Intercepts of the momentum correlation functions in $\mu$-Bose gas model and their asymptotics

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The so-called $\mu$-deformed oscillator (or $\mu$-oscillator) introduced by A. Jannussis, though possesses rather exotic properties with respect to other better known deformed oscillator models, also has good potential for diverse physical applications. In this paper, the corresponding $\mu$-Bose gas model based on $\mu$-oscillators is developed. Within this model, the intercepts $\lambda^{(2)}(K)$ and $\lambda^{(3)}(K)$ of two- and three-particle momentum correlation functions are calculated with the goal of possible application for modeling the non-Bose type behavior of the intercepts of two- and three-pion correlations, observed in the experiments on relativistic heavy ion collisions. In derivation of intercepts, a fixed order of approximation in the deformation parameter $\mu$ is assumed. For the asymptotics of the intercepts $\lambda^{(2)}(K)$ and $\lambda^{(3)}(K)$, we derive exact analytical formulas. The results for $\lambda^{(2)}(K)$ are compared with experimental data, and with earlier known results drawn using other deformed Bose gas models.

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1. INTRODUCTION

For the last two decades, diverse models of deformed oscillators have received much attention due to both their unusual properties, as compared with the standard quantum harmonic oscillator, and their great potential for being applied in description of miscellaneous physical systems showing essential nonlinearities. The diversity of applications ranges from, say, quantum optics and Landau problem to high energy quantum particle phenomenology, modern quantum field theory and D-brane (string) theory [1–9]. Among the best known, extensively studied deformed oscillators we encounter such as the $q$-deformed Arik-Coon (AC) one [10] and Biedenharn-Macfarlane (BM) one [11, 12], as well as the two-parameter $p,q$-deformed oscillator [13]. Besides, there exists the $q$-deformed Tamm-Dancoff (TD) oscillator [14, 15], also explored though to much lesser extent [16]. Unlike all these, there is very modest knowledge concerning the so-called $\mu$-deformed oscillator. Introduced in [17] more than a decade and half ago, the $\mu$-oscillator essentially differs from the models already mentioned and exhibits rather unusual properties [18].

An important direction of applying deformed oscillators (deformed bosons) is the elaboration on their base of the respective deformed versions of Bose gas model, like the $q$- or $p,q$-Bose gas models in [19–21], [6, 8]. The $q$-deformed extensions of Bose gas are used in modern approaches to $^4$He theory, see e.g. [22]. On the other hand, $q$-Bose and $p,q$-Bose gas models provide an efficient ground for successful description of the observed, in the experiments on heavy ion collisions, unusual behavior of two-pion and three-pion momentum correlations. As shown in [6, 8, 19, 21, 23] the intercept $\lambda^{(2)}(K)$, see formula (10) below, of the two-particle momentum correlation function derived in the deformed Bose gas models based on the AC- or BM-type $q$-deformed oscillators, as well as on the $p,q$-oscillators, lead to a successful modeling of the nontrivial, non-Bose type (such that basically differ from bosonic case) properties of pion correlation intercepts in the RHIC and CERN SPS experiments on relativistic collisions of nuclei.

Inspired by those results, in this paper we undertake an analogous study basing on $\mu$-oscillators ([17], [18]). Central place in our treatment is given to calculation, within the $\mu$-Bose gas model involving $\mu$-bosons ($\mu$-oscillators), of the two- and three-particle correlation function intercepts $\lambda^{(2)}(K)$ and $\lambda^{(3)}(K)$. For these we establish asymptotically constant type of behavior (i.e., asymptotical approaching to a constant given solely by $\mu$). In addition, we make some comparison of the obtained results for intercepts with the existing data.

The paper is organized as follows. In Section 2 main facts about the $\mu$-oscillator are given. For our goals we use the concept of the deformation structure function [24–26] denoted as $\varphi(N)$. Here we also sketch some unusual properties of the $\mu$-oscillator such as the possibility of accidental degeneracy of energy levels and the non-Fibonacci nature of its energy spectrum. Section 3 is devoted to calculation of the intercepts of two- and three-particle momentum correlation functions. Namely, within the $\mu$-Bose gas model based on $\mu$-oscillators we calculate these intercepts, for fixed order of approximation. Derivation of their large momentum asymptotics which is one of our main results is given in Section 4. Next 5th Section is devoted to a comparison with some experimental data, and with analogous results obtained earlier within other ($q$- or $p,q$-) deformed Bose gas models. The paper ends with concluding remarks. Appendix A contains the data for the mean values or thermal averages of diverse orders (for powers of the number operator), as well as some interesting properties of the coefficients appearing in the mean values.
2. SETUP OF THE $\mu$-OSCILLATOR

