{\partial R(z; x, L)}{\partial y}\varphi(L) - z \int_{-L}^{L} R(z; x, y)\varphi(y) \, dy + \varphi(x).
\]
From the explicit form (3.12) of \( R(z; x, y) \) we deduce
\[
\frac{\partial}{\partial y}R(z; x, \pm L) = \frac{\sin(\sqrt{z}(x \pm L))}{\sin(2\sqrt{z}L)},
\]
which implies (3.40).

(b) For Formula (3.41) we use (2.22) along with (3.40),
\[
0 = (R(z)(z\mathbb{1} + \nabla^2)D(z))(x, y)
\]
\[
= D(z; x, y) - \frac{\sin(\sqrt{z}(x + L))}{2\sin(\sqrt{z}L)\cos(\sqrt{z}L)} D(z; L, y)
\]
\[
+ \frac{\sin(\sqrt{z}(x - L))}{2\sin(\sqrt{z}L)\cos(\sqrt{z}L)} D(z; -L, y).
\]
By the Dirichlet boundary conditions
\[(R(z)\nabla[\nabla, R(z)])(\pm L, y) = \int_{-L}^{L} R(z; \pm L, y')(\nabla[\nabla, R(z)])(y', y) \, dy' = 0.\]
Hence, by definition (2.21) and the explicit form (3.12) of \(R(z)\) we get
\[D(z; \pm L, y) = \pm \frac{L}{2} \frac{\sin(\sqrt{z}(y \pm L))}{\sin(\sqrt{z}L) \cos(\sqrt{z}L)}. \]
Putting everything together we obtain
\[D(z; x, y) = \frac{L}{8} \frac{\sin(\sqrt{z}(x + L)) \sin(\sqrt{z}(y + L)) + \sin(\sqrt{z}(x - L)) \sin(\sqrt{z}(y - L))}{\sin^2(\sqrt{z}L) \cos^2(\sqrt{z}L)}, \]
which implies the statement via the usual trigonometric formulae. \(\square\)

3.4. Perturbed resolvent

Since the perturbed operator enters only through the T-operator and the operator \(\Omega(z)\) (cf. (2.3) and (2.7)) we have a closer look at suchlike operators. Recall from Section 2 that we already know those operators to exist for \(z \notin \mathbb{R}\).

What is new herein is that the bounds hold true uniformly on the entire Fermi parabola \(\Gamma_N\) including the Fermi energy \(\nu_N\) (cf. (3.5), (3.4)).

Lemma 3.7. Let \(V \in L^1(\mathbb{R})\) and assume in addition
\[q_\Omega := \frac{4}{\sqrt{\nu_N}} \|V\|_1 < 1. \quad (3.42)\]
Then, the operators \(\Omega(z)\) exist for all \(z \in \Gamma_N\) and are uniformly bounded with
\[\|\Omega(z)\| \leq \frac{1}{1 - q_\Omega} =: C_\Omega. \quad (3.43)\]
In particular, \(\nu_N \notin \sigma(H_V)\). If in addition \(V \in L^\infty(\mathbb{R})\) then the T-operator, \(T(z)\) (see (2.3)), exists and is bounded with \(\|T(z)\| \leq C_\Omega \|V\|_\infty.\)

Proof. Using \(\|J\| = 1\) we obtain from (3.29) the bound
\[\|\sqrt{|V|}R(z)\sqrt{|V|}J\| \leq \frac{4}{\sqrt{\nu_N}} \|V\|_1 < 1. \]
Hence, a Neumann series argument shows that \(\Omega(z)\) exists and is bounded with (3.43). Since \(\nu_N \notin \sigma(H)\) by construction the remark after (2.8) shows \(\nu_N \notin \sigma(H_V)\). Furthermore,
\[\|T(z)\| = \|\sqrt{|V|}J\Omega(z)\sqrt{|V|}\| \leq \|\sqrt{|V|}\|_\infty^2 \|J\| \|\Omega(z)\| = \|V\|_\infty \|\Omega(z)\| \]
completes the proof. \(\square\)

We had seen in Section 3.1 that it is advantageous to work with the operators \(R_{\infty}^{\pm}(\nu_N, s)\) (cf. (3.20)) instead of the resolvent \(R(z)\). Likewise, we employ the operators \(\Omega_{\infty}^{\pm}(\nu_N, s),\)
\[\Omega_{\infty}^{\pm}(\nu_N, s) := (1 - \sqrt{|V|}R_{\infty}^{\pm}(\nu_N, s)\sqrt{|V|}J)^{-1} \quad (3.44)\]
instead of $\Omega(z)$. In view of the rank-two operator in (3.20) it is reasonable to define (cf. (3.15))
\[
\Phi(\nu_N) := (1 - \sqrt{|V|}K(\nu_N)\sqrt{|V|})^{-1}, \quad K(\nu_N) := \frac{1}{2\sqrt{\nu_N}}G(\nu_N).
\]
(3.45)
The operator $K(\nu_N)$ is closely related to the resolvent of the free Schrödinger operator defined on the whole of $\mathbb{R}$. That is why it replaces $G(\nu_N)$.

**Lemma 3.8.** Let $V \in L^1(\mathbb{R})$. If
\[
q_\infty := \frac{3}{2\sqrt{\nu_N}} \|V\|_1 < 1, \quad q_\Phi := \frac{1}{2\sqrt{\nu_N}} \|V\|_1 < 1,
\]
(3.46)
then the operators $\Omega_\infty^{\pm}(\nu_N, s)$ and $\Phi(\nu_N)$ (defined in (3.44), (3.45)) exist and are bounded with
\[
\|\Omega_\infty^{\pm}(\nu_N, s)\| \leq \frac{1}{1 - q_\infty} := C_{\Omega_\infty}, \|\Phi(\nu_N)\| \leq \frac{1}{1 - q_\Phi} := C_\Phi.
\]
(3.47)
Furthermore, let $z \in \Gamma_N$. Then, (see (2.7))
\[
\|\Omega(z) - \Omega_\infty(\nu_N, Ls)\| \leq C_{\Omega_\infty}^\prime [\|V_L(0) + V_L^{(2)}(2s)\|V_L(2|s|)^{1\over 2}],
\]
(3.48)
where $\Omega_\infty$ stands for $\Omega_\infty^{\pm}$, $\Omega_\infty^{\pm}$ depending on whether $N$ in $\nu_N$ is even or odd. The constant is (see (3.43))
\[
C_{\Omega_\infty}^\prime := C_T C_{\Omega_\infty} \frac{3}{\sqrt{\nu_N}} \max\{1, \frac{1}{2\sqrt{\nu_N}}\}.
\]

**Proof.** We know from (3.31) and the assumption (3.46) that
\[
\|\sqrt{|V|}R_\infty^{\pm}(\nu_N, s)\sqrt{|V|}\| \leq \frac{3}{2\sqrt{\nu_N}} \|V\|_1 < 1.
\]
A Neumann series argument shows that $\Omega_\infty^{\pm}(\nu_N, s)$ exists and is bounded with (3.47). The operator $\Phi(\nu_N)$ is treated in like manner. For (3.48) note
\[
\Omega(z) - \Omega_\infty(\nu_N, Ls) = \Omega(z) \sqrt{|V|}(R_\infty(\nu_N, Ls) - R(z)) \sqrt{|V|}\Omega_\infty(\nu_N, s),
\]
where $\Omega_\infty$ is $\Omega_\infty^{\pm}$ for even $N$ in $\nu_N$ and $\Omega_\infty^{\pm}$ for odd $N$ and $R_\infty$ likewise. Hence, (3.43), (3.32), and the first part (3.47) conclude the proof.

