GAMMA FACTORS FOR ASAI REPRESENTATIONS OF GL₂

SHIH-YU CHEN, YAO CHENG, AND ISAO ISHIKAWA

Abstract. Let $E$ be a quadratic semisimple extension of a local field $F$ of characteristic zero. We determine explicit relation between gamma factors for Asai representations of $R_{E/F} GL_{2/E}$ defined by the Weil-Deligne representations and local zeta integrals. When $E = F 	imes F$, the results were due to Henniart and Jacquet. We completed the theory in this article based on explicit calculation.

1. Introduction

1.1. Main results. Let $F$ be a local field of characteristic zero and $E$ be a quadratic semisimple $F$-algebra. Fix a non-trivial additive character of $F$, and let $\omega$ be the homomorphism $Gal(F/F)$ associated to the parameters associated to $\pi$ via the local Langlands correspondence. Denote $\psi$ be the Asai representation defined by the parameters associated to $\pi$.

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functoriality of $\pi$
When $n = 2$, we consider the local factors $L_{PSR}(s, A\pi \otimes \tau)$ and $\varepsilon_{PSR}(s, A\pi \otimes \tau, \psi, \xi)$ defined by the local zeta integrals of Piatetski-Shapiro and Rallis when both $\pi$ and $\tau$ are generic (Cf. [PSR87] and [Ike89]). The main results of this paper are the following

**Theorem A** (Corollary 2.2 and Theorem 3.1). Let $n = 1$. Assume $\pi$ is generic. We have

$$L_{RS}(s, A\pi \otimes \tau) = L_{Gal}(s, A\pi \otimes \tau),$$
$$\varepsilon_{RS}(s, A\pi \otimes \tau, \psi, \xi) = \omega_\pi(\xi)\omega_\tau(\xi^2)|\xi^2|_{F}^{-1/2}\lambda_{E/F}(\psi)^{-1}\varepsilon_{Gal}(s, A\pi \otimes \tau, \psi).$$

Here $\lambda_{E/F}(\psi)$ is the Langlands constant for $E/F$ with respect to $\psi$.

**Theorem B** (Theorem 4.1). Let $n = 2$. Assume both $\pi$ and $\tau$ are generic, and $\tau$ is a subquotient of a principal series representation. We have

$$\gamma_{PSR}(s, A\pi \otimes \tau, \psi, \xi) = \omega(4\xi^2)^{-1}|4\xi^2|_{F}^{-2s+1}\omega_{E/F}(-1)\gamma_{Gal}(s, A\pi \otimes \tau, \psi).$$

Here $\omega = \omega_\pi|_{F^\times} \cdot \omega_\tau$ and $\omega_{E/F}$ is the quadratic character of $F^\times$ associated with $E/F$ by local class field theory.

**Remark 1.**

1. Let $n = 1$. When $E = F \times F$, Theorem A was proved in [Jac72] and [Hen00]. When $E$ is a field and $F$ is non-archimedean, the equality for $L$-factors follow from the results in [AR05] Theorem 1.6], [Hen10] Section 1.5, Théorème], and [Mat10] Theorem 4.2].
2. Let $n = 2$. When $E = F \times F$, Theorem B was proved in [Ike89] Theorem 3].

**Remark 2.** Our theorems has an important application for construction of twisted triple product $p$-adic $L$-functions ([LS17]). We obtain an explicit interpolation formula for the twisted triple product $p$-adic $L$-function along Hida families through explicit computation of the local period integrals in Ichino’s formula ([Ich08]). Theorem A and Theorem B are essential to confirm the non-triviality and the good $p$-adical behavior of the local period integrals.

Combining Theorem B with [Gan08] Theorem 1.2], we can prove the dichotomy on trilinear forms.

**Theorem C** (Corollary 4.3). Assume $\omega_\pi|_{F^\times} \cdot \omega_\tau = 1$. Then $\text{Hom}_{GL_2(F)}(\pi \otimes \tau, C) \neq 0$ if and only if

$$\omega_{E/F}(-1)\varepsilon_{Gal}\left(\frac{1}{2}, A\pi \otimes \tau\right) = 1.$$ 

Here $\omega_{E/F}$ is the quadratic character of $F^\times$ associated with $E/F$ by local class field theory.

**Remark 3.** When $F$ is non-archimedean, except when both $\pi$ and $\tau$ are supercuspidal, the result is proved by Prasad in [Pra92]. On the other hand, when $F$ is archimedean, it follows from comparing the value of epsilon factors with a result of Loke in [Lok01].

Now we switch to global situation. Let $F$ be a number field and $E$ be a quadratic semisimple $F$-algebra. Fix a non-trivial additive character $\psi$ of $A_E/F$ and an element $\xi \in E^\times$ such that $tr_{E/F}(\xi) = 0$. Let $\pi$ and $\tau$ be irreducible cuspidal automorphic representations of $GL_2(A_E)$ and $GL_2(A_E)$ with central characters $\omega_\pi$ and $\omega_\tau$, respectively. Put $\omega = \omega_\pi|_{A_E^\times} \cdot \omega_\tau$.

**Theorem D** (Theorem 4.4). For each place $v$ of $F$, we have

$$L_{PRS}(s, A\pi_v \otimes \tau_v) = L_{Gal}(s, A\pi_v \otimes \tau_v),$$
$$\varepsilon_{PRS}(s, A\pi_v \otimes \tau_v, \psi_v, \xi) = \omega_v(4\xi^2)^{-1}|4\xi^2|_{F_v}^{-2s+1}\omega_{E_v/F_v}(-1)\varepsilon_{Gal}(s, A\pi_v \otimes \tau_v, \psi_v).$$

**Remark 4.** When $E = F \times F$, Theorem D was proved in [Rama00] Theorem 4.4.1 as an application of the functoriality of $\pi$ to $GL_4(A_E)$. 

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1.2. Structure of the article. The proof for Theorem A is separated into two cases according to $F$ is archimedean or not.

In §2 we consider the case when $F$ is non-archimedean. In §2.1 we recall the definition of Asai local factors defined by local zeta integrals and state the main results. To prove Theorem A, the problem boils down to two logically independent problems: (i) the case when is supercuspidal, (ii) multiplicative of gamma factors. By a standard global-to-local argument in Corollary 2.2, the case when $\pi$ is supercuspidal follow from the functoriality of global Asai transfer to $\text{GL}_4$ proved in [Kri03] and standard unramified calculation of local zeta integrals. The main innovation is the proof of multiplicative of gamma factors which is given in §2.2.

In §3 we consider the case when $F$ is archimedean. Indeed, we develop the theory concerning the analytic properties of the zeta integrals associated to the Asai representation for the case $E = \mathbb{C}$ and $F = \mathbb{R}$, and prove analogy results (Cf. Theorem 6.1) to that of [Jac72] Theorem 17.2 and [Jac09] Theorem 2.1. To the best of our knowledge, unlike the non-archimedean case, the analytic properties of the zeta integrals for this case can not be found in the literatures.

The proofs for Theorems B-D are given in §4. In §4.1 and §4.2 we recall some notation and definitions of the intertwining operators on $\text{GSp}_4$ and good sections for induced representations on $\text{GSp}_4$. In §4.3 we recall the definition of twisted Asai local factors defined by local zeta integrals and state the main results. The proof of Theorems C and D are based on global-to-local arguments, the functoriality of global Asai transfer to $\text{GL}_4$, and Theorem B. The proof of Theorem B is given by §4.4. We give a detailed proof following the idea of Ikeda in [Ike89] Theorem 3]. Finally, in the Appendix, we prove some results which are used in the proofs in §4.

1.3. Notation. Denote $B$ the standard Borel subgroup of $\text{GL}_2$ consisting of upper triangular matrices and $U$ be its unipotent radical.

Let $F$ be a local field of characteristic zero. When $F$ is non-archimedean, denote $\mathcal{O}_F$ the ring of integers of $F$, $\varpi_F$ a prime element, and $\text{ord}_F$ be the valuation on $F$ normalized so that $\text{ord}_F(\varpi_F) = 1$. Let $|\cdot|_F$ be the absolute value on $F$ normalized so that $|\varpi_F|_F^{-1}$ is equal to the cardinality of $\mathcal{O}_F/\varpi_F \mathcal{O}_F$. When $F$ is archimedean, let $|\cdot|_F = |\cdot|_\mathbb{R}$ be the usual absolute value on $\mathbb{R}$ and $|z|_\mathbb{C} = z\overline{z}$ on $\mathbb{C}$. For a finite dimensional vector space $V$ over $F$, denote $\mathcal{S}(V)$ the space of Bruhat-Schwartz functions on $V$.

An additive character $\psi$ of $F$ means a continuous homomorphism $\psi : F \to \mathbb{C}^\times$. For $a \in F^\times$, let $\psi^a$ be the additive character defined by $\psi^a(x) = \psi(ax)$. When $F$ is non-archimedean, we denote $c(\psi)$ the smallest integer such that $\psi$ is trivial on $\varpi_F^{(c(\psi))}\mathcal{O}_F$.

A character $\chi$ of $F^\times$ means a continuous homomorphism $\chi : F^\times \to \mathbb{C}^\times$. When $F$ is non-archimedean, we denote $c(\chi)$ the smallest non-negative integer such that $\chi$ is trivial on $1 + \varpi_F^{(c(\chi))}\mathcal{O}_F$. Denote

$$L(s, \chi), \quad \varepsilon(s, \chi, \psi), \quad \gamma(s, \chi, \psi)$$

the local factors of a character $\chi$ of $F^\times$ with respect to an additive character $\psi$ of $F$ defined in [Tat79]. When $\chi = 1$, we denote $\zeta_F(s) = L(1, s)$. Then

$$\zeta_F(s) = \begin{cases} 
(1 - |\varpi_F|_F^{-s})^{-1} & \text{if } F \text{ is non-archimedean}, \\
\pi^{-s/2}\Gamma(s/2) & \text{if } F = \mathbb{R}, \\
2(2\pi)^{-s}\Gamma(s) & \text{if } F = \mathbb{C}.
\end{cases}$$

Here $\Gamma(s)$ is the gamma function.

All representations of the $F$-points of a linear algebraic group over $F$ is assumed to be smooth. Let $\pi$ be a representation of $\text{GL}_2(F)$ with finite length. Let $\psi$ be an additive character of $F$. The space of Whittaker functionals of $\pi$ with respect to $\psi$ has dimension less than or equal to one. When non-zero Whittaker functional exist, we denote $\mathcal{W}(\pi, \psi)$ to be the corresponding space of Whittaker functions. Recall that $\mathcal{W}(\pi, \psi)$ consists of smooth functions $W : \text{GL}_2(F) \to \mathbb{C}$ such that

$$W \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \psi(x)W(g)$$

for $x \in F$ and $g \in \text{GL}_2(F)$. Moreover the right translation of $\text{GL}_2(F)$ on $\mathcal{W}(\pi, \psi)$ is equivalent to $\pi$. Recall that when $F$ is non-archimedean, smoothness means locally constant. Let $K$ be a maximal compact subgroup of $\text{GL}_2(F)$. Denote by $\mathcal{W}(\pi, \psi)_0$ the subspace of right $K$-finite functions. Note that $\mathcal{W}(\pi, \psi)_0 = \mathcal{W}(\pi, \psi)$ when $F$ is non-archimedean.
Let $\mu$ and $\nu$ be characters of $F^\times$. We recall the definition of the principal series representation $\text{Ind}_{B(F)}^{GL_2(F)}(\mu, \nu)$ of $GL_2(F)$. Denote $\mathcal{B}(\mu, \nu)$ the space of smooth functions $f : GL_2(F) \to \mathbb{C}$ such that

$$ f \left( \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) g = \mu(a)\nu(d) \frac{|a|^2}{|d|^{1/2}} f(g), $$

for $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(F)$ and $g \in GL_2(F)$. Denote $\rho$ the right translation of $GL_2(F)$ on $\mathcal{B}(\mu, \nu)$. The representation $(\rho, \mathcal{B}(\mu, \nu))$ of $GL_2(F)$ is denoted by $\text{Ind}_{B(F)}^{GL_2(F)}(\mu, \nu)$. Similarly, let $\mathcal{B}(\mu, \nu)_0$ be the subspace of right $K$-finite functions. Assume $F$ is non-archimedean, for $f \in \mathcal{B}(\mu, \nu)$, define $W_{\psi, f} \in \mathcal{W}(\pi, \psi)$ by

$$ W_{\psi, f}(g) := \lim_{n \to \infty} \int_{\pi^{-n} \mathcal{O}_F} f \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x) dx. $$

2. Asai local factors for the non-archimedean case

In this section, we consider the case $n = 1$. Let $F$ be a non-archimedean local field of characteristic zero and $q$ be the cardinality of the residual field of $F$. Fix a non-trivial additive character $\psi$ of $F$.

2.1. Asai local factors via Rankin-Selberg integrals. Let $E$ be a quadratic semisimple $F$-algebra, namely, $E$ is equal to a quadratic extension field over $F$ or $F \times F$. If $E = F \times F$, we embed $F$ into $E$ via the diagonal embedding. Fix an element $\xi \in E^\times$ such that $\text{tr}_{E/F}(\xi) = 0$. Let $\psi_\xi$ be an additive character of $E$ defined by $\psi_\xi(x) = \psi(\text{tr}_{E/F}(\xi x))$.

Let $\pi$ be an irreducible generic representation of $GL_2(E)$ with central character $\omega$. For any $\Phi \in \mathcal{G}(F^2)$ and $W \in \mathcal{W}(\pi, \psi_\xi)$, we define a function on $s \in \mathbb{C}$ by

$$ (2.1.1) \quad Z(s, W, \Phi) := \int_{U(F) \backslash GL_2(F)} W(g) \Phi((0,1)g) |\det(g)|_F^s \, dg. $$

where we normalize the invariant measure so that $\text{vol}(GL_2(\mathcal{O}_F), dg) = 1$. We note that $Z(s, W, \Phi)$ converges absolutely for sufficiently large $Re(s)$, and is analytically continued to the whole complex plane as a meromorphic function. Moreover, it is an element of $\mathbb{C}[q^s, q^{-s}]$. The $\mathbb{C}$-vector space generated by $Z(s, W, \Phi)$'s for $W \in \mathcal{W}(\pi, \psi_\xi)$ and $\Phi \in \mathcal{G}(F^2)$ is actually a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ containing $1$. Thus, there exists $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$ such that $P(q^{-s})^{-1}$ is a generator of this ideal (see [Kab03] p.801 or [F93] Appendix, Theorem).

We define the Asai $L$-function by

$$ L_{RS}(s, As \pi) := \frac{1}{P(q^{-s})}. $$

More generally, for any character $\chi : F^\times \to \mathbb{C}^\times$, we define

$$ L_{RS}(s, As \pi \otimes \chi) := L_{RS}(s, As(\pi \otimes \chi)) $$

where $\chi^\times : E^\times \to \mathbb{C}^\times$ is a character such that

$$ \chi |_{E^\times} = \chi. $$

We note that this definition is independent of the choice of $\chi$. This function satisfies the following functional equation (see [F93] Appendix, Theorem): For any $W \in \mathcal{W}(\pi, \psi_\xi)$, we have

$$ (2.1.2) \quad \frac{Z(1-s, W \otimes \chi^{-1} \omega^{-1}, \Phi)}{L_{RS}(1-s, As \pi \otimes \chi^{-1})} = \varepsilon_{RS}(s, As \pi \otimes \chi, \psi, \xi) \frac{Z(s, W \otimes \chi, \Phi)}{L_{RS}(s, As \pi \otimes \chi)}, $$

where there exists $c \in \mathbb{C}^\times$ and $m \in \mathbb{Z}$ depending only on $\pi, \psi, \xi$ such that

$$ \varepsilon_{RS}(s, As \pi \otimes \chi, \psi, \xi) := cq^{-ms}, $$

and

$$ (2.1.3) \quad \widetilde{\Phi}(x, y) := \int_{F \times F} \Phi(u, v) \psi(ux - vy) \, du \, dv. $$

Here $du \, dv$ is the self-dual measure associated with $F \times F \to \mathbb{C}; (x, y) \to \psi(x + y)$. We note that the epsilon factor $\varepsilon_{RS}(s, As \pi, \psi, \xi)$ is the same as $\varepsilon(s, r(\pi), \psi_\xi)$ appeared in [F93] Appendix, Theorem] and different
from $\varepsilon(s, \pi, As, \psi_\xi)$ only by $\omega(-1)$ appeared in [Kab04, Theorem 3]. In the case that $\chi$ is trivial, we denote $\varepsilon_{RS}(s, As \pi \otimes \chi, \psi, \xi)$ by just $\varepsilon_{RS}(s, As \pi, \psi, \xi)$. Put

$$\gamma_{RS}(s, As \pi \otimes \sigma, \psi, \xi) = \varepsilon_{RS}(s, As \pi \otimes \sigma, \psi, \xi)\lambda_{RS}(1-s, As \pi' \otimes \sigma')\lambda_{RS}(s, As \pi \otimes \sigma)^{-1}.$$

**Remark 5.**

1. In the case of $E = F \times F$, let $\xi = (\xi_0, -\xi_0)$ and $\pi = \pi_1 \otimes \pi_2$ where $\pi_i$ is an irreducible generic representation of $GL_2(F)$ with central character $\chi_i$. The definition of Asai $L$-function is the same as that defined in [Jac72, Theorem 14.8, (1)]. With the epsilon factors, the relation of $\varepsilon_{RS}$ and one defined in [Jac72, Theorem 14.8, (3)] is given as follows:

$$\varepsilon_{RS}(s, As \pi, \psi, \xi) = \omega(-1)\omega(\xi_0)|\xi_0|_F^{2s-1}\varepsilon(s, \pi_1 \otimes \pi_2, \psi)$$

where the epsilon factor $\varepsilon(s, \pi_1 \otimes \pi_2, \psi)$ in the right hand side is defined in [Jac72, Theorem 14.8, (3)].

2. If we drop the assumption that $\pi$ is irreducible, but assume that $\pi$ has finite length and the space of Whittaker functionals of $\pi$ with respect to $\psi$ is one-dimensional. Then, by [Kab04, Theorem 1], we can define Asai local factors for $\pi$ following the same definition. Moreover, if $\pi'$ is any irreducible generic subquotient of $\pi$, then

$$\gamma_{RS}(s, As \pi', \psi, \xi) = \gamma_{RS}(s, As \pi, \psi, \xi).$$

We note that for any $a \in F^\times$, we have

$$\varepsilon_{RS}(s, As \pi, \psi^a, \xi) = |a|^{\frac{1}{2}-2s}\varepsilon_{RS}(s, As \pi, \psi, \xi),$$

(2.1.4) $$\varepsilon_{RS}(s, As \pi, \psi, a\xi) = \omega(a)|a|^{\frac{1}{2}+2s}\varepsilon_{RS}(s, As \pi, \psi, \xi).$$

Let $\sigma$ be the non-trivial automorphism of $E$ over $F$. If $\chi$ is a character of $E^\times$, let $\chi^\sigma$ be a character of $E^\times$ defined by $\chi^\sigma(x) = \chi(x^\sigma)$.

Following theorem is the main result of this section.

**Theorem 2.1** (Multiplicativity of gamma factors). Assume $E$ is a field, and $c(\psi) = c(\psi_\xi) = 0$. Let $\pi$ be an irreducible generic subquotient of a principal series representation $\text{Ind}^{GL_2(F)}_{B(E)}(\mu, \nu)$. We have

$$\gamma_{RS}(s, As \pi, \psi, \xi) = \nu(-1)\gamma(s, \mu|_{F^\times}, \psi)\gamma(s, \nu|_{F^\times}, \psi)\gamma(s, \mu|_{F^\times}, \psi_\xi)\gamma(s, \nu|_{F^\times}, \psi_\xi).$$

**Corollary 2.2.** We have

$$\varepsilon_{RS}(s, As \pi, \psi, \xi) = \omega(\xi)|\xi|_F^{\frac{1}{2}}\lambda_{E/F}(\psi)^{-1}\varepsilon_{\text{Gal}}(s, As \pi, \psi)$$

Here $\lambda_{E/F}(\psi)$ is the Langlands constant for $E/F$ with respect to $\psi$.

**Proof.** As we mentioned in Remarks 11(1) and 12(1), the equality for $\varepsilon$-factors is equivalent to the equality for $\gamma$-factors and it is suffices to consider the case when $E$ is a field.

First we assume $\pi$ be an irreducible generic subquotient of a principal series representation $\text{Ind}^{GL_2(F)}_{B(E)}(\mu, \nu)$. By (2.1.4), we may assume $c(\psi) = c(\psi_\xi) = 0$. By the property of Asai representation defined by (11.11) (Cf. [Pra92, Lemma 7.1 (d)]), we have

$$\gamma_{\text{Gal}}(s, As \pi, \psi) = \lambda_{E/F}(\psi)\gamma(s, \mu|_{F^\times}, \psi)\gamma(s, \nu|_{F^\times}, \psi)\gamma(s, \mu|_{F^\times}, \psi_\xi)\gamma(s, \nu|_{F^\times}, \psi_\xi).$$

The last equality follows from the fact that $|\xi|_E = |\xi|_F^{-c(E/F)}$, since we assume that $c(\psi) = c(\psi_\xi) = 0$. The assertion then follows from Theorem 2.3.

Assume $\pi$ is a supercuspidal representation. Let $E/F$ be a quadratic extension of number fields such that there exist a finite place $v_0$ of $F$ such that $E_{v_0} = E$ and $F_{v_0} = F$. Fix a non-trivial additive character $\psi$ of $\mathbb{A}_F/F$ and an element $\xi \in E^\times$ such that $\text{tr}_{E/F}(\xi) = 0$. By [Sha90, Proposition 5.1], there exist an irreducible cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A}_E)$ such that

- $\pi_{v_0} = \pi$.
- $\pi_v$ is spherical for any finite place $v \neq v_0$.  

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Denote \( \omega \) the central character of \( \pi \). By Theorem 2.1 and (3.2.2), for all places \( v \neq v_0 \) of \( F \), we have
\[
\gamma_{RS}(s, \pi_v, \psi_v, \xi) = \omega_v(\xi)\xi^{s-1/2}\lambda_{E_v/F_v}(\psi)^{-1}\gamma_{Gal}(s, \pi_v, \psi_v).
\]

On the other hand, by [Kri03, Theorem 6.7], the irreducible admissible representation \( \pi = \otimes_v \pi_v \) is an isobaric automorphic representation of \( GL_2(A_F) \). Since \( \pi \) is isobaric, it follows from the global functional equation for Rankin-Selberg \( L \)-functions that the global automorphic \( L \)-function \( L_{Gal}(s, \pi) = \prod_v L_{Gal}(s, \pi_v) \) has meromorphic continuation to \( s \in \mathbb{C} \) and satisfies the functional equation
\[
L_{Gal}(s, \pi) = \varepsilon_{Gal}(s, \pi)L_{Gal}(1-s, \pi^\vee).
\]

The assertion then follows from (2.1.4), (2.1.5), (2.1.6), and the global functional equation for \( L_{RS}(s, \pi) \). This completes the proof.

\[\square\]

2.2. Proof of Theorem 2.1

As explained in Remark 5-(2), we may assume \( \pi = \text{Ind}_{E/F}^{GL_2(E)}(\mu, \nu) \) even though the induced representation might not be irreducible.

It suffices to prove that there exist \( W \in \mathcal{W}(\pi, \psi) \) and \( \Phi \in \mathcal{S}(F^2) \) such that \( Z(s, W, \Phi) \neq 0 \) and
\[
Z(1-s, W, \Phi) = \nu(-1)^r\gamma(s, \mu|_{F^\times}, \psi)\gamma(s, \nu|_{F^\times}, \psi)\gamma(s, \nu|_{E^\times}, \psi)Z(s, W, \Phi).
\]

We have the following three cases:

(i) Exactly one of \( \mu \) and \( \nu \) is unramified,

(ii) Both \( \mu \) and \( \nu \) are ramified,

(iii) Both \( \mu \) and \( \nu \) are unramified.

In case (iii), take \( W \in \mathcal{W}(\pi, \psi) \) be a non-zero Whittaker function fixed by \( GL_2(O_E) \), \( \Phi \in \mathcal{S}(F^2) \), and \( \chi = 1 \). With this datum, we can calculate both sides of (2.1.2) and deduce the formula for gamma factors. The calculation is standard and is left to the readers.

2.2.1. The proof of case (i).

Proof. We may assume that \( \mu \) is ramified and \( \nu \) is unramified. Twist by \( \nu^{-1} \), we may further assume \( \nu = 1 \). Put
\[
r := [c(\mu)/e_{E/F}].
\]

Here \( e_{E/F} \) is the ramified index for \( E/F \). Let \( f \in \mathcal{B}(\mu, 1) \) be a section characterized by
\[
f \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) = \mu^{-1}(x)|x|_E^{-\delta}x_1(x) \quad (x \geq 1) \quad (x \geq 1).
\]

Note that \( \rho(k)f = f \) for \( k = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}(O_E) \) with \( c \in \mathcal{O}_E(c(\mu)) \) (Cf. [Sc02, Section 2.1]).

