INTEGRATION OF VOEVODSKY MOTIVES

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ABSTRACT. In this paper, we construct a theory of integration of Voevodsky motives over a perfect field \( k \), and show that it circumvents some of the complications of motivic integration, leading to new arithmetic and geometric results concerning K-equivalent \( k \)-varieties. One main application is that up to direct summing a common Chow motive, K-equivalent smooth projective \( k \)-varieties have the same \( \mathbb{Z}[1/p] \)-Chow motives (\( p \) is the characteristic exponent of \( k \)), partially answering a conjecture of Chin-Lung Wang [59]. In addition to generalizing a theorem of Kontsevich on the equality of Hodge numbers of K-equivalent smooth projective complex varieties, we show that such varieties have isomorphic integral singular cohomology groups. On the arithmetic side, we show that K-equivalent smooth \( k \)-varieties have isomorphic \( \ell \)-adic Galois representations up to semi-simplification. Furthermore, we connect this theory of integration of Voevodsky motives to the existence of motivic \( t \)-structures for geometric Voevodsky motives; we show that if the expected motivic \( t \)-structure on rational geometric Voevodsky motives exists, then K-equivalent smooth projective varieties, in particular birational Calabi-Yau smooth projective varieties over a field admitting resolution of singularities, have equivalent rational (Chow) motives. We also connect this to a conjecture of Orlov concerning bounded derived categories of coherent sheaves. This makes progress on showing that all cohomology theories (considering only their group structures) should agree for K-equivalent varieties.

1. Introduction

Let \( k \) be a perfect field and \( R \) a commutative ring. Suppose throughout that \( k \) has exponential characteristic \( p \) invertible in the coefficient ring \( R \). For our purposes, \( R \) will be \( \mathbb{Z}[1/p] \) or \( \mathbb{Q} \) most of the time. Furthermore, a variety is said to be Calabi-Yau if it has trivial canonical bundle. In this paper, we take inspiration from classical motivic integration and its wealth of applications to birational geometry to develop a theory of integration for Voevodsky’s mixed motives. Though the core goals of the theory are the same, we construct it so that it circumvents some of the complications of classical motivic integration.

The primary application in which we are interested is the following question in birational geometry.
**Question 1.1.** If $X$ and $Y$ are birational smooth projective Calabi-Yau varieties, what properties do they have in common? How can we distinguish non-isomorphic birational Calabi-Yau varieties?

As we will soon recall, there are tools such as classical motivic integration that give results in this direction. If we are concerned with the group structure of cohomology theories, the ultimate (conjectural) answer is that birational smooth projective Calabi-Yau varieties have isomorphic integral Chow motives. Indeed, this is the conjecture of Chin-Lung Wang [58] (see conjecture 1.4 below).

Via realization functors, this gives us the equivalence of many cohomology theories for such varieties. We try to come as close as possible to answering this aspect of the above question.

In this paper, after setting up the foundations, we use our theory to connect the theory of Voevodsky motives to birational geometry and classical questions on derived categories of coherent sheaves. The first main application in this paper is the following.

**Theorem 1.2.** If $X$ and $Y$ are $K$-equivalent smooth $k$-varieties, then $[M(X)] = [M(Y)]$ in $K_0(DM_{	ext{eff}}^{gm}(k;\mathbb{Z}[1/p]))$.

Recall that two smooth $k$-varieties $X$ and $Y$ are said to be $K$-equivalent if there is a smooth $k$-variety $Z$ along with proper birational morphisms $f: Z \to X$ and $g: Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$. Examples of $K$-equivalent complex varieties are birational smooth projective complex varieties with nef canonical divisors. If $k$ admits resolution of singularities, then birational smooth projective Calabi-Yau $k$-varieties are $K$-equivalent.

As a consequence, we obtain the following stronger result using Bondarko’s weight structures [11], [12].

**Theorem 1.3.** If $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, then $M(X) \oplus P \simeq M(Y) \oplus P$ in $DM_{	ext{eff}}^{gm}(k;\mathbb{Z}[1/p])$ for some Chow motive $P$.

Note that (effective) Chow motives sit inside (effective) geometric Voevodsky motives (corollary 4.2.6 of Voevodsky’s [53]), and so the above is also a statement about effective Chow motives with coefficients in $R$. This gives a partial answer to the following conjecture of Chin-Lung Wang.

**Conjecture 1.4** (Chin-Lung Wang [58]). If $X$ and $Y$ are $K$-equivalent smooth projective complex varieties, then they have isomorphic integral Chow motives.

What theorem 1.3 says is that at least after direct summing with a common Chow motive, this conjecture is true. We will come back to this conjecture later in the introduction.

Note that theorem 1.2 has, as a particular consequence, the equality of Hodge numbers of $K$-equivalent complex varieties, which is a theorem due to Kontsevich [37]. In fact, we have that they have equivalent rational mixed Hodge structures due to the semisimplicity of the abelian category of polarizable rational mixed Hodge structures. We also prove the following.

**Corollary 1.5.** If $k$ is of characteristic zero and $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, then $H^*(X^{an};\mathbb{Z}) \simeq H^*(Y^{an};\mathbb{Z})$.

Of course, a special case of this corollary is when $X$ and $Y$ are birational Calabi-Yau complex varieties. Note that this is a refinement of Batyrev’s result on the equality of Betti numbers of birational smooth projective Calabi-Yau complex varieties [5].

**Remark 1.6.** Mark McLean has recently claimed a proof that birational Calabi-Yau complex manifolds admit Hamiltonians with isomorphic Hamiltonian Floer cohomology algebras after a certain change of Novikov rings. From this, he deduces that birational Calabi-Yau complex projective manifolds have isomorphic integral singular cohomology groups. See theorem 1.2 and corollary 1.3 of [43].
We also have the following new arithmetic result.

**Corollary 1.7.** If $X$ and $Y$ are $K$-equivalent smooth $k$-varieties, then $H^*(X_\bar{k}; \mathbb{Z}_\ell)^{ss} \simeq H^*(Y_\bar{k}; \mathbb{Z}_\ell)^{ss}$ as graded Galois representations (up to semi-simplification).

As a particular consequence, if $k = \mathbb{F}_q$, then $X$ and $Y$ have the same zeta functions. If $\mathbb{F}_q$ admits resolution of singularities, then two birational Calabi-Yau $\mathbb{F}_q$-varieties have the same $\ell$-adic Galois representations (up to semi-simplification), and so also the same zeta functions.

Recall the notion of a Krull-Schmidt category.

**Definition 1.8 (Krull-Schmidt category).** An $R$-linear additive category $\mathcal{C}$ is said to be a Krull-Schmidt category if every object is a finite direct sum of objects with local endomorphism rings.

The Krull-Schmidt theorem says that an object in a Krull-Schmidt category has a local endomorphism ring if and only if it is indecomposable. Furthermore, it also says that any object is uniquely, up to permutation, a direct sum of indecomposable objects. Many examples of Krull-Schmidt categories come from abelian categories in which every object has finite length. A concrete example is the category of finitely generated modules over a finite $R$-algebra, where $R$ is a commutative Noetherian local complete ring (e.g. $\mathbb{Z}_\ell$). It is not known if the category of effective Chow motives over a field $k$ with $R$-coefficients $\text{Chow}^\text{eff}(k; R)$ is a Krull-Schmidt category, even if $k$ is of characteristic zero and $R = \mathbb{Q}$. It is known, however, that integral Chow motives do not form a Krull-Schmidt category (see example 32 of Chernousov and Merkurev’s paper[13]).

Using theorem 1.3, we can prove the following.

**Theorem 1.9.** Suppose $\text{Chow}^\text{eff}(k; R)$ is a Krull-Schmidt category. Then for $X$ and $Y$ $K$-equivalent smooth projective $k$-varieties, the Chow motives of $X$ and $Y$ in $\text{Chow}^\text{eff}(k; R)$ are equivalent.

As stated above, example 32 of [13] shows that $\text{Chow}^\text{eff}(k; \mathbb{Z})$ is not a Krull-Schmidt category, and so this line of argument will not prove (the integral version of) conjecture 1.4.

On the other hand, if there is a motivic $t$-structure on rational geometric Voevodsky motives (see subsection 4.2 for a discussion of motivic $t$-structures), then $\text{Chow}(k; \mathbb{Q})$ is a Krull-Schmidt category. Consequently, we will show the prove the following.

**Theorem 1.10.** Suppose the motivic $t$-structure conjecture is true for $\text{DM}_{gm}(k; \mathbb{Q})$. If $X$ and $Y$ are two $K$-equivalent smooth projective $k$-varieties, then $M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q}$.

The existence of such a motivic $t$-structure for rational geometric Voevodsky motives is a central conjecture in Voevodsky’s theory of motives, and if it is not true then his theory is inadequate. Note that the category of rational (effective) Chow motives embeds fully faithfully into the triangulated category of rational (effective) geometric Voevodsky motives (Voevodsky [53], corollary 4.2.6), and so the conclusion is equivalent to the equivalence of rational Chow motives. As particular consequences, this theorem implies that the existence of a suitable motivic $t$-structure on $\text{DM}_{gm}(k; \mathbb{Q})$ implies that $K$-equivalent smooth projective $k$-varieties have isomorphic rational noncommutative motives, rational $\ell$-adic Galois representations, rational mixed Hodge structures (if defined), rational Chow groups, and rational algebraic $K$-groups.

Recall the following standard definition.

**Definition 1.11 (D-equivalence).** Two $k$-varieties $X$ and $Y$ are said to be $D$-equivalent if their bounded derived categories of coherent sheaves are equivalent as $k$-linear triangulated categories.

Another consequence of the integration of Voevodsky motives is that we can prove a theorem pertaining to a conjecture of Orlov [47]. Indeed, we have the following.
Corollary 1.12. If $X$ and $Y$ are $D$-equivalent smooth projective complex varieties such that either $\kappa(X) = \dim X$ (general type) or $\kappa(X, -K_X) = \dim X$, then $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{DM}_{\text{eff}}(\text{gm})(k; \mathbb{Z})$ for some Chow motive $P \in \text{Chow}_{\text{eff}}(k; \mathbb{Z})$.

In particular, if rational Chow motives form a Krull-Schmidt category, then we have $M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q})$. This would follow if there is a motivic $t$-structure.

Remark 1.13. It is known that if $X$ and $Y$ are $D$-equivalent $k$-varieties with ample or anti-ample canonical bundles, then $X$ and $Y$ are isomorphic [9]. When they have ample canonical bundles for example, we are in the setting of varieties of general type.

The proof of the above is an easy consequence of our main theorem. Indeed, for smooth projective complex varieties of general type, Kawamata [32] has proved that $D$-equivalence implies $K$-equivalence. See theorem 4.26. As far as the author knows, this is the first time the existence of a motivic $t$-structure or Chow motives forming a Krull-Schmidt category has been connected to this conjecture of Orlov.

One reason we obtain stronger concrete results inaccessible to other theories of motivic integration not based on Voevodsky motives is that our theory comes equipped with a natural injective map

$$c_R : K_0(\text{DM}_{\text{eff}}(\text{gm})(k; R)) \to M(k; R),$$

where $M(k; R)$ is the abelian group in which our integrals will land. In classical motivic integration, the analogous injectivity statement is only conjectural, and so our path to more refined information encounters great complications. In our theory of integration, however, $M(k; R)$ is constructed fine enough so that our theory comes equipped with a transformation rule analogous to that in classical motivic integration, but not too fine so that it would be difficult to have the injectivity of $c_R$ above.

As stated above, we use this to show that $K$-equivalent smooth $k$-varieties have motives whose classes are equal in $K_0(\text{DM}_{\text{eff}}(\text{gm})(k; R))$. In the final section, using our main theorem, we deduce concrete information about the geometry and number theory of $K$-equivalent varieties. We note that the strongest result in this direction previously known is that in characteristic zero a theory of motivic integration due to Cluckers and Loeser gives the equality of classes of $K$-equivalent complex varieties in the localization of $K_0(\text{Var}_k)$ with respect to the classes of affine line, $\mathbb{G}_m$, and all projective spaces [17]. Consequently, in some sense, our theory not only gives stronger motivic results in characteristic zero, but it also works in positive characteristics. Another advantage of working with Voevodsky motives is that we are now able to use Bondarko’s weight structures ([11] and [12]) to deduce results about Chow motives that were inaccessible to classical motivic integration. Furthermore, we are able to connect important questions in the theory of motives to $K$-equivalence.

Let us recall why the idea of motivic integration is important by recalling the history behind the theory. To mathematicians as well as physicists, the classification of Calabi-Yau varieties is important. Batyrev proved that two birationally equivalent smooth projective Calabi-Yau complex varieties have the same Betti numbers [5]. This result dating to 1996 was used by Beauville to explain the Yau-Zaslow formula counting the number of rational curves on $K3$ surfaces [6]. Such results are also used to bound Hodge numbers of elliptic Calabi-Yau varieties [22], used to prove the log canonical threshold formula [40],[44], and prove transfer principles that allow the study of the Fundamental Lemma in the Langlands program [18], [19].

Roughly, Batyrev’s proof of the equality of Betti numbers of birational smooth projective Calabi-Yau complex varieties goes as follows. Choose a lift of the two varieties to the maximal compact subring $B$ of an appropriately chosen local field, and suppose its maximal ideal is $q$ and its residue field is isomorphic to the finite field $\mathbb{F}_q$ of characteristic $p$. Count the number of $\mathbb{F}_q$-points on the
varieties using $p$-adic integration with respect to canonical measures induced by gauge forms on the two varieties (gauge forms exist because the varieties are assumed to be Calabi-Yau). By showing that the $p$-adic integrals in this setup can be computed on dense open subsets of the varieties, he showed via the transformation rule for Haar integrals that the reduction modulo $q$ of the models have the same number of $\mathbb{F}_q$-points. By doing the same process with cyclotomic extensions of $B$, he showed that they have the same number of $\mathbb{F}_{q^n}$-points. Concisely, the zeta functions of the reductions modulo $q$ are the same. By using the Weil conjectures proved by Deligne [20], he concluded that they have the same Betti numbers. The arithmetic nature of this proof for a complex-geometric result suggests the existence of a deeper underlying motivic reason for the validity of this theorem.

On the other hand, it was conjectured that two such varieties in fact have the same Hodge numbers. This generalizes the result of Batyrev; indeed, given this result, the decomposition theorem in Hodge theory implies the result of Batyrev. On December 7 1995, Kontsevich gave a lecture at Orsay envisioning a theory of motivic integration to prove the stronger statement that two such complex varieties have the same Hodge numbers [37]. In fact, he proved the following more general result.

**Theorem 1.14** (Kontsevich [37], Denef-Loeser [21]). *If $X$ and $Y$ are $K$-equivalent smooth projective complex varieties, then they have the same Hodge numbers.*

Kontsevich’s idea of using motivic integration in the proof of his theorem is based on the following observation. Hodge numbers are encoded by the Deligne-Hodge polynomial. In the case of a smooth projective complex variety $X$, the polynomial is given by

$$E(X) = \sum_i \sum_j (-1)^{i+j} h^i(X; \Omega^j_X) u^i v^j \in \mathbb{Z}[u,v].$$

For general smooth complex varieties, this is defined using mixed Hodge structures. The Deligne-Hodge polynomial has the property that it can be viewed as a function on the Grothendieck ring of complex varieties $K_0(\text{Var}_\mathbb{C})$. In his lecture in Orsay, Kontsevich envisioned a theory of motivic integration that allows us to prove that the classes in a completion $\hat{M}_k$ of $K_0(\text{Var}_\mathbb{C})[L^{-1}]$ ($L := [\mathbb{A}^1]$ the Lefschetz motive) of two $K$-equivalent complex varieties are equal. The Deligne-Hodge polynomial extends to this completion, from which the result follows.

In 1996, a preprint of Denef and Loeser containing such a construction of motivic integration was circulating in the mathematical community. This construction was published in 1999 [21]. The advantage of this proof is that it not only circumvents the usage of the Weil conjectures proved by Deligne and used by Batyrev, but it gives a stronger result in the sense that two such complex varieties have the same value on any function on the aforementioned completion of the Grothendieck ring of varieties with the Lefschetz motive inverted. We remark that later on Chin-Lung Wang in 2002 [58] and Tetsushi Ito in 2004 [31] independently proved this result on Hodge numbers by using $p$-adic Hodge theory to refine Batyrev’s proof [31].

Motivic integration is by now a well-developed theory with many applications to birational geometry. In the classical theory of motivic integration, motivic integrals take values in the completion $\hat{M}_k$ for suitable $k$. Since not all motivic measures factor through this completion, we cannot deduce all the results we want. For example, the counting measure $C_q : K_0(\text{Var}_{\mathbb{F}_q})[L^{-1}] \to \mathbb{Q}$ does not factor through $\hat{M}_k$. Indeed, $q^n/L^n \to 0$ in the topology of the completion while $C_q(q^n/L^n) = 1$. This is one reason it is difficult to deduce arithmetic information using classical motivic integration. Therefore, often it is needed to know that the motivic integrals computed take values is a proper subring of $\hat{M}_k$. Unfortunately, it is unknown if the natural map from $K_0(\text{Var}_{\mathbb{F}_q})[L^{-1}]$ to its completion $\hat{M}_k$ is injective. On the other hand, there have been improvements to classical motivic
integration in characteristic zero in this direction; however, the integrals still take values in the localization of $K_0(\text{Var}_k)$ with respect to the classes of the affine line and the classes of projective spaces [17]. Such problems have been central to the theories of motivic integration developed using the Grothendieck ring of varieties.

The elements of $K_0(\text{Var}_k)$ are called virtual motives because this ring encapsulates the idea of cutting and pasting $k$-varieties. On the other hand, it is by now clear that the formalism of mixed motives à la Voevodsky is a better framework for thinking about motives because they are ∞-categories carrying a very rich structure. For example, they are stable ∞-categories that come with six functor formalisms. Moreover, there are realization functors to derived categories of ℓ-adic sheaves, mixed Hodge structures, etc. The fact that Voevodsky motives are categorical as opposed to ring-theoretic roughly means that they are richer in information and flexibility. It is this flexibility that allows us to develop a theory of integration for Voevodsky motives with better properties.

