**L^p SPACES IN VECTOR LATTICES AND APPLICATIONS**

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**Abstract.** L^p spaces are investigated for vector lattice-valued functions, with respect to filter convergence. As applications, some classical inequalities are extended to the vector lattice context, and some properties of the Brownian Motion and the Brownian Bridge are studied, to solve some stochastic differential equations.

1. Introduction

Function spaces, in particular L^p spaces, play a central role in many problems in Mathematical Analysis and have lots of applications in several branches. The L^p spaces are perhaps the most useful and important examples of Banach spaces; of independent and higher interest is the L^2 space, whose origins are related with fundamental investigations and developments in Fourier analysis (see for example [5, 6, 24]), in reconstruction of signals, integral and discrete operators (see for example [1, 2, 4, 7–9, 29, 30, 38, 50, 51]) and Stochastic Integration (see also [10, 34–37, 40, 46–49]).

In this paper some fundamental properties of L^p spaces in the vector lattice setting are investigated, continuing a research initiated by the authors in [12–14, 16, 21, 22] and developed later in [15]. The range of the involved functions is a vector lattice endowed with filter/ideal convergence (for a related literature, see also [3, 17–20, 25, 27]). Thanks to the triangle inequality, it is possible to view the space L^p as a metric space endowed with a distance of the type \( d(f, g) = \| f - g \|_p \). Note that, in general, this space is not complete, as Example 3.8 shows. As applications, several inequalities are given, as well as some approximation results concerning processes related to the Brownian Motion.

The paper is organized as follows. In Section 2 basic notions and results are given, recalling some main properties of vector lattices, filters, modulars and a
Vitali-type Theorem. In Section 3 some new results on $L^p$ spaces, for $p \in \mathbb{N}$ in the vector lattice context and a Minkowski inequality are given, together with an example in which it is shown that in general these spaces are not complete. Section 4 is divided into two subsections; in the first one some inequalities (Hermite-Hadamard, Féjer, Jensen and Schwartz) are proved. In the second subsection, applications to the Brownian Motion and the Brownian Bridge are given, leading to the solution of some particular stochastic differential equations and to the reconstruction of a perturbed signal.

2. Preliminaries

Some basic properties of vector lattices and filter convergence are recalled first. For the basic subjects and fundamental tools used in the vector lattice theory see, for instance, [41, 53]. A vector lattice $X$ is said to be Dedekind complete iff every nonempty subset $A \subset X$, bounded from above, has a lattice supremum in $X$, denoted by $\bigvee A$. From now on, $X$ is a Dedekind complete vector lattice, $X^+$ is the set of all strictly positive elements of $X$, and $X^+_0 = X^+ \cup \{0\}$. For each $x \in X$, let $|x| := x \lor (-x)$. An extra element $+\infty$ will be added to $X$, extending order and operations in a natural way, set $X = X \cup \{+\infty\}$, $X^+_0 = X^+ \cup \{+\infty\}$, and assume $0 \cdot (+\infty) = 0$. A sequence $(p_n)_n$ in $X$ is called $(o)$-sequence iff it is decreasing and $\bigwedge_n p_n = 0$. An order unit of $X$ is an element $e$, such that for every $x \in X$ there is a positive real number $c$ with $|x| \leq ce$.

Let $X_1, X_2, X$ be Dedekind complete vector lattices. We say that $(X_1, X_2, X)$ is a product triple iff a product $\cdot : X_1 \times X_2 \rightarrow X$ is defined, satisfying natural conditions of compatibility (see for instance [14, Assumption 2.1]).

Remark 2.1. A vector lattice $X$ is called an $f$-algebra (see also [53, Definition 140.8]) iff there exists in $X$ an associative multiplication, satisfying the usual algebraic properties, with $xy \geq 0$ whenever $x, y \geq 0$ and such that $x \land y = 0$ implies $(x \cdot z) \land y = 0$ whenever $x, y \in X$ and $z \in X^+_0$. The following condition will be required in the paper.

$(H_0)$ $(X_1, X_2, X)$ is a product triple, and $X, X_1$ are endowed with order units $e, e_1$ respectively.

Note that every lattice $X$ equipped with an order unit is an $f$-algebra. Indeed, by the Maeda-Ogasawara-Vulikh representation theorem 3.6 (see also [52]), $X$ is algebraically and lattice isomorphic to the space $C(\Omega)$ of all continuous real-valued functions defined on a suitable compact and extremely disconnected topological space $\Omega$. So, it is not difficult to deduce that $X$ is an $f$-algebra since $\mathbb{R}$ is.

Given any fixed countable set $Z$, a class $\mathcal{F}$ of subsets of $Z$ is called a filter of $Z$ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and $B \supset A$ it is $B \in \mathcal{F}$. The symbol $\mathcal{F}_{\text{cofin}}$ denotes the filter of all cofinite subsets of $Z$. A filter $\mathcal{F}$ of $Z$ is said to be free iff it contains $\mathcal{F}_{\text{cofin}}$. An example of free filter is the filter
of all subsets of \( \mathbb{N} \), having asymptotic density one. If \( \mathcal{F} \) is a filter of \( Z \), then let \( \mathcal{F} \otimes \mathcal{F} \) be the product filter of \( Z \times Z \), defined by

\[
\mathcal{F} \otimes \mathcal{F} := \{ C \subset Z \times Z : \text{there exist } A, B \in \mathcal{F} \text{ with } A \times B \subset C \}. \tag{2.1}
\]

**Definition 2.2.** Let \( \mathcal{F} \) be any filter of \( Z \). A sequence \( (x_z)_{z \in \mathbb{Z}} \) in \( X \) \( (o_F) \)-converges to \( x \in R \) \( (x_z \xrightarrow{o_F} x) \) iff there exists an \( (o) \)-sequence \( (\sigma_p)_p \) in \( X \) such that for all \( p \in \mathbb{N} \) the set \( \{ z \in \mathbb{Z} : |x_z - x| \leq \sigma_p \} \) belongs to \( \mathcal{F} \).

A sequence \( (x_z)_{z \in \mathbb{Z}} \) in \( X \) \( (r_F) \)-converges to \( x \in X \) \( (x_z \xrightarrow{r_F} x) \) iff there exist a \( u \in X^+ \) and an \( (o) \)-sequence \( (\varepsilon_p)_p \) in \( \mathbb{R}^+ \) such that for every \( p \in \mathbb{N} \) the set \( \{ z \in \mathbb{Z} : |x_z - x| \leq \varepsilon_p u \} \) is an element of \( \mathcal{F} \).

A sequence \( (x_z)_z \) in \( X \) \( (o) \)-converges to \( x \in X \) \( (x \xrightarrow{o} x) \) in the classical sense iff it \((o_{\mathcal{F}, \text{class}})\)-converges to \( x \) (see also [20]).

