Perturbative corrections to semileptonic $b$ decay distributions

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Abstract

We compute $O(\alpha_s)$ and $O(\alpha_s^2 \beta_0^{-1})$ (BLM) corrections to the five structure functions relevant for $b \to q\ell\bar{\nu}$ decays and apply the results to the moments of a few distributions of phenomenological importance. We present compact analytic one-loop formulae for the structure functions, with proper subtraction of the soft divergence.

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1 Introduction

The experimental study of inclusive semileptonic $B$ decays has reached an unprecedented level of precision. The total branching ratio is now known to better than 2% [1], while various moments of the lepton energy and invariant hadronic mass distributions have been measured by the Delphi, BaBar, Belle, CLEO, and CDF Collaborations [2] with impressive accuracy. The first field of application of this wealth of data is the extraction of the CKM matrix element $V_{cb}$, but the semileptonic distributions provide us with important information on the structure of the $B$ meson and on the quark masses that can be employed in many applications.

From the theoretical point of view, inclusive semileptonic $B$ decays can be described in the framework of the Operator Product Expansion (OPE) in inverse powers of the heavy $b$ mass. The dominant term in this expansion represents the decay of a free bottom quark and the non-perturbative corrections to the parton model result are suppressed by at least two powers of the energy release, $E_{rel} \approx m_b - m_c$ [3]. The $O(1/m_b^2)$ [3, 4] and $O(1/m_b^3)$ [5] corrections are known in terms of the $B$ meson expectation values of a limited set of dimension five and six operators. These non-perturbative parameters can be extracted from the moments of the measured spectra of semileptonic and radiative inclusive decays, although complementary information is also provided by a number of heavy quark relations and sum rules (see e.g. [6]). Recent fits to the moments of the distributions lead to consistent results and allow for a determination of $|V_{cb}|$ to better than 2% [1, 7, 8], with theoretical uncertainties starting to play a dominant role [9].

At the current level of experimental accuracy it becomes crucial to have perturbative corrections under control. In this respect, a lot of work has been done in the past 25 years. The one-loop corrections to a few differential distributions have been known for a long time [10, 11] and the correction to the total width is known in terms of polylogarithms since 1989 [12]. Analytic expressions for the $O(\alpha_s)$ corrections to the moments of the hadron spectra have been computed in [13, 14], while numerical results for the leptonic and hadronic moments with a lower cut on the charged lepton energy can be found in [15] and [16, 14]. However, the complete one-loop corrections to the five structure functions that enter the triple differential distribution for $b \rightarrow c\ell\bar{\nu}$ have become available only recently [17, 18]. In the limit of massless leptons, just three of these structure functions are actually relevant, but their knowledge is necessary in order to compute the moments of a generic distribution with arbitrary experimental cuts and it is therefore phenomenologically important. At the level of physical observables, but not of structure functions, semileptonic $b \rightarrow u$ decays can be obtained as the $m_c \rightarrow 0$ limit of our results. In this case the expressions simplify considerably, and the complete structure functions are known since a few years [19].

Beyond the one-loop approximation, several results are already available. In particular, the $O(\alpha_s^2\beta_0)$ (i.e. the so-called BLM [20]) corrections to the total width have been computed in [21, 22]. The equivalent corrections to the lepton energy spectrum can be found in [23], while the first results of a comprehensive analysis of hadronic moments have been reported in [18]. The BLM corrections are related to the running of the coupling constant in the loop contributions, and are practically most easily computed using one-loop diagrams calculated with a finite gluon mass [24]. Starting from this result, it is moreover possible to compute and even resum higher order BLM contributions, $O(\alpha_s^{n+1}\beta_0^n)$ [22, 25]. Complete two-loop results
are known for specific kinematic points only (at zero, maximal, and intermediate recoil [26]), from which one can interpolate over the entire $q^2$ range and estimate the correction to the total width, which turns out to be well approximated by the BLM result. Recently, even three-loop $O(\alpha_s^3)$ corrections have become available for a particular kinematic configuration, the extreme zero-recoil limit [27].

In this paper, we calculate analytically the complete one-loop corrections to the five structure functions relevant for the decay $b \to c\ell\bar{\nu}$, paying particular attention to the proper subtraction of the soft singularities, which is important for an accurate numerical evaluation. In this respect, our calculation differs from [17, 18]. In fact, our aim is to provide a reliable tool to compute the QCD corrections to the moments of $B \to X_c\ell\bar{\nu}$ in the most general experimental setting, including the case $\ell = \tau$. Whenever possible, we give our results in analytic form; this in particular applies to the five structure functions and to a few moments with an arbitrary cut on the lepton energy. Our calculation represents an important check of previous results and provides new analytic formulae that allow for an easy and accurate numerical implementation. We find good agreement with previous results whenever a comparison is possible.

Using the technique of refs. [24, 28], we also compute the $O(\alpha_s^2/\alpha_0^{-1})$ corrections to the triple differential distribution and to its moments. To this end, we have performed analytically the calculation of the structure functions with a finite gluon mass, and integrate numerically the results over the gluon mass, applying the technique to the differential rate. We present numerical results for the one-loop and the BLM corrections in a few cases of practical interest.

This paper is organized as follows: in Section 2 we set our notation, briefly describe the calculation of the next-to-leading order contributions, and discuss the subtraction of the soft singularity; we also report at the end of this section the complete expressions for the one-loop corrections to the structure functions. In Section 3 we describe the technique used to calculate the BLM corrections. Section 4 contains some reference numerical result and a comparison with the literature. Finally, we briefly summarize our results in Section 5. Details on the kinematics involved to next-to-leading order are given in Appendix A, while in Appendix B we illustrate the computation of the one-loop corrections. Appendix C gives analytic expressions for a few moments of the hadronic invariant mass and energy with an arbitrary cut on the lepton energy.

2 The differential decay width to next-to-leading order

The differential rate for the process

$$b(p) \to \ell(p_\ell) + \bar{\nu}(p_\nu) + X_c(r)$$

is given by

$$\frac{d\Gamma}{dq^2 du dE_\ell} = \frac{G_F^2 |V_{cb}|^2}{16\pi^3 m_b^2} L_{\mu\nu}(p_\ell, p_\nu) W^{\mu\nu}(p, q),$$

(2.1)

where $q = p_\ell + p_\nu$, $r = p - q$, $u = r^2 - m_c^2$. $E_\ell$ is the charged lepton energy in the rest frame of the decaying $b$ quark, and for massless neutrinos

$$L^{\mu\nu}(p_\ell, p_\nu) = p_\ell^\mu p_\nu^\nu - p_\ell p_\nu g^{\mu\nu} + p_\ell^\nu p_\nu^\mu - i\epsilon^{\mu\nu\alpha\beta} p_\alpha p_\beta.$$  

(2.2)
Details on the kinematics involved are given in Appendix A. The decay rate is often expressed in terms of \( q_0 = (m_b^2 + q^2 - u - m_c^2)/(2m_b) \), instead of \( u \), which we have chosen for later convenience. The hadronic tensor \( W^{\mu\nu} \) has the following, general decomposition, based on Lorentz covariance and CP symmetry:

\[
W^{\mu\nu}(p, q) = -W_1(q^2, \hat{u}) g^{\mu\nu} + W_2(q^2, \hat{u}) v^{\mu} v^{\nu} + iW_3(q^2, \hat{u}) \epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma + W_4(q^2, \hat{u}) \hat{q}^{\mu} \hat{q}^{\nu} + W_5(q^2, \hat{u}) (v^{\mu} \hat{q}^{\nu} + v^{\nu} \hat{q}^{\mu}),
\]

(2.3)

where \( v = p/m_b \), \( \hat{q} = q/m_b \), and \( \hat{u} = u/m_b \) (in the following, we denote with a hat dimensionless quantities, normalized to the \( b \) quark mass). In the limit of massless leptons the terms proportional to \( W_4 \) and \( W_5 \) do not contribute to the decay rate, since in that case \( q_\mu L^{\mu\nu} = q_\nu L^{\mu\nu} = 0 \). One finds

\[
\frac{L^{\mu\nu} W^{\mu\nu}}{m_b^2} = q^2 W_1(q^2, \hat{u}) - \left[ 2(v \hat{p}_\ell)^2 - 2v \hat{p}_\ell v \hat{q} + \frac{\hat{q}^2}{2} \right] W_2(q^2, \hat{u}) + q^2 \left( 2v \hat{p}_\ell - v \hat{q} \right) W_3(q^2, \hat{u}).
\]

(2.4)

The squared matrix element depends on the charged lepton energy \( E_\ell \) only through \( v p_\ell = E_\ell \).

The leading contribution to the hadronic tensor \( W^{\mu\nu} \) describes the process

\[
b(p) \to \ell(p_\ell) + \bar{\nu}(p_\bar{\nu}) + c(p')
\]

(2.5)

at the tree level. One finds

\[
W^{\mu\nu}_{(0)}(p, q) = \left[ -T_1 g^{\mu\nu} + T_2 v^{\mu} v^{\nu} + iT_3 \epsilon^{\mu\nu\rho\sigma} v_\rho \hat{q}_\sigma + T_4 \hat{q}^{\mu} \hat{q}^{\nu} + T_5 (v^{\mu} \hat{q}^{\nu} + v^{\nu} \hat{q}^{\mu}) \right] \pi \delta(\hat{u}),
\]

(2.6)

where

\[
T_1(q^2) = -\hat{\omega} \quad T_2(q^2) = 4 \quad T_3(q^2) = 2 \quad T_4(q^2) = 0 \quad T_5(q^2) = -2,
\]

(2.7)

and we have defined

\[
\hat{\omega} = \hat{q}^2 - 1 - \rho; \quad \rho = \frac{m_c^2}{m_b^2}.
\]

(2.8)

The hadronic tensor receives contributions of order \( \alpha_s \) from one-gluon emission at the tree level,

\[
b(p) \to \ell(p_\ell) + \bar{\nu}(p_\bar{\nu}) + c(p') + g(k),
\]

(2.9)

integrated over the charm-gluon phase space (in this case, \( r = p' + k \)), and from one-loop virtual corrections to the process in eq. (2.5):

\[
W^{\mu\nu}_{(1)}(p, q) = W^{\mu\nu}_{(1)R}(p, q) + W^{\mu\nu}_{(1)V}(p, q).
\]

(2.10)

The computation can be performed either by computing the relevant squared matrix elements and integrating over the corresponding phase space measures, or by taking the imaginary part of the \( bW^* \to bW^* \) forward amplitude. We calculated the structure functions in both ways; in Appendix B we give some details about the first method only. We specifically consider here only the weak decay structure functions and assume the corresponding kinematics, \( i.e. \ m_b^2 \geq q^2 \geq 0 \) and \( q_0 \geq \sqrt{q^2} \).
Let us begin with the contribution from real emission. We have performed the computation with the help of the algebraic manipulation programs Maxima [29] and Mathematica [30] for an arbitrary gluon mass $\lambda$. This is necessary for two reasons: first, the gluon mass regulates the logarithmic infrared divergence that arises in the phase space integration from the soft part of the gluon spectrum, and second, the full dependence of the one-loop corrections on the gluon mass allows us to compute the order-$\alpha_s^0\beta_0^{-1}$ corrections. The calculation is straightforward, and is already discussed in the literature in different contexts [11]; the only point that deserves a careful discussion is the isolation of the infrared-singular term, that is canceled by a similar contribution from virtual diagrams. Both for conceptual reasons and for the numerical implementation, it is important that the final result is manifestly independent of the gluon mass in the $m_g \to 0$ limit.

