Stability data and t-structures on a triangulated category

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We propose the notion of stability on a triangulated category that is a generalization of the T. Bridgeland’s stability data. We establish connections between stabilities and t-structures on a category and as application we get the classification of bounded t-structures on the category $D^b(Coh \mathbb{P}^1)$.

1 Introduction

For quite a number of years the authors believed that certain generalization of stability would be desirable to use in the context of derived and triangulated categories, but could not come to the satisfactory definition.

Recently in his article \cite{6}, T. Bridgeland, following not so rigorously presented, but inspiring physical ideas of Michael R. Douglas (\cite{9}) provided the definition of stability for a triangulated category. Bridgeland also showed how the key properties of stability can be reformulated in this context and worked on describing all the stabilities for a given category via a kind of moduli space of stabilities. Although it is not clear to us where the approach to constructing the moduli space of stabilities proposed by T. Bridgeland will lead, we believe that his definition of the stability makes the breakthrough.

In this paper we propose the definition of stability for a triangulated category or, in short, t-stability which is the modification and in fact the generalization of the definition given by T. Bridgeland. We believe we have taken away “unnecessary details” keeping the core features intact. In short, we exclude all about the “central charge” (in the sense of \cite{6}) from the definition of t-stability, and we do not demand that the semi-stable subcategories are ordered according the real numbers assigned to them as indexes. But we keep the way to generalize the Harder-Narasimhan filtration that was proposed by T. Bridgeland.

We begin in Section 2 reminding the basic properties of the stability for an abelian category. Section 3 is devoted to the definition of t-stability for a triangulated category. We also discuss basic properties of t-stability and several basic examples or constructions of a t-stability on a triangulated (or derived) category.

We show (p. 7) that the natural Gieseker stability for torsion free coherent sheaves extend to the derived category of coherent sheaves, and that the helix generalization (\textsuperscript{11}) of the Beilinson theorem about the basis of the derived category $D^b(Coh \mathbb{P}^n)$ of coherent sheaves on the projective space $\mathbb{P}^n$ (\textsuperscript{2}) leads to the remarkable t-stabilities for this category (Proposition 3.7).

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In Section 4 we discuss the Postnikov systems that we start to call in our context by t-filtrations. We show that the choice of t-stability provides each object with the canonical Harder-Narasimhan t-filtration and we describe properties of these HN-filtrations.

In Section 5 we develop several methods to connect t-stabilities and t-structures on a category, as well as to make more coarse or refined t-stabilities.

For the case \( \mathbb{P}^1 \) this enables us to get the full classification of t-stabilities on \( D^b(\text{Coh} \mathbb{P}^1) \) that we present in Subsection 6.4. As a consequence in Subsection 6.10 we obtain the full list of bounded t-structures on \( D^b(\text{Coh} \mathbb{P}^1) \) (considered as a triangulated category).

In Section 7 we show how to describe all t-stabilities for coherent sheaves on an elliptic curve.

2 Stability data on abelian categories

We begin with a remark about the stability of coherent sheaves. This notion arose as a tool for construction moduli spaces of coherent torsion free sheaves on varieties and came from the invariant theory. The moduli space is obtained as an orbit space for a reductive group action on a vector space that parameterizes sheaves with fixed topological invariants. In this approach moduli space points corresponding to closed orbits are well defined. Closed orbits and sheaves containing in such orbits are called stable. Unfortunately, in many cases the set of stable (closed) orbits is not compact. To compactify it we should add orbits, whose closure does not contain the zero vector. Such orbits and corresponding sheaves are called semistable.

There exist numerical criteria of stability for orbits (and sheaves) (see, for example [10, 13], and [14]). Roughly speaking, we correspond a vector \( \mu(F) \) (or, simply, number) to each a torsion free sheaf \( F \). This vector is called a slope of the sheaf. The criterium says that a torsion free sheaf \( F \) is semistable iff for any nonzero subsheaf \( E \subset F \) we have \( \mu(E) \leq \mu(F) \) (the vectors are compared lexicographically). The stability of a sheaf \( F \) means that \( F \) is semistable and simple (i.e. \( \text{Hom}(F,F) = \mathbb{C} \)).

As a rule, each a component of a vector slope is a ratio of additive functions on \( K_0(\text{Coh}) \), the Grothendieck group of a coherent sheaves category. Therefore slope satisfies the following seesaw condition:

\[
\text{for an exact sequence of torsion free sheaves } \quad 0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0
\]

we have

\[
\begin{align*}
\mu(E) &< \mu(F) \iff \mu(F) < \mu(G), \\
\mu(E) &= \mu(F) \iff \mu(F) = \mu(G), \\
\mu(E) &> \mu(F) \iff \mu(F) > \mu(G).
\end{align*}
\]

Besides the moduli space construction, stability of coherent sheaves have two more useful applications, following from the seesaw condition:

(i) for any semistable sheaves \( E \) and \( F \) an inequality \( \mu(E) > \mu(F) \) implies \( \text{Hom}(E,F) = 0 \);

(ii) any torsion free sheaf \( X \) has a canonical Harder–Narasimhan filtration

\[
X = F^0 X \rightarrow F^1 X \rightarrow F^2 X \rightarrow \cdots \rightarrow F^n X \rightarrow F^{n+1} X = 0,
\]

\[
\text{with } G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \cdots \rightarrow G_n.
\]
where each vertical epimorphism is a part of the short exact sequence

\[ 0 \to F^i X \to F^{i-1} X \to G_i \to 0 \]

with semistable \( G_i \) and \( \mu(G_i) < \mu(G_j) \) whenever \( i < j \).

So, we see that a stability of coherent sheaves gives a powerful filtration but for torsion free sheaves only. We would like to define a stability in such a way that any nonzero object has a Harder–Narasimhan filtration. The first abstract definition of stability on an abelian category was done in [15] as follows.

**Definition 2.1.** Let us say that a stability structure on an abelian category \( \mathcal{A} \) is given if there is a preorder on \( \mathcal{A} \) such that for an exact sequence of nonzero objects

\[ 0 \to A \to B \to C \to 0 \]

we have the seesaw property:

- either \( A \preceq B \iff A \preceq C \iff B \preceq C \),
- or \( A \succ B \iff A \succ C \iff B \succ C \),
- or \( A \asymp B \iff A \asymp C \iff B \asymp C \).

We’ll call a nonzero object \( A \in \mathcal{A} \) semistable if \( B \preceq A \) whenever \( 0 \neq B \subseteq A \).

In [15] the following theorem was proved.

**Theorem 2.2.** 1. If objects \( A, B \) are semistable and \( A \prec B \), then \( \text{Hom}(B, A) = 0 \).

2. Suppose \( \mathcal{A} \) is weakly-artinian and weakly-noetherian, then for an object \( X \in \mathcal{A} \) there exists a unique Harder–Narasimhan filtration

\[ X = F^0 X \leftarrow F^1 X \leftarrow F^2 X \leftarrow \cdots \leftarrow F^n X \leftarrow F^{n+1} X = 0 \]

with semistable quotients \( G_i \)'s such that \( G_i \prec G_j \) whenever \( i < j \).

Thus, having a preorder on an abelian category satisfying the seesaw property and finiteness conditions, we obtain the set of semistable objects and Harder–Narasimhan filtration for each object. But there appears a question: how one can order the objects? In all known examples such an order is obtained with the help of slope. Therefore we propose an abstract definition of slope on an abelian category, generalizing Bridgeland’s central charge.

**Definition 2.3.** Let \( \mathcal{A} \) be an abelian category. A linearly independent system of additive functions \( (x_0, \ldots, x_{r-1}) \) on \( K_0(\mathcal{A}) \) (the Grothendieck group of \( \mathcal{A} \)) is called positive if for any \( A \in \mathcal{A} \) the following holds

\[ x_0(A) \geq 0, \quad \text{and} \]
\[ x_0(A) = 0 \Rightarrow x_1(A) \geq 0, \quad \text{and} \]
\[ x_0(A) = x_1(A) = 0 \Rightarrow x_2(A) \geq 0, \quad \text{and} \]
\[ \ldots \]
\[ x_0(A) = \cdots = x_{r-2}(A) = 0 \Rightarrow x_{r-1}(A) > 0. \]

If, in addition

\[ x_0(A) = \cdots = x_{r-1}(A) = 0 \Rightarrow A = 0, \]
the positive system is called a positive base.

Let $s = \min_i \{ x_i(A) \neq 0 \}$ and

$$\gamma(A) = \left( \frac{1}{s}, \ldots, \frac{1}{s}, \nu \left( \frac{-x_{s+1}(A)}{x_s(A)} \right), \nu \left( \frac{-x_{s+2}(A)}{x_s(A)} \right), \ldots, \nu \left( \frac{-x_r(A)}{x_s(A)} \right) \right),$$

where

$$\nu \left( \frac{a}{b} \right) = \begin{cases} \frac{1}{\pi} \arctg \frac{a}{b}, & b > 0, \\ 1, & b = 0. \end{cases}$$

We call $\gamma(A)$ the slope of an object $A \in A$, determined by the positive system $(x_0, \ldots, x_{r-1})$ (not necessary a base).

For example, the base $(\text{rk}, \text{deg})$ on the category $\text{Coh}_C$ of coherent sheaves on a smooth algebraic curve $C$ has the positivity property, the base $(\text{rk}, \text{deg}, \chi(\mathcal{O}_S, \cdot))$ on the category $\text{Coh}_S$ of coherent sheaves on a smooth algebraic surface $S$ with Picard’s number 1 has the positivity property as well.

Since the slope $\gamma$ is formed via additive functions on $K_0(A)$, the ordering induced by $\gamma$ ($A \preceq B \iff \gamma(A) \leq \gamma(B)$) satisfies the seesaw property. Therefore, we obtain a stability structure and, consequently, Harder–Narasimhan filtration for each object.

Another way to consider stability on an abelian category is to take the properties (i) and (ii) above as a definition. Namely,

**Definition 2.4.** Let $A$ be an abelian category and $\Phi$ be a linearly ordered set. Suppose that for any $\varphi \in \Phi$ a subcategory $\Pi_\varphi \subset A$ is determined and $\Pi_\varphi$ are closed under extensions. If the following properties are valid

(i) $\text{Hom}_A(\Pi_{\varphi'}, \Pi_{\varphi''}) = 0$ for $\varphi' > \varphi''$;

(ii) each nonzero object $X \in A$ has a Harder–Narasimhan filtration

$$X = F^0 X \longrightarrow F^1 X \longrightarrow F^2 X \longrightarrow \cdots \longrightarrow F^n X \longrightarrow F^{n+1} X = 0$$

with $G_i = F^i X/F^{i+1} X \in \Pi_{\varphi_i}$ and $\varphi_i < \varphi_j$ for $i < j$;

then we call the data $(\Phi, \{ \Pi_\varphi \}_{\varphi \in \Phi})$ stability data (or stability) on $A$.

Thus we come to three definitions of stability on an abelian category:

1. via ordering objects (let us call it order-stability);

2. via base of $K^*(A)$ with positivity property (slope-stability);

3. via collection of semistable categories (stability data).

It is obvious, that any slope-stability on an abelian category induces an order-stability, and the last one induces a stability data. But we don’t know: if one can reconstruct the slope-stability (or the order-stability) starting with given stability data on an abelian category. The question is interesting.
3 Definition and basic examples

In Section 2 we considered a stability on an abelian categories. Keeping in mind the three
definitions, we see that the best for extending a stability on a triangulated category is the last
one (def. 2.4). We need the generalization of the Harder–Narasimhan filtration to a triangulated
category. The first step in this direction was done by T. Bridgeland in [6]. We modify the
Bridgeland’s definition of stability data, and exclude any reference to a central charge in order
to make them more general.

The classical notion of a filtration is based on subobjects and quotients. For a triangulated
category Bridgeland proposed a natural generalization ([6]), namely a Postnikov’s system

\[ \begin{array}{c}
X_{\phi_0} & \xrightarrow{p_1} & X_{\phi_1} & \xrightarrow{p_2} & \cdots & \xrightarrow{p_n} & X_{\phi_n} \\
\phi & \downarrow & \phi & \downarrow & \cdots & \downarrow & \phi \\
F^0X & \xleftarrow{p_1} & F^1X & \xleftarrow{p_2} & \cdots & \xleftarrow{p_n} & F^{n+1}X = 0
\end{array} \]

(where each a triangle \( F^iX \xrightarrow{p} F^{i+1}X \) is distinguished). We call such a system a filtration
of an object \( X \) in a triangulated category or in short ”t-filtration”. It is natural to call the
objects \( X_{\phi_i} \) quotients and \( F_iX \) terms of the t-filtration. Often we need information only about
quotients of a t-filtration on \( X \). In this case we write: \( X \rightsquigarrow (X_0, X_1, \ldots, X_n) \) as a notation
for the t-filtration.

We are ready to formulate the main definition.

