Abstract

When designing large-scale distributed controllers, the information sharing constraints between sub-controllers, as defined by a communication topology interconnecting them, are as important as the controller itself. Controllers implemented using dense topologies typically outperform those implemented using sparse topologies, but it is also desirable to minimize the cost of controller deployment. Motivated by the above, we introduce a compact but expressive graph recurrent neural network (GRNN) parameterization of distributed controllers that is well suited for distributed controller and communication topology co-design. Our proposed parameterization enjoys a local and distributed architecture, similar to previous Graph Neural Network (GNN)-based parameterizations, while further naturally allowing for joint optimization of the distributed controller and communication topology needed to implement it. We show that the distributed controller/communication topology co-design task can be posed as an $\ell_1$-regularized empirical risk minimization problem that can be efficiently solved using stochastic gradient methods. We run extensive simulations to study the performance of GRNN-based distributed controllers and show that (a) they achieve performance comparable to GNN-based controllers while having fewer free parameters, and (b) our method allows for performance/communication density tradeoff curves to be efficiently approximated.

1 Introduction

Future large-scale autonomous systems, such as intelligent transportation, power, and communication networks, will be composed of dynamically interconnected subsystems and data-driven decision makers. When designing the distributed control policies for such systems, the controller architectures, as defined by the actuation, sensing, and communication topologies of the controller, can no longer be taken for granted, as controllers with denser architectures will typically outperform controllers implemented using less resources. However, it is also desirable to minimize the cost of deploying controllers, leading to a non-trivial tradeoff between controller performance and architectural complexity.

A rich body of work exists addressing the controller architecture co-design problem in the context of (distributed) linear optimal control. When no communication constraints are imposed on

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the controller, i.e., when the control problem is centralized, it has been shown that several metrics defined in terms of the system controllability/observability Gramians are (sub)modular. This allows for efficient algorithms to be deployed for actuator/sensor selection that enjoy approximation guarantees, see for example Pequito et al. (2015); Tzoumas et al. (2016); Summers and Lygeros (2014); Summers et al. (2015) and references therein. When communication constraints are present, augmenting the distributed control problem with a suitable regularizer to encourage simplicity in the controller architecture has proven to be successful. For example, $\ell_1$ regularization is used to co-design sparse static state feedback gains in Lin et al. (2013) and synchronization topologies in Fardad et al. (2014); group norm penalties are used to co-design actuation/sensing schemes in Matni and Chandrasekaran (2016); Dhingra et al. (2014); and a specialized atomic norm (Chandrasekaran et al., 2012) is used to design communication delay constraints that are well-suited to $\mathcal{H}_2$ distributed optimal control in Matni (2015).

In this paper, we assume that the actuation and sensing architecture is fixed, and focus on co-designing the communication topology of the distributed controller. In particular, we model a distributed controller as a collection of sub-controllers, each equipped with a fixed set of actuators and sensors, that exchange their respective measurements with each other subject to communication delays imposed by the to-be-designed underlying communication topology. In general, one can select communication topologies that range in density from completely decentralized, i.e., that allow no communication between sub-controllers, to “virtually centralized,” i.e., that allow instantaneous communication between all sub-controllers.

However, there are now well understood limits on the simplicity of the designed communication topology (Rotkowitz and Lall, 2005; Rotkowitz et al., 2010; Lessard and Lall, 2011) if we wish for the resulting distributed control synthesis task to be convex. In particular a sufficient, and under mild assumptions necessary, condition for a distributed linear optimal controller to be specified by the solution to a convex optimization problem is that the communication delays between sub-controllers allow for information to be exchanged between them at least as quickly as their control actions propagate through the plant (Rotkowitz et al., 2010). This has limited the subset of the Pareto frontier that could be explored in previous work to communication topologies that are at least as dense as the topology of the underlying plant (see for example, the base QI communication graph in Matni (2015, §4.2)).

Motivated by these limitations on tractability and topology sparsity, this paper proposes an alternative approach. We draw upon the recent successes of graph neural network (GNN)-based controllers in the context of distributed control (Tolstaya et al., 2020; Gama et al., 2020a; Khan et al., 2020; Gama and Sojoudi, 2021), and propose a novel distributed controller and communication topology co-design procedure. We propose a compact graph recurrent neural network (GRNN) parameterization for distributed controllers, and show that the control parameters and communication topology, as encoded by the graph shift operator defining the GRNN-based distributed controller, can be jointly optimized via an $\ell_1$-regularized empirical risk minimization problem. Finally, through extensive empirical evaluations, we show that (a) distributed controllers based on our GRNN parameterization can achieve performance comparable to previously proposed GNN-based parameterizations while having fewer free parameters, and (b) our method allows for performance/communication density tradeoff curves to be efficiently approximated, even in the extremely sparse

\footnote{For a more detailed overview of the relationship between information exchange constraints and the convexity of distributed optimal control problems, we refer the reader to Bamieh and Voulgaris (2005); Mahajan et al. (2012) and the references therein.}
communication topology regime.

