We say that a Cellular Automata (CA) is coalescing when its execution on two distinct (random) initial configurations in the same asynchronous mode (the same cells are updated in each configuration at each time step) makes both configurations become identical after a reasonable time. We prove coalescence for two elementary rules, non coalescence for two other, and show that there exists infinitely many coalescing CA. We then conduct an experimental study on all elementary CA and show that some rules exhibit a phase transition, which belongs to the universality class of directed percolation.

*Key words:* synchronization, coalescence, asynchronism, asynchronous cellular automata, directed percolation, coupling, robustness, elementary cellular automata, stochastic process, discrete dynamical system, phase transition
1 INTRODUCTION

The coalescence phenomenon, as we call it, has been observed for the first time by Fates [1], in the context of asynchronous cellular automata. Coalescing CA exhibit the following behavior: starting from two different initial random configurations and running the same updating sequence (the same cells are updated at each time step in both configurations), the configurations quickly become identical, i.e. both configurations not only reach the same attractor, but they also both synchronize their orbits. This of course appears in trivial situations, for example if the CA converges to a single fixed point, but in [1] it is also observed it in a case where the asymptotic orbit is absolutely non trivial.

The goal of this paper is to explore this rather strange emergent phenomenon in which the asymptotic behavior seems to be only related to the (random) sequence of update of the cells and not to the initial configuration. This work shows that, in some cases, the randomness used during evolution is as important as the one used during initialization: this stochastic dynamic, with high entropy, is perfectly insensitive to initial condition. There is thus no chaos here.

The results presented here are of two kinds. First, we prove the existence of infinitely many different (we precise this notion) non trivial coalescent CA. We also prove the existence of non trivial non coalescent CA. Secondly, we study by simulation the behavior of all elementary CA (the ECA are the CA with 1 dimension, two states, and two nearest neighbors) with regards to this coalescence property, in an asynchronous context in which at each step each cell has a fixed probability $\alpha$ to be updated. We show that over the 88 different ECA, six situations occur: a/ 37 ECA never coalesce; b/ 21 always coalesce in a trivial way (they converge to a unique fixed point); c/ 5 always coalesce on non trivial orbits; d/ 15 combine a/, b/ and c/ depending on $\alpha$; e/ 7 enter either full agreement (coalescence) or full disagreement; last, f/ 3 combine e/ with either a/ or c/.

We also study the transition between non coalescence and coalescence when $\alpha$ varies for the ECA that combine a/ and c/: there is a phase transition belonging to the universality class of directed percolation. We thus get a new model of directed percolation, with a few variants.

Unlike many directed percolation models, the limit of the sub-critical regime is neither a single absorbing state, nor a set of fixed points, but a non trivially evolving phase.

This paper is an extended version of [2].
The paper is organized as follows. Section 2 gives definitions and notations. Section 3.1 proves non coalescence for two CA. We prove in Section 3.2 that, under certain conditions, CA 6 and 7 (using the Wolfram’s numbering of ECA) are coalescing and show how to construct from them coalescing CA with arbitrarily many states. We also prove that CA 15 and 170 either coalesce or enter total disagreement, each case occurring with probability $\frac{1}{2}$.

In Section 4, we describe the exhaustive simulation study of all ECA and then check the directed percolation hypothesis. Moreover, we observe that some CA exhibit two phase transitions: one for small $\alpha$ and one for high $\alpha$.

## 2 DEFINITIONS AND NOTATIONS

**Definition.** An asynchronous finite CA is a tuple $(Q, d, V, \delta, n, \mu)$ where

- $Q$ is the set of states;
- $d \in \mathbb{N}^*$ is the dimension;
- $V = \{v_1, \ldots, v_{|V|}\}$, the neighborhood, is a finite set of vectors in $\mathbb{Z}^d$;
- $\delta : Q^{|V|} \to Q$ is the transition rule;
- $n \in \mathbb{N}^*$ is the size;
- $U := (\mathbb{Z}/n\mathbb{Z})^d$ is the cell space (with periodic boundary condition);
- $\mu$, the synchronism, is a probability measure on $\{0, 1\}^U$.

A configuration specifies the state of each cell, and so is a function $c : U \to Q$.

If $x \in \{0, 1\}^U$, let $|x|_1$ be the number of 1 in the coordinates of $x$.

The dynamic on such CA is then the following. Let $c_t$ denote the configuration at time $t$, ($c_0$ is the initial configuration). Let $\{M_t \mid t \in \mathbb{N}\}$ be a sequence of independent identically distributed random variables with distribution $\mu$. The configuration at time $t+1$ is obtained by

$$c_{t+1}(z) := \begin{cases} c_t(z) & \text{if } M_t(z) = 0 \\ \delta(c_t(z + v_1), \ldots, c_t(z + v_{|V|})) & \text{if } M_t(z) = 1 \end{cases}.$$

In other words, for each cell $z$, we apply the usual transition rule if $M_t(z) = 1$ and freeze it (keep its state) if $M_t(z) = 0$.

