ORNSTEIN–UHLENBECK SEMIGROUPS ON STAR GRAPHS

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Dedicated to the memory of Rosa Maria Mininni

Abstract. We prove first existence of a classical solution to a class of parabolic problems with unbounded coefficients on metric star graphs subject to Kirchhoff-type conditions. The result is applied to the Ornstein–Uhlenbeck and the harmonic oscillator operators on metric star graphs. We give an explicit formula for the associated Ornstein–Uhlenbeck semigroup and give the unique associated invariant measure. We show that this semigroup inherit the regularity properties of the classical Ornstein–Uhlenbeck semigroup on $\mathbb{R}$.

1. Introduction

The aim of this note is to present some preliminary results in the study of elliptic operators with unbounded coefficients on non-compact metric graphs. For the sake of simplicity, we here restrict first and foremost to the case of graphs with the simplest possible topology, i.e., metric star graphs $S_m$ consisting of $m < \infty$ halflines; and to the best understood class of operators with unbounded coefficients, viz the Ornstein–Uhlenbeck operators: these are, on $\mathbb{R}$, the operators associated with the Ornstein–Uhlenbeck stochastic process, i.e., they are defined by

\begin{equation}
Af(x) = \frac{1}{2}f''(x) - xf'(x), \quad x \in \mathbb{R}.
\end{equation}

The theory of second-order differential operators on compact metric graphs $\mathcal{G}$ is classical and goes back to Lumer [11, 12] and Pavlov–Faddeev [21]. Shortly afterwards, Roth [22] presented an explicit formula for the heat kernel – the integral kernel of the semigroup generated by the plain Laplacian with natural (i.e., continuity and Kirchhoff-type) boundary conditions; this was later extended to more general vertex conditions [19], to (possibly) infinite equilateral graphs [4], and recently to (possibly) infinite graphs of bounded geometry [2], to which the case of $S_m$ can be reduced by elementary arguments. It is known that, just like its counterpart on $\mathbb{R}$, this semigroup can be associated with the Brownian motion on $\mathcal{G}$ [7]. Qualitative properties of
differential operators of order two [6] and three [18] on $S_m$ have been recently studied, too; metric graphs including semi-infinite intervals appear in the study of linear scattering theory ([3, Section 5.4] and references therein) and nonlinear Schrödinger equations, ever since [20]. In virtually all of these investigations, the relevant functional setting is the Hilbert space $L^2(\mathcal{G})$ with respect to the measure on $\mathcal{G}$ canonically induced by the Lebesgue measure on each halfline $\mathbb{R}_+$. As usual in the theory of operators with unbounded coefficients, we will instead introduce an appropriate measure adapted to our setting; this will turn out to be the invariant measure for the Ornstein–Uhlenbeck semigroup.

Let us now describe our main results and the structure of the paper. In Section 2 we will recall the basic notions in the theory of metric graphs and introduce the relevant function spaces. In Section 3 we will introduce the class of operators we are going to study and prove that they drive well-posed evolution equations on $S_m$. In a certain sense, our approach here is similar to that of [2]: our proofs are based on a kind of transference principle, as we extend the properties of the semigroup’s integral kernel on $\mathbb{R}$ to explicitly define the integral kernel of the semigroup on $S_m$, thus proving existence of a classical solution for a certain class of initial data. In this article, this is done by making good use of the symmetries of the metric star graph and invariance properties of the semigroup on $\mathbb{R}$. This strategy can probably be pursued in greater generality, as long as the existence of an integral kernel for the relevant semigroup is known.

In Section 4 we turn to the issue of studying the operator theoretical properties of the semigroup associated with this integral kernel. To this purpose, we focus on an especially interesting special case and study its realizations in the space of bounded continuous functions as well as on Lebesgue spaces, either with respect to the Lebesgue measure or to a suitable alternative measure – the invariant measure associated with the Ornstein–Uhlenbeck process on $S_m$. In this way, we are especially able to prove the existence of a consistent family of analytic, positive, compact Ornstein–Uhlenbeck semigroups on $L^p_{\mu}$-spaces; we can also determine their spectra.

Finally, as an application of our results, in Section 5 we briefly discuss the behavior of the harmonic oscillator on a metric star graph; indeed, it is already well-known that on $\mathbb{R}$ the relevant Hamiltonian is similar to the Ornstein–Uhlenbeck operator on $L^2_{\mu}(\mathbb{R})$, and in particular they have equal spectrum. To our knowledge, the properties of this physical model on graphs have never been studied in the literature; although we wish to mention a well-known model of irreversible quantum graphs due to Smilansky and Solomyak that boils down to coupling a Laplacian on $\mathcal{G}$ with a harmonic oscillator on $\mathbb{R}$ [23, 24], thus defining an operator on $L^2(\mathcal{G}) \oplus L^2(\mathbb{R}) \simeq L^2(\mathcal{G} \times \mathbb{R})$.

2. General setting

Object of our investigations here is a metric star graph, $S_m$, with $m$ rays of semi-infinite length, $m \in \mathbb{N}$; i.e., $S_m$ is the quotient space

$$\bigsqcup_{i=1}^{m} [0, \infty) / \sim$$

that consists of $m$ disjoint half-lines $[0, \infty)$ whose origins are identified with one common zero point, 0.
Because $d\mu$ is absolutely continuous with respect to the Lebesgue measure $dx$, one sees that $d\mu$ is locally finite with respect to the Euclidean distance, too: we conclude that $\mathcal{G}$ is a metric-measure space with respect to the path metric and the direct sum measure induced by the measure $d\mu$. Clearly, also $\mathcal{S}_m^n$ are metric-measure spaces. In particular, this allows us to consider, without ambiguity, functions spaces based on topological and measure-theoretical notions: in particular, the spaces $C_b(\mathcal{S}_m)$ (resp., $BUC(\mathcal{S}_m)$) of bounded (resp., bounded uniformly continuous) functions on $\mathcal{S}_m$; and the Lebesgue spaces $L^p(\mathcal{S}_m)$ (resp., $L^p_\mu(\mathcal{S}_m)$) with respect to the Lebesgue measure (resp., to the measure $\mu$). Likewise, one defines the Sobolev space $W^{1,p}(\mathcal{S}_m)$ (resp., $W^{1,p}_\mu(\mathcal{S}_m)$) as the space of functions in $C(\mathcal{S}_m) \cap L^p(\mathcal{S}_m)$ (resp., $C(\mathcal{S}_m) \cap L^p_\mu(\mathcal{S}_m)$) that are weakly differentiable with a weak derivative in $L^p(\mathcal{S}_m)$ (resp., $L^p_\mu(\mathcal{S}_m)$). By definition of $\mathcal{S}_m$ as a disjoint union, any function $f : \mathcal{S}_m \to \mathbb{K}$ can be equivalently regarded as a family $(f_i)_{1 \leq i \leq m}$, where $f_i : \mathbb{R}_+ \to \mathbb{K}$.

