CONFORMAL MAPPING OF EXTERIOR REGIONS VIA A DENSITY CORRESPONDENCE FOR THE DOUBLE-LAYER POTENTIAL

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Abstract. We derive a representation formula for harmonic polynomials in terms of densities of the double-layer potential on bounded piecewise smooth and simply connected domains. From this result, we obtain a method for the numerical computation of conformal maps for the exterior of such regions onto the exterior of the unit disk. We present analysis and numerical experiments supporting the accuracy and broad applicability of the method.

Key words. conformal map, potential theory, Faber polynomial, exterior map, high-order methods

AMS subject classifications. 65E05, 30C30, 65R20

1. Introduction. The numerical construction of a function that maps the exterior of a simply connected region conformally onto the exterior of some other region arises in a number of applications including aerodynamics (for example, the calculation of potential flow around an airfoil), iterative methods [11,12,32], initial value problems [24], and approximation of matrix functions and analytic functions [4,14]. A number of the latter applications are enabled by the exterior map’s close relationship with Faber polynomials [13] and their approximation properties.

We focus on the case where the domain is the exterior of a Jordan curve Γ and the target region is the exterior of the unit circle. Most techniques for conformal mapping of exterior regions, much like their interior counterparts, rely on computing a boundary correspondence function θ between Γ and the boundary of the target domain. From the boundary correspondence, the mapping function can be derived via a Cauchy integral [19, p. 381]. As a technical tool for this work, we study the representation of harmonic polynomials defined on the interior Jordan domain Ω+ as double-layer potentials

\[ D\varphi(x) = -\frac{1}{2\pi} \int_\Gamma \varphi(y) \frac{\partial \log |y-x|}{\partial n(y)} ds(y), \quad x \in \Omega^+. \]

We characterize the real-valued density functions \( \varphi \) that give rise to harmonic polynomials in terms of the exterior mapping function. Specifically, we find that all such densities for a polynomial of degree \( n \) take the form

\[ \varphi(y) = \sum_{k=0}^{n} \lambda_k \cos(k\theta(y) + \mu_k) \]

where \( \theta \) is the boundary correspondence. Letting \( n = 1 \), this leads to a uniquely solvable, second kind integral equation for the boundary correspondence. Furthermore, we demonstrate how the Nyström discretization [26] resulting from the application of high-order accurate quadrature rules in turn achieves high-order accuracy. Our method is of practical interest because of the availability of fast solvers for second kind equations involving the double-layer potential.

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From a mathematical standpoint, the problem of exterior and interior mapping are equivalent via an inversion of the extended complex plane $\mathbb{C} \cup \{\infty\}$ (see Section 2.2). While methods for interior mapping can also be used for exterior mapping, they tend to encounter numerical difficulty when applied in this setting [15, p. 12], and thus a number of methods have been developed specifically for the exterior case. Examples of further methods employing the boundary correspondence include the method of Murid et. al [25] using an integral equation based on the Kerzman-Stein kernel, and DeLillo and Elcrat [9] who use an iterative method based on Timman’s method to solve a nonlinear integral equation for the boundary correspondence.

Other methods used for this problem include Symm’s method, which is a based on an integral equation involving the logarithmic potential [21,33]. Amano’s method [1] is similar to Symm’s method but uses an approximation to the logarithmic potential constructed by placing point charges inside the domain. Papamichael and Kokkinos have applied the Ritz method, based on the minimum area principle, and the Bergman kernel method, based on constructing an approximation to the Bergman kernel, to the exterior mapping function [27]. Rabinowitz [29] uses sequences of orthogonal polynomials defined on the boundary, whose ratios converge to the exterior mapping function.

For methods for the inverse problem, that is, finding a conformal map from the exterior of the unit disk onto the exterior of a given domain, see, for instance [8,9,17]. For a comprehensive overview of methods for interior and exterior mapping see Wegmann [34], Gaier [15], or Henrici [19].

The remainder of this paper is organized as follows: In Section 2.1, we recall some facts about harmonic functions defined on $\Gamma \cup \Omega^+$, in particular relating to potential theory and the Cauchy integral, and in Section 2.2 we discuss properties of the exterior mapping function. Based on these preliminaries, we introduce our main technical result regarding the representation of harmonic polynomials by double-layer potentials in Section 3. This allows us to develop method for exterior mapping and a high-order discretization method for the same, in Section 4. We close in Section 5 with some numerical experiments on smooth and non-smooth domains.

