INVERSE SPECTRAL PROBLEMS FOR DIRAC OPERATORS WITH SUMMABLE POTENTIALS

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Dedicated to B. M. Levitan, one of the pioneers of this subject

Abstract. The spectral properties of Dirac operators on \((0,1)\) with potentials that belong entrywise to \(L^p(0,1)\), for some \(p \in [1,\infty)\), are studied. The algorithm of reconstruction of the potential from two spectra or from one spectrum and the corresponding norming constants is established, and a complete solution of the inverse spectral problem is provided.

1. Introduction

The main aim of the present article is to solve the direct and inverse spectral problems for one-dimensional Dirac operators on a finite interval under possibly least restrictive assumptions on their potentials. Namely, the Dirac operators under consideration are generated by the differential expressions

\[ \ell_Q := B \frac{d}{dx} + Q(x) \]

and some boundary conditions, where

\[ B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} q_1(x) & q_2(x) \\ q_2(x) & -q_1(x) \end{pmatrix}, \]

and \(q_1\) and \(q_2\) are real-valued functions from \(L^p(0,1)\), \(p \in [1,\infty)\). To simplify unessential technicalities, we shall only consider the boundary conditions that correspond to the Neumann–Dirichlet and Neumann ones in the case of Sturm–Liouville equations, although other boundary conditions can be treated in a similar manner (cf. the study of Sturm–Liouville operators with nonsmooth potentials and various boundary conditions in [26, 52]).

The corresponding Dirac operators \(\mathcal{A}_1\) and \(\mathcal{A}_2\) in the Hilbert space \(\mathbb{H} := L^2(0,1) \times L^2(0,1)\) act according to the formula \(\mathcal{A}_j u = \ell_Q u\) on the domains

\[ \text{dom} \mathcal{A}_j := \{ u = (u_1, u_2) \mid u_1, u_2 \in AC(0,1), \ell_Q u \in \mathbb{H}, u_2(0) = u_j(1) = 0 \}. \]

It is well known [36] that the operators \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are selfadjoint in \(\mathbb{H}\) and have simple discrete spectra accumulating at \(-\infty\) and \(+\infty\). Our primary goal is two-fold: firstly, to give a complete description of the spectra of \(\mathcal{A}_1\) and \(\mathcal{A}_2\) for potentials \(Q\) of the form (1.1) with \(q_1, q_2 \in L^p(0,1)\) for some \(p \in [1,\infty)\)—i.e., to solve the direct spectral problem,—and, secondly, to give an algorithm of reconstruction of these operators from their spectra or from one spectrum and the corresponding norming constants—i.e., to solve the inverse spectral problem.
Ever since P. Dirac suggested in 1929 the equation (later named after him) modelling the evolution of spin-$\frac{1}{2}$ particles in the relativistic quantum mechanics \cite{Dirac1928}, its range of applicability in various areas of physics and mathematics has been continuously expanding. In particular, in 1973 Ablowitz, Kaup, Newell, and Segur \cite{Ablowitz1973} discovered that the Dirac equation is related to a nonlinear wave equation (the “modified Korteweg–de Vries equation”, a member of the AKNS–ZS hierarchy, see \cite{Ablowitz1973, Zakharov1974}) in the same manner as the Schrödinger equation is related to the KdV equations, and this stimulated the increasing interest in direct and inverse problems for Dirac operators in both physical and mathematical literature. Earlier in 1966, Gasymov and Levitan solved the inverse problems for Dirac operators on $\mathbb{R}_+$ by using the spectral function \cite{Gasymov1966} and by the scattering phase \cite{Levitan1966}. Their investigations were continued and further developed in many directions. The reference list is so vast, that we can only mention those papers, which, in our opinion, are most pertinent to our topic and refer the reader to the bibliography cited therein for further material. The books by Levitan and Sargsjan \cite{Levitan1982} and by Thaller \cite{Thaller1992} may serve as a good introduction to the (respectively mathematical and physical part of the) theory of Dirac operators.

The inverse scattering theory was developed for Dirac operators on the axis in \cite{Ablowitz1973, Ablowitz1974, Zakharov1974, Kato1975}, for Dirac systems of order $2n$ on semiaxis in \cite{Kato1975}, and for more general canonical systems on $\mathbb{R}$ in \cite{Kato1977}. The nonselfadjoint case was treated in \cite{Krein1971} and nonstationary scattering, including point interactions, in \cite{Krein1971b} and \cite{Krein1971c}, respectively. Reconstruction from the spectral function on semiaxis was done in \cite{Kato1975b} for a general boundary condition at $x = 0$ and in \cite{Kato1977b} in the case of an interface condition in an interior point; the general first order systems in $L_2(\mathbb{R}_+, \mathbb{C}^{2n})$ were recently treated in \cite{Kordyukov2015}. Inverse problems in the periodic case were studied in \cite{Kordyukov2017, Kordyukov2018}, and the Weyl–Titchmarsh $m$-function was used to recover the potential of the Dirac operator in \cite{Kordyukov2015} and of the Dirac systems of order $2n$ in \cite{Kordyukov2017, Kordyukov2018b, Kordyukov2019} (see also the detailed reference lists therein).

The inverse problems for Dirac operators on a finite interval have also been studied in detail. Reconstruction of a continuous potential from two spectra was carried out in \cite{Gasymov1966}, from one spectrum and the norming constants (in the presence of a Coulomb-type singularity) in \cite{Albeverio1997}, and from the spectral function in \cite{Albeverio2000}. Explicit formulae for solutions (based on the degenerate Gelfand–Levitan–Marchenko equation) in the case where finitely many spectral data are perturbed were given in \cite{Albeverio2001}. Uniqueness results for other types of inverse problems were established—e.g., for mixed spectral \cite{Albeverio2002} or interior \cite{Albeverio2003} data, nonseparated boundary conditions \cite{Albeverio2004}, or for the weighted Dirac equations \cite{Albeverio2005}. Ambartsumyan-type theorems were proved in \cite{Albeverio2006} and for the matrix case in \cite{Albeverio2007}. Finally, uniqueness of the inverse problem for general Dirac-type systems of order $2n$ was recently established in \cite{Albeverio2008, Albeverio2009}.

We observe that in the above-cited papers the inverse spectral problems for Dirac operators on a finite interval were considered for continuous potentials only, which excludes, e.g., the important case of piecewise constant potentials. We remove this restriction by allowing potentials belonging entrywise to $L_p(0, 1)$, $p \in [1, \infty)$, and completely solve the inverse spectral problem for Dirac operators in this class (see Theorems \ref{thm:main1} \ref{thm:main2} \ref{thm:main3} \ref{thm:main4} and \ref{thm:main5}). The main idea of the proof rests on the fact that the transformation operators for Dirac operators under consideration satisfy not only the classical Gelfand–Levitan–Marchenko equation \eqref{eq:GLM}, but also its counterpart \eqref{eq:Krein}, which was used by Krein \cite{Krein1971} in the study of the inverse problem for impedance Sturm–Liouville equations (see also \cite{Kostyuchenko1982, Kostyuchenko1983}). This “Krein equation” survives the passage to
the limit in the $L_p$-topology and thus allows us to treat potentials belonging to $L_p(0, 1)$ entrywise.

The paper is organized as follows. In Section 2 transformation operators are constructed and some of their properties are established. Based on this, in Section 3 we find the asymptotics of eigenvalues and norming constants for the operators $\mathcal{A}_1$ and $\mathcal{A}_2$. The Gelfand–Levitan–Marchenko and Krein equations, which relate the spectral data and the transformation operators, are derived in Section 4 and the solution of the inverse spectral problem is given in Section 5. Finally, two appendices contain some facts related to harmonic analysis and the factorisation theory in operator algebras.

Throughout the paper, we shall denote by $\langle \cdot, \cdot \rangle$ the scalar product in $H$ and by $\mathcal{M}_2$ the algebra of $2 \times 2$ matrices with complex entries endowed with the operator norm $| \cdot |$ of the Euclidean space $\mathbb{C}^2$. Where no confusion arises, we abbreviate $L_p(0, 1)$ to $L_p$ and write $L_p(\mathcal{M}_2)$ for the space $L_p((0, 1), \mathcal{M}_2)$ of $\mathcal{M}_2$-valued functions on $(0, 1)$ with complex-valued entries and the norm
\[
\|V\|_{L_p} := \left( \int_0^1 |V(t)|^p \, dt \right)^{1/p}.
\]
Also, $(x, y)^t$ shall stand for the column-vector in $\mathbb{C}^2$ with components $x$ and $y$.

2. Transformation operators

Assume that $Q \in L_p(\mathcal{M}_2)$, $p \in [1, \infty)$, is of the form (1.1) and denote by $U(\cdot) = U(\cdot, \lambda)$ the Cauchy matrix corresponding to the equation $\ell_Q u = \lambda u$. In other words, $U$ is a $2 \times 2$ matrix-valued function satisfying the equation
\[
B \frac{dU}{dx} + QU = \lambda U
\]
and the initial condition $U(0) = I := \text{diag}(1, 1)$. Denoting by $c(\cdot, \lambda) := (c_1(\cdot, \lambda), c_2(\cdot, \lambda))^t$ and $s(\cdot, \lambda) := (s_1(\cdot, \lambda), s_2(\cdot, \lambda))^t$ the solutions of the equation $\ell_Q u = \lambda u$ satisfying the initial conditions $c_1(0, \lambda) = s_2(0, \lambda) = 1$ and $c_2(0, \lambda) = s_1(0, \lambda) = 0$, we find that
\[
U(x, \lambda) = \begin{pmatrix} c_1(x, \lambda) & s_1(x, \lambda) \\ c_2(x, \lambda) & s_2(x, \lambda) \end{pmatrix}.
\]
Our next aim is to derive an integral representation for $U$ of a special form.

**Theorem 2.1.** Assume that $Q \in L_p(\mathcal{M}_2)$, $p \in [1, \infty)$. Then
\[
U(x, \lambda) = e^{-\lambda x B} + \int_0^x e^{-\lambda(x-s)B} P(x, s) \, ds,
\]
where the matrix-valued function $P = P_Q$ has the following properties:

(a) for every $x \in [0, 1]$ the function $P(x, \cdot)$ belongs to $L_p(\mathcal{M}_2)$;
(b) the mapping $P_Q : x \mapsto P_Q(x, \cdot) \in L_p(\mathcal{M}_2)$ is continuous on $[0, 1]$;
(c) the function $P_Q$ depends continuously in $C([0, 1], L_p(\mathcal{M}_2))$ on $Q \in L_p(\mathcal{M}_2)$.