We only need the matrix elements with respect to the functions $\omega_{s,c}(\nu_N)$ from (3.24). To this end, we introduce the $2 \times 2$ matrices
\[
\Omega_\infty^{\pm}(\nu_N, s) := \begin{pmatrix}
\omega_s(\nu_N), & J\Omega_\infty^{\pm}(\nu_N, s)\omega_s(\nu_N) \\
\omega_c(\nu_N), & J\Omega_\infty^{\pm}(\nu_N, s)\omega_c(\nu_N)
\end{pmatrix}
\]
(3.49)
and
\[
\hat{\Phi}(\nu_N) := \begin{pmatrix}
\omega_s(\nu_N), & J\hat{\Phi}(\nu_N)\omega_s(\nu_N) \\
\omega_c(\nu_N), & J\hat{\Phi}(\nu_N)\omega_c(\nu_N)
\end{pmatrix}
\]
(3.50)
Note that $\hat{\Phi}(\nu_N)^* = \hat{\Phi}(\nu_N)$ since $J\Phi(\nu_N)$ is self-adjoint. In this one-dimensional case the above $2 \times 2$ matrices correspond to the eigenspace decomposition of angular momentum in higher dimensions.
Lemma 3.9. Let $V \in L^1(\mathbb{R})$ satisfy in addition (see (3.47))
\[
\frac{1}{\sqrt{\nu_N}}\|V\|_1 \leq 1. 
\tag{3.51}
\]
Then, the $2 \times 2$ matrices $\hat{Z}^\pm(\nu_N, s)$,
\[
\hat{Z}^\pm(\nu_N, s) := (1 \mp \frac{1}{2\sqrt{\nu_N}} \hat{\Phi}(\nu_N)\hat{\tau}(\pm s))^{-1}, \ s \in \mathbb{R},
\tag{3.52}
\]
even exist. Here (for $\tau$ see (3.19)) for $s \in \mathbb{R}$,
\[
\hat{\tau}(s) := \text{diag}(\tau(s), -\tau(-s)), \ \lim_{s \to \infty} \hat{\tau}(s) = -i\mathbb{1}
\tag{3.53}
\]
Furthermore, we have
\[
\hat{\Omega}_\infty^\pm(\nu_N, s) = \hat{Z}^\pm(\nu_N, s)\hat{\Phi}(\nu_N) = \hat{\Phi}(\nu_N)\hat{Z}^\pm(\nu_N, -s)^*.
\tag{3.54}
\]
Proof. The operators $\Omega_\infty^\pm(\nu_N, s)$ have the form $(A - a_1(f_1, \cdot)g_1 - a_2(f_2, \cdot)g_2)^{-1}$
with an invertible operator $A$, vectors $f_{1,2}$, $g_{1,2}$, and $a_{1,2} \in \mathbb{C}$. Computing
the inverse on the vectors $g_{1,2}$ amounts to solving the equations
\[
(A - a_1(f_1, \cdot)g_1 - a_2(f_2, \cdot)g_2)h_k = g_k, \ k = 1, 2,
\]
for $h_{1,2}$. The matrix elements $(f_j, h_k)$, in particular, satisfy
\[
(f_j, h_k) - a_1(f_1, h_k)(f_j, A^{-1}g_1) - a_2(f_2, h_k)(f_j, A^{-1}g_2) = (f_j, A^{-1}g_k)
\]
for $j, k = 1, 2$. Introducing the $2 \times 2$-matrices
\[
\hat{B} := ((f_j, h_k))_{j,k=1,2}, \ \hat{A} := (f_j, A^{-1}g_k)_{j,k=1,2}, \ \hat{a} := \text{diag}(a_1, a_2)
\]
we can write this as
\[
\hat{B} - \hat{A}\hat{a}\hat{B} = \hat{A}
\]
which can easily be solved for $\hat{B}$. Now, for $\Omega^+(\nu, s)$ put
\[
a_1 = \frac{\tau(s)}{2\sqrt{\nu_N}}, \ a_2 = \frac{\tau(-s)}{2\sqrt{\nu_N}}, \ f_{1,2} = J^*\omega_{s,c}(\nu_N), \ g_{1,2} = \omega_{s,c}(\nu_N),
\]
to obtain the first equality in (3.54). The second follows from
\[
\hat{\Phi}(\nu_N)\hat{Z}^+(\nu_N, -s)^* = \hat{\Phi}(\nu_N)(1 - \frac{1}{2\sqrt{\nu_N}}\hat{\tau}(-s)^*\hat{\Phi}(\nu_N))^{-1}
\[
= (1 - \frac{1}{2\sqrt{\nu_N}}\hat{\Phi}(\nu_N)\hat{\tau}(s))^{-1}\hat{\Phi}(\nu_N)
\]
where we used the next to last relation in (3.53). The relations for $\hat{\tau}(s)$ are obvious. In order to show that $\hat{Z}^+(\nu_N, s)$ is well-defined we look at the entries
\[
|\omega_{s,c}(\nu_N), J\Phi(\nu_N)\omega_{s,c}(\nu_N)| \leq \|V\|_1\|\Phi(\nu_N)\|
\]
With the maximum norm $\|\cdot\|_\infty$ for matrices we thus get
\[
\frac{1}{2\sqrt{\nu_N}}\|\hat{\Phi}(\nu_N)\hat{\tau}(s)\|_\infty \leq \frac{1}{\sqrt{\nu_N}}\|V\|_1\|\Phi(\nu_N)\| \leq \frac{1}{\sqrt{\nu_N}}\|V\|_1\|C_\Phi < 1.
\]
Now a Neumann series argument proves the statement. The matrix $\hat{Z}^-(\nu_N, s)$
is treated likewise. □
The Neumann series was the only abstract tool we used in proving invertibility of operators. Therefore, the conditions put on the potential $V$ might be too restrictive. For example, the operator $\mathbb{1} - VR(z)$ is known to be invertible for all $z \in \mathbb{C} \setminus M$ with $M$ being a discrete set (see [16], p.114). Thus, more advanced tools could possibly help to allow for larger classes of potentials. But that is not our main concern here.

3.5. Perturbed eigenvalues

One important consequence of Lemma 3.7 is that the spectrum $\sigma(H_V)$ of the operator $H_V$ on $L^2[-L,L]$ can be decomposed with respect to the Fermi energy $\nu_N$,

$$\sigma(H_V) = \sigma_1(H_V) \cup \sigma_2(H_V),$$

$$\sigma_1(H_V) := \{ \mu_j \mid \mu_j < \nu_N \}, \quad \sigma_2(H_V) := \{ \mu_j \mid \mu_j > \nu_N \}. \quad (3.55)$$

Equivalently, there is an $M = M(N)$ with

$$\mu_j < \nu_N, \quad j = 1, \ldots, M, \quad \mu_j > \nu_N, \quad j \geq M + 1. \quad (3.56)$$

Exactly $N$ free eigenvalues lie below $\nu_N$. We need to know how many perturbed eigenvalues do so which amounts to estimating $M$. For the upper bound we modify Bargmann’s inequality on negative eigenvalues (cf. [15], Thm. XIII.9).

**Proposition 3.10.** Let $V_- := \min\{V, 0\}$ satisfy

$$|V_-(x)| \leq \frac{C_\alpha}{(1 + |x|)^{\alpha+1}} \quad (3.57)$$

with $\alpha > 0$ and $C_\alpha \geq 0$. Then, for all $E > 0$

$$M := \#\{\mu_j \mid \mu_j < E\} \leq \frac{2L}{\pi} \sqrt{E} + C_E,$$

$$C_E := \frac{1}{2E} \left( \frac{2C_\alpha}{\alpha \pi} (\|V_-\|_\infty + E)^{\frac{1}{2}} + \|V_-\|_\infty \right). \quad (3.58)$$

In particular, with $E = \nu_N$ being the Fermi energy the bound becomes

$$M \leq N + \frac{1}{2} + C_{\nu_N}. \quad (3.59)$$

**Proof.** By the variational principle, the number of eigenvalues $M = M(V)$ satisfies $M(V) \leq M(V_-)$. We may therefore assume that $V \leq 0$ and hence $V = -|V|$. By a shift of the spectrum $M$ equals the number of negative eigenvalues of

$$-\psi'' - (|V| + E)\psi = \tilde{\mu}\psi, \quad \psi(-L) = 0 = \psi(L).$$

The eigenfunction $\psi_M$ corresponding to $\tilde{\mu}_M$ has exactly $M + 1$ roots,

$$-L \leq x_0 < x_1 < \cdots < x_M = L.$$

Let us abbreviate

$$I_k := [x_k, x_{k+1}], \quad V_k := \sup_{x \in I_k} |V(x)|, \quad k = 0, \ldots, M - 1.$$
Apparently, $\tilde{\mu}_M$ is a negative eigenvalue for the Dirichlet problem on each $I_k$. We want a lower bound for the distance of two consecutive roots. To this end, we estimate

$$\int_{I_k} (|V(x)| + E)|\psi_M(x)|^2 \, dx \leq (V_k + E) \int_{I_k} |\psi_M(x)|^2 \, dx$$

$$\leq (V_k + E) \left( \frac{x_{k+1} - x_k}{\pi} \right)^2 \int_{I_k} |\psi_M'(x)|^2 \, dx,$$

where we used Wirtinger’s inequality (see [4]) or in other words, the variational principle for the lowest Dirichlet eigenvalue. If we had

$$(V_k + E) \left( \frac{x_{k+1} - x_k}{\pi} \right)^2 \leq 1$$

the differential equation and the Dirichlet conditions would imply

$$\int_{I_k} (|V(x)| + E)|\psi_M(x)|^2 \, dx$$

$$\leq \int_{I_k} |\psi_M'(x)|^2 \, dx = \int_{I_k} (|V(x)| + E)|\psi(x)|^2 \, dx + \tilde{\mu}_M \int_{I_k} |\psi_M(x)|^2 \, dx.$$

This is impossible for $\tilde{\mu}_M < 0$ and thus

$$1 \leq \left( \frac{x_{k+1} - x_k}{\pi} \right)^2 (V_k + E). \quad (3.60)$$

Since $V_k \leq \|V\|_\infty$ we obtain a first rough but uniform bound

$$x_{k+1} - x_k \geq \frac{\pi}{(\|V\|_\infty + E)^{\frac{1}{2}}} =: \delta, \quad 0 \leq k \leq M - 1.$$

Now we estimate in (3.60),

$$\pi \leq (x_{k+1} - x_k)(V_k + E)^{\frac{1}{2}} \leq (x_{k+1} - x_k)\sqrt{E} \left( 1 + \frac{V_k}{2E} \right).$$

This can be cast into the form

$$1 \leq \frac{\sqrt{E}}{\pi} (x_{k+1} - x_k) + \frac{V_k}{2E} \frac{1}{1 + \frac{V_k}{2E}} \leq \frac{\sqrt{E}}{\pi} (x_{k+1} - x_k) + \frac{V_k}{2E}.$$

Summing up from 0 to $M - 1$ we obtain

$$M \leq \frac{2L}{\pi} \sqrt{E} + \frac{1}{2E} \sum_{k=0}^{M-1} V_k.$$

Using (3.57) we compare the sum with the integral of the majorant of $V$

$$\sum_{k=0}^{M-1} V_k \leq \frac{1}{\delta} \sum_{0 \leq k \leq M-2, x_{k+2} \leq 0} (x_{k+2} - x_{k+1})V_k + \frac{1}{\delta} \sum_{1 \leq k \leq M-1, x_{k-1} > 0} (x_k - x_{k-1})V_k + \|V\|_\infty$$

$$\leq \frac{1}{\delta} \int_{-L}^{L} \frac{C_\alpha}{(1 + |x|)^{\alpha+1}} \, dx + \|V\|_\infty,$$
where \( \|V\|_\infty \) is due to the summand that was left out. This proves (3.58). Finally, (3.59) is an immediate consequence of the definition (3.41) of \( \nu_N \). □

An upper bound on the eigenvalues gives a lower bound on their number.

