Realistic mathematics education on teaching functions to develop algebraic thinking skills

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Abstract. This study discusses how the notion of a function can be developed for high school students through the use of various contexts that are close to students’ daily life or make sense to them. The Realistic Mathematics Education approach provides the framework that guides the discussions. The concept of function has become one of the basic ideas in modern mathematics that underlies almost all disciplines. But it proved to be one of the most difficult concepts in learning mathematics. The concept of function is often taught without connecting to everyday context. Formal mathematical symbolism of concept such as $f(x)$ introduced prematurely to students which can lead to learning obstacle about functions. The study provides mathematics educators a hypothetical learning trajectory as a platform which facilitates a Realistic Mathematics Education approach to teaching function especially to develop algebraic thinking skills.

1. Introduction

Algebra is a branch of mathematics related to the study of the quantity, relationships, and structures that are formed. To learn these things, use symbols to represent something uncertain, in this case variables, parameters, or something unknown. Algebra relates to expressions using symbols and with extended numbers to resolve equations, analyze functional relationships, and to determine the structure of a representational system consisting of expressions and relationship. Activities such as resolving equations, analyzing functional relationships, and determining the structure of a representational system are not the purpose of algebra but rather a tool to model real-world phenomena and solve problems that related to various situations [1-3]. In general, students still have difficulty in learning algebra [4-6]. Students with low algebraic thinking skills tend to fail in solving algebraic problems, weak in simplifying algebraic equations and expressions, and have difficulty in interpreting graphs of functions. These difficulties make students learn by memorization without understanding the concept. As a result, students only able to solve low-level routine problems.

In everyday life, students often find situations related to functional relationships. Therefore they can bring a lot of relevant knowledge into the classroom. This knowledge can help students reason through algebraic problems. However, if the mathematical understanding that students bring to class is not associated with formal algebra learning, then it will not support new learning [7]. The concept of function has become one of the basic ideas in modern mathematics that underlies almost all disciplines. But it proved to be one of the most difficult concepts to master in learning mathematics in high school [8,9]. The concept of function is often taught without connecting to everyday contexts. Formal
mathematical symbolism of concepts such as \( f(x) \) is sometimes introduced prematurely to students which can lead to misconceptions about functions. When the concept of function is taught using a mathematical context that is not understood by students, students may have difficulty understanding the concept of the function.

There are several frameworks that cover contextual and conceptual approaches. The concept of the emerging model has a starting point from a specific problem situation, which is then modeled [10]. Giving a problem/context at the beginning of learning provides an opportunity for students to develop specific situation methods and symbolization. Furthermore, methods and symbols are modeled from a mathematical perspective and in this sense models emerge from student activities. The model first appears as a situation, then gradually becomes an entity and is finally used as a model for mathematical reasoning. The shift presented from 'model of' to 'model for' should occur with changes in the way students perceive and think about models, from models derived from the context of the situation to thinking about mathematical content.

Learning functions can include the main components of algebraic thinking skills. In a broad sense, algebraic thinking includes a series of understandings that are needed to translate information, or events into mathematical language to explain and predict a phenomenon. To develop true understanding, students must work with problem situations that arise in various contexts that are close to students' daily lives or make sense to them. This is in accordance with the characteristics of the RME.

This study discusses how the notion of a function can be developed for high school students through the use various contexts that are close to students’ daily life or make sense to them. The Realistic Mathematics Education approach provides the framework that guides the discussions. The study provides mathematics educators a hypothetical learning trajectory as a platform which facilitates a Realistic Mathematics Education approach to teaching function especially to develop algebraic thinking skills.

2. Literature review

2.1. Teaching function

Functions can be written and described in many ways, not only by formal definition. The general way of describing functions is with arrow diagrams, tables, graphs, algebraic symbols, input and output boxes, ordered pairs, words, and problem situations [7,8]. Different representations of functions can help students to gain insight into one of the big ideas in mathematics [11,12]. When students move between different contexts and representations of the concept of functions, they can realize that through mathematical ideas that might seem different are actually connected to each other and can interpret the same ideas. One of the serious obstacles in learning function is the reluctance to emphasize the visual aspects of concepts [8]. Students tend to think of the concept of function only in symbolic mode of representation. Related functions and ideas are not visually understood. Non-visual approach to learning related to this function inhibits the development of a person to understand the function.

