Quantum vacuum interaction between two sine-Gordon kinks

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Abstract
We calculate the quantum vacuum interaction energy between two kinks of the sine-Gordon equation. Using the $TGTG$ formula, the problem is reduced to the known formulas for quantum fluctuations in the background of a single kink. This interaction induces an attractive force between the kinks in parallel to the Casimir force between conducting mirrors.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Topological solitons belong to the most interesting topics in quantum field theory as models for extended objects like strings, domain walls and baryons. The interaction of these objects, especially the scattering on one another, follows from their properties as classical fields. In addition, there is an interaction due to the vacuum fluctuations of quantum fields coupled to the background of the topological solitons. This is a vacuum quantum effect in complete analogy to the Casimir effect between conducting surfaces. The investigation of this interaction was so far quite difficult due to the inherent ultraviolet divergences present in the intermediate steps and the complicated procedure of their removal. Some years ago, the situation changed with the appearance of the new $T$-matrix (or scattering) representation of the vacuum interaction energy [1, 2]. It is also called the $TGTG$ formula and we follow this notation. It allows us to compute the Casimir force between objects of complicated shape with moderate computational effort. In [3], the method was generalized to background fields that give rise to potentials with compact support. In the present paper we are going to apply this method to the vacuum interaction of two kinks of the sine-Gordon (SG) equation.

Since the appearance of the Dashen–Hasslacher–Neveu formula in the mid-70s ([4, 5]), where the one-loop correction to the mass of the $\lambda\phi^4$ kink was first computed, much progress...

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1 The name of the $TGTG$ formula is motivated by the structure of the formula in terms of the free field propagator ($G$) and the $T$ operator of scattering theory.
has been made in developing computational methods to calculate one-loop corrections to the masses of topological solitons. Heat kernel and zeta function techniques [6–8] have been very powerful to compute one-loop corrections to the masses of topological defects even when the explicit classical solution for the topological defect is not known. A complete account for the vacuum energy in a continuous background field in terms of scattering data was given in [9]. Remarkable recent results using zeta function techniques are given in the papers by Alonso-Izquierdo, Guílarté et al [10–13].

The classical interaction between topological objects in field theory is a very well known topic, for references see [14]. As related to the present paper we mention [15, 16], where the interaction between two SG kinks was considered, the classical and quantum (in terms of collective coordinate) scattering for instance.

The interaction between classical objects like kinks receives contributions from the vacuum energy of the quantum fluctuations of these fields. For one object, these are the quantum corrections to the mass mentioned above. If one has two objects, their common mass receives similar corrections. We are interested in that part, which depends on the separation and which is responsible for the force acting between them. Of course, this part can be calculated starting from the complete quantum energy of the compound object. An example is the vacuum interaction between two delta functions in [17]. However, this makes it necessary to solve the spectral problem in the background of two kinks. This is possible, in principle, but quite cumbersome. Using the TG TG formula, this problem can be solved using the knowledge from the problem for a single object. In this way, the problem becomes much simpler, especially in the case where the scattering for a single object is known like in the SG model which we consider here.

The motivation of this work is to generalize the TG TG formula to study the quantum vacuum interaction between extended topological objects. The simplest model in which such objects appear is the SG model. The kinks of the SG model are very well known and when they are interpreted as domain walls, the quantum vacuum interaction between SG kinks can be understood as a quantum vacuum force between domain walls. Moreover, the generalization of the TG TG formula to a topological extended object will allow us in the future to study quantum vacuum forces between extended topological objects appearing in many areas of physics: cosmic strings and type II superconductor vortices (Abrikosov–Nielsen–Olesen vortices) are two significant examples from cosmology and condensed matter physics.

In the next section we introduce the necessary notation and basic formulas of the SG model. In section 3 we consider two kinks and their classical interaction. In section 3, we first define the setup for the quantum fluctuations, including the TG TG formula, and then calculate the quantum interaction. The conclusions are given in the last section.

### 2. Units and dimensions in the SG model

We are interested in the calculation of the quantum corrections to classical energy. Hence it is useful to take a system of units in which

\[ c = 1 \quad \text{and} \quad \hbar \neq 1. \]

Therefore \( L = T \) and \( M \neq L \). With this system of units, for a \((1 + 1)\)-dimensional field theory, the dimension of the Lagrangian density is \([\mathcal{L}_{1+1}] = M \cdot L^{-1}\). The most general scalar field theory is given by a Lagrangian density of the form

\[ \mathcal{L}(\bar{\phi}) = \frac{1}{2} \partial_{\mu} \bar{\phi}^{\mu} \partial^{\nu} \bar{\phi} - U(\bar{\phi}). \]

Quantities with units will be marked by an overline.

