A MEAN CURVATURE FLOW ALONG A KÄHLER-RICCI FLOW

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Abstract. Let \((M, g)\) be a Kähler surface, and \(\Sigma\) an immersed surface in \(M\). The Kähler angle of \(\Sigma\) in \(M\) is introduced by Chern-Wolfson [9]. Let \((M, \overline{\nabla}_t)\) evolve along the Kähler-Ricci flow, and \(\Sigma_t\) in \((M, \overline{\nabla}_t(x))\) evolve along the mean curvature flow. We show that the Kähler angle \(\alpha(t)\) satisfies the evolution equation:

\[
\frac{\partial}{\partial t} - \Delta \cos \alpha = |\nabla J_{\Sigma_t}|^2 \cos \alpha + R \sin^2 \alpha \cos \alpha,
\]

where \(R\) is the scalar curvature of \((M, \overline{\nabla}_t(x))\).

The equation implies that, if the initial surface is symplectic (Lagrangian), then along the flow, \(\Sigma_t\) is always symplectic (Lagrangian) at each time \(t\), which we call a symplectic (Lagrangian) Kähler-Ricci mean curvature flow.

In this paper, we mainly study the symplectic Kähler-Ricci mean curvature flow.

1. Introduction

Suppose that \((M, J, \omega, \overline{\nabla})\) is a Kähler surface. Let \(\Sigma\) be a compact oriented real surface which is smoothly immersed in \(M\), the Kähler angle \(\alpha\) of \(\Sigma\) in \(M\) is defined by Chern-Wolfson ([9])

\[
\omega|_\Sigma = \cos \alpha d\mu_\Sigma
\]

where \(d\mu_\Sigma\) is the area element of \(\Sigma\) of the induced metric from \(\overline{\nabla}\). We say that \(\Sigma\) is a symplectic surface if \(\cos \alpha > 0\), \(\Sigma\) is a holomorphic curve if \(\cos \alpha \equiv 1\) and \(\Sigma\) is a Lagrangian surface if \(\cos \alpha \equiv 0\).

If \(M\) is a Kähler-Einstein surface, a symplectic mean curvature flow in \(M\) was studied by Chen-Tian [6], Chen-Li [4], Wang [32] and Han-Li [18], etc. The main point is that, along the mean curvature flow, the Kähler angle satisfies a parabolic equation. However, without the Einstein condition, one can not have the nice equation. In this paper we find that, if \(M\) evolves along the Kähler-Ricci flow, the Kähler angle satisfies the same parabolic equation.

Now let \(\overline{\nabla}(t)\) evolve along the Kähler-Ricci flow on \(M\) and \(\Sigma\) evolve along the mean curvature flow in \((M, \overline{\nabla}(t))\), that is,

\[
\begin{align*}
\frac{\partial}{\partial t}\overline{\nabla}(t, \cdot) &= -\overline{\text{Ric}}(g(t, \cdot)) + \overline{\nabla}(t, \cdot) \\
\frac{d}{dt}F_t &= \overline{H} \\
\overline{\nabla}(0, \cdot) &= \overline{\nabla}_0 \\
F(\cdot, 0) &= F_0
\end{align*}
\]
where $F_0: \Sigma_0 \to (M, g_0)$ is the initial immersion and $\vec{H}$ is the mean curvature vector of $\Sigma_t$ in $(M, \vec{g}(t))$, and $r$ is a constant. We call it a Kähler-Ricci mean curvature flow, denoted by $(M, \vec{g}(t), \Sigma_t)$.

The Ricci flow was introduced by Hamilton [12] in order to study the famous Poincaré conjecture, which was finally achieved by Perelman ([24], [25], [26]). The Kähler-Ricci flow was introduced by Cao [1] to study Calabi conjecture, which was studied by many authors (see [8], [30], [28], [29]). The mean curvature flow was intensively studied by Huisken [16], [17]. Recall that, if $C_1(M) < 0 (= 0, > 0)$, choosing $r = -2 (0, 2)$ and the initial Kähler form with $c_1(M)$ as its Kähler class, Cao [1] proved that the Kähler-Ricci flow exists globally, and converges to a Kähler-Einstein metric at infinity in the case that $C_1(M) \leq 0$.

The main point of this paper is to derive the evolution equation of the Kähler angle along the Kähler-Ricci mean curvature flow. The purpose is to find symplectic minimal surface, and especially holomorphic curves in Kähler surfaces.

Assume that $\Sigma_t$ evolves along the Kähler-Ricci mean curvature flow in $(M, \vec{g}(t))$. We show that the evolution equation of $\cos \alpha$ is

$$
\left( \frac{\partial}{\partial t} - \Delta \right) \cos \alpha = |\nabla J_{\Sigma_t}|^2 \cos \alpha + R \sin^2 \alpha \cos \alpha,
$$

where $\vec{R}$ is the scalar curvature of $(M, \vec{g}(t))$, and $|\nabla J_{\Sigma_t}|^2$ will be defined in (4.1).

The same equation is obtained independently by Chen-Li [4] and Wang [32] (also see [6]), for a symplectic mean curvature flow in a Kähler-Einstein surface. By the maximum principle for parabolic equations, we see that, if the initial surface is symplectic (Lagrangian), then along the Kähler-Ricci mean curvature flow $\Sigma_t$ is always symplectic (Lagrangian), which we call a symplectic (Lagrangian) Kähler-Ricci mean curvature flow. We will show in a forthcoming paper [20] that, Lagrangian is preserved by Kähler-Ricci mean curvature flow in any dimension.

In this paper, we will show that a symplectic Kähler-Ricci mean curvature flow does not develop Type I singularity under the assumption that Kähler-Ricci flow does not develop any singularity. If the Kähler surface is sufficiently close to a Kähler-Einstein surface and the initial surface is sufficiently close to a holomorphic curve, then the symplectic Kähler-Ricci mean curvature flow exists globally and converges to a holomorphic curve in a Kähler-Einstein surface at infinity.

Suppose that $M = M_1 \times M_2$ where $M_1, M_2$ are Riemann surfaces with unit Kähler forms $\omega_1, \omega_2$. If $(M_1, \vec{g}_1(0))$ and $(M_2, \vec{g}_2(0))$ have the same average scalar curvature, and the initial surface is a graph with $\langle e_1 \times e_2, \omega_1 \rangle > \frac{\sqrt{2}}{2}$ where $e_1, e_2$ is an orthonormal frame of the initial surface, then the Kähler-Ricci mean curvature flow exists globally. In addition, if the scalar curvature of $M_1, M_2$ is positive, then the Kähler-Ricci mean curvature flow converges to a totally geodesic surface at infinity. The same result was also proved by Chen-Li-Tian [7] and Wang [32] in the case that $M_1, M_2$ have the same constant curvature.

Throughout this paper we will adopt the following ranges of indices:

$$A, B, \ldots = 1, \ldots, 4,$$
\[ \alpha, \beta, \gamma, \cdots = 3, 4, \]
\[ i, j, k, \cdots = 1, 2. \]

2. Short Time Existence

In this section, we show the short time existence for the Kähler-Ricci mean curvature flow (1.2).

**Theorem 2.1.** The evolution equation (1.2) has a solution \((M, g(t), \Sigma_t)\) for a short time with any smooth compact initial surface \(\Sigma_0\) at \(t = 0\), that is, the solution of (1.2) exists on a maximum time interval \([0, T)\).

**Proof.** It is well-known that there exists \(T_1 > 0\) such that the Ricci flow exists on \([0, T_1)\). Let \(\Sigma\) evolves by the mean curvature flow in \((M, g(t))\) for \(t < T_1\). Now we use a trick of De Turck [11] to prove the short time existence of \(g(t)\).

Recall the Gauss-Weingarten equation
\[ \frac{\partial^2 F^\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial F^\alpha}{\partial x^k} + \Gamma^\alpha_{\rho\sigma} \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j} = h^\alpha_{ij} v^\alpha, \]
\(\Gamma^k_{ij}\) is the Christoffel symbol of \(g(t)\) and \(\Gamma^\alpha_{\rho\sigma}\) is the Christoffel symbol of \(\bar{g}(t)\). Note that,
\[ \Delta_{g(t)} F = g^{ij} \nabla_i \nabla_j F \]
\[ = g^{ij} \left( \frac{\partial^2 F}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial F}{\partial x^k} \right). \]

So the mean curvature flow equation can be written as
\[ \frac{\partial F^\alpha}{\partial t} = \Delta_{g(t)} F^\alpha + g^{ij} \Gamma^\alpha_{\rho\sigma}(t) \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j} \]
which is not the strictly parabolic equation. In order to apply the standard theory of strictly parabolic equation to get short time existence, we use a trick of De Turck by modifying the flow through a diffeomorphism of the parameter space of \(\Sigma\). We consider the following equation
\[ \frac{\partial \tilde{F}^\alpha}{\partial t} = \Delta_{g(t)} \tilde{F}^\alpha + g^{ij} \Gamma^\alpha_{\rho\sigma}(t) \frac{\partial \tilde{F}^\rho}{\partial x^i} \frac{\partial \tilde{F}^\sigma}{\partial x^j} + v^k \nabla_k \tilde{F}^\alpha, \quad (2.1) \]
where the vector field \(v^k\) will be chosen to make the equation strictly parabolic. In fact, if \(\tilde{F}(y, t)\) is a solution of (2.1), then
\[ F(x, t) = \tilde{F}(y(x, t), t) \]
satisfies the equation
\[ \frac{\partial F^\alpha}{\partial t} = \frac{\partial \tilde{F}^\alpha}{\partial t} + \nabla_k \tilde{F}^\alpha \cdot \frac{dy^k}{dt} \]
\[ = \Delta_{g(t)} \tilde{F}^\alpha + g^{ij} \Gamma^\alpha_{\rho\sigma}(t) \frac{\partial \tilde{F}^\rho}{\partial x^i} \frac{\partial \tilde{F}^\sigma}{\partial x^j} + (v^k + \frac{dy^k}{dt}) \nabla_k \tilde{F}^\alpha \]
\[ = \Delta_{g(t)} F^\alpha + g^{ij} \Gamma^\alpha_{\rho\sigma}(t) \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j}. \]
by choosing
\begin{align}
\begin{cases}
\frac{dy^k}{dt} &= -v^k(x, t), \\
y^k|_{t=0} &= x^k.
\end{cases}
\end{align}
(2.2)

