Nonextensive Viscous Dark Energy: A Bayesian Analysis

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By assuming a fluid description for dark energy, we propose a phenomenological approach which takes into account microscopic effects based on the nonextensive (non-additive) thermodynamics. These effects lead to a tiny perturbation in the perfect fluid which is the thermodynamics part of the standard model ($\Lambda$CDM model). The extra quantity arises from the microscopic correlations captured through the distribution function of the 4-momentum of the components of dark energy fluid. These strong statistical correlations are taken into account as a nonextensive effect which is also related to resistance to expansion of volume element of the dark energy fluid. This mechanism provides a way to introduce a nonextensive bulk viscosity, which should be null in the extensive (additive) limit recovering the perfect fluid description. From the phenomenological standpoint, we motivate models of bulk viscosity for dark energy in the context of the nonextensive formulation. In order to test this approach, we perform a Bayesian analysis based on the data of CMB Distance priors, Baryon Acoustic Oscillations Measurements, Cosmic Chronometers, and SNe Ia distance measurements.

I. INTRODUCTION

The accelerated expansion of the universe has widely been corroborated by the greater amount of observational data, such as type Ia Supernovae [1, 2], Baryon Acoustic Oscillations (BAO) [3] and Cosmic Microwave Background (CMB) anisotropies [4, 5]. These observations converge to the standard model, the $\Lambda$CDM model, where cosmological constant $\Lambda$ is responsible by acceleration of the universe and CDM refers to the Cold Dark Matter. Although this model has been confirmed as the standard cosmological model, a theoretical explanation of the physical mechanism responsible by cosmic acceleration has been a significant challenge in the modern cosmology [6]. From the observational standpoint, there is a tension associated with the measurements of Hubble parameter at $z = 0$ by CMB anisotropies [4, 5], and Cepheids and Supernovae [7–10]. The other reported tension is related to measurements of the growth of matter density fluctuations between late-time observations and CMB anisotropies (see more details in Ref. [11]). There are different approaches to solve these puzzles. Typically, they can be divided into modified general relativity [12] and dark energy models [13]. The first case assumes modification in the standard general relativity based on some physical phenomena. The latter case proposes a new description for dark energy, or scalar field within the general relativity framework.

Another idea addressing the dark energy has focused on the fluid description, with the thermodynamics being the core of this scenario (see, e.g., [14] and references therein). Many cosmological models, which are extensions of $\Lambda$CDM, have typically addressed dissipative process like the bulk viscosity in order to provide a thermodynamical framework [15]. More recently, by considering that the dark energy presents a bulk viscosity mechanism, the dark energy models have been analyzed in the context of fluid [16], in the context scalar field [17, 18] and in the modified general relativity framework [19].

Nowadays, the thermodynamics and its microscopic approach (statistical mechanics and kinetic theory) have been extended in order to face the so-called complex systems [20]. By summarizing, the non-additive framework is based on the parameterization of the entropy formula which depends on a free parameter $q$ (also called entropic parameter), and provides the Boltzmann-Gibbs entropy in the additive limit $q \to 1$. In the context of cosmology, many connections have been investigated, e.g. the entropic cosmology for a generalized black hole entropy [21], black holes formation [22, 23], the modified Friedmann equations from Verlinde theory [24], the role of the $q$-statistics on the light dark matter fermions [25], the new perspective for the holographic dark energy [26]. Indeed, there are many connections considering the nonextensive framework (see, e.g., [27] and references therein).

A recent study addressed a connection between dissipative processes and nonextensive framework [28, 29]. The principle behind this connection is based on the so-called Nonextensive/Dissipative Correspondence (NexDC), being associated with the microscopic description of the fluid through the Tsallis distribution function which captures strong statistical correlations among the 4-momenta of the particles [31]. Specifically, the NexDC has been implemented to describe viscous dark matter [28]. In addition, by using the nonextensive effect in the Verlinde’s theory standpoint, a general model was proposed in order to investigate viscous dark matter [29]. Here, we particularly are interested in a new interpretation of viscous dark energy by taking into the account the microscopic statistical correlations for a dark energy fluid. In this regard, we are looking for a mechanism which can explain the viscous dark energy (cosmological bulk viscosity) through the nonextensive framework that emerges from the Tsallis entropy. Indeed, we shall show that this mechanism introduces an energy–momentum tensor given by $T^{\mu\nu}_q = T^{\mu\nu}_{eq} + (q - 1)\Delta T^{\mu\nu}$, where $q$ is the nonextensive parameter, $T^{\mu\nu}_{eq}$ is the equilibrium energy–momentum tensor and $(q - 1)\Delta T^{\mu\nu}$ provides the nonextensive dissipative effect. The nonextensive fluid mimics a perfect fluid plus dissipative flux within the Maxwell–Boltzmann–Juttner statistics. In order to implement this mechanism associated with the nonextensive
viscous dark energy, we assume some simplifications: i) We consider a fluid description for the dark energy; ii) By considering the cosmological principle, most of dissipative effects should be ignored, e.g. shear forces, heat conduction and diffusion. Therefore, the only scalar dissipative component which is consistent with the cosmological principle is the bulk viscosity; iii) By assuming a possible deviation from ΛCDM behavior, we consider that the bulk viscosity is small. Therefore, taking into account these simplifications, we propose a thermodynamical interpretation for the viscous dark energy based on the nonextensive effect. By introducing this mechanism, we shall reinterpret some models of viscous dark energy [16, 17, 28–30], making a Bayesian Analysis 1.

