Analytic solutions and Singularity formation for the Peakon b–Family equations

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Abstract. Using the Abstract Cauchy-Kowalewski Theorem we prove that the b-family equation admits, locally in time, a unique analytic solution. Moreover, if the initial data is real analytic and it belongs to $H^s$ with $s > 3/2$, and the momentum density $u_0 - u_{0,xx}$ does not change sign, we prove that the solution stays analytic globally in time, for $b \geq 1$. Using pseudospectral numerical methods, we study, also, the singularity formation for the b-family equations with the singularity tracking method. This method allows us to follow the process of the singularity formation in the complex plane as the singularity approaches the real axis, estimating the rate of decay of the Fourier spectrum.

1. Introduction

In 2003 Holm and Staley \cite{13,14} introduced, as a one-dimensional version of active fluid transport, the b-family equation

\begin{equation}
\frac{m}{t} + \frac{u}{x} m_x + \frac{b}{x} m_x x = 0, \quad t > 0, \quad x \in \mathbb{R},
\end{equation}

where $b$ is a real parameter, $u = u(x,t)$ the fluid velocity, and $m = u - u_{xx}$ the momentum density. The quadratic terms in (1.1) represent the competition and balance, in fluid convection between nonlinear transport and amplification due to $b$-dimensional stretching. The parameter $b$ characterizes the equations because it is the ratio of stretching to convective transport and also the number of covariant dimensions associated with the momentum density $m$.

Within this family of evolutionary equations we have both the Camassa-Holm equation \cite{4,15} in correspondence of $b = 2$ and the Degasperis-Procesi equation \cite{8} in correspondence of $b = 3$ as special cases. Using the method of asymptotic integrability, it has been shown that these are the unique two cases in which (1.1) is completely integrable \cite{8}. A quite complete list of references on the Camassa-Holm, the Degasperis-Procesi equations and the b–family equations can be found in \cite{6,7,25,26,27,28}.

We can rewrite (1.1) is the following third order form in terms only of the unknown $u$

\begin{equation}
-u_{txx} + (b + 1) uu_x = bu_x x + uu_{xxx}, \quad t > 0, \quad x \in \mathbb{R}.
\end{equation}

Moreover, (1.1) and (1.2) are also equivalent to the following hyperbolic elliptic system

\begin{equation}
\frac{u}{t} + uu_x + P_x = 0, \quad -P_{xx} + P = \frac{b}{2} u^2 + \frac{3 - b}{2} u_x^2 .
\end{equation}
Since $e^{-|x|}/2$ is the Green's function of the Helmholtz operator $1 - \partial_x^2$ we can rewrite (1.3) in the following integro-differential form

$$u_t + uu_x + \int \frac{e^{-|x-y|}}{2} \text{sign}(y-x) \left( \frac{b}{2} u^2(y, t) + \frac{3-b}{2} u_x^2(y, t) \right) dy = 0.$$ 

In this paper we are concerned with the initial value problem associated to (1.2), therefore we augment the equation with the initial condition

$$(1.4) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}. $$

Assuming that

$$u_0 \in H^s(\mathbb{R}), \quad s > \frac{3}{2}, \quad u_0 - u_{0,xx} \text{ has constant sign}$$

using the Kato's theory and giving a precise description of the blow-up scenario, Gui, Liu, and Tian proved the existence of global in time solutions.

The rest of the paper is organized as follows. In Section 2.1 we introduce the functional spaces in which our results hold. Section 2.2 is devoted to the local in time existence of solutions for the Cauchy problem associated to (1.2). The existence of global in time analytical solution for (1.2) is proved in Section 3. In Section 4 we analyze the singularity formation using the singularity tracking method for different values of $b > -1$.

