A constructive version of the Boyle-Handelman theorem on the spectra of nonnegative matrices

May 7, 2010

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Abstract

A constructive version of the celebrated Boyle-Handelman theorem on
the non-zero spectra of nonnegative matrices is presented.

2010 Mathematics Subject Classification: 15A18, 15A29, 15A42,
15B36, 15B48

Key words: inverse eigenvalue problem for nonnegative matrices

1 Introduction.

Let

\[ \sigma = (\lambda_1, \ldots, \lambda_n) \]

be a list of complex numbers and let

\[ s_k := \lambda_1^k + \ldots + \lambda_n^k, \quad k = 1, 2, 3, \ldots \]

The nonnegative inverse eigenvalue problem (NIEP) asks for necessary and
sufficient conditions on \( \sigma \) in order that it be the spectrum of an entry-wise
nonnegative matrix. If this occurs, we say that \( \sigma \) is realizable, and we call a
nonnegative matrix \( A \) with spectrum \( \sigma \) a realizing matrix for \( \sigma \).

A necessary condition for realizability coming from the Perron-Frobenius
theorem [2] is that there exists \( j \) with \( \lambda_j \) real and \( \lambda_j \geq |\lambda_i| \), for all \( i \). Such a
\( \lambda_j \) is called the Perron root of \( \sigma \).
A more obvious necessary condition is that all the \( s_k \) are nonnegative. A stronger form of this condition was found independently by Loewy and London [11] and Johnson [8], namely:

\[(JLL)\quad n^{k-1}s_{km} \geq s_{m}^k, \text{ for all positive integers, } k \text{ and } m.\]

In terms of \( n \), a complete solution of the NIEP is available only for \( n \leq 4 \). The solution for \( n = 4 \), expressed in terms of inequalities for the \( s_k \), appears in the PhD thesis of Meehan [12] and a solution in terms of the coefficients of the characteristic polynomial has been published more recently by Torre-Mayo, Abril-Raymundo, Alarcia-Estevez, Marijuan, and Pisanero [14].

However, the same problem in which we may augment the list \( \sigma \) by adding an arbitrary number \( N \) of zeros was solved by Boyle and Handelman [4]. Using a range of tools coming from linear algebra, dynamical systems, ergodic theory, and graph theory, they proved the remarkable result that if

1. \( \sigma \) has a Perron element \( \lambda_1 > |\lambda_j| \) (all \( j > 1 \)) and
2. \( s_k \geq 0 \) for all positive integers \( k \) (and \( s_m = 0 \) for some \( m \) implies \( s_d = 0 \) for all positive divisors \( d \) of \( m \)), then

\[\sigma_N := (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0) \text{ (} N \text{ zeros)}\]

is realizable for all sufficiently large \( N \).

Under these assumptions, a realizing matrix can be chosen to be primitive. See Friedland [6] for an extension to the irreducible case.

The proof of the Boyle-Handelman result is not constructive and does not provide a bound on the minimal number \( N = N(\sigma) \) of zeros required for realizability.

Finding a constructive proof, with a bound on the minimum number \( N \) of zeros required, has been an area of much research, and a number of special cases have been resolved. In particular, a best possible result in the case that \( \text{Re}(\lambda_j) \leq 0 \), for all \( j > 1 \), has been obtained by Šmigoc and the author [9] and, when \( \sigma \) is real and has exactly two positive entries, a constructive proof with a bound on \( N \) has also been found [10].

In the case that \( \sigma \) is real and has just one positive entry, then the inequality \( s_1 \geq 0 \) is necessary and sufficient for realizability. This was proved by Suleimanova [13] and this is often viewed as the first result on the NIEP. Friedland [5] re-proved her result by showing that the companion matrix with spectrum \( \sigma \) has nonnegative entries, and matrices related to companion matrices are used in the cited work with Šmigoc.

