THE DIVIDED POWERS ALGEBRA OF A
FINITE-DIMENSIONAL NICHOLS ALGEBRA OF
DIAGONAL TYPE

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Abstract. Let \( B_q \) be a finite-dimensional Nichols algebra of diagonal
type corresponding to a matrix \( q \). We consider the graded dual \( L_q \)
of the
distinguished pre-Nichols algebra \( \tilde{B}_q \) from \[A3\] and the divided powers
algebra \( U_q \), a suitable Drinfeld double of \( L_q \# k\theta \). We provide basis and
presentations by generators and relations of \( L_q \) and \( U_q \), and prove that
they are noetherian and have finite Gelfand-Kirillov dimension.

1. Introduction

We fix an algebraically closed field \( k \) of characteristic zero. Let \( g \) be
a finite-dimensional simple Lie algebra and \( q \in k \) a root of 1 (with some
restrictions depending on \( g \)). In the theory of quantum groups, there are
several Hopf algebras attached to \( g \) and \( q \):

- The Frobenius-Lusztig kernel (or small quantum group) \( u_q(g) \).
- The \( q \)-divided powers algebra \( U_q(g) \), see \[L1\] \[L2\].
- The quantized enveloping algebra \( U_q(g) \), see \[DK\] \[DKP\] \[DP\].

These Hopf algebras have the following features:

- They admit triangular decompositions, e.g. \( u_q(g) \simeq u_q^+(g) \otimes u_q^0(g) \otimes u_q^-(g) \).
- The 0-part of this triangular decomposition is a Hopf subalgebra, actually
  a group algebra.
- The positive and negative parts are not Hopf subalgebras, but rather Hopf
  algebras in braided tensor categories, braided Hopf algebras for short.
- There are morphisms \( u_q^+(g) \hookrightarrow U_q^+(g) \), \( U_q^+(g) \twoheadrightarrow u_q^+(g) \) of braided Hopf
  algebras, and ditto for the full Hopf algebras.
- The full Hopf algebras can be reconstructed from the positive part by
  standard procedures (bosonization, the Drinfeld double).
- The positive part \( u_q^+(g) \) has very special properties— it is a Nichols algebra.

Indeed, \( u_q^+(g) \) is completely determined by the matrix \( q = (q^{d_i a_{ij}}) \), where
\( a_{ij} \) is the Cartan matrix of \( g \) and \( d_i \in \{1, 2, 3\} \) make \( (d_i a_{ij}) \) symmetric. In
other words, \( u_q^+(g) \) is the Nichols algebra of diagonal type associated to \( q \).

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The knowledge of the finite-dimensional Nichols algebras of diagonal type is crucial in the classification program of finite-dimensional Hopf algebras [AS]. Two remarkable results on these Nichols algebras are:

(a) The explicit classification [H2].
(b) The determination of their defining relations [A1, A2].

Let \( q \in k^{\mathfrak{g} \times \mathfrak{g}} \) with Nichols algebra \( B_q \) and assume that \( \dim B_q < \infty \). There are several reasons to consider the analogues of the braided Hopf algebras \( U_q^+(g) \) and \( U_q^+(g) \), for \( B_q \), motivated by the classification of Hopf algebras with finite Gelfand-Kirillov dimension and by representation theory. The analogue \( \tilde{B}_q \) of \( U_q^+(g) \) was introduced in [A2] and studied in [A3] under the name of distinguished pre-Nichols algebra. The definition of \( \tilde{B}_q \) is by discarding some of the relations in [A2]. The purpose of this paper is to study the analogue \( L_q \) of \( U_q^+(g) \); this is the graded dual of \( \tilde{B}_q \) and although it could be called the distinguished post-Nichols algebra of \( q \), we prefer to name it the Lusztig algebra as in [AAGTV], where mentioned in passing.

The paper is organized as follows. Section 2 is devoted to preliminaries and Section 3 to Nichols algebras of diagonal type and distinguished pre-Nichols algebras. In Section 4 we discuss Lusztig algebras: we provide a basis and a presentation by generators and relations, and prove that they are noetherian and have finite Gelfand-Kirillov dimension. In Section 5 we introduce the divided powers algebra \( U_q \), that is a suitable Drinfeld double of \( L_q \# kZ^\theta \); we also provide a presentation by generators and relations, and prove that it is noetherian and has finite Gelfand-Kirillov dimension.

**Remark 1.1.** The quantum divided power algebras were introduced and studied in [GH, Hu]; they correspond to Nichols algebras of Cartan type \( A_1 \times \cdots \times A_1 \).

## 2. Preliminaries and conventions

**2.1. Conventions.** If \( \theta \in \mathbb{N} \), then we set \( \mathbb{I}_\theta := \{1, 2, ..., \theta\} \); or simply \( \mathbb{I} \) if no confusion arises. If \( \Gamma \) is a group, then \( \hat{\Gamma} \) is its group of characters, that is, one-dimensional representations.

Let \( S_n \) and \( B_n \) be the symmetric and braid groups in \( n \) letters, with standard generators \( \tau_i = (i \ i + 1) \), respectively \( \sigma_i, \ i \in \mathbb{I}_{n-1} \). Let \( s : S_\theta \to B_\theta \) be the (Matsumoto) section of the projection \( \pi : B_\theta \to S_\theta, \ \pi(\sigma_i) = \tau_i, \ i \in \mathbb{I}_{n-1} \), given by \( s(\omega) = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_j} \), whenever \( \omega = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_j} \in S_\theta \) has length \( j \).

We consider the \( q \)-numbers in the polynomial ring \( \mathbb{Z}[q] \), \( n \in \mathbb{N}, \ 0 \leq i \leq n, \)

\[
(n)_q = \sum_{j=0}^{n-1} q^j, \quad (n)q^! = \prod_{j=1}^{n}(j)q, \quad \binom{n}{i}_q = \frac{(n)q^!}{(n-i)q^!(i)q}. 
\]

If \( q \in k \), then \( (n)_q, (n)q^!, \binom{n}{i}_q \) are the respective evaluations at \( q \).
We use the Heynemann-Sweedler notation for coalgebras and comodules; the counit of a coalgebra is denoted by \(\varepsilon\), and the antipode of a Hopf algebra, by \(S\). All Hopf algebras in this paper have bijective antipode.

Let \(H\) be a Hopf algebra. A Yetter-Drinfeld module \(V\) over \(H\) is a \(H\)-module and a \(H\)-comodule satisfying the compatibility condition

\[
\delta(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}, \quad h \in H, v \in V.
\]

Morphisms of Yetter-Drinfeld modules preserve the action and the coaction. Thus Yetter-Drinfeld modules over \(H\) form a braided tensor category \(\mathbb{H}_H YD\), with braiding \(c_W(v \otimes w) = v_{(-1)} : w \otimes v_{(0)}\), \(v, w \in V, w \in W\).

The full subcategory of finite-dimensional objects is rigid.

2.2. Braided vector spaces and Nichols algebras. A braided vector space is a pair \((V, c)\) where \(V\) is a vector space and \(c \in \text{Aut}(V \otimes V)\) is a solution of the braid equation \((c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(c \otimes \text{id})\).

If \(V\) is a vector space, then we identify \(V^* \otimes V^* \otimes V^* \) with a subspace of \((V \otimes V)^*\) by \((f \otimes g, v \otimes w) = \langle f, g, v \rangle,\) \(v, w \in V, f, g \in V^*\). If \((V, c)\) is a finite-dimensional braided vector space, then \((V^*, c')\) is its dual braided vector space, where \(c' : V^* \otimes V^* \otimes V^* \to V^* \otimes V^* \otimes V^*\) is \(\langle c'(f \otimes g), v \otimes w \rangle = \langle f \otimes g, c(w \otimes v) \rangle\).

We refer to [11] for the basic theory of braided Hopf algebras. If \(R = \bigoplus_{n \geq 0} R^n\) is a graded braided Hopf algebra with \(\dim R^n < \infty\) for all \(n\), then its graded dual \(R^d = \bigoplus_{n \geq 0} (R^n)^*\) is again a graded braided Hopf algebra.

We use the variation of the Sweedler notation \(\Delta(X) = X^{(1)} \otimes X^{(2)}\) for the coproducts in braided Hopf algebras.

The Nichols algebra of a braided vector space \((V, c)\) is a graded braided Hopf algebra \(B_q = \oplus_{n \geq 0} B^n_q\) with very rigid properties. There are several alternative definitions of Nichols algebras, see [AS]. We recall now two of these definitions.

Let \(T(V) = \oplus_{n \geq 0} T^n(V)\) be the tensor algebra of \(V\); it has a braiding \(c\) induced from \(V\). Let \(T(V) \otimes T(V) = T(V \otimes T(V)\) with the multiplication \((m \otimes m)(\text{id} \otimes c \otimes \text{id})\) and let \(\Delta : T(V) \to T(V) \otimes T(V)\) be the unique algebra map such that \(\Delta(v) = v \otimes 1 + 1 \otimes v\), for all \(v \in V\). Then \(T(V)\) is a (graded) braided Hopf algebra with respect to \(\Delta\). Dually, consider the cotensor coalgebra \(T^c(V)\) which is isomorphic to \(T(V)\) as a vector space. It bears a multiplication making \(T^c(V)\) a braided Hopf algebra with an analogous property, see e.g. [RG, AG]. There exists only one morphism of braided Hopf algebras \(\Theta : T(V) \to T^c(V)\) that it is the identity on \(V\). The image of \(\Theta\) is the Nichols algebra \(B_q\) of \(V\).