2. Local in time result

2.1. Functional settings and the ACK Theorem. In this section we introduce the functional spaces where local in time analyticity for the $b$-family equations will be proved (see [17,18,22]). We consider $x$ a complex variable, with $\Re x$ and $\Im x$ respectively its real and imaginary part. We consider $f(x)$ a complex function analytic in

$$(2.1) \quad D(\rho) = \{ x \in \mathbb{C} : |\Im x| \leq \rho \}$$

and $f \in L^2(\Gamma(\Im x))$ for $\Im x \in (-\rho, \rho)$. The path where the $L^2$ integration is performed is:

$$(2.2) \quad \Gamma(b) = \{ x \in \mathbb{C} : |\Im x| = b \}.$$ 

This means that if $\Im x$ is inside $(-\rho, \rho)$, then $f(\Re x + i\Im x)$ is a square integrable function of $\Re x$. We denote by $H^{0,\rho}$ the set of complex functions $f(x)$ with the above properties and such that

$$(2.3) \quad |f|_\rho = \sup_{\Im x \in (-\rho, \rho)} \|f(\cdot + i\Im x)\|_{L^2(\Gamma(\Im x))} < \infty$$

is satisfied.

We denote by $H^{k,\rho}$ the space of all complex functions $f(x)$ such that

$$(2.4) \quad \partial^j_x f \in H^{0,\rho}, \quad \text{for} \quad 0 \leq j \leq k; \quad \text{and} \quad \|f\|_{k,\rho} = \sum_{0 \leq j \leq k} \|\partial^j_x f\|_\rho.$$

In term of the Fourier transform $\hat{f}(\xi)$ of $f$, the previous norm can be rewritten as

$$(2.5) \quad \|f\|_{k,\rho} = \left( \int e^{2\rho|\xi|} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi \right)^{1/2}.$$ 

We denote by $H^{1,\rho}_{p,T}$ the space of all functions $f(x,t) \in C^1([0,T], H^{1,\rho})$ and

$$|f|_{1,\rho,\beta,T} = \sup_{0 \leq t \leq T} |f(\cdot, t)|_{0,\rho-\beta t} + \sup_{0 \leq t \leq T} \|\partial_x f(\cdot, t)|_{0,\rho-\beta t} + \sup_{0 \leq t \leq T} \|\partial_x f(\cdot, t)|_{0,\rho-\beta t}$$

is finite. In an analogous way, $H^{k,\rho}_{p,T}$ is the space of all functions $f \in C^k([0,T], H^{k,\rho})$ and $|f|_{k,\rho,\beta,T} < \infty$. 
We conclude this section recalling the ACK theorem in the form given by Safonov [21].

Consider the problem
\[ u = F(u, t), \]
where \( u \in X_\rho \) and \( X_\rho \) is a Banach space with norm denoted by \( | \cdot |_\rho \).

**Theorem 1. ACK Theorem:** Suppose that \( \exists R > 0, \rho_0 > 0, \) and \( \beta_0 > 0 \) such that if \( 0 < t \leq \rho_0/\beta_0 \), the following properties hold:

1. for all \( 0 < \rho' < \rho \leq \rho_0 \) and \( u \in X_\rho \) such that \( \sup_{0 \leq t \leq T} |u(\cdot, t)|_\rho \leq R \), the map \( F(u, t) : [0, T] \rightarrow X_{\rho'} \) is continuous;
2. for all \( 0 < \rho < \rho_0 \) the function \( F(0, t) : [0, \rho_0/\beta_0] \rightarrow \{ u \in X_\rho : \sup_{0 \leq t \leq T} |u(\cdot, t)|_\rho \leq R \} \) is continuous and \( |F(0, t)|_\rho \leq \rho_0 < R \);
3. for all \( 0 < \rho' < \rho(s) \leq \rho_0 \) and for all \( w \) and \( u \in \{ u \in X_\rho : \sup_{0 \leq t \leq T} |u(\cdot, t)|_{\rho - \beta_0 t} \leq R \} \), results
\[ |F(u, t) - F(w, t)|_{\rho'} \leq C \int_0^t |u - w|_{\rho(s)} ds, \]
then there exists \( \beta > \beta_0 \) such that for all \( 0 < \rho < \rho_0 \) equation () has a unique solution \( u(\cdot, t) \in X_\rho \) with \( t \in [0, (\rho_0 - \rho)/\beta_0] \) and moreover
\[ \sup_{\rho - \beta_0 t \leq t \leq \rho_0} |u(\cdot, t)|_\rho \leq R. \]