**Proof.** The standard variation of constant arguments show that $U$ satisfies the equivalent integral equation (recall that $B^2 = -I$)
\[
U(x, \lambda) = e^{-\lambda x B} + \int_0^x e^{-\lambda(x-t)B} BQ(t)U(t) \, dt,
\]
which can be solved by the method of successive approximations. Namely, with

\[(2.4) \quad U_0(x) := e^{-\lambda x B} \quad \text{and} \quad U_{n+1}(x) := \int_0^x e^{-\lambda(x-t)B} B Q(t) U_n(t) \, dt \quad \text{for} \quad n \geq 0,\]

the solution of (2.3) formally equals \(\sum_{n=0}^{\infty} U_n\). Assume that we have proved that

\[(2.5) \quad \sum_{n=0}^{\infty} \|U_n\|_\infty < \infty,\]

where \(\|U_n\|_\infty := \sup_{x \in [0,1]} |U_n(x)|\). Differentiating then the recurrence relations (2.4), we find that

\[U'_n(x) = -\lambda B U_n(x) + B Q(x) U_{n-1}(x),\]

which in view of (2.5) shows that the series \(\sum_{n=0}^{\infty} U_n\) converges in the topology of the space \(W^1_p((0,1), \mathcal{M}_2)\) to some \(\mathcal{M}_2\)-valued function \(V\). This function \(V\) solves (2.1) and satisfies the initial condition \(V(0) = I\), and hence it coincides with the Cauchy matrix \(U\).

To justify (2.5), we use the identity

\[(2.6) \quad e^{-\lambda x B} Q(t) = Q(t) e^{\lambda x B}, \quad x, t \in [0,1],\]

in the recurrence relations (2.4) and derive the formula

\[(2.7) \quad U_n(x) = \int_{\Pi_n(x)} e^{-\lambda(x-2\xi_n(t))B} B Q(t_1) \cdots B Q(t_n) \, dt_1 \cdots dt_n,\]

in which we have set

\[\Pi_n(x) = \{t := (t_1, \ldots, t_n) \in \mathbb{R^n} \mid 0 \leq t_n \leq \cdots \leq t_1 \leq x\}, \quad \xi_n(t) = \sum_{l=1}^{n} (-1)^{l+1} t_l.\]

Upon the change of variables \(s = \xi_n(t)\), \(y_l = t_{l+1}, \ l = 1, 2, \ldots, n-1\), we recast the integral in (2.7) as

\[U_n(x) = \int_0^x e^{-\lambda(x-2s)B} P_n(x, s) \, ds,\]

where \(P_1(x, s) \equiv B Q(s)\) and, for all \(n \in \mathbb{N}\) and \(0 \leq s \leq x \leq 1,\)

\[(2.8) \quad P_{n+1}(x, s) = \int_{\Pi_n(x,s)} B Q(s + \xi_n(y)) B Q(y_1) \cdots B Q(y_n) \, dy_1 \cdots dy_n,\]

with

\[\Pi_n^*(x, s) = \{y = (y_1, \ldots, y_n) \in \mathbb{R^n} \mid 0 \leq y_n \leq y_{n-1} \leq \cdots \leq y_1 \leq s + \xi_n(y) \leq x\}.\]

For convenience, we extend the functions \(P_n\), \(n \geq 2\), to the whole square \([0,1] \times [0,1]\) by setting \(P_n(x, s) = 0\) for \(0 \leq x < s \leq 1\).
Using the Hölder inequality and Fubini’s theorem, we find that, for every $n \in \mathbb{N}$, the function $P_{n+1}(x, \cdot)$ belongs to $L_p(\mathcal{M}_2)$ and that

$$\|P_{n+1}(x, \cdot)\|_{L_p}^p = \int_0^1 |P_{n+1}(x, s)|^p ds$$

$$\leq (n!)^{1-p} \int_0^1 \int_{\Pi^*_n(x,s)} |Q(s + \xi_{n-1}(y))|^p |Q(y_1)|^p \cdots |Q(y_n)|^p dy_1 \cdots dy_n ds$$

$$= (n!)^{1-p} \int_{\Pi_{n+1}(x)} |Q(t_1)|^p \cdots |Q(t_{n+1})|^p dt_1 \cdots dt_{n+1}$$

$$= \frac{1}{(n!)^p(n+1)!} (\int_0^x |Q|^p)^{n+1} \leq \frac{\|Q\|_{L_p}^{(n+1)p}}{(n!)^p}.$$ 

Henceforth with $C := \max_{x \in [-1,1]} |e^{-\lambda x B}|$ we have

$$|U_n(x)| \leq C \int_0^x |P_n(x, s)| ds \leq C \|P_n(x, \cdot)\|_{L_p} \leq C \frac{\|Q\|_{L_p}^n}{(n-1)!},$$

and (2.3) follows.

Moreover, the above inequality implies that the series $\sum_{n=1}^{\infty} P_n(x, \cdot)$ converges in $L_p(\mathcal{M}_2)$ to some function $P(x, \cdot)$ and yields the estimate

$$\|P(x, \cdot)\|_{L_p} \leq \sum_{n=1}^{\infty} \frac{\|Q\|_{L_p}^n}{(n-1)!} = \|Q\|_{L_p} \exp\{\|Q\|_{L_p}\}$$

for all $x \in [0, 1]$. This establishes (a).

Assume that $\tilde{Q}$ is another potential in $L_p(\mathcal{M}_2)$ and denote by $\tilde{P}_n$ the corresponding functions constructed as above but for $\tilde{Q}$ instead of $Q$; then similar calculations on account of the inequality

$$\left| \prod_{k=1}^{n} a_k - \prod_{k=1}^{n} b_k \right|^p \leq \left( \sum_{k=1}^{n} |a_k - b_k| \prod_{j \neq k} (|a_j| + |b_j|) \right)^p$$

$$\leq n^{p-1} \sum_{k=1}^{n} |a_k - b_k|^p \prod_{j \neq k} (|a_j| + |b_j|)^p$$

lead to the estimate

$$\|P_{n+1}(x, \cdot) - \tilde{P}_{n+1}(x, \cdot)\|_{L_p}^p \leq \left( \frac{n+1}{n!} \right)^p \|Q - \tilde{Q}\|_{L_p}^p \left( \|Q\|_{L_p} + \|\tilde{Q}\|_{L_p} \right)^{np}.$$ 

It follows that

$$\|P_Q(x, \cdot) - \tilde{P}_Q(x, \cdot)\|_{L_p} \leq (1 + 2r)e^{2r}\|Q - \tilde{Q}\|_{L_p}$$

as soon as $r$ is such that $\|Q\|_{L_p}, \|\tilde{Q}\|_{L_p} \leq r$.

Observe that if $\tilde{Q} \in C([0, 1], \mathcal{M}_2)$, then the functions $\tilde{P}_n$, $n \geq 2$, are continuous in the square $[0, 1] \times [0, 1]$, and, moreover,

$$\max_{0 \leq s \leq 1} |\tilde{P}_n(x, s)| \leq \frac{\|\tilde{Q}\|_{L_p}^n}{n!},$$
so that the function \([0, 1] \ni x \mapsto P_0(x, \cdot) \in L_p(\mathcal{M}_2)\) is continuous. Since the potential \(Q \in L_p(\mathcal{M}_2)\) is the limit in \(L_p(\mathcal{M}_2)\) of potentials \(Q_n \in C([0, 1], \mathcal{M}_2)\), estimate (2.11) yields both assertions (b) and (c). The proof is complete. \(\Box\)

**Corollary 2.2.** Assume that \(Q \in L_p(\mathcal{M}_2)\) and set

\[
P^+ = \sum_{n=1}^{\infty} P_{2n}, \quad P^- = \sum_{n=1}^{\infty} P_{2n-1},
\]

where the functions \(P_n\) are given by formula (2.8). Set \(c_0(x, \lambda) := (\cos \lambda x, \sin \lambda x)^t\) and

\[
R(x, t) = R_Q(x, t) := P^+(x, t) + P^-(x, t)J,
\]

\[
K(x, t) = K_Q(x, t) := \frac{1}{2} \left[ R(x, \frac{x-t}{2}) + R(x, \frac{x+t}{2})J \right],
\]

where \(J = \text{diag}(1, -1)\). Then the vector-function \(c(\cdot, \lambda)\) is given by

\[
c(x, \lambda) = c_0(x, \lambda) + \int_0^x K(x, t)c_0(t, \lambda) \, dt.
\]

**Proof.** Using (2.6) and (2.8), we conclude that

\[
e^{-\lambda(x-2s)B}P_{2n}(x, s) = P_{2n}(x, s)e^{-\lambda(x-2s)B},
\]

\[
e^{-\lambda(x-2s)B}P_{2n-1}(x, s) = P_{2n-1}(x, s)e^{\lambda(x-2s)B}.
\]

Therefore equality (2.2) can be written as

\[
U(x, \lambda) = e^{-\lambda x B} + \int_0^x P^+(x, s)e^{-\lambda(x-2s)B} \, ds + \int_0^x P^-(x, s)e^{\lambda(x-2s)B} \, ds.
\]

Observing that

\[
e^{-\lambda x B} = \begin{pmatrix} \cos \lambda x & -\sin \lambda x \\ \sin \lambda x & \cos \lambda x \end{pmatrix}
\]

and taking the first column of the above equality, we get

\[
c(x, \lambda) = c_0(x, \lambda) + \int_0^x R(x, s)c_0(x-2s, \lambda) \, ds.
\]

Since

\[
\int_0^{x/2} R(x, s)c_0(x-2s, \lambda) \, ds = \frac{1}{2} \int_0^x R(x, \frac{x-t}{2})c_0(t, \lambda) \, dt
\]

\[
\int_{x/2}^x R(x, s)c_0(x-2s, \lambda) \, ds = \int_{x/2}^x R(x, s)Jc_0(2s-x, \lambda) \, ds
\]

\[
= \frac{1}{2} \int_0^x R(x, \frac{x+t}{2})Jc_0(t, \lambda) \, dt,
\]

the required relation follows. \(\Box\)

Equality (2.13) shows that the operator \(\mathcal{I} + \mathcal{K}\) defined by

\[
(\mathcal{I} + \mathcal{K})u(x) = u(x) + \int_0^x K(x, t)u(t) \, dt
\]

transforms the solution of the equation \(\ell_0u = \lambda u\) (i.e., with a potential \(Q\) equal to zero identically) subject to the initial conditions \(u_1(0) = 1, u_2(0) = 0\) into the solution
of the equation $\ell_Q u = \lambda u$ satisfying the same initial conditions. Denote by $A_Q$ the operator in $H$ acting as $A_Q u = \ell_Q u$ on the domain
\[
\text{dom}(A_Q) := \{ u = (u_1, u_2)^t \in W^1_2(0,1) \times W^1_2(0,1) \mid u_2(0) = 0 \};
\]
then $\mathcal{I} + \mathcal{K}$ is in fact the transformation operator for $A_Q$ and $A_0$, i.e., $A_Q(\mathcal{I} + \mathcal{K}) = (\mathcal{I} + \mathcal{K})A_0$, see Theorem 2.4.