Here is the second description of \(B_q\). Let \(S\) be the partially ordered set of homogeneous Hopf ideals of \(T(V)\) with trivial intersection with \(k \oplus V\). Then \(S\) has a maximal element \(J_q\) and \(B_q = T(V)/J_q\) [AS].

2.3. Pre- and post-Nichols algebras. For several purposes, it is useful to consider braided Hopf algebras \(T(V)/I\), for various \(I \in S\). These are called pre-Nichols algebras [Ma]. Indeed, \(\Psi_{\text{pr}}(V) = \{T(V)/I : I \in S\}\) is a poset
with ordering given by the surjections; so that it is isomorphic to \((\mathcal{G}, \subseteq)\). The minimal element in \(\mathpzc{Pre}(V)\) is \(T(V)\), and the maximal is \(\mathcal{B}_q\). Dually, the poset \(\mathpzc{Post}(V)\) consists of graded Hopf subalgebras \(S = \bigoplus_{n \geq 0} S^n\) of \(T^c(V)\) such that \(S^1 = V\), ordered by the inclusion. Now the minimal element is \(\mathcal{B}_q\) and the maximal is \(T^c(V)\). We shall call them post-Nichols algebras.

Remark 2.1. The map \(\Phi : \mathpzc{Pre}(V) \to \mathpzc{Post}(V^*)\), \(\Phi(R) = R^d\), is an anti-isomorphism of posets.

Proof. If \(R = T(V)/I \in \mathpzc{Pre}(V)\), then \(R^d = I^\perp\); hence, \(\Phi\) is well-defined and it reverses the order. Also \(\Phi\) is surjective, because for a given \(S \in \mathpzc{Post}(V^*)\), \(I = S^\perp\) is a graded Hopf ideal of \(T(V)\) and \(S = (T(V)/I)^d\).  

3. Nichols algebras of diagonal type

A braided vector space \((V, c)\) is of diagonal type if there exist a basis \(x_1, \ldots, x_n\) of \(V\) and a matrix \(q = (q_{ij}) \in M_n(\mathbb{k})\) such that \(c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i\) for all \(i, j \in \mathbb{I} = \{1, \ldots, n\}\). Let \(H\) be a Hopf algebra, \(\chi_i \in \text{Hom}_{\text{alg}}(H, \mathbb{k})\) and \(g_j \in G(H)\) central in \(H\) such that \(\chi_j(g_i) = q_{ij}, i, j \in \mathbb{I}\). Then \((V, c)\) is realized in \(H^\mathcal{YD}\) by \(h \cdot x_i = \chi_i(h) x_i\) and \(\rho(x_i) = g_i \otimes x_i\). If \(i \in \mathbb{I}\), \(h \in H\).

We will only consider the case when \(H = k\mathbb{Z}^\theta\). \(g_i = \alpha_i\) and \(\chi_j \in \mathbb{Z}^\mathbb{Z}\) is given by \(\chi_j(\alpha_i) = q_{ij}, i, j \in \mathbb{I}\). Here \(\alpha_1, \ldots, \alpha_\mathbb{I}\) is the canonical basis of \(\mathbb{Z}^\mathbb{Z}\).

Let \(W = V^* \in k^{Z^\mathbb{Z}}_\mathbb{Z}^\mathbb{Z}\mathcal{YD}\); it is also a braided vector space of diagonal type, with matrix \(q^t\).

Since \(T(V)\) and \(\mathcal{B}_q\) are Hopf algebras in \(k^{Z^\mathbb{Z}}_\mathbb{Z}^\mathbb{Z}\mathcal{YD}\), we may consider the bosonizations \(T(V) \# k\mathbb{Z}^\mathbb{Z}\) and \(\mathcal{B}_q \# k\mathbb{Z}^\mathbb{Z}\). We refer to [AS] [1.5] for the definition of the adjoint action of a Hopf algebra, respectively the braided adjoint \(\text{ad}_c\) action of a Hopf algebra in \(k^{Z^\mathbb{Z}}_\mathbb{Z}^\mathbb{Z}\mathcal{YD}\). Then \(\text{ad}_c x \otimes \text{id} = \text{ad}(x \# 1)\) if \(x \in T(V)\) or \(\mathcal{B}_q\), see [AS] (1-21).

Now the matrix \(q\) gives rise to a bilinear form \(\Xi : \mathbb{Z}^\mathbb{Z} \times \mathbb{Z}^\mathbb{Z} \to \mathbb{k}^\times\) by \(\Xi(\alpha_j, \alpha_k) = q_{jk}\) for all \(j, k \in \mathbb{I}\). If \(\alpha, \beta \in \mathbb{Z}^\mathbb{Z}\), we also set

\[
q_{\alpha\beta} = \Xi(\alpha, \beta).
\]

The algebra \(T(V)\) is \(\mathbb{Z}^\mathbb{Z}\)-graded. If \(x, y \in T(V)\) are homogeneous of degrees \(\alpha, \beta \in \mathbb{Z}^\mathbb{Z}\) respectively, then their braided commutator is

\[
[x, y]_c = xy - \text{multiplication} \circ c(x \otimes y) = xy - q_{\alpha\beta}yx.
\]

Note that \(\text{ad}_c(x)(y) = [x, y]_c\) whenever \(x\) is primitive. We say that \(x\) q-commutes with a family \((y_i)_{i \in I}\) of homogeneous elements if \([x, y_i]_c = 0\), for all \(i \in I\). Same considerations are valid in any braided graded Hopf algebra.

Define a matrix \((c^3_{ij})_{i,j \in I}\) with entries in \(\mathbb{Z} \cup \{\infty\}\) by \(c^3_{ii} = 2\),

\[
c^3_{ij} := -\min\{n \in \mathbb{N}_0 : (n + 1)q_{ij}(1 - q_{ji}^2q_{ij}) = 0\}, \quad i \neq j.
\]

We assume from now on that \(\dim \mathcal{B}_q < \infty\). Then \(c^3_{ij} \in \mathbb{Z}\) for all \(i, j \in \mathbb{I}\) [Ro] and we may define the reflections \(s^3_i \in GL(\mathbb{Z}^\mathbb{Z})\), by \(s^3_i(\alpha_j) = \alpha_j - c^3_{ij}\alpha_i\),
The definitions involve the assignments of the Weyl groupoid and the generalized root system; the definitions involve the assignments \( q \sim \rho_i(q) \) described above. For our purposes, we just need to recall that

\[
\Delta_q^+ \text{ is the set of positive roots of } B_q, \tag{5}
\]

3.1. Drinfeld doubles. Let \((V,c)\) be our fixed braided vector space of diagonal type with matrix \( q \), realized in \( \mathbf{k}\mathbb{Z}^\theta \mathcal{YD} \) as above. In this Subsection, the hypothesis on the dimension of the Nichols algebra is not needed. We describe here the Drinfeld doubles of the bosonizations \( T(V) \# \mathbf{k}\mathbb{Z}^\theta, B_q \# \mathbf{k}\mathbb{Z}^\theta \) with respect to suitable bilinear forms. This construction goes back essentially to Drinfeld [Dr], and was adapted to different settings in various papers; here we follow [H3].

**Definition 3.1.** The Drinfeld double \( U_q \) of \( T(V) \# \mathbf{k}\mathbb{Z}^\theta \) is the algebra generated by elements \( E_i, F_i, K_i, K_i^{-1}, L_i, L_i^{-1}, i \in I \), with defining relations

\[
XY = YX, \quad X,Y \in \{K_i^{\pm 1}, L_i^{\pm 1} : i \in I\},
\]

\[
K_i K_i^{-1} = L_i L_i^{-1} = 1, \quad E_i F_j - F_j E_i = \delta_{i,j} (K_i - L_i).
\]

\[
K_i E_j = q_{ij} E_j K_i, \quad L_i E_j = q_{ij}^{-1} E_j L_i,
\]

\[
K_i F_j = q_{ij}^{-1} F_j K_i, \quad L_i F_j = q_{ij} F_j L_i.
\]

Then \( U_q \) is a \( \mathbb{Z}^\theta \)-graded Hopf algebra, where the comultiplication and the grading are given, for \( i \in I \), by

\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,
\]

\[
\Delta(L_i^{\pm 1}) = L_i^{\pm 1} \otimes L_i^{\pm 1}, \quad \Delta(F_i) = F_i \otimes L_i + 1 \otimes F_i.
\]

\[
\deg(K_i) = \deg(L_i) = 0, \quad \deg(E_i) = \alpha_i = -\deg(F_i).
\]

