How to Find New Characteristic-Dependent Linear Rank Inequalities using Binary Matrices as a Guide

Victor Peña* and Humberto Sarria†
Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

In Linear Algebra over finite fields, a characteristic-dependent linear rank inequality is a linear inequality that holds by ranks of subspaces of a vector space over a finite field of determined characteristic, and does not in general hold over other characteristics. In this paper, we show a method to produce these inequalities using binary matrices with suitable ranks over different fields. In particular, for each \( n \geq 7 \), we produce \( 2 \left\lfloor \frac{n-1}{2} \right\rfloor - 4 \) characteristic-dependent linear rank inequalities over \( n \) variables. Many of the inequalities obtained are new but some of them imply the inequalities presented in [1,8].

Keywords: Linear rank inequality, complementary vector space, binary matrix.

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1 Introduction

A linear rank inequality is a linear inequality that is always satisfied by ranks (dimensions) of subspaces of a vector space over any field. Information inequalities are a sub-class of linear rank inequalities [9]. The Ingleton inequality is an example of a linear rank inequality which is not information inequality [7]. Other inequalities have been presented in [3,7]. A characteristic-dependent linear rank inequality is as a linear rank inequality but this is always satisfied by vector spaces over fields of certain characteristic and does not in general hold over other characteristics. In Information Theory, especially in linear network coding, all these inequalities are useful to calculate the linear capacity of communication networks [11]. It is remarkable that the linear capacity of a network depends on the characteristic of the scalar field associated to the vector space of the network codes, as an example, the Fano network [2,4]. Therefore, when we study linear capacities over specific fields, characteristic-dependent linear rank inequalities are more useful than usual linear rank inequalities.

Characteristic-dependent linear rank inequalities have been presented by Blasiak, Klemberg and Lubetzky [1], Dougherty, Freiling and Zeger [4], and E. Freiling [5]. The technique used by Dougherty et al. to produce these inequalities used as a guide the flow of some matroidal network to obtain restriction over linear solubility of these and it is different from the technique used by Blasiak et al. which is based on the dependency relations of the Fano and non-Fano Matroids. In [8], we show some inequalities using the ideas of Blasiak and present some applications to network coding that improve some existing results in [11,5].

Organization of the work and contributions. We show a general method to produce characteristic-dependent linear rank inequalities using as a guide binary matrices with suitable rank over different fields. We try to find as many inequalities as the method can produce: For each \( n \geq 7 \), we explicitly produce \( 2 \left\lfloor \frac{n-1}{2} \right\rfloor - 4 \) characteristic-dependent linear rank inequalities in \( n \) variables of which half are true over characteristics in sets of primes of the form \( \{ p : p \mid t \} \) and the other half are true over characteristics in sets of primes of the form \( \{ p : p \notmid t \} \), where \( 2 \leq t \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \), but we note

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* e-mail: vbpenam@unal.edu.co
† e-mail: hsarriaz@unal.edu.co
that more inequalities can be produced. Also, for the first class of inequalities, we prove that all are independent of each other and they can not be recovered from any of our inequalities in a greater number of variables. We remark that to date such number of inequalities of this type in \( n \) variables were not known. In addition, the inequalities presented in [8] can be recovered when \( n \) is of the form \( 2m + 3 \) and \( t \) is equal to \( m \).

## 2 Entropy in Linear Algebra

Let \( A, A_1, \ldots, A_n, B \) be vector subspaces of a finite dimensional vector space \( V \) over a finite field \( \mathbb{F} \). Let \( \sum_{i \in I} A_i \) denote the span or sum of \( A_i \). The sum \( A + B \) is a direct sum if and only if \( A \cap B = O \), the notation for such a sum is \( A \oplus B \). Subspaces \( A_1, \ldots, A_n \) are called mutually complementary subspaces in \( V \) if every vector of \( V \) has a unique representation as a sum of elements of \( A_1, \ldots, A_n \). Equivalently, they are mutually complementary subspaces in \( V \) if and only if \( V = A_1 \oplus \cdots \oplus A_n \). In this case, \( \pi_S \) denotes the canonical projection function \( V \to \bigoplus_{i \in S} A_i \). \( \{ e_i \} \) is the canonical bases in \( V \) and \( e_S \) is the vector whose inputs are 1 in the components in \( S \) and 0 in another case.

