Real algebraic curves on real del Pezzo surfaces of degree 1 and 2

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Abstract

The study of the topology of real algebraic varieties dates back to the work of Harnack, Klein and Hilbert in the 19th century; in particular, the isotopy type classification of real algebraic curves in real toric surfaces is a classical subject that has undergone considerable evolution. On the other hand, not much is known for more general ambient surfaces. We take a step forward in the study of topological types classification of real algebraic curves on non-toric surfaces focusing on real del Pezzo surfaces of degree 1 and 2 with multi-components real part. We use degeneration methods and real enumerative geometry in combination with variations of classical methods to give obstructions to topological types and to give constructions of real algebraic curves with prescribed topology.

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1 Introduction

The study of topology of real algebraic varieties dates back to the work of Harnack, Klein and Hilbert in the 19-th century ([Har76], [Hil02], [Kle22]). A real algebraic variety $(X, \sigma)$ is a compact complex algebraic variety equipped with an anti-holomorphic involution $\sigma : X \to X$, called real structure. The real part $\mathbb{R}X$ of $(X, \sigma)$ is the set of points fixed by the involution $\sigma$. Hilbert proposed in the first part of his 16-th problem to classify the isotomy types of real algebraic curves (resp. surfaces) of degree 6 (resp. degree 4) in $\mathbb{R}P^2$ (resp. in $\mathbb{R}P^3$). The classification on $\mathbb{R}P^2$ had been achieved by Gudkov ([Gud69]) at the end of the 60’s and the other one by Kharchev ([Kha76], [Kha78]) around ten years later. At the moment, the classification of real algebraic surfaces (resp. curves) in the real projective space (resp. in the real projective plane) is known only up to degree 4 (resp. 7; [Vir84a], [Vir84b]). There are two main directions in the study of topology of real algebraic varieties. The first is to give obstructions for topologies of real algebraic varieties. The second direction is to provide constructions of real algebraic varieties with prescribed topology. From the 70’s, especially thanks to the work of Arnold and Rokhlin ([Arn71], [Rok72], [Rok74], [Rok78], [Rok80]), many general obstructions had been discovered. On the other hand, the construction techniques had remained relatively elementary for a long time. In 1979, Viro provided a breakthrough in the construction direction by inventing the patchworking method ([Vir84a], [Vir84b]). Such method and its generalizations still remain the most powerful tools to construct real algebraic hypersurfaces with prescribed topology in real algebraic toric varieties with the standard real structure. Exploiting the patchworking technique, several construction results have been achieved on real algebraic toric varieties. On the other hand, only few works have been devoted to classification problems on real non-toric varieties ([Mik98]) or on toric varieties with the non-standard real structure (for example, [CS80], [MH04], [DZ99], [Man19]).

In this paper, we take a step forward in the classification of embedded topology of real algebraic curves on real non-toric surfaces studying the topological types classifications of real algebraic curves on real del Pezzo surfaces of degree 1 and 2 with real part respectively homeomorphic to $\mathbb{R}P^2 \bigsqcup_{i=1}^k S^2$ and $\bigsqcup_{i=1}^k S^2$, with $0 \leq k \leq 4$. We call such del Pezzo surfaces $k$-spheres real del Pezzo surfaces of degree 1 and 2. In 1998, Mikhalkin ([Mik98]) was the first to face the issue of classifying real algebraic curves on real algebraic surfaces with non-connected real part. In particular, he studied the topology of the real part of transverse intersections of real quadratic surfaces with real cubic surfaces in $\mathbb{C}P^3$ equipped with the standard real structure. To our knowledge, there are no other classifications on real algebraic surfaces with non-connected real part.

First of all, let us present some general definitions and known results about real algebraic curves. Let $(X, \sigma)$ be any real algebraic non-singular compact curve. A very useful tool is Harnack-Klein’s inequality ([Har76], [Kle22]), which bounds the number $l$ of connected components of $\mathbb{R}X$ by the genus $g$ of $X$ plus one. We say that $(X, \sigma)$ is a $M$-curve or a maximal curve, if $l = g + 1$. If $X \setminus \mathbb{R}X$ is connected, we say that $X$ is of type II or non-separating, otherwise of type I or separating ([Kle22]). Looking at the real part of the curve and its position with respect to its complexification gives us information about $l$ and viceversa. For example, we know that if $X$ is maximal, then $X$ is of type I. Or, if $X$ is of type I then $l$ has the parity of $g + 1$. Moreover, if $X$ is of type I, the two halves of $X \setminus \mathbb{R}X$ induce two opposite orientations on $\mathbb{R}X$ called complex orientations of the curve ([Rok72]).

The anti-(bi)canonical system exhibits the $k$-spheres real del Pezzo surfaces of degree 2 (resp. degree 1) as double covers of $\mathbb{C}P^2$ ramified along real non-singular quartics (resp. as a double cover of quadratic cones in $\mathbb{C}P^3$ ramified along real cubic sections). We mainly focus on the topological types classification of real algebraic curves in 4-spheres real del Pezzo surfaces of degree 2 and of degree 1 which are real minimal$^1$ surfaces. On such surfaces, every real curve realizes in homology an integer multiple of the anti-canonical class of the surface. In combination with variations of classical classification methods, the main tools of construction of real curves rely on degeneration methods, resp. of obstructions of topological types rely on real enumerative

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$^1$We say that a real algebraic variety $(X, \sigma)$ is minimal, or $R$-minimal, if every real degree 1 holomorphic function $f : X \to Y$ to a real algebraic surface $(Y, \tau)$ is a biholomorphism.
geometry. Later, we apply the classification tools developed for 4-spheres del Pezzo surfaces to $k$-spheres del Pezzo surfaces, with $k < 4$, to classify the topological types of real algebraic curves realizing in homology an integer multiple of the anti-canonical class of the surface. The classifications in $k$-spheres real del Pezzo surfaces of degree $2$ are in Section 3 and the classifications of those of degree $1$ in Section 4.

1.1 $k$-spheres real degree 2 del Pezzo surfaces

In Sections 3.1, 3.2 we give definitions, notations and state the main results about $k$-spheres real del Pezzo surfaces of degree $2$. The proofs of the main statements are in Sections 3.3, 3.4, 3.5 and 3.7. Let us denote by $X^k$ a $k$-spheres del Pezzo surfaces and let $c_1(X^k)$ denote the anti-canonical class of $X^k$. We say that a real algebraic curve in $X^k$ is of class $d$, if it realizes in homology $dc_1(X^k)$, for some integer $d \in \mathbb{Z}_{\geq 0}$. We say that an arrangement of $l$ embedded circles in $\bigcup_{i=1}^d S^2$ is a topological type in class $d$, if $l \leq d(d - 1) + 2$. Moreover, we say that a real algebraic curve in $X^k$ (resp. a topological type in class $d$ in $\mathbb{R}X^k$) is symmetric if it can be realized via the anti-canonical map of $X^k$ (resp. is realizable by a symmetric real algebraic curve of class $d$ in $X^k$); see Definition 3.6. First of all, we classify non-singular real algebraic curves of class $d \in \{1, 2\}$.

Result 1.1 (Proposition 3.8). For any topological type $S$ in class $d = 1, 2$ which is not prohibited by Harnack-Klein’s inequality, there exists a $k$-spheres real del Pezzo surface $X^k$ and a symmetric real algebraic curve of class $d$ in $X^k$ realizing $S$.

We construct $k$-spheres real del Pezzo surfaces $X^k$ of degree $2$ and real algebraic curves in $X^k$ with prescribed topology combining the action of the anti-canonical map of $X^k$ and construction techniques on $\mathbb{R}P^2$. Later, we focus on real algebraic curves of class $d \geq 3$. First of all, Harnack-Klein’s inequality does not give a complete set of restrictions anymore. Furthermore, most of all classical obstructions do not seem to apply; for example we did not find applications of the results of [GM77] to our setting. In Section 3.3 we show how to use Welschinger invariants ([Wel05], [Shu14], [IKS15]) from real enumerative geometry, to obtain Bézout-type obstructions on topological types for every integer $d$. We use also variations of classical Bézout’s type restrictions, which are powerful only for small $d$.

For $d = 3$, dealing with real maximal curves only, we are able to obtain a partial classification on 4-spheres real del Pezzo surfaces of degree 2.

Result 1.2 (Theorem 3.9). There are 74 topological types with 8 ovals in class 3 on $\mathbb{R}X^4$ which are not prohibited by Bézout’s type restrictions and Harnack-Klein inequality. Moreover 48 among the 74 topological types are such that: for each of them there exist a 4-spheres real del Pezzo surface $X^4$ of degree 2 and real curve of class 3 in $X^4$ realizing it.

Among the 48 realized topological types in class 3, at least 19 are symmetric because they are realized combining the action of the anti-canonical map with classical construction methods on the real projective plane (Proposition 3.21). The other topological types (Proposition 3.35) are realized using some degeneration methods, which found recent applications in enumerative geometry ([BP13]). We do not know yet if there are non-symmetric topological types realizable in class 3 and 4. But, for any integer $d \geq 5$ we have the following result.

Result 1.3 (Proposition 3.11). For any integer $d \geq 5$, there exists a non-symmetric topological type $S$ in class $d$ consisting of $2d + 1$ connected components) such that there exists a 4-spheres real del Pezzo surface $X^4$ of degree 2 and a non-symmetric real algebraic curve of class $d$ in $X^4$ realizing $S$. Furthermore $S$ is not realizable in class $d$ by any symmetric real algebraic curve of class $d$ in any 4-spheres real del Pezzo surfaces of degree 2.

Moreover, among the 26 topological types still unrealized in class 3, there is one that we can realize by a non-singular real symplectic curve (Proposition 3.10) in a real symplectic degree 2 del Pezzo surface with real part composed by 4 spheres. We do not know yet whether this topological type is realizable algebraically. Using the obstructions in Section 3.3 and combining the construction methods previously adopted to the case of 4-spheres real del Pezzo surfaces of degree 2, we obtain...
the following classification result on $k$-spheres real del Pezzo surfaces of degree 2, with $k < 4$.

**Result 1.4** (Proposition 3.12). *There are 69 (resp. 56, resp. 27) topological types with 8 ovals in class 3 on $\mathbb{R}X^3$ (resp. $\mathbb{R}X^2$, resp. $\mathbb{R}X^1$) which are not prohibited by Bézout’s type restrictions and Harnack-Klein inequality. Moreover 49 (resp. 38, resp. 17) among the 69 (resp. 56, resp. 27) topological types are such that: for each of them there exist a 3-spheres (resp. 2-spheres, resp. 1-spheres) real del Pezzo surface $X^3$ (resp. $X^2$, resp. $X^1$) of degree 2 and real curve of class 3 in $X^3$ (resp. $X^2$, resp. $X^1$) realizing it.*

We end this section giving an idea of the main construction method. The combination of the action of the anti-canonical map with classical construction methods on the real projective plane does not seem to be enough to realize all topological types in class $d \geq 3$ in $k$-spheres real del Pezzo surfaces of degree 2. Indeed, we use degeneration methods to realize some topological types in class 3 (resp. the non-symmetric topological types in class $d \geq 5$) listed in Theorem 3.9 and Proposition 3.35 (resp. Proposition 3.11). We start with a real nodal degree 2 del Pezzo surface $X_0$ with real part composed by $1 \leq k \leq 3$ two-dimensional connected components; as example, see on the right of Fig. 1. Then, we degenerate $X_0$ to the union of a real ruled surface $T$ and a $k$-spheres real del Pezzo surface $S$ intersecting transversely along a curve $E$; see in the center of Fig. 1 (Proposition 3.26). We construct real algebraic curves

![Figure 1: An example of degeneration of $\mathbb{R}X_0$ into $\mathbb{R}T \cup \mathbb{R}S$ and patchworking of surfaces. $\mathbb{R}E \simeq S^1$ in white, $\mathbb{R}T$ in black and $\mathbb{R}S$ in gray.](image)

$C_T,C_S$ of given topology and homology class separately on $T$ and on $S$. Then, to end the construction, we use the version of patchworking developed by Shustin and Tyomkin ([ST06a], [ST06b]) which allows us, under some transversality conditions, to "glue" such surfaces and curves to realize real algebraic curves on a $k$-spheres (resp. $(k+1)$-spheres) real del Pezzo surface of degree 2 (Fig. 1 on the right), with topology prescribed by the arrangement of the triplet $(\mathbb{R}T \cup \mathbb{R}S, \mathbb{R}E, \mathbb{R}C_T \cup \mathbb{R}C_S)$. The patchworking technique presented in [ST06a], [ST06b] has been recently exploited in [BDIM18] to construct real algebraic curves whose real part consists of a finite number of points in $CP^2$ and in the quadric ellipsoid.

### 1.2 $k$-spheres real degree 1 del Pezzo surfaces

In Sections 4.1 and 4.2, we give definitions, notations and state the main results about $k$-spheres real del Pezzo surfaces of degree 1. The proofs of the main statements are in Sections 4.3, 4.6.1, 4.6.2 and 4.6.3. Let us denote by $Y^k$ a $k$-spheres del Pezzo surfaces and let $c_1(Y^k)$ denote the anti-canonical class of $Y^k$. We say that a real algebraic curve in $Y^k$ is of class $d$, if it realizes in homology $d c_1(Y^k)$, for some integer $d \in \mathbb{Z}_{\geq 0}$. We say that an arrangement of $l$ embedded circles in $RP^2 \bigsqcup_{i=1}^{l} S^2$ is a topological type in class $d$, if $l \leq \frac{d(d-1)}{2} + 2$. First of all, we prove that the intersection form on $Y^k$ and Harnack-Klein’s inequality (Proposition 4.11) gives a complete set of restrictions for the topological types classifications of real algebraic curves of class $d \in \{1, 2, 3\}$ in $Y^k$.

**Result 1.5** (Theorem 4.3 and Proposition 4.10). *For any topological type $S$ in class $d$, with $d \in \{1, 2, 3\}$ which is not prohibited by Proposition 4.11 there exists a $k$-
spheres real del Pezzo surface $Y^k$ of degree 1 and a real algebraic curve of class $d$ in $Y^k$ realizing $S$.

Moreover, only in the case of $Y^4$, one can label the spheres of $\mathbb{R}Y^4$ as positive and negative via the anti-bicanonical map of $Y^4$ (Section 4.3). It follows that, for any fixed non-negative integer $d$, there are two topological types classifications for real algebraic curves of class $d$ in $Y^4$. Up to class 3, we show that these two classifications are the same. Let us consider the disjoint union of a real projective plane and four spheres. Then, label two spheres as positive $S_2^+$ and the others as negative $S_2^-$. We call a topological type in class $d$ in $\mathbb{R}P^2 \bigcup_{i=1}^2 S_2^+ \bigcup_{i=1}^2 S_2^-$ a refined topological type in class $d$.

**Result 1.6 (Theorem 4.8).** For any refined topological type $S$ in class $d \in \{1, 2, 3\}$ which is not prohibited by Proposition 4.14, there exists a 4-spheres real del Pezzo surface $Y^4$ of degree 1 and a real algebraic curve of class $d$ in $Y^4$ realizing $S$.

Furthermore, in Proposition 4.13 we give Bézout’s type restrictions on the topology of real algebraic curves of class $d \geq 4$, using the intersection form on $Y$.

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## 2 Preliminaries

### 2.1 Encoding topological types

Let $(X, \sigma)$ be a real algebraic surface. Let $\sigma_+ : H_2(X; \mathbb{Z}) \rightarrow H_2(X; \mathbb{Z})$ be the group homomorphism induced by $\sigma$ and let $H_2^-(X; \mathbb{Z})$ be the $(-1)$-eigenspace of $\sigma_+$. In the following, for a fixed homology class $\alpha \in H_2^-(X; \mathbb{Z})$, we are interested in the classification of the topological types of the pair $(\mathbb{R}X, \mathbb{R}A)$ up to homeomorphism, where $A \subset X$ is a non-singular real algebraic curve realizing $\alpha$ in $H_2(X; \mathbb{Z})$. The real part of $A$ is homeomorphic to a union of circles embedded in $\mathbb{R}X$, and can be embedded in $\mathbb{R}X$ in different ways. For the purpose of this paper, we only need to explain how to encode the embedding of a given collection $\bigcup_{i=1}^l B_i$ of $l$ disjoint circles in $\mathbb{R}P^2$ and in $S^2$. An embedded circle realizing the trivial-class in $H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z})$ or $H_1(S^2; \mathbb{Z}/2\mathbb{Z})$ is called oval, otherwise is called pseudo-line.

Let us call oval any circle embedded in $\mathbb{R}^2$. Let us consider a collection $\bigcup_{i=1}^l B_i$ of circles embedded in $\mathbb{R}^2$, resp. in $\mathbb{R}P^2$. An oval $B_i$ in $\mathbb{R}^2$, resp. in $\mathbb{R}P^2$, separates two disjoint non-homeomorphic connected components: the connected component homeomorphic to a disk is called interior of the oval; the other one is called exterior of the oval. For each pair of ovals, if one is in the interior of the other we speak about an injective pair, otherwise a non-injective pair. We shall adopt the following notation to encode a given topological pair $(\mathbb{R}^2, \bigcup_{i=1}^l B_i)$, resp. $(\mathbb{R}P^2, \bigcup_{i=1}^l B_i)$.

An empty union of ovals is denoted by 0. We say that a union of $l$ ovals realizes $l$ if there are no injective pairs. The symbol $\langle S \rangle$ denotes the disjoint union of a collection of ovals realizing $S$, and an oval forming an injective pair with each oval of the collection.

We use the following notation only in Proposition 3.11 and in the proof of the Proposition 3.11 in Section 3.7.2. Let $h$ be a non-negative integer. The symbol $N(h, S)$ denotes

- $\langle S \rangle$, if $h = 0$;
- $\langle N(h - 1, S) \rangle$, otherwise.

Finally, the disjoint union of any two collections of ovals, realizing respectively $S'$ and $S''$ in $\mathbb{R}^2$ (resp. $\mathbb{R}P^2$), is denoted by $S' \sqcup S''$ if none of the ovals of one collection forms an injective pair with the ovals of the other one. Moreover, a pseudo-line in
$\mathbb{R}P^2$ is denoted by $\mathcal{J}$. Since $\mathbb{R}^2$ is homeomorphic to $S^2$ deprived of a point, we say that the pair $(S^2, \bigsqcup_{i=1,\ldots,l} B_i)$ realizes $\mathcal{S}$ if there exists a point $p \in S^2 \setminus \bigsqcup_{i=1,\ldots,l} B_i$ such that $(S^2 \setminus \{p\}, \bigsqcup_{i=1,\ldots,l} B_i)$ realizes $\mathcal{S}$.

**Example 2.1.** Assume to have 8 disjoint ovals $B_i$ in $S^2$ as depicted in a) of Fig. [2] (resp. 7 disjoint embedded circles $B'_i$ in $\mathbb{R}P^2$ as depicted in b) of Fig. [3]. We say that the pair $(S^2, \bigsqcup_{i=1,\ldots,8} B_i)$ realizes $\mathcal{J} \sqcup (\langle \{1\} \rangle \sqcup \langle 2 \rangle)$. Let $S$ and $A$ circles embedded in $\mathbb{R}$ realizes $S$ (resp. $\mathbb{R}$).

Finally, we need some more definitions for particular collections of ovals in $\mathbb{R}P^2$ and $S^2$.

**Definition 2.2.** Let $(X, \sigma)$ be a real algebraic surface. An arrangement $\mathcal{S}$ of disjoint circles embedded in $\mathbb{R}X$ is called real scheme. Let $A \subset X$ be a real curve. We say that $\mathcal{A}$ has real scheme $\mathcal{S}$ if the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $\mathcal{S}$.

2.2 Hirzebruch surfaces

A Hirzebruch surface is a compact complex surface which admits a holomorphic fibration over $\mathbb{C}P^1$ with fiber $\mathbb{C}P^1$ ([Brau3]). Every Hirzebruch surface is biholomorphic to exactly one of the surfaces $\Sigma_n = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \mathbb{C})$ for $n \geq 0$. The surface $\Sigma_n$ admits a natural fibration

$$
\pi_n : \Sigma_n \to \mathbb{C}P^1
$$

with fiber $\mathbb{C}P^1 =: F_n$. Denote by $B_n$, resp. $E_n$, the section $\mathbb{P}(\mathcal{O}_{\mathbb{C}P^1}(n) \oplus \{0\})$, resp. $\mathbb{P}(\{0\} \oplus \mathbb{C})$. The self-intersection of $B_n$ (resp. $E_n$ and $F_n$) is $n$ (resp. $-n$ and 0). When $n \geq 1$, the exceptional divisor $E_n$ determines uniquely the Hirzebruch surface since it is the only irreducible and reduced algebraic curve in $\Sigma_n$ with negative self-intersection.

