The Tribonacci and ABC Representations of Numbers are Equivalent

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Abstract

It is shown that the unique representation of positive integers in terms of tribonacci numbers and the unique representation in terms of iterated $A$, $B$ and $C$ sequences defined from the tribonacci word are equivalent. These sequences are studied in detail.

1 Introduction and Synopsis

The quintessence of many applications of the tribonacci sequence $T = \{T(n)\}_{n=0}^{\infty}$ \[A000073\] \[5\], \[6\], \[2\] \[1\] is the ternary substitution sequence $2 \rightarrow 0$, $1 \rightarrow 02$ and $0 \rightarrow 01$. Starting with 2 this generates an infinite (incomplete) binary tree with ternary node labels called $T$Tree. See Fig 1 for the first 6 levels $l = 0, 1, ..., 5$ denoted by $TTree_5$. The number of nodes on level $l$ is the tribonacci number $T(l+2)$, for $l \geq 0$. In the limit $n \rightarrow \infty$ the last level $l = n$ of $TTree_n$ becomes the infinite self-similar tribonacci word $T$Word. The nodes on level $l$ are numbered by $N = 0, 1, ..., T(l+2) - 1$.

The left subtree, starting with 0 at level $l = 1$ will be denoted by $T$Tree$L$, and the right subtree, starting with 2 at level $l = 0$ is named $T$Tree$R$. The number of nodes on level $l$ of the left subtree $TTree$L is $T(l+1)$, for $l \in \mathbb{N}$; the number of nodes on level $l$ of $TTree$R is 1 for $l = 0$ and $T(l+1)$, for $l \in \mathbb{N}$.

$T$Word considered as ternary sequence $t$ is given in \[4\] \[A080843\] (we omit the OEIS reference henceforth if $A$ numbers for sequences are given): $\{0, 1, 0, 2, 0, 1, 0, 0, 1, 0, 2, 0, 1, ...\}$. See also Table 1. This is the analogue of the binary rabbit sequence \[A005614\] in the Fibonacci case. Like in the Fibonacci case with the complementary and disjoint Wythoff sequences $A = \[A000201\]$ and $B = \[A001950\]$ recording the positions of 1 and 0, respectively, in the tribonacci case the sequences $A = \[A278040\]$, $B = \[A278039\]$, and $C = \[http://www.itp.kit.edu/~wl\]
record the positions of 1, 0, and 2, respectively. These sequences start with \( A = \{1, 5, 8, 12, 14, 18, 21, 25, 29, 32, \ldots \} \), \( B = \{0, 2, 4, 6, 7, 9, 11, 13, 15, 17, \ldots \} \), and \( C = \{3, 10, 16, 23, 27, 34, 40, 47, 54, 60, \ldots \} \). See also Table 1.

The present work is a generalization of the theorem given in the Fibonacci case for the equivalence of the Zeckendorf- and Wythoff-representations of numbers in \( \mathbb{N} \).

Note that there are other complementary and disjoint tribonacci \( A, B \) and \( C \) sequences given in OEIS. They use the same ternary sequence \( t = A080843 \) (which has offset 0), with 0 → a, 1 → b and 2 → c, however with offset 1, and record the positions of a, b and c by \( A = A003144 \), \( B = A003145 \) and \( C = A003146 \), respectively. In [2] and [1] they are called \( a, b, \) and \( c \). This tribonacci \( ABC \)-representation is given in \( A317206 \). The relation between these sequences (we call them now \( a, b, c \)) is: \( a(n) = B(n-1) + 1, b(n) = A(n-1) + 1, \) and \( c(n) = C(n-1)+1, \) for \( n \geq 1 \). We used \( B(0) = 0 \) in analogy to the Wythoff-representation in the Fibonacci case.

From the uniqueness of the ternary sequence \( t \) (with offset 0) it is clear that the three sequences \( A, B \) and \( C \) cover the nonnegative integers \( \mathbb{N}_0 \) completely, and they are disjoint.

In contrast to the Fibonacci case where the Wythoff sequences are Beatty sequences [7] for the irrational number \( \phi = A001622 \), the golden section, and are given by \( A(n) = \lfloor n \phi \rfloor \) and \( B(n) = \lfloor n \phi^2 \rfloor \), for \( n \in \mathbb{N} \) (with \( A(0) = 0 = B(0) \)), no such formulae for the complementary sequences \( A, B \) and \( C \) in the tribonacci case are considered. The definition given above in terms of \( TWord \), or as sequence \( t \), is not burdened by numerical precision problems.

Note that the irrational tribonacci constant \( \tau = 1.83928675521416... = A058265 \), the real solution of characteristic cubic equation of the tribonacci recurrence \( \lambda^3 - \lambda^2 - \lambda - 1 = 0 \), defines, together with \( \sigma = \frac{\tau}{\tau - 1} = 2.19148788395311... = A316711 \) the complementary and disjoint Beatty sequences \( At := \lfloor n \tau \rfloor \) and \( Bt := \lfloor n \sigma \rfloor \), given in \( A158919 \) and \( A316712 \), respectively.

![Figure 1: Tribonacci Tree TTree₅](image.png)
The analogue of the unique Zeckendorf-representation of positive integers is the unique tribonacci-representation of these numbers.

\[(N)_T = \sum_{i=0}^{I(N)} f_i T(i + 3), \quad f_i \in \{0, 1\}, \quad f_i f_{i+1} f_{i+2} = 0, \quad f_{I(N)} = 1.\]  

(1)

The sum should be ordered with falling \(T\) indices. This representation will also be denoted by \((Z\) as a reminder of Zeckendorf\)

\[ZT(N) = \Pi_{i=0}^{I(N)} f_{I(N)-i} = f_{I(N)} f_{I(N)-1} \ldots f_0.\]  

(2)

The product with concatenation of symbols is here denoted by \(\Pi\), and the concatenation symbol \(\circ\) is not written. This product has to be read from the right to the left with increasing index \(i\). This representation is given in \(A278038\)\((N)\), for \(N \geq 1\). See also Table 3 for \(ZT(N)\) for \(N = 1, 2, \ldots, 100\).

E.g., \((1)_T = T(3)\), \((8)_T = T(6) + T(3)\), \((ZT(8)) = 1001\). The length of \(ZT(N)\) is \(#ZT(N) = I(N) + 1 = \{1, 2, 3, 6, 11, 20, 37\ldots\} = \{A278044\}(N)\}_{N \geq 1}.\) The number of numbers \(n\) is given by \(\{1, 2, 3, 6, 11, 20, 37\ldots\} = \{A001590\}(n + 2)\}_{n \geq 1}\). These are the companion tribonacci numbers of \(T = A000073\) with inputs 0, 1, 0 for \(n = 0, 1, 2\), respectively.

\(ZT(N)\) can be read off any finite \(TTree_n\) with \((n + 2) \geq N\) after all node labels 2 have been replaced by 1. See Figure 1 for \(n = 5\) (with \(2 \rightarrow 1\)) and numbers \(N = 0, 1, \ldots, 12\). The branch for \(N\) is read from bottom to top, recording the labels of the nodes, ending with the last 1 label. Then the obtained binary string is reversed in order to obtain the one for \(ZT(N)\). E.g., \(N = 9\) leads to the string 0101 which after reversion becomes \(ZT(9) = 1010\).

The analogue of the Wythoff-representation of nonnegative integers is the tribonacci \(ABC\)-representation using iterations of the sequences \(A, B\) and \(C\).

\[(N)_{ABC} = \left(\Pi_{j=1}^{J(N)} X(N)_{j}^{k(N)_{j}}\right) B(0), \quad \text{with} \quad N \in \mathbb{N}_0, \quad k(N)_{j} \in \mathbb{N}_0,\]  

(3)

again with an ordered concatenation product. Here \(X(N)_{j} \in \{A, B, C\}\), for \(j = 1, 2, \ldots, J(N) - 1\), with \(X(N)_{j} \neq X(N)_{j+1}\), and \(X(N)_{J(N)} \in \{A, C\}\). Powers of \(X(N)_{j}\) are also to be read as concatenations. Concatenation means here iteration of the sequences. The exponents can be collected in \(k(N) := (k(1), \ldots, k(N)_{J(N)})\). For the equivalence proof only positive integers \(N\) are considered. If exponents vanish the corresponding \(A, B, C\) symbols are not present \((X(N)_{j})^0\) is of course not 1). If all exponents vanish, the \(\Pi\) is empty, and \(N = 0\) could be represented by \((0)_{ABC} = B(0) = 0\) (but this will not be used for the equivalence proof).

