HYPERBOLIC STRUCTURES ON CLOSED SPACELIKE MANIFOLDS

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ABSTRACT. In this paper, we study the intrinsic mean curvature flow on certain closed spacelike manifolds, and prove the existence of hyperbolic structures on them.

1. INTRODUCTION

Recall that a Riemannian manifold \((M, g)\) is hyperbolic if it has constant negative sectional curvature. These manifolds all come from the quotient of hyperbolic space \(\mathbb{H}^n\) by discrete isometry groups. However, it is difficult to find a good intrinsic characterization on the existence of hyperbolic structures on a given manifold. First, we know that some negatively pinched Riemannian manifolds cannot admit hyperbolic metric. In [7], for \(n \geq 4\), the counterexample contrasts sharply with the pinching theorem of positively curved manifolds. In [14], it was shown that for \(n \geq 10\) the space of negatively curved metric on some \(n\)-manifold is highly non-connected. This implies that for a given negatively curved metric, it is not always possible to deform it into a metric with constant negative curvature by any geometric flows.

In this paper, motivated by Lorentzian geometry, we will show that the hyperbolic structure exists naturally on a large class of spacelike manifolds. The motivation is the following. It is well known that the imaginary unit sphere of Minkowski space \(\mathbb{R}^{1,n}\) is the model of hyperbolic spaces, where under Cartesian coordinates \((x^0, x^1, \ldots, x^n)\) on \(\mathbb{R}^{1,n}\), the Minkowski metric is

\[
g = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^n)^2
\]

and the equation of imaginary unit sphere is

\[
-(x^0)^2 + (x^1)^2 + \cdots + (x^n)^2 = -1.
\]

This can be seen from Gauss-Codazzi equations

\[
\begin{align*}
R_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}) &= 0, \\
\nabla_i h_{jk} - \nabla_j h_{ik} &= 0,
\end{align*}
\]

Date: September 26, 2008.

Key words and phrases. hyperbolic structure, intrinsic mean curvature flow, closed spacelike manifold.
where \( h_{ij} \) is the second fundamental form, and \( h_{ij} \) equals to \( g_{ij} \) on the imaginary unit sphere. In this paper, we are interested in an intrinsic generalization of this model.

**Definition 1.1.** We call a triple \((M, g_{ij}, h_{ij})\) a spacelike manifold, if \((M, g_{ij})\) is a Riemannian manifold, and \( h_{ij} \) is a symmetric tensor satisfying the Gauss-Codazzi equations

\[
\begin{align*}
R_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}) &= 0 \\
\nabla_i h_{jk} - \nabla_j h_{ik} &= 0.
\end{align*}
\]

Now we state the main theorem of this paper in the following.

**Theorem 1.2.** Let \((M, g, h)\) be an \(n\)-dimensional (\(n \geq 4\)) closed spacelike manifold with \( h_{ij} > 0 \), then \( M \) admits a hyperbolic metric.

The idea is to use geometric flows. We define an intrinsic mean curvature flow of \((g, h)\):

\[
\begin{align*}
\frac{\partial g_{ij}}{\partial t} &= -2R_{ij} + 2h_{im}h_{nj}g^{mn} \\
\frac{\partial h_{ij}}{\partial t} &= \Delta h_{ij} - R_{im}h_{nj}g^{mn} - R_{jm}h_{ni}g^{mn} \\
&\quad + 2h_{ik}h_{lm}h_{nj}g^{kl}g^{mn} - |A|^2 h_{ij},
\end{align*}
\]

with \( g_{ij}(x, 0) = \tilde{g}_{ij}(x) \), \( h_{ij}(x, 0) = \tilde{h}_{ij}(x) \), where \( \tilde{g}_{ij}(x) \) is the initial metric on \( M \) and \( \tilde{h}_{ij}(x) \) is the initial data of \( h_{ij} \) and \( |A|^2 = g^{ik}g^{jl}h_{ij}h_{kl} \).

Mean curvature flow has been intensively studied in recent years (see [3] for Euclidean ambient space and [10] for Minkowski ambient space). Notice that in extrinsic mean curvature flow (with ambient space \( \mathbb{R}^{1,n} \)), we deform the position vector \( F \) by the evolution equation

\[
\frac{\partial F}{\partial t} = -H,
\]

and (1.1) is just the equations of the metric and the second fundamental form. Here, our observation is that (1.1) itself is also an intrinsically defined evolution system of \((g, h)\), and it has its own right to be studied. In this paper, we solve (1.1) intrinsically and show that the solution exists for all time \([0, \infty)\) and converges (after normalization) to a hyperbolic metric.

**Acknowledgement** I am grateful to my advisor Professor B.L.Chen for his guidance.

2. **Short-Time Existence and Uniqueness**

Since (1.1) is not a strictly parabolic system, in order to apply theory of strictly parabolic equation to get short time existence, we use a trick of De Turck by combining our evolution equation (1.1) with the harmonic map flow.
Let \((M^n, g_{ij}(x))\) and \((N^m, s_{\alpha\beta}(y))\) be two Riemannian manifolds, \(F : M^n \to N^m\) be a map. The harmonic map flow is the following evolution equation for maps from \(M^n\) to \(N^m\),

\[
\begin{cases}
\frac{\partial}{\partial t} F(x, t) = \triangle F(x, t), & \text{for } x \in M^n, t > 0, \\
F(x, 0) = F(x), & \text{for } x \in M^n,
\end{cases}
\]

(2.1) where \(\triangle\) is defined by using the metrics \(g_{ij}(x)\) and \(s_{\alpha\beta}(y)\) as follows

\[
\triangle F^\alpha(x, t) = g^{ij}(x) \nabla_i \nabla_j F^\alpha(x, t),
\]

and

\[
\nabla_i \nabla_j F^\alpha(x, t) = \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial F^\alpha}{\partial x^k} + \hat{\Gamma}_y^\alpha \frac{\partial F^\beta}{\partial x^j} \frac{\partial F^\gamma}{\partial x^i},
\]

(2.2) Here we use \(\{x^i\}\) and \(\{y^\alpha\}\) to denote the local coordinates of \(M^n\) and \(N^m\) respectively, \(\Gamma^k_{ij}\) and \(\hat{\Gamma}_y^\alpha\) the corresponding Christoffel symbols of \(g_{ij}\) and \(s_{\alpha\beta}\). The harmonic map flow is strictly parabolic, so for any initial data, there exists a short time smooth solution.