**2.2. First main theorem: short time existence of analytic solution.** In this section we prove the following first main theorem:

**Theorem 2. Local in time analyticity:** Let the initial data for the b-family \( u_0 \in H^{1, \rho_0} \). Then there exists \( \beta > 0 \) such that for any \( \rho, \) with \( 0 < \rho < \rho_0 \), there exists a unique continuously differentiable w.r.t. time solution \( u \) of the b-family equation with the following property:

- \( u(\cdot, t) \in H^{1, \rho} \) and \( \partial_t u(\cdot, t) \in H^{1, \rho} \), when \( t \in \left[ 0, \frac{\rho_0 - \rho}{\beta_0} \right] \).

**Proof.** We apply the abstract Cauchy-Kovalevskaya (ACK) theorem to the b-family equation to prove the short time existence of analytic solution. We refer to [18] for the details and here we give only a sketch of the proof.

To apply the ACK theorem to the b-family equation, we write the b-family equation as:
\[ u_t + uu_x = -A^{-2} \left( \frac{b}{2} u^2 + \frac{3 - b}{2} u_x^2 \right)_x, \]
where \( A^2 = (1 - \partial_t^2) \), and, with an integration in time, we transform (2.6) in (2.7)
\[ u = F(u, t), \quad \text{with} \]
\[ F(u, t) = u_0 - \int_0^t \left[ uu_x + A^{-2} \left( \frac{b}{2} u^2 + \frac{3 - b}{2} u_x^2 \right)_x \right] dt'. \]
It is clear that properties 1 and 2 of the ACK theorem are satisfied by the operator \( F(u, t) \) given in (2.7). For the validity of hypothesis 3, one considers first the norm \( |F(u, t)|_{1, \rho'} \), where the bound is easily obtained using the following estimates which are a direct consequence of the Cauchy estimate on the derivative of analytic function (17, 22):

**Lemma 1.** Let \( u, w \in H^{1, \rho''} \) and \( \rho' < \rho'' \), then \( \|uw_x - uw_x\|_{1, \rho'} \leq c \frac{|u - w|_{1, \rho''}}{\rho'' - \rho'}. \)

Finally, the bound on \( |\partial_x F(u, t)|_{1, \rho'} \) is obtained using the fact that the operator \( A^{-2} \) is bounded. \( \square \)
3. Global in time analyticity

In this section we prove that the \( b \)-family equation admits a strong solution in Gevrey-class functional space of index 1. This analysis gives an explicit lower bound on the shrinking of the radius of analyticity of the solution of the \( b \)-family equation with given analytical initial data, and assuming suitable regularity in Sobolev space.

We consider the space \( D(A^r e^{\rho A}) \), which consists of the real restriction of complex analytic functions on the strip \( D(\rho) \). The operator \( A^r e^{\rho A} \) is the Gevrey operator with \( A^2 = (1 - \partial_x^2) \). The space \( D(A^r e^{\rho A}) \) is equivalent to the space \( H^{r, \rho} \) defined in section 2.1, with norm given by (2.5), then we define the Gevrey class \( G^1 \) of index 1 by (see [16]):

\[
G^1 = \bigcup_{\rho > 0} H^{r, \rho}.
\]

We suppose that the initial data \( u_0 \in D(A^r e^{\rho_0 A}) \), with \( \rho_0 \) and \( r \) positive.

In this section we prove the following second main theorem:

**Theorem 3.** Global in time analyticity: Let \( b \geq 1 \) and \( A^2 = (1 - \partial_x^2) \). Let \( u_0 \in D(A^r e^{\rho_0 A}) \), with \( r > 3/2, \rho_0 > 0 \) and \( m_0 = u_0 - u_{0x} \) does not change sign. Then the unique solution \( u \) of the \( b \)-family equation lies in Gevrey class of index 1 globally in time.

