ON THE CONVERGENCE OF THE ORTHOGONAL SPECTRAL SEQUENCE

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Abstract. We show that the orthogonal spectral sequence introduced by the second author is strongly convergent in Voevodsky’s triangulated category of motives DM over a field k. In the context of the Morel-Voevodsky $\mathbb{A}^1$-stable homotopy category we provide concrete examples where the spectral sequence is not strongly convergent, and give a criterion under which the strong convergence still holds. This criterion holds for Voevodsky’s slices, and as a consequence we obtain a spectral sequence which converges strongly to the $E_1$-term of Voevodsky’s slice spectral sequence.

1. Introduction

1.1. In order to study finite filtrations on the Chow groups of a smooth projective variety $Y$ over a field $k$ which satisfy some of the properties of the still conjectural Bloch-Beilinson-Murre filtration [2], [4], [15], the second author introduced a tower of triangulated functors $bc_{\leq \bullet}$:

\[
\cdots \to bc_{\leq n-1} \to bc_{\leq n} \to bc_{\leq n+1} \cdots \to
\]

in Voevodsky’s triangulated category of motives $DM$.

The filtration on the Chow groups with coefficients in a commutative ring $R$ is defined by evaluating the tower (1.1.1) in the motive of a point $1_R$ and then mapping $M(Y)(-q)[-2q]$ into $bc_{\leq \bullet}(1_R)$, where $M(Y)$ is the motive of $Y$, and $(-q)$ (resp. $-2q$) is defined in terms of the Tate twist (resp. suspension) in $DM$, see [23]. This process gives a filtration in the Chow groups since [23]:

\[
CH^n(Y)_R \cong \text{Hom}_{DM}(M(Y)(-q)[-2q], 1_R).
\]

Given $A, B \in DM$, one may as well evaluate the tower (1.1.1) in $A$ and then map $B$ into $bc_{\leq \bullet}(A)$. Then one obtains a spectral sequence:

\[
E^1_{p,q} = \text{Hom}_{DM}(B,(bc_{p/p-1}A)[q-p]) \Rightarrow \text{Hom}_{DM}(B,A)
\]

where $bc_{p/p-1}A$ is defined in terms of a canonical distinguished triangle in $DM$ [19, 3.2.8]:

\[
bc_{p-1}A \to bc_{p}A \to bc_{p/p-1}A
\]

The goal of this paper is to study the convergence properties of the spectral sequence (1.1.2).

Our main result (4.1.3) shows that the spectral sequence (1.1.2) is strongly convergent for $B = M(X)(r)[s]$, with $r, s \in \mathbb{Z}$ and $X$ an arbitrary smooth scheme.
of finite type over $k$. We show as well that the analogous result does not hold in the Morel-Voevodsky $A^1$-stable homotopy category $\mathcal{SH}$ by providing explicit counterexamples \(5.1.1\)–\(5.1.3\). On the other hand, we verify that under suitable conditions for $A \in \mathcal{SH}$ the spectral sequence is strongly convergent \(5.1.4\), and that this conditions are satisfied for Voevodsky’s slices $s_n G \in \mathcal{SH}$. As a direct consequence we obtain a spectral sequence converging strongly to the $E_1$-term of Voevodsky’s slice spectral sequence \(5.1.6\).

In a future work, we will apply the spectral sequence \(1.1.2\) and its convergence properties to describe the higher terms for the filtration \([19, 6.14]\) mentioned above on the Chow groups of smooth projective varieties with rational coefficients. We refer the reader to \([12\) and \([6\) for filtrations which are constructed by a related process.

2. Preliminaries

In this section we fix the notation that will be used throughout the rest of the paper and collect together facts from the literature that will be necessary to establish our results.

2.1. Definitions and Notation. We fix a base field $k$. We will write $\text{Sch}_k$ for the category of $k$-schemes of finite type and $\text{Sm}_k$ for the full subcategory of $\text{Sch}_k$ consisting of smooth $k$-schemes regarded as a site with the Nisnevich topology.

We will use the following notation in all the categories under consideration: $0$ will denote the zero object, and $\cong$ will denote that a map (resp. a functor) is an isomorphism (resp. an equivalence of categories).

We shall use freely the language of triangulated categories. Our main reference will be \([17\). Given a triangulated category, we will write $[1]$ (resp. $[-1]$) to denote its suspension (resp. desuspension) functor; and for $n > 0$, $[n]$ (resp. $[-n]$) will be the composition of $[1]$ (resp. $[-1]$) iterated $n$-times. If $n = 0$, $[0]$ will be the identity functor. Given an inductive system \(\cdots \to T_n \to T_{n+1} \to \cdots\), its homotopy colimit, $\text{hocolim}_{n \to \infty} T_n$, will be defined as in \([17\).

