Partial regularity to the Landau-Lifshitz equation with spin accumulation

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Abstract

In this paper, we consider a model for the spin-magnetization system that takes into account the diffusion process of the spin accumulation. This model consists of the Landau-Lifshitz equation describing the precession of the magnetization, coupled with a quasi-linear parabolic equation describing the diffusion of the spin accumulation. This paper establishes the global existence and uniqueness of weak solutions for large initial data in $\mathbb{R}^2$. Moreover, partial regularity is shown. In particular, the solution is regular on $\mathbb{R}^2 \times (0, \infty)$ with the exception of at most finite singular points.

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1 Introduction

In this paper, we consider the following coupled system modeling the spin-magnetization in ferromagnetic multilayers, where the diffusion process of the spin accumulation through the multilayers is taken into account. The spin accumulation $s$ is described by a system of quasilinear diffusion equations and the precession of the magnetization $m$ is described by the Landau-Lifshitz equation. The coupled system is given by

$$\begin{cases}
\partial_t s = -\text{div} J_s - D_0(x)s - D_0(x)s \times m \\
\partial_t m = -m \times (h + s) + \alpha m \times \partial_t m,
\end{cases} \quad (1.1)$$

where $s = (s_1, s_2, s_3) \in \mathbb{R}^3$ is the spin accumulation, $m = (m_1, m_2, m_3) \in S^2$ is the precession of the magnetization, and $J_s$ is the spin current given by

$$J_s = m \otimes J_e - D_0(x) [\nabla s - \beta m \otimes (\nabla s \cdot m)],$$
where $J_e$ is the applied electric current, and the local field $h$ can be derived from the Landau-Lifshitz energy

$$E(m) = \int \Phi(m) + \frac{1}{2} \int |\nabla m|^2 - \frac{1}{2} \int h_d \cdot m,$$

by

$$h = -\frac{\delta E(m)}{\delta m} = -\nabla m \Phi + \Delta m + h_d.$$ 

In the above system, $D_0(x) > 0$ is the diffusion coefficient of the spin accumulation which is assumed to be a measurable function bounded from above and below, $0 < \beta < 1$ is the spin polarization parameter, $\alpha > 0$ is the Gilbert damping parameter and the term $\alpha m \times \partial_t m$ is usually referred to as the Gilbert damping. The additional term in the LLG equation corresponds to the interaction $F_0[s, m] = -\int m \cdot sdx$. For more physics background, the interested readers may refer to [13, 27, 37] for more details.

To get rid of unimportant factors for the study in this paper, we set $J_e \equiv 0$, $D_0(x) \equiv 1$, and only keep $h = \Delta m$ is the magnetization field. These simplification will not influence the results of this paper substantially, but will do simplify the presentation of this paper significantly. In this paper, we will concentrate on the two dimensional case, i.e., we let $x \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$, and regard $(s, m) \in \mathbb{R}^3 \times S^2$ as functions of $(x, t) \in \mathbb{R}^2 \times \mathbb{R}^+$, and leave the three dimensional case in a forthcoming paper, since they are handled differently.

The equation for the spin accumulation $s$ in (1.1) can then be rewritten as

$$\partial_t s - \text{div}(A(m) \nabla s) + s + s \times m = 0,$$  (1.2)

where the coefficient of the principal part depends on the magnetization field $m$ by

$$A(m) = \begin{pmatrix} 1 - \beta m_1^2 & -\beta m_2 m_1 & -\beta m_1 m_3 \\ -\beta m_2 m_1 & 1 - \beta m_2^2 & -\beta m_2 m_3 \\ -\beta m_3 m_1 & -\beta m_3 m_2 & 1 - \beta m_3^2 \end{pmatrix}.  \quad (1.3)$$

Since $0 < \beta < 1$ and $|m| \equiv 1$, $A(m)$ is strictly positively definite with

$$(1 - \beta)|\xi|^2 \leq \xi^T A(m) \xi \leq |\xi|^2 \quad (1.4)$$

and equation (1.2) is strongly parabolic. On the other hand, since $|m| = 1$, the second equation of (1.1) can also be rewritten in the following two equivalent forms

$$(1 + \alpha^2) \frac{\partial m}{\partial t} = -m \times (\Delta m + s) - \alpha m \times (m \times (\Delta m + s)) \quad (1.5)$$

or

$$(1 + \alpha^2) \frac{\partial m}{\partial t} - \alpha \Delta m = \alpha |\nabla m|^2 m - m \times (\Delta m + s) - \alpha m \times (m \times s). \quad (1.6)$$

When the spin accumulation $s$ is not considered, the system (1.1) reduces to the Landau-Lifshitz equation, which is a fundamental equation describing the evolution of ferromagnetic spin chain and was proposed on the phenomenological ground in studying the dispersive theory of magnetization of ferromagnets in 1935 by Landau and Lifshitz [19]. An equivalent form of the Landau-Lifshitz equation was proposed by Gilbert in 1955 [14], and $\alpha$ is called the Gilbert damping coefficient. Hence the Landau-Lifshitz equation is also called the Landau-Lifshitz-Gilbert (LLG) equation in the literature.

The Landau-Lifshitz equation is interesting in both mathematics and physics, not only because it is closely related to the famous heat flow of harmonic maps (formally when the Gilbert damping parameter $\alpha \to \infty$) [5, 11, 12, 29, 30] and to the Schrödinger flow on the sphere (when the Gilbert damping parameter $\alpha \to 0$) [2, 10, 15], but also because it has concrete physics background in the study of the magnetization in ferromagnets. In recent years, there has been lots
of interesting studies for the Landau-Lifshitz equation, concerning its existence, uniqueness and regularities of various kinds of solutions. In the sequel, we list only a few of the literature that are closely related to our work in the present paper.

For the Landau-Lifshitz equation on two dimensional compact manifold $\mathcal{M}$ without boundary, Guo and Hong \[15\] proved global existence and uniqueness of smooth solutions under small energy assumptions. Note that in the 2D case, the Landau-Lifshitz equation is energy critical. Moreover, they showed the partial regularity of weak solutions, in the spirit of the Struwe’s treatment of the heat flow of harmonic maps on two dimensional compact manifold without boundary \[29\]. They showed that for any initial data in $H^1$, there exists a unique solution that is regular with exception of finitely many singular points on $\mathcal{M} \times (0, \infty)$. Global existence of weak solutions in 3D was also considered in their paper by Ginzburg-Landau approximation. In $\mathbb{R}^3$, Alouges and Soyeur proved the existence of weak solutions by Ginzburg-Landau approximation for the Landau-Lifshitz-Gilbert equation in the paper \[1\], where nonuniqueness is also shown.

In $\mathbb{R}^3$, the Landau-Lifshitz equation becomes energy supercritical, and therefore uniqueness and regularity problems become more delicate. Global existence of classical solutions with small initial data was obtained by Melcher \[22\] by deriving a covariant Ginzburg-Landau equation and using the Coulomb gauge, inspired by recent developments in the context of Schrödinger maps \[2\]. We also note that in the one dimensional case, the global existence of classical solutions to the Landau-Lifshitz equation without Gilbert damping (i.e. the one dimensional schrödinger maps flow) for any smooth initial data was obtained the seminal paper \[38\], where the moving frame method was introduced for the first time to study the Landau-Lifshitz equation.

For regularity problems for the Landau-Lifshitz equation in higher dimensions, Moser \[24\] showed that the weak solutions of the Landau-Lifshitz equation of the ferromagnetic spin chain are smooth in an open set with complement of vanishing $d$-dimensional Hausdorff measure respect to the parabolic metric in $\mathbb{R}^d$ for $d \leq 4$, when the solution is stationary, in the spirit of Feldman’s result \[12\] for stationary weak solutions of the heat flow of harmonic maps. Slightly later, Liu \[21\] studied the partial regularity of stationary weak solutions for the Landau-Lifshitz equation, by obtaining a generalized monotonicity inequality. Melcher \[22\] established the existence of partially regular weak solutions for the Landau-Lifshitz equation in $\mathbb{R}^3$ without stationary assumptions, based on the Ginzburg-Landau approximation with trilinear estimates. Wang \[33\] also studied the partial regularity of the Landau-Lifshitz equation, obtaining the existence of a global weak solution for smooth initial data, which is smooth off a set with locally finite $d$-dimensional parabolic Hausdorff measure for $d \leq 4$. Meanwhile, Ding and Wang \[9\] studied the finite time singularity of the Landau-Lifshitz equation in dimensions three and four, for suitably chosen initial data. Other regularity or blow up results to the Landau-Lifshitz-Maxwell equations were studied in \[7, 8\], to list only a few.

However, for the spin-magnetization system \[11\] that takes into account the diffusion process of the accumulation, there are few mathematical studies in the literature. The first mathematical result is due to García-Cervera and Wang \[13\], who firstly studied such a coupled system and obtained global existence of global weak solutions in a 3D bounded domain. Nonuniqueness was also discussed in their paper. Global existence and uniqueness of smooth solutions in 2D when the initial data is small \[16\] and in 1D for any smooth initial data were studied in \[29\]. But we don’t know whether the weak solutions in 2D are regular when the initial data is not small. In this paper, we show that the weak solutions are indeed unique and regular with the exception of finitely many points in $\mathbb{R}^2 \times (0, \infty)$ for any initial data $(s_0, \mathbf{m}_0) \in L^2(\mathbb{R}^2) \times H^1_4(\mathbb{R}^2)$. See precise statement of the results in Theorem \[14\] below. Similar result can be generalized to the periodic case. The partial regularity result in $\mathbb{R}^3$ and global existence of small solutions under smallness conditions will be presented in forthcoming papers.

For a given constant vector $\mathbf{a} \in S^2$ and a positive integer $k$, we define

$$H^k_a(\mathbb{R}^2, S^2) = \{ \mathbf{m} : \mathbf{m} - \mathbf{a} \in H^k(\mathbb{R}^2, S^2), |\mathbf{m}| = 1, a.e., \text{ in } \mathbb{R}^2 \}.$$
Then our main results are stated as follows:

**Theorem 1.1.** Assume that the initial data $s_0 \in L^2(\mathbb{R}^2;\mathbb{R}^3)$ and $m_0 \in H^1_a(\mathbb{R}^2;\mathbb{S}^2)$. Then there exists a unique global weak solution $(s,m)$ of the system (1.1) which is smooth in $\mathbb{R}^2 \times ((0, \infty) \setminus \{ T_l \}_{l=1}^L)$ with a finite number of singular points $(x^l_i,T_l)$, $1 \leq l \leq L$. Moreover, there are two constants $\varepsilon_0 > 0$ and $R_0 > 0$ such that each singular point $(x^l_i,T_l)$ is characterized by

$$\limsup_{t \uparrow T_l} \int_{B_R(x^l_i)} |\nabla m(\cdot,t)|^2 dx > \varepsilon_0$$

for any $0 < R \leq R_0$.

The strategy basically follows the seminal work of Struwe for the heat flow of harmonic maps. But there are something new in this paper. First, the Sobolev space $s$ that the components of the magnetization field $m$ and for the spin polarization field $s$. From Theorem 1.1 we can see that we only require $m_0 \in H^1_a(\mathbb{R}^2;\mathbb{S}^2)$ and $s_0 \in L^2(\mathbb{R}^2;\mathbb{R}^3)$, and the regularity of $s$ is very low. The main difficulty caused by this fact is that we don’t have any $L^\infty$-estimates of the spin polarization $s$, different from that of the magnetization $m \in \mathbb{S}^2$, whose $L^\infty$-estimate is obvious. The inherent structure restricts us from copying/mimicking the arguments of any presenting literature. Secondly, with such a low regularity, the uniqueness of weak solutions becomes a real problem. In this paper, we prove the uniqueness under the help of Littlewood-Paley theory and the techniques of Besov spaces, presented in Section 3.