**Proof.** We start the energy type estimate on \( D(A^r e^{\rho A}) \) for the \( b \)-family equation. Supposing that \( \rho \) is a decreasing, positive \( C^1 \) function of the time \( t \) with \( \rho(0) = \rho_0 \), we consider the \( L^2 \)-scalar product of the \( b \)-family equation (2.6) with \( A^{2r-2} e^{2\rho A} u \):

\[
\frac{1}{2} \frac{d}{dt} \| A^r e^{\rho A} u \|^2 - \dot{\rho} \| A^{r+1/2} e^{\rho A} u \|^2 = \]

\[
= |\langle A^r e^{\rho A} uu_x, A^r e^{\rho A} u \rangle + \langle A^{r-1} A^r e^{\rho A} \left( \frac{b}{2} u^2 + \frac{3 - b}{2} u_x^2 \right)_x, A^{r-1} e^{\rho A} u \rangle |
\]

\[
\leq |\langle A^r e^{\rho A} uu_x, A^r e^{\rho A} u \rangle| + \left| \frac{b}{2} \langle A^{r-1} e^{\rho A} (u_x^2)_x, A^{r-1} e^{\rho A} u \rangle \right| + \frac{|b - 3|}{2} \langle A^{r-1} e^{\rho A} (u_x^2)_x, A^{r-1} e^{\rho A} u \rangle |.
\]

(3.2)

Now, we start to estimate the three terms on the second member of (3.2).

\( I_1 = |\langle A^r e^{\rho A} uu_x, A^r e^{\rho A} u \rangle|, \ I_2 = |\langle A^{r-1} e^{\rho A} (u_x^2)_x, A^{r-1} e^{\rho A} u \rangle| \) and

\( I_3 = |\langle A^{r-1} e^{\rho A} (u_x^2)_x, A^{r-1} e^{\rho A} u \rangle|. \)

From Lemma 3 below, we have

\[
I_1 \leq C_1 \| A^r u \|^3 + C_2 \rho \| A^r e^{\rho A} u \| \| A^{r+1/2} e^{\rho A} u \|^2.
\]

(3.3)

Using Cauchy estimate and Lemma 3 below we obtain:

\[
I_2 \leq \langle A^r e^{\rho A} u_x, A^{r-1} e^{\rho A} u \rangle \leq \| A^r e^{\rho A} u_x \|^2 \| A^{r-1} e^{\rho A} u \|
\]

\[
\leq \| A^r e^{\rho A} u_x \|^2 \| A^{r-1} e^{\rho A} u \| \leq \| A^r e^{\rho A} u \|^3
\]

\[
\leq C_1 \| A^r u \|^3 + \rho C_2 \| A^{r+1/2} e^{\rho A} u \|^3
\]

\[
\leq C_1 \| A^r u \|^3 + \rho C_2 \left( \| A^r e^{\rho A} u \|^{1/3} \| A^{r+1/2} e^{\rho A} u \|^{2/3} \right)^3
\]

\[
\leq C_1 \| A^r u \|^3 + \rho C_2 \| A^r e^{\rho A} u \| \| A^{r+1/2} e^{\rho A} u \|^2;
\]

(3.4)
(3.5) \[ I_3 \leq \langle A^{-1} e^{\rho A} u_x, A e^{\rho A} u \rangle \leq \| A^{-1} e^{\rho A} u_x \| \| A e^{\rho A} u \| \]
\[ \leq \| A^{-1} e^{\rho A} u_x \| \| A e^{\rho A} u \| \leq \| A e^{\rho A} u \|^2 \]
\[ \leq C_1 \| A u \|^3 + \| A^{r+1/2} e^{\rho A} u \|^3 \]
\[ \leq C_1 \| A u \|^3 + C_2 \left( \| A e^{\rho A} u \|^{1/3} \| A^{r+1/2} e^{\rho A} u \|^{2/3} \right)^3 \]
\[ \leq C_1 \| A u \|^3 + C_2 \| A e^{\rho A} u \| A^{r+1/2} e^{\rho A} u \|^2. \]

