A Finiteness Theorem for Quaternionic-Kähler Manifolds with Positive Scalar Curvature

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Abstract

We study the topology and geometry of those compact Riemannian manifolds \((M^{4n}, g), n \geq 2\), with positive scalar curvature and holonomy in \(Sp(n) \cdot Sp(1)\). Up to homothety, we show that there are only finitely many such manifolds of any dimension \(4n\).

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Let \((M^\ell, g)\) be a connected Riemannian manifold. If \(x \in M\) is an arbitrary base-point, one defines the holonomy group \(\mathcal{H}(M, g, x) \subset \text{End}(T_x M)\) of \((M, g, x)\) to be the set of linear maps \(T_x M \to T_x M\) obtained by Riemannian parallel transport of tangent vectors around piece-wise smooth loops based at \(x\); the reduced holonomy group \(\mathcal{H}_0(M, g, x)\) is similarly defined, but now using only null-homotopic loops. The latter is automatically a connected Lie subgroup of the orthogonal transformations of the tangent space \(T_x M\), and so may be identified with a Lie subgroup of \(SO(\ell)\) by choosing an orthonormal basis for \(T_x M\); and the conjugacy class of this subgroup is unaffected by a change of basis or base-point \(x\). As was first pointed out by Berger, relatively few groups can actually arise in this way. If we exclude the so-called locally reducible manifolds, meaning those which are locally Riemannian Cartesian products of lower-dimensional manifolds, and the locally symmetric manifolds, meaning those for which the curvature tensor is covariantly constant, the only possibilities\([2]\) for the reduced holonomy of \(M^\ell\) are those which appear on the following list:

| \(\ell\) | \(\mathcal{H}_0\) | geometry          |
|---------|----------------|------------------|
| \(\ell\) | \(SO(\ell)\) | generic          |
| \(2m \geq 4\) | \(U(m)\) | Kähler           |
| \(2m \geq 4\) | \(SU(m)\) | Ricci-flat Kähler |
| \(4n \geq 8\) | \(Sp(n)\) | hyper-Kähler     |
| \(4n \geq 8\) | \(Sp(n) \times Sp(1)/\mathbb{Z}_2\) | quaternionic-Kähler |
| 7       | \(G_2\)   | imaginary Cayley |
| 8       | \(Spin(7)\) | octonionic       |

In particular, the universal cover of any complete Riemannian manifold is the product of globally symmetric spaces and manifolds whose holonomy appears on this list. Each of these possible holonomies may
Therefore be thought of as representing a “fundamental geometry,” each of which has its own peculiar flavor.

Of these fundamental geometries, the quaternionic-Kähler or $Sp(n) \cdot Sp(1) := Sp(n) \times Sp(1)/\mathbb{Z}_2$ possibility is in some ways the most enigmatic. It is the only holonomy geometry which forces the metric to be Einstein but not Ricci-flat. In particular, the scalar curvature of a quaternionic-Kähler manifold is constant, and the sign of this constant turns out to have a decisive influence on the nature of manifolds in question. Thus, while complete, non-compact, non-symmetric quaternionic-Kähler manifolds of negative scalar curvature exist in profusion [3][7][14], a complete quaternionic-Kähler manifold of positive scalar curvature must be compact, and the only known examples of such manifolds are symmetric. Indeed, Poon and Salamon [19], generalizing earlier work of Hitchin [10], have proved that there are no others in dimension 8, and it is therefore tempting to conjecture that this situation persists in higher dimensions. This article will indicate some new evidence supporting such a conjecture. In particular, a complete proof the following result is given:

**Theorem A (Finiteness Theorem)** For any $n$, there are, modulo isometries and rescalings, only finitely many compact quaternionic-Kähler 4n-manifolds of positive scalar curvature.

Of course, this doesn’t predict that every such manifold is symmetric. But the other main result which will be described in detail herein does allow one to draw such a conclusion for certain topological types:

**Theorem B (Strong Rigidity)** Let $(M, g)$ be a compact quaternionic-Kähler manifold of positive scalar curvature. Then $\pi_1(M) = 0$ and

$$\pi_2(M) = \begin{cases} 0 & (M, g) = HP_n \\ \mathbb{Z} & (M, g) = Gr_2(\mathbb{C}^{n+2}) \\ \text{finite} \supset \mathbb{Z}_2 & \text{otherwise} \end{cases}$$

It might be particularly emphasized that these result rely very heavily on the global aspects of the problem. In particular, if one chooses to consider compact orbifolds rather than manifolds, both fail [3], even though the more general metrics involved are still smooth everywhere—at least when viewed from the skewed perspective of certain multi-valued coordinate charts.
Many of the results described herein were obtained as part of a joint research project with Simon Salamon, and further details will appear in our joint paper [16].

