GLOBAL EXISTENCE FOR THE MHD SYSTEM IN CRITICAL SPACES

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Abstract. In this article, we show that the magneto-hydrodynamic system (MHD) in \( \mathbb{R}^N \) with variable density, variable viscosity and variable conductivity has a local weak solution in the Besov space \( \dot{B}_{p_1,1}^{\frac{N}{p_1}}(\mathbb{R}^N) \times \dot{B}_{p_2,1}^{\frac{N}{p_2}-1}(\mathbb{R}^N) \) for all \( 1 < p_2 < +\infty \) and some \( 1 < p_1 \leq \frac{2N}{N+3} \) if the initial density approaches a positive constant. Moreover, this solution is unique if we impose the restrictive condition \( 1 < p_2 \leq 2N \). We prove also that the constructed solution exist globally in time if the initial data are small enough. In particular, this allows us to work in the frame of Besov space with negative regularity indices and this fact is particularly important when the initial data are strong oscillating.

1. Introduction.

In this paper we study existence and uniqueness of solutions for the magneto-hydrodynamic system with variable viscosity and variable density, which describes the coupling between the inhomogeneous Navier-Stokes system and the Maxwell equation:

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0 \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - 2\text{div}(\mu(\rho)\mathcal{M}) + \nabla (\Pi + \frac{B^2}{2}) &= \rho f + \text{div}(B \otimes B) \\
\partial_t B - \text{div}\left(\frac{\nabla B}{\sigma(\rho)}\right) &= B \cdot \nabla u - u \cdot \nabla B \\
\text{div} u &= \text{div} B = 0 \\
(\rho, u, B)|_{t=0} &= (\rho_0, u_0, B_0),
\end{align*}
\]

(MHD)

where \( \mathcal{M} = \frac{1}{2}(\nabla u + (\nabla u)^T) \) is the symmetrical part of the gradient, the external force \( f \) is given, \( \mu(\cdot) > 0 \) is the viscosity of the fluid, \( \sigma(\cdot) > 0 \) is the conductivity and \( \Pi(t, x) \) is the pressure in the fluid. Moreover, we suppose that \( \sigma \) and \( \mu \) are \( C^\infty \) functions and that

\[
0 < \sigma \leq \frac{1}{\sigma} \leq \bar{\sigma} < \infty \quad \text{and} \quad 0 < \underline{\mu} \leq \mu.
\]

The homogeneous case (\( \rho = \text{const.} \)) of the (MHD) system was studied by G. Duvaut and J.-L. Lions [12]. They established local existence and uniqueness of a solution in the classical Sobolev spaces \( H^s(\mathbb{R}^N) \), \( s \geq N \). They proved also global existence of the solution for small initial data.
The inhomogeneous case has been studied by many authors. Let us mention J.-F. Gerbeau and C. Le Bris [15] and also B. Desjardins and C. Le Bris [11] who studied global existence of weak solutions of finite energy in $\mathbb{R}^3$ and in the torus $T^3$. On the other hand, local existence of strong solutions was recently considered by H. Abidi and T. Hmidi [2]. They proved also global existence of strong solutions when the initial data are small in some Sobolev spaces.

The principal aim of this paper is to study the strong solutions in some Sobolev-Besov critical spaces of negative regularity index. Working with initial data in Besov spaces of negative regularity allows us to choose the initial velocity and the initial magnetic field to be very irregular (even discontinuous) functions. On the other hand, working in spaces of negative regularity allows us to prove that the MHD system is globally well-posed for strongly oscillating initial data.

In the following, we suppose that the initial density verifies $\inf_x \rho_0(x) > 0$ and thus, by the maximum principle for the transport equation, we have $\inf_x \rho(t, x) > 0$. We also suppose that the density of the fluid is a small perturbation of a constant density which we choose to be equal to 1. This implies that we can use the transform $a = \frac{1}{\rho} - 1$ which allows us to work with the following system:

\[
\begin{aligned}
\partial_t a + u \cdot \nabla a &= 0 \\
\partial_t u + u \cdot \nabla u + (1 + a) \left\{ \nabla \Pi + \nabla (\frac{B^2}{2}) - 2 \text{div} (\tilde{\mu}(a) \mathcal{M}) \right\} &= f + (1 + a)B \cdot \nabla B \\
\partial_t B - \text{div} (\tilde{\sigma}(a) \nabla B) &= B \cdot \nabla u - u \cdot \nabla B \\
\text{div} u &= \text{div} B = 0 \\
(a, u, B)|_{t=0} &= (a_0, u_0, B_0),
\end{aligned}
\]

where $\tilde{\mu}(a) = \frac{\mu(\frac{1}{1+a})}{a(\frac{1}{1+a})}$ and $\tilde{\sigma}(a) = \frac{1}{a(\frac{1}{1+a})}$ are regular functions.

Let us recall the theorem proved by H. Abidi et T. Hmidi in their recent paper [2].

We denote by $\mathcal{P}$ the Leray projector on the divergence free vector fields and by $Q = I - \mathcal{P}$ the projector on the gradient type vector fields. The Besov spaces are defined in the next section.

**Theorem 1.1.** [2] Let $1 < p < 6$. There exists a constant $c$ depending on $p$ and on the functions $\mu$ and $\sigma$ such that, for $u_0, B_0 \in \dot{B}^{\frac{3}{p} - 1}_{p,1}(\mathbb{R}^3)$ with $\text{div} u_0 = \text{div} B_0 = 0$, $f \in L^1(\mathbb{R}_+; \dot{B}^{\frac{3}{p} - 1}_{p,1}(\mathbb{R}^3))$ with $Qf$ belonging to $L^2_{\text{loc}}(\mathbb{R}_+; \dot{B}^{\frac{3}{p} - 2}_{p,1}(\mathbb{R}^3))$ and $a_0 \in \dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ where

\[
\|a_0\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \leq c,
\]
then, there exists $T \in (0, +\infty]$ “such that” the system $(\text{MHD})$ has a solution $(a, u, B, \nabla \Pi)$

$$a \in C_b([0, T); \dot{B}_{p,1}^{\frac{2}{p}-1} \cap L^\infty([0, T); \dot{B}_{p,1}^{\frac{2}{p}}); u, B \in C_b([0, T); \dot{B}_{p,1}^{\frac{2}{p}-1} \cap L^1(0, T; \dot{B}_{p,1}^{\frac{2}{p}+1})).$$

Moreover, there is a sufficiently small constant $c_1 > 0$ such that, if

$$\|u_0\|_{\dot{B}_{p,1}^{\frac{2}{p}+1}}, \|B_0\|_{\dot{B}_{p,1}^{\frac{2}{p}-1}}, \|f\|_{L^1(\mathbb{R}^+; \dot{B}_{p,1}^{\frac{2}{p}-1})} \leq c_1 \inf(\mu^1, \sigma^1),$$

with $\mu^1 = \mu(1)$ et $\sigma^1 = \sigma(1)$, then $T = +\infty$. If $1 < p \leq 3$, then this solution is unique.

This result can be easily generalized to the case of fluid evolving in the whole space $\mathbb{R}^N$. However, the result does not provide uniqueness for $N < p \leq 2N$, which would allow one to conclude that the system $(\text{MHD})$ is globally well-posed for strongly oscillating initial data. Addressing the issue of uniqueness is the principal motivation of our work.

In order to have a more clear idea of uniqueness, let us note that the system $(\text{MHD})$ can be written as a coupled system of a transport equation for the density and a Navier-Stokes type equation for the couple $(u, B)$. Let us note also that the stabilizing effect of strongly oscillating initial data is well known for the classical homogeneous Navier-Stokes equation. Indeed, for the Navier-Stokes system in the homogeneous case $(\rho, B = \text{const.})$, i.e.

$$\begin{cases}
\partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = 0 \\
\text{div} \, u = 0 \\
u|_{t=0} = u_0,
\end{cases}$$

it is classical to obtain global existence and uniqueness of solutions for small initial data in the Besov space $\dot{B}_{p,1}^{\frac{2}{p}+1-\frac{N}{p}}(\mathbb{R}^N)$ for all $1 < p < \infty$ (see [4]). The Cannone-Meyer-Planchon result generalizes the classical theorem by Fujita-Kato [14], which gives existence and uniqueness of solutions in the framework of classical Sobolev spaces $\dot{H}^{\frac{N}{p}-1}(\mathbb{R}^N)$, to Besov spaces of negative regularity index. The interest in such a result comes from the fact that initial data which are large in $\dot{H}^{\frac{N}{p}-1}(\mathbb{R}^N)$ become small in the presence of oscillations in the norm of the space $\dot{B}_{p,1}^{\frac{2}{p}+1-\frac{N}{p}}$ when $N < p < +\infty$. In particular, we find that the very fast oscillations of the initial data stabilize the Navier-Stokes system in the sense that the solution exists globally in time.

**Theorem 1.2.** (Cannone-Meyer-Planchon [4]) Let $1 < p < +\infty$ and let $u_0 \in \dot{B}_{p,1}^{\frac{2}{p}+1-\frac{N}{p}}(\mathbb{R}^N)$ be a divergence free vector field. There then exists a time $T > 0$ such that system $(\text{NS}_\mu)$ has a unique solution.

$$u \in C_b([0, T); \dot{B}_{p,1}^{\frac{2}{p}+1-\frac{N}{p}}) \cap L^1(0, T; \dot{B}_{p,1}^{\frac{2}{p}+1}).$$
Moreover, there is a constant $c > 0$ small enough such that if
\[ \|u_0\|_{\dot{B}^{\frac{N}{p_1} - 1}_{p_1,1}} \leq c \mu, \]
then $T = \infty$.

In this article we will show the existence and uniqueness of global solution for system (MHD) for strongly oscillating initial data. For that it will be necessary to work in spaces with negative index of regularity. Let us note that the result of [2] does not make it possible to construct a unique global solution for the data in spaces of negative index, since one has uniqueness of the solution only in the case when $1 < p \leq N$. Also let us note that one has existence of a global weak solution when $N < p < 2N$ for small data. In this paper, we prove in fact that the (MHD) system is globally well-posed for oscillating initial data, when $\frac{1}{p_1} - 1 \in \dot{B}^{\frac{N}{p_2} - 1}_{p_2,1}$ and $u_0, B_0 \in \dot{B}^{\frac{N}{p_2} - 1}_{p_2,1}$ with $p_1 \leq p_2$ and $\frac{1}{p_1} \leq \frac{1}{p_2} + \frac{1}{N}$ and $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{2}{N}$. Note in particular that we obtain the H. Abidi and T. Hmidi results as a particular case of our theorem by taking $p_1 = p_2$. The improvement obtained in our result is due directly to the fact that we work with the density, velocity field, and magnetic field in the spaces of Besov built on different spaces of Lebesgue. The method of the proof is based on the regularizing effect for the heat equation (for more precise details, see [6]). To be more precise, we point out a result of harmonic analysis due to R. Danchin [8], which is an inequality of the type of the Poincaré-type inequality for functions localized in frequencies. That enables us to gain two derivatives of the solution from the heat equation starting from the Laplacian, and thus, for initial data in $\dot{B}^{\frac{1}{p_1} + \frac{N}{r}}_{p_1,1}(\mathbb{R}^N)$ we find that the solution belongs to the space $L^1([0, T]; \dot{B}^{\frac{1}{p_1} + \frac{N}{r}}_{p_1,1}(\mathbb{R}^N))$ which is a subspace of $L^1(Lip(\mathbb{R}^N))$. This is the principal reason for why one cannot work with the initial data $u_0 \in \dot{B}^{\frac{1}{p_1} + \frac{N}{r}}_{p_1,1}(\mathbb{R}^N)$ for $r > 1$.

We prove an existence result in critical Besov spaces (for the definition see the next section). Our principal result is as follows:

**Theorem 1.3.** Let $1 < p_1 \leq p_2 < +\infty$ be such that $\frac{1}{p_1} \leq \frac{1}{p_2} + \frac{1}{N}$ and $\frac{1}{N} < \frac{1}{p_1} + \frac{1}{p_2}$. There exists a positive constant $c$ depending on $p$ and on functions $\mu, \sigma$ such that, for $u_0, B_0 \in \dot{B}^{\frac{N}{p_2} - 1}_{p_2,1}(\mathbb{R}^N)$ with div $u_0 = \text{div} B_0 = 0$, $f \in L^1_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N}{p_2} - 1}_{p_2,1}(\mathbb{R}^N))$ with $Qf \in L^2_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N}{p_2} - 2}_{p_2,1}(\mathbb{R}^N))$ and $a_0 \in \dot{B}^{\frac{N}{p_1}}_{p_1,1}(\mathbb{R}^N)$ where
\[ \|a_0\|_{\dot{B}^{\frac{N}{p_1}}_{p_1,1}} \leq c, \]
then there exists $T(u_0, B_0; f) > 0$ such that the system ($\text{MHD}$) has a solution $(a, u, B, \nabla \Pi)$ with

$$a \in C_b([0, T]; \dot{B}^{\frac{N}{p_2}}_{p_2, 1}) \cap \tilde{L}^\infty(0, T); \dot{B}^{\frac{N}{p_1}}_{p_1, 1});$$

$$u, B \in C_b([0, T); \dot{B}^{\frac{N}{p_2} - 1}_{p_2, 1} \cap L^1(0, T; \dot{B}^{\frac{N}{p_2} + 1}_{p_2, 1})$$

and $\nabla \Pi \in L^2_T(\dot{B}^{\frac{N}{p_2} - 1 - \eta}_{p_2, 1})$, with $0 \leq \eta < \inf(1, \frac{2N}{p_2})$ and $\frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}$.

