BMO-ESTIMATION AND ALMOST EVERYWHERE EXPONENTIAL SUMMABILITY OF QUADRATIC PARTIAL SUMS OF DOUBLE FOURIER SERIES

U. GOGINAVA, L. GOGOLADZE AND G. KARAGULYAN

Abstract. It is proved a BMO-estimation for quadratic partial sums of two-dimensional Fourier series from which it is derived an almost everywhere exponential summability of quadratic partial sums of double Fourier series.

1. Introduction

Let \( T := [-\pi, \pi] = \mathbb{R}/2\pi \) and \( \mathbb{R} := (-\infty, \infty) \). We denote by \( L_1 (T) \) the class of all measurable functions \( f \) on \( \mathbb{R} \) that are 2\( \pi \)-periodic and satisfy
\[
\| f \|_1 := \int_T |f| < \infty.
\]
The Fourier series of the function \( f \in L_1 (T) \) with respect to the trigonometric system is the series
\[
\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx},
\]
where
\[
\hat{f}(n) := \frac{1}{2\pi} \int_T f(x) e^{-inx} dx
\]
are the Fourier coefficients of \( f \).

Denote by \( S_n(x,f) \) the partial sums of the Fourier series of \( f \) and let
\[
\sigma_n(x,f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(x,f)
\]
be the \((C,1)\) means of \( f \). Fejér [1] proved that \( \sigma_n(f) \) converges to \( f \) uniformly for any 2\( \pi \)-periodic continuous function. Lebesgue in [15] established

2010 Mathematics Subject Classification: 40F05, 42B08

Key words and phrases: Fourier series, Strong Summability, Quadratic sums.

The research of U. Goginava was supported by Shota Rustaveli National Science Foundation grant no.31/48 (Operators in some function spaces and their applications in Fourier analysis)
almost everywhere convergence of \((C,1)\) means if \(f \in L_1(\mathbb{T})\). The strong summability problem, i.e. the convergence of the strong means

\[
\frac{1}{n+1} \sum_{k=0}^{n} |S_k(x,f) - f(x)|^p, \quad x \in \mathbb{T}, \quad p > 0,
\]

was first considered by Hardy and Littlewood in [11]. They showed that for any \(f \in L_r(\mathbb{T})\) \((1 < r < \infty)\) the strong means tend to 0 a.e., if \(n \to \infty\). The trigonometric Fourier series of \(f \in L_1(\mathbb{T})\) is said to be \((H,p)\)-summable at \(x \in \mathbb{T}\), if the values (2) converge to 0 as \(n \to \infty\). The \((H,p)\)-summability problem in \(L_1(\mathbb{T})\) has been investigated by Marcinkiewicz [17] for \(p = 2\), and later by Zygmund [26] for the general case \(1 \leq p < \infty\). K. I. Oskolkov in [19] proved the following

**Theorem A.** Let \(f \in L_1(\mathbb{T})\) and let \(\Phi\) be a continuous positive convex function on \([0, +\infty)\) with \(\Phi(0) = 0\) and

\[
\ln \Phi(t) = O \left(\frac{t}{\ln \ln t}\right) \quad (t \to \infty).
\]

Then for almost all \(x\)

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi(|S_k(x,f) - f(x)|) = 0.
\]

It was noted in [19] that V. Totik announced the conjecture that (4) holds almost everywhere for any \(f \in L_1(\mathbb{T})\), provided

\[
\ln \Phi(t) = O(t) \quad (t \to \infty).
\]

In [20] V.Rodin proved

**Theorem B.** Let \(f \in L_1(\mathbb{T})\). Then for any \(A > 0\)

\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} (\exp (A|S_k(x,f) - f(x)|) - 1) = 0
\]

for a.e. \(x \in \mathbb{T}\).

