NEW REALIZATION OF ∆QUANTUM GROUPS VIA ∆-HALL ALGEBRAS

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Abstract. For an essentially small hereditary abelian category \( A\), we define a new kind of algebra \( H_\Delta(A)\), called the \( \Delta\)-Hall algebra of \( A\). The basis of \( H_\Delta(A)\) is the isomorphism classes of objects in \( A\), and the \( \Delta\)-Hall numbers calculate certain three-cycles of exact sequences in \( A\). We show that the \( \Delta\)-Hall algebra \( H_\Delta(A)\) is isomorphic to the 1-periodic derived Hall algebra of \( A\). By taking suitable extension and twisting, we can obtain the \( \Delta\)Hall algebra and the semi-derived Hall algebra associated to \( A\) respectively.

When applied to the the nilpotent representation category \( A = \text{rep}^{\text{nil}}(k\mathcal{Q})\) for an arbitrary quiver \( \mathcal{Q}\) without loops, the (resp. extended) \( \Delta\)-Hall algebra provides a new realization of the (resp. universal) \( \Delta\)quantum group associated to \( \mathcal{Q}\).

1. Introduction

The Hall algebra approach to quantum groups is a hot topic in representation theory of algebras. In 1990, Ringel [22] constructed a Hall algebra associated to a quiver \( \mathcal{Q}\) over a finite field, and identified its generic version with the positive part of quantum groups when \( \mathcal{Q}\) is Dynkin type. Later, Green [9] generalized it to Kac-Moody setting.

Since then, people made efforts to the realization of the whole quantum groups via Hall algebras. Xiao [25] constructed the Drinfeld double of Hall algebras. Bridgeland [6] considered the Hall algebra of 2-periodic complexes of projective modules for a hereditary algebra. Lu-Peng [15] generalized Bridgeland’s construction to the hereditary abelian categories (perhaps without projective objects). The Hall type Lie algebras of root categories can also be used to realize Kac-Moody algebras, see Peng-Xiao [21] and Lin-Peng [14].

Ringel’s version of Hall algebra has found further generalizations and improvements which allow more flexibilities. Toën [24] and Xiao-Xu [26] introduced the derived Hall algebras for triangulated categories satisfying certain homological finiteness conditions. Xu-Chen [27] constructed derived Hall algebras for odd periodic triangulated categories. Gorsky [8] investigated semi-derived Hall algebras for Frobenius categories. Lu defined semi-derived Hall algebras for the so-called weakly 1-Gorenstein categories in [20, Appendix A].

The quantum group is a generalization of quantum groups, which arises from the construction of quantum symmetric pairs by Letzter [11]; see [10] for an extension to Kac-Moody type. A striking breakthrough of the quantum groups is the discovery of canonical basis by Bao-Wang [2]. As outlined by Bao-Wang, most of the fundamental constructions of quantum groups should admit generalizations in the setting of quantum groups, see [1, 2, 3] for generalizations of (quasi) \( R\)-matrix and canonical bases, and also see [4, 12, 19] for geometric realizations and [5] for KLR type categorification.

In recent years, Lu-Wang [20, 18] have developed \( \Delta\)Hall algebras of \( \Delta\)quivers to realize the universal quasi-split quantum groups of Kac-Moody type. Let’s briefly recall the construction
of $\mathfrak{t}$Hall algebras. For a hereditary abelian category $\mathcal{A}$ over a finite field $\mathbb{F}_q$, we have the Ringel-Hall algebra $\mathcal{H}(\mathcal{C}_1(\mathcal{A}))$ of the category $\mathcal{C}_1(\mathcal{A})$ of 1-periodic complexes over $\mathcal{A}$, and the $\mathfrak{t}$Hall algebra $\tilde{\mathcal{H}}(\mathcal{A})$ is then obtained from $\mathcal{H}(\mathcal{C}_1(\mathcal{A}))$ by successively applying quotient, localization and twisting. The construction is difficult, but it turns out that the $\mathfrak{t}$Hall algebra accesses a nice basis and a simplified multiplication formula, c.f. [18].

Recall that the $\mathfrak{t}$quantum group is a quotient of the universal $\mathfrak{t}$quantum group by a certain ideal generated by central elements. It has been proved in [7] that the 1-periodic derived Hall algebra of $\mathcal{A}$ is a quotient algebra of the $\mathfrak{t}$Hall algebra $\tilde{\mathcal{H}}(\mathcal{A})$, which produces a realization for the split $\mathfrak{t}$quantum group.