The paper follows the sequence: in Sec. II we introduce our microscopic approach as a possible physical explanation for viscous dark energy and interpret three models [16, 17, 28–30] within of our approach. In order to constrain parameters and compare models, we make a Bayesian Analysis based on the data of CMB Distance priors, Baryon Acoustic Oscillations Measurements, Cosmic Chronometers, and SNe Ia distance measurements, in Sec. III. The main results and discussion concerning our approach for the viscous dark energy models are presented in Sec. IV.

II. DYNAMICS OF NONEXTENSIVE VISCOUS DARK ENERGY

In order to introduce the mechanism which attempts to explain the viscous dark energy, we will consider that the dynamics of a homogeneous, isotropic and flat universe is described by Friedmann’s equations 2:

\[ H = \frac{\dot{a}}{a} = \frac{8\pi G}{3} \rho, \]  

(1)

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p), \]  

(2)

where \( a, \rho, p \) and \( H \) are the scale factor, energy densities and pressures of the fluids, for example, relativistic radiation, dark matter, baryon matter, dark energy, etc., respectively, and \( H \) is Hubble parameter. By assuming the cosmological principle, i.e., a homogeneous and isotropic background, only bulk viscosity is allowed for cosmic fluids.

On the other hand, by considering the nonextensive relativistic kinetic theory, the energy-momentum tensor is written via the relativistic q-distribution function [31, 33]

\[ T^\mu_\nu = \frac{1}{(2\pi)^3} \int \frac{d^3p}{E} p^\mu p^\nu f^q(x, p), \]  

(3)

where 4-momentum is \( p^\mu = (E, p) \) and the energy \( E = \sqrt{p^2 + m^2} \), with \( m \) the mass of the relativistic particles. The generalized distribution \( f(x, p) \) has been proposed through the proof of relativistic H-theorem [34]

\[ f(x, p) = [1 - (1 - q)(\alpha + \beta_\mu p^\mu)]^{1/1-q}, \]  

(4)

where \( \alpha \) and \( \beta_\mu \) are scalar and 4-vector, respectively. These constants are properly chosen in order to give the power-law generalization of Juttner’s exponential distribution [34]. Here, let us consider the first-order Taylor expansion of \( f^q(x, p) \) around \( q = 1 \) [35]

\[ f^q(x, p) \approx f_{q=1}(x, p) + \frac{1}{2}(q-1) \left( \frac{p^\mu u_\mu}{T} \right) \left( -2 + \frac{p^\mu u_\mu}{T} \right) f_{q=1}(x, p), \]  

(5)

where \( f_{q=1}(x, p) \) represents the Juttner’s exponential distribution. The conditions of validity of the expansion for the Tsallis distribution are \( |1-q|\left( \frac{p^\mu u_\mu}{T} \right) < 1 \) and \( |1-q|^2 \left( \frac{p^\mu u_\mu}{T} \right) < 2 \) [31, 35]. Then, substituting the Eq. (5) within (3), we obtain

\[ T^\mu_\nu = T^\mu_\nu_{eq} + (q - 1) \Delta T^\mu_\nu, \]  

(6)

where \( T^\mu_\nu_{eq} \) is the equilibrium energy-momentum tensor and \( (q - 1) \Delta T^\mu_\nu \) gives the nonextensive dissipative effect. From the microscopic standpoint, the dissipative part is given by

\[ \Delta T^\mu_\nu = \frac{1}{2} \frac{1}{(2\pi)^3} \int \frac{d^3p}{E} p^\mu p^\nu \left( \frac{p^\mu u_\mu}{T} \right) \left( -2 + \frac{p^\mu u_\mu}{T} \right) f_{q=1}(x, p) \]  

(7)

Here, we assume that the format of the dissipative term \( \Delta T^\mu_\nu \) is based on the Eckart theory for relativistic bulk viscosity [36], i.e., in the following form

\[ \Delta T^\mu_\nu = -\xi (g^\mu_\nu - u^\mu u^\nu) \nabla_\mu u_\nu, \]  

(8)

where \( \xi \) is the bulk viscosity coefficient, \( g^\mu_\nu \) is the metric and \( u^\mu \), four-velocity. In a Friedmann-Lemaître-Robertson-Walker universe, \( \nabla_\mu u^\mu = 3H \). We will choose a reference frame in which the hydrodynamics four-velocity \( u^\mu \) is unitary, \( u^\mu u_\mu = -1 \) then, by replacing Eq. (8) in (6), we obtain

\[ T^\mu_\nu = (\rho + P_{eff})u^\mu u^\nu + P_{eff} g^\mu_\nu \]  

(9)

where \( \rho \) is the energy density and the effective pressure reads \( P_{eff} = P_{eq} + \Pi \), with \( \Pi = -3(q - 1)\xi H \), the bulk viscosity pressure. By considering the Eq. (9), the energy conservation is given by

3 The expansion using \( f^q \) has been proposed in order to investigate the particle spectra at LHC [35]. We have used the first-order Taylor expansion of \( f(x, p) \) in Ref. [28]. However, both expansion leads to similar conclusions upon the bulk viscosity.