Collecting all the previous estimates (3.3), (3.5) and (3.6), we obtain

(3.6) \[ \frac{1}{2} \frac{d}{dt} \| A e^{\rho A} u \|^{2} \leq (\dot{\rho} + C_2 \rho \| A e^{\rho A} u \|) \| A^{r+1/2} e^{\rho A} u \|^2 + C_1 \| A u \|^3, \]
where constants \( C_1 \) and \( C_2 \) depend only on \( b \) and \( \rho \). The last inequality (3.6) implies that \( u \in D(A e^{\rho A}) \) with

(3.7) \[ \rho(t) = \rho_0 e^{-C_2 \int_0^t (\| A e^{\rho A} u_0 \| + C_1 (\int_0^t \| A^{r+1/2} e^{\rho A} u \|^2 dt') \) dt'. \]

To prove the global in time analyticity of the solution \( u \), it is necessary to have an a priori Sobolev estimate on \( \| A^r u \| \). To do that we use the global existence result for the \( b \)-family equation, given by the result of Lemma (4) below.

**Lemma 2.** Let \( \rho \geq 0 \) and let \( u, v \in D(A^r e^{\rho A}) \) with \( r > 1/2 \); and let \( w \in D(A^{r+s} e^{\rho A}) \), with \( r, s \geq 0 \). There exists constants \( C, C_1 \) and \( C_2 \), depending only on \( r \), such that

\[ \| A^r e^{\rho A} (uw) \| \leq C \| A^r e^{\rho A} u \| \| A^r e^{\rho A} v \| ; \]
\[ \| A^r e^{\rho A} w \| \leq C_1 \| A^r u \| + C_2 \| A^{r+s} e^{\rho A} w \|. \]

where \( \| \cdot \| \) is the usual \( L^2 \) norm.

**Proof.** See [10]. \( \square \)

**Lemma 3.** Let \( v \in D(A^r e^{\rho A}) \) and \( u \in D(A^{r+1/2} e^{\rho A}) \), with \( \rho \geq 0 \) and \( r > 3/2 \), then there exists constants \( C_1 \) and \( C_2 \), depending only on \( r \), such that

\[ \| A^r e^{\rho A} u_x, A^r e^{\rho A} u \| \leq C_1 \| A^r e^{\rho A} u \| \| A^r e^{\rho A} u \| \leq C_2 \rho \| A^r e^{\rho A} u \| \| A^r+1/2 e^{\rho A} u \|. \]

**Proof.** See [11] and [18]. \( \square \)

**Lemma 4.** Let \( b \geq 1 \) and let \( u_0 \in H^s(\mathbb{R}) \cap L^1(\mathbb{R}) \), with \( s > 3/2 \) and \( m_0 = u_0 - u_{0xx} \) does not change sign, then the corresponding solution does exists globally.

**Proof.** See [11] for the proof. \( \square \)

4. Singularity formation for the \( b \)-family equation

In this section we present the analysis of the singularity formation for the \( b \)-family equation using the singularity tracking method for different values of the parameter \( b \). The singularity tracking method is based on the following Laplace asymptotic formula (see [5]) for the Fourier spectrum of a complex function \( u(z) \):

(4.1) \[ \hat{u}_k \sim C |k|^{-(1+\alpha)} \exp(-\delta |k|) \exp(i |k| z^*) \]
where \( z^* = x^* + i \delta \) is the nearest complex singularity to the real axis, \( \alpha \) is the algebraic type, and at the singularity \( u(z) \sim (z - z^*)^\alpha \). Therefore an estimation of \( \delta \) and \( x^* \) gives the complex location of the singularity. Resolving the rate of algebraic decay \( 1 + \alpha \), one can classify the singularity type. In particular, when one deals with a PDE with solution \( u(t) \), if at a critic time \( t_c \) results that \( \delta(t_c) = 0 \) then the width of analyticity of the solution is zero and one can observe a blow up of the solution or a blow up of its derivatives. The reader is invited to see [3, 9, 12, 10, 20, 23, 24] for more details on this method.
To apply the singularity tracking method to the $b$-family, we numerically solve the equations in the spatial domain $[-\pi, \pi]$, employing periodic boundary conditions. We write the $b$-family equations in pseudodifferential form

$$u_t + uu_x = -A^{-2} \left( \frac{b}{2} u^2 + \frac{3 - b}{2} u_x^2 \right)_x,$$

and the dynamic of the $k$th Fourier mode of $u$ is described by the following ODE

$$(4.2) \quad \partial_t \hat{u}_k = - \left( \frac{i k}{1 + k^2} \left( \frac{b}{2} (u^2)_k + \frac{3 - b}{2} (u_x^2)_k \right) \right).$$