2.2. Triangulated categories. Let $\mathcal{T}$ be a compactly generated triangulated category in the sense of Neeman \([16, \text{Def. 1.7}\) with set of compact generators $G$. For $G' \subseteq G$, let $\text{Loc}(G')$ denote the smallest full triangulated subcategory of $\mathcal{T}$ which contains $G'$ and is closed under arbitrary (infinite) coproducts.

Definition 2.2.1. Let $\mathcal{T}' \subseteq \mathcal{T}$ be a triangulated subcategory. We will write $\mathcal{T}' \perp$ for the full subcategory of $\mathcal{T}'$ consisting of the objects $E \in \mathcal{T}$ such that for every $K \in \mathcal{T}'$: $\text{Hom}_{\mathcal{T}}(K, E) = 0$.

If $\mathcal{T}' = \text{Loc}(G')$ and $E \in \mathcal{T}' \perp$, we will say that $E$ is $G'$-orthogonal.

2.3. Slice and orthogonal towers. As in \((2.2\). Consider a family of subsets of $\mathcal{G}$: $\mathcal{S} = \{G_n\}_{n \in \mathbb{Z}}$ such that $G_{n+1} \subseteq G_n \subseteq \mathcal{G}$ for every $n \in \mathbb{Z}$.

Thus, we obtain a tower of full triangulated subcategories of $\mathcal{T}$:

\[\cdots \subseteq \text{Loc}(G_{n+1}) \subseteq \text{Loc}(G_n) \subseteq \text{Loc}(G_{n-1}) \subseteq \cdots\]

We will call \((2.3.1)\) the slice tower determined by $\mathcal{S}$. The reason for this terminology is \([21, 9, 26, \text{p. 18}\). If we consider the orthogonal categories $\text{Loc}(G_n) \perp$ \((2.2.1)\), we obtain a tower of full triangulated subcategories of $\mathcal{T}$:

\[\cdots \subseteq \text{Loc}(G_{n+1}) \perp \subseteq \text{Loc}(G_n) \perp \subseteq \text{Loc}(G_{n-1}) \perp \subseteq \cdots\]
2.3.3. **Orthogonal covers.** Recall [19, 2.1.7(3)] that the inclusion, \( j_n : \text{Loc}(G_n)^{\perp} \to T \) admits a right adjoint:
\[
p_n : T \to \text{Loc}(G_n)^{\perp},
\]
which is also a triangulated functor. We define \( bc_{\leq n} = j_{n+1} \circ p_{n+1} \).

2.3.4. The counit \( bc_{\leq n} = j_{n+1}p_{n+1} \theta_{\leq n} \) id of the adjunction in (2.3.3) satisfies the following universal property (by an argument parallel to [19, 3.2.4]):

For any \( A \) in \( T \) and for any \( B \in \text{Loc}(G_{n+1})^{\perp} \), the map \( \theta_{bc}^{A} : bc_{\leq n}A \to A \) in \( T \) induces an isomorphism of abelian groups:
\[
\text{Hom}_T(B, bc_{\leq n}A) \xrightarrow{\theta_{bc}^{A}} \text{Hom}_T(B, A)
\]

2.3.5. Observe that by construction \( bc_{\leq n}A \) is in \( \text{Loc}(G_{n+1})^{\perp} \) (2.3.3) and in addition \( \text{Loc}(G_{n+1})^{\perp} \subseteq \text{Loc}(G_{n+2})^{\perp} \) (2.3.2). Thus, it follows from (2.3.4) that there exists a canonical natural transformation \( bc_{\leq n} \to bc_{\leq n+1} \) and that \( bc_{\leq n} \circ bc_{\leq n+1} \cong bc_{\leq n} \) (the argument works for any compactly generated triangulated category). Hence, for every \( A \) in \( T \) there is a functorial tower in \( T \) (19, 3.2.14, 3.2.15):

\[
\cdots \to bc_{\leq n}A \to bc_{\leq n+1}A \to \cdots \to \text{hocolim}_{n \to \infty} bc_{\leq n}A
\]

(2.3.6)

where all the triangles commute. We will call (2.3.6) the orthogonal tower of \( A \).

**Definition 2.3.7.** For \( A, B \) in \( T \) we consider the increasing filtration \( F_{\bullet} \) on \( \text{Hom}_T(B, A) \) (resp. \( \text{Hom}_T(B, \text{hocolim}_{n \to \infty} bc_{\leq n}A) \)), where \( F_{p} \) is given by the image of\n\[
\theta_{p}^{A} : \text{Hom}_T(B, bc_{\leq p}A) \to \text{Hom}_T(B, A)
\]

(resp. \( \lambda_{p}^{A} : \text{Hom}_T(B, bc_{\leq p}A) \to \text{Hom}_T(B, \text{hocolim} bc_{\leq n}A) \))

where \( \lambda_{p}^{A} : bc_{\leq p}A \to \text{hocolim}_{n \to \infty} bc_{\leq n}A \) is the canonical map into the homotopy colimit (2.3.6).