Let \( G \) be any infinite set, \( \mathcal{P}(G) \) be the family of all subsets of \( G \), \( A \subset \mathcal{P}(G) \) be an algebra and \( \mu : A \to [X_2]_\sigma^\mathbb{R} \) be a finitely additive measure. The symbol \( \mu^* \) denotes the outer measure associated to \( \mu \), namely \( \mu^*(B) := \wedge_{A \in A, A \supset B} \mu(A) \), \( B \in \mathcal{P}(G) \), and \( A_b \) is the family of the sets \( B \in A \) with \( \mu(B) \in X_2 \). For every \( A \in A \) and \( u \in X_1 \), let \( u \cdot 1_A : G \to X_1 \) be the function whose values are \( u \) when \( t \in A \) and 0 otherwise.

As in [15, Subsection 2.1], the modulars in the vector lattice setting are introduced (for the classical case and a related literature, see e.g. [9, 39, 43, 45]).

Let \( T \) be a linear sublattice of \( X_1^2 \), such that \( e_1 \cdot 1_A \in T \) for every \( A \in A_b \).

A functional \( \rho : T \to X_1^\mathbb{R} \) is said to be a modular on \( T \) iff it satisfies the following properties.

\[
\begin{align*}
\rho(0) & = 0; \\
\rho(-f) & = \rho(f) \text{ for every } f \in T; \\
\rho(\alpha_1 f + \alpha_2 h) & \leq \rho(f) + \rho(h) \text{ for every } f, h \in T \text{ and for any } \alpha_1, \alpha_2 \geq 0 \text{ with } \alpha_1 + \alpha_2 = 1.
\end{align*}
\]

Moreover, some additional conditions will be required.

\[
\begin{align*}
\rho_m & \text{ A modular } \rho \text{ is monotone iff } \rho(f) \leq \rho(h) \text{ for every } f, h \in T \text{ with } |f| \leq |h|. \text{ In this case, if } f \in T, \text{ then } |f| \in T \text{ and hence } \rho(f) = \rho(|f|). \\
\rho_{co} & \text{ A modular } \rho \text{ is convex iff } \rho(\alpha_1 f_1 + \alpha_2 f_2) \leq \alpha_1 \rho(f_1) + \alpha_2 \rho(f_2) \text{ for all } f_1, f_2 \in T \text{ and for any real numbers } \alpha_1, \alpha_2 \geq 0 \text{ with } \alpha_1 + \alpha_2 = 1. \\
\rho_f & \text{ A modular } \rho \text{ is finite iff for every } A \in A_b \text{ and every } (o)\text{-sequence } (\varepsilon_p)_p \text{ in } \mathbb{R}^+, \text{ the sequence } (\rho(\varepsilon_p 1_A))_p \text{ is } (r)\text{-convergent to 0 (for the case } X = X_1 = X_2 = \mathbb{R}, \text{ see for instance [9]).}
\end{align*}
\]

The concept of (equi-) absolute continuity in the context of modulars and filter convergence is introduced here.
(a.ₚ) A map \( f \in T \) is (\( \alpha_f \))-absolutely continuous with respect to the modular \( \rho \) (shortly, absolutely continuous) iff there is a positive real constant \( \alpha \), satisfying the following properties.

(a.ₚ(1)) For each \( (\alpha) \)-sequence \( (\sigma_p)_p \) in \( X^+_1 \) there exists an \( (\alpha) \)-sequence \( (w_p)_p \) in \( R^+ \) such that for all \( p \in \mathbb{N} \) and whenever \( \mu(B) \leq \sigma_p \) it is \( \rho(\alpha f 1_B) \leq w_p \);

(a.ₚ(2)) there is an \( (\alpha) \)-sequence \( (z_m)_m \) in \( X^+ \) such that to each \( m \in \mathbb{N} \) there corresponds a set \( B_m \in \mathcal{A}_b \) with \( \rho(\alpha f 1_{G \setminus B_m}) \leq z_m \).

(acₚ) Given a modular \( \rho \) and any free filter \( \mathcal{F} \) of \( Z \), a sequence \( f_z : G \to \mathbb{R} \), \( z \in Z \), is said to be \( \rho, \mathcal{F} \)-equi-absolutely continuous, or in short \( \rho \)-\( \mathcal{F} \)-absolutely continuous, iff there is \( \alpha \in \mathbb{R}^+ \), satisfying the following two conditions.

(acₚ(1)) For every \( (\alpha) \)-sequence \( (\sigma_p)_p \) in \( X^+_1 \) there are an \( (\alpha) \)-sequence \( (w_p)_p \) in \( X^+ \) and a sequence \( (\Lambda^p)_p \in \mathcal{F} \) with \( \rho(\alpha f_z 1_B) \leq w_p \) whenever \( z \in \Lambda^p \) and \( \mu(B) \leq \sigma_p \), \( p \in \mathbb{N} \);

(acₚ(2)) there are an \( (\alpha) \)-sequence \( (r_m)_m \) in \( X^+ \) and a sequence \( (B_m)_m \in \mathcal{A}_b \) such that, for all \( m \in \mathbb{N} \), it is

\[
\Lambda^m := \{ z \in Z : \rho(\alpha f_z 1_{G \setminus B_m}) \leq r_m \} \in \mathcal{F}.
\]

(2.2)

The concepts of filter uniform convergence and convergence in measure are recalled (see also [13]). Let \( \mathcal{F} \) be any fixed free filter of \( Z \).

**Definitions 2.3.**

2.3.1) A sequence of functions \( (f_z)_{z \in Z} \) in \( X^+_1 \) is said to \( (\mathcal{F}, \mathcal{G}) \)-converge uniformly (shortly, converge uniformly) to \( f \), iff there exists an \( (\alpha) \)-sequence \( (\varepsilon_p)_p \) in \( R^+ \) with

\[
\{ z \in Z : \bigvee_{t \in G} |f_z(t) - f(t)| \leq \varepsilon_p e_1 \} \in \mathcal{F} \quad \text{for any } p \in \mathbb{N}.
\]

In this case one can write:

\[
(\mathcal{F}, \mathcal{G}) \lim z \left( \bigvee_{t \in G} |f_z(t) - f(t)| \right) = 0.
\]

2.3.2) Given a sequence \( (f_z)_z \in X^+_1 \) and \( f \in X^+_1 \), we say that \( (f_z)_z \) \( (\mathcal{F}, \mathcal{G}) \)-converges in measure (shortly, \( \mu \)-converges) to \( f \), iff there are two \( (\alpha) \)-sequences, \( (\varepsilon_p)_p \) in \( R^+ \), \( (\sigma_p)_p \) in \( X^+_1 \), and a double sequence \( (\Lambda^p_z)_{(z,p) \in Z \times \mathbb{N}} \) in \( \mathcal{A} \) such that \( \Lambda^p_z \supset \{ u \in G : |f_z(u) - f(v)| \leq \varepsilon_p e_1 \} \) for every \( z \in Z \) and \( p \in \mathbb{N} \), and \( \{ z \in Z : \mu(\Lambda^p_z) \leq \sigma_p \} \in \mathcal{F} \) for all \( p \in \mathbb{N} \).