It should be pointed out that it is not clear to the author how the two theories of motivic integration relate to one another beyond both being the same in philosophy. One main goal of motivic integration theories based on $K_0(\text{Var}_k)[L^{-1}]$ is proving equality of classes in this ring; however, as previously mentioned, to a great extent this is not achieved. Contrary to classical motivic integration, we are able to show that our theory comes equipped with a natural injective map $c_R : K_0(\text{DM}_{\text{gm}}^\text{eff}(k; R)) \to M(k; R)$ into the target $M(k; R)$ of our theory of integration (see proposition 3.8). We will be able to use this injectivity result to obtain new information about classes of geometric motives in $K_0(\text{DM}_{\text{gm}}^\text{eff}(k; R))$, not just in $M(k; R)$. If we want to study concrete structures like ℓ-adic Galois representations and mixed Hodge structures, having equality in either $K_0(\text{DM}_{\text{gm}}^\text{eff}(k; R))$ (or $K_0(\text{DM}_{\text{gm}}(k; R))$) or $K_0(\text{Var}_k)[L^{-1}]$ is good enough. We must note that however, the natural morphism

$$K_0(\text{Var}_C)[L^{-1}] \to K_0(\text{DM}_{\text{gm}}(C; \mathbb{Q}))$$

induced by sending $X$ to $\pi^X_{\text{fr}}1_X$, where $\pi^X : X \to \text{Spec} \mathbb{C}$ is the structure morphism of $X$, has a nontrivial kernel. Indeed, for $g \geq 2$, there are abelian $g$-folds $A$ such that $\text{End}(A) \cong \mathbb{Z}$ and $A \not\cong \widehat{A} := \text{Pic}^0(A)$. $[A] - [\widehat{A}]$ is a nonzero element of $K_0(\text{Var}_C)[L^{-1}]$ that maps to zero because $A$ and $\widehat{A}$ are isogenous and so have equivalent rational Chow motives. (For details, see [30].) As a result, equality in $K_0(\text{Var}_C)[L^{-1}]$ is more refined than equality in $K_0(\text{DM}_{\text{gm}}(C; \mathbb{Q}))$. That being said, what is clear is that using Voevodsky motives allows us to circumvent many of the deficiencies of classical motivic integration that prevent us from obtaining stronger concrete results in geometry and arithmetic.

This paper grew out of an attempt to construct a categorified motivic integration taking values in some refinement of $\text{DM}_{\text{gm}}(k; R)$ and not just in a Grothendieck-group-like construction associated to a triangulated category. This categorified version, though, requires a better understanding of the geometric side of motivic integration, and is an ongoing project. If we manage to develop such a categorified motivic integration, then we will be able to make theorem 1.10 independent of the existence of a motivic t-structure. We may even be able to construct an integral version, hence proving the aforementioned conjecture of Chin-Lung Wang.

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2. Conventions and Preliminaries

Throughout this paper, the pair \((k, R)\) will be such that \(k\) is a perfect field with exponential characteristic \(p\) invertible in the commutative ring \(R\). For us, a Calabi-Yau \(k\)-variety \(X\) is one whose canonical bundle \(\omega_X\) is trivial. Henceforth, we will use the language of \(\infty\)-categories as developed by Lurie in [34] and [35]. In particular, homotopy (co)limits will be called (co)limits. Note that if \(\mathcal{C}_0\) is an ordinary category, we abuse notation and still write \(\mathcal{C}_0\) instead of its (classical) nerve \(N(\mathcal{C}_0)\) which is an \(\infty\)-category. Classical (co)limits in \(\mathcal{C}_0\) correspond to \(\infty\)-categorical/homotopy (co)limits in \(N(\mathcal{C}_0)\), and so this convention should not cause confusion when dealing with ordinary categories.

In the rest of this section, we recall the definition of Jet schemes and prove some of its properties. Furthermore, we discuss the essentials of the stable \(\infty\)-category of mixed motives in the sense of Voevodsky, and discuss its properties that will be important in this paper. Everything in this section is known, and is discussed here only for the convenience of the reader and to fix the notation.

2.1. Jet Schemes. In this subsection, we give the definition and some of the properties of Jet schemes.

**Proposition 2.1.** Suppose \(X\) is a \(k\)-scheme of finite type. For each \(n \geq 0\), there is a \(k\)-scheme \(J_n(X)\) of finite type representing the functor
\[
Z \mapsto \text{Mor}_k(Z \times_k \text{Spec} k[[t]]/(t^{n+1}), X).
\]
For a proof, see [26] and [27]. For \(n \geq m\), \(k[[t]]/(t^{n+1}) \to k[[t]]/(t^{m+1})\) induces a morphism of \(k\)-schemes
\[
\pi^m_n : J_n(X) \to J_m(X).
\]
These morphisms are affine morphisms, and so we have a \(k\)-scheme
\[
J_\infty(X) := \lim_{\leftarrow n} J_n(X).
\]
For brevity, we sometimes write \(X_\infty\) instead of the cumbersome \(J_\infty(X)\). We shall call this the Jet scheme of \(X\), and we let
\[
\pi^X_n : J_\infty(X) \to J_n(X)
\]
be the natural projection. We know that \(J_\infty(X)\) represents the functor
\[
Z \mapsto \text{Mor}_k(Z \times_k \text{Spec} k[[t]], X),
\]
and that \(\pi^X_n\) is induced by the ring homomorphism \(k[[t]] \to k[[t]]/(t^{n+1})\).

Jet schemes are higher order versions of tangent bundles. In particular, if \(X\) is a \(k\)-variety, then \(J_1(X) \simeq TX\) is the tangent bundle of \(X\).

We now recall some known properties of Jet schemes.

**Proposition 2.2.** Let \(X \to Y\) be an étale morphism of \(k\)-schemes of finite type. The following natural square is a pullback square:
\[
\begin{array}{ccc}
J_m(X) & \longrightarrow & J_m(Y) \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y.
\end{array}
\]
In other words, \( \mathcal{J}_m(X) \simeq \mathcal{J}_m(Y) \times_Y X \).

**Proof.** We show the isomorphism on the level of the corresponding functor of points. Precisely, we want to show that

\[
\text{Mor}_k(-, \mathcal{J}_m(X)) \simeq \text{Mor}_k(- \times_k \text{Spec} k[[t]]/(t^{m+1}), X)
\]

and

\[
\text{Mor}_k(-, \mathcal{J}_m(Y) \times_Y X) \simeq \text{Mor}_k(-, \mathcal{J}_m(Y)) \times_{\text{Mor}_k(-, Y)} \text{Mor}_k(-, X)
\]

are equivalent. Let \( Z \) be a \( k \)-scheme and consider the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma} & Y \\
\downarrow{p} & & \downarrow{\theta} \\
Z & \longrightarrow & Z \times_k \text{Spec} k[[t]]/(t^{m+1})
\end{array}
\]

Since \( X \to Y \) is étale, and so formally étale, there is a one-to-one correspondence between \( \gamma \in \text{Mor}_k(Z \times_k \text{Spec} k[[t]]/(t^{m+1}), X) \) as in the diagram above, and morphisms \( \theta \) and \( p \) making the above diagram commute. The conclusion follows. \( \square \)

From this, we obtain the following important corollary.

**Corollary 2.3.** Let \( X \) be a smooth \( k \)-scheme of finite type of pure dimension \( d \). Then \( \mathcal{J}_m(X) \) is an \( \mathbb{A}^{dm} \)-bundle over \( X \). In particular, \( \mathcal{J}_m(X) \) is smooth of pure dimension \( d(m+1) \). Consequently, \( \pi_{m+1}^m : \mathcal{J}_{m+1}(X) \to \mathcal{J}_m(X) \) is an \( \mathbb{A}^d \)-bundle.

**Proof.** It suffices to check this locally. By the étale invariance above and the fact that every smooth morphism is locally a composition of an étale morphism followed by an affine projection, it suffices to check this for \( X = \mathbb{A}^d \), which is a simple computation. \( \square \)

2.2. **Mixed Motives.** In the mid-eighties, Beilinson and Deligne conjectured the existence of a triangulated category with a \( t \)-structure whose heart is the conjectural abelian category of mixed motives. Voevodsky constructed triangulated categories with the hope that (with rational coefficients) their full subcategory of constructable objects is the derived category of the hypothetical abelian category of mixed motives. Though this property is, as of today, conjectural, these triangulated categories have had many applications. For example, the Bloch-Kato conjecture about the relation of Galois cohomology to Milnor K-theory was proved by Voevodsky \([51],[52],[54],[56],[57]\). We now have a theory of motivic cohomology, a spectral sequence from motivic cohomology to algebraic K-theory (analogue of the Atiyah-Hirzebruch spectral sequence for complex K-theory), and a six functor formalism among many other things. These tools give us a flexibility that allows us to study motivic phenomena within a very rich framework. In our case, the theory of Voevodsky motives is indispensable.

In this subsection, we briefly recall the definition of the main stable \( \infty \)-category of Voevodsky motives. We will then discuss a part of the six functor formalism, localization sequences, and purity, all of which will be essential to our work. Our exposition is terse, and so we recommend the reader to look at \([14]\) and \([15]\) for the basic definitions and theorems.

There are many variants of the construction of motives in the spirit of Voevodsky. One way the stable \( \infty \)-category of Voevodsky motives is constructed is using smooth correspondences. It is defined using the Nisnevich topology, partly because algebraic K-theory satisfies Nisnevich descent. We say that a morphism \( f : X \to Y \) is *completely decomposed* at \( y \in Y \) if there is an \( x \in X \) above \( y \) such that the residual field extension \( k(y) \to k(x) \) is an isomorphism. A Nisnevich covering of an
S-scheme $Y$ in this topology is a finite family $\{f_j : U_j \to Y\}$ of étale $S$-morphisms with the property that for every point $y \in Y$, there is a $j$ such that $f_j$ is completely decomposed at $y$. This gives us the site $(\text{Sm}/S)_{\text{Nis}}$ of smooth $S$-schemes with the Nisnevich topology. In this setting, a Nisnevich sheaf of $R$-modules with transfers is a contravariant functor on the category of correspondences $\text{Cor}_S$ to the category of $R$-modules that is a Nisnevich sheaf once we restrict it to $\text{Sm}/S$. Let us denote this category of Nisnevich sheaves of $R$-modules with transfers by $\text{Sh}^\text{tr} (S; R)$. Consider the derived $\infty$-category of chain complexes of $R$-modules $\text{D}(\text{Sh}^\text{tr} (S; R))$ on the category of Nisnevich sheaves with transfers, and invert the chain complexes of the form

$$\ldots \to 0 \to h_{X \times \mathbb{A}^1} \to h_X \to 0 \to \ldots,$$

where $h_T$ denotes the representable Nisnevich sheaf with transfers associated to $T$. This gives us the stable $\infty$-category $\text{DM}^\text{eff} (S; R)$ of effective Voevodsky motives. The object in $\text{DM}^\text{eff} (S; R)$ associated to $h_X$ will be denoted by $M_S (X)$, and will be called the motive of $X$. Voevodsky motives in general are obtained by stabilizing with respect to the Tate twist, which we now describe. The structure morphism $\mathbb{P}_S^1 \to S$ induces the morphism of motives

$$M_S (\mathbb{P}_S^1) \to M_S (S).$$

Its fiber shifted by $[-2]$ is denoted by $1_S (1)$ and called the Tate twist. Stabilizing with respect to the Tate twist gives us

$$\text{DM} (S; R) := \text{colim}_n \left( \text{DM}^\text{eff} (S; R) \xrightarrow{-\otimes 1_S (1)} \text{DM}^\text{eff} (S; R) \xrightarrow{-\otimes 1_S (1)} \ldots \right),$$

where the colimit is taken in the $\infty$-category of presentable $\infty$-categories with morphisms left adjoint functors. This is the stable $\infty$-category of Voevodsky motives (without the effectiveness condition). The smallest stable subcategory of $\text{DM}^\text{eff} (S; R)$ containing $M_S (X)$ is denoted by $\text{DM}^\text{gm} (S; R)$ and called the $\infty$-category of geometric effective Voevodsky motives. $\text{DM}^\text{gm} (S; R)$, called the $\infty$-category of geometric Voevodsky motives, is obtained by inverting $1_S (1)$ in the category of geometric effective Voevodsky motives $\text{DM}^\text{eff} (S; R)$.

In our case, however, we will need to work with schemes of finite type, not just the smooth ones. The construction above does not allow us to do this because of its restriction to smooth schemes. Therefore, we may consider another variant of the above construction. Instead of working with smooth correspondences, we may work with the category $\text{Sch}^{\text{cor}} / S$ consisting of separated finite type $S$-schemes with morphisms finite $S$-correspondences. Given a topology $\tau$, we may consider the big site $(\text{Sch}/S)_{\tau}$ of finite type $S$-schemes with the $\tau$-topology. Considering presheaves of $R$-modules on $\text{Sch}^{\text{cor}} / S$ that restrict to $\tau$-sheaves on $(\text{Sch}/S)_{\tau}$ gives us the category $\text{Sh}^{\text{cor}} (S; R)$ of $\tau$-sheaves of $R$-modules with transfers. Taking the derived category, $A^1$-localizing, and inverting the Tate twist as before gives us the category $\text{DM}^{\tau} (S; R)$ of $\tau$-motives. In this general context, any separated finite type $S$-scheme $X$ defines an object $M_S (X) \in \text{DM}^{\tau} (S; R)$. When we do not use the symbol $\tau$, we mean that $\tau = \text{Nis}$. Therefore, $\text{DM} (S; R) := \text{DM}^{\text{Nis}} (S; R)$. We denote by $\text{DM}^{\text{gm}} (S; R)$ the smallest stable $\infty$-category containing motives of the form $M_S (X)(n), n \geq 0$ and $X$ smooth $S$-scheme. Its stabilization with respect to the Tate twist will be denoted by $\text{DM}^{\text{gm}} (S; R)$. Considering the largest localizing full subcategory of $\text{DM}^{\tau} (S; R)$ generated by motives $M_S (X)(n)$ for $X$ smooth $S$-schemes and $n \in \mathbb{Z}$ gives us the category $\text{DM}^{\tau} (S; R)$. From now on, we let $S$ be Noetherian. Taking $\tau = \text{cdh}$ gives us the categories $\text{DM}^{\text{cdh}} (S; R)$ and $\text{DM}^{\text{cdh}} (S; R)$ that turn out to have some nice properties proved in [15].

For all such categories, we have basic functors $f^*, f_*, \otimes, \text{Hom}(-, -)$. For smooth $f : X \to Y$, we also have a left adjoint $f_#$ to $f^*$. If $X$ is a smooth $S$-scheme with structure morphism $\pi : X \to S$, then $M_S (X) = \pi^{X,1}_S (1_X)$. 
The functor that will be of greatest importance to us is \( f_i \), which is, up to equivalence, given by \( p_*j_\# \) for \( f = p \circ j \) any factorization of \( f \) into an open embedding \( j \) followed by a proper morphism \( p \) (such a factorization exists by Nagata compactification [45]).

Essential to our work will be localization sequences. By theorem 5.11 of Cisinski and Déglise in [15], since by assumption \( k \) has characteristic exponent invertible in \( R \), for each closed embedding \( i : Z \hookrightarrow X \) of \( k \)-varieties with open complement \( j : U \hookrightarrow X \), there is a cofiber sequence

\[
    j_\# j^* 1_X \to 1_X \to i_* i^* 1_X \to j_\# j^* 1_X [1]
\]

in \( \text{DM}_{\text{cdh}}(k; R) \). Applying the exact functor \( \pi_1^\text{gm} \) to it, we obtain the cofiber sequence

\[
    \pi_1^\text{gm} 1_U \to \pi_1^\text{gm} 1_X \to \pi_1^\text{gm} 1_Z \to \pi_1^\text{gm} 1_U [1]
\]

in \( \text{DM}_{\text{cdh}}(k; R) \). Note that we are not assuming smoothness, which is an advantage of working with the cdh topology. Note, however, that by corollary 5.9 of Cisinski and Déglise [15], the natural morphism \( \text{DM}(X; R) \to \text{DM}_{\text{cdh}}(X; R) \) is an equivalence of symmetric monoidal stable \( \infty \)-categories if \( X \) is a regular \( k \)-scheme. Furthermore, using proposition 8.1(c) of Cisinski and Déglise [15] for the field \( k \) we have that \( \text{DM}(k; R) \cong \text{DM}_{\text{cdh}}(k; R) = \text{DM}_{\text{cdh}}(k; R) \), and so we can work in the larger category \( \text{DM}_{\text{cdh}}(k; R) \) and pass to \( \text{DM}(k; R) \).

Furthermore, we also have purity, a part of which says that if \( f : X \to Y \) is a smooth separated morphism of finite type over \( k \) of relative dimension \( d \), then \( f_! \cong f_\# (-d)[-2d] \) (see theorem 11.4.5 of [14]). We will take advantage of these properties when proving the well-definedness of our motivic measure.

Note that we could make our constructions with cdh-motives from the onset, and use the above equivalences between Nisnevich and cdh-motives to conclude our results about Nisnevich motives.

3. Integration of Motives

In this section, we define integration in the setting of geometric Voevodsky motives. In order to do so, we first define the category of completed geometric effective motives \( \text{DM}_{\text{gm}}^\text{eff,∞}(k; R) \). This will be defined in subsection 3.1. We also define the notion of convergent motives that will be used in the definition of the group \( \mathcal{M}(k; R) \) in which our integrals will land. We show that \( K_0(\text{DM}_{\text{gm}}^\text{eff}(k; R)) \) naturally injects into \( \mathcal{M}(k; R) \). In order to define our integrals, we need a notion of measure; we define a measure on the Jet scheme \( J_\infty(X) \) taking values in \( \text{DM}_{\text{gm}}^\text{eff,∞}(k; R) \). We do so by first defining the measure on the so-called stable subschemes of \( J_\infty(X) \), and then we extend this by defining the notions of good and measurable subsets along with their measures. Given this measure, we define the integration of Voevodsky motives, and specialize to a particular class of functions, called measurable, that will be of greatest interest to us. In the subsection on computations, we set the stage for the transformation rule by doing some computations demonstrating the way integrals change with respect to blowups along smooth centers. In the final subsection, we prove the transformation rule, a formula describing how an integral changes with respect to birational morphisms. The transformation rule is the device that allows us to deduce results in birational geometry; it is a birational-geometric variant of the change of variables formula in differential geometry.