**Definition 3.1.** Let \( \mathcal{T} \) be a triangulated category, \( \Phi \) be a linearly ordered set. Suppose that for
each \( \phi \in \Phi \) a strongly full, extension-closed\(^2\) nonempty subcategory \( \Pi_\phi \subset \mathcal{T} \) is defined. The pair
\((\Phi, \{\Pi_\phi\}_{\phi \in \Phi})\) is called stability data (or simply t-stability) if

1. the shift of the triangulated category acts on the set \( \{\Pi_\phi\}_{\phi \in \Phi} \) in the following sense: there
exists \( \tau \in \text{Aut} \Phi \) such that \( \Pi_\phi[1] = \Pi_{\tau(\phi)} \) and \( \tau(\phi) \geq \phi \);
2. \( \forall \psi > \phi \in \Phi \quad \text{Hom}^{\leq 0}(\Pi_\psi, \Pi_\phi) = 0 \);
3. for any nonzero object \( X \in \mathcal{T} \) there exists a finite Postnikov’s system:

\[ \begin{array}{c}
X_{\phi_0} & \xrightarrow{p_1} & X_{\phi_1} & \xrightarrow{p_2} & \cdots & \xrightarrow{p_n} & X_{\phi_n} \\
\phi & \downarrow & \phi & \downarrow & \cdots & \downarrow & \phi \\
F^0X & \xleftarrow{p_1} & F^1X & \xleftarrow{p_2} & \cdots & \xleftarrow{p_n} & F^{n+1}X = 0
\end{array} \]

with nonzero \( X_{\phi_i} \in \Pi_\phi \) and \( \phi_i < \phi_{i+1} \).

We shall call such a system Harder–Narasimhan system (HN-system or HN-filtration) of \( X \), the
objects \( X_{\phi_i} \) are \( \Phi \)-semistable quotients (w.r.t stability data \((\Phi, \{\Pi_\phi\}_{\phi \in \Phi})\)), the subcategories \( \Pi_\phi \)
are semistable subcategories of slope \( \phi \).

Fairly often we denote stability data \((\Phi, \{\Pi_\phi\}_{\phi \in \Phi})\) simply by \( \Phi \).

\(^2\)An extension-closed subcategory \( A \subset \mathcal{T} \) means that whenever \( A \rightarrow B \rightarrow C \rightarrow A[1] \) is a distinguished
triangle, with \( A \in A \) and \( C \in A \), then \( B \in A \) also.
To construct some trivial, but important examples, recall (3) that a t-structure on a triangulated category $\mathcal{T}$ is a pair $(D_{\leq 0}, D_{\geq 0})$ of strictly full subcategories, satisfying the following conditions:

1. $D_{\leq 0} \subset D_{\leq 0}[-1]$ and $D_{\geq 0} \supset D_{\geq 0}[-1]$;
2. $\text{Hom}^0(D_{\leq 0}, D_{\geq 0}[-1]) = 0$;
3. $\forall X \in \mathcal{T}$ there exists a distinguished triangle

$$X \xrightarrow{a} X_{\geq 1} \xrightarrow{p} X_{\leq 0}$$

with $X_{\geq 1} \in D_{\geq 0}[-1]$ and $X_{\leq 0} \in D_{\leq 0}$.

If the following property holds, then the t-structure is called bounded (6):

4) $\forall X \in \mathcal{T}$ there exists $m, n \in \mathbb{Z}$ such that $X \in D_{\geq 0}[-m] \cap D_{\leq 0}[-n]$

We shall use the standard notation: $D_{\leq n} = D_{\leq 0}[-n]$ and $D_{\geq n} = D_{\geq 0}[-n]$.

**Lemma 3.2.** Suppose $(D_{\leq 0}, D_{\geq 0})$ is a t-structure on $\mathcal{T}$ such that $D_{\leq 0} = D_{\leq 1} = D_{\leq 0}[-1]$, $D_{\geq 0} = D_{\geq 1} = D_{\geq 0}[-1]$. Let

$$\Phi = \{0, 1\}, \quad \Pi_0 = D_{\geq 0}[-1], \quad \Pi_1 = D_{\leq 0}, \quad \tau(0) = 0, \quad \tau(1) = 1.$$  

Then $(\Phi, \{\Pi_{\varphi}\}_{\varphi \in \Phi})$ determines stability data on $\mathcal{T}$.

**Proof:** follows immediately from the definitions. $\blacksquare$

**Lemma 3.3.** Let $(D_{\leq 0}, D_{\geq 0})$ be a bounded t-structure on a triangulated category $\mathcal{T}$.

Let $\Pi_i = A[i] = D_{\leq i} \cap D_{\geq i}$, then $(\mathbb{Z}, \{\Pi_i\}_{i \in \mathbb{Z}})$ makes stability data on $\mathcal{T}$.

**Proof:** can go as follows:

Notice that if $p < q$, then $\text{Hom}^{0}(D_{\leq -p}, D_{\geq -q}) = 0$.

Since the t-structure $(D_{\leq 0}, D_{\geq 0})$ is bounded, then for each nonzero object $X \subset \mathcal{T}$ there exist $n_+(X) \geq n_-(X) \in \mathbb{Z}$ such that

$$\text{Hom}^0(X, D_{\geq -n_-(X)}) \neq 0, \quad \text{Hom}^{0}(X, D_{\leq -n_+(X)}) \neq 0,$$

$$\text{Hom}^0(D_{\leq -n_+(X)}, X) \neq 0, \quad \text{Hom}^0(D_{\leq -k}, X) = 0 \text{ for } k > n_+(X).$$

Denote $n_-(X)$ by $n_0$ and consider a distinguished triangle

$$X \xrightarrow{a} Y_0 \xrightarrow{p} X_0$$

with $X_0 \in D_{\geq -n_0}$, $Y_0 \in D_{\leq -(n_0+1)}$ (it exists by the definition of t-structure). Clearly we get
Hom \leq 0(Y_0, X_0) = 0;
• X_0 \neq 0;
• X_0 \in D^{\geq -n_0} \cap D^{\leq -n_0} = \Pi_{n_0},

and Y_0 serves as the first term of the Harder–Narasimhan filtration.

Now one can finish the proof by induction. ■

We would like to note that Definition 2.4 and Lemma 3.3 allow us to extend a stability from an abelian category \(A\) onto the bounded derived category of \(A\). Let us formulate slightly more general fact.

**Proposition 3.4.** Let \(T\) be a triangulated category with a bounded t-structure \((D^{\geq 0}, D^{\leq 0})\) (t-category). Suppose that on the core \(A = D^{\geq 0} \cap D^{\leq 0}\) of the t-structure stability data \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) are given. Consider the set \(\Psi = \mathbb{Z} \times \Phi\) with the lexicographic order by \(\Psi\) put \(P_{(i, \varphi)} = \Pi_\varphi[i]\). Then \((\Psi, \{P_{(i, \varphi)}\})_{(i, \varphi) \in \Psi}\) constitutes stability data on the category \(T\).

**Proof.** There is only one non-obvious moment in the proof. Namely, the existence of the finite HN-system for each nonzero object. But this follows directly from Definition 2.4 of stability data on an abelian category, Lemma 3.3 and Proposition 4.3 that we prove in the next section. ■

**Corollary 3.5.** Any stability structure \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})\) on an abelian category \(A\) induces a t-stability \((\mathbb{Z} \times \Phi, \{\Pi_\varphi[i]\})_{(i, \varphi) \in \mathbb{Z} \times \Phi}\) on the bounded derived category \(D^b(A)\).

It was be shown in article [15] that Gieseker stability of torsion free coherent sheaves on a projective variety \(X\) one can extend to a stability structure on the category \(\text{Coh} X\) of all coherent sheaves on \(X\). Therefore due to the previous corollary we have an extension of Gieseker stability to a t-stability on the bounded derived category \(D^b(\text{Coh} X)\). A detailed discussion of this t-stability is the subject of another article.
Then for any object \( X \in \mathcal{T} \) there exists a canonical Postnikov system

\[
\begin{array}{cccccc}
X & \overset{V_0^* \otimes E_0}{\rightarrow} & F^1 X & \overset{V_1^* \otimes E_1}{\rightarrow} & F^2 X & \rightarrow \ldots \rightarrow F^n X & \overset{V_n^* \otimes E_n}{\rightarrow} 0
\end{array}
\]

Under the assumption of Theorem 3.6 let us denote by \( \Upsilon_i \) the full extension-closed subcategory in \( \mathcal{T} \), generated by objects \( E_i[z] \) with \( z \in \mathbb{Z} \).

Since the collection \( (E_0, E_1, \ldots, E_n) \) is exceptional, the subcategories \( \Upsilon_i \) satisfy the property \( \text{Hom}^{\leq 0}(\Upsilon_j, \Upsilon_i) = 0 \) whenever \( j > i \). Taking into account Theorem 3.6 we obtain stability data on \( \mathcal{T} \). More exactly the following proposition takes place.

**Proposition 3.7.** Let \( \mathcal{T} \) be a linear triangulated category, generated by an exceptional collection \( (E_0, E_1, \ldots, E_n) \). Then \( (\Delta, \{\Upsilon_i\}_{i \in \Delta}) \) makes stability data on \( \mathcal{T} \), where \( \Delta \) is the natural ordering set \( \{0, 1, \ldots, n\} \).

In the next section we show that HN-system has the familiar properties of the Harder–Narasimhan filtration defined for coherent sheaves in respect to Gieseker or Mumford–Takemoto stability.

## 4 Properties of HN-system

We fix stability data \( (\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}) \) on a triangulated category \( \mathcal{T} \). The important property of HN-system is that it is a canonical \( t \)-filtration.

**Theorem 4.1.** The HN-system for any object \( X \in \mathcal{T} \) is determined up to a unique isomorphism of Postnikov’s systems.

**Proof.** Let

\[
\begin{array}{cccccc}
X = F^0 X & \overset{\varphi_0}{\rightarrow} & F^1 X & \overset{\varphi_1}{\rightarrow} & F^2 X & \rightarrow \ldots \rightarrow F^n X & \overset{\varphi_n}{\rightarrow} F^{n+1} X = 0
\end{array}
\]  

(4.1)

and

\[
\begin{array}{cccccc}
X = Q^0 X & \overset{\psi_0}{\rightarrow} & Q^1 X & \overset{\psi_1}{\rightarrow} & Q^2 X & \rightarrow \ldots \rightarrow Q^m X & \overset{\psi_m}{\rightarrow} Q^{m+1} X = 0
\end{array}
\]  

(4.2)

be two HN-systems for an object \( X \in \mathcal{T} \). We have show that \( n = m \), \( \varphi_i = \psi_i \ \forall i \), and the
identical morphism \( id_X \in \text{Hom} \tau(X, X) \) induces a unique isomorphism of t-filtrations:

\[
\begin{array}{cccccc}
X & \xrightarrow{X_{\varphi_0}} & \xrightarrow{X_{\varphi_1}} & \cdots & \xrightarrow{X_{\varphi_n}} & X \\
F^1X & \xrightarrow{\varphi_1} & F^2X & \cdots & F^nX & F^{n+1}X \\
Q^1X & \xrightarrow{\psi_1} & Q^2X & \cdots & Q^nX & Q^{n+1}X
\end{array}
\]

To do this we prove some additional properties of HN-systems:

**Proposition 4.2.** Let

\[
X = F^0X \xrightarrow{X_{\varphi_0}} F^1X \xrightarrow{X_{\varphi_1}} \cdots \xrightarrow{X_{\varphi_n}} F^{n+1}X = 0
\]

be a HN-system for \( X \). Then

1. \( \text{Hom}^{\leq 0}(X, \Pi_{\varphi}) = 0 \) whenever \( \varphi < \varphi_0 \),
2. \( \text{Hom}^{\leq 0}(F^iX, \Pi_{\varphi}) = 0 \) whenever \( \varphi \leq \varphi_i \),
3. \( \text{Hom}^{\leq 0}(\Pi_{\psi}, X) = 0 \) whenever \( \psi > \varphi_n \);
4. if

\[
Y = F^0Y \xrightarrow{Y_{\psi_0}} F^1Y \xrightarrow{Y_{\psi_1}} \cdots \xrightarrow{Y_{\psi_m}} F^{m+1}Y = 0
\]

is a HN-system for \( Y \) such that \( \varphi_n < \psi_0 \), then \( \text{Hom}^{\leq 0}(Y, X) = 0 \).

**Proof of the proposition** is easy obtained via the application of functors \( \text{Hom}(\cdot, \Pi_{\varphi}) \), \( \text{Hom}(\Pi_{\psi}, \cdot) \), and \( \text{Hom}(Y, \cdot) \) to each of the triangle

\[
\xrightarrow{X_{\varphi_1}}
\]

\[
F^iX \xleftarrow{F^{i+1}X}
\]

One should notice that \( F^nX \in \Pi_{\varphi_n} \). We leave the details to the reader. ■

Returning to the proof of the theorem, consider the first triangle of HN-system for \( X \) and any \( Y \in \Pi_{\varphi_0} \). Applying the functor \( \text{Hom}(\cdot, Y) \) to the triangle, we have

\[
\text{Hom}^{-1}(F^1X, Y) \rightarrow \text{Hom}^0(X_{\varphi_0}, Y) \xrightarrow{h_Y} \text{Hom}^0(X, Y) \rightarrow \text{Hom}^0(F^1X, Y)
\]

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It follows from Proposition 4.2(2), that the very left and right terms of the above exact sequence are zero. Hence,
\[ \text{Hom}^0(X_{\varphi_0}, Y) \cong \text{Hom}^0(X, Y) \quad \forall Y \in \Pi_{\varphi_0}. \]  
(4.3)
In other words, the object \( X_{\varphi_0} \) represents the functor \( \text{Hom}^0(X, \cdot) : \Pi_{\varphi_0} \to \text{Vect} \).