There is a correspondence between linear rank inequalities and information inequalities associated to certain class of random variables induced by vector spaces [9, Theorem 2], we explain that: Let \( f \) be chosen uniformly at random from the set of linear functions from \( V \) to \( \mathbb{F} \), for \( A_1, \ldots, A_n \) define the random variables \( X_1 = f \mid_{A_1}, \ldots, X_n = f \mid_{A_n} \), then

\[
H(X_i : i \in I) = \log |\mathbb{F}| \dim \left( \sum_{i \in I} A_i \right),\quad I \subseteq [n] := \{1, \ldots, n\}.
\]

The difference between entropy and dimension is a fixed positive multiple scalar. Therefore, any inequality satisfied by entropies, it is an inequality satisfied by dimensions of vector spaces; for simplicity, we identify these parameters, i. e. the entropy of \( A_1, \ldots, A_n \) is

\[
H(A_i : i \in I) \overset{\text{def.}}{=} \dim \left( \sum_{i \in I} A_i \right).
\]

So, we can think \( A_1, \ldots, A_n \) as a tuple of random variables induced in described form, such random variables are called linear random variables over \( \mathbb{F} \).

The mutual information of \( A \) and \( B \) is given by \( I(A; B) = \dim (A \cap B) \). If \( B \) is a subspace of a subspace \( A \), then we denote the codimension of \( B \) in \( A \) by \( \text{codim}_A(B) := H(A) - H(B) \). We have that \( H(A | B) = \text{codim}_A(A \cap B) \). In a similar way conditional mutual information is expressed.

We formally define the inequalities that concern this paper:

**Definition 1.** Let \( m \) be a positive integer, let \( P \) be a set of primes, and let \( S_1, \ldots, S_k \) be subsets of \( \{1, \ldots, m\} \). Let \( \alpha_i \in \mathbb{R} \) for \( 1 \leq i \leq k \). A linear inequality of the form

\[
\sum_{i=1}^{k} \alpha_i H(A_j : j \in S_i) \geq 0
\]

- is called a characteristic-dependent linear rank inequality if it holds for all jointly distributed linear random variables \( A_1, \ldots, A_m \) over finite fields with characteristic in \( P \).
- is called a linear rank inequality if it is a characteristic-dependent linear rank inequality with \( P \) is equal to the collection of all prime numbers.
- is called an information inequality if the inequality holds for all jointly distributed random variables.

The following inequality is the first linear rank inequality which is not information inequality.
Example 2. (Ingleton’s inequality [6]) For any $A_1, A_2, A_3$ and $A_4$ vector subspaces of a finite dimensional vector space, $I(A_1; A_2) + I(A_1; A_2 | A_3) + I(A_1; A_2 | A_4) + I(A_3; A_4)$.

We are interested in finding interesting characteristic-dependent linear rank inequalities i.e. where $P$ is a proper subset of primes.

### 2.1 Producing inequalities: How to find and use a suitable binary matrix

The following theorem is the principal theorem of this paper and shows a method to produce pairs of characteristic-dependent linear rank inequalities from suitable binary matrices. The demonstration is presented in subsection 2.2. We use this notation: $[e_n, e_m] = \{e_i : n \leq i \leq m\}$, $[e_n, e_m] = \{e_i : n \leq i < m\}$ and $[e_n] := [e_1, e_n] = \{e_i\}$; for a binary matrix $B$, we denote $B = (B') = (e_{S_i})$, with $S_i = \{j : B(j,i) = 1\}$.

**Theorem 3.** Let $B = (B') = (e_{S_i})$ be a $n \times m$ binary matrix over $\mathbb{F}$, $m \leq n$ and $t \geq 2$ integer. We suppose that rank$B = m$ if char$\mathbb{F}$ does not divide $t$, and rank$B = m - 1$ in other cases. Let $A_e$, $e \in [e_n], B_{e_{S_i}}, e \in B'$ and $C$ be vector subspaces of a finite dimensional vector space $V$ over $\mathbb{F}$. Then

(i) The following inequality is a characteristic-dependent linear rank inequality over fields whose characteristic divides $t$,

$$H(A_e, B_{e_{S_i}}, C : e_{S_i} \in B', e_j \in B'', C \in B''') + (|B''| | B' | + |B''|) H(C) \leq (m - 1) I(A_{[e_n]}, C)$$

$$+ \sum_{e_{S_k} \in B'} \left[ H\left(E_{e_{S_k}} | A_{e_i} : i \in S_k\right) + H\left(E_{e_{S_k}} | A_{e_i}, C : i \notin S_k\right)\right] + (|B'| + 1) \sum_{e_i \in B''} H(A_{e_i})$$

$$+ (|B''| | B' | + |B''| + |B'|) \left[ H(C | A_{[e_n]}) + \sum_{e_i \in [e_n]} I(A_{[e_n]} - e_i : C)\right]$$

$$+ \sum_{e_{S_k} \in B'} \left[ \nabla (A_{e_i} : i \in S_k, e_i \notin B'') + \nabla (A_{e_i} : i \notin S_k, e_i \notin B'')\right],$$