In particular, if $f \in C(\mathcal{S}_m)$, then in agreement with the above convention we write $f(0) := \lim_{x \to 0} f_i(x), 1 \leq i \leq m$.

### 3. Operators with Unbounded Coefficients on Metric Star Graphs

We want to study first the Kolmogorov operator

$$Lf(x_i) = q(|x_i|)f''(x_i) + b(|x_i|)f'(x_i) + c(|x_i|)f(x_i), \quad |x_i| \geq 0, \quad i = 1, \ldots, m,$$

on $C_b(\mathcal{S}_m)$, where $q, b, c \in C^{\alpha}_{\text{loc}}((0, \infty))$ for some $\alpha \in (0, 1)$, $b(0) = 0$, $q(x) > 0$ for all $x \in [0, \infty)$ and $\sup c \leq c_0$ for some $c_0 \in \mathbb{R}$. We equip it with continuity along with Kirchhoff-type condition in zero by defining it on the domain

$$D(L) = \{ f \in C_b(\mathcal{S}_m) \cap \bigcap_{1 \leq p < \infty} \widehat{W}^{2,p}_{\text{loc}}(\mathcal{S}_m) : \sum_{i=1}^{m} f'(0_i) = 0 \text{ and } Lf \in C_b(\mathcal{S}_m) \},$$
where

\[(3.2) \quad \widetilde{W}_{loc}^{k,p}(S_m) := \bigoplus_{i=1}^{m} W_{loc}^{k,p}(\mathbb{R}^+), \quad k \in \mathbb{N}.\]

(Note that, unlike for \(W^{1,p}(S_m)\) defined in Section 2, we are not imposing continuity at 0 on the functions in \(\widetilde{W}_{loc}^{k,p}(S_m)\).) We associate with the operator \(L\) a further operator \(\tilde{L}\), acting on the function space \(C_b(\mathbb{R})\), defined by

\[\tilde{L}f(x) = \tilde{q}(x)f''(x) + \tilde{b}(x)f'(x) + \tilde{c}(x)f(x)\]

with domain

\[D(\tilde{L}) = \{f \in C_b(\mathbb{R}) \cap \bigcap_{1 \leq p < \infty} W_{loc}^{2,p}(\mathbb{R}) : \tilde{L}f \in C_b(\mathbb{R})\},\]

where

\[\tilde{q}(x) = q(x), \quad \tilde{b}(x) = b(x), \quad \tilde{c}(x) = c(x) \text{ if } x \geq 0 \quad \text{and}\]

\[(3.3) \quad \tilde{q}(x) = q(-x), \quad \tilde{b}(x) = -b(-x), \quad \tilde{c}(x) = c(-x) \text{ if } x \leq 0.\]

In this section we are mainly interested in the existence and uniqueness of solutions to the parabolic problem

\[(P_\Lambda) \quad \begin{cases} 
\partial_t u(t, \cdot) = \Lambda u(t, \cdot), & t > 0, \\
u(0, \cdot) = f(\cdot),
\end{cases}\]

where the subscript in \((P_\Lambda)\) always indicates which operator \(\Lambda\) is currently under consideration.

The following remark is crucial for our study.

**Remark 3.1.** We observe that every function \(f \in C_b(S_m)\) uniquely determines \(m\) functions \(\tilde{f}_i \in C_b(\mathbb{R})\) given by

\[(3.4) \quad \tilde{f}_i(x) := \begin{cases} 
f(x_i), & \text{if } x_i = x, \\
\frac{2}{m} \sum_{1 \leq j \leq m} f(-x_j) - f(-x_i), & \text{if } x_i = -x, \quad i = 1, \ldots, m.
\end{cases}\]

Classical solutions to \((P_L)\) for \(L\) as defined in [3.1], are defined as follows.

**Definition 3.2.** A function \(u \in C_b([0, \infty) \times S_m)\) is called classical solution of \((P_L)\) if \(u(\cdot, x) \in C^1((0, \infty))\) for every \(x \in S_m\), \(u(t, \cdot) \in D(L)\) for every \(t > 0\) and \(u\) satisfies \((P_L)\).

The main result of this section concerns existence of solution to the problem \((P_L)\).

**Theorem 3.3.** Assume that \(q, b, c\) are in \(C^\alpha_{loc}([0, \infty))\) for some \(\alpha \in (0, 1)\), \(q(x) > 0\) for all \(x \in [0, \infty)\), \(\sup c \leq c_0\) for some \(c_0 \in \mathbb{R}\) and \(b(0) = 0\). Then, for every function \(f \in C_b(S_m)\), there exists at least one classical solution of \((P_L)\).
Furthermore, if the solution to \((P_L)\) is unique then so is the solution of \((P_{\tilde{L}})\). In that case the semigroup \((T_m(t))_{t \geq 0}\) generated by \(\tilde{L}\) on \(C_b(S_m)\) is given by

\[
T_m(t)f(x_i) = \int_{(\mathbb{R}^+,i)} (k(t, |x_i|, |y_i|) - k(t, |x_i|, -|y_i|)) f(y_i) \, dy_i \\
+ \sum_{j=1}^m \int_{(\mathbb{R}^+,j)} \frac{2}{m} k(t, |x_i|, -|y_j|) f(y_j) \, dy_j,
\]

where \(k\) is the integral kernel of the semigroup \((T(t))_{t \geq 0}\) generated by \(\tilde{L}\). Moreover if \(c \equiv 0\) and \(T(t_0)1 = 1\) for some \(t_0 > 0\), then \(T_m(t)1 = 1\) for all \(t \geq 0\) (i.e., \(T_m(\cdot)\) is conservative).

At the danger of being redundant, we stress that the integral kernel \(k\) depends on \(q, b,\) and \(c\). Also, we do not expect \(T_m(\cdot)\) to be strongly continuous.

