Slowly, rotating non-stationary, fluid solutions of Einstein’s equations and their match to Kerr empty space-time

RJ Wiltshire*
Division of Mathematics and Statistics,
The University of Glamorgan,
Pontypridd, CF37 1DL.
email: rjwiltsh@glam.ac.uk

March 24, 2022

Abstract

A general class of solutions of Einstein’s equation for a slowly rotating fluid source, with supporting internal pressure, is matched using Lichnerowicz junction conditions, to the Kerr metric up to and including first order terms in angular speed parameter. It is shown that the match applies to any previously known non-rotating fluid source made to rotate slowly for which a zero pressure boundary surface exists. The method is applied to the dust source of Robertson-Walker and in outline to an interior solution due to McVittie describing gravitational collapse. The applicability of the method to additional examples is transparent. The differential angular velocity of the rotating systems is determined and the induced rotation of local inertial frame is exhibited.

1 Introduction

In the period since the discovery of the Kerr [1] metric which describes analytically, the asymptotically flat, vacuum gravitational field outside a rotating source in terms of Einstein’s field equations, there have been many attempts to find closed interior solutions which match the exterior smoothly. In general terms attempts to find solutions have proved unsuccessful as has been described by Pichon and Lynden-Bell [2]. One difficulty has been the considerable mathematical complexity in solving Einstein’s equations, see for example, Krasinski [3], Chinea & Gonzalez-Romero [4]. This has led to an ‘embarrassing hiatus’ according to Bradley et al [5] in the number of potential interior solutions available for matching which in turn has contributed to a lack in the development in the theory of differentially rotating fluid bodies in general relativity. Even for the case of the remarkable and much quoted, Wahlquist [6] closed form interior
there is no possible fit to the Kerr exterior as has been recently been shown by Bradley et al [5]. Only for the important case of thin super-massive rotating discs, supported by internal pressure have analytic sources for the Kerr metric been found (Pichon and Lynden-Bell [2]). Yet it is important to develop further the relativistic theory of rotation since it has considerable potential application in astrophysics, for example, in the description of the gravitational collapse of rotating matter, quasars, or potential sources for gravitational radiation.

To this end one way forward is perhaps to adopt a linearised perturbation technique wherein, in the first instance, the interior source is rotating only very slowly. Following the successful match to the Kerr exterior one would proceed to develop higher order perturbation methods to describe sources with higher angular velocity. The use of perturbation techniques in general relativity are of course common. In the context of rotation they have been applied successfully by Hartle [7] to equilibrium configurations of cold stars. In the case of non-equilibrium configurations Kegeles [8] has applied the method to Robertson-Walker dust sources up to the first order in angular velocity parameter although, the results are somewhat restrictive and are not suitable for application to sources supported by internal pressure.

The appropriate junction conditions for solutions of Einstein’s equations are extensively considered in the literature for example, Misner et al [9], Mars & Senovilla [10], Stephani [11], Hernandez-Pastora et al [12] where the main focus of attention concerns the methods of Darmois [13] and Lichnerowicz [14] which have been shown to be equivalent by Bonnor and Vickers [15]. In the Darmois approach it is necessary that the components of the metric tensor, and also the extrinsic curvature for the Kerr exterior and the interior source are continuous at the boundary surface. A common coordinate description of source and exterior is not required. On the other hand, the Lichnerowicz approach requires a common ‘admissible’ coordinate system wherein both the metric tensor and its first partial derivatives are continuous at the boundary surface. Although flexibility and covariance has ensured the extensive use of the Darmois approach in the literature, the requirement to define the ‘admissible’ system by Lichnerowicz has proven to be of benefit for the analysis below.

The main aim here is therefore to develop the perturbation method for rotating systems up to and including first order terms in the angular velocity parameter and to find general solutions of Einstein’s equations for perfect fluid bodies which fit the Kerr solution. The solution of Einstein’s equations for the non-rotating system will be assumed given as for example, the comoving cases presented by McVittie [15], Kustaanheimo [16], Bonnor & Faulkes [17], Chakravarty et al [18] and other described by Kramer et al [19].