In the theory of deformed oscillators it is both convenient and efficient to use the concept of deformation structure function, either in the $\varphi(N)$ version [24, 25] or in the $F(N)$ version [26]. In our present treatment we prefer to use the former variant $\varphi(N)$. Each specialized structure function determines the particular deformed oscillator model through its corresponding oscillator algebra, a unital algebra given in terms of the elements $a, a^\dagger, N$. These generating elements of the algebra of a particular model of linear (usual) or nonlinear (deformed) quantum oscillator obey the defining relations

\[ [a, a^\dagger] = F(N), \]

\[ [N, a^\dagger] = a^\dagger, \quad [N, a] = -a. \]

From (2) it follows that $[N, aa^\dagger] = [N, a^\dagger a] = 0$. The relation (1) involves $F(N)$, function of the number operator $N$. For the usual quantum oscillator, $a^\dagger a = N, aa^\dagger = N + 1$ and $F(N) = 1$. For general deformed oscillator, like in [24, 25], we introduce the deformation structure function $\varphi(N)$ according to the relations

\[ a^\dagger a = \varphi(N), \quad aa^\dagger = \varphi(N + 1). \]

Due to (1) and (3), the basic commutation relation takes the form

\[ aa^\dagger - a^\dagger a = F(N) = \varphi(N + 1) - \varphi(N). \]

That is, the defining commutation relation is unambiguously given by structure function. The Hamiltonian $H$ will be taken in the standard form $H = \frac{1}{2}(aa^\dagger + a^\dagger a)$. Here and below we put $\hbar \omega = 1$. In terms of structure function the Hamiltonian is

\[ H = \frac{1}{2}(\varphi(N + 1) + \varphi(N)). \]

We use the $q$-analog of Fock space. Within it, from the ground state obeying

\[ a|0\rangle = 0, \quad N|0\rangle = 0, \quad \langle 0|0\rangle = 1, \]

all the $n$-particle excited states are generated such that

\[ N|n\rangle = n|n\rangle, \quad \varphi(N)|n\rangle = \varphi(n)|n\rangle. \]

The $n$-particle normalized state is given by the formula

\[ |n\rangle = \frac{(a^\dagger)^n}{\sqrt{\varphi(n)!}}|0\rangle, \quad \varphi(n)! = \varphi(n)\varphi(n-1)\ldots\varphi(1), \quad \varphi(0)! = 1. \]

The creation and annihilation operators $a^\dagger, a$ when acting on the $n$-particle state give

\[ a^\dagger|n\rangle = \sqrt{\varphi(n + 1)}|n + 1\rangle, \quad a|n\rangle = \sqrt{\varphi(n)}|n - 1\rangle. \]

In what follows we focus on the so called $\mu$-deformed oscillator [17] for which the generating elements $a, a^\dagger$ and $N$ of the corresponding algebra obey (2) and (4), and the defining structure function is

\[ \varphi(N) = \varphi_\mu(N) = \frac{N}{1 + \mu N}. \]

where $\mu$ is the deformation parameter, $\mu \geq 0$. With this, the basic relation and the energy spectrum for the Hamiltonian (5) of $\mu$-oscillator take the form

\[ aa^\dagger - a^\dagger a = \varphi_\mu(N + 1) - \varphi_\mu(N) = \frac{N + 1}{1 + \mu(N + 1)} - \frac{N}{1 + \mu N}. \]

\[ E_n = \frac{1}{2} \left( \frac{n + 1}{1 + \mu(n + 1)} + \frac{n}{1 + \mu n} \right). \]

Setting $\mu = 0$ recovers the algebra and the energies of the usual harmonic oscillator.

Note that the linear structure function $\varphi(N) = N$, which determines the standard quantum oscillator, stems as particular case in the limit of deformation parameter(s), such as $q \rightarrow 1$ for $q$-oscillators or $\mu \rightarrow 0$ in the present case of $\mu$-oscillator, from each model of deformed oscillators (a kind of consistency).

Unusual properties of $\mu$-oscillator

The ground state energy and the large $n$ asymptotics for the energy of $\mu$-oscillator are respectively $E_0 = \frac{1}{2(1+\mu)}$ and $E_\infty = \frac{1}{\mu}$. Both of these values basically differ from the analogous value and limit for the usual oscillator, $E_0 = \frac{1}{2}$ and $E_n \stackrel{n\rightarrow\infty}{\longrightarrow} \infty$. Moreover, the ground state $E_0$ of the $\mu$-oscillator differs also from that of the Fibonacci oscillators such as $p,q$-deformed family and its one-parameter $q$-deformed subfamilies (AC, BM, TD), for which zeroth level energy is $E_0 = \frac{1}{2}$ exactly as for the standard oscillator.