1 The Bayesian analysis has been used in order to investigate the bulk viscous model with different forms of the bulk viscous coefficient [32].

2 Here we have set \( c = \hbar = k_B = 1 \)
\[ \dot{\rho} + 3H(\rho + P_{\text{eff}}) = 0. \] (10)

As each component of the cosmic fluid is individually conserved, we obtain the conservation of the viscous dark energy component

\[ \dot{\rho}_{\text{de}} + 3H(\rho_{\text{de}} + \dot{\rho}_{\text{de}}) = 0, \] (11)

where \( H \) is the Hubble parameter, which describes the background evolution of the universe, and \( \rho_{\text{de}} \) is the dark energy density, \( \dot{\rho}_{\text{de}} = \rho_{\text{de}} + \dot{\Pi} \) is the effective pressure, \( \Pi = -3\xi \rho H \), the bulk viscosity pressure of the nonextensive viscous dark energy with \( \xi = (q-1)\xi \), the bulk viscosity parameter. It is worth mentioning that the inhomogeneous equation of state for nonextensive viscous dark energy is a consequence of the nonextensive effect, i.e., by using the microscopic description of the energy-momentum tensor, (3), we interpret the bulk viscosity as a consequence of the strong correlations among 4-momentum of components of dark energy fluid. Furthermore, this argument is based on the idea that the strong statistical correlations mimic a resistance to expansion of volume element of the dark energy fluid. In particular, in the limit \( q \to 1 \), the bulk viscosity becomes null, and the perfect fluid description based on the Juttner statistics is recovered. Next, we will study some models for viscous dark energy, which have been introduced through different physical motivations, namely: a phenomenological description [16], in the context of dark matter\(^4\) [28, 30] and non-minimal derivative coupling scalar field approach [17]. However, we are following the microscopic approach based on the nonextensive framework. It is worth mentioning, that these models, with the exception of [30], were also proposed in the context of background and used the Eckart theory. Therefore, these models are compatible with the nonextensive approach for the viscous dark energy.

Let us reinterpret the bulk viscosity for the dark energy proposed in [16, 17, 28, 30] through the microscopic mechanism related to nonextensive framework. The choice of bulk viscosity coefficient \( \xi \) generates different nonextensive viscous dark energy models. The general case, \( \xi \) is not constant, and in the literature there are different approaches to determining how bulk viscosity evolves. We consider three different bulk viscosity functions in our analysis: (i) bulk viscosity being proportional to the Hubble parameter, \( \dot{\xi} = \xi_0 H \); (ii) bulk viscosity proportional to energy density and inversely proportional to Hubble parameter; (iii) the usual ansatz for the bulk viscosity, a function for thermodynamical state, i.e., energy density of the fluid, in the case \( \xi = \xi_0 \dot{\rho}_{\text{de}} \).

### A. Model I

The first model analyzed is the bulk viscosity proportional to the Hubble parameter. The model was studied in Ref. [16].

The ansatz choosing for bulk viscosity evolution is

\[ \dot{\xi} = \xi_0 H, \] (12)

and, the effective pressure reads

\[ \dot{\rho}_{\text{de}} = \rho_{\text{de}} - 3H^2 \xi_q, \] (13)

where \( \xi_q = (q-1)\xi \) is the current value for bulk viscosity and \( \rho_{\text{de}} = \omega \rho_{\text{de}} \). Firstly, we consider parameter of equation of state, \( \omega = -1 \), consequently, the effective pressure is \( \dot{\rho}_{\text{de}} = -\rho_{\text{de}} - 3\xi \rho H^2 \). We call this Model Ia. The second case, we consider \( \omega \) as free parameter, this model is called Model Ib. Afterwards, combining Eqs. (11) and (13), the Friedmann equation for Model Ia is given by

\[
\frac{H^2}{H_0^2} = \Omega_b (1+z)^3 + \Omega_{\gamma} (1+z)^4 + \frac{\Omega_{\text{dm}}}{1+\xi_q} (1+z)^3 \\
+ \left(1 - \frac{\Omega_{\text{dm}}}{1+\xi_q}\right)(1+z)^3 \xi_q,
\] (14)

where \( z \) is the redshift, \( \Omega_{\text{dm}} \) is the matter density parameter today and \( \xi_q \) is dimensionless nonextensive bulk viscosity. For Model Ib, the Friedmann equations reads

\[
\frac{H^2}{H_0^2} = \Omega_b (1+z)^3 + \Omega_{\gamma} (1+z)^4 + \frac{\omega \Omega_{\text{dm}}}{\omega - \xi_q} (1+z)^3 \\
+ \left(1 - \frac{\omega \Omega_{\text{dm}}}{\omega - \xi_q}\right)(1+z)^{3(1+\omega-\xi_q)}.
\] (15)

Notice that, this nonextensive viscous model reduces to the standard \( \Lambda \)CDM in the limit \( q \to 1 \) with \( \omega = -1 \).