Thus in the following the spherically symmetric source will be a given solution of Einstein’s equations described by means of the metric

\[ ds^2 = e^{2\lambda} d\eta^2 - e^{2\mu} d\xi^2 - r^2 d\Omega^2 , \]
\[ d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 , \]

where \( \lambda = \lambda (\xi, \eta) , \mu = \mu (\xi, \eta) , r = r (\xi, \eta) \). In the following the components of this metric will denoted by the tensor \( g_{ab} \). The source boundary surface will
be described by the equation:

\[ \xi_b = F(\eta) \]  

(2)

where \( F(\eta) \) is some function of \( \eta \) alone and where the suffix ‘b’ will denote evaluation at the boundary throughout. Lichnerowicz conditions will be used to join the interior to the Schwarzschild exterior which will be written in the form:

\[
d\sigma^2 = e^N d\Pi^2 - e^{-N} d\Sigma^2 - \Sigma^2 d\bar{\Omega}^2 ,
\]

\[
d\bar{\Omega}^2 = d\bar{\theta}^2 + \sin^2 \bar{\theta} d\bar{\phi}^2 ,
\]

(3)

where:

\[ e^N = 1 - \frac{2m}{\Sigma} > 0 . \]  

(4)

It will be assumed that the coordinate description \((\xi, \theta, \phi, \eta)\) is a suitable admissible system where the boundary conditions apply and that this is related to the \((\Sigma, \bar{\theta}, \bar{\phi}, \Pi)\) description of the exterior by means of the transformation:

\[
\Sigma = \Sigma(\xi, \eta) \quad , \quad \bar{\theta} = \theta \quad , \quad \bar{\phi} = \phi \quad , \quad \Pi = \Pi(\xi, \eta) .
\]

(5)

It follows that (3) can be transformed to the form:

\[
d\sigma^2 = (e^N \Pi^2 - e^{-N} \Sigma^2) d\eta^2 + 2 (e^N \Pi_\xi \Pi_\eta - e^{-N} \Sigma_\xi \Sigma_\eta) d\xi d\eta
- (e^{-N} \Sigma^2 - e^N \Pi^2) d\xi^2 - \Sigma^2 (\xi, \eta) d\Omega^2
\]

(6)

where the suffices \( \xi \) and \( \eta \) mean partial derivatives with respect to \( \xi \) and \( \eta \) respectively. The metric components of (6) will be denoted by \( \gamma_{ab} \). In addition units are chosen such that \( c = 1 = G \).

The approach adopted here therefore will be at first to apply the Lichnerowicz junction conditions, in which the components of \( g_{ab} \) and, their first partial derivatives describing a spherically symmetric non stationary fluid sphere is matched continuously to \( \gamma_{ab} \) and, their first partial derivatives describing the Schwarzschild metric. The conditions will then be considered in the context of slowly rotating systems. The results are used to construct new solutions of Einstein’s equations which are applicable for slow rotation and as an example a rotating dust source is considered and the results of Kegeles [8]. In addition in a further example the McVittie [15] source is ‘set’ into slow rotation.

2 Application of the Lichnerowicz junction conditions

By means of (1) and (3) the continuity of the metric components \( g_{kl} \) and \( \gamma_{kl} \), \( kl = 22 \) and first partial derivatives across the boundary imply that

\[
\Sigma_b = r_b \quad , \quad \{\Sigma_\xi\}_b = \{r_\xi\}_b \quad , \quad \{\Sigma_\eta\}_b = \{r_\eta\}_b
\]

(7)
where, for any function $X = X (\xi, \eta)$:

$$X_b = (X)_{\xi_b=F(\eta)} , \quad \{X_\xi\}_b = \left\{ \frac{\partial X}{\partial \xi} \right\}_{\xi_b=F(\eta)} , \quad \{X_\eta\}_b = \left\{ \frac{\partial X}{\partial \eta} \right\}_{\xi_b=F(\eta)}$$  \hspace{1cm} (8)

However, for any $X_b$:

$$\frac{dX_b}{d\eta} = \{X_\eta\}_b + F_\eta \{X_\xi\}_b ,$$  
$$\{X_\eta\}_b = \frac{d\{X_\xi\}_b}{d\eta} - F_\eta \{X_\xi\}_b$$  
$$\{X_\eta\}_b = \frac{d\{X_\eta\}_b}{d\eta} - F_\eta \frac{d\{X_\xi\}_b}{d\eta} + F^2_\eta \{X_\xi\}_b$$  \hspace{1cm} (9)

and so applying this to $\{\Sigma_\eta\}_b$ and $\{\Sigma_\eta\}_b$ it also follows that:

$$\{\Sigma_\eta\}_b = \{r_\eta\}_b + F_\eta \{r_\xi\}_b - F_\eta \{\Sigma_\xi\}_b ,$$  
$$\{\Sigma_\eta\}_b = \{r_\eta\}_b - F^2_\eta \{r_\xi\}_b + F^2_\eta \{\Sigma_\xi\}_b .$$  \hspace{1cm} (10)

Thus equations (10) may be used to determine $\{\Sigma_\eta\}_b$ and $\{\Sigma_\eta\}_b$ once $F$ and $\{\Sigma_\xi\}_b$ have been determined. In particular, when $F_\eta = 0$ then $\{\Sigma_\eta\}_b = \{r_\eta\}_b$ and $\{\Sigma_\eta\}_b = \{r_\eta\}_b$.