Let us substitute \( X_{\varphi_0} \) for \( Y \) in (4.3) and denote the morphism \( h_{X_{\varphi_0}}(id_{X_{\varphi_0}}) \in \text{Hom}^0(X, X_{\varphi_0}) \) by \( q_0 \). It can be shown in the usual way (see, for example, [8, Lemma IV.4.5]) that the distinguished triangle
\[
\begin{array}{ccc}
X_{\varphi_0} & \xrightarrow{q_0} & F^1X \\
\downarrow & & \downarrow \\
X & \xrightarrow{id_X} & X
\end{array}
\]
where \( X_{\varphi_0} \in \Pi_{\varphi_0} \) and \( \text{Hom}^0(F^1X, X_{\varphi_0}) = 0 \), is determined up to a unique isomorphism of triangles.

Note that \( q_0 \neq 0 \). Therefore, \( \text{Hom}^0(X, \Pi_{\varphi_0}) \neq 0 \). Further, starting with the first triangle of HN-system (4.2), we get \( \text{Hom}^0(X, X_{\varphi_0}) \neq 0 \). Applying now Proposition 4.2(1) to \( X, \Pi_{\varphi_0} \) and \( \Pi_{\psi_0} \), we conclude that \( \varphi_0 = \psi_0 \). Moreover, it follows from the uniqueness of the object representing a functor, that the identical map \( id_X \in \text{Hom}_T(X, X) \) extends to a canonical isomorphism of triangles:
\[
\begin{array}{ccc}
X_{\varphi_0} & \cong & X_{\psi_0} \\
X & \cong & X \\
F^1X & \cong & Q^1X
\end{array}
\]
The rest of the proof is done by induction. ■

The last proposition that we prove in this section deals with properties of a general t-filtration.

**Proposition 4.3.**

1. Let \( X \) has a t-filtration \( X \rightsquigarrow (Y_0, Y_1, \ldots, Y_n) \) and each \( Y_s \) has one too \( Y_s \rightsquigarrow (X_{(s,0)}, \ldots, X_{(s,k_s)}) \) \( \forall s \). Then one can construct a combined t-filtration
\[ X \rightsquigarrow (X_{(0,0)}, \ldots, X_{(0,k_0)}, X_{(1,0)}, \ldots, X_{(1,k_1)}, \ldots, X_{(n,0)}, \ldots, X_{(n,k_n)}). \]

2. Let
\[ X \rightsquigarrow (X_{(0,0)}, \ldots, X_{(0,k_0)}, X_{(1,0)}, \ldots, X_{(1,k_1)}, \ldots, X_{(n,0)}, \ldots, X_{(n,k_n)}) \]
be a t-filtration. Then there exist t-filtrations \( X \rightsquigarrow (Y_0, Y_1, \ldots, Y_n) \) and \( Y_s \rightsquigarrow (X_{(s,0)}, \ldots, X_{(s,k_s)}) \) for each \( s \).

3. Let \( X \rightsquigarrow (X_0, X_1, \ldots, X_n) \) and \( Y \rightsquigarrow (Y_0, Y_1, \ldots, Y_m) \) be t-filtrations. Then for each shuffle permutation of the quotients, i.e. linear ordering of the set
\[ \{Z_s\}_{s=0,\ldots,n+m+1} = \{X_i\}_{i=0,\ldots,n} \bigcup \{Y_j\}_{j=0,\ldots,m} \]
that respects the initial orders on \( \{X_i\}_{i=0,\ldots,n} \) and \( \{Y_j\}_{j=0,\ldots,m} \) there exists a t-filtration
\[
X \oplus Y \rightsquigarrow (Z_0, Z_1, \ldots, Z_{n+m+1}).
\]
Proof. We start with the second statement. Let $t$-filtration for $X$ in the statement be determined by the sequence of distinguished triangles:

$$
\begin{align*}
X_{(i,j)} & \xrightarrow{F_{(i,j)} X} F_{(i,j+1)} X, & \text{where } i = 0, \ldots, n; j = 0, \ldots, k_i - 1; \text{ and} \\
X_{(i,k_i)} & \xrightarrow{F_{(i,k_i)} X} F_{(i+1,0)} X, & \text{where } i = 0, \ldots, n; \\
F^{(0,0)} X & = X; \text{ and } F^{(n+1,0)} X = 0.
\end{align*}
$$

Denote the composition $p_{(i+1,0)} p_{(i,k_i)} \cdots p_{(i,1)}: F^{(i+1,0)} X \rightarrow F^{(i,0)} X$ by $p_i$. In these notations we have got a new filtration for $X$ that consists of triangles:

$$
Y_i 
\begin{align*}
Y_i & \xrightarrow{F^{(i,0)} X} F^{(i+1,0)} X. 
\end{align*}
$$

Therefore, we need to show that for any $i = 0, \ldots, n$ there exists a $t$-filtration $Y_i \leadsto (X_{(i,0)}, \ldots, X_{(i,k_i)})$.

It is clear that there is a $t$-filtration

$$
F^{(i,0)} X \leadsto (X_{(i,0)}, \ldots, X_{(i,k_i)}, X_{(i+1,0)}, \ldots, X_{(n,k_n)}),
$$

and the statement follows from the lemma:

**Lemma 4.4.** Let

$$
Z = F^0 Z \xrightarrow{h_1} F^1 Z \xrightarrow{h_2} F^2 Z \xrightarrow{\cdots} F^m Z \xrightarrow{h_{m+1}} F^{m+1} Z = 0
$$

be a $t$-filtration for an object $Z$. For each $s, 1 \leq s \leq m+1$ denote by $c_s$ the composition $c_s = h_s h_{s-1} \cdots h_1: F^s Z \rightarrow Z$ and consider distinguished triangles

$$
Z \xrightarrow{c_s} F^s Z.
$$

Then for each $s$ there exists a $t$-filtration $Q_s \leadsto (Z_0, Z_1, \ldots, Z_{s-1})$.

**Proof of the lemma** is by induction on $s$. For $s = 1$ there is nothing to prove. In the general case consider the diagram of distinguished triangles:

$$
\begin{align*}
Z_s & \xrightarrow{Q_s[-1]} F^s Z \xrightarrow{c_s} Z. 
\end{align*}
$$
Applying the octahedron axiom to the diagram

\[
\begin{array}{c}
Q_s[-1] \xrightarrow{h_{s+1}} F^s Z \xrightarrow{c_s} Z, \\
\end{array}
\]

we obtain distinguished triangles:

\[
\begin{array}{c}
Q_{s+1} \xrightarrow{c_{s+1}} F^{s+1} Z, \\
Q_s \xrightarrow{F^1 Z} Z.
\end{array}
\]

By the induction hypothesis, \( Q_s \leadsto (Z_0, \ldots, Z_{s-1}) \). In particular, we have the first distinguished triangle of a t-filtration for \( Q_s \)

\[
\begin{array}{c}
Z_0 \xrightarrow{F^1 Q_s} Q_s \xrightarrow{F^{s+1} Q_s} Z_{s-1}.
\end{array}
\]

and \( F^1 Q_s \leadsto (Z_1, \ldots, Z_{s-1}) \) the rest of the t-filtration. We shall construct the filtration \( F^{s+1} Q_s \leadsto (Z_0, Z_1, \ldots, Z_{s-1}, Z_s) \).

First we apply to the diagram of distinguished triangles

\[
\begin{array}{c}
Z_0 \xrightarrow{F^1 Q_s} Q_s \xrightarrow{F^{s+1} Q_s} Z_{s-1}.
\end{array}
\]

the octahedron axiom:

\[
\begin{array}{c}
Z_0 \xrightarrow{F^1 Q_s} F^{s+1} Q_s \xrightarrow{Z_s} Q_{s+1} \xrightarrow{F^1 Q_s} Z_s.
\end{array}
\]

and get \( F^1 Q_{s+1} \) and distinguished triangles

\[
\begin{array}{c}
F^1 Q_s \xrightarrow{F^1 Q_{s+1}} Z_s, \\
F^1 Q_{s+1} \xrightarrow{Z_{s+1}} Q_s \xrightarrow{F^1 Q_s} Z_s.
\end{array}
\]

By the induction hypothesis, it follows from the first triangle that there exists a t-filtration \( F^1 Q_{s+1} \leadsto (Z_1, \ldots, Z_{s-1}, Z_s) \). Therefore, one can include the second triangle into the t-filtration

\[
\begin{array}{c}
Q_{s+1} = F^0 Q_{s+1} \xrightarrow{F^1 Q_{s+1}} F^2 Q_{s+1} \xrightarrow{\cdots} F^s Q_{s+1} \xrightarrow{F^{s+1} Q_{s+1}} Z_{s+1} = 0.
\end{array}
\]
This completes the proof of the lemma and the second statement of the proposition. ■

Arguing very similarly, one can prove the first statement of the proposition. To prove the last one, consider the t-filtrations:

\[ X_0 \xrightarrow{g} X_1 \xrightarrow{g} \cdots \xrightarrow{g} X_n = F_0 X \xrightarrow{q_0} F_1 X \xrightarrow{q_1} \cdots \xrightarrow{q_n} F_{n+1} X = 0, \]

\[ Y_0 \xrightarrow{g} Y_1 \xrightarrow{g} \cdots \xrightarrow{g} Y_m = F_0 Y \xrightarrow{p_0} F_1 Y \xrightarrow{p_1} \cdots \xrightarrow{p_m} F_{m+1} Y = 0. \]

Suppose,

\[ (Z_0, \ldots, Z_{n+m+1}) = (X_0, \ldots, X_{i_1}, Y_0, \ldots, Y_{j_1}, X_{i_1+1}, \ldots, X_{i_2}, Y_0, \ldots, Y_{j_2}, X_{i_2+1}, \ldots, \ldots, X_{i_m}, Y_0, \ldots, Y_{j_m}). \]

One constructs the t-filtration

\[ X \oplus Y \xrightarrow{h_0} F^1(X \oplus Y) \xrightarrow{h_1} F^2(X \oplus Y) \xrightarrow{h_2} F^3(X \oplus Y) \xrightarrow{h_3} \cdots \]

in the following way:

\[ h_0 = q_0 \oplus 0, \ldots, h_{i_1} = q_{i_1} \oplus 0; \]

\[ F^1(X \oplus Y) = F^1 X \oplus F^1 Y, \ldots, F^{i_1}(X \oplus Y) = F^{i_1} X \oplus Y; \]

\[ h_{i_1+1} = 0 \oplus p_0, \ldots, h_{i_1+j_1} = 0 \oplus p_{j_1}; \]

\[ F^{i_1+1}(X \oplus Y) = F^{i_1} X \oplus F^1 Y, \ldots, F^{i_1+j_1}(X \oplus Y) = F^{i_1} X \oplus F^{i_1} Y; \]

and so on. ■

**Corollary 4.5.** Let \( X \in T \) be a semistable object of slope \( \varphi \). If \( X = X^1 \oplus X^2 \oplus \cdots \oplus X^k \), then each summand \( X^i \) is semistable of slope \( \varphi \) as well.

**Proof.** It is sufficient to prove the statement for \( k = 2 \). Consider the HN-systems for the direct summands

\[ X^1 \leadsto (X_0^1, X_1^1, \ldots, X_n^1), \quad X^2 \leadsto (X_0^2, X_1^2, \ldots, X_m^2). \]

Ordering the quotients in compliance with the order on \( \Phi \), unless \( n = m = 0 \) and \( \varphi_0 = \psi_0 \) one can construct the nontrivial HN-system for \( X \) (see Proposition 4.3). But \( X \) has only the trivial HN-system because it is semistable (Theorem 4.1). This concludes the proof. ■

5 Connections between t-stabilities and t-structures

It was shown in Section 3 that a bounded t-structure on a triangulated category induces stability data (Lemma 3.3). On the other hand a t-stability gives a family or collection of t-structures. Before formulation more exact statement, let us introduce some convenient notations.
For a subset $S$ of a triangulated category $T$ we denote by $\langle S \rangle$ the minimal full extension-closed subcategory of $T$, containing the subset $S$. Note that if we have a t-stability $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ on $T$ and $\Psi \subset \Phi$, then $\langle \Pi_\varphi | \varphi \in \Psi \rangle$ consists of objects whose have HN-system $X \sim (X_{\psi_0}, \ldots, X_{\psi_k})$ with $\psi_i \in \Psi$.