2 Preliminaries

2.1 Distributed Optimal Control

Consider a system composed of \( N \) subsystems, with the \( i \)-th subsystem described by a state vector \( x_i(t) \in \mathbb{R}^p \) and a control input \( u_i(t) \in \mathbb{R}^q \). We model inter-subsystem interactions through a time-invariant unweighted digraph \( G_{\text{dyn}} = (\mathcal{V}, \mathcal{E}_{\text{dyn}}) \). Each element in the vertex set \( \mathcal{V} = \{v_1, v_2, \ldots, v_N\} \) corresponds to a subsystem. The edge set \( \mathcal{E}_{\text{dyn}} \subset \mathcal{V} \times \mathcal{V} \) contains an element \( (i,j) \in \mathcal{E}_{\text{dyn}} \) only when subsystem \( i \) directly influences the states of subsystem \( j \) through its states or control inputs. We assume linear time-invariant dynamics, allowing us to write the dynamics at subsystem \( i \) as

\[
x_i(t + 1) = \sum_{j: (j,i) \in \mathcal{E}_{\text{dyn}}} A_{ij} x_j(t) + B_{ij} u_j(t),
\]

for suitable matrices describing local interactions between subsystems. Equivalently, the dynamics of the full system can be expressed in terms of the joint state \( x(t) := (x_i(t))_{i=1}^N \) and control \( u(t) := (u_i(t))_{i=1}^N \) vectors as

\[
x(t + 1) = Ax(t) + Bu(t),
\]

where matrices \( A \) and \( B \) are constructed such that the full dynamics (2.2) are consistent with the subsystem dynamics in (2.1). Note that the full system matrices \( (A,B) \) are sparse when the subsystems are sparsely interconnected, i.e., when the graph \( G_{\text{dyn}} \) is sparse.

Our goal is to efficiently explore the space of distributed controllers composed of sub-controllers that have access to a combination of instantaneous local information and delayed global information. Formally, the sub-controller for subsystem \( i \) at time \( t \) can directly observe a local information set \( J_{i,t} \), which might fail to capture the full state of system (2.2). For example, \( J_{i,t} \) might only include the state of subsystem \( i \), i.e. \( x_i(t) \), but not those of other subsystems. We assume that local information can be exchanged between subsystems, subject to a delay defined by a communication topology, modeled as a time-invariant unweighted digraph \( G_{\text{c}} = (\mathcal{V}, \mathcal{E}_{\text{c}}) \) (note that the communication topology \( G_{\text{c}} \) need not equal the interaction topology \( G_{\text{dyn}} \)). If \( (i,j) \in \mathcal{E}_{\text{c}} \), subsystem \( i \) can share its local information set \( J_{i,t} \) with subsystem \( j \), with a 1-step delay, i.e., subsystem \( j \) gains access to \( J_{i,t} \) at time \( t + 1 \). Subsystem \( j \) can then relay this information to its neighbors in \( G_{\text{c}} \) with a further time delay.

Denoting the directed distance from vertex \( i \) to vertex \( j \) in a digraph \( \mathcal{G} \) as \( \text{dist}_{\mathcal{G}}^i(j \rightarrow i) \), and vertex \( i \)'s \( d \)-hop incoming neighbors as \( \text{in}^i_{\mathcal{G}}(d) := \{ v_j \, | \, \text{dist}_{\mathcal{G}}^i(v_j \rightarrow v_i) \leq d \in \mathbb{N} \} \), the information sharing constraints imposed by the communication topology \( G_{\text{c}} \) mean that at time \( t \), the total information available to subsystem \( i \) is given by

\[
I_{i,t}(G_{\text{c}}) := \bigcup_{d=1}^{t} \left( \bigcup_{j \in \text{in}^i_{\mathcal{G}}(d)} J_{j,t} \right). \tag{2.3}
\]

This then restricts the action taken by subsystem \( i \) at time \( t \) to be of the form

\[
u_i(t) = \gamma_{i,t}(I_{i,t}(G_{\text{c}})), \tag{2.4}
\]
for $\gamma_{i,t}$ a suitable map from the total information set $I_{i,t}(G_c)$ to control actions. This formulation also makes clear that by varying the density of the communication topology defined by $G_c$, we can vary the total amount of information available to each subsystem $i$.

In the following, we summarize recent results showing that GNNs offer a natural means of parameterizing such controllers, and introduce our proposed parameterization for control and communication delay co-design.