For example $\Sigma_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$. The Hirzebruch surface $\Sigma_1$ is the complex projective plane blown-up at a point, and $\Sigma_2$ is the quadratic cone with equation $Q : X^2 + Y^2 - Z^2 = 0$ blown-up at the node in $\mathbb{C}P^3$. The fibration of $\Sigma_2$ (resp. of $\Sigma_1$) is the extension of the projection from the blown-up point to a hyperplane section (resp. to a line) which does not pass through the blown-up point.
The group $H_2(\Sigma_n; \mathbb{Z})$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ and is generated by the classes $[B_n]$ and $[F_n]$. An algebraic curve $C$ in $\Sigma_n$ is said to be of bidegree $(a, b)$ if it realizes the homology class $a[B_n] + b[F_n]$ in $H_2(\Sigma_n; \mathbb{Z})$. Note that $[E_n] = [B_n] - n[F_n]$ in $H_2(\Sigma_n; \mathbb{Z})$. An algebraic curve of bidegree $(3, 0)$ on $\Sigma_n$ is called a trigonal curve.

We can obtain $\Sigma_{n+1}$ from $\Sigma_n$ via a birational transformation $\beta_n^n : \Sigma_n \to \Sigma_{n+1}$ which is the composition of a blow-up at a point $p \in E_n \subset \Sigma_n$ and a blow-down of the strict transform of the fiber $\pi_n^{-1}(\pi_n(p))$.

The surface $\Sigma_n$ is also the projective toric surface which corresponds to the polygon of vertices $(0,0), (0,1), (1,1), (n+1,0)$, depicted in Fig. 3(a) where the number labeling an edge corresponds to its integer length. The Newton polygon of an algebraic curve $C$ of bidegree $(a,b)$ on $\Sigma_n$, lies inside the trapeze with vertices $(0,0), (0,a), (b,a), (an+b,0)$ as in Fig. 3(b). The surface $\Sigma_n$ is canonically endowed by a real structure induced by the standard complex conjugation in $(\mathbb{C})^2$. For this real structure $\mathbb{R}\Sigma_n$ is a torus if $n$ is even and a Klein bottle if $n$ is odd. We will depict $\mathbb{R}\Sigma_n$ as a square whose opposite sides are identified in a suitable way. Moreover, the horizontal sides will represent $\mathbb{R}E_n$. Moreover, let $C$ be any type I real algebraic curve in $\Sigma_n$, the depicted orientation on $\mathbb{R}C$ will denote a complex orientation of the curve.

The restriction of $\pi_n$ to $\mathbb{R}\Sigma_n$ defines an $S^1$-bundle over $S^1$ that we denote by $L$. We are interested in the isotopy types with respect to $L$ of real algebraic curves in $\mathbb{R}\Sigma_n$.

**Definition 2.5.**

- An arrangement $\eta$ of immersed circles and points in $\mathbb{R}\Sigma_n$ is called a real scheme. We say that a real algebraic curve $C \subset \mathbb{R}\Sigma_n$ has real scheme $\eta$ if the pair $(\mathbb{R}\Sigma_n, \mathbb{R}C)$ realizes $\eta$.

- Two arrangements of immersed circles and points in $\mathbb{R}\Sigma_n$ are $L$-isotopic if there exists an isotopy of $\mathbb{R}\Sigma_n$ which brings one arrangement to the other, each line of $L$ to another line of $L$ and whose restriction to $\mathbb{R}E_n$ is an isotopy of $\mathbb{R}E_n$.

- An arrangement of circles and points immersed in $\mathbb{R}\Sigma_n$ up to $L$-isotopy of $\mathbb{R}\Sigma_n$ is called an $L$-scheme.

- An $L$-scheme is realizable by a real algebraic curve of bidegree $(a,b)$ in $\Sigma_n$ if there exists such a curve whose real part is $L$-isotopic to the arrangement of circles and points in $\mathbb{R}\Sigma_n$.

- A trigonal $L$-scheme is an $L$-scheme in $\mathbb{R}\Sigma_n$ which intersects each fiber in 1 or 3 real points counted with multiplicities and which does not intersect $\mathbb{R}E_n$.

### 2.3 Dessins d’enfant

Orevkov in [Ore03] has formulated the existence of real algebraic trigonal curves realizing a given trigonal $L$-scheme in $\mathbb{R}\Sigma_n$ in terms of the existence of a real rational graph on $\mathbb{C}P^1$. In the proof of Lemmas 3.32, 3.33 (Section 3) and of Proposition 4.24 (Section 4), we use this construction technique.

**Definition 2.6.** Let $n$ be a fixed positive integer. We say that a graph $\Gamma$ is a real trigonal graph of degree $n$ if
• it is a finite oriented connected graph embedded in $\mathbb{C}P^1$, invariant under the standard complex conjugation of $\mathbb{C}P^1$;

• it is decorated with the following additional structure:
  
  – every edge of $\Gamma$ is colored solid, bold or dotted;
  – every vertex of $\Gamma$ is $\bullet$, $\circ$, $\times$ (said essential vertices) or monochrome

and satisfying the following conditions:

1. any vertex is incident to an even number of edges; moreover, any $\circ$-vertex (resp. $\bullet$-vertex) to a multiple of 4 (resp. 6) number of edges;

2. for each type of essential vertices, the total sum of edges incident to the vertices of a same type is $12n$;

3. the orientations of the edges of $\Gamma$ form an orientation of $\partial(\mathbb{C}P^1 \setminus \Gamma)$ which is compatible with an orientation of $\mathbb{C}P^1 \setminus \Gamma$ (see Fig. 5);

4. all edges incidents to a monochrome vertex have the same color;

5. $\times$-vertices are incident to incoming solid edges and outgoing dotted edges;

6. $\circ$-vertices are incident to incoming dotted edges and outgoing bold edges;

7. $\bullet$-vertices are incident to incoming bold edges and outgoing solid edges.

Let $n$ be a positive integer and let $c(x, y) = y^3 + b_2(x)y + b_3(x)$ be a real polynomial, where $b_i(x)$ has degree in $x$. By a suitable change of coordinates in $\Sigma_n$, any trigonal curve $C$ in $\Sigma_n$ can be put into this form. Denote by $\Delta = -4b_2^3 + 27b_3^2$ the discriminant of $c(x, y)$ with respect to the variable $y$. The knowledge of the arrangement of the real roots of the real polynomials $\Delta = -4b_2^3 + 27b_3^2$, $27b_3^2$ and $-4b_2^3$ in $\mathbb{R}\Sigma_n$ allows to recover the trigonal $\mathcal{L}$-scheme realized by $C$ in $\mathbb{R}\Sigma_n$. Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be the homogenized discriminant, i.e. the rational function defined by $f := \frac{\Delta}{27b_3}$. Orevkov’s method allows to construct real polynomials $c(x, y)$ which have prescribed arrangements of the real roots and the construction is based on a consideration of the arrangement of the graph given by $f^{-1}(\mathbb{R}P^1)$ with the coloring and orientation induced by those of $\mathbb{R}P^1$ as depicted in a) of Fig 4. In this section, we only give an algorithmic way to encode any trigonal $\mathcal{L}$-scheme in $\mathbb{R}\Sigma_n$ into a colored oriented graph on $\mathbb{R}P^1 \subset \mathbb{C}P^1$ just looking at the intersections of the fibers of $\mathcal{L}$ with $\eta$; for details see [Bru04], [Ore95].

![Diagram](image)

Figure 4: a) Colored oriented $\mathbb{R}P^1$. b) $\mathcal{L}_{|_I}$ in $\mathbb{R}\Sigma_n$.

**Definition 2.7.** Let $\eta$ be a trigonal $\mathcal{L}$-scheme. For any fixed interval of points $I := \{(x, y) : y \in \mathbb{R}, x \in [a, a + b) \subset \mathbb{R}, a \neq b\} \subset \mathbb{R}\Sigma_n$, we denote with $\mathcal{L}_{|_I}$ the fibers of $\mathcal{L}$ containing the points of $I$ (see b) of Fig. 7). Thanks to $\pi_{|_I,|_I}$, we can encode $\eta$ into a colored oriented graph $\Gamma$ on $\mathbb{R}P^1 \subset \mathbb{C}P^1$ as follows (in Fig. 3 the dashed lines denote fibers of $\mathcal{L}$):

1. To each fiber of $\mathcal{L}$ intersecting $\eta$ in two points we associate a $\times$-vertex on $\mathbb{R}P^1$.

2. Fixed an interval $I$, let $F_1, F_2$ be two fibers of $\mathcal{L}_{|_I}$ intersecting $\eta$ in two points such that $\eta$, up to $\mathcal{L}$-isotopy, is locally as depicted in b) or c) of Fig. 7. Let $F_3$ be another fiber between $F_1, F_2$. Then, we associate to $F_3$ a $\circ$-vertex on $\mathbb{R}P^1$. Moreover, if between $F_1$ and $F_2$ each other fiber intersects $\eta$ in only one real point (as in b) of Fig. 7), then we associate to a fiber between $F_1$ and $F_3$ (resp. $F_3$ and $F_2$) a $\bullet$-vertex on $\mathbb{R}P^1$. Between $\bullet$ and $\circ$-vertices we put bold edges.
3. For all intervals $I$, except for the fibers of $\mathcal{L}_1$, to which we associate essential vertices and bold edges, we associate dotted (resp. solid) edges on $\mathbb{R}P^1$ to the fibers of $\mathcal{L}_1$, which intersect $\eta$ in three distinct real points (resp. only one real point).

4. The orientations of the edges incident to a vertex are in an alternating order. In particular, the orientations of the edges incident to an essential vertex are respectively as described in 5., 6., 7. of Definition 2.6.

The graph $\Gamma$, called real graph, is considered up to isotopy of $\mathbb{R}P^1$, namely it is determined by the order of its colored vertices since the edges are determined by the color of their adjacent vertices.

We say that $\Gamma$ is complementable in degree $n$ if there exists a complete real trigonal graph $\Gamma$ of degree $n$ such that $\Gamma \cap \mathbb{R}P^1 = \Gamma$.

![Figure 5: Local topology of trigonal $\mathcal{L}$-schemes and their corresponding real graphs.](image)

**Theorem 2.8 ([Ore03], [Deg12]).** A trigonal $\mathcal{L}$-scheme on $\mathbb{R}\Sigma_n$ is realizable by a real algebraic trigonal curve if and only if its real graph is complementable in degree $n$.

Given a real graph $\Gamma$, we depict only the completion to a real trigonal graph $\Gamma$ on a hemisphere of $\mathbb{C}P^1$ since $\Gamma$ is symmetric with respect to the standard complex conjugation. Moreover, we can omit orientations in figures representing real trigonal graphs because each vertex is adjacent to an even number of edges oriented in an alternating order as, for example, depicted in Fig. 6, and such orientations are compatible with each others.

![Figure 6: Colored vertices of a real trigonal graph.](image)

3 Real curves on $k$-spheres real del Pezzo surfaces of degree 2

3.1 Definitions

Let $X$ be $\mathbb{C}P^2$ blown up at seven points in generic position; then, the surface $X$ is a del Pezzo surface of degree 2 (see [BCC+08], pag. 289-312, Chapter 8). The anti-canonical system $\phi : X \rightarrow \mathbb{C}P^2$ exhibits $X$ as a double ramified cover of $\mathbb{C}P^2$; the branch locus of $\phi$ consists of an irreducible non-singular quartic $Q$ defined by a homogeneous polynomial $f(x, y, z)$. By construction, the first Chern class $c_1(X)$ is the pull back via $\phi$ of the class of a line in $\mathbb{C}P^2$ ([DIK00]). Moreover, the surface $X$ is isomorphic to the real hypersurface in $\mathbb{C}P(1, 1, 1, 2)$ defined by the weighted polynomial equation $f(x, y, z) = w^2$, with coordinates $x$, $y$, $z$ and $w$ respectively of
weights 1 and 2. Conversely, any double cover of $\mathbb{CP}^2$ ramified along a non-singular algebraic quartic yields a del Pezzo surface of degree 2.

If one equips $X$ with a real structure $\sigma$, the quartic $\overline{Q}$ is real and $f(x, y, z)$ can be chosen with real coefficients and so that the real surface $(X, \sigma)$ is isomorphic to the real hypersurface in $\mathbb{CP}(1,1,1,2)$ of equation $f(x, y, z) = w^2$. It follows that the double cover $\phi$ projects $\mathbb{R}X$ into the region

$$\Pi_+ := \{ [X : Y : Z] \in \mathbb{RP}^2 : f(x, y, z) \geq 0 \}.$$

Conversely, the double cover of $\mathbb{CP}^2$ ramified along a non-singular real quartic $\overline{Q} \subset \mathbb{CP}^2$ and a choice of a real polynomial equation $f(x, y, z)$ of $\overline{Q}$ yields a real del Pezzo surface $X$. The surface $X$ is $\mathbb{R}$-minimal if and only if $\mathbb{R}X$ is homeomorphic either to $\bigsqcup_{i=1}^k S^2$ or to $\bigsqcup_{i=1}^{2k} S^2$. Moreover $X$ has $\mathbb{R}X$ homeomorphic to $\bigsqcup_{i=1}^k S^2$, with $1 \leq k \leq 4$, if and only if $\overline{Q}$ is a non-singular real quartic realizing the real scheme $k$ in $\mathbb{RP}^2$ and $\Pi_+$ is orientable; see [DK02] and, as example for $k = 4$, Fig. 7 (where $\Pi_+$ is in gray on the right).

**Definition 3.1.** Let $(X, \sigma)$ be a real degree 2 del Pezzo surface. If $\mathbb{R}X$ is homeomorphic to $\bigsqcup_{i=1}^k S^2$, with $1 \leq k \leq 4$, we say that $X$ is a $k$-spheres real del Pezzo surface of degree 2.

**Notation 3.2.** Let $X$ be a $k$-spheres real del Pezzo surface of degree 2. We denote the connected components of $\mathbb{R}X$ with $X_1, \ldots, X_k$.

The lifting of a non-singular real algebraic curve $C \subset \mathbb{CP}^2$ of degree $d$ via $\phi$ is a real algebraic curve $A \subset X$ realizing $dc_1(X)$ in $H_2(X; \mathbb{Z})$. Moreover, from the topological arrangement of the triplet $(\mathbb{RP}^2, \overline{Q}, \mathbb{R}C)$, one recovers the topological arrangement of the pair $(\mathbb{R}X, \mathbb{R}A)$. As example, assume that $d = 1$, the quartic $\overline{Q}$ is maximal, $\Pi_+$ is orientable and $\mathbb{R}Q \cup \mathbb{R}C$ is arranged in $\mathbb{RP}^2$ as depicted in Fig. 4 on the right; then the pair $(\mathbb{R}X, \mathbb{R}A)$ has topological arrangement as pictured on the left of Fig. 4.

![Figure 7: Example: $\phi : \mathbb{R}X \mapsto \Pi_+$](image)

**Definition 3.3.** Let $X$ be a $k$-spheres real del Pezzo surface of degree 2, with $1 \leq k \leq 4$, and let $A \subset X$ be a non-singular real algebraic curve. Then, we say that $A$ has class $d$ on $X$ if $A$ realizes $dc_1(X)$ in $H_2(X; \mathbb{Z})$.

If $X$ is a 4-spheres real del Pezzo surface of degree 2, any real algebraic curve has class $d$, where $d$ is some non-negative integer ([BCC+08, pag. 289-312]). Combining Harnack-Klein’s inequality and the adjunction formula, one obtains the following immediate result.

**Proposition 3.4.** Let $A$ be a real algebraic curve of class $d$ in a $k$-spheres real del Pezzo surface $X$ of degree 2, with $1 \leq k \leq 4$. Then, the number $l$ of ovals of $\mathbb{R}A$ is bounded as follows:

$$l \leq d(d - 1) + 2.$$

### 3.2 Main results

**Definition 3.5.** Let $X$ be a $k$-spheres real del Pezzo surface of degree 2 with $1 \leq k \leq 4$. Let $S$ be a real scheme on $\mathbb{R}X$. We say that $S$ is in class $d$ if the number of connected components of $S$ does not exceed $d(d - 1) + 2$. Moreover, we say that $S$ is realizable in class $d$, if there exist a real algebraic curve $A \subset X$ of class $d$, such that the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $S$. 

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Definition 3.6. (1) Let $A$ be a class $d$ algebraic curve in a $k$-spheres real degree 2 del Pezzo surface $X$, with $1 \leq k \leq 4$. Let $\phi : X \to \mathbb{C}P^2$ be the anti-canonical map. We say that $A$ is symmetric if there exists a degree $d$ algebraic curve $B \subset \mathbb{C}P^2$ such that $A = \phi^{-1}(B)$. Otherwise, we say that $A$ is non-symmetric.

(2) Let $S$ be a real scheme in class $d$ in $\mathbb{R}X$. We say that $S$ is symmetric in class $d$ if there exists a symmetric real algebraic curve $A \subset X$ of class $d$ realizing it. Otherwise, we say that $S$ is non-symmetric in class $d$.

See Section [2.1] for real schemes notation. In order to simplify the presentation of Proposition [3.8] and Proposition [3.12], let us introduce one more definition.

Definition 3.7. Let $S_1 : \ldots : S_k$ be an arrangement of ovals in the disjoint union of $k$ spheres and let $X$ be a $j$-spheres real del Pezzo surface of degree 2 with $1 \leq j < k \leq 4$. If $s$ among the $S_s$’s are 0, where $s$ is at least $k - j$, we say that $S_1 : \ldots : S_k$ is realizable in class $d$ on $\mathbb{R}X$ if there exist a real algebraic curve $A \subset X$ of class $d$, such that the pair $(\mathbb{R}X, \mathbb{R}A)$ realizes $S_1 : \ldots : S_j$, where $\{S_1, \ldots, S_k\} = \{0, S'_1, \ldots, S'_j\}$.

Proposition [3.8] gives a complete real schemes classification in class 1 and 2 on $k$-spheres real del Pezzo surfaces of degree 2.

Proposition 3.8 (Class 1 and 2). Let $A$ be a non-singular real algebraic curve of class $d = 1, 2$ in a $k$-spheres real del Pezzo surface $X$ of degree 2, with $1 \leq k \leq 4$. Then, the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the following real schemes:

(i) if $k = 4, 3, 2, 1$ and

(1) $d = 1$:

- $\alpha : 0 : 0 : 0$, with $0 \leq \alpha \leq 2$;

(2) $d = 2$:

- $\alpha : 0 : 0 : 0$, with $0 \leq \alpha \leq 4$;
- $\langle \alpha \rangle \sqcup \langle \beta \rangle : 0 : 0 : 0$, with $0 \leq \alpha + \beta \leq 2$.

(ii) if $k = 4, 3, 2$ and

(1) $d = 1$:

- $1 : 1 : 0 : 0$;

(2) $d = 2$:

- $\alpha : \beta : 0 : 0$, with $0 \leq \alpha + \beta \leq 4$;
- $\langle \alpha \rangle \sqcup \langle \beta \rangle : \gamma : 0 : 0$, with $0 \leq \alpha + \beta + \gamma \leq 2$.

(iii) if $k = 4, 3$ and $d = 2$:

- $\alpha : \beta : \gamma : 0$, with $0 \leq \alpha + \beta + \gamma \leq 4$.

(iv) if $k = 4$ and $d = 2$:

- $1 : 1 : 1 : 1$.

Furthermore, for each real scheme $S$ respectively listed in (i) – (iv), there exist a $k$-spheres real degree 2 del Pezzo surface $X$ and a symmetric real algebraic curve of class $d$ in $X$ realizing $S$.

Proof. All real schemes listed above are those non-prohibited by Proposition 3.4. In Section 3.3 we construct class 1, 2 real curves in $k$-spheres real del Pezzo surfaces of degree 2 realizing all real schemes listed above.