Using these assumptions and notations, in [15, Theorem 2.3] a Vitali-type theorem was obtained. For a historical overview on this topic see also [19, 28] and their bibliographies.
Theorem 2.4. (Vitali) Let \( \rho \) be a monotone and finite modular, and \( \mathcal{F} \) be a fixed free filter of \( Z \). If \((f_z)_z\) is a sequence in \( T \), \( \mu \)-convergent to 0 and equi-absolutely continuous, then there is a positive real number \( \alpha \) with
\[
(\alpha \mathcal{F}) \lim \rho(\alpha f_z) = 0.
\]
As an application, a Cauchy-type property for \( \rho \)-convergence of function sequences can be obtained.

Theorem 2.5. Let \( \mathcal{F} \) and \( \rho \) be as in Theorem 2.4, and suppose that \((f_n)_n \in \mathbb{N}\) is a sequence in \( T \), such that the double sequence \((f_h - f_q)_{h,q}(\alpha \mathcal{F})\)-converges in measure to 0. If \((f_n)_n \in \mathbb{N}\) is equi-absolutely continuous, then there is \( \alpha > 0 \) with
\[
(\alpha \mathcal{F}) \lim_{h,q} \rho(\alpha f_h - f_q) = 0.
\]

Proof. It is enough to replace, in Theorem 2.4, \( Z \) with \( \mathbb{N} \times \mathbb{N} \) and \( \mathcal{F} \) with \( \mathcal{F} \otimes \mathcal{F} \), respectively. \( \square \)

As a consequence it follows

Corollary 2.6. Let \( \mathcal{F} \) and \( \rho \) be as in Theorem 2.4, and suppose that \((f_n)_n \in \mathbb{N}\) is \((\alpha \mathcal{F})\)-convergent in measure to 0. If there exist an absolutely continuous function \( g \) in \( T \) and an element \( F_0 \in \mathcal{F} \), such that \(|f_z(t)| \leq g(t)\) for all \( z \in F_0 \) and \( t \in G \), then there is a positive real number \( \alpha \) with \((\alpha \mathcal{F}) \lim \rho(\alpha f_z) = 0.
\]

3. \( L^p \) Spaces

Some definitions which will be used in the sequel are recalled here for the sake of simplicity.

Definition 3.1. [15, Definition 3.2] A function \( f \in X^G_1 \) is said to be simple iff \( f(G) \) is a finite set and \( f^{-1}\{x\} \in \mathcal{A} \) for every \( x \in X_1 \). The space of all simple functions is denoted by \( \mathcal{S} \).

Let \( L^* \) be the set of all simple functions \( f \in \mathcal{S} \) vanishing outside a set of finite \( \mu \)-measure. If \( f \in L^* \), its usual integral is denoted by \( \int_G f(t) d\mu(t) \). It is not difficult to see that the functional \( \iota : L^* \to X \) defined as
\[
\iota(f) := \int_G |f(t)| d\mu(t), \quad f \in L^*.
\]
(3.1)
is a monotone finite modular, and it is also linear and additive on positive functions and constants (see also [15, Remark 3.3]).

Definition 3.2. ( [15, Definition 3.4]) A positive function \( f \in X^G_1 \) is integrable iff there exist an equi-absolutely continuous sequence of functions \((f_n)_n \in L^* \), \( \mu \)-convergent to \( f \), and a map \( l : \mathcal{A} \to X \), with
\[
(\alpha \mathcal{F}) \lim \mathcal{V}_{A \in \mathcal{A}} \mathcal{A} \left| \int_A f_n(t) d\mu(t) - l(A) \right| = 0
\]
(3.2)
(the sequence \((f_n)_n\) is said to be defining). In this case, we say that
\[
l(A) = (\alpha_F) \lim_n \int_A f_n(t) \, d\mu(t)
\]
uniformly with respect to \(A \in \mathcal{A}\).

Note that \(l(A)\) is independent of the choice of the defining sequence. If \(f \in X^G\), then \(f\) is said to be integrable iff the functions \(t \mapsto f(t) \vee 0\) and \(t \mapsto (-f(t)) \vee 0\) are integrable.

The \(L^p\) spaces in the vector lattice setting will be introduced now. Assume that \(X_1 = X\) (\(X\) is a lattice ordered algebra with complete multiplication, see also [53]), \(X_2 = \mathbb{R}\).

**Definition 3.3.** Let \(f \in X^G\). We say that \(f \in L^p\) iff both \(f\) and \(f^p\) belong to \(L\) according with Definition 3.2 with a common basic defining sequence \((f_n)_n\) (this means that \((f_n)_n\) is a defining sequence for \(f\) and \((f^p_n)_n\) is a defining sequence for \(f^p\), respectively).

It is not difficult to see that \(L^p\) is a linear space. Since homogeneity is straightforward, we just prove that \(L^p\) is stable with respect to addition. The \(\sigma\)-finiteness property for measurable functions will be proved first.

**Lemma 3.4.** Let \((f_n)_n\) be a sequence of simple functions, \(\mu\)-converging to a mapping \(f\). Then there exist an \((o)\)-sequence \((\beta_k)_k\) in \(\mathbb{R}^+\), an increasing sequence \((N_k)_k\) of positive integers and a sequence \((H_k)_k\) in \(\mathcal{A}\) with \(\mu(H_k) \leq \beta_k\) and \(\{t \in G : |f(t)| \leq N_k e\} \subset H_k\) for every \(k \in \mathbb{N}\).

**Proof.** Thanks to \(\mu\)-convergence, there exist an \((o)\)-sequence \((\sigma_k)_k\) in \(\mathbb{R}^+\), an \((o)\)-sequence \((\varepsilon_k)_k\) in the real interval \([0, 1]\) and a sequence \((F^k)_k\) in \(\mathcal{F}\) such that, for every integer \(k\) and every \(z \in F^k\) there exists an element \(A^k_z \in \mathcal{A}\) satisfying \(\mu(A^k_z) \leq \sigma_k\) and \(\{t \in G : |f(t) - f_z(t)| \leq \varepsilon_k e\} \subset A^k_z\).

Now for every \(k\) let us denote by \(z_k\) the least element of \(F^k\) and by \(N_k\) any positive integer such that \(|f_{z_k}(t)| \leq N_k e\) for every \(t \in G\), and set \(H_k := A^k_{z_k}\). Then for every integer \(k\) it is \(\mu(H_k) \leq \sigma_k\) and \(\{t \in G : |f(t) - f_{z_k}(t)| \leq e\} \subset H_k\), and therefore \(\{t \in G : |f(t)| \leq (N_k + 1)e\} \subset H_k\), from which the assertion follows, just replacing \(\sigma_k\) with \(\beta_k\) and \(N_k + 1\) with \(N_k\). \(\square\)

**Proposition 3.5.** If \(f, g \in L^p\), then \(f + g \in L^p\).

**Proof.** Let \((f_n)_n\) and \((g_n)_n\) be two defining sequences for \(f, g\), respectively. From the properties of \(\mu\)-convergence it follows that the sequence \((f_n + g_n)_n\) is \(\mu\)-convergent to \(f + g\). Moreover, since \((f_n)_n\) and \((g_n)_n\) are equi-absolutely continuous (with respect to the modular \(l\)), it is not difficult to check that \((f_n + g_n)_n\) is too.