2.4. **The orthogonal spectral sequence.** Let \( A, B \) be in \( T \). By an argument parallel to [19] Thm. 3.2.8 there exist canonical triangulated functors \( bc_{p/p-1} : T \to T \), \( p \in \mathbb{Z} \) which fit in a natural distinguished triangle in \( T \):
\[
bc_{\leq p-1}A \to bc_{\leq p}A \to bc_{p/p-1}A
\]

Then (2.3.6) induces a spectral sequence of homological type [19] Thm. 3.2.16:
\[
E_{p,q}^{1} = \text{Hom}_T(B, (bc_{p/p-1}A)[q-p]) \Rightarrow \text{Hom}_T(B, A)
\]

with differentials \( d_r : E_{p,q}^{r} \to E_{p-r,q-r+1}^{r} \) and where the abutment is given by the associated graded group for the increasing filtration (2.3.7) \( F_{\bullet} \) of \( \text{Hom}_T(B, A) \).

Similarly, the horizontal row in (2.3.6) induces a spectral sequence of homological type:
\[
E_{p,q}^{1} = \text{Hom}_T(B, (bc_{p/p-1}A)[q-p]) \Rightarrow \text{Hom}_T(B, \text{hocolim}_{n \to \infty} bc_{\leq n}A)
\]
with exactly the same differentials as (2.4.2) and where the abutment is given by the associated graded group for the increasing filtration (2.3.7) \( F_n \to \Hom_T(B, \hocolim bc_{\leq n} A) \).

Now, we observe that the map \( c \) in (2.3.6) induces a map of spectral sequences (2.4.3) \( \to (2.4.2) \) which is the identity on the \( E_1 \)-terms:

\[
E^1_{p,q} = \Hom_T(B, (bc_{p/p-1} A)(q - p)) \Rightarrow \Hom_T(B, \hocolim bc_{\leq n} A) \\
E^1_{p,q} = \Hom_T(B, (bc_{p/p-1} A)(q - p)) \Rightarrow \Hom_T(B, A)
\]

(2.4.4)

2.5. Voevodsky’s triangulated category of motives. We will only consider motives with \( R \)-coefficients, where \( R = \mathbb{Z}[\frac{1}{p}] \) and \( p \) is the exponential characteristic of the base field \( k \).

Let \( \text{Cor}_k \) be the Suslin-Voevodsky category of finite correspondences over \( k \), i.e. the category with the same objects as \( \text{Sm}_k \) and morphisms \( c(U, V) \) given by the \( R \)-module of finite relative cycles on \( U \times_k V \) over \( U \) [21] with composition as in [25, p. 673 diagram (2.1)]. The graph of a morphism in \( \text{Sm}_k \) induces a functor \( \Gamma : \text{Sm}_k \to \text{Cor}_k \). A Nisnevich sheaf with transfers is an additive contravariant functor \( F \) from \( \text{Cor}_k \) to the category of \( R \)-modules such that the restriction \( F \circ \Gamma \) is a Nisnevich sheaf. Let \( \text{Shv}^{\text{tr}} \) be the category of Nisnevich sheaves with transfers which is an abelian category [14, 13.1]. Given \( X \in \text{Sm}_k \), we will write \( \text{Z}_{\text{tr}}(X) \) for the Nisnevich sheaf with transfers represented by \( X \) [14, 2.8 and 6.2].

Consider the category of chain complexes (unbounded) on \( \text{Shv}^{\text{tr}} \), \( K(\text{Shv}^{\text{tr}}) \), equipped with the injective model structure [3, Prop. 3.13], and let \( D(\text{Shv}^{\text{tr}}) \) be its homotopy category. Let \( K^b(\text{Shv}^{\text{tr}}) \) be the left Bousfield localization [7, 3.3] of \( K(\text{Shv}^{\text{tr}}) \) with respect to the set of maps \( \{ \text{Z}_{\text{tr}}(X \times_k \mathbb{A}^1)[n] \to \text{Z}_{\text{tr}}(X)[n] : X \in \text{Sm}_k; n \in \mathbb{Z} \} \) induced by the projections \( p : X \times_k \mathbb{A}^1 \to X \). Voevodsky’s triangulated category of effective motives \( DM^{\text{eff}} \) is the homotopy category of \( K^b(\text{Shv}^{\text{tr}}) \) [22].

Let \( T \in K^b(\text{Shv}^{\text{tr}}) \) denote the chain complex of the from \( \mathbb{Z}_{\text{tr}}(\mathbb{G}_m)[1] \) [14, 2.12], where \( \mathbb{G}_m \) is the \( k \)-scheme \( \mathbb{A}^1 \setminus \{0\} \) pointed by \( 1 \). We consider the category of symmetric \( T \)-spectra on \( K^b(\text{Shv}^{\text{tr}}) \), \( Spt_T(\text{Shv}^{\text{tr}}) \), equipped with the model structure defined in [8, 8.7 and 8.11], [1, Def. 4.3.29]. Voevodsky’s triangulated category of motives \( DM \) is the homotopy category of \( Spt_T(\text{Shv}^{\text{tr}}) \) [22].

