A note on Adrian Lewis’ result on permutation invariant functions

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Abstract

We obtain a generalisation of Adrian Lewis’ result on the subdifferential of permutation invariant convex function to functions invariant under finite reflection groups.

1 Introduction

The beautiful result of Adrian Lewis [4] that underlies his work on the eigenvalue optimisation and features prominently in the overview work [2], states that if a function $f : \mathbb{R}^n \to \mathbb{R}$ is invariant with respect to the permutation of coordinates, then $y \in \partial f(x)$ if and only if

$$y^\dagger \in \partial f(x^\dagger) \quad \text{and} \quad \langle x^\dagger, y^\dagger \rangle = \langle x, y \rangle,$$

(1)

where $\langle x, y \rangle = x^T y$ is the scalar product, and $x^\dagger$ denotes the nonincreasing reordering of coordinates.

Let $G$ be a finite reflection group. For each $x \in \mathbb{R}^n$ by $\tilde{x}$ denote the unique intersection point of the orbit of $x$ and a fundamental chamber of $G$. The goal of this paper is to show that for a convex function $f : \mathbb{R}^n \to \mathbb{R}$ invariant under $G$, $y \in \partial f(x)$ if and only if

$$\tilde{y} \in \partial f(\tilde{x}) \quad \text{and} \quad \langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle.$$

(2)

The result of Adrian Lewis [4] is a special case of our developments, when the group $G$ is the group of permutations, and $\tilde{x} = x^\dagger$ is the nonincreasing arrangement of coordinates.

We begin with recalling the basic notions and results related to finite reflection groups in Section 2 then in Section 3 we obtain several technical results concerning the subdifferential of functions invariant under group actions. Section 4 contains the precise formulation and proof of the main result. Throughout the paper, we let $\langle x, y \rangle = x^T y$, and $\|x\| = \sqrt{x^T x}$. By $S_{n-1}$ we denote the unit sphere in $\mathbb{R}^n$.

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2 Finite reflection groups

A finite reflection group $G$ on $\mathbb{R}^n$ is a finite subgroup of the orthogonal linear group $O(n)$ generated by a family of reflections. In two dimensions all finite reflection groups are generated by the reflections through two mirrors in a plane positioned at an angle $2\pi/n$, where $n$ is a positive integer and are representations of dihedral groups. For example, the symmetry group $\text{Dih}_6$ (snowflake symmetry) is generated by a pair of mirrors located at the angle $\pi/6$ (see Fig. 1).

![Figure 1: A snowflake and the two mirrors that generate the symmetry group](image)

A mirror is a hyperplane in $\mathbb{R}^n$ with associated unit normal $u \in S_{n-1}$ and the reflection represented by the Householder transformation $H_u = I - 2uu^T$, which fixes this hyperplane. We denote this mapping by $h_u$, i.e. $h_u(x) = H_u x$. Note that the linear isometries generated by a set of finite reflections are not all reflections. For example, the symmetry group of the square $\text{Dih}_4$ has the representation as a finite reflection group generated by two reflections at a $\pi/4$ angle (one of which goes through one of the square’s diagonals). This group consists of four reflections (see Fig. 2) and four rotations (by $\pi/2$, $\pi$, $3\pi/2$ and the identity). We can work this out by direct computation, picking an arbitrary pair of adjacent mirrors (normals), for example, $u_1 = (0, 1)$ and $u_2 = (1/\sqrt{2}, -1/\sqrt{2})$. The corresponding Householder
transformations are
\[ R_1 = I - 2u_1u_1^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R_2 = I - 2u_2u_2^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

The remaining six transformations can be computed by the direct multiplication of the elementary reflections and checking the completeness of the relevant Cayley table.

Observe that the mirrors, or the reflection lines, of the two-dimensional reflection groups split the space into similar wedges, and each can be obtained from any other by consecutive reflections via adjacent sides. Moreover, we can pick our ‘generating wedge’ in arbitrary way, and make sure that the conic hull of the relevant normals that correspond to the mirrors bounding the wedge form the dual cone to the wedge. Even though finite reflection groups in higher dimensions have more complicated structure than the dihedral groups, these two observations are still valid and form the core of the theory of finite reflection groups, and of more general Coxeter groups as well.

For a finite reflection group the space can be subdivided into chambers bounded by mirror hyperplanes that play the same role as wedges in the planar groups. We can choose any (closed) chamber and call it the fundamental chamber. A core result in the theory of finite groups is that the fundamental chamber contains one and only one representative of each of the orbits (the sets formed by all images of a given point under all group actions), i.e. a fundamental chamber is a fundamental domain of the group action. Before we state this result, we formalise the notation.