Dividing the time interval $[0, T]$ in $N$ sub-intervals of size $\Delta t = T/N$, we approximate the solution $u$ as

$$u(x, n\Delta t) \approx \sum_{k=-K/2}^{K/2} \hat{u}_n^k e^{ikx};$$

and we solve the system of ODEs (4.2) using explicit Runge–Kutta method of the 4th order with initial conditions given by

$$\hat{u}_0^k = \hat{u}_0^k,$$

where $\hat{u}_0^k$ are the Fourier coefficients of the initial data $u(x, 0) = u_0(x)$. The reader is referred to [9] for the analysis of the convergence properties of this numerical scheme for the Camassa–Holm equation case, which is the $b$-family equation with $b = 2$.

The estimation of the parameters $C$, $\alpha$ and $\delta$ in the Laplace asymptotic formula (4.1) is performed with the technique of the sliding–fitting with length 3 (see [3, 9, 19, 20, 23] for details). This procedure is based on the local estimation of the wanted values for each $k$ mode, by the formulas:

$$\alpha(k) = \frac{\log \left( \frac{\hat{u}_{k-1} \hat{u}_{k+1}}{\hat{u}_{k}^2} \right)}{\log \left( \frac{k}{(k-1)(k+1)} \right)},$$

$$\delta(k) = \left[ \log \left( \frac{\hat{u}_k}{\hat{u}_{k+1}} \right) + \alpha \log \left( \frac{k}{k+1} \right) \right],$$

$$\log C(k) = \log |\hat{u}_k| + \alpha(k) \log (k) + k\delta(k).$$

The asymptotic values for large $k$ can be computed with an extrapolation process using the epsilon algorithm of Wynn (see [19]), giving the wanted values $C$, $\alpha$ and $\delta$.

For all the numerical simulations we shall present in this work, we have used up to 4096 Fourier modes calculated with 32-digit (using the ARPREC package).
Here we present the numerical results of the Degasperi–Procesi equation, which corresponds to the \( b \)-family equation with \( b = 3 \) with respect to initial data of type I and II.

![Figure 1](image1.png)

**Figure 1.** The behavior in time of the numerical solution of the Degasperi–Procesi equation with initial condition of type I. At time \( t_s = 0.8295 \) a singularity forms as a blow up of the first derivative.

![Figure 2](image2.png)

**Figure 2.** The behavior in time of the numerical solution of the Degasperi–Procesi equation with initial condition of type II. At time \( t_s = 0.8875 \) a singularity forms as a blow up of the first derivative.

The behaviors in time of the numerical solutions and their derivatives of the Degasperi–Procesi equation with initial datum of type I and II, are shown respectively in Fig.1 and Fig.2. The critical times when singularities develop are respectively \( t_s = 0.8295 \) and \( t_s = 0.8875 \). At these critical times, it is evident a blow up of the first derivatives of the solutions.

The analysis of the singularities formation and their characterization is demanded in the sequel, where we perform a complete analysis for all the \( b \)-family equations.

5. Classifications of complex singularities: initial data \( u_0 = \sin(x) \).

In this section we report the results obtained by applying the singularity tracking method to the \( b \)-family equation with initial datum of type I.

In Fig.3 it is shown the behavior in time of the spectrum of the Degasperi–Procesi numerical solution \((b = 3)\), starting at time \( t = 0.35 \) up to singularity time.
$t_s = 0.8295$ with increments of 0.05: at critical time $t_s$ the spectrum totally loss exponential decay, as a singularity of cubic–root type hits the real axis, with the blow–up of the first derivative of the solution. The extrapolation of $\delta(t)$ and $\alpha(t)$ (on Fig.4) is obtained by the sliding fitting procedure of length 3.