2. Preliminaries. In this paper, we work with a simple closed positively oriented curve $\Gamma$. The following conventions will be used throughout the paper:

- We refer to the inner component of the curve $\Gamma$ as $\Omega^+$ and to $\Omega^-$ as the outer component.
- We define the exterior (Riemann) map $\Psi$ as the unique complex analytic bijection that maps $\Omega^-$ onto the exterior of the unit disk for which $\lim_{z \to \infty} \Psi(z) = \infty$ and $\lim_{z \to \infty} \Psi'(z)$ is real and positive.
- The circle of radius $R$ centered at 0 is denoted as $C_R$.
- The unit disk is $\mathbb{D}^+$ and the exterior of the unit disk is $\mathbb{D}^-$.
- Carathéodory’s theorem [28] implies $\Psi$ has a continuous extension onto $\Gamma \cup \Omega^-$. This extension establishes a one-to-one correspondence between $\Gamma$ and the unit circle $C_1$. The real-valued mapping $\theta(w)$, relating points $w \in \Gamma$ on the curve to points $\exp(i\theta(w))$ on the unit circle, as in $\theta(w) = \arg\Psi(w)$ is known as the boundary correspondence.

2.1. Harmonic functions and potential theory. Any function $f : \Gamma \cup \Omega^+ \to \mathbb{R}$ that is harmonic in $\Omega^+$ and whose restriction to $\Gamma$ is continuous admits a represen-
tation as a double-layer potential

\[ f(x) = D\varphi(x) = -\frac{1}{2\pi} \int_\Gamma \varphi(y) \frac{\partial \log |y - x|}{\partial n(y)} \, ds(y), \quad x \in \Omega^- \quad (2.1) \]

associated with a unique, continuous real valued density function \( \varphi : \Gamma \to \mathbb{R} \) [23 Thm. 6.22]. Here \( \frac{\partial}{\partial n(y)} \) denotes the derivative with respect to the outward-facing unit normal.

As \( |y - x| \) is nonzero, one can take a complex analytic branch of \( \log(y - x) \). The Cauchy-Riemann equations imply the relation between the normal derivative \( \frac{\partial}{\partial n(y)} \) and the derivative with respect to the unit tangential vector to the curve, \( \frac{\partial}{\partial \tau(y)} \),

\[ \frac{\partial \log |y - x|}{\partial n(y)} = \text{Im} \frac{\partial \log(y - x)}{\partial \tau(y)}. \]

Since

\[ \frac{\partial \log(y - x)}{\partial \tau(y)} = \lim_{h \to 0} \frac{1}{h} \left[ \log (y - x + h\tau(y)) - \log(y - x) \right] = \frac{\tau(y)}{y - x}, \]

the double-layer potential \( D\varphi \) can be written as

\[ f(x) = D\varphi(x) = -\frac{1}{2\pi} \text{Im} \int_\Gamma \varphi(y) \frac{\tau(y)}{y - x} \, ds(y). \quad (2.2) \]

The kernel appearing in [22] is also referred to the Neumann kernel [19 Def. 15.9-4]. [22] also makes clear the relationship between the double-layer potential and the Cauchy integral operator. Since \( \tau(y) ds(y) = dy \) and \( \text{Re} \, i\alpha = -\text{Im} \alpha \), we have

\[ f(x) = -\text{Re} \frac{1}{2\pi} \int_\Gamma \frac{\varphi(y)}{y - x} \, dy, \quad x \in \Omega^+ \]

and thus, for real-valued densities \( \varphi : \Gamma \to \mathbb{R} \), the double-layer potential coincides with the real part of the Cauchy integral operator applied to \( \varphi \) [23 eqn. (7.35)]. This derivation continues to hold in the case of piecewise smooth \( \Gamma \) [19 p. 276].