We divide this problem into the following two cases:

(1) \( \mu|_{F^\times} \) is ramified,

(2) \( \mu|_{F^\times} \) is unramified.

Case (1): \( \mu|_{F^\times} \) is ramified

Denote \( c_\mu = c(\mu|_{F^\times}) \) and
\[
\delta := \begin{cases} 0 & \text{if } c(\mu)/e_{E/F} \in \mathbb{Z}, \\ 1 & \text{if } c(\mu)/e_{E/F} \notin \mathbb{Z}. \end{cases}
\]

Note that when \( E/F \) is ramified, \( c(\chi) \) is even for any character \( \chi \) of \( E^\times \) with \( \chi|_{F^\times} = 1 \). In particular, if \( \delta = 1 \), then \( c_\mu = r \).

Let
\[
g := |\omega^r|_F^{-1} \int_{\omega^r \mathcal{O}_E} \rho \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) f \, dx.
\]

We explicitly determine \( W_{\psi, \theta} \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \). Let \( \theta \in \mathcal{O}_E \) be an element such that \( \mathcal{O}_E = \mathcal{O}_F[\theta] \) and
\[
\psi(c + d\theta) = \psi(d)
\]
for $c, d \in \mathcal{O}_F$. By straightforward computation, for $a \in F^\times$, we have

$$W_{\psi, g} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = |a|_F \int_{E} \sum_{u \in \mathcal{O}_F/\mathcal{O}^{-r-c}_{F} \mathcal{O}_F} \mu(x^{-1})|x|_F^{-1} f \left( \begin{pmatrix} x^{-1} & 1 \\ \varphi(x) & 1 \end{pmatrix} \right) \psi(-ax) dx$$

$$= |a|_F \sum_{u \in \mathcal{O}_F/\mathcal{O}^{-r-c}_{F} \mathcal{O}_F} \mu^{-1}(1 + \varphi(x)u) \psi(-ax)$$

$$= |a|_F \sum_{m=0}^{r-c} \int_{\mathcal{O}_F} A_m(x) \psi(-ax) dx$$

$$= |a|_F \sum_{m=1}^{r-c} \int_{\mathcal{O}_F} A_m(x\theta) \psi(-ax) dx$$

where for $m = 0, \ldots, r - c$, we put

$$A_m(x) := \sum_{u \in (\mathcal{O}_F/\mathcal{O}^{-r-c}_{F} \mathcal{O}_F)^\times} \mu^{-1}(1 + \varphi(x)u).$$

Note that if $m < r - c$ and $a \notin \mathcal{O}^{-m-c}_{F} \mathcal{O}_F$, we have

$$\int_{\mathcal{O}_F} A_m(x) \psi(-ax) dx = 0.$$ 

Therefore, if $\delta = 1$, for $a \in F^\times$ we have

$$W_{\psi, g} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = |a|_F \mathcal{I}_{\mathcal{O}_F}(a).$$

Suppose $\delta = 0$. Let $m = r - c$, then for $a \in F^\times$ we have

$$\int_{\mathcal{O}_F} A_m(x\theta) \psi(-ax) dx = \mathcal{I}_{\mathcal{O}_F}(a).$$

Let $0 \leq m < r - c$. Since $A_m(xv) = A_m(x)$ for any $v \in \mathcal{O}_F^\times$, we have

$$\int_{\mathcal{O}_F} A_m(x\theta) \psi(-ax) dx = \int_{\mathcal{O}_F} A_m(x) \psi(-\varphi(x)v) dx$$

$$= \int_{\mathcal{O}_F} A_m(x) \frac{\varphi(x)^{r-m-c}}{1 - \varphi(x)^r} \sum_{w \in (\mathcal{O}_F/\mathcal{O}^{-m-c}_{F} \mathcal{O}_F)^\times} \psi(-\varphi(x)^m v) dx \cdot \mathcal{I}_{\mathcal{O}_F^\times}(a).$$

Since for any $k \in \mathbb{Z}$, $x \in \mathcal{O}_F$ with $r - k \geq 1$,

$$\frac{\varphi(x)^{r-k}}{1 - \varphi(x)^r} \sum_{w \in (\mathcal{O}_F/\mathcal{O}^{-k}_{F} \mathcal{O}_F)^\times} \psi(-\varphi(x)^{r-k} w) = \begin{cases} 0 & x \notin \mathcal{O}^{-r-k-1}_{F} \mathcal{O}_F, \\ -((\varphi(x)_F)^{r-k-1} - 1)^{-1} & x \in \mathcal{O}^{-r-k-1}_{F} \mathcal{O}_F, \\ 1 & x \in \mathcal{O}^{-k}_{F} \mathcal{O}_F. \end{cases}$$
Hence, for \( a \in F^\times \) we have
\[
\int_{O_F} A_a(x\theta)\psi(-ax) \, dx
\]
\[
= \left( 1 - |\varpi_F|/F \right) \|\varpi_F|^{r-m-c_\mu} \sum_{u \in (O_F/\varpi_F^{r-m-c_\mu}F)_{\times}} \mu^{-1}(1 + \varpi_F^{-1}u) \right] \|O_{\varpi_F^{r+c_\mu-r}O_F^\times}(a)
\]
\[
= \left( 1 - |\varpi_F|/F + |\varpi_F|/|F| \right) \|O_{\varpi_F^{r+c_\mu-r}O_F^\times}(a)
\]
\[
= \|O_{\varpi_F^{r+c_\mu-r}O_F^\times}(a).
\]
Thus, for \( a \in F^\times \) we have
\[
W_{\psi, g} \left( \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \right) = |a| F \|O_{\varpi_F^{r+c_\mu-r}O_F^\times}(a).
\]
Therefore, in any case, for \( a \in F^\times \) we have
\[
(2.2.1) \quad W_{\psi, g} \left( \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \right) = |a| F \|O_{\varpi_F^{r+c_\mu-r}O_F^\times}(a).
\]
Note that for \( x \in O_E \) and \( y \in E^\times \), we have
\[
\rho(w_1)W_{\psi, f} \left( \left( \begin{array}{cc} y & 0 \\ x & 1 \end{array} \right) \right) \omega(y)^{-1} = \varepsilon(1/2, \pi, \psi) |\varpi_E^{s} y|^{1/2} \|O_E(\varpi_F^{\delta}) y).
\]
Therefore, for \( x \in O_F \) and \( y \in F^\times \), we have
\[
(2.2.2) \quad \rho(w_1)W_{\psi, g} \left( \left( \begin{array}{cc} y & 0 \\ x & 1 \end{array} \right) \right) \omega(y)^{-1} = |\varpi_F|^{r+s} \varepsilon(0, \pi, \psi)|y| F \|O_{\varpi_F^{r+c_\mu-r}O_F^\times}(y).
\]
Let \( W = W_{\psi, f} \Phi(x, y) = \varpi_F^{\delta(\pi, \psi)} \|x_1 + \varpi_F^c O_F(y) \), and \( \chi = 1 \). With this datum, the functional equation (2.2.2) reduces to
\[
\int_{O_F} \int_{F^\times} \rho(w_1)W_{\psi, f} \left( \left( \begin{array}{cc} y & 0 \\ x & 1 \end{array} \right) \right) \omega^{-1}(y) |y|^{-s} \frac{y}{F} \, dy \, dx
\]
\[
= \left| \varpi_F \right|^{r+s} \varepsilon(0, \pi, \psi) \|x_1 + \varpi_F^c O_F(y) \, dy \, dx
\]
By formulae (2.2.1) and (2.2.2), the above equation reduces to
\[
|\varpi_F|^{r(1-s)} \varepsilon(x, \mu, \psi) \|x_1 + \varpi_F^c O_F(x) \, dy \, dx
\]
In other words, we have
\[
\gamma_{RS}(x, \pi, \mu, \psi) = \gamma(s, \mu | F^\times, \psi) \gamma(s, 1, \psi) \gamma(s, \mu, \psi).
\]
This completes the proof of case (1).

**Case (2): \( \mu | F^\times \) is unramified** Note that in this case, \( c(\mu) \) is always even when \( E/F \) is ramified extension. Let
\[
h := \int_{GL_2(O_F)} \rho(k) \, dk
\]
\[
= \sum_{u \in O_F/\varpi_F^{-1}O_F} \rho \left( \left( \begin{array}{cc} 1 \varpi_F u & 0 \\ u & 1 \end{array} \right) \right) f + \sum_{u \in O_F/\varpi_F^c O_F} \rho \left( w_1 \left( \begin{array}{cc} 1 & 0 \\ u & 1 \end{array} \right) \right) f.
\]
be a GL_2(O_F)-invariant vector. We directly compute \( W_{\psi, h} \left( \left( \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right) \right) \) for \( a \in F^\times \). Let
\[
h_1 := \sum_{u \in O_F/\varpi_F^{-1}O_F} \pi \left( \left( \begin{array}{cc} 1 \varpi_F & 0 \\ 0 & 1 \end{array} \right) \right) f, \quad h_2 := \sum_{u \in O_F/\varpi_F^c O_F} \pi \left( w_1 \left( \begin{array}{cc} 1 & 0 \\ u & 1 \end{array} \right) \right) f.
The function $W_{\psi, \eta}$ is computed in the totally same way with that for $W_{\psi, \eta}$ in the previous case, and we have

$$W_{\psi, \eta} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = |a|_F \mathbb{I}_{\mathcal{O}_F \to \mathcal{O}_F}(a).$$

We compute $W_{\psi, \eta} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$ for $a \in F^\times$.

$$W_{\psi, \eta} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= |a|_F \sum_{u \in \mathcal{O}_F / \mathcal{O}_F^+} \int_F \mu^{-1}(x + u)x + u|_E^{-1} \mathbb{I}_{x \geq 1}(x + u)\psi(-ax) dx$$

$$= \mu(a)|a|_F |\mathcal{O}_F|^{-1} \varepsilon(1, \mu, \psi) \mathbb{I}_{\mathcal{O}_F \to \mathcal{O}_F}(a).$$

By [Del76, Théorème 3.2], we have

$$\varepsilon(1, \mu, \psi) = \mu(\mathcal{O}_F)|\mathcal{O}_F|^r \mu(\mathcal{O}_F).$$

Therefore, for $a \in F^\times$ we have

$$(2.2.3)$$

$$W_{\psi, \eta} \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) = \mu(a)|a|_F \mu(\mathcal{O}_F)|\mathcal{O}_F|^{-1} \varepsilon(1, \mu, \psi) \mathbb{I}_{\mathcal{O}_F \to \mathcal{O}_F}(a).$$

Let $W = W_{\psi, \eta}$, $\Phi = \mathbb{I}_{\mathcal{O}_F \to \mathcal{O}_F}$, and $\chi = 1$. With this datum, the functional equation (2.1.2) reduces to

$$L(2s, \mu^{-1}|_{F^\times}) \int_{\text{GL}_2(\mathcal{O}_F)} \int_{F^\times} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \mu^{-1}(y)|y|_F^{-s} d_F^r y dy dk$$

$$= L(2s, \mu|_{F^\times}) \gamma_{RS}(s, \text{As} \pi, \psi, \xi) \int_{\text{GL}_2(\mathcal{O}_F)} \int_{F^\times} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) |y|_F^{-s} d_F^r y dy.$$
For $f \in \mathcal{B}(\mu, \nu)$ and $m, n \in \mathbb{Z}_{\geq 0}$, define the integral

$$I_{m,n}(s, \mu, \nu; f) := \int_{F} \int_{F} \int_{E} f \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \right) \times \psi(-z) |y|_{F}^{-1} \nu^{-1}(x) \, d_{F}x \, d_{F}y \, d_{E}z.$$

Note that the limit

$$\lim_{n \to \infty} \left( \lim_{m \to \infty} I_{m,n}(s, \mu, \nu; f) \right) \quad \text{(resp. } \lim_{m \to \infty} \left( \lim_{n \to \infty} I_{m,n}(1-s, \nu^{-1}, \mu^{-1}; f) \right))$$

is equal to the integral in the right (resp. left) hand side of the formula (2.2.4). By direct computation, we have

$$I_{m,n}(s, \mu, \nu; f) = \int \int \frac{\nu^{-1}(y) |y|z^{-1} f \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \right) \times \phi_{m,n}(x, y, z) \, d_{F}x \, d_{F}y \, d_{E}z}{z^{-1} f \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \right)} \quad \text{(2.2.5)}$$

Now we specify $f$. Let $\theta \in \mathcal{O}_{E}$ be an element such that

$$\mathcal{O}_{E} = \mathcal{O}_{F}[\theta], \quad \psi(a + b \theta) = \psi(b)$$

for any $a, b \in F$. For $x \in F$, we write $x = a_{x} + b_{x} \theta$ for a unique pair $(a_{x}, b_{x}) \in F$. Fix a choice of $\eta \in F^{\times}$ and $M \in \mathbb{Z}_{\geq 0}$ such that

$$|\eta|_{F} M < |\eta^{2}|_{F} M < |\mathcal{W}_{F}^{c(\mu, \nu)/E}|_{F}, \quad |\mathcal{W}_{F}|_{F} M \leq |\mathcal{W}_{F}|_{F}^{c}.$$

Let $\Phi_{n,M} \in \mathcal{G}(E)$ be a Bruhat-Schwartz on $E$ defined by

$$\Phi_{n,M}(x) = \psi(\eta a_{x}) \prod \mathcal{O}_{F}(a_{x}) \cdot \prod \mathcal{O}_{F}(b_{x}).$$

Let $N = N(\eta, M) \in \mathbb{Z}_{\geq 0}$ such that for $z \in E$ with $\Phi_{n,M}(z) \neq 0$, we have

$$|\mathcal{W}_{E}|_{E}^{N} \leq |z|_{E} \leq |\mathcal{W}_{E}|_{E}^{-N}.$$

Define $f \in \mathcal{B}(\mu, \nu)$ be the section characterized by

$$f \left( \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right) = \Phi_{n,M}(x).$$

Take $m, n$ sufficiently large so that

$$|\mathcal{W}_{F}|_{F}^{-N} |\eta|_{F} \leq |\mathcal{W}_{F}|_{F}^{c} \leq |\mathcal{W}_{F}|_{F}^{m+N} |\eta|_{F}, \quad |\mathcal{W}_{E}|_{E}^{-N} |\eta|_{E} \leq |\mathcal{W}_{E}|_{F}^{c} \leq |\mathcal{W}_{E}|_{E}^{n+N} |\eta|_{E}.$$
With this choice of \(m, n,\) and \(f\) in (2.2.6), we have
\[
I_{m,n}(s, \mu, \nu; f) = \frac{\zeta_F(1)}{\zeta_E(1)} \int \int \int \nu^{-1}(x) [x]^{-\epsilon} \psi(\nu x) \cdot \nu(\eta)^{-1} |\eta|^{-\epsilon} \cdot \psi(y) \cdot \mu \nu^\sigma(z) \cdot |z|^{1}_E d_F^\nu d_F^\epsilon d_F^\eta.
\]
\[
= \frac{\zeta_F(1)}{\zeta_E(1)} \int \int \int \nu^{-1}(x) [x]^{-\epsilon} \psi(\nu x) \cdot \nu(\eta)^{-1} |\eta|^{-\epsilon} \cdot \psi(y) \cdot \mu \nu^\sigma(z) \cdot |z|^{1}_E d_F^\nu d_F^\epsilon d_F^\eta.
\]
The first equality follows from a change of variable from \(\psi\) to \(\nu\). The last equality follows from (2.2.8) and the well-known formula that for \(k \in \mathbb{Z}\), we have
\[
\int_{\mathbb{R}} \nu(y) |\eta|^{-\epsilon} \psi(y) dy = \begin{cases} 0 & \text{if } k \neq -c_\nu, \\ \epsilon(1 - s, \nu |^{-1}_F, \psi) & \text{if } k = -c_\nu.
\end{cases}
\]
Thus, we conclude that
\[
\lim_{m \to \infty} \lim_{n \to \infty} I_{m,n}(s, \mu, \nu; f) = \nu(1-1) |\eta|^{-\epsilon} \frac{\zeta_F(1)}{\zeta_E(1)} \int_E \Phi_{\eta,M}(z) \mu \nu^\sigma(z) |z|^{1}_E d_F^\epsilon.
\]
Similarly, we also have
\[
\lim_{m \to \infty} \lim_{n \to \infty} I_{m,n}(1-s, \nu^{-1}, \mu^{-1}; f) = \mu(1-1) |\eta|^{-\epsilon} \frac{\zeta_F(1)}{\zeta_E(1)} \int_E \Phi_{\eta,M}(z) \nu \mu^\sigma(z) |z|^{1}_E d_F^\epsilon.
\]
By (2.2.6) and the condition that \(\eta \in \mathbb{M}\), one can show that the Fourier transform of \(\Phi_{\eta,M}\) with respect to \(\psi\) satisfies
\[
\hat{\Phi}_{\eta,M}(z) = \Phi_{\eta,M}(z^\sigma).
\]
Therefore, by the functional equation for the character \(\mu \nu^\sigma\), we have
\[
\lim_{m \to \infty} \lim_{n \to \infty} I_{m,n}(1-s, \nu^{-1}, \mu^{-1}; f) = \mu \nu(1-1) \gamma(s, \mu \nu^\sigma, \psi) \lim_{m \to \infty} \lim_{n \to \infty} I_{m,n}(s, \mu, \nu; f).
\]
Therefore, what remains to prove is that the integral
\[
\int_{E^*} \Phi_{\eta,M}(z) \mu \nu^\sigma(z) |z|^{1}_E d_F^\epsilon
\]
is non-zero. By direct computation, we have
\[
\int_{E^*} \Phi_{\eta,M}(z) \mu \nu^\sigma(z) |z|^{1}_E d_F^\epsilon
\]
\[
= \zeta_E(1) \int_{\pi^\mathbb{M}_F \cap \pi^\mathbb{M}_F} \psi(\eta a) \mu \nu^\sigma(a + b \theta)|a + b \theta|^{1}_E d_F^\nu d_F^\epsilon d_F^\eta
\]
\[
= \zeta_E(1) \mu \nu^\sigma(\eta) \eta^{1}_E \int_{\pi^\mathbb{M}_F \cap \pi^\mathbb{M}_F} \psi(\eta a) \mu \nu^\sigma(a + \theta)|a + \theta|^{1}_E d_F^\nu d_F^\epsilon d_F^\eta
\]
\[
= \zeta_E(1) \mu \nu^\sigma(\eta) \eta^{1}_E \int_{\pi^\mathbb{M}_F \cap \pi^\mathbb{M}_F} \psi(\eta^2 a) \mu \nu^\sigma(a + \theta)|a + \theta|^{1}_E d_F^\nu d_F^\epsilon d_F^\eta
\]
\[
= \zeta_E(1) \mu \nu^\sigma(\eta) \eta^{1}_E \int_{\pi^\mathbb{M}_F \cap \pi^\mathbb{M}_F} \psi(\eta^2 a) \mu \nu^\sigma(a + \theta)|a + \theta|^{1}_E d_F^\nu d_F^\epsilon d_F^\eta
\]
\[
\times \left[ \int_{\pi^\mathbb{M}_F} \psi(\eta^2 a) \mu \nu^\sigma(a + \theta) d_F^\nu + \sum_{r=-M-\text{ord}_F(\eta)}^{\text{ord}_F(\eta)-1} |\mathbb{W}_F^r| d_F^\nu \right].
\]

It suffices to show that
\[
\int_{\pi^\mathbb{M}_F} \psi(\eta^2 a) \mu \nu^\sigma(a + \theta) d_F^\nu \neq 0
\]
We denote by \( (3.1.2) \) the functions \( W_u \) for some \( -M - \text{ord}_F(\eta) \leq r \leq -1 \). By \( (2.2.7) \), \( -M - \text{ord}_F(\eta) \leq (-2\text{ord}_F(\eta) - c_\omega) \leq -1 \) and

\[
\int_{\mathbb{A}_F^* \mathbb{O}_F^* \mathbb{A}_F^*} \psi(\eta^2 a) \mu(\sigma(\alpha + \theta) d_F a = \int_{\mathbb{A}_F^* \mathbb{O}_F^* \mathbb{A}_F^*} \psi(\eta^2 a) \omega(\alpha) d_F a
\]

\[
= \omega(\eta)^{-2} |\eta|_{\mathbb{F}}^2 \begin{cases} 
\varepsilon(0, \omega^{-1} |F_\psi, \psi) & \text{if } c_\omega > 0, \\
\xi_F(1)^{-1} & \text{if } c_\omega = 0.
\end{cases}
\]

This completes the proof.

\[\square\]

3. ASAI LOCAL FACTORS FOR THE ARCHIMEDEAN CASE

In this section, we develop the theory concerning the analytic properties of the zeta integrals defined by \( (2.1.1) \) for the case \( E = \mathbb{C} \) and \( F = \mathbb{R} \), and prove analogy results to that of \( \text{[Jac74, Theorem 17.2]} \) and \( \text{[Jac90, Theorem 2.1]} \). Instead of concerning the Harish-Chandra modules, we consider the Harish-Chandra representations.

3.1. Harish-Chandra representation. Let \((\pi, V)\) be an infinite-dimensional irreducible Harish-Chandra representation of \( \text{GL}_2(\mathbb{C}) \) defined in \( \text{[Cas89]} \). Then \( V \) is a Frechet space and the representation on \( V \) is smooth. Moreover, \((\pi, V)\) is of moderate growth, a notion that we now recall. We follow the notation in \( \text{[Cas89]} \). For \( g \in \text{GL}_2(\mathbb{C}) \), we set

\[
\|g\|_H = \text{Tr} \left( g^t \tilde{g} \right) + \text{Tr} \left( g^{-1} t \tilde{g}^{-1} \right).
\]

Then for every continuous semi-norm \( \| \cdot \| \) on \( V \), there is a positive integer \( M \) and another semi-norm \( \| \cdot \|' \) such that for every \( v \in V \) and \( g \in \text{GL}_2(\mathbb{C}) \),

\[
\|\pi(g)v\| \leq \|g\|_H \|v\|'.
\]

Let \( V_0 \) denote the subspace of \( U_2(\mathbb{R}) \)-finite elements in \( V \). Then \( V_0 \) is a Harish-Chandra module. Its a result of Casselman and Wallach that \( V \) is determined, up to topological equivalent by the equivalent class of \( V_0 \). In other words, \( V \) is the canonical Casselman-Wallach completion of \( V_0 \). In \( \text{[Jac90]} \), Jacquet called \( V \) a Casselman-Wallach representation.

Classification of the Harish-Chandra module \( V_0 \) can be founded in \( \text{[JL70, Theorem 6.2]} \). Combining this result with \( \text{[Cas89, Propositions 4.1, 4.4]} \), we see that \( \pi = \text{Ind}_{\text{B}(\mu, \nu)}^{\text{GL}_2(\mathbb{C})}(\mu, \nu) \) is an induced representation. Therefore \( V \) can be realized as \( \mathcal{B}(\mu, \nu) \) and the representation is now by the right translation.

We describe the topology on \( \mathcal{B}(\mu, \nu) \). By the Iwasawa decomposition for \( \text{GL}_2(\mathbb{C}) \), we can identify the space of functions in \( \mathcal{B}(\mu, \nu) \) to the space of their restriction to \( U_2(\mathbb{R}) \), which we denote by \( \mathcal{B}(\mu, \nu)|_{U_2(\mathbb{R})} \). The topology of \( \mathcal{B}(\mu, \nu)|_{U_2(\mathbb{R})} \) is the one given by the semi-norms

\[
\text{Sup}_{\psi \in \mathcal{U}_2(\mathbb{R})} \|\rho(\psi)f(k)\|,
\]

where \( X \) ranges over the universal enveloping algebra of the complexified Lie algebra of \( U_2(\mathbb{R}) \).

We let \((\pi', V')\) be a representation of \( \text{GL}_2(\mathbb{C}) \) which isomorphic to \( \text{Ind}_{\text{B}(\mu, \nu)}^{\text{GL}_2(\mathbb{C})}(\mu^{-1}, \nu^{-1}) \). Then the Harish-Chandra modules \( V_0 \) and \( V'_0 \) are dual to each other. By our definition, we have \( (\pi')^\vee = \pi \).

Let \( \psi' \) be a non-trivial additive character of \( \mathbb{C} \). By \( \text{[Wal92, Theorem 15.4.1]} \), there is a non-zero continuous functional \( \lambda_{\psi'} \) on \( \mathcal{B}(\mu, \nu) \), and within a scalar factor, a unique one, such that for every \( f \in \mathcal{B}(\mu, \nu) \) and \( u \in U(\mathbb{C}) \), we have

\[
\lambda_{\psi'}(\pi(u)f) = \psi'(z)\lambda_{\psi'}(f).
\]

Here \( u = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \) with \( z \in \mathbb{C} \).