**Proposition 3.11.** Let \( V_+ := \max\{V,0\} \in L^1(\mathbb{R}) \). Then, the perturbed eigenvalues satisfy

\[
\sqrt{\mu_k} \leq \frac{k\pi}{2L} + \frac{1}{k\pi} \|V_+\|_1.
\]

Moreover, for \( E > 0 \) satisfying

\[
E \geq \frac{2}{L} \|V_+\|_1
\]

the number of eigenvalues below \( E \) has the lower bound

\[
M := \#\{\mu_k | \mu_k \leq E\} \geq \frac{2L}{\pi^2} \sqrt{E} - \frac{1}{\sqrt{E}} - 1.
\]

In particular, with \( E = \nu_N \) being the Fermi energy this becomes

\[
M \geq N - \frac{1}{2} \frac{2\|V_+\|_1}{\mu_1} \sqrt{\nu_N}.
\]

**Proof.** By the variational principle, the eigenvalues \( \mu_j = \mu_j(V) \) and the number of eigenvalues \( M = M(V) \) satisfy \( \mu_j(V) \leq \mu_j(V_+) \) and \( M(V) \geq M(V_+) \). Thus, we may assume \( V \geq 0 \). In (3.9) we use the modified Prüfer variables

\[
0 \neq \left( \frac{1}{\mu} \psi' \right) = r \left( \cos \vartheta \sin \vartheta \right).
\]

The phase function \( \vartheta \) satisfies the initial value problem

\[
\vartheta' = \sqrt{\mu} - \frac{V}{\mu} \sin^2 \vartheta, \ \vartheta(-L,\mu) = 0.
\]

Integrating yields

\[
\vartheta(L,\mu) = 2L\sqrt{\mu} - \frac{1}{\sqrt{\mu}} \int_{-L}^{L} V(y) \sin^2 \vartheta(y,\mu) dy.
\]

To give a solution of (3.9) is equivalent to \( \vartheta(L,\mu_k) = k\pi, \ k \in \mathbb{N} \). We show that \( \vartheta(x,\mu) \) is strictly increasing in \( \mu \) or more precisely

\[
\Theta(x,\mu) := \frac{\partial}{\partial \mu} \vartheta(x,\mu) > 0.
\]

From (3.66) we deduce

\[
\Theta' = \frac{1}{2\sqrt{\mu}} (1 + \frac{V}{\mu} \sin^2 \vartheta) - \frac{V}{\sqrt{\mu}} \sin(2\vartheta) \Theta, \ \Theta(-L,\mu) = 0.
\]

With the abbreviation

\[
a(x) := -\frac{1}{\sqrt{\mu}} \int_{-L}^{x} V(y) \sin 2\vartheta(y) dy
\]
we obtain
\[ \Theta(x, \mu) = \frac{1}{2\sqrt{\mu}} e^{a(x)} \int_{-L}^{x} e^{-a(y)} \left[ 1 + \frac{V(y)}{\mu} \sin^2 \vartheta(y) \right] dy > 0. \]

Furthermore, from (3.66) it is obvious that
\[ \limsup_{\mu \to +0} \vartheta(L, \mu) \leq 0, \quad \liminf_{\mu \to +\infty} \vartheta(L, \mu) = \infty. \]

We conclude that \( \mu_k \) is the unique solution of the eigenvalue equation
\[ k\pi = 2L\sqrt{\mu} - \frac{1}{\sqrt{\mu}} \int_{-L}^{L} V(y) \sin^2 \vartheta(y) \, dy. \]

This implies the bound (3.61) since \( \mu_k \geq \lambda_k \). A lower bound for \( M \) is thus given by the largest \( k \) such that
\[ \frac{k\pi}{2L} + \frac{||V||_1}{k\pi} \leq \sqrt{E} \]

which can be written equivalently
\[ \left( k - \frac{L\sqrt{E}}{\pi} \right)^2 \leq \frac{L^2 E}{\pi^2} - \frac{2L}{\pi^2} ||V||_1 =: r_{E,L}^2. \]

The righthand side is positive by (3.62). Solving for \( k \) yields two inequalities
\[ \frac{2L||V||_1}{\pi^2} \frac{1}{L\sqrt{E} + r_{E,L}} \leq k \leq \frac{2L\sqrt{E}}{\pi} - \frac{2L||V||_1}{\pi^2} \frac{1}{L\sqrt{E} + r_{E,L}}. \]

These are surely satisfied when
\[ \frac{2||V||_1}{\pi} \frac{1}{\sqrt{E}} \leq k \leq \frac{2L}{\pi} \sqrt{E} - \frac{2||V||_1}{\pi} \frac{1}{\sqrt{E}} \]

which makes sense because of (3.62). The righthand side differs from the next smaller integer by at most one which proves (3.63).

\( \square \)

4. Delta-estimate

An integral containing Dirac’s delta function reduces to a point evaluation of the integrand. A similar effect will be employed in Proposition 5.2. The necessary estimates are dubbed delta estimates for that reason. To any bounded function \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) we associate the transformed function
\[ f^*(L) := \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} f(s) \int_{-L}^{L} e^{s|x|} \, dx \, ds, \quad a > 0. \]  

(4.1)

The inner integral is motivated by the estimate (3.20).

**Lemma 4.1.** Let \( W \in L^1(\mathbb{R}) \) satisfy \( X^nW \in L^\infty(\mathbb{R}) \) with some \( n \in \mathbb{N}_0 \) and define \( W_L \) as in (3.25). Let \( g \geq 0 \) be bounded and weakly differentiable with \( g' \leq 0 \). Then,
\[ \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} g(s) W_L^{(m)}(s) \, ds \leq L^{m-1} g(0) n ||W||_1 + L^{m-n} g^*(L) ||X^nW||_\infty \]

(4.2)
for all \( m \in \mathbb{N}_0 \). Moreover, let \( h \geq 0 \) be bounded and weakly differentiable with \( h(0) = 0 \) and \( h' \leq g \). For all \( m \in \mathbb{N}_0 \),

\[
\int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} h(s) W_L(s) \, ds \leq \frac{n^2}{L^2} g(0) \| W \|_1 + \frac{1}{L^n} \left[ \frac{n}{L} g^+(L) + h^+(L) \right] \| X^n W \|_{\infty}. \tag{4.3}
\]

**Proof.** Let \( f \geq 0 \) be weakly differentiable and bounded. Integration by parts and dropping the negative term that appears yields the following inequality

\[
L \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} f(s) W_L^{(p)}(s) \, ds \leq f(0) W_L^{(p)}(0) + \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} f'(s) W_L^{(p)}(s) \, ds + \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} f(s) W_L^{(p+1)}(s) \, ds. \tag{4.4}
\]

(a) When \( f = g \) in (4.4), the integral containing \( g' \) becomes non-positive and can be dropped. Iterating the resulting inequality \( n \)-times yields

\[
L \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} g(s) W_L^{(m)}(s) \, ds
\leq g(0) \sum_{k=0}^{n-1} \frac{1}{L^k} W_L^{(m+k)}(0) + \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} g(s) W_L^{(m+n)}(s) \, ds.
\]

Using the estimates (3.20) with \( p = m + k, q = 0 \) in the sum and \( p = m, q = n \) in the integral we obtain (4.2).

(b) With \( f = h \) in (4.4) we get

\[
\int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} h(s) W_L^{(k)}(s) \, ds
\leq \frac{1}{L} \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} g(s) W_L^{(k)}(s) \, ds + \frac{1}{L} \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} h(s) W_L^{(k+1)}(s) \, ds.
\]

After iterating we obtain

\[
\int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} h(s) W_L(s) \, ds
\leq \frac{1}{L} \sum_{k=0}^{n-1} \frac{1}{L^k} \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} g(s) W_L^{(k)}(s) \, ds + \frac{1}{L^n} \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} h(s) W_L^{(n)}(s) \, ds
\leq \frac{1}{L} \sum_{k=0}^{n-1} \frac{1}{L^k} \left[ nL^{k-1} \| W \|_1 g(0) + L^{k-n} g^+(L) \| X^n W \|_{\infty} \right] + \frac{1}{L^n} h^+(L) \| X^n W \|_{\infty}
\]

where we estimated the integrals in the sum via (a) and the remaining integral by (3.26) with \( p = n, q = 0 \). That concludes the proof.

