We show that it is possible to achieve a perfect impedance matching by designing an antireflection temporal medium, which is omnidirectional and frequency-independent in an ultra-wide band. Our approach is an extension of the antireflection temporal coating by Pacheco-Peña et al [Optica 7, 323 (2020)]. We demonstrate that a specially-engineered multi-stage discrete or continuous temporal changing in the permittivity of the temporal medium allows ultra-wideband reflectionless wave propagation, which has been confirmed analytically and numerically. As an illustrative example for application, the proposed approach is applied to match a dielectric slab with free space for an ultra-wideband pulse.

**Keyword:** temporal medium, multi-stage temporal transformer, antireflection coating
Electromagnetic wave will be partially reflected when impinging from free space into a dielectric medium [2]. This reflection is often undesirable in electromagnetic components and devices, thus removing reflection by antireflection coating has always been an important topic in both the scientific and engineering communities. The simplest antireflection coating is a spatial quarter-wave impedance transformer with refractive index intermediate between those of free space and the dielectric medium [3]. The underlying physical principle is that waves reflected back from two interfaces cancel each other in a destructive interference. However, its performance is actually quite limited because it is typically designed only for normal incidence at a specific frequency. Though significant efforts have been undertaken to extend the single-frequency quarter-wave antireflection to a wide frequency band, such as utilizing spatial multiple layers [4], variable refractive index profiles [5], [6], and surface textures [7], these approaches still suffer from narrow frequency band as well as strongly limited range of incident angles. In 2018, [8] proposed a universal antireflection layer which is theoretically independent of polarization, frequency and incident angle. However, the proposed layer is still a challenge to be physically created due to the specific spatiotemporal dispersion.

Recently, temporal media, whose constitutive parameters vary with time, has been attracting much attention. Compared with the traditional time-invariant naturally occurring media or artificial metamaterials, temporal media offer a significant potential for furtherly boosting the degree of manipulating of wave propagation. They have been suggested in various applications, such as energy accumulation [9], time reversal [10], inverse prism [11]-[12], magnetless nonreciprocity [13]-[15], wave pattern engineering [16], temporal crystal [17]-[18], and impedance matching [8]. In 2020, V. Pacheco-Peña et al proposed a
temporal analogue of the quarter-wave impedance transformer technique, achieving both a perfect impedance matching and frequency conversion for a desired frequency [1]. It should be noted that such an antireflection temporal coating is inherently independent of incident angle. This novel design initiates a new research direction for achieving perfect impedance matching.

In this paper, we extend the quarter-wave temporal transformer to multi-stage transformer, and then to continuously variable one. We demonstrate analytically and numerically that a specially-engineered multi-stage discrete or continuous changing in the permittivity of the temporal medium allows an ultra-wideband reflectionless wave propagation. The reflection spectra for various temporal media are numerically derived to evaluate its figure of merit. As an illustrative example for application, the proposed approach is applied to match a dielectric slab with free space for an ultra-wideband pulse.

We start with the significant feature of temporal media is the preservation of the wavenumber $k$ and the wavelength $\lambda$ as the parameters of medium alter [19]-[20]. Therefore, the frequency of the electromagnetic (EM) wave is changed with the wave velocity, which is $\omega = kc/\sqrt{\varepsilon}$, $c$ being the speed of EM wave in the vacuum. In a consecutively changed medium, $\omega_{\varepsilon_0} = \omega(t)\sqrt{\varepsilon(t)}$, which means the final frequency is dependent on the initial and final permittivity of the medium.

The temporal transmission and reflection coefficients at a temporal boundary can be calculated based on the continuity of vector $D$ and $B$. Considering non-magnetized ($\mu=1$) material, the electric field transmission index $A$ and reflection index $B$ in can be derived as [19]
In [1], an intermediate stage of permittivity is inserted into 2 unmatched stages to suppress the reflection, named of “temporal quarter-wave transformer”. As shown in Fig. 1(a). At 2 temporal boundaries, 2 reflection waves obtain the same amplitude but inverted phases. Therefore, the total reflection is zero. It is obvious the “quarter-wave” is for one frequency, and such transformer is congenitally frequency-dependent. It is not applicable to matching wideband signal.

In spatial scenarios, multi-section transmission line is realized by cascading several ununiform impedance transmission lines to broaden the reflectionless bandwidth. It is also analogous to the multilayered medium model. Inspired by the multi-section line model, we propose a multi-stage transformer to achieve wideband temporal matching.