For each \( f \in \mathcal{B}(\mu, \nu) \), we set

\[
(3.1.1) \quad W_{\psi,f}(g) = \lambda_{\psi'}(\rho(g)f), \quad g \in \text{GL}_2(\mathbb{C}).
\]

We denote by \( \mathcal{U}(\pi, \psi') \) the space spanned by the functions \( W_{\psi,f} \). By \( \text{[JL70, Section 6]} \), its also spanned by the functions

\[
(3.1.2) \quad W_{\psi'}(g) = \mu(\text{det}(g))|\text{det}(g)|^\frac{1}{4} \int_{\mathbb{C}^2} \omega_{\psi'}(g)\Psi(z, z^{-1}) \mu^{-1}(z) d^\times z.
\]

Here \( \Psi \) is an element in the space \( \mathcal{S}(\mathbb{C}^2) \), and \( (\omega_{\psi'}, \mathcal{S}(\mathbb{C}^2)) \) is the Weil representation of \( \text{GL}_2(\mathbb{C}) \) defined in the first section of \( \text{[JL70]} \).
Note that the space \( \mathcal{W}(\pi', \psi') \) is spanned by the functions \((W \otimes \omega^{-1}) (g) := W(g) \omega(\det(g))^{-1}\) with \( W \in \mathcal{W}(\pi, \psi')\).

### 3.2. Main results for archimedean case

We state our main results of this section. Let

\[
\mu(z) = |z|^{\lambda_1 - \frac{s}{2}} e^{cz_1} \quad \text{and} \quad \nu(z) = |z|^{\lambda_2 - \frac{s}{2}} e^{cz_2},
\]

for some \( \lambda_1, \lambda_2 \in \mathbb{C} \) and \( n_1, n_2 \in \mathbb{Z} \). Let \( \psi(x) = e^{2\pi i ax} \) for some \( a \in \mathbb{R} \). Note that \( \xi = c\sqrt{-1} \) for some non-zero real number \( c \).

Define a subspace of \( \mathcal{G}(\mathbb{R}^2) \)

\[
\mathcal{G}(\mathbb{R}^2, \psi) = \left\{ p(x, y) e^{-\pi|a|(x^2+y^2)} \mid p(x, y) \in \mathbb{C}[x, y] \right\}.
\]

We note that \( \mathcal{G}(\mathbb{R}^2, \psi) \) is invariant under the Fourier transform \( \Phi \mapsto \hat{\Phi} \) defined by (2.1.3).

Following [Jac09], we let \( \mathcal{L}(As, \pi) \) be the space of meromorphic functions \( f(s) \) which are holomorphic multiples of \( L(s, As, \pi) \) and furthermore satisfy the following condition. Let \( P(s) \in \mathbb{C}[s] \) be a polynomial such that \( P(s)L(s, As, \pi) \) is holomorphic in the strip \( a \leq \text{Re}(s) \leq b \). Then \( P(s)f(s) \) is bounded in the same strip.

We put

\[
L_{\text{Gal}}(s, As, \pi) = L(s, \mu|_{\mathbb{R}^x}) L(s, \nu|_{\mathbb{R}^x}) L(s, \mu^\sigma),
\]

\[
L_{\text{Gal}}(s, As \pi') = L(s, \mu^{-1}|_{\mathbb{R}^x}) L(s, \nu^{-1}|_{\mathbb{R}^x}) L(s, \mu^{-1}(\nu^{-1})^\sigma),
\]

\[
\varepsilon_{\text{Gal}}(s, As, \pi, \psi) = \lambda_{\mathbb{C}/\mathbb{R}}(\psi) \varepsilon(s, \mu|_{\mathbb{R}^x}, \psi) \varepsilon(s, \nu|_{\mathbb{R}^x}, \psi) \varepsilon(s, \mu\nu^\sigma, \psi_C).
\]

Notice that the Langlands constant is given by \( \lambda_{\mathbb{C}/\mathbb{R}}(\psi) = \text{sgn}(a)\sqrt{-1} \).

**Theorem 3.1.** Let \( W \in \mathcal{W}(\pi, \psi_\xi) \) and \( \Phi \in \mathcal{G}(\mathbb{R}^2) \).

1. The zeta integral \( Z(s, W, \Phi) \) converges absolutely when \( Re(s) > 2 \max \{-Re(\lambda_1), -Re(\lambda_2)\} \) and has a meromorphic continuation to the whole complex plane. In fact, \( Z(s, W, \Phi) \) defines an element in the space \( \mathcal{L}(As, \pi) \). Moreover, if \( W \in \mathcal{W}(\pi, \psi_0) \) and \( \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \), then \( Z(s, W, \Phi) \) is of the form

\[
|a|^{-2s} |\xi|_C^{-s/2} P(s) L_{\text{Gal}}(s, As, \pi),
\]

for some \( P(s) \in \mathbb{C}[s] \).

2. The functional equation

\[
\frac{Z\left(1-s, W \otimes \omega^{-1}, \Phi\right)}{L_{\text{Gal}}(1-s, As \pi')} = \varepsilon_{\text{RS}}(s, As, \pi, \psi, \xi) \frac{Z(s, W, \Phi)}{L_{\text{Gal}}(s, As, \pi)},
\]

holds in the sense of analytic continuation. Moreover, we have the following relation

\[
\omega^{-1}(\xi)|\xi|_C^{-s+1/2} \lambda_{\mathbb{C}/\mathbb{R}}(\psi) \varepsilon_{\text{RS}}(s, As, \pi, \psi, \xi) = \varepsilon_{\text{Gal}}(s, As, \pi, \psi).
\]

3. There exist \( W \in \mathcal{W}(\pi, \psi_\xi) \) and \( \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \) such that

\[
Z(s, W, \Phi) = |a|^{-2s} |\xi|_C^{-s/2} L_{\text{Gal}}(s, As, \pi).
\]

**Remark 6.** Notice that if \( L(s) \) is a meromorphic function on \( \mathbb{C} \) which satisfies (1) of Theorem 3.1 and if there exist \( W \in \mathcal{W}(\pi, \psi_\xi) \) and \( \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \) so that \( Z(s, W, \Phi) = q(s) L(s) \) for some holomorphic function \( q(s) \), then \( L(s) = h(s) L(s, As, \pi) \) for some holomorphic function \( h(s) \) without zeros.

Following the proof of our proof for Theorem 3.1 whose proof will occupy in (3.4)

**Lemma 3.2.** There exist \( W \in \mathcal{W}(\pi, \psi_\xi) \) and \( \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \) such that

\[
Z(s, W, \Phi) = c |a|^{-2s} |\xi|_C^{-s/2} L_{\text{Gal}}(s, As, \pi),
\]

\[
Z\left(1-s, W \otimes \omega^{-1}, \Phi\right) = c^\vee |a|^{-2s+2s} |\xi|_C^{-(1+s)/2} L_{\text{Gal}}(1-s, As, \pi'),
\]

for some non-zero constants \( c \) and \( c^\vee \).
Suppose Theorem 3.1 holds for some \( \psi \) and \( \xi \), we see what happen if we replace \( \psi \) by \( \psi^b(x) := \psi(bx) \) and \( \xi \) by \( \xi' = c\xi \) for some \( b, c \in \mathbb{R}^* \). Consider the following \( \mathbb{C} \)-linear isomorphisms \( \mathcal{W}(\pi, \psi) \xrightarrow{\sim} \mathcal{W}(\pi, \psi^b) \), \( W \to W^{bc} \) and \( \mathcal{S}(\mathbb{R}^2) \xrightarrow{\sim} \mathcal{S}(\mathbb{R}^2) \), \( \Phi \to \Phi^b \), where

\[
W^{bc}(g) = W\left( \begin{pmatrix} bc & 0 \\ 0 & 1 \end{pmatrix} g \right) \quad \text{and} \quad \Phi^b(x, y) = \Phi\left(|b|^{1/2}x, |b|^{1/2}y\right).
\]

Notice that \( \Phi \to \Phi^b \) induces an isomorphism between the spaces \( \mathcal{S}(\mathbb{R}^2, \psi) \) and \( \mathcal{S}(\mathbb{R}^2, \psi^b) \).

One check that

\[
\hat{\Phi}^b(x, y) = \hat{\Phi}\left(|b|^{1/2}x, |b|^{1/2}y\right),
\]

where the Fourier transform on the LHS is with respect to \( \psi^b \), while the Fourier transform on the RHS is with respect to \( \psi \). Moreover, we have

\[
Z(s, W^{bc}, \Phi^b) = \omega\left(|b|^{1/2}\right)^{-s} Z(s, W, \Phi),
\]

\[
Z\left(s, W^{bc} \otimes \omega^{-1}, \frac{\hat{\Phi}^b}{\Phi}\right) = \omega\left(|b|^{1/2}\right)^{-s} Z\left(s, W \otimes \omega^{-1}, \frac{\hat{\Phi}}{\Phi}\right).
\]

From these we obtain the following relation

\[
\varepsilon_{\text{RS}}(s, A, \pi, \psi^b, c\xi) = \omega\left(|b|^2\right)^{2s-1} \varepsilon_{\text{RS}}(s, A, \pi, \psi, \xi).
\]

Since

\[
\varepsilon(s, A, \pi, \psi^b) = \text{sgn}(b)\omega(|b|) |b|^{s-1} \varepsilon(s, A, \pi, \psi),
\]

this shows that if Theorem 3.1 holds for some \( \psi \) and \( \xi \), then it holds for all non-trivial additive character \( \psi \) and all non-zero element \( \xi \) with \( \text{tr}_{\mathbb{C}/\mathbb{R}}(\xi) = 0 \).

3.3. Proof of Theorem 3.1. Taking Lemma 3.2 for granted in this section, we prove Theorem 3.1 except the relation (3.2.2), whose proof will be given in 3.3. By the remark at the end of 3.2.2, we may assume \( \psi(x) = e^{2\pi i x} \) and \( \xi = \sqrt{-1} \). Also, it suffices to prove first two assertions, as the last is already contained in Lemma 3.2. Let \( \mu \) and \( \nu \) be as in the equation (3.2.1). Since the character \( \psi_\xi \) has fixed, we will suppress the notation \( \psi_\xi \) in \( W_{\psi_\xi} \) and \( W_{\psi_\xi} \) from now on. Put \( n_0 = |n_1 - n_2| \).

3.3.1. Convergence. We show that the zeta integral \( Z(s, W, \Phi) \) converges absolutely when

\[
\Re(s) > 2 \max \{-\Re(\lambda_1), -\Re(\lambda_2)\}.
\]

It will be convenient to use the following notation

\[
A \ll x, y, z, \ldots \ll B
\]

to indicate there is a constant \( C > 0 \) depending at most upon \( x, y, z, \ldots \) so that \( |A| \leq C|B| \).

Let \( r = \Re(s) \), \( r_1 = \Re(\lambda_1) \) and \( r_2 = \Re(\lambda_2) \). Formally we have

\[
Z(s, W, \Phi) = \int_{\mathbb{S}_2(\mathbb{R})} \int_{\mathbb{R}^*} W(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k) |y|^{s-1} \int_{\mathbb{R}^*_+} \Phi((0, t)k) \omega(t) t^{2s} t^x t^y dk,
\]

(3.3.1)

\[
= \int_{\mathbb{S}_2(\mathbb{R})} \int_{\mathbb{R}^*} W(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k) |y|^{s-1} d^x d^y \left( \int_{\mathbb{R}^*_+} \Phi((0, t)k) \omega(t) t^{2s} t^x t^y dk \right),
\]

by the Iwasawa decomposition.

We may assume \( W = W_{\psi} \) for some \( \Psi \in \mathcal{S}(\mathbb{C}^2) \). Then by equation (3.3.2), we have

\[
\int_{\mathbb{R}^*} W(\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k) |y|^{s-1} d^x d^y = \int_{\mathbb{R}^*} \mu(y) |y|^{s} \int_{\mathbb{C}^*} \omega_{\psi_\xi}(k) \Psi(yz, z^{-1}) \mu^{-1} (z) d^x d^y.
\]

(3.3.2)

Since \( \mathbb{S}_2(\mathbb{R}) \) is compact, by the continuity of the Weil representation and [Wei, Lemme 5], there exists \( \Psi_0 \in \mathcal{S}(\mathbb{C}^2) \) such that

\[
\omega_{\psi_\xi}(k) \Psi \ll_{\Psi} \Psi_0,
\]

for all \( k \in \mathbb{S}_2(\mathbb{R}) \). On the other hand, for every positive integer \( N \), we have

\[
\Psi_0(yz, z^{-1}) \ll_N (1 + |yz|^c + |z|^{-1})^{-N},
\]

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From this observation we may assume
\[ 3.3.2. \]
claim. (3.3.3) and (3.3.4) depend on the choice of Haar measures, but different choice certainly do not affect our
\[ N \gg 0 \] is arbitrary, our assertion follows at once. Of course the constants appeared in our estimate in
(3.3.5)
by \cite[Lemma 3.1 (i)]{Jac09}. It follows that
\[
\begin{align*}
\int_{\mathbb{R}^n} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k \right) |y|^{s-1} dx y & \ll_{\Phi, N} \int_{\mathbb{R}^n} \int_{\mathbb{C}^n} \frac{|y|^{r+2r_1}|z|^{r_2-r_2}}{(1 + |yz|_C + |z|^2_C)^N} d^x z d^x y \\
& \ll_{\Phi, N} \int_{\mathbb{R}^n} \int_{\mathbb{C}^n} \frac{|y|^{r+2r_1} t^{2r_1-2r_2}}{(1 + y^2 t^2 + t^{-2})^N} d^x y d^x t \\
& \ll_{\Phi, N} \int_{0}^{\infty} \int_{0}^{\infty} \frac{|y|^{r+2r_1-1} t^{r+2r_2-1}}{(1 + y^2 + t^2)^N} dy dt \\
& \ll_{\Phi, N} \int_{0}^{\infty} \int_{0}^{\infty} \frac{|y|^{r+2r_1-1} t^{r+2r_2-1}}{(1 + y^2)^{N/2}(1 + t^2)^{N/2}} dy dt.
\end{align*}
\]
\[ (3.3.3) \]
In the third line, we change the variables \( y \mapsto t^{-1} y, t^{-1} \mapsto t \), while in the last line, we use the inequality
\[ (1 + y^2 + t^2)^2 \geq (1 + y^2)(1 + t^2). \]
Applying \cite[Lemma 3.1 (i)]{Jac09} again, we see that
\[ 3.3.4 \]
\[
\int_{\mathbb{R}^n} \Phi((0, t) k) \omega(t) t^{2r} dt \ll_{\Phi, N} \int_{0}^{\infty} \frac{t^{2r+2r_1+2r_2-1}}{(1 + t^2)^N} dt.
\]
By equations (3.3.3) and (3.3.4), we find that \( Z(s, W, \Phi) \) converges absolutely when
\[ 0 < r + 2r_1 < N, \quad 0 < r + 2r_2 < N, \quad 0 < 2r + 2r_1 + 2r_2 < 2N. \]
Since \( N > 0 \) is arbitrary, our assertion follows at once. Of course the constants appeared in our estimate in
(3.3.3) and (3.3.4) depend on the choice of Haar measures, but different choice certainly do not affect our claim.

3.3.2. Reduction to the K-finite datum. Using the results in \cite[Section 4]{Jac09}, we reduce the proof of Theorem 3.1 to the K-finite datum. More precisely, we show that if the assertions of Theorem 3.1 hold for
\( W \in \mathcal{W}(\pi, \psi_\xi)_0 \) and \( \Phi \in \mathcal{S}(\mathbb{R}^n, \psi) \), then they also hold for \( W \in \mathcal{W}(\pi, \psi_\xi) \) and \( \Phi \in \mathcal{S}(\mathbb{R}^n) \).

Observed that if Theorem 3.1 holds for \( \pi \), then it also holds for \( \pi \otimes | \xi |^2 \) for every \( u \in \mathbb{C} \). Here \( \pi \otimes | \xi |^2 \) denote the representation of \( \text{GL}_2(\mathbb{C}) \) on the same space \( V \) with the action
\[
(\pi \otimes | \xi |^2)(g)v = |\det(g)|_C^2 \pi(g)v, \quad g \in \text{GL}_2(\mathbb{C}), \quad v \in V.
\]
From this observation we may assume \( Z(s, W, \Phi) \) converges absolutely when \( \text{Re}(s) \geq 0 \).

Zeta integral can be written as
\[
Z(s, W, \Phi) = \int_{\mathbb{R}^n} |y|^s \int_{U(\mathbb{R}) \setminus \text{SL}_2(\mathbb{R})} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} g \right) \Phi((0, 1) g) dg dx y.
\]
Since \( Z(s, W, \Phi) \) converges absolutely when \( \text{Re}(s) \geq 0 \), following integrals are also absolute convergent
\[ 3.3.5 \]
\[
\begin{align*}
\varphi(y) &= \omega(\xi) \int_{U(\mathbb{R}) \setminus \text{SL}_2(\mathbb{R})} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} g \right) \Phi((0, 1) g) dg, \\
\hat{\varphi}(y) &= \int_{U(\mathbb{R}) \setminus \text{SL}_2(\mathbb{R})} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} g \right) \omega^{-1}(y) \Phi((0, 1) g) dg.
\end{align*}
\]
Recall that \( \xi = \sqrt{-1} \).

Suppose Theorem 3.1 holds for \( W \in \mathcal{W}(\pi, \psi_\xi)_0 \) and \( \Phi \in \mathcal{S}(\mathbb{R}^n, \psi) \). In this case, \( Z(s, W, \Phi) \) is of the form
\[
P(s) L_{\text{Gal}}(s, As \pi),
\]
for some \( P(s) \in \mathbb{C}[s] \). We note that \( Z(s, W, \Phi) \) is an element of the space \( \mathcal{L}(As \pi) \). Indeed, this follows immediately from the asymptotic behaviour of the gamma function
\[
\Gamma(x + iy) \sim (2\pi)^\frac{1}{2} |y|^{-\frac{1}{2}} e^{-\frac{1}{2} |y|}
\]
for \( x \) fixed and \( |y| \to \infty \), together with the Phragmen-Lindelof principle.

Let \( \phi \) be the representation of \( W_k \) defined by
\[
\phi = \mu|_{\mathbb{R}^n} \oplus \nu|_{\mathbb{R}^n} \oplus \text{Ind}_{W_k}^{W_\mathbb{C}}(\mu^\sigma).
\]
In [Jac09] Section 4, Jacquet defined a notion of so called \((\phi, \psi)\) pairs, and he used this to reduce the proofs to the \(K\)-finite datum.

Let \(\varphi, \tilde{\varphi}\) be the functions given by equation (3.3.5) with \(W = W_f \in \mathcal{W}(\pi, \psi_\xi)_0\) and \(\Phi \in \mathcal{S}(\mathbb{R}^2, \psi)\). Here \(f \in \mathcal{B}(\mu, \nu)_0\). By our assumption and Jacquet Proposition 4.2, we have

\[
\int_{\mathbb{R}^\times} \tilde{\varphi}(y) \tilde{h}(y) dy = \int_{\mathbb{R}^\times} \varphi(y) h(y) dy,
\]

for every \((\phi, \psi)\) pair \((h, \tilde{h})\). By equation (3.3.6), this is equal to the following equality

\[
\int_{U(\mathbb{R})/GL_2(\mathbb{R})} W_f \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) g \omega^{-1}(\det(g)) \tilde{\Phi}((0,1)g) \tilde{h}(\det(g)) dg
\]

\[
= \omega(\xi) \int_{U(\mathbb{R})/GL_2(\mathbb{R})} W_f \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) \Phi((0,1)g) h(\det(g)) dg,
\]

for every \((\phi, \psi)\) pair \((h, \tilde{h})\), and for every \(f \in \mathcal{B}(\mu, \nu)_0\), \(\Phi \in \mathcal{S}(\mathbb{R}^2, \psi)\).

Notice that both sides of (3.3.6) are absolute convergent for every \(f \in \mathcal{B}(\mu, \nu)\) and \(\Phi \in \mathcal{S}(\mathbb{R}^2, \psi)\). Indeed, for every integer \(N\), we have

\[
h(y) \ll_{h,N} |y|^N \quad \text{and} \quad \tilde{h}(y) \ll_{\tilde{h},N} |y|^N,
\]

for all \(y \in \mathbb{R}^\times\). Combining these with [Jac09] Lemma 3.5, Proposition 3.3, we see that, when \(h, \tilde{h}\) are fixed, both sides in (3.3.6) are continuous functions of \((f, \Phi)\). As the equality holds for all \(f \in \mathcal{B}(\mu, \nu)_0\) and \(\Phi \in \mathcal{S}(\mathbb{R}^2, \psi)\), we find that it also holds for all \(f \in \mathcal{B}(\mu, \nu)\) and \(\Phi \in \mathcal{S}(\mathbb{R}^2)\) by the continuity and an argument of density. Since \(h, \tilde{h}\) are arbitrary \((\phi, \psi)\) pair, we can now apply [Jac09] Proposition 4.1, and then Theorem 3.1 follows.

Because of this observation, for the rests of this section, we are devoted to prove Theorem 3.1 for the \(K\)-finite datum.

3.3.3. Notation. We introduce some notations which will be used in our proof of Theorem 3.1 and Lemma 3.2 for the \(K\)-finite datum. Let \(g = \text{Lie}(GL_2(\mathbb{R})) \otimes_{\mathbb{R}} \mathbb{C}\). Then \(\mathcal{S}(\mathbb{R}^2, \psi)\) is a \((g, O_2(\mathbb{R}))\)-module with the actions

\[
\rho(X)\Phi(x,y) = \frac{d}{dt} \Phi((x,y)e^{tX})|_{t=0}, \quad \rho(k)\Phi(x,y) = \Phi((x,y)k),
\]

for \(X \in g\) and \(k \in O_2(\mathbb{R})\).

For a pair \(\underline{a} = (a_1, a_2)\) of non-negative integers, we set

\[
\Phi_{\underline{a}}(x,y) = (x + iy)^{a_1} (x - iy)^{a_2} e^{-\pi(x^2 + y^2)} \quad \text{and} \quad \Psi_{\underline{a}}(z) = z^{a_1} \bar{z}^{a_2} e^{2\pi \bar{z}z},
\]

for \(x, y \in \mathbb{R}\) and \(z \in \mathbb{C}\).

These functions have the following properties

\[
\rho(k(\theta))\Phi_{\underline{a}} = e^{i(a_1 - a_2)\theta} \Phi_{\underline{a}} \quad \text{and} \quad \Psi_{\underline{a}}(ze^{i\theta}) = e^{i(a_1 - a_2)\theta} \Psi_{\underline{a}}(z).
\]

Here

\[
k(\theta) := \begin{pmatrix}
\cos\theta & -\sin\theta \\
\sin\theta & \cos\theta
\end{pmatrix} \in SO_2(\mathbb{R}).
\]

There is a similar space attached to \(\mathbb{C}^2\) and \(\psi_\xi\) given by

\[
\mathcal{S}(\mathbb{C}^2, \psi_\xi) = \left\{ \left. p(z, \bar{z}, w, \bar{w}) e^{-2\pi(z\bar{z} + w\bar{w})} \right| p(x_1, x_2, x_3, x_4) \in \mathbb{C}[x_1, x_2, x_3, x_4] \right\}.
\]

Let \(\underline{b} = (b_1, b_2)\) be another pair of non-negative integers and define

\[
\Psi_{\underline{a}} \otimes \Psi_{\underline{b}}(z, w) := \Psi_{\underline{a}}(z) \Psi_{\underline{b}}(w), \quad z, w \in \mathbb{C}.
\]

Then we have

\[
\mathcal{S}(\mathbb{R}^2, \psi) = \bigoplus_{\underline{a}} C \Phi_{\underline{a}} \quad \text{and} \quad \mathcal{S}(\mathbb{C}^2, \psi_\xi) = \bigoplus_{\underline{a}, \underline{b}} C \Psi_{\underline{a}} \otimes \Psi_{\underline{b}},
\]

where \(\underline{a}, \underline{b}\) and \(\underline{a}\) run through all pairs of non-negative integers.
3.3.4. Haar measures. Although the choice of Haar measures on $GL_2(\mathbb{R})$ and $U(\mathbb{R})$ are not really important, we fix the choice on various groups in this and the next section for the sake of convenience.

The Haar measure $dx$ on $\mathbb{R}$ is the usual Lebesgue measure and the Haar measure $d^x x$ on $\mathbb{R}^\times$ is given by $d^x x = |x|^{-1} dx$. Since $U(\mathbb{R}) \cong \mathbb{R}$, the measure on $U(\mathbb{R})$ is defined. Haar measure $dz$ on $\mathbb{C}$ is twice of the usual Lebesgue measure and the Haar measure $d^x z$ on $\mathbb{C}^\times$ is $d^x z = |z|^{-1} dz$.

The Haar measure $dg$ on $GL_2(\mathbb{R})$ is given by

$$dg = \frac{dt \, dx \, dy}{t \, |y|^2}$$

for $g = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} k$ with $x, y \in \mathbb{R}, t \in \mathbb{R}^+, k \in SO_2(\mathbb{R})$, where $dx, dy, dt$ are the usual Lebesgue measures and $dk$ is the Haar measure on $SO_2(\mathbb{R})$ such that $\text{Vol}(SO_2(\mathbb{R}), dk) = 1$.