### 2.2 Graph Neural Networks for Distributed Control

GNNs are a class of neural networks, built around the notion of a graph filter, that are well-suited for processing and representing data with graph structure. They enjoy many additional desirable properties, such as being permutation equivariant and Lipschitz continuous to changes in the network (Gama and Sojoudi, 2021; Gama et al., 2020b), making them natural candidates for distributed controllers. Following conventions in the GNN literature, we rewrite the joint state $x(t) \in \mathbb{R}^{Np}$ and control vectors $u(t) \in \mathbb{R}^{Nq}$ as

$$X(t) = \begin{bmatrix} x_1^T(t) \\ \vdots \\ x_N^T(t) \end{bmatrix} \in \mathbb{R}^{N \times p}, \quad U(t) = \begin{bmatrix} u_1^T(t) \\ \vdots \\ u_N^T(t) \end{bmatrix} \in \mathbb{R}^{N \times q},$$

as this representation more naturally allows for the required graph convolutions to be defined. In what follows, we tailor our discussion to using GNNs in the context of distributed control.

#### 2.2.1 Graph Convolutions

The notation in (2.5) allows one to express linear information exchanges as a matrix multiplication while enforcing the communication topology as a constraint on the multiplicand. Specifically, consider a matrix $S \in \mathbb{R}^{N \times N}$ which satisfies the property that $S_{ij} \neq 0$ only if $(j, i) \in E_c$, and define the matrix $U \in \mathbb{R}^{N \times p}$ as $U = SX$. Then,

$$U_{ij} = \sum_{k=1}^N S_{ik}X_{kj} = \sum_{k: (k,i) \in E_c} S_{ik}X_{kj} = \sum_{k:v_k \in \text{in}_{G_i}^G(1)} S_{ik}X_{kj},$$

where the first equality follows from the definition of matrix multiplication, the second equality follows the definition of nonzero elements of $S$, and the third from the definition of the vertex $i$’s 1-hop incoming neighbors. This latter definition emphasizes that the $i$-th row of $U$ can be computed by subsystem $i$ using information received from 1-hop incoming neighbors in $G_c$, i.e., only the $X_{kj}$ in (2.6) such that $v_k \in \text{in}_{G_i}^G(1)$ are needed.

In general, left multiplying a joint state or input vector by the matrix $S$ corresponds to communication and aggregation of data between 1-hop neighbors, as defined by the communication topology $G_c$. By associating a unit delay with such a communication and aggregation step, we can model the spatiotemporal propagation of information across a graph, making this operator a natural tool for encoding information sharing constraints in a distributed control setting. We note that such an operation is analogous to the time-shift operator in classical signal processing, and is thus referred to as the graph shift operator. Examples of valid graph shift operators on a graph $G$ include its adjacency matrix and its Laplacian. For the rest of the paper, we denote the set of valid graph shift operators on graph $G$ as $S_G \subset \mathbb{R}^{N \times N}$. 

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Graph convolution extends the graph shift operation by aggregating information over multiple time steps. Consider a graph $G$ and a corresponding graph shift operator $S \in S_G$. We can define a linear distributed controller $U(t)$ in terms of the graph convolution of the joint state $X(t)$ over $K$ steps specified by the set of weights $H = \{H_k\}_{k=0}^{K-1}, H_k \in \mathbb{R}^{p \times q}$ as

$$U(t) = H_G \ast X(t) = \sum_{k=0}^{K-1} S^k X(t-k)H_k,$$

where each row of resulting control signal $U(t) \in \mathbb{R}^{N \times q}$ can be computed at subsystem $i$ using $k$-step delayed information received from subsystems $j$ such that $\text{dist}_G(j \to i) = k \leq K$. To see this, we first note that, by the properties of graph shift operators, $S^k_{ij} \neq 0$ only if $v_j \in \text{in}_G^k(i)$, i.e., node $j$ is a $k$-hop incoming neighbor of node $i$ in $G$. Thus, for each $k \in \{0, \ldots, K-1\}$ in the sum, the $i$-th row of $U(t)$ aggregates $k$-step delayed information from $k$-hop incoming neighbors of subsystem $i$, with information set given by $\mathcal{J}_{i,t} = \{x_i(t)\}$. We therefore have that the total information set needed to compute the $i$-th row of $U(t)$ is given by

$$\mathcal{I}_{i,t}(Gc) = \{x_i(t)\} \cup \left( \bigcup_{k=1}^{K-1} \{x_j(t-k) \mid j \in \text{in}_i^G(k)\} \right),$$

i.e., the collection of $k$-step delayed states from $k$-hop neighbors, for $k = 0, \ldots, K-1$. We highlight that the graph convolution encodes both spatial and temporal structure, as captured by the propagation of signals by repeated application of the graph shift and time delay operator within the convolution, making it well-suited for parameterizing controllers subject to information sharing constraints as in (2.4). Finally, we note that a filter $H_k$ is applied uniformly across all rows of its multiplicand. Thus, the result from right multiplying $H_k$ to the aggregated signal $S^kX(t-k)$ can be computed locally for any matrix $H_k$.