Indeed, if \((\sigma_k)_k\) is any \((o)\)-sequence in \(\mathbb{R}^+\), then there are two \((o)\)-sequences
Thus also the double sequence \((w_k)_k\) and \((w'_k)_k\) in \(X\) and two sequences \((\Xi_k)_k\), \((\Xi'_k)_k\) in \(F\), such that for every \(k \in \mathbb{N}\) it is \(\iota(f_z \cdot 1_{B}) \leq w_k\), \(\iota(g_z \cdot 1_{B}) \leq w'_k\) as soon as \(z \in \Xi_k \cap \Xi'_k\) and \(\mu(B) \leq \sigma_k\).

Thus, choosing \(w'_k = w_k + w'_k\) and \(\Xi'_k = \Xi_k \cap \Xi'_k\), it is \(\iota((f_n + g_n) \cdot 1_{B}) \leq w'_k\) as soon as \(\mu(B) \leq \sigma_k\) and \(z \in \Xi'_k\). This proves property \((ac_p(1))\).

Moreover, there are two \((\sigma)-\)sequences \((r_m)_m\), \((r'_m)_m\) in \(X^+\) and two sequences \((B_m)_m\), \((B'_m)_m\) in \(A\) such that the sets \(\Lambda_m := \{z \in Z : \iota(f_z \cdot 1_{G}) \leq r_m\}\), \(\Lambda'_m := \{z \in Z : \iota(g_z \cdot 1_{G}) \leq r'_m\}\) belong to \(F\). So, taking \(r^*_m = r_m + r'_m\) and \(B^*_m = B_m \cup B'_m\), for every \(z \in \Lambda_m \cap \Lambda'_m\) it is

\[
\iota((f_z + g_z) \cdot 1_{G \setminus B^*_m}) \leq \iota(f_z \cdot 1_{G \setminus B^*_m}) + \iota(g_z \cdot 1_{G \setminus B^*_m}) \leq \iota(f_z \cdot 1_{G \setminus B^*_m}) + \iota(g_z \cdot 1_{G \setminus B^*_m}) \leq r^*_m,
\]

which proves \((ac_p(2))\).

Now for the same reason, since \((f^n_p)_n\) and \((g^n_p)_n\) are defining sequences for \(f^p, g^p\), respectively, we deduce that \(((f^n \cdot 1_{A^k})^p))\) is an equi-absolutely continuous sequence (we recall that the absolute continuity is essentially a condition on \(|f|\)). From this, since \(|f^n + g^n| \leq (|f^n| + |g^n|)^p \leq 2^p(|f^n| \vee |g^n|)^p \leq 2^p(|f^n| + |g^n|)^p\), it is clear that the sequence \((f^n + g^n)^p\) is equi-absolutely continuous. The next step is to prove that the sequence \(((f^n + g^n)^p)_n\) is \(\mu\)-convergent to \((f+g)^p\). Thanks to \(\mu\)-convergence of the four sequences \((f^n)_n\), \((g^n)_n\), \((f^n'_p)_n\), \((g^n'_p)_n\), there exist an \((\sigma)-\)sequence \((\varepsilon_k)_k\) in \([0,1]\) and a sequence \((F_k)_k\) in \(F\) such that for every integer \(k\) and every \(z \in F_k\) there is \(A^k_z \in A\) such that \(\mu(A^k_z) \leq \varepsilon_k\), and furthermore

\[
\{t \in G : |f_z(t) - f(t)| \leq \varepsilon_k e\} \cup \{t \in G : |g_z(t) - g(t)| \leq \varepsilon_k e\} \subset A^k_z,
\]

\[
\{t \in G : |f'^n_z(t) - f^n(t)| \leq \varepsilon_k e\} \cup \{t \in G : |g'^n_z(t) - g^n(t)| \leq \varepsilon_k e\} \subset A^k_z.
\]

As in Lemma 3.4, without loss of generality, the quantities \(\beta_k\), \(N_k\), \(H_k\) here obtained can be considered the same for both \(f, g\).

Now a subsequence \((\varepsilon'_k)_k\) of \((\varepsilon_k)_k\) will be found such that \(\varepsilon'_k \leq ( kp^2 \cdot 1)\) for every \(k\). In correspondence with \((\varepsilon'_k)_k\), we denote by \((\sigma_k)_k\) and \((F'_k)_k\) the subsequences of \((\sigma_k)_k\) and \((F_k)_k\), respectively. Fix \(k \in \mathbb{N}\), choose an element \(z \in F'_k\) and let \(t \notin A^k_z \cup H_k\). Then it is

\[
(f_z(t) + g_z(t))^p - (f(t) + g(t))^p = [(f_z(t) - f(t)) + (g_z(t) - g(t))]
\]
\[
((f_z(t) + g_z(t))^{p-1} + (f_z(t) + g_z(t))^{p-2}(f(t) + g(t)) + \ldots + (f(t) + g(t))^{p-1}),
\]

and therefore \(|f_z(t) + g_z(t)|^p - (f(t) + g(t))^p| \leq 2\varepsilon'_k e\) \((N_k + 1)^p\) \(\leq \frac{2}{k} e \downarrow 0\). Since \(\mu(A^k_z \cup H_k) \leq \beta_k + \sigma_k\), this is sufficient to conclude the proof of \(\mu\)-convergence. Thus also the double sequence \((n, k) \mapsto (f_n + g_n)^p - (f_k + g_k)^p\) is equi-absolutely continuous and \(\mu\)-convergent to 0 (with respect to the product filter \(\mathcal{F} \otimes \mathcal{F}\)).

From this, thanks to [15, Theorem 2.3], the integrals

\[
\int_G (|f_n(t) + g_n(t)|^p - (f_k(t) + g_k(t))^p) d\mu(t)
\]
converge to 0 as \( n, k \to \infty \), and so

\[
\int_E \left( (f_n(t) + g_n(t))^p - (f_k(t) + g_k(t))^p \right) d\mu(t)
\]

converge to 0 uniformly with respect to \( E \), which in turn implies that the limit

\[
(o_F) \lim_n \int_E (f_n(t) + g_n(t))^p d\mu(t)
\]

exists uniformly with respect to \( E \) (see also [18, Proposition 2.14]), thus proving that \( ((f + g)^p)_n \) is a defining sequence for \( (f + g)^p \).

One easily sees that, if \( f \in L^p \), also \(|f|\) is and the map \( f \mapsto \int_G |f|^p d\mu \) is a monotone and finite modular in \( L^p \).