G. Karagulyan [12] proved that the following is true.

**Theorem C.** Suppose that a continuous increasing function \(\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0\), satisfies the condition

\[
\limsup_{t \to +\infty} \frac{\log \Phi(t)}{t} = \infty.
\]

Then there exists a function \(f \in L_1(\mathbb{T})\) for which the relation

\[
\limsup_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} \Phi(|S_k(x,f)|) = \infty
\]

holds everywhere on \(\mathbb{T}\).
In fact, Rodin in [20] has obtained a BMO estimate for the partial sums of Fourier series and his theorem stated above is obtained from that estimate by using John-Nirenberg theorem. Recall the definition of BMO $[0, 1]$ space. It is the Banach space of functions $f \in L_1[0, 1]$ with the norm

$$
\|f\|_{\text{BMO}} = \mathfrak{R}(f) + \left| \int_0^1 f(t) dt \right|
$$

where

$$
\mathfrak{R}(f) = \sup_I (|f - f_I|)_I, f_I = \frac{1}{|I|} \int_I f(t) dt
$$

and the supremum is taken over all intervals $I \subset [0, 1]$ ([H], chap. 6). Let \( \{\xi_n : n = 0, 1, 2, \ldots\} \) be an arbitrary sequence of numbers. Taking $\delta^n_k = [k/(n+1), (k+1)/(n+1)]$, we define

$$
\text{BMO} \left[\xi_n\right] = \sup_{0 \leq n < \infty} \left\| \sum_{k=0}^n \xi_k \mathbb{I}_{\delta^n_k}(t) \right\|_{\text{BMO}}
$$

where $\mathbb{I}_{\delta^n_k}(t)$ is the characteristic function of $\delta^n_k$. Notice that the expressions

(6)  \( \text{BMO} \left[ \widetilde{S}_n(x, f) \right], \quad \text{BMO} \left[ S_n(x, f) \right], \quad f \in L_1(\mathbb{T}), x \in \mathbb{T} \)

define a sublinear operators, where $\widetilde{S}_n(x, f)$ is the conjugate partial sum. The following theorem is proved by Rodin in [20].

**Theorem D.** The operators (6) are of weak type $(1, 1)$, i.e. the inequalities

(7)  \( |\{x \in \mathbb{T} : \text{BMO} \left[ S_n(x, f) \right] > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{T}} |f(t)| dt \)

and

(8)  \( |\{x \in \mathbb{T} : \text{BMO} \left[ \widetilde{S}_n(x, f) \right] > \lambda\}| \leq \frac{c}{\lambda} \int_{\mathbb{T}} |f(t)| dt \)

hold for any $f \in L_1(\mathbb{T})$.

In this paper we study the question of exponential summability of quadratic partial sums of double Fourier series. Let $f \in L_1(\mathbb{T}^2)$, be a function with Fourier series

(9)  \( \sum_{m,n=-\infty}^{\infty} \hat{f}(m, n) e^{i(mx+ny)} \),

where

$$
\hat{f}(m, n) = \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} f(x, y) e^{-i(mx+ny)} dxdy
$$
are the Fourier coefficients of the function $f$. The rectangular partial sums of $f$ are defined as follows:

$$S_{MN}(x, y, f) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} \hat{f}(m, n) e^{i(mx+ny)}.$$  

We denote by $L \log L(T^2)$ the class of measurable functions $f$, with

$$\iint_{T^2} |f| \log^+ |f| < \infty,$$

where $\log^+ u := \log(1+u)$ for $u > 0$. For quadratic partial sums of two-dimensional trigonometric Fourier series Marcinkiewicz [18] has proved, that if $f \in L \log L(T^2)$, then

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} (S_{kk}(x, y, f) - f(x, y)) = 0$$

for a. e. $(x, y) \in T^2$. L. Zhizhiashvili [24] improved this result showing that class $L \log L(T^2)$ can be replaced by $L_1(T^2)$.

From a result of S. Konyagin [14] it follows that for every $\varepsilon > 0$ there exists a function $f \in L \log^{1-\varepsilon}(T^2)$ such that

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} |S_{kk}(x, y, f) - f(x, y)| \neq 0 \quad \text{for a. e.} \quad (x, y) \in T^2.$$  

The main result of the present paper is the following.

**Theorem 1.** If $f \in L \log L(T^2)$, then

$$\left| \{(x, y) \in T^2 : \text{BMO} \left[ S_{nn}(f, x, y) \right] > \lambda \} \right| \leq \frac{c}{\lambda} \left( 1 + \iint_{T^2} |f| \log^+ |f| \right)$$

for any $\lambda > 0$, where $c$ is an absolute positive constant.