(ii) The following inequality is a characteristic-dependent linear rank inequality over fields whose characteristic does not divide $t$,

$$H(C) \leq \frac{1}{m} H(A_e, B_{e_{S_i}}, C : e_{S_i} \in B', e_j \in B'', C \in B''') + H\left(C | A_{[e_n]}\right) + \sum_{e_i \in [e_n]} I(A_{[e_n]} - e_i : C)$$

$$+ \sum_{e_{S_k} \in B'} \left[ H(C | A_{e_i}, B_{e_{S_k}} : i \notin S_k) + H\left(B_{e_{S_k}} | A_{e_i} : i \in S_k\right) + \nabla (A_{e_i} : i \notin S_k) + \nabla (A_{e_i} : i \in S_k)\right],$$

where $B' = \{e_{S_i} : 1 < |S| < n\}$; $B'' = \{e_{S_i} : |S| = 1\}$; $B''' = \{C\}$ if there exists $e_{S_i}$ in $B$ such that $|S_i| = n$, and $B'''$ is empty in other case; and $\nabla$ is a finite sum of entropies given by

$$\nabla (A_{e_i} : e_i \in T \leq [n]) \overset{\text{def.}}{=} I\left(A_{e_{k_1}, e_{k_1+1}} ; A_{e_{k_1}, e_{k_2}}\right) + \cdots + I\left(A_{e_{k_1+1}, e_{k_1}} \right)$$

where $k_1 \leq k_2 \leq \cdots \leq k_l$ give a partition in intervals, with maximum length, of $T$.

The first inequality does not hold in general over vector spaces whose characteristic does not divide $t$ and the second inequality does not hold in general over vector spaces whose characteristic divides $t$. A counter example would be in $V = GF(p)^n$, take the vector spaces $A_{e_i} = \langle e_i \rangle$, $e_i \in [e_n], B_{e_{S_j}} = \langle e_{S_j} \rangle$, $e_{S_j} \in B'$, and $C = \langle \sum e_i \rangle$. Then, when $p$ does not divide $t$, first inequality does not hold; and when $p$ divides $n$, second inequality does not hold.
Corollary 4. If the dimension of vector space $V$ is at most $n - 1$, then inequalities implicated in Theorem 3 are true over any field.

Corollary 5. If some vector space in Theorem 3 is the zero space, the inequalities implicated are linear rank inequalities.

Below is shown the class of $\left\lfloor \frac{n-1}{2} \right\rfloor - 2$ inequalities that are true over finite sets of primes (i.e. sets of the form $\{ p : p \mid t \}$), and another class of $\left\lfloor \frac{n-1}{2} \right\rfloor - 2$ inequalities that are true over co-finite sets of primes (i.e. sets of the form $\{ p : p \nmid t \}$).

Example 6. Taking $n \geq 7$ and setting $t$ integer such that $2 \leq t \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1$ and $M(n, t) = n - t - 2$, the following inequalities are produced using as a guide square matrices $B_{M(n, t)}^{t}$ with column vectors of the form $B_{i} := B_{c|M(n, t)|} = c - e_{i}$, $A_{i} := A_{e_{i}} = e_{i}$, with $c = \sum_{j \in [M(n, t)\rangle} e_{j}$, as described in figure [4] left side. The rank of $B_{M(n, t)}^{t}$ is $M(n, t)$ when $\text{char} \mathbb{F}$ does not divide $t$ and is $M(n, t) - 1$ in other case. We remark that in [5] we used the case $M(n, t) = t + 1$, so the columns of the matrices were only of the form presented in figure [4] right side. Let $A_{1}, A_{2}, \ldots, A_{M(n, t)}, B_{1}, B_{2}, \ldots, B_{t+1}, C$ be subspaces of a finite-dimensional vector space $V$ over a scalar field $\mathbb{F}$. We have:

(a) If $\text{char} (\mathbb{F})$ divides $t$,

\[
H \left( B_{t+1}, A_{M(n, t) - t+1} \right) + (t + 2) (M(n, t) - t - 1) H(C) \leq (M(n, t) - 1) I \left( A_{M(n, t)}; C \right)
\]

\[
+ (t + 2) \sum_{i=t+2}^{M(n, t)} H(A_{i}) + [(t + 2) (M(n, t) - t) - 1] \left( H(C) + I \left( A_{M(n, t)}; A_{M(n, t) - 1}; C \right) \right)
\]

\[
+ \sum_{i=1}^{t+1} \left( H(B_{i} \mid A_{M(n, t) - i}) + H(B_{i} \mid A_{i}, C) + I \left( A_{i}; A_{i+1}; A_{i} \right) + I \left( A_{i-1}; A_{i} \right) \right).
\]

(b) If $\text{char} (\mathbb{F})$ does not divide $t$,

\[
H(C) \leq \frac{1}{M(n, t)} H \left( B_{t+1}, A_{M(n, t) - t+1} \right) + H \left( C \mid A_{M(n, t)} \right) + \sum_{i=1}^{M(n, t)} I \left( A_{M(n, t) - i}; C \right)
\]

\[
+ \sum_{i=2}^{t+1} I \left( A_{i-1}; C \right) + \sum_{i=1}^{t+1} \left( H(C) \mid A_{i}, B_{i} \right) + H \left( B_{i} \mid A_{M(n, t) - i} \right) + I \left( A_{i}; A_{M(n, t) - i} \right) \right).
\]

Corollary 5 shows that each inequality, presented in example 6, can not be deduced from a higher order inequality by nullifying some variables. In fact, using Corollary 4 we can say more about the class (a) of these inequalities.

