INVARIANTS AND INFINITESIMAL TRANSFORMATIONS FOR CONTACT SUB-LORENTZIAN STRUCTURES ON 3-DIMENSIONAL MANIFOLDS.

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Abstract. In this article we develop some elementary aspects of a theory of symmetry in sub-Lorentzian geometry. First of all we construct invariants characterizing isometric classes of sub-Lorentzian contact 3 manifolds. Next we characterize vector fields which generate isometric and conformal symmetries in general sub-Lorentzian manifolds. We then focus attention back to the case where the underlying manifold is a contact 3 manifold and more specifically when the manifold is also a Lie group and the structure is left invariant.

1. Introduction

1.1. Basic notions and motivation. For all details and facts concerning sub-Lorentzian geometry, the reader is referred to [8] and the references therein (see also [15] [13]). In this section we only present the notions that are required for the formulation of the main results of the paper.

Let \( M \) be a smooth manifold. A sub-Lorentzian structure on \( M \) is a pair \((H, g)\) where \( H \) is a bracket generating distribution of constant rank on \( M \), and \( g \) is a Lorentzian metric on \( H \). A triple \((M, H, g)\), where \((H, g)\) is a sub-Lorentzian structure on \( M \), will be called a sub-Lorentzian manifold.

For any \( q \in M \), a vector \( v \in H_q \) will be called horizontal. A vector field \( X \) on \( M \) is horizontal if it takes values in \( H \). We will denote the set of all local horizontal vector fields by \( \Gamma(H) \). To be more precise, \( X \in \Gamma(H) \) if and only if \( X \) is a horizontal vector field defined on some open subset \( U \subset M \).

A nonzero vector \( v \in H_q \) is said to be timelike (resp. spacelike, null, non-spacelike) if \( g(v, v) < 0 \) (resp. \( g(v, v) > 0 \), \( g(v, v) = 0 \), \( g(v, v) \leq 0 \)), moreover the zero vector is defined to be spacelike. Similarly, vector fields are categorized analogously according to when their values lie in exactly one of the four categories mentioned above. An absolutely continuous curve \( \gamma : [a, b] \rightarrow M \) is called horizontal if \( \dot{\gamma}(t) \in H_{\gamma(t)} \) a.e. on \([a, b] \). A horizontal curve \( \gamma : [a, b] \rightarrow M \) is timelike (spacelike, null, non-spacelike) if \( \dot{\gamma}(t) \) is timelike (spacelike, null, non-spacelike) a.e. on \([a, b] \).

If \((H, g)\) is a sub-Lorentzian metric on \( M \) then, as shown in [10], \( H \) can be represented as a direct sum \( H = H^- \oplus H^+ \) of subdistributions such that rank \( H^- = 1 \) and the restriction of \( g \) to \( H^- \) (resp. to \( H^+ \)) is negative (resp. positive) definite. This type of decomposition will be called a causal decomposition of \( H \). Now we say
that \((M, H, g)\) is time (resp. space) orientable if the vector bundle \(H^- \to M\) (resp. 
\(H^+ \to M\)) is orientable. It should be noted that although causal decompositions 
are not unique, the definition of time and space orientation is independent of a 
particular choice of a causal decomposition. Since a line bundle is orientable if and 
only if it is trivial, time orientability of \((M, H, g)\) is equivalent to the existence of a 
continuous timelike vector field on \(M\). A choice of such a timelike field is called 
a time orientation of \((M, H, g)\).

Suppose that \((M, H, g)\) is time oriented by a vector field \(X\). A nonspacelike \(v \in H_q\) will be called future (resp. past) directed if \(g(v, X(q)) < 0\) (resp. \(g(v, X(q)) > 0\)). A horizontal curve \(\gamma : [a, b] \to M\) is called timelike future (past) directed if 
\(\dot{\gamma}(t)\) is timelike future (past) directed a.e. Similar classifications can be made for 
other types of curves, e.g. nonspacelike future directed etc. If \(q_0 \in M\) is a point 
and \(U\) is a neighborhood of \(q_0\), then by the future timelike (nonspacelike, null) 
reachable set from \(q_0\) relative to \(U\) we mean the set of endpoints of all timelike 
(nonspacelike, null) future directed curves that start from \(q_0\) and are contained in 
\(U\).

Now we define a very important notion that will play a crucial role in the sequel. As 
it is known [S], any sub-Lorentzian structure \((H, g)\) determines the so-called 
geodesic Hamiltonian which is defined as follows. The existence of the structure 
\((H, g)\) is equivalent to the existence of the fiber bundle morphism \(G : T^*M \to H\) 
covering the identity such that if \(v, w\) are any horizontal vectors, then 
\(g(v, w) = \langle \xi, G\eta \rangle = \langle \eta, G\xi \rangle\) whenever \(\xi \in G^{-1}(v), \eta \in G^{-1}(w)\). The geodesic Hamiltonian is 
the map \(h : T^*M \to \mathbb{R}\) defined by 
\[
h(\lambda) = \frac{1}{2} \langle \lambda, G\lambda \rangle.
\]
If \(X_1, \ldots, X_k\) is an orthonormal basis for \((H, g)\) with a time orientation \(X_1\), then 
\[
h|_{T^*_qM}(\lambda) = -\frac{1}{2} \langle \lambda, X_1(q) \rangle^2 + \frac{1}{2} \sum_{i=2}^k \langle \lambda, X_i(q) \rangle^2.
\]
A horizontal curve \(\gamma : [a, b] \to M\) is said to be a Hamiltonian geodesic if 
there exists \(\Gamma : [a, b] \to T^*M\) such that \(\dot{\Gamma} = \frac{d}{dt} \Gamma(t) = \gamma(t)\) on 
\([a, b]\): by \(\pi : T^*M \to M\) we denote the canonical projections, and \(h\) is the Hamiltonian 
vector field corresponding to \(h\).

Let \(\gamma : [a, b] \to M\) be a nonspacelike curve. The non-negative number 
\(L(\gamma) = \int_a^b |g(\dot{\gamma}(t), \dot{\gamma}(t))|^{1/2} dt\) is called the sub-Lorentzian length of a curve \(\gamma\). If \(U \subset M\) is an open subset, then the (local) sub-Lorentzian distance relative to \(U\) is the function 
\(d[U] : U \times U \to [0, +\infty)\) defined as follows: For \(q_1, q_2 \in U\), let \(\Omega_{q_1,q_2}^{nspc}(U)\) denote 
the set of all nonspacelike future directed curves contained in \(U\) which join \(q_1\) to 
\(q_2\), then 
\[
d[U](q_1, q_2) = \left\{ \sup \left\{ L(\gamma) : \gamma \in \Omega_{q_1,q_2}^{nspc}(U) \right\} : \Omega_{q_1,q_2}^{nspc}(U) \neq \emptyset \right\}.
\]
If \(\Omega_{q,q}^{nspc}(U)\) is non-empty for a \(q \in U\) then \(d[U](q, q) = +\infty\). A nonspacelike future 
directed curve \(\gamma : [a, b] \to U\) is called a \(U\)-maximizer if \(d[U](\gamma(a), \gamma(b)) = L(\gamma)\). 
It can be proved (see [S]) that every sufficiently small subarc of every nonspacelike 
future directed Hamiltonian geodesic is a \(U\)-maximizers for suitably chosen \(U\).
Suppose now that we are given two sub-Lorentzian manifolds \((M_i, H_i, g_i), i = 1, 2\). A diffeomorphism \(\varphi: M_1 \rightarrow M_2\) is said to be a sub-Lorentzian isometry, if \(d\varphi(H_1) \subset H_2\) and for each \(q \in M_1\), the mapping \(d\varphi_q: (H_1)_q \rightarrow (H_2)_{\varphi(q)}\) is a linear isometry, i.e., for every \(v_1, v_2 \in (H_1)_q\) it follows that
\[
g_1(v_1, v_2) = g_2(d\varphi_q(v_1), d\varphi_q(v_2)).
\]
Of course, any isometry maps timelike curves from \(M_1\) to timelike curves on \(M_2\). The same for spacelike and null curves. Moreover isometries preserve the sub-Lorentzian length of nonspacelike curves. If \((M_i, H_i, g_i), i = 1, 2\), are both time- and space-oriented, then we can distinguish among all isometries those that preserve both orientations. More precisely, suppose that \(\varphi: M_1 \rightarrow M_2\) is an isometry. Let \(H_1 = H_1^- \oplus H_1^+\) be a causal decomposition. Let \((H_1^-)_{\varphi(q)} = d\varphi_q(H_1^-)_q\) and \((H_2^+)_{\varphi(q)} = d\varphi_q(H_2^)_q, q \in M\). Then \(H_2 = H_2^- \oplus H_2^+\) is again a causal decomposition and we say that \(\varphi\) preserves time (resp. space) orientation if the vector bundle morphism \(d\varphi|_{H_1^-}: H_1^- \rightarrow H_2^-\) (resp. \(d\varphi|_{H_1^+}: H_1^+ \rightarrow H_2^+\)) is orientation preserving. An isometry that preserves time and space orientation will be called a \(ts\)-isometry. It is clear that any \(ts\)-isometry preserves Hamiltonian geodesics, maximizers, and local sub-Lorentzian distance functions. Notice furthermore that the set of all isometries \((M, H, g) \rightarrow (M, H, g)\) is a Lie group and the set of all \(ts\)-isometries forms a connected component containing the identity.