E.g., \((30)_{ABC} = (BCBA)B(0) = B(C(B(A(B(0))))), J(30) = 4, \tilde{k}(30) := (k_1, k_2, k_3, k_4) = (1, 1, 1, 1), X(30)^{k_1} = B^1, X(30)^{k_2} = C^1, X(30)^{k_3} = B^1, X(30)^{k_4} = A^1\) (sometimes the arguments \((N)\) are skipped).
The number of $A, B$ and $C$ sequences present in this representation of $N$ is $\sum_{j=1}^{J(N)} k(N)_j + 1 = A316714(N)$. This representation is also written as

$$ABC(N) = \left( H_{j=1}^{J(N)} x(N)_j^{k(N)_j} \right) 0, \text{ with } k(N)_j \in \mathbb{N}_0,$$

and $x(N)_j \in \{0, 1, 2\}$, for $j = 1, 2, ..., J(N)−1$, with $x(N)_j \neq x(N)_{j+1}$, and $x(N)_{J(N)} \in \{1, 2\}$. Here $x = 0, 1, 2$ replaces $X = B, A, C$, respectively.

E.g., $ABC(0) = 0, J(1) = 0$ (empty product); $ABC(30) = 02010$.

For this $ABC$-representation see A319195. Another version is A316713 (where for a technical reason $B, A, \text{ and } C$ are represented by 1, 2 and 3 (not 0, 1 and 2), respectively). See also Table 3 for $ABC(N)$, for $N = 1, 2, ..., 100$.

The number of $B$s, $A$s and $C$s in the $ABC$-representation of $N$ is given in sequences A316715, A316716 and A316717, respectively. The length of this representation is given in A3167174.

**A) From ZT(N) to N_{ABC}**

For the proof of the equivalence of these two representations $(N)_T$ and $(N)_{ABC}$ for positive integers $N$ one uses for the first part, $(N)_T \rightarrow (N)_{ABC}$, in the version $ZT(N) \rightarrow (N)_{ABC}$, the reversed word $ZT(N)$ with a concatenated 0 at the beginning and at the end. This intermediate step will be called $\overline{ZT}(N)$. E.g., $\overline{ZT}(1) = 010$ from $ZT(1) = 1$; $\overline{ZT}(30) = 00110010$ from $ZT(30) = 100110$. I.e., $\overline{ZT}(N) = 0\overline{ZT}(N)0$, with the reversed word $\overline{ZT}(N) := (ZT(N))_{\text{reversed}}$.

This simple definition of $\overline{ZT}(N)$ for given $N$ becomes somewhat complicated if a compact explicit notation is used for general $N \in \mathbb{N}$.

$$\overline{ZT}(N) = \overline{ZT}(N; P(N), j_A(N,p), j_C(N,p)) = 0H_{p=1}^{P(N)} \left( H_{k=1}^{J_A(N,p)} 0^{j_{A,k}(N,p)} 1 \right) \left( H_{k=1}^{J_C(N,p)} 0^{j_{C,k}(N,p)} 11 \right) 0. \quad (5)$$

Several explanations follow, and rules are needed to avoid the appearance of 111 in this binary word. Uniqueness requires rules for the separation between neighboring $p$-words.

**Explanations:**

1) Vanishing $p$–dependent ordered concatenation products are indicated by $J_A = 0$ or $J_C = 0$ (we omit sometimes the arguments $(N,p)$). In this case undefined products arise (which are here not set to 1, of course). Not both products are allowed to vanish for any $p$, i.e., $J_A = 0 = J_C$ is forbidden.

2) Exponents of 0 indicate the multiplicity. A vanishing exponent means disappearance of the 0s. One could use another notation like $0_{j_{A,k}}$ and $0_{j_{C,k}}$.

3) The separation of consecutive $p$-words is done such that $P(N)$ becomes minimal. For this the following two rules apply.

   i) If $J_A(N,p+1) \geq 1$ (part $A$ present for $p+1$) then $J_C(N,p) \neq 0$.

   ii) $J_C(N,p) = 0 = J_A(N,p+1)$ is forbidden.

4) To avoid the appearance of the subword 111 some rule is needed:
The exponents of 0 are collected in $\vec{j}_A$ and $\vec{j}_C$. In general they satisfy $j_{A,k} \in \mathbb{N}$ and $j_{C,k} \in \mathbb{N}$, i.e., positive powers of 0 appear. But there are exceptions for which these exponents may vanish.

The first exceptions apply to the start of the $p$–product. It may start with 1 or with 11.

**Exception 1**

i) $j_{A,1}(N,1) \in \mathbb{N}_0$

ii) $j_{C,1}(N,1) \in \mathbb{N}_0$ if $j_{A}(N,1) = 0$

The remaining exception applies for $p \geq 2$. Then one has to make sure that no 111 appears in the transition from a $p-1$ to $p$.

**Exception 2**

If $j_{C}(N,p-1) = 0$ then $j_{A,1}(N,p) \in \mathbb{N}_0$, for $2 < p < P(N)$.

Some examples may illustrate these explanations and exceptions.

**Examples 1**

1) $\widehat{ZT}(N) = 0101010^211010$. Minimal $P(N)$ (Explanation 3) is obtained with $P(N) = 2$, $j_{A}(N,1) = 3$, $j_{C}(N,1) = 1$, $j_{A}(N,2) = 1$, $j_{C}(N,2) = 0$. Here $j_{A,1}(N,1) = 0$ (exception 1)i)). E.g., the separation 01|01|0011|010 with $P = 4$ is forbidden. For this $j_{C}(N,1) = 0$, $j_{A}(N,2) = 2$ and $j_{C}(N,2) = 0 = j_{A}(N,3)$. The last separation is the only one for the given minimal $P(N)$ solution.

2) $\widehat{ZT}(N) = 010^210^211011010$. This is an instance $P(N) = 2$, $j_{A}(N,1) = 2$, $j_{C}(N,1) = 2$, $j_{C}(N,2) = 0$ and

$$
\vec{j}_A(N,1) = (0, 2), \vec{j}_C(N,1) = (2, 1), \vec{j}_A(N,2) = (1).
$$

Here the minimal $P$ is 2, and exception 1)i) (start with 1) applies. Also explanation 1) is needed because for $p = 2$ part $C$ is missing but not part $A$. The corresponding $ZT(N)$ is 1011011001001, with $I(N) = 12$, and $N = 1705 + 504 + 274 + 81 + 44 + 7 + 1 = 2616$.

3) $\widehat{ZT}(N) = 0110^211010$. Here exception 1)ii) (start with 11) occurs and $P(N) = 2$ with $j_{A}(N,1) = 0$ (explanation 1), $j_{C}(N,1) = 2$ and $j_{A}(N,2) = 1$, $j_{C}(N,2) = 0$ (explanation 1). $ZT(N)$ is 101100011, with $I(N) = 7$, and $N = 81 + 24 + 13 + 2 + 1 = 121$.

The translation from $\widehat{ZT}(N)$ to $(N)_{ABC}$ is now performed, in an intermediate step introducing two new symbols and $\bullet$ and $\times$ with the help of four substitution rules in the word $w(N)$, used here as abbreviation for $w(N) := \widehat{ZT}(N) = H_{i=1}^{\#w(N)} w(N)_i$ with $\#w(N) = I(N) + 3 = A278044(N) + 2$. This intermediate representation will be denoted by $(N)_{AB\times}$. The following rules depend on the neighbors of $w(N)_i$ for $i = 1, 2, \ldots, \#w(N) - 1$. To mark the position $i$, the number (letter) $w(N)_i$ to be substituted is given in the rules in boldface and underlining. $w(N)_1$ has no left neighbor denoted in the following by $\emptyset$. This $\emptyset$ is also used to signal the end of each word $w(N)$ after 10.

