DEGENERATION OF CURVES
AND ANALYTIC DEFORMATIONS

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§0 Introduction

Let $\Delta$ be a complex disk $\{ z \in \mathbb{C} \mid |z| < 1 \}$ and $p : D \to \Delta$ be a proper flat morphism of relative dimension one. It is called a minimal degeneration of genus $g$ curves, if (1) $D$ is smooth, (2) the restriction $p : p^{-1}(\Delta - \{0\})$ is a smooth family of genus $g$ curves, and (3) the fiber $p^{-1}(0)$ contains no smooth rational curves of the first kind. By choosing a metric $\mu$ on $p^{-1}(\Delta - \{0\})$, a loop $\gamma$ around 0 with a base point $\delta \in \Delta - \{0\}$ induces a differentiable automorphism $\Gamma(p, \mu, \gamma)$ of $p^{-1}(\delta)$, whose class $\Gamma'(p) \in MC(p^{-1}(\delta))$ is independent of the choice of the metric $\mu$ and the homotopy class of $\gamma$. Let $S_g$ be an oriented closed differentiable surface of genus $g$ and $D : p^{-1}(\delta) \to S_g$ be a diffeomorphism. Then the conjugacy class $\Gamma(p)$ of $D \circ \Gamma'(p) \circ D^{-1}$ in the mapping class group $MC_g = MC(S_g)$ of $S_g$ is independent of the diffeomorphism $D$. Then $\Gamma(p)$ is known to be a pseudo-periodic with negative Dehn twists. It is also known by [MM] that for any pseudo-periodic conjugacy class $\Gamma$ with negative Dehn twists, there exists a minimal degeneration of genus $g$ curves $p : D \to \Delta$ such that $\Gamma(p) = \Gamma$ unique up to diffeomorphism over $\Delta$. If $p_1 : D_1 \to \Delta$ and $p_2 : D_2 \to \Delta$ are obtained by a smooth holomorphic deformation of the degeneration $p_2 : D_2 \to \Delta$, then $D_1$ and $D_2$ are diffeomorphic over $\Delta$ and $\Gamma(p_1)$ is equal to $\Gamma(p_2)$. In this paper, we prove that if two minimal degeneration of curves $p_1 : D_1 \to \Delta$ and $p_2 : D_2 \to \Delta$ satisfies $\Gamma(p_1) = \Gamma(p_2)$, then these two degenerations $p_1, p_2$ are contained in the same equivalence class generated by smooth holomorphic deformations.

The contents of this paper is organized as follows. We introduce general notations for stable curves and stable curves with an action of cyclic group. Most of the terminology appeared here are introduced in [MM] in the topological context. In the next section, we introduce several moduli problems related to level structures and group actions. Here we use the notion of the type of a group action introduced in §1 and the beginning of §2. The main result of this section is the connectedness of moduli of curves with an action of cyclic group (Proposition 2.2 (2)). In §3, we study the group action of the local moduli space. Here we introduce a coordinate on local moduli compatible with the action of the cyclic group. In §4, we study the local moduli map attached to the stabilization of the degeneration of genus $g$ curves. The screw numbers introduced by [MM] appears here as an analytic invariant. The statement and the main theorem are given in §5.

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§1 A stable curve with an action of a cyclic group

In this section, we introduce several notations concerning stable curves with an action of a cyclic group. Let \( g \) be a natural number greater than 1. Let \( S \) be an analytic space and \( f : \tilde{C} \to S \) be a proper flat morphism of relative dimension one with connected fibers. The morphism \( f \) is called a stable curve of genus \( g \) ([DM]), if

1. \( \dim H^1(f^{-1}(s), \mathcal{O}_{f^{-1}(s)}) = g \) for all \( s \in S \).

2. All the fibers \( f^{-1}(s) \) are reduced curves whose singularities are at most nodes. Here a point \( p \in f^{-1}(s) \) is called a node if there exists an open set \( U \) of \( p \) such that \( U \) is isomorphic to \( \{ (x, y) \in \mathbb{C}^2 \mid |x| < 1, |y| < 1, xy = 0 \} \).

We do not impose that \( f^{-1}(s) \) is irreducible for \( s \in S \).

The genus of the normalization \( \tilde{C}_i \) of a component \( C_i \) in \( f^{-1}(s) \) is called the genus of the irreducible component \( C_i \).

3. Any non-singular genus 0 component meets with another components at least 3 points.

For any stable curve \( C \) of genus \( g \) over a point, we can associate a graph called dual graph \( \tau \) of \( C \) as follows. The sets of vertices \( V(\tau) \) and edges \( E(\tau) \) of the graph \( \tau \) is given by the set of irreducible components of \( C \) and the singular points of \( C \) respectively. A vertex \( v \) is connected to an edge \( e \) if and only if the corresponding singularity \( s_e \) is an element of the corresponding component \( C_v \). The normalization of the curve \( C_v \) is denoted by \( \pi_v : \tilde{C}_v \to C_v \) and the inverse image of singular points under \( \pi_v \) are called special points of \( \tilde{C}_v \). The set of special points in \( \tilde{C} \) is denoted by \( Sp(C) \). The genus of \( \tilde{C}_v \) is denoted by \( g(v) \). It is a function from the set of vertices \( V(\tau) \) to the set of natural numbers. An edge \( e \) has two extreme \( e^{(1)}, e^{(2)} \) and they are called flags. The set of flags is denoted by \( F(\tau) \) and it is identified with \( Sp(C) \). A flag \( f \) is called a tail of \( v \) if \( f \) is connected to the vertex \( v \) and the set of tails of \( v \) is denoted by \( T(v) \). The number of the tails of \( v \) is denoted by \( t(v) \). A graph is called numbered if the index set of vertices \( V(\tau) \) and flags \( F(\tau) \) are numbered by \( I_v \) and \( I_f \) respectively. The union of \( I_v \) and \( I_f \) is denoted by \( I \).

Definition. Let \( \tau \) be a connected graph and \( g : V(\tau) \to \mathbb{N} \) be a function from the set of vertices to the set of non-negative integers. A pair \( (\tau, g) \) is called a stable graph if any vertex \( v \) with \( g(v) = 0 \) satisfies \( t(v) \geq 3 \).

An automorphism \( \sigma \) of \( C \) of finite order \( m \) induces an automorphism of the dual graph \( \tau \) of \( C \) and it is denoted by \( \sigma_\ast \). If we fix a numbering of \( \tau \), \( \sigma_\ast \) can be identified with a permutation of the index set \( I \). It is easy to see that the function \( g : V(\tau) \to \mathbb{N} \) is preserved by \( \sigma_\ast \).

