CHARACTERIZATIONS OF FAMILIES OF RECTANGULAR, FINITE IMPULSE RESPONSE, PARA-UNITARY SYSTEMS

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ABSTRACT. We here study Finite Impulse Response (FIR) rectangular, not necessarily causal, systems which are (para)-unitary on the unit circle (=the class \( \mathcal{U} \)). First, we offer three characterizations of these systems. Then, introduce a description of all FIRs in \( \mathcal{U} \), as copies of a real polytope, parametrized by the dimensions and the McMillan degree of the FIRs.

Finally, we present six simple ways (along with their combinations) to construct, from any FIR, a large family of FIRs, of various dimensions and McMillan degrees, so that whenever the original system is in \( \mathcal{U} \), so is the whole family.

A key role is played by Hankel matrices.

1. Introduction

This work is on the crossroads of Operator and Systems theory from the mathematical side and Control, Signal Processing and Communications theory from the engineering side. It addresses problems or employs tools from all these areas. Thus, it is meant to serve as a bridge between the corresponding communities. We start by formally laying out the set-up.

1.1. Finite Impulse Response. We here focus on \( p \times m \)-valued polynomials of a complex variable \( z \), of the form

\[
F(z) = z^q (z^{-1}B_1 + \ldots + z^{-n}B_n),
\]

where the natural \( n \) and the integer \( q \) are parameters. Hereafter, we relate to Laurent polynomials when the powers of \( z \) may be positive or negative (or both).

The polynomial \( F(z) \) in (1.1) may be viewed as the (two-sided) \( Z \)-transform of \( \Phi(t) \) a (discrete) time sequence,

\[
\Phi(t) = \delta_K(t - q + 1)B_1 + \ldots + \delta_K(t - q + n)B_n \quad t \text{ integral variable,}
\]

where \( \delta_K \) is the Kronecker delta,

\[
\delta_K(\beta) = \begin{cases} 1 & \beta = 0, \\ 0 & \beta \neq 0, \end{cases} \quad \beta \text{ integer.}
\]

Hence, in Engineering terminology \( \Phi(t) \) (and often \( F(z) \)) is referred to as Finite Impulse Response (i.e. the support of \( \Phi(t) \) is finite).

Moreover \( F(z) \) will be called \textit{causal} whenever \( 1 \geq q \) (i.e. \( \Phi(t) \equiv 0 \) for all \( 0 > t \)) and \textit{strictly causal} if \( 0 \geq q \). Similarly, \( \textit{(strictly) anti-causal} \) when \( (q \geq n + 1) \ q \geq n \).

\textit{Key words and phrases.} Finite Impulse Response, Laurent polynomials, isometry, co-isometry, realization, Blaschke-Potapov, Hankel operator

\textit{AMS 2010 subject classification index:} 11C08, 11C20, 20H05, 26C05, 47A15, 47B35, 51F25, 93B20, 94A05, 94A08, 94A11, 94A12.

This research is partially supported by the BSF grant no. 2010117.

D. Alpay thanks the Earl Katz family for endowing the chair which supported his research.
When anti-causal, $F(z)$ in (1.1) is a usual polynomial with non-negative powers of the variable $z$.

Finite Impulse Response functions (=Laurent polynomials) are of numerous applications in communications control and signal processing see e.g. [12], [13], [14], [15], [27], [28], [39], [42].

1.2. Unitary symmetry. Let $\mathbb{T}$ be the unit circle,

$$\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}.$$  

We denote by $\mathcal{U}$ the class of $p \times m$-valued rational functions having unitary symmetry on the unit circle, i.e.

(1.3)

$$\mathcal{U} := \left\{ F(z) : \begin{array}{ll}
(F(z))^* F(z) = I_m & p \geq m \text{ isometry} \\
F(z) (F(z))^* = I_p & m \geq p \text{ co-isometry} 
\end{array} \forall z \in \mathbb{T} \right\}.$$  

In engineering terminology, if in addition all poles of $F(z)$ are within the open unit disk (=Schur stable), the function $F(z)$ is called lossless\footnote{Passive electrical circuits are either dissipative or lossless.}, see e.g. [17], [40, Section 14.2] or all-pass\footnote{For example, in studying classical filters a “high-pass” could be viewed as an “all-pass” minus a “low-pass”}. For a given $p \times m$-valued rational function $F(z)$ we here define the $m \times p$-valued conjugate rational function as,

$$F^\#(z) := \left( F\left( \frac{1}{z} \right) \right)^*.$$  

Note that

$$F^\#(z)_{|z \in \mathbb{T}} = \left( F(z)_{|z \in \mathbb{T}} \right)^*.$$  

It is well known, see e.g. [1, Eq. (3.1)], [23, Définition 35], that for rational functions condition (1.3) is equivalent to the following,

$$F \in \mathcal{U} \iff \forall z \in \mathbb{C} \begin{array}{ll}
F^\#(z) F(z) = I_m & p \geq m \text{ isometry}, \\
F(z) F^\#(z) = I_p & m \geq p \text{ co-isometry}.
\end{array}$$  

The interest in the class $\mathcal{U}$ stems from various aspects: For realization see e.g. [1], [2], [8] [17] and [41] for factorization see e.g. [33] and for some signal processing applications see e.g. [10].

1.3. The current work. The interest in Finite Impulse Response functions within $\mathcal{U}$ (=para-unitary, in signal processing “dialect”) is vast, see e.g. the books [11], [25], [29] Section 7.3], [37] Section 5.2], [40] Section 6.5], the theses [23], [31] and the papers [4], [5], [7], [10], [16], [22], [26], [32], [34], [36] and [43]. For example, the classical spectral factorization of self-adjoint matrices and the singular values decomposition of rectangular matrices (for constant matrices see e.g. [21] Theorems 2.5.6, 4.1.5] and [21] Theorem 7.3.5], respectively) have been generalized to matrix valued polynomials (where at least one of the factors is para-unitary). These extension have several signal processing applications and were studied in [39] and [38].
Similarly, the Q-R factorization of constant matrices has been generalized to matrix-valued polynomials (one of which is para-unitary). In [12], [13] this was studied along with applications to OFDM communications.

This work focuses on characterizations of families of rectangular (not necessarily causal) Finite Impulse Response (FIR) functions within $\mathcal{U}$.

This work is aimed at three different communities: mathematicians interested in classical analysis, signal processing engineers and system and control engineers. Thus adopting the terminology familiar to one audience, may intimidate or even alienate the other. For example, as already mentioned, matrix-valued Laurent polynomials (powers of various signs) and not necessarily causal Finite Impulse Response systems, are virtually the same entity seen by a different community. Similarly, what is known to engineers as McMillan degree also arises in geometry of loop groups as an index.

Books like [11], [19], [37], and the theses [23], [31] have made an effort to be at least “bi-lingual”. Lack of space prevents us from providing even a concise dictionary of relevant terms. Thus, we shall try to employ only basic concepts.

This work is organized as follows.

In Subsection 2.1 we construct the Hankel matrix $H$ associated with a polynomial $F(z)$ in (1.1) and show how that McMillan degree can be extracted from it. In Subsection 2.2 we show how to construct from a single Laurent polynomial a whole family of Laurent polynomials of various powers and degrees. Moreover, whenever the original polynomial is in $\mathcal{U}$, so is the whole resulting family. This construction is based on the Hankel matrix $H$. The details are relegated to the Appendix.

In subsection 3.1 we present a characterization of causal Schur stable rational functions in $\mathcal{U}$ through their minimal realization matrix. In subsection 3.2 we present the Blaschke-Potapov characterization of rational functions in $\mathcal{U}$. Here, the rational functions may have poles everywhere (including infinity) except the unit circle. In Theorem 3.5 we introduce three convenient formulations of this result, in the framework of FIRs.

As a by-product, this enables us to offer an easy-to-use description of all FIRs in $\mathcal{U}$ with McMillan degree and dimensions as parameters. In fact, for causal systems, it turns out to be a genuine real polytope and in general these are copies of this polytope. This is in particular convenient if one wishes to: (i) Design FIRs within $\mathcal{U}$ through optimization, see e.g. [34]. (ii) Iteratively apply para-unitary similarity, see e.g. [23, Section 3.3], [30], [36]. In signal processing literature, this is associated with channel equalization and in communications literature with decorrelation of signals or (iii) Iteratively apply Q-R factorization in the framework of communications, see e.g. [12], [13].

In Subsection 4.1 we return to the Hankel $H$ and show that more information is “encoded” in it: Out of $H$ one can check whether or not $F(z)$ is in $\mathcal{U}$ (see Theorem 4.1). Moreover, just from the singular values of $H$ one can deduce whether $F(z)$ is square or rectangular (see Proposition 4.2). It turns out that the conditions there are closely related to those in the Nehari problem where one approximates an anti-causal rational polynomial, by a causal one, see e.g. [19, Section 12.8].
A concluding remark is given at the end.

2. THE HANKEL MATRIX AND FIR

In this section we present our first use of Hankel matrices. They will be re-appear in Section 4 and the Appendix.

2.1. Realization of FIR and the Hankel matrix.

In this subsection we focus on the McMillan degree a $p \times m$-valued polynomial polynomial

\[ F(z) = z^q (z^{-1}B_1 + \ldots + z^{-n}B_n), \]

where the natural $n$ and the integer $q$ are parameters.

We here specialize textbook material on state space realization (of not necessarily causal systems) and the corresponding Hankel matrices, see e.g. [19, Subsection 12.8.1].