**The four substitution rules**

\begin{align*}
(S1) & \quad 00 & \rightarrow & \bullet & \text{and} & \quad x00 & \rightarrow & B, & \text{for } x \in \{\emptyset, 0\}, \\
(S2) & \quad 011 & \rightarrow & \times & \text{and} & \quad 010 & \rightarrow & A, \\
(S3) & \quad 11 & \rightarrow & \times, \\
(S4) & \quad 101 & \rightarrow & \bullet & \text{and} & \quad 10x & \rightarrow & B, & \text{for } x \in \{\emptyset, 0\}. (6)
\end{align*}
These rules suffice and are not in conflict which each other. Eg. 110 is not needed because if the word ends in the numbers 110 then rule (S4), part two, with \( x = \emptyset \), applies for the substitution of the last 1 becoming a B. Otherwise it is either 1100 or 1101 in which case also (S4) applies either with part two and \( x = 0 \) or with part one.

E.g., \( w(N) = 0101010011010 \) with \( \#w = 13 \) translates to \( (N)_{AB\times} = A\bullet A\bullet A B\bullet \times \times \bullet AB \) with length \( \#w - 1 = 12 \).

\( w(N) \) ends always in 10. This last substitution of 1 uses part two of rule (S4) with \( x = \emptyset \).

In the final step the translation into \( (N)_{ABC} \) is obtained by omitting all \( \bullet \)s and substituting \( \times \rightarrow C \). This reduces the length to \( \text{A316714}(N) \), the one of \( (N)_{ABC} \).

The preceding example thus gives \( (N)_{ABC} = A^3BCAB \) which represents \( N = 752 \) corresponding to the given \( ZT(752) = 10110010101 = 504 + 149 + 81 + 13 + 4 + 1 \).

\begin{figure}

\centering

\includegraphics[width=\textwidth]{figure2.png}

\caption{ABC-representation with the ABCTree_5}

\end{figure}

In Figure 2 the tribonacci tree \( TTree_5 \) from Figure 1 has been used with labeling the edges (branches) with symbols \( A, B, C, \bullet, \times \) in a special way. It is called \( ABCTree_5 \). The new branch decorated infinite tree is denoted by \( ABCTree \).

The \( ABC \)-representation of \( N \) is obtained directly by reading the branches from bottom to top. If there are two edge labels like \( A \) and \( \times \), or \( B \) and \( \bullet \), for an edge going out from from a node, the choice is fixed from the direction from which the previous (lower) edge reached the node. If it reached the node from the right-hand side, the label on the right-hand side of the outgoing edge has to be chosen, and similarly for the left-hand side. If one considers a finite \( ABCTree_n \) with levels \( l = 0, 1, ..., n \) having no incoming edges from the next level \( l = n + 1 \), one chooses always the left variant for the outgoing edges from nodes of the last level \( l = n \) of \( ABCTree_n \). The \( ABC \)-representation ends always in \( AB(0) \) or \( CB(0) \) which means that coming from the left subtree one stops after reaching the first node on the outermost branch with only \( B \) edges. Only in the right subtree one has to go all the way up
to node 2 at level \( l = 0 \). The \( N = 0 \) case is not considered in the equivalence proof, but the tree shows that \( N = 0 \) would be represented by \( B = B(0) \) (Figure 2 the \( B \) emerging from the first node labeled 0 at level \( l = 5 \)).

E.g., for \( N = 8 \) one has from \( ABCTree_5 \) the edge labels from bottom to top \( A \bullet B A B \) corresponding to \( ABAB = A(B(A(B(0)))) = 8 \). Here one sees why the tree started with node 2 at level \( l = 0 \). For \( N = 6 \) the path is \( B \times \times B \rightarrow BCB = B(C(B(0))) \) ending at node 0 at level \( l \).

Note that if one adds a level \( n + 1 \) to \( ABCTree_n \), thus obtaining \( ABCTree_{n+1} \), the first numbers \( N = 0, 1, ..., T(3 + l) - 1 \) related to level \( l + 1 \) of the left subtree \( TTree_{L_{l+1}} \) have the same \( ABC \)-representations like the those compiled starting from level \( n \) of \( ABSTree_n \). This is because one stays in the left subtree \( ABSTree_{L_{l+1}} \) and one reaches at most the 0 from level 1.

B) From \((N)_{ABC}\) to \(ZT(N)\)

The reverse part of the equivalence proof starts with the representation \((N)_{ABC}\) eq. 3, and constructs \(ZT(N)\) eq. 5. After erasing the 0 at the beginning and end, and reversing the remaining word one obtains the binary word \(ZT(N)\) eq. 2 and from this \((N)_{T}\) eq. 1.

The first task is to find the intermediate \((N)_{AB\bullet x}\) version from \((N)_{ABC}\). For this one derives from the substitution rules eq. 6 how \( A, B \) and \( C \) can appear. \( A \) is reached uniquely from \( 010 \). For \( B \) one has to distinguish two types, called \( BI \) and \( BII \). \( BI \) originates from a substituted \( 0 \) either at the start from \( 000 \) or from \( 000 \). \( BII \) originates from a substituted \( 1 \) either at the end from \( 100 \) or from \( 100 \). Finally, \( C \), represented by \( \times \times \), originates from substituting 0 in \( 011 \) leading to \( \times \), and the following substitution for 1 produces the second \( \times \). Note that \( \times \times \) obtained from substituting 11 would need in fact 111 which is forbidden. Therefore \( C \) can appear only from a 0110 string starting substitutions with the first 0.

Consider now \((N)_{ABC}\) from eq. 3. It turns out that the transition between the blocks of powers of \( A, B \) and \( C \) is important in order to find out the correct \((N)_{AB\bullet x}\) representation. The final \( B(0) \) in \( N_{ABC} \) will only at the end be added as a final \( B \). There is never a final \( B \)-block for \( j = J(N) \) in eq. 3 from the uniqueness requirement of the representation. The following statements then follow.

**Step 1 replacements**

**Step1A** A block \( A^n \), for \( n \in \mathbb{N} \), \((i.e., X(N)_j = A, k(N)_j = n \) in eq. 3), appearing alone \((J(N) = 1)\) or at the end \((j = J(N))\) or if followed by a \( B \)-block is replaced by \((A\bullet)^{n-1}A\) (remember that \((A\bullet)^0\) means disappearance). The \( B \) following an \( A^n \)-block is always of type \( BII \) (in the cases \( J(N) = 1 \) or \( j = J(N) \) this means that last omitted \( B \) is of type \( BII \)).

If the \( A^n \)-block is followed by a \( C \)-block then it is replaced by \((A\bullet)^n\).

**Step1B** A block \( B^n \), for \( n \in \mathbb{N} \), which can never appear alone, stays \( B^n \) if it begins with a \( B \) of type \( BI \) (especially if \( X(N)_1 = B \)). If the block \( B^n \) begins with a \( B \) of type \( BII \) then \( B^n \) is replaced by \( B\bullet B^{n-1}. \)

**Step1C** A block \( C^n \) followed by an \( A \)-block is replaced by \((\bullet \times \times)^n \). If \( C^n \) is followed by a \( B \)-block starting with a \( B \) of type \( BII \) then it is replaced by \( \times \times \). This applies also if
a C-block appears alone \((J(N) = 1)\). A C-block is never followed by a B-block beginning with a B of type BI.

(7)

In order to obtain the \((N)_{AB\times}\) representation one adds after these **Step1 replacements** the final B. Some examples are in order:

**Examples 2**

1) \(N_{ABC} = B^3AB\). The starting \(B^3\) remains \(B^3\) because the first B is of type I (it comes from 0(00)). Because the \(A^1\) (the last block) is followed by a B (always type II) it remains an A. After appending the omitted last B one obtains \((N)_{AB\times} = B B B A B, i.e., here no \bullet\) appears.

2) \(N_{ABC} = A^3BCAB\). The starting A-block is replaced via **Step1A** by \((A\bullet)^2A\). The following block \(B^1\) is replaced by \(B\bullet\) because the B after an A is always of type II. The next block \(C^1\) followed by the last A-block \(A^1\) is replaced by \(\bullet \times \times\) The last A remains an A. After adding the final B one obtains \((N)_{AB\times} = A \bullet A \bullet A B \bullet \times \times \bullet A B\). This is the representation found above in part A) from \(ZT(752)\)

The translation from \((N)_{AB\times}\) to \(\hat{ZT}(N)\) is simply done by starting with an extra 0 and appending the \((N)_{AB\times}\) string by replacing \(A \rightarrow 1, B \rightarrow 0, \bullet \rightarrow 0\) and \(\times \rightarrow 1\).