The just mentioned unusual feature of the $\mu$-oscillator is tightly connected with the next its peculiarity: the $\mu$-oscillator unlike the $p,q$-deformed one does not belong to the class of Fibonacci oscillators. Namely, as shown in detail in the recent paper [18] the $\mu$-oscillator is typical representative of the wide class of so-called quasi-Fibonacci oscillators.

It is important that the $\mu$-oscillator admits nontrivial extensions, in particular through combining it with other ($q$- or $p,q$-) deformed oscillators. Any such extension will inherit the peculiarities of those simpler deformed oscillators from which it is composed, as shown in [18]. As can be demonstrated for positive $\mu$ such that $\mu < 1$, the $\mu$-oscillator admits occurrence of accidental energy level degeneracy if it is extended by at least one deformation parameter. However, unlike the case of Tamm-Dancoff $q$-deformed oscillator [16], here not all types of pairwise energy level degeneracies are possible.
In other words, direct two- or three-parameter extensions of \( \mu \)-oscillator can exhibit the phenomena of accidental degeneracy (arising in the absence of any obvious underlying symmetry) of energy levels. In more detail this property will be studied elsewhere (see however \[16, 27-29\] where the ability to display accidental energy levels degeneracy has been demonstrated for the \( p,q \)-deformed and some \( q \)-deformed oscillators).

3. THE USE OF \( \mu \)-OSCILLATORS IN \( \mu \)-BOSE GAS MODEL

Any real system of particles is far from being ideal. Therefore, a necessity arises to make complete account of inter-particle interactions. Besides, the nonzero proper volume or substructure of particles may also be of importance. For all these issues it is known that the system admits effective description by means of deformed oscillators \[30-32\]. Due to that, the system can be treated as an ideal one, but consisting of unconventional appropriately deformed "bosons". As result, account of any of the indicated complications or reasons turns out to be effectively incorporated just in the particular deformation, with its specific deformation parameter(s).

Such an effective picture which uses the \( q \)-deformed oscillators has been applied to develop the related \( q \)-Bose gas model, with subsequent successful application to superfluid \( ^4\text{He} \) \[22\] or to an effective description \[19-21, 23\] of the observed non-Bose type behavior of the intercept \( \lambda(m, K) \) of two-particle momentum correlation function of pions produced and registered in the experiments on heavy-ion collisions. "Non-Bose" means that the existing data for the intercept clearly demonstrate the trend and the values that principally differ (see e.g. \[33-35\]) from the would-be constant intercept equal to 1 at any momentum, for the pions viewed as pure bosons. More advanced picture which involves the two-parameter \( p,q \)-deformed oscillators and treats \( n \)-particle correlation function intercepts within the \( p,q \)-Bose gas model, along with analogous application, has been developed in \[6, 8\]. Below we exploit similar ideas for the case of \( \mu \)-Bose gas model.

3.1. Intercept of two-particle correlation function

In the experiments on heavy ion-collisions, the colliding (primary) particles generate, as the products of collisions, a miriads of secondary particles, e.g. pions, which are then detected. As known \[36-38\], correlation functions carry important information concerning geometry and dynamics of the emitting sources. The probability for a joint observation of two particles with momenta \( k_1 \) and \( k_2 \) is given by \( P_2(k_1, k_2) \). The two-particle momentum correlation function is defined, see e.g. \[36-38\], as

\[
C^{(2)}(k_1, k_2) = \gamma \frac{P_2(k_1, k_2)}{P_1(k_1)P_1(k_2)} \tag{9}
\]

where \( P_1(k_i), i = 1, 2 \), means pure single-particle probability distribution for \( i \)-th particle and \( \gamma \) is normalizing factor. We put \( \gamma = 1 \) in what follows. It is common to denote the difference of 4-momenta of the pair by \( Q = k_1 - k_2 \) and the mean 4-momentum by \( K = (k_1 + k_2)/2 \). In the case of two identical (of mass \( m \)) bosons and coinciding momenta, the \( Q \) is vanishing. Then the two-particle correlation function in variables \( K, Q \), and the reduced case \( Q = 0 \) of it, is

\[
C^{(2)}(Q, K) \xrightarrow{k_1=k_2} C^{(2)}(Q=0, K) = 1 + \lambda^{(2)}(m, K) \tag{10}
\]

where \( \lambda^{(2)}(m, K) \) does appear. The entity \( \lambda^{(2)}(m, K) \) at an early stage was usually called the "coherence" or (non-) chaoticity parameter. On the other hand, since \( C^{(2)}(Q, K) \) at \( k_1 = k_2 \) attains its maximal value, \( \lambda^{(2)}(K) \) is also termed the intercept of the correlation function \( C^{(2)}(Q, K) \) (and similarly for \( n \)-th order correlation functions \( C^{(n)} \)). We will prefer to use the latter term because, as stressed in \[8\], even in case of complete chaocity of sources the intercept can attain the values differing from the would-be purely bosonic ones \[39\]. The intercept can be expressed \[36\] in terms of particle (whose momentum is \( K \)) creation and annihilation operators as

\[
\lambda^{(2)}(K) = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} - 1, \quad K = k_1 = k_2. \tag{11}
\]

A natural way to obtain after calculations the nontrivial (non-constant) form of the intercept is to use some version of deformed oscillator and to develop on this base the corresponding deformed Bose gas model.