Furthermore from (1) and (6) continuity of the component $g_{kl}$ and $\gamma_{kl}$, $kl = 44, 14, 11$ of the metric across the boundary gives rise to three further relationships that:

$$\{\Pi_\eta\}_b = \left\{ e^{-N} \left( e^{2\lambda} + e^{-N} \Sigma_\eta^2 \right) \right\}_b^{\frac{1}{2}},$$
$$\{\Pi_\eta\}_b = \left\{ e^{-N} \left( e^{-2N} \Sigma_\xi^2 - e^{2\mu} \right) \right\}_b^{\frac{1}{2}}.$$  \hspace{1cm} (11)

and the physical restriction:

$$\left\{ e^{2\lambda} \Sigma_\xi^2 - e^{2(\mu+\lambda)+N} - e^{2\mu} \Sigma_\eta^2 \right\}_b = 0 ,$$  \hspace{1cm} (12)

where:

$$\{e^N\}_b = 1 - \frac{2\bar{m}}{\Sigma_b} = 1 - \frac{2\bar{m}}{r_b} .$$  \hspace{1cm} (13)

Equation (12) is merely the condition:

$$\{m (\xi, \eta)\}_b = \bar{m} .$$  \hspace{1cm} (14)

where the mass function $m (\xi, \eta)$ is as usual defined, in terms of the Riemann tensor through:

$$m (\xi, \eta) = \frac{r}{2} R^3_{232} = \frac{r}{2} \left( 1 + e^{-2\lambda} r^2_\eta - e^{-2\mu} r^2_\xi \right).$$  \hspace{1cm} (15)
This condition may be used to simplify (11) with the result that:

\[
\{\Pi_\eta\}_b = \left\{ e^{-N} e^{\lambda - \mu} \Sigma_\xi \right\}_b , \quad \{\Pi_\xi\}_b = \left\{ e^{-N} e^{\mu - \lambda} \Sigma_\eta \right\}_b .
\] (16)

Consider now the continuity of the partial derivatives of the metric components \(g_{kl}\), and \(\gamma_{kl}\) for \(kl = 44, 11\) across the boundary surface. It follows immediately that:

\[
\{\Pi_{\eta\eta}\}_b = \left\{ \frac{\partial}{\partial \eta} \left[ e^{-\frac{N}{2}} \left( e^{2\lambda} + e^{-N} \Sigma_\eta^2 \right)^{\frac{1}{2}} \right] \right\}_b , \\
\{\Pi_{\xi\xi}\}_b = \left\{ \frac{\partial}{\partial \xi} \left[ e^{-\frac{N}{2}} \left( e^{-N} \Sigma_\xi^2 - e^{2\mu} \right)^{\frac{1}{2}} \right] \right\}_b ,
\] (17)

and:

\[
\{\Pi_{\eta\xi}\}_b = \left\{ \frac{\partial}{\partial \xi} \left[ e^{-\frac{N}{2}} \left( e^{2\lambda} + e^{-N} \Sigma_\eta^2 \right)^{\frac{1}{2}} \right] \right\}_b = \left\{ \frac{\partial}{\partial \eta} \left[ e^{-\frac{N}{2}} \left( e^{-N} \Sigma_\xi^2 - e^{2\mu} \right)^{\frac{1}{2}} \right] \right\}_b .
\] (18)

Furthermore direct expansion of the consistency relation \(\{\partial \Pi_\eta/\partial \xi\}_b = \{\partial \Pi_\xi/\partial \eta\}_b\) in (18) gives rise to:

\[
\{\Sigma_{\eta\xi} - \mu_\eta r_\xi - \lambda_\xi r_\eta\}_b = 0
\] (19)

and, using (11) may be expanded in terms of the Einstein tensor component \(G_{41}\), to give:

\[
\left\{ \frac{r_\gamma e^{2\lambda} G_{41}}{2} \right\}_b + F_\eta \left\{ r_{\xi\xi} - \Sigma_{\xi\xi}\right\}_b = 0 .
\] (20)