Let \((g_{ij}(x, t), h_{ij}(x, t))\) be a complete smooth solution of our evolution equation (1.1), then the harmonic map flow coupled with our evolution equation is the following equation:

\[
\begin{cases}
\frac{\partial}{\partial t} F(x, t) = \triangle_i F(x, t), & \text{for } x \in M^n, t > 0, \\
F(x, 0) = \text{identity}, & \text{for } x \in M^n,
\end{cases}
\]

(2.3) where \(\triangle_i\) is defined by using the metrics \(g_{ij}(x, t)\) and \(s_{\alpha\beta}(y)\).

Let \((F^{-1})^*g\) and \((F^{-1})^*h\) be the one-parameter families of pulled back metrics and pull back tensors on the target \((N^m, s_{\alpha\beta})\). Denote \(\hat{g}_{\alpha\beta}(y, t) = ((F^{-1})^*g)_{\alpha\beta}(y, t)\) and \(\hat{h}_{\alpha\beta}(y, t) = ((F^{-1})^*h)_{\alpha\beta}(y, t)\). Then by direct calculations, \(\hat{g}_{\alpha\beta}(y, t)\) and \(\hat{h}_{\alpha\beta}(y, t)\) satisfy the following evolution equation:

\[
\begin{cases}
\frac{\partial \hat{g}_{\alpha\beta}}{\partial t}(y, t) = -2 \hat{R}_{\alpha\beta}(y, t) + 2 \hat{h}_{\alpha\sigma} \hat{h}_{\rho\beta} \hat{g}^{\sigma\rho} + \nabla_\alpha V_\beta + \nabla_\beta V_\alpha \\
\frac{\partial \hat{h}_{\alpha\beta}}{\partial t}(y, t) = \Delta \hat{h}_{\alpha\beta}(y, t) - \hat{R}_{\alpha\sigma} \hat{h}_{\rho\beta} \hat{g}^{\sigma\rho} - \hat{R}_{\beta\sigma} \hat{h}_{\rho\alpha} \hat{g}^{\sigma\rho} + 2 \hat{h}_{\alpha\lambda} \hat{h}_{\mu\rho} \hat{g}^{\lambda\rho} \hat{g}^{\mu\nu} - |\hat{A}|^2 \hat{h}_{\alpha\beta} \\
\quad + \hat{h}_{\beta\gamma} \nabla_\alpha V_\gamma + \hat{h}_{\beta\gamma} \nabla_\beta V_\gamma
\end{cases}
\]

(2.4) where \(V^\alpha = g^{\beta\gamma} (\Gamma^\alpha_{\beta\gamma}(\hat{g}) - \hat{\Gamma}^\alpha_{\beta\gamma}(s))\), \(\Gamma^\alpha_{\beta\gamma}(\hat{g})\) and \(\hat{\Gamma}^\alpha_{\beta\gamma}(s)\) are the Christoffel symbols of the metrics \(\hat{g}_{\alpha\beta}(y, t)\) and \(s_{\alpha\beta}(y)\) respectively. Here we analysis the principle part of the right side of (2.4). One can see

\[
-2 \hat{R}_{\alpha\beta}(y, t) + 2 \hat{h}_{\alpha\sigma} \hat{h}_{\rho\beta} \hat{g}^{\sigma\rho} + \nabla_\alpha V_\beta + \nabla_\beta V_\alpha
\]

\[
= \hat{g}^{\mu\nu} \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\mu \partial y^\nu} + \text{(lower order terms)}
\]
and
\[
\begin{align*}
\triangle \hat{h}_{\alpha\beta}(y,t) - \hat{R}_{\alpha\sigma} \hat{h}_{\rho\beta} \hat{g}^{\sigma\rho} - \hat{R}_{\beta\sigma} \hat{h}_{\rho\alpha} \hat{g}^{\sigma\rho} \\
+ 2 \hat{h}_{\alpha\lambda} \hat{h}_{\rho\mu} \hat{h}_{\beta\sigma} \hat{g}^{\lambda\mu} \hat{g}^{\rho\sigma} - |\hat{\Lambda}|^2 \hat{h}_{\alpha\beta} + \hat{h}_{\beta\gamma} \nabla_\alpha V^\gamma + \hat{h}_{\alpha\gamma} \nabla_\beta V^\gamma \\
= \hat{g}^{\mu\nu} \left( \frac{\partial^2 \hat{h}_{\alpha\beta}}{\partial y^\mu \partial y^\nu} - \frac{\partial \Gamma^\sigma_{\alpha\mu}}{\partial y^\nu} \hat{h}_{\sigma\beta} - \frac{\partial \Gamma^\sigma_{\beta\mu}}{\partial y^\nu} \hat{h}_{\sigma\alpha} \right) \\
- \hat{g}^{\mu\nu} \left( - \frac{\partial \Gamma^\sigma_{\alpha\mu}}{\partial y^\nu} + \frac{\partial \Gamma^\sigma_{\mu\alpha}}{\partial y^\nu} \right) \hat{h}_{\sigma\beta} - \hat{g}^{\mu\nu} \left( - \frac{\partial \Gamma^\sigma_{\beta\mu}}{\partial y^\nu} + \frac{\partial \Gamma^\sigma_{\mu\beta}}{\partial y^\nu} \right) \hat{h}_{\sigma\alpha} \\
+ \hat{g}^{\mu\nu} \frac{\partial \Gamma^\gamma_{\mu\nu}}{\partial y^\alpha} \hat{h}_{\gamma\beta} + \hat{g}^{\mu\nu} \frac{\partial \Gamma^\gamma_{\mu\nu}}{\partial y^\beta} \hat{h}_{\gamma\alpha} + \text{(lower order terms)} \\
= \hat{g}^{\mu\nu} \frac{\partial^2 \hat{h}_{\alpha\beta}}{\partial y^\mu \partial y^\nu} + \text{(lower order terms)}.
\end{align*}
\]

