Discovering exact, gauge-invariant, local energy-momentum conservation laws for the electromagnetic gyrokinetic system by high-order field theory on heterogeneous manifolds

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Abstract

Gyrokinetic theory is arguably the most important tool for numerical studies of transport physics in magnetized plasmas. However, exact local energy-momentum conservation laws for the electromagnetic gyrokinetic system have not been found despite continuous effort. Without such local conservation laws, energy and momentum can be instantaneously transported across spacetime, which is unphysical and casts doubt on the validity of numerical simulations based on the gyrokinetic theory. The standard Noether procedure for deriving conservation laws from corresponding symmetries does not apply to gyrokinetic systems because the gyrocenters and electromagnetic field reside on different manifolds. To overcome this difficulty, we develop a high-order field theory on heterogeneous manifolds for classical particle-field systems and apply it to derive exact, local conservation laws, in particular the energy-momentum conservation laws, for the electromagnetic gyrokinetic system. A weak Euler-Lagrange equation is established to replace the standard Euler-Lagrange equation for the particles. It is discovered that an induced weak Euler-Lagrange current enters the local conservation laws. And it is the new physics captured by the high-order field theory on heterogeneous manifolds. A recently developed gauge-symmetrization method for high-order electromagnetic field theories using the electromagnetic displacement-potential tensor is applied to render the derived energy-momentum conservation laws electromagnetic gauge-invariant.
I. INTRODUCTION

Gyrokinetic theory, gradually emerged since the 1960s [1–6], has become an indispensable tool for analytical and numerical studies [7–14] of instabilities and transport in magnetized plasmas, with applications to magnetic fusion and astrophysics. Modern gyrokinetic theory has been developed to systematically derive more accurate governing equations. It began with Littlejohn’s treatment of the guiding center dynamics [15–18] using the Lie perturbation method [19–22]. Dubin et al. [23] applied the Hamiltonian Lie perturbation method to derive the gyrokinetic equations for low frequency drift wave perturbations, followed by Hahm et al. [24–26] and Brizard [27, 28]. Qin et al. [29–35] extended the gyrokinetic model to treat high-frequency dynamics [32] and MHD perturbations [29–31]. Sugama introduced the field theoretical approach for the gyrokinetic models [36], which has been widely adopted since [34, 35, 37–39]. Present research on gyrokinetic theories focuses on endowing the models with more physical structures and conservation properties using modern geometric method [33–35], with the goal of achieving improved accuracy [40–43] and fidelity for describing magnetized plasmas. For example, the Euler-Poincare reduction procedure [44], Hamiltonian structure [45, 46] and explicit gauge independence [47] have been constructed for gyrokinetic systems. These studies closely couple with the investigation of structure-preserving geometric algorithms of the guiding center dynamics [48–59] for gyrokinetic simulations with long term accuracy and fidelity.

One conservation property of fundamental importance for theoretical models in physics is the energy-momentum conservation. The gyrokinetic theory is no exception. For tokamak physics, the exact energy conservation law was used to analysis the energy flux and transport property [60]. The mean flows and radial electric field, crucial for tokamak equilibrium and stability, are determined by the momentum conservation [61, 62]. Exact conservation laws also serve as tests for the accuracy of numerical simulations [63–67].

However, exact local energy-momentum conservation laws for the gyrokinetic system with fully self-consistent time-dependent electromagnetic field are still unknown. It is worthwhile to emphasize that we are searching for local conservation laws instead of the weaker global ones. If a theoretical model does not admit local energy-momentum conservation law, energy and momentum can be instantaneously transported across spacetime, which is unphysical and detrimental for the purpose of studying energy and momentum transport in magnetized
To derive conservation laws, there are two ways to proceed. One can construct conservation laws by taking various moments of the gyrokinetic equation system [23, 24, 27]. This approach is effective for simple systems such as the standard Vlasov-Maxwell (VM) system in the laboratory phase space, where the moments of energy-momentum and forms of conservation can be easily guessed based on physical intuition. However, for more sophisticated systems such as the gyrokinetic systems, it is difficult to know what moments are involved for the exact conservation laws.

A better approach is to start from variational principles, or field theories, and derive conservation laws by identifying first the underpinning symmetries admitted by the Lagrangians of the systems. This is the familiar Noether procedure. Low [68] presented the first variational principle of Vlasov-Maxwell system, where the dynamics of particles is Lagrangian and that of the electromagnetic field is Eulerian. Using Low’s variational approach for the 6D distribution function, Sugama et al. [69] derived flux surface averaged conservation laws of energy and toroidal angular momentum for a toroidally confined plasma satisfying the Vlasov-Poisson-Ampere approximation under the Coulomb gauge.

In principle, such a field theoretical methodology can also be adopted for gyrokinetic systems or the guiding-center drift kinetic system. A thorough review of the existing literature shows that the following work have been done in this regard. i) A local momentum conservation law for the guiding-center drift kinetic system [70] was derived by Sugama et al. using an Eulerian variational formulation through the Euler-Poincare reduction procedure [44, 71]. Using the same procedure, a local energy-momentum conservation law for the guiding-center drift kinetic system was also recently derived by Hirvijoki et al. [71]. ii) Brizard [72] developed another Eulerian variational principle which requires a constrained variation of the distribution function on an 8D phase space. With this formalism, energy and momentum conservation laws for the guiding-center drift kinetic system [39] and the gyrokinetic Vlasov-Poisson system [73] were derived, as well as global energy conservation for the electromagnetic gyrokinetic system [74]. iii) Very recently, Brizard derived a local energy conservation law for the perturbed electromagnetic field and distribution function of the electromagnetic gyrokinetic system when the background field is time-independent [75, 76].

Despite these advances, as mentioned above, exact local energy-momentum conservation
laws for the general gyrokinetic Vlasov-Maxwell system remain elusive. The technical difficulties involved can be viewed from two different angles. For the Eulerian formalism for gyrokinetic models, the Euler-Lagrange equation assumes a different form because the field variations are constrained, and the derivation of conservation laws from symmetries does not follow the standard Noether procedure for unconstrained variations. In particular, the well-established infinitesimal symmetry condition, prolongation and integration by parts in the jet space \([\mathcal{J}]\) cannot be applied without modification to constrained variations. Since constrained variations assume different formats for different applications, there is no established general formulation for the Noether procedure in the case of constrained variations. For Low’s type of variational principles with mixed Lagrangian and Eulerian variations, particles (gyrocenters in this case) and the electromagnetic field reside on different manifolds. The electromagnetic field is defined on spacetime, but the particles are defined on the time axis only. This differs from the standard Noether procedure. These difficulties are not unique to the gyrokinetic theory. They appear in other systems too. For example, if we choose to derive the energy-momentum conservation laws for the Vlasov-Maxwell system or the Vlasov-Poisson system in the laboratory phase space \((x, v)\) from the corresponding spacetime translation symmetry, we would encounter exactly the same difficulties. Admittedly, these difficulties are more prominent for the gyrokinetic system because its Lagrangian depends on high-order derivatives of the field and the phase space coordinates for gyrocenters are non-fibrous \([34, 35]\). For the Vlasov-Maxwell system in the laboratory phase space \((x, v)\), we don’t need to go through the symmetry analysis to derive the energy-momentum conservation, since it can be guessed and proved directly. But to derive exact conservation laws for gyrokinetic systems, symmetry analysis seems to be the only viable approach.

Recently, this difficulty is overcome by the development of an alternative field theory for the classical particle-field system \([78-80]\). This new field theory embraces the fact that different components, i.e., particles and electromagnetic field, reside on heterogeneous manifolds, and a weak Euler-Lagrange equation was derived to replace the standard Euler-Lagrange equation for particles. It was shown that under certain conditions the correspondence between symmetries and conservation laws is still valid, but with a significant modification. The weak Euler-Lagrange equation introduces a new current in the corresponding conservation law. This new current, called weak Euler-Lagrange current, represents the new physics captured by the field theory on heterogeneous manifolds \([80]\).
The field theory on heterogeneous manifolds has been successfully applied to find local conservation laws in the Vlasov-Poisson system and the Vlasov-Darwin system that were previously unknown [78, 80]. In particular, the previous well-known momentum conservation law for the Vlasov-Darwin system written down by Kaufman and Rostler [81] in 1971 without derivation was found to be erroneous, and a correct momentum conservation was systematically derived using the field theory for particle-field system on heterogeneous manifolds [80].

In this paper, we extend the field theory for particle-field system on heterogeneous manifolds to systems with high-order field derivatives in non-canonical phase space coordinates and apply it to systematically derive local conservation laws for the electromagnetic gyrokinetic system from the underpinning spacetime symmetries. In particular, the exact local energy-momentum conservation laws for the electromagnetic gyrokinetic system are derived. For gyrokinetic systems, the Finite-Larmor-Radius (FLR) effect is important, and the Lagrangian density must include derivatives of the field up to certain desired orders. Therefore, extending the field theory on heterogeneous manifolds to systems with high-order field derivatives is a necessary first step. We first extend the theory to include arbitrary high-order field derivatives, and then derive the energy-momentum conservation law for the electromagnetic gyrokinetic system. When the derivatives above the first order are ignored, the Lagrangian density does not contain any derivatives of the electromagnetic field \( E \) and \( B \), and system reduces to the guiding-center drift kinetic system.

Another difference between the present work and previous studies [75, 76] is that we don’t separate the electromagnetic field into perturbed and background parts. The field theory and conservation laws are expressed in terms of the total distribution functions and the 4-potential \( (\varphi (t, \mathbf{x}), \mathbf{A} (t, \mathbf{x})) \). This ensures that the Lagrangian density does not explicitly depend on the spacetime coordinates \( \mathbf{x} \) and \( t \), and always admits exact energy-momentum conservation laws. In previous studies [75, 76], the magnetic field are separated into perturbed and background parts, and conservation laws were derived for the perturbed fields. However, such conservation laws exist only when the background field is symmetric with respect to certain spacetime coordinates. In particular, in the tokamak geometry, the momentum conservation cannot be established in these previous studies because the background magnetic field is inhomogeneous.

In the present study, we also adopt a systematic approach to remove the electro-
magnetic gauge dependence from the electromagnetic gyrokinetic system using a gauge-symmetrization method recently developed for classical charged particle-electromagnetic field theories \[82\]. For field theories involving the electromagnetic field, it is well known that the Energy-Momentum Tensor (EMT) derived by the Noether procedure from the underpinning spacetime translation symmetry is neither gauge invariant (a.k.a. gauge symmetric) nor symmetric with respect to its tensor indices. The standard Belinfante-Rosenfeld method \[83–85\] symmetrizes the EMT using a super-potential associated with the angular momentum but does not necessarily make the EMT gauge invariant for a general field theory. The result reported in Ref. \[82\] shows that a third order tensor called electromagnetic displacement-potential tensor can be constructed to explicitly remove the gauge dependency of the EMT for high-order electromagnetic field theories. This method is applied here to render the exact, local energy-momentum conservation laws derived for the electromagnetic gyrokinetic system gauge invariant.

This paper is organized as follows. In Sec. II we extend the field theory for particle-field systems on heterogeneous manifolds to systems, such as the gyrokinetic system, with high-order field derivatives in non-canonical phase space coordinates. The weak EL equation is developed as necessitated by the fact that classical particles and fields live on different manifolds. Symmetries for the systems and the links between the symmetries and conservation laws are established. In Sec. III the general theory developed is applied to derive the exact, gauge-invariant, local energy-momentum conservation laws induced by spacetime translation symmetries for the electromagnetic gyrokinetic system.

