Abstract

The generalization of standard and generalized distributions has become one of the concerns that the statistical theory depends on to obtain more flexible distributions. In this article, a new distribution that is considered a generalization of the generalized inverted exponential distribution called the Type II Topp Leone Generalized Inverted Exponential (TIITLGIE) distribution is introduced. Some statistical properties of this distribution were obtained. The quantile function, median, moments, moment generating function, Reliability function, hazard function, mode, harmonic mean, mean and median deviation are derived. Furthermore, important measures such Rnyi entropy and the Maximum Likelihood (ML) estimation are deduced for parameters. Conduct a Monte Carlo simulation to study behavior of parameter estimates. Finally, applications on three real data sets are discussed.

Keywords: Generalized inverted exponential distribution; Type II Topp Leone; Rnyi entropy; maximum likelihood estimation; Monte Carlo simulation.

1 Introduction

Some complex real phenomena need new lifetime distributions to be modeled with. For this reason, the researchers in the area of statistics and distribution theory have been attracted to generate different lifetime models. The great importance of these models can be seen and found in many fields of study such as finance, insurance, reliability engineering and survival analysis. In the last
few years, the authors have proposed new families of distribution by adding an additional parameter using generator or combining existing distributions. Some of these families are: Exponentiated-G family [1], Lomax-G family [2], the type I Topp-Leone-G family [3], Topp Leone Exponentiated-G Family [4], Type II power Topp-Leone generated family [5]. Recently [6], introduced the type II Topp-Leone-G family (for short TIITL-G) using the half logistic distribution as a generator instead of the gamma generator in the CDF of Ristic-Balakrishnan-G. This family was characterized by more flexibility, which made it of interest to many researchers. Many lifetime models were generated using this family. For example: type II topp-leone generalized inverse Rayleigh distribution [7], type II topp-leone power Lomax distribution [8] and type II topp-leone inverse exponential distribution [9]. The CDF of TIITL distribution is given by:

\[ F(x) = 1 - [1 - G^2(x)]^\alpha \] \hspace{1cm} (1.1)

The corresponding PDF of (1.1) is given by:

\[ f(x) = 2\alpha g(x)G(x)[1 - G^2(x)]^{\alpha - 1} \] \hspace{1cm} (1.2)

where \( \alpha > 0 \) is a shape parameter, \( G(x) \) and \( g(x) \) are the CDF and PDF of a baseline distribution respectively.

A two-parameter Generalized Inverted exponential (GIE) distribution was proposed by [10] as a generalization of the Inverted Exponential (IE) distribution which is better than the Inverted Exponential when goodness of fit was assessed using the Likelihood Ratio and Kolmogorov-Smirnov tests. Recently, there are many authors who studied the GIE distribution. For example: [11].

The probability density function (PDF) of a tow parameter GIE distribution is given by:

\[ g(x) = \left( \frac{\theta \lambda}{x} \right) \exp\left( \frac{\lambda}{x} \right) [1 - \exp\left( \frac{-\lambda}{x} \right)]^{\theta - 1}, x > 0, \lambda, \theta > 0, \] \hspace{1cm} (1.3)

and the cumulative distribution function (CDF) is given by:

\[ G(x) = 1 - [1 - \exp\left( \frac{-\lambda}{x} \right)]^\theta, x > 0, \lambda, \theta > 0, \] \hspace{1cm} (1.4)

where \( \theta \) is the shape parameter and \( \lambda \) is the scale parameter.

This article aims to combine the works of [10] and [6] in order to define and provide the basic statistical properties of our new model called Type II Topp-Leone Generalized Inverted Exponential Distribution (as short TIITLGIE). This new model shows that it is more flexible in real applications using three different real data sets.

In Section 2, we introduce the TIITLGIE distribution. Statistical properties of the model are derived in Section 3. Rnyi entropy derived in Section 4. In Section 5, Maximum Likelihood estimators of parameters are derived. We will provide simulation study in Section 6. Finally, three real data sets will be applied in Section 7. Various conclusions are addressed in Section 8.

## 2 The Type II Topp-Leone Generalized Inverted Exponential Distribution

In this section, we derived three parameter TIITLGIE Distribution. The CDF and PDF of TIITLGIE distribution with three parameters \( (\alpha, \lambda, \theta) \) is obtained by inserting (1.3) and (1.4) in (1.1) and (1.2):

\[ F(x) = 1 - [1 - [1 - \exp\left( \frac{-\lambda}{x} \right)]^\theta]^\alpha, x > 0, \lambda, \alpha > 0, \] \hspace{1cm} (2.1)
and
\[
\begin{align*}
f(x) &= \frac{2\alpha \theta \lambda}{x^2} \exp\left(-\frac{\lambda}{x}\right) \left[1 - \exp\left(-\frac{\lambda}{x}\right)^\theta - 1\right] \left[1 - \exp\left(-\frac{\lambda}{x}\right)^{\theta - 1}\right] \\
&= \lambda \alpha \theta \lambda x^2 \exp\left(-\lambda x\right) \left[1 - \left[1 - \exp\left(-\frac{\lambda}{x}\right)^2\right]^{\theta - 1}\right], \quad x > 0, \lambda, \theta, \alpha > 0,
\end{align*}
\]

(2.2)

where, \(\lambda\) is scale parameter and \(\theta, \alpha\) are shape parameters.