There are many possibilities to asynchronously run a CA. We use the two most classical ones ([3]), which are defined as follow.
• The Partially Asynchronous Dynamic. Let $0 < \alpha \leq 1$. For each cell, we update it with probability $\alpha$, independently from its neighbors. $\mu$ is thus the product measure of Bernoulli distributions: $\mu(x) := \alpha^{|x|_1} (1 - \alpha)^{|x|_0}$ (with $0^0 = 1$). The case $\alpha = 1$ corresponds to the synchronous dynamic.

• The Fully Asynchronous Dynamic. At each step, we choose one cell and update it, which defines $\mu(x)$ as $1/n$ if $|x|_1 = 1$ and $\mu(x) := 0$ otherwise.

This last dynamic can be regarded as the limit, when $\alpha \to 0$, of the partially asynchronous dynamic (however, to simulate $t$ steps of this dynamic with the partially asynchronous one, we need more than $nt^2$ steps).

It is also equivalent (for the order of updates) to the following. Consider a continuous time and give a clock to each cell. Each clock measure the time before the next update: when a clock reaches zero, its owner is updated. The clock is then reset to a random time (to choose the next update date), according to an exponential law.

The usual way to visualize the dynamics of a cellular automaton, asynchronous or not, is the space-time diagram.

**Definition** (space-time diagram). Figure 1 is an example. The space-time diagram is obtained by stacking up the successive configurations, i.e., each configuration is plotted horizontally, above the previous one. Time thus goes upwards.

Since we use pairs of configurations and study the differences between both, we will also use two superimposed space-time diagrams, like in Figure 6. On such a diagram, we sometimes restrict our attention to cells that agree (plotted light) and cells that don’t (plotted dark). The resulting quotient diagram is called the agreement/disagreement space-time diagram.

Let us now introduce the definition of coalescing CA to formalize the observation of [1]. The principle is to use two initial configurations, and to let them evolve with the same outcome of the random variables $\{M_t \mid t \in \mathbb{N}\}$. In other words, we use two copies of the CA, and at each time step, we update the same cells in both copies. This comes down to using the same source of randomness for both copies.
Definition. An asynchronous finite CA is coalescing if, for any two initial configurations, applying the same sequence of updates leads both configurations to become identical within polynomial expected time (with respect to \( n \)).

Note that it is required for the CA to coalesce quickly enough, where “quickly” means in polynomial time. Section 3.1 will exhibit a CA that does not coalesce in polynomial time but nevertheless always coalesce if given enough time.

Any nilpotent CA (converging towards a configuration where all states are identical) is coalescing if it converges in polynomial time. But as we will see, and this makes the interest of our study, there exist non nilpotent coalescing CA, which we call non trivial.

The term “coalescing” (rather than synchronizing) is used to avoid any confusion with the manipulation of the synchronisms.

In the following, we will often focus on the simplest CA, namely the Elementary CA: one dimension \((d = 1)\), 2 states \((Q = \{0, 1\})\), nearest neighbors \((V = \{-1, 0, 1\})\). There are \(2^8 = 256\) possible rules, 88 after symmetry considerations. We use the notation introduced by S. Wolfram, numbering the rules from 0 to 255. In this notation, a rule \( \delta \) is denoted by the number \(2^7 \delta(1, 1, 1) + 2^6 \delta(1, 1, 0) + 2^5 \delta(1, 0, 1) + 2^4 \delta(1, 0, 0) + 2^3 \delta(0, 1, 1) + 2^2 \delta(0, 1, 0) + 2^1 \delta(0, 0, 1) + 2^0 \delta(0, 0, 0)\). For instance, the code for the rule
“minority” or “take the least present state of my neighborhood” (which is \textit{23}) is obtained by reading the last line of Table 1 as a base 2 number. We simply write “23” (with this font) for “the rule which has code 23”.

Table 1
Transition table of \textit{23}

| neighborhood | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|--------------|-----|-----|-----|-----|-----|-----|-----|-----|
| result of $\delta$ | 0   | 0   | 0   | 1   | 0   | 1   | 1   | 1   |

Having defined our objects, the next two sections will prove that there exists both (non trivial) coalescing CA and non coalescing CA.

3 ANALYTICAL STUDY

3.1 Formal proof of non coalescence

Quite obviously, \textit{204}, which means “identity”, cannot coalesce: disagreeing cells disagree forever. We would be more interested in a rule that is not coalescing because it does not reach agreement in polynomial time, but nevertheless always reaches configuration where all cells agree if given enough time. In this part, we will prove that this is the case for \textit{60}. The proof uses specific properties of \textit{affine} rules, which we will introduce first.

\textbf{Definition.} A rule is said to be \textit{affine} if it can be expressed as

$$\delta(q_1, q_2, q_3) = \varepsilon + \sum_{i \in I} q_i \mod 2$$

where $I \subseteq \{1, 2, 3\}$ and $\varepsilon \in \{0, 1\}$.