II. HIGH-ORDER FIELD THEORY ON HETEROGENEOUS MANIFOLDS

Before specializing to the electromagnetic gyrokinetic system, we develop a general high-order field theory on heterogeneous manifolds for particle-field systems using noncanonical phase space coordinates. A weak Euler-Lagrange equation is derived. Exact local conservation laws are established from the underpinning symmetries. The weak Euler-Lagrange current in the conservation laws induced by the weak Euler-Lagrange equation is the new physics predicted by the field theory on heterogeneous manifolds.
A. Weak Euler-Lagrangian equation

We start from the action of particle-field systems and revisit the field theory on heterogeneous manifolds developed in Refs. [78–80]. We extend the theory to include high order field derivatives and use noncanonical phase space coordinates \((X_a, U_a)\) for particles. The action of gyrokinetic systems assumes the following form with the field derivatives up to the \(n\)-th order,

\[
A = \sum_a \int L_a \left( t, X_a, \dot{X}_a, U_a, \dot{U}_a; \text{pr}^{(n)} \psi (t, X_a) \right) dt + \int L_F \left( t, x, \text{pr}^{(n)} \psi (t, x) \right) dtd^3x. \tag{1}
\]

In this section, we will work out the field theory for this general form of action without specializing to gyrokinetic models. The subscript \(a\) labels particles, \((X_a(t), U_a(t))\) is the trajectory of the \(a\)-th particle in phase space over the time axis. \(X_a(t)\) takes value in the 3D laboratory space, and \(\psi(t, x)\) is a vector (or 1-form) field defined on spacetime. For gyrokinetic system, \(\psi\) will be the 4-potentials of the electromagnetic field, i.e., \(\psi = (\varphi, A)\). \(L_a\) is Lagrangian of the \(a\)-th particle, including the interaction between the particle and fields. \(L_F\) is the Lagrangian density for the field \(\psi\). Here, \(\text{pr}^{(n)} \psi (t, x)\) as a vector field on the jet space is the prolongation of the field \(\psi (t, x)\) \[77\], which contains \(\psi\) and its derivatives up to the \(n\)-th order, i.e.,

\[
\text{pr}^{(n)} \psi (t, x) := (\psi, \partial_{\mu_1} \psi, \ldots, \partial_{\mu_2} \partial_{\mu_2} \ldots \partial_{\mu_n} \psi), \tag{2}
\]

where \(\partial_{\mu_i} \in \{\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}\), \((i = 1, 2, \ldots, n)\), represents a derivative with respect to one of the spacetime coordinates.

The difference in the domains of the field and particles is clear from Eq. (1). The fields \(\psi\) is defined on the 4D spacetime, whereas each particle’s trajectory as a field is just defined on the 1D time axis. The integral of the Lagrangian density \(L_F\) for the field \(\psi\) is over spacetime, and the integral of Lagrangian \(L_a\) for the \(a\)-th particle is over the time axis only. Because of this fact, Noether’s procedure of deriving conservation laws from symmetries is not applicable without modification to the particle-field system defined by the action \(A\) in Eq. (1).

To overcome this difficulty, we multiply the first part on the right-hand side of Eq. (1) by the identity

\[
\int \delta_a d^3x = 1, \tag{3}
\]
where $\delta_a \equiv \delta(x - X_a(t))$ is Dirac’s $\delta$-function. The action $\mathcal{A}$ in Eq. (1) is then transformed into an integral over spacetime,

$$
\mathcal{A} = \int L dt d^3x, \quad L = \sum_a \mathcal{L}_a + \mathcal{L}_F, \tag{4}
$$

$$
\mathcal{L}_a \left(t, x, X_a, \dot{X}_a, U_a, \dot{U}_a; \text{pr}^{(n)} \psi(t, X_a)\right) = L_a \left(t, X_a, \dot{X}_a, U_a, \dot{U}_a; \text{pr}^{(n)} \psi(t, X_a)\right) \delta_a. \tag{5}
$$

Note that the Lagrangian of the $a$-th particle $L_a$ is transformed to the Lagrangian density $\mathcal{L}_a$ by multiplying $\delta_a$. Obviously, the variation of the action we constructed here will not have any constraints, which will make the variational process easier. We now calculate how the action given by Eq. (4) varies in response to the field variations $\delta X_a$, $\delta U_a$ and $\delta \psi$,

$$
\delta \mathcal{A} = \sum_a \left\{ \int E_{X_a}(\mathcal{L}) d^3x \cdot \delta X_a + \int E_{U_a}(\mathcal{L}) d^3x \cdot \delta U_a \right\} dt + \int E_\psi(\mathcal{L}) \cdot \delta \psi dt d^3x, \tag{6}
$$

where

$$
E_{X_a} \equiv \frac{\partial}{\partial X_a} - \frac{D}{Dt} \frac{\partial}{\partial \dot{X}_a}, \tag{7}
$$

$$
E_{U_a} \equiv \frac{\partial}{\partial U_a} - \frac{D}{Dt} \frac{\partial}{\partial \dot{U}_a}, \tag{8}
$$

$$
E_\psi \equiv \frac{\partial}{\partial \psi} + \sum_{j=1}^n (-1)^j D_{\mu_1} \cdots D_{\mu_j} \frac{\partial}{\partial \mu_1 \cdots \partial \mu_j} \psi, \tag{9}
$$

are Euler operators with respect to $X_a$, $U_a$ and $\psi$, respectively. In Eq. (6), the terms $\delta X_a$ and $\delta U_a$ can be taken out from the space integral because they are fields just defined on the time axis. Applying Hamilton’s principle to Eq. (6), we immediately obtain the equations of motion for particles and fields

$$
E_\psi(\mathcal{L}) = 0, \tag{10}
$$

$$
\int E_{X_a}(\mathcal{L}) d^3x = 0, \tag{11}
$$

$$
\int E_{U_a}(\mathcal{L}) d^3x = 0, \tag{12}
$$

by the arbitrariness of $\delta X_a$, $\delta U_a$ and $\delta \psi$. Equation (10) is the EL equation for fields $\psi$. Equations (11) and (12) are called submanifold Euler-Lagrange equations for $X_a$ and $U_a$ because they are defined only on the time axis after integrating over the spatial dimensions $[78, 80]$. We can easily prove that the submanifold EL equations (11) and (12) are equivalent to the standard EL equations of $L_a$,

$$
E_{X_a}(L_a) = 0, \quad E_{U_a}(L_a) = 0, \tag{13}
$$
by substituting the Lagrangian density (5).

Our next goal is to derive an explicit expression for $E_{U_a}(\mathcal{L})$ and $E_{X_a}(\mathcal{L})$. From the EL equation (13),

$$E_{U_a}(\mathcal{L}) = E_{U_a}(L_a) \delta_a = 0$$

(14)

because $\delta_a$ doesn’t depend on $U_a$. However, $E_{X_a}(\mathcal{L})$ is not zero but a total divergence [78–80],

$$E_{X_a}(\mathcal{L}) = \frac{D}{Dx} \cdot \left( \dot{X}_a \frac{\partial L_a}{\partial \dot{X}_a} - \mathcal{L}_a I \right).$$

(15)

To prove Eq. (15), we calculate

$$E_{X_a}(\mathcal{L}) = \frac{\partial (L_a \delta_a)}{\partial X_a} - \frac{D}{Dt} \frac{\partial (L_a \delta_a)}{\partial \dot{X}_a}$$

$$= \left( \frac{\partial L_a}{\partial X_a} \frac{D}{Dt} \frac{\partial L_a}{\partial X_a} \right) \delta_a + L_a \frac{\partial \delta_a}{\partial X_a} - \frac{\partial L_a}{\partial X_a} \frac{D \delta_a}{Dt}$$

$$= E_{X_a}(L_a) \delta_a - L_a \frac{D \delta_a}{Dt} \frac{\partial L_a}{\partial X_a} + X_a \cdot \frac{D \delta_a}{Dt} \frac{\partial L_a}{\partial X_a}$$

$$= \frac{D}{Dx} \cdot \left( \dot{X}_a \frac{\partial L_a}{\partial \dot{X}_a} \delta_a - L_a \delta_a I \right)$$

$$= \frac{D}{Dx} \cdot \left( \dot{X}_a \frac{\partial L_a}{\partial X_a} - \mathcal{L}_a I \right).$$

We will refer to Eq. (15) as weak Euler-Lagrange equation. The qualifier “weak” here indicates that the spatial integral of $E_{X_a}(\mathcal{L})$, instead of $E_{X_a}(\mathcal{L})$ itself, is zero [78–80]. The weak EL equation plays a crucial role in connecting symmetries and local conservation laws for the field theory on heterogeneous manifolds. The non-vanishing right-hand-side of the weak EL equation (15) will induce a new current in conservation laws [78–80]. This new current is called the weak Euler-Lagrange current, and it is the new physics associated with the field theory on heterogeneous manifolds.

**B. General symmetries and conservation laws**

We now discuss the symmetries and conservation laws. A symmetry of the action $\mathcal{A}$ is a group of transformations,

$$g_e : (t, x, X_a(t), U_a(t), \psi(t, x)) \mapsto \left( \tilde{t}, \tilde{x}, \tilde{X}_a(\tilde{t}), \tilde{U}_a(\tilde{t}), \tilde{\psi}(\tilde{t}, \tilde{x}) \right),$$

(16)
such that
\[
\int L \left( t, x, X_a(t), \dot{X}_a(t), U_a(t), \dot{U}_a(t) ; \text{pr}^{(n)} \psi (t, x) \right) dt dx = 
\int L \left( \tilde{t}, \tilde{x}, \tilde{X}_a(\tilde{t}), \dot{\tilde{X}}_a(\tilde{t}), \tilde{U}_a(\tilde{t}), \dot{\tilde{U}}_a(\tilde{t}) ; \text{pr}^{(n)} \tilde{\psi} (\tilde{t}, \tilde{x}) \right) d\tilde{t} d\tilde{x} \tag{17}
\]
for every subdomain. Here, \( g_\epsilon \) constitutes a continuous group of transformations parameterized by \( \epsilon \). Equation (17) is called symmetry condition. To derive a local conservation law, an infinitesimal version of the symmetry condition is required. For this purpose, we take the derivative of Eq. (17) with respect to \( \epsilon \) at \( \epsilon = 0 \),
\[
\frac{d}{d\epsilon} \bigg|_0 \int L \left( \tilde{t}, \tilde{x}, \tilde{X}_a(\tilde{t}), \dot{\tilde{X}}_a(\tilde{t}), \tilde{U}_a(\tilde{t}), \dot{\tilde{U}}_a(\tilde{t}) ; \text{pr}^{(n)} \tilde{\psi} (\tilde{t}, \tilde{x}) \right) d\tilde{t} d\tilde{x} = 0 \tag{18}
\]
Following the procedures in Ref. [77], the infinitesimal criterion derived from Eq. (18) is
\[
\text{pr}^{(1,n)} \mathbf{v} (\mathcal{L}) + \mathcal{L} \left( \frac{D\xi^t}{Dt} + \frac{D}{Dx} \cdot \mathbf{\xi} \right) = 0, \tag{19}
\]
\[
\mathbf{v} := \frac{d}{d\epsilon} \bigg|_0 g_\epsilon (t, x, X_a, U_a, \psi) = \xi^t \frac{\partial}{\partial t} + \mathbf{\xi} \cdot \frac{\partial}{\partial \mathbf{x}} + \sum_a \mathbf{\theta}_a \cdot \frac{\partial}{\partial X_a} + \sum_a \mathbf{\zeta}_a \cdot \frac{\partial}{\partial U_a} + \mathbf{\phi} \cdot \frac{\partial}{\partial \psi}, \tag{20}
\]
\[
\text{pr}^{(1,n)} \mathbf{v} := \frac{d}{d\epsilon} \bigg|_0 \text{pr}^{(1,n)} g_\epsilon (t, x, X_a, U_a, \psi) = \frac{d}{d\epsilon} \bigg|_0 \left( \tilde{t}, \tilde{x}, \tilde{X}_a, \frac{d\tilde{X}_a}{dt}, \tilde{U}_a, \frac{d\tilde{U}_a}{dt} ; \text{pr}^{(n)} \tilde{\psi} (\tilde{t}, \tilde{x}) \right). \tag{21}
\]
Here, \( \mathbf{v} \) is the infinitesimal generator of the group of transformations and the vector field \( \text{pr}^{(1,n)} \mathbf{v} \) is the prolongation of \( \mathbf{v} \) defined on the jet space, which can be explicitly expressed as
\[
\text{pr}^{(1,n)} \mathbf{v} = \mathbf{v} + \sum_a \mathbf{\theta}_a \cdot \frac{\partial}{\partial X_a} + \sum_a \mathbf{\zeta}_a \cdot \frac{\partial}{\partial U_a} + \sum_{j=1}^{n} \phi_{\mu_1 \cdots \mu_j}^\alpha \frac{\partial}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} \psi^\alpha \right)}, \tag{22}
\]
\[
\mathbf{\theta}_a = \xi^t \tilde{X}_a + \mathbf{q}_a, \quad \mathbf{\zeta}_a = \xi^t \tilde{U}_a + \mathbf{p}_a, \quad \phi_{\mu_1 \cdots \mu_j}^\alpha = \xi^\nu D_{\mu_1} \cdots D_{\mu_j} (D_\nu \psi^\alpha) + D_{\mu_1} \cdots D_{\mu_j} Q^\alpha, \tag{23}
\]
where
\[
\mathbf{q}_a = \mathbf{\theta}_a - \xi^t \tilde{X}_a, \quad \mathbf{p}_a = \mathbf{\zeta}_a - \xi^t \tilde{U}_a, \quad Q^\alpha = \phi^\alpha - \xi^\nu D_\nu \psi^\alpha \tag{24}
\]
are the characteristics of the infinitesimal generator \( \mathbf{v} \). The superscript \( \alpha \) is the index of the fields \( \mathbf{\phi} \) and \( \psi \). The formulations and proofs of Eqs. (22)-(24) can be found in Ref. [77].

