Third formulation of the space-time finite element method

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Abstract. The paper explains differences existing between the space-time finite elements, which fundamentals were presented by the Kączkowski in 1975 and 1976, and its modification. The replacement of load with concentrated forces and moments (which are to satisfy equilibrium conditions at nodes) was given up, namely in the latter one; instead, a requirement was made that functions approximating the solution would satisfy differential equation in inter-node areas in a specified integral way. This leads to a system of algebraic equations, in which kinematic parameters existing in the solution approximating functions are the unknowns. Displacements and their derivatives (of any order) at nodes are those parameters.

1. Introduction
The space-time finite element method is an adaptation of the classical finite element method [4,17,18] to a time-dependent problem. The first formulation of the space-time finite element method, the so-called stationary presented by Kączkowski in the works [8-10]. The stationary formulation of the method was generalized in the doctoral dissertation by Kacprzyk [6]. In 1983, Kączkowski presented the second formulation of the method in which he applied the non-stationary division of space-time [11]. This approach was developed by Bajer in the works [1-3]. The method was used and developed by many researchers [7,14-16].

Below we present a modified space-time element method, in which we will give up replacing loads acting on an element with a system of loads concentrated at nodes. Functions approximating the solution, i.e. satisfying approximately a system of differential equations in the finite element area, we will make dependent on any number of kinematic node parameters. Displacement values and any number of consecutive derivatives of functions approximating the solution will be those parameters. Each increase in those parameters number (without changing the nodes grid) results in a significant increase in the solution accuracy.

2. General relationships
Let us assume that in the space-time area $\Omega$ the motion of a body is described by a differential matrix equation

$$\mathbf{L} \mathbf{u}(\mathbf{x}, t) = \mathbf{p}(\mathbf{x}, t)$$

in which $\mathbf{L}$ is a matrix composed of differential operators multiplied by coefficients dependent on physical parameters characterising the material, $\mathbf{u}$ is the displacement vector, and $\mathbf{p}$ - the vector of external load. Number $n$ depends on the number of kinematic parameters describing the motion of the
considered body. Equation (1) multiplied by the volume of infinitely small element \(d\Omega\) expresses the equilibrium condition for impulses acting on this element.

Equation (1) may be also considered as a mathematical notation of the Newton’s third law; the right side of equation (1) means all loads acting on the considered body from outside, while the left side - an equal but oppositely directed action (reaction) of the body.

After transferring load \(p\) to the left side of the equal sign, equation (1) gets the following form:

\[
Lu - p = 0
\]

(2)

If vector \(u\) is an exact solution of equation (1), then the product of any functional matrix \(\bar{u}^T(x,t)\) and equation (2) as well as an integral of this product in area \(\Omega\) are as an identity equal to zero:

\[
\iint_{\Omega} [\bar{u}^T(Lu - p)]d\Omega = 0
\]

(3)

Components of matrix \(\bar{u}\) may be treated as virtual increments in displacements arranged in matrix \(u\), and equation (3) - as an equation of virtual work-time.

Let us present an approximate solution of differential equation (1) in the following form

\[
v(x,t) = N(x,t)C
\]

(4)

So-called shape functions, satisfying specific geometrical conditions, which we will discuss below, are components of matrix \(N\). Sub-matrices \(C_j (j = 1,2,3,...,k)\) are components of one-column matrix \(C\), each of them consists of geometrical parameters expressing values of function \(v\) and its derivatives of the first and possibly higher orders at node \(j\) of coordinates

\[
x,t = x_j, t_j, \quad j = 1,2,3,...,k
\]

(5)

One of components of function matrix \(N\) is assigned to each of constants forming vector \(C\). If elements featuring the shape of a rectangle or cuboid are used, the shape functions usually get the shape of basic functions, each of which depends only on one space or time variable and on the approximation order. Table 1 specifies examples of basic functions of orders \(I, II, III,\) and \(IV\). These functions are components of the following one-row matrices

\[
N' = [\rho], \quad N'' = [\rho, \omega], \quad N''' = [\rho, \omega, \chi], \quad N'''' = [\rho, \omega, \chi, \lambda]
\]

(6)

The following notation has been used in the table 1

\[
\vartheta = \vartheta(\xi) = \frac{1-\xi^2}{4}, \quad \xi_i = \pm 1, \quad \xi \in (-1,+1)
\]

(7)