All mathematical concepts have a formal definition. Most of these definitions were introduced to high school students or students at a time. On the other hand, students do not have to use definitions when determining whether a given mathematical object is an example or not an example of a concept. In most cases, students decide based on the concept image (concept image) that is the set of all mental images that are related in the student's mind with the name of the concept, along with the traits that characterize it. This means all kinds of representations of images, symbolic shapes, diagrams, graphs, and so on. Students' concept images of functions may differ from their mathematical definition [11].

2.2. Realistic mathematics education

Realistic Mathematics Education (RME) is a learning theory specifically for mathematics developed in the Netherlands. As a theory, RME has a philosophy and characteristics. According to Freudenthal, the emergence of RME is based on a philosophy about mathematics as a human activity [10]. Human activity is first and foremost in mathematics, where doing mathematics is more important than seeing
mathematics as a ready-made product. Mathematics as a human activity can mean two things. First, mathematics is constructed from human activities. Second, mathematics can be implemented in human activities. According to Freudenthal, mathematics learning is the process of doing mathematics which leads to mathematics as a product [10]. This condition is contrary to traditional mathematics learning that uses the results of other people's activities as a starting point for learning. In mathematics education, teaching mathematics does not have to be the same as how mathematics was discovered by mathematicians.

The characteristic of RME is the "realistic" situation given in the learning process. This situation serves as a source to begin the development of mathematical concepts, tools and procedures, as well as a context in which students can apply their mathematical knowledge at a later stage which gradually becomes more formal and more general [13]. The "realistic" situation referred to in RME implies that the problem that will be presented to students in its context cannot only be obtained from the real world. The fairy-tale fantasy world and even the formal world of mathematics can be a suitable context for the problem presented, provided that the context is real in the minds of students [14-16]. According to Freudenthal mathematics must be connected with reality, stay close to students, and relevant to society, in order to have human values. In mathematics education, the focus is on the activities and processes of mathematics [10].

Webb visualizes the process of mathematization as the process of forming an iceberg [17]. The iceberg model is a visual metaphor that describes the process of progressive formalization through informal, pre-formal, and formal representation. The iceberg model consists of the tip of the iceberg and a much larger area below it is defined as "floating capacity". The tip of the iceberg represents formal procedures or symbolic representations. Before reaching this formal level, students must have the opportunity to deal with informal contexts and pre-formal representations. These informal and pre-formal representations and strategies play an important role in understanding formal mathematics. Students initially meet a realistic context which then motivates the use of mathematical language. Next they use a more structured pre-formal model that supports their understanding of formal notation.

2.3. Algebraic thinking skills
Radford explained that the history of algebra thought originated from a proportional thinking as an alternative way of resolving "non-practical" problems [3]. A practical problem is the problem that relates to the practical needs of human beings, for example how to calculate the area of a certain land plot, how to calculate the loan interest, how to solve the problems related to inheritance, how calculating prices from different commodities. While the non-practical problems are problems that do not have a direct relationship with practical needs. For example, how to determine the area of land that is twice as long as the width. Based on that opinion, algebraic thinking begins with a person's sensitivity of something uncertain (unknown, variable, parameter), and then analysis of the objects, then models it with certain symbols.

The algebraic thinking was organized in two main components: the development of mathematical thinking tools and the study of basic algebraic ideas [18]. Mathematical thinking tools are a habit of analytic thinking, covering 3 topics: problem solving ability, representation ability, and quantitative reasoning ability. The basic idea of algebraic represents the content of the domain in which mathematical thinking tools develop, including: algebra as a generalized arithmetic, algebraic as a language, and algebraic as a tool for reviewing functions and mathematical modelling.

Kieran explains that thinking algebra can be interpreted as an approach to quantitative situations emphasizing aspects of general relations with tools that do not necessarily require a letter symbol, but can eventually use as a cognitive support to introduce and retain more traditional school algebra material [1]. This study adopts algebraic thinking activities proposed by Kieran [1] as the basis for describing students' algebraic thinking skills. This is because the opinion already includes the opinions of other experts about the algebraic thinking skills. Indicators for each activity in algebraic thinking skills are presented in Table 1.
**Table 1.** Indicators of algebraic thinking skills.

| Activity       | Indicators                                                                                                                                 |
|----------------|---------------------------------------------------------------------------------------------------------------------------------------------|
| **Generational** | Students are able to use relationship patterns to analyze situation.                                                                          |
| **Transformational** | Students are able to move between different representation.                                                                                           |
|                 | Students are able to make and use symbolic, visual or spatial notations, and words or sentences in solving mathematical problems.              |
| **Global Meta-Level** | Students are able to explore problem solving related to other topics or fields of science.                                                      |
|                 | Students are able to model and solve the problem.                                                                                               |