Having derived the weak EL Eq. (15) and infinitesimal symmetry criterion (19), we now can establish the conservation law. We cast the infinitesimal criterion (19) into an equivalent
form,
\[
\partial_\nu \left[ \mathcal{L}_\nu + \sum_a \mathcal{P}_a^\nu \delta_a + \mathbb{P}_F^\nu \right] + \frac{D}{Dt} \left[ \sum_a \frac{\partial \mathcal{L}}{\partial \dot{X}_a} \cdot q_a + \sum_a \frac{\partial \mathcal{L}}{\partial U_a} \cdot p_a \right] \\
+ \sum_a \left[ E_{X_a} (\mathcal{L}) \cdot q_a + E_{U_a} (\mathcal{L}) \cdot p_a \right] + E (\mathcal{L}) \cdot Q = 0, \tag{25}
\]

where the 4-vector fields \( \mathcal{P}_a^\nu \) and \( \mathbb{P}_F^\nu \) contain high-order derivatives of the field \( \psi \). They are the boundary terms calculated by integration by parts,
\[
\begin{align*}
\mathcal{P}_a^\nu &= (\mathcal{P}_a^0, \mathcal{P}_a) = \sum_{j=1}^n \mathcal{P}_a^{\nu (j)}, \quad \mathcal{P}_a^{\nu (j)} = \sum_{k=1}^j \mathcal{P}_a^{\nu (j), k}, \\
\mathbb{P}_F^\nu &= (\mathbb{P}_F^0, \mathbb{P}_F) = \sum_{j=1}^n \mathbb{P}_F^{\nu (j)}, \quad \mathbb{P}_F^{\nu (j)} = \sum_{k=1}^j \mathbb{P}_F^{\nu (j), k}.
\end{align*}
\tag{26}
\]

Here, the terms \( \mathcal{P}_a^{\nu (j), k} \) and \( \mathbb{P}_F^{\nu (j), k} \) in Eq. (26) are defined by
\[
\begin{align*}
\mathcal{P}_a^{\nu (j), k} &= Q^\alpha \frac{\partial L_0}{\partial (\partial_\mu \psi^\alpha)}, \quad k = j = 1, \\
\mathbb{P}_F^{\nu (j), k} &= Q^\alpha \frac{\partial L_0}{\partial (\partial_\mu \psi^\alpha)}, \\
\mathcal{P}_a^{\nu (j), k} &= (-1)^{k+1} D_{\mu_{k+1}} \cdots D_{\mu_j} Q^\alpha \left[ \frac{\partial L_0}{\partial (\partial_\mu \partial_{\nu_{k+1}} \cdots \partial_{\nu_j} \psi^\alpha)} \right], \quad 1 < k < j, \\
\mathbb{P}_F^{\nu (j), k} &= (-1)^{k+1} D_{\mu_{k+1}} \cdots D_{\mu_j} Q^\alpha \left[ \frac{\partial L_0}{\partial (\partial_\mu \partial_{\nu_{k+1}} \cdots \partial_{\nu_j} \psi^\alpha)} \right],
\end{align*}
\tag{27}
\]

\[
\begin{align*}
\mathcal{P}_a^{\nu (j), k} &= (-1)^{k+1} D_{\mu_{k+1}} \cdots D_{\mu_j} Q^\alpha \left[ D_{\mu_1} \cdots D_{\mu_{k-1}} \frac{\partial L_0}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_{k-1}} \partial_{\nu_j} \psi^\alpha)} \right], \quad 1 < k < j, \\
\mathbb{P}_F^{\nu (j), k} &= (-1)^{k+1} D_{\mu_{k+1}} \cdots D_{\mu_j} Q^\alpha \left[ D_{\mu_1} \cdots D_{\mu_{k-1}} \frac{\partial L_0}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_{k-1}} \partial_{\nu_j} \psi^\alpha)} \right],
\end{align*}
\tag{28}
\]

\[
\begin{align*}
\mathcal{P}_a^{\nu (j), k} &= (-1)^{k+1} Q^\alpha \left[ D_{\mu_1} \cdots D_{\mu_{k-1}} \frac{\partial L_0}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_{k-1}} \partial_{\nu_j} \psi^\alpha)} \right], \quad 1 < k = j, \\
\mathbb{P}_F^{\nu (j), k} &= (-1)^{k+1} Q^\alpha \left[ D_{\mu_1} \cdots D_{\mu_{k-1}} \frac{\partial L_0}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_{k-1}} \partial_{\nu_j} \psi^\alpha)} \right],
\end{align*}
\tag{29}
\]

The last two terms in Eq. (25) vanish due to the EL equations (10) and (14), while the third term is not zero because of the weak EL equation (15) and induces a new current for system.

If the characteristic \( q_a \) is independent of \( x \), the local conservation law of the symmetry is finally established as
\[
\frac{D}{Dt} \left[ \sum_a \frac{\partial \mathcal{L}}{\partial \dot{X}_a} \cdot q_a + \sum_a \frac{\partial \mathcal{L}}{\partial U_a} \cdot p_a + \mathcal{L}^\xi + \sum_a \mathcal{P}_a^0 \delta_a + \mathbb{P}_F^0 \right] \\
+ \frac{D}{Dt} \left[ \mathcal{L}^\xi + \sum_a \left( \dot{X}_a \frac{\partial \mathcal{L}}{\partial X_a} - L_a I \right) \cdot q_a + \sum_a \mathcal{P}_a \delta_a + \mathbb{P}_F \right] = 0. \tag{31}
\]

Here, the terms \( \dot{X}_a \) and \( \dot{U}_a \) are regarded as functions of \( (X_a(t), U_a(t)) \) through the EL equation (13).
C. Statistical form of the conservation laws

The local conservation law (31) is written in terms of particle’s phase space coordinates \((X_a(t), U_a(t))\) and field \(\psi(t, x)\). To express it in the statistical form in terms of distribution functions of particles and field, we classify the particles into several species by their invariants such as mass and charge. A particle indexed by the subscript \(a\) can be regarded as the \(p\)-th particle of the \(s\)-species, i.e., \(a\) is equivalent to a pair of indices,

\[
a \sim sp.
\]

(32)

For each species, the Klimontovich distribution function is defined to be

\[
F_s(t, x, u) \equiv \sum_p [\delta(x - X_{sp}) \delta(u - U_{sp})].
\]

(33)

Functions \(L_a, q_a\) and \(P^\nu_a\) in Eq. (31) distinguished by the index \(a \sim sp\) are same functions in phase space for the same species. For such a function \(g_a(x, u)\), the label \(a \sim sp\) can be replaced just by \(s\), i.e.,

\[
g_a = g_{sp} = g_s.
\]

(34)

In the conservation law (31), the summations in the form of \(\sum_a g_a(X_a(t), U_a(t)) \delta_a\) can be expressed in terms of the distribution functions \(F_s(t, x, u)\),

\[
\sum_a g_a(X_a(t), U_a(t)) \delta_a = \sum_s \int [F_s(t, x, u) g_s(x, u)] d^3u.
\]

(35)

Using Eq. (35), the conservation law (31) can be equivalently written in the statistical form in terms of the distribution functions \(F_s(t, x, u)\) and field \(\psi(t, x)\) as

\[
\frac{D}{Dt} \left[ \sum_s \int F_s \left( \frac{\partial L_s}{\partial X_s} \cdot q_s + \frac{\partial L_s}{\partial U_s} \cdot p_s + L_s \xi^t + P^0_s \right) d^3u + \mathcal{L}_F \xi^t + P^F \right]
\]

\[
+ \frac{D}{Dx} \cdot \left\{ \sum_s \int \left[ \left( \dot{X}_s \frac{\partial L_s}{\partial X_s} - L_s I \right) \cdot q_s + L_s \xi + P^s \right] d^3u + \mathcal{L}_F \xi + P^F \right\} = 0,
\]

(36)

where \(L_s, q_s, p_s, P^\nu_s, \dot{X}_s, \dot{U}_s\) and \(\partial L_s/\partial \dot{X}_s\) are the functions in phase space, evaluated at \((t, x, u)\).

Note that in Eq. (36), the index for individual particles \(a\) has been absorbed by the Klimontovich distribution function \(F_s(t, x, u)\), which serves as the bridge between particle representation using \((X_a(t), U_a(t))\) and distribution function representation. In Sec. III local conservation laws for the electromagnetic gyrokinetic system will be first established using the particle representation in the form of Eq. (31). They are then transformed to the statistical form in the form of Eq. (36) using this technique.
III. EXACT, GAUGE-INVARIANT, LOCAL ENERGY-MOMENTUM CONSERVATION LAWS FOR THE ELECTROMAGNETIC GYROKINETIC SYSTEM

In this section, we apply the field theory on heterogeneous manifolds for particle-field systems developed in Sec. II to the electromagnetic gyrokinetic system, and derive the exact, gauge-invariant, local energy-momentum conservation laws of the system from the underpinning spacetime translation symmetries. For the general electromagnetic gyrokinetic system specified by the Lagrangian density in Eq. (37), the final conservation laws are given by Eqs. (96) and (123). The derivation is explicitly illustrated using the first-order system specified by the Lagrangian density in Eq. (57).

A. The Electromagnetic gyrokinetic system

When the field theory on heterogeneous manifolds developed in Sec. II is specialized to the electromagnetic gyrokinetic theory, \( X_a \) is the gyrocenter position, \( U_a = (u_a, \mu_a, \theta_a) \) consists of parallel velocity, magnetic moment and gyrophase, and the field \( \psi (t, x) = (\varphi (t, x), A (t, x)) \) is the 4-potential. As in the general case, the Lagrangian density of the system \( \mathcal{L} \) is composed of the field Lagrangian density \( \mathcal{L}_F \) and particle Lagrangian \( \mathcal{L}_a \),

\[
\mathcal{L} = \mathcal{L}_F + \sum_a \mathcal{L}_a, \quad (37)
\]

\[
\mathcal{L}_a = L_a \delta(x - X_a). \quad (38)
\]

For the general electromagnetic gyrokinetic system, \( \mathcal{L}_F \) is the standard Lagrangian density of the Maxwell field theory,

\[
\mathcal{L}_F = \frac{1}{8\pi} \left( E^2 - B^2 \right), \quad E = -\frac{1}{c} \partial_t A - \nabla \varphi, \quad B = \nabla \times A. \quad (39)
\]

For particles,

\[
L_a = L_{0a} + \delta L_a = L_{0a} + L_{1a} + \ldots, \quad (40)
\]

\[
\mathcal{L}_a = L_a \delta(x - X_a) = \mathcal{L}_{0a} + \delta \mathcal{L}_a = \mathcal{L}_{0a} + \mathcal{L}_{1a} + \ldots, \quad (41)
\]

where \( L_{0a} \) is the leading order of the Lagrangian \( L_a \) of the \( a \)-th particle, \( L_{1a} \) is the first order, etc. And \( \delta L_a \) represents all high-order terms of of \( L_a \). The expressions of \( L_{0a} \) and \( L_{1a} \) are
give by Eqs. (58) and (59), respectively. The expansion parameter is the small parameter of the gyrokinetic ordering, i.e.,
\[ \epsilon = \max(\rho k, \omega/\Omega) \ll 1. \] (42)
Here, \( k \) and \( \omega \) measure the spacetime scales of the electromagnetic field \( E \) and \( B \) associated the total total 4-potential \( (\varphi, A) \), and \( \rho \) and \( \Omega \) are the typical gyro-radius and gyro-frequency of the particles.