We start by recalling some classical facts concerning realization of $p \times m$-valued rational function $F(z)$. With a slight abuse of notation, an arbitrary $p \times m$-valued rational function $F(z)$ may be written as

\[ F(z) = F_l(z) + D + F_r(z) \]

where $D$ is a constant matrix and

\[
\lim_{z \to 0} F_l(z) = 0_{p \times m}, \quad \lim_{z \to \infty} F_r(z) = 0_{p \times m},
\]

(the subscripts stand for “left” and “right”). Note that $F_l(z)$, $F_r(z)$ may be viewed as the (two-sided) $Z$-transform of strictly anti-causal, strictly causal (discrete) time sequences, respectively.

Recall that $\tilde{F}_r(z)$ a $p \times m$-valued rational function with no poles at infinity is given by,

\[ \tilde{F}_r(z) := F_r(z) + D = C(zI_n - A)^{-1}B + D. \]

Sometimes it is convenient to present $\tilde{F}_r(z)$ in (2.3) by its $(\nu+p) \times (\nu+m)$ realization matrix $R$, i.e.

\[ R := \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \]

A realization of $\tilde{F}_r(z)$ in (2.3), is called minimal if in (2.4), $\nu$ the dimension of $A$, is the smallest possible. This $\nu$ is called the McMillan degree of $\tilde{F}_r(z)$ in (2.3).

As already mentioned, in the polynomial framework in (2.1), $F(z)$ is causal when $1 \geq q$ and strictly causal when $0 \geq q$. A special attention will be devoted to the case where in (2.1) $q = 0$, i.e.

\[ F_0(z) := \tilde{F}_r(z)|_{q=0} = F(z)|_{q=0} = z^{-1}B_1 + \ldots + z^{-n}B_n, \]

\[ = \text{Bounded at infinity and in engineering called causal or colloquially proper.} \]
For future reference, we introduce the following notation

\[
B_\eta := \begin{pmatrix}
  0_{n \times p} & B_1 & \cdots & B_n \\
  \vdots & B_1 & \cdots & B_n \\
  \vdots & \vdots & \ddots & \vdots \\
  0_{p \times m} & B_1 & \cdots & B_n
\end{pmatrix}, \quad \eta = 0, 1, 2, \ldots
\]

and by \( J_k \) the following \( kp \times kp \) block-shift matrix,

\[
J_k = \begin{pmatrix}
  I_p & \cdots & I_p \\
  \vdots & \ddots & \vdots \\
  0_{p \times p} & \cdots & 0_{p \times p}
\end{pmatrix}.
\]

With this notation, specializing the realization matrix (2.4) to the case of \( F_\eta(z) \) in (2.5), yields

\[
R := \begin{pmatrix}
  J_n & B_0 \\
  I_p & 0_{p \times (n-1)p} & 0_{p \times m}
\end{pmatrix},
\]

where \( B_0 \) is as in (2.6) (with \( \eta = 0 \)).

This classical approach has two limitations:

(i) Often the realization in (2.7) is not minimal. Moreover, the actual McMillan degree (bounded from above by \( np \)) is not apparent from (2.7).

(ii) Strictly speaking (2.4) and its special case in (2.7) are realization around zero, suitable for causal systems. Realizations for the anti-causal case (around \( z = \infty \)), are fairly common as well. Although known, it is more challenging to write down realizations of polynomials \( F(z) \) containing powers of mixed signs, i.e. when in (2.1) \( q \in \{2, n-1\} \).

We next show that representing realizations of FIR systems through Hankel matrices\(^5\) circumvent both limitations:

It is suitable for all \( q \) in (2.1) and the McMillan degree is apparent.

In Subsection 4.1 below these Hankel matrices will be used to introduce a characterization of Finite Impulse Response functions within \( U \).

To study the Hankel matrix representation of \( F(z) \) in (2.1),

\[
F(z) = z^q \left( z^{-1}B_1 + \ldots + z^{-n}B_n \right), \quad q \text{ integral parameter},
\]

we find it convenient to separately consider two extremes possibilities, and an intermediate case:

(i) \( q \geq n + 1 \) so \( F(z) \) is strictly anti-causal and

(ii) \( 0 \geq q \) so \( F(z) \) is strictly causal,

(iii) \( q \in [1, n] \) so \( F(z) \) is a genuine Laurent polynomial.

(i) Assume now that \( F(z) \) in (2.1) is strictly causal, \( -q := \eta \geq 0 \). Here (2.1) takes the form

\[
F(z) = F_r(z) = z^{-(1+\eta)}B_1 + \ldots + z^{-(n+\eta)}B_n \quad \eta \geq 0.
\]

\(^4\)In the sequel, boldface characters will stand for block-structured matrices.

\(^5\)In general, the Hankel operator is infinite, but since we here focus on \( F(z) \) in (2.2) with a Finite Impulse Response, the corresponding Hankel matrix is finite and no truncation is needed.
We shall denote by $H_\eta$ (recall, block-structured matrices are represented by boldface characters) the associated $p(n + \eta) \times m(n + \eta)$ Hankel matrix,

\begin{equation}
H_\eta = \begin{pmatrix}
B_1 & \cdots & B_n \\
\vdots & \ddots & \vdots \\
B_n \\
\end{pmatrix}.
\end{equation}

The $p \times m$ (block) elements of $H_\eta$ are known as the Markov parameters of $F_r(z)$ and in particular, the first (block) row of $H_\eta$ is the impulse response of $F_r(z)$.

For completeness we add that the Hankel matrix can be obtained as a product of the observability and controllability matrices here it takes the form

\begin{equation}
H_\eta = \begin{pmatrix} J^0 & J^1 & \cdots & J^{n+\eta-1} \end{pmatrix} \begin{pmatrix} B_n & \cdots & B_1 \end{pmatrix},
\end{equation}

where $B_\eta$ is as in (2.6), $J = J_{n+\eta}$ and $J^0 = I_{(n+\eta)p}$.

Associating the Hankel matrix $H_\eta$ in (2.8) with the polynomial $F_r(z)$ is fairly classical and goes back at least to [18 Eq. (7)].

(ii) The other extreme is where $F(z)$ is strictly anti-causal, i.e. in (2.10) $q \geq n + 1$ so $F(z)$ is a genuine polynomial with $\eta := q - n - 1 \geq 0$,

\begin{equation}
F(z) = F_r(z) = z^{n+\eta}B_1 + \ldots + z^{1+\eta}B_n, \quad \eta \geq 0.
\end{equation}

Then, the corresponding $p(n + \eta) \times m(n + \eta)$ Hankel matrix $\hat{H}_\eta$ (hat for left polynomial) takes the form of

\begin{equation}
\hat{H}_\eta = \begin{pmatrix} B_n & \cdots & B_1 \\
B_\eta \\
\vdots \\
B_1 \\
\end{pmatrix}.
\end{equation}

(iii) In the intermediate case, where $q \notin [1, n]$, $F(z)$ in (2.11) is a genuine Laurent polynomial and we shall write it in the form of (2.2) (with $B_\eta = D$) as

\begin{equation}
\begin{align*}
F(z) &= F_r(z) + D + F_r(z) \\
F_r(z) &= z^{-1}B_{q+1} + z^{-2}B_{q+2} + \ldots + z^{-n}B_n, \quad q \in [1, n], \\
F_l(z) &= z^{n-1}B_1 + z^{n-2}B_2 + \ldots + z\eta B_{\eta-1}.
\end{align*}
\end{equation}

The corresponding Hankel matrices for $F_r(z)$ and $F_l(z)$, respectively are

\begin{equation}
H = \begin{pmatrix} B_{q+1} & B_{q+2} & \cdots & B_{n-1} & B_n \\
B_{q+2} & B_{q+3} & \cdots & B_n \\
\vdots & \vdots & \ddots & \vdots \\
B_n \\
\end{pmatrix} \quad \text{and} \quad \hat{H} = \begin{pmatrix} B_{q+1} & B_{q+2} & \cdots & B_1 \\
B_{q+2} & B_{q+3} & \cdots & B_1 \\
\vdots & \vdots & \ddots & \vdots \\
B_1 \\
\end{pmatrix}.
\end{equation}

It is well known, see e.g. [9 Theorem 4.5], that the McMillan degree of $F(z)$ in (2.2) and in (2.10) is equal to the sum of the ranks of $\hat{H}$ and $H$, the associated Hankel matrices. Thus we can now state the main result of this subsection.

**Observation 2.1.** Let us denote by $d$ the McMillan degree of the $p \times m$-valued (possibly Laurent) polynomial $F(z)$ in (2.1),

\[ F(z) = z^q \left( z^{-1}B_1 + \ldots + z^{-n}B_n \right). \]
Let $H, \hat{H}$ be the Hankel matrices associated with $F_r(z), F_l(z)$, respectively. Then,

$$d = \begin{cases} 
\text{rank}(H) & 0 \geq q \quad \text{Eq. (2.8)} \\
\text{rank}(\hat{H}) & q \geq n+1 \quad \text{Eq. (2.9)} \\
\text{rank}(H) + \text{rank}(\hat{H}) & q \in [1, n] \quad \text{Eq. (2.11)} 
\end{cases}$$

So far, we have used the Hankel matrix $H$ to obtain the McMillan degree of a given FIR $F(z)$. In Subsection 4.1 below we show that this $F(z)$ is in $\mathcal{U}$ if and only if $H^*H$ (or $HH^*$) have a certain invariant subspace.