Now we pick
\[ v^k = g^{ij}(\Gamma^k_{ij}(t) - \Gamma^k_{ij}(0)). \]

The equation (2.1) becomes
\[ \frac{\partial \tilde{F}_\alpha}{\partial t} = g^{ij}\left\{ \frac{\partial^2 \tilde{F}_\alpha}{\partial x^i \partial x^j} - \Gamma^k_{ij}(0) \frac{\partial \tilde{F}_\alpha}{\partial x^k} \right\} + g^{ij}\Gamma^\rho_{\rho\sigma}(t) \frac{\partial \tilde{F}_\rho}{\partial x^i} \frac{\partial \tilde{F}_\sigma}{\partial x^j} \]
which is a strictly parabolic equation. Since \( \Gamma^\rho_{\rho\sigma}(t) \) is uniformly bounded in \([0, T)\), then by the standard theory of parabolic equations we get the short time existence of the mean curvature flow (c.f. [22]). Thus there exists \( T > 0 \) such that the Kähler-Ricci mean curvature flow exists on \([0, T)\).

Q. E. D.

If the Kähler-Ricci mean curvature flow blows up at \( T \), there are two possibilities. One is the Kähler-Ricci flow blows up at \( T \), another one is the mean curvature flow blows up at \( T \). Let’s state some fundamental results regarding the singularity of the Ricci flow and the mean curvature flow.

**Theorem 2.2.** [27] If the Ricci curvature is uniformly bounded under the Ricci flow \( \frac{\partial}{\partial t} g_{ij} = -2R_{ij} \) for all times \( t \in [0, T) \), then the solution can be extended beyond \( T \).

**Theorem 2.3.** [16] If the second fundamental form is uniformly bounded under the mean curvature flow \( \frac{\partial F}{\partial t} = \vec{H} \) for all times \( t \in [0, T) \), then the solution can be extended beyond \( T \).

### 3. Evolution Equations

In this section the evolution equations of the metric and the second fundamental form of \( \Sigma_t \) will be derived along the Kähler-Ricci mean curvature flow (1.2). In terms of coordinates \( \{x^i\} \) on \( \Sigma_t \) and coordinates \( \{y^A\} \) on \((M, \overline{g}(t))\), the metric of \( \Sigma \) can be expressed as follows:
\[ g_{ij}(x, t) = \overline{g}_{AB}(F(x, t), t) \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j}. \]

**Lemma 3.1.** The metric of \( \Sigma_t \) satisfies the evolution equation
\[ \frac{\partial}{\partial t} g_{ij} = -2H^\alpha h_{ij}^\alpha - \overline{R}_{ij} + \frac{r}{2} g_{ij}. \]

**Proof.** It is clear that
\[ \frac{\partial}{\partial t} g_{ij} = \overline{g}_{AB,C} \frac{\partial F^C}{\partial t} \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j} + \frac{\partial}{\partial t} \overline{g}_{AB} \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j} + 2 \overline{g}_{AB}(t) \frac{\partial}{\partial t} \left( \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j} \right) \]
\[ = -\text{Ricc}(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j}) + \frac{r}{2} \overline{g}_{AB} \frac{\partial F^A}{\partial x^i} \frac{\partial F^B}{\partial x^j} - 2H^\alpha h_{ij}^\alpha. \]
\[ \frac{d}{dt} [\tilde{R}_{i\bar{j}}^\alpha] = -2\tilde{R}_{i\bar{j}k}^{\alpha}h_{\bar{k}}^\alpha + \Delta h_{i\bar{j}}^\alpha + \nabla_k \tilde{R}_{\alpha ik} + \nabla_{\bar{j}} \tilde{R}_{\alpha ik} \]

Q. E. D.

From the evolution equation of the metric we can get the evolution equation of the area element.

**Corollary 3.2.** Set \( \tilde{R} = \frac{1}{2} g^{i\bar{j}} \tilde{R}_{i\bar{j}} \), the area element of \( \Sigma_t \) satisfies the following equation.

\[ \frac{d}{dt} d\mu_t = (-|H|^2 - \tilde{R} + r) d\mu, \]  (3.2)

and consequently,

\[ \frac{d}{dt} (e^{-rt} \int_{\Sigma_t} d\mu_t) = -e^{-rt} \int_{\Sigma_t} (|H|^2 + \tilde{R}) d\mu. \]  (3.3)

**Lemma 3.3.** Under the flow (1.2), the second fundamental form \( h_{i\bar{j}}^\alpha \) satisfies the following equation.

\[ \begin{align*}
\frac{d}{dt} h_{i\bar{j}}^\alpha &= \Delta h_{i\bar{j}}^\alpha + \nabla_k \tilde{R}_{\alpha ik} + \nabla_{\bar{j}} \tilde{R}_{\alpha ik} \\
&\quad - 2\tilde{R}_{i\bar{jk}}h_{\bar{k}}^\alpha + 2\tilde{R}_{\alpha\bar{jk}}h_{\bar{k}}^\alpha + 2\tilde{R}_{\alpha\beta\bar{jk}}h_{i\bar{k}}^\beta \\
&\quad - \tilde{R}_{\alpha\beta\bar{k}}h_{i\bar{k}}^\beta - \tilde{R}_{i\bar{k}}h_{\alpha\bar{k}} + \tilde{R}_{\alpha\beta\bar{k}}h_{i\bar{k}}^\beta \\
&\quad - h_{i\bar{m}}(H\gamma h_{m\bar{j}}^\gamma - h_{m\bar{k}}h_{i\bar{j}}^\gamma) \\
&\quad - h_{i\bar{m}}(h_{m\bar{j}}h_{i\bar{k}}^\gamma - h_{m\bar{k}}h_{i\bar{j}}^\gamma) \\
&\quad - h_{i\bar{k}}(h_{i\bar{\alpha}}h_{i\bar{k}} - h_{i\bar{\beta}}h_{i\bar{j}}) \\
&\quad - h_{i\bar{j}}(h_{i\bar{\alpha}k}h_{i\bar{k}} - h_{i\bar{\beta}k}h_{i\bar{j}}) \\
&\quad - h_{i\bar{k}}(h_{i\bar{\alpha}k}H + h_{i\bar{\beta}k}b_{\alpha\beta}^\gamma) \\
&\quad - \tilde{R}_{\alpha\beta\bar{j}}h_{i\bar{j}}^\beta + \frac{r}{2} h_{i\bar{j}}^\alpha - \frac{1}{2}(\nabla_i \tilde{R}_{\bar{j}\alpha} + \nabla_{\bar{j}} \tilde{R}_{i\alpha} - \nabla_{\bar{i}} \tilde{R}_{\alpha\bar{j}}). 
\end{align*} \]  (3.4)

where \( \nabla \) is the covariant derivative of \( (M, \tilde{g}(t)) \) and \( b_{\alpha\beta}^\gamma = \langle \frac{d}{dt} e_{\alpha}, e_{\beta} \rangle \). In particular, \( |A|^2 \) satisfies the following equation along the flow (1.2).

\[ \frac{d}{dt} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2[\nabla_k \tilde{R}_{\alpha ik} + \nabla_{\bar{j}} \tilde{R}_{\alpha ik}]h_{i\bar{j}}^\alpha \\
- 4\tilde{R}_{i\bar{jk}}h_{\bar{k}}^\alpha h_{i\bar{j}}^\alpha + 8\tilde{R}_{\alpha\bar{jk}}h_{\bar{k}}^\alpha h_{i\bar{j}}^\alpha - 4\tilde{R}_{\alpha\beta\bar{k}}h_{i\bar{k}}^\beta h_{i\bar{j}}^\alpha + 2\tilde{R}_{\alpha\beta\bar{k}}h_{i\bar{\gamma}}h_{i\bar{\gamma}}^\alpha \\
+ 2 \sum_{\alpha, \gamma, i, m} (h_{i\bar{m}}h_{\bar{k}}^\gamma - h_{m\bar{k}}h_{i\bar{k}}^\gamma)^2 + 2 \sum_{i, j, m, k} (h_{i\bar{m}}h_{i\bar{m}}^\alpha h_{i\bar{m}}^\alpha)^2 \\
+ 2\tilde{R}_{i\bar{k}}h_{i\bar{k}}h_{i\bar{k}}^\alpha - 2\tilde{R}_{\alpha\beta\bar{k}}h_{i\bar{k}}^\beta h_{i\bar{k}}^\alpha - r|A|^2 \\
- \frac{1}{2}h_{i\bar{j}}(\nabla_i \tilde{R}_{\bar{j}\alpha} + \nabla_{\bar{j}} \tilde{R}_{i\alpha} - \nabla_{\bar{i}} \tilde{R}_{\alpha\bar{j}}). \]  (3.5)

More generally, we have

\[ \frac{d}{dt} |\nabla^m A|^2 = \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A \ast \nabla^j A \ast \nabla^k A \ast \nabla^m A, \]  (3.7)
where we denote by $\nabla^i A * \nabla^j A * \nabla^k A$ the linear combination of $\nabla^i A, \nabla^j A$ and $\nabla^k A$.