\[ ^2 \text{Quantities with units will be marked by an overline.} \]
Hence, a scalar field in a $(1+1)$-dimensional spacetime will have the units $[\phi_{1+1}] = M^{1/2} L^{1/2}$, and the energy

$$E = \int d\mathbf{x} \left[ \frac{1}{2} \left( (\partial_t \phi)^2 + (\partial_\mathbf{x} \phi)^2 \right) + U(\phi) \right]$$

has units of mass: $[E] = M$. Following the notation used in [18] the SG model is described by the Lagrangian density

$$L = \frac{1}{2} \partial \mu \phi \partial_\mu \phi + \frac{m^4}{\lambda} \left[ \cos \left( \frac{\sqrt{\lambda}}{m} \phi \right) - 1 \right].$$

In the system of units selected and taking into account that the SG model is a $(1+1)$-dimensional field theory, the constants appearing in the SG model have the following units:

$$[\lambda] = M^{-1} L^{-3}; \quad [m] = L^{-1}.$$

To perform numerical calculations it is useful to rewrite the SG model in terms of pure non-dimensional quantities $\{\phi, x, t\}$. Again following [18], we can introduce the non-dimensional quantities using the constants of the problem:

$$x \equiv m \bar{x}; \quad t \equiv m \bar{t}; \quad \phi \equiv \frac{\sqrt{\lambda}}{m} \phi.$$

(2.1)

Hence, the dimensional Lagrangian can be written as

$$\mathcal{L} = \frac{m^4}{\lambda} \mathcal{L},$$

(2.2)

$\mathcal{L}$ being the dimensionless Lagrangian written in terms of non-dimensional quantities:

$$\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) + \left( \cos(\phi) - 1 \right).$$

(2.3)

The energy functional written in terms of non-dimensional quantities takes the form

$$E = \frac{m^3}{\lambda} \int d\mathbf{x} \left[ \frac{1}{2} \left( (\partial_t \phi)^2 + (\partial_\mathbf{x} \phi)^2 \right) - \left( \cos(\phi) - 1 \right) \right]$$

(2.4)

and the action functional is now

$$S(\phi) = \frac{m^2}{\lambda} \int d\mathbf{x} dt \mathcal{L}.$$  

(2.5)

The constants of the original Lagrangian allow us to define the characteristic action and energy scale for the SG model,

$$\mathcal{S}_c = \frac{m^2}{\lambda}, \quad \epsilon_c = \frac{m^3}{\lambda}.$$

(2.6)

These characteristic scales are used to write the loop expansion for quantum fluctuations around any background field, in terms of dimensionless coefficients. Since the loop expansion is a series in powers of $\hbar$, the loop expansion of the energy of a quantum configuration ($E_Q$), written in terms of dimensionless coefficients and quantities, has the form

$$E_Q = \frac{m^3}{\lambda} \sum_{k=0}^{\infty} \lambda \left( \frac{\hbar \lambda}{m^2} \right)^k E_\lambda(\phi_Q).$$

(2.7)

In this expression, $E_0$ is the classical energy corresponding to the classical configuration that gives rise to quantum fluctuations and $E_1$ is the one-loop correction to the classical energy. At the classical level, the SG model has a dimensional coupling constant, but at the quantum level the loop expansion has a dimensionless coupling given by $\hbar \lambda / m^2$, which is assumed to be small in order to give the loop expansion a sense. Hence we assume that $\hbar \lambda / m^2 \ll 1$. Our purpose is to compute the distance dependence of one-loop correction on the force between two kinks in the large separation regime, using the $TGTG$ formula. We mention that the same dimensional analysis is valid for the $\lambda \phi^4$ since it is just a truncation of the cosine series for the potential in the SG model.
3. Kink configurations in the classical SG model

Following the notation used in [18] the dimensionless action for the SG model is given by

\[ S(\phi) = \int dx\, dt \left( \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + (\cos(\phi) - 1) \right). \] (3.1)

The equation of motion arising from the preceding action functional is the very well known SG equation

\[ \partial^2 \phi + \sin(\phi) = 0. \] (3.2)

From this equation, we obtain the kink and anti-kink solutions:

\[ \phi_K(x) = 4\arctan(e^{x}) \] (3.3)

\[ \phi_{K'}(x) = -4\arctan(e^{x}) = -\phi_K(x). \] (3.4)

The energy functional for the SG model is given by equation (2.4). For static field configurations the dimensionless energy functional can be rewritten as

\[ E[\phi] = \int dx \left[ \frac{1}{2}(\partial_\phi \phi)^2 + 2\sin^2 \left( \frac{\phi}{2} \right) \right] = \frac{1}{2} \int dx \left( \partial_\phi \phi - 2\sin \left( \frac{\phi}{2} \right) \right)^2 - 4 \left[ \cos \left( \frac{\phi}{2} \right) \right]_\infty^{\infty}. \] (3.5)

The last equality is the Bogomolnyi expression for the energy functional, and gives rise to the first-order equations for stable classical solutions with finite energy

\[ \partial_\phi \phi - 2\sin \left( \frac{\phi}{2} \right) = 0. \] (3.6)