Those rules where introduced in [4], a detailed study is in [5].

\textbf{Lemma 1.} For both asynchronous dynamics, affine rules are exactly the rules for which the agreement/disagreement space-time diagram is the space-time diagram of a cellular automaton.

We call the latter CA the \textit{quotient} rule.

\textbf{Proof.} First, it is clear that if the rule is affine, we have what we want: if $q_i$ are the state of one configuration and $q_i'$ are the states of the second, the agreement/disagreement state is 1 if $q_i = q_i'$ and 0 otherwise, which we
write \( q_i \oplus q_i' \). In the next configuration, the agreement/disagreement state is
\[
\delta(q_1, q_2, q_3) \oplus \delta(q_1', q_2', q_3') = \varepsilon + \sum_{i \in I} q_i + \varepsilon + \sum_{i \in I} q_i' \mod 2 = \sum_{i \in I} (q_i \oplus q_i') \mod 2,
\]
which is the result of the application of a CA rule \( \delta' \) on the states of the previous configuration. We even see that the quotient rule \( \delta' \) is
\[
\delta'(q_1, q_2, q_3) = \sum_{i \in I} q_i \mod 2,
\]
e.i. the linear part of \( \delta \).

Reciprocally, let us assume that the agreement/disagreement diagram is still the diagram of a cellular automaton. If the rule is constant, \( I = \emptyset \) is suitable.

Otherwise, there is some configuration \((q_1, q_2, q_3)\) for which \( \delta \) changes if a specific \( q_i \) changes. We can choose it to be \( q_1 \) (the other cases are symmetrical) i.e. \( \delta(q_1, q_2, q_3) \neq \delta(1 - q_1, q_2, q_3) \). Thanks to the initial assumption, this inequality has to be true for all \( q_2 \) and \( q_3 \) (indeed, if there would exist \( q_2' \) and \( q_3' \) such that \( \delta(q_1, q_2', q_3') = \delta(1 - q_1, q_2', q_3') \), we would have two pairs of configurations with the same agreement/disagreement pattern at step 0, but distinct agreement/disagreement pattern at step 1, violating our hypothesis).

The rule can thus be expressed as \( \delta(q_1, q_2, q_3) = \varepsilon + q_1 + \delta'(q_2, q_3) \mod 2 \). Since this is true for any neighbor which can influence the outcome of \( \delta \), the result follows.

**Proposition 2.** For both asynchronous dynamics, 60 is not coalescing.

**Proof.** 60 is affine, so we can reason on the agreement/disagreement diagram. The rule governing this space-time diagram is still 60. This rule means “XOR between me and my left neighbor”.

Let us distinguish two cases:

- If all cells are in the disagreement state, the total agreement occurs if and only if all cells update, which happens with probability \( \alpha^n \).
- Otherwise, there is a disagreement cell with a left neighbor in the agreement state. Whether it updates or not, this cell will stay in the disagreement state at the next step.

The expected time to reach total agreement is thus greater than \( \frac{1}{\alpha^n} \), which is asymptotically greater than any polynomial in \( n \) for \( 0 < \alpha < 1 \).

**Remark.** If given enough time, 60 always reaches total agreement. Indeed, from any configuration, it is possible (though unlikely) to reach agreement: first update exactly the agreement cells having a disagreement left neighbor (those cell thus go into the disagreement state), and repeat until all cells are in the disagreement state. Then update all the cells at once, and total agreement
occurs. Thus, at each time step, there is a (tiny) non-zero probability to reach total agreement.

3.2 Formal proof of coalescence

In this section, we will prove that there are infinitely many coalescing CA. For that, we will prove the coalescence of two particular CA and show how to build an infinite number of coalescing CA from one of them.

An easy way to do that last point would be to extend a coalescing CA by adding states that are always mapped to one state of the original CA, regardless of their neighbors. However, we consider such a transformation to be artificial since it leads to a CA that is in some sense identical to the original one. To avoid this, we focus on state minimal CA: CA in which any state can be reached (but not necessarily any configuration). Note that among ECA, only 0 and 255 are not state-minimal.

We will first exhibit two state-minimal coalescing CA (proposition 3); then, using this result, we will deduce the existence of an infinite number of such CA (theorem 4); finally we will describe the coalescent behavior of two others ECA (proposition 5).

Proposition 3. 6 and 7 are coalescing for the fully asynchronous dynamic when n is odd.

Proof. We call number of zones the number of patterns 01 in a configuration, which is the number of “blocks” of consecutive 1 (those blocks are the zones). We first consider only one copy (one configuration).

Table 2
Transition table of 6

| Neighbors | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| New state  | 0   | 0   | 0   | 0   | 0   | 0   | 1   | 1   |

Table 2 shows the transition table of 6. Since one cell at a time is updated, and since updating the central cell of 101 or 010 does not change its state, zones cannot merge, i.e. the number of zones cannot decrease.