Before carrying out the detailed derivation of the energy-momentum conservation laws, we shall point out a few features of the electromagnetic gyrokinetic system defined by Eq. (37). In the gyrokinetic formalism adopted by most researchers, the electromagnetic potentials (fields) are separated into perturbed and background parts,
\[ A(t, x) = A_0(t, x) + A_1(t, x), \] (43)
\[ \varphi(t, x) = \varphi_0(t, x) + \varphi_1(t, x), \] (44)
where subscript “0” indicates the background part, and subscript “1” the perturbed part. Here, \( A_1 \sim \epsilon A_0 \) and \( \varphi_1 \sim \epsilon \varphi_0 \). Let \( k_1 \) and \( \omega_1 \) denote the typical wave number and frequency of the electromagnetic field associated the perturbed 4-potential \( (\varphi_1, A_1) \). While gyrokinetic theory requires Eq. (42), it does allow
\[ \rho k_1 \sim \omega_1/\Omega \sim 1. \] (45)
The energy conservation law derived in Refs. [75, 76] is for the perturbed field \( (\varphi_1, A_1) \) when the background field \( (\varphi_0, A_0) \) does not depend on time explicitly. Because the background magnetic field \( B_0(x) = \nabla \times A_0 \) depends on \( x \), the momentum conservation law in terms of \( (\varphi_1, A_1) \) cannot be established in general, except for the case where \( B_0(x) \) is symmetric with respect to specific spatial coordinates.

In the present study, we do not separate the electromagnetic potentials (fields) into perturbed and background parts, and the theory and the energy-momentum conservation laws are developed for the total field \( (\varphi, A) \). Therefore, it is guaranteed that the Lagrangian density \( \mathcal{L} \) defined in Eq. (37) does not explicitly depend on the spacetime coordinate \( (t, x) \), and that the exact local energy-momentum conservation laws always exist.

It is important to observe that condition (45) is consistent with the gyrokinetic ordering (42), because the amplitude of the perturbed field is smaller by one order of \( \epsilon \). Since our theory is developed for the total field \( (\varphi, A) \), only the gyrokinetic ordering (42) is required,
and it is valid for cases with condition (15). To express the FLR effects of the gyrokinetic systems using the total field \( (\varphi, \mathbf{A}) \), it is necessary and sufficient to include high-order field derivatives in the Lagrangian density \( \mathcal{L} \), which is the approach we adopted. The general theory developed include field derivatives to all orders, and we explicitly work out the first-order theory, which includes field derivatives up to the second order.

Without specifying the explicit form of \( \mathcal{L}_F \) and \( L_a \), the equations of motion for \( \varphi \) and \( \mathbf{A} \) derived directly from the Eq. (10) are

\[
E_\varphi (\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \varphi} - \frac{D}{D\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \nabla \varphi} + \sum_{j=1}^{n-1} (-1)^{j+1} \left( \frac{D}{D\chi_1^{\mu_1}} \cdots \frac{D}{D\chi_j^{\mu_j}} \frac{D}{D\mathbf{x}} \right) \frac{\partial \mathcal{L}}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} \nabla \varphi \right)}
\]

\[
= -\frac{D}{D\mathbf{x}} \cdot \frac{\partial \mathcal{L}_F}{\partial \nabla \varphi} + \frac{\partial}{\partial \varphi} \left( \sum_a L_a \right) + \frac{D}{D\mathbf{x}} \cdot \left\{ \sum_a \left[ -\frac{\partial L_a}{\partial \nabla \varphi} + \sum_{j=1}^{n-1} (-1)^{j+1} \left( \frac{D}{D\chi_1^{\mu_1}} \cdots \frac{D}{D\chi_j^{\mu_j}} \right) \frac{\partial L_a}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} \nabla \varphi \right)} \right] \right\}
\]

\[
= \frac{1}{4\pi} \nabla \cdot \mathbf{E} - \rho_g + \nabla \cdot \mathbf{P} = 0,
\]

(46)

\[
E_\mathbf{A} (\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \mathbf{A}} - \frac{D}{Dt} \frac{\partial \mathcal{L}}{\partial \mathbf{A}_t} - \frac{D}{D\mathbf{x}} \cdot \frac{\partial \mathcal{L}}{\partial \nabla \mathbf{A}} + \sum_{j=1}^{n-1} (-1)^{j+1} \left( \frac{D}{D\chi_1^{\mu_1}} \cdots \frac{D}{D\chi_j^{\mu_j}} \frac{D}{D\mathbf{x}} \right) \frac{\partial \mathcal{L}}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} \mathbf{A}_t \right)}
\]

\[
= -\frac{D}{Dt} \frac{\partial \mathcal{L}_F}{\partial \mathbf{A}_t} - \frac{D}{D\mathbf{x}} \cdot \frac{\partial \mathcal{L}_F}{\partial \nabla \mathbf{A}} + \frac{\partial}{\partial \mathbf{A}} \left( \sum_a L_a \right) + \frac{D}{Dt} \left\{ \sum_a \left[ -\frac{\partial L_a}{\partial \mathbf{A}_t} + \sum_{j=1}^{n-1} (-1)^{j+1} \left( \frac{D}{D\chi_1^{\mu_1}} \cdots \frac{D}{D\chi_j^{\mu_j}} \right) \frac{\partial L_a}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} \mathbf{A}_t \right)} \right] \right\}
\]

\[
+ \frac{D}{D\mathbf{x}} \left\{ \sum_a \left[ -\frac{\partial L_a}{\partial \nabla \mathbf{A}} + \sum_{j=1}^{n-1} (-1)^{j+1} \left( \frac{D}{D\chi_1^{\mu_1}} \cdots \frac{D}{D\chi_j^{\mu_j}} \right) \frac{\partial L_a}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} \nabla \mathbf{A} \right)} \right] \right\},
\]

\[
= -\frac{1}{4\pi} \left[ -\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} \right] + \mathbf{j}_g + \frac{1}{c} \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} = 0, \tag{47}
\]

where

\[
\rho_g = -\frac{\partial}{\partial \varphi} \left( \sum_a L_a \right), \quad \mathbf{j}_g = \frac{\partial}{\partial \mathbf{A}} \left( \sum_a L_a \right), \tag{48}
\]

16
\[ P = \sum_a \left[ \frac{\partial L_a}{\partial E} + \sum_{j=1}^{n-1} (-1)^j D_{\mu_1} \cdots D_{\mu_j} \frac{\partial L_a}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} E \right)} \right], \quad (49) \]

\[ M = \sum_a \left[ \frac{\partial L_a}{\partial B} + \sum_{j=1}^{n-1} (-1)^j D_{\mu_1} \cdots D_{\mu_j} \frac{\partial L_a}{\partial \left( \partial_{\mu_1} \cdots \partial_{\mu_j} B \right)} \right]. \quad (50) \]

The following equations
\[ \frac{\partial E}{\partial \nabla \varphi} = c \frac{\partial E}{\partial A}, \quad \frac{\partial B}{\partial \nabla A} = \frac{\partial}{\partial \nabla A} (\varepsilon \cdot \nabla A) = \varepsilon, \quad (51) \]
\[ \frac{\partial \mathbf{L}_a}{\partial \nabla \varphi} = \frac{c}{c} \frac{\partial \mathbf{L}_a}{\partial \nabla A} = -\frac{\partial \mathbf{L}_a}{\partial \mathbf{E}}, \quad (52) \]
\[ \frac{\partial \mathbf{L}_a}{\partial D_{\mu_1} \cdots D_{\mu_j} \nabla \varphi} = c \frac{\partial \mathbf{L}_a}{\partial D_{\mu_1} \cdots D_{\mu_j} A} = -\frac{\partial \mathbf{L}_a}{\partial D_{\mu_1} \cdots D_{\mu_j} E}, \quad j = 1, 2, \ldots, n-1, \quad (53) \]
\[ \frac{\partial \mathbf{L}_a}{\partial \nabla A} = \varepsilon \cdot \frac{\partial \mathbf{L}_a}{\partial \nabla B}, \quad \frac{\partial \mathbf{L}_a}{\partial D_{\mu_1} \cdots D_{\mu_j} \nabla A} = \varepsilon \cdot \frac{\partial \mathbf{L}_a}{\partial D_{\mu_1} \cdots D_{\mu_j} B}, \quad j = 1, 2, \ldots, n-1 \quad (54) \]

are used in the last steps of Eqs. (46) and (47), and \( \varepsilon \) in Eq. (54) is the Levi-Civita symbol in the Cartesian coordinates. In Eq. (48), \( \rho_g \) and \( j_g \) are charge and current densities of gyrocenter, and \( P \) and \( M \) in Eqs. (49) and (50) are polarization and magnetization, which contain field derivatives up to the \( n \)-th order. Using Eqs. (46) and (47), the equation of motion for fields \( (\varphi, A) \) are then transformed into
\[ \nabla \cdot (\mathbf{E} + 4\pi \mathbf{P}) = 4\pi \rho_g, \quad (55) \]
\[ \nabla \times (\mathbf{B} - 4\pi \mathbf{M}) - \frac{1}{c} \frac{\partial}{\partial t} (\mathbf{E} + 4\pi \mathbf{P}) = 4\pi \mathbf{j}_g. \quad (56) \]

We will derive the exact, gauge-invariant, local energy-momentum conservation laws for the general electromagnetic gyrokinetic system specified by the Lagrangian density in Eq. (37). The final conservation laws are given by Eqs. (96) and (123). To simplify the presentation, we only give the detailed derivation for the following first-order electromagnetic gyrokinetic theory which only keeps \( L_{1a} \) in \( \delta L_a \) [35],

\[ \mathcal{L}_a = \mathcal{L}_{a0} + \mathcal{L}_{a1} = (L_{0a} + L_{1a}) \delta(x - X_a), \quad (57) \]
\[ L_{0a} = \frac{q_a}{c} \mathbf{A}_a^\dagger \cdot \dot{X}_a - q_a H_a, \quad (58) \]
\[ L_{1a} = -\frac{m_ac}{q_a} \mu_a \mathbf{R}_a \cdot \dot{X}_a - \frac{m_ac}{q_a} \left\{ \left( \mathbf{E}_{a\perp}^\dagger - \frac{u_a}{c} \mathbf{B}_a^\dagger \times \mathbf{b} \right) \cdot \frac{\mu_a c}{2B b_{\parallel}} \nabla B \right\} + \frac{\mu_a u_a}{2} \mathbf{b} \cdot \nabla \times \mathbf{b} - \frac{\mu_a c}{2B} (\nabla \cdot \mathbf{E} - bb : \nabla \mathbf{E}) - \frac{\mu_a c}{m_a} \mathbf{R}_a^0, \quad (59) \]
\[ R_a = (\nabla c_a) \cdot a_a = R_a (u_a, w_a), \quad R_a^0 = -\frac{1}{c} \partial_t c_a \cdot a_a = R_a^0 (u_a, w_a), \]

\[ c_a = \frac{w_a}{w_a}, b = \frac{B}{B}, a_a = b \times c_a, \]

\[ E_a^i = -\nabla \varphi_a^i - \frac{1}{c} \partial_t A_a^i, \quad B_a^i = \nabla \times A_a^i, \]

\[ A_a^i = A + \frac{m_a c}{q_a} u_a b + \frac{m_a c}{q_a} D, \quad \varphi_a^i = \varphi + \frac{\mu_a}{q_a} B, \]

\[ H_a = \frac{1}{2} m_a \left( u_a^2 + D^2 \right) + \frac{\mu_a B}{q_a} + \varphi, \quad \mu_a = \frac{m_a w_a^2}{2B}, \quad D = \frac{cE \times B}{B^2}. \]

where \( m_a \) and \( q_a \) are mass and charge of the \( a \)-th particle, and \( w_a \) is the perpendicular velocity. The Routh reduction has been used to decouple the gyrophase dynamics. Note that the first order Lagrangian \( L_{1a} \) contains second-order spacetime derivatives of the electromagnetic 4-potential \((\varphi, A)\). The prolongation field involved is thus \( \text{pr}^{(2)} \psi (t, x) \).