Moreover, there exists a positive constant $c_1$ such that if

$$\|u_0\|_{\dot{B}^{\frac{N}{p_2} - 1}_{p_2, 1}} + \|B_0\|_{\dot{B}^{\frac{N}{p_2} - 1}_{p_2, 1}} + \|f\|_{L^1(\mathbb{R}^+; \dot{B}^{\frac{N}{p_2} - 1}_{p_2, 1})} \leq c_1 \inf(\mu^1, \sigma^1),$$

with $\mu^1 = \bar{\mu}(1)$, $\sigma^1 = \bar{\sigma}(1)$, then $T = +\infty$.

If, in addition, we have that $1 < p_2 \leq 2N$, and $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{2}{N}$ then such a solution is unique.

The proof of Theorem 1.3 is carried out in two stages. Firstly, we show the uniqueness result that is based on a logarithmic estimate combined Osgood lemma. Secondly, for the existence part we proceed as follows: we regularize both initial data and (MHD) system, for which we establish the existence of solutions. After we show we can bound from below the time existence. Finally we prove that the regularization solutions converge to a solution satisfying our initial problem.

**Remark 1.** In the case of variable viscosity and variable conductivity, we need the more restrictive condition $p_1 \leq p_2$. This condition does not appear in the case where the viscosity is constant (see our paper 3).

**Remark 2.** This theorem allows us to construct a solution (local in time in general, respectively global in time when the initial data is small compared with viscosity), for $u_0$, $B_0 \in \dot{B}^{-\frac{1}{p_2}}_{p_2, 1}(\mathbb{R}^N)$ and all $1 < p_2 < +\infty$. In fact, it is enough for example to consider the density such that $a_0 = \rho_0^{\frac{1}{p_2}} - 1 \in \dot{B}^{1}_{N, 1}(\mathbb{R}^N)$ when $N \leq p_2 < +\infty$. In the case when $1 < p_2 < N$ we take for example $p_1 = p_2$ (other choices are possible, it suffices for example to take $p_1$ which verifies $\sup(1, \frac{Np_2}{N + p_2}) < p_1 \leq p_2$).

On the other hand, we obtain a unique solution for all $u_0$, $B_0 \in \dot{B}^{-\frac{1}{p_2}}_{p_2, 1}(\mathbb{R}^N)$ for all $1 < p_2 \leq 2N$. In order to obtain this, it suffices to consider for example $a_0 = \rho_0^{\frac{1}{p_2}} - 1 \in \dot{B}^{1}_{p_2, 1}(\mathbb{R}^N)$ with $p_1 = \frac{2N}{p_2}$ when $N \leq p_2 \leq 2N$, and and it suffices to take $\sup(1, \frac{Np_2}{N + p_2}) < p_1 \leq p_2$ when $1 < p_2 < N$. 


Remark 3. In particular, Theorem 1.3 implies existence of a unique global solution for the \( \widetilde{MHD} \) system, when the initial data \((\rho_0, u_0, B_0)\) have the particular form

\[
a_0 = \rho_0^{-1} - 1 \in \mathcal{S}(\mathbb{R}^3); \quad u_0 = \varepsilon^{-\alpha} \sin \left( \frac{x_3}{\varepsilon} \right) (-\partial_2 \phi^1, \partial_1 \phi^1, 0);
\]

\[
B_0 = \varepsilon^{-\beta} \sin \left( \frac{x_3}{\varepsilon} \right) (-\partial_2 \phi^2, \partial_1 \phi^2, 0)
\]

with \( \alpha, \beta \in [0, 1) \), \( \inf_{x \in \mathbb{R}^3} \rho_0 > 0 \) and \( \phi^i \in \mathcal{S}(\mathbb{R}^3) \), with \( a_0 \) small and \( \varepsilon > 0 \) small enough. Indeed, it is easy to verify the following assertion. Let \( \phi \in \mathcal{S}(\mathbb{R}^3) \), \( k \in \mathbb{R}^3 \), \( |k| \neq 0 \) and \((\sigma, p, r) \in \mathbb{R}_+^* \times [1, \infty)^2 \). Then, the function \( \phi_{\varepsilon}(x) = \phi(x) e^{ixk/\varepsilon} \) is small in the space \( \dot{B}^{-\sigma}_{p, r} \). More precisely, we have

\[
\| \phi_{\varepsilon} \|_{\dot{B}^{-\sigma}_{p, r}} \leq C(\phi)\varepsilon^\sigma,
\]

where \( C(\phi) = \| \phi \|_{\dot{B}^\sigma_{p, r}} \).

2. Preliminaries.

2.1. Notation. Let \( X \) be a Banach space and \( p \in [1, \infty] \). We denote by \( L^p(0, T; X) \) the set of measurable functions \( f : (0, T) \to X \), such that \( t \mapsto \| f(t) \|_X \) belongs to \( L^p(0, T) \), and we denote by \( C([0, T]; X) \) we denote the space of continuous functions on \([0, T]\) with values in \( X \), \( C_b([0, T); X) := C([0, T); X) \cap L^\infty(0, T; X) \). Let \( \mu^1 = \mu(1), \tilde{\mu}(a) = \mu(\frac{1}{1+a}), \tilde{\sigma}(a) = \frac{1}{\sigma(1+a)}, \sigma^1 = \tilde{\sigma}(1) \) and for \( 1 \leq p \leq \infty \), we denote by \( p' \) the conjugate exponent of \( p \) given by \( \frac{1}{p} + \frac{1}{p'} = 1 \).

2.2. Littlewood-Paley theory. In this section, we briefly recall the Littlewood-Paley theory and we define the functional spaces in which we will work. To this order, we use a unit dyadic (see for example [3]). Let \( \mathcal{C} \subset \mathbb{R}^N \) be the annulus centered in 0, with the small radius \( \frac{4}{3} \), and the big radius \( \frac{8}{3} \). There exist two positive radially symmetric functions \( \chi \) and \( \varphi \) belonging respectively to \( C_0^\infty(B(0, \frac{4}{3})) \) and to \( C_0^\infty(\mathcal{C}) \) such that:

\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \neq 0 \quad \text{et} \quad \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}^N.
\]

We define the following operators.

\[
\Delta_q u = \varphi(2^{-q}D) u \quad \forall q \in \mathbb{Z} \quad \text{et} \quad S_q u = \sum_{p \leq q-1} \Delta_p v \quad \forall q \in \mathbb{Z}.
\]

Moreover, we have:

\[
u = \sum_{q \in \mathbb{Z}} \Delta_q u \quad \forall u \in \mathcal{S}'(\mathbb{R}^N)/\mathcal{P}[\mathbb{R}^N],
\]
where $\mathcal{P} [\mathbb{R}^N]$ is the set of polynomials (see for example [17]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

\[ \Delta_k \Delta_q u \equiv 0 \quad \text{if} \quad |k - q| \geq 2 \quad \text{and} \quad \Delta_k (S_{q-1} u \Delta_q u) \equiv 0 \quad \text{if} \quad |k - q| \geq 5. \]

**Definition 2.1.** For $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ and $u \in S' (\mathbb{R}^N)$, we denote

\[ \| u \|_{\dot{B}^s_{p r}} := \left( \sum_{q \in \mathbb{Z}^2} 2^{rq} \| \Delta_q u \|_{L^p}^r \right)^{\frac{1}{r}} \]

with the usual change for the case $r = +\infty$. Then for $s < \frac{N}{p}$ and $s \leq \frac{N}{p}$, $r = 1$ we define

\[ \dot{B}^s_{p r} := \left\{ u \in S' (\mathbb{R}^N) \mid \| u \|_{\dot{B}^s_{p r}} < \infty \right\}, \]

otherwise, we define $\dot{B}^s_{p r}$ like the adherence in $S'$ of functions belonging to the Schwartz space, for the norm $\| \cdot \|_{\dot{B}^s_{p r}}$.

Let us recall also the Bernstein inequality (see for example [5]) which allows us to obtain some embeddings of spaces.

**Lemma 2.2.** (BERNSTEIN) Let $(r_1, r_2)$ be a couple of nonnegative real numbers such that $r_1 < r_2$. Then there exists a nonnegative constant $C$ such that for any integer $k$, any couple $(a, b)$ such that $1 \leq a \leq b \leq \infty$ and every function $u$ in $L^a (\mathbb{R}^N)$, we have

\[ \text{Supp} \mathcal{F} u \in B(0, \lambda r_1) \implies \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^b} \leq C^k \lambda^{k+N(\frac{1}{b} - \frac{1}{a})} \| u \|_{L^a}, \]

\[ \text{Supp} \mathcal{F} u \in C(0, \lambda r_1, \lambda r_2) \implies C^{-k} \lambda^k \| u \|_{L^a} \leq \sup_{|\alpha| = k} \| \partial^\alpha u \|_{L^b} \leq C^k \lambda^k \| u \|_{L^a}. \]

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use the spaces $\dot{L}^\rho_T (\dot{B}^s_{p r})$ introduced by J.-Y. Chemin and N. Lerner in [7].

**Definition 2.3.** Let $s \leq \frac{N}{p}$ (respectively $s \in \mathbb{R}$), $(r, \rho, p) \in [1, +\infty]^3$ and $T \in [0, +\infty]$. We say then that $f \in \dot{L}^\rho_T (\dot{B}^s_{p r})$, if

\[ \| f \|_{\dot{L}^\rho_T (\dot{B}^s_{p r})} := \left( \sum_{q \in \mathbb{Z}^2} 2^{qr} \left( \int_0^T \| \Delta_q f(t) \|_{L^p}^\rho dt \right)^{\frac{p}{\rho}} \right)^{\frac{1}{p}} < \infty. \]

with the usual change if $r = \infty$.

For $\theta \in [0, 1]$, we have

\[ \| u \|_{\dot{L}^\rho_T (\dot{B}^s_{p r})} \leq \| u \|_{\dot{L}^\rho_T (\dot{B}^{s_1}_{p r})}^{1 - \theta} \| u \|_{\dot{L}^\rho_T (\dot{B}^{s_2}_{p r})}^\theta \]

where $s_1, s_2$ are such that $s = \frac{s_1 + s_2}{2}$. In particular when $\{ s_1, s_2 \}$ are admissible, then $s$ is admissible.
with \( \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \) and \( s = \theta s_1 + (1-\theta)s_2 \).

Note that the Minkowski inequality implies that
\[
\|u\|_{\dot{L}_t^p(B_{p,r}^s)} \leq \|u\|_{L_t^p(\dot{B}_{p,r}^s)} \quad \text{if } \rho \leq r \quad \text{and} \quad \|u\|_{L_t^p(\dot{B}_{p,r}^s)} \leq \|u\|_{\dot{L}_t^p(B_{p,r}^s)} \quad \text{if } r \leq \rho.
\]

We give now the product laws in Besov spaces based on different Lebesgue spaces. This product laws are studied in detail in the paper [3].

**Proposition 2.4.** Let \((p,p_1,p_2,r,\lambda_1,\lambda_2) \in [1,\infty]^6\) such that \(\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}, p_1 \leq \lambda_2, p_2 \leq \lambda_1, \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1} \leq 1\) or \(\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2} \leq 1\). Then, we have the following inequality:

If \(s_1 + s_2 + N \inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0, s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}\) and \(s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}\). Then
\[
(4) \quad \|uv\|_{\dot{B}_{p,r}^{s_1+s_2-N(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{p})}} \leq \|u\|_{\dot{B}_{p,r}^{s_1}} \|v\|_{\dot{B}_{p,r}^{s_2}},
\]
when \(s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}\) (respectively \(s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}\)) we replace \(\|u\|_{\dot{B}_{p,r}^{s_1}} \|v\|_{\dot{B}_{p,r}^{s_2}}\) (respectively \(\|v\|_{\dot{B}_{p,r}^{s_2}}\)) by \(\|u\|_{\dot{B}_{p,r}^{s_1}} \|v\|_{\dot{B}_{p,r}^{s_2}}\) (respectively \(\|v\|_{\dot{B}_{p,r}^{s_2}}\)), if \(s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}\) and \(s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}\) we take \(r = 1\).

If \(s_1 + s_2 = 0, s_1 \in \left(\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}\right)\) and \(\frac{1}{p_1} + \frac{1}{p_2} \leq 1\), then
\[
(5) \quad \|uv\|_{\dot{B}_{p,\infty}^{s_1+s_2-N(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{p})}} \leq \|u\|_{\dot{B}_{p,1}^{s_1}} \|v\|_{\dot{B}_{p,\infty}^{s_2}}.
\]

If \(|s| < \frac{N}{p}\) for \(p \geq 2\) and \(-\frac{N}{p} < s < \frac{N}{p}\) otherwise, we have
\[
(6) \quad \|uv\|_{\dot{B}_{p,r}^s} \leq \|u\|_{\dot{B}_{p,r}^s} \|v\|_{\dot{B}_{p,\infty}^{\frac{N}{p}} \cap L^\infty}.
\]

**Remark 4.** In the following, \(p\) will be equal to \(p_1\) or to \(p_2\) and \(\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}\) if \(p_1 \leq p_2\), respectively \(\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}\) if \(p_2 \leq p_1\).