Let \( U_q^+ \) (respectively, \( U_q^- \)) be the subalgebra of \( U_q \) generated by \( E_i \) (respectively, \( F_i \)), \( i \in I \). Let \( W = V^* \) as above. There are isomorphisms \( \psi^+: T(V) \to U_q^+, \psi^-: T(W) \to U_q^- \) of Hopf algebras in \( \mathbf{k}\mathbb{Z}^\theta \mathcal{YD} \). Let

\[
\mathbb{U}_q = U_q/(\psi^-(J_q) + \psi^+(J_q));
\]

this is the Drinfeld double of \( B_q \# \mathbf{k}\mathbb{Z}^\theta \). We denote by \( E_i, F_i, K_i, L_i \) the elements of \( \mathbb{U}_q \) that are images of their homonymous in \( U_q \). Let \( u^0 \) (respectively, \( u_q^+, u_q^- \)) be the subalgebra of \( \mathbb{U}_q \) generated by \( K_i, L_i \), (respectively, by \( E_i \), by \( F_i \)), \( i \in I \). Then \( u^0 \simeq \mathbf{k}\mathbb{Z}^\theta \);

- there is a triangular decomposition \( \mathbb{U}_q \simeq u^+_q \otimes u^0 \otimes u^-_q \);
- \( u^+_q \simeq B_q, u^-_q \simeq B_q^* \).
3.2. Lusztig isomorphisms and PBW bases. G. Lusztig defined automorphisms of the quantized enveloping algebra $U_q(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$, see [L2]. These automorphisms satisfy the relations of the braid group covering the Weyl group of $\mathfrak{g}$; they are instrumental in the construction of Poincaré-Birkhoff-Witt (PBW) bases of $U_q(\mathfrak{g})$. These results were extended to the Drinfeld double of a finite-dimensional Nichols algebra of diagonal type in [H3], with the role of the Weyl group played here by the Weyl groupoid. The definition of the Lusztig isomorphisms in [H3] requires some hypotheses on the matrix groupoid. The definition of the Lusztig isomorphisms in [H3] requires some hypotheses on the matrix groupoid. So, let $(V, c)$ and $q$ as above; recall that we assume that $\dim B_q < \infty$. Fix $i \in \mathbb{I}$. We first recall the definition of the isomorphisms $u_q \to u_{\rho_i(q)}$ [H3]. For $i \neq j \in \mathbb{I}$ and $n \in \mathbb{N}_0$, define the elements of $u_q$

$$E_{j,n} = (\text{ad } E_j)^n E_j, \quad F_{j,n} = (\text{ad } F_j)^n F_j.$$

Let $E_j$, $F_j$, $K_j$, $L_j$ be the generators of $u_{\rho_i(q)}$. Set

$$a_j(q) := (-c_{ij}^q)^{1-c_{ij}^{-1}} \prod_{s=0}^{\infty} (q_{i,s}^{q} q_{i,s} q_{ij} - 1), \quad j \neq i.$$

**Theorem 3.2.** [H3 6.11] There are algebra isomorphisms $T_i : u_q \to u_{\rho_i(q)}$ uniquely determined, for $h, j \in \mathbb{I}$, $j \neq i$, by

$$T_i(K_h) = K^{-c_{ih}^q} K_h, \quad T_i(E_i) = E_j L_j^{-1}, \quad T_i(E_j) = E_j - c_{ij}^q,$$

$$T_i(L_h) = L_i^{-c_{ih}^q} L_h, \quad T_i(F_i) = K_i^{-1} E_i, \quad T_i(F_j) = \frac{1}{a_j(q) E_j - c_{ij}^q}. \quad \square$$

Let $w \in \mathcal{W}_q$ be an element of maximal length and fix a reduced expression $w = \sigma_i^q \sigma_{i_2} \cdot \cdot \cdot \sigma_{i_M}$. If $k \in \mathbb{I}_M$ and $\mathbf{h} = (h_1, \ldots, h_M) \in \mathbb{N}_0^M$, set

$$\beta_k = s_i^q \cdot \cdot \cdot s_{i_{k-1}}(\alpha_{i_k})$$

$$E_{\beta_k} = T_1 \cdot \cdot \cdot T_{i_{k-1}}(E_{i_k}) \in (u_q^+)_{\beta_k},$$

$$E_{h} = E_{h_M} E_{h_{M-1}} \cdot \cdot \cdot E_{h_1}.$$

By [CH] Prop. 2.12, $\Delta_+^q = \{ \beta_k | 1 \leq k \leq M \}$. Thus, we set

$$N_\beta = N_k = \text{ord } q \beta \in \mathbb{N} \cup \{\infty\}, \quad \text{if } \beta = \beta_k \in \Delta_+^q.$$

**Theorem 3.3.** [HY2 4.5, 4.8, 4.9] The following set is a basis of $u_q^+$:

$$\{ E_{\beta} \mid \mathbf{h} \in \mathbb{N}_0^M, \ 0 \leq h_k < N_k, \ k \in \mathbb{I}_M \}. \quad \square$$

3.3. Distinguished pre-Nichols algebra. We now recall the definition of the distinguished pre-Nichols algebra from [A3]. Let $\mathfrak{g}, V$ be as above. First, $i \in \mathbb{I}$ is a Cartan vertex of $\mathfrak{g}$ if

$$q_{ij} q_{ji} = q_{ij}^3, \quad \text{for all } j \neq i.$$
recall (3). Then the set of Cartan roots of \( q \) is
\[
\mathcal{O}_q = \{ s_{i_1}^q s_{i_2} \ldots s_{i_k} : i \in \mathbb{I} \text{ is a Cartan vertex of } \rho_{i_k} \ldots \rho_{i_2} \rho_{i_1} (q) \}. 
\]

A set of defining relations of the Nichols algebra \( \mathcal{B}_q \), i.e. generators of the ideal \( \mathcal{I}_q \), was given in [A3, Theorem 3.1]. We now consider the ideal \( \mathcal{I}_q \subset \mathcal{J}_q \) of \( T(V) \) generated by all the relations in loc. cit., but

- we exclude the powers root vectors \( E^N_\alpha, \alpha \in \mathcal{O}_q \),
- we add the quantum Serre relations \( (\text{ad}_c E_i)^{1-c_{ij}} E_j \) for those \( i \neq j \)
such that \( q_{ij}^c = q_{ji} q_{ki} = q_{ii} \).

**Definition 3.4.** [A3, 3.1] The distinguished pre-Nichols algebra of \( V \) is
\[
\tilde{\mathcal{B}}_q = T(V)/\mathcal{I}_q.
\]

Let \( \tilde{u}_q = U_q/(\psi^{-}(\mathcal{I}_q) + \psi^{+}(\mathcal{I}_q)) \); this is the Drinfeld double of \( \tilde{\mathcal{B}}_q \# \mathbb{k} \mathbb{Z}^0 \). It was shown in [A3] that there is a triangular decomposition \( \tilde{u}_q \simeq \tilde{u}_q^+ \otimes \tilde{u}^0 \otimes \tilde{u}_q^- \) as above, with \( \tilde{u}^0 \simeq u^0 \simeq \mathbb{k} \mathbb{Z}^{\geq 0} \).

If \( \beta_k \) is as in (7), \( k \in \mathbb{I}_M \), then we set \( \bar{N}_k = \begin{cases} N_k & \text{if } \beta_k \notin \mathcal{O}_q, \\ \infty & \text{if } \beta_k \in \mathcal{O}_q. \end{cases} \)

For simplicity, we introduce
\[
(12) \quad \mathbb{H} = \{ h \in \mathbb{N}_0^M : 0 \leq h_k < \bar{N}_k, \text{ for all } k \in \mathbb{I}_M \}
\]

**Theorem 3.5.**

(a) [A3, 3.4] There exist algebra isomorphisms \( \tilde{T}_i : \tilde{u}_q \to \tilde{u}_{\rho_i(q)} \) inducing the isomorphisms \( T_i : u_q \to u_{\rho_i(q)} \). Thus the elements \( E_{\beta_k}, E^h \) in [8], (9) make sense in \( u_q \).