**Lemma 5.1.** Let $(\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi})$ makes stability data on a triangulated category $T$. Suppose there exists a decomposition of the set $\Phi$ in two disjoint parts: $\Phi = \Phi^- \sqcup \Phi^+$ such that for any $\varphi^- \in \Phi^-$ and $\varphi^+ \in \Phi^+$ we have $\varphi^- < \varphi^+$. Then the pair of subcategories

$$T^{\geq 0} = \langle \Pi_\varphi | \varphi \in \tau(\Phi^-) \rangle, \quad T^{\leq 0} = \langle \Pi_\varphi | \varphi \in \Phi^+ \rangle$$

defines a t-structure on $T$ (recall that $\tau$ is an automorphism of the linearly ordered set $\Phi$, that corresponds to the shift of $T$).

**Corollary 5.2.** Each element $\varphi \in \Phi$ gives a t-structure

$$T^{\geq 0}_\varphi = \langle \Pi_\psi | \psi \leq \tau(\varphi) \rangle, \quad T^{\leq 0}_\varphi = \langle \Pi_\psi | \psi > \varphi \rangle.$$

**Proof of the lemma.** Firstly let us recall that by the definition of t-stability $\Pi_\varphi[1] = \Pi_{\tau^{-1}(\varphi)}$ and $\tau(\varphi) \geq \varphi$, whence the first axiom of a t-structure is valid:

$$T^{\leq 0} \subset T^{\leq 0}[1] \quad \text{and} \quad T^{\geq 0} \supset T^{\geq 0}[1].$$

Besides $\text{Hom}^{\leq 0}(\Pi_\varphi, \Pi_\psi) = 0$ whenever $\varphi > \psi$. Therefore the axiom

$$\text{Hom}^{\leq 0}(T^{\leq 0}, T^{\geq 0}[1]) = 0$$

of a t-structure follows from the statement:

Let $S, S'$ be subsets of objects of $T$ such that $\text{Hom}^{\leq 0}(X, Y) = 0$ for each $X \in S$ and $Y \in S'$. Then $\text{Hom}^{\leq 0}(\langle S \rangle, \langle S' \rangle) = 0$,

which is evidently true.

To verify the last axiom consider HN-system for an object $0 \neq X \in T$:

$$X = \cdots \xleftarrow{F^2} F^1 X \xleftarrow{F^0} X \xrightarrow{F^1} \cdots \xrightarrow{F^{n+1}} F^n X = 0.$$

If $\varphi_n \in \Phi_-$ or $\varphi_0 \in \Phi_+$, then the distinguished triangle of the axiom is trivial. Suppose there exists $k$ such that $\varphi_k \in \Phi_-$ and $\varphi_{k+1} \in \Phi_+$. Completing the morphism $X_{\leq 0} = F^{k+1}X \xrightarrow{p_{k+1} \ldots p_1} X$ by a cone $X_{\geq 1}$, we get the needed distinguished triangle

$$\begin{array}{c}
X_{\geq 1} \\
X_{\leq 0}
\end{array}$$

of the axiom (see Proposition 4.3).
Remark. Note that the t-structure, constructed above, is bounded if and only if
\[ Φ = \left( \bigcup_{n \in \mathbb{Z}_{\geq 0}} τ^{-n}(Φ_+) \right) \quad \text{and} \quad Φ = \left( \bigcup_{n \in \mathbb{Z}_{\geq 0}} τ^n(Φ_-) \right). \]

As a result, we have got the connection between t-structures and t-stabilities on a given triangulated category.

It is easy to observe, that a given t-structure can be induced by several stability data. Let us consider a t-stability \((Φ, \{Π_ϕ\} _{ϕ \in Φ})\) on \(T\) such that Φ is a disjoin union \(Φ = \biguplus _{ψ ∈ Ψ} Φ_ψ\) of nonempty subsets with the following properties:

1. if \(ϕ_i ∈ Φ_ψ_i\) (i = 1, 2) and \(ϕ_1 < ϕ_2\), then for any \(ϕ'_i ∈ Φ_ψ_i\) the inequality \(ϕ'_1 < ϕ'_2\) holds.
2. \(∀ ψ ∈ Ψ \ \exists ψ' ∈ Ψ\) such that \(τ(Φ_ψ) = Φ_ψ'\).

The properties of the disjoin union allow to determine an order on Ψ and \(τ ∈ \text{Aut}(Ψ)\), corresponding to \(τ ∈ \text{Aut}(Φ)\) in the obvious way.

Further we define \(P_ψ\) for \(ψ ∈ Ψ\) as \(P_ψ = \langle Π_ϕ | ϕ ∈ Φ_ψ \rangle\). The fact that \((Ψ, \{P_ψ\} _{ψ ∈ Ψ})\) is a t-stability follows immediately from the definition and Proposition 4.3.

We have constructed another t-stability \((Ψ, \{P_ψ\} _{ψ ∈ Ψ})\), but a t-structure induced by a decomposition of Ψ, can be obtained also from a decomposition of Φ. The t-stability Φ induces all t-structures that Ψ does, and more. We would like to say that the stability data Φ are finer then Ψ, and Ψ are weaker. Generalizing this construction, let us formulate the definition.

Definition 5.3. Let \((Φ, \{Π_ϕ\} _{ϕ ∈ Φ})\) and \((Ψ, \{P_ψ\} _{ψ ∈ Ψ})\) be t-stabilities on a triangulated category \(T\) with automorphisms \(τ_Φ\) and \(τ_Ψ\), corresponding to the shift on \(T\). We say that \(Φ\) is finer then \(Ψ\) (and \(Ψ\) is weaker then \(Φ\)) and denote this by \(Φ \preceq Ψ\) if there exists a surjection \(r : Φ → Ψ\) such that

1. \(rτ_Φ = τ_Ψ r;\)
2. \(ϕ' > ϕ'' ⇔ r(ϕ') ≥ r(ϕ'');\)
3. \(∀ ψ ∈ Ψ \ P_ψ = \langle Π_ϕ | ϕ ∈ r^{-1}(ψ) \rangle.\)

Clearly this gives a partial order on the set of all t-stabilities on a given triangulated category. Therefore, some of stability data can be minimal w.r.t the order. They seem to contain the maximal information about t-structures. We call such t-stabilities the finest.

Now we give conditions of comparability and being the finest for stability data.

Proposition 5.4. Let \((Φ, \{Π_ϕ\} _{ϕ ∈ Φ}, τ_Φ)\) and \((Ψ, \{P_ψ\} _{ψ ∈ Ψ}, τ_Ψ)\) make stability data on \(T\). Then

1. \(Φ \preceq Ψ\) if and only if
   (i) any \(Φ\)-semistable object is \(Ψ\)-semistable,
   (ii) for \(Π_ϕ_i ⊂ P_ψ_i\) (i = 1, 2) the condition \(ϕ_1 < ϕ_2\) implies \(ψ_1 ≤ ψ_2\),
   (iii) if \(Π_ϕ ⊂ P_ψ\), then \(Π_{τ_Φ(ϕ)} ⊂ P_{τ_Ψ(ψ)}\).
2. Suppose that for each \( \varphi \in \Phi \) we have

\[
\forall X, Y \in \Pi_\varphi \quad \text{Hom}_T(X, Y) \neq 0 \quad \text{and} \quad \text{Hom}_T(Y, X) \neq 0.
\]

Then the t-stability \((\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}, \tau_\Phi)\) is the finest.

**Proof.** The fact that the assumption \( \Phi \preceq \Psi \) implies conditions (i)-(iii) follows immediately from the definition. To prove the converse it is sufficient to show that each semistable subcategory \( P_\psi \) is generated by some collection of subcategories \( \Pi_\varphi \). In other words, if \( X_\psi \in P_\psi \) has a HN-system \( X_\psi \rightsquigarrow (X_{\varphi_0}, \ldots, X_{\varphi_n}) \) w.r.t t-stability \( \Phi \), then \( \Pi_{\varphi_i} \subset P_\psi \) for each \( i \).

By condition (i) any \( X_{\varphi_i} \) is \( \Psi \)-semistable. Therefore, we can find \( \psi_i \in \Psi \) such that \( X_{\varphi_i} \in P_{\psi_i} \). Since \( \text{Hom}^0(X_{\varphi_i}, X_{\varphi_0}) \neq 0 \) and \( X_{\varphi_0} \in P_{\psi_0} \), we have \( \psi \leq \psi_0 \) (Proposition 4.2). On the other hand, \( \text{Hom}^0(X_{\varphi_i}, X_{\varphi_0}) \neq 0 \), consequently, \( \psi_0 \leq \psi \). Besides, \( \varphi_0 < \varphi_1 < \cdots < \varphi_n \), so (by condition (ii)) \( \psi_0 \leq \psi_1 \leq \cdots \leq \psi_n \). This is possible only if \( \psi_i = \psi_n \).

To verify the second statement of the proposition, we note that if \( (\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}) \) is not the finest, then there exists some \( \varphi \in \Phi \) and nonempty subcategories \( \Pi_-, \Pi_+ \) such that \( \Pi_\varphi = (\Pi_-, \Pi_+) \), where \( \text{Hom}^{\leq}(\Pi_+, \Pi_-) = 0 \). This contradicts the assumption of the statement. \( \blacksquare \)

**Remark.** As we shall see later, a pair of t-stabilities can satisfy the condition (i) and be noncomparable. The second condition is significant.

As the conclusion of the section we formulate a statement that we shall use later for the classification of bounded t-structures.

**Proposition 5.5.** Let \( T \) be a triangulated category with the property that for any t-stability \( \Psi \) there exists the finest t-stability \( \Phi \preceq \Psi \). Then for any bounded t-structure \((T^{\leq 0}, T^{> 0})\) there exists a t-stability \( \Phi = (\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}) \) and a decomposition \( \Phi = \Phi_- \sqcup \Phi_+ \) in two disjoint parts such that:

\[
T^{\leq 0} = (\{\Pi_\varphi| \varphi \in \Phi_+\}, \quad T^{> 0} = (\{\Pi_\varphi| \varphi \in \tau(\Phi_-)\}).
\]

Without loss of generality we can assume that \( \Phi \) is one of the finest t-stabilities on \( T \).

**Proof** follows directly from Lemmas 3.3 and 5.1 and assumptions of the proposition. \( \blacksquare \)

### 6 Stability data for \( D^b(\text{Coh} \, \mathbb{P}^1) \)

Here we study t-stabilities and bounded t-structures on the bounded derived category \( D^b(\text{Coh} \, \mathbb{P}^1) \) of coherent sheaves on the projective line. At first we describe two types of the finest t-stabilities for \( D^b(\text{Coh} \, \mathbb{P}^1) \): standard and exceptional. Then we show that any t-stability for \( D^b(\text{Coh} \, \mathbb{P}^1) \) is weaker than either standard one or exceptional one. This enables us to classify all bounded t-structures for \( D^b(\text{Coh} \, \mathbb{P}^1) \).

In this section we identify sheaves with 0-complexes of \( D^b(\text{Coh} \, \mathbb{P}^1) \).

#### 6.1 Standard t-stability.** It follows from Lemma 3.3 that \( (\mathbb{Z}, \{(\text{Coh} \, \mathbb{P}^1)[i]\}_{i \in \mathbb{Z}}) \) makes stability data for \( D^b(\text{Coh} \, \mathbb{P}^1) \). Obviously, they are not the finest. To construct the finest t-stability \((\mathcal{M}, \{\Pi_\mu\}_{\mu \in \mathcal{M}} \preceq (\mathbb{Z}, \{(\text{Coh} \, \mathbb{P}^1)[i]\}_{i \in \mathbb{Z}})\), recall that each sheaf \( F \) on \( \mathbb{P}^1 \) decomposes in a finite direct sum

\[
F = \bigoplus_j \Xi_{x_j} \oplus \left( \bigoplus_i \mathcal{O}(n_i) \right),
\]
where $\Xi_{x_j}$ is a torsion sheaf concentrated at the point $x_j$, and $O(n_i)$ is an invertible sheaf of degree $n_i$. Let us define semistable subcategories as

$$
\Pi_{(i,x)} = \langle O_x[i] \rangle \quad \text{and} \quad \Pi_{(j,n)} = \langle O(n)[j] \rangle
$$

with $x \in \mathbb{P}^1$, and $n, i, j \in \mathbb{Z}$. This yields the set of slopes $\mathcal{M} = \mathbb{Z} \times (\mathbb{Z} \sqcup \mathbb{P}^1)$.

Now we have to introduce a linear order on $\mathcal{M}$ in a way

$$
\mu > \nu \Rightarrow \text{Hom}^{\leq}(\Pi_\mu, \Pi_\nu) = 0.
$$

This condition implies:

- $(i, \alpha) > (j, \beta)$ for $i > j$, $\alpha, \beta \in \mathbb{Z} \cup \mathbb{P}^1$;
- $(i, n) > (i, m)$ for $n > m \in \mathbb{Z}$, $i \in \mathbb{Z}$;
- $(i, x) > (i, n)$ for $x \in \mathbb{P}^1$, $i, n \in \mathbb{Z}$.

Besides, for each $i, j, q \in \mathbb{Z}$, $\text{Hom}^q(O_x[i], O_y[j]) = 0$ unless $x = y$. Thus, defining a linear order on $\mathbb{P}^1$ in arbitrary way, we get the linearly ordered set $\mathcal{M}$.