2.4. The hypothetical learning trajectory about function using realistic mathematics education approach

The hypothetical learning trajectory about functions are prepared with attention to the principles and characteristics of Realistic Mathematics Education. The context is presented at the beginning, middle and end of learning. The context at the beginning of learning serves to trigger a good learning process. The context in the middle of learning serves to encourage students to work in mathematics or "doing mathematics". While the context at the end of learning serves as an application or a follow up to the learning process. Contextual problems begin the existence of a model of a situation, then the model for develops when students have directed the model to solve problems mathematically so that students can be at the level of formal mathematics. A visualization of the mathematical process in learning function is illustrated as follows.

![Figure 1. Iceberg model on learning function.](image)

2.4.1. Understand the concept of function. To understand the concept of function, the teacher does not directly provide a formal definition of function. Students are guided to discover their own concept of function through the exploration of contextual problems. For example students are asked to observe plants in the school yard. As a group, students collect data about the type of plant along with the leaf colour, root type, and height of each plant. Then students are asked to discuss and explain the relationship between the plants found with the root type, leaf colour, and height. From the data obtained
will be found several examples of relationships that are both functions and not functions. The teacher guides students to explain the concept of function according to their understanding. When listing the types and properties of plants that are observed, students will get information that one type of plant has a certain height, and has a certain type of root. Students will also discover the fact that a type of plant may have more than one colour. When finding plants that do not have leaves, students may have difficulty explaining the relationship between the type of plant and the leaf colour of the leaves of the plant. As an anticipation, the teacher can guide students to find the concept of "not function".

2.4.2. Represent functions. The teacher asks students to present the data obtained in the previous activity. In representing the relationship between plant types and leaf colour and root types some students may present data obtained from observations in tabular form, because students are accustomed to seeing or even presenting data in tabular form since elementary school. Other students might represent it in the form of an arrow diagram. If this conjecture does not arise, the teacher must facilitate by guiding students based on their understanding of the concept of function so that students are able to represent the relationship in the form of an arrow diagram. In representing the relationship between types of plants with height, students with teacher guidance are strived to be able to present data obtained from observations in the form of tables, arrow diagrams, cartesian diagrams, histograms, or graphs. The teacher guides students to pay attention to the measurement scale used in the diagram or graph created. Some other contextual problems can be given by the teacher to develop students' skills in presenting functions in various representations as well as translating between different representations. For example by providing data on plant height changes from week to week, changes in student body height, changes in the population of Indonesia from year to year, and other contextual issues.

2.4.3. Analyzing situations using relationship patterns to build a function and determine the function value. In this activity the teacher can make a narration, for example about the change in height of a plant from week to week then display the data in tabular form. Students are asked to present the data in different representations. The teacher asks students to express various phenomena that arise related to changes in plant height. Students will notice that from week to week plant height continues to increase, the highest in a particular week (according to the data provided). Students are asked to predict the likelihood of what happens in the following weeks. Predictions that appear for example in the following week the plant will increase in height or may decrease in height. These different predictions will lead students to functions that may also be different. Some other contextual problems can be given or raised by students themselves. The teacher can guide students using relationship patterns by analyzing problems or given situations to build a function and determine the value of the function.

3. Conclusion
The concept of function is often taught without connecting to everyday contexts. The formal mathematical symbolism of concepts is introduced prematurely to students which can lead to misconceptions about functions. When the concept of function is taught by using a mathematical context that is not understood by students, students may have difficulty understanding the concept of the function. Difficulties of students in learning functions need to be overcome by the teacher's efforts in planning, implementing, and evaluating the learning process. This role is performed by the teacher before, during and after the learning process takes place. Every teacher is required to be able to design mathematics learning that is suitable for students' needs. To realize this, learning and programs that are designed must be based on long-term coherent teaching-learning trajectories. One of learning approach that can facilitate the development of students' algebraic thinking skills on learning functions is the Realistic Mathematics Education approach. In this paper HLT on learning functions are formulated which consists of three learning objectives as follows: understanding the concept of functions, representing functions, and build a function as well as determining the value of a function. HLT is prepared with a Realistic Mathematics Education approach. The implementation of the HLT is expected can be develop students' algebraic thinking skills.
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