Let \( G \) be the subgroup of \( \text{Aut}(C) \) generated by \( \sigma \). Let \( G(\nu), G(f) \) and \( G(e) \) be the stabilizer of a vertex \( \nu \), a flag \( f \) and an edge \( e \) respectively. Then we have \( G(e) \supset G(f) \) and \( \#[G(e) : G(f)] \) is 1 or 2. An edge \( e \) is called amphidrome (resp. non-amphidrome) if \( \#[G(e) : G(f)] = 2 \) (resp. \( G(e) = G(f) \)). An element \( \sigma \in G(\nu) \) (resp. \( \sigma \in G(f) \), \( \sigma \in G(e) \)) acts on the normalization \( \tilde{C}_v \) of the component \( C_v \) (resp. a neighbourhood of the special point, a neighbourhood of the singular point). For a point \( p \) in the normalization \( \tilde{C} \) of \( C \), the character of the stabilizer \( G(p) \) of \( p \) induced on the tangent space \( T_p\tilde{C} \) of \( \tilde{C} \) at \( p \) is called the local representation at \( p \) and denoted by \( \rho(p) : G(p) \to \mathbb{C}^\times \). If the point \( p \) corresponds to a flag \( f \), \( \rho_f \) is denoted by \( \rho_f \). Note that this is an injective homomorphism. If \( p \) and \( q \) are in the same orbit under \( G \), we have \( C(p) = C(q) \) and \( e(p) = e(q) \). A point \( p \in \tilde{C} \) is called
a ramification point for the action of $G$ if $G(p) \neq 1$. The order of the group $G(p)$ is called the ramification index and denoted by $m_p$. If $p$ corresponds to a flag $f$, $m_p$ is written by $m_f$. The order of $G(e)$ is denoted by $m_e$. For the local representation $\rho(p) : G(p) \to \mathbb{C}^\times$, at $p$, put $\rho(e(\frac{a}{m_p})) = e(\frac{b}{m_p})$. Then since $(b, m_p) = 1$, there exists $0 \leq a < m_p$ such that $a \cdot b \equiv 1 \pmod{m_p}$. The rational number $\frac{a}{m_e}$ is called the valency at $p$ for the action of $\mu_m$ and denoted by $\text{val}(p)$. For a flag $f$, the valency of the corresponding special point is denoted by $\text{val}(f)$ and the restriction of $\text{val}$ to $T(v)$ is denoted by $\text{val}_v$. The set of ramification points in $\tilde{C}$ is denoted by $R$. Then $R$ is a finite set. Denote $R \cap \text{Sp}(C) = R_1$ and $R - R_1 = R_0$. An element of $R_1$ (resp. $R_0$) is called a singular ramification point (resp. a smooth ramification).

Let $\tilde{C}_v$ be a component of the normalization of $\tilde{C}$. An element

$$r_v = \sum_{q \in (R_0 \cap \tilde{C}_v) / G(v)} [\text{val}(q)] \in \oplus_{\alpha \in \mathbb{Q} \cap [0,1]} \mathbb{Z}[[\alpha]]$$

is called the type of smooth ramification for $v$. The type of the action $\sigma$ on $C$ is defined by the collection $T = (r_v(v \in V(\tau)), \text{val}_v(f)(f \in T(v)))_v$. It is easy to see the following lemma.

**Lemma 1.1.**

1. $$\sum_{q \in (R \cap \tilde{C}_v) / G(v)} \text{val}(q)$$

is an integer.

2. $$\tilde{g}_v = \frac{1}{2} \left( \frac{1}{#G(v)}(2g - 2) - \sum_{p \in (R \cap \tilde{C}_v) / G(v)} (1 - \frac{1}{m_p}) \right) + 1$$

coincides with the genus of the quotient $\tilde{C}_v / G(v)$, and it is a non-negative integer.

**Definition.**

1. For an element $r = \sum_{\alpha} u_{\alpha} [\alpha] \in \oplus_{\alpha \in \mathbb{Q}} \mathbb{Z}[[\alpha]]$, $\sum_{\alpha} u_{\alpha} < \alpha >$ is denoted by $< r >$.

2. For a positive rational number $\alpha$, $\text{Cor}(\alpha) = 1 - \frac{1}{d}$, where $d$ is the denominator of $\alpha$. For an element $r = \sum_{\alpha} u_{\alpha} [\alpha]$, we define $\text{Cor}(r) = \sum_{\alpha} u_{\alpha} \cdot \text{Cor}(\alpha)$.

**§2 Several moduli problems, smoothness and connectedness**

In this section we study the moduli space of smooth marked curves. Let $g \geq 0$ and $m \geq 1$ be natural numbers. The module $\oplus_{\alpha \in \frac{1}{m} \mathbb{Z} \cap [0,1]} \mathbb{Z}[[\alpha]]$ is denoted by $B(m)$. Let $r = \sum_{\alpha} u_{\alpha} [\alpha] \in B(m)$ with $u_{\alpha} \geq 0$, $T$ be a finite set on which $\mu_m$ acts and $\text{val} : T \to \frac{1}{m} \mathbb{Z} \cap [0,1]$ be an invariant function under the action of $\mu_m$.

**Definition.** The pair $(r, \text{val})$ is realized if

1. $< r > + \sum_{f \in T / \mu_m} \text{val}(f)$ is an integer.

2. $$\tilde{g} = \frac{1}{2} \left( \frac{1}{m}(2g - 2) - \text{Cor}(r) - \sum_{f \in T / \mu_m} \text{Cor}(\text{val}(f)) \right) + 1$$
is a non-negative integer.

(3) If \( \bar{g} = 0 \), no proper divisor \( m' \) of \( m \) has the following property:

\[
    t \in B(m') \text{ and } Im(val) \in \frac{1}{m'} \mathbb{Z}.
\]

From now on, we fix a sufficiently big prime number \( l \) such that (1) \( \mu_m \subset \mathbb{F}_l^\times \), and (2) \( Aut(C) \to Aut(Pic^0(C)_l) \) is injective for all curves of genus \( g \). We define a symplectic structure and \( \mu_m \)-action on \( V = \mathbb{F}_l^{2g} \) as follows. The representation of \( \mu_m \) on \( \mathbb{F}_l^\times \) defined by \( \mu_m \ni x \mapsto x^t \in \mathbb{F}_l^\times \) \((0 \leq t < m)\) is denoted by \( \mathbb{F}^t(t) \). Let \( V^{01} \) and \( V^{10} \) be \( \mu_m \)-modules defined by \( V^{01} = \oplus_{t=1}^{m-1} \mathbb{F}^t(t)^{h^{01}(t)} \) and \( V^{10} = \oplus_{t=1}^{m-1} \mathbb{F}^t(t)^{h^{10}(t)} \), where

\[
    h^{01}(t) = g - 1 + < t \cdot r > + \sum_{\bar{f} \in T/\mu_m} < t \cdot \text{val}(\bar{f}) >
\]

\[
    h^{10}(t) = g - 1 + < -t \cdot r > + \sum_{\bar{f} \in T/\mu_m} < -t \cdot \text{val}(\bar{f}) >
\]

Then there exists a natural perfect pairing \((,): V^{01} \times V^{10} \to \mathbb{F}_l\). Let \( V_p = V^{01} \oplus V^{10} \). This vector space has a symplectic structure \( \psi_p \) defined by

\[
    \psi_p(a, a') = \psi_p(b, b') = 0 \quad (a, a' \in V^{01}, b, b' \in V^{10})
\]

\[
    \psi_p(a, b) = -\psi_p(b, a) = (a, b) \quad (a \in V^{01}, b \in V^{10}).
\]

Let \( \bar{V} \) be the vector space over \( \mathbb{F}_l \) with the base \( \alpha_1, \ldots, \alpha_{\bar{g}}, \beta_1, \ldots, \beta_{\bar{g}} \). The symplectic form \( \bar{\psi} \) on \( \bar{V} \) is given by

\[
    \bar{\psi}(\alpha_i, \alpha_j) = \bar{\psi}(\beta_i, \beta_j) = 0, \bar{\psi}(\alpha_i, \beta_j) = \delta_{i,j}.
\]

Then the direct sum \( V = \bar{V} \oplus V_p \) has a symplectic form \( \bar{\psi} \oplus \psi_p \).