For real schemes in class $d \geq 3$ in $k$-spheres real del Pezzo surfaces of degree 2, Proposition 3.4 does not provide a complete system of restrictions anymore. In this paper, we mainly focus on classifications in 4-spheres del Pezzo surfaces of degree 2: in Section 3.3 we show how to use Welschinger-type invariants to obtain Bézout-type obstructions for real schemes in class $d$. Moreover, we construct real algebraic curves via the anti-canonical map (Section 3.5) and degeneration methods (Section 3.7). We use such obstructions and constructions to give a partial classification of real schemes in class 3 with 8 ovals (Theorem 3.9), and to realize some non-symmetric real schemes for each class $d \geq 5$ with $2d + 1$ ovals in 4-spheres del Pezzo surfaces of degree 2. After, we use the classification tools adopted on 4-spheres del Pezzo surfaces of degree 2 in the case of $k$-spheres real degree 2 del Pezzo surfaces, with $k < 4$ (Proposition 3.12).
Theorem 3.9 (Class 3 and \( k = 4 \)). Let \( A \) be a non-singular real algebraic maximal curve of class 3 in a 4-spheres real del Pezzo surface \( X \) of degree 2. Then, the pair \((RA,RX)\) realizes one of the real schemes listed in Tables 1 and 2. Moreover, for each real scheme \( S \) labeled with \((*)\) (resp. \((\circ)\)) in Tables 1 and 2, there exist a 4-spheres real degree 2 del Pezzo surface \( X \) and a (symmetric) real maximal curve of class 3 in \( X \) realizing \( S \).

Table 1: Real schemes in class 3 in \( k \)-spheres real del Pezzo surfaces of degree 2.

```
\begin{align*}
\text{Table 1: Real schemes in class 3 in } \mathbb{R}^3 \\
\text{real degree } \mathbb{R}^3 \\
\hline
\text{realizes one of the real schemes listed in Tables 1 and 2. Moreover, for each real scheme } \mathbb{R}^3 \\
\hline
\text{real del Pezzo surface } \mathbb{R}^3, \text{ realized by } \mathbb{R}^3, \text{ in a } \mathbb{R}^3-\text{spheres real del Pezzo surfaces of degree } \mathbb{R}^3.
\end{align*}
```

```
\begin{align*}
\text{Table 2: Real schemes in class 3 in } \mathbb{R}^3 \text{-spheres real del Pezzo surfaces of degree 2.}
\end{align*}
```

```
\begin{align*}
\text{Theorem 3.9 (Class 3 and } k = 4 \text{). Let } A \text{ be a non-singular real algebraic maximal curve of class 3 in a 4-spheres real del Pezzo surface } X \text{ of degree 2. Then, the pair } (RA,RX) \text{ realizes one of the real schemes listed in Tables 1 and 2. Moreover, for each real scheme } S \text{ labeled with } (*) \text{ (resp. } (\circ)\text{) in Tables 1 and 2, there exist a 4-spheres real degree 2 del Pezzo surface } X \text{ and a (symmetric) real maximal curve of class 3 in } X \text{ realizing } S.\\
\end{align*}
```
Proof. All real schemes in Tables 1 and 2 are those non-prohibited by Propositions 3.4, 3.14 and Lemmas 3.16, 3.18 (see Proposition 3.19). The constructions of (symmetric) class 3 real maximal curves in 4-spheres real del Pezzo surfaces of degree 2 realizing all real schemes labeled with \((\ast)\) (resp. with \((\circ)\)) in Tables 1 and 2 are in the proof of Proposition 3.35 (resp. Proposition 3.21).

The real scheme \(2 \sqcup\langle X\rangle\) in class 3 of Table 1 is symplectically realizable. The proof of this is in Section 3.6.

**Proposition 3.10.** There exist a 4-spheres real symplectic degree 2 del Pezzo surface \(X\) and a non-singular real symplectic curve of class 3 in \(X\) realizing the real scheme \(2 \sqcup\langle X\rangle\) in class 3 of Table 1.

For each class \(d \geq 5\), we realize at least two non-symmetric real schemes in class \(d\). The proof of the following statement is in Section 3.7.

**Proposition 3.11.** Let \(d, k_1, k_2, h_1, h_2, h_3, h_4\) be non-negative integers such that

- \(d \geq 5\);
- \(k_1 + k_2 = d - 4\);
- \(\sum_{i=1}^{4} h_i = d - 1\);
- \(h_j \neq 2\), for \(j = 3, 4\).

Then, each of the following real schemes \(S\) is non-symmetric in class \(d\) and there exists a 4-spheres real del Pezzo surface \(X\) of degree 2 and a class \(d\) real algebraic curve \(A \subset X\) such that the pair \((\mathbb{R}X, RA)\) realizes \(S\).

1. for \(k_1 = 0\) and \(h_1 \geq 1\)
   - \((i)\) \(1 \sqcup N(h_1+1,0) : 1 \sqcup N(k_2,1 \sqcup \langle 1 \rangle) \sqcup N(h_2,0) : N(h_3,0) : N(h_4,0)\)
   - \((ii)\) \(1 \sqcup \langle 1 \rangle \sqcup N(h_1+1,0) : 1 \sqcup N(k_2,1) \sqcup N(h_2,0) : N(h_3,0) : N(h_4,0)\)

2. for \(d \geq 6\) and \(k_1, k_2 \neq 0\)
   - \((i)\) \(1 \sqcup N(k_1,1) \sqcup N(h_1,0) : N(k_2,1 \sqcup \langle 1 \rangle) \sqcup N(h_2,0) : N(h_3,0) : N(h_4,0)\)
   - \((ii)\) \(1 \sqcup N(k_2,1 \sqcup \langle 1 \rangle) \sqcup N(h_2,0) : N(k_1,1) \sqcup N(h_1+1,0) : N(h_3,0) : N(h_4,0)\)

3. for \(k_1 = 0\) and \(h_1 \neq 1\)
   - \((i)\) \(1 \sqcup N(h_1,0) \sqcup N(h_2+1,0) : N(k_2+1,1 \sqcup \langle 1 \rangle) : N(h_3,0) : N(h_4,0)\)
   - \((ii)\) \(1 \sqcup \langle 1 \rangle \sqcup N(h_1,0) \sqcup N(h_2+1,0) : N(k_2+1,1) : N(h_3,0) : N(h_4,0)\)

4. for \(d \geq 6\) and \(k_1, k_2 \neq 0\)
   - \((i)\) \(N(k_2,1 \sqcup \langle 1 \rangle) \sqcup N(h_2+1,0) \sqcup N(h_1,0) : N(k_1+1,1) : N(h_3,0) : N(h_4,0)\)
   - \((ii)\) \(N(k_1+1,1) \sqcup N(h_2+1,0) \sqcup N(h_1,0) : N(k_2,1 \sqcup \langle 1 \rangle) : N(h_3,0) : N(h_4,0)\)

The list of real schemes in class 3 in Table 1 and on the left of Table 2 is not complete in the case of \(k\)-spheres real del Pezzo surface of degree 2, with \(k = 3, 2, 1\) (see Section 3.3). In this paper, we prove the following result.
Proposition 3.12 (Class 3 and k = 3, 2, 1). Let A be a non-singular real algebraic maximal curve of class 3 in a k-spheres real del Pezzo surface X of degree 2, with 1 ≤ k ≤ 3. Then, the pair \((R, A, RX)\) realizes one of the real schemes listed in Tables 1, 2 and 3. Moreover, for any 1 ≤ k ≤ 3, and for each real scheme S labeled with \((\ast)\) (resp. \((\ast)\)) in Tables 1 and on the left of Table 2, there exists a k-spheres real degree 2 del Pezzo surface X and a (symmetric) real maximal curve of class 3 in X realizing S.

Proof. All real schemes in class 3 listed in Tables 1, 2 and 3 are those non-prohibited by Proposition 3.14 for k = 3, 2 and Lemma 3.15 for k = 1 (see Proposition 3.19). The construction of (symmetric) real algebraic curves of class 3 realizing all real schemes labeled with \((\ast)\) (resp. \((\ast)\)) in Tables 1 and on the left of Table 2 is in the proof of Proposition 3.15 (resp. Proposition 3.22). \(\square\)

\[
\begin{align*}
& k = 3, 2, 1 \\
& (1) \sqcup (1) \sqcup (1) \sqcup (1) : 0 : 0 : 0 \quad (1) \sqcup (1) \sqcup (1) : (1) : 1 : 0 \\
& k = 3, 2 \\
& (1) \sqcup (1) : (1) \sqcup (1) : 0 : 0 \quad (1) \sqcup (1) : 2 : 0 : 0 \\
& 1 \sqcup (2) : (1) \sqcup (1) : 0 : 0 \quad 1 \sqcup (2) : (1) : 1 : 0 \\
& 1 \sqcup (1) \sqcup (1) : (1) : 0 : 0 \quad 1 \sqcup (1) : 3 : 1 : 0 \\
& (1) \sqcup (2) : (1) : 0 : 0
\end{align*}
\]

Table 3: Real schemes in class 3 in k-spheres real del Pezzo surfaces of degree 2, with \(k = 3, 2, 1\).

3.3 Obstructions based on Welschinger-type invariants

Welschinger invariants can be regarded as real analogues of genus zero Gromov-Witten invariants. They were introduced in [Wel05] and count, with appropriate signs, the real rational curves which pass through a given real collection of points in a given real rational algebraic surface. In the case of k-spheres real del Pezzo surfaces of degree 2, the Welschinger invariants, as well as their generalizations to higher genus ([Shu14]), can be used to prove the existence of interpolating real curves of genus 0 ≤ g ≤ k − 1; see [IKS15] and [Shu14].

Proposition 3.13. ([Shu14], Propositions 4 and 5) Let s be an integer greater than 1 and \(r_1, r_2\) be two non-negative odd integers such that \(r_1 + r_2 = 2s\). Let \(P\) be a generic configuration of \(2s + j\) real points, with \(j = 2, 1, 0\), on a k-spheres real del Pezzo surface X of degree 2, where \(k = 2 + j\), such that

- \(X_i\) contains \(r_i\) points of \(P\), with \(i = 1, 2\);
- \(X_i\) contains one point of \(P\), if \(i \neq 1, 2\).

Then, there exists a real algebraic curve T of class s and genus \(j + 1\) in X passing through P. Furthermore, the points of P belong to the one-dimensional connected components of \(R \overline{T}\).

We use the results of Proposition 3.13 to prove the following proposition.

Proposition 3.14. Let s be an integer strictly greater than 1 and \(r_1, r_2\) be two non-negative odd integers such that \(r_1 + r_2 = 2s\). Moreover, let A be a non-singular real algebraic curve of class d in a k-spheres real del Pezzo surface X of degree 2, with \(k = 4, 3, 2\). Let t denote the number of connected components of RX to which RA belongs. Assume that RA has \(r_i\) disjoint nests \(N_h\) of depth \(j_h\) on \(X_i\), respectively with \(1 ≤ h ≤ r_i\) for \(i = 1\), and with \(r_1 + 1 ≤ h ≤ 2s\) for \(i = 2\).

1. If \(r_1, r_2 > 1\), then \(\sum_{h=1}^{2s} j_h ≤ ds − (t − 2)\);

2. If \(r_1 = 2s − 1\) and \(r_2 = 1\), then \(\sum_{h=1}^{2s−1} j_h ≤ ds − (t − 1)\).
Proof. Assume that \( r_1 \) and \( r_2 \) are strictly greater than 1. It follows that \( \mathbb{R} \mathcal{A} \) has at least 3 disjoint nests on \( X_i \), with \( i = 1, 2 \). In order to prove inequality (1), let us choose a generic collection \( \mathcal{P} \) of \( 2k + j \) real points, with \( j = 2, 1, 0 \), in the following way. On each boundary of the \( r_1 \) (resp. \( r_2 \)) disks in \( X_1 \setminus \bigcup_{h=1}^{r_1} N_h \) (resp. \( X_1 \setminus \bigcup_{h=r_1+1}^{2k} N_h \)), pick a point. Moreover, pick a point on every connected component \( X_i \), with \( i = 3, 4 \), such that the point belongs to \( \mathbb{R} \mathcal{A} \) any time the real algebraic curve has at least one oval on \( X_i \). Then, Proposition 3.13 assures the existence of a real algebraic curve \( T \) of class \( s \) and genus \( j + 1 \) on \( X \) passing through \( \mathcal{P} \). Furthermore, the points of \( \mathcal{P} \) belong to the one-dimensional connected components of \( \mathbb{R} \mathcal{T} \). Thus, the number of real intersection points of \( A \) with \( T \) is at least \( 2(\sum_{h=1}^{2k} j_h + (t - 2)) \). Inequality (1) follows directly from the fact that the intersection number \( A \circ T = 2dk \) is greater or equal than the number of real intersection points of \( A \) with \( T \).

The proof of (2) is similar to the previous one.

The following statement gives more topological obstructions for real curves in 1-spheres real del Pezzo surfaces of degree 2.

**Lemma 3.15.** Let \( A \) be a real algebraic curve in class \( d \) in a 1-spheres real del Pezzo surface \( X \) of degree 2. Assume that \( \mathbb{R} \mathcal{A} \) has three nests \( N_h \) of depth \( j_h \) on \( X_1 \). Then,

\[
\sum_{i} j_i + j_2 + j_3 \leq 2d.
\]

**Proof.** The statement follows from

- the existence of a real algebraic curve \( T \) of class 2 and genus 0 on \( X \) passing through a given real configuration of 3 distinct points on \( X_1 \) (see [IKS15, Table 1, Section 2.2]) and

- an argument similar to that used in the proof of Proposition 3.14.

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We have one more Bézout-type restriction on the topology of real curves in 4-spheres real del Pezzo surfaces of degree 2.

**Lemma 3.16.** Let \( A \) be a real algebraic curve in class \( d \) in a 4-spheres real del Pezzo surface \( X \) of degree 2. Assume that \( \mathbb{R} \mathcal{A} \) has a nest \( N_1 \) of depth \( j_1 \) on \( X_1 \) and a nest \( N_2 \) of depth \( j_2 \) on \( X_2 \). Let \( t \) denote the number of connected components of \( \mathbb{R} X \) to which \( \mathbb{R} \mathcal{A} \) belongs. Then,

\[
\sum_{i} j_i + j_2 \leq 2d - (t - 2).
\]

**Proof.** Unlike in the proof of Proposition 3.14, we do not need Proposition 3.13. In fact, for a given configuration \( \mathcal{P} \) of 6 points in \( X \) there always exists an algebraic curve \( T \) of class 2 and genus 3 passing through \( \mathcal{P} \). Moreover, if \( \mathcal{P} \) consists of only real points such that at least one point of \( \mathcal{P} \) lies on \( X_i \), \( \forall i \in \{1, 2, 3, 4\} \), the curve \( T \) is real and \( \mathbb{R} \mathcal{T} \) has exactly 4 connected components. Let us choose \( \mathcal{P} \) as follows. On each boundary of the 2 disks in \( X_i \setminus N_i \), with \( i = 1, 2 \), pick a point. In addition, pick a point on the connected components \( X_i \), with \( i = 3, 4 \), such that the point belongs to \( \mathbb{R} \mathcal{A} \) any time \( A \) has at least one oval on \( X_i \). Then, there exists a real algebraic curve \( T \) of class 2 and genus 3 passing through \( \mathcal{P} \) and \( \mathbb{R} \mathcal{T} \) has a connected component on each \( X_i \), with \( i = 1, 2, 3, 4 \), and it has at least a connected component of dimension 1 on \( X_i \), with \( i = 1, 2 \). Thus, the number of real intersection points of \( A \) with \( T \) is at least \( 2(j_1 + j_2 + (t - 2)) \). Since the intersection number \( A \circ T = 4d \) is greater or equal than the number of real intersection points of \( A \) with \( T \), we prove the inequality. See Example 3.17.

-------------------------------------------------------------------------------------------------------------------------------

**Example 3.17 (Application of the proof of Proposition 3.14).** Let us consider the real scheme \( \mathcal{S} := \{ 2 \} \cup \{ 1 \} : 1 \cup \{ 1 \} : 0 \) in a 4-spheres real del Pezzo surface \( X \) of degree 2. Let us illustrate the proof of the inequality in Lemma 3.16 for \( \mathcal{S} \); see Fig. 8. Assume that there exists a real algebraic curve \( A \) of class 3 in \( X \) realizing \( \mathcal{S} \). The real scheme \( \mathcal{S} \) has a nest of depth 3 on \( X_1 \), a nest of depth 3 in \( X_2 \) and an oval on \( X_3 \). Let us choose a configuration \( \mathcal{P} \) of 6 real points \( p_1, \ldots, p_6 \) as depicted in Fig. 8; then, there exists a real algebraic curve \( T \) of class 2 in \( X \) passing through \( \mathcal{P} \) and
RT has a single connected component on each $X_i$, with $i = 1, 2, 3, 4$. It follows that $T$ and $A$ intersect in at least 14 points, but they can not intersect in more than 12 points. In conclusion, the real scheme $S$ is unrealizable in class 3.

![Figure 8: Example of unrealizable real scheme in class 3. The real part of $A$ (resp. $T$) in thick black (resp. gray).](image)

A variant of the technique used in proof of Lemma 3.16 leads to prohibit a particular real scheme in class 3 in a 4-spheres real del Pezzo surface of degree 2.

**Lemma 3.18.** There is no real algebraic maximal curve of class 3 in a 4-spheres real del Pezzo surface $X$ of degree 2 realizing the real scheme $S := \{1\} \cup \{1\} \cup \{1\} : 0 : 0 : 0$.

**Proof.** Assume that there exists a non-singular real algebraic curve $A$ of class 3 realizing $S$ in $X$. Let us choose a configuration $P$ of 6 real points as follows. On each boundary of the 4 disks in $X_1 \setminus RA$, pick a point. Moreover, pick a point on the connected components $X_2$ and $X_3$. Then, there exists a non-singular real algebraic curve $T$ of class 2 passing through $P$ and $T$ has at most two ovals on $X_1$ and one oval on both $X_2$ and $X_3$; see Fig. 9. Thus, the number of real intersection points of $A$ with $T$ is at least 14. But the intersection number $A \circ T$ is 12.

![Figure 9: Unrealizability of $\{1\} \cup \{1\} \cup \{1\} : 0 : 0 : 0$ in class 3.](image)

Propositions 3.4, 3.14 and Lemmas 3.16, 3.18 imply the following proposition.

**Proposition 3.19.** Let $A$ be a non-singular real algebraic maximal curve of class 3 in a 4-spheres (resp. $k$-spheres) real del Pezzo surface $X$ of degree 2 (with $1 \leq k \leq 3$). Then, the pair $(\mathbb{R}A, \mathbb{R}X)$ realizes one of the real schemes in Tables 1 and 2 (resp. Tables 1, 2 and 3).

### 3.4 Class 1 and 2

Let us construct some symmetric real curves of class 1 and 2 on $k$-spheres real del Pezzo surfaces of degree 2.

**End of Proof of Proposition 3.8 (Class 1 and 2).** Fix a non-negative integer $k \leq 4$ and any real scheme $S$ in class $d = 1, 2$ listed in Proposition 3.8. It is easy to construct a non-singular real plane quartic $Q$, with real scheme $k$, and a line (resp. conic) $C$ in $\mathbb{CP}^2$ such that, via the double cover of $\mathbb{CP}^2$ ramified along $Q$, the lifting of $C$ is a class 1 (resp. 2) real algebraic curve realizing $S$ in a $k$-spheres real del Pezzo surface of degree 2. See Example 3.20.

**Example 3.20.** Fix $k = 4$. From the quartics and lines (resp. conics) arranged in $\mathbb{RP}^2$ as depicted in Fig. 10 (resp. 11), one construct 4-spheres real del Pezzo surfaces $X$ of degree 2 and class 1 (resp. 2) real algebraic curves in $X$ realizing all real schemes in class 1 (resp. all real schemes in class 2 with 4 ovals) listed in Proposition 3.8.
3.5 Class 3

Let us realize the symmetric real schemes in class 3 of Theorem 3.9 and Proposition 3.12 in $k$-spheres real del Pezzo surfaces of degree 2.

Proposition 3.21. For each real scheme $\mathcal{S}$ in Tables 1 and 2 labeled with $(\circ)$ (resp. $(\oplus)$), there exist a 4-spheres (resp. $k$-spheres) real degree 2 del Pezzo surface $X$ (with $1 \leq k \leq 3$) and a symmetric real algebraic maximal curve of class 3 in $X$ realizing $\mathcal{S}$.

Proof. In [Ore02], Orevkov has constructed real maximal quartics $\mathcal{Q}$ and cubics $\mathcal{C}$ arranged, up to isotopy, in $\mathbb{R}P^2$ as depicted in Fig. 12. To each such pair corresponds a real algebraic curve of class 3 in a 4-spheres real del Pezzo surface of degree 2 realizing one of the real schemes labeled with $(\circ)$ in Table 1 but the real schemes

$$
2 : 2 : 2 : 2, \quad 2 \quad (4) : 1 : 0 : 0, \quad 1 \quad (6) : 0 : 0 : 0,
$$

$$
6 : 2 : 0 : 0, \quad 1 \quad (4) : 2 : 0 : 0,
$$

$$
\langle 1 \rangle \quad (5) : 0 : 0 : 0 \quad \text{and} \quad 3 \quad (1) \quad (1) : 1 : 0 : 0
$$

Let us realize the real scheme $2 : 2 : 2 : 2$ in class 3. There exist a real cubic $\tilde{\mathcal{C}}$ and a real line $L$ in $\mathbb{C}P^2$ arranged in $\mathbb{R}P^2$ as represented in a) of Fig. 13.
\[ \tilde{p}(x, y, z) = 0 \quad \text{(resp.} \quad l(x, y, z) = 0) \] be a real polynomial equation defining \( \tilde{C} \) (resp. \( L \)). Pick three real lines \( L_1, L_2, L_3 \), as those depicted in dashed in \( a \) of Fig. 13.