![Figure 3](image1.png)

**Figure 3.** The behavior of the spectrum in time of the Degasperi–Procesi equation numerical solution with initial datum of type I, starting at time $t = 0.35$ up to singularity time $t_s = 0.8295$ with increments of 0.05.

![Figure 4](image2.png)

**Figure 4.** The tracking singularity method results of the Degasperi–Procesi equation numerical solution with initial datum of type I. On the top, the evolution in time of the width of the analyticity strip $\delta$. On the bottom, the evolution in time of the algebraic character $\alpha$. The algebraic character at singularity time is $\alpha(t_s) = 0.33$.

We have also performed several other numerical simulations for the initial condition I, by varying the parameter $b > -1$. We have obtained that in each case a singularity develops in a finite time, and in Fig.5 and Fig.6 the algebraic character of the singularities and the times of singularity formation are shown with respect to the parameter $b$. We remember that for $b = -1$ the solution for the initial datum of type I is a traveling wave solution, and therefore in this case the solution is globally analytic in time.
6. Classifications of complex singularities: initial data \( u_0 = 1 + \sin(x) \).

In this section we report the results obtained by applying the singularity tracking method to the \( b \)-family equation with initial datum of type II. In Fig.7 it is shown the behavior in time of the spectrum of of the Degasperi–Procesi numerical solution, starting at time \( t = 0.4 \) up to singularity time \( t_s = 0.8875 \) with increments of 0.05. On Fig.8 the evolutions in time of the width of the analyticity strip \( \delta \) and the algebraic character \( \alpha \) are shown. One can see that at the singularity time \( t_s = 0.8875 \) the algebraic character is \( \alpha(t_s) = 0.41 \), meaning that a blow–up of the first derivative of the solution occurs. The results are also obtained by applying the sliding fitting procedure with length 3.

Also for the initial condition of type II we have performed several other numerical simulations by varying the parameter \( b \). The singularity time and the algebraic character of the singularities are shown in Fig.9 and Fig.10 w.r.t the parameter \( b \). As observed in the case of the initial condition of type I, for \( b = -1 \) the solution for the initial data of type II is a traveling wave solution, and therefore the solution is globally analytic in time.
Figure 7. The behavior in time of the spectrum of the Degasperi–Procesi equation numerical solution with initial datum of type II, starting at time $t = 0.4$ up to singularity time $t_s = 0.8875$ with increments of 0.05.

Figure 8. The tracking singularity method results of the Degasperi–Procesi equation numerical solution with initial datum of type II up to singularity time $t_s = 0.8875$. On the top, the evolution in time of the width of the analyticity strip $\delta$. On the bottom, the evolution in time of the algebraic character $\alpha$. The algebraic character at singularity time is $\alpha(t_s) = 0.41$.

7. Conclusions

In this paper we have considered the $b$-family equations, introduced by Holm and Staley \cite{13, 14}. Our main interest has been to show the well-posedness of the $b$-family equation in analytic function spaces. Using the Abstract Cauchy-Kowalewski Theorem, we prove that the $b$-family equation admits a unique analytic solution, both locally in time (Theorem 2) then globally in time (Theorem 3). An interesting problem in the theory of PDEs is the singularity formation and how to detect it from a numerical point of view. One of the most effective methods to achieve this goal is the singularity tracking method. It consists to follow the singularity in the complex plane before the appearance as a blow up of the solution. The distance of the singularity from the real axis can be estimated through the study of the exponential rate of decay of the Fourier spectrum. The algebraic character of the singularity is determinate, instead, by the study of the algebraic rate of decay of the Fourier spectrum. We use numerical pseudospectral methods to
study the process of singularity formation for the $b$-family equations, w.r.t the parameter $b > -1$, and the singularity time and the algebraic characters are shown in Sections 5 and 6. We analysed also the cases for $b < -1$ and also in these cases the solutions develop a singularity, but it was complicated to determine their algebraic character. Very high numeric precision is needed and it will be the subject of future work.

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