(b) [A3, 3.6] \( \{ E^h | h \in \mathbb{H} \} \) is a basis of \( \tilde{u}_q^+ \). \( \square \)

As before, we have an isomorphism \( \tilde{\psi} : \tilde{\mathcal{B}}_q \to \tilde{u}_q^+ \) of Hopf algebras in \( \mathbb{k} \mathbb{Z}^0 \mathbb{V} \mathcal{D} \), so we define
\[
x_{\beta_k} = \tilde{\psi}^{-1}(E_{\beta_k}), \quad k \in \mathbb{I}_M; \quad x^h = \tilde{\psi}^{-1}(E^h), \quad h \in \mathbb{H}.
\]

Note that \( E_{\beta_k} \) is a well-defined sequence of braided commutators in the elements \( E_i, i \in \mathbb{I} \); then \( x_{\beta_k} \) is the same sequence of braided commutators in the \( x_i \)'s. Also, \( x^h = x^h_{\beta_M} x^h_{\beta_{M-1}} \cdots x^h_{\beta_1} \) and
\[
\mathcal{B} = \{ x^h | h \in \mathbb{H} \}
\]
is a basis of \( \tilde{\mathcal{B}}_q \). The Hilbert series of a graded vector space \( V = \bigoplus_{n \in \mathbb{N}_0} V^n \) is \( \mathcal{H}_V = \sum_{n \in \mathbb{N}_0} \dim V^n T^n \in \mathbb{Z}[[T]] \). It follows from Theorem 3.5 (b) that
\[
(13) \quad \text{GKdim} \tilde{\mathcal{B}}_q = |\mathcal{O}_q|, \quad \mathcal{H}_{\tilde{\mathcal{B}}_q} = \prod_{\beta_k \in \mathcal{O}_q} \frac{1}{1 - T^{\deg \beta}}, \quad \prod_{\beta_k \notin \mathcal{O}_q} \frac{1}{1 - T^{\deg \beta}}.
\]
4. Lusztig algebras

Let \( q = (q_{ij}) \in M_d(k^*) \), \((V, c)\) the corresponding braided vector space of diagonal type and \( W = V^* \). We still assume that \( B_q \) is finite-dimensional. As in [AAGTV 3.3.4], we define the Lusztig algebra \( L_q \) of \((V, c)\) as the graded dual of the distinguished pre-Nichols algebra \( \tilde{B}_q \); thus, \( B_q \subseteq L_q \). In this Section we establish some basic properties of this algebra.

4.1. Gelfand-Kirillov dimension. Since \( \mathcal{H}_{L_q} = \mathcal{H}_{\tilde{B}_q} \) is a rational function by [13], the following result implies that \( \text{GKdim } L_q = |\mathcal{O}_q| \).

**Theorem 4.1.** [KL 12.6.2] Assume that the Hilbert series \( \mathcal{H}_A \) of an infinite dimensional algebra \( A \) is a rational function. Then the radius of convergence \( r \) of \( \mathcal{H}_A(T) \) is \( \leq 1 \), and

- either \( r < 1 \), in which case \( A \) has exponential growth,
- or \( r = 1 \), and then \( \mathcal{H}_A(T) = \frac{P(T)}{1 - \beta(T)} \), for some polynomial \( P(T) \) with \( P(1) \neq 0 \); and \( \text{GKdim } A = d \).

**Remark 4.2.** The same argument shows that a pre-Nichols algebra \( R \) has finite GKdim if and only the post-Nichols algebra \( R^d \) has finite GKdim.

4.2. Presentation. If \( h \in H \), then define \( y_h \in \tilde{B}_q^* \) by \( \langle y_h, x^j \rangle = \delta_{h,j}, j \in H \). Then \( y_h \in L_q \) and \( \{y_h \mid h \in H\} \) is a basis of \( L_q \).

Let \((h_k)_{k \in \mathbb{M}}\) denote the canonical basis of \( \mathbb{Z}^M \). If \( k \in \mathbb{M} \) and \( \beta = \beta_k \in \Delta_k^\# \), then we denote the element \( y_{\beta_k} \) by \( \gamma_{\beta_k} \).

We recall some notation and results from [AY] and [AY]. For \( i \in \mathbb{M} \), let

\[
B_i = \langle \{ x_{\beta_i}^{h_1} \cdots x_{\beta_i}^{h_j} \mid 0 \leq h_j < N_j \} \rangle \subseteq B_q^*,
\]

\[
B_i^j = \langle \{ x_{\beta_i}^{h_j} x_{\beta_{j+1}}^{h_{j+1}} \cdots x_{\beta_i}^{h_N} \mid 0 \leq h_j < N_j \} \rangle \subseteq B_q^*,
\]

\[
\tilde{B}_i = \langle \{ x_{\beta_i}^{h_1} \cdots x_{\beta_i}^{h_j} \mid 0 \leq h_j < N_j \} \rangle \subseteq \tilde{B}_q^*,
\]

\[
\tilde{B}_i = \langle \{ x_{\beta_i}^{h_j} \cdots x_{\beta_i}^{h_N} \mid 0 \leq h_j < N_j \} \rangle \subseteq \tilde{B}_q^*,
\]

We also denote by \( \tilde{L}_i \) and \( \tilde{L}_i^j \) the analogous subspaces of \( L_q \):

\[
\tilde{L}_i = \langle \{ y_{\beta_i}^{(h_1)} \cdots y_{\beta_i}^{(h_j)} \mid 0 \leq h_j < N_j \} \rangle \subseteq L_q,
\]

\[
\tilde{L}_i^j = \langle \{ y_{\beta_i}^{(h_j)} \cdots y_{\beta_i}^{(h_N)} \mid 0 \leq h_j < N_j \} \rangle \subseteq L_q.
\]

**Proposition 4.3.** \(\bullet\) [AY 4.2, 4.11] \( B_i^j \) (respectively \( B_i^j \)) is a right (respectively left) coideal subalgebra of \( \tilde{B}_q^* \).

\(\bullet\) \[ A^2 4.1 \] If \( \beta \in \mathcal{O}_q \), then \( x_{\beta}^{N_\beta} \) \( q \)-commutes with every element of \( \tilde{B}_q^* \).

\(\bullet\) \[ A^2 4.9 \] If \( \beta_i \in \mathcal{O}_q \), then there exist \( X(n_1, \ldots, n_{i-1}) \in \tilde{B}_q^* \) such that

\[
\Delta(x_{\beta_i}^{N_{\beta_i}}) = x_{\beta_i}^{N_{\beta_i}} \otimes 1 + 1 \otimes x_{\beta_i}^{N_{\beta_i}} + \sum_{n_k \in \mathbb{N}_0} x_{\beta_{i-1}}^{n_i N_{\beta_i-1}} \cdots x_{\beta_{i-1}}^{N_{\beta_i-1}} \otimes X(n_1, \ldots, n_{i-1}). \]
Let $Z^+_q$ be the subalgebra of $\tilde{B}_q$ generated by $x_\beta^N$, $\beta \in \mathcal{O}_q$. 

**Theorem 4.4.** [A3 4.10, 4.13] $Z^+_q$ is a braided normal Hopf subalgebra of $\tilde{B}_q$. Moreover $Z^+_q = \text{com} \tilde{B}_q$. \(\Box\)

**Lemma 4.5.** Let $x$, $x_1$ and $x_2$ be elements in the PBW basis $B$ of $\tilde{B}_q$. If $x_1 \otimes x_2$ has a nonzero coefficient in the expression of $\Delta(x)$ in $B$ and $x_1x_2$ is also in $B$, then $x = cx_1x_2$ for some $c \in k$.

**Proof.** Let $x = x^h \in \tilde{B}_q$. We proceed by induction on $\text{ht}(h) := \sum_{i \in I_M} h_i$. If $x = x_\beta$, then $\Delta(x) \in x \otimes 1 + 1 \otimes x + \tilde{B}^i \otimes \tilde{B}^i$. This was proved in [AY 4.10] for Nichols algebras, but the same argument works for the distinguished pre-Nichols. Thus $x_1x_2$ is an ordered monomial if and only if $x_1 = 1$, $x_2 = x$ or $x_1 = x$, $x_2 = 1$. In both cases we have that $x_1x_2 = x$.

Let $x = x_1^h \cdots x_\beta^h$ with $h_i \neq 0$. We write $x' = x_1^{h_1-1} \cdots x_\beta^{h_\beta} = x^h$, thus $x = x_\beta x'$ and $\text{ht}(h') < \text{ht}(h)$.

Furthermore, $\Delta(x) = \Delta(x_\beta)\Delta(x')$. Since $\Delta(x_\beta) \in x_\beta \otimes 1 + 1 \otimes x_\beta + \tilde{B}^i \otimes \tilde{B}^i$, we have the following three cases:

(i) \( \begin{cases} x_1 = x_\beta x_1', \\ x_2 = x_2' \end{cases} \) (ii) \( \begin{cases} x_1 = x_1', \\ x_2 = x_\beta x_2' \end{cases} \) (iii) \( \begin{cases} x_1 \in \tilde{B}^i x_1', \\ x_2 \in \tilde{B}^i x_2' \end{cases} \)

for some $x_1', x_2'$ such that $x_1' \otimes x_2'$ appears in $\Delta(x')$. In the first case, $x_1x_2$ belongs to $B$ only if $x_1' \not\subseteq x_\beta$ and $x_2' \not\subseteq x_\beta$ do. Hence $x' = cx_\beta x_2'$ and $x = cx_1x_2$, $c \in k$ by inductive hypothesis. As $x' \in \tilde{B}^i$, Proposition 4.3 implies that $x_1' \in \tilde{B}^i$. Then, in (ii), $x_1x_2$ is ordered only if $x_1' = x_\beta$. This implies $x = x_1x_2$. In the last case $x_1x_2$ cannot be an element of $B$ unless $x = x_1x_2 = x_\beta$. \(\Box\)

**Corollary 4.6.** $\tilde{B}^i$ is a right coideal subalgebra of $\tilde{B}_q$. \(\Box\)

**Corollary 4.7.** If $\beta \in \Delta_4^i$, then

(14) \[ y_\beta = \frac{y_\beta^{(r)}}{y_\beta^{(r)}} \] \quad $r < N_\beta = \text{ord} q \beta$;

(15) \[ y_\beta = (y_\beta^{(N_\beta)})^s y_\beta^{(r)} \] \quad $\beta \in \mathcal{O}_q$, $n = sN_\beta + r$, $r < N_\beta$.