While tree-level and virtual contributions are characterized by a $\delta(u)$ that defines their kinematic structure (production of an on-shell charm quark without gluons), the contributions from real emission diagrams are defined for values of the variable $\hat{u}$ between

$$\hat{u}_- = 2\hat{\lambda}\sqrt{\rho} + \hat{\lambda}^2 \quad \text{and} \quad \hat{u}_+ = (1 - \sqrt{\rho^2})^2 - \rho. \quad (2.11)$$

In this range, the real emission contributions to the structure functions are regular as $\hat{\lambda} \to 0$, but practical applications require an integration over $\hat{u}$. In the physical limit $\hat{\lambda} \to 0$, when the integration is performed close to the kinematic boundary $\hat{u} = \hat{u}_- = 0$ the real emission contributions develop a $\log \hat{\lambda}$ singularity. As mentioned above, this singularity is canceled in physical quantities by a corresponding term originating from the virtual corrections. Since the singularity is localized at $\hat{u} = 0$, it is possible to extract it and write it in the form of a term proportional to $\delta(\hat{u}) \log \hat{\lambda}$, making the cancellation of the infrared divergences explicit already before integration.

This can be done as follows. The full result can be written as

$$W_{(1)R}^{\mu\nu}(p, q) = W_{(1)R, \text{reg}}^{\mu\nu}(p, q) + W_{(1)R, \text{sing}}^{\mu\nu}(p, q) \quad (2.12)$$

where we have isolated in $W_{(1)R, \text{sing}}^{\mu\nu}(p, q)$ all the terms that give rise to a soft singularity when integrated over $u$. There are obviously different ways to perform the splitting between the two above contributions, that differ in $W_{(1)R, \text{sing}}^{\mu\nu}$ by terms that do not give rise to singularities upon integration over $u$. Let us first concentrate on $W_{(1)R, \text{sing}}^{\mu\nu}(p, q)$, which we define as

$$W_{(1)R, \text{sing}}^{\mu\nu}(p, q) = C_F \alpha_s \left[ \hat{\omega} g^{\mu\nu} + 4 v^\mu v^\nu + 2i \epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta - 2(v^\mu q^\nu + v^\nu q^\mu) \right] D(q^2, \hat{u}, \hat{\lambda}^2), \quad (2.13)$$

where $C_F = 4/3$, and

$$D(q^2, \hat{u}, \hat{\lambda}^2) = \frac{1}{\hat{u}} \frac{\hat{\omega}}{\sqrt{\lambda_b}} \log \frac{\hat{t}_+ - \hat{t}_-}{\hat{t}_+ - \hat{t}_-} - \frac{\rho}{\hat{u}^2} \frac{\hat{t}_+ - \hat{t}_-}{\sqrt{\lambda_b}}, \quad (2.14)$$

$$\hat{t}_+ = \hat{\lambda}^2 + \frac{(\hat{u} + \hat{\lambda}^2)(\hat{\omega} - \hat{u}) \pm \sqrt{\lambda_b} \sqrt{\lambda_c}}{2(\hat{u} + \rho)} \quad (2.15)$$

$$\lambda_b = (\hat{\omega} - \hat{u})^2 - 4(\hat{u} + \rho) \quad (2.16)$$

$$\lambda_c = (\hat{u} - \hat{\lambda}^2)^2 - 4\hat{\lambda}^2 \rho. \quad (2.17)$$

\footnote{On general grounds, the structure functions are actually proportional to $\theta(\hat{u} - \hat{u}_-)$. In the following, we omit all $\theta$ functions from the explicit expressions, as it should not cause confusion.}
In the limit $\lambda \to 0$

$$D(q^2, \hat{u}, 0) = \frac{D_0(q^2, \hat{u})}{\hat{u}}$$ (2.18)

$$D_0(q^2, \hat{u}) = \frac{\hat{\omega}}{\sqrt{\lambda_0^{\rho}}} \log \frac{\hat{\omega} - \hat{u} + \sqrt{\lambda_0^{\rho}}}{\hat{\omega} - \hat{u} - \sqrt{\lambda_0^{\rho}}} - \frac{\hat{u} + 2\rho}{\hat{u} + \rho},$$ (2.19)

so that the integral over $\hat{u}$ is logarithmically divergent. The two above equations are sufficient to isolate the divergent term, but not to evaluate the finite contribution. The problem is that $D(q^2, \hat{u}, \lambda)$ is not analytic when $\hat{u} = \hat{u}_-$ and $\lambda = 0$: indeed,

$$D(q^2, \hat{u}, \hat{\lambda}^2) = D_s(q^2, \hat{u}, \hat{\lambda}^2) + O(\hat{u} - \hat{u}_-, \hat{\lambda})$$ (2.20)

$$D_s(q^2, \hat{u}, \hat{\lambda}^2) = \frac{1}{\hat{u}} \frac{\hat{\omega}}{\sqrt{\lambda_0^{\rho}}} \log \frac{1 + \frac{\sqrt{\lambda_0^{\rho}}}{\hat{\omega}} \frac{\sqrt{u_0^2 - u_-^2}}{u}}{1 - \frac{\sqrt{\lambda_0^{\rho}}}{\hat{\omega}} \frac{\sqrt{u_0^2 - u_-^2}}{u}} - \frac{\sqrt{u_0^2 - u_-^2}}{\hat{u}^2} - \frac{\sqrt{u_0^2 - u_-^2}}{\hat{u}^2},$$ (2.21)

where $\lambda_0^\rho = \hat{\omega}^2 - 4\rho$. The integration over $\hat{u}$ becomes non-trivial when it starts from the endpoint $\hat{u} = \hat{u}_-$. We now add and subtract to $D(q^2, \hat{u}, \hat{\lambda}^2)$ a term proportional to $\delta(\hat{u})$:

$$D(q^2, \hat{u}, \hat{\lambda}^2) = [D(q^2, \hat{u}, \hat{\lambda}^2) - A(q^2, \hat{\lambda}) \delta(\hat{u})] + A(q^2, \hat{\lambda}) \delta(\hat{u})$$ (2.22)

where we have defined

$$A(q^2, \hat{\lambda}) = \int_{\hat{u}_-}^{\hat{u}_+} d\hat{u} D_s(q^2, \hat{u}, \hat{\lambda}^2).$$ (2.23)

The $\hat{u}$ integration of the term in square brackets in eq. (2.22) is regular for $\hat{\lambda} \to 0$ over any range, while the soft singularity is entirely contained in $A(q^2, \hat{\lambda})$. For instance, for a generic test function $F(\hat{u})$ regular in $\hat{u} = 0$, we have

$$\int_{\hat{u}_-}^{\hat{u}_+} d\hat{u} F(\hat{u}) D(q^2, \hat{u}, \hat{\lambda}^2) = \int_{\hat{u}_-}^{\hat{u}_+} d\hat{u} \left[ F(\hat{u}) D(q^2, \hat{u}, \hat{\lambda}^2) - F(0) D_s(q^2, \hat{u}, \hat{\lambda}^2) \right] + F(0) A(q^2, \hat{\lambda}),$$ (2.24)

The integration range in eq. (2.24) is irrelevant and may be restricted by the shape of the test function. We have chosen it to coincide with the allowed kinematic range $\hat{u}_- \leq \hat{u} \leq \hat{u}_+$ so that for $F(\hat{u}) = 1$ the integral in eq. (2.24) is the one which is relevant for the computation of the total rate.

One can make notation more compact by defining

$$D(q^2, \hat{u}, \hat{\lambda}^2) = \left[ D(q^2, \hat{u}, \hat{\lambda}^2) \right]_+ + A(q^2, \hat{\lambda}) \delta(\hat{u}),$$ (2.25)

where the plus prescription is\(^2\)

$$\int_{\hat{u}_-}^{\hat{u}_+} d\hat{u} \left[ D(q^2, \hat{u}, \hat{\lambda}^2) \right]_+ F(\hat{u}) = \int_{\hat{u}_-}^{\hat{u}_+} d\hat{u} \left[ D(q^2, \hat{u}, \hat{\lambda}^2) F(\hat{u}) - D_s(q^2, \hat{u}, \hat{\lambda}^2) F(0) \right].$$ (2.26)

\(^2\)This is a generalization of the standard 'plus' distribution.
In the limit $\hat{\lambda} \to 0$, we have
\[ \left[ D(\hat{q}^2, \hat{u}, \hat{\lambda}^2) \right]_+ = D_0(\hat{q}^2, \hat{u}) \left( \frac{1}{\hat{u}} \right)_+ , \] (2.27)

where the distribution $(1/\hat{u})_+$ is defined by
\[ \int_0^{\hat{u}_+} d\hat{u} \left( \frac{1}{\hat{u}} \right)_+ f(\hat{u}) = \int_0^{\hat{u}_+} \frac{d\hat{u}}{\hat{u}} [f(\hat{u}) - f(0)]. \] (2.28)