**Proof.** To construct a solution for the initial data \(f \in C_b(S_m)\) we first consider problem \((P_L)\) on the truncated stars \(S_m^n, n \in \mathbb{N}\), with initial data \(f|_{S_m^n}\) and with Dirichlet boundary conditions on the endpoints \((n, i)\) for each \(i = 1, \ldots, m\). For each \(n \in \mathbb{N}\) and \(i = 1, \ldots, m\) we consider the Cauchy–Dirichlet problem

\[
\begin{align*}
\partial_t u^n_i(t, \cdot) &= \tilde{L} u^n_i(t, \cdot), \quad t > 0, \\
u^n_i(0, x) &= \tilde{f}_i(x), \quad x \in (-n, n), \\
u^n_i(t, n) &= \tilde{f}_i(n), \quad x = n,
\end{align*}
\]

(3.6)

where \(\tilde{f}_i\) is the function given by (3.4). By classical results, cf. [10] Theorem 9.4.1, for parabolic Cauchy problems in bounded domains we know that the above problem has a unique solution \(u^n_i \in C([0, \infty) \times (-n, n)) \cap C^{1+\frac{\alpha}{2}, 2+\alpha}_{loc}((0, \infty) \times [-n, n]), i = 1, \ldots, m,\)

We now define a function \(\hat{u}^n\) on \([0, \infty) \times S_m^n\) by

\[
\hat{u}^n(t, x_i) := u^n_i(t, |x_i|), \quad i = 1, \ldots, m, \quad |x_i| \geq 0, \quad t \geq 0.
\]

In order to prove that \(\hat{u}^n\) is a “classical” solution of problem \((P_{\tilde{L}})\) on \(S_m^n\), we only have to verify that \(\hat{u}^n(t, \cdot) \in D(L_{|c_{ij}(S_m^n)})\) for all \(t > 0, i.e.,\)

\[
(\hat{u}^n)'(t_0) = \sum_{i=1}^m (\hat{u}^n)'(t_0, i) = 0, \quad t > 0.
\]

Given the functions \(\tilde{f}_i\) constructed according to (3.4), \(i = 1, \ldots, m\), let us define functions \(\tilde{f}_{i,j} : \mathbb{R} \rightarrow \mathbb{R}\) by \(\tilde{f}_{i,j}(x) := \tilde{f}_i(x) - \tilde{f}_j(-x)\) for each \(i, j = 1, \ldots, m\). Now, each such \(\tilde{f}_{i,j}\) is odd, since by construction \(\tilde{f}_{i,j} = \tilde{f}_{j,i}\). Therefore, using the definition of the functions \(\tilde{q}\) and \(\tilde{b}\) we deduce that the unique solution \(v^n_{ij}(t, x) := u^n_i(t, x) - u^n_j(t, -x)\) of (3.6) with initial data \(\tilde{f}_{i,j}\) is odd and especially \(v^n_{ij}(0, 0) = 0\) for all \(t \geq 0\). This and (3.7) imply the continuity of \(\hat{u}^n\), i.e., the first condition in (3.8). To prove the second condition in (3.8) one considers the solution \(v^n(t, x) = \sum_{i=1}^m u^n_i(t, x)\) of (3.6) with initial data the function \(\tilde{f}(x) = \sum_{i=1}^m \tilde{f}_i(x)\), which is even.
Thus, again by (3.3), one deduces that \( v^n(t, x) = v^n(t, -x) \). This proves the second condition in (3.8).

Now, using Schauder interior estimates and a compactness argument, cf. [8, Theorem 2.2.5], we know that for each \( i = 1, \ldots, m \), the function \( u_i : [0, \infty) \times \mathbb{R} \to \mathbb{R} \)

\[
  u_i(t, x) := \lim_{n \to \infty} u_i^n(t, x)
\]
even exists for any \( t \geq 0 \) and any \( x \in \mathbb{R} \), and belongs to \( C([0, \infty) \times \mathbb{R}) \cap C^{1+\frac{\alpha}{2}, 2+\alpha}_{\text{loc}}((0, \infty) \times \mathbb{R}) \).

Moreover for each \( i = 1, \ldots, m \), \( u_i \) is a classical solution of \((\mathcal{P}_L)\) on \((0, \infty) \times \mathbb{R}\) with initial data \( f_i \) and satisfies

\[
  (3.4) \quad |u_i(t, x)| \leq e^{\alpha t} \| f_i \|_{\infty}, \quad t > 0, \quad x \in \mathbb{R}, \quad i = 1, \ldots, m.
\]

If we set \( T(t) \tilde{f}_i(x) := u_i(t, x) \), then \((T(t))_{t \geq 0}\) is a semigroup on \( C_b(\mathbb{R})\) satisfying

\[
  T(t)f(x) = \int_{\mathbb{R}} k(t, x, y)f(y) \, dy, \quad f \in C_b(\mathbb{R}), \quad t > 0, \quad x \in \mathbb{R},
\]

where the kernel \( k \) is strictly positive, \( k(t, \cdot, \cdot) \) and \( k(t, x, \cdot) \) are measurable for any \( t > 0, \quad x \in \mathbb{R} \)
and for a.e. fixed \( y \in \mathbb{R}, \quad k(\cdot, \cdot, y) \in C^{1+\frac{\alpha}{2}, 2+\alpha}_{\text{loc}}((0, \infty) \times \mathbb{R}) \) and it is a solution of \( \partial_t u - \tilde{L}u = 0 \), cf. [8, Theorem 2.2.5].

Defining now the function \( T_m(t) f : \mathcal{S}_m \to \mathbb{R} \) by

\[
  T_m(t) f(x_i) := T(t) \tilde{f}_i(|x_i|), \quad i = 1, \ldots, m, \quad |x_i| \geq 0, \quad t \geq 0,
\]

and using (3.8), we arrive at the desired classical solution of \((\mathcal{P}_L)\) on \( \mathcal{S}_m \). Moreover, using (3.9), we deduce that \((T_m(t))_{t \geq 0}\) is a semigroup of contractions on \( C_b(\mathcal{S}_m) \). On the other hand, by (3.1), we have

\[
  T_m(t) f(x_i) = \int_{\mathbb{R}_+} k(t, |x_i|, y)f(y) \, dy + \int_{\mathbb{R}_-} k(t, |x_i|, y)f(y) \, dy
\]

\[
  = \int_{(\mathbb{R}_+, i)} (k(t, |x_i|, |y_i|) - k(t, |x_i|, -|y_i|)) f(y_i) \, dy_i
\]

\[
  + \sum_{j=1}^{m} \int_{(\mathbb{R}_+, j)} \frac{2}{m} k(t, |x_i|, -|y_j|) f(y_j) \, dy_j.
\]