**Proof.** Arguing inductively, we may suppose that $y_\beta^{r-1} = (r-1)_{q \beta^r}y_\beta^{(r-1)}$. If $x = x^h \in \tilde{B}_q$, then

$\langle y_\beta, x \rangle = \langle y_\beta^{r-1}(1), y_\beta^{(2)} \rangle \neq 0 \implies x^{(1)} = c_1x_\beta^{r-1}, x^{(2)} = c_2x_\beta$, for some scalars $c_1$, $c_2$. By Lemma 4.3 $x = x_\beta'$. Then

$\langle y_\beta, x_\beta' \rangle = \langle y_\beta^{(r-1)}, (x_\beta')^{(1)} \rangle \langle y_\beta, (x_\beta')^{(2)} \rangle = (r-1)^{q \beta^r} = (r)^{q \beta^r}$.

The second equation follows immediately since $\langle y_\beta^{(N_\beta)} y_\beta^{(r)}, x_\beta^{N_\beta+r} \rangle = 1$. \(\Box\)

The next lemma is crucial for the presentation of the algebra $\mathcal{L}_q$ by generators and relations.
Lemma 4.8. Let $i \in \mathbb{I}_M$, $h_i < \tilde{N}_{\beta_i}$ and $\mathbf{h} = (h_1, \ldots, h_M) \in \mathbb{N}^M$, then

\[
y_{\mathbf{h}} = y_{\beta_M}^{(h_M)} \cdots y_{\beta_1}^{(h_1)}.
\]

Hence $\{y_{\beta_M}^{(h_M)} \cdots y_{\beta_1}^{(h_1)} | 0 \leq h_i < \tilde{N}_{\beta_i}\}$ is a basis of $\mathcal{L}_q$.

Proof. The proof is again by induction on $ht(\mathbf{h})$. If $ht(\mathbf{h}) = 1$ then $y_{\mathbf{h}} = y_{\beta}$ for some $\beta \in \Delta^q_+$ and the claim follows by definition.

Let $1 \leq i_j < \cdots < i_1 \leq M$, $n_k < \tilde{N}_{\beta_k}$ and $n_1 = sN_{\beta_1} + r \neq 0$ where $r < N_{\beta_1}$. Let $y = y_{\beta_1}^{(n_1)} \cdots y_{\beta_j}^{(n_j)} \in \mathcal{L}_q$. Since $\{y_{\mathbf{h}} | \mathbf{h} \in \mathbb{H}\}$ is a basis of $\mathcal{L}_q$, we can express $y$ as the linear combination $y = \sum_{\mathbf{h} \in \mathbb{H}} c_{\mathbf{h}} y_{\mathbf{h}}$. Notice that $c_{\mathbf{h}} \neq 0$ if and only if $\langle y, x^\mathbf{h} \rangle \neq 0$.

If $r \neq 0$, then we write $y = y_{\beta_1}^{(n_1)} y'$ where $y' = y_{\beta_1}^{(n_1)} \cdots y_{\beta_j}^{(n_j)}$ and $q = q_{\beta_1} \beta_1$. Then $\langle y, x^\mathbf{h} \rangle = \frac{r}{q} y_{\beta_1}^{(n_1)} (y')$, $\langle y', x^\mathbf{h} \rangle$. By inductive hypothesis and Lemma 4.5, $c_{\mathbf{h}} \neq 0$ if and only if $\mathbf{h} = (0, \ldots, n_1, \ldots, n_k, 0, \ldots)$. Moreover, the nonzero $c_{\mathbf{h}}$ is equal to 1 and the proof in this case is completed.

If $r = 0$, $n_1 = sN_{\beta_1}$, then we write $y = y_{\beta_1}^{(n_1)} y'$. Arguing as above, (16) follows. Hence $\{y_{\beta_M}^{(h_M)} \cdots y_{\beta_1}^{(h_1)} | 0 \leq h_i < \tilde{N}_{\beta_i}\}$ is a basis of $\mathcal{L}_q$ because so is $\{y_{\mathbf{h}} | \mathbf{h} \in \mathbb{H}\}$ by definition.

We seek for a presentation of $\mathcal{L}_q$. Let us consider the algebra $\mathbb{L}$ presented by generators $y_{\beta}^{(n)}, \beta \in \Delta^q_+, n \in \mathbb{N}$ with relations

\[
y_{\beta}^{(N_{\beta})} = 0, \quad \beta \in \Delta^q_+ - O_q; \tag{17}
\]

\[
y_{\beta}^{(h)} y_{\beta}^{(j)} = \binom{h+j}{j} y_{\beta}^{(h+j)}, \quad \beta \in \Delta^q_+, h, j \in \mathbb{N}; \tag{18}
\]

\[
[y_{\alpha}^{(h)}, y_{\beta}^{(j)}]_c = \sum_{m \in M(\alpha, \beta, h, j)} \kappa_m \, m, \quad \alpha < \beta \in \Delta^q_+, 0 < h < N_{\alpha}, 0 < j < N_{\beta}; \tag{19}
\]

\[
[y_{\alpha}^{(N_{\alpha})}, y_{\beta}^{(N_{\beta})}]_c = \kappa_{\gamma} y_{\gamma}^{(N_{\gamma})} + \sum_{0 < l < N_{\beta}, 0 < l < N_{\alpha}} \sum_{m \in M(\alpha, \beta, N_{\alpha} - i, N_{\beta} - l)} \kappa_{m,l} y_{\beta}^{(l)} y_{\alpha}^{(i)}, \quad \alpha, \beta, \gamma \in O_q, \alpha < \gamma < \beta; \tag{20}
\]

\[
[y_{\alpha}^{(N_{\alpha})}, y_{\beta}^{(j)}]_c = \sum_{0 < i < N_{\alpha}, m \in M(\alpha, \beta, N_{\alpha} - i, j)} \kappa_{m,0} y_{\alpha}^{(i)}, \quad \alpha \in O_q, \beta \in \Delta^q_+, 0 < j < N_{\beta}. \tag{21}
\]

Here we set

\[
M(\alpha, \beta, h, j) = \{m = y_{\beta_r}^{(h_r)} \cdots y_{\beta_k}^{(h_k)} \in \mathbb{L}^\beta \cap \mathbb{L}^\alpha : \deg m = \deg y_{\alpha}^{(h)} + \deg y_{\beta}^{(j)}\};
\]

\[
k_{m,l} = \langle y_{\alpha}^{(h)}, y_{\beta}^{(j)}, x_{\beta}^{(h_r)} x_{\beta}^{(h_{r-1})} \cdots x_{\beta}^{(h_k)} x_{\alpha}^{(i)}\rangle;
\]

\[
\kappa_{m} = \langle y_{\alpha}^{(N_{\alpha})}, y_{\beta}^{(N_{\beta})}, x_{\alpha}^{(N_{\alpha})}\rangle, \quad \deg y_{\gamma}^{(N_{\gamma})} = \deg y_{\alpha}^{(N_{\alpha})} + \deg y_{\beta}^{(N_{\beta})}.
\]
Theorem 4.9. There is an algebra isomorphism $\Upsilon : \mathbb{L} \rightarrow \mathcal{L}_q$ given by

$$\Upsilon(y_{\beta}^{(n)}) = y_{\beta}^{(n)}, \quad \beta \in \Delta_q^+,$$  

$n < \tilde{N}_\beta$.

Proof. We first prove that $\Upsilon$ is well-defined, i.e. that (17), ..., (21) are satisfied by the elements $y_{\beta}^{(n)} \in \mathcal{L}_q$. Relation (17) is trivial since $x_{\beta}^{N_\beta} = 0$ if $\beta \not\in \mathcal{O}_q$ and (18) is clear from (14).

For the other relations, given $\alpha < \beta$ and $h, j \in \mathbb{N}$, we write $y_{\alpha}^{(h)} y_{\beta}^{(j)} = \sum_{h \in \mathbb{N}} c_h y_h$. Then

$$\Delta(y_{\alpha}^{(h)} y_{\beta}^{(j)}) = (\langle y_{\alpha}^{(h)} y_{\beta}^{(j)} \rangle, x^h) = (\langle y_{\alpha}^{(h)} \rangle, (x^h)^{(1)})(\langle y_{\beta}^{(j)} \rangle, (x^h)^{(2)})$$

is the coefficient of $x_{\alpha}^{h} \otimes x_{\beta}^{j}$ in the expression of $\Delta(x^h)$ as linear combination of elements of the PBW basis in both sides of the tensor product.