We can rewrite the CDF & PDF of THTLGE distribution using following infinite power series as follows:

\[
F(x) = 1 - \sum_{s=0}^{\infty} \nu_s \exp\left(-\frac{\lambda}{x}s\right),
\]

(2.3)

where

\[
\nu_s = \sum_{j=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^{j+k+s}}{(s+1)(2j+1)(\theta k + 1)\beta(j+1, \alpha - j + 1)\beta(k+1, 2j-k+1)\beta(s + 1, \theta k - s + 1)}. \quad \text{(2.4)}
\]

and

\[
f(x) = \frac{\lambda}{x^2} \sum_{s=0}^{\infty} (s+1) \psi_s \exp\left(-\frac{\lambda}{x}(s+1)\right),
\]

(2.5)

where

\[
\psi_s = \sum_{j=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^{j+k+s}}{(j+1)(k+1)(s+1)\beta(j+1, \alpha - j)\beta(k+1, 2(j+1) - k)\beta(s + 1, (k+1)\theta - s + 1)}. \quad \text{(2.6)}
\]

The PDF of THTLGED can be rewrite as

\[
f(x) = \sum_{s=0}^{\infty} (s+1) \psi_s g_\lambda s(x),
\]

(2.7)

where \(g_\lambda s(x)\) is the PDF of IE distribution with scale parameter \(\lambda(s+1)\).

**Some Ideal Sub Models as Special Cases from Our Proposed Distribution:**

- For \(\theta = 1\), our distribution in Equation (2.1) converts to Type II Topp Leone Inverse Exponential (TITLIE) distribution.
- For \(\theta = 1\) and \(\lambda = 1\), our distribution reduces to Type II Topp Leone Standard Inverse Exponential (TITLSIE) distribution.
- For \(\lambda = 1\), our distribution reduces to Type II Topp Leone Generalized Standard Inverse Exponential (TITLGSIE) distribution.

Fig. 1 shows that shape of the PDF is positively skewed and unimodal for different values of the parameters.
3 Properties of TIITLGIE Distribution

3.1 Quantile and Median

The Quantile function of random variable X of TIITLGIE distribution is given by:

\[ x = Q(u) = -\lambda \log \left[ 1 - \left( 1 - \left( 1 - u \right)^\frac{1}{2} \right)^\frac{1}{2} \right] \tag{3.1} \]

where \( u \sim \text{uniform} \ (0,1) \), we can derive the median of TIITLGIE distribution by setting \( u = 0.5 \) in Equation (3.1). The median (M) is given by:

\[ M = -\lambda \log \left[ 1 - \left( 1 - \left( 0.5 \right)^\frac{1}{2} \right)^\frac{1}{2} \right] \tag{3.2} \]

3.2 Moments and Moment Generating Function

The \( r^{th} \) moment of TIITLGIE distribution random variable X is given by:

\[ \mu_r' = \lambda \sum_{s=0}^{\infty} (s + 1) \psi_s \int_{0}^{\infty} x^{r-2} \exp \left( -\frac{\lambda}{x} (s + 1) \right) dx, \tag{3.3} \]

where \( \psi_s \) is defined in Equation (2.6).
By setting \( u = \frac{1}{x} (s + 1) \),
we obtain the \( r \)-th moment of TIITLGIE distribution:

\[
\mu_r = \lambda^r \sum_{s=0}^\infty (s+1)^r \psi_s [E_r(1) + \sum_{n=0}^\infty \frac{(-1)^n}{(n-r+1)n!}] ,
\]

where \( E_r(1) \) is the integration exponential function, and \( \psi_s \) was known in Equation (2.6)
Substituting \( r = 1 \) in Equation (3.4), we obtain the mean of TIITLGIE Distribution as follows:

\[
\mu = \lambda \sum_{s=0}^\infty (s+1) \psi_s [E_1(1) + \sum_{n=0}^\infty \frac{(-1)^n}{n!}] .
\]

The moment generating function (MGF) of TIITLGIE distribution is given by:

\[
M_X(t) = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r)
= \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{t^r}{r!} \lambda^r (s+1)^r \psi_s [E_r(1) + \sum_{n=0}^\infty \frac{(-1)^n}{(n-r+1)n!}] .
\]

### 3.3 Skewness and Kurtosis

By using quantiles, the skewness and kurtosis of TIITLGIE distribution can be defined.