Let us first show that the number of zones actually increase until there are no patterns 000 or 111. On each pattern 111, the central cell can be updated (leading to the pattern 101) before its neighbors with probability \( \frac{1}{3} \) and with expected time \( n \). On each pattern 0001, the sequence of Table 3 is possible.
Table 3
A possible update sequence of $6$.

\[
\cdots 0\ 0\ 0\ 1\ \cdots \\
\cdots 0\ 0\ 1\ 1\ \cdots \\
\cdots 0\ 1\ 1\ 1\ \cdots \\
\cdots 0\ 1\ 0\ 1\ \cdots 
\]

It happens without other update of the four cells with probability $1/4^3$ and with an expected time of $3n$. So, as long as there are patterns $000$ or $111$, the number of zones increases with an expected time $O(n)$. Since there are $O(n^2)$ zones, the total expected time of this increasing phase is $O(n^2)$.

The configuration is then regarded as a concatenation of words on $\{0, 1\}^*$. Separation between words are chosen to be the middle of each pattern $00$ and $11$, so we get a sequence of words that have no consecutive identical letters, each word being at least two letter long (that is, words of the language $\langle(01)^+0? | (10)^+1?\rangle$). We now show that borders between these words follow a one way random walk (towards right) and meet, in which case a word disappear with positive probability. The CA evolves therefore towards a configuration with only one word. Updating the central cell of $100$ does not change its state, so the borders cannot move towards left more than one cell. On the other hand, updating the central cell of $001$ or $110$ makes the border move. One step of this random walk takes an expected time $O(n)$. The length of a word also follows a (non-biased) random walk, which reaches 1 after (on average) $O(n^3)$ steps, leading to the pattern $000$ or $111$. This pattern disappears with a constant non zero probability like in the increasing phase. The expected time for $O(n)$ words to disappear is then $O(n^4)$.

Since $n$ is odd, the two letters at the ends of the words are the same, i.e. there is one single pattern $00$ or $11$, still following the biased random walk. We now consider again the two copies. Since this pattern changes the phase in the sequence $(01)^+$, it is therefore a frontier between a region where both configurations agree and a region where they do not. The pattern in the other configuration let us come back to the region where the configurations have coalesced.

Let us now study the length of the (single) region of disagreement. It follows a non biased random walk determined by the moves of both patterns.
Table 4
Possible update sequences of 6.

\[ \cdots 1 0 1 0 0 1 0 1 \cdots \cdots 0 1 0 1 0 0 1 0 \cdots \cdots 0 1 0 1 1 0 1 0 \cdots \]

a. b. c.

When this length reaches \( n \), as in Table 4.a, the only change happens when the fourth cell is updated, and it decreases the length. So, the random walk cannot indefinitely stay in state \( n \). On the other hand, when the length reaches 1, one possibility is Table 4.b, where updating the fifth cell leads to coalescence. The other possibility is Table 4.c, where updating the fifth then the fourth cell leads to the former possibility. In each case, coalescence happens with a constant non zero probability. One step of this random walk takes an expected time \( O(n) \), the total expected time of the one word step is thus \( O(n^3) \) (details on expected time can be found in [6]).

Therefore, 6 is coalescing.

The proof for 7 is identical, unless that 000 leads to 010, which does not affect the proof (only the increasing phase is faster). \( \square \)

**Remark.** If \( n \) is even, the proof is valid until there is only one word, at which point we get a configuration without 00 nor 11, i.e. \((01)^{n/2}\) or \((10)^{n/2}\). If both copies have the same parity, it is coalescence, otherwise both copies perfectly disagree (definitively). Both happen experimentally.

**Theorem 4.** For the fully asynchronous dynamic, there are non trivial state-minimal coalescing cellular automata with an arbitrarily large number of states, and therefore infinitely many non trivial state-minimal coalescing CA.

**Proof.** Let \( \mathcal{A}^2 \) be the product of a CA \( \mathcal{A} = (Q, d, V, \delta, n, \mu) \) by itself, defined as \((Q^2, d, V, \delta^2, n, \mu)\) where

\[ \delta^2((a,b), (c,d), (e,f)) := (\delta(a,c,e), \delta(b,d,f)) \]

Intuitively, \( \mathcal{A}^2 \) is the automaton we get by superposing two configurations of \( \mathcal{A} \) and letting both evolve according to \( \delta \), but with the same \( M_t \). If \( \mathcal{A} \) is state-minimal, so is \( \mathcal{A}^2 \).

Let \( \mathcal{A} \) be a coalescing CA. Then \( \mathcal{A}^2 \) converges in polynomial expected time towards a configuration of states all in \( \{(q,q) \mid q \in Q\} \). From this point, \( \mathcal{A}^2 \) simulates \( \mathcal{A} \) (by a mere projection of \( Q^2 \) to \( Q \)) and is therefore coalescing (with an expected time at most twice as long); and so are \((\mathcal{A}^2)^2\), \((\mathcal{A}^2)^2)^2\), ...
etc, that form an infinite sequence of CA with increasing size.

