Abstract—This paper presents a linear complexity iterative rake detector for the recently proposed orthogonal time frequency space (OTFS) modulation scheme. The basic idea is to extract and coherently combine the received multipath components of the transmitted symbols in the delay-Doppler grid using maximal ratio combining (MRC) to improve the SNR of the combined signal. We reformulate the OTFS input-output relation in the vector form by placing guard null symbols in the delay-Doppler grid and exploiting the resulting circulant property of the blocks of the channel matrix. Using this vector input-output relation we propose a low complexity iterative decision feedback equalizer (DFE) based on MRC. The performance and complexity of the proposed detector favourably compares with the state of the art message passing detector. We further propose a Gauss-Seidel based over-relaxation parameter in the rake detector to improve the performance and the convergence speed of the iterative detection. We also show how the MRC detector can be combined with outer error-correcting codes to operate as a turbo-DFE scheme to further improve the error performance. All results are compared with a baseline orthogonal frequency division multiplexing (OFDM) scheme employing a single tap minimum mean square error (MMSE) equalizer.

Index Terms—OTFS, Detector, Decoder, Rake, Maximal Ratio Combining, Delay-Doppler channel, turbo, DFE, Gauss Seidel, Successive Over Relaxation.

I. INTRODUCTION

Orthogonal time frequency and space (OTFS) is a new two dimensional (2D) modulation technique that transforms information symbols in the delay-Doppler domain to the familiar time-frequency domain by spreading all the information symbols (e.g., QAM) over both time and frequency to achieve maximum effective diversity [1], [2]. As a result, a time-frequency selective channel due to multipath fading and mobility, is converted into a separable and quasi-orthogonal interaction, where all received information symbols experience roughly the same localized impairment [1]. Hence, for each information symbol, the received components in all the delay-Doppler diversity branches can be separated and coherently combined.

OTFS can also be interpreted as a two-dimensional code division multiple access (CDMA) scheme, where information symbols are spread in both time and frequency, differently from conventional CDMA systems [1]. In direct sequence CDMA operating in a multipath fading channel, a rake receiver works by combining the delayed components (or echoes) of the transmitted symbols extracted by using matched filters tuned to the respective delays. Similarly, in the case of OTFS, the received delay and Doppler-shifted components of the transmitted information symbols can be extracted and coherently combined using linear diversity combining techniques to improve the SNR of the accumulated signal.

Diversity combining techniques are well studied in the literature starting from Brennan’s paper on linear diversity combining [4]. Rake receivers for time domain combining using a variety of linear combining schemes like maximum ratio combining (MRC), equal gain combining (EGC) and selection combining (SC) are discussed in [5], [6]. MRC is shown to be optimal in the case of correlated and uncorrelated branches, even for unequal noise and interference power in the branches [7]. Moreover, iterative rake combining schemes and its variants are shown to combat inter-symbol interference better and are well investigated in the literature for CDMA systems [8].

In this paper, we propose an iterative rake receiver for the reduced cyclic prefix (CP) OTFS system using the maximal ratio combining scheme. We start from the matrix input-output relation following [9] and then group the delay-Doppler grid symbols into vectors according to their delay index and reformulate the input-output relation between the transmitted and received frames in terms of these transmitted and received vectors. By placing some null symbols (similar to a zero-padded (ZP) guard band) in the delay-Doppler domain we arrive at a reduced input-output relation, which allows the use of the maximal ratio combining to design a low complexity detector for OTFS. The overhead of the null guard symbols, needed for the proposed detection scheme, also allows to insert pilot symbols at no additional cost [10]. OTFS with the ZP guard band as mentioned above is similar to the Doppler resilient Orthogonal Signal Division Multiplexing (D-OSDM) scheme recently proposed in [11] for under water acoustic (UWA) channels [12] which is modelled as relatively faster time-varying as compared to the vehicular channel model assumption [13]. Even though the information symbols in both the schemes are transmitted in the delay-Doppler domain, the

1 Effective diversity introduced for OTFS in [2] is a more meaningful measure of the actual diversity at practical SNR values, when the number of transmitted symbols are large.

2 Only one CP is added for the entire frame, unlike in OFDM where a CP is added for every OFDM symbol in the frame.
main advantage of the general OTFS transceiver structure is the provision to insert arbitrary frequency domain windowing, which is not a part of the OSDM scheme. Windowing allows OTFS to select a subset of sub-carriers for transmission and reception, which is particularly useful in multi-user communication schemes.

The rest of the paper is organized as follows. In Section II, we discuss the system model and derive the input-output relation in the vector form. In Section III, the proposed MRC based iterative rake detector, its low complexity implementation and the conditions for convergence are described. In Section IV, we propose further improvements to the rake detector providing faster convergence and better error performance. The simulation results are given in Section V followed by a discussion on the complexity of the proposed algorithm in Section VI. Section VII contains our concluding remarks and future research directions.

II. OTFS SYSTEM MODEL

A. Notations

The following notations will be followed in this paper: \( a, \mathbf{a} \) represent a scalar, vector, and matrix, respectively; \( a(n) \) and \( \mathbf{a}(m,n) \) represents the \( n^{th} \) and \( (m,n)^{th} \) element of \( a \) and \( \mathbf{a} \) respectively; \( \mathbf{A}^{H}, \mathbf{A}^{*} \) and \( \mathbf{A}^{n} \) represent the Hermitian transpose, complex conjugate and \( n^{th} \) power of \( \mathbf{A} \). The set of \( M \times N \) dimensional matrices with complex entries in denoted by \( \mathbb{C}^{M \times N} \). Let \( \otimes \) represent circular convolution, \( \circ \) the Hadamard product (the element wise multiplication) and \( \odot \) the Hadamard division (the element wise division), \( \mathbf{A} \) represents a scalar, vector, and matrix, respectively; \( \mathbf{I}_{M} \) and \( \mathbf{I}_{N} \) denote a \( N \times N \) and \( M \times M \) identity matrix. The vectors \( \mathbf{0}_{N} \) and \( \mathbf{0}_{M} \) denote a \( N \times 1 \) and \( M \times 1 \) column vector of zeros and ones, respectively.

B. Transmitter and Receiver in delay-Doppler domain

The transmitter and receiver operate as described in [9], [14]. We will be using the following matrix/vector representation throughout the paper. Let \( \mathbf{X}, \mathbf{Y} \in \mathbb{C}^{M \times N} \) be the transmitted and received two-dimensional delay-Doppler grid, forming a frame of \( M \times N \) Q-QAM symbols, with unit average energy. Let \( \mathbf{x}_{m}, \mathbf{y}_{m} \in \mathbb{C}^{N \times 1} \) be column vectors containing the symbols in the \( m^{th} \) row of \( \mathbf{X} \) and \( \mathbf{Y} \), respectively: \( \mathbf{x}_{m} = [\mathbf{X}(m,0), \mathbf{X}(m,1), \cdots, \mathbf{X}(m,N-1)]^{T} \) and \( \mathbf{y}_{m} = [\mathbf{Y}(m,0), \mathbf{Y}(m,1), \cdots, \mathbf{Y}(m,N-1)]^{T} \), where \( m \) and \( n \) denotes the delay and Doppler indices respectively, in the two-dimensional grid. The total frame duration and bandwidth of the transmitted OTFS signal frame are \( T_{f} = NT \) and \( B = M\Delta f \), respectively. We consider the case where \( T\Delta f = 1 \), i.e., the OTFS signal is critically sampled for any pulse shaping waveform.

C. Channel

Consider a channel with \( P \) propagation paths, where \( h_{i}, l_{i} \) and \( k_{i} \) are the complex path gain, delay and Doppler shift index associated with the \( i^{th} \) path. The delay and Doppler-shift for the \( i^{th} \) path is given by \( \tau_{i} = \frac{l_{i}}{c\Delta f} \), \( \nu_{i} = \frac{k_{i}}{c\Delta f} \). We define \( l_{\text{max}} = \max\{l_{i}\} \) and \( k_{\text{max}} = \max\{k_{i}\} \). We assume that the maximum delay of the channel is \( \tau_{\text{max}} = l_{\text{max}}T/M \) and that the channel is under-spread, i.e., \( l_{\text{max}} < M \) and \( k_{\text{max}} < N \). Since the number of channel coefficients, representing different scatterers, in the delay-Doppler domain is typically limited the channel response has a sparse representation [1], [9]:

\[
h(\tau, \nu) = \sum_{i=1}^{P} h_{i} \delta(\tau - \tau_{i}) \delta(\nu - \nu_{i})
\]

(1)

In this paper, we assume that \( N \) and \( M \) are sufficiently large that the effect of fractional Doppler and delay on the receiver performance is negligible [22]. Then, the corresponding time-varying channel impulse response function can be written as

\[
g(\tau, t) = \sum_{l \in \mathcal{L}} g_{l}(t) \delta(\tau - \tau_{l})
\]

(2)

where \( \mathcal{L} = \{l_{i}\} \) is the set of \( L = |\mathcal{L}| \) unique delay tap indices among the \( P \) received paths in the delay-Doppler domain. The time varying channel gain for the \( l \)-th delay tap, \( g_{l}(t) \), is related to the delay-Doppler channel coefficients \( (h_{i}) \) as

\[
g_{l}(t) = \sum_{i \in \mathcal{K}_{l}} h_{i} e^{j2\pi\nu_{i}t}
\]

(3)

where \( \mathcal{K}_{l} = \{1 \leq i \leq P | l = l_{i}\} \) is the set of paths with delay index \( l \) and distinct Dopplers \( \nu_{i} \). Note that \( \mathcal{K}_{l} \) is an empty set if \( l \notin \mathcal{L} \).