1 Preliminaries

Definition 1 Let \((M, g)\) be a connected Riemannian \(4n\)-manifold, \(n \geq 2\). We will say that \((M, g)\) is a quaternionic-Kähler manifold iff the holonomy group \(H(M, g)\) is conjugate to \(H \cdot Sp(1)\) for some Lie subgroup \(H \subset Sp(n) \subset SO(4n)\).

Example. The quaternionic projective spaces

\[
HP_n = Sp(n + 1)/(Sp(n) \times Sp(1))
\]

are quaternionic-Kähler manifolds. So are the complex Grassmannians

\[
Gr_2(C^{n+2}) = SU(n + 2)/SU(n) \times U(2)
\]

and the oriented real Grassmannians

\[
Gr_4(R^{n+4}) = SO(n + 4)/SO(n) \times SO(4).
\]

In fact, these examples very nearly exhaust the compact homogeneous examples of quaternionic-Kähler manifolds. Indeed [1], every such homogeneous space is a symmetric space, and [23] there is exactly one such symmetric space for each compact simple Lie algebra. They can be constructed as follows: let \(G\) be a compact simple centerless group, and let \(Sp(1)\) be mapped to \(G\) so that its root vector is mapped to a root of highest weight. If \(H\) is the centralizer of this \(Sp(1)\), then the symmetric space \(M = G/(H \cdot Sp(1))\) is quaternionic-Kähler, and every compact homogeneous quaternionic-Kähler manifold arises this way.

How typical are these symmetric examples? One geometric feature of any irreducible symmetric space is that it must be Einstein, with non-zero scalar curvature. This, it turns out, also happens for quaternionic-Kähler manifolds:

Proposition 1 (Berger) Every quaternionic-Kähler manifold is Einstein, with non-zero scalar curvature.
For details, see [6]. In particular, a complete quaternionic-Kähler manifold has constant scalar curvature.

**Definition 2** We will say that a quaternionic-Kähler manifold is positive if it is complete and has positive scalar curvature.

It is now an immediate consequence of Myers’ theorem that a positive quaternionic-Kähler manifold is compact and has finite fundamental group. Unfortunately, however, the only known positive quaternionic-Kähler manifolds are the previously mentioned symmetric spaces! The main objective of the present article will be to explain why this situation is hardly surprising. The main tool in our investigation will be the next result:

**Theorem 1** (Salamon [20]; Bérard-Bergery [5]) Let \((M^{4n}, g)\) be a quaternionic-Kähler manifold. Then there is a complex manifold \((Z, J)\) of complex dimension \(2n + 1\), called the twistor space of \((M, g)\), such that

- there is a smooth fibration \(\varphi : Z \to M\) with fiber \(S^2\);
- each fiber of \(\varphi\) is a complex curve in \((Z, J)\) with normal bundle holomorphically isomorphic to \([O(1)]^{\oplus 2n}\), where \(O(1)\) is the point-divisor line bundle on \(\mathbb{CP}_1\); and
- there is a complex-codimension 1 holomorphic sub-bundle \(D \subset TZ\) which is maximally non-integrable and transverse to the fibers of \(\varphi\).

Moreover, if \((M, g)\) is positive, then \(Z\) carries a Kähler-Einstein metric of positive scalar curvature such that

- \(\varphi\) is a Riemannian submersion;
- \(D\) is the orthogonal complement of the vertical tangent bundle of \(\varphi\); and
- the induced metric on each fiber of \(\varphi\) has constant curvature.

If \((M, g)\) is instead negative, there is an indefinite Kähler-Einstein pseudo-metric on \(Z\) with all these properties.

In particular, the twistor space \(Z\) of a positive quaternionic-Kähler manifold is **Fano**.
Definition 3 A Fano manifold is a compact complex manifold \( Z \) such that \( c_1(Z) \) can be represented by a positive \((1,1)\)-form.

That is, a Fano manifold is a compact complex manifold which admits Kähler metrics of positive Ricci curvature. Every Fano manifold is simply connected, since \( c_1 > 0 \Rightarrow \chi(\mathcal{O}) = h^0(\mathcal{O}) = 1 \) by the Kodaira vanishing theorem, thus forbidding the possibility that the manifold might have a finite cover. Applying the exact homotopy sequence of \( Z \to M \), we now conclude the following:

Proposition 2 Any positive quaternionic-Kähler manifold is compact and simply connected.

A completely different and extremely important feature of our twistor spaces is the holomorphic hyperplane distribution \( D \), which gives a so-called complex contact structure to \( Z \). Such structures will be discussed systematically in §2.