Using the explicit expression for the energy functional, it is very easy to compute its dimensionless numerical value for a static field configuration given by the addition of two kinks with different centers of mass separated by a distance \( a \):

\[ \phi_{2K}(x; a) = \phi_K(x + a/2) + \phi_K(x - a/2). \] (3.7)

Assuming \( \phi_1(x) = \phi_K(x + a/2) \) and \( \phi_2(x) = \phi_K(x - a/2) \), and knowing that from the Bogomolny first-order equation \( \partial_\phi \phi_K = 2\sin(\phi_K/2) \) if we take into account that \( E[\phi_1] = E[\phi_2] = E_K \) (where \( E_K \) is the energy of one kink which is independent of the position of the center of mass), the classical energy for the configuration \( \phi_{2K} \) is given by

\[ E[\phi_1 + \phi_2] = 2E_K + 4 \int dx \sin \left( \frac{\phi_1}{2} \right) \sin \left( \frac{\phi_2}{2} \right) \left( 1 + \cos \left( \frac{\phi_1 + \phi_2}{2} \right) \right). \] (3.8)

Only the last term on the right-hand side of (3.8) depends on the distance between kinks. Therefore this term determines the classical interaction energy between two kinks \( \Delta, E_{\text{int}}(\phi_1, \phi_2) \):

\[ \Delta, E_{\text{int}}(\phi_1, \phi_2) = 4 \int dx \sin \left( \frac{\phi_1}{2} \right) \sin \left( \frac{\phi_2}{2} \right) \left( 1 + \cos \left( \frac{\phi_1 + \phi_2}{2} \right) \right). \] (3.9)

Using the explicit expression for the kink solution given above and simplifying, we obtain the classical interaction energy density between two kinks (see figure 1),

\[ \Delta, \epsilon_{\text{int}}(K, K) = 8 \frac{\sinh^2(x)}{\cosh^2(x + a/2) \cosh^2(x - a/2)}. \] (3.10)

Integrating the energy density we obtain the classical interaction energy (see figure 1)

\[ \Delta, E_{\text{int}}(K, K) = 16 \frac{\sinh(a) - a}{\sinh(a) (\cosh(a) - 1)}. \] (3.11)
The same calculation can be performed for the interaction between a kink and an anti-kink (note that $E_K = E_{\bar{K}}$) just replacing $\phi_K(x - a/2)$ by $-\phi_K(x - a/2) = \phi_{\bar{K}}(x - a/2)$ giving rise to the classical interaction energy density (see figure 2)

$$\Delta_c \epsilon_{\text{int}}(K, \bar{K}) = -8 \frac{\cosh^2(x)}{\cosh^2(x + a/2) \cosh^2(x - a/2)}. \quad (3.12)$$

After integrating the energy density, the classical interaction energy between a kink and an anti-kink is obtained (see figure 2):

$$\Delta_c E_{\text{int}}(K, \bar{K}) = -16 \frac{\sinh(a) + a}{\sinh(a) (\cosh(a) + 1)}. \quad (3.13)$$

We mention that equation (3.7) is not the only way to introduce two kinks which are not solutions of the equation of motion. Another possibility, aimed from the exact solution with two moving kinks, was used in [15], equation (3.1),

$$\phi_{2K} = 4 \arctan \left( e^{\alpha - \frac{\alpha}{2}} - e^{-\alpha + \frac{\alpha}{2}} \right), \quad (3.14)$$

which has, however, a larger classical energy than the configuration (3.7) considered in this paper.
4. Quantum vacuum interaction of two kinks

In the TG TG method for calculating the vacuum interaction energy the background potential is assumed to be a sum of two,

\[ V(x) = V_1(x) + V_2(x). \]  \hspace{1cm} (4.1)

Originally, it was assumed that both parts of the potential have non-intersecting compact supports. In fact, this restriction can be released. In that case one cannot reduce the problem to the scattering data but since we have explicit formulas for the scattering at any separation, this restriction is not effective in our case. In the original papers the method was used in (3+1) dimensions. So we will give the corresponding formulas for (1+1) dimensions below. These are easier, but specific since we have a scattering problem on the whole axis in place of the half-axis of a radial variable.

