BOUND ON THE MAXIMAL BOCHNER-RIESZ MEANS FOR ELLIPTIC OPERATORS

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ABSTRACT. We investigate $L^p$ boundedness of the maximal Bochner-Riesz means for self-adjoint operators of elliptic type. Assuming the finite speed of propagation for the associated wave operator, from the restriction type estimates we establish the sharp $L^p$ boundedness of the maximal Bochner-Riesz means for the elliptic operators. As applications, we obtain the sharp $L^p$ maximal bounds for the Schrödinger operators on asymptotically conic manifolds, the harmonic oscillator and its perturbations or elliptic operators on compact manifolds.

1. INTRODUCTION

Convergence of the Bochner-Riesz means and boundedness of the associated maximal operators on Lebesgue $L^p$ spaces are among the most classical problems in harmonic analysis. The study on the Bochner-Riesz means can be seen as an attempt to justify the Fourier inversion. We begin with recalling the Bochner-Riesz means on $\mathbb{R}^n$ which are defined by, for $\alpha \geq 0$ and $R > 0$,

$$S_{\alpha,R}f(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)^\alpha \hat{f}(\xi), \quad \forall \xi \in \mathbb{R}^n. \tag{1.1}$$

Here $(x)_+ = \max\{0, x\}$ for $x \in \mathbb{R}$ and $\hat{f}$ denotes the Fourier transform of $f$. The associated maximal function which is called ‘maximal Bochner-Riesz operator’ is given by

$$S_{\alpha}^* f(x) = \sup_{R > 0} |S_{\alpha,R}f(x)|. \tag{1.2}$$

The problem of characterizing the optimal range of $\alpha$ for which $S_{\alpha}^*$ (and $S_{\alpha,R}^*$) is bounded on $L^p(\mathbb{R}^n)$ is known as the Bochner-Riesz (and maximal Bochner-Riesz) conjecture. It has been conjectured that, for $1 \leq p \leq \infty$ and $p \neq 2$, $S_{\alpha}^*$ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\alpha > \alpha(p) = \max\left\{n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right\}. \tag{1.3}$$

We refer the reader to [14], Stein’s monograph [44, Chapter IX] and Tao [48] for historical background and more on the Bochner-Riesz conjecture. It was shown by Herz that for a given $p$ the above condition on $\alpha$ is necessary, see [22]. Carleson and Sjölin [8] proved the conjecture when $n = 2$. Afterward substantial progress has been made [49, 26, 3, 19], but the conjecture still remains open for $n \geq 3$.

Concerning the $L^p$ boundedness of $S_{\alpha}^*$, for $p \geq 2$ it is natural to expect that $S_{\alpha}^*$ is bounded on $L^p$ on the same range where $S_{\alpha,R}^*$ is bounded, see e.g. [26, 28]. This was shown to be true by Carbery [7] when $n = 2$. In dimensions greater than two partial results are known. Christ [10] showed that...
$S_\alpha^\ast$ is bounded on $L^p$ if $p \geq 2(n+1)/(n-1)$ and $\alpha > \alpha(p)$, and the range of $p$ was extended by the second named author to the range $p > 2(n+2)/n$ in [26] and see [27] for the most recent progress. In this paper we focus on the case $p \geq 2$ but it should be mentioned that, for $p < 2$, the range of $\alpha$ where $S_\alpha^\ast$ is bounded on $L^p$ is different from that of $S_\alpha^\alpha$. Tao [46] showed that the additional restriction $\alpha \geq (2n-1)/(2p) - n/2$ is necessary. Besides, when $n = 2$ he obtained an improved estimate over the classical result [47].

**Bochner-Riesz means for elliptic operators.** Since the Bochner-Riesz means are radial Fourier multipliers, they can be defined in terms of the spectral resolution of the standard Laplace operator $\Delta = \sum_{\nu=1}^{n} \partial_x^2$. This point of view naturally allows us to extend the Bochner-Riesz means and the maximal Bochner-Riesz operator to arbitrary positive self-adjoint operator. For this purpose suppose that $(X, d, \mu)$ is a metric measure space with a distance $d$ and a measure $\mu$, and that $L$ is a non-negative self-adjoint operator acting on the space $L^2(X)$. Such an operator admits a spectral resolution

$$L = \int_0^{\infty} \lambda dE_L(\lambda).$$

Now, the Bochner-Riesz mean of order $\alpha \geq 0$ can be defined by

$$S_\alpha^\ast(L)f(x) = \left( \int_0^{\infty} \left(1 - \frac{\lambda}{R^2}\right)^\alpha dE_L(\lambda)f\right)(x), \quad x \in X$$

and the associated maximal operator is given by

$$S_\alpha^\ast(L)f(x) = \sup_{R > 0} \left|S_\alpha^\ast(L)f(x)\right|.$$

If we set $L = -\Delta$, the operators $S_\alpha^\ast(-\Delta)$ and $S_\alpha^\ast(-\Delta)$ coincide with the classical $S_R$ and $S_\alpha^\alpha$, respectively. In this paper we aim to investigate $L^p$-boundedness of the maximal Bochner-Riesz given by a certain class of self-adjoint operators.

**Restriction estimates.** The celebrated Stein-Tomas restriction estimate to the sphere played an important role in the development of Bochner-Riesz problem (see [44]). This estimate can be reformulated in terms of spectral decomposition of the standard Laplace operator. Indeed, for $\lambda > 0$ let $R_\lambda$ be the restriction operator given by $R_\lambda(f)(\omega) = \hat{f}(|\lambda\omega|)$, where $\omega \in S^{n-1}$ (the unit sphere). Then

$$dE_{\sqrt{-\Delta}}(\lambda) = (2\pi)^{-n}\lambda^{n-1}R^*_\lambda R_\lambda.$$

Thus, putting $L = -\Delta$, the Stein-Tomas theorem ([44, p. 386]) is equivalent to the estimate

$$\|dE_{\sqrt{-\Delta}}(\lambda)\|_{p \to p'} \leq C \lambda^{(\frac{n}{2} - \frac{1}{p'}) - 1}, \quad \lambda > 0$$

for $1 \leq p \leq 2(n+1)/(n+3)$. In [20] Guillarmou, Hassell and the third named author showed that the estimate (1.6) remains valid for the Schrödinger type operators on asymptotically conic manifolds.

It is easy to check that (1.6) is equivalent to the following estimate:

$$\|F(\sqrt{L})\|_{p \to 2} \leq CR^{\left(\frac{1}{p} - \frac{1}{2}\right)}\|\delta RF\|_2$$

for any $R > 0$ and all Borel functions $F$ supported in $[0, R]$, where the dilation $\delta RF$ is defined by

$$\delta_R F(x) = F(Rx)$$

(see [9, Proposition 1.4]).

Observation regarding relation between restriction estimate and the sharp $L^p$-boundedness (the boundedness of $S_\alpha^\ast$ in $L^p$ for $\alpha$ satisfying (1.3)) of the Bochner-Riesz means goes back as far as Stein [17] (and also see [44]). The argument in [17] and the Stein-Tomas restriction estimate give the sharp $L^p$ estimates for $S_\alpha^\ast(-\Delta)$ for $p$ satisfying $\max(p, p') \geq 2(n+1)/(n-1)$. Likewise, it
is natural to suspect if there is a similar connection between \((R_p)\) and the sharp \(L^p\) bound for \(S^\alpha_R(L)\) when \(L\) is a general elliptic operator. This question was explored in [9]. In fact, it was shown in [9, Corollary I.6] that if the operator \(L\) satisfies the finite speed of propagation property and the condition \((R_p)\), then the Bochner-Riesz means are bounded on \(L^p(X)\) spaces for \(p\) on the range where \((R_p)\) holds if \(\alpha > \max(0, n|1/p - 1/2| - 1/2)\).

Our first result is the maximal generalization of the aforementioned result in [9].

**Theorem A.** Let \(B(x, r) = \{y \in X : d(x, y) < r\}\) and \(V(x, r) = \mu(B(x, r))\). Suppose that

\[(1.7) \quad C^{-1}r^n \leq V(x, r) \leq Cr^n\]

holds for all \(x \in X\), and \(L\) satisfies the finite speed of propagation property (see, Definition 2.1) and the condition \((R_{p_0})\) for some \(1 \leq p_0 < 2\). Then the operator \(S^\alpha_L\) is bounded on \(L^p(X)\) whenever

\[(1.8) \quad 2 \leq p < p', \quad \text{and} \quad \alpha > \alpha(p_0) = \max \left\{ n \left( \frac{1}{p_0} - \frac{1}{2} \right) - \frac{1}{2}, 0 \right\} .\]

As a consequence, if \(f \in L^p(X)\), then for \(p\) and \(\alpha\) satisfying (1.8),

\[
\lim_{R \to \infty} S^\alpha_R(L)f(x) = f(x), \quad \text{a.e.}
\]

Later, we will see that the condition (1.7) can be replaced by the doubling condition (2.2).

**Cluster estimates.** It is not difficult to see that the condition \((R_p)\) implies that the set of point spectrum of \(L\) is empty. Indeed, one has, for \(0 \leq a < R\), \(\|\mathbb{I}_a(\sqrt{L})\|_{p \to 2} \leq CR^{n(1/p - 1/2)}\|\mathbb{I}_a(R)\|_{2} = 0\), and thus \(\mathbb{I}_a(\sqrt{L}) = 0\). Since \(\sigma(L) \subseteq [0, \infty)\), it is clear that the point spectrum of \(L\) is empty. In particular, \((R_p)\) does not hold for elliptic operators on compact manifolds or for the harmonic oscillator. In order to treat these cases as well we need to modify the estimate \((R_p)\) as follows: For a fixed natural number \(K\) and for all \(N \in \mathbb{N}\) and all even Borel functions \(F\) supported in \([-N, N]\),

\[(SC^K_p) \quad \|F(\sqrt{L})\|_{p \to 2} \leq CN^{K(1/p - 1/2)}\|\delta_N F\|_{N^\times 2},\]

where

\[(1.9) \quad \|F\|_{N^\times 2} := \left( \frac{1}{2N} \sum_{\ell = 1 - N}^{N} \sup_{|\lambda - \frac{\ell}{2N}| < \frac{1}{2N}} |F(\lambda)|^2 \right)^{1/2}\]

for \(F\) with \(\text{supp } F \subset [-1, 1]\). The norm \(\|F\|_{N^\times 2}\) already appeared in [13, 15] in the study of spectral multipliers, see also [9].

As shown in [9, Proposition I.14], the condition \((SC^1_p)\) is equivalent to the following \((p, p')\) spectral cluster estimate \((S_p)\) introduced by Sogge (see [40, 41, 42]): For all \(\lambda \geq 0\),

\[(S_p) \quad \|E\sqrt{\lambda}(\lambda, \lambda + 1)\|_{p \to p'} \leq C(1 + \lambda)^{n(1/p - 1/p')} - 1.\]

In this context we shall prove the following result.

**Theorem B.** Suppose that the condition

\[(1.10) \quad \mu(X) < \infty \quad \text{and} \quad C^{-1} \min(r^n, 1) \leq V(x, r) \leq C \min(r^n, 1)\]

is valid for all \(x \in X\) and \(r > 0\). And suppose that the operator \(L\) satisfies the finite speed of propagation property (see, Definition 2.1) and the condition \((SC^1_{p_0})\). Then the operator \(S^\alpha_L\) is
bounded on $L^p(X)$ whenever (1.8) is satisfied. As a consequence, if $f \in L^p(X)$, then for $p$ and $\alpha$ satisfying (1.8)
\[
\lim_{R \to \infty} S_R^\alpha(L)f(x) = f(x), \quad a.e.
\]

We now consider the case $\mu(X) = \infty$ with the property (1.7). Motivated by the harmonic oscillator $L = -\Delta + |x|^2$ we obtain the following variant of Theorem B.

**Theorem C.** Suppose that condition (1.7) holds, and the operator $L$ satisfies the finite speed of propagation property and the condition (SC)$p_0^\alpha$ for some $1 \leq p_0 < 2$ and some positive integer $\kappa$. In addition, we assume that there exists $\nu \geq 0$ such that
\[
\|(1 + L)^{-\gamma/2}\|_{p_0' \to 2} \leq C, \quad \gamma = n(\kappa - 1)(1/p_0 - 1/2) + \kappa\nu.
\]
Then the operator $S_R^\alpha(L)$ is bounded on $L^p(X)$ whenever
\[
2 \leq p < p_0', \quad \text{and} \quad \alpha > \nu + \max\left\{n\left(\frac{1}{p_0} - \frac{1}{2}\right) - \frac{1}{2}, 0\right\}.
\]
As a consequence, if $f \in L^p(X)$, then for $p$ and $\alpha$ satisfying (1.12),
\[
\lim_{R \to \infty} S_R^\alpha(L)f(x) = f(x), \quad a.e.
\]

We shall show that in dimension $n \geq 2$, (1.11) holds with $\kappa = 2$ and each $\nu > 0$ for the harmonic oscillator $L = -\Delta + |x|^2$ and $L = -\Delta + V(x)$ with the potential $V$ satisfying (6.2) below. The restriction estimates (SC)$p_0^\alpha$ for those operator were obtained by Kardzhojv [24], Thangavelu [52], Koch and Tataru [25]. Combining these estimates with Theorem C, we are able to obtain the sharp $L^p$ bounds for the associated maximal Bochner-Riesz operators. See Section 6.3.

In order to prove Theorems A, B, and C, we make use of the square function which has been utilized to control the maximal Bochner-Riesz operators (see [43, 7, 10, 26]). The square function estimates in Proposition 4.2 and Proposition 5.6 also have other applications. In particular, those estimates can be used to deduce smoothing properties for the Schrödinger and the wave equations and also spectral multiplier theorems of Hörmander-Mihlin type, see [28, 29] for such implications when $L = -\Delta$. However, unlike the classical case $L = -\Delta$, for the general elliptic operators we don’t have the typical properties of Fourier multipliers such as translation and scaling invariances. Also, the associated heat kernels are not necessarily smooth. This requires to refine the classical argument in various aspects. In particular we will use a new variant of Calderón–Zygmund technique for the square functions, see for example [1, 2].

Roughly speaking, we show that the estimate (R$p_0$) (equivalently (1.6)) or its variant implies the $L^p$ boundedness of the maximal Bochner-Riesz operators assuming the finite speed of propagation property. Main advantage of this approach is that we can handle large class of elliptic operators. Since the restriction type estimates are better understood now, it is possible to extend part of this argument to general setting of the homogeneous spaces, and also to include operators such as harmonic oscillator or operators acting on compact manifolds.

The Bochner-Riesz means operator for various classes of self-adjoint operators have been extensively studied (see [9, 15, 21, 23, 24, 32, 36, 39, 40, 42, 50, 51, 52] and references therein). However, as far as the authors are aware, there is no result that proves, on the range of $p$ up to that of restriction type estimate, the sharp $L^p$ boundedness of the maximal Bochner-Riesz operator other than the standard Laplacian and Fourier multipliers (see [3, 7, 8, 10, 18, 26, 27, 28, 29, 37, 45]).
Organization of the paper. In Section 2 we provide some prerequisites, which we need later, mostly on the restriction type estimate and the finite speed of propagation property. In Section 3 we consider the maximal bounds under less restrictive assumptions which includes more general elliptic operators though they don’t give the sharp bounds. The proof of Theorem A will be given in Section 4. The proof of Theorems B and C will be given in Section 5. In Section 6 we discuss some examples of applications of Theorems A, B, C which include the harmonic oscillator and its perturbation, Schrödinger operators on asymptotically conic manifolds, elliptic operators on compact manifolds and the radial part of the standard Laplace operator.