D. Input-Output Relation in Delay-Doppler domain

The OTFS delay-Doppler domain discrete system can be expressed as

\[
\mathbf{y} = \mathbf{H} \cdot \mathbf{x} + \mathbf{w};
\]

(4)

where \( \mathbf{x}, \mathbf{y}, \mathbf{w} \in \mathbb{C}^{NM \times 1} \) and \( \mathbf{H} \in \mathbb{C}^{NM \times NM} \) is the OTFS channel matrix when transmitted and received symbol-vectors, \( \mathbf{x}_{m}, \mathbf{y}_{m} \in \mathbb{C}^{NM \times 1} \) are grouped and stacked as \( \mathbf{y} = [\mathbf{y}_{0}^{T}, \mathbf{y}_{1}^{T}, \cdots, \mathbf{y}_{M-1}^{T}]^{T} \), \( \mathbf{x} = [\mathbf{x}_{0}^{T}, \mathbf{x}_{1}^{T}, \cdots, \mathbf{x}_{M-1}^{T}]^{T} \) and \( \mathbf{w} = [\mathbf{w}_{0}^{T}, \mathbf{w}_{1}^{T}, \cdots, \mathbf{w}_{M-1}^{T}]^{T} \) is iid AWGN noise with variance \( \sigma_{n}^{2} \).

Following [9], the input-output relation for the ideal (i.e., perfectly bi-orthogonal) pulse shaping waveforms can be written as a two dimensional circular convolution between \( \mathbf{X} \) and the channel, i.e.,

\[
\mathbf{Y}(m,n) = \sum_{i=1}^{P} h_{i} \mathbf{X}([m-l_{i}]_{M}, [n-k_{i}]_{N}) + W(m,n)
\]

(5)

where \( W(m,n) \) is the AWGN noise vector. In practice, the pulse shaping waveforms are not ideal, and the imperfect bi-orthogonality introduces extra phase shifts \( \alpha_{m,n} \) to each of the channel coefficients \( h_{i} \). We assume a rectangular transmit and receive pulse shaping waveform as described in [9], [14].

Following [9], the input-output relation for the rectangular pulse shaping waveform case can be written as a two dimensional convolution in the form (omitting the AWGN noise vector for brevity)

\[
\mathbf{Y}(m,n) = \sum_{i=1}^{P} h_{i} \alpha_{i}(m,n) \mathbf{X}([m-l_{i}]_{M}, [n-k_{i}]_{N})
\]

(6)
For correction equations reduce to the last with a varying channel due to the phase terms in $W$. We note that in this case we have a circular convolution of $X$ with a varying channel due to the phase terms in $\alpha_l(m, n)$.

First, following the notations described in the above Section II-B, we can rewrite (6) in vector form by replacing $Y(n, n)$ with $x_m(n)$ and $X(n) = X(n - l_i)$ with $x_m(n - l_i)$ as

$$y_m(n) = \sum_{i=1}^{N} h_i \alpha_l(m, n)x_{m-l_i}(n - k_i)$$

Equation (7) gives two cases for the phase shifts introduced by the rectangular pulse shaping waveform. The first case, for phase shifts with $m < l_i$, is dependent on both $m$ and $n$, whereas the second equation for $m \geq l_i$ depends only on $m$. We may ignore the first case in (7), which has a dependency on $n$ (Doppler index), by placing null symbol-vectors $x_m$ in the last $l_{\text{max}}$ rows of $X$ such that, for all $l_i \leq l_{\text{max}}$,

$$h_i \alpha_l(m, n)x_{m-l_i}(n - k_i) = 0 \quad \text{if} \quad m < l_i$$

Hence, we can set

$$x_m(n) = 0 \quad \text{if} \quad m \geq M - l_{\text{max}}$$

Fig. 1 shows the $NM \times NM$ vectorized channel matrix $H$ for OFTS for $N = M = 8$ and $l_{\text{max}} = 3$. As shown in Fig. 1, the transmitted and received symbol-vectors, $x_m$ and $y_m$ respectively, are stacked in a column according to the respective delay indices ($m$). At the transmitter, the shaded vectors $(x_0, x_1, x_2, x_3, x_4)$ denote valid symbol-vectors and the non-shaded vectors $(x_5, x_6, x_7)$ denote null symbol-vectors ($0_N$). Thus $H$ only contains the shaded blocks in Fig. 1.

Inserting the null symbols as described above the phase correction equations reduce to

$$\alpha_l'(m) = \begin{cases} z^{k(m-l_i)}, & \text{if} \quad m \geq l_i \\ 0, & \text{otherwise} \end{cases}$$

For $m = 0, \ldots, M - 1$ and $k = 0, \ldots, N - 1$, let us define the vectors $\phi_m \in \mathbb{C}^{N \times 1}$ as the phase correction vector containing the phases $\alpha_l'(m)$ introduced by the non ideal pulse shaping waveform (rectangular in this case), with entries:

$$\phi_m(k) = \begin{cases} z^{km}, & \text{if} \quad 0 \leq k \leq N/2 - 1 \\ z^{-(N-k)\nu}, & \text{if} \quad N/2 \leq k \leq N - 1 \end{cases}$$

Let $v_{0,l} \in \mathbb{C}^{N \times 1}$ be the channel Doppler spread vector of the $l$-th delay tap at OTFS grid delay index $m = 0$, with entries:

$$v_{0,l}(k) = \begin{cases} h_l, & \text{if} \quad l = l_i \quad \text{and} \quad k = [k_i]_N \\ 0, & \text{otherwise} \end{cases}$$

We can now rewrite (8), for $m < M - l_{\text{max}}$, by replacing the channel coefficients $h_l$ and the reduced phase corrections $\alpha_l'(m)$ with the channel Doppler spread vectors for ideal pulses $v_l$ and phase correction vector $\phi_m$ introduced by the rectangular pulses.

$$y_m(n) = \sum_{l \in L} \sum_{k = 0}^{N-1} v_{0,l}(k)\phi_m(k)x_{m-l}([n - k]_N)$$

Now this can be written as the sum of one-dimensional circular convolutions between the vectors $v_{m,l} \cdot x_{m-l} \in \mathbb{C}^{N \times 1}$, where $v_{m,l} = [v_{m,l}(0), v_{m,l}(1), \ldots, v_{m,l}(N - 1)]$

$$y_m = \sum_{l \in L} v_{m,l} \circledast x_{m-l} \quad \text{for} \quad m = 0, \ldots, M - 1$$

where

$$v_{m,l}(k) = \begin{cases} v_{0,l}(k)\phi_m(k), & \text{if} \quad l \in L, m \geq l \quad \text{otherwise} \end{cases}$$

Note that for the ideal pulse shaping waveform case, $\phi_m = 1_N$, and hence $v_{m,l} = v_{0,l}$ for all $m \in \{0, \ldots, M - 1\}$. Referring to the vectorized form shown in Fig. 1, we convert the circular convolution between two vectors into the product of a matrix and a vector by defining $K_{m,l} \in \mathbb{C}^{N \times N}$ to be a banded matrix for $l \in L$ and an all zero matrix otherwise

$$K_{m,l} = \text{circ}[v_{m,l}(0), \ldots, v_{m,l}(N - 1)]$$

From (13) we note that the band width of each submatrix $K_{m,l}$ of $H$ is equal to the maximum Doppler spread $k_{\text{max}} < N$ and the full channel matrix $H$ has a band width equal to $N(l_{\text{max}} + 1)$. We can then write (15) as

$$y_m = \sum_{l \in L} K_{m,l} \cdot x_{m-l}$$

Note that $K_{m,l}$ (or $v_{m,l}$) can be considered as the LTV channel between the receiver delay index $m$ and transmitter delay index $m - l$ in the OTFS delay-Doppler grid. Now (15) and (17) gives us a very simple equation relating the transmitted and received symbol-vectors that we defined at the start of this section. This is a much more compact form, compared to the input-output relation we began with. The vector relations shows how the symbol-vector transmitted at delay index $m - l$ is impaired by the channel Doppler spread vector $v_{m,l}$ (or matrix $K_{m,l}$) at the delay tap with index $l$. 