Finally, the measure on the quotient space $U(\mathbb{R})\backslash GL_2(\mathbb{R})$ is the unique quotient measure induced from the measure $dg$ on $GL_2(\mathbb{R})$ and the measure $dx$ on $U(\mathbb{R})$.

We stress that when we compute the Fourier transform $\hat{\Phi}$ of an element $\Phi \in \mathcal{S}(\mathbb{R}^2)$, the Haar measures $du, dv$ should be self-dual with respect to $\psi$. Since we have assumed that $\psi(x) = e^{2\pi i x}$, the measures $du, dv$ are just the usual Lebesgue measure on $\mathbb{R}$.

3.3.5. Meromorphic continuation. We show that the zeta integral $Z(s, W, \Phi)$ admits a meromorphic continuation to whole complex plane.

By [1107.0 Section 6], the space $\mathcal{W}(\pi, \psi_0)$ is spanned by the functions $W_\Psi$ (Cf. equation (3.1.2)) with $\Psi \in \mathcal{S}(\mathbb{C}^2, \psi_\xi)$. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two pairs of non-negative integers. We put

$$W_{(a,b)} = W_a \otimes w_b.$$ 

Recall that $W_a \otimes w_b$ is given by (3.3.4) and (3.3.0). Also, the elements $W_{(a,b)}$ span $\mathcal{W}(\pi, \psi_0)$ by (3.3.11).

Let $s \in \mathbb{C}, W \in \mathcal{W}(\pi, \psi_0)$ and $\chi$ be a character of $\mathbb{R}^\times$. We define an integral attached to $W$ and $\chi$, which is analogly to the classical Tate integral.

$$\zeta(s, W, \chi) = \int_{\mathbb{R}^\times} W \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(y)|y|^{s-1} d^x y.$$ 

Lemma 3.3. We have

1. If $a_1 - a_2 + n_1 \neq b_1 - b_2 + n_2$, then $W_{(a,b)} = 0$.
2. Suppose $\chi(y) = |y|^a \text{sgn}^m(y)$ for some $\lambda \in \mathbb{C}$ and $m \in \{0, 1\}$. The integral $\zeta(s, W, \chi)$ converges absolutely for $\text{Re}(s) > 2 \max\{-\text{Re}(\lambda_1), -\text{Re}(\lambda_2)\} - \text{Re}(\lambda)$ and has meromorphic continuation to the whole complex plane. More precisely, if $W = W_{(a,b)}$ with $a_1 - a_2 + n_1 = b_1 - b_2 + n_2$, then

$$\zeta(s, W_{(a,b)}, \chi) = -2^{-1} (1 + (-1)^{n_1+m+a_1+a_2}) (2\pi)^{-s+2a_1+2a_2+2}\frac{\Gamma\left(\frac{1}{2}(s + \lambda + 2 \lambda_1 + a_1 + a_2)\right) \Gamma\left(\frac{1}{2}(s + \lambda + 2 \lambda_2 + b_1 + b_2)\right)}{\Gamma\left(\frac{1}{2}(s + \lambda + 2 \lambda_1 + a_1 + a_2)\right)} \Gamma\left(\frac{1}{2}(s + \lambda + 2 \lambda_2 + b_1 + b_2)\right).$$

(3.3.14)

In particular, the integral $\zeta(s, W_{(a,b)}, \chi)$ vanishes if $n_1 + m + a_1 + a_2$ is odd.

Proof. Recall that $W_{(a,b)} = 0$ if and only if $W_{(a,b)} \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) = 0$ for all $\alpha \in \mathbb{C}^\times$. By equation (3.1.2) and the relation (3.3.3), we have

$$W_{(a,b)} \left( \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right) = \mu(\alpha)|\alpha|^{\frac{1}{2}} \int_{\mathbb{C}^\times} \Psi_\alpha(az)\Psi_\beta(z^{-1}) \mu^{-1}(z) d^x z$$

(3.3.15)

$$= 2\mu(\alpha)|\alpha|^{\frac{1}{2}} \int_{\mathbb{R}^+} \Psi_\alpha(at)\Psi_\beta(t^{-1}) \mu^{-1}(t) \int_0^{2\pi} e^{i(a_1-a_2+b_1-b_2+n_1-n_2)t} d\theta d^x t.$$ 

In particular, $W_{(a,b)} = 0$ if $a_1 - a_2 + n_1 \neq b_1 - b_2 + n_2$. This proves (1).

To prove (2), we may assume $W = W_{(a,b)}$ with $a_1 - a_2 + n_1 = b_1 - b_2 + n_2$. Let $y \in \mathbb{R}^\times$. By equation (3.3.11),

$$W_{(a,b)} \left( \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \right) = 4\pi \mu(y)|y| \int_{\mathbb{R}^+} \Psi_\alpha(yt)\Psi_\beta(t^{-1}) \mu^{-1}(t) d^x t.$$
Formally, we have
\[
\zeta(s, W_{(a,b)}, \chi) = 4\pi \int_{\mathbb{R}^x} \mu(y)|y|^c_2 \int_{\mathbb{R}^y_+} \Psi_a(yt)\Psi_b(t^{-1})\mu\nu^{-1}(t)d^xtd^xy
\]
\[
= 4\pi \left( \int_{\mathbb{R}^y} \Psi_a(y)\mu\nu(y)|y|^c_2d^x y \right) \left( \int_{\mathbb{R}^y_+} \Psi_b(t)\nu\chi(t)t^{2s}d^x t \right),
\]
after changing variables. Since these two are the Tate integrals, our assertion (2) follows at once. \(\square\)

We now show that \(Z(s, W, \Phi)\) has a meromorphic continuation to whole complex plane. Of course this is a direct consequence of what we have proved in the remark above (Cf. equation (3.3.22)). But we still give a proof here as we need more explicit results.

By equation (3.3.1), and the right \(SO_2(\mathbb{R})\)-finiteness of \(W\) and \(\Phi\), we see that it suffices to prove the assertion for the integral
\[
(3.3.16) \quad \int_{\mathbb{R}^x} W_{(a,b)} \left( \begin{array}{c} y \\ 0 \\ 1 \end{array} \right) |y|^{s-1}d^x y \int_{\mathbb{R}^y_+} \Phi \left( (0, t) \right) \omega(t)t^{2s}d^x t,
\]
where \((a, b) = (a_1, a_2), (b, b_2) = (b_1, b_2)\) and \((c, c_2)\) are three pairs of non-negative integers with \(a_1 - a_2 + n_1 = b_1 - b_2 + n_2\). First integral is take care by Lemma 3.3 while the second integral is the Tate integral, whose properties are well-known.

3.3.6. G.C.D. of poles. We show that there is a meromorphic function \(L_{RS}(s, As\pi)\) such that if \(W \in \mathcal{W}(\pi, \psi_\xi)_0\) and \(\Phi \in \mathcal{G}(\mathbb{R}^2, \psi)\), then \(Z(s, W, \Phi)\) is of the form \(P(s) L_{RS}(s, As\pi)\) for some \(P(s) \in \mathbb{C}[s]\). Furthermore, we show that there exist \(W_j \in \mathcal{W}(\pi, \psi_\xi)_0\) and \(\Phi_j \in \mathcal{G}(\mathbb{R}^2, \psi)\) such that
\[
(3.3.17) \quad \sum_j Z(s, W_j, \Phi_j) = L_{RS}(s, As\pi).
\]
Notice that these two conditions characterized \(L_{RS}(s, As\pi)\) up to non-zero constants.

Recall that \(n_0 = |n_1 - n_2|\). First we have.

Lemma 3.4. Let \((c, c_2)\) be a pair of non-negative integers. Suppose \(c_1 + c_2 \not\equiv n_0 \pmod{2}\), then the zeta integrals \(Z(s, W, \Phi_\xi)\) vanish for all \(W \in \mathcal{W}(\pi, \psi_\xi)_0\).

Proof. Let \(n\) be a non-negative integer and \(\rho_n\) denote the \(n\)-th symmetric power of the standard two-dimensional representation of \(GL_2(\mathbb{C})\) on \(\mathbb{C}^2\). By [JL70] (Theorem 6.2), we have
\[
\mathcal{W}(\pi, \psi_\xi)_0 = \bigoplus_{n \geq n_0, n \equiv 0 \pmod{2}} \mathcal{W}(\pi, \psi_\xi; \rho_n)
\]
as \(SU(2)\)-modules. Here \(\mathcal{W}(\pi, \psi_\xi; \rho_n)\) is the \(\rho_n\) isotypic component. From this we find that if \(W'\) is a non-zero element in \(\mathcal{W}(\pi, \psi_\xi)\) and \(\ell\) is an integer such that \(\rho\circ(k(\theta))W' = e^{i\theta}W'\), then \(\ell \equiv n_0 \pmod{2}\).

Now let \(W \in \mathcal{W}(\pi, \psi_\xi)_0\). For \(Re(s) \gg 0\), we have
\[
Z(s, W, \Phi_\xi) = \int_{\mathbb{R}^x} W' \left( \begin{array}{c} y \\ 0 \\ 1 \end{array} \right) |y|^{s-1} \int_{\mathbb{R}^y_+} \Phi_\xi \left( (0, t) \right) \omega(t) t^{2s} d^x t d^y,
\]
by equation (3.3.1). Here
\[
W'(g) := \frac{1}{2\pi} \int_{0}^{2\pi} W(gk(\theta)) e^{i(c_1 - c_2)\theta} d\theta.
\]
By right \(SO_2(\mathbb{R})\)-finiteness of \(W\), we have \(W' \in \mathcal{W}(\pi, \psi_\xi)_0\). Moreover, it follows immediately from the definition that \(\rho\circ(k(\theta))W' = e^{i(c_2 - c_1)\theta}W'\) for every \(k(\theta)\). This implies \(W' = 0\) if \(c_1 + c_2 \not\equiv n_0 \pmod{2}\) as desired. \(\square\)

By Lemma 3.3 and Lemma 3.4, we see that \(Z(s, W, \Phi)\) is a linear combination of the integrals given by the equation (3.3.16) with
\[
a_1 - a_2 + n_1 = b_1 - b_2 + n_2,
\]
\[
a_1 + a_2 \equiv n_1 \quad \text{(mod 2)},
\]
\[
c_1 + c_2 \equiv n_0 \quad \text{(mod 2)}.
\]
Notice that the first two imply \( b_1 + b_2 \equiv n_2 \) (mod 2). Applying equation (3.3.14), it turns out that \( Z(s, W, \Phi) \) is a linear combination of the functions

\[
2^{-s} \pi^{-2s} \Gamma \left( \frac{1}{2} (s + 2\lambda_1 + a_1 + a_2) \right) \Gamma \left( \frac{1}{2} (s + 2\lambda_1 + b_1 + b_2) \right) \Gamma \left( s + 2\lambda_1 + \lambda_2 + \frac{c_1 + c_2}{2} \right),
\]

where \( a_1, a_2, b_1, b_2 \) and \( c_1, c_2 \) are non-negative integers which satisfy the conditions (3.3.18).

Let \( \epsilon_j \in \{0, 1\} \) so that \( n_j \equiv \epsilon_j \) (mod 2) for \( j = 0, 1, 2 \). Define a meromorphic function

\[
E_\pi(s) = \zeta_R(s + 2\lambda_1 + \epsilon_1) \zeta_R(s + 2\lambda_2 + \epsilon_2) \zeta_C(s + \lambda_1 + \lambda_2 + \epsilon_0/2).
\]

Let \( I(\pi) \) denote the subspace of the space of meromorphic functions spanned by the following set

\[
\left\{ \frac{Z(s, W, \Phi)}{E_\pi(s)} \mid W \in \mathcal{W}(\pi, \psi_\xi)_0, \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \right\}.
\]

By equation (3.3.19), one sees that \( I(\pi) \) is a subspace of \( \mathbb{C}[s] \). In fact, \( I(\pi) \) is an ideal of \( \mathbb{C}[s] \). To see this, let \( \Phi'(x, y) = \frac{d}{dt} \Phi((x, y)e^{itZ}) |_{t=0} \) where \( J_2 \) is the identity element in \( \mathfrak{g} \). We have the following relation

\[
2(s + \lambda_1 + \lambda_2)Z(s, W, \Phi) + Z(s, W, \Phi') = 0.
\]

Being an ideal of \( \mathbb{C}[s] \), there exists \( P_\pi(s) \in I(\pi) \) such that \( I(\pi) = P_\pi(s)\mathbb{C}[s] \). Lemma [3.2] implies \( P_\pi(s) \neq 0 \). Put

\[
L_{RS}(s, As \pi) = E_\pi(s)P_\pi(s).
\]

Similarly, we let

\[
E_{\pi^\vee}(s) = \zeta_R(s - 2\lambda_1 + \epsilon_1) \zeta_R(s - 2\lambda_2 + \epsilon_2) \zeta_C(s - \lambda_1 - \lambda_2 + \epsilon_0/2),
\]

and \( I(\pi^\vee) = P_{\pi^\vee}(s)\mathbb{C}[s] \) be the ideal of \( \mathbb{C}[s] \) spanned by the set

\[
\left\{ \frac{Z(s, W \otimes \omega^{-1}, \Phi)}{E_{\pi^\vee}(s)} \mid W \in \mathcal{W}(\pi, \psi_\xi)_0, \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \right\}.
\]

We put

\[
L_{RS}(s, As \pi^\vee) = E_{\pi^\vee}(s)P_{\pi^\vee}(s).
\]

Notice that \( L_{RS}(s, As \pi) \) satisfies the two conditions stated at the beginning of (3.3.6). We will show that, up to a non-zero constant, \( L_{RS}(s, As \pi) \) is equal to \( L_{Gal}(s, As \pi) \) in (3.3.8).

### 3.3.7. The functional equation

Functional equation between zeta integrals is indeed a consequence of a work of Loke [Lok01, Theorem 1.3].

Let \( s \in \mathbb{C}, \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \) and \( g \in GL_2(\mathbb{R}) \). Define two integrals

\[
f_\Phi^{(s)}(g) = |det(g)|^s \int_{\mathbb{R}^\times} \Phi((0, t)g)\omega_0(t)|t|^{2s}d^xt,
\]

\[
\tilde{f}_\Phi^{(s)}(g) = \omega_0(det(g))^{-1}|det(g)|^s \int_{\mathbb{R}^\times} \Phi((0, t)g)\omega_0^{-1}(t)|t|^{2s}d^xt.
\]

Here \( \omega_0 := \omega|_{\mathbb{R}^\times} \). These two integrals converge absolutely for \( \text{Re}(s) \gg 0 \) and have meromorphic continuations to whole complex plane. Moreover, we have

\[
f_\Phi^{(s)} \in \mathcal{B} \left( \left| \frac{s}{2} \right|^{1/2}, \left| \frac{1}{2} \right|, \left| \frac{s-1}{2} \right|, \left| \frac{s+1}{2} \right| \right)_0 \quad \text{and} \quad \tilde{f}_\Phi^{(s)} \in \mathcal{B} \left( \omega_0^{-1} | s-1/2 |, | 1/2-s | \right)_0,
\]

and the linear maps \( \Phi \mapsto f_\Phi^{(s)} \) and \( \Phi \mapsto \tilde{f}_\Phi^{(s)} \) are surjective except for countable many \( s \).

Let

\[
Mf_\Phi^{(s)}(g) := \int_{\mathbb{R}} f_\Phi^{(s)} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} \right) g \ dx \quad \text{and} \quad M^* := \omega_0(-1)^\gamma (2s - 1, \omega_0, \psi) M.
\]

The integral converges absolutely for \( \text{Re}(s) \gg 0 \) and admits a meromorphic continuation to whole complex plane. It defines an intertwining operator

\[
M^* : \mathcal{B} \left( \left| \frac{s}{2} \right|^{1/2}, \left| \frac{1}{2} \right|, \left| \frac{s-1}{2} \right|, \left| \frac{s+1}{2} \right| \right)_0 \rightarrow \mathcal{B} \left( \omega_0^{-1} | 1/2-s |, | s-1/2 | \right)_0,
\]

between two Harish-Chandra modules and one check that

\[
(3.3.21) \quad M^* f_\Phi^{(s)} = \tilde{f}_\Phi^{(1-s)}.
\]
These facts can be founded in [GJ79, section 4].
By equations (3.3.1), (3.3.20) and (3.3.21), we can write
\[
Z(s, W, \Phi) = \int_{\mathbb{R}^n \setminus \text{GL}_2(\mathbb{R})} W(g) f^{(s)}_\Phi(g) dg,
\]
(3.3.22)
\[
Z \left(1 - s, W \otimes \omega^{-1}, \Phi \right) = \int_{\mathbb{R}^n \setminus \text{GL}_2(\mathbb{R})} W(g) M^* f^{(s)}_\Phi(g) dg.
\]
Both integrals appeared in the RHS of (3.3.22) define \( (\text{g}, \text{O}(2)) \)-invariant forms on the space
\[
\mathcal{W}(\pi, \psi, \xi, 0) \otimes \mathcal{B} \left( \frac{|s-1/2|, \omega_0^{-1}|}{1/2-s} \right).
\]
By a result of [Lok01, Theorem 1.3], there is a meromorphic function \( \gamma_{RS} \), independent of \( W \) and \( \Phi \) such that
\[
\gamma_{RS} \left(1 - s, W \otimes \omega^{-1}, \Phi \right) = \gamma_{RS} \left(1 - s, W, \Phi \right) Z(s, W, \Phi).
\]
Then we obtain the functional equation
\[
\frac{Z \left(1 - s, W \otimes \omega^{-1}, \Phi \right)}{L_{RS} \left(1 - s, \As{\pi} \nu \right)} = \varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) L_{RS} \left(1 - s, \As{\pi} \nu \right),
\]
for \( W \in \mathcal{W}(\pi, \psi, \xi, 0) \) and \( \Phi \in \mathcal{S}(\mathbb{R}^2, \psi) \) as desired.

We show that \( \varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) \in \mathbb{C} \). In fact, by equations (3.3.17) and (3.3.23), we see that
\[
\varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) \in \mathbb{C} \left[s \right].
\]
On the other hand, if we apply the functional equation twice and using again equation (3.3.17), we find that
\[
\varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) \varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) = 1.
\]
This shows our claim.

3.3.8. Matching \( L \)-factors. We show that, up to non-zero constants, \( L_{RS} \left(1 - s, \As{\pi} \right) \) is equal to \( L_{\text{Gal}} \left(1 - s, \As{\pi} \right) \). Recall that \( \epsilon_j \in \{0, 1\} \) so that \( n_j \equiv \epsilon_j \pmod{2} \) for \( j = 0, 1, 2 \), and we have
\[
L_{RS} \left(1 - s, \As{\pi} \nu \right) = P_{\pi}(s) E_\nu(s) \quad \text{and} \quad L_{RS} \left(1 - s, \As{\pi} \nu \right) = P_{\pi}(s) E_{\nu}(s),
\]
with \( P_{\pi} \in \mathbb{C}[s] \) (resp. \( P_{\pi} \in \mathbb{C}[s] \)) is a generator of the ideal \( I(\pi) \) (resp. \( I(\nu) \)) defined in (3.3.6). Here
\[
E_\nu(s) = \zeta_\nu(s - 2\lambda_1 + \epsilon_1) \zeta_\nu(s + 2\lambda_2 + \epsilon_2) \zeta_\nu(s + \lambda_1 + \lambda_2 + \nu + 2),
\]
\[
E_{\nu}(s) = \zeta_\nu(s - 2\lambda_1 + \epsilon_1) \zeta_\nu(s - 2\lambda_2 + \epsilon_2) \zeta_\nu(s - \lambda_1 - \lambda_2 + 2).
\]
On the other hand,
\[
L_{\text{Gal}} \left(1 - s, \As{\pi} \right) = \zeta_\nu(s - 2\lambda_1 + \epsilon_1) \zeta_\nu(s + 2\lambda_2 + \epsilon_2) \zeta_\nu(s + \lambda_1 + \lambda_2 + 2),
\]
\[
L_{\text{Gal}} \left(1 - s, \As{\pi} \nu \right) = \zeta_\nu(s - 2\lambda_1 + \epsilon_1) \zeta_\nu(s - 2\lambda_2 + \epsilon_2) \zeta_\nu(s - \lambda_1 - \lambda_2 + 2).
\]
Let \( n_0 = \epsilon_0 + 2m \) for some non-negative integer \( m \). By Lemma 3.2 and the functional equation (3.3.23), we obtain the following equality
\[
(3.3.24) \quad c^\nu \prod_{j=0}^{m-1} \frac{(1 - s - \lambda_1 - \lambda_2 + \epsilon_0/2 + j)}{P_{\pi}(1 - s)} = c \varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) \prod_{j=0}^{m-1} \frac{(s + \lambda_1 + \lambda_2 + \epsilon_0/2 + j)}{P_{\pi}(s)}.
\]
Here we understand that \( \prod_{j=0}^{m-1} = 1 \) if \( m = 0 \). Since \( \varepsilon_{RS} \left(1 - s, \As{\pi} \nu \right) \) is a non-zero constant, we see that both sides of the equation (3.3.24) are elements of \( \mathbb{C}[s] \) and they have no common roots. This implies
\[
P_{\pi}(s) = \alpha \prod_{j=0}^{m-1} (s + \lambda_1 + \lambda_2 + \epsilon_0/2 + j) \quad \text{and} \quad P_{\pi}(1 - s) = \alpha^\nu \prod_{j=0}^{m-1} (1 - s - \lambda_1 - \lambda_2 + \epsilon_0/2 + j),
\]
for some non-zero constants \( \alpha \) and \( \alpha^\nu \). This proves the assertion.
3.4. Proof of Lemma \[3.2\] and the relation between epsilon factors. The purpose of this section is to prove Lemma \[3.2\]. As a consequence, we obtain the relation \[3.2.2\]. To do this, we need some preparations. We adopt the notations defined in \[3.3.3\] as well as the Haar measures given in \[3.3.4\]. As in the proof of Lemma \[3.4\] for a non-negative integer \(n\), let \(\rho_n\) denote the \(n\)-th symmetric power of the standard two-dimensional representation of \(GL_2(\mathbb{C})\) on \(\mathbb{C}^2\). We realize \(\rho_n\) as \((\rho, L_n(\mathbb{C}))\) with

\[
L_n(\mathbb{C}) = \bigoplus_{j=0}^{n-1} \mathbb{C}X^jY^{n-j} \quad \text{and} \quad \rho(g)P(X,Y) = P((X,Y)g),
\]

for \(P(X,Y) \in L_n(\mathbb{C})\) and \(g \in GL_2(\mathbb{C})\). Let \(\langle \cdot, \cdot \rangle_n\) denote the non-degenerated bilinear pairing on \(L_n(\mathbb{C})\) defined by

\[
\langle X^jY^{n-j}, X^\ell Y^{n-n}\rangle_n = \begin{cases} (-1)^j \binom{n}{j}^{-1} & \text{if } j + \ell = n, \\ 0 & \text{if } j + \ell \neq n. \end{cases}
\]

One checks that

\[
\langle \rho(g)P(X,Y), \rho(g)Q(X,Y)\rangle_n = \det^n(g)\langle P(X,Y), Q(X,Y)\rangle_n.
\]

In particular, \(\langle \cdot, \cdot \rangle_n\) defines an \(SU_2(\mathbb{R})\)-invariant bilinear pairing on \(L_n(\mathbb{C})\).

3.4.1. Whittaker function of type \(\rho_n\). Let \(\mu, \nu\) be as in the equation \[3.2.1\]. Recall that \(n_0 = |n_1 - n_2|\). Let \(n \geq n_0\) be an integer which has the same parity with \(n_0\). By \[\text{[JL70, Lemma 6.1]}\], we have

\[
\dim_{\mathbb{C}}\text{Hom}_{SU_2(\mathbb{R})}(\rho_n, \pi) = 1.
\]

As a consequence, there is a unique (up to constants) non-zero element \(\tilde{W}^{(n)}_\pi(\mathbb{C}) \in \mathcal{W}(\pi, \psi_\xi)_0 \otimes L_n(\mathbb{C})\) characterized by the following two conditions:

1. \(\tilde{W}^{(n)}_\pi(gu) = \rho_n(u)^{-1}\tilde{W}^{(n)}_\pi(g)\) for every \(g \in GL_2(\mathbb{C})\) and \(u \in SU_2(\mathbb{R})\).
2. For every \(v \in L_n(\mathbb{C})\), the function \(W_v(g) := \langle \tilde{W}^{(n)}_\pi(g), v \rangle_n\) belongs to \(\mathcal{W}(\pi, \psi_\xi; \rho_n)\), the isotypic component of \(\rho_n\) in \(\mathcal{W}(\pi, \psi_\xi)_0\).

Following \[\text{[Jac72, section 18]}\], we refer to \(\tilde{W}^{(n)}_\pi\) as the Whittaker function of type \(\rho_n\) attached to \(\pi\). When \(n = n_0\), we simply write \(\tilde{W}^{(n_0)}_\pi\) for \(\tilde{W}^{(n_0)}_\pi\).

3.4.2. Construction of \(\tilde{W}^{(n)}_\pi\). By the uniqueness of the space \(\mathcal{W}(\pi, \psi_\xi)_0\), we may assume \(n_1 \geq n_2\), so that

\(n_0 = n_1 - n_2 \geq 0\).