Fig. 1(b) describes the schematic of multi-stage transformer. (2) lists the total reflection coefficients of multi-stage transformer with \( N+1 \) temporal boundaries. They are defined as \( N \)-order multi-stage transformer, since \( N \) stages are inserted.

Denoting \( \Gamma_n = B_{n+1}/A_{n+1} = (Z_{n+1} - Z_n)/(Z_{n+1} + Z_n) \), for temporal quarter-wave (1-order), 2-order and 3-order transformers, the total reflection can be approximated as

\[
\begin{align*}
\Gamma^{(1)} &= A_2 A_{2}^{-2\phi} = A_1 A_{1}^{-2\phi} \left( \Gamma_{0} + \Gamma_{1}^{-2\phi} \right) \\
\Gamma^{(2)} &= A_4 A_{4}^{-4\phi} \left( \Gamma_{0} + \Gamma_{1}^{-2\phi} + \Gamma_{2}^{-2(2\phi+\phi)} \right) \\
\Gamma^{(3)} &= A_4 A_{4}^{-4\phi} \left( \Gamma_{0} + \Gamma_{1}^{-2\phi} + \Gamma_{2}^{-2(2\phi+\phi)} + \Gamma_{3}^{-2(3\phi+\phi)} \right)
\end{align*}
\]

By parity for reasoning, it can be generalized that

\[
\Gamma^{(N)} \propto \Gamma_{0} + \Gamma_{1} e^{2\phi} + \Gamma_{2} e^{2(2\phi+\phi)} + \ldots + \Gamma_{N} e^{\sum_{i=1}^{N} 2\phi}
\]

where \( \phi_n = \omega_n \tau_n \), \( \tau_n \) being the duration of the \( n \)-th stage. To note that
\[ \gamma(\phi) = (1 + e^{j2\phi})^N = \sum_{n=0}^{N} C_n^* e^{j2\phi} \]  

possesses the band-stop property around \( \phi = \pi/2 \), and \( |\gamma(\pi/2)| \) equals to zero [3]. (4) can be the prototype function for a reflectionless wideband system. Comparing (3) and (4), \( \forall n, \phi_n = \phi \) should be satisfied. For the incident center frequency \( \omega_0 \) as well as its subsequent converted frequency \( \omega_n \), we have \( \omega_n \tau_n = \pi/2 \). The value of \( \tau_n \) should satisfy

\[ \Gamma_n = \frac{(Z_{n+1} - Z_n)}{(Z_{n+1} + Z_n)} = PC_N^w, \]

where \( P \) is the normalization constant. \( P \) can be determined by

\[ \ln(Z_{n+1}/Z_0) = \prod_{n=0}^{N} \frac{(1 + PC_N^w)}{(1 - PC_N^w)} \]

The impedance \( Z_1 = Z_0 (1 + PC_N^w) / (1 - PC_N^w) \) and so on. The permittivity of each stage, the subsequent converted center frequencies, and the duration of each stage can be determined as

\[
\begin{cases}
\epsilon_m = \epsilon_0 Z^2_0 / Z_m^2 \\
\omega_m = \omega_0 \sqrt{\epsilon_m} / \sqrt{\epsilon_0} \\
\tau_m = \pi / 2\omega_m
\end{cases}
\]

respectively. So far, the parameters of a multi-stage transformer are obtained. For the passband with maximum tolerance of reflection amplitude \( r \), the relative bandwidth is

\[ RBW = 4\text{arctan}\left[\sqrt{4r^{2(N-1)} - r^{4(N-1)}} / (2 - r^{2(N-1)})\right] / \pi \]

from which the minimum order of the transformer can be determined for a signal with known bandwidth.
**Fig. 1.** The schematic diagram of temporal quarter-wave (a) and multi-stage (b) impedance transformers. Functions of permittivity for 1-, 5- and 9-order multi-stage transformers with a center frequency of 2 GHz (c) and 4 GHz (d). (e), (f) calculated reflection spectral of the transformers shown in (c) and (d), respectively.

In Fig. 1(c), 1/5/9-order multi-stage transformers are designed based on (5) to (7). The initial center frequency is 2 GHz and \( \varepsilon_r \) changes from 1 to 4. Fig. 1(e) shows the calculated reflection spectral response based on (3). It can be observed that higher order leads to a wider passband. Similar to the temporal quarter-wave transformer, the multi-stage transformer has periodical passbands as well, which occur at \((2a-1)\omega_0\), \(a\) being a positive integer. Fig. 1(d) and 1(f) shows the 1/5/9-order transformers designed at 4 GHz center frequency and their reflection spectral responses. The difference from Fig. 1(d) to 1(c) is merely the half-reduced duration of each stage.
Fig. 2. The waveform of Gaussian pulse (a) with a center frequency of 2 GHz and a spectral standard deviation of 0.5 GHz, and its spectrum (b). (c) simulated spatial distributions of the reflected and transmitted electric fields when the Gaussian pulse passes through the 1-order multi-stage transformers at t=3.45 ns and 9-order multi-stage transformers at t=2.92 ns, and the enlarged view of reflected electric fields (d).