In the example 1) this produces \(\hat{ZT}(N) = 000010\). The example 2) gives 0101010011010.

The final translation from \(\hat{ZT}(N)\) to \(ZT(N)\) is then trivial: omit the two boundary 0s and reverse the remaining binary string.

The two examples give: 1) 00011 = 1000, which is \(ZT(7)\), and 2) 101010011011 = 10110010101 which is \(ZT(752)\). This was used above as start of the example for the proof in the other direction.

## 2 Equivalence of representations \(ZT(N)\) and \(ABC(N)\)

First the uniqueness of the tribonacci-representation \(ZT(N)\) of eq. 2 is considered.

It is clear that every binary sequence starting with 1, without three consecutive 1s, represents some \(N \in N\). An algorithm for finding such a representation for every \(N \in N\) is given to prove the following lemma.

**Lemma 1.** The tribonacci-representation \(ZT(N)\) of eq.2 is unique.

**Proof:**

The recurrence of the tribonacci sequence \(T := \{T(l)\}_{l=3}^\infty\), with inputs \(T(3) = 1, T(4) = 2\) and \(T(5) = 4\), shows that this sequence is strictly increasing. Define the floor function 
\(floor(T; n)\), for \(n \in N\), giving the largest member of \(T\) smaller or equal to \(n\). The corresponding index of \(T\) will then be called \(Ind(floor(T; n))\). Define the finite sequence 
\(Nseq := \{N_j\}_{j=1}^{j_{\text{max}}}\) recursively by

\[
N_j = N_{j-1} - floor(T; N_{j-1}), \quad \text{for } j = 1, 2, ..., j_{\text{max}},
\]  

(8)
with \( N_0 = N \) and \( N_{j_{\text{max}}} = 0 \).

It is clear that this recurrence reaches always 0. Define the finite sequences \( fTN := \{\text{floor}(T; N_j)\}_{j=0}^{j_{\text{max}}-1} \) and \( I fTN := \{\text{Ind}(fTN_j)\}_{j=0}^{j_{\text{max}}-1} \). Then \( I(N) \) in eq. 2 is given by

\[
I(N) = I fTN_0 \quad \text{and the finite sequence } fseq = \{fI(N) - k\}_{k=0}^{j_{I(N)}} \text{ is given by}
\]

\[
fI(N) - k = \begin{cases} 
1 & \text{if } I(N) - k \in I fTN; \\
0 & \text{otherwise}.
\end{cases}
\]  

(9)

\[\square\]

**Example 3** \( N = 263 \). \( Nseq = \{263, 144, 33, 92, 0\} \), \( fTN = \{149, 81, 24, 7, 2\} \), \( I fTN = \{8, 7, 5, 3, 1\} \), \( I(N) = 8 \), \( fseq = \{1, 1, 0, 1, 0, 1, 0, 1\} \).

Next follows the lemma on the uniqueness of the \( ABC \)-representation given in eq. 3.

**Lemma 2.** The tribonacci \( ABC \)-representation \( (N)_{ABC} \) of eq. 3, for \( N \in \mathbb{N}_0 \), is unique.

**Proof:**

From the definition of the \( A-, B- \) and \( C-\)sequences (each with offset 0) based on the value 1, 0 and 2, respectively, of \( t(n) \), for \( n \in \mathbb{N}_0 \), it is clear that these sequences are disjoint and \( \mathbb{N}_0 \)-complementary. 0 is represented by \( B(0) \). Therefore the \( n \)-fold iteration \( B^{[n]}(0) \) (written as \( B^n(0) \)) is allowed only for \( n = 1 \), and any representation ends in \( B(0) \). Iterations acting on 0 are encoded by words over the alphabet \( \{A, B, C\} \), and \( n \)-fold repetition of a letter \( X \) is written as \( X^n \), named \( X \)-block, where \( n = 0 \) means that no such \( X \)-block is present. Then any word consisting of consecutive different non-vanishing \( X \)-blocks ending in the \( B \)-block \( B^1 \) represents a number \( N \in \mathbb{N}_0 \).

In order to prove that with such representations every \( N \in \mathbb{N}_0 \) is reached the following algorithm is used. Replace any number \( n \in \mathbb{N}_0 \), which is \( n = X_n(k) \) with \( X_n \in \{A, B, C\} \) and \( k \in \mathbb{N}_0 \), by the 2-list \( L(n) = [L(n)_1, L(n)_2] := [X_n, k(n)] \). Define the recurrence

\[
L(j) = [L(L(j - 1)_2)_1, L(L(j - 1)_2)_2], \quad \text{for } j = 1, 2, \ldots, j_{\text{max}},
\]  

(10)

with input \( L(0) = [X_n, k(N)] \), and \( j_{\text{max}} \) is defined by \( L(j_{\text{max}}) = [B, 0] \).

Then the word is \( w(N) = H_{j=0}^{j_{\text{max}}} L(j)_1 \) (a concatenation product), and read as iterations acting on 0 this becomes the representation \( (N)_{ABC} \). The length of the word \( w(N) \) is \( j_{\text{max}} + 1 \).

\[\square\]

**Example 4** \( N = 38 \). \( L(0) = [A, 11] \), \( L(1) = [B, 6] \), \( L(2) = [B, 3] \), \( L(3) = [C, 0] \), and \( L(4) = [B, 0] \); hence \( j_{\text{max}}(38) = 4 \), \( w(38) = ABBCB \), and \( (38)_{ABC} = ABBCB(0) \), to be read as \( A(B(B(C(B(0)))))) \).

After these preliminaries the main theorem can be stated.

**Theorem.** The tribonacci-representation \( ZT(N) \) of eq. 1, is equivalent to the tribonacci \( ABC \)-representation \( (N)_{ABC} \) eq. 3, for \( N \in \mathbb{N} \).
Proof:
Part A): The proof of the map \( ZT(N) \rightarrow (N)_{ABC} \) is performed in three steps:

\[
\begin{align*}
\text{Step 1:} & \quad ZT(N) \rightarrow \widehat{ZT}(N) := 0(ZT(N)_{\text{reverse}})0, \\
\text{Step 2:} & \quad \widehat{ZT}(N) \rightarrow (N)_{AB\bullet\times}, \\
\text{Step 3:} & \quad (N)_{AB\bullet\times} \rightarrow (N)_{ABC}.
\end{align*}
\]

(11)

Step 1 is clear.
For Step 2 one uses eq. 5 and the Explanations 1) to 4) with Exception 1) and 2). See also Example 1. The four substitution rules \((S1), (S2), (S3)\) and \((S4)\) of eq. 6 are then applied to obtain \((N)_{AB\bullet\times}\). See also the example for \(N = 752\) there.
In Step 3 the symbols \(\bullet\) in \((N)_{AB\bullet\times}\) are omitted and the pair of symbols \(\times\times\) (\(\times\) always appears as a pair) is replaced by \(C\).

Part B): The proof of the map \((N)_{ABC} \rightarrow ZT(N)\) is performed also in three steps:

\[
\begin{align*}
\text{Step 1:} & \quad (N)_{ABC} \rightarrow (N)_{AB\bullet\times}, \\
\text{Step 2:} & \quad (N)_{AB\bullet\times} \rightarrow \widehat{ZT}(N), \\
\text{Step 3:} & \quad \widehat{ZT}(N) \rightarrow ZT(N).
\end{align*}
\]

(12)

Step 1 is a bit tricky. The representation \((N)_{ABC}\) of eq. 3 without the final \(B(0)\) consists of blocks of powers of \(A\), \(B\) or \(C\) with the restriction that a \(B\)-block never appears alone or at the end (because \(B^{n+1}(0) = 0\), for \(n \in \mathbb{N}\), the uniqueness of the representation would be violated). Then the \textbf{Step 1 replacements} of eq. 7 are applied to the \(A\)-, \(B\)-, and \(C\)-blocks, called there \textit{Step1A}, \textit{Step1B} and \textit{Step1C}. The omitted final \(B\) is again appended. See also Example 2.
In \textit{Step 2} the replacements \(A \rightarrow 1\), \(B \rightarrow 0\), \(\bullet \rightarrow 0\) and \(\times \rightarrow 1\) are applied and an extra 0 is added at the beginning of the thus obtained binary string. This is \(\widehat{ZT}(N)\).