This paper is organized as below. In the next section, we give some a priori estimates. In Section 3 and 4, we show existence and uniqueness of the weak solutions and finally in Section 5, we prove a local well-posedness result. Throughout this article, $C$ denotes a constant depending on $\alpha$ or $\beta$, which may be different from line to line.

## 2 A priori Estimates

In this section, we show some a priori estimates for the system (1.1). As in [29], we introduce the following Sobolev spaces. For $0 \leq T < T$, let

$$V(\tau,T) := \left\{ m : \mathbb{R}^2 \times [\tau,T] \to \mathbb{S}^2 \mid m \in H^1_a(\mathbb{R}^2,\mathbb{S}^2) \text{ for a.e. } t \in [\tau,T], \right. \quad \text{ess sup}_{\tau \leq t \leq T} \left[ \int_{\mathbb{R}^2} |\nabla m(\cdot,t)|^2 dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla^2 m|^2 + |\partial_t m|^2 dxdt < \infty \right\}.$$  

(2.1)

and

$$W(\tau,T) := \left\{ s : \mathbb{R}^2 \times [\tau,T] \to \mathbb{R}^3 \mid s \text{ is measurable, } \right. \quad \text{ess sup}_{\tau \leq t \leq T} \left[ \int_{\mathbb{R}^2} |s(\cdot,t)|^2 dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla s|^2 dxdt < \infty \right\}.$$  

(2.2)

By the same proof as in Lemma 3.1 of [29], we have

**Lemma 2.1.** There exist some absolute constants $C$, $R_0 > 0$ such that for any function $f$ in $W(0,T)$, and any $R \in (0,R_0]$ the following estimate holds

$$\int_{\mathbb{R}^2 \times [0,T]} |f|^4 dxdt \leq C \cdot \text{ess sup}_{0 < t < T} \int_{B_R(x)} |f(\cdot,t)|^2 dx \cdot \left( \int_0^T \int_{\mathbb{R}^2} |\nabla f|^2 dxdt + R^{-2} \int_0^T \int_{\mathbb{R}^2} |f|^2 dxdt \right).$$  

(2.3)
For simplicity, we denote that
\[
E_0 = E_0^* + \alpha E_0^m, \quad E_0^* = \int_{\mathbb{R}^2} |s_0|^2 dx, \quad E_0^m = \int_{\mathbb{R}^2} |\nabla m_0|^2 dx,
\]
\[
E_R(x_0, t) = E_R^*(x_0, t) + \alpha E_R^m(x_0, t) = \int_{B_R(x_0)} |s(x, t)|^2 dx + \alpha \int_{B_R(x_0)} |\nabla m(x, t)|^2 dx,
\]
\[
E(t) = E^*(t) + \alpha E^m(t) = \int_{\mathbb{R}^2} |s|^2(\cdot, t) + \alpha |\nabla m|^2(\cdot, t) dx.
\]

At first, we have the following basic energy type inequalities.

**Lemma 2.2.** Assume that \((s, m) \in W(0, T) \times V(0, T)\) is a solution of the system (1.1). Then there holds the following estimates
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} |s|^2(\cdot, t) dx + 2(1 - \beta) \int_0^T \int_{\mathbb{R}^2} |\nabla s|^2 dx dt + 2 \int_0^T \int_{\mathbb{R}^2} |s|^2 dx dt \leq E_0^*, \quad (2.4)
\]
and
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} |s|^2(\cdot, t) + \alpha |\nabla m|^2(\cdot, t) dx + \int_0^T \int_{\mathbb{R}^2} \left(|s|^2 + 2(1 - \beta)|\nabla s|^2 + \alpha^2 |\partial_t m|^2\right) dx dt \leq \int_{\mathbb{R}^2} |s_0|^2 + \alpha |\nabla m_0|^2 dx, \quad (2.5)
\]
which is \(E(t) \leq E_0\) for all \(0 < t \leq T\).

**Proof.** Multiplying equation (1.5) by \(\partial_t m\) and then integrating on \(\mathbb{R}^2 \times [0, T]\) yield that
\[
(1 + \alpha^2) \int_0^T \int_{\mathbb{R}^2} |\partial_t m|^2 dx dt = - \int_0^T \int_{\mathbb{R}^2} m \times (\Delta m + s) \cdot \partial_t m \, dx dt - \alpha \int_0^T \int_{\mathbb{R}^2} m \times (m \times (\Delta m + s)) \cdot \partial_t m \, dx dt.
\]

Applying the vector cross product formula \(a \times (b \times c) = (a \cdot c)b - (a \cdot b)c\) and noticing that \(m \cdot \partial_t m = 0\), we have
\[
(1 + \alpha^2) \int_0^T \int_{\mathbb{R}^2} |\partial_t m|^2 dx dt + \frac{\alpha}{2} \int_0^T \int_{\mathbb{R}^2} \frac{d}{dt} |\nabla m|^2 dx dt = \alpha \int_0^T \int_{\mathbb{R}^2} s \cdot \partial_t m dx dt - \int_0^T \int_{\mathbb{R}^2} m \times (\Delta m + s) \cdot \partial_t m dx dt. \quad (2.6)
\]

On the other hand, it follows from the second equation of (1.1) that
\[
\int_0^T \int_{\mathbb{R}^2} |\partial_t m|^2 dx dt = - \int_0^T \int_{\mathbb{R}^2} \partial_t m \cdot (m \times (\Delta m + s)) dx dt \quad (2.7)
\]
Thus using the Hölder inequality
\[
\alpha \left| \int_0^T \int_{\mathbb{R}^2} s \cdot \partial_t m dx dt \right| \leq \frac{\alpha^2}{2} \int_0^T \int_{\mathbb{R}^2} |\partial_t m|^2 + \frac{1}{2} \int_0^T \int_{\mathbb{R}^2} |s|^2,
\]

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which combines (2.6) and (2.7) implies that
\[
\alpha^2 \int_0^T \int_{\mathbb{R}^2} |\partial_t m|^2 dx dt + \alpha \int_0^T \int_{\mathbb{R}^2} \frac{d}{dt} |\nabla m|^2 dx dt \leq \int_0^T \int_{\mathbb{R}^2} |s|^2 dx dt. \tag{2.8}
\]

Furthermore, it follows from the equation of \(s\) (1.2) that
\[
\int_0^T \int_{\mathbb{R}^2} |s|^2 dx dt + 2(1 - \beta) \int_0^T \int_{\mathbb{R}^2} |\nabla s|^2 dx dt + 2 \int_0^T \int_{\mathbb{R}^2} |s|^2 dx dt = 0
\]
which and (2.4) yield the required inequality. \(\square\)

**Remark 2.1.** Under the assumptions of Lemma 2.2, the estimate (2.3) implies that
\[
\int_0^T \int_{\mathbb{R}^2} |s|^4 dx dt \leq C \cdot \text{ess sup}_{(x,t) \in \mathbb{R}^2 \times [0,T]} E_R^s(x_0,t) \left( \int_0^T \int_{\mathbb{R}^2} |\nabla s|^2 dx dt + TR^{-2}E_0 \right), \tag{2.9}
\]
and
\[
\int_0^T \int_{\mathbb{R}^2} |\nabla m|^4 dx dt \leq C \cdot \text{ess sup}_{(x,t) \in \mathbb{R}^2 \times [0,T]} E_R^m(x_0,t) \left( \int_0^T \int_{\mathbb{R}^2} |\nabla^2 m|^2 + TR^{-2}E_0 \right). \tag{2.10}
\]

**Lemma 2.3.** Let \((s,m) \in W(0,T) \times V(0,T)\) be a solution of the system (1.1) with initial data \((s_0,m_0) \in L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\). There exist constants \(\varepsilon > 0\) and \(R_0 > 0\) such that if
\[
\text{ess sup}_{\tau \leq t \leq T, x_0 \in \mathbb{R}^2} E_R^m(x_0,t) < \varepsilon,
\]
for any \(R \in (0,R_0)\) and \(0 < \tau < T\), then we have
\[
\int_{\mathbb{R}^2 \times [\tau,T]} |\nabla^2 m|^2 + |\nabla s|^2 dx dt \leq CE_0 + C\varepsilon(T - \tau)R^{-2}E_0, \tag{2.11}
\]
and
\[
\int_{\mathbb{R}^2 \times [\tau,T]} |\nabla m|^4 + |s|^4 dx dt < C\varepsilon(1 + (T - \tau)R^{-2})E_0. \tag{2.12}
\]

**Proof.** Without loss of generality, we can assume that \(\tau = 0\), since the system (1.1) is translation invariant. Multiplying equation (1.6) by \(-\Delta m\), integrating over \(\mathbb{R}^2 \times [0,T]\) and using Hölder inequality, we have
\[
\frac{1 + \alpha^2}{2} \int_0^T \int_{\mathbb{R}^2} \frac{d}{dt} |\nabla m|^2 dx dt + \alpha \int_0^T \int_{\mathbb{R}^2} |\Delta m|^2 dx dt \\
\leq C \int_0^T \int_{\mathbb{R}^2} |\Delta m||\nabla m|^2 dx dt + C \int_0^T \int_{\mathbb{R}^2} |s||\Delta m| dx dt \\
\leq \frac{\alpha}{2} \int_0^T \int_{\mathbb{R}^2} |\Delta m|^2 dx dt + C \int_0^T \int_{\mathbb{R}^2} |\nabla m|^4 dx dt + C \int_0^T \int_{\mathbb{R}^2} |s|^2 dx dt \tag{2.13}
\]
by virtue of \((m \times \Delta m) \cdot \Delta m = 0\) and \(|m \times s| \leq |s|\), which implies that
\[
\alpha \int_0^T \int_{\mathbb{R}^2} |\Delta m|^2 dx dt \leq (1 + \alpha^2) \int_{\mathbb{R}^2} |\nabla m_0|^2 dx + C \int_0^T \int_{\mathbb{R}^2} |\nabla m|^4 dx dt + C \int_0^T \int_{\mathbb{R}^2} |s|^2 dx dt.
\]
But from Remark 2.1, it follows that
\[
\int_{\mathbb{R}^2 \times [0,T]} |\nabla m|^4 dx dt \leq C\varepsilon \left( \int_{\mathbb{R}^2 \times [0,T]} |\nabla^2 m|^2 dx dt + R^2 \int_{\mathbb{R}^2 \times [0,T]} |\nabla m|^2 dx dt \right) \leq C\varepsilon (1 + TR^{-2}).
\]
which and Lemma 2.2 yield that
\[
\int_{\mathbb{R}^2 \times [0,T]} |\nabla^2 m|^2 + |\nabla s|^2 dx dt \leq C E_0 + C\varepsilon TR^{-2} E_0,
\]
and
\[
\int_{\mathbb{R}^2 \times [0,T]} |\nabla m(\cdot, t)|^4 + |s(\cdot, t)|^4 dx dt \leq C\varepsilon (1 + TR^{-2}) E_0.
\]
The proof is complete.

**Lemma 2.4.** Let \((s, m) \in W(0, T) \times V(0, T)\) be a solution of (1.1) with the initial data \((s_0, m_0) \in L^2(\mathbb{R}^2) \times H^1_a(\mathbb{R}^2)\), then
\[
\int_{B_R(x_0)} \left( |\nabla m|^2 + |s|^2 \right)(\cdot, t) dx \leq \int_{B_{2R}(x_0)} \left( |\nabla m_0|^2 + |s_0|^2 \right) dx + C \frac{t}{R^2} E_0 + Ct E_0, \tag{2.14}
\]
for any \(x_0 \in \mathbb{R}^2\) and \(0 < t < T\).