Hence
\[
\begin{align*}
\frac{\partial \hat{g}_{\alpha\beta}}{\partial t}(y,t) &= \hat{g}^{\mu\nu} \frac{\partial^2 \hat{g}_{\alpha\beta}}{\partial y^\mu \partial y^\nu} + \text{(lower order terms)} \\
\frac{\partial \hat{h}_{\alpha\beta}}{\partial t}(y,t) &= \hat{g}^{\mu\nu} \frac{\partial^2 \hat{h}_{\alpha\beta}}{\partial y^\mu \partial y^\nu} + \text{(lower order terms)}
\end{align*}
\]

and we know (2.4) is a strictly parabolic system. By theory of strictly parabolic equations, for any initial data (2.4) exists a smooth short time solution.

So we can recover the solution \((g, h)\) for the original evolution equations from the solution \((\hat{g}, \hat{h})\) as following. Let \((N^n, s_{\alpha\beta}) = (M^n, g_{\alpha\beta}(\cdot, 0))\) and since
\[
V^\alpha = g^{\beta\gamma}(\Gamma^\alpha_{\beta\gamma}(\hat{g}) - \hat{\Gamma}^\alpha_{\beta\gamma}(s)) = -(\triangle F \circ F^{-1})^\alpha,
\]
thus
\[
\frac{\partial F}{\partial t} = -V \circ F.
\]

Now once having \(\hat{g}_{\alpha\beta}\), we know \(V\) and we can solve (2.7) which is just a system of ordinary differential equations on the domain \(M\). Hence \((g, h)\) can be recovered as the pull-back \(g = F^* \hat{g}\) and \(h = F^* \hat{h}\).

Now we claim the solutions of (1.1) with given smooth initial conditions on a compact manifold are unique. For suppose \((g_1, \hat{h}_1)\) and \((g_2, \hat{h}_2)\) are two solutions which agree at \(t = 0\). We can solve the coupled harmonic map flow (2.3) for maps \(F_1\) and \(F_2\) with the metrics \(g_1\) and \(g_2\) on \(M\) into the same target \(N\) with the same fixed \(s\), and starting at the same initial data. Then we have two solutions \(\hat{g}_1\) and \(\hat{g}_2\) on \(N\) with the same initial metric. By the standard uniqueness result for strictly parabolic equations, we have \((\hat{g}_1, \hat{h}_1) = (\hat{g}_2, \hat{h}_2)\). Hence by (2.6) the corresponding vector fields \(V_1 = V_2\). Then the solutions of
the two ODE systems
\[
\frac{\partial F_1}{\partial t} = -V_1 \circ F_1 \quad \text{and} \quad \frac{\partial F_2}{\partial t} = -V_2 \circ F_2
\]
with the same initial values must coincide, and hence two solutions of (1.1)
\[
(g_1, h_1) = F^*(\hat{g}_1, \hat{h}_1) \quad \text{and} \quad (g_2, h_2) = F^*(\hat{g}_2, \hat{h}_2)
\]
must agree.

3. Preserving Gauss-Codazzi Equations

In this section, we will show that the Gauss-Codazzi equations are preserved under (1.1). Let \( G_{ijkl} = R_{ijkl} - (h_i h_j - h_i h_j) \) and \( C_{ijk} = \nabla_i h_{jk} - \nabla_j h_{ik} \).

**Proposition 3.1.** If the tensor \( h_{ij} \) satisfies Gauss’s equation and Codazzi’s equation
\[
\begin{align*}
R_{ijkl} - (h_i h_j - h_i h_j) &= 0 \\
\nabla_i h_{jk} - \nabla_j h_{ik} &= 0
\end{align*}
\]
at time \( t = 0 \), then it remains so for \( t > 0 \).

**Proof.** By direct calculations, we have
\[
\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left\{ \nabla_j \left( \frac{\partial}{\partial t} g_{il} \right) + \nabla_i \left( \frac{\partial}{\partial t} g_{jl} \right) - \nabla_l \left( \frac{\partial}{\partial t} g_{ij} \right) \right\}
\]
\[
\frac{\partial}{\partial t} R_{ijkl} = \nabla_i \left( \frac{\partial}{\partial t} \Gamma^k_{jl} \right) - \nabla_j \left( \frac{\partial}{\partial t} \Gamma^k_{il} \right)
\]
\[
\frac{\partial}{\partial t} R_{ijkl} = g_{hk} \frac{\partial}{\partial t} R_{ijkl} + \frac{\partial g_{hk}}{\partial t} R_{ijkl}.
\]

With these identities we get
\[
\frac{\partial}{\partial t} R_{ijkl} = \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} - \nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik} - \nabla_i \nabla_k (h_{jm} h_{nl} g^{mn}) + \nabla_j \nabla_l (h_{jm} h_{nk} g^{mn})
\]
\[
+ \nabla_j \nabla_k (h_{im} h_{nl} g^{mn}) - \nabla_j \nabla_l (h_{im} h_{nk} g^{mn}) - R_{ijks} (R_{tl} - h_{tm} h_{nl} g^{mn}) g^{st} + R_{ijks} (R_{lk} - h_{lm} h_{nk} g^{mn}) g^{st}
\]
and the following identity
\[
\Delta R_{ijkl} = -2 (B_{ijkl} - B_{ijlk} - B_{ikjl} + B_{ikjl})
\]
\[
+ \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} - \nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik} + R_{mijkl} R_{nij} g^{mn}
\]
where \( B_{ijkl} = R_{mij} R_{nklt} g^{mn} g^{st} \).