Proof. By (7.4) in [32], the Laplacian of $h^\alpha_{ij}$ satisfies
\[
\Delta h^\alpha_{ij} = H^\alpha_{ij} - (\nabla_k \bar{R})_{aijk} - (\nabla_j \bar{R})_{aijk} + 2\mathcal{R}_{lijk} h^\alpha_{ik} - 2\mathcal{R}_{\alpha\beta jk} h^\beta_{jk} - 2\mathcal{R}_{\alpha\beta ik} h^\beta_{jk} + (\mathcal{R}_{j} - 2\bar{g}_{ij}) h^\alpha_{ik} + (\mathcal{R}_{i} - 2\bar{g}_{ij}) h^\alpha_{ik} + h^\alpha_{im} (H^\gamma h^\gamma_{mj} - h^\gamma_{mk} h^\gamma_{jk}) + h^\alpha_{mk} (h^\gamma_{mj} h^\gamma_{ik} - h^\gamma_{mk} h^\gamma_{ij}) + h^\beta_{ik} (h^\beta_{ij} h^\alpha_{ik} - h^\beta_{ik} h^\alpha_{ij}).
\]
(3.8)

Now we compute $\frac{d}{dt} h^\alpha_{ij}$. Since $h^\alpha_{ij} = \langle \nabla_i e_j, e_\alpha \rangle_\mathcal{g} = \bar{g}_{\alpha A} (\nabla_i e_j)^A e_\alpha$, we get
\[
\frac{d}{dt} h^\alpha_{ij} = \frac{d}{dt} \bar{g}_{\alpha A} (\nabla_i e_j)^A e_\alpha + \langle \frac{\partial}{\partial t} (\nabla)(e_i e_j), e_\alpha \rangle + \langle \nabla_H \nabla_i e_j, e_\alpha \rangle + \langle \nabla_i e_j, \frac{\partial}{\partial t} e_\alpha \rangle.
\]
(3.9)

Using (7.5) in [32], we have
\[
\langle \nabla_H \nabla_i e_j, e_\alpha \rangle = H^\alpha_{ij} - H^\beta h^\beta_{ik} h^\alpha_{jk} - H^\beta \bar{R}_{\beta jia}.
\]

Using (2.24) in [10], we get that
\[
\langle \frac{\partial}{\partial t} \nabla)(e_i e_j), e_\alpha \rangle = -\frac{1}{2} (\nabla_i \mathcal{R}_{jia} + \nabla_j \mathcal{R}_{ia} - \nabla_\alpha \mathcal{R}_{ij}).
\]

Set $\langle \frac{d}{dt} e_\alpha, e_\beta \rangle = b^\beta_\alpha$. Putting these equations into (3.9) we obtain that
\[
\frac{d}{dt} h^\alpha_{ij} = -\mathcal{R}_{\alpha \beta} - \frac{r}{2} \bar{g}_{\alpha \beta} h^\beta_{ij} - \frac{1}{2} (\nabla_i \mathcal{R}_{jia} + \nabla_j \mathcal{R}_{ia} - \nabla_\alpha \mathcal{R}_{ij}) + H^\alpha_{ij} - H^\beta h^\beta_{ik} h^\alpha_{jk} - H^\beta \bar{R}_{\beta jia} + b^\beta_\alpha.
\]
(3.10)

Combining equations (3.8) and (3.10), we get the parabolic equation (3.4) for $h^\alpha_{ij}$. Since $|A|^2 = g^{ik} g^{\alpha j} h^\alpha_{ij} h^\alpha_{ji}$, by (3.1) we have,
\[
\frac{d}{dt} |A|^2 = 2 (\frac{d}{dt} g^{ik}) h^\alpha_{ij} h^\alpha_{ji} + 2 (\frac{d}{dt} h^\alpha_{ij}) h^\alpha_{ij} = 2 (2H^\beta h^\beta_{ik} + \mathcal{R}_{ik} - r g_{ik}) h^\alpha_{ij} h^\alpha_{ji} + 2 h^\alpha_{ij} \Delta h^\alpha_{ij} + \nabla_k \mathcal{R}_{aijk} + \nabla_j \mathcal{R}_{aijk} - 2\mathcal{R}_{ij k} h^\alpha_{ik} + 2\mathcal{R}_{\alpha jk} h^\beta_{jk} + 2\mathcal{R}_{\alpha ik} h^\beta_{jk} - \mathcal{R}_{l ijk} h^\alpha_{ij} - \mathcal{R}_{l ijk} \mathcal{R}_{l \alpha jk} + \mathcal{R}_{l \alpha jk} \mathcal{R}_{l \alpha jk} - h^\alpha_{im} (H^\gamma h^\gamma_{mj} - h^\gamma_{mk} h^\gamma_{jk}) - h^\alpha_{mk} (h^\gamma_{mj} h^\gamma_{ik} - h^\gamma_{mk} h^\gamma_{ij}) - h^\beta_{ik} (h^\beta_{ij} h^\alpha_{ik} - h^\beta_{ik} h^\alpha_{ij}) - h^\beta_{jik} (h^\beta_{ij} H^\beta + h^\beta_{ij}).
\[-R_{\alpha\beta} h^\beta_{ij} + \frac{r}{2} h^\alpha_{ij} - \frac{1}{2}(\nabla_i R_{j\alpha} + \nabla_j R_{i\alpha} - \nabla_{\alpha} R_{ij}).\]

Using
\[
\Delta(h^\alpha_{ij})^2 = 2|\nabla A|^2 + 2 h^\alpha_{ij} \Delta h^\alpha_{ij},
\]
and the antisymmetric of \(b^\alpha_{\beta}, \langle e^\beta, \nabla_H e_\alpha \rangle \), we get that
\[
\frac{d}{dt}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2 R^\alpha_{ij} h^\alpha_{ik} h^\alpha_{kj} - 2 R^\alpha_{\beta jk} h^\alpha_{ik} + 2 R^\alpha_{\alpha kik} h^\alpha_{jk} - \frac{1}{2}(\nabla_i R_{j\alpha} + \nabla_j R_{i\alpha} - \nabla_{\alpha} R_{ij}).
\]

The fourth order terms can be calculated as in [32]. The equation (3.7) can be proved similarly.

Q. E. D.

The following theorem follows easily from Theorem 2.2, Theorem 2.3 and (3.7).

**Theorem 3.4.** If the Ricci curvature of \(\mathcal{F}(t)\) is uniformly bounded, and the second fundamental form of \(\Sigma_t\) is uniformly bounded under the Kähler-Ricci mean curvature flow for all time \(t \in [0, T)\), then the solution can be extended beyond \(T\).

4. THE EVOLUTION OF THE KÄHLER ANGLE ALONG THE FLOW

This is the main section of this paper, in which we will derive the evolution equation of the Kähler angle along the Kähler-Ricci mean curvature flow.

Let \((M, J)\) be a Kähler surface. Let \(\mathcal{F}(t)\) evolve along the Kähler-Ricci flow on \(M\), and \(\Sigma_t\) evolve along the mean curvature flow in \((M, \mathcal{F}(t))\). Choose an orthonormal basis \(\{e_1, e_2, e_3, e_4\}\) on \((M, \mathcal{F}(t))\) along \(\Sigma_t\) such that \(\{e_1, e_2\}\) is the basis of \(\Sigma_t\). Let \(J_{\Sigma_t}\) be an almost complex structure in a tubular neighborhood of \(\Sigma_t\) on \((M, \mathcal{F}(t))\) with
\[
\begin{align*}
J_{\Sigma_t} e_1 &= e_2 \\
J_{\Sigma_t} e_2 &= -e_1 \\
J_{\Sigma_t} e_3 &= e_4 \\
J_{\Sigma_t} e_4 &= -e_3.
\end{align*}
\]

It is proved in [4] that
\[
|\nabla J_{\Sigma_t}|^2 = |h^4_{ik} + h^3_{2k}|^2 + |h^4_{2k} - h^3_{1k}|^2 \\
\geq \frac{1}{2} |H|^2.
\]
Theorem 4.1. Let \((M, \overline{\mathcal{g}}(t), \Sigma_t)\) be a Kähler-Ricci mean curvature flow. Then the evolution equation of \(\cos \alpha\) is
\[
\left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha = |\nabla J_{\Sigma_t}|^2 \cos \alpha + \overline{R} \sin^2 \alpha \cos \alpha,
\]
where \(\overline{R}\) is the scalar curvature of \((M, \overline{\mathcal{g}}(t))\). As a corollary, if the initial surface \(\Sigma_0\) is symplectic, then along the flow, at each time \(t\), \(\Sigma_t\) is symplectic, and if the initial surface \(\Sigma_0\) is Lagrangian, then along the flow, at each time \(t\), \(\Sigma_t\) is Lagrangian.

Proof. Choose an orthonormal basis \(\{e_1, e_2, e_3, e_4\}\) on \((M, \overline{\mathcal{g}}(t))\) along \(\Sigma_t\) such that \(\{e_1, e_2\}\) is the basis of \(\Sigma_t\) and \(\omega_t\) takes the form
\[
\omega_t = \cos \alpha u_1 \wedge u_2 + \cos \alpha u_3 \wedge u_4 + \sin \alpha u_1 \wedge u_3 - \sin \alpha u_2 \wedge u_4,
\]
where \(\{u_1, u_2, u_3, u_4\}\) is the dual basis of \(\{e_1, e_2, e_3, e_4\}\).