As mentioned, the explicit form of \( \lambda^{(2)}(K) \) has been earlier obtained in \[6, 19-21, 28\] in the framework of deformed versions of Bose gas model based on the popular AC-, BM-, \( p,q \)-type deformed oscillator models. In the case of \( p,q \)-bosons this result was extended \[6, 8\] to the \( n \)-particle correlation function intercepts \( \lambda^{(n)}(K) \).

Here we use the set of \( \mu \)-oscillators to develop the corresponding \( \mu \)-Bose gas model. Within it, we derive the intercepts of the two-particle and three-particle correlation functions. For that goal we take \( 11 \) as our starting point and in view of \( 3 \) make account of the defining structure function of the chosen \( \mu \)-oscillator, namely

\[
a^\dagger a = \varphi_\mu(N) = [N]_\mu = \frac{N}{1 + \mu N}. \tag{12}
\]

From eq. \( 2 \), for a function \( f(N) \) the property follows:

\[
a f(N) = f(N+1)a, \quad a^\dagger f(N) = f(N-1)a^\dagger. \tag{13}
\]

From \( 12 \) we have one- and two-particle distributions

\[
\langle a^\dagger a \rangle = \langle [N]_\mu \rangle = \left\langle \frac{N}{1 + \mu N} \right\rangle,
\]

\[
\langle a^\dagger a^\dagger a a \rangle = \langle a^\dagger [N]_\mu a \rangle = \langle a^\dagger a[N-1]_\mu \rangle = \langle [N]_\mu [N-1]_\mu \rangle. \tag{14}
\]

Let us point out the principal difference between deformed and usual, non-deformed, approaches. Due to
the usual oscillator, we have \(a^\dagger a = N\), one-particle distribution is given by \(\langle a^\dagger a \rangle\), and not by \(\langle N \rangle\).

Account of the both relations (14) in eq. (11) implies

\[
\lambda^{(2)}_\mu(K) = \frac{\langle [N]\mu [N-1]\mu \rangle}{\langle [N]\mu \rangle^2} - 1. \tag{15}
\]

Assuming that \(\mu\) is positive and sufficiently small we present the ratio (12) as

\[
[N]_\mu = N - \mu N^2 + \mu^2 N^3 \ldots = N \sum_{s=0}^{\infty} (-\mu N)^s. \tag{16}
\]

After substitution of the expression (16) in eq. (15) the formula for intercept of two-particle momentum correlation function reads:

\[
\lambda^{(2)}_\mu(K) = \frac{\langle N(N-1) \sum_{s=0}^{\infty} (-\mu N)^s \sum_{s=0}^{\infty} (-\mu N-1)^s \sum_{s=0}^{\infty} (-\mu N+1)^s \sum_{s=0}^{\infty} (-\mu N-2)^s \ldots \rangle}{\langle [N]\mu \rangle^2 \langle [N]\mu \rangle^2} - 1. \tag{17}
\]

Let us note that we cannot obtain single Taylor series for the ratio \(\lambda^{(2)}_\mu(K)\) in (15), since the numerator and denominator imply their own averages and, besides, we have no general formula for necessary coefficients.

So, in calculating \(\lambda^{(2)}_\mu(K)\) we deal with series and hence use some approximation. Let us restrict ourselves with the approximation in (16) up to \(\mu^5\). In this case the last formula (17) can be cast to the form

\[
\lambda^{(2)}_\mu(K)|_{\mu^5} = \frac{\sum_{s=0}^{7} (-1)^s \mu^s N^{s+1}}{\langle [N]\mu \rangle^2} - 1. \tag{18}
\]

with the coefficients \(\alpha_s\) as follows:

\[
\begin{align*}
\alpha_1 &= -1 - \mu - 5\mu^2 - \mu^3 - 5\mu^4 - 5\mu^5, \\
\alpha_2 &= 1 + 3\mu + 4\mu^2 + 5\mu^3 + 6\mu^4 + 7\mu^5, \\
\alpha_3 &= -2\mu - 6\mu^2 - 10\mu^3 - 15\mu^4 - 21\mu^5, \\
\alpha_4 &= 3\mu^2 + 10\mu^3 + 20\mu^4 + 35\mu^5, \\
\alpha_5 &= -4\mu^3 - 15\mu^4 - 35\mu^5, \\
\alpha_6 &= 5\mu^4 + 21\mu^5, \\
\alpha_7 &= -6\mu^5.
\end{align*}
\]