A sub-Lorentzian manifold \((M, H, g)\) is called a contact sub-Lorentzian manifold, if \(H\) is a contact distribution on \(M\). Among sub-Lorentzian manifolds, those which are contact seem to be the easiest to study and hence well known. Contact sub-Lorentzian manifolds are studied for instance in papers [6] [7] [9] [12] [13] [14] [15] [16]. The investigations go in two directions. The first addresses global aspects in the group case, e.g., in [9] [7] the Heisenberg sub-Lorentzian metric is treated. More precisely, the future timelike, nonspacelike and null reachable sets from a point are computed, and a certain estimate on the distance function is given. Moreover, it is shown that the future timelike conjugate locus of the origin is zero, while the future null conjugate locus equals the union of the two null future directed Hamiltonian geodesics starting from the origin. In turn, in [15] and [21] it is proved that the set reachable from the origin by future directed timelike Hamiltonian geodesics coincides with the future timelike reachable set from the origin. In [15] the authors also study the set reachable by spacelike Hamiltonian geodesics and prove the uniqueness of geodesics in the Heisenberg case. Next, in the papers [15] [16] the so-called \(hh\)-type groups (i.e. higher dimensional analogues of the 3D Heisenberg group) with suitable sub-Lorentzian metrics are studied, and the main emphasis is put on the problem of connectivity by geodesics, i.e. given two points \(q_1, q_2\), figure out how many geodesics joining \(q_1\) to \(q_2\) exist. A similar problem is also dealt with in [14]. On the other hand, in [14] the group \(SL(2, \mathbb{R})\) with the sub-Lorentzian metric is studied. As it will become clear below, the cases of the Heisenberg group and that of \(SL(2, \mathbb{R})\) are especially interesting for us because these are exactly the cases that arise when the invariant \(h\) (defined below) vanishes.

As one can see, problems connected with isometric and conformal symmetry have not been examined in an explicit sense although in broader contexts such as parabolic geometry and Cartans's equivalence, there are applicable results. The aim of this paper is to embark on filling this gap. More precisely, first we construct invariants for contact sub-Lorentzian manifolds \((M, H, g)\) with \(\dim M = 3\), more
or less in the way as it is done in the contact sub-Riemannian case - cf. [4]. Our invariants are: a \((1,1)\)-tensor \(\tilde{\eta}\) on \(H\) and a smooth function \(\kappa\) on \(M\). Then, we consider in some detail the case that \(M\) is a 3-dimensional Lie group such that \(\tilde{\eta} = 0\). It turns out that in such a case \(M\) is locally either the Heisenberg group or the universal cover of \(SL(2,\mathbb{R})\). In these two cases we describe infinitesimal isometries and more generally infinitesimal conformal transformations.

1.2. The content of the paper. In section 2 we construct invariants for \(ts\)-oriented contact sub-Lorentzian metrics on 3D manifolds. The construction follows the ideas of [4], however the full analogy does not exist due to the special character of indefinite case. Our main invariants for a manifold \((M, H, g)\) are: a smooth \((1,1)\)-tensor \(\tilde{\eta}\) on \(H\) and a smooth function \(\kappa\) on \(M\). These invariants provide necessary conditions for two contact sub-Lorentzian manifolds to be locally \(ts\)-isometric. We also consider another invariant \(\chi\) arising from the eigenvalues of \(\tilde{\eta}\) which to a lesser extent also distinguishes the structure.

In section 3 we define and prove basic properties of infinitesimal sub-Lorentzian isometries and conformal transformations. Then we notice that the invariant \(\tilde{\eta}\) can be expressed in terms of the restricted Lie derivative of the metric \(g\) in the direction of the Reeb vector field. The immediate consequence of this latter fact is that the Reeb vector field \(X_0\) is an infinitesimal isometry if and only if \(\tilde{\eta}\) vanishes identically.

Section 4 covers some other implications of certain combinations of the invariants vanishing. In particular we demonstrate (see proposition [16]) that without any assumptions on orientation, the condition \(\chi = 0\) and \(\tilde{\eta} \neq 0\) implies the existence of line sub-bundle \(L \to M\) of \(H\) on which the metric \(g\) is equal to zero. We then begin to focus on the condition \(\tilde{\eta} = 0\) where \(\kappa\) comes to the fore. For example, when \(M\) is a simply connected Lie group, we show that \(\tilde{\eta} = 0\) and \(\kappa = 0\) implies that \(M\) is the Heisenberg group - cf. corollary [11] and \(\tilde{\eta} = 0\) and \(\kappa \neq 0\) implies that \(M\) is the universal cover of \(SL(2,\mathbb{R})\) - see corollary [12]. This contrasts with the sub-Riemannian case where a third group, namely \(SU(2)\), also appears.

Section 6 is devoted to computing infinitesimal isometries and infinitesimal conformal transformations using Cartan’s equivalence method and Section 7 presents an example of an isometrically rigid sub-Lorentzian structure.

Finally, the appendix presents possible applications of our invariants to a non-contact case.

2. Constructing the invariants.

2.1. Preliminaries. Let \((M, H, g)\) be a contact sub-Lorentzian manifold, \(\dim M = 3\), which is supposed to be both time and space oriented or \(ts\)-oriented for short. Since \(H\) is of rank 2, any causal decomposition \(H = H^- \oplus H^+\) splits \(H\) into a direct sum of line bundles. So in this case a space orientation is just a continuous spacelike vector field, and consequently \(H\) admits a global basis. Let us fix an orthonormal basis \(X_1, X_2\) for \((H, g)\), i.e.

\[
g(X_1, X_1) = -1, \quad g(X_1, X_2) = 0, \quad g(X_2, X_2) = 1,
\]

where \(X_1\) (resp. \(X_2\)) is a time (resp. space) orientation. From now on we will work with \(ts\)-invariants, i.e. with invariants relative to \(ts\)-isometries. However, when reading the text the reader will see that the space orientation is only an auxiliary notion here and most of the results do not depend on it (some of them do not depend on an orientation at all).
Let $\omega$ be a contact 1-form such that $H = \ker \omega$. Without loss of generality we may assume that $\omega$ is normalized so that

\[(2.1) \quad d\omega(X_1, X_2) = \omega([X_2, X_1]) = 1.\]

Next, denote by $X_0$ the so-called Reeb vector field on $M$ which is defined by

\[(2.2) \quad \omega(X_0) = 1, \quad d\omega(X_0, \cdot) = 0.\]

It is seen that $X_0$ is uniquely determined by $ts$-oriented sub-Lorentzian structure. Using (2.2) it is seen that the action of $ad_{X_0}$ preserves the horizontality of vector fields, i.e.

\[(2.3) \quad ad_{X_0}(\Gamma(H)) \subset \Gamma(H).\]

Now (similarly as in [4]) we introduce the structure functions. Thanks to (2.3) and (2.2) we have

\[(2.4) \quad [X_1, X_0] = c^1_{01} X_1 + c^2_{01} X_2, \]
\[(2.5) \quad [X_2, X_0] = c^1_{02} X_1 + c^2_{02} X_2, \quad [X_2, X_1] = c^1_{12} X_1 + c^2_{12} X_2 + X_0.\]

Let $\nu_0, \nu_1, \nu_2$ be the dual basis of 1-forms: $\langle \nu_i, X_j \rangle = \delta_{ij}$, $i, j = 0, 1, 2$. Rewriting (2.4) in terms of $\nu_i$’s we have

\[(2.6) \quad c^1_{01} + c^2_{02} = 0.\]

2.2. Induced bilinear form and linear operator. In the introduction we defined the geodesic Hamiltonian $h$ which can be written as

\[h = -\frac{1}{2} h_1^2 + \frac{1}{2} h_2^2,\]

where $h_i(\lambda) = \langle \lambda, X_i \rangle$, $i = 1, 2$. We also consider the function $h_0(\lambda) = \langle \lambda, X_0 \rangle$ and observe that by definition both $h$ and $h_0$ are invariant with respect to $ts$-oriented structure $(H, g)$. Therefore, it is the same with their Poisson bracket $\{h, h_0\}$ which, when evaluated at $q \in M$, gives a symmetric bilinear form

\[\{h, h_0\}_q : T_q^*M \times T_q^*M \rightarrow \mathbb{R}.\]

If $\lambda \in T_q^*M$ then $\lambda = \sum_{i=0}^3 h_i(\lambda)\nu_i(q)$ and

**Lemma 2.1.** $\{h, h_0\}_q = -c^1_{01} h_1^2 + (c^2_{01} - c^2_{02}) h_1 h_2 + c^2_{02} h_2^2.$

**Proof.** The formula follows from $\{h, h_0\} = -h_1 \{h_1, h_0\} + h_2 \{h_2, h_0\}$, where we substitute $\{h_i, h_0\}(\lambda) = \langle \lambda, [X_i, X_0] \rangle$, and then use (2.4). $\square$