### 2.2.2 Graph Neural Networks

A graph convolution as defined above can be viewed as a two-step process: first, a set of delayed information is aggregated at each node using the graph shift operator $S$; then, the aggregated information is used to synthesize an output by using the filter weights $\{H_k\}$. GNNs (Bruna et al., 2013; Kipf and Welling, 2016; Atwood and Towsley, 2015; Gama et al., 2018) seek to expand the expressiveness of graph convolutions by repeatedly applying one or both steps in combination with nonlinearities.

One architecture that achieves this is delayed aggregation GNNs (AGNNs) (Tolstaya et al., 2020; Gama et al., 2018). Informally, an AGNN first aggregates information through a graph convolution over the states. The output is then fed through a standard convolution neural network (CNN). As the CNN is acting on signals synthesized from locally available information, such an architecture naturally encodes the information sharing specified by the graph shift operator of the graph convolution layer. In the interest of space, we omit a detailed description of AGNNs, and instead refer the reader to Gama et al. (2020a) for a more detailed overview of their use in the context of distributed control.

When information exchange constraints are defined only in terms of spatial constraints, and not delays, one can repeated apply both the aggregation and the convolution steps to parameterize the
corresponding distributed controller using a graph convolution neural network (GCNN). A GCNN controller with $L$ layers can be written recursively as

$$
\begin{align*}
U(t) &= X_L \\
X_l &= \sigma \left( \sum_{k=0}^{K_l} S^k X_{l-1} H_{l,k} \right), \quad l = 1, \ldots, L \\
X_0 &= X(t),
\end{align*}
$$

(2.8)

where $\sigma$ is applied element-wise and $H_{l,k}$ is a set of trainable parameters. Note that while the application of the graph shift operator maintains locality of information exchange, the sub-controller at subsystem $i$ will need to instantaneously collect the local information $J_{j,t}$ from all neighboring subsystems $j$ satisfying $\text{dist}^{GC}(i,j) \leq \sum_{l=0}^{L} K_l$ – thus one must be careful to balance the depth $L$ of the network, the density of the graph shift operator $S$, and the aggregation horizon $K_l$, to ensure that locality is indeed preserved. The size of parameter matrices $H_{l,k} \in \mathbb{R}^{r_l \times c_l}$ can be arbitrary as long as internal state dimensions are consistent, i.e., $r_l = c_{l-1}$ and $r_0 = p, c_L = q$. The total number of parameters of a GNN is thus a tunable hyperparameter totaling $\sum_{l=0}^{L} (K_l + 1) r_l c_l$. We end by noting that distributed controllers parameterized as GCNNs will outperform comparable delayed AGNNs defined in terms of the same graph shift operator $S$, because the former does not enforce delay constraints on information sharing, whereas the latter does. As such, we use GCNN-based controllers as a baseline for comparison in our experiments.

### 2.2.3 Graph Recurrent Neural Networks

For the problem of communication topology co-design, we propose parameterizing distributed controllers using GRNNs – GRNNs have been successfully used for decentralized control in the context of imitation learning in Gama et al. (2020a). GRNNs extend the aforementioned GNN-based controller parameterizations by introducing a local hidden state $z_i(t) \in \mathbb{R}^h$ at each sub-controller, resulting in a dynamic distributed controller. We further allow sub-controllers to communicate both their local state and internal state with neighboring subsystems, i.e., in this parameterization, the local information set of subsystem $i$ is given by $J_{i,t} := \{x_i(t), z_i(t)\}$. Let $Z(t) := (z_i(t)^\top)_{i=1}^N \in \mathbb{R}^{N \times h}$ denote the full internal state obtained by stacking the subsystem internal states as in equation (2.5). While more general forms of GRNNs exist and can be used, we propose using the following compact GRNN parameterization, where an update step can be computed as

$$
\begin{align*}
Z(t) &= \sigma(S \ Z(t-1) \ W + X(t) \ F), \\
U(t) &= Z(t) \ G,
\end{align*}
$$

(2.9)

where $F \in \mathbb{R}^{p \times h}, G \in \mathbb{R}^{h \times q}, W \in \mathbb{R}^{h \times h}$ are trainable parameters.

We selected this architecture because it enjoys certain properties that make it naturally suited for encoding and optimizing communication delay structure. At each time step, the update step (2.9) applies the graph shift operator $S$ once on the internal states, which ensures that information within the controller is consistent with information flow across the communication network. Thus a controller parameterized by equation (2.9) satisfies the information sharing constraints defined in

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2. By comparable, we mean with comparably chosen hyper-parameters such that the overall number of free parameters of the function classes are approximately the same.
(2.3) and (2.4) with the information set $\mathcal{J}_{t,t} = \{x_i(t), z_i(t)\}$.