To prove that $(\mathcal{M}, \{\Pi_{\mu}\}_{\mu \in \mathcal{M}})$ is a t-stability it is sufficient to verify that each nonzero object has HN-system. This directly follows from Proposition \ref{prop:existence_of_stability_data} and the fact that $Coh \mathbb{P}^1[i] = \langle \Pi_{(i,\alpha)} \rangle \alpha \in \mathbb{Z} \cup \mathbb{P}^1$.

Finally, note that all the semistable subcategories $\Pi_{(i,\alpha)}$ ($\alpha \in \mathbb{Z} \cup \mathbb{P}^1$) satisfy the assumption of Proposition \ref{prop:existence_of_stability_data}(2). Therefore the stability data $(\mathcal{M}, \{\Pi_{\mu}\}_{\mu \in \mathcal{M}})$ is the finest one.

Notice, that the t-stability $(\mathcal{M}, \{\Pi_{\mu}\}_{\mu \in \mathcal{M}})$ depends on an order on the slope set $\mathcal{M}$. Hence we have got various finest incomparable t-stabilities with the same semistable subcategories and slopes. Each of them we shall call standard.

### 6.2. Exceptional t-stability

The construction of the next type of stability data is based on Proposition \ref{prop:existence_of_stability_data}. The category $D^b(Coh \mathbb{P}^1)$ is generated by the exceptional collection $(O(k), O(k+1))$ ($k$ is an arbitrary fixed integer number). Therefore we have stability data $\{\{0,1\}, \{\Pi^k_i\}_{i \in \{0,1\}}\}$, for each integer number $k$, where

$$
\Pi^k_0 = \langle O(k)[j] \mid j \in \mathbb{Z} \rangle,
\Pi^k_1 = \langle O(k+1)[i] \mid i \in \mathbb{Z} \rangle.
$$

Since $\text{Hom}^{\leq}(O(n)[i], O(n)[j]) = 0$ for $n, i, j \in \mathbb{Z}$ whenever $i > j$, the t-stability defined above is not finest. To refine it we consider semistable subcategories and slopes

$$
\Pi^k_{(i,0)} = \langle O(k)[i] \rangle, \quad \Pi^k_{(i,1)} = \langle O(k+1)[i] \rangle, \quad (i,0); (i,1) \in \mathcal{E} = \mathbb{Z} \times \{0,1\}.
$$

We are going to refine the stability data $\{\{0,1\}, \{\Pi^k_i\}_{i \in \{0,1\}}\}$ to some finest $\left(\mathcal{E}, \{\Pi^k_{\varepsilon}\}_{\varepsilon \in \mathcal{E}}\right)$. A linear order on the set $\mathcal{E}$ and automorphism $\tau \in \text{Aut} \mathcal{E}$ should be coordinated with the order and the automorphism above. Therefore, we come to

- $(i, 0) < (j, 1);
- (i, 0) < (i + 1, 0), \ (i, 1) < (i + 1, 1);
- \tau(i, 0) = (i + 1, 0), \ \tau(i, 1) = (i + 1, 1)$

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for arbitrary $i,j \in \mathbb{Z}$.

Finally we need to obtain a finite HN-system w.r.t. $\mathcal{E}$ for each nonzero object $X$ of the derived category. Note that homological dimension of $\text{Coh}\mathbb{P}^1$ is 1. We shall use the following result well-known among specialists, but it seems there is no reference.

**Proposition 6.3.** If homological dimension of an abelian category $\mathcal{A}$ equals 1, then each object $X$ of $D^b(\mathcal{A})$ is isomorphic to a finite sum $X = \bigoplus_i A_i[-i]$, where $A_i \in \mathcal{A}$ and $A_i[-i]$ denotes $i$-complex, i.e., a complex with the unique nonzero term $A_i$ at the place $i$.

For convenience of the reader we provide the proof.

**Proof.** Suppose $X \in D^b(\mathcal{A})$ is represented by a complex

$$
C^\bullet = C^n \xrightarrow{d_n} C^{n+1} \xrightarrow{d_{n+1}} \cdots \xrightarrow{d_{n-1}} C^{m-1} \xrightarrow{d_m} C^m
$$

such that $H^i(C^\bullet) = 0$ for $i < s$ and $H^s(C^\bullet) \neq 0$. The exact sequence of complexes

$$
\begin{align*}
K^\bullet &= C^n \xrightarrow{d_n} C^{n+1} \cdots \xrightarrow{d_{s-1}} C^s \xrightarrow{id} \text{Im} \ x \cdots \xrightarrow{id} \text{Im} d_{s-1} \xrightarrow{id} 0 \\
\text{C}^\bullet &= C^n \xrightarrow{d_n} C^{n+1} \cdots \xrightarrow{d_{s-1}} C^s \xrightarrow{id} \text{Im} d_{s-1} \xrightarrow{id} \text{Im} d_{s-1} \xrightarrow{id} 0 \\
B^\bullet &= 0 \xrightarrow{d_s} B^s \xrightarrow{d_s} \xrightarrow{id} C^s \xrightarrow{id} \cdots
\end{align*}
$$

determines the distinguished triangle in $D^b(\mathcal{A})$

$$
\begin{array}{c}
\text{Z} \\
\downarrow \downarrow \downarrow \\
\text{X} & \text{Y}
\end{array}
$$

with $Y \simeq K^\bullet$ and $Z \simeq B^\bullet$. Since the complex $K^\bullet$ is acyclic, $Y \simeq 0$ (see [8]), i.e. $X \simeq Z \simeq B^\bullet$.

Now we shall prove the proposition by induction on the number $h(B^\bullet)$ of nonzero cohomologies of the complex $B^\bullet$.

The base of induction ($h = 1$) was proved in [8]. Further, let us complete the inclusion $(\text{Ker} \ d_s)[-s] \rightarrow B^\bullet$ to a distinguished triangle

$$
\begin{array}{c}
D^\bullet \\
\downarrow \downarrow \downarrow \\
B^\bullet & (\text{Ker} \ d_s)[-s]
\end{array}
$$

(6.2)

Since $h(D^\bullet) < h(B^\bullet)$, we can use the induction hypothesis: $D^\bullet \simeq \bigoplus_{i>s} A_i[-i]$ with $A_i \in \mathcal{A}$.

Applying to the triangle the functor $\text{Hom}(\cdot, (\text{Ker} \ d_s)[-s])$, we get exact sequence

$$
\begin{align*}
\text{Hom}^0(B^\bullet, (\text{Ker} \ d_s)[-s]) & \rightarrow \text{Hom}^0((\text{Ker} \ d_s)[-s], (\text{Ker} \ d_s)[-s]) \\
& \rightarrow \bigoplus_{i>s} \text{Hom}^1(A_i[-i], (\text{Ker} \ d_s)[-s]).
\end{align*}
$$

It follows from the inequality $i > s$ that $q = 1 + i - s \geq 2$ and we have

$$
\text{Hom}^1(A_i[-i], (\text{Ker} \ d_s)[-s]) = \text{Ext}_A^q(A_i, \text{Ker} \ d_s).
$$
On the other hand, homological dimension of $A$ equals 1, so $\bigoplus_{i > s} \text{Hom}^1(A_i[-i], (\text{Ker} \, \bar{d}_s)[-s]) = 0$ and the triangle (6.2) splits. This completes the proof. ■

In addition, as we already marked, a sheaf on $\mathbb{P}^1$ is direct sum of invertible sheaves and torsion sheaves with support at point. Besides, the shift of the derived category acts on the set of distinguished triangles. Therefore, according to Proposition 4.3, it is sufficient to construct HN-system for an invertible sheaf $O(n)$ and a torsion sheaf $\Xi_x$ concentrated at a point $x$. The needed HN-filtrations are obtained from the exact sequences

$$
0 \to (n - k - 1)O(k) \to (n - k)O(k + 1) \to O(n) \to 0, \quad \text{if } n > k + 1,
$$

$$
0 \to O(n) \to (k - n + 1)O(k) \to (k - n)O(k + 1) \to 0, \quad \text{if } n < k,
$$

$$
0 \to dO(k) \to dO(k + 1) \to \Xi_x \to 0 \quad \text{where } d = \deg \Xi_x,
$$

and they have the form

$$
(n - k - 1)O(k)[1] \quad \text{if } n > k + 1, \quad (6.3)
$$

$$
(k - n + 1)O(k) \quad \text{if } n < k, \quad (6.4)
$$

$$
dO(k)[1] \quad \Xi_x \quad dO(k + 1). \quad (6.5)
$$

Thus, we have got the finest stability data $\left(\mathcal{E}, \left\{\Pi^k_{\mathcal{E}}\right\}_{\mathcal{E} \in \mathcal{E}}\right)$ (see Prop. 5.4).

As we saw before, changing an order of the slopes set, one can get another $t$-stability. Let us research this possibility for the stability data $\left(\mathcal{E}, \left\{\Pi^k_{\mathcal{E}}\right\}_{\mathcal{E} \in \mathcal{E}}\right)$. Any linear order on the set $\mathcal{E}$ must satisfy the conditions

$$
\varepsilon'' < \varepsilon' \Rightarrow \text{Hom}^{\leq 0}(\Pi^k_{\mathcal{E}}, \Pi^k_{\varepsilon''}) = 0, \quad (6.6)
$$

$$
(j', i') < (j'', i'') \Leftrightarrow (j' + 1, i') < (j'' + 1, i'') \quad \text{for } (j', i'), (j'', i'') \in \mathcal{E}. \quad (6.7)
$$

Identifying the set $\mathcal{E}$ with generators of the semistable subcategories, we see that the first condition implies

$$
O(k)[i] < O(k)[j] \quad \text{and } O(k + 1)[i] < O(k + 1)[j] \quad \text{whenever } i < j.
$$

On the other hand, HN-system (6.4) yields $O(k) < O(k + 1)[-1]$. Therefore due to the second condition we have

$$
O(k)[i] < O(k + 1)[i - 1] \quad \forall i \in \mathbb{Z}.
$$

---

3When $X$ is an object of an additive category we use notation $mX$ for the object $X^\oplus m$. 

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Now, using the transitivity of an order, we see that linear orders on \( \mathcal{E} \), satisfying (6.6) and (6.7), depend on a parameter \( p \in \mathbb{N} \cup \{0, +\infty\} = \bar{\mathbb{N}} \) and have the form

\[
\mathcal{O}(k)[i+p] < \mathcal{O}(k+1)[i] < \mathcal{O}(k)[i+p+1] \quad \forall i \in \mathbb{Z}, \quad \text{if } p \in \mathbb{N} \cup \{0\},
\]

\[
\mathcal{O}(k)[i] < \mathcal{O}(k+1)[j] \quad \forall i, j \in \mathbb{Z}, \quad \text{if } p = +\infty.
\]

Let us denote the set \( \mathcal{E} \) with the order corresponding to \( p \in \bar{\mathbb{N}} \) by \( \mathcal{E}_p \). Since the existence of finite HN-system for each object of \( D^b(\text{Coh} \, \mathbb{P}^1) \) is obvious, we conclude that \( (\mathcal{E}_p, \{\Pi^k_\varepsilon\}_{\varepsilon \in \mathcal{E}_p}) \) are the finest stability data. We call them exceptional.

### 6.4. The set of all stability data for \( D^b(\text{Coh} \, \mathbb{P}^1) \)

In this subsection we prove the following classification result.

**Theorem 6.5.** Any finest t-stability on \( D^b(\text{Coh} \, \mathbb{P}^1) \) is either a standard one or an exceptional one, thus one of the finest t-stabilities constructed in the previous sections.

**Proof.** Let \( (\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}) \) be a t-stability on \( D^b(\text{Coh} \, \mathbb{P}^1) \) that incomparable with any standard one. We show that \( (\Phi, \{\Pi_\varphi\}_{\varphi \in \Phi}) \) is weaker then one of the exceptional stabilities.

Recall that the standard stability data \( (\mathcal{M}, \{\Pi_\mu\}_{\mu \in \mathcal{M}}) \) depends on a linear order on \( \mathcal{M} \) in the notations of Subsection 6.1. Suppose every \( \mathcal{M} \)-semistable object is \( \Phi \)-semistable. Since each possible linear order on \( \mathcal{M} \) gives one of the standard t-stabilities, we obtain that \( \Phi \) is weaker then \( \mathcal{M} \). Therefore if \( \Phi \) is not comparable with \( \mathcal{M} \), then there exists a \( \mathcal{M} \)-semistable object that is not \( \Phi \)-semistable. By definition semistable subcategories are closed under extension. Consequently, without loss of generality, we can assume such \( \mathcal{M} \)-semistable object is one of the generators of \( \Pi_\mu \): \( \mathcal{O}_x[i] \) or \( \mathcal{O}(n)[i] \).

