Fake Instability in the Euclidean Formalism

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Abstract

We study the path-integral formalism in the imaginary-time to show its validity in a case with a metastable ground state. The well-known method based on the bounce solution leads to the imaginary part of the energy even for a state that is only metastable and has a simple oscillating behavior instead of decaying. Although this has been argued to be the failure of the Euclidean formalism, we show that proper account of the global structure of the path-space leads to a valid expression for the energy spectrum, without the imaginary part. For this purpose we use the proper valley method to find a new type of instanton-like configuration, the “valley instantons”. Although valley instantons are not the solutions of equation of motion, they have dominant contribution to the functional integration. A dilute-gas approximation for the valley instantons is shown to lead to the energy formula. This method extends the well-known imaginary-time formalism so that it can take into account the global behavior of the theory.

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I. INTRODUCTION

The imaginary-time formalism has been successful for studies of the various quantum tunneling phenomena in the semi-classical regime. This is because of the existence of the solution of the Euclidean equation of motion, around which we could evaluate the relevant functional integration in Gaussian and higher order approximations. In the systems with finite degrees of freedom, this is known to lead to WKB results.

More specifically, in case when there are perturbatively degenerate vacua (or ground states), the instanton calculation takes into account the tunneling between them and gives the nonperturbative contribution to the energy splitting. On the other hand, when a perturbative vacuum is unstable due to the tunneling to lower energy states, the bounce solution leads to the imaginary part of the energy, thus the decay rate. This formalism provides us with a good calculational tool, valid in a wide range of physical systems.

One subtle feature of these calculations can be seen for a one-dimensional quantum mechanical model with a potential, $V(\phi)$, illustrated in Fig. 4. (We denote the coordinate by $\phi$ in this letter.) If one restricts the wave functions
to have only the outgoing component at $\phi \gg \phi_{\text{ESC}}$, the hermiticity of the Hamiltonian is violated and the energy eigenvalues become complex; this imaginary part is a direct consequence of the instability of the localized wave packet at $\phi \sim 0$. In the imaginary-time formalism, this complex energy is thought to be evaluated by an analytic continuation of the ill-defined divergent Gaussian integral over the negative mode direction at the bounce solution \[ \Box \].

We may then ask ourselves what are the eigenvalues for an unbalanced double-well potential illustrated in Fig.2. (In order to distinguish this case from the previous one, we call this ‘metastable’ case and the previous one ‘unstable’.) As far as we stick to the classical solution and take into account only the contribution of its infinitesimal neighborhood for the path integral, the result is the same as the unstable case; we have a similar bounce solution with a negative mode and obtain complex eigenvalues.

This imaginary part is obviously a fake. The perturbative ground state in the left well is only metastable in the sense that any wave packet that tunnels to the right well oscillates between the two wells. This oscillation generates the splitting behavior similar to the degenerate case, $\epsilon = 0$, but not to the decay rate. In other words, we can only take wave functions with decaying exponential at $\phi \gg \phi_0$ for the potential in Fig.2 and the hermiticity of the Hamiltonian cannot be violated.

It was claimed that Euclidean formalism is doomed due to the fact that it only uses the information around the infinitesimal neighborhood of the solutions \[ \Box \]. As an alternative, the complex-time method was proposed, in which the real-time part of the classical trajectory takes into account the correct boundary condition for $\phi \geq \phi = \phi_{\text{ESC}}$. Although this is a very exciting development by itself \[ \Box \Box \], it is not clear whether it should replace the Euclidean formalism.

In this letter we show that the proper treatment of the imaginary-time path integral leads us to the correct behavior of the energy eigenvalues for the metastable potential. The essence of the improvement is the use of the proper valley method, which was developed independently by Silvestrov \[ \Box \Box \] and two of the present authors \[ \Box \Box \]. It teaches us how to enlarge the set of background configurations besides the classical solutions in order to take into account the global structure of the functional space. Using this method, we construct “valley instanton”, which should replace the bounce solution. Interestingly, it has a zero mode and this expedites the calculation of its determinant and Jacobean as was the case of the instanton. It is a well-localized configuration with respect to the imaginary time and their dilute gas sum generates the reasonable energy corrections of the lowest states instead of the imaginary part. We will also show that it converges analytically to the instanton in the limit of $\epsilon \to 0$ and all the results reproduce the well-known instanton results.