Furthermore using (17), (18) and it follows that the continuity of the partial derivatives of the metric components \(g_{kl}\), and \(\gamma_{kl}\) for \(kl = 14\) can also be written as:

\[
0 = \left\{ \frac{\partial}{\partial x} \left( e^{-2\mu} \Sigma_\xi^2 + e^{N} - e^{-2\lambda} \Sigma_\eta^2 \right) \right\}_b
\] (21)

where \(x = \eta, \xi\). With the aid of (11), these conditions may also be written in terms of the Einstein tensor components \(G_{41}\) and \(G_{44}\). Thus:

\[
\left\{ r_{\xi\xi} - \Sigma_{\xi\xi}\right\}_b \left\{ r_\xi e^{-2\mu} + F_\eta r_\eta e^{-2\lambda} \right\}_b = \frac{r_\nu}{2} \left\{ G_{41}^{1r_\eta} - G_{44}^{4r_\xi}\right\}_b , \\
F_\eta \left\{ G_{41}^{4r_\eta} - G_{44}^{4r_\xi}\right\}_b + \left\{ G_{44}^{4r_\xi} - G_{41}^{1r_\eta}\right\}_b = 0 .
\] (22)

The second equation in (22) may also be expressed in terms of the mass function to give:

\[
\left\{ F_\eta m_\xi + m_\eta\right\}_b = \left\{ \frac{dm}{d\eta} \right\}_b = 0 .
\] (23)
The equations (20), and (22) may be substituted one into the other to obtain the following simplification:

\[
\left\{ G_1 - F_\eta G_4 \right\}_b = 0 , \quad \left\{ G_1^4 - F_\eta G_4^4 \right\}_b = 0
\]

(24)

\[
\left\{ r e^{2\mu} G_4 \right\}_b + 2 \left\{ \Sigma \xi \xi - r \xi \xi \right\}_b = 0 .
\]

(25)

These equations may be used to determine \( F, \left\{ \Sigma \xi \xi \right\}_b \) and also express the restriction (23).

Note that the above results have been obtained without reference to the nature of the energy momentum tensor \( T^k_l \). However if it is now supposed that the source is a perfect fluid then (24), (23) become:

\[
p_b = 0 , \quad \left\{ u^1 - F_\eta u^4 \right\}_b = 0
\]

(26)

as expected, where \( p(\xi, \eta) \) is the source pressure and \( u^k \) are the components of the velocity four-vector. Further in a comoving description where \( u^1 = 0 \) then \( F(\eta) \) is a constant, again as expected.

Thus direct application of the boundary conditions specifies conditions or limitations on the functions \( \Sigma(\xi, \eta) \) and \( \Pi(\xi, \eta) \) but in no way defines them uniquely. However, the nature of these transformation functions must be established to complete the definition of the admissible system. It is straightforward to check that the transformation functions:

\[
\Sigma(\xi, \eta) = r(\xi, \eta) \left[ 1 - \left\{ e^{2\mu} G_4 \right\}_b (\xi - F)^2 \right] + \sum_{n=3}^{\infty} D_n (\xi - F)^n
\]

(27)

and

\[
\Pi(\xi, \eta) = \Pi_b + \left\{ \Pi_\xi \right\}_b (\xi - F) + \left\{ \Pi_\xi \xi \right\}_b (\xi - F)^2 + \sum_{n=3}^{\infty} E_n (\xi - F)^n
\]

(28)

with

\[
\Pi_b = \int \left( \left\{ \Pi_\eta \right\}_b + F_\eta \left\{ \Pi_\xi \right\}_b \right) d\eta ,
\]

(29)

and \( D_n = D_n(\eta), E_n = E_n(\eta), n \geq 3 \), are consistent with equations (7), (10), (14), (17), (18) and (25). The ambiguity of \( \Sigma(\xi, \eta) \) and \( \Pi(\xi, \eta) \) is clear through the arbitrary definition of \( D_n \) and \( E_n \).

### 3 Junction conditions for slowly rotating systems

Consider now a general first order rotating source given by

\[
d\sigma^2 = e^{2\lambda} d\eta^2 - e^{2\mu} d\xi^2 - r^2 d\Omega^2 - 2r^2 \sin^2 (\theta) q (Y d\xi d\phi + X d\phi d\eta)
\]