List of notation.

- $(X, d, \mu)$ denotes a metric measure space with a distance $d$ and a measure $\mu$.
- $L$ is a non-negative self-adjoint operator acting on the space $L^2(X)$.
- For $x \in X$ and $r > 0$, $B(x, r) = \{y \in X : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$.
- $\delta_r F$ is defined by $\delta_r F(x) = F(Rx)$ for $R > 0$ and Borel function $F$ supported on $[0, R]$.
- $[t]$ denotes the integer part of $t$ for any positive real number $t$.
- $\mathbb{N}$ is the set of positive integers.
- For $p \in [1, \infty]$, $p' = p/(p - 1)$.
- For $1 \leq p \leq \infty$ and $f \in L^p(X, d\mu)$, $\|f\|_p = \|f\|_{L^p(X, \mu)}$.
- $\langle , \rangle$ denotes the scalar product of $L^2(X, \mu)$.
- For $1 \leq p, q \leq +\infty$, $\|T\|_{p \to q}$ denotes the operator norm of $T$ from $L^p(X, \mu)$ to $L^q(X, \mu)$.
- If $T$ is given by $Tf(x) = \int K(x, y)f(y)d\mu(y)$, we denote by $K_T$ the kernel of $T$.
- Given a subset $E \subseteq X$, we denote by $\chi_E$ the characteristic function of $E$.
- For $1 \leq r < \infty$, $\mathcal{M}_r$, denote the uncentered $r$-th maximal operator over balls in $X$, that is
  \[
  \mathcal{M}_r f(x) = \sup_{y \in B} \left( \frac{1}{\mu(B)} \int_{B} |f(y)|^r d\mu(y) \right)^{1/r}.
  \]
  For simplicity we denote by $\mathcal{M}$ the Hardy-Littlewood maximal function $\mathcal{M}_1$.

2. Preliminaries

We say that $(X, d, \mu)$ satisfies the doubling property (see Chapter 3, [11]) if there exists a constant $C > 0$ such that
\[
V(x, 2r) \leq CV(x, r) \quad \forall r > 0, x \in X.
\]
If this is the case, there exist $C, n$ such that for $\lambda \geq 1$ and $x \in X$
\[
V(x, \lambda r) \leq C\lambda^n V(x, r).
\]
In the Euclidean space with Lebesgue measure, $n$ corresponds to the dimension of the space. Observe that if $X$ satisfies (2.1) and has finite measure then it has finite diameter. Therefore, if $\mu(X)$ is finite, then we may assume that $X = B(x_0, 1)$ for some $x_0 \in X$.

2.1. Finite speed of propagation property and elliptic type estimates. To formulate the finite speed of propagation property for the wave equation corresponding to an operator $L$, we set
\[
\mathcal{D}_r = \{(x, y) \in X \times X : d(x, y) \leq r\}.
\]
Given an operator $T$ from $L^p(X)$ to $L^q(X)$, we write
\[
\text{supp} \ K_T \subseteq \mathcal{D}_r
\]
if \( \langle T f_1, f_2 \rangle = 0 \) whenever \( f_1 \in L^p(B(x_1, r_1)), f_2 \in L^q(B(x_2, r_2)) \) with \( r_1 + r_2 + r < d(x_1, x_2) \). Note that if \( T \) is an integral operator with a kernel \( K_T \), then (2.3) coincides with the standard meaning of \( \text{supp} \, K_T \subseteq D_r \), that is \( K_T(x, y) = 0 \) for all \((x, y) \notin D_r \).

**Definition 2.1.** Given a non-negative self-adjoint operator \( L \) on \( L^2(X) \), we say that \( L \) satisfies the finite speed of propagation property if

\[
(\text{FS}) \quad \text{supp} \, K_{\cos(t \sqrt{L})} \subseteq D_t, \quad \forall t > 0.
\]

Property (FS) holds for most of second order self-adjoint operators and is equivalent to celebrated Davies-Gaffney estimates, see for example [12] and [38].

**Lemma 2.2.** Assume that \( L \) satisfies the property (FS) and that \( F \) is an even bounded Borel function with Fourier transform \( \hat{F} \in L^1(\mathbb{R}) \) and that \( \text{supp} \, \hat{F} \subseteq [-r, r] \). Then

\[
\text{supp} \, K_{F(\sqrt{L})} \subseteq D_r.
\]

**Proof.** If \( F \) is an even function, then by the Fourier inversion formula,

\[
F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(t) \cos(t \sqrt{L}) \, dt.
\]

But \( \text{supp} \, \hat{F} \subseteq [-r, r] \), and the lemma follows then from (FS). \( \square \)

Since our discussion covers general elliptic operators, we need some related estimates which are slightly more technical. We start with defining the multiplication operator. For any function \( W : X \to \mathbb{R} \), we define \( M_w \) by

\[
(M_w f)(x) = W(x) f(x).
\]

In what follows, we shall identify the operator \( M_w \) with the function \( W \). This means that, if \( T \) is a linear operator, we shall denote by \( W_1 T, T W_2, W_1 T W_2 \), the operators \( M_{W_1} T, T M_{W_2}, M_{W_1} T M_{W_2} \), respectively.

We can now formulate the weighted \( L^p - L^2 \) estimates (Sobolev type conditions). Firstly we consider

\[
(\text{EV}_{p,2}) \qquad \sup_{t > 0} \| e^{-t^2 L} V_t^{1/p - 1/2} \|_{p \to 2} < +\infty,
\]

where \( V_t(x) = V(x, t) \) and \( 1 \leq p < 2 \). An detailed and systematic discussion on the condition (EV_{p,2}) can founded in [4]. The following condition which was introduced in [9]:

\[
(G_{p,2}) \qquad \| e^{-t^2 L} \chi_{B(x,s)} \|_{p \to 2} \leq C V(x, s)^{1-p} (s^{m+1/p})
\]

holds for all \( x \in X \) and \( s \geq t > 0 \).

**Lemma 2.3.** Let \( 1 \leq p < 2 \). Suppose that \( L \) satisfies the property (FS). Then the following are equivalent:

(i) (EV\(_{p,2}\)) holds.

(ii) (G\(_{p,2}\)) holds.

(iii) For every \( N > n(1/p - 1/2) \) there exists \( C \) such that

\[
\| (I + t \sqrt{L})^{-N} V_t^{1/p} \|_{p \to 2} \leq C.
\]
(iv) For all \( x \in X \) and \( r \geq t > 0 \) we have
\[
\| (I + t \sqrt{L})^{-N} \chi_{B(x,r)} \|_{p \to 2} \leq CV(x, r)^{\frac{1}{2} + \frac{1}{p}} \left( \frac{r}{t} \right)^{\frac{1}{2} + \frac{1}{p}}.
\]

Proof. The equivalence of the conditions (ii) and (iv) was verified in [9, Proposition I.3]. The similar argument shows that the conditions (i) and (iii) are also equivalent. Thus it is enough to show equivalence between (iii) and (iv).

First we prove that (iii) implies (iv). Note that by the doubling condition for all \( y \in B(x, r) \) one has \( V(x, r) \sim V(y, r) \). Hence for all \( x \in X \) and \( r \geq t > 0 \),
\[
\left\| (I + t \sqrt{L})^{-N} \chi_{B(x,r)} \right\|_{p \to 2} \leq C \left\| (I + t \sqrt{L})^{-N} \chi_{B(x,r)} V_{r}^{\frac{1}{2} - \frac{1}{p}} \right\|_{p \to 2} V(x, r)^{\frac{1}{2} + \frac{1}{p}} \left( \frac{r}{t} \right)^{\frac{1}{2} + \frac{1}{p}}
\]
\[
\leq C \left\| (I + t \sqrt{L})^{-N} \chi_{B(x,r)} V_{t}^{\frac{1}{2} - \frac{1}{p}} \right\|_{p \to 2} V(x, r)^{\frac{1}{2} + \frac{1}{p}} \left( \frac{r}{t} \right)^{\frac{1}{2} + \frac{1}{p}}
\]
\[
\leq C \left\| (I + t \sqrt{L})^{-N} V_{t}^{\frac{1}{2} - \frac{1}{p}} \right\|_{p \to 2} V(x, r)^{\frac{1}{2} + \frac{1}{p}} \left( \frac{r}{t} \right)^{\frac{1}{2} + \frac{1}{p}}.
\]

By the assumption (iii) it follows that
\[
\left\| (I + t \sqrt{L})^{-N} \chi_{B(x,r)} \right\|_{p \to 2} \leq CV(x, r)^{\frac{1}{2} + \frac{1}{p}} \left( \frac{r}{t} \right)^{\frac{1}{2} + \frac{1}{p}},
\]
where we used (iii) in the last inequality.

We now show that (iv) implies (iii). Let us recall the well known identity, for \( a > 0 \),
\[
C_{a} \int_{0}^{\infty} \left( 1 - \frac{x^2}{s} \right)_{+} e^{-s/4} s^{a} \, ds = e^{-x^2/4}
\]
with some suitable \( C_{a} > 0 \). Taking the Fourier transform on both sides of the above equality yields
\[
\int_{0}^{\infty} F_{a}(\sqrt{s} \lambda) s^{a + \frac{1}{2}} e^{-s/4} \, ds = e^{-\lambda^2},
\]
where \( F_{a} \) is the Fourier transform of the function \( t \to (1 - t^2)^{a}_{+} \) multiplied by the appropriate constant. Hence, by spectral theory,
\[
\int_{0}^{\infty} F_{a}(\sqrt{s} \lambda) s^{a + \frac{1}{2}} e^{-s/4} \, ds = e^{-\lambda L}.
\]

Using this and Minkowski’s inequality give
\[
\left\| e^{-tL} V_{t}^{\frac{1}{p} - \frac{1}{p}} \right\|_{p \to 2} \leq \int_{0}^{\infty} \left\| F_{a}(\sqrt{s} \lambda) V_{t}^{\frac{1}{p} - \frac{1}{p}} \right\|_{p \to 2} s^{a + \frac{1}{2}} e^{-s/4} \, ds
\]
\[
\leq C \int_{0}^{\infty} \left\| F_{a}(\sqrt{s} \lambda) V_{t}^{\frac{1}{p} - \frac{1}{p}} \right\|_{p \to 2} (\sqrt{s} + \frac{1}{\sqrt{s}})^{\frac{1}{p} - \frac{1}{2}} s^{a + \frac{1}{2}} e^{-s/4} \, ds,
\]
hence, with \( a \) large enough,
\[
(2.4) \quad \sup_{r > 0} \left\| e^{-tL} V_{t}^{\frac{1}{p} - \frac{1}{p}} \right\|_{p \to 2} \leq C' \sup_{r > 0} \left\| F_{a}(\sqrt{t} L) V_{t}^{\frac{1}{p} - \frac{1}{p}} \right\|_{p \to 2}.
\]

We note that \( \Phi = F_{a} \) satisfies the assumptions of Lemma 2.2. Thus \( \sup_{r > 0} F_{a}(r \sqrt{L}) \subseteq D_{r}, \ \forall \ r > 0 \). Hence, by [4, Lemma 4.1.2]
\[
(2.5) \quad \left\| F_{a}(r \sqrt{L}) V_{t}^{\frac{1}{p} - \frac{1}{p}} \right\|_{p \to 2} \leq C \sup_{x \in M} \left\| F_{a}(r \sqrt{L}) V_{t}^{\frac{1}{p} - \frac{1}{p}} \chi_{B(x,r)} \right\|_{p \to 2}.
\]
Observe that
\[
\|F_a(r \sqrt{L})V_r^{\frac{1}{2}-\frac{1}{p}} \chi_{B(x,r)}\|_{p \rightarrow 2} \leq \|F_a(r \sqrt{L})(1 + r \sqrt{L})^{\frac{1}{p}}(1 + r \sqrt{L})^{-N}V_r^{\frac{1}{2}-\frac{1}{p}} \chi_{B(x,r)}\|_{p \rightarrow 2} \\
\leq \|F_a(r \sqrt{L})(1 + r \sqrt{L})^{N}\|_{2 \rightarrow 2}\|(1 + r \sqrt{L})^{-N}V_r^{\frac{1}{2}-\frac{1}{p}} \chi_{B(x,r)}\|_{p \rightarrow 2} \\
\leq C\|(1 + r \sqrt{L})^{-N}V_r^{\frac{1}{2}-\frac{1}{p}} \chi_{B(x,r)}\|_{p \rightarrow 2}.
\]

From this and (iv) with \(r = t\), we get
\[
\|F_a(r \sqrt{L})V_r^{\frac{1}{2}-\frac{1}{p}} \chi_{B(x,r)}\|_{p \rightarrow 2} \leq CV(x, r)^{\frac{1}{p} - \frac{1}{2}}\|(1 + r \sqrt{L})^{-N} \chi_{B(x,r)}\|_{p \rightarrow 2} \leq C.
\]

Combining this with (2.4) and (2.5) shows (EV_{p,2}) which is equivalent with (iii).

Recall that \(L\) is a non-negative self-adjoint operator on \(L^2(X)\) and that the semigroup \(e^{-tL}\), generated by \(-L\) on \(L^2(X)\), has the kernel \(p_t(x, y)\) which satisfies the following Gaussian upper bound:

\[
(\text{GE}) \quad |p_t(x, y)| \leq \frac{C}{V(x, \sqrt{t})} \exp\left(-c \frac{d^2(x, y)}{t}\right)
\]

for all \(t > 0\), and \(x, y \in X\), where \(C\) and \(c\) are positive constants. The estimate (GE) follows from (FS) and (EV_{1,2}). Indeed, (EV_{1,2}) is equivalent to the standard Gaussian heat kernel estimate which is valid for a broad class of second order elliptic operators, see e.g. [4].

It is not difficult to see that, for \(1 \leq p < 2\), both the conditions (FS) and (EV_{p,2}) follow from the Gaussian estimate (GE). But the converse is not true in general. For some \(1 < p < 2\), there are operators which fail to satisfy (GE) while (FS) and (EV_{p,2}) hold for them. Examples for such operators are provided by the Schrödinger operators with inverse-square potential, see [12] and the second order elliptic operators with rough lower order terms, see [30].

2.2. Stein-Tomas restriction type condition. Let \(1 \leq p < 2\) and \(2 \leq q \leq \infty\). Following [9], we say that \(L\) satisfies the Stein-Tomas restriction type condition if for any \(R > 0\) and all Borel functions \(F\) supported in \([0, R]\),

\[
(\text{ST}_{p,2}^q) \quad \|F(\sqrt{L}) \chi_{B(x,r)}\|_{p \rightarrow 2} \leq CV(x, r)^{\frac{1}{p} - \frac{1}{2}}(Rr)^{\frac{n}{2} - \frac{1}{p}}\|\delta_r F\|_q
\]

for all \(x \in X\) and all \(r \geq 1/R\). To motivate this definition we state the following two lemmas.