Bowley’s skewness is based on quantiles [12]. It was calculated as follows:

\[
B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}.
\]

Moors’ kurtosis [13] is based on octiles, and could be written as:

\[
M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(3/8) - Q(2/8)} ,
\]

where \( Q(\cdot) \) is the quantile function defined in Equation (3.1).

### 3.4 Mode

The mode of TIITLGIE distribution can be found by solving the following equation:

\[
\frac{df(x)}{dx} = 0.
\]

By using Equation (2.2), we get:

\[
f(x) \left( \frac{-2}{x} + \frac{\lambda}{x^2} - \left[ \frac{\lambda}{x^2} (\theta - 1) (1 - exp(\frac{-\lambda}{x}))^{-1} \right] \right. \] 
\times \left. [1 - (1 - exp(\frac{-\lambda}{x}))^{\theta-1} - (\alpha - 1) (1 - (1 - exp(\frac{-\lambda}{x}))^{\theta}) - 2 \frac{x}{\lambda} x^2 exp(\frac{-\lambda}{x}) (1 - exp(\frac{-\lambda}{x}))^{\theta}] \right) = 0 .
\]

The mode is obtained from the following equation:

\[
\frac{-2}{x} + \frac{\lambda}{x^2} - \left[ \frac{\lambda}{x^2} (\theta - 1) (1 - exp(\frac{-\lambda}{x}))^{-1} \right. \] 
\times \left. [1 - (1 - exp(\frac{-\lambda}{x}))^{\theta-1} - (\alpha - 1) (1 - (1 - exp(\frac{-\lambda}{x}))^{\theta}) - 2 \frac{x}{\lambda} x^2 exp(\frac{-\lambda}{x}) (1 - exp(\frac{-\lambda}{x}))^{\theta}] \right) = 0 .
\]

Equation (3.10) is a nonlinear equation and it can not be found analytically. Further, the mode of TIITLGIE distribution can be found numerically by solving Equation (3.10) using Newton-Raphson method.
3.5 Reliability Function and Hazard function

The reliability function and hazard function are very important properties of lifetime distribution. The reliability function is the probability of the non-failure occurring before time $t$. While the hazard function is the instantaneous rate of failure at a given time $t$. The reliability function of TIITLGIE distribution is denoted by $R(x)$, also known as survival function obtained as follows:

$$R(x) = 1 - F(x), \quad (3.11)$$

The survival function of TIITLGIE distribution is obtained by substituting (2.1) in (3.11) to deduce:

$$R(x) = [1 - [1 - (1 - \exp(-\lambda x))^{\theta}]^{2}]^{\alpha}, \quad (3.12)$$

and the corresponding hazard function of TIITLGIE distribution is defined as follows:

$$h(x) = \frac{f(x)}{1 - F(x)}, \quad (3.13)$$

then the hazard function can be written as:

$$h(x) = \frac{2\alpha \lambda \theta}{x^2} \exp\left(-\frac{\lambda}{x}\right) \left[1 - \exp\left(-\frac{\lambda}{x}\right)^{\theta - 1}\left[1 - \exp\left(-\frac{\lambda}{x}\right)^{\theta}\right]\right] \times \left[1 - \left[1 - (1 - \exp(-\lambda x))^{\theta} \right]^{2} - 1 \right]. \quad (3.14)$$

Fig. 2 shows that the reliability curves are decreasing for different values of parameters for the TIITLGIE distribution, while Fig. 3 shows that the hazard function of the TIITLGIE distribution is increasing at first for different values of parameters, then decreasing in shape. These kind of models are useful in survival analysis. The TIITLGIE distribution gives good statistical behavior based on these two functions.

![Fig. 2. Plots of Reliability function of TIITLGIE distribution for different values of the parameters when (a,d) $\alpha$ increases, (b) $\lambda$ increases, (c) $\theta$ increases](image)

3.6 Harmonic Mean

The Harmonic Mean defined as reciprocal of the arithmetic mean of the reciprocal of the values $x_1, x_2, \ldots, x_N$ and could be written as:

$$H_m(x) = \frac{1}{E(\frac{1}{x})} = \left[\int_{0}^{\infty} x^{-1} f(x) dx\right]^{-1}. \quad (3.15)$$

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Using Equation (2.5), the harmonic mean of TIITLGIE distribution can be derived as follows:

Let:

\[ I = \int_0^\infty x^{-1} f(x) dx \]

\[ = \lambda \sum_{s=0}^{\infty} (s+1) \psi_s \int_0^\infty x^{-3} \exp\left(\frac{-\lambda x}{x(s+1)}\right) dx \]

By setting \( u = \frac{x}{x(s+1)} \),
we obtain the harmonic mean of TIITLGIE distribution :

\[ H_m(x) = \left[ \sum_{s=0}^{\infty} \frac{1}{\lambda(s+1) \psi_s} \right]^{-1}. \tag{3.16} \]

In Table 1 we observe that the behavior of the TIITLGIE distribution affects by changing the values of its parameters. We notice that if \( \alpha \) and \( \theta \) are increasing the mode, median, mean and harmonic mean are decreasing, else the skewness and kurtosis are decreasing. By increasing the scale parameter \( \lambda \) the mode, median and mean are increasing but the harmonic mean is decreasing and the skewness and kurtosis remain constant.