Also, we consider nonextensive bulk viscosity effects on the non-flat universe. To make this, we add curvature density parameter evolution in the Model Ia, and we name Model Ic. Friedmann equation for this model is given by

\[
\frac{H^2}{H_0^2} = \Omega_b (1+z)^3 + \Omega_{\gamma} (1+z)^4 + \frac{\Omega_{\text{dm}}}{1-\xi_q} (1+z)^3 \\
+ \frac{2\Omega_b}{2+3\xi_q} (1+z)^2 + \left(1 - \frac{2\Omega_b}{2+3\xi_q} - \frac{\Omega_{\text{dm}}}{1+\xi_q}\right)(1+z)^{-3\xi_q},
\] (16)

where \( \Omega_k \) is the today curvature density parameter. Again, in the extensive limit, i.e. \( q \to 1 \) and \( \Omega_k = 0 \), we recover \( \Lambda \)CDM model. For the three models, the dimensionless bulk viscosity is defined by

\[
\xi_q = \frac{8\pi G \xi \rho_0}{H_0}.
\] (17)

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\(^4\) Primarily, this model has been investigated in the context of dark matter [30]. Here, we will use in the description of the dark energy.
where \( \xi \) is the present-day bulk viscosity and \( \rho_{de} \), dark energy density. The effective pressure for this model is

\[
\tilde{p}_{de} = -\rho_{de} - 3\xi \sqrt{\rho_{de}},
\]

where \( \xi_q = (q-1)\xi \). For this ansatz, nonextensive bulk viscosity of dark energy is insignificant in early Universe (when dark matter dominates). Any one way, the bulk viscosity increases in late Universe [17].

From Eqs. (11) and (19) in the flat Universe, the evolution of the Friedmann equation for nonextensive bulk viscosity model is given by

\[
\frac{H^2}{H_0^2} = \Omega_b(1+z)^3 + \Omega_r(1+z)^4 + \Omega_{dm}(1+z)^3 + \Omega_{de}(1+z)\xi_q - \Omega_r(1+z)^4 + \Omega_{dm}(1+z)^3 + \Omega_{de}(z) \xi_q)^{1/2},
\]

where \( \xi_q \) is the dimensionless nonextensive bulk viscosity coefficient defined by

\[
\xi_q = \frac{24\pi G\xi_{q0}}{H_0^2}.
\]

We use the normalization condition \( \Omega_{de} = 1 - \Omega_b - \Omega_{dm} - \Omega_r \).

### B. Model II

Another interesting functional form for nonextensive bulk viscosity is a ratio between corresponding energy density and expansion rate given by [17]

\[
\xi = 3\xi_0 \frac{\sqrt{\rho_{de}}}{H},
\]

where \( \xi_0 \) is the present-day bulk viscosity and \( \rho_{de} \), dark energy density. The effective pressure for this model is

\[
\tilde{p}_{de} = -\rho_{de} - 3\xi \sqrt{\rho_{de}},
\]

where \( \xi_q = (q-1)\xi \). For this ansatz, nonextensive bulk viscosity of dark energy is insignificant in early Universe (when dark matter dominates). Any one way, the bulk viscosity increases in late Universe [17].

From Eqs. (11) and (19) in the flat Universe, the evolution of the Friedmann equation for nonextensive bulk viscosity model is given by

\[
\frac{H^2}{H_0^2} = \Omega_b(1+z)^3 + \Omega_r(1+z)^4 + \Omega_{dm}(1+z)^3 + \Omega_{de}(1+z)^3 + \Omega_{de}(z),
\]

where \( \Omega_b, \Omega_r, \Omega_{dm} \) are baryonic, radiation and dark matter density parameter today, respectively. In goal to determine functional form of \( \Omega_{de} \) we have to solve conservation equation (11) with the effective pressure, Eq. (23), then

\[
\frac{d\Omega_{de}}{dz} = \frac{3\Omega_{de}}{1+z}(1 + \omega) - \frac{\xi_q}{1+z} \left[ \Omega_b(1+z)^3 + \Omega_r(1+z)^4 + \Omega_{dm}(1+z)^3 + \Omega_{de}(z) \right]^{1/2},
\]

where we define a dimensionless nonextensive bulk viscosity parameter as

\[
\tilde{\xi}_q = \frac{24\pi G\xi_{q0}}{H_0^2}
\]

in which is valid for Eq. (25). The initial condition for the differential equation is \( \Omega_{de}(0) = 1 - \Omega_b - \Omega_{dm} - \Omega_r \).

### III. DATA CONSTRAINTS AND BAYESIAN ANALYSIS

In this section, we will present the data and technique used in this work. To constrain parameters and compare models, we perform Bayesian Analysis based on the presented data. Recently, Bayesian Analysis has been extensively used to constraint and compare cosmological models [29, 37–41]. In our analysis, we consider CMB Distance priors derived from Planck 2015, the eight baryon acoustic oscillations measurements [51–55], 24 cosmic chronometers measurements from Ref. [65] and 1048 SNe Ia distance measurements of the Pan-STARRS (Pantheon) dataset [68].