The operator $\mathcal{K}$ possesses some important properties, which we now establish. Denote by $G_p(M_2)$ the set of measurable $2 \times 2$ matrix-valued functions $K$ on $[0,1] \times [0,1]$ having the property that, for each $x$ and $t$ in $[0,1]$, the matrix-valued functions $K(x, \cdot)$ and $K(\cdot, t)$ belong to $L_p(M_2)$ and, moreover, the mappings
\[
[0,1] \ni x \mapsto K(x, \cdot) \in L_p(M_2), \quad [0,1] \ni t \mapsto K(\cdot, t) \in L_p(M_2)
\]
are continuous (i.e., they coincide a.e. with some continuous mappings from $[0,1]$ into $L_p(M_2)$). The set $G_p(M_2)$ becomes a Banach space under the norm
\[
(2.14) \quad \|K\|_{G_p} := \max_{x \in [0,1]} \|K(x, \cdot)\|_{L_p}, \max_{t \in [0,1]} \|K(\cdot, t)\|_{L_p}.
\]
We also denote by $\mathcal{G}_p(M_2)$ the set of the integral operators $\mathcal{K}$ in $H$ with kernels $K$ from $G_p(M_2)$. Under the induced norm $\|\mathcal{K}\|_{\mathcal{G}_p} := \|K\|_{G_p}$, the set $\mathcal{G}_p(M_2)$ becomes an algebra. The algebra $\mathcal{G}_p(M_2)$ is continuously embedded into the algebra $\mathcal{B}(H)$ of all bounded operators in $H$ since the functions $K$ belonging to $G_p(M_2)$ have finite Holmgren norm [22]; moreover, for $\mathcal{K} \in \mathcal{G}_p(M_2)$ the inequality $\|\mathcal{K}\|_{\mathcal{B}(H)} \leq \|\mathcal{K}\|_{\mathcal{G}_p}$ holds true.

**Theorem 2.3.** Assume that $Q \in L_p(M_2)$; then the integral operator $\mathcal{K} = \mathcal{K}_Q$ with kernel $K$ of (2.12) belongs to $\mathcal{G}_p(M_2)$ and, moreover, the mapping $L_p(M_2) \ni Q \mapsto \mathcal{K}_Q \in \mathcal{G}_p(M_2)$ is continuous.

**Proof.** In view of relations (2.12), we have
\[
(2.15) \quad K(x,t) = \frac{1}{2} \left[ P^+(x, \frac{x-t}{2}) + P^{-}(x, \frac{x-t}{2})J + P^+(x, \frac{x-t}{2})J + P^-(x, \frac{x-t}{2}) \right],
\]
and hence it suffices to prove the assertions of the theorem for the operators with kernels $P^+(x, \frac{x-t}{2})$ and $P^-(x, \frac{x-t}{2})$. Since the proof is analogous for all four functions, we shall give it for the function $P^+(x, \frac{x-t}{2}) =: \hat{P}(x,t)$ only.

It follows from the proof of Theorem 2.1 that the function $P^+$ enjoys the properties (a)–(c) of that theorem. Simple arguments based on the change of variables justify the validity of the properties (a)–(c) for the kernel $\hat{P}$. It thus remains to establish similar properties of $\hat{P}$ with respect to the variable $t$.

Assume first that $Q$ is continuous. Changing the variables $\eta = s + \xi(y)$, $\tilde{y}_1 = y_2$, $\ldots$, $\tilde{y}_{n-1} = y_n$ in integral (2.8), we arrive at the relation
\[
P_{n+1}(x,s) = \int_s^x BQ(\eta)P_n(\eta, \eta - s) \, d\eta,
\]
which yields
\[
P^+(x,s) = \int_s^x BQ(\eta)P^-(\eta, \eta - s) \, d\eta.
\]
It follows that
\[
\norm{\hat{P}(\cdot, t)}_{L^p} = \int_{t}^{1} \left| \int_{\frac{x-t}{2}}^{x} BQ(\eta)P^{-}(\eta, \eta - \frac{x-t}{2}) \, d\eta \right|^p \, dx
\]
\[
\leq \int_{0}^{1} d\eta |Q(\eta)|^p \int_{t}^{2\eta + t} |P^{-}(\eta, \eta - \frac{x-t}{2})|^p \, dx
\]
\[
\leq \norm{Q}_{L^p} \max_{\eta \in [0,1]} \norm{P^{-}(\eta, \cdot)}_{L^p} \leq \norm{Q}_{L^p} \exp\{p \|Q\|_{L^p}\}.
\]
cf. (2.9). If \(~Q\) is another continuous potential, then we find analogously that
\[
\norm{\hat{P}(\cdot, t) - \hat{P}(\cdot, t)}_{L^p} \leq 2^{p-1} \norm{Q - \tilde{Q}}_{L^p} \max_{\eta \in [0,1]} \norm{P^{-}(\eta, \cdot)}_{L^p}
\]
\[
+ 2^{p-1} \norm{\tilde{Q}}_{L^p} \max_{\eta \in [0,1]} \norm{P^{-}(\eta, \cdot) - P^{-}(\eta, \cdot)}_{L^p}.
\]
Recalling inequality (2.10), we conclude that the function \(t \mapsto \hat{P}(\cdot, t)\), which belongs to \(C([0,1], L^p(\mathcal{M}_2))\) if \(Q\) is continuous, depends therein continuously on \(Q \in C([0,1], \mathcal{M}_2)\) with respect to the topology of \(L^p(\mathcal{M}_2)\). Since the space \(C([0,1], \mathcal{M}_2)\) is dense in \(L^p(\mathcal{M}_2)\), we show by continuity that, for every \(Q \in L^p(\mathcal{M}_2)\), the function \(\hat{P}(\cdot, t)\) belongs to \(L^p(\mathcal{M}_2)\) for every fixed \(t \in [0,1]\), that the mapping \(t \mapsto \hat{P}(\cdot, t)\) is continuous, and that the continuous \(L^p(\mathcal{M}_2)\)-valued function of \(t\), \(t \mapsto \hat{P}(\cdot, t)\), depends continuously in \(C([0,1], L^p(\mathcal{M}_2))\) on \(Q \in L^p(\mathcal{M}_2)\).

Summing up, we have shown that the function \(\hat{P} = \hat{P}_Q\) belongs to \(G_p(\mathcal{M}_2)\) and depends in \(G_p(\mathcal{M}_2)\) continuously on \(Q \in L^p(\mathcal{M}_2)\). This establishes the theorem.

**Theorem 2.4.** Assume that \(Q \in L^p(\mathcal{M}_2)\) and let \(\mathcal{H}\) be an integral operator with kernel \(K\) of (2.12). Then \(\mathcal{I} + \mathcal{H}\) is the transformation operator for the pair \(\mathcal{A}_Q\) and \(\mathcal{A}_0\), i.e., \(\mathcal{A}_Q(\mathcal{I} + \mathcal{H}) = (\mathcal{I} + \mathcal{H})\mathcal{A}_0\).

**Proof.** Since \(\mathcal{H}\) belongs to \(B_p(\mathcal{M}_2)\) by Theorem 2.3 and its kernel \(K\) is lower-diagonal, it follows that \(\mathcal{H}\) is a Volterra operator in \(\mathbb{H}\) and hence \(\mathcal{I} + \mathcal{H}\) is a homeomorphism of \(\mathbb{H}\).

Write \(\hat{\mathcal{A}}_Q := (\mathcal{I} + \mathcal{H})^{-1}\mathcal{A}_Q(\mathcal{I} + \mathcal{H})\). In view of (2.13), \(c_0(\cdot, \lambda)\) is an eigenvector of the operator \(\hat{\mathcal{A}}_Q\) corresponding to the eigenvalue \(\lambda\) for every \(\lambda \in \mathbb{C}\). Denote by \(\mathcal{L}\) the linear hull of the system \(\{c_0(\cdot, \lambda) \mid \lambda \in \mathbb{C}\}\); then the restrictions of the operators \(\hat{\mathcal{A}}_Q\) and \(\mathcal{A}_0\) onto \(\mathcal{L}\) coincide. Since \(\mathcal{L}\) is a core of \(\mathcal{A}_0\) (see a similar result in [3, Theorem 3.3]) and \(\hat{\mathcal{A}}_Q\) is closed, it follows that \(\mathcal{A}_0 \subset \hat{\mathcal{A}}_Q\). It remains to observe that \(\hat{\mathcal{A}}_Q\) cannot be a proper extension of \(\mathcal{A}_0\) since otherwise \(\hat{\mathcal{A}}_Q\) and \(\hat{\mathcal{A}}_Q\) by similarity — would have a two-dimensional nullspace, which would contradict the uniqueness of solutions to the equation \(\ell_Q u = \lambda u\). Thus \(\hat{\mathcal{A}}_0 = \hat{\mathcal{A}}_Q\), and the proof is complete. 

### 3. Direct spectral problem

The aim of this section is to perform the direct spectral analysis for the Dirac operators \(\mathcal{A}_1\) and \(\mathcal{A}_2\). The main tool of our investigations will be the transformation operators constructed in the previous section.

**Theorem 3.1.** Assume that \(Q \in L^p(\mathcal{M}_2)\); then the eigenvalues \((\lambda_n)_{n \in \mathbb{Z}}\) and \((\mu_n)_{n \in \mathbb{Z}}\) of \(\mathcal{A}_1\) and \(\mathcal{A}_2\) respectively can be enumerated so that they satisfy the interlacing condition
\[
\lambda_{n-1} < \mu_n < \lambda_n, \quad n \in \mathbb{Z},
\]
and the asymptotics

\begin{align}
\lambda_n &= \pi(n + \frac{1}{2}) + e_n(g_1), \\
\mu_n &= \pi n + e_n(g_2),
\end{align}

where \( g_1, g_2 \in L_p \) and \( e_n(g) := \int_0^1 e^{-2\pi nx} g(x) \, dx \) are the Fourier coefficients of a function \( g \).

**Proof.** The fact that the spectra of \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) interlace (i.e., that between two consecutive eigenvalues of one operator, there is exactly one eigenvalue of the other operator) is well known (see, e.g., [14]). It is easily seen that if we have an enumeration of \( \lambda_n \) and \( \mu_n \) obeying (3.2) for some \( g_1, g_2 \in L_p \), then (3.1) holds for all \( n \) with sufficiently large \( |n| \). Since the two spectra interlace, we can permute a finite number of indices if necessary in such a way that (3.1) becomes valid for all \( n \in \mathbb{Z} \). This reordering amounts to adding trigonometric polynomials of finite degree to the functions \( g_1 \) and \( g_2 \), and the modified functions \( g_1 \) and \( g_2 \) will remain in \( L_p \). Therefore it suffices to establish (3.2).

The equation \( \ell_Q u = \lambda u \) subject to the initial conditions \( f_1(0) = 1, \, f_2(0) = 0 \) has the solution

\[
c(x, \lambda) = c_0(x, \lambda) + \int_0^x K(x, s) c_0(s, \lambda) \, ds,
\]

where \( K := (k_{jl})_{j,l=1}^2 \) is the kernel of the transformation operator constructed in Corollary 2.2. The numbers \( \lambda_n \) are zeros of the function \( c_1(1, \lambda) \), which, after simple transformations, takes the form

\[
c_1(1, \lambda) = \cos \lambda + \int_0^1 \left[ k_{11}(s) \cos(\lambda s) + k_{12}(s) \sin(\lambda s) \right] \, ds
\]

(3.3)

\[
= \cos \lambda + \int_{-1}^1 f_1(s) e^{i\lambda s} \, ds,
\]

where

\[
f_1(s) := \begin{cases} 
\frac{1}{2} \left[ k_{11}(1, s) - ik_{12}(1, s) \right], & s \geq 0, \\
\frac{1}{2} \left[ k_{11}(1, -s) + ik_{12}(1, -s) \right], & s < 0
\end{cases}
\]

is a function in \( L_p(-1, 1) \).