4.1. Vacuum interaction between extended objects in (1 + 1)-dimensional field theories

A scalar field theory over the real line is described by the dimensionless action functional

\[ S(\phi) = \int d^2 x \left( \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right). \]

If \( \Phi_0 \) is a classical field configuration, the formula for the vacuum energy of the small quantum oscillations around this classical background is given by

\[ E_0 = i \frac{1}{2} \int_0^\infty \frac{d\omega}{\pi} \text{Tr} \ln (G_\omega^{(V)}) \]  \hspace{1cm} (4.2)

(see, e.g., equation (3.112) in [19]), where \( V(x) \) is the effective potential defined by the background classical field

\[ V(x) = \left. \frac{d^2 U}{d\phi^2} \right|_{\phi_0} \]  \hspace{1cm} (4.3)

and \( G_\omega^{(V)} \) is the Green functional associated with the Schrödinger problem defined by the effective potential for the quantum fluctuations around the classical background. The Euclidean formula for the vacuum energy is given by

\[ E_0 = -\frac{1}{2} \int_0^\infty \frac{d\xi}{\pi} \text{Tr} \ln (G_\xi^{(V)}). \]  \hspace{1cm} (4.4)

**Remark on the Euclidean formulation.** When the quantum mechanical system for one-particle states arising from the quantum field theory has bound states we must be careful in all the manipulations that involve free field operators. All calculations must be done in the Euclidean rotated system in order to avoid complicated expressions in which operators acting on different Hilbert spaces are multiplied (these multiplications are not defined). When the Euclidean rotation is taken all the spectra are continuous and the only restriction we must take into account is the possibility of a lower bound for the imaginary frequency \( \xi \).

In order to use just a potential that goes to 0 when \( x \to \pm \infty \) we define the scattering potential \( \tilde{V}(x) \) in terms of \( m^2 \equiv \lim_{x \to \pm \infty} V(x) \) as

\[ \tilde{V}(x) \equiv V(x) - m^2. \]  \hspace{1cm} (4.5)

3 The original TG TG method was developed to compute the vacuum interaction energy between dielectric bodies of arbitrary shape. In this picture each of those objects was represented by a classical potential with compact support.
The Euclidean Schrödinger problem associated with the one-particle states of the field theory is described by the differential equation
\[
\left( \xi^2 - \frac{d^2}{dx^2} + \tilde{V}(x) + m^2 \right) \phi_\xi(x) = 0, \tag{4.6}
\]
and the kernel \( G^{(V)}_{\xi \xi}(x, y) \) of \( G^{(V)}_{\xi \xi} \) is given by the equation
\[
\left( \xi^2 - \frac{d^2}{dx^2} + \tilde{V}(x) + m^2 \right) G^{(V)}_{\xi \xi}(x, y) = \delta(x - y). \tag{4.7}
\]

**Observation.** Note that the Green functions for the potentials \( V(x) \) and \( \tilde{V}(x) \) are related by the equality
\[
G^{(V)}_{\xi \xi}(x, y) = G^{(\tilde{V})}_{\xi \xi}(x, y) \quad \text{and} \quad \xi = \sqrt{\xi^2 + m^2}. \tag{4.8}
\]

Taking this into account from now on we will only manipulate the Green function associated with \( \tilde{V}(x) \) and keep in mind that the vacuum energy for the background defined by \( V(x) \) can be computed in terms of the Green function associated with \( \tilde{V}(x) \) changing the integration interval in \( \xi \):
\[
E_0[V] = -\frac{1}{2} \int_m^{\infty} \text{d}\xi \frac{\xi}{\sqrt{\xi^2 - m^2}} \text{Tr} \ln (G^{(V)}_{\xi \xi}). \tag{4.9}
\]

To obtain an expression for the kernel \( G^{(V)}_{\omega \omega}(x, y) \) it is useful to go back to the non-Euclidean theory and study the standard scattering problem. If \( u_k(x) \) and \( v_k(x) \) are two independent solutions of the problem defined by
\[
\left( -\frac{d^2}{dx^2} + \tilde{V}(x) \right) \psi(x) = k^2 \psi(x), \tag{4.10}
\]
then the kernel \((\omega^2 = k^2)\) for the scattering potential \( \tilde{V} \) \( G^{(V)}_{\omega \omega}(x, y) \) of \( G^{(V)}_{\omega \omega} \) is given by
\[
G^{(V)}_{\omega \omega}(x, x') = -\frac{u_k(x) v_k(x')}{W[u_k, v_k]}, \tag{4.11}
\]
where \( W[u_k, v_k] = u_k(x) v'_k(x) - u'_k(x) v_k(x) \) being the Wronskian of the two solutions. From scattering theory, if \( k \) is in the continuum spectrum, then \( u_k(x) \) and \( v_k(x) \) correspond to the right- and left-handed scattering solutions, whose asymptotic behavior is given by

\[
u_k(x) = \begin{cases} e^{i k x} + r_r e^{-i k x}, & x \to -\infty, \\ i e^{i k x}, & x \to \infty, \end{cases}
\]
\[
u_k(x) = \begin{cases} l_l e^{-i k x}, & x \to -\infty, \\ i e^{i k x} + e^{-i k x}, & x \to \infty. \end{cases}
\]