Take a small perturbation of \( \tilde{C} \) replacing \( \tilde{p}(x, y, z) \) with \( p(x, y, z) := \tilde{p}(x, y, z) + \varepsilon l_1(x, y, z)l_2(x, y, z)l_3(x, y, z) \), where \( l_i(x, y, z) \) is a real polynomial defining \( L_i \), with \( i = 1, 2, 3 \), and \( \varepsilon \neq 0 \) is a sufficient small real number. Up to a choice of the sign of \( \varepsilon \), the real curve \( C \), defined by \( p(x, y, z) = 0 \), is a real cubic arranged in \( \mathbb{R}P^2 \) as depicted (in thick black) in \( b \) of Fig. 13. Let \( \bigcup_{i=1}^3 L_i \) be the union of four non-real lines pairwise complex conjugated and defined by a real polynomial \( u(x, y, z) \). Take a small
perturbation of $\tilde{C} \cup L$ replacing $\tilde{p}(x, y, z)l(x, y, z)$ with $\tilde{p}(x, y, z)l(x, y, z) + \delta u(x, y, z) = 0$, where $\delta \neq 0$ is a sufficient small real number. Up to the choice of the sign of $\delta$, such equation defines a non-singular real plane maximal quartic $\overline{Q}$ such that $Q \cup C$ is arranged in $\mathbb{R}P^2$ as pictured in $c$ of Fig. 13. It follows that there exists a 4-spheres real degree 2 del Pezzo surface $X$ and a real algebraic curve of class 3 in $X$ realizing the real scheme $2 : 2 : 2 : 2$.

Finally, for $k = 4$ (resp. $1 \leq k \leq 3$), we construct the remaining (resp. all) real schemes in Tables 1 and 2 labeled with (c) (resp. (c)). There exist a real quartic $\overline{Q}$ with real scheme $k$, where $2 \leq k \leq 4$ (resp. $1 \leq k \leq 4$), and three real lines arranged in $\mathbb{R}P^2$ as pictured in $a$, $c$, $d$ of Fig. 14 and in $b$ of Fig. 15 (resp. $b$ of Fig. 14 and $a$) of Fig. 15; where we depict only the ovals of $\mathbb{R}Q$ (in gray) intersecting the three lines (in thick black). Perturb the union of the three lines into a non-singular real cubic $C$ such that $C \cup \overline{Q}$ is arranged in $\mathbb{R}P^2$ respectively as depicted in $e) - h)$ of Fig. 14 and $c) - d)$ of Fig. 15; we depict only the ovals of $\mathbb{R}Q$ intersecting the cubic (in thick black). From $C \cup \overline{Q}$, one constructs a real algebraic curve of class 3 in a $k$-spheres real degree 2 del Pezzo surface with $2 \leq k \leq 4$ (resp. $1 \leq k \leq 4$) respectively realizing

$$2 \sqcup \langle 4 \rangle : 1 : 0 : 0, \quad 3 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 1 : 0 : 0,$$

$$6 : 2 : 0 : 0 \quad \text{and} \quad 1 \sqcup \langle 4 \rangle : 2 : 0 : 0$$

(resp. $1 \sqcup \langle 6 \rangle : 0 : 0 : 0$ and $\langle 1 \rangle \sqcup \langle 5 \rangle : 0 : 0 : 0$).

\[ \square \]

3.6 Symplectic curve on a 4-spheres real symplectic degree 2 del Pezzo surface

There exists a certain mutual arrangement in $\mathbb{R}P^2$ of a real symplectic cubic and a real symplectic quartic which is unrealizable algebraically; see [Ore02]. Analogously to the algebraic case, one can construct from such arrangement in $\mathbb{R}P^2$ a real symplectic del Pezzo surface of degree 2 and a real symplectic curve of class 3 on it with topology prescribed from the arrangement on the real projective plane.

**Proof of Proposition 3.10.** Let us consider $(\mathbb{C}P^2, \omega_{std}, \text{conj})$, where $\omega_{std}$ is the symplectic Fubini-Study 2-form on $\mathbb{C}P^2$ and $\text{conj} : \mathbb{C}P^2 \to \mathbb{C}P^2$ is the standard real structure on $\mathbb{C}P^2$. Let $\text{conj}^* : H^2(\mathbb{C}P^2, \mathbb{Z}) \to H^2(\mathbb{C}P^2, \mathbb{Z})$ be the group homomorphism map induced by $\text{conj}$. It follows that $\text{conj}^* \omega_{std} = -\omega_{std}$. Due to [Ore02], there exist a non-singular real symplectic maximal quartic $\overline{Q}$ and a non-singular real symplectic maximal cubic $\overline{C}$ which are mutually arranged in $\mathbb{R}P^2$ as depicted in Fig. 16. The double cover $\overline{\sigma} : \overline{X} \to \mathbb{C}P^2$ ramified along $\overline{Q}$ carries a natural symplectic structure $\omega$ such that $\omega = \overline{\sigma}^* \omega_{std}$ ([Gro13], [Aur00]). Let $\sigma$ be one of the two lifts of $\text{conj}$ via the double ramified cover. Since $\overline{\sigma} \circ \sigma = \text{conj} \circ \overline{\sigma}$, we have that $\sigma^* \omega = -\omega$; namely $\sigma : \overline{X} \to \overline{X}$ is a real structure of $\overline{X}$. Then, up to choose $\sigma$, the surface $(\overline{X}, \omega, \sigma)$ is real diffeomorphic to a 4-spheres real del Pezzo surface of degree 2 and, from $\overline{C}$, we construct a real symplectic curve of class 3 in $\overline{X}$ realizing

$$2 \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle : 0 : 0 : 0.$$

\[ \square \]

![Figure 16](image-url)
3.7 Constructions via patchworking

In this section, we present a construction method that allows to construct (non-symmetric) real algebraic curves with prescribed topology in $k$-spheres real del Pezzo surfaces of degree 2. First of all, let us give some definitions. Let $Bl_{p_1, \ldots, p_r} : S \to \mathbb{C}P^2$ be the blow-up of $\mathbb{C}P^2$ at a collection of 7 points $p_1, \ldots, p_r$ subject to the condition that all of them do not belong to a conic, 6 of them belong to a conic, no 3 of them belong to a line. Then, the strict transform of the conic passing through 6 points of the collection is a smooth rational curve $E_S \subset S$ of self-intersection $(-2)$ in $S$. Assume from now on, that $S$ contains a unique smooth rational curve of self intersection $(-2)$. The pair $(S, E_S)$ is called a nodal degree 2 del Pezzo pair. The anti-canonical system $\phi'$ of $S$ decomposes into a regular map $S \to S'$ of degree 1 which contracts the $(-2)$-curve of $S$, and a double cover $S' \to \mathbb{C}P^2$ ramified along a quartic $Q$ with a double point as only singularity. The surface $S'$ is called a nodal del Pezzo surface of degree 2. Conversely, the minimal resolution of the double cover of $\mathbb{C}P^2$ ramified along a quartic with a double point as only singularity is a nodal degree 2 del Pezzo pair.

Let us equip $S$ with a real structure $\sigma'$, then $E_S$ is real. Assume that $\mathbb{R}S$ is homeomorphic to $\bigcup_{j=1}^{7} S^2$. It follows that the quartic $Q \subset \mathbb{C}P^2$ is real, it has a real non-degenerate double point as only singularity and that $\mathbb{R}Q$ consists of $j$ connected components of dimension 1. Conversely, given such a quartic, one can construct a nodal degree 2 del Pezzo pair $(S, E_S)$ where $S$ is equipped with a real structure such that $\mathbb{R}S$ is homeomorphic to $\bigcup_{j=1}^{7} S^2$; see [DIK00] and see, as example, Fig. 17, where $q$ (resp. $p$) denotes the real non-degenerate double point of the quartic $Q$ (resp. the nodal degree 2 del Pezzo surface $S'$). The homology group $H_2^*(S; \mathbb{Z})$ is generated by $c_1(S)$ and $E_S$ ([BCC+08] pag. 289-312).

Let us give some more definitions.

\begin{align*}
\phi' & \\
a) (\mathbb{R}P^2, \mathbb{R}Q, q) & \quad \quad b) (\mathbb{R}S', p) & \quad \quad c) (\mathbb{R}S, \mathbb{R}E_S)
\end{align*}

Figure 17: Example: Action of the anti-canonical map of a real nodal degree 2 del Pezzo pair.

**Definition 3.22.**
- Let $(S, E_S)$ be a nodal degree 2 del Pezzo pair. Let $S$ be equipped with a real structure $\sigma'$. If $\mathbb{R}S$ is homeomorphic to $\bigcup_{j=1}^{7} S^2$, we say that $(S, E_S)$ is a $j$-spheres real nodal degree 2 del Pezzo pair.
- Let $A \subset S$ be a real algebraic curve realizing the class $dc_1(S) + k[E_S] \in H_2(S; \mathbb{Z})$. Then, we say that $A$ has bi-class $(d, k)$.

**Notation 3.23.** Let $s$ be a non-negative integer greater or equal to 2. We denote with $\vee_{j=1}^{s} S^n$ a bouquet of $s$ $n$-dimensional spheres.

**Topological construction:** Let $X'_0$ be a real reducible surface given by the union of two real algebraic surfaces $S$ and $T$, where

1. $T$ is a non-singular real quadric surface;
2. $S$ contains a unique smooth rational $(-2)$-curve $E_S \subset S$ such that $(S, E_S)$ is a $j$-spheres real nodal degree 2 del Pezzo pair, with $1 \leq j \leq 3$;
3. $S$ and $T$ intersect transversely along a real curve $E$ which is a bidegree $(1,1)$ real curve in $T$ and $E_S$ in $S$.

Let $C_S \subset S$ and $C_T \subset T$ be non-singular real algebraic curves respectively of bi-class $(d, k)$ and of class $kE$ in $H_2(T; \mathbb{Z})$, such that $C_T$ and $C_S$ intersect transversely $E$ in
a real configuration \( \{ p_1, ..., p_{2k} \} \) of \( 2k \) distinct points.

If \( T \) is the quadric ellipsoid (resp. \( T \) is the real quadric surface with empty real part) and \( \mathbb{R}E = \emptyset \), the arrangement realized by \( (\mathbb{R}S \cup \mathbb{R}T, \mathbb{R}C_S \cup \mathbb{R}C_T) \) is an arrangement of ovals in \( \bigsqcup_{j=1}^{j+1} S^2 \) (resp. \( \bigsqcup_{j=1}^j S^2 \)). Otherwise, if \( T \) is the quadric ellipsoid (resp. \( T \) is the quadric hyperboloid) and \( \mathbb{R}E \simeq S^1 \) from the arrangement realized by the pair \( (\mathbb{R}S \cup \mathbb{R}T, \mathbb{R}C_S \cup \mathbb{R}C_T) \), we can realize an arrangement \( S \) of ovals in \( \bigsqcup_{j=1}^{j+1} S^2 \) (resp. \( \bigsqcup_{j=1}^j S^2 \)). See, as example, Fig. 19.

By Theorem 3.24, such topological construction is realizable algebraically. The proof of Theorem 3.24 requires the existence of a real flat one-parameter family whose general fibers are \((j+1\text{-spheres}, \text{resp. } j\text{-spheres})\) real degree 2 del Pezzo surfaces and whose central fiber is \( X'_0 \). We prove the existence of such family in Corollary 3.27.

**Theorem 3.24.** The real scheme \( S \) is realizable in class \( d \) by a non-singular real algebraic curve \( A \) in a \( j+1\text{-spheres} \) (resp. \( j\text{-spheres} \)) real degree 2 del Pezzo surface \( X \).

**Proof.** Due to Corollary 3.27 below, we can put \( X'_0 \) in a real flat one-parameter family \( \tilde{\pi}' : X' \to D(0) \), where \( X' \) is a 3-dimensional real algebraic variety and \( D(0) \subset \mathbb{C} \) is a real disk centered at 0, such that the fibers \( X'_t = \tilde{\pi}'^{-1}(t) \) are \((j+1\text{-spheres}, \text{resp. } j\text{-spheres})\) non-singular degree 2 del Pezzo surfaces, for \( t \neq 0 \) (and \( t \) real), and the central fiber is \( X'_0 \). By Ramanujan’s Vanishing theorem ([Dol12, Section 8])

\[
H^1(X'_0; \mathcal{O}_{X'_0}(C_0)) = 0;
\]

then, [ST06a, Theorem 2.8] assures the existence of an open neighborhood \( U(0) \subset D(0) \) and a deformation \( C_t \) in \( \tilde{\pi}'^{-1}(t) \) such that \( C_t \) are non-singular (real) curves in
for $t$ (real) in $U(0) \setminus \{0\}$. In particular, there exists a real $\tilde{t} \in U(0) \setminus \{0\}$ such that the pair $(R X'_t, R C'_t)$ realizes the real scheme $S$.

**Example 3.25** (Application of the proof of Theorem 3.24). Assume that the arrangement of ovals of $S$ in $\bigsqcup_{j=1}^{2+h} S^2$ realizes $2 : 2 : 0 : 0$, with $0 \leq h \leq 2$. Moreover, assume that $R C_S$ and $R C_T$ consists of one oval respectively in $RS$ and $RT$ such that the arrangement of the pair $(RS \cup RT, R C_S \cup R C_T)$ is locally as depicted in a) of Fig. 20. Then, the proof of Theorem 3.24 assure the existence of a real $\tilde{t}$ such that the arrangement of the pair $(R X'_t, R C'_t)$ is as depicted in b) of Fig. 20 and realizes $S$.

![Figure 20](image_url)

To prove Corollary 3.27, we need the following proposition. In the proof we make use of a type of construction presented in [Ati58], which found recent applications in real enumerative geometry; see [BPT13]. In particular, we prove that given a real algebraic quartic $\tilde{Q}$ in $\mathbb{CP}^2$ with a non-degenerate double point as only singularity, one can put $\tilde{Q}$ in a real flat one-parameter family in which $\tilde{Q}$ is the only singular fiber. Then, from the family of quartics one can construct a particular real flat one-parameter family of del Pezzo surfaces.

**Proposition 3.26.** Let $\tilde{Q}$ be a real quartic in $\mathbb{CP}^2$ with a real non-degenerate double point $q$ as only singularity and such that $RQ$ is homeomorphic to either $\bigsqcup_{j=1}^{j-1} S^1 \cup \{q\}$ or to $\bigsqcup_{j=1}^{j-1} S^1 \cup \bigvee_{j=1}^{2} S^1$, where $1 \leq j \leq 3$. Then, there exists a real flat one-parameter family of non-singular ($k$-spheres real) degree 2 del Pezzo surfaces (with $k = j$, resp. with $k = j + 1$) but the central fiber which is a real reducible surface $X'_0$ equal to the union of two real algebraic surfaces $S \cup T$, where

1. $T$ is either a real quadric hyperboloid or a real quadric surface with empty real part (resp. a real quadric ellipsoid);
2. $S$ is the minimal resolution of the double cover of $\mathbb{CP}^2$ ramified along $\tilde{Q}$ and it contains a unique smooth rational curve $E_S \subset S$ such that $(S, E_S)$ is a $j$-spheres nodal degree 2 del Pezzo pair, with $1 \leq j \leq 3$;
3. $S$ and $T$ intersect transversely along a curve $E$ which is a bidegree $(1,1)$ real curve in $T$ and the $(-2)$-curve $E_S$ in $S$.

**Proof.** Let $f(x, y, z) = 0$ be a real polynomial equation defining the real quartic $\tilde{Q}$ in $\mathbb{CP}^2$. Up to multiply $f(x, y, z)$ by $-1$, we can always put $\tilde{Q}$ in a real flat one-parameter family $\pi : \Omega \to D(0)$, where

- $D(0) \subset \mathbb{C}$ equipped with the standard real structure of $\mathbb{C}$, is a real disk centered at 0;
- $\Omega \subset \mathbb{CP}^2 \times D(0)$ is defined by $f(x, y, z) + \varepsilon z^4 t^2 = 0$, with $\varepsilon \in \{1, -1\}$

and such that
• the fibers $\overline{Q}_t := \pi^{-1}(t)$ are non-singular (real) quartics (with real part homeomorphic to either $\bigcup_{j=1}^t S^1$ or to $\bigcup_{j=1}^{t+1} S^1$, and $\Pi_+$ is orientable) for $t \neq 0$ (and $t$ real);

• $Q_0 = \tilde{Q}$.

From the family of quartics, we can construct a real flat one-parameter family $\tilde{\pi} : \tilde{X} \to D(0)$ such that

• $\tilde{X}$ is the double cover of $\mathbb{C}P^2 \times D(0)$ ramified along $\mathcal{Q}$ and $\tilde{X}$ is isomorphic to the hypersurface in $\mathbb{C}P(1,1,1,2) \times D(0)$ defined by the polynomial equation $f(x,y,z) + \varepsilon z^4 t^2 = w^2$;

• $X_0$ is the double cover of $\mathbb{C}P^2$ ramified along $\tilde{Q}$. Depending on the real scheme realized by the pair $(\mathbb{R}^2, \mathbb{R}Q)$, the real part of $X_0$ is homeomorphic either to $\bigcup_{j=1}^t S^2 \sqcup \{ pt \}$ or to $\bigcup_{j=1}^{t-1} S^2 \sqcup \sqrt{2} = S^2$, where $\{ pt \}$ is a point.

• the fibers $\tilde{\pi}^{-1}(t) := X_t$ are non-singular ($k$-spheres real) degree 2 del Pezzo surfaces (with either $k = j$ or $k = j + 1$ depending on $\mathbb{R}Q_t$), for $t \neq 0$ (and $t$ real).

Now, performing the blow up $\text{Bl}_p : \mathcal{X}' \to \tilde{X}$ at the node $p$ of $\tilde{X}$, we obtain a real flat one-parameter family $\tilde{\pi}' : \mathcal{X}' \to D(0)$ such that $\text{Bl}_p^{-1}(p) =: T$ is a real quardic surface with real structure dependent on $f(x,y,z)$ and $\varepsilon$, the fibers $\tilde{\pi}'^{-1}(t) := X_t'$ are non-singular ($k$-spheres real) del Pezzo surfaces of degree 2 (with either $k = j$ or $k = j + 1$ depending on $\mathbb{R}X_t'$), for $t \neq 0$ (and $t$ real), and $X_0'$ is equal to the union of two real algebraic surfaces $S \cup T$, where $S$, $T$ and $E$ are as described in (1) - (3).

**Corollary 3.27.** Let $X_0'$ be a real reducible surface equal to the union of two real algebraic surfaces $S \cup T$, where

1. $T$ is either a real quadric hyperboloid or a real quadric surface with empty real part (resp. a real quadric ellipsoid);

2. $S$ contains a unique smooth rational $(-2)$-curve $E_S$ such that $(S, E_S)$ is a $j$-spheres nodal degree 2 del Pezzo pair, with $1 \leq j \leq 3$;

3. $S$ and $T$ intersect transversely along a curve $E$ which is a bidegree $(1,1)$ real curve in $T$ and the $(-2)$-curve $E_S$ in $S$.

Then, there exists a real flat one-parameter family $\tilde{\pi}' : \mathcal{X}' \to D(0)$, where $D(0) \subset \mathbb{C}$ is a real disk centered in 0, the fibers $\tilde{\pi}'^{-1}(t) := X_t'$ are non-singular ($k$-spheres real) del Pezzo surfaces of degree 2 (with $k = j$, resp. $k = j + 1$), for $t \neq 0$ (and $t$ real), and the central fiber is $X_0'$.

**Proof.** The anti-canonical system exhibits $S$ as the minimal resolution of the double cover of $\mathbb{C}P^2$ ramified along a real algebraic quartic $Q$ with a real non-degenerate double point $q$ as only singularity and such that $\mathbb{R}Q$ is homeomorphic to either $\bigcup_{j=1}^t S^1 \sqcup \{ q \}$ or to $\bigcup_{j=1}^{t-1} S^1 \sqcup \sqrt{2} = S^1$, where $1 \leq j \leq 3$. Applying the proof of Proposition 3.26 to $Q$, we prove the statement.