**Proof.** (i) Let \(\varphi \in C_0^\infty(B_{2R}(x_0))\) satisfy \(0 \leq \varphi \leq 1\), \(\varphi \equiv 1\) on \(B_{R}(x_0)\), \(|\nabla \varphi| \leq \frac{C}{R}\). Multiplying equation (1.1) by \(\partial_t m \varphi^2\) and integrating over \(\mathbb{R}^2\), we obtain
\[
(1 + \alpha^2) \int_0^t \int_{\mathbb{R}^2} |\partial_t m|^2 \varphi^2 dx dt + \frac{\alpha}{2} \int_0^t \int_{\mathbb{R}^2} \frac{d}{dt}(|\nabla m|^2 \varphi^2) dx dt + \int_0^t \int_{\mathbb{R}^2} \partial_t m \cdot (m \times (\Delta m + s)) \varphi^2 dx dt \leq \alpha \int_0^t \int_{\mathbb{R}^2} |\nabla m||\partial_t m||\nabla \varphi| \varphi dx dt + \alpha \int_{\mathbb{R}^2} |s||\partial_t m| \varphi^2 dx dt.
\]
By the second equation in (1.1),
\[
\int_0^t \int_{\mathbb{R}^2} \partial_t m \cdot (m \times (\Delta m + s)) \varphi^2 dx dt = - \int_0^t \int_{\mathbb{R}^2} |\partial_t m|^2 \varphi^2 dx dt,
\]
thus we can deduce from (2.14)
\[
\alpha^2 \int_0^t \int_{\mathbb{R}^2} |\partial_t m|^2 \varphi^2 dx dt + \frac{\alpha}{2} \int_0^t \int_{\mathbb{R}^2} \frac{d}{dt}(|\nabla m|^2 \varphi^2) dx dt \leq \alpha^2 \int_0^t \int_{\mathbb{R}^2} |\partial_t m|^2 \varphi^2 dx dt + C \int_0^t \int_{\mathbb{R}^2} |\nabla m|^2 |\nabla \varphi|^2 dx dt + C \int_0^t \int_{\mathbb{R}^2} |s|^2 \varphi^2 dx dt. \tag{2.15}
\]
Finally, by Remark 2.1 and Lemma 2.2
\[
E_R^m(x_0, t) \leq \int_0^t \int_{\mathbb{R}^2} |\nabla m|^2 \varphi^2(\cdot, t) dx dt = \int_{\mathbb{R}^2} |\nabla m_0|^2 \varphi^2 dx + \int_0^t \int_{\mathbb{R}^2} \frac{d}{dt}(|\nabla m|^2 \varphi^2) dx dt \leq \int_{\mathbb{R}^2} |\nabla m_0|^2 \varphi^2 dx + C R^{-2} E_0 t + C E_0 t \leq E_R^m(x_0, 0) + C \frac{t}{R^2} E_0 + Ct E_0.
\]
(ii) We then multiply the equation with \( s \varphi^2 \) and integrate over \( \mathbb{R}^2 \) to obtain

\[
\int_{\mathbb{R}^2} \partial_t s \cdot s \varphi^2 dx - \int_{\mathbb{R}^2} \text{div}(A(m)\nabla s) \cdot s \varphi^2 + \int_{\mathbb{R}^2} |s|^2 \varphi^2 dx = 0.
\]

Noting that

\[
- \int_{\mathbb{R}^2} \text{div}(A(m)\nabla s) \cdot s \varphi^2 dx = \int_{\mathbb{R}^2} a_{ij}(m) \partial_j s \cdot \partial_i s \varphi^2 dx + 2 \int_{\mathbb{R}^2} a_{ij}(m) \partial_j s \cdot s \varphi \partial_i \varphi dx \\
\geq (1 - \beta) \int_{\mathbb{R}^2} |\nabla s|^2 \varphi^2 dx - 2 \int_{\mathbb{R}^2} |\nabla s||\varphi||\nabla \varphi| dx \\
\geq \frac{(1 - \beta)}{2} \int_{\mathbb{R}^2} |\nabla s|^2 \varphi^2 dx - C R^{-2} \int_{\mathbb{R}^2} |s|^2 dx,
\]

where \( a_{ij} \) are the entries of the matrix \( A(m) \). Integrating over \([0, t]\), one obtains

\[
\int_{B_R(x)} |s(\cdot, t)|^2 dx + (1 - \beta) \int_0^t \int_{\mathbb{R}^2} |\nabla s|^2 \varphi^2 dx + 2 \int_0^t \int_{\mathbb{R}^2} |s|^2 \varphi^2 dx \\
\leq \int_{B_R(x)} |s_0(\cdot, t)|^2 dx + C t R^{-2} \int_{\mathbb{R}^2} |s_0|^2 dx,
\]

which and (2.10) yield (2.14). The proof is complete. \( \square \)

**Lemma 2.5.** Let \((s, m) \in W(0, T) \times V(0, T)\) be a solution of (1.1) with the initial data \((s_0, m_0) \in L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)\). Assume that there exist constants \( \varepsilon > 0 \) and \( R_0 > 0 \) such that

\[
\sup_{x \in \mathbb{R}^2, 0 \leq t \leq T} \int_{B_R(x)} |\nabla m(x, t)|^2 dx < \varepsilon,
\]

for any \( R \in (0, R_0] \). Then for any \( t \in [\tau, T] \) for \( \tau > 0 \), we have

\[
\int_{\mathbb{R}^2} |\nabla^2 m(\cdot, t)|^2 + |\nabla s(\cdot, t)|^2 dx + \int_{\tau}^T \int_{\mathbb{R}^2} |\Delta m|^2 + |\Delta s|^2 dx dt \leq C(\tau, T, E_0, \frac{T}{R^2}), \quad (2.16)
\]

and

\[
\int_{\tau}^T \int_{\mathbb{R}^2} |\nabla^2 m|^4 + |\nabla s|^4 dx dt \leq C(\tau, T, E_0, \frac{T}{R^2}). \quad (2.17)
\]

**Proof.** **Step 1. Estimate for \( s \).** We take the inner product of equation (1.1) with \(-\Delta s\) to obtain

\[
- \int_{\mathbb{R}^2} \partial_t s \cdot \Delta s dx + \int_{\mathbb{R}^2} \text{div}(A(m)\nabla s) \cdot \Delta s dx - \int_{\mathbb{R}^2} s \cdot \Delta s dx - \int_{\mathbb{R}^2} (s \times m) \cdot \Delta s dx = 0.
\]

By integration by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla s|^2 dx + \frac{1 - \beta}{2} \int_{\mathbb{R}^2} |\Delta s|^2 dx \leq \int_{\mathbb{R}^2} |s|^2 dx + C \int_{\mathbb{R}^2} |\nabla m|^2 |\nabla s|^2 dx \\
\leq E_0 + C \|\nabla m\|_{L^4(\mathbb{R}^2)}^2 \|\nabla s\|_{L^2(\mathbb{R}^2)}^2 \|\nabla^2 s\|_{L^2(\mathbb{R}^2)}^2,
\]

where we used Lemma 2.2 and the Gagliardo-Nirenberg interpolation inequality. By Gronwall’s inequality we have

\[
\sup_{\tau < t < T} \int_{\mathbb{R}^2} |\nabla s|^2 dx + (1 - \beta) \int_{\tau}^T \int_{\mathbb{R}^2} |\Delta s|^2 dx dt \leq C(T, E_0, \|\nabla m\|_{L^4(\mathbb{R}^2 \times (0, T))}) \int_{\mathbb{R}^2} |\nabla s|^2(\cdot, s) dx,
\]

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where \( s \in (0, \tau) \) and we can choose \( s \) such that
\[
\int_{\mathbb{R}^2} |\nabla s|^2(\cdot, s)\,dx \leq \tau^{-1} \int_{\mathbb{R}^2 \times (0, \tau)} |\nabla s|^2\,dxdt.
\]
Hence using Lemma 2.2 and Lemma 2.3 we get
\[
\sup_{\tau < t < T} \int_{\mathbb{R}^2} |\nabla s|^2\,dx + (1 - \beta) \int_{\tau}^{T} \int_{\mathbb{R}^2} |\Delta s|^2\,dxdt \leq C(\tau, T, E_0, \frac{T}{R^2}). \tag{2.18}
\]
By the interpolation inequality, it then gives the estimate
\[
\int_{\tau}^{T} \int_{\mathbb{R}^2} |\nabla s|^4\,dxdt \leq C(\tau, T, E_0, \frac{T}{R^2}). \tag{2.19}
\]
**Step 2. Estimate for \( m \).**
Applying \( \Delta \) to equation (1.6) and then taking inner product with \( \Delta m \), we have
\[
(1 + \alpha^2) \int_{\mathbb{R}^2} \partial_t \Delta m \cdot \Delta m\,dx + \alpha \int_{\mathbb{R}^2} |\nabla \Delta m|^2\,dx
= \alpha \int_{\mathbb{R}^2} \Delta m \cdot \Delta (|\nabla m|^2 m)\,dx - \int_{\mathbb{R}^2} \Delta m \cdot \Delta (m \times \Delta m)\,dx
- \int_{\mathbb{R}^2} \Delta m \cdot \Delta [(m \times s) + \alpha m \times (m \times s)]\,dx =: I_1 + I_2 + I_3.
\]
For the term \( I_1 \), we have
\[
|I_1| \leq 2\alpha \int_{\mathbb{R}^2} |\nabla \Delta m| |\nabla m| |\nabla^2 m|\,dx + \alpha \int_{\mathbb{R}^2} |\nabla \Delta m| |\nabla m|^3\,dx
\leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla \Delta m|^2\,dx + C \int_{\mathbb{R}^2} |\nabla m|^2 |(\nabla^2 m|^2 + |\nabla m|^4)\,dx \tag{2.20}
\leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla \Delta m|^2\,dx + C \|\nabla m\|_{L^4(\mathbb{R}^2)}^2 \|\nabla^2 m\|_{L^2(\mathbb{R}^2)} \|\nabla^3 m\|_{L^2(\mathbb{R}^2)},
\]
where we used \( \Delta m \cdot m = -|\nabla m|^2 \) and Gagliardo-Nirenberg interpolation inequality. The term \( I_2 \) is estimated in a similar way:
\[
|I_2| \leq \|\nabla m\|_{L^4(\mathbb{R}^2)} \|\nabla^2 m\|_{L^4(\mathbb{R}^2)} \|\nabla^3 m\|_{L^2(\mathbb{R}^2)}
\leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla \Delta m|^2\,dx + C \|\nabla m\|_{L^4(\mathbb{R}^2)}^2 \|\nabla^2 m\|_{L^2(\mathbb{R}^2)} \|\nabla^3 m\|_{L^2(\mathbb{R}^2)}. \tag{2.21}
\]
For \( I_3 \), by Hölder inequality and Lemma 2.3 we have
\[
|I_3| \leq CE_0 \left[ \|
abla \Delta m\|_{L^4(\mathbb{R}^2)} + \|
abla \Delta m\|_{L^4(\mathbb{R}^2)} \|
abla s\|_{L^4(\mathbb{R}^2)} \right] + C \|
abla \Delta m\|_{L^4(\mathbb{R}^2)} \|
abla s\|_{L^2(\mathbb{R}^2)} \tag{2.22}
\]
Using Gagliardo-Nirenberg interpolation inequality again, we have
\[
I_3 \leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla \Delta m|^2\,dx + C(E_0) \|
abla \Delta m\|_{L^2(\mathbb{R}^2)}^2 + \|
abla s\|_{L^2(\mathbb{R}^2)}^4 + \|
abla s\|_{L^2(\mathbb{R}^2)}^4. \tag{2.23}
\]
Therefore, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\Delta m|^2\,dx + \int_{\mathbb{R}^2} |\nabla \Delta m|^2\,dx
\leq C(E_0)(1 + \|
abla m\|_{L^4(\mathbb{R}^2)}^4) \|
abla m\|_{L^2(\mathbb{R}^2)}^2 + C \|
abla s\|_{L^2(\mathbb{R}^2)}^4 + C \|
abla s\|_{L^2(\mathbb{R}^2)}^4, \tag{2.24}
\]
which combines (2.19) and Lemma 2.5 yields that
\[
\int_{\mathbb{R}^2} |\nabla^2 m(\cdot, t)|^2 \, dx + \int_0^T \int_{\mathbb{R}^2} |\nabla \Delta m|^2 \, dx \, dt \leq C(\tau, T, E_0, \frac{T}{R^2}),
\]
(2.25)
due to the Gronwall’s inequality.