Let \(m \in \mathbb{Z}\) such that \(n = n_0 + 2m\). We review the construction of \(\tilde{W}^{(n)}_\pi\) given in \[\text{[Jac72, section 18]}\].

Let \(\Psi \in \mathcal{S}(\mathbb{C}^2, \psi_\xi)\) and define the partial Fourier transform \(\Psi^\sim\) of \(\Psi\) by

\[
\Psi^\sim(z, w) = \int_{\mathbb{C}} \Psi(z, u)\psi_\xi(wu)du.
\]

Here \(du\) is twice of the usual Lebesgue measure on \(\mathbb{C}\). One can define the partial Fourier transform of elements in \(\mathcal{S}(\mathbb{C}^2, \psi_\xi) \otimes L_n(\mathbb{C})\) in an obvious way.

Let \(\tilde{\Psi}^{(n)}_\pi \in \mathcal{S}(\mathbb{C}^2, \psi_\xi) \otimes L_n(\mathbb{C})\) be the element such that

\[
\tilde{\Psi}^{(n)}_\pi(z, w)^\sim = (wX - zY)^m \tilde{\Psi}^{(n)}_\pi(z, -z^{-1}) e^{-2\pi(z\bar{z} + w\bar{w})}.
\]

Then

\[
\tilde{W}^{(n)}_\pi(g) = \mu(\det(g))|\det(g)|^{\frac{1}{2}} \int_{\mathbb{C}^n} \omega_{\psi_\xi}(g)\tilde{\Psi}^{(n)}_\pi(z, z^{-1}) \mu^{-1}(z)d^\times z, \quad g \in GL_2(\mathbb{C}).
\]

Here we use the same notation \(\omega_{\psi_\xi}\) to indicate the representation \(\omega_{\psi_\xi} \otimes 1\) of \(GL_2(\mathbb{C})\) on the space \(\mathcal{S}(\mathbb{C}^2) \otimes L_n(\mathbb{C})\). The Haar measure \(d^\times z = |z|^{-1}_C dz\), where \(dz\) is twice of the usual Lebesgue measure on \(\mathbb{C}\).
When \( n = n_0 \), we have

\[
\tilde{\Psi}^{(n_0)}_\pi (z, w) = \sum_{\ell=0}^{n_0} \left( \frac{n_0}{\ell} \right) z^\ell w^{n_0-\ell} e^{-2\pi (z\bar{z} + w\bar{w})} X^\ell Y^{n_0-\ell}
\]

\[
= \sum_{\ell=0}^{n_0} \left( \frac{n_0}{\ell} \right) \Psi_{(0,\ell)} \otimes \Psi_{(n_0-\ell,0)} (z, w) X^\ell Y^{n_0-\ell},
\]

in our previous notation (Cf. §3.3.3). As a consequence, we find that (Cf. equation (3.3.1))

\[
W_\pi = \sum_{\ell=0}^{n_0} \left( \frac{n_0}{\ell} \right) W_{((0,\ell), (n_0-\ell,0))} X^\ell Y^{n_0-\ell}.
\]

We also need the case when \( n_0 = 0 \) and \( n = 2 \). In this case, we have

\[
\tilde{\Psi}^{(2)}_\pi (z, w) = -z\bar{w} e^{-2\pi (z\bar{z} + w\bar{w})} X^2 + \left( \frac{1}{2\pi} - z\bar{z} + w\bar{w} \right) e^{-2\pi (z\bar{z} + w\bar{w})} XY
\]

\[ -zw e^{-2\pi (z\bar{z} + w\bar{w})} Y^2.\]

Therefore we find that

\[
W^{(2)}_\pi = -W_{((0,1),(0,1))} X^2 + \left( \frac{1}{2\pi} W_{((0,0),(0,0))} - W_{((1,1),(0,0))} + W_{((0,0),(1,1))} \right) XY - W_{((1,0),(1,0))} Y^2.
\]

3.4.3. List of the results. We list the choices of \( W \) and \( \Phi \) in Lemma 3.2 and the resulting epsilon factors in the following. There are five cases which depend on the parities of \( n_0 \) and \( n_1 \).

Case 1. \( n_0 \geq 0 \) is even and \( n_1 \) is even.

Choose

\[
W = (\tilde{W}_\pi, (X + iY)^{n_0} (X - iY)^{n_0}) \text{ and } \Phi(x, y) = e^{-\pi(x^2+y^2)}.
\]

Then we have

\[
Z(s, W, \Phi) = \frac{\pi}{2} L_{\text{Gal}} (s, As \pi) \text{ and } Z \left( 1 - s, W \otimes \omega^{-1}, \hat{\Phi} \right) = \frac{\pi}{2} L_{\text{Gal}} (1 - s, As \pi^\vee).
\]

As a consequence,

\[
\varepsilon_{\text{RS}} (s, As \pi, \psi, \xi) = 1.
\]

Case 2. \( n_0 \geq 2 \) is even and \( n_1 \) is odd.

Choose

\[
W = (\tilde{W}_\pi, (X + iY)^{n_0+1} (X - iY)^{n_0-1}) \text{ and } \Phi(x, y) = (x - iy)^2 e^{-\pi(x^2+y^2)}.
\]

Then we have

\[
Z(s, W, \Phi) = \sqrt{-1} \pi L_{\text{Gal}} (s, As \pi) \text{ and } Z \left( 1 - s, W \otimes \omega^{-1}, \hat{\Phi} \right) = \sqrt{-1} \pi L_{\text{Gal}} (1 - s, As \pi^\vee).
\]

As a consequence,

\[
\varepsilon_{\text{RS}} (s, As \pi, \psi, \xi) = 1.
\]

Case 3. \( n_0 \) is odd and \( n_1 \) is even.

Choose

\[
W = (\tilde{W}_\pi, (X + iY)^{n_0+1} (X - iY)^{n_0-1}) \text{ and } \Phi(x, y) = (x - iy)e^{-\pi(x^2+y^2)}.
\]

Then we have

\[
Z(s, W, \Phi) = \frac{\pi}{2\sqrt{-1}} L_{\text{Gal}} (s, As \pi) \text{ and } Z \left( 1 - s, W \otimes \omega^{-1}, \hat{\Phi} \right) = \frac{\pi}{2} L_{\text{Gal}} (1 - s, As \pi^\vee).
\]

As a consequence,

\[
\varepsilon_{\text{RS}} (s, As \pi, \psi, \xi) = \sqrt{-1}.
\]
Case 4. \( n_0 \) is odd and \( n_1 \) is odd.
Choose
\[
W = \langle \bar{W}_\pi, (X + iY)_{\frac{n+1}{2}} (X - iY)_{\frac{n-1}{2}} \rangle_{n_0} \quad \text{and} \quad \Phi(x, y) = (x - iy)e^{-\pi(x^2 + y^2)}.
\]
Then we have
\[
Z(s, W, \Phi) = -\frac{\pi}{2} L_{\text{Gal}}(s, A\pi) \quad \text{and} \quad Z \left(1 - s, W \otimes \omega^{-1}, \hat{\Phi} \right) = \frac{\sqrt{-1}}{2} L_{\text{Gal}}(1 - s, A\pi). 
\]
As a consequence,
\[
\varepsilon_{\text{RS}}(s, A\pi, \psi, \xi) = -\sqrt{-1}.
\]

Case 5. \( n_0 = 0 \) and \( n_1 \) is odd.
Choose
\[
W = \langle \bar{W}_\pi^{(2)}, (X + iY)(X - iY) \rangle_2 \quad \text{and} \quad \Phi(x, y) = e^{-\pi(x^2 + y^2)}.
\]
Then we have
\[
Z(s, W, \Phi) = -\frac{\pi}{2} L_{\text{Gal}}(s, A\pi) \quad \text{and} \quad Z \left(1 - s, W \otimes \omega^{-1}, \hat{\Phi} \right) = -\frac{\pi}{2} L_{\text{Gal}}(1 - s, A\pi). 
\]
As a consequence,
\[
\varepsilon_{\text{RS}}(s, A\pi, \psi, \xi) = 1.
\]

**Remark 7.** In the last case, one might wonder why we do not choose \( W = \bar{W}_\pi \). The reason is \( Z(s, \bar{W}_\pi, \Phi) = 0 \) for all \( \Phi \in \mathcal{G}(\mathbb{R}^2, \psi) \).

With the recipes above, the relation \[\text{(3.4.2)}\] follows immediately. We remain that
\[
\varepsilon_{\text{Gal}}(s, A\pi, \psi) = (\sqrt{-1})^{1+\epsilon_1+\epsilon_2+n_0}.
\]
Here \( \epsilon_j \in \{0, 1\} \) such that \( n_j \equiv \epsilon_j \pmod{2} \) for \( j = 1, 2 \).

3.4.4. **Explicit computations.** In the following, we prove the assertions listed in \[\text{(3.3.3)}\] We only check case 3 and case 5 as the other cases are similar. We need a simple result.

**Lemma 3.5.** Let \( N \) be a non-negative integer and \( z, w \in \mathbb{C} \). We have
\[
\sum_{\ell=0}^{N} \left( \begin{array}{c} N \\ \ell \end{array} \right) \Gamma(z + \ell) \Gamma(w - \ell) = \frac{\Gamma(z)\Gamma(w - N)\Gamma(z + w)}{\Gamma(z + w - N)}.
\]

**Proof.** Induction on \( N \). \( \square \)

First note that in any case, we have
\[
W(gk(\theta))\Phi((x, y)k(\theta)) = W(g)\Phi(x, y), \quad g \in \text{GL}_2(\mathbb{C}), \quad k(\theta) \in \text{SO}_2(\mathbb{R}) \quad \text{and} \quad x, y \in \mathbb{R}.
\]
Similarly the function \( (W \otimes \omega^{-1})(g)\hat{\Phi}(x, y)g) \) is also right \( \text{SO}_2(\mathbb{R}) \)-invariant. As a consequence, we obtain
\[
Z(s, W, \Phi) = \left( \int_{\mathbb{R}^2} W \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) |y|^{s-1} d^x y \right) \left( \int_{\mathbb{R}^2} \Phi((0, t)\omega(t)t^{2s} d^x t) \right),
\]
\[
(3.4.7)
\]
\[
Z \left(1 - s, W \otimes \omega^{-1}, \hat{\Phi} \right) = \left( \int_{\mathbb{R}^2} W \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) \omega^{-1}(y)|y|^{-s} d^x y \right) \left( \int_{\mathbb{R}^2} \hat{\Phi}(0, t)\omega^{-1}(t)t^{2-2s} d^x t \right).
\]

In our notation
\[
\zeta(s, W, 1) = \int_{\mathbb{R}^2} W \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) |y|^{s-1} d^x y,
\]
\[
(3.4.8)
\]
\[
\zeta(1 - s, W, \omega_0^{-1}) = \int_{\mathbb{R}^2} W \left( \begin{array}{cc} y & 0 \\ 0 & 1 \end{array} \right) \omega^{-1}(y)|y|^{-s} d^x y.
\]
Here 1 means the identity character of \( \mathbb{R}^2 \) and \( \omega_0 := \omega|_{\mathbb{R}^2} \).
Assume $n_0$ is odd and $n_1$ is even. By equations (3.3.14), (3.4.2), (3.4.5) and Lemma 3.5 again, we find that

$$
\zeta(s, W, 1) = \sum_{\ell=0}^{n_0-1} \left( \frac{n_0-1}{\ell} \right) \zeta \left( s, W, (0,2\ell, (n_0-2\ell,0)) \right)
$$

$$
= (2\pi)^{-s+\lambda_1+\lambda_2+n_0/2} \prod_{\ell=0}^{n_0-1} \left( \frac{n_0-1}{\ell} \right) \Gamma \left( \frac{s}{2} + \lambda_1 + \ell \right) \Gamma \left( \frac{s}{2} + \lambda_2 + \frac{n_0}{2} - \ell \right)
$$

$$
= (2\pi)^{-s+\lambda_1+\lambda_2+n_0/2} \Gamma \left( \frac{s}{2} + \lambda_1 + \frac{s}{2} \right) \Gamma \left( \frac{s}{2} + \lambda_2 + \frac{s}{2} \right) \Gamma \left( \frac{s}{2} + \lambda_1 + \lambda_2 + \frac{s}{2} \right).
$$

It then follows from the equation (3.4.7) that

$$
Z(s, W, \Phi) = (2\pi)^{-s+\lambda_1+\lambda_2+n_0/2} \Gamma \left( \frac{s}{2} + \lambda_1 + \frac{s}{2} \right) \Gamma \left( \frac{s}{2} + \lambda_2 + \frac{s}{2} \right) \Gamma \left( \frac{s}{2} + \lambda_1 + \lambda_2 + \frac{s}{2} \right)
$$

$$
\times \left( -2\sqrt{-1} \right) \pi^{-(s+\lambda_1+\lambda_2+1/2)} \Gamma \left( s + \lambda_1 + \lambda_2 + \frac{1}{2} \right)
$$

$$
= \frac{\pi}{2\sqrt{-1}} L_{\text{Gal}} (s, As \pi).
$$

On the other hand, by using equations (3.3.14), (3.4.2), (3.4.5) and Lemma 3.5 again, we obtain

$$
\zeta \left( 1-s, W, \omega_0^{-1} \right) = \left( -\sqrt{-1} \right) \sum_{\ell=0}^{n_0-1} \left( \frac{n_0-1}{\ell} \right) \zeta \left( 1-s, W, (0,2\ell+1, (n_0-2\ell-1,0)) \right), \omega_0^{-1}
$$

$$
= (2\pi)^{-s+n_0/2-\lambda_1-\lambda_2} \prod_{\ell=0}^{n_0-1} \left( \frac{n_0-1}{\ell} \right) \Gamma \left( 1 - \frac{s}{2} - \lambda_2 + \ell \right) \Gamma \left( \frac{n_0 - s}{2} - \lambda_1 - \ell \right)
$$

$$
= (\sqrt{-1}) (2\pi)^{-s+n_0/2-\lambda_1-\lambda_2} \Gamma \left( \frac{1}{2} - \frac{s}{2} \right) \Gamma \left( 1 - \frac{s}{2} - \lambda_2 \right) \Gamma \left( 1 - s - \lambda_1 - \lambda_2 - \frac{s}{2} \right).
$$

Since $\hat{\Phi} = -\Phi$, by using the equation (3.4.7) again, we find that

$$
Z \left( 1-s, W \otimes \omega^{-1}, \hat{\Phi} \right) = (\sqrt{-1}) (2\pi)^{-s+n_0/2-\lambda_1-\lambda_2} \Gamma \left( \frac{1}{2} - \frac{s}{2} \right) \Gamma \left( 1 - \frac{s}{2} - \lambda_2 \right) \Gamma \left( 1 - s - \lambda_1 - \lambda_2 - \frac{s}{2} \right)
$$

$$
\times \left( 2\sqrt{-1} \right) \pi^{-(3/2-s-\lambda_1-\lambda_2)} \Gamma \left( \frac{3}{2} - s - \lambda_1 - \lambda_2 \right)
$$

$$
= \frac{\pi}{2} L_{\text{Gal}} (1-s, As \pi^\vee).
$$

These show the assertions for case 3.

We now consider case 5. Hence we assume $n_0 = 0$ and $n_1$ is odd. By equations (3.3.14), (3.4.2) and (3.4.6), we have

$$
\zeta(s, W, 1) = -\zeta \left( s, W, (0,1, (0,1)) \right) - \zeta \left( s, W, (1,0, (1,0)) \right)
$$

$$
= -2(2\pi)^{-s+\lambda_1+\lambda_2} \Gamma \left( \frac{s+1}{2} + \lambda_1 \right) \Gamma \left( \frac{s+1}{2} + \lambda_2 \right).
$$

Therefore by equation (3.5.7), we find that

$$
Z(s, W, \Phi) = -\frac{\pi}{2} L_{\text{Gal}} (s, As \pi).
$$

Similarly, we have

$$
\zeta \left( 1-s, W, \omega_0^{-1} \right) = -\zeta \left( s, W, (0,1, (0,1)) \right) - \zeta \left( s, W, (1,0, (1,0)) \right), \omega_0^{-1}
$$

$$
= -2(2\pi)^{-1-s-\lambda_1-\lambda_2} \Gamma \left( 1 - \frac{s}{2} - \lambda_1 \right) \Gamma \left( 1 - \frac{s}{2} - \lambda_2 \right),
$$

and hence

$$
Z \left( 1-s, W \otimes \omega^{-1}, \hat{\Phi} \right) = -\frac{\pi}{2} L_{\text{Gal}} (1-s, As \pi^\vee).$$
This proves Lemma 3.2 and hence completes the proof of Theorem 3.1.

4. Twisted Asai local factors

In this section, we consider the twisted Asai local factors. The main result is Theorem 4.1 whose proof uses the functoriality of global Asai transfer proved in [Kri03] and Theorem 4.1. When \( E = F \times F \), Theorem 4.1 is a theorem in [Ike89]. As mentioned by Ikeda in the proof of [Ike92, Lemma 2.2], the result should be true when \( E \) is a field. Following the idea of Ikeda and combine our results in the case \( n = 1 \), we give a detailed proof of it.

4.1. Notation. For \( n \in \mathbb{Z}_{>0} \), let \( \text{GSp}_n \) be the linear algebraic group over \( \mathbb{Q} \) defined by

\[
\text{GSp}_n = \left\{ g \in \text{GL}_{2n} \mid g \left( \begin{array}{cc} 0_n & 1_n \\ -1_n & 0_n \end{array} \right) \cdot g = \nu(g) \left( \begin{array}{cc} 0_n & 1_n \\ -1_n & 0_n \end{array} \right), \nu(g) \in \mathbb{G}_m \right\}.
\]

The map \( \nu : \text{GSp}_n \to \mathbb{G}_m \) is called the scale map. Let \( P_n \) and \( B_n \) be parabolic subgroups of \( \text{GSp}_n \) defined by

\[
P_n = \left\{ \left( \begin{array}{cc} A & * \\ 0_n & \nu^t A^{-1} \end{array} \right) \in \text{GSp}_n \mid A \in \text{GL}_n, \nu \in \mathbb{G}_m \right\},
\]

\[
B_n = \left\{ \left( \begin{array}{cc} A & * \\ 0_n & \nu^t A^{-1} \end{array} \right) \in \text{GSp}_n \mid A \text{ is upper triangular, } \nu \in \mathbb{G}_m \right\}.
\]

Let \( U_n \) and \( N_n \) be the unipotent subgroups of \( P_n \) and \( B_n \), respectively. Then

\[
U_n = \left\{ \left( \begin{array}{cc} 1_n & X \\ 0_n & 1_n \end{array} \right) \in P_n \mid X = \nu X \right\},
\]

\[
N_n = \left\{ \left( \begin{array}{cc} A & * \\ 0_n & \nu^t A^{-1} \end{array} \right) \in B_n \mid A \text{ is unipotent upper triangular, } \nu \in \mathbb{G}_m \right\}.
\]

Let \( M_n \) and \( T_n \) be Levi subgroups of \( P_n \) and \( B_n \) given by

\[
M_n = \left\{ \left( \begin{array}{cc} A & 0_n \\ 0_n & \nu^t A^{-1} \end{array} \right) \in P_n \mid A \in \text{GL}_n, \nu \in \mathbb{G}_m \right\},
\]

\[
T_n = \left\{ \left( \begin{array}{cc} A & 0_n \\ 0_n & \nu^t A^{-1} \end{array} \right) \in B_n \mid A \text{ is diagonal, } \nu \in \mathbb{G}_m \right\}.
\]

Let

\[
K_n = \begin{cases} 
\text{GSp}_n(F) \cap \text{GL}_{2n}(\mathbb{O}_F) & \text{if } F \text{ is nonarchimedean}, \\
\text{GSp}_n(\mathbb{R}) \cap \text{O}_{2n}(\mathbb{R}) & \text{if } F = \mathbb{R}
\end{cases}
\]

be a maximal compact subgroup of \( \text{GSp}_n(F) \). Let \( K_n^1 = \text{Sp}_n(F) \cap K_n \). For \( A \in \text{GL}_n, \nu \in \mathbb{G}_m, \) and \( X \in M_n \) with \( X = \nu X \), let

\[
m(A, \nu) = \left( \begin{array}{cc} A & 0_n \\ 0_n & \nu^t A^{-1} \end{array} \right),
\]

\[
u(X) = \left( \begin{array}{cc} 1_n & X \\ 0_n & 1_n \end{array} \right),
\]

\[
u_-(X) = \left( \begin{array}{cc} 1_n & 0_n \\ X & 1_n \end{array} \right).
\]

4.2. Intertwining operators. Let \( N(T_3) \) be the normalizer of \( T_3 \) in \( \text{GSp}_3 \), and \( W = N(T_3)/T_3 \) be the corresponding Weyl group. By abuse of notation, we write \( w \) for the coset \( wT_3 \in W \). Let \( J_3 \in \text{GL}_3 \) be the anti-diagonal matrix with non-zero entries all equal to 1, and \( w_3 \in W \) be a Weyl element given by

\[
w_3 = \left( \begin{array}{cc} 0_3 & -J_3 \\ J_3 & 0_3 \end{array} \right).
\]

For \( i = 1, 2, 3 \), let \( x_i \) be the character of \( T_3 \) defined by

\[
x_i \left( \left( \begin{array}{cc} t_1 & 0 \\ 0 & t_2 \\ \nu \end{array} \right) \right) = t_i.
\]

The set of simple roots of \( T_3 \) with respect to \( B_3 \) is given by \( \Delta_3 = \{ \alpha_1, \alpha_2, \alpha_3 \} \) with

\[
\alpha_1 = x_1 - x_2, \quad \alpha_2 = x_2 - x_3, \quad \alpha_3 = 2x_3.
\]
For $i = 1, 2, 3$, let $\kappa_{(i)}$ be the embedding of $GL_2$ into $GSp_3$ defined by

$$\kappa_{(1)} : GL_2 \rightarrow GSp_3, \quad g \mapsto \begin{pmatrix} g & 1 \\ t g^{-1} & 1 \end{pmatrix},$$

$$\kappa_{(2)} : GL_2 \rightarrow GSp_3, \quad g \mapsto \begin{pmatrix} g & 1 \\ t g^{-1} & 1 \end{pmatrix},$$

$$\kappa_{(1)} : GL_2 \rightarrow GSp_3, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} (ad - bc) & 0 & 0 & 0 & 0 \\ 0 & (ad - bc) & 0 & 0 & 0 \\ 0 & 0 & a & 0 & b \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & c & 0 & d \end{pmatrix}.$$

Denote $w_{(i)} = \kappa_{(i)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N(T_3)$. Note that the image of $\kappa_{(i)}$ is equal to the centralizer of the connected component of $Ker \alpha_i$ in $GSp_3$.

For $\chi$ be a character of $T_3$. Define $\text{Ind}_{B_3(F)}^{GSp_3(F)}(\chi)$ to be the space consisting of functions $f : GSp_3(F) \rightarrow \mathbb{C}$ which satisfies the following conditions:

- $f$ is right $K_3$-finite.
- For $g \in GSp_3(F)$ and $b \in B_3(F)$, we have
  $$f(bg) = \chi(b)\delta_{B_3}(b)^{1/2}f(g).$$

Here $\delta_{B_3}$ is the modulus character of $B_3$.

For $w \in W$ and $\chi$ a character of $T_3$, we define the intertwining operator

$$I_w : \text{Ind}_{B_3(F)}^{GSp_3(F)}(\chi) \rightarrow \text{Ind}_{B_3(F)}^{GSp_3(F)}(\chi^w),$$

$$I_w f(g) = \int_{N_3(F)\cap wN_3^{-}(F)w^{-1}} f(w^{-1}nh)dn.$$ 

Here $N_3^-$ is the unipotent radical of the opposite Borel subgroup of $B_3$. The Haar measure $dn$ is defined to be the product measure of the self-dual Haar measure with respect to $\psi$ of each coordinate. The integral is absolutely convergent if $\chi$ belongs to some open subset and can be meromorphically continued to all $\chi$.