II. THE VALLEY BOUNCE AND THE VALLEY INSTANTON

Consider the quantum mechanical system with the following Euclidean action;

$$ S_E[\phi] = \int d\tau \left( \frac{1}{2} \left( \frac{d\phi}{d\tau} \right)^2 + V(\phi) \right), $$

with the potential,

$$ V(\phi) = \frac{1}{2} \phi^2 (1 - g\phi)^2 - \epsilon(4g^3\phi^3 - 3g^4\phi^4), \tag{1} $$

where the coupling constants $g$ and $\epsilon$ are positive. The potential (1) has a local minimum, $V(0) = 0$ and a global minimum at $\phi_0 = 1/g$, where $V(1/g) = -\epsilon$. (Fig.2 is plotted for $g = 0.3$ and $\epsilon = 0.25.$) The potential $V(\phi)$ in Eq.(1) is a canonical form of quartic potentials with two minima, since any such potential can be cast into this form by suitable linear transformations on $\phi$ and $\tau$. In this sense the following analysis is quite a general one. In the following, we consider the cases with small coupling $g \ll 1$, but not necessarily small $\epsilon$.

As mentioned in the introduction, the naive application of the semi-classical approximation around the bounce solution of this potential leads us to the fake imaginary part of the energy eigenvalues. In order to circumvent this, we construct the different type of configurations around which we expand the action. Such configurations are most straightforwardly identified by the valley methods \[ \Box \Box \], or more specifically, by the proper valley method \[ \Box \Box \Box \]. The latter has many advantages over the previous one. It has been applied to the Borel summability problem \[ \Box \Box \], induced bubble nucleation problem \[ \Box \Box \], the instanton in gauge-Higgs system \[ \Box \Box \], and the baryon number violation problem \[ \Box \Box \Box \] successfully.

For the current model, the advantage of using the proper valley method is that we can see how the action behaves if we go along the negative mode direction at the bounce, the direction most important to evaluate the path integral correctly. As we will see below, it reaches to $\phi \sim \phi_0$, and see the behavior of the potential around there.
The proper valley configurations are given by the following new valley equations:

\[-\partial_{\tau}^2 \phi + V'(\phi) = F,\]
\[\left(-\partial_{\tau}^2 + V''(\phi)\right) F = \lambda F.\]  

(2)

(3)

For \( F = 0 \), this set of equations reduces to the ordinary equation of motion. Thus, any solution of the equation of motion is the solution of the new valley equations. Otherwise, eliminating \( F \) one finds that the left-hand side of Eq. (2), which is \( \delta S/\delta \phi \), is the eigenvector of the \( \delta^2 S/\delta \phi \delta \phi(= D) \) with the eigenvalue \( \lambda \). The general solutions can be parametrized by the eigenvalue \( \lambda \), or any arbitrary function \( \alpha \) of \( \lambda \). We denote this one-parameter family of the solutions, the valley trajectory, by \( \phi_\alpha(\tau) \). The contribution of such a configuration to the vacuum transition amplitude,

\[\langle \phi = 0, \tau = +\infty | \phi = 0, \tau = -\infty \rangle = N \int_{\phi(\pm \infty) = 0}^{\alpha} D\phi e^{-S[\phi]},\]

is evaluated around this valley by inserting the following triviality:

\[1 = \Delta \int d\alpha \int d\tau (\phi(\tau) - \phi_\alpha(\tau)) \frac{\delta S}{\delta \phi_\alpha}.\]

(5)

This forces the expansion of \( \phi - \phi_\alpha \) in the subspace orthogonal to \( \delta S/\delta \phi \), enabling the Gaussian integration without the linear terms. At the leading order of \( \hbar \), we obtain:

\[\langle \phi = 0, \tau = +\infty | \phi = 0, \tau = -\infty \rangle = \int d\alpha \int d\tau \phi_\alpha e^{-S[\phi_\alpha]},\]

(6)

where \( \det D' \) is the usual determinant less the eigenvalue \( \lambda \) in the new valley equation.\(^3\) and

\[J_\alpha \equiv \int d\tau \frac{d\phi_\alpha}{d\alpha} F = \frac{dS[\phi_\alpha]}{d\alpha}.\]

(7)

This way we are able to perform the integral for the negative mode direction on a more rigid basis than the subtle Gaussian integral. The factor \( J_\alpha \) is the Jacobian for this change of the integration variable. We also note that Eq. (6) is apparently invariant under any local reparametrization in \( \alpha \).