Our definition of quaternionic has carefully avoided the case of \( n = 1 \); after all, \( Sp(1) \cdot Sp(1) \) is all of \( SO(4) \), so such a holonomy restriction says nothing at all. Instead, we choose the our definition in order to insure that Theorem 1 remains valid:

Definition 4 A Riemannian 4-manifold \((M, g)\) is called quaternionic-Kähler if it is Einstein, with non-zero scalar curvature, and half-conformally flat.

2 Complex Contact Manifolds

Definition 5 A complex contact manifold is a pair \((X, D)\), where \( X \) is a complex manifold and \( D \subset TX = T^{1,0}X \) is a codimension-one holomorphic sub-bundle which is maximally non-integrable in the sense that the O’Neill tensor

\[
D \times D \to TX/D
(v, w) \mapsto [v, w] \mod D
\]

is everywhere non-degenerate.
Example. Let $Y_{n+1}$ be any complex manifold, and let $X_{2n+1} = P(T^*Y)$ be its projectivated holomorphic cotangent bundle; dually stated, $X$ is the Grassmann bundle of complex $n$-planes in $TY$. Let $\pi : X \to Y$ be the canonical projection, and let $D \subset TX$ be the sub-bundle defined by $D|_P := \pi^{-1}_*(P)$ for all complex $n$-planes $P \subset TY$. Then $D$ is a complex contact structure on $X$.

The condition of non-integrability has a very useful reformulation, which we shall now describe. Given a codimension-one holomorphic sub-bundle $D \subset TX$, let $L := TX/D$ denote the quotient line bundle. Letting $\theta : TX \to L$ be the tautological projection, we may think of $\theta$ as a line-bundle-valued 1-form $
abla \in \Gamma(X, \Omega^1(L))$, and so attempt to form its exterior derivative $d\theta$. Unfortunately, this ostensibly depends on a choice of local trivialization; for if $\vartheta$ is any 1-form, $d(f\vartheta) = fd\vartheta + df \wedge \vartheta$. However, it is now clear that $d\theta|_D$ is well defined as a section of $L \otimes \wedge^2 D^*$, and an elementary computation, which we leave to the reader, shows that $d\theta|_D$, thought of in this way, is exactly the O’Neill tensor mentioned above. Now if the skew form $d\theta|_D$ is to be non-degenerate, $D$ must have positive even rank $2n$, so that $X$ must have odd complex dimension $2n + 1 \geq 3$. Moreover, the non-degeneracy exactly requires that

$$\theta \wedge (d\theta)^\wedge n \in \Gamma(X, \Omega^{2n+1}(L^{n+1}))$$

is nowhere zero. But this provides a bundle isomorphism between $L^{\otimes(n+1)}$ and the anti-canonical line bundle $\kappa^{-1} = \wedge^{2n+1} T^{1,0} X$.

Conversely, let $X$ be a simply-connected compact complex $(2n+1)$-manifold, and suppose that $c_1(X)$ is divisible by $n + 1$. Then there is a unique holomorphic line bundle $L := \kappa^{-1/(n+1)}$ such that $L^{\otimes(n+1)} \cong \kappa^{-1}$. If we are then given a twisted holomorphic 1-form

$$\theta \in \Gamma(X, \Omega^1(\kappa^{-1/(n+1)}))$$

we may then construct

$$\theta \wedge (d\theta)^\wedge n \in \Gamma(X, \Omega^{2n+1}(\kappa^{-1})) = \Gamma(X, \mathcal{O}) = \mathbb{C}.$$

If this constant is non-zero, $D = \ker \theta$ is then a complex contact structure.

This simple observation has powerful consequences:
Proposition 3 Let $X_{2n+1}$ be a simply connected compact complex manifold, and let $\mathcal{G}$ denote the identity component of the group of biholomorphisms $X \to X$. Then $\mathcal{G}$ acts transitively on the set of complex contact structures on $X$.

Proof. We may assume that there is at least one complex contact structure on $X$, since otherwise there is nothing to prove. In this case, the canonical line bundle $\kappa$ has a root $\kappa^{1/(n+1)}$, and there is only one such root because $H^1(X, \mathbb{Z}_{n+1}) = 0$. Thus any complex contact structure is determined by a class $[\theta] \in P\Gamma(X, \Omega^1 \otimes (\kappa^{-1/(n+1)}))$ satisfying $\theta \wedge (d\theta)^n \neq 0$. The group $\mathcal{G}$ acts on this projective space $\mathbb{P}m$ in a manner preserving the hypersurface $S$ defined by $\theta \wedge (d\theta)^n = 0$, and so partitions $\mathbb{P}m - S$ into orbits; since $\mathbb{P}m - S$ is connected, it therefore suffices to prove that each orbit is open, and for this it would be enough to prove that the holomorphic vector fields generating the action of the the Lie algebra of $\mathcal{G}$ on $\mathbb{P}m - S$ span the tangent space at each point.