\textit{Step 3} is trivial: omit the two bordering 0s of \(\widehat{ZT}(N)\) and reverse the binary string to obtain \(ZT(N)\).
\(\square\)

3 Investigation of the A-, B- and C- sequences

In this section a detailed investigation of the \(A\)-, \(B\)- and \(C\)- sequences is presented.
The starting point is the infinite tribonacci word \(T\text{Word}\), written as a sequence \(t = \text{A080843}\).
Its self-similarity leads to the following definitions and lemmata.

**Definition 3.** The tribonacci words \(tw(l)\) over the alphabet \(\{0, 1, 2\}\) of length \#\(tw(l) = T(l + 2)\) are defined recursively by concatenations (we omit the concatenation symbol \(\circ\)) as

\[
tw(l) = tw(l - 1) tw(l - 2) tw(l - 3), \quad \text{with} \quad tw(1) = 0, tw(2) = 01, tw(3) = 0102. \quad (13)
\]
Also \(tw(0) = 2\) is used.

The substitution map acting on tribonacci words and other strings with characters \(\{0, 1, 2\}\) is defined as a concatenation homomorphism by \(\sigma : 0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0\). The inverse map is \(\sigma^{-1}\) (One replaces first each 01 and 02 then the left over 0). With \(\sigma\) the words \(tw(l)\) are generated iteratively from \(tw(0) = 2, \sigma(tw(l)) = tw(l + 1)\), for \(l \in \mathbb{N}_0\), and \(\lim_{l \to \infty} \sigma^l(0) = TWord\). Self-similarity of \(TWord\) means \(\sigma(TWord) = TWord\).

Substrings of \(TWord\) of length \(n\), starting with the first letter (number) \(t(0) = 0\), are denoted by \(s_n := \mathbb{H}^{n-1}_j t(n)\). If \(n = T(l + 2)\), for \(l \in \mathbb{N}_0\), then \(s_n = tw(l)\) (the string becomes a tribonacci word), and the numbers of \(s_n\) map to the node labels of the last level of \(TTree_i\) read from the left-hand side.

Also substrings of \(TWord\) not starting with \(t(0)\) are used, like \(\hat{s}_2 = 02 = \sigma(1)\), starting with \(t(2)\).

**Lemma 4.**

**A)** With \(s_{13} = 0102010010201 = tw(5)\), \(s_{11} = 01020100102\) and \(s_7 = 0102010 = tw(4)\) define

\[
t_1 = s_{13}s_{11}s_{13}s_{11}s_{13}s_{11}s_{13}s_{13}s_{13}s_{13}s_{13}s_{13}s_{13}... = \mathbb{H}_j^{\infty} s_{\varepsilon(t(j))},
\]

where \(\varepsilon(0) = 13, \varepsilon(1) = 11\) and \(\varepsilon(2) = 7\).

**B)** With \(s_7 = 0102010 = tw(4)\), \(s_6 = 010201\) and \(s_4 = 0102 = tw(3)\) define

\[
t_2 = s_7s_6s_7s_4s_7s_6s_7s_6s_7s_7s_4s_7... = \mathbb{H}_j^{\infty} s_{\pi(t(j))},
\]

where \(\pi(0) = 7, \pi(1) = 6\) and \(\pi(2) = 4\).

**C)** With \(s_4 = 0102 = tw(3)\), \(s_3 = 010\) and \(s_2 = 01 = tw(2) = \sigma(0)\) define

\[
t_3 = s_4s_3s_4s_2s_4s_3s_4s_4s_3s_4s_2s_4s_4... = \mathbb{H}_j^{\infty} s_{\tau(t(j))},
\]

where \(\tau(0) = 4, \tau(1) = 3\) and \(\tau(2) = 2\).

**D)** With \(s_2 = 01, \hat{s}_2 = 02\) and \(s_1 = 0 = tw(1) = \sigma(2)\) define

\[
t_4 = s_2\hat{s}_2s_2s_1s_2\hat{s}_2s_2s_2s_2s_2s_1...
\]

Here the string follows \(t\) with \(s_2, \hat{s}_2\) and \(s_1\) playing the rôle of 0, 1 and 2, respectively. Then

\[
t_1 = t_2 = t_3 = t_4 = TWord.
\]

**Proof:**

**D:** The definition of \(\sigma^{-1}\) shows that \(\sigma^{-1}(t_4) = TWord\). Hence \(t_4 = \sigma(TWord) = TWord\).

**C:** Because \(\sigma(s_2) = s_4, \hat{\sigma}(s_2) = s_3\) and \(\sigma(s_1) = s_2\) it follows that \(t_3 = \sigma(t_4) = TWord\).

**B:** Because \(\sigma(s_4) = s_7, \sigma(s_3) = s_6\) and \(\sigma(s_2) = s_4\) it follows that \(t_2 = \sigma(t_3) = TWord\).

**A:** Because \(\sigma(s_7) = s_{13}, \sigma(s_6) = s_{11}\) and \(\sigma(s_4) = s_7\) it follows that \(t_1 = \sigma(t_2) = TWord\).

\(\square\)

Using eq.16 a formula for sequence entry \(A(n) = A_{278040}(n)\) in terms of \(z(n) := \sum_{j=0}^{n} t(j)\) is derived. This sequence \(\{z(j)\}_{j=0}^{\infty}\) is given in \(A_{319198}\).
Proposition 5.

\[ A(n) = 4n + 1 - z(n-1), \text{ for } n \in \mathbb{N}_0, \text{ with } z(-1) = 0. \]  \hspace{1cm} (19)

Proof:
Define \( \Delta A(k+1) := A(k+1) - A(k) \). Consider the word \( t_2 \) of eq. 16. The distances between the 1s in the pairs \( s_4 s_3, s_3 s_4, s_4 s_2, s_2 s_4 \) and \( s_4 s_4 \) are 4, 3, 4, 2, 4. Therefore, the sequence of these distances is 4, 3, 4, 2, 4, 3, 4, 3, 4, 2, \ldots. Thus, because the \( s \)-string \( t_2 \) follows the pattern of \( t \), i.e., of \( T \text{word} \),

\[ \Delta A(k+1) = 4 - t(k), \text{ for } k = 0, 1, \ldots. \]  \hspace{1cm} (20)

Then the telescopic sum produces the assertion, using \( A(0) = 1 \).

\[ A(n) = A(0) + \sum_{k=0}^{n-1} \Delta A(k+1) = 1 + 4n - z(n-1), \text{ with } z(-1) = 0. \]  \hspace{1cm} (21)

The \( B \)-numbers \( A278039 \), giving the increasing indices \( k \) with \( t(k) = 0 \), come in three types: \( B0 \)-numbers form the sequence of increasing indices \( k \) of sequence \( t \) with \( t(k) = 0 = t(k+1) \). Similarly the \( B1 \)-sequence lists the increasing indices \( k \) with \( t(k) = 0, t(k+1) = 1 \) and for the \( B2 \)-sequence the indices \( k \) are such that \( t(k) = 0, t(k+1) = 2 \).

These numbers \( B0(n), B1(n) \) and \( B2(n) \) are given by \( A319968(n+1), A278040(n) - 1, \) and \( A278041(n) - 1 \), respectively.

Before giving proofs we define the counting sequences \( z_A(n), z_B(n) \) and \( z_C(n) \) to be the numbers of \( A, B \) and \( C \) numbers not exceeding \( n \in \mathbb{N} \), respectively. If these counting functions appear for \( n = -1 \) they are set to 0.

These sequences are given by \( A276797(n+1), A276796(n+1) \) and \( A276798(n+1) - 1 \) for \( n \geq -1 \).

