Wave Function of the Universe in
Topological and in Einstein 2-form Gravity†

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ABSTRACT

We clarify the relation between 2-form Einstein gravity and its topological version. The physical space of the topological version is contained in that of the Einstein gravity. Moreover a new vector field is introduced into 2-form Einstein gravity to restore the large symmetry of its topological version. The wave function of the universe is obtained for each model.

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1. Introduction

Topological gravity is expected to elucidate the global (topological) aspects of gravity. In Ref.[1], a topological version of four dimensional Einstein gravity is proposed. This topological version is obtained by modifying an alternative formulation of gravity enlightened by Capovilla et al.[2], in which anti-self-dual 2-forms are used as fundamental variables. This formulation, which we call 2-form Einstein gravity, leads to the canonical formalism discovered by Ashtekar [3]. Investigating the relation between 2-form Einstein gravity and its topological version, we found that, a unique quantum state in the latter turns out to be one of the physical states in the former. This physical state is interpreted as the wave function of the universe.

Moreover in Ref.[4], a new vector field is introduced into 2-form Einstein gravity to restore the large symmetry of its topological version. We also obtain the wave function of the universe in the new system.

2. Symmetries

Einstein Gravity

The action of (Euclidean) 2-form Einstein gravity is given in terms of 2-form $\Sigma^k$ and SU(2) spin connection 1-form $\omega^k$ in the presence of a cosmological constant $\Lambda$,

$$ S = \int \Sigma^k \wedge R_k - \frac{\Lambda}{24} \Sigma^k \wedge \Sigma_k + \frac{\alpha}{2} \psi_{kl} \Sigma^k \wedge \Sigma^l, \quad (1) $$

where $R_k \equiv d\omega_k + (\omega \times \omega)_k$, $\psi_{kl}$ is a symmetric trace-free Lagrange multiplier field, and $\alpha$ is a constant parameter. The SU(2) indices $i, j, k, \cdots = 1, 2, 3$, in the fields imply that they transform under the chiral local-Lorentz representation $(1,0)$ of
SU(2)×SU(2) [5]. In this formulation, the metric $g_{\mu\nu}$ is defined as

$$g^{\frac{1}{2}}g_{\mu\nu} = -\frac{1}{12} \epsilon^{\alpha\beta\gamma\delta} \Sigma_{\mu\alpha} \cdot (\Sigma_{\beta\gamma} \times \Sigma_{\delta\nu}) , \quad g \equiv \text{det}(g_{\mu\nu}) . \quad (2)$$

Using this definition, we find that the action (1) is equivalent to the usual Einstein-Hilbert action [2,6].

Since the action (1) describes general relativity, it is invariant under the local-Lorentz transformation and diffeomorphism,

$$\delta \omega^k = D \theta_0^k + \mathcal{L}_\xi \omega^k , \quad \delta \Sigma^k = [\Sigma, \theta_0]^k + \mathcal{L}_\xi \Sigma^k , \quad (3)$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to a vector field $\xi^\mu$, and the local-Lorentz transformation corresponds to the SU(2) gauge transformation with a parameter $\theta_0^k$.

**Topological Gravity**

From the equations of motion, the multiplier field $\psi_{kl}$ is determined to be proportional to the anti-self-dual part of the Weyl tensor, which governs the modes of the gravitational wave. Therefore the topological version of the theory is obtained by simply dropping the last term in the action (1), that is, by setting $\alpha = 0$ [1]:

$$S_{\alpha=0} = \int \Sigma^k \wedge R_k - \frac{\Lambda}{24} \Sigma^k \wedge \Sigma_k . \quad (4)$$

In this case, a new symmetry generated by a parameter 1-form $\theta_i^k$ emerges in addition to diffeomorphism and the local-Lorentz (with $\theta_0^k$) symmetries,

$$\delta \omega^k = D \theta_0^k + \frac{\Lambda}{12} \theta_1^k , \quad \delta \Sigma^k = 2(\Sigma \times \theta_0)^k + D \theta_1^k . \quad (5)$$