Now we define moduli stack \( \mathcal{M}_{g,t} \), \( \mathcal{M}_{g,t}(l) \), \( \mathcal{M}_{g,t}(r, \text{val}) \) and \( \mathcal{M}_{g,t}(r, \text{val}, l) \). An object of \( \mathcal{M}_{g,t} \) is a pair \((C \to S, P_f(f \in T))\) consisting of a smooth curve \( C \to S \) of genus \( g \) over an analytic variety \( S \) and a set of sections \( P_f: S \to C \) indexed by the set \( T \) of the cardinality \( t \). We assume that \( P_f(s) \neq P_{f'}(s) \) for all \( s \in S \) if \( f \neq f' \).

A morphism from \((C_1 \to S_1, P_{1f})\) to \((C_2 \to S_2, P_{2f})\) is a following commutative diagram

\[
    \begin{array}{c}
    C_1 \longrightarrow C_2 \\
    \downarrow \quad \downarrow \\
    S_1 \longrightarrow S_2
    \end{array}
\]

\((*)\)

which is (1) cartesian, and (2) preserving sections indexed by \( T \).

An object of \( \mathcal{M}_{g,t}(l) \) is a triple

\[
    (C \to S, P_f(f \in T), \phi: Pic^0(C/S)_l \simeq V),
\]

where \( C \to S \) and \( P_f \) are as before and \( \phi \) is an isomorphism of local system of \( \mathbb{F}_l \) vector spaces compatible with the symplectic structure. The morphism in \( \mathcal{M}_{g,t}(l) \)
is defined in the same way. In this case, we impose that the cartesian product induces an isomorphism of local system $\text{Pic}^0_l$ compatible with the third data $\phi$. It is known that $\mathcal{M}_{g,t}(l)$ is connected and representable if $l \geq 3$. Let $(r, \text{val})$ be a realized pair. An object of $\mathcal{M}_{g,t}(r, \text{val})$ is a triple $(\mathcal{C} \to S, P_f, \psi : \mu_m \to \text{Aut}(\mathcal{C}/S))$ such that $\mathcal{C} \to S$ and $P_f$ are as before and the action of $\mu_m$ on $\mathcal{C}$ via $\psi$ preserve that set of sections $\{P_f\}_{f \in T}$. For an action $\psi$, we define valency $\text{val} : R \to \frac{1}{m} \mathbb{Z} \cap [0,1)$ as before. We impose that $r = \sum \text{val}(p)$, and the restriction of val to $T$ is equal to the given map $\text{val}$. A pair $(r, \text{val})$ is called the type of the action $\psi : \mu_m \to \text{Aut}(\mathcal{C}/S)$. A morphism in $\mathcal{M}_{g,t}(r, \text{val})$ is a cartesian product (*) compatible with the action of $\mu_m$. An object in $\mathcal{M}_{g,t}(r, \text{val}, l)$ is a quadraple $(\mathcal{C}, \{P_f\}_{f \in T}, \psi, \phi)$, where $\mathcal{C} \to S$ and $P_f$ ($f \in T$) are the same as before and $\psi : \mu_m \to \text{Aut}(\mathcal{C})$ and $\phi : \text{Pic}^0(\mathcal{C}/S)(l) \to V$, where the type of the action $\psi$ is $(r, \text{val})$ and $\phi$ is an isomorphisms of local system compatible with $\mu_m$-actions and the symplectic structures.

By forgetting structure, we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{M}_{g,t}(r, \text{val}, l) & \xrightarrow{c} & \mathcal{M}_{g,t}(l) \\
\downarrow a & & \downarrow b \\
\mathcal{M}_{g,t}(r, \text{val}) & \xrightarrow{d} & \mathcal{M}_{g,t}
\end{array}
$$

**Proposition 2.1.**

1. $\mathcal{M}_{g,t}$ and $\mathcal{M}_{g,t}(l)$ are smooth algebraic stack and $\mathcal{M}_{g,t}(l)$ is representable.
2. The morphism $a$ and $b$ are etale.
3. The morphism $c$ is an immersion. Especially, $\mathcal{M}_{g,t}(r, \text{val}, l)$ is representable.

**Proof.** For the statement (1) and the etality of the morphism $a$, see [DM]. To prove the etality of $b$, we consider an object over $S$; $S \to \mathcal{M}_{g,t}(r, \text{val})$. Let $(\mathcal{C} \to S, \{P_f\}_{f \in T}, \phi : \mu_m \to \text{Aut}(\mathcal{C}/S))$ be the corresponding object. Then $S \times \mathcal{M}_{g,t}(r, \text{val})$ is represented by

$$
\text{Isoms}_{S-gp, \text{symplectic}, \mu_m-\text{module}}(\text{Pic}^0(\mathcal{C}/S)_l, V),
$$

which is etale over $S$. This proves the etality of $b$. To prove the locally closedness of $c$, we take an object $(\mathcal{C} \to S, \{P_f\}, \psi : \text{Pic}^0(\mathcal{C}/S)_l \to V)$. Consider the natural map

$$
P_l : \text{Aut}_S(\mathcal{C}) \to \text{Aut}_S(\text{Pic}^0(\mathcal{C}/S)_l)
$$

They are finite and unramified over $S$. Therefore $P_l$ is also finite and unramified. Moreover it is injective by the choice of $l$. Therefore, $P_l$ is closed for sufficiently large $l$. Therefore the morphism

$$
\text{Hom}_S(\mu_m, \text{Aut}_S(\mathcal{C})) \to \text{Hom}_S(\mu_m, \text{Aut}_S(\text{Pic}^0(\mathcal{C}/S)_l))
$$

$$
\to \text{Hom}_S(\mu_m, \text{Aut}_S(V))
$$

is also closed. Let $\text{nat}$ be the element of $\text{Hom}_S(\mu_m, \text{Aut}_S(V))$ corresponding to the action of $\mu_m$ to $V$. Then $Z = (\text{Ad}(\phi) \circ P_l)^{-1}(\text{nat})$ is a closed subvariety of $S$. It is easy to see that $\mathcal{M}_{g,t}(r, \text{val}, l) \times_S Z$ is a subfunctor of $Z$. Consider the
corresponding $\mu_m$-action on $C | Z \to Z$. Since the type is a constructible function on $Z$, the strata $Z(r, \text{val})$ corresponding to the type $(r, \text{val})$ is a locally closed subvariety of $Z$. This proves the statement (3). The stack $\mathcal{M}_{g,t}(r, \text{val}, l) \times_{\mathcal{M}_{g,t}} S$ is represented by $Z(r, \text{val})$. As a consequence, $\mathcal{M}_{g,t}(r, \text{val}, l)$ is an algebraic stack.