**Proposition 5.** 15 and 170, for both asynchronous dynamics, either coalesce or end in total disagreement, each case with probability $\frac{1}{2}$ (with respect to the outcome of initial configuration and $(M_t)$).

**Proof.** 170 (shift) means “copy your right neighbor”. If one runs two configurations in parallel, they agree on a cell if and only if they agreed on the right neighbor before this cell was updated. So (as seen in lemma 1), the agreement/disagreement space-time diagram is the space-time diagram of a CA. This CA is still 170. 170 converges in polynomial time towards $0^n$ (corresponding to coalescence) or $1^n$ (full disagreement) [6]. By symmetry, each case has probability $\frac{1}{2}$.

15 means “take the state opposed to the one of your right neighbor”, and the proof is identical (the quotient CA is still 170).

### 4 EXPERIMENTAL STUDY AND PHASE TRANSITION

In this section, we will describe experimental results in the context of partially asynchronous dynamic. We will show that many ECA exhibit coalescence, then make a finer classification. Specifically, we will observe that some ECA undergo a phase transition for this property when $\alpha$ varies. We will experimentally show that this phase transition belongs always but in one case to the universality class of directed percolation.

#### 4.1 Classification of CA with respect to coalescence

4.1.1 Protocol

We call run the temporal evolution of a CA when all parameters (rule, size, $\alpha$ and an initial configuration) are chosen. We stop the run when the CA has coalesced, or when a predefined maximum running time has been reached.

Let us describe the parameters we used.

1. Number of cells $n$. The main point when choosing $n$ is to check that the results do not depend on a particular choice of $n$. Some authors (like [7]) suggest that small is enough ($n = 30$), others (like [8]) state the opposite, and we follow the latter. A similar problem studied in [9] shows a stable behavior for $n \geq 200$. We set $n = 2000$ and check that the results do not change for $n = 500$. 
2. Number of computation steps. To measure the asymptotic density, we let the automaton run for 200,000 steps, then measure the density averaged over 10,000 steps. (A few cases had to be checked with longer times.)

Remark. According to directed percolation theory, near a phase transition, the automaton can take an arbitrarily long time before settling down to the asymptotic density. So, for a few choices of \( \alpha \), those parameters are not sufficient to measure the true asymptotic density. However, they are big enough to detect that there is a transition point, and then study more precisely what happens there.

3. Update rate \( \alpha \). We try to sample the entire set of possible update rates (indeed, we will find transitions distributed among this set). For each of the 88 rules, we do 999 runs: one for each value of \( \alpha \) ranging from 0.001 to 0.999.

One might want to average over many runs. To show that we do not need to, we plot the asymptotic density \( \rho \) versus \( \alpha \). The smoothness of the resulting curve (Figures 2 and 3) shows that the variance between runs is low.

The random seed for deciding which cells to update at each step is distinct for each value of \( \alpha \).

4. The initial configuration is random and distinct for each value of \( \alpha \).

4.1.2 Results

We get the following empirical classes of behaviour:

a/ Some CA never coalesce (or take a too long time to be observed): 4, 5, 12, 13, 25, 28, 29, 33, 36, 37, 41, 44, 45, 51, 54, 60, 72, 73, 76, 77, 78, 90, 94, 104, 105, 108, 122, 132, 140, 142, 150, 156, 164, 172, 200, 204, 232.

One can be a bit more precise by plotting the asymptotic density \( \rho \) versus \( \alpha \). No coalescence means that this curve is always above the \( x \) axis. We most of the time get a noisy curve, close to a horizontal line usually around 0.5 (Figure 2.a). This noise shows the precision of the measure. It is to be compared to Figure 2.b (204, which means “identity”) showing the variance in the initial configuration: for this rule, the asymptotic density is the same as the initial density. The fact that 150 (Figure 2.c) is “mixing” and has a density converging to 0.5
Coalescing Cellular Automata

seems hard to prove. Figures 2.d and 2.e are the other types of plots that arise.

The case of 142 (Figure 2.f) can be solved analytically. In this rule, a cell can change its state only if it is in the configuration 001 or 110. The neighboring cells cannot be in one of these configurations and cannot change their states. Thus, the number of 1* blocks (or equivalently, the number of occurrences of the word 01) is conserved. So the automaton can coalesce only if those numbers are equal in both initial configurations. The probability for this to occur tends to 0 as \( n \) grows.

Experimentally, when it does not coalesce, there is no asymptotic density for this rule: the density keeps evolving. The rule still has correlation between both configurations, with large zones of agreement and large zones of disagreement.

b-c/ Some CA coalesce rapidly.

b/ The trivial way to do this is to converge to a unique fixed point. One can consider the two copies independently, and wait for them to reach the fixed point, the CA has then coalesced. This is the case for 0, 2, 8, 10, 24, 32, 34, 38, 40, 42, 56, 74, 128, 130, 134, 136, 138, 152, 160, 162, 168.