As anticipated, the last term in eq. (2.25) has the same kinematic structure as the tree-level and virtual contributions. The integral $A(\hat{q}^2, \hat{\lambda})$ can be computed analytically for generic values of $\hat{\lambda}$; in the small-$\hat{\lambda}$ limit we have
\[ A(\hat{q}^2, \hat{\lambda}) = \left( 2 - \frac{1}{a} \log \frac{1+a}{1-a} \right) \log \frac{\hat{\lambda} \sqrt{\rho}}{\hat{u}_+} + \frac{1}{2a} \left[ \text{Li}_2 \left( \frac{2a}{\hat{\lambda}} \right) - \text{Li}_2 \left( \frac{2a}{\hat{\lambda}+1} \right) \right] + \frac{1}{2a} \log \frac{1+a}{1-a} + 1 + O(\hat{\lambda}^2) , \] (2.29)

where $a = \sqrt{\lambda_b^0/\hat{\omega}}$. Finally we have
\[ W_{(1)R,sing}(p,q) = C_F \alpha_s \left[ \hat{\omega} g^{\mu\nu} + 4 v^\mu v^\nu + 2i \epsilon^{\mu\nu\alpha\beta} v^\beta q_\alpha q_\beta - 2(v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu) \right] \left[ D(\hat{q}^2, \hat{u}, \hat{\lambda}^2) \right]_+ + \frac{C_F \alpha_s}{\pi} W_{(0)}(p,q) A(\hat{q}^2, \hat{\lambda}) . \] (2.30)

We now consider the regular part of the real emission contribution. It can be written in terms of structure functions as in eq. (2.3):
\[ W_{(1)R,reg}(p,q) = C_F \alpha_s \left\{ - R_1(\hat{q}^2, \hat{u}) g^{\mu\nu} + R_2(\hat{q}^2, \hat{u}) v^\mu v^\nu + i R_3(\hat{q}^2, \hat{u}) \epsilon^{\mu\nu\alpha\beta} v^\beta q_\alpha q_\beta \right. \right. \]
\[ \left. \left. + R_4(\hat{q}^2, \hat{u}) \hat{q}^\mu \hat{q}^\nu + R_5(\hat{q}^2, \hat{u}) (v^\mu \hat{q}^\nu + v^\nu \hat{q}^\mu) \right\} . \] (2.31)

The analytic expressions of the structure functions $R_i$ for $\hat{\lambda} = 0$ are
\[ R_1(\hat{q}^2, \hat{u}) = \frac{\hat{u}(3\hat{u} + 2\rho)}{4(\hat{u} + \rho)^2} + \frac{\hat{u}^2(\omega - \hat{u})}{2(\hat{u} + \rho) \lambda_b} + \frac{6\hat{u}}{\lambda_b} \]
\[ + \frac{4\hat{u}^2 - (6\hat{\omega} + \lambda_b)\hat{u} + 2\lambda_b\hat{\omega}}{2\lambda_b \sqrt{\lambda_b}} \log \tau \] (2.32)
\[ R_2(\hat{q}^2, \hat{u}) = \frac{\hat{\omega} \hat{u}(8 - 9\hat{u}) - 8\hat{\omega}^3 + 15\hat{\omega}^2 \hat{u} + 2\hat{u}(\hat{u}^2 - 8\hat{u} + 8)}{\lambda_b^2 (\hat{u} + \rho)^2} + \frac{64(1 + \hat{\omega} - \hat{u}) \rho}{\lambda_b^2} \]
\[ + \frac{\hat{\omega} \hat{u}(\hat{\omega} - \hat{u})^3}{\lambda_b^2 (\hat{u} + \rho)^2} - \frac{2(2\hat{\omega}^4 + \hat{\omega}^2 \hat{u}(7 + 4\hat{u}) - 5\hat{\omega}^4 \hat{u} - 9\hat{\omega} \hat{u}^2 + 2\hat{u}^3 - \hat{\omega} u^3)}{\lambda_b^2 (\hat{u} + \rho)} \]
\[ + \frac{2}{\lambda_b^2 \sqrt{\lambda_b}} \left[ 2\lambda_b^2 + \lambda_b(26\hat{u} - 2\hat{u}^2 + \hat{\omega}(8 + 3\hat{u}) + 16\rho) \right. \] (2.33)
\[ \left. - 6\hat{u} \left( 3\hat{u}^2 + 2\hat{u}(\rho - 5) - 12\rho - \hat{\omega}(3 + 4\hat{u} + 3\rho) \right) \right] \log \tau \]
\[ - 6\hat{u} \left( 3\hat{u}^2 + 2\hat{u}(\rho - 5) - 12\rho - \hat{\omega}(3 + 4\hat{u} + 3\rho) \right) \] (2.34)
\[ R_3(q^2, \hat{u}) = \frac{(\hat{\omega} - 2\hat{u} - \hat{\omega}^2 - 4(\hat{\omega} + \rho) + \lambda_b \log \tau - \frac{\rho}{2(\hat{u} + \rho)^2}}{\lambda_b \sqrt{\lambda_b}} \]

\[ R_4(q^2, \hat{u}) = \frac{3\hat{\omega} + \rho}{2(\hat{\omega} + \rho)(\hat{u} + \rho)} - \frac{8\hat{\omega} + 5\hat{\omega}^2 - 3\hat{\omega}\hat{u} + 4\rho + 4\hat{\omega}\rho - 2\hat{u}\rho}{\lambda_b (\hat{\omega} + \rho)} \]

\[ R_5(q^2, \hat{u}) = \frac{8(2\hat{\omega}^2 - 2\hat{u} - 5\hat{\omega}\hat{u} + 3\hat{u}^2)}{\lambda_b^2} + \frac{64\rho}{\lambda_b^2} + \frac{\hat{u}^2(\hat{\omega} - \hat{u})^3}{\lambda_b^2 (\hat{u} + \rho)} \left[ \frac{3\hat{u}(2\hat{u} - 3\hat{\omega})}{\lambda_b} \right] \]

\[ \tau = \frac{\hat{\omega} - \hat{u} + \sqrt{\lambda_b}}{\hat{\omega} - \hat{u} - \sqrt{\lambda_b}} \]

We now turn to the virtual contribution to the differential width. We calculate the one-loop corrections to the decay of an on-shell $b$ quark into an on-shell $c$ quark and a virtual $W$ boson, parameterizing the result in terms of pole masses. Some detail of the calculation is given in Appendix B. The result has the form

\[ W_{(1)\nu}(p, q) = -C_F \alpha_s \left\{ -V_1(\hat{q}, \hat{\lambda}) g^{\mu\nu} + V_2(\hat{q}, \hat{\lambda}) \gamma^\mu v^\nu + iV_3(\hat{q}, \hat{\lambda}) e^{\mu\nu} v^\alpha \gamma_\alpha \right\} \delta(\hat{u}) - \frac{C_F \alpha_s}{\pi} W_{(0)\nu}(p, q) V_0(q^2, \hat{\lambda}), \]

where

\[ V_0(q^2, \hat{\lambda}) = \frac{1}{2} (1 + \hat{I}_0 - 2\hat{J} - 2\hat{I}_1) \]

\[ V_1(q^2, \hat{\lambda}) = -\hat{\omega}\hat{K} - (\hat{\omega}^2 - \hat{\omega} - 2\rho)\hat{I}_x - (\hat{\omega}^2 - \hat{\omega} \rho - 2\rho)\hat{I}_y \]

\[ V_2(q^2, \hat{\lambda}) = -4 \left[ (\hat{\omega} + \rho + 1)\hat{I}_{xy} - \hat{\omega}(\hat{I}_x + \hat{I}_y) \right] \]

\[ V_3(q^2, \hat{\lambda}) = 2K + 2(\hat{\omega} - 1)\hat{I}_x + 2(\hat{\omega} - \rho)\hat{I}_y \]

\[ V_4(q^2, \hat{\lambda}) = 4\hat{I}_x - \hat{I}_{xy} \]

\[ V_5(q^2, \hat{\lambda}) = 2 \left[ (\hat{\omega} + 2)\hat{I}_{xy} - (\hat{\omega} + 1)\hat{I}_x - (\hat{\omega} - \rho)\hat{I}_y \right], \]

and the integrals $\hat{I}_0$, $\hat{J}$, etc. are defined in Appendix B, eqs. (B.8) and (B.13). In the limit $\hat{\lambda} \to 0$ we have

\[ \hat{J} = -1 - \log \frac{\lambda_b^2}{\sqrt{\rho}}; \quad \hat{K} = \frac{1}{2}; \quad \hat{I}_0 = \frac{\sqrt{\lambda_b^0}}{2q^2} \log \frac{z_+}{z_-} - \frac{1 - \rho}{2q^2} \log \rho \]

8
\[ I_1 = \frac{\hat{\omega}}{\sqrt{\lambda_0}} \left[ \frac{1}{2} \log \frac{z_+ - \lambda_0}{z_-} + \log \frac{z_+}{\sqrt{\rho}} \right] \log \frac{z_+ - 1 - z_+}{z_+ - 1 - z_-} \] 
\[ + \log \frac{z_-}{\sqrt{\rho}} \left( \frac{1}{2} \frac{z_+ - 1 - z_+}{z_- - 1 - z_-} \right) - \text{Li}_2 \left( \frac{1}{2} - \frac{z_-}{z_+ - 1 - z_-} \right) \] 
\begin{equation}
(2.42)
\end{equation}

\[ I_x = \frac{1}{2q^2} \left[ \log \frac{1}{\rho} - \frac{\hat{\omega} + 2\rho}{\sqrt{\lambda_0}} \log \frac{z_+}{z_-} \right] ; \quad I_y = \frac{1}{2q^2} \left[ \log \rho - \frac{\hat{\omega} + 2\rho}{\sqrt{\lambda_0}} \log \frac{z_+}{z_-} \right] \] 
\begin{equation}
(2.43)
\end{equation}

\[ I_{xy} = \frac{(1 - \rho)^2 - q^2(1 + \rho)}{4q^4 \sqrt{\lambda_0}} \log \frac{z_+}{z_-} + \frac{\rho - 1}{4q^2} \log \rho - \frac{1}{2q^2}, \] 
\begin{equation}
(2.44)
\end{equation}

where
\[ z_\pm = -\frac{\hat{\omega} + \sqrt{\lambda_0}}{2} = -\frac{\hat{\omega}}{2} (1 \pm a). \] 
\begin{equation}
(2.45)
\end{equation}

The soft singularity arising from loop integration resides in the function \( V_0(\hat{q}^2, \lambda) \),
\[ V_0(\hat{q}^2, \lambda) = \left( 2 - \frac{1}{a} \log \frac{1 + a}{1 - a} \right) \log \lambda + \ldots, \] 
\begin{equation}
(2.46)
\end{equation}

where the dots stand for terms regular in the limit \( \lambda \to 0 \). It cancels exactly the singularity we have found in eq. (2.29).

