A New Unicity Theorem and Erdös’ Problem for Polarized Semi-Abelian Varieties

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§1 Introduction

1.1 Nevanlinna’s unicity theorem.

**Theorem 1.1**

(Unicity Theorem) Let \( f, g : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \) be two non-constant meromorphic functions.

If \( \exists a_i \in \mathbb{P}^1(\mathbb{C}), 1 \leq i \leq 5, \) distinct such that
\[
\text{Supp } f^* a_i = \text{Supp } g^* a_i, 1 \leq i \leq 5,
\]
then \( f \equiv g. \)

This follows from Nevanlinna’s Second Main Theorem (SMT):

**Theorem 1.2**

(SMT) Let \( f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \) be a meromorphic function, and \( a_i \in \mathbb{P}^1(\mathbb{C}), 1 \leq i \leq q, \) be distinct \( q \) points. Then
\[
(q - 2) T_f(r) \leq \sum_{i=1}^{q} N(r, \text{Supp } f^* a_i) + \text{small-term}.
\]
Proof of Theorem 1.1.

By Nevanlinna’s SMT 1.2 we have

\[(5 - 2 = 3) \, T_{f(\text{ or } g)}(r) \leq \sum_{i=1}^{5} N(r, \text{Supp } f^*(\text{ or } g^*)a_i) + \text{small-term.}\]

Suppose \( f \not\equiv g \). Then the assumption implies that

\[\sum_{i=1}^{5} N(r, \text{Supp } f^*a_i) \leq N(r, (f - g)_0) \leq T_{f-g}(r) + O(1)\]

\[\leq T_f(r) + T_g(r) + O(1) \leq \frac{2}{3} \sum_{i=1}^{5} N(r, \text{Supp } f^*a_i) + \text{small-term.}\]

Thus, \( 1 \leq \frac{2}{3} \); a contradiction.
Remark.

The number 5 in the above unicity theorem is optimal for the following trivial reason: Set
\[ f(z) = e^z, \ g(z) = e^{-z}; \ a_1 = 0, a_2 = \infty, a_3 = 1, a_4 = -1. \]
Then \( f^*a_i = g^*a_i, \ 1 \leq i \leq 4. \)

Note that by setting \( \sigma(w) = w^{-1} \) and \( D = \sum_{1}^{4} a_i \) we have
\[ \sigma^*D = D, \ \ \sigma \circ f = g; \ \ f(z), g(z) \in \mathbb{C}^*. \]

**Theorem 1.3**

(E.M. Schmid 1971) *Let \( E \) be an elliptic curve, \( a_i \in E, 1 \leq i \leq 5, \) distinct points. Let \( f, g : \mathbb{C} \to E \) be holomorphic maps. If \( \text{Supp} \ f^*a_i = \text{Supp} \ g^*a_i, \ 1 \leq i \leq 5, \) then \( f \equiv g. \)
Theorem 1.4

(H. Fujimoto (1975)) Let \( f, g : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C}) \) be holomorphic curves such that at least one of them is linearly non-degenerate; \( \{H_j\}_{j=1}^{3n+2} \) be hyperplanes in general position. If \( f^*H_j = g^*H_j, 1 \leq j \leq 3n+2 \) (as divisors, counting multiplicities), then \( f \equiv g \).

Schmid’s and Fujimoto’s theorems are deduced from some SMT’s in the corresponding cases.

The following is a kind of unicity problem in arithmetic theory, which is sometimes called a “support problem”:

Erdös’ Problem (1988). Let \( x, y \) be positive integers. Is it true that

\[
\{p; \text{prime}, p| (x^n - 1)\} = \{p; \text{prime}, p| (y^n - 1)\}, \forall n \in \mathbb{N}
\]

\[\iff x = y \, ?\]
The answer is Yes:

**Theorem 1.5**

(Corráles-Rodrigáñez and R. Schoof, JNT 1997)

1. Suppose that except for finitely many prime $p \in \mathbb{Z}$

\[ y^n \equiv 1 \pmod{p} \text{ whenever } x^n \equiv 1 \pmod{p}, \forall n \in \mathbb{N}. \]

Then, $y = x^h$ with $\exists h \in \mathbb{N}$.