If furthermore \( c \equiv 0 \) and \( T(t_0) \mathbbm{1} = \mathbbm{1} \) for some \( t_0 > 0 \), then, by [8, Proposition 4.1.10], \( T(t) \mathbbm{1} = \mathbbm{1} \)
for all \( t \geq 0 \), and hence \( \int_{\mathbb{R}} k(t, x, y) \, dy = 1 \) for all \( t > 0, \quad x \in \mathbb{R} \). This implies that

\[
  \int_{(\mathbb{R}_+, i)} k(t, |x_i|, |y_i|) \, dy_i + \int_{(\mathbb{R}_+, j)} k(t, |x_i|, -|y_j|) \, dy_j = 1, \quad \forall i, j = 1, \ldots, m, \quad x \in \mathcal{S}_m, \quad t > 0,
\]

holds. So, it follows from (3.3) that the semigroup \((T_m(t))_{t \geq 0}\) is conservative.

Finally, if \( u \) is a solution of \((\mathcal{P}_L)\) with initial data \( f \equiv 0 \), then for each \( i = 1, \ldots, m \), \( u_i \) defined by (3.4) is a solution of \((\mathcal{P}_L)\) with \( u_i(0, \cdot) = 0 \). Thus, the uniqueness of the solution to \((\mathcal{P}_L)\)
implies that \( u_i \equiv 0 \) for each \( i = 1, \ldots, m \), and hence \( u \equiv 0 \). \( \square \)
Remark 3.4.  

(a) The formula (3.5) shows that the semigroup \((T_m(t))_{t \geq 0}\) is positive provided
\[
k(t, |x_i|, |y_i|) \geq k(t, |x_i|, -|y_i|), \quad t > 0, x_i, y_i \in S_m.
\]
This is especially the case for the Ornstein–Uhlenbeck kernel
\[
k_{OU}(t, x, y) := \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \exp[-(1 - e^{-2t})^{-1}(e^{-t}x - y)^2], \quad t > 0, x, y \in \mathbb{R}.
\]

(b) The representation (3.5) allows us to extend the semigroup to the space \(B(S_m)\) of bounded and measurable functions. Moreover this semigroup has the strong Feller property, since \(T_m(t)f(x_i) = T(t)f_i(\{x_i\})\) and \(T(\cdot)\) has the strong Feller property, cf. [8, Proposition 2.2.12].

4. The Ornstein–Uhlenbeck Semigroup on Metric Star Graphs

As already mentioned in the introduction, a particularly interesting special case of the class operators studied above is the Ornstein–Uhlenbeck type operator
\[
Af(x_i) = \frac{1}{2} f''(x_i) - |x_i| f'(x_i), \quad |x_i| \geq 0, \quad i = 1, \ldots, m,
\]
with Kirchhoff-type condition in zero encoded in the domain
\[
D(A) = \{ f \in C_b(S_m) \cap \bigcap_{1 \leq p < \infty} W^{2,p}_{\text{loc}}(S_m) : \sum_{i=1}^{m} f'(0_i) = 0 \text{ and } Af \in C_b(S_m) \}.
\]
For \(m = 1\) we have the Ornstein–Uhlenbeck operator on \(\mathbb{R}^+\) with Neumann boundary condition in zero and for \(m = 2\) the Ornstein–Uhlenbeck operator on \(\mathbb{R}\). Therefore our setting can be regarded as a generalization of these well known cases.

As a consequence of Theorem 3.3 and Remark 3.4 we have the following, where we denote by \(S(\cdot)\) the classical Ornstein–Uhlenbeck semigroup on \(C_b(\mathbb{R})\).

Proposition 4.1. For every \(f \in C_b(S_m)\) there exists a unique bounded, classical solution \(u\) of
\((P_A)\). This solution is given by the so-called Ornstein–Uhlenbeck semigroup on \(S_m\)
\[
(S_m(t)f)(x_i) := u(t, x_i) = S(t)f_i(\{x_i\})
\]
\[
= \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \int_{(\mathbb{R}+i)} \left( \exp[-(1 - e^{-2t})^{-1}(e^{-t}|x_i| - |y_i|)^2] \right.
- \exp[-(1 - e^{-2t})^{-1}(e^{-t}|x_i| + |y_i|)^2]) f(y_i) \, dy_i
\]
\[
+ \frac{2}{m \sqrt{\pi(1 - e^{-2t})}} \sum_{1 \leq j \leq m} \int_{(\mathbb{R}+i)} \exp[-(1 - e^{-2t})^{-1}(e^{-t}|x_i| + |y_j|)^2] f(y_j) \, dy_j
\]
for \(1 \leq i \leq m\). Furthermore, \(S_m(\cdot)\) is irreducible, conservative, contractive and has the strong Feller property.
Proof. It suffices to prove that \( S_m(\cdot) \) is irreducible and contractive. To prove the contractivity of \( S_m(\cdot) \) we take \( f \in C_b(S_m) \) and \( t \geq 0 \). Now, by Remark 3.4(a), \( S_m(\cdot) \) is positive and so, \( S_m(t)1 = 1 \) implies that

\[
|S_m(t)f| \leq S_m(t)|f| \leq \|f\|_\infty S_m(t)1 = \|f\|_\infty.
\]

To show the irreducibility of \( S_m(\cdot) \), let us consider \( 0 \leq f \in C_b(S_m) \) such that \( f \not\equiv 0 \). Assume, by contradiction, that there is \( x_i \in S_m \) and \( t > 0 \) such that \( S_m(t)f(x_i) = 0 \). So, by (4.3) and Remark 3.4(a), we have

\[
\int_{(\mathbb{R},j)} \exp\left[-(1 - e^{-2t})^{-1}(e^{-t}|x_i| + |y_j|)^2\right]f(y_j) \, dy_j = 0, \quad \forall j = 1, \ldots m.
\]

Thus, \( f \equiv 0 \), which is a contradiction. \( \square \)