Consequently, (2.25) and (2.18) imply the required inequality (2.16). The inequality (2.17) follows from (2.16) via Gagliardo-Nirenberg interpolation inequality. The proof is complete. \(\square\)

Indeed, using the above idea by induction, one can prove the following

**Corollary 2.1.** Assume that \((s, m) \in W(0, T) \times V(0, T)\) is a solution of (1.1) with the initial data \((s_0, m_0) \in L^2(\mathbb{R}^2) \times H^1_0(\mathbb{R}^2).\) Then there is a constant \(\varepsilon_1\) such that for any \(R \in (0, R_0],\) if
\[
\text{ess sup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla m(\cdot, t)|^2 \, dx < \varepsilon,
\]
then for all \(t \in (\tau, T)\) with \(\tau \in (0, T),\) for all \(l \geq 1,\) it holds that
\[
\int_{\mathbb{R}^2} |\nabla^{l+1} m(\cdot, t)|^2 + |\nabla^l s|^2(\cdot, t) \, dx + \int_0^t \int_{\mathbb{R}^2} |\nabla^{l+2} m|^2 + |\nabla^{l+1} s|^2 \, dx \, dt \leq C \left(l, \tau, T, E_0, \frac{T}{R^2}\right).
\]
(2.26)
Moreover, \(m\) and \(s\) are regular for all \(t \in (0, T).\)

**Proof.** The case \(l = 1\) is proved in Lemma 2.5. Now we consider the case \(l = 2.\)

**Step I. Estimate for \(s.\)** We first improve the regularity of \(s.\) Taking \(\Delta\) to the equation (1.1) satisfied by \(s\) and then taking inner product with \(\Delta s,\) we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\Delta s|^2 \, dx - \int_{\mathbb{R}^2} \Delta \text{div}(A(m) \nabla s) \cdot \Delta s \, dx + \int_{\mathbb{R}^2} |\Delta s|^2 \, dx + \int_{\mathbb{R}^2} \Delta(s \times m) \cdot \Delta s = 0.
\]
(2.27)
For the second term on the left, by \(\Delta m \cdot m = -|\nabla m|^2\) we have
\[
\int_{\mathbb{R}^2} \Delta \text{div}(A(m) \nabla s) \cdot \Delta s \, dx + (1 - \beta)|\nabla^3 s|^2_{L^2(\mathbb{R}^2)} \\
\leq C \left( \int_{\mathbb{R}^2} |\nabla m|^2 |\nabla^2 s|^2 \, dx + \int_{\mathbb{R}^2} |\nabla^2 m||\Delta s||\nabla^3 s| \, dx + \int_{\mathbb{R}^2} |\nabla m|^2 |\nabla s||\nabla^3 s| \, dx \right) \\
\leq C \|\nabla m\|_{L^4(\mathbb{R}^2)} \|\nabla^2 s\|_{L^4(\mathbb{R}^2)} \|\nabla^3 s\|_{L^2(\mathbb{R}^2)} + C \|\nabla^2 m\|_{L^4(\mathbb{R}^2)} \|\nabla^3 s\|_{L^4(\mathbb{R}^2)} \|\nabla^2 s\|_{L^4(\mathbb{R}^2)}.
\]
Using Gagliardo-Nirenberg interpolation inequality for the term \(\|\nabla^2 s\|_{L^4(\mathbb{R}^2)},\) we get
\[
\int_{\mathbb{R}^2} \Delta \text{div}(A(m) \nabla s) \cdot s(\cdot, t) \, dx + \frac{(1 - \beta)}{4} \|\nabla^3 s\|^2_{L^2(\mathbb{R}^2)} \\
\leq C \|\nabla m\|_{L^4(\mathbb{R}^2)} \|\nabla^2 s\|^2_{L^4(\mathbb{R}^2)} + C \|\nabla^2 m\|^2_{L^4(\mathbb{R}^2)} \|\nabla s\|^2_{L^4(\mathbb{R}^2)}.
\]
Moreover, we have
\[
\int_{\mathbb{R}^2} \Delta(s \times m) \cdot \Delta s \leq C \left( \|\nabla^2 m\|_{L^4(\mathbb{R}^2)} \|s\|_{L^4(\mathbb{R}^2)} + \|\nabla m\|_{L^4(\mathbb{R}^2)} \|\nabla s\|_{L^4(\mathbb{R}^2)} \|\nabla^3 s\|_{L^2(\mathbb{R}^2)} \right) \\
\leq \frac{(1 - \beta)}{4} \|\nabla^3 s\|_{L^2(\mathbb{R}^2)} + C \left( \|\nabla^2 m\|_{L^4(\mathbb{R}^2)} \|s\|^2_{L^4(\mathbb{R}^2)} + \|\nabla m\|^2_{L^4(\mathbb{R}^2)} \|\nabla s\|^2_{L^4(\mathbb{R}^2)} \right).
\]
Hence, it follows from (2.27), that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\Delta s|^2 \, dx + \|\nabla^3 s\|^2_{L^2(\mathbb{R}^2)} \leq C \|\nabla s\|^2_{L^4(\mathbb{R}^2)} + C \|\nabla^2 s\|^2_{L^4(\mathbb{R}^2)} + C \|\nabla^2 m\|^2_{L^4(\mathbb{R}^2)} \|\nabla s\|^2_{L^4(\mathbb{R}^2)}
\]
\[
+ C \left( \|\nabla^2 m\|^2_{L^4(\mathbb{R}^2)} |s|_{L^4(\mathbb{R}^2)} + \|\nabla m\|^2_{L^4(\mathbb{R}^2)} \|\nabla s\|^2_{L^4(\mathbb{R}^2)} \right)
\]
for \( t \in (\tau, T) \). Using Lemma 2.3 and Lemma 2.5, Gronwall’s inequality implies that
\[
\sup_{\tau \leq t \leq T} \|\nabla^2 s\|^2_{L^2} + \int_{\tau}^{T} \|\nabla^3 s\|^2_{L^2} \, dt \leq C(\tau, T, E_0, \frac{T}{R^2}). \tag{2.28}
\]

**Step II. Estimate for \( m \).** Next we improve the regularity of \( m \). First we note that by taking \( \Delta \) to (1.6) and then taking inner product of the resultant with \( \Delta^2 \), we obtain that
\[
(1 + \alpha^2) \int_{\mathbb{R}^2} \partial_t \nabla \Delta m \cdot \nabla \Delta m \, dx + \alpha \int_{\mathbb{R}^2} |\nabla^4 m|^2 \, dx
\]
\[
= \alpha \int_{\mathbb{R}^2} \nabla \Delta m \cdot \nabla (|\nabla m|^2) \, dx - \int_{\mathbb{R}^2} \nabla \Delta m \cdot \nabla (m \times \Delta m) \, dx
\]
\[
- \int_{\mathbb{R}^2} \nabla \Delta m \cdot \nabla [(m \times s) + \alpha m \times (m \times s)] \, dx =: I_1' + I_2' + I_3'.
\]

For the term \( I_1' \), we have
\[
|I_1'| \leq C \int_{\mathbb{R}^2} |\nabla^4 m| |\nabla m| |\nabla^3 m| \, dx + C \int_{\mathbb{R}^2} |\nabla^4 m| |\nabla^2 m|^2 \, dx
\]
\[
\leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla^4 m|^2 \, dx + C |\nabla^2 m|_{L^4(\mathbb{R}^2)}^4 + C |\nabla m|_{L^4(\mathbb{R}^2)}^2 |\nabla^3 m|_{L^2(\mathbb{R}^2)} \|\nabla^4 m\|_{L^2(\mathbb{R}^2)},
\]
where we used \( \Delta m \cdot m = -|\nabla m|^2 \) and Gagliardo-Nirenberg interpolation inequality. Then
\[
|I_1'| \leq \frac{\alpha}{4} \int_{\mathbb{R}^2} |\nabla^4 m|^2 \, dx + C |\nabla^2 m|_{L^4(\mathbb{R}^2)}^4 + C(\tau, T, E_0, \frac{T}{R^2}) \left( 1 + |\nabla m|_{L^4(\mathbb{R}^2)}^4 |\nabla^3 m|_{L^2(\mathbb{R}^2)}^2 \right).
\]

The term \( I_2' \) is estimated in a similar way since
\[
|I_2'| \leq C \int_{\mathbb{R}^2} |\nabla^4 m| |\nabla m| |\nabla^3 m| \, dx.
\]

For \( I_3' \), by (2.28) we get
\[
|I_3'| \leq C \int_{\mathbb{R}^2} |\nabla^4 m|(|\nabla^2 m| |s| + |\nabla m| |\nabla s| + |\nabla^2 s|) \, dx
\]
\[
\leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla^4 m|^2 \, dx + C(\tau, T, E_0, \frac{T}{R^2}) + C \left( |\nabla^2 m|_{L^4(\mathbb{R}^2)}^2 |s|_{L^4(\mathbb{R}^2)}^2 + |\nabla m|_{L^4(\mathbb{R}^2)}^2 |\nabla s|_{L^4(\mathbb{R}^2)}^2 \right).
\]

Therefore, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |\Delta m|^2 \, dx + \alpha \int_{\mathbb{R}^2} |\nabla \Delta m|^2 \, dx
\]
\[
\leq C |\nabla^2 m|_{L^4(\mathbb{R}^2)}^4 + C(\tau, T, E_0, \frac{T}{R^2})(1 + |\nabla m|_{L^4(\mathbb{R}^2)}^4 |\nabla^3 m|_{L^2(\mathbb{R}^2)})
\]
\[
+ C(\tau, T, E_0, \frac{T}{R^2}) + C |\nabla^2 m|_{L^4(\mathbb{R}^2)}^2 |s|_{L^4(\mathbb{R}^2)}^2 + C |\nabla m|_{L^4(\mathbb{R}^2)}^2 |\nabla s|_{L^4(\mathbb{R}^2)}^2,
\]
which combines Lemma 2.3 and Lemma 2.5 yields that
\[
\int_{\mathbb{R}^2} |\nabla^3 m(\cdot, t)|^2 dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla^4 m|^2 dx dt \leq C(\tau, T, E_0, \frac{T}{R^2}),
\]
due to the Gronwall’s inequality.