The following theorem shows that the quadratic sums of two-dimensional Fourier series of a function $f \in L \log L(T^2)$ are almost everywhere exponentially summable to the function $f$. It will be obtained from the previous theorem by using John-Nirenberg theorem.

**Theorem 2.** Suppose that $f \in L \log L(T^2)$. Then for any $A > 0$

$$\lim_{m \to \infty} \frac{1}{m+1} \sum_{n=0}^{m} (\exp(A |S_{nn}(x, y, f) - f(x, y)|) - 1) = 0$$

for a. e. $(x, y) \in T^2$.

According to a Lemma of L. D. Gogoladze [9], this theorem can be formulated in more general settings.
Theorem 3. Let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing function, satisfying the conditions

\[
\lim_{u \to 0} \psi(u) = \psi(0) = 0, \quad \limsup_{u \to \infty} \frac{\log \psi(u)}{u} < \infty.
\]

Then for any \( f \in L \log L(T^2) \) we have

\[
\lim_{m \to \infty} \frac{1}{m+1} \sum_{n=0}^{m} \psi(|S_{nn}(x,y,f) - f(x,y)|) = 0
\]

almost everywhere on \( T^2 \).

The results on Marcinkiewicz type strong summation for the Fourier series have been investigated in [2, 3, 10, 5, 7, 6, 8, 16, 23, 27, 28, 24].

2. Notations and Lemmas

The relation \( a \lesssim b \) bellow stands for \( a \leq c \cdot b \), where \( c \) is an absolute constant. The conjugate function of a given \( f \in L_1(T) \) is defined by

\[
\tilde{f}(x) = \text{p.v.} \frac{1}{\pi} \int_T \frac{f(x+t)}{2\tan(t/2)} dt = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\epsilon < |t| < \pi} \frac{f(x+t)}{2\tan(t/2)} dt.
\]

According to Kolmogorov’s and Zygmund’s inequalities (see [26], chap. 7), we have

\[
|\{|x \in T : |\tilde{f}(x)| > \lambda\}| \lesssim \frac{\|f\|_{L_1(T)}}{\lambda},
\]

(12)

\[
\int_T |\tilde{f}(x)| dx \lesssim 1 + \int_T |f(x)| \log^+ |f(x)| dx.
\]

(13)

It will be used two simple properties of \( \text{BMO} \) norm below. First one says, if \( \xi_n = c, \ n = 1, 2, \ldots \), then \( \text{BMO} [\xi_n] = |c| \). The second one is, the bound \( \text{BMO} [\xi_n] \leq 3 \sup_n |\xi_n| \).

We shall consider the operators

\[
U_n(x,f) = \text{p.v.} \frac{1}{\pi} \int_T \frac{\cos nt}{2\tan(t/2)} f(x+t) dt.
\]

The following lemma is an immediate consequence of Theorem D.

Lemma 1. The inequality

\[
|\{|x \in T : \text{BMO} [U_n(x,f)] > \lambda\}| \lesssim \frac{\|f\|_{L_1(T)}}{\lambda}
\]

holds for any \( f \in L_1(T) \).
Proof. For the conjugate Dirichlet kernel we have

\[
\tilde{D}_n(t) = \frac{\cos(t/2) - \cos(n + 1/2)t}{2 \sin(t/2)} = \frac{1}{2 \tan(t/2)} + \frac{\sin nt}{2} - \frac{\cos nt}{2 \tan(t/2)}
\]

and we get

\[
\tilde{S}_n(x,f) = \frac{1}{\pi} \int_{\mathbb{T}} \tilde{D}_n(t) f(x+t) dt = \tilde{f}(x) + \frac{1}{2 \pi} \int_{\mathbb{T}} f(x+t) \sin nt dt - U_n(x,f).
\]

Thus, applying simple properties of BMO norm, we obtain

\[
\text{BMO}[U_n(x,f)] \leq |\tilde{f}(x)| + \frac{1}{2 \pi} \int_{\mathbb{T}} |f(t)| dt + \text{BMO}[\tilde{S}_n(x,f)].
\]

Applying the bound (12) and Theorem D, the last inequality completes the proof of lemma.