**Remark 5.** Note that for \(p_1 = p_2\) we obtain the classical product laws. On the other hand, if \(s_i < \frac{N}{p_i}\), \(s_1 + s_2 > 0\) and \(p_1 \leq p_2\) we obtain that \(uv \in \dot{B}_{p_2,1}^{s_1+s_2-N/p_1}\), otherwise, if \(s_i < \frac{N}{p_2}\) we obtain \(uv \in \dot{B}_{p_1,1}^{s_1+s_2-N/p_2}\). The interpretation of this facts, is that in a product law we can a smaller number of derivatives than usual, if we measure these derivatives with a \(L^p\) Lebesgue space with small \(p \geq 1\).

**Remark 6.** The Proposition [2.4] also holds in \(\dot{L}_t^p(\dot{B}_{p,r}^s)\). For example inequality (6) becomes
\[
\|uv\|_{\dot{B}_{p,r}^s} \lesssim \|u\|_{\dot{B}_{p,r}^s} \|v\|_{\dot{B}_{p,\infty}^{\frac{N}{p}} \cap L^\infty}
\]
whenever \(|s| < \frac{N}{p}\) for \(p \geq 2\) and \(-\frac{N}{p} < s < \frac{N}{p}\), \(1 \leq \rho, \rho_1, \rho_2 \leq \infty\) and \(1/\rho = 1/\rho_1 + 1/\rho_1\).
3. Estimates for the transport and Stokes equations.

We note that the MHD system with variable density consists of a transport equation for the density and a Stokes equation for the velocity vector-field. We begin by giving the necessary estimates for the transport and for the non-stationary Stokes equations (for the proofs, see the paper [3]):

**Proposition 3.1.** Let \((p_1, p_2) \in [1, +\infty]^2\), \(-1 - N\inf(\frac{1}{p_2}, \frac{1}{p_1}) < s < 1 + N\inf(\frac{1}{p_1}, \frac{1}{p_2})\) where \(p'_1\) is the conjugate exponent of \(p_1\) (respectively \(s = 1 + N\inf(\frac{1}{p_1}, \frac{1}{p_2})\) and \(r \in [1, +\infty]\) (respectively \(r = 1\)). Let \(u\) a free-divergence vector field such that \(\nabla u \in L^1(0, T; \dot{B}_{p^*_2}^{n+1} \cap L^\infty)\) (respectively \(u \in L^1(0, T; \dot{B}_{p^*_2}^{n+1})\)). We suppose that \(\rho_0 \in \dot{B}_{p^*_1}^{n+1}, f \in L^1(0, T; \dot{B}_{p^*_1}^{n+1})\). Let \(\rho \in L^\infty(0, T; \dot{B}_{p^*_1}^{n+1}) \cap C([0, T]; S')\) be a solution of the following system

\[\begin{cases}
\partial_t \rho + u \cdot \nabla \rho = f, \\
\rho|_{t=0} = \rho_0.
\end{cases}\]

Then there exists a non-negative constant \(C\) depending on \(N\) and \(s\) such that

\[
\|\rho\|_{L_T^\infty(\dot{B}_{p^*_1}^{n+1})} \leq e^{CU(t)}(\|\rho_0\|_{\dot{B}_{p^*_1}^{n+1}} + \int_0^T \|f(\tau)\|_{\dot{B}_{p^*_1}^{n+1}} d\tau),
\]

where \(U(t) = \int_0^t \|\nabla u(\tau)\|_{\dot{B}_{p^*_1}^{n+1}} d\tau\). (respectively \(U(t) = \int_0^t \|u(\tau)\|_{\dot{B}_{p^*_1}^{n+1}} d\tau\)).

**Proposition 3.2.** Let \(p \in ]1, \infty[\) and \(-1 - N\inf(\frac{1}{p}, \frac{1}{p}) < s < \frac{N}{p}\), where \(p\) is the conjugate exponent of \(p\). Let \(u_0\) be a divergence free vector field with the components in \(\dot{B}_{p}^{n+1}\) and \(g\) a vector field with the components in \(L_T^1(\dot{B}_{p}^{n+1})\). Let \(u\) and \(v\) be two divergence free vector fields such that \(\nabla v\) has the coefficients in \(L^1(0, T; \dot{B}_{p^*_2}^{n+1} \cap L^\infty)\) (respectively \(L^1(0, T; \dot{B}_{p^*_2}^{n+1})\)) and \(u \in C([0, T]; \dot{B}_{p^*_1}^{n+1} \cap L_T^1(\dot{B}_{p^*_1}^{n+2}))\). Let \(u\) be a solution of the non stationary Stokes system

\[
(L) \quad \begin{cases}
\partial_t u + v \cdot \nabla u - \nu \Delta u + \nabla \Pi = g \\
\text{div} u = 0 \\
u|_{t=0} = u_0.
\end{cases}
\]

Then there exists \(C > 0\) depending on \(N\) and \(s\) such that \(u\) verifies the following estimate

\[
\|u\|_{L_T^\infty(\dot{B}_{p^*_1}^{n+1})} + \nu \|u\|_{L_T^2(\dot{B}_{p^*_2}^{n+2})} + \|\nabla \Pi\|_{L_T^1(\dot{B}_{p^*_1}^{n+1})} \leq e^{C\|\nabla v\|_{L_T^1(\dot{B}_{p^*_1}^{n+1}) \cap L^\infty}} \left\{ \|u_0\|_{\dot{B}_{p^*_1}^{n+1}} + C\|g\|_{L_T^1(\dot{B}_{p^*_1}^{n+1})} \right\}.
\]

Moreover, if \(2 \leq p\) and \(-1 - \frac{N}{p}\), then we have the following estimate:

\[
\|u\|_{L_T^\infty(\dot{B}_{p^*_1}^{n+1})} + \nu \|u\|_{L_T^2(\dot{B}_{p^*_2}^{n+2})} + \|\nabla \Pi\|_{L_T^1(\dot{B}_{p^*_1}^{n+1})} \leq e^{C\|\nabla v\|_{L_T^1(\dot{B}_{p^*_1}^{n+1}) \cap L^\infty}} \left\{ \|u_0\|_{\dot{B}_{p^*_1}^{n+1}} + C\|g\|_{L_T^1(\dot{B}_{p^*_1}^{n+1})} \right\}.
\]
Let us recall the Osgood Lemma (see [13]), which allows us to infer uniqueness of the solution in the critical case (see the uniqueness section).

**Lemma 3.3.** (Osgood)

Let $\rho \geq 0$ be a measurable function, $\gamma$ be a locally integrable function and $\mu$ be a positive, continuous and non decreasing function which verifies the following condition

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty.$$ 

Let also $a$ be a positive real number and let $\rho$ satisfy the inequality

$$\rho(t) \leq a + \int_0^t \gamma(s) \mu(\rho(s)) ds.$$ 

Then if $a$ is equal to zero, the function $\rho$ vanishes. If $a$ is not zero, then we have

$$-M(\rho(t)) + M(a) \leq \int_0^t \gamma(s) ds,$$ 

with

$$M(x) = \frac{1}{\int x dr} \mu(r).$$

Finally, we recall the following result of logarithmic interpolation (see [10] Proposition 2.8).

**Lemma 3.4.** Let $(p, \lambda) \in [1, +\infty]^2$, $s \in \mathbb{R}$, $t \in \mathbb{R}_+$, $\varepsilon \in (0, 1]$ and $u \in \tilde{L}^\lambda_t(\dot{B}^{s+\varepsilon}_{p, \infty}) \cap \tilde{L}^\lambda_t(\dot{B}^s_{p, 1}) \cap \tilde{L}^\lambda_t(\dot{B}^{s+\varepsilon}_{p, \infty})$. Then

$$\|u\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p, 1})} \lesssim \frac{\|u\|_{\tilde{L}^\lambda_t(\dot{B}^{s+\varepsilon}_{p, \infty})}}{\varepsilon} \log \left( 1 + \frac{\|u\|_{\tilde{L}^\lambda_t(\dot{B}^{s+\varepsilon}_{p, \infty})}}{\|u\|_{\tilde{L}^\lambda_t(\dot{B}^s_{p, 1})}} \right).$$

4. **Proof of the Theorem 1.3.**

We will proceed in two steps. First we prove the uniqueness of the solution which is principally based on a logarithmic estimate and on the Osgood Lemma which is useful in the case of logarithmic estimates. The second part is devoted to the proof of existence of the solution.

4.1. **Uniqueness.** Let $1 \leq p_2 \leq 2N$ and $1 < p_1 \leq p_2$ be such that $\frac{1}{p_1} + \frac{1}{p_2} \geq \frac{2}{N}$ and $\frac{1}{p_1} \leq \frac{1}{p_2} + \frac{1}{N}$. We denote by $(a^i, u^i, \nabla \Pi^i)$ for $1 \leq i \leq 2$ two solutions of the (MHD) system. We define

$$(\mathcal{M}^1, \delta \mathcal{M}) := \left( \frac{1}{2} (\nabla u^1 + i \nabla u^1), \mathcal{M}^2 - \mathcal{M}^1 \right)$$

and $$(\delta a, \delta u, \nabla \delta \Pi, \delta B) := (a^2 - a^1, u^2 - u^1, \nabla \Pi^2 - \nabla \Pi^1, B^2 - B^1).$$
We can easily check that
\[
\begin{cases}
\partial_t \delta a + u^2 \cdot \nabla \delta a = -\delta u \cdot \nabla a^1 \\
\partial_t \delta u + u^2 \cdot \nabla \delta u - \mu^1 \Delta \delta u + \nabla \delta \Pi = H(a^i, u^i, \nabla \Pi^i, B^i) \\
\partial_t \delta B + u^2 \cdot \nabla \delta B - \sigma^1 \Delta \delta B = G(a^i, u^i, B^i)
\end{cases}
\]
where
\[
H(a^i, u^i, \nabla \Pi^i, B^i) = -\delta u \cdot \nabla u^1 + a^1(\mu^1 \Delta \delta u - \nabla \delta \Pi) + \delta a(\mu^1 \Delta u^2 - \nabla \Pi^2) \\
+ 2 \text{div} \left[ (\bar{\mu}(a^2) - \mu^1) \delta M \right] + 2 \delta a \text{div} \left[ (\bar{\mu}(a^2) - \mu^1) M^2 \right] + 2 a^1 \text{div} \left[ (\bar{\mu}(a^1) - \mu^1) \delta M \right] \\
+ 2 \text{div} \left[ (\bar{\mu}(a^2) - \bar{\mu}(a^1)) M^1 \right] + 2 a^1 \text{div} \left[ (\bar{\mu}(a^2) - \bar{\mu}(a^1)) M^2 \right] - \frac{1}{2} \delta a \nabla (B^2)^2 \\
- \frac{1}{2} (1 + a^1) \nabla ((B^2)^2 - (B^1)^2) + (1 + a^1)(B^2 \cdot \nabla B + \delta B \cdot \nabla B^1) + \delta a B^2 \cdot \nabla B^2
\]
and
\[
G(a^i, u^i, B^i) = B^2 \cdot \nabla \delta u + \delta B \cdot \nabla u^1 - \delta u \cdot \nabla B^1 + \text{div} \left\{ (\bar{\sigma}(a^2) - \bar{\sigma}(a^1)) \nabla B^2 \right\} \\
+ \text{div} \left\{ (\bar{\sigma}(a^1) - \sigma^1) \nabla \delta B \right\}.
\]

In our discussion we will distinguish between two cases: the first case deals with the situation where \( \frac{1}{p_1} + \frac{1}{p_2} > \frac{2}{n} \) and the second case concerns \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{2}{n} \). The distinction between the two cases appears on the level of the product laws that we use.

**The case where** \( N \geq 3, 1 \leq p_2 < 2N \) and \( \frac{1}{p_1} + \frac{1}{p_2} > \frac{2}{N} \). We have established the following result.

**Proposition 4.1.** Let \((a^i, u^i, \nabla \Pi^i, B^i)\), with \( i \in \{1, 2\} \), be two solutions of \((\overline{\text{MHD}})\) system, corresponding to the same initial data \( a_0 \in \dot{B}_{p_1,1}^{\frac{N}{p_1}} \cap L^{\infty}(\mathbb{R}^N) \), \( u_0, B_0 \in \dot{B}_{p_2}^{\frac{N}{p_2} - 1} \) with \( \text{div} u_0 = \text{div} B_0 = 0 \) and the external forcing term \( f \) belonging to \( L^{1}_{\text{loc}}([0, T^*]; \dot{B}_{p_2}^{\frac{N}{p_2} - 1}) \) such that \( \mathcal{Q}f \) belongs to \( L^{1}_{\text{loc}}([0, T^*]; \dot{B}_{p_2}^{\frac{N}{p_2} + 1}) \). Assume that for \( i = 1, 2 \) we have
\[
\begin{align*}
a^i & \in C([0, T^*]; \dot{B}_{p_1,1}^{\frac{N}{p_1}}(\mathbb{R}^N)), \\
u^i & \in C([0, T^*]; \dot{B}_{p_2}^{\frac{N}{p_2} - 1}) \cap L^{1}_{\text{loc}}([0, T^*]; \dot{B}_{p_2,1}^{\frac{N}{p_2} + 1}), \\
B^i & \in C([0, T^*]; \dot{B}_{p_2}^{\frac{N}{p_2} - 1}) \cap L^{1}_{\text{loc}}([0, T^*]; \dot{B}_{p_2,1}^{\frac{N}{p_2} + 1}), \\
\nabla \Pi^i & \in L^{1}_{\text{loc}}([0, T^*]; \dot{B}_{p_2,1}^{\frac{N}{p_2} - 1}).
\end{align*}
\]
There exists a positive constant $c$ such that if we have
\[ \|a^1\|_{L^\infty_t(B^1_{p_1,\infty} \cap L^\infty)} \leq c, \]
then $(a^2, u^2, \nabla \Pi^2, B^2) = (a^1, u^1, \nabla \Pi^1, B^1)$.