**Step III. The case \( l > 2 \).** We’ll do it by induction. Assume that (2.26) holds for \( l \leq k \) with \( k \geq 2 \), and we are aimed to prove the case \( k + 1 \) also holds. At this time, by Sobolev embedding inequality we have
\[
\int_{\mathbb{R}^2} |\nabla^{l+1} m(\cdot, t)|^2 + |\nabla^l s|^2(\cdot, t) dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla^{l+2} m|^2 + |\nabla^{l+1} m|^4 dx dt
\]
\[
+ \int_\tau^T \int_{\mathbb{R}^2} |\nabla^{k+1} s|^2 + |\nabla^k s|^4 dx dt \leq C\left( k, \tau, T, E_0, \frac{T}{R^2} \right), \quad 0 \leq l \leq k \quad (2.29)
\]
and
\[
\|\nabla^{l-1} m\|_{L^\infty(\mathbb{R}^2 \times (\tau, T))} + \|\nabla^{l-2} s\|_{L^\infty(\mathbb{R}^2 \times (\tau, T))} \leq C\left( k, \tau, T, E_0, \frac{T}{R^2} \right), \quad 2 \leq l \leq k \quad (2.30)
\]
Taking \( \Delta \) to the equation (1.1) satisfied by \( s \) and then taking inner product with \( \nabla^{k+1} s \), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |\nabla^{k+1} s|^2 dx - \int_{\mathbb{R}^2} \nabla^{k+1} \text{div}(A(m) \nabla s) \cdot \nabla^{k+1} s dx
\]
\[
+ \int_{\mathbb{R}^2} |\nabla^{k+1} s|^2 dx + \int_{\mathbb{R}^2} \nabla^{k+1} (s \times m) \cdot \nabla^{k+1} s = 0 \quad (2.31)
\]
For the second term on the left, by \( \Delta m \cdot m = -|\nabla m|^2 \) and \( \nabla m \in L^\infty \) we have
\[
- \int_{\mathbb{R}^2} \nabla^{k+1} \text{div}(A(m) \nabla s) \cdot \nabla^{k+1} s dx
\]
\[
\geq (1 - \beta) \|\nabla^{k+2} s\|^2_{L^2(\mathbb{R}^2)} - C \|\nabla^{k+1} s\|^2_{L^2(\mathbb{R}^2)} - C \|\nabla^2 m\|^2_{L^4(\mathbb{R}^2)} \|\nabla^k s\|^2_{L^4(\mathbb{R}^2)}
\]
\[
- C(\|\nabla^2 m\|^2_{L^4(\mathbb{R}^2)} + \|\nabla^3 m\|^2_{L^4(\mathbb{R}^2)}) \|\nabla^{k-1} s\|^2_{L^4(\mathbb{R}^2)} - \sum_{j=4}^{k-1} \|\nabla^j A(m)\|^2_{L^4(\mathbb{R}^2)} \|\nabla^{k+2-j} s\|^2_{L^4(\mathbb{R}^2)},
\]
where the last term is bounded by \( C\left( k, \tau, T, E_0, \frac{T}{R^2} \right) \) due to (2.26). The last term of (2.31) is estimated in the same way. Like the arguments in Step I, by Gronwall’s inequality one can obtain
\[
\int_{\mathbb{R}^2} |\nabla^{k+1} s|^2(\cdot, t) dx + \int_\tau^T \int_{\mathbb{R}^2} |\nabla^{k+2} s|^2(\cdot, t) + |\nabla^{k+1} s|^4 dx dt \leq C\left( k, \tau, T, E_0, \frac{T}{R^2} \right)
\]
Similarly, taking \( \nabla^{k+2} \) to (1.6) and then taking inner product of the resultant with \( \nabla^{k+2} m \), we obtain that
\[
(1 + \alpha^2) \int_{\mathbb{R}^2} \partial_t \nabla^{k+2} m \cdot \nabla^{k+2} m dx + \alpha \int_{\mathbb{R}^2} |\nabla^{k+3} m|^2 dx
\]
\[
= \alpha \int_{\mathbb{R}^2} \nabla^{k+2} m \cdot \nabla^{k+2} |(\nabla^2 m) dx - \int_{\mathbb{R}^2} \nabla^{k+2} m \cdot \nabla^{k+2} (m \times \Delta m) dx
\]
\[
- \int_{\mathbb{R}^2} \nabla^{k+2} m \cdot \nabla^{k+2} [(m \times s) + \alpha m \times (m \times s)] dx =: I''_1 + I''_2 + I''_3.
\]
For the term $I''$, by (3.20) we have
\[
|I''| \leq \frac{\alpha}{8} \int_{\mathbb{R}^2} |\nabla^{k+3} m|^2 dx + C\|\nabla^{k+2} m\|^2_{L^2(\mathbb{R}^2)} + C\|\nabla^{k+1} m\|^2_{L^2(\mathbb{R}^2)} + C\|\nabla^{k} m\|^2_{L^2(\mathbb{R}^2)} + C,
\]
and other terms are handled in the same way. Hence the case $k + 1$ for the inequality (3.20) holds.

Using estimates in Step III and trading spatial derivatives with time derivatives, one can finish the proof of the Corollary.

\[\square\]

3 Existence of global weak solution

Next we complete the proof of the existence part in Theorem 1.1, see similar arguments in [17, 20, 29, 34]. We sketch its steps for completeness.

**Proof of Theorem 1.1.** For any data $(s_0, m_0) \in L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, one can approximate it by a sequence of smooth maps $(s_0^k, m_0^k)$ in $L^2(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$, and we can assume that $s_0^k \in H^4(\mathbb{R}^2; \mathbb{R}^2)$ and $m_0^k \in H^2(\mathbb{R}^2; \mathbb{S}^2)$ (see [28]). Due to the absolute continuity property of the integral, for any $\epsilon_1 > 0$, there exists $R_0 \geq R_1 > 0$ such that
\[
\sup_{x \in \mathbb{R}^2} \int_{B_{R_1}(x)} |\nabla m_0^k|^2 + |s_0^k|^2 dx \leq \epsilon_1,
\]
and by the strong convergence of $m_0^k$ and $s_0^k$,
\[
\sup_{x \in \mathbb{R}^2} \int_{B_{R_1}(x)} |\nabla m_0^k|^2 + |s_0^k|^2 dx \leq 2\epsilon_1
\]
for a sufficient large $k$. Without loss of generality, we assume that it holds for all $k \geq 1$.

For the data $m_0^k$, by Theorem 1.1 there exists a time $T^k$ and a strong solution $(s^k, m^k)$ such that
\[
s^k, \nabla m^k \in C \left([0, T^k]; H^4(\mathbb{R}^2)\right).
\]
Hence there exists $T_0^k \leq T^k$ such that
\[
\sup_{0 < t < T_0^k, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla m^k(y, t)|^2 dy \leq (8 + \frac{1}{\alpha})\epsilon_1,
\]
where $R \leq R_0 < 1$ and $\epsilon_1 < \epsilon$. However, by the local monotonic inequality in Lemma 2.4 we have $T_0^k \geq \frac{\epsilon_1 R^2}{C(\epsilon_1, E_0)} = T_0 > 0$ uniformly. For any $0 < \tau < T_0$, by the estimates in Corollary 2.1 for any $l \geq 1$ we get
\[
\sup_{\tau < t < T_0} \int_{\mathbb{R}^2} |\nabla^{l+1} m^k|_2^2 dy + \int_{\tau}^{T_0} \int_{\mathbb{R}^2} |\nabla^{l+2} m^k(\cdot, s)|^2 + |\nabla^{l+1} s^k(\cdot, s)|^2 dx ds \leq C(l, \epsilon_1, E_0, \tau, T_0, \frac{T_0}{R^2}).
\]
Moreover, the energy inequality in Lemma 2.2 a priori estimates in Lemma 2.3 and the equation (1.1) yield that
\[
E(t) \leq E_0, \quad 0 < t < T^k,
\]
and
\[
\int_{\mathbb{R}^2 \times [0, T^k]} \left(|\nabla^2 m|^2 + |\nabla m|^2 + |\partial_t m|^2 + |\nabla m|^4 + |s^k|^4\right) dx dt \leq C(\epsilon_1, C_0, E_0).
\]
Hence the above estimates (3.1)- (3.3) and Aubin-Lions Lemma yield that there exists a solution \((s, m - a) \in W^{1,0}_2(\mathbb{R}^2 \times [0, T_0]; \mathbb{R}^3) \times W^{2,1}_2(\mathbb{R}^2 \times [0, T_0]; \mathbb{R}^3)\) such that (at most up to a subsequence)
\[
m^k - a \to m - a, \quad \text{locally in} \quad W^{2,1}_2(\mathbb{R}^2 \times (0, T_0); \mathbb{R}^3).
\]

By (3.2), \(s(t) \to s_0\) and \(\nabla m(t) \to \nabla m_0\) weakly in \(L^2(\mathbb{R}^2)\), thus \(E_0 \leq \liminf_{t \to 0} E(t)\). On the other hand, by the energy estimates of \((m^k)\), we have
\[
E_0 \geq \limsup_{t \to 0} E(t).
\]

Hence, \(s(t) \to s_0\) and \(\nabla m(t) \to \nabla m_0\) strongly in \(L^2(\mathbb{R}^2)\) and \(m\) is the solution of the equation (1.1) with the initial data \(m_0\). From the weak limit of regular estimates (3.1), we know that \((s, m) \in C^\infty(\mathbb{R}^2 \times (0, T_0))\) and \(\nabla^i s(\cdot, T_0), \nabla^i+1 m(\cdot, T_0) \in L^2(\mathbb{R}^2)\) for any \(i \geq 1\). By Theorem 5.1 there exists a unique smooth solution of (1.1) with the initial data \((s, m)(\cdot, T_0)\), which is still written as \((s, m)\), and blow-up criterion yields that if \((s, m)\) blows up at finite time \(T^*\), then
\[
\|s\|_{L^\infty(\mathbb{R}^2)}(t) + \|\nabla m\|_{L^\infty(\mathbb{R}^2)}(t) \to \infty, \quad \text{as} \quad t \to T^*.
\]

As a result, we have
\[
|\nabla^3 s|(x, t) + |\nabla^4 m|(x, t) \notin L^\infty_t L^2((T_0, T^*) \times \mathbb{R}^2) \quad (3.4)
\]

We assume that \(T_1\) is the first singular time of \((s, m)\), then we have
\[
(s, m) \in C^\infty(\mathbb{R}^2 \times (0, T_1); \mathbb{R}^3) \quad \text{and} \quad (s, m) \notin C^\infty(\mathbb{R}^2 \times (0, T_1); \mathbb{R}^3);
\]

and by Corollary 2.1 and (3.4), there exists \(\epsilon_0 > 0\) such that
\[
\limsup_{t \uparrow T_1} \sup_{x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla m|^2(\cdot, t) \geq \epsilon_0, \quad \forall R > 0.
\]

Finally, since \(m - a \in C^0([0, T_1], L^2(\mathbb{R}^2))\) by the interpolation inequality (similarly see P330, 201), we can define
\[
m(T_1) - a = \lim_{t \uparrow T_1} m(t) - a \quad \text{in} \quad L^2(\mathbb{R}^2).
\]

Also, \(s \in C^0([0, T_1], H^{-1}(\mathbb{R}^2))\) and we can define
\[
s(T_1) = \lim_{t \uparrow T_1} s(t) \quad \text{in} \quad H^{-1}(\mathbb{R}^2)
\]
in the distribution sense. On the other hand, by the energy inequality \(s, \nabla m \in L^\infty(0, T_1; L^2(\mathbb{R}^2))\), hence \(\nabla m(t) \to \nabla m(T_1)\). Similarly we can extend \(T_1\) to \(T_2\) and so on. It’s easy to check that the energy loss at every singular time \(T_i\) for \(i \geq 1\) is at least \(\epsilon_1\), thus the number \(L\) of the singular time is finite. Moreover, singular points at every singular time are finite by similar arguments as in 29, since \(\partial_{x_i} u \in L^2_{x,i}\) in Lemma 2.2 and the local monotonicity inequality in Lemma 2.4 hold. Assume that singular points are \((x^*_1, T_i)\) with \(1 \leq j \leq L_1\) and \(i \leq L\), and we have
\[
\limsup_{t \uparrow T_i} \int_{B_R(x^*_i)} |\nabla m|^2(\cdot, t) \geq \epsilon_0, \quad \forall R > 0.
\]

The proof is complete. \(\square\)
4 Uniqueness of weak solutions

In this section, we prove the following uniqueness result.