Here, a constructive approach to the Boyle-Handelman result is presented. It is shown that a certain kind of patterned matrix is “universal” for the realization of spectra with power sums \( s_k > 0 \), \( (k = 1, 2, 3, \ldots) \). in the sense that all such spectra satisfying the Perron condition (i) above can, with sufficiently many zeros added, be realized as the spectrum of a primitive nonnegative matrix with that pattern.
2 A matrix related to Newton’s identities

Let

\[ \tau = (\mu_1, \ldots, \mu_n), \]
\[ x_k := \mu_1^k + \ldots + \mu_n^k, \quad k = 1, 2, 3, \ldots \]
\[ q(x) := \Pi_{i=1}^n (x - \mu_i) \]
\[ = x^n + q_1 x^{n-1} + \ldots + q_n. \]

Let \( X_n = \)
\[
\begin{pmatrix}
  x_1 & 1 & 0 & \ldots & 0 \\
  x_2 & x_1 & 2 & 0 & \ldots & 0 \\
  x_3 & x_2 & x_1 & 3 & 0 & \ldots & 0 \\
  \vdots & x_3 & \vdots & \vdots & \ddots & \vdots & \vdots \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
  x_{n-1} & x_{n-2} & \ldots & x_2 & x_1 & n-1 \\
  x_n & x_{n-1} & \ldots & x_3 & x_2 & x_1
\end{pmatrix}
\]

The matrix \( X_n \) occurs in the context of the Newton identities relating the coefficients of a polynomial to the power sums of its roots. If we use Cramer’s rule to express the \( q_i \) in terms of the \( x_j \), we get

\[ \det(X_n) = (-1)^n n! q_n. \]

However, the matrix \( X_n \) itself, as distinct from its determinant, does not appear to have been widely investigated. A key observation is:

**Proposition 1** The characteristic polynomial of \( X_n \) is

\[ Q(x) = x^n + nq_1 x^{n-1} + n(n-1)q_2 x^{n-2} + \ldots + n!q_n. \]

Since \( X_n \) has nonnegative entries if the \( x_i \) are nonnegative, it follows that the spectrum of \( Q(x) \) is realizable if the \( x_i, \ (i = 1, 2, \ldots, n) \), are nonnegative.

Suppose that we are given a list \( \sigma = (\lambda_1, \ldots, \lambda_n) \) that we wish to realize as the spectrum of a nonnegative matrix.

Let
\[ f(x) := \Pi_{i=1}^n (x - \lambda_i) \]
\[ = x^n + p_1 x^{n-1} + \ldots + p_n. \]
Let
\[ q(x) := x^n + q_1x^{n-1} + \ldots + q_n \]
where
\[ q_i = \frac{p_i}{n(n-1)} \ldots \frac{1}{(n-i+1)} \cdot i = 1, 2, \ldots, n. \]

Then the corresponding \( Q(x) \) is \( f(x) \). Now the power sums \( x_i \) of the roots of \( q(x) \) are nonnegative if and only if that holds for
\[ x^n + nq_1x^{n-1} + \ldots + n^aq_n. \]
Hence we have

**Theorem 2** \( \sigma \) is realizable by the matrix \( X_n \) if the \( j \) th power sum of the roots of the polynomial
\[ J_n(f(x)) := x^n + p_1x^{n-1} + \frac{n}{n-1}p_2x^{n-2} + \frac{n^2}{(n-1)(n-2)}p_3x^{n-3} + \ldots + \frac{n^{n-1}}{(n-1)!}p_n \]
is nonnegative for \( j = 1, 2, 3, \ldots, n \).

But now suppose that we choose \( N > n \) and ask for the realizability of \( \sigma \) with \( N - n \) zeros added. This amounts to replacing \( f(x) \) by \( x^{N-n}f(x) \) and \( J_n(f(x)) \) by \( x^{N-n}J_N(f(x)) \), where
\[ J_N(f(x)) := x^n + p_1x^{n-1} + \frac{N}{N-1}p_2x^{n-2} + \frac{N^2}{(N-1)(N-2)}p_3x^{n-3} + \ldots + \frac{N^{n-1}}{(N-1)(N-2)\ldots(N-n+1)}p_n. \]

So \( \sigma \) with \( N - n \) zeros added is realizable by the matrix \( X_N \) if the \( j \) th power sums of the roots of the polynomial \( J_N(f(x)) \) are nonnegative for \( j = 1, 2, 3, \ldots, N \).