### 3.7 Probability Weighted Moments

The PWMs can be calculated from the following:

\[ \tau_{r,\beta} = E[x^r F(x)^\beta] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^\beta dx. \tag{3.17} \]
Table 1. The mode, median, mean, Harmonic Mean, skewness and kurtosis of TIITLGIE distribution

| α  | θ  | λ  | mode | median | mean  | Harmonic Mean | skewness | kurtosis |
|----|----|----|------|-------|-------|---------------|----------|----------|
| 1.5| 1.5| 2  | 1.56289 | 2.60944| 3.97169| 0.875         | 0.302401| 0.91689  |
| 2  | 1.5| 2  | 1.47466 | 2.21353| 2.94902| 0.622135      | 0.259407| 0.761012 |
| 2.5| 2  | 2  | 1.38028 | 2.03437| 2.68548| 0.437337      | 0.230537| 0.663612 |
| 1.5| 2  | 2  | 1.38028 | 2.03437| 2.68548| 0.6875        | 0.256803| 0.753663 |
| 2  | 2  | 2  | 1.30144 | 1.76805| 2.15337| 0.6255   | 0.220002| 0.630509 |
| 2.5| 2  | 2  | 1.24234 | 1.60362| 1.87148| 0.600074     | 0.195195| 0.551819 |
| 1.5| 2.5| 2  | 1.25663 | 1.72018| 2.11884| 0.64596      | 0.19408 | 0.548956 |
| 2  | 2.5| 2  | 1.18516 | 1.51786| 1.7653  | 0.664596     | 0.19408 | 0.548956 |
| 2.5| 2.5| 2  | 1.1321  | 1.39066| 1.56731| 0.720075     | 0.171876| 0.481    |
| 1  | 1.5| 1.5| 1.12039 | 1.92522| 3.13949| 0.611111     | 0.320047| 0.988074 |
| 1.5| 1.5| 1.5| 1.03521 | 1.52578| 2.01411| 0.916667     | 0.256803| 0.753663 |
| 2  | 2  | 1.5| 0.931755| 1.20271| 1.40361| 0.800999     | 0.195195| 0.551819 |
| 1  | 2  | 1.5| 1.49385 | 2.56696| 4.18599| 0.458333     | 0.320047| 0.988074 |
| 1.5| 2  | 1.5| 1.38028 | 2.03437| 2.68548| 0.6875        | 0.256803| 0.753663 |
| 2  | 2  | 2  | 1.24234 | 1.60362| 1.87148| 0.600074     | 0.195195| 0.551819 |
| 1  | 2  | 2.5| 1.86732 | 3.2087 | 5.23248| 0.366667     | 0.320047| 0.988074 |
| 1.5| 2  | 2.5| 1.72535 | 2.54296| 3.35684| 0.55         | 0.256803| 0.753663 |
| 2  | 2  | 2.5| 1.55292 | 2.00452| 2.3935  | 0.48006      | 0.195195| 0.551819 |

By using Equations (2.1) and (2.2) we get:

\[
\tau_{r,\beta} = 2\alpha\beta x^{\beta - 2} \exp\left(\frac{-\lambda}{x}\right) \left(1 - \exp\left(\frac{-\lambda}{x}\right)\right)^{\beta - 1} \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^\beta \int_0^\infty \left[1 - \exp\left(\frac{-\lambda}{x}\right)\right]^\beta dx.
\]

Hence, we get the PWMs of TIITLGIE distribution.

\[
\psi_{ikts} = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \left(\frac{\beta}{i}\right) \left(\frac{\alpha(1 + i) - 1}{k}\right) \left(\frac{2k + 1}{t}\right) \left(\frac{\theta(t + 1) - 1}{s}\right) (-1)^{i+k+t+s}.
\]

3.8 The Mean Deviation and the Median Deviation

The mean deviation and the median deviation are measures of dispersion. They are derived as follows

3.8.1 The mean deviation about the mean

The mean deviation about the mean can be defined as

\[
D(\mu) = E|x - \mu| = \int_0^\infty |x - \mu| f(x) dx
\]