We can now formulate the delta estimate.
Proposition 4.2. Let $W \in L^1(\mathbb{R})$ satisfy $X^nW \in L^\infty(\mathbb{R})$ for some $n \in \mathbb{N}_0$ and define $W_L$ as in (3.25). Assume that $f_L : \mathbb{R}^+ \to \mathbb{R}^+$ obey
\[ f_L(s) \leq s\Theta(L) \quad \text{and} \quad f_L(s) \leq \vartheta(L), \quad L > 0, \quad s \in \mathbb{R}^+, \quad \text{(4.5)} \]
with functions $\vartheta, \Theta : \mathbb{R}^+ \to \mathbb{R}^+$. Then,
\[
\int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} f_L(s)W_L(s) \, ds \leq \frac{n^2\Theta(L)}{L^2} \|W\|_1
\]
\[ + \frac{2}{L^n} \left[ \frac{n\Theta(L)}{L} (1 + \sqrt{2} \ln(L + 1)) + \frac{1}{2} L\vartheta(L)^2 + \sqrt{2} \ln \left( \frac{\Theta(L)}{\vartheta(L)} + 1 \right) \right] \|X^nW\|_\infty. \quad \text{(4.6)}
\]

Proof. We want to apply (4.3) in Lemma 4.1. To this end, we define $g_L(s) := \Theta(L)$ and
\[
h_L(s) := \begin{cases} s\Theta(L) & \text{for } s \leq \eta, \\ \vartheta(L) & \text{for } s > \eta, \end{cases}
\]
where $\eta := \vartheta(L)/\Theta(L)$ for short. Obviously, $f_L \leq h_L$ and $h_L' \leq g_L$. Thus, we only need to estimate $g_L^*$ and $h_L^*$. For any admissible $f$ (e.g. bounded)
\[
\frac{1}{2} f^*(L) = \int_0^\infty \frac{e^{-Ls}}{\sqrt{a + s^2}} s f(s) \int_0^{sL} e^{-x} \, dx \, ds
\leq L \int_0^a \frac{f(s)}{\sqrt{a + s^2}} \, ds + \int_a^\infty \frac{f(s)}{\sqrt{a + s^2}} \, ds
\]
with some $a > 0$. For $f = g_L$ we choose $a := 1/L$ and substitute $s \mapsto 1/s$ in the second integral. Then,
\[
\frac{1}{2} g_L^*(L) \leq \Theta(L) \left[ L \int_0^\frac{1}{L} ds + \sqrt{2} \int_0^L \frac{1}{1 + s} \, ds \right] = \Theta(L)(1 + \sqrt{2} \ln(1 + L)).
\]
When $f = h_L$ we put $a = \eta$ which matches the definition of $h_L$. Then,
\[
\frac{1}{2} h_L^*(L) \leq L\Theta(L) \int_0^\eta s \, ds + \sqrt{2} \vartheta(L) \int_0^\frac{1}{L} \frac{1}{1 + s} \, ds
\leq \frac{1}{2} L\vartheta(L)^2 + \sqrt{2} \vartheta(L) \ln \left( 1 + \frac{\Theta(L)}{\vartheta(L)} \right)
\]
where we employed the same substitution as above. \hfill \square

We will need the delta-estimate only for $n = 2$ and two choices of $\vartheta$ and $\Theta$. The resulting estimates for the integral $I_L$ in (4.6) are
\[
\vartheta(L) = \ln L, \quad \Theta(L) = L : \quad I_L \leq C_1(W) \left[ \frac{1}{L} + \frac{\ln L}{L^2} + \frac{\ln^2 L}{L^2} \right] \quad \text{(4.7)}
\]
\[
\vartheta(L) = 1, \quad \Theta(L) = L^\frac{3}{2} : \quad I_L \leq C_2(W) \left[ \frac{1}{L^\frac{3}{2}} + \frac{\ln L}{L^2} + \frac{\ln L}{L^2} \right] \quad \text{(4.8)}
\]
for $L \to \infty$ with constants $C_1(W), C_2(W)$ that depend only on $W$. 
5. Asymptotics

In the thermodynamic limit the particle density, \( \rho \), is kept constant. Usually, that would be \( N/(2L) \). However, taking

\[
\rho := \frac{N + \frac{1}{2} }{2L} > 0, \quad \nu := \pi^2 \rho^2
\]

will make our formulae handier since \( \nu_N = \nu \) is constant then. We start with combining Propositions 2.1, 2.2, and 3.6 and write

\[
\text{tr} \left[ P_N (1 - \Pi_M) \right] = \frac{1}{2\pi i} \int_{\Gamma_N} \left[ \frac{1}{2z} \text{tr} A_N(z) dz + \frac{1}{z} \text{tr} B_N(z) \right] dz.
\]

(5.2)

Here \( M = M(N) \) according to the decomposition of the spectrum \( \sigma(H_V) \). The Fermi parabola \( \Gamma_N \) is defined in (3.5). The operators in (5.2) are

\[
A_N(z) := P_N R(z)T(z)(R(z) - C(z))T(z), \quad B_N(z) := P_N R(z)T(z)D(z)T(z),
\]

(5.3)

with the operators \( P_N, R(z), C(z), D(z), \) and \( T(z) \) defined in (3.7), (3.11), (2.21), (2.3), respectively. The traces can be treated further. For \( A_N(z) \) we use the \( \varphi_j \)'s and write

\[
\text{tr} A_N(z) = \sum_{j=1}^{N} \frac{1}{z - \lambda_j} (\varphi_j, T(z)(R(z) - C(z))T(z)\varphi_j).
\]

(5.4)

For \( B_N(z) \) we recall the definition \( S_N(z) = P_N R(z) \), the rank-one operators \( P_{s,c}(z) \) from (3.15), the operator \( \Omega(z) \) from (2.7), and (3.41) for \( D(z) \). Then,

\[
\text{tr} B_N(z) = d_{s,L}(z)(J\Omega(\bar{z})\omega_s(\bar{z}), \sqrt{|V|S_N(z)} \sqrt{|V|J\Omega(z)\omega_s(z)})
\]

\[
+ d_{c,L}(z)(J\Omega(\bar{z})\omega_c(\bar{z}), \sqrt{|V|S_N(z)} \sqrt{|V|J\Omega(z)\omega_c(z)}),
\]

(5.5)

with \( \omega_{s,c}(z) \) as in (3.24) and the abbreviation

\[
d_{s,L}(z) := \frac{L}{4\sin^2(L\sqrt{z})}, \quad d_{c,L}(z) := \frac{L}{4\cos^2(L\sqrt{z})}.
\]

(5.6)

The complex conjugates in (5.5) are due to the sesquilinearity of the scalar product. We will see that both \( \text{tr} A_N(z) \) and \( \text{tr} B_N(z) \) decay sufficiently fast on the Fermi parabola such that the integrals can be treated separately.

5.1. Subdominant term

We discuss the subdominant term arising directly from the integral formula. Additional corrections will appear in Section 5.2.

Proposition 5.1. Let \( V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfy (3.42). Furthermore, assume \( X^2 V \in L^1(\mathbb{R}) \) with the operator \( X \) from (3.1). Then,

\[
\left| \int_{\Gamma_N} \frac{1}{2z} \text{tr} A_N(z) dz \right| \leq C_{sub} \sqrt{N + \frac{1}{L}} (\nu_N^{-\frac{1}{2}} + \nu_N^{-\frac{3}{2}})
\]

(5.7)

with a constant \( C_{sub} \geq 0 \). The integral converges absolutely. The operator \( A_N(z) \) is defined in (5.3) and the Fermi parabola \( \Gamma_N \) in (3.5).
Proof. We estimate \( \text{tr} A_N(z) \) for \( z \in \Gamma_N \) (see (3.3)). From (5.4) we obtain
\[
| \text{tr} A_N(z) | \leq \sum_{j=1}^{N} \frac{1}{|z - \lambda_j|} \left| (\varphi_j, T(z)R(z)T(z)\varphi_j) + (\varphi_j, T(z)C(z)T(z)\varphi_j) \right|. \tag{5.8}
\]
The matrix elements can be estimated with the aid of Lemmas 3.3 and 3.7 as follows
\[
| (\varphi_j, T(z)R(z)T(z)\varphi_j) | = | (\varphi_j, \sqrt{|V|J}\Omega(z)\sqrt{|V|}R(z)\sqrt{|V|J}\Omega(z)\sqrt{|V|}\varphi_j) | \leq \| \sqrt{|V|\varphi_j}^2 \| \Omega(z)^2 \| K(z) \| \leq \frac{C_1}{L} \frac{1}{\sqrt{\nu_N + s^2}},
\]
and
\[
| (\varphi_j, T(z)C(z)T(z)\varphi_j) | \leq \| \sqrt{|V|\varphi_j}^2 \| \Omega(z)^2 \| \sqrt{|V|C(z)\sqrt{|V|}} \| \leq \frac{C_2}{L}
\]
with constants
\[
C_1 := 4\|V\|^2 C_3^2, \quad C_2 := 8\|V\|^2 \|X^2V\|_1 \frac{1}{L}. \tag{5.9}
\]
In order to treat the remaining sum in (5.8) we bound (3.6) from below via
\[
|z - \lambda_j| \geq (\nu_N + s^2)^{\frac{1}{2}} ((\sqrt{\nu_N} - \sqrt{\lambda_j})^2 + s^2)^{\frac{1}{2}} \geq \sqrt{2}(\nu_N + s^2)^{\frac{1}{2}} (\sqrt{\nu_N} - \sqrt{\lambda_j})^{\frac{3}{2}} \sqrt{|s|}. \]
With the aid of (A.1) we obtain
\[
\sum_{j=1}^{N} \frac{1}{|z - \lambda_j|} \leq \frac{1}{\sqrt{2}(\nu_N + s^2)^{\frac{3}{2}}} \frac{1}{\sqrt{|s|}} \frac{\sqrt{2L}}{\sqrt{\pi}} \sum_{j=1}^{N} \left( \frac{1}{N + \frac{1}{2} - j} \right)^{\frac{3}{2}} \leq \frac{4}{\sqrt{\pi} (\nu_N + s^2)^{\frac{3}{2}} \sqrt{|s|}}
\]
for \( s \neq 0 \) and thus
\[
| \text{tr} A_N(z) | \leq \frac{4}{\sqrt{\pi}} \frac{\sqrt{N + 1}}{\sqrt{L}} \frac{1}{(\nu_N + s^2)^{\frac{3}{2}}} \sqrt{|s|} \left( \frac{C_1}{(\nu_N + s^2)^{\frac{3}{2}}} + C_2 \right). \tag{5.9}
\]
Parametrizing the Fermi parabola as usual (see (3.5)), we estimate in (5.7)
\[
\left| \int_{\mathbb{R}} \frac{1}{\sqrt{\nu_N + is}} \text{tr} A_N(z(s)) \, ds \right| \leq \frac{4}{\sqrt{\pi}} \frac{\sqrt{N + 1}}{\sqrt{L}} \left[ \int_{\mathbb{R}} \left( \frac{C_1}{(\nu_N + s^2)^{\frac{3}{2}}} \right) \sqrt{|s|} \, ds + \int_{\mathbb{R}} \frac{C_2}{(\nu_N + s^2)^{\frac{3}{2}}} \sqrt{|s|} \, ds \right],
\]
where we used (5.9). For \( \alpha \in \left\{ \frac{3}{2}, 1 \right\} \) the integral
\[
\int_{\mathbb{R}} \frac{1}{(\nu_N + s^2)^{\alpha}} \sqrt{|s|} \, ds = 4\nu_N^{\frac{1}{2} - \alpha} \int_{0}^{\infty} \frac{1}{(1 + s^4)^{\alpha}} \, ds
\]
exists. Thus, the integral in (5.7) converges absolutely and satisfies the bound given there with an appropriate constant. \( \square \)
5.2. Dominant term

To begin with, we single out the dominant part of the integral over \( \text{tr} \, B_N(z) \).