To prove the last statement, let $\theta \in \Gamma(X, \Omega^1 \otimes (\kappa^{-1/(n+1)}))$ be any contact form, and let $\phi \in \Gamma(X, \Omega^1 \otimes (\kappa^{-1/(n+1)}))$ be any other section. If $D$ denotes the kernel of $\theta$, $d\theta|_D : D \to D \otimes \kappa^{-1/(n+1)}$ is an isomorphism of holomorphic vector bundles, so we can define a holomorphic vector field $v \in \Gamma(X, \mathcal{O}(D))$ by $v = (d\theta|_D)^{-1}(\phi)$. We then have $L_\xi \theta \equiv \xi \cdot d\theta \equiv \phi \mod \theta$, so that action of the Lie algebra of $\mathcal{G}$ spans the tangent space of $\mathbb{P}\Gamma(X, \Omega^1 \otimes (\kappa^{-1/(n+1)}))$ at $[\theta]$, thus proving the proposition.

Corollary 1 Two simply-connected compact complex manifolds are complex-contact isomorphic iff the underlying complex manifolds are biholomorphically equivalent.

This will now yield a result which is crucial for our purposes.

Definition 6 We will say that two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$ are homothetic if there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^*g_2 = cg_1$ for some constant $c > 0$. Such a map $\Phi$ will be called a homothety.

Proposition 4 Two positive quaternionic-Kähler manifolds are homothetic iff their twistor spaces are biholomorphic.
Proof. Let \((M, g)\) and \((\tilde{M}, \tilde{g})\) be two given quaternionic-Kähler manifolds, \(\varphi : Z \to M\) and \(\tilde{\varphi} : \tilde{Z} \to \tilde{M}\) their twistor spaces, \(h\) and \(\tilde{h}\) the Kähler-Einstein metrics of \(Z\) and \(\tilde{Z}\). We also suppose that a biholomorphism \(\Phi : Z \to \tilde{Z}\) is given to us. For some positive constant \(c > 0\), \(h\) and \(c\tilde{h}\) have the same scalar curvature; and notice that replacing \(\tilde{h}\) with \(c\tilde{h}\) just corresponds to replacing \(\tilde{g}\) with \(c\tilde{g}\). Now \(\Phi^* c\tilde{h}\) is a Kähler-Einstein metric on \(Z\) with the same scalar curvature as \(h\), and the Bando-Mabuchi theorem [4] on the uniqueness of Kähler-Einstein metrics now asserts that there exists a biholomorphism \(\Psi : Z \to Z\) such that \(\Psi^*(\Phi^* c\tilde{h}) = h\).

Let \(N \subset \Gamma(Z, \Omega^1(\kappa^{-1/(n+1)}))\) be defined by \(\phi \wedge (d\phi)^n = 1\). Proposition 3 implies that a finite connected cover \(G\) of the connected component of the automorphism group of \((Z, J)\) acts transitively on \(N\), since, in the notation of the proof of that proposition, \(N \to \mathbb{P}_m - S\) is a finite covering. Because \(h\) is Kähler-Einstein, with positive scalar curvature, the Killing fields are a real form of the algebra of holomorphic vector fields, and a finite cover \(G\) of the connected component of the isometry group of \((Z, h)\) is therefore a compact real form of \(G\). Morse theory now predicts that one orbit of the action of \(G\) on \(N\) is precisely the set of critical points of the \(G\)-invariant strictly plurisubharmonic function \(f : N \to \mathbb{R}\) given by \(\phi \mapsto \|\phi\|^2_{L^2,h}\). On the other hand, the derivative of \(f\) at a contact form \(\phi\) in the direction of a real-holomorphic vector field \(\xi\) on \(Z\) is given by

\[df(\xi)_\phi = \frac{1}{(2n)!} \int_X d(\xi \wedge \omega) \wedge |\phi|^2_h \left( \beta_\phi - \frac{n+2}{n+1} \omega^{2n} \right),\]