Let \( X \) be one of the generators. Since \( X \) is not \( \Phi \)-semistable, it has a nontrivial HN-system w.r.t. \( \Phi \). Consider the first distinguished triangle of the system

\[
X \xrightarrow{\varphi_0} \mathcal{O}(k)[0] \xrightarrow{\varphi_0} \mathcal{O}(k)[1] \xrightarrow{} F^1X.
\]

We know that \( X_{\varphi_0} \) is \( \Phi \)-semistable and

\[
\text{Hom} \leq 0(F^1X, X_{\varphi_0}) = 0.
\]

The shift of the derived category acts on the set of all distinguished triangles and on semistable subcategories. Besides the condition (6.9) is invariant under the shift too. Thus, we can assume that \( X \) is either \( \mathcal{O}_x \) or \( \mathcal{O}(n) \).

We need the following lemma.

**Lemma 6.6.** Let \( X \in D^b(\text{Coh} \, \mathbb{P}^1) \) is \( \mathcal{O}_x \) or \( \mathcal{O}(n) \), then there exists \( k \in \mathbb{Z} \) such that the distinguished triangle (6.8), satisfying the condition (6.9), is isomorphic to one of the triangles (6.3)–(6.5).

Since the distinguished triangle (6.8) is the first triangle of HN-system, we obtain that \( x_0\mathcal{O}(k)[0] \) or \( x_0\mathcal{O}(k)[1] \) is \( \Phi \)-semistable for some natural \( x_0 \). Using Corollary 4.5 and Definition

\[\text{We identify each sheaf } X \in \text{Coh} \, \mathbb{P}^1 \text{ with } 0 \text{-complex } X[0] \in D^b(\text{Coh} \, \mathbb{P}^1).\]
We conclude that the objects $\mathcal{O}(k)[i]$ are $\Phi$-semistable for all $i \in \mathbb{Z}$, and the distinguished triangle \([6.8]\) should have the form

$$
\xymatrix{
X 
& x_0 \mathcal{O}(k)[j] \ar[dr] 
& y_1 \mathcal{O}(k+1)[j-1] \\
& \ar[ur] x_0 \mathcal{O}(k)[j] & F^2X 
}
$$

where $x_0, y_1$ are natural numbers and $j = 0, 1$.

Let us show, that $y_0 \mathcal{O}(k+1)[j-1]$ is $\Phi$-semistable as well. Suppose not, then the HN-system for $X$ continues:

$$
\xymatrix{
X 
& x_0 \mathcal{O}(k)[j] \ar[dr] 
& y_0 \mathcal{O}(k+1)[j-1] \\
& \ar[ur] x_0 \mathcal{O}(k)[j] & F^2X 
}
$$

It follows from the definition of HN-system and Proposition \([4.2]\) that

$$
\text{Hom}^{<0}(F^2X, X_{\varphi_1}) = 0, \\
\text{Hom}^{<0}(X_{\varphi_1}, x_0 \mathcal{O}(k)[j]) = 0.
$$

The condition \([6.11]\) and Lemma \([6.6]\) allow us to conclude, that there exist an integer $m$ and natural numbers $x_1, y_2$ such that

either $X_{\varphi_1} = x_1 \mathcal{O}(m)[j-1]$, $F^2X = y_2 \mathcal{O}(m+1)[j-2]$, if $k < m - 1$,

or $X_{\varphi_1} = x_1 \mathcal{O}(m)[j]$, $F^2X = y_2 \mathcal{O}(m+1)[j-1]$, if $k > m$.

In the first case ($k < m - 1$)

$$
\text{Hom}^0(X_{\varphi_1}, x_0 \mathcal{O}(k)[j]) = \text{Ext}^1_{\text{Coh}\, \mathbb{P}^1}(x_1 \mathcal{O}(m), x_0 \mathcal{O}(k)) \neq 0.
$$

This contradicts \([6.12]\). In the second one ($k > m$)

$$
\text{Hom}^0(X_{\varphi_1}, x_0 \mathcal{O}(k)[j]) = \text{Hom}_{\text{Coh}\, \mathbb{P}^1}(x_1 \mathcal{O}(m), x_0 \mathcal{O}(k)) \neq 0.
$$

This contradicts \([6.12]\) again.

Thus there cannot be the continuation of the HN-system, whence $y_0 \mathcal{O}(k+1)[j-1]$ is $\Phi$-semistable. Hence for any $j \in \mathbb{Z}$ the object $\mathcal{O}(k+1)[j]$ is $\Phi$-semistable.

In such a way, we got $\Phi$-semistable objects $\mathcal{O}(k)[i]$ and $\mathcal{O}(k+1)[i]$ ($i \in \mathbb{Z}$). Considering more fine stability data (if necessary), we can assume that $\langle \mathcal{O}(k)[i] \rangle$ and $\langle \mathcal{O}(k+1)[i] \rangle$ are $\Phi$-semistable subcategories. Let us show that each $\Phi$-semistable object belongs to one of them.

Indeed, any object $X$ of the derived category $D^b(\text{Coh}\, \mathbb{P}^1)$ could be decomposed into a direct sum of $i$-complexes $X_i[i]$ (Prop. \([6.3]\)), where $X_i$ is a direct sum of invertible sheaves and torsion sheaves concentrated each in one point. Each these direct summand of $X$ has HN-system w.r.t. $\Phi$ with $\mathcal{O}(k)[i]$ and $\mathcal{O}(k+1)[i]$ as quotients (see \([6.3]-[6.4]\)). Therefore, according to Proposition \([4.3]\) one can construct HN-system for any object $X$ with only objects $a\mathcal{O}(k)[i]$, $b\mathcal{O}(k+1)[i]$ as quotients. Now, it follows from the uniqueness of the HN-system that only objects $x_0 \mathcal{O}(k)[i]$ and $x_1 \mathcal{O}(k+1)[i]$ are $\Phi$-semistable. Consequently the collection of $\Phi$-semistable subcategories coincides with the such collection for some exceptional $t$-stability.
The action of the shift on the \( \Phi \)-semistable subcategories is uniquely determined. Finally, any admissible linear order on the collection of \( \Phi \)-semistable subcategories gives us the exceptional t-stability. This concludes the proof of the theorem. ■

**Corollary 6.7.** Stability data on \( D^b(\text{Coh P}^1) \) is weaker than either the standard or the exceptional t-stability.

**Proof of Lemma 6.6.** We start with an observation.

**Remark 6.8.** Suppose in the triangle \( (6.8) \)
\( X_{\psi} = X_1 \oplus X_2 \) and \( f = f_1 \oplus f_2 \), where
\( f_i \in \text{Hom}(X, X_i) \). If \( f_2 = 0 \), then \( F^1 X = Y \oplus X_2 \) and \( \text{Hom}^{\leq 0}(F^1 X, X_{\psi_0}) \neq 0 \).

Let \( X \) from triangle \( (6.8) \) be one of the objects \( \mathcal{O}_x[0] \) or \( \mathcal{O}(n)[0] \). Taking into account Proposition 6.3 and the previous remark, we can assume that
\[
X_{\psi} = X_0[0] \oplus X_1[1] \quad \text{with} \quad X_i \in \text{Coh P}^1, \\
f = f_0 \oplus f_1 \quad \text{with} \quad f_0 \in \text{Hom}_{\text{Coh P}^1}(X, X_0), \\
f_1 \in \text{Ext}^1_{\text{Coh P}^1}(X, X_1), \\
f_i = 0 \iff X_i = 0.
\]

Then \( F^1 X \) is isomorphic to \( C[-1] \), where \( C \) is the cone of the morphism \( f \).

Consider the extension corresponding to \( f_1 \in \text{Ext}^1_{\text{Coh P}^1}(X, X_1) \):
\[
0 \longrightarrow X_1 \longrightarrow E \xrightarrow{d_{-1}} X \longrightarrow 0. \tag{6.13}
\]

The object \( X_1[1] \) is isomorphic to the complex
\[
0 \longrightarrow E \xrightarrow{d_{-1}} X \longrightarrow 0,
\]
and the morphism \( f_1 \) can be represented as the morphism of the complexes
\[
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow & & \downarrow \text{id}_X \\
0 & \longrightarrow & E \xrightarrow{d_{-1}} X \\
\end{array}
\]

Hence the morphism \( f : X \longrightarrow X_0[0] \oplus X_1[1] \) is following:
\[
\begin{array}{ccc}
0 & \longrightarrow & X \\
\downarrow & & \downarrow \text{id}_X \oplus f \\
0 & \longrightarrow & E \xrightarrow{d_{-1} \oplus 0} X \oplus X_0 \\
\end{array}
\]

The cone of this morphism is the complex
\[
0 \longrightarrow X \oplus E \xrightarrow{\delta_{-1}} X \oplus X_0 \longrightarrow 0,
\]
where \( \delta_{-1} \) one can write as the matrix
\[
\delta_{-1} = \begin{pmatrix} \text{id}_X & d_{-1} \\ f_0 & 0 \end{pmatrix}.
\]

\[22\]
The cone complex is isomorphic to \((\text{Ker} \delta_{-1})[1] \oplus (\text{Coker} \delta_{-1})[0]\) in the derived category.

Consider the commutative diagram in the abelian category \(\text{Coh} \mathbb{P}^1\).

\[
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & X_1 & E & X & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \text{Ker} \delta_{-1} & X \oplus E & \delta_{-1} & X \oplus X_0 & \text{Coker} \delta_{-1} & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & \text{Ker} f_0 & X & f_0 & X_0 & \text{Coker} f_0 & 0 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 & 0 & \\
\end{array}
\]

We obtain that \(\text{Coker} \delta_{-1} = \text{Coker} f_0\) and the condition \(6.9\) has the form

\[
\text{Hom}^{\leq 0} \left( \left( \text{Ker} \delta_{-1} \right)[0] \oplus (\text{Coker} f_0)[-1], X_0[0] \oplus X_1[1] \right) = 0,
\]

whence

\[
\text{Hom}^{\leq 0} \left( \left( \text{Ker} \delta_{-1} \right)[0], X_0[0] \right) = 0, \quad (6.14)
\]

\[
\text{Hom}^{\leq 0} \left( \left( \text{Ker} \delta_{-1} \right)[0], X_1[1] \right) = 0. \quad (6.15)
\]

Assume that \(X = \mathcal{O}_x\) and \(X_0 \neq 0\). Then \(\text{Ker} f_0 = 0\), i.e. \(\text{Ker} \delta_{-1} = X_1\). This contradicts \(6.15\). Therefore \(X_0 = 0\) and the triangle is obtained from the exact sequence \(6.13\). In particular, \(C[-1] = E\). So, the condition \(6.9\) means \(\text{Hom}^{\leq 0}(E, X_1[1]) = 0\), i.e. \(\text{Hom}_{\text{Coh} \mathbb{P}^1}(E, X_1) = 0\) and \(\text{Ext}^1_{\text{Coh} \mathbb{P}^1}(E, X_1) = 0\). Besides, by construction, \(\text{Hom}_{\text{Coh} \mathbb{P}^1}(X_1, E) \neq 0\). Now the proof in the case \(X = \mathcal{O}_x\) follows from the remark:

**Remark 6.9.** If sheaves \(E\) and \(X_1\) on \(\mathbb{P}^1\) satisfy the conditions

\[
\text{Hom}_{\text{Coh} \mathbb{P}^1}(E, X_1) = 0, \quad \text{Ext}^1_{\text{Coh} \mathbb{P}^1}(E, X_1) = 0, \quad \text{Hom}_{\text{Coh} \mathbb{P}^1}(X_1, E) \neq 0,
\]

then there exist an integer \(k\) and natural numbers \(e, x_1\) such that \(E = e\mathcal{O}(k+1), X_1 = x_1\mathcal{O}(k)\).

**Proof** is left to the reader. ■

To continue consider \(X = \mathcal{O}(n)\). Suppose at first that neither \(X_0 \neq 0\) nor \(X_1 \neq 0\). Then \(X_1\) has an invertible direct summand \(L\), since in the converse case \(f_1 = 0\), i.e. \(X_1 = 0\) (see Rem. 6.8). Hence \(\text{Ker} \delta_{-1}\) has an invertible direct summand \(L'\) as well.

Assuming \(\text{Ker} f_0 = 0\), we get \(\text{Ker} \delta_{-1} = X_1\), which contradicts \(6.15\). Therefore \(\text{Ker} f_0 \neq 0\), i.e. \(\text{Im} f_0\) is a torsion sheaf, and \(X_0\) has a nonzero torsion \(T\). Nevertheless the fact that the space \(\text{Hom}_{\text{Coh} \mathbb{P}^1}(L', T) \neq 0\) contradicts \(6.14\). Thus in the case \(X = \mathcal{O}(n)\) we have

either \(X_1 = 0\) or \(X_0 = 0\).

If \(X_1 = 0\) the triangle \(6.8\) is obtained from the exact sequence

\[
0 \rightarrow \text{Ker} f_0 \rightarrow \mathcal{O}(n) \xrightarrow{f_0} X_0 \rightarrow \text{Coker} f_0 \rightarrow 0
\]
and has the form

\[
\begin{array}{c}
X_0[0] \\
\downarrow f_0 \\
0(n) & \rightarrow & (\text{Ker } f_0)(0) \oplus (\text{Coker } f_0)[-1].
\end{array}
\]

The condition (6.9) gives

\[
\begin{align*}
\text{Hom}^{\leq 0}((\text{Ker } f_0)[0], X_0[0]) &= 0, \\
\text{Hom}^{\leq 0}((\text{Coker } f_0)[-1], X_0[0]) &= 0.
\end{align*}
\] (6.17) (6.18)

As above, the assumption \( \text{Ker } f_0 \neq 0 \) contradicts (6.17). Therefore \( \text{Ker } f_0 = 0 \). Furthermore, the condition (6.18) and Remark 6.8 imply that the triangle (6.16) is isomorphic to (6.4) for some integer \( k > n \).