We will write \( M(X) \) for the image of \( \mathbb{Z}_{\text{tr}}(X) \in D(\text{Shv}^{\text{tr}}) \), \( X \in \text{Sm}_k \) under the \( \mathbb{A}^1 \)-localization map \( D(\text{Shv}^{\text{tr}}) \to DM^{\text{eff}} \). Let \( \Sigma^\infty : DM^{\text{eff}} \to DM \) be the suspension functor [8, 7.3] (denoted by \( F_0 \) in \textit{loc. cit.}, we will abuse notation and simply write \( E \) for \( \Sigma^\infty E, E \in DM^{\text{eff}} \). Given a map \( f : X \to Y \) in \( \text{Sm}_k \), we will write \( f : M(X) \to M(Y) \) for the map induced by \( f \) in \( DM \).

Notice that, \( DM^{\text{eff}} \) and \( DM \) are tensor triangulated categories [1] Thm. 4.3.76 and Prop. 4.3.77 with unit \( 1 = M(\text{Spec}(k)) \). We will write \( A(1) \) for \( A \otimes \mathbb{Z}_{\text{tr}}(\mathbb{G}_m)[-1], A \in DM \) and inductively \( A(n) = (A(n-1))(1), n \geq 0 \). We observe that the functor \( DM \to DM, A \mapsto A(1) \) is an equivalence of categories [8, 8.10], [1] Thm. 4.3.38; we will write \( A \to A(-1) \) for its inverse, and inductively \( A(-n) = (A(-n+1))(-1), n > 0 \). By convention \( A(0) = A \) for \( A \in DM \).
2.5.1. **Generators.** It is well known that $DM$ is a compactly generated triangulated category \((\text{2.2})\) with compact generators \([1 \text{ Thm. 4.5.67}]\):

\[(2.5.2)\quad G_{DM} = \{M(X)(p) : X \in Sm_k; p \in \mathbb{Z}\}.

Let $G^{\text{eff}} \subseteq G_{DM}$ be the set consisting of compact objects of the form:

\[(2.5.3)\quad G^{\text{eff}} = \{M(X)(p) : X \in Sm_k; p \geq 0\}.

If $n \in \mathbb{Z}$, we will write $G^{\text{eff}}(n) \subseteq G_{DM}$ for the set consisting of compact objects of the form:

\[(2.5.4)\quad G^{\text{eff}}(n) = \{M(X)(p) : X \in Sm_k; p \geq n\}.

2.5.5. By Voevodsky’s cancellation theorem \([25]\), the suspension functor $\Sigma^\infty : DM^{\text{eff}} \to DM$ induces an equivalence of categories between $DM^{\text{eff}}$ and the full triangulated subcategory $\text{Loc}(G^{\text{eff}})$ of $DM$ \((\text{2.2})\). We will abuse notation and write $DM^{\text{eff}}$ for $\text{Loc}(G^{\text{eff}})$. Strictly speaking \([25 \text{ Cor. 4.10}]\) is only stated for perfect base fields, but by the work of Suslin \([20 \text{ Cor. 4.13, Thm. 4.12 and Thm. 5.1}]\) it follows that the result holds as well for non-perfect base fields.

2.5.6. We will write $DM^{\text{eff}}(n)$ for the full triangulated subcategory $\text{Loc}(G^{\text{eff}}(n))$ of $DM$ \((\text{2.2})\), and $DM^+(n)$ for the orthogonal category $\text{Loc}(G^{\text{eff}}(n)) \perp G_{DM}$ \((\text{2.3.1})\). Notice that $DM^{\text{eff}}(n)$ is compactly generated with set of generators $G^{\text{eff}}(n)$ \([16 \text{ Thm. 2.1(2.1.1})]\). Notice that $DM^{\text{eff}}(n)$ is compactly generated with set of generators $G^{\text{eff}}(n)$ \([16 \text{ Thm. 2.1(2.1.1})]\).

2.5.7. We will consider \((\text{2.3.1}), (\text{2.3.2}), (\text{2.3.6}), (\text{2.3.7}), (\text{2.4.2}), (\text{2.4.3})\) and \((\text{2.4.4})\) in $DM$ for the family $S = \{G^{\text{eff}}(n)\}_{n \in \mathbb{Z}}$ of subsets of $G_{DM}$ \((\text{2.5.3})\).

2.6. **The Morel-Voevodsky $A^1$-stable homotopy category.** We refer the reader to \([10 \text{ §, Thm. 4.15}]\) for the construction of the stable model structure on the category of symmetric $T$-spectra. We will write $SH$ for its homotopy category, which is the Morel-Voevodsky $A^1$-stable homotopy category.