(30)

where \( X = X(\xi, \eta), Y = Y(\xi, \eta) \) and \( q \) is a small angular speed parameter whose square terms and higher are negligible. Note that up to and including
this order the source boundary may still be written as $\xi = \xi_b$. The exterior metric now the Kerr solution which is transformed to the form:

$$d\sigma^2 = \left( e^N \Pi^2_\eta - e^{-N} \Sigma^2_\eta \right) d\eta^2 + 2 \left( e^N \Pi_\xi \Pi_\eta - e^{-N} \Sigma_\xi \Sigma_\eta \right) d\xi d\eta$$

$$- \left( e^{-N} \Sigma^2_\xi - e^N \Pi^2_\xi \right) d\xi^2 - \Sigma^2 (\xi, \eta) d\Omega^2$$

$$- 2 \sin^2 (\theta) a \left( \frac{2m}{\Sigma} \Pi_\xi d\xi d\phi + \frac{2\bar{m}}{\Sigma} \Pi_\eta d\phi d\eta \right)$$

(31)

where, $a$ is the angular speed parameter with negligibly small quadratic terms.

The additional continuity conditions applied to $X = X (\xi, \eta)$ and $Y = Y (\xi, \eta)$ are obtained by comparing (30) with (31) so that with $a = q$:

$$X_b = \left\{ \frac{2\bar{m} \Pi_\eta}{\Sigma r^2} \right\}_b, \quad Y_b = \left\{ \frac{2\bar{m} \Pi_\xi}{\Sigma r^2} \right\}_b, \quad (32)$$

Moreover from (32) the continuity conditions applied to the derivative of the metric tensor require:

$$\{X_\xi\}_b = \left\{ \frac{\partial}{\partial \xi} \left( \frac{2\bar{m} \Pi_\eta}{\Sigma r^2} \right) \right\}_b, \quad \{X_\eta\}_b = \left\{ \frac{\partial}{\partial \eta} \left( \frac{2\bar{m} \Pi_\eta}{\Sigma r^2} \right) \right\}_b, \quad (33)$$

and:

$$\{Y_\xi\}_b = \left\{ \frac{\partial}{\partial \xi} \left( \frac{2\bar{m} \Pi_\xi}{\Sigma r^2} \right) \right\}_b, \quad \{Y_\eta\}_b = \{Y_\eta\}_q = \left\{ \frac{\partial}{\partial \eta} \left( \frac{2\bar{m} \Pi_\xi}{\Sigma r^2} \right) \right\}_b \quad (34)$$

and so using (15):

$$\{Y_\eta - X_\xi\}_b = \left\{ \frac{6\bar{m} e^{\lambda + \mu}}{r^4} \right\}_b. \quad (35)$$

4 Solution of Einstein’s equations for slowly rotating systems

It will now be shown that solutions of Einstein’s equations satisfying the junction conditions do exist. Suppose that Einstein’s equations for a perfect fluid source are written in the form:

$$G^a_b = -8\pi T^a_b, \quad T^a_b = (\rho + p) u^a u_b - \delta^a_b p$$

(36)

where $\rho$, $p$ are the source density and pressure and $u^a$ are the components of the velocity four-vector with the property that $u^a u_a = 1$ and further suppose, for simplicity that comoving fluid spheres satisfying $G^1_1 = G^2_2$ and $G^4_4 = 0$ are given for a non-rotating source. In addition, it is noted that the source density and supporting internal pressure for the metric (30) are not affected by the addition of a rotation speed parameter up to and including order $q$. However, for the slowly rotating systems it is necessary to ensure that the components of the velocity four-vector satisfy $u^1 = u^2$. Now it is straightforward to show that
\[ G_3^2 = 0 = G_2^2, G_1^2 = 0 = G_1^3 \] identically for the metric (30), whilst \[ G_3^1 = 0 = G_1^3 \] will be satisfied so long as:

\[ (Y_{\eta\eta} - X_{\xi\eta}) = (Y_\eta - X_\xi) \left( \lambda_\eta + \mu_\eta - \frac{4r_\eta}{r} \right) \]. \tag{37} 

Thus Einstein’s equations are satisfied whenever:

\[ Y_\eta = X_\xi + \frac{h(\xi)e^{\lambda+\mu}}{r^4} \tag{38} \]

where \( h \) is an arbitrary function of \( \xi \). So comparing (35) with (38) it is seen that the boundary conditions are consistent with Einstein’s equations provided that:

\[ h_b = 6\bar{m}. \tag{39} \]