A norm in the space \( L^p \) can be defined in the following way:

\[
\|f\|_p := \inf \left\{ \varepsilon > 0 : \int_G \left( \frac{|f(t)|}{\varepsilon} \right)^p d\mu(t) \leq \varepsilon \right\} = \left\| \int_G |f(t)|^p d\mu(t) \right\|_e \frac{1}{\varepsilon},
\]

where \( \|x\|_e = \inf \{ \varepsilon > 0 : |x| \leq \varepsilon e \} \) is the \( M \)-norm in \( X \) associated with \( e \) (see also [26]).

It is not difficult to verify that \( \| \cdot \|_p \) is positively homogeneous; in order to prove the Minkowski inequality the Maeda-Ogasawara-Vulikh representation theorem for vector lattices is needed (see also [32,41,52]).

**Theorem 3.6.** (MOV representation [52, §1]) Given a Dedekind complete vector lattice \( X \) with order unit \( e \), there exists a compact extremely disconnected topological space \( \Omega \), unique up to homeomorphisms, such that \( X \) is algebraically and lattice isomorphic to \( C(\Omega) := \{ h \in \mathbb{R}^\Omega : h \text{ is continuous} \} \).

This allows to prove the following

**Proposition 3.7.** (Minkowski inequality) For every \( p \in \mathbb{N} \) and \( f, g \in L^p \), it is

\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

**Proof.** Let \( \Omega \) be as in Theorem 3.6. By Remark 2.1 and [42, Lemma], for every \( \alpha \in ]0,1[ \), \( t \in G \) and \( \omega \in \Omega \) it is

\[
(|f(t)(\omega)| + |g(t)(\omega)|)^p \leq \alpha^{1-p}|f(t)(\omega)|^p + (1 - \alpha)^{1-p}|g(t)(\omega)|^p.
\]

Since, by [32, Corollary 2.20] and Theorem 3.6 the representation preserves the multiplication, from (3.3) it is for each \( \alpha \in ]0,1[ \) and \( t \in G \) it is

\[
(|f(t)| + |g(t)|)^p \leq \alpha^{1-p}|f(t)|^p + (1 - \alpha)^{1-p}|g(t)|^p.
\]
Example 3.8. Following [11, Example 2.2], let $L$ shows.

In this setting the space $\mu$ implies that $\mu$ and by the monotonicity of the $M$-norm, the conclusion follows.

Moreover one can see that, when $\|f\|_p = 0$, then $\int_G |f(t)|^p d\mu(t) = 0$, which implies that $\mu(E) = 0$ for each $\varepsilon > 0$ and every $E \in \mathcal{A}$ such that $f(t) \geq \varepsilon e$ for all $t \in E$. This property can be expressed by saying that $f$ is essentially $\mu$-null. The essentially $\mu$-null functions in $L^p$ form a subspace, which allows us to introduce an equivalence relation in $L^p$, by setting

$$f \sim g \text{ iff } f - g \text{ is essentially } \mu\text{-null.}$$

There is a large literature on completeness of $L^p$ spaces, see for example [11,31]. In this setting the space $L^p$ is not complete in general, as the following example shows.

Example 3.8. Following [11, Example 2.2], let $G = \mathbb{N}$ and $\mathcal{A}$ be the family of all finite or cofinite sets. Observe that the completion of $\mathcal{A}$ in the sense of [11, Section 1.6] is $\mathcal{P}(\mathbb{N})$. Let

$$\mu(A) := \left\{ \begin{array}{ll}
\mu(A) = \sum_{i \in A} 2^{-i}, & \text{if } A \text{ is finite,} \\
\mu(A) = 2 - \sum_{i \in A^c} 2^{-i}, & \text{if } A \text{ is cofinite.}
\end{array} \right. \quad (3.4)$$

In this setting, the $\mu$-convergence of a sequence $f_n$ to $f$ can be expressed in the following way: for every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that, for every $n > n(\varepsilon)$, it is $\mu^\ast(\{s \in \mathbb{N} : |f_n(s) - f(s)| \geq \varepsilon\}) \leq \varepsilon$, where $\mu^\ast$ denotes the usual outer measure.

Let $A_n = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. The sequence $(1_{A_n})_n \subset L^\ast$ and then $1_{A_n} \in L^p$ for every $p \in \mathbb{N}$. Note that $(1_{A_n})_n$ is a Cauchy sequence in $L^p$. In fact, for every $m, n \in \mathbb{N}$, it is $\int_{\mathbb{N}} |1_{A_n}(s) - 1_{A_m}(s)|^p d\mu(s) = e^p \mu(A_n \triangle A_m)$. Suppose, by
In N will show that f converges to contradiction, that L_{10} A BOCCUTO, D. CANDELORO, AND A. R. SAMBUCINI

A function Definition 4.3. for every v \in X for each s \in X such that \mu for every v \in X Definition 4.1. the following notions and results will be given.

Theorem 4.4. uniformly continuous on [a,b] X Then (\alpha < \beta \leq b) then the following characterization of convex uniform derivative X and \mathbb{R}^+, respectively, such that \|f(t) - f(t_2)\| \leq \sigma_p u whenever t_1, t_2 \in \mathbb{R}^+ and p \in \mathbb{N} satisfy |t_1 - t_2| \leq \delta_p.

Definition 4.2. ( [15, Definition 3.11]) Let I \subset \mathbb{R} be a connected set, and f : I \to X. The function f is said to be uniformly differentiable on I iff there exist a bounded function g : I \to X and two (o)-sequences, (\sigma_p)p and (\delta_p)p in X and \mathbb{R}^+, respectively, with

\[ \left\{|\frac{f(v) - f(u)}{v - u} - g(x)| : u \leq x \leq v, 0 < v - u \leq \delta_p \right\} \leq \sigma_p e \]

for every p \in \mathbb{N}. In this case g is said to be the uniform derivative of f in I.

Definition 4.3. A function f : X_1 \to X is said to be a convex function, iff for every v \in X_1 there exists \beta_v \in X_1 with

\[ f(s) \geq f(v) + \beta_v(s - v) \]

for each s \in X_1. When X_1 = \mathbb{R}, this means

\[ f(t) \leq f(t_1) + \frac{f(t_2) - f(t_1)}{t_2 - t_1} (t - t_1) \]

whenever t, t_1, t_2 \in \mathbb{R}, t_1 < t < t_2.

Let [a,b] be a compact subinterval of the real line. If f : [a,b] \to X is uniformly continuous on [a,b], then the following characterization of convex vector lattice-valued functions will be obtained.

Theorem 4.4. Assume that f : [a,b] \to X is a uniformly continuous function. Then f is convex if and only if for every \alpha < \beta \leq b it is

\[ f \left( \frac{\alpha + \beta}{2} \right) \leq \frac{f(\alpha) + f(\beta)}{2}. \]

4. Applications

4.1. Inequalities. In order to investigate some main inequalities in L^p spaces, the following notions and results will be given.
Proof. We begin with proving the "only if" part. We can show that (4.1) implies that \( f(\alpha + (1-c)\beta) \leq cf(\alpha) + (1-c)f(\beta) \) for all \( \alpha, \beta \) in \([a, b]\) and \( c \in [0, 1] \).