Here diffeomorphism with a vector field $\xi^\mu$ is implicitly included in the above local-Lorentz and $\theta_1^k$ transformations as we can see by setting $\theta_0^k = \xi^\nu \omega^k_{\nu}$ and

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\[\text{SU(2)}\times\text{SU(2)}\]  
$\text{SU(2)}$ indices, $F \cdot G \equiv F^i G_i$ and $(F \times G)^i \equiv \varepsilon_{ijk} F^j G^k$, where $\varepsilon_{ijk}$ is the structure constant of $\text{SU(2)}$. 

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$^1$ We use the notation for the $\text{SU(2)}$ indices, $F \cdot G \equiv F^i G_i$ and $(F \times G)^i \equiv \varepsilon_{ijk} F^j G^k$, where $\varepsilon_{ijk}$ is the structure constant of $\text{SU(2)}$. 

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\( \theta^k_{1\mu} = 2\xi^\nu \Sigma^k_{\nu\mu} \). The theory turns out to be on-shell reducible in the sense that the transformation laws (5) are invariant, modulo the equations of motion, under

\[
\delta \theta^k_0 = -\frac{\Lambda}{12} \epsilon^k_0, \quad \delta \theta^k_1 = D\epsilon^k_0. \tag{6}
\]

This means that not all of the parameters in (5) are independent.

**New System**

From the viewpoint of the topological gravity, one can see that the large \( \theta^k_1 \) -symmetry is partially broken in Einstein gravity leaving only diffeomorphism and local-Lorentz symmetries intact and, as a result, the modes of the gravitational wave are induced. The obstruction for the \( \theta^k_1 \) -symmetry is the last term in the action (1). We can restore the symmetry by introducing a vector field (1-form) \( \eta^k \) in the last term as follows,

\[
\frac{\alpha}{2} \int \psi_{kl}\Sigma^k \wedge \Sigma^l \Rightarrow \frac{\alpha}{2} \int \psi_{kl}\hat{\Sigma}^k \wedge \hat{\Sigma}^l, \quad \hat{\Sigma}^k \equiv \Sigma^k - D\eta^k + \frac{\Lambda}{12}(\eta \times \eta)^k. \tag{7}
\]

This makes our new system invariant under the \( \theta^k_1 \) -transformation in (5) with \( \delta \eta^k = \theta^k_1 \), together with diffeomorphism and the local-Lorentz transformation:

\[
\delta \omega^k = D\theta^k_0 + \mathcal{L}_\xi \omega^k + \frac{\Lambda}{12} \theta^k_1, \quad \delta \Sigma^k = [\Sigma, \theta_0]^k + \mathcal{L}_\xi \Sigma^k + D\theta^k_1.
\]

\[
\delta \eta^k = [\eta, \theta_0]^k + \mathcal{L}_\xi \eta^k + \theta^k_1. \tag{8}
\]

They are all independent and hence there is no reducibility in the system. It is easily shown that the new system is equivalent to Einstein gravity, by choosing the gauge condition, \( \eta^k = 0 \), for the \( \theta^k_1 \)-symmetry. Indeed physical degrees of freedom of the new system is 2: the modes of the gravitational wave.
3. Physical states

Topological Gravity

In the topological case ($\alpha = 0$), the action (4) becomes in canonical form:

$$S = \int dt \int d^3x [\dot{\omega}_a \cdot B^a - \omega_0 \cdot \varphi - \Sigma_{a0} \cdot \phi^a] .$$

(9)

The canonical variables are $\omega^k_a$ and their conjugate momenta $B^a_k \equiv \epsilon^{abc} \Sigma^k_{bc}$, which are the spatial components of the spin connection $\omega^k$ and the 2-form $\Sigma^k$. Varying (9) with respect to their time components $\omega^k_0$ and $\Sigma^k_{a0}$, we get two sets of constraints,

$$\varphi^a_k \equiv -D_a B^a_k \approx 0 , \quad \phi^a_k \equiv 2 (\epsilon^{abc} R^k_{bc} - \frac{\Lambda}{12} B^a_k) \approx 0 .$$