Next step of obtaining \(\lambda^{(2)}_\mu(K)\) consists in finding mean values of powers of \(N\). Say, \(\langle N \rangle\) yields (like in [40]):

\[
\langle N \rangle = \frac{Tr Ne^{-\beta \sum_k H_k}}{Tr e^{-\beta \sum_k H_k}} = \sum_{n} \langle n | Ne^{-\beta \sum_k H_k} | n \rangle = \sum_{n} \langle n | e^{-\beta \sum_k H_k} | n \rangle
\]

\[
= \sum_{n} n e^{-\beta \omega n} = \frac{1}{e^{\beta \omega} - 1} \tag{19}
\]

where \(\beta = \frac{1}{kT}\), \(k = 1\). Since in what follows we will use the developed model for description of the intercepts of pion correlation functions, we assume the particle’s energy to be \(\omega = (m^2 + K^2)^{1/2}\). Similarly to (19) we can derive the mean values \(\langle N^k \rangle\), \(k \geq 2\). Let us present those of them which are needed in (18):

\[
\begin{align*}
\langle N \rangle &= \frac{1}{x}, \\
\langle N^2 \rangle &= \frac{1}{x^2} + \frac{2}{x^2}, \\
\langle N^3 \rangle &= \frac{1}{x^3} + \frac{6}{x^3} + \frac{6}{x^3}, \\
\langle N^4 \rangle &= \frac{1}{x^4} + \frac{14}{x^4} + \frac{36}{x^4} + \frac{24}{x^4}, \\
\langle N^5 \rangle &= \frac{1}{x^5} + \frac{30}{x^5} + \frac{150}{x^5} + \frac{240}{x^5} + \frac{120}{x^5}, \\
\langle N^6 \rangle &= \frac{1}{x^6} + \frac{62}{x^6} + \frac{540}{x^6} + \frac{1560}{x^6} + \frac{1800}{x^6} + \frac{720}{x^6}, \\
\langle N^7 \rangle &= \frac{1}{x^7} + \frac{126}{x^7} + \frac{1806}{x^7} + \frac{8400}{x^7} + \frac{16800}{x^7} + \frac{15120}{x^7} + \frac{5040}{x^7}.
\end{align*}
\]

The expressions for mean values of these and higher order powers \(N^m\) are of general form

\[
\langle N^m \rangle = \sum_{r=1}^{m} B^{(m)}_r x^{-r}, \quad x \equiv e^{\beta \omega} - 1. \tag{20}
\]

The coefficients \(B^{(m)}_r\), both for \(m \leq 7\) and for \(8 \leq m \leq 14\), see Appendix A, show very interesting properties, also mentioned in Appendix A: say, they are arranged in a remarkable analog of the well-known Pascal’s triangle.

To summarize: the explicit expression for \(\lambda^{(2)}(m, K)\) in the \(\mu^5\) approximation is given by eq. (18), with the above mean values \(\langle N^k \rangle\). This result covers the lower \(\mu^3\) and \(\mu^4\) orders as well. Using data in Appendix A, similar result can be given for the order \(\mu^6\), and so on.

### 3.2. Intercept of three-particle correlation function

Like in the preceding Subsection, the intercept of three-particle momentum correlation function is expressible in terms of creation and annihilation operators:

\[
\lambda^{(3)}_\mu(K) = \frac{\langle a^\dagger a^\dagger a^\dagger a a a \rangle}{\langle a^\dagger a \rangle^3} - 1, \quad K = k_1 = k_2 = k_3. \tag{21}
\]

Using (13) and (12) the intercept \(\lambda^{(3)}_\mu(K)\) takes the form

\[
\lambda^{(3)}_\mu(K) = \frac{\langle [N]\mu [N-1]\mu [N-2]\mu \rangle}{\langle [N]\mu \rangle^3} - 1. \tag{22}
\]

Account of (16) in (22) leads to general formula for intercept of three-particle momentum correlation function:
Like before, we consider the order $\mu^5$ approximation and obtain the expression for $\lambda^{(3)}_\mu(K)$ in the form

$$
\lambda^{(3)}_\mu(K)|_{\mu^5} = \frac{\sum_{k=1}^8 \eta_k \langle N^k \rangle}{\langle \sum_{r=0}^5 (-1)^r \mu^r N^{r+1} \rangle^3} - 1
$$