**Lemma 2.4.** Assume that \(C^{-1}r^n \leq V(x, r) \leq Cr^n\) for all \(x \in X\) and \(r > 0\). Then (ST_{p,2}^2) is equivalent to \((R_p)\).

**Lemma 2.5.** Assume that a metric measure space \((X, d, \mu)\) satisfies the doubling condition (2.2). Then (ST_{p,2}^\omega) is equivalent to (EV_{p,2}) or any other condition listed in Lemma 2.3.

For the proofs of these Lemmas and more on the condition (ST_{p,2}^q) we refer the reader to [9], especially [9, Proposition I.3] and [9, Proposition I.4].

The following result for the spectral multipliers of non-negative self-adjoint operators was one of the main results obtained in [9, Theorem I.16, Corollary I.6]. Fix a non-trivial auxiliary function \(\eta \in C_c^\infty(0, \infty)\).

**Proposition 2.6.** Assume that \(L\) satisfies the property (FS) and the condition (ST_{p,2}^q) for some \(p, q\) satisfying \(1 \leq p < 2\) and \(2 \leq q \leq \infty\).
(i) Then for any bounded Borel function \( F \) such that \( \sup_{r>0} \| \eta \delta_t F \|_{W^{p,q}} < \infty \) for some \( \beta > \max(n(1/p - 1/2), 1/q) \) the operator \( F(\sqrt{L}) \) is bounded on \( L'(X) \) for all \( p < r < p' \). In addition,

\[
\| F(\sqrt{L}) \|_{p \to r} \leq C_{\beta} \left( \sup_{r>0} \| \eta \delta_t F \|_{W^{p,q}} + |F(0)| \right).
\]

(ii) For all \( \alpha > n(1/p - 1/2) - 1/q \) we have the uniform bound, for \( R > 0 \),

\[
\left\| \left( I - \frac{L}{R^2} \right)^{\alpha} \right\|_{p \to p} \leq C.
\]

Finally, we state a standard weighted inequality for the Littlewood-Paley square function, which we shall use in what follows. For its proof, we refer the reader to [6, 16] for \( p = 1 \), and [2] for the general \( 1 \leq p < 2 \) on the Euclidean space \( \mathbb{R}^n \). The estimate remains valid on spaces of homogeneous type.

**Proposition 2.7.** Assume that \( L \) satisfies the property (FS) and the condition (EV\(_{p,2}\)) for some \( 1 \leq p < 2 \). Let \( \psi \) be a function in \( \mathcal{S}(\mathbb{R}) \) such that \( \psi(0) = 0 \), and let the fractional functional be defined by

\[
G_L(f)(x) = \left( \sum_{j \in \mathbb{Z}} |\psi(2^j \sqrt{L})f(x)|^2 \right)^{1/2}
\]

for \( f \in L^2(X) \). Then for any \( w \in A_1 \) (i.e., the Muckenhoupt \( A_1 \) weight), \( G_L \) is bounded on \( L'(w, X) \) for all \( p < r < p' \).

3. **Plancherel estimate and maximal Bochner-Riesz operator**

In this section we will discuss the case \( p = 1 \) for the condition (ST\(_{1,2}^q\)). In Corollary 3.5 and Proposition 3.4 below we state a version of Theorem A which deals with the case \( p = 1 \). In this case the proofs of results are significantly simpler. We also describe some other observations which will be useful for results in full generality. Following [15], we will call the estimate (ST\(_{1,2}^q\)) the Plancherel estimate.

Assume that \( (X, d, \mu) \) satisfies the doubling condition (2.2). We start with the following lemma.

**Lemma 3.1.** Let \( L \) satisfy the Gaussian bound (GE) and let \( m \) be a bounded Borel function such that \( \text{supp} m \subseteq [-2, 2] \). If \( \| m \|_{W^s_1} < \infty \) for some \( s > n + 1/2 \), for all \( x \in X \),

\[
\sup_{t>0} |m(tL)f(x)| \leq C \| m \|_{W^s_1} W(f)(x).
\]

As a consequence, if \( \alpha > n \), then \( S_\alpha^V(L) \) is a bounded operator on \( L^p(X) \) for all \( 1 < p < \infty \).

**Proof.** Let \( H(t) := m(\sqrt{t})e^t \). By the Fourier inversion formula \( H(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{H}(\tau)e^{it\tau}d\tau \), we have

\[
m(t\sqrt{L}) = H(t^2 L)e^{-t^2 L} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{H}(\tau)e^{-\tau(1-i\tau)L}d\tau.
\]

Let \( z := t^2(1 - i\tau) \) and \( \theta = \arg z \). From (GE), it is well known (see [34, Theorem 7.2]) that there exist positive constants \( C, c \) such that for all \( z \in \mathbb{C}^+ \) and a.e. \( x, y \in X \),

\[
|p_z(x, y)| \leq \frac{C (\cos \theta)^{-n}}{\sqrt{V(x, \sqrt{\frac{|z|}{\cos \theta}})} V(y, \sqrt{\frac{|z|}{\cos \theta}})} \exp \left( -c \frac{d^2(x, y)}{|z| \cos \theta} \right).
\]
By the doubling properties of the space $X$, we use a standard argument to obtain
\[
|e^{-\mathcal{L}}f(x)| \leq C(1 + \tau^2)^{n/2} \int_X \frac{1}{V(x, \sqrt{\cos \theta})} \exp \left(-\frac{d(x, y)^2}{2c \|z\| \cos \theta}\right) |f(y)|d\mu(y)
\]
\[
\leq C(1 + \tau^2)^{n/2} \mathcal{M}f(x).
\]
Then, from this and (3.1) it follows that, for any $\varepsilon \in (0, 1)$,
\[
|m(t \sqrt{\mathbb{L}})f(x)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} e^{-\mathcal{L}}f(x)\hat{\mathcal{L}}(\tau)d\tau \leq C\mathcal{M}(f)(x) \int_{\mathbb{R}} |\hat{\mathcal{L}}(\tau)(1 + \tau^2)^{n/2}d\tau
\]
(3.3)
\[
\leq C\|\mathcal{M}\|_{W^2(2n+1/2\varepsilon)} \mathcal{M}(f)(x) \leq C\|m\|_{W^2(2n+1/2\varepsilon)} \mathcal{M}(f)(x).
\]
Because $\|\mathcal{M}\|_{W^2(2n+1/2\varepsilon)} \leq C\|m\|_{W^2(2n+1/2\varepsilon)}$ since supp $m \subset [-2, 2]$. This gives the desired inequality.

Finally, we notice that $(1 - t^2)^{\alpha}_+ \in W^2_\alpha$ if and only if $\alpha > s - 1/2$. From (3.3), $L^p$-boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$, we see that for $\alpha > n$, $S^\alpha_*(L)$ is a bounded operator on $L^p(X)$ for $1 < p < \infty$.

In Lemma 3.1 the order of $\alpha$ for which $S^\alpha_*(L)$ is bounded on $L^p(X)$ for $1 < p < \infty$ is relatively large. This is mainly because the maximal bound is obtained by the pointwise estimate. The bound can be improved by making use of the spectral theory. For this, let us first recall that the Mellin transform of the function $F: \mathbb{R} \to \mathbb{C}$ is defined by
\[
\mathcal{M}_F(u) = \frac{1}{2\pi} \int_0^{\infty} F(\lambda)\lambda^{-1-iu}d\lambda, \quad u \in \mathbb{R}.
\]
Moreover the inverse transform is given by the following formula
\[
F(\lambda) = \int_\mathbb{R} \mathcal{M}_F(u)\lambda^{iu}du, \quad \lambda \in [0, \infty).
\]

**Lemma 3.2.** Suppose that $L$ satisfies the property (FS) and the condition (EV$_{p, 2}$) for some $1 \leq p < 2$. Let $p < r < p'$ and $s > n|1/p - 1/2|$. Suppose also that $F: \mathbb{R} \to \mathbb{C}$ is a bounded Borel function such that
\[
\int_\mathbb{R} |\mathcal{M}_F(u)|(1 + |u|)^sdu = C_{F,s} < \infty.
\]
Then the maximal operator
\[
F^*(L)f(x) = \sup_{r > 0} |F(tL)f(x)|
\]
is a bounded operator on $L^r(X)$ with $\|F^*(L)\|_{r \to r} \leq C_{F,s}$. In particular, if $\text{supp } F \subseteq [-2, 2]$ and $\|F\|_{W^2} < \infty$ for some $s > n|1/p - 1/2| + 1/2$, then $F^*(L)$ is bounded on $L^r(X)$ for $p < r < p'$. As a consequence, if $\alpha > n|1/p - 1/2|$, then $S^\alpha_*(L)$ is bounded on $L^r(X)$ for $p < r < p'$.

**Proof.** By (3.4), (3.5) it follows that
\[
F(tL) = \int_0^\infty F(\lambda)dE_L(\lambda) = \int_0^\infty \int_\mathbb{R} \mathcal{M}_F(u)(t\lambda)^{iu}du dE_L(\lambda)
\]
\[
= \int_\mathbb{R} \int_0^\infty \mathcal{M}_F(u)(t\lambda)^{iu}dE_L(\lambda)du = \int_\mathbb{R} \mathcal{M}_F(u)(t\lambda)^{iu}L^{iu}du.
\]
Hence $F^*(L)f(x) = \sup_{r > 0} |F(tL)f(x)| \leq \int_\mathbb{R} |\mathcal{M}_F(u)||L^{iu}f(x)|du$. And we get
\[
\|F^*(L)f\|_r \leq C\|f\|_r \int_\mathbb{R} |\mathcal{M}_F(u)||L^{iu}|_{r \to r, du}.
\]
From Proposition 2.6, we have that the imaginary power $L^{it}$ of $L$ is bounded on $L'(X)$, $p < r < p'$ with the bound $\|L^{it}\|_{r \to r'} \leq C(1 + |t|)^s$ for any $s > n|1/p - 1/2|$. This, together with (3.7), gives
\begin{equation}
\|F^*(L)f\|_r \leq C\|f\|_r \int_R |m_F(u)|(1 + |u|)^s du.
\end{equation}

Set $F(t) = (1 - t^2)^\alpha_\ast$. Substituting $\lambda = e^\nu$ in (3.4), we notice that $m$ is the Fourier transform of $G(v) = F(e^\nu)$. Since $(1 - t^2)^\alpha_\ast$ is compactly supported in $[-1, 1]$, we get
\begin{equation}
\int_R |m_F(u)|(1 + |u|)^s du \leq C\|G\|_{W_{s+1/2}^2} \leq C\|F\|_{W_{s+1/2}^2}
\end{equation}
for any $\epsilon > 0$. On the other hand, $(1 - t^2)^\alpha_\ast \in W_{s+1/2+\epsilon}^2$ if and only if $\alpha > s + \epsilon$. From this, we know that if $\alpha > s|1/p - 1/2|$, then $S^\alpha_\ast(L)$ is a bounded operator on $L'(X)$ for $p < r < p'$. \hfill \square

As a consequence of Lemma 3.2, we have the following which gives essentially sharp $L^2$ maximal bound for the Bochner-Riesz means.

**Corollary 3.3.** Suppose that $L$ satisfies the property (FS) and the condition (EV_{p,2}) for some $1 \leq p < 2$. If $\alpha > 0$, then $S^\alpha_\ast(L)$ is bounded on $L^2(X)$.

In Lemma 3.1 we obtained the pointwise estimate for the maximal function under the Gaussian bound (GE) only. In what follows we additionally impose the condition (ST^q_{1,2}). This significantly improves the regularity assumption on $F$ so that this allows us to essentially recover the sharp maximal bounds for the Bochner-Riesz means for $p = \infty$, see Corollary 3.5. The proof of Proposition 3.4 was inspired by an argument of Thangavelu [50, Theorem 4.2].

**Proposition 3.4.** Let $q \in [2, \infty]$. Let $L$ satisfy the Gaussian bound (GE) and let $F$ be a Borel function such that $\text{supp} F \subseteq [1/4, 1]$ and $\|F\|_{W_2^q} < \infty$ for some $s > n/2$. Suppose that the condition (ST^q_{1,2}) holds, then we have, for each $2 \leq r < \infty$,
\begin{equation}
F^*(L)f(x) \leq C\|F\|_{W_2^q} \mathfrak{M}_r f(x).
\end{equation}

Hence, $F^*(L)$ is a bounded operator on $L^p(X)$ for all $2 < p < \infty$.

**Proof.** Let $r' \in (1, 2]$ such that $1/r + 1/r' = 1$ and fix $R > 0$. Consider a partition of $X$ into the dyadic annuli $A_k = \{y : 2^kR - 1 < d(x, y) \leq 2^kR - 1\}$, for $k \in \mathbb{N}$. For a given $f$ we set
\begin{equation}
f_o(y) = f(y)\chi_{\{y : d(x, y) \leq R\}}, \quad f_k(y) = f(y)\chi_{A_k}(y), \quad k \in \mathbb{N}.
\end{equation}

Then, note that $|F(L/R^2)f(x)| \leq \sum_{k=0}^\infty |F(L/R^2)f_k(x)|$. By Hölder’s inequality, for $f \in L^2(X) \cap L^p(X)$,
\begin{equation}
|F(L/R^2)f(x)| \leq \left(\int_{d(x,y) \leq R^{-1}} |K_{F(L/R^2)}(x, y)|^{r'} d\mu(y)\right)^{1/r'} \|f_o\|_r + \sum_{k=1}^\infty \left(\int_{A_k} |K_{F(L/R^2)}(x, y)|^{r'} d\mu(y)\right)^{1/r'} \|f_k\|_r.
\end{equation}

We also note that
\begin{equation}
\left(\int_{A_k} |K_{F(L/R^2)}(x, y)|^{r'} d\mu(y)\right)^{1/r'} \leq \sum_{k=0}^\infty V(x, 2^kR^{-1})^{1/2} \left(\int_{A_k} |K_{F(L/R^2)}(x, y)|^{r} d\mu(y)\right)^{1/2}
\end{equation}
and $\|f_k\|_r \leq CV(x, 2^kR^{-1})^{1/2} \mathfrak{M}_r f(x)$. Combining all these inequalities gives
\begin{equation}
|F(L/R^2)f(x)| \leq C \mathfrak{M}_r f(x) \sum_{k=0}^\infty 2^{-k\lambda} V(x, 2^kR^{-1})^{1/2} \mathfrak{M}_r f(x),
\end{equation}
where
\begin{equation*}
\mathfrak{M}_r f(x) = \sup_{R > 0} \left(\int_{d(x,y) \leq R} |K_{F(L/R^2)}(x, y)|^{r} d\mu(y)\right)^{1/r}.
\end{equation*}
where
\[ \mathcal{Y}(R) = \left( \int_X |K_{F/L; R^2}(x, y)|^2 (1 + Rd(x, y))^{2s} d\mu(y) \right)^{1/2}. \]

Notice that \( L \) satisfies the condition (ST_{1,2}^q) for some \( q \in [2, \infty] \). By [15, Lemma 4.3] we have
\[ \int_X |K_{F/L; R^2}(x, y)|^2 (1 + Rd(x, y))^{2s} d\mu(y) \leq CV(x, R^{-1})^{-1} ||F||_{W_{s+\epsilon}^q}^2, \quad \forall \epsilon > 0. \]

Hence, this and (3.10) yield
\[ |F(L/R^2)f(x)| \leq C||F||_{W_{s+\epsilon}^q} \sum_{k=0}^{\infty} 2^{-ks} \left( \frac{V(x, 2^k R^{-1})}{V(x, R^{-1})} \right)^{1/2} \mathcal{M}_r f(x). \]

Since \( s > n/2 \), we get
\[ |F(L/R^2)f(x)| \leq C||F||_{W_{s+\epsilon}^q} \sum_{k=0}^{\infty} 2^{(\frac{s}{2}-\delta)k} \mathcal{M}_r f(x) \leq C||F||_{W_{s+\epsilon}^q} \mathcal{M}_r f(x). \]

From this and \( L^p \)-boundedness of the Hardy-Littlewood maximal operator \( \mathcal{M}_r \) for \( p > r \), we obtain that \( ||F^*(L)||_{p\to p} \leq C \) for \( p \in (2, \infty) \). This completes the proof.