Then we obtain
\[
\left( \frac{\partial}{\partial t} - \Delta \right) R_{ijkl} - 2(B_{ijkl} - B_{ijlk} - B_{ilkj} + B_{ikjl}) = 0
\]
\[
= -R_{ijks} (R_{kl} - h_{tn} h_{nt} g^{mn}) g^{st} - R_{ijkl} (R_{tk} - h_{tn} h_{nk} g^{mn}) g^{st}
- R_{sjkl} (R_{ti} - h_{tn} h_{ni} g^{mn}) g^{st} - R_{ijkl} (R_{ti} - h_{tn} h_{nj} g^{mn}) g^{st}
- \nabla_i \nabla_k (h_{jm} h_{nt} g^{mn}) + \nabla_i \nabla_l (h_{jm} h_{nk} g^{mn})
+ \nabla_j \nabla_k (h_{im} h_{nt} g^{mn}) - \nabla_j \nabla_l (h_{im} h_{nk} g^{mn})
\]
(3.1)

To simplify the evolution equations, we will use a moving frame trick. More precisely, let us pick an abstract vector bundle \( V \) over \( M \) isomorphic to the tangent bundle \( TM \). Choose an orthonormal frame \( F_a = F_a^{ij} \frac{\partial}{\partial x^i}, a = 1, \ldots, n \) of \( V \) at \( t = 0 \), then evolve \( F_i^a \) by the equation
\[
\frac{\partial}{\partial t} F_i^a = g^{ij} (R_{jk} - h_{jm} h_{nk} g^{mn}) F_k^a.
\]
Then the frame \( F = \{ F_1, \ldots, F_a, \ldots, F_n \} \) will remain orthonormal for all time. In the following we will use indices \( a, b, \ldots \) on a tensor to denote its components in the evolving orthonormal frame. In this frame we have the following:
\[
\left( \frac{\partial}{\partial t} - \Delta \right) R_{abcd} - 2(B_{abcd} - B_{abdc} - B_{adcb} + B_{acbd}) = 0
\]
\[
= -R_{sbcd} h_{tm} h_{na} g^{mn} g^{st} - R_{ascd} h_{tm} h_{nb} g^{mn} g^{st}
- \nabla_a \nabla_c (h_{bm} h_{nd} g^{mn}) + \nabla_a \nabla_d (h_{bm} h_{nc} g^{mn})
+ \nabla_b \nabla_c (h_{am} h_{nd} g^{mn}) - \nabla_b \nabla_d (h_{am} h_{nc} g^{mn})
\]
(3.2)

and
\[
\left( \frac{\partial}{\partial t} - \Delta \right) h_{ab} = -|A|^2 h_{ab}.
\]
(3.3)

By calculations, we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) \{ R_{abcd} - (h_{ad} h_{bc} - h_{ac} h_{bd}) \} = 0
\]
\[
= 2(B_{abcd} - B_{abdc} - B_{adcb} + B_{acbd})
- R_{sbcd} h_{tm} h_{na} g^{mn} g^{st} - R_{ascd} h_{tm} h_{nb} g^{mn} g^{st}
- \nabla_a \nabla_c (h_{bm} h_{nd} g^{mn}) + \nabla_a \nabla_d (h_{bm} h_{nc} g^{mn})
+ \nabla_b \nabla_c (h_{am} h_{nd} g^{mn}) - \nabla_b \nabla_d (h_{am} h_{nc} g^{mn})
+ 2|A|^2 (h_{ad} h_{bc} - h_{ac} h_{bd})
+ 2(\nabla_m h_{ad} \nabla_n h_{bc} - \nabla_m h_{ac} \nabla_n h_{bd}) g^{mn}.
\]
(3.4)

Then we want to replace \( B_{abcd} \) by
\[
\bar{B}_{abcd} = \{ R_{mabs} - (h_{ms} h_{ab} - h_{mb} h_{as}) \} \{ R_{mcds} - (h_{ms} h_{cd} - h_{md} h_{cs}) \} g^{mn} g^{st}
\]
(6)
and replace terms including $\nabla h, \nabla\nabla h$ by $C$ and $\nabla C$ respectively. That is

$$B_{abcd} - B_{adbc} - B_{adcb} + B_{acbd} = \tilde{B}_{abcd} - \tilde{B}_{adbc} - \tilde{B}_{adcb} + \tilde{B}_{acbd}$$

$$- R_{abcd} h_{dn} h_{tc} g^{mn} g^{st} - R_{mdca} h_{bn} h_{ta} g^{mn} g^{st} + R_{mbdc} h_{cn} h_{td} g^{mn} g^{st} - R_{macc} h_{dn} h_{tb} g^{mn} g^{st} - R_{abcd} h_{nt} h_{cb} g^{mn} g^{st} + R_{mdca} h_{bn} h_{ta} g^{mn} g^{st} - R_{macc} h_{dn} h_{tb} g^{mn} g^{st} + R_{mbdc} h_{cn} h_{td} g^{mn} g^{st} - R_{macc} h_{dn} h_{tc} g^{mn} g^{st} + h_{am} h_{bs} h_{cn} h_{dt} g^{mn} g^{st} + h_{am} h_{bs} h_{dn} h_{ct} g^{mn} g^{st} + h_{ad} h_{bc} |A|^2 - h_{am} h_{ds} h_{nt} h_{tc} g^{mn} g^{st} - h_{bm} h_{cs} h_{nt} h_{ad} g^{mn} g^{st} - h_{ac} h_{bd} |A|^2 + h_{am} h_{cs} h_{nt} h_{bd} g^{mn} g^{st} + h_{bm} h_{ds} h_{nt} h_{ac} g^{mn} g^{st}$$