We have carried out the numerical analysis and obtained the solutions plotted in Fig.\(^4\). This is a simple extension of the previous analysis for the meta-stability problem in a quantum field theory by two of the current authors, H. A. and S. W.\(^5\). The solid line, \( \alpha \), is the bounce solution of the equation of motion. The rest have \( F \neq 0 \) do not satisfy the equation of motion. We call these solutions (including the bounce solution) “valley bounce”. The values of the action, \( S \), and the eigenvalue, \( \lambda \), for the valley bounce are plotted in Fig.\(^4\). The bounce solution lies at the top of the line of the action in Fig.\(^4\) corresponding to the fact that it has a negative eigenvalue. This negative eigenvalue can be read from the corresponding point of the plot of \( \lambda \) in the lower half of Fig.\(^4\). The most notable feature is that the large valley bounce is a clean interior, where \( \phi = \phi_0 \) and \( F = 0 \). (This property is shared by the higher dimensional configurations, \( i.e. \), the valley bubbles.) The effect of this is apparent in the behavior of the action in Fig.\(^4\). the action decreases linearly with large \( |\phi| \), which is proportional to the size of the valley bounce. Thus, the valley bounce provides a natural way for the evaluation of the contribution of large regions of the true minimum \( \phi_0 \).

In the large valley bounces, \( c, d, \) and \( e \) in Fig.\(^4\), we notice that the shape of the wall, \( i.e. \), the transition region from \( \phi \sim \phi_0 \) to \( \phi \sim 0 \), is almost identical to each other. That is, they overlap with each other very well when translated in \( \tau \). Thus these large valley bounces can be approximated by simply connecting the walls by a flat region, \( \phi = \phi_0 \), at various separation. The shape of the wall can be most readily identified when the size \( \langle |\phi| \rangle \equiv \int d\tau |\phi| \) of the valley bounce becomes \( \infty \). In this limit, the wall is simply a localized transition from \( \phi = 0 \) to \( \phi = \phi_0 \) (or vise-versa). Such a solution is an analogue of the instanton (or anti-instanton). The difference is that now it is not a solution of equation of motion, but is a solution of the new valley equation. This kind of configuration is called the “valley instanton”.\(^6\)

In the following, we evaluate the properties of the valley instantons, in order to approximate the large valley bounces by a pair of valley instanton \( (\phi = 0 \rightarrow \phi_0) \) and a valley anti-instanton \( (\phi = \phi_0 \rightarrow 0) \).

Since we define the valley instantons at the large-size limit, \( |\phi| \rightarrow \infty \), of the valley bounce, the plot of the eigenvalue in Fig.\(^4\) implies that the eigenvalue \( \lambda \) of the valley instanton is exactly zero. This is not a trivial property. A solution
of equation of motion is guaranteed to have zero modes corresponding to its symmetry transformation, such as a time translation. Arbitrary background configurations do not have this property in general. However, we can prove the existence of the zero mode as follows: Take a derivative of Eq. (2) with respect to \( \tau \), multiply \( F \), and integrate over \( \tau \). After partial integrations (which surface terms vanish), we then find that

\[
\lambda \int_{-\infty}^{\infty} F \dot{\phi} d\tau = \int_{-\infty}^{\infty} F \ddot{\phi} d\tau .
\]  
(8)

The integral in the left-hand side is generally non-zero due to the boundary conditions of the valley instantons. This can be seen from the behavior of \( \phi \) and \( F \) in the walls in Fig. 3. Since the right-hand side is zero, we find that \( \lambda = 0 \). (For the valley bounces, the integral in the left-hand side is zero since \( \phi(\tau) \) and \( F(\tau) \) are even functions. Therefore, \( \lambda \neq 0 \) is allowed.)

We have carried out numerical investigation of the valley instanton with \( \lambda = 0 \) and have successfully obtained the solutions, which have turned out to be almost identical to the wall regions in Fig. 3. Although we know of no exact analytical expression of the valley instanton, it can be constructed analytically for small \( \epsilon \) solutions, which have turned out to be almost identical to the wall regions in Fig. 3. Although we know of no exact analytical expression of the valley instanton, it can be constructed analytically for small \( g^2 \) in a manner used in the construction of the constrained instanton \( \[15\] \) as well as other types of the valley instantons \( \[13\] \). Consider the valley instanton in \( \tau \in [-T/2,T/2] \) \((T \gg 1)\). We define its central coordinate to be at the origin, \( \tau = 0 \), by \( \phi(0) = 1/2g \).