2.2. Families of FIR systems in $\mathcal{U}$. In this subsection we show how to produce, out of a given $p \times m$-valued Laurent polynomial,

$$(2.12) \quad F(z) = z^q \left( z^{-1}B_1 + \ldots + z^{-n}B_n \right),$$

a whole family of Laurent polynomials of various dimensions and powers. Moreover, whenever the original one is in $\mathcal{U}$, then so is all the resulting family of polynomials.

Clearly, when $F(z)$ in (2.12) is in $\mathcal{U}$, so are $F(z)U_m$ and $U_pF(z)$, where $U_m$ and $U_p$ are arbitrary $m \times m$ and $p \times p$ constant unitary matrices.

Note also that if for some value of the parameter $q$, $F(z)$ in (2.12) is in $\mathcal{U}$, then this is the case for all $q$. Thus, without loss of generality, we can take all polynomials to be causal, i.e. in (2.12) $0 \geq q$.

We here illustrate six versions of the newly generated polynomials. Obviously, to further enrich the variety, they may be combined.

I. The reverse polynomial,

$$F_{rev}(z) := z^{-1}B_n + z^{-2}B_{n-1} + \ldots + z^{1-n}B_2 + z^{-n}B_1.$$  

II. Preserving the McMillan degree.

For $-1 \geq q$ and $j = 1, \ldots, 1 - q$, one can generate $jp \times jm$-valued polynomials. For example, taking in (2.12) $n = 4$ one may obtain:

a. For $q = -1$

$$F(z) = z^{-1}(z^{-1}B_1 + z^{-2}B_2 + z^{-3}B_3 + z^{-4}B_4), \quad p \times m- \text{ valued}$$

$$F(z) = z^{-1} \left( \begin{array}{c} 0 \\ B_1 \end{array} \right) + z^{-2} \left( \begin{array}{c} B_2 \\ B_3 \end{array} \right) + z^{-3} \left( \begin{array}{c} B_4 \\ 0 \\ 0 \end{array} \right), \quad 2p \times 2m- \text{ valued},$$

b. For $q = -2$

$$F(z) = z^{-2}(z^{-1}B_1 + z^{-2}B_2 + z^{-3}B_3 + z^{-4}B_4), \quad p \times m- \text{ valued}$$

$$F(z) = z^{-1} \left( \begin{array}{c} 0 \\ 0 \\ B_1 \end{array} \right) + z^{-2} \left( \begin{array}{c} B_2 \\ B_3 \end{array} \right) + z^{-3} \left( \begin{array}{c} B_4 \\ 0 \\ 0 \end{array} \right), \quad 2p \times 2m- \text{ valued}$$

$$F(z) = z^{-1} \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ B_1 \end{array} \right) + z^{-2} \left( \begin{array}{c} B_2 \\ B_3 \end{array} \right) + z^{-3} \left( \begin{array}{c} B_4 \\ 0 \\ 0 \end{array} \right), \quad 3p \times 3m- \text{ valued}.$$  

We shall see that both polynomials in a share the same McMillan degree. A similar statement holds for the three polynomials in b.

III. Doubling the powers.

For a pair of parameters $a$ integer and $\gamma$ natural,

$$(2.13) \quad F(z) = z^a \left( z^{-1}\gamma B_1 + z^{-2}\gamma B_2 + \ldots + z^{-n}\gamma B_n \right).$$
In particular, for $a = 0$, $F(z)$ in (2.14) may be written as $F(z^\gamma)$. Rational functions within $\mathcal{U}$, of this structure, suit what is known in signal processing as filter banks, see e.g. the books [37, 40] and the papers [3, 6, 14, 32, 20, 34, 32, 43].

In a similar way, one can obtain richer structures, e.g. of the form

\begin{equation}
F(z) = z^{-1}B_1 + z^{-2}B_2 + z^{-3}B_3 + z^{-9}B_4 + z^{-10}B_5 + z^{-11}B_6.
\end{equation}

IV. Rectangular polynomials. Let $\rho$ be a parameter so that $\frac{n}{\rho}$ is natural.

\begin{equation}
F(z) = z^{a\rho} \left( z^{-1-\gamma\rho}B_1 + z^{-2-\gamma\rho}B_2 + \ldots + z^{-n-\gamma\rho}B_n \right).
\end{equation}

a. $F(z)$ is a $\rho \times m$-valued polynomial with coefficients,

\[ B_1 = \begin{pmatrix} B_1 \\ \vdots \\ B_p \end{pmatrix}, \quad B_2 = \begin{pmatrix} B_{p+1} \\ \vdots \\ B_{2\rho} \end{pmatrix}, \quad \ldots, \quad B_{n} = \begin{pmatrix} B_{n+1-\rho} \\ \vdots \\ B_{n} \end{pmatrix}. \]

b. $F(z)$ is a $p \times \rho m$-valued polynomial with coefficients,

\[ B_1 = (B_1 \ldots B_p), \quad B_2 = (B_{p+1} \ldots B_{2\rho}), \quad \ldots, \quad B_n = (B_{n+1-\rho} \ldots B_n). \]

V. Composition of polynomials. Out of the pair of polynomials,

\begin{equation}
F_b(z) = z^{-1}B_1 + \ldots + z^{-n}B_n, \quad p_b \times m_b \\
F_c(z) = z^{-1}C_1 + \ldots + z^{-l}C_l, \quad p_c \times m_c
\end{equation}

craft the following third polynomial

\[ F_d(z) = z^{-1}D_1 + \ldots + z^{-n}D_n, \]

in three different forms.

a. $F_d(z)$ is $(p_b + p_c) \times (m_b + m_c)$-valued polynomial with coefficients,

\[ D_1 = \begin{pmatrix} B_1 & 0 \\ 0 & C_1 \end{pmatrix}, \quad D_{i+1} = \begin{pmatrix} B_{i+1} & 0 \\ 0 & C_{i+1} \end{pmatrix}, \quad \ldots, \quad D_n = \begin{pmatrix} B_n & 0 \\ 0 & C_n \end{pmatrix}, \]

or

\[ D_1 = \begin{pmatrix} 0 & B_1 \\ C_1 & 0 \end{pmatrix}, \quad \ldots, \quad D_{i+1} = \begin{pmatrix} 0 & B_{i+1} \\ C_{i+1} & 0 \end{pmatrix}, \quad \ldots, \quad D_n = \begin{pmatrix} 0 & B_n \\ C_n & 0 \end{pmatrix}. \]

If both $F_b(z)$ and $F_c(z)$ are isometries on $T$, then so is $F_d(z)$.

If both $F_b(z)$ and $F_c(z)$ are co-isometries on $T$, then so is $F_d(z)$.

b. For $m_c \geq m_b$, $F_d(z)$ is $(p_b + p_c) \times m_c$-valued polynomial with coefficients,

\[ D_1 = \begin{pmatrix} \sqrt{\alpha}B_1 & 0_{p_b \times (m_c - m_b)} \\ 0_{\sqrt{1-\alpha}C_1} & 0 \end{pmatrix}, \quad \ldots, \quad D_{i+1} = \begin{pmatrix} \sqrt{\alpha}B_{i+1} & 0_{p_b \times (m_c - m_b)} \\ 0_{\sqrt{1-\alpha}C_{i+1}} & 0 \end{pmatrix}, \quad \ldots, \quad D_n = \begin{pmatrix} \sqrt{\alpha}B_n & 0_{p_b \times (m_c - m_b)} \\ 0_{\sqrt{1-\alpha}C_n} & 0 \end{pmatrix}, \\]

\[ \alpha \in [0, 1] \] parameter.

If both $F_b(z)$ and $F_c(z)$ are isometries on $T$, then so is $F_d(z)$.

\[ \text{For given } n \text{ and } \rho, \text{ one can always find } \zeta \in [0, \rho - 1] \text{ so that } \frac{n + \alpha}{\rho} \text{ is natural. Then, the last part of } B_n \text{ in (2.15) is comprised of zeros.} \]
There are several variants of characterizations of $F_d(z)$ is $p_b \times (m_b + m_c)$-valued polynomial with coefficients
\[
D_1 = \begin{pmatrix}
\sqrt{\alpha B_1} & \sqrt{1-\alpha} C_1 \\
0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c}
\end{pmatrix} \quad \ldots \quad D_i = \begin{pmatrix}
\sqrt{\alpha B_i} & \sqrt{1-\alpha} C_i \\
0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c}
\end{pmatrix} \quad \alpha \in [0, 1] \text{ parameter.}
\]
\[
D_{i+1} = \begin{pmatrix}
\sqrt{\alpha B_{i+1}} & 0_{m_b \times m_c} \\
0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c}
\end{pmatrix} \quad \ldots \quad D_n = \begin{pmatrix}
\sqrt{\alpha B_n} & 0_{m_b \times m_c} \\
0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c}
\end{pmatrix}
\]
If both $F_b(z)$ and $F_c(z)$ are co-isometries on $\mathbb{T}$, then so is $F_d(z)$.

VI. Product of the polynomials.

\[
F_b(z) = z^{-1} B_1 + \ldots + z^{-n} B_n \quad p_b \times \rho
\]
\[
F_c(z) = z^{-1} C_1 + \ldots + z^{-l} C_l \quad \rho \times m_c
\]
i.e. $F_d(z)$ is the following $p_b \times m_c$-valued polynomial
\[
F_b(z) F_c(z) := F_d(z) = z^{-1} \left( z^{-1} D_1 + \ldots + z^{-(n+l-1)} D_{n+l-1} \right),
\]
where a straightforward computation yields that the coefficients $D_1, \ldots, D_{n+l-1}$ are given by,
\[
D_{1,n+l-1} = H \cdot T_{n+l, \rho} \cdot C_{1,l}
\]
with $C_{1,l}$ and $D_{1,n+l-1}$ in the spirit of (2.10), $H$ is the $(n+l)p \times (n+l)m$ Hankel matrix as in (2.3) (with $\eta = l$) and
\[
T_{n+l, \rho} := \begin{pmatrix}
I_{\rho} & I_p & \cdots \\
I_p & I_{\rho} & \cdots \\
\vdots & \ddots & \ddots
\end{pmatrix}.
\]
In the Appendix below we show how each of the new polynomials, in the above items I through VI, is constructed. A key tool there will be the associated Hankel matrix.