3.1. Completed and Convergent Motives. Integration of Voevodsky motives will deal with the category of effective geometric Voevodsky motives \( \text{DM}_{\text{gm}}^\text{eff}(k; R) \) and will take values in an abelian group \( \mathcal{M}(k; R) \) close to the Grothendieck group of some categorical limit \( \text{DM}_{\text{gm}}^\text{eff,∞}(k; R) \) of Verdier quotients of \( \text{DM}_{\text{gm}}^\text{eff}(k; R) \) by \( \otimes \)-ideals. In this subsection, we define this categorical completion and the notion of convergent motives, and discuss their relations to the usual categories of Voevodsky’s
motives.

Note that by theorem 1.1.4.4 of [35], the \( \infty \)-category \( \mathcal{C}\text{at}_{\infty}^{\text{ex}} \) of small stable \( \infty \)-categories with exact functors is closed under small limits. The \( \infty \)-category of commutative algebra objects in it, denoted by \( \mathcal{C}\text{at}_{\infty,\mathbb{Q}}^{\text{ex}} \), is also closed under small limits. In fact, limits in \( \mathcal{C}\text{at}_{\infty,\mathbb{Q}}^{\text{ex}} \) can be computed in \( \mathcal{C}\text{at}_{\infty}^{\text{ex}} \). See section 3.2.2. of [35].

**Definition 3.1.** Define the \( \infty \)-category of completed effective (geometric) Voevodsky motives with \( R \)-coefficients as the following limit in the \( \infty \)-category \( \mathcal{C}\text{at}_{\infty,\mathbb{Q}}^{\text{ex}} \) of small stable symmetric monoidal \( \infty \)-categories with exact functors:

\[
\text{DM}^{\text{eff,}\wedge}(k; R) := \lim_{n} \text{DM}^{\text{eff}}(k; R)/\text{DM}^{\text{eff}}(k; R)(n),
\]

where \( \text{DM}^{\text{eff}}(k; R)(n) \) is the full sub-\( \infty \)-category of \( \text{DM}^{\text{eff}}(k; R) \) generated by effective (geometric) motives Tate-twisted \( n \) times. The transition functors \( \text{DM}^{\text{eff}}(k; R)/\text{DM}^{\text{eff}}(k; R)(n + 1) \to \text{DM}^{\text{eff}}(k; R)/\text{DM}^{\text{eff}}(k; R)(n) \) are the localization functors.

Let us set some notation that we will use soon. Let

\[
L_n : \text{DM}^{\text{eff}}(k; R) \to \text{DM}^{\text{eff}}(k; R)/\text{DM}^{\text{eff}}(k; R)(n)
\]

be the natural localization functor. Since \( \text{DM}^{\text{eff}}(k; R) \) is presentable, this localization functor has a fully faithful right adjoint that we denote by

\[
i_n : \text{DM}^{\text{eff}}(k; R)/\text{DM}^{\text{eff}}(k; R)(n) \to \text{DM}^{\text{eff}}(k; R).
\]

By the universal property of limits, there is a natural functor \( L_\infty : \text{DM}^{\text{eff}}(k; R) \to \text{DM}^{\text{eff,}\wedge}(k; R) \). We also denote the restriction of \( L_\infty \) to \( \text{DM}^{\text{eff}}_{\text{gm}}(k; R) \) by \( L_\infty \).

Let us prove a lemma that will be used in the proof of subsequent lemmas.

**Lemma 3.2.** Suppose \( N \in \text{DM}^{\text{eff}}_{\text{gm}}(k; R) \), and \( A \in \text{DM}^{\text{eff}}(k; R) \). Then for \( n \gg 0 \) (depending on \( N \)), \( A(n) \to N \) is trivial.

**Proof.** \( \text{DM}^{\text{eff}}(k; R) \) is compactly generated by motives of the form \( M(X)(n) \) for \( n \geq 0 \) and \( X \) smooth projective \( k \)-varieties. Since \( N \) is geometric it is generated by finitely many such \( M(X_i)(n_i)[m_i] \), \( X_i \) smooth projective \( k \)-varieties of dimension \( d_i \). Suppose \( A \) is a colimit of objects \( A_j = M(Y_j)(k_j)[t_j] \), where \( Y_j \) are smooth projective \( k \)-varieties and \( k_j \geq 0 \). Consider

\[
\text{Map}_{\text{DM}^{\text{eff}}(k; R)}(A_j(n), N)
\]

\[
\simeq \colim_i \text{Map}_{\text{DM}^{\text{eff}}(k; R)}(M(Y_j)(k_j + n)[t_j], M(X_i)(n_i)[m_i])
\]

\[
\simeq \colim_i \text{Map}_{\text{DM}(k; R)}(M(Y_j)(k_j + n - n_i)[t_j - m_i], M(X_i))
\]

\[
\overset{(1)}{\simeq} \colim_i \text{Map}_{\text{DM}(k; R)}(M(Y_j)(k_j + n - n_i)[t_j - m_i], M(X_i))
\]

\[
\overset{(2)}{\simeq} \colim_i \text{Map}_{\text{DM}(k; R)}(M(Y_j)(k_j + n - n_i)[t_j - m_i], M(X_i)[d_i][2d_i])
\]

\[
\overset{}{\simeq} \colim_i \text{Map}_{\text{DM}(k; R)}(M(Y_j \times_k X_i)(k_j + n - n_i)[t_j - m_i], 1_k(d_i)[2d_i]),
\]

where (1) follows from the fact that the \( X_i \) are smooth projective, and (2) follows from \( M^c(X) \simeq M(X)[d][2d] \), where \( d \) is the dimension of smooth \( k \)-variety \( X \) and the \( \wedge \) denotes dualization. The last mapping space is contractible if \( k_j + n - n_i > d_i \), that is, when \( n > n_i + d_i - k_j \) (this follows from Voevodsky’s paper [55]). We can take \( n \geq \text{max}_i\{n_i + d_i + 1\} \), which depends only on \( N \) and ranges over finitely many \( i \) by the assumption that \( N \) is geometric. Consequently, \( \text{Map}_{\text{DM}^{\text{eff}}(k; R)}(A(n), N) \simeq \lim_j \text{Map}_{\text{DM}^{\text{eff}}(k; R)}(A_j(n), N) \simeq 0 \) for \( n \geq \text{max}_i\{n_i + d_i + 1\} \), that is, for \( n \gg 0 \). The conclusion follows. \( \square \)
A lemma that will be of importance to us later is the following. It demonstrates the natural expectation that completed effective geometric Voevodsky motives contain usual effective geometric Voevodsky motives as a full subcategory.

**Lemma 3.3.** The natural functors

\[ \text{DM}^\text{eff}_{gm}(k; R) \to \text{DM}^\text{eff,\wedge}_{gm}(k; R) \]

and

\[ \text{DM}^\text{eff,\wedge}_{gm}(k; R) \to \text{DM}^\text{eff,\wedge}(k; R) \]

are fully faithful.

**Proof.** Consider the natural commutative diagram

\[
\begin{array}{ccc}
\text{DM}^\text{eff}_{gm}(k; R) & \longrightarrow & \text{DM}^\text{eff,\wedge}_{gm}(k; R) \\
\downarrow & & \downarrow \\
\text{DM}^\text{eff}(k; R) & \longrightarrow & \text{DM}^\text{eff,\wedge}(k; R)
\end{array}
\]

of functors. In order to show that \( \text{DM}^\text{eff}_{gm}(k; R) \to \text{DM}^\text{eff,\wedge}_{gm}(k; R) \) is fully faithful, we show that \( \text{DM}^\text{eff,\wedge}_{gm}(k; R) \to \text{DM}^\text{eff,\wedge}(k; R) \) and \( L_\infty : \text{DM}^\text{eff}_{gm}(k; R) \to \text{DM}^\text{eff}(k; R) \to \text{DM}^\text{eff,\wedge}(k; R) \) are fully faithful functors.

First, let us show that \( \text{DM}^\text{eff,\wedge}_{gm}(k; R) \to \text{DM}^\text{eff,\wedge}(k; R) \) is fully faithful. Since the limit of fully faithful functors is fully faithful, it suffices to show that for each \( n \in \mathbb{Z} \),

\[ \text{DM}^\text{eff}_{gm}(k; R)/\text{DM}^\text{eff}_{gm}(k; R)(n) \to \text{DM}^\text{eff}(k; R)/\text{DM}^\text{eff}(k; R)(n) \]

is fully faithful. However, this is a consequence of theorem 2.1 of [46].

We now show that the composition \( \text{DM}^\text{eff}_{gm}(k; R) \to \text{DM}^\text{eff}(k; R) \to \text{DM}^\text{eff,\wedge}(k; R) \) is fully faithful. Suppose \( M \) and \( N \) are geometric motives in \( \text{DM}^\text{eff}_{gm}(k; R) \). We are to show that

\[ \text{Map}_{\text{DM}^\text{eff}_{gm}(k; R)}(M, N) \to \text{Map}_{\text{DM}^\text{eff,\wedge}(k; R)}(L_\infty M, L_\infty N) \]

is a weak equivalence. However,

\[
\text{Map}_{\text{DM}^\text{eff,\wedge}(k; R)}(L_\infty M, L_\infty N) \simeq \lim_n \text{Map}_{\text{DM}^\text{eff}(k; R)/\text{DM}^\text{eff}(k; R)(n)}(L_n M, L_n N) \\
\simeq \lim_n \text{Map}_{\text{DM}^\text{eff}(k; R)}(M, i_n L_n N).
\]

We claim that since \( N \) is geometric, for \( n \gg 0 \), \( N \xrightarrow{\varphi} i_n L_n N \) is an equivalence, from which the result will follow. The cokernel of \( \varphi \) lies in \( \text{DM}^\text{eff}(k; R)(n) \). By lemma 3.2, for \( n \gg 0 \), all maps \( A(n) \to N, A \in \text{DM}^\text{eff}(k; R) \) are trivial. Consider the distinguished triangle

\[ N \to i_n L_n N \to \text{coker} \varphi \to N[1] \]

for \( n \gg 0 \). \( \text{coker} \varphi \text{DM}^\text{eff}(k; R)(n) \), and so \( \text{coker} \varphi \to N[1] \) is 0 for \( n \gg 0 \). Consequently, the sequence splits, and so \( N \oplus \text{coker} \varphi \simeq i_n L_n N \). Since \( \text{Map}(\text{coker} \varphi, i_n L_n N) \simeq \text{Map}(L_n \text{coker} \varphi, L_n N) \simeq 0 \), \( \text{coker} \varphi \to i_n L_n N \) is 0. Therefore, \( \text{coker} \varphi \) must be 0. Therefore, \( \varphi \) is an equivalence for \( n \gg 0 \). We conclude that the composition

\[ \text{Map}_{\text{DM}^\text{eff}_{gm}(k; R)}(M, N) \to \text{Map}_{\text{DM}^\text{eff,\wedge}(k; R)}(L_\infty M, L_\infty N) \simeq \text{Map}_{\text{DM}^\text{eff}(k; R)}(M, N) \]

is the identity morphism, as required.

\[ \square \]
Remark 3.4. Note that it is not true that $L_\infty : \DM_{\eff}(k; R) \to \DM_{\eff, \wedge}(k; R)$ is fully faithful. Indeed, Ayoub (lemma 2.4 of [2]) constructs a (phantom) motive $F = \hocolim_{n \in \mathbb{N}} Q(n)[n]$ in $\DM_{\eff}(k; Q)$, $k$ a field of infinite transcendence degree over its prime field, that is not equivalent to 0. Clearly, this maps to 0 under $L_\infty$. His construction is as follows. Note that $H^n(\text{Speck}; Q(n)) = K_n^M(k) \otimes Q$, where $K_n^M(k)$ is the $n$-th Milnor $K$-group of $k$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements of $k^\times$ that are algebraically independent. Let $a_n : Q(n)[n] \to Q(n+1)[n+1]$ be the map corresponding to $a_n \in k^\times$ modulo the isomorphism $\text{Hom}(Q(n)[n], Q(n+1)[n+1]) \simeq \text{Hom}(Q(0), Q(1)[1]) = H^1(\text{Speck}; Q(1)) = k^\times \otimes Q$. This gives an $\mathbb{N}$-inductive system $\{Q(n)[n]\}_{n \in \mathbb{N}}$, using which we obtain the object $F := \hocolim_{n \in \mathbb{N}} Q(n)[n]$. Ayoub shows that $F$ is nonzero by showing that the natural map $\alpha_\infty : Q(0) \to F$ is nonzero. Indeed, since $Q(0)$ is compact, $\alpha_\infty$ is zero if and only if for some $n$ the natural map $\alpha_n : Q(0) \to Q(n)[n]$ is zero. But this corresponds, under the identification $\text{Hom}(Q(0), Q(n)[n]) = K_n^M(k) \otimes Q$, to the symbol $\{a_0, \ldots, a_n\} \in K_n^M(k) \otimes Q$. This is nonzero as a result of the assumption that the $a_i$ are algebraically independent. As a side remark, it is called a phantom motive because its Betti realization is 0. This example also shows that the conservativity conjecture is false if we do not restrict to geometric motives.

3.2. $\mathcal{M}(k; R)$ and injectivity from $K_0(\DM_{\eff_{\text{gm}}}(k; R))$. In this subsection, we define the abelian group $\mathcal{M}(k; R)$ in which our integrals will land. We also show that $K_0(\DM_{\eff_{\text{gm}}}(k; R))$ naturally injects into this group. First, let us fix some notation.

Suppose $\mathcal{A}$ is a stable $\infty$-category. Denote by $F(\mathcal{A})$ the free abelian group on equivalence classes $[X]$ of objects $X$ of $\mathcal{A}$. Denote by $E(\mathcal{A})$ the subgroup of $F(\mathcal{A})$ generated by elements of the form $[X \oplus Y] - [X] - [Y]$. Let $T(\mathcal{A})$ be the subgroup of $F(\mathcal{A})$ generated by elements of the form $[A] - [B] + [C]$, where

$$A \to B \to C \to A[1]$$

is a cofiber sequence in $\mathcal{A}$. Define $G(\mathcal{A}) := F(\mathcal{A})/E(\mathcal{A})$ and $K_0(\mathcal{A}) := F(\mathcal{A})/T(\mathcal{A})$. Clearly, there is a surjection $G(\mathcal{A}) \to K_0(\mathcal{A})$. Let $I(\mathcal{A})$ be its kernel.

Definition 3.5. An effective convergent motive $X$ is an object of $\DM_{\eff}(k; R)$ with the property that for $N \gg 0$, $\iota_N L_N X$ is in the image of $\DM_{\eff_{\text{gm}}}(k; R) \to \DM_{\eff}(k; R)$. Let $\DM_{\text{conv}}(k; R)$ be the full subcategory of the stable $\infty$-category $\DM_{\eff}(k; R)$ consisting of effective convergent motives. We call this stable $\infty$-category the $\infty$-category of effective convergent motives.

An example of an effective convergent motive is $M(BG_{\text{gm}}) = \bigoplus_{i=0}^{\infty} \mathbb{1}_k(i)[2i]$. We remark that there is a natural functor $\DM_{\text{conv}}(k; R) \to \DM_{\eff, \wedge}(k; R)$ with essential image in $\DM_{\eff, \wedge}(k; R)$. Note that the phantom motive $F$ in remark 3.4 is convergent and effective. Furthermore, its image under $\DM_{\text{conv}}(k; R) \to \DM_{\eff, \wedge}(k; R)$ is 0, and so this functor is not fully faithful. On the other hand, $\bigoplus_{N=1}^{\infty} \mathbb{1}_k$ is effective but not convergent.

We define here the notion of virtual dimension; it will recur throughout this paper.

Definition 3.6. The virtual dimension of an object $M$ of $\DM(k; R)$, denoted by $\text{vdim } M$, is the largest $n \in \mathbb{Z} \cup \{-\infty\}$ such that $M \in \DM_{\eff}(k; R)[n]$.

Remark 3.7. Note that $\text{vdim } 0 = +\infty$. Also, for $F$ the phantom motive in remark 3.4, we have $\text{vdim } F = \infty$, while $F \neq 0$. The value $-\infty$ can also be attained: $\text{vdim } \bigoplus_{n=-\infty}^{\infty} \mathbb{1}_k(n)[2n] = -\infty$.

Suppose for each $i \geq 0$,

$$X^{(i)}_0 \to X^{(i)}_1 \to \ldots \to X^{(i)}_n \to \ldots$$

is a tower of objects in $\DM_{\eff_{\text{gm}}}(k; R)$ such that the cofibers $X^{(i)}_n/X^{(i)}_{n-1}$ have virtual dimensions going to infinity as $n$ goes to infinity. Furthermore, assume that the virtual dimensions of $X^{(i)}_n$ go to
infinity as $i$ goes to infinity. Denote by $T_\infty(DM_{\text{conv}}^\text{eff}(k; R))$ the image in $F(DM_{\text{gm}}^\text{eff,^}\wedge(k; R))$ of the smallest subgroup of $F(DM_{\text{conv}}^\text{eff}(k; R))$ containing $T(DM_{\text{conv}}^\text{eff}(k; R))$ and elements of the form
\[
\bigoplus_i \colim_n X_n^{(i)} - \bigoplus_i \bigoplus_n X_n^{(i)}/X_{n-1}^{(i)}
\]
coming from all towers as above. Define the abelian groups
\[
\mathcal{M}(k; R) := F(DM_{\text{gm}}^\text{eff,^}\wedge(k; R))/T_\infty(DM_{\text{conv}}^\text{eff}(k; R))
\]
and
\[
\mathcal{M}_0(k; R) := F(DM_{\text{gm}}^\text{eff,^}\wedge(k; R))/E(DM_{\text{conv}}^\text{eff}(k; R)),
\]
where we are abusing notation and viewing $E(DM_{\text{conv}}^\text{eff}(k; R))$ as its image in $F(DM_{\text{gm}}^\text{eff,^}\wedge(k; R))$. Note that $\mathcal{M}(k; R)$ and $\mathcal{M}_0(k; R)$ are modules over the rings $K_0(DM_{\text{conv}}^\text{eff}(k; R))$ and $G(DM_{\text{conv}}^\text{eff}(k; R))$, respectively. There is a natural $K_0(DM_{\text{gm}}^\text{eff}(k; R))$-linear morphism
\[
c_R : K_0(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}(k; R),
\]
as well as a natural $G(DM_{\text{gm}}^\text{eff}(k; R))$-linear morphism
\[
G(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}_0(k; R).
\]
Our integrals will take values in $\mathcal{M}(k; R)$. Consequently, if we want to prove results about classes of geometric motives in $K_0(DM_{\text{gm}}^\text{eff}(k; R))$ using integration, it will be very useful to know the following injectivity result.