Let $\operatorname{USp}(V)$ be the group

$$\{\phi \in \operatorname{Aut}(V) \mid (\phi v, \phi w) = (v, w) \text{ for all } v, w \in V, \quad \phi(gv) = g\phi(v) \text{ for all } g \in \mu_m, v \in V\}.$$ 

For an object $a = (C \to S, \{P_f\}, \psi : \mu_m \to \operatorname{Aut}(C), \phi : \operatorname{Pic}^0(C/S)_l \to V)$ in $\mathcal{M}_{g,t}(r, \text{val}, l)$, and an element $\sigma \in \operatorname{USp}(V)$, we can define an object $\sigma(a)$ in $\mathcal{M}_{g,t}(r, \text{val}, l)$ by

$$(C \to S, \{P_f\}, \psi : \mu_m \to \operatorname{Aut}(C), \sigma \circ \phi : \operatorname{Pic}^0(C/S)_l \to V)$$

This action gives rise to an action of $\operatorname{USp}(V)$ on $\mathcal{M}_{g,t}(r, \text{val}, l)$.

**Proposition 2.2.**

1. The scheme $\mathcal{M}_{g,t}(r, \text{val}, l)$ is smooth.
2. The group $\operatorname{USp}(V)$ acts transitively to the set of connected components of $\mathcal{M}_{g,t}(r, \text{val}, l)$.

**Proof.** Let $p$ be a point of $\mathcal{M}_{g,t}(r, \text{val}, l)$ and $U$ be a sufficiently small neighbourhood of $p$. Let $C \to U$, $\phi : \mu_m \to \operatorname{Aut}(C/U)$, and $\psi : \operatorname{Pic}^0(C/U)_l \to V$ be the corresponding curve $C$, automorphism of $C$, and isomorphism of etale sheaves on $U$. By taking quotient $D = C/\mu_m$, we get a genus $\tilde{g}$ curve with a level structure $\psi_{\mu_m} : \operatorname{Pic}^0(D/U)_l \simeq \operatorname{Pic}^0(C/U)_{\mu_m} \simeq V_{\mu_m}$. This gives a morphism $U \xrightarrow{\tilde{g}} \mathcal{M}_{g,1}(l)$. By taking sufficiently small $U$, we may assume that the image $\alpha(U)$ is contained in a sufficiently small neighbourhood $W$ of $\alpha(p)$ in $\mathcal{M}_{g,1}(l)$. Let $D_W \to W$ be the corresponding curve over $W$. Then we have $D \simeq D_W \times_W U$. Let $\operatorname{Sym}(D_W/W) = \prod \operatorname{Sym}^{u_{\alpha}}(D_W/W) \times_W D_W^l$, where $r = \sum_{\alpha \in \mathbb{Q}} u_{\alpha}[\alpha]$ and $t = \#(T/\mu_m)$. Then the branch locus of $D$ gives a natural morphism $\beta : U \to \operatorname{Sym}(D_W/W)$ from $U$ to $\operatorname{Sym}(D_W/W)$. More precisely, let $L_U = (\pi_s \mathcal{O}_C)(\omega)$, where $\omega : \mu_m \to C^\times$ is the natural inclusion. Then $L_U^{\otimes m} = \mathcal{O}_D(-R)$, where

$$R = \sum_{\alpha} (m\alpha)b_{\alpha} + \sum_{\{\overline{f} \in T/\mu_m\}} m \text{val}(\overline{f}) \cdot p_{\overline{f}},$$

where $b_{\alpha} : U \to \operatorname{Sym}^{u_{\alpha}}(D_W/W)$ and $p_{\overline{f}} : U \to D_W$. Let $\tilde{D}$ be a sufficiently small neighbourhood of $\beta(p)$ containing the image $\beta(U)$ of $\beta$ by changing $U$ if necessary. Let $D_{\tilde{W}} \to \tilde{W}$ be the fiber product $D_W \times_W \tilde{W}$ and $R$ be the divisor on $D_W$ corresponding to the map $\tilde{W} \to \operatorname{Sym}(D_{\tilde{W}}/W)$. Since $\tilde{W}$ is sufficiently small, we can find a line bundle $L_{\tilde{W}}$ on $D_{\tilde{W}}$ such that

1. $p^*L_{\tilde{W}} = ((\pi_p)_* \mathcal{O}_{C_p})(\chi)$, where $p : \{p\} \to U \to \tilde{W}$ and $\pi_p : C_p \to C_p/\mu_m \simeq D_p$.
2. $L_{U}^{\otimes m} \simeq \mathcal{O}(-R)$ on $D_W$. 

...
Let $\mathcal{C}_W = \text{Spec}(O_W \oplus \mathcal{L}_W \oplus \cdots \oplus \mathcal{L}_W^{\otimes(m-1)})$ be the scheme whose algebra structure is defined by $\mathcal{L}_W^{\otimes m} \simeq O(-R)$. One can define $\mu_m$-action on $\mathcal{C}_W$ by $\phi(\sigma)^* |_{\mathcal{C}_W} = \omega(\sigma)$. If we take $\tilde{W}$ to be simply connected, we can find an isomorphism of etale sheaves $\phi: \text{Pic}^0(\mathcal{C}_W) \cong V$ and an isomorphism $p^*\mathcal{C}_W \cong \mathcal{C}_p$ compatible with $\phi$. This defines a morphism $\tilde{W} \rightarrow \mathcal{M}_{g,t}(r, \text{val}, l)$ such that $\gamma \circ \beta: U \rightarrow \tilde{W} \rightarrow \mathcal{M}_{g,t}(r, \text{val}, l)$ is the natural inclusion. By changing $\tilde{W}$ sufficiently small, we may assume $\beta^{-1}(\tilde{W}) = U$. In this situation, $\beta$ is an isomorphism and $\tilde{W}$ is smooth. This proves the smoothness of $\mathcal{M}_{g,t}(r, \text{val}, l)$.