Using the evolution equation of the metric (3.1) and (4.4), we get that
\[
\frac{\partial}{\partial t} \cos \alpha = \frac{\partial}{\partial t} \frac{\omega(e_1, e_2)}{\sqrt{\det(g_{ij})}} = \frac{\partial}{\partial t} \frac{\langle J e_1, e_2 \rangle_{\overline{\mathcal{g}}}^\alpha e_2}{\sqrt{\det(g_{ij})}}
\]
\[
= - \overline{Ric}(J e_1, e_2) + \frac{r}{2} \cos \alpha + \sin \alpha (H^4, 1 + H^3, 2) - |H|^2 \cos \alpha
\]
\[
+ \frac{1}{2} \cos \alpha (\overline{R}_{11} + \overline{R}_{22}) - \frac{r}{2} \cos \alpha + |H|^2 \cos \alpha
\]
\[
= - \overline{Ric}(J e_1, e_2) + \sin \alpha (H^4, 1 + H^3, 2) + \frac{1}{2} \cos \alpha (\overline{R}_{11} + \overline{R}_{22}).
\]

Recall the equation in Proposition 3.1 and Lemma 3.2 in [19] for \(\cos \alpha\),
\[
\Delta \cos \alpha = - |\nabla J_{\Sigma_t}|^2 \cos \alpha + \sin \alpha (H^4, 1 + H^3, 2) - \sin^2 \alpha \overline{Ric}(J e_1, e_2).
\]
Thus we have
\[
\left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha = |\nabla J_{\Sigma_t}|^2 \cos \alpha - \cos^2 \alpha \overline{Ric}(J e_1, e_2) + \frac{1}{2} \cos \alpha (\overline{R}_{11} + \overline{R}_{22}).
\]
Using the equation in Lemma 3.2 in [19], \(\overline{Ric}(J e_1, e_2) = \frac{1}{\cos \alpha} (\overline{R}_{1212} + \overline{R}_{1234})\) we have
\[
- \cos^2 \alpha \overline{Ric}(J e_1, e_2) + \frac{1}{2} \cos \alpha (\overline{R}_{11} + \overline{R}_{22}) = - \cos \alpha (\overline{R}_{1212} + \overline{R}_{1234})
\]
\[
+ \frac{1}{2} \cos \alpha (2 \overline{R}_{1212} + \overline{R}_{333} + \overline{R}_{444})
\]
\[
= \frac{1}{2} \cos \alpha (\overline{R}_{333} + \overline{R}_{444} - 2 \overline{R}_{1234})
\]
\[
= \overline{R} \cos \alpha \sin^2 \alpha,
\]
where the last equality was derived in Lemma 3.2 of [23]. Therefore we proved the theorem.

Then by the parabolic minimum principle, we obtain that

**Theorem 4.2.** Suppose the smooth solution of (1.2) exists on \([0, T]\). Let \((M, g(0))\) be a Kähler surface with nonnegative scalar curvature. If \(\cos \alpha(x, 0) \geq c_0 > 0\), then

\[
\cos \alpha(x, t) \geq c_0,
\]

for all \(t \in [0, T]\).

**Proof.** Recall the evolution equation of the scalar curvature of \(M\) under the Kähler-Ricci flow,

\[
\frac{\partial}{\partial t} R = \frac{1}{2} \Delta R + |Ric|^2 - r R^2.
\] (4.5)

Thus by the parabolic minimum principle, if the scalar curvature of the initial surface is nonnegative, then the scalar curvature of \(R(t)\) is nonnegative for all \(t \in [0, T]\). Using the parabolic minimum principle again and (4.3), we proved the theorem. Q. E. D.

**Corollary 4.3.** We can rewrite the equation (4.3) as

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \sin^2(\alpha/2) = -|\nabla J_{\Sigma_t}|^2 \cos \alpha - 4R \sin^2(\alpha/2) \cos^2(\alpha/2) \cos \alpha,
\] (4.6)

which yields that, along the flow (1.2), the Kähler angle decrease to zero exponentially if \(R \geq \delta > 0\) and \(\cos \alpha(\cdot, 0) \geq c_0\).

**Remark 4.4.** The same evolution equation for \(\cos \alpha\) along the mean curvature flow in the case that \(M\) is Kähler-Einstein surface was proved by Chen-Li [4] and Wang [32].

**Lemma 4.5.**

\[
|\nabla \alpha|^2 \leq |\nabla J_{\Sigma_t}|^2.
\]

**Proof.** Using the frame in Theorem 4.1 and (4.4), it is easy to see that

\[
\nabla_1 \cos \alpha = \omega(\nabla e_1, e_2) + \omega(e_1, \nabla e_2) = (h_{11}^4 + h_{12}^4) \sin \alpha.
\] (4.7)

Similarly, we can get that

\[
\nabla_2 \cos \alpha = (h_{22}^3 + h_{12}^4) \sin \alpha.
\] (4.8)

Therefore,

\[
|\nabla \alpha|^2 \leq |\nabla J_{\Sigma_t}|^2.
\]

Q. E. D.
5. Monotonicity Formula

In this section we assume that the Kähler-Ricci flow exists on $[0, T]$, i.e., the Ricci curvature of $(M, \overline{g}(t))$ is uniformly bounded on $[0, T]$ (see [27]). We only consider the singularity of the mean curvature flow, we assume that it blows up at $T$. The monotonicity formula for the mean curvature flow was essentially proved by Huisken [17] and Hamilton [14]. The weighted monotonicity formula for the symplectic Kähler-Ricci mean curvature flow was proved by Chen-Li in [4]. In this section, we prove one weighted monotonicity formula for the flow (1.2). Since the Kähler-Ricci flow exists $[0, T]$, thus the injective of $(M, \overline{g}(t))$ is uniformly bounded from below on $[0, T)$ [30]. We can therefore choose a cut-off function on $(M, \overline{g}(t))$ to prove the monotonicity formula for the symplectic Kähler-Ricci mean curvature flow.

Let $H(X, X_0, t, t_0)$ be the backward heat kernel on $\mathbb{R}^4$. Define

$$
\rho(X, t) = (4\pi(t_0 - t))H(X, X_0, t, t_0) = \frac{1}{4\pi(t_0 - t)} \exp - \frac{|X - X_0|^2}{4(t_0 - t)}
$$

for $t < t_0$, such that

$$
\frac{d}{dt}\rho = -\Delta \rho - \rho \left( |H + \frac{(X - X_0)^\perp}{2(t_0 - t)}|^2 - |H|^2 \right).
$$

where $(X - X_0)^\perp$ is the normal component of $X - X_0$.

Let $i_M$ be the lower bound of the injective radius of $(M, \overline{g}(t))$. We choose a cut-off function $\phi \in C_0^\infty(B_{2\overline{r}}(X_0))$ with $\phi \equiv 1$ in $B_{\overline{r}}(X_0)$, where $X_0 \in M$, $0 < 2\overline{r} < i_M$. Choose a normal coordinates in $B_{2\overline{r}}(X_0)$ in $(M, \overline{g}(t))$ and express $F$ using the coordinates $(F^1, F^2, F^3, F^4)$ as a function in $\mathbb{R}^4$. Set $v(x, t) = e^{\overline{R} t} \cos \alpha(x, t)$, where $\overline{R}_0 = \max\{0, -\overline{R}\}$ and $\overline{R}$ is the scalar curvature of $M$. We define

$$
\Phi(F, X_0, t, t_0) = \int_{\Sigma_t} \frac{1}{v}\phi(F)\rho(F, t)d\mu_t.
$$

**Proposition 5.1.** There are positive constants $c_1$ and $c_2$ depending only on $(M, \overline{g}(t))$, $F_0$ and $\overline{r}$ such that

$$
\frac{\partial}{\partial t} e^{c_1\sqrt{t_0-t}}\Phi(F, X_0, t, t_0) \leq - e^{c_1\sqrt{t_0-t}} \left( \int_{\Sigma_t} \frac{1}{v}\phi(F)\rho(F, t)|H + \frac{(F - X_0)^\perp}{2(t_0 - t)}|^2 d\mu_t 
+ \int_{\Sigma_t} \frac{1}{v} |\nabla J_{\Sigma_t}|^2 \phi(F, t)d\mu_t + \int_{\Sigma_t} \frac{2|\nabla \cos \alpha|^2}{v^3}\phi(F, t)d\mu_t \right) 
+ c_2(t_0 - t). 
$$

(5.1)

**Proof.** By the evolution equation of the Kähler angle in Theorem 4.1, we have

$$
(\frac{\partial}{\partial t} - \Delta) \frac{1}{v} \leq - \frac{|\nabla J|^2}{v} - \frac{2|\nabla \alpha|^2}{v^3}.
$$

So,

$$
\frac{\partial}{\partial t} \Phi(F, X_0, t, t_0) = \int_{\Sigma_t} \frac{\partial}{\partial t} \frac{1}{v}\phi(F, t)d\mu_t + \int_{\Sigma_t} \frac{1}{v} \rho(F, t)\phi d\mu_t.
$$
$$- \int_{\Sigma_t} \frac{1}{v} \phi \rho(F, t)|H|^2 d\mu_t - \int_{\Sigma_t} \frac{1}{v} \phi(\tilde{R} - r) \rho(F, t)$$

$$\leq \int_{\Sigma_t} \frac{1}{v} \phi (\partial_t + \Delta) \rho(F, t) d\mu_t - \int_{\Sigma_t} \frac{1}{v} \phi \rho(F, t) |H|^2 d\mu_t$$

$$- \int_{\Sigma_t} \frac{\nabla J_{\Sigma_t}}{v} \phi \rho(F, t) d\mu_t - \int_{\Sigma_t} \frac{2|\nabla v|^2}{v^3} \phi \rho(F, t) d\mu_t$$

$$+ \int_{\Sigma_t} \frac{1}{v} \Delta \phi \rho(F, t) d\mu_t + 2 \int_{\Sigma_t} \frac{1}{v} \nabla \phi \nabla \rho(F, t) d\mu_t$$

$$- \int_{\Sigma_t} \frac{1}{v} (\tilde{R} - r) \phi \rho(F, t) d\mu_t.$$  

In $[0, T)$, $\tilde{R}$ is uniformly bounded, thus

$$\left| \int_{\Sigma_t} \frac{1}{v} \phi(\tilde{R} - r) \rho(F, t) \right| \leq C$$

and by (3.2)

$$\frac{\partial}{\partial t} \int_{\Sigma_t} d\mu_t \leq C \int_{\Sigma_t} d\mu_t,$$

so,

$$\text{Area}(\Sigma_t) \leq e^{CT} \text{Area}(\Sigma_0).$$

where the constant $C$ depends on $(M, \overline{g}(t)).$

Straight computation leads to

$$\frac{\partial}{\partial t} \rho(F, t) = \left( \frac{1}{t_0 - t} - \frac{\langle H, F - X_0 \rangle}{2(t_0 - t)} - \frac{|F - X_0|^2}{4(t_0 - t)^2} \right) \rho(F, t)$$

and

$$\Delta \exp\left( -\frac{|F - X_0|^2}{4(t_0 - t)} \right) = \exp\left( -\frac{|F - X_0|^2}{4(t_0 - t)} \right) \left( \frac{\langle F - X_0, \nabla F \rangle^2}{4(t_0 - t)} - \frac{\langle F - X_0, \Delta F \rangle}{2(t_0 - t)} - \frac{|\nabla F|^2}{2(t_0 - t)} \right).$$