Using the Lippmann–Schwinger equations we can write the differential equation for the propagator \( G^{(V)}_{\xi \xi}(x, x') \) in an integral form, in terms of the potential \( \tilde{V}(x) \) and the free Green function \( G^{(0)}_{\xi \xi}(x, x') \):
\[
G^{(V)}_{\xi \xi}(x, x') = G^{(0)}_{\xi \xi}(x, x') - \int \text{d}x_1 G^{(0)}_{\xi \xi}(x, x_1) \tilde{V}(x_1) G^{(V)}_{\xi \xi}(x_1, x'). \tag{4.14}
\]

\footnote{In order to simplify the notation, from now on and unless it induces errors, we will denote the Euclidean rotated operator kernels and functions just with the subindex \( \xi \) instead of the subindex \( i\xi \). We will use the subindex \( \omega \) to denote the real frequencies, and hence we will refer to the standard theory (non-Wick rotation) instead of its Euclidean version.}

\footnote{Observe that since \( u_k(x) \) and \( v_k(x) \) are independent solutions with the same eigenvalue, their Wronskian is constant and non-zero [20].}
Defining the $T$ operator for the potential $\tilde{V}$ in coordinates as

$$G_\xi^{(\tilde{V})}(x,x') = G_\xi^{(0)}(x,x') - \int dx_1 \, dx_2 \, G_\xi^{(0)}(x,x_1)T_\xi^{(\tilde{V})}(x_1,x_2)G_\xi^{(0)}(x_2,x'),$$

one deduces that

$$\tilde{V}(x)G_\xi^{(\tilde{V})}(x,x') = \int dx_1 T_\xi^{(\tilde{V})}(x,x_1)G_\xi^{(0)}(x_1,x'),$$

and, in operator notation,

$$G_\xi^{(\tilde{V})} = G_\xi^{(0)} - G_\xi^{(0)} \cdot T_\xi \cdot G_\xi^{(0)} = G_\xi^{(0)} - G_\xi^{(0)} \cdot \tilde{V} \cdot G_\xi^{(0)},$$

$$\Rightarrow G_\xi^{(\tilde{V})} = \frac{1}{1 + G_\xi^{(0)} \cdot \tilde{V}} \cdot G_\xi^{(0)} = G_\xi^{(0)} \cdot \frac{1}{1 + \tilde{V} \cdot G_\xi^{(0)}},$$

(4.17)

Suppose now that $\tilde{V}(x)$ is split into two parts, $\tilde{V}(x) = \tilde{V}_1(x) + \tilde{V}_2(x)$. In the next section we are going to apply this splitting to the potential of two kinks,

$$V(x) = V_1(x) + V_2(x) = m^2 + \tilde{V}_1(x) + \tilde{V}_2(x),$$

(4.18)

where $m^2 = m_1^2 + m_2^2$ with $m_i \equiv \lim_{x \to \infty} V_i(x)$. Using the preceding formula, we can easily write

$$G_\xi^{(\tilde{V}_1 + \tilde{V}_2)} = \frac{1}{1 + \tilde{V}_1 \cdot G_\xi^{(0)}} \cdot \left(1 - \frac{1}{1 + \tilde{V}_2 \cdot G_\xi^{(0)}} \cdot \tilde{V}_1 \right) \cdot \frac{1}{1 + \tilde{V}_2 \cdot G_\xi^{(0)}} \cdot \tilde{V}_2 \cdot G_\xi^{(0)},$$

(4.19)

Introducing expression (4.19) into equation (4.9) we obtain an expression for the vacuum energy of a classical background given by a potential $V(x)$ that splits into two parts $V(x) = V_1(x) + V_2(x)$,

$$E_0 = -\frac{1}{2} \int_{m}^{\infty} \frac{d\xi}{\pi} \frac{\xi}{\sqrt{\xi^2 - m^2}} \left(\text{Tr} \ln \left(\frac{G_\xi^{(0)}}{1 + G_\xi^{(0)} \cdot V_1}\right) + \text{Tr} \ln \left(\frac{G_\xi^{(0)}}{1 + G_\xi^{(0)} \cdot V_2}\right) - \text{Tr} \ln \left(G_\omega^{(0)}\right) - \text{Tr} \ln \left(1 - M_\xi\right)\right),$$

(4.20)

where the operator $M_\xi$ is defined as

$$M_\xi = N_\xi^{(1)} \cdot N_\xi^{(2)}; \quad N_\xi^{(i)} = \frac{1}{1 + G_\omega^{(0)} \cdot \tilde{V}_i} \cdot G_\omega^{(0)} \cdot \tilde{V}_i.$$

(4.21)

Looking at equation (4.20) one notices that the first three terms do not depend on the distance between the objects that the potentials $V_1(x)$ and $V_2(x)$ represent. Hence, they will not contribute to the Casimir force between them, i.e. they do not enter in the vacuum interaction between the objects represented by $V_1(x)$ and $V_2(x)$. Therefore the last term of equation (4.20) gives the vacuum interaction energy between the objects represented by $V_1(x)$ and $V_2(x)$,

$$E_{\text{int}}^{(0)} = \frac{1}{2} \int_{m}^{\infty} \frac{d\xi}{\pi} \frac{\xi}{\sqrt{\xi^2 - m^2}} \text{Tr} \ln (1 - M_\xi).$$

(4.22)

Knowing that in operator notation equation (4.16) takes the form $\tilde{V} \cdot G_\xi^{(0)} = T_\xi^{(\tilde{V})} \cdot G_\xi^{(0)}$, we can rewrite the operator $M_\xi$ just in terms of the two $T$-operators (one for each potential), and the vacuum Green functions using the relation

8
for the $\mathcal{N}^{(i)}_{\xi}$ operators.