Analogously the eigenvalues of \( \mathcal{A}_2 \) are zeros of the entire function \( c_2(1, \lambda) \), which has the form

\[
c_2(1, \lambda) = \sin \lambda + \int_{-1}^1 f_2(s) e^{i\lambda s} \, ds
\]

(3.4)

for some \( f_2 \in L_p(-1, 1) \).

The required asymptotics for zeros of \( c_1(1, \lambda) \) and \( c_2(1, \lambda) \) follows now from [27]. □

**Definition 3.2.** We denote by \( \text{SD}_p \) the set of all pairs \( \{(\lambda_n)_{n \in \mathbb{Z}}, (\mu_n)_{n \in \mathbb{Z}}\} \), in which \( (\lambda_n) \) and \( (\mu_n) \) are sequences of real numbers that obey the interlacing condition (3.1) and the asymptotics (3.2).

Theorem 3.1 shows that, for any real-valued \( Q \in L_p(\mathcal{M}_2) \), the spectra of the operators \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) form an element of \( \text{SD}_p \). In the reverse direction, Theorem 3.1 claims that any element of \( \text{SD}_p \) is composed of the spectra of Dirac operators \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) for some \( Q \in L_p(\mathcal{M}_2) \). The reconstruction algorithm uses in fact the spectrum (\( \lambda_n \)) of
and the sequence of corresponding norming constants $\alpha_n := \|c(\cdot, \lambda_n)\|^{-2}$, whose properties we are going to study next.

**Theorem 3.3.** Assume that $Q \in L_p(\mathcal{M}_2)$; then the norming constants $\alpha_n = \|c(\cdot, \lambda_n)\|^{-2}$ have the asymptotics

$$
\alpha_n = 1 + \epsilon_n(g),
$$

where $g \in L_p$.

Set $\phi(\lambda) := c_1(1, \lambda)$ and $\psi(\lambda) := c_2(1, \lambda)$. We observe that the functions $\phi(\lambda)$ and $\psi(\lambda)$ can be reconstructed from their zeros as follows [27].

**Proposition 3.4.** The following equalities hold:

$$
\phi(\lambda) = \text{V.p.} \prod_{n=-\infty}^{\infty} \frac{\lambda_n - \lambda}{\pi(n + \frac{3}{2})}, \quad \psi(\lambda) = (\lambda - \mu_0)\text{V.p.} \prod_{n=-\infty}^{\infty} \frac{\mu_n - \lambda}{\pi n}
$$

(where the prime means that the factor corresponding to the index $n = 0$ should be omitted); moreover, the products converge uniformly on compact sets.

It turns out that $\phi$ and $\psi$ (i.e., that the two spectra $(\lambda_n)_{n \in \mathbb{Z}}$ and $(\mu_n)_{n \in \mathbb{Z}}$) determine the norming constants $\alpha_n$ as follows.

**Lemma 3.5.** The norming constants $\alpha_n$ satisfy the following relation:

$$
\alpha_n = \frac{1}{\phi(\lambda_n)\psi(\lambda_n)}.
$$

**Proof.** Since the system of eigenvectors $(c_n)_{n \in \mathbb{Z}}$, $c_n := c(\cdot, \lambda_n)$, of the operator $\mathcal{A}_1$ is an orthogonal basis of $\mathbb{H}$, we have

$$(\mathcal{A}_1 - \lambda)^{-1}u = \sum_{n=-\infty}^{\infty} \frac{\alpha_n(c_n, c_n)c_n}{\lambda_n - \lambda}.$$

In particular, the residue of this expression at $\lambda = \lambda_n$ equals $-\alpha_n(c_n, c_n)c_n$.

On the other hand, $(\mathcal{A}_1 - \lambda)^{-1}$ can be calculated as

$$
(\mathcal{A}_1 - \lambda)^{-1}u(x) = \left[\int_{0}^{1} \langle u(t), s^+(t, \lambda) \rangle c_z dt \right] \frac{c(x, \lambda)}{W(\lambda)}
+ \left[\int_{0}^{x} \langle u(t), c(t, \lambda) \rangle c_z dt \right] \frac{s^+(x, \lambda)}{W(\lambda)}.
$$

where $s^+(\cdot, \lambda)$ is a solution of the equation $\ell_Q(u) = \lambda^2 u$ subject to the terminal conditions $u_1(1) = 0$ and $u_2(1) = 1$, and $W(\lambda)$ is the Wronskian of the solutions $c(\cdot, \lambda)$ and $s^+(\cdot, \lambda)$, i.e.,

$$
W(\lambda) := c_1(x, \lambda)s_2^+(x, \lambda) - c_2(x, \lambda)s_1^+(x, \lambda).
$$

Since the right-hand side of the above equality does not depend on $x$, we see that

$$
W(\lambda) = c_1(1, \lambda) = \phi(\lambda).
$$

Thus the residue of $(\mathcal{A}_1 - \lambda)^{-1}u$ at $\lambda = \lambda_n$ is equal to

$$
\left[\int_{0}^{x} \langle u(t), s^+(t, \lambda_n) \rangle c_z dt \right] \frac{c(x, \lambda_n)}{\phi(\lambda_n)} + \left[\int_{0}^{x} \langle u(t), c(t, \lambda_n) \rangle c_z dt \right] \frac{s^+(x, \lambda_n)}{\phi(\lambda_n)}.
$$
We observe now that the vector-functions \( s^+(x, \lambda_n) \) and \( c(x, \lambda_n) \) are collinear, namely,
\[
s^+(\cdot, \lambda_n) = \frac{s_2^+(1, \lambda_n)}{c_2(1, \lambda_n)} c(\cdot, \lambda_n) = \frac{1}{\psi(\lambda_n)} c(\cdot, \lambda_n),
\]
and thus expression (3.8) simplifies to \( \langle u, c_n \rangle c_n / (\phi(\lambda_n) \psi(\lambda_n)) \). Equating this with the above expression for the residue of \( (\mathcal{A}_1 - \lambda)^{-1} u \) at \( \lambda = \lambda_n \), we obtain relation (3.7) for \( a_n \). The proof is complete.

**Proof of Theorem 3.3.** In view of equality (3.7) and Propositions A.1 and A.2 it suffices to prove that the numbers \( a_n := -\dot{\phi}(\lambda_n) \) and \( b_n := \psi(\lambda_n) \) can be represented as \((-1)^n (1 + \bar{a}_n)\) and \((-1)^n (1 + \bar{b}_n)\) respectively, where \( \bar{a}_n \) and \( \bar{b}_n \) are \( n \)-th Fourier coefficients of some functions from \( L_p \).

Using formulae (3.3) and (3.4), we show that the numbers \( a_n \) and \( b_n \) are of the form
\[
\sin \lambda_n + \int_{-1}^1 f(s) e^{i\lambda_n s} \, ds
\]
with \( f(s) = -is f_1(s) \in L_p(-1, 1) \) for \( a_n \) and \( f(s) = f_2(s) \in L_p(-1, 1) \) for \( b_n \). Hence it remains to show that \((-1)^n \sin \lambda_n = 1 + e_n(\tilde{g}_1)\) for some \( \tilde{g}_1 \in L_p \) and that
\[
(-1)^n \int_{-1}^1 f(s) e^{i\lambda_n s} \, ds = e_n(\tilde{g}_2)
\]
for some \( \tilde{g}_2 \in L_p \).

Since by Theorem 3.1 the eigenvalues \( \lambda_n \) satisfy the relation \( \lambda_n = \pi(n + \frac{1}{2}) + \tilde{\lambda}_n \), where \( \tilde{\lambda}_n = e_n(\tilde{g}_1) \) for some function \( g_1 \in L_p \), we find that \( \sin \lambda_n = (-1)^n \cos \tilde{\lambda}_n \), so that
\[
(-1)^n \sin \lambda_n - 1 = \sum_{k=1}^{\infty} (-1)^k \frac{\tilde{\lambda}_n^{2k}}{(2k)!}.
\]
Applying Proposition A.1 to the element \( x := (\tilde{\lambda}_n)_{n \in \mathbb{Z}} \in X_p \), we see that there exists a function \( \tilde{g}_1 \) in \( L_p \) such that the sum on the right-hand side of the above equality equals \( e_n(\tilde{g}_1) \).

Changing the variables \( s \mapsto 1 - 2t \) in the integral, we get
\[
\int_{-1}^1 f(s) e^{i\lambda_n s} \, ds = \int_0^1 2f(1 - 2t) e^{i\lambda_n (1 - 2t)} \, dt = (-1)^n \int_0^1 \tilde{f}(t) e^{i\tilde{\lambda}_n (1 - 2t)} e^{-2\pi n i t} \, dt
\]
with \( \tilde{f}(t) := 2if(1 - 2t)e^{-\pi i t} \). Developing the function \( e^{i\tilde{\lambda}_n (1 - 2t)} \) into the Taylor series and then changing summation and integration order (which is allowed in view of the absolute convergence of the Taylor series and the integral), we find that
\[
(-1)^n \int_{-1}^1 f(s) e^{i\lambda_n s} \, ds = \sum_{k=0}^{\infty} \frac{\tilde{\lambda}_n^k}{k!} e_n(V^k \tilde{f}),
\]
where \( V \) is the operator of multiplication by \( i(1 - 2t) \). In virtue of Proposition A.1 and the fact that \( V \) has norm 1 in \( L_p \), the right-hand side of the above equality gives the \( n \)-th Fourier coefficient of some function \( \tilde{g}_2 \) from \( L_p \), and the proof is complete. \( \square \)
Write \( c_n(\cdot) = c(\cdot, \lambda_n) \) and \( \alpha_n := \| c_n \|_{\mathbb{H}}^{-2} \). Since the functions \( \{ c_n \}_{n \in \mathbb{Z}} \) form an orthogonal basis of \( \mathbb{H} \), we have

\[
\begin{align*}
\text{s-lim}_{k \to \infty} \sum_{n = -k}^{k} \alpha_n \langle \cdot, c_n \rangle c_n &= \mathcal{I},
\end{align*}
\]

where \( \mathcal{I} \) is the identity operator in \( \mathbb{H} \). On the other hand, \( c_n = (\mathcal{I} + \mathcal{K}) v_n \) with \( v_n(x) := c_0(x, \lambda_n) = (\cos \lambda_n x, \sin \lambda_n x)^{\top} \), so that the previous relation can be rewritten as

\[
(\mathcal{I} + \mathcal{K}) \left[ \text{s-lim}_{k \to \infty} \sum_{n = -k}^{k} \alpha_n \langle \cdot, v_n \rangle v_n \right] (\mathcal{I} + \mathcal{K}^*) = \mathcal{I},
\]

which implies

\[
\text{s-lim}_{k \to \infty} \sum_{n = -k}^{k} \alpha_n \langle \cdot, v_n \rangle v_n = (\mathcal{I} + \mathcal{K})^{-1} (\mathcal{I} + \mathcal{K}^*)^{-1}.
\]