We conclude this section by summarizing our results for the structure functions to order \( \alpha_s \) for massless gluon. We have found:
\[ W_i(q^2, \hat{u}) = \pi \left\{ T_i(q^2) \delta(\hat{u}) + \frac{\alpha_sC_F}{\pi} \left[ T_i(q^2) S(q^2) - V_i(q^2) \right] \delta(\hat{u}) \right. \] 
\[ + \left. \frac{\alpha_sC_F}{\pi} \left[ T_i(q^2) D_0(q^2, \hat{u}) \left( \frac{1}{\hat{u}} \right)_+ + R_i(q^2, \hat{u}) \right] \right\}, \] 
\begin{equation}
(2.47)
\end{equation}

where we have defined
\[ S(q^2) = \lim_{\lambda \to 0} \left[ A(q^2, \lambda) - V_0(q^2, \lambda) \right]. \] 
\begin{equation}
(2.48)
\end{equation}

The \( T_i \) are given in eq. (2.7), \( A \) in eq. (2.29), \( V_i \) in eq. (2.40), \( R_i \) in eqs. (2.32-2.38), and \( D_0 \) in eq. (2.19).

Ref. [17] also reports the results of the calculation of the structure functions \( W_i \). However, a direct comparison of our results for \( W_i \) with ref. [17] is possible only before subtraction of the soft singularity. We agree with all the real emission contributions of ref. [17], but the virtual contributions to \( W_3, W_4, W_5 \) (in the notation of ref. [17]) are different from our results. The discrepancy, however, is such that our \( V_1, V_2, V_3 \) are reproduced correctly. Therefore, a difference should be observed when computing the decay rate into massive leptons only.

### 3 Generalized BLM – \( O(\alpha_s^n \beta_0^{n-1}) \) corrections

The full perturbative expansion for the total rate can be written as
\[ \Gamma = \Gamma_0 \left[ 1 + a_1(0) \frac{\alpha_s}{\pi} + \left( \frac{\alpha_s}{\pi} \right)^2 \sum_{n=0}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n a_{n+2} \right], \] 
\begin{equation}
(3.1)
\end{equation}
that the real emission term vanishes when 
\[ \alpha \]
 can be computed in terms of the first perturbative coefficient \( a \)

The dominant term \( u \)

\[ n \]

in the number of light fermions \( n_f \), or equivalently in the first coefficient of the \( \beta \)

function for \( \alpha, \beta = 11 - 2n_f/3 \):

\[ a_{n+2} = a_{n+2}^{\text{BLM}} \beta_0^{n+1} + O(\beta_0^n). \]

The dominant term \( a_{n+2}^{\text{BLM}} \) in the large-\( \beta_0 \) limit, usually called the (extended) BLM approximation [20], can be computed in terms of the first perturbative coefficient \( a_1(\lambda^2) \) [24]; in the \( \overline{\text{MS}} \)

scheme for the coupling constant the result reads [9, 24]

\[ a_{n+2}^{\text{BLM}} = - \frac{1}{4n+1} \sum_{k=0}^{n/2} (-1)^k \left( \begin{array}{c} n+1 \\ 2k+1 \end{array} \right) A_{nk}. \] (3.4)

\[ A_{nk} = \int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \left[ a_1(\lambda^2) - a_1(0) \frac{\mu^2}{\mu^2 + \lambda^2} \right]; \quad \mu^2 = m_b^2 e^{5/3}. \] (3.5)

We will now obtain an expression similar to eq. (3.4) for the BLM correction to the differential rate. As we have seen in the previous section, the order-\( \alpha_s \) correction to the total rate is obtained by summing the virtual-soft contribution proportional to \( \delta(u) \) and the real contribution:

\[ a_1(\lambda^2) = a_1^{\text{VS}}(\lambda^2) + a_1^{\text{R}}(\lambda^2). \] (3.6)

After integration over the lepton energy \( E_\ell \), with or without kinematic cuts, we have

\[ a_1^{\text{VS}}(\lambda^2) = \int_0^{\delta^2} dq^2 \frac{da_1^{\text{VS}}(q^2, \lambda^2)}{dq^2}. \] (3.7)

\[ a_1^{\text{R}}(\lambda^2) = \theta(\delta - \lambda) \int_0^{(\delta - \lambda)^2} dq^2 \int_{u_-}^{u_+} du \frac{da_1^{\text{R}}(q^2, u, \lambda^2)}{dq^2 du}. \] (3.8)

where we have defined \( \delta = m_b - m_c \). The integration bounds on \( u = r^2 - m_c^2 \) are

\[ u_- = (m_c + \lambda)^2 - m_c^2; \quad u_+ = (m_b - \sqrt{q^2})^2 - m_c^2. \] (3.9)

The \( u \) integration in the virtual-soft term is trivial, since it is proportional to \( \delta(u) \). Also note that the real emission term vanishes when \( \lambda > \delta \); this is implicitly taken into account in the definition of \( a_1^{\text{VS}} \), which contains the soft contribution of the real emission term. We have

\[ A_{nk} = \int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \left[ a_1^{\text{VS}}(\lambda^2) + a_1^{\text{R}}(\lambda^2) - \frac{\mu^2}{\lambda^2 + \mu^2} \left( a_1^{\text{VS}}(0) + a_1^{\text{R}}(0) \right) \right] \]

\[ = \int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \int_0^{\delta^2} dq^2 \left[ \frac{da_1^{\text{VS}}(q^2, \lambda^2)}{dq^2} - \frac{da_1^{\text{VS}}(q^2, 0)}{dq^2} \frac{\mu^2}{\lambda^2 + \mu^2} \right] \]

\[ + \int_0^{\delta^2} \frac{d\lambda^2}{\lambda^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \int_0^{(\delta - \lambda)^2} dq^2 \int_{u_-}^{u_+} du \frac{da_1^{\text{R}}(q^2, u, \lambda^2)}{dq^2 du} \]

\[ - \int_0^{\infty} \frac{d\lambda^2}{\lambda^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \int_0^{\delta^2} dq^2 \int_0^{u_+} du \frac{da_1^{\text{R}}(q^2, u, 0)}{dq^2 du} \frac{\mu^2}{\lambda^2 + \mu^2}. \] (3.10)
Now we observe that
\[ \int_0^{\delta^2} d\lambda^2 \int_0^{(\delta-\lambda)^2} dq^2 \int_{u_-}^{u_+} du = \int_0^{\delta^2} dq^2 \int_0^{u_+} du \int_0^{\Lambda^2} d\lambda^2, \]  
(3.11)
with
\[ \Lambda^2 = (\sqrt{r^2} - m_c)^2. \]  
(3.12)

Thus,
\[ A_{nk} = \int_0^\infty \frac{d\lambda^2}{\lambda^2} \int_0^{\delta^2} dq^2 \left[ \frac{da_1^{\text{NS}}(q^2, \lambda^2)}{dq^2} - \frac{da_1^{\text{NS}}(q^2, 0)}{dq^2} \right] \frac{\mu^2}{\lambda^2 + \mu^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \\
+ \int_0^{\delta^2} dq^2 \int_0^{u_+} du \int_0^{\Lambda^2} d\lambda^2 \left[ \frac{da_1^{\text{NS}}(q^2, u, \lambda^2)}{dq^2 du} - \frac{da_1^{\text{NS}}(q^2, u, 0)}{dq^2 du} \right] \frac{\mu^2}{\lambda^2 + \mu^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \\
- \int_0^{\delta^2} dq^2 \int_0^{u_+} du \frac{da_1^{\text{NS}}(q^2, u, 0)}{dq^2 du} \int_0^{\Lambda^2} d\lambda^2 \frac{\mu^2}{\lambda^2 + \mu^2} \log^{n-2k} \frac{\mu^2}{\lambda^2}. \]  
(3.13)

The \( \lambda^2 \) integral in the last term can be expressed in terms of polylogarithms; for example,
\[ \int_0^\infty \frac{d\lambda^2}{\lambda^2} \frac{\mu^2}{\lambda^2 + \mu^2} = \log \frac{\lambda^2 + \mu^2}{\lambda^2} \]
\[ \int_0^\Lambda d\lambda^2 \frac{\mu^2}{\lambda^2 + \mu^2} \log \frac{\mu^2}{\lambda^2} = \text{Li}_2 \left( -\frac{\mu^2}{\Lambda^2} \right) - \log \frac{\lambda^2 + \mu^2}{\lambda^2}. \]  
(3.14)

From eq. (3.13), we can read off an expression for the BLM corrections to the differential rate:
\[ \frac{dA_{nk}}{dq^2 du} = \int_0^\infty \frac{d\lambda^2}{\lambda^2} \delta(u) \left[ \frac{da_1^{\text{NS}}(q^2, \lambda^2)}{dq^2} - \frac{da_1^{\text{NS}}(q^2, 0)}{dq^2} \right] \frac{\mu^2}{\lambda^2 + \mu^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \\
+ \int_0^{\Lambda^2} \frac{d\lambda^2}{\lambda^2} \left[ \frac{da_1^{\text{NS}}(q^2, u, \lambda^2)}{dq^2 du} - \frac{da_1^{\text{NS}}(q^2, u, 0)}{dq^2 du} \right] \frac{\mu^2}{\lambda^2 + \mu^2} \log^{n-2k} \frac{\mu^2}{\lambda^2} \\
- \frac{da_1^{\text{NS}}(q^2, u, 0)}{dq^2 du} \int_0^{\Lambda^2} \frac{d\lambda^2}{\lambda^2 + \mu^2} \log^{n-2k} \frac{\mu^2}{\lambda^2}. \]  
(3.15)

Eq. (3.15) can be used for a numerical computation of the BLM coefficients, since each integral is separately convergent. Any kind of kinematic cuts can be implemented using appropriate step-functions in the integrands.