Obviously,

\[ z(n) = z_A(n) + 0 z_B(n) + 2 z_C(n) = z_A(n) + 2 z_C(n), \text{ for } n = -1, 0, 1, \ldots. \]  \hspace{1cm} (22)

These counting functions are obtained by partial sums of the corresponding characteristic sequences for the \( A\), \( B\) and \( C\) numbers (or \( 0-, 1-, \) and \( 2- \) numbers in \( t \)), called \( k_A, k_B \) and \( k_C \), respectively.

\[ z_X(n) = \sum_{k=0}^{n} k_X(k), \text{ for } X \in \{ A, B, C \}. \]  \hspace{1cm} (23)

The characteristic sequences members \( k_A(n), k_B(n) \) and \( k_C(n) \) are given in \( A276794(n+1), A276793(n+1) \) and \( A276791(n+1) \), for \( n \in \mathbb{N}_0 \), and they are, in terms of \( t \), obviously given by

\[ k_A(n) = t(n)(2 - t(n)), \]  \hspace{1cm} (24)

\[ k_B(n) = \frac{1}{2}(t(n) - 1)(t(n) - 2), \]  \hspace{1cm} (25)

\[ k_C(n) = \frac{1}{2}t(n)(t(n) - 1). \]  \hspace{1cm} (26)
By definition it is trivial that (note the offset 0 of the $A$, $B$, $C$ sequences)
\[
z_X(X(k)) = k + 1, \text{ for } X \in \{A, B, C\} \text{ and } k \in \mathbb{N}.
\] (27)

**Proposition 6.**

For $n \in \mathbb{N}_0$:

- **B0)** $B_0(n) = 13n + 6 - 2[z_A(n-1) + 3z_C(n-1)] = 2C(n) - n$, \hspace{1cm} (28)
- **B1)** $B_1(n) = 4n - z(n-1) = 4n - [z_A(n-1) + 2z_C(n-1)] = A(n) - 1$, \hspace{1cm} (29)
- **B2)** $B_2(n) = 7n + 2 - [z_A(n-1) + 3z_C(n-1)] = \frac{1}{2} (B_0(n) + n - 2) = C(n) - 1$, \hspace{1cm} (30)
- **B)** $B(n) = 2n - zC(n-1)$. \hspace{1cm} (31)

**Proof:**

**B0:** Part 1: Define $\triangle B_0(k+1) := B_0(k+1) - B_0(k)$ and consider the word $t_1$ of eq. 14. The distances between pairs of 00 in $s_{13}s_{11}$, $s_{11}s_{13}$, $s_{13}s_7$, $s_7s_{13}$ and $s_{13}s_{13}$ are 13, 11, 13, 7, 13. Note that $S_7$ has no substring 00, however because $S_7$ is always followed by $S_{13}$ the last 0 of $s_7$ and the first of $s_{13}$ build the 00 pair. Similarly, in the $s_{13}s_7$ case the last 0 of $s_7$ is counted as a beginning of a 00 pair. Therefore, the sequence of these distances is 13, 11, 13, 7, 13, 11, 13, 13, 11, 13, 7, ... Because the $s$-string $t_1$ follows the pattern of $t$ the defect from 13 is 0, -2, -6 if $t(k) = 0, 1, 2$, hence
\[
\triangle B_0(k+1) = 13 - t(k) (t(k) + 1), \text{ for } k \in \mathbb{N}_0.
\] (32)

The telescopic sum gives, with $B_0(0) = 6$,
\[
B_0(n+1) = B_0(0) + \sum_{k=0}^{n} \triangle B_0(k+1)
= 6 + 13 (n + 1) - [(1^2 z_A(n)) + 2^2 z_C(n)) + z(n)]
= 13n + 19 - 2(z_A(n) + 3z_C(n)).
\] (33)

In the last step $z(n)$ has been replaced by eq. 22. Substituting $n \to n - 1$ proves the first part of **B0**. The proof of part 2 follows later from **B2**.

**B1:** With $\triangle B_1(k+1) := B_1(k+1) - B_1(k)$ and $t_2$ of eq. 15 one finds for the distances between consecutive 1s similar to the above argument
\[
\triangle B_1(k+1) = 4 - t(k), \text{ for } k \in \mathbb{N}_0.
\] (34)

The telescopic sum gives, with $B_1(0) = 0$,
\[
B_1(n+1) = 4(n + 1) - z(n),
\] (35)

and with $n \to n - 1$ this becomes the first part of **B1**, which shows, with eq 19, also the third one. The second part uses eq. 22.
Note that $B1(n) = A(n) - 1$ is trivial because 1 in the tribonacci word $TWord$ can only come from the substitution $\sigma(0) = 01$, and $TWord$ (and $t$) starts with 0. Therefore, one could directly prove $B1$ from eqs. 19 and 22 without first computing $\triangle B1(k + 1)$.

**B2:** Because 2 in $TWord$ appears only from $\sigma(1) = 02$, it is clear that $B2(n) = C(n) - 1$.

Now one finds a formula for $C$ by looking first at $\triangle C(k + 1) := C(k + 1) - C(k)$ using again $t_2$ of eq. 15. The distances between consecutive 2s in the five pairs $s_7s_6$, $s_6s_7$, $s_7s_4$, $s_4s_7$ and $s_7s_7$ is 7, 6, 7, 4, 7, respectively, and

$$\triangle C(k + 1) = 7 - \frac{1}{2} t(k)(t(k) + 1), \text{ for } k \in \mathbb{N}_0. \quad (36)$$

The telescopic sum leads here, using $C(0) = 3$, $z(n)$ from eq. 22 and letting $n \to n - 1$, to

$$C(n) = 7n + 3 - [z_A(n - 1) + 3 z_C(n - 1)], \text{ for } k \in \mathbb{N}_0. \quad (37)$$

This proves $B2$, and also the second part of $B0$.

**B:** Here $t_4$ of eq. 17 can be used. The differences of 0s in the five pairs $s_2\hat{s}_2$, $s_2s_1$, $s_1s_2$ and $s_2s_2$ is 2, 2, 2, 1, 2. Thus

$$\triangle B(k + 1) := B(k + 1) - B(k) = 2 - \frac{1}{2} t(k)(t(k) - 1) = 2 - k_C(n), \text{ for } k \in \mathbb{N}_0. \quad (38)$$

In the last step $k_C$ from eq. 26 has been used. By telescoping, using $B(0) = 0$, eliminating $z(n - 1)$ with eq. 19, and letting $n \to n - 1$, proves the assertion. □

Eqs. 36 and 38 show that $\triangle C(k + 1) - \triangle B(k + 1) = 5 - t(k)$, for $k \in \mathbb{N}_0$. Telescoping leads to the result, obtained directly from eqs. 37 and 31, with eq. 22,

$$C(n) - B(n) = 5n + 3 - z(n - 1), \text{ for } k \in \mathbb{N}_0, \quad (39)$$

and with $A$ from eq. 19 this becomes

$$C(n) - (A(n) + B(n)) = n + 2, \text{ for } k \in \mathbb{N}_0. \quad (40)$$

This equation can be used to eliminate $C$ from the equations.

Next the formulae for $z_X$ for $X \in \{A, B, C\}$ are listed, valid for $n = -1, 0, 1, ...$

**Proposition 7.**

$$z_A(n) = 2B(n + 1) - A(n + 1) + 1, \quad (41)$$

$$z_B(n) = A(n + 1) - B(n + 1) - (n + 2), \quad (42)$$

$$z_C(n) = 2(n + 1) - B(n + 1). \quad (43)$$

**Proof:** Version 1. The inputs $z_X(-1) = 0$, for $X \in \{A, B, C\}$, follow from eqs. 19 and 31. The first differences $\triangle z_X(n) := z_X(n) - z_X(n - 1)$ produce with the claimed formulae, and $\triangle A(n + 1)$ and $\triangle B(n + 1)$ from eqs. 20 and 38, the trivial results given in eqs. 24 to 26. Therefore $z_X(n)$ from eq. 23 holds.
Version 2. Besides eq. \text{22} the trivial formula

\[ z_A(n) + z_B(n) + z_C(n) = n + 1 \]  \hspace{1cm}(44)

can be used.

\( z_A(n) \) is computed from the difference of \( 3(z_A(n-1) + 2z_C(n-1)) \) from eq. \text{30}, with \( C(n) \) from eq. \text{40}, and \( 2(z_A(n-1) + 3z_C(n-1)) \) from eq. \text{29}. This difference leads to the claim eq. \text{41}.

\[ 2z_C(n) = -A(n+1) + 4n + 5 - z_A(n) \]  \hspace{1cm} \text{from eq. \text{29}. Inserting the proven } z_A(n) \text{ formula}

leads to the claim eq. \text{43}.