To investigate the performance of the multi-stage transformers, a Gaussian pulse is used as the input. Its waveform and spectrum are shown in Fig. 2(a) and 2(b), respectively. To note that the spectral standard deviation \( \sigma_f \) is 0.5 GHz. Alternatively speaking, the 3-dB bandwidth of such signal is 0.833 GHz.

The full-wave simulations are implemented in COMSOL Multiphysics 5.5. The signal propagates in the parallel plate waveguide structure, whose upper and lower boundaries are PEC, and the left and the right sides are scattering boundaries. The TEM wave propagates along \( x \)-direction. The electric field is polarized in \( y \)-direction, perpendicular to PEC boundary, and magnetic field is oriented in \( z \)-direction. Inside the waveguide is the spatially homogeneous temporal medium whose permittivity subjects to the 1- and 9-order transformer as shown in Fig. 1(c)

Fig. 2(c) and 2(d) shows the normalized electric fields distributed after the conversion by multi-stage transformers with 1-order and 9-order. The time ranges of the simulation are such that the backward waves and forward waves are completely separated. Compared
to the quarter-wave temporal transformer (1-order), the 9-order multi-stage transformer suppresses the amplitude of backward wave by 29.8 dB.

Fig. 3. The schematic diagram of multi-stage discrete (a) and continuously varying (b) impedance transformers. The sigmoidal functions of permittivity with different $\alpha$ (c) and their reflection spectra (d).

The analysis above shows the effectiveness of multi-stage transformer in wideband matching purpose. However, to realize a step variation of permittivity is difficult. Then, can we achieve wideband temporal matching with the continuous permittivity variations? To address this problem, the continuous temporal transformer is studied. Fig. 3(a) and 3(b) demonstrate the evolution from multi-stage transformer to continuous temporal transformer. To analyze the reflection spectral response of continuous transformer, the time-dimension is firstly divided into multiple infinitesimal time interval $\Delta t$, and reflection $\Delta \Gamma[n]$ at the moment $n\Delta t$ is obtained.

$$
\Delta \Gamma[n] = \exp(-j 2 \sum_{l=0}^{n} \omega_l [l\Delta t] \cdot \Delta t) \cdot (Z([n+1]\Delta t) - Z[n\Delta t])/2Z[n\Delta t])
$$

$$
= \exp(-j \omega_0 \Delta t \sum_{l=0}^{n} \sqrt{\frac{\varepsilon_{\infty}}{\varepsilon_{\infty}[l\Delta t]}} \cdot \left(\varepsilon_{\infty}\frac{[n+1]\Delta t}{2\varepsilon_{\infty}[n\Delta t]} \right)
$$

(8)

in which $\omega[n\Delta t] = \omega_0 \sqrt{\varepsilon_{\infty}/[\varepsilon_{\infty}[n\Delta t]]}$ and $Z[n\Delta t] = \sqrt{\mu_{0}/\varepsilon_{\infty}[n\Delta t]}$ are used. $h[n\Delta t] = \varepsilon_{\infty}[n\Delta t]$ for simplification, the total reflection can be written into definite integral form
\[
\Gamma(h, \omega_0) = \int_0^\infty \left[ \exp\left(-j\omega_0 \int_0^t \frac{h(s)}{h(t_0)} ds \right) \cdot \frac{h'(t)}{2h(t)} \right] dt
\]

(9)

fixed endpoint value \( h(t_0) \equiv \varepsilon_{r_0}^{-1/2} \) and \( h(t_1) \equiv \varepsilon_{r_1}^{-1/2} \). Thus, the reflection calculation would be a functional of \( h(t) \). Each waveform of permittivity variation \( \varepsilon_r(t) \) or \( h(t) \) result in a reflection spectral response.