### 3.7.1 Intermediate constructions: Constructions on real quadric surfaces and on $j$-spheres real nodal del Pezzo pairs of degree 2

In Section 3.7.2 we use Theorem 3.24 to end the proof of Theorem 3.9 Proposition 3.12 and to prove Proposition 3.11. In order to do that we need some intermediate constructions. Therefore, the aim of this section is to

• construct real algebraic curves with prescribed topology and intersection with a given real curve in the quadric ellipsoid (Proposition 3.29), resp. in the quadric hyperboloid (Proposition 3.31);

• construct real algebraic curves with prescribed topology on $j$-spheres real nodal degree 2 del Pezzo pairs (Proposition 3.34).
Notation 3.28. Whenever we talk about interior and exterior of an oval \( O \) in \( S^2 \) is with respect to an opportune chosen point \( p \in S^2 \setminus O \); see Section 2.1. In Figures 21 - 26 we depict \( S^2 \) projected from \( p \) on a plane.

In the proofs of Propositions 3.29, 3.30, 3.31 we use a variant of Harnack’s construction method (Har76).

Proposition 3.29. Let \( T \) be the quadric ellipsoid and let \( E_T \) be a non-singular real algebraic curve of bidegree \((1, 1)\) in \( T \). Then, for any real configuration \( P_{2k} \) of \( 2k \) distinct points in \( E_T \) fixed as follows, there exists a non-singular real algebraic curve \( C_T \) of bidegree \((k, k)\) on \( T \), intersecting transversely \( E_T \) in the \( 2k \) points and such that the triplet \((RT, RE_T, RC_T)\) is arranged as depicted:

1. in a) of Fig. 21 for \( k = 2 \) and 4 fixed real points;
2. in b) of Fig. 21 for \( k = 3 \) and no fixed real points;
3. in c) of Fig. 21 for \( k = 3 \) and 2 fixed real points;

![Diagram](image)

Figure 21: \( RE_T \) in dashed.

Proof. For any real configuration \( P_4 \subset RE_T \) let us construct a real curve \( \tilde{H} \) of bidegree \((2, 2)\) passing through \( P_4 \) and such that the arrangement of \( \tilde{H} \cup RE_T \) is as depicted in a) of Fig. 21. First of all, remind that for any 2 fixed distinct points on \( E_T \), there exists a bidegree \((1, 1)\) real algebraic curve passing through them.

Let \( P_0(x, y)P_1(x, y) = 0 \) be a real polynomial equation defining the union of \( E_T \) and a bidegree \((1, 1)\) real curve \( H \) such that the points of \( P_4 \) belong to one connected component \( E \) of \( RE_T \setminus RH \). Let \( H_1 \) and \( H_2 \) be two bidegree \((1, 1)\) real curves such that \( H_1 \cup H_2 \) contains \( P_4 \). Replace the left side of the equation \( P_0(x, y)P_1(x, y) = 0 \) with \( P_0(x, y)P_1(x, y) + \varepsilon f_1(x, y)f_2(x, y) \), where \( f_i(x, y) = 0 \) is an equation for \( H_i \) and \( \varepsilon \) is a sufficient small real number. Up to a choice of the sign of \( \varepsilon \), one constructs a small perturbation \( \tilde{H} \) of \( E_T \cup H \), where \( \tilde{H} \) is a bidegree \((2, 2)\) non-singular real curve such that \( \bigcup_{i=1}^2 H_i \cap E_T = \tilde{H} \cap E_T \) and the triplet \((RT, RE_T, \tilde{H})\) is arranged as depicted in a) of Fig. 21.

Now, for any real configuration \( P_6 \subset E_T \setminus RE_T \) (resp. \( P_6' \subset E_T \), whose exactly 2 are real), we want to construct real curves \( C_T \) of bidegree \((3, 3)\) passing through \( P_6 \) (resp. \( P_6' \)) and such that the arrangement of \( RC_T \cup RE_T \) is as depicted in b) of Fig. 21 on the left (resp. in c) of Fig. 21). One can construct \( C_T \) applying a small perturbation to \( E_T \cup \tilde{H} \) (resp. to \( E_T \cup H_2 \)), where \( \tilde{H}_2 \) is a real curve of bidegree \((2, 2)\) such that:

- the triplet \((RT, RE_T, \tilde{H}_2)\) is arranged as depicted in a) of Fig. 21;
- the real points of \( P_6' \) belong to an interior connected component, resp. an exterior connected component, of \( RE_T \setminus RH_2 \), with respect to some point \( p \in RT \setminus RE_T \cup RH_2 \); see Notation 3.28.

Let us end the proof constructing real curves \( C_T \) of bidegree \((3, 3)\)
• containing a given real configuration $\mathcal{P}' = \{p_1, \overline{p}_1, p_2, \overline{p}_2, p_3, \overline{p}_3\}$ of points on $E_T \setminus \mathbb{R}E_T$, where $p_i$ and $\overline{p}_i$ are complex conjugated points;
• such that the arrangement of $\mathbb{R}C_T \cup \mathbb{R}E_T$ is as depicted in b) of Fig. 21 respectively in the center and on the right.

Let $\Pi_i$ be a pencil of hyperplanes with base points $p_i$ and $\overline{p}_i$, with $i = 1, 2, 3$. We want to show that one can always construct a non-singular real algebraic curve $C_T$ of bidegree $(3, 3)$ as perturbation of the union of three hyperplanes respectively of $\Pi_1$, $\Pi_2$ and $\Pi_3$ such that the arrangement of the triplet $(RT, RE_T, RC_T)$ is arranged as depicted on the left (resp. on the right) of Fig. 22. Namely, we prove in the following that one can always find three hyperplanes respectively of $\Pi_1$, $\Pi_2$ and $\Pi_3$ whose union and real arrangement with respect to $RE_T$ is as depicted on the left (resp. on the right) of a) in Fig. 22. First of all, remark that on each of the two connected components of $\mathbb{R}T \setminus \mathbb{R}E_T$, the real part of the real hyperplanes of the pencil $\Pi_i$ vary from a real point $q_i$ to $RE_T$, with $i = 1, 2, 3$. Moreover, the real points $q_1$, $q_2$ and $q_3$ are distinct points.

Proposition 3.30. Let $T$ be the quadric ellipsoid and let $E_T$ be a non-singular real algebraic curve of bidegree $(1, 1)$ in $T$. Then,

• for any integer $k \geq 5$,

• for any real configuration $\mathcal{P}_{2k}$ of $2k$ distinct points, whose exactly 2 are real, in $E_T$,
for any given non-negative integers $k_1, k_2$ such that $k_1 + k_2 = k - 4$

there exists a non-singular real algebraic curve $C_T$ of bidegree $(k, k)$ on $T$, intersecting transversely $E_T$ in $\mathcal{P}_{2k}$ and such that the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}C_T)$ is arranged as depicted in Fig. 24.

Proof. The first step is to prove the statement for $k = 5$. After we end the proof by induction on $k$. Let us introduce some notation. Let us call $\hat{H}_i$ any real curve of bidegree $(i, i)$ such that only one oval of $\mathbb{R}H_i$, denoted with $O_i$, intersects $\mathbb{R}E_T$. Moreover, we denote with $\mathcal{F}_1, \ldots, \mathcal{F}_j$ and $\hat{\mathcal{F}}_1, \ldots, \hat{\mathcal{F}}_j$ the connected components of $\mathbb{R}E_T \setminus O_i$ respectively in the interior and in the exterior of $O_i$ with respect to some point $p \in \mathbb{R}T \setminus \mathbb{R}E_T \cup \mathbb{R}H_i$ (see Notation 3.28).

Fix $\mathcal{P}_{10} \subset E_T$ and denote with $p_1, p_2$ the two real points of $\mathcal{P}_{10}$. We start with the construction of a real curve $\hat{H}_i$ of bidegree $(5, 5)$ passing through $\mathcal{P}_{10}$ and such that the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}\hat{H}_i)$ is arranged as depicted in a) of Fig. 25 (resp. b) of Fig. 25). If one can construct a real curve $\hat{H}_i$ of bidegree $(4, 4)$ such that

1. $\mathbb{R}\hat{H}_i \cap \mathbb{R}E_T = O_i \cap \mathbb{R}E_T$ consists of 2 points;
2. $\{p_1, p_2\} \subset \mathcal{F}_1$ of $O_i$;
3. the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}\hat{H}_i)$ is arranged as depicted in c) of Fig. 26

then, the real curve $\hat{H}_i$ exists as small perturbation of $\hat{H}_i \cup E_T$; see the proof of Proposition 4.2.29 for details on small perturbation method.

Let us construct $\hat{H}_i$. Fix a configuration $\mathcal{P}_4$ of 4 points on $\mathbb{R}E_T$ such that $\{p_1, p_2\}$
belong to the same connected components of $\mathbb{R}E_T \setminus \mathcal{P}_4$. Via small perturbation, we construct a real curve $\hat{H}_2$ of bidegree $(2, 2)$ such that

- the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}\hat{H}_2)$ is arranged as depicted in a) of Fig. 26;
- $\mathbb{R}\hat{H}_2 \cap \mathbb{R}E_T = \mathcal{O}_2 \cap \mathbb{R}E_T = \mathcal{P}_4$;
- $\{p_1, p_2\} \subset \hat{\mathcal{F}}_1$ of $\mathcal{O}_2$.

Now, fix a real configuration $\mathcal{P}_6$ of 6 points on $E_T$ such that (see b) of Fig. 26

- exactly 4 points are real;
- one of the fixed real points belongs to $\mathcal{F}_i \subset \mathcal{O}_2$, for $i = 1, 2$;
- two of the fixed real points are on $\hat{\mathcal{F}}_1 \subset \mathcal{O}_2$ and $p_1, p_2$ belong to the same connected component of $\mathcal{F}_1 \setminus \mathcal{O}_2$.

One can construct a small perturbation $\hat{H}_3$ of $E_T \cup \hat{H}_2$, where $\hat{H}_3$ is a bidegree $(3, 3)$ non-singular real curve passing through $\mathcal{P}_6$ and such that the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}\hat{H}_3)$ is arranged as depicted in c) of Fig. 26. To end the construction of $\hat{H}_4$, fix 8 points on $E_T$ such that exactly 2 points are real and belong to two different connected components of $\hat{\mathcal{F}}_1 \setminus \{p_1, p_2\} \subset \mathcal{O}_3$; see d) of Fig. 26. One obtains $\hat{H}_4$ as small perturbation of $\hat{H}_1 \cup E_T$.

Let us proceed by induction to end the proof. Assume that the statement hold for $k - 1$. Now, for any given non-negative integers $k_1, k_2$ such that $k_1 + k_2 = k - 4$, let us construct a bidegree $(k, k)$ real algebraic curve $C_T$ passing through a given real configuration $\mathcal{P}_{2k} \subset E_T$, whose exactly 2 points are real and such that the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}C_T)$ is arranged as depicted in Fig. 24.

By induction, for any choice of $k_1, k_2$ and $\mathcal{P}$, there exists a bidegree $(k - 1, k - 1)$ real algebraic curve $\hat{H}_{k-1}$ such that

- $\hat{H}_{k-1}$ passes through a given real configuration $\mathcal{P}_{2k-2} \subset E_T$;
- the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}\hat{H}_{k-1})$ is arranged as depicted in Fig. 24, where $k_1 = \hat{k}_1$ and $k_2 = \hat{k}_2 - 1$;
- the connected component $\mathcal{F}_1 \subset \mathcal{O}_{k-1}$ contains the 2 real points of $\mathcal{P}_{2k}$.

Finally, opportunely applying the small perturbation method to $E_T \cup \hat{H}_{k-1}$, one construct the wanted real curve $C_T$ of bidegree $(k, k)$.

**Proposition 3.31.** Let $T$ be the quadric hyperboloid and let $E_T$ be a non-singular real algebraic curve of bidegree $(1, 1)$ in $T$. Then, for any real configuration of 2k distinct points in $E_T$ fixed as follows, there exists a non-singular real algebraic curve $C_T$ of bidegree $(k, k)$ on $T$, intersecting transversely $E_T$ in the 2k points and such that the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}C_T)$ is arranged as depicted:

1. (1) in a), resp. b), resp. c) of Fig. 27 for $k = 3$ and 2 fixed real points;
2. (2) in d) of Fig. 27 for $k = 3$ and no fixed real points;
3. (3) in e), resp. f) of Fig. 27 for $k = 2$ and 4 fixed real points.

**Proof.** For any configuration $\mathcal{P}_4$ of 4 fixed points on $\mathbb{R}E_T$ let us construct real curves $\hat{H}$ of bidegree $(2, 2)$ passing through $\mathcal{P}$ and such that the arrangements of $\mathbb{R}H \cup \mathbb{R}E_T$ are respectively as depicted in e) and f) of Fig. 27. Let $P_0(x, y)P_1(x, y) = 0$ be a real polynomial equation defining the union of $E_T$ and a bidegree $(1, 1)$ real curve $H$ such that the points of $\mathcal{P}_4$ belong to one connected component $\mathcal{E}$ of $\mathbb{R}E_T \setminus \mathbb{R}H$. Let $H_1$ and $H_2$ be two bidegree $(1, 1)$ real curves such that $H_1 \cup H_2$ contains $\mathcal{P}_4$. Replace the left side of the equation $P_0(x, y)P_1(x, y) = 0$ with $P_0(x, y)P_1(x, y) + \epsilon f_1(x, y)f_2(x, y)$, where $f_1(x, y) = 0$ is an equation for $H_4$ and $\epsilon$ is a sufficient small real number. Up to a choice of the sign of $\epsilon$, one constructs a small perturbation $\hat{H}$ of $E_T \cup H$, where $\hat{H}$ is a bidegree $(2, 2)$ non-singular real curve such that $\bigcup_{i=1}^2 H_i \cap E_T = H \cap E_T$ and the triplet $(\mathbb{R}T, \mathbb{R}E_T, \mathbb{R}\hat{H})$ is arranged as depicted in c) of Fig. 27 resp in f) of Fig. 27. Analogously, via small perturbation method we can construct real algebraic curves of bidegree $(3, 3)$ as described in (2) – (3) and end the proof.
In order to accomplish some particular constructions in the proof of Proposition 3.34, we need the two following lemmas.

**Lemma 3.32.** There exist real algebraic curves \( \tilde{Q} \) and \( C \) respectively of degree 4 and 3 in \( \mathbb{C}P^2 \) with a unique real non-degenerate double singularity at a point \( q \), such that the triplet \( (\mathbb{R}P^2, \mathbb{R}\tilde{Q}, \mathbb{R}C) \) realizes the real scheme depicted in a) of Fig. 28 (resp. in b), resp. in c), resp. d), resp. e) of Fig. 28).
Proof. The blow-up of $\mathbb{C}P^2$ at the point $q$ is the first Hirzebruch surface $\Sigma_1$ (Section 2.2). Then, in order to prove the statement, it is sufficient to construct a reducible real algebraic curve $K_1$ (resp. $K_2$, resp. $K_3$, resp. $K_4$, resp. $K_5$) of bidegree $(3,4)$ in $\Sigma_1$ as union of two non-singular real algebraic curves $Q$ and $A$ respectively of bidegree $(2,2)$ and $(1,2)$ in $\Sigma_1$ such that the pair $(\mathbb{R}Q, \mathbb{R}Q \cup \mathbb{R}A)$ realizes the $\mathcal{L}$-scheme depicted in a) of Fig. 29 (resp. in b), resp. in c), resp. in d), resp. in e) of Fig. 29.

Let $\tilde{\eta}_1$, $\tilde{\eta}_2$, $\tilde{\eta}_3$, $\tilde{\eta}_4$ and $\tilde{\eta}_5$ be trigonal schemes in $\mathbb{R}\Sigma_5$ respectively as depicted in a), b), c), d) and e) of Fig. 30. Due to Theorem 2.3, if the real graph associated to $\tilde{\eta}_i$ is completable in degree 5 to a real trigonal graph, then there exists a real algebraic trigonal curve $\tilde{K}_i$ realizing $\tilde{\eta}_i$, for $i = 1, 2, 3, 4, 5$. Therefore, the completion $\Gamma_i$ of the real graph associated to $\tilde{\eta}_i$ depicted in a), b), c) and d) of Fig. 31 respectively for $i = 1, 2, 3$ and $i = 4, 5$, proves the existence of such $\tilde{K}_i$; see Section 2.3. The trigonal curve $\tilde{K}_i$ is reducible because it has 12 non-degenerate double points and its normalization has 4 real connected components. In particular $\tilde{K}_i$ has to be the union of a real curve of bidegree $(2,0)$ and a real curve of bidegree $(1,0)$.

Let us consider the birational transformation $\Xi := \beta^{-1}_1 \beta^{-1}_2 \beta^{-1}_3 \beta^{-1}_4 : (\Sigma_1, \tilde{K}_i) \rightarrow (\Sigma_1, K_i)$, defined as in Section 2.2 where the points $p_j$'s, with $j = 1, 2, 3, 4$, are the real double points of $\tilde{K}_i$ as depicted in a) of Fig. 30 (resp. b), resp. c), resp. d), resp. e) of Fig. 30), and the dashed real fibers are those intersecting the $p_j$'s. The image via $\Xi$ of the reducible real trigonal curve $\tilde{K}_i$ is a reducible curve $K_i$ of bidegree $(3,4)$ which is the union of two non-singular real curves $Q$ and $A$, respectively of bidegree $(2,2)$ and $(1,2)$ in $\Sigma_1$. Moreover, the $\mathcal{L}$-scheme of $K_i$ (resp. $K_2$, resp. $K_3$, resp. $K_4$, resp. $K_5$) is as depicted in a) of Fig. 29 (resp. in b), resp. in c), resp. in e), resp. in f) of Fig. 29).

Lemma 3.33. There exist real algebraic curves $\hat{Q}$ and $C$ respectively of degree 4 and 3 in $\mathbb{C}P^2$ with a unique real non-degenerate double singularity at a point $q$, such that the triplet $(\mathbb{R}P^2, \mathbb{R}Q, \mathbb{R}C)$ realizes the real scheme depicted in Fig. 32.
Proof. The proof is similar to that of Proposition 3.32; see the construction of cubics and quartics arranged in $\mathbb{R}P^2$ respectively as depicted in a) – e) of Fig. 29.

We end this section giving the following intermediate constructions.

**Proposition 3.34.** (i) For every arrangement of ovals in $\bigsqcup_{s=1}^p S^2$ respectively depicted in Fig. 33, Fig. 34 and Fig. 35, for every $s \leq j \leq 3$, there exists a $j$-spheres real nodal del Pezzo pair $(S,E_S)$, and real algebraic curve $C_S \subset S$ of bi-class $(d,k)$ such that $R_{C_S} \cup R_{E_S}$ is arranged in $R_S$ as depicted in

1. Fig. 33 for $d = 3$ and $k = 2$;
2. Fig. 34 and in Fig. 35 for $d = 3$ and $k = 3$.

(ii) Let $d, k_1, k_2$ and $h_1, h_2, h_3, h_4$ be non-negative integers such that

- $d \geq 5$;
- $k_1 + k_2 = d - 4$;
- $\sum_{i=1}^4 h_i = d - 1$;
- $h_j \neq 2$, for $j = 3, 4$.

Then, there exist a $3$-spheres real nodal del Pezzo pair $(S,E_S)$ and a bi-class $(d,d)$ real curve $C_S \subset S$ such that $R_{C_S} \cup R_{E_S}$ is arranged in $R_S$ as depicted in Fig. 36.

**Proof.** Let $\tilde{Q}$ be a real quartic with a real non-degenerate and non-isolated double point at $q$ as only singularity and let $C$ be a real curve of degree $d$ with one $k$-fold singularity at $q$. To a pair $(\tilde{Q}, C)$ correspond a real nodal del Pezzo pair $(S,E_S)$ and a real algebraic curve $C_S \subset S$ of bi-class $(d,k)$ with topology described by the topology of the triplet $(\mathbb{R}P^2, \mathbb{R}\tilde{Q}, \mathbb{R}C)$.