All the one-loop results of Section 2, as well as the above expressions, have been derived in the on-shell scheme for the \( b \) and \( c \) quark masses. The use of pole masses is known to induce large higher order perturbative corrections and is not adequate to precision studies. In most practical \( B \) physics applications (see e.g. [31]) it is therefore appropriate to employ well-defined short distance mass parameters, such as the so-called kinetic masses, namely heavy quark parameters which are renormalized \( a \ la \) Wilson and depend explicitly on a ‘hard’ normalization scale \( \mu \) which is chosen close to 1 GeV [25, 32]. The change of scheme required to express \( O(\alpha_s^{n+1} \beta_0^n) \) corrections in terms of short-distance parameters is elementary, once the relation between the latter and their on-shell counterpart is known at the very same order.
Table 1: Leptonic moments without $E_\ell$ cut.

| $n$ | $L_n^{(0)}$ | $L_n^{(1)}$ | $L_n^{(2)}$ | $L_n^{(3)}$ |
|-----|-------------|-------------|-------------|-------------|
| 0   | 1.000000    | -1.777588   | -1.917079   | -2.827995   |
| 1   | 0.307202    | -0.551243   | -0.617666   | -0.938365   |
| 2   | 0.103000    | -0.187687   | -0.217405   | -0.338429   |
| 3   | 0.036524    | -0.067828   | -0.080868   | -0.128695   |

4 Numerical results

The primary field of application of the results described in the previous sections is the calculation of the perturbative contributions to the first few moments of various differential decay distributions. The moments most frequently employed are those of the charged lepton energy spectrum, and of the invariant mass of the hadronic system in the final state [2]. As experimental cuts are usually applied, the moments of phenomenological interest are actually truncated. A typical case is a lower cut on the charged lepton energy. At the one-loop level, a fully analytic approach is possible even with a cut on the lepton energy, but the integration over the phase space is a tedious task. We have performed the phase space integration analytically for several of the partonic moments that represent the building blocks of the first three hadronic mass moments (see [13, 31]), in the case of an arbitrary cut on the lepton energy. A few representative examples of the results are reproduced in Appendix C. However, in general, a strategy for an accurate and fast numerical evaluation is necessary, especially when computing higher order corrections in the BLM approximation.

We have implemented these calculations in a FORTRAN code, which is available from the authors upon request. The subtraction of soft singularities described in Section 2 allows us to compute a generic moment and to apply arbitrary kinematic cuts.

In this section we provide a few tables of reference numbers obtained with our code for massless leptons; they have been checked in a number of ways, both analytical and numerical. All results are in the pole mass scheme.

In particular, we focus here on the leptonic moments,

\[ L_n = \frac{1}{\Gamma_0} \int dq^2 du dE_\ell \, E_\ell^n \frac{d^3\Gamma}{dq^2 du dE_\ell} \]  

\[ = L_n^{(0)} + \frac{\alpha_s}{\pi} L_n^{(1)} + \frac{\alpha_s^2}{\pi^2} L_n^{(2)} + \frac{\alpha_s^3}{\pi^3} L_n^{(3)} + \ldots \]  

and on the hadronic moments of the form

\[ H_{ij} = \frac{1}{\Gamma_0} \int dq^2 du dE_\ell \, \hat{E}_0^i \hat{E}_0^j \frac{d^3\Gamma}{dq^2 du dE_\ell} \]  

\[ = H_{ij}^{(0)} + \frac{\alpha_s}{\pi} H_{ij}^{(1)} + \frac{\alpha_s^2}{\pi^2} H_{ij}^{(2)} + \frac{\alpha_s^3}{\pi^3} H_{ij}^{(3)} + \ldots \]  

where $\Gamma_0$ is the total tree-level width of eq. (3.2), and $E_0 = (m_b^2 + r^2 - q^2)/2m_b$ is the energy of the hadronic system at the partonic level in the $b$ quark rest frame. We employ the on-shell...
Table 2: Leptonic moments with $E_\ell > 1$ GeV.

| n | $L_n^{(0)}$ | $L_n^{(1)}$ | $L_n^{(2)}$ | $L_n^{(3)}$ |
|---|---|---|---|---|
| 0 | 0.814810 | -1.439726 | -1.600023 | -2.407834 |
| 1 | 0.277642 | -0.497511 | -0.567236 | -0.870422 |
| 2 | 0.097933 | -0.178501 | -0.208715 | -0.326798 |
| 3 | 0.035615 | -0.066185 | -0.079299 | -0.126665 |

Table 3: Hadronic moments without $E_\ell$ cuts.

| i | j | $H_{ij}^{(0)}$ | $H_{ij}^{(1)}$ | $H_{ij}^{(2)}$ | $H_{ij}^{(3)}$ |
|---|---|---|---|---|---|
| 0 | 0 | 1. | -1.777615 | -1.917079 | -2.827995 |
| 0 | 1 | 0.422009 | -0.719047 | -0.732727 | -1.028913 |
| 0 | 2 | 0.183191 | -0.291969 | -0.275917 | -0.362931 |
| 0 | 3 | 0.081475 | -0.117703 | -0.099753 | -0.118659 |
| 1 | 0 | 0. | 0.090094 | 0.161513 | 0.304195 |
| 1 | 1 | 0. | 0.047004 | 0.081646 | 0.148907 |
| 1 | 2 | 0. | 0.0250925 | 0.0421838 | 0.0743690 |
| 2 | 0 | 0. | 0.0091060 | 0.0124670 | 0.0163852 |
| 2 | 1 | 0. | 0.0053384 | 0.0071132 | 0.0090464 |
| 3 | 0 | 0. | 0.0018101 | 0.0021753 | 0.0023946 |

masses $m_b = 4.6$ GeV and $m_c = 1.15$ GeV. These numerical values do not correspond to the actual $b$ and $c$ pole masses, but they simplify the comparison with [18, 31].

In table 1 we present the results of $L_n^{(j)}$ for $n = 0, 1, 2, 3$ and $j = 0, 1, 2, 3$ when no lower cut is applied on the charged lepton energy. The first row refers to the total width: we reproduce the results of [12, 21, 22] for the four entries. The second to fourth rows refer to the first few moments of the leptonic energy; we have checked that we reproduce with good accuracy the results of [15, 23] for the first two entries, while the third and fourth entries have not been reported so far.

Table 2 contains the same leptonic moments as table 1, but a 1 GeV lower cut on the lepton energy is applied. Here the second entry in each row confirms results by [23], while the third and fourth entries are new.

Reference values for the hadronic moments are given in table 3 for the case without lepton energy cut, and in table 4 for $E_\ell \geq 1$ GeV. The values of tables 3 and 4 reproduce the tree-level and one-loop results presented in refs. [13] and [16], taking into account the different input values. The same holds for a few entries of the third column (second-order BLM) given in [16]. We always find acceptable numerical agreement with [18]. More difficult is a comparison with the one-loop numerical results presented in [17], as they refer to the physical hadronic invariant mass moments, and are expressed in the 1S $b$ mass scheme.
Table 4: Hadronic moments with $E_\ell > 1$ GeV.

5 Summary

We have performed a calculation of the $O(\alpha_s)$ and $O(\alpha_s^2 \beta_0^{-1})$ corrections to the differential rate for inclusive semileptonic $b$ decays. We have calculated all five structure functions; only three of them contribute to the decays into massless leptons, while they are all relevant for $B \to X_c \tau \bar{\nu}$. Our one-loop calculation checks several existing results, while the BLM calculation is far more general than those present in the literature. Since we explicitly subtract the soft singularity from the gluon emission contribution, our results allow for an accurate numerical implementation, which we provide in a FORTRAN code, available upon request. The code computes $O(\alpha_s)$ and $O(\alpha_s^2 \beta_0^{-1})$ corrections to the triple differential distribution and integrates over the phase space with arbitrary experimental cuts. A few numerical results are presented in Section 4 for the case of massless leptons.

For what concerns the analyses of semileptonic moments, relevant to the determination of $|V_{cb}|$, the present work permits the evaluation of the perturbative contributions to hadronic moments at the same level as for the leptonic moments, since all $O(\alpha_s)$ and $O(\alpha_s^2 \beta_0^{-1})$ contributions can be now computed. Practical applications have been touched upon in [18] and will be expanded in a future publication.

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A Kinematics

Let us consider the phase space element for the processes

\[ b(p) \rightarrow \ell(p_\ell) + \bar{\nu}(p_\bar{\nu}) + X_c, \]  

(A.1)

where the states \( X_c \) contain a charm quark with momentum \( p' \), plus a collection of \( n \) partons with momenta \( k_1, \ldots, k_n \) and masses \( m_1, \ldots, m_n \). It can be decomposed as

\[ d\phi_{3+n}(p; p_\ell, p_\bar{\nu}, p', k_i) = \frac{dq^2}{2\pi} \frac{dr^2}{2\pi} d\phi_2(p; q, r) d\phi_2(q; p_\ell, p_\bar{\nu}) d\phi_{n+1}(r; p', k_i), \]  

(A.2)

where

\[ q_-^2 \leq q^2 \leq q_+^2; \quad q_-^2 = (m_\ell + m_\bar{\nu})^2 = 0; \quad q_+^2 = (m_b - m_c - m_1 - \ldots - m_n)^2 \]  

(A.3)

\[ r_-^2 \leq r^2 \leq r_+^2; \quad r_-^2 = (m_c + m_1 + \ldots + m_n)^2; \quad r_+^2 = (m_b - \sqrt{q^2})^2. \]  

(A.4)

In the rest frame of the decaying \( b \) quark, the direction of the momentum \( \vec{q} \) of the lepton pair is irrelevant, and its modulus is fixed for fixed \( q^2, r^2 \). Therefore, the first two-body phase space in eq. (A.2) gives simply an overall factor:

\[ d\phi_2(p; q, r) = \frac{d^3q}{(2\pi)^32q_0} \frac{d^3r}{(2\pi)^32r_0} (2\pi)^4\delta^4(p - q - r) = \frac{1}{4\pi m_b} |\vec{q}|. \]  

(A.5)

where

\[ |\vec{q}|^2 = \frac{\lambda(q_-^2, r_-^2, m_b^2)}{4m_b^2}; \quad \lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \]  

(A.6)

We now turn to the phase space factor for the lepton pair. For a given value of \( q^2 \), the kinematics is completely fixed by the energy of one of the two leptons, for example the charged lepton energy \( E_\ell \):

\[ d\phi_2(q; p_\ell, p_\bar{\nu}) = \frac{d^3p_\ell}{(2\pi)^32E_\ell} \frac{d^3p_\bar{\nu}}{(2\pi)^32E_\bar{\nu}} (2\pi)^4\delta^4(q - p_\ell - p_\bar{\nu}) \]

\[ = \frac{1}{8\pi} \frac{E_\ell}{|\vec{q} - \vec{p}_\ell|} \frac{dE_\ell \, d\cos \theta}{|\vec{q} - \vec{p}_\ell|} \delta(q^0 - E_\ell - |\vec{q} - \vec{p}_\ell|). \]  

(A.7)

where \( \theta \) is the angle formed by the directions of \( \vec{q} \) and \( \vec{p}_\ell \). The delta function can be used to perform the angular integration. We obtain

\[ d\phi_2(q; p_\ell, p_\bar{\nu}) = \frac{1}{8\pi} \frac{dE_\ell}{|\vec{q}|}, \]  

(A.8)

with

\[ \cos \theta = \frac{2q^0E_\ell - q^2}{2|q|E_\ell}; \quad q^0 = \sqrt{q^2 + |q|^2} = \frac{m_b^2 + q^2 - r^2}{2m_b}. \]  