Taking into consideration the fact that function \(\vartheta\) for both extreme values of \(\xi\) is equal zero, and function \(\frac{1+\xi \xi}{2}\) only at \(\xi = \xi_i\) takes the value of 1, based on table 1 it is easy to determine boundary conditions, satisfied by individual components of matrices specified in formulae (6).
Table 1. Basic shape functions

| Order | \( f \) |
|-------|--------|
| I     | \( \rho \) \( \frac{1 + \xi_i \xi_j}{2} \) |
| II    | \( \rho \) \( \frac{1 + \xi_i \xi_j + \xi_i \xi_j \partial}{2} \) \( \omega \) \( - (\xi_i + \xi_j) \partial \) |
| III   | \( \rho \) \( \frac{1 + \xi_i \xi_j + \xi_i \xi_j (\partial + 3 \partial^2)}{2} \) \( \omega \) \( - (\xi_i + \xi_j) \partial - (\xi_i + 3 \xi_j) \partial^2 \) \( \zeta \) \( (1 + \xi_i \xi_j) \partial^2 \) |
| IV    | \( \rho \) \( \frac{1 + \xi_i \xi_j + \xi_i \xi_j (\partial + \partial^2) + 10 \xi_i \xi_j \partial^3}{2} \) \( \omega \) \( - (\xi_i + \xi_j) \partial - (\xi_i + 3 \xi_j) \partial^2 - 2 (\xi_i + 5 \xi_j) \partial^3 \) \( \zeta \) \( (1 + \xi_i \xi_j) \partial^2 + (2 + 4 \xi_i \xi_j) \partial^3 \) \( \lambda \) \( - \frac{2}{3} (\xi_i + \xi_j) \partial^3 \) |

Having substituted an approximate solution (4) to differential equation (2) this equation will be satisfied only approximately

\[
\mathbf{L} \mathbf{v}(\mathbf{x},t) - \mathbf{p}(\mathbf{x},t) = \mathbf{L} \mathbf{N}(\mathbf{x},t) \mathbf{C} - \mathbf{p}(\mathbf{x},t) = \mathbf{e}(\mathbf{x},t) \approx 0 \quad (8)
\]

Let us multiply equation (25) by a function matrix

\[
\mathbf{v}^T(\mathbf{x},t) = \overline{\mathbf{C}}^T \mathbf{N}^T(\mathbf{x},t) \quad (9)
\]

i.e. by the function matrix consisting of shape functions \( \mathbf{N} \), defined above, multiplied by any coefficients being components of vector \( \overline{\mathbf{C}} \). Although the product of function (9) and equation (8) is different from zero, however, we can demand this product integral within boundaries of area \( \Omega \) to be equal to zero

\[
\overline{\mathbf{C}}^T [\mathbf{K} \mathbf{C} - \mathbf{P}] = 0 \quad (10)
\]

In this equation symbols \( \mathbf{K} \) and \( \mathbf{P} \) stand for the following quantities

\[
\mathbf{K} = \int_{\Omega} \mathbf{N}^T \mathbf{L} \mathbf{N} d\Omega, \quad \mathbf{P} = \int_{\Omega} \mathbf{N}^T d\Omega \quad (11)
\]

To satisfy equation (10) irrespective of the number of matrix \( \overline{\mathbf{C}} \) components, the expression in brackets must be equal to zero

\[
\mathbf{K} \mathbf{C} - \mathbf{P} = 0 \quad (12)
\]

In this way we have received a system consisting of \( m \) equations depending on \( m \) parameters making vector \( \mathbf{C} \). Part of those parameters is known from boundary and initial conditions, and also from possible kinematic conditions at points situated within area \( \Omega \). Having split vector \( \mathbf{C} \) into two sub-vectors: \( ^u \mathbf{C} \) composed of unknown parameters and \( ^i \mathbf{C} \) - of known parameters, and after an appropriate regrouping of matrix \( \mathbf{K} \) columns and rows, the system of equations (12) gets the following form:
Hence unknown parameters $^uC$ will be determined from the following formula:

$$^uC = -K_{mai}^{-1}k_{ab}^C + K_{ma}^{-1}^aP$$  \hspace{1cm} (14)$$

and after substituting them and components of vector $^kC$ to formula (4) we obtain an approximate solution of the problem. The presented procedure may be applied not only in the space-time element method but also in other numerical methods used in mechanics. Below, we will test the derived formulae on a problem from dynamics.