Remark 4.2. In view of (4.3), an equivalent formula for the Ornstein–Uhlenbeck semigroups \( (S_m(t))_{t \geq 0} \) is as follows:

\[
(S_m(t)f)(x_i) = \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \int_{(\mathbb{R},i)} e^{-\frac{(e^{-t}|x_i| - |y_j|)^2}{(1 - e^{-2t})}} f(y_i) \, dy_i + \frac{1}{\sqrt{\pi(1 - e^{-2t})}} \sum_{1 \leq j \leq m} \int_{(\mathbb{R},j)} \sigma_{ij} e^{-\frac{(e^{-t}|x_i| + |y_j|)^2}{(1 - e^{-2t})}} f(y_j) \, dy_j,
\]

where \( \Sigma = (\sigma_{ij}) \) is the scattering matrix defined by

\[
\sigma_{ij} := \begin{cases} 
2 - m, & \text{if } i = j, \\
m, & \text{otherwise}.
\end{cases}
\]

In other words, the integral kernel of \( (S_m(t))_{t \geq 0} \) can be obtained as the overlapping of the plain Ornstein–Uhlenbeck kernel on \( \mathbb{R} \) (corresponding to the unscattered movement of a particle between two points of the same edge of \( S_m \)) and the weighted sum of the paths between \( x \) and a point \( y \) on different edges (with weight \( \frac{2}{m} \)), or else on the same edge with \( y \) reached only after the particle has been reflected in the center of the star (with weight \( \frac{2 - m}{m} \)). Observe that if \( m = 2 \), then no reflection is possible and the above formula yields just the usual Ornstein–Uhlenbeck semigroup \( S(\cdot) \) on \( \mathbb{R} \). Following the ideas of [22], we can conjecture that this formula may be generalized to an arbitrary graph \( G \) as follows:

\[
(\tilde{S}(t)f)(x) = \int_G p(t, x, y) f(y) \, dy \quad \text{for all } t > 0 \text{ and } x \in G,
\]

where, for any two \( x, y \in G \), \( \mathcal{P}_{x,y} \) denotes the set of all paths from \( x, y \),

\[
p(t, x, y) := \sum_{\mathcal{P} \in \mathcal{P}_{x,y}} \sigma(\mathcal{P}) G_1(t, \text{dist}(e^{-t}x, y)), \quad t > 0, \ x, y \in G,
\]
Remark 4.3. The usual properties of the Ornstein–Uhlenbeck semigroup, cf. [8, Sections 9.2 and 9.4], hold (with the same proofs):

- Since \( S_m(t)f(x_i) = S(t)\tilde{f}_i(x_i) \) and \( C_0(\mathbb{R}) \) is invariant for \( S(t) \), it follows that \( S_m(t) \) maps \( C_0(S_m) \) into \( C_0(S_m) \) for all \( t \geq 0 \).
- \( S_m(t) \) is not compact on \( C_b(S_m) \). This can be proven as in [8, Theorem 5.1.11].
- \( S_m(\cdot) \) is not strongly continuous on \( C_b(S_m) \). More specifically, \( \lim_{t \to 0} \|S_m(t)f - f\|_{\infty} = 0 \) iff \( f \in BUC(S_m) \) and \( \lim_{t \to 0} |f(e^{-t}x_i) - f(x_i)| = 0 \) uniformly with respect to \( x_i \in S_m \).
- From [1,3] one deduces that \( S_m(\cdot) \) extrapolates to a consistent family of strongly continuous semigroups on \( L^p(S_m) \) for all \( 1 \leq p < \infty \).

The following result gives the unique invariant measure of \( S_m(\cdot) \).

**Theorem 4.4.** There exists a unique invariant probability measure \( \mu_m \) for the Ornstein–Uhlenbeck semigroup \( S_m(\cdot) \). This measure has density

\[
(4.5) \quad \mu_m(\,dx_i) = \frac{2}{m\sqrt{\pi}} e^{-|x_i|^2} \, dx_i, \quad i = 1, \ldots, m,
\]

with respect to Lebesgue measure.

**Proof.** Let \( f \in C_b(S_m) \), \( \tilde{f}_i \in C_b(\mathbb{R}) \) as in (3.4), \( i = 1, \ldots, m \). Let \( S(\cdot) \) be the Ornstein–Uhlenbeck semigroup on \( \mathbb{R} \), \( \mu \) the Gaussian measure on \( \mathbb{R} \), \( \mu(\,dx) = \frac{1}{\sqrt{2\pi}} e^{-|x|^2} \, dx \), and \( S_m(\cdot) \) be the Ornstein–Uhlenbeck semigroup on \( S_m \). We know \( \mu \) is the invariant measure of \( S(\cdot) \), hence

\[
\int_{\mathbb{R}} S(t) \sum_{i=1}^{m} \tilde{f}_i(x) \mu(\,dx) = \int_{\mathbb{R}} \sum_{i=1}^{m} \tilde{f}_i(x) \mu(\,dx) \quad \text{for all } t > 0,
\]

and, because \( \sum_{i=1}^{m} \tilde{f}_i(x) \) and therewith \( S(t) \sum_{i=1}^{m} \tilde{f}_i(x) \) are even functions on \( \mathbb{R} \), we infer that \( \frac{1}{\sqrt{2\pi}} e^{-|x|^2} \, dx \), \( i = 1, \ldots, m \), defines an invariant measure for \( S_m(\cdot) \). After normalizing this measure we may conclude that \( \mu_m \) is indeed an invariant probability measure for \( S_m(\cdot) \).

Uniqueness follows along the same lines as usual from the ergodicity of the invariant measure. \( \square \)

**Remark 4.5.** As in Remark [4,3], the regularity properties of the classical Ornstein–Uhlenbeck semigroup on \( \mathbb{R} \), cf. [8, Sections 9.3], hold for the semigroup \( S_m(\cdot) \) on \( L^p_{\mu_m}(S_m) \):
For any \( p \in (1, \infty) \), \( S_m(\cdot) \) is analytic in \( L^p_{\mu_m}(S_m) \) and consistent, i.e. \( S_m(t) \) on \( L^p_{\mu_m}(S_m) \) and on \( L^q_{\mu_m}(S_m) \) coincide on \( L^p_{\mu_m}(S_m) \cap L^q_{\mu_m}(S_m) \) for all \( p, q \in (1, \infty) \) and \( t \geq 0 \).

- For any \( p \in (1, \infty) \), \( W^{1,p}_{\mu_m}(S_m) \) is compactly embedded in \( L^p_{\mu_m}(S_m) \).
- The semigroup \( S_m(t) \) maps \( L^p_{\mu_m}(S_m) \) into \( W^{1,p}_{\mu_m}(S_m) \) and hence \( S_m(t) \) is compact in \( L^p_{\mu_m}(S_m) \), \( 1 < p < \infty \), for any \( t > 0 \).