Further, the parameterization (2.9) is an affine mapping of the graph shift operator $S$ composed with a point-wise nonlinearity $\sigma$. The simplicity with which $S$ appears in parameterization (2.9) makes it well suited for treating $S$ as an optimization variable. In particular, whereas existing GNN-based distributed controllers (Tolstaya et al., 2020; Gama et al., 2020a; Khan et al., 2020; Gama and Sojoudi, 2021) assume a fixed graph shift operator $S$, we take a slight departure and jointly optimize the GRNN-based distributed controller parameter values and graph shift operator sparsity pattern and parameter values. Doing so allows us to co-design a distributed controller and the communication graph topology $\mathcal{G}_c$ needed to implement it.

3 Co-Design via Regularized Empirical Risk Minimization

Given a distributed linear system (2.1), our goal is to explore the design space of distributed controllers satisfying communication constraints of the form (2.4) while minimizing a cost function over a finite horizon $T \in \mathbb{Z}^+$. We consider two problem formulations. First, we show how to solve the distributed controller design problem on a given communication network $\mathcal{G}_c$ using GRNN with an empirical risk minimization (ERM) approach akin to that proposed in Gama and Sojoudi (2021). Next, we build on this formulation and show how to jointly design a distributed controller and the communication topology $\mathcal{G}_c$ needed to implement it through the use of $\ell_1$-regularized ERM.

3.1 Distributed GRNN Control Design: Given Topology

We consider the linear system (2.2), which can be equivalently rewritten as

$$ \text{vec}(\mathbf{X}(t + 1)) = A \text{vec}(\mathbf{X}(t)) + B \text{vec}(\mathbf{U}(t)). $$

(3.1)

Let $\mathcal{X} \subseteq \mathbb{R}^{N \times p}$ be a set, $\mathcal{D}$ be a distribution over $\mathcal{X}$, and the initial conditions of our system be distributed as $\mathbf{X}(0) \sim \mathcal{D}$. Suppose the quality of a system trajectory is characterized by the cost function $J(\{\mathbf{X}(t)\}_{t=0}^T, \{\mathbf{U}(t)\}_{t=0}^T)$. We can then pose our optimal control problem as

$$ \begin{align*}
\text{minimize} & \quad E_{\mathbf{X}(0) \sim \mathcal{D}} \left[ J(\{\mathbf{X}(t)\}_{t=0}^T, \{\mathbf{U}(t)\}_{t=0}^T) \right] \\
\text{subject to} & \quad \text{dynamics (3.1)}, \\
& \quad u_i(t) = \gamma_{i,t}(\mathcal{I}_{i,t}(\mathcal{G}_c)), \quad i = 1, ..., N, \quad t = 0, ..., T
\end{align*} $$

(3.2)

In general, this problem is non-convex even if the cost function is quadratic and the distribution $\mathcal{D}$ is Gaussian (corresponding to LQG control) due to the communication constraints (2.3) and (2.4) (encoded as $u_i(t) = \gamma_{i,t}(\mathcal{I}_{i,t}(\mathcal{G}_c))$). We thus aim to approximate the optimal solution to problem (3.2) by using the GRNN parameterization (2.9) for distributed controllers, and leveraging ERM.

As presented in Section 2.2.3, GRNN models can encode the communication constraints (2.3) and (2.4) induced by topology $\mathcal{G}_c$ by restricting the graph shift operator to satisfy $S \in \mathcal{S}_{\mathcal{G}_c}$, i.e., by enforcing that $S$ be a valid graph shift operator for the communication topology $\mathcal{G}_c$. In practice, we enforce this constraint by restricting the support of $S$ to be consistent with that of the adjacency matrix of the graph $\mathcal{G}_c$. Thus, we draw initial conditions $\{\mathbf{X}_p(0)\}_{p=1}^k \sim \mathcal{D}^k$ and pose the distributed GRNN control design problem as the following ERM problem:

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3. We note that $x_i(t)$ is in fact only ever needed at subsystem $i$, but we include it in $\mathcal{J}_{i,t}$ to be consistent with equations (2.3) and (2.4).
\[
\begin{align*}
\text{minimize} & \quad \frac{1}{k} \sum_{p=0}^{k} \left[ J \left( \{X_p(t)\}_{t=0}^T, \{U_p(t)\}_{t=0}^T \right) \right] \\
\text{s.t.} & \quad \text{dynamics (3.1),} \\
& \quad Z_p(t) = \sigma(S Z_p(t-1) W + X_p(t) F), \\
& \quad U_p(t) = Z_p(t) G, \ p = 0, \ldots, k, \\
& \quad S \in \mathcal{S}_{\mathcal{G}_c}, \ (S, F, G, W) \in \Theta. 
\end{align*}
\]

where the set \( \Theta \) is a user-specified set constraining the overall expressivity of the model to prevent over-fitting. A typical example would be to impose Frobenius norm bounds on the parameters, i.e.,

\[
\Theta = \{(S, F, G, W) : \|S\|_F^2 \leq B_S, \|F\|_F^2 \leq B_F, \|G\|_F^2 \leq B_G, \|W\|_F^2 \leq B_W\}.
\]

Following Gama and Sojoudi (2021), this problem can be approximately solved in a self-supervised manner using stochastic gradient descent. During training, we simulate the initial conditions forward with the controllers parameterized with the current parameter iterates \( \hat{S}, \hat{F}, \hat{G}, \hat{W} \), and compute their cost. The gradient of the cost with respect to the parameters can then be found efficiently via back propagation through time (Goodfellow et al., 2016, §10.2.2). We note that in contrast to prior work (Tolstaya et al., 2020; Gama et al., 2020a; Khan et al., 2020; Gama and Sojoudi, 2021), we propose optimizing the parameters of the graph shift operator \( S \) in addition to the parameters \( \{W, F, G\} \).