2.2. Properties of the exterior mapping function. The existence of the exterior map \( \Psi : \Omega^- \to \mathbb{D}^- \) is a direct consequence of the Riemann mapping theorem for bounded simply connected domains on the complex plane. In fact, the exterior map is related to an associated interior (Riemann) map \( \hat{\Psi} : \hat{\Omega}^+ \to \mathbb{D}^+ \), where \( \hat{\Omega}^+ \) is the inner component of the inverted curve \( \frac{1}{\Gamma} = \{ \frac{1}{z} \mid z \in \Gamma \} \). The relation is given by [19 p. 381]:

\[ \hat{\Psi}(z) = 1/\Psi(1/z), \quad z \in \hat{\Omega}^+ \]
\[ \Psi(z) = 1/\hat{\Psi}(1/z), \quad z \in \Omega^- \]

In particular, this means that many regularity theorems typically stated for the case of interior maps, such as the Carathéodory theorem, also continue to hold in an analogous sense for the exterior map, via inversion.

Let \( R > 0 \) sufficiently large so that the domain \( \Omega^+ \) is contained within a disk of radius \( R \) around 0. Then, as \( \Psi \) is univalent for \(|w| > R\), it follows that \( \Psi \circ \frac{1}{w} \) is
2.1. Conformal mapping of the exterior of the unit square onto the exterior of the unit circle.

2.2. Conformal mapping of the exterior of a complicated domain onto the exterior of the unit circle.

univalent for $0 < |w| < \frac{1}{R}$. Then from the argument principle the pole at $w = 0$ is simple and $\Psi\left(\frac{1}{w}\right)$ has the Laurent expansion

$$\Psi\left(\frac{1}{w}\right) = \frac{\alpha_1}{w} + \alpha_0 + \alpha_{-1}w + \alpha_{-2}w^2 + \cdots, \quad 0 < |w| < R.$$  

This implies that $\Psi$ has the representation

$$\Psi(z) = \alpha_1z + \alpha_0 + \frac{\alpha_{-1}}{z} + \frac{\alpha_{-2}}{z^2} + \cdots, \quad |z| > R.$$  

(2.3)

2.2.1. The inverse map $\Psi^{-1}$ and the continuity of its derivative at the boundary. For purposes of Lemma 3.1 we briefly recall the inverse function $\Psi^{-1} : \Omega^- \to \Omega^-$ and describe the regularity properties concerning the derivative $[\Psi^{-1}]'$ of the inverse exterior map. $\Psi^{-1}$ is a Riemann map that maps the exterior of the unit
Proposition 2.1. (follows from [19, Sec. 16.4], [28, Thm. 3.9]) Suppose that \( \Gamma \) is piecewise \( C^2 \) and has no inward facing cusps (corners with outward angle 0). Then the function \( [\Psi^{-1}'] : D^- \to \Omega^- \) is analytic for \( |z| > 1 \), is continuous and nonzero up to and on \( |z| = 1 \) except at the pre-images of corner points, and \( [[\Psi^{-1}']] \) is integrable on \( |z| = 1 \).

We note that, generally speaking, more smoothness in \( \Gamma \) leads to better regularity of \( \Psi^{-1} \) and its derivatives, a fact that is useful in the analysis of the Nyström method with composite trapezoid rule (see Section 4.2). In a \( C^2 \) curve without the presence of corner points, \( [\Psi^{-1}'] \) has an extension to the unit circle that is continuous and nonzero [28, Thm. 3.6].

3. Harmonic Polynomials and the Double-Layer Potential. The following lemma identifies the double-layer density functions that correspond to harmonic polynomials.

Lemma 3.1. Let \( \Gamma \) be piecewise \( C^2 \) with no inward facing cusps. A function \( f : \Omega^+ \to \mathbb{R} \) is a harmonic polynomial of degree \( n \) if and only if \( f \) has the representation \( f = D \varphi \) and the associated double-layer density function \( \varphi \) takes the form

\[
\varphi(z) = \sum_{k=0}^{n} \lambda_k \cos(k\theta(z) + \mu_k), \quad z \in \Gamma \tag{3.1}
\]

for some set of real coefficients \( \{\lambda_k \mid 0 \leq k \leq n\} \cup \{\mu_k \mid 0 \leq k \leq n\} \).