Then clearly (4.2) will follow immediately, taking \( c = 1/2 \). To prove the claim, fix \( \alpha, \beta, c \) as above, and take \( v = \alpha + (1-c)\beta \). Then from (4.1) it follows \( f(\beta) \geq f(v) + \beta_v c(\beta - \alpha) \), and \( f(\alpha) \geq f(v) + \beta_v (1-c)(\alpha - \beta) \), by choosing first \( s = \beta \) and then \( s = \alpha \). Now it is \((1-c)f(\beta) \geq (1-c)f(v) + \beta_v c(1-c)(\beta - \alpha), \)
and \( cf(\alpha) \geq cf(v) + \beta_v c(1-c)(\alpha - \beta) \). Summing up the two last inequalities, the implication follows.

We now prove the "if" part. By usual techniques, from (4.2) it follows that \( f(q\alpha + (1-q)\beta) \leq qf(\alpha) + (1-q)f(\beta) \) for each \( \alpha, \beta \in [a, b] \) with \( \alpha < \beta \), and for any dyadic rational number \( q \in [0, 1] \). Since the set of all dyadic rationals of \([0, 1]\) is dense in \([0, 1]\), by continuity of \( f \) it is not difficult to deduce that

\[
f(\alpha + (1-c)\beta) \leq cf(\alpha) + (1-c)f(\beta) \tag{4.3}
\]

for any \( \alpha, \beta \in [a, b] \) with \( \alpha < \beta \), and \( c \in [0, 1] \).

Now we claim that \( f \) is convex. Indeed, if \( t, t_1, t_2 \in [a, b] \) and \( t_1 < t < t_2 \), one can write \( t = dt_1 + (1-d)t_2 \), where \( d = \frac{t_2 - t}{t_2 - t_1} \). A simple application of the inequality in (4.3) yields \( f(t) \leq f(t_1) + \frac{f(t_2) - f(t_1)}{t_2 - t_1} (t - t_1) \). This ends the proof. \( \square \)

Some inequalities for convex vector lattice-valued functions will be proven now.

**Theorem 4.5.** If \( f : [a, b] \to X \) is uniformly continuous and \( \varphi : X \to X \) is convex, then \( \varphi \circ f \) is uniformly continuous too.

**Proof.** By uniform continuity of \( f \) there are a positive element \( u \in X \) and two \((a)\)-sequences \((\sigma_p)_{p}, (\delta_p)_{p}\) in \( X \) and \( \mathbb{R}^+ \), respectively, such that \( |f(t_1) - f(t_2)| \leq \sigma_p u \) for all \( t_1, t_2 \in \mathbb{R}^+ \) and \( p \in \mathbb{N} \) satisfying \( |t_1 - t_2| < \delta_p \). Using (4.1) with \( v = f(t_1) \), it is \( \varphi(f(t_2)) \geq \varphi(f(t_1)) + \beta_{f(t_1)}(f(t_2) - f(t_1)) \), while for \( v = f(t_2) \) one has \( \varphi(f(t_1)) \geq \varphi(f(t_2)) + \beta_{f(t_2)}(f(t_1) - f(t_2)) \). Setting \( \alpha := |\beta_{f(t_1)}| \lor |\beta_{f(t_2)}| \), finally it follows \( |\varphi(f(t_1)) - \varphi(f(t_2))| \leq \alpha |f(t_1) - f(t_2)| \leq \alpha u \sigma_p \) whenever \( |t_1 - t_2| < \delta_p \). \( \square \)

**Theorem 4.6.** (Jensen inequality) If \( f : [a, b] \to X \) is uniformly continuous and \( \varphi : X \to X \) is convex, then

\[
\varphi(\int_a^b f(t)d\mu(t)) \leq \int_a^b \varphi(f(t))d\mu(t)
\]

whenever \( \mu([a, b]) = 1 \).
Proof. By [15, Proposition 3.12] and Proposition 4.5, both \( f \) and \( \varphi \circ f \) are bounded and integrable. Let \( \tau := \int_a^b f(t) d\mu(t) \) and \( m, M \in X \) be such that \( m \leq f(t) \leq M \) for every \( t \in [a, b] \). Let \( \beta_\tau \) be the element associated to \( \tau \) according to Definition 4.3. Then
\[
\varphi(f(t)) \geq \varphi(\tau) + \beta_\tau(f(t) - \tau).
\]
Integrating both members, it is
\[
\int_a^b \varphi(f(t)) d\mu(t) \geq \varphi(\int_a^b f(t) d\mu(t)) + \int_a^b \beta_\tau(f(t) - \tau)d\mu(t) = \varphi(\int_a^b f(t) d\mu(t)),
\]
that is the assertion. \( \square \)

Theorem 4.7. Let \( f : [a, b] \to X \) be a uniformly differentiable and convex function, and set
\[
r(t) := f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right), \quad t \in [a, b]. \tag{4.4}
\]
Then \( f(t) \geq r(t) \) for every \( t \in [a, b] \).

Proof. Fix arbitrarily \( t_1, t_2 \in [a, b] \) with \( t_1 < t_2 \). By convexity of \( f \), for every \( t \in [t_1, t_2] \), it is
\[
\frac{f(t) - f(t_1)}{t - t_1} \leq \frac{f(t_2) - f(t_1)}{t_2 - t_1}, \quad \frac{f(t) - f(t_2)}{t - t_2} \geq \frac{f(t_2) - f(t_1)}{t_2 - t_1}. \tag{4.5}
\]
Thanks to uniform differentiability of \( f \), (4.5) implies
\[
f'(t_1) \leq \frac{f(t_2) - f(t_1)}{t_2 - t_1} \leq f'(t_2). \tag{4.6}
\]
Using the first inequality in (4.6) with \( t_1 = \frac{a+b}{2}, \quad t_2 = t \), it follows
\[
f(t) \geq f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(t - \frac{a+b}{2}\right) = r(t).
\]
By applying the second inequality in (4.6) with \( t_1 = t, \quad t_2 = \frac{a+b}{2} \), we obtain again \( f(t) \geq r(t) \). The assertion follows from arbitrariness of \( t_1 \) and \( t_2 \). \( \square \)