In the assertion of lemma 2.1 and in many other places below we write $c^i_{jk}(q)$ for $c^i_{jk}(q).$
It follows that \( \{h, h_0\}_q (\lambda, \cdot) = 0 \) whenever \( \lambda \in H^+_q \) (by definition \( H^+_q \) is the set of such covectors \( \lambda \in T^*_q M \) that \( \langle \lambda, v \rangle = 0 \) for every \( v \in H_q \)), so in fact

\[
\{h, h_0\}_q : T^*_q M / H^+_q \times T^*_q M / H^+_q \longrightarrow \mathbb{R}.
\]

Let us recall (mutually inverse) musical isomorphisms determined by the metric \( g \): these are \( ^t \alpha : H^* \longrightarrow H \) and \( ^b \beta : H \longrightarrow H^* \), where by definition \((\nu_1)^2 = -X_1, (\nu_2)^2 = X_2, (X_1)^\nu = -\nu_1, (X_2)^\nu = \nu_2 \). Now it is easy to see that the bundle morphism \( G : T^* M \longrightarrow H \) from the introduction induces for each \( q \) a natural identification

\[
F_q : T^*_q M / H^+_q \longrightarrow H_q, \quad F([\alpha]) = (\alpha_{|H_q})^t
\]

where \([\alpha] \) stands for the class of \( \alpha \in T^*_q M \) modulo \( H^+_q \); more precisely, \( H^+_q \) is spanned by \( \nu_0 \), and \( F([\nu_0]) = -X_1, F([\nu_2]) = X_2 \). This permits us to define a bilinear symmetric form \( \tilde{h}_q : H_q \times H_q \longrightarrow \mathbb{R} \) by

\[
\tilde{h}_q (v, w) = \{h, h_0\}_q (F_q^{-1}(v), F_q^{-1}(w)).
\]

Its matrix in the basis \( X_1(q), X_2(q) \) is

\[
(2.7) \quad \begin{pmatrix}
-c_{01}^1 & -\frac{1}{2} (c_{02}^1 - c_{01}^2) \\
-\frac{1}{2} (c_{02}^1 - c_{01}^2) & c_{02}^2
\end{pmatrix}.
\]

Finally we define a linear mapping \( \tilde{h}_q : H_q \longrightarrow H_q \) by the following formula:

\[
\tilde{h}_q (v) = (\tilde{h}_q(v, \cdot))^t.
\]

Using (2.7), it is seen that the matrix of the operator \( \tilde{h}_q \) in the basis \( \{X_1(q), X_2(q)\} \) is equal to

\[
(2.8) \quad \begin{pmatrix}
c_{01}^1 & \frac{1}{2} (c_{02}^1 - c_{01}^2) \\
-\frac{1}{2} (c_{02}^1 - c_{01}^2) & c_{02}^2
\end{pmatrix}.
\]

2.3. The \( ts \)-invariants. By our construction, the eigenvalues and determinant of \( \tilde{h}_q \) as well as \( \tilde{h}_q \) itself, are all invariants for the \( ts \)-oriented structure \((H, g)\). Clearly \( \det \tilde{h}_q = c_{01}^2 c_{02}^1 + \frac{1}{2} (c_{02}^1 - c_{01}^2)^2 = - (c_{01}^1)^2 + \frac{1}{4} (c_{02}^1 - c_{01}^2)^2 \). Since, in view of (2.6), the trace of \( \tilde{h}_q \) is equal to 0, the eigenvalues of \( \tilde{h}_q \) are equal to \( \pm \sqrt{-(c_{01}^1)^2 + \frac{1}{4} (c_{02}^1 - c_{01}^2)^2} \). We can choose

\[
\chi = - (c_{01}^1)^2 + \frac{1}{4} (c_{02}^1 - c_{01}^2)^2
\]

as a functional \( ts \)-invariant for our structure. In analogy with the sub-Riemannian case ([1], [3] and [4]), we consider the functional \( ts \)-invariant defined as follows:

\[
(2.9) \quad \kappa = X_2 (c_{12}^1) + X_1 (c_{12}^2) - (c_{12}^2)^2 \quad \text{for} \quad \chi = - (c_{01}^1)^2 + \frac{1}{4} (c_{02}^1 - c_{01}^2)^2.
\]

Unlike the sub-Riemannian case where \( \chi \) and \( \kappa \) play the crucial role, it is \( \tilde{h} \) and \( \kappa \) that play the crucial role in the sub-Lorentzian setting.

**Proposition 2.1.** \( \kappa \) is indeed a \( ts \)-invariant.
2.3.1. **Proof of proposition 2.1.** Let $X_1, X_2$ is an orthonormal basis with a time orientation $X_1$ and a space orientation $X_2$, and let $c^i_{jk}$ be structures functions determined by this basis. Next, let $\theta = \theta(q)$ be a smooth functions and consider an orthonormal basis $Y_1, Y_2$ given by

\begin{align}
Y_1 &= X_1 \cosh \theta + X_2 \sinh \theta \\
Y_2 &= X_1 \sinh \theta + X_2 \cosh \theta .
\end{align}

(2.10)

Then $Y_1$ ($Y_2$) is a time (space) orientation, and of course

\begin{align}
X_1 &= Y_1 \cosh \theta - Y_2 \sinh \theta \\
X_2 &= -Y_1 \sinh \theta + Y_2 \cosh \theta .
\end{align}

(2.11)

Let $d^i_{jk}$ be the structure functions determined by the basis $Y_1, Y_2$, i.e.

\begin{align}
[Y_1, Y_0] &= d^i_{01} Y_1 + d^i_{02} Y_2 \\
[Y_2, Y_0] &= d^i_{02} Y_1 + d^i_{02} Y_2 \\
[Y_2, Y_1] &= d^i_{12} Y_1 + d^i_{12} Y_2 + X_0
\end{align}

In order to prove proposition 2.1 we need the following lemma.

**Lemma 2.2.** The following formulas hold true:

\begin{align}
d^0_{02} &= -X_0(\theta) + c^1_{02} \cosh^2 \theta - c^2_{02} \sinh^2 \theta + (c^1_{01} - c^2_{01}) \sinh \theta \cosh \theta \\
d^0_{02} &= (c^1_{01} - c^2_{02}) \sinh \theta \cosh \theta + c^2_{02} \cosh^2 \theta - c^1_{01} \sinh^2 \theta \\
d^0_{01} &= c^1_{01} \cosh^2 \theta - c^2_{02} \sinh^2 \theta + (c^2_{02} - c^1_{01}) \sinh \theta \cosh \theta \\
d^0_{01} &= -X_0(\theta) + c^2_{01} \cosh^2 \theta - c^1_{02} \sinh^2 \theta + (c^2_{02} - c^1_{01}) \sinh \theta \cosh \theta \\
d^1_{12} &= (c^1_{12} - X_1(\theta)) \cosh \theta - (X_2(\theta) + c^2_{12}) \sinh \theta \\
d^1_{12} &= (X_1(\theta) - c^1_{12}) \sinh \theta + (X_2(\theta) + c^2_{12}) \cosh \theta .
\end{align}

(2.12)

**Proof.** All formulas are proved by direct calculations. For instance, using (2.10) we write

$$[Y_2, Y_1] = [X_1 \sinh \theta + X_2 \cosh \theta, X_1 \cosh \theta + X_2 \sinh \theta] = -X_1(\theta)X_1 + X_2(\theta)X_2 + [X_2, X_1] .$$

Then using (2.10) and (2.11) we arrive at

$$-X_1(\theta)X_1 + X_2(\theta)X_2 + c^1_{12}X_1 + c^1_{12}X_2 + X_0 =$$

$$-X_1(\theta) (Y_1 \cosh \theta - Y_2 \sinh \theta) + X_2(\theta) (-Y_1 \sinh \theta + Y_2 \cosh \theta) +$$

$$c^1_{12} (Y_1 \cosh \theta - Y_2 \sinh \theta) + c^1_{12} (-Y_1 \sinh \theta + Y_2 \cosh \theta) + X_0 ,$$

from which the fifth and sixth equations in (2.12) follow.

Now, using lemma 2.2, we see that

\begin{align}
\frac{1}{2} (d^0_{01} + d^0_{02}) &= -X_0(\theta) + \frac{1}{2} (c^2_{01} + c^1_{02}) .
\end{align}

(2.13)

and

\begin{align}
(d^1_{12})^2 - (d^2_{12})^2 &= (X_1(\theta) - c^1_{12})^2 - (X_2(\theta) + c^2_{12})^2 .
\end{align}

(2.14)

Finally, we compute $Y_2(d^1_{12}) + Y_1(d^2_{12})$. To this end let us write

\begin{align}
Y_2(d^1_{12}) + Y_1(d^2_{12}) &= I + II ,
\end{align}

(2.15)
where
\[
I = X_2(c_{12}^1) + X_1(c_{12}^2) - [X_2, X_1](\theta)
\]
\[
= X_2(c_{12}^1) + X_1(c_{12}^2) - c_{12}^1X_1(\theta) - c_{12}^2X_2(\theta) - X_0(\theta)
\]
and
\[
II = -c_{12}^1X_1(\theta) - X_2^2(\theta) - c_{12}^2X_2(\theta) + X_1^2(\theta).
\]
Combining (2.13), (2.14) and (2.15) completes the proof of Proposition 2.1.

In summary, our basic ts-invariants are: a smooth function \(\kappa\) on \(M\) and a \((1,1)\) tensor \(h\) on \(H\).