We consider the square partial sums

\[
S_{nn}(x,y,f) = \frac{1}{\pi^2} \int_{\mathbb{T}^2} \frac{\sin(n + 1/2)t \sin(n + 1/2)s}{4 \sin(t/2) \sin(s/2)} f(x+t,y+s) dtds
\]

and their modification, defined by

\[
S^*_{nn}(x,y,f) = \frac{1}{\pi^2} \int_{\mathbb{T}^2} \frac{\sin nt \sin ns}{4 \tan(t/2) \tan(s/2)} f(x+t,y+s) dtds.
\]

Lemma 2. If \( f \in L \log L(\mathbb{T}^2) \), then

\[
\int_{\mathbb{T}^2} \sup_n |S_{nn}(x,y,f) - S^*_{nn}(x,y,f)| \, dxdy \lesssim 1 + \int_{\mathbb{T}^2} |f| \log^+ |f|.
\]

Proof. Substituting the expression for Dirichlet kernel

\[
D_n(t) = \frac{\sin(n + 1/2)t}{2 \sin(t/2)} = \frac{\sin nt}{2 \tan(t/2)} + \frac{\cos nt}{2}
\]
in (15), we get

\[
S_{nn}(x, y, f) - S_{nn}^r(x, y, f) = \frac{1}{\pi^2} \int_{T^2} \int \left( \sin nt \cdot \cos ns \cdot \frac{f(x + t, y + s)}{4 \tan(t/2)} \right) dt ds
+ \frac{1}{\pi^2} \int_{T^2} \int \left( \cos nt \cdot \sin ns \cdot \frac{f(x + t, y + s)}{4 \tan(s/2)} \right) dt ds
+ \frac{1}{4\pi^2} \int_{T^2} \int \left( \cos nt \cdot \cos ns \cdot f(x + t, y + s) \right) dt ds
= S_{nn}^{(1)}(x, y, f) + S_{nn}^{(2)}(x, y, f) + S_{nn}^{(3)}(x, y, f).
\]

It is clear, that

\[
|S_{nn}^{(3)}(x, y, f)| \lesssim \|f\|_{L^1(T^2)} \lesssim 1 + \int_{T^2} |f| \log^+ |f|.
\]

Everywhere below the notation

\[
p.v. \int_{T^2} \int f(t, s) dt ds
\]

stands for either

\[
p.v. \int_{T} \left( p.v. \int_{T} f(t, s) dt \right) ds, \ \text{or} \ \ p.v. \int_{T} \left( p.v. \int_{T} f(t, s) ds \right) dt
\]

and in each cases we have equality of these two iterated integrals. To observe that we will need just the fact that \( f \in L \log L(T^2) \) implies \( \tilde{f} \in L_1(T) \). Hence,
making simple transformations and then changing the variables, we get

\begin{equation}
S_{mn}^{(1)} (x, y, f) \tag{17}
= p.v. \frac{1}{2\pi^2} \int_{T^2} \frac{\sin n(t + s)}{2 \tan(t/2)} f(x + t, y + s) \, ds dt
+ p.v. \frac{1}{2\pi^2} \int_{T^2} \frac{\sin n(t - s)}{2 \tan(t/2)} f(x + t, y + s) \, ds dt
= p.v. \frac{1}{2\pi^2} \int_{T^2} \frac{\sin nu \cdot f(x + v, y + u - v)}{2 \tan(v/2)} \, dv du \quad (u = t + s, v = t)
+ p.v. \frac{1}{2\pi^2} \int_{T^2} \frac{\sin nu \cdot f(x + v, y + v - u)}{2 \tan(v/2)} \, dv du \quad (u = t - s, v = t)
= \frac{1}{2\pi} \int_{T} \sin nu \left( p.v. \frac{1}{\pi} \int_{T} \frac{f(x + v, y + u - v)}{2 \tan(v/2)} \, dv \right) du
+ \frac{1}{2\pi} \int_{T} \sin nu \left( p.v. \frac{1}{\pi} \int_{T} \frac{f(x + v, y + v - u)}{2 \tan(v/2)} \, dv \right) du.
\end{equation}