From Eqs. (49) and (50), we can obtain the polarization \( P \) and magnetization \( M \) for the first-order theory as

\[ P = P_0 + P_1, \]

\[ P_0 = \sum_a \frac{\partial L_{0a}}{\partial E} = \sum_a \frac{m_a c \delta_a}{B} \left[ b \times (\dot{X}_a - D) \right], \]

\[ P_1 = \sum_a \left[ \frac{\partial L_{1a}}{\partial E} - D_{\mu} \frac{\partial L_{1a}}{\partial (\partial_\mu E)} \right], \]

\[ M = M_0 + M_1, \]

\[ M_0 = \sum_a \frac{\partial L_{0a}}{\partial B} \]

\[ = \sum_a \frac{m_a c \delta_a}{B} \left[ \frac{u_a}{c} \dot{X}_a - \frac{\mu_a B}{m_a c} b - \frac{E}{B} \times (\dot{X}_a - D) - \frac{2}{c} \left[ (\dot{X}_a - D) \cdot D \right] b \right], \]

\[ M_1 = \sum_a \left[ \frac{\partial L_{1a}}{\partial B} - D_{\mu} \frac{\partial L_{1a}}{\partial (\partial_\mu B)} \right]. \]

The detailed derivations of Eq. (66) and (69) are shown in Appendix A.

**B. Time translation symmetry and local energy conservation law**

First, we look at the local energy conservation. It is straightforward to verify that the action for the gyrokinetic system specified by the Lagrangian density in Eq. (37) is invariant under the time translation,

\[ g_\epsilon : (t, x, X_a, U_a, \varphi, A) \mapsto (\tilde{t}, \tilde{x}, \tilde{X}_a, \tilde{U}_a, \tilde{\varphi}, \tilde{A}) = (t + \epsilon, x, X_a, U_a, \varphi, A), \quad \epsilon \in \mathbb{R}, \]

(71)
because the Lagrangian density doesn’t contain the time variables explicitly. Using Eqs. (20) and (22), the infinitesimal generator and its prolongation of the group transformation are calculated as

$$v = pr^{(1,2)}v = \frac{\partial}{\partial t},$$  \hspace{1cm} (72)

where $\xi^t = 1$, $\xi = 0$ and $\theta_{ai} = \phi^a_{\mu_1 \cdots \mu_j} = 0$ (see Eqs. (20)-(23)). The infinitesimal criterion (19) is reduced to

$$\frac{\partial L}{\partial \dot{X}_a} = 0,$$ \hspace{1cm} (73)

which is indeed satisfied as the Lagrangian density doesn’t depend on time explicitly. Because the characteristic of the infinitesimal generator $q_a = \theta_a - \xi^t \dot{X}_a = -\dot{X}_a$ is independent of $x$, the infinitesimal criterion (73) will induce a conservation law by calculating terms in Eq. (31). Using Eqs. (24) and (26)-(30), these terms for the first-order theory specified by Eq. (57) are

$$q_a = -\dot{X}_a, \quad p_a = -\dot{U}_a, \quad Q = (-\varphi, -A),$$ \hspace{1cm} (74)

$$\frac{\partial L}{\partial X_a} = \frac{q_a}{c} A^a + \frac{\partial L_1}{\partial X_a},$$ \hspace{1cm} (75)

$$\sum_a \mathcal{R}^\nu_{a(1)} \delta_a + \mathbb{P}^\nu_{F(1)} = \frac{1}{4\pi} \left( \frac{1}{c} (E + 4\pi P_0) \cdot A, (E + 4\pi P_0) \varphi, t + A_t \times (B - 4\pi M_0) \right)$$

$$+ \sum_a \mathcal{R}^\nu_{a(1)} \delta_a, \quad (76)$$

$$\mathcal{R}^\nu_{1a(1)} = \left( \frac{1}{c} \frac{\partial L_{1a}}{\partial E} \cdot A, \frac{\partial L_{1a}}{\partial E} \varphi, t - A_t \times \frac{\partial L_{1a}}{\partial B} \right),$$ \hspace{1cm} (77)

$$\mathbb{P}^\nu_{F(2)} = 0,$$ \hspace{1cm} (78)

$$\mathcal{R}^\nu_{a(2)} = \left[ -\left[ \frac{\partial L_{1a}}{\partial (\partial E)} \right] \cdot \partial_t E - \left[ \frac{\partial L_{1a}}{\partial (\partial B)} \right] \cdot \partial_t B - \frac{1}{c} \left[ \frac{D_\mu}{\partial (\partial E)} \right] \cdot A, t, \right.$$

$$- \left[ \frac{\partial L_{1a}}{\partial (\nabla E)} \right] \cdot \partial_t E - \left[ \frac{\partial L_{1a}}{\partial (\nabla B)} \right] \cdot \partial_t B - \left[ \frac{\partial L_{1a}}{\partial (\nabla B)} \right] \cdot \partial_t B$$

$$- \left[ \frac{D_\mu}{\partial (\partial E)} \right] \varphi, t + A_t \times \left[ \frac{D_\mu}{\partial (\partial B)} \right] \right).$$ \hspace{1cm} (79)

The detailed derivations of Eqs. (76)-(79) are shown in Appendix B. The velocity $\dot{X}_a$, as a function of $(X_a(t), U_a(t))$, is determined by the equation of motion of the $a$-th particle [35], which can be obtained by the EL equation (13). Substituting Eqs. (74)-(79) into Eq. (31), we obtain the local energy conservation law

$$\frac{D}{Dt} \left[ \sum_a q_a H_a \delta_a - \frac{1}{8\pi} (E^2 - B^2) - \frac{1}{4\pi c} (E + 4\pi P_0) \cdot A, t + \sum_a \frac{\partial L_1}{\partial X_a} \cdot \dot{X}_a - L_1 - \sum_a \mathcal{R}^0_{1a} \delta_a \right]$$
\[ \frac{D}{Dt} \left\{ \sum_a q_a H_a \dot{X}_a - \frac{1}{4\pi} (E + 4\pi P_0) \varphi, t - \frac{1}{4\pi} [A, t \times (B - 4\pi M_0)] - \sum_a P_{1a} \delta_a \right\} + \sum_a \left( \dot{X}_a \frac{\partial L_{1a}}{\partial \dot{X}_a} - L_{1a} I \right) \cdot \dot{X}_a = 0, \] (80)

where

\[ P_{1a}^0 = P_{1a(1)}^0 + P_{1a(2)}^0 = \frac{1}{c} p_{1a} \cdot A, t - \left[ \frac{\partial L_{1a}}{\partial (\partial_t E)} \right] \cdot \partial_t E - \left[ \frac{\partial L_{1a}}{\partial (\partial_t B)} \right] \cdot \partial_t B, \] (81)

\[ P_{1a} = P_{1a(1)} + P_{1a(2)} \]

\[ = \varphi, t p_{1a} - A, t \times m_{1a} - \left[ \frac{\partial L_{1a}}{\partial (\nabla E)} \right] \cdot \partial_t E - \left[ \frac{\partial L_{1a}}{\partial (\nabla B)} \right] \cdot \partial_t B, \] (82)

\[ p_{1a} = \frac{\partial L_{1a}}{\partial E}, \quad m_{1a} = \frac{\partial L_{1a}}{\partial B} - \left[ D_\mu \frac{\partial L_{1a}}{\partial (\partial_\mu E)} \right]. \] (83)

Here, \( p_{1a} \) and \( m_{1a} \) in Eq. (83) are first-order polarization and magnetization for the \( a \)-th particle. And \( p_{1a} \) and \( m_{1a} \) are obviously gauge invariant.

Because electromagnetic field in the field theory is represented by the 4-potential \((\varphi, A)\), the conservation laws depends on gauge explicitly. To remove the explicit gauge dependency from the Noether procedure, we can add the identity

\[ \frac{D}{Dt} \left\{ \frac{D}{Dx} \cdot \left[ - \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) \varphi \right] \right\} + \frac{D}{Dx} \cdot \left\{ D \left[ \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right] \varphi \right\} = 0 \] (84)

to Eq. (80), and rewrite the two terms on the left-hand side of Eq. (84) as follows,

\[ \frac{D}{Dx} \cdot \left[ - \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) \varphi \right] = \frac{\partial L_0}{\partial \varphi} \varphi + \frac{\partial L_0}{\partial E} \cdot \nabla \varphi - \left( \frac{\partial L_1}{\partial E} - D_\mu D_\nu \frac{\partial L_1}{\partial (\partial_\mu E)} \right) \cdot \nabla \varphi \]

\[ = - \frac{1}{4\pi} (E + 4\pi P_0) \cdot \nabla \varphi - \sum_a q_a \varphi \delta_a - \sum_a (p_{1a} \cdot \nabla \varphi) \delta_a, \] (85)

\[ \frac{D}{Dt} \left[ \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) \varphi \right] = \frac{\partial L_0}{\partial \varphi} \varphi, t + \left( \frac{\partial L_1}{\partial E} - D_\mu D_\nu \frac{\partial L_1}{\partial (\partial_\mu E)} \right) \varphi, t + c \nabla \varphi \times \frac{\partial L_0}{\partial B} \]

\[ + c \nabla \varphi \times \left[ \frac{\partial L_1}{\partial B} - D_\mu \frac{\partial L_1}{\partial (\partial_\mu B)} \right] - c \frac{\partial L_0}{\partial A} \varphi - c \nabla \times \left\{ \varphi \left[ \frac{\partial L}{\partial B} - D_\mu \frac{\partial L}{\partial (\partial_\mu B)} \right] \right\} \]

\[ = \frac{1}{4\pi} \varphi, t (E + 4\pi P_0) - \sum_a q_a \varphi \dot{X}_a + \sum_a \varphi, t p_{1a} \delta_a - \frac{c}{4\pi} \nabla \varphi \times (B - 4\pi M_0) \]

\[ + c \nabla \varphi \times \sum_a m_{1a} \delta_a - c \nabla \times \left\{ \varphi \left[ \frac{\partial L}{\partial B} - D_\mu \frac{\partial L}{\partial (\partial_\mu B)} \right] \right\}. \] (86)