Theorem 4.8. (Hermite-Hadamard inequality) Let \( f : [a, b] \to X \) be a uniformly differentiable and convex function. Then
\[
f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) d\mu(t) \leq \frac{f(a) + f(b)}{2}.
\]
Proof. Let \( r \) be as in (4.4), and set
\[
    r^*(t) = f(a) + \frac{f(b) - f(a)}{b - a} (t - a), \quad t \in [a, b]. \tag{4.7}
\]
By [15, Proposition 3.12], \( f \in L^1(\lambda) \) and
\[
    \int_a^b r(t) \, d\mu(t) = (b - a) f\left(\frac{a + b}{2}\right) \leq \int_a^b f(t) \, d\mu(t).
\]
Since \( f \) is convex, \( f(t) \leq r^*(t) \) for every \( t \in [a, b] \) and
\[
    \int_a^b f(t) \, d\mu(t) \leq \int_a^b r^*(t) \, d\mu(t) = (b - a) \frac{f(a) + f(b)}{2}.
\]
\[\square\]

**Theorem 4.9.** (Féjer inequality) Let \( f : [a, b] \to \mathbf{X} \) be a uniformly differentiable and convex function and let \( w : [a, b] \to \mathbb{R}^{+} \) be a uniformly continuous map such that \( w(a + t) = w(b - t) \) for every \( 0 \leq t \leq \frac{a + b}{2} \). Then
\[
    f\left(\frac{a + b}{2}\right) \int_a^b w(t) \, d\mu(t) \leq \int_a^b f(t) w(t) \, d\mu(t) \leq \frac{f(a) + f(b)}{2} \int_a^b w(t) \, d\mu(t).
\]

Proof. Again by [15, Proposition 3.12], \( f w \) is in \( L^1(\lambda) \). Let \( r, r^* \) be as in (4.4) and (4.7), respectively. Then
\[
    r(t) w(t) \leq f(t) w(t) \leq r^*(t) w(t), \quad t \in [a, b]. \tag{4.8}
\]
By integrating all members in (4.8) the assertion follows, since
\[
    \frac{b + a}{2} \int_a^b w(t) \, d\mu(t) = \int_a^b tw(t) \, d\mu(t).
\]
\[\square\]

In this setting the case \( p = 2 \) is particularly interesting. Indeed it is:

**Theorem 4.10.** (Schwartz inequality) If \( f, g \in L^2 \), then \( fg \) is integrable. Moreover it is
\[
    \left( \int_G |f(t)g(t)| \, d\mu(t) \right)^2 \leq \left( \int_G (f(t))^2 \, d\mu(t) \right) \left( \int_G (g(t))^2 \, d\mu(t) \right),
\]
\[
    \sqrt{\left( \int_G |f(t)g(t)| \, d\mu(t) \right)^2} \leq \|f\|_2 \|g\|_2.
\]

Proof. Since \( f, g \in L^2 \), there exist defining sequences \( (f_n)_n \) and \( (g_n)_n \), related to \( f \) and \( g \), such that \( (f_n^2)_n \) and \( (g_n^2)_n \) are defining for \( f^2, g^2 \) respectively. Then, \( f + g \in L^2 \) and \( (f_n + g_n)^2 \) is defining for \( (f + g)^2 \) (see also Proposition 3.5). This means that \( f_n g_n := 1/2 ((f_n + g_n)^2 - f_n^2 - g_n^2) \), \( n \in \mathbb{N} \), is a defining sequence for
Now (without loss of generality, we can take the same partition for both mappings). Therefore, \( f \) functions thanks to the definition of integral, it is sufficient to prove it just for simple functions \( f \) and \( g \). So, assume that

\[
f = \sum_{i=1}^{n} c_i 1_{E_i}, \quad g = \sum_{i=1}^{n} d_i 1_{E_i}
\]

(without loss of generality, we can take the same partition for both mappings).

Now

\[
\int_{G} f(t)g(t)\,d\mu(t) = \sum_{i=1}^{n} c_i d_i \mu(E_i), \quad \int_{G} (f(t))^2\,d\mu(t) = \sum_{i=1}^{n} c_i^2 \mu(E_i),
\]

\[
\int_{G} (g(t))^2\,d\mu(t) = \sum_{i=1}^{n} d_i^2 \mu(E_i).
\]

Therefore,

\[
\left(\int_{G} f(t)g(t)\,d\mu(t)\right)^2 = \sum_{i=1}^{n} c_i^2 d_i^2 \mu(E_i) + 2 \sum_{i<j} c_i c_j d_i d_j \mu(E_i)\mu(E_j),
\]

\[
\left(\int_{G} (f(t))^2\,d\mu(t)\right)^2 \left(\int_{G} (g(t))^2\,d\mu(t)\right)^2 = \sum_{i} c_i^2 d_i^2 \mu(E_i)\mu(E_j) + \sum_{i<j} c_i^2 d_j^2 \mu(E_i)\mu(E_j) + \sum_{i<j} c_j^2 d_i^2 \mu(E_i)\mu(E_j).
\]

Comparing the last two formulas, it is clear that

\[
\left(\int_{G} (f(t))^2\,d\mu(t)\right)^2 \cdot \left(\int_{G} (g(t))^2\,d\mu(t)\right)^2 - \left(\int_{G} f(t)g(t)\,d\mu(t)\right)^2 = \sum_{i<j} (c_i^2 d_j^2 + c_j^2 d_i^2 - 2c_i c_j d_i d_j) \mu(E_i)\mu(E_j).
\]

Since \( c_i^2 d_j^2 + c_j^2 d_i^2 - 2c_i c_j d_i d_j = (c_i d_j - c_j d_i)^2 \) for all \( i, j \), easily it follows

\[
\left(\int_{G} f(t)g(t)\,d\mu(t)\right)^2 \leq \left(\int_{G} (f(t))^2\,d\mu(t)\right)^2 \left(\int_{G} (g(t))^2\,d\mu(t)\right)^2.
\]

Taking the norm, we finally deduce

\[
\left\| \left(\int_{G} f(t)g(t)\,d\mu(t)\right)^2 \right\|_e \leq \left\| \left(\int_{G} (f(t))^2\,d\mu(t)\right)^2 \left(\int_{G} (g(t))^2\,d\mu(t)\right)^2 \right\|_e \leq \left\| \left(\int_{G} (f(t))^2\,d\mu(t)\right)^2 \right\|_e \cdot \left\| \left(\int_{G} (g(t))^2\,d\mu(t)\right)^2 \right\|_e.
\]
where the last inequality follows from the idempotence of $e$. In conclusion
\[
\sqrt{\left\| \left( \int_G f(t)g(t)d\mu(t) \right) \right\|_e} \leq \|f\|_2\|g\|_2.
\]

\[\square\]

4.2. The Brownian Motion. In order to obtain a concrete application in Stochastic Integration, we assume that $B := (B_t)_{0 \leq t < T} : [0, T] \rightarrow L^2(\Omega)$ (with $T < +\infty$) is the standard Brownian Motion defined on a probability space $(\Omega, \Sigma, P)$. As observed in [15], $L^2$ has not an order unit, but thanks to the well-known Maximum Principle (see e.g. [23]) there is a positive element $Z \in L^2$ with $|B(t)| \leq Z$ for all $t \in [0, T]$. Moreover, there exists a positive random variable $W$ in $L^2$ such that $|B(t + h) - B(t)| \leq |h|^{1/4}W$ whenever $t, t + h \in [0, T]$ (see e.g. [33, 44]). Thus, taking $X$ as the (complete) subspace of $L^2$ generated by all elements dominated by some real multiple of $W + Z$, we see that $X$ has an order unit (i.e. $W + Z$), and $B$ is a uniformly continuous $X$-valued function defined on $[0, T]$.