Let $\Sigma^\infty X_+ \in SH$, $X \in Sm_k$ denote the infinite suspension of the simplicial presheaf represented by $X$ with a disjoint base point (written $F_0(X_+)$ in \([10 \text{ p. 506}]\). By \([10 \text{ Prop. 4.19}]\), $SH$ is a tensor triangulated category with unit $1 = \Sigma^\infty \text{Spec } k_+$. We will write $E(1)$ for $E \otimes \Sigma^\infty (G_m)[-1]$, $E \in SH$ and inductively $E(n) = (E(n-1))(1)$, $n \geq 0$. We observe that the functor $SH \to SH$, $E \mapsto E(1)$ is an equivalence of categories \([8 \text{ 8.10}], [1 \text{ Thm. 4.3.38}]\); we will write $E \mapsto E(-1)$ for its inverse, and inductively $E(-n) = (E(-n + 1))(-1)$, $n > 0$. By convention $E(0) = E$ for $E \in SH$.

As in the case of $DM$, it follows from \([1 \text{ Thm. 4.5.67}]\) that $SH$ is a compactly generated triangulated category \((\text{2.2})\) with compact generators:

\[(2.6.1)\quad G_{SH} = \{\Sigma^\infty X_+(p) : X \in Sm_k; p \in \mathbb{Z}\}.

For $n \in \mathbb{Z}$, we will write $G^{\text{eff}}_{SH}(n) \subseteq G_{SH}$ for the set consisting of compact objects of the form:

\[(2.6.2)\quad G^{\text{eff}}_{SH}(n) = \{\Sigma^\infty X_+(p) : X \in Sm_k; p \geq n\}.

2.6.3. Let $SH^{\text{eff}}(n)$ be the full triangulated subcategory $\text{Loc}(G^{\text{eff}}_{SH}(n))$ of $SH$ \((\text{2.2})\), and $SH^+(n)$ be the orthogonal category $\text{Loc}(G^{\text{eff}}_{SH}(n)) \perp G_{SH}$ \((\text{2.3.1})\). Notice that $SH^{\text{eff}}(n)$ is compactly generated with set of generators $G^{\text{eff}}_{SH}(n)$ \([16 \text{ Thm. 2.1(2.1.1})]\).
2.6.4. We will consider (2.3.1), (2.3.2), (2.3.6), (2.3.7), (2.4.2), (2.4.3) and (2.4.4) in $\mathcal{SH}$ for the family $S = \{G_{\mathcal{SH}}(n)\}_{n \in \mathbb{Z}}$ of subsets of $G_{\mathcal{SH}}$ (2.6.2). These were constructed in [18].

3. Orthogonality and Duality

3.1. Recall that we are working with $\mathbb{Z}[\frac{1}{p}]$-coefficients (2.5). In this section we will consider $Y \in \text{Sm}_k$ connected of dimension $d$, and $s, t \in \mathbb{Z}$.

Proposition 3.1.1. With the notation and conditions of (3.1). Then:

$$M(Y)(s)[t] \in DM_{\perp}(d+s+1).$$

(see (2.5.6) and (2.2.1)).

Proof. By [19, 2.1.2] it suffices to show that $\text{Hom}_{DM}(M(X)(a)[b], M(Y)(s)[t]) = 0$, for every $X \in \text{Sm}_k$, $a, b \in \mathbb{Z}$ such that $a \geq d+s+1$. So, by [20] Cor. 4.13, Thm. 4.12 and Thm. 5.1] we may assume that the base field $k$ is perfect. Now, if the base field $k$ admits resolution of singularities, it follows from [22, Thm. 4.3.7] that:

$$\text{Hom}_{DM}(M(X)(a)[b], M(Y)(s)[t]) \cong \text{Hom}_{DM}(M(X) \otimes M^c(Y)(e)[f], 1)$$

where $M^c(Y) \in DM_{\text{eff}}$ is the motive of $Y$ with compact supports [22, §4.1, Cor. 5.1], $e = a - s - d$ and $f = b - t - 2d$. For a perfect base field of positive characteristic, we obtain the same conclusion by [11, Thm. 5.5.14 and Lem. 5.5.6]. Therefore, by [19, 5.1.1] it suffices to check that $M(X) \otimes M^c(Y)(e)[f] \in DM_{\text{eff}}(1)$ (2.5.10), which holds by hypothesis: $e = a - s - d \geq 1$. □

Corollary 3.1.2. With the notation and conditions of (3.1). Let $E = M(Y)(s)[t] \in DM$. Then:

(1) The natural map (2.3.4):

$$\theta_{bc \leq d+s}^E : bc_{\leq d+s}E \to E$$

is an isomorphism in $DM$.

(2) For any $A \in DM$, and any map $f : E \to A$ in $DM$, there exists a unique lifting $g : E \to bc_{\leq d+s}A$ such that the following diagram commutes in $DM$:

$$\begin{array}{ccc}
E & \xrightarrow{f} & A \\
\downarrow^g & & \\
bc_{\leq d+s}A & \xrightarrow{\theta_{bc \leq d+s}^A} & A
\end{array}$$

(3.1.3)

(3) The map $f$ in (3.1.3) is zero if and only if the map $g$ in (3.1.3) is zero.