where \(\omega\) is the Kähler form of \((Z, J, h)\) and \(\beta_\phi\) is the \((2n, 2n)\) form obtained by orthogonally extending the restriction of \(\omega^{2n}\) from \(D = \ker \phi\) to \(TZ\). For the canonical contact form \(\theta\) associated with the quaternionic-Kähler metric \(g\) by the twistor construction, \(\theta|_h\) is constant and \(\beta_\theta = \varphi^*(2n)!dvol_g\) is closed, so that \(\theta\) is a critical point of \(f\); but the same argument applies equally to the contact form of \(\tilde{Z}\), and hence to the pull-back of this contact form via the holomorphic isometry \(\Phi\). Hence there is a holomorphic isometry \(\Xi : Z \to \tilde{Z}\) sending the first of these contact structures to the second, and \(\Phi\Xi \Xi : Z \to \tilde{Z}\) is a then biholomorphism which sends \(h\) to \(c\tilde{h}\) and \(D\) to \(\tilde{D}\). Since the vertical tangent spaces of \(\varphi\) and \(\tilde{\varphi}\) are the orthogonal complements of \(D\) and \(\tilde{D}\) with respect to \(h\) and \(\tilde{h}\), respectively, it follows that \(\Phi\Xi \Xi\) sends fibers of \(\varphi\) to fibers of \(\tilde{\varphi}\), and so covers a diffeomorphism \(F : M \to \tilde{M}\).
Moreover, since the \( \varphi \) and \( \tilde{\varphi} \) are Riemannian submersions, one has \( F^*c\tilde{g} = g \), and \( F \) is thus a homothety between \( (M, g) \) and \( (\tilde{M}, \tilde{g}) \).

For less precise but more broadly applicable theorems on the invertibility of the twistor construction, cf. \[13\] [3].

**Definition 7** Let \( (X_{2n+1}, D) \) be a complex contact manifold. An \( n \)-dimensional submanifold \( \Sigma_n \subset X_{2n+1} \) is called Legendrian if \( T\Sigma \subset D \).

**Lemma 1** Let \( (X_{2n+1}, D) \) be a complex contact manifold, and let \( \pi : X \to Y_{n+1} \) be a proper holomorphic submersion with Legendrian fibers. Then \( X \cong \mathbb{P}(T^*Y) \) as complex contact manifolds.

**Proof.** Define \( \Psi : X \to \text{Gr}_n(TY) \) by \( x \to \pi^*(D_x) \). This map preserves the contact structure; and since the pull-back of the contact form of \( \text{Gr}_n(TY) \) via \( \Psi \) is the contact form of \( X \), \( \Psi^* \) induces an isomorphism between forms of top degree. In other words, \( \Psi \) is a submersion onto its image, and, in particular, induces a submersion from each fiber of \( X \) onto its image in the fiber of \( \text{Gr}_n(TY) = \mathbb{P}(T^*Y) \). By the properness assumption, \( \Psi \) is fiber-wise therefore a covering map. But the fibers of \( \mathbb{P}(T^*Y) \) are projective spaces, and so simply connected. Hence \( \Psi \) is an injective holomorphic submersion, and so biholomorphic.

**Definition 8** If \( (X, D) \) is a complex contact manifold such that \( X \) is Fano, we will say that \( (X, D) \) is a Fano contact manifold.

**Lemma 2** Let \( \varpi : \mathcal{X} \to \mathcal{B} \) be a holomorphic family of Fano contact manifolds with smooth connected parameter space— that is, let \( \mathcal{B} \) be a connected complex manifold, \( \varpi \) a proper holomorphic submersion with Fano fibers, and assume that \( \mathcal{X} \) is equipped with a maximally non-integrable, complex codimension 1 sub-bundle \( D \subset \ker \varpi_* \) of the vertical tangent bundle. Then any two fibers \( (X_0, D|_{X_0}) \) and \( (X_t, D|_{X_t}) \) are isomorphic as complex contact manifolds.
Proof. Since any two points in $B$ can be joined by a finite chain of holomorphic images of the unit disk $\Delta \subset \mathbb{C}$, it suffices to prove the lemma when the base $B$ is a disk $\Delta$.

We now proceed as in [12]. By Darboux’s theorem, any complex contact structure in dimension $2n + 1$ is locally complex-contact isomorphic to the one on $\mathbb{C}^{2n+1}$ determined by the 1-form

$$\vartheta = dz^{2n+1} + \sum_{j=1}^{n} z^j dz^{n+j},$$

so we may cover our family $\varpi : \mathcal{X} \to \Delta$ by Stein sets $U_j$ on which we have holomorphic charts $\Phi_j : U_j \to \mathbb{C}^{2n+1} \times \Delta$ such that the last coordinate is given by $\varpi$ and the fiber-wise contact structure on $\mathcal{X}$ agrees with that induced by $\Phi_j^* \vartheta$. Letting $t$ denote the standard complex coordinate on $\Delta$, we lift $d/dt$ to each $U_j$ as the vector field $v_j := (\Phi_j^{-1})_* d/dt$, and observe that the $t$-dependent vertical vector field $w_{jk} := v_j - v_k$ satisfies $\mathcal{L}_{w_{jk}} \vartheta \propto \vartheta$.