But observe that as \( N \to \infty, J_N(f(x)) \to f(x) \), since \( n \) is fixed.

Suppose that the power sums \( s_j \) of the elements of \( \sigma \) are positive for all \( j \geq 1 \). Then, on continuity grounds, one might expect that for sufficiently large \( N \), the power sums of the roots of \( J_N(f(x)) \) would also be positive. However, this is not true in general, but it is true if \( \sigma \) has its Perron element
\[ \lambda_1 \geq |\lambda_j| \ (j = 1, 2, \ldots, n). \]

In this case, \( \sigma \) with sufficiently many zeros added is the spectrum of a nonnegative matrix \( X_N \).

Since, we only require that the \( j \) th power sum of the roots of \( J_N(f(x)) \) be nonnegative for \( j = 1, 2, \ldots, N \), one can obtain a bound on the minimal number of zeros required.
3 Main Theorem

We now state the main result of this paper.

Theorem 3 Let
\[ \sigma = (\lambda_1, \ldots, \lambda_n), \]
be a list of complex numbers with corresponding power sums
\[ s_k := \lambda_1^k + \ldots + \lambda_n^k, \quad k = 1, 2, 3, \ldots. \]

Suppose that
(i) \( \lambda_1 > |\lambda_j|, \quad (all \ j > 1) \)
(ii) \( s_1 \geq 0, \quad and \quad s_m > 0, \quad for \ all \ m \geq 2. \)

Let
\[ f(x) = \Pi_{i=1}^{n}(x - \lambda_i) \]
\[ = x^n + p_1x^{n-1} + \ldots + p_n. \]
\[ \gamma = 2 \max(1, |p_1|, |p_2|^{1/2}, \ldots, |p_n|^{1/n}). \]
\[ \lambda_0 = \max\{|\lambda_j|: j > 1\}, \]
\[ R = \frac{(\lambda_1 - \lambda_0)}{4}, \quad \ell = \frac{3\lambda_1 + \lambda_0}{\lambda_1 + 3\lambda_0}, \quad r = \min(R, 1), \]
\[ m = \max\{1, \lambda_1\}, \quad N_0 = \left\lceil \frac{\ln(2n - 2)}{\ln(\ell)} \right\rceil, \]
\[ M = \min\{1, s_2, \ldots, s_{N_0}\}, \]
and
\[ N = \left\lceil 2 \left( \frac{16\gamma n N_0 (m + r)^{N_0 - 1}}{3^{1/2} M r} \right)^n \right\rceil. \]

Then \( \sigma \) with \( N - n \) zeros added is the spectrum of the nonnegative matrix \( X_N \), with \( x_k := \mu_1^k + \ldots + \mu_n^k, k = 1, 2, 3, \ldots, N, \) where
\[ J_N(f(x)) = (x - N\mu_1)(x - N\mu_2)\ldots(x - N\mu_n). \]

Given a list \( \sigma \) satisfying the hypotheses, it is relatively easy to find \( N \) for which \( J_N(f(x)) \) has the corresponding power sums nonnegative, so one obtains a reasonably efficient constructive algorithm. However, the number of zeros required in the construction is not optimal in general.
4 Proofs of the results

Let

\[
P = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
q_1 & 1 & 0 & 0 & \cdots & 0 \\
q_2 & q_1 & \frac{q_2}{2} & 0 & 0 & \cdots & 0 \\
q_3 & q_2 & \frac{q_3}{2} & \frac{q_2}{6} & 0 & \cdots & 0 \\
\vdots & \vdots & \frac{q_3}{2} & \frac{q_2}{6} & \frac{q_1}{24} & \cdots & 0 \\
q_{n-2} & q_{n-3} & \cdots & \cdots & 1 & \cdots & 0 \\
q_{n-1} & q_{n-2} & \cdots & \cdots & \frac{1}{(n-2)!} & 0 & \cdots & 0 \\
q_n & q_{n-1} & \frac{q_{n-2}}{2} & \cdots & \cdots & \frac{1}{(n-1)!} & \cdots & 0 \\
\end{pmatrix}
\]

and let

\[
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
-n!q_n & \cdots & \cdots & \cdots & -n!q_1 \\
\end{pmatrix}
\]

be the companion matrix of \(Q(x) = x^n + nq_1x^{n-1} + n(n-1)q_2x^{n-2} + \ldots + n!q_n\).