**Proposition 3.8.** The natural homomorphism
\[
c_R : K_0(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}(k; R)
\]
is injective.

**Proof.** Prior to showing that $c_R$ is injective, we show that
\[
G(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}_0(k; R)
\]
is injective. Every object of $G(DM_{\text{gm}}^\text{eff}(k; R))$ is of the form $[X]$ for some $X \in DM_{\text{gm}}^\text{eff}(k; R)$. Suppose $[X] = 0$ in $\mathcal{M}_0(k; R)$. Then there is an effective convergent motive $Z$ such that $X \oplus Z \simeq Z$ in $DM_{\text{gm}}^\text{eff}(k; R) \to DM_{\text{gm}}^\text{eff,^}\wedge(k; R)$. Since $X$ is geometric, by lemma 3.2, for $N \gg 0$, $X \to i_N L_N X$ is an equivalence (see proof of lemma 3.3). Furthermore, since $Z$ is convergent, we may choose $N$ so that it additionally satisfies $i_N L_N Z \in DM_{\text{gm}}^\text{eff}(k; R)$. Applying the exact functor $i_N L_N$ to both sides, we obtain $X \oplus i_N L_N Z \simeq i_N L_N Z$ as objects in $DM_{\text{gm}}^\text{eff}(k; R)$. This implies that $[X] = 0$ in $G(DM_{\text{gm}}^\text{eff}(k; R))$. This argument establishes the injectivity of
\[
G(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}_0(k; R).
\]
We now show that $K_0(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}(k; R)$ is injective. Note that there is a natural quotient map $\mathcal{M}_0(k; R) \to \mathcal{M}(k; R)$. Denote its kernel by $J$. The injectivity of $K_0(DM_{\text{gm}}^\text{eff}(k; R)) \to \mathcal{M}(k; R)$ is equivalent to showing that $G(DM_{\text{gm}}^\text{eff}(k; R)) \cap J = I(DM_{\text{gm}}^\text{eff}(k; R))$. The inclusion $I(DM_{\text{gm}}^\text{eff}(k; R)) \subseteq G(DM_{\text{gm}}^\text{eff}(k; R)) \cap J$ is clear. Now suppose $[X]$ is in $G(DM_{\text{gm}}^\text{eff}(k; R)) \cap J$. Then $[X] = [A] + [C] - [B] + [\bigoplus_i \colim_n X_n^{(i)}] - \bigoplus_i \bigoplus_n X_n^{(i)}/X_{n-1}^{(i)}$ in $\mathcal{M}_0(k; R)$ for $A \to B \to C \to A[1]$ a cofiber sequence with $A, B, C$ (in the image of) effective convergent motives in $DM_{\text{gm}}^\text{eff,^}\wedge(k; R)$, and $X_0^{(i)} \to X_1^{(i)} \to \ldots$ a sequence of towers as above. Therefore, there is an effective convergent motive $U$ such that
\[
X \oplus B \oplus U \bigoplus_i \bigoplus_n X_n^{(i)}/X_{n-1}^{(i)} \simeq A \oplus C \oplus U \bigoplus_i \colim X_n^{(i)}.
\]
As in the previous argument, we may apply \( i_N L_N \) for \( N \gg 0 \) such that \( X \to i_N L_N X \) is an equivalence and \( i_N L_N A, i_N L_N B, i_N L_N C \), and \( i_N L_N U \) are geometric motives. The exactness of \( i_N L_N \) gives us the equivalence

\[
X \oplus i_N L_N B \oplus i_N L_N U \oplus \bigoplus_i \bigoplus_n i_N L_N (X_n^{(i)}/X_{n-1}^{(i)}) \approx i_N L_N A \oplus i_N L_N C \oplus i_N L_N U \oplus \bigoplus_i \colim_n i_N L_N X_n^{(i)}
\]

in \( \text{DM}^\text{eff}(k; R) \). Note that since the virtual dimensions of the \( X_n^{(i)}/X_{n-1}^{(i)} \) go to infinity as \( n \) goes to infinity, for fixed \( N \) and for \( n \gg 0 \) we have that \( L_N(X_n^{(i)} \to L_N(X_n^{(i)}) \) is an equivalence. Consequently, \( i_N \colim_n L_N X_n^{(i)} \approx \colim_n i_N L_N X_n^{(i)} \), which is what we used above. Note that \( i_N \) is a right adjoint, and so we cannot conclude that it commutes with colimits. Let \( k \) be such that the virtual dimensions of \( X_n^{(i)} \) and \( X_n^{(i)}/X_{n-1}^{(i)} \) are at least \( N \) for \( i, n \geq k \). Then, for \( M \gg 0 \), we have

\[
L_M \left( X \oplus i_N L_N B \oplus i_N L_N U \oplus \bigoplus_{i,n \leq k} X_n^{(i)}/X_{n-1}^{(i)} \right) \approx L_M \left( i_N L_N A \oplus i_N L_N C \oplus i_N L_N U \oplus \bigoplus_{i=0}^k X_n^{(i)} \right).
\]

Using lemma 3.3, we obtain that

\[
X \oplus i_N L_N B \oplus i_N L_N U \oplus \bigoplus_{i,n \leq k} X_n^{(i)}/X_{n-1}^{(i)} \approx i_N L_N A \oplus i_N L_N C \oplus i_N L_N U \oplus \bigoplus_{i=0}^k X_n^{(i)}.
\]

Since \( i_N L_N A \to i_N L_N B \to i_N L_N C \to i_N L_N A[1] \) is a cofiber sequence in \( \text{DM}^\text{eff}_{gm}(k; R) \), it follows that

\[
[X] = [i_N L_N A] + [i_N L_N C] - [i_N L_N B] + \left[ \bigoplus_{i \leq k} X_n^{(i)} \right] - \left[ \bigoplus_{i,n \leq k} X_n^{(i)}/X_{n-1}^{(i)} \right],
\]

an element of \( I(\text{DM}^\text{eff}_{gm}(k; R)) \), as required.

The injectivity of the analogous result for virtual motives in unknown and is a source of complications in classical motivic integration. Indeed, in classical motivic integration, integrals take values in a completion \( \tilde{M}_k \) of \( K_0(\text{Var}_k)[L^{-1}] \), and it is not known if the completion map \( K_0(\text{Var}_k)[L^{-1}] \to \tilde{M}_k \) is injective. Therefore, motivic integration has its complications when we try to show equality of classes in \( K_0(\text{Var}_k)[L^{-1}] \). In our case, however, integrals take values in \( \mathcal{M}(k; R) \), and so the above proposition will help us deduce results about classes in \( K_0(\text{DM}^\text{eff}_{gm}(k; R)) \). Let me point out that the the I believe that at least for \( R = \mathbb{Q} \), the natural morphism

\[
K_0(\text{DM}^\text{eff}_{gm}(k; R)) \to K_0(\text{DM}^\text{eff}_{gm}(k; R))
\]

is injective. In fact, I believe the stronger result that the composition

\[
K_0(\text{DM}^\text{eff}_{gm}(k; R)) \to K_0(\text{DM}^\text{eff}_{gm}(k; R)) \to \lim_n K_0(\text{DM}^\text{eff}_{gm}(k; R)/\text{DM}^\text{eff}_{gm}(k; R)(n))
\]

is injective. Not being able to prove this is the reason \( \mathcal{M}(k; R) \) is considered instead of \( \text{DM}^\text{eff}_{gm}(k; R) \).

One main reason for developing the integration of Voevodsky motives is to have such an injective map so that we can deduce stronger concrete results in geometry and arithmetic.

### 3.3. Motivic measure and measurable subsets

In this section, we define the motivic measure on the Jet scheme \( \mathcal{J}_\infty(X) \) of a smooth \( k \)-variety \( X \). We first define stable subschemes of the Jet scheme and define a measure on such subschemes. We will then define good and measurable subsets of the Jet scheme and extend our measure to such subsets.

**Definition 3.9.** Let \( X \) be a smooth \( k \)-scheme. A subscheme \( A \subseteq \mathcal{J}_\infty(X) \) is said be stable if there is an \( m \in \mathbb{N} \) such that \( A_m := \pi_m(A) \) is a locally closed subscheme of \( \mathcal{J}_m(X) \) and \( A = \pi_m^{-1}(A_m) \). We shall say that such an \( A \) is stable at least at the \( m \)th level.
In order to define the motivic measure on stable subschemes, we need the following lemma.

**Lemma 3.10.** Suppose $X$ is a smooth $k$-variety of dimension $d$. If $A$ is a stable subscheme of $\mathcal{J}_\infty(X)$ that is stable at the $N$th level, then for every $m \geq N$,

$$\pi^{A_{m+1}}_1 A_{m+1} ((m+1)d)[2(m+1)d] \simeq \pi^A_1 A_m (md)[2md]$$

in $\text{DM}_{gm}(k; R)$.

**Proof.** First note that both objects are constructible motives in $\text{DM}(k; R)$ by corollary 4.2.12 of [14], and so they are geometric. Note that by the smoothness of $X$, for every $m \geq N$

$$A_{m+1} \to A_m$$

is an $\mathbb{A}^d$-bundle. By purity, we know that $\pi^{m+1}_m A_{m+1} \simeq \pi^m_1 A_{m+1} (-d)[-2d]$. Therefore, we obtain

$$\pi^{A_{m+1}}_1 A_{m+1} ((m+1)d)[2(m+1)d] \simeq \pi^A_1 \pi^{m+1}_m A_{m+1} ((m+1)d)[2(m+1)d]$$

$$\simeq \pi^A_1 \pi^m_1 A_{m+1} (md)[2md]$$

$$\simeq \pi^{A_{m+1}}_1 A_m (md)[2md],$$

where the last equivalence follows from $\mathbb{A}^1$-homotopy invariance and the fact that $\pi^{m+1}_m A_{m+1}$ is an $\mathbb{A}^d$-bundle implying that $\pi^m_1 A_{m+1} \simeq \pi^A_1 A_m$. The conclusion follows. \qed

Therefore, on such stable subschemes, we can define a measure $\mu_X$ as follows.

**Definition 3.11.** For $A$ a stable subscheme of $\mathcal{J}_\infty(X)$, we define its volume by

$$\mu_X (A) := \pi^A_1 A_m ((m+1)d)[2(m+1)d] \in \text{DM}_{gm}(k; R)$$

for sufficiently large $m$.

By lemma 3.10, this is independent of $m$ for sufficiently large $m$. Furthermore, it is also geometric. For effectivity, we used the fact that $\pi^Y_1 Y \in \text{DM}_{gm}(k; R)(-d_Y)$, where $d_Y := \dim Y$. We show this in the following lemma.

**Lemma 3.12.** Suppose $X$ is a $k$-variety of dimension $d$. Then $\text{vdim} \pi^X_1 X \geq -d$.

**Proof.** First assume that $X$ is a smooth $k$-variety. By purity, $\pi^X_1 X \simeq M(X)(-d)[-2d]$, and so $\text{vdim} \pi^X_1 X \geq -d$. If $\text{vdim} \pi^X_1 X > -d$, then $\text{vdim} M(X) \geq 1$, that is $M(X) \in \text{DM}_{gm}(k; R)(1)$. From this, we obtain $M(X_L) \in \text{DM}_{gm}(L; R)(1)$ for any finite extension $L | k$. Take a point $\text{Spec} L \to X_L$ for some large enough $L$. From this, we see that $1_L$ splits off $M(X_L)$ as a direct summand. As a result, $M(X_L)$ has virtual dimension 0. Consequently, $\text{vdim} \pi^X_1 X = -d$.

We now prove the lemma when $X$ need not be smooth. We do so by inducting on the dimension of $X$. The lemma is true if $X$ is of dimension 0. Suppose the lemma is true for dimensions $< d$. Let $\text{Sing} X$ be the singular locus of $X$. Then $\dim \text{Sing} X < d$, and so by the inductive hypothesis, $\text{vdim} \pi^\text{Sing} X \text{Sing} X = -\dim \text{Sing} X > -d$. By the smoothness of $X \setminus \text{Sing} X$, we have $\text{vdim} \pi^X_{\text{Sing} X} \text{Sing} X \geq -d$. Consider the localization cofiber sequence

$$\pi^X_{\text{Sing} X} X_{\text{Sing} X} \to \pi^X_1 X \to \pi^\text{Sing} X_{\text{Sing} X} X_{\text{Sing} X} \to \pi^X_{\text{Sing} X} X_{\text{Sing} X} [1]$$

in $\text{DM}_{cdh}(k; R)$. Twisting it $d$ times, we obtain the cofiber sequence

$$\pi^X_{\text{Sing} X} X_{\text{Sing} X} (d) \to \pi^X_1 X (d) \to \pi^\text{Sing} X_{\text{Sing} X} X_{\text{Sing} X} (d) \to \pi^X_{\text{Sing} X} X_{\text{Sing} X} (d)[1]$$

in $\text{DM}_{cdh}(k; R)$. In the above cofiber sequence, $\pi^X 1_X (d)$ is an extension of two effective motives, and so is itself an effective cdh-motive. Consequently, $\text{vdim} \pi^X_1 X \geq -d$. Assume to the contrary
that $\text{vdim } \pi_1^X 1_X > -d$. Then we have $\text{vdim } \pi_1^X \mathbb{1}_{X \setminus \text{Sing} X} > -d$ from the localization sequence above, a contradiction. As a result, $\text{vdim } \pi_1^X 1_X = -d$, as required. Note that we are using corollary 5.3.9 of Kelly’s thesis [33] saying that the right adjoint of the canonical functor

$$\text{DM}^\text{eff}(k; R) \to \text{DM}^\text{eff}_{\text{cdh}}(k; R)$$

is an equivalence of categories for any perfect field $k$ of characteristic exponent $p \in R^\times$.

Later, we will extend $\mu_X$ to a larger collection of subsets of $\mathcal{J}_\infty(X)$ called measurable subsets. The following corollary will be used in proving that our measure on measurable subsets is well-defined up to equivalence.

**Corollary 3.13.** Suppose $A \subseteq \bigcup_{i=1}^N A_i \subseteq \mathcal{J}_\infty(X)$, where $A$ and the $A_i$ are stable subschemes. Then

$$\text{vdim } \mu_X(A) \geq \min_{1 \leq i \leq N} \text{vdim } \mu_X(A_i).$$

**Proof.** This is an easy consequence of lemma 3.12, and the fact that if $X \subseteq \bigcup_{i=1}^N X_i$ is a covering of a subscheme by a finite collection of subschemes, then $\dim X \leq \max_{1 \leq i \leq N} \dim X_i$. □

As a sanity check, we have additivity of $\mu_X$ in the following weak sense.

**Lemma 3.14.** $\mu_X$ is additive on finite disjoint unions of stable subschemes of $\mathcal{J}_\infty(X)$. More precisely, if the subscheme $A = \bigcup_{i=1}^k A_i$ is a finite disjoint union of stable subschemes of $\mathcal{J}_\infty(X)$, then

$$\mu_X(A) \simeq \bigoplus_{i=1}^k \mu_X(A_i).$$

**Proof.** By the stability of the $A_i$, there is an $N$ such that every $A_i$ stabilizes from the $N$th level onward. Then it is easy to see that $A_N = \bigcup_{i=1}^k A_i,N$, and so

$$\mu_X(A) \simeq \pi_1^{A_N} 1_{A_N}((N+1)d)[2(N+1)d]$$

$$\simeq \pi_i^{A_N} 1_{\bigcup_{i=1}^k A_i,N}((N+1)d)[2(N+1)d]$$

$$\simeq \bigoplus_{i=1}^k \pi_1^{A_i,N} 1_{A_i,N}((N+1)d)[2(N+1)d]$$

$$\simeq \bigoplus_{i=1}^k \mu_X(A_i).$$

Another lemma that will be important in the proof of the well-definedness of the measure $\mu_X$ once we extend it to measurable subsets, to be defined later, is the following “compactness” result.

**Lemma 3.15.** Suppose $D$ and $D_n$, $n \in \mathbb{N}$, are stable subschemes of $\mathcal{J}_\infty(X)$ such that $D \subseteq \bigcup_{n \in \mathbb{N}} D_n$ and the virtual dimensions of $\mu_X(D_n)$ go to infinity. Then $D$ is contained in the union of finitely many of the $D_n$.

**Proof.** The proof is the same as that of lemma 2.3 of [42] but made scheme-theoretic. Suppose $D = \pi_n^{-1} \pi_n(D)$, where $\pi_n(D)$ is a locally closed $k$-subscheme of $\mathcal{J}_n(X)$. Such an $n$ exists because $D$ is assumed to be a stable subscheme. Assume to the contrary that $D$ cannot be covered by a finite subcollection. Since $\lim_{n \to \infty} \text{vdim } \mu_X(D_n) = \infty$, there is a $k \in \mathbb{N}$ such that $\text{vdim } \mu_X(D_i) > (n+2)d$ for every $i > k$. By assumption, $D \setminus \bigcup_{i \leq k} D_i \neq \emptyset$, and so we may choose $x_{n+1} \in \pi_{n+1}(D \setminus \bigcup_{i \leq k} D_i)$. Then the (scheme-theoretic) fiber $\pi_n^{-1}(x_{n+1}) \subset D$. This set is not covered by finitely many of the $D_i$. Indeed, $\pi_n^{-1}(x_{n+1})$ is not covered by the $D_i$ when $i \leq k$. Furthermore, $\pi_n^{-1}(x_{n+1}) \cap D_i$ has positive codimension in $D_i$ for $i > k$. 