**Proposition 5.2.** Let \( V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) satisfy \((3.42), (3.46)\), and \( X^pV \in L^\infty(\mathbb{R}) \), \( p = 2, 3 \). The following integral over \( B_N \) (see \((5.5)\)) converges absolutely and behaves in the thermodynamic limit according to \((5.1)\) asymptotically as

\[
\frac{1}{2\pi i} \int_{\Gamma_N} \frac{1}{z} \text{tr} \, B_N(z) \, dz = \kappa_N \gamma_L(\nu) + O(1), \quad N, L \to \infty. \tag{5.10}
\]

Here, \( \kappa_N \) is from Proposition [3.3] and \( \gamma_L(\nu) := \gamma_{s,L}(\nu) + \gamma_{c,L}(\nu) \),

\[
\gamma_{s,c,L}(\nu) := \frac{1}{\pi \sqrt{\nu}} \int_{\mathbb{R}} d_{s,c,L}(s) (\Omega_{\infty}(\nu, -L s) \omega(\nu), F(\nu) \Omega_{\infty}(\nu, L s) \omega(\nu)) \, ds \tag{5.11}
\]

with the bounded operator

\[
F(\nu) := J^* \sqrt{|V|}(P_{s}(\nu) + P_{c}(\nu)) \sqrt{|V|} J. \tag{5.12}
\]

\( \Omega_{\infty} \) stands for \( \Omega_{\infty}^+, \Omega_{\infty}^- \) depending on whether \( N \) is even or odd.

**Proof.** We proceed in three steps. First, we show that the integral converges absolutely. Then, we weed out the non-essential parts of the integral with the aid of the delta-estimate, Proposition [4.2]. Finally, we keep only the dominant part of the truncated resolvent \( S_N(\nu) \).

(a) We bound \( \text{tr} \, B_N(z) \) for \( z \in \Gamma_N \) (see \((5.5)\)). With the aid of Lemmas 3.1 [3.2] and 3.4 we infer from \((5.5)\)

\[
| \text{tr} \, B_N(z) | \leq \| \Omega(z) \| \| \Omega(z) \| \sqrt{|V|} |S_N(z)\sqrt{|V|}| \times
\]

\[
\times (|d_s(z)| \| \omega_s(z) \| \| \omega_s(z) \| + |d_c(z)| \| \omega_c(z) \| \| \omega_c(z) \|)
\]

\[
\leq 32 \pi C^2 \Omega \| V \|_1 \frac{e^{-2L|s|}}{\sqrt{\nu} + s^2} V_L(2s) \ln(N + 1). \tag{5.13}
\]

Parametrizing the Fermi parabola as in \((5.5)\), we conclude that

\[
\frac{1}{2\pi i} \int_{\Gamma_N} \frac{1}{z} \text{tr} \, B_N(z) \, dz = \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\sqrt{\nu + is}} \text{tr} \, B_N(z(s)) \, ds \tag{5.14}
\]

converges absolutely because of \((4.2)\) with \( g \equiv 1 \) and \( m = n = 0 \).

(b) The following calculations look alike for \( \delta_{s,c,L}(s) := d_{s,c}(z(s)) \) and the corresponding quantities. Therefore, we simply write \( \delta_L(s) \) etc. to denote either case. We evaluate the integral in \((5.14)\) by successively simplifying

\[
(J \Omega(z(s))\omega(z(s)), \sqrt{|V|} S_N(z(s))\sqrt{|V|} J \Omega(z(s))\omega(z(s))) \tag{5.15}
\]

in the integrand with the aid of the delta-estimate, Proposition [4.2].
(i) At first we replace $S_N(z)$ in \((5.15)\) by $S_N(\nu)$ which results in the error
\[
e^{(1)}_L := \left| \int_R \frac{\delta_L(s)}{\sqrt{\nu + i s}} \times (J\Omega(\tilde{z}(s))\omega(\tilde{z}(s)), \sqrt{|V|}(S_N(z(s)) - S_N(\nu))\sqrt{|V|}J\Omega(z(s))\omega(z(s)))\right| ds \leq 4C_1^2 L \int_R \frac{e^{-2L|s|}}{\sqrt{\nu + s^2}} V_L(2s)\|\sqrt{|V|}(S_N(z(s)) - S_N(\nu))\|\sqrt{|V|}\| ds.
\]
Note that $N = N(L)$.

Recalling Lemma 3.4, we use Proposition 4.2 with $f_L(s) = \|\sqrt{|V|}(S_N(z(s)) - S_N(\nu))\|\sqrt{|V|}$, $\vartheta(L) = \ln L$, $\Theta(L) = L$ and obtain the error (cf. \((4.7)\))
\[
e^{(1)}_L \leq C_1(1 + \frac{\ln L}{L} + \frac{\ln^2 L}{L^2}).
\]

(ii) Now we replace the right $\omega(z)$ in \((5.15)\) by $\omega(\nu)$ resulting in the error
\[
e^{(2,r)}_L \leq 4C_1^2 \int_R \frac{e^{-2L|s|}}{\sqrt{\nu + s^2}} \|\omega(\tilde{z}(s))\|\|\omega(z(s)) - \omega(\nu)\| ds \|\sqrt{|V|}S_N(\nu)\|\sqrt{|V|}.
\]
By virtue of \((3.27)\) we can estimate
\[
\|\omega(\tilde{z}(s))\|\|\omega(z(s)) - \omega(\nu)\| \leq (V_L(2|s|)^{\frac{1}{2}} + V_L(0)^{\frac{1}{2}})V_L(2|s|)^{\frac{1}{2}} \leq 2V_L(2|s|).
\]
Alternatively, \((3.28)\) along with \((3.28)\) yields
\[
\|\omega(\tilde{z}(s))\|\|\omega(z(s)) - \omega(\nu)\| \leq |s|V_L(2|s|)^{\frac{1}{2}}V_L(2|s|)^{\frac{1}{2}} \leq |s|L^{\frac{1}{2}} V_L(2|s|)^{\frac{1}{2}}V_L^{(1)}(2|s|)^{\frac{1}{2}}.
\]
Define $W$ by $W(x) := \max\{|V(x)|, |xV(x)|\}$ and note $X^2 W \in \mathcal{L}^\infty(\mathbb{R})$. Then,
\[
\|\omega(\tilde{z}(s))\|\|\omega(z(s)) - \omega(\nu)\| \leq f_L(s)W_L(2|s|).
\]
Thus, Proposition 4.8 applies with $\vartheta(L) = 1$, $\Theta(L) = L^{\frac{1}{2}}$. The left $\omega(z(s))$ in \((5.15)\) and the corresponding error $e^{(2)}_L$ can be treated in like manner when one uses, for the sake of convenience, the same bound for $\omega(\nu)$ as for $\omega(z(s))$ (see \((3.26)\)). Thus, the total error $e^{(2)}_L := e^{(2,r)}_L + e^{(2,l)}_L$ made in this section can be bounded
\[
e^{(2)}_L \leq C_2 \left( \frac{1}{L^2} + \frac{\ln L}{L^{\frac{1}{2}}} + \frac{\ln L}{L^{2}} \right) \ln L
\]
where the rightmost logarithm is from the truncated resolvent (Lemma 3.4).