Notice that, in the induced metric on $\Sigma_t$,

$$|\nabla F|^2 = 2$$

and

$$\Delta F^\alpha = H^\alpha - g^{ij} \Gamma_{\rho\sigma}^{\alpha} \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j},$$

then we have

$$\left( \frac{\partial}{\partial t} + \Delta \right) \rho(F, t)$$

$$= -\left( \frac{\langle F - X_0, H \rangle}{(t_0 - t)} + \frac{|(F - X_0)^2|^2}{4(t_0 - t)^2} + \frac{\langle F - X_0, g^{ij} \Gamma_{\rho\sigma}^{\alpha} \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j} e_\alpha \rangle}{2(t_0 - t)} \right) \rho(F, t).$$

Note that $\Delta \phi = 0$, $\nabla \phi = 0$ in $B_r(X_0)$, we can see that

$$|\Delta \phi \rho(F, t)| \leq C$$

and

$$|\nabla \phi \nabla \rho(F, t)| \leq C.$$
Hence
\[
\int_{\Sigma_t} \frac{1}{v} \Delta \phi \rho(F, t) d\mu_t \leq C \int_{\Sigma_t} d\mu_t \leq C
\]
\[
\int_{\Sigma_t} \frac{1}{v} \nabla \phi \nabla \rho(F, t) d\mu_t \leq C \int_{\Sigma_t} d\mu_t \leq C.
\]

Since we choose a normal coordinates in \(B_{2\tilde{r}}(X_0)\) in \((M, \varphi(t))\), we have \(\Gamma_{\rho\sigma}^{\alpha}(X_0, t) = 0\), and \(|g^{ij} \Gamma_{\rho\sigma}^{\alpha} \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j} e_\alpha| \leq C|F - X_0|\), thus
\[
\langle F - X_0, g^{ij} \Gamma_{\rho\sigma}^{\alpha} \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j} e_\alpha \rangle \leq C \frac{|F - X_0|^2}{2(t_0 - t)}.
\]

Hence
\[
\langle F - X_0, g^{ij} \Gamma_{\rho\sigma}^{\alpha} \frac{\partial F^\rho}{\partial x^i} \frac{\partial F^\sigma}{\partial x^j} e_\alpha \rangle \leq c_2 \frac{\rho(F, t)}{\sqrt{t_0 - t}} + c_3.
\]

It concludes that
\[
\frac{\partial}{\partial t} \Phi(F, X_0, t, t_0) \leq -\int_{\Sigma_t} \frac{1}{v} \phi \rho(F, t) |H + \frac{(F - X_0)^+}{2(t_0 - t)}|^2 d\mu_t
\]
\[
-\int_{\Sigma_t} \frac{\nabla J_{\Sigma_t}}{v} \phi \rho(F, t) d\mu_t - \int_{\Sigma_t} \frac{2|\nabla v|^2}{v^3} \phi \rho(F, t) d\mu_t
\]
\[
+ \frac{c_1}{\sqrt{t_0 - t}} \Phi(F, X_0, t_0) + c_2.
\]

The proposition follows.

Q. E. D.

6. NO TYPE I SINGULARITY

We assume in this section that the Kähler-Ricci flow exists globally. We study the singularity of the mean curvature flow if it blows up at \(T\). Since the Kähler-Ricci flow exists for all time, the Ricci curvature is uniformly bounded, and the evolution equation of the Kähler angle along the Kähler-Ricci mean curvature flow is the same as that of the mean curvature flow in Kähler-Einstein surface (c.f (4.3) and Proposition 3.2 in [4]), thus the analysis of the singularity of the mean curvature flow is the same as that of the mean curvature flow in Kähler-Einstein surface [4]. For completeness, we give some details below.

We recall the classification of the singularities of the mean curvature flow. We say the mean curvature flow has type I singularity at \(T > 0\), if
\[
\lim_{t \to T} (T - t) \max_{\Sigma_t} |A|^2 \leq C,
\]
for some positive constant \(C\). Otherwise, we say the mean curvature flow has type II singularity.
Lemma 6.1. Let $U(t) = \max_{\Sigma_t} |A|^2$. If the mean curvature flow blows up at $T > 0$, then there is a positive constant $c$ depending only on the bound of the curvature $(M, \mathcal{G}(t))$ such that, if $0 < T - t < \pi/16\sqrt{\mathcal{C}}$, the function $U(t)$ satisfies
\[
U(t) \geq \frac{1}{4\sqrt{2}(T-t)}.
\]

Proof. By Lemma 3.3 and the parabolic maximum principle, we have
\[
\frac{\partial}{\partial t} U(t) \leq 2(U(t))^2 + c_1 U(t) + c_2 \sqrt{U(t)}\]
\[
\leq 4(U(t))^2 + 4c,
\]
where $c_1, c_2$ are constants which depend only on the bounds of the curvature and its covariant derivatives of $(M, \mathcal{G}(t))$. This implies the desired inequality. Q. E. D.

Theorem 6.2. Assume that the Kähler-Ricci flow exists globally. The symplectic Kähler-Ricci mean curvature flow has no type I singularity at any $T > 0$.

Proof. Suppose that the mean curvature flow has a type I singularity at $t_0 > 0$. Assume that
\[
\lambda_k^2 = |A|^2(x_k, t_k) = \max_{t \leq t_k} |A|^2
\]
and $x_k \to p \in \Sigma$, $t_k \to t_0$ as $k \to \infty$. We choose a local coordinate system on $(M, \mathcal{G}(t))$ around $F(p, t_0)$ such that $F(p, t_0) = 0$. And we rescale the mean curvature flow,
\[
F_k(x, t) = \lambda_k F(x, \lambda_k^{-2}t + t_k) - F(p, t_k), \quad t \in [-\lambda_k^{-2}t_k, 0].
\]
Denote by $\Sigma^k_t$ the rescaled surface $F_k(\cdot, t)$. By Lemma 6.1, we have
\[
\frac{C}{t_0 - t_k} \geq |A|^2(x_k, t_k) \geq \frac{c}{t_0 - t_k}
\]
for some uniform constants $c$ and $C$ independent of $k$. Therefore,
\[
|A_k|^2(F(x_k, t_k) - F(p, t_k)) = |A|^2(F_k(x_k, 0)) = \lambda_k^{-2} |A|^2(x_k, t_k) \geq \lambda_k^{-2} \frac{c}{(t_0 - t_k)}.
\]
Since the mean curvature flow has type I singularity at $t_0 > 0$, we have
\[
\lambda_k^2(t_0 - t_k) \leq C.
\]
So,
\[
|A_k|^2(0) \geq c > 0,
\]
for some uniform constants $c$. It is easy to see that
\[
|A_k|^2(x, t) \leq 1,
\]
thus there is a subsequence of $F_k$ which we also denote by $F_k$, such that $F_k \to F_\infty$ in any ball $B_R(0) \subset \mathbb{R}^4$, and $F_\infty$ satisfies
\[
\frac{\partial F_\infty}{\partial t} = H_\infty \text{ with } 1 \geq |A_\infty|(0) \geq c > 0.
\]
By the monotonicity formula (5.1), we know that \( \lim_{t \to t_0} e^{c_1 \sqrt{t_0 - t}} \Phi \) exists. Using the equality

\[
\int_{\Sigma^k_t} \frac{1}{v} e^{c_1 \sqrt{t_0 - t}} \phi_{\lambda k r}(F_k) \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - t)}\right) d\mu_t^k
\]

we get that, for any \(-\infty < s_1 < s_2 < 0\),

\[
e^{c_1 \sqrt{t_k - (t_k + \lambda_k^{-2}s_1)}} \int_{\Sigma^k_{s_2}} \frac{1}{v} e^{c_1 \sqrt{t_0 - s_2}} \phi_{\lambda k r}(F_k) \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s_2)}\right) d\mu_{s_2}^k
\]

\[
e^{-c_1 \sqrt{t_k - (t_k + \lambda_k^{-2}s_2)}} \int_{\Sigma^k_{s_1}} \frac{1}{v} e^{c_1 \sqrt{t_0 - s_1}} \phi_{\lambda k r}(F_k) \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s_1)}\right) d\mu_{s_1}^k
\]

\[
\to 0 \text{ as } k \to \infty.
\]

Integrating (5.1) from \( t_k + \lambda_k^{-2}s_1 \) to \( t_k + \lambda_k^{-2}s_2 \), we get that

\[
|\nabla J_{\Sigma_k^t}|^2 \to 0,
\]

and

\[
|H_k + \frac{(F_k + \lambda_k F(p, t_k))^+}{2(0 - t)}|^2 \to 0 \text{ as } k \to \infty.
\]