2. Let $E$ be an elliptic curve defined over a number field $k$, and let $P, Q \in E(k)$. Suppose that except for finitely many prime $p \in O(k)$

\[ nQ = 0 \text{ whenever } nP = 0 \text{ in } E(k_p). \]

Then either $Q = \sigma(P)$ with $\exists \sigma \in \text{End}(E)$, or both $P, Q$ are torsion points.
Yamanoi’s unicity theorem

Yamanoi proved in Forum Math. 2004 the following striking unicity theorem:

**Theorem 1.6**

Let $A_i, i = 1, 2$, be abelian varieties; $D_i \subset A_i$ be irreducible divisors such that

$$\text{St}(D_i) = \{a \in A_i; a + D_i = D_i\} = \{0\};$$

$f_i : \mathbb{C} \to A_i$ be (algebraically) nondegenerate entire holomorphic curves. Assume that $f_1^{-1}D_1 = f_2^{-1}D_2$ as sets. Then $\exists$ isomorphism $\phi : A_1 \to A_2$ such that

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^*D_2.$$
\section*{\textbf{N.B.}}

1. The new point is that we can determine not only $f$, but the moduli point of a polarized abelian variety $(A, D)$ through the distribution of $f^{-1}D$ by a nondegenerate $f : \mathbb{C} \to A$.

2. The assumptions for $D_i$ to be irreducible and the triviality of $\text{St}(D_i)$ are not restrictive. There is a way of reduction.

3. For simplicity we assume them here.
§2 Main Results

We want to uniformize the results in the previous section. Therefore we deal with semi-abelian varieties.

Let $A_i, i = 1, 2$ be semi-abelian varieties:

$$0 \rightarrow (\mathbb{C}^*)^{t_i} \rightarrow A_i \rightarrow A_{0i} \rightarrow 0.$$ 

Let $D_i \subset A_i, i = 1, 2$, be irreducible divisors such that $\text{St}(D_i) = \{0\}$ (for simplicity).

For real-valued functions $\phi(r)$ and $\psi(r)$ ($r > 1$), we write $\phi(r) \leq \psi(r)\|_E$ if $E \subset [1, \infty)$, Borel, $m(E) < \infty$, and $\phi(r) \leq \psi(r), r \notin E$.

$$\phi(r) \sim \psi(r)\| \iff \exists E, \exists C > 0, C^{-1}\phi(r) \leq \psi(r) \leq C\phi(r)\|_E.$$
Main Results

Main Theorem

Main Theorem 2.1

Let \( f_i : C \to A_i \) (\( i = 1, 2 \)) be non-degenerate holomorphic curves. Assume that

\[(2.2) \quad \text{Supp} \ f_1^* D_1 \subset \text{Supp} \ f_2^* D_2 \quad \text{(germs at } \infty), \]
\[(2.3) \quad N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2). \]

Here \( N_1(r, f_1^* D_1) = N(r, \text{Supp} \ f_1^* D_1) \).

Then there is a finite étale morphism \( \phi : A_1 \to A_2 \) such that

\[ \phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2. \]

If equality holds in (2.2), then \( \phi \) is an isomorphism and \( D_1 = \phi^* D_2 \).

N.B. Assumption (2.3) is necessary by example.
Main Results

The following corollary follows immediately from the Main Theorem 2.1.

Corollary 2.4

1. Let \( f : \mathbb{C} \to \mathbb{C}^* \) and \( g : \mathbb{C} \to E \) with an elliptic curve \( E \) be holomorphic and non-constant. Then

\[
\{1\}_\infty \neq \{0\}_\infty.
\]

2. If \( \dim A_1 \neq \dim A_2 \) in the Main Theorem 2.1, then

\[
f_1^{-1}D_1_\infty \neq f_2^{-1}D_2_\infty.
\]
The first statement means that the difference of the value distribution property caused by the quotient $\mathbb{C}^* \rightarrow \mathbb{C}^*/\langle \tau \rangle = E$ cannot be recovered by any later choice of $f$ and $g$, even though they are allowed to be \textit{arbitrarily transcendental}.

\[
\begin{array}{ccc}
\mathbb{C} & \overset{f}{\rightarrow} & \mathbb{C}^* \\
\downarrow g & & \downarrow /\langle \tau \rangle \\
\end{array}
\]

The second statement implies that the distribution of $f_i^{-1}D_i$ about $\infty$ contains the \textit{topological informations} such as $\dim A_i$ and the compactness or non-compactness of $A_i$. 

\[E\]
Example.

Set $A_1 = \mathbb{C}/\mathbb{Z} (\cong \mathbb{G}_m)$ and let $D_1 = 1$ be the unit element of $A_1$. Let $f_1: \mathbb{C} \to A_1$ be the covering map.

Take a number $\tau \in \mathbb{C}$ with $\Im \tau \neq 0$.

Set $A_2 = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$, which is an elliptic curve.

Let $D_2 = 0 \in A_2$ and $f_2: \mathbb{C} \to A_2$ be the covering map.

Then $f_1^{-1}D_1 = \mathbb{Z} \subset \mathbb{Z} + \tau\mathbb{Z} = f_2^{-1}D_2$: assumption (2.2) of the Main Theorem 2.1 is satisfied.