(A.9)

From the condition \(|\cos \theta| \leq 1\) we get the limits on \( E_\ell \):

\[ E_-^\ell \leq E_\ell \leq E_+^\ell; \quad E_\ell^\pm = \frac{q^0 \pm |\vec{q}|}{2}. \]  

(A.10)
Collecting eqs. (A.2, A.5, A.8) we obtain

\[
d\phi_{3+n}(p; p_\ell, p_\nu, p', k) = \left[ \frac{1}{128\pi^4 m_b} dE_\ell \, dq^2 \, dr^2 \right] d\phi_{n+1}(r; p', k_1, \ldots, k_n). \tag{A.11}
\]

For fixed \(q^2\) and \(E_\ell\), the variable \(r^2\) is in one-to-one correspondence with the neutrino energy \(E_\nu\):

\[
r^2 = (p - q)^2 = m_b^2 + q^2 - 2m_b(E_\ell + E_\nu). \tag{A.12}
\]

Similarly, the invariant mass of the lepton pair \(q^2\) can be traded for the energy \(E_{X_e}\) of the hadronic system in the final state; they are related through

\[
q^2 = (p - r)^2 = m_b^2 + r^2 - 2m_b E_{X_e}. \tag{A.13}
\]

The differential decay width for the processes in eq. (A.1) is given by

\[
\frac{d^3\Gamma}{dq^2 dr^2 dE_\ell} = \frac{1}{2 \, 2m_b \, 128\pi^4 m_b} \sum_n \int d\phi_{n+1}(r; p', k_1, \ldots, k_n) |M_n(p; p_\ell, p_\nu, p', k_i)|^2. \tag{A.14}
\]

At leading order in the weak interactions, we have

\[
M_n(p; p_\ell, p_\nu, p', k_i) = -\frac{i}{q^2 - m_W^2} M^\mu_n(p; q, p', k_i) M_\nu(p; p_\ell, p_\nu) \simeq \frac{i}{m_W^2} M^\mu_n(p; q, p', k_i) M_\nu(p; p_\ell, p_\nu), \tag{A.15}
\]

where \(M^\mu_n(p; q, p', k_i)\) and \(M_\nu(p; p_\ell, p_\nu)\) are the amplitudes for the processes

\[
\begin{align*}
b(p) \to W^*(q) + X_e & \tag{A.16} \\
W^*(q) \to \ell(p_\ell) + \bar{\nu}(p_\nu) & \tag{A.17}
\end{align*}
\]

respectively. We define a leptonic tensor \(L^{\mu\nu}\) and a hadronic tensor \(W^{\mu\nu}\) through

\[
\begin{align*}
g^2 \, L^{\mu\nu}(p_\ell, p_\nu) &= M^\mu(p; q, p', k_i) M^{\nu*}(q; p_\ell, p_\nu) & \tag{A.18} \\
g^2 \, |V_{cb}|^2 \, W^{\mu\nu}(p, q) &= \sum_n \int d\phi_{n+1}(r; p', k_1, \ldots, k_n) \, M^\mu_n(p; q, p', k_i) M^{\nu*}_n(p; q, p', k_i). \tag{A.19}
\end{align*}
\]

In terms of \(L^{\mu\nu}\) and \(W^{\mu\nu}\), the decay rate is given by

\[
\frac{d^3\Gamma}{dq^2 dr^2 dE_\ell} = \frac{1}{2 \, 2m_b \, 128\pi^4 m_b} \frac{g^4 |V_{cb}|^2}{m_W^4} L^{\mu\nu}(p_\ell, p_\nu) W^{\mu\nu}(p, q) = \frac{G_F^2 |V_{cb}|^2}{16\pi^4 m_b^2} L^{\mu\nu}(p_\ell, p_\nu) W^{\mu\nu}(p, q). \tag{A.20}
\]

Finally, we compute the phase space for the hadron system in the cases \(n = 0, 1\). For \(n = 0\), which is the case at order \(\alpha_s^0\) and for the virtual contribution at order \(\alpha_s\), we have simply

\[
d\phi_1(r; p') = \frac{d^3p'}{(2\pi)^3 2p_0} (2\pi)^4 \delta^{(4)}(r - p') = 2\pi \delta(r^2 - m_c^2). \tag{A.21}
\]
In this Appendix, we present in full detail the computation of the order-$\alpha_s$ Virtual corrections to the process in eq. (2.5). We have

$$d\phi_2(r; p', k) = \frac{d^3k}{(2\pi)^32p_0} \frac{d^3k}{(2\pi)^32k_0} (2\pi)^4\delta^{(4)}(r - p' - k) = \frac{1}{8\pi}\frac{|\vec{k}|}{\sqrt{r^2}} d\cos\theta,$$  \hspace{1cm} (A.22)

with

$$|\vec{k}|^2 = \frac{\lambda(r^2, m_e^2, \lambda^2)}{4r^2}; \quad k_0 = \sqrt{|\vec{k}|^2 + \lambda^2} = \frac{r^2 - m_e^2 + \lambda^2}{2\sqrt{r^2}}. \hspace{1cm} (A.23)$$

It will be convenient to introduce the variable

$$t = (p - k)^2 - m_b^2 = \lambda^2 - 2(p_0k_0 - |\vec{p}||\vec{k}|\cos\theta); \quad d\cos\theta = \frac{dt}{2|\vec{p}||\vec{k}|},$$ \hspace{1cm} (A.24)

where

$$|\vec{p}|^2 = \frac{\lambda(q^2, r^2, m_b^2)}{4r^2}; \quad p^0 = \sqrt{|\vec{p}|^2 + m_b^2} = \frac{m_b^2 + r^2 - q^2}{2\sqrt{r^2}} \hspace{1cm} (A.25)$$

because of the energy conservation constraint

$$\sqrt{|\vec{p}|^2 + m_b^2} = \sqrt{r^2} + \sqrt{|\vec{p}|^2 + q^2}. \hspace{1cm} (A.26)$$

Hence

$$d\phi_2(r; p', k) = \frac{1}{8\pi}\frac{dt}{\sqrt{\lambda(q^2, r^2, m_b^2)}} \hspace{1cm} (A.27)$$

$$t_+ \leq t \leq t_-; \quad t_\pm = \lambda^2 - 2(p_0k_0 \pm 2|\vec{p}||\vec{k}|). \hspace{1cm} (A.28)$$

### B Virtual corrections

In this Appendix, we present in full detail the computation of the order-$\alpha_s$ contribution to the differential rate due to one-loop corrections to the process in eq. (2.5). We have

$$W_{(1)\nu}^{\mu\nu}(p, q) = -i\frac{C_F g_s^2}{8} 2\pi\delta(u) \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - \lambda^2} \frac{1}{[(\ell + p)^2 - m_b^2][(\ell + p')^2 - m_e^2]}$$

\[ \times \text{Tr} \left[ (p' + m_c)\gamma_\mu(f + p' + m_c)\gamma^\nu(1 - \gamma_5)(f + \phi + m_b)\gamma^\rho(\phi + m_b)\gamma^\nu(1 - \gamma_5) \right. \]

\[ + (\phi + m_b)\gamma_\mu(f + p' + m_c)\gamma^\nu(1 - \gamma_5)(f + p' + m_c)\gamma^\rho(\phi + m_c)\gamma^\nu(1 - \gamma_5) \bigg] \]

\[ - i \left( Z_b + Z_c \right) W_{(0)\nu}^{\mu\nu}(p, q), \hspace{1cm} (B.1) \]

where

$$Z_i = \left. \frac{d\Sigma(p, m_i)}{dp} \right|_{p = m_i}; \quad \Sigma(p, m_i) = C_F g_s^2 \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - \lambda^2} \frac{\gamma_\mu(f + \phi + m_i)\gamma^\rho}{(f + p)^2 - m_i^2}, \hspace{1cm} (B.2)$$

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and $p' = p - q$. With standard techniques, we obtain

$$ W_{(1)\nu}^{\mu}(p, q) = -i C_F \frac{g_s^2}{\pi} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{1}{2} \left[ \int \frac{d^d \ell}{(2\pi)^d} \frac{N_1^{\mu\nu}}{(\ell^2 - M^2)^3} + \int \frac{d^d \ell}{(2\pi)^4} \frac{N_2^{\mu\nu}}{(\ell^2 - M^2)^3} \right] \pi \delta(u) + \int \frac{d^d \ell}{(2\pi)^d} \frac{2 - d}{(\ell^2 - M^2)^2} - \int \frac{d^d \ell}{(2\pi)^4} \frac{8m_b^2 x(1 + x)}{(\ell^2 - M_b^2)^3} \right\} W_{(0)}^{\mu\nu}(p, q) + \int \frac{d^d \ell}{(2\pi)^d} \frac{2 - d}{(\ell^2 - M_c^2)^2} - \int \frac{d^d \ell}{(2\pi)^4} \frac{8m_c^2 x(1 + x)}{(\ell^2 - M_c^2)^3} \right\} W_{(0)}^{\mu\nu}(p, q), $$

where we have collected in $N_1^{\mu\nu}$ all terms quadratic in the loop momentum $\ell$, while $N_2^{\mu\nu}$ is $\ell$-independent. Furthermore,

$$ M^2 = m_b^2 x^2 + m_c^2 y^2 - \omega xy + \lambda^2 (1 - x - y) \quad (B.3) $$
$$ M_c^2 = m_c^2 x^2 + \lambda^2 (1 - x), \quad (B.4) $$

where $\omega = q^2 - m_b^2 - m_c^2$. Ultraviolet divergences are regulated by dimensional regularization, with $d = 4 - 2\epsilon$. Using the properties of symmetric integration, we can replace

$$ N_1^{\mu\nu} \to 8 \ell^2 \frac{(2 - d)^2}{d} \left[ \omega g^{\mu\nu} + 4 p^\mu p^{\nu} + 2i e^{\mu\nu\alpha\beta} p_\alpha q_\beta - 2(q^\mu p^{\nu} + q^\nu p^{\mu}) \right], \quad (B.5) $$

and hence, by comparison with eq. (2.6),

$$ N_1^{\mu\nu} \pi \delta(u) \to 8 \ell^2 \frac{(2 - d)^2}{d} W_{(0)}^{\mu\nu}(p, q). \quad (B.6) $$