For most “captive” automata (in the ECA case, captive rules are rules for which 000 leads to 0 and 111 leads to 1), it has been proven in [6] whether the rule converges to 0\(^n\) or not. Those rules that provably tend to 0\(^n\) independently, and thus are coalescing, are: 128, 130, 136, 138, 152, 160, 162, 168.

c/ The non trivial rules are 3, 19, 35, 46, 154. Rule 19 converge especially slowly for small \( \alpha \).

d/ Some CA combine the previous behaviors, depending on \( \alpha \) (Figure 3). 6, 18, 26, 106, 146 combine a/ and b/; 50 combines b/ and c/; 1, 9, 11, 27, 57, 62, 110, 126 combine a/ and c/ (see Figure 6); 58 combines a/, b/ and c/.

e/ Some CA end in either full agreement between configurations (coalescence) or full disagreement, depending on the outcome of \( (M_t) \) and the initial configuration: 14, 15, 23, 43, 170, 178, 184.

These CA take a longer time to reach this asymptotic behavior (this time experimentally seems to be \( \Theta(n^2) \)), so the asymptotic density versus \( \alpha \) plots are not two horizontal lines, one at \( \rho = 0 \) and the other at
Figure 2
Asymptotic density of disagreement cells versus $\alpha$ for some non coalescing rules. The scale of both axis is $[0; 1]$. The first case is typical.
Figure 3
Same as Figure 2, but for some rules which coalesce or not, depending on $\alpha$. The first case is the most common. Those rules, undergoing a transition, are studied in greater detail in Section 4.3.
ρ = 1 (Figure 4). They take an even longer time for some specific values of α (thus the “bow tie” plot: the closer α is to this specific value, the longer the density stays close to 0.5): low α (Figure 4.a), α = 0.5 (Figure 4.b) or elsewhere (Figure 4.c).

f/ 22 and 30 combine the previous point (for small α) with a/7, 7 combines it (for small α) with c/. Note that 30 also shows a varying asymptotic density as it goes from a/ to c/, but due to the difficulties mentioned in e/ this asymptotic density has not been measured.

Let us now study the phase transition “coalescence or not” when α changes (point d/), that is, 1, 9, 11, 27, 57, 58, 62, 110, 126 and 6, 18, 26, 106, 146. Some rules (9, 58, 110, 126) show two phase transitions, one for low α, denoted by a subscript ℓ like in 9ℓ, one for high α, denoted by a subscript h like in 9h.

The next subsection (4.2) recalls the useful background before studying this phase transition: it briefly recalls what a phase transition is, it presents a conjecture implying that our transition belongs to the directed percolation universality class, and describes this directed percolation class.

4.2 Phase transition and directed percolation

4.2.1 Phase transition

A phase transition is an abrupt change in macroscopic properties of a system with only a small change of control a parameter, say T, around a critical value Tc. This paper is concerned only with second order phase transitions, or continuous phase transitions, which can be characterized by critical exponents. If one let the parameter T vary near the phase transition (occuring at T = Tc), all other variables being fixed, a measurable quantity C has a power law behaviour $C \propto |T - T_c|^{\beta}$ at least on one side of Tc. Several exponents are defined, depending on the quantity measured.

Remarkably, many systems with no a priori relation turn out to have the same critical exponents. A universality class is defined as all the systems having the same set of critical exponents.

4.2.2 A conjecture on damage spreading

Chaos theory deals with the sensitivity to initial condition of deterministic systems. To also study the influence of small perturbations on stochastic systems, [10] introduced damage spreading. In this model, two copies of a stochastic model are run in parallel with the same source of random bits, starting from different initial configurations (often they are set to differ in exactly one site).
Figure 4
Same as Figure 2, but for some rules ending either in full agreement or full disagreement.
One measures the temporal evolution of the proportion of differing sites, called the Hamming distance. If this goes to zero, i.e. if both copies become identical, the initial “damage” has “healed”, otherwise the damage is said to spread.

There is a conjecture by [11] stating that, if a transition occurs between healing and spreading in a stochastic spin model, the universality class of this phase transition is always the same, namely the one of directed percolation, which will be presented in the next paragraph.

There are some conditions for this conjecture:

1. Only short range interactions in time and space,
2. translational invariance,
3. non vanishing probability for a site to become healed locally,
4. the transition does not coincide with another phase transition.

Points 1 and 2 are easily fullfilled for CA. We will discuss points 3 and 4 in Section 4.3.1.

4.2.3 The Model of Directed Percolation

An more detailed introduction to directed percolation can be found in [3] (note that this papers cites a different conjecture of Grassberger than the one we deal with). A survey of directed percolation is contained in [12], which also covers damage spreading.