If $h < N_\alpha$ and $j < N_\beta$, then $y_{\alpha}^{(h)} y_{\beta}^{(j)} \in \mathcal{B}_q$. If $c_h \neq 0$ then $x^h$ appears in the expression of $x_{\alpha}^{h} x_{\beta}^{j}$ in elements of the PBW basis, see [A1, Section 3]. Hence, by [HY2, 4.8] $x^h \in \mathcal{B}^\alpha \cap \mathcal{B}^\beta$, and relation (19) is clear.

Let $\alpha, \beta \in \mathcal{O}_q$, $h = N_\alpha$ and $j = N_\beta$. Suppose that there is $h = (h_1, \ldots, h_M)$ such that $c_h \neq 0$ and $h_i \geq N_i$ for some $i \in \mathbb{I}_M$. As $x_{\beta}^{N_\beta}$ q-commutes with every element of $\tilde{B}_q$, we have $x^h = c x_{\beta}^{N_\beta} x^{h'}$, where $h' = (h_1, \ldots, h_i - N_i, \ldots, h_M)$ and $c = \Xi(h_M \beta_M + \cdots + h_i+1 \beta_{i+1}, N_i \beta_i) \in \mathbb{K}$.

Then $\Delta(x^h) = \mathcal{L}(\alpha, \beta)$ and hence $x^h = x_{\beta}^{N_\beta}$ by Proposition 4.3. For the remaining $j$ such that $c_j \neq 0$ we have $j_i < N_i$ for all $i \in \mathbb{I}_M$. We write $x_{\alpha}^{N_\alpha} \otimes x_{\beta}^{N_\beta} = \xi(1 \otimes x_{\beta}^{N_\beta})(x_{\alpha}^{N_\alpha} \otimes x_{\beta}^{N_\beta} - n)(x_{\alpha}^{(m)} \otimes 1)$ where $\xi = \Xi^{-1}((N_\alpha - m)\alpha, m(\beta - n)\beta)$. Therefore, arguing as in the proof of (19) for $y_{\alpha}^{(N_\alpha - m)} y_{\beta}^{(N_\beta - n)}$, we obtain that $y_j = y_{\beta}^{(n)} y_{\alpha}^{(m)} m \in \tilde{\mathcal{L}}^\alpha \cap \mathcal{L}^\beta$.

Here, either $m = N_\alpha n = N_\beta$ so $y_j = \Xi(N_\alpha \alpha, N_\beta \beta) y_{\beta}^{(N_\beta)} y_{\alpha}^{(N_\alpha)}$, or else $m < N_\alpha n < N_\beta$. Hence relation (20) follows up to consider the correct degree for $y_n$.

For (21), $c_h \neq 0$ implies $x^h \in \mathcal{B}_q$ by the same argument above, since $Z_q^+$ is a braided Hopf subalgebra by Theorem 4.4.

Hence, $\Upsilon$ is a morphism of algebras. By the presentation of $\mathbb{L}$ we can prove that $\{y_{\beta}^{(h_M)} \cdots y_{\beta}^{(h_1)} : h_i < N_i\}$ is a basis of $\mathbb{L}$. So, $\Upsilon$ maps a basis to a basis by Lemma 4.8 and then it is bijective. $\square$

Example 4.10. Let $\theta = 3 \leq N$, $q \in \mathbb{k}^\times$, ord $q = N$. We consider a diagonal braiding (of super type $A$) given by a matrix $q = (q_{ij})_{i,j \in \mathbb{I}_q}$ such that

$$q_{11} = q_{23} q_{32} = q, \quad q_{12} q_{21} = q^{-1}, \quad q_{22} = q_{33} = -1, \quad q_{13} q_{31} = 1.$$ 

Let $\alpha_{jk} = \sum_{j \leq i \leq k} \alpha_i$; then $\Delta_q^+ = \{\alpha_{jk} : 1 \leq j \leq k \leq 3\}$, $\mathcal{O}_q^+ = \{\alpha_1, \alpha_{23}, \alpha_{13}\}$.

The Lusztig algebra $\mathcal{L}_q$ is presented by generators $y_{jk}^{(n)}$, $1 \leq j \leq k \leq 3$, $n \in \mathbb{N}$ and relations:
Corollary 4.11. The algebra \( \mathcal{L}_q \) is finitely generated.

Proof. By \([12]\), it is generated by \( \{ y_\beta : \beta \in \Delta^+_q \} \cup \{ y_\alpha^{(N)} : \alpha \in \mathcal{O}_q \} \).
Remark 4.12. Actually, the subalgebra $B_q \subset \mathcal{L}_q$ is generated by its primitive elements \( \{y_\alpha : \alpha \in \Pi_q\} \) where $\Pi_q$ denotes the set of simple roots $\alpha_1, \ldots, \alpha_q$. Moreover, $y_{\gamma}^{(N_{\gamma})} \in k^\ast [y_{\alpha}^{(N_{\alpha})}, y_{\beta}^{(N_{\beta})}]$ if and only if $x_{\alpha}^{N_{\alpha}} \otimes x_{\beta}^{N_{\beta}}$ appears with nonzero coefficient in $\Delta(x_{\gamma}^{N_{\gamma}})$. Hence,
\[
\{y_\alpha : \alpha \in \Pi_q\} \cup \{y_{\gamma}^{(N_{\gamma})} : \alpha \in \mathcal{O}_q, x_{\alpha}^{N_{\alpha}} \in \mathcal{P}(\tilde{B}_q)\}
\]
generates $\mathcal{L}_q$ as an algebra.

Proposition 4.13. $\tilde{L}^i$ is a left coideal subalgebra of $\mathcal{L}_q$.

Proof. From Theorem 4.9 we have that $y_{\beta_i}^{(n)} y_{\beta_j}^{(m)} \in \tilde{L}^i$ for $i < j$, thus $\tilde{L}^i$ is a subalgebra of $\mathcal{L}_q$. On the other hand, we know that $(y_{\beta_i}^{(n)}, xx') = \langle (y_{\beta_i}^{(n)})^{(1)}, x \rangle \langle (y_{\beta_i}^{(n)})^{(2)}, x' \rangle$. Therefore $y_h \otimes y_j$ appears with nonzero coefficient in $\Delta(y_{\beta_i}^{(n)})$ if and only if $x_h^{n} \otimes x_j$ appears with nonzero coefficient in the expression of $x_h^{n} y_j$ in the PBW basis. The last condition implies that $x_h \in \tilde{B}^\beta$ and $x_j \in \tilde{B}^\beta$. Hence,
\[
\Delta(y_{\beta_i}^{(n)}) \in \sum_{i=0}^{n} y_{\beta_i}^{(i)} \otimes y_{\beta_i}^{(n-i)} + \tilde{L}^\beta \otimes \tilde{L}^\beta.
\]
Hence $\Delta(y_{\beta_{M}}^{(n_{M})} \cdots y_{\beta_i}^{(n_i)}) = \Delta(y_{\beta_{M}}^{(n_{M})} \cdots y_{\beta_{i+1}}^{(n_{i+1})}) \Delta(y_{\beta_i}^{(n_i)}) \in \mathcal{L}_q \otimes \tilde{L}^i$ and the proof is complete. \(\square\)

4.3. The algebra $\mathcal{L}_q$ is Noetherian. We argue as in the pre-Nichols case [A3, Section 3.4], cf. [DP]. Let us consider the lexicographic order in $\mathbb{N}_0^M$, so that $h_M < \cdots < h_1$, where $(h_j)_{j \in \mathbb{I}_M}$ denotes the canonical basis of $\mathbb{Z}^M$.

Lemma 4.14. Let $\mathcal{L}_q(h)$ be the subspace of $\mathcal{L}_q$ generated by $y_j$, with $j \leq h$. Then $\mathcal{L}_q(h)$ is an $\mathbb{N}_0^M$-algebra filtration of $\mathcal{L}_q$.

Proof. It is enough to prove that $y_h y_j \in \mathcal{L}_q(h+j)$ for all $h, j \in \mathbb{N}$. First we consider the case when $h = n h_k + m h_l$, $k, l \in \mathbb{I}_M$, $n, m \in \mathbb{N}$. We claim that $y_{\beta_{k}}^{(n)} y_{\beta_{l}}^{(m)} \in \mathcal{L}_q(n h_k + m h_l)$. This follows by definition when $l \leq k$. If $k < l$, then $[y_{\beta_{k}}^{(n)}, y_{\beta_{l}}^{(m)}]_c \in \sum_{j < k} \tilde{L}_{k+1}^{j} y_{\beta_{k}}^{(j)}$ by Theorem 4.9, thus
\[
y_{\beta_{k}}^{(n)} y_{\beta_{l}}^{(m)} \in \mathcal{L}_q(n h_k + m h_l) \quad \text{since} \quad \sum_{j=k+1}^{M} a_j h_j < n h_k + m h_l.
\]
The Lemma follows by reordering the factors of $y_h y_j$, for any $h, j \in \mathbb{N}_0^M$. \(\square\)

We now consider the corresponding graded algebra
\[
gr \mathcal{L}_q = \oplus_{h \in \mathbb{N}_0^M} gr^h \mathcal{L}_q, \quad \text{where} \quad gr^h \mathcal{L}_q = \mathcal{L}_q(h) / \sum_{j < h} \mathcal{L}_q(j).
\]
Lemma 4.15. The algebra $\text{gr } L_q$ is presented by generators $y^{(n)}_k$, $k \in \mathbb{I}_M$, $n \in \mathbb{N}$, and relations

$$y^{(N_k)}_k = 0, \quad \beta_k \not\in O_q;$$

$$y^{(n)}_k y^{(m)}_k = \binom{n + m}{m} y^{(n+m)}_k,$$

$$[y^{(n)}_k, y^{(m)}_l]_c = 0, \quad k < l.$$

Proof. Let $\mathcal{G}$ be the algebra presented by the generators and relations above and $\pi : \mathcal{G} \to \text{gr } L_q$ given by $y^{(n)}_k \mapsto y^{(n)}_k$. By Theorem 4.9, the relations above hold in $\text{gr } L_q$. By a direct computation, $\mathcal{G}$ has a basis

$$\{y^{(h_1)}_1 \ldots y^{(h_i)}_i : h_i < N_i, i \geq 1\}.$$

On the other hand, $y_h \in L_q(h) - \sum_{j<h} L_q(j)$. Hence the projection of the PBW basis of $L_q$ is a basis of $\text{gr } L_q$ and $\pi$ is an isomorphism. \hfill $\square$

Proposition 4.16. The algebra $L_q$ is Noetherian.