3. SUB-LORENTZIAN INFINITESIMAL ISOMETRIES AND CONFORMAL TRANSFORMATIONS.

In this section \((M, H, g)\) is a fixed sub-semi-Riemannian manifold, rank \(H\) and \(\dim M\) are arbitrary.

**Definition 3.1.** A a diffeomorphism \(f : M \rightarrow M\) is called a conformal transformation of \((M, H, g)\) if \((i)\) \(d_qf : H_q \rightarrow H_{f(q)}\) for every \(q \in M\), \((ii)\) there exists a function \(\rho \in C^\infty(M), \rho > 0,\) such that
\[
g(d_qf(v), d_qf(w)) = \rho(q)g(v, w)
\]
for every \(q \in M\) and every \(v, w \in H_q\). Of course, if \(\rho = 1\) then \(f\) is an isometry of \((M, H, g)\).

Along with conformal transformations and isometries we consider their infinitesimal variants.

**Definition 3.2.** A vector field \(Z\) on \((M, H, g)\) is called an infinitesimal conformal transformation (resp. infinitesimal isometry) if its flow \(h^t\) consists of conformal transformations (isometries).

Let us note a simple lemma.

**Lemma 3.1.** Let \(Z\) be a vector field on \(M\) and denote by \(h^t\) its flow. Then the following conditions are equivalent:
\[(a)\ ad_Z : \Gamma(H) \rightarrow \Gamma(H);
(b)\ d_qh^t : H_q \rightarrow H_{h^t(q)}\] for every \(q \in M\) and every \(t\) such that \(h^t\) is defined around \(q\).

**Proof.** Although the result is known, we give a proof for the sake of completeness.

\((a) \Rightarrow (b)\) Following [19], we fix a point \(q\) and consider a basis \(X_1, ..., X_k\) of \(H\) defined on a neighborhood \(U\) of \(q\). By our assumption, there exist smooth functions \(\alpha_{ij}, i, j = 1, ..., k\), such that \(ad_ZX_i = \sum_{j=1}^k \alpha_{ij}X_j\) on \(U\) and it follows that if \(v_i(t) = (h^t_iX_i)(q) = dh^tX_i(h^{-t}q)\) then
\[
v_i(t) = (h^t_iad_ZX_i)(q) = \sum_{j=1}^k (\alpha_{ij} \circ h^{-t})(q) (h^t_jX_j)(q) = \sum_{j=1}^k \beta_{ij}(t)v_j(t),
\]
where \(\beta_{ij}(t) = (\alpha_{ij} \circ h^{-t})(q)\). For any covector \(\lambda \in T_q^*M\) which annihilates \(H_q\), i.e. \(\langle \lambda, v \rangle = 0\) for every \(v \in H_q\), we obtain a system of linear differential equations...
for the functions $w_i(t) = \langle \lambda, v_i(t) \rangle$, $i = 1, ..., k$:

$$\dot{w}_i(t) = \sum_{j=1}^{k} \beta_{ij}(t)w_j(t)$$

with initial conditions $w_i(0) = 0$, $i = 1, ..., k$, since $v_i(0) = X_i(q) \in H_q$. Therefore $v_i(t) = 0$ and $(h^t_i X_i)(q) \in H_q$ every $t$ for which $v_i(t)$ is defined, $i = 1, ..., k$.

(b) $\Rightarrow$ (a) Take a point $q$, then for every $t$ such that $|t|$ is sufficiently small, we have $(h^t_i X)(q) \in H_q$ and it follows that $(\text{ad}_Z X)(q) = \frac{d}{dt}|_{t=0} (h^{-t}_i X)(q) \in H_q$. $\Box$

Suppose now that $f : M \rightarrow M$ is a diffeomorphism such that $df(H) = H$ and let $T$ be a tensor of type $(0,2)$ on $H$. We define a pull-back $\tilde{f}^* : \Gamma(H) \times \Gamma(H) \rightarrow C^\infty(M)$ by

$$(\tilde{f}^* T)_q (X,Y) = Tf(q)(d_q f(X), d_q f(Y)),$$

where $X,Y \in \Gamma(H)$ (tilde indicates that we restrict to horizontal vector fields).

We can now reformulate the definition of conformal transformations in a manner consistent with semi-Riemannian geometry:

$f$ is a conformal transformation of $(M, H, g)$ if and only if there exists a function $\rho \in C^\infty(M)$, $\rho > 0$, such that $\tilde{f}^* g = \rho g$ (if $\rho = 1$, $f$ is an isometry).

Suppose that $Z$ is a vector field on $M$ such that $\text{ad}_Z : \Gamma(H) \rightarrow \Gamma(H)$ and let $h^t$ denote the (local) flow of $Z$. Using lemma 3.1 again by analogy to the classical geometry, we can define a local operator $\tilde{L}_Z T : \Gamma(H) \times \Gamma(H) \rightarrow C^\infty(M)$ which will be called the restricted Lie derivative:

$$(\tilde{L}_Z T)(q) = \frac{d}{dt}|_{t=0} \big((\tilde{h}^t)^* T \big)(q).$$

It turns out that

**Proposition 3.1.** A vector field $Z$ is an infinitesimal conformal transformation of $(M, H, g)$ if and only if the following conditions hold:

(i) $\text{ad}_Z : \Gamma(H) \rightarrow \Gamma(H)$, and

(ii) there exists a function $\lambda \in C^\infty(M)$ such that $\tilde{L}_Z g = \lambda g$.

**Proof.** Remembering that we use only horizontal vector fields, the proof is the same as in the classical geometry. Again $h^t$ is the flow of $Z$.

$``\Rightarrow``$ By lemma 3.1 we know that (i) is satisfied. If $(\tilde{h}^t)^* g = \rho_t g$, where for each $t$ the function $\rho_t$ is smooth and positive, then it follows that

$$\tilde{L}_Z((\tilde{h}^t)^* g) = \frac{d}{dt}|_{s=0} (\tilde{h}^s)^* (\tilde{h}^t)^* g = \frac{d}{dt}(\tilde{h}^t)^* g = \frac{d}{dt}(\rho_t g) = \dot{\rho} g.$$

On the other hand, we also have that

$$\tilde{L}_Z(\tilde{h}^t)^* g = \tilde{L}_Z(\rho_t g) = X(\rho_t)g + \rho_t(\tilde{L}_Z g),$$

and so we see that $\tilde{L}_Z g = \lambda g$ where

$$\lambda = \frac{\dot{\rho}_t - X(\rho_t)}{\rho_t}$$

(note that $\rho_0 = 1$).

$``\Leftarrow``$ From lemma 3.1 we know that $dh^t$ preserves $H$. From (ii) and (3.1) we have

$$\frac{d}{dt}(\tilde{h}^t)^* g = (\tilde{h}^t)^* \left( \tilde{L}_Z g \right) = (\tilde{h}^t)^* (\lambda g) = (\lambda \circ h^t)(\tilde{h}^t)^* g,$$
which implies that $\tilde{h}^*g = \rho g$ where
\[ \rho_t(q) = \exp \int_0^t \lambda(h^*(q))ds. \]

By direct calculation we obtain
\[ (3.2) \quad (\tilde{L}Zg)(X,Y) = Z(g(X,Y)) - g(\text{ad}ZX, Y) - g(X, \text{ad}ZY) \]
for every $X,Y \in \Gamma(H)$, which gives the following two Corollaries:

**Corollary 3.1.** $Z$ is an infinitesimal conformal transformation of $(M, H, g)$ if and only if there exists a function $\lambda \in C^\infty(M)$ such that for every $X,Y \in \Gamma(H)$
\[ Z(g(X,Y)) = g(\text{ad}ZX, Y) + g(X, \text{ad}ZY) + \lambda g(X,Y). \]

**Corollary 3.2.** $Z$ is an infinitesimal isometry of $(M, H, g)$ if and only if for every $X,Y \in \Gamma(H)$
\[ Z(g(X,Y)) = g(\text{ad}ZX, Y) + g(X, \text{ad}ZY). \]

Furthermore:

**Corollary 3.3.** If $Z$ is an infinitesimal conformal transformation or isometry of $(M, H, g)$ then for every $n \geq 2$ and every $X,Y \in \Gamma(H)$
\[ (3.3) \quad \sum_{k=2}^n \binom{n}{k} g(\text{ad}^kZX, \text{ad}^{n-k}ZY) = 0. \]

*Proof.* Fix a point $q \in M$. Under the above notation, for any $n \in \mathbb{N}$ and sufficiently small $|t|$

\[ \rho_t(h^tq)g(X(h^tq), Y(h^tq)) = g(d_{h^tq}h^{-1}(X), d_{h^tq}h^{-1}(Y)) = \]
\[ = g \left( \sum_{k=0}^n \frac{t^k}{k!} (\text{ad}_Z X)(q), \sum_{m=0}^n \frac{t^m}{m!} (\text{ad}_Z Y)(q) \right) + o(t^n). \]

Using corollary 3.1, we can remove from (3.2) terms of order 0 and 1 with respect to $t$. What we obtain is
\[ \sum_{k=2}^n \frac{t^k}{k!} \sum_{i+j=k} 1_{ij}! g \left( (\text{ad}_Z X)(q), (\text{ad}_Z Y)(q) \right) + o(t^n) = 0 \]
for $|t|$ sufficiently small, which gives (3.3). □

4. Some properties of invariants

In this section we assume all sub-Lorentzian manifolds to be $ts$-oriented. Let us start from an obvious observation.