By (6.2), we get that

\[
DJ_{\infty} \equiv 0,
\]

where \( D \) is the derivative in \( \mathbb{R}^4 \), which implies that

\[
H_{\infty} \equiv 0.
\]

Since

\[
|F(p, t_k)| \leq \int_{t_k}^{t_0} \left| \frac{\partial F_k}{\partial t} \right| dt \leq \int_{t_k}^{t_0} |H| dt \leq C \sqrt{t_0 - t_k} \leq \frac{C}{\lambda_k},
\]

thus \( \lambda_k F(p, t_k) \to q \) as \( k \to \infty \). Hence by (6.3), we have

\[
(F_{\infty} + q)^+ \equiv 0,
\]

this implies that, for \( \alpha = 3, 4 \),

\[
det((h_{\infty})_{ij}^\alpha) = 0.
\]

Since \( H_\infty = 0 \), we also have, for \( \alpha = 3, 4 \),

\[
tr((h_{\infty})_{ij}^\alpha) = 0.
\]

This yields that \( (h_{\infty})_{ij}^\alpha = 0 \) for all \( i, j = 1, 2, \alpha = 3, 4 \) which implies that \( |A_{\infty}| \equiv 0 \). This contradicts with (6.1). Q. E. D.
7. Graph Case

In this section we study the Kähler-Ricci mean curvature flow (1.2) in a special case. Suppose that $M$ is a product of compact Riemann surfaces $M_1, M_2$, i.e., $(M, \overline{g}) = (M_1 \times M_2, \mathcal{J}_1 \oplus \mathcal{J}_2)$. We denote by $r_1, r_2$ the average scalar curvature of $M_1, M_2$, we assume that $r_1 = r_2$. Then the Kähler-Ricci flow on $M$ can be split into the Kähler-Ricci flow on $M_1, M_2$ respectively. It is well known that the Kähler-Ricci flow on surface exists for long time and converges to the surface with constant curvature at infinity. Suppose that $\Sigma$ is a graph in $M$. By the work of Hamilton [15] and Chow [2], we know that, for any initial metrics $\{g_1, g_2\}$, $\{\omega_1, \omega_2\}$ is an orthonormal frame on $\Sigma$.

**Theorem 7.1.** Let $(M_1, \mathcal{J}_1, \omega_1)$ and $(M_2, \mathcal{J}_2, \omega_2)$ be Riemann surfaces which have the same average scalar curvature. Suppose that $M_1 \times M_2$ evolves along the Ricci flow and $\Sigma_0$ evolves along the mean curvature flow in $M_1 \times M_2$. If $v(\cdot, 0) > \frac{\sqrt{2}}{2}$, then the Kähler-Ricci mean curvature flow exists for all time.

**Proof.** Because $M_1$ and $M_2$ have the same average scalar curvature, the metric $\overline{g} = \mathcal{J}_1 \oplus \mathcal{J}_2$ on $M$ evolves along the Kähler-Ricci flow is equivalent to $\mathcal{J}_1$ and $\mathcal{J}_2$ evolves along the Ricci flow respectively. When $n = 2$, $R_{ij} = \frac{1}{2}R g_{ij}$. Thus, $\mathcal{J}_1(t)$, $\mathcal{J}_2(t)$ satisfy the evolution equations:

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t}(\mathcal{J}_1)_{ij} = -\frac{1}{2}(R_{ij} - r)(\mathcal{J}_1)_{ij}, \\
(\mathcal{J}_1)_{ij}(0) = (\mathcal{J}_1)_{ij},
\end{array} \right.
$$

and

$$
\left\{ \begin{array}{l}
\frac{\partial}{\partial t}(\mathcal{J}_2)_{ij} = -\frac{1}{2}(R_{ij} - r)(\mathcal{J}_2)_{ij}, \\
(\mathcal{J}_2)_{ij}(0) = (\mathcal{J}_2)_{ij},
\end{array} \right.
$$

By the work of Hamilton [15] and Chow [2], we know that, for any initial metrics the flows exist for long time and converge to $M_1^\infty \times M_2^\infty$ with constant curvatures at infinity.

Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $M$ along $\Sigma_t$ such that $\{e_1, e_2\}$ is the basis of $\Sigma_t$. Set $u_1 = \langle e_1 \wedge e_2, \omega_1 + \omega_2 \rangle$ and $u_2 = \langle e_1 \wedge e_2, \omega_1 - \omega_2 \rangle$ where $\omega_2$ is an unit Kähler form on $M_2$. Since both $\omega_1 + \omega_2$ and $\omega_1 - \omega_2$ are parallel Kähler forms on $M_1 \times M_2$, we see that Theorem 4.1 is applicable. Therefore,

$$
(d - \Delta)u_1 = [(h_3^2 + h_4^2) - (h_2^2 - h_4^2)]u_1 + R_1 u_1 (1 - u_1^2). 
$$

(7.1)

By switching $e_3$ and $e_4$, we get that

$$
(d - \Delta)u_2 = [(h_3^2 + h_4^2) - (h_2^2 - h_4^2)]u_2 + R_2 u_2 (1 - u_2^2). 
$$

(7.2)

It is easy to see that $\langle e_1 \wedge e_2, \omega_1 \rangle^2 + \langle e_1 \wedge e_2, \omega_2 \rangle^2 \leq 1$. The initial condition $v(x, 0) \geq \frac{\sqrt{2}}{2}$ implies that $u_i(x, 0) \geq v(x, 0) - \frac{\sqrt{2}}{2} \geq c_0 > 0, i = 1, 2$. By (7.1) and (7.2), applying the maximum principle for parabolic equations, we see that $u_i(x, t)$ have positive lower
bounds at any finite time. Suppose that \( u_i \geq \delta \) for \( 0 \leq t < t_0 \). Then we claim that the flow \( F \) can be extended smoothly to \( t_0 + \varepsilon \) for some \( \varepsilon \).

Set \( u = u_1 + u_2 \). Adding (7.1) into (7.2), we get that
\[
\left( \frac{d}{dt} - \Delta \right) u = u|A|^2 + 2(u_1 - u_2)h_{1k}^3 h_{2k}^4 - 2(u_1 - u_2)h_{1k}^3 h_{2k}^4 + R_1 u_1(1 - u_1^2) + R_2 u_2(1 - u_2^2).
\]
(7.3)

Since \( u \geq 2\delta + |u_1 - u_2| \), using the Cauchy-Schwarz inequality, we get that
\[
\left( \frac{d}{dt} - \Delta \right) u \geq 2\delta |A|^2 - C,
\]
(7.4)
where \( C \) is the lower bound of the scalar curvature of \( (M, \overline{\eta}_1(t) \oplus \overline{\eta}_2(t)) \).

Assume that \( (X_0, t_0) \) is a singularity point. As in the proof of Proposition 5.1, we can derive a weighted monotonicity formula for \( \int_{\Sigma_t} \phi \frac{1}{u} \rho(F, X_0, t, t_0) d\mu_t \), where \( \phi \) is the cut-off function in Proposition 5.1.

\[
\frac{d}{dt} \int_{\Sigma_t} \phi \frac{1}{u} \rho(F, X_0, t, t_0) d\mu_t \\
\leq \int_{\Sigma_t} \phi \rho \frac{1}{u} \bigg( -2 \delta \int_{\Sigma_t} \phi \frac{|A|^2}{u^2} \rho d\mu_t - 2 \int_{\Sigma_t} \phi \frac{|\nabla u|^2}{u^3} \rho d\mu_t \\
+ \int_{\Sigma_t} \phi \frac{C}{u^2} d\mu_t - \int_{\Sigma_t} \phi \frac{1}{u} \left( \Delta \rho + \left( \left| H + \frac{(F - X_0)_1}{2(t_0 - t)} \right|^2 - |H|^2 \right) \rho \right) d\mu_t \\
- \int_{\Sigma_t} \phi \frac{|H|^2}{u} \rho d\mu_t - \int_{\Sigma_t} \phi \frac{R - r}{u} \rho d\mu_t \bigg) \\
\leq \int_{\Sigma_t} \phi \rho \left( \frac{2}{u^3} |\nabla u|^2 + \frac{1}{u} \left| H + \frac{(F - X_0)_1}{2(t_0 - t)} \right|^2 + 2\delta \frac{|A|^2}{u^2} d\mu_t \right) \\
- \int_{\Sigma_t} \phi \frac{R - r}{u} \rho d\mu_t + \int_{\Sigma_t} \Delta \phi \frac{1}{u} \rho d\mu_t + 2 \int_{\Sigma_t} \frac{1}{u} \nabla \phi \cdot \nabla \rho d\mu_t \\
\leq C - 2\delta \int_{\Sigma_t} \phi \frac{|A|^2}{u^2} \rho(F, X_0, t_0) d\mu_t,
\]
(7.5)
where \( C \) depends on the scalar curvature of \( \overline{R}_1, \overline{R}_2 \) and the bound of \( |\nabla \phi|, |\Delta \phi| \). From this we see that \( \lim_{t \to t_0} \int_{\Sigma_t} \phi \frac{1}{u} \rho d\mu_t \) exists.

Let \( 0 < \lambda_i \to \infty \) and let \( F_i \) be the blow up sequence:
\[
F_i(x, s) = \lambda_i (F(x, t_0 + \lambda_i^{-2}s) - X_0).
\]
Let \( d\mu_s \) denote the induced volume form on \( \Sigma_t^i \) by \( F_i \). It is obvious that,
\[
\int_{\Sigma_t^i} \phi \frac{1}{u} \rho(F, X_0, t, t_0) d\mu_t = \int_{\Sigma_t^i} \phi \frac{1}{u} \rho(F_i, 0, 0) d\mu_s^i.
\]
Therefore we get that,
\[
\frac{d}{ds} \int_{\Sigma_t^i} \phi \frac{1}{u} \rho(F_i, 0, s, 0) d\mu_s^i
\]
\[
\frac{C}{\lambda_i^2} - 2\delta \int_{\Sigma^i_s} \phi \frac{|A_i|^2}{u^2} \rho(F_i, 0, s, 0) d\mu_{s}^i
\]

Note that \( t_0 + \lambda_i^{-2} s \to t_0 \) for any fixed \( s \) as \( i \to \infty \) and that \( \lim_{i \to t_0} \int_{\Sigma} \phi \frac{1}{u^2} \rho \) exists.