In addition, the angular velocity \( L(\xi, \eta) \) of the source is given by:

\[ L(\xi, \eta) = \frac{u^3}{u^4} = -\frac{qe^{\lambda-\mu}h_\xi}{16\pi r^4(\rho + p)} - qX \tag{40} \]

and, a particle moving in the field of (30) will have zero angular momentum whenever \( u_3 = 0 \), so that the quantity:

\[ \frac{u_3}{u_4} = \frac{q \sin^2 \theta e^{-\lambda-\mu}h_\xi}{16\pi r^2(\rho + p)} \tag{41} \]

will also be zero for such a particle. It follows that the induced angular velocity \( \Omega(\xi, \eta) \) of the inertial frame is given by:

\[ \Omega(\xi, \eta) = -qX \tag{42} \]

and that with (16) and (32):

\[ \Omega_b = -q \left\{ \frac{2\bar{m}e^{-N}e^{\lambda-\mu}\Sigma_\xi}{\Sigma r^2} \right\}_b. \tag{43} \]

This confirms that the ‘frame dragging’ effect decreases inversely with the cube of \( r \) as is well known (for example, Schutz [20]).

Clearly it is now necessary to determine those functions \( X(\xi, \eta) \) and \( Y(\xi, \eta) \) satisfying the junction conditions which also satisfy (38). These are established by writing:

\[ X(\xi, \eta) = X_b + (\xi - \xi_b) \{X_\xi\}_b + \Psi(\xi, \eta), \tag{44} \]

\[ Y(\xi, \eta) = Y_b + \Phi(\xi, \eta), \tag{44} \]

where \( \{\Psi_\xi\}_b = 0 \) and using (35) and (39) in (38) gives rise to:

\[ \Phi(\xi, \eta) = \int \left[ \Psi_\xi + \frac{h(\xi)e^{\lambda+\mu}}{r^4} - \left\{ \frac{6me^{\lambda+\mu}}{r^4} \right\}_b \right] d\eta. \tag{45} \]

Thus (44) with (15) is solution for \( X(\xi, \eta) \) in terms of \( Y(\xi, \eta) \) and satisfying the necessary boundary conditions (33) to (35).
5 A slowly rotating dust cloud

Suppose that a slowly rotating dust cloud is described by a perturbed Robertson Walker metric of the type:

$$dσ^2 = dη^2 - R^2(η) \left( \frac{dz^2}{1 - kξ^2} + ξ^2dθ^2 + ξ^2\sin^2(θ)dφ^2 \right) - 2ξ^2\sin^2(θ)R^2q(Ydξdφ + Xdφdη) \tag{46}$$

It is easy to see that \(\{m(ξ,η)\}_b = \bar{m}\) from (14) and \(p_b = 0\) from (26) give rise to:

$$R^2(η) = \frac{2\bar{m}}{ξ_b^3R} - k \tag{47}$$

The boundary relations for \(Σ(ξ,η)\) are:

$$Σ_b = ξ_b R, \quad (Σ_ξ)_b = R, \quad (Σ_η)_b = ξ_b Rη$$

$$\{Σ_ξξ\}_b = -\frac{3\bar{m}}{ξ_b^2(1 - kξ^2)}, \quad \{Σ_ηη\}_b = -\frac{2\bar{m}}{Rξ_b} \tag{48}$$

whilst for \(Π(ξ,η)\):

$$Π_b = (1 - kξ^2)^{\frac{1}{2}} \int e^{-N}dη, \quad \{Π_ξ\}_b = \frac{e^{-N}}{1 - kξ^2}ξ_b Rη$$

$$\{Π_η\}_b = \frac{e^{-N}}{(1 - kξ^2)^{\frac{1}{2}}}R, \quad \{Π_ξξ\}_b = \frac{∂}{∂η} \left[ e^{-N}RΣ_η \right]_b, \quad \{Π_ξη\}_b = \frac{∂}{∂η} \left[ e^{-N} RΣ_ξ \right]_b \quad \{Π_ξξ\}_b = \left[ e^{-N}R \right]_b$$

(49)

Furthermore the final solution (44) with (45) may be written as:

$$X(ξ,η) = \frac{2\bar{m}(1 - kξ^2)^{\frac{1}{2}}}{ξ_b^2R^3} \left[ e^{-N} \right]_b + (ξ - ξ_b) \left[ \frac{∂}{∂ξ} \left( \frac{2\bar{m}e^{-N}(1 - kξ^2)^{\frac{1}{2}}Σ_ξ}{Σ^2R^3} \right) \right]_b + Ψ(ξ,η) \quad \{e^{-N}\}_b = 1 - \frac{2\bar{m}}{Rξ_b} \tag{50}$$

$$Y(ξ,η) = \frac{2\bar{m}Rη}{ξ_b^2(1 - kξ^2)^{\frac{1}{2}}R^2} + Φ(ξ,η)$$

(50)

with:

$$Φ(ξ,η) = \int Ψξdη + \left( \frac{h(ξ)}{ξ^4(1 - kξ^2)^{\frac{1}{2}}} - \frac{6\bar{m}}{ξ_b^4(1 - kξ^2)^{\frac{3}{2}}} \right) \int dη \tag{51}$$