By using the CDF from Equation (2.3), the mean deviation of TIITLGIE distribution can be derived as:

\[
D(\mu) = 2 \int_0^\mu \left[1 - \sum_{s=0}^{\infty} \nu_s \exp\left(\frac{-\lambda}{s}\right)\right] dx,
\]
where $\nu_s$ is defined in Equation (2.4).

\[
2\left[\int_0^\mu dx - \sum_{s=0}^\infty \nu_s \int_0^\mu \exp\left(-\frac{\lambda}{x} s\right)dx\right].
\]

Then, the mean deviation about the mean is given by:

\[
= 2\mu - 2 \sum_{s=0}^\infty \nu_s \times [\mu \exp\left(-\frac{\lambda}{\mu} s\right) - \lambda \mu \Gamma(0, -\frac{\lambda}{\mu} s)],
\]

where $\Gamma(0, -\frac{\lambda}{\mu} s)$ is the incomplete gamma function.

### 3.8.2 The mean deviation about the median

The mean deviation about the median can be defined as

\[
D(m) = E|x - m| = \int_0^\infty |x - m|f(x)dx.
\]

By using the CDF from Equation (2.3), we obtain

\[
D(m) = \mu - m + 2 \int_0^m [1 - \sum_{s=0}^\infty \nu_s \exp\left(-\frac{\lambda}{x} s\right)dx].
\]

The mean deviation about the median of TIITLGIE distribution can be obtained as the following:

\[
D(m) = \mu + m - 2 \sum_{s=0}^\infty \nu_s [m \exp\left(-\frac{\lambda}{m} s\right) - \lambda s \Gamma(0, -\frac{\lambda}{m} s)],
\]

where $\Gamma(0, -\frac{\lambda}{m} s)$ was known in Equation (3.19).

### 3.9 Order Statistics

If $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from the TIITLGIE distribution with CDF $F(j)$ and PDF $f(j)$, then the PDF of $X_{(j)}$ is given by:

\[
f(x_{(j)}) = \frac{n!}{(j-1)!(n-j)!} f(x) F(x)^{j-1} [1 - F(x)]^{n-j}.
\]

The PDF of the jth order statistic of TIITLGIE distribution is given by:

\[
f(x_{(j)}) = \frac{n!}{(j-1)!(n-j)!} 2\pi \alpha \theta \exp\left(-\frac{\lambda}{x_{(j)}}\right) [1 - \exp\left(-\frac{\lambda}{x_{(j)}}\right)]^{\theta-1} [1 - \exp\left(-\frac{\lambda}{x_{(j)}}\right)]^\theta \times [1 - [1 - (1 - \exp\left(-\frac{\lambda}{x_{(j)}}\right))^{\theta} 2\alpha j^{-1}] [1 - (1 - \exp\left(-\frac{\lambda}{x_{(j)}}\right))^{\theta} 2\alpha (1+n-j)^{-1}],
\]

$x_{(j)} > 0$.

Therefore, the PDF of the largest order statistic $X_{(n)}$ is given by:

\[
f(x_{(n)}) = \frac{2\pi \alpha \theta}{x_{(n)}^{n-1}} \exp\left(-\frac{\lambda}{x_{(n)}}\right) [1 - \exp\left(-\frac{\lambda}{x_{(n)}}\right)]^{\theta-1} [1 - \exp\left(-\frac{\lambda}{x_{(n)}}\right)]^\theta \times [1 - (1 - \exp\left(-\frac{\lambda}{x_{(n)}}\right))^{\theta} 2\alpha n^{-1}] [1 - (1 - \exp\left(-\frac{\lambda}{x_{(n)}}\right))^{\theta} 2\alpha n^{n-1}],
\]

$x_{(n)} > 0$.
And the PDF of the smallest order statistic $X_{(1)}$ is:

$$f(x_{(1)}) = \frac{2n\alpha\lambda^2}{x_{(1)}^2} \exp\left(-\frac{\lambda}{x_{(1)}}\right)[1 - \exp\left(-\frac{\lambda}{x_{(1)}}\right)]^{\theta-1}[1 - \exp\left(-\frac{\lambda}{x_{(1)}}\right)]^{\theta}$$

$$\times [1 - (1 - \exp\left(-\frac{\lambda}{x_{(1)}}\right)]^{\theta}\right)^{2n-1},$$

$x_{(1)} > 0$.