(24)

where the coefficients $\eta_k$ are the following:

\begin{align*}
\eta_1 &= 2 + 6\mu + 14\mu^2 + 30\mu^3 + 62\mu^4 + 126\mu^5, \\
\eta_2 &= -3 - 15\mu - 45\mu^2 - 115\mu^3 - 273\mu^4 - 623\mu^5, \\
\eta_3 &= +1 + 12\mu + 55\mu^2 + 180\mu^3 + 511\mu^4 + 1344\mu^5, \\
\eta_4 &= -3\mu - 30\mu^2 - 145\mu^3 - 525\mu^4 - 1652\mu^5, \\
\eta_5 &= 6\mu^2 + 60\mu^3 + 315\mu^4 + 1260\mu^5, \\
\eta_6 &= -10\mu^3 - 105\mu^4 - 602\mu^5, \\
\eta_7 &= 15\mu^4 + 168\mu^5, \\
\eta_8 &= -21\mu^5.
\end{align*}

4. ASYMPTOTICS OF INTERCEPTS OF TWO-AND THREE-PARTICLE CORRELATION FUNCTIONS

Here we study the asymptotical behavior at $\beta \omega \to \infty$ (i.e. at $|K| \to \infty$, constant $T$) of the intercepts $\lambda^{(2)}_\mu$ and $\lambda^{(3)}_\mu$ of the two- and three-particle correlation functions.

Let us find the asymptotical expression for $\lambda^{(2)}_\mu$ within the order $\mu^5$ approximation, see (18). With the account of relevant averages from Appendix A both in the numerator and in the denominator of (18), we derive the result

$$
\lambda^{(2)}_{axs}(\mu)|_{\mu^5} = \frac{1 - 4\mu + 11\mu^2 - 26\mu^3 + 57\mu^4 - 120\mu^5}{1 - 2\mu + 3\mu^2 - 4\mu^3 + 5\mu^4 - 6\mu^5}.
$$

(25)

This expression can be written in the form

$$
\lambda^{(2)}_{axs}(\mu) = \frac{3\mu^8}{1^{\prime}}
$$

where $k = \mu$, covered by agreement (about eq. (18)) we have $\mu^4$ and $\mu$ at $k = 3$; $\mu^2$, covered by agreement (about eq. (18)) we have $\mu^4$ and $\mu$ at $k = 3$; $\mu^2$, covered by agreement at $k = 3$.

Now consider the asymptotics of the three-particle correlation function intercept. Applying the limit $\beta \omega \to \infty$ to eq. (24), with the $\langle N^s \rangle$ taken from Appendix A, we obtain the desired asymptotic formula in $\mu^5$ approximation:

$$
\lambda^{(3)}_{axs}(\mu)|_{\mu^5} = \frac{6(1 - 6\mu + 25\mu^2 - 90\mu^3 + 301\mu^4 - 966\mu^5)}{1 - 3\mu + 6\mu^2 - 10\mu^3 + 15\mu^4 - 21\mu^5} - 1.
$$

(28)

Using this expression (which also contains the order $\mu^3$, $\mu^4$ results) we find the exact asymptotical formula

$$
\lambda^{(3)}_{axs}(\mu) = \frac{6 \sum_{s=0}^{\infty} (-1)^s \left( \frac{1}{2}(3^{s+2} + 1 - 2^{s+2}) \right) \mu^s}{\sum_{r=0}^{\infty} (-1)^r \frac{1}{2} \left( 7(r+1)(r+2) \right) \mu^r} - 1 = \frac{6}{(1 + \mu)(1 + 3\mu)}.
$$

(29)

In fig. 1 we present the two- and three-particle correlation function intercepts $\lambda^{(2)}_\mu$, $\lambda^{(3)}_\mu$ for the order $\mu^5$ approximation, with their asymptotes shown explicitly.

![Graph showing intercepts](image-url)
5. INTERCEPTS IN $\mu$-BOSE GAS MODEL COMPA-RED WITH OTHER MODELS AND EXPERIMENTAL DATA

It is of interest to compare the behavior of the intercepts $\lambda^{(2)}$ and $\lambda^{(3)}$ obtained in the $\mu$-Bose gas, see fig. 1, with main features of such intercepts found within q-Bose gas models \cite{8, 19–21, 28} and plotted in the corresponding figures of those works. As seen from the comparison, the picture is qualitatively the same: with increasing mean momentum, in each model we observe the occurrence of minimum, convexity downwards, then the transition through inflection point to the convexity upwards, and finally asymptotical tending to a constant value strictly lesser than 1 if $q \neq 1$ or $\mu \neq 0$.