**Proposition 4.1.** Let $(M_i, H_i, g_i)$, $i = 1, 2,$ be a contact 3-dimensional $ts$-oriented sub-Lorentzian manifolds. Denote by $\chi_i$, $\kappa_i$, $\tilde{h}_i$ the corresponding objects defined by $(H_i, g_i)$, $i = 1, 2$. If there exists a local $ts$-isometry $\varphi : (M_1, H_1, g_1) \rightarrow (M_2, H_2, g_2)$, then $\chi_1 = \varphi^* \chi_2$, $\kappa_1 = \varphi^* \kappa_2$, and $\tilde{h}_1 = \varphi^* \tilde{h}_2$. 

Fix a contact 3-dimensional sub-Lorentzian manifold \((M, H, g)\). First of all let us notice how the invariant \(\hat{h}\) can be expressed in terms of the restricted Lie derivative of the metric \(g\) in the direction of the Reeb field. Indeed, knowing (3.2) it is clear that for every \(q \in M\) and every \(v, w \in H_q\)

\[
\hat{h}_q(v, w) = \frac{1}{2}(\bar{L}_v g)(q)(v, w).
\]

(4.1) Such an approach allows to define higher order invariants, namely ones that correspond to the bilinear forms

\[
\hat{h}^{(l)}_q(v, w) = \frac{1}{2}(\bar{L}_v g)(q)(v, w), \quad l = 2, 3, ...
\]

In this way, however, we will not obtain any formulas involving the structure functions \(c_{12}\), so \(\kappa\) is unavoidable.

Using (4.1) we are in a position to prove the propositions below.

**Proposition 4.2.** If the Reeb vector field \(X_0\) is an infinitesimal isometry for \((H, g)\) then \(\chi = 0\) everywhere.

**Proposition 4.3.** The Reeb vector field \(X_0\) is an infinitesimal isometry for \((H, g)\) if and only if \(\hat{h}_q = 0\) for every \(q \in M\).

Proposition 4.3 shows one of the ways how to produce sub-Lorentzian isometries. This is important because we know very little examples of such maps.

Next we study the effect on the invariants when we dilate the structure. To this end suppose that we have a sub-Lorentzian ts-oriented structure \((H, g)\) which is given by an orthonormal frame \(X_1, X_2\) with a time (resp. space) orientation \(X_1\) (resp \(X_2\). Let \(\lambda > 0\) be a constant. Consider the sub-Lorentzian structure \((H', g')\) defined by assuming the frame \(X'_1 = \lambda X_1, X'_2 = \lambda X_2\) to be orthonormal with the time (resp. space) orientation \(X'_1\) (resp. \(X'_2\)). The normalized one form \(\omega'\) which defines \(H'\) is given by \(\omega' = \frac{1}{\lambda^2} \omega\), i.e.,

\[
d\omega'(X'_1, X'_2) = \omega'([X'_2, X'_1]) = 1.
\]

It follows that the Reeb field is now \(\lambda^2 X_0\). Then it is easy to see that (2.4) can be rewritten as

\[
[X'_1, X'_0] = c'_{01} X'_1 + c'_{01} X'_2
\]

(4.2) \[
[X'_2, X'_0] = c'_{02} X'_1 + c'_{02} X'_2,
\]

\[
[X'_2, X'_1] = c'_{12} X'_1 + c'_{12} X'_2 + X'_0
\]

where \(c'_{jk} = \lambda c_{jk}\). As a corollary we obtain

**Proposition 4.4.** Let \(\chi, \kappa, \hat{h}\) (resp. \(\chi', \kappa', \hat{h}'\)) be the ts-invariants of the sub-Lorentzian structure defined by an orthonormal basis \(X_1, X_2\) (resp. by \(X'_1 = \lambda X_1, X'_2 = \lambda X_2\)). Then

\[
\chi' = \lambda^2 \chi, \quad \kappa' = \lambda^2 \kappa, \quad \hat{h}' = \lambda \hat{h}.
\]

### 4.1. The case \(\chi = 0\)

Next let us assume that \(\chi(q) = 0\) but \(\hat{h}_q \neq 0\) (i.e. \(c_{01} \neq 0\)) everywhere. As we shall see we are given an additional structure in this case. Indeed, the correspondence \(q \rightarrow \ker \hat{h}_q\) defines an invariantly given field of directions. We can distinguish two cases: (i) \(c_{01} = \frac{3}{2} (c_{02} - c_{01})\), and (ii) \(c_{01} = -\frac{1}{2} (c_{02} - c_{01})\). In the first case the matrix of \(\hat{h}_q\) is of the form

\[
\begin{pmatrix}
c_{01} & c_{01} \\
-c_{01} & -c_{01}
\end{pmatrix}
\]
and \( \ker \tilde{h}_q \) is spanned by \( X_1(q) - X_2(q) \) for each \( q \). In the second case the matrix of \( \tilde{h}_q \) is equal to

\[
\begin{pmatrix}
  c_{01}^1 & -c_{01}^2 \\
  c_{01}^2 & -c_{01}^1
\end{pmatrix}
\]

and \( \ker \tilde{h}_q \) is spanned by \( X_1(q) + X_2(q) \). Thus in the considered case there exists a line sub-bundle \( L \to M \) of \( H \) on which \( g \) is equal to zero. Of course this result is trivial under assumption on \( ts\)-orientation because then \( H \) admits a global orthonormal basis \( X_1, X_2 \) and we have in fact two such subbundles, namely \( \text{Span}\{X_1 + X_2\} \) and \( \text{Span}\{X_1 - X_2\} \). What is interesting here is that the condition \( \chi = 0, \tilde{h} \neq 0 \) does not depend on the assumption on orientation. Indeed, notice that if we change a time (resp. space) orientation keeping space (resp. time) one then \( \tilde{h} \) is multiplied by \(-1\) (because so is \( X_0 \)). Moreover the condition \( \chi = 0, \tilde{h} \neq 0 \) means that \( \tilde{h} \) is a non-zero map with vanishing eigenvalues, the fact being independent of possible multiplication by \(-1\). Therefore the condition \( \chi = 0, \tilde{h} \neq 0 \) makes sense even for an unoriented contact sub-Lorentzian structures. In this way we are led to the following proposition.

**Proposition 4.5.** Suppose that \( (M, H, g) \) is a contact sub-Lorentzian manifold (we don’t make any assumptions on orientation). If \( \chi(q) = 0 \) and \( \tilde{h}_q \neq 0 \) for every \( q \in M \) then there exists a line sub-bundle \( L \to M \) of \( H \) on which \( g \) is equal to zero.

**Proof.** Fix an arbitrary point \( q \in M \). Let \( Y_1, Y_2 \) be an orthonormal basis for \( (H, g) \) defined on a neighborhood \( U \) of \( q \), where \( Y_1 \) is timelike and \( Y_2 \) is spacelike. Supposing \( Y_1 \) (resp. \( Y_2 \)) to be a time (resp. space) orientation we can apply the above construction of \( ts \)-invariants obtaining the corresponding objects \( \chi_U \) and \( \tilde{h}_U \). By our assumption and the above remark \( \chi_U = 0, \tilde{h}_U \neq 0 \) on \( U \), and we get an invariantly defined line subbundle \( L_U \to U: U \ni q \to \ker(\tilde{h}_U)_q =: L_U(q) \). We repeat the same construction around any point \( q \in M \), which results in the family \( \{L_U \to U\}_{U \subset M} \) of line sub-bundles, indexed by elements \( U \) of an open covering of \( M \). By construction \( L_U(q) = L_{U'}(q) \) for any \( q \in U \cap U' \).

Let us note that if \( M \) is simply connected, then the assertion of proposition 4.5 holds true no matter the values of \( \chi \) and \( \tilde{h} \) are, because in this case the metric \((H, g)\) admits a global orthonormal frame, see [10].

4.2. The case \( \tilde{h} = 0 \). We begin with the following observation:

**Proposition 4.6.** If \( \tilde{h} = 0 \) then \( X_0(\kappa) = 0 \), i.e., \( \kappa \) is constant along the trajectories of \( X_0 \).

**Proof.** This is clear because \( X_0 \) is an infinitesimal isometry. \( \square \)

By (2.8) the assumption \( \tilde{h} = 0 \) implies \( c_{01}^1 = c_{02}^2 = 0, c_{02}^1 = c_{01}^2 \). We will write \( c = c_{02}^1 = c_{01}^2 \). Now (2.4) takes the form

\[
\begin{align*}
[X_1, X_0] &= cX_2 \\
[X_2, X_0] &= cX_1 \\
[X_2, X_1] &= c^1_2 X_1 + c^2_1 X_2 + X_0
\end{align*}
\]
Rewriting as above (4.3) in terms of \( \nu \)'s we arrive at

\[
\begin{align*}
\theta_0 &= \nu_1 \wedge \nu_2 \\
\theta_1 &= c \nu_0 \wedge \nu_2 + c_1^2 \nu_1 \wedge \nu_2 \\
\theta_2 &= c \nu_0 \wedge \nu_1 + c_1^2 \nu_1 \wedge \nu_2 
\end{align*}
\]

\[ (4.4) \]

**Lemma 4.1.** The following identities hold

\[
\begin{align*}
-X_1(c) - c_1^2 c + X_0(c_1^2) &= 0, \\
X_2(c) - c_1^2 c + X_0(c_1^2) &= 0.
\end{align*}
\]

**Proof.** The lemma is obtained upon applying the exterior differential to both sides of the second and the third equation in (4.4). \( \square \)

Our next aim, which will be achieved in the next subsection, is to find a hyperbolic rotation of our frame \( X_1, X_2 \) so that (4.3) significantly simplifies. More precisely we want to kill the terms \( c_1^2, i = 1, 2 \). To this end let us introduce the following 1-form

\[ \eta = (\kappa + c) \nu_0 + c_1^2 \nu_1 - c_1^2 \nu_2 \]

\[ (4.5) \]

whose significance will become evident below.