### 3.2 Communication Topology Co-Design

Unlike the previous setting, here we assume that the topology \( \mathcal{G}_c \) is also to be designed. In particular, we aim to develop a methodology that allows us to efficiently explore the tradeoff space between communication complexity, as measured by the cardinality of the edge set \( |\mathcal{E}_c| \), and controller performance. To do so, we propose a two step process.

First, we co-design the communication topology \( \mathcal{G}_c \) by solving the \( \ell_1 \)-regularized ERM problem

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{k} \sum_{p=0}^{k} \left[ J \left( \{X_p(t)\}_{t=0}^T, \{U_p(t)\}_{t=0}^T \right) \right] + \lambda \|S\|_1 \\
\text{s.t.} & \quad \text{dynamics (3.1),} \\
& \quad Z_p(t) = \sigma(S Z_p(t-1) W + X_p(t) F), \\
& \quad U_p(t) = Z_p(t) G, \ p = 0, \ldots, k, \\
& \quad (S, F, G, W) \in \Theta.
\end{align*}
\]

Note that in contrast to optimization problem (3.3), we have dropped the constraint that the graph shift operator satisfy \( S \in \mathcal{S}_{\mathcal{G}_c} \), and instead added an \( \ell_1 \) regularizer to the objective function, which is known to promote sparsity (Tibshirani, 1996; Donoho, 2006). Thus, by varying the regularization weight \( \lambda \), we can trade off sparsity of the network against controller performance.

Denote by \( \hat{S}(\lambda) \) the solution to optimization problem (3.4) for a given value of \( \lambda \), and let \( \hat{A}(\lambda) = \text{supp}(\hat{S}(\lambda)) \) be the corresponding adjacency matrix defined by its support. The designed graph \( \hat{\mathcal{G}}(\lambda) \) is then taken to be the graph induced by the adjacency matrix \( \hat{A}(\lambda) \). We then proceed with a standard refinement step, and solve optimization problem (3.3) subject to information sharing constraints imposed by the designed communication topology \( \hat{\mathcal{G}}(\lambda) \).

Note that in practice, we apply a thresholding procedure to determine adjacency matrix, i.e., \( \hat{A}_{ij} = 1 \) if an only if \( |\hat{S}_{ij}| \geq \varepsilon \), for some small numerical threshold \( \varepsilon \).

\[\text{supp}(\hat{S}(\lambda))\]
4 Numerical Experiments

All code needed to reproduce the examples found in this section will be made available at the following repository: https://github.com/unstable-zeros/grnn-comms-codesign.

We consider the distributed linear quadratic regulator (LQR) problem over $N = 20$ subsystems and a finite time horizon $T = 50$. Specifically, we consider the case where the state and control actions of the subsystems are scalars, i.e., $p = q = 1$. Under this assumption, $X(t) = x(t)$, $U(t) = u(t)$, and the problem can be written as

\begin{align}
\text{minimize} \quad & \mathbb{E}_{x(0) \sim \mathcal{N}(0, I)} \left[ \sum_{t=0}^{T-1} x(t)^\top Q x(t) + u(t)^\top R u(t) + x(T)^\top P x(T) \right], \\
\text{s.t.} \quad & x(t+1) = Ax(t) + Bu(t), \quad t = 0, 1, \ldots, T-1, \\
& u_i(t) = \gamma_{i,t}(\mathcal{I}_{i,t}(G_c)) \\
& \text{for } i = 1, \ldots, N, \quad t = 0, \ldots, T, \tag{4.1}
\end{align}

for dynamics matrices $A, B \in \mathbb{R}^{N \times N}$, $N \times N$ symmetric cost matrices $Q \succeq 0, R \succ 0, P \succeq 0$, and communication topology $G_c$.