Furthermore, as already pointed out, we shall see that if the original polynomial $F(z)$ in (2.12) is in $\mathcal{U}$, so are all the resulting polynomials.

To this end, in Subsections 3.1, 3.2 and 4.1 respectively, we present three characterizations of Laurent polynomials in $\mathcal{U}$.

3. Minimal realization of (co-)isometric FIR

3.1. Characterization through realization matrices of Schur stable systems. In this subsection we characterize, through a corresponding realization matrix, rational functions, analytic outside the open unit disk (Schur stable), which are in $\mathcal{U}$.

There are several variants of characterizations of $F \in \mathcal{U}$ through its minimal realization matrix $R$. The square case ($m = p$) was addressed in [17] Lemma 2 & Theorem 3. Another version of it appeared in [1] Theorem 3.1 and [2] Theorem 2.1. Below we cite and adapted form of [40] Theorem 14.5.1. A more general case was studied in [8] Theorem 4.5. [1] in fact, in [1, 2, 8] and [17] they address the indefinite inner product case where $(F(z))^\ast J_p F(z) = J_m$ with $J_p, J_m$ signature matrices, i.e. diagonal matrices satisfying $J_p^2 = I_p$ and $J_m^2 = I_m$. 
Theorem 3.1. Let $F(z)$ be a $p \times m$-valued rational function whose poles are within the open unit disk and let $R$ be a corresponding $(\nu+p) \times (\nu+m)$ minimal realization matrix \((2.4)\)

$$R := \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

I. Assume that $p \geq m$. $F(z)$ is in $\mathcal{U}$ if and only if,

\[ R^* \text{diag} \{ I_{\nu} \ I_m \} R = \text{diag} \{ I_{\nu} \ I_m \}. \tag{3.1} \]

II. Assume that $m \geq p$. $F(z)$ is in $\mathcal{U}$ if and only if,

\[ R \text{diag} \{ I_{\nu} \ I_p \} R^* = \text{diag} \{ I_{\nu} \ I_p \}. \tag{3.2} \]

We now recall the notion of Controllability and Observability Gramians, see e.g. \cite[Eqns (12.8.17), (12.8.37)].

Assuming that the spectrum of $A$ (the upper left block in $R$ in (2.4)) is within the open unit disk (Schur stable), $W_{\text{cont}}, W_{\text{obs}}$, the associated $\nu \times \nu$ Controllability and Observability Gramians, respectively, are given by the solution to the respective Stein equations

\[ W_{\text{cont}} - AW_{\text{cont}} A^* = BB^* \quad W_{\text{obs}} - A^* W_{\text{obs}} A = C^* C. \tag{3.3} \]

We can now state the following whose proof is given in \cite{5}.

Lemma 3.2. Let $F(z)$ be a $p \times m$-valued rational function whose poles are within the open unit disk and denote by $W_{\text{cont}}, W_{\text{obs}}$ the associated controllability and observability gramians, respectively.

Assume that $F(z)$ is in $\mathcal{U}$.

I. If $p \geq m$, $F(z)$ admits a state space realization $R$ in (3.1) so that

\[ (I_{\nu} - W_{\text{cont}}) \text{ positive semidefinite} \quad W_{\text{obs}} = I_{\nu}. \]

II. If $m \geq p$, $F(z)$ admits a state space realization $R$ in (3.2) so that

\[ W_{\text{cont}} = I_{\nu} \quad (I_{\nu} - W_{\text{obs}}) \text{ positive semidefinite}. \]

III. If $p = m$, $F(z)$ admits a state space realization $R$ in (3.1), (3.2) so that

\[ W_{\text{cont}} = I_{\nu} \quad W_{\text{obs}} = I_{\nu}. \]

This result will be used in the proof of Proposition 4.2 below.

We conclude this subsection by illustrating the results of Observation 2.1, Theorem 3.1 and of Lemma 3.2

Example 3.3. I. Consider the $2 \times 2$-valued polynomial $F(x)$ in (1.1) with $n = 3$ i.e.

\[ F(z) = z^q (z^{-1} B_1 + z^{-2} B_2 + z^{-3} B_3), \]

where

\[ B_1 = \frac{1}{2} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad B_2 = \frac{1}{2} \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}, \quad B_3 = \frac{1}{2} \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{3.4} \]

and $q$ is a parameter assuming the values 2 and 1.

(i) For $q = 2$, $F(z)$ takes the form,

\[ F(z) = z B_1 + B_2 + z^{-1} B_3. \]
Namely, as in (2.10), and here (2.11) takes the form of $H = B_3$ and $\hat{H} = B_1$ so the McMillan degree is $d = \text{rank}(\hat{H}) + \text{rank}(H) = 2$.

A minimal realization (i.e. of dimension 2) is

$$F(z) = zv_1v_1^* + B_2 + \frac{1}{z}v_3v_3^*$$

with $v_1 = \sqrt{2} \sqrt{5} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_3 = \sqrt{2} \sqrt{5} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(ii) For $q = 1$, $F(z)$ is causal and takes the form,

$F(z) = B_1 + z^{-1}B_2 + z^{-2}B_3$.

A straightforward dimension 4 realization of the form (2.7) is

$$\tilde{R}_r = \begin{pmatrix} 0 & I_2 & B_2 \\ 0 & 0 & B_3 \\ I_2 & 0 & B_1 \end{pmatrix}.$$ 

A closer scrutiny reveals that here (2.11) takes the form of $\hat{H}$ empty, and

$$H = \begin{pmatrix} B_2 & B_3 \\ B_3 & 0 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 & 0 & 3 & 2 \\ 3 & 0 & -2 & 2 \\ -2 & -2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{pmatrix}.$$ 

Thus the McMillan degree is $d = \text{rank}(\hat{H}) + \text{rank}(H) = 0 + 2 = 2$.

Indeed, a minimal realization is

$$\tilde{R}_r = \frac{1}{5} \begin{pmatrix} -2 & 0 & 3 & 2 \\ -2 & 0 & 3 & 2 \\ -2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$ 

Recall that from both parts of Theorem 3.1 it follows that $R_rR_r^* = \text{diag} \{I_2, I_2\} = R_r^*R_r$.

II. Consider the $1 \times 2$-valued polynomial

(3.5) $F(z) = z^q (z^{-1} (0 - \frac{1}{z}) + z^{-2} (\frac{1}{z} 0)),$

where the parameter $q$ assume the values 0 and 1.

For $q = 0$, $F(z)$ is strictly causal. The associated Hankel matrix $H$ in (2.8) is given by,

(3.6) $H = \frac{1}{4} \begin{pmatrix} 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

and the Hankel singular values are 1 and 0.8. A minimal realization is

$$R = \begin{pmatrix} 0 & \frac{4}{5} & 0 & -\frac{3}{5} \\ \frac{4}{5} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ 

Indeed, from part II of Theorem 3.1 it follows that $R$ is co-isometry, i.e. $RR^* = \text{diag} \{I_2, 1\}$ and from Lemma 3.2 we have that the Observability Gramian is $W_{\text{obs}} = \text{diag} \{1 \frac{16}{25} \frac{16}{25} \}$.

Substituting in (3.5) $q = 1$ yields the causal polynomial,

$F(z) = (0 - \frac{1}{z}) + z^{-1} (\frac{1}{z} 0)$.

A corresponding minimal realization is

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{z} & 0 & 0 & -\frac{1}{z} \end{pmatrix}.$$ 

Indeed, $RR^* = \text{diag} \{1 \ 1\}$ and $W_{\text{obs}} = \frac{16}{25} \frac{16}{25}$. □
In this subsection we have assumed that the system is causal and in particular Schur stable. In the sequel, we remove this restriction.

3.2. A characterization through the Blaschke-Potapov product. We first need some preliminaries. We shall use the fact that a $k \times k$ rank one orthogonal projection,

$$P^* = P = P^2 \quad \text{rank}(P) = 1,$$

can always be written as

$$P = vv^* \quad v^* v = 1 \quad v \in \mathbb{C}^k.$$  \hfill (3.7)

Using (3.7), a rank $k - 1$ orthogonal projection $Q$ i.e.

$$Q^* = Q^2 = Q \quad \text{rank}(Q) = k - 1,$$

can always be written as

$$Q := I_k - vv^*. \quad \hfill (3.8)$$

We can now cite the classical Blaschke-Potapov product result (as appeared in [5]).