For $m \in \mathbb{N}$ and $p$ prime, the function that counts all the powers of $p$ less than or equal to $m$ is denoted by $\varphi (m, p)$. In example 6, $\varphi \left( \left\lfloor \frac{n-1}{2} \right\rfloor, 2, p \right)$ inequalities in $n$ variables, which are true over fields whose characteristic is $p$, are produced. By Corollary 4 each of these inequalities holds over any characteristic when the dimension of $V$ is at most $n - t - 3$. Also, each inequality is determined by $t$ and this number can run the powers of $p$ less than or equal to $\left\lfloor \frac{n-1}{2} \right\rfloor - 2$. This means that each inequality is true in at least one vector space where the other inequalities are not true. Therefore, any of these inequalities can not be deduced from the other inequalities, much less if they are combined with linear rank inequalities, without violating this property. We have the next corollary.

Corollary 7. For each $n \geq 7$ and $p$ prime. There are at least $\varphi \left( \left\lfloor \frac{n-1}{2} \right\rfloor, 2, p \right)$ independent inequalities in $n$ variables which are characteristic-dependent linear rank inequalities that are true over fields whose characteristic is $p$.

1One can use software such as Xitip to note that they must be Shannon information inequalities.
2.2 Proof of Theorem 3

In a general way, we show how to build characteristic-dependent linear rank inequalities from dependency relations in certain type of binary matrices. We show this in three steps:

A. Finding an equation.

B. Conditional characteristic-dependent linear rank inequalities.

C. Characteristic-dependent linear rank inequalities.

First of all, we show how to abstract an equation as presented in [8, Lemma 3]. Second, how to define “conditional-linear rank inequalities” as presented in [8, Lemma 5 and 6]. Third, the technique of upper bounds used in [1, for a particular case] and improved in [8, for a family of binary matrices] is applied.

A. Finding an equation: Let $\mathbb{F}^n = \langle e_1 \rangle \oplus \cdots \oplus \langle e_n \rangle$ and $c = e_1 + \cdots + e_n$. Let $B = (B^i) = (e_{S_i})$ be a $n \times m$ binary matrix over $\mathbb{F}$, $m \leq n$. We make the following correspondence between the columns of $B$ and the canonical projection functions on $\mathbb{F}^n$:

$$e_{S_m} \longleftrightarrow \pi_{S_m} \quad \text{where} \quad S_m = \{j : B_{(j,i)} = 1\}.$$

We suppose that rank $B = m$ if char $\mathbb{F}$ does not divide $t$, and the rank $B = m - 1$ if char $\mathbb{F}$ divides $t$, for $t \geq 2$. Having account the previous correspondence, we can define the following propositions whose proof is omitted:

$$e_{S_m} = \sum_{i=1}^{m-1} \alpha_i e_{S_i} \quad \text{if and only if} \quad \pi_{S_m} = \sum_{i=1}^{m-1} \alpha_i \pi_{S_i}$$

$\{e_{S_i}\}_{i=1}^r$ is an independent set if and only if $\sum_{i=1}^r \pi_{S_i}(\langle c \rangle)$ is a direct sum

We get an equation of the form: $H(\pi_{S_j}(\langle c \rangle) : j \in [m]) = \begin{cases} (m - 1)H(\langle c \rangle) & \text{if char}(\mathbb{F}) \mid t \\ mH(\langle c \rangle) & \text{if char}(\mathbb{F}) \nmid t. \end{cases}$

Previous argument can be easily generalized to vector subspaces $A_{e_1}, \ldots, A_{e_n}, C$ of a vector space $V$ over a field $\mathbb{F}$, where $A_{e_1}, A_{e_n}, \ldots, A_{e_n}$ are mutually complementary in $V$ and $C$ is such that the sum of $\bigoplus_{i \neq k} A_{e_i}$ and $C$ is a direct sum for all $k$, such a collection of spaces is called tuple that satisfies condition of complementary vector spaces. Formally, we have:

Claim 8. When $B$ exists, a tuple that satisfies condition of complementary vector spaces holds

$$H(\pi_{S_j}(C) : j \in [m]) = \begin{cases} (m - 1)H(C) & \text{if char}(\mathbb{F}) \mid t \\ mH(C) & \text{if char}(\mathbb{F}) \nmid t. \end{cases}$$

$$B_1 \cdots B_{i+1} A_{i+2} \cdots A_m A_{n,t} \left( \begin{array}{cccc} 0 & \cdots & 1 & \cdots & 0 \\ 1 & \vdots & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \vdots & 1 & 0 & \vdots \\ 1 & \vdots & 0 & 0 & \vdots \\ 1 & \vdots & 1 & 1 & \vdots \\ 1 & \vdots & 0 & 0 & \vdots \\ 1 & \vdots & 1 & \vdots & \vdots \\ 1 & \vdots & 1 & 0 & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 1 \end{array} \right)$$

$$B_1 \cdots B_{n+1} \left( \begin{array}{cccc} 0 & \cdots & 1 \\ 1 & \vdots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \vdots & 1 \\ 1 & \vdots & 0 \\ 1 & \cdots & 1 & 0 \end{array} \right)$$

Figure 1: Matrix $D^t_M(n,t)$ and matrix $D^n_M(2n+3,m)$ used in [8].
B. Conditional characteristic-dependent linear rank inequalities: In the previous step we noticed that dependence relations of \( B \) can be expressed using projections of a suitable space \( C \). In fact, we can derive more properties as follows, from \( e_{S_i} = \sum_{j \in S_i} e_j = c - \sum_{j \notin S_i} e_j \), we derive

\[
\pi_{S_i}(C) = \bigoplus_{e_j \in S_i} A_{e_j} \cap \left( C \oplus \bigoplus_{e_j \notin S_i} A_{e_j} \right).
\]

This equality is easily proven. The following claim use this to find inequalities that depend on the characteristic of \( \mathbb{F} \), and the involved spaces have some dependency relationships expressed by \( B \). We denote by \( B' = \{ e_{S_i} : 1 < |S_i| < n \}; B'' = \{ e_{S_i} : |S_i| = 1 \}; B''' = \{ C \} \) if there exists \( e_{S_i} \) in \( B \) such that \(|S_i| = n\), and \( B''' \) is empty in other case.

**Claim 9.** For a tuple of vector subspaces \( (A_{e_1}, B_{e_{S_j}}) : e_{S_j} \in B', e_i \in B'' \) such that \( (A_{e_1}, \ldots, A_{e_n}, C) \) satisfies the condition of complementary vector subspaces, consider the following conditions:

(i) \( A_{e_i} \leq A_{[e_n] - e_i} \oplus C \) for \( i \) such that \( e_i \in B'' \).

(ii) \( B_{e_{S_i}} \leq \bigoplus_{j \in S_i} A_{e_j} \) for \( e_{S_i} \in B' \).

(iii) \( B_{e_{S_i}} \leq \bigoplus_{j \notin S_i} A_{e_j} \oplus C \) for \( e_{S_i} \in B' \).

We have that

a. If condition (i), (ii) and (iii) hold over a fields whose characteristic divides \( t \), then

\[
H(A_{e_1}, B_{e_{S_j}}, C : e_{S_j} \in B', e_i \in B'', C \in B''') \leq (m - 1) H(C).
\]

b. If condition (ii) and (iv) hold over a fields whose characteristic does not divide \( t \), then

\[
mH(C) \leq H(A_{e_1}, B_{e_{S_j}}, C : e_{S_j} \in B', e_i \in B'', C \in B''').
\]

C. Characteristic-dependent linear rank inequalities: We find vector subspaces that satisfy conditions of previous claim. Let \( (A_{e_1}, B_{e_{S_j}}, C : e_{S_j} \in B', e_i \in B'' \) a tuple of arbitrary vector subspaces of a finite dimensional vector space \( V \) over a finite field \( \mathbb{F} \).

From \( A_{e_1}, \ldots, A_{e_n}, \) and \( C \), we obtain a tuple that satisfies condition of complementary spaces \( (A'_{e_1}, \ldots, A'_{e_n}, C) \) as obtained in \([1]\) which holds:

\[
codim_{A_{e_k}}(A'_{e_k}) = \mathbb{I}(A_{[e_n] - e_i}; A_{e_k}), \text{ for all } k,
\]

\[
codim_C(C) \leq \nabla(C) \overset{\text{def}}{=} H(C | A_{[e_n]}) + \sum_{e_i \in [e_n]} \mathbb{I}(A_{[e_n] - e_i}; C).
\]

Additionally, for \( T \subseteq [e_n] \), we can take some elements \( e_{k_1}, \ldots, e_{k_l} \) with \( k_1 \leq k_2 \leq \cdots \leq k_l \) such that it is possible to built a partition in intervals \([e_{k_1}, e_{k_l}]\), with maximum length, of \( T \). So,

\[
codim_{C} \sum_{e \in T} A_{e} \bigoplus A_{e} \leq \nabla(A_e : e \in T) \overset{\text{def}}{=} \mathbb{I}(A_{[e_1, e_{k_1}]}; A_{[e_{k_1}, e_{k_2}]}) + \cdots + \mathbb{I}(A_{[e_{k_1}, e_{k_l}]}; A_{[e_{k_{l-1}}, e_{k_l}]})
\]