Observe, that the functions

\begin{align*}
F_1(x, y, u) &= p.v. \frac{1}{\pi} \int_{T} \frac{f(x + v, y + u - v)}{2 \tan(v/2)} \, dv \\
F_2(x, y, u) &= p.v. \frac{1}{\pi} \int_{T} \frac{f(x + v, y + v - u)}{2 \tan(v/2)} \, dv
\end{align*}

are defined for almost all triples \((x, y, u)\). Moreover, we shall prove that

\begin{equation}
\iiint_{T^3} |F_i(x, y, u)| \, dx dy du \lesssim 1 + \iint_{T^2} |f| \log^+ |f|, \quad i = 1, 2. \tag{18}
\end{equation}

Consider the function \(h(t, s, u) := f(t + s, t + u - s)\). Substituting \(x = t + s\) and \(y = t - s\) in the expression of \(F_1\), we get

\[F_1(t + s, t - s, u) = p.v. \frac{1}{\pi} \int_{T} \frac{h(t, s + v, u)}{2 \tan(v/2)} \, dv.\]

Thus, first using the inequality \([13]\) for variable \(s\), then integrating by \(t\) and \(u\), we obtain

\[\iiint_{T^3} |F_1(t + s, t - s, u)| \, ds dt du \lesssim 1 + \iint_{T^3} |h(t, s, u)| \log^+ |h(t, s, u)| \, dt ds du.\]
After the change of variables $t = (x + y)/2$ and $s = (x - y)/2$ in the integrals, we get (18) in the case $i = 1$. The case $i = 2$ may be proved similarly. On the other hand, from (17) it follows that

$$|S_{nn}^{(1)}(x, y, f)| \leq \frac{1}{2\pi} \int_{T} |F_1(x, y, u)| du + \frac{1}{2\pi} \int_{T} |F_2(x, y, u)| du.$$ 

Combining this inequality with (18), we obtain

$$\int_{T^2} \sup_{n} |S_{nn}^{(1)}(x, y, f)| dx dy \lesssim 1 + \int_{T^2} |f| \log^+ |f|.$$ 

Similarly we can get the same bound for $S_{nn}^{(2)}(x, y, f)$, which together with (16) completes the proof of lemma.

\[ \square \]

3. Proof of Theorems

Proof of Theorem 1. From Lemma 2 we obtain

$$|S_{nn}(x, y, f) - S_{nn}^*(x, y, f)| \leq \phi(x, y), \quad n = 1, 2, \ldots,$$

where the function $\phi(x, y) \geq 0$ satisfies the bound

$$\int_{T^2} \phi(x, y) dx dy \lesssim 1 + \int_{T^2} |f| \log^+ |f|.$$ 

Thus we get

$$BMO[S_{nn}(x, y, f)] \leq BMO[S_{nn}^*(x, y, f)] + 3\phi(x, y).$$

Hence, the theorem will be proved, if we obtain BMO weak $(1, 1)$ estimate for modified partial sums. We have

$$S_{nn}^*(x, y, f)$$

\[ \begin{align*}
= & \frac{1}{2\pi^2} \int_{T^2} f \cos n(t - s) \cdot f(x + t, y + s) \frac{4 \tan (t/2) \tan (s/2)}{dtds} \\
= & \frac{1}{2\pi^2} \int_{T^2} f \cos n(t + s) \cdot f(x + t, y + s) \frac{4 \tan (t/2) \tan (s/2)}{dtds} \\
= & \frac{1}{2\pi^2} \int_{T^2} f \cos nu \cdot f(x + u + v, y + v) \frac{4 \tan ((u + v)/2) \tan (v/2)}{dudv} \quad (u = t - s, v = s) \\
= & \frac{1}{2\pi^2} \int_{T^2} f \cos nu \cdot f(x + u + v, y - v) \frac{4 \tan ((u + v)/2) \tan (v/2)}{dudv} \quad (u = t + s, v = -s) \\
= & I_n(x, y, f) - J_n(x, y, f).
\end{align*} \]
Using a simple and an important identity