In Fig. 1(c) and 1(d), the discrete waveform of multi-stage transformer is sigmoid-like. Based on this clue, we study the sigmoid functions \( \varepsilon_r(t) = \varepsilon_{r_0} + (\varepsilon_{r_1} - \varepsilon_{r_0})/(1 + e^{-\alpha t}) \) as the variation of permittivity, in which \( \alpha \) depicts the transition rapidness of the permittivity. Fig. 3(c) show the sigmoidal permittivity variation waveforms with different \( \alpha \), in which \( \varepsilon_r \) is gradually changed from 1 to 4. In Fig. 3(d), the reflection spectral responses of the continuous transformer in Fig. 3(c) are calculated based on (8). All of them manifest great flatness in reflectionless passband starting from a “cutoff frequency” to infinite frequency, which means such transformer possesses high-pass property. That’s because (9) is the limiting form of (3), and since the stage duration \( \Delta t \) approaches to zero, the designed center frequency is equivalent to infinite.

Fig. 4. Simulated spatial distributions of the reflected and transmitted electric fields when the Gaussian pulse passes through the sigmoidal continuously varying transformers with different \( \alpha \), \( \alpha=1.6 \text{ GHz at } t=11 \text{ ns (a), } \alpha=3.2 \text{ GHz at } t=6 \text{ ns (b), and } \alpha=6.4 \text{ GHz at } t=3.5 \text{ ns (c). (d)-(f) the enlarged view of reflected electric fields in three cases.}
With the same full-wave simulation setup, the same Gaussian pulse in Fig. 2(a) is also used as the input signal to test the sigmoidal continuous transformer with $\alpha=1.6\text{GHz}$, $3.2\text{GHz}$ and $6.4\text{GHz}$, respectively. The distributed electric field after the conversions are shown in Fig. 4(a)-(c). It can be seen that the proposed continuous transformers successfully achieve the reflection suppression when the input signal spectrum lies in the passband of the transformer. Fig. 4(d)-(f) show the zoom-in view of the total reflection of 3 continuous transformers.

![Fig. 5](image)

**Fig. 5.** The waveform of a composite pulse consisting of 3 Gaussian pulses and its spectrum (b). Simulated spatial distributions of the reflected and transmitted electric fields when the composite pulse passes through the 9-order multi-stage transformers at $t=3.5\text{ ns}$ (c), and passes through the sigmoidal continuously varying transformers with $\alpha=3.2\text{ GHz}$ at $t=6\text{ ns}$ (d). The inset shows the enlarged view of the reflected electric fields.

Fig. 5 shows the comparison between multi-stage transformer and continuous transformer. A new pulse acting as the input signal, consists of 3 in-phase Gaussian pulses respectively centered at 2 GHz, 4 GHz and 6 GHz, with same spectral standard deviation $\sigma_f = 0.5\text{ GHz}$. The waveform and the spectrum of such pulse are presented in Fig. 5(a) and 5(b), respectively. After the conversion of 9-order multi-stage transformer and 3.2-GHz-$\alpha$ continuous transformer, the electric field distributions are plotted in Fig. 5(c)-(d). It can be
seen that for the multi-stage transformer, the Gaussian components centered at 2 GHz and 6 GHz can be matched. But the 4 GHz-center Gaussian component lies in the stopband of the transformer. Thus, a noticeable reflection can be detected. However, all of these Gaussian components can be matched by the continuous transformer, since there exist no stopbands beyond the “cutoff-frequency”.

Fig. 6. Spatial impedance matching with the presence of dielectric slab by temporal impedance transformer. (a) The permittivity variation of each spatial section. (b) The propagation of the new pulse with temporal transformer sections. The transformer sections are used to match the free space to the dielectric slab. (c) The propagation of the new pulse in the presence of a dielectric slab without temporal transformer sections.

The discussions above are based on spatial homogeneous condition. Nevertheless, the wideband temporal transformer is more practical for spatiotemporal matching, because most of signals are with finite length, thus, wideband. As an illustration, we use the temporal continuous transformers to achieve spatial-inhomogeneous wideband impedance matching. As shown in Fig. 6, an $\varepsilon_r = 4$ dielectric slab is inserted in the $\varepsilon_r = 1$ free space. The pulse of Fig. 5(a) is used as the input. Two parts of 3.2-GHz-$\alpha$ continuous transformers
are placed at the both sides of the slab. The permittivity temporal variations of all spatial parts are indicated in Fig. 6(a). It is seen that in the whole matching process, there is no reflection by means of temporal wideband matching. For comparison, without the temporal matching sections, the multiple reflections of the pulse due to the spatial boundaries can be detected as shown in Fig. 6(c).