Now consider some experimental data, e.g., taken from \cite{33}. It is nice that the experimental data for the two-pion correlation intercepts (4 points corresponding to 4 momentum bins) respect the basic features mentioned in the previous paragraph (the data show stable qualitative picture or "trend", for different colliding nuclei, energies of collision and centralities). In our fig. 2 (left), for illustration, we make a comparison of the curve $\lambda^{(2)}$ from (18) with experimental points \cite{33} corresponding to the most central 62.4 GeV Au+Au collision. In principal, we could get besides qualitative, also better quantitative agreement with data within the considered $\mu$-Bose gas model, although by taking lesser temperature and slightly greater value of $\mu$. Note again that the fig. 2 (left) serves for illustration only, and more detailed and realistic comparative analysis will be given elsewhere.

What concerns the existing data for three-pion correlation intercept, these are obviously insufficient. For that reason we postpone confronting with data of our analytical results on $\lambda^{(3)}(K)$ for future work as well.

Before ending this section, let us note that an important function $r^{(3)}(K)$ was introduced in [41]. It is composed from $\lambda^{(2)}(K)$ and $\lambda^{(3)}(K)$ according to the formula
\[
r^{(3)}(K) = \frac{1}{2} \frac{\lambda^{(3)}(K) - 3\lambda^{(2)}(K)}{(\lambda^{(2)}(K))^{3/2}}.
\]

Due to special form of the ratio, the function (30) provides improved purity (on the experimental side). Indeed, many unwanted distortions which especially for low $K$ may affect $\lambda^{(2)}(K)$ and $\lambda^{(3)}(K)$ when considered separately, in $r^{(3)}(K)$ are mutually canceled.

In case of q-Bose gas this function was studied in \cite{8} by one of us. From our analytical results (18) and (24) derived in the $\mu$-Bose gas model, the formula for the function $r^{(3)}_{\mu}(K)$ reads
\[
r^{(3)}_{\mu}(K) = \frac{\sum_{\nu} \nu \langle N^k \rangle}{\sum_{\nu=0}^{\nu-1} \nu \langle N^k \rangle} - 3 \frac{\sum_{\nu} \alpha_k \langle N^k \rangle}{\sum_{\nu=0}^{\nu-1} \nu \langle N^k \rangle} + 2 \left( \frac{\sum_{\nu} \alpha_k \langle N^k \rangle}{\sum_{\nu=0}^{\nu-1} \nu \langle N^k \rangle} - 1 \right)^{3/2}.
\]

Finally, using exact asymptotical formulas (27) and (29) we find the exact expression for the $r^{(3)}_{\mu}(K)$ asymptotics:
\[
r_{\mu}^{(3)}(\mu) = \frac{1 - \mu}{1 + 3\mu} \sqrt{1 + 2\mu}.
\]

Fig. 2, right, gives the plot of $r^{(3)}_{\mu}(K)$ along with its asymptotics. As seen, small raise of the parameter $\mu$ results in basic change of the properties of $r^{(3)}_{\mu}(K)$: from positive to negative sign. But, as seen from (32) its asymptotical value is always positive since $\mu < 1$.

6. CONCLUSIONS

The $\mu$-deformed oscillator model is a peculiar one possessing novel interesting features with regards to the other, more popular and better known models of deformed oscillators. The specifics of the $\mu$-oscillator is rooted in its defining structure function whose non-polynomial, rational dependence on the number operator $N$ causes nontrivial properties of this oscillator especially those mentioned at the end of sect. 2.

The energy spectrum of $\mu$-oscillator does not form \cite{18} the usual Fibonacci sequence. For that reason we treated the $\mu$-oscillator as the one obeying a generalization of Fibonacci relation and called it quasi-Fibonacci oscillator \cite{18}. We hope that the deformed oscillators possessing quasi-Fibonacci property may be useful not only in the present context of multiparticle correlations of pions (kaons) produced and registered in the experiments at RHIC or LHC, but also, say, for description of the properties of some quasi-periodic chains \cite{42}.

In the present paper, the $\mu$-oscillators are employed for elaborating on their base the corresponding $\mu$-Bose gas model. The results concerning one-, two-particle distribution functions and the two-particle correlation function intercept obtained in $\mu$-Bose gas model are derived by using definite ($\mu^5$) order of approximation. From the
We believe the situation will soon improve. Needed in order to make choice in favor of a particular case, see fig. 2, experimental data on intercepts of pions, as shown in specific expression have been obtained for the intercept \( \lambda^{(3)} \) of three-particle correlation function.

Let us emphasize ones more that the obtained analytical formulas for \( \lambda^{(2)}(K) \), \( \lambda^{(3)}(K) \) and \( r^{(3)}(K) \) provide explicit and nontrivial dependence on the particles’ mean momentum. This allows to make a comparison with experimental data on intercepts of pions, as shown in special case, see fig. 2, left. Certainly, more detailed data are needed in order to make choice in favor of a particular model and we believe the situation will soon improve.