### 4 Rnyi Entropy of TIITLGIE

The Rnyi entropy was introduced by [14]. It is one of the several generalizations of Shannons entropy [15], they are measures of variation of uncertainty. Entropy theory is used in variety applications such as in information theory, engineering and physics. For the density function $f(x)$ the Rnyi entropy is defined by:

$$R_\delta(X) = \frac{1}{1-\delta} \log[J(\delta)], \quad (4.1)$$

where

$$J(\delta) = \int_0^\infty f^\delta(x)dx; \delta > 0 \text{ and } \delta \neq 1,$$

$$J(\delta) = (2\alpha\lambda)^\delta \int_0^\infty \exp\left(-\frac{\lambda}{x}\right)\left[1 - \exp\left(-\frac{\lambda}{x}\right)\right]^{(\theta-1)\delta}[1 - (1 - \exp\left(-\frac{\lambda}{x}\right))]^{\delta}\right|^2(\alpha-1)^\delta dx.$$

Let $u = \frac{\lambda}{x}$

$$J(\delta) = \lambda(2\alpha\theta u^2)\int_0^\infty \exp(-u\delta)[1 - \exp(-u)]^{(\theta-1)\delta}[1 - (1 - \exp(-u))]^{\delta}\right]^2(\alpha-1)^\delta \frac{1}{u^2} du.$$

Now by applying the binomial expansion, we get

$$J(\delta) = \lambda(2\alpha\theta u^2)\sum_{i=0}^\infty \left(\frac{(\alpha - 1)\delta}{i}\right) \int_0^\infty \exp(-u\delta)[1 - \exp(-u)]^{(\theta-1)\delta}[1 - (1 - \exp(-u))]^{\delta}\right]^2(\alpha-1)^\delta \frac{1}{u^2} du.$$

We get the Rnyi entropy for TIITLGIE as

$$R_\delta(X) = \frac{1}{1-\delta} \log[\lambda(2\alpha\theta)^\delta \sum_{i=0}^\infty \sum_{k=0}^\infty \sum_{l=0}^\infty \left(\frac{(\alpha - 1)\delta}{i}\right) \left(\frac{2i + \delta + k}{k}\right) \left(\frac{\theta(\delta + k) - \delta}{t}\right)\right]^{-\frac{1}{(\delta + 1)[2\delta - 1]}} \Gamma(2\delta - 1)].$$

### 5 Maximum Likelihood Estimation Method

The maximum likelihood estimators of the unknown parameters for the TIITLGIE distribution are discussed. Let $x_1, x_2, ..., x_n$ be a realization of a random sample of size $n$ from TIITLGIE distribution then the likelihood function is written as follows:

$$L = \prod_{i=0}^n f(y_i),$$
and the log-likelihood function is given as follows

\[ \ell = \log(L) = n \log(2) + n \log(\alpha) + n \log(\theta) + n \log(\lambda) - 2 \sum_{i=1}^{n} \log(x_i) - 2 \left( \sum_{i=1}^{n} \frac{\lambda}{x_i} \right) + (\theta - 1) \sum_{i=1}^{n} \log(1 - \exp(-\lambda x_i)) + \sum_{i=1}^{n} \log(1 - (1 - \exp(-\lambda x_i))^\theta) + (\alpha - 1) \sum_{i=1}^{n} \log(1 - (1 - \exp(-\lambda x_i))^\theta) \]

Differentiating (\ell) with respect to each of the parameters \( \alpha, \theta \) and \( \lambda \) gives:

\[ \frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1 - (1 - \exp(-\lambda x_i))^\theta^2) = 0, \tag{5.1} \]

\[ \frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log(1 - \exp(-\lambda x_i)) \left( \frac{(1 - \exp(-\lambda x_i))^\theta \log(1 - \exp(-\lambda x_i))}{1 - (1 - \exp(-\lambda x_i))^\theta} \right) + 2(\alpha - 1) \sum_{i=1}^{n} \frac{(1 - (1 - \exp(-\lambda x_i))^\theta)(1 - \exp(-\lambda x_i))^\theta \log(1 - \exp(-\lambda x_i))}{1 - (1 - \exp(-\lambda x_i))^\theta^2} = 0, \tag{5.2} \]

\[ \frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{1}{x_i} + (\theta - 1) \sum_{i=1}^{n} \frac{\exp(-\lambda x_i)}{x_i(1 - \exp(-\lambda x_i)^\theta)} - \theta \sum_{i=1}^{n} \frac{\exp(-\lambda x_i)(1 - \exp(-\lambda x_i))^\theta - 1}{x_i(1 - (1 - \exp(-\lambda x_i))^\theta)} \right) + 2\theta(\alpha - 1) \sum_{i=1}^{n} \frac{\exp(-\lambda x_i)(1 - \exp(-\lambda x_i))^\theta - 1}{x_i(1 - (1 - \exp(-\lambda x_i))^\theta^2)} = 0. \tag{5.3} \]

The MLE of parameters \( \hat{\alpha}, \hat{\theta} \) and \( \hat{\lambda} \) can be found numerically by equating the derivatives Equations in (5.1), (5.2) and (5.3) to zero and solve them using Mathematica (V.10.2).