In conclusion, we firstly extend the temporal quarter-wave impedance transformer to multi-stage transformer. All the parameters of such transformer are determined. By increasing the order of transformer, the width of the passband can be broadened. For practicality, the continuous temporal impedance transformer is investigated. We derive the analytical reflection spectral response applicable to any waveform of continuous permittivity variation. Specially, sigmoidal permittivity variations provide great flatness in the passband. Compared with multi-stage transformer with periodical passband, the continuous transformer has consecutive passband in higher frequency. The cutoff-frequency is dependent on the transition rapidness. Finally, we use 2 parts of continuous temporal transformer to achieve the spatiotemporal impedance matching purpose.

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Conflicts of Interest

The authors declare no conflicts of interest.

Reference:

[1] V. Pacheco-Peña and N. Engheta, "Antireflection temporal coatings," Optica, vol. 7, no. 4, pp. 323-331, 2020.
[2] L. D. Landau, The classical theory of fields. Elsevier, 2013.
[3] D. M. Pozar, Microwave engineering. John Wiley & Sons, 2011.
[4] H. A. Macleod and H. A. Macleod, Thin-film optical filters. CRC press, 2010.
[5] Q. Tang, S. Ogura, M. Yamasaki, and K. Kikuchi, "Experimental study on intermediate and gradient index dielectric thin films by a novel reactive sputtering method," Journal of Vacuum Science & Technology A: Vacuum, Surfaces, and Films, vol. 15, no. 5, pp. 2670-2672, 1997.
[6] H. K. Raut, V. A. Ganesh, A. S. Nair, and S. Ramakrishna, "Anti-reflective coatings: A critical, in-depth review," Energy & Environmental Science, vol. 4, no. 10, pp. 3779-3804, 2011.
[7] P. Spinelli, M. Verschuuren, and A. Polman, "Broadband omnidirectional antireflection coating based on subwavelength surface Mie resonators," Nature communications, vol. 3, no. 1, pp. 1-5, 2012.
[8] K. Im, J.-H. Kang, and Q.-H. Park, "Universal impedance matching and the perfect transmission of white light," Nature Photonics, vol. 12, no. 3, pp. 143-149, 2018.
[9] K. A. Lurie and V. V. Yakovlev, "Energy accumulation in waves propagating in space-and time-varying transmission lines," IEEE Antennas and Wireless Propagation Letters, vol. 15, pp. 1681-1684, 2016.
[10] V. Bacot, M. Labousse, A. Eddi, M. Fink, and E. Fort, "Time reversal and holography with spacetime transformations," Nature Physics, vol. 12, no. 10, pp. 972-977, 2016.
[11] A. Akbarzadeh, N. Chamanara, and C. Caloz, "Inverse prism based on temporal discontinuity and spatial dispersion," Optics letters, vol. 43, no. 14, pp. 3297-3300, 2018.
[12] C. Caloz and Z.-L. Deck-Léger, "Spacetime metamaterials—part I: general concepts," IEEE Transactions on Antennas and Propagation, vol. 68, no. 3, pp. 1569-1582, 2019.
[13] Z. Yu and S. Fan, "Complete optical isolation created by indirect interband photonic transitions," Nature photonic, vol. 3, no. 2, pp. 91-94, 2009.
[14] X. Guo, Y. Ding, Y. Duan, and X. Ni, "Nonreciprocal metasurface with space–time phase modulation," Light: Science & Applications, vol. 8, no. 1, pp. 1-9, 2019.
[15] A. E. Cardin et al., "Surface-wave-assisted nonreciprocity in spatio-temporally modulated metasurfaces," Nature communications, vol. 11, no. 1, pp. 1-9, 2020.
[16] V. Pacheco-Peña and N. Engheta, "Temporal aiming," Light: Science & Applications, vol. 9, no. 1, pp. 1-12, 2020.
[17] J. S. Martinez-Romero, O. Becerra-Fuentes, and P. Halevi, "Temporal photonic crystals with modulations of both permittivity and permeability," Physical Review A, vol. 93, no. 6, p. 063813, 2016.
[18] A. M. Shaltout, J. Fang, A. V. Kildishev, and V. M. Shalaev, "Photonic time-crystals and momentum band-gaps," in CLEO: QELS Fundamental Science, 2016: Optical Society of America, p. FM1D. 4.
[19] C. Caloz and Z.-L. Deck-Léger, "Spacetime metamaterials—part II: theory and applications," IEEE Transactions on Antennas and Propagation, vol. 68, no. 3, pp. 1583-1598, 2019.

[20] F. R. Morgenthaler, "Velocity modulation of electromagnetic waves," IRE Transactions on microwave theory and techniques, vol. 6, no. 2, pp. 167-172, 1958.