We hope that the above \( \mu \)-Bose gas model constructed by employing \( \mu \)-oscillators will find interesting applications in different branches of nonlinear quantum physics: in description of high-energy quantum physics phenomena, in \(^4\text{He} \) theory, in quantum optics, etc.

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APPENDIX A. EXPLICIT EXPRESSIONS FOR THE MEAN VALUES $\langle N^m \rangle$

The expressions for mean values of the powers $N^m$ are given as $\langle N^m \rangle = \sum_{r=1}^{m} B_r^{(m)} x^{-r}$, $x = e^{\beta \omega} - 1$, with the coefficients $B_r^{(m)}$, $m = 8, ..., 14$, placed in the Table 1. For $m=1, ..., 7$ see the expressions preceding eq. (20).

**TABLE 1. The coefficients $B_r^{(m)}$ for mean values of the powers $N^m$ (8 ≤ m ≤ 14)**

| $r \setminus m$ | 8    | 9    | 10   | 11   | 12   | 13   | 14   |
|------------------|------|------|------|------|------|------|------|
| 1                | 1    | 1    | 1    | 1    | 1    | 1    | 1    |
| 2                | 254  | 510  | 1022 | 2046 | 4094 | 8190 | 16382|
| 3                | 5796 | 18150| 55980| 171006| 519156| 1569750| 4733820|
| 4                | 40824| 186480| 818520| 3498000| 14676024| 60780720| 249401880|
| 5                | 126000| 834120| 5103000| 29607600| 165528000| 901020120| 4809004200|
| 6                | 191520| 1905120| 16435440| 129230640| 953029440| 6711344640| 45674188560|
| 7                | 141120| 2328480| 29635200| 322494480| 3162075840| 28805736960| 248619571200|
| 8                | 40320| 1451520| 30240000| 479001600| 6411968640| 76592355840| 843184742400|
| 9                | 362880| 1632960| 419126400| 8083152000| 130456085760| 1863435974400|
| 10               | 362880| 19958400| 6187104000| 142702560000| 2731586457600|
| 11               | 39916800| 2634508800| 2637143308800|
| 12               | 479001600| 37362124800| 1612798387200|
| 13               | 6227020800| 566658892800|
| 14               | 87178291200|

The coefficients $B_r^{(m)}$ in $\langle N^x \rangle$ and an analog of Pascal’s triangle

Here we examine in some detail the coefficients appearing in the mean values of powers of $N$, which determine the intercepts $\lambda^{(2)}$ or $\lambda^{(3)}$. Let us specialize (20) for $m = 1, 2, ..., 7$. Then, e.g., for $m = 5$ we have: $B_1^{(5)} = 1$, $B_2^{(5)} = 30$, $B_3^{(5)} = 150$, $B_4^{(5)} = 240$, $B_5^{(5)} = 120$. Consider the correspondence between the coefficients $B_r^{(m)}$ in $\langle N^m \rangle$ and the entries of some analog of Pascal’s triangle. For the classical Pascal’s triangle [43], there is a "rule": each coefficient in $m$-th row equals the sum of two adjacent coefficients in the $(m - 1)$-th row. Now arrange our coefficients as the entries of some triangle. Then, the triangle with entries $B_r^{(m)}$, $r = 1, ..., m$, for seven rows ($m \leq 7$) looks as

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 6 & 36 & 240 & 1560 & 1800 & 720 \\
1 & 6 & 150 & 240 & 1800 & 720 \\
1 & 126 & 1800 & 8400 & 16800 & 15120 & 5040 \\
\end{array}
\]

With regards to the famous Pascal’s triangle, here we have two essential distinctions. First, now we encounter unities only on the left hand side. Second, the above mentioned "rule" is to be modified. That is, each entry $B_r^{(m)}$ in $m$-th row equals the sum of two adjacent entries above it, multiplied by $r$, the position number of $B_r^{(m)}$:

\[
B_r^{(m)} = r (B_{r-1}^{(m-1)} + B_{r-2}^{(m-1)}).
\]

It is worth to note that in the mean value $\langle N^m \rangle$, the last coefficient fixed by $r = m$ is given by the formula

\[
B_r^{(m)} = m!.
\]

Here $m$ is both the power of $N$ and the row number, while $r$ is the number of entry in $m$-th row. Likewise, the entries with $r = m - 1$ in the $m$-th row, and the second entry of each $m$-th row are given as

\[
B_{r=m-1}^{(m)} = \frac{(m-1)}{2} m!, \quad B_{r=2}^{(m)} = 2(2^{m-1} - 1) = 2^m - 2.
\]
Let us note that (33) and the first expression in (34) coincide with the Lah numbers $L(m, 1)$ and $L(m, 2)$ respectively, while the second formula in (34) coincides with the Mersenne numbers (see [43]).