For later purposes we propose to characterize the domain of the realization, \( A_2 \) of \( A \) in \( L^2_{\mu_m}(S_m) \). Here we recall that

\[
L^2_{\mu_m}(S_m) = \bigoplus_{i=1}^m L^2_{\mu_m}((\mathbb{R}_+, i)) \text{ endowed with the norm }
\|
\|_L^2_{\mu_m}(S_m) = \sum_{i=1}^m \| f_i \|^2_{L^2_{\mu_m}((\mathbb{R}_+, i))} \text{ for } f = (f_i)_{1 \leq i \leq m}.
\]

Like in \( (3.2) \), we define the weighted Sobolev spaces

\[
\tilde{H}^k_{\mu_m}(S_m) := \bigoplus_{i=1}^m H^k_{\mu_m}((\mathbb{R}_+, i)), \quad k \in \mathbb{N}.
\]

On

\[
D(a) := H^1_{\mu_m}(S_m) := \{ f \in \tilde{H}^1_{\mu_m}(S_m) : f_i(0) = f_j(0) \text{ for all } i, j = 1, \ldots, m \}
\]

we define the sesquilinear form

\[
a(f, g) := \frac{1}{2} \sum_{i=1}^m \int_{(\mathbb{R}_+, i)} f_i'(x_i) g_i'(x_i) \mu_m(\text{d}x_i), \quad f, g \in D(a).
\]

Since \( a \) is densely defined, symmetric, accretive, continuous and closed sesquilinear form, we can associate the self-adjoint operator

\[
D(B) := \left\{ f \in D(a) : \exists g \in L^2_{\mu_m}(S_m) \text{ s.t. } a(f, \phi) = \langle g, \phi \rangle_{L^2_{\mu_m}(S_m)} \forall \phi \in D(a) \right\},
\]

\[
Bf := g.
\]

We can finally describe the realization \( A_2 \) of the Ornstein–Uhlenbeck operator in \( L^2_{\mu_m}(S_m) \).

**Proposition 4.6.** The generator \( A_2 \) of the Ornstein–Uhlenbeck semigroup on \( L^2_{\mu_m}(S_m) \) is given by

\[
D(A_2) = \left\{ f \in \tilde{H}^2_{\mu_m}(S_m) : f_i(0) = f_j(0) \text{ for all } i, j = 1, \ldots, m \text{ and } \sum_{i=1}^m f_i'(0) = 0 \right\}
\]

\[
(A_2 f)_i(x) = \frac{1}{2} f_i''(x) - x f_i'(x), \quad \text{for all } f = (f_i)_{1 \leq i \leq m} \in D(A_2).
\]
Proof. Let $f \in D := \{ f \in \widetilde{H}^2_{\mu_m}(S_m) : f_i(0) = f_j(0) \text{ for all } i, j = 1, \ldots, m \text{ and } \sum_{i=1}^{m} f_i'(0) = 0 \}$. Then $f \in D(a)$ and integrating by part one obtains
\[ \langle -A_2 f, \phi \rangle_{L^2_{\mu_m}(S_m)} = a(f, \phi), \ \forall \phi \in D(a). \]
So, $(-A_2, D) \subseteq (B, D(B))$.

Now, let $f = (f_i)_{1 \leq i \leq m} \in D(B)$. Then, there is $g \in L^2_{\mu_m}(S_m)$ such that
\[ (4.6) \quad a(f, \phi) = \langle g, \phi \rangle_{L^2_{\mu_m}(S_m)}, \ \forall \phi \in D(a). \]
For any fixed $j \in \{1, \ldots, m\}$ consider the function $\phi = (\phi_i)_{1 \leq i \leq m}$ with $\phi_j \in C^\infty_c(\mathbb{R}_+, i)$ and $\phi_i \equiv 0$ for $i \neq j$. Applying (4.6) with $\phi$ as above, one can see that $f_j \in H^2_{\mu_m}(\mathbb{R}_+, j)$ and $-\frac{1}{2} f_j'' + x_j f_j' = g_j$. Thus, $f \in H^2_{\mu_m}(S_m)$. So, we can integrate by part in (4.6) and obtain, for any $\phi \in D(a)$,
\[ a(f, \phi) = \frac{1}{2} \sum_{i=1}^{m} \int_{(\mathbb{R}_+, i)} f_i'(x_i) \overline{\phi_i(x_i)} \mu_m(\mathrm{d}x_i) = \sum_{i=1}^{m} \int_{(\mathbb{R}_+, i)} \left( -\frac{1}{2} f_i''(x_i) + x_i f_i'(x_i) \right) \overline{\phi_i(x_i)} \mu_m(\mathrm{d}x_i) + \frac{1}{2} \sum_{i=1}^{m} f_i'(0) \overline{\phi_i(0)} = \langle g, \phi \rangle_{L^2_{\mu_m}(S_m)} + \frac{1}{2} \sum_{i=1}^{m} f_i'(0) \overline{\phi_i(0)}. \]
By choosing now $\phi \in D(a)$ such that $\phi(0) \neq 0$, one obtains $f \in D$. Hence, $(-A_2, D) = (B, D(B))$. \hfill \square

Before characterizing the spectrum of $A$, we need a preparatory lemma. The following seems to be folklore, but we could not find an appropriate reference in the literature. Because $\mathbb{R} \simeq S_2$, with a slight abuse of notation we still denote by $A_2$ the realization of the Ornstein–Uhlenbeck operator on $L^2_{\mu}(\mathbb{R}) \simeq L^2_{\mu_2}(S_2)$.

**Lemma 4.7.** The realization of the Ornstein–Uhlenbeck operator on $L^2_{\mu_1}(\mathbb{R}_+)$ has purely point spectrum given by
\[
\begin{cases}
\{ -2k : k \in \mathbb{N}_0 \}, & \text{if Neumann conditions are imposed at } 0, \\
\{ -2k - 1 : k \in \mathbb{N}_0 \}, & \text{if Dirichlet conditions are imposed at } 0.
\end{cases}
\]

**Proof.** It is well known \cite{15} Theorem 3.1] that the spectrum of the Ornstein–Uhlenbeck operator on $L^2_{\mu}(\mathbb{R})$ consists precisely of the simple eigenvalues $k = 0, -1, -2, \ldots$, and that the corresponding eigenfunctions are given by the Hermite polynomials $H_k$, where
\[ H_k(x) := (-1)^k e^{-x^2} D^k e^{-x^2}, \quad x \in \mathbb{R}, \ k \in \mathbb{N}_0. \]
This information can be reformulated: since we know that $A_2$ leaves invariant the mutually orthogonal subspaces $L^2_{\text{odd}}$ and $L^2_{\text{even}}$ (of odd and even $L^2_{\mu}(\mathbb{R})$-functions, respectively), the spectrum of $A$ can be described as the disjoint union of two subsets: the spectrum of the restrictions of $A$ to $L^2_{\text{odd}}$ and $L^2_{\text{even}}$. In turn, these restrictions are unitarily equivalent (and isospectral) with
the realizations $A_D$ and $A_N$ of the Ornstein–Uhlenbeck operator on $L^2_{\mu_{\mu}}(\mathbb{R}_+)$ with Dirichlet and Neumann conditions, respectively.