We conclude this section with the following result which covers a special case of Theorem A. In fact, this shows the case \( p_0 = 1 \) in Theorem A if we take \( q = 2 \) in the following.

**Corollary 3.5.** Let \( L \) satisfy the Gaussian bounds (GE). Suppose that the condition (ST_{1,2}^q) holds for some \( q \in [2, \infty] \). If \( \alpha > n/2 - 1/q \), then for each \( 2 \leq r < \infty \),
\[ S^\alpha_r(L)f(x) \leq C\mathcal{M}_r f(x). \]

As a consequence, \( S^\alpha_r(L) \) is a bounded operator on \( L^p(X) \) for all \( 2 \leq p \leq \infty \).

**Proof.** Let \( S^\alpha(t) = (1 - t^2)^\alpha \). We set
\[ S^\alpha(t) = S^\alpha(t)\phi(t^2) + S^\alpha(t)(1 - \phi(t^2)) =: S^{\alpha,1}(t) + S^{\alpha,2}(t^2), \]
where \( \phi \in C^\infty(\mathbb{R}) \) is supported in \( \{ \xi : |\xi| \geq 1/4 \} \) and \( \phi = 1 \) for all \( |\xi| \geq 1/2 \). Define the maximal Bochner-Riesz operators \( S^{\alpha,i}_r(L) \), \( i = 1, 2 \) by
\[ S^{\alpha,i}_r(L)f(x) = \sup_{R>0} |S^{\alpha,i}(L/R^2)f(x)|, \quad i = 1, 2. \]

Note that by Lemma 3.1, \( S^{\alpha,2}_r(L)f(x) \leq C\mathcal{M}_r f(x) \). For the operator \( S^{\alpha,1}_r(L) \), we choose \( n/2 < s < \alpha + 1/q \), and notice that \( S^{\alpha,1}_r \in W_{s+\epsilon}^q \) if and only if \( s + \epsilon < \lambda + 1/q \). Taking \( \epsilon \) small enough, we apply Proposition 3.4 to obtain that \( S^{\alpha,1}_r(L)f(x) \leq C||S^{\alpha,1}_r||_{W_{s+\epsilon}^q} \mathcal{M}_r f(x) \leq C\mathcal{M}_r f(x) \) for all \( 2 \leq r < \infty \). Hence \( S^\alpha_r(L) \) is bounded on \( L^p(X) \) for all \( p > 2 \). This, together with Corollary 3.3, finishes the proof of Corollary 3.5.

\[ \square \]

### 4. Spectral restriction estimate and maximal bound

The aim of this section is to prove Theorem A. However, we would like to describe a slightly more general result which remains valid for the spaces of homogeneous type. For this end, we assume that \( (X, d, \mu) \) satisfies the doubling condition, that is (2.2). In this section, we will prove the following result, which yields Theorem A as a special case with \( q = 2 \) and the uniform volume estimate (1.7).
**Theorem 4.1.** Suppose that \((X,d,\mu)\) satisfies the doubling condition \((2.2)\). Suppose that \(L\) satisfies the property \((FS)\) and the condition \((ST_{p_0,2}^{\alpha})\) for some \(1 \leq p_0 < 2\) and \(2 \leq q \leq \infty\). Then the operator \(S_\alpha^\ast(L)\) is bounded on \(L^p(X)\) whenever

\[
2 \leq p < p_0' \quad \text{and} \quad \alpha > \max \left\{ n \left( \frac{1}{p_0} - \frac{1}{2} \right) - \frac{1}{q}, 0 \right\}.
\]

As a consequence, if \(f \in L^p(X)\), for \(p\) and \(\alpha\) in the range of \((4.1)\),

\[
\lim_{R \to \infty} S_{\alpha}^\ast(R)f(x) = f(x), \quad \text{a.e.}
\]

In order to prove Theorem 4.1 we use the classical approach which makes use of the square function to control the maximal operator (see \([7, 10, 26]\)). Here we should mention that we may assume that

\[
n \left( \frac{1}{p_0} - \frac{1}{2} \right) - \frac{1}{q} \geq 0.
\]

Otherwise, by \([9, \text{Corollary I.7}]\) it follows that \(L = 0\). Thus Theorem 4.1 trivially holds. We assume the condition \((4.2)\) for the rest of this section.

### 4.1 Reduction to square function estimate.

Let us recall the well known identity, for \(\alpha > 0\),

\[
\left( 1 - \frac{|m|^2}{R^2} \right) \alpha = C_{\alpha,\rho} R^{-2\alpha} \int_{|m|}^R (R^2 - r^2)^{\alpha - \rho} \left( 1 - \frac{|m|^2}{r^2} \right)^\rho dt
\]

where \(C_{\alpha,\rho} = 2 \Gamma(\alpha + 1)/\Gamma(\rho + 1) \Gamma(\alpha - \rho)\). By the spectral theory, we use an argument in \([45, \text{p.278–279}]\) to obtain

\[
S_\alpha^\ast(L)f(x) \leq C_{\alpha,\rho} \sup_{0 < R < \infty} \left( \frac{1}{R} \int_0^R |S_\rho^\ast(L)f(x)|^2 dt \right)^{1/2}
\]

provided that \(\rho > -1/2\) and \(\alpha > \rho + 1/2\).

By dyadic decomposition, we write \(x_\rho = \sum_{k \in \mathbb{Z}} 2^{-k}\phi(2^k x)\) for some \(\phi \in C_0^\infty(1/4,1/2)\). Thus

\[
(1 - |x|^2)^\rho =: \phi_0^\rho(x) + \sum_{k=1}^\infty 2^{-2k} \phi_0^\rho_2(k)
\]

where \(\phi_0^\rho_2 = \phi(2^k(1 - |x|^2))\), \(k \geq 1\) and

\[
\text{supp } \phi_0^\rho \subseteq \{ |x| \leq \frac{3}{4} \},
\]

\[
\text{supp } \phi_0^\rho_2 \subseteq \{ 1 - 2^{-k} \leq |x| \leq 1 - 2^{-k-2} \}.
\]

By \((4.3)\), for \(\alpha > \rho + 1/2\)

\[
\| S_\alpha^\ast(L)f \|_p \leq C \left\| \left( \sup_{0 < R < \infty} \frac{1}{R} \int_0^R \left| \phi_0^\rho \left( \frac{\sqrt{t}}{t} \right) f(x) \right|^2 dt \right)^{1/2} \right\|_p
\]

\[
+ C \sum_{k=1}^\infty 2^{-2k} \left\| \left( \int_0^\infty \left| \phi_0^\rho \left( \frac{\sqrt{t}}{t} \right) f(x) \right|^2 dt \right)^{1/2} \right\|_p
\]

By Lemma 3.2, for the first term we have

\[
\left\| \sup_{0 < t < \infty} \left| \phi_0^\rho \left( t \sqrt{L} \right) f(x) \right| \right\|_p \leq C \| f \|_p, \quad p_0 < p < p_0'.
\]
Now, in order to prove Theorem 4.1, by (4.5) it is sufficient to show the following.

**Proposition 4.2.** Let \( \phi \) be a fixed \( C^\infty \) function supported in \([-1/2, 1/2], |\phi| \leq 1 \). For every \( 0 < \delta \leq 1 \), define

\[
T_\delta f(x) = \left( \int_0^\infty \left| \phi \left( \delta^{-1} \left( 1 - \frac{L_f}{t^2} \right) \right) f(x) \right|^2 dt \right)^{1/2}.
\]

Suppose that \( L \) satisfies the property (FS) and the condition \((ST_{p_0,2})\) for some \( 1 \leq p_0 < 2 \) and \( 2 \leq q \leq \infty \). Then for all \( 2 \leq p < p'_0 \) and \( 0 < \delta \leq 1 \),

\[
||T_\delta f||_p \leq C(p)\delta^{1/2 + n(1 - \frac{1}{p})} ||f||_p.
\]

Before we start the proof of Proposition 4.2, we show that Theorem 4.1 is a straightforward consequence of Proposition 4.2.

**Proof of Theorem 4.1.** Substituting (4.6) and (4.8) with \( \delta = 2^{-k} \) back into (4.5) yields that, for a small enough \( \delta > 0 \),

\[
||S_\delta(L)f||_p \leq C||f||_p + C \sum_{k=1}^\infty 2^{-k(a - \frac{1}{2} - \epsilon)} 2^{-k(1/2 + n(1 - \frac{1}{p}))} ||f||_p \leq C||f||_p
\]

provided that \( 2 \leq p < p'_0 \) and \( \alpha > n(1/p_0 - 1/2) - 1/q \). This gives Theorem 4.1. \( \square \)

In order to prove Proposition 4.2, let us verify (4.8) for \( p = 2 \) first. Note that \( \phi \) is a fixed \( C^\infty \) function supported in \([-1/2, 1/2], |\phi| \leq 1 \). It follows from the spectral theory \([53]\) that, for any \( f \in L^2(X) \),

\[
||T_\delta f||_2 = \left( \int_0^\infty \left( \phi^2 \left( \delta^{-1} \left( 1 - \frac{L_f}{t^2} \right) \right) f, f \right) dt \right)^{1/2} \leq \left( \int_0^\infty \phi^2 \left( \delta^{-1} \left( 1 - \frac{L_f}{t^2} \right) \right) \frac{dt}{t} \right)^{1/2} ||f||_2
\]

(4.9)

\[
\leq C\delta^{1/2} ||f||_2.
\]

Since \( \delta \in (0, 1] \) and we assume the condition (4.2), the estimate (4.8) for \( p = 2 \) follows from (4.9).

For proof of Proposition 4.2 for \( 2 < p < p'_0 \), we make use of a weighted inequality which reduces the desired inequality to \( L^2 \) weighted estimate. See \([7, 10, 28]\).

**4.2. Weighted inequality for the square function.** Let \( r_0 \) be a number such that \( 1/r_0 = 2/p_0 - 1 \).

**Lemma 4.3.** Suppose that \( L \) satisfies the property (FS) and the condition \((ST_{p_0,2})\) for some \( 1 \leq p_0 < 2 \) and \( 2 \leq q \leq \infty \). For any \( 0 \leq w \) and \( 0 < \delta \leq 1 \),

\[
\int_X |T_\delta f(x)|^2 w(x) d\mu(x) \leq C\delta^{1/2 + n(1 - \frac{2}{p_0})} \int_X |f(x)|^2 \mathcal{M}_{r_0} w(x) d\mu(x).
\]

Again before we prove the lemma we show that it concludes the proof of Proposition 4.2. For every \( 2 < p < p'_0 \), we take \( w \in L^r \) with \( ||w||_r \leq 1 \) where \( 1/r + 2/p = 1 \). Since \( r_0 < r \), we have

\[
\int_X |T_\delta f(x)|^2 w(x) d\mu(x) \leq C\delta^{1/2 + n(1 - \frac{2}{p_0})} \int_X |f(x)|^2 \mathcal{M}_{r_0} w(x) d\mu(x)
\]

\[
\leq C\delta^{1/2 + n(1 - \frac{2}{p_0})} ||f||_p^2 ||\mathcal{M}_{r_0} w||_p \leq C\delta^{1/2 + n(1 - \frac{2}{p_0})} ||f||_p^2.
\]

Hence

\[
||T_\delta f||_p^2 \leq C\delta^{1/2 + n(1 - \frac{2}{p_0})} ||f||_p^2
\]

for some constant \( C > 0 \) independent of \( f \) and \( \delta \). This finishes the proof of Proposition 4.2.
Proof of Lemma 4.3. We start with Littlewood-Paley decomposition associated to the operator $L$. Fix a function $\varphi \in C^\infty$ supported in $\{1 \leq |s| \leq 3\}$ such that $\sum_{-\infty}^{\infty} \varphi(2^k s) = 1$ on $\mathbb{R}\setminus\{0\}$. Let

$$\varphi_k(\sqrt{L}) f = \varphi(2^{-k} \sqrt{L}) f, \quad k \in \mathbb{Z}. \tag{4.11}$$

By the spectral theory we have that, for any $f \in L^2(X)$,

$$\sum_k \varphi_k(\sqrt{L}) f = f. \tag{4.12}$$

By (4.12) we have that, for $f \in L^2(X) \cap L^p(X)$,

$$|T_{\delta} f(x)|^2 \leq \sum_k \int_0^\infty \left| \varphi \left( \delta^{-1} \left( 1 - \frac{L}{t^2} \right) \right) \varphi_k(\sqrt{L}) f(x) \right|^2 \frac{dt}{t} \tag{4.13}$$

For given $0 < \delta \leq 1$, we set $j_0 = -\lfloor \log_2 \delta \rfloor - 1$. Fix an even function $\eta \in C_0^\infty$, identically one on $\{|s| \leq 1\}$ and supported on $\{|s| \leq 2\}$. Let us set

$$\zeta_{j_0}(s) = \eta(2^{-j_0} s), \quad \zeta_j(s) = \eta(2^{-j} s) - \eta(2^{-j+1} s), \quad j > j_0 \tag{4.14}$$

so that

$$1 = \sum_{j \geq j_0} \zeta_j(s), \quad \forall s > 0. \tag{4.15}$$

Then we set $\phi_{\delta}(s) = \varphi \left( \delta^{-1} \left( 1 - |s|^2 \right) \right)$, and set, for $j \geq j_0$

$$\phi_{\delta,j}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_j(u) \overline{\varphi_{\delta}(u)} \cos(su) du. \tag{4.16}$$

Note that $\zeta_j$ is a dilate of a fixed smooth compactly supported function, supported away from 0 when $j > j_0$, hence

$$|\phi_{\delta,j}(s)| \leq \begin{cases} C_N 2^{(j_0-j)N}, & |s| \in [1/4, 8]; \\ C_N 2^{j_0} (1 + 2|s| - 1)^{-N}, & \text{otherwise} \end{cases} \tag{4.17}$$

for any $N$ and all $j \geq j_0$ (see [10, page 18]). By the Fourier inversion formula,

$$\phi \left( \delta^{-1} \left( 1 - s^2 \right) \right) = \sum_{j \geq j_0} \phi_{\delta,j}(s), \quad s > 0. \tag{4.18}$$

Set

$$d_j = 2^{j+1}/t. \tag{4.19}$$

By Lemma 2.2,

$$\text{supp } K_{\phi_{\delta,j}(\sqrt{L}/t)} \subseteq \mathcal{D}_{d_j} = \left\{ (x, y) \in X \times X : d(x, y) \leq 2^{j+1}/t \right\}. \tag{4.19}$$