Note that for $x, y, z, w, u, t \in F$, we have

$$\kappa_{(3)} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} w_{(3)}^{-1} \kappa_{(2)} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} w_{(2)}^{-1} \kappa_{(1)} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} w_{(1)}^{-1} \times \kappa_{(3)} \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix} w_{(3)}^{-1} \kappa_{(2)} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} w_{(2)}^{-1} \kappa_{(3)} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} w_{(3)}^{-1} = u_{-} \begin{pmatrix} x & y & z \\ y & w & u \\ z & u & t \end{pmatrix} m \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w_{3}^{-1}.$$

Therefore, for $\chi$ a character of $T_3$ and $f \in \text{Ind}_{B_3(F)}^{GSp_3(F)}(\chi)$, we have

$$I_{w_3} f = \chi \left( m \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) I_{w_{(3)}} I_{w_{(3)}} I_{w_{(3)}} I_{w_{(3)}} I_{w_{(3)}} I_{w_{(3)}} I_{w_{(3)}} f.$$
Let $\chi$ be a character of $T_3$, $\mu$ be a character of $F^\times$, and $f \in \text{Ind}_{B_3(F)}^{\text{GSp}_3(F)}(\chi)$. Define functions $W_{w(1)} f$, $W_{w(3)} f$, and $Z_{(1)}(s, \mu, W_{w(1)} f, -)$ on $\text{GSp}_3(F)$ by

$$W_{w(1)} f(g) = \int_F f\left( w_{(1)}^{-1} \chi(1) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \psi(-x) dx,$$

$$W_{w(3)} f(g) = \int_F f\left( w_{(3)}^{-1} \chi(3) \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \psi(-x) dx,$$

$$Z_{(1)}(s, \mu, W_{w(1)} f, g) = \int_{F \times} W_{w(1)} f(\kappa(1) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g) \mu(a)|a|^{-1/2} d^\times a.$$ 

The integrals defining $W_{w(1)} f$ and $W_{w(3)} f$ are absolutely convergent for $\chi$ in some open sets and can be meromorphically continued to all $\chi$. The integral defining $Z_{(1)}(s, \mu, W_{w(1)} f, -)$ are absolutely convergent for $\mu$, and $s$ in some open subsets and can be meromorphically continued to all $\mu$ and $s$.

Let $\omega$ be a character of $F^\times$. Define $I(\omega, s)$ be the space consisting of functions $f : \text{GSp}_3(F) \rightarrow \mathbb{C}$ which satisfies the following conditions:

- $f$ is right $K_3$-finite.
- For $g \in \text{GSp}_3(F)$, $A \in \text{GL}_3(F)$, and $\nu \in F^\times$, we have

$$\int f(m(A, \nu) g) = \omega^{-2}(\nu)|\nu|^{-3s-3/2}\omega(\det(A))|\det(A)|^{2s+1} f(g).$$

Note that $I(\omega, s) \subseteq \text{Ind}_{B_3(F)}^{\text{GSp}_3(F)}(\chi_{\omega, s})$, where $\chi_{\omega, s}$ is a character of $T_3$ defined by

$$\chi_{\omega, s} = \omega^{-2}(\nu)|\nu|^{-3s+3/2}\omega(t_1)|t_1|^{2s-2}\omega(t_2)|t_2|^{2s-1}\omega(t_3)|t_3|^{2s}.$$ 

For $f \in I(\omega, s)$, we have $\omega(\nu(g)) I_{w_3} f(g) \in I(\omega^{-1}, 1 - s)$. Let $I_{w_3}^* : I(\omega, s) \rightarrow I(\omega^{-1}, 1 - s) \otimes (\omega^{-1} \circ \nu)$ be the normalized intertwining operator defined by

$$I_{w_3}^* f = \gamma(2s - 2, \omega, \psi) \gamma(4s - 3, \omega^2, \psi) I_{w_3} f.$$

Note that $I_{w_3}^*$ is well-defined except for countably many values of $s$.

We say that a function $(s, g) \mapsto f^{(s)}(g)$ on $\text{GSp}_3(F) \times \mathbb{C}$ is a holomorphic section of $I(\omega, s)$ if $f^{(s)}$ satisfies the following conditions:

- $f^{(s)}(g)$ is right $K_3$-finite.
- For each $s \in \mathbb{C}$, $f^{(s)}(g)$ belongs to $I(\omega, s)$ as a function of $g \in \text{GSp}_3(F)$.
- For each $g \in \text{GSp}_3(F)$, $f^{(s)}(g)$ is holomorphic in $s$.

A function $(s, g) \mapsto f^{(s)}(g)$ on $\text{GSp}_3(F) \times \mathbb{C}$ is called a meromorphic section of $I(\omega, s)$ if there exist a non-zero entire function $\beta$ such that $\beta(s) f^{(s)}(g)$ is a holomorphic section of $I(\omega, s)$. We can define holomorphic sections and meromorphic sections of $\text{Ind}_{B_3(F)}^{\text{GSp}_3(F)}(\chi_{\omega, s})$ in a similar way.

A meromorphic section $f^{(s)}(g)$ of $I(\omega, s)$ is called a good section if it satisfies the following condition:

$$a_{\omega}(2s - 1)^{-1} I_{w_3} f^{(s)}(g)$$

is holomorphic for all $w \in \Omega_3$.

Here $\Omega_3 \subset N(T_3)$ is certain subset defined in [Ike92] page 191 and $a_{\omega}(s)$ is a meromorphic functions defined in [Ike92] page 195 for each $w \in \Omega_3$. By [Ike92] Lemma 1.2], $f^{(s)}(g)$ is a good section of $I(\omega, s)$ if and only if $\omega(\nu(g)) I_{w_3} f^{(s)}(g)$ is a good section of $I(\omega^{-1}, 1 - s)$. In particular, if $f^{(s)}$ is a good section of $I(\omega, s)$, then

$$(4.2.2) L(2s + 1, \omega)^{-1} L(4s, \omega^2)^{-1} f^{(s)}(g)$$

is a holomorphic section of $I(\omega, s)$.

Also, a holomorphic section $f^{(s)}$ of $I(\omega, s)$ is a good section (Cf. [Ike92] Lemma 1.3]).
4.3. Twisted Asai local factors via local zeta integrals. Define algebraic groups over $F$

$$G = \{ (g^{(1)}, g^{(2)}) \in (R_{E/F}\text{GL}_{2/E}) \times \text{GL}_{2/F} \mid \det(g^{(1)}) = \det(g^{(2)}) \},$$

$$U_0 = \{ (u(x), u(y)) \in (R_{E/F}\text{GL}_{2/E}) \times \text{GL}_{2/F} \mid \text{tr}_{E/F}(x) + y = 0 \},$$

$$U' = \{ (u(x), u(y)) \in U_0 \mid y = 0 \},$$

$$\text{GL}_2 = \{ g \in R_{E/F}\text{GL}_{2/E} \mid \det(g) \in \mathbb{G}_m/F \},$$

$$U^o = \{ u(x) \in R_{E/F}\text{GL}_{2/E} \mid \text{tr}_{E/F}(x) = 0 \}.$$

Let

$$K^o = \begin{cases} 
\text{GL}_2^o(F) \cap \text{GL}_2(O_E) & \text{if } F \text{ is nonarchimedean}, \\
\text{GL}_2^o(\mathbb{R}) \cap U_2(\mathbb{R}) & \text{if } F = \mathbb{R} 
\end{cases}$$

be an open compact subgroup of $\text{GL}_2^o(F)$.

Fix $\xi \in E^\times$ such that $\text{tr}_{E/F}(\xi) = 0$. Define embeddings

$$\iota_2, \xi : \text{GL}_2^o(F) \longrightarrow \text{GSp}_2(F)$$

$$(a \ b \ c \ d) \overset{\iota_2, \xi}{\longrightarrow} \begin{pmatrix} a_1 & a_2 & 2b_1 & 2\xi^2b_2 \\ \xi^2a_2 & a_1 & 2\xi^2b_2 & 2\xi^2b_1 \\ c_1/2 & c_2/2 & d_1 & \xi^2d_2 \\ c_2/2 & \xi^{-2}c_1/2 & d_2 & d_1 \end{pmatrix},$$

$$\gamma : \text{GSp}_2(F) \longrightarrow \text{GSp}_3(F)$$

$$(A \ B \ C \ D) \overset{\gamma}{\longrightarrow} \begin{pmatrix} A & 0 & B & 0 \\ 0 & \nu & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

here $a = a_1 + \xi a_2, b = b_1 + \xi b_2, c = c_1 + \xi c_2, d = d_1 + \xi d_2,$ and $\nu = \nu \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right).$ Define $\eta, \eta' \in \text{Sp}_3(\mathbb{Z})$ by

$$\eta = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad \eta' = \eta^\gamma \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}.$$

Note that

$$(4.3.1) \quad \eta_\xi(g)\eta^{-1} = \begin{pmatrix} d_1 & -c_2/2 & c_1/2 & -c_1/2 & -\xi^2d_2 & 0 \\ -2\xi^2b_2 & a_1 & -\xi^2a_2 & \xi^2a_2 & 2\xi^2b_1 & 0 \\ -b' & 0 & a' & 0 & 0 & b' \\ -2b_1 - b' & a_2 & -a_1 + a' & a_1 & 2\xi^2b_2 & b' \\ -d_2 & \xi^2c_1/2 & -c_2/2 & c_2/2 & d_1 & 0 \\ d_1 - d' & -c_2/2 & c_1/2 + c' & -c_1/2 & -\xi^2d_2 & d' \end{pmatrix}.$$
In particular, for $x, y, z \in F, a \in F^\times$, 
\[
\eta z \left( \begin{pmatrix} 1 & 0 \\ z + \xi x & 1 \end{pmatrix}, \begin{pmatrix} a & 0 \\ ay & a^{-1} \end{pmatrix} \right) \eta^{-1} = m \left( \begin{pmatrix} 1 & -x/2 & az/2 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix}, 1 \right) u \left( \begin{pmatrix} -z/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) 
\times u_+ \left( \begin{pmatrix} 0 & 0 & a-1 \\ 0 & -z^2/2 & -ax/2 \\ a-1 & -ax/2 & a^2y + a^2z/2 \end{pmatrix} \right).
\]

(4.3.2)

Let $\pi_1$ and $\pi_2$ be generic irreducible admissible representations of $GL_2(E)$ and $GL_2(F)$ with central characters $\omega_1$ and $\omega_2$, respectively. Put $\omega = \omega_1|_{F^\times} \cdot \omega_2$. Let $f^{(s)}$ be a good section of $I(\omega, s)$, and $W_1 \in \mathcal{W}(\pi_1, \psi_1), W_2 \in \mathcal{W}(\pi_2, \psi_2)$. Define the local zeta integral (Cf. \cite{PSR87} and \cite{Ike90}) 
\[
\Psi(f^{(s)}, W_1, W_2) = \int_{F \times \mathcal{U}_0(F) \backslash G(F)} f^{(s)}(\eta \xi(g)) W_1(g(1)) W_2(g(2)) dg.
\]

In \cite{PSR87} and \cite{Ike90}, it is proved that the local zeta integral is absolutely convergent for $Re(s) > 0$ and can be meromorphically continued to $s \in \mathbb{C}$. Moreover, there exist a meromorphic function $\gamma(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi)$ such that the following functional equation holds
\[
\Psi(I^{(s)}_{\omega} f^{(s)}, W_1, W_2) = \gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) \Psi(f^{(s)}, W_1, W_2).
\]

Note that for $a \in F^\times$, we have
\[
\gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi^a, \xi) = \omega(a)^4 |a|_F^{s-4} \gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi), \\
\gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, a\xi) = \omega(a)^{-2} |a|_F^{4s+2} \gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi).
\]

Assume $F$ is non-archimedean. Let $q$ be the cardinality of the residue field of $F$. By \cite{PSR87} Appendix 3 to §3, the ideal generated over $\mathbb{C}[q^s, q^{-s}]$ by $\Psi(f^{(s)}, W_1, W_2)$ for good sections $f^{(s)}$ and $W_1 \in \mathcal{W}(\pi_1, \psi_1), W_2 \in \mathcal{W}(\pi_2, \psi_2)$ is a fractional ideal of $\mathbb{C}[q^s, q^{-s}]$ containing $1$. Therefore, there is a unique generator of the form $P(q^{-s})$ with $P(X) \in \mathbb{C}[X]$ and $P(0) = 1$. Put
\[
L_{PSR}(s, \omega, \pi_1 \otimes \pi_2) = P(q^{-s})^{-1},
\]
\[
\varepsilon_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) = \gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) L_{PSR}(s, \omega, \pi_1 \otimes \pi_2) L_{PSR}(1 - s, \omega, \pi_1 \otimes \pi_2)^{-1}.
\]

By (4.3.3), $\varepsilon_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) \in \mathbb{C}[q^s, q^{-s}]^\times$.

Assume $F$ is archimedean. One can deduce from the proofs of \cite{Ike90} Proposition 5.1 and the following proposition that there exist a meromorphic function $\alpha(s)$ without zeros such that $\alpha(s)^{-1} Z(f^{(s)}, W_1, W_2)$ is holomorphic for any good section $f^{(s)}$ and $W_1 \in \mathcal{W}(\pi_1, \psi_1), W_2 \in \mathcal{W}(\pi_2, \psi_2)$. As explained in \cite{Ike92} page 228, it follows that, up to holomorphic functions without zeros, there exist a unique meromorphic function $L_{PSR}(s, \omega, \pi_1 \otimes \pi_2)$ without zeros satisfying the following conditions:

- $L_{PSR}(s, \omega, \pi_1 \otimes \pi_2)^{-1} Z(f^{(s)}, W_1, W_2)$ is holomorphic for any good section $f^{(s)}$ and $W_1 \in \mathcal{W}(\pi_1, \psi_1), W_2 \in \mathcal{W}(\pi_2, \psi_2)$.
- For each $s_0 \in \mathbb{C}$, there exist a good section $f^{(s)}$ and $W_1 \in \mathcal{W}(\pi_1, \psi_1), W_2 \in \mathcal{W}(\pi_2, \psi_2)$ such that $L_{PSR}(s, \omega, \pi_1 \otimes \pi_2)^{-1} Z(f^{(s)}, W_1, W_2)$ is non-zero at $s = s_0$.

Put
\[
\varepsilon_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) = \gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) L_{PSR}(s, \omega, \pi_1 \otimes \pi_2) L_{PSR}(1 - s, \omega, \pi_1 \otimes \pi_2)^{-1}.
\]

It is well-defined up to holomorphic function without zeros. By the properties characterizing the $L$-factors and (4.3.3), $\varepsilon_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi)$ is a holomorphic function without zeros.

Following the idea in the proof of \cite{Ike90} Theorem 3, we prove the following proposition.

**Theorem 4.1.** Assume $\pi_2$ is a subquotient of a principal series representation $\text{Ind}_{B(F)}^{GL_2(F)}(\mu_2, v_2, v_2, v_2)$. Then
\[
\gamma_{PSR}(s, \omega, \pi_1 \otimes \pi_2, \psi, \xi) = \omega(4\xi^4)^{-1} |\xi|^4 |\omega|_{F_2}^{2s+1} \gamma_{RS}(s + v_2, \omega, \pi_1 \otimes \mu_2, \psi, \xi) \gamma_{RS}(s - v_2, \omega, \pi_1 \otimes v_2, \psi, \xi)
\]
\[
= \omega(4\xi^4)^{-1} |\xi|^4 |\omega|_{F_2}^{2s+1} \omega_{E/F}(-1) \gamma_{Gal}(s, \omega, \pi_1 \otimes \pi_2, \psi).
\]

Here $\omega_{E/F}$ is the quadratic character of $F^\times$ associated to $E/F$ by local class field theory.
If $\pi_1$ and $\pi_2$ are both unitary, let $\Lambda(\As_1 \otimes \pi_2)$ be the non-negative real number defined in [Ike92 page 228].

**Corollary 4.2.** Assume $\pi_1$ and $\pi_2$ are both unitary and $\Lambda(\As_1 \otimes \pi_2) < 1/2$. If $\pi_2$ is a subquotient of a principal series representation, then

\[ L_{\text{PRS}}(s, \As_1 \otimes \pi_2) = L_{\text{Gal}}(s, \As_1 \otimes \pi_2), \]

\[ \varepsilon_{\text{PRS}}(s, \As_1 \otimes \pi_2, \psi, \xi) = \omega(4\xi^2)^{-1}[4\xi^2]^{2s+1}\omega_{E/F}(-1)\varepsilon_{\text{Gal}}(s, \As_1 \otimes \pi_2, \psi). \]

**Proof.** By [Ike92 Lemma 2.1], for any good section $f(s)$ and $W_1 \in \mathcal{W}(\pi_1, \psi_E)_0$, $W_2 \in \mathcal{W}(\pi_2, \psi)_0$, the local zeta integral defining $Z(f(s), W_1, W_2)$ is absolutely convergent for $\Re(s) > \Lambda(\As_1 \otimes \pi_2)$. Therefore, $L_{\text{PRS}}(s, \As_1 \otimes \pi_2)$ is holomorphic for $\Re(s) > \Lambda(\As_1 \otimes \pi_2)$. Similarly, $L_{\text{PRS}}(1-s, \As_1 \otimes \pi_2^\vee)$ is holomorphic for $\Re(s) < 1 - \Lambda(\As_1 \otimes \pi_2)$. The assumption $\Lambda(\As_1 \otimes \pi_2) < 1/2$ then implies that $L_{\text{PRS}}(s, \As_1 \otimes \pi_2)$ and $L_{\text{PRS}}(1-s, \As_1 \otimes \pi_2^\vee)$ have no common pole. On the other hand, it can be verified directly that $L_{\text{Gal}}(s, \As_1 \otimes \pi_2)$ and $L_{\text{Gal}}(1-s, \As_1 \otimes \pi_2^\vee)$ have no poles for $\Re(s) > \Lambda(\As_1 \otimes \pi_2)$ and $\Re(s) < 1 - \Lambda(\As_1 \otimes \pi_2)$, respectively. The assertion then follows from Theorem 4.1. \hfill $\square$

**Remark 8.** If $E = F \times F$, then Corollary 4.2 holds for any $\pi_1$ and $\pi_2$ by the results of [Rama00].

Assume $F$ is non-archimedean and $E$ is a field. When both $\pi_1$ and $\pi_2$ are supercuspidal representations, Corollary 4.2 can be proved by using a global-to-local argument and the global functoriality of the Asai transfer to GL4 (Cf. The proof of Corollary 1.3). When $\pi_1$ is not supercuspidal and $\pi_2$ is supercuspidal, so far we are not able to prove Corollary 4.2 in this case. Note that the last mentioned case was assumed to be true in the proof of [Rama02 Lemma 7.3].

**Corollary 4.3.** Assume $\omega = 1$. Then $\text{Hom}_{\text{GL}_2(F)}(\pi_1 \otimes \pi_2, \mathbb{C}) \neq 0$ if and only if

\[ \omega_{E/F}(-1)\varepsilon_{\text{Gal}}\left(\frac{1}{2}, \As_1 \otimes \pi_2\right) = 1. \]

Here $\omega_{E/F}$ is the quadratic character of $F^\times$ associated with $E/F$ by local class field theory.

**Proof.** Except when both $\pi_1$ and $\pi_2$ are supercuspidal, the result is proved in [Pra92]. Therefore, we assume both $\pi_1$ and $\pi_2$ are supercuspidal.

Note that

\[ L_{\text{Gal}}(s, \As_1 \otimes \pi_2) = L_{\text{PRS}}(s, \As_1 \otimes \pi_2) = 1, \]

\[ L_{\text{Gal}}(s, \As_1^\vee \otimes \pi_2^\vee) = L_{\text{PRS}}(s, \As_1^\vee \otimes \pi_2^\vee) = 1. \]

By [Gan08 Theorem 1.2], it suffices to prove

\[ \gamma_{\text{PRS}}(s, \As_1 \otimes \pi_2, \psi, \xi) = [4\xi^2]^{2s+1}\omega_{E/F}(-1)\gamma_{\text{Gal}}(s, \As_1 \otimes \pi_2, \psi). \]

Note that the definition of $\varepsilon$-factor in [Gan08] is different from the one defined in this article by $\omega_{E/F}(-1)$ (Cf. Gan08 p. 13]).

Let $E/F$ be a quadratic extension of number fields such that there exist a finite place $v_0$ of $F$ such that $E_{v_0} = E$ and $F_{v_0} = F$. Fix a non-trivial additive character $\psi$ of $\mathbb{A}_F/F$ and an element $\xi \in E^\times$ such that $\text{tr}_{E/F}(\xi) = 0$. By [Sha90 Proposition 5.1], there exist irreducible cuspidal automorphic representations $\pi_1$ and $\pi_2$ of $\text{GL}_2(\mathbb{A}_E)$ and $\text{GL}_2(\mathbb{A}_F)$ respectively such that

- $\pi_{1,v_0} = \pi_1$, $\pi_{2,v_0} = \pi_2$
- $\pi_{1,v}$ and $\pi_{2,v}$ are spherical for any finite place $v \neq v_0$.

Put $\omega = \omega_{\pi_1}|_{\mathbb{A}_F} \cdot \omega_{\pi_2}$. By Theorem 4.1 for each place $v \neq v_0$ of $F$, we have

\[ (4.3.5) \quad \gamma_{\text{PRS}}(s, \As_1 \otimes \pi_2, \psi, \xi) = \omega_{v}(4\xi^2)^{-1}[4\xi^2]^{2s+1}\omega_{E/F}(1-s)\gamma_{\text{Gal}}(s, \As_1 \otimes \pi_2, \psi_v). \]

On the other hand, by [Kri03 Theorem 6.7], the irreducible admissible representation $\As_1 = \otimes_v \As_{1,v}$ is an isobaric automorphic representation of $\text{GL}_4(\mathbb{A}_F)$. Since $\As_1$ is isobaric, it follows from the global functional equation for Rankin-Selberg $L$-functions that the global automorphic $L$-function

\[ L_{\text{Gal}}(s, \As_1 \otimes \pi_2) := \prod_v L_{\text{Gal}}(s, \As_{1,v} \otimes \pi_2) \]
has meromorphic continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$(4.3.6) \quad L_{\text{Gal}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2) = \varepsilon_{\text{Gal}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2)L_{\text{Gal}}(1-s,\mathbb{A} \otimes \pi_1^\vee \otimes \pi_2^\vee).$$

The assertion then follows from $[\text{PSR}87, \S 5], [\text{IK}89, \S 6]$, and the global functional equation for $L_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2)$. This completes the proof.

Now we switch to global setting. Let $F$ be a quadratic extension of number fields. Fix a non-trivial additive character $\psi$ of $\mathbb{A}_F/F$ and $\xi \in E^\times$ with $\text{tr}_{E/F}(\xi) = 0$. Let $\pi_1$ and $\pi_2$ be irreducible cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$ and $GL_2(\mathbb{A}_F)$, respectively. Let $\omega_1$ and $\omega_2$ be the central characters of $\pi_1$ and $\pi_2$, respectively. Put $\omega = \omega_1|_{\mathbb{A}_F^\times} \cdot \omega_2$. Define

$$L_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2) := \prod_v L_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2,v),$$

$$\varepsilon_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2) := \prod_v \varepsilon_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2,v,\psi_v,\xi).$$

In $[\text{PSR}87, \S 5]$ and $[\text{IK}89, \S 6]$, it is proved that $L_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2)$ can be meromorphically continued to $s \in \mathbb{C}$ and satisfies the functional equation

$$(4.3.7) \quad L_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2) = \varepsilon_{\text{PSR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2)L_{\text{PSR}}(1-s,\mathbb{A} \otimes \pi_1^\vee \otimes \pi_2^\vee).$$

**Theorem 4.4.** For each place $v$ of $F$, we have

$$L_{\text{PR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2,v) = L_{\text{Gal}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2,v),$$

$$\varepsilon_{\text{PR}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2,v,\psi_v,\xi) = \omega_v(4\xi^2)^{-1}|4\xi^2|_{F_v}^{-2s+1}\omega_{E_v/F_v}(1-1)\varepsilon_{\text{Gal}}(s,\mathbb{A} \otimes \pi_1 \otimes \pi_2,v,\psi_v).$$

**Proof.** We may assume both $\pi_1$ and $\pi_2$ are unitary. By the result of $[\text{KM}02]$, for any place $v$ of $F$, we have

$$\Lambda(\mathbb{A} \otimes \pi_1 \otimes \pi_2,v) < 1/2.$$

By $[\text{KM}03]$ Theorem 6.7, the irreducible admissible representation $\mathbb{A} \otimes \pi_1 \otimes \pi_2$ is an isobaric automorphic representation of $GL_4(\mathbb{A}_F)$.

Fix a place $v_0$ of $F$. If $\pi_{2,v_0}$ is a subquotient of a principal series representation, then the theorem follows from Corollary 4.1. Assume $v_0$ is a finite place and $\pi_{2,v_0}$ is a supersingular representation. By $[\text{Sha}90]$ Proposition 5.1, there exist an irreducible unitary cuspidal automorphic representation $\sigma$ of $GL_2(\mathbb{A}_F)$ such that

- $\sigma_{v_0} = \pi_{2,v_0}$.
- $\sigma_v$ is spherical for any finite place $v \neq v_0$.