**Generating Problem Instances:** An instance of the LQR problem (4.1) is fully defined by the tuple $(G_c, A, B, Q, R, P)$. We generate problem instances using the same process for all experiments. For each instance, we start by creating a communication topology $G_c$ by randomly sampling $N$ numbers $\{u_i\}_{i=1}^N \sim U[0, 1]$, and creating a bi-directional link between $v_i$ and each of its 5 nearest points as defined by the topology on the interval $[0, 1]$ under the metric $d(v_i, v_j) = |u_i - u_j|$. Following Gama and Sojoudi (2021), we generate the dynamics matrices $A$ and $B$ to share the same eigenvectors as the normalized adjacency matrix of $G_c$, and sample their eigenvalues i.i.d. from the standard normal distribution – hence both $A$ and $B$ are symmetric matrices. Then, to induce further sparsity in the dynamics, we set entries $A_{ij} = 0$ and $B_{ij} = 0$ whenever $\text{dist}^{G_c}(i \rightarrow j) > 3$. Finally, we normalize the matrices $A$ and $B$ to have the prescribed norms for the experiments being run. We take the cost matrices $Q = R = I_N$. Finally, the terminal state cost matrix $P$ is taken to be the solution to the Discrete Algebraic Riccati Equation (DARE)

\begin{equation}
A^\top PA - P - A^\top PB(B^\top PB + R)^{-1}B^\top PA = -Q. \tag{4.2}
\end{equation}

This ensures that the centralized solution is stabilizing. We end by noting that communication topologies $G_c$ generated this way have, on average, 141 directed edges (out of a possible 400).

In all of the following experiments, we use the ADAM algorithm (Kingma and Ba, 2014) to approximately solve the resulting ERM problems with a batch size of 20, and forgetting factors of 0.9 and 0.999. For the GRNN models, we choose the learning rate to be 0.02, and apply cosine annealing on top of ADAM for learning rate scheduling. For the GCNN models, we use a learning rate of 0.01, and decay the learning rate by a factor of 0.9 every 10 batches, as specified in Gama et al. (2020a). The initial conditions in each batch are sampled i.i.d. from $\mathcal{N}(0, I_N)$. We train the models with 750 batches of initial conditions.

4.1 Performance Benchmarks

In this set of experiments, we compare the performance of different variants of the GRNN parameterization (2.9) with a GCNN benchmark from Gama and Sojoudi (2021) on the LQR problem. We note that we focus exclusively on GNN based controllers in our study, as Gama and Sojoudi (2021)
has already conducted extensive studies comparing GNN-based distributed controllers against more traditional (fully-connected) neural network parameterizations of distributed controllers.

**Architectures:** We consider four different variants of the GRNN-based controller (2.9). All the GRNN variants considered share the same architecture with a hidden state dimension of 5. However, they differ in their training objective functions and constraints.

- **(GRNN)** approximately solves the LQR problem (4.1) using the ERM problem (3.3).

- **(GRNN-Dense)** does not place constraints on the graph shift operator $S$, and can thus design a centralized controller.

- **(GRNN-Sparse)** co-designs a communication topology and distributed controller using (3.4) to identify a topology $\hat{G}$ and (3.3) restricted to the identified topology for the subsequent refinement step.

- **(GRNN-Fixed)** restricts $S$ to be constant and only optimizes over the parameters $F,G$, and $W$.

- **GCNN** is the GCNN architecture in Gama and Sojoudi (2021), which we take as a benchmark. We take the graph shift operator $S$ to be the normalized adjacency matrix of $G_c$. The GCNN model has 2 layers with 5 and 1 filter taps, respectively and uses a total of 192 parameters.

**Training and evaluation:** We run our experiments across 50 random LQR instances. We report the costs achieved by each controller normalized with respect to that achieved by the centralized LQR optimal controller $K_{LQR} = -(B^T P B)^{-1}B^T P A$ computed via the solution to the DARE (4.2). We note that the centralized cost represents a fundamental lower limit on the achievable performance, and thus a normalized cost of 1 is optimal.

We run this experiment on 50 randomly generated pairs of communication networks $G_c$ and dynamics matrices $(A,B)$. Every 10 batches, we evaluate the controllers over a validation set of 100 initial conditions and report the average performance. To test the performance of the GRNN method on both stable and unstable environments, we repeat the process on two scenarios, one with the norm of the dynamics matrix $\|A\|_2 = 0.995$, the other with $\|A\|_2 = 1.05$. When training the GRNNs, we also apply weight-decay (WD) (Frobenius-norm regularization) on the parameters $(S,F,W,G)$, when applicable. When training the GCNNs, we experimented with no WD on the filter-taps, as specified in Gama and Sojoudi (2021), as well as with weight-decay: we found the results with and without WD to be qualitatively and quantitatively similar, and present here only those without WD. When solving the co-design problem (3.4), we set the regularization penalty $\lambda$ to 1 and 2 for $\|A\|_2 = 0.995$ and $\|A\|_2 = 1.05$, respectively: these values were selected based off of the co-design experiments we describe next. We report the median relative cost for all methods in Table 1, and plot the learning curves of the controllers during the training in Figure 1.