From (4.13), (4.18) and Minkowski’s inequality, it follows that for every function $w \geq 0$,

$$\int_X |T_{\delta} f(x)|^2 w(x) d\mu(x) \leq C \sum_k \left[ \sum_{j \geq j_0} \left( \int_{2^{j-1}}^{2^{j+1}} \left| \phi_{\delta,j} \left( \frac{\sqrt{L}}{t} \right) \varphi_k(\sqrt{L}) f \right|^2 \frac{dt}{t} \right)^{1/2} \right]^2. \tag{4.20}$$
For a given $k \in \mathbb{Z}$, $j \geq j_0$, set $\rho = 2^{j-k+2} > 0$. Following an argument as in [20], we can choose a sequence $(x_m) \in X$ such that $d(x_m, x_t) > \rho/10$ for $m \neq \ell$ and $\sup_{x \in X} \inf_m d(x, x_m) \leq \rho/10$. Such sequence exists because $X$ is separable. Let $B_m = B(x_m, 3\rho)$ and define $\overline{B}_m$ by the formula

$$\overline{B}_m = \overline{B}\left( x_m, \frac{\rho}{10} \right) \cup \bigcup_{\ell < m} \overline{B}\left( x_\ell, \frac{\rho}{10} \right),$$

where $\overline{B}(x, \rho) = \{y \in X : d(x, y) \leq \rho\}$. Note that for $m \neq \ell$, $B(x_m, \frac{\rho}{20}) \cap B(x_\ell, \frac{\rho}{20}) = \emptyset$. Hence, by the doubling condition (2.2)

$$K = \sup_{m} \#\{\ell : d(x_m, x_\ell) \leq 2\rho\} \leq \sup_{x} \frac{V(x, (2 + \frac{1}{20})\rho)}{V(x, \frac{\rho}{20})} < C(41)^n.$$ 

It is not difficult to see that

$$D_{\rho} \subset \bigcup_{\{m : d(x_m, x_\ell) < 2\rho\}} \overline{B}_\ell \times \overline{B}_m \subset D_{4\rho}.$$ 

Recall that $1/r_0 + 2/p_0' = 1$ and $d_j = 2^{j+1}/t$. It follows by (4.19) and Hölder’s inequality that, for every $j, k$ and any test function $w \geq 0$,

$$\left\| \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) \varphi_k(\sqrt{L}) f \right\|_2^2 = \left\| \sum_{\ell, m : d(x_\ell, x_m) < 2d_j} X_{\overline{B}_\ell} \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) X_{\overline{B}_m} \varphi_k(\sqrt{L}) f \right\|_2^2.$$ 

Using (4.21), we have

$$\left\| \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) \varphi_k(\sqrt{L}) f \right\|_2^2 = \sum_{\ell} \left\| \sum_{m : d(x_\ell, x_m) < 2d_j} X_{\overline{B}_\ell} \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) X_{\overline{B}_m} \varphi_k(\sqrt{L}) f \right\|_2^2 
\leq K \sum_{\ell} \sum_{m : d(x_\ell, x_m) < 2d_j} \left\| X_{\overline{B}_\ell} \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) X_{\overline{B}_m} \varphi_k(\sqrt{L}) f \right\|_2^2.$$ 

By Hölder’s inequality it follows that

$$\left( \left\| \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) \varphi_k(\sqrt{L}) f \right\|_2^2, w \right) \leq K^2 \sum_{m} \|X_{B_m} w\|_{p_0} \left\| X_{B_m} \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) X_{\overline{B}_m} \varphi_k(\sqrt{L}) f \right\|_{p_0}^2.$$ 

Since $\phi_{\delta, j}$ is not compactly supported, we choose an even function $\theta \in C_0(-4, 4)$ such that $\theta(s) = 1$ for $s \in (-2, 2)$. Set

$$\psi_{0, \delta}(s) = \theta(\delta^{-1}(1 - s)) \quad \text{and} \quad \psi_{\ell, \delta}(s) = \theta(2^{-\ell} \delta^{-1}(1 - s)) - \theta(2^{-\ell+1} \delta^{-1}(1 - s))$$ 

for all $\ell \geq 1$ such that $1 = \sum_{\ell=0}^{\infty} \psi_{\ell, \delta}(s)$, and so $\phi_{\delta, j}(s) = \sum_{\ell=0}^{\infty} \psi_{\ell, \delta}(s)\phi_{\delta, j}(s)$ for all $s > 0$. From this, we apply (4.22) to write

$$\left( \int_{2^{k+1}}^{2^{k+2}} \left( \left\| \phi_{\delta, j} \left( \frac{\sqrt{L}}{t} \right) \varphi_k(\sqrt{L}) f \right\|_2^2, w \right) \frac{dt}{t} \right)^{1/2}$$

$$\leq \sum_{\ell = 0}^{[-\log_2\delta]} \sum_{m} \|X_{B_m} w\|_{p_0} \left( \int_{2^{k+1}}^{2^{k+2}} \left\| X_{B_m} \left( \psi_{\ell, \delta} \phi_{\delta, j} \right) \left( \frac{\sqrt{L}}{t} \right) X_{\overline{B}_m} \varphi_k(\sqrt{L}) f \right\|_{p_0}^2 \frac{dt}{t} \right)^{1/2}$$

$$+ \sum_{\ell = [-\log_2\delta]+1}^{\infty} \sum_{m} \|X_{B_m} w\|_{p_0} \left( \int_{2^{k+1}}^{2^{k+2}} \left\| X_{B_m} \left( \psi_{\ell, \delta} \phi_{\delta, j} \right) \left( \frac{\sqrt{L}}{t} \right) X_{\overline{B}_m} \varphi_k(\sqrt{L}) f \right\|_{p_0}^2 \frac{dt}{t} \right)^{1/2}$$

$$= I(j, k) + II(j, k).$$ 

As to be seen later, the first term $I(j, k)$ is the major one.
Estimate for $I(j, k)$. For $k \in \mathbb{Z}$ and $\lambda = 0, 1, \ldots, \lambda_0 = [8/\delta] + 1$, we set

$$I_k = \left[2^{k-1} + \lambda 2^{k-1} \delta, 2^{k-1} + (\lambda + 1) 2^{k-1} \delta\right],$$

so that $[2^{k-1}, 2^{k+2}] \subseteq \bigcup_{\lambda=0}^{\lambda_0} I_k$. Define

$$\eta_\lambda(s) = \eta \left( \lambda + \frac{2^{k-1} - s}{2^{k-1} \delta} \right),$$

where $\eta \in C_0^\infty(-1, 1)$ and $\sum_{\lambda \in \mathbb{Z}} \eta(-\lambda) = 1$. Observe that for every $t \in I_\lambda$, it is possible that $\psi_{t, \delta}(s/t) \eta_\lambda(s) \neq 0$ only when $\lambda - 2^{\ell+6} \leq \lambda' \leq \lambda + 2^{\ell+6}$. Hence, for $t \in I_\lambda$,

$$\left(\psi_{t, \delta}(\phi_{\delta, j})\right) \left(\frac{\sqrt{L}}{t}\right) = \sum_{\lambda' = \lambda - 2^{\ell+6}}^{\lambda + 2^{\ell+6}} \left(\psi_{t, \delta}(\phi_{\delta, j})\right) \left(\frac{\sqrt{L}}{t}\right) \eta_{\lambda'}(\sqrt{L}),$$

so

$$I(j, k) \leq \sum_{\ell=0}^{-\log_2 \delta} \sum_{m} \left\| \chi_{B_m} w \right\|_{l_0} \sum_{\lambda} \int_{j_k} \left(\sqrt{t} \sum_{\lambda' = \lambda - 2^{\ell+6}}^{\lambda + 2^{\ell+6}} \left\| \chi_{B_m}(\psi_{t, \delta}(\phi_{\delta, j})\right) \left(\frac{\sqrt{L}}{t}\right) \eta_{\lambda'}(\sqrt{L})[\chi_{B_m} \varphi_k(\sqrt{L})f] \right\|_{l_0}^2 dt \right)^{1/2}.$$

Note that

$$\text{supp } \psi_{t, \delta} \subseteq (1 - 2^{\ell+2} \delta, 1 + 2^{\ell+2} \delta).$$

Moreover, if $\ell \geq 1$, then $\psi_{t, \delta}(s) = 0$ for $s \in (1 - 2^{\ell} \delta, 1 + 2^{\ell} \delta)$. By the Stein-Tomas restriction type condition (ST$^q_{p_0, 2}$), we have, for $0 \leq \ell \leq [-\log_2 \delta]$,

$$\left\| \chi_{B_m}(\psi_{t, \delta}(\phi_{\delta, j})\right) \left(\frac{\sqrt{L}}{t}\right) \left\|_{l_0, p_0} = \left\| \left(\psi_{t, \delta}(\phi_{\delta, j})\right) \left(\frac{\sqrt{L}}{t}\right) \chi_{B_m}\right\|_{l_0, p_0} \leq C \left(2^{j(1 + 2^{\ell+2} \delta)} \right)^{\frac{n}{2}} \mu(B_m)^{\frac{1}{2}} \mu(\psi_{t, \delta}(\phi_{\delta, j}))/((1 + 2^{\ell+2} \delta) \cdot ) \|_q \leq C 2^{j(\frac{n}{2} + \frac{1}{2})} \mu(B_m)^{\frac{1}{2}} \mu(\psi_{t, \delta}(\phi_{\delta, j}))/((1 + 2^{\ell+2} \delta) \cdot ) \|_q.$$

From the definition of the function $\psi_{t, \delta}$, it follows by (4.17) that, for any $N < \infty$,

$$\left\| \psi_{t, \delta}(\phi_{\delta, j})(1 + 2^{\ell+2} \delta) \cdot \right\|_q \leq C \begin{cases} \frac{1}{\delta^{2^{(j_0 - j)N}}}, & \ell = 0, \\ \frac{1}{\delta^{2^{(j_0 - j)N}2^{j_{n-2}}}} & 1 \leq \ell \leq [-\log_2 \delta], \\ \frac{1}{\delta^{2^{(j_0 - j)N}2^{j_{n-2}}}} & 0 \leq \ell \leq [-\log_2 \delta]. \end{cases}$$

By this we have

$$\left\| \chi_{B_m}(\psi_{t, \delta}(\phi_{\delta, j})\right) \left(\frac{\sqrt{L}}{t}\right) \eta_{\lambda'}(\sqrt{L})[\chi_{B_m} \varphi_k(\sqrt{L})f] \left\|_{l_0} \leq C \delta^{\frac{1}{2}} 2^{(j_0 - j)N} \delta^{\frac{1}{2}} 2^{j_{n-2}} \mu(B_m)^{\frac{1}{2}} \left\| \eta_{\lambda'}(\sqrt{L})[\chi_{B_m} \varphi_k(\sqrt{L})f] \left\|_2. \right.$$
This, together with estimates (4.28) and (4.30), the fact that $1/r_0 + 2/p_0' = 1$ and

$$
\|\chi_{B_m} w\|_{r_0} \leq C \mu(B_m)^{\frac{1}{p_0'} - 1} \inf_{x \in B_m} \mathfrak{M}_{r_0} w(x),
$$

show that

$$
I(j, k) \leq C \delta^{\frac{1}{2} + \frac{1}{2j_{m-N}}} 2^{j_{m-N}(\frac{1}{p_0'} - 1)} \sum_{\ell} 2^{-\ell(N-1)} \left( \sum_{m} \mu(B_m)^{\frac{1}{p_0'} - 1} \|\chi_{B_m} w\|_{r_0} \int_{\mathfrak{M}_{r_0}} |\varphi_k(\sqrt{L}) f|^2 d\mu(x) \right)^{1/2}
$$

$$
\leq C \delta^{\frac{1}{2} + \frac{1}{2j_{m-N}}} 2^{j_{m-N}(\frac{1}{p_0'} - 1)} \left( \sum_{m} \int_{\mathfrak{M}_{r_0}} |\varphi_k(\sqrt{L}) f(x)|^2 \mathfrak{M}_{r_0} w(x) d\mu(x) \right)^{1/2}
$$

$$
\leq C \delta^{\frac{1}{2} + \frac{1}{2j_{m-N}}} 2^{j_{m-N}(\frac{1}{p_0'} - 1)} \left( \int_{X} |\varphi_k(\sqrt{L}) f(x)|^2 \mathfrak{M}_{r_0} w(x) d\mu(x) \right)^{1/2}.
$$

**Estimate for II(j, k).** Next we show bounds for the term II(j, k). For compactly supported function the $L^q$ norm is majorized by the supremum norm, so it follows from (ST$_{pq}$) that

$$
\left\| \chi_{B_m}(\psi_{\ell,\delta} \phi_{\delta,j}) \right\|_{L^q} \left( \frac{\sqrt{L}}{t} \right) \left\| \psi_{\ell,\delta} \phi_{\delta,j} \right\|_{L^q_{p_0}} \leq C(2j(1 + 2^{\ell+2}\delta)^{\frac{1}{2j_{m-N}}} \mu(B_m)^{\frac{1}{p_0'} - 1} \|\chi_{B_m} w\|_{r_0} \|\chi_{B_m} \varphi_k(\sqrt{L}) f\|_2)
$$

From the definition of the function $\psi_{\ell,\delta}$, it follows by (4.17) that, for $\ell \geq [-\log_2\delta] + 1$,

$$
\|\psi_{\ell,\delta} \phi_{\delta,j}((1 + 2^{\ell+2}\delta) \cdot)\|_\infty \leq C N 2^{-j_{m-N}(2j_{m-N})^{-N}}
$$

for any $N < \infty$. Therefore,

$$
II(j, k) \leq C \sum_{\ell = [-\log_2\delta] + 1}^{\infty} 2^{-j_{m-N}(2j_{m-N})^{-N}} \left( \sum_{m} \mu(B_m)^{\frac{1}{p_0'} - 1} \|\chi_{B_m} w\|_{r_0} \|\chi_{B_m} \varphi_k(\sqrt{L}) f\|_2 \right)^{1/2}
$$

$$
\leq C \delta^{\frac{1}{2} + \frac{1}{2j_{m-N}}} 2^{-j_{m-N}(2j_{m-N})^{-N+1}} \left( \int_{X} |\varphi_k(\sqrt{L}) f(x)|^2 \mathfrak{M}_{r_0} w(x) d\mu(x) \right)^{1/2}.
$$

(4.31)

Collecting the estimates of the terms $I(j, k)$ and $II(j, k)$, together with (4.20) and (4.24), we arrive at the conclusion that

$$
\int_{X} |T_{\delta} f(x)|^2 w(x) d\mu(x) \leq C \delta \left( \sum_{j \geq j_{m-N}} (\delta^{\frac{1}{2} + \frac{1}{2j_{m-N}}} + 2^{-j_{m-N}(2j_{m-N})^{-N+1}}) \right)^2 \sum_{k} \int_{X} |\varphi_k(\sqrt{L}) f(x)|^2 \mathfrak{M}_{r_0} w(x) d\mu(x)
$$

$$
\leq C \delta^{1 + \frac{1}{2} + n(1 - \frac{1}{p_0})} \int_{X} \sum_{k} |\varphi_k(\sqrt{L}) f(x)|^2 \mathfrak{M}_{r_0} w(x) d\mu(x)
$$

$$
\leq C \delta^{1 + \frac{1}{2} + n(1 - \frac{1}{p_0})} \int_{X} |f(x)|^2 \mathfrak{M}_{r_0} w(x) d\mu(x)
$$

whenever $N > (1/p_0 - 1/2) + 1$. The last inequality follows by Proposition 2.7 for the weighted inequality for the square function, since $\mathfrak{M}_{r_0} w$ is an $A_1$ weight. This proves Lemma 4.3 and completes the proof of Theorem 4.1. □
Throughout this section, we assume that \((X, d, \mu)\) is a metric measure space satisfying the conditions (1.7) or (1.10).