Let $f_{jk} := \vartheta(w_{jk}) \in \Gamma(U_j \cap U_k, \mathcal{O}(L))$, and notice that the collection \{\(f_{jk}\}\} is a Čech cocycle representing an element of $H^1(\mathcal{X}, \mathcal{O}(L))$. On the other hand, since $L$ is a fiber-wise $(n+1)^{st}$-root of the vertical anti-canonical bundle $\kappa^{-1}$, and since each fiber $X_t$ of $\varpi$ is assumed to be a Fano manifold, the bundle $\kappa^{-1} \otimes L$ is fiber-wise positive, and $H^1(X_t, \mathcal{O}(L)) = 0 \forall t \in \Delta$ by the Kodaira vanishing theorem. Thus the first direct image sheaf $\varpi_! \mathcal{O}(L)$ is zero. Since $\Delta$ is Stein, the Leray spectral sequence now yields $H^1(\mathcal{X}, \mathcal{O}(L)) = 0$. Hence there exist sections $h_j \in \Gamma(U_j \mathcal{O}(L))$ such that $f_{jk} = h_j - h_k$ on $U_j \cap U_k$.

On $U_j$ there is now a unique vertical holomorphic vector field $u_j$ such that $\vartheta(u_j) = h_j$ and $\mathcal{L}_{u_j} \vartheta \propto \vartheta$. Indeed, taking a local trivialization of $L$ so as to locally represent $\vartheta$ by a holomorphic 1-form $\vartheta$, a vector field $u$ satisfies $\mathcal{L}_u \vartheta \propto \vartheta$ iff

$$u \llcorner d \vartheta \equiv -d(u \llcorner \vartheta) \mod \vartheta,$$

so that such a field is uniquely determined by an arbitrary local function $f = \vartheta(u) = u \llcorner \vartheta$. We therefore conclude that $v_j - v_k = w_{jk} = u_j - u_k$ on $U_j \cap U_k$, and the vector field $v = v_j - u_j$ is therefore globally defined. Since $\mathcal{L}_{v_j} \vartheta = 0$ and $\mathcal{L}_{u_j} \vartheta \equiv 0 \mod \vartheta, dt$, the flow of $v = v_j - u_j$ preserves the fiberwise contact structure on $\mathcal{X}$. And since $\varpi$ is a proper map, we can now integrate the flow of our lift $v$ of $d/dt$ to produce a fiber-wise contact biholomorphism between $\mathcal{X}$ and
$X_0 \times \Delta$. In particular, any fiber $X_\ell$ is complex-contact equivalent to the central fiber $X_0$.

3 Mori Theory

Mori’s theory of extremal rays [17] has led to a startling series of advances in the classification of complex algebraic varieties, especially in the Fano case which interests us. One beautiful consequence of this is the so-called contraction theorem: if $X$ is a Fano manifold, there is always a map $\Upsilon : X \to Y$ to some other variety $Y$ which decreases the second Betti number $b_2$ by one, and where the kernel of $\Upsilon_* : H_2(X, \mathbb{R}) \to H_2(Y, \mathbb{R})$ is generated by the class of a rational holomorphic curve $\mathbb{P} \subset X$. (The positive half of such a one-dimensional subspace $\ker \Upsilon_* \subset H_2(X, \mathbb{R})$ is called an “extremal ray”.) If $b_2(X) = 1$, this tells us next to nothing, because we can take $Y$ to be a point; but for $b_2(X) \geq 2$, it is quite a powerful tool. In particular, it gives rise to the following very useful result of Wiśniewski [22]:

**Theorem 2 (Wiśniewski)** Let $X$ be a Fano manifold of dimension $2r - 1$ for which $r|c_1$. Then $b_2(X) = 1$ unless $X$ is one of the following: (i) $\mathbb{C}P_{r-1} \times Q_r$; (ii) $\mathbb{P}(T^*\mathbb{C}P_r)$; or (iii) $\mathbb{C}P_{2r-1}$ blown up along $\mathbb{C}P_{r-2}$.

Here $Q_r \subset \mathbb{C}P_{r+1}$ denotes the $r$-quadric, while the projectivization of a bundle $E \to Y$ is defined by $\mathbb{P}(E) := (E - 0_Y)/(\mathbb{C} - 0)$. The essence of the proof is that, since the rational curves collapsed by the Mori contraction have, in these circumstances, normal bundles of rather large index, they are so mobile that they sweep out projective spaces of comparatively large dimension, and these must therefore be the fibers of the contraction map.

The following is now an easy consequence:

**Corollary 2** Let $(X_{2n+1}, D)$ be a Fano contact manifold. If $b_2(X) > 1$, then $X = \mathbb{P}(T^*\mathbb{C}P_{n+1})$.