Direct multiplication, using the Newton identities, yields \(PC = X_nP\). This proves the proposition.

To obtain the desired bound we use the following refinement by Bhatia, Elsner and Krause [3] of a classical result of Ostrowski.

**Theorem 4** Let \(f(x) = x^n + a_1x^{n-1} + \ldots + a_n\) and \(g(x) = x^n + b_1x^{n-1} + \ldots + b_n\) be real polynomials with roots \(\alpha_1, \ldots, \alpha_n\) and \(\beta_1, \ldots, \beta_n\), respectively. Then there is a labelling of \(\beta_1, \ldots, \beta_n\) such that

\[
\max\{|\alpha_i - \beta_i| : 1 \leq i \leq n\} \leq \left(\frac{16}{3\sqrt{3}}\right) \left(\sum_{k=1}^{n} |a_k - b_k| \gamma^{n-k}\right)^{1/n},
\]

where \(\gamma = 2 \max\{|a_k|^{1/k}, |b_k|^{1/k} : 1 \leq k \leq n\}\).

[The original Ostrowski result had the factor \((2n - 1)\) in place of \(\left(\frac{16}{3\sqrt{3}}\right)\).]
Now let
\[ f(x) = (x - \lambda_1) \ldots (x - \lambda_n) \]
\[ = x^n + p_1 x^{n-1} + \ldots + p_n \]

and
\[ g(x) = x^n + p_1 x^{n-1} + \left( \frac{N}{N-1} \right) p_2 x^{n-2} + \ldots + \left( \frac{N^{n-1}}{(N-1) \ldots (N-n+1)} \right) p_n. \]

We note that if \( g(x) \) has nonnegative Newton power sums, then the corresponding matrix \( X_N \) is nonnegative and has spectrum \( N\lambda_1, \ldots, N\lambda_n \).

Suppose that \( \lambda_1 > | \lambda_j | \) (all \( j > 1 \)) and let \( \lambda_0 = \max(| \lambda_j | : j = 2, \ldots, n) \) and \( R = \frac{\lambda_1 - \lambda_0}{4} \). Let \( \ell = \frac{\lambda_1 + \lambda_0}{\lambda_1 - R} \), so \( \ell < 1 \). Let \( r = \min\{1, R\} \).

Let \( s_k = \lambda_1^k + \ldots + \lambda_n^k \) for \( k = 1, 2, \ldots \). Assume that \( s_1 \geq 0 \), and that \( s_k > 0 \), for all \( k > 1 \). Let
\[ M = \min\{s_k : k = 2, 3, \ldots\}. \]

Let \( \mu_1, \ldots, \mu_n \) be the roots of \( g(x) = 0 \) and suppose that
\[ \max\{| \lambda_j - \mu_j | : j = 1, 2, \ldots, n\} < \delta, \quad (*) \]

where
\[ \delta = \frac{Mr}{nN_0(m + r)N_0^{-1}}, \]

with
\[ m = \max\{1, \lambda_1\}, \quad N_0 = \left\lceil \frac{\ln(2(n-1))}{\ln(1/\ell)} \right\rceil. \]

Then \( | \mu_1 | \) is greatest among all the \( | \mu_j | \), and, since \( g(x) \) has real coefficients, \( \mu_1 \) is real and, since \( \lambda_1 \) is positive, so is \( \mu_1 \). Let
\[ S_k = \mu_1^k + \ldots + \mu_n^k. \]

Then \( | s_k - S_k | \leq \sum_{i=1}^n | \lambda_i^k - \mu_i^k | \). Now
\[ \begin{align*}
| \lambda_i^k - \mu_i^k | &= | \lambda_i - \mu_i || \lambda_i^{k-1} + \lambda_i^{k-2} \mu_i + \ldots + \mu_i^{k-1} | \\
&< \delta k(\lambda_1 + r)^{k-1}.
\end{align*} \]