Proof. The first step of the proof consists in proving that $(\delta a, \delta u, \nabla \delta \Pi, \delta B) \in F^p_T$, where
\[ F^p_T : = C([0, T]; \dot{B}^\infty_{p,1}) \times (L^1_T(B^\infty_{p_1,1}) \cap C([0, T]; \dot{B}^\infty_{p_2,1}^{-2})) \times (L^1_T(\dot{B}^\infty_{p_1,1}^{-2})) \times L^1_T(\dot{B}^\infty_{p_2,1}^{-2}) \cap C([0, T]; \dot{B}^\infty_{p_2,1}^{-2}). \]

We define for all $t \leq T$ the quantity
\[ \gamma(t) := \| (\delta a, \delta u, \nabla \delta \Pi, \delta B) \|_{F^p_T} = \| \delta a \|_{L^\infty_t(B^\infty_{p,1})} + \| \delta u \|_{L^\infty_t(B^\infty_{p_1,1})} + \mu^1 \| \delta u \|_{L^1_t(B^\infty_{p_2,1})} + \| \nabla \delta \Pi \|_{L^1_t(B^\infty_{p_2,1})} + \| \delta B \|_{L^1_t(B^\infty_{p_2,1})}. \]

In order to prove that the solution belongs to the space $F^p_T$, it suffices to have $(a^i - a_0, \bar{u}^i, \nabla \Pi^i, \bar{B}^i) \in F^p_T$, where we have defined $(\bar{u}^i, \nabla \Pi^i, \bar{B}^i)$ by $u^i = u_L + \bar{u}^i$, $\nabla \Pi^i = \nabla \Pi^i_L + \nabla \Pi^i$ et $B^i = B_L + \bar{B}^i$. The quantities $u_L, \nabla \Pi_L$ and $B_L$ are defined by the system given bellow:

\[
\begin{aligned}
\partial_t u_L - \mu^1 \Delta u_L + \nabla \Pi_L &= f \\
\partial_t B_L - \sigma^1 \Delta B_L &= 0 \\
div u_L &= div B_L = 0 \\
(u_L, B_L)_{|t=0} &= (u_0, B_0).
\end{aligned}
\]

Indeed, we have by Proposition 2.1 of [6] that $u_L$ and $B_L$ have their components in the space $C([0, T]; \dot{B}^\infty_{p_2,1}^{-1}) \cap L^1(0, T; \dot{B}^\infty_{p_2,1}^{-1})$ and $\nabla \Pi_L \in L^1(0, T; \dot{B}^\infty_{p_2,1}^{-1})$. The quantities $(\bar{u}^i, \nabla \Pi^i, \bar{B}^i)$ verify

\[
(\text{MHD}_{\text{mod}}) \quad \begin{cases}
\partial_t \bar{u}^i - \mu^1 \Delta \bar{u}^i + \nabla \Pi^i = K(a^i, u^i, \nabla \Pi^i, B^i) \\
\partial_t \bar{B}^i - \sigma^1 \Delta \bar{B}^i = L(u^i, B^i) \\
div \bar{u}^i = div \bar{B}^i = 0 \\
(\bar{u}^i, \bar{B}^i)_{|t=0} = (0, 0),
\end{cases}
\]

where
\[
K(a^i, u^i, \nabla \Pi^i, B^i) = -u^i \cdot \nabla u^i + a^i (\mu^1 \Delta u^i - \nabla \Pi^i) + (1 + a^i) \div \left[ (\bar{\mu}(a^i) - \mu^1) \mathcal{M}^i \right] - \frac{1}{2} (1 + a^i) \nabla B^i + (1 + a^i) B^i \cdot \nabla B^i
\]
and
\[ L(u^i, B^i) = B^i \cdot \nabla u^i - u^i \cdot \nabla B^i + \text{div} \left\{ (\tilde{\sigma}(a^i) - \sigma^1) \nabla B^i \right\}. \]

We apply the operator \( \mathcal{P} \) to the first equation of the system \( \text{MHD}_{\text{mod}} \) and we obtain
\[ \partial_t \overline{u}^i - \mu^1 \Delta \overline{u}^i = \mathcal{P} \left( K(a^i, u^i, \nabla \Pi^i, B^i) \right). \]

In the same manner, the divergence operator applied to the same equation gives
\[ \text{div} \left( (1 + a^i) \nabla \Pi^i \right) = \text{div} Qf - \text{div} \left( u^i \cdot \nabla u^i + \frac{1}{2} (1 + a^i) \nabla B^i \cdot (1 + a^i) B^i \cdot \nabla B^i \right) \]
\[ + \text{div} \left( \mu^1 (a^i \Delta u^i) + (1 + a^i) \text{div} \left( (\tilde{\mu}(a^i) - \mu^1) M^i \right) \right). \]

Combining the inequality (3) together with the hypothesis concerning the solutions stated at the beginning, we find \( u^i, B^i \in L^2_T(B^{rac{N}{p_2}}_{p_2,1}). \) On the other hand the inequality (6) gives that \( u^i \otimes u^i, B^i \otimes B^i \) and \( B^i \in L^2_T(B^{rac{N}{p_2},-2}) \) for \( p_2 < 2N, N \geq 3 \) and \( \frac{1}{p_1} + \frac{1}{p_2} > \frac{2}{N}. \) Inequality (4) then implies that
\[ u^i \cdot \nabla u^i, a^i \Delta u^i \in L^2_T(B^{rac{N}{p_2}}_{p_2,1}), (1 + a^i) B^i \cdot \nabla B^i \text{ and } (1 + a^i) \nabla B^i \in L^2_T(B^{rac{N}{p_2},-2}). \]

Now the inequality (4) and Taylor’s formula with a remainder in the integral form imply
\[ \left\| (1 + a^i) \text{div} \left[ (\tilde{\mu}(a^i) - \mu^1) M^i \right] \right\|_{L^2_T(B^{rac{N}{p_2},-2})} \lesssim (1 + \| a^i \|_{L^\infty_T(B^{rac{N}{p_1},\infty} \cap L^\infty)}) \]
\[ \cdot \left\| (\tilde{\mu}(a^i) - \mu^1) M^i \right\|_{L^2_T(B^{rac{N}{p_2},-1})} \lesssim \| a^i \|_{L^\infty_T(B^{rac{N}{p_1},\infty} \cap L^\infty)} \| u^i \|_{L^2_T(B^{rac{N}{p_2},1})}. \]

We conclude also that the left-hand side term of equality (\ref{eq:11}) belongs to \( L^2_T(B^{rac{N}{p_2},-2}). \)

On the other hand, inequality (\ref{eq:11}) gives
\[ \| a^i \nabla \Pi^i \|_{L^2_T(B^{rac{N}{p_2},-2})} \leq \| a^i \|_{L^\infty_T(B^{rac{N}{p_1},\infty} \cap L^\infty)} \| \nabla \Pi^i \|_{L^2_T(B^{rac{N}{p_2},1})}. \]

Consequently, the smallness condition on \( a^i \) together with (\ref{eq:11}) give that
\( \nabla \Pi^i \in L^2_T(B^{rac{N}{p_2},-2}). \) This allows us to obtain, using the hypothesis concerning \( a^i \) and the inequality (4), that \( a^i \nabla \Pi^i \in L^1_T(B^{rac{N}{p_2},-2}). \) So we conclude that \( K(a^i, u^i, \nabla \Pi^i, B^i) \) belongs to \( L^1_T(B^{rac{N}{p_2},-2}). \) In the similar manner we have \( L(u^i, B^i) \in L^1_T(B^{rac{N}{p_2},-2}). \) Since the operator \( \mathcal{P} \) is continuous on the spaces \( B^a_{p,r}, \) the terms at the left-hand side
of equality (10) belong to $L^1_T(\dot{B}^{N-2}_{p_2,1})$. Consequently, applying Proposition 2.1 of [6], we obtain that $\bar{\nabla}^i \bar{U} \in L^1_T(\dot{B}^{N-2}_{p_2,1}) \cap C([0,T]; \dot{B}^{N-2}_{p_2,1})$ and $\nabla \bar{\Pi}^i \in L^1_T(\dot{B}^{N-2}_{p_2,1})$. For $a^i$, we write $\partial_t a^i = -u^i \cdot \nabla a^i$. Since $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p_2}$, therefore, the product laws (1) allow us to see that $\partial_t a^i$ belongs to $L^1_T(\dot{B}^{\frac{N}{p_1}-1}_{p_1,1})$, which gives by the inequality of Cauchy-Schwarz that $(a^i - a_0) \in C([0,T]; \dot{B}^{\frac{N}{p_1}-1}_{p_1,1})$. Finally we have $(\delta a, \delta u, \nabla \delta \Pi, \delta B) \in F^p_2$.

Using these Propositions 3.1 and 3.2 we prove successively that for all $t \leq T$

$$\|\delta a\|_{L^1_T(\dot{B}^{\frac{N}{p_1}-1}_{p_1,1})} \lesssim \frac{C\|u\|}{L^1_T(\dot{B}^{\frac{\infty}{p_1}-1}_{p_1,1})} \|\delta u\|_{L^1_T(\dot{B}^{\frac{\infty}{p_1}-1}_{p_1,1})} \|\nabla a^1\|_{L^1_T(\dot{B}^{\frac{\infty}{p_1}-1}_{p_1,1})},$$

$$\|\delta u\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} + \mu^1 \|\delta u\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} + \|\nabla \delta \Pi\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} \lesssim \frac{C\|u\|}{L^1_T(\dot{B}^{\frac{\infty}{p_2}-1}_{p_2,1})} \|G(a^i, u^i, B^i)\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})},$$

and

$$\|\delta B\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} + \sigma^1 \|\delta B\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} \lesssim \frac{C\|u\|}{L^1_T(\dot{B}^{\frac{\infty}{p_2}-1}_{p_2,1})} \|G(a^i, u^i, B^i)\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})}.$$ We will estimate next the term $H(a^i, u^i, \nabla \Pi^i, B^i)$. Inequalities (6) and (1) give

$$\left\| - \delta u \cdot \nabla u^i + a^1(\mu^1 \Delta \delta u - \nabla \delta \Pi) + \delta a(\mu^1 \Delta u^2 - \nabla \Pi^2) \right\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} \lesssim \|\delta u\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-1}_{p_2,1})},$$

$$\times \|u^1\|_{L^2_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})} + \|a^1\|_{L^2_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})} \left( \|\Delta \delta u\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} + \|\nabla \delta \Pi\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}-2}_{p_2,1})} \right)$$

$$+ \|\delta a\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})} \left( \|\Delta u^2\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})} + \|\Pi^2\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})} \right).$$

Owing to (4) and Taylor’s formula with a remainder in the integral form, one finds

$$\left\| \text{div} \left[ (\bar{\mu}(a^1) - \mu^1) \delta M \right] + a^1 \text{div} \left[ (\bar{\mu}(a^1) - \mu^1) \delta M \right] \right\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})} \lesssim \|a^1\|_{L^\infty_T(\dot{B}^{\frac{\infty}{p_1}1}_{p_1,1})} \|\delta u\|_{L^1_T(\dot{B}^{\frac{\infty}{p_2}1}_{p_2,1})}$$

for $p_1 \leq p_2$. Using once more the inequality (4), Taylor’s formula, inequality (6), and the fact that the space of Besov is stable by the action of a $C^\infty$-function (see
for example [16]), one obtains

$$\left\| \text{div} \left[ (\tilde{\mu}(a^2) - \bar{\mu}(a^1))M^2 \right] \right\|_{L^1_T(B_{p_2,1}^{\infty, -2})} \lesssim \int_0^T \| \bar{\mu}(a^2) - \bar{\mu}(a^1) \|_{B_{p_1,1}^{\infty, -1}} \| \nabla u^2 \|_{B_{p_2,1}^{\infty}} \, dt$$

$$\lesssim \int_0^T \| \delta a \|_{B_{p_1,1}^{\infty, -1}} \| u^2 \|_{B_{p_2,1}^{\infty, +1}} \, dt.$$  

Combining the inequality (1) together with an interpolation result in the temporal variable, we prove that

$$\left\| \delta a \nabla (B^2)^2 \right\|_{L^1_T(B_{p_2,1}^{\infty, -2})} \lesssim \| \delta a \|_{L^\infty_T(B_{p_1,1}^{\infty, -1})} \| (B^2)^2 \|_{L^1_T(B_{p_2,1}^{\infty})}$$

$$\lesssim \| \delta a \|_{L^\infty_T(B_{p_1,1}^{\infty, -1})} \| B^2 \|_{L^1_T(B_{p_2,1}^{\infty, +1})} \| B^2 \|_{L^1_T(B_{p_2,1}^{\infty, +1})}.$$  

In the same manner we find

$$\left\| (1 + a^1) \nabla ((B^2)^2 - (B^1)^2) \right\|_{L^1_T(B_{p_2,1}^{\infty, -2})} \lesssim \left( \| B^1 \|_{L^\infty_T(B_{p_1,1}^{\infty, -1})} + \| B^2 \|_{L^\infty_T(B_{p_2,1}^{\infty, +1})} \right) \| B^2 \|_{L^1_T(B_{p_2,1}^{\infty, -1})}$$

$$\lesssim \sum_{i=1}^2 \| B^i \|_{L^\infty_T(B_{p_2,1}^{\infty, -2})} \left( \| B^i \|_{L^\infty_T(B_{p_2,1}^{\infty, -2})} + \| \delta B \|_{L^1_T(B_{p_2,1}^{\infty})} \right).$$