Set

\[
(4.1) \quad H(s) := \text{V.p.} \sum_{n = -\infty}^{\infty} \left( \alpha_n e^{-2\lambda_n sB} - e^{-\pi(2n+1)sB} \right),
\]

where for \( p = 1 \) the summation is understood in the Cesàro sense, see Lemma 5.2. Observing that \( \langle \cdot, c_0(\cdot, \lambda) \rangle c_0(\cdot, \lambda) \) is an integral operator with kernel

\[
\frac{1}{2} \left[ e^{-\lambda(x-t)B} + e^{-\lambda(x+t)B} J \right]
\]

and that

\[
\text{s-lim}_{k \to \infty} \sum_{n = -k}^{k} \langle \cdot, v_{0,n} \rangle v_{0,n} = \mathcal{I}
\]

with \( v_{0,n} := c_0(\cdot, \pi(n + \frac{1}{2})) \), we conclude that

\[
(4.2) \quad (\mathcal{I} + \mathcal{K})^{-1} (\mathcal{I} + \mathcal{K}^*)^{-1} = \mathcal{I} + \mathcal{F},
\]

where \( \mathcal{F} \) is an integral operator in \( \mathbb{H} \),

\[
(\mathcal{F} u)(x) = \int_{0}^{1} F(x,s) u(s) \, ds,
\]

with kernel

\[
(4.3) \quad F(x,s) := \frac{1}{2} \left[ H(\frac{x-t}{2}) + H(\frac{x+t}{2}) J \right].
\]

Since \( \mathcal{K}^\ast \) is an integral Volterra operator with upper-diagonal kernel, the operator \( \mathcal{I} + \mathcal{K}^\ast \) is invertible and its inverse can be written in the form \( \mathcal{I} + \mathcal{K}^\ast \), where \( \mathcal{K}^\ast \) is an integral operator in \( \mathbb{H} \) with upper-diagonal kernel. By (4.2) one gets

\[
(\mathcal{I} + \mathcal{K}^\ast)(\mathcal{I} + \mathcal{K}) = (\mathcal{I} + \mathcal{K}^\ast)^{-1} \mathcal{I} + \mathcal{K}^\ast;
\]

spelling out this equality in terms of the kernels \( K \) and \( F \) for \( x > t \), we arrive at the Gelfand–Levitan–Marchenko (GLM) equation

\[
(4.4) \quad K(x,t) + F(x,t) + \int_{0}^{x} K(x,s) F(s,t) \, ds = 0.
\]
If $Q$ is continuous, then such is also $K$, and one has the formula (36) Lemma 12.1.1
(4.5) \[ Q(x) = K(x, x)B - BK(x, x) \]
relating the potential and the kernel of the corresponding transformation operator. This suggests the following algorithm of solution of the inverse spectral problem: given the spectral data \( \{(\lambda_n, (\alpha_n)\) \}, one constructs first the kernel $F$ via (4.3) and (4.1), then solves the GLM equation (4.4) for $K$, and, finally, recovers the potential via (4.5). However, if $Q \in L_p(M_2)$, then relation (4.5) becomes meaningless since, by Theorem 2.3, in this case the kernel $K$ belongs to $G_p(M_2)$ but can be neither continuous nor well defined on subsets of $[0, 1] \times [0, 1]$ of Lebesgue measure zero.

It turns out that for $Q \in L_p(M_2)$ the restriction $R(x, x)$ of the kernel $R$ of (2.12) to the diagonal does determine a matrix-function with entries in $L_p(0, 1)$. If $Q$ is continuous, then (4.5) together with (2.12) and the commutator relations
(4.6) \[ BR(x, t) = R(x, t)B, \quad BJ = -JB \]
yields the equality
(4.7) \[ Q(x) = R(x, x)JB. \]
Equation (4.7) retains sense also for $Q \in L_p(M_2)$ as an equality in $L_p(M_2)$ and thus can be used to recover $Q$ in this situation.

Our next task is to explain how the kernel $R$ can be determined from the spectral data. We do this by deriving below the Krein equation (4.9), an analogue of the GLM equation for $R$ [10, 32].

Applying the operator $B$ to equality (4.4) from both sides, we obtain its counterpart,
(4.8) \[ BK(x, t)B + BF(x, t)B + \int_0^x BK(x, s)F(s, t)B ds = 0. \]
With regard to the commutator relations (1.0) and $BH(x) = H(x)B$, we see that
\[
BK(x, t)B = \frac{1}{2}[-R(x, \frac{x-t}{2}) + R(x, \frac{x+t}{2})]J,
\]
\[
BF(x, t)B = \frac{1}{2}[-H(\frac{x-t}{2}) + H(\frac{x+t}{2})]
\]
and hence
\[
K(x, t) + BK(x, t)B = R(x, \frac{x+t}{2})J,
\]
\[
F(x, t) + BF(x, t)B = H(\frac{x+t}{2})J.
\]
It also follows that
\[
K(x, s)F(s, t) + BK(x, s)F(s, t)B = \frac{1}{2}[R(x, \frac{x-s}{2})H(\frac{x+t}{2})J + R(x, \frac{x+s}{2})JH(\frac{x-t}{2})].
\]
Adding now (4.4) and (4.8), combining the above formulae in the resulting expression, and using the relation $JH(x) = H(-x)J$, we arrive at the equation
\[
R(x, \frac{x+t}{2}) + H(\frac{x+t}{2}) + \int_0^x R(x, s)H(\frac{x+t}{2} - s) ds = 0, \quad 0 \leq t \leq x \leq 1.
\]
Subtracting (4.8) from (4.4) and performing similar transformations, we arrive at the above formula with $\frac{x+t}{2}$ replaced by $\frac{x-t}{2}$, and both can now be combined together to give
\[
R(x, t) + H(t) + \int_0^x R(x, s)H(t - s) ds = 0, \quad 0 \leq t \leq x \leq 1.
\]
We see that the function $\tilde{R}(x,t) := R(x,x-t)$ satisfies the following Krein equation:

$$(4.9) \quad \tilde{R}(x,t) + H(x-t) + \int_0^x \tilde{R}(x,s)H(s-t)\,ds = 0.$$ 

We observe that as soon as a kernel $H$ is given by (4.1) with $\lambda_n$ and $\alpha_n$ obeying the proper asymptotics (guaranteeing that the series for $H$ converges in $L_p(\mathcal{M}_2)$), the integral operator $\mathcal{H}$ with kernel $H(x-t)$ belongs to the algebra $\mathcal{G}_p(\mathcal{M}_2)$ introduced in Section 2 and $\mathcal{I} + \mathcal{H}$ is positive in $\mathcal{H}$ (see Lemma 5.4). The Krein equation (4.9) is then uniquely soluble and its solution belongs to $G_p(\mathcal{M}_2)$, see Appendix C. In particular, $\tilde{R}(x,0) = R(x,x)$ is in $L_p(0,1)$ entrywise indeed.

**Remark 4.1.** We notice that the GLM equation (4.4) is the even part of the Krein equation (4.9) in the sense that if $\tilde{R}$ is a solution to (4.9), then the function

$$(4.10) \quad K(x,t) := \frac{1}{2}\left[\tilde{R}(x, \frac{x+t}{2}) + \tilde{R}(x, \frac{x-t}{2})\right]$$

solves (4.4). Moreover, the condition $\mathcal{I} + \mathcal{H} > 0$ implies that the operator $\mathcal{I} + \mathcal{F}$ is positive in $\mathcal{H}$ and thus, in view of the results of Appendix C, guarantees that the GLM equation (4.4) with $F$ of (1.3) is uniquely soluble for $K$ and the solution belongs to $G_p(\mathcal{M}_2)$.

5. **Inverse spectral problem**

The purpose of this section is, firstly, to show by limiting arguments that formula (4.7) remains valid if the matrix potential $Q$ belongs to $L_p(\mathcal{M}_2)$ and, secondly, to justify the algorithm reconstructing the potential $Q$ from the spectral data. Namely, we shall prove the following theorem, which constitutes the main result of the paper.

**Theorem 5.1.** Assume that $\{(\lambda_n)_{n \in \mathbb{Z}}, (\mu_n)_{n \in \mathbb{Z}}\}$ is an arbitrary element of $\text{SD}_p$, $p \in [1, \infty)$. Then there exists a unique potential $Q \in L_p(\mathcal{M}_2)$ such that $(\lambda_n)$ and $(\mu_n)$ are eigenvalues of the corresponding Dirac operators $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. The potential $Q$ is equal to $R(x,x)JB = \tilde{R}(x,0)JB$, where $\tilde{R}$ is the solution of the Krein equation (4.9) with $H$ of (4.1) and $\alpha_n$ given by (5.7).

The reconstruction algorithm proceeds as follows. Given an arbitrary element of $\text{SD}_p$, we construct functions $\phi$ and $\psi$ via relations (3.6) and then determine the constants $\alpha_n$ by (5.7). Since the $\lambda_n$’s and $\mu_n$’s interlace, it is easily seen that all $\alpha_n$’s are positive. By virtue of the results of [27] there exist functions $f_1$ and $f_2$ in $L_p(-1,1)$ such that

$$\phi(\lambda) = \cos \lambda + \int_{-1}^1 f_1(s)e^{i\lambda s}\,ds,$$

$$\psi(\lambda) = \sin \lambda + \int_{-1}^1 f_2(s)e^{i\lambda s}\,ds.$$ 

Therefore the proof of Theorem 5.3 remains valid, and thus the numbers $\alpha_n$ satisfy the asymptotics $\alpha_n = 1 + e_n(g)$ for some $g \in L_p(0,1)$. We shall now prove that the series (4.1) converges in $L_p(\mathcal{M}_2)$, so that the function $H$ is well defined and belongs to $L_p(\mathcal{M}_2)$.

**Lemma 5.2.** Assume that the numbers $\lambda_n$ and $\alpha_n$, $n \in \mathbb{Z}$, satisfy the asymptotics of Theorems 3.1 and 3.3 for some $p \in [1, \infty)$. Then the series (4.1) converges in the space $L_p((-1,1), \mathcal{M}_2)$ (in the Cesàro sense if $p = 1$).
Proof. Since the matrix $B$ is skew-adjoint and has eigenvalues $\pm i$, it suffices to prove that the scalar series

$$h(s) := \text{V.p.} \sum_{n=-\infty}^{\infty} (\alpha_n e^{2\lambda_n is} - e^{(2n+1)\pi is}) \tag{5.2}$$

converges in $L_P(-1, 1)$ (in the Cesàro sense if $p = 1$). We shall justify the convergence on $(0, 1)$; that on $(-1, 0)$ will then follow if we replace $\alpha_n$ with $\alpha_{-1-n}$ and $\lambda_n$ with $-\lambda_{-1-n}$.

By assumption, $\alpha_n = 1 + e_n(g)$ for some $g \in L^p$, and classical theorems of harmonic analysis [28, Sec. I.2] show that the series

$$\text{V.p.} \sum_{n=-\infty}^{\infty} (\alpha_n - 1)e^{2n\pi is}$$

converges in $L^p$ to $g$ in the required sense. It remains to establish that the series

$$\text{V.p.} \sum_{n=-\infty}^{\infty} \alpha_n (e^{2\lambda_n is} - e^{(2n+1)\pi is}) \tag{5.3}$$

is convergent in $L^p$. We shall treat only the case $p = 1$; the arguments remain the same for $p > 1$ but with partial Cesàro sums replaced by the ordinary partial sums.