Since $H_k$ is a polynomial, $AH_k = -kH_k$; furthermore, $H_k(0) = 0$ if and only if $k$ is even, whereas $H_k(0) = 0$ if and only if $k$ is odd. It follows that $H_k$ is an eigenfunction of $A_N$ whenever $k$ is even and $H_k$ is an eigenfunction of $A_D$ whenever $k$ is odd. This yields the claim. \hfill $\square$

We now characterize the spectrum of $A_p$.

**Theorem 4.8.** The spectrum of the realization, $A_p$, $p \in (1, \infty)$, of $A$ in $L^p_{\mu_{m}}(S_m)$ consists of isolated eigenvalues and is independent of $p \in (1, \infty)$. Moreover,

$$\sigma(A_p) = \{-k : k \in \mathbb{N}_0\}, \quad p \in (1, \infty),$$

where all even eigenvalues have multiplicity 1, whereas all odd eigenvalues have multiplicity $m - 1$.

**Proof.** By Remark 4.5, we know that $S_m(t)$ is compact in $L^p_{\mu_{m}}(S_m)$ for any $t > 0$. Hence, the spectrum $\sigma(A_p)$ of $A_p$ consists of a sequence of eigenvalues. By standard arguments, see the proof of [13, Proposition 2.10], one deduces that $\sigma(A_p)$ is independent of $p$, cf. [11, Section 7.2.2]. Anyway for the reader’s convenience we give some details. From Remark 4.5 we know that $R(\lambda, A_p) = R(\lambda, A_q)$ on $L^p_{\mu_{m}}(S_m) \cap L^q_{\mu_{m}}(S_m)$ for any $\lambda > 0$. Since $\sigma(A_p)$ and $\sigma(A_q)$ consist of isolated eigenvalues, $\mathbb{C} \setminus (\sigma(A_p) \cup \sigma(A_q))$ is a connected open set in $\mathbb{C}$. Hence, $R(\lambda, A_p) = R(\lambda, A_q)$ on $L^p_{\mu_{m}}(S_m) \cap L^q_{\mu_{m}}(S_m)$ for any $\lambda \in \mathbb{C} \setminus (\sigma(A_p) \cup \sigma(A_q))$.

Let us now fix $\lambda_0 \in \sigma(A_p)$. So, $\lambda_0$ is isolated in $\sigma(A_p) \cup \sigma(A_q)$. Thus, there is $\varepsilon > 0$ small enough such that $B_\varepsilon(\lambda_0) \setminus \{\lambda_0\} \subset \mathbb{C} \setminus (\sigma(A_p) \cup \sigma(A_q))$. Let $P$ be the spectral projection associated with $\lambda_0$, which is defined by

$$Pf = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(\lambda_0)^+} R(\lambda, A_p)f \, dx, \quad f \in L^p_{\mu_{m}}(S_m).$$

If $\lambda_0 \notin \sigma(A_q)$, then we have

$$P\varphi = \frac{1}{2\pi i} \int_{\partial B_\varepsilon(\lambda_0)^+} R(\lambda, A_q)\varphi \, dx = 0$$

for all $\varphi \in C_c^\infty(S_m)$. Thus, by density, $P \equiv 0$, which is a contradiction. Therefore, $\sigma(A_p) \subset \sigma(A_q)$ and hence $\sigma(A_p) = \sigma(A_q)$, since $p$ and $q$ have been arbitrarily fixed. In particular we have $\sigma(A_p) = \sigma(A_2)$.

Let us now turn to the task of describing the spectrum of $A_2$ on $L^2_{\mu_{m}}(S_m)$. We will adapt a method which the first-named author has learned from Pavel Kurasov: it was, e.g., already used in [13, Section 3.5] to solve the problem of determining the spectrum of the Laplacian with natural vertex conditions on equilateral star graphs.

To begin with, we observe that $A_2$ leaves invariant the the mutually orthogonal subspaces $L^2_{\text{odd}}$ and $L^2_{\text{even}}$ of odd and even $L^2_{\mu_{m}}(S_m)$-functions, respectively [13], up to minor modifications,

\begin{itemize}
  \item \textit{even}, if $f(x_i) = f(x_j)$ for all $i, j = 1, \ldots, m$,
\end{itemize}
this can be proved as in [17, Proposition 6.88]. In fact, more is true: if we denote by \( R \) the bounded, unitary operator on \( L^2_{\mu_m}(S_m) \) defined by
\[
R : (f_1, \ldots, f_{m-1}, f_m) \mapsto (f_2, \ldots, f_m, f_1),
\]
then it is easy to see that \( R \) commutes with \( A_2 \). Accordingly, by the Spectral Theorem for normal operators \( A_2 \) and \( R \) can be simultaneously diagonalized: i.e., any eigenfunction of \( A_2 \) turns out to be an eigenfunction of \( R \), and vice versa. So, what are the eigenfunctions of \( R \)?