**Proof.** Setting $r = n + 1$, we notice that the existence of a contact structure implies that $(n + 1)|c_1$. We may therefore invoke Theorem 2. On the other hand, spaces (i) and (iii) aren’t complex contact
manifolds, since $\Gamma(\mathbb{C}P_{r-1}, \Omega^1(1)) = 0$ and therefore the obvious foliations by $\mathbb{C}P_{r-1}$'s would necessarily have Legendrian leaves, implying (by Lemma 3) that these spaces would then have to be of the form $\mathbb{P}(T^*Y)$, where $Y$ is the leaf space $Q_r$ or $\mathbb{C}P_r$—a contradiction. So the only candidate left is (ii), and this is in fact a contact manifold.

**Theorem 3** \[15\] Let $(M, g)$ be a compact quaternionic-Kähler 4n-manifold with $s > 0$. Then either

(a) $b_2(M) = 0$; or else

(b) $M = Gr_2(\mathbb{C}^{n+2})$ with its symmetric-space metric.

**Proof.** By the Leray-Hirsch theorem on sphere bundles, the second Betti numbers of $M^{4n}$ and its twistor space $Z_{2n+1}$ are related by $b_2(Z) = b_2(M) + 1$. Since $Z$ is a Fano contact manifold, $b_2(M) > 0 \Rightarrow Z = \mathbb{P}(T^*\mathbb{C}P_{n+1})$ by Corollary 2. But this is the twistor space of $Gr_2(\mathbb{C}^{n+2})$. The result therefore follows by Proposition 4.

**Theorem B (Strong Rigidity)** Let $M$ be a compact quaternionic-Kähler manifold of positive scalar curvature. Then $\pi_1(M) = 0$ and $H_2(M, \mathbb{Z}) = \begin{cases} 0 & M = \mathbb{H}P_n \\ \mathbb{Z} & M = Gr_2(\mathbb{C}^{n+2}) \\ \text{finite } \supset \mathbb{Z}_2 & \text{otherwise.} \end{cases}$

**Proof.** If $(M, g)$ is not homothetic to the symmetric space $Gr_2(\mathbb{C}^{n+2}) = SU(n+2)/S(U(n) \times U(2))$, $b_2(M) = 0$ by Theorem 3, so that $H^2(M, \mathbb{Z}) = 0$ and $H_2(M, \mathbb{Z})$ is finite. Since we also know that $H_1(M, \mathbb{Z}) = 0$, $H^2(M, \mathbb{Z})$ is exactly the the 2-torsion of $H_2(M, \mathbb{Z})$ by the universal coefficients theorem. If, on the other hand, $(M, g)$ is not homothetic to the symmetric space $\mathbb{H}P_n$, the class $\varepsilon \in H^2(M, \mathbb{Z}_2)$ must be non-zero [24], and the finite group $\pi_2(M) = H_2(M, \mathbb{Z})$ must therefore contain an element of order 2.
4 The Finiteness Theorem

**Theorem 4** Up to biholomorphism, there are only finitely many Fano contact manifolds of any given dimension $2n + 1$.

**Proof.** By Wisniewski’s theorem, we may restrict our attention to Fano manifolds with $b_2 = 1$. A theorem of Nadel \cite{18} then asserts that there are only a finite number of deformation types of any fixed dimension.

For any fixed deformation type, we may embed each Fano manifold in a fixed projective space $\mathbb{CP}_N$ in such a manner that the restriction of the generator $\alpha \in H^2(\mathbb{CP}_N, \mathbb{Z})$ is a fixed multiple $\ell c_1(Z)/q$ of the anti-canonical class, and we may freely choose the positive integers $\ell$ and $q$ as long as $q \mid c_1$ and $\ell$ is sufficiently large. Thus, let $F$ denote the set of all complex submanifolds $Z \subset \mathbb{CP}_N$ of some fixed dimension $m$ and degree $d$, and with the additional property that, for fixed integers $\ell, q$, the restriction of the hyperplane class is $\ell c_1(Z)/q$. Thus $F$ is a Zariski-open subset in a component of the Chow variety, and so, in particular, is quasi-projective. There is now a tautological family

$$ Z \leftrightarrow F \times \mathbb{CP}_N $$

such that the fiber of $Z \in F$ is the submanifold $Z \subset \mathbb{CP}_N$.

We now assume moreover that the dimension $m$ is an odd number $2n + 1$, take $q = n + 1$, and choose $\ell \gg 0$ such that $\gcd(n + 1, \ell) = 1$. Letting $V \to Z$ denote the vertical tangent bundle $\ker \pi_*$, the vertical anti-canonical line bundle $\kappa^{-1} := \wedge^{2n+1} V$ has a consistent fiber-wise $(n + 1)^{st}$-root $L \to Z$; indeed, using the Euclidean algorithm to write $1 = a(n + 1) + b \ell$, we may define $L$ by $L = \kappa^{-a} \otimes \mathcal{H}^b$, where $\mathcal{H}$ is the pull-back of $O(1)$ from $\mathbb{CP}_N$ to $Z$.