Consequently, we can inductively construct a sequence \((x_m)_{m>n}\) such that for every \(m > n\) \(x_m \in \mathcal{J}_m(X)\), \(x_{m+1}\) is above \(x_m\), and \(\pi^{-1}_m(x_m)\) is not coverable by finitely many of the \(D_i\). This determines an element \(x \in \mathcal{J}_\infty(X)\) such that \(\pi_n(x) \in \pi_n(D)\). Consequently, \(x \in D\) because \(D\) is stable at least at level \(n\). Since \(D\) is covered by the \(D_i\), there is a \(j\) such that \(x \in D_j\). \(D_j\) is stable, and so is stable at least at some level \(m > n\). This implies that \(\pi^{-1}_m(x_m) \subseteq D_j\), contradicting the fact that \(\pi^{-1}_m(x_m)\) is not finitely coverable by the \(D_i\).

\[\square\]

The subsets of \(\mathcal{J}_\infty(X)\) that will show up in the integration of Voevodsky motives will not necessarily be stable subschemes; they will come from subsets of \(\mathcal{J}_\infty(X)\), called measurable subsets, that can be approximated by stable subschemes in the following sense.

**Definition 3.16.** A subset \(C \subseteq \mathcal{J}_\infty(X)\) is said to be good if there is a monotonic sequence (the inclusions are locally closed embeddings of \(k\)-schemes) of stable subschemes

\[C_0 \supseteq \ldots \supseteq C_n \supseteq C_{n+1} \supseteq \ldots \ (\text{or } C_0 \subseteq \ldots \subseteq C_n \subseteq C_{n+1} \subseteq \ldots)\]

of \(\mathcal{J}_\infty(X)\) containing (resp. contained in) \(C\), and stable \(C_{n,i}, i, n \in \mathbb{N}\) such that for every \(n\)

\[C_n \setminus C \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i} \left(\text{resp. } C \setminus C_n \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i}\right),\]

and for every \(n, i\),

\[n \leq \text{vdim } \mu_X(C_{n,i})\]

and

\[\text{vdim } \mu_X(C_{n,i}) \xrightarrow{i \to \infty} \infty.\]

We then define the volume of \(C\) as the object

\[\mu_X(C) := (\mu_X(C_n))_n \in \text{DM}_{\text{gm}}^{\text{eff,^}}(k; R).\]

We call a pair \((C, S)\) consisting of a subset \(C \subseteq \mathcal{J}_\infty(X)\) and a finite decomposition \(S\) \((C = \sqcup_{i \in S} C_i, C_i \subseteq \mathcal{J}_\infty(X))\) measurable if each \(C_i\) is a good subset. We then let

\[\mu_X(C, S) := \bigoplus_{i \in S} \mu_X(C_i).\]

We view a good subset \(C\) without a prescribed decomposition as a measurable subset with the trivial decomposition \(S = \{C\}\).

Just as in classical measure theory, the question arises as to whether this measure is well-defined (up to equivalence).

**Proposition 3.17.** This measure \(\mu_X\) is well-defined up to equivalence, that is, for a fixed measurable subset \((C, S)\), any two sets of data in the above definition give rise to volumes that are equivalent in \(\text{DM}_{\text{gm}}^{\text{eff,^}}(k; R)\).

**Proof.** Since the decomposition is fixed, we may assume without loss of generality that our measurable subset \((C, S)\) is a good subset with the trivial decomposition. Suppose \((C_n)_n\) and \((D_n)_n\) are two monotonic sequences as in the definition above. We are to show that

\[(\mu_X(C_n))_n \simeq (\mu_X(D_n))_n\]

in \(\text{DM}_{\text{gm}}^{\text{eff,^}}(k; R)\).

Suppose first that both are decreasing sequences. Note that \((C_n \cap D_n)_n\) is also a decreasing sequence satisfying the properties in the definition above. Indeed, \((C_n \cap D_n) \setminus C \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i}\). Therefore, we may assume without loss of generality that for each \(n\), \(C_n \subseteq D_n\). If both sequences are increasing, then a similar argument shows that we may assume without loss of generality that \(C_n \subseteq D_n\). If
We show that there are equivalences \( \mu_X(C_n) \simeq \mu_X(D_n) \) in \( \text{DM}_{\text{gm}}^\text{eff}(k; R)/\text{DM}_{\text{gm}}^\text{eff}(k; R)(n) \) that are compatible with each other as \( n \) varies. This can be done by first writing \( C_n \hookrightarrow D_n \) as the composition of an open embedding \( C_n \hookrightarrow K_n \) followed by a closed embedding \( K_n \hookrightarrow D_n \), where \( K_n \) is the inverse image under \( \pi_m \) of a locally closed \( k \)-subvariety of \( J_m(X) \) for some \( m \). Then, we may use the localization sequences. Indeed, in \( \text{DM}_{\text{gm}}^\text{eff}(k; R) \) we have cofiber sequences

\[
\mu_X(C_n) \to \mu_X(K_n) \to \mu_X(K_n \setminus C_n) \to \mu_X(C_n)[1]
\]

\[
\mu_X(D_n \setminus K_n) \to \mu_X(D_n) \to \mu_X(K_n) \to \mu_X(D_n \setminus K_n)[1]
\]

from localization sequences. Here, we can assume that they are cofiber sequences in \( \text{DM}(k; R) \) because we have the cofiber sequences in \( \text{DM}_{\text{cdh}}^\text{eff}(k; R) \simeq \text{DM}(k; R) \) (see subsection 2.2). Furthermore, \( \text{DM}_{\text{gm}}^\text{eff}(k; R) \) is a triangulated full subcategory of \( \text{DM}(k; R) \) and the objects in the above cofiber sequences are in \( \text{DM}_{\text{gm}}^\text{eff}(k; R) \) by the discussion immediately after definition 3.11.

Note that \( K_n \setminus C_n \subseteq D_n \setminus C_n \). If \( (C_n) \) and \( (D_n) \) are both increasing, then \( D_n \setminus C_n \subseteq C \setminus C_n \subseteq \bigcup_{i \in \mathbb{N}} C_{n,i} \). \( K_n \setminus C_n \) is a stable subscheme of \( J_{\infty}(X) \), and so lemma 3.15 implies that there are finitely many \( C_{n,i} \) such that

\[
K_n \setminus C_n \subseteq \bigcup_{i=1}^{N} C_{n,i}.
\]

Corollary 3.13 implies that the virtual dimension of \( \mu_X(K_n \setminus C_n) \) is at least the minimum of the virtual dimensions of the \( \mu_X(C_{n,i}) \) which is at least \( n \). Therefore, \( \mu_X(K_n \setminus C_n) \in \text{DM}_{\text{gm}}^\text{eff}(k; R)(n) \). As a result,

\[
\mu_X(C_n) \to \mu_X(K_n)
\]

is an equivalence in \( \text{DM}_{\text{gm}}^\text{eff}(k; R)/\text{DM}_{\text{gm}}^\text{eff}(k; R)(n) \). Similarly, using \( D_n \setminus K_n \subseteq C \setminus K_n \subseteq C \setminus C_n \), we deduce that

\[
\mu_X(D_n) \to \mu_X(K_n)
\]

is also an equivalence in \( \text{DM}_{\text{gm}}^\text{eff}(k; R)/\text{DM}_{\text{gm}}^\text{eff}(k; R)(n) \). Consequently, we have equivalences

\[
\begin{array}{ccc}
\mu_X(C_n) & \sim & \mu_X(K_n) \\
\downarrow & & \downarrow \\
\mu_X(D_n) & \sim & \mu_X(D_n)
\end{array}
\]

in \( \text{DM}_{\text{gm}}^\text{eff}(k; R)/\text{DM}_{\text{gm}}^\text{eff}(k; R)(n) \). We can similarly construct morphisms

\[
\mu_X(C_n) \to \mu_X(C_{n+1})
\]
that are equivalences in $\text{DM}_{\text{gm}}^{\text{eff}}(k; R)/\text{DM}_{\text{gm}}^{\text{eff}}(k; R)(n)$. As a result, we have a diagram

$$
\mu_X(C_n) \sim \mu_X(C_{n+1})
$$

in $\text{DM}_{\text{gm}}^{\text{eff}}(k; R)/\text{DM}_{\text{gm}}^{\text{eff}}(k; R)(n)$. We can fill in the dotted arrows (with necessarily an equivalence in $\text{DM}_{\text{gm}}^{\text{eff}}(k; R)/\text{DM}_{\text{gm}}^{\text{eff}}(k; R)(n)$) so that the diagram commutes. We have shown that if $(C_n)$ and $(D_n)$ are both increasing, then we can find equivalences $\mu_X(C_n) \sim \mu_X(D_n)$ in $\text{DM}_{\text{gm}}^{\text{eff}}(k; R)/\text{DM}_{\text{gm}}^{\text{eff}}(k; R)(n)$ that are compatible as $n$ varies. Equivalently, we have established the existence of an equivalence $(\mu_X(C_n))_n \sim (\mu_X(D_n))_n$ in $\text{DM}_{\text{gm}}^{\text{eff}, \wedge}(k; R)$, as required. The other two cases can be easily dealt with via easy variants of this proof.

Some examples of measurable subsets of $\mathcal{J}_\infty(X)$ are stable subschemes, subschemes of the form $\mathcal{J}_\infty(Y)$ for any $Y \subseteq X$ a $k$-subvariety, and pairs of the form $(\pi_n^{-1}Z, \pi_n^{-1}S)$, where $(Z, S)$ is a constructible subset of $\mathcal{J}_n(X)$ for some $n$ with a prescribed finite decomposition $S$ into locally closed subschemes. Here, if $S = \{S_1, \ldots, S_k\}$, then $\pi_n^{-1}S := \{\pi_n^{-1}S_1, \ldots, \pi_n^{-1}S_k\}$. Another example of a measurable subset of $\mathcal{J}_\infty(X)$ is the scheme-theoretic disjoint union of countably many stable subschemes whose virtual dimensions go to $\infty$ and such that the union of the the first $N$ objects, $N$ any integer, is also a stable subscheme; see lemma 3.19. In the following lemma, we compute the measure of $\mathcal{J}_\infty(Y)$ when $Y$ is a locally closed $k$-subvariety of $X$ of dimension less than that of $X$.

**Lemma 3.18.** Suppose $Y$ is a subvariety of a smooth $k$-scheme $X$ of strictly positive codimension. Then $\mathcal{J}_\infty(Y)$ is good (and so measurable) with $\mu_X(\mathcal{J}_\infty(Y)) \simeq 0$ in $\text{DM}_{\text{gm}}^{\text{eff}, \wedge}(k; R)$.

**Proof.** Note that

$$
\mathcal{J}_\infty(Y) = \bigcap_{n=0}^{\infty} \pi_n^{-1}\mathcal{J}_n(Y).
$$

which is the intersection of the following decreasing sequence

$$
\pi_0^{-1}\mathcal{J}_0(Y) \supseteq \pi_1^{-1}\mathcal{J}_1(Y) \supseteq \ldots
$$

of stable subschemes of $\mathcal{J}_\infty(X)$ containing $\mathcal{J}_\infty(Y)$.

By the main result of [28], there is a positive integer $e$ such that for all $m$ sufficiently large,

$$
\pi_{[m/e]}\mathcal{J}_\infty(Y) = \pi_m^{-1}\mathcal{J}_m(Y).
$$

Choose the decreasing sequence

$$
\pi_0^{-1}\mathcal{J}_0(Y) \supseteq \pi_e^{-1}\mathcal{J}_e(Y) \supseteq \pi_{2e}^{-1}\mathcal{J}_{2e}(Y) \supseteq \pi_{3e}^{-1}\mathcal{J}_{3e}(Y) \supseteq \ldots
$$

We show that for each $n$, $\mu_X(\pi_{ne}\mathcal{J}_{ne}(Y)) \in \text{DM}_{\text{gm}}^{\text{eff}}(k; R)(nc)$, where $c$ is the codimension of $Y$ in $X$. We follow a well-known argument in classical motivic integration. By lemma 4.3 of [21],
\[\dim \pi_n(\mathcal{J}_\infty(Y)) \leq (n+1) \dim Y.\] As a result, for \(m > 0\)
\[
\dim \mathcal{J}_m(Y) \leq \dim \pi_{(m/e)}^m \mathcal{J}_m(Y) + (m - \lfloor m/e \rfloor) \dim X
= \dim \pi_{(m/e)}^m \mathcal{J}_\infty(Y) + (m - \lfloor m/e \rfloor) \dim X
\leq (\lfloor m/e \rfloor + 1) \dim Y + (m - \lfloor m/e \rfloor) \dim X
= (m + 1) \dim X - (\lfloor m/e \rfloor + 1) (\dim X - \dim Y)
\]

Using this inequality, for \(n > 0\)
\[
\text{vdim } \mu_X(\pi_{ne}^{-1} \mathcal{J}_{ne}(Y)) = \text{vdim } \pi_{ne} \mathcal{J}_{ne}(Y) \bigotimes_{\text{dim } \mathcal{J}_{ne}(Y)} ((ne + 1) \dim X)[2(ne + 1) \dim X]
= (ne + 1) \dim X - \dim \mathcal{J}_{ne}(Y)
\geq (ne + 1) \dim X - (ne + 1) \dim X + (n + 1)c
= (n + 1)c.
\]

Consequently, \(\mu_X(\pi_{ne}^{-1} \mathcal{J}_{ne}(Y)) \in \text{DM}_{\text{gm}}^{\text{eff}}(k; R)(nc)\). Since by assumption \(c \geq 1\), \(\mu_X(\mathcal{J}_\infty(Y)) \simeq 0\) in \(\text{DM}_{\text{gm}}^{\text{eff,\wedge}}(k; R)\).

\[\square\]

\textbf{Lemma 3.19.} Suppose \(C = \bigsqcup_{n=0}^\infty C_n\) as subschemes of \(\mathcal{J}_\infty(X)\), where \(C_n\) are stable subschemes of \(\mathcal{J}_\infty(X)\) with \(\text{vdim } \mu_X(C_n) \to \infty\) and with \(\cup_{i \leq N} C_i\) stable subscheme for each \(N\). Then \(C\) is measurable and
\[
\mu_X(C) \simeq \bigoplus_{n=0}^\infty \mu_X(C_n).
\]

\textit{Proof.} Choose the sequence \(C_0 \subseteq C_0 \cup C_1 \subseteq \ldots \subseteq C_0 \cup C_1 \cup \ldots \cup C_n \subseteq \ldots \subseteq C\) of stable subschemes of \(\mathcal{J}_\infty(X)\), and use lemma 3.14. \(\square\)

We end this subsection by making the observation that if \(C\) is a stable subscheme with a finite (locally closed) decomposition \(\mathcal{S}\) into stable subschemes \(C_i\) given by \(C = \cup_{i \in \mathcal{S}} C_i\), then \(\mu_X(C)\) is given by a sequence of extensions of \(\mu_X(C_i)\), while \(\mu_X(C, \mathcal{S}) = \bigoplus \mu_X(C_i)\) is given by the trivial extensions. Therefore, \(\mu_X(C)\) is up to a sequence of extensions, \(\mu_X(C)\) and \(\mu_X(C, \mathcal{S})\) are the same.

\section{Measurable functions, integrals, and computations.}

After the previous preparatory sections, we give here the definition of measurable functions and define our integrals. We then do some computations that allow us to anticipate the transformation rule that is at the heart of the theory of integration of Voevodsky motives.

\textbf{Definition 3.20.} A function \(F : \mathcal{J}_\infty(X) \to N_{\geq 0} \cup \{\infty\}\) is measurable if for each \(s \in N_{\geq 0}\), \(F^{-1}(s)\) is measurable and \(F^{-1}(\infty)\) has measure 0.

An important example of a measurable function to us is \(\text{ord}_Y\), where \(i : Y \to X\) is a closed \(k\)-subvariety of \(X\) of positive codimension. This function is defined as follows. The associated ideal sheaf \(\mathcal{I}_{Y/X}\) fits into a short exact sequence
\[0 \to \mathcal{I}_{Y/X} \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0.\]

Given \(\gamma \in \mathcal{J}_\infty(X)\), we may view it as a morphism \(\gamma : \text{Spec}(\gamma)[[t]] \to X\) or \(\gamma^* : \mathcal{O}_X \to k(\gamma)[[t]]\), where \(k(\gamma)\) is the residue field of \(\gamma \in \mathcal{J}_\infty(X)\). Then \(\text{ord}_Y(\gamma)\) is defined as the largest number \(e \in N_{\geq 0} \cup \{\infty\}\) such that the composition
\[
\mathcal{O}_X \xrightarrow{\gamma^*} k(\gamma)[[t]] \to k(\gamma)[[t]]/(t^e)
\]
sends \(\mathcal{I}_{Y/X}\) to zero. Note that by definition, the order function depends only on the isomorphism class of the ideal sheaf \(\mathcal{I}_{Y/X}\). The fact that \(\text{ord}_Y^{-1}(\infty)\) has measure zero when \(Y\) has strictly positive codimension in \(X\) immediately follows from lemma 3.18.
**Definition 3.21.** Given a measurable function $F : \beta_{\infty}(X) \to \mathbb{N}_{\geq 0} \cup \{\infty\}$, define the motivic integral

$$\int_{X_{\infty}} 1_k(F)[2F]d\mu_X := \left( \bigoplus_{s=0}^{\infty} \mu_X(F^{-1}(s))[2s] \right)$$

viewed as an element of $\mathcal{M}(k;R)$.