Let \(1 \leq p < 2\) and \(2 \leq q \leq \infty\). Following [9], we say that \(L\) satisfies the Sogge spectral cluster condition: If for a fixed natural number \(\kappa\) and for all \(N \in \mathbb{N}\) and all even Borel functions \(F\) such that \(\text{supp} \, F \subseteq [-N, N]\),

\[
(\text{SC}^q_{p, \kappa}) \quad \|F(\sqrt{L})\|_{p \to 2} \leq CN^{\kappa(\frac{1}{q} - \frac{1}{2})}\|F(N\cdot)\|_{\Lambda_{N, q}}
\]

for all \(x \in X\) where

\[
\|F\|_{\Lambda_{N, q}} = \left(\frac{1}{2N} \sum_{\ell = 1}^{N} \sup_{\lambda \in \left[\frac{\ell - \frac{1}{2}}{N}, \frac{\ell + \frac{1}{2}}{N}\right]} |F(\lambda)|^q\right)^{1/q}
\]

for \(F\) supported in \([-1, 1]\). For \(q = \infty\), we may put \(\|F\|_{\Lambda_{N, \infty}} = \|F\|_\infty\) (see also [13, 15]).

Both Theorems B and C stated in Introduction are a special case of the following statement with \(q = 2\).

**Theorem 5.1.** Suppose that \(L\) satisfies the property (FS) and the condition \((\text{SC}^q_{p, \kappa})\) for some \(1 \leq p_0 < 2\), \(2 \leq q \leq \infty\) and for some \(\kappa \in \mathbb{N}\). In addition, we assume that there exists \(\nu \geq 0\) such that (1.11) holds. Then the operator \(S^\alpha_{\nu}(L)\) is bounded on \(L^p(X)\) whenever

\[
2 \leq p < p^*_0, \quad \text{and} \quad \alpha > \nu + \max \left\{n \left(\frac{1}{p_0} - \frac{1}{2}\right) - \frac{1}{q}, 0\right\}.
\]

As a consequence, if \(f \in L^p(X)\), then for \(p\) and \(\alpha\) satisfying (5.1),

\[
\lim_{R \to \infty} S^\alpha_{\nu}(L)f(x) = f(x), \quad \text{a.e.}
\]

**Remark 5.2.** Note that if \((X, d, \mu)\) satisfies (1.10), then by Hölder’s inequality and \((1 + L)^0 = Id\), the condition (1.11) holds with \(\gamma = 0\).

**Remark 5.3.** Taking into account the condition \((\text{ST}^q_{p_0, 2})\) one could consider the following estimate introduced in [9]:

\[
\|F(\sqrt{L})\chi_{B(x, r)}\|_{p \to 2} \leq CV(x, r)^{\frac{1}{2} - \frac{1}{q}} (N\cdot)^{\kappa(\frac{1}{q} - \frac{1}{2})}\|F(N\cdot)\|_{\Lambda_{N, q}}.
\]

However, one can easily check that the above condition under assumption (1.7) or (1.10) is equivalent to \((\text{SC}^q_{p, \kappa})\), so here we only discuss the latter only.

**Remark 5.4.** Note that condition \((\text{SC}^q_{p_0, \kappa})\) is weaker than \((\text{ST}^q_{p_0, 2})\) and we need a priori estimate (1.11) in Theorem 5.1. Recall that in [9, Theorem I.10], one can obtain \(L^p\) bounds for Bochner-Riesz means under the assumption \((\text{AB}^q_{p_0})\) instead of estimate (1.11). Following [9], we say that \(L\) satisfies the condition \((\text{AB}^q_{p_0})\) if for each \(\varepsilon > 0\), there exists constant \(C_\varepsilon > 0\) such that for all \(N \in \mathbb{N}\) and even Borel functions \(H\) with \(\text{supp} \, H \subseteq [-N, N]\),

\[
(\text{AB}^q_{p_0}) \quad \|H(\sqrt{L})\|_{p_0 \to p_0} \leq C_\varepsilon N^{\kappa(\frac{1}{q} - \frac{1}{2}) + \varepsilon}\|H(N\cdot)\|_{\Lambda_{N, q}}.
\]

(see also [13, Theorem 3.6] and [15, Theorem 3.2] for related results). Once (1.11) is proved for some \(p_0 \in [1, 2]\) and all \(\nu > 0\), it is not difficult to check that \((\text{SC}^q_{p_0})\) implies \((\text{AB}^q_{p_0})\). Indeed, we apply (1.11) and \((\text{SC}^q_{p_0})\) to obtain

\[
\|H(\sqrt{L})\|_{p_0 \to \infty} \leq \|H(\sqrt{L} (1 + L)^{\frac{1}{2} - \frac{1}{q} + \varepsilon})\|_{p_0 \to 2}\|(1 + L)^{\frac{1}{2} - \frac{1}{q} + \varepsilon}\|_{2 \to p_0}
\]
and 4.1 is trivially true. We assume the condition (5.2) is a consequence of the following.

This shows that we may assume the condition

$$\frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{p_0}\right) - \nu \leq 0.$$  

Because, otherwise, $L = 0$ and Theorem 5.1 is trivially true. We assume the condition (5.2) for the rest of this section. As in Theorem 4.1, Theorem 5.1 is a consequence of the following.

**Proposition 5.5.** Under the same assumption as in Theorem 5.1, we have the uniform bound

$$\| (I - \frac{L}{R^2})_{\alpha} \|_{p_0 \to p_0} \leq C,$$

for $\alpha > \nu + n(1/p_0 - 1/2) - 1/q$ and $R > 0$. As a consequence, if $1/q > n(1/p_0 - 1/2) + \nu$ for some $q \geq 2$ and $1 \leq p_0 < 2$, then $L = 0$.

This shows that we may assume the condition

$$\frac{1}{q} + n\left(\frac{1}{2} - \frac{1}{p_0}\right) - \nu \leq 0.$$  

Recall that for every $0 < \delta \leq 1$ $T_\delta$ is defined by (4.6). Then for all $2 \leq p < p'_0$ and $0 < \delta \leq 1$,

$$\| T_\delta f \|_p \leq C(p)\delta^{\frac{1}{2} + n(\frac{1}{p_0} - \frac{1}{p}) - \nu} \| f \|_p.$$  

The estimate (5.3) for $p = 2$ follows from (4.9) and the condition (5.2). To show (5.3) for $2 < p < p'_0$, for $0 < \delta \leq 1$ we write

$$T_\delta f(x) = \left( \int_0^\infty \left| \phi \left( \delta^{-1} \left( 1 - \frac{L}{t^2} \right) \right) f(x) \right|^2 \frac{dt}{t} \right)^{1/2} \leq T_\delta^{(1)} f(x) + T_\delta^{(2)} f(x) + T_\delta^{(3)} f(x),$$

where

$$T_\delta^{(1)} f(x) : = \left( \int_0^\infty \left| \phi \left( \delta^{-1} \left( 1 - \frac{L}{t^2} \right) \right) f(x) \right|^2 \frac{dt}{t} \right)^{1/2},$$

$$T_\delta^{(2)} f(x) : = \left( \int_1^{1/\sqrt{\delta}} \left| \phi \left( \delta^{-1} \left( 1 - \frac{L}{t^2} \right) \right) f(x) \right|^2 \frac{dt}{t} \right)^{1/2},$$

$$T_\delta^{(3)} f(x) : = \left( \int_{1/\sqrt{\delta}}^\infty \left| \phi \left( \delta^{-1} \left( 1 - \frac{L}{t^2} \right) \right) f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$  

It is clear that to prove Proposition 5.6 it is sufficient to show the following Lemmas 5.7 and 5.8.

**Lemma 5.7.** Suppose the operator $L$ satisfies the property (FS) and condition (SC$_{p_0}$) for some $p_0$ such that $1 \leq p_0 < 2$, $2 \leq q \leq \infty$ and some $\kappa \in \mathbb{N}^+$. In addition, we assume that (1.11) holds for some $\nu \geq 0$. Then for all $2 \leq p \leq p'_0$ and $0 < \delta \leq 1$, we have

$$\| T_\delta^{(1)} f \|_p \leq C\delta^{1/2} \| f \|_p.$$
and

\( \|T^{(2)}_\delta f\|_p \leq C\delta^{\frac{1}{q} + \frac{1}{q'}} + (4 - \frac{1}{p_0})^{-\gamma}\|f\|_p. \)

**Lemma 5.8.** Suppose the operator \( L \) satisfies the property (FS) and the condition (SC\(^{q,k}\)) for some \( p_0 \) such that \( 1 < p_0 < 2, 2 \leq q \leq \infty \) and some \( k \in \mathbb{Z}^+ \). Then for all \( 2 \leq p < p'_0 \) and \( 0 < \delta \leq 1, \)

\[ \|T^{(3)}_\delta f\|_p \leq C(p)\delta^{\frac{1}{q} + \frac{1}{q'}} + (4 - \frac{1}{p_0})^{-\gamma}\|f\|_p. \]

**5.1. Proof of Lemma 5.7.** From (4.9), the proof reduces to showing (5.5) and (5.6) for \( p = p'_0 \) by interpolation. By (4.12), we have that for \( f \in L^2(X) \cap L^p(X), \)

\( |T^{(1)}_\delta f(x)|^2 \leq 5 \sum_{k \leq 0} \int_{2^{k+1}}^{2^{k+3}} \|\phi (\delta^{-1} \left( 1 - \frac{L}{t^2} \right)) \varphi_k(\sqrt{L}) f(x) \|^2 \frac{dt}{t}. \)

Write

\[ \Phi_{t,\delta}(\sqrt{L}) := \phi (\delta^{-1} \left( 1 - \frac{L}{t^2} \right)). \]

Similarly as in Section 4 for \( k \in \mathbb{Z} \) and \( \lambda = 0, 1, \ldots, \lambda_0 = \lfloor 8/\delta \rfloor + 1 \) let \( I_\lambda \) and \( J_\lambda \) be defined by (4.25) and (4.26), respectively. Observe that for every \( t \in I_\lambda \), if \( \Phi_{t,\delta}(s) \eta_k(s) \neq 0 \), then \( \lambda - \lambda \delta - 3 \leq \lambda' \leq \lambda + \lambda \delta + 3 \). Hence, we see that, for every \( t \in I_\lambda \),

\[ \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) = \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) \eta_{\lambda'}(\sqrt{L}), \]

and thus

\[ \int \left| \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) f \right|^2 \frac{dt}{t} = \sum_{\lambda} \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \int_{I_\lambda} \left| \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) f \right|^2 \frac{dt}{t} \leq C \sum_{\lambda} \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \int_{I_\lambda} \left| \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) f \right|^2 \frac{dt}{t}. \]

By Minkowski’s inequality,

\[ \|T^{(1)}_\delta f\|_{p'_0} \leq C \left\| \left( \sum_{k \leq 0} \int_{2^{k+1}}^{2^{k+3}} \left| \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{p'_0} \leq C \left\| \sum_{\lambda} \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \int_{I_\lambda} \left| \Phi_{t,\delta}(\sqrt{L}) \eta_{\lambda'}(\sqrt{L}) f \right|^2 \frac{dt}{t} \right\|_{p'_0} \leq C \left( \sum_{k \leq 0} \sum_{\lambda} \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \int_{I_\lambda} \left\| \Phi_{t,\delta}(\sqrt{L}) \eta_{\lambda'}(\sqrt{L}) f \right\|^2 \frac{dt}{t} \right)^{1/2}. \]

Note that \( t \leq 1 \) and by (SC\(^{q,k}\))

\[ \|\Phi_{t,\delta}(\sqrt{L})(1 + L)^{\gamma/2}\|_{2 \rightarrow p'_0} = \|\Phi_{t,\delta}(\sqrt{L})(1 + L)^{\gamma/2}\|_{p_0 \rightarrow 2} \leq C 2^{\left(\frac{1}{p_0} - \frac{1}{2}\right)} \|\Phi_{t,\delta}(2\cdot)\|_{2 \rightarrow \gamma} \leq C. \]
Hence \( \| \Phi_{t,\delta}(\sqrt{L}) \varphi_k(\sqrt{L}) \eta_{t'}(\sqrt{L}) f \|_{p_0} \leq C \| \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L})(1 + L)^{-\gamma/2} f \|_2 \). From this it is easy to see

\[
\| T^{(1)}_\delta f \|_{p_0'} \leq C \left( \sum_{k \leq 0} \sum_{d} \sum_{d' = d - 10}^{d + 10} \left\| \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L})(1 + L)^{-\gamma/2} f \right\|_2 \int_{I_k} \frac{dt}{t} \right)^{1/2}
\leq C \left( \delta \sum_{k \leq 0} \sum_{d} \left\| \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L})(1 + L)^{-\gamma/2} f \right\|_2 \right)^{1/2}
\leq C \delta^{1/2} \| (1 + L)^{-\gamma/2} f \|_2
\leq C \delta^{1/2} \| f \|_{p_0'},
\]

where for the last inequality we use (1.11). Thus we get (5.5).