The operator notation has been very useful to obtain a general expression for the vacuum interaction energy. However, in order to explicitly compute the quantum vacuum interaction energy between two extended objects in one-dimensional quantum field theory, we must give an expression for the energy in terms of the kernels for the corresponding operators. As usual, we will denote the kernel of an operator $K$ by the corresponding capital letter $K(x,x')$. Hence,

$$M_{\xi}(x,x') = \int d\xi' N^{(1)}_{\xi}(x,x') N^{(2)}_{\xi}(x',x') \quad (4.24)$$

and

$$N^{(i)}_{\xi}(x,x') = \int d\xi' G^{(0)}_{\xi}(x,x') T^{(i)}_{\xi}(x',x'). \quad (4.25)$$

Since we are in the perturbative regime of the field theory we can assume that $||\mathcal{M}_{\xi}|| < 1$, and hence write $6 \ln(1 - \mathcal{M}_{\xi}) \simeq -\mathcal{M}_{\xi} - \mathcal{M}_{\xi}^2/2 + \mathcal{O}(||\mathcal{M}_{\xi}||^3)$. Under this assumption we can write the interaction energy as

$$E^{(0)}_{\text{int}} \simeq -\frac{1}{2} \int^{\infty}_{m} \frac{d\xi}{\pi} \frac{\xi}{\sqrt{\xi^2 - m^2}} \text{Tr}(\mathcal{M}_{\xi}) - \frac{1}{4} \int^{\infty}_{m} \frac{d\xi}{\pi} \frac{\xi}{\sqrt{\xi^2 - m^2}} \text{Tr}(\mathcal{M}_{\xi}^2)$$

$$= -\frac{1}{2} \int^{\infty}_{m} \frac{d\xi}{\pi} \frac{\xi}{\sqrt{\xi^2 - m^2}} \int dx M_{\xi}(x,x)$$

$$- \frac{1}{4} \int^{\infty}_{m} \frac{d\xi}{\pi} \frac{\xi}{\sqrt{\xi^2 - m^2}} \int dx \int dx' M_{\xi}(x,x') M_{\xi}(x',x) + \mathcal{O}(||\mathcal{M}_{\xi}||^3). \quad (4.26)$$

Assuming

$$E^{\text{int}}_n = -\int^{\infty}_{m} \frac{d\xi}{\pi n} \frac{\xi}{\sqrt{\xi^2 - m^2}} \text{Tr}(\mathcal{M}^{(n)}_{\xi}), \quad (4.27)$$

we can write the whole series expansion in powers of $\mathcal{M}_{\xi}$ as $E^{(0)}_{\text{int}} = \sum_{n=1}^{\infty} E^{\text{int}}_n/2$. Using expressions (4.24) and (4.25), the general term of the series expansion in powers of $\mathcal{M}_{\xi}$ can

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6 The condition $||\mathcal{M}_{\xi}|| < 1$ is closely related to the existence of bound states in the ordinary quantum mechanical system for the one-particle states of the corresponding quantum field theory.
be written as

\[ E_{\text{int}} = -\frac{1}{n} \int_{m}^{\infty} \frac{\xi}{\sqrt{\xi^2 - m^2}} \int d\mathbf{x}_n d\mathbf{y}_n dt_n \delta(x_1 - x) \delta(x_{n+1} - x) \]

\[ \times \prod_{k=1}^{n} G^{(0)}_\xi(x_k, y_k) T^{(1)}_\xi(y_k, z_k) G^{(0)}_\xi(z_k, t_k) T^{(2)}_\xi(z_k, x_{k+1}), \]

(4.28)

where \( d\mathbf{x}_n = dx_1 \ldots dx_{n+1} \) whereas \( d\mathbf{y}_n = dy_1 \ldots dy_n \) and similarly for \( dt_n \) and \( dz_n \). Therefore with this series expansion the loop expansion for the interaction energy up to first order in \( \hbar \lambda/m^2 \) is written as

\[ E_{\text{int}} = -\frac{m^3}{\lambda} \left[ \Delta_n E_{\text{int}} + \frac{h \lambda}{2m^2} \left( E_{1\text{int}}^\prime + E_{2\text{int}}^\prime + O(\mathcal{M}_\xi \parallel_3) \right) + O \left( \frac{h \lambda}{m^2} \right) \right]. \]