**Theorem 4.1.** Let \((s_1, m_1)\) and \((s_2, m_2)\) be two weak solutions of (1.1) in \(\mathbb{R}^2\) with the same initial data \((s_0, m_0)\) as stated in Theorem 1.1 then we have

\[(s_1, m_1) = (s_2, m_2)\]

for any \(t \in [0, \infty)\).

4.1 Littlewood-Paley theory and nonlinear estimates

Let us recall some basic facts on Littlewood-Paley theory (see [4] for more details). Choose two nonnegative radial functions \(\chi, \phi \in \mathcal{S}(\mathbb{R}^n)\) supported respectively in \(\{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}\) and \(\{\xi \in \mathbb{R}^n, \frac{4}{3} \leq |\xi| \leq \frac{8}{3}\}\) such that for any \(\xi \in \mathbb{R}^n\),

\[\chi(\xi) + \sum_{j \geq 0} \phi(2^{-j}\xi) = 1.\]

The frequency localization operator \(\Delta_j\) and \(S_j\) are defined by

\[\Delta_j f = \phi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y)f(x-y)dy, \quad \text{for } j \geq 0,\]

\[S_j f = \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y)f(x-y)dy,\]

\[\Delta_{-1} f = S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2;\]

where \(h = \mathcal{F}^{-1}\phi\) and \(\tilde{h} = \mathcal{F}^{-1}\chi\). With this choice of \(\phi\), it is easy to verify that

\[(4.1) \Delta_j \Delta_k f = 0, \quad \text{if } |j-k| \geq 2; \quad \Delta_j(S_{k-1}f \Delta_k f) = 0, \quad \text{if } |j-k| \geq 5.\]

In terms of \(\Delta_j\), the norm of the inhomogeneous Besov space \(B^s_{p, q}\) for \(s \in \mathbb{R}\), and \(p, q \geq 1\) is defined by

\[\|f\|_{B^s_{p, q}} := \left\| \left\{ 2^{js} \|\Delta_j f\|_p \right\}_{j \geq -1} \right\|_{l^q},\]

and

\[\|f\|_{B^s_{p, \infty}} := \sup_{j \geq -1} \left\{ 2^{js} \|\Delta_j f\|_p \right\}.\]

The Bony’s decomposition from [3] is given by

\[uv = T_u v + T_v u + R(u, v), \quad (4.2)\]

where

\[T_u v = \sum_j S_{j-1} u \Delta_j v \quad \text{and} \quad R(u, v) = \sum_{|j-j'| \leq 1} \Delta_j u \Delta_{j'} v.\]

We will constantly use the following Bernstein’s inequality [4].
Lemma 4.1. Let \( c \in (0, 1) \) and \( R > 0 \). Assume that \( 1 \leq p \leq q \leq \infty \) and \( f \in L^p(\mathbb{R}^n) \). Then

\[
\text{supp} \hat{f} \subset \{ |\xi| \leq R \} \Rightarrow \| \partial^\alpha f \|_q \leq CR^{\frac{\alpha}{\alpha+n}+n(\frac{1}{\beta}-\frac{1}{p})} \| f \|_p,
\]

\[
\text{supp} \hat{f} \subset \{ cR \leq |\xi| \leq R \} \Rightarrow \| f \|_p \leq CR^{1-\alpha} \sup_{|\beta|=\alpha} \| \partial^\beta f \|_p,
\]

where the constant \( C \) is independent of \( f \) and \( R \).

We need the following nonlinear estimates, seeing [35] for more details.

**Lemma 4.2.** Let \( \beta \in (0, 1) \). For any \( j \geq -1 \), there holds

\[
\| \Delta_j(fg) \|_2 \leq C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| g \|_{H^1} + C2^{\frac{\beta(\alpha+1)}{4}} \| g \|_4 \| f \|_{B^j_{2,\infty}} \sum_{|j'|+|j''| \leq |j|} \| \Delta_{j'} f \|_2 \| \Delta_{j''} g \|_2.
\]

**Corollary 4.1.** Let \( \beta \in (0, 1) \) and \( j \geq -1 \).

1. When \( f \in H^1 \), \( g \in L^\infty \cap H^1 \) (for example \( f = s \) and \( g = m \)), we have

\[
\| \Delta_j(fg) \|_2 \leq C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| \nabla g \|_{L^2} + C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| g \|_{\infty}.
\]

2. When \( g \in H^1 \), \( f \in L^\infty \cap H^1 \) (for example \( f = m \) and \( g = s \)), we get

\[
\| \Delta_j(fg) \|_2 \leq C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| g \|_{H^1}.
\]

**Proof.** Similar to the proof of Lemma 4.2 in [35], we sketch the proof.

1. By Bony’s composition [12], we have

\[
\Delta_j(fg) = \Delta_j(T_j g + T_j f + R(f, g))
\]

We get (4.1) and Lemma 4.1 that

\[
\| \Delta_j(T_j g) \|_2 \leq C \sum_{|j'-j| \leq 4} \| S_{j'-j} f \|_\infty \| \Delta_{j'} g \|_2
\]

\[
\leq C \sum_{|j'-j| \leq 4} \sum_{|j''| \leq j'-2} \| \Delta_j f \|_\infty \| \Delta_{j'} g \|_2
\]

\[
\leq C \sum_{|j'-j| \leq 4} \sum_{|j''| \leq j'-2} 2^j(1+\beta) \| f \|_{B^j_{2,\infty}} \| \Delta_{j'} g \|_2
\]

\[
\leq C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| \nabla g \|_{L^2},
\]

where we have used \( j' \geq 0 \), and

\[
\| \Delta_j T_j f \|_2 \leq C \sum_{|j'-j| \leq 4} \| S_{j'-j} g \|_\infty \| \Delta_{j'} f \|_2
\]

\[
\leq C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| g \|_{\infty}.
\]

Note that \( \Delta_j(\Delta_j f \Delta_j g) = 0 \) if \( |j' - j''| \leq 1 \) and \( \max\{j', j''\} = j - 3 \). Hence,

\[
\| \Delta_j R(f, g) \|_2 \leq C2^{j\beta} \sum_{j', j'' \geq j-3, j'' \geq 0, |j'-j''| \leq 1} \| \Delta_{j'} f \|_2 \| \Delta_{j''} g \|_2
\]

\[
+ C \sum_{j', j'' \geq j-3, j'' < 0, |j'-j''| \leq 1} \| \Delta_{j'} f \|_2 \| \Delta_{j''} g \|_{\infty}
\]

\[
\leq C2^{j\beta} \sum_{j' \geq j-3} 2^{j\beta} 2^{-j'\beta} \| \Delta_{j'} f \|_2 \sum_{j'' \geq j-3, |j'-j''| \leq 1} 2^{-j''} 2^{j' \beta} \| \Delta_{j''} g \|_2
\]

\[
+ C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| g \|_{\infty}
\]

\[
\leq C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| \nabla g \|_{L^2} + C2^{j\beta} \| f \|_{B^j_{2,\infty}} \| g \|_{\infty}.
\]
(2) When \( g = s \in H^1 \), \( f = m \in L^\infty \cap H^1 \), the first term \( \| \Delta_j (T_j g) \|_2 \) is similar, and we consider other terms.

\[
\| \Delta_j T_j f \|_2 \leq C 2^j \sum_{|j' - j| \leq 4} \| S_{j'} g \|_2 \| \Delta_j f \|_2 \\
\leq C 2^{j \beta} \| f \|_{B^{1-\beta}_{2,\infty}} \| g \|_{L^2}.
\]

Note that \( \Delta_j (\Delta_j f \Delta_j g) = 0 \) if \( |j' - j''| \leq 1 \) and \( \max \{j', j''\} \leq j - 3 \). Hence,

\[
\| \Delta_j R(f, g) \|_2 \leq C 2^j \sum_{j' \geq j - 3, |j' - j''| \leq 1} \| \Delta_j f \|_2 \| \Delta_j g \|_2 \\
\leq C 2^j \sum_{j' \geq j - 3} 2^{j' \beta} 2^{-j' \beta} \| \Delta_j f \|_2 \sum_{j'' \geq j - 3, |j' - j''| \leq 1} 2^{-j'' \beta} \| \Delta_j g \|_2 \\
\leq C 2^{j \beta} \| f \|_{B^{1-\beta}_{2,\infty}} \| g \|_{H^1}.
\]

The proof is complete. \( \square \)

**Lemma 4.3.** Let \( \beta \in (0, 1) \). For any \( j \geq -1 \), we have

\[
\| \Delta_j (fgh) \|_2 \leq C 2^{j \beta} (\| f \|_\infty + \| \nabla f \|_2) \| g \|_{B^{1-\beta}_{2,\infty}} \| h \|_2.
\]

**Lemma 4.4.** Let \( \beta \in (0, 1) \). For any \( j \geq -1 \), it holds that

\[
\| \nabla f \|_2 \leq C 2^{j \beta} \| \nabla f \|_2 \| g \|_{B^{1-\beta}_{2,\infty}} \sum_{|j' - j| \leq 4} 2^{j' \beta} \| \Delta_j g \|_2 \bar{\beta} + C 2^j \| g \|_{B^{1-\beta}_{2,\infty}} (\| f \|_\infty + \| \nabla^2 f \|_2).
\]

### 4.2 Proof of Theorem 4.1

Let \( s = s_1 - s_2 \), \( m = m_1 - m_2 \), then from the system (1.1) we have

\[
\partial_t s - \text{div}(A(m_1) \nabla s) = \text{div}(A(m_1) - A(m_2)) \nabla s_2) - \text{div}(A(m_1) - A(m_2) - s \times m_1 - s_2 \times m)
\]

and

\[
(1 + \alpha^2) \partial_t m - \alpha \Delta m = \alpha |\nabla m_1|^2 m + \alpha ((\nabla m_1 + \nabla m_2) : \nabla m) m_2 \\
- (m_1 \times \Delta m + m \times \Delta m_2) - (m_1 \times s + m \times s_2) \\
- \alpha (m \times (m_1 \times s_1) + m_2 \times (m \times s_1) + m_2 \times (m_2 \times s))
\]

(4.4)

For \( \beta \in (0, 1/2) \), let

\[
W_j(t) = \left[ \| \Delta_j s \|_{L^2(R^2)}^2 + \| \Delta_j m \|_{L^2(R^2)}^2 + \| \Delta_m m \|_{L^2(R^2)}^2 \right]
\]

and

\[
W(t) = \| s(\cdot, t) \|_{B^{1-\beta}_{2,\infty}(R^2)}^2 + \| m \|_{B^{1-\beta}_{2,\infty}(R^2)}^2 = \sup_{j \geq -1} 2^{-2j \beta} W_j(t).
\]