Recall that for a group $G$ acting on $\mathbb{R}^n$, and a point $x \in \mathbb{R}^n$, the orbit of $x$ is the set of all its images under $G$: \[ O(x) = \{ y \in \mathbb{R}^n \mid \exists g \in G, y = gx \}. \]

A set $\Phi \subset S_{n-1}$ is a root system of a finite reflection group $G$ if \[ \Phi = \{ u \in S_{n-1} \mid h_u \in G \}, \]
i.e. it consists of all normals (positive and negative) to all reflection hyperplanes in the group.

A set $U \subset \Phi$ is a positive root system if there exists a linear mapping $f(x): \mathbb{R}^n \to \mathbb{R}$ with $f(u) > 0$ for all $u \in U$, and $f(u) < 0$ for all $u \in \Phi \setminus U$. Observe that it is always possible to construct a positive root system, since the group $G$ is finite.

Given a positive root system $U$ of a group $G$ the closed fundamental chamber $C$ of $G$ is the dual cone of the positive root system
\[ C = (\text{cone } U)^* = \{ x \in \mathbb{R}^n \mid \langle x, u \rangle \geq 0 \quad \forall u \in U \}. \]

We have the following well-known result (for the proof see [3, Theorem (a), page 22]).

**Lemma 1.** For any finite reflection group $G$ and any $x \in \mathbb{R}^n$ the set \[ O(x) \cap C, \]
where $O(x)$ is the orbit of $x$, is a singleton.
By Lemma 1 we can map any point \( x \in \mathbb{R}^n \) to its unique intersection \( \tilde{x} \) with the fundamental chamber, i.e.

\[
\{ \tilde{x} \} = C \cap O(x).
\]

We will need one more technical result that concerns group stabilisers. Recall that a stabiliser (isotropy group, pointwise centraliser) \( C_G(x) \) of \( x \in \mathbb{R}^n \) is the subset of the group action that fixes \( x \), i.e.

\[
C_G(x) = \{ h \in G \mid hx = x \}.
\]

**Lemma 2.** Let \( G \) be a finite reflection group with a positive root system \( U \), and assume \( C_G(x) \) is a stabiliser of \( x \in \mathbb{R}^n \). Then \( C_G(x) \) is generated by those reflections that \( C_G(x) \) contains.

**Proof.** Follows directly from Theorem 12.6 in [1].

3 Subdifferentials of functions invariant under finite reflection groups

Recall that given a convex function \( f : \mathbb{R}^n \to \mathbb{R} \) its Moreau-Rockafellar subdifferential \( \partial f(x) \) (see [5]) is the set of such \( v \in \mathbb{R}^n \) (called subgradients) that

\[
f(y) \geq f(x) + \langle v, y - x \rangle \quad \forall y \in \mathbb{R}^n.
\]

The next result is well-known and follows directly from the subdifferential chain rule. We provide the proof for convenience.

**Lemma 3.** Let \( x \in \mathbb{R}^n \), and assume \( f : \mathbb{R}^n \to \mathbb{R}^n \) is convex and invariant under a reflection group \( G \). Then for any \( y \in \partial f(x) \) and any \( g \in G \)

\[
\partial f(x) = g^* \partial f(gx) \quad \text{and so} \quad g \partial f(x) = f(gx).
\] (3)

Moreover, if \( x \) is such that \( x = hx \) for some \( h \in G \), then

\[
\partial f(x) = h \partial f(x).
\] (4)

**Proof.** First observe that every reflection \( g \in G \) is surjective, hence, we can apply the subdifferential chain rule (see [5, Theorem 23.9]):

\[
\partial (f \circ g)(x) = g^* \partial f(gx).
\]

Since in our case \( f \) is \( G \)-invariant, we have \( (f \circ g)(x) = f(x) \), hence, \( \partial (f \circ g)(x) = \partial f(x) \), and we get (3). The relation (5) follows from (3) by substituting \( x = gx \) in the right hand side.

We conclude this section with two technical results that connect the subdifferential theory with finite reflection groups.

**Lemma 4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be invariant under a finite reflection group action \( G \), and let \( U \) be a positive root system of \( G \). For \( x \in \mathbb{R}^n \) let

\[
V(x) := \{ v \in U \mid h_v \in C_G(x) \},
\]

where \( C_G(x) \) is the stabiliser of \( x \). Then for any \( y \in \partial f(x) \) and \( u \in U \setminus V \) we have \( \langle y, u \rangle \geq 0 \).
Proof. Fix $x \in \mathbb{R}^n$ and pick an arbitrary $y \in \partial f(x)$ and $u \in U \setminus V$. From the convexity and invariance of $f$ we have for the reflection $h_u$ represented by the Householder transformation $H_u = I - 2uu^T$:

$$f(x) = \frac{1}{2} f(x) + \frac{1}{2} f(h_u x) \geq f \left( \frac{x + h_u x}{2} \right) = f(x - \langle u, x \rangle u) \geq f(x) - \langle u, x \rangle \langle u, y \rangle,$$

hence,

$$\langle u, x \rangle \langle u, y \rangle \geq 0.$$

Since $x \in \mathcal{C}$, and $\langle x, u \rangle \neq 0$ (recall that $h_u$ is not in the stabiliser), we have $\langle u, x \rangle > 0$, and therefore $\langle u, y \rangle \geq 0$.