Proof. [11]: This follows directly by combining (3.1.1) with [19, 3.2.7].

(2) and (3) follow from (3.1.4) and the universal property of $\theta_{bc \leq d+s}^E$ (2.3.4). □

4. Convergence

4.1. In this section we will consider objects $A, B \in DM$, where $B$ is of the form $B = M(X)(s)[t]$ for $X \in \text{Sm}_k$ and $s, t \in \mathbb{Z}$.

Theorem 4.1.1. With the notation and conditions of (4.1). Then the spectral sequence (2.4.3) is strongly convergent [5, Def. 5.2(iii)].
ON THE CONVERGENCE OF THE ORTHOGONAL SPECTRAL SEQUENCE

Proof. Since $B = M(X)(s)[t]$ is compact in $DM_{(2.5.1)}$, it follows from [16, Lem. 2.8] that

$$\text{Hom}_{DM}(B, \text{hocolim}_{n \to \infty} bc_{\leq n} A) \cong \text{colim}_{n \to \infty} \text{Hom}_{DM}(B, bc_{\leq n} A)$$

which implies that the filtration (2.3.7) $F_{\bullet}$ on $\text{Hom}_{DM}(B, \text{hocolim}_{n \to \infty} bc_{\leq n} A)$ is exhaustive.

Now, we observe that $B \in DM_{(2.5.6)}$ for every $t \in \mathbb{Z}$ and by construction $bc_{\leq n} A \in DM_{(2.3.3)}$, so we deduce that $\text{Hom}_{DM}(B, bc_{\leq n} A) = 0$ for all $n \leq s - 1$ and every $t \in \mathbb{Z}$. Hence, applying the distinguished triangle (2.4.1) we conclude that $E_{p,q}^{1} = 0$ for $p \leq s - 1$. Then, [5, Thm. 6.1(a)] implies that the spectral sequence is strongly convergent since the differentials are of the form $d_{r} : E_{p,q}^{r} \to E_{p-r,q-r+1}^{r}$ (notice that our notation is homological while Boardman’s is cohomological, see [5, (12.1) and Thm. 12.2] for an explicit comparison). □

Corollary 4.1.2. With the notation and conditions of (4.1). Assume that the canonical map $c : \text{hocolim}_{n \to \infty} bc_{\leq n} A \to A$ induces an isomorphism of abelian groups:

$$c_{\ast} : \text{Hom}_{DM}(B, \text{hocolim}_{n \to \infty} bc_{\leq n} A) \cong \text{Hom}_{DM}(B, A).$$

Then the spectral sequence (2.4.2) is strongly convergent [5, Def. 5.2(iii)].

Proof. Follows directly by combining (2.4.4) with (4.1.1). □

The following is the main theorem:

Theorem 4.1.3. With the notation and conditions of (4.1). Then the canonical map

$$c : \text{hocolim}_{n \to \infty} bc_{\leq n} A \cong A$$

is an isomorphism in $DM$. Hence, the spectral sequence (2.4.2) is strongly convergent.

Proof. By (4.1.2) it is enough to show that $c$ is an isomorphism in $DM$. In order to prove this, it suffices to see (2.5.1) that for every $a, b \in \mathbb{Z}$ and every connected $Y \in Sm_{k}$ the induced map:

$$c_{\ast} : \text{Hom}_{DM}(M(Y)(a)[b], \text{hocolim}_{n \to \infty} bc_{\leq n} A) \to \text{Hom}_{DM}(M(Y)(a)[b], A)$$

is an isomorphism of abelian groups.

First we show that $c_{\ast}$ is surjective. In effect given $f : M(Y)(a)[b] \to A$ we obtain a lifting by (3.1.2(2)):

$$\begin{array}{ccc}
M(Y)(a)[b] & \xrightarrow{f} & A \\
\downarrow & & \\
bc_{\leq d+a} A & \xrightarrow{\theta_{a+b}} & A
\end{array}$$

where $d$ is the dimension of $Y$. Then the surjectivity follows by (2.3.7).

Finally we consider the injectivity. Let $f : M(Y)(a)[b] \to \text{hocolim}_{n \to \infty} bc_{\leq n} A$ such that the composition $c_{\ast}(f) : M(Y)(a)[b] \to A$ is zero. Since $M(Y)(a)[b]$ is compact in $DM_{(2.5.1)}$, we conclude that [16, Lem. 2.8]:

$$\text{Hom}_{DM}(M(Y)(a)[b], \text{hocolim}_{n \to \infty} bc_{\leq n} A) \cong \text{colim}_{n \to \infty} \text{Hom}_{DM}(M(Y)(a)[b], bc_{\leq n} A)$$
Thus we may assume that $f$ factors as:

$$\xymatrix{ & bc_{\leq n}A \\
M(Y)(a)[b] \ar[ru]^{f'} \ar[d]_f & \ar[d] \ar[d]
\text{hocolim}_{n\to\infty} bc_{\leq n}A \\
& bc_{\leq n}A \\
M(Y)(a)[b] \ar[ru]^{f''} \ar[d]_f & \ar[d] \ar[d]
}$$

for some $n \geq d + a$. Now, applying again 3.1.2(2) we may factor $f'$ as follows:

$$\xymatrix{ & bc_{\leq d+a}A \cong bc_{\leq d+a}(bc_{\leq n}A) \\
M(Y)(a)[b] \ar[ru]^{f''} \ar[d]_f & \ar[d] \ar[d]
bc_{\leq n}A \\
& bc_{\leq n}A \\
M(Y)(a)[b] \ar[ru]^{f'} \ar[d]_f & \ar[d] \ar[d]
}$$

where the isomorphism follows from [19, 3.2.6] since $d + a \leq n$. Thus it suffices to show that $f''$ is zero. But this follows from 3.1.2(3) since

$$\theta_{d+a}^A \circ f'' = c_\ast(f) = 0$$

where the first equality follows from (2.3.6) and the two commutative triangles above while the second equality follows by hypothesis.

\[\square\]

**Remark 4.1.4.** We observe that (4.1.1) and (4.1.2) hold for a compactly generated triangulated category $\mathcal{T}$ with compact generators $G$ and any choice of a family of subsets of $G$: $S = \{G_n\}_{n \in \mathbb{Z}}$ satisfying the conditions in (2.3) and in addition $\bigcup_{n \in \mathbb{Z}} G_n = G$.

However, (4.1.3) does not hold for a general compactly generated triangulated category as we will see in the next section.

## 5. The $\mathbb{A}^1$-stable homotopy category

### 5.1. In this section we show that (4.1.3) does not hold for the sphere spectrum in $\mathcal{SH}$. On the other hand, we show that (4.1.3) holds for objects in $\mathcal{SH}^+(n)$, $n \in \mathbb{Z}$.

As a direct consequence we obtain a spectral sequence converging to the $E_1$-term of Voevodsky’s slice spectral sequence.

**Proposition 5.1.1.** The canonical map $c : \text{hocolim}_{n\to\infty} bc_{\leq n}1 \to 1$ (2.3.10) is not an isomorphism in $\mathcal{SH}$.

**Proof.** We proceed by contradiction, and assume that $c$ is an isomorphism in $\mathcal{SH}$. Then, since $1 \in \mathcal{SH}$ is compact we conclude [15, Lem. 2.8]:

$$\xymatrix{ & bc_{\leq n}1 \\
\text{colim}_{n\to\infty} \text{Hom}_{\mathcal{SH}}(1, bc_{\leq n}1) \ar[ru]^{c} & \ar[l]_{c_\ast} \ar[l] \ar[l] \ar[l] \ar[l] \ar[l] \ar[l] \ar[l]
\text{Hom}_{\mathcal{SH}}(1, 1) \\
1 \ar[ru]_{id} & \ar[d]_1 \\
& 1 \\
}$$

Thus, for some $n \in \mathbb{Z}$ the identity map for $1$ factors as in the following commutative diagram in $\mathcal{SH}$:
Hence, \( 1 \oplus E \cong \Sigma_{bc}^{<n} 1 \in \mathcal{SH}^{+}(n+1) \) \(^{\text{(2.3.3)}}\) for some \( E \in \mathcal{SH} \). Since \( \mathcal{SH}^{+}(n+1) \) is closed under direct summands \(^{\text{(2.2.1)}}\), we deduce that \( 1 \in \mathcal{SH}^{+}(n+1) \). But this is a contradiction since it implies that for every \( m \geq n \), the slice functors of Voevodsky \(^{\text{[24] Thm. 2.2}}\) vanish \( s_m 1 = 0 \) for the sphere spectrum, which is not the case \(^{\text{[24] Conj. 9], [13] p. 350}}\).

\[ \square \]

**Remark 5.1.2.** The argument above shows that \(^{\text{(5.1.1)}}\) holds for any compact object \( A \in \mathcal{SH} \) such that for every \( n \in \mathbb{Z} \) there exists \( m \geq n \) with \( s_m A \neq 0 \).

**Corollary 5.1.3.** Consider the spectral sequence \(^{\text{(2.4.2)}}\) in \( \mathcal{SH} \) for \( A = 1 \). Then the spectral sequence is not strongly convergent for every \( B = \Sigma_{r} X(s)[t], X \in Sm_{k}, s,t \in \mathbb{Z} \).