\[ \hat{\alpha} = \left[ -\frac{1}{n} \sum_{i=1}^{n} \log(1 - (1 - \exp(-\lambda x_i))^\theta^2) \right]^{-1}. \tag{5.4} \]

### 6 Simulation Study

In this section, we will conduct simulation to study behavior of unknown parameters \((\alpha, \theta, \lambda)\) for TIITLIE using Mathematica (V.10.2). We generate samples of size \( n = 10; 30; 50; 100; 200; 500 \) and 1000 from TIITLIE distribution for some selected combination of parameters. This process will be repeated \( N = 1000 \) times. In each process, estimates of the parameters that will be obtained by mean estimate, mean squared error and bias. Then, the estimates of \( R(x_0) \) and \( h(x_0) \) from Equations (3.12) and (3.14) at point \( x_0 = 0.5 \) were also evaluated using the estimated parameters. We can observe from Tables (2, 3) that, if the sample size increases, the bias (BIAS) and mean squared error (MSE) decreases in all cases.
Table 2. The MLE, BIAS and MSE of TIITLGIE distribution for true values ($\alpha = 1$, $\theta = 1$, $\lambda = 1$, $x_0 = 0.5$)

| $n$  | Parameters | MLE   | BIAS  | MSE   |
|------|------------|-------|-------|-------|
| 10   | $\alpha$  | 2.46842 | 1.46842 | 8.38966 |
|      | $\theta$  | 2.06023 | 1.06023 | 5.9919  |
|      | $\lambda$ | 1.47638 | 0.476379 | 1.00841 |
|      | $R(x)$    | 0.975781 | -0.00190381 | 0.000733155 |
|      | $h(x)$    | 0.129346 | -0.0199127 | 0.0197592 |
| 30   | $\alpha$  | 2.00886 | 1.00886 | 5.1536  |
|      | $\theta$  | 1.73775 | 0.737747 | 3.09241 |
|      | $\lambda$ | 1.15384 | 0.153842 | 0.13499 |
|      | $R(x)$    | 0.98223 | 0.000545434 | 0.000188478 |
|      | $h(x)$    | 0.13552 | -0.0017385 | 0.00650461 |
| 50   | $\alpha$  | 1.80555 | 0.80555 | 4.0329  |
|      | $\theta$  | 1.74504 | 0.745041 | 2.84547 |
|      | $\lambda$ | 1.11085 | 0.110846 | 0.114978 |
|      | $R(x)$    | 0.981389 | 0.000154577 | 0.000119655 |
|      | $h(x)$    | 0.142212 | -0.000704724 | 0.000431343 |
| 100  | $\alpha$  | 1.74406 | 0.74406 | 3.3529  |
|      | $\theta$  | 1.4306 | 0.430605 | 1.01672 |
|      | $\lambda$ | 1.05897 | 0.0589683 | 0.0612171 |
|      | $R(x)$    | 0.982026 | 0.00034169 | 0.0000707223 |
|      | $h(x)$    | 0.14276 | -0.00034984 | 0.00230038 |

Table 3. The MLE, BIAS and MSE of TIITLGIE distribution for true values ($\alpha = 1$, $\theta = 1$, $\lambda = 1$, $x_0 = 0.5$)

| $n$  | Parameters | MLE   | BIAS  | MSE   |
|------|------------|-------|-------|-------|
| 200  | $\alpha$  | 1.45024 | 0.450238 | 2.04912 |
|      | $\theta$  | 1.41734 | 0.417339 | 1.32852 |
|      | $\lambda$ | 1.04602 | 0.0460214 | 0.042614 |
|      | $R(x)$    | 0.981799 | 0.000114878 | 0.0000405748 |
|      | $h(x)$    | 0.146533 | -0.00272554 | 0.000120306 |
| 500  | $\alpha$  | 1.20161 | 0.201606 | 0.68862 |
|      | $\theta$  | 1.21196 | 0.211957 | 0.599059 |
|      | $\lambda$ | 1.02016 | 0.0201573 | 0.0220294 |
|      | $R(x)$    | 0.981555 | 0.000129064 | 0.0000299308 |
|      | $h(x)$    | 0.14883 | -0.000429294 | 0.000532922 |
| 1000 | $\alpha$  | 1.10397 | 0.103975 | 0.305711 |
|      | $\theta$  | 1.09735 | 0.0973547 | 0.22821 |
|      | $\lambda$ | 1.08855 | 0.0885523 | 0.0109286 |
|      | $R(x)$    | 0.981538 | 0.000146694 | 0.0000109141 |
|      | $h(x)$    | 0.14946 | -0.000200975 | 0.000289528 |

7 Applications

In this Section, three sets of data are presented to demonstrate the utility of using the TIITLGIE distribution. We compared the TIITLGIE distribution with, TIITLSIE distribution, TIITLGSIE distribution and Topp Leone Generalized Inverted Exponential (TLGIE) distribution [16]. The parameters are estimated using maximum likelihood method, and computed using Mathematica (V.10.2).
The following statistical measures were calculated: log-likelihood (LL), Akaike information criterion (AIC), Consistent Akaike information criteria (CAIC) [17] and Hannan-Quinn information criterion (HQIC) [18].