We now show (5.6) for \( p = p_0' \). By (4.12), we have that for \( f \in L^2(X) \cap L^p(X) \),

\[
\| T^{(2)}_\delta f(x) \| \leq C \sum_{0 < k \leq 1 - \log_{2 \delta}} \sum_{\nu = 2^{k+1}}^{2^{k+2}} \left\| \phi \left( \delta^{-1} \left( 1 - \frac{L}{t^2} \right) \right) \varphi_k(\sqrt{L}) f(x) \right\|_2 \frac{dt}{t}.
\]

Again, for \( k \in \mathbb{Z} \) and \( t \in [2^{k-1}, 2^{k+2}] \) and \( \lambda = 0, 1, \cdots, A_0 = [8/\delta] + 1 \), we consider the interval \( I_k \) and the function \( \eta_k \) which are given by (4.25) and (4.26), respectively. Observe that for every \( t \in I_k \), if \( \Phi_{t,\delta}(\eta_{t'}) \neq 0 \), then \( \lambda - \lambda \delta - 3 \leq \lambda' \leq \lambda + \lambda \delta + 3 \). Hence, as before it follows that, for every \( t \in I_k \), (5.8) holds and we have (5.9). Putting this in (5.10) and Minkowski’s inequality (twice) give

\[
\| T^{(2)}_\delta f \|_{p_0'} \leq C \left( \sum_{0 < k \leq 1 - \log_{2 \delta}} \sum_{\nu = 2^{k+1}}^{2^{k+2}} \left\| \Phi_{t,\delta}(\sqrt{L}) \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L}) f \right\|_2 \frac{dt}{t} \right)^{1/2}
\leq C \left( \delta \sum_{0 < k \leq 1 - \log_{2 \delta}} \sum_{\nu = 2^{k+1}}^{2^{k+2}} \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \left\| \Phi_{t,\delta}(\sqrt{L}) \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L}) f \right\|_2 \frac{dt}{t} \right)^{1/2}.
\]

We claim that

\[
\| \Phi_{t,\delta}(\sqrt{L})(1 + L)^{\gamma/2} \|_{2 \to p_0'} \leq C \delta^{1/2 - \gamma} \left( \frac{t}{\delta^{\gamma}} \right)^{1/2} - \nu.
\]

Assuming this for the moment, we complete the proof. From (5.10) and (5.11) we have

\[
\| T^{(2)}_\delta f \|_{p_0'} \leq C \delta^{1/2 - \gamma} \left( \frac{t}{\delta^{\gamma}} \right)^{1/2} - \nu \left( \sum_{0 < k \leq 1 - \log_{2 \delta}} \sum_{\nu = 2^{k+1}}^{2^{k+2}} \sum_{\lambda' = \lambda - 10}^{\lambda + 10} \left\| \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L})(1 + L)^{-\gamma/2} f \right\|_2 \frac{dt}{t} \right)^{1/2}
\]

Thus, it is easy to see that

\[
\| T^{(2)}_\delta f \|_{p_0'} \leq C \delta^{1/2 - \gamma} \left( \frac{t}{\delta^{\gamma}} \right)^{1/2} - \nu \left( \sum_{0 < k \leq 1 - \log_{2 \delta}} \sum_{\nu = 2^{k+1}}^{2^{k+2}} \sum_{\lambda'} \left\| \eta_{t'}(\sqrt{L}) \varphi_k(\sqrt{L})(1 + L)^{-\gamma/2} f \right\|_2 \right)^{1/2}
\leq C \delta^{1/2 - \gamma} \left( \frac{t}{\delta^{\gamma}} \right)^{1/2} - \nu \| (1 + L)^{-\gamma/2} f \|_2
\leq C \delta^{1/2 - \gamma} \left( \frac{t}{\delta^{\gamma}} \right)^{1/2} - \nu \| f \|_{p_0'}.
\]

For the last inequality we use (1.11). This gives the desired estimate.

It remains to show (5.11). Let \( N = 8[\delta] + 1 \). Note that \( \text{supp} \Phi_{t,\delta} \subset [-N, N] \). From (SC\(_{p_0}^{0,k}\))

\[
\| \Phi_{t,\delta}(\sqrt{L})(1 + L)^{\gamma/2} \|_{2 \to p_0'} = \| \Phi_{t,\delta}(\sqrt{L})(1 + L)^{\gamma/2} \|_{p_0 \to 2}
\]
\[ \leq CN^{n(\frac{1}{p_0} - \frac{1}{2})}\|\Phi_{t,\delta}(Nu)(1 + N^2 u^2)^{\gamma/2}\|_{N \cdot q}. \]

We estimate \(\|\Phi_{t,\delta}(Nu)(1 + N^2 u^2)^{\gamma/2}\|_{N \cdot q} \). Set \(H(\lambda) = \Phi_{t,\delta}(\lambda)(1 + \lambda)^{\gamma/2}. \) Let \(\xi \in C_c^\infty\) be an even function such that \(\text{supp} \xi \subset [-1, 1], \xi(0) = 1\) and \(\xi^{(k)}(0) = 0\) for all \(1 \leq k \leq [\beta] + 2.\) Write \(\xi_N = N\xi(Nu).\) Then
\[ \|\Phi_{t,\delta}(Nu)(1 + N^2 u^2)^{\gamma/2}\|_{N \cdot q} \leq \|(H - \xi_{N^{-1}} * H)(Nu)\|_{N \cdot q} + \|(\xi_{N^{-1}} * H)(Nu)\|_{N \cdot q}. \]

To estimate the first in the right hand side, we make use of the following (for its proof, see [13, (3.29)]) or [15, Proposition 4.6]): If \(supp G \subset [-1, 1]\), then
\[ (5.12) \quad \|G - \xi_N * G\|_{N \cdot q} \leq CN^{-\beta}\|G\|_{W^{\beta,q}} \]
for all \(\beta > 1/q\) and any \(N \in \mathbb{N}^\ast\). Since \((H - \xi_{N^{-1}} * H)(Nu) = H(Nu) - (\xi_{N} * (H(N\cdot))(u). For \(\beta > n(1/2 - 1/p_0)^n\), we get
\[ (5.13) \quad \|(H - \xi_{N^{-1}} * H)(Nu)\|_{N \cdot q} \leq CN^{-\beta}\|H(N \cdot)\|_{W^{\beta,q}} = CN^{-\beta}\|\Phi_{t,\delta}(Nu)(1 + N^2 u^2)^{\gamma/2}\|_{W^{\beta,q}} \leq CN^{-\beta + \gamma}\delta_{\pi}^{1/\gamma} \delta_{\pi}^{-\beta}. \]

For the second one, note that
\[ \|(\xi_{N^{-1}} * H)(Nu)\|_{N \cdot q} = \left( \frac{1}{Nx} \sum_{i=1-Nx}^{Nx} \sup_{\lambda \in [i/Nx]} |(\xi \ast H(\cdot/Nx^{-1}))(\lambda)|^q \right)^{1/q} \leq \left( \frac{1}{Nx} \sum_{i=1-Nx}^{Nx} \sup_{\lambda \in [i-1, i]} |(\xi \ast H(\cdot/Nx^{-1}))(\lambda)|^q \right)^{1/q}. \]

Using \(|\xi \ast h(\lambda)|^q \leq C|\xi|^q \int_{-1}^{1+1} |h(u)|^q du,
\[ \|(\xi_{N^{-1}} * H)(Nu)\|_{N \cdot q} \leq C \left( \frac{1}{Nx} \sum_{i=1-Nx}^{Nx} \sup_{\lambda \in [i-1, i]} \int_{\lambda-1}^{\lambda+1} |H(u/Nx^{-1})|^q du \right)^{1/q} \leq C \left( \frac{1}{Nx} \sum_{i=1-Nx}^{Nx} \int_{i-2}^{i+1} |H(u/Nx^{-1})|^q du \right)^{1/q} \leq CN^{-\gamma}\|H(\cdot/Nx^{-1})\|_{q} \leq CN^{-\gamma}(t^\gamma \delta_{t^\gamma} \delta_{t^\gamma} \leq C\delta_{t^\gamma}^{1/\gamma} \delta_{t^\gamma}^{-\gamma}. \]

Combining (5.13) and the above, and noting that \(1 \leq t \leq 1/\sqrt{\delta}, \) we have
\[ \|\Phi_{t,\delta}(\sqrt{L})(1 + L)^{\gamma/2}\|_{2 \rightarrow r'_0} \leq CN^{n(\frac{1}{r_0} - \frac{1}{2})}(N^{-\beta + \gamma}\delta_{\pi}^{1/\gamma} \delta_{\pi}^{-\beta} + t^\gamma \delta_{t^\gamma}^{1/\gamma}) \leq C\delta_{t^\gamma}^{1/\gamma} \delta_{t^\gamma}^{-\gamma - n(1/p_0 - 1/2) + \gamma}. \]

Here, we use the relation \(\gamma = n(\kappa - 1)(1/p_0 - 1/2) + \kappa \nu. \) This gives (5.11), and completes the proof of (5.6).

\[ \square \]

5.2. Proof of Lemma 5.8. As in Proposition 4.2, the proof of Lemma 5.8 reduces to showing the following lemma.

Lemma 5.9. For any \(0 \leq w\) and \(0 < \delta \leq 1,
\[ \int_{X} |T^{(3)}_{\delta} f(x)|^w w(x)d\mu(x) \leq C\delta^{1/\gamma + \gamma(\frac{1}{r_0} - 1)} \int_{X} |f(x)|^2 \Re \eta_{r_0} w(x)d\mu(x), \]
where \(1/r_0 + 2/p'_0 = 1.\)
Proof. We prove Lemma 5.9 by modifying that of Lemma 4.3. By (4.12), we have that, for \( f \in L^2(X) \cap L^p(X) \),

\[
|T^{(3)}_\delta f(x)|^2 \leq C \sum_{k > 1 - \log_2 \delta} \int_{2^{k-1}}^{2^{k+2}} \left| \phi(\delta^{-1} \left( 1 - \frac{L}{t^2} \right)) \varphi_k(\sqrt{L}f(x)) \right|^2 dt \cdot \frac{1}{t}.
\]

For given \( 0 < \delta \leq 1 \), we let \( \delta \in [2^{-j_0-1}, 2^{-j_0}) \) for some \( j_0 \in \mathbb{Z} \). As in the proof Lemma 4.3 we fix a cutoff function \( \eta \in C^\infty_0 \), identically one on \(|\eta| \leq 1 \) and supported on \(|\eta| \leq 2 \). For \( j \geq j_0 \) we define \( \zeta_j \) by (4.14) so that (4.15) holds. Then let \( \phi_{\delta,j} \) be defined by (4.16) so that (4.18) holds. From (5.14) and (4.18), it follows that for every function \( w \geq 0 \),

\[
\int_X |T^{(3)}_\delta f(x)|^2 w(x) d\mu(x) \leq C \sum_{k > 1 - \log_2 \delta} \left[ \sum_{j \leq j_0} \left( \int_{2^{k-1}}^{2^{k+2}} \left| \phi_{\delta,j} \left( \frac{\sqrt{L}}{t} \right) \varphi_k(\sqrt{L}f)^2 \right| w \, dt \right) \right]^{1/2} \cdot \frac{1}{t}.
\]

For \( \ell \geq 0 \) let \( \psi_{\ell,\delta} \) be defined by (4.23). So, \( 1 = \sum_{\ell=0}^\infty \psi_{\ell,\delta}(s) \), and so \( \phi_{\delta,j}(s) = \sum_{\ell=0}^\infty \psi_{\ell,\delta}(s) \) for all \( s > 0 \). Similarly as in (4.24) we get

\[
\left( \sum_{\ell=0}^{[-\log_2 \delta]} \left( \sum_m \left\| \chi_{B_m} w \right\| \int_{2^{k-1}}^{2^{k+2}} \left\| \chi_{B_m} \psi_{\ell,\delta} \phi_{\delta,j} \right\| \left( \frac{\sqrt{L}}{t} \right) \chi_{B_m} \varphi_k(\sqrt{L}f)^2 \, dt \right) \right)^{1/2}
\]

\[
= I(j, k) + \Pi(j, k).
\]

As in Section 4, the first term \( I(j, k) \) is the major one. We handle \( \Pi(j, k) \) first.

**Estimates for \( \Pi(j, k) \).** Note that \( |F|_{L^2} \leq |F|_{L^{\infty}} = |F|_{L^{\infty}} \) so that \( F \in C^\infty_0 \) implies \( \text{ST}_{p_0,2} \) for all functions \( F \) with \( \text{supp} F \subset (b, R) \). Hence we can repeat the same argument used for the proof of (4.31) to show that for any \( N < \infty \)

\[
\Pi(j, k) \leq C \delta 2^{j \left[ \frac{\left( 1 + 2^j \right)}{\left( 1 + 2^j \right)} - N + 1 \right]} \left( \int_X \varphi_k(\sqrt{L}f)^2 w(x) d\mu(x) \right)^{1/2}.
\]

**Estimates for \( I(j, k) \).** As before (see Section 4), for \( k \in \mathbb{Z} \) and \( \ell \in [2^{k-1}, 2^{k+2}] \) and \( \lambda = 0, 1, \ldots, \lambda_0 = [8/\delta] + 1 \), we consider the interval \( I_1 \) and the function \( \eta_1 \) which are given by (4.25) and (4.26), respectively. For \( t \in I_1 \), \( \lambda - 2^{\ell+\delta} \leq \lambda - \lambda + 2^{\ell+\delta} \) \( \psi_{\ell,\delta}(s/t) \eta_1(s) \neq 0 \). Thus, for \( t \in I_1 \), we have (4.27). Using this we get

\[
I(j, k) \leq \sum_{\ell=0}^{[-\log_2 \delta]} \left[ \sum_m \left\| \chi_{B_m} w \right\| \int_{I_1} \left( \sum_{s = 1 - 2^{\ell+2\delta}}^{1 + 2^{\ell+2\delta}} \left\| \chi_{B_m} \psi_{\ell,\delta} \phi_{\delta,j} \right\| \left( \frac{\sqrt{L}}{t} \right) \eta_1(\sqrt{L}) \chi_{B_m} \varphi_k(\sqrt{L}f)^2 \, dt \right) \right]^{1/2}.
\]

Note that \( \text{supp} \psi_{\ell,\delta} \subset (1 - 2^{\ell+2\delta}, 1 + 2^{\ell+2\delta}) \). Moreover, if \( \ell \geq 1 \), then \( \psi_{\ell,\delta}(s) = 0 \) for \( s \in (1 - 2^{\ell+2\delta}, 1 + 2^{\ell+2\delta}) \), and so \( \text{supp} (\psi_{\ell,\delta} \phi_{\delta,j}) (s/t) \subset [t(1 - 2^{\ell+2\delta}), t(1 + 2^{\ell+2\delta})] \). Let \( R = [t(1 + 2^{\ell+2\delta})] + 1 \). By the
condition \((\text{SC}^\alpha\ell_{p_0})\), we have that, for \(0 \leq \ell \leq \lfloor -\log_2 \delta \rfloor\),
\[
\left\| X_{B_n}(\psi_{\ell,\delta}(\phi_{j})) \left( \frac{\nabla L}{t} \right) \right\|_{L^2 \to L^p_0} = \left\| \left( \frac{\nabla L}{t} \right) X_{B_n}(\psi_{\ell,\delta}(\phi_{j})) \right\|_{L^p_0 \to L^p_0} 
\leq C \left( 2^{\ell(1 + 2^{\ell+2} \delta)^{\frac{1}{2} \mu(B_m)^{\frac{1}{2} - \frac{1}{2}}} \mu(B_m)^{\frac{1}{2} - \frac{1}{2}}} \right) \| (\psi_{\ell,\delta}(\phi_{j}))(R \cdot / t) \|_{L^\infty} \bigg)_{R^\infty} \delta. \tag{5.17}
\]

We note that
\[
\text{supp} (\psi_{\ell,\delta}(\phi_{j}))(R \cdot / t) \subset \left[ \frac{t(1 - 2^{\ell+2} \delta)}{R}, \frac{t(1 + 2^{\ell+2} \delta)}{R} \right].
\]
This, in combination with the fact that \(R^\infty \delta \geq 1\), gives
\[
\| (\psi_{\ell,\delta}(\phi_{j}))(1 + 2^{\ell+2} \delta) \|_{L^\infty} \leq \| \psi_{\ell,\delta}(\phi_{j}) \|_{\infty} \| X_{n = 1^{\ell+2} \delta 2^{\ell+2} \delta}} \| \| X_{B_n} \psi_k(\sqrt{L}) \|_{L^q} \leq C \| \psi_{\ell,\delta}(\phi_{j}) \|_{\infty} \left( \frac{2^{\ell+3} t \delta}{R} \right)^{1/q}.
\]
From this and (4.29) with \(q = \infty\) we see that
\[
\| (\psi_{\ell,\delta}(\phi_{j}))(1 + 2^{\ell+2} \delta) \|_{L^\infty} \leq C N 2^{(j_0 - j)N} 2^{\ell(2^\ell \delta)^{\frac{1}{2}}}. \tag{5.17}
\]

Thus (5.17) and the above inequality yield
\[
\| X_{B_n}(\psi_{\ell,\delta}(\phi_{j})) \left( \frac{\nabla L}{t} \right) \eta(L) [X_{B_n} \psi_k(\sqrt{L})] f \|_{L^p_0} \leq C \delta^{-\frac{1}{4} j_0 - j/2} \eta(L) [X_{B_n} \psi_k(\sqrt{L})] f \|_{L^p_0} \leq \frac{1}{4} j_0 - j/2 \int X |\phi_k(\sqrt{L}) f(x)|^2 \eta_{\nu_0} w(x) d\mu(x). \tag{5.18}
\]

Once (5.18) is obtained, we may repeat the lines of argument in the proof of Lemma 4.3 to get
\[
I(j, k) \leq C \delta^{\frac{1}{4} j_0 - j/2} 2^{\ell(N - 1)/2} \delta^{\frac{1}{2} \mu(B_m)^{\frac{1}{2} - \frac{1}{2}}} \int X |\phi_k(\sqrt{L}) f(x)|^2 \eta_{\nu_0} w(x) d\mu(x).
\]

Finally, combining the estimates for \(I(j, k)\) and \(I^2(j, k)\), together with (5.15) and (5.16), we get
\[
\int X |T^{(3)}_{\delta} f(x)|^2 w(x) d\mu(x) \leq C \delta^{\frac{1}{4} j_0 + n(1 - 1/2)} \int X |f|^2 \eta_{\nu_0} w(x) dx
\]
whenever \(N > n(1/p_0 - 1/2) + 1\). This completes proof of Lemma 5.9.