Let $F_j \subset F$ denote the locus

$$ F_j := \left\{ Z \in F \mid h^0(Z, \Omega^1_Z \otimes \kappa^{-1/(n+1)}) \geq j \right\} $$

where the space of candidate contact forms has dimension at least $j$.

Since $\Omega^1_Z \otimes \kappa^{-1/(n+1)}$ is just the restriction of $V^* \otimes L$ to the appropriate

\[\text{[1] It is now known \cite{1} that this is true even without the restriction } b_2 = 1.\]
fiber of \( \varpi \), and since \( \varpi \) is a flat morphism, it follows from the semi-continuity theorem \([9]\) that each \( \mathcal{F}_j \) is a Zariski-closed subset of the quasi-projective variety \( \mathcal{F} \), and so, in particular, has only finitely many components.

Let \( \tilde{\mathcal{F}}_j := \mathcal{F}_j - \mathcal{F}_{j-1}, j \geq 1 \). Since \( \tilde{\mathcal{F}}_j \) is a quasi-projective variety, it is a finite union of irreducible strata \( \mathcal{F}_{jk} \), each of which is a connected complex manifold. On each stratum \( \mathcal{F}_{jk} \), define a vector bundle \( E_{jk} \) as the zero-th direct image \( \mathcal{O}(E_{jk}) = \varpi^*_0 \mathcal{O}(V^* \otimes L) \) of the fiber-wise 1-forms with values in \( L \). Let \( \mathcal{L} \) denote the line bundle \( \varpi^*_0 \mathcal{O}(L^{n+1} \otimes \wedge^{2n+1} V) \) on \( \mathcal{F} \). Then \( \theta \mapsto \theta \wedge (d\theta)^n \) is defines a canonical holomorphic section of the symmetric-product bundle \( \mathcal{L} \otimes \mathcal{O}(E_{jk}^*) \); let \( \mathcal{E}_{jk} \) denote the open subset in the total space of \( E_{jk} \rightarrow \mathcal{F}_{jk} \) where this homogeneous function is non-zero. Thus each \( \mathcal{E}_{jk} \) is either a connected complex manifold or is empty. Each \( \mathcal{E}_{jk} \) may now be viewed as the smooth parameter space of a connected family of Fano contact manifolds by taking the fiber over \( \theta \in \Gamma(Z, \Omega^1(L)) \) to be the pair \((Z, \ker \theta)\). On the other hand, every Fano contact manifold \((Z, D)\), where \( Z \) is of the fixed deformation type, appears in one of these families— albeit many times. Applying Lemma \( 2 \), each of these families is of constant contact type. Since we must construct \( \mathcal{F} \) only for a finite number of degrees in order to account for all Fano deformation types of the given dimension, and since, for each \( \mathcal{F} \) we only have a finite number of contact families \( \mathcal{E}_{jk} \), the result now follows.

**Theorem A (Finiteness Theorem)** Up to homothety, there are only finitely many compact quaternionic-Kähler manifolds of positive scalar curvature in any given dimension \( 4n \).

**Proof.** By Proposition \( 4 \), two positive quaternionic-Kähler manifolds are homothetic iff their twistor spaces are biholomorphic. Since the twistor space of any such manifold is a Fano contact manifold, the result now follows immediately from Theorem \( 3 \).

5 Other Results

We have seen in \( \S 3 \) that the second homology of a positive quaternionic-Kähler manifold is far from arbitrary, and may by itself contain enough
information to determine the metric up to isometry. Recent calculations of Salamon show that the higher homology groups are similarly constrained, in the following remarkable manner:

**Theorem 5 (Salamon)** Let \((M^{4n}, g)\) be a compact quaternionic-Kähler manifold with positive scalar curvature. Then the “odd” Betti numbers \(b_{2k+1}\) of \(M\) vanish, and the “even” Betti numbers \(b_{2k} = b_{2(2n-k)}\) are subject to the linear constraint

\[
\sum_{k=0}^{n} a_k b_{2k} = 0 ,
\]

where \(a_k = \begin{cases} 1 + 2k + 2k^2 - 4n/3 - 2kn + n^2/3 & k < n \\ (n^2 - n)/6 & k = n. \end{cases} \)

The proof of this result involves an intricate interplay between the Kodaira vanishing theorem and the Penrose transform. Details will appear elsewhere \[16\].

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