Fig. 1. The delay-Doppler domain channel matrix $H$ after adding null symbols only contains the shaded blocks for $N = M = 8$ and $l_{\text{max}} = 3$. 

where $z = e^{i2\pi}$ and

$$\alpha_l(m, n) = \begin{cases} e^{-i2\pi \nu l(m - l_i)M}, & \text{if} \quad m < l_i \\ e^{-i2\pi \nu (m - l_i)M}, & \text{if} \quad m \geq l_i \\ 0, \quad \text{otherwise} \end{cases}$$

\begin{equation} \text{(7)} \end{equation}
E. Input-Output Relation in Delay-Time domain

The matrices $K_{m,l}$ in the delay-Doppler domain can be diagonalized to $\tilde{K}_{m,l}$ in the corresponding Fourier domain (delay-time domain) as

$$K_{m,l} = F_N \cdot \tilde{K}_{m,l} \cdot F_N^H,$$

$$\implies \tilde{K}_{m,l} = \text{diag}[\tilde{y}_{m,l}(0), \cdots, \tilde{y}_{m,l}(N-1)]$$

where $\tilde{y}_{m,l} = F_N^H v_{m,l}$

thereby transforming the delay-Doppler domain channel matrix $H$ into the delay-time domain channel matrix $\tilde{H}$ by replacing the sub-matrices $K_{m,l}$ in $H$ with $\tilde{K}_{m,l}$. Given the input-output relation in (4) was simplified in (17) by placing null symbols in the delay-Doppler grid as given in (10), the strictly upper triangular blocks of $\tilde{H}$ can also be set to zero. The input-output relation in the delay-time domain, illustrated in Fig. 2 can then be written in the matrix form as

$$\tilde{y} = \tilde{H} \cdot \tilde{x} + \tilde{w};$$

where

$$\tilde{y} = (I_M \otimes F_N^H) \cdot y, \quad \tilde{x} = (I_M \otimes F_N^H) \cdot x$$

$$\tilde{H} = (I_M \otimes F_N^H) \cdot H \cdot (I_M \otimes F_N)$$

and $\tilde{w}$ is the time domain AWGN vector. In this domain, the complexity of matrix multiplication is significantly reduced as the sparsity $L/N$ of $\tilde{H}$ is less than or equal to the sparsity $P/N$ of $H$, where $L$ is the number of unique delay taps and $P$ is the total number of propagation paths. The delay-time domain channel matrix $\tilde{H}$ is a banded block matrix (with a bandwidth of $N_{\text{max}} + 1$), where $K_{m,l} \in C^{N \times N}$ are non-zero diagonal matrices for $m \geq l$ and $l \in L$ and zero matrices otherwise. Consequently, the delay-Doppler domain input-output relation in (15) becomes

$$\tilde{y}_m = \sum_{l \in L} \tilde{y}_{m,l} \circ \tilde{x}_{m-l}$$

in the delay-time domain, where $\tilde{x} = [\tilde{x}_0^T, \cdots, \tilde{x}_{M-1}^T]^T$ and $\tilde{y} = [\tilde{y}_0^T, \cdots, \tilde{y}_{M-1}^T]^T$.

F. Input-Output Relation in Time Domain

Here we show how delay-Doppler or delay-time domain signals can be converted to the time domain for transmission over the physical channel. We also show how the time domain input-output relation is connected to the delay-Doppler and the delay-time domain input-output relations.

Let $s, r \in C^{NM 	imes 1}$ be the transmitted and received discrete time domain signal samples for one frame, respectively. For the case of rectangular pulse shaping waveforms, these can be related to the delay-Doppler domain information symbols as

$$s = \text{vec}(X \cdot F_N^H) \quad \text{and} \quad r = \text{vec}(Y \cdot F_N^H)$$

(21)

The operation in (21) in the literature is known as the inverse discrete Zak transform (15).

The delay-time vectors $\tilde{x}$ and $\tilde{y}$ in (18) are simply shuffled versions of the time domain transmitted and received vectors $s$ and $r$, respectively. Let $s$ and $r$ be split into $N$ blocks each of size $M$, such that $s = [s_0^T, \cdots, s_{N-1}^T]^T$ and $r = [r_0^T, \cdots, r_{N-1}^T]^T$. Then $\tilde{x}_m = [s_0(m), \cdots, s_{N-1}(m)]^T$ and $\tilde{y}_m = [r_0(m), \cdots, r_{N-1}(m)]^T$.

Let

$$P = \begin{bmatrix} E_{1,1} & E_{2,1} & \cdots & E_{M,1} \\ E_{1,2} & E_{2,2} & \cdots & E_{M,2} \\ \vdots & \ddots & \ddots & \vdots \\ E_{1,M} & E_{2,M} & \cdots & E_{M,M} \end{bmatrix} \in C^{NM \times NM}$$

(22)

be the row-column interleave permutation matrix such that $\tilde{x} = P \cdot s$ and $\tilde{y} = P \cdot r$ where $E_{i,j} \in C^{N \times N}$ is defined as

$$E_{i,j}(i', j') = \begin{cases} 1, & \text{if } i' = i \text{ and } j' = j \\ 0, & \text{otherwise} \end{cases}$$

(23)

Such permutation is known in the literature as a perfect shuffle, and has the following property (16): given square matrices $A$ and $B$

$$A \otimes B = P \cdot (B \otimes A) \cdot P^T$$

(24)

It can be noted from (25) that $E_{i,j} = E_{j,i}^T$ and hence $P = P^T$. Since the inverse of a permutation matrix is its transpose and
be the channel impaired signal component of $\mathbf{x}_m$ in the received $y_{m+l}$ vector at delay index $m+l$ after removing the interference of the other transmitted symbol-vectors $\mathbf{s}_k$ for $k \neq m$. Assuming we have the estimates of symbol-vectors $\hat{x}_m$ from previous iterations, we can then write $b_m^l$ for $l \in \mathcal{L}$ as

$$b_m^l = y_{m+l} - \sum_{\ell \in \mathcal{L}, \ell \neq l} \mathbf{K}_{m+l, \ell} \cdot \hat{x}_{m+l-\ell}$$  \hspace{1cm} (31)$$

Then from (30) and (31) for $l \in \mathcal{L}$, we have $L$ equations for the symbol-vector estimates $\hat{x}_m$ given as

$$b_m^l = \mathbf{K}_{m+l, l} \cdot \hat{x}_m + w_{m+l} + \text{interference}$$  \hspace{1cm} (32)$$

in the delay branch with index $l$ due to error in the current estimates of the interfering symbol-vectors $\hat{x}_{m+l-\ell}$ for $l \neq \ell$. In the proposed scheme, instead of estimating the transmitted symbol-vector $\hat{x}_m$ separately from each of the $L$ equations in (32), we maximal ratio combine the estimates $b_m^l$ (33) and then decode vectors $\hat{x}_m$ symbol-by-symbol by using (36). The vector output of the maximal ratio combiner, $c_m \in \mathbb{C}^{N \times 1}$, is given by

$$c_m = D_m^{-1} \cdot g_m$$  \hspace{1cm} (33)$$

where

$$D_m = \sum_{l \in \mathcal{L}} \mathbf{K}_{m+l, l}^H \cdot \mathbf{K}_{m+l, l}$$  \hspace{1cm} (34)$$

$$g_m = \sum_{l \in \mathcal{L}} \mathbf{K}_{m+l, l}^H \cdot b_m^l$$  \hspace{1cm} (35)$$

and the hard estimates are given by

$$\hat{x}_m(n) = \arg \min_{c_m \in \mathbb{Q}} |a_j - c_m(n)|.$$  \hspace{1cm} (36)$$

where $a_j$ is signal from the QAM alphabet $\mathbb{Q}$, with $j = 1, \ldots, |\mathbb{Q}|$ and $n = 0, \ldots, N-1$. Let $D(.)$ denote the decision on the estimate $c_m$ in every iteration such that $\hat{x}_m(n) = D(c_m(n))$. Hard-decision function $D(c)$ is given by the ML criterion in (36). Once we update the estimate $\hat{x}_m$, we increment

Algorithm 1: MRC in delay-Doppler domain

\begin{algorithm}
\begin{algorithmic}
\STATE \textbf{Input:} $H$, $D_m$, $y_m$, $\mathbf{x}_m = 0_N$ \hspace{0.5cm} $\forall m = 0, \ldots, M - 1$
\FOR{$i = 1 \text{: max iterations}$}
\FOR{$m = 0 : M' - 1$}
\FOR{$l \in \mathcal{L}$}
\STATE $b_m^l = y_{m+l} - \sum_{\ell \neq l} \mathbf{K}_{m+l, \ell} \cdot \hat{x}_{m+l-\ell}$
\END
\STATE $g_m = \sum_{l \in \mathcal{L}} \mathbf{K}_{m+l, l}^H \cdot b_m^l$
\STATE $c_m = D_m^{-1} \cdot g_m$
\STATE $\hat{x}_m = D(c_m)$ (or $\hat{x}_m = c_m$)
\END
\END
\STATE \textbf{Output:} $\hat{x}_m$
\end{algorithmic}
\end{algorithm}

$m$ and repeat the same to estimate all $M' = M - l_{\text{max}}$ information symbol-vectors $\hat{x}_m$ using the updated estimated\(^3\) of the previously decoded symbol-vectors in the form of a

\(^3\)Alternatively, a soft estimate can also be used in conjunction with an outer coding scheme as described in Section VIII.
decision feedback equalizer (DFE). Note that the DFE action leads to sequential updates whereas alternatively, using only the previous iteration estimates leads to parallel updates. We verified experimentally that parallel updates result in slower convergence. Algorithm 1 shows the delay-Doppler domain MRC operation.