and

$$\nabla_a \nabla_c (h_{bm} h_{nd} g^{mn}) + \nabla_a \nabla_d (h_{bn} h_{nc} g^{mn}) + \nabla_b \nabla_c (h_{am} h_{nd} g^{mn}) - \nabla_b \nabla_d (h_{am} h_{nc} g^{mn}) + 2(\nabla_m h_{ad} \nabla_n h_{bc} g^{mn} - \nabla_m h_{ac} \nabla_n h_{bd} g^{mn})$$

$$= - \nabla_c (\nabla_a h_{bm} - \nabla_b h_{am}) h_{nd} g^{mn} - \nabla_a (\nabla_c h_{dm} - \nabla_d h_{cm}) h_{nb} g^{mn} + \nabla_d (\nabla_a h_{bm} - \nabla_b h_{am}) h_{nc} g^{mn} - \nabla_b (\nabla_c h_{dm} - \nabla_d h_{cm}) h_{na} g^{mn} - (\nabla_a h_{bn} - \nabla_b h_{an}) (\nabla_c h_{dn} - \nabla_d h_{cn}) g^{mn} - (\nabla_a h_{dm} - \nabla_m h_{ad}) \nabla_c h_{bn} g^{mn} - (\nabla_d h_{am} - \nabla_m h_{ad}) \nabla_b h_{cn} g^{mn} + (\nabla_a h_{cm} - \nabla_m h_{ac}) \nabla_d h_{bn} g^{mn} + (\nabla_c h_{am} - \nabla_m h_{ac}) \nabla_b h_{dn} g^{mn} + (\nabla_m h_{bc} - \nabla_b h_{mc}) \nabla_n h_{ad} g^{mn} - (\nabla_m h_{bd} - \nabla_d h_{mb}) \nabla_n h_{ac} g^{mn} - (\nabla_m h_{bd} - \nabla_b h_{md}) \nabla_n h_{ac} g^{mn} - (\nabla_m h_{bd} - \nabla_b h_{md}) \nabla_n h_{ac} g^{mn} - R_{acbm} h_{ns} h_{td} g^{mn} g^{st} - R_{acms} h_{nd} h_{tb} g^{mn} g^{st} + R_{bcam} h_{ns} h_{td} g^{mn} g^{st} + R_{bcms} h_{nd} h_{ta} g^{mn} g^{st} + R_{adbm} h_{ns} h_{tc} g^{mn} g^{st} + R_{adms} h_{nc} h_{tb} g^{mn} g^{st} - R_{bdam} h_{ns} h_{tc} g^{mn} g^{st} - R_{bdms} h_{nc} h_{ta} g^{mn} g^{st}.$$ 

Let us denote curvature tensor by $Rm$ and denote any tensor product of two tensors $S$ and $T$ by $S * T$ when we do not need the precise expression. Therefore, if we replace terms including $Rm * h * h$ by term $G * h * h$, with (3.4)(3.5)(3.6) and by some calculation we obtain

$$\frac{\partial}{\partial t} - \Delta) G = G * G + G * h * h + \nabla C * h + C * \nabla h + C * C,$$
where $G_{ijkl} = R_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl})$ and $C_{ijk} = \nabla_i h_{jk} - \nabla_j h_{ik}$.

Since we have

$$\frac{\partial}{\partial t} \nabla_i h_{jk} = \nabla_i \left( \frac{\partial}{\partial t} h_{jk} \right) - \left( \frac{\partial}{\partial t} \Gamma^l_{ij} \right) h_{lk} - \left( \frac{\partial}{\partial t} \Gamma^l_{ik} \right) h_{lj}$$

$$= \nabla_i (\Delta h_{jk} - R_{jmn} h_{ikg} h_{mn} - R_{km} \nabla_l h_{njg} + 2h_{jmn} h_{lsk} g_{mn} g_{st} - |A|^2 h_{jk})$$

$$- \left( \frac{\partial}{\partial t} \Gamma^l_{ij} \right) h_{lk} + \nabla_i R_{km} h_{njg} + \nabla_i R_{km} h_{njg} - \nabla_m R_{ijk} h_{njg}$$

$$- \nabla_i h_{km} h_{ns} h_{ljg} + \nabla_i h_{ms} h_{nk} h_{ljg} + \nabla_i h_{im} h_{ns} h_{ljg} + \nabla_i h_{im} h_{ns} h_{ljg}$$

$$= \nabla_i (\Delta h_{jk} - R_{jmn} h_{ikg} h_{mn} + 2(R_{mijs} \nabla_n h_{tk} + R_{niks} \nabla_n h_{tij}) g_{mn} g_{st}$$

$$+ \nabla_j R_{km} h_{njg} - \nabla_m R_{ijk} h_{njg} + \nabla_k R_{im} h_{njg} - \nabla_{m} R_{ik} h_{njg}. $$

So we get

$$\left( \frac{\partial}{\partial t} - \Delta \right) \nabla_i h_{jk} + \left( \frac{\partial}{\partial t} \Gamma^l_{ij} \right) h_{lk} = - R_{jmn} \nabla_i h_{njg} + 2h_{jmn} h_{lsk} g_{mn} g_{st} - |A|^2 h_{jk}$$

$$- \nabla_i R_{km} h_{njg} - \nabla_j R_{km} h_{njg}$$

(3.8)