Finally suppose \( X_0 = 0 \). Then the triangle (6.8) is obtained from the exact sequence

\[
0 \rightarrow X_1 \rightarrow E \xrightarrow{d_1} 0(n) \rightarrow 0
\]

and has the form

\[
\begin{array}{c}
X_1[1] \\
\downarrow \\
0(n) & \rightarrow & E
\end{array}
\]

with \( \text{Hom}^{\leq 0}(E, X_1[1]) = 0 \). Using Remark 6.9 again, we get that the triangle (6.19) is isomorphic to (6.3) for an integer \( k < n - 1 \). This completes the proof of the lemma. ■

### 6.10. Classification of t-structures on \( D^b(Coh \mathbb{P}^1) \).

In this subsection we give the list of all bounded t-structures on the bounded derived category of coherent sheaves on \( \mathbb{P}^1 \). Notice that the group \( \text{Aut}(D^b(Coh \mathbb{P}^1)) \) of autoequivalences of the category acts on t-structures and we write only representatives of t-structures modulo this action. According to [4] the group \( \text{Aut}(D^b(Coh \mathbb{P}^1)) \) is generated by the shift, the automorphisms group of \( \mathbb{P}^1 \) and Picard’s group Pic\( \mathbb{P}^1 \).

As it is well-known, a t-structure \( (D^{\geq 0}, D^{\leq 0}) \) is uniquely reconstructed by any its half. Therefore we shall indicate both parts of a t-structure only if the reconstruction is not obvious.

The first series of t-structures is obtained from the standard t-stability (Subsection 6.1) \((\mathcal{M}, \{\Pi_\mu\}_\mu \in \mathcal{M})\).

We start with the tautological t-structure of the derived category:

\[
\mathcal{A}^{\leq 0} = \langle (\text{Coh } \mathbb{P}^1)[j], j \geq 0 \rangle.
\]

Its core\(^5\) is \( \mathcal{A}^0 = \text{Coh } \mathbb{P}^1 \). This could be shown on a picture.

\(^5\)Here and further on we denote the core of a t-structure \( (D^{\geq 0}, D^{\leq 0}) \) by \( D^0 \).
By definition the tilting the torsion pair \((\mathcal{A}, \mathcal{B})\) is cotilting, if any object in \(\mathcal{A}\) is a quotient of an object in \(\mathcal{B}\).

The torsion pair \((\mathcal{A}, \mathcal{B})\) defines a t-structure on \(D^b(\mathcal{A})\) by

\[
\begin{align*}
D^b(\mathcal{A})_{\leq 0} &= \{ A \in D^b(\mathcal{A}) | H^0(A) \in \mathcal{A} \}, \\
D^b(\mathcal{A})_{\geq 0} &= \{ A \in D^b(\mathcal{A}) | H^{-1}(A) \in \mathcal{A} \}.
\end{align*}
\]

By definition the tilting \(D^b(\mathcal{A})_{\leq 0}\) of \(\mathcal{A}\) w.r.t. \((\mathcal{A}, \mathcal{B})\) is the core of this t-structure. In the case when the torsion pair \((\mathcal{A}, \mathcal{B})\) is cotilting, the derived categories coincide: \(D^b(\mathcal{A})_{\leq 0} = D^b(\mathcal{A})_{\geq 0}\).

For \(D^b(\text{Coh } \mathbb{P}^1)\) we have the following cotilting torsion pairs and t-structures.

\[
\begin{align*}
\mathcal{B}_{\leq 0} &= \langle \mathcal{O}(n), n < 0 \rangle, \quad \mathcal{B}_1 = \langle \mathcal{O}(n), n \geq 0; \mathcal{O}_x, x \in \mathbb{P}^1 \rangle, \\
\mathcal{B}^0 &= \langle \mathcal{O}(n)[i], n \geq 0, i \geq 0; \mathcal{O}_x[i], x \in \mathbb{P}^1, i \geq 0; \mathcal{O}(n)[j], n < 0, j \geq 1 \rangle, \\
\mathcal{B}^{\geq 0} &= \langle \mathcal{O}(n)[0], n \geq 0; \mathcal{O}_x[0], x \in \mathbb{P}^1; \mathcal{O}(n)[1], n < 0 \rangle.
\end{align*}
\]

The last cotilting torsion pair for \(\text{Coh } \mathbb{P}^1\) depends on an arbitrary nonempty subset \(P \subset \mathbb{P}^1\).

\[
\begin{align*}
\mathcal{D}(P)_{\leq 0} &= \langle \mathcal{O}_x, x \in P \rangle, \quad \mathcal{D}(P)_{\geq 0} = \langle \mathcal{O}(n), n \in \mathbb{Z} \rangle, \\
\mathcal{D}(P)^{\leq 0} &= \langle \mathcal{O}_x[i], x \in P, i \geq 0; \mathcal{O}(n)[j] n \in \mathbb{Z}, j \geq 1 \rangle, \\
\mathcal{D}(P)^{\geq 0} &= \langle \mathcal{O}_x[0], x \in \mathbb{P}^1; \mathcal{O}(n)[1], n \in \mathbb{Z} \rangle.
\end{align*}
\]
It is obvious that t-structures \( \mathcal{D}(P) \sim \mathcal{D}(P') \) modulo \( \text{Aut}^b(\text{Coh} \mathbb{P}^1) \) iff \( P' = \varphi(P) \) for some \( \varphi \in \text{Aut} \mathbb{P}^1 \).

The following t-structures are induced by the exceptional stability data \( (\mathcal{E}_p, \{\Pi^{(i)}_\varepsilon\}_{\varepsilon \in \mathcal{E}_p}) \) (see Subsection 6.2), where ordering, as we already know, depends on \( p \in \bar{\mathbb{N}} \). They are sorted into two groups: bounded (\( \mathcal{E}(p) \) and \( \mathcal{F}(p) \)) and unbounded (\( \mathcal{G}, \mathcal{H}, \) and \( \mathcal{I} \)), and defined as follows.

\[
\mathcal{E}(p)^{\leq 0} = \langle \mathcal{O}[i], i \geq p; \mathcal{O}(1)[j], j \geq -2 \rangle,
\mathcal{E}(p)^{0} = \langle \mathcal{O}(p)[0], \mathcal{O}(1)[-2] \rangle,
\]

where \( p \) is a nonnegative integer.

\[
\begin{array}{c}
\mathcal{E}(p)^{\leq 0} \\
\cdots \mathcal{O}[p-1] \mathcal{O}(1)[-2] \mathcal{O}[p] \mathcal{O}(1)[-1] \mathcal{O}[p+1] \mathcal{O}(1)[0] \cdots \\
\mathcal{E}(p)^{\geq 0}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{F}(p)^{\leq 0} \\
\cdots \mathcal{O}[p-1] \mathcal{O}(1)[-2] \mathcal{O}[p] \mathcal{O}(1)[-1] \mathcal{O}[p+1] \mathcal{O}(1)[0] \cdots \\
\mathcal{F}(p)^{\geq 0}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{G}^{\leq 0} = \langle \mathcal{O}[i], i \geq 0; \mathcal{O}(1)[j], j \in \mathbb{Z} \rangle,
\mathcal{G}^{\geq 0} = \langle \mathcal{O}[i], i \leq 0 \rangle,
\mathcal{G}^{0} = \langle \mathcal{O}[0] \rangle.
\end{array}
\]

\[
\begin{array}{c}
\mathcal{G}^{\geq 0} \\
\cdots \mathcal{O}[-1] \mathcal{O}[0] \mathcal{O}[1] \cdots \mathcal{O}(1)[-1] \mathcal{O}(1)[0] \mathcal{O}(1)[1] \cdots \\
\mathcal{G}^{\leq 0}
\end{array}
\]

\[
\begin{array}{c}
\mathcal{O}[j], j \in \mathbb{Z} \\
\mathcal{O}(1)[j], j \in \mathbb{Z}
\end{array}
\]
\[ H^{\leq 0} = \langle \mathcal{O}(1)[j], j \geq 0 \rangle, \]
\[ H^{\geq 0} = \langle \mathcal{O}[i], i \in \mathbb{Z}; \mathcal{O}(1)[j], j \leq 0 \rangle, \]
\[ H^0 = \langle \mathcal{O}(1)[0] \rangle. \]

\[ I^{\leq 0} = \langle \mathcal{O}(1)[j], j \in \mathbb{Z} \rangle, \]
\[ I^{\geq 0} = \langle \mathcal{O}[i], i \in \mathbb{Z} \rangle, \]
\[ I^0 = \langle 0 \rangle. \]

**Remark 6.11.** It can be easily checked that the cores of the t-structures induced by the exceptional stability data are following:

- \( \mathcal{E}(p)^0 \) \( \forall p \geq 0 \) and \( \mathcal{F}(p)^0 \) \( \forall p \geq 1 \) are equivalent to direct sum of the categories of vector spaces \( \text{Vect} \oplus \text{Vect} \);

- \( \mathcal{F}(0)^0 \) is equivalent to the category of the quiver representations;

- \( \mathcal{G}^0 \) and \( H^0 \) are equivalent to \( \text{Vect} \); and

- \( I^0 \) is the zero category.

Since the category \( \text{Vect} \) is splitting, in the case of t-structures induced by the exceptional stability data only for \( \mathcal{A} = \mathcal{F}(0)^0 \) the derived category \( D^b(\mathcal{A}) \) is equivalent to \( D^b(\text{Coh} \mathbb{P}^1) \) (see [9]).

Now we recall, that a given bounded t-structure on \( D^b(\text{Coh} \mathbb{P}^1) \) induces weak stability data (Lemma [3.3]). By Corollary [6.7] this t-stability is weaker either the standard (\( \mathcal{M} \)) one or the exceptional (\( \mathcal{E} \)) one. Therefore according to Proposition [5.5] any given bounded t-structure can obtained by subdividing the slope set \( \mathcal{M} \) or \( \mathcal{E} \) in two disjoint parts. Thereby we proved the following theorem.

**Theorem 6.12.** The defined above t-structures \( \mathcal{A} = \mathcal{F}(p) \) are the only bounded t-structures on \( D^b(\text{Coh} \mathbb{P}^1) \) modulo the group of autoequivalences of the category.
7 Stability data for a curve of positive genus

In this section we study stability data for the bounded derived category of coherent sheaves on a smooth curve $C_g$ of a positive genus $g$. We denote the triangulated category $D^b(Coh C_g)$ by $\mathcal{T}_g$.

At first we define the set of standard stability structures extending the Mumford–Takemoto stability. Then we show that a t-stability for $\mathcal{T}_g$ is induced by a stability on the abelian category $Coh C_g$. In the case of elliptic curve $C_1$ we prove that there exists a unique type of the finest t-stability on $\mathcal{T}_1$. Namely, the standard t-stability depending on ordering $\mathbb{Q}$ copies of the curve. Finally we list all bounded t-structures on $\mathcal{T}_1$.

It was shown in Proposition 6.3 that a t-stability on a derived category of an abelian category $\mathcal{A}$ can be constructed as an extension of a stability on $\mathcal{A}$. For the abelian category $\mathcal{A}$ we may define stability structure via a positive base of $K^b_{\mathcal{A}}(\mathcal{A})$ (Definition 2.3). In the case $\mathcal{A} = Coh C_g$ the base $(\text{rk}, \text{deg})$ (rk is rank and deg is degree of a sheaf) is positive. So we have a slope $\gamma(F)$ for every sheaf $F \in Coh C_g$, where

$$\gamma(F) = \begin{cases} \frac{1}{\pi} \arctg \left(-\frac{\deg F}{\text{rk} F}\right) & \text{when } \text{rk} F \neq 0, \\ 1 & \text{otherwise}. \end{cases}$$

Since $\arctg x$ is a strictly decreasing function, this slope is equivalent to

$$\bar{\mu}(F) = \begin{cases} \frac{\deg F}{\text{rk} F} & \text{when } \text{rk} F \neq 0, \\ +\infty & \text{otherwise}. \end{cases}$$

Recall that in these terms a sheaf $E$ is $\bar{\mu}$-semistable, if for any $0 \neq F \subset E$ the inequality $\bar{\mu}(F) \leq \bar{\mu}(E)$ is valid. Therefore, any torsion sheaf is $\bar{\mu}$-semistable, and a torsion free sheaf on $C_g$ is $\bar{\mu}$-semistable if and only if it is MT-semistable (for Mumford–Takemoto stability).