\begin{equation}
\frac{1}{\tan((u + v)/2) \tan (v/2)} = \frac{1}{\tan(u/2) \tan(v/2)} - \frac{1}{\tan(u/2) \tan((u + v)/2)} - 1,
\end{equation}

we obtain

\[
I_n(x, y, f) = \text{p.v.} \frac{1}{2\pi^2} \iint_{T^2} \cos nu \cdot f(x + u + v, y + v) \frac{dudv}{4 \tan(u/2) \tan(v/2)}
\]

\[
- \text{p.v.} \frac{1}{2\pi^2} \iint_{T^2} \cos nu \cdot f(x + u + v, y + v) \frac{dudv}{4 \tan((u + v)/2) \tan((u + v)/2)}
\]

\[
- \frac{1}{2\pi^2} \iint_{T^2} f(x + t, y + s) dtds
\]

\[
= \text{p.v.} \frac{1}{2\pi} \int_T \frac{\cos nu}{2 \tan(u/2)} \left( \text{p.v.} \frac{1}{\pi} \int_T \frac{f(x + u + v, y + v)}{2 \tan(v/2)} dv \right) du
\]

\[
- \text{p.v.} \frac{1}{2\pi} \int_T \frac{\cos nu}{2 \tan(u/2)} \left( \text{p.v.} \frac{1}{\pi} \int_T \frac{f(x + u + v, y + v)}{2 \tan((u + v)/2)} dv \right) du
\]

\[
- \frac{1}{2\pi^2} \iint_{T^2} f(t, s) dtds = I_n^{(1)}(x, y, f) - I_n^{(2)}(x, y, f) - I_n^{(0)},
\]

where

\begin{equation}
|I_n^{(0)}| = \frac{1}{2\pi^2} \iint_{T^2} |f(t, s)| dtds \lesssim 1 + \iint_{T^2} |f(x, y)| \log^+ |f(x, y)| dxdy.
\end{equation}

Observe that

\[
I_n^{(1)}(x, y, f) = \frac{1}{2} \cdot U_n(x, A(\cdot, y))
\]

where

\[
A(x, y) = \text{p.v.} \frac{1}{\pi} \int_T \frac{f(x + v, y + v)}{2 \tan(v/2)} dv.
\]

Denoting \(g(t, s) := f(t + s, t - s)\) and substituting \(x = t + s\) and \(y = t - s\) we get

\[
A(t + s, t - s) = \text{p.v.} \frac{1}{\pi} \int_T \frac{g(t + v, s)}{2 \tan(v/2)} dv.
\]
Using the inequality (13) for variable $t$ and then integrating by $s$, we obtain
\[ \int \int_{\mathbb{T}^2} |A(t+s,t-s)| dsdt \lesssim 1 + \int \int_{\mathbb{T}^2} |g(t,s)| \log^+ |g(t,s)| dtds. \]

After the changing back of variables $t = (x+y)/2$ and $s = (x-y)/2$ we get
\[ \int \int_{\mathbb{T}^2} |A(x,y)| dxdy \lesssim 1 + \int \int_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dxdy. \]

Hence, applying the Lemma 1, we conclude
\[ |\{(x,y) \in \mathbb{T}^2 : \text{BMO } [I_n^{(1)}(x,y,f)] > \lambda\}| \lesssim \frac{1}{\lambda} \left( 1 + \int \int_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dxdy \right). \]

After the changing of variable $u+v \rightarrow \nu$ in the inner integral of the expression of $I_n^{(2)}(x,y,f)$ we get
\[ I_n^{(2)}(x,y,f) = \text{p.v.} \frac{1}{2\pi} \int_{\mathbb{T}} \cos nu \left( \text{p.v.} \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+\nu, y+\nu-u)}{2 \tan(\nu/2)} d\nu \right) du, \]

and then analogously we can prove that
\[ |\{(x,y) \in \mathbb{T}^2 : \text{BMO } [I_n^{(2)}(x,y,f)] > \lambda\}| \lesssim \frac{1}{\lambda} \left( 1 + \int \int_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dxdy \right). \]

