A Result on Fractional $(a, b, k)$-critical Covered Graphs

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Abstract A fractional $[a, b]$-factor of a graph $G$ is a function $h$ from $E(G)$ to $[0, 1]$ satisfying $a \leq d_G^h(v) \leq b$ for every vertex $v$ of $G$, where $d_G^h(v) = \sum_{e \in E(v)} h(e)$ and $E(v) = \{ e = uv : u \in V(G) \}$. A graph $G$ is called fractional $[a, b]$-covered if $G$ contains a fractional $[a, b]$-factor $h$ with $h(e) = 1$ for any edge $e$ of $G$. A graph $G$ is called fractional $(a, b, k)$-critical covered if $G - Q$ is fractional $[a, b]$-covered for any $Q \subseteq V(G)$ with $|Q| = k$. In this article, we demonstrate a neighborhood condition for a graph to be fractional $(a, b, k)$-critical covered. Furthermore, we claim that the result is sharp.

Keywords graph; neighborhood; fractional $[a, b]$-factor; fractional $[a, b]$-covered graph; fractional $(a, b, k)$-critical covered graph

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1 Introduction

All graphs discussed are assumed to be finite, undirected and simple. For a graph $G$, the set of vertices in $G$ is denoted by $V(G)$, and the set of edges in $G$ is denoted by $E(G)$. For $v \in V(G)$, the degree of $v$ in $G$ is the number of edges of $G$ incident with $v$ and is denoted by $d_G(v)$, and the neighborhood of $v$ in $G$ is the set of vertices of $G$ adjacent to $v$ and is denoted by $N_G(v)$. Distinctly, $d_G(v) = |N_G(v)|$. For any subset $X$ of $V(G)$, the subgraph of $G$ induced by $X$ is denoted by $G[X]$ and the neighborhood of $X$ in $G$ is denoted by $N_G(X)$. Setting $G - X = G[V(G) \setminus X]$. A subset $X$ of $V(G)$ is called independent if $G[X]$ does not possess edges. The minimum degree of $G$ is defined by $\delta(G) = \min \{ d_G(v) : v \in V(G) \}$. Assume that $c$ is a real number. Recall that $\lfloor \ldots \rfloor$ is the greatest integer satisfying $\lfloor \ldots \rfloor \leq c$.

For positive integers $a$ and $b$ satisfying $a \leq b$, an $[a, b]$-factor of a graph $G$ is a spanning subgraph $F$ of $G$ satisfying $a \leq d_F(v) \leq b$ for all $v \in V(G)$. Assume that $a = b = r$, then we call it an $r$-factor. A fractional $[a, b]$-factor of a graph $G$ is a function $h$ from $E(G)$ to $[0, 1]$ satisfying $a \leq d_G^h(v) \leq b$ for every vertex $v$ of $G$, where $d_G^h(v) = \sum_{e \in E(v)} h(e)$ and $E(v) = \{ e = uv : u \in V(G) \}$. If $a = b = r$, then we call it a fractional $r$-factor. A graph $G$ is called fractional $[a, b]$-covered if $G$ contains a fractional $[a, b]$-factor $h$ with $h(e) = 1$ for any edge $e$ of $G$. If $a = b = r$, then we call $G$ being a fractional $r$-covered graph. A graph $G$ is called fractional $(a, b, k)$-critical covered if $G - Q$ is fractional $[a, b]$-covered for any $Q \subseteq V(G)$ with $|Q| = k$ where $k$ is a nonnegative integer. When $a = b = r$, a fractional $(a, b, k)$-critical covered graph is a fractional $(r, k)$-critical covered graph.