Proof. Let $Z^+$ be the subalgebra of $\text{gr } L_q$ generated by $\{y^{(N_\beta)}_\beta : \beta \in O_q\}$. Then $Z^+$ is a quantum affine space and $\text{gr } L_q$ is a finitely generated free $Z^+$-module. Hence $\text{gr } L_q$ is Noetherian and so is $L_q$. \hfill $\square$

5. Divided powers algebras

5.1. Definition. Let $q$, $(V, c)$ be as above with $\dim B_q < \infty$. Let $W = V^*$, with braiding $c'$ as usual, and let $\{z^{(n)}_\beta : \beta \in \Delta_q, n \in \mathbb{N}\}$ be the generators of $L_{q'}$. In this section we define the divided powers algebra $U_q$ of $(V, c)$ and we establish some of its basic properties.

Let $\Gamma$ and $\Lambda$ be two copies of $\mathbb{Z}_+$, generated by $(K_i)_{i \in \mathbb{I}}$ and $(L_i)_{i \in \mathbb{I}}$ respectively; so that $(K^{\pm 1}_i)_{i \in \mathbb{I}}$ and $(L^{\pm 1}_i)_{i \in \mathbb{I}}$ are the generators of $k\Gamma$ and $k\Lambda$, respectively. Set $K_\alpha = K^{a_1}_1 \ldots K^{a_\theta}_\theta$ and $L_\alpha = L^{a_1}_1 \ldots L^{a_\theta}_\theta$ for $\alpha = (a_1, \ldots, a_\theta) \in \mathbb{Z}_+^\theta$. Then $L_q \in k\Gamma \mathcal{YD}$, $L_{q'} \in k\Lambda \mathcal{YD}$ with structure determined by the formulae

$$K^{\pm 1}_\alpha \cdot y^{(n)}_\beta = q^{\pm n}_{\alpha\beta} y^{(n)}_\beta, \quad \rho(y^{(n)}_\beta) = K^n_{\beta} \otimes y^{(n)}_\beta;$$

$$L^{\pm 1}_\alpha \cdot z^{(n)}_\beta = q^{\pm n}_{\beta\alpha} z^{(n)}_\beta, \quad \rho(z^{(n)}_\beta) = L^n_{\beta} \otimes y^{(n)}_\beta.$$

Therefore, we can consider the bosonizations $L_q \# k\Gamma$ and $L_{q'} \# k\Lambda$.

We define next the quantum double of $L_q \# k\Gamma$ and $L_{q'} \# k\Lambda$ following [J, 3.2.2]. For this we need a Hopf pairing between them.

Lemma 5.1. There is a unique bilinear form $(\cdot, \cdot) : T^c(V) \times T^c(W) \to k$ such that $(1|1) = 1$,

$$\langle y_i|z_j \rangle = \delta_{ij}, \quad i, j \in \mathbb{I};$$

$$\langle y|z z' \rangle = (y_1|z)(y_2|z'), \quad y \in T^c(V), z, z' \in T^c(W);$$

$\langle j| \cdot \rangle = 0, \quad i \not\in \mathbb{I}$.
\[(yy'z) = (y'z(z(1))y'z(z(2))), \quad y, y' \in T^c(V), z \in T^c(W);\]
\[(yz) = 0, \quad |y| \neq |z|, y \in T^c(V), z \in T^c(W).\]

**Proof.** Let \( T^n = \sum_{\sigma \in S_n} s(\sigma) : (T^c)^n(W) \to T^n(W), \) where \( s : S_n \to \mathbb{B}_n \) is the Matsumoto section, see [AG, §3.2]. Let \( \langle , \rangle : T^c(V) \otimes T(W) \to k \) be the evaluation map. We define \((11) = 1,\)
\[(yz) = \langle y, T^n(z)\rangle, \quad y \in (T^c)^n(V), z \in (T^c)^n(W)\]
\[(yz) = 0, \quad y \in (T^c)^n(V), z \in (T^c)^n(W), n \neq m.\]

Note that \( T^{i+j} = T_{i,j}(T^i \otimes T^j) \) with \( T_{i,j} = \sum s(\sigma^{-1}) \) where the sum is over all \((i,j)\)-shuffles \( \sigma. \) Then, for \( y \in (T^c)^n(V), z \in (T^c)^i(W), z' \in (T^c)^{n-i}(W),\)
\[(yzz') = \langle y, T^n(zz')\rangle = \langle y, T_{i,n-i}(T^i \otimes T^{n-i})(zz')\rangle = \langle y, T_{i,n-i}(T^i(z) \otimes T^{n-i}(z'))\rangle = (y_{(1)}(1), T^i(z), T^{n-i}(z')) = (y_{(1)}(1), T^i(z), T^{n-i}(z'))\]
\[= (y(1)) = (y(2)) = (y(3))\]
The other conditions are clear. \( \square \)

This bilinear form restricts to \( L_q \times L_q \) and then it can be extended to a bilinear form between their bosonizations. Then we may define a Hopf pairing between \( L_q \# k\Gamma \) and \( L_q \# k\Lambda, \) or equivalently:

**Corollary 5.2.** There is a unique skew-Hopf pairing

\[\langle , \rangle : L_q \# k\Gamma \times (L_q \# k\Lambda)^{\text{cop}} \to k\]

such that for all \( y^{(n)}_\alpha \in L_q, K_\alpha \in kZ^\theta, z^{(m)}_\beta \in L_q \) and \( L_\beta \in kZ^\theta,\)
\[(y^{(n)}_\alpha)z^{(m)}_\beta = \delta_{n\alpha, m\beta}, \quad (y^{(n)}_\alpha)L_\beta = 0, \quad (K_\alpha)(1)z^{(m)}_\beta = 0, \quad (K_\alpha)L_\beta = q_{\alpha\beta}.\]

Moreover, this pairing satisfies the equation \( yK|zL = (y|z)(K|L). \) \( \square \)

Let \( U_q \) be the Drinfeld double of \( L_q \# k\Gamma \) and \( (L_q \# k\Lambda)^{\text{cop}} \) with respect to the skew-Hopf pairing in Corollary 5.2. In other words:

**Definition 5.3.** Let \( U_q \) be the unique Hopf algebra such that

1. \( U_q = (L_q \# k\Gamma) \otimes (L_q \# k\Lambda) \) as vector spaces,
2. the maps \( Y \mapsto Y \otimes 1 \) and \( Z \mapsto 1 \otimes Z \) are Hopf algebra morphisms,
3. the product is given by
   \[ (Y \otimes Z)(Y' \otimes Z') = (Y'(1)|S(Z(1)))Y(2) \otimes Z(2)Z'(Y'(3)|Z(3)) \]
   for all \( Y, Y' \in L_q \# k\Gamma \) and \( Z, Z' \in (L_q \# k\Lambda)^{\text{cop}}. \)

By the construction of \( U_q, \) there is a triangular decomposition, via the multiplication, \( U_q \simeq U_q^+ \otimes U_0 \otimes U_q^- \) where
\[ U_q^+ \simeq L_q, \quad U_q^- \simeq L_q, \quad U_0 \simeq k(Z^\theta \times Z^\theta). \]