Hence, using (21), (23) and (24), we obtain
\[ |\{(x,y) \in \mathbb{T}^2 : \text{BMO } [I_n(x,y,f)] > \lambda\}| \lesssim \frac{1}{\lambda} \left( 1 + \int \int_{\mathbb{T}^2} |f(x,y)| \log^+ |f(x,y)| dxdy \right). \]

Using the absolutely same process we may get the analogous estimate for $J_n(x,y,f)$ and therefore for $S_{nn}^\ast(x,y,f)$. The theorem is proved. $\square$

Let $X$ be either $[0,1]$ or $\mathbb{T}^2$ and $L_M = L_M(X)$ is the Orlicz space of functions on $X$, generated by Young function $M$, i.e. $M$ is convex continuous even function such that $M(0) = 0$ and
\[ \lim_{t \to 0^+} \frac{M(t)}{t} = \lim_{t \to \infty} \frac{t}{M(t)} = 0. \]
It is well known that $L_M$ is a Banach space with respect to Luxemburg norm

$$ \| f \|_{(M)} := \inf \left\{ \lambda : \lambda > 0, \int_X M \left( \frac{|f|}{\lambda} \right) \leq 1 \right\} < \infty. $$

We will need some basic properties of Orlicz spaces (see \[13\]).

1) According to a theorem from (\[13\], chap. 2, theorem 9.5) we have

$$ \| f \|_{(M)} \leq 1 \Rightarrow \int_X M (|f|) \leq \| f \|_{(M)}, $$

2) From this fact we may deduce, that

$$ 0,5 \left( 1 + \int_X M (|f|) \right) \leq \| f \|_{(M)} \leq 1 + \int_X M (|f|) $$

provided $\| f \|_{(M)} = 1$.

3) From the definition of norm $\| \cdot \|_{(M)}$ immediately follows that $|f(x)| \leq |g(x)|$ implies $\| f \|_{(M)} \leq \| g \|_{(M)}$. Besides, for any measurable set $E$ we have

$$ \| I_E \|_{(M)} = o(1) \text{ as } |E| \to 0 \ (\[13\], (9.23)). $$

4) If $M$ satisfies $\Delta_2$-condition, that is

$$ M(2t) \leq cM(t), t > t_0, $$

and $X = \mathbb{T}^2$, then the set of two variable trigonometric polynomials on $\mathbb{T}^2$ is dense in $L_M$ (\[13\], §10).

5) From (25) it follows that for any sequence of functions $f_n$ the condition $\| f_n \|_{(M)} \to 0$ implies $\int_X M (|f_n|) \to 0$.

**Proof of Theorem 2.** We will deal with two $M$-functions

$$ \Phi(t) = t \log^+ t, $$

$$ \Psi(t) = \exp t - 1. $$

We consider two Orlicz spaces $L_\Phi = L_\Phi(\mathbb{T}^2)$ and $L_\Psi = L_\Psi(0,1)$. Combining (26) with Theorem 1 we may obtain

$$ |\{(x,y) \in \mathbb{T}^2 : \text{BMO} [S_{nn}(x,y,f)] > \lambda \} | \lesssim \frac{\| f \|_{(\Phi)}}{\lambda}. $$

Indeed, at first we deduce the case when $\| f \|_{(\Phi)} = 1$, then, using a linearity principle, we get the inequality in the general case.

The inequality

$$ \| f \|_{(\Psi)} \lesssim \| f \|_{\text{BMO}} $$
proved in [20]. It is an immediate consequence of the John-Nirenberg theorem. Denote

$$Bf(x, y) = \sup_{0 \leq n < \infty} \left\| \sum_{k=0}^{n} S_{kk}(x, y, f) \delta_k^f(t) \right\|_{(\Psi)}.$$  

Notice, that by the definition we have

$$\text{BMO} \left[ S_{nn}(f, x, y) \right] = \sup_{0 \leq n < \infty} \left\| \sum_{k=0}^{n} S_{kk}(x, y, f) \delta_k^f(t) \right\|_{\text{BMO}}.$$  

So, taking into account (27) and (28) we obtain

$$|\{(x, y) \in T^2 : Bf(x, y) > \lambda \}| \lesssim \frac{\|f\|_{(\Phi)}}{\lambda}.$$  