### A. Reduced complexity delay-time domain implementation

In (31), for each symbol-vector \( \mathbf{x}_m \), we need to compute \( L \) vectors \( \mathbf{b}_m^l \). This operation requires \( L(L-1) \) products between matrices \( \mathbf{K}_{m,l} \) and estimated symbol-vectors \( \hat{\mathbf{s}}_{m-l} \). We can take advantage of the redundant operations to reduce the complexity. Let us define the residual noise plus interference (RNPI) term in the \( l^{th} \) iteration

\[
\Delta y_m^{(l)} = y_m - \sum_{l \in \mathcal{L}} \mathbf{K}_{m,l} \cdot \hat{s}_{m-l}^{(l)}
\]

which can be considered as the residual error in the reconstructed received delay-Doppler domain symbols due to error in transmission of the transmitted symbols. Note that symbol-vectors \( \hat{\mathbf{s}}_m \) are estimated in increasing order for \( m = 0, \ldots, M' - 1 \). Therefore, for estimating the symbol-vector \( \mathbf{x}_m \), only the symbol-vectors \( \mathbf{x}_{m+p}, \) for \( p < 0 \), have updated estimates available in the current iteration. For \( p \geq 0 \), the previous iteration estimates are used. From (31) and (37), \( \mathbf{b}_m^l \) computation for estimating the symbol-vector \( \mathbf{x}_m \) in the \( l^{th} \) iteration can be written as

\[
\mathbf{b}_m^l = \Delta y_m^{(l)} + \mathbf{K}_{m+l,l} \cdot \hat{s}_{m-l}^{(l-1)}
\]

Substituting (33) for \( \mathbf{b}_m^l \) in (38), the direct computation of \( \mathbf{b}_m^l \) can be avoided by writing \( \mathbf{g}_m^{(l)} \) for the \( l^{th} \) iteration as

\[
\mathbf{g}_m^{(l)} = \sum_{l \in \mathcal{L}} \mathbf{K}_{m+l,l}^H \Delta y_m^{(l)} + \left( \sum_{l \in \mathcal{L}} \mathbf{K}_{m+l,l}^H \cdot \mathbf{K}_{m+l,l} \right) \cdot \hat{s}_{m-l}^{(l-1)}
\]

Then from (33) and (39), the MRC output at the \( l^{th} \) iteration can be written as

\[
\mathbf{c}_m^{(l)} = \frac{\mathbf{g}_m^{(l-1)} + \mathbf{d}_m \cdot \Delta \mathbf{g}_m^{(l)}}{\sum_{l \in \mathcal{L}} \mathbf{K}_{m+l,l}^H \Delta y_m^{(l)} + \mathbf{d}_m \cdot \Delta \mathbf{g}_m^{(l)}}
\]

where

\[
\Delta \mathbf{g}_m^{(l)} = \sum_{l \in \mathcal{L}} \mathbf{K}_{m+l,l}^H \Delta y_m^{(l)}
\]

The vector \( \Delta \mathbf{g}_m^{(l)} \) in (41) is the maximal ratio combining of the RNPI’s in all the delay branches \( \{y_{m+l} \mid l \in \mathcal{L}\} \) having a component of \( \mathbf{x}_m \) in them.

In the \( l^{th} \) iteration, for every estimated symbol-vector \( \mathbf{x}_m \), L RNI vector products \( \Delta y_m^{(l)} \) need to be updated. which costs \( L^2 \) matrix-vector products. However, the complexity of (37) can be reduced by storing and updating the initial RNPI vectors \( \Delta y_m^{(0)} \). The L RNI vectors which have a component of the most recently estimated symbol-vector are updated as follows,

\[
\Delta y_m^{(l)} \leftarrow \Delta y_m^{(l)} - \mathbf{K}_{m+l,l} \cdot (\hat{s}_m^{(l)} - \hat{s}_m^{(l-1)})
\]

The number of matrix-vector products required to compute \( \Delta y_m^{(l)} \) has now been reduced from \( L^2 \) in (37) to \( L \) in (42).

Moreover, as described in Section II-E, the matrix-vector products in (41) and (42) are products between circulant matrices \( \mathbf{K}_{m,l} \in \mathbb{C}^{N \times N} \) and column vectors \( \mathbf{x}_m \) or \( \Delta \mathbf{y}_m \in \mathbb{C}^{N \times 1} \) which can be converted to element-wise products of vectors \( \hat{\mathbf{v}}_{m,l} \circ \hat{\mathbf{s}}_m \) or \( \hat{\mathbf{v}}_{m,l} \circ \hat{\mathbf{y}}_m \) respectively, in the delay-time domain with a complexity of \( N \) complex multiplications. The subscript \( \circ \) here denotes the N-IFFT of the vector \( \mathbf{a} \) (i.e., \( \hat{\mathbf{a}} = \mathbf{F}_N^{-1} \cdot \mathbf{a} \)). The equations (40), (41) and (42) can now be written in corresponding delay-time domain as

\[
\hat{\mathbf{c}}_m^{(l)} = \hat{\mathbf{s}}_m^{(l-1)} + \Delta \mathbf{g}_m^{(l)} \circ \hat{\mathbf{d}}_m
\]

\[
\Delta \mathbf{g}_m^{(l)} = \sum_{l \in \mathcal{L}} \hat{\mathbf{v}}_{m+l,l}^* \circ \Delta \mathbf{y}_m^{(l)}
\]

\[
\Delta \mathbf{y}_m^{(l)} \leftarrow \Delta \mathbf{y}_m^{(l)} - \hat{\mathbf{v}}_{m+l,l} \circ (\hat{\mathbf{s}}_m^{(l)} - \hat{\mathbf{s}}_m^{(l-1)})
\]

where

\[
\hat{\mathbf{d}}_m = \sum_{l \in \mathcal{L}} \hat{\mathbf{v}}_{m+l,l} \circ \hat{\mathbf{y}}_{m+l,l}
\]

which can be computed in only NL complex multiplications.

### Algorithm 2: Reduced complexity MRC in delay-time domain

1. **Input**: \( \hat{\mathbf{H}}, \hat{\mathbf{d}}_m, \mathbf{x}_m^{(0)}, \hat{\mathbf{s}}_m \) for \( m = 0, \ldots, M - 1 \)
2. **for** \( m = 0 : M' - 1 \) **do**
3. \( \Delta \mathbf{y}_m^{(0)} = \mathbf{y}_m - \sum_{l \in \mathcal{L}} \hat{\mathbf{v}}_{m+l,l} \circ \mathbf{x}_m^{(0)} \)
4. **end**
5. **for** \( i = 1: \text{max iterations} \) **do**
6. \( \Delta \mathbf{y}_m^{(i)} = \Delta \mathbf{y}_m^{(i-1)} \)
7. **for** \( m = 0 : M' - 1 \) **do**
8. \( \Delta \mathbf{g}_m^{(i)} = \sum_{l \in \mathcal{L}} \hat{\mathbf{v}}_{m+l,l}^* \circ \Delta \mathbf{y}_m^{(i)} \)
9. \( \hat{\mathbf{c}}_m^{(i)} = \hat{\mathbf{s}}_m^{(i-1)} + \Delta \mathbf{g}_m^{(i)} \circ \hat{\mathbf{d}}_m \)
10. \( \hat{\mathbf{s}}_m^{(i)} = \mathbf{F}_N \cdot \mathbf{D}(\mathbf{F}_N \cdot \hat{\mathbf{c}}_m^{(i)}) \) (or \( \hat{\mathbf{s}}_m^{(i)} = \hat{\mathbf{s}}_m^{(i-1)} \))
11. **for** \( l \in \mathcal{L} \) **do**
12. \( \Delta \mathbf{y}_m^{(i)} \leftarrow \Delta \mathbf{y}_m^{(i)} - \hat{\mathbf{v}}_{m+l,l} \circ (\hat{\mathbf{s}}_m^{(i)} - \hat{\mathbf{s}}_m^{(i-1)}) \)
13. **end**
14. **end**
15. **if** \( ||\Delta \mathbf{y}_m^{(i)}|| \geq ||\Delta \mathbf{y}_m^{(i-1)}|| \) **then** EXIT
16. **end**
17. **Output**: \( \hat{\mathbf{s}}_m = \mathbf{D}(\mathbf{F}_N \cdot \hat{\mathbf{s}}_m) \)