Let $\omega'_1$ be the central character of $\sigma$. Put $\omega' = \omega'_1|_{\mathbb{A}_F^\times} \cdot \omega'_2$. By Theorem 4.1, for each place $v \neq v_0$ of $F$, we have

$$(4.3.8) \quad \gamma_{\text{PR}}(s,\mathbb{A} \otimes \sigma_v,\psi_v,\xi) = \omega_v(4\xi^2)^{-1}|4\xi^2|_{F_v}^{-2s+1}\omega_{E_v/F_v}(1-1)\gamma_{\text{Gal}}(s,\mathbb{A} \otimes \sigma_v,\psi_v).$$

On the other hand, for each finite place $v$ of $F$, the gamma factor $\gamma_{\text{Gal}}(s,\mathbb{A} \otimes \sigma_v,\psi_v)$ is equal to the gamma factor of defined by the Rankin-Selberg local zeta integrals (Cf. $[\text{BH}99]$ and $[\text{JPSS}83]$). Since $\mathbb{A} \otimes \sigma$ is isobaric, it follows from the global functional equation for Rankin-Selberg $L$-functions that the global automorphic $L$-function $L_{\text{Gal}}(s,\mathbb{A} \otimes \sigma) = \prod_v L_{\text{Gal}}(s,\mathbb{A} \otimes \sigma_v)$ has meromorphic continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$(4.3.9) \quad L_{\text{Gal}}(s,\mathbb{A} \otimes \sigma) = \varepsilon_{\text{Gal}}(s,\mathbb{A} \otimes \sigma)L_{\text{Gal}}(1-s,\mathbb{A} \otimes \sigma^\vee).$$

By $[\text{PSR}87]$ with $\sigma$ in place of $\pi_2$, $[4.3.8]$, and $[4.3.9]$, we have

$$\gamma_{\text{PR}}(s,\mathbb{A} \otimes \sigma,\psi_v,\xi) = \omega_v'(4\xi^2)^{-1}|4\xi^2|_{F_v}^{-2s+1}\omega_{E_v/F_v}(1-1)\gamma_{\text{Gal}}(s,\mathbb{A} \otimes \sigma,\psi_v).$$

Since $\Lambda(\mathbb{A} \otimes \sigma,\psi_v) < 1/2$, following the same argument as in the proof of Corollary 4.2, we conclude that

$$L_{\text{PR}}(s,\mathbb{A} \otimes \sigma,\psi_v) = L_{\text{Gal}}(s,\mathbb{A} \otimes \sigma,\psi_v),$$

$$\varepsilon_{\text{PR}}(s,\mathbb{A} \otimes \psi_v,\xi) = \omega_v'(4\xi^2)^{-1}|4\xi^2|_{F_v}^{-2s+1}\omega_{E_v/F_v}(1-1)\varepsilon_{\text{Gal}}(s,\mathbb{A} \otimes \psi_v).$$

This completes the proof.

\[\square\]
4.4. Proof of Theorem 4.1

Proof. The case $E = F \times F$ is proved in [Ike89] Theorem 3. We assume $E$ is a field. For brevity, we denote

$$\iota = t_\xi, \xi = \nu_\xi, \text{and } a(\nu) = \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}_2.$$

Let $f(s)$ be a good section of $I(\omega, s)$, and $W_1 \in \mathcal{W}(\pi_1, \psi_E)$, $W_2 \in \mathcal{W}(\pi_2, \psi)$. Fix a holomorphic section $h_2^{(u)}$ of $B(\mu_2| |_{F_1} \nu_2| |_{F_2})_0$ such that $W_2 = W_\psi h_2^{(u)}$. Denote $W_2^{(u)} = W_\psi h_2^{(u)}$.

If $\pi_1$ is a subquotient of $\text{Ind}_{B(E)}^{GL_2(E)}(\mu_1| |_{F_1} \nu_1| |_{F_2})_0$ such that $W_1 = W_\psi h_2^{(u)}$. Denote $\pi_1^{(u)} = \text{Ind}_{B(E)}^{GL_2(E)}(\mu_1| |_{F_1} \nu_1| |_{F_2})$ and $W_1^{(u)} = W_\psi h_2^{(u)}$.

Let $W_1^{(u)} \in \mathcal{W}(\pi_1^{(u)}| |_{F_1}, \psi_\xi)$ defined by

$$W_1^{(u)}(g) = W_1^{(u)}(a(\xi)ga(\xi)^{-1}).$$

After twisting $\pi_1$ and $\pi_2$ by unramified characters if necessary, we may assume $\mu_i, \nu_i$ are unitary for $i = 1, 2$.

Let $D, D'$ be subsets of $\mathbb{C}^3$ defined by

$$D = \left\{ (s, u_1, u_2) \in \mathbb{C}^3 \mid \text{Re}(s) > 2|\text{Re}(u_1)| + |\text{Re}(u_2)| \right\},$$

$$D' = \left\{ (s, u_1, u_2) \in \mathbb{C}^3 \mid \text{Re}(1 - s) > 2|\text{Re}(u_1)| + |\text{Re}(u_2)| \right\},$$

$$\mathcal{D} = \left\{ (s, u_1, u_2) \in \mathbb{C}^3 \mid \text{Re}(u_2) > 2|\text{Re}(u_1)| + |\text{Re}(u_2) - \frac{1}{2} | \frac{1}{2} \right\}.$$

By [Ike92] Lemma 2.1, the integrals defining $\Psi(f(s), W_1^{(u_1)}, W_2^{(u_2)})$ and $\Psi(f(s), W_1^{(u_1)}, W_2^{(u_2)})$ are absolutely converge for $(s, u_1, u_2) \in D$ and $(s, u_1, u_2) \in D'$, respectively. If furthermore $\text{Re}(u_2) > 0$, a similar estimation shows that the function

$$G(F) \longrightarrow \mathbb{C},$$

$$g = (g_1, g_2) \longmapsto f(s)(\eta(G))W_1^{(u_1)}(g_1)h_2^{(u_2)}(w_1g_2)$$

define an integrable function on $F^\times U_0(F) \setminus G(F)$. Therefore, for $(s, u_1, u_2) \in D$ and $\text{Re}(u_2) > 0$, we have

$$\Psi(f(s), W_1^{(u_1)}, W_2^{(u_2)})$$

$$= \int_{F^\times U_0(F) \setminus G(F)} f(s)(\eta(G))W_1^{(u_1)}(g_1)h_2^{(u_2)}(w_1g_2)dg_2dg_1.$$
respectively. As functions of $g \in \text{GL}_2(F)$ and $(s, u_2) \in \mathbb{C}^2$,

$$F_{f,h_2}^{s,u_2}(-,g_1) \in \mathcal{B}(\nu_2) \left| F \left| s-u_2-1/2, \omega_1 \nu_2^{-1} \right| \right| F \left| s+u_2+1/2 \right|,$$

$$F_{I_{2,3}^s f,h_2}^{s,u_2}(-,g_1) \in \mathcal{B}(\omega_1^{-1} \mu_2^{-1}) \left| F \left| s-u_2-1/2, \mu_2 \right| \right| F \left| s+u_2-1/2 \right|$$

are meromorphic sections. By [Fl93, Appendix, Theorem] and Proposition A, we have

$$\Psi(f(s), W_1^{(u_1)}, W_2^{(u_2)}) = \gamma(2s-2u_2-1, \omega_1 \nu_2^2, \psi) \gamma_{RS}(s-u_2, As \pi_1^{(u_1)} \otimes \nu_2, \psi, \xi)^{-1}$$

$$\times \int_{\text{SL}_2(F) \backslash \text{SL}_2(E)} Z(M F_{f,h_2}^{s,u_2}, \rho(g_1) W_1^{(u_1)}) dg_1,$$

$$\Psi(I_{w_3}^s f(s), W_1^{(u_1)}, W_2^{(u_2)}) = \gamma(-2s-2u_2+1, \omega_1^{-1} \mu_2^{-2}, \psi) \gamma_{RS}(1-s-u_2, As \pi_1^{(u_1)} \otimes \omega_1^{-1} \mu_2^{-1}, \psi, \xi)^{-1}$$

$$\times \int_{\text{SL}_2(F) \backslash \text{SL}_2(E)} Z(M F_{I_{w_3}^s f,h_2}^{s,u_2}, \rho(g_1) W_1^{(u_1)}) dg_1$$

for

$$\text{Re}(s) > 2|\text{Re}(u_1)| + |\text{Re}(u_2)| \text{ and } \text{Re}(u_2) > 0,$$

$$\text{Re}(1-s) > 2|\text{Re}(u_1)| + |\text{Re}(u_2)| \text{ and } \text{Re}(u_2) > 0,$$

respectively. Here $M$ denote the intertwining operator defined in [63.3.7]. Note that the local zeta integrals defining $Z(M F_{f,h_2}^{s,u_2}, \rho(g_1) W_1^{(u_1)})$ and $Z(M F_{I_{w_3}^s f,h_2}^{s,u_2}, \rho(g_1) W_1^{(u_1)})$ are absolutely converge for

$$\text{Re}(1-s+u_2) > 2|\text{Re}(u_1)| \text{ and } \text{Re}(s+u_2) > 2|\text{Re}(u_1)|,$$

respectively. Therefore, we conclude that

$$\Psi(f(s), W_1^{(u_1)}, W_2^{(u_2)}) = \gamma(2s-2u_2-1, \omega_1 \nu_2^2, \psi) \gamma_{RS}(s-u_2, As \pi_1^{(u_1)} \otimes \nu_2, \psi, \xi)^{-1}$$

$$\times \int_{\text{SL}_2(F) \backslash \text{SL}_2(E)} \left( \int_{\text{U}(F) \backslash \text{GL}_2(F)} MF_{f,h_2}^{s,u_2}(g, g_1) W_1^{(u_1)}(gg_1) dg \right) dg_1,$$

$$\Psi(I_{w_3}^s f(s), W_1^{(u_1)}, W_2^{(u_2)}) = \gamma(-2s-2u_2+1, \omega_1^{-1} \mu_2^{-2}, \psi) \gamma_{RS}(1-s-u_2, As \pi_1^{(u_1)} \otimes \omega_1^{-1} \mu_2^{-1}, \psi, \xi)^{-1}$$

$$\times \int_{\text{SL}_2(F) \backslash \text{SL}_2(E)} \left( \int_{\text{U}(F) \backslash \text{GL}_2(F)} MF_{I_{w_3}^s f,h_2}^{s,u_2}(g, g_1) W_1^{(u_1)}(gg_1) dg \right) dg_1$$

for

$$\text{Re}(u_2) > 0 \text{ and } 2|\text{Re}(u_1)| < \text{Re}(s-u_2) < 1 - 2|\text{Re}(u_1)|,$$

$$\text{Re}(u_2) > 0 \text{ and } 2|\text{Re}(u_1)| < \text{Re}(1-s-u_2) < 1 - 2|\text{Re}(u_1)|,$$

respectively.

Define functions $\mathcal{F}_{f,h_2}^{s,u_2}$ and $\mathcal{F}_{I_{w_3}^s f,h_2}^{s,u_2}$ on $\text{GL}_2(F)$ by

$$\mathcal{F}_{f,h_2}^{s,u_2}(g) = MF_{f,h_2}^{s,u_2}(a(\nu(g)), a(\nu(g))^{-1} a(\xi)^{-1} ga(\xi)),$$

$$\mathcal{F}_{I_{w_3}^s f,h_2}^{s,u_2}(g) = MF_{I_{w_3}^s f,h_2}^{s,u_2}(a(\nu(g)), a(\nu(g))^{-1} a(\xi)^{-1} ga(\xi)),$$

for $g \in \text{GL}_2(F)$. Put

$$\mathcal{F}_{f,h_2}^{s,u_2}(g) = \int_F \mathcal{F}_{f,h_2}^{s,u_2}(u(x)g) \psi(2x) dx,$$

$$\mathcal{F}_{I_{w_3}^s f,h_2}^{s,u_2}(g) = \int_F \mathcal{F}_{I_{w_3}^s f,h_2}^{s,u_2}(u(x)g) \psi(2x) dx,$$
for \( g \in \text{GL}_2^0(F) \). By (4.4.1), formally we have

\[
\Psi(f^{(s)}, W_1^{(u_1)}, W_2^{(u_2)}) = \gamma(2s - 2u_2 - 1, \omega_1 \nu_2^2, \psi) \gamma_{\text{RS}}(s - u_2, A_1 \pi_1^{(u_1)} \otimes \nu_2, \psi, \xi)^{-1} \\
\times \int_{F \times U(E) \backslash \text{GL}_2^0(F)} \mathcal{F}_{f,h_2}(g) W_1^{(u_1)}(g) \, dg,
\]

(4.4.3)

\[
\psi(I_{w_3}^{(s)} f^{(s)}, W_1^{(u_1)}, W_2^{(u_2)}) = \gamma(-2s - 2u_2 + 1, \omega_1^{-1} \mu_2^{-2}, \psi) \gamma_{\text{RS}}(1 - s - u_2, A_1 \pi_1^{(u_1)} \otimes \omega_1^{-1} \mu_2^{-1}, \psi, \xi)^{-1} \\
\times \int_{F \times U(E) \backslash \text{GL}_2^0(F)} \mathcal{F}_{I_{w_3}^{(s)} f,h_2}(g) W_1^{(u_1)}(g) \, dg.
\]

In the following we shall prove that \( \mathcal{F}_{f,h_2} \) and \( \mathcal{F}_{I_{w_3}^{(s)} f,h_2} \) can be meromorphically continued to \((s, u_2) \in \mathbb{C}^2 \) and satisfying functional equation (4.4.4). Then we will prove that the integrals in (4.4.3) are both absolutely converge for \((s, u_1, u_2) \in \mathcal{D} \).

Fix a compact subset \( \Omega \subseteq \text{GL}_2^0(F) \) consisting of a set of representatives of the image of the map

\[
F^x \backslash E^x \times K^o \longrightarrow F^x \backslash \text{GL}_2^0(F) \\
(F^x a, k) \longmapsto F^x \left( \begin{array}{cc}
a & 0 \\
0 & a^{-1}
\end{array} \right) k.
\]

Note that when \( F = \mathbb{R} \), we can and will assume \( \nu_2(\Omega) \subseteq K_2 \).

Define functions \( G_{f,h_2}^{(s,u_2)} \) and \( G_{I_{w_3}^{(s)} f,h_2}^{(s,u_2)} \) on \( \text{GSp}_2(F) \) by

\[
G_{f,h_2}^{(s,u_2)}(g) = \int_{F \backslash \text{SL}_2(F)} f^{(s)} \left( u - \left( \begin{array}{cc} 0 & 0 \\
0 & x \\
\end{array} \right) \right) \eta'(1, g_2) \gamma(g) \, h_2^{(u_2)}(w_1 g a(\nu(g))) \, dg \, dx,
\]

\[
G_{I_{w_3}^{(s)} f,h_2}^{(s,u_2)}(g) = \int_{F \backslash \text{SL}_2(F)} I_{w_3}^{(s)} f^{(s)} \left( u - \left( \begin{array}{cc} 0 & 0 \\
0 & x \\
\end{array} \right) \right) \eta'(1, g_2) \gamma(g) \, h_2^{(u_2)}(w_1 g a(\nu(g))) \, dg \, dx.
\]

Note that the integrals defining \( G_{f,h_2}^{(s,u_2)} \) and \( G_{I_{w_3}^{(s)} f,h_2}^{(s,u_2)} \) are absolutely converge for

\[
\text{Re}(s) > 0, \text{Re}(s - u_2) > \frac{1}{2}, \text{ and } \text{Re}(s + u_2) > -1,
\]

\[
\text{Re}(1 - s) > 0, \text{Re}(1 - s - u_2) > \frac{1}{2}, \text{ and } \text{Re}(1 - s + u_2) > -1,
\]

respectively. By the \( K_3 \)-finiteness of \( f^{(s)} \) and (4.3.2), we have the following properties:

- \( G_{f,h_2}^{(s,u_2)} \) and \( G_{I_{w_3}^{(s)} f,h_2}^{(s,u_2)} \) are right \( K_2 \)-finite.

- For \( g \in \text{GSp}_2(F) \) and \( p = \left( \begin{array}{ccc} a_1 & 0 & * \\
* & a_2 & * \\
0 & 0 & \nu a_1^{-1} \end{array} \right) \), we have

\[
G_{f,h_2}^{(s,u_2)}(pg) = \nu_2 \omega_1^{-1}(\nu) \nu_{\nu}^{-s - u_2 - 1}(a_1) a_1 [2] \mu_2 \nu_2^{-1}(a_2) [2] \nu_{\nu}^{2u_2 + 2} G_{f,h_2}^{(s,u_2)}(g),
\]

\[
G_{I_{w_3}^{(s)} f,h_2}^{(s,u_2)}(pg) = \nu_2(\nu) \nu_{\nu}^{-1}(1 - s - u_2 - 1)(a_1) a_1 [2] \mu_2 \nu_2^{-1}(a_2) [2] \nu_{\nu}^{2u_2 + 2} G_{I_{w_3}^{(s)} f,h_2}^{(s,u_2)}(g).
\]
By \textbf{[Ike89]},
\begin{align*}
G_{f,h_2}(s,u_2) &= \int_{K_1} h_2^{(u_2)}(w_1 ka(\nu(g))) \int_{F} \int_{F} \omega_1 \nu_2(a) |a|^2 s - 2 u_2 - 1 \nonumber \\
& \quad \times f(s) \left( u_- \left( \begin{array}{ccc}
0 & 0 & a \\
0 & x & y \\
a & x & y
\end{array} \right) \right) \eta'(1,k) \gamma(g) \ dy dx d^\infty adk, \nonumber \\
G_{I_{w_1}^*,f,h_2}(s,u_2) &= \int_{K_1} h_2^{(u_2)}(w_1 ka(\nu(g))) \int_{F} \int_{F} \omega_1^{-1} \mu_2^{-2}(a) |a|^2 (1-s) - 2 u_2 - 1 \nonumber \\
& \quad \times I_{w_1}^* f(s) \left( u_- \left( \begin{array}{ccc}
0 & 0 & a \\
0 & x & y \\
a & x & y
\end{array} \right) \right) \eta'(1,k) \gamma(g) \ dy dx d^\infty adk. \nonumber 
\end{align*}

Note that for $x,y,a \in F$,
\begin{align*}
\left( \begin{array}{ccc} 0 & 0 & a \\ 0 & x & y \\ a & x & y \end{array} \right) &= \kappa(3) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right) \left( \begin{array}{ccc} w_3^{-1} \kappa(2) & w_3^{-1} \kappa(2) & w_3^{-1} \kappa(2) \\ w_3^{-1} \kappa(1) & w_3^{-1} \kappa(1) & w_3^{-1} \kappa(1) \\ w_3^{-1} \kappa(1) & w_3^{-1} \kappa(1) & w_3^{-1} \kappa(1) \end{array} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array} \right)
\end{align*}

and
\begin{align*}
I_{w_1} f(s) \in B(\omega | \int_{F} |^{2s-4}, \omega | \int_{F} |^{2s-1/2})_0, \\
I_{w_1}^* I_{w_3} f(s) \in B(\omega | \int_{F} |^{2s-2}, \omega | \int_{F} |^{2s-5/2})_0
\end{align*}

are meromorphic sections. Therefore, by \textbf{[Ike89]} Lemma 5.1, we have
\begin{align*}
G_{f,h_2}(s,u_2) &= \gamma(2s - 2 u_2 - 1, \omega_1 \nu_2, \psi)^{-1} \int_{K_1} h_2^{(u_2)}(w_1 ka(\nu(g))) \nonumber \\
& \quad \times Z(1) \left( 2 u_1 + \frac{5}{2} \mu_2 \nu_2 , W_{w_1} I_{w_1} f(s), w_1 w_2 w_3 u_- \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right) \right) \eta'(1,k) \gamma(g) 
\end{align*}

and
\begin{align*}
G_{I_{w_1}^*,f,h_2}(s,u_2) &= \gamma(-2s - 2 u_2 + 1, \omega_1^{-1} \mu_2^{-2}, \psi)^{-1} \int_{K_1} h_2^{(u_2)}(w_1 ka(\nu(g))) \nonumber \\
& \quad \times Z(1) \left( 2 u_1 + \frac{5}{2} \mu_2 \nu_2 , W_{w_1} I_{w_1}^* I_{w_3} f(s), w_1 w_2 w_3 u_- \left( \begin{array}{ccc} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right) \right) \eta'(1,k) \gamma(g) 
\end{align*}

In particular, both $G_{f,h_2}$ and $G_{I_{w_1}^*,f,h_2}$ can be meromorphically continued to $(s,u_2) \in \mathbb{C}^2$.

By an argument similar to \textbf{[Ike89]} page 209–212, we have
\begin{equation}
\mathcal{F}_{f,h_2}(s,u_2) = \omega_1(-1)|2 \xi^{2}|_{F} G_{f,h_2}^{(s,u_2)} \left( \begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{array} \right) \eta_2(a(\xi) w_1 a(\xi)^{-1} g) \right),
\end{equation}

(4.4.4)

\begin{equation}
\mathcal{F}_{I_{w_1}^*,f,h_2}(s,u_2) = \omega_1(-1)|2 \xi^{2}|_{F} G_{I_{w_1}^*,f,h_2}^{(s,u_2)} \left( \begin{array}{ccc} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{array} \right) \eta_2(a(\xi) w_1 a(\xi)^{-1} g) \right).
\end{equation}

By \textbf{[Ike89]} (5.2.11) and (5.2.12) and the properties of $G_{f,h_2}$ and $G_{I_{w_1}^*,f,h_2}$, we deduce that
\begin{align*}
|\mathcal{F}_{f,h_2}(a(\nu) u(x) g)| & \ll_{s,u_2} |\nu|^2_{F} \Re (u_2 - s + 1) N(2 \xi^{2}, 2 - 1 \xi^{2}, \xi^{2} - 2 x) \Re (a(\nu) u_2) N(2 \xi^{2}, 2 - 1 \xi^{2}, \xi^{2} - 2 x) \Re (s - u_2) \Re (s - 1)
\end{align*}
for \( g \in \Omega, \nu \in F^x, \) and \( x \in F \). Therefore, the integrals

\[
\int_{F^x U^o(F) \setminus \text{GL}_2^o(F)} F_{f,h_2}^{(s,u_2)}(g) W_1^{(u_1)}(g) dg \quad \text{and} \quad \int_{F^x U^o(F) \setminus \text{GL}_2^o(F)} F_{t_{u_3} f,h_2}^{(s,u_2)}(g) W_1^{(u_1)}(g) dg,
\]

are absolutely converge for

\[
\text{Re}(s) > \frac{1}{2}, \text{Re}(u_2) > 2|\text{Re}(u_1)| + \text{Re}(s) - 1,
\]

\[
\text{Re}(1 - s) > \frac{1}{2}, \text{Re}(u_2) > 2|\text{Re}(u_1)| + \text{Re}(1 - s) - 1,
\]

respectively.

Define functions \( \overline{G}_{f,h_2}^{(s,u_2)} \) and \( \overline{G}_{t_{u_3} f,h_2}^{(s,u_2)} \) on \( \text{GSp}_2(F) \) by

\[
\overline{G}_{f,h_2}^{(s,u_2)}(g) = \int_F G_{f,h_2}^{(s,u_2)} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} u \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} g \psi(-x) dx,
\]

\[
\overline{G}_{t_{u_3} f,h_2}^{(s,u_2)}(g) = \int_F G_{t_{u_3} f,h_2}^{(s,u_2)} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} u \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} g \psi(-x) dx.
\]

By the properties of \( \overline{G}_{f,h_2}^{(s,u_2)} \) and \( \overline{G}_{t_{u_3} f,h_2}^{(s,u_2)} \), we have the following properties.