Isotropic percolation was first defined when studying propagation of a fluid through a porous medium. It has been mathematically modelled as an infinite square grid where each site has the four nearest sites as neighbors. Each bond between two neighbors can be open (letting the fluid go through) with probability $p$ or closed with probability $1 - p$, independently of all other bonds.

The question is whether the fluid inserted at one point will pass through the medium, i.e. whether this point is part of an infinite network of sites connected by open bonds.

Directed percolation appears when one adds gravity to the model, i.e. when the fluid is only allowed to travel in one direction (Figure 5). Static 2D directed percolation can also be seen as a 1D dynamical model where some sites are “active” (where active can mean wet, infected, etc.). An active cell can stay active or die (become inactive), and make its neighbors active. Depending on the probabilities of these possibilities, active regions spread
or disappear. Cells can only have an influence on the future states of their neighbors, thus the directed percolation.

The order parameter measured is the density of active states as a function of $p$ and time, $\rho(p, t)$. It is zero in one phase and non-zero in the other. There exists a critical probability $p_c$ which is the limit between two phases.

- For $p < p_c$, the asymptotic density $\rho(p, \infty)$ is 0;
- for $p > p_c$ we have a power law $\rho(p, \infty) \propto (p - p_c)^\beta$;
- for $p = p_c$ the density goes to 0 as $\rho(p_c, t) \propto t^{-\delta}$.

![Isotropic bond percolation](image1.png) ![Directed bond percolation](image2.png)

Figure 5
Isotropic (left) and directed (right) percolation. Figure reprinted from [12].

4.3 Directed percolation in our model

Our active sites are the cells where the configurations disagree. Density of such sites is written $\rho(\alpha, t)$, or $\rho(\alpha)$ for the asymptotic density. The pairs of configurations where all cells agree constitute the absorbing set. Percolation transition (coalescence or not) appears when varying $\alpha$, see Figure 6. Note that there is no direct relation between $\alpha$ and $p$. The aim is thus to identify $\beta$ assuming that $\rho(\alpha) \propto |\alpha - \alpha_c|^\beta$ for some $\alpha_c$. Like many authors, we will focus on $\beta$ and consider it as sufficient to test directed percolation.

We deal with finite configurations and can thus be subject to finite size effects. One effect of particular importance is the following. Take a rule with a phase transition, there is an update rate $\alpha$ for which the rule is coalescing, i.e. reaches total agreement in polynomial time. In the non coalescing regime,
Rule 110, \( n = 500 \). Time goes upwards, during 500 steps. Sites where both configurations disagree are dark, coalesced sites are light (with light blue standing for state 1, white for 0). Left: sub-critical phase (\( \alpha = 0.47 < \alpha_c \simeq 0.566 \)), branches die. Right: supercritical phase (\( \alpha = 0.65 > \alpha_c \)), active sites spread.

with low probability, the outcome of the random bits determining which cells get updated can make the CA simulate the coalescing regime for a fixed number of steps. So, if a CA can coalesce for a given \( \alpha \), it can coalesce for any \( \alpha \) in \((0; 1)\). The true asymptotic regime is thus always coalescence.

However, the long term behaviour we’re interested in, and the one we can observe with practical simulations, is after a polynomial time only. It is a “long transient” as opposed to the short transient happening after initialization. The system stays in this “long transient” regime for a super-polynomial time, then reaches the true asymptotic regime. (It can be compared to a living system where one studies reactions to and transient regime after various events in life, but where the asymptotic regime is always death.) Note that coalescence was precisely defined as coalescence in polynomial expected time.

4.3.1 Measure of \( \alpha_c \)

To measure \( \beta \), we use the method described in [12] that advise to first measure \( \alpha_c \) before doing a fit to measure \( \beta \), instead of fitting \( \alpha \) and \( \beta \) at the same time.

To measure \( \alpha_c \), the method implies plotting the density \( \rho \) of active sites versus time in logarithmic scale and finding the \( \alpha \) value for which one gets
a straight line (for $\alpha < \alpha_c$, the AC coalesce faster, for $\alpha > \alpha_c$, it has a positive asymptotic $\rho$). We used random initial configuration with each state equiprobable. To get readable plots we needed up to $n = 10^6$ cells and $10^7$ time steps. We get (recall that $\alpha_c$ is not universal, it is just used to compute $\beta$):

| rule | 1 | 6 | 9 | 11 | 18 | 26 | 27 | 57 |
|------|---|---|---|----|----|----|----|----|
| $\alpha_c > ...$ | 0.101 | 0.06 | 0.073 | 0.7575 | 0.9575 | 0.7138 | 0.4747 | 0.856 | 0.749 |
| $\alpha_c < ...$ | 0.103 | 0.08 | 0.074 | 0.758 | 0.9583 | 0.7141 | 0.4751 | 0.858 | 0.750 |

| rule | 58 | 62 | 106 | 110 | 126 | 126 | 146 |
|------|----|----|-----|-----|-----|-----|-----|
| $\alpha_c > ...$ | 0.4745 | 0.8408 | 0.598 | 0.8143 | 0.073 | 0.566 | 0.101 | 0.720 | 0.6750 |
| $\alpha_c < ...$ | 0.4748 | 0.8412 | 0.599 | 0.8147 | 0.075 | 0.567 | 0.103 | 0.721 | 0.6753 |

Note that the $\alpha_c$ of 1 and 126, like 9 and 110, are very close, and may be equal. Also, $\alpha_c$ for 6 and 11 are quite close to 0 and 1.