The reason it makes sense for (the image of the convergent motive) $\bigoplus_{s=0}^{\infty} \mu_X(F^{-1}(s))[2s]$ (in $\text{DM}^\text{eff,gm}(k;R)$) to be viewed as an object of $\text{DM}^\text{eff,gm}(k;R)$ is that $\mu_X(F^{-1}(s))[2s] \in \text{DM}^\text{eff,gm}(k;R)(s)$. Note that $\text{DM}^\text{eff,gm}(k;R)$ is not closed under arbitrary small colimits just as $\text{DM}^\text{eff}(k;R)$ is not. Also, note that $1_k(F)[2F]$ is an analogue of functions of the form $L^{-F}$ in classical motivic integration, and that the classical motivic integral of $L^{-F}$ is defined by

$$\int_{X_{\infty}} L^{-F}d\mu_X := \sum_{s=0}^{\infty} \mu_X(F^{-1}(s))L^{-s}$$

as an element of $\widehat{M}_k$, where $\mu_X$ here is the measure in classical motivic integration. Note that in classical motivic integration, one reason we must complete $K_0(\text{Var}_k)[L^{-1}]$ is so that we can talk about infinite sums as above. In our case, we do not complete, but work categorically so that we have a notion of infinite direct sums. This infinite direct sum cannot be taken in a presentable category like $\text{DM}^\text{eff}(k;R)$ because passing to $K_0$ would give us a value group that is zero; this is the reason we consider completed Voevodsky motives. Finally, note that both definitions of integration above are completely analogous to the way Lebesgue integration is defined in real analysis.

Of great importance to us in this paper is a formula that will allow us to understand how integrals change via resolution of singularities. In fact, in the next section, we shall prove that if $f : X \to Y$ is a proper birational morphism of smooth $k$-varieties with $K_{X/Y}$ its relative canonical divisor, and $D \subseteq Y$ is an effective divisor, then

$$\int_{Y_{\infty}} 1_k(\text{ord}_D)[2\text{ord}_D]d\mu_Y = \int_{X_{\infty}} 1_k(\text{ord}_{f^{-1}D+K_{X/Y}})[2\text{ord}_{f^{-1}D+K_{X/Y}}]d\mu_X.$$ 

This is the analogue of the transformation rule in classical motivic integration:

$$\int_{Y_{\infty}} L^{-\text{ord}_D} d\mu_Y = \int_{X_{\infty}} L^{-\text{ord}_{f^{-1}D+K_{X/Y}}} d\mu_X.$$ 

Prior to proving the transformation rule in our setting in the next subsection, we do some computations and compare them to their analogues in classical motivic integration.

**Example 3.22.** Set $X$ to be a smooth $k$-variety and $F = \text{ord}_\emptyset$. Then

$$\int_{X_{\infty}} 1_k(\text{ord}_\emptyset)[2\text{ord}_\emptyset]d\mu_X = [M(X)].$$

Indeed, $\text{ord}_\emptyset^{-1}(s) = 0$ for $s > 0$ and $\text{ord}_\emptyset^{-1}(0) = \beta_{\infty}(X)$. Therefore,

$$\int_{X_{\infty}} 1_k(\text{ord}_\emptyset)[2\text{ord}_\emptyset]d\mu_X = \left( \bigoplus_{s=0}^{\infty} \mu_X(\text{ord}_\emptyset^{-1}(s))[2s] \right) = [\mu_X(\beta_{\infty}(X))] = [M(X)]$$

in $\mathcal{M}(k;R)$. This is analogous to the calculation in classical motivic integration that

$$\int_{X_{\infty}} L^{-\text{ord}_\emptyset} d\mu_X = [X]$$

in $\widehat{M}_k$. 

Example 3.23. Suppose $X$ is a smooth $k$-variety of dimension $d$ and $Y$ is a smooth divisor. Let $n$ be a positive integer. We claim that

$$\int_{X} 1_{k}(\text{ord}_{Y} n)[2 \text{ord}_{Y} n]d\mu_{X} = [M(X \setminus Y)] + [\pi_{Y}^{*} 1_{Y}(d + n)[2(d + n)]/[M(\mathbb{P}^{n})].$$

Indeed, we have $\text{ord}_{Y}^{-1}(0) = \beta_{\infty}(X \setminus Y)$, and for $s > 0$ $\text{ord}_{Y}^{-1}(s) = \text{ord}_{Y}^{-1}([s, \infty)) = \pi_{s-1}^{-1} \beta_{s-1}(Y) \setminus \pi_{s}^{-1} \beta_{s}(Y) = \pi_{s}^{-1}((\pi_{s-1}^{-1}(\beta_{s-1}(Y)) \setminus \beta_{s}(Y))$. As a result,

$$\int_{X} 1_{k}(\text{ord}_{Y} n)[2 \text{ord}_{Y} n]d\mu_{X}$$

$$= \left[ \bigoplus_{s=0}^{\infty} \mu_{X}(\text{ord}_{Y}^{-1}(s))(ns)[2ns] \right]$$

$$= \left[ \mu_{X}(\beta_{\infty}(X \setminus Y)) \oplus \bigoplus_{s=1}^{\infty} \mu_{X}(\pi_{s}^{-1}((\pi_{s-1}^{-1}(\beta_{s-1}(Y)) \setminus \beta_{s}(Y)))((s + 1)d)[2(s + 1)d](ns)[2ns] \right]$$

$$= [M(X \setminus Y)] + \left[ \bigoplus_{s=1}^{\infty} \pi_{s}^{\cdot} \beta_{s}(Y)((s + 1)d)[2(s + 1)d](ns)[2ns] \right]$$

$$= [M(X \setminus Y)] + \left[ \bigoplus_{s=1}^{\infty} \pi_{s}^{\cdot} \beta_{s}(Y)((s + 1)d)[2(s + 1)d](ns)[2ns] \right]$$

$$= [M(X \setminus Y)] + [\pi_{Y}^{*} 1_{Y}(d - 1)[2(d - 1)] Y \otimes \bigoplus_{s=1}^{\infty} 1_{k}((n + 1)s)[2(n + 1)s][2$$

If $n = 1$ and $Y$ is a divisor, then this is analogous to the computation in classical motivic integration that

$$\int_{X} \mathbb{L}^{-\text{ord}_{Y} n}d\mu_{X} = [X \setminus Y] + [Y]/[\mathbb{P}^{1}].$$

Example 3.24. Let $\pi : \tilde{X} \to X$ be the blowup morphism of $X$ along a smooth center $Y$ of codimension $c$. Let $K_{\tilde{X}/X}$ be the canonical divisor of $\pi$. We claim that

$$\int_{\tilde{X}} 1_{k}(\text{ord}_{K_{\tilde{X}/X}} n)[2 \text{ord}_{K_{\tilde{X}/X}} n]d\mu_{\tilde{X}} = [M(X)].$$
Indeed, let $E$ be the exceptional divisor of the blowup. Then $K_{\tilde{X}/X} = (c-1)E$. Using the computations in the previous example, we obtain
\begin{align*}
\int_{\tilde{X}} 1_k(\text{ord}_{K_{\tilde{X}/X}})[2\text{ord}_{K_{\tilde{X}/X}}]d\mu_{\tilde{X}} \\
= \int_{\tilde{X}} 1_k(\text{ord}_{(c-1)E})[2\text{ord}_{(c-1)E}]d\mu_{\tilde{X}} \\
= [M(\tilde{X} \setminus E)] + [\pi^F 1_E(d + c - 1)(2(d + c - 1))] / [M(\mathbb{P}^{c-1})] \\
= [M(\tilde{X} \setminus E)] + [\pi^Y 1_Y d] / [M(\mathbb{P}^{c-1})] \\
= [M(X \setminus Y)] + [\pi^Y 1_Y d] = [M(X)].
\end{align*}
This is analogous to the formula in classical motivic integration that
\[ \int_{\tilde{X}} \mathbb{L}^{-\text{ord}_{K_{\tilde{X}/X}}} d\mu_{\tilde{X}} = [X]. \]
This example, in combination with example 3.22 gives us
\[ \int_{\tilde{X}} 1_k(\text{ord}_{K_{\tilde{X}/X}})[2\text{ord}_{K_{\tilde{X}/X}}]d\mu_{\tilde{X}} = \int_{X} 1_k(\text{ord}_0)[2\text{ord}_0]d\mu_X, \]
which is analogous to the equality
\[ \int_{\tilde{X}} \mathbb{L}^{-\text{ord}_{K_{\tilde{X}/X}}} d\mu_{\tilde{X}} = \int_{X} \mathbb{L}^{-\text{ord}_0} d\mu_X \]
in classical motivic integration. In the next subsection, we generalize this last example.

3.5. The transformation rule. The aim of this subsection is to prove the following transformation rule. Recall that we are assuming that $k$ has characteristic exponent invertible in the coefficient ring $R$.

**Theorem 3.25.** Suppose $f : X \to Y$ is a proper birational morphism of smooth $k$-varieties with $K_{X/Y}$ its relative canonical divisor, and let $D \subseteq Y$ be an effective divisor. Then
\[ \int_{Y} 1_k(\text{ord}_D)[2\text{ord}_D]d\mu_Y = \int_{X} 1_k(\text{ord}_{f^{-1}D + K_{X/Y}})[2\text{ord}_{f^{-1}D + K_{X/Y}}]d\mu_X. \]
As we will shortly see, this is an analogue of the change of variables formula in calculus. In the next section, this formula will be crucial to our applications regarding $K$-equivalent varieties.

Prior to proving the transformation rule above, we recall the first fundamental exact sequence for Kähler differentials and related topics.

Given a morphism of smooth $k$-schemes $f : X' \to X$, we have an $\mathcal{O}_{X'}$-linear morphism $f^*\Omega^1_X \xrightarrow{df} \Omega^1_{X'}$, (of locally free sheaves on $X'$) that fits into the exact sequence
\[ f^*\Omega^1_X \xrightarrow{df} \Omega^1_{X'} \to \Omega^1_{X'/X} \to 0 \]
of locally free sheaves. Suppose now that $f$ is *birational* and let $d = \dim X = \dim X'$ as before. Since $f$ is birational, $X$ and $Y$ have the same function fields. The stalk of $\Omega^1_{X'/X}$ at the generic point is isomorphic to $\Omega^1_{K(X')/K(X)} = 0$. Therefore, taking stalks at the generic point for $f^*\Omega^1_X \xrightarrow{df} \Omega^1_{X'}$, gives
a surjection of finite-dimensional vector spaces of the same dimension \((X \text{ and } X')\) are birational, and so an isomorphism. Therefore, when \(f\) is birational, we obtain the short exact sequence

\[0 \to f^*\Omega^1_X \xrightarrow{df} \Omega^1_{X'} \to \Omega^1_{X'/X} \to 0.\]

Taking the \(d\)th exterior power of \(df\), we obtain the exact sequence

\[0 \to f^*\omega_X \to \omega_{X'},\]

of line bundles, where \(\omega_X = \Omega^d_X\) and \(\omega_{X'} = \Omega^d_{X'}\) denote the canonical bundles of \(X\) and \(X'\) (over \(k\)). Tensoring by \(\omega^{-1}_{X'}\), we obtain the exact sequence

\[0 \to f^*\omega_X \otimes \omega^{-1}_{X'} \to \mathcal{O}_{X'}\]

of line bundles. Therefore, we may view \(f^*\omega_X \otimes \omega^{-1}_{X'}\) as a locally principal ideal of \(\mathcal{O}_{X'}\), which we denote by \(J_{X'/X}\). Let \(K_{X'/X}\) be the Cartier divisor given locally by the vanishing of \(J_{X'/X}\). This is called the relative canonical divisor of \(f\). Note that \(J_{X'/X}\) is locally given by the vanishing of \(\det df\).

Suppose now that \(L|k\) is a field extension. Let \(\gamma : \Spec L[[t]] \to X'\) be an \(L\)-point of \(\mathcal{O}_\infty(X')\) with \(\ord_{K_{X'/X}}(\gamma) = e\). This is equivalent to \(\gamma^*J_{X'/X} = (t^e) \subset L[[t]]\). Since \(J_{X'/X}\) is locally defined by the vanishing of \(\det df\), this is equivalent to \(\gamma^*(\det df) = (t^e)\), or equivalently, \(\det(\gamma^*(df)) = (t^e)\).

The pillar on which the proof of the transformation rule rests is the following proposition due to Denef and Loeser. From context, it will be clear what \(\pi_m\) and \(\pi_m^n\) mean; we abuse notation in the following lemma.

**Proposition 3.26.** *(Lemma 3.4 of [21])* Suppose \(f : X' \to X\) is a proper birational morphism of smooth \(k\)-varieties. Let \(C'_e := \ord^{-1}_{K_{X'/X}}(e)\), where \(K_{X'/X}\) is the relative canonical divisor of \(f\). Let \(C_e := f_\infty C'_e\). Let \(L|k\) be a field extension. Let \(\gamma \in C'_e(L)\) be an \(L\)-point, that is, a map \(\gamma^* : \mathcal{O}_{X'} \to L[[t]]\) such that \(\gamma^*(J_{X'/X}) = (t^e)\). Then for \(m \geq 2e\) one has:

(a) For all \(\xi \in \mathcal{O}_\infty(X)\) such that \(\pi_m(\xi) = f_m(\pi_m(\gamma))\), there is \(\gamma' \in J_{\infty}(X')\) such that \(f_\infty(\gamma') = \xi\) and \(\pi_{m-e}(\gamma') = \pi_{m-e}(\gamma)\). In particular, the fiber of \(f_m\) over \(f_m(\gamma_m)\) lies in the fiber of \(\pi_{m-e}\) over \(\pi_{m-e}(\gamma)\).

(b) \(\pi_m(C'_e)\) is a union of fibers of \(f_m\).

(c) The map \(f_m : \pi_m(C'_e) \to \pi_m(C_e)\) is a piecewise trivial \(A^e\)-fibration.

For the proof of the transformation rule, we also need the following two lemmas.

**Lemma 3.27.** If \(f : X' \to X\) is a proper birational morphism of smooth \(k\)-varieties, then away from a measure zero closed subscheme, \(f_\infty : \mathcal{O}_\infty(X') \to \mathcal{O}_\infty(X)\) is bijective.

**Proof.** The proof is more or less that given in classical motivic integration. Let \(Z \subset X'\) be a proper closed \(k\)-subvariety on whose complement \(f\) is an isomorphism. Given \(\gamma \in \mathcal{O}_\infty(X)\), we denote by \(k(\gamma)\) its residue field. Then \(\gamma\) can be viewed as a morphism \(\gamma : \Spec(k(\gamma)[[t]]) \to X\). We show that any \(\gamma\) which does not lie entirely in \(f(Z)\) uniquely lifts to an arc in \(X'\). We apply the valuative criterion for properness to show this. Consider the following diagram.

\[
\begin{array}{ccc}
\Spec(k(\gamma)[[t]])(0) & \xrightarrow{\gamma} & X' \setminus Z \\
\downarrow & & \downarrow \\
\Spec(k(\gamma)[[t]]) & \xrightarrow{\gamma} & X \setminus f(Z).
\end{array}
\]

By assumption, the generic point \(\Spec(k(\gamma)[[t]])(0)\) is in \(X \setminus f(Z)\). Since \(f\) is an isomorphism from \(X' \setminus Z \to X \setminus f(Z)\), this generic point lifts to \(X'\) uniquely (the upper left horizontal dashed arrow).
We can similarly define
\[ \text{ord}_{\mathcal{J}}(X') \setminus \mathcal{J}_{\infty}(Z) \to \mathcal{J}_{\infty}(X) \setminus \mathcal{J}_{\infty}(f(Z)), \]
of \( k \)-schemes is bijective.

**Lemma 3.28.** If \( f : X' \to X \) is a proper birational morphism of smooth \( k \)-varieties, then for every \( m, f_m : \mathcal{J}_m(X') \to \mathcal{J}_m(X) \) is surjective.

**Proof.** Suppose \( Z \subset X \) is a proper closed \( k \)-subvariety such that \( f : X' \setminus f^{-1}(Z) \to X \setminus Z \) is an isomorphism of schemes. From the previous lemma 3.27, we know that if \( \gamma \notin \mathcal{J}_\infty(Z) \), then it lies in the image of \( f_{\infty} \). If \( \gamma \in \mathcal{J}_m(X) \), then \( \pi_{m,1}(\gamma) \) cannot be contained entirely in \( \mathcal{J}_\infty(Z) \) because the latter has measure zero. As a result, there is an element in \( \pi_{m,1}(\gamma) \) that is in the image of \( f_{\infty} \). In particular, there is an element in \( \mathcal{J}_m(X') \) that maps to \( \gamma \in \mathcal{J}_m(X) \), and so \( f_m \) is surjective.

**Proof of the transformation rule.** Using the above proposition and lemmas, we are now ready to deduce the transformation rule. Let \( C'_\leq = \text{ord}_f^{-1}(\{0, e]\}) \) and \( C'_e = \text{ord}_f^{-1}(e) \). Note that \( \mathcal{J}_{\infty}(K_{X'/X}) \) has measure 0 by lemma 3.18. Up to (removing) a measure zero subscheme, therefore,

\[ C'_{\leq 0} \subseteq C'_{\leq 1} \subseteq \ldots \subseteq \mathcal{J}_{\infty}(X') \]
is a filtration of \( \mathcal{J}_{\infty}(X') \) by locally closed subschemes. We can refine the filtration into even smaller locally closed subschemes according to the order of contact along \( f^{-1}D \): set

\[ C'_{\leq e,k} := C'_{\leq e} \cap \text{ord}_f^{-1}(k, \infty) \] and \( C'_{e,k} := f_\infty(C'_{e,k}) \).

We can similarly define \( C_{e,k}, C_{e,k}', C'_{e,k}, \) and \( C_{e,k} \). We have the filtration (up to measure zero) of \( \text{ord}_D^{-1}(k, \infty) \):

\[ (C_{\leq 0,k}, S_{\leq 0,k}) \subseteq (C_{\leq 1,k}, S_{\leq 1,k}) \subseteq \ldots \subseteq (C_{\leq n,k}, S_{\leq n,k}) \subseteq \ldots \subseteq \text{ord}_D^{-1}(k, \infty), \]

where the \( S_{\leq e,k} \), as \( e \geq 0 \) varies, form an increasing sequence of finite decompositions of the constructable subsets \( C_{\leq e,k} \) into stable subschemes. By this assumption, we can define \( S_{e,k} := S_{\leq e,k} \setminus S_{\leq e-1,k} \). For each pair \( (k, k+1) \), we can take refinements \( S_{e,k} \) of \( S_{e,k} \) so that \( S_{e,k} = S_{e,k+1} \). Define \( S_{e,k} := S_{e,k} \setminus S_{e,k+1} \). Though the notation may suggest otherwise, the refinements chosen for the pairs \( (k, k+1) \) need not be compatible. Define \( S_{e,k} \) to be \( \bigcup_{i \geq e} S_{e,i} \).