The proof of Theorem 4.1 is based on the following two propositions. To state them neatly, we introduce the function

\[
\bar{h}(t) = 1 + \| (s_1, s_2, \nabla m_1, \nabla m_2) \|_4^4 + \| (\partial_t m_1, \partial_t m_2) \|_2^2 + \| (s_1, s_2, \nabla m_1, \nabla m_2) \|_{H^1}^2.
\]

Since \( (s_1, m_1) \) and \( (s_2, m_2) \) are both Struwe type weak solutions and \( T_1 \) is the first blow-up time, we have \( \bar{h}(t) \in L^1(0, T_1 - \theta) \) for any \( \theta > 0 \).
Proposition 4.1. For any \( j \geq -1 \) and \( \epsilon > 0 \), it holds that
\[
\frac{d}{dt} \left[ \| \Delta_j s \|^2_{L^2(\mathbb{R}^2)} + \| \Delta_j \nabla m \|^2_{L^2(\mathbb{R}^2)} \right] + \frac{\alpha}{2} \| \Delta_j \Delta m \|^2_2 + \frac{\lambda}{2} \| \Delta_j \nabla s \|^2_2 \\
\leq C 2^{2j} \bar{h}(t) W(t) + \epsilon \sum_{l=j-4}^{j+4} 2^{2l} \| \Delta_j s \|^2_2 + \epsilon \sum_{l=j-4}^{j+4} 2^{2l} \| \Delta_j m \|^2_2.
\]

Proposition 4.2. It holds that
\[
\frac{d}{dt} \| \Delta_{-1} m \|^2_2 \leq C \bar{h}(t) W(t).
\]

Then by Gronwall's inequality, we get \( W(t) = 0 \) for \( t \in [0, T_1 - \theta] \) for any \( \theta > 0 \). Using similar arguments as in [35, 36] and [26], one can complete the proof and we omitted the details.

4.3 Proof of Proposition 4.1 and 4.2

In what follows, we prove Proposition 4.1 and 4.2.

Proof of Proposition 4.1. We write \( \| \cdot \|_{L^2(\mathbb{R}^2)} \) as \( \| \cdot \|_2 \) and \( \int_{\mathbb{R}^2} fg dx \) as \( \langle f, g \rangle \) for simplicity. From the identity (4.3) and (4.4), we have
\[
\frac{1}{2} \partial_t \| \Delta_j s \|^2_2 + A(m_1) \| \nabla \Delta_j s \|^2 + \| \Delta_j s \|^2_2 \\
= - \langle [\Delta_j, A(m_1)] \nabla s, \nabla \Delta_j s \rangle - \langle [\Delta_j(A(m_1) - A(m_2))] \nabla s, \nabla \Delta_j s \rangle \\
- \langle \Delta_j (s \times m_1), \Delta_j s \rangle - \langle \Delta_j (s_2 \times m), \Delta_j s \rangle = I_1 + \cdots + I_4
\]
and
\[
\frac{1}{2} \partial_t \| \Delta_j \nabla m \|^2_2 + \alpha \| \Delta_j \Delta m \|^2_2 \\
= - \alpha \langle \Delta_j (|\nabla m|^2 m), \Delta_j \Delta m \rangle - \alpha \langle \Delta_j ((\nabla m_1 + \nabla m_2) : \nabla m_2), \Delta_j \Delta m \rangle \\
+ \langle \Delta_j (m_1 \times \Delta m), \Delta_j \Delta m \rangle + \langle \Delta_j (m_2 \times \Delta m), \Delta_j \Delta m \rangle + \langle \Delta_j (m_1 \times s), \Delta_j \Delta m \rangle \\
+ \langle \Delta_j (m_2 \times s), \Delta_j \Delta m \rangle + \alpha \langle \Delta_j (m_2 \times (m_1 \times s_1), \Delta_j \Delta m \rangle \\
- \langle \Delta_j (m_2 \times (m_2 \times s)), \Delta_j \Delta m \rangle = I_1 + \cdots + I_5.
\]

Now we want to estimate all the terms on the right hand step by step.
- **Estimate of \( I_1 \).** We have by Lemma 4.4 that
\[
\| [\Delta_j, f] \nabla g \|^2_2 \leq C 2^{j} \bar{h}(t)^{1/2} \| g \|_{B_{2,2}^{-j}} \sum_{|j' - j| \leq 4} 2^{j'} \| \Delta_j g \|_2 + C 2^{j} \bar{h}(t) \| g \|_{B_{2,2}^{-j}}.
\]
Hence, for \( f = A(m_1) \) and \( g = s \) we have
\[
I_1 \leq \epsilon \| \nabla \Delta_j s \|^2_2 + C 2^{j} \bar{h}(t)^{1/2} \| s \|_{B_{2,2}^{-j}} \sum_{|j' - j| \leq 4} 2^{j'} \| \Delta_j s \|_2 + C 2^{j} \bar{h}(t) \| s \|_{B_{2,2}^{-j}},
\]
where \( \epsilon > 0 \) is to be determined.

Note that \( \langle m_1 \times \Delta_j \Delta m, \Delta_j \Delta m \rangle = 0 \). Similarly, using Lemma 4.4 again, for the term \( II_3 \) we have
\[
II_3 \leq \epsilon \| \Delta_j \Delta m \|^2_2 + C 2^{j} \bar{h}(t)^{1/2} \| m \|_{B_{2,2}^{-j}} \sum_{|j' - j| \leq 4} 2^{j'} \| \Delta_j \nabla m \|_2 \\
+ C 2^{j} \bar{h}(t) \| m \|_{B_{2,2}^{-j}} + C \delta_{-1,j} \| \Delta_{-1} m \|^2_2.
\]
• **Estimate of** $I_2$. Let $f = (m_1, m_2)$, $g = m$ and $h = \nabla s_2$. By Lemma \ref{lemma:estimate_I2}, we have

$$
\|\Delta(fgh)\|_2 \leq C2^j\beta (\|f\|_\infty + \|\nabla f\|_2)\|g\|_{B^1_{2,\infty}}\|h\|_2.
$$

Hence

$$
\|\Delta(fgh)\|_2 = \|\Delta((A(m_1) - A(m_2))\nabla s_2)\|_2
\leq C2^{j\beta} (\|(m_1, m_2)\|_\infty + \|\nabla (m_1, m_2)\|_2)\|m\|_{B^1_{2,\infty}}\|\nabla s_2\|_2.
\quad (4.8)
$$

and

$$
I_2 \leq C\hat{h}(t)\|m\|_{B^1_{2,\infty}}^2 + \epsilon\|\nabla \Delta_j s\|_2^2.
$$

Furthermore, choosing $f = 1$, $h = |\nabla m_1|^2 + |\nabla m_2|^2$ or $h = |(\Delta m_1, \Delta m_2)|$, by Lemma \ref{lemma:estimate_I2} we have

$$
II_1 + II_4 \leq C2^{2j\beta}\hat{h}(t)\|m\|_{B^1_{2,\infty}}^2.
$$

• **Estimate of** $I_3$. By (1) of Corollary \ref{corollary:estimate_I3} we have

$$
I_3 \leq C2^{2j\beta}\|s\|_{B^1_{2,\infty}}^2 + \epsilon\|\Delta_j s\|_2^2.
$$

Similarly, we have

$$
II_5 + II_7 \leq C2^{2j\beta}|s|_{B^1_{2,\infty}}^2 + \epsilon\|\Delta_j \Delta m\|_2^2 + C\delta_{-1,j}\|\Delta_{-1}m\|_2^2.
$$

• **Estimate of** $I_4$. By (2) of Corollary \ref{corollary:estimate_I3} we have

$$
I_4 \leq C2^{2j\beta}\hat{h}(t)\|m\|_{B^1_{2,\infty}}^2 + \epsilon\|\Delta_j \nabla s\|_2^2.
$$

Similarly, we have

$$
II_6 + II_8 \leq C2^{2j\beta}\hat{h}(t)\|m\|_{B^1_{2,\infty}}^2 + \epsilon\|\Delta_j \Delta m\|_2^2 + C\delta_{-1,j}\|\Delta_{-1}m\|_2^2.
$$

• **Estimate of** $II_2$. By Lemma \ref{lemma:estimate_II2} we have

$$
II_2 \leq C2^{2j\beta}\hat{h}(t)\|m\|_{B^1_{2,\infty}}^2 + \epsilon\sum_{|j^1 - j| \leq 4}\|\Delta_{j^1} \Delta m\|_2^2 + \epsilon\|\Delta_j \Delta m\|_2^2 + C\delta_{-1,j}\|\Delta_{-1}m\|_2^2.
$$

Collecting the above estimates, by choosing a smaller $\epsilon$ than $\alpha$ or $\lambda$, one can complete the roof of Proposition \ref{proposition:evolution}

Compared with $W_j(t)$, only the term $\|\Delta_{-1}m\|_2^2$ is not estimated in Proposition \ref{proposition:evolution} Now we estimate the evolution of it.

**Proof of Proposition** \ref{proposition:evolution} By direct computation, we have

$$
\frac{1}{2} \partial_t \|\Delta_{-1} m\|_2^2 + \alpha \|\nabla \Delta_{-1} m\|_2^2
\quad (4.9)
$$

\begin{align*}
= & \alpha \langle \Delta_{-1}(\nabla m_1 M^2), \Delta_{-1}m \rangle + \alpha \langle \Delta_{-1}(\nabla m_1 + \nabla m_2) : \nabla m_2 \rangle, \Delta_{-1}m \\
& - \langle \Delta_{-1}(m_1 \times \Delta m), \Delta_{-1}m \rangle - \langle \Delta_{-1}(m \times \Delta m_2), \Delta_{-1}m \rangle - \langle \Delta_{-1}(m_1 \times s), \Delta_{-1}m \rangle \\
& - \langle \Delta_{-1}(m \times s_2), \Delta_{-1}m \rangle - \alpha \langle \Delta_{-1}(m \times (m_1 \times s_1)), \Delta_{-1}m \rangle \\
& - \alpha \langle \Delta_{-1}(m_2 \times (m \times s_1)), \Delta_{-1}m \rangle - \alpha \langle \Delta_{-1}(m_2 \times (m_2 \times s)), \Delta_{-1}m \rangle \\
= & II_1 + \cdots + II_9
\end{align*}

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It’s sufficient to consider the term $II'_3$, while other terms are handled similarly to those in Proposition 4.1.