On the other hand we have

$$\frac{1}{n+1} \sum_{k=0}^{n} (\exp A|S_{kk}(x, y, f) - f(x, y)| - 1)$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} \Psi(A|S_{kk}(x, y, f) - f(x, y)|)$$

$$= \int_{0}^{1} \Psi \left( A \sum_{k=0}^{n} |S_{kk}(x, y, f) - f(x, y)| \delta_k^f(t) \right) dt.$$  

Thus, according the property 5) of Orlicz spaces, to prove the theorem it is enough to prove that

$$\left\| \sum_{k=0}^{n} (S_{kk}(x, y, f) - f(x, y)) \delta_k^f(t) \right\|_{(\Psi)} \to 0,$$

almost everywhere on $T^2$ as $n \to \infty$, for any $f \in L\Phi$. It is easy to observe, that (31) holds if $f$ is a real trigonometric polynomial in two variables. Indeed, if $P(x, y)$ is a polynomial of degree $m$, then we have

$$S_{kk}(x, y, P) - P(x, y) \equiv 0, \quad k \geq m.$$  

Therefore, if $n \geq m$, then we get

$$\left| \sum_{k=0}^{n} (S_{kk}(x, y, P) - P(x, y)) \delta_k^f(t) \right| \leq C \cdot \|P\|_{[0, m/(n+1)]}(t),$$

where $C$ is a constant, depending on $P$. Then, applying the property 3) of Orlicz spaces, we conclude that (31) holds if $f = P$. To prove the general
case, we consider the set

$$(32) \quad G_\lambda = \{(x, y) \in T^2 : \limsup_{n \to \infty} \left\| \sum_{k=0}^n (S_{kk}(x, y, f) - f(x, y)) I_{\delta_k}(t) \right\| > \lambda \}. $$

To complete the proof of theorem, it enough to prove that $|G_\lambda| = 0$ if $\lambda > 0$. It is easy to check that $\Phi(t)$ satisfies the $\Delta_2$-condition. Therefore, according the property 4), we may chose a polynomial $P(x, y)$ such that $\|f - P\|_{(\Phi)} < \varepsilon$.

Using the definition of $(\Phi)$-norm, we get

$$\int_{T^2} \Phi \left( \left| \frac{f - P}{\varepsilon} \right| \right) < 1. $$

From Chebishev’s inequality, one can easily deduce

$$|\{(x, y) \in T^2 : |f(x, y) - P(x, y)| > \lambda\}| \leq \frac{1}{\Phi(\lambda/\varepsilon)}, \quad \lambda > 0. $$

Thus, using (30) for any $\lambda > 0$ we get

$$|G_\lambda| = |\{(x, y) \in T^2 : \limsup_{n \to \infty} \left\| \sum_{k=0}^n (S_{kk}(x, y, f - P) - f(x, y) + P(x, y)) I_{\delta_k}(dt) \right\| > \lambda \}| \leq \frac{\|f - P\|_{(\Phi)}}{\lambda} + \frac{1}{\Phi(\lambda/\varepsilon)} \leq \frac{\varepsilon}{\lambda} + \frac{1}{\Phi(\lambda/\varepsilon)}. $$

Since $\varepsilon > 0$ may be taken sufficiently small, we conclude $|G_\lambda| = 0$ if $\lambda > 0$. 

**Acknowledgement.** The authors would like to thank the referees for helpful suggestions.

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U. GOGINAVA, DEPARTMENT OF MATHEMATICS, FACULTY OF EXACT AND NATURAL SCIENCES, IV. JAVAKHISHVILI TBILISI STATE UNIVERSITY, CHAVCHAVADZE STR. 1, TBILISI 0128, GEORGIA

E-mail address: zazagoginava@gmail.com
L. Gogoladze, Department of Mathematics, Faculty of Exact and Natural Sciences, Iv. Javakhishvili Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia
E-mail address: lgogoladze1@hotmail.com

G. Karagulyan, Institute of Mathematics of Armenian National Academy of Science, Bughramian Ave. 24/5, 375019, Yerevan, Armenia
E-mail address: g.karagulyan@yahoo.com