By the definition of summability in the sense of Cesàro we have to show that the sequence of partial sums

$$S_N(x) := \sum_{k=1}^{N} \sum_{n=-k}^{k} \alpha_n (e^{2\lambda_n ix} - e^{(2n+1)\pi ix})$$

is a Cauchy sequence in $L^1$. Recalling that $\lambda_n = \pi(n + \frac{1}{2}) + \tilde{\lambda}_n$ with $\tilde{\lambda}_n = e_n(f)$ for some $f \in L^1$ and developing $e^{2\lambda_n ix}$ into Taylor series around $(2n + 1)\pi ix$, we see that

$$S_N(x) = \sum_{k=1}^{N} \sum_{n=-k}^{k} \alpha_n e^{(2n+1)\pi ix} \sum_{m=1}^{\infty} \frac{(2\tilde{\lambda}_n ix)^m}{m!}$$

$$= e^{\pi ix} \sum_{m=1}^{\infty} \frac{(2ix)^m}{m!} \sum_{k=1}^{N} \sum_{n=-k}^{k} \alpha_n \tilde{\lambda}_n^m e^{2n\pi ix}$$

$$= e^{\pi ix} \sum_{m=1}^{\infty} \frac{(2ix)^m}{m!} \sigma_N(x, f_m),$$

where

$$\sigma_N(x, f_m) := \sum_{k=1}^{N} \sum_{n=-k}^{k} e_n(f_m)e^{2n\pi ix}$$

is the partial Cesàro sum for the function

$$f_m := \underbrace{f * \cdots * f}_{m} + \underbrace{f * \cdots * f * g}_{m}, \quad m \geq 1$$

(see Appendix A). As is well known (cf. [28, Ch. II]), for any $f \in L^1$, the partial Cesàro sums $\sigma_N(\cdot, f)$ converge to $f$ in $L^1$ and $\|\sigma_N(\cdot, f)\|_{L^1} \leq \|f\|_{L^1}$. Since

$$\|f_m\|_{L^1} \leq (1 + \|g\|_{L^1}) \|f\|_{L^1},$$
the Lebesgue dominated convergence theorem shows that the limit
\[
\lim_{N \to \infty} S_N(x) = e^{-\pi i x} \sum_{m=1}^{\infty} \frac{(2ix)^m}{m!} f_m(x)
\]
exists in \(L_1\), whence the series (5.3) converges in \(L^1\) in the Cesàro sense. The lemma
is proved. \(\square\)

Next we show that the GLM equation (4.4) and the Krein equation (4.9) have unique
solutions that belong to the space \(G_p(M_2)\). To this end it suffices to show (see Appen-
dix B and Remark 4.1 for details) that the operator \(I + H\) is positive in \(H\), where \(H\) is a Wiener–Hopf operator given by
\[
H u(x) := \int_0^1 H(x-s)u(s) \, ds.
\]
As a preliminary, we show that certain systems of functions form Riesz bases in \(L^2\) or \(H\).

Lemma 5.3. Assume that the sequence \((\lambda_n)\) is as in Theorem 5.1. Then
(a) the system \((e^{i\lambda_n x})_{n \in \mathbb{Z}}\) forms a Riesz basis of \(L_2(-1,1)\);
(b) the system \((v_n)_{n \in \mathbb{Z}}\), with \(v_n := c_0(\cdot, \lambda_n)\), forms a Riesz basis of \(H\).

Proof. In view of relation (3.3) the numbers \(\lambda_k\) are zeros of the exponential function of
type with indicator diagram \([-1,1]\) [34], so that item (a) follows from [6, Proposition II.4.3].

Assume now that \(g := (g_1, g_2)^t\) is an arbitrary element of \(H\). We extend \(g_1\) and
\(g_2\) to functions \(\tilde{g}_1\) and \(\tilde{g}_2\) on \((-1,1)\) in the even and odd way, respectively, and set
\(\tilde{g} := \tilde{g}_1 + i\tilde{g}_2\). By (a), there exists a unique sequence \((a_n)_{n \in \mathbb{Z}}\) in \(\ell_2\), for which
\[
\tilde{g} = \sum_{n=-\infty}^{\infty} a_n e^{i\lambda_n x},
\]
the series being convergent in \(L_2(-1,1)\). Taking the even and odd parts of the above
equality, we arrive at the relations
\[
\tilde{g}_1(x) = \sum_{n=-\infty}^{\infty} a_n \cos(\lambda_n x), \quad i\tilde{g}_2(x) = i \sum_{n=-\infty}^{\infty} a_n \sin(\lambda_n x),
\]
i.e., at the representation
\[
g = \sum_{n=-\infty}^{\infty} a_n v_n
\]
in \(H\). We observe that there is a constant \(C\) independent of \(\tilde{g}\) such that
\[
C^{-1} \sum_{n=-\infty}^{\infty} |a_n|^2 \leq \|\tilde{g}\|_2^2 = 2\|g\|^2_H \leq C \sum_{n=-\infty}^{\infty} |a_n|^2.
\]
Hence the system \((v_n)_{n \in \mathbb{Z}}\) is a Riesz basis of \(H\). \(\square\)

Lemma 5.4. Assume that \(\{(\lambda_n), (\mu_n)\}\) is an arbitrary element of \(SD_p\) and that the
function \(H\) is constructed as explained above. Then the corresponding operator \(I + H\)
is positive in \(H\).
Proof. We notice that the matrix $B$ is skew-adjoint and has eigenvalues $\pm i$. It is clear that the subspaces $\ker(B \pm i) \otimes L_2(0, 1)$ are invariant subspaces of $\mathcal{H}$. Therefore $\mathcal{H}$ is unitarily equivalent to the direct sum $\mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$(\mathcal{H}_\pm g)(s) = \int_0^1 h(\pm(x - s))g(s) \, ds, \quad g \in L_2,$$

and $h$ is the function of (5.2). Thus positivity of $\mathcal{I} + \mathcal{H}$ is equivalent to that of both $I + \mathcal{H}_+$ and $I + \mathcal{H}_-$. We shall prove only that the operator $I + \mathcal{H}_+$ is positive in $L_2$; the positivity of the other one is established analogously.

Since the systems $(e^{i(2n+1)\pi is})_{n \in \mathbb{Z}}$ and $(e^{i2\lambda_n is})_{n \in \mathbb{Z}}$ constitute respectively an orthonormal and a Riesz basis of $L_2$ (cf. Lemma 5.3 and [6, Sect. II.4.2]), it is easy to see that

$$(I + \mathcal{H}_+)f, f = (f, f) + \lim_{k \to \infty} \sum_{n = -k}^k |\alpha_n|(f, e^{i2\lambda_n is})^2 - |(f, e^{i(2n+1)\pi is})|^2$$

$$= \sum_{n = -\infty}^{\infty} \alpha_n |(f, e^{i2\lambda_n is})|^2.$$

Since the numbers $\alpha_n$ are uniformly bounded away from zero, we conclude that there is a $C > 0$ such that $(I + \mathcal{H}_+)f, f \geq C\|f\|^2$ for all $f \in L_2$, i.e., the operator $I + \mathcal{H}_+$ is (uniformly) positive in $L_2$.

According to Theorem 3.2 positivity of the operator $\mathcal{I} + \mathcal{H}$ implies that the GLM equation (1.4) with $F$ given by (1.3) and the Krein equation (1.9) have unique solutions $K, \tilde{R} \in G_p(\mathcal{M}_2)$. We have shown in Section 4 that in the smooth case the solution $K$ is the kernel of the transformation operator $\mathcal{I} + \mathcal{H}$ for the pair $(\mathcal{A}_Q, \mathcal{A}_0)$ with $Q := \tilde{R}(\cdot, 0)JB$. Based on this result, we treat here the general case by a limiting procedure.

**Theorem 5.5.** Assume that $\{(\lambda_n), (\mu_n)\}$ is an arbitrary element of $\text{SD}_p$ and that $H \in L_p(\mathcal{M}_2)$ is a function of (1.1) constructed as explained above. Let also $K$ and $\tilde{R}$ be the solutions of the GLM equation (1.4) and the Krein equation (1.9) respectively. Denote by $\mathcal{K}$ the integral operator with kernel $K$. Then there exist a unique $Q \in L_p(\mathcal{M}_2)$—namely, $Q = \tilde{R}(\cdot, 0)JB$—such that the operator $\mathcal{I} + \mathcal{K}$ is the transformation operator for the pair $\mathcal{A}_Q$ and $\mathcal{A}_0$.

Proof. We shall approximate the function $H$ in the norm of $L_p(\mathcal{M}_2)$ by a sequence $(H_l)_{l=1}^\infty$ of real-valued, smooth (say, infinitely differentiable) $\mathcal{M}_2$-valued functions so that the following holds:

(a) for every $l \in \mathbb{N}$, the GLM equation (1.4) with $H$ replaced by $H_l$ has a unique solution $K_l$, and the corresponding integral operators $\mathcal{K}_l$ converge to $\mathcal{K}$ as $l \to \infty$ in the uniform operator topology of $\mathcal{H}$;

(b) for every $l \in \mathbb{N}$, there exists $Q_l \in L_p(\mathcal{M}_2)$ of the form (1.1) such that $\mathcal{I} + \mathcal{K}_l$ is a transformation operator for the pair $\mathcal{A}_Q$ and $\mathcal{A}_0$;

(c) the matrix-functions $Q_l$ converge to $Q := \tilde{R}(\cdot, 0)JB$ in $L_p(\mathcal{M}_2)$.

If (a)–(c) hold, then by Theorems 2.3 and 2.3 the operators $\mathcal{I} + \mathcal{K}_l$ converge in $G_p(\mathcal{M}_2)$ (and hence in the uniform operator topology of $\mathcal{H}$) to an operator $\mathcal{I} + \mathcal{K}_Q$, which is the transformation operator for the pair $(\mathcal{A}_Q, \mathcal{A}_0)$. Thus $\mathcal{K} = \mathcal{K}_Q$ yielding the result. The uniqueness of $Q$ is obvious.
The details are as follows. Using $(\lambda_n)$ and $(\mu_n)$, we construct the sequence of constants $\alpha_n$ and set
\[
H_l(s) := \sum_{n=-l}^{l} \left( \alpha_n e^{-2\lambda_n sB} - e^{-\pi(2n+1)sB} \right)
\]
(for $p = 1$, we replace $H_l$ by the corresponding partial Cesàro sum $\frac{1}{l} \sum_{k=1}^{l} H_k$); then $H_l \rightarrow H$ in $L_p(\mathcal{M}_2)$ as $l \rightarrow \infty$ by Lemma 5.2.

We observe that this choice of $H_l$ corresponds to setting $\alpha_n = 1$ and $\lambda_n = \pi(n + \frac{1}{2})$ for all $n$ with $|n| > l$, so that in view of Lemma 5.4 the Wiener–Hopf operators $\mathcal{H}_l$ with symbol $H_l$ satisfy the condition $\mathcal{I} + \mathcal{H}_l > 0$. Hence by Corollary B.3 and Remark 4.1 the GLM equation (4.4) with $H$ replaced by $H_l$ has a unique solution $K_l$. This solution $K_l$ belongs to $G_p(\mathcal{M}_2)$, and hence the corresponding integral operator $\mathcal{K}_l$ is bounded. Since the relation $H_l \rightarrow H$ in $L_p(\mathcal{M}_2)$ as $l \rightarrow \infty$ implies that $\mathcal{H}_l \rightarrow \mathcal{H}$ in $G_p(\mathcal{M}_2)$, we conclude that $\mathcal{H}_l \rightarrow \mathcal{H}$ in the uniform operator topology of $\mathbb{H}$, see Appendix B. This establishes (a).