Proof of (i): First of all, Lemmas 3.32 and 3.33 immediately implies the existence of $j$-spheres real nodal degree 2 del Pezzo pairs $(S,E_S)$ and real curves $C_S \subset S$ of
Let us realize the arrangements in (2). It is easy to see that there exist a real plane quartic $\tilde{Q}_1$ with a real non-degenerate double point $q_1$ as only singularity and real part homeomorphic to $\bigsqcup_{i=1}^{2} S^1 \cup \bigsqcup_{j=1}^{2} S^1$ and a pencil of lines $L_{q_1} \subset \mathbb{C}P^2$, centered at $q_1$, such that $\tilde{Q}_1 \cup L_{q_1}$ is arranged as depicted in a) of Fig. 37 (resp. in b) of Fig. 37). The union of any three distinct lines of $L_{q_1}$ is a cubic $C_1$ with a triple point at $q_1$. From $\tilde{Q}_1$ and $C_1$ one can construct $j$-spheres real nodal degree 2 del Pezzo pairs $(S, E_S)$ and curves $C_S \subset S$ of bi-class $(3, 3)$ such that the triplet $(\mathbb{R}S, \mathbb{R}E_S, \mathbb{R}C_S)$ is arranged as depicted in Fig. 34 (resp. Fig. 35) where we depict only the non-empty spheres of $\mathbb{R}S$.

Proof of (ii): Assume that $\tilde{Q}_1$ has real part homeomorphic to $\bigsqcup_{i=1}^{2} S^1 \cup \bigsqcup_{j=1}^{2} S^1$. The union of line $a \subset L_{q_1}$ (in thick black) and other $d - 1$ distinct lines (in dashed) of
Figure 37: The set of points of \( \mathbb{R}Q \) homeomorphic to \( \bigsqcup_{i=1}^{j-1} S^1 \) is depicted in dashed.

Figure 38: \( L \) and other \( d - 1 \) lines (in dashed) of the pencil \( L_{q_1} \).

\( L_{q_1} \) as depicted in a) of Fig. 38 (resp. b) of Fig. 38) is a degree \( d \) real curve \( C_d \) with a \( d \)-fold singularity at \( q_1 \). From \( Q_1 \) and \( C_d \) one can construct 3-spheres real nodal degree 2 del Pezzo pairs \( (S, E_S) \) and real curves \( C_S \subset S \) of bi-class \((d, d)\) such that the triplet \( (\mathbb{R}S, \mathbb{R}E_S, \mathbb{R}C_S) \) is arranged as depicted in a) of Fig. 36 (in b) of Fig. 36). □

3.7.2 Final constructions

We end the proof of Theorem 3.9 and Proposition 3.12. Moreover, we prove Proposition 3.11. The proofs combine the results and constructions of Theorem 3.24 and Propositions 3.29, 3.30, 3.31, 3.34.

**Proposition 3.35.** For each real scheme \( S \) labeled with \( (\ast) \) (resp. with \( (\ast) \)) in Tables 1 and 2 there exist a 4-spheres (resp. \( k \)-spheres) real del Pezzo surface \( X \) of degree 2 (with \( 1 \leq k \leq 3 \)) and a real algebraic maximal curve of class 3 in \( X \) realizing \( S \).

**Proof.** Pick any \( j \)-spheres real nodal degree 2 del Pezzo pair \( (S, E_S) \) and real algebraic curve \( C_S \subset S \) of bi-class \((3, 2)\) constructed as in proof of Proposition 3.34. Due to Corollary 3.27 there exist a real algebraic surface \( X' \) as union of \( S \) and \( T \), intersecting along \( E \), and a real algebraic curve \( C_0 \) as union of \( C_S \) and \( C_T \), intersecting along 4 points of \( E \); where \( T \) is the quadric ellipsoid and \( C_T \subset T \) is the real algebraic curve of bidegree \((2, 2)\) constructed as in proof of Proposition 3.29. Then, thanks to Theorem 3.24 there exist a \( k \)-spheres real degree 2 del Pezzo surface \( X \), with \( k = j + 1 \), and a non-singular real algebraic maximal curve \( A \subset X \) of class 3 such that the arrangement of the pair \((\mathbb{R}X, \mathbb{R}A)\) realizes the real scheme

\[
2 \hspace{1em} \sqcup \hspace{1em} \langle 2 \rangle : 3 : 0 : 0
\]

(resp. \( 4 : 4 : 0 : 0 \) and \( 5 : 3 : 0 : 0 \),

resp. \( 1 \hspace{1em} \sqcup \hspace{1em} \langle 2 \rangle : 2 : 2 : 0 \),

resp. \( 1 \hspace{1em} \sqcup \hspace{1em} \langle 1 \rangle \hspace{1em} \sqcup \hspace{1em} \langle 1 \rangle : 3 : 0 : 0 \),

resp. \( 1 \hspace{1em} \sqcup \hspace{1em} \langle 3 \rangle : \langle \langle 1 \rangle \rangle : 0 : 0 \) and \( 1 \hspace{1em} \sqcup \hspace{1em} \langle 2 \rangle : 1 \hspace{1em} \sqcup \hspace{1em} \langle 2 \rangle : 0 : 0 \).

See Example 3.36 (1).

In order to realize the remaining real schemes labeled with \( (\ast) \) (resp. \( (\ast) \)) on the right.
of Table 1 and in Table 2 we apply the previous construction as follows. First of all, pick any \( j \)-spheres real nodal degree 2 del Pezzo pair \((S, E_S)\) and real algebraic curve \( C_S \subset S \) of bi-class \((3, 2)\) constructed as in proof of Proposition 3.34. Due to Corollary 3.27 there exist a real algebraic surface \( X_0' \) as union of \( S \) and \( T \), intersecting along \( E \), and a real algebraic curve \( C_T \) as union of \( C_S \) and \( C_T \), intersecting along 6 points of \( E \); where \( T \) is the quadric hyperboloid and \( C_T \subset T \) a real algebraic curve of bidegree \((3, 3)\) constructed as in proof of Proposition 3.29. Then, thanks to Theorem 3.24 for each real scheme \( S \) labeled with \((\ast)\) (resp. \((\ast)\)) in Tables 1 on the right and 2 there exist a \( k \)-spheres real degree 2 del Pezzo surface \( X \), with \( k = j + 1 \), and a non-singular real algebraic maximal curve \( A \subset X \) of class 3 such that the arrangement of the pair \((\mathbb{R}X, \mathbb{R}A)\) realizes \( S \). See Example 3.36 (2).

Let us end the proof realizing the real schemes labeled with \((\ast)\) on the left of Table

\[ \begin{align*}
\text{Figure 39: } \mathbb{R}E_S &\simeq \mathbb{S}^1 \text{ in dashed.} \\
1 \quad \text{Pick any } j \text{-spheres real nodal degree 2 del Pezzo pair } (S, E_S) \text{ and a real algebraic curve } C_S \subset S \text{ of bi-class } (3, 2) \text{ (resp. } (3, 3)\text{) constructed as in proof of Proposition 3.34 such that the triplet } (\mathbb{R}S, \mathbb{R}E_S, \mathbb{R}C_S) \text{ is arranged as depicted respectively in } a), b), d) \text{ and } e) \text{ of Fig. 33 (resp. in Fig. 39), where we depict only the non-empty spheres of } \mathbb{R}S. \text{ Due to Corollary 3.27 there exist a real algebraic surface } X_0' \text{ as union of } S \text{ and } T', \text{ intersecting along } E, \text{ and a real algebraic curve } C_T \text{ as union of } C_S \text{ and } C_T, \text{ intersecting along 4 (resp. 6) points of } E; \text{ where } T \text{ is the quadric hyperboloid and } C_T \subset T \text{ a real algebraic curve of bidegree } (2, 2) \text{ (resp. } (3, 3)\text{) constructed as in proof of Proposition 3.31. Then, thanks to Theorem 3.24 for each of the following real schemes } S: \\
8 : 0 : 0 : 0, \quad 3 \sqcup \{4\} : 0 : 0 : 0, \quad 1 \sqcup \{2\} \sqcup \{3\} : 0 : 0 : 0, \\
4 \sqcup \{1\} \sqcup \{1\} : 0 : 0 : 0, \quad 1 \sqcup \{1\} \sqcup \{1\} \sqcup \{2\} : 0 : 0 : 0, \\
1 \sqcup \{1\} \sqcup \{1\} \sqcup \{2\} : 0 : 0 : 0, \quad 1 \sqcup \{1\} \sqcup \{2\} \sqcup \{1\} : 0 : 0 : 0
\end{align*} \]

(resp. the remaining real schemes \( S \) labeled with \((\ast)\) on the left of Table 1, there exist a \( k \)-spheres real degree 2 del Pezzo surface \( X \), with \( k = j \), and a non-singular real algebraic maximal curve \( A \subset X \) of class 3 such that the arrangement of the pair \((\mathbb{R}X, \mathbb{R}A)\) realizes \( S \). See Example 3.36 (3).

Example 3.36. (1) We construct a real algebraic curve of class 3 in a 4-spheres real del Pezzo surface of degree 2 as in proof of Proposition 3.35 with real scheme 2 \( \sqcup \{2\} : 3 : 0 : 0 \) (resp. 1 \( \sqcup \{2\} : 2 : 2 : 0 \)).

- Let \( C_S \subset S \) be the real algebraic curve such that \( \mathbb{R}E_S \cup \mathbb{R}C_S \) is arranged in \( \mathbb{R}S \) as pictured in a) of Fig. 33 (resp. b) of Fig. 39.
- Let \( T \) be the quadric ellipsoid and let \( C_T \subset T \) be the real algebraic curve of bidegree \((2, 2)\) such that \( \mathbb{R}E_T \cup \mathbb{R}C_T \) is arranged in \( \mathbb{R}T \) as depicted in a) of Fig. 40.
- Thanks to Theorem 3.22 there exists a real algebraic curve \( A \) of class 3 in a real del Pezzo surface \( X \) of degree 2 realizing the real scheme 2 \( \sqcup \{2\} : 3 : 0 : 0 \) (resp. 1 \( \sqcup \{2\} : 2 : 2 : 0 \)). See b) of Fig. 40 (resp. c) of Fig. 40.

(2) We construct a real algebraic curve of class 3 in a 4-spheres real del Pezzo surface of degree 2 as in proof of Proposition 3.35 with real scheme \( \{1\} \sqcup \{2\} : 3 : 0 : 0 \).

Let \( C_S \subset S \) be the real algebraic curve such that \( \mathbb{R}E_S \cup \mathbb{R}C_S \) is arranged in \( \mathbb{R}S \) as pictured in a) of Fig. 41.
• Let $T$ be the quadric ellipsoid and let $C_T \subset T$ be the real algebraic curve of bidegree $(3, 3)$ such that $RE_T \cup RC_T$ is arranged in $RT$ as depicted in b) of Fig. 41.

• Thanks to Theorem 3.24 there exists a real algebraic curve $A$ of class 3 in a real del Pezzo surface $X$ of degree 2 realizing the real scheme $(1) \sqcup (2) : 3 : 0 : 0$. See c) Fig. 41.

(3) We construct real algebraic curves of class 3 in a $k$-spheres real del Pezzo surface of degree 2, with $k = 3, 2, 1$, as in proof of Proposition 3.35 with real scheme $b \sqcup \langle a + 1 \rangle \sqcup \langle 1 \rangle : 0 : 0 : 0$, where $a, b$ denotes number of ovals and $a + b = 3$.

• Let $C_S \subset S$ be the real algebraic curve such that $RE_S \cup RC_S$ is arranged in $RS$ as pictured in a) of Fig. 42 where we depict only the non-empty spheres of $RS$.

• Let $T$ be the quadric hyperboloid and $C_T \subset T$ be the real algebraic curve of bidegree $(3, 3)$ such that $RE_T \cup RC_T$ is arranged in $RT$ as depicted in a), resp. b), of Fig. 27.

• Thanks to Theorem 3.24 for any values of $a, b$, there exists a real algebraic curve $A$ of class 3 in a $k$-spheres real del Pezzo surface $X$ of degree 2 realizing the real scheme $b \sqcup \langle a + 1 \rangle \sqcup \langle 1 \rangle : 0 : 0 : 0$. See c) of Fig. 42 where we depict only the non-empty spheres of $RX$.

We use the following definition in the proof of Proposition 3.11.

**Definition 3.37.** Let $O$ be an oval of a real scheme $S$ in $S^2$. We say that $O$ is trivial if at least one connected component of $S^2 \setminus O$ is a connected component of $S^2 \setminus S$.

**Proof of Proposition 3.11.** Let $S$ be any real scheme in class $d$ listed in Proposition 3.11 for some fixed integers $d, k_1, k_2$ and $h_1, h_2, h_3, h_4$. First of all, the real scheme $S$ is not prohibited by Propositions 3.4 and 3.11. Let us prove by contradiction that $S$ is non-symmetric in class $d$. Assume that $S$ is symmetric in class $d$. Then, there exist a real maximal quartic $Q$ and a degree $d$ real algebraic curve $B \subset \mathbb{C}P^2$ such

![Figure 40](image-url)\[\text{Figure 40: } \mathbb{R} E \simeq S^1 \text{ in thick dashed.}\]

![Figure 41](image-url)\[\text{Figure 41: } \mathbb{R} E \simeq S^1 \text{ in thick dashed.}\]
that the pair \((\mathbb{R}X, \mathbb{R}\phi^{-1}(B))\) realizes \(\mathcal{S}\), where \(\phi : X \to \mathbb{C}P^2\) is the double cover of \(\mathbb{C}P^2\) ramified along \(\overline{Q}\). Remark that for each oval \(O \subset \mathbb{R}\phi^{-1}(B)\), none of the two connected components of \(X_j \setminus O\) contains two trivial ovals of \(\mathcal{S}\), where \(O \subset X_j\), for some \(j \in \{1, 2, 3, 4\}\); see Definition 3.37. It follows that every oval of \(\mathbb{R}B\) is either intersecting \(\mathbb{R}Q\) or forms a non-injective pair with each oval of \(\mathbb{R}Q\). Since \(\mathcal{S}\) has \(2d + 1\) ovals, the real curve \(\mathcal{B}\) has to intersect \(\overline{Q}\) in at least \(4d + 2\) real points: that contradicts Bézout’s theorem. It follows that \(\mathcal{S}\) is non-symmetric in class \(d\).

In order to construct non-symmetric real algebraic curves of class \(d\) in a 4-spheres real del Pezzo surface of degree 2 with real schemes as listed in Proposition 3.31 let us proceed as in the proof of Proposition 3.35.

- Let \(C_S \subset S\) be the real algebraic curve of bi-class \((d, d)\) constructed as in proof of Proposition 3.34. We have that \(\mathbb{R}E_S \cup \mathbb{R}C_S\) is arranged in \(\mathbb{R}S\) as pictured in a) of Fig. 36 (resp. b) of Fig. 36.

- Let \(T\) be the quadric ellipsoid and let \(C_T \subset T\) be the real algebraic curve of bidegree \((d, d)\) constructed as in proof of Proposition 3.30. Then \(\mathbb{R}E_T \cup \mathbb{R}C_T\) is arranged in \(\mathbb{R}T\) as depicted in Fig. 24.

- Thanks to Theorem 3.24, there exist a 4-spheres real del Pezzo surface \(X\) of degree 2 and a class \(d\) real algebraic curve \(A \subset X\) such that the pair \((\mathbb{R}X, \mathbb{R}A)\) realizes \(S\). See Example 3.38.

\[\square\]

**Example 3.38.** For any fixed non-negative integers \(d, k_1, k_2, h_1, h_2, h_3, h_4\) such that

- \(d \geq 6;\)
- \(k_1 + k_2 = d - 4\) and \(k_1, k_2 \neq 0;\)
- \(\sum_{i=1}^{4} h_i = d - 1;\)
- \(h_j \neq 2, \text{ for } j = 3, 4,\)

we construct a real algebraic curve of class \(d\) on a 4-spheres real del Pezzo surface of degree 2, with real scheme 1 \(\sqsubset N(k_1, 1) \sqsubset N(h_1, 0) : N(k_2, 1) \sqcup \{1\}\) \(\sqcup N(h_2 + 1, 0) : N(h_3, 0) : N(h_4, 0)\) (as listed in (2) (i) of Proposition 3.17).

- Let \(C_S \subset S\) be the real algebraic curve of bi-class \((d, d)\) such that \(\mathbb{R}E_S \cup \mathbb{R}C_S\) is arranged in \(\mathbb{R}S\) as pictured in a) of Fig. 36.

- Let \(C_T \subset T\) be the real algebraic curve of bidegree \((d, d)\) such that \(\mathbb{R}E_T \cup \mathbb{R}C_T\) is arranged in \(\mathbb{R}T\) as depicted in Fig. 24.

- Thanks to Theorem 3.24, there exists a real algebraic curve \(A\) of class \(d\) in a real del Pezzo surface \(X\) of degree 2 realizing the real scheme 1 \(\sqsubset N(k_1, 1) \sqsubset N(h_1, 0) : N(k_2, 1) \sqcup \{1\}\) \(\sqcup N(h_2 + 1, 0) : N(h_3, 0) : N(h_4, 0);\) see Fig. 45.

\[\square\]
4 Real curves on $k$-spheres real del Pezzo surfaces of degree 1

4.1 Definitions

Let $Y$ be $\mathbb{C}P^2$ blown up at eight points in generic position; then, the surface $Y$ is a del Pezzo surface of degree 1 (see [BCC+08, pag. 289-312], [Dol12, Chapter 8]). The anti-bicanonical map $\psi: Y \to \mathbb{C}P^3$ exhibits $Y$ as a double ramified cover of an irreducible singular quadric $Q$ in $\mathbb{C}P^3$; the branch locus of $\psi$ consists of the node $V$ of $Q$ and a non-singular cubic section $\tilde{S}$ on $Q$ disjoint from $V$. Conversely, any such double covering is a del Pezzo surface of degree 1.

By construction, the first Chern class $c_1(Y)$ is the pull back via $\psi$ of the class of a line on $Q$. Let $Q_1$ and $Q_2$ be two distinct disjoint unions of connected components of $\mathbb{R}Q \setminus (\mathbb{R}S \cup \{V\})$ such that each $Q_i$ is bounded by $\mathbb{R}S \cup \{V\}$. There exist two lifts to $Y$ of the real structure of $Q$ via the double cover $\psi$ and the real part of $Y$ is the double of one of the $Q_i$’s. Let $\sigma$ be a lifting to $Y$ of the standard real structure on $Q$. Then $\mathbb{R}Y$ is homeomorphic to $\mathbb{R}P^2 \bigsqcup_{j=1}^k S^2$, with $0 \leq k \leq 3$, if and only if the pair $(\mathbb{R}Q, \mathbb{R}S)$ realizes the real scheme depicted in (a) of Fig. 45.

Moreover $(Y, \sigma)$ is $\mathbb{R}$-minimal if and only if $\mathbb{R}Y$ is homeomorphic to $\mathbb{R}P^2 \bigsqcup_{j=1}^4 S^2$; see [DK02, DIK00] and Fig. 44.

![Figure 43: $\mathbb{R}E \simeq S^1$ in thick dashed.](image)

![Figure 44: Real part of the ramified double cover map.](image)

**Definition 4.1.** Let $Y$ be a real degree 1 del Pezzo surface equipped with a real structure $\sigma$. If $\mathbb{R}Y$ is homeomorphic to $\mathbb{R}P^2 \bigsqcup_{j=1}^k S^2$, with $0 \leq k \leq 4$, we say that $Y$ is a $k$-spheres real del Pezzo surface of degree 2.

**Notation 4.2.** Let $Y$ be a $k$-spheres real del Pezzo surface of degree 1, with $0 \leq
Figure 45: \((\mathbb{R}Q, \mathbb{R}\tilde{S})\) respectively for \(k = 0, 1, 2, 3\)

\(k \leq 4\). We denote the connected components of \(\mathbb{R}Y\) with \(Y_0, \ldots, Y_k\), where \(Y_0\) is homeomorphic to \(\mathbb{R}P^2\) and \(Y_j\), with \(j \geq 1\), is homeomorphic to \(S^2\).

**Definition 4.3.** Let \(Y\) be a \(k\)-spheres real del Pezzo surface of degree 1, with \(0 \leq k \leq 4\), and let \(B \subset Y\) be a non-singular real algebraic curve. Then, we say that \(B\) has class \(d\) on \(Y\) if \(B\) realizes \(d c_1(Y)\) in \(H_2(Y; \mathbb{Z})\).

Let \(Y\) be a 4-spheres real del Pezzo surface of degree 1, then \(H_2^{-}\)\((Y; \mathbb{Z})\) is generated by \(c_1(Y)\) (BCC"08 pag. 289-312]). We are interested in the classification up to homeomorphism of the pairs \((\mathbb{R}Y, \mathbb{R}B)\), where \(B\) is a non-singular real algebraic curve of class \(d\) in a \(k\)-spheres real del Pezzo surface \(Y\) of degree 1, with \(0 \leq k \leq 4\).