The ΛCDM model is assumed as reference model and is parameterized with following set of cosmological parameters: the dimensionless Hubble constant \( h \), the baryon density parameter, \( \Omega_b \), the cold dark matter density parameter \( \Omega_{dm} \). The parameters of the other models are listed in the Table I together with their priors. We choose uniform priors on baryon parameter \( \Omega_b \) and cold dark matter parameter \( \Omega_{dm} \). For dimensionless Hubble parameter \( h \) we consider a range 10 times wider than value obtained in Ref. [7]. For curvature parameter \( \Omega_k \) and \( \Omega_w \) we adopt 1σ values reported by Planck Results [4] and uniform prior, respectively. The prior for nonextensive bulk viscosity is based in recent results [28–30]. We fix \( \Omega_b h^2 = 2.469 \times 10^{-5} \) [66], \( \Omega_r h^2 = 1.698 \Omega_r \) [67].

The most important quantity for Bayesian model comparison is the Bayesian evidence, or marginal likelihood, and
is obtained here by implementing the PyMultiNest [44], a Python interface for MultiNest [45], the Bayesian tool based on the nested sampling [46] in which calculates the evidence, but still allows constrain parameters with consequence. We plot the results using GetDist [47].

We following the standard description (see Refs. [29, 37, 41, 42]), the posterior distribution \( P(\Theta | D, M) \) is given by

\[
P(\Theta | D, M) = \frac{L(D|\Theta, M)\pi(\Theta|M)}{\mathcal{E}(D|M)},
\]

(27)

where \( L(D|\Theta, M) \), \( \pi(\Theta|M) \) and \( \mathcal{E}(D|M) \), the likelihood, the prior and Bayesian evidence with \( \Theta \) denotes the parameters set, \( D \) the cosmological data and \( M \) the model. The evidence can be written in the continuous parameter space \( \Omega \) as

\[
\mathcal{E} = \int_{\Omega} L(D|\Theta, M)\pi(\Theta|M)d\Theta.
\]

(28)

In order to compare two models, \( M_i \) and \( M_j \), we compute the ratio of the posterior probabilities, given by [42]

\[
P(M_i | D) / P(M_j | D) = B_{ij} P(M_i) / P(M_j),
\]

(29)

where \( B_{ij} \) is known as the Bayes factor, defined as

\[
B_{ij} = \frac{\mathcal{E}_i}{\mathcal{E}_j}.
\]

(30)

The Bayes factor of the model \( i \) relative to the model \( j \) (here, we assumed to be the \( \Lambda \)CDM model). It is emphasized pointing out that the Bayesian evidence rewards models that balance the quality of fit and complexity [48]. Indeed, the larger the number of free parameter, not required by the data, the penalization of the model will be greater than the other. The usual interpretation of the Bayes factor is related to Jeffreys’ scale. We use an alternative version of Jeffreys’ scale suggested in Ref. [42].

| Parameter | Model | Prior | Reference |
|-----------|-------|-------|-----------|
| \( h \) | All | \( \mathcal{U}(0.5584, 0.9064) \) | [7] |
| \( \Omega_0 \) | All | \( \mathcal{U}(0.0005, 0.1) \) | - |
| \( \Omega_{\text{dm}} \) | All | \( \mathcal{U}(0.001, 0.99) \) | - |
| \( \Omega_k \) | Model ic | \( \mathcal{N}(-0.05, 0.05) \) | [4] |
| \( \omega \) | \( \omega\text{CDM} \), Model lb | \( \mathcal{U}(-2.0, 0.0) \) | - |
| \( \xi_{\text{eq}} \) | All except \( \Lambda \)CDM | \( \mathcal{N}(0.0, 0.1) \) | [28–30] |

A. Baryon Acoustic Oscillations Measurements

The interaction between gravitational force and primordial relativistic plasma generates acoustic oscillations at the recombination epoch, which leave their signature in every epoch of the Universe. The measurements of BAO provide an independent standard ruler to constrain cosmological models.

The BAO measurements are given in terms of angular scale and the redshift separation, this is obtained from the calculation of the spherical average of the BAO scale measurement, and it is given by [49, 50]

\[
d_z = \frac{r_s(z_{\text{drag}})}{D_V(z)},
\]

(31)

in which \( D_V(z) \) is volume-averaged distance given by

\[
D_V(z) = \left[ (1 + z)^2 D_A(z)^2 \frac{cz}{H(z)} \right]^{1/3},
\]

(32)

where \( c \) is the speed of light and \( D_A \) is the angular diameter distance given by

\[
D_A(z) = \frac{c}{1 + z} \int_0^\infty \frac{dz}{H(z)}.
\]

(33)

\( r_s(z_{\text{drag}}) \) is the comoving size of the sound horizon calculated in redshift at the drag epoch defined by

\[
r_s(z_{\text{drag}}) = \int_{z_{\text{drag}}}^{\infty} c_s \frac{dz}{H(z)},
\]

(34)

in which \( c_s(z) = \frac{c}{\sqrt{\Omega_{\text{dm}}}} \) is the sound speed of the photon-baryon fluid and \( R = \frac{1}{4} D_{\Omega_{\text{dm}}} \). We consider the redshift at the drag epoch \( z_{\text{drag}} \) given by [50]

\[
z_{\text{drag}} = \frac{1291(\Omega_m h^2)^{0.251}}{1 + 0.659(\Omega_m h^2)^{0.828}} \left[ 1 + b_1(\Omega_m h^2)^{b_2} \right].
\]

(35)

where \( b_1 = 0.313(\Omega_m h^2)^{-0.419}[1 + 0.607(\Omega_m h^2)^{-0.674}], b_2 = 0.238(\Omega_m h^2)^{0.223} \).