**Proof.**

Let \( m \) be a non-negative integer. The exterior map \( \Psi : \Omega^- \to D^- \) raised to the \( m \)th power takes the form

\[
\Psi(z)^m = (\alpha_1 z + \alpha_0 + \alpha_{-1} z^{-1} + \alpha_{-2} z^{-2} + \cdots)^m \tag{3.2}
\]

for \( |z| > R \), where \( R \) is sufficiently large so that the domain \( \Omega^+ \) is contained inside the disk of radius \( R \) centered at 0. Thus, we may write

\[
\Psi(z)^m = \sum_{k=0}^{m} \gamma_k z^k + \sum_{k=1}^{\infty} \frac{\gamma_{-k}}{z^k} = P_m(z) + Q_m(z), \quad |z| > R
\]

so that \( P_m(z) \) and \( Q_m(z) \) are the polynomial and non-polynomial parts of the Laurent expansion for (3.2), respectively. The set \( \{P_m\}_{m=0}^{\infty} \), known as the Faber polynomials [16, p. 42], forms a basis for all complex polynomials. In turn, every harmonic polynomial in the plane is the real or imaginary part of a complex polynomial. So, to prove the lemma, it suffices to show that the real and imaginary parts of the Faber polynomials, restricted to the domain \( \Omega^+ \) have the representation \( D \varphi \) given in (3.1).

For the case \( m = 0 \), \( P_0(z) = \Psi(z)^0 = 1 \) for all \( z \), so that, when \( z_0 \in \Omega^+ \), \( P_0(z_0) = D \varphi(z_0) \) where \( \varphi \equiv -1 \) [23, eqn. (6.23)]. For \( m > 0 \), we show that the \( m \)th Faber polynomial \( P_m \) has the representation

\[
P_m(z_0) = -2D(\varphi_{1,m}(z_0) + i\varphi_{2,m}(z_0)), \quad z_0 \in \Omega^+ \tag{3.3}
\]

where

\[
\varphi_{1,m}(z) = \cos(m\theta(z)), \quad \varphi_{2,m}(z) = \sin(m\theta(z)), \quad z \in \Gamma.
\]

Then (3.1) follows from the linearity of \( D \) and the linear combination formula for sine and cosine.
Let $P_m$ be a Faber polynomial, $m > 0$. Let $z_0 \in \Omega^+$. First, we show that

$$P_m(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi(z)^m}{z - z_0} \, dz. \quad (3.4)$$

(The argument is essentially that of Gaier [16]). What distinguishes (3.4) from the Cauchy integral formula is that $\Psi(z)$ is only guaranteed to be analytic in $\Omega^-$ (but not in $\Omega^+$). However, the Cauchy integral formula implies

$$P_m(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{P_m(z)}{z - z_0} \, dz. \quad (3.5)$$

The function $\frac{Q_m(z)}{z - z_0}$ is $O(1/|z|^2)$ as $|z| \to \infty$, hence we also have

$$\lim_{r \to \infty} \left| \frac{1}{2\pi i} \int_{C_r} \frac{Q_m(z)}{z - z_0} \, dz \right| = 0$$

so that, using Cauchy’s theorem,

$$\frac{1}{2\pi i} \int_{C_R} \frac{Q_m(z)}{z - z_0} \, dz = 0. \quad (3.6)$$

In the region lying between $\Gamma$ and $C_R$, the function $\frac{\Psi(z)^m}{z - z_0}$ is a quotient of analytic functions (recall that $z_0 \in \Omega^+$). So, from Cauchy’s theorem and (3.5) and (3.6) we get

$$P_m(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi(z)^m}{z - z_0} \, dz = \frac{1}{2\pi i} \int_{C_R} \frac{P_m(z) + Q_m(z)}{z - z_0} \, dz. \quad (3.7)$$

Now that we have shown (3.4), we would like to rewrite (3.4) as a double-layer potential. To this end, consider the integral

$$I(z_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi(z)^m}{z - z_0} \, dz. \quad (3.8)$$

Recall the inverse function $\Psi^{-1} : \mathbb{D}^- \to \Omega^-$. From Proposition 2.1, although $|\Psi^{-1}'|$ may have finitely many singularities on the unit circle, the function $|\Psi^{-1}'|$ remains integrable when $|w| = 1$. We may carry out the substitution $\Psi^{-1}(w) = z$ to get

$$I(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{1}{w^m} \frac{|\Psi^{-1}'(w)|}{\Psi^{-1}(w) - z_0} \, dw. \quad (3.9)$$