Using proposition 3.26, we know that \( C_{e,k} \) are constructable subsets of \( \mathcal{J}_{\infty}(X) \). Note that \( f_\infty : C'_{e,k} \to C_{e,k} \) is a piecewise trivial \( \mathcal{A}_e \)-bundle by proposition 3.26. Since \( C_{e,k} \) is constructable, choose a finite decomposition into stable subschemes, say \( \mathcal{S} = \{C_i\}_i \), such that atop each \( C_i \), \( f_\infty : C'_{e,k} \to C_{e,k} \) is a trivial \( \mathcal{A}_e \)-bundle. Furthermore, \( S' := \{ f_\infty^{-1}(C_i) \}_i \) is a decomposition of \( C'_{e,k} \) into stable subschemes. We may assume without loss of generality that above each object of \( S_{e,k} \), \( f_\infty \) is an \( \mathcal{A}_e \)-bundle.

Consider the filtration

\[ (C_{\leq 0,k}, S_{\leq 0,k}) \subseteq (C_{\leq 1,k}, S_{\leq 1,k}) \subseteq \ldots \subseteq (C_{\leq n,k}, S_{\leq n,k}) \subseteq \ldots \subseteq \text{ord}_D^{-1}(k, \infty), \]

From this filtration, we obtain the tower

\[ \bigoplus_k \mu_X(C_{\leq 0,k}, S_{\leq 0,k})(k)[2k] \to \bigoplus_k \mu_X(C_{\leq 1,k}, S_{\leq 1,k})(k)[2k] \to \ldots \]

Note that \( \mu_X(\text{ord}_D^{-1}(k, \infty)) \) is a sequential colimit of measures of finite unions of objects in \( \cup_{e} S_{e,k} \). The reason this is so is because \( \text{ord}_D^{-1}(k, \infty) \) is a closed subset of \( \mathcal{J}_{\infty}(X) \), and the objects of \( \cup_{e} S_{e,k} \) are stable subschemes covering it. Indeed, the closure of each object of \( \cup_{e} S_{e,k} \) (which is locally
closed) is stable and contained in $\text{ord}^{-1}_D([k, \infty])$ which is the union of the stable subschemes in $\cup e S_{e, \geq k}$ whose virtual dimensions go to infinity. By lemma 3.15, the closure of each object of $\cup e S_{e, \geq k}$ is contained in the union of finitely many of the objects of $\cup e S_{e, \geq k}$. Furthermore, the measures of the objects of $S_{e, \geq k}$ go to 0 as $e$ goes to infinity. This implies that we can assume without loss of generality that we have an ordering on the objects of $\cup e S_{e, \geq k}$ so that $\text{ord}^{-1}_D([k, \infty])$ is a sequential colimit of measures of finite unions of the first $N$ objects in $\cup e S_{e, \geq k}$ as $N$ varies.

The above tower has slices $\bigoplus_k \mu_X(C_{e, \geq k}, S^t_{e, \geq k})(k)[2k]$ with $e$ varying. Therefore, in $M(k; R)$, we have

$$\int_{X_\infty} 1_k(\text{ord}_D)[2\text{ord}_D]d\mu_X = \bigoplus_k \mu_X(\text{ord}^{-1}_D(k))(k)[2k] = \bigoplus_k \left( \bigoplus_e \mu_X(C_{e, \geq k}, S^t_{e, \geq k})(k)[2k] \right)$$

By assumption, atop each object of $S_{e, k}$, $f_\infty$ is an $A^e$-bundle. Consequently,

$$\mu_X(C'_{e, k}, f_\infty^{-1} S_{e, k})(e)[2e] = \bigoplus_k \mu_X'(C'_e(e), e)[2e] \simeq \bigoplus_k \mu_X(C) = \mu_X(C_{e, k}, S_{e, k}).$$

From this, we obtain

$$\int_{X_\infty} 1_k(\text{ord}_D)[2\text{ord}_D]d\mu_X = \bigoplus_k \mu_X(\text{ord}^{-1}_D(k))(k)[2k] = \bigoplus_k \left( \bigoplus_e \mu_X(C_{e, k}, S_{e, k})(k)[2k] \right)$$

$$= \bigoplus_k \left( \bigoplus_{k, e} \mu_X'(C'_{e, k}, f_\infty^{-1}(S_{e, k}))(k + e)[2(k + e)] \right)$$

$$= \bigoplus_t \left( \bigoplus_{k + e = t} \mu_X'(C'_{e, k}, f_\infty^{-1}(S_{e, k}))(t)[2t] \right)$$

$$= \bigoplus_t \mu_X'(\text{ord}^{-1}_{f^{-1} D + K_{X'/X}}(t))(t)[2t]$$

$$= \int_{X_\infty} 1_k(\text{ord}_{f^{-1} D + K_{X'/X}})[2\text{ord}_{f^{-1} D + K_{X'/X}}]d\mu_X.$$. 
4. Applications

Now that we have constructed a theory of integration of Voevodsky motives, we apply it to answer a few questions related to the number theory and geometry of K-equivalent varieties. In particular, we obtain new results in the case of birational Calabi-Yau varieties and birational minimal varieties.

4.1. Classes of motives of K-equivalent varieties. In this subsection, we prove the following theorem using our theory of motivic integration.

**Theorem 4.1.** Suppose X and Y K-equivalent smooth k-varieties. Then in $K_0(\text{DM}_{gm}^{\text{eff}}(k;\mathbb{Z}[1/p]))$, $[M(X)] = [M(Y)]$.

**Proof.** Since X and Y are K-equivalent, there is a smooth k-variety Z as well as proper birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z \\
& \searrow & \downarrow \scriptstyle{g} \\
& \swarrow & \\
& Y
\end{array}
\]

Applying the transformation rule to each of $f$ and $g$, we obtain

\[
[M(X)] = \int_{X_{\infty}} 1_k(\text{ord}_{\emptyset})[2\text{ord}_{\emptyset}]d\mu_X = \int_{Z_{\infty}} 1_k(\text{ord}_{K_{Z/X}})[2\text{ord}_{K_{Z/X}}]d\mu_Z
\]

and

\[
[M(Y)] = \int_{Y_{\infty}} 1_k(\text{ord}_{\emptyset})[2\text{ord}_{\emptyset}]d\mu_Y = \int_{Z_{\infty}} 1_k(\text{ord}_{K_{Z/Y}})[2\text{ord}_{K_{Z/Y}}]d\mu_Z.
\]

Since X and Y are K-equivalent,

\[K_{Z/X} = K_Z - f^*K_X = K_Z - g^*K_Y = K_{Z/Y},\]

and so

\[
[M(X)] = \int_{Z_{\infty}} 1_k(\text{ord}_{K_{Z/X}})[2\text{ord}_{K_{Z/X}}]d\mu_Z = \int_{Z_{\infty}} 1_k(\text{ord}_{K_{Z/Y}})[2\text{ord}_{K_{Z/Y}}]d\mu_Z = [M(Y)]
\]

in $M(k;\mathbb{Z}[1/p])$. Consequently, by proposition 3.8 $[M(X)] = [M(Y)]$ in $K_0(\text{DM}_{gm}^{\text{eff}}(k;\mathbb{Z}[1/p]))$, as required. □

When X and Y are smooth projective k-varieties, we have the following stronger result.

**Theorem 4.2.** Suppose X and Y are K-equivalent smooth projective k-varieties, then there is an effective Chow motive $P \in \text{Chow}^{\text{eff}}(k;\mathbb{Z}[1/p])$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k;\mathbb{Z}[1/p])^{op} \hookrightarrow \text{DM}_{gm}^{\text{eff}}(k;\mathbb{Z}[1/p])$.

**Proof.** By proposition 3.1.3 of Bondarko’s paper [12] (characteristic $p > 0$) and theorem 6.4.2 of Bondarko’s [11] (characteristic zero), we know that the inclusion

\[\text{Chow}^{\text{eff}}(k;\mathbb{Z}[1/p])^{op} \hookrightarrow \text{DM}_{gm}^{\text{eff}}(k;\mathbb{Z}[1/p])\]

induces an isomorphism after taking $K_0$. As a consequence of theorem 4.1 and the fact that X and Y are smooth projective, $[M(X)] = [M(Y)]$ in $\text{Chow}^{\text{eff}}(k;\mathbb{Z}[1/p])$, from which it follows that there is an effective Chow motive $P \subseteq \text{Chow}^{\text{eff}}(k;\mathbb{Z}[1/p])$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k;\mathbb{Z}[1/p])^{op} \hookrightarrow \text{DM}_{gm}^{\text{eff}}(k;\mathbb{Z}[1/p])$. □

This gives a good partial answer to the following conjecture. Wang conjectured it for complex varieties, while we conjecture it in general [59].
Conjecture 4.3. If $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, then $M(X) \simeq M(Y)$ (integrally), that is, they have equivalent integral Chow motives.

Birational compact hyperkähler manifolds have isomorphic Hodge structures. In particular, they have the same Hodge and Betti numbers. Furthermore, they have isomorphic integral singular cohomology rings. However, the isomorphism between the integral singular cohomology rings is not only the result of them being $K$-equivalent smooth projective complex varieties. It is not in general true that $K$-equivalent smooth projective complex varieties have isomorphic integral cohomology rings. For example, Nam-Hoon Lee and Keiji Oguiso have jointly constructed birational Calabi-Yau complex threefolds with non-isomorphic integral singular cohomology rings [41]. It has been conjectured that if we take into account quantum corrections, then the (quantum) cohomology rings are isomorphic, that is, $K$-equivalent smooth projective complex varieties have isomorphic quantum cohomology rings in the extended Kähler moduli space. In the hyperkähler case, the usual cup product coincides with its quantum-corrected cup product which gives the isomorphism of singular cohomology rings. This begs the question of how can one bring in the data of quantum corrections into our theory of motivic integration, if possible? Admittedly, this is a very open-ended question.

In three dimensions, Reid and Mori’s classification of three dimensional singularities has allowed Kollár and Mori to completely understand the relation between birational minimal models. Via a complete classification of three dimensional flops and flips, it can be shown that three dimensional birational minimal models have isomorphic integral singular (and intersection) cohomology groups and mixed (and intersection pure) Hodge structures. These integral results point to the possibility that the above conjecture may be valid.

As a small corollary to theorem 4.2, let us mention the following which is a shadow of the proof underlying Batyrev’s proof of the equality of Betti numbers of birational Calabi-Yau complex varieties.

Corollary 4.4. If $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, then their rational numerical motives are isomorphic.

Proof. The proof is a simple consequence of theorem 4.2 and the fact that rational numerical motives form a semisimple abelian category, a theorem due to Jannsen [36].

As another corollary to theorem 4.2, we obtain the following.

Corollary 4.5. If $k$ admits resolution of singularities and $X$ and $Y$ are birational smooth projective Calabi-Yau $k$-varieties, then there is an effective Chow motive $P \in \text{Chow}^{\text{eff}}(k; \mathbb{Z}[1/p])$ such that $M(X) \oplus P \simeq M(Y) \oplus P$ in $\text{Chow}^{\text{eff}}(k; \mathbb{Z}[1/p]) \hookrightarrow \text{DM}^{\text{eff}}_\text{gm}(k; \mathbb{Z}[1/p])$.

Proof. If $X$ and $Y$ are birational smooth projective Calabi-Yau $k$-varieties, choose a common resolution of singularities (possible since $k$ admits resolution of singularities)

\[
\begin{array}{ccc}
Z & \xleftarrow{f} & X \\
\downarrow{g} & & \downarrow{}
\end{array}
\]

Since $K_X = 0$ and $K_Y = 0$, two such birational Calabi-Yau $k$-varieties are therefore $K$-equivalent.

Remark 4.6. The same corollary is true more generally for birational smooth projective $k$-varieties with nef canonical bundles.
An advantage of obtaining motivic results is that through realization functors we can deduce results of interest to geometers and number theorists. Before mentioning some concrete implications of the above motivic results, we make a few comments regarding some important realization functors.

(1) Let $\text{Vec}_{\mathbb{Q}}$ and $\text{Vec}_{\mathbb{Q}}$ be the category of finite dimensional $\mathbb{Q}$- and $\mathbb{Q}$-vector spaces. For each prime $\ell$ coprime to the characteristic of the field $k$, there is an $\ell$-adic realization functor $r_{\ell} : \text{DM}_{\text{gm}}(k; \mathbb{Q}) \to D^b(\text{Vec}_{\mathbb{Q}})$ induced by sending $M(X)$, a $k$-variety $X$, to $H^*_c(X_k; \mathbb{Q}_\ell)^*$. If $k$ is a field of characteristic zero with an embedding $\sigma : k \hookrightarrow \mathbb{C}$, we have the Betti realization functor $r_{\sigma} : \text{DM}_{\text{gm}}(k; \mathbb{Q}) \to D^b(\text{Vec}_{\mathbb{Q}})$ induced by sending $M(X)$, $X$ any $k$-variety, to $H^*_c(X^{an}; \mathbb{Q})$. There is also an integral contravariant version of the Betti realization $B_{\sigma} : \text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z})^{\text{op}} \to D^b(\mathbb{Z})$ given by sending $M(X)$ to $H^*_c(X^{an}; \mathbb{Z})$.

(2) ($\ell$-adic realization) Voevodsky’s triangulated or stable $\infty$-category of geometric effective motives $\text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z})$ is based on the Nisnevich topology, a topology coarser than the \'{e}tale topology. For each $n$, reducing modulo $\ell^n$ and sheafifying with respect to the \'{e}tale topology gives us functors

$$\text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z}) \to \text{DM}_{\text{gm}, \text{et}}^{\text{eff}}(k; \mathbb{Z}/\ell^n) \hookrightarrow \text{DM}_{\text{et}}^{\text{eff}}(k; \mathbb{Z}/\ell^n).$$

By rigidity (see Cisinski-Déglise’s [16] for the case that $k$ does not have finite cohomological dimension), the latter category is equivalent to the derived category of sheaves of $\mathbb{Z}/\ell^n$-modules on the small \'{e}tale site of $k$, that is $D(k_{\text{et}}, \mathbb{Z}/\ell^n)$. These functors assemble into the $\ell$-adic realization functor

$$\text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z}) \to \hat{D}(k_{\text{et}}; \mathbb{Z}),$$

where $\hat{D}(k_{\text{et}}; \mathbb{Z})$ is the derived category of $\ell$-adic sheaves. This latter category is equivalent to the derived category of continuous $\ell$-adic Galois representations $D(\text{Rep}_{\text{cnt}}(G_k; \mathbb{Z}_\ell))$. We know by work of Cisinski and Déglise [16], for example, that the above functor factors through the bounded derived category of constructable $\ell$-adic sheaves. We can also rationalize everything (by inverting $\ell$) if we want to work over $\mathbb{Q}_\ell$.

(3) (Hodge realization) Recall that an integral mixed Hodge structure consists of a finitely generated abelian group $V_\mathbb{Z}$ together with a finite increasing filtration $W_i$ on $V_\mathbb{Q} := V_\mathbb{Z} \otimes \mathbb{Q}$, called a weight filtration, and a finite decreasing filtration $F^p$ on $V_\mathbb{C} := V_\mathbb{Z} \otimes \mathbb{C}$, called a Hodge filtration, with some compatibility conditions. This is an abstraction of the algebraic structure one obtains from Hodge theory on integral cohomology groups of every complex algebraic variety [25]. There is an abelian category of integral mixed Hodge structures $\text{MHS}_\mathbb{Z}$. Consider the category $\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Z})$ of geometric Voevodsky motives over $\text{Spec}\mathbb{C}$. Lecomte and Wach [39] have constructed an integral Hodge realization functor

$$\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Z})^{\text{op}} \to D^b(\text{MHS}_{\mathbb{Z}}),$$

to the derived category of integral mixed Hodge structures. It sends $M(X)$ to $H^*(X^{an}; \mathbb{Z})$ with its canonical integral mixed Hodge structure. We could also rationalize to pass to (polarizable) rational mixed Hodge structures.

As a corollary to theorem 4.2, we have the following result.

**Theorem 4.7.** If $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties with $k$ of characteristic zero with $\sigma : k \hookrightarrow \mathbb{C}$, then $H^*(X^{an}; \mathbb{Z}) \simeq H^*(Y^{an}; \mathbb{Z})$ as graded abelian groups.

**Proof.** Indeed, there is the Betti realization functor $B_{\sigma} : \text{DM}_{\text{gm}}^{\text{eff}}(k; \mathbb{Z})^{\text{op}} \to D^b(\mathbb{Z})$ into the bounded derived category of finitely generated abelian groups. This is given by sending $M(X)$ to the graded abelian group $\bigoplus_n H^n(X^{an}; \mathbb{Z})[−n] \in D^b(\mathbb{Z})$. Applying theorem 4.2, we deduce that

$$H^*(X^{an}; \mathbb{Z}) \oplus H^*(B_{\sigma}(P)) \simeq H^*(Y^{an}; \mathbb{Z}) \oplus H^*(B_{\sigma}(P))$$
as finitely generated graded abelian groups. Since we are landing in finitely generated abelian groups, the classification of finitely generated abelian groups allows us to cancel $H^*(B\sigma(P))$ from both sides to obtain $H^*(X^{an};\mathbb{Z}) \cong H^*(Y^{an};\mathbb{Z})$ as graded abelian groups.

\[\square\]

**Remark 4.8.** The cohomology rings need not be isomorphic. In fact, Nam-Hoon Lee and Keiji Oguiso have jointly constructed birational Calabi-Yau complex threefolds with non-isomorphic integral singular cohomology rings [41]. For birational smooth projective Calabi-Yau complex manifolds $X$ and $Y$, Mark McLean has claimed a proof of the above theorem by identifying the quantum cohomologies of $X$ and $Y$ up to tensoring with a suitable Novikov ring. His proof uses Hamiltonian Floer cohomology.