In this analysis we consider the BAO measurements from diverse surveys. Furthermore, we also include three measurements from WiggleZ Survey [51]: \( d_s(z = 0.44) = 0.073, d_s(z = 0.6) = 0.0726, \) and \( d_s(z = 0.73) = 0.0592 \). These measurements are correlated by following inverse covariance matrix

\[
C^{-1} = \begin{pmatrix}
1040.3 & -807.5 & 336.8 \\
-807.5 & 3720.3 & -1551.9 \\
336.8 & -1551.9 & 2914.9
\end{pmatrix}.
\]

(36)
For the WiggleZ data, the chi-squared function is
\[ \chi^2_{\text{WiggleZ}} = \mathbf{D}^T \mathbf{C}^{-1} \mathbf{D}, \]
where \( \mathbf{D} = \mathbf{d}^{\text{obs}} - \mathbf{d}^{\text{mod}} \) and \( \mathbf{C}^{-1} \) is the covariance matrix given by Eq. (36).

The chi-squared function related with each survey is given by
\[ \chi^2_{\text{Survey}} = \left[ \frac{d_i^{\text{obs}}(z) - d_i^{\text{mod}}(z)}{\sigma_i} \right]^2, \]
where \( d_i^{\text{obs}} \) is the observed ratio value, \( d_i^{\text{mod}} \) is theoretical ratio value and \( \sigma_i \) is the uncertainties in the measurements for each data point.

Then, the BAO \( \chi^2 \) function contribution is defined as
\[ \chi^2_{\text{BAO}} = \chi^2_{\text{WiggleZ}} + \chi^2_{\text{Survey}}. \]

### B. CMB Distance Priors

CMB distance priors can be derived from data, such as Planck collaboration or WMAP from the full Boltzmann analysis of CMB data. In Refs. [56–58], they discussed the possibility to compress CMB likelihood in few numbers: CMB shift parameter \( \mathcal{R} \) [59], the angular scale of the sound horizon at last scattering \( \ell_A \), they are important to deal with the late-time expansion history, and baryon density today \( \Omega_b h^2 \), it is important to study the late-time universe but not sensitive to the cosmological models.

CMB shift parameter is defined as
\[ \mathcal{R} = \sqrt{\Omega_m (H_0^2 r(z_*) / c)}, \]
where \( r(z_*) = \frac{c}{H_0} \int_0^{z_*} \frac{dz}{H(z)} \) and angular scale of the sound horizon at last scattering
\[ \ell_A = \pi r(z_*) / r_s(z_*), \]
where \( r_s(z_*) \) is comoving size of the sound horizon calculated in the redshift of decoupling epoch given by [60]

\[ z_* = 1048[1 + 0.00124(\Omega_b h^2)^{-0.738}]^2 [1 + g_1(\Omega_m h^2)^{0.82}], \]
where
\[ g_1 = \frac{0.0783 \Omega_b^{0.238}}{1 + 39.5(\Omega_b h^2)^{0.705}}, \]
\[ g_2 = \frac{0.560}{1 + 21.1(\Omega_b h^2)^{1.81}}. \]

Then, the \( \chi^2 \) function of the CMB prior is defined as
\[ \chi^2_{\text{CMB}} = \mathbf{x}_{\text{CMB}}^T \mathbf{C}_{\text{CMB}}^{-1} \mathbf{x}_{\text{CMB}}, \]
where \( \mathbf{x}_{\text{CMB}} = (\mathcal{R}(z_*), \ell_A(z_*), \Omega_b h^2) - (\mathcal{R}^{\text{obs}}, \ell_A^{\text{obs}}, \Omega_b h^2^{\text{obs}}) \) with \( \mathcal{R}^{\text{obs}} = 1.7488, \ell_A^{\text{obs}} = 301.76, \Omega_b h^2^{\text{obs}} = 0.02228 \) and covariance matrix \( \mathbf{C} \) from Planck Results [4].

### C. Cosmic Chronometers

Another analysis considered in this work are the cosmic chronometers obtained through the differential age method. The cosmic chronometer is a method to determine the Hubble parameter values at different redshifts taking the relative age of passively evolving galaxies [61–63]. The method calculates \( dz / dt \) and hence the Hubble parameter is given by
\[ H(z) = -\frac{1}{1 + z} \frac{dz}{dt}. \]

Here, the theoretical values of \( H(z) \) are given by Eqs. (14), (15), (16), (20), (24). The measurement of \( dz / dt \) is obtained through spectroscopic data with high accuracy, then for a precise measurement of the Hubble parameter, it is necessary to measure the differential age evolution \( dt \) of such galaxies, and hence cosmic chronometers are considered to be model independent. A detailed description about the cosmic chronometer method can be found in Ref. [7, 64]. Here we use the 24 measurements of the Hubble parameter in the redshift interval 0.1 < \( z < 1.2 \), which are listed in Table III [65]. The choose of this measures is motivate by the following argument, the expansion history data of the universe might no be smooth outside the quoted redshift range [64].