\( z_B(n) \) can then be computed from eq. \text{44}.

Finally all formulae for compositions of the types \( X(Y(k)+1) \) and \( X(Y(k)) \), for \( X, Y \in \{A, B, C\} \) and \( k \in \mathbb{N}_0 \) shall be given. They are of interest in connection with the tribonacci \( ABC \)-representation given in the preceding section. For this one needs first the results for the compositions \( z(X(k)) \). The formulae will be given in terms of \( A \) and \( B \) (with \( C \) eliminated by eq. \text{40}).

Proposition 8.

\[ z(A(k)) = 2(A(k) - B(k)) - k - 1, \]  \hspace{1cm}(45)

\[ z(B(k)) = -A(k) + 3B(k) - k + 1, \]  \hspace{1cm}(46)

\[ z(C(k)) = B(k) + 2k + 3. \]  \hspace{1cm}(47)

\textbf{Proof:} \( z(X(k)) \) will be found from the self-similarity properties given in eqs. \text{16, 17 and 15}, for \( X = A, B \) and \( C \), respectively. These strings \( t_3, t_4 \) and \( t_2 \) are chosen because the relevant numbers 1, 0 and 2, respectively, appear precisely once in all \( s \)-substrings. For \( z(X(k)) = \sum_{j=0}^{X(k)} t(j) \) one has to sum all the numbers of the first \( k \) substrings \( s \) but in the last one only the numbers up to the number standing for \( X \) are summed.

A) In the \( t_3 \) substrings \( s_4 = 0102, s_3 = 010 \) and \( s_2 = 01 \) the number 1 appears just once. In all three substrings the sum up to the relevant number 1 (for \( A \)) is \( 0 + 1 = 1 \), so for the last \( s \) one has always to add 1. Because \( s_4, s_3 \) and \( s_2 \), with sums 3, 1 and 1, play the rôle of 0, 1 and 2, respectively, in \( t_3 \) one obtains \( z(A(k)) = 3z_B(k-1) + 1(z_A(k-1) + z_C(k-1)) + 1 \). With the identity eq. \text{44} this becomes \( 2z_B(k-1) + k + 1 \), and with the \( z_B \) formula eq. \text{42} this leads to the claim eq. \text{45}.

B) In \( t_4 \) the sums of the substrings \( s_2, s_1 \) are 1, 2, 0, respectively, and because all three begin with the relevant number 0 nothing to be summed for the last \( s \). Thus \( z(B(k)) = 1z_B(k-1) + 2z_A(k-1) + 0 + 0 \). Using eqs. \text{42} and \text{41} this becomes the claim.

C) In \( t_2 \) the sums are 4 for \( s_7, s_6 \) and 3 for \( s_4 \). The sums up to the relevant number 2 are 3 for each case. Therefore \( z(C(k)) = 4(z_B(k-1) + z_A(k-1)) + 3z_C(k-1) + 3 = z_B(k-1) + z_A(k-1) + 3k + 3 = B(k) + 2k + 3 \), with eqs. \text{44, 42 and 41}. \hspace{1cm} \square

Proposition 9.

\[ A(A(k)+1) = 2(A(k) + B(k)) + k + 6, \quad A(A(k)) = A(A(k)+1) - 3, \]  \hspace{1cm}(48)

\[ A(B(k)+1) = A(k) + B(k) + k + 4, \quad A(B(k)) = A(B(k)+1) - 4, \]  \hspace{1cm}(49)

\[ A(C(k)+1) = 4A(k) + 3B(k) + 2(k + 5), \quad A(C(k)) = A(C(k)+1) - 2. \]  \hspace{1cm}(50)
\[
B(A(k) + 1) = A(k) + B(k) + k + 3, \quad B(A(k)) = B(A(k) + 1) - 2, \quad (51)
\]
\[
B(B(k) + 1) = A(k) + 1, \quad B(B(k)) = B(B(k) + 1) - 2, \quad (52)
\]
\[
B(C(k) + 1) = 2(A(k) + B(k)) + k + 5, \quad B(C(k)) = B(C(k) + 1) - 1. \quad (53)
\]
\[
C(A(k) + 1) = 4A(k) + 3B(k) + 2(k + 6), \quad C(A(k)) = C(A(k) + 1) - 6, \quad (54)
\]
\[
C(B(k) + 1) = 2(A(k) + B(k)) + k + 8, \quad C(B(k)) = C(B(k) + 1) - 7, \quad (55)
\]
\[
C(C(k) + 1) = 7A(k) + 6B(k) + 4(k + 5), \quad C(C(k)) = C(C(k) + 1) - 4. \quad (56)
\]

**Proof:**

The two versions are related by \( \triangle X(n + 1) = X(n + 1) - X(n) \) given in eqs. 20, 38, 36, for \( X \in \{A, B, C\} \), respectively, and \( n \) replaced by \( Y(k) \) with \( Y \in \{A, B, C\} \). For \( C(n) \) eq. 40 is always used.

A) This follows from \( A(n + 1) \) given from eq. 19 with \( z(Y(k)) \) from eqs. 45, 46 and 47.

B) One proves \( B(A(k)) = A(k) + B(k) + k + 1 \) from which \( B(A(k) + 1) \) follows. With eqs. 40 and 30 this means that

\[ B(A(k)) = C(k) - 1 = B2(k). \quad (57) \]

After applying \( z_B \) on both sides, using eq. 27 this is equivalent to

\[ A(k) + 1 = z_B(C(k) - 1)) = z_B(C(k)). \quad (58) \]

The second equality is trivial. This is now proved. From eq. 22 \( z_B(n) = n + 1 - z(n) + z_C(n) \). Hence \( z_B(C(k)) = C(k) + 1 - z(C(k)) + (k + 1) \), with eq. 27. This is \( C(k) - k - 1 - B(k) \) from eq. 47, and replacing \( C(k) \) gives \( A(k) + 1 \).

One proves \( B(B(k)) = A(k) + 1 \) or, after application of \( z_B \) on both sides, \( B(k) + 1 = z_B(A(k) + 1) = z_B(A(k)) \), where the second equality is trivial. But from eqs. 44 and 27 follows \( z_B(A(k)) = A(k) + 1 - (k + 1) - z_C(A(k)) \). Applying eq. 43 and the just proven \( B(A(k) + 1) \) formula shows that

\[ z_B(A(k)) = B(k) + 1. \quad (59) \]

The \( B(C(k)) \) claim can be written in terms of \( C \) from eqs. 40 and 28 as

\[ B(C(k)) = 2C(k) - k = B0(k). \quad (60) \]

Indeed, eqs. 31, 27 imply for \( B(C(k)) = 2C(k) - z_C(C(k) - 1) = 2C(k) - (z_C(C(k)) - 1) = 2C(k) - k \). The second equality is trivial.

C) This follows immediately from \( C(n + 1) \) of eq. 40 and the already proved formulae for \( A(Y(k) + 1) \) and \( B(Y(k) + 1) \).

\( \square \)

The collection of the results for \( Z_X(Y(k)) \) is, for \( k \in \mathbb{N}_0 \):

\[ \]
Proposition 10.

\[ z_A(A(k)) = k + 1, \]
\[ z_A(B(k)) = A(k) - B(k) - (k + 1) = z_C(A(k)), \]
\[ z_A(C(k)) = B(k) + 1. \]  \hspace{1cm} (61)

\[ z_B(A(k)) = B(k) + 1 = z_A(C(k)) \]
\[ z_B(B(k)) = k + 1, \]
\[ z_B(C(k)) = A(k) + 1. \]  \hspace{1cm} (62)

\[ z_C(A(k)) = A(k) - B(k) + (k + 1) = z_A(B(k)) \]
\[ z_C(B(k)) = 2B(k) - A(k) + 1, \]
\[ z_C(C(k)) = k + 1. \]  \hspace{1cm} (63)

Proof:
That \( z_X(X(k)) = k + 1 \) has been noted already in eq. 27.
The other claims follow from the \( z_X(n) \) results after replacing \( n \) by \( Y(k) \neq X(k) \), and application of the formulae from Proposition 9. \( \square \)

Many of the formulae from section 3 appear in [2] and [1] with the above mentioned translation between their sequences \( a, b, \) and \( c \) to our \( B, A, \) and \( C \). For example, Theorem 13 of [2], p. 57, for the nine twofold iterations (in our notation \( X(Y(k) \) of Proposition 9) can be checked.