The details of the derivation of Eqs. (85) and (86) can be found in Ref. [82]. The resulting energy conservation is

\[ \frac{D}{Dt} \left\{ \sum_a \left[ \frac{1}{2} m_a (u_a^2 + D^2) + \mu_a B \right] \delta_a + \frac{1}{8\pi} (E^2 + B^2) + P_0 \cdot E \right\} \]
larization are the zeroth-order polarization and magnetization for particles of the s-species. The power
we have
the limit of guiding-center drift kinetics, the first-order terms in Eq. (88) are neglected, and
In Eqs. (87), \( \dot{X}_a \) is drift velocity of the guiding center, and it is a function of \((X_a, t, U_a(t))\)
determined by the EL equation (13). The detailed expression of \( \dot{X}_a \) can be found in Ref. [35].
Following the procedure in Sec. II C, Eq. (87) can be expressed in terms of the Klimont-

tovich distribution function \( F_s(t, x, u) \) and the electromagnetic field,
\[
\frac{D}{Dt} \left\{ \sum_s \int d^3u F_s \left[ \frac{1}{2} m_s \left( u_r^2 + D^2 \right) + \mu_B + E \cdot p_{0s} \right] + \frac{1}{8\pi} \left( E^2 + B^2 \right) \right. \\
+ \sum_s \int d^3u F_s \left\{ \frac{\partial L_{1s}}{\partial \dot{X}_s} \cdot \dot{X}_s - L_{1s} + p_{1s} \cdot E + \left[ \frac{\partial L_{1s}}{\partial (\partial_t E)} \right] \cdot \partial_t E + \left[ \frac{\partial L_{1s}}{\partial (\partial_t B)} \right] \cdot \partial_t B \} \right. \\
+ \left. \frac{D}{Dx} \cdot \left\{ \sum_s \int d^3u F_s \left[ \frac{1}{2} m_s \left( u_r^2 + D^2 \right) + \mu_B - c E \cdot m_{0s} \right] \dot{X}_s + \frac{c}{4\pi} E \times B + \sum_s \int d^3u F_s \times \left[ \left( \frac{\partial L_{1s}}{\partial X_s} - L_{1s} I \right) \cdot \dot{X}_s + \left[ \frac{\partial L_{1s}}{\partial (\nabla E)} \right] \cdot \partial_t E + \left[ \frac{\partial L_{1s}}{\partial (\nabla B)} \right] \cdot \partial_t B \} \right. \\
\left. \right\} = 0, \tag{88} \]
where
\[
p_{0s} = \frac{\partial L_{0s}}{\partial E} = \sum_a \frac{m_c}{B} \left[ b \times (\dot{X}_s - D) \right], \tag{89} \]
\[
m_{0s} = \frac{\partial L_{0s}}{\partial B} = \sum_a \frac{m_c}{B} \left[ u_s \dot{X}_{a \perp} - \frac{\mu_B}{m_c} b - \frac{E}{B} \times (\dot{X}_s - D) - \frac{2}{c} \left( (\dot{X}_s - D) \cdot D \right) b \right] \tag{90} \]
are the zeroth-order polarization and magnetization for particles of the s-species. The polariza-
tion \( P_1 \) and magnetization \( M_1 \) are contained in the first-order terms of Eq. (88). In
the limit of guiding-center drift kinetics, the first-order terms in Eq. (88) are neglected, and
we have
\[
\frac{D}{Dt} \left\{ \sum_s \int d^3u F_s \left[ \frac{1}{2} m_s \left( u_r^2 + D^2 \right) + \mu_B + E \cdot p_{0s} \right] + \frac{1}{8\pi} \left( E^2 + B^2 \right) \right. \\
+ \left. \frac{D}{Dx} \cdot \left\{ \sum_s \int d^3u F_s \left[ \frac{1}{2} m_s \left( u_r^2 + D^2 \right) + \mu_B - c E \cdot m_{0s} \right] \dot{X}_s + \frac{c}{4\pi} E \times B \right\} = 0. \tag{91} \]
In the limit of guiding-center drift kinetics, if the \( E \times B \) term \( D \) in \( L_a \) is also ignored,

namely,
\[ \mathcal{L}_a = \left[ \left( \frac{q_a}{c} \mathbf{A} + m_a u_a \mathbf{b} \right) \cdot \dot{\mathbf{X}}_a - \left( \frac{1}{2} m_a u_a^2 + \mu_a B + \varphi \right) \right] \delta_a, \quad (92) \]

then the polarization vector field \( \mathbf{P}_0 \) and magnetization vector field \( \mathbf{M}_0 \) reduce to
\[ \mathbf{P}_0 = 0, \quad \mathbf{M}_0 = \sum_a m_{0a} \delta_a, \quad m_{0a} = \frac{m_a u_a}{B} \dot{X}_a - \mu_a \mathbf{b}. \quad (93) \]

Thus, the energy conservation law is further reduced to
\[
\frac{D}{Dt} \left\{ \sum_a \left[ \frac{1}{2} m_a u_a^2 + \mu_a B \right] \delta_a + \frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{B}^2 \right) \right\} + \frac{D}{Dx} \cdot \left\{ \sum_a \left[ \frac{1}{2} m_a u_a^2 + \mu_a B \right] \delta_a \dot{\mathbf{X}}_a + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right\} = 0, \quad (94)\]

which, in terms of the distribution function and the electromagnetic field, is
\[
\frac{D}{Dt} \left\{ \sum_s \int d^3 u F_s \left[ \frac{1}{2} m_s (u_s^2 + D^2) + \mu B + \mathbf{E} \cdot \mathbf{p}_{0s} \right] + \frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{B}^2 \right) \right\} + \frac{D}{Dx} \cdot \left\{ \sum_s \int d^3 u F_s \left[ \frac{1}{2} m_s (u_s^2 + D^2) + \mu B - cE \times m_{0s} \right] \dot{\mathbf{X}}_s + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right\} = 0. \quad (95)\]

Equation (95) agrees with the result of Brizard et al. [39] for guiding-center drift kinetics. Note that before the present study, local energy conservation law was not known for the high-order electromagnetic gyrokinetic systems. Our local energy conservation law for the electromagnetic gyrokinetic systems (88) and (96) recover the previous known results for the first-order guiding-center Vlasov-Maxwell system and the drift kinetic system as special cases.

The above derivation of local energy conservation law is for the first-order theory specified by Eq. (57). For the general electromagnetic gyrokinetic system of arbitrary high order specified by Eq. (47), an exact, gauge-invariant, local energy conservation law can be derived using the same method. It is listed here without detailed derivation,
\[
\frac{D}{Dt} \left\{ \sum_s \int d^3 u F_s \left[ \frac{1}{2} m_s (u_s^2 + D^2) + \mu B + \mathbf{E} \cdot \mathbf{p}_{0s} \right] + \frac{1}{8\pi} \left( \mathbf{E}^2 + \mathbf{B}^2 \right) \right\} + \frac{D}{Dx} \cdot \left\{ \sum_s \int d^3 u F_s \left[ \frac{1}{2} m_s (u_s^2 + D^2) + \mu B - cE \times m_{0s} \right] \dot{\mathbf{X}}_s + \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right\} \]
\[
\sum_s \int d^3 u F_s \left[ \left( \frac{\partial \delta L_s}{\partial \dot{\mathbf{X}}_s} - \delta L_s I \right) \cdot \dot{\mathbf{X}}_s + [-cE \times \delta m_s - \delta J_s] \right] = 0, \quad (96)\]
\[
\delta p_s = \frac{\partial \delta L_s}{\partial \mathbf{E}} + \sum_{j=1}^{n-1} (-1)^j D_{\mu_1} \cdots D_{\mu_j} \frac{\partial \delta L_s}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_j} \mathbf{E})},
\]
\[
\delta m_s = \frac{\partial \delta L_s}{\partial \mathbf{B}} + \sum_{j=1}^{n-1} (-1)^j D_{\mu_1} \cdots D_{\mu_j} \frac{\partial \delta L_s}{\partial (\partial_{\mu_1} \cdots \partial_{\mu_j} \mathbf{B})},
\]
\[
\delta J_s^0 = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \partial t D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{E}} \right] \cdot (D_{\mu_{j+1}} \cdots D_{\mu_i} \partial_t \mathbf{E})
+ \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \partial t D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{B}} \right] \cdot (D_{\mu_{j+1}} \cdots D_{\mu_i} \partial_t \mathbf{B}),
\]
\[
\delta J_s = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \mathbf{\nabla} D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{E}} \right] \cdot (D_{\mu_{j+1}} \cdots D_{\mu_i} \partial_t \mathbf{E})
+ \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \mathbf{\nabla} D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{B}} \right] \cdot (D_{\mu_{j+1}} \cdots D_{\mu_i} \partial_t \mathbf{B}).
\]

C. Space translation symmetry and momentum conservation law

We now discuss the space translation symmetry and momentum conservation. It is straightforward to verify that the action of the gyrokinetic system specified by Eq. (37) is unchanged under the space translation
\[
(\tilde{t}, \tilde{\mathbf{x}}, \tilde{\mathbf{X}}_a, \tilde{\mathbf{U}}_a, \tilde{\varphi}, \tilde{\mathbf{A}}) = (t, \mathbf{x} + \epsilon \mathbf{h}, \mathbf{X}_a + \epsilon \mathbf{h}, \mathbf{U}_a, \varphi, \mathbf{A}),
\]
where \( \mathbf{h} \) is an arbitrary constant vector. Note that this symmetry group transforms both \( \mathbf{x} \) and \( \mathbf{X}_a \).

It is worthwhile to emphasize again that in order for the system to admit spacetime translation symmetry and thus local energy-momentum conservation laws, we do not separate the electromagnetic field into background and perturbed components. This is different from other existing studies in gyrokinetic theory, which separate the background magnetic field from the perturbed magnetic field, and as a result no momentum conservation law can be established in these studies for the plasmas dynamics in tokamaks or devices with inhomogeneous background magnetic fields.
The infinitesimal generator corresponding to Eq. (101) is
\[ v = h \cdot \frac{\partial}{\partial x} + \sum_a h \cdot \frac{\partial}{\partial X_a}. \] (102)

Because \( \xi^t = 0, \xi = \theta_a = h \) and \( \theta_{a1} = \phi^\alpha_{\mu_1 \ldots \mu_j} = 0 \) (see Eqs. (20)-(23)), the prolongation of \( v \) is the same as \( v \),
\[ \text{pr}^{(1,2)}v = v. \]

The infinitesimal criterion (19) is then satisfied since
\[ h \cdot \left( \frac{\partial L}{\partial x} + \sum_a \frac{\partial L}{\partial X_a} \right) = 0, \] (103)
where used is made of the fact that \( \partial \delta_a/\partial x = -\partial \delta_a/\partial X_a \). The characteristics of the infinitesimal generator (102) is
\[ q_a = h, \ p_a = 0, \ Q = -h \cdot \nabla \psi = (-h \cdot \nabla \varphi, -h \cdot \nabla A). \] (104)

The infinitesimal criterion (103) thus implies a conservation law because \( q_a \) is a constant vector field independent of \( x \).

We now calculate each term in Eq. (31) for the first-order theory specified by Eq. (57) to obtain the conservation law. Using the definitions of \( P^\nu_a \) and \( P^\nu_F \) (see Eqs. (26)-(30)), the most complicated terms \( \sum_a P^\nu_{a(1)} \delta_a + P^\nu_{F(1)} \) and \( \sum_a P^\nu_{a(2)} \delta_a + P^\nu_{F(2)} \) in the conservation law can be explicitly written as
\[ \sum_a P^\nu_{a(1)} \delta_a + P^\nu_{F(1)} = \frac{1}{4\pi} \left( \frac{1}{c} (E + 4\pi P_0) \cdot (\nabla A)^T, (E + 4\pi P_0) \nabla \varphi \right) - \varepsilon : \left[ (B - 4\pi M_0) (\nabla A)^T \right] \cdot h + \left( \sum_a \sigma^\nu_{a(1)} \delta_a \right) \cdot h, \] (105)
\[ \sigma^\nu_{a(1)} = \left( \frac{1}{c} \frac{\partial L_{1a}}{\partial E} \cdot (\nabla A)^T, \frac{\partial L_{1a}}{\partial E} \nabla \varphi + \frac{\partial L_{1a}}{\partial B} \times (\nabla A)^T \right), \] (106)
\[ \sum_a P^\nu_{a(2)} \delta_a + P^\nu_{F(2)} = \left( \sum_a \sigma^\nu_{a(2)} \delta_a \right) \cdot h, \] (107)
\[ \sigma^\nu_{a(2)} = \left( -\nabla E \cdot \left[ \frac{\partial L_{1a}}{\partial (\nabla E)} \right] - \nabla B \cdot \left[ \frac{\partial L_{1a}}{\partial (\nabla B)} \right] - \left[ \frac{1}{c} D^\mu \frac{\partial L_{1a}}{\partial (\nabla E)} \right], (\nabla A)^T, \right), \]
\[ - \frac{\partial L_{1a}}{\partial (\nabla E)} \cdot (\nabla E)^T \cdot (\nabla B)^T \]
\[ - \left[ D^\mu \frac{\partial L_{1a}}{\partial (\nabla E)} \right] \nabla \varphi - \left[ D^\mu \frac{\partial L_{1a}}{\partial (\nabla B)} \right] \times (\nabla A)^T \right). \] (108)
The detailed derivations of Eqs. (105)-(108) are shown in Appendix B. Substituting Eqs. (105)-(108) into Eq. (31), we obtain the momentum conservation laws as

\[
\frac{D}{Dt} \left[ \sum_a \frac{q_a}{c} A^a \delta_a + \frac{1}{4\pi c} (E + 4\pi P_0) \cdot (\nabla A)^T + \sum_a \frac{\partial L_1}{\partial X_a} + \sum_a \sigma_a \delta_a \right] \\
+ \frac{D}{Dt} \cdot \left\{ \sum_a \frac{q_a}{c} \dot{X}_a A^a \delta_a + \frac{E^2 - B^2}{8\pi} I - \frac{B - 4\pi M_0}{4\pi} \times (\nabla A)^T \right. \\
+ \frac{E + 4\pi P_0}{4\pi} \nabla \varphi + \sum_a \left( \dot{X}_a \frac{\partial L_{1a}}{\partial X_a} \right) + \sum_a \sigma_a \delta_a \right\} = 0,
\]