1) **Computation complexity per iteration**: Overall complexity per iteration for calculating \( \Delta \mathbf{g}_m^{(i)}, \hat{\mathbf{c}}_m^{(i)} \) and \( \Delta \mathbf{y}_m^{(i)} \) for all symbol-vectors is \( M'(2L + 1)N \) complex multiplications. The redundant FFT computations can be avoided by storing the Fourier transform of the \( M' \) Doppler spread vectors \( \mathbf{v}_{m,l} \), the \( M' \) initial symbol-vector estimates \( \mathbf{x}_m^{(0)} \) and the RNI vectors \( \Delta \mathbf{y}_m^{(0)} \) in (42). The hard decision estimates require the delay-time vectors to be transformed into the delay-Doppler domain and needs \( 2N \log_2(N) \) complex multiplications per symbol-vector. Algorithm 2 shows the low complexity delay-time domain MRC implementation. The detector iterations are stopped when the overall RNI error \( \| \Delta \mathbf{y}_m \| \geq \| \Delta \mathbf{y}_m^{(0)} \| \) due to the estimation error in symbol-vectors stops reducing.
2) Initial computation complexity: In the proposed detector, the initial computations include generating all the entries of the matrices $\mathbf{H}$ and $\tilde{\mathbf{H}}$, which requires computing the vectors $\nu_{m,l}$ and their Fourier transform $\tilde{\nu}_{m,l}$ for all $m = 0, \ldots, M-1$ and $l \in L$. Assuming the delay-Doppler channel parameters $(h_i, k_i, \tau_i)$ are known for $i = 1, 2, \ldots, P$, the channel Doppler spread vectors $\nu_{m,l}$ can be easily computed using the relations given in (13) and (16).

Let $k_l$ be the number of non-zero channel coefficients in each vector $\nu_{m,l}$ (or paths with different Doppler shift in the same delay bin $l \in L$) such that total number of channel coefficients or propagation paths as seen by the OTFS receiver is $P = \sum_{l \in L} k_l$. The number of complex multiplications required to compute the $M'L$ vectors $\nu_{m,l}$ using (16) is $M' \sum_{l \in L} k_l = M'P$. The OTFS channel matrix $\mathbf{H}$ (or equivalently the $\nu_{m,l}$) can then be generated in $M'P$ complex multiplications.

For the delay-time domain MRC operation in Algorithm 2 $\tilde{\nu}_{m,l}$ (N-IFFT of $\nu_{m,l}$) can be computed in \( \min\{Nk_i, N \log_2(N)\} \) complex multiplications, since there are only $k_l$ non-zero channel coefficients in each delay tap $l$. Then, the number of complex multiplications required to compute $\tilde{\mathbf{H}}$ (or equivalently all the $\tilde{\nu}_{m,l}$) is upper bounded by $M'N \sum_{l \in L} k_l = M'NP$. Alternatively, $\tilde{\nu}_{m,l}$ can be generated directly from the channel gains, delays, and Doppler shifts $(h_i, v_i, \tau_i)$ of the $P$ paths, using (2), (3) and (29) with $M'NP$ complex multiplications.

B. Low complexity initial estimate

In Algorithm 1 and 2, we initially assume that all the $Q$-QAM signals $a_j$ are equally likely and the mean of $a_j$’s is zero and so we initialize $\hat{x}_m^{(0)} = 0_N$, for all $m$. The MRC detector complexity per iteration is of the order $O(NM'L)$ and the overall complexity scales linearly with the number of iterations.

However, a better initial estimate of the OTFS symbols instead of $\hat{x}_m^{(0)} = 0_N$ may reduce the required number of MRC iterations and to reach convergence. Assuming ideal pulse shaping waveform $(\phi_m = 1_N$ in (12)), a single tap equalizer in the time-frequency domain can provide an improved low complexity initial estimate.

Define $\mathbf{H}_{dd} \in \mathbb{C}^{M \times N}$, the delay-Doppler domain channel impulse response matrix for the ideal pulse shaping waveform case,

$$\mathbf{H}_{dd}(m,n) = \begin{cases} h_i, & \text{if } m = i, n = [k_i]_N \\ 0, & \text{otherwise}. \end{cases} \quad (47)$$

From (13), the ideal channel response can also be written in terms of the Doppler spread vectors as $\mathbf{H}_{dd} = [\nu_{0,0}, \nu_{0,1}, \cdots, \nu_{0,M-1}]^T$. The corresponding time-frequency channel response for the ideal pulse shaping waveform is obtained by an ISFFT operation on the delay-Doppler channel as

$$\mathbf{H}_{df} = \mathbf{F}_M \cdot \mathbf{H}_{dd} \cdot \mathbf{F}_N^H$$

$$= \mathbf{F}_M \cdot [\nu_{0,0}, \nu_{0,1}, \cdots, \nu_{0,M-1}]^T \cdot \mathbf{F}_N^H$$

$$= \mathbf{F}_M \cdot [\tilde{\nu}_{0,0}, \tilde{\nu}_{0,1}, \cdots, \tilde{\nu}_{0,M-1}]^T \quad (49)$$

Similarly, the received time-frequency vectors can be obtained by the ISFFT operation on the received delay-Doppler domain samples as

$$y_{df} = \mathbf{F}_M \cdot Y \cdot \mathbf{F}_N^H = \mathbf{F}_M \cdot [\tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_{M-1}]^T \quad (50)$$

Since in the ideal pulse shaping waveform case, circular convolution of the channel and transmitted symbols in the delay-Doppler domain transforms to element-wise product in the time-frequency domain, we estimate the transmitted symbols in the time-frequency domain by a single tap minimum mean square error (MMSE) equalizer

$$\hat{x}_{df}(m,n) = \frac{\mathbf{H}_{df}^* (m,n) \cdot y_{df}(m,n)}{|\mathbf{H}_{df}(m,n)|^2 + \sigma_w^2} \quad (51)$$

for $m = 0, \ldots, M-1$ and $n = 0, \ldots, N-1$.

The time-delay domain initial estimates of the OTFS symbol-vectors can then be obtained by the Heisenberg transform operation on the time-frequency domain estimates as

$$[\hat{x}_0^{(0)}, \hat{x}_1^{(0)}, \cdots, \hat{x}_{M-1}^{(0)T}] = \mathbf{H}_M^H \cdot \hat{x}_{df} \quad (52)$$

Note that $v_{0,l} = 0_N$ for $l \notin L$ and hence the operation in (49) can be computed in $\min\{NM' L, NM \log_2(M)\}$ complex multiplications. Since we have already computed $\tilde{v}_{m,l}$, and $\tilde{y}$ is just a shuffled version of the received time-domain samples, the overall number of computations (for the steps in (49), (50), (51) and (52)) required for the initial estimate is upper bounded by $NM(L + 2 \log_2(M) + 3)$, which is comparable to the complexity of one detector iteration $N'M'(2L + 1)$.

C. Condition for Detector Convergence

In this section, we cast the delay-time algorithm (Algorithm 2) in the time-domain with the purpose of analysing the detector convergence using the properties of Jacobi and Gauss Seidel iterative methods for solving linear equations [18, 19]. The basic principle of iterative MRC operation in the delay-time domain with sequential updates given in (43)-(45) can be compactly expressed as

$$\tilde{x}^{(i)} = \tilde{x}^{(i-1)} + \hat{D}^{-1} \tilde{H}^H (\tilde{y} - \tilde{H} \tilde{x}^{(i-1)}) \quad (53)$$

when using parallel updates (i.e. without DFE), where $\hat{D}$ is the matrix containing diagonal elements of $\tilde{H} \tilde{H}^H$. The rows and columns of the delay-time channel matrix $\tilde{H}$ are perfectly shuffled using the permutation matrix $P$ to obtain a similar, block diagonal time-domain channel matrix $G$ as explained in Section II-F. This allows the equivalent operation in (53) to be split and executed in parallel for each independent time domain block $G_n$ as

$$s_n^{(i)} = s_n^{(i-1)} + \hat{D}_n^{-1} G_n^H (r_n - G_n s_n^{(i-1)}) \quad (54)$$

where $D_n$ is the matrix containing the diagonal elements of $G_n^H G_n$. Equation (54) can be written in the form

$$s_n^{(i)} = -T_n^j \cdot s_n^{(i-1)} + Q_n^j \cdot z_n \quad (55)$$

where $L_n$ and $L_n^H$ are the matrices containing the strictly lower and upper triangular parts of the Hermitian matrix
The linear system in (59) is convergent, if \( \rho \) is singular and hence positive definite Hermitian.