7.1 Data set 1

The first data set, is the numbers (in million Riyals) of credit facilities provided to micro enterprises in Saudi Arabia From Q1 2018 to Q2 2021. This data is downloaded from (https://data.gov.sa/Data/en/dataset/credit-facilities-provided-to-smes).

7.2 Data set 2

The second data set represents the number of daily COVID-19 cases in Jeddah, Saudi Arabia from 2nd May to 6th July. These data were taken from the website of the Saudi Ministry of Health with URL: https://covid19.moh.gov.sa/.

7.3 Data set 3

The data set shows the seasonal (July 1 - June 30) rainfall in inches recorded at Los Angeles Civic Center from 1962 to 2012. These data were taken from the website of Los Angeles Almanac with URL: http://www.laalmanac.com/weather/we13.php reported by United States National Weather Service.

In Tables (4-6) shows that the TITITLGE distribution has smaller values for measure, (LL, AIC, CAIC and HQIC) compared with the values of others models for the three data sets.

Figs. 4-6 shows the empirical distribution and estimated CDF of the models for three data sets.

| Table 4. Statistical measures for data set 1 |
|-----------------|-----------------|-----------------|-----------------|
| Model           | Parameters      | LL   | AIC   | CAIC  | HQIC  |
| TITITLGIE       | $\hat{\alpha} = 4.19523$ | $\theta = 12.9575$ | $\lambda = 4.78025$ | -2.11004 | 10.2201 | 13.6486 | 9.46762 |
| TITITLSIE       | $\hat{\alpha} = 3.37693$ | -10.3583 | 22.7105 | 23.161 | 22.4657 |
| TITITLGSIE      | $\hat{\alpha} = 15.6049$ | $\theta = 0.392864$ | $\lambda = 9.31184$ | -9.31184 | 22.6237 | 24.1237 | 22.122 |
| TLGIE           | $\hat{\alpha} = 1.42755$ | $\theta = 0.5$ | $\lambda = 0.992072$ | -15.2647 | 36.5295 | 39.958 | 35.777 |

| Table 5. Statistical measures for data set 2 |
|-----------------|-----------------|-----------------|-----------------|
| Model           | Parameters      | LL   | AIC   | CAIC  | HQIC  |
| TITITLGIE       | $\hat{\alpha} = 15.6834$ | $\theta = 0.962963$ | $\lambda = 483.436$ | -406.108 | 818.216 | 818.603 | 820.812 |
| TITITLSIE       | $\hat{\alpha} = 0.197486$ | -553.007 | 1108.01 | 1108.08 | 1108.88 |
| TITITLGSIE      | $\hat{\alpha} = 8.5854$ | $\theta = 0.067094$ | $\lambda = 525.562$ | -525.562 | 1055.12 | 1055.32 | 1056.86 |
| TLGIE           | $\hat{\alpha} = 1.39053$ | $\theta = 0.5$ | $\lambda = 211.596$ | -450.282 | 906.564 | 906.951 | 909.16 |

| Table 6. Statistical measures for data set 3 |
|-----------------|-----------------|-----------------|-----------------|
| Model           | Parameters      | LL   | AIC   | CAIC  | HQIC  |
| TITITLGIE       | $\hat{\alpha} = 13.8009$ | $\theta = 0.510292$ | $\lambda = 12.4307$ | -168.324 | 342.648 | 343.17 | 344.832 |
| TITITLSIE       | $\hat{\alpha} = 0.504938$ | -218.012 | 438.023 | 438.107 | 438.752 |
| TITITLGSIE      | $\hat{\alpha} = 6.98918$ | $\theta = 0.164037$ | $\lambda = 205.072$ | -205.072 | 414.144 | 414.399 | 415.6 |
| TLGIE           | $\hat{\alpha} = 1.33702$ | $\theta = 0.5$ | $\lambda = 8.6618$ | -186.422 | 378.844 | 379.366 | 381.028 |
8 Conclusions

In this study, we derived a three parameter Type II Topp Leone Generalized Inverted Exponential Distribution. We deduced from the behavior of its hazard and reliability function that this kind of model is useful in survival analysis. It shows good statistical behavior based on these two functions. Else, we concluded from the result of the measures of (LL, AIC, CAIC, HQIC) that our model is more flexible than other related distributions using three real data sets.

Competing Interests

Authors have declared that no competing interests exist.

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