**Lemma 3.4.** A $p \times m$-valued rational function $F(z)$, of McMillan degree $d$, is in $U$, (1.3), if and only if

$$p \geq m \quad F(z) = \left( \prod_{j=1}^d \left( I_p + \left( \frac{1-\alpha j z}{z-\alpha j} - 1 \right) v_j v_j^* \right) \right) U_{iso} \quad v_j \in \mathbb{C}^p \quad v_j^* v_j = 1$$

$$m \geq p \quad F(z) = U_{coiso} \left( \prod_{j=1}^d \left( I_m + \left( \frac{1-\alpha j z}{z-\alpha j} - 1 \right) v_j v_j^* \right) \right) \quad v_j \in \mathbb{C}^m \quad v_j^* v_j = 1$$

Recall $\prod_{j=1}^0 := I$

In Theorem 3.5 below, we specialize this result to the FIR framework. To this end, we recall that

$$\left( I + \left( \frac{1-\alpha z}{z-\alpha} - 1 \right) vv^* \right)^{-1} = \left( I + \left( \frac{1-\alpha z}{1-\alpha z} - 1 \right) vv^* \right) \quad v^* v = 1$$

We can now formulate the main result of this subsection.

**Theorem 3.5.** Let $F(z)$ be a $p \times m$-valued Finite Impulse Response function of McMillan degree $d$.

I. Assuming $p \geq m$, $F(z)$ is isometrically in $U$ if and only if it can be written in one, and hence in each, of these three equivalent forms,

$$F(z) = \begin{cases}
\prod_{\gamma=1}^d \left( I_p + (z-1)v_j v_j^* \right) & \prod_{\gamma=1}^d \left( I_p + (1-\gamma v_j v_j^*) \right) U_{iso} \\
\prod_{j=1}^d \left( I_p + (z-1)v_j v_j^* \right) & \prod_{j=1}^d \left( I_p + (1-\gamma v_j v_j^*) \right) \quad v_j \in \mathbb{C}^p \quad v_j^* v_j = 1 \quad U_{iso} U_{iso} = I_m \\
\prod_{j=1}^d \left( I_p + (1-\gamma v_j v_j^*) \right) & \prod_{j=1}^d \left( I_p + (1-\gamma v_j v_j^*) \right) \quad \gamma \in [0, d].
\end{cases} \hfill (3.11)$$

II. Assuming $m \geq p$, $F(z)$ is coisometrically in $U$ if and only if it can be written in one, and hence in each, of these three equivalent forms,

$$F(z) = \begin{cases}
\prod_{\gamma=1}^d \left( I_m + \left( \frac{1-\alpha j z}{z-\alpha j} - 1 \right) v_j v_j^* \right) U_{iso} \\
\prod_{j=1}^d \left( I_m + \left( \frac{1-\alpha j z}{z-\alpha j} - 1 \right) v_j v_j^* \right) \quad v_j \in \mathbb{C}^m \quad v_j^* v_j = 1 \quad U_{coiso} U_{coiso} = I_m \\
\prod_{j=1}^d \left( I_m + \left( \frac{1-\alpha j z}{z-\alpha j} - 1 \right) v_j v_j^* \right) \quad \gamma \in [0, d].
\end{cases} \hfill (3.12)$$

III. Assuming $p = m$, $F(z)$ is coisometrically in $U$ if and only if it can be written in one, and hence in each, of these three equivalent forms,
II. For \( m \geq p \), \( F(z) \) is co-isometrically in \( \mathcal{U} \) if and only if it can be written in one, and hence in each, of these three equivalent forms,\(^{13}\)

\[
F(z) = \begin{cases} 
U_{\text{coiso}} \prod_{j=1}^{\gamma} \left( I_m + (\frac{1}{z} - 1)v_j v_j^* \right) & \text{if } \alpha = 0 \\
U_{\text{coiso}}^* \prod_{j=1}^{\gamma} \left( I_m + (1 - z)v_j v_j^* \right) & \text{if } \alpha = d
\end{cases}
\]

\[(3.12)\]

In particular, the function \( F(z) \) in \((3.11),(3.12)\) is causal for \( \gamma = 0 \) and anti-causal for \( \gamma = d \).

The first version in both \((3.11)\) and \((3.12)\) is a re-writing of \((3.9)\) with \( \alpha_1 = \ldots = \alpha_\gamma = \infty \) and \( \alpha_{\gamma+1} = \ldots = \alpha_d = 0 \). Using \((3.10)\), the other two versions follow from the first.

Note that it is only for the causal case, \( \gamma = 0 \), or the anti-causal case, \( \gamma = d \), that in \((3.11)\) a product of the form,

\[
\prod_{j=1}^{\gamma} \left( I_p + (z-1)v_j v_j^* \right) \prod_{j=\gamma+1}^{d} \left( I_p + (1-z)v_j v_j^* \right) U_{\text{iso}} \quad \text{if } \alpha = 0
\]

\[(3.13)\]

is of McMillan degree \( d \) for any choice of the projections \( v_1, \ldots, v_d \). For \( \gamma \in [1, d-1] \), the McMillan degree of the expression in \((3.13)\) is at most \( d \). For example, repetitive use of \((3.10)\) reveals that substituting in \((3.13)\)

\[v_j v_j^* = v_{d+1-j} v_{d+1-j}^* \quad j = 1, \ldots, \gamma \quad d = 2\gamma \quad \gamma \text{ natural},\]

yields the zero degree function \( I_p \).

Theorem \((3.5)\) asserts that whenever \( F \in \mathcal{U} \) is of McMillan degree \( d \), there exist rank one orthogonal projections \( v_1^*, \ldots, v_d^* \) satisfying \((3.11)\) and \((3.12)\).

Recall that employing optimization for design of FIRs is fairly common, see e.g. \([14, 15, 20, 39] \) and \([42] \) and for FIRs within \( \mathcal{U} \) see e.g. \([33] \). This motivates a convex description of large families of FIRs. To this end, we next specialize \([5, \text{Observation 4.3}] \).

**Corollary 3.6.** All \( p \times m \)-valued FIR rational functions of McMillan degree \( d \) in \( \mathcal{U} \) may be parametrized by,

\[
(d+1) \cdot [0, 2\pi)^{(2p-m-1)(m+d)+d(m-1)+m} \quad p \geq m
\]

\[(3.14)\]

The causal subset (here, =lossless subset), may be parametrized by,

\[0, 2\pi)^{(2p-m-1)(m+d)+d(m-1)+m} \quad p \geq m
\]

\[(3.15)\]

There are a few parametrizations of all FIRs in \( \mathcal{U} \), see e.g. \([23, \text{Propri´ et´ e 41}] \), \([40, \text{Section 14.4}] \) and the more detailed study in \([16] \). The above choice is advantageous as the set of parameters in \((3.15)\) is not only convex, but in fact a polytope. This corresponds to having in \((3.11)\) or in \((3.12)\), \( \gamma = 0 \). As the integral parameter \( \gamma \)
attains values in [0, d], the parameter set in (3.14) are d + 1 copies of this polytope. This is in particular convenient if one wishes to:
(i) Design FIRs within \( \mathcal{U} \) through optimization, see e.g. [33].
(ii) Iteratively apply para-unitary similarity, see e.g. [23, Section 3.3], [30], [36]. In signal processing literature, this is associated with channel equalization and in communications literature with decorrelation of signals or
(iii) Iteratively apply Q-R factorization in the framework of communications, see e.g. [12], [13].

We now establish a connection between Blaschke-Potapov description of \( F(z) \) in (3.11), (3.12) (with \( \gamma = 0 \)) and \( B_1, \ldots, B_n \) the coefficients of the polynomial \( F(z) \) in (3.16) with \( q = 1 \), i.e.

\[
F(z) = B_1 + z^{-1}B_2 + \ldots + z^{1-n}Z_n .
\]

**Observation 3.7.** Assume that the (causal) \( p \times m \)-valued polynomial \( F(z) \) in (3.16) is in \( \mathcal{U} \) and of McMillan degree \( d \).

Then if \( m = p \) up to multiplication by a unitary matrix from the left or from the right, the square coefficient matrices \( B_1, \ldots, B_n \), are given by,

\[
\begin{align*}
B_1 &= Q_1 \ldots Q_d \\
B_2 &= \sum_{j=1}^{d} Q_1 \ldots Q_{j-1} v_jv_j^* Q_{j+1} \ldots Q_d \\
B_3 &= \sum_{k=j+1}^{d} \sum_{j=1}^{d-1} Q_1 \ldots Q_{j-1} v_jv_j^* Q_{j+1} \ldots Q_kv_k^* Q_{k+1} \ldots Q_d \\
B_4 &= \sum_{q=k+1}^{d} \sum_{k=j+1}^{d-2} \sum_{j=1}^{d-2} Q_1 \ldots Q_{j-1} v_jv_j^* Q_{j+1} \ldots Q_{k-1}v_kv_k^* Q_{k+1} \ldots Q_{q-1}v_q^* Q_{q+1} \ldots Q_d \\
&\vdots \\
B_n &= v_1v_1^* \ldots v_dv_d^* 
\end{align*}
\]

where \( v_1v_1^* , \ldots, v_dv_d^* \), are rank one orthogonal projections (3.7) and \( Q_j := I - v_jv_j^* \), \( j = 1, \ldots, d \), see (3.8).

If \( p \geq m \), the above \( p \times p \) coefficient matrices \( B_1, \ldots, B_n \) are multiplied from the right by a \( p \times m \) isometry \( U_{iso} \), \( (U_{iso}^* U_{iso} = I_m) \).

If \( m \geq p \), the above \( m \times m \) coefficient matrices \( B_1, \ldots, B_n \) are multiplied from the left by a \( p \times m \) co-isometry \( U_{coiso} \), \( (U_{coiso}^* U_{coiso} = I_p) \).

In particular this implies that

\[
F(1) = B_1 + \ldots + B_n = \begin{cases} 
U_{iso} & F \text{ isometry} \\
U_{coiso} & F \text{ co-isometry} 
\end{cases}
\]

We conclude this section by pointing out that so far we have focused on characterizations of Finite Impulse Response functions in \( \mathcal{U} \) through their minimal realization. In the sequel this restriction is removed.