This is a general representation of a collapsing, slowly rotating, Robertson-Walker dust cloud and generalises the results given by Kegeles (1978) who does not give transparent forms \(X(ξ,η)\) and \(Y(ξ,η)\).
6 Example with non-zero internal pressure

Consider now, in outline a representative of a broad class of solutions in which the rotating source has non-zero supporting pressure. The chosen source is the McVittie [15] solution although any spherically symmetric, comoving solution of Einstein’s equations with an added rotation term could have been used. The only restriction on their use is that any solution must have a boundary surface where the mass function is constant, or, equivalently the supporting pressure is zero.

In this case consider (30) with:

\[ e^{2\mu} = \frac{S^2}{\xi^4} \left( 1 + \frac{\bar{m}}{2\xi S} \right)^4, \quad e^{2\lambda} = \frac{(1 - \frac{\bar{m}}{2\xi S})^2}{(1 + \frac{\bar{m}}{2\xi S})^2}, \]

\[ r(\xi, \eta) = \frac{\xi S}{\xi^2} \left( 1 + \frac{\bar{m}}{2\xi S} \right)^2, \quad f(\xi) = (1 - c_1 \xi^2)^{\frac{1}{2}} \]

where \( S = S(\eta) \) and \( c_1 \) is constant. The respective internal density and supporting pressure are given by:

\[ 8\pi \rho = \frac{K^5 S^2 + 128\xi^3 S^5 \left( f^2 - 1 \right)}{K^5 S^2}. \]

\[ 8\pi p = -\frac{2K^6 S S_{\eta\eta} + 12\xi K^5 S S_{\eta} - 5K^6 S_{\eta}^2 + 256\xi^4 S^6 \left( f^2 - 1 \right)}{K^5 S^2 \left( 4\xi S - K \right)}, \]

where \( K = 2\xi S + \bar{m} f \). Since this metric is described by a comoving observer then \( F_{\eta} = 0 \) and the boundary surface is given by the constant \( \xi = \xi_b \). This surface is defined through the constancy of the mass function (14) at the boundary or by differentiating this with respect to \( \eta \) to obtain the zero pressure boundary condition (24). Using the equations one may calculate:

\[ S_{\eta} = \left\{ \frac{8\sqrt{2} \left[ \xi^3 S^5 \left( 2\xi S - f^2 K \right) \right]^{\frac{1}{2}}}{K^3} \right\}_b. \]

The transformation functions, \( \Sigma_b, \{ \Sigma_\xi \}_b, \{ \Sigma_\eta \}_b, \{ \Sigma_{\xi\eta} \}_b \) and \( \{ \Sigma_{\xi\xi} \}_b \) may be calculated directly using (7), (11) and (23) whilst \( \Pi_b, \{ \Pi_\xi \}_b, \{ \Pi_\eta \}_b, \{ \Pi_{\xi\eta} \}_b, \{ \Pi_{\xi\xi} \}_b \) and \( \{ \Pi_{\xi\eta} \}_b \) may be determined easily from (16), (17) and (18). Some simplification of the results will occur through the use of (15), (55). The resulting somewhat lengthy expressions need not be reproduced here.

To obtain the corresponding rotating solution it is necessary to calculate each of \( X_b, \{ X_\xi \}_b, \{ X_\eta \}_b, Y_b, \{ Y_\xi \}_b, \{ Y_\eta \}_b \) to ensure a smooth match to empty space-time of which \( X_b, \{ X_\xi \}_b, Y_b, \) and \( \{ Y_\xi \} \) are also required for the explicit determination of the solution (44) and (45). These calculations are straightforward.
Also note that to obtain the final solution it is necessary to evaluate the integral (45) for which the integrand contain terms of the type:

$$h(\xi) e^{\lambda + \mu} r^{-4}$$

where, \( h_b = 6\bar{m} \). Clearly the resulting solution will therefore not have a closed form. None the less the solution for a rotating version of the McVittie (1933) has been established formally, although it is interesting to note that a solution so elegant in the absence of rotation has such an awkward form when slow rotation is included.

## 7 Conclusion

Solutions of Einstein’s equations for slowly rotating time varying sources supported by internal pressure have been presented. It has been shown that any known collapsing, or expanding, fluid source known by a comoving observer not to be rotating may be ’made’ to rotate slowly and also matched smoothly to the Kerr exterior at all times provided that a zero pressure boundary surface exists. In each case the source rotates with an angular velocity which is inversely proportional to the sum of its internal density and pressure and the induced rotation of the inertial frame of reference is explicitly clear.