(2) Let $P_1, P_2$ be points in $\mathcal{M}_{g,t}(r, \text{val}, l)$. We prove that there exists an element $\sigma$ such that $P_1$ and $\sigma(P_2)$ are connected by a path in $\mathcal{M}_{g,t}(r, \text{val}, l)$. Let $(C_i, \{P_{f_i}\}, \phi_i, \psi_i)$ be the quadruple corresponding to the point $P_i$ ($i = 1, 2$). Let $D_i = C_i/\mu_m$ and $E_i$ be the maximal unramified covering of $D_i$. Let $\mu_m' \subset \mu_m$ be the subgroup of $\mu_m$ corresponding to $E_i$. Let $Q_i$ be the point of $\mathcal{M}_{\tilde{g}}(l)$ corresponding to $D_i$ and the isomorphism $\text{Pic}^0(D_i)_l \simeq \text{Pic}^0(C_i)^{\mu_m} \simeq V^{\mu_m} = (F^2\tilde{g})$. Since $\mathcal{M}_{\tilde{g}}(l)$ is connected there exists a path $\gamma_1$ connecting $Q_1$ and $Q_2$. Taking a lift $\tilde{\gamma}_1$ of $\gamma_1$ in $\mathcal{M}_{g,t}(r, \text{val}, l)$, we may assume $Q_1 = Q_2$. The pair $(E_i, \mu_m/\mu_m' \rightarrow \text{Aut}(E_i))$ corresponds to a surjective homomorphism from $\pi_1(D_i)$ to $\mu_m/\mu_m'$. Since $T_{\tilde{g}} \rightarrow S_p(2\tilde{g}, F_l) \times S_p(2\tilde{g}, m\mathbf{Z}/m'\mathbf{Z})$ is surjective, we can choose a path $\gamma_2$ with the base point $Q_1 = Q_2$ such that the pair $(E_1, \mu_m/\mu_m' \rightarrow \text{Aut}(E_1))$ is analytically continued to the pair $(E_2, \mu_m/\mu_m' \rightarrow \text{Aut}(E_2))$ along the path $\gamma_2$. By taking a lift $\tilde{\gamma}_2$ of a path $\gamma_2$ in $\mathcal{M}_{g,t}(r, \text{val}, l)$ again, we may assume the pair $(E_1, \mu_m/\mu_m' \rightarrow \text{Aut}(E_1))$ is isomorphic to $(E_2, \mu_m/\mu_m' \rightarrow \text{Aut}(E_2))$. The curve $E_1$ and $D_1$ are denoted by $E$ and $D$ respectively. Let $f_{i,1}, f_{i,2}$ and $f_{i,3}$ be the natural morphisms

$$f_{i,1}: C_i \rightarrow E, f_{i,2}: E \rightarrow D, f_{i,3}: C_i \rightarrow D.$$  

Let $\omega$ be the natural homomorphism $\mu_m \rightarrow \mathbf{C}^\times$. Then the character $\omega' = \omega^{m'}$ is considered as a character of $\mu_m/\mu_m' = \text{Gal}(E/D)$. Let $\mathcal{L}_i = f_{i,3*}\mathcal{O}_{C_i}(\omega)$ and $\mathcal{A} = f_{2*}\mathcal{O}_E(\omega')$. Then we have the natural homomorphism $\mathcal{L}_i^{\otimes m'} \rightarrow f_{i,3*}\mathcal{O}_{C_i}(\omega') \rightarrow \mathcal{A}$, and this composite morphism is denoted by $\theta_i$. Since $f_2$ is unramified, $\mathcal{A}^{\otimes(m/m')} \simeq \mathcal{O}_D$. Therefore we have a morphism $\theta_i^{\otimes(m/m')} : \mathcal{L}_i^{\otimes m} \rightarrow \mathcal{O}_D$. This morphism defines an effective divisor $\tilde{R}_i$. As in the proof of (1), $\tilde{R}_i$ can be written as

$$\tilde{R}_i = \sum_{\alpha}(m\alpha)b_\alpha + \sum_{f \in T/\mu_m} m\text{val}(\hat{f}) \cdot p_f,$$

where $b_\alpha \in \text{Sym}^{m\alpha}(D)$ and $p_f \in D$. The largest common divisor of $m\alpha \ (u_\alpha \neq 0)$ and $m\text{val}(\hat{f}) \ (\hat{f} \in T/\mu_m)$ is equal to $m/m'$. Therefore $\tilde{R}_i$ is divisible by $(m/m')$ and $(m'/m)\tilde{R}_i$ is denoted by $R_i$. This is equal to the effective divisor defined by $\theta$ and we have $\mathcal{A} = \mathcal{L}_i^{\otimes m'} \otimes \mathcal{O}_D(R_i)$. For an element $\xi = (b_\alpha, p_f)$, we define $R(\xi)$ by

$$R(\xi) = \sum_{\alpha}(m'\alpha)b_\alpha + \sum_{f \in T/\mu_m} m'\text{val}(\hat{f}) \cdot p_f.$$  

Let $r = \sum_{\alpha} m'\alpha u_\alpha + \sum_{f \in T/\mu_m} m'\text{val}(\hat{f})$. Let $F_A$ be the fiber product $\text{Sym}(D) \times_{\text{Pic}^{m'\tau}(D)} \text{Pic}'(D)$ for morphisms $\text{Sym}(D) \ni \xi \mapsto \mathcal{O}_D(-R(\xi)) \in \text{Pic}^{m'\tau}(D)$ and $\text{Pic}'(D) \ni \mathcal{L} \mapsto \mathcal{L}^{\otimes m'} \otimes A^{-1} \in \text{Pic}^{m'\tau}(D)$. Then the pair $(R_i, \mathcal{L}_i)$ defines a point of $F_A$. 
Proposition 2.3. The fiber product $F_A$ is connected.

Proof. Since $Pic^r(D)$ is an etale covering of $Pic^{m'r}(D)$ associated to $m' H_1(Pic^{m'r}(D), \mathbb{Z})$, it is enough to show that the composite

\[
\pi_1(Sym(D)) \to \pi_1(Pic^{m'r}(D)) \to \pi_1(Pic^{m'r}(D)) \otimes \mathbb{Z}/m'\mathbb{Z}
\]

is surjective. Consider the natural map $D^s \to Sym(D)$, where $s = \sum \alpha u_\alpha + \#(T/\mu_m)$, and write the composite $D^s \to Pic^{m'r}(D)$ as $(d_i)_{i=1,\ldots,s} \mapsto \mathcal{O}_D(-\sum_i \mu_i d_i)$, where $\mu_i$ is the multiplicity for the $i$-th component. To get the surjectivity of (*), it is enough to prove the surjectivity of

\[
H_1(D, \mathbb{Z})^G \to H_1(Sym(D), \mathbb{Z}) \to H_1(Pic^{m'r}(D), \mathbb{Z}) \otimes \mathbb{Z}/m'\mathbb{Z}.
\]

Since the map $H_1(D, \mathbb{Z}) \to H_1(Pic^m(D), \mathbb{Z})$ induced by the map $p \mapsto \mathcal{O}(\mu_i p)$ is given by $\mu_i$-multiplication, and $gcd(\mu_i, m') = 1$, we get the required surjectivity.

§3 $\mu_m$-action for a local moduli space.

Now we return to the stable curve $C = \cup_{v \in V(\tau)} C_v$. Consider a $\mu_m$-action on $C$ whose type is $(r_v, val_v)_v$. Let $G = \mu_m$ and $G(v)$ be the stabilizer of $v$ in $G$. Let $X \to \hat{M}_g$ be an etale covering of $M_g$ with a representable stack $X$ and $\tilde{p} : Spec(C) \to \hat{M}_g$ be the point of $\hat{M}_g$ corresponding to $C$. Then there exists a point $p \to X$ such that $\tilde{p} = \pi \circ p$. Let $U$ be a neighbourhood of $p \in X$ and $C \to U$ be the corresponding curve on $U$. By [DM], the inverse image of $\hat{M}_g - M_g$ under the map $U \to \hat{M}_g$ is a normal crossing divisor $D$ and by taking sufficiently small $U$, we may assume $D = \cup_{e \in E(\tau)} D_e$. Let $W$ be a closed analytic set defined by $\cap_{e \in E(\tau)} D_e$. It is easy to see that for a point $w \in W$, the dual graph of the fiber $f^{-1}(w)$ at $w$ is isomorphic to $\tau$. Moreover, by taking sufficiently small $U$, the inverse image $C_W$ of $W$ under the map $f$ has an irreducible decomposition $C_W = \cup_{v \in V(\tau)} C_{W,v}$. Let $f_{W,v} : \tilde{C}_{W,v} \to W$ be the normalization of $C_{W,v}$ and $\tilde{C}_{W,v} = f_{W,v}^{-1}(q)$ for $q \in W$.