6. Applications

As applications of our theorems we discuss several examples of important elliptic operators. Our results, Theorems 4.1 and 5.1 have applications to all the examples which are discussed in [15] and [9]. Those include elliptic operators on compact manifolds, the harmonic oscillator, radial Schrödinger operators with inverse square potentials and the Schrödinger operators on asymptotically conic manifolds.

6.1. Laplace-Beltrami operator on compact manifolds. Let \(\Delta_g\) be the Laplace-Beltrami operator on a compact smooth Riemannian manifold \((M, g)\) of dimension \(n\). It was shown by Sogge that the condition \((\text{SC}^\alpha\ell_{p_0})\) holds with \(L = -\Delta_g\) in the standard range of Stein-Tomas restriction theorem, that is to say, for \(1 \leq p \leq 2(n + 1)/(n + 3)\), see [40, 41]. Hence we can apply Theorem 5.1 and obtain the following.
Corollary 6.1. Suppose that $\Delta_g$ is the Laplace-Beltrami operator on a compact smooth Riemannian manifolds $(M, g)$ of dimension $n$. Then the operator $S^\alpha_n(\Delta_g)$ is bounded on $L^p(M)$ whenever
\begin{equation}
    p \geq \frac{2(n+1)}{n-1}, \quad \text{and} \quad \alpha > \max \left\{ n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right\}.
\end{equation}

The corollary can be extended to the Laplace-Beltrami operator on a certain class of compact manifolds with boundaries if one combines Theorem 5.1 and the results in Sogge [42]. As far as we are aware, Corollary 6.1, especially in view of its generality, has not appeared in any literature before. However, we should mention that in [33] Mockenhaupt, Seeger and Sogge showed that the sharp maximal Bochner-Riesz bounds for $p \geq 2$ holds when $(M, g)$ is a compact Riemannian manifold of dimension 2 with periodicity assumption for the geodesic flow.

6.2. Schrödinger operator on asymptotically conic manifolds. Scattering manifolds or asymptotically conic manifolds are defined as interiors of a compact manifold with boundary $M$, and the metric $g$ is smooth on the interior $M^\circ$ and has the form
\[ g = \frac{dx^2}{x^2} + \frac{h(x)}{x^2} \]
in a collar neighbourhood near $\partial M$, where $x$ is a smooth boundary defining function for $M$ and $h(x)$ a smooth one-parameter family of metrics on $\partial M$; the function $r := 1/x$ near $x = 0$ can be thought of as a radial coordinate near infinity and the metric there is asymptotic to the exact metric cone $((0, \infty), \times \partial M, dr^2 + r^2h(0))$.

The restriction estimate (1.6) and Bochner-Reisz sumability results for a class of Laplace type operators on on asymptotically conic manifolds were obtained in [20]. Our approach allows us to complement these results with the following concerning the maximal Bochner-Riesz operator.

Corollary 6.2. Let $(M, g)$ be an asymptotically conic nontrapping manifold of dimension $n \geq 3$, and let $x$ be a smooth boundary defining function of $\partial M$. Let $L := -\Delta_g + V$ be a Schrödinger operator with $V \in x^3C^\infty(M)$ and assume that $L$ has no $L^2$-eigenvalues and that $0$ is not a resonance. Then the operator $S^\alpha_n(L)$ is bounded on $L^p(M)$ whenever
\begin{equation}
    p \geq \frac{2(n+1)}{n-1}, \quad \text{and} \quad \alpha > \max \left\{ n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right\}.
\end{equation}

Proof. Corollary 6.2 follows from restriction estimates (1.6) established in [20, Theorem 1.2] and Theorem A. □

Corollary 6.2 includes a class of operators which are 0-th order perturbations of the Laplacian on nontrapping asymptotically conic manifolds. In particular, our results cover the following settings: the Schrödinger operators, i.e. $-\Delta + V$ on $\mathbb{R}^n$, where $V$ smooth and decaying sufficiently at infinity; the Laplacian with respect to metric perturbations of the flat metric on $\mathbb{R}^n$, again decaying sufficiently at infinity; and the Laplacian on asymptotically conic manifolds, see [20].

6.3. The harmonic oscillator. In this section we focus on the Schrödinger operators such as the harmonic oscillator $-\Delta + |x|^2$ on $L^2(\mathbb{R}^n)$ for $n \geq 2$. Bochner-Riesz summability results for the harmonic oscillator were studied and sharp results were obtained by Karadzhov [24] and Thangavelu in [51, 52]). Here we establish the corresponding result for the maximal Bochner-Riesz operator. However, we consider the class Schrödinger operators $L = -\Delta + V(x)$ with a positive potential $V$ which satisfies the following condition
\begin{equation}
    V(x) \sim |x|^2, \quad |\nabla V(x)| \sim |x|, \quad |\partial^2_x V(x)| \leq 1.
\end{equation}
Clearly this class includes the harmonic oscillator.

A restriction type result for this class of operators was established by Koch and Tataru in [25, Theorem 4], which states that, for $\lambda \geq 0$ and $1 \leq p \leq 2n/(n+2)$,

$$\|E_\lambda[A^2, A^2 + 1]\|_{p \rightarrow 2} \leq C(1 + \lambda)^{(n+1)/2 - 1}.$$  

It is not difficult to show that the above condition is equivalent to condition $(\text{SC}_p^2)$ for $\kappa = 2$ and $1 \leq p \leq 2n/(n+2)$, see [9].

As a consequence of Theorem 5.1 we establish boundedness of the associated maximal Bochner-Riesz operator.

**Corollary 6.3.** Let $L = -\Delta + V(x)$ with a positive potential $V(x)$ satisfying (6.2). Then the operator $S_\alpha^a(L)$ is bounded on $L^p(\mathbb{R}^n)$ whenever

$$p \geq \frac{2n}{n-2} \quad \text{and} \quad \alpha > \max \left\{ \frac{1}{p}, \frac{1}{2}, \frac{1}{2} \right\}. \quad (6.3)$$

**Proof.** As we just mentioned, the condition $(\text{SC}_p^2)$ for $\kappa = 2$ and $1 \leq p' \leq \frac{2n}{n+2}$ follows from [25, Theorem 4] and [9, Theorem III.9]. Hence by Theorem C it is enough to show that if $V(x) \sim |x|^2$ is a positive potential and $L = -\Delta + V$, then

$$\|(1 + L)^{-\gamma/2}\|_{2 \rightarrow p'} \leq C, \quad \gamma = n(1/p - 1/2) + 2\nu \quad (6.4)$$

for $1 \leq p' \leq \frac{2n}{n+2}$ and all $\nu > 0$. The proof of (6.4) for $p = 1$ is given in [15, Lemma 7.9]. We give a brief proof of this for completeness.

Now fix $\nu$ as a positive number. To prove (6.4), we put $M = M_{\sqrt{1+\nu}}$. Then we note that

$$\|(1 + L)^{1/2}f\|_2^2 = \langle (1 + L)f, f \rangle \geq \langle M^2f, f \rangle = \|Mf\|_2^2.$$  

By the L"owner-Heinz inequality for any quadratic forms $B_1$ and $B_2$, if $B_1 \geq B_2 \geq 0$, then $B_1^{\alpha} \geq B_2^{\alpha}$ for $0 \leq \alpha \leq 1$. Hence,

$$\langle (1 + L)^{\alpha}f, f \rangle \geq \langle M^{2\alpha}f, f \rangle.$$  

Thus, for $\alpha \in [0, 1]$,

$$\|M^{\alpha}(1 + L)^{-\alpha/2}\|_{2 \rightarrow 2} \leq C. \quad (6.5)$$

For $\alpha = 1$ the operator $M^{\alpha}(1 + L)^{-\alpha/2}$ is of a first order Riesz transform type and a standard argument yields, for any $q \in (1, 2]$,

$$\|M(1 + L)^{-1/2}\|_{q \rightarrow q} \leq C, \quad (6.6)$$

see [38, Theorem 11]. Then by Hölder’s inequality, for any $q_1 \geq q_2 \geq 1$ with $s = (1/q_2 - 1/q_1)^{-1}$,

$$\|M^{-\alpha}\|_{q_1 \rightarrow q_2} \leq C \left( \int_{\mathbb{R}^n} (1 + V(x))^{-s\alpha/2} dx \right)^{1/(s\alpha)}. \quad (6.7)$$

Recall that $\gamma = n(1/p - 1/2) + 2\nu$. Write

$$(1 + L)^{-\gamma/2} = (M^{-1}M(1 + L)^{-1/2})^{\gamma/2} = \gamma M^{\gamma} M^{-\gamma} M^{\gamma} (1 + L)^{[(\gamma - \gamma)/2]}. \quad (6.8)$$

Because of $V(x) \sim |x|^2$, choose $s = (n + \varepsilon)/\alpha$ in (6.7) with $\varepsilon = 2\nu/(1/p' - 1/2) > 0$. Denote $p_0$ by $1/p_0 = (\gamma - [\gamma])/(n + \varepsilon) + 1/2$ and for each $1 \leq i \leq [\gamma] - 1$ we define $p_i$ by putting $1/p_{i+1} - 1/p_i =
1/(n + s), so \( p_{\gamma} = p' \). Now multiple composition of operators from (6.5), (6.6) and (6.7), in combination with (6.8), yield

\[
\|(1 + L)^{-\gamma/2}\|_{2 \to p'} \leq \|M^{\gamma-i\gamma}(1 + L)^{(\gamma-i\gamma)/2}\|_{2 \to 2}\|M^{\gamma-i\gamma}\|_{2 \to p_0} \prod_{i=0}^{[\gamma]-1} \|M^{-1}M(1 + L)^{-1/2}\|_{p_i \to p_{i+1}} \leq C.
\]

This finishes the proof of (6.4), and completes the proof of Corollary 6.3.

6.4. Operators \( \Delta_n + c/r^2 \) acting on \( L^2((0, \infty), r^{n-1}dr) \). In this section we consider a class of the Schrödinger operators on \( L^2((0, \infty), r^{n-1}dr) \). These operators generate semigroups but do not have the classical Gaussian upper bound for the heat kernel.

Fix \( n > 2 \) and \( c > -(n-2)^2/4 \) and consider the space \( L^2((0, \infty), r^{n-1}dr) \). For \( f, g \in C_c^\infty(0, \infty) \) we define the quadratic form

\[
Q_{n,c}^{(0,\infty)}(f, g) = \int_0^\infty f'(r)g'(r)r^{n-1}dr + \int_0^\infty \frac{c}{r^2}f(r)g(r)r^{n-1}dr.
\]

Using the Friedrichs extension one can define the operator \( L_{n,c} = \Delta_n + c/r^2 \) as the unique self-adjoint operator corresponding to \( Q_{n,c}^{(0,\infty)} \), acting on \( L^2((0, \infty), r^{n-1}dr) \). In the sequel we will write \( L \) instead of \( L_{n,c} \), which is formally given by the following formula

\[
Lf = (\Delta_n + \frac{c}{r^2})f = -\frac{d^2}{dr^2}f - \frac{n-1}{r} \frac{d}{dr}f + \frac{c}{r^2}f.
\]

The classical Hardy inequality

\[
-\Delta \geq \frac{(n-2)^2}{4}|x|^{-2},
\]

shows that for all \( c > -(n-2)^2/4 \), the self-adjoint operator \( L \) is non-negative. Such operators can be seen as radial Schrödinger operators with inverse-square potentials. It follows by Theorem 3.3 of [12] that \( L \) satisfies the property (FS).

Now for \( -(n-2)^2/4 < c < 0 \), we set \( p^*_c = n/\sigma \) where \( \sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c} \). Note that \( 2 < \frac{2n}{n-2} < p_c^* \). Liskevich, Sobol and Vogt [30] proved that, for \( t > 0 \) and \( p \in ((p^*_c)'', p_c^*) \),

\[
\|e^{-tL}\|_{p \to p} \leq C.
\]

They also proved that the range of \( p \), \( ((p^*_c)'', p_c^*) \) is optimal in the sense that, if \( p \notin ((p^*_c)'', p_c^*) \), the semigroup does not act on \( L^p((0, \infty), r^{n-1}dr) \) (see also [12]).

**Corollary 6.4.** Suppose that \( n > 2 \) and \( -(n-2)^2/4 < c \). Set

\[
p_c^* = \begin{cases} \frac{2n}{n-2}, & c < 0; \\ \sigma, & c \geq 0, \end{cases}
\]

where \( \sigma = (n-2)/2 - \sqrt{(n-2)^2/4 + c} \). Then the operator \( S^\alpha_p(L) \) is bounded on \( L^p((0, \infty), r^{n-1}dr) \) whenever

\[
\frac{2n}{n-1} < p < p_c^* \quad \text{and} \quad \alpha > \max \left\{ n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right\}.
\]

**Proof.** It was shown in [9, Proposition III.10] that the condition \( ST_p^\alpha \) for the operators \( \Delta_n + c/r^2 \) holds for \( p \in ((p^*_c)'', \frac{2n}{n+1}) \) for \( c < 0 \); for \( p \in [1, \frac{2n}{n+1}) \) for \( c \geq 0 \). Now the corollary follows from Theorem 4.1. \( \square \)
Remark 6.5. In the proof of Corollary 6.4 one has to use condition \((ST^{2})\) because the condition \((R_{p})\) is no longer valid in this setting.

Finally we mention that our approach can be also applied to a class of sub-Laplacians on Heisenberg \(H\)-type group considered in [31], for the class of inverse square potentials considered in [5] and a class of Schrödinger type operators investigated in [35].

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