4. **Hankel matrices - revisited**

4.1. **Isometric FIR - a Hankel matrix characterization.** Let \( F(z) \), (1.1), be a Finite Impulse Response (possibly rectangular) rational function,

\[
F(z) = z^{q}(z^{-1}B_1 + \ldots, z^{-n}B_n) \quad q \text{ parameter.}
\]
Clearly, the above $F(z)$ is in $\mathcal{U}$, if and only if $F_o(z)$ in (2.5),
$$F(z)|_{z=0} = F_o(z) := z^{-1}B_1 + \ldots + z^{-n}B_n,$$
is in $\mathcal{U}$.

Thus, in the sequel we find it convenient to focus on $F_o(z)$. Subsequently, substituting in (2.6) and in (2.8) $\eta = 0$ one obtains the matrices
$$B_0 = \begin{pmatrix} B_1 \\ \vdots \\ B_n \end{pmatrix}, \quad H_0 = \begin{pmatrix} B_1 & \cdots & B_n \\ \vdots & \ddots & \vdots \\ B_n & \cdots & B_1 \end{pmatrix}.
$$

We shall also find it convenient to write $F_o(z)$ in (2.5) in each of the two following forms
(4.1) $$F_o(z) = ZB_0 = \hat{B}Z,$$
where $B_0$ is as before and
(4.2) $$\hat{B} := (B_1, \cdots, B_n),$$
and
(4.3) $$Z := (z^{-1} \cdots z^{-n}) \otimes I_p = (z^{-1}I_p \cdots z^{-n}I_p)$$
where $\otimes$ denotes the usual Kronecker (=tensor) product, see e.g. [22, Section 4.2].

We now state the main result of this subsection.

**Theorem 4.1.** Let $F_0(z)$ be a $p \times m$ polynomial in (2.5)
$$F_0(z) = z^{-1}B_1 + \ldots + z^{-n}B_n,$$
and let $H_0$ be the corresponding Hankel matrix.

The polynomial $F_o(z)$ is in $\mathcal{U}$ if and only if
(4.4) $$\begin{pmatrix} I_{nm} - H_0^\dagger H_0 \\ 0_{m(n-1) \times m} \end{pmatrix} \begin{pmatrix} I_m \\ 0_p \end{pmatrix} = 0_{pm \times m} \quad \text{for } p \geq m$$
$$\begin{pmatrix} I_p \\ 0_{p \times (n-1)p} \end{pmatrix} \begin{pmatrix} I_{np} - H_0 H_0^\dagger \\ 0_{np \times m} \end{pmatrix} = 0_{p \times np} \quad \text{for } m \geq p.$$

**Proof:** A straightforward substitution of $F_o(z)$ in the definition of $\mathcal{U}$ (1.3) yields the following characterization,
(4.5) $$\begin{pmatrix} \sum_{j=1}^n B_j^* B_j \\ \sum_{j=1}^{n-1} B_{i+j}^* B_j \\ \vdots \\ \sum_{j=1}^n B_n^* B_{i+j} \end{pmatrix} = \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \quad \text{for } p \geq m$$
$$\begin{pmatrix} \sum_{j=1}^n B_j^* B_j + \sum_{j=1}^{n-1} B_{i+j}^* B_j + \cdots + \sum_{j=1}^n B_{i+n-j}^* B_j \\ B_n^* B_{i+j} \end{pmatrix} = \begin{pmatrix} I_p \\ 0_{p \times (n-1)p} \end{pmatrix} \quad \text{for } m \geq p.$$
For the square case, see e.g. [23, Propriété 37].

Using the matrices \( H_0 \) and \( B_0 \) this may be equivalently written as,

\[
H_0^* B_0 = \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \quad p \geq m,
\]

(4.6)

\[
\hat{B} H_0^* = \begin{pmatrix} I_p \\ 0_{p \times (n-1)p} \end{pmatrix} \quad m \geq p.
\]

where \( \hat{B} \) is as in (4.2).

Now, since

\[
B_0 = H_0 \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} \quad \text{and} \quad \hat{B} = \begin{pmatrix} I_p \\ 0_{p \times (n-1)p} \end{pmatrix} H_0
\]

the relation in (4.6) is equivalent to

\[
H_0^* H_0 \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix} = \begin{pmatrix} I_m \\ 0_{m(n-1) \times m} \end{pmatrix}
\]

\[
(\hat{B} H_0^*) = (I_p \\ 0_{p \times (n-1)p}) H_0 H_0^* = (I_p \\ 0_{p \times (n-1)p}),
\]

which in turn can be written as (4.4). □

An alternative proof of the same result is given in [5, Theorem 5.2].

Theorem 4.1 offers a characterization of a Laurent polynomial \( F(z) \) in \( \mathcal{U} \) through the existence of a certain invariant subspace of \( H_0^* H_0 \) (or \( H_0 H_0^* \)), see (4.3). A closer scrutiny reveals that whenever \( F(z) \) is in \( \mathcal{U} \), more can be said about these matrices.

**Proposition 4.2.** Let \( F_o(z) \) be a \( p \times m \)-valued polynomial in (2.5)

\[
F_o(z) = z^{-1}B_1 + \ldots + z^{-n}B_n,
\]

and let \( H_0 \) be the corresponding Hankel matrix.

Assume that \( F(z) \) is in \( \mathcal{U} \).

I. For \( p \geq m \),

\[
I_{pn} - H_0^* H_0 = \begin{pmatrix} 0_{m \times m} & 0 \\ 0 & \Delta_{pn-m} \end{pmatrix},
\]

where \( \Delta_{pn-m} \) is a \((pn-m) \times (pn-m)\) positive semi-definite weak contraction.

Moreover, if \( p = m \) then \( \Delta_{pn-m} \) is an orthogonal projection.

II. For \( m \geq p \),

\[
I_{mn} - H_0^* H_0 = \begin{pmatrix} 0_{p \times p} & 0 \\ 0 & \Delta_{mn-p} \end{pmatrix},
\]

where \( \Delta_{mn-p} \) is a \((mn-p) \times (mn-p)\) positive semi-definite weak contraction.

Moreover, if \( p = m \) then \( \Delta_{mn-p} \) is an orthogonal projection.

**Proof:** The structure of the right hand side of (4.7) and (4.8), is immediate from (4.4). All is left to verify is the spectrum of \( \Delta \).

Recall that the square of the Hankel singular values, are the positive eigenvalues of \( H_0^* H_0 \) (or of \( H_0 H_0^* \)) which in turn are equal to the positive eigenvalues of the product of the Gramians \( W_{\text{cont}} W_{\text{obs}} \), appeared in (3.3), see e.g. [19, Eq. (12.8.43)].
From Lemma 3.2 we know that in the rectangular case, the spectrum of \((W_{\text{cont}}W_{\text{obs}})\) lies in the interval \([0, 1]\) and in the square case the non-zero eigenvalues of \((W_{\text{cont}}W_{\text{obs}})\) are all ones.

Finally, note that the eigenvalues of the right hand side of \((4.7)\) and \((4.8)\) are just 1 minus the eigenvalues of \(H_0^*H_0\) or of \(H_0H_0^*\), respectively. □

It is interesting to point out that the condition of having the quantities in \((4.7)\) and \((4.8)\) respectively,

\[
I_{pn} - H_0^*H_0 \quad I_{mn} - H_0H_0^*
\]

positive semidefinite (as we indeed do), commonly appears in the context of Nehari’s problem where one approximates an anti-causal polynomial (positive powers of \(z\)) by a causal rational function (no pole at infinity), see e.g. [19, Section 12.8].

The following example illustrates the above results.

**Example 4.3.** Consider part II of Example 3.3. As \(H\) in \((4.4)\) satisfies condition \((4.7)\), from Theorem 4.1 it follows that the polynomial in \((3.5)\) is in \(\mathcal{U}\). □

The above results in Theorem 4.1 and in Proposition 4.2 were formulated in the language of \(H_0^*H_0\) or \(H_0H_0^*\), where \(H_0\) is a Hankel matrix. We now show that these results can be equivalently formulated in terms of block-triangular Toeplitz matrices. Indeed, using \(T\) from \((2.18)\) note that

\[
H_0^*H_0 = (T_{n,p}H_0)^* (T_{n,p}H_0) = \begin{pmatrix} B_n & \cdots & \cdots & B_n \\ B_1 & \cdots & \cdots & B_1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
B_n & \cdots & \cdots & B_n \end{pmatrix}^* \begin{pmatrix} B_n & \cdots & \cdots & B_n \\ B_1 & \cdots & \cdots & B_1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
B_n & \cdots & \cdots & B_n \end{pmatrix}
\]

\[
H_0H_0^* = (H_0T_{n,m})(H_0T_{n,m})^* = \begin{pmatrix} B_n & \cdots & \cdots & B_n \\ B_1 & \cdots & \cdots & B_1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
B_n & \cdots & \cdots & B_n \end{pmatrix} \begin{pmatrix} B_n & \cdots & \cdots & B_n \\ B_1 & \cdots & \cdots & B_1 \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
B_n & \cdots & \cdots & B_n \end{pmatrix}^*
\]

Technically, this relation is well known. The problem of finding a factorization of (block) positive semi-definite matrix to (block) triangular Toeplitz is classical, see e.g. [25]. In Subsection 2.1 we saw that Toeplitz operator are better established in system theory.