(10)

The Poisson brackets among them are given by

$$\{\varphi^a_i (x), \varphi^b_j (y)\} = -2 \epsilon_{ijk} \varphi^a_k (x) \delta^3 (x - y) , \quad \{\phi^a_i (x), \phi^b_j (y)\} = 0 ,$$

$$\{\varphi^a_i (x), \phi^b_j (y)\} = -2 \epsilon_{ijk} \phi^a_k (x) \delta^3 (x - y) .$$

(11)

All the constraints are of first class and the algebra is closed. The constraints $\varphi^a_k$ and $\phi^a_k$ generate the local-Lorentz and $\theta^k_1$- transformations in (5) respectively. In this canonical formulation, the on-shell reducibility (6) appears as a linear dependence of the constraints,

$$D_a \phi^a_k - \frac{\Lambda}{6} \varphi_k = 0 .$$

(12)

In the Dirac approach for quantization, one has to impose quantum conditions to choose physical wave functional $\Psi$. In the topological case with $\Lambda \neq 0$, these conditions can be expressed using the constraints (10) in $\omega^k_a$ representation,

$$\varphi_k (\omega, \delta/\delta \omega) \Psi (\omega) = i D_a (\delta/\delta \omega^k_a) \Psi (\omega) = 0 ,$$

$$\phi^a_k (\omega, \delta/\delta \omega) \Psi (\omega) = 2 (\epsilon^{abc} R^k_{bc} + i \frac{\Lambda}{12} \delta/\delta \omega^k_a) \Psi (\omega) = 0 .$$

(13)

Since these equations are linear differential equations, we can easily solve them to
obtain the unique functional of $\omega_a^k$:

$$\Psi(\omega) = \exp(\frac{6i}{\Lambda} I_{C-S}) \, , \quad I_{C-S} \equiv \int d^3 x \varepsilon^{abc} \omega_a \cdot (\partial_b \omega_c + \frac{2}{3}(\omega_b \times \omega_c)) \, , \quad (14)$$

where $I_{C-S}$ is the Chern-Simons term on the three-dimensional boundary. This type of solution is also found in a different version of topological gravity [7]. The functional $\Psi(\omega)$ is the wave function of the universe in four dimensional (Euclidean) topological gravity. It can also be considered as the BRST invariant vacuum, because it is the unique representative annihilated by the BRST operator [1].

**Einstein Gravity**

On the other hand in the Einstein gravity ($\alpha \neq 0$), we have to solve the constraint equations, which can be considered as five linear equations for nine Lagrange multipliers $\Sigma_{ab}^k$ [2]. The solution is expressed with four arbitrary variables $N^a$ (shift vector) and $N$ (lapse density of weight $-1$),

$$\Sigma_{ab}^k = -\frac{1}{4} \varepsilon_{abc} [N^b B^c_k + N (B^b \times B^c)^k] \, . \quad (15)$$

Substituting this result for the canonical action (9), we now have four constraints, together with $\varphi_k$ in (10), which are associated with the Lagrange multipliers $N^a$ and $N$,

$$C_a \equiv \frac{1}{4} \varepsilon_{abc} B^b \cdot \phi^c = B^b \cdot R_{ab} \approx 0 \, ,$$

$$C \equiv \frac{1}{4} \varepsilon_{abc} (B^a \times B^b) \cdot \phi^c = (B^a \times B^b) \cdot (R_{ab} - \frac{\Lambda}{24} \varepsilon_{abc} B^c) \approx 0 \, . \quad (16)$$