We now focus on the sequential update method given in Algorithm 1 and 2 based on the DFE iteration. Note that, in Algorithm 2 the linear matrix equation in (59) is solved block-wise with low complexity, where the latest estimates of the symbol-vectors calculated in the current iteration are used in estimating the next symbol-vector as in a DFE

\[
\mathbf{s}_n^{(i)} = \mathbf{s}_n^{(i-1)} + \mathbf{D}_n^{-1}(\mathbf{z}_n - \mathbf{L}_n \mathbf{s}_n^{(i)}) - \mathbf{L}_n^H\mathbf{s}_n^{(i-1)}
\]

(56)

where \( a \) and \( b \) denotes the contribution of the current and previous-iteration estimates, respectively. We can modify (56) for the DFE iterative method in (56) as

\[
\mathbf{s}_n^{(i)} = - \mathbf{T}_n \cdot \mathbf{s}_n^{(i-1)} + \mathbf{Q}_n \cdot \mathbf{z}_n \\
\mathbf{T}_n = (\mathbf{D}_n + \mathbf{L}_n)^{-1} \cdot \mathbf{L}_n^H \\
\mathbf{Q}_n = (\mathbf{D}_n + \mathbf{L}_n)^{-1}
\]

(57)

and observe that Algorithm 2 coincides with the well studied Gauss Seidel (GS) method available in the literature [18], [19]. Algorithm 3 shows the equivalent time domain GS method implementing Algorithm 2.

Algorithm 3: MRC time-delay domain operation principle
in the form of time domain Gauss-Seidel method

1. **Input:** \( r, G \)
2. for \( n = 0 : N - 1 \) do
3. \( \mathbf{R}_n = \mathbf{G}^H \cdot \mathbf{G}_n \)
4. \( \mathbf{z}_n = \mathbf{G}^H \cdot \mathbf{r}_n \)
5. \( \mathbf{L}_n = \text{strictly lower triangular part}(\mathbf{R}_n) \)
6. \( \mathbf{T}_n = (\mathbf{D}_n + \mathbf{L}_n)^{-1} \cdot \mathbf{L}_n^H \)
7. \( \mathbf{Q}_n = (\mathbf{D}_n + \mathbf{L}_n)^{-1} \)
8. end
9. \( \hat{\mathbf{s}}^{(0)} = \mathbf{P} \cdot (\mathbf{I}_M \otimes \mathbf{F}_N^H) \cdot \mathbf{s}^{(0)} \)
10. for \( i = 1 : \text{max iterations} \) do
11. \( \text{for } n = 0 : N - 1 \) do
12. \( \hat{\mathbf{s}}_n^{(i)} = - \mathbf{T}_n \cdot \mathbf{s}_n^{(i-1)} + \mathbf{Q}_n \cdot \mathbf{z}_n \)
13. \( \text{end} \)
14. if \( (||\mathbf{r} - \mathbf{G} \cdot \hat{\mathbf{s}}^{(i)}|| \geq ||\mathbf{r} - \mathbf{G} \cdot \hat{\mathbf{s}}^{(i-1)}||) \) then EXIT
15. end
16. **Output:** \( \hat{\mathbf{s}} = (\mathbf{I}_M \otimes \mathbf{F}_N) \cdot (\mathbf{P} \cdot \hat{\mathbf{s}}^{(i)}) \)

Both Jacobi and GS methods are used to iteratively find the least squares solution

\[
\hat{\mathbf{s}}_n = \min_{\mathbf{s}_n} ||\mathbf{z}_n - \mathbf{R}_n \mathbf{s}_n||^2
\]

(58)
of the \( M \)-dimensional linear system of equations

\[
\mathbf{z}_n = \mathbf{R}_n \cdot \mathbf{s}_n + \bar{\mathbf{w}}_n
\]

(59)

where \( \mathbf{R}_n \in \mathbb{C}^{M \times M} \) and \( \bar{\mathbf{s}}_n, \mathbf{z}_n \in \mathbb{C}^{M \times 1} \). We further assume that the time-domain correlation matrix \( \mathbf{R}_n = \mathbf{G}^H \mathbf{G}_n \) is non-singular and hence positive definite Hermitian.

In [18], [19], it is shown that the iteration method (55) for the linear system in (59) is convergent, if \( \rho(\mathbf{T}_n) < 1 \), where \( \rho(\mathbf{T}_n) \) is the spectral radius of the square matrix \( \mathbf{T}_n \) [18], [19]. For the Jacobi method, \( \rho(\mathbf{T}_n^1) < 1 \) if \( \mathbf{R}_n \) is diagonally dominant, which depends on the channel and cannot be guaranteed. However, the GS method is known to converge faster and convergence is guaranteed under more general conditions than the Jacobi method [18], [19]. In Appendix A we prove the following lemma

**Lemma 1.** The GS iterative method for the solution of (59) is converging (i.e., \( \rho(\mathbf{T}_n) < 1 \)) if \( \mathbf{R}_n \) is a positive definite Hermitian matrix. Furthermore, \( \rho(\mathbf{T}_n) = 1 \) if \( \mathbf{R}_n \) is a positive semi-definite Hermitian matrix.

We note that the algorithm may still converge even for some channels that result in a positive semi-definite Hermitian matrix \( \mathbf{R}_n \) (i.e., \( \rho(\mathbf{T}_n) = 1 \)), but this is not guaranteed.

Even though the implementation of the iterative MRC detector in Algorithm 3 looks simpler than the one in Algorithm 2, the complexity of initial computations for directly calculating \( \mathbf{R}_n, \mathbf{T}_n \) and \( \mathbf{Q}_n \) is \( O(NML^2) \) complex multiplications since \( \mathbf{G}_n \) is a banded matrix with \( L \) non-zero elements in each row. However, in Algorithm 2 the circulant property of the blocks of the channel matrix \( \mathbf{H} \) (due to the placement of null symbols in the OTFS grid as shown in Fig. 1) is utilized to reduce the overall complexity of the initial computations to \( O(NML) \) complex multiplications as explained in Section III-A.

IV. FURTHER IMPROVEMENTS

A. Successive Over Relaxed (SOR) Iterative Rake Detector

In time domain, the proposed iterative Rake detector is similar to doing \( N \) parallel GS iterations on the matched filtered received waveform, as shown in Section III-C. GS and its variants such as successive over-relaxation (SOR) method are well presented in [18–20]. The SOR method is obtained by introducing a relaxation parameter \( \omega \) in the GS method (59) as

\[
\mathbf{s}_n^{(i)} = \mathbf{s}_n^{(i-1)} + \omega \mathbf{D}_n^{-1}(\mathbf{z}_n - \mathbf{L}_n \mathbf{s}_n^{(i)}) - \mathbf{L}_n^H \mathbf{s}_n^{(i-1)}
\]

(60)

The corresponding GS iteration matrix \( \mathbf{T}_n \) and \( \mathbf{Q}_n \) in Algorithm 3 can be modified as

\[
\mathbf{T}_n^{\omega} = (\mathbf{D}_n + \omega \mathbf{L}_n)^{-1} \cdot ((\omega - 1)\mathbf{D}_n + \omega \mathbf{L}_n^H)
\]

(61)

\[
\mathbf{Q}_n^{\omega} = (\mathbf{D}_n + \omega \mathbf{L}_n)^{-1}
\]

(62)

In Appendix B we prove the following lemma.
Lemma 2. The SOR GS iterative method for the solution of (59) is converging (i.e., $\rho(T_{\omega}) < 1$) if $R_n$ is a positive definite Hermitian matrix and $0 < \omega < 2$.

We can then simply modify the proposed delay-time detector Algorithm 2 by rewriting (43) as

$$\hat{c}_m^{(i)} = \hat{c}_m^{(i-1)} + \omega(\Delta q_m^{(i-1)} \otimes \hat{d}_m)$$  \hspace{1cm} (63)

Note that when $\omega = 1$, (63) coincides with (43). The relaxation parameter when $\omega > 1$ is called the over-relaxation parameter and when $\omega < 1$ is called the under relaxation parameter. The computation of the optimal SOR parameter $\omega = \omega_{opt}$ which minimizes the spectral radius $\rho(T_{\omega})$ requires computing the eigenvalues of the iteration matrix $T_{\omega}$. 18, 19.