By the above monotonicity formula, we have, for any fixed \( s_1 \) and \( s_2 \),

\[
0 \leq \int_{\Sigma^i_{s_1}} \phi \frac{1}{u^2} \rho(F_i, 0, s_1, 0) d\mu_{s_1}^i - \int_{\Sigma^i_{s_2}} \phi \frac{1}{u^2} \rho(F_i, 0, s_2, 0) d\mu_{s_2}^i
\]

\[
\geq 2\delta \int_{s_1}^{s_2} \int_{\Sigma^i_s} \phi \frac{|A_i|^2}{u^2} \rho(F_i, 0, s, 0) d\mu_{s}^i.
\]

Since \( u \) is bounded below, we have

\[
\int_{s_1}^{s_2} \int_{\Sigma^i_s} |A_i|^2 \rho(F_i, 0, s, 0) \to 0 \quad \text{as} \quad i \to \infty.
\]

Therefore, for any ball \( B_R(0) \subset \mathbb{R}^4 \),

\[
\int_{\Sigma^i_{s} \cap B_R(0)} |A_i|^2 \to 0 \quad \text{as} \quad i \to \infty. \tag{7.6}
\]

Because \( u \) has a positive lower bound, we see that \( \Sigma_i \) can locally be written as the graph of a map \( f_i : \Omega \subset M_1 \to M_2 \) with uniformly bounded \( |df_i| \). Consider the blow up of \( f_{t_0 + \lambda_i^{-2} s} \),

\[
f_i(y) = \lambda_i f_{t_0 + \lambda_i^{-2} s}(\lambda_i^{-1} y).
\]

It is clear that \( |df_i| \) is also uniformly bounded and \( \lim_{i \to \infty} f_i(0) = 0 \). By Arzella theorem, \( f_i \to f_{\infty} \) in \( C^\alpha \) on any compact set. By the inequality (29) in [21], we have

\[
|A_i| \leq |\nabla df_i| \leq C(1 + |df_i|^3)|A_i|,
\]

where \( \nabla df_i \) is measured with respect to the induced metric on \( \Sigma_{s_i}^i \). From equation (7.6) it follows that, for any ball \( B_R(0) \subset \mathbb{R}^4 \),

\[
\int_{\Sigma^i_{s} \cap B_R(0)} |\nabla df_i|^2 \to 0 \quad \text{as} \quad i \to \infty,
\]

which implies that \( f_i \to f_{\infty} \) in \( C^\alpha \cap W_{loc}^{1,2} \) and the second derivative of \( f_{\infty} \) is 0. It is then clear that \( \Sigma_i^i \to \Sigma_{\infty} \) and \( \Sigma_{\infty} \) is the graph of a linear function. Therefore,

\[
\lim_{i \to \infty} \int \phi \rho(F_i, 0, s_i, 0) d\mu_{s_i}^i = \int \rho(F_{\infty}, 0, -1, 0) d\mu_{\infty}^i = 1,
\]

We therefore have

\[
\lim_{t \to t_0} \int \rho(F, X_0, t, t_0) = \lim_{i \to \infty} \int \phi \rho(F, X_0, t_0 + \lambda_i^{-2} s_i, t_0)
\]

\[
= \lim_{i \to \infty} \int \phi \rho(F_i, 0, s_i, 0) d\mu_{s_i}^i = 1. \tag{7.7}
\]

By the White’s regularity theorem [31], we know that \( (X_0, t_0) \) is a regular point. This proves the theorem.

**Theorem 7.2.** Under the same assumption as in Theorem 7.1. If the scalar curvature of \( M_1, M_2 \) is positive, then it converges to a totally geodesic surface at infinity.
Proof. Since the scalar curvature of $M$ is positive, by (7.1) and (7.2),
\[
(\frac{\partial}{\partial t} - \Delta)(1 - u_1) \leq -\bar{R}_1 u_1(1 + u_1)(1 - u_1) \leq -c_1 \bar{R}_1 (1 - u_1),
\]
where $c_1$ depends only on the lower bound of $u_1$. Applying the maximum principle, we get that $1 - u_1 \leq ce^{-c_1 \bar{R}_1 t}$. Similarly, $1 - u_2 \leq ce^{-c_2 \bar{R}_2 t}$, where $c_1, c_2$ depends only on the lower bound of $u_1, u_2$. By (7.3), for any $\varepsilon > 0$, there exists $T$ such that as $t > T, u_1 > 1 - \varepsilon$, $u_2 > 1 - \varepsilon$, $|u_1 - u_2| < \varepsilon$ and
\[
(\frac{\partial}{\partial t} - \Delta)u \geq (1 - \varepsilon)|A|^2.
\]
From (3.6) we see that
\[
(\frac{\partial}{\partial t} - \Delta)|A|^2 \leq -2|\nabla A|^2 + C_1|A|^4 + C_2|A|^2 + C_3,
\]
where $C_1, C_2, C_3$ are constants that depend on the bounds of the curvature tensor and its covariant derivatives of $(M, \bar{g}(t))$.

Let $p > 1$ be a constant to be fixed later. Now we consider the function $\frac{|A|^2}{e^{\nu}}$.

\[
(\frac{\partial}{\partial t} - \Delta)\frac{|A|^2}{e^{\nu}} = 2\nabla(\frac{|A|^2}{e^{\nu}}) \cdot \nabla e^{\nu} + \frac{1}{e^{2\nu}}[e^{\nu}(\frac{\partial}{\partial t} - \Delta)|A|^2 - |A|^2(\frac{\partial}{\partial t} - \Delta)e^{\nu}]
\]
\[
\leq 2pe^{\nu} - \frac{1}{e^{2\nu}}[e^{\nu}(C_1|A|^4 + C_2|A|^2 + C_3) - p|A|^2e^{\nu}[(1 - \varepsilon)|A|^2 - p|\nabla u|^2]].
\]

From (4.7) and (4.8), it follows that,
\[
|\nabla u_1|^2 \leq 2(1 - u_2^2)((h_{1k}^4)^2 + (h_{2k}^3)^2),
|\nabla u_2|^2 \leq 2(1 - u_2^2)((h_{1k}^3)^2 + (h_{2k}^4)^2).
\]

So, for $t$ is sufficiently large, we have
\[
|\nabla u_1|^2 \leq \varepsilon|A|^2, \ |\nabla u_2|^2 \leq \varepsilon|A|^2.
\]
and
\[
|\nabla u|^2 \leq 2(|\nabla u_1|^2 + |\nabla u_2|^2) \leq 4\varepsilon|A|^2.
\]
Therefore,
\[
(\frac{\partial}{\partial t} - \Delta)\frac{|A|^2}{e^{\nu}} \leq 2pe^{\nu} - \frac{1}{e^{2\nu}}[(C_1 - p(1 - \varepsilon) + 4p^2\varepsilon)|A|^4 + C_2|A|^2 + C_3].
\]

Set $p^2 = 1/\varepsilon$, then
\[
C_1 - p(1 - \varepsilon) + 4p^2\varepsilon = C_1 - \varepsilon^{-\frac{1}{2}} + \varepsilon^\frac{3}{2} + 4.
\]
As $t$ is sufficiently large, i.e. $\varepsilon$ is sufficiently close to 0, we have

$$(C_1 - \varepsilon^{-\frac{t}{2}} + \varepsilon^\frac{t}{2} + 4) \leq -1.$$  

So,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \frac{|A|^2}{e^{pu}} \leq 2p \nabla \left( \frac{|A|^2}{e^{pu}} \right) \cdot \nabla u - \frac{|A|^4}{e^{2pu}} + C_2 \frac{|A|^2}{e^{pu}} + C_3$$

Applying the maximum principle for parabolic equations, we conclude that $\frac{|A|^2}{e^{pu}}$ is uniformly bounded, thus $|A|^2$ is also uniformly bounded. Thus $F(\cdot,t)$ converges to $F_\infty$ in $C^2$ as $t \to \infty$. Since

$$1 - u_1 \leq ce^{-c_1 R_1 t},$$

and

$$1 - u_2 \leq ce^{-c_1 R_2 t},$$

So, we have $u_1 \equiv 1$ and $u_2 \equiv 2$ at infinity. By (7.3), we see that the second fundamental form is zero identically, in other words, the limiting surface $F_\infty$ is totally geodesic.

Q. E. D.

8. Stability of Kähler-Ricci Mean Curvature Flow

Let $(M,J)$ be a Kähler surface with $c_1(M) > 0$. Now we choose $r = 2$. Suppose $(M,J)$ is pre-stable and the Futaki invariant of the class $2\pi c_1(M)$ vanishes. Chen-Li [3] proved that for any $\gamma$, there exists a small positive constant $\varepsilon(\gamma)$ such that for any metric $\bar{g}$ in the subspace of Kähler metrics

$$\{\omega_\bar{g} \in 2\pi c_1(M) | |Rm|(\omega_\bar{g}) \leq \gamma, |Ric(\omega_\bar{g}) - \omega_\bar{g}| \leq \varepsilon\},$$

the Kähler Ricci flow with the initial metric $\omega_\bar{g}$ will converge exponentially fast to a Kähler-Einstein metric.