### 4.2 Real schemes

**Definition 4.4.** Let \(Y\) be a \(k\)-spheres real del Pezzo surface of degree 1, with \(0 \leq k \leq 4\). Let \(\mathcal{S}\) be a real scheme on \(\mathbb{R}Y\). We say that \(\mathcal{S}\) is in class \(d\) if the number of ovals of \(\mathcal{S}\) does not exceed \(\frac{d(d-1)}{2} + 2\). Moreover, we say that \(\mathcal{S}\) is realizable in class \(d\), if there exists a real algebraic curve \(B \subset Y\) of class \(d\), such that the pair \((\mathbb{R}Y, \mathbb{R}B)\) realizes \(\mathcal{S}\).

See Section 2.1 for real schemes notation.

### 4.3 Positive and negative connected components

Only in the case of 4-spheres real del Pezzo surface of degree 1, one can label the spheres of the real part of the surface as positive and negative as follows. Let \(Y\) be a 4-spheres real del Pezzo surface of degree 1, the anti-bicanonical map \(\psi\) exhibits \(Y\) as a double cover of \(Q\) ramified along a real maximal cubic section \(\tilde{S}\). Independently from the choice of a complex orientation on \(\mathbb{R}\tilde{S}\), we can distinguish the connected components of \(\mathbb{R}\tilde{S}\) on \(\mathbb{R}Q\) in the following way. Let us consider \(Q \setminus \{V\}\). There are four connected components, called ovals, of \(\mathbb{R}\tilde{S}\) realizing the trivial class in \(H_1(\mathbb{R}Q \setminus \{V\}; \mathbb{Z}/2\mathbb{Z})\); while the connected component of \(\mathbb{R}\tilde{S}\) realizing the non-trivial class in \(H_1(\mathbb{R}Q \setminus \{V\}; \mathbb{Z}/2\mathbb{Z})\), is called long-component. If the union of an oval and the long-component of \(\mathbb{R}\tilde{S}\) in \(\mathbb{R}Q \setminus \{V\}\) bounds an oriented surface, the oval is called positive; otherwise negative.

Each oval of \(\mathbb{R}\tilde{S}\) is either positive or negative, independently from the choice of a complex orientation on \(\mathbb{R}\tilde{S}\); moreover, two ovals are positive and two are negative. Via \(\psi\) we can label the connected components of \(\mathbb{R}Y\) homeomorphic to \(S^2\) as positive and negative; see Fig. 46, where the preimage of negative and positive ovals, is depicted respectively in dashed and thick black. It follows that for any fixed non-negative integer \(d\), we have two topological classifications of real algebraic curves of class \(d\) in 4-spheres real del Pezzo surfaces of degree 1.
Notation 4.5. We use the convention that the connected components $Y_1$ and $Y_2$ are positive and $Y_3$ and $Y_4$ negative.

Definition 4.6. Let $Y$ be a 4-spheres real del Pezzo surface of degree 1. An arrangement $S = S_0 | S_1 : S_2 : S_3 : S_4$ of disjoint circles embedded in $R^4$ is called a refined real scheme if each $S_k$ encodes the ovals arrangement on $Y_k$. We say that a refined real scheme $S$ is realizable in class $d$ if there exists a class $d$ real algebraic curve $B \subset Y$ such that the pair $(R, R_B)$ realizes $S$.

Example 4.7. Let $Y$ be a 4-spheres real del Pezzo surface of degree 1. Let $J \sqcup (\{1\} \mid 1 : 1 : 0 : 0)$ be a real scheme in $R$. Assume that there exists a class $d$ real curve $A_1 \subset Y$ (resp. $A_2 \subset Y$) such that the pair $(R, R_{A_1})$ (resp. $(R, R_{A_2})$) realizes the refined real scheme $J \sqcup (\{1\} \mid 0 : 1 : 1 : 0$ (resp. $J \sqcup (\{1\} \mid 1 : 1 : 0 : 0$). We say that the real scheme $J \sqcup (\{1\} \mid 1 : 1 : 0 : 0$ is realized in class $d_{\text{ref}}$ by the curves $A_1$ and $A_2$. Moreover, we say that the refined real scheme $J \sqcup (\{1\} \mid 1 : 1 : 0 : 0$ is realized in class $d_{\text{ref}}$ by $A_2$, and it is not realized by $A_1$.

4.4 Main results

In this paper, we prove that the real schemes classification and the refined real schemes classification are the same up to class 3 in 4-spheres real del Pezzo surfaces of degree 1 (Theorem 4.8). Furthermore, we classify real algebraic curves up to class 3 in $k$-spheres real del Pezzo surfaces of degree 1, where $k < 4$ (Proposition 4.10).

Theorem 4.8 (Classification of (refined) real schemes in class $d \in \{1, 2, 3\}$). Let $B$ be a real algebraic curve of class $d \in \{1, 2, 3\}$ in a 4-spheres real del Pezzo surface of degree 1. Then $B$ realizes one of the following real schemes:

(1) If $d = 1$:

(i) $J \sqcup 1 \mid 0 : 0 : 0 : 0$;
(ii) $J \mid 1 : 0 : 0 : 0$;
(iii) $J \mid 0 : 0 : 0 : 0$.

(2) If $d = 2$:

(i) $0 \sqcup 0 : 0 : 0 : 0$;
(ii) $\alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : 0 : 0$, with $0 \leq \alpha + \beta + \gamma + \delta \leq 2$;
(iii) $0 \sqcup \alpha \sqcup \langle \beta \rangle : \gamma : \delta : 0$, with $0 \leq \alpha + \beta + \gamma + \delta \leq 2$;
(iv) $\langle \{1\} \rangle \mid 0 : 0 : 0 : 0$.

(3) If $d = 3$:

(i) $J \sqcup \alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : \epsilon : 0$, with $0 \leq \alpha + \beta + \gamma + \delta + \epsilon \leq 3$;
(ii) $J \sqcup 1 \mid 1 : 1$ \sqcup \langle 1 \rangle : 0 : 0 : 0$;
(iii) $J \sqcup \alpha \sqcup \langle \beta \rangle \mid \gamma : 0 : 0 : 0$, with $0 \leq \alpha + \beta + \gamma \leq 2$;
(iv) $J \mid \alpha \sqcup \langle \beta \rangle \sqcup \langle \gamma \rangle : \delta : \epsilon : 0$, with $0 \leq \alpha + \beta + \gamma + \delta + \epsilon \leq 2$;
(v) $J \sqcup \langle \{1\} \rangle \mid 0 : 0 : 0 : 0$;
(vi) $J \sqcup \langle \{1\} \rangle \sqcup \langle \{1\} \rangle \mid 0 : 0 : 0 : 0$;
(vii) $J \sqcup \langle 1 \sqcup \langle 1 \rangle \rangle \mid 0 : 0 : 0 : 0$;
(viii) $J \mid 1 : 1 : 1 : 1$;
(ix) $J \mid 1 : 0 : 0 : 0$;
(x) $J \mid 0 : 0 : 0 : 0$.

Moreover, let $S_0 \mid S_1 : S_2 : S_3 : S_4$ be any real scheme in the above list. For any permutation $\sigma \in S_4$, there exist a 4-spheres real del Pezzo surface $Y$ of degree 1 and a real algebraic curve of class $d$ in $Y$ realizing the refined real scheme $S_0 \mid S_{\sigma(1)} : S_{\sigma(2)} : S_{\sigma(3)} : S_{\sigma(4)}$. 

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**Proof.** All real schemes in the above list are those non-prohibited by Proposition 4.12. We realize all the refined real schemes in class \(d \in \{1, 2, 3\}\) in 4-spheres real del Pezzo surfaces of degree 1 in the proofs of Proposition 4.18, Proposition 4.21 and Corollary 4.25.

In order to simplify the presentation of Proposition 4.10, let us introduce one more definition.

**Definition 4.9.** Let \(S_0 \mid S_1 : \ldots : S_k\) be an arrangement of ovals in the disjoint union of a real projective plane and \(k\) spheres and let \(Y\) be a \(j\)-spheres real del Pezzo surface of degree 1 with \(0 \leq j < k \leq 4\). If \(s \geq k - j\) among the \(S_i\)'s, with \(i \neq 0\), are 0, we say that \(S_0 \mid S_1 : \ldots : S_k\) is realizable in class \(d\) on \(\mathbb{R}Y\) if there exists a real algebraic curve \(B \subset Y\) of class \(d\), such that the pair \((\mathbb{R}Y, \mathbb{R}B)\) realizes \(S_0 \mid S_1 : \ldots : S_j\), where \(\{S_0, S_1, \ldots, S_k\} = \{0, S_0, S_1, \ldots, S_j\}\).

**Proposition 4.10** \((k = 3, 2, 1, 0\) and \(d \in \{1, 2, 3\}\)). Let \(B\) be a real algebraic curve of class \(d \in \{1, 2, 3\}\) in a \(k\)-spheres real del Pezzo surface of degree 1, with \(0 \leq k \leq 3\). Then \(B\) realizes one of the following real schemes:

- **if** \(k = 3, 2, 1, 0\) and
  
  - \((1)\) \(d = 1:\)
    - \((i)\) \(J \sqcup 1 \mid 0 : 0 : 0;\)
    - \((ii)\) \(J \mid 0 : 0 : 0.\)
  - \((2)\) \(d = 2:\)
    - \((i)\) \(0 \mid 0 : 0 : 0;\)
    - \((ii)\) \(\alpha \sqcup \langle \beta \rangle \mid 0 : 0 : 0, \text{ with } 0 \leq \alpha + \beta \leq 2;\)
    - \((iii)\) \(\langle \langle 1 \rangle \rangle \mid 0 : 0 : 0.\)
  - \((3)\) \(d = 3:\)
    - \((i)\) \(J \sqcup \alpha \sqcup \langle \beta \rangle \mid 0 : 0 : 0, \text{ with } 0 \leq \alpha + \beta \leq 3;\)
    - \((ii)\) \(J \sqcup \alpha \sqcup \langle \langle \beta \rangle \rangle \mid 0 : 0 : 0, \text{ with } 0 \leq \alpha + \beta \leq 2;\)
    - \((iii)\) \(J \sqcup \langle \langle \langle 1 \rangle \rangle \rangle \mid 0 : 0 : 0;\)
    - \((iv)\) \(J \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \mid 0 : 0 : 0;\)
    - \((v)\) \(J \mid 0 : 0 : 0.\)

- **if** \(k = 3, 2, 1\) and
  
  - \((1)\) \(d = 1:\)
    - \((i)\) \(J \mid 1 : 0 : 0.\)
  - \((2)\) \(d = 2:\)
    - \((i)\) \(\alpha \sqcup \langle \beta \rangle \mid \gamma : 0 : 0, \text{ with } 0 \leq \alpha + \beta + \gamma \leq 2;\)
    - \((ii)\) \(0 \mid \alpha \sqcup \langle \beta \rangle : 0 : 0, \text{ with } 0 \leq \alpha + \beta \leq 2.\)
  - \((3)\) \(d = 3:\)
    - \((i)\) \(J \sqcup \alpha \sqcup \langle \beta \rangle \mid \gamma : 0 : 0, \text{ with } 0 \leq \alpha + \beta + \gamma \leq 3;\)
    - \((ii)\) \(J \sqcup \alpha \sqcup \langle \langle \beta \rangle \rangle \mid \gamma : 0 : 0, \text{ with } 0 \leq \alpha + \beta + \gamma \leq 2;\)
    - \((iii)\) \(J \sqcup \langle \beta \rangle \sqcup \langle \gamma \rangle : 0 : 0, \text{ with } 0 \leq \alpha + \beta + \gamma \leq 2;\)
    - \((iv)\) \(J \sqcup 1 \sqcup 1 \sqcup \langle 1 \rangle : 0 : 0;\)
    - \((v)\) \(J \mid 1 : 0 : 0.\)

- **if** \(k = 3, 2\) and
  
  - \((1)\) \(d = 2:\)
    - \((i)\) \(\alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : 0, \text{ with } 0 \leq \alpha + \beta + \gamma + \delta \leq 2;\)
    - \((ii)\) \(0 \mid \alpha : \beta : 0, \text{ with } 0 \leq \alpha + \beta \leq 3.\)
  - \((2)\) \(d = 3:\)
    - \((i)\) \(J \sqcup \alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : 0, \text{ with } 0 \leq \alpha + \beta + \gamma + \delta \leq 3;\)
    - \((ii)\) \(J \mid \alpha \sqcup \langle \beta \rangle : \gamma : 0, \text{ with } 0 \leq \alpha + \beta + \gamma \leq 2.\)
• If \( k = 3 \) and

\( (1) \) \( d = 2: \)

\( (i) \ 0 \mid 1 : 1 : 1. \)

\( (2) \) \( d = 3: \)

\( (i) \ J \sqcup \alpha \mid \beta : \gamma : \delta, \) with \( 0 \leq \alpha + \beta + \gamma + \delta \leq 4. \)

Moreover, for each real scheme \( S \) in the above list, there exists a \( k \)-spheres real del Pezzo surface of degree 1 and a class \( d \) real algebraic curve \( B \subset Y \) such that the pair \((\mathbb{R}Y, \mathbb{R}B)\) realizes \( S \).

**Proof.** All real schemes in the above list are those non-prohibited by Proposition 4.12. We realize all of them in the proofs of Proposition 4.20, Proposition 4.22 and Corollary 4.26.

\[ \square \]

### 4.5 Obstructions

The number of pseudo-lines (see Section 2.1) of a real algebraic curve of class \( d \) in a real minimal del Pezzo surface \( Y \) of degree 1, is determined by \( d \).

**Proposition 4.11.** Let \( B \) be a non-singular real algebraic curve of class \( d \) in a real minimal del Pezzo surface \( Y \) of degree 1. Then, the real scheme realized by \( \mathbb{R}B \) has one and only one pseudo-line if \( d \equiv 1 \) (mod 2) and no pseudo-lines otherwise.

**Proof.** Let \( d \) be odd (resp. even). Since the value modulo 2 of the intersection form on \( H_1(Y; \mathbb{Z}) \) descends on \( H_1(\mathbb{R}Y; \mathbb{Z}/2\mathbb{Z}) \approx H_1(\mathbb{R}P^2; \mathbb{Z}/2\mathbb{Z}) \), it follows that \( \mathbb{R}B \) has an odd (resp. even) number of pseudo-lines. Moreover, the real part of \( B \) has at most one pseudo-line since any two pseudo-lines meet in at least one point.

The number of connected components of a real curve in a \( k \)-spheres real del Pezzo surface of degree 1 is bounded as follows.

**Proposition 4.12.** Let \( B \) be a non-singular real algebraic curve of class \( d \) in a \( k \)-spheres real del Pezzo surface \( Y \) of degree 1, with \( 0 \leq k \leq 4 \). Then, the number \( l \) of connected components of \( \mathbb{R}B \) is bounded as follows:

\[
\varepsilon \leq l \leq \frac{d(d-1)}{2} + 2,
\]

where \( \varepsilon \in \{0, 1\} \) is such that \( \varepsilon \equiv d \) (mod 2).

**Proof.** The right inequality follows from Harnack-Klein’s inequality and the adjunction formula; while the left one follows from Proposition 4.11. \( \square \)

The system of obstructions given by Proposition 4.12 is complete for (refined) real schemes in class 1, 2 and 3; see Section 4.6. The next statement provides an example of additional obstructions for real schemes in class \( d > 3 \) in 4-spheres real del Pezzo surfaces of degree 1.

**Proposition 4.13.** Let \( B \) be a non-singular real algebraic curve of class \( d = 2s + \varepsilon \) in a 4-spheres real del Pezzo surface \( Y \) of degree 1, where \( \varepsilon \in \{0, 1\} \) and \( s \in \mathbb{Z}_{\geq 1} \). Assume that all connected components of \( \mathbb{R}B \) lie on \( Y_0 \) and on \( t \) of the \( Y_j \)’s, for \( j = 1, 2, 3, 4 \). Assume that \( N_h \), with \( h = 1, 2, 3 \), are three nests of depth \( i_h \) of \( \mathbb{R}B \). Moreover, assume that \( i_1 \leq i_2 \) and \( N_1, N_2 \) form a disjoint pair of nests in \( Y_0 \); while \( N_3 \) lies on some \( Y_j \), where \( j \neq 0 \). Then, we have the following restrictions on the depths of the nests:

\[
i_1 + i_2 \leq 3s + \varepsilon - t; \quad (1)
\]

\[
i_2 + i_3 \leq 3s + \varepsilon - (t - 1). \quad (2)
\]

**Proof.** For any given collection \( \mathcal{P} \) of 6 distinct points on \( Y \), there exists an algebraic curve \( T \) of class 3 on \( Y \) passing through \( \mathcal{P} \). If all points of \( \mathcal{P} \) are real and such that each connected component of \( \mathbb{R}Y \) contains at least one point of \( \mathcal{P} \), then \( T \) is real and \( \mathbb{R}T \) has exactly one connected component on each connected component of \( \mathbb{R}Y \).

Let us choose such a collection \( \mathcal{P} \) in the following way. On each boundary of the
two disks in \( Y_0 \setminus \bigcup_{n=1}^{2} N_n \), pick a point. Moreover, pick a point on every connected component \( Y_j \), with \( j \geq 1 \), such that the point belongs to \( \mathbb{R}B \) any time the real algebraic curve has at least one oval on \( Y_j \). Then, there exists a real algebraic curve \( T \) of class 3 passing through \( \mathcal{P} \). Thus, the number of real intersection points of \( \mathbb{R}B \) with \( \mathbb{R}T \) is at least \( 2(i_1 + i_2 + \varepsilon) \). Therefore, inequality \((1)\) follows directly from the fact that the intersection number \( \mathbb{B} \circ T = 3(2s + \varepsilon) \) is greater or equal to the number of real intersection points of \( \mathbb{B} \) with \( T \).

The proof of \((2)\) is similar to the previous one. \( \square \)

**Example 4.14.** Let \( Y \) be a 4-spheres real del Pezzo surface of degree 1. Let us illustrate the proof of the inequality \((1)\) in Proposition 4.13 for the real scheme \( \mathcal{S} := (1) \sqcup (1) | 2 : 1 : 1 : 0 \) in class 4 on \( \mathbb{R}Y \); see Fig. 47. Assume that there exists a real curve \( \mathbb{B} \subset Y \) of class 4 such that \((\mathbb{R}Y, \mathbb{R}B)\) realizes \( \mathcal{S} \). The real scheme \( \mathcal{S} \) has two nests of depth 2 on \( Y_0 \) and ovals on \( Y_1, Y_2 \) and \( Y_3 \). Choose a configuration \( \mathcal{P} \) of 6 real points \( p_1, \ldots, p_6 \) as depicted in Fig. 47. Then, there exists a real algebraic curve \( T \) passing through \( \mathcal{P} \) as depicted in dashed in Fig. 47. It follows that \( T \) and \( B \) intersect in at least 14 points, but they can not intersect in more than 12 points. In conclusion \( \mathcal{S} \) is unrealizable in class 4.

![Figure 47: Example of unrealizable real scheme in class 4.](image)

### 4.6 Constructions

Let \( \text{Bl}_V : \Sigma_2 \to Q \) be the blow-up of the quadratic cone \( Q \) at the node \( V \); see Section 2.2.

**Definition 4.15.** Let \( k \) and \( l \) be two non-negative integers. We say that an algebraic curve \( \mathcal{C} \) on \( Q \) has bidegree \((k, l)\) if \( \text{Bl}_V^{-1}(\mathcal{C}) \) has bidegree \((k, l)\) in \( \Sigma_2 \).

Let \( \tilde{S} \) be a real cubic section on \( Q \), namely a real curve of bidegree \((3, 0)\), such that the double cover \( \psi : Y \to Q \) ramified along \( \tilde{S} \) is a \( 4 \)-spheres real del Pezzo surface of degree 1. Let \( Z \) be any real algebraic curve of bidegree \((s, \varepsilon)\) on \( Q \), with \( \varepsilon \in \{0, 1\} \). From the arrangement of the triplet \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z)\), we recover the real scheme realized by the real algebraic curve \( \psi^{-1}(Z) \) of class \( 2s + \varepsilon \) in \( Y \). See as example Fig. 48 on the left, the arrangement of a triplet \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z)\), where \( \tilde{S} \) is maximal and we have fixed one of the two complex orientations on \( \mathbb{R}\tilde{S} \); on the right, the induced refined real scheme \( \mathcal{J} \sqcup 1 \mid 1 : 0 : 1 : 2 \) on \( \mathbb{R}Y \).