(4.29)

For our purpose, the case in which the potentials \( V_1 \) and \( V_2 \) are the same potential with a given displacement, i.e. \( V_2(x) = V_1(x - a) \), is of special interest. In this case we have

\[ N^{(2)}_\xi(x, x') = N^{(1)}_\xi(x - a, x' - a) \]

(4.30)

and taking into account that for the free field propagator \( G^{(0)}_\xi(x, x') = G^{(0)}_\xi(x - x') \) holds, we can immediately write the two first terms of the vacuum interaction energy as

\[ E_{1\text{int}} = -\frac{1}{2m} \int B_{\xi} \left( \frac{\xi}{\sqrt{\xi^2 - m^2}} \int d\mathbf{y} N^{(1)}_\xi(x, y) N^{(1)}_\xi(y - a, x - a) \right) \]

(4.31)

\[ E_{2\text{int}} = -\frac{1}{2m} \int B_{\xi} \left( \frac{\xi}{\sqrt{\xi^2 - m^2}} \int d\mathbf{y} dy_2 dy_3 N^{(1)}_\xi(x, y_1) N^{(1)}_\xi(y_1 - a, y_2 - a) \right) \]

\[ \times N^{(1)}_\xi(y_2, y_3) N^{(1)}_\xi(y_3 - a, x - a), \]

(4.32)

where now

\[ N^{(1)}_\xi(x, x') = \int dy G^{(0)}_\xi(x - y) T^{(1)}_\xi(y, x') = G^{(0)}_\xi(x, x') V_1(x). \]

(4.33)

Hence, using the last equality in expression (4.33), we can rewrite each term of the vacuum interaction energy in terms of the propagator for the potential \( V_1(x) \) also. Also, it should be mentioned at this point that these expressions allow us to perform numerical computations in a very friendly way because the calculation of vacuum energy with this method requires only performing a numerical integration.

4.2. TGTG calculation for the vacuum energy between two kinks: first order

In order to be able to obtain an expression for the quantum vacuum interaction energy between two kinks or a kink and an anti-kink, we must first compute the propagator. The potential that governs the dynamical behavior (up to second order) of the small fluctuations around the kink and anti-kink configuration is the well-known transparent Pöschl–Teller potential,

\[ V(x) = 1 - \frac{2}{\cosh^2(x)} \Rightarrow \tilde{V}(x) = -\frac{2}{\cosh^2(x)}. \]

(4.34)

The lower bound in the integrations over the Euclidean frequency in this particular case will be given by \( m = \sqrt{2} \) since each Pöschl–Teller potential gives a contribution of 1 to \( m^2 \). This

7 The spectrum of small quantum fluctuations around the kink and the anti-kink configuration is governed by the same potential: the Pöschl–Teller transparent potential. This means that at the one-loop level both configurations have the same quantum properties. Therefore, the one-loop correction for the mass of the kink and the anti-kink is the same, as well as the quantum vacuum interaction between the kink and the anti-kink, and between two kinks.
potential has been deeply studied in many areas of physics, and the Schrödinger problem arising for the corresponding scattering potential

\[
\left( -\frac{d^2}{dx^2} - \frac{2}{\cosh^2(x)} \right) \psi_k = \lambda_k \psi_k
\]

is perfectly known and solved. The corresponding eigenfunctions and spectrum are given by

\[
\psi_{\pm k} = e^{\pm ik} (\tanh(x) \mp ik), \quad \lambda_k = k^2 \quad \text{(scattering states)},
\]

\[
\psi_{\pm \xi} = e^{\mp \xi} (\tanh(x) \pm k), \quad \lambda_{\xi} = -1 \quad \text{(semi-bounded state)}.
\]

The corresponding Wronskian of two independent scattering solutions with the same eigenvalue is given by

\[
W(\psi_k, \psi_{-k}) = \psi_k \psi'_{-k} - \psi'_{k} \psi_{-k} = -2ik(1 + k^2) = W(k).
\]

For \(k^2 = 0\) and \(k^2 = -1\), there is only one independent solution. For the other cases we have two independent solutions for a given eigenvalue \(k^2\). The Fourier projection of the Green function in the time coordinate is given by

\[
G_\omega^{\psi}(x, y) = \frac{1}{W(k)} \left( \theta(x - y) \psi_k(x) \psi_{-k}(y) + \theta(y - x) \psi_k(y) \psi_{-k}(x) \right),
\]

where \(\theta(x)\) is the Heaviside step function. From the general expression using the eigenfunctions written above, we obtain the expression for the Green function of the Pöschl–Teller potential,

\[
G_\omega^{\psi}(x, y) = -\frac{e^{i|y-x|}}{2ik(k^2 + 1)} (\tanh(x) \tanh(y) + i|\tanh(x) - \tanh(y)| + k^2),
\]