In this paper, we try to analyse the effect of $\omega$ by simulation. Fig. 4 and Fig. 5 show the BER plot for 64-QAM, and the required SNR to achieve a BER of $10^{-3}$ for different modulation sizes, respectively, for different values of $\omega \in [1, 1.5]$. It can be seen that the optimum $\omega$ for the 9 paths EVA channel model 13 consistently lies in the interval $[1.2, 1.3]$. We can observe that there is a 2.5 dB and 17dB gain at a BER of $10^{-3}$ for 16-QAM and 64-QAM respectively, due to just the over-relaxation parameter with almost no extra computational complexity.

The optimization of $\omega$ with low complexity, for different SNR, channel profiles and number of multipaths will be investigated in future work.

B. Iterative Rake Turbo Decoder

In order to improve FER performance, the turbo decoder principle shown in Fig. 6 is proposed. The encoded bits are random interleaved in the frame so as to enhance the delay-Doppler diversity.

The detector output bit log likelihood ratios (LLR) after random de-interleaving is fed to the LDPC decoder. The hard decision coded bits from the LDPC decoder after interleaving and QAM modulation is then fed back to the MRC detector as the input symbol-vector estimates and the process repeats. Overall, one turbo iteration involves one iteration of MRC detector, de-interleaver, LDPC decoder, interleaver, and the QAM modulator. As shown in Fig. 6 for the first iteration, the initial estimate of the QAM symbols is provided by the low complexity MMSE equalizer as explained in Section III-B, after which the initial estimate comes form the LDPC decoder.

From (40), the soft estimate of the delay-Doppler domain symbol-vector $c_m$ after MRC combining can be written as

$$\hat{c}_m = x_m + e_m \quad m = 0, \ldots, M' - 1$$  \hspace{1cm} (64)

where $x_m$ is the transmitted symbol-vector at delay index $m$ and $e_m$ denotes the normalized post MRC NPI vector. We assume that $e_m$ follows a zero mean Gaussian distribution with variance $\sigma_m^2$. This assumption becomes more accurate as the number of interfering terms increases. Then, the LLR $L_{m,n,b}^{(i)}$ of bit $b$ of the $n^{th}$ transmitted symbol in the estimated symbol-vector $c_m^{(i)}$ in the $i^{th}$ iteration can be obtained by

$$L_{m,n,b}^{(i)} = \log \frac{Pr(b = 0 | c_m^{(i)}(n))}{Pr(b = 1 | c_m^{(i)}(n))}$$

$$= \log \frac{\sum_{q \in Q_0} \exp(-|c_m^{(i)}(n) - q|^2/\sigma_m^2)}{\sum_{q' \in Q_1} \exp(-|c_m^{(i)}(n) - q'|^2/\sigma_m^2)}$$  \hspace{1cm} (65)

where $Q_0$ and $Q_1$ are the subsets of QAM symbols, where the $b^{th}$ bit of the symbol is 0 and 1, respectively. The complexity of LLR calculation can be reduced by the max-log approximated LLR obtained as

$$\tilde{L}_{m,n,b}^{(i)} = \frac{1}{\sigma_m^2} \min_{q \in Q_0} \left[ |c_m^{(i)}(n) - q|^2 - \min_{q' \in Q_1} |c_m^{(i)}(n) - q'|^2 \right]$$  \hspace{1cm} (66)

In order to compute the bit LLRs, an estimate of the post MRC NPI variance $\sigma_m^2$ is required. Accurate estimation of $\sigma_m^2$ is not straightforward and requires knowledge of the correlation between all the estimated symbol-vectors and RNPI vectors which changes every iteration as well. Since the entries of channel Doppler spread vectors $v_{n,l}$ can be assumed to be zero mean, i.i.d. and normal distributed 13, the channel Doppler spread for different delay taps can be assumed to be uncorrelated. i.e., $E[v_{n,l}v_{n',l'}^H] = 0$ for $l \neq l'$. Furthermore, for the purpose of a simple estimate of the post MRC NPI variance, we assume that RNPI $\Delta\Sigma_m^{(i)}$ in the different delay branches are uncorrelated (i.e., $E[\Delta\Sigma_m^{(i)}\Delta\Sigma_k^{(i)}] = 0$ for $m \neq k$ in all iterations) and follows Gaussian distribution. The covariance matrix of the delay-time RNPI vector $\Delta\Sigma_m^{(i)}$ in the $i^{th}$ iteration

$$\Sigma_m^{(i)}(j,k) = \langle \Delta\Sigma_m^{(i)}(j) - E[\Delta\Sigma_m^{(i)}] \rangle (\Delta\Sigma_m^{(i)}(k) - E[\Delta\Sigma_m^{(i)}])$$  \hspace{1cm} (67)
for $j, k = 0, \ldots, N - 1$ and $E\{\Delta y_m^{(i)}\} = \frac{1}{N} \sum_{n=1}^{N} \Delta y_m^{(i)}(n)$. Since Fourier transformation is a unitary transformation, the NPI variance remains the same in both domains, and we approximate the post MRC NPI variance for the symbol-vector soft estimate $\tilde{e}_m^{(i)}$ in the $i^{th}$ iteration as

$$
\sigma_m^{2(i)} = \text{Var}(\tilde{e}_m^{(i)}) \approx \frac{1}{N} \sum_{l \in L} \eta_{m,l} |\tilde{y}_{m,l} \otimes \tilde{d}_m|^{2}
$$

where $\eta_{m,l} = ||\tilde{y}_{m,l} \otimes \tilde{d}_m||^{2}$ is the normalized post MRC channel power in the different delay branches selected for combining.

V. SIMULATION RESULTS AND DISCUSSION

For simulations we generate OTFS frames for $N = 128$ and $M = 512$. The sub-carrier spacing $\Delta f$ is taken as 15 kHz. The maximum delay spread (in terms of integer taps) is taken to be 32 ($l_{\text{max}} = 31$) which is approximately 4 $\mu$s. The channel delay model is generated according to the standard Extended Vehicular A (EVA) model\(^5\) (with a speed of 120km/h) with the Doppler shift for the $i^{th}$ path generated from a uniform distribution $U(0, \gamma_{\text{max}})$, where $\gamma_{\text{max}}$ is the maximum Doppler shift\(^6\). We consider one Doppler shifted path per delay tap with $L = 9, l_{\text{max}} = 31$ and $l_{\text{max}} = 16$. For our simulations, we assume perfect knowledge of the channel state information at the receiver (see\(^7\) for practical channel estimation in OTFS). For BER plots, 10$^5$ frames are send for every point in the BER curve and for FER plots, all simulations run for a minimum of 10$^4$ frames or until 100 OTFS frame errors are encountered. BER is plotted to show uncoded performance, while FER is used when an outer coding scheme is applied.

Fig. 7 shows the BER plot for the MRC detector, with and without the initial estimate in Section III-B, for 4-QAM modulated OTFS waveform with a maximum of 10 iterations\(^8\). Performance is compared with the state of the art message passing algorithm (MPA) described in\(^21, 22\) (labeled as OTFS-MPA in Fig. 7 and 8) with a maximum of 10 iterations\(^8\) and the OFDM single tap MMSE equalizer. It can be seen that with the initial estimate (labeled as OTFS-MRC with Init Est), there is a $\approx 1$ dB gain over the MPA algorithm at a BER of $10^{-3}$. This gain is contributed by the improved SNR due to the MRC operation (or matched-filtering) at the receiver and the initial time-frequency MMSE estimate, which is more reliable for lower modulation sizes like BPSK and 4-QAM thereby increasing the convergence speed.

Note that the same initial estimates could also be used to improve the performance of MPA. However, the estimates need to be transformed into the delay-Doppler domain and Q-QAM alphabet probabilities for all the information symbols

\(^5\)The EVA channel power-delay profile is given by $[0.1.5, -1.4, -3.6, -0.6, -9.1, -7.0, -12.0, -16.9]$ dB with excess tap delays $[0, 30, 150, 310, 370, 710, 1090, 1730, 2510]$ ns

\(^6\)The MPA stopping criteria is based on the convergence of the estimated symbol probabilities\(^21\).
need to be calculated. This would incur a high complexity just to get the improved initial estimate. Moreover, similar to MRC detection, MPA can also be applied on the matched-filtered system matrix $H^H H$ instead of $H$, but this approximately doubles the MPA complexity, which scales linearly with the number of non-zero elements in the matrix. \cite{21, 22}.

Fig. \ref{fig:BER} shows the BER plot for the MRC detector for 16-QAM modulation with maximum 15 iterations compared to the MPA-based detector with maximum 30 iterations. It can be seen that with the over-relaxed iterative detection (labeled as OTFS-SOR-MRC with init Est ($\omega = 1.25$)), the BER performance is improved by around 2.5 dB at BER $= 10^{-3}$. Moreover, the SOR-iterative algorithm converges on average in less than 8 iterations for SNR $> 15$ dB. We can see from Fig. \ref{fig:Error} and \ref{fig:Error2} that the SOR parameter has more impact at higher modulation schemes, where the initial low complexity estimate is less accurate and the convergence is generally slow without SOR.