We can identify $\varphi_k, C_a$ and $C$ with the independent first class constraints corresponding to the generators of the local-Lorentz transformation, spatial diffeomorphism and temporal diffeomorphism respectively.
An important observation is that all the constraints in the Einstein Gravity are linear combinations of those in the topological version. Especially four diffeomorphism generators $C^a$ and $C$ in (16) are linearly dependent on nine ‘new-type’ generators $\phi^a_k$ in (10). We see that $\Psi(\omega)$ in (14) becomes a special solution of all quantum constraints in the Einstein gravity if the operator ordering is arranged as in (16). This ordering is consistent with the commutation relations among the constraint operators [8]. This $\Psi(\omega)$ is nothing but the Euclidean version of the wave functional discovered by Kodama [9,10]. Therefore the physical space of the topological gravity is contained in that of the Einstein gravity.

New System

With the vector field $\eta^k$, the action becomes in canonical form,

$$S = \int dt \int d^3x [\dot{\omega}_a \cdot B^a - \omega_0 \cdot \varphi - \Sigma_{a0} \cdot \phi^a - 2\alpha \psi_{kl} \hat{\Sigma}_a^k \hat{B}_l^a] ,$$

where $\hat{B}_a^k \equiv \epsilon^{abc} \hat{\Sigma}_b^c$ is the spatial components of the 2-form $\hat{\Sigma}^k$. Again we have to solve the constraint equation derived by varying (17) with respect to $\psi_{kl}$. The solution is expressed by using four arbitrary variables $N^a$ and $\tilde{N}^a$,

$$\Sigma_{a0}^k = -\frac{1}{4} \epsilon^{abc}[N^b \hat{B}_c^k + N(\hat{B}_b \times \hat{B}_c^a)] - \frac{1}{2} \hat{\eta}_a - (\omega_0 \times \eta_a)^k + \frac{1}{2} \hat{D}_a \eta_0^k .$$

Substituting this result for the canonical action (17), we get four sets of constraints:

$$\phi^a_k \equiv -D_a B^a_k - 2(\eta_a \times \eta^a)^k \approx 0 ,$$

$$\hat{\phi}^a_k \equiv 2(\epsilon^{abc} \hat{R}^k_b - \frac{\Lambda}{12} B^a_k) - 2 \eta^a \pi^a \approx 0 ,$$

$$H_a \equiv \frac{1}{4} \epsilon^{abc} \hat{B}_b \cdot (\hat{\phi}^c + 2 \eta^c \pi^c) \approx 0 ,$$

$$\mathcal{H} \equiv \frac{1}{4} \epsilon^{abc}(\hat{B}_a \times \hat{B}_b) \cdot (\hat{\phi}^c + 2 \eta^c \pi^c) \approx 0 .$$

The fields $\pi^a_k$ are the conjugate momenta of the spatial components of $\eta^a_k$. Next
we redefine the constraint $H_a$ as
\[ H_a = 2H_a + \omega_a \cdot \dot{\phi} - \frac{1}{2} \epsilon_{abc} (\hat{B}^b - B^b) \cdot \dot{\phi}^c. \] (20)

The new constraint $H_a$ generates the spatial diffeomorphism. The non-zero Poisson brackets among the constraints are given by
\[
\{ \hat{\phi}^*[g_1], \hat{\phi}^*[g_2] \} = -2 \hat{\phi}^*[ (g_1 \times g_2) ] ,
\{ \hat{\phi}^*[g], \hat{\phi}^*[h_a] \} = -2 \hat{\phi}^*[ (g \times h_a) ] ,
\{ \mathcal{H}[N], \mathcal{H}[M] \} = H_a[L^a] - \hat{\phi}[L^a \omega_a] + \frac{1}{2} \hat{\phi}^*[\epsilon_{abc} L^b \hat{B}^c - B^c] ,
\] (21)

where $\hat{\phi}^*[g_1] \equiv \int d^3x g_k^k \hat{\phi}_k$, $\hat{\phi}^*[h_a] \equiv \int d^3x h_k^a \hat{\phi}_k$, $L^a \equiv \hat{B}^a \cdot \hat{B}(M \partial_b N - N \partial_b M)$.