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Table 1: Sequences $t$, $A$, $B$, $C$, for $n = 0, 1, \ldots, 79$

| n  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $t$ | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 1  | 0  | 2  | 0  | 1  | 0  |
| $A$ | 1  | 5  | 8  | 12 | 14 | 18 | 21 | 25 | 29 | 32 | 36 | 38 | 42 | 45 | 49 | 52 | 56 | 58 | 62 | 65 |
| $B$ | 0  | 2  | 4  | 6  | 7  | 9  | 11 | 13 | 15 | 17 | 19 | 20 | 22 | 24 | 26 | 28 | 30 | 31 | 33 | 35 |
| $C$ | 3  | 10 | 16 | 23 | 27 | 34 | 40 | 47 | 54 | 60 | 67 | 71 | 78 | 84 | 91 | 97 | 104| 108| 115| 121|
| $n$ | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| $t$ | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 0  | 2  | 0  | 1  | 0  | 2  | 0  | 1  | 0  |
| $A$ | 69 | 73 | 76 | 80 | 82 | 86 | 89 | 93 | 95 | 99 | 102| 106| 110| 113| 117| 119| 123| 126| 130| 133|
| $B$ | 37 | 41 | 43 | 44 | 46 | 48 | 50 | 51 | 53 | 55 | 57 | 59 | 61 | 63 | 64 | 66 | 68 | 70 | 72 |
| $C$ | 128| 135| 141| 148| 152| 159| 165| 172| 176| 183| 189| 196| 203| 209| 216| 220| 227| 233| 240| 246|
| $n$ | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| $t$ | 2  | 0  | 1  | 0  | 0  | 1  | 0  | 0  | 1  | 0  | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 1  | 0  | 1  |
| $A$ | 137| 139| 143| 146| 150| 154| 157| 161| 163| 167| 170| 174| 178| 181| 185| 187| 191| 194| 198| 201|
| $B$ | 74 | 75 | 77 | 79 | 81 | 83 | 85 | 87 | 88 | 90 | 92 | 94 | 96 | 98 | 100| 101| 103| 105| 107| 109|
| $C$ | 253| 257| 264| 270| 277| 284| 290| 297| 301| 308| 314| 321| 328| 334| 341| 345| 352| 358| 365| 371|
| $n$ | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
| $t$ | 2  | 0  | 1  | 0  | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 2  | 0  | 1  | 0  | 1  | 0  | 1  | 0  | 2  |
| $A$ | 205| 207| 211| 214| 218| 222| 225| 229| 231| 235| 238| 242| 244| 248| 251| 255| 259| 262| 266| 268|
| $B$ | 111| 112| 114| 116| 118| 120| 122| 124| 125| 127| 129| 131| 132| 134| 136| 138| 140| 142| 144| 145|
| $C$ | 378| 382| 389| 395| 402| 409| 415| 422| 426| 433| 439| 446| 450| 457| 463| 470| 477| 483| 490| 494|
Table 2: ZT(N), for N = 1, 2, ..., 100

| N  | ZT(N) | N  | ZT(N) | N  | ZT(N) | N  | ZT(N) | N  | ZT(N) |
|----|-------|----|-------|----|-------|----|-------|----|-------|
| 1  | 1     | 21 | 11001 | 41 | 110100| 61 | 1010100| 81 | 10000000|
| 2  | 10    | 22 | 11010 | 42 | 110101| 62 | 1010101| 82 | 10000001|
| 3  | 11    | 23 | 11011 | 43 | 110110| 63 | 1010110| 83 | 10000010|
| 4  | 100   | 24 | 100000| 44 | 1000000| 64 | 1011000| 84 | 10000001|
| 5  | 101   | 25 | 100001| 45 | 1000001| 65 | 1011001| 85 | 10000100|
| 6  | 110   | 26 | 100010| 46 | 1000010| 66 | 1011010| 86 | 10001001|
| 7  | 1000  | 27 | 100011| 47 | 1000111| 67 | 1011011| 87 | 10000110|
| 8  | 1001  | 28 | 100100| 48 | 1001000| 68 | 1100000| 88 | 10001000|
| 9  | 1010  | 29 | 100101| 49 | 1001010| 69 | 1100001| 89 | 10001001|
| 10 | 1011  | 30 | 100110| 50 | 1001100| 70 | 1100100| 90 | 10010010|
| 11 | 10100 | 31 | 101000| 51 | 1010000| 71 | 1100111| 91 | 10010111|
| 12 | 10101 | 32 | 101010| 52 | 1010100| 72 | 1101010| 92 | 10010110|
| 13 | 101001| 33 | 1010010| 53 | 1010010| 73 | 1101010| 93 | 10011001|
| 14 | 1010010| 34 | 1010010| 54 | 1010010| 74 | 1101010| 94 | 10010000|
| 15 | 1010010| 35 | 1010010| 55 | 1010010| 75 | 1101010| 95 | 10010001|
| 16 | 1010010| 36 | 1010010| 56 | 1010010| 76 | 1101010| 96 | 10010010|
| 17 | 1010010| 37 | 1010010| 57 | 1010010| 77 | 1101010| 97 | 10010101|
| 18 | 1010010| 38 | 1010010| 58 | 1010010| 78 | 1101010| 98 | 10010110|
| 19 | 1010010| 39 | 1010010| 59 | 1010010| 79 | 1101010| 99 | 10010110|
| 20 | 1000000000 | 40 | 110011 | 60 | 1010010 | 80 | 1101010 | 100 | 10010110 |
Table 3: ABC(N), for N = 1, 2, ..., 100

| N | ABC(N) | N | ABC(N) | N | ABC(N) | N | ABC(N) | N | ABC(N) |
|---|--------|---|--------|---|--------|---|--------|---|--------|
| 1 | 10     | 21| 1020   | 41| 00120  | 61| 001110 | 81| 00000010 |
| 2 | 010    | 22| 0120   | 42| 1120   | 62| 11110  | 82| 1000010  |
| 3 | 20     | 23| 220    | 43| 0220   | 63| 02110  | 83| 0100010  |
| 4 | 0010   | 24| 000010 | 44| 0000010| 64| 00210  | 84| 200010   |
| 5 | 110    | 25| 100010 | 45| 100010 | 65| 10210  | 85| 0010010  |
| 6 | 020    | 26| 010010 | 46| 010010 | 66| 01210  | 86| 110010   |
| 7 | 00010  | 27| 20010  | 47| 20010  | 67| 2210   | 87| 020010   |
| 8 | 1010   | 28| 001010 | 48| 001010 | 68| 000020 | 88| 0010010  |
| 9 | 0110   | 29| 11010  | 49| 11010  | 69| 100020 | 89| 101010   |
| 10| 210    | 30| 02010  | 50| 02010  | 70| 010020 | 90| 011010   |
| 11| 0020   | 31| 000110 | 51| 000110 | 71| 20020  | 91| 210010   |
| 12| 120    | 32| 10110  | 52| 10110  | 72| 001020 | 92| 002010   |
| 13| 000010 | 33| 01110  | 53| 01110  | 73| 11020  | 93| 120010   |
| 14| 10010  | 34| 2110   | 54| 2110   | 74| 02020  | 94| 0001010  |
| 15| 01010  | 35| 00210  | 55| 00210  | 75| 000120 | 95| 1001010  |
| 16| 2010   | 36| 1210   | 56| 1210   | 76| 10120  | 96| 0101010  |
| 17| 00110  | 37| 000020 | 57| 0000110| 77| 01120  | 97| 201010   |
| 18| 1110   | 38| 10020  | 58| 100110 | 78| 2120   | 98| 0011010  |
| 19| 0210   | 39| 01020  | 59| 010110 | 79| 00220  | 99| 111010   |
| 20| 00020  | 40| 2020   | 60| 20110  | 80| 1220   | 100| 021010 |

Here 0, 1 and 2 stand for B, A and C, respectively. E.g., ABC(6) = BCB = B(C(B(0))).