The previous works have implied that there is a close relationship between neighborhood and the existence of factors and fractional factors in graphs. Amahashi and Kanou[3], Berge and Las Vergnas[3], independently, derived a neighborhood condition for graphs having $[1, b]$-factors. Kano[7] posed a neighborhood condition for the existence of $[a, b]$-factors in graphs. Zhou, Pu

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and Xu\textsuperscript{[20]} claimed a sharp neighborhood condition for a graph possessing a fractional $r$-factor. For more results on factors and fractional factors of graphs, see [2, 4–6, 9–19, 21–25].

A neighborhood condition for the existence of fractional $r$-factors in graphs was demonstrated by Zhou, Pu and Xu\textsuperscript{[20]}, which is the following result.

**Theorem 1.1**\textsuperscript{[20]}. Let $r \geq 1$ be an integer, and let $G$ be a graph of order $n$ with $n \geq 6r - 12 + \frac{6}{r}$. Assume, for every subset $X$ of $V(G)$, that

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{rn}{2r - 1} \right\rfloor;$$

$$|N_G(X)| \geq \frac{2r - 1}{r}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{rn}{2r - 1} \right\rfloor.$$

Then $G$ possesses a fractional $r$-factor.

In this article, we generalize Theorem 1.1, and claim a neighborhood condition for graphs being fractional $(a, b, k)$-critical covered graphs.

**Theorem 1.2**. Let $a, b$ and $k$ be integers with $b \geq a \geq 2$ and $k \geq 0$, and let $G$ be a graph of order $n$ with $n \geq \frac{(a + b - 2)(2a + b - 3) + 2}{b}$. Assume, for every vertex subset $X$ of $G$, that

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{(b(n - 1) - bk)n - 2(n - 1)}{(a + b - 1)(n - 1)} \right\rfloor;$$

$$|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{(b(n - 1) - bk)n - 2(n - 1)}{(a + b - 1)(n - 1)} \right\rfloor.$$

Then $G$ is fractional $(a, b, k)$-critical covered.

The following result holds for $k = 0$ in Theorem 1.2.

**Corollary 1.3**. Let $a, b$ be integers with $b \geq a \geq 2$, and let $G$ be a graph of order $n$ with $n \geq \frac{(a + b - 2)(2a + b - 3) + 2}{b}$. Assume, for every vertex subset $X$ of $G$, that

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{bn - 2}{a + b - 1} \right\rfloor;$$

$$|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - 2}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{bn - 2}{a + b - 1} \right\rfloor.$$

Then $G$ is fractional $[a, b]$-covered.

We can acquire the following result when $a = b = r$ in Theorem 1.2.

**Corollary 1.4**. Let $r$ and $k$ be integers with $r \geq 2$ and $k \geq 0$, and let $G$ be a graph of order $n$ with $n \geq 6r - 12 + \frac{8}{r} + \frac{r^2}{r - 1}$. Assume, for every vertex subset $X$ of $G$, that

$$N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{(r(n - 1) - rk)n - 2(n - 1)}{(2r - 1)(n - 1)} \right\rfloor;$$

$$|N_G(X)| \geq \frac{(2r - 1)(n - 1)}{r(n - 1) - rk - 2}|X| \quad \text{if} \quad |X| < \left\lfloor \frac{(r(n - 1) - rk)n - 2(n - 1)}{(2r - 1)(n - 1)} \right\rfloor.$$

Then $G$ is fractional $(r, k)$-critical covered.

The following result is obtained if $a = b = r$ in Corollary 1.3.
Corollary 1.5. Let \( r \) be an integer with \( r \geq 2 \), and let \( G \) be a graph of order \( n \) with \( n \geq 6r - 12 + \frac{2}{r} \). Assume, for every vertex subset \( X \) of \( G \), that

\[
N_G(X) = V(G) \quad \text{if} \quad |X| \geq \left\lfloor \frac{rn - 2}{2r - 1} \right\rfloor;
\]

\[
|N_G(X)| \geq \frac{(2r - 1)(n - 1)}{r(n - 1) - 2} |X| \quad \text{if} \quad |X| < \left\lfloor \frac{rn - 2}{2r - 1} \right\rfloor.
\]

Then \( G \) is fractional \( r \)-covered.