**Proposition 4.7.** \( d\eta = dk \wedge \nu_0 \).

**Proof.** Computations give

\[
\begin{align*}
\theta_0 &= d\kappa \wedge \nu_0 + (-X_1(c) - c_1^2 c + X_0(c_1^2)) \nu_0 \wedge \nu_1 \\
&\quad + (-X_2(c) + c_1^2 c - X_0(c_1^2)) \nu_0 \wedge \nu_2 \\
&\quad + (\kappa + c - X_2(c_1^2) - X_1(c_1^2)) \nu_1 \wedge \nu_2.
\end{align*}
\]

To end the proof we use lemma 4.1 and the definition of \( \kappa \). \( \square \)

## 4.3. The simply-connected Lie group case.

Suppose that our contact sub-Lorentzian manifold \((M, H, g)\) is such that \( M \) is a simply-connected Lie group and \( H, g \) are left-invariant; this means that left translations of \( M \) are sub-Lorentzian isometries (note that any left invariant bracket generating distribution on a 3-dimensional Lie group is necessarily contact). In such a case, clearly, \( \chi \) and \( \kappa \) are constant. We also remark that unlike the general situation, the assumption on \( ts \)-orientation is no longer restrictive since groups are parallelizable manifolds. As above, assume that \( h = 0 \) everywhere.

Recalling our aim formulated in the previous subsection we prove the following lemma.

**Lemma 4.2.** There exists a smooth function \( \theta : M \rightarrow \mathbb{R} \) such that \( X_1(\theta) = c_1^2, X_2(\theta) = -c_1^2 \).

**Proof.** Suppose that such a function \( \theta \) exists. Then

\[
\begin{align*}
X_0(\theta) &= [X_2, X_1](\theta) - c_1^2 X_1(\theta) - c_1^2 X_2(\theta) \\
&= X_2(c_1^2) + X_1(c_1^2) - (c_1^2)^2 + (c_1^2)^2 = \kappa + c,
\end{align*}
\]

and it follows that

\[
\begin{align*}
d\theta &= X_0(\theta) \nu_0 + X_1(\theta) \nu_1 + X_2(\theta) \nu_2 = (\kappa + c) \nu_0 + c_1^2 \nu_1 - c_1^2 \nu_2 = \eta,
\end{align*}
\]
where $\eta$ is defined by \[4.5\]. Thus to prove the existence of $\theta$ it is enough to show that $\eta$ is exact. Since $M$ is simply-connected we must show that $d\eta = 0$. This is however clear by proposition \[4.7\] and the fact that $\kappa$ is a constant. □

Now we apply to our frame $X_1, X_2$, the hyperbolic rotation by the angle $\theta$ specified above. As a result, the frame $Y_1, Y_2$ given by \[2.10\] satisfies Proposition 4.8.

\[
[Y_1, X_0] = -\kappa Y_2 \\
[Y_2, X_0] = -\kappa Y_1 \\
[Y_2, Y_1] = X_0
\]

Proof. It follows directly from facts proved in subsections 4.2, 4.3, from \[4.3\], and from lemma 2.2. □

Corollary 4.1. If $M$ is a simply-connected Lie group such that $\tilde{h}$ and $\kappa$ vanish identically, then $M$ is the Heisenberg group.

When $\kappa \neq 0$ we have:

Corollary 4.2. If $M$ is a simply-connected Lie group such that $\tilde{h}$ vanishes and $\kappa \neq 0$, then it is isometric to a sub-Lorentzian structure on $\tilde{SL}_2(\mathbb{R})$ induced by the Killing form.

Before proving corollary \[4.2\] let us recall some basic facts about the Killing form and Cartan decompositions. For any Lie algebra the Killing form is the symmetric bilinear form defined by $K(X, Y) = \text{Trace}(\text{ad}_X \text{ad}_Y)$. The Killing form has the following invariance properties:

1. $K([X, Y], Z) = K(X, [Y, Z])$ and
2. $K(T(X), T(Y)) = K(X, Y)$ for all $T \in \text{Aut}(g)$.

If $g$ is simple then any symmetric bilinear form satisfying the first invariance condition is a scalar multiple of the Killing form and Cartan’s criterion states that a Lie algebra is semisimple if and only if the Killing form is non-degenerate.

A Cartan involution is any element $\Theta \in \text{Aut}(g)$ such that $\Theta^2 = I$ and $\langle X, Y \rangle_\Theta = -K(X, \Theta(Y))$ is positive definite. Corresponding with $\Theta$ we have a Cartan decomposition $g = t \oplus p$ where $t$ and $p$ are the eigenspaces corresponding with the eigenvalues 1 and $-1$ respectively. Since $\Theta$ is an automorphism, it follows that $[t, t] \subseteq t$, $[t, p] \subseteq p$ and $[p, p] \subseteq t$. Moreover, the Killing form is negative definite on $t$ and positive definite on $p$.

The standard Cartan involution on $\mathfrak{sl}_2$ is given by $\Theta(A) = -A^T$. In this case we have that $t = \text{span}\{f_1\}$ and $p = \text{span}\{f_2, f_0\}$ where

\[
f_0 = \frac{1}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad f_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

and the Lie brackets are

\[
[f_2, f_1] = f_0, \quad [f_1, f_0] = f_2, \quad [f_2, f_0] = f_1.
\]

The Killing form for $\mathfrak{sl}_2$ is given by $K(A, B) = 4Tr(AB)$ and so the bilinear form $B(A, B) = \frac{1}{4}K(A, B)$ satisfies

\[
B(f_0, f_0) = 1, \quad B(f_1, f_1) = -1, \quad B(f_2, f_2) = 1, \quad B(f_0, f_1) = 0, \quad B(f_0, f_2) = 0, \quad B(f_1, f_2) = 0.
\]
Thus we have two choices: 1. \( \mathcal{H}_c = \text{span}\{f_1, f_2\} \) or 2. \( \mathcal{H}_c = \text{span}\{f_1, f_0\} \). In each case, by left translation, we obtain left invariant sub-Lorentzian structures on \( SL_2(\mathbb{R}) \) satisfying \( \tilde{h} = 0 \). An isometry between these two structures is induced by Lie algebra automorphism \( T \) where \( T f_0 = f_2, T f_1 = -f_1 \) and \( T f_2 = f_0 \).

**Proof.** (corollary 4.2) First we observe that the matrices

\[
\begin{align*}
e_0 &= \frac{1}{2} \begin{pmatrix} \kappa & 0 & 0 \\ 0 & -\kappa & 0 \\ 0 & 0 & 0 \end{pmatrix}, & e_1 &= \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ -\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & e_2 &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\kappa & 0 & 0 \end{pmatrix},
\end{align*}
\]

form a basis of \( \mathfrak{sl}_2 \) and the bracket relations are \([e_2, e_1] = e_0, [e_1, e_0] = -\kappa e_2, [e_2, e_0] = -\kappa e_1\). Furthermore \( K(e_1, e_1) = 2\kappa, K(e_2, e_2) = -2\kappa, \) and \( K(e_1, e_2) = 0 \). Since we assume the sub-Lorentzian structure on \( M \) is left invariant, the metric must be the left translation of the metric \( B(A, B) = -\frac{1}{2\kappa} K(A, B) \) on \( T_M M = \mathfrak{sl}_2 \).

Since \( M \) is simply connected it must be the universal cover of \( SL_2(\mathbb{R}) \). \( \square \)

We remark that in general

\[
\begin{align*}
K(e_0, e_0) &= 2\kappa^2, & K(e_1, e_1) &= 2\kappa, & K(e_2, e_2) &= -2\kappa, \\
K(e_0, e_1) &= 0, & K(e_0, e_2) &= 0, & K(e_1, e_2) &= 0,
\end{align*}
\]

and so the corresponding Cartan involution is given by

\[
\Theta(e_0) = -e_0, \quad \Theta(e_1) = -e_1, \quad \Theta(e_2) = e_2.
\]

Hence \( \mathfrak{g} = \text{span}\{e_2\} \) and \( \mathfrak{p} = \text{span}\{e_1, e_0\} \). If \( |\kappa| \neq 1 \), then since \( K(e_0, e_0) = 2\kappa^2 \), the only choice we have for a sub-Lorentzian structure induced by the Killing form is \( \mathcal{H}_c = \text{span}\{e_1, e_2\} \).