(iii) Finally, we replace $\Omega(z(s))$ in \((5.15)\) by $\Omega_\infty(\nu, L)$). Here, $\Omega_\infty$ stands for $\Omega^+_{\infty}$, $\Omega^-_{\infty}$ when $N$ is even or odd, respectively. The inequality
\[
\|\Omega(z(s)) - \Omega_\infty(\nu, L)\| \leq f_L(s)W_L(2|s|)
\]
follows from \((3.47)\) and \((3.48)\) with the same $W$ and the same simplifications as in (ii). The functions $f_L$ satisfy the assumptions of Proposition 4.2 with $\vartheta(L) = 1$ and $\Theta(L) = L^{\frac{1}{2}}$. Hence, $e^{(3)}_L$ can be bounded as in (ii).
(iv) It is easy to replace $\sqrt{\nu + s^2}$ in the integral (5.14) by $\sqrt{\nu}$ which gives the error

$$e_L^{(4)} \leq 4C^2_{\Omega^\infty} \|\omega(\nu)\|^2 L \int_\mathbb{R} \frac{e^{-2L|s|}}{\sqrt{\nu(\nu + s^2)}} ds \times \|\sqrt{V} S_N(\nu)\sqrt{V}\| \leq C_4 \frac{\ln L}{L}. $$

(c) We decompose $S_N(\nu)$ according to Proposition 3.5 and find

$$\|\sqrt{V} S_{j,N}(\nu)\| \leq C, \quad j = 0, 1, \quad \|\sqrt{V} S_{1,N}(\nu)\| \leq C $$

because of the estimates (3.37) and (3.38). Thus, we are left with

$$\left| \int_\mathbb{R} \delta_L(s) \left( J\Omega^\infty(\nu, -Ls) \omega(\nu), J\Omega^\infty(\nu, Ls) \omega(\nu) \right) ds \right| \leq 4C^2_{\Omega^\infty} \|V\|_1.$$ 

Hence, the dominant term is given through $S_{0,N}(\nu)$. Writing it as in (3.39) gives the operator $F(\nu)$, which is obviously bounded, and thus $\gamma_L(\nu) = \gamma_{s,L}(\nu) + \gamma_{c,L}(\nu)$ with $\gamma_{s,c,L}(\nu)$ as in (5.11). Summing up the errors made in (i) through (iv) and in (c) gives the overall error

$$|e_L| \leq C \left( 1 + \frac{\ln L}{L} + \frac{\ln^2 L}{L^2} + \frac{\ln^2 L}{L^2} + \frac{\ln^2 L}{L^2} \right) $$

which proves (5.10).

\[ \square \]

The coefficient $\gamma_L(\nu)$ in (5.10) seems to depend still on $L$. We will see that this is actually not so.

**Theorem 5.3.** Let $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy the assumptions of Propositions 5.1, 5.2 and in addition (3.51) as well as (3.57) with some $\alpha > 0$. In the thermodynamic limit according to (5.1) with Fermi energy $\nu$, the Anderson integral (1.2) has the leading asymptotics

$$I_{N,L} = \gamma(\nu) \ln N + O(1), \quad N, L \to \infty,$$

(5.16)

with the constant

$$\gamma(\nu) := \frac{1}{4\pi^2 \nu} \mathrm{tr} \left[ (1 + \frac{1}{4\nu} \hat{\Phi}(\nu)^2)^{-1} \hat{\Phi}(\nu)^2 \right] \geq 0 \quad (5.17)$$

and the $2 \times 2$ matrix $\hat{\Phi}(\nu)$ as in (3.50).

**Proof.** (a) We evaluate the integral in (5.11). First of all note the operators $\Omega^\pm_\infty$. It will turn out that both $\Omega^+_\infty$ and $\Omega^-_\infty$ eventually yield the same $\gamma(\nu)$. Therefore, we restrict ourselves to $\Omega^+_\infty$ and drop the superscript for the sake of convenience. We recall the definition (5.6) of $d_{s,c,L}$ along with (3.17) and make a change of variables, $s = t/L$. Then,

$$\gamma_{s,c}(\nu) = \frac{1}{8\pi^2 \nu} \int_\mathbb{R} \frac{2}{(\cosh t \pm i \sinh t)^2} \left( \Omega^\infty(\nu, -t) \omega_{s,c}(\nu), F(\nu) \Omega^\infty(\nu, t) \omega_{s,c}(\nu) \right) dt.$$
where we dropped the index $L$ since there is no explicit $L$-dependence any longer. With the definition (5.12) of $F(\nu)$ we obtain

\[
(\Omega_{\infty}(\nu, -t)\omega_{s,c}(\nu), F(\nu)\Omega_{\infty}(\nu, t)\omega_{s,c}(\nu))
\]

\[
= (J\Omega_{\infty}(\nu, -t)\omega_{s,c}(\nu), \sqrt{|P_\nu|}J\Omega_{\infty}(\nu, t)\omega_{s,c}(\nu))
\]

\[
+ (J\Omega_{\infty}(\nu, -t)\omega_{s,c}(\nu), \sqrt{|P_\nu|}J\Omega_{\infty}(\nu, t)\omega_{s,c}(\nu))
\]

\[
= (J\Omega_{\infty}(\nu, -t)\omega_{s,c}(\nu), \omega_{s}(\nu), J\Omega_{\infty}(\nu, t)\omega_{s,c}(\nu))
\]

\[
+ (J\Omega_{\infty}(\nu, -t)\omega_{s,c}(\nu), \omega_{c}(\nu), J\Omega_{\infty}(\nu, t)\omega_{s,c}(\nu)).
\]

With the aid of the matrices $\hat{\Omega}_{\infty}(\nu, t)$ and $\hat{\tau}(t)$ (cf. (3.49) and (3.53)) we can write the integrand of $\gamma_s(\nu) + \gamma_c(\nu)$ as the trace of $2 \times 2$ matrices which leads to the integral

\[
I := i \int_{\mathbb{R}} \text{tr} \left[ \hat{\tau}'(t)\hat{\Omega}_{\infty}(\nu, -t)^*\hat{\Omega}_{\infty}(\nu, t) \right] dt
\]

\[
= i \int_{\mathbb{R}} \text{tr} \left[ \hat{\tau}'(t)\hat{Z}(\nu, t)\hat{\Phi}(\nu)\hat{Z}(\nu, t)\hat{\Phi}(\nu) \right] dt.
\]

Here we used both equalities in (3.54) to express $\hat{\Omega}_{\infty}(\nu, t)$ through $\hat{\Phi}(\nu)$ and $\hat{Z}(\nu, t)$. By the cyclicity of the trace,

\[
I = i \int_{\mathbb{R}} \text{tr} \left[ \hat{Z}(\nu, t)\hat{\Phi}(\nu)\hat{\tau}'(t)\hat{Z}(\nu, t)\hat{\Phi}(\nu) \right] dt
\]

\[
= 2\sqrt{\nu} i \int_{\mathbb{R}} \text{tr} \left[ \hat{\tau}'(t)\hat{\Phi}(\nu) \right] dt
\]

\[
= 2\sqrt{\nu} i \lim_{t \to \infty} \text{tr} \left[ (\hat{Z}(\nu, t) - \hat{Z}(\nu, -t))\hat{\Phi}(\nu) \right].
\]

We compute the difference

\[
\hat{Z}(\nu, t) - \hat{Z}(\nu, -t) = \frac{1}{2\sqrt{\nu}}\hat{Z}(\nu, t)\hat{\Phi}(\nu)(\hat{\tau}(t) - \hat{\tau}(-t))\hat{Z}(\nu, -t)
\]

\[
= \frac{2i}{2\sqrt{\nu}} \text{Im} \tau(t) \hat{Z}(\nu, t)\hat{\Phi}(\nu)\hat{Z}(\nu, -t).
\]

Thus, our integral becomes

\[
I = -2 \lim_{t \to \infty} \text{Im} \tau(t) \text{tr} \left[ \hat{Z}(\nu, t)\hat{\Phi}(\nu)\hat{Z}(\nu, -t)\hat{\Phi}(\nu) \right]
\]

\[
= -2 \lim_{t \to \infty} \text{Im} \tau(t) \text{tr} \left[ \hat{Z}(\nu, t)\hat{\Phi}(\nu)^2\hat{Z}(\nu, t)^* \right],
\]

where we used (3.54). The limit can be computed via (3.19) and (3.53). Then,

\[
I = 2 \text{tr} \left[ (1 - \frac{i}{2\sqrt{\nu}}\hat{\Phi}(\nu))^{-1}(1 + \frac{i}{2\sqrt{\nu}}\hat{\Phi}(\nu))^{-1}\hat{\Phi}(\nu)^2 \right]
\]

\[
= 2 \text{tr} \left[ (1 + \frac{1}{4\nu}\hat{\Phi}(\nu)^2)^{-1}\hat{\Phi}(\nu)^2 \right].
\]

Apart from the prefactor this is the coefficient $\gamma(\nu)$ in (5.17). From $\hat{\Phi}(\nu)^* = \hat{\Phi}(\nu)$ we infer that $\gamma(\nu)$ is the trace of the product of two non-negative matrices. Hence $\gamma(\nu) \geq 0$. 


(b) The integral formula \(5.2\) involves \(\Pi_M\) instead of \(\Pi_N\) and therefore differs from the actual Anderson integral \(I_{N,L}\) by
\[
| \text{tr} \, P_N(1 - \Pi_N) - \text{tr} \, P_N(\Pi_N - \Pi_M) | = | \text{tr} \, P_N(\Pi_N - \Pi_M) | \leq |N - M|.
\]
From Propositions 3.10 and 3.11 we deduce
\[
N - \frac{1}{2} - \frac{1}{\nu} \pi \|V_+\|_1 \leq M \leq N + \frac{1}{2} + \frac{1}{2\nu} \left( \frac{2C_\alpha}{\alpha\pi} (\|V_-\|_\infty + \nu)^{1/2} + \|V_-\|_\infty \right).
\]
Therefore, replacing \(\Pi_N\) by \(\Pi_M\) causes an error that is bounded by a constant.