2 The Proof of Theorem 1.2

The proof of Theorem 1.2 relies on the following theorem, which is a special case of fractional \((g,f)\)-covered graph theorem obtained by Li, Yan and Zhang [8].

Theorem 2.1 [8]. Let \( a, b \) be nonnegative integers satisfying \( b \geq a \), and let \( G \) be a graph. Then \( G \) is fractional \([a,b]\)-covered if and only if

\[
\theta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \geq \varepsilon(S)
\]

for any vertex subset \( S \) of \( G \) and \( T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a\} \), where \( \varepsilon(S) \) is defined by

\[
\varepsilon(S) = \begin{cases} 
 2, & \text{if } S \text{ is not independent}, \\
 1, & \text{if } S \text{ is independent, and there is an edge joining } S \text{ and } V(G) \setminus (S \cup T) \text{ or there is an edge } e = uv \text{ joining } S \text{ and } T \text{ satisfying } d_{G-S}(v) = a \text{ for } v \in T, \\
 0, & \text{otherwise}.
\end{cases}
\]

Proof of Theorem 1.2. Let \( D \subseteq V(G) \) with \( |D| = k \), and let \( H = G - D \). It suffices to claim that \( H \) is fractional \([a,b]\)-covered. Assume that \( H \) is not fractional \([a,b]\)-covered. Then using Theorem 2.1, we possess

\[
\theta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \leq \varepsilon(S) - 1,
\]

for a vertex subset \( S \) of \( H \), where \( T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq a\} \). Evidently, \( T \neq \emptyset \) by (2.1) and \( \varepsilon(S) \leq |S| \). Thus, we define

\[
\beta = \min\{d_{H-S}(x) : x \in T\}.
\]

By virtue of the definition of \( T \), we acquire

\[
0 \leq \beta \leq a.
\]

Claim 1. \( \delta(H) \geq \frac{(a-1)n+b-(a-1)k+2}{a+b-1} \).

Proof. Let \( v \in V(G) \) such that \( d_G(v) = \delta(G) \), and set \( W = V(G) \setminus N_G(v) \). Distinctly, \( v \notin N_G(W) \), and so \( N_G(W) \neq V(G) \). Then using the condition of Theorem 1.2, we deduce

\[
|N_G(W)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2} |W|.
\]

Note that \( |W| = n - \delta(G) \) and \( |N_G(W)| \leq n - 1 \). Thus, we derive

\[
n - 1 \geq |N_G(W)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2} |W| = \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2} (n - \delta(G)),
\]
which hints that
\[ \delta(G) \geq \frac{(a-1)n+b+bk+2}{a+b-1}. \tag{2.2} \]

In light of (2.2), \( H = G - D \) and \( |D| = k \), we possess
\[ \delta(H) \geq \delta(G) - k \geq \frac{(a-1)n+b-(a-1)k+2}{a+b-1}. \]

We finish the proof of Claim 1. \( \Box \)

Note that \( \beta = \min\{d_{H-S}(x) : x \in T\} \). Then there exists \( x_1 \in T \) such that \( d_{H-S}(x_1) = \beta \). By virtue of Claim 1 and \( \delta(H) \leq d_H(x_1) \leq d_{H-S}(x_1) + |S| = \beta + |S| \), we acquire
\[ |S| \geq \delta(H) - \beta \geq \frac{(a-1)n+b-(a-1)k+2}{a+b-1} - \beta. \tag{2.3} \]

The following proof is divided into four cases by virtue of the value of the \( \beta \).

**Case 1.** \( \beta = a. \)

It follows from (2.1) and \( \varepsilon(S) \leq |S| \) that
\[ \varepsilon(S) - 1 \geq \theta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + \beta|T| - a|T| = b|S| \geq |S| \geq \varepsilon(S), \]
which is a contradiction.