3. Example - A Free Motion of Mass Concentrated in the Gravitational Field

We will follow the procedure in the modified space-time element method using the simplest example of mass $m$ motion in a free-fall. In the space-time a concentrated mass is a one-dimensional object, referred to as the mass life line. As long as the mass is in rest, its life line is a straight line, parallel to the time axis. The life line of a mass moving under the influence of the force of gravity is a curve described by a differential equation

$$m \frac{d^2u}{dt^2} = mg$$  \hspace{1cm} (15)$$

After dividing the mass life line into elements $2h$ long (measured in time units) and after introducing dimensionless quantities

$$\tau = \frac{t}{h}, \quad \nu = \frac{u}{gh^2}$$  \hspace{1cm} (16)$$

equation (15) gets the form

$$\dot{\nu} = 1, \quad \text{where} \quad \ddot{\nu} = \frac{d^2\nu}{d\tau^2}$$  \hspace{1cm} (17)$$

We look for the integral of this equation in the area of element comprised between points $a$ (ante) and $p$ (post) (see figure 1) in the form of

$$\nu(\tau) = \left[ N_a(\tau) \quad N_p(\tau) \right] \left[ C_a \quad C_p \right]$$  \hspace{1cm} (18)$$

where

$$N_i(\tau) = \left[ \rho_i(\tau) \quad \omega_i(\tau) \right], \quad C_i = \left[ v_i \right], \quad i = a, p$$  \hspace{1cm} (19)$$

Figure 7. Dimensionless space-time element.
We assumed that order II functions are components of the shape matrix (cf. table 1). The application of order I shape function is impossible, because the substitution of a linear function to an inhomogeneous differential equation (17) would result in a contradiction.

With the symbols used here the components of matrices $K$ and $P$ (cf. formulae (11) and (13) have the following structure

$$K_{ij} = \frac{1}{30} \begin{bmatrix} -18\tau_i\tau_j & 18\tau_i + 15\tau_j \\ 3\tau_j & -3 - 5\tau_i\tau_j \end{bmatrix}, \quad P_{ij} = \frac{1}{3} \begin{bmatrix} 3 \\ -\tau_i \end{bmatrix}$$

where $i, j = a, p$

In particular it is

$$K_{pp} = \frac{1}{30} \begin{bmatrix} -18 & 33 \\ 3 & -8 \end{bmatrix}, \quad K_{pa} = \frac{1}{30} \begin{bmatrix} 18 & 3 \\ -3 & 2 \end{bmatrix}, \quad P_p = \frac{1}{3} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Then we calculate successively

$$K^{-1}_{pp} = \frac{2}{3} \begin{bmatrix} 8 & 33 \\ 3 & 18 \end{bmatrix}, \quad -K^{-1}_{pp}K_{pa} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad K^{-1}_{pp}P_p = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Hence we will find the next vectors of dimensionless displacements based on a recurrent formula (14)

$$\begin{bmatrix} v_i \\ \dot{v}_i \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{i-1} \\ \dot{v}_{i-1} \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and - having returned to denominated quantities - we will obtain

$$\begin{bmatrix} u_i \\ \dot{u}_i \end{bmatrix} = \begin{bmatrix} gh^2 & 0 \\ 0 & gh \end{bmatrix} \begin{bmatrix} v_i \\ \dot{v}_i \end{bmatrix} + \begin{bmatrix} 2gh^2 \\ 2gh \end{bmatrix}$$

Let us notice that - irrespective of the element dimension - we obtain exact results. If we assume that at moment $t = 0$ the displacement $u_0$ and the velocity of mass motion $\dot{u}_0$ are equal to zero, then at moments $t = 2h$ and $t = 4h$, based on formula (24), we will obtain

$$u_1 = 2gh^2 = \frac{g(2h)^2}{2}, \quad \dot{u}_1 = g(2h); \quad u_2 = 8gh^2 = \frac{g(4h)^2}{2}, \quad \dot{u}_2 = g(4h)$$

Also it is easy to check that not only at nodes but also at any point between nodes $a$ and $p$ the displacement:

$$u = gh^2\left[ v_n\rho_{-1}(\tau) + \dot{v}_n\omega_{-1}(\tau) + v_p\rho_{-1}(\tau) + \dot{v}_p\omega_{-1}(\tau) \right]$$

exactly satisfies the differential equation of the problem.
4. Conclusions
The basic difference between the finite element method and the modification presented above consists in the fact, that the replacement of load with concentrated forces and moments (which are to satisfy equilibrium conditions at nodes) was given up, instead, a requirement was made that functions approximating the solution would satisfy differential equations in inter-node areas in a specified integral way. This leads to a system of algebraic equations, in which kinematic parameters existing in the solution approximating functions are the unknowns. Displacements and their derivatives (of any order) at nodes are those parameters. The modified space-time element method was tested on a few experimental numerical examples - in this article, we present one simple example.

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