Then, the \( \chi^2 \) function of the cosmic chronometers is defined as
\[ \chi^2_{\text{CC}} = \sum_{i=1}^{24} \left( \frac{H^{\text{obs}}(z_i) - H^{\text{mod}}(z_i)}{\sigma^*_H} \right)^2, \]
where the \( \sigma^*_i \) uncertainties in the \( H(z) \) measurements for each data point \( i \).
The observational distance moduli of SNe \( \mu_{\text{obs}} \) of 1048 measurements in the redshift range 0 < \( z < 2.3 \) in Refs. [68, 69]). Then, to calculate the observational distance moduli could not be used to constrain parameters. To alleviate this problem, Ref. [68] proposes a method to calibrated SNe Ia named BEAMS with Bias Corrections (BBC) [68, 69]. The Pantheon sample is calibrated using the BBC method, reducing the photometric calibration uncertainties (see more details in Refs. [68, 69]). Then, to calculate the observational distance moduli we subtract \( M_B \) from the apparent magnitude \( m^*_{\text{B,corr}} \) and do not need the color and stretch corrections because now they are equal zero.

The theoretical distance modulus \( \mu_{\text{th}} \) for a given supernova in redshift \( z \) is expressed as

\[
\mu_{\text{th}} = m^*_B + \alpha X_1 - \beta C - M_B, \tag{48}
\]

where \( m^*_B \), \( X_1 \) and \( C \) are the B-band apparent magnitude, the stretch factor and color parameter, respectively. \( M_B \) is the absolute magnitude. \( \alpha \) and \( \beta \) characterize the stretch and color-luminosity relationships, respectively. Commonly, \( \alpha \) and \( \beta \) are considered as free parameters and are constrained jointly with cosmological parameters. Nonetheless, this approach is model dependent, thus the distance calibrated by a cosmological could not be used to constrain parameters. To alleviate this problem, Ref. [68] proposes a method to calibrated SNe Ia named BEAMS with Bias Corrections (BBC) [68, 69]. The Pantheon sample is calibrated using the BBC method, reducing the photometric calibration uncertainties (see more details in Refs. [68, 69]). Then, to calculate the observational distance moduli we subtract \( M_B \) from the apparent magnitude \( m^*_B, \text{corr} \) and do not need the color and stretch corrections because now they are equal zero.

The theoretical distance modulus \( \mu_{\text{th}} \) for a given supernova in redshift \( z \) is expressed as

\[
\mu_{\text{th}} = 5 \log_{10} \frac{d_L}{M_{\text{pc}}} + 25, \tag{49}
\]

where \( d_L = (c/H_0)D_L \) is the luminosity distance, with \( c \) is the speed of light, \( H_0 \) is the Hubble constant. Hubble-free luminosity distance is given by

\[
D_L = (1 + z_{\text{hel}}) \int_0^{z_{\text{CMB}}} \frac{dz}{E(z)}, \tag{50}
\]

where \( E(z) = H(z)/H_0 \), \( z_{\text{CMB}} \) and \( z_{\text{hel}} \) is the dimensionless Hubble parameter, is the CMB frame and heliocentric redshift, respectively. From Eq. (48) with \( \alpha \) and \( \beta \) equal zero, the observed distance moduli reads [68]

\[
\mu_{\text{obs}} = m^*_B - M, \tag{51}
\]

with \( m^*_B \) the B-band apparent magnitude and \( M \) is nuisance parameter that encompasses absolute magnitude \( M_B \) and the Hubble constant \( H_0 \). The \( \chi^2 \) function from Pantheon data is given by

\[
\chi^2_{\text{Pan}} = X^T_{\text{Pan}} \cdot C^{-1}_{\text{Pan}} \cdot X_{\text{Pan}}, \tag{52}
\]

where for the \( i \)-th SNe Ia, \( X_{\text{Pan}} = \mu_{\text{obs},i} - \mu_{\text{th},i} \) and \( C \) is the covariance matrix. We can rewrite Eq. (52) as

\[
\chi^2_{\text{Pan}} = \Delta m^T \cdot C^{-1} \cdot \Delta m, \tag{53}
\]

with \( \Delta m = m_B - m_{\text{mod}}, \) and

\[
m_{\text{mod}} = 5 \log_{10} D_L + M, \tag{54}
\]

in which \( H_0 \) in \( d_L \) can be absorbed into \( M \), while the total covariance matrix \( C \) is given by

\[
C = D_{\text{stat}} + C_{\text{sys}}, \tag{55}
\]

where \( D_{\text{stat}} \) is the diagonal covariance matrix of the statistical uncertainties and \( C_{\text{sys}} \) is the covariance matrix of systematics errors [68]. The nuisance parameter \( M \) could be marginalized following steps in Ref. [70]. Then, after the marginalization over \( M \), we define the following quantities

\[
a = \Delta m^T \cdot C^{-1} \cdot \Delta m, \tag{56}
\]

\[
b = \Delta m^T \cdot C^{-1} \cdot 1, \tag{57}
\]

\[
c = 1^T \cdot C^{-1} \cdot 1, \tag{58}
\]

where \( \Delta m = m_B - m_{\text{mod}} \) and \( 1 \) is a vector of unitary elements, finally, the \( \chi^2 \) function is reads

\[
\chi^2_{\text{Pan}} = a - \frac{b^2}{c} + \ln \frac{c}{2\pi}. \tag{59}
\]