Moreover, by the well-known result for continuous potentials [36, Ch. 12.3–4] the operator $\mathcal{I} + \mathcal{H}_l$ is the transformation operator for the pair $(\mathcal{A}_Q, \mathcal{A}_0)$ with $Q_l(x) := K_l(x,x)B - BK_l(x,x)$. As was explained in Section 4 this yields the relation $Q_l(\cdot, \cdot) = R_l(\cdot, 0)JB$, where $R_l$ is the solution to the Krein equation (4.9) for $H = H_l$. Thus (b) is fulfilled.

To prove (c), we observe that, according to the results of Appendix B the solution $R_l$ to the Krein equation (4.9) depends continuously in the norm of the space $G_p(\mathcal{M}_2)$ on the function $H \in L_p(\mathcal{M}_2)$. Therefore the sequence $R_l(\cdot, 0)$ converges in $L_p(\mathcal{M}_2)$ to the function $R(\cdot, 0)$. The proof is complete.\n
The last step of the reconstruction procedure is to show that the numbers $\lambda_n$ and $\mu_n$ we have started with are the very eigenvalues of the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ with the potential $Q$ just found.

Since the solution $K$ to the GLM equation (4.4) generates a transformation operator $\mathcal{I} + \mathcal{K}$ for the pair $(\mathcal{A}_Q, \mathcal{A}_0)$, the functions $c(\cdot, \lambda) := (\mathcal{I} + \mathcal{K})c_0(\cdot, \lambda)$ belong to dom $\mathcal{A}_Q$ and satisfy the relation $\mathcal{A}_Qc(\cdot, \lambda) = \lambda c(\cdot, \lambda)$. We set $c_k := c(\cdot, \lambda_k)$ and show that these functions are orthogonal and that the $\alpha_k$ are the corresponding norming constants.

**Lemma 5.6.** The system of functions $\{c_k\}_{k \in \mathbb{Z}}$ is an orthogonal basis of $\mathbb{H}$. Moreover, for the above numbers $\alpha_k$ (defined at the beginning of this section), we have\n\[
\langle c_k, c_l \rangle = \alpha_k^{-1} \delta_{kl},
\]
where $\delta_{kl}$ is the Kronecker delta.

**Proof.** Denoting by $v_n$ the function $c_0(\cdot, \lambda_n)$ and recalling that the integral operator $\mathcal{F}$ with kernel $F$ of (4.3) is related to $\mathcal{K}$ by $(\mathcal{I} + \mathcal{K})(\mathcal{I} + \mathcal{F})(\mathcal{I} + \mathcal{K})^* = \mathcal{I}$ (see Appendix B for details), we conclude that\n\[
\langle c_k, c_l \rangle = \langle (\mathcal{I} + \mathcal{K})^*(\mathcal{I} + \mathcal{K})v_k, v_l \rangle = \langle (\mathcal{I} + \mathcal{K})^{-1}v_k, v_l \rangle.
\]
Reverting the arguments of Section 4 we see that the operator $\mathcal{I} + \mathcal{F}$ is equal to\n\[
s\text{-lim}_{k \to \infty} \sum_{n=-k}^{k} \alpha_n \langle \cdot, v_n \rangle v_n.
\]
Since the sequence \((v_n)_{n \in \mathbb{Z}}\) is a Riesz basis of \(\mathbb{H}\) in view of Lemma 5.3, the inverse of \(\mathcal{I} + \mathcal{F}\) can be represented as

\[
(\mathcal{I} + \mathcal{F})^{-1} = \operatorname{s-lim}_{n \to \infty} \sum_{m=-n}^{n} \alpha_{m}^{-1} \langle \cdot, \tilde{v}_{m} \rangle \tilde{v}_{m},
\]

where \((\tilde{v}_{m})\) is a basis biorthogonal to \((v_{m})\) (see [19, Ch. VI]). Therefore

\[
\langle (\mathcal{I} + \mathcal{F})^{-1} v_{k}, v_{l} \rangle = \operatorname{s-lim}_{n \to \infty} \sum_{m=-n}^{n} \alpha_{m}^{-1} \langle v_{k}, \tilde{v}_{m} \rangle \langle \tilde{v}_{m}, v_{l} \rangle = \sum_{m=-\infty}^{\infty} \alpha_{m}^{-1} \delta_{km} \delta_{ml} = \delta_{kl}.
\]

Completeness of the system \(\{c_{n}\}_{n \in \mathbb{Z}}\) immediately follows from the fact that the system \(\{v_{n}\}_{n \in \mathbb{Z}}\) is complete and that \(\mathcal{I} + \mathcal{F}\) is a homeomorphism of \(\mathbb{H}\), and the lemma is proved.

Next we show that the numbers \(\lambda_{n}\) are indeed the eigenvalues of the operator \(\mathcal{A}_{1}\). According to what was said above, it suffices to show that \(c_{1}(1, \lambda_{k}) = 0\). For the operator \(\mathcal{A}_{Q}\), one has the Green formula

\[
\langle \mathcal{A}_{Q} c(\cdot, \lambda), c(\cdot, \mu) \rangle - \langle c(\cdot, \lambda), \mathcal{A}_{Q} c(\cdot, \mu) \rangle = c_{2}(1, \lambda) c_{1}(1, \mu) - c_{1}(1, \lambda) c_{2}(1, \mu);
\]

taking therein \(\lambda = \lambda_{k}\) and \(\mu = \lambda_{n}\) and using the previous lemma, we arrive at the relation

\[(5.4) \quad c_{2}(1, \lambda_{k}) c_{1}(1, \lambda_{n}) = c_{1}(1, \lambda_{k}) c_{2}(1, \lambda_{n}).\]

Assume first that none of the numbers \(c_{1}(1, \lambda_{k})\) vanishes. Relation (5.4) implies that there is a constant \(\gamma\) such that, for all \(k \in \mathbb{Z}\),

\[c_{2}(1, \lambda_{k})/c_{1}(1, \lambda_{k}) = \gamma.\]

Then \(c_{k}\) are eigenvectors corresponding to the eigenvalues \(\lambda_{k}\) of the operator that is the restriction of \(\mathcal{A}_{Q}\) by the boundary condition \(u_{2}(1) = \gamma u_{1}(1)\). In other words, the numbers \(\lambda_{n}\) are zeros of the function

\[c_{2}(1, \lambda) - \gamma c_{1}(1, \lambda) = \sin \lambda - \gamma \cos \lambda + \int_{-1}^{1} \tilde{f}(s) e^{i\lambda s} \, ds\]

for some \(\tilde{f} \in L_{p}(-1, 1)\). However, the standard arguments based on Rouché’s theorem (see, e.g., [11, Ch. 1.3]) show that the zeros \(\tilde{\lambda}_{n}\) of the function \(c_{2}(1, \lambda) - \gamma c_{1}(1, \lambda)\) obey the different asymptotics \(\tilde{\lambda}_{n} = \pi n + \arctan \gamma + o(1)\), which leads to a contradiction.

Therefore there is a \(k \in \mathbb{Z}\) such that \(c_{1}(1, \lambda_{k}) = 0\). Then \(c_{2}(1, \lambda_{k}) \neq 0\) due to the uniqueness of solutions to the equation \(\ell_{Q}(u) = \lambda_{k} u\), and relation (5.4) implies that \(c_{2}(1, \lambda_{n}) = 0\) for all \(n \in \mathbb{Z}\). In other words, the functions \(c_{n}\) are the eigenvectors of the operator \(\mathcal{A}_{1}\) corresponding to the eigenvalues \(\lambda_{n}\). Since by Lemma 5.6 the system \(\{c_{n}\}_{n \in \mathbb{Z}}\) is complete in \(\mathbb{H}\), the operator \(\mathcal{A}_{1}\) has no other eigenvalues.

It remains to prove that \(\mu_{n}\) are the eigenvalues of \(\mathcal{A}_{2}\). We denote by \(\tilde{\mu}_{n}\) these eigenvalues and construct the corresponding function \(\tilde{\psi}\) of (3.6). Recalling expression (3.7), we conclude that \(\psi(\lambda_{k}) = \tilde{\psi}(\lambda_{k})\) for all \(k \in \mathbb{Z}\). Since the function \(\tilde{\psi}\) has the representation

\[\tilde{\psi}(\lambda) = \sin \lambda + \int_{-1}^{1} \tilde{f}_{2}(s) e^{i\lambda s} \, ds\]
for some $\tilde{f}_2 \in L_p(-1,1)$ (see the proof of Theorem 3.1) and $\psi$ has a similar representation with some $f_2 \in L_p(-1,1)$ instead of $\tilde{f}$ by (5.1), we see that the function $\omega := f_2 - \tilde{f}_2$ is such that
\[
\int_{-1}^{1} \omega(s)e^{i\lambda_n s} \, ds = 0
\]
for all $n \in \mathbb{Z}$. Recalling that the system of functions $\{e^{i\lambda_n s}\}_{n \in \mathbb{Z}}$ is closed $L_p(-1,1)$ (this follows from \[35\] Theorem III] for $p > 1$ and from \[27\] Lemma 3.3] for $p = 1$), we conclude that $\omega = 0$. Thus the numbers $\mu_n = \tilde{\mu}_n$ are the eigenvalues of the operator $A_2$, and the reconstruction procedure is complete. As the spectral data determine the function $H$ (and thus the transformation operator $I + \mathcal{K}$) unambiguously, the potential $Q$ is unique. The proof of Theorem 5.1 is complete.

We observe that, in passing, we have solved the inverse spectral problem of reconstructing the potential $Q$ of the Dirac operator from the spectrum of $A_1$ and the corresponding sequence of norming constants. Namely, the following is true.

**Theorem 5.7.** Sequences of real numbers $(\lambda_n)_{n \in \mathbb{Z}}$ and positive numbers $(\alpha_n)_{n \in \mathbb{Z}}$ are respectively the sequences of eigenvalues and norming constants of an operator $A_1$ for some $Q \in L_p(\mathcal{M}_2)$, $p \in [1, \infty)$, if and only if the following holds:

(i) the numbers $\lambda_n$ strictly increase and obey the asymptotics of Theorem 3.1;

(ii) the numbers $\alpha_n > 0$ obey the asymptotics of Theorem 3.3.

If (i) and (ii) hold, then $Q$ is given by $R(x,x)JB = \tilde{R}(x,0)JB$, where $\tilde{R}$ is the solution of the Krein equation (4.9) with $H$ of (4.1).

The reconstruction algorithm proceeds as the previous one, except that the first and the last step (related to the spectrum $(\mu_n)$) should be omitted.

In a similar manner we can also treat the inverse spectral problem for the operator $A_2$ (or operators corresponding to arbitrary separated boundary conditions).