Let $\tilde{V}(\tau) \subset V(\tau)$ be a representative of $\mu_m$-orbit of the set of vertices $V(\tau)$. We consider a $\mathbb{F}_1$-vector space $V_v$ for each $v \in V$ as in the last section. For $v \in \tilde{V}(\tau)$, we choose a marking $\psi_{p,v} : Pic^0(\tilde{C}_{p,v})_l \simeq V_v$ compatible with the action of $G(v)$. Then the quadruple $(\tilde{C}_{p,v}, P(f \in T(v)), \phi_v : G(v) \to Aut(\tilde{C}_{p,v}), \psi_{v,p} : Pic^0(\tilde{C}_{p,v})_l \simeq V_v)$ defines a point $p_v \in M_v(r_v, val_v, l) \subset M_v(l)$, where $M_v(r_v, val_v, l) = M_{g(v), t(v)}(r_v, val_v, l)$ and $M_v(l) = M_{g(v), t(v)}(l)$. By using isomorphism $\sigma_{i^{-1}} (i = 1, \ldots, \#(G/G(v)))$, we define a point $\sigma(p_v) \in M_{\sigma_{i^{-1}}(v)}(r_v, val_v, l) \subset M_{\sigma_{i^{-1}}(v)}(l)$ and as a consequence, we define a point

\[
p' \in \prod_{v \in \tilde{V}(\tau)/G} \prod_{i=1}^{\#(G/G(v))} M_{\sigma_{i^{-1}}(v)}(r_v, val_v, l) \subset \prod_{v \in \tilde{V}(\tau)} M_v(l).
\]

By choosing a family of marking $\psi_v : Pic^0(\tilde{C}_{W,v})_l \simeq V_v$ extending $\psi_{v,p}$, we get an etale morphism $W \to \prod_v M_v(l)$. For a sufficiently small $U$, this map is an open immersion and $p$ is identified with $p'$. The fiber product $W \times \prod M_v(l) \prod_{v \in \tilde{V}(\tau)} M_v(r_v, val_v, l)$ is denoted by $Z$. We can define the action of $G$ on $\prod M_v(l)$ as follows. An element $g \in G(v)$ acts on the space $M_v(l)$ by $(\tilde{g}, \psi_{v,p}) \mapsto (\tilde{g} \circ \psi_{v,p})$.
Therefore on the product, $\mathcal{M}_v \times G$ the group $G(v)$ acts diagonally. The quotient $(\mathcal{M}_v \times G)/G(v)$ is denoted by $\text{Ind}_{G(v)}^G \mathcal{M}_v(l)$ and via the action on the second factor, the group $G$ acts on $\text{Ind}_{G(v)}^G \mathcal{M}_v(l)$. Using natural isomorphism $\prod_{v \in V(\tau)} \mathcal{M}_v(l) \simeq \prod_{v \in V(\tau)/G} \text{Ind}_{G(v)}^G \mathcal{M}_v(l)$, we get the action of $G$ on $\prod_v \mathcal{M}_v(l)$. The restriction of $f$ to $C_Z = f^{-1}(Z)$ is denoted by $f_Z$.

\[
C \subset C_Z \subset C_W \subset C
\]

\[
p \in Z \subset W \subset U
\]

Since $C \to U$ is a versal deformation of $C$, and automorphism of stable curves are discrete, for an automorphism $g \in \text{Aut}(C)$, there exists a sufficiently small neighbourhood $U_1$ of $p$ and a map $g_* : U_1 \to U$ such that the automorphism $g$ is induced by the pull of $C \to U$ by $g_*$. Since $\text{Aut}(C)$ is a finite group, we may assume that $U_1$ is stable under the map $g_*$, $g \in \text{Aut}(C)$. This defines an action of $\text{Aut}(C)$ on $U$ and by restricting this representation, we get an action of $G$ on $U$. It is easy to see that the subspace $W$ is stable under the action of $G$. Moreover, the restriction of this action to $W$ coincides with the action on $\prod_v \mathcal{M}_v(l)$ given before. Therefore, the subspace $Z$ is fixed part of $W$ under the action of $G = \langle \sigma \rangle$.

We introduce a coordinate of $U$ as follows. Since $\sigma(D_e) = D_{\sigma(e)}$, and $\cup_{e \in E(\tau)} D_e$ is normal crossing, we can take local equations $t_e$ of $D_e$ such that $t_{\sigma(e)}$ is a constant multiple of $\sigma^*(t_e)$. By this choice of $\{t_e\}_{e \in E(\tau)}$,

\[
G(e) \ni \sigma \mapsto \frac{\sigma^*(t_e)}{t_e} \in \mathbb{C}^\times
\]

defines a character $\chi_e$ of $G(e)$. Since the subspace $W$ is smooth, we can take a coordinate $\{t_i\}_{i=1, \ldots, \dim W}$ such that $\sigma^*(t_i) = \chi_i(\sigma)t_i$, $(\sigma \in G)$ for some character $\chi_i$ of $G$. As a whole, $\{t_i\}_{i=1, \ldots, \dim W} \cup \{t_e\}_{e \in E(\tau)}$ forms a local coordinate for $U$. The character $\chi_e$ is computed from $(r_e,\val_v)_v$ as follows.

Let $e^{(1)}$ and $e^{(2)}$ be the two extremes of the edge $e$ and $u_1$ and $u_2$ be the local coordinate of $\tilde{C}$ corresponding to $e^{(1)}$ and $e^{(2)}$. We may assume that for any $\sigma \in G(e^{(i)})$, $\sigma^*(u_i)$ is a constant multiple of $u_i$. For a small neighbourhood of $p_e \in C$, the local equation is written as $u_1u_2 = t_e$. The group $G(e^{(1)}) = G(e^{(2)})$ is denoted by $G(f)$.

(1) Amphidrome case. Suppose that there exists $\sigma \in G(e)$ such that $\sigma(e^{(1)}) = e^{(2)}$. In this case, $\#G(e) : G(f) = 2$ and for $\sigma \in G(f)$, $\chi_f(\sigma) = \sigma^*(u_1)/u_1 = \sigma^*(u_2)/u_2$ defines a character of $G(f)$. For $\sigma \in G(e)$, we have

\[
\chi_e(\sigma) = \frac{\sigma^*(t_e)}{t_e} = \frac{\sigma^*(u_1u_2)}{u_1u_2} = \rho_f(\sigma^2)
\]

(2) Non-amphidrome case. If $G(e) = G(f)$, $\chi_{e^{(1)}}(\sigma) = \sigma^*(u_1)/u_1$ and $\chi_{e^{(2)}}(\sigma) = \sigma^*(u_2)/u_2$ defines a character of $G(e)$. By the same argument, we have $\chi_e = \rho_{e^{(1)}}\rho_{e^{(2)}}$.