Ultraviolet divergences are immediately seen to cancel, and setting $d = 4$ we are left with

$$ W_{(1)\nu}^{\mu}(p, q) = -\frac{C_F \alpha_s}{2\pi} \left\{ (1 + I_0 - 2J) W_{(0)}^{\mu\nu}(p, q) + \frac{\pi}{8} \delta(u) \int_0^1 dx \int_0^{1-x} dy \frac{N_2^{\mu\nu}}{M^2} \right\}, \quad (B.7) $$

where we have defined

$$ I_0 = \int_0^1 dx \int_0^{1-x} dy \log \frac{M^4}{M_b^2 M_c^2} $$
$$ J = \int_0^1 dx \int_0^{1-x} dy \left( \frac{m_b^2}{M_b^2} + \frac{m_c^2}{M_c^2} \right). \quad (B.8) $$

After some algebra we find

$$ N_2^{\mu\nu} = -16 \omega \left[ \omega g^{\mu\nu} + 4 p^\mu p^{\nu} + 2i e^{\mu\nu\alpha\beta} p_\alpha q_\beta - 2(p^\mu q^{\nu} + p^{\nu} q^\mu) \right] \left[ \omega M_b^2 + (\omega^2 - \omega m_b^2 - 2m_b^2 m_c^2)x + (2\omega^2 - \omega m_c^2 - 2m_b^2 m_c^2)y \right] g^{\mu\nu} $$
$$ -64 \left[ (\omega + m_c^2 - m_b^2)xy - \omega(x + y) \right] p^\mu p^{\nu} $$
$$ +32 \left[ M_b^2 + (\omega - m_b^2)x + (\omega - m_c^2)y \right] i e^{\mu\nu\alpha\beta} p_\alpha q_\beta $$
$$ +64m_b^2 x(1 - y) q^\mu q^{\nu} $$
$$ +32 \left[ (\omega + 2m_b^2)xy - (\omega + m_b^2)x - (\omega - m_c^2)y \right] (p^\mu q^{\nu} + p^{\nu} q^\mu), \quad (B.9) $$

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where
\[ M_0^2 = m_0^2 x^2 + m_c^2 y^2 - \omega x y. \] (B.10)

Our final result for the virtual contribution is therefore
\[
W_{(1)\nu}^{\mu\nu}(p, q) = -C_F \alpha_s \left\{ V_0(q^2, \hat{\lambda}) \left[ \hat{\omega} g^{\mu\nu} + 4t^\mu t^\nu + 4i \epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta - 2(q^\mu v^\nu + q^\nu v^\mu) \right] 
- V_1(q^2, \hat{\lambda}) g^{\mu\nu} + V_2(q^2, \hat{\lambda}) v^\mu v^\nu + iV_3(q^2, \hat{\lambda}) \epsilon^{\mu\nu\alpha\beta} v_\alpha q_\beta 
+ V_4(q^2, \hat{\lambda}) q^\mu q^\nu + V_5(q^2, \hat{\lambda}) (v^\mu q^\nu + q^\mu v^\nu) \right\} \delta(\hat{u}), \] (B.11)

where
\[
V_0(q^2, \hat{\lambda}) = \frac{1}{2} (1 + I_0 - 2J - 2I) \\
V_1(q^2, \hat{\lambda}) = -\frac{1}{m_b^2} \left[ \omega K + (\omega^2 - \omega m_b^2 - 2m_b^2 m_c^2) I_x + (\omega^2 - \omega m_c^2 - 2m_b^2 m_c^2) I_y \right] \\
V_2(q^2, \hat{\lambda}) = -4 \left[ (\omega + m_b^2 + m_c^2) I_{xy} - \omega (I_x + I_y) \right] \\
V_3(q^2, \hat{\lambda}) = 2 \left[ K + (\omega - m_b^2) I_x + (\omega - m_c^2) I_y \right] \\
V_4(q^2, \hat{\lambda}) = 4m_b^2 (I_x - I_{xy}) \\
V_5(q^2, \hat{\lambda}) = 2 \left[ (\omega + 2m_b^2) I_{xy} - (\omega + m_b^2) I_x - (\omega - m_c^2) I_y \right] \] (B.12)

and
\[
K = \int_0^1 dx \int_0^{1-x} dy \frac{M_0^2}{M^2} \\
I_1 = \int_0^1 dx \int_0^{1-x} dy \frac{\omega}{M^2} \\
I_x = \int_0^1 dx \int_0^{1-x} dy \frac{x}{M^2} \\
I_y = \int_0^1 dx \int_0^{1-x} dy \frac{y}{M^2} \\
I_{xy} = \int_0^1 dx \int_0^{1-x} dy \frac{xy}{M^2} \] (B.13)

The integrals \( I_1 \) and \( J \) are divergent for \( \lambda = 0 \); this is precisely the soft singularity which is needed in order to cancel the analogous divergent terms in the real emission contribution. Also in this case, it is convenient to isolate the soft logarithm. To this purpose, we perform the change in the integration variables
\[
x = x_1 x_2 \quad \text{(B.14)} \\
y = x_1 (1 - x_2) \quad \text{(B.15)} \\
dx \, dy = x_1 \, dx_1 \, dx_2; \quad 0 \leq x_1 \leq 1; \quad 0 \leq x_2 \leq 1. \] (B.16)

In terms of the new variables, we have
\[
M^2 = P(x_2) x_1^2 + \lambda^2 (1 - x_1). \] (B.17)
where

$$P(x_2) = m_b^2 x_2^2 + m_c^2 (1 - x_2)^2 - \omega x_2 (1 - x_2).$$  \hfill (B.18)

Thus,

$$I_1 = \frac{\omega}{2} \int_0^1 dx_2 \int_0^1 dx_1 \frac{x_1}{P(x_2) x_1^2 + \lambda^2 (1 - x_1)} \frac{2P(x_2) x_1 - \lambda^2}{P(x_2) x_1^2 + \lambda^2 (1 - x_1)} + \frac{\omega}{2} \int_0^1 dx_2 \int_0^1 dx_1 \frac{\lambda^2}{P(x_2) x_1^2 + \lambda^2 (1 - x_1)}. \hfill (B.19)$$

The coefficient of $\log \lambda^2$ can be computed explicitly. We find

$$-\frac{\omega}{2} \int_0^1 \frac{dx_2}{P(x_2)} = \frac{1}{2a} \log \frac{1 + a}{1 - a}, \hfill (B.20)$$

where $a = \sqrt{\lambda_b^0/\omega}$. Hence,

$$I_1 = \frac{1}{2a} \log \frac{1 + a}{1 - a} \log \frac{\lambda^2}{m_b m_c}$$

$$+ \frac{\omega}{2} \int_0^1 dx_2 \frac{P(x_2)}{m_b m_c} + \frac{\omega}{2} \int_0^1 \frac{dx_2}{P(x_2)} \int_0^1 dx_1 \frac{\lambda^2}{P(x_2) x_1^2 + \lambda^2 (1 - x_1)}. \hfill (B.21)$$

Similarly,

$$\int_0^1 dx \frac{m_b^2 x (1 - x^2)}{m_b^2 x^2 + \lambda^2 (1 - x)} = \frac{1}{2} \int_0^1 dx \frac{2m_b^2 x - \lambda^2}{m_b^2 x^2 + \lambda^2 (1 - x)} + \frac{1}{2} \int_0^1 dx \frac{\lambda^2 - 2m_b^2 x^3}{m_b^2 x^2 + \lambda^2 (1 - x)}$$

$$= \frac{1}{2} \log \frac{\lambda^2}{m_b^2} + \frac{1}{2} \int_0^1 dx \frac{\lambda^2 - 2m_b^2 x^3}{m_b^2 x^2 + \lambda^2 (1 - x)} \hfill (B.22)$$

and therefore

$$J = -\log \frac{\lambda^2}{m_b m_c} + \frac{1}{2} \int_0^1 dx \left[ \frac{\lambda^2 - 2m_b^2 x^3}{m_b^2 x^2 + \lambda^2 (1 - x)} + \frac{\lambda^2 - 2m_c^2 x^3}{m_c^2 x^2 + \lambda^2 (1 - x)} \right]. \hfill (B.23)$$

### C Analytic expressions for one-loop moments

We present here a few examples of $O(\alpha_s)$ corrections to the moments of $u^i E_{\ell}^0$ at arbitrary values of the lower cut on the charged lepton energy. Keeping in mind eqs. (3.2) and (4.2) we define

$$M_{ij} = \frac{\Gamma_0}{C_F \Gamma_u} H_{ij}^{(1)},$$

with $\Gamma_u = G_F^2 |V_{cb}|^2 m_b^5 / (192 \pi^3)$, $\xi = 2E_{\ell}^{cut}/m_b$, and $L_\xi = \ln(1 - \xi)$. 

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The moments with \( i > 0 \) receive contributions from bremsstrahlung diagrams only and are significantly easier to express in analytic form. Here are the lowest moments:

\[
M_{10} = -\frac{2873 \rho^5}{1800} + \frac{\rho^4 (1 - 62 \xi + 108 \xi^2 - 93 \xi^3)}{24 (1 - \xi)^2} + (1 - \xi)^2 \left( \frac{1 - \xi^2}{600} - \frac{91 - 298 \xi - 52 \xi^2 + 9 \xi^3}{6} \right) \\
+ \frac{\rho^3 (17 + 109 \xi - 83 \xi^2 + 18 \xi^3)}{6 - 6 \xi} + \frac{\rho^2 (-79 - 438 \xi + 222 \xi^2 + 25 \xi^3)}{18} \\
+ \frac{\rho (1 - \xi) (71 + 107 \xi + 45 \xi^2 - 51 \xi^3)}{24} + \left[ \frac{\rho^3 (63 - 37 \xi - 14 \xi^2)}{3 - 3 \xi} \right] \\
+ \frac{\rho^2 (-146 + 195 \xi - 60 \xi^2 + 2 \xi^3)}{6} - \frac{(1 - \xi)^2 (96 + 17 \xi - 62 \xi^2 + 9 \xi^3)}{120} \\
+ \frac{\rho^4 (-60 + 137 \xi - 118 \xi^2 + 17 \xi^3)}{24 (1 - \xi)^2} + \frac{\rho (-9 - 20 \xi + 18 \xi^3 - 7 \xi^4)}{6} \\
+ \frac{\rho^4 (5 - 10 \xi + 6 \xi^3)}{2 (1 - \xi)^2} + \rho^2 \left( \frac{5}{3} - 8 \xi + 5 \xi^2 - \frac{\xi^3}{6} \right) \\
+ \frac{3 \rho (1 - 2 \xi^3 + 4 \xi^4)}{2} \ln \rho \tag{C.1}
\]