![Figure 48: Preimage via \( \psi \) of a particular triplet \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z)\). In dashed the negative ovals (resp. spheres) of \( \mathbb{R}\tilde{S} \) (resp. of \( \mathbb{R}Y \)). In thick black \( \mathbb{R}Z \) and \( \psi^{-1}(\mathbb{R}Z) \).](image)

**Remark 4.16.** Let \( \tilde{S} \) and \( Z_1 \) be non-singular real algebraic curves on \( Q \) respectively of bidegree \((3, 0)\) and \((1, 0)\). Let \( F \) be any real line on \( Q \). On can apply small perturbation
method to \( Z_1 \cup F \) in order to construct a bidegree \((1,1)\) non-singular real algebraic curve \( Z_2 \) on \( Q \). In particular, from the arrangement of the triplet \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z_1 \cup \mathbb{R}F)\) one can recover the arrangement of the triplet \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z_2)\).

**Example 4.17.** Let us illustrate via an example the construction of Remark 4.16. Let \( S, Z_1 \) and \( F \) be non-singular real algebraic curves on \( Q \) respectively of bidegree \((3,0)\), \((1,0)\) and \((0,1)\). Let the topological type of \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z_1 \cup \mathbb{R}F)\) be as depicted in Fig. 49 on the left. One can perturb \( Z_1 \cup F \) in order to construct a bidegree \((1,1)\) non-singular real algebraic curve \( Z_2 \) on \( Q \) such that the arrangement of the triplet \((\mathbb{R}Q, \mathbb{R}\tilde{S}, \mathbb{R}Z_2)\) is as depicted on the right of Fig. 49.

![Figure 49: Application of construction in Remark 4.16](image)

4.6.1 Small perturbation method on \( Q \)

In Propositions 4.18 and 4.20 we realize some of the real schemes respectively listed in Theorem 4.8 and Proposition 4.10.

**Proposition 4.18.** Let \( S_0 \mid S_1 : S_2 : S_3 : S_4 \) be any real scheme in class \( d \) listed below. Then, for any permutation \( \sigma \in S_4 \), there exists a 4-spheres real del Pezzo surface \( Y \) of degree 1 and a real algebraic curve \( B \) of class \( d \) such that the pair \((\mathbb{R}Y, \mathbb{R}B)\) realizes the refined real scheme \( S_0 \mid S_{\sigma(1)} : S_{\sigma(2)} : S_{\sigma(3)} : S_{\sigma(4)} \).

1. For \( d = 1 \):
   - (i) \( J \sqcup 1 \mid 0 : 0 : 0 : 0 ; \)
   - (ii) \( J \mid 1 : 0 : 0 : 0 ; \)
   - (iii) \( J \mid 0 : 0 : 0 : 0 . \)

2. For \( d = 2 \):
   - (i) \( 0 \mid 0 : 0 : 0 : 0 ; \)
   - (ii) \( \alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : 0 : 0 , with \ 0 \leq \alpha + \beta + \gamma + \delta \leq 2 ; \)
   - (iii) \( \langle 1 \rangle \mid 0 : 0 : 0 : 0 . \)

3. For \( d = 3 \):
   - (i) \( J \sqcup \alpha \sqcup \langle \beta \rangle \mid \gamma : \delta : 0 : 0 , with \ 0 \leq \alpha + \beta + \gamma + \delta + \varepsilon \leq 3 ; \)
   - (ii) \( J \sqcup 1 \mid 1 \mid 1 \mid (1) : 0 : 0 : 0 ; \)
   - (iii) \( J \sqcup \alpha \sqcup \langle \beta \rangle \mid \gamma : 0 : 0 : 0 , with \ 0 \leq \alpha + \beta + \gamma \leq 2 ; \)
   - (iv) \( J \sqcup \langle 1 \rangle \mid 0 : 0 : 0 : 0 ; \)
   - (v) \( J \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \mid 0 : 0 : 0 : 0 ; \)
   - (vi) \( J \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle \mid 0 : 0 : 0 : 0 ; \)
   - (vii) \( J \mid 1 : 0 : 0 : 0 ; \)
   - (viii) \( J \mid 0 : 0 : 0 : 0 . \)

**Proof.** There exists a real algebraic maximal curve \( \tilde{S} \) of bidegree \((3,0)\) on \( Q \) with real part as depicted in a) of Fig. 50 (resp. b) of Fig. 50). Choose one of the two complex orientations on \( \mathbb{R}\tilde{S} \). Any real line \( F \) on \( Q \) lifts to a real algebraic curve of class \( 1 \) on a 4-spheres real del Pezzo surface of degree 1; in Fig. 51 from the depicted real lines on \( Q \), one recovers all refined real schemes in class \( 1 \). For any two lines \( F_1, F_2 \) on \( Q \) there exists an hyperplane section \( H \subset \mathbb{C}P^3 \) passing
through $V$ such that $Q \cap H = F_1 \cup F_2$. Moving slightly $H$, one constructs a real algebraic curve $Z_1$ of bidegree $(1, 0)$ on $Q$ such that from the arrangement of the triplet $(RQ, \tilde{S}, RF_1 \cup RF_2)$ one can recover the arrangement of the triplet $(RQ, \tilde{S}, RZ_1)$. See as example $a)$ of Fig. 52 where we have depicted a bidegree $(0, 2)$ real algebraic curve as intersection of $Q$ with an hyperplane section $H$ passing through $V$; resp. in $b)$ of Fig. 52, moving slightly $H$, we have constructed a bidegree $(1, 0)$ real algebraic curve on $Q$. Considering all the possible arrangements of $\tilde{S}$ and two lines, one can construct real algebraic curves $Z_1$ of bidegree $(1, 0)$ on $Q$ such that from the arrangement of the triplet $(RQ, \tilde{S}, RZ_1)$ one can recover the arrangement of the triplet $(RQ, \tilde{S}, RZ_1)$.

Example 4.19. Assume that there exist real curves $H$, $Z_1$ and $Z_2$ as in the proof of Proposition 4.18 such that the triplet $(RQ, \tilde{S}, RH)$ is arranged as depicted in $a)$ of Fig. 52, resp. $(RQ, \tilde{S}, RZ_1)$ as in $b)$ of Fig. 52 and resp. $(RQ, \tilde{S}, RZ_2)$ as in $c)$ of Fig. 52. We deduce that there exist a 4-spheres real del Pezzo surfaces $Y$ of degree 1 and real curves $B_1$ of class 2 realizing the refined real scheme $1 \sqcup (1) | 0 : 0 : 0 : 0$ (resp. $J \sqcup (1) \sqcup (1) | 0 : 0 : 0 : 0$).

Proposition 4.20. Let $S$ be any real scheme in class $d$ listed below. Then, there exist a $k$-spheres real del Pezzo surface $Y$ of degree 1 and a class $d$ real curve $B \subset Y$ such that the pair $(RY, RB)$ realizes $S$.

- For $k = 3, 2, 1, 0$ and
  
  (1) $d = 1$:
  
  (i) $J \sqcup 1 | 0 : 0 : 0$;
  
  (ii) $J | 0 : 0 : 0$.
  
  (2) $d = 2$:
  
  (i) $0 | 0 : 0 : 0$;
  
  (ii) $\alpha \sqcup (\beta) | 0 : 0 : 0$, with $0 \leq \alpha + \beta \leq 2$;

Figure 50:

a) 

b) 

Figure 51:

\[ RF \]

\[ RH \cap RQ \]

\[ RZ_1 \]

\[ RZ_2 \]

Figure 52: a) $(RQ, \tilde{S}, RH)$. b) After moving the hyperplane section $H$. c) Application of construction in Remark 4.16.
(3) \( d = 3: \)
   (i) \( J \sqcup \alpha \sqcup \langle \beta \rangle | 0 : 0 : 0, \) with \( 0 \leq \alpha + \beta \leq 3; \)
   (ii) \( J \sqcup \alpha \sqcup \langle \langle \beta \rangle \rangle | 0 : 0 : 0, \) with \( 0 \leq \alpha + \beta \leq 2; \)
   (iii) \( J \sqcup \langle \langle 1 \rangle \rangle | 0 : 0 : 0; \)
   (iv) \( J \sqcup (1) \sqcup \langle 1 \rangle | 0 : 0 : 0; \)
   (v) \( J \sqcup \langle 1 \rangle \sqcup \langle 1 \rangle | 0 : 0 : 0; \)
   (vi) \( J | 0 : 0 : 0. \)

\( \bullet \) For \( k = 3, 2, 1 \) and

(1) \( d = 1: \)
   (i) \( J | 1 : 0 : 0. \)

(2) \( d = 2: \)
   (i) \( \alpha \sqcup \langle \beta \rangle | \gamma : 0 : 0, \) with \( 0 \leq \alpha + \beta + \gamma \leq 2. \)

(3) \( d = 3: \)
   (i) \( J \sqcup \alpha \sqcup \langle \beta \rangle | \gamma : 0 : 0, \) with \( 0 \leq \alpha + \beta + \gamma \leq 3; \)
   (ii) \( J \sqcup \alpha \sqcup \langle \langle \beta \rangle \rangle | \gamma : 0 : 0, \) with \( 0 \leq \alpha + \beta + \gamma \leq 2; \)
   (iii) \( J \sqcup 1 | 1 \sqcup \langle 1 \rangle | 0 : 0 : 0; \)
   (iv) \( J | 1 : 0 : 0. \)

\( \bullet \) For \( k = 3, 2 \) and

(1) \( d = 2: \)
   (i) \( \alpha \sqcup \langle \beta \rangle | \gamma : \delta : 0, \) with \( 0 \leq \alpha + \beta + \gamma + \delta \leq 2. \)

(2) \( d = 3: \)
   (i) \( J \sqcup \alpha \sqcup \langle \beta \rangle | \gamma : \delta : 0, \) with \( 0 \leq \alpha + \beta + \gamma + \delta \leq 3. \)

\( \bullet \) For \( k = 3 \) and \( d = 3: \)

(i) \( J \sqcup 1 | 1 : 1 : 1. \)

**Proof.** The proof is similar to that of Proposition 4.18.

### 4.6.2 Harnack’s construction method on \( \mathbb{Q} \)

In the proof of Propositions 4.21 and 4.22, we use a variant of Harnack’s construction method to realize some (refined) real schemes respectively listed in Theorem 4.8 and Proposition 4.10.

**Proposition 4.21.** Let \( S_0 : S_1 : S_2 : S_3 : S_4 \) be any real scheme in class \( d \) listed below. For any permutation \( \sigma \in S_4 \), there exists a 4-spheres real del Pezzo surface \( Y \) of degree 1 and a real algebraic curve \( B \subset Y \) of class \( d \) such that the pair \((R_Y, R_B)\) realizes the refined real scheme \( S_0 : S_{\sigma(1)} : S_{\sigma(2)} : S_{\sigma(3)} : S_{\sigma(4)}. \)

(1) For \( d = 2: \)
   (i) \( 0 | \alpha : \beta : \gamma : 0, \) with \( 0 \leq \alpha + \beta + \gamma \leq 3. \)

(2) For \( d = 3: \)
   (i) \( J | \alpha \sqcup \langle \beta \rangle : \gamma : \delta, \) with \( 0 \leq \alpha + \beta + \gamma + \delta \leq 3. \)

**Proposition 4.22.** Let \( S \) be any real scheme in class \( d \) listed below. Then, there exist a \( k \)-spheres real del Pezzo surface \( Y \) of degree 1 and a class \( d \) real curve \( B \subset Y \) such that the pair \((R_Y, R_B)\) realizes \( S. \)

\( \bullet \) For \( k = 3, 2, 1 \) and

(1) \( d = 2: \)
   (i) \( 0 | \alpha : 0 : 0, \) with \( 0 \leq \alpha + \beta \leq 3. \)
(2) \(d = 3:\)

(i) \(\mathcal{J} \mid \alpha \cup \langle \beta \rangle : 0 : 0, \text{ with } 0 \leq \alpha + \beta \leq 3.\)

- For \(k = 3, 2\) and

(1) \(d = 2:\)

(i) \(0 \mid \alpha : \beta : 0, \text{ with } 0 \leq \alpha + \beta \leq 3.\)

(2) \(d = 3:\)

(i) \(\mathcal{J} \mid \alpha \cup \langle \beta \rangle : \gamma : 0, \text{ with } 0 \leq \alpha + \beta + \gamma \leq 3.\)

- For \(k = 3\) and

(1) \(d = 2:\)

(i) \(0 \mid \alpha : \beta : \gamma, \text{ with } 0 \leq \alpha + \beta \leq 3.\)

(2) \(d = 3:\)

(i) \(\mathcal{J} \mid \alpha \cup \langle \beta \rangle : \gamma : \delta, \text{ with } 0 \leq \alpha + \beta + \gamma + \delta \leq 3.\)

Proof of Proposition \[4.21\] and Proposition \[4.22\]: Fix a non-singular real algebraic curve \(Z_1\) of bidegree \((1, 0)\) on \(Q\). Pick any other real algebraic curve \(L_1\) of bidegree \((1, 0)\) on \(Q\) such that \(Z_1 \cap L_1\) consists of two distinct real points. Let \(P_0(x, y)P_1(x, y) = 0\) be a polynomial equation defining the union of \(Z_1\) and \(L_1\) in some local affine chart of \(Q\). Choose \(4\) real lines \(F_1\) on \(Q\), with \(i = 1, 2, 3, 4\), intersecting transversely \(Z_1 \cup L_1\). Replace the left side of the equation \(P_0(x, y)P_1(x, y) = 0\) with \(P_0(x, y)P_1(x, y) + \varepsilon f_1(x, y)f_2(x, y)f_3(x, y)f_4(x, y)\), where \(f_1(x, y) = 0\) is an affine equation of the line \(F_i\) and \(\varepsilon \neq 0\) is a sufficient small real number. Up to a choice of the sign of \(\varepsilon\), one constructs a small perturbation \(L_2\) of \(Z_1 \cup L_1\), where \(L_2\) is a non-singular real curve of bidegree \((2, 0)\) such that \(\bigcup_{i=1}^4 F_i \cap Z_1 = L_2 \cap Z_1\). In Fig. \[53\] we have depicted an example of such a construction before and after a small perturbation, respectively pictured in \(a)\) and \(b)\); the dashed segments are the lines \(F_i\)’s.

Analogously, by a small perturbation of \(Z_1 \cup L_2\), one can construct a non-singular real algebraic curve \(\hat{S}\) of bidegree \((3, 0)\) on \(Q\). In Fig. \[53\] we have depicted an example of such a construction before and after a small perturbation, respectively pictured in \(b)\) and \(c)\). Finally, one can construct real algebraic curves \(\hat{S}\) on \(Q\) and bidegree \((1, 0)\), resp. \((1, 1)\) thanks to Remark \[4.16\] real algebraic curves \(Z_1\), resp. \(Z_2\), on \(Q\) such that, from the arrangement of the triplets \((RQ, R\hat{S}, RZ_1)\), resp. \((RQ, R\hat{S}, RZ_2)\), we deduce that there exist \(k\)-spheres del Pezzo surfaces \(Y\) of degree \(1\) and real curves \(B_1 \subset Y\) of class \(2\), resp. \(B_2 \subset Y\) of class \(3\), realizing all (refined) real schemes listed in Propositions \[4.21\] and \[4.22\]. See Example \[4.23\].
4.6.3 Five particular constructions

With Corollary 4.25 (resp. Corollary 4.26) we end the proof of Theorem 4.8 (resp. Proposition 4.10). First of all, we give some intermediate constructions in Proposition 4.24 whose proofs rely on Viro’s patchworking method ([Vir84a], [Vir84b], [Vir89], [Vir06]) and constructions via dessins d’enfant (Section 2.3).

**Proposition 4.24.** There exist real algebraic curves of bidegree $(3,0)$ on $Q$ respectively with charts as depicted in a), b), c), d) and e) of Fig. 54.

![Figure 54: Charts of real algebraic curves.](image)

![Figure 55: Charts of real algebraic curves.](image)

![Figure 56: Intermediate constructions.](image)

**Proof.** First of all, let us construct real algebraic curves of bidegree $(2,2)$ in $\Sigma_2$ with charts respectively as depicted in a), b), c), d) and e) of Fig. 55. Let $\tilde{\eta}_1$, $\tilde{\eta}_2$, $\tilde{\eta}_3$, $\tilde{\eta}_4$ and $\tilde{\eta}_5$ be trinodal $\mathcal{L}$-schemes on $\mathbb{R}\Sigma_4$ respectively as depicted in a), b), c), d) and e) of Fig. 56. By Theorem 2.28 the completions of the real graphs associated to the $\tilde{\eta}_i$’s respectively depicted in a), b), c), d) and e) of Fig. 57 prove the existence of real trinodal curves $\tilde{D}_i$’s $\Sigma_4$ respectively realizing the $\tilde{\eta}_i$’s. Moreover, the $\tilde{D}_i$’s are reducible because they have 8 non-degenerate double points and their normalizations have 5 real connected components. In addition, the $\tilde{D}_i$’s have to be the union of a real curve of bidegree $(2,0)$ and a real curve of bidegree $(1,0)$.
schemes and there exists a non-singular real algebraic curve of class
arrangements with respect to the coordinates axis
Finally, we can apply Viro’s patchworking method to the polynomials and charts
construct bidegree
in
a
Proof. In Proposition 4.24, we have constructed bidegree
real scheme
of degree
in
a
Q
gebraic curves
\( \tilde{\Sigma} \)
and also bidegree
\( \Sigma \)
Corollary 4.26.
For any
\( \tilde{\Sigma} \)
real schemes
\( J \mid \langle \langle \langle \langle 1 \rangle \rangle \rangle \rangle \)
\( 0 : 0 : 0 : 0 \)
(resp.
Corollary 4.26).
Finally, we can apply Viro’s patchworking method to the polynomials and charts
of the \( D_i \)'s and a non-singular real algebraic curve of class
(1,0) in \( \Sigma_2 \). Moreover, the charts of the \( D_i \)'s are respectively as depicted
in a), b), c), d) and e) of Fig. 55
Corollary 4.25.
There exist a 4-spheres real del Pezzo surface \( Y \) of degree 1 and
non-singular real algebraic curves of class 2 (resp. 3) in \( Y \), realizing the real schemes
\( 0 \mid \langle \langle 1 \rangle \rangle : 0 : 0 : 0 \) and \( 0 : 0 : 0 : \langle \langle 1 \rangle \rangle : 0 \) (resp.
\( J \mid \langle \langle \langle \langle 1 \rangle \rangle \rangle \rangle : 0 : 0 : 0 \) and
\( J \mid 0 : 0 : 0 : \langle \langle 1 \rangle \rangle : 0 \).
Proof. In Proposition 4.24 we have constructed bidegree (3,0) non-singular real algebraic
maximal curves \( \tilde{S} \) and also bidegree (1,0) non-singular real algebraic maximal
curves \( Z_1 \) on \( Q \); see the charts of \( \tilde{S} \) respectively depicted in a) and b) of Fig. 54
Let \( Y \)
be the double cover of \( Q \) ramified along \( \tilde{S} \). Therefore, there exist non-singular real
algebraic curves of class 2 in \( Y \) respectively realizing the real schemes \( 0 \mid \langle \langle 1 \rangle \rangle : 0 : 0 : 0 \)
and \( 0 : 0 : 0 : \langle \langle 1 \rangle \rangle : 0 \). Applying the construction in Remark 4.16 to the curves
\( Z_1 \), one can construct non-singular real algebraic curves of class 3 in \( Y \) respectively with
real schemes \( J \mid \langle \langle \langle \langle 1 \rangle \rangle \rangle \rangle : 0 : 0 : 0 \) and
\( J \mid 0 : 0 : 0 : \langle \langle 1 \rangle \rangle : 0 \).
Corollary 4.26. For any \( k \in \{3,2,1\} \), there exist a \( k \)-spheres real del Pezzo surface
of degree 1 and a non-singular real algebraic curve of class 2 (resp. 3) realizing the real
scheme \( 0 \mid \langle \langle 1 \rangle \rangle : 0 : 0 \) (resp. \( J \mid \langle \langle \langle \langle 1 \rangle \rangle \rangle \rangle : 0 : 0 \).
Proof. In Proposition 4.24 we have constructed bidegree (3,0) non-singular real algebraic
curves \( \tilde{S} \) and also bidegree (1,0) non-singular real algebraic curves \( Z_1 \) on
\( Q \); see the charts of \( \tilde{S} \) respectively depicted in c), d) and e) of Fig. 54
Therefore, for any \( k \in \{3,2,1\} \), there exists a \( k \)-spheres real del Pezzo surface of degree 1
and there exists a non-singular real algebraic curve of class 2 in \( Y \) realizing the real
schemes \( 0 \mid \langle \langle 1 \rangle \rangle : 0 : 0 : 0 \). Applying the construction in Remark 4.16 to \( Z_1 \), one

Figure 57: Intermediate constructions.
can construct a non-singular real algebraic curve of class 3 in $Y$ with real schemes $J \mid \langle\langle 1 \rangle\rangle : 0 : 0 : 0$.

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