- \( \overline{G}_{f,h_2}^{(s,u_2)} \) is right \( K_2 \)-finite.
- For \( k_2 \in K_2 \), as functions in \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(F) \), we have

\[
\overline{G}_{f,h_2}^{(s,u_2)} \begin{pmatrix} a & 0 & b & 0 \\ 0 & \text{det}(g) & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} k_2 \in \mathcal{W}(\mu_2 | F^{s+u_2+1/2}, \mu_2 \omega^{-1} | F^{-s+u_2+3/2})_0,
\]

\[
\overline{G}_{t_{u_3} f,h_2}^{(s,u_2)} \begin{pmatrix} a & 0 & b & 0 \\ 0 & \text{det}(g) & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} k_2 \in \mathcal{W}(\mu_2 \omega^{-1} | F^{-s+u_2+3/2}, \mu_2 | F^{s+u_2+1/2})_0.
\]

By \( \text{[3.3.2]} \),

\[
\overline{G}_{f,h_2}^{(s,u_2)}(g) = \int_{k \in K_1^u} h_2^{(u_2)}(w_1 k a(\nu(g))) \int_{F^x} \int_F \int_F \int_F \psi(z) \omega_1 \nu_2^2(a) |a|_{F^{s-u_2}}^{2s-2u_2-1} f^{(s)}(u) \begin{pmatrix} 0 & 0 & a-1 \\ 0 & z & x \\ a-1 & x & y \end{pmatrix} \eta'(1,k) w(1) \gamma(g) dz dy dx \operatorname{adk},
\]

\[
\overline{G}_{t_{u_3} f,h_2}^{(s,u_2)}(g) = \int_{k \in K_1^u} h_2^{(u_2)}(w_1 k a(\nu(g))) \int_{F^x} \int_F \int_F \int_F \psi(z) \omega_1^{-1} \mu_2^{-2} |a|_{F^{1-s}}^{2(1-s)-2} f^{(s)}(u) \begin{pmatrix} 0 & 0 & a-1 \\ 0 & z & x \\ a-1 & x & y \end{pmatrix} \eta'(1,k) w(1) \gamma(g) dz dy dx \operatorname{adk}.
\]
Note that for $x, y, z, a \in F$,
\[
  u_-(\begin{pmatrix} 0 & 0 & a \\ 0 & z & x \\ a & x & y \end{pmatrix}) = \kappa_{(1)} \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) w_{(3)}^{-1} \kappa_{(2)} \left( \begin{pmatrix} 1 & 0 \\ 1 & x \end{pmatrix} \right) w_{(2)}^{-1} \kappa_{(3)} \left( \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \right) \times w_{(1)}^{-1} \kappa_{(1)} (u(-a)) w_{(1)} w_{(2)} w_{(3)}.
\]

Therefore, by [Ike89] Lemma 5.1, we have
\[
  G_{f,h_2}^{(s,u_2)}(g) = \gamma(2s - 2u_2 - 1, \omega_1 \nu_2, \psi)^{-1} \int_{\mathcal{K}_I^1} h_{(u_2)}^{(u_2)}(w_1 \kappa a(\nu(g)))
  Z(1) \left( 2u_2 + \frac{5}{2}, \mu_2 \nu_2^{-1}, W_{w(1)} W_{w(3)} I_{w(2)} I_{w(3)} f^{(s)}, \right.
  w_{(3)} w_{(1)} w_{(2)} w_{(3)} u_-(\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}) \eta'(1,k) w_{(1)} \gamma(g) \right) dk;
\]
\[
  G_{I_{w_3} f,h_2}^{(s,u_2)}(g) = \gamma(-2s - 2u_2 + 1, \omega_1^{-1} \mu_2^{-2}, \psi)^{-1} \int_{\mathcal{K}_I^1} h_{(u_2)}^{(u_2)}(w_1 \kappa a(\nu(g)))
  Z(1) \left( 2u_2 + \frac{5}{2}, \mu_2 \nu_2^{-1}, W_{w(1)} W_{w(3)} I_{w(2)} I_{w(3)} I_{w_3}^* f^{(s)}, \right.
  w_{(3)} w_{(1)} w_{(2)} w_{(3)} u_-(\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}) \eta'(1,k) w_{(1)} \gamma(g) \right) dk.
\]

In particular, both $G_{f,h_2}^{(s,u_2)}$ and $G_{I_{w_3} f,h_2}^{(s,u_2)}$ can be meromorphically continued to $(s,u_2) \in \mathbb{C}^2$. By [Ike89] and proceeding as in [Ike89] page 215, we have
\[
  W_{w(1)} W_{w(3)} I_{w(2)} I_{w(3)} f^{(s)} = \omega(-1) W_{w(1)} W_{w(3)} I_{w(2)} I_{w(3)} I_{w_3}^* f^{(s)}.
\]

If follows that
\[
  G_{I_{w_3} f,h_2}^{(s,u_2)} = \omega(-1) \gamma(2s - 2u_2 - 1, \omega_1 \nu_2, \psi) \gamma(-2s - 2u_2 + 1, \omega_1^{-1} \mu_2^{-2}, \psi)^{-1} G_{f,h_2}^{(s,u_2)}.
\]

Note that for $x \in F$, we have
\[
  \left( \begin{array}{ccc} 0 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 0 \end{array} \right) \tau_2(a(\xi) w_1 a(\xi)^{-1} u(x)) = m \left( \begin{array}{ccc} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{array} \right) u \left( \begin{array}{ccc} 0 & 0 & 2 \xi - 2 \\ 0 & 0 & \xi - 2 \\
-1 & 0 & 0 \end{array} \right) m \left( \begin{array}{ccc} 2 \xi^2 & 0 & 2 \xi - 2 \\ 0 & -2^{-1} \xi - 2 \\ 0 & 0 \end{array} \right) , -1
\]
\[
  \times \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{array} \right) u \left( \begin{array}{ccc} -2 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right) m \left( \begin{array}{ccc} 1 & 0 \\ 0 & -1 \end{array} \right) , -1 \right).
\]

Therefore, for $g \in \text{GL}_2^0(F)$
\[
  G_{f,h_2}^{(s,u_2)}(g) = \left| 2^{|F^*|} |F|^{-1} \right| 2 \xi^2 |F|^{-2s-2u_2-1} \mu_2 (-1) \omega_1 \nu_2 \mu_2^{-1} \nu_2 (2 \xi^2) G_{f,h_2}^{(s,u_2)} \left( m \left( \begin{array}{ccc} 1 & 0 \\ 0 & 1 \end{array} \right) , -1 \right) , -1 \right) \tau_2(g) \right),
\]
\[
  G_{I_{w_3} f,h_2}^{(s,u_2)}(g) = \left| 2^{|F^*|} |F|^{-1} \right| 2 \xi^2 |F|^{-2s-2u_2-1} \omega_1 \mu_2 (-1) \omega_1^{-1} \mu_2^{-1} \nu_2 (2 \xi^2) G_{I_{w_3} f,h_2}^{(s,u_2)} \left( m \left( \begin{array}{ccc} 1 & 0 \\ 0 & -1 \end{array} \right) , -1 \right) \tau_2(g) \right).
\]

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By (4.4.5) and (4.4.6), we have

\[ \mathcal{F}_{f,h_2}^{(s,u_2)} = \omega(4\xi^4)^{-1}(4\xi^4)^{-2s+1}\gamma(2s-2u_2+1,\omega_1\mu_2^{-2},\psi)\gamma(-2s-2u_2+1,\omega_1^{-1}\mu_2^{-2},\psi)^{-1}\mathcal{F}_{f,h_2}. \]

By the properties of \( \mathcal{G}_{f,h_2}^{(s,u_2)} \) and \( \mathcal{F}_{f,h_2}^{(s,u_2)} \), and (4.4.6), for each \( \epsilon > 0 \) there exist \( \varphi \in S(F) \) depending on \( \epsilon, s \) and \( u_2 \) such that

\[ \mathcal{F}_{f,h_2}^{(s,u_2)}(a(\nu)g) \ll_{s,u_2} |\nu|_F^{u_2+3/2-|\Re(s)-1/2|-\epsilon}\varphi(\nu), \]

\[ \mathcal{F}_{f,h_2}^{(s,u_2)}(a(\nu)g) \ll_{s,u_2} |\nu|_F^{u_2+3/2-|\Re(s)-1/2|-\epsilon}\varphi(\nu) \]

for \( g \in \Omega \) and \( \nu \in F^\times \). Therefore, the integrals

\[ \int_{F \times U(E) \backslash GL_2(F)} \mathcal{F}_{f,h_2}^{(s,u_2)}(g)W_1^{(u_1)}(g)dg, \]

\[ \int_{F \times U(E) \backslash GL_2(F)} \mathcal{F}_{f,h_2}^{(s,u_2)}(g)W_1^{(u_1)}(g)dg \]

are both absolutely convergent for \((s, u_1, u_2) \in \mathcal{D}\). It follows that (4.4.3) holds for \((s, u_1, u_2) \) in the nonempty open set \( D \cap D^\vee \cap \mathcal{D} \). By [F93, Appendix, Theorem] and Theorem 3.1-(1),

\[ L_{RS}(s+u_2, As\pi_1^{(u_1)} \otimes \mu_1) L_{RS}(s-u_2, As\pi_1^{(u_1)} \otimes \nu_2) \]

and

\[ L_{RS}(1-s-u_2, As\pi_1^{(u_1)} \otimes \omega_1^{-1}\mu_2^{-1}) L_{RS}(1-s+u_2, As\pi_1^{(u_1)} \otimes \omega_1^{-1}\nu_2^{-1}) \]

are non-zero and has no poles in \( D \) and \( D^\vee \), respectively. Therefore, by the uniqueness of the analytic continuation, (4.3.3), (4.4.3), and (4.4.7), we conclude that both

\[ \Psi((s), W_1^{(u_1)}, W_2^{(u_2)}) \]

\[ L_{RS}(1-s-u_2, As\pi_1^{(u_1)} \otimes \omega_1^{-1}\mu_2^{-1}) L_{RS}(1-s+u_2, As\pi_1^{(u_1)} \otimes \omega_1^{-1}\nu_2^{-1}) \]

and

\[ \Psi((s), W_1^{(u_1)}, W_2^{(u_2)}) \]

\[ L_{RS}(s+u_2, As\pi_1^{(u_1)} \otimes \mu_2) L_{RS}(s-u_2, As\pi_1^{(u_1)} \otimes \nu_2) \]

can be analytically continued to \( D \cup D^\vee \), and for \((s, u_1, u_2) \in D \cup D^\vee \)

\[ \Psi((s), W_1^{(u_1)}, W_2^{(u_2)}) \]

\[ L_{RS}(s+u_2, As\pi_1^{(u_1)} \otimes \mu_2) L_{RS}(s-u_2, As\pi_1^{(u_1)} \otimes \nu_2) \]

\[ \Psi((s), W_1^{(u_1)}, W_2^{(u_2)}) \]

\[ L_{RS}(1-s-u_2, As\pi_1^{(u_1)} \otimes \omega_1^{-1}\mu_2^{-1}) L_{RS}(1-s+u_2, As\pi_1^{(u_1)} \otimes \omega_1^{-1}\nu_2^{-1}) \]

(4.4.8)

\[ = \omega(4\xi^4)^{-1}(4\xi^4)^{-2s+1}\epsilon(s+u_2, As\pi_1^{(u_1)} \otimes \rho_2, \psi, \xi, \xi)\epsilon(s-u_2, As\pi_1^{(u_1)} \otimes \nu_2, \psi, \xi) \]

\[ \times L_{RS}(s+u_2, As\pi_1^{(u_1)} \otimes \mu_2) L_{RS}(s-u_2, As\pi_1^{(u_1)} \otimes \nu_2). \]

Note that \( \epsilon(s+u_2, As\pi_1^{(u_1)} \otimes \rho_2, \psi, \xi, \xi) = \omega(4\xi^4)^{-1}(4\xi^4)^{-2s+1}\gamma_{RS}(s+u_2, As\pi_1^{(u_1)} \otimes \nu_2, \psi, \xi)\gamma_{RS}(s-u_2, As\pi_1^{(u_1)} \otimes \nu_2, \psi, \xi). \)

The second assertion follows from Corollary 2.2 and Theorem 3.1. This completes the proof. \( \square \)

**APPENDIX**

Let \( F \) be a local field of characteristic zero and \( E/F \) be a quadratic field extension. Fix an element \( \xi \in E^\times \) such that \( \text{tr}_{F/E}(\xi) = 0 \). Let \( \psi \) be a non-trivial additive character of \( F \).

Let \( \mu, \nu \) be two characters of \( E^\times \). Let \( \pi = \text{Ind}_{B(E)}^{GL_2(E)}(\mu, \nu) \) be an infinite-dimensional irreducible Harish-Chandra representation of \( GL_2(E) \) if \( F = \mathbb{R} \), and a smooth admissible representation of \( GL_2(E) \) if \( F \) is non-archimedean. Let \( W \in \mathcal{W}(\pi, \psi_\xi) \) and \( \Phi \in \mathcal{S}(F^2, \psi) \). Here \( \mathcal{S}(F^2, \psi) = \mathcal{S}(F^2) \) if \( F \) is non-archimedean.

It is easy to verify that

\[ Z(s, W, \Phi) = \int_{\mathbb{R} \times U(\mathbb{R}) \backslash GL_2(\mathbb{R})} W(g)\mathcal{F}_{\Phi}^{(s)}(g)dg, \]
where \( f^{(s)}_\Phi \) is the Godement section associated to \( \Phi \in \mathcal{S}(F^2, \psi) \) defined by the integral
\[
f^{(s)}_\Phi(g) = |\det(g)|^s \int_{F^\times} \Phi((0, t)g)\omega_0(t)|t|_F^{2s} d^x t.
\]
Recall that
\[
f^{(s)}_\Phi \in I(s, \omega_0)_0 \quad \text{with} \quad I(s, \omega) = \mathcal{B} \left( |s-1/2|, \omega_0^{-1} | |s-1/2| \right).
\]
Here \( \omega_0 = \mu \nu|_{\mathbb{R}^s} \).

In this appendix, we show that the zeta integrals have meromorphic continuations to the whole complex plane, and satisfy the functional equation even when we replace \( f^{(s)}_\Phi \) by a holomorphic section \( f^{(s)} \) of \( I(s, \omega_0) \).

By a holomorphic section of \( I(s, \omega_0) \) we mean a function \( f^{(s)}(g) : GL_2(F) \times \mathbb{C} \to \mathbb{C} \) which satisfies following two conditions:
- for each \( s \in \mathbb{C}, f^{(s)}(g) \in I(s, \omega_0) \).
- for each \( g \in GL_2(F) \), \( f^{(s)}(g) \) is a holomorphic function in \( s \).

Notice that we do not require \( f^{(s)} \) to be right \( O_2(\mathbb{R}) \)-finite when \( F = \mathbb{R} \). Let \( W \in \mathcal{W}(\pi, \psi) \) and \( f^{(s)} \) be a holomorphic section of \( I(s, \omega_0) \). Define
\[
Z(f^{(s)}, W) = \int_{F^x U(F) \backslash GL_2(F)} W(g)f^{(s)}(g)dg.
\]

Analogous to \( \text{[3.3.7]} \) we define the intertwining operator
\[
M^* : \mathcal{B} \left( |s-1/2|, \omega_0^{-1} | |s-1/2| \right) \to \mathcal{B} \left( \omega_0^{-1} | |1/2-s|, |s-1/2| \right)
\]
by
\[
M f^{(s)}(g) := \int_F f^{(s)} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx \quad \text{and} \quad M^* := \omega_0(-1)\gamma(2s - 1, \omega_0, \psi) M.
\]
The integral converges absolutely for \( \Re(s) \gg 0 \) and admits a meromorphic continuation to whole complex plane. Moreover, \( M^* f^{(s)} \) is a holomorphic section.

The purpose of this appendix is to prove following results, which are used in \( \S 4 \).

**Proposition A.** Write
\[
\mu = \chi_1 | |\lambda_1, \nu = \chi_2 | |\lambda_2\in \mathbb{C}.
\]
for some unitary characters \( \chi_1, \chi_2 \) of \( E^\times \) and some \( \lambda_1, \lambda_2 \in \mathbb{C} \).

Let \( W \in \mathcal{W}(\pi, \psi) \) and \( f^{(s)} \) be a holomorphic section of \( I(s, \omega_0) \). The integral \( Z(f^{(s)}, W) \) converges absolutely when \( \Re(s) > 2 \max \{ -\Re(\lambda_1), -\Re(\lambda_2) \} \) and has a meromorphic continuation to the whole complex plane. Moreover, it satisfies the functional equation
\[
Z(M^* f^{(s)}, W) = \gamma_{BS} (s, \pi, \psi, \xi, M) Z(f^{(s)}, W).
\]

**Proof.** When \( F \) is non-archimedean, the assertions are easy to prove. Since any holomorphic section of \( I(s, \omega_0) \) is right \( GL_2(O_F) \)-finite.

Assume \( F = \mathbb{R} \). Let \( \chi \) be a character of \( \mathbb{R}^\times \). For \( W \in \mathcal{W}(\pi, \psi) \), let \( \zeta(s, W, \chi) \) be the integral defined by the equation \( \text{[3.3.13]} \). We first show that the second assertion of Lemma \( \text{[3.3]} \) holds for \( W \in \mathcal{W}(\pi, \psi) \). This can be argued as follows: Certainly we can assume \( W = W_{\Psi} \) for some \( \Psi \in \mathcal{S}(\mathbb{C}^2) \). Then by equation \( \text{[3.1.2]} \) and by changing the variables as in the proof of Lemma \( \text{[3.3]} \) we arrive
\[
\zeta(s, W_{\Psi}, \chi) = 2 \int_{\mathbb{R}^x} \int_{\mathbb{R}^2} \Psi'(y, t) \mu^{\chi}(y)|y|^s \nu^{\chi}(t)|t|^s d^x y d^x t,
\]
where we define, for \( \Psi \in \mathcal{S}(\mathbb{C}^2), y, t \in \mathbb{R} \), an element \( \Psi' \in \mathcal{S}(\mathbb{R}^2) \) by
\[
\Psi'(y, t) = \int_{0}^{2\pi} \Psi(y e^{i\theta}, t e^{-i\theta}) \mu^{\nu^{-1}}(e^{i\theta}) d\theta.
\]
One check easily that
\[
\Psi'(y, -t) = \mu^{-1}(-1)\Psi'(-y, t).
\]
It follows that
\[
\zeta(s, W_{\Psi}, \chi) = \int_{\mathbb{R}^x} \int_{\mathbb{R}^2} \Psi'(y, t) \mu^{\chi}(y)|y|^s \nu^{\chi}(t)|t|^s d^x y d^x t.
\]
Our assertion now follows immediately from \cite{Henniart:Les-constantes-locales-de-l-equation-fonctionnelle-de-la} Lemma 5.15.1. This proves the claim.

We show the integral \( Z(f^{s}, W) \) converges absolutely when
\[
\text{Re}(s) > 2 \max \{-\text{Re}(\lambda_{1}), -\text{Re}(\lambda_{2})\}
\]
and has a meromorphic continuation to the whole complex plane. Indeed, by the Iwasawa decomposition, we have
\[
Z(f^{s}, W) = \int_{SO_{2}(\mathbb{R})} f^{s}(k)\zeta(s, \rho(k)W, 1)dk,
\]
where 1 is the trivial character of \( \mathbb{R}^{\times} \). The first assertion of Proposition A follows immediately from the claim and the compactness of \( SO_{2}(\mathbb{R}) \).

Next we prove the functional equation. Its known that \( M^{s}f^{s}(g) \) converges absolutely for \( \text{Re}(s) \) is large and extends to a meromorphic function on \( \mathbb{C} \). Moreover, except countable many \( s \), \( M^{s}f^{s}(\cdot) \) defines an element in \( B(\omega_{0}^{-1}|1/2-s|, |s-1/2|) \). Same argument as above shows the integral \( Z(M^{s}f^{s}, W) \) has a meromorphic continuation to whole complex plane.

On the other hand, except countable many \( s \), both integrals \( Z(f^{s}, W) \) and \( Z(M^{s}f^{s}, W) \) define continuous \( GL_{2}(\mathbb{R}) \)-invariant functional on
\[
B(\mu, \nu) \otimes I(s, \omega_{0}).
\]
Here we index \( W = W_{f} \) by an element \( f \in B(\mu, \nu) \). Since \( B(\mu, \nu) \) is dense in \( B(\mu, \nu) \) (resp. \( I(s, \omega_{0}) \)) the uniqueness result of \cite{Henniart:Les-constantes-locales-de-l-equation-fonctionnelle-de-la} Theorem 1.3] implies these two functional are proportion. Therefore there is a meromorphic function \( \gamma(s) \) which is independent of \( f^{s} \) and \( W \) such that
\[
Z(M^{s}f^{s}, W) = \gamma(s)Z(f^{s}, W),
\]
for all such \( f^{s} \) and \( W \in \mathcal{W}(\pi, \psi_{\xi}) \). It remains to show that \( \gamma(s) = \gamma_{RS}(s, As, \pi, \psi, \xi) \). But this follows immediately if we let \( f^{s} = L(2s, \omega_{0}^{-1}f_{\Phi}^{s}) \) and choice any \( W \in \mathcal{W}(\pi, \psi_{\xi}) \) so that \( Z(s, W, \Phi) \) is non-vanishing. The section assertion of Proposition A then follows from equation (5.3.22). \( \square \)

**References**

[Ana08] Anandavardhanan, U. K., Root numbers of Asai L-functions, Int. Math. Res. Not. (2008), Art. ID rnn125, 25 pp.

[AR05] Anandavardhanan, U. K., Rajan, C. S., Distinguished representations, base change, and reducibility for unitary groups, Int. Math. Res. Not. (2005), no. 14, 841-854.

[Bu97] Bump, D., Automorphic forms and representations, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997.

[BH99] Bushnell, C. J. and Henniart, G., Calculs de facteurs epsilon de paires pour \( GL_{n} \), Bull. Lond. Math. Soc. 31 (1999), 534-542.

[Cas89] Casselman, W., Canonical extensions of Harish-Chandra modules to representations of \( G \), Can. J. Math. 41 (1989), no.3, 385-438.

[Del76] Deligne, P., Les constantes locales de l’équation fonctionnelle de la fonction \( L \) d’Artin d’une représentation orthogonale, Invent. math. 35 (1976), 299-316.

[FI88] Flicker, Y. Z., Twisted tensors and Euler products, Bull. Soc. Math. France 116 (1988), no. 3, 295-313.

[FI93] Flicker, Y. Z., On zeroes of the twisted tensor \( L \)-function, Math. Ann. 297 (1993), no. 2, 199-219.

[Gan08] W. T. Gan, Trilinear forms and Triple product epsilon factors, Int. Math. Res. Not. Volume 2008, 1 January 2008, rnn058, https://doi.org/10.1093/imrn/rnn058.

[GJ79] Gelbart, S., Jacquet, H. Forms of \( GL(2) \) from the analytic point of view, In automorphic forms, representations, and \( L \)-functions, 33 (1979), part 1, 213-251.

[Hen00] Henniart, G., Une preuve simple des conjectures de Langlands pour \( GL(n) \) sur un corps \( p \)-adique, Invent. math. 139 (2000), 439-455.

[Hen08] Henniart, G., Correspondance de Langlands et fonctions \( L \) des carrés extérieur et symétrique, Int. Math. Res. Not. (2010), no. 4, 633-673.

[Ich00] Ichino, A., Trilinear forms and the central values of triple product \( L \)-functions, Duke Math. J. 145 (2008), no. 2, 281-307.

[Ike89] Ikeda, T., On the functional equations of the triple \( L \)-functions, J. Math. Kyoto Univ. 29-2 (1989), 175-219.

[Ike92] Ikeda, T., On the location of poles of the triple \( L \)-functions, Compositio, Math. 83 (1992), 187-237.

[Ish17] Ishikawa, I., On the construction of twisted triple product \( p \)-adic \( L \)-functions for unbalanced weights, preprint (2017).

[Jac72] Jacquet, H., Automorphic forms on \( GL(2) \). Part II, Lecture Notes in Mathematics, Vol. 278, Springer-Verlag, Berlin-New York, 1972.

[Jac90] Jacquet, H., Archimedean Rankin-Selberg integrals, Contemp. Math. 489 (2009), 57-172.

[JL70] Jacquet, H., Langlands, R., Automorphic forms on \( GL(2) \), Part I, Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, 1970.

[JPSS83] Jacquet, H., Piatetskii-Shapiro, I. I., Shalika, J. A., Rankin-Selberg convolutions, Amer. J. Math. 105 (1983), no. 2, 367-464.
[Kab04] Kable, A. C., *Asai L-functions and Jacquet’s conjecture*, Amer. J. Math. 126 (2004), no. 4, 789-820.

[KM02] Kim, H. H. and Shahidi, F., *Functorial products for GL₂ × GL₃ and the symmetric cube for GL₂*, Ann. of Math. 155 (2002), no. 2, 837-893.

[Kri03] Krishnamurthy, M. *The Asai transfer to GL₄ via the Langlands-Shahidi method*, Int. Math. Res. Not. (2003), no. 41, 2221-2254.

[Lok01] Loke, Y. H. *Trilinear forms of gl₂*, Pacific. J. Math. 197 (2001), no.1, 119-144.

[Mat09] Matringe, M., *Conjectures about distinction and local Asai L-functions*, Int. Math. Res. Not. (2009), no. 9, 1699-1741.

[Mat10] Matringe, N., *Distinguished representations and exceptional poles of the Asai-L-function*, Manuscripta Math. 131 (2010), no. 3-4, 415-426.

[PSR87] Piatetski-Shapiro, I.I. and Rallis, S., *Rankin triple L-functions*, Compositio. Math. 64 (1987), 31-115.

[Pra92] Prasad, D., *Invariant forms for representations of GL₂ over a local field*, Amer. J. Math. 114 (1992), no. 6, 1317-1363.

[Rama00] Ramakrishnan, D., *Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2)*, Ann. of Math. 152 (2000), 45-111.

[Rama02] Ramakrishnan, D., *Modularity of Solvable Artin Representations of GO(4)-Type*, Int. Math. Res. Not. 2002 (2002), no. 2, 1-54

[Sc02] Schmidt, R., *Some remarks on local newforms for GL(2)*, J. Ramanujan Math. Soc. 17 (2002), no. 2, 115-147.

[Ser79] Serre, J.-P. *Local fields*, Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin (1979), ISBN: 0-387-90424-7.

[Sha84] Shahidi, F., *Fourier transforms of intertwining operators and Plancherel measures for GL(n)*, Amer. J. Math. 106 (1984), no. 1, 67-111.

[Sha90] Shahidi, F., *A proof of Langlands’ conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) 132 (1990), no. 2, 273-330

[Tat79] J. Tate, *Number theoretic background*, Automorphic forms, representations, and L-functions. Proceedings of symposia in pure mathematics Vol. 33, Part 2, 3-26.

[Wal92] Casselman, W., *Real reductive groups II*, Academic Press, Pure and applied mathematics. 132 (1992), no.3.

[Wei64] Weil, A., *Sur certains groupes d’operateurs unitaires*, Acta Math. 111 (1964), 143-211.

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