\[
M_{20} = \frac{6341 \rho^6}{10800} + \frac{\rho^2 (1 - \xi) (-101 - 201 \xi + 61 \xi^2 + 41 \xi^3)}{48} + \frac{\rho^3 (43 + 528 \xi - 285 \xi^2 + 47 \xi^3)}{54} \\
+ \frac{(1 - 3 \xi) (-125 + 2013 \xi - 474 \xi^2 + 62 \xi^3)}{10800} + \frac{\rho^4 (29 + 139 \xi - 186 \xi^2 + 134 \xi^3)}{-48 + 48 \xi} \\
+ \frac{\rho (1 - \xi) (-137 + 536 \xi - 561 \xi^2 + 262 \xi^3)}{600} + \frac{\rho^5 (925 - 984 \xi - 443 \xi^2 + 734 \xi^3)}{600 (1 - \xi)^2} \\
+ \frac{\rho^5 (2 - 3 \xi)}{10} + \frac{\rho^4 (-6 + 3 \xi + \xi^2)}{2} \ln \rho \\
+ \frac{\rho^3 (-7 \rho^6 + 4 \xi^2 + 2 \xi^3)}{45} + \frac{\rho^2 (15 + 32 \xi - 21 \xi^2 + 3 \xi^3)}{10 (1 - \xi)^2} \\
+ \frac{\rho^3 (38 + 24 \xi^2 - 24 \xi^2 + 29 \xi^3)}{18} - \frac{5 \rho^2 (1 - 2 \xi^3 + 4 \xi^4)}{4} \ln \rho \\
+ \frac{\rho^5 (866 - 1921 \xi + 1484 \xi^2 - 309 \xi^3)}{600 (1 - \xi)^2} + \frac{\rho^3 (176 - 207 \xi + 84 \xi^2 - 26 \xi^3)}{18} \\
+ \frac{\rho (1 - \xi)^2 (-162 + 111 \xi - 16 \xi^2 + 7 \xi^3)}{120} + \frac{\rho^2 (-1 + \xi) (25 - 53 \xi + 9 \xi^2 + 9 \xi^3)}{12} \\
+ \frac{(1 - \xi)^3 (398 - 153 \xi - 78 \xi^2 + 13 \xi^3)}{1800} + \frac{\rho^4 (84 - 63 \xi - 22 \xi^2 + 25 \xi^3)}{-24 + 24 \xi} \ln \rho \tag{C.2}
\]

\[
M_{11} = -\frac{23 \rho^6}{80} + \frac{\rho^3 (6 + 20 \xi - 16 \xi^2 + 11 \xi^3)}{6} + \frac{\rho^5 (1457 - 4383 \xi + 5481 \xi^2 - 6356 \xi^3 + 2604 \xi^4)}{3600 (1 - \xi)^3}
\]
\[
\begin{align*}
&\rho(1-\xi) \left(209 + 449\xi - 231\xi^2 - 76\xi^3 + 4\xi^4\right) \\
&+ \frac{\rho^4}{48(1-\xi)^2} \left(-49 - 340\xi + 570\xi^2 - 172\xi^3 + 37\xi^4\right) \\
&+ \frac{\rho^5}{10} \left(-\frac{1}{10} + \rho^2(-1 + \xi)^3 + \rho^3(-11 + 8\xi - \xi^2) + \frac{\rho^4(-6 - 2\xi + \xi^2)}{4}\right) L_\xi \ln \frac{1 - \xi}{\rho} \\
&+ \frac{\rho^6}{10} \left(\frac{1}{4} + \rho^3(2 + 44\xi - 10\xi^2 + 3\xi^3)\right) + \frac{\rho^4(1 + 4\xi - 6\xi^2 + 3\xi^3 - 4\xi^4)}{4(1 - \xi)^2} \\
&+ \frac{\rho^2(5 + 12\xi - 30\xi^2 + 7\xi^3 + 3\xi^4)}{12} + \frac{\rho^5(-23 + 57\xi - 39\xi^2 - 31\xi^3 + 24\xi^4)}{120(-1 + \xi)^3} \\
&- \frac{3\rho(-7 + 20\xi^3 - 15\xi^4 + 2\xi^5)}{40} \ln \rho + \frac{\rho^5(1 - 63\xi + 213\xi^2 - 91\xi^3)}{300(-1 + \xi)^3} \\
&+ \frac{\rho^3(40 - 184\xi + 38\xi^2 - 9\xi^3)}{12} + \frac{\rho^2(13 - 30\xi + 17\xi^2 + 3\xi^3 - 2\xi^4)}{4} \\
&+ \frac{\rho^2(-1 - \xi)(12 - 28\xi + 10\xi^2 + 4\xi^3 - \xi^4)}{12} - \frac{(1 - \xi)^2(526 - 218\xi - 237\xi^2 + 124\xi^3 - 15\xi^4)}{1200} \\
&+ \frac{\rho^4(-438 + 766\xi - 337\xi^2 + 8\xi^3 + 25\xi^4)}{48(1 - \xi)^2} \right] L_\xi \tag{C.3}
\end{align*}
\]

References

[1] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 93 (2004) 011803 [arXiv:hep-ex/0404017].

[2] Delphi, babar hadr exp results; B. Aubert et al. [BABAR Collaboration], Phys. Rev. D 69 (2004) 111104 [arXiv:hep-ex/0403030]; A. H. Mahmood et al. [CLEO Collaboration], Phys. Rev. D 70 (2004) 032003 [arXiv:hep-ex/0403053]; K. Abe et al. [BELLE Collaboration], arXiv:hep-ex/0408139 and arXiv:hep-ex/0409015; D. Acosta [CDF Collaboration], arXiv:hep-ex/0502003.

[3] I. Bigi, N. Uraltsev and A. Vainshtein, Phys. Lett. B293 (1992) 430 and Phys. Rev. Lett. 71 (1993) 496; B. Blok, L. Koyrakh, M. Shifman and A. Vainshtein, Phys. Rev. D49 (1994) 3356.

[4] A. V. Manohar and M. B. Wise, Phys. Rev. D 49 (1994) 1310 [arXiv:hep-ph/9308246].

[5] M. Gremm and A. Kapustin, Phys. Rev. D55 (1997) 6924.

[6] N. Uraltsev, Phys. Lett. B 545 (2002) 337 [arXiv:hep-ph/0111166].

[7] M. Battaglia et al., Phys. Lett. B 556 (2003) 41 [arXiv:hep-ph/0210319].

[8] C. W. Bauer, Z. Ligeti, M. Luke, A. V. Manohar and M. Trott, Phys. Rev. D 70 (2004) 094017 [arXiv:hep-ph/0408002].
[9] D. Benson, I. I. Bigi, T. Mannel and N. Uraltsev, Nucl. Phys. B 665 (2003) 367 [arXiv:hep-ph/0302262].

[10] A. Ali and E. Pietarinen, Nucl. Phys. B 154 (1979) 519; G. Altarelli, N. Cabibbo, G. Corbo, L. Maiani and G. Martinelli, Nucl. Phys. B 208 (1982) 365.

[11] M. Jezabek and J. H. Kuhn, Nucl. Phys. B 314 (1989) 1; Nucl. Phys. B 320 (1989) 20; A. Czarnecki, M. Jezabek and J. H. Kuhn, Acta Phys. Polon. B 20 (1989) 961; A. Czarnecki and M. Jezabek, Nucl. Phys. B 427 (1994) 3 [arXiv:hep-ph/9402326].

[12] Y. Nir, Phys. Lett. B 221 (1989) 184.

[13] A. F. Falk, M. E. Luke and M. J. Savage, Phys. Rev. D 53 (1996) 2491.

[14] C. W. Bauer and B. Grinstein, Phys. Rev. D 68 (2003) 054002 [arXiv:hep-ph/0212164].

[15] M. B. Voloshin, Phys. Rev. D 51 (1995) 4934.

[16] A. F. Falk and M. E. Luke, Phys. Rev. D 57 (1998) 424.

[17] M. Trott, Phys. Rev. D 70 (2004) 073003 [arXiv:hep-ph/0402120].

[18] N. Uraltsev, arXiv:hep-ph/0403166, to appear in IJMPA.

[19] F. De Fazio and M. Neubert, JHEP 9906 (1999) 017 [arXiv:hep-ph/9905351].

[20] S. J. Brodsky, G. P. Lepage and P. B. Mackenzie, Phys. Rev. D 28 (1983) 228.

[21] M. E. Luke, M. J. Savage and M. B. Wise, Phys. Lett. B 345 (1995) 301 [arXiv:hep-ph/9410387].

[22] P. Ball, M. Beneke and V. M. Braun, Phys. Rev. D 52 (1995) 3929 [arXiv:hep-ph/9503492].

[23] M. Gremm and I. Stewart, Phys. Rev. D 55 (1997) 1226.

[24] B. H. Smith and M. B. Voloshin, Phys. Lett. B 340 (1994) 176 [arXiv:hep-ph/9405204]; P. Ball, M. Beneke and V. M. Braun, Nucl. Phys. B 452 (1995) 563 [arXiv:hep-ph/9502300]; Y. L. Dokshitzer, G. Marchesini and B. R. Webber, Nucl. Phys. B 469 (1996) 93 [arXiv:hep-ph/9512336].

[25] N. Uraltsev, Nucl. Phys. B 491 (1997) 303 [arXiv:hep-ph/9610425].

[26] A. Czarnecki, Phys. Rev. Lett. 76 (1996) 4124 [arXiv:hep-ph/9603261]; A. Czarnecki and K. Melnikov, Phys. Rev. Lett. 78 (1997) 3630 [arXiv:hep-ph/9703291]; Phys. Rev. D 59 (1999) 014036 [arXiv:hep-ph/9804215].

[27] J. P. Archambault and A. Czarnecki, Phys. Rev. D 70 (2004) 074016 [arXiv:hep-ph/0408021].

[28] N. Uraltsev, Mod. Phys. Lett. A17 (2002) 2317.
[29] see http://maxima.sourceforge.net

[30] S. Wolfram, The MATHEMATICA book, Wolfram Media, 2003.

[31] P. Gambino and N. Uraltsev, Eur. Phys. J. C 34 (2004) 181 [arXiv:hep-ph/0401063].

[32] I. I. Y. Bigi, M. A. Shifman, N. Uraltsev and A. I. Vainshtein, Phys. Rev. D 56 (1997) 4017 [arXiv:hep-ph/9704245] and Phys. Rev. D 52 (1995) 196 [arXiv:hep-ph/9405410].