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**APPENDIX A. FOURIER TRANSFORM IN $L_p(0,1)$**

For any $f \in L_p(0,1)$, we denote by $e_n(f)$, $n \in \mathbb{Z}$, its $n$-th Fourier coefficients, i.e.,
\[
e_n(f) = \int_0^1 f(x)e^{-2\pi i nx} \, dx.
\]
We also denote by $e(f)$ the sequence $(e_n(f))_{n \in \mathbb{Z}}$ and put
\[
X_p := \{e(f) \mid f \in L_p(0,1)\}.
\]
The vector space $X_p$ is algebraically embedded into $\ell_\infty(\mathbb{Z})$ and becomes a Banach space under the induced norm
\[
\|e(f)\|_{X_p} := \|f\|_{L_p}.
\]
For any $x = (x_n)$ and $y = (y_n)$ in $\ell_\infty(\mathbb{Z})$ we shall denote by $xy$ the entrywise product of $x$ and $y$, i.e., the element of $\ell_\infty(\mathbb{Z})$ with the $n$-th entry $x_ny_n$. 
Proposition A.1. $X_p$ is a commutative Banach algebra under the entrywise multiplication, i.e.,

\begin{equation}
\|xy\|_{X_p} \leq \|x\|_{X_p} \|y\|_{X_p}.
\end{equation}

Indeed, inequality (A.1) follows from the fact that $e_n(f)e_n(g) = e_n(f * g)$, where

\[(f * g)(x) := \int_{0}^{1} f(x-t)g(t) \, dt\]

is the convolution of $f$ and $g$ ($f$ being periodically extended to $(-1, 1)$ by $f(x+1) = f(x), \ x \in (0, 1)$), and from the inequality

\[\|f * g\|_{L_p} \leq \|f\|_{L_p} \|g\|_{L_p}.

The following statement is an analogue of the well-known Wiener lemma.

Proposition A.2. Assume that $f \in L_p(0, 1)$, where $p \in [1, \infty)$. If $1 + e_n(f) \neq 0$ for all $n \in \mathbb{N}$, then there exists a function $g \in L_p(0, 1)$ such that

\[(1 + e_n(f))^{-1} = 1 + e_n(g).

Proof. To begin with, we adjoin to $X_p$ the unit element $\delta$ (with all components equal to 1) and denote the resulting unital algebra by $\hat{X}_p$. Assume that the assumptions of the lemma hold and denote by $x$ an element of $\hat{X}_p$ with components $x_n := 1 + e_n(f)$. We shall prove below that $x$ is invertible in $\hat{X}_p$; since $e_n(f) \to 0$ as $n \to \infty$, this will imply that $x^{-1} = \delta + y$ for some $y \in X_p$ as required.

As is well known [16], the element $x$ is invertible in the unital Banach algebra $\hat{X}_p$ if and only if $x$ does not belong to any maximal ideal of $\hat{X}_p$. Proceeding by contradiction, assume that there exists a maximal ideal $m$ of $\hat{X}_p$ containing $x$. Since $\hat{X}_p$ includes all finite sequences and none of $x_n$ vanishes, $m$ also contains all finite sequences. Finite sequences form a dense subset of $X_p$ because the set of all trigonometric polynomials is dense in $L_p(0, 1)$. Recalling that maximal ideals are closed, we conclude that $X_p \subseteq m$. Next we observe that $X_p$ is a proper subset of $m$ (e.g., $x$ belongs to $m \setminus X_p$) and that $X_p$ has codimension 1 in $\hat{X}_p$. Hence $m = \hat{X}_p$, which contradicts our assumption that $m$ is a maximal ideal of $\hat{X}_p$. As a result, $x$ is not contained in any maximal ideal of $\hat{X}_p$ and thus is invertible in $\hat{X}_p$ indeed. The lemma is proved. \hfill \Box

Appendix B. The GLM equation and factorisation of Fredholm operators

In this appendix, we shall explain relationships between solubility of the GLM equation and factorisation of related Fredholm operators in some special algebras. We refer the reader to the books [13] [20] for related concepts and basic facts.

Write $\mathbb{H} := L_2((0, 1), \mathbb{C}^2)$ and denote by $\mathcal{B}$ (by $\mathcal{B}_\infty$) the Banach algebra of all bounded (compact) operators in $\mathbb{H}$. Denote also by $P_t, t \in [0, 1]$, the operator in $\mathbb{H}$ of multiplication by $\chi_{[0,t]}$, the characteristic function of the interval $[0, t]$. Set

\[\mathcal{B}_\infty^+ := \{ B \in \mathcal{B}_\infty \mid \forall t \in [0, 1], \ P_tB(I - P_t) = 0 \},\]

\[\mathcal{B}_\infty^- := \{ B \in \mathcal{B}_\infty \mid \forall t \in [0, 1], \ (I - P_t)BP_t = 0 \};\]

then $\mathcal{B}_\infty^\pm$ are closed subspaces of $\mathcal{B}_\infty$ and $\mathcal{B}_\infty^+ \cap \mathcal{B}_\infty^- = \{ 0 \}$. We also observe that the operators in $\mathcal{B}_\infty^\pm$ are Volterra ones.
Recall that \( G_p(\mathcal{M}_2) \), \( p \in [1, \infty) \), stands for the algebra in \( B_\infty \) of integral operators over \((0,1)\) with kernels in the space \( G_p(\mathcal{M}_2) \) introduced in Section 2. The sets
\[
G_p^+ (\mathcal{M}_2) = G_p(\mathcal{M}_2) \cap B_\infty^+, \quad G_p^- (\mathcal{M}_2) := G_p(\mathcal{M}_2) \cap B_\infty^-,
\]
are subalgebras of \( G_p(\mathcal{M}_2) \) consisting of operators with lower- and upper-triangular kernels respectively and \( G_p(\mathcal{M}_2) = G_p^+ (\mathcal{M}_2) + G_p^- (\mathcal{M}_2) \).

We say that an operator \( \mathcal{I} + \mathcal{L} \), \( \mathcal{L} \in G_p(\mathcal{M}_2) \), admits a factorization in \( G_p(\mathcal{M}_2) \) if
\[
(\mathcal{I} + \mathcal{L}) = (\mathcal{I} + \mathcal{K}^+)^{-1}(\mathcal{I} + \mathcal{K}^-)^{-1}
\]
with some \( \mathcal{K}^\pm \in G_p^\pm (\mathcal{M}_2) \).

The following two theorems were established in [43, 44] for the space \( L_2(0,1) \), however, their generalisation to our situation is straightforward.

**Theorem B.1.** If \( \mathcal{I} + \mathcal{L} \) admits a factorization in \( G_p(\mathcal{M}_2) \), then the operators \( \mathcal{K}^\pm = \mathcal{K}^\pm (\mathcal{L}) \) are unique. Moreover, the set \( \Phi_p \) of operators \( \mathcal{L} \in G_p(\mathcal{M}_2) \), for which the operator \( \mathcal{I} + \mathcal{L} \) is factorisable, is open in \( G_p(\mathcal{M}_2) \), and the functions
\[
\Phi_p \ni \mathcal{L} \mapsto \mathcal{K}^\pm (\mathcal{L}) \in G_p(\mathcal{M}_2)
\]
are continuous.

**Theorem B.2.** Assume that \( \mathcal{L} \in G_p(\mathcal{M}_2) \). For the operator \( \mathcal{I} + \mathcal{L} \) to admit a factorisation in \( G_p(\mathcal{M}_2) \), it is necessary and sufficient that the operators \( I + P_t \mathcal{L} P_t \) have a trivial kernel in \( \mathbb{H} \) for each \( t \in [0,1] \).

We remark that for a self-adjoint operator \( \mathcal{L} \in G_p(\mathcal{M}_2) \) the requirement that the operators \( I + P_t \mathcal{L} P_t \) have a trivial kernel in \( \mathbb{H} \) for all \( t \in [0,1] \) is equivalent to positivity of \( \mathcal{I} + \mathcal{L} \) in \( \mathbb{H} \).

Assume that \( \mathcal{I} + \mathcal{L} \) is factorisable in \( G_p(\mathcal{M}_2) \), so that (B.1) holds. Applying \( \mathcal{I} + \mathcal{K}^+ \) to both sides of this equality and using the fact that \((\mathcal{I} + \mathcal{K}^-)^{-1} = \mathcal{I} + \mathcal{K}^- \) for some \( \mathcal{K}^- \in B_\infty^- \), we derive the relation
\[
\mathcal{K}^+ + \mathcal{P}^+ \mathcal{L} + \mathcal{P}^+(\mathcal{K}^+ \mathcal{L}) = 0,
\]
where \( \mathcal{P}^+ \) denotes the projection operator of \( G_p(\mathcal{M}_2) \) onto \( G_p^+ (\mathcal{M}_2) \) parallel to \( G_p^- (\mathcal{M}_2) \).

This relation is an abstract analogue of the Gelfand–Levitan–Marchenko (GLM) equation; indeed, in terms of the kernels \( K^+ \) and \( L \) of the operators \( \mathcal{K}^+ \) and \( \mathcal{L} \) we get
\[
K^+(x,t) + L(x,t) + \int_0^x K^+(x,s) L(s,t) \, ds = 0, \quad 0 \leq t \leq x \leq 1,
\]
and
\[
K^+(x,t) + L(x,t) = 0.
\]

We see that if an operator \( \mathcal{I} + \mathcal{L} \) is factorisable in \( G_p(\mathcal{M}_2) \), then the abstract GLM equation (B.2) has a solution \( \mathcal{K}^+ = \mathcal{K}^+ (\mathcal{L}) \in G_p^+ (\mathcal{M}_2) \). Conversely, if \( \mathcal{K}^+ \in G_p^+ (\mathcal{M}_2) \) is a solution of equation (B.2), then \( \mathcal{P}^+(\mathcal{K}^+ \mathcal{L} + \mathcal{K}^+ \mathcal{L}) = 0 \), i.e., \( \mathcal{K} := \mathcal{K}^+ + \mathcal{L} + \mathcal{K}^+ \mathcal{L} \) belongs to \( G_p^- (\mathcal{M}_2) \), and the relation
\[
(\mathcal{I} + \mathcal{K}^+) (\mathcal{I} + \mathcal{L}) = \mathcal{I} + \mathcal{K}
\]
holds. Since \( \mathcal{I} + \mathcal{K} \) has the form \((\mathcal{I} + \mathcal{K}^-)^{-1} \) for \( \mathcal{K}^- := (\mathcal{I} + \mathcal{K})^{-1} - \mathcal{I} \in G_p^- (\mathcal{M}_2) \), we conclude that \( \mathcal{I} + \mathcal{L} \) is factorisable in \( G_p(\mathcal{M}_2) \).

Summing up, we derive the following assertion on the solubility of the GLM equation (B.3).
Corollary B.3. Assume that $\mathcal{L}$ is a selfadjoint operator in $G_p(\mathcal{M}_2)$ with kernel $L$ such that $I + \mathcal{L}$ is positive. Then equation (B.3) is uniquely soluble, and the solution $K^+$ belongs to $G_p(\mathcal{M}_2)$ and depends continuously therein on $L \in G_p(\mathcal{M}_2)$.

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