Before continuing, we need the following three statements:

**Claim 10.** Tuple \( (\tilde{A}_{e_1}, \ldots, \tilde{A}_{e_n}, \tilde{C}) \) defined by

\[
\tilde{A}_{e_k} := A'_{e_k} \cap \left( \bar{C} + \bigoplus_{e_i \notin D''} A'_{e_i} + \bigoplus_{e_i \in D'', i < k} \tilde{A}_{e_i} + \bigoplus_{e_i \in D'', i > k} A'_{e_i} \right), \text{ for } e_k \in B''
\]

\[
\tilde{A}_{e_k} := A'_{e_k}, \text{ for } e_k \notin B''
\]

satisfies condition of complementary spaces, condition (i), \( H(\tilde{A}_{e_k}) = H(\tilde{C}) \leq 1 \) and

\[
codim_{A_{e_k}}(\tilde{A}_{e_k}) \leq H(A_{e_k}) - H(C) + \nabla(C), \text{ for } e_k \in B''.
\]
Proof. We obviously have that $\bar{A}_{e_k} \leq \bar{C} + \bigoplus_{e_i \notin B''} A'_{e_i} + \bigoplus_{e_i \in B'', i < k} \bar{A}_{e_i} + \bigoplus_{e_i \in B'', i \geq k} A'_{e_i}$. Now, for $k$, we show that
\[
\bar{C} \leq \bigoplus_{e_i \notin B''} A'_{e_i} + \bigoplus_{e_i \in B'', i < k} \bar{A}_{e_i} + \bigoplus_{e_i \in B'', i \geq k} A'_{e_i}
\] (5)

In effect, we show case $k = l = \min \{ i : e_i \in B'' \}$, the general case is by induction. We note that $\bar{A}_{e_i} = O$ if and only if $\bar{C} = O$, so this case is trivial. Otherwise, there exists $a_{e_i} \neq O$ in $\bar{A}_{e_i}$ and $c \neq O$ in $\bar{C}$ such that $a_{e_i} = c + \sum a_{e_i}$, then $c \in \bigoplus_{i \neq l} A'_{e_i} + \bar{A}_{e_i} + \bigoplus_{e_i \in B'', i > l} A'_{e_i}$. The affirmation is obtained noting that $c$ generates $\bar{C}$ [Remark 4]. Taking $k = \max \{ i : e_i \in B'' \}$, we obtain that $\bar{C} \leq \bigoplus \bar{A}_{e_i}$, so condition of complementary spaces is satisfied. Also, we have the equation:
\[
H(\bar{A}_{e_k}) = I \left( A'_{e_k} : \bar{C}, \bigoplus_{e_i \in B''} A'_{e_i}, \bigoplus_{e_i \in B'', i < k} \bar{A}_{e_i}, \bigoplus_{e_i \in B'', i \geq k} A'_{e_i} \right)
\]
\[
= H(A'_{e_k}) + H \left( \bigoplus_{i \notin B''} A'_{e_i}, \bigoplus_{i \in B'', i < k} \bar{A}_{e_i}, \bigoplus_{i \in B'', i \geq k} A'_{e_i} \right) + H \left( \bar{C}, \bigoplus_{e_i \notin B''} A'_{e_i}, \bigoplus_{e_i \in B'', i < k} \bar{A}_{e_i}, \bigoplus_{e_i \in B'', i \geq k} A'_{e_i} \right)
\]
\[
= H(\bar{C}) \text{ [using condition of complementary spaces]}
\]
which also implies the desired upper bound on $\text{codim}_{A_{e_k}} (\bar{A}_{e_k})$. Now, condition (i) is straightforward because $\bar{C}$ and each $\bar{A}_{e_k}$, $e_k \in B''$, have the same dimension. \hfill $\square$

Claim 11. For $e_{S_k} \in B'$, we define $\bar{B}_{e_{S_k}} := B_{e_{S_k}} \cap \left( \bigoplus_{e_i \in S_k} \bar{A}_{e_i} \right) \cap \left( \bigoplus_{e_i \notin S_k} \bar{A}_{e_i} \oplus \bar{C} \right)$. We have that tuple $(\bar{A}_{e_1}, \ldots, \bar{A}_{e_l}, \bar{B}_{e_{S_k}}, \bar{C} : e_{S_k} \in B')$ satisfies condition (i), condition (ii), condition (iii) and

$$
\text{codim}_{B_{e_{S_k}}} \bar{B}_{e_{S_k}} \leq H \left( B_{e_{S_k}} \mid A_{e_i} : i \in S_k \right) + H \left( B_{e_{S_k}} \mid A_{e_i} : i \notin S_k \right) + \sum_{e_i \in B''} H(A_{e_i}) + \nabla(A_{e_i})_{i \notin S_k, e_i \notin B''} + \left( |B''| + 1 \right) \nabla(C) - |B''| H(C)
$$