### 5. A CONCLUDING REMARK

Although modest in size, the family of para-unitary Finite Impulse Response systems (=co-isometric Laurent polynomials) is of great interest in various fields.

In Theorems 3.1, 3.5 and 4.1 we have offered three characterizations of this family. In Corollary 3.6 we introduced an easy-to-use parameterization of this set.

Finally, in Subsection 2.2 we suggested six ways (along with their combinations) to construct from a given para-unitary FIR system, a whole family of systems, of various dimensions and powers, all with this property. This may raise the following open problem of going in the opposite direction: Given a “complicated” FIR in \(\mathcal{U}\). How to “factorize” it, following items I to VI, to simpler building blocks.

The nature of this work suggests that the same set appears, under possibly different terminology, in additional fields and further connections may be established.
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Appendix: Construction of families of FIRs

For each of the items I through VI from Subsection 2.2 we here fill-in the following details:
1. Construct new polynomials from the original one.
2. Show that whenever the original polynomial was in \( \mathcal{U} \), so are the newly constructed polynomials.
I. Reverse polynomial

Using $T$ from (2.18),

$$F_{rev}(z) := z^{-1}B_n + z^{-2}B_{n-1} + \ldots + z^{1-n}B_2 + z^{-n}B_1 = ZT_{n,p}B_0 = BT_{n,m}Z.$$ 

The relation with the corresponding Hankel matrix is straightforward and thus omitted.

II. Preserving the McMillan degree.

For simplicity of exposition we consider as an illustrative example the polynomial $F(z)$ in (2.11) with $n = 4$ and the parameter $q$ attaining the values $-1$ and $-2$.

a. For $q = -1$ the corresponding Hankel matrix, $H_1$ from (2.13) can be, without loss of generality, extended with a row and a column of zeros so it is $6p \times 6m$. Now the resulting Hankel matrix may be partitioned in two forms,

$$H_1 = \begin{pmatrix}
0 & B_1 & B_2 & B_3 & B_4 & 0 \\
B_1 & B_2 & B_3 & B_4 & 0 & 0 \\
B_2 & B_3 & B_4 & 0 & 0 & 0 \\
B_3 & B_4 & 0 & 0 & 0 & 0 \\
B_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & B_1 & B_2 & B_3 & B_4 & 0 \\
B_1 & B_2 & B_3 & B_4 & 0 & 0 \\
B_2 & B_3 & B_4 & 0 & 0 & 0 \\
B_3 & B_4 & 0 & 0 & 0 & 0 \\
B_4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

which produce the two polynomials in item II a.

Following Theorem 4.1 note that

$$(I_{6m} - H_1^*H_1) \begin{pmatrix} I_{2m} \\ 0_{4m \times 2m} \end{pmatrix} = 0_{6m \times 2m} \quad p \geq m$$

$$(I_{2p} \ 0_{2p \times 4p}) (I_{6p} - H_1H_1^*) = 0_{2p \times 6p} \quad m \geq p,$$

so indeed the polynomial is in $\mathcal{U}$.

b. Substituting in $F(z)$ in (2.12) $q = -2$ yield a Hankel matrix $H_2$ with three partitionings

$$H_2 = \begin{pmatrix}
0 & 0 & B_1 & B_2 & B_3 & B_4 && \\
B_1 & B_2 & B_3 & B_4 & 0 & 0 \\
B_2 & B_3 & B_4 & 0 & 0 & 0 \\
B_3 & B_4 & 0 & 0 & 0 & 0 \\
B_4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & B_1 & B_2 & B_3 & B_4 && \\
B_1 & B_2 & B_3 & B_4 & 0 & 0 \\
B_2 & B_3 & B_4 & 0 & 0 & 0 \\
B_3 & B_4 & 0 & 0 & 0 & 0 \\
B_4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

so that the three polynomials in item II b are obtained.

These three polynomials are in $\mathcal{U}$, as following Theorem 4.1 one obtains,

$$(I_{6m} - H_2^*H_2) \begin{pmatrix} I_{3m} \\ 0_{3m \times 3m} \end{pmatrix} = 0_{6m \times 3m} \quad p \geq m$$

$$(I_{3p} \ 0_{3p \times 3p}) (I_{6p} - H_2H_2^*) = 0_{3p \times 6p} \quad m \geq p.$$
To make the last construction more realistic take for example, \( m = p = 2 \) and
\[
B_1 = \frac{1}{\sqrt{3}} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \\ \end{array} \right) \quad B_2 = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1 \\ -1 & 1 \\ \end{array} \right) \quad B_3 = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -2 \\ 1 & 2 \\ \end{array} \right) \quad B_4 = \frac{1}{\sqrt{3}} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \\ \end{array} \right).
\]

In items III through VI we produce more elaborate structures out of a given Hankel matrix. To this end, we find it convenient to introduce the following notation. Let \( \alpha, \beta \geq 0 \) and \( \eta, \delta > 0 \) be integers. One can construct the following \((\alpha + \beta + \eta) \delta \times \eta \delta\) isometry, i.e. \( U_{\text{Iso}} U_{\text{Iso}}^* = I_{\eta \delta} \),
\[
U_{\text{Iso}}(\alpha, \beta, \eta, \delta) = I_{\eta} \otimes \left( \begin{array}{ccc} 0_{\beta \times \delta} & I_\delta & 0_{\eta \times \delta} \\ 0_{\alpha \times \delta} & I_\delta & 0_{\eta \times \delta} \\ \end{array} \right).
\]

Similarly, \( U_{\text{Coiso}} \) is the following \( \eta \delta \times (\alpha + \beta + \eta) \delta \) co-isometry, i.e. \( U_{\text{Coiso}} U_{\text{Coiso}}^* = I_{\eta \delta} \),
\[
U_{\text{Coiso}}(\alpha, \beta, \eta, \delta) = I_{\eta} \otimes \left( \begin{array}{ccc} 0_{\delta \times \beta} & I_\delta & 0_{\delta \times \alpha} \\ 0_{\delta \times \beta} & I_\delta & 0_{\delta \times \alpha} \\ \end{array} \right).
\]

Let \( \rho \in \{1, 2, \ldots, n\} \) be so that \( \frac{n}{\rho} \) is natural. Substitute in \( U_{\text{Iso}} \) and in \( U_{\text{Coiso}} \) see (5.1), (5.2) respectively: \( \alpha = \frac{ap}{\rho}, \beta = \frac{bp}{\rho} \), with \( a, b \geq 0 \) integers, \( \eta = \frac{n}{\rho}, \delta = \rho p \) and consider the pair of products,
\[
U_{\text{Iso}} B = \left( \begin{array}{c} 0_{b \rho p \times m} \\ B_1 \\ \vdots \\ B_\rho \\ 0_{(a+b) \rho p \times m} \\ B_{\rho+1} \\ \vdots \\ 0_{(a+b) \rho p \times m} \\ B_{n+1-p} \\ \vdots \\ 0_{a \rho p \times m} \\ \end{array} \right),
\]
and
\[
B U_{\text{Coiso}} = (0_{p \times \rho pm}, B_1 \cdots B_\rho, 0_{p \times (a+b) \rho pm}) B_{\rho+1} \cdots B_{2p} \cdots 0_{p \times (a+b) \rho pm} \cdots B_{n+1-p} \cdots B_n,\quad 0_{p \times a \rho pm}).
\]
Both cases yield the same \((a + b + 1)np \times (a + b + 1)nm\) Hankel matrix, denoted by \(H(a, b, \rho)\). For example,

\[
(5.5) \quad H(0, 2, 1) = \begin{pmatrix}
B_1 & B_2 & \cdots & B_n \\
B_1 & \cdots & & \\
& \ddots & \ddots & \\
& & \ddots & B_{n-1} \ B_n
\end{pmatrix},
\]

or

\[
(5.6) \quad H(2, 2, 2) = \begin{pmatrix}
B_1 & B_2 & \cdots & B_n \\
B_1 & \cdots & & \\
& \ddots & \ddots & \\
& & \ddots & B_{n-1} \ B_n
\end{pmatrix},
\]

or for \(n = 6\)

\[
(5.7) \quad H(0, 1, 3) = \begin{pmatrix}
B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \\
B_1 & \cdots & & & & \\
& \ddots & \ddots & & & \\
& & \ddots & & \ddots & \\
& & & \ddots & \ddots & \\
& & & & \ddots & \ddots \\
& & & & & \ddots & B_6
\end{pmatrix}.
\]

**III.** Doubling the powers.
First note that \(H(a, b, 1)\) (i.e. when \(\rho = 1\)) corresponds to \(p \times m\)-valued polynomial in \((2.13)\) with \(\gamma := a + b + 1\).

As another example, the above Hankel matrix \(H(0, 1, 3)\) is associated with the polynomial \(F(z)\) in \((2.14)\).

**IV.** Rectangular polynomials.
Another sample of a Hankel matrix associated with \(U_{\text{Iso}} \mathbf{B}_\alpha\) in \((5.3)\) (or \(\hat{\mathbf{B}}_0 U_{\text{Coiso}}\) in \((5.4)\)) is obtained when the parameters are \(a = 2, b = 2, \rho = 2\), i.e. Now, multiplying \(H(a, b, \rho)\) from the right by \(U_{\text{Iso}}\) in \((5.1)\) with the parameters \(\alpha = (\rho - 1)m, \beta = 0, \eta = \frac{a}{\rho}(a + b + 1)\) and \(\delta = m\) yields the following \((a + b + 1)np \times (a + b + 1)\frac{a}{\rho}m\)
Hankel matrix (here $a = 2, b = 2, \rho = 2$) (5.7)

\[
H(2, 2, 2)U_{\text{Iso}} = 
\begin{bmatrix}
B_1 & B_2 & \cdots & B_{n-1} & B_n \\
B_2 & B_3 & \cdots & B_{n-1} & B_n \\
& B_3 & \cdots & B_{n-1} & B_n \\
& & \ddots & \cdots & \cdots \\
& & & B_{n-1} & B_n
\end{bmatrix}
\]

which corresponds to the $p\rho \times m$-valued polynomial in IV a.