The naive perturbation in \( \epsilon g^2 \) yields the following perturbative valley instanton solution;

\[
\phi = \phi' + 3g^2 \tau \phi' + O((\epsilon g^2)^2),
\]

\[
F = -6g^2 \phi' + 36g^5 \tau \phi' + O((\epsilon g^2)^3),
\]

where \( \phi' \) is the instanton solution for \( \epsilon = 0 \);

\[
\phi' = \frac{1}{g} \frac{1}{1+\epsilon^{-\tau}}.
\]

From Eq.(3), it is apparent that this naive perturbation is valid only in the region close to the instanton center, \(|\tau| \ll 1/(\epsilon g^2)\). On the other hand, in the asymptotic regions, \( \tau \to \pm \infty \), we linearize the new valley equation and find the general solutions valid for \(|\tau| \gg 1 \). Coefficients of the general solutions are fixed by matching it with the inner solution Eq.(3) in the intermediate region, \( 1 \ll |\tau| \ll 1/(\epsilon g^2) \). We have found that this procedure can be done consistently. The resulting asymptotic behaviors are for \( \tau \to +\infty \);

\[
\phi \simeq \frac{1}{g} \left(1 - \left(1 + \frac{3g^2 \tau}{\omega_-} \right) e^{-\omega_- \tau}\right),
\]

\[
F \simeq -6\epsilon g e^{-\omega_- \tau},
\]

where \( \omega_- = V''(\phi_0) = 1 + 12g^2 \), and for \( \tau \to -\infty \);

\[
\phi \simeq \frac{1}{g} \left(1 + 3g^2 \tau \right) e^{\tau},
\]

\[
F \simeq -6\epsilon g e^{\tau}.
\]

This way, the valley instanton is constructed in all regions of \( \tau \) for small \( \epsilon \).

The action of the valley instanton is given by \( S_I = 1/6g^2 + \epsilon(-T/2 + 1/2) + O(\epsilon^2) \). This action is divided to the volume part and the remaining (proper) part as \( S_I = -\epsilon T/2 + S^I \). From the construction above, we find that \( S^I = 1/6g^2 + \epsilon/2 + \ldots \). However, there is a subtlety on this point: In the following we integrate over the position coordinate of the instantons and anti-instantons in the dilute gas approximation. These coordinates are originally the valley parameters \((O/s)\) of the valley bounces (and their central coordinates). The \( O(\epsilon) \) term in \( S^I \) depends on the definition of these valley parameters, since the definition of the volume \((T)\) is affected by it. Therefore, careful study of the small valley bounces is needed to fix this term. This term, however, has only the nonleading contribution. Therefore, we will not pursue this problem any further here.

The Jacobian for the instanton position is given by,
\[
J^I = \frac{\epsilon}{\sqrt{\int d\tau F^2}}.
\]  

Since the contribution to the integration is dominated by the central region, the leading term of Eq. (13) for small \( \epsilon \) is evaluated by the use of Eq. (12). The result is that \( J^I = 1/\sqrt{6g^2(1+O(\epsilon g^2))} \). The first term is the instanton action for \( \epsilon = 0 \). Therefore, in the limit \( \epsilon \to 0 \), the Jacobian of the valley instanton reduces to that of the ordinary instanton.

The determinant, \( \det D' \), can be calculated by extending the Coleman's method [3], in spite of the fact that the valley instanton is not the solution of the equation of motion. This is due to the fact that the valley instanton possesses the exact zero mode \( F(\tau) \). We define the asymptotic coefficients \( F_{\pm} \) by \( F(\tau) \approx F_{\pm} e^{\mp \omega \pm \tau} \), where for the sake of notation we introduced \( \omega_+ = V''(0) = 1 \). After some calculation, we find that the ratio of the determinants is given by the following:

\[
\det'(-\partial^2 + V''(\phi^I)) = \kappa e^{(\omega_+ - \omega_-)(T/2 - \tau_0)}, \quad \kappa \equiv \frac{1}{2\omega_+ \omega_- F_+ F_-} \int_{-\infty}^{\infty} F^2.
\]

The exponential factor is the perturbative contribution to the zero energy at the true minima, \( \phi_0 \). The factor \( \kappa \) is the ‘proper’ instanton contribution. From Eq. (11) and Eq. (12), we find that \( \kappa \) reduces to the ordinary instanton determinant for \( \epsilon \to 0 \).