(109)

where

\[
\sigma_a = \sigma_{a(1)} + \sigma_{a(2)} = \frac{1}{c} \frac{q_a}{c} \cdot (\nabla A)^T - \nabla E \cdot \left[ \frac{\partial L_{1a}}{\partial (\partial_\mu E)} \right] - \nabla B \cdot \left[ \frac{\partial L_{1a}}{\partial (\partial_\mu B)} \right],
\]

(110)

\[
\sigma_a = \sigma_{a(1)} + \sigma_{a(2)} = \cdots
\]

(111)

Akin to the situation of Eq. (83) in Sec. III B, Eq. (109) is gauge dependent. We can add in the following identity

\[
\frac{D}{Dt} \left\{ \frac{D}{Dt} \cdot \left[ - \frac{1}{c} \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) A \right] \right\} + \frac{D}{Dt} \cdot \left\{ \frac{D}{Dt} \left[ - \frac{1}{c} \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) A \right] \right\} = 0
\]

(112)

to remove the explicit gauge dependency (see Ref. [82]). The two terms in Eq. (112) can be rewritten as

\[
\frac{D}{Dt} \cdot \left[ - \frac{1}{c} \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) A \right] = - \frac{1}{c} \frac{\partial L_0}{\partial A} A - \frac{1}{c} \left( \frac{\partial L_0}{\partial E} \right) \cdot \nabla A \\
- \frac{1}{c} \left[ \frac{\partial L_1}{\partial E} - D_\mu \frac{\partial L_1}{\partial (\partial_\mu E)} \right] \cdot \nabla A = \cdots
\]

(113)

\[
\frac{D}{Dt} \left[ - \frac{1}{c} \left( \frac{\partial L}{\partial E} - D_\mu \frac{\partial L}{\partial (\partial_\mu E)} \right) A \right] = - \frac{\partial L_0}{\partial A} A - \left( \frac{\partial L_0}{\partial B} \right) \times \nabla A + \frac{1}{c} \left( \frac{\partial L_0}{\partial E} \right) A, \\
- \left[ \frac{\partial L_1}{\partial B} - D_\mu \frac{\partial L_1}{\partial (\partial_\mu B)} \right] \times \nabla A + \frac{1}{c} \left[ \frac{\partial L_1}{\partial E} - D_\mu \frac{\partial L_1}{\partial (\partial_\mu E)} \right] A, + \nabla \times \left[ \left( \frac{\partial L}{\partial B} - D_\mu \frac{\partial L}{\partial (\partial_\mu B)} \right) A \right] \\
= \cdots
\]

(114)
Details of the derivation is shown in Ref. [82]. Substituting Eqs. (112)-(114) into Eq. (109), we obtain

\[
\frac{D}{Dt} \left\{ \sum_a m_a (u_a b + D) \delta_a + \frac{(E + 4\pi P_0) \times B}{4\pi c} + \sum_a \frac{\partial L_{1a}}{\partial X_a} \right. \\
+ \sum_a \frac{1}{c} (p_{1a} \times B) \delta_a - \left[ \frac{\partial L_{1a}}{\partial (\partial_t E)} \right] \cdot (\nabla E)^T \delta_a - \left[ \frac{\partial L_{1a}}{\partial (\partial_t B)} \right] \cdot (\nabla B)^T \delta_a \right\} \\
+ \frac{D}{Dx} \cdot \left\{ \sum_a \dot{X}_a (m_a u_a b + m_a D) \delta_a + \sum_a p_{1a} E \delta_a + \left[ B m_{1a} - (B \cdot m_{1a}) \right] \delta_a \right. \\
- \left[ \frac{\partial L_{1a}}{\partial (\nabla E)} \right] \cdot (\nabla E)^T \delta_a - \left[ \frac{\partial L_{1a}}{\partial (\nabla B)} \right] \cdot (\nabla B)^T \delta_a \right\} = 0,
\]  

(115)

where used is made of the following equations

\[
(E + 4\pi P_0) \cdot (\nabla A)^T - \nabla A = (E + 4\pi P_0) \times B,
\]

(116)

\[
(B - 4\pi M_0) \times (\nabla A - (\nabla A)^T) = [(B - 4\pi M_0) \cdot B] I - B (B - 4\pi M_0),
\]

(117)

Here, the drift velocity \( \dot{X}_a \) of the guiding center in Eq. (115) determined by the EL equation (13), which is regarded as a function of \((X_a(t), U_a(t))\). Using the procedure in Sec. II C, the momentum conservation can be expressed in terms of the the Klimontovich distribution function \( F_s(t, x, u) \) and the electromagnetic field,

\[
\frac{D}{Dt} \left\{ \sum_s \int d^3 u F_s \left[ m_s (u_b + D) + \frac{1}{c} p_{os} \times B \right] + \frac{E \times B}{4\pi c} + \sum_s \int d^3 u F_s \left[ \frac{\partial L_{1s}}{\partial X_s} + \frac{1}{c} (p_{1s} \times B) \right] \right. \\
- \nabla E \cdot \left[ \frac{\partial L_{1s}}{\partial (\partial_t E)} \right] \right. \\
+ \left[ \frac{\partial L_{1s}}{\partial (\partial_t B)} \right] \right. \\
\left. + \frac{B m_{0s} - (m_{0s} \cdot B) I - p_{0s} E} {8\pi} \right] \left[ \frac{E^2 + B^2}{4\pi} \right] I - \frac{E E + B B}{4\pi} + \sum_s \int d^3 u F_s \times \\
\left. \left[ \dot{X}_s \frac{\partial L_{1s}}{\partial X_s} \right] \right. \\
- \left[ \frac{\partial L_{1s}}{\partial (\nabla E)} \right] \cdot (\nabla E)^T - \left[ \frac{\partial L_{1s}}{\partial (\nabla B)} \right] \cdot (\nabla B)^T \right\} = 0.
\]

(118)

For the special case of guiding-center drift kinetics, the first-order Lagrangian density \( L_{1a} \) is neglected, and we have

\[
\frac{D}{Dt} \left\{ \sum_s \int d^3 u F_s \left[ (u_b + m_s D) + \frac{1}{c} p_{os} \times B \right] + \frac{E \times B}{4\pi c} \right\} + \frac{D}{Dx} \cdot \left\{ \sum_s \int d^3 u F_s \left[ m_s \dot{X}_s (u_b + D) \right] \right.
\]
\[ + B m_{0s} - (m_{0s} \cdot B) I - p_{0s} E \] + \left( \frac{E^2 + B^2}{8\pi} \right) I - EE + BB \right\} = 0. \quad (119) \]

In the limit of guiding-center drift kinetics, if the \( E \times B \) term \( D \) in \( \mathcal{L}_a \) is also ignored (see Eq. (92)), then the momentum conservation is further reduced to

\[
\frac{D}{Dt} \left\{ \sum_a m_a u_a b \delta_a + \frac{E \times B}{4\pi c} \right\} + \frac{D}{Dx} \cdot \left\{ \sum_a m_a u_a \dot{X}_a b \delta_a + \left( \frac{E^2 + B^2}{8\pi} \right) I - EE + BB \right\} = 0. \quad (120) \]

Substituting the polarization vector field \( P \) and magnetization vector field \( M \) of the drift kinetic system (see Eq. (93)) into Eq. (120), we have

\[
\frac{D}{Dt} \left\{ \sum_a m_a u_a \int \delta^3 u F_s \left[ m_s (u_{\parallel} b + D) + \frac{1}{c} p_{0s} \times B \right] + \frac{E \times B}{4\pi c} \right\} + \frac{D}{Dx} \cdot \left\{ \sum_a m_a u_a \int \delta^3 u F_s \left[ \dot{X}_a \cdot b + b \dot{X}_a \right] + \left( \frac{E^2 + B^2}{8\pi} \right) I - EE + BB \right\} = 0. \quad (121) \]

In terms of the distribution function \( F_s(t, x, u) \) and the electromagnetic field \( (E(t, x), B(t, x)) \), Eq. (121) is

\[
\frac{D}{Dt} \left\{ \sum_s m_s \int \delta^3 u F_s \left[ m_s (u_{\parallel} b + D) + \frac{1}{c} p_{0s} \times B \right] + \frac{E \times B}{4\pi c} \right\} + \frac{D}{Dx} \cdot \left\{ \sum_s m_s \int \delta^3 u F_s \left[ \dot{X}_s \cdot b + b \dot{X}_s \right] + \left( \frac{E^2 + B^2}{8\pi} \right) I - EE + BB \right\} = 0. \quad (122) \]

Equation (122), as a special case of the gyrokinetic momentum conservation law (118), is consistent with the result shown by Brizard et al. [39] for the drift kinetics.

This completes our derivation and discussion of the momentum conservation law for the first-order theory.

For the general electromagnetic gyrokinetic system defined by Eq. (37), the following exact, gauge-invariant, local momentum conservation law can be derived using a similar method,

\[
\frac{D}{Dt} \left\{ \sum_s \int \delta^3 u F_s \left[ m_s \left( u_{\parallel} b + D \right) + \frac{1}{c} p_{0s} \times B \right] + \frac{E \times B}{4\pi c} \right\} \right. + \frac{D}{Dx} \cdot \left\{ \sum_s \int \delta^3 u F_s \left[ \dot{X}_s \left( u_{\parallel} b + D \right) \right] \right. + \frac{E^2 + B^2}{8\pi} \right\} I - EE + BB \right\} = 0. \]
\[ + \sum \int d^3 u F_s \left[ X_s \frac{\partial \delta L_s}{\partial X_s} - \delta p_s \mathbf{E} + B \delta m_s - (B \cdot \delta m_s) I + \delta K \right] = 0, \quad (123) \]

where \( \delta p_s \) and \( \delta m_s \) are defined in Eqs. (97) and (98), and

\[ \delta K = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^j \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \partial t \partial D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{E}} \right] \cdot \left[ D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{(\nabla E)^T} \right] \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^j \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \partial t \partial D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{B}} \right] \cdot \left[ D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{(\nabla B)^T} \right], \quad (124) \]

\[ \delta K = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^j \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \mathbf{\nabla D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{E}}} \right] \cdot \left[ D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{(\nabla E)^T} \right] \]

\[ + \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^j \left[ D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \delta L_s}{\partial D_{\mu_1} \cdots D_{\mu_{j-1}} \mathbf{\nabla D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{B}}} \right] \cdot \left[ D_{\mu_{j+1}} \cdots D_{\mu_i} \mathbf{(\nabla B)^T} \right]. \quad (125) \]

IV. CONCLUSION

We have established the exact, gauge-invariant, local energy-momentum conservation laws for the electromagnetic gyrokinetic system from the underpinning spacetime translation symmetries of the system. Because the gyrocenter and electromagnetic field are defined on different manifolds, the standard Noether procedure for deriving conservation laws from symmetries does not apply to the gyrokinetic system without modification.