Lemma 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function invariant under a finite reflection group action $G$, and assume $x \in \mathcal{C}$. Then for any $g \in G$

$$\partial f(gx) = g(C_G(x)(\partial f(x) \cap \mathcal{C})).$$

Proof. By Lemma 3 we have $\partial f(gx) = g\partial f(x)$, hence, it is sufficient to show that

$$\partial f(x) = C_G(x)(\partial f(x) \cap \mathcal{C}) \quad \forall x \in \mathcal{C}. \quad (5)$$

Let $x \in \mathcal{C}$. If $y \in \partial f(x)$, then by Lemma 3 we have $hy \in \partial f(x)$ for all $h \in C_G(x)$, hence,

$$C_G(x)(\partial f(x) \cap \mathcal{C}) \subset \partial f(x).$$

To show the reverse inclusion, let $y \in \partial f(x)$. From the positive root system $U$ choose the subsystem $V = \{ v \in U \mid h_v \in C_G(x) \}$. By Lemma 2 $V$ is the positive root system of $C_G(x)$ defined by the same linear mapping as $U$. By Lemma 1 the intersection of each orbit with the fundamental chamber is unique, hence, there exists $h \in C_G(x)$ such that $\langle hy, v \rangle \geq 0$ for all $v \in V$ i.e. $hy \in \mathcal{O}(y) \cap (\text{cone } V)^*$. For all $u \in U \setminus V$ Lemma 1 yields $\langle hy, u \rangle \geq 0$. Observe that also by Lemma 3 we have $hy \in \partial f(x)$. We hence conclude that $hy \in \mathcal{C} \cap \partial f(x)$, and hence $y = h^* \cdot (hy) \in C_G(x)(\partial f(x) \cap \mathcal{C})$. By the arbitrariness of $y$ this yields the desired inclusion

$$\partial f(x) \subset C_G(x)(\partial f(x) \cap \mathcal{C}).$$

$\square$

4 Proof of the main result

Theorem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function invariant under a finite reflection group $G$. Then $y \in \partial f(x)$ if and only if

$$\tilde{y} \in \partial f(x) \quad \text{and} \quad \langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle,$$

where by $\tilde{x}$ (resp. $\tilde{y}$) we denote the unique point that belongs to the intersection of the orbit $O_G(x)$ of $x$ (resp. $O_G(y)$ of $y$) with the fundamental chamber $\mathcal{C}$ of $G$. 


Proof. Let \( y \in \partial f(x) \). There exists \( g \in G \) such that \( x = g \bar{x} \). From Lemma \( 6 \)

\[
y \in g(C_G(\bar{x})(\partial f(\bar{x}) \cap C)),
\]

therefore, there exists \( h \in C_G(\bar{x}) \) and \( y' \in \partial f(\bar{x}) \cap C \) such that \( y = gh y' \). Thus \( y' \in O(y) \cap C \).

By Lemma \( 1 \) such \( y' \) is unique and must coincide with \( \tilde{y} \). Hence \( y = gh y' = gh \tilde{y} \).

Since \( h \) fixes \( \bar{x} \), we also have \( x = g \bar{x} = gh \bar{x} \).

Applying Lemma \( 3 \) we have

\[
\tilde{y} = h^* g^* y \in h^* g^* \partial f(x) = h^* g^* \partial f(gh \bar{x}) = \partial f(\bar{x}).
\]

Finally,

\[
\langle x, y \rangle = \langle gh \bar{x}, gh \tilde{y} \rangle = \langle \bar{x}, \tilde{y} \rangle.
\]

Now assume that (6) holds for some \( x, y \in \mathbb{R}^n \). Then there exists \( g \in G \) such that \( \bar{x} = gx \) and Lemma \( 3 \) gives

\[
\tilde{y} \in \partial f(\bar{x}) = g \partial f(x).
\]

Thus there exists \( y' \in \partial f(x) \) with \( \tilde{y} = gy' \), and \( y' \in \partial f(x) \). As \( y' \in O(\tilde{y}) = O(y) \), there is a \( k \in G \) with \( ky = y' \).

As \( y' \in \partial f(x) \) via the subgradient inequality (and invariance of \( f \)) we have for all \( z \) that

\[
f(z) - f(x) = f(kz) - f(x) \geq \langle y', kz - x \rangle
\]

\[
= \langle ky, kz \rangle - \langle y', x \rangle
\]

\[
= \langle y, z \rangle - \langle g^* \tilde{y}, g^* \bar{x} \rangle
\]

\[
= \langle y, z \rangle - \langle y, x \rangle
\]

\[
= \langle y, z - x \rangle,
\]

hence \( y \in \partial f(x) \).

\[ \square \]

References

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