We shall use the notation $\bar{\mathbb{Q}}$ for the slope set $\mathbb{Q} \cup \{+\infty\}$ with the natural order. Thus we obtain the stability $(\bar{\mathbb{Q}}, \{\Pi_\mu\}_{\mu \in \bar{\mathbb{Q}}})$ for the abelian category $Coh C_g$ and, furthermore, a t-stability for the derived category $\mathcal{T}_g$. It is clear that the constructed t-stability is not the finest.

Let us refine $\bar{\mathbb{Q}}$. Note that for any rational number $q = \frac{a}{r}$ with coprime $d \neq 0$ and $r > 0$ (or $d = 0$, $r = 1$) there exists a bundle on $C_g$ of rank $r$ and degree $d$ that is stable w.r.t. Mumford–Takemoto stability. Denote the set of all such MT-stable bundles by $\mathcal{M}_q$. It is well-known, that for $F, G \in \mathcal{M}_q$ $\Hom(F, G) = \Hom(G, F) = 0$ whenever $F \neq G$. If in addition we denote by $\mathcal{M}_\infty$ the curve $C_g$ parameterizing torsion sheaves $O_x$, we obtain the set $\mathcal{M}_g$ of semistable subcategories $\Pi_{(q, F)} = \langle F \rangle$, where $q \in \bar{\mathbb{Q}}$, and $F \in \mathcal{M}_q$. Choosing a linear order on each $\mathcal{M}_q$ in arbitrary way and extending lexicographically the orders to an order on $\mathcal{M}_g$, we construct the finest stability $(\mathcal{M}_g, \{\Pi_\mu\}_{\mu \in \mathcal{M}_g})$ on the abelian category $Coh C_g$ (because the existence of Harder–Narasimhan filtration is obvious).

Furthermore, introducing as the slope set $D(\mathcal{M}_g) = \mathbb{Z} \times \mathcal{M}_g$ with its lexicographic order and defining as the semistable subcategories $\Pi_{(i, q, F)} = \langle F[i] \rangle$, where $F \in \mathcal{M}_q$, we get the finest stability data on $\mathcal{T}_g$ (Proposition 5.4 and Proposition 3.4). We shall call various t-stabilities obtained this way the standard t-stabilities.

Note that in the case of an elliptic curve $C_1$ any set $\mathcal{M}_g$ of MT-semistable sheaves with coprime $r$ and $d$ consists of indecomposable simple bundles and is naturally isomorphic to $C_1$ ($\Pi_1$). Therefore the slope set $\mathcal{M}_1$ of a standard stability on $Coh C_1$ is the direct product $\mathbb{Q} \times C_1$.

Further, we prove the following proposition.

**Proposition 7.1.** Any a t-stability on $D^b(Coh C_g)$ for $g > 0$ is induced by a stability on $Coh C_g$. 


**Proof.** Since homological dimension of $\text{Coh}_g$ is 1, the proposition is true iff any sheaf $E \in \text{Coh}_g$ considered as a 0-complex has a HN-system (w.r.t. the t-stability) $E \rightsquigarrow (E_{\varphi_0}, \ldots, E_{\varphi_n})$ such that $E_{\varphi_i} \in \text{Coh}_g \forall i$.

Let us consider a distinguished triangle

$$
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
E & \rightarrow & Y
\end{array}
$$

(7.1)

with

$$\text{Hom}^{\leq 0}(Y, X) = 0.$$  

(7.2)

To verify that all quotients of HN-system for a sheaf are sheaves as well, it is sufficient to show that the objects $X$ and $Y$ from the triangle are sheaves, whenever $E$ is.

Now the proposition immediately follows from the following lemma.

**Lemma 7.2.** Suppose $E \in \text{Coh}_g$ is included in the triangle (7.1), satisfying the condition (7.2). Then $X, Y \in \text{Coh}_g$ and $\text{Hom}_{\text{Coh}_g}(Y, X) = 0$.

**Proof of the lemma.** Since homological dimension of $T_g$ equals 1, taking into account Remark 6.8 we can assume that $X = X_0[0] \oplus X_1[1]$, where $X_0, X_1 \in \text{Coh}_g$, and $f = f_0 \oplus f_1$ with $f_0 \in \text{Hom}_{\text{Coh}_g}(E, X_0)$, $f_1 \in \text{Ext}^1_{\text{Coh}_g}(E, X_1)$. Moreover, the condition $f_i = 0$ is equivalent to $X_i = 0$.

Consider the extension

$$0 \rightarrow X_1 \rightarrow G \xrightarrow{d-1} E \rightarrow 0,$$

(7.3)

corresponding to $f_1$, and realize the object $X_1[1]$ as a complex $G \xrightarrow{d-1} E$. Then $f_1$ can be represented as the morphism of complexes:

$$
\begin{array}{ccc}
0 & \rightarrow & E \\
\downarrow & & \downarrow \text{id}_E \\
G & \xrightarrow{d-1} & E
\end{array}
$$

Hence the direct sum of maps $f_0 \oplus f_{-1}$ is the morphism:

$$
\begin{array}{ccc}
0 & \rightarrow & E \\
\downarrow & & \downarrow \text{id}_E \oplus f_0 \\
G & \xrightarrow{d-1} & E \oplus X_0
\end{array}
$$

A cone $C(f_0 \oplus f_1)$ becomes equal to a complex of the form

$$E \oplus G \xrightarrow{d-1} E \oplus X_0,$$

where $\delta_{-1}$ is determined by the matrix

$$
\begin{pmatrix}
\text{id}_E & d_{-1} \\
0 & f_0
\end{pmatrix}.$$
Now it follows from the commutative diagram

\[
\begin{array}{cccccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \rightarrow & X_1 & \rightarrow & G & \rightarrow & E & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & H^{-1} & \rightarrow & E \oplus G & \rightarrow & E \oplus X_0 & \rightarrow & H_0 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & \text{Ker } f_0 & \rightarrow & E & \rightarrow & X_0 & \rightarrow & \text{Coker } f_0 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

that the cone \( C(f_0 \oplus f_1) \) is isomorphic to \( H^{-1}[1] \oplus \text{Coker } f_0[0] \). In this notation the condition (7.2) can be rewritten as

\[
\text{Hom}^{\leq 0}(H^{-1}[0] \oplus \text{Coker } f_0[-1], X_0[0] \oplus X_1[1]) = 0.
\]

In particular,

\[
\begin{align*}
\text{Hom}^0(\text{Coker } f_0[-1], X_0[0]) &= \text{Ext}^1_{\text{coh } C_g}(\text{Coker } f_0, X_0) = 0, \\
\text{Hom}^0(H^{-1}[0], X_1[1]) &= \text{Ext}^1_{\text{coh } C_g}(H^{-1}, X_1) = 0.
\end{align*}
\]

By definition of a cokernel we have that \( \text{Hom}_{\text{coh } C_g}(X_0, \text{Coker } f_0) \neq 0 \) unless \( \text{Coker } f_0 = 0 \). On the other hand, the canonical divisor \( K_g \) on a curve of positive genus is either zero (\( g = 1 \)) or effective (\( g > 1 \)). Therefore, \( \text{Hom}_{\text{coh } C_g}(X_0, \text{Coker } f_0(K_g)) \neq 0 \) also. Whence by Serre duality theorem we get that \( \text{Ext}^1_{\text{coh } C_g}(\text{Coker } f_0, X_0) \neq 0 \). This contradicts the condition (7.4), and we conclude that \( \text{Coker } f_0 = 0 \).

We see from the commutative diagram that in the case \( X_1 \neq 0 \) the space \( \text{Hom}_{\text{coh } C_g}(X_1, H^{-1}) \) is not trivial, consequently \( \text{Hom}_{\text{coh } C_g}(X_1, H^{-1}(K_g)) \neq 0 \). Using Serre duality theorem again we obtain \( \text{Ext}^1_{\text{coh } C_g}(H^{-1}, X_1) \neq 0 \), which contradicts the condition (7.5). Thus \( X_1 = 0 \) and the triangle (7.1) reduces to

\[
\begin{array}{cccc}
X_0 & \rightarrow & E & \rightarrow & \text{Ker } f_0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & & E & \rightarrow & \text{Ker } f_0
\end{array}
\]

This completes the proofs of the lemma and the proposition. ■

As a corollary of the proposition we get the following theorem.

**Theorem 7.3.** Any stability data \((\Phi, \{\Pi_\phi\}_{\phi \in \Phi})\) on the bounded derived category \( T_1 \) of coherent sheaves on a smooth elliptic curve \( C_1 \) one can refine to one of the standard stabilities \((D(M_1), \{\Pi_\mu\}_{\mu \in D(M_1)})\). In particular, any finest \( t \)-stability on \( T_1 \) is standard.

**Proof.** Since any stability data on \( T_1 \) is induced by a stability on \( \text{Coh } C_1 \), it is sufficient to show, that each indecomposable sheaf \( E \) (a generator of a standard semistable subcategory) is \( \Phi \)-semistable. Suppose that it is not true. Then by Lemma 7.2 the sheaf \( E \) is included in an exact sequence

\[
0 \rightarrow Y \rightarrow E \rightarrow X \rightarrow 0
\]
with \( \text{Hom}_{\text{Coh}C_1}(Y,X) = 0 \). It follows from Serre duality theorem that \( \text{Ext}^1_{\text{Coh}C_1}(X,Y) = 0 \). Therefore the exact sequence splits and \( E = X \oplus Y \).

Finally, using various admissible decompositions of the slope set \( \mathcal{M}_1 \), we list all bounded t-structures on \( T_1 \) modulo the group \( \text{Aut} T_1 \). Each of them is determined by a cotilting torsion pair. Therefore we shall describe only the pairs.

Let \( X \) denotes an indecomposable simple sheaf on \( C_1 \). Denote by \( I \) the union of rational numbers from the segment \([0,1)\) with \( \{ \infty \} \). Then each \( q \in I \) and a subset \( P \subset \mathfrak{M}_q \simeq C_1 \) of the moduli space of slope \( q \) MT-stable sheaves on \( C_1 \) (may be empty) give the cotilting torsion pair \( \left( \mathcal{A}(q,P)_1, \mathcal{A}(q,P)_0 \right) \), where

\[
\mathcal{A}(q,P)_0 = \{ X | \bar{\mu}(X) < q \text{ or } \bar{\mu}(X) = q \text{ and } X \in P \},
\]

\[
\mathcal{A}(q,P)_1 = \{ X | \bar{\mu}(X) > q \text{ or } \bar{\mu}(X) = q \text{ and } X \notin P \}.
\]

Note that the standard t-structure is obtained when \( q = 0 \) and \( P = \emptyset \).

Thus we have the theorem.

**Theorem 7.4.** Any bounded t-structure on the bounded derived category \( D^b(\text{Coh}C_1) \) of coherent sheaves on a smooth elliptic curve (modulo \( \text{Aut}^b(\text{Coh}C_1) \)) is determined by one of the cotilting torsion pairs \( \left( \mathcal{A}(q,P)_1, \mathcal{A}(q,P)_0 \right) \) described above.

**References**

[1] Atiyah, M.F. *Vector Bundles Over an Elliptic Curve*. Proc. Lond. Math. Soc, VII (1957) 414-452.

[2] Beilinson, A.A. *Coherent Sheaves on \( \mathbb{P}^n \) and Problems in Linear Algebra*. Funk. An., 12 (1978), p. 68–69.

[3] A.I. Bondal. *Helices, Representations of Quivers and Koszul Algebras*. Helices and Vector Bundles: Seminare Rudakov. London Math. Soc., Lect. N. Ser. 148 (1990). p. 75–95.

[4] A. Bondal, D. Orlov. *Reconstruction of a variety from the derived category and groups of autoequivalences*. Compositio Math. 125 (2001), no. 3, 327–344.

[5] A. Bondal and M. Van den Bergh. *Generators and representability of functors in commutative and noncommutative geometry*. Moscow Mathematical Journal v. 3 (2003) N 1. (It is available also in arXiv:math.AG/0204218 v2.)

[6] Tom Bridgeland. *Stability conditions on triangulated categories*. arXiv:math.AG/0212237.

[7] Tom Bridgeland. *Stability conditions on K3 surfaces*. arXiv:math.AG/0307164 v1.

[8] S.I. Gelfand and Yu.I. Manin: *Methods of Homological Algebra*, Springer-Verlag (1996).

[9] Michael R. Douglas. *Dirichlet branes, homological mirror symmetry, and stability*. arXiv:math.AG/0207021.
[10] Gieseker, D.: *On the moduli of vector bundles on an algebraic surface*. Ann. of Math. 106, 45 (1977).

[11] A. Gorodentsev, S. Kuleshov: *Helix theory*. Preprint MPI 2001 (97).

[12] D. Happel, I. Reiten, and S. Smalø. *Tilting in abelian categories and quasitilted algebra*, Memoirs of the AMS, vol. 575, Amer. Math. Soc., 1996.

[13] Maruyama, M.: *Moduli of stable sheaves I*. J. Math. Kyoto 17, 91 (1977).

[14] Maruyama, M.: *Moduli of stable sheaves II*. J. Math. Kyoto 18, 557 (1978).

[15] A. Rudakov. *Stability for an abelian category*. J. Algebra 197 (1997), no. 1, 231–245