Furthermore, the integrand in (3.9) is analytic for $|w| > 1$. It follows that

$$I(z_0) = \lim_{r \to 1^+} \int_{C_r} \frac{1}{w^m} \frac{|\Psi^{-1}'(w)|}{\Psi^{-1}(w) - z_0} \, dw.$$ 

The function $\Psi^{-1}$ is a Riemann map from the exterior of the unit disk onto $\Omega^-$, also with the representation [28 eqn. (1.2.1)]

$$|\Psi^{-1}'(w)| = \delta_1 w + \delta_0 + \frac{\delta_1}{w} + \frac{\delta_2}{w^2} + \cdots, \quad |w| > 1.$$
Since
\[ \lim_{|w| \to \infty} \frac{|\Psi^{-1}(w)|}{|w|} = |\delta_1| \]
the integrand in (3.9) must be \( O(1/|w|^{m+1}) \) as \(|w| \to \infty\).

Since the integrand in (3.9) must be analytic for \(|w| > 1\), we have from Cauchy's theorem that
\[ I(z_0) = \lim_{r \to 1} \frac{1}{2\pi i} \int_{C_r} \frac{1}{w^m} \frac{[\Psi^{-1}]'(w)}{\Psi^{-1}(w) - z_0} \, dw \]
and
\[ = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{C_r} \frac{1}{w^m} \frac{[\Psi^{-1}]'(w)}{\Psi^{-1}(w) - z_0} \, dw \]
\[ = 0. \]

Finally, since
\[ \Psi(z)^m + \frac{1}{\Psi(z)^m} = e^{im\theta(z)} + e^{-im\theta(z)} = 2 \cos(m\theta(z)), \quad z \in \Gamma \]
it follows that
\[ \text{Re} P_m(z_0) = \text{Re} \frac{1}{2\pi i} \int_{\Gamma} \frac{\Psi^m(z)}{z - z_0} \, dz + I(z_0) \]
\[ = \text{Re} \frac{1}{2\pi i} \int_{\Gamma} \frac{2 \cos(m\theta(z))}{z - z_0} \, dz \]
\[ = -2D\varphi_{1,m}(z_0). \quad (3.10) \]

Repeating the above arguments in the case of the function \( iP_m(z) \), we have
\[ \text{Re} iP_m(z_0) = \text{Re} \frac{1}{2\pi i} \int_{\Gamma} \frac{i\Psi^m(z)}{z - z_0} \, dz - I(z_0) \]
\[ = \text{Re} \frac{1}{2\pi i} \int_{\Gamma} \frac{-2 \sin(m\theta(z))}{z - z_0} \, dz \]
\[ = 2D\varphi_{2,m}(z_0). \]

Therefore
\[ P_m(z_0) = \text{Re} P_m(z_0) - i \text{Re} iP_m(z_0) = -2D(\varphi_{1,m}(z_0) + i\varphi_{2,m}(z_0)). \]

We briefly note three related results in the literature. (3.3) can also be derived from Lemma 18.2d of Henrici [19, p. 524], although our proof does not rely on this lemma. Equation (2.20) of Gaier [15, p. 14] resembles the case \( m = 1 \) of (3.10), but Gaier’s derivation only applies when the target domain is the exterior of a horizontal slit. Finally, Murid et al [23, (3.12)] derive a related integral equation involving the derivative \( \Psi' \) and the adjoint Neumann kernel.

4. A Method for the Computation of Exterior Maps.
4.1. Outline of method. We assume we are given a closed, piecewise \( C^2 \), positively oriented curve \( \Gamma \). (A further simplifying assumption we make is that 0 is contained inside the inner domain \( \Omega^+ \), but this assumption may be dispensed with.) This section describes a method to compute the boundary correspondence \( \theta(z) \) for the exterior map \( \Psi \).