Suppose that \( k \geq N_0 \). Then \( S_k \geq (\lambda_1 - r)^k - (n-1)(\lambda_0 + r)^k = (\lambda_1 - r)^k(1 - (n-1)(\frac{\lambda_0 + r}{\lambda_1 - r})^k) > (\frac{1}{2})(\lambda_1 - r)^k > 0 \). For \( k \leq N_0 \),
\[ \begin{align*}
| s_k - S_k | &\leq \delta(1 + 2(\lambda_1 + r) + \ldots + N_0(\lambda_1 + r)^{N_0-1}) \\
&< \delta nN_0(m + r)^{N_0-1} = Mr \leq M.
\end{align*} \]
So \( S_k \geq 0 \), for all \( k \geq 2 \). Also, \( S_1 = s_1 \geq 0 \). This shows that if we can choose \( N \) so that the inequality
\[
\max\{ |\lambda_j - \mu_j | : j = 1, 2, \ldots, n \} < \delta
\]
holds for that \( \delta \), then the corresponding \( X_N \) will be a nonnegative matrix with spectrum \( \lambda_1, \ldots, \lambda_n \) and \( N - n \) zeros. Now,
\[
\max\{ |\lambda_j - \mu_j | : j = 1, 2, \ldots, n \} \leq \left( \frac{16}{3\sqrt{3}} \right) \left( \sum_{k=1}^{n} p_k \left( \frac{N^{k-1}}{(N-1)\ldots(N-k+1)} - 1 \right) \right)^{1/n} \gamma^{n-k}.
\]
But
\[
\frac{N^{k-1}}{(N-1)\ldots(N-k+1)} - 1 \leq \frac{2n^2}{N}, \text{ if } N > n^2.
\]
By definition, \( \gamma = 2 \max\{1, |p_k|^{1/k}, k = 1, 2, \ldots, n\} \). Hence
\[
\max\{ |\lambda_j - \mu_j | : j = 1, 2, \ldots, n \} \leq \left( \frac{16\gamma}{3\sqrt{3}} \right) \left( \frac{2n^3}{N} \right)^{1/n}.
\]
But \( n^{1/n} \leq 3^{1/3} \). Hence
\[
\max\{ |\lambda_j - \mu_j | : j = 1, 2, \ldots, n \} \leq \frac{16\gamma 2^{1/n}}{\sqrt{3}N^{1/n}} \leq \delta,
\]
provided
\[
N \geq \frac{2^{4n+1} \gamma^n}{3n^{2/3} \delta^n} = \frac{2(16\gamma n N_0 (m+r)^{N_0-1})^n}{3^{n/2}M^{n/r}}.
\]
This gives the required bound.

There are variations of the Ostrowski bound, some using the Bombieri norm in place of the \( \ell_2 \) one, available though the work of Beauzamy [1], Galantai and Hegedus [7], and these may lead to better bounds for \( N \) in certain circumstances. However, the main interest is that such a bound exists, and the general form it has.

When the Perron root \( \lambda_1 = 1 \), a nonnegative matrix \( A \) with the given nonzero spectrum can be made stochastic. In this case \( r \) and \( \ell \) are measures of the spectral gap, which control the rate at which the powers of \( A \) converge to the stationary state of the corresponding Markov process. The size of \( N_0 \) is inversely related to \( r \) and \( \ell \).

The number \( M \) measures how close to zero the power sums can get, and we see its appearance (as \( M^n \)) in the denominator of the bound.

We conclude with an example involving the realization of a spectrum with three nonzero entries.
Example 5  \( \sigma = (\rho, \exp(\frac{\pi}{10}), \exp(-\frac{\pi}{10})) \) has all its power sums positive if \( \rho > \sqrt{2} \cos(\pi/10) = 1.07 \ldots \). If we take \( \rho = 1.1 \), and carry out the algorithm, we find that \( \sigma \) with 125 zeros added is the spectrum of an \( 128 \times 128 \) nonnegative matrix of the form of \( X \) above. The least number of zeros required to be added to \( \sigma \) to ensure realizability does not appear to be known in this case.

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