There is, however, no non-constant morphism $\phi: A_1 \to A_2$. Note that

$$N_1(r, f_1^*D_1) \sim r, \quad N_1(r, f_2^*D_2) \sim r^2.$$

Thus, $N_1(r, f_1^*D_1) \nsubseteq N_1(r, f_2^*D_2)$: assumption (2.3) fails.
For a closed subscheme $Z \subset X$ (\textit{compact} complex space) and holomorphic $f : C \to X$, $f(C) \not\subset \text{Supp } Z$, we write
\[
T_f(r, \omega_Z) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* \omega_Z,
\]
\[
f^*Z_{k,a} = \min \{\deg_a f^* Z, k\} \quad (k \leq \infty),
\]
\[
N_k(r, f^* Z) = \int_1^r \frac{dt}{t} \left( \sum_{a \in \Delta(t)} f^*Z_{k,a} \right),
\]
\[
N(r, f^* Z) = N_{\infty}(r, f^* Z) < T_f(r, \omega_Z) + O(1) \quad \text{(FMT)}.
\]

Let
\begin{itemize}
  \item $A$ be a semi-abelian variety,
  \item $f : C \to A$ be a holomorphic curve.
\end{itemize}

Set
\begin{itemize}
  \item $J_k(A) \cong A \times C^{nk}$: the $k$-jet bundle over $A$;
  \item $J_k(f) : C \to J_k(A)$: the $k$-jet lift of $f$;
  \item $X_k(f)$: the Zariski closure of the image $J_k(f)(C)$ in $J_k(A)$.
\end{itemize}
Theorem 3.1

(N.-Winkelmann-Yamanoi, Acta 2002 & Forum Math. 2008, Yamanoi Forum Math. 2004)

Let $f : C \to A$ be algebraically non-degenerate.

(i) Let $Z$ be an algebraic reduced subvariety of $X_k(f)$ ($k \geq 0$). Then

$\exists \overline{X}_k(f)$, compactification of $X_k(f)$ such that

$\left(3.2\right) \quad T_{J_k(f)}(r; \omega_Z) = N_1(r; J_k(f)^* Z) + o(T_f(r))||$.

(ii) Moreover, if $\text{codim} \ X_k(f) Z \geq 2$, then

$\left(3.3\right) \quad T_{J_k(f)}(r; \omega_Z) = o(T_f(r))||$.

(iii) If $k = 0$ and $Z$ is an effective divisor $D$ on $A$, then $\overline{A}$ is smooth, equivariant, and independent of $f$; furthermore, $\left(3.2\right)$ takes the form

$\left(3.4\right) \quad T_f(r; L(\overline{D})) = N_1(r; f^* D) + o(T_f(r; L(\overline{D})))||$. 
Let me first recall

**Theorem 4.1**

(Log Bloch-Ochiai, Nog. 1977 Hiroshima Math.J./81 Nagoya Math.J.)

Let $f : C \to A$ be a holomorphic curve into a semi-abelian variety $A$. Then $\overline{f(C)}^{\text{Zar}}$ is a translate of a subgroup.

**Proof of Main Theorem 2.1.** With the given $f_i : C \to A_i \ (i = 1, 2)$ we set

$g = (f_1, f_2) : C \to A_1 \times A_2$ ;

$A_0 = g(C)^{\text{Zar}}$ (semi-abelian variety by Log Bloch-Ochiai);

$p_i : A_0 \to A_i$ be the projections;

$E_i = p_i^*D_i$. 
It follows that

\[ T_{f_1}(r) \sim T_{f_2}(r) \sim T_g(r) = T(r). \]

By N. Math. Z. (1998) and a translation we may assume \( g(0) = 0 \in E_1 \).

Let \( E_i = \sum_{\nu}(F_i + a_{i\nu}) \) be the irred. decomp. and \( F_i \ni 0 \).

If \( F_1 \neq F_2 \), then \( \text{codim}_{A_0} F_1 \cap F_2 \geq 2 \). It follows from SMT Theorem 3.1 that

\[ T(r) \sim N_1(r, f_1^*D_1) \sim N_1(r, g^*(F_1 \cap F_2)) = o(T(r))|.| \]

This is a contradiction. Therefore we see that \( F_1 = F_2 \). Moreover, we deduce that

1. \( E_1 \subset E_2 \),
2. \( \text{St}(E_1) \subset \text{St}(E_2) \), and are finite,
3. \( p_i \) are isogenies,
4. \( A_1 \cong A_0/\text{St}(E_1) \xrightarrow{\phi} A_0/\text{St}(E_2) \cong A_2 \). \( \square \)
§5 Arithmetic Recurrence.

Due to the well-known correspondence between Number Theory and Nevanlinna Theory, it is tempting to give a number-theoretic analogue of Theorem 2.1 as Pál Erdős Problem–Corrales-Rodrigáñez&Schoof Theorem.

A related problem asks to classify the cases where $x^n - 1$ divides $y^n - 1$ for infinitely many positive integers $n$.