Then in the moving frame we obtain

$$\left( \frac{\partial}{\partial t} - \Delta \right) \nabla_a h_{bc} + |A|^2 \nabla_a h_{bc} + \left( \frac{\partial}{\partial t} \Gamma^l_{ij} \right) h_{lk} F_a^i F_b^j F_c^k$$

$$= - \nabla_a h_{cm} h_{ns} h_{tb} g_{mn} g_{st} - \nabla_a h_{mb} h_{ns} h_{tc} g_{mn} g_{st}$$

$$- \nabla_m h_{bc} h_{ns} h_{ta} g_{mn} g_{st} + 2\nabla_a h_{cm} h_{ns} h_{tc} g_{mn} g_{st}$$

$$+ 2\nabla_a h_{cm} h_{ns} h_{tb} g_{mn} g_{st} + 2\nabla_a h_{ms} h_{nb} h_{tc} g_{mn} g_{st}$$

$$- \nabla_a h_{ms} h_{nb} h_{tc} g_{mn} g_{st} - \nabla_a h_{cm} h_{ns} h_{tb} g_{mn} g_{st}$$

$$+ \nabla_m h_{as} h_{mb} h_{tc} g_{mn} g_{st} + \nabla_m h_{as} h_{nb} h_{tc} g_{mn} g_{st}$$

$$- 2R_{mabc} \nabla_n h_{tc} g_{mn} g_{st} - 2R_{macs} \nabla_n h_{tb} g_{mn} g_{st}.$$
Combing (3.7) (3.10), we obtain
\[
\frac{\partial}{\partial t} - \triangle (|G|^2 + |C|^2) \\
\leq C_1(|G|^2 + |C|^2) - 2|\nabla G|^2 - 2|\nabla C|^2 \\
+ \langle G, G \ast G + G \ast h \ast h + \nabla C \ast h + C \ast \nabla h + C \ast C \rangle \\
+ \langle C, -|A|^2 C + C \ast h \ast h + C \ast Rm + G \ast \nabla h \rangle \\
\leq C_2(|G|^2 + |C|^2)
\]
where we use Cauchy-Schwarz inequality, and for \(0 \leq t < \delta\) we have bounded \(|Rm|, |A|, |\nabla h|\). Thus, by the standard maximum principle
\[
\frac{d}{dt}(|G|^2 + |C|^2)_{\text{max}} \leq C_2(|G|^2 + |C|^2)_{\text{max}},
\]
we get
\[
(|G|^2 + |C|^2)_{\text{max}}(t) \leq e^{C_2t}(|G|^2 + |C|^2)_{\text{max}}(0).
\]
Since \((|G|^2 + |C|^2)_{\text{max}}(0) = 0\), the Gauss-Codazzi equations are preserved as long as the solution exists. \(\square\)

In the following we will still call \(h_{ij}(x, t)\) the second fundamental form and its trace \(H\) the mean curvature.

4. EVOLUTION OF METRIC AND CURVATURE

Using Gauss-Codazzi equations, we rewrite our evolution equations in the following

**Proposition 4.1.**

(4.1a) \(\frac{\partial}{\partial t} g_{ij} = 2H h_{ij}\)

(4.1b) \(\left(\frac{\partial}{\partial t} - \triangle\right) h_{ij} = 2H h_{im} h_{nj} g^{mn} - |A|^2 h_{ij}\)

(4.1c) \(\left(\frac{\partial}{\partial t} - \triangle\right) H = -H |A|^2\)

(4.1d) \(\left(\frac{\partial}{\partial t} - \triangle\right) |A|^2 = -2|\nabla A|^2 - 2|A|^4\).

Since \(h_{ij}\) is positive at \(t = 0\) and \(M\) is compact, there are some \(\varepsilon > 0\) and \(\beta > 0\), such that \(\beta H g_{ij} \geq h_{ij} \geq \varepsilon H g_{ij}\) at \(t = 0\) holds on \(M\). We want to show that inequality remains true as long as the solution of our evolution equation (1.1) exists. For this purpose we need the following maximum principle for tensor on manifolds, which is proved in [1].

Let \(u^k\) be a vector field and let \(M_{ij}\) and \(N_{ij}\) be symmetric tensors on a compact manifold \(M\) which may all depend on time \(t\). Assume that \(N_{ij} = p(M_{ij}, g_{ij})\) is a polynomial in \(M_{ij}\) formed by contracting products of \(M_{ij}\) with itself using the metric. Furthermore, let this polynomial...
satisfy a null-eigenvector condition, i.e. for any null-eigenvector \( X \) of \( M_{ij} \) we have \( N_{ij}X^iX^j \geq 0 \). Then we have

**Theorem 4.2** (Hamilton). Suppose that on \( 0 \leq t < T \) the evolution equation

\[
\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij}
\]

holds, where \( N_{ij} = p(M_{ij}, g_{ij}) \) satisfies the null-eigenvector condition above. If \( M_{ij} \geq 0 \) at \( t = 0 \), then it remains so on \( 0 \leq t < T \).

An immediate consequence is

**Proposition 4.3.** If \( \varepsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij} \), and \( H > 0 \) at \( t = 0 \), then these remain so as long as the solution of (1.1) exists.

**Proof.** First, by using maximum principle on

\[
\left( \frac{\partial}{\partial t} - \Delta \right) H = -H|A|^2,
\]

we know \( H > 0 \) as long as the solution of (1.1) exists.