Similarly, multiplying $H(a, b, \rho)$ from the left by $U_{\text{Coiso}}$ in (5.2) with the parameters $\alpha = (\rho - 1)\rho, \beta = 0, \eta = \frac{1}{\rho}(a + b + 1)$ and $\delta = p$ yields the following $(a + b + 1)\frac{\rho}{p} \times (a + b + 1)nm$ Hankel matrix (here $a = 2, b = 2, \rho = 2$) (5.8)

\[
U_{\text{Coiso}}H(2, 2, 2) = 
\begin{bmatrix}
B_1 & B_2 & \cdots & B_{n-1} & B_n \\
& B_2 & B_3 & \cdots & B_{n-1} \\
& & B_3 & \cdots & B_{n-1} \\
& & & \ddots & \cdots \\
& & & & B_{n-1}
\end{bmatrix}
\]

which corresponds to the $p \times \rho m$-valued polynomial in IV b.

It is easy to verify that if $H_0$ satisfies the first line in (4.4) then so do all Hankel matrices of the form $H_0U_{\text{Iso}}$ (5.7).

Similarly, if $H_0$ satisfies the second line in (4.4) then so do all Hankel matrices of the form $U_{\text{Coiso}}H_0$ (5.8).

In the sequel, we shall adjust our previous notation in (2.8) of the Hankel matrix associated with

\[F(z) = z^{-(1+\eta)}B_1 + \cdots + z^{-(n+\eta)}B_n \quad \eta \geq 0,\]

to $H_{B, n, \eta}$ (in 2.8) the subscripts $B$ and $n$ were omitted, as so far they were evident from the context). For example, with the polynomial

\[z^{-(1+\eta)}C_1 + \cdots + z^{-(l+\eta)}C_l \quad \eta = 0, 1, 2\ldots \]
one can associate the \((l + \eta)p_c \times (l + \eta)m_c\) Hankel matrix,

\[
H_{C,l,\eta} = \left( \begin{array}{cccc}
C_1 & \cdots & C_1 \\
\vdots & \ddots & \vdots \\
C_1 & \cdots & C_1
\end{array} \right).
\]

V. Composition of polynomials.

With the pair of polynomials in (2.16) one can associate the Hankel matrices \(H_{B,n,0}\) and \(H_{C,l,0}\), which are of dimensions \(np_b \times nm_b\) and \(lp_c \times nm_c\), respectively.

Out of this pair, one can construct (at least) the three following Hankel matrices, all of the form \(H_{D,n,0}\):

- A \(n(p_b + p_c) \times n(m_b + m_c)\) Hankel matrix

\[
\begin{pmatrix}
B_1 & C_1 & B_2 & C_2 & \cdots & B_l & C_l & B_{l+1} & 0_{p_c \times m_c} & \cdots & 0_{p_c \times m_c} \\
B_2 & C_2 & B_3 & C_3 & \cdots & B_{l+1} & 0_{p_c \times m_c} & \cdots & 0_{p_c \times m_c} \\
B_3 & C_3 & B_4 & C_4 & \cdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
B_l & C_l & B_{l+1} & 0_{p_c \times m_c} & \cdots & \vdots & \vdots & \vdots & \vdots \\
B_{l+1} & 0_{p_c \times m_c} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
B_n & 0_{p_c \times m_c} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]

- Or another \(n(p_b + p_c) \times n(m_b + m_c)\) Hankel matrix

\[
\begin{pmatrix}
C_1 & B_1 & C_2 & B_2 & \cdots & C_l & B_l & 0_{p_c \times m_c} & \cdots & 0_{p_c \times m_c} & B_n \\
C_2 & B_2 & C_3 & B_3 & \cdots & C_{l+1} & 0_{p_c \times m_c} & \cdots & \vdots & \vdots & \vdots \\
C_3 & B_3 & C_4 & B_4 & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
C_l & B_l & B_{l+1} & 0_{p_c \times m_c} & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0_{p_c \times m_c} & B_{l+1} & 0_{p_c \times m_c} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0_{p_c \times m_c} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
B_n & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\]
b. For \( m_c \geq m_b \) a \( n(p_b + p_c) \times nm_c \) Hankel matrix

\[
\begin{pmatrix}
\sqrt{\alpha B_1} & 0_{p_b \times (m_c-m_b)} & \sqrt{\alpha B_{l+1}} & 0_{p_b \times (m_c-m_b)} & \sqrt{\alpha B_n} & 0_{p_b \times (m_c-m_b)} \\
\sqrt{1-\alpha C_1} & 0_{p_b \times (m_c-m_b)} & \sqrt{1-\alpha C_l} & 0_{p_b \times (m_c-m_b)} & 0_{p_b \times m_c} \\
& \vdots & \vdots & \vdots & \vdots \\
& 0_{(p_b-p_c) \times m_b} & 0_{(p_b-p_c) \times m_b} & 0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c} \\
& 0_{(p_b-p_c) \times m_b} & 0_{(p_b-p_c) \times m_b} \\
& 0_{(p_b-p_c) \times m_b} \\
& 0_{p_b \times m_c} \\
\end{pmatrix}
\]

For \( p_b \geq p_c \) a \( np_b \times (m_b + m_c) \) Hankel matrix

\[
\begin{pmatrix}
0 & \sqrt{\alpha B_1} & 0_{p_b \times m_b} & \sqrt{\alpha B_{l+1}} & 0_{p_b \times m_b} & \sqrt{\alpha B_n} & 0_{p_b \times m_b} \\
0 & 0_{(p_b-p_c) \times m_b} & \sqrt{1-\alpha C_1} & 0_{p_b \times m_b} & \sqrt{1-\alpha C_l} & 0_{p_b \times m_b} & 0_{p_b \times m_b} \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& 0 & 0_{(p_b-p_c) \times m_b} & 0_{(p_b-p_c) \times m_b} & 0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c} & 0_{p_b \times m_c} \\
& 0 & 0_{(p_b-p_c) \times m_b} & 0_{(p_b-p_c) \times m_b} & 0_{p_b \times m_c} \\
& 0 & 0_{(p_b-p_c) \times m_b} \\
& 0_{p_b \times m_c} \\
\end{pmatrix}
\]

VI. Product of polynomials
Recall that out of

\[
F_b(z) = z^{-1}B_1 + \ldots + z^{-n}B_n \quad p_b \times \rho
\]

\[
F_c(z) = z^{-1}C_1 + \ldots + z^{-l}C_l \quad \rho \times m_c
\]

the following \( p_b \times m_c \) valued polynomial was obtained

\[
F_d(z) := F_b(z)F_c(z) = z^{-1}\left( z^{-1}D_1 + \ldots + z^{-(n+l-1)}D_{n+l-1} \right).
\]

where the coefficients \( D_1, \ldots, D_{n+l-1} \) were explicitly given in (2.17).

Expressing, (2.17) in terms of corresponding Hankel matrices yields

\[
H_{D_{n+l-1,1}} = H_{B_{n+l,l}}T_{m+l,\rho}H_{C_{l,n}} - H_{B_{n+l,l}}H_{B_{n,l}}T_{m+l,\rho}H_{C_{l,n}}.
\]

where the Hankel matrices \( H_{B_{n,l}} \) and \( H_{D_{n+l-1,1}} \) are \( (n+l) p_b \times (n+l) \rho \), \( (n+l) \rho \times (n+l) m_c \) and \( (n+l) p_b \times (n+l) m_c \) respectively, while \( T_{m+l,\rho} \) is the permutation matrix as in (2.18).

Next, to establish the fact that \( F_d(z) \) is in \( U \), we go through the following steps.

First, from (5.10) note that

\[
H_{D_{n+l-1,1}} = H_{C_{l,n}}T_{m+l,\rho} \cdot \text{diag} \{ \Delta_{(n-1)\rho} \} \cdot \Delta_{(n-1)\rho} \cdot T_{m+l,\rho} H_{C_{l,n}}.
\]

Assuming now that \( F_b \in U \), it follows from (5.4) that

\[
H_{D_{n+l-1,1}} = H_{C_{l,n}}T_{m+l,\rho} \cdot \text{diag} \{ I_{(l+1)\rho} \} \cdot \Delta_{l+1} \rho \cdot H_{C_{l,n}}.
\]

where \( \Delta_{(n-1)\rho} \) is \( (n-1) \rho \times (n-1) \rho \) positive semi-definite (weak) contraction.
Assuming now that also $F_c \in \mathcal{U}$, (carefully following the dimensions) it follows from (4.3) that

$$H_{D, n+l-1, 1}^* H_{D, n+l-1, 1} = \text{diag}\{I_{2p_b} \hat{\Delta}_{(n+l-2)p_b}\},$$

where $\hat{\Delta}_{(n+l-2)p_b}$ is a $(n + l - 1)p_b \times (n + l - 1)p_b$ positive semi-definite (weak) contraction, this part is established and indeed $F_d \in \mathcal{U}$.

Showing the relation for $H_{D, n+l-1, 1}^* H_{D, n+l-1, 1}$ is quite similar and thus omitted. \(\square\)

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