\section{Period of degeneration and monodromy}

In this section, we study the period map for the stabilization of a degeneration of curves. Let $p : \mathcal{D} \to \Delta$ be a proper morphism of dimension one whose restriction...
to $p^{-1}(\Delta - \{0\})$ is smooth. If the fiber $p^{-1}(0)$ at 0 contains no rational curves of the first kind, it is called a minimal degeneration of genus $g$ curves. Then by the stable reduction theorem [DM], there exists a covering $\pi_m: \Delta_m \to \Delta; t \mapsto t^m = \tau$ of degree $m$ such that the fiber product $\mathcal{D}_m = \mathcal{D} \times_{\Delta} \Delta_m$ has stable reduction. Let $\mathcal{C} \to \Delta_m$ be the stable model of $\mathcal{D}_m$. The special fiber $C$ of $\mathcal{C}$ defines a point $p$ of $\mathcal{M}_g$. Let $U$ be a neighborhood of $p$ which is representable and sufficiently small. The corresponding curve on $U$ is denoted by $\mathcal{C}_U \to U$. By the functoriality of stable model, the action of $\text{Gal}(\Delta_m/\Delta)$ acts on $\mathcal{C}$ which is compatible with the natural action on $\Delta_m$. Restricting this action to the special fiber, we get an action of $\text{Gal}(\Delta_m/\Delta)$ on $C$. By the versality of $\mathcal{C}_U \to U$, the group $\text{Gal}(\Delta_m/\Delta)$ acts on $U$ by taking sufficiently small $U$ as in Section 3. We get the following cartesian diagram compatible with the action of $\text{Gal}(\Delta_m/\Delta)$ by changing $U$ by sufficiently small neighborhood of $p$.

$$
\begin{array}{ccc}
\mathcal{C} & \longrightarrow & \mathcal{C}_U \\
p_m \downarrow & & \downarrow p_U \\
\Delta_m & \phi & \longrightarrow & U
\end{array}
$$

We choose a coordinate $(t_i, t_e)$ introduced in Section 3. Using this coordinate the morphism $\Phi$ can be written as $\Phi(t) = (f_i(t), f_e(t))$, where $f_i$ and $f_e$ are holomorphic function on $\Delta_m$. Since the morphism $\Phi$ is equivariant under the action of $G = \text{Gal}(\Delta_m/\Delta)$, we have the following functional equation for $f_i$ and $f_e$.

$$
f_i(\sigma t) = x_i(\sigma) f(t), f_e(\sigma t) = e_{\sigma, e} f_{\sigma e}(t),
$$

where $e_{\sigma, e} = \sigma^* t_e / t_{\sigma e} \in \mathbb{C}^\times$. As a consequence, there exist germs of holomorphic functions $\tilde{f}_i$ and $\tilde{f}_e$ such that

$$
f_i(t) = t^{e_i} \tilde{f}_i(t^m), f_e(t) = t^{m_e} \tilde{f}_e(t^{m_e}),
$$

where $0 \leq b_e < 1$ and $\rho_f(e(\frac{1}{m_f})) = e(b_e)$ if $e$ is amphidrome and $\rho_{f(1)}(e(\frac{1}{m_{\sigma(1)}})) \rho_{f(2)}(e(\frac{1}{m_{\sigma(2)}})) = e(b_e)$ if $e$ is non-amphidrome. Now we consider the minimal resolution of $\mathcal{C}/G$. Note that the singularity of $C$ is contained in singular locus of the special fiber $C$. Let $x$ be a point in $C$ fixed by some non-trivial element of $G$, i.e. $G(x) = \text{Stab}_x(G) \neq 1$.

(1) The case where $x$ is contained in the smooth part of $C$. In this case, $G(x)$ acts on the tangent space of $C$ at $x$ via the local representation $\rho_x$. Therefore the action of $G(x)$ on the tangent space of $C$ is equivalent to the direct sum $\rho_x \oplus \text{nat}$ of $\rho_x$ and the natural representation nat.

Therefore the resolution process depends only on $\text{val}(x)$ and does not depend on the map $\Phi$.

(2) The case where $x$ is contained in the singular locus of $C$. The local equation of $\mathcal{C}$ at $x$ is given by

$$
u_1 u_2 = t^{m_e} \tilde{f}_e(t^{m_e})
$$

in the space $(u_1, u_2, t)$. The action of $G(x) = G(e)$ is given as follows.

(2-1) Amphidrome case. Let $\sigma = e(\frac{1}{m_{\sigma}})$ be the generator of $G(e)$ (see §3). Then by changing coordinate, the action of $\sigma$ is given as

$$
u_1 \mapsto \nu_2, \nu_2 \mapsto \chi_e(e(\frac{1}{m_{\sigma}})) \nu_1, t \mapsto e(\frac{1}{m_{\sigma}}) t.
$$
Therefore the resolution process depends only on \( \chi_e \) and the order of \( f_e \) with respect to the parameter \( \tau_e = t^{m_e} \).

(2-1) Non-amphidrome case. Using the same notation as in (2-1), the action of \( G_x \) is given as

\[
u_1 \to \chi_{e(1)}(\sigma)\nu_1, \quad \nu_2 \to \chi_{e(2)}(\sigma)\nu_2, \quad \nu = e(\frac{1}{m_e})\mu.
\]

In this case, the resolution process also depends only on \( \chi_{e(1)}, \chi_{e(2)} \) and the order of \( \tilde{f}_e \) with respect to the parameter \( \tau = t^{m_e} \).

**Proposition 4.1.** Let \( \Phi_j : \Delta_m \to U \) (\( j = 1, 2 \)) be a holomorphic map which satisfies \( \Phi_j(\sigma t) = \sigma^*(\Phi_j(t)) \) and write \( \Phi_e = (f_i^{(j)}, f_e^{(j)}) \) be using the coordinate \( (t_i, t_e) \). Suppose that \( \text{ord}_t f_i^{(1)}(t) = \text{ord}_t f_i^{(2)}(t) \) for all \( t \in E(\tau) \). Then there exists a smooth family \( h : D \to \Delta \) and \( \pm \epsilon \in \Delta \) such that \( D_1 = h^{-1}(-\epsilon) \) and \( D_2 = h^{-1}(\epsilon) \) are the minimal resolution of \( \Phi_1^*(C_U)/G \) and \( \Phi_2^*(C_U)/G \) respectively.