Then we consider

\[
M_{ij} = h_{ij} - \varepsilon H g_{ij}
\]

\[
\frac{\partial M_{ij}}{\partial t} = \frac{\partial h_{ij}}{\partial t} - \varepsilon \frac{\partial H}{\partial t} g_{ij} - \varepsilon H \frac{\partial g_{ij}}{\partial t}
\]

\[
= \Delta h_{ij} + 2H h_{im} h_{nj} g^{mn} - |A|^2 h_{ij}
\]

\[
- \varepsilon(\Delta H - |A|^2 H) g_{ij} - \varepsilon H(2H h_{ij})
\]

\[
= \Delta M_{ij} + 2H h_{im} h_{nj} g^{mn}
\]

\[
- |A|^2 (h_{ij} - \varepsilon H g_{ij}) - 2\varepsilon H^2 h_{ij}
\]

For any null vector \( v^i \) of \( M_{ij} \), we have

\[
[2H h_{im} h_{nj} g^{mn} - |A|^2 (h_{ij} - \varepsilon H g_{ij}) - 2\varepsilon H^2 h_{ij}] v^j
\]

\[
= 2H h_{im} g^{mn} (\varepsilon H v_n) - 2\varepsilon H^2 (\varepsilon H v_i)
\]

\[
= 2H(\varepsilon H v_i) \varepsilon H - 2\varepsilon H^2 (\varepsilon H v_i)
\]

\[
= 0
\]

Thus, \( \varepsilon H g_{ij} \leq h_{ij} \) follows from theorem 4.2 Then \( h_{ij} \leq \beta H g_{ij} \) follows in the same way. \( \square \)

Finally, we state the higher derivative estimate in the following proposition.

**Proposition 4.4.** There exist constants \( C_m, m = 1, 2, \cdots \), such that if the second fundamental form of a complete solution to our evolution equation is bounded by

\[
|A| \leq M
\]


up to time \( t \) with \( 0 < t \leq 1/M \), then the covariant derivative of the second fundamental form is bounded by
\[
|∇A| \leq C_1 M/\sqrt{t}
\]
and the \( m \)th covariant derivative of the second fundamental form is bounded by
\[
|∇^mA| \leq C_m M/t^m.
\]
Here the norms are taken with respect to the evolving metric.

Proof. By direct calculation, for any \( m \) we have an equation
\[
\left( \frac{∂}{∂t} - \triangle \right) |∇^mA|^2 = -2|∇^{m+1}A|^2 + \sum_{i+j+k=m} \nabla^i A \ast \nabla^j A \ast \nabla^k A \ast \nabla^mA.
\]
So we can follow the same way using a somewhat standard Bernstein estimate in PDEs to get our theorem (see [4] for Ricci flow). \( \square \)

5. **Monotonicity formula and Long time behaviors**

First, by positivity of \( h_{ij} \) we have
\[
H^2/n \leq |A|^2 < H^2.
\]
Then from (4.1c) we get
\[
-H^3 < \left( \frac{∂}{∂t} - \triangle \right) H \leq -\frac{H^3}{n}.
\]
Thus by maximum principle we obtain
\[
\frac{1}{\sqrt{2t + \frac{1}{H^2_{\min}(0)}}} < H(t) \leq \frac{1}{\sqrt{\frac{2t}{n} + \frac{1}{H^2_{\max}(0)}}}.
\]
With applying maximum principle on (4.1d) again, we have
\[
|A|^2(t) \leq \frac{1}{2t + \frac{1}{|A|^2_{\max}(0)}}.
\]
Since
\[
\frac{1}{2nt + \frac{n}{H^2_{\min}(0)}} < H^2(t)/n \leq |A|^2(t),
\]
we get
\[
\frac{1}{2nt + \frac{n}{H^2_{\min}(0)}} < |A|^2(t) \leq \frac{1}{2t + \frac{1}{|A|^2_{\max}(0)}}.
\]
In particular, (5.2) implies
\[
|A| \to 0 \quad as \quad t \to +\infty.
\]
Combining with our derivatives estimate (Proposition 4.4) we know the solution of our evolution equation (1.1) exists for all the time.

We need the following monotonicity formula to understand the long time behaviors of the solution to (1.1).
Proposition 5.1. If \((g_{ij}(t), h_{ij}(t))\) is the solution of (1.1), then we have the formula
\[
\frac{\partial}{\partial t} \int_M H^n d\mu = -n(n-1) \int_M \frac{|\nabla H|^2}{H^2} H^n d\mu - n \int_M |h_{ij} - \frac{1}{n} H g_{ij}|^2 H^n d\mu.
\]

Proof. It follows from the evolution equations of Proposition 4.1 and direct calculation. \(\square\)

From proposition 5.1 we know
\begin{equation}
0 < \int_M H^n d\mu < C
\end{equation}
for all \(t \in [0, +\infty)\).

This implies
\[
\begin{align*}
&\int_0^\infty \int_M \frac{|\nabla H|^2}{H^2} H^n d\mu < \infty \\
&\int_0^\infty \int_M |h_{ij} - \frac{1}{n} H g_{ij}|^2 H^n d\mu < \infty.
\end{align*}
\]

In particular, there is a sequence \(t_k \to +\infty\) such that
\begin{equation}
t_k \int_M \frac{|\nabla H|^2}{H^2} H^n d\mu \to 0 \quad \text{as} \quad k \to \infty
\end{equation}
and
\begin{equation}
t_k \int_M |h_{ij} - \frac{1}{n} H g_{ij}|^2 H^n d\mu \to 0 \quad \text{as} \quad k \to \infty.
\end{equation}

Denote by
\[
\epsilon_k = \frac{1}{|A|_{max}(t_k)}.
\]

We parabolically scale the solution and shift the time \(t_k\) to the origin 0,
\[
\begin{align*}
g_{ij}^k(\cdot, \tilde{t}) &= \epsilon_k^{-2} g_{ij}(\cdot, t_k + \epsilon_k^2 \tilde{t}), \\
h_{ij}^k(\cdot, \tilde{t}) &= \epsilon_k^{-1} h_{ij}(\cdot, t_k + \epsilon_k^2 \tilde{t}),
\end{align*}
\]
where \(\tilde{t} \in [-t_k/\epsilon_k^2, +\infty)\).