We have \(\frac{1}{p_1} + \frac{1}{p_2} > \frac{2}{N}, p_2 < 2N\) and \(\frac{1}{p_1} - \frac{1}{p_2} \leq \frac{1}{N}\), so the inequalities (4) and (6) imply

$$\| \delta a B^2 \cdot \nabla B^2 \|_{L^1_T(B_{p_2,1}^{\infty, -2})} \lesssim \| \delta a \|_{L^\infty_T(B_{p_1,1}^{\infty, -1})} \| B^2 \|_{L^\infty_T(B_{p_2,1}^{\infty, -1})} \| B^2 \|_{L^1_T(B_{p_2,1}^{\infty, +1})}.$$  

Since one has \(\frac{1}{p_1} - \frac{1}{p_2} < \frac{1}{N}\), \(\frac{1}{p_1} + \frac{1}{p_2} > \frac{2}{N}\) and \(p_1 \leq p_2\), Lemmas 3.1 and 3.2 of [1] remain valid. Thus, combining the preceding inequalities with these Lemmas, we find

$$\| H(a^i, u^i, \nabla \Pi^i, B^i) \|_{L^1_T(B_{p_2,1}^{\infty, -2})} \lesssim \gamma(t) \left\{ \| (u^1, u^2) \|_{L^1_T(B_{p_2,1}^{\infty, -1})} + \| \nabla \Pi^2 \|_{L^1_T(B_{p_2,1}^{\infty, +1})} \right\}$$

$$+ \| (B^1, B^2) \|_{L^1_T(B_{p_2,1}^{\infty, +1})} + \| (B^1, B^2) \|_{L^1_T(B_{p_2,1}^{\infty, +1})} + \| a^1 \|_{L^\infty_T(B_{p_1,1}^{\infty, -1} \cap L^\infty)}$$

$$+ \int_0^T \| \delta a(t) \|_{B_{p_1,1}^{\infty, -1}} \| u^2 \|_{B_{p_2,1}^{\infty, +1}} \, dt.$$
We need now to estimate $G(a^i, u^i, B^i)$. Since $\text{div} \, B^2 = 0$, then using the inequalities of Bernstein and (3) together with an interpolation argument we obtain
\[
\| B^2 \cdot \nabla \delta u \|_{L^1_t(B^{N-2}_{p,2,1})} \lesssim \| B^2 \otimes \delta u \|_{L^1_t(B^{N}_{p,2,1}^{-1})}
\lesssim \| B^2 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} \| \delta u \|_{L^\infty_t(B^{N}_{p,2,1})}
\lesssim \| B^2 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} \left( \| \delta u \|_{L^\infty_t(B^{N}_{p,2,1})} + \| \delta u \|_{L^1_t(B^{N}_{p,2,1})} \right).
\]

In the same manner, we have
\[
\| \delta B \cdot \nabla u^1 - \delta u \cdot \nabla B^1 \|_{L^1_t(B^{N-2}_{p,2,1})} \lesssim \| u^1 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} \left( \| \delta B \|_{L^\infty_t(B^{N}_{p,2,1})} + \| \delta B \|_{L^1_t(B^{N}_{p,2,1})} \right) + \| B^1 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} \left( \| \delta u \|_{L^\infty_t(B^{N}_{p,2,1})} + \| \delta u \|_{L^1_t(B^{N}_{p,2,1})} \right).
\]

Arguing similarly to the case of inequality (12), one finds that
\[
\left\| \text{div} \left( (\tilde{\sigma}(a^1) - \sigma^1) \nabla \delta B \right) \right\|_{L^1_t(B^{N-2}_{p,2,1})} \lesssim \| a^1 \|_{L^\infty_{t,\infty} \cap L^\infty_t} \| \delta B \|_{L^1_t(B^{N}_{p,2,1})}.
\]

Using the above estimates and Lemmas 3.1, 3.2 of [1], and arguing in the same manner as for the $H$ term, we obtain finally that
\[
\| G(a^i, u^i, \nabla \Pi^i, B^i) \|_{L^1_t(B^{N-2}_{p,2,1})} \lesssim \gamma(t) \left( \| (u^1, u^2) \|_{L^1_t(B^{N-1}_{p,2,1}) \cap L^2_t(B^{N}_{p,2,1})} + \| \nabla \Pi^2 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} + \| B^1, B^2 \|_{L^1_t(B^{N+1}_{p,2,1}) \cap L^2_t(B^{N}_{p,2,1})} + \| a^1 \|_{L^\infty_{t,\infty} \cap L^\infty_t} \right) + \int_0^T \| \delta a(t) \|_{B^{N-1}_{p,1}} \| B^2 \|_{B^{N+1}_{p,2,1}} dt.
\]

Thus, one finds for $t \leq T$ that
\[
\gamma(t) \lesssim \gamma(t) \left( \| (u^1, u^2) \|_{L^1_t(B^{N+1}_{p,2,1}) \cap L^2_t(B^{N}_{p,2,1})} + \| \nabla \Pi^2 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} + \| a^1 \|_{L^\infty_{t,\infty} \cap L^\infty_t} \right) + \int_0^T \gamma(t) \| (u^2, B^2) \|_{B^{N+1}_{p,2,1}} dt.
\]

We choose a small time $T_1 \leq T$ such that we have for a constant $c > 0$ small enough the following inequality
\[
\| (u^1, u^2) \|_{L^1_t(B^{N+1}_{p,2,1}) \cap L^2_t(B^{N}_{p,2,1})} + \| \nabla \Pi^2 \|_{L^1_t(B^{N}_{p,2,1}^{-1})} \leq c
\]
and
\[
\| (B^1, B^2) \|_{L^1_t(B^{N+1}_{p,2,1}) \cap L^2_t(B^{N}_{p,2,1})} \leq c.
\]
Using the assumption that \( \|a^1\|_{L^\infty_x(B^{p_1}_{p_1,1})} \leq c \), we have \( \forall t \leq T_1 \)
\[
\gamma(t) \leq C \int_0^t \gamma(t) \|u^2, B^2\|_{B^{p_2}_{p_2,1}} dt.
\]
Since the function \( t \mapsto \|u^2\|_{B^{p_2}_{p_2,1}} + \|B^2\|_{B^{p_2}_{p_2,1}} \) is locally integrable, we deduce by Lemma 3.3 that \( \gamma(t) = 0 \) for all \( t \in [0, T_1] \). It is easy to see that this property is conserved on the whole time interval and we obtain finally that \( \gamma(t) = 0 \) for all \( t \in [0, T] \). Thus the proof is complete in the case \( p_2 < 2N \). The above calculations are available for \( p \neq 1 \) (since they are based on Proposition 3.2). The case \( p = 1 \) is deduced by injection. \( \square \)

The case \( \frac{1}{p_1} + \frac{1}{p_2} = \frac{2}{N} \) or \( N = 2 \) or \( p_2 = 2N \). In this case the condition
\[
\|a^1\|_{L^\infty_x(B^{p_1}_{p_1,1})} \leq c \]

is not sufficient. To show uniqueness, one needs to suppose that \( \|a^1\|_{L^\infty_x(B^{p_1}_{p_1,1})} \leq c \). More precisely, we have the following proposition.

**Proposition 4.2.** Let \( (a^1, u^1, \nabla \Pi, B^1) \) and \( (a^2, u^2, \nabla \Pi^2, B^2) \) be two solutions of the (MHD) corresponding to the initial data \( a_0 \in B^{p_1}_{p_1,1} \), \( u_0 \in B^{N}_{p_2,1} \) where \( \text{div} \, u_0 = \text{div} \, B_0 = 0 \) and \( f \) is such that its components are in \( L^1_{\text{loc}}([0, T^*]; B^{N}_{p_2,1}) \) and \( Qf \) belongs to \( L^1_{\text{loc}}([0, T^*]; B^{N}_{p_2,1}) \). We assume that for \( i = 1, 2 \) we have
\[
\begin{align*}
a^i &\in C([0, T^*]; L^\infty_{\text{loc}}([0, T^*]; B^{p_1}_{p_1,1})), \\
u^i &\in C([0, T^*]; B^{p_2,1}_{N,1}) \cap L^1_{\text{loc}}([0, T^*]; B^{N+1}_{p_2,1}), \\
\nabla \Pi^i &\in C([0, T^*]; B^{p_2,1}_{N,1}) \cap L^1_{\text{loc}}([0, T^*]; B^{N+1}_{p_2,1}),
\end{align*}
\]

\( \nabla \Pi^i \subseteq L^1_{\text{loc}}([0, T^*]; B^{N}_{p_2,1}) \).

Then there exists a positive constant \( c \) which does not depend on these solutions such that the inequality
\[
\|a^1\|_{L^\infty_x(B^{p_1}_{p_1,1})} \leq c
\]
implies \( (a^2, u^2, \nabla \Pi^2, B^2) = (a^1, u^1, \nabla \Pi^1, B^1) \).

**Proof.** We need to prove first that \( (\delta a, \delta u, \nabla \delta \Pi, \delta B) \in G_T \), where
\[
G_T := L^\infty_{\text{loc}}(B^{p_1}_{p_1,1}) \times \tilde{L}^1_{T}(B^{p_2}_{p_2,1}) \cap L^\infty_{\text{loc}}(B^{N}_{p_2,1}) \times L^1_{T}(B^{N+1}_{p_2,1}) \times L^1_{T}(B^{N}_{p_2,1}) \cap L^\infty_{\text{loc}}(B^{N}_{p_2,1}).
\]
The estimates in this space would allow us to obtain uniqueness of the solution by the Osgood Lemma. We define

\[
\gamma(t) := \|\delta u\|_{L^p_t\left(\mathcal{B}^{-1/2\cdot\frac{N}{p_2}}_{p_2,\infty}\right)} + \|\delta u\|_{\tilde{L}^1_t\left(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty}\right)} + \|\nabla \delta \Pi\|_{\tilde{L}^1_t\left(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty}\right)} + \|\delta B\|_{L^p_t\left(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty}\right)} + \|\delta B\|_{\tilde{L}^1_t\left(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty}\right)}.
\]

The term $G_T$ is dealt with in the same way as in the first case. The only difference to be noted appears in the treatment of the products of the type $a^i \nabla \Pi^i$. Here inequality \((5)\) should be used to ensure that the left-hand side term of equality \((10)\) belongs to $L^2_T(\mathcal{B}^{-1/2}_{3,\infty})$. Thus Proposition 2.1 of \([6]\) implies that $(\delta a, \delta u, \nabla \delta \Pi, \delta B) \in G_T$.

In this case it is enough to study the case $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$, since one can deduce the other cases from this one. Indeed, if $p_2 = 2N$, then $p_1 = \frac{2N}{3}$, since $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p_2}$ and $\frac{2}{N} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Therefore it is a particular case of $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$. For $N = 2$, one starts with $p_2 = 4$ and $p_1 = \frac{4}{3}$. Afterwards by injection, one will have uniqueness for $1 \leq p_1 \leq \frac{4}{3}$ and $1 \leq p_2 \leq 4$, the same for $1 \leq p_1 \leq 4$ and $1 \leq p_2 \leq \frac{4}{3}$. Hence one can suppose that $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p_2}$.

Moreover, one can suppose that $p_2 \geq 2$ since inequality \((5)\) is valid for $p \geq 2$. The case $p_2 \leq 2$ follows by injection. Using Propositions \([3.1]\) and \([3.2]\), we have

\[
\|\delta a\|_{L^\infty_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})} \leq \frac{C\|\nabla u^2\|_{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})}}{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})} \|\delta u\|_{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})},
\]

\[
\|\delta u\|_{L^\infty_t(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty})} + \mu^1 \|\delta u\|_{\tilde{L}^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})} + \|\nabla \delta \Pi\|_{\tilde{L}^1_t(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty})} \leq Ce^{\frac{C\|\nabla u^2\|_{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})}}{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})}} \times \|H(a^i, u^i, \nabla \Pi^i, B^i)\|_{L^1_t(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty})},
\]

and

\[
\|\delta B\|_{L^\infty_t(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty})} + \sigma^1 \|\delta u\|_{\tilde{L}^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})} \leq Ce^{\frac{C\|\nabla u^2\|_{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})}}{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})}} \|G(a^i, u^i, B^i)\|_{L^1_t(\mathcal{B}^{-2+\frac{N}{p_2}}_{p_2,\infty})}.
\]

Combining the estimates of $\delta a$, inequality \((4)\), the Bernstein and Minkowski inequalities, we obtain

\[
\|\delta u \cdot \nabla a^1\|_{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})} \leq \|\delta u\|_{L^1_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})} \|a^1\|_{L^\infty_t(\mathcal{B}^{\frac{N}{p_2}}_{p_2,\infty})}.
\]
By Lemma 3.4 one has

$$\|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \lesssim \|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \log \left( e + \frac{\|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}}{\|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}} \right)$$

$$\lesssim \|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \log \left( e + \frac{\|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}}{\|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}} \right).$$