**Proof.** By the assumption, \( f_i^{(j)}(t) \) and \( f_e^{(j)}(t) \) can be written as

\[
f_i^{(j)}(t) = t^{e_i}f_i^{(j)}(t^{m_i}), \quad f_e^{(j)}(t) = t^{m_e}f_e^{(j)}(t^{m_e}).
\]

By the assumption we can take two variable function \( F_i(\tau, u) \) and \( F_e(\tau, u) \) on \( \Delta \times \Delta \) such that

\[
F_i(t, -\epsilon) = f_i^{(1)}(t), F_i(t, \epsilon) = f_i^{(2)}(t), \quad F_e(t, -\epsilon) = f_e^{(1)}(t), F_e(t, \epsilon) = f_e^{(2)}(t),
\]

and the order of \( F_e(\tau, u) \) with respect to \( \tau \) is constant for all \( u \in U \) and if \( e \) and \( e' \) are in the same orbit under the action of \( \mu_m, F_e \) is a constant multiple of \( F_{e'} \). By pulling back by the morphism

\[
\Delta_m \times \Delta \ni (t, u) \mapsto (t^{e_i}F_i(t^{m_i}, u), t^{m_e}F_e(t^{m_e}, u)) \in U,
\]

we get a family of curves on \( \Delta \times \Delta \) by taking quotient and take a resolution, we get the required smooth family \( \tilde{D} \to \Delta \).

§5 Mapping class group and the main theorem

First we recall several definitions of mapping class group. Let \( D \to \Delta \) be a minimal degeneration of genus \( g \) curves. Let \( 0 < \epsilon < \delta < 1 \). By choosing a metric on \( \sigma \Delta \to \Delta \), and a path around 0 with the base point \( \delta \), we obtain a \( C^\infty \) automorphism of \( \delta \) and it defines an element \( \Gamma(p) \in MC(p^{-1}(\delta)) \) of the mapping class group of \( p^{-1}(\delta) \). Let \( S_g \) be an oriented closed \( C^\infty \) surface. By taking a diffeomorphism \( D : p^{-1}(\delta) \to S_g \) from \( p^{-1}(\delta) \) to \( S_g \), we get an element \( \Gamma_D(p) = D \circ \Gamma(p) \circ D^{-1} \) of the mapping class group \( MC_g = MC(S_g) \) of \( S_g \). The conjugacy class of \( \Gamma_D(p) \) does not depend on the choice of \( \epsilon, \delta \) and \( D \) and it is denoted by \( \Gamma(p) \).

We review several results of [MM] from the analytic point of view. Let \( D \to \Delta \) be a minimal degeneration of genus \( g \) curves. Let \( m \) be the minimal degree of \( \Delta \to \Delta \) for which \( D \times \Delta \) has stable reduction. By stable reduction theorem, it
is equal to the minimal $m_i$ such that $\det(1 - x\Gamma(p)^m | H_1(p^{-1}(\delta), \mathbb{Z})) = (1 - x)^{2g}$. Therefore $m$ depends only on the conjugacy class $\Gamma(p)$ of $\Gamma'(p)$ in the mapping class group. Let $p_m : C \to \Delta_m$ be the stable model of $D \times_{\Delta} \Delta_m$ and $C$ be the special fiber $p_m^{-1}(0)$ of $C$. Then by the functoriality of the stable model the action of $G = \text{Gal}(\Delta_m/\Delta)$ extends to the action of $C$ and as a consequence, we have an action of $G$ on the closed fiber $C$ and the smooth part $C^0$ of $C$. It is easy to see that $C^0$ is homeomorphic to $p^{-1}(\delta) - \text{Cir}$, where $\text{Cir}$ is the minimal set of simple closed curve on $p^{-1}$ such that the restriction of $\Gamma(p)$ to $p^{-1}(\delta) - \text{Cir}$ is periodic. Moreover under this homeomorphism, the action of $G$ on $C^0$ is homotopically equivalent to that of $\Gamma(p)$ on $p^{-1}(\delta) - \text{Cir}$. Therefore the valency defined in §2 is equal to that given in [MM]. Let $U$ be the versal deformation of the special fiber $C$. We use the same notation $(t_i, t_e), (f_i(t), f_e(t))$, etc. as in §3. Let $\text{ord}_i(f_e(t))$ be the order of $f_e$ with respect to the parameter $t$. Then $\text{ord}_i(f_e(t))/m_e$ of $f_e$ coincides with the screw number defined in [MM] and depends only on the conjugacy class of the mapping class group $\Phi \in \text{Aut}(\pi_1(D_n))$ arising from the family $D \to \Delta$ of curves.

Now we can prove the following main theorem.

**Theorem 5.1.** Let $p_i : D_i \to \Delta$ $(i = 1, 2)$ be degenerations of genus $g$ curves. If $\Gamma(p_1) = \Gamma(p_2)$, then there exists a sequence of proper flat morphisms $e_i : E_i \to \Delta \times D_i$ $(i = 1, \ldots, k)$ of dimension 1 such that

1. The composite $pr_2 \circ C_i : E_i \to D_i$ is smooth.
2. The restriction $e_i^{-1}(\Delta^0 \times D_i) \to \Delta^0 \times D_i$ is smooth.
3. There exists an isomorphism $(pr_2 \circ e_i)^{-1}(\varepsilon) \cong (pr_2 \circ e_{i+1})^{-1}(-\varepsilon)$ compatible with the projection to $\Delta$.
4. $(pr_2 \circ e_1)^{-1}(\varepsilon) \to \Delta$ and $(pr_2 \circ e_i)^{-1}(\varepsilon) \to \Delta$ are isomorphic to $p_1 : D_1 \to \Delta$ and $p_2 : D_2 \to \Delta$ respectively.

In other words, if two degenerations of curves are topologically isomorphic at each other, they are equivalent under analytic deformations.

**Proof.** By the assumption the minimal degree of $\Delta_m \to \Delta$ for which $D_1 \times_{\Delta} \Delta_m$ has stable reduction and that for $D_2 \times_{\Delta} \Delta_m$ coincides and we denote it $m$. Let $C_i$ be the stable model of $D_i \times_{\Delta} \Delta_m$ and $C_i$ be the special fiber of $C_i$. Then the stable graph $(\tau_i, g)$ associated to $C_i$ is isomorphic to each other. Moreover the type $(r_{i,v}, \text{val}_{i,v})$ of the ramification of $C_i$ for the action of $\mu_m$ is also equal and it is denoted by $(r_v, \text{val}_v)$. The point in $M_g$ defined by the special fiber $C_1$ and $C_2$ are denoted by $p_1$ and $p_2$ respectively. By Proposition 2.2, there exist liftings $\tilde{p}_i$ of $p_i$ which belong to the same connected component $K$ of $\coprod_{v \in V(\tau)/\mu_m} M_v(r_v, \text{val}_v, l) \subset \tilde{M}_g(l)$. Let $U_i$ $(i = 1, \ldots, k)$ be a sequence of open sets of $\tilde{M}_g(l)$ with $U_i \cap U_{i+1} \cap K \neq \emptyset$ for $i = 1, \ldots, k - 1$ and $\tilde{p}_1 \in U_1 \cap K$ and $\tilde{p}_2 \in U_k \cap K$. Since the screw number for $C_1 \to \Delta_m$ and $C_2 \to \Delta_m$ coincides for all $e \in E(\tau)$, we get the required sequence of morphism $e_i : E_i \to \Delta \times D_i$ by Proposition 4.1.

**References**

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