**Case 2.** \( a-1 \geq \beta \geq 2. \)

By virtue of (2.1), (2.3), \( |S| + |T| + k \leq n \) and \( \varepsilon(S) \leq 2 \), we acquire
\[ 1 \geq \varepsilon(S) - 1 \geq \theta_H(S,T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + \beta|T| - a|T| = b|S| - (a-\beta)|T| \geq b|S| - (a-\beta)(n-k-|S|) = (a+b-\beta)|S| - (a-\beta)(n-k) \geq (a+b-\beta)\left(\frac{(a-1)n+b-(a-1)k+2}{a+b-1} - \beta\right) - (a-\beta)(n-k). \]

Let \( \varphi(\beta) = (a+b-\beta)(\frac{(a-1)n+b-(a-1)k+2}{a+b-1} - \beta) - (a-\beta)(n-k) \). Then we get \( \varphi(\beta) \leq 1. \)

It follows from \( a-1 \geq \beta \geq 2 \) and \( n \geq \frac{(a+b-2)(2a+b-3)+2}{b} + \frac{bk}{b-1} \geq \frac{(a+b-2)(2a+b-3)+2}{b} + k \) that
\[ \frac{d\varphi}{d\beta} = -\left(\frac{(a-1)n+b-(a-1)k+2}{a+b-1} - \beta\right) - (a+b-\beta) + (n-k) \]
\[ = 2\beta + \frac{b(n-k)-b-2}{a+b-1} - a-b \]
\[ \geq 4 + \frac{(a+b-2)(2a+b-3)+2-b-2}{a+b-1} - a-b \]
\[ = a-1 + \frac{1}{a+b-1} > 0. \]
Hence, we easily see that $\varphi(\beta)$ attains its minimum value at $\beta = 2$, that is,

$$\varphi(\beta) \geq \varphi(2). \quad (2.5)$$

According to (2.4), (2.5) and $n \geq \frac{(a+b-2)(2a+b-3)+2}{b} + \frac{bk}{b-1} \geq \frac{(a+b-2)(2a+b-3)+2}{b} + k$, we deduce

$$1 \geq \varphi(\beta) \geq \varphi(2) = (a + b - 2)\left(\frac{(a-1)n+b-(a-1)k+2}{a+b-1} - 2\right) - (a-2)(n-k)$$

$$= \frac{b(n-k)-(a+b-2)(2a+b-4)}{a+b-1} \geq \frac{(a+b-2)(2a+b-3)+2-(a+b-2)(2a+b-4)}{a+b-1} = \frac{a+b}{a+b-1} > 1,$$

a contradiction.

**Case 3.** $\beta = 1$.

**Subcase 3.1.** $|T| > \left\lfloor \frac{(b(n-1)-bk)n-2(n-1)}{(a+b-1)(n-1)} \right\rfloor$.

In terms of the integrity of $|T|$, we have

$$|T| \geq \left\lfloor \frac{(b(n-1)-bk)n-2(n-1)}{(a+b-1)(n-1)} \right\rfloor + 1. \quad (2.6)$$

Note that $d_{H-S}(x_1) = \beta = 1$ and $x_1 \in T$. Then we easily see that

$$x_1 \notin NG(T \setminus NG(x_1)). \quad (2.7)$$

It follows from (2.6), $T \cap (S \cup D) = \emptyset$ and $H = G - D$ that

$$\begin{align*}
|T \setminus NG(x_1)| &= |T \setminus NG-D(x_1)| = |T \setminus NH(x_1)| = |T \setminus NH-S(x_1)| \\
&\geq |T| - |NH-S(x_1)| = |T| - d_{H-S}(x_1) = |T| - 1 \\
&\geq \left\lfloor \frac{(b(n-1)-bk)n-2(n-1)}{(a+b-1)(n-1)} \right\rfloor.
\end{align*}$$

Combining this with the assumption of Theorem 1.2, we gain

$$NG(T \setminus NG(x_1)) = V(G),$$

which contradicts (2.7).