We will now estimate the term $H(a^i, u^i, \nabla \Pi^i, B^i)$. Since $\text{div} \, \delta u = 0$ the inequalities of Bernstein, (6) for $p_2 < 2N$ and (5) for $(p_2 = 2N)$ imply

$$\|\delta u \cdot \nabla u^1\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \lesssim \|\delta u \otimes u^1\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}$$

$$\lesssim \|u^1\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}$$

$$\lesssim \|u^1\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \left( \|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} + \|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \right).$$

Since $p_1 \leq p_2, \frac{2}{p} \leq \frac{1}{p_1} + \frac{1}{p_2} \leq 1$, thanks to inequalities (6) and using the Bernstein inequality, we have

$$\left\| a^1 \left( \mu^1 \Delta u - \nabla \Pi \right) + \delta a \left( \mu^1 \Delta u^2 - \nabla \Pi^2 \right) \right\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \lesssim \left\| a^1 \right\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})}$$

$$\times \left( \|\delta u\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} + \|\nabla \Pi\|_{L^1_t(B^{1+\frac{N}{p^2}}_{p^2,\infty})} \right)$$

$$+ \int_0^t \left\| \delta a \right\|_{B^{1+\frac{N}{p_1}}_{p_1,\infty}} \left( \|u^2\|_{B^{1+\frac{N}{p_2}}_{p_2,\infty}} + \|\nabla \Pi^2\|_{B^{1+\frac{N}{p_2}}_{p_2,\infty}} \right) d\tau.$$
This and the inequality of Minkowski, \(^{(5)}\), the Bernstein inequality, \(^{(4)}\), Taylor’s formula and \(^{(6)}\) give

\[
\left\| a^1 \text{div}\left[ (\tilde{\mu}(a^2) - \tilde{\mu}(a^1)) \mathcal{M} \right] \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{-2+\frac{N}{p_2}})} \lesssim \int_0^t \left\| a^1 \right\|_{\dot{B}_{p_1,1}^{\frac{N}{p_1}}} \left\| (\tilde{\mu}(a^2) - \tilde{\mu}(a^1)) \mathcal{M} \right\|_{\dot{B}_{p_2,\infty}^{-1+\frac{N}{p_2}}} \, d\tau
\]

\[
(19)
\]

In the same manner we obtain the following estimates

\[
\left\| a^1 \text{div}\left[ (\tilde{\mu}(a^1) - \mu^1) \delta \mathcal{M} \right] \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{-2+\frac{N}{p_2}})} \lesssim \left\| a^1 \right\|_{L_t^\infty(\dot{B}_{p_1,1}^{\frac{N}{p_1}})} \left\| (\tilde{\mu}(a^1) - \mu^1) \delta \mathcal{M} \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{-1+\frac{N}{p_2}})} \lesssim \left\| a^1 \right\|_{L_t^\infty(\dot{B}_{p_1,1}^{\frac{N}{p_1}})} \left\| \delta u \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{\frac{N}{p_2}})}
\]

and

\[
(20) \quad \left\| \text{div}\left[ (\tilde{\mu}(a^2) - \tilde{\mu}(a^1)) \mathcal{M} \right] \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{-2+\frac{N}{p_2}})} \lesssim \int_0^t \left\| \delta a \right\|_{B_{p,1}^{-1+\frac{N}{p}}} \left\| u^2 \right\|_{B_{p_2,1}^{-1+\frac{N}{p}}} \, d\tau.
\]

Using the Minkowski inequality, \(^{(4)}\), the fact that \(\dot{B}_{p_2,1}^{\frac{N}{p}}\) is an algebra, and interpolation, we obtain

\[
\left\| \delta a \nabla (B^2)^2 \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{-2+\frac{N}{p_2}})} \lesssim \int_0^t \left\| \delta a \right\|_{B_{p_1,\infty}^{-1+\frac{N}{p}}} \left\| B^2 \right\|_{\dot{B}_{p_2,1}^{+\frac{N}{p}}} \, d\tau
\]

\[
\lesssim \int_0^t \left\| \delta a \right\|_{B_{p_1,\infty}^{-1+\frac{N}{p}}} \left\| B^2 \right\|_{B_{p_2,1}^{-1+\frac{N}{p}}} \left\| B^2 \right\|_{B_{p_2,1}^{+\frac{N}{p}}} \, d\tau.
\]

Since \(\text{div} B^2 = 0\), in an analogous manner, we obtain

\[
\left\| \delta a B^2 \cdot \nabla B^2 \right\|_{L_t^1(\dot{B}_{p_2,\infty}^{-2+\frac{N}{p_2}})} \lesssim \int_0^t \left\| \delta a \right\|_{B_{p_1,\infty}^{-1+\frac{N}{p}}} \left\| B^2 \right\|_{B_{p_2,1}^{-1+\frac{N}{p}}} \left\| B^2 \right\|_{B_{p_2,1}^{+\frac{N}{p}}} \, d\tau.
\]
Thanks to inequalities (5), (6) and a classical interpolation argument, we can write

$$\left\| (1 + a^1) \nabla (B^2) - (B^1)^2 \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \lesssim \left( 1 + \left\| a^1 \right\|_{L^\infty_t(B_{p_1, 1}^{N, N})} \right) \left\{ (B^2)^2 - (B^1)^2 \right\}_{L^1_t(B_{p_2, \infty}^{N, N})}$$

$$\lesssim 2 \left\| B^i \right\|_{L^2_t(B_{p_2, 1}^{N, N})} \left\| \delta B \right\|_{L^2_t(B_{p_2, \infty}^{N, N})}$$

$$\lesssim 2 \left\| B^i \right\|_{L^2_t(B_{p_2, 1}^{N, N})} \left( \left\| \delta B \right\|_{L^\infty_t(B_{p_2, \infty}^{N, N})} + \left\| \delta B \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \right).$$

Since \( \text{div} \delta B = \text{div} B^2 = 0 \), one will have in the same way

$$\left\| (1 + a^1) (\delta B \cdot \nabla B^1 + B^2 \cdot \nabla \delta B) \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \lesssim \sum_{i=1}^2 \left\| B^i \right\|_{L^2_t(B_{p_2, 1}^{N, N})} \left( \left\| \delta B \right\|_{L^\infty_t(B_{p_2, \infty}^{N, N})} + \left\| \delta B \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \right).$$

Combining all these estimates, we are able to establish

$$\left\| H(a^i, u^i, \nabla \Pi^i, B^i) \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \lesssim \gamma(t) \left( \left\| (u^1, u^2) \right\|_{L^1_t(B_{p_2, 1}^{N, N}) \cap L^2_t(B_{p_2, 1}^{N, N})} + \left\| a^1 \right\|_{L^\infty_t(B_{p_1, 1}^{N, N})} + \left\| (B^1, B^2) \right\|_{L^1_t(B_{p_2, 1}^{N, N}) \cap L^2_t(B_{p_2, 1}^{N, N})} \right),$$

(21)

$$+ \int_0^t \left\| \delta a \right\|_{L^{1+1, N}_{p_1, \infty}} g(\tau) d\tau,$$

where \( g \) is a locally integrable function.

We give now the estimates for \( G \). Using the Bernstein inequality and (10) for \( p_2 < 2N \), (5) for \( p_2 = 2N \), we obtain by interpolation

$$\left\| B^2 \cdot \nabla \delta u + \delta B \cdot \nabla u^1 - \delta u \cdot \nabla B^1 \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \lesssim \left\| B^2 \otimes \delta u + \delta B \otimes u^1 - \delta u \otimes B^1 \right\|_{L^1_t(B_{p_2, \infty}^{N, N})}$$

$$\lesssim \left\| (B^1, B^2) \right\|_{L^2_t(B_{p_2, 1}^{N, N})} \left\| \delta u \right\|_{L^2_t(B_{p_2, \infty}^{N, N})} + \left\| \delta B \right\|_{L^2_t(B_{p_2, \infty}^{N, N})} \left\| u^1 \right\|_{L^2_t(B_{p_2, 1}^{N, N})}$$

$$\lesssim \left\| (B^1, B^2, u^1) \right\|_{L^2_t(B_{p_2, 1}^{N, N})} \times \left( \left\| \delta B \right\|_{L^\infty_t(B_{p_2, \infty}^{N, N})} + \left\| \delta B \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} + \left\| \delta u \right\|_{L^\infty_t(B_{p_2, \infty}^{N, N})} + \left\| \delta u \right\|_{L^1_t(B_{p_2, \infty}^{N, N})} \right).$$
We obtain identically to (20) and (18) that
\[
\| \text{div} \left[ (\tilde{\sigma}(a^2) - \tilde{\sigma}(a^1)) \nabla B^2 \right] \|_{L^1_t(B^{-2+\frac{N}{p_2}}_{p_1,\infty})} \lesssim \int_0^t \| \delta a \|_{B^{-1+\frac{N}{p_1}}_{p_1,\infty}} \| B^2 \|_{B^{1+\frac{N}{p_2}}_{p_1,1}} d\tau
\]
and
\[
\| \text{div} \left[ (\tilde{\sigma}(a^1) - \sigma^1) \nabla \delta B \right] \|_{L^1_t(B^{-2+\frac{N}{p_2}}_{p_1,\infty})} \lesssim \| a^1 \|_{L^\infty_t(B^{-\frac{N}{p_1}}_{p_1,1})} \| \delta B \|_{L^1_t(B^{-\frac{N}{p_2}}_{p_2,\infty})}.
\]

We deduce from these estimates that
\[
\| G(a^i, u^i, B^i) \|_{L^1_t(B^{-2+\frac{N}{p_2}}_{p_2,\infty})} \lesssim \gamma(t) \left( \left\| (u^1, B^1, B^2) \right\|_{L^1_t(B^{1+\frac{N}{p_2}}_{p_1,1}) \cap L^\infty_t(B^{\frac{N}{p_2}}_{p_1,1})} + \| a^1 \|_{L^\infty_t(B^{\frac{N}{p_1}}_{p_1,1})} \right)
+ \int_0^t \| \delta a \|_{B^{-1+\frac{N}{p_1}}_{p_1,\infty}} \| B^2 \|_{B^{1+\frac{N}{p_2}}_{p_1,1}} d\tau.
\]

Using the above estimate together with those given by (21), we have
\[
\gamma(t) \lesssim \gamma(t) \left( \left\| (u^1, u^2, B^1, B^2) \right\|_{L^1_t(B^{1+\frac{N}{p_2}}_{p_1,1}) \cap L^\infty_t(B^{\frac{N}{p_2}}_{p_1,1})} + \| a^1 \|_{L^\infty_t(B^{\frac{N}{p_1}}_{p_1,1})} \right)
+ \int_0^t \| \delta a \|_{B^{-1+\frac{N}{p_1}}_{p_1,\infty}} \| B^2 \|_{B^{1+\frac{N}{p_2}}_{p_1,1}} d\tau.
\]

Using the above estimate, we may choose a sufficiently small time $T_1$ so that using inequalities (13), (16), (17) and the smallness of $a^1$, we obtain for all $t \in [0, T_1]$
\[
\gamma(t) \lesssim \int_0^t \log \left( e + \frac{\alpha(T)}{\| \delta u \|_{L^\infty_t(B^{-\frac{N}{p_2}}_{p_2,\infty})}} \right) \| \delta u \|_{L^1_t(B^{-\frac{N}{p_2}}_{p_2,\infty})} g(\tau) d\tau,
\]
with $\alpha(T) = \sum_{i=1}^2 T \left\| u^i \right\|_{L^\infty_t(B^{-1+\frac{N}{p_2}}_{p_2,1})} + \| u^1 \|_{L^1_t(B^{1+\frac{N}{p_2}}_{p_2,1})}$. Owing to the fact that $x \mapsto x \log(e + \frac{\alpha(x)}{x})$ is an increasing function on $\mathbb{R}_+$, we have for all $t \in [0, T_1]$
\[
\gamma(t) \lesssim \int_0^t \gamma(\tau) \log \left( e + \frac{\alpha(T)}{\gamma(\tau)} \right) g(\tau) d\tau.
\]

So by Lemma 3.3, we deduce that that $\gamma(t) = 0$, for all $t \in [0, T_1]$. This gives by inequality (13), that $\delta a = 0$. Standard arguments now yield the required conclusion. We note that the method used in this section (the logarithmic interpolation argument and the application of the Osgood lemma) is inspired by the proofs of the uniqueness given by Danchin [?] and was used by the authors in [3].
4.2. **Existence.** Throughout this section we assume that \( p_1 \leq p_2, \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{N} \) and \( \frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p_2} \).

The proof of existence of a solution is performed in a standard manner. We begin by solving an approximate problem and we prove that the solutions are uniformly bounded. The last step consists in studying the convergence to a solution of the initial equation.

**Construction of a regular approximate solution.** Let us recall first the following result (see [1], Lemma 4.2).

**Lemma 4.3.** Assume that \( s_i \in \mathbb{R} \) and \( (p_i, r_i) \in [1, \infty]^2 \) for \( i = 1, 2 \). Let \( G \in \dot{B}_p^{s_1}(\mathbb{R}^N) \). Then there exists \( G^n \in H^\infty(\mathbb{R}^N) \), such that for all \( \varepsilon > 0 \) there is \( n_0 \) such that

\[
\|G^n - G\|_{\dot{B}_p^{s_1}} \leq \varepsilon \quad \forall \ n \geq n_0.
\]

If we have \( \text{div} \ G = 0 \) and \( \mathcal{Q}G \in \dot{B}_p^{s_2}(\mathbb{R}^N) \), then we can choose \( G^n \) such that \( \text{div} \ G^n = 0 \) and \( \mathcal{Q}G^n \) is uniformly bounded with respect to \( n \) in the space \( \dot{B}_p^{s_2}(\mathbb{R}^N) \).