All the constraints in the system are of first class. Among them, $\hat{\phi}_k$ and $\hat{\phi}_k^a$ are identified with the generators of the local-Lorentz and $\theta_k^k$- transformations in (8) respectively. The constraint $\mathcal{H}$ generates temporal diffeomorphism while $H_a$ the spatial one.

As in the previous cases, quantum conditions are
\[
\hat{\phi}_k(\omega, \delta/\delta \omega, \eta, \delta/\delta \eta) \Psi(\omega, \eta) = 0 , \quad \hat{\phi}_k^a(\omega, \delta/\delta \omega, \eta, \delta/\delta \eta) \Psi(\omega, \eta) = 0 ,
\mathcal{H}_a(\omega, \delta/\delta \omega, \eta, \delta/\delta \eta) \Psi(\omega, \eta) = 0 , \quad \mathcal{H}(\omega, \delta/\delta \omega, \eta, \delta/\delta \eta) \Psi(\omega, \eta) = 0 .
\] (22)

These linear differential equations are solved as the following [11]:
\[
\Psi(\omega, \eta) = \exp[ \frac{6i}{\Lambda} I_{C-S} + \beta(I_{C-S} - \frac{\Lambda}{6} I_{New})] ,
I_{New} \equiv \int d^3x \varepsilon^{abc} \eta_a \cdot R_{bc} - \frac{\Lambda}{24} \int d^3x \varepsilon^{abc} \eta_a \cdot D_b \eta_c + \frac{4}{3} (\frac{\Lambda}{24})^2 \int d^3x \varepsilon^{abc} \eta_a \cdot (\eta_b \times \eta_c) ,
\] (23)

where $\beta$ is a constant parameter. If we choose it as $-\frac{6}{\Lambda}i$, the wave function of the Universe becomes
\[
\Psi(\omega, \eta) = \exp(i I_{New}) .
\] (24)

This solution is suitable for both $\Lambda = 0$ and $\Lambda \neq 0$ cases.
4. Summary

We have clarified the relation between the 2-form Einstein gravity and its topological version. The physical space of the topological version is contained in that of the Einstein gravity.

Moreover the vector field $\eta^k$ is introduced into 2-form Einstein gravity to restore the large symmetry of its topological version. Since this new model has the modes of the gravitational wave, it is equivalent to the Einstein gravity. It may be a good strategy in quantum gravity to study models with large symmetry, in addition to local Lorentz and diffeomorphisms.

We have obtained the wave function of the universe for each model.

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REFERENCES

1. H.Y. Lee, A. Nakamichi, and T. Ueno, ‘Topological 2-form Gravity in Four Dimensions’, Phys. Rev. D47 (1993) 1563.

2. R. Capovilla, J. Dell, T. Jacobson and L. Mason, Class. Quantum Grav. 8 (1991) 41.

3. A. Ashtekar, Phys. Rev. D36 (1987) 1587; ‘New Perspectives in Canonical Gravity’, (Bibliopolous, Naples, Italy, 1988).

4. A. Nakamichi and T. Ueno, ‘New Vector Field and BRST Charges in 2-form Einstein Gravity’, TIT/HEP-206/COSMO-24, February 1993.

5. R. Penrose and W. Rindler, ‘Spinors and Space-time’ Vols. I, II (Cambridge University Press, Cambridge, 1984).

6. T. Jacobson and L. Smolin, Class. Quantum Grav. 5 (1988) 583.

7. G.T. Horowitz, Commun. Math. Phys. 125 (1989) 417.

8. A. Ashtekar, Phys. Rev. Lett. 57 (1986) 2244.

9. H. Kodama, Phys. Rev. D42 (1990) 2548.

10. H. Ikemori, in ‘Proceeding of the Workshop on Quantum Gravity and Topology’, edited by I. Oda (Institute for Nuclear Study, University of Tokyo, 1991).

11. A. Nakamichi and T. Ueno, in preparation.