**Subcase 3.2.** $|T| \leq \left\lfloor \frac{(b(n-1)-bk)n-2(n-1)}{(a+b-1)(n-1)} \right\rfloor$.

**Claim 2.** $|T| \leq \frac{b(n-1)-bk-2}{a+b-1}$.

**Proof.** Let $|T| > \frac{b(n-1)-bk-2}{a+b-1}$. According to (2.3) and $\beta = 1$, we gain

$$|S| + |T| + k > \frac{(a-1)n+b-(a-1)k+2}{a+b-1} - 1 + \frac{b(n-1)-bk-2}{a+b-1} + k = n - 1.$$

Combining this with $|S| + |T| + k \leq n$ and the integrity of $|S| + |T| + k$, we have

$$|S| + |T| + k = n. \quad (2.8)$$
From (2.8), \(\varepsilon(S) \leq 2\) and \(|T| \leq \left\lceil \frac{(b(n-1) - bk) n - 2(n-1)}{(a+b-1)(n-1)} \right\rceil \leq \frac{(b(n-1) - bk) n - 2(n-1)}{(a+b-1)(n-1)}\), we refer

\[
\theta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + |T| - a|T|
\]

\[
= b(n - k - |T|) - (a - 1)|T| = b(n - k) - (a + b - 1)|T|
\]

\[
\geq b(n - k) - (a + b - 1) \cdot \frac{(b(n-1) - bk)n - 2(n-1)}{(a+b-1)(n-1)}
\]

\[
= \frac{nbk}{n-1} - bk + 2 \geq 2 \geq \varepsilon(S),
\]

which contradicts (2.1). This finishes the proof of Claim 2.

Let \(\lambda = |\{x : x \in T, d_{H-S}(x) = 1\}|\). Evidently, \(\lambda \geq 1\) and \(|T| \geq \lambda\). Using (2.3), Claim 2, \(b \geq a \geq 2\), \(\beta = 1\) and \(\varepsilon(S) \leq 2\), we acquire that

\[
\theta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + 2|T| - \lambda - a|T|
\]

\[
= b|S| - (a - 2)|T| - \lambda
\]

\[
\geq b\left(\frac{(a - 1)n + b - (a - 1)k}{a + b - 1} - 1\right) - (a - 2) \cdot \frac{b(n-1) - bk - 2}{a + b - 1} - \lambda
\]

\[
= 2 + \frac{b(n-1) - bk - 2}{a + b - 1} \geq 2 + |T| - \lambda \geq 2 \geq \varepsilon(S),
\]

which conflicts with (2.1).

**Case 4.** \(\beta = 0\).

Let \(d = |\{x : x \in T, d_{H-S}(x) = 0\}|\). Distinctly, \(d \geq 1\). Set \(Z = V(H) \setminus S\). Thus, \(N_G(Z) \neq V(G)\) since \(\beta = 0\). By the assumption of Theorem 1.2, we get

\[
n - d \geq |N_G(Z)| \geq \frac{(a + b - 1)(n - 1)}{b(n-1) - bk - 2}|Z| = \frac{(a + b - 1)(n - 1)}{b(n-1) - bk - 2}(n - k - |S|),
\]

which implies

\[
|S| \geq n - k - \frac{(n - d)(b(n-1) - bk - 2)}{(a+b-1)(n-1)}. \tag{2.9}
\]

By virtue of \(n \geq \frac{(a+b-2)(2a+b-3)+2}{b} + \frac{bk}{b-1}\), we easily verify that

\[
\frac{b(n-1) - bk - 2}{n - 1} > 1. \tag{2.10}
\]