Proof. In effect,
\[
\text{codim}_{B_{e_{S_k}}} \bar{B}_{e_{S_k}} \leq \text{codim}_{B_{e_{S_k}}} \left( \left( \bigoplus_{i \in S_k} \bar{A}_{e_i} \right) \cap B_{e_{S_k}} \right) + \text{codim}_{B_{e_{S_k}}} \left( \bigoplus_{i \notin S_k} \bar{A}_{e_i} \oplus \bar{C} \right) \cap B_{e_{S_k}} \right)
\]
\[
= \text{codim}_{B_{e_{S_k}}} \left( \bigoplus_{i \in S_k} A_{e_i} \right) \cap B_{e_{S_k}} \right) + \text{codim}_{B_{e_{S_k}}} \left( \bigoplus_{i \notin S_k} A_{e_i} \oplus \bar{C} \right) \cap B_{e_{S_k}} \right)
\]
\[
+ \text{codim} \left( \sum_{i \in S_k} A_{e_i} \right) \cap B_{e_{S_k}} \right) + \text{codim} \left[ \sum_{i \notin S_k} A_{e_i} \oplus \bar{C} \right] \cap B_{e_{S_k}} \right)
\]
\[
\leq H \left( B_{e_{S_k}} \mid A_{e_i} : i \in S_k \right) + H \left( B_{e_{S_k}} \mid C, A_{e_i} : i \notin S_k \right) + \text{codim} \left( \sum_{i \in S_k} A_{e_i} \right) \cap B_{e_{S_k}} \right) + \text{codim} \left( \sum_{i \notin S_k} A_{e_i} \oplus \bar{C} \right) \cap B_{e_{S_k}} \right)
\]
\[
\text{codim} \sum_{i \in S_k, e_i \in B'} A_{e_i} + \sum_{i \notin S_k, e_i \in B''} A_{e_i} + \text{codim}_C \bar{C} \\
\leq H \left( B_{e_{S_k}} \mid A_{e_i} : i \in S_k \right) + H \left( B_{e_{S_k}} \mid A_{e_i} : C : i \notin S_k \right) + \sum_{i \notin S_k, e_i \in B''} H(A_{e_i}) \\
+ \left| \{ e_i \in B'' : i \in S_k \} \right| (\nabla(C) - H(C)) + \nabla(A_{e_i} : i \in S_k, e_i \notin B'') + \sum_{i \notin S_k, e_i \in B''} H(A_{e_i}) \\
+ \left| \{ e_i \in B'' : i \notin S_k \} \right| (\nabla(C) - H(C)) + \nabla(A_{e_i} : i \notin S_k, e_i \notin B'') + \nabla(C) \\
\text{[from Lemmas 1 and 2 of [8], inequalities 13 and 14].}
\]

Claim 12. For \( e_{S_k} \in B' \), we define \( \hat{B}_{e_{S_k}} := B_{e_{S_k}} \cap \bigoplus_{j \in S_k} A'_{e_j} \) and \( \hat{C} := C \cap \left( \bigoplus_{e_i \notin S_k} A'_{e_i} + \hat{B}_{e_{S_k}} \right) \). We have that \( \left( A'_{e_i}, \hat{B}_{e_{S_j}}, \hat{C} : e_i \in B'', e_{S_i} \in B' \right) \) satisfies condition of complementary spaces, conditions (ii), (iv) and
\[
\text{codim}_{B_{e_{S_k}}} \hat{B}_{e_{S_k}} \leq H \left( B_{e_{S_k}} \mid A_{e_i} : i \in S_k \right) + \nabla(A_{e_i} : i \in S_k), \quad (6)
\]
\[
\text{codim}_{C} \hat{C} \leq \nabla(C) + \sum_{e_{S_k} \in B'} \left[ H(C \mid A_{e_i}, B_{S_k} : i \notin S_k) + H \left( B_{e_{S_k}} \mid A_{e_i} : i \in S_k \right) \right] \\
+ \sum_{e_{S_k} \in B'} \left[ +\nabla(A_{e_i} : i \notin S_k) + \nabla(A_{e_i} : i \in S_k) \right]. \quad (7)
\]