We would like to deal with the case of a semi-abelian variety with a given divisor, i.e., a polarized semi-abelian variety.

In the present situation, We can prove an analogue of the Main Theorem 2.1 only in the linear toric case, but not in the general case of semi-abelian varieties, that is left to be a Conjecture.
Here is our result in the arithmetic case.

**Theorem 5.1**

Let
\[ \mathcal{O}_S \] be a ring of \( S \)-integers in a number field \( k \);
\( \mathbf{G}_1, \mathbf{G}_2 \) be linear tori;
\( g_i \in \mathbf{G}_i(\mathcal{O}_S) \) be elements generating Zariski-dense subgroups.
\( D_i \) be reduced divisors defined over \( k \), with defining ideals \( \mathcal{I}(D_i) \), such that each irreducible component has a finite stabilizer and \( \text{St}(D_2) = \{0\} \).

Suppose that for infinitely many \( n \in \mathbb{N} \),

\[
(g_1^n)^* \mathcal{I}(D_1) \supset (g_2^n)^* \mathcal{I}(D_2).
\]

Then \( \exists \) étale morphism \( \phi : \mathbf{G}_1 \to \mathbf{G}_2 \), defined over \( k \), and \( \exists h \in \mathbb{N} \) such that \( \phi(g_1^h) = g_2^h \) and \( D_1 \subset \phi^*(D_2) \).
N.B.

1. Theorem 5.1 is deduced from the main results of Corvaja-Zannier, Invent. Math. 2002.
2. By an example we cannot take $h = 1$ in general.
3. By an example, the condition on the stabilizers of $D_1$ and $D_2$ cannot be omitted.
4. Note that inequality (inclusion) (5.2) of ideals is assumed only for an infinite sequence of $n$, not necessarily for all large $n$. On the contrary, we need the inequality of ideals, not only of their supports, i.e. of the primes containing the corresponding ideals.
5. One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is of a weaker form.
In S. Lang’s “Introduction to Transcendental Numbers”, Addison-Wesley, 1966, he wrote at the last paragraph of Chap. 3

“Independently of transcendental problem one can raise an interesting question of algebraic-analytic nature, namely given a 1-parameter subgroup of an abelian variety (say Zariski dense), is its intersection with a hyperplane section necessarily non-empty, and infinite unless this subgroup is algebraic?”

In 6 years later, J. Ax (Amer. J. Math. (1972)) took this problem:

**Theorem 6.1**

Let \( \theta \) be a reduced theta function on \( \mathbb{C}^m \). Let \( L \) be a 1-dimensional affine subspace of \( \mathbb{C}^m \). Then either \( (\theta|L) \) is constant or has an infinite number of zeros; \( |\{(\theta|L) = 0\} \cap \Delta(r)| \sim r^2 \).

N.B. It seems to be still open that \( |\{(\theta|L) = 0\}/\Gamma| = \infty \) unless \( f(\mathbb{C}) \) is algebraic.
Theorem 6.2

Let \( f : \mathbb{C} \to A \) be a 1-parameter analytic subgroup in a semi-abelian variety \( A \) with \( v = f'(0) \).

Let \( D \) be a reduced divisor on \( A \).

1. If \( A \) is abelian and \( H(\cdot, \cdot) \) denotes the Riemann form associated with \( D \), then we have

\[
N(r; f^*D) = H(v, v) \pi r^2 + O(\log r),
\]

\[
= (1 + o(1))N_1(r; f^*D).
\]

2. Assume that \( \dim A \geq 2 \). If \( f \) is algebraically non-degenerate and if \( \text{St}(D) \) is finite, there is an irreducible component \( D' \) of \( D \) such that then \( f(\mathbb{C}) \cap D' \) is Zariski dense in \( D' \); in particular, \( |f(\mathbb{C}) \cap D| = \infty \).

N.B. In fact, the second statement holds for an arbitrary algebraically non-degenerate holomorphic curve \( f : \mathbb{C} \to A \).
Proof.

(i) Note that the first Chern class $c_1(L(D))$ is represented by $i\partial\bar{\partial}H(w, w)$. It follows from our SMT Theorem 3.1 that

$$N(r; f^*D) = T_f(r; L(D)) + O(\log r)$$
$$= \int_0^r \frac{dt}{t} \int_{\Delta(t)} iH(v, v)dz \wedge d\bar{z} + O(\log r)$$
$$= H(v, v)\pi r^2 + O(\log r)$$
$$= (1 + o(1))N_1(r, f^*D).$$

(ii) If the claim does not hold, $\exists$ an algebraic subset $E$ such that $f(C) \cap D \subset E \subsetneq D$ and $\text{codim}_AE \geq 2$. Then our SMT Theorem 3.1 yields that

$$N(r, f^*E) = o(r^2) = N(r, f^*D) \sim r^2|| \ (\text{contradiction}).$$
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