Let \( z \in \Gamma \), where \( z \) is not a corner point. From formula (3.3) for \( m = 1 \), and from the inner jump relation for the double-layer potential [23, Thm. (6.18)], we have the integral equation

\[
\alpha_1 z + \alpha_0 = \frac{1}{\pi} \int_\Gamma \Psi(y) \frac{\partial \log(|y - z|)}{\partial n(y)} \, ds(y) - \Psi(z), \quad z \in \Gamma
\]

or

\[
\alpha_1 z + \alpha_0 = -2 \left( D - \frac{1}{2} \right) \Psi(z), \quad z \in \Gamma
\]

on which we base our numerical method. Lacking knowledge of \( \alpha_1 \) and \( \alpha_0 \), we do not solve this equation directly, however. Instead, since the operator \( D \) is linear and for constants \( c \), \( (D - \frac{1}{2})c = -c \) [23, Example 6.17], we rewrite (4.1) as

\[
z = \left( D - \frac{1}{2} \right) \left[ -\frac{2}{\alpha_1} (\Psi(z) - \alpha_0) \right], \quad z \in \Gamma.
\]

Let \( \varphi(z) = -\frac{2}{\alpha_1} (\Psi(z) - \alpha_0) \). Knowing \( \Gamma \), (4.2) can be solved for \( \varphi \) computationally. It is a second kind integral equation with a unique solution continuously dependent on the data [23].

From the density \( \varphi \), we can recover \( \Psi \) and the boundary correspondence as follows. Assume \( 0 \in \Omega^+ \). Using the definition of \( \varphi \), from the Cauchy integral formula and from (3.7) we get that

\[
\frac{1}{2\pi i} \int_\Gamma \frac{\varphi(z)}{z} \, dz = \frac{1}{2\pi i} \left[ -\int \frac{2\Psi(z)}{\alpha_1} \frac{dz}{z} + \int \frac{\alpha_0}{\alpha_1} \frac{dz}{z} \right]
\]

\[
= -\frac{2\alpha_0}{\alpha_1} + \frac{\alpha_0}{\alpha_1}
\]

\[
= -\frac{\alpha_0}{\alpha_1}.
\]

Let \( \tilde{\varphi}(z) \) denote

\[
\tilde{\varphi}(z) = \varphi(z) + \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(z)}{z} \, dz = -\frac{2}{\alpha_1} \Psi(z).
\]

Now, it only remains to normalize \( \tilde{\varphi} \) to get

\[
\Psi(z) = -\frac{\tilde{\varphi}(z)}{||\tilde{\varphi}(z)||} \quad \text{and} \quad \theta(z) = \arg \left\{ -\frac{\tilde{\varphi}(z)}{||\tilde{\varphi}(z)||} \right\}.
\]

Finally, let \( w : [0, L] \to \Gamma \) be an arc length parametrization. For some purposes, such as computing the inverse map [19, p. 381] the derivative of the boundary correspondence \( \theta'(s) = \frac{d}{ds} \theta(w(s)) \) may be desired. Writing \( \hat{\varphi}(s) \) for \( \tilde{\varphi}(w(s)) \), (4.3) implies that

\[
\theta'(s) = -i \frac{\hat{\varphi}'(s)}{\hat{\varphi}(s)}.
\]

To summarize, the numerical tasks for computing \( \theta \) are as follows:
1. Solve the equation (4.2) for the density $\varphi$.
2. Compute the integral \( \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(z)}{z} \, dz \) and form $\tilde{\varphi}$ via (4.3).
3. Compute the boundary correspondence via $\tilde{\varphi}$ and (4.4).

### 4.2. Discretization

Next, we discuss the numerical realization of steps 1 and 2. For simplicity, we assume that the boundary is $C^{m+2}$, with a parametric representation $p(t), 0 \leq t \leq \beta$. We first consider the problem of solving equation (4.2). The Nyström discretization of (4.2) with the trapezoid rule and $n$ equispaced points takes the form

\[
\varphi^*(t_i) - \frac{\beta}{n} \sum_{j=1}^{n} k(t_i, t_j) \varphi^*(t_j) = -2p(t_i), \quad 1 \leq i \leq n
\]

(4.6)

where \[23, Exercise 6.1]\]

\[
k(t, \tau) = \begin{cases} 
\frac{1}{\pi} \text{Im} \frac{p'(t)}{p(t) - p(\tau)} & t \neq \tau \\
-\frac{1}{2\pi} \text{Im} \frac{p''(t)}{p'(t)} & t = \tau.
\end{cases}
\]

The Nyström method also provides a natural interpolation formula of the form

\[
\varphi^*(t) = -2p(t) + \frac{\beta}{n} \sum_{j=1}^{n} k(t, t_j) \varphi^*(t_j)
\]

which is useful for computing $\varphi^*$ at points that are not discretization points.