Fig. \ref{fig:ErrorRake} and \ref{fig:ErrorRake2} shows the frame error performance of the plain and SOR-turbo-Rake decoder with initial low complexity estimate for 16 and 64 QAM modulation respectively, compared with bit interleaved coded OFDM with MMSE detection scheme (labeled as OFDM BICM decoder). A half-rate LDPC code of length $N_c = 3840$ bits from \cite{24} is used and every OTFS frame contains $\lceil NM \log_2(|Q|)/N_c \rceil$ codewords.

Turbo iterations are stopped when all the decoded codewords within the frame satisfy the LDPC parity check. It can be observed that just 1 iteration of turbo MRC detector (labeled as Turbo-Rake 1 iter) is required to achieve better error performance than the bit interleaved coded MMSE OFDM. Moreover, with the over-relaxation parameter $\omega = 1.25$ (labeled as SOR-Turbo-Rake), a gain of $\approx 0.2$ dB for 16 QAM with 3 turbo iterations and $\approx 1$ dB for 64 QAM with 3 turbo iterations) is achieved in the FER performance. The overall detector complexity in terms average number of iterations to converge is significantly reduced by using turbo iterations along with the initial estimates from the time-frequency single tap equalizer.

Fig. \ref{fig:FER} shows the FER performance of the proposed detector vs BICM-OFDM for different codeword lengths: long (labeled as SOR-Turbo-Rake-3840) and short (labeled as SOR-Turbo-Rake-672). For a fair comparison with the OFDM scheme, the FER plot for a single turbo iteration is also plotted alongside. It can be observed that, the proposed detector with single turbo iteration has a gain of $\approx 3$ dB and $\approx 4$ dB for codeword length of 3840 and 672 respectively, as compared to the OFDM scheme at a FER of $10^{-5}$. It can be noted that more iterations are required for short codewords to achieve the same performance as long codewords.

VI. DETECTOR COMPLEXITY

In the table below, we summarize and compare the overall complexity of the iterative Rake receiver (in terms of complex multiplications), including initial computations and Fourier domain transformations as discussed in Section \ref{sec:appendix}.

| Computations per iteration | (I) | $NM'(2L + 1 + 2 \log_2(N))$ |
|----------------------------|-----|----------------------------|
| Initial computations       | (II)| $NM'(P + 2L)$               |
| (III) | $NM[L + 2 \log_2(M) + 3]$ |

Term (I) accounts for the computations inside each detector iteration, which includes calculating $\Delta_m^{(l)}$, $\Delta_m^{(l)}$, and $\Delta_m^{(l)}$, and the symbol-vector hard decision estimates $s_m$ in Algorithm \ref{alg:algo2}. Term (II) is for initial computations, which involves calculating $M'L$ delay-time Doppler spread vectors $\tilde{v}_{m,t}$, initial $M'$ residual vectors $\tilde{\Delta}_m^{(0)}$ in (45), and $M'$ vectors $\tilde{d}_m$ and term (III) is to compute the low complexity initial time-frequency estimate $s_m^{(0)}$ in (51).

Linear complexity detectors for OTFS with non ideal pulse shaping waveform (rectangular) are discussed in \cite{21, 23}. The complexity of MPA detector per iteration scales with alphabet size $|Q|$ and has a complexity of the order of $O(NMP|Q|)$ \cite{21}. The storage requirement for the MRC detector is of the order $O(NML)$, whereas for MPA it is $O(NMP|Q|)$ \cite{21}. The LMMSE detector proposed in \cite{23} even though is a non iterative detector has a computational complexity of $O(NMk_{\text{max}}P^2)$ where $k_{\text{max}}$ is the maximum Doppler spread, whereas the proposed detector has a complexity of $O(SNML)$ where $L \leq P$.

VII. CONCLUSION

We reformulated the OTFS input-output relation and proposed a linear complexity iterative rake detector algorithm for OTFS modulation based on the maximal ratio combining scheme. We show that the MRC detector along with a low complexity initial estimate of symbol-vectors can achieve similar BER performance as compared to MPA detector but with lower complexity and storage requirements. Based on the well studied GS method, we introduced a successive over relaxation parameter in the proposed detector for improved error performance and faster convergence. The MRC detector performance was further improved by performing turbo iterations with the aid of an outer error control coding scheme.

VIII. APPENDIX

A. Proof of Lemma \ref{lem:lemma1}

Consider the $M$ dimensional linear system of equations $z_n = R_n \cdot s_n$ without the noise term in (59). The positive definite
Hermitian system matrix $R_n$ can be split as $D_n + L_n + L_n^H$, where $D_n$ and $L_n$ $\in \mathbb{C}^{M \times M}$ are the matrices containing the diagonal and strictly lower-triangular elements, respectively. Pre and post-multiplying both sides of (59) by $D_n^{-1/2}$ and $D_n^{1/2}$, respectively, we get the re-scaled system of equations

$$ z'_n = R'_n \cdot s'_n \quad (69) $$

where

$$ R'_n = D_n^{-1/2} \cdot R_n \cdot D_n^{1/2}, \quad z'_n = D_n^{-1/2} \cdot z_n, \quad s'_n = D_n^{1/2} \cdot s_n \quad (70) $$

$R'_n$ is the re-scaled system matrix, which can be split as

$$ R'_n = I_M + L'_n + L_n^H \quad (71) $$

where $L'_n = D_n^{-1/2} \cdot L_n \cdot D_n^{-1/2}$.

Since $R'_n$ is a positive definite Hermitian matrix, any non-zero vector $u$ such that $u^H \cdot u = \beta > 0$ satisfies,

$$ u^H \cdot (I_M + L'_n + L_n^H) \cdot u > 0 \implies \beta + 2 \Re [u^H \cdot L'_n \cdot u] > 0 \quad (72) $$

The inequality in (72) can now be written as

$$ a = \Re [u^H \cdot L'_n \cdot u] = \Re [u^H \cdot L_n^H \cdot u] > -\frac{\beta}{2} \quad (73) $$

where $\Re [\cdot]$ denotes the real part. Also note that

$$ b = \Im [u^H \cdot L'_n \cdot u] = -\Im [u^H \cdot L_n^H \cdot u] \quad (74) $$

where $\Im [\cdot]$ denotes the imaginary part.

Solving (59) is equivalent to solving the linear system of equations in (69) and re-scaling its solution vector as given in (70). The equivalent GS iteration matrix $T_n$ for (70) can be written as

$$ T_n = (I_M + L'_n)^{-1} \cdot L_n^H \quad (75) $$

Now, the GS method for the system equation given in (57) is guaranteed to converge if $|\lambda(T_n)| < 1$, where $\lambda(T_n)$ denotes any eigenvalue of $T_n$, which satisfy $T_n \cdot v = \lambda(T_n) v$, for the corresponding eigenvectors $v$, i.e.,

$$ (I_M + L'_n)^{-1} \cdot L_n^H \cdot v = \lambda(T_n) v \quad (76) $$

After multiplying both sides of (76) by $v^H \cdot (I_M + L'_n)$, we can write $\lambda(T_n)$ as

$$ \lambda(T_n) = \frac{v^H \cdot L_n^H \cdot v_n}{\beta + v^H \cdot L'_n \cdot v_n} = \frac{|a - jb|}{|\beta + a + jb|} = \sqrt{\frac{a^2 + b^2}{(\beta + a)^2 + b^2}} \quad (77) $$

From (73), (74) and (77), it can be seen that $|\lambda(T_n)| < 1$. Similarly for the case when $R_n$ is positive semi-definite i.e., (73) becomes $a \geq -\beta/2$, the eigenvalue inequality becomes $|\lambda(T_n)| \leq 1$. Since $\rho(T_n)$ is equal to the largest absolute value of the eigenvalues of $T_n$, the positive definiteness of $R_n$ ensures that $\rho(T_n) < 1$.

**B. Proof of Lemma (2)**

Following the steps in Appendix I, the equation in (77) can be modified for the eigenvalues of the SOR-GS iteration matrix $T_n^\omega$ defined in (61) as

$$ \lambda(T_n^\omega) = \frac{(\omega - 1)(v^H \cdot v) + \omega(v^H \cdot L_n^H \cdot v_n)}{v^H \cdot v + \omega(v^H \cdot L_n^H \cdot v)} \quad (78) $$

The condition for eigenvalues $\lambda(T_n)$ in (77) can then be modified for the SOR case as

$$ \lambda(T_n^\omega) = \frac{\sqrt{(\omega - 1)^2 + (\omega b)^2}}{\sqrt{\beta + (\omega a)^2 + (\omega b)^2}} \quad (79) $$

It can be seen from (79) that $|\lambda(T_n^\omega)| < 1$, if $|\omega(1-\beta + \omega a)| < |\beta + \omega a|$, which is guaranteed if $0 < \omega < 2$.

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