For joint analysis, we consider the likelihood of each probe, namely

\[
L_{\text{joint}} = L_{\text{BAO}} \times L_{\text{CC}} \times L_{\text{CMB}} \times L_{\text{Pan}}.
\]

IV. RESULTS AND CONCLUSIONS

In this work, we proposed a viscous dark energy model assuming a microscopic interpretation for the bulk viscosity based on the nonextensive approach. By considering this new interpretation, we revisited some models for bulk viscosity.
Table IV. Confidence limits for the cosmological parameters using the BAO + CMB + CC + SNe Ia. The columns shows the constrains on each model whereas the rows shows each parameter considering in this analysis. In the last rows we have the Bayesian evidence, Bayes’ factor and the interpretation.

| Parameter | $\Lambda$CDM | $\omega$CDM | Model Ia | Model Ib | Model Ic | Model II | Model III |
|-----------|--------------|-------------|----------|----------|----------|----------|----------|
| $h$       | 0.675 ± 0.005| 0.679 ± 0.008| 0.675 ± 0.006| 0.681 ± 0.009| 0.686 ± 0.009| 0.679 ± 0.008| 0.675 ± 0.005|
| $\Omega_b$| 0.049 ± 0.001| 0.048 ± 0.001| 0.048 ± 0.001| 0.048 ± 0.001| 0.047 ± 0.001| 0.048 ± 0.001| 0.049 ± 0.001|
| $\Omega_{dm}$| 0.267 ± 0.007| 0.266 ± 0.007| 0.267 ± 0.006| 0.264 ± 0.007| 0.267 ± 0.006| 0.266 ± 0.007| 0.266 ± 0.006|
| $\Omega_k$ | – | – | – | – | – | – | – |
| $\omega$ | – | −1.027 ± 0.040 | – | −1.059 ± 0.059 | – | – | −1.012 ± 0.035 |
| $\xi_v$ | – | – | −0.0004 ± 0.008 | −0.0097 ± 0.013 | −0.026 ± 0.020 | 0.012 ± 0.019 | −0.002 ± 0.008 |
| $\ln \mathcal{E}$ | −534.675 ± 0.025 | −537.491 ± 0.008 | −537.168 ± 0.008 | −539.232 ± 0.006 | −537.060 ± 0.029 | −536.199 ± 0.007 | 535.902 ± 0.599 |
| $\ln B$ | – | −2.816 ± 0.008 | −2.493 ± 0.008 | −4.557 ± 0.006 | −2.385 ± 0.029 | −1.524 ± 0.007 | −1.227 ± 0.599 |

Figure 1. Confidence regions and PDFs for the parameters $h$, $\Omega_b$, $\Omega_{dm}$, $\Omega_k$, $\omega$ and $\xi_v$, for all the models studied considering combining data BAO + CMB + CC + SNe Ia.

evolution proposed in the literature. To analyze these models, we performed a Bayesian analysis in terms of the Jeffreys’ Scale that evaluating the strength of evidence when comparing models [42]. To achieve this analysis, we adopted the prior described in Table I and considered distinct background data such as CMB priors distance, BAO measurements, cosmic chronometers, Pantheon Type Ia Supernova.

The main results of joint analysis (CMB + CC + BAO + SNe Ia) were summarized in Table IV, including the mean and corresponding $1\sigma$ error of parameters for each model. Fig. 1 shows the posterior distributions and $1\sigma$, $2\sigma$ and $3\sigma$ contours regions for models studied. In the Table IV, the dimensionless Hubble parameter converged for value obtained in the last Planck results [4, 5]. It is easy to see that $\Omega_b$ and $\Omega_{dm}$, were little affected by the test considered. For the Model Ic, we found that the spatial curvature $\Omega_k = −0.053 ± 0.036$ was not com-
In summary, we showed that the nonextensive viscous dark energy is compatible with the cosmological observations. However, the statistical constraints on the model parameters imply that the standard ΛCDM is recovered, i.e., nonextensive bulk viscosity is zero. Moreover, we concluded from Bayesian analysis standpoint that our model has moderate and weak disfavored evidence compared with ΛCDM. We concluded that ΛCDM still has the best efficiency to explain the data used in this work; this conclusion is dependent either by analyzing the parameters, or the Bayesian evidence.

Finally, it is worth emphasizing that in order to obtain a robust formulation (theoretical and observational), we need to investigate both background expansion and perturbations effects. This issue will be investigated in the future work.

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