Observe that \( R_m \) is the identity operator of \( L^2_{\mu_m}(S_m) \), so its eigenvalues are precisely the \( m \)-th roots of unity: \( e^{2j\pi i/m} \), \( j = 0, \ldots, m-1 \). A direct computation shows that the corresponding (infinite-dimensional) \( j \)-th eigenspace of \( R \) is
\[
E_j := (1, z^j, z^{2j}, \ldots, z^{j(m-1)}) \otimes L^2_{\mu_m}(R_+),
\]
where \( z := e^{2\pi i/m} \). Observe that the eigenspace \( E_0 \) agrees with the space \( L^2_{\text{even}} \) of even \( L^2_{\mu_m}(S_m) \)-functions, whereas the remaining \( m-1 \) eigenspaces of \( R \) consist of odd functions: we conclude that
\[
L^2_{\text{odd}} = \bigoplus_{j=1}^{m-1} E_j.
\]

Let us first study the restriction of \( A_2 \) to \( L^2_{\text{odd}} \), or equivalently \( \bigoplus_{j=1}^{m-1} A_{E_j} \): for continuous functions on \( S_m \) (like those in \( D(A_2) \)), oddness induces, by
\[
0 = \sum_{i=1}^{m} f(0_i) = mf(0),
\]
Dirichlet boundary conditions at 0: we conclude that the spectrum of \( A_{E_j} \) is
\[
\{-2k-1 : k \in \mathbb{N}_0 \} \quad \text{for all } j = 1, \ldots, m-1.
\]

Likewise, the restriction of \( A_2 \) to the space \( E_0 \) of even functions on \( S_m \) is isomorphically equivalent, hence isospectral, with the realization \( A_N \) of the Ornstein–Uhlenbeck operator on \( L^2_{\mu_1}(R_+) \) with Neumann conditions at 0: we already know from Lemma [4.7] that the corresponding eigenvalues form the set
\[
\{-2k : k \in \mathbb{N}_0 \}.
\]
This concludes the proof.

We have just seen that the Ornstein–Uhlenbeck semigroup generated by \( A_2 \) in \( L^2_{\mu_m}(S_m) \) is compact. In fact, more can be said.

**Proposition 4.9.** The Ornstein–Uhlenbeck semigroup generated by \( A_2 \) on \( L^2_{\mu_m}(S_m) \) is of trace class.

**Proof.** It suffices to observe that the \( L^2_{\mu_m}(S_m) \)-eigenvalues of \( (\lambda - A_2)^{-1} \), \( \lambda > 0 \), are square summable for all \( m \in \mathbb{N} \). Accordingly, \( A_2 \) has Hilbert–Schmidt resolvent, and the trace class property of the semigroup follows. \( \square \)

In the case of \( m = 2 \), the assertion in Proposition [4.9] is a direct consequence of [14, Theorem 3.3].

- odd, if \( f(x_1) + \ldots + f(x_m) = 0 \).
5. The harmonic oscillator

Let us discuss a further example: the harmonic oscillator.

\[(5.1)\quad B f(x_i) = \frac{1}{2} \left( f''(x_i) - |x_i|^2 f(x_i) + f(x_i) \right), \quad |x_i| \geq 0, \quad i = 1, \ldots, m, \]
on the star graph \(S_m\), again with Kirchhoff-type conditions in 0, i.e.,

\[(5.2)\quad D(B) = \left\{ f \in C_b(S_m) \cap \bigcap_{1 \leq p < \infty} W^{2,p}_{\text{loc}}(S_m) : \sum_{i=1}^{m} f'(0_i) = 0 \right\} \cap C_b(S_m). \]

For \(m = 2\) we have the classical harmonic oscillator on \(\mathbb{R}\): we refer to [5, Section 4.3] for basic facts about it. In particular, it is known that \(B\) generates on \(L^2(\mathbb{R})\) an ultracontractive semigroup whose heat kernel is given by

\[(5.3)\quad k(t, x, y) := \frac{1}{\sqrt{\pi(1 - e^{-2t})}} e^{\frac{4x e^{-t} - (x^2 + y^2)(1 + e^{-2t})}{2(1 - e^{-2t})}} \]
by the celebrated Mehler formula.

Applying Theorem 3.3 with \(q(x) = \frac{1}{2}, b(x) = 0\) and \(c(x) = -\frac{1}{2}(x^2 - 1)\) and (5.3) we deduce the following.

**Corollary 5.1.** For every function \(f \in C_b(S_m)\), there exists a unique classical solution \(u\) of \((P_B)\) given by the semigroup \(U_m(\cdot)\)

\[(5.4)\quad u(t, x_i) = (U_m(t)f)(x_i) = \frac{2}{\sqrt{\pi(1 - e^{-2t})}} \int_{(\mathbb{R}^+)^1} \left( \exp \left[ -\left( 1 + e^{-2t} \right) \frac{2|x_i||y_i|e^{-t}}{1 - e^{-2t}} \right] \sinh \left( \frac{2|x_i||y_i|e^{-t}}{1 - e^{-2t}} \right) f(y_i) \, dy_i \right) \]
\[+ \frac{2}{m \sqrt{\pi(1 - e^{-2t})}} \sum_{1 \leq j \leq m} \int_{(\mathbb{R}^+)^1} e^{\frac{-4|x_i| |y_j| e^{-t} - (|x_i|^2 + |y_j|^2)(1 + e^{-2t})}{2(1 - e^{-2t})}} f(y_j) \, dy_j\]

for \(1 \leq i \leq m\).

As in the previous section we describe the realization \(B_2\) of the harmonic oscillator \(B\) in

\[L^2(S_m) = \bigoplus_{i=1}^{m} L^2(\mathbb{R}_+, i)\]
endowed with the norm

\[\|f\|_{L^2(S_m)}^2 = \sum_{i=1}^{m} \|f_i\|_{L^2(\mathbb{R}_+, i)}^2 \quad \text{for } f = (f_i)_{1 \leq i \leq m}.\]

To this purpose we consider the isometry

\[T : L^2_{\mu_m}(S_m) \to L^2(S_m)\]

\[f \mapsto (\sqrt{c_m} e^{-\frac{x^2}{2}} f_i),\]
where \( c_m := \frac{2}{m \sqrt{\pi}} \). An easy computation shows that \( B = T A T^{-1} \) and so by Proposition 4.6 and Theorem 4.8 we have the following result.

**Proposition 5.2.** The generator \( B_2 \) of the harmonic oscillator semigroup \( (U_m(\cdot)) \) on \( L^2(S_m) \) is given by

\[
D(B_2) = \left\{ f \in \tilde{H}^2(S_m) : f_j(0) = f_i(0) \text{ for all } i, j = 1, \ldots, m \text{ and } \sum_{i=1}^{m} f_i'(0) = 0 \right\}
\]

\[
(B_2 f)_i(x) = \frac{1}{2} \left( f_i''(x) - x^2 f_i(x) + f_i(x) \right), \quad \text{for all } f = (f_i)_{1 \leq i \leq m} \in D(B_2).
\]

Moreover, \( U_m(\cdot) = TS_m(\cdot)T^{-1} \) and

\[
\sigma(B_2) = \{-k : k \in \mathbb{N}_0\},
\]

where all even eigenvalues have multiplicity 1, whereas all odd eigenvalues have multiplicity \( m - 1 \).

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