and its Euclidean version,

\[
G_\xi^{\psi}(x, y) = -\frac{e^{-\kappa|y-x|}}{2\kappa(\kappa^2 - 1)} (\tanh(x) \tanh(y) - \kappa|\tanh(x) - \tanh(y)| - \kappa^2),
\]

with \(\kappa^2 \equiv \xi^2\). Introducing expression (4.41) in the expression for \(N_1\) given in the last equality of equation (4.33), we obtain from equations (4.31) and (4.32) two integral expressions for the first two contributions to the vacuum energy that can be computed numerically. The kernel of the operator \(N_{\xi}^{(K)}\) for a single kink is obtained from the last equality in equation (4.33),

\[
N_{\xi}^{(K)}(x, y) = \frac{\text{sech}^2(x) e^{-\kappa|y-x|} (-\kappa|\tanh(x) - \tanh(y)| + \tanh(x) \tanh(y) - \kappa^2)}{\kappa(\kappa^2 - 1)}.
\]

Using the expression for \(N_{\xi}^{(K)}\) given in the preceding equation, it is very easy to compute numerically the first two contributions to the vacuum energy by using formulas (4.31) and (4.32). In order to obtain the correct result we must take into account that the asymptotic value of the Pöschl–Teller potential is 1 so the integration interval over \(\xi\) is \((\sqrt{2}, \infty)\). After numerical calculations we obtain the behavior shown in figure 4. As can be seen, the contribution of \(E_{1\text{int}}^{\text{int}}\) is much smaller than \(E_{1\text{int}}^{\text{ext}}\). This fact means that in this case the series expansion in powers of operator \(M\) is convergent, and hence \(||M_{\xi}|| < 1\) justifies the expansion of the logarithm.

\[8\] Note that the expressions of the Green functions for the Pöschl–Teller potential with mass term \((V(x) = 1 - 2\cosh^{-2}(x))\) will be exactly the same as the ones we have in expressions (4.40) and (4.41) in terms of \(k\). The difference would be in the dispersion relation: for the case of the Pöschl–Teller potential with mass term, the dispersion relation is given by \(\omega^2 = k^2 + 1\) for the non-Euclidean case and \(\xi^2 = k^2 - 1\) for the Euclidean case.
5. Conclusions and outlook

In the foregoing section we have computed the quantum vacuum interaction of two kinks of the SG equation. We have extended the TGTG method to compute the corresponding interaction energy of non-compact objects represented by smooth classical backgrounds. This allowed us to reduce the problem, which initially required the calculation of the spectrum of the quantum fluctuation in the background of two kinks. That is possible, at least numerically, but constitutes a quite complicated problem. In contrast, for the TGTG method, it is sufficient to know the quantum corrections in the background of a single kink. For this problem, explicit formulas are available. In this way, the final formula, equation (4.29) with (4.31), (4.32) and (4.42) inserted, involves threefold, respectively fivefold, integrals over explicit functions. We find an attractive force.

The TGTG formula gives exact results for non-overlapping potentials. In our case, the potentials do overlap. However, since the potentials reach their asymptotic values exponentially fast, the overlap is exponentially small as soon as the separation becomes larger than the size of the kinks. Therefore our results are a good approximation for such separations, for the large ones especially: in the SG model the two-kink configuration is asymptotically given by $\phi_2(x; a)$ (section 3) for large kink separations. For this reason our approximation is good for large kink separations.

In contrast to the Casimir effect for conductors, where there is no classical force between the mirrors, here we have a classical force between the kinks. It can be calculated easily, see equation (3.11) for instance. For large separation its energy decreases according to

$$\Delta E_{\text{int}}(K, K) \sim 32 e^{-a}. \quad (5.1)$$

For the quantum interaction we observe the same exponential decrease, with a smaller factor in front\(^9\), however. For all separations, the classical force between two kinks is repulsive and the quantum force is attractive, but small. This can be seen in figure 5.

Further we mention that the quantum correction is small on the background of the classical part. There are two sources for smallness. The one is the factor in front, see equation (2.7). The second is the numerical smallness. Here one needs to compare the classical energy shown in figure 3, with the quantum one shown in figure 4, which is smaller by a numerical factor of approximately 6. This feature is quite similar to the one with the quantum correction to the mass of the kink. Another remark concerns the expansion of the logarithm in (4.22). We

\(^9\) The numerical values for the factors appearing in front of the exponential decay of $E_1^{\text{int}}$ and $E_2^{\text{int}}$ are $-1.1436/\pi$ and $-0.0274/\pi$, respectively.
calculated the first two orders of its expansion. We have seen that the second order is more than one order smaller than the first one. Since the expansion of the logarithm is equivalent to a perturbative expansion in powers of the background field, we can conclude that such a perturbative expansion works well in the considered example.

We conclude with the remark that the techniques demonstrated in this paper have the potential to simplify calculations of the quantum interaction for more complicated objects like strings and other extended and topological objects.

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