Owing to the above Lemma there exist \( a^n_0, u^n_0, b^n_0 \in H^\infty(\mathbb{R}^N) \) and \( f^n \in L^1_{\text{loc}}(\mathbb{R}^+; H^\infty(\mathbb{R}^N)) \) such that we have

\[
\|a^n_0\|_{L^\infty} \lesssim \|a_0\|_{L^\infty}, \quad \text{div} \ u^n_0 = \text{div} \ b^n_0 = 0
\]

and

\[
\|\mathcal{Q}f^n\|_{L^1_{\text{loc}}(\mathbb{R}^+; \dot{B}_p^{s_2})} \lesssim \|\mathcal{Q}f\|_{L^1_{\text{loc}}(\mathbb{R}^+; \dot{B}_p^{s_2})}.
\]

Now, owing to [2, Theorem 1.1], we deduce that system (MHD) with the initial data \((a^n_0, u^n_0, b^n_0, f^n)\) admits a unique local in time solution \((a^n, u^n, \nabla \Pi^n, b^n)\) verifying

\[
a^n \in C([0, T^n); H^{s+1}(\mathbb{R}^N)), \quad u^n, b^n \in C([0, T^n); H^s(\mathbb{R}^N)) \cap \dot{L}^1_{T^n}(H^{s+2})
\]

and

\[
\nabla \Pi^n \in L^1([0, T^n); H^s(\mathbb{R}^N)) \quad \text{with} \quad s > \frac{N}{2} - 1.
\]

**Estimates of the regularized solution.** Let \( T \in [0, +\infty] \) be defined as \( \inf_{n \in \mathbb{N}} T^n \). Our first goal is to prove that \( T > 0 \) such that \((a^n, u^n, \nabla \Pi^n, b^n)\) belongs to and is uniformly bounded in the space

\[
E_T = \left( \dot{L}^\infty_T(\dot{B}^{s_1}_{p_1, 1}) \right) \times \left( L^1_T(\dot{B}^{s+1}_{p_2, 1}) \cap \dot{L}^\infty_T(\dot{B}^{-1}_{p_2, 1}) \right) \times L^1_T(\dot{B}^{-1}_{p_2, 1}) \times \left( L^1_T(\dot{B}^{s+1}_{p_2, 1}) \cap \dot{L}^\infty_T(\dot{B}^{-1}_{p_2, 1}) \right).
\]
Let \((u^n_L, \Pi^n_L)\) be a solution of the following non-stationary Stokes system

\[
\begin{aligned}
\partial_t u^n_L - \mu^1 \Delta u^n_L + \nabla \Pi^n_L &= f^n \\
\partial_t B^n_L - \sigma^1 \Delta B^n_L &= 0 \\
\text{div } u^n_L &= \text{div } B^n_L = 0 \\
(u^n_L, B^n_L)|_{t=0} &= (u^n_0, B^n_0).
\end{aligned}
\]

By construction, \(u^n_0, B^n_0 \in \dot{B}^{\frac{N}{p_2} -1}_{p_2,1} \cap H^s\) and \(f^n \in L^1_{loc}(\mathbb{R}^+; \dot{B}^{\frac{N}{p_2} -1}_{p_2,1} \cap H^s)\). So following Proposition 2.3 from [3], we have \((u^n_L, \nabla \Pi^n_L, B^n_L) \in L^\infty_t(\dot{B}^{\frac{N}{p_2} -1}_{p_2,1} \cap H^s) \times L^1_t(\dot{B}^{\frac{N}{p_2} -1}_{p_2,1} \cap H^s) \times L^\infty_t(\dot{B}^{\frac{N}{p_2} -1}_{p_2,1} \cap H^s)\) and moreover \(u^n_L, B^n_L \in L^1_t(\dot{B}^{\frac{N}{p_2} +1}_{p_2,1})\) for all \(t > 0\).

Let \(u^n = u^n_L + \overline{a}^n, \nabla \Pi^n = \nabla \Pi^n_L + \nabla \Pi^n\) and \(B^n = B^n_L + \overline{B}^n\). Then

\[
(a^n, \overline{a}^n, \nabla \Pi^n, \overline{B}^n) \in C\left(([0, T^n); H^{s+1}(\mathbb{R}^N)) \times \left(C([0, T^n); H^s(\mathbb{R}^N)) \times L^1_{T^n}(H^s(\mathbb{R}^N)) \times C\left([0, T^n); H^s(\mathbb{R}^N))\right)
\]

and verifies

\[
\begin{aligned}
\partial_t a^n + u^n \cdot \nabla a^n &= 0 \\
\partial_t \overline{a}^n + u^n \cdot \nabla \overline{a}^n - \mu^1 \Delta \overline{a}^n + \nabla \Pi^n = H(a^n, u^n, \nabla \Pi^n, B^n) \\
\partial_t \overline{B}^n + u^n \cdot \nabla \overline{B}^n - \sigma^1 \Delta \overline{B}^n &= -\text{div} \left(\overline{\sigma}(a^n) - \sigma^1\right) \nabla B^n + B^n \cdot \nabla u^n - u^n \cdot \nabla B^n_L \\
\text{div } \overline{a}^n &= \text{div } \overline{B}^n = 0 \\
(a^n, \overline{a}^n, \overline{B}^n)|_{t=0} &= (a^n_0, 0, 0),
\end{aligned}
\]

where

\[
H(a^n, u^n, \nabla \Pi^n, B^n) = -u^n \cdot \nabla u^n_L + a^n(\mu^1 \Delta u^n - \nabla \Pi^n)
\]

\[
+ 2(1 + a^n)\text{div}\left\{\left(\tilde{a}(a^n) - \mu^1\right)\mathcal{M}^n\right\} + (1 + a^n)(B^n \cdot \nabla B^n - \frac{1}{2} \nabla B^n^2)
\]

with \(\mathcal{M}^n = \frac{1}{2}(\nabla u^n + t \nabla u^n)\). We find that \((a^n, \overline{a}^n, \nabla \Pi^n, \overline{B}^n)\) belongs to \(E_{T^n}\) by following the arguments as in [3].

Now we are in a position to prove that \(T > 0\) such that \((a^n, u^n, \nabla \Pi^n, B^n)\) is bounded in \(E_T\).

In what follows, we will use the notation

\[
U^n(t) := \|\overline{a}^n\|_{L^\infty_t(\dot{B}^{\frac{N}{p_2} -1}_{p_2,1})} + \|\overline{a}^n\|_{L^1_t(\dot{B}^{\frac{N}{p_2} +1}_{p_2,1})} + \|\nabla \Pi^n\|_{L^1_t(\dot{B}^{\frac{N}{p_2} -1}_{p_2,1})}
\]

and

\[
B^n(t) := \|\overline{B}^n\|_{L^\infty_t(\dot{B}^{\frac{N}{p_2} -1}_{p_2,1})} + \|\overline{B}^n\|_{L^1_t(\dot{B}^{\frac{N}{p_2} +1}_{p_2,1})}.
\]
Since \( \frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p_2} \), then according to Proposition 3.1, we have
\[
\|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \leq e^{L^1_T(B_{p_2,1}^{\frac{N}{p^2}+1})} \|a_0^n\|_{B_{p_1,1}^{\frac{N}{p}}},
\]
\[
\leq e^{L^1_T(B_{p_2,1}^{\frac{N}{p^2}+1})} \|a_0^n\|_{B_{p_1,1}^{\frac{N}{p}}}. \]

Moreover, Proposition 3.2 implies that
\[
U^n(T^n) \leq Ce^{L^1_T(B_{p_2,1}^{\frac{N}{p^2}+1})} \|H(a^n, u^n, \nabla \Pi^n, B^n)\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}+1})}. \]

Since \( \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{N} \), then the inequality (4) implies that
\[(22)\]
\[
\|a^n(\mu^1 \Delta u^n - \nabla \Pi^n)\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}})} \lesssim \|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \left( \|u^n\|_{L^2_T(B_{p_2,1}^{\frac{N}{p^2}+1})} + \|\nabla \Pi^n\|_{L^2_T(B_{p_2,1}^{\frac{N}{p^2}+1})} \right).
\]

From the Bernstein inequality, (4), and a classical interpolation argument, we may infer that
\[(23)\]
\[
\|u^n \cdot \nabla u^n\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}})} \lesssim \|a^n \otimes u^n\|_{L^2_T(B_{p_1,1}^{\frac{N}{p}})} \lesssim \|u^n\|_{L^2_T(B_{p_2,1}^{\frac{N}{p^2}})} \|u^n\|_{L^2_T(B_{p_2,1}^{\frac{N}{p^2}})}.
\]

Since \( p_1 \leq p_2 \) and \( \frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{N} \), then the Bernstein inequality, estimate (4), and Taylor’s formula imply that
\[
\left\|(1 + a^n)\text{div}\left\{ (\bar{\mu}(a^n) - \mu^1) \mathcal{M}^n \right\} \right\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}})} \lesssim \left( 1 + \|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \right) \times \left( \|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \|\bar{\mu}(a^n) - \mu^1\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}})} \right) \lesssim \left( 1 + \|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \right) \left\|a^n\right\|_{L^2_T(B_{p_1,1}^{\frac{N}{p}})} \left\|u^n\right\|_{L^2_T(B_{p_2,1}^{\frac{N}{p^2}+1})}, \]
\[
\left\|(1 + a^n)(B^n \cdot \nabla B^n - \frac{1}{2} \nabla B^{n^2}) \right\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}})} \lesssim \left( 1 + \|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \right) \times \left( \|B^n \otimes B^n\|_{L^1_T(B_{p_2,1}^{\frac{N}{p}})} + \|B^{n^2}\|_{L^1_T(B_{p_2,1}^{\frac{N}{p^2}})} \right) \lesssim \left( 1 + \|a^n\|_{L^\infty_T(B_{p_1,1}^{\frac{N}{p}})} \right) \left\|B^n\right\|_{L^2_T(B_{p_1,1}^{\frac{N}{p}})} \left\|B^n\right\|_{L^2_T(B_{p_2,1}^{\frac{N}{p^2}+1})}, \]
\]
For $\overline{B}^n$, we have
\[
\partial_t \overline{B}^n + u^n \cdot \nabla \overline{B}^n - \sigma^1 \Delta \overline{B}^n = -\text{div} \left[ (\overline{\sigma}(a^n) - \sigma^1) \nabla \overline{B}^n \right] + \overline{B}^n \cdot \nabla u^n + B^n_L \cdot \nabla u^n - u^n \cdot \nabla B^n_L.
\]

By Proposition 3.2 and inequalities (4) and (6) that for $t \in [0, T^n]$
\[
B^n(t) \leq C e^{t^1(B_{p,1}^\infty, B_{p,1}^\infty)} \left\{ \|B^n_t\|_{L^2_t(B_{p,1}^\infty)} \|u^n\|_{L^2_t(B_{p,1}^\infty)} + \|a^n\|_{L^\infty_t(B_{p,1}^\infty)} \|B^n\|_{L^1_t(B_{p,1}^\infty)} \right\},
\]

So, by interpolation, we have
\[
\|v\|_{L^2_t(B_{p,1}^\infty)} \leq \|v\|_{L^\infty_t(B_{p,1}^\infty)} \|v\|_{L^\infty_t(B_{p,1}^\infty, B_{p,1}^\infty)} \quad \forall \ v \in L^1_t(B_{p,1}^\infty) \cap L^\infty_t(B_{p,1}^\infty),
\]

thus
\[
B^n(t) \leq C e^{t^1(B_{p,1}^\infty, B_{p,1}^\infty)} \left\{ \|B^n_t\|^{1/2}_{L^\infty_t(B_{p,1}^\infty)} \|u^n\|^{1/2}_{L^\infty_t(B_{p,1}^\infty)} + \|a^n\|_{L^\infty_t(B_{p,1}^\infty)} \|B^n\|_{L^1_t(B_{p,1}^\infty)} \right\}.
\]

In the same manner, we have
\[
U^n(t) \leq e^{t^1(B_{p,1}^\infty, B_{p,1}^\infty)} \left[ \|u^n\|_{L^\infty_t(B_{p,1}^\infty)} \|u^n\|_{L^\infty_t(B_{p,1}^\infty)} + \|a^n\|_{L^\infty_t(B_{p,1}^\infty)} \left( \|u^n\|_{L^1_t(B_{p,1}^\infty)} + \|\nabla \Pi^n\|_{L^1_t(B_{p,1}^\infty)} \right) + \|B^n\|_{L^\infty_t(B_{p,1}^\infty)} \|B^n\|_{L^1_t(B_{p,1}^\infty)} \right].
\]

Let $\zeta$ be a small positive real number. Then there exists $T_1 > 0$ such that
\[
\|(u_L, B_L)\|_{L^1_t(B_{p,1}^\infty)} + \|\nabla \Pi L\|_{L^1_t(B_{p,1}^\infty)} \leq \zeta
\]

and (see Proposition 2.3 of [6])
\[
\|u_L\|_{\overline{L}^{\infty} (B_{p,1}^\infty)} \leq \|u_0\|_{\overline{L}^{\infty} (B_{p,1}^\infty)} + \|P f\|_{L^1_t(B_{p,1}^\infty)} = U_0.
\]