It has been shown that rotating solution may be expressed in terms of the expression \( \int e^{\lambda + \mu} r^{-4} d\eta \) and thus it follows that this must be integrable in closed form for the existence of analytic solutions. It is interesting to speculate that solutions of Einstein’s equations have previously been found without the need to give the nature of this expression any consideration. For example it has previously been common to express comoving systems in terms of the isotropic coordinate \( r = \xi e^{\mu} \) which may not necessarily be an appropriate choice for the development of rotating descriptions. This matter is currently under investigation.

The analysis here has been presented here with accuracy up to to an including first order terms in the angular speed parameter. The natural extension of the work to the matching of second order case with Kerr space-time, where the source boundary is more complex in nature, will be reported in the near future. Extensions of this approach to higher orders may result in an analytic description of sources supported by internal pressure and which are surrounded by empty space-time.

Finally, although the focus here has been upon slowly rotating compact bodies it is interesting to note that the Robertson-Walker metric with the added rotation term (46) may be considered as a cosmological model. In this case the junctions conditions need not be applied. The expressions for pressure and density remain homogeneous and the angular velocity term (40) may also be
made independent of $\xi$ by choosing:

$$h(\xi) = \int \frac{\xi^4 d\xi}{(1 - k\xi^2)^{\frac{3}{2}}}$$  \hfill (57)

and $X = X(\eta)$. The solution of Einstein’s equations (58) then may be written as:

$$Y_\eta = \frac{h(\xi)}{\xi^4 (1 - k\xi^2)^{\frac{3}{2}} R^3}.$$  \hfill (58)

8 Acknowledgments

I would like to record my sincere thanks for the helpful encouragement of Professor Bill Bonnor at Queen Mary & Westfield College, London during the past year and also, Professors Leonid Grishchuk, Mike Edmunds and Peter Blood for making me so welcome in the Physics and Astronomy Department in Cardiff. I am also much indebted to those at the University of Glamorgan who made my visit to Cardiff possible.

References

[1] Kerr, RP. 1963. Phys. Rev. Lett.,11, 237.
[2] Pichon, C & Lynden-Bell. 1996. Mon.Not. Roy. Astron. Soc., 280, 1007.
[3] Krasinski, A. 1998. J. Maths. Phys., 39,4, 2148.
[4] Chinea, FJ & Gonzalez-Romero, LM (1993), Rotating objects and relativistic physics. Lecture Notes in Physics 423. Springer Verlag.
[5] Bradley, M, Fodor, G,Marklund,M & Perjes, Z. 2000. Class,Quant Grav,17,351.
[6] Wahlquist, HD. 1968. Phys. Rev. 172. 1291.
[7] Hartle, JB. 1967. Astro. Phys. J., 150. 1005.
[8] Kegeles, L.S. 1978. Physical Review D, 18,4,1020.
[9] Misner, CW, Thorne, KS & Wheeler, JA. 1973. Gravitation. WH Freeman & Company
[10] Mars, M & Senovilla, JMM. 1993. Class. Quantum. Grav.,10, 1865.
[11] Stephani, H.1994. General Relativity. Cambridge University Press.
[12] Hernandez-Pastora,JL, Martin, J & Ruiz, E. 2001. gr-qc/0109031
[13] Darmois, G. 1927. Mémorial des Sciences Mathématiques, Vol 25, (Paris: Gautier-Villars)
[14] Lichnerowicz, A. 1955. *Théories relativistes de la gravitation et de l’Electromagnetisme*, Masson & Cie.

[15] Bonnor, WB & Vickers, PA, 1981, Gen. Rel Grav., 13, 29.

[16] McVittie, GC, 1933. Mon. Not. Roy. Astron. Soc., 93, 325.

[17] Kustaanheimo, P & Qvist, B. 1948. Comment. Phys. Math., Helsingf. 13, 12, 1.

[18] Bonnor, WB & Faulkes, MC, 1967. Mon. Not. Roy. Astron. Soc., 137, 239.

[19] Chakravarty, N, Choudhury, SBD & Baberjee, A. 1976. Aust. J. Phys. 29, 113.

[20] Kramer, D, Stephanie, H, MacCallum, M & Herlt, E. 1980. *Exact solutions of Einstein’s field equations*. Cambridge University Press.

[21] Schutz, B. 2000. *A first course on General Relativity*. Cambridge University Press.