From theorem 4.1, $X$ and $Y$ that are $K$-equivalent smooth $k$-varieties satisfy $[M(X)] = [M(Y)]$ at least in $K_0(DM_{gm}(k;\mathbb{Z}[1/p]))$, where $p$ is the exponent character of $k$. Applying the $\ell$-adic realization functor described above, we deduce that $X$ and $Y$ have the same $\ell$-adic Galois representations (up to semi-simplification). In particular, if $k = \mathbb{F}_q$, then taking traces of the Frobenius we deduce that they have the same zeta functions: $\zeta_X(t) = \zeta_Y(t)$. Summarizing, we have

**Theorem 4.9.** If $X$ and $Y$ are $K$-equivalent smooth $k$-varieties, then they have the same $\ell$-adic Galois representations (up to semi-simplification). In particular, in the case $k = \mathbb{F}_q$, two such $\mathbb{F}_q$-varieties have the same zeta functions. If $\mathbb{F}_q$ admits resolution of singularities, then all this is true for $X$ and $Y$ that are birationally equivalent smooth projective Calabi-Yau $\mathbb{F}_q$-varieties.

Using the Hodge realization functor, we deduce the following corollary of theorem 4.1.

**Corollary 4.10.** If $X$ and $Y$ are $K$-equivalent complex varieties, then they have the same classes of integral mixed Hodge structures in $K_0(MHS_{\mathbb{Z}})$. Two such varieties have the same rational mixed Hodge structures.

**Proof.** The first statement is an direct consequence of theorem 4.1, while the second part is a consequence of the semisimplicity of the abelian category of polarizable mixed Hodge structures. \[\square\]

This recovers the well-known result of Kontsevich [37] mentioned earlier (and that of Batyrev [5]) as a consequence.

### 4.2. Motivic $t$-structure, Krull-Schmidt categories, and rational motives

In this section, we show how the notion of Krull-Schmidt categories could improve theorem 4.2 above, and so prove conjecture 4.3 of Wang. We also study how the existence of a motivic $t$-structure on $DM_{gm}(k;\mathbb{Q})$ with the expected properties implies the equivalence of rational Voevodsky motives of $K$-equivalent smooth projective $k$-varieties.

Recall the following definition.

**Definition 4.11** (Krull-Schmidt category). An $R$-linear additive category $\mathcal{C}$ is said to be a Krull-Schmidt category if every object is a finite direct sum of objects with local endomorphism rings.

The Krull-Schmidt theorem says that an object in a Krull-Schmidt category has a local endomorphism ring if and only if it is indecomposable. Furthermore, it also say that any object is uniquely, up to permutation, a direct sum of indecomposable objects. Many examples of Krull-Schmidt categories come from abelian categories in which every object has finite length. A concrete example is the category of finitely generated modules over a finite $R$-algebra, where $R$ is a commutative Noetherian local complete ring (e.g. $\mathbb{Z}_l$). It is not known if the category of effective Chow motives over a field $k$ with $R$-coefficients $\text{Chow}^{\text{eff}}(k;R)$ is a Krull-Schmidt category, even if $k$ is of characteristic zero and $R = \mathbb{Q}$. When $R = \mathbb{Z}$, this is known to be false due to example 32 of Chernousov and Merkurjev in [13].
A direct consequence of theorem 4.2 is the following.

**Theorem 4.12.** Suppose \( \text{Chow}^{\text{eff}}(k; R) \) is a Krull-Schmidt category, \( k \) and \( R \) as in our conventions set at the beginning of this introduction. Then for \( X \) and \( Y \) \( K \)-equivalent smooth projective \( k \)-
varieties, the Chow motives of \( X \) and \( Y \) in \( \text{Chow}^{\text{eff}}(k; R) \) are equivalent.

When working with rational coefficients, we can approach the rational version of conjecture 4.3 from the point of view of motivic \( t \)-structures, which is what we pursue now.

Let \( \mu \) be a \( t \)-structure on \( DM_{\text{gm}}(k; \mathbb{Q}) \) with \( DM_{\text{gm}}(k; \mathbb{Q}) \leq 0 \) and \( DM_{\text{gm}}(k; \mathbb{Q}) \geq 0 \) its positive and negative parts, respectively. We denote by \( DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright} := DM_{\text{gm}}(k; \mathbb{Q}) \leq 0 \cap DM_{\text{gm}}(k; \mathbb{Q}) \geq 0 \) the heart of \( \mu \). Furthermore, let \( \mu H^* : DM_{\text{gm}}(k; \mathbb{Q}) \to DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright} \) be the cohomology functors with respect to the motivic \( t \)-structure \( \mu \). We say that \( \mu \) is a non-degenerate \( t \)-structure if the functors \( \{\mu H^a\}_a \) form a conservative family of functors.

**Definition 4.13.** \( \mu \) is said to be a motivic \( t \)-structure if it is non-degenerate and compatible with \( \otimes \) and \( r \), that is, \( \otimes \) and \( r \) are \( t \)-exact. Here, \( r \) is either \( r_{\mathbb{Q}_\ell} \) or \( r_\sigma \) as described above.

A notoriously difficult conjecture in the theory of Voevodsky motives states the following.

**Conjecture 4.14.** *(Motivic \( t \)-structure conjecture)* There is a motivic \( t \)-structure on the stable \( \infty \)-category \( DM_{\text{gm}}(k; \mathbb{Q}) \) whose heart \( DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright} \) has semisimple part the category \( \text{Num}(k; \mathbb{Q}) \) of rational numerical motives, and such that every motive has a filtration by rational numerical motives. Additionally, for each smooth projective \( k \)-variety \( X \), each \( \mu H^a M(X)_\mathbb{Q} \) is a semisimple object of the heart. In characteristic zero, this last condition follows from the existence of a motivic \( t \)-structure (proposition 1.5 of Beilinson’s [7]).

We now show a concrete consequence of the validity of this conjecture. First a proposition.

**Proposition 4.15.** Suppose the motivic \( t \)-structure conjecture is true for \( DM_{\text{gm}}(k; \mathbb{Q}) \). Then for \( X \) and \( Y \) smooth projective \( k \)-varieties such that \( [M(X)_\mathbb{Q}] = [M(Y)_\mathbb{Q}] \) in \( K_0(DM_{\text{gm}}(k; \mathbb{Q})) \), \( M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q} \).

**Proof.** Then the inclusions \( \text{Num}(k; \mathbb{Q}) \hookrightarrow DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright} \) and \( DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright} \hookrightarrow DM_{\text{gm}}(k; \mathbb{Q}) \) induce isomorphisms

\[
K_0(\text{Num}(k; \mathbb{Q})) \to K_0(DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright})
\]

and

\[
K_0(DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright}) \to K_0(DM_{\text{gm}}(k; \mathbb{Q})).
\]

The latter isomorphism follows from the theorem of the heart [4]. The second morphism has inverse given by \( [M] \mapsto \sum_a (-1)^a [\mu H^a M] \). Since \( [M(X)_\mathbb{Q}] = [M(Y)_\mathbb{Q}] \) in \( K_0(DM_{\text{gm}}(k; \mathbb{Q})) \), we obtain that

\[
\left( \oplus_{\text{odd}} \mu H^a M(X)_\mathbb{Q} \oplus \left( \oplus_{\text{even}} \mu H^a M(Y)_\mathbb{Q} \right) \right) = \left( \oplus_{\text{odd}} \mu H^a M(X)_\mathbb{Q} \oplus \left( \oplus_{\text{even}} \mu H^a M(Y)_\mathbb{Q} \right) \right)
\]

in \( K_0(DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright}) \). By assumption, for each smooth projective \( k \)-variety \( Z \), \( \mu H^a M(Z) \) is semisimple, and so a numerical motive by the assumption that the semisimple part of the heart is the category of rational numerical motives. Therefore, the last equality holds in \( K_0(\text{Num}(k; \mathbb{Q})) \). By Jannsen’s theorem [36], \( \text{Num}(k; \mathbb{Q}) \) is a semisimple abelian category, and so

\[
\left( \oplus_{\text{even}} \mu H^a M(X)_\mathbb{Q} \oplus \left( \oplus_{\text{odd}} \mu H^a M(Y)_\mathbb{Q} \right) \right) \simeq \left( \oplus_{\text{odd}} \mu H^a M(X)_\mathbb{Q} \oplus \left( \oplus_{\text{even}} \mu H^a M(Y)_\mathbb{Q} \right) \right)
\]

as numerical motives, and so also in \( DM_{\text{gm}}(k; \mathbb{Q})^{\vartriangleright} \). By proposition 1.7 of [7], this forces us to have \( \mu H^a M(X)_\mathbb{Q} \simeq \mu H^a M(Y)_\mathbb{Q} \) for every \( a \). By proposition 1.4 of Beilinson’s [7], for each smooth projective \( k \)-variety \( Z \), \( M(Z)_\mathbb{Q} \simeq \oplus_a \mu H^a M(Z)_\mathbb{Q}[-a] \). Consequently, \( M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q} \). \( \square \)
Remark 4.16. We do not need to actually identify the semisimple part of the heart with rational numerical motives to make the above proof work. We just need to have that the motivic cohomologies of smooth projective varieties are semisimple. Therefore, there is no need to invoke Jannsen’s theorem.

Consequently, we have the following theorem.

Theorem 4.17. Suppose the motivic t-structure conjecture is true for $\text{DM}_{\text{gm}}(k; \mathbb{Q})$. Then if $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, $M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q}$.

Proof. Combine proposition 4.15 and theorem 4.1. □

All of these suggest the following conjecture, which is a weaker version of conjecture 4.3 above.

Conjecture 4.18. If $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, then $M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q}$.

For rational coefficients, it is highly unexpected that the motivic t-structure conjecture above is false, and so theorem 4.17 suggests that it is very likely that this last conjecture is true. The goal for the future is to refine the construction of motivic integration in order to unconditionally prove this last conjecture and possibly its more general integral version.

4.3. Relation to D-equivalence and Orlov’s conjecture. In this section, we study the implications of our theorems for the following two important conjectures in noncommutative geometry due to Bondal-Orlov and Orlov. The first is as follows.

Conjecture 4.19 (Bondal-Orlov [8]). If $X$ and $Y$ are birationally equivalent smooth projective Calabi-Yau complex varieties, then $D^b\text{Coh}(X) \simeq D^b\text{Coh}(Y)$.

Though true for birational Calabi-Yau varieties of dimension at most 3 and birational symplectic 4-folds, this conjecture is vastly open. The latter claim can be found in a preprint of J.Wierzba [60].

Note that there is a functor

$$
\text{DM}_{\text{gm}}(k; \mathbb{Q}) \rightarrow \text{KMM}(k)_{\mathbb{Q}}
$$

sending $M(X)_\mathbb{Q}$ to $NM(X) \in \text{KMM}(k)_{\mathbb{Q}}$, where $\text{KMM}(k)_{\mathbb{Q}}$ is Kontsevich’s category of rational noncommutative motives. See [49] for details. We thus obtain the following corollary of theorem 4.17.

Corollary 4.20. Suppose the motivic t-structure conjecture is true for $\text{DM}_{\text{gm}}(k; \mathbb{Q})$. Then if $X$ and $Y$ are $K$-equivalent smooth projective $k$-varieties, $NM(X) \simeq NM(Y)$ in $\text{KMM}(k)_{\mathbb{Q}}$.

We know this to be true unconditionally if we replace K-equivalence with D-equivalence (combine proposition 1 of [8] with the functor in [48] or [49]). In light of conjecture 4.19, it is still open if K-equivalence, at least in the setting of birational smooth projective complex varieties, implies D-equivalence. What this last corollary states is that if the expected motivic t-structure on rational geometric Voevodsky motives exists, then K-equivalence implies the equivalence of noncommutative motives in the sense of Kontsevich. Consequently, many of the noncommutative cohomology theories agree for K-equivalent smooth projective varieties. The second conjecture is the relation between the bounded derived category of coherent sheaves on smooth projective complex varieties and their rational motives.

Conjecture 4.21 (Orlov [47]). If $X$ and $Y$ are smooth projective complex varieties such that $D^b\text{Coh}(X) \simeq D^b\text{Coh}(Y)$ as $k$-linear triangulated categories, then $M(X)_\mathbb{Q} \simeq M(Y)_\mathbb{Q}$ in $\text{DM}_{\text{gm}}(\mathbb{C}, \mathbb{Q})$.

Equivalences of such bounded derived categories are given by Fourier-Mukai transforms, and under certain conditions on the kernel of this transform, the latter conjecture is true [47]. However, the unconditional conjecture is still vastly open.
Remark 4.22. Conjecture 4.21 is false if we require the equivalence of integral motives in the conclusion. Indeed, there are $D$-equivalent Calabi-Yau threefolds with non-isomorphic third integral singular cohomology groups [1]. Furthermore, the converse is also false. Indeed, take $X$ to be $\mathbb{P}^2_k$ blown-up at a point and $Y$ to be $\mathbb{P}^1_k \times_k \mathbb{P}^1_k$. Both have motives $1_k \oplus 1_k(1)[2]^{\oplus 2} \oplus 1_k(2)[4]$ and are both Fano. If they were $D$-equivalent, then they would be isomorphic (use result of Bondal and Orlov [10]), which is not true.

Remark 4.23. It is a result of Balmer [3] in $\otimes$-triangular geometry that quasi-compact and quasi-separated schemes $X$ can be recovered from their $\otimes$-triangulated categories of perfect complexes $D_{\text{perf}}(X)$. This is not true if we forget the $\otimes$-structure, as shown by Mukai that there are abelian varieties $A$ such that $A \not\cong \hat{A} := \text{Pic}^0(A)$ but are $D$-equivalent. Therefore, Orlov’s conjecture states that if we forget the $\otimes$-structure on $D_{\text{perf}}(X) \simeq D^b\text{Coh}(X)$, for $X$ any smooth projective complex variety, we can at least recover the rational (Chow) motive of $X$. From another perspective, we have Gabriel’s theorem stating that if $\text{Coh}(X) \cong \text{Coh}(Y)$ as abelian categories, then $X$ and $Y$ are isomorphic [24]. Therefore, another perhaps more enlightening interpretation of Orlov’s conjecture is that considering derived equivalence on the left hand side forces us to consider equivalence in Voevodsky’s category of rational geometric motives, conjecturally a bounded derived category of mixed motives.

In any case, the combination of the above two conjectures suggests that if $X$ and $Y$ are birational smooth projective Calabi-Yau complex varieties, then $M(X)_\mathbb{Q} \cong M(Y)_\mathbb{Q}$, giving more evidence for the validity of the rational version of the conjecture of Wang.

A priori, there is no obvious connection between the existence of a motivic $t$-structure on $\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Q})$ and conjecture 4.21 of Orlov. However, a theorem of Kawamata states the following.

Theorem 4.24 (Part (2) of Theorem 2.3 of [32]). Suppose $X$ and $Y$ are $D$-equivalent smooth projective complex varieties such that $X$ is of general type ($\kappa(X) = \dim X$, i.e. maximal Kodaira dimension) or $\kappa(X, -K_X) = \dim X$. Then they are $K$-equivalent.

In combination with theorem 4.2 and theorem 4.17, we obtain the following theorems.

Theorem 4.25. Then for $X$ and $Y$ smooth projective complex varieties such that $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$, $D^b\text{Coh}(X) \cong D^b\text{Coh}(Y) \implies M(X) \oplus P \cong M(Y) \oplus P$ for some integral Chow motive $P$.

We also obtain the following conditional result.

Theorem 4.26. Suppose the motivic $t$-structure conjecture is true for $\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Q})$. Then for $X$ and $Y$ smooth projective complex varieties such that $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$, $D^b\text{Coh}(X) \cong D^b\text{Coh}(Y) \implies M(X)_{\mathbb{Q}} \cong M(Y)_{\mathbb{Q}}$.

In other words, if the motivic $t$-structure conjecture is true, then Orlov’s conjecture 4.21 is true for smooth projective complex varieties with $\kappa(X) = \dim X$ or $\kappa(X, -K_X) = \dim X$.

Remark 4.27. We remark that when $X$ and $Y$ are smooth projective complex varieties such that $X$ has ample or anti-ample canonical bundle, then a $\mathbb{C}$-linear equivalence of bounded derived categories of coherent sheaves implies an isomorphism of $X$ and $Y$. This is a theorem due to Bondal and Orlov [9].

In a good sense, the vast majority of smooth projective complex varieties are of general type. For example, in the moduli space of curves, the space of curves of Kodaira dimension $-\infty$ (genus 0) is a point, the space of curves of Kodaira dimension 0 (genus 1) is 1-dimensional, while the space of curves of genus $g \geq 2$ (general type) has dimension $3g - 3$. Also, a hypersurface of degree $d$ in $\mathbb{P}^n$ is of general type if and only if $d > n + 1$, and so most hypersurfaces are of general type.
Consequently, conditional on the existence of the expected motivic $t$-structure on $\text{DM}_{\text{gm}}(\mathbb{C}; \mathbb{Q})$, we have established conjecture 4.21 for the vast majority of smooth projective complex varieties. Most likely, any possible counterexample to Orlov’s conjecture can only be found among Calabi-Yau varieties. As mentioned at the end of the previous subsection, if we develop a more refined theory of integration for Voevodsky motives, we may be able to unconditionally prove that $K$-equivalent smooth projective varieties have equivalent (Chow) motives, in which case we will make unconditional the theorems and corollaries above that are conditional on the existence of a motivic $t$-structure or the Krull-Schmidt property. This technique of passing through $K$-equivalence to show that $D$-equivalence implies equivalence of rational motives will not work for all smooth projective varieties since $D$-equivalence does not in general imply $K$-equivalence. Uehara has provided an example of two birational smooth projective complex varieties that are $D$-equivalent but not $K$-equivalent [50].

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