To establish the connection between energy-momentum conservation and spacetime translation symmetry for the electromagnetic gyrokinetic system, we first extended the field theory for classical particle-field system on heterogeneous manifolds \([78-80]\) to include high-order field derivatives and using noncanonical phase space coordinates in a general setting without specializing to the gyrokinetic system. The field theory on heterogeneous manifolds embraces the fact that for classical particle-field systems, particles and fields reside on different manifolds, and a weak Euler-Lagrange equation was developed to replace the standard Euler-Lagrange equation for particles. The weak Euler-Lagrange current, induced by the weak Euler-Lagrange equation, is the new physics associated with the field theory on heterogeneous manifolds, and it plays a crucial role in the connection between symmetries and conservation laws when different components of the system are defined on different manifolds.
The high-order field theory on heterogeneous manifolds developed was then applied to the electromagnetic gyrokinetic system to derive the exact, local energy-momentum conservation laws from the spacetime translation symmetries admitted by the Lagrangian density of the system. And, finally, the recently developed gauge-symmetrization procedure [82] using the electromagnetic displacement-potential tensor was applied to render the conservation laws electromagnetic gauge invariant.

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Appendix A: Derivations of polarization and magnetization in Eqs. (66) and (69)

In this appendix, we give the derivations of zeroth order polarization $P_0$ and magnetization $M_0$. From the definition of $P_0$, $M_0$ and Lagrangian density of the $a$-th particle (see Eqs. (66), (69), (57) and (5)), they are derived as follows

$$P_0 = \sum_a \frac{\partial L_{0a}}{\partial \dot{E}}$$

$$= \sum_a \left\{ \frac{\partial}{\partial \dot{E}} \left[ \frac{m_a c \delta_a}{B} \dot{E} \cdot (b \times \dot{X}_a) \right] - \frac{\partial}{\partial \dot{E}} \left( \frac{1}{2} m_a D^2 \delta_a \right) \right\}$$

$$= \sum_a \left\{ \left[ \frac{m_a c \delta_a}{B} \frac{\partial \dot{E}}{\partial \dot{E}} \cdot (b \times \dot{X}_a) \right] - m_a \delta_a \frac{\partial D}{\partial \dot{E}} \cdot D \right\}$$

$$= \sum_a \left\{ \left[ \frac{m_a c \delta_a}{B} I \cdot (b \times \dot{X}_a) \right] - m_a c \delta_a \left[ I \times \left( \frac{b}{B} \right) \right] \cdot D \right\}$$

$$= \sum_a \left\{ \left[ \frac{m_a c \delta_a}{B} (b \times \dot{X}_a) \right] - m_a c \delta_a \left[ \left( \frac{b}{B} \right) \times D \right] \right\}$$

$$= \sum_a \frac{m_a c \delta_a}{B} \left[ b \times (\dot{X}_a - D) \right]$$

(A1)
and

\[ M_0 = \sum_a \frac{\partial \mathcal{L}_{0a}}{\partial \dot{B}} \]

\[ = \sum_a \frac{\partial}{\partial \dot{B}} \left[ m_a u_a \delta_a b \cdot \dot{X}_a + m_a \delta_a D \cdot \dot{X}_a - \frac{1}{2} m_a \delta_a D^2 - \mu_a \delta_a B \right] \]

\[ = \sum_a \left[ m_a u_a \delta_a \frac{\partial b}{\partial \dot{B}} \cdot \dot{X}_a + m_a \delta_a \frac{\partial D}{\partial \dot{B}} \cdot \dot{X}_a - \frac{1}{2} m_a \delta_a \frac{\partial D^2}{\partial \dot{B}} - \mu_a \delta_a \frac{\partial B}{\partial \dot{B}} \right] \]

\[ = \sum_a \left[ m_a u_a \delta_a \left( \frac{I - bb}{B} \right) \cdot \dot{X}_a - \frac{m_a c \delta_a}{B^2} [(I - 2bb) \times E] \cdot (\dot{X}_a - D) - \mu_a \delta_a b \right] \]

\[ = \sum_a \left[ m_a u_a \dot{X}_a \times \mu_a \delta_a b + \left[ -\frac{m_a c \delta_a}{B^2} (I \times E) - \frac{2m_a \delta_a}{B} bb \cdot D \right] \cdot (\dot{X}_a - D) \right] \]

\[ = \sum_a \frac{m_a c \delta_a}{B} \left[ \frac{u_a}{c} \dot{X}_a \times \mu_a B - \frac{E}{B} \times (\dot{X}_a - D) - \frac{2}{c} [(\dot{X}_a - D) \cdot D] b \right]. \tag{A2} \]

In obtaining Eqs. \([A1]\) and \([A2]\), the following equations were used

\[ \frac{\partial D}{\partial E} = c \frac{\partial}{\partial E} \left( \frac{E \times B}{B^2} \right) = c \frac{\partial E}{\partial E} \times \left( \frac{b}{B} \right) = c I \times \left( \frac{b}{B} \right), \tag{A3} \]

\[ \frac{\partial B}{\partial B} = \frac{\partial \sqrt{B^2}}{\partial B} = \frac{1}{\sqrt{B^2}} \frac{\partial B^2}{\partial B} = \frac{B}{B} = b. \tag{A4} \]

\[ \frac{\partial b}{\partial B} = \frac{\partial}{\partial B} \left( \frac{B}{B} \right) = \left[ \frac{1}{B^2} \frac{\partial B}{\partial B} + \frac{\partial}{\partial B} \left( \frac{1}{B} \right) B \right] = \left[ \frac{I}{B} - \frac{1}{B^2} \frac{\partial B}{\partial B} B \right] = \frac{I - bb}{B}, \tag{A5} \]

\[ \frac{\partial D}{\partial B} = -c \frac{\partial}{\partial B} \left( \frac{B \times E}{B^2} \right) = -c \frac{\partial}{\partial B} \left( \frac{B}{B^2} \right) \times E \]

\[ = -c \left\{ \left[ \frac{\partial}{\partial B} \left( \frac{1}{B^2} \right) B + \frac{1}{B^2} \frac{\partial B}{\partial B} \right] \times E \right\} = -\frac{c}{B^2} (I - 2bb) \times E. \tag{A6} \]

Appendix B: Derivations of Eqs. (76)-(79) and Eqs. (105)-(108)

In this appendix, we show the detailed derivations of Eqs. (76)-(79) and Eqs. (105)-(108), which are boundary terms induced by time and space translation symmetries. For the time translation symmetry, using Eqs. (24) and (26)-(30), equations (76)-(79) can be proved as follows

\[ \sum_a \mathcal{P}_a(1) \delta_a + P_{F(1)}^\nu = Q^\alpha \frac{\partial \mathcal{L}_0}{\partial (\partial_\tau \psi^\alpha)} + Q^\alpha \frac{\partial \mathcal{L}_1}{\partial (\partial_\tau \psi^\alpha)} \]

\[ = \left( -\frac{\partial \mathcal{L}_0}{\partial (\partial_\tau \varphi)} \varphi, t + \frac{\partial \mathcal{L}_0}{\partial (\partial_\tau \mathbf{A})} \cdot \mathbf{A}, t \right) - \frac{\partial \mathcal{L}_0}{\partial (\nabla \varphi)} \varphi, t - \frac{\partial \mathcal{L}_0}{\partial (\nabla \mathbf{A})} \cdot \mathbf{A}, t \right) + \sum_a \mathcal{P}_a(1) \delta_a \]

30
\[
Eqs. (26)-(30)), \text{ equations (105)-(108) are demonstrated as follows}
\]
\[
\mathcal{P}_{1a(1)} = Q^a \frac{\partial L_0}{\partial (\partial_t \varphi)} - \frac{\partial L_1}{\partial (\partial_t A)}, \quad \mathcal{P}_{F(1)} = \frac{1}{4\pi} \left( \frac{1}{c} (E + 4\pi P_0) \cdot A_t, (E + 4\pi P_0) \varphi_t + A_t \times (B - 4\pi M_0) \right) + \sum_a \mathcal{P}_{1a(1)} \delta_a.
\] (B1)
\[
\partial_t \varphi_t - \frac{\partial L_1}{\partial \varphi} \varphi_t = \frac{\partial L_1}{\partial A} \varphi_t - \frac{\partial L_1}{\partial (\partial_t A)} \cdot A_t
\] (B2)
\[
\Pi_{F(2)} = D_\mu Q^a \left[ \frac{\partial L_F}{\partial (\partial_\mu \varphi)} \right] - Q^a \left[ D_\mu \frac{\partial L_F}{\partial (\partial_\mu \varphi)} \right] = 0.
\] (B3)
\[
\mathcal{P}_{a(2)} = D_\mu Q^a \left[ \frac{\partial L_a}{\partial (\partial_\mu \varphi)} \right] - Q^a \left[ D_\mu \frac{\partial L_a}{\partial (\partial_\mu \varphi)} \right]
\] (B4)

Similarly, for the space translation symmetry, using the definitions of \( \mathcal{P}_{a} \) and \( \Pi_{F} \) (see Eqs. (26)-(30)), equations (105)-(108) are demonstrated as follows

\[
\sum_a \mathcal{P}_{a(1)} \delta_a + \Pi_{F(1)} = \frac{1}{4\pi} \left( \frac{1}{c} (E + 4\pi P_0) \cdot A_t, (E + 4\pi P_0) \varphi_t + A_t \times (B - 4\pi M_0) \right) + \sum_a \mathcal{P}_{1a(1)} \delta_a = Q^a \frac{\partial L_0}{\partial (\partial_t \varphi)} + Q^a \frac{\partial L_1}{\partial (\partial_t A)}
\]
\[
= \left( - \frac{\partial L_0}{\partial (\partial_t \varphi)} \varphi_t - \frac{\partial L_0}{\partial (\partial_t A)} \cdot A_t - \frac{\partial L_0}{\partial (\partial_t \varphi)} \varphi_t - \frac{\partial L_0}{\partial (\partial_t A)} \cdot \nabla A \right) \cdot h.
\]
\[
+ \frac{1}{4\pi} \left( \frac{1}{c} (E + 4\pi P_0) \cdot (\nabla A)^T, (E + 4\pi P_0) \nabla \varphi - \varepsilon : [B - 4\pi M_0] (\nabla A)^T \right) \cdot h + \left( \sum_a \sigma_{a(1)} \right) \cdot h
\] (B5)
\[
\sigma_{a(1)} = - \frac{\partial L_1}{\partial (\partial_t \varphi)} \varphi_t = \left( \sigma_{a(1)}^0, \sigma_{a(1)}^1 \right)
\]
\[
\sigma_{a(1)}^0 = - \frac{\partial L_1}{\partial (\partial_t \varphi)} \varphi_t = \left( \sigma_{a(1)}^0, \sigma_{a(1)}^1 \right)
\]
\[
\sigma_{a(1)}^1 = - \frac{\partial L_1}{\partial (\partial_t \varphi)} \varphi_t = \left( \sigma_{a(1)}^0, \sigma_{a(1)}^1 \right)
\]
\[ \sum_a \mathcal{F}_a^\nu \delta_a + \mathcal{F}_F^\nu = D_\mu Q^\nu \left[ \frac{\partial L}{\partial (\partial_\mu \varphi)} \right] - Q^\nu \left[ \frac{\partial L}{\partial (\partial_\mu \psi_\alpha)} \right] = \left( \sum_a \sigma_\alpha^\nu \delta_a \right) \cdot h, \]

(B6)

\[ \sigma_\alpha^\nu = - \left[ \frac{\partial L_{1a}}{\partial (\partial_\mu \varphi)} \right] D_\mu (\nabla \psi_\alpha) + \left[ D_\mu \frac{\partial L_{1a}}{\partial (\partial_\mu \psi_\alpha)} \right] (\nabla \psi_\alpha) = \left( \sigma_\alpha^\nu, \sigma_\alpha^\nu \right) \]

(B7)

\[ \frac{1}{c} \left\{ \varphi \left[ \frac{\partial L_1}{\partial (\partial_\mu \varphi)} \right] \frac{\partial L_1}{\partial (\partial_\mu \varphi)} \right\} - \left[ \frac{\partial L_1}{\partial (\partial_\mu \varphi)} \right] \nabla_\mu (\varphi) + \left[ \frac{\partial L_1}{\partial (\partial_\mu \varphi)} \right] \nabla \varphi + \left[ \frac{\partial L_1}{\partial (\partial_\mu \varphi)} \right] \nabla \varphi - \left[ \frac{\partial L_1}{\partial (\partial_\mu \varphi)} \right] \nabla \varphi = \left( \nabla \varphi, \nabla \varphi \right) \]

(B8)

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