**Proof.** We proceed by contradiction and assume that the spectral sequence is strongly convergent for every \( B \) as above. Then combining \(^{\text{(2.3.4)}}\) and \(^{\text{(4.1.1)}}\) (which also holds in \( \mathcal{SH} \) \(^{\text{(4.1.4)}}\)) we conclude that

\[ c_{\ast} : \text{Hom}_{\mathcal{SH}}(B, \text{hocolim} bc_{\leq n} 1) \to \text{Hom}_{\mathcal{SH}}(B, 1) \]

is an isomorphism. But this implies that \( c \) is an isomorphism since \( \mathcal{SH} \) is a compactly generated category with generators \( G_{\mathcal{SH}} \) \(^{\text{(2.6.1)}}\).

However, the spectral sequence \(^{\text{(2.4.2)}}\) is strongly convergent for a large class of objects in \( \mathcal{SH} \):

**Proposition 5.1.4.** Let \( A \in \mathcal{SH} \) such that for some \( r \in \mathbb{Z}, A \in \mathcal{SH}^{+}(r) \) \(^{\text{(2.6.3)}}\). Then the canonical map

\[ c : \text{hocolim}_{m \to \infty} bc_{\leq n} A \xrightarrow{\cong} A \]

is an isomorphism in \( \mathcal{SH} \). Hence, the spectral sequence \(^{\text{(2.4.2)}}\) is strongly convergent for every \( B = \Sigma_{r} X(s)[t], X \in Sm_{k}, s,t \in \mathbb{Z} \).

**Proof.** By \(^{\text{(4.1.2)}}\) (which holds as well in \( \mathcal{SH} \) \(^{\text{(4.1.4)}}\)) it is enough to show that \( c \) is an isomorphism in \( \mathcal{SH} \).

Let \( m \geq r \) be an arbitrary integer. It follows from \(^{\text{(2.3.2)}}\) that \( A \in \mathcal{SH}^{+}(m) \), so by the universal property \(^{\text{(2.3.3)}}\) we conclude that \( \theta_{m}^{A} : bc_{\leq m} A \to A \) is an isomorphism in \( \mathcal{SH} \) (see \(^{\text{[19] 2.3.7}}\)). But this implies that the canonical map \( c : \text{hocolim}_{n \to \infty} bc_{\leq n} A \to A \) \(^{\text{(2.3.7)}}\) is an isomorphism in \( \mathcal{SH} \).

**5.1.5.** Let \( B \in \mathcal{SH} \) with \( B = \Sigma_{r} X(s)[t], X \in Sm_{k}, s,t \in \mathbb{Z} \). Consider Voevodsky’s slice spectral sequence \(^{\text{[24] \S 7}}\) for \( G \in \mathcal{SH} \):

\[ E_{1}^{m,n} = \text{Hom}_{\mathcal{SH}}(B, s_{m} G[m + n]) \Rightarrow \text{Hom}_{\mathcal{SH}}(B, G) \]

A direct consequence of \(^{\text{(5.1.4)}}\) is the fact that we obtain a spectral sequence which converges strongly to the \( E_{1} \)-term of Voevodsky’s slice spectral sequence and which is compatible with the differentials \( d_{1} : E_{1}^{m,n} \to E_{1}^{m+1,n} \).

**Corollary 5.1.6.** With the notation and conditions of \(^{\text{(5.1.5)}}\). Then the spectral sequence \(^{\text{(2.4.2)}}\) for \( A = s_{m} G[m + n] \) converges strongly to the \( E_{1} \)-term of Voevodsky’s slice spectral sequence \( E_{1}^{m,n} = \text{Hom}_{\mathcal{SH}}(B, s_{m} G[m + n]) \). In addition, the differential \( d_{1} : E_{1}^{m,n} \to E_{1}^{m+1,n} \) in Voevodsky’s slice spectral sequence induces a
map between the spectral sequences:

\[
E_{p,q}^1 = \text{Hom}_{\SH}(B, (bc_{p/p-1}s_mG[m+n])\lbrack q-p \rbrack) \Longrightarrow E_{1}^{m,n}
\]

(5.1.7)

\[
E_{p,q}^1 = \text{Hom}_{\SH}(B, (bc_{p/p-1}s_{m+1}G[m+1+n])\lbrack q-p \rbrack) \Longrightarrow E_{1}^{m+1,n}
\]

Proof. By construction \(s_mG \in \mathcal{SH}^\perp(m+1)\) [24 Thm. 2.2(3)]. Thus the strong convergence of (2.4.2) for \(A = s_mG[m+n]\) follows directly from (5.1.4).

We observe that the differential \(d_1: E_{1}^{m,n} \rightarrow E_{1}^{m+1,n}\) in the slice spectral sequence is induced by the map \(\partial[m+n]\) in \(\mathcal{SH}\) where \(\partial\) is the following composition [24 Thm. 2.2(1)]:

\[
s_mG \xrightarrow{\sigma_m} f_{m+1}G[1] \xrightarrow{\pi_{m+1}[1]} s_{m+1}G[1].
\]

Since the tower (2.3.6) is functorial in \(\mathcal{SH}\) we conclude that \(\partial[m+n]\) induces the desired map of spectral sequences (5.1.7).

\[\square\]

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