Proof. We only show last inequality:
\[
\text{codim}_{C} \hat{C} \leq \text{codim}_{C} \bar{C} + \sum_{e_{S_k} \in B'} \text{codim}_{C} \left( C \cap \left[ \bigoplus_{i \notin S_k} A'_{e_i} + \hat{B}_{e_{S_k}} \right] \right) \\
= \text{codim}_{C} \bar{C} + \sum_{e_{S_k} \in B'} \text{codim}_{C} \left( C \cap \left[ \bigoplus_{i \notin S_k} A_{e_i} + B_{e_{S_k}} \right] \right) \\
+ \sum_{e_{S_k} \in B'} \text{codim} \left[ C \cap \left( \bigoplus_{i \notin S_k} A_{e_i} + B_{e_{S_k}} \right) \right] \\
\leq \text{codim}_{C} \bar{C} + \sum_{e_{S_k} \in B'} \text{codim}_{C} \left( C \cap \left[ \bigoplus_{i \notin S_k} A_{e_i} + B_{e_{S_k}} \right] \right) + \sum_{e_{S_k} \in B'} \text{codim} \sum_{i \in S_k} A_{e_i} \bigoplus_{i \notin S_k} A'_{e_i} \\
+ \sum_{e_{S_k} \in B'} \text{codim}_{B_{e_{S_k}}} \hat{B}_{e_{S_k}} \text{ [from Lemmas 1 and 2 of [8] and inequality 13]} \\
\leq \nabla(C) + \sum_{e_{S_k} \in B'} \left[ H(C \mid A_{e_i}, B_{S_k} : i \notin S_k) + H \left( B_{e_{S_k}} \mid A_{e_i} : i \in S_k \right) \right] \\
+ \sum_{e_{S_k} \in B'} \left[ +\nabla(A_{e_i} : i \notin S_k) + \nabla(A_{e_i} : i \in S_k) \right] \text{ [from 13]}
\]

\qed
We can finally build the inequalities of our theorem:

By Claims 10 and 11 tuple \( \tilde{A}_{e_i}, \tilde{B}_{eS_j}, \tilde{C} : e_i \in [e_n], e_s \in B' \) satisfies hypothesis of the Claim 9 with conditions (i), (ii) and (iii) in a finite field \( F \) whose field characteristic divides \( t \), we get

\[
H(\tilde{B}_{eS_j}, \tilde{A}_{e_i}, \tilde{C} : e_S, e_j \in B', e_j, e_i \in B'' \leq B''') \leq (m-1)H(\tilde{C}).
\]

(8)

On the other hand,

\[
H(\tilde{C}) \leq I(A_{[e_n]}; C) \quad \text{[from } \tilde{C} \leq C],
\]

(9)

codim \sum_{e_s \in B'} B_{eS_j} + \sum_{e_i \in B''} A_{e_i} \left( \sum_{e_s \in B'} B_{eS_j} + \sum_{e_i \in B''} A_{e_i} \right) \leq \sum_{e_s \in B'} H(\tilde{B}_{eS_j} | A_{e_i} : i \in S_k)

\[
\sum_{e_s \in B'} H(\tilde{B}_{eS_j} | A_{e_i}, C : i \notin S_k) + (|B'| + 1) \sum_{e_i \in B''} H(A_{e_i})
\]

\[
+ (|B'| |B'| + |B''| + |B'|) \nabla (C) - (|B''| |B'| + |B''|) H(C)
\]

\[
+ \sum_{e_s \in B'} \nabla (A_{e_i} : i \in S_k, e_i \notin B''' \leq B''').
\]

From (8), (9), (10) and last inequality, we can obtain the desired characteristic-dependent linear rank inequality over fields whose characteristic divides \( t \).

By Claims 12 tuple \( \tilde{A}'_{e_i}, \tilde{B}_{eS_j}, \tilde{C} : e_i \in [e_n], e_s \in B' \) satisfies hypothesis of the Claim 9 with conditions (ii) and (iv) in a finite field \( F \) whose field characteristic does not divide \( t \), we get

\[
mH(\tilde{C}) \leq H(\tilde{A}'_{e_i}, \tilde{B}_{eS_j}, \tilde{C} : e_s \in B', e_i \in B'', \tilde{C} \in B''').
\]

(10)

On the other hand,

\[
H(\tilde{A}'_{e_i}, \tilde{B}_{eS_j}, \tilde{C} : e_s \in B', e_i \in B'', \tilde{C} \in B''') \leq H(\tilde{A}_{e_i}, \tilde{B}_{eS_j}, C : e_s \in B', e_i \in B'', C \in B'''').
\]

(11)

From (10), (7) and last inequality, we can obtain the desired characteristic-dependent linear rank inequality over fields whose characteristic does not divide \( t \).

Remark 13. In case that the dimension of \( V \) is at most \( n-1 \), there exists some \( A'_{e_i} = 0 \) in the demonstration above. Therefore, the equation given by the matrix used as a guide is trivial. This implies Corollary 4 Corollary 5 is obtained in a similar way.

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