**III. PATH INTEGRAL IN THE “DILUTE-GAS” VALLEY INSTANTON APPROXIMATION**

Combining all factors evaluated in the previous section, we find the expression of the finite time \( (T) \) vacuum transition amplitude to be the following:

\[
Z(T) \equiv \frac{\langle \phi = 0, \tau = T | \phi = 0, \tau = 0 \rangle}{\langle \phi = 0, \tau = T | \phi = 0, \tau = 0 \rangle_0} = \sum_{n=0}^{\infty} \alpha^{2n} I_n,
\]

where the amplitude \( \langle \ldots \rangle_0 \) is for the harmonic oscillator with \( \omega_+ \), \( n \) is the number of the valley instanton pairs, and the factor \( \alpha \) is the product of the proper contributions of the action, the determinant ratio and the Jacobian; \( \alpha = (J^I/\sqrt{\kappa}) e^{-S^I} \). The actual integrations over the positions of the valley instantons are in the factors \( I_n \):

\[
I_n(T) \equiv \begin{cases} 
1, & \text{for } n = 0, \\
\int_0^T d\tau_{2n} \int_{\tau_{2n}-1}^{\tau_{2n}} \ldots \int_0^{\tau_{2}} d\tau_{1} e^{\epsilon(\tau_{2n} - \tau_{2n-1}\ldots + \tau_2 - \tau_1)}, & \text{for } n \geq 1,
\end{cases}
\]

where zero-energy contributions of the determinants are absorbed in \( \epsilon \) by \( \bar{\epsilon} \equiv \epsilon - (\omega_- - \omega_+)/2 \).

The infinite series can be summed by the use of the generating function method [16]: From Eq. (16), we find that the following differential equation is satisfied by \( Z(T) \):

\[
Z(T)'' - \bar{\epsilon} Z(T)' - \alpha^2 Z(T) = 0.
\]

Also, \( Z(0) = 1 \), and \( Z'(0) = 0 \). Therefore, we find that

\[
Z(T) = \frac{k_+ e^{-k_+ T} - k_- e^{-k_- T}}{k_+ - k_-},
\]

where

\[
k_{\pm} \equiv -\frac{\bar{\epsilon}}{2} \pm \sqrt{\frac{\bar{\epsilon}^2}{4} + \alpha^2}.
\]

Thus we find that the energies of the two lowest states, \( E_{\pm} \) is given by

\[
E_{\pm} = \frac{\omega_+}{2} \pm k_{\pm} = \frac{\omega_+}{2} - \frac{\bar{\epsilon}}{2} \pm \sqrt{\frac{\bar{\epsilon}^2}{4} + \alpha^2}.
\]
Furthermore, from the coefficients of the respective exponents of Eq. (18), we find that $|\langle \phi = 0 | E_\pm \rangle|^2 = \pm k_+/ (k_+ - k_-)$
This is the main conclusion of this letter. There appear no fake imaginary parts in the energy spectrum. We observe that $E_\pm$ are equal to the eigenvalues of the following matrix;

\[ H = \begin{pmatrix} \frac{\omega_+}{2} & \alpha \\ \alpha & -\epsilon + \frac{\omega_-}{2} \end{pmatrix}. \]

(21)

Furthermore, the weights of the state localized in the left well agree with Eq. (18): Denoting the eigenvectors of Eq. (21) $V_\pm$ with eigenvalues $E_\pm$,

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{\frac{k_+}{k_+ - k_-}} V_+ + \sqrt{\frac{-k_-}{k_+ - k_-}} V_- \].

(22)

This allows a simple explanation of the result. The energy spectrum we obtain is the same as the two-level system made of the perturbative ground state at $\phi = 0$ and $\phi = \phi_0$, with the tunneling matrix element $\alpha$.