Consequently we have
\[
\|u^n_L\|_{L^1_t(B_{p,1}^\infty)} + \|\nabla \Pi^n L\|_{L^1_t(B_{p,1}^\infty)} \leq C\zeta \quad \text{et} \quad \|u^n_L\|_{\overline{L}^{\infty} (B_{p,1}^\infty)} \leq C U_0
\]
and
\begin{align}
\|B^n\|_{L^1_t(B^p_{p2,1})} & \leq C \zeta & \text{and} & \|B^n\|_{L^1_t(B^p_{p2,1})} & \leq C \|B_0\|_{B^p_{p2,1}}.
\end{align}

In the following we can suppose that $T^n \leq T_1$, otherwise we take a smaller $T^n$. Let $t \leq T^n$, then
\begin{align}
B^n(t) & \leq Ce^{\left(\zeta + \|\overline{u}\|_{L^1_t(B^p_{p2,1})}\right)} \left\{ \frac{1}{2} \left(U_0 + \|\overline{u}\|_{L^\infty_t(B^p_{p2,1})}\right)^{\frac{1}{2}} \left(\zeta + \|\overline{u}\|_{L^1_t(B^p_{p2,1})}\right)^{\frac{1}{2}} + \|a^n\|_{L^\infty_t(B^p_{p2,1})} \left(\zeta + \|\overline{u}\|_{L^1_t(B^p_{p2,1})}\right) + \zeta \left(U_0 + \|\overline{u}\|_{L^\infty_t(B^p_{p2,1})}\right) \right\}
\end{align}
and
\begin{align}
\|a^n\|_{L^\infty_t(B^p_{p2,1})} & \leq Ce^{\left(\zeta + \|\overline{u}\|_{L^1_t(B^p_{p2,1})}\right)} \|a_0\|_{B^p_{p1,1}}.
\end{align}

Let $T_2 \leq T^n$ such that
\begin{align}
\exp \left(C \left(\zeta + \|\overline{u}\|_{L^1_t(B^p_{p2,1})}\right)\right) & < 2.
\end{align}
So if
\begin{align}
16C^2 \|a_0\|_{B^p_{p1,1}} & \leq 1,
\end{align}
then
\begin{align}
\|a^n\|_{L^\infty_t(B^p_{p2,1})} & \leq 2C \|a_0\|_{B^p_{p1,1}}
\end{align}
and
\begin{align}
B^n(T_2) & \leq 4C \left\{ \zeta \left(U_0 + \|\overline{u}\|_{L^\infty_t(B^p_{p2,1})}\right)^{\frac{1}{2}} \left(\zeta + \|\overline{u}\|_{L^1_t(B^p_{p2,1})}\right)^{\frac{1}{2}} + 2C\zeta \|a_0\|_{B^p_{p1,1}} + \zeta \left(U_0 + \|\overline{u}\|_{L^\infty_t(B^p_{p2,1})}\right) \right\}.
\end{align}
Using inequalities (24) and (31) satisfied by $B^3 = B^3_L + \overline{B}^3$, we obtain that
\begin{align}
U^n(T_2) & \leq C \left\{ \zeta (U^n(T_2) + U_0) + 2C \|a_0\|_{B^p_{p1,1}} \left(1 + 2C \|a_0\|_{B^p_{p1,1}}\right) \left(\zeta + U^n(T_2)\right) + \zeta \|B_0\|_{B^p_{p2,1}} \left(1 + 2C \|a_0\|_{B^p_{p1,1}}\right) \left(U_0^2 + \zeta^2 + U^n(T_2)^2\right) \right\}.
\end{align}
Using (30) and the smallness of $a_0$, we obtain for $\zeta$ small enough,

\[(32) \quad U^n(T_2) \leq \zeta C \left( U_0, \|a_0\|_{B^p_{1,1}}, \|B_0\|_{B^p_{1,1}} \right).\]

Taking $\zeta$ small enough we observe that inequality (29) is satisfied. Consequently, a standard argument then yields that $T_2 = T^n$. The same type of reasoning allows one to show that $T^n = T^1$, with uniform control.

We give in what follows a precise estimate of the pressure term. Namely, we prove the following

**Lemma 4.4.** Let $0 < \eta < \inf(1, \frac{2N}{p_2})$ be such that $\frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}$. Then $\nabla \Pi^n$ is uniformly bounded in $L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})$.

**Proof.** Applying the divergence operator to the equation containing the pressure term, we obtain

\[
\text{div}\left( (1 + a^n) \nabla \Pi^n \right) = \text{div}\left\{ (1 + a^n) \left( \text{div}\{ \tilde{\mu}(a^n) \mathcal{M}^n \} + B^n \cdot \nabla B^n - \frac{1}{2} \nabla B^{n,2} \right) + Qf^n - u^n \cdot \nabla u^n \right\}.
\]

By construction of $f^n$ and by interpolation, we have that $Qf^n$ is uniformly bounded in $L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})$. By interpolation, we have that $u^n$ is uniformly bounded in $L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})$. Since $\eta < \frac{2N}{p_2}$, inequality (3) implies the estimate

\[
\|u^n \cdot \nabla u^n\|_{L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})} \lesssim \|u^n \otimes u^n\|_{L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})}
\]

\[
\lesssim \|u^n\|_{L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})} \|u^n\|_{L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})},
\]

which shows that $u^n \cdot \nabla u^n$ is uniformly bounded in $L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})$. In the same way $\left(1 + a^n\right)\text{div}\{ \tilde{\mu}(a^n) \mathcal{M}^n \}$, for $p_1 \leq p_2$, and $\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{N}$, the Bernstein inequality and (4) imply that $\text{div}\{ \tilde{\mu}(a^n) \mathcal{M}^n \}$ is uniformly bounded in $L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})$. So, by an interpolation argument, we obtain that $\text{div}\{ \tilde{\mu}(a^n) \mathcal{M}^n \}$ is uniformly bounded in $L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta})$. Since $\frac{1}{N} + \frac{\eta}{N} < \frac{1}{p_1} + \frac{1}{p_2}$, the inequality (3) implies that

\[
\left(1 + a^n\right)\text{div}\{ \tilde{\mu}(a^n) \mathcal{M}^n \} \text{ uniformly bounded in } L^{\frac{2}{1-\eta}}_{T_1} (\dot{B}_{p_2,1}^{\frac{N}{p_2} - \eta}) \]

in the same way as for $\left(1 + a^n\right)(B^n \cdot \nabla B^n - \frac{1}{2} \nabla B^{n,2})$. So $\nabla \Pi^n$ is also uniformly bounded because we have $\|a^n\|_{L^\infty_{T_1}(\dot{B}_{p_2,1}^{\frac{N}{p_2} - 1})} \leq 2C\|a_0\|_{L^\infty_{T_1}(\dot{B}_{p_2,1}^{\frac{N}{p_2} - 1})} < 1$. □

By the construction of the time of existence, then $T_1 = \infty$, provided that

\[
\|u_0\|_{\dot{B}_{p_2,1}^{\frac{N}{p_2} + 1}} + \|B_0\|_{\dot{B}_{p_2,1}^{\frac{N}{p_2} + 1}} + \|f\|_{L^1(\mathbb{R}_+; \dot{B}_{p_2,1}^{\frac{N}{p_2} - 1})} \leq c' \inf(\mu^1, \sigma^1).
\]
**Passage to the limit.** Let us note first that by construction of \((u^n_0, f^n)\), the sequence \((u^n_L, \nabla \Pi^n, B^n_L)\) converges strongly to the solution \((u_L, \nabla \Pi_L, B_L)\) of the system \((L)\). However, to show that the weak limit of \((a^n, \overline{u}^n, \nabla \Pi^n, \overline{B}^n)\) is a solution to the system \((NL)\), we need to use some compactness arguments.

We have already established that \((a^n, \overline{u}^n, \nabla \Pi^n, \overline{B}^n)\) is uniformly bounded in

\[
\tilde{L}^\infty_{T_1}(\hat{B}^N_{p_1,1}) \times \tilde{L}^\infty_{T_1}(\hat{B}^N_{p_2,1}) \cap L^1_{T_1}(\hat{B}^{N+1}_{p_2,1}) \times L^1_{T_1}(\hat{B}^{N-1}_{p_2,1}) \cap L^1_{T_1}(\hat{B}^{N+1}_{p_2,1}),
\]

Moreover \(\nabla \Pi^n\) is uniformly bounded in \(L^{2-\eta}_{T_1}(\hat{B}^{N-1}_{p_2,1})\).

So, in order to use the Ascoli theorem, it suffices to estimate the time derivative of \(a^n\), \(\overline{u}^n\) and \(\overline{B}^n\) (see for example [9]). Following the proof of Lemma 4.4, the following lemma is shown to hold true.

**Lemma 4.5.**

(i) The sequence \((\partial_t a^n)_{n \in \mathbb{N}}\) is uniformly bounded in \(L^2_{T_1}(\hat{B}^{N}_{p_1,1})\).

(ii) The sequence \((\partial_t \overline{u}^n)_{n \in \mathbb{N}}\) is uniformly bounded in \(L^{2-\eta}_{T_1}(\hat{B}^{N-1}_{p_2,1})\) for \(0 < \eta < \inf(1, \frac{2N}{p_2})\) and \(\frac{1}{N} + \frac{n}{N} < \frac{1}{p_1} + \frac{1}{p_2}\).

(iii) The sequence \((\partial_t \overline{\Pi}^n)_{n \in \mathbb{N}}\) is uniformly bounded in \(L^{2-\eta}_{T_1}(\hat{B}^{N-2}_{p_2,1})\) for \(0 < \eta < \inf(1, \frac{2N}{p_2})\) and \(\frac{1}{N} + \frac{n}{N} < \frac{1}{p_1} + \frac{1}{p_2}\).

From the above lemma, the Cauchy-Schwarz inequality and Hölder’s inequality, we deduce the following corollary.

**Corollary 1.**

(i) The sequence \((a^n)_{n \in \mathbb{N}}\) is uniformly bounded in \(C^\frac{1}{2}([0, T_1]; \hat{B}^{N}_{p_1,1})\).

(ii) The sequence \((\overline{\Pi}^n)_{n \in \mathbb{N}}\) is uniformly bounded in \(C^\frac{1}{2}([0, T_1]; \hat{B}^{N-1}_{p_2,1})\) for all \(\eta\) belonging to \(]0, \inf(1, \frac{2N}{p_2})[\) and \(\frac{1}{N} + \frac{n}{N} < \frac{1}{p_1} + \frac{1}{p_2}\).

(iii) The sequence \((\overline{B}^n)_{n \in \mathbb{N}}\) is uniformly bounded in \(C^\frac{1}{2}([0, T_1]; \hat{B}^{N-2}_{p_2,1})\) for all \(\eta\) belonging to \(]0, \inf(1, \frac{2N}{p_2})[\) and \(\frac{1}{N} + \frac{n}{N} < \frac{1}{p_1} + \frac{1}{p_2}\).

We recall that the injection of \(\hat{B}^{s+\varepsilon}_{p,q,loc}\) in \(B^s_{p,q,loc}\) (the inhomogeneous Besov space \(B^s_{p,q,loc}\)) is compact for all \(\varepsilon > 0\) (see for example [18]).

So, there exists a subsequence (which is still denoted by \((a^n, \overline{\Pi}^n, \nabla \Pi^n, \overline{B}^n)\)) which converges to \((a, \overline{\Pi}, \nabla \Pi, \overline{B})\). Consequently, \((a, u, \nabla \Pi, B)\) is a solution of the \((\text{MHD})\) system belonging to

\[
\tilde{L}^\infty_{T_1}(\hat{B}^{N}_{p_1,1}) \times \tilde{L}^\infty_{T_1}(\hat{B}^{N}_{p_2,1}) \cap L^1_{T_1}(\hat{B}^{N+1}_{p_2,1}) \times L^1_{T_1}(\hat{B}^{N-1}_{p_2,1}) \cap L^1_{T_1}(\hat{B}^{N+1}_{p_2,1}),
\]
Concerning the continuity of \( u \), we have used the fact that
\[
(H) \quad \begin{cases} 
\partial_t u - \mu^1 \Delta u = H(a, u, \nabla \Pi, B) \\
u(0) = u_0,
\end{cases}
\]
where
\[
H(a, u, \nabla \Pi, B) = f - u \cdot \nabla u - (1 + a)(\nabla \Pi + \frac{1}{2} \nabla B^2 - B \cdot \nabla B) + 2(1 + a) \text{div}\left\{ \left( \tilde{\mu}(a) - \mu^1 \right) \mathcal{M} \right\} + \mu^1 a \Delta u.
\]

Since \( (a, u, \nabla \Pi, B) \in \tilde{L}^{\infty}_{T_1}(B_{p_1}^{\frac{N}{p_1}-1}) \times \tilde{L}^{\infty}_{T_1}(B_{p_2}^{\frac{N}{p_2}+1}) \cap L^1_{T_1}(B_{p_1}^{\frac{N}{p_1}-1}) \times \tilde{L}^{\infty}_{T_1}(B_{p_2}^{\frac{N}{p_2}+1}) \cap L^1_{T_1}(B_{p_2}^{\frac{N}{p_2}-1}), \)
then Proposition 2.4 implies that \( H(a, u, \nabla \Pi, B) \in L^1_{T_1}(B_{p_2}^{-1}) \). And consequently, Proposition 2.1 [6], ensured the continuity in time of \( u \), in the same way for \( B \).

To prove that \( a \) is continuous and that the \( L^\infty \)-norm is conserved, we use that \( a = a_0 \circ \Psi^{-1} \) where \( \Psi \) is the flow of \( u \). This completes the proof of Theorem 1.3. \( \square \)

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