Now, Propositions 5.1 and 5.2 along with the asymptotics for \(\kappa_N\) in Lemma A.1 prove (5.16). □

The coefficient \(\gamma(\nu)\) can be given a scattering theoretical interpretation. Recall that in this one-dimensional case the S-matrix is indeed a 2 \(\times\) 2-matrix,
\[
S(\nu) = \begin{pmatrix} t(\sqrt{\nu}) & r_2(\sqrt{\nu}) \\ r_1(\sqrt{\nu}) & t(\sqrt{\nu}) \end{pmatrix} \tag{5.18}
\]
with the transmission coefficient \(t(\sqrt{\nu})\) and the reflection coefficients \(r_{1,2}(\sqrt{\nu})\) (e.g. [3], in particular pp. 143–146 for the formulae needed herein). In what follows we drop the \(\nu\) in the argument of operators and vectors which makes the formulae look a little less ornate. To begin with, we decompose \(K\) into a Lippmann-Schwinger like operator and a rank two operator
\[
\sqrt{|V|} K \sqrt{|V|} = -\sqrt{|V|} K_+ \sqrt{|V|} + \frac{1}{2\sqrt{\nu}} (\omega_c, \cdot) \omega_s - \frac{1}{2\sqrt{\nu}} (\omega_s, \cdot) \omega_c
\]
by using the addition theorem for the sine. The operator \(K_+\) has the kernel
\[
K_+(x,y) := \frac{1}{\sqrt{\nu}} \chi(y-x) \sin(\sqrt{\nu}(x-y)), \; x, y \in \mathbb{R},
\]
with the Heaviside function \(\chi\) being zero for \(x < 0\) and one elsewhere. We define further
\[
\Phi_+ := (\mathbb{1} + \sqrt{|V|} K_+ \sqrt{|V|} J)^{-1}, \; \hat{\Phi}_+ := \begin{pmatrix} (\omega_s, J\Phi_+ \omega_s) \\ (\omega_c, J\Phi_+ \omega_c) \end{pmatrix}.
\]
We will see below that the entries of \(\hat{\Phi}_+\) can be computed explicitly with the aid of the transmission and reflection coefficients. We want to express \(\Phi\) through \(\Phi_+\), which amounts to solving the equation
\[
(1 + \sqrt{|V|} K_+ \sqrt{|V|} J) \psi - \frac{1}{2\sqrt{\nu}} (\omega_c, J\psi) \omega_s + \frac{1}{2\sqrt{\nu}} (\omega_s, J\psi) \omega_c = \omega
\]
for \(\psi\). Here, \(\omega\) equals \(\omega_s\) or \(\omega_c\). Since we are only interested in \(\hat{\Phi}\) we take scalar products and obtain after some elementary calculations
\[
(1 + \frac{1}{2\sqrt{\nu}} \hat{\Phi}_+ W) \hat{\Phi} = \hat{\Phi}_+ \text{ with } W := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; W^2 = -\mathbb{1}.
\]
We assume the first matrix to be invertible,
\[
\hat{\Phi} = (1 + \frac{1}{2\sqrt{\nu}} \hat{\Phi}_+ W)^{-1} \hat{\Phi}_+ = \hat{\Phi}_+ (1 + \frac{1}{2\sqrt{\nu}} W \hat{\Phi}_+)^{-1}.
\]
and obtain
\[ 4\pi^2\nu\gamma = \text{tr} \left[ (1 + \frac{1}{4\nu}\hat{\Phi}^2)^{-1}\hat{\Phi}^2 \right] = \text{tr} \left[ \hat{\Phi}_+ (1 + \frac{1}{2\sqrt{\nu}}\hat{\Phi}_+ W + \frac{1}{2\sqrt{\nu}}W\hat{\Phi}_+)^{-1}\hat{\Phi}_+ \right]. \]

Scattering theory in general uses exponential functions,
\[ e_{\pm}(x) := \sqrt{|V(x)|} e^{\pm i\sqrt{\nu}x}, \]
rather than the trigonometric functions as in \( \omega_{s,c} \). Thus, we introduce
\[ \tilde{\Phi}_+ := \left( \begin{array}{cc} e_+, J\hat{\Phi}_+ e_+ & e_+, J\hat{\Phi}_+ e_- \\ e_-, J\hat{\Phi}_+ e_+ & e_-, J\hat{\Phi}_+ e_- \end{array} \right) = 2i\sqrt{\nu} \left( \begin{array}{cc} 1 - \frac{1}{t} & -\frac{r_2}{t} \\ \frac{r_2}{t} & \frac{1}{t} - 1 \end{array} \right) \]
with \( r_{1,2} \) and \( t \) from (5.18) (see [3], pp. 145, 146). We transform our matrices
\[ \hat{\Phi}_+ = \frac{1}{2} U^* \hat{\Phi}_+ U, \quad W = iU^* IU \quad \text{with} \quad U := \frac{1}{\sqrt{2}} \left( \begin{array}{cc} -i & 1 \\ i & 1 \end{array} \right), \quad I := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \]
and the trace becomes
\[ 4\pi^2\nu\gamma = \frac{1}{4} \text{tr} \left[ (1 + \frac{i}{4\sqrt{\nu}}I\hat{\Phi}_+ + \frac{i}{4\sqrt{\nu}}\hat{\Phi}_+ I)^{-1}\tilde{\Phi}_+^2 \right]. \]

The inverse simplifies considerably since \( I\tilde{\Phi}_+ + \tilde{\Phi}_+ I \) is diagonal. Thereby,
\[ (1 + \frac{i}{4\sqrt{\nu}}I\hat{\Phi}_+ + \frac{i}{4\sqrt{\nu}}\hat{\Phi}_+ I)^{-1} = \left( \begin{array}{cc} t & 0 \\ 0 & \frac{1}{t} \end{array} \right). \]

Furthermore,
\[ \tilde{\Phi}_+^2 = -4\nu \left( (1 - \frac{1}{t})^2 - \left| \frac{r_2}{t} \right|^2 \quad \left( \frac{1}{t} - 1 \right)^2 - \left| \frac{r_2}{t} \right|^2 \right) \]
where the off-diagonal elements are not needed. Finally,
\[ 4\pi^2\nu\gamma = -2\nu \text{Re} \left\{ t [ (1 - \frac{1}{t})^2 - \left| \frac{r_2}{t} \right|^2 ] \right\} = 4\nu \text{Re}(1 - t) \]
where we used \( |t|^2 + |r_2|^2 = 1 \) which is due to the unitarity of the S-matrix. We summarize what we have found.

**Corollary 5.4.** The coefficient \( \gamma(\nu) \) in Theorem 5.3 can be written
\[ \gamma(\nu) = \frac{1}{\pi^2} (1 - \text{Re} t(\sqrt{\nu})) \]
where \( t(\sqrt{\nu}) \) is the transmission coefficient with wave number \( \sqrt{\nu} \).

In [6], Theorem 2.4, the lower bound
\[ \mathcal{I}_{N,L} \geq \gamma'(\nu) \ln N, \quad \gamma'(\nu) = \frac{1}{(2\pi)^2} \text{tr} \left[ (S(\nu) - 1)^*(S(\nu) - 1) \right] \]
has been derived where \( S(\nu) \) is the S-matrix at energy \( \nu \). By Corollary 5.4, \( \gamma'(\nu) = \gamma(\nu) \) in one-dimension.
5.3. Determinant

The asymptotics in Theorem 5.3 can be used to derive lower and upper bounds for the transition probability $D_N$ from (1.1). Standard reasoning yields

$$D_N = \det P_N \Pi_N P_N = \exp(\text{tr}(P_N \Pi_N P_N))$$  \hspace{1cm} (5.19)

where the determinant is to be taken with respect to ran $P_N$ otherwise it would be zero. Using Wouk’s integral formula [20] for the operator logarithm (see also [13]) we obtain

$$D_N = \exp \left[ -\int_0^1 \text{tr} \left[ P_N (1 - P_N \Pi_N P_N)(1 - t(1 - P_N \Pi_N P_N))^{-1} \right] dt \right],$$  \hspace{1cm} (5.20)

which immediately yields the inequalities

$$\exp \left[ -(1 - \|P_N(P_N - \Pi_N)P_N\|)^{-1} \text{tr} P_N(1 - \Pi_N) \right] \leq D_N \leq \exp \left[ -\text{tr} P_N(1 - \Pi_N) \right].$$  \hspace{1cm} (5.21)

The upper bound was already derived by Anderson [2] using Hadamard’s and Bessel’s inequality as well as an inequality for the logarithm. The lower bound, of course, holds only when $\|P_N(P_N - \Pi_N)P_N\| < 1$. Such operator-norm estimates are studied in the realm of so-called subspace perturbation problems. However, those results either depend on the size of the spectral gap (see [11]) or require perturbations that are off-diagonal with respect to $P_N$ (see [12]). Both conditions are not met here wherefore we present a new approach.