**Discussion:** First, we note that the GRNN parametrization outperforms the the GCNN benchmark on both the stable and unstable dynamics despite the latter parameterization not enforcing communication delay constraints. This suggests that the GRNN model can leverage the recurrent structure to aggregate information without needing a large number of filter taps. We further observe that all GRNN architectures enjoy reduced variance as compared to the GCNN architecture.\footnote{We take the threshold value for determining the adjacency matrix support to be $\varepsilon = 0.004$.}

\footnote{The GCNN architecture with WD has only slightly less variance at the expense of even worse performance.}
Table 1: Median normalized cost for the performance benchmarking experiments described in §4-A over 50 randomly generated problem instances. The GCNN baseline is that described in Gama and Sojoudi (2021).

| Architecture      | $\|A\|_2 = 0.995$ | $\|A\|_2 = 1.05$ |
|-------------------|---------------------|---------------------|
| Autonomous        | 2.534               | 69.294              |
| GRNN              | 1.112               | 1.153               |
| GRNN-Dense        | 1.091               | 1.110               |
| GRNN-Sparse       | 1.093               | 1.124               |
| GRNN-Fixed        | 1.267               | 2.210               |
| GCNN              | 1.212               | 1.715               |

Interestingly, we also observe that the GRNN-Sparse controllers, for which the communication topology is co-designed, are able to outperform the GRNN models with a prescribed communication topology that is designed to be well aligned with the system dynamics (via their eigenspaces). Furthermore, the GRNN-Sparse controllers use on average, 53.4 and 56.1 directed communication links for the stable and unstable dynamics, respectively, far fewer than the average number of links in the given communication topology. This indicates that our co-design algorithm was able to identify key links in the communication topology needed for control, and further suggests that by restricting the controller parameterization to a simpler (sparser) model class, less data is required to learn a high-performing controller. We leave formalizing these observations in the context of the ERM framework to future work.

4.2 Controller-Topology Co-design

We show how our approach allows us to efficiently explore the trade-off between the sparsity of the communication topology of a distributed controller and the performance that it achieves via our co-design problem (3.4). We run our GRNN co-design algorithm for four families of LQR instances, each characterized by the operator norm of the $A$ matrix defining the system dynamics, with $\|A\|_2$ ranging from stable (less than 1) to unstable (greater than 1). For each value of $\|A\|_2$, we solve the co-design problem (3.4) with regularization weight $\lambda$ varying across several orders of magnitude,
sweeping out a curve $\hat{G}(\lambda)$ in communication topology space: a representative example of such a curve is illustrated in Fig. 3. We record the cost achieved by the co-designed distributed controller, as well as number of directed links needed to implemented it as captured by the cardinality of the designed edge set $|\mathcal{E}_c|$ of the communication topology over 30 realizations for each value of $\|A\|_2$. We plot the resulting trade-off curves in Figure 2.

Discussion: As the results show, there is a clear trade-off between the sparsity of the communication topology and the performance of the system. We observe that the trade-off becomes more pronounced as the norm of $A$ grows, i.e., as the open-loop system becomes more unstable and consequently, difficult to control. Interestingly, we observe that for all values of $\|A\|_2$, our co-design algorithm is able to explore the entire design space from dense (virtually centralized) to (nearly) completely decentralized. We emphasize that such (nearly) completely decentralized communication topologies do not satisfy the conditions needed for convexity/tractability of linear distributed optimal control such as quadratic invariance (Rotkowitz and Lall, 2005; Rotkowitz et al., 2010) or funnel causality (Bamieh and Voulgaris, 2005), and yet reasonably well-performing distributed optimal controllers are identified. While prior work (Matni and Chandrasekaran, 2016) identifies conditions under which such co-designed architectures are optimal, we are not aware of such conditions in the context of data-driven distributed control. We view our results as strong empirical evidence that such an investigation should be undertaken.

![Figure 2: Trade-off curves of the normalized cost versus communication topology complexity (characterized via the number of directed edges $|\mathcal{E}|$) for varying values of $\|A\|_2$. For each value of $\|A\|_2$, we plot the median values across 30 randomly generate problem instances, with horizontal and vertical error bars denoting the quartiles in number of edges and normalized cost, respectively.](image)

5 Conclusion & Future Work

Jointly designing communication topologies with distributed controllers is difficult in general. In this work, we proposed a data-driven method to solve the co-design problem by parametrizing the distributed controllers as a GRNN and solving an $\ell_1$-regularized ERM problem that allows for a principled tradeoff between control cost and communication topology complexity. We demonstrated empirically that our proposed GRNN parametrization can generate distributed controllers
with good performance and our method allows us to efficiently explore the trade-off space between the sparsity of the designed communication network and the performance of the distributed controllers. Future work will look to provide generalization bounds on the learned controllers by leveraging recent results that allow for uniform convergence guarantees to be applied to dynamical systems under suitable stability assumptions (Tu et al., 2021) along with quadratic constraint based conditions for the stability of dynamical systems modelled as recurrent neural networks (Revay et al., 2020).

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