Comparison to a simpler model In [3], a quite similar problem is studied. The author chooses the more classical model with only one configuration, but still the same asynchronous updating. The “1” is taken as the active state, and $0^n$ as the absorbing dead state. This article showed experimentally that some rules ($6, 18, 26, 50, 58, 106$ and $146$) belong to the directed percolation class: they converge to $0^n$ for small enough $\alpha < \alpha_c$ for some $\alpha_c$, and the asymptotic density $d'$ of cells in the state “1” near $\alpha_c$ is $C(\alpha - \alpha_c)^\beta$ for some constant $C$.

Such rules are clearly trivially coalescing for low $\alpha$: both configurations independently converge to $0^n$. So, if one of these rules is not coalescing for high $\alpha$, it will undergo a phase transition in our model. Here is a review of those rules:

- 50 is always coalescing, either trivially or not, and thus isn’t of interest in the present work. The other rules are not always coalescing.

- 58 has a different $\alpha_c$ than in our work and is studied in Section 4.3.2.

- 6 also has a different $\alpha_c$. However, near $\alpha_c$, both configurations quickly becomes mostly composed of periodic regions repeating the pattern “01”. Depending on the parity, both configurations agree or disagree on the whole overlap of such regions. Inside such an overlap, a cell state does not change, whether updated or not. So, point 3 of Grassberger’s conjecture is not fullfilled: a single site cannot always heal. Indeed,
the density versus time plot is not a power law in the form $t^{-\delta}$, as one
would expect for directed percolation. See Figure 7.

- Finally, $18, 26, 106$ and $146$ seem to have the same $\alpha_c$ than the one
measured in [3] (at least up to the available precision $±3.10^{-4}$).

In our model with two configurations, if we assume both configurations
to be independent*, then the density of disagreement cells is $2d'(1-d')$
(as defined in the previous paragraph, $d'$ is the density of “1” in
the model with only one configuration). $2d'(1-d') = 2C(\alpha - \alpha_c)\beta (1 - C(\alpha - \alpha_c)\beta) = 2C(\alpha - \alpha_c)\beta - 2C^2(\alpha - \alpha_c)^{2\beta}$. The second term
is negligible (second order) near $\alpha_c$, so (under the assumption of inde-
pendence) the rule still belongs to the directed percolation class. Note
however that this is harder to observe experimentally because of the
higher precision needed (precision that we do not currently reach, al-
though we observe lower $\beta$ for those rules, and greater sensitivity to
the sampling interval for $\alpha$). Anyway, those rules do not meet point 4
of Grassberger’s conjecture, since there is another phase transition hap-
pening in the underlying, single configuration, CA.

For both reasons, we leave those rules for future work.

4.3.2 Measure of $\beta$

Let us now plot $\rho$ versus $\alpha$ near $\alpha_c$ (Figure 8). We actually plot $\log \rho$
versus $\log(\alpha - \alpha_c)$ and fit a straight line, the slope of which is an estimator
of $\beta$. It is important to do the fit against $\log \rho$ (and not $\rho$), so that all errors
get the same weight when fitting a line on the log-log plot. It is also possible
to adjust $\alpha_c$ to get a straight line. This method experimentally has the same
computing time/precision ratio as determining $\alpha_c$ beforehand.

Protocol The protocol is only semi-automatic. We choose $n$ between
10 000 and 1 000 000 to get a reasonably smooth line on the density versus
time plot. We visually check that the density has reached a steady state, then
average the density over at least half a decade. This yields one measure point.
We repeat this process for several values of $\alpha$ near $\alpha_c$.

The fit gives the following ranges, taking into account uncertainty about
$\alpha_c$ and which points to keep for the fit. Experimental value for $\beta$ measured
on other systems is 0.276.

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*The present article studies specifically the cases where both configurations are equal, but
since there is only trivial coalescence, or no coalescence for those rules, this assumption is not
trivially false.
Figure 7
Density versus time for a rule in the directed percolation class (58, top) and for 6 (bottom). The asymptotic density goes to 0 as $\alpha \to \alpha_c$ (and is 0 afterwards, i.e. coalescence happens).
As expected, all models remaining at the end of Section 4.3.1 seem to belong to the universality class of directed percolation. We cannot definitely conclude about 11 and 27, due to higher noise and thus lack of precision.

5 ACKNOWLEDGMENTS

We would like to thanks Peter Grassberger for useful advice, and Nazim Fates for sharing early results.

Source code is available on cimula.sf.net.

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