The null lines in \( \mathfrak{sl}_2 \) are span\{e_1 - e_2\} and span\{e_1 + e_2\}. Furthermore if we set

\[
n_0 = e_3, \quad n_1 = \frac{1}{\sqrt{2}}(e_1 - e_2), \quad n_2 = \frac{1}{\sqrt{2}}(e_1 + e_2)
\]

then

\[
[n_2, n_1] = n_0, \quad [n_1, n_0] = \kappa n_1, \quad [n_2, n_0] = -\kappa n_2.
\]

If we set \( \mathcal{H}_c = \text{span}\{n_1, n_2\} \) and define

\[
B(n_1, n_1) = -1, \quad B(n_2, n_2) = 1, \quad B(n_1, n_2) = 0,
\]

then the induced left invariant structure on \( \tilde{SL}_2(\mathbb{R}) \) is isometrically distinct from the cases above, indeed

\[
\tilde{h} = \kappa \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\]

5. **Infinitesimal Sub-Lorentzian Transformations on Groups with \( \tilde{h} = 0 \).**

5.1. **Introduction.** In this section we determine the conformal and isometry groups for the Heisenberg group and the universal cover of \( SL_2 \). In particular we will see that in both cases the local infinitesimal conformal transformations are given by \( \mathfrak{sl}_3 \).

In the context of this paper it would be natural to construct the vector fields using the criteria developed in section 3, however this leads to complicated systems of p.d.e. which we cannot provide explicit proof concerning solutions. Instead we apply Cartan’s equivalence method which leads to the general solution without having to solve p.d.e.
In work in preparation with Alexandr Medvedev we explore further the application of the Cartan approach to sub-Lorentzian geometry. In particular the invariants discussed here appear in a much more systematic manner.

5.2. Application of the equivalent method. The structure equations for the Cartan connection associated with the conformal symmetries of the sub-Lorentzian structure on the group given in proposition 4.8 are:

\[
\begin{align*}
    d\Theta_1 &= \Pi_1 \wedge \Theta_1 + \Pi_2 \wedge \Theta_2 + \Pi_3 \wedge \Theta_3 \\
    d\Theta_2 &= \Pi_1 \wedge \Theta_2 + \Pi_2 \wedge \Theta_1 + \Pi_4 \wedge \Theta_3 \\
    d\Theta_3 &= 2\Pi_1 \wedge \Theta_3 + \frac{1}{2} \Theta_1 \wedge \Theta_2 \\
    d\Pi_1 &= -\frac{1}{4} \Pi_3 \wedge \Theta_2 - \frac{1}{4} \Pi_4 \wedge \Theta_1 - \Omega \wedge \Theta_3 \\
    d\Pi_2 &= \frac{3}{4} \Pi_3 \wedge \Theta_1 - \frac{3}{4} \Pi_4 \wedge \Theta_2 \\
    d\Pi_3 &= \Pi_3 \wedge \Pi_1 - \Pi_4 \wedge \Pi_2 - \Omega \wedge \Theta_1 \\
    d\Pi_4 &= \Pi_4 \wedge \Pi_1 - \Pi_3 \wedge \Pi_2 - \Omega \wedge \Theta_2 \\
    d\Omega &= \frac{1}{2} \Pi_4 \wedge \Pi_3 + 2\Omega \wedge \Pi_1,
\end{align*}
\]

and for the isometries the structure equations are:

\[
\begin{align*}
    d\Theta_1 &= \Pi_2 \wedge \Theta_2 \\
    d\Theta_2 &= \Pi_2 \wedge \Theta_1 \\
    d\Theta_3 &= \frac{1}{2} \Theta_1 \wedge \Theta_2 \\
    d\Pi_2 &= -\frac{\kappa}{2} \Theta_1 \wedge \Theta_2.
\end{align*}
\]

Consequently, for all \(\kappa\), the conformal symmetries are given by \(SL_3(\mathbb{R})\) and the isometries are always 4 dimensional.

The structure equations are obtained by staring with the ordered basis \(\{\omega_1, \omega_2, \omega_0\}\) such that

\[
\begin{align*}
    d\omega_1 &= \kappa \omega_2 \wedge \omega_0 \\
    d\omega_2 &= \kappa \omega_1 \wedge \omega_0 \\
    d\omega_0 &= \frac{1}{2} \omega_1 \wedge \omega_2
\end{align*}
\]

an applying Cartan’s equivalence method with structure group \(G\) given by the matrices:

\[
\begin{pmatrix}
    e^r \cosh(t) & e^r \sinh(t) & a \\
    e^r \sinh(t) & e^r \cosh(t) & b \\
    0 & 0 & e^{2r}
\end{pmatrix}.
\]

Note that \(G\) is the subgroup of \(GL_3\) that leaves the sub-Lorentzian metric \(\omega_2^2 - \omega_1^2\) conformally invariant modulo terms of the form \(\eta \odot \omega_0\).
To begin we define ones forms $\Theta_i$ on $M \times G$ by setting

$$
\begin{pmatrix}
\Theta_1 \\
\Theta_2 \\
\Theta_3
\end{pmatrix} =
\begin{pmatrix}
e^r \cosh(t) & e^r \sinh(t) & a \\
e^r \sinh(t) & e^r \cosh(t) & b \\
0 & 0 & e^{2r}
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2 \\
\omega_0
\end{pmatrix}
$$

and augment the set $\{\Theta_1, \Theta_2, \Theta_3\} \subset T^*(M \times G)$ with the following forms:

$$
\Pi_1 = \alpha_1 - \frac{1}{4}e^{-2r}b\Theta_1 + \frac{1}{4}e^{-2r}a\Theta_2 - B_1\Theta_3
$$

$$
\Pi_2 = \alpha_2 + \frac{3}{4}e^{-2r}a\Theta_1 - \frac{3}{4}e^{-2r}b\Theta_2 - B_2\Theta_3
$$

$$
\Pi_3 = \alpha_3 - (B_1 + \frac{1}{2}e^{-4r}ab)\Theta_1 + (\frac{1}{2}e^{-4r}a^2 - B_2 + e^{-2r}\kappa)\Theta_2 - B_3\Theta_3
$$

$$
\Pi_4 = \alpha_4 + (e^{-2r}\kappa - B_2 - \frac{1}{2}e^{-4r}b^2)\Theta_1 + (\frac{1}{2}e^{-4r}ab - B_1)\Theta_2 - B_4\Theta_3.
$$

The coefficients of the $\Theta_i$ in $\Pi_j$ are determined by absorbing torsion and the $\alpha_j$ are the Maurer–Cartan forms:

$$
\alpha_1 = dr, \quad \alpha_2 = dt, \quad \alpha_3 = (da - adr - bdt)e^{-2r}, \quad \alpha_4 = (db - bdr - adt)e^{-2r}.
$$

In particular the coefficients $B_1, \ldots, B_4$ are undetermined parameters from absorption and so a prolongation is required. We write

$$
\begin{align*}
\Pi_1 &= \omega_1 - B_1\Theta_3 \\
\Pi_2 &= \omega_2 - B_2\Theta_3 \\
\Pi_3 &= \omega_3 - B_1\Theta_1 - B_2\Theta_2 - B_3\Theta_3 \\
\Pi_4 &= \omega_4 - B_2\Theta_1 - B_1\Theta_2 - B_4\Theta_3.
\end{align*}
$$

(5.4)

and consider the equivalence problem for the ordered basis

$$
\{\Theta_1, \Theta_2, \Theta_3, \omega_1, \omega_2, \omega_3, \omega_4\}
$$

with structure group $G^{(1)}$ consisting of matrices of the form

$$
\begin{pmatrix}
1 & 0 \\
R & 1
\end{pmatrix}
$$

where

$$
R =
\begin{pmatrix}
0 & 0 & -B_1 \\
0 & 0 & -B_2 \\
-B_1 & -B_2 & -B_3 \\
-B_2 & -B_1 & -B_4
\end{pmatrix}.
$$

The forms $\Theta_1, \Theta_2, \Theta_3, \Pi_1, \Pi_2, \Pi_3, \Pi_4$ are now viewed as forms on the 11 dimensional manifold $M \times G \times G^{(1)}$ and again are augmented by forms $\{\Omega_1, \Omega_2, \Omega_3, \Omega_4\} \subset T^*(M \times G \times G^{(1)})$. We get the following reductions of the structure group $G^{(1)}$:

$$
B_2 = \frac{3}{8}a^2 e^{-4r} + \frac{3}{4}c^{-2r}\kappa - \frac{3}{8}b^2 e^{-4r}
$$

$$
B_3 = \frac{1}{4}b^3 e^{-6r} - \frac{1}{2}b e^{-4r}\kappa - \frac{1}{4}a^2 b e^{-6r}
$$

$$
B_4 = -\frac{1}{4}a^3 e^{-6r} - \frac{1}{2}a e^{-4r}\kappa + \frac{1}{4}ab^2 e^{-6r}
$$

and so $M \times G \times G^{(1)}$ becomes an 8 dimensional manifold and we only need $\Omega = \Omega_1$ to augment. Finally after absorption we arrive at the structure equations (5.1).
The structure equations for the isometries are obtained similarly but do not require prolongation.