Example 4.11. Fix now $T > a > 1$, and set
\[
f(t) := \begin{cases} 
0, & \text{if } 0 \leq t \leq a \text{ or } t \geq T, \\
(t - T)(B_t - B_a), & \text{if } a \leq t \leq T.
\end{cases}
\]

Note that $f$ is a uniformly continuous function from $[0, +\infty[$ to $L^2(\Omega)$. Then
\[
\Phi(x, s) = \frac{x}{s^x} \int_0^s f(t)t^{x-1}d\mu(t) \quad (4.9)
\]
is a solution of the partial differential equation $s \frac{\partial^2 \Phi}{\partial x \partial s} + x \frac{\partial \Phi}{\partial x} + \Phi(x, s) = f(s)$.

Indeed, it is sufficient to differentiate with respect to $x$ the equation
\[
s \frac{\partial \varphi(x, s)}{\partial s} + x \varphi(x, s) = xf(s), \quad (4.10)
\]
taking into account that it admits the solution $\varphi(x, s) = \frac{1}{s^x} \left( c + x \int_0^s f(t)t^{x-1}d\mu(t) \right)$, where $c$ is an arbitrary real constant.

A more general solution for the previous partial differential equation is
\[
\Phi(x, s) = F(s) + \frac{G(x)}{s^x} + \frac{x}{s^x} \int_0^s f(t)t^{x-1}d\mu(t),
\]
where $F$ and $G$ are arbitrary sufficiently regular functions.

In the following two figures, the graphs of the square of the Brownian Bridge $f$ and its corresponding surface $\varphi(x, s)$ are represented, respectively.
The two graphs show also that the surface $\varphi(x, s)$ uniformly converges to $f(s)$ as $x$ tends to $+\infty$, according to [15, Proposition 3.13]: this could be verified, comparing the graph of Figure 1 with the section of the surface in Figure 2 corresponding to $x = 50$.

The previous example suggests a different type of operator, acting with respect to stochastic differentials. Indeed, with the same notations as above,
Example 4.12. Let

\[ \Psi(x, s) := \frac{x}{s^x} \int_0^s t^{x-1} df(t), \]

where now \( df(t) \) is the stochastic differential of the Brownian Bridge \( f(t) \), namely \( df(t) = (B_t - B_a) dt + (t - T) dB_t \), when \( a \leq t \leq T \).

The existence of the above integral is ensured by the fact that \( s \mapsto s^{x-1} \) is a function of class \( C^1 \), and the trajectories of \( f(t) \) are continuous maps. Moreover, the well-known formula of integration by parts leads to

\[ \Psi(x, s) = \frac{x}{s^x} \left( s^{-1} x f(s) - \int_0^s f(t) dt^{x-1} \right) = \frac{x}{s} f(s) - \frac{x(x-1)}{s^x} \int_0^s f(t) t^{x-2} dt. \]

So, in conclusion

\[ \Psi(x, s) = \frac{x}{s} (f(s) - \Phi(x-1, s)), \]

(4.11)

where \( \Phi \) is as in (4.9). This leads to a stochastic differential equation satisfied by \( \Psi \).

However, in order to avoid confusion, we point out that expressions like \( d\Psi \) or \( d\Phi \) always refer to the variable \( s \), since \( x \) is just a parameter.

Differentiating (4.11), it follows

\[ d\Psi(x, s) = -\frac{x}{s} (f(s) - \Phi(x-1, s)) ds + \frac{x}{s} (df(s) - \Phi'(x-1, s) ds). \]

(Here the stochastic differential involved is \( df \), rather than \( dB \)).

Thanks to (4.11) and (4.10), it is

\[ d\Psi(x, s) = -\frac{1}{s} \Psi(x, s) ds + \frac{x}{s} df(s) - \frac{x(x-1)}{s} \Phi(x-1, s) ds, \]

and, again by (4.11), \( d\Psi(x, s) = -\frac{1}{s} \Psi(x, s) ds + \frac{x}{s} df(s) - \frac{x-1}{s} \Psi(x, s) ds \). In conclusion, after a final simplification, \( \Psi(x, s) \) satisfies the stochastic equation

\[ d\Psi(x, s) = -\frac{x}{s} \Psi(x, s) ds + \frac{x}{s} df(s). \]

(4.12)

A more general solution of the last equation is

\[ \Psi(x, s) = \frac{1}{s^x} \left( k + x \int_0^s t^{x-1} df(t) \right), \]

where \( k \) is an arbitrary constant.

Another interesting application of this type of operator can be found in detecting a regular signal, when it is distorted by a random noise. More precisely
Example 4.13. Let \( h : [0, +\infty[ \to \mathbb{R} \) be any \( C^1 \) map, satisfying the condition that \( h(x) = h'(x) = 0 \) for all \( x \in [0, a] \) and \( x \geq T \), where \( a, T \) are fixed positive numbers, \( 1 < a < T \). As above, let \( f \) denote the Brownian Bridge, like in Example 4.11, and fix any positive number \( \varepsilon \). Then set \( G(s) = h(s) + \varepsilon f(s) \), for \( s \in [0, +\infty[ \), and

\[
\Psi_G(x, s) = \frac{x}{s^x} \int_0^s t^{x-1} dG(s),
\]

i.e.

\[
\Psi_G(x, s) = \frac{x}{s^x} \int_0^s t^{x-1} h'(t) dt + \frac{x}{s^x} \int_0^s \varepsilon t^{x-1} df(t).
\]

Thanks to (4.10) and (4.11),

\[
\Psi_G(x, s) = \Phi_{h'}(x, s) + \frac{\varepsilon x}{s} \left( f(s) - \Phi_f(x - 1, s) \right),
\]

where the operator \( \Phi \) has the same meaning as in Example 4.11. Now, thanks to [15, Proposition 3.13],

\[
\lim_{x \to +\infty} \Phi_{h'}(x, s) = h'(s)
\]

and

\[
\lim_{x \to +\infty} \left( f(s) - \Phi_f(x - 1, s) \right) = 0,
\]

both uniformly with respect to \( s \). So, choosing \( x = \varepsilon^{-1} \) in the above quantities, it is

\[
\lim_{\varepsilon \to 0} \Psi_G(\varepsilon^{-1}, s) = h'(s).
\]

Conclusions

After an abstract introduction of \( L^p \) spaces for vector lattice-valued functions, some applications are found, first extending some classical inequalities to the vector lattice setting, and then finding approximations for stochastic processes like the Brownian Motion and the Brownian Bridge, based on the moment operator; as an outcome, this method leads to solve some stochastic differential equations.

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