Before discussing the implication of the result Eq. (20), we examine the validity of the dilute gas approximation we used. The mean size of the bounce, which is the mean distance, $R$, between the instanton and the anti-instanton located at the right of the instanton, can be obtained as the expectation value, $\langle \tau_2 - \tau_1 \rangle$ in the amplitude, Eq. (15) and Eq. (16). (Any other $\langle \tau_m - \tau_{m-1} \rangle$ with integer $m$ would yield the same result for $T \to \infty$.) Using the same generating function method as above, we obtain the following;

\[ R = \frac{1}{\alpha^2} \left( \frac{\tilde{\epsilon}}{2} + \sqrt{\frac{\tilde{\epsilon}^2}{4} + \alpha^2} \right). \]

(23)

Similarly, the mean distance, $d$, between the anti-instanton and the instanton located at the right of the anti-instanton ($\langle \tau_3 - \tau_2 \rangle$) is given by,

\[ d = \frac{1}{\alpha^2} \left( -\frac{\tilde{\epsilon}}{2} + \sqrt{\frac{\tilde{\epsilon}^2}{4} + \alpha^2} \right). \]

(24)

For $\epsilon \gg \alpha$, $d \sim 1/\tilde{\epsilon}$. On the other hand, the thickness of the instanton is $O(1/\sqrt{\epsilon})$ for $\epsilon \gg 1$, which can be seen by a simple scaling argument on the new valley equations, Eq. (3) and Eq. (4). This means that the dilute gas approximation is valid for $\tilde{\epsilon} < 1$.

For $\epsilon \ll \alpha$, the energy spectrum Eq. (20) gives the following;

\[ E_\pm = \frac{\omega_+ + \omega_-}{4} \pm \frac{\epsilon}{2} - O \left( \frac{\epsilon^2}{\alpha} \right). \]

(25)

In the limit $\epsilon \to 0$, this result reduces to the well-known instanton result for the degenerate case. In addition, we have the average of the perturbative zero-point energies of the left-well ($\omega_+/2$) and the right-well ($\omega_-/2 - \epsilon$). Since the wavefunction is distributed evenly at the zeroth order of the $\epsilon$ expansion, this is the correct formula for the two lowest energy eigenvalues. Since the perturbative contribution to the energy splitting, $\delta E = E_+ - E_-$, is of order $\epsilon g^4$, even a purist [8] would retain our result, $\delta E = 2\alpha$. For $\alpha \ll \epsilon < 1$, Eq. (20) leads to,

\[ E_+ = \frac{\omega_+}{2} + \frac{\alpha^2}{\epsilon} + O \left( \frac{\alpha^4}{\epsilon^3} \right), \quad E_- = -\epsilon + \frac{\omega_-}{2} - \frac{\alpha^2}{\epsilon} + O \left( \frac{\alpha^4}{\epsilon^3} \right). \]

(26)

The lower energy, $E_-$ corresponds to the state almost localized in the right well.

IV. CONCLUSION AND DISCUSSION

In this letter we have applied the proper valley method to take into account the large nonperturbative configurations in the path integral. We have found and constructed a new type of instanton, the valley instanton, both numerically and analytically. We further evaluated its action, Jacobian, and determinant and found that these have a smooth limit to the ordinary instanton for the degenerate case. The dilute valley-instanton approximation to the path integral
has lead us to the energy formula Eq.[20]. Thus we have successfully shown that proper treatment of the imaginary time formalism does not lead to any contradiction and in fact yields the valid energy formula.

Although the dilute valley instanton gas approximation fails for $\epsilon > 1$, this does not limit the applicability of the proper valley method. For such a case, the valley bounces on the background $\phi_0$ are expected to become important. Such valley bounces are known to exist from the analysis of [12] and allows us to take into account the contribution of the configurations with $\phi < \phi_0$.

In view of the current development, the ordinary calculation of the imaginary part of the energy in the unstable cases (Fig.1) needs to be examined under the new light. This is under way and will be reported elsewhere.

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FIG. 1. A potential that is flat in the asymptotic direction, $V(\infty) = -\epsilon$.

FIG. 2. An asymmetric double-well potential, in which $\phi = 0$ is only metastable.
FIG. 3. Valley bounce solutions \((\phi(\tau), F(\tau))\) of the new valley equations. Center of all of the configurations are chosen to be at the origin, \(\tau = 0\), around which the solutions are symmetric. The solid line \(a\) in the upper figure is the usual bounce solution, which has \(F(\tau) = 0\). The other lines, \(b-e\), are unique to the new valley equations.

FIG. 4. The values of the action, \(S\), and the eigenvalue, \(\lambda\), of the valley bounces. The peak of the action is given by the bounce solution, the line \(a\) in Fig.3. The points corresponding to the valley bounces in Fig.3 are plotted with circle. The solid lines are drawn as the guide for eyes. The valley parameter is chosen to be \(|\phi| \equiv \int d\tau \phi\).