The Killing form for the conformal structure equations is

\[
K = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
0 & 0 & 0 & 12 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Since \( \det K \neq 0 \) and \( K \) is indefinite with signature \(+ + + + - - - -\), the Lie algebra must be \( \mathfrak{sl}_3 \), i.e., the only 8 dimensional simple Lie algebras are \( \mathfrak{sl}_3, \mathfrak{su}_3 \) and \( \mathfrak{su}_{2,1} \), however \( \mathfrak{su}_3 \) and \( \mathfrak{su}_{2,1} \) are ruled out by the indefiniteness and signature. Alternatively one can simply compute the Lie brackets of the vector fields dual to the system of one forms and check that it isomorphic to \( \mathfrak{sl}_3 \).

The fact in that the universal cover of \( SL_2(\mathbb{R}) \) and the Heisenberg group both have \( SL_3(\mathbb{R}) \) as the conformal symmetry group implies that they are all conformally equivalent, see [17] proposition 2.3.2. and [5] section 2.5. We thus have the following Theorem conformal Darboux theorem.

**Theorem 5.1.** All left invariant sub-Lorentzian structures on the universal cover of \( SL_2(\mathbb{R}) \) such that \( \tilde{h} = 0 \) are locally conformally equivalent to the sub-Lorentzian Heisenberg group.

6. Rigid example.

As it is seen from the previous sections, in particular (5.2), the dimension of the algebra of infinitesimal isometries for left-invariant sub-Lorentzian structures is at least 3 and at most 4. The goal of this section is to show that this is not the case for general, not necessarily left-invariant, contact sub-Lorentzian structures. Since the computations based on considerations from Section 3 rely on solving systems of PDE’s, even in the simplest cases, they are very complicated and do not lead to explicit solutions. We will use instead known facts from the geometry of second order ODE’s.

Consider an equation

\[(6.1)\quad u'' = Q(x, u, u')\]

where the right hand side is smooth. In terms of differential forms, equation (6.1) is encoded by by the coframe:

\[
\omega^1 = du - pdx, \quad \omega^2 = dp - Q(x, u, p)dx, \quad \omega^3 = dx,
\]

defined on the first jet space \( J^1 = J^1(\mathbb{R}, \mathbb{R}) \) with coordinates \((x, u, p)\), where \( p = u' \). More precisely, a curve \( \gamma(x) = (x, u(x), p(x)) \) in the space \( J^1 \) defines a solution to (6.1) if and only if \( \gamma^* \omega^i = 0, \quad i = 1, 2 \) (one can easily show that the vanishing of the two pull-backs is equivalent to \( u'(x) = p(x) \) and in turn to \( u''(x) = Q(x, u(x), u'(x)) \)). Consider a diffeomorphism \( \Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R} ; \Phi \) is called a point transformation or a point symmetry of (6.1) if and only if it maps graphs of solutions to (6.1) onto graphs of solutions to (6.1). It can be proved that
a necessary and sufficient condition for $\Phi$ to be a point symmetry of (6.1) is the existence of smooth functions $a_i, i = 1, ..., 5$, such that

$$\hat{\Phi}^*\omega^1 = a_1\omega^1, \hat{\Phi}^*\omega^2 = a_2\omega^1 + a_3\omega^2, \hat{\Phi}^*\omega^3 = a_4\omega^1 + a_5\omega^3,$$

where $\hat{\Phi} : J^1 \rightarrow J^1$ is the prolongation of $\Phi$ to the jet space $J^1$. One of the most important problems in the geometric theory of ODE’s is to classify ODE’s with respect to point transformations. It is known that the group of point symmetries of a given second order ODE has dimension varying between 0 and 8. E.g. the flat equation $u'' = 0$ has its symmetry group of dimension 8 and the equation

$$u'' = ((x + x^2)e^u)'$$

do not have (nontrivial) symmetries at all, see [13] page 182.

Any equation (6.1) defines a conformal class of contact sub-Lorentzian metrics in terms of the forms (6.2). Indeed, let $H = \ker \omega^1$, $L_1 = \ker \omega^1 \cap \ker \omega^2$, $L_2 = \ker \omega^1 \cap \ker \omega^3$. Clearly, $H$ is a contact distribution that splits into the union of line bundles: $H = L_1 \oplus L_2$. Similarly as in the classical situation the splitting can be viewed as the field of null cones for a Lorentzian metric on $H$. Of course all such metrics are conformally equivalent. In particular it is seen that all solutions to (6.1) are determined by the trajectories of the null field $\frac{\partial}{\partial x} + p\frac{\partial}{\partial p} + Q(x, u, p)\frac{\partial}{\partial x}$ spanning $L_1$.

Consider now any sub-Lorentzian structure $(U, H, g)$, $U$ being a neighborhood of 0 in $\mathbb{R}^3$, associated, in the just described manner, with the equation (6.3). It is easy to notice that $(U, H, g)$ does not admit nontrivial infinitesimal isometries. Indeed, suppose that $X$ is an infinitesimal isometry. Then its flow, say $h^t$, preserves $H$ and moreover (since it is isotopic with identity) $h^t(L_i) = L_i, i = 1, 2$. But then $h^t$ is induced by a point symmetry of the equation (6.3), therefore $h^t = id$ and, what follows, $X = 0$.

7. Appendix.

In this appendix we would like to draw the reader’s attention to some possible applications of invariants in non-contact cases. Consider the simplest such case, namely the Martinet case. Martinet sub-Lorentzian structures (of Hamiltonian type) were studied in [11]. Let $(M, H, g)$ be a sub-Lorentzian manifold where $(H, g)$ is a Martinet sub-Lorentzian structure (or a metric). That is, there exists a hypersurface $S$, the so-called Martinet surface, with the following properties:

1. $H$ is a contact structure on $M \setminus S$,

2. $\dim(H_q \cap T_q S) = 1$ for every $q \in M$,

3. the field of directions $L : S \ni q \rightarrow L_q = H_q \cap T_q S$ is timelike.

It is a standard fact that trajectories of $L$ are abnormal curves for the distribution $H$. Obviously our construction of the invariants can be carried out on the contact sub-Lorentzian manifold $(M \setminus S, H|_{M \setminus S}, g|_{M \setminus S})$. In this way we can produce necessary conditions for two Martinet sub-Lorentzian structures to be ts-isometric. More precisely, let $(M_i, H_i, g_i)$ be Martinet sub-Lorentzian manifolds such that $(H_i, g_i)$ are ts-oriented Martinet sub-Lorentzian metrics for $i = 1, 2$. Suppose that $\varphi : (M_1, H_1, g_1) \rightarrow (M_2, H_2, g_2)$ is a ts-isometry, then since abnormal curves are preserved by diffeomorphisms, $\varphi(S_1) = S_2$ where $S_i$ is the Martinet...
surface for \( H_i, i = 1, 2 \). It follows that \( \varphi \) induces a ts-isometry \( \tilde{\varphi} = \varphi|_{M_i \setminus S_i} : (M_1 \setminus S_1, H_1|_{M_1 \setminus S_1}, g_1|_{M_1 \setminus S_1}) \rightarrow (M_2 \setminus S_2, H_2|_{M_2 \setminus S_2}, g_2|_{M_2 \setminus S_2}) \). Therefore, using results from section 4 we arrive at

\[
\chi_i = \tilde{\varphi}^* \chi_2, \quad \kappa_i = \tilde{\varphi}^* \kappa_2, \quad \text{and} \quad \tilde{h}_i = \tilde{\varphi}^* \tilde{h}_2,
\]

where \( \chi_i, \kappa_i, \tilde{h}_i \) are the corresponding invariants for \( (M_i \setminus S_i, H_i|_{M_i \setminus S_i}, g_i|_{M_i \setminus S_i}) \), \( i = 1, 2 \).

As one might expect, the invariants become singular when one approaches the Martinet surface. Indeed, let us look at the following example.

**Example 7.1.** Consider the simplest Martinet sub-Lorentzian structure, namely the flat one (cf. [11]). This structure is defined on \( \mathbb{R}^3 \) by the orthonormal frame

\[
X_1 = \frac{\partial}{\partial x} + \frac{1}{2} y^2 \frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} - \frac{1}{2} xy \frac{\partial}{\partial z},
\]

where we assume \( X_1 \) (resp. \( X_2 \)) to be a time (resp. space) orientation. The Martinet surface in this case is \( S = \{ y = 0 \} \), and we can write \( H = \text{Span}\{ X_1, X_2 \} = \ker \omega \) for \( \omega \) defined as \( \omega = \frac{4}{3} y dz - \frac{1}{3} y dx + \frac{1}{3} xydy \). Clearly, \( d\omega(X_1, X_2) = 1 \), and as usual we define the Reeb field \( X_0 \) on \( \mathbb{R}^3 \setminus S \) with equations \( d\omega(X_0, \cdot) = 0, \omega(X_0) = 1 \).

Easy computations yield

\[
X_0 = -\frac{1}{y} \frac{\partial}{\partial x} + \frac{\partial}{\partial z},
\]

Moreover

\[
[X_2, X_1] = \frac{1}{y} X_1 + X_0, \quad [X_1, X_0] = 0, \quad [X_2, X_0] = \frac{1}{y^2} X_1
\]

from which we finally obtain

\[
\tilde{h} = \begin{pmatrix}
0 & \frac{1}{2 y^2} \\
-\frac{1}{2 y^2} & 0
\end{pmatrix}, \quad \chi = \frac{1}{4} \frac{1}{y^4}, \quad \text{and} \quad \kappa = -\frac{5}{2} \frac{1}{y^2}.
\]

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