Using (2.9), (2.10), \(|S| + |T| + k \leq n\), \(b \geq a \geq 2\) and \(\varepsilon(S) \leq 2\), we deduce

\[
\theta_H(S, T) = b|S| + d_{H-S}(T) - a|T| \geq b|S| + |T| - d - a|T|
\]

\[
= (a + b - 1)|S| - (a - 1)(n - k) - d
\]

\[
\geq (a + b - 1)\left(n - k - \frac{(n - d)(b(n-1) - bk - 2)}{(a+b-1)(n-1)}\right) - (a - 1)(n - k) - d
\]

\[
= b(n - k) - \frac{(n - d)(b(n-1) - bk - 2)}{n - 1} - d
\]

\[
\geq b(n - k) - \frac{(n - 1)(b(n-1) - bk - 2)}{n - 1} - 1
\]

\[
= b + 1 > 2 \geq \varepsilon(S),
\]

which conflicts with (2.1). Theorem 1.2 is justified. \(\Box\)
3 Remark

Now, we show that the condition on neighborhood in Theorem 1.2 is sharp, that is, we cannot replace it by \( N_G(X) = V(G) \) or

\[
|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|
\]

for all \( X \subseteq V(G) \).

Let \( a, b, k, t \) be nonnegative integers such that \( b \geq a \geq 2 \), \( t \) is sufficiently large, and \( \frac{t+1}{2} \) and \( \frac{(a-1)t+2}{b} \) are two integers. We construct a graph \( G = K_{\frac{t+1}{2} + k} \) or \( \frac{(a-1)t+2}{b} \). Set \( n = |V(G)| = \frac{(a-1)t+2}{b} + k + t + 1 \), \( A = V(K_{\frac{t+1}{2} + k}) \), \( B = V(\frac{t+1}{2} K_2) \) and \( D \subseteq A \) with \( |D| = k \). Next, we show that either \( N_G(X) = V(G) \) or

\[
|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|
\]

holds for all \( X \subseteq V(G) \).

We easily see that \( N_G(X) = V(G) \) if \( |X \cap A| \geq 2 \), or \( |X \cap A| = 1 \) and \( |X \cap B| \geq 1 \). Of course, if \( |X| = 1 \) and \( X \subseteq A \), then we easily claim

\[
|N_G(X)| = |V(G)| - 1 = n - 1 = \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|.
\]

Hence, we may assume that \( X \subseteq B \). In this case, we possess

\[
|N_G(X)| = |A| + |X| = \frac{(a-1)t+2}{b} + k + |X|.
\]

Therefore,

\[
|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|
\]

holds if and only if

\[
\frac{(a-1)t+2}{b} + k + |X| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|.
\]

This inequality is equivalent to \( |X| \leq t \). Thus if \( X \subseteq B \), then we derive

\[
|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|.
\]

If \( X = B \), then we easily see that \( N_G(X) = V(G) \). Consequently, \( N_G(X) = V(G) \) or

\[
|N_G(X)| \geq \frac{(a + b - 1)(n - 1)}{b(n - 1) - bk - 2}|X|
\]

holds for all \( X \subseteq V(G) \).

Finally, we demonstrate that \( G \) is not fractional \((a, b, k)\)-critical covered. Let \( H = G - D \).

For above \( A \setminus D \) and \( B \), we admit \( |A \setminus D| = \frac{(a-1)t+2}{b} \), \( |B| = t + 1 \), \( d_{H-(A \setminus D)}(B) = t + 1 \) and \( \varepsilon(A \setminus D) = 2 \). Thus, we acquire

\[
\theta_H(A \setminus D, B) = |A \setminus D| + d_{H-(A \setminus D)}(B) - a|B| = (a-1)t + 2 + t + 1 - a(t + 1) = 3 - a \leq 1 < 2 = \varepsilon(A \setminus D).
\]

By Theorem 2.1, \( H \) is not fractional \([a, b]\)-covered, and so, \( G \) is not fractional \((a, b, k)\)-critical covered.

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