- Estimate of $II'_3$: Obviously,

$$II'_3 = \langle \Delta^{-1}(m_1 \times \nabla, m), \Delta^{-1}\nabla, m \rangle + \langle \Delta^{-1}(\nabla, m_1 \times \nabla, m), \Delta^{-1}m \rangle$$

$$= \langle \Delta^{-1}(m_1 \times \nabla, m), \Delta^{-1}\nabla, m \rangle - \langle \Delta^{-1}(\Delta m_1 \times m), \Delta^{-1}m \rangle$$

$$= III_1 + \cdots + III_3$$

For the first term $III_1$, by (1) of Corollary 4.1, we have

$$III_1 \leq C \|\nabla m\|^2_{B_{2,\infty}^2} + \epsilon \|\Delta^{-1}\nabla m\|^2_2.$$  

By Lemma 5.3 for the second and third term we have

$$III_2 + III_3 \leq C \hat{h}(t) \|m\|^2_{B_{2,\infty}^2} + \epsilon \|\Delta^{-1}m\|^2_2 + \epsilon \|\Delta^{-1}\nabla m\|^2_2.$$  

Hence the proof is complete. 

\[\square\]

5 Local well-posedness

In this subsection we will consider the local well-posedness of the spin polarized Landau-Lifshitz equation (1.1). For the Landau-Lifshitz equation, the local solvability in appropriate Sobolev spaces has been investigated by authors in \[10, 18, 23\]. The local well-posedness can be obtained via the method of mollification \[31, 32\]. Let us fix the magnetization at infinity $a \in S^2$ and set

$$H^\sigma(\mathbb{R}^3, S^2) = \{m : \mathbb{R}^3 \to S^2 : m - a \in H^\sigma(\mathbb{R}^3, \mathbb{R}^3)\},$$

where $H^\sigma(\mathbb{R}^3) = (I - \Delta)^{-\sigma/2}L^2(\mathbb{R}^3)$ is the usual Sobolev space. For the initial data, we assume that $(s_0, m_0) \in H^{\sigma-1}(\mathbb{R}^3, \mathbb{R}^3) \times H^{\sigma}(\mathbb{R}^3, S^2)$. We have the following local well-posedness result stated in the general space dimension.

**Theorem 5.1.** Let $\sigma > n/2 + 2$. There exists a time $T^* > 0$ and a unique solution $(s, m)$ such that

$$s \in C^0([0, T]; H^{\sigma-1}(\mathbb{R}^3, \mathbb{R}^3)) \cap C^1([0, T]; H^{\sigma-3}(\mathbb{R}^3, \mathbb{R}^3))$$

and

$$m \in C^0([0, T]; H^{\sigma}(\mathbb{R}^3, S^2)) \cap C^1([0, T]; H^{\sigma-2}(\mathbb{R}^3, S^2))$$

for all $T < T^*$ with $(s(0), m(0)) = (s_0, m_0)$. Moreover, if $T^* < \infty$, then

$$\limsup_{t \to T^*} \int_0^t \|\{s(s), \nabla m(s)\}\|_{L^\infty} ds = \infty.$$  

Indeed, when the initial data is smooth, the solution $(s, m)$ is in fact a classical solution and

$$(s, m) \in C^0(0, T^*); H^\infty(\mathbb{R}^3, \mathbb{R}^3) \times H^\infty(\mathbb{R}^3, S^2),$$

where $H^\infty = \bigcup_{\sigma \in \mathbb{Z}} H^\sigma$.

The following inequalities will be used in the sequel (see [4] for example).
Lemma 5.1. Let $\alpha$, $\beta$ and $\gamma$ be multi-indices, there holds that
\[
\|\partial^{\alpha}(fg)\|_{L^2} \leq C \sum_{|\gamma|=|\alpha|} (\|f\|_{L^\infty}\|\partial^{\gamma}g\|_{L^2} + \|g\|_{L^\infty}\|\partial^{\gamma}f\|_{L^2}),
\]
and
\[
\|\partial^{\alpha}f\partial^{\beta}g\|_{L^2} \leq C \left( \sum_{|\gamma|=|\alpha|+|\beta|} \|\partial^{\gamma}f\|_{L^2}\|g\|_{L^\infty} + \sum_{|\gamma|=|\alpha|+|\beta|-1} \|\nabla f\|_{L^\infty}\|\partial^{\gamma}g\|_{L^2} \right),
\]
for all $f, g \in C_0^\infty(\mathbb{R}^n)$.

Proof of Theorem 5.7. To prove this local result, we first note that since $m \in S^2$ is on the unit sphere, we have the following identities
\[
m \times \Delta m = \nabla \cdot (m \times \nabla m) \quad \text{and} \quad -m \times (m \times \Delta m) = \Delta m + |\nabla m|^2 m.
\]
Therefore, the system (1.1) is a quasilinear parabolic system in divergence form and the $m$-part can be rewritten in terms of $u = m - a$ as
\[
(1 + \alpha^2)\partial_t u = \nabla \cdot (B(u)\nabla u) + C(u, s, \nabla u),
\]
where $\langle \xi, B(u)\xi \rangle = \alpha|\xi|^2$ for every $u \in \mathbb{R}^3$ and $\xi \in \mathbb{R}^{3 \times 3}$. Therefore, together with the equation satisfied by $s$, the system (1.1) can be written in divergence form
\[
\partial_t U = \nabla \cdot (A(U)\nabla U) + C(U, \nabla U)
\]
for $U = (s, m)$, whose local well-posedness can be obtained via the modified Galerkin’s method as in [31][32]. For this purpose, we need the following higher order energy estimates as well the stability estimates.

Lemma 5.2. Let $\sigma > n/2 + 2$ and $(s, m)$ be a smooth solution to the system (1.1) over $[0, T]$, then
\[
\|(\nabla m(T), s(T))\|_{H^{\sigma - 1}} + \frac{\alpha}{2} \int_0^T \|(\nabla m(T), s(T))\|_{H^{\sigma - 1}}^2 d\tau \leq e^{C(T)} \|(\nabla m(0), s(0))\|_{H^{\sigma - 1}}^2, \quad (5.1)
\]
where, for a universal constant $c > 0$ that only depends on $\alpha$, $\sigma > n/2 + 2$,
\[
C(t) = c(\alpha, \sigma) \int_0^t \left( 1 + \|(s, \nabla m)(\tau)\|_{L^\infty}^2 \right) d\tau.
\]

Proof. First, we consider the $L^2$ estimates for $m - a$. Directly use the equation to obtain
\[
\frac{1}{2} \frac{d}{dt} \|m - a\|_{L^2}^2 = \langle \partial_t m, m - a \rangle \\
\leq C(1 + \|(\nabla m)\|_{L^\infty}) \|(\nabla m)\|_{H^1} \|m - a\|_{L^2} + C\|m - a\|_{L^2} \|s\|_{L^2} \quad (5.2)
\]
Let $\alpha$ be a multiindex and $1 \leq |\alpha| \leq \sigma$. We have
\[
\partial^{\alpha}(m \times \nabla m) = m \times \partial^{\alpha} \nabla m + [\partial^{\alpha}, m \times] \nabla m
\]
where $[\cdot, \cdot]$ is the commutator and the last term bounded by
\[
\|[\partial^{\alpha}, m \times] \nabla m\|_{L^2} \leq C\|\nabla m\|_{L^\infty} \|\nabla m\|_{H^{\sigma - 1}},
\]

where we have used the inequalities in Lemma \[5.1\]. Moreover, we have

\[ \| \partial^\alpha (|\nabla m|^2 m) \|_{L^2} \leq C \| \nabla m \|_{L^\infty} \| \nabla m \|_{H^{\sigma}} + C \| \nabla m \|_{H^{\sigma}}^2 \| \nabla m \|_{H^{\sigma-1}} \] (5.3)

and

\[ \| \partial^\alpha (m \times s) \|_{L^2} + \| \partial^\alpha (m \times (m \times s)) \|_{L^2} \leq C \left( \| \nabla s \|_{H^{\sigma-1}} + \| s \|_{L^\infty} \| \nabla m \|_{H^{\sigma-1}} + \| \nabla m \|_{L^\infty} \| s \|_{H^{\sigma-1}} \right). \] (5.4)

Applying \( \partial^\alpha \) to the \( m \)-part of system \[1.1\] and taking inner product with \( \partial^\alpha m \) in \( L^2 \), we obtain by integration by parts that

\[ \frac{d}{dt} \| \partial^\alpha m \|_{L^2}^2 + \| \partial^\alpha \nabla m \|_{L^2}^2 \leq C(1 + \| \nabla m \|_{L^\infty}^2) \| \nabla m \|_{H^{\sigma-1}}^2 \]

\[ + C \| \nabla m \|_{L^\infty} \| \nabla m \|_{H^{\sigma-1}} \| \nabla m \|_{H^{\sigma-1}} + \| s \|_{L^\infty} \| \nabla m \|_{H^{\sigma-1}} \| \nabla m \|_{H^{\sigma-1}}. \] (5.5)

Summing all possible \( \alpha \) with \( 1 \leq |\alpha| \leq \sigma \), we obtain, upon using \( \varepsilon \)-Young’s inequality, that

\[ \frac{d}{dt} \| \nabla m \|_{H^{\sigma-1}}^2 + \| \nabla m \|_{H^{\sigma-1}}^2 \leq C(1 + \| \nabla m \|_{L^\infty}^2 + \| s \|_{L^\infty}) \| \nabla m \|_{H^{\sigma-1}}^2 + \frac{1}{4} \| m \|_{H^\sigma}^2. \] (5.6)

Now, we consider the \( s \)-part of system \[1.1\]. For \( 0 \leq |\alpha| \leq \sigma - 1 \), we have

\[ \partial^\alpha (A(m) \nabla s) = A(m) \partial^\alpha \nabla s + [\partial^\alpha, A(m)] \nabla s \]

and using Lemma \[5.1\] again we have for the commutator

\[ \| [\partial^\alpha, A(m)] \nabla s \|_{L^2} \leq C \| \nabla m \|_{L^\infty} \| s \|_{H^{\sigma-1}} + C \| s \|_{L^\infty} \| \nabla m \|_{H^{\sigma-1}}. \]

Similar estimates as for the \( m \)-part yield the following

\[ \frac{1}{2} \frac{d}{dt} \| s \|_{H^{\sigma-1}}^2 + \| \nabla s \|_{H^{\sigma-1}}^2 + \| s \|_{H^\sigma}^2 \leq C(1 + \| s \|_{L^\infty}^2 + \| \nabla m \|_{L^\infty}^2) \| \nabla m \|_{H^{\sigma-1}}^2 + \frac{1}{2} \| s \|_{H^\sigma}^2, \] (5.7)

Therefore we obtain the following higher order estimates for \( \sigma > n/2 + 2 \),

\[ \frac{d}{dt}(\| s \|_{H^{\sigma-1}}^2 + \| \nabla m \|_{H^{\sigma-1}}^2) + (\| s \|_{H^\sigma}^2 + \| \nabla m \|_{H^\sigma}^2) \leq C(1 + \| \nabla m, s \|_{L^\infty}^2) \| \nabla m \|_{H^{\sigma-1}}^2 + \| s \|_{H^\sigma}^2. \] (5.8)

The proof of Lemma \[5.2\] is complete. \(\square\)

Now, we consider the stability in \( L^2 \). Let \((s_1, m_1)\) and \((s_2, m_2)\) be two solutions. After similar computation as above, on can obtain the following

\[ \| (s_1 - s_2, m_1 - m_2) (t) \|_{L^2}^2 \leq C(t) \| (s_1 - s_2, m_1 - m_2) (0) \|_{L^2}^2, \]

where \( C(t) \) depends on the solutions \((s_1, m_1)\) and \((s_2, m_2)\).

Using mollification \( J_\varepsilon \) of functions \( v \in L^p(\mathbb{R}^n), \) \( 1 \leq p \leq \infty \), defined by

\[ (J_\varepsilon v)(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} \rho \left( \frac{x - y}{\varepsilon} \right) v(y) dy, \quad \varepsilon > 0, \]

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for a given radial function

\[ \rho(|x|) \in C_0^\infty(\mathbb{R}^n), \quad \rho > 0, \quad \int_{\mathbb{R}^n} \rho \, dx = 1, \]

one can prove the local existence results in Theorem 5.1. The blow up criterion follows from the higher order energy estimates and uniqueness follows from stability estimates. The details are hence omitted here and one can find similar treatment in [23,31,32] for Landau-Lifshitz equation or general parabolic equations, or our recent paper for a similar model in [26]. This completes the proof of Theorem 5.1. \( \Box \)

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