From Corollary 10.19 of Kress \[23\], to analyze the pointwise error $\|\varphi^* - \varphi\|_\infty$, it suffices to look at the order of accuracy of the quadrature rule (4.6). The convergence order of the trapezoid rule depends on the smoothness of the integrand. On a $C^{m+2}$ curve, both the kernel $k(t, \tau)$ and the density possess at least $m$ continuous derivatives \[28, Thm 3.6\]. This implies that \[23, p. 205\] we expect to obtain $\|\varphi^* - \varphi\|_\infty$ with an error of $O(h^m)$, where $h = \frac{\beta}{n}$ is the spacing. If the integrand is analytic, as is the case with an analytic domain \[28\], then the error is at most $O(e^{-O(n)})$.

For computing the derivative of the Nyström interpolant as required for (4.5), a similar analysis may be applied.

The analysis of the trapezoid rule for the Cauchy integral

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi^*(z)}{z} \, dz \approx \frac{1}{2\pi i n} \sum_{j=1}^{n} \frac{\varphi^*(p(t_j))p'(t_i)}{p(t_i)}
\]

also proceeds in the same manner. Thus we expect an overall order of convergence of $O(h^m)$.

In the case of a non-smooth boundary, the error analysis is more involved. However, we note (and illustrate below) that it is still possible to use the Nyström method to achieve high accuracy \[6\].

Once the discrete boundary correspondence is computed, it remains to compute $\Psi(z)$ at points that are not the boundary points. This can be done via the Cauchy integral formula \[19, p. 391\]. Note that naive evaluation of the Cauchy integral near $\Gamma$ can be numerically unstable due to the nearby singularity. However, a number of methods exist that effectively remedy this problem \[2,3,22\].
5. Numerical experiments.

5.1. Smooth domains. We implemented the method described in Section 4.2. Following Murid et al., we used two smooth test domains, namely the ellipse, for which the parametrization and the corresponding exterior map are given by

$$p(t) = a \cos(t) + i \sin(t)$$

$$\Psi(p(t)) = e^{it}$$

and the oval of Cassini, for which the parametrization and the corresponding exterior map are given by

$$p(t) = \left( \cos(2t) + \sqrt{a^4 - \sin^2(2t)} \right)^{1/2} e^{it},$$

$$\Psi(p(t)) = (p(t)^2 - 1)^{1/2}/a.$$  

Starting with a Nyström discretization of $n$ points and the composite trapezoid rule, we estimated $\|\hat{\Psi}(z) - \Psi(z)\|_\infty$ by computing the exterior map value $\hat{\Psi}(z)$ at 36 equispaced points on the parameter domain. Results for these domains can be found in Tables 5.1 and 5.2.
For the unit square, we evaluated the inverse boundary correspondence $\Psi^{-1}$ at 36 equispaced points on the unit circle, using the MATLAB SC Toolbox \cite{10}. We then tested the accuracy by computing $\tilde{\Psi}$ at those points.

The results are in Figure 5.2. We began with a square partitioned into 64 equispaced panels. At each refinement level, we split the panels that were closest to a corner point of the square into two panels of equal length. We used 8th order Gaussian quadrature on each panel.

6. Conclusions. This paper makes two contributions.

Firstly, we characterize the density functions $\varphi$ that give rise to harmonic polynomials represented as double-layer potentials $D\varphi$ on the interior of a piecewise smooth Jordan domain. We show how these density functions relate to the exterior Riemann map associated with the domain. In addition to the described application to conformal mapping, this work may be of mathematical interest for those studying the behavior of the double-layer potential and related numerical methods.

Secondly, we derive an integral equation whose solution allows us to recover the boundary correspondence for the exterior mapping function. From a practical standpoint, our equation is second-kind, uniquely solvable and has a continuous real-valued kernel, which leads to a robust and simple discretization with the Nyström method. Our experiments demonstrate that the method achieves high order on smooth domains and can be made accurate in the presence of corners. A major advantage of the double-layer potential is the availability of existing fast solvers, such as those in \cite{5,7,20,30,31}.

A number of issues remain unexplored. From an implementation standpoint, our current implementation does not make use of the fast algorithms mentioned above. From a mathematical standpoint, it would be interesting to generalize the results of Lemma 3.1 to other potentials (such as the single-layer potential), or to determine the most general geometries under which the results still hold.

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