That is

$$||Ric(g(t,\cdot)) - \bar{g}(t,\cdot)||_{\bar{g}(t,\cdot)} \leq C\varepsilon e^{-\beta t},$$

for some positive constants $C$ and $\beta$ which depend only on $\gamma$ and $\varepsilon$. In this case, we show that, if the initial surface is sufficiently close to a holomorphic curve, then the Kähler-Ricci mean curvature flow exists for all time (the idea is similar to that in [18]) and converges to $(M,\bar{g}_\infty,\Sigma_\infty)$ at infinity.

Note that, since $\Sigma_0$ is smooth, it is well-known that

$$\lim_{r \to 0} \int_{\Sigma_0} \phi(F) \frac{1}{4\pi r^2} e^{\frac{|F - X_0|^2}{4r^2}} d\mu_0 = 1$$

for any $X_0 \in \Sigma_0$. So we can find a sufficiently small $r_0$ such that

$$\int_{\Sigma_0} \phi(F) \frac{1}{4\pi r_0^2} e^{\frac{|F - X_0|^2}{4r_0^2}} d\mu_0 \leq 1 + \varepsilon_0/2$$  (8.1)
for all $X_0 \in M$, where $\varepsilon_0$ is the constant in White’s Theorem.

**Theorem 8.1.** There exist sufficiently small constant $\varepsilon_1, \varepsilon_2$ such that, if $(\varepsilon_1 + \varepsilon_2)/r_0^6 \ll \varepsilon_0$ where $r_0$ is defined in (8.1) and $\varepsilon_0$ is a constant in White’s theorem, and at any time $t,$

$$\|\overline{\text{Ric}}(g(t, \cdot)) - \overline{\mathcal{g}}(t, \cdot)\|_{\overline{g}(t, \cdot)} \leq C\varepsilon_1 e^{-\beta t},$$

and the Kähler angle of the initial surface satisfies $\cos \alpha_0 \geq 1 - \varepsilon_2$, then Kähler-Ricci mean curvature flow exists globally and it converges to $(M, \overline{\mathcal{g}}_{\infty}, \Sigma_{\infty})$ at infinity. Furthermore, $\Sigma_{\infty}$ is the holomorphic curve in $(M, \overline{\mathcal{g}}_{\infty})$ and $(M, \overline{\mathcal{g}}_{\infty})$ is Kähler-Einstein surface.

**Proof.** Since along the Ricci flow, at any time $t$ we have,

$$|\overline{\text{Ric}} - \omega| \leq C\varepsilon_1 e^{-\beta t},$$

then at any time $t$,

$$|\overline{R} - 2| \leq C\varepsilon_1 e^{-\beta t}. \quad (8.2)$$

By (4.3), along the mean curvature flow, $\cos \alpha$ increase, i.e, $\cos \alpha \geq 1 - \varepsilon_2$. Using the equation of (4.6) we can obtain that

$$(\frac{\partial}{\partial t} - \Delta) \sin^2 \alpha/2 \leq -2c\varepsilon_2(1 + \varepsilon_2) \sin^2 \alpha/2.$$

So,

$$\sin^2 \alpha/2 \leq \sin^2 \alpha_0/2 e^{-ct} \leq \varepsilon_2 e^{-ct}. \quad (8.3)$$

By (3.2),

$$\frac{d}{dt} \int_{\Sigma_t} d\mu_t \leq \int_{\Sigma_t} |\overline{R} - 2| d\mu_t \leq C\varepsilon_1 e^{-\beta t} \int_{\Sigma_t} d\mu_t,$$

thus,

$$\text{Area}(\Sigma_t) \leq C\text{Area}(\Sigma_0).$$

Because $\omega(t)$ is always closed, we can see that

$$\int_{\Sigma_t} \cos \alpha d\mu_t = \int_{\Sigma_t} \omega$$

is constant under the continuous deformation in $t$. Thus,

$$\frac{\partial}{\partial t} \int_{\Sigma_t} (1 - \cos \alpha) d\mu_t = -\int_{\Sigma_t} |H|^2 d\mu_t - \int_{\Sigma_t} (\overline{R} - 2) d\mu_t. \quad (8.4)$$

Integrating the above inequality from $t$ to $t + 1$ we obtain that

$$\int_{t}^{t+1} \int_{\Sigma_t} |H|^2 d\mu_t dt \leq \int_{\Sigma_t} \sin^2 \alpha/2 d\mu_t + \int_{t}^{t+1} \int_{\Sigma_s} |\overline{R} - 2| d\mu_s ds$$

$$\leq C\text{Area}(\Sigma_0)(\varepsilon_2 e^{-ct} + \varepsilon_1 e^{-\beta t})$$

$$\leq C(\varepsilon_1 + \varepsilon_2)e^{-\lambda t},$$
where $\lambda = \min\{c, \beta\}$. From this we can derive an $L^1$-estimate of the mean curvature vector.

$$\int_0^t \int_{\Sigma_s} |H| d\mu_s ds = \sum_{k=0}^{t-1} \int_k^{k+1} \int_{\Sigma_s} |H| d\mu_s ds$$

$$\leq \sum_{k=0}^{t-1} \left( \int_k^{k+1} \int_{\Sigma_s} |H|^2 d\mu_s ds \right)^{1/2} \left( \int_k^{k+1} Area \Sigma_t \right)^{1/2}$$

$$\leq C Area(\Sigma_0)^{1/2} \sum_{k=0}^{t-1} \left( \int_k^{k+1} \int_{\Sigma_s} |H|^2 d\mu_s ds \right)^{1/2}$$

$$\leq C (\varepsilon_1 + \varepsilon_2)^{1/2} \sum_{k=0}^{t-1} e^{-\frac{\lambda S}{2}}$$

$$\leq C \left( \varepsilon_1 + \varepsilon_2 \right)^{1/2} \frac{1 - e^{-\frac{\lambda T}{2}}}{1 - e^{-\frac{\lambda T}{2}}}$$

where $C$ depends only on the area of the initial surface $\Sigma_0$.

Now we explore a possible singularity $(X_0, T)$. We study the density in White’s local regularity theorem [31]. Recall that it is defined by

$$\Phi(X, X_0, t, t - r^2) = \int_{\Sigma_{t-r^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F - X_0|^2}{4r^2}} d\mu_{t-r^2},$$

where $\phi$ is a cut off function around $X_0$ on $M$ such that $\phi \equiv 1$ in $B_r(X_0)$. Differentiating this equation with respect to $t$ we get that

$$\frac{\partial}{\partial t} \int_{\Sigma_{t-r^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F - X_0|^2}{4r^2}} d\mu_{t-r^2} = \int_{\Sigma_{t-r^2}} \nabla \phi \cdot H \frac{1}{4\pi r^2} e^{-\frac{|F - X_0|^2}{4r^2}} d\mu_{t-r^2}$$

$$- \int_{\Sigma_{t-r^2}} \phi(F) \frac{1}{8\pi r^4} \langle F - X_0, H \rangle d\mu_{t-r^2}$$

$$- \int_{\Sigma_{t-r^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F - X_0|^2}{4r^2}} |H|^2 d\mu_{t-r^2}$$

$$- \int_{\Sigma_{t-r^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F - X_0|^2}{4r^2}} (\tilde{R} - 2) d\mu_{t-r^2}.$$ (8.5)

Integrating (8.5) from $r_0^2$ to $T$ we get that

$$\int_{\Sigma_{r_0^2}} \phi(F) \frac{1}{4\pi r_0^2} e^{-\frac{|F - X_0|^2}{4r_0^2}} d\mu_{t-r_0^2} \leq \int_{\Sigma_0} \phi(F) \frac{1}{4\pi r_0^2} e^{-\frac{|F - X_0|^2}{4r_0^2}} d\mu_0$$

$$+ \int_{r_0}^{T} \int_{\Sigma_{t-r_0^2}} \nabla \phi ||H|| \frac{1}{4\pi r^2} e^{-\frac{|F - X_0|^2}{4r^2}} d\mu_{t-r_0^2} dt$$

$$+ \int_{r_0}^{T} \int_{\Sigma_{t-r_0^2}} \phi(F) \frac{1}{8\pi r_0^2} e^{-\frac{|F - X_0|^2}{4r_0^2}} |F - X_0||H| d\mu_{t-r_0^2} dt$$
\[ + \int_{r_0}^{T} \int_{\Sigma_{t-r_0^2}} \frac{\phi}{4\pi r_0^2} e^{-\frac{|F-X_0|^2}{4r_0^2}} |\tilde{R} - 2|d\mu_{t-r_0^2}dt. \]

Using the $L^1$-estimate of the mean curvature vector and (8.2), note that $|\nabla \phi| \leq C$, we get that

\[
\int_{\Sigma_{T-r_0^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}} d\mu_{t-r^2} \leq 1 + \varepsilon_0/2 + \frac{C}{4\pi r_0^2} \left( \frac{\varepsilon_1 + \varepsilon_2}{1 - e^{-\lambda/2}} \right) + \frac{C}{8\pi r_0^2} \left( \frac{\varepsilon_1 + \varepsilon_2}{1 - e^{-\lambda/2}} \right) + \frac{C\varepsilon_1}{4\pi r_0^2}. \]

If $(\varepsilon_1 + \varepsilon_2)/r_0^6 \ll \varepsilon_0$, then

\[
\int_{\Sigma_{T-r_0^2}} \phi(F) \frac{1}{4\pi r^2} e^{-\frac{|F-X_0|^2}{4r^2}} d\mu_{t-r^2} \leq 1 + \varepsilon_0. \]

Applying White's theorem we obtain an uniform estimate of the second fundamental from which implies the global existence and convergence of the mean curvature flow. From (8.3) we see that $\cos \alpha_{\infty} = 1$, that is $\Sigma_{\infty}$ is a holomorphic curve. This proves the theorem.

Q. E. D

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