We give a presentation of the algebra \( U_q \) by generators and relations. The tensor product signs in elements of \( U_q \) will be omitted.
Proposition 5.4. The algebra $\mathcal{U}_q$ is generated by the elements $y_{\beta}^{(n)}$, $z_{\beta}^{(n)}$, $K_{\beta}^{\pm 1}$, $L_{\beta}^{\pm 1}$ for $\beta \in \Delta_+^q$, $n \in \mathbb{N}$; and relations (17), ..., (21) between the $y_{\beta}^{(n)}$'s, similar relations for the $z_{\beta}^{(n)}$'s plus the relations

\begin{equation}
K_{\beta}K_{\beta}^{-1} = L_{\beta}^{-1}L_{\beta} = 1, \quad K_{\beta}^{\pm 1}L_{\alpha}^{\pm 1} = L_{\alpha}^{\pm 1}K_{\beta}^{\pm 1}
\end{equation}

\begin{equation}
K_\alpha y_{\beta}^{(n)} = q_{\alpha \beta}^{n} y_{\beta}^{(n)} K_\alpha, \quad L_\alpha y_{\beta}^{(n)} = q_{\beta \alpha}^{-n} y_{\beta}^{(n)} L_\alpha,
\end{equation}

\begin{equation}
K_\alpha z_{\beta}^{(n)} = q_{\alpha \beta}^{-n} z_{\beta}^{(n)} K_\alpha, \quad L_\alpha z_{\beta}^{(n)} = q_{\beta \alpha}^{n} z_{\beta}^{(n)} L_\alpha,
\end{equation}

\begin{equation}
zy = (y_{\beta}^{(1)})S(z_{\beta}^{(3)}) (K_2 K_3 | L_3^{-1}) (y_{\beta}^{(3)}) (z_{\beta}^{(1)}) y_{\beta}^{(2)} L_3 z_{\beta}^{(2)} L_3,
\end{equation}

for all $\alpha, \beta \in \Delta_+^q$, $n, m \in \mathbb{N}$. Here in (25) $y = y_{\beta}^{(n)} \in \mathcal{L}_q$, $z = z_{\alpha}^{(m)} \in \mathcal{L}_q^t$, and denote $K_i = (y_{\beta}^{(1)})_{(-1)}$ and $L_i = (z_{\beta}^{(1)})_{(-1)}$ for the coactions of $k\Gamma$ and $k\Lambda$ respectively.

Note that if $y = y_{\alpha i}$, $z = z_{\alpha j}$ with $\alpha_i, \alpha_j \in \Pi_q$, then $y$, $z$ are primitives and relation (25) is $zy - yz = \delta_{ij}(K_i - L_i)$.

5.2. Basic properties. Proceeding as in [DP] A3, we will prove that the algebra $\mathcal{U}_q$ is Noetherian. For each $h, j \in \mathbb{H}$, $K \in \Gamma$, $L \in \Lambda$, set

$$d_1(y_{h} K L z_{j}) = \sum_{i \in \Lambda M} (h_i + j_i) ht(\beta_i),$$

$$d(y_{h} K L z_{j}) = \left( d_1(y_{h} K L z_{j}), h_1, \ldots, h_M, j_1, \ldots, j_M \right) \in \mathbb{N}_0^{2M+1}.$$  

Consider the lexicographic order in $\mathbb{N}_0^{2M+1}$. If $u \in \mathbb{N}_0^{2M+1}$, then we set $\mathcal{U}_q(u) = \text{span of } \{ y_{h} K L z_{j} : h, j \in \mathbb{H}, K \in \Gamma, L \in \Lambda, d(y_{h} K L z_{j}) \leq u \}$.

Lemma 5.5. ($\mathcal{U}_q(u))_{u \in \mathbb{N}_0^{2M+1}}$ is an $\mathbb{N}_0^{2M+1}$-algebra filtration of $\mathcal{U}_q$.

Proof. It is enough to prove that $(y_{h} K L z_{j})(y_{h'} K'L z_{j'}) \in \mathcal{U}_q(u + u')$ for all $h, j, h', j' \in \mathbb{H}$, $K, K' \in \Gamma$ and $L, L' \in \Lambda$ where $d(y_{h} K L z_{j}) = u$ and $d(y_{h'} K'L z_{j'}) = u'$.

First we claim that

$$d_1(z_{\beta}^{(n)} y_{\alpha}^{(m)} - y_{\alpha}^{(m)} z_{\beta}^{(n)}) < m \text{ht}(\alpha) + n \text{ht}(\beta).$$

Indeed, since the coproduct in $\mathcal{L}_q$ (resp. $\mathcal{L}_q^t$) is graded, we have that $d_1((y_{\alpha}^{(m)})^{(2)}) < m \text{ht}(\alpha)$ if $(y_{\alpha}^{(m)})^{(1)} \neq 1$ (resp. $d_1((z_{\beta}^{(n)})^{(2)}) < n \text{ht}(\beta)$ if $(z_{\beta}^{(n)})^{(1)} \neq 1$). Hence, for $K \in \Gamma$ and $L \in \Lambda$ we have

$$d_1((y_{\alpha}^{(m)})^{(2)} K L (z_{\beta}^{(n)})^{(2)}) \leq m \text{ht}(\alpha) + n \text{ht}(\beta)$$

and by Proposition 5.4 the claim follows.

Since $K$, $L$ q-commutes with all elements of $\mathcal{L}_q$ and $\mathcal{L}_q^t$ for all $K \in \Gamma$ and $L \in \Lambda$. We proceed as in Lemma 4.13 and we reduce the proof to the
product between $z_{\beta_i}^{(n)}$ and $y_{\beta_j}^{(m)}$. It follows directly by [26] that

$$z_{\beta_i}^{(n)} y_{\beta_j}^{(m)} \in \mathcal{U}_q(m \operatorname{ht}(\beta_j) + n \operatorname{ht}(\beta_i), \delta_j, \delta_i).$$

Proof. Let $\mathcal{U}_q = \oplus_{\nu \in \mathbb{N}^{2M}} \mathcal{U}_q^{\nu}$ where $\mathcal{U}_q^{\nu} = \mathcal{U}_q(\nu)/\sum_{\nu < \nu} \mathcal{U}_q(\mu)$.

**Corollary 5.6.** The algebra $\mathfrak{gr} \mathcal{U}_q$ is presented by generators $y_{j}^{(n)}$, $z_{j}^{(n)}$, $K_j^{\pm 1}$, $L_j^{\pm 1}$, $j \in I_M$, $n \in \mathbb{N}$ and relations

\begin{align*}
RS &= SR, & R, S &\in \{K_j^{\pm 1}, L_j^{\pm 1} : j \in I_M\} \\
K_{\alpha} K_{\beta}^{-1} &= L_{\beta} L_{\beta}^{-1} = 1 & y_{k}^{(n)} z_{l}^{(m)} &= z_{l}^{(m)} y_{k}^{(n)} \\
y_{k}^{(N_k)} &= 0, & z_{k}^{(N_k)} &= 0, \\
y_{k}^{(n)} y_{k}^{(m)} &= \left(\begin{array}{c} n + m \\
m \end{array}\right)_{q_{\delta_k} \delta_k} & z_{k}^{(n)} z_{k}^{(m)} &= \left(\begin{array}{c} n + m \\
m \end{array}\right)_{q_{\delta_k} \delta_k} \\
[y_{k}^{(n)}, y_{l}^{(m)}]_{c} &= 0, & [z_{k}^{(n)}, z_{l}^{(m)}]_{c} &= 0, \\
K_{\alpha} y_{\beta}^{(n)} &= q_{\alpha \beta}^{n} y_{\beta}^{(n)} K_{\alpha}, & K_{\alpha} z_{\beta}^{(n)} &= q_{\alpha \beta}^{-n} z_{\beta}^{(n)} K_{\alpha}, \\
L_{\alpha} y_{\beta}^{(n)} &= q_{\beta \alpha}^{-n} y_{\beta}^{(n)} L_{\alpha}, & L_{\alpha} z_{\beta}^{(n)} &= q_{\beta \alpha}^{n} z_{\beta}^{(n)} L_{\alpha}.
\end{align*}

**Proof.** The proof of this statement is similar to the proof of Lemma 4.15 if we check that $y_{k}^{(n)} z_{l}^{(m)} = z_{l}^{(m)} y_{k}^{(n)}$ for all $y_{k}^{(n)} \in \mathcal{L}_q$ and $z_{l}^{(m)} \in \mathcal{L}_q$, but this follows by [26].

**Proposition 5.7.** The algebra $\mathcal{U}_q$ is Noetherian and $\text{GKdim} \mathcal{U}_q = 2|\mathcal{O}_q| + 2\theta$.

**Proof.** Let $\mathcal{Z}$ be the subalgebra of $\mathfrak{gr} \mathcal{U}_q$ generated by $\{K_i, L_i : i \in I\}$ and $\{y_{\beta}^{(N_\beta)}, z_{\beta}^{(N_\beta)} : \beta \in \mathcal{O}_q\}$. Then $\mathcal{Z}$ is the localization of a quantum affine space and $\mathfrak{gr} \mathcal{U}_q$ is a free $\mathcal{Z}$-module of rank $\prod_{i \in I_M} N_i$. Therefore $\mathfrak{gr} \mathcal{U}_q$ is Noetherian and so is $\mathcal{U}_q$. Moreover, by [KL, Prop. 6.6],

$$\text{GKdim} \mathcal{U}_q = \text{GKdim} \mathfrak{gr} \mathcal{U}_q = \text{GKdim} \mathcal{Z} = 2|\mathcal{O}_q| + 2\theta.$$

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