**Theorem 5.5.** Let $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy (3.42). Moreover, assume that the assumptions of Propositions 3.10 and 3.11 are satisfied such that

$$\frac{1}{2\nu_N} \left( \frac{2C_\alpha}{\alpha \pi} \left( \|V_+\|_\infty + E \right)^{\frac{1}{2}} + \|V_--\|_\infty \right) < \frac{1}{2} \text{ and } \frac{1}{\sqrt{\nu_N}} \frac{2\|V_+\|_1}{\pi} < \frac{1}{2} \hspace{1cm} (5.22)$$

with some $\alpha > 0$. Then,

$$\|P_N(P_N - \Pi_N)P_N\| \leq \frac{16C_\Omega}{\sqrt{\nu_N}}\|V\|_1.$$  \hspace{1cm} (5.23)

**Proof.** Because of (5.22) and Propositions 3.10 and 3.11 we may compute the matrix elements $a_{jk} := (\varphi_j, (P_N - \Pi_N)\varphi_k)$ via the integral formula (2.13)

$$a_{jk} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\sqrt{\nu_N} + is}{(z(s) - \lambda_j)(z(s) - \lambda_k)} (\varphi_j, \sqrt{|V|}J\Omega(z(s))\sqrt{|V|}\varphi_k) \, ds$$

where $z(s) \in \Gamma_N$ (see (3.3)). By (3.8) and (3.43) these can be estimated

$$|a_{jk}| \leq C_L \int_0^\infty \frac{1}{\sqrt{1 + s^2}} \left( (1 - \frac{i}{N+\frac{1}{2}})^2 + s^2 \right)^{\frac{1}{2}} \left( (1 - \frac{k}{N+\frac{1}{2}})^2 + s^2 \right)^{\frac{1}{2}} \, ds$$

where $z(s) \in \Gamma_N$ (see (3.3)). By (3.8) and (3.43) these can be estimated

$$|a_{jk}| \leq C_L \int_0^\infty \frac{1}{\sqrt{1 + s^2}} \left( (1 - \frac{i}{N+\frac{1}{2}})^2 + s^2 \right)^{\frac{1}{2}} \left( (1 - \frac{k}{N+\frac{1}{2}})^2 + s^2 \right)^{\frac{1}{2}} \, ds$$

where $z(s) \in \Gamma_N$ (see (3.3)). By (3.8) and (3.43) these can be estimated

$$|a_{jk}| \leq C_L \int_0^\infty \frac{1}{\sqrt{1 + s^2}} \left( (1 - \frac{i}{N+\frac{1}{2}})^2 + s^2 \right)^{\frac{1}{2}} \left( (1 - \frac{k}{N+\frac{1}{2}})^2 + s^2 \right)^{\frac{1}{2}} \, ds$$

By the variational principle

$$\|A\| \leq C_L \|B\|, \quad A := (a_{jk})_{j,k=1,\ldots,N}, \quad B := (b_{jk})_{j,k=1,\ldots,N}.$$
We introduce the integral operator $k_N : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$ with kernel

$$k_N(s, t) := \frac{1}{(1 + s^2)^{\frac{1}{2}}(1 + t^2)^{\frac{1}{2}}} \sum_{j=1}^{N} \frac{1}{((1 - \frac{j}{N + \frac{1}{2}})^2 + s^2)^{\frac{1}{2}}} \frac{1}{((1 - \frac{j}{N + \frac{1}{2}})^2 + t^2)^{\frac{1}{2}}}.$$

Simple algebra shows that each eigenvalue of $B$ is an eigenvalue of $k_N$ as well. It is therefore enough to bound the operator norm $\|k_N\|$. To this end, we drop the prefactor and estimate the sum by an integral

$$k_N(s, t) \leq 2 \int_0^{N + \frac{1}{2}} \frac{1}{(1 - \frac{u}{N + \frac{1}{2}})^2 + s^2} \frac{1}{((1 - \frac{u}{N + \frac{1}{2}})^2 + t^2)^{\frac{1}{2}}} \, du$$

$$= 2(N + \frac{1}{2}) \int_0^{1} \frac{1}{(u^2 + s^2)^{\frac{1}{2}}(u^2 + t^2)^{\frac{1}{2}}} \, du$$

$$=: 2(N + \frac{1}{2}) k(s, t).$$

Once again, it is enough to bound $\|k\|$. To this end, we estimate the quadratic form of $k$ by using Cauchy’s inequality along with Hilbert’s trick

$$|(f, kf)| = \left| \int_0^\infty \int_0^\infty \left( \frac{s}{t} \right)^{\frac{1}{4}} f(s) \sqrt{k(s, t)} \left( \frac{t}{s} \right)^{\frac{1}{4}} f(t) \sqrt{k(s, t)} \, dt \, ds \right|$$

$$\leq \int_0^\infty f(s)^2 \int_0^\infty \left( \frac{s}{t} \right)^{\frac{1}{4}} k(s, t) \, dt \, ds.$$

Note that $k(s, t) = k(t, s)$. We evaluate the $t$-integral

$$\int_0^\infty \left( \frac{s}{t} \right)^{\frac{1}{4}} k(s, t) \, dt = s^{\frac{1}{4}} \int_0^1 \frac{1}{(u^2 + s^2)^{\frac{1}{2}}} \int_0^\infty \frac{1}{(u^2 + t^2)^{\frac{1}{2}}} \frac{1}{t^\frac{1}{2}} \, du \, dt$$

$$= \int_0^1 \frac{1}{(1 + u^2)^{\frac{1}{2}}} \frac{1}{u^\frac{1}{2}} \int_0^\infty \frac{1}{(1 + t^2)^{\frac{1}{2}}} \frac{1}{t^\frac{1}{2}} \, dt$$

$$\leq 4 \left[ \int_0^\infty \frac{1}{(1 + t^4)^{\frac{1}{2}}} \, dt \right]^2.$$

The last integral could be expressed with the aid of the gamma function. However, since a bound is enough we estimate the integrand by means of $1 + t^2$ to obtain $\|k\| \leq 2\pi^2$. This concludes the proof. \qed

**Corollary 5.6.** Let the conditions of Theorems 5.3 and 5.5 be satisfied. Assume further that $\|V\|_1$ and $\nu$ are such that $\|P_N(P_N - \Pi_N)P_N\| < 1$ in (5.23). Then, the transition probability $D_{N, L}$ (cf. (1.1)) satisfies in the thermodynamic limit (cf. (5.1))

$$\tilde{C} N^{-\tilde{\gamma}(\nu)} \leq D_{N, L} \leq C N^{-\gamma(\nu)}$$

with appropriate constants $\tilde{C}, C > 0$, $\gamma(\nu)$ from Theorem 5.3 and $\tilde{\gamma}(\nu) > 0$.

**Proof.** The upper bound follows from (5.21) and Theorem 5.3. For the lower bound one needs in addition Theorem 5.5 which also gives $\tilde{\gamma} > 0$. \qed
Appendix A. Estimates

At various points we need estimates which are not directly related to our main subject. To begin with, we mention the following sums

\[
\sum_{j=1}^{N} \frac{1}{(N + \frac{1}{2} - j)^{\alpha}} = \sum_{j=0}^{N-1} \frac{2^{\alpha}}{(2j + 1)^{\alpha}} \leq 4^{\alpha} \sum_{j=1}^{N} \frac{1}{(j+1)^{\alpha}} \leq 4^{\alpha} \int_{0}^{N} \frac{1}{(t+1)^{\alpha}} \, dt.
\]

Evaluating the integral yields

\[
\sum_{j=1}^{N} \frac{1}{(N + \frac{1}{2} - j)^{\alpha}} \leq 4^{\alpha} \begin{cases} 
\frac{1}{\alpha - 1}, & \text{for } \alpha > 1, \\
\frac{1}{1-\alpha}(N+1)^{1-\alpha}, & \text{for } 0 \leq \alpha < 1, \\
\ln(N+1), & \text{for } \alpha = 1.
\end{cases}
\]

The constant \(\zeta_N\) in Proposition 3.5 requires more reasoning.

**Lemma A.1.** Let \(M \in \mathbb{R}, \ w \in \mathbb{C}\) such that \(\Re w + \frac{1}{2} - M > 0\). Then,

\[
\int_{0}^{\infty} e^{-wt} \frac{\cosh(Mt)}{\cosh \frac{t}{2}} \, dt \leq \frac{2}{w + \frac{1}{2} - |M|}.
\]

In particular, \(|\zeta_N| \leq 4\) in Proposition 3.5. Furthermore,

\[
\int_{0}^{\infty} e^{-wt} \frac{\sinh(Mt)}{\sinh \frac{t}{2}} \, dt = \frac{M}{(w + \frac{1}{2})^{2} - M^{2}} + \ln \frac{w + \frac{1}{2} + M}{w + \frac{1}{2} - M} + 8M(w + \frac{1}{2}) \int_{0}^{\infty} \frac{y}{[(w + \frac{1}{2} + M)^{2} + y^{2}][(w + \frac{1}{2} - M)^{2} + y^{2}]} \frac{1}{e^{2\pi y} - 1} \, dy.
\]

In particular, this yields the asymptotics for all \(N \in \mathbb{N}\)

\[
\zeta_N = \int_{0}^{\infty} e^{-(N + \frac{1}{2})t} \frac{\sinh((N + \frac{1}{2})t)}{\sinh \frac{t}{2}} \, dt = \ln(4N + 3) + c_N, \ 0 \leq c_N \leq 2.
\]

**Proof.** For the first inequality one estimates \(\cosh(t/2)\) by the exponential function. For the second inequality, we write for \(t > 0\)

\[
\frac{1}{\sinh \frac{t}{2}} = 2e^{-\frac{t}{2}} - 1 \quad \frac{1}{1 - e^{-t}} = 2e^{-\frac{t}{2}} - \sum_{j=0}^{\infty} e^{-jt},
\]

and integrate termwise which yields

\[
\int_{0}^{\infty} e^{-wt} \frac{\sinh(Mt)}{\sinh \frac{t}{2}} \, dt = 2M \sum_{j=0}^{\infty} \frac{1}{(w + \frac{1}{2} + j)^{2} - M^{2}}.
\]
We apply the Abel-Plana summation formula \[8, \text{Th. 4.9c}\]
\[
\frac{1}{2M} \int_0^\infty e^{-wt} \frac{\sinh(Mt)}{\sinh \frac{t}{2}} \, dt = \frac{1}{2} \left( \frac{1}{(w + \frac{1}{2})^2 - M^2} + \int_0^\infty \frac{1}{(w + \frac{1}{2} + x)^2 - M^2} \, dx \right) + i \int_0^\infty \left[ \frac{1}{(w + \frac{1}{2} + iy)^2 - M^2} - \frac{1}{(w + \frac{1}{2} - iy)^2 - M^2} \right] \frac{1}{e^{2\pi y} - 1} \, dy
\]
which implies the formula. Finally,
\[
0 \leq \frac{y}{e^{2\pi y} - 1} \leq \frac{1}{2\pi}
\]
yields the estimate. \(\square\)

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