We can check that \((\tilde{g}_{ij}^k(\cdot, \tilde{t}), \tilde{h}_{ij}^k(\cdot, \tilde{t}))\) is still a solution to (1.1).

Since
\[
|\tilde{A}^k(\cdot, \tilde{t})|^2 = \frac{|A(\cdot, t_k + \epsilon_k^2 \tilde{t})|^2}{|A|_{max}^2(t_k)},
\]
and (5.2), it follows that
\begin{equation}
\frac{1}{C_1} < |\tilde{A}^k(\cdot, \tilde{t})|^2 < C_1 \quad \text{for} \quad \tilde{t} \in [-t_k/2\epsilon_k^2, 0],
\end{equation}
where the constant \(C_1\) is independent of \(k\).

By our derivatives estimate (Proposition 4.4), the uniform bound of the second fundamental form \(|\tilde{A}^k(\cdot, \tilde{t})|\) implies the uniform bound on
all the derivatives of the second fundamental form at \( \tilde{t} = 0 \) for all \( k \).

By Gauss equation we have uniform bound of the curvature and all the derivatives of the curvature at \( \tilde{t} = 0 \) for all \( k \).

By (5.3) we know

\[
\int_M (\tilde{H}^k(\cdot, 0))^n d\tilde{\mu}_{0} < C_2.
\]

Combining with (5.1) it follows

\[(5.7) \quad \text{Vol}(M, \tilde{g}^k_{ij}(\cdot, 0)) < C_3.\]

On the other hand, by Proposition 4.3, (5.2) and Gauss equation we have

\[(5.8) \quad 0 > -\frac{1}{C_4} \geq \sec(M, \tilde{g}^k_{ij}(\cdot, 0)) > -1.\]

With (5.8) and (5.7), we can get the uniform upper bound on their diameters and uniform lower bound on their volumes by using the following theorem.

**Theorem 5.2 (Gromov\[8\]).** Let \( M \) be an \( n \)-dimensional closed Riemannian manifold of negative curvature and \( \text{Sec}(M) \geq -1 \). If \( n \geq 8 \), then \( \text{Vol}(M) \geq C(1 + d(M)) \) and for \( n = 4, 5, 6, 7 \), \( \text{Vol}(M) \geq C(1 + d^{1/3}(M)) \), where we denote volume of \( M \) by \( \text{Vol}(M) \), diameter of \( M \) by \( d(M) \) and the constant \( C > 0 \) depends only on \( n \).

Now we know \((M, \tilde{g}^k_{ij}(\cdot, 0), \tilde{h}^k_{ij}(\cdot, 0))\) is a sequence which have uniform bound on sectional curvature, uniform upper bound on diameters and uniform lower bound on volumes. Using cheeger’s Lemma in [6] we apply the same argument of Hamilton’s compactness theorem in [2] to extract a convergent subsequence \((M, \tilde{g}^k_{ij}(\cdot, 0), \tilde{h}^k_{ij}(\cdot, 0))\) from \((M, \tilde{g}^k_{ij}(\cdot, 0), \tilde{h}^k_{ij}(\cdot, 0))\). More precisely, there is a triple \((M_\infty, \tilde{g}^\infty_{ij}(\cdot, 0), \tilde{h}^\infty_{ij}(\cdot, 0))\) and a sequence of diffeomorphisms \(f_l : M_\infty \rightarrow M_l\). Notice that \(M_\infty\) is diffeomorphism to \(M\), since we have uniform diameter bound. And the pull-back metrics \((f^*_l \tilde{g}^k_{ij}(\cdot, 0))\) and the pull-back second fundamental forms \((f^*_l \tilde{h}^k_{ij}(\cdot, 0))\) converge in \(C^\infty\) topology to \((\tilde{g}^\infty_{ij}(\cdot, 0), \tilde{h}^\infty_{ij}(\cdot, 0))\).

From (5.4) and (5.5) we obtain

\[
t_k \epsilon_{k_l}^{-2} \int_M \frac{\nabla \tilde{H}^k_{ij}(\cdot, 0)^2}{\tilde{H}^k_{ij}(\cdot, 0)^2}(\tilde{H}^k_{ij}(\cdot, 0))^{n}(0) d\tilde{\mu}_{t_{k_l}} \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty
\]

and

\[
t_k \epsilon_{k_l}^{-2} \int_M \frac{\nabla \tilde{H}^k_{ij}(\cdot, 0)^2}{\tilde{H}^k_{ij}(\cdot, 0)^2}(\tilde{H}^k_{ij}(\cdot, 0))^{n}(0) d\tilde{\mu}_{t_{k_l}} \rightarrow 0 \quad \text{as} \quad l \rightarrow \infty.
\]

Here the norm is taken with respect to \(\tilde{g}^k_{ij}(0)\).

Notice that \(t_k \epsilon_{k_l}^{-2}\) and \(|\tilde{H}^k_{ij}(0)|\) and \(\text{Vol}(M, \tilde{g}^k_{ij}(\cdot, 0))\) have uniform lower
bound, we have
\( |\tilde{\nabla} \tilde{H}^k_l|(0) \to 0 \) as \( l \to \infty \)
and
\( |\tilde{h}^{k_l}_{ij} - \frac{1}{n} \tilde{H}^k_l \tilde{g}^l_{ij}(0)| \to 0 \) as \( l \to \infty \).
Therefore, by Gauss equation, we know the sectional curvature of \((M_\infty, \tilde{g}_\infty^{ij}(\cdot, 0), \tilde{h}_\infty^{ij}(\cdot, 0))\) is a constant \(\equiv -1/n\).

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