The goal of this paper is to construct new algebras in a more direct way to realize (universal) $\mathfrak{t}$quantum groups. Namely, we define the $\Delta$-Hall algebra $H_\Delta(\mathcal{A})$ associated to $\mathcal{A}$. Following Ringel’s original construction, the basis of $H_\Delta(\mathcal{A})$ is the isomorphism classes of objects in $\mathcal{A}$, while the Hall number is replaced by the so-called $\Delta$-Hall number. Roughly speaking, the $\Delta$-Hall number $\hat{F}_{AB}^M$ for $A, B, M \in \mathcal{A}$ calculates the number of three-cycles of exact sequences as follow:

$$
\begin{array}{c}
\text{I} \\
\downarrow \quad \downarrow \\
A & B \\
\downarrow \quad \downarrow \\
N \leftarrow M \leftarrow L,
\end{array}
$$

where each two arrows on the same line form a short exact sequence.

The associativity for the $\Delta$-Hall numbers relies on Green’s formula in $\mathcal{A}$. The main results of this paper indicate that the $\Delta$-Hall algebra $H_\Delta(\mathcal{A})$ is isomorphic to the 1-periodic derived Hall algebra $\mathcal{D}\mathcal{H}_1(\mathcal{A})$, while the extended $\Delta$-Hall algebra $\tilde{H}_\Delta(\mathcal{A})$ is isomorphic to the $\mathfrak{t}$Hall algebra $\tilde{\mathcal{H}}(\mathcal{A})$. Moreover, by twisting on $\tilde{H}_\Delta(\mathcal{A})$ we recover the semi-derived Hall algebras. When applied to the the nilpotent representation category $\mathcal{A} = \text{rep}^{\text{nil}}(kQ)$ for an arbitrary quiver $Q$ without loops, we obtain new realizations of the quantum group $\mathfrak{t}U_{|v=v}$ and the universal quantum group $\tilde{\mathfrak{t}}U_{|v=v}$ associated to $Q$. We summarized the relations between these algebras in the following commutative diagram:

$$
\begin{align*}
\tilde{\mathfrak{t}}U_{|v=v} & \xrightarrow{[18, \text{Thm } 9.6]} \tilde{\mathcal{H}}(kQ) \\
& \xrightarrow{\text{Prop } 4.5} \tilde{H}_\Delta(kQ) \\
\mathfrak{t}U_{|v=v} & \xrightarrow{[7, \text{Thm } 3.4]} \mathcal{D}\mathcal{H}_1(kQ) \\
& \xrightarrow{\text{Prop } 4.6} \mathcal{H}_\Delta(kQ)
\end{align*}
$$

The paper is organized as follows. In Section 2, we prove the associativity for the $\Delta$-Hall numbers and define the $\Delta$-Hall algebras. We show that $\Delta$-Hall algebras are isomorphic to derived Hall algebras in Section 3, which provide a new realization for $\mathfrak{t}$quantum groups. Section 4 is devoted to formulate the isomorphism between extended $\Delta$-Hall algebras and $\mathfrak{t}$Hall algebras, hence provide a new way to realize the universal quantum groups. In Section 5, by considering the twisting on $\Delta$-Hall algebras we recover the semi-derived Hall algebras.
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For the convenience of readers, we list different algebras (with multiplication and basis), associated to an essentially small hereditary abelian category $\mathcal{A}$, involved in this paper as follows:

- $(\mathcal{H}(\mathcal{A}), \odot)$ — Ringel-Hall algebra of $\mathcal{A}$ with the basis $\{[M] \mid [M] \in \text{Iso}(\mathcal{A})\}$;
- $(\mathcal{H}_\Delta(\mathcal{A}), \ast)$ — $\Delta$-Hall algebra of $\mathcal{A}$ with the basis $\{[M] \mid [M] \in \text{Iso}(\mathcal{A})\}$;
- $(\mathcal{H}_1(\mathcal{A}), \ast)$ — 1-periodic derived Hall algebra of $\mathcal{A}$ with the basis $\{u_{[M]} \mid [M] \in \text{Iso}(\mathcal{D}_1(\mathcal{A}))\}$;
- $(\text{SDH}(C_1(\mathcal{A})), \odot)$ — semi-derived Hall algebra of $C_1(\mathcal{A})$ with the basis $\{[M] \odot [K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$;
- $(\tilde{\mathcal{H}}_\Delta(\mathcal{A}), \ast)$ — $\tilde{\mathcal{H}}$-Hall algebra of $C_1(\mathcal{A})$ with the basis $\{[M] * [K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$;
- $(\tilde{\mathcal{H}}_\Delta(\mathcal{A}), \ast)$ — another extended version of $\mathcal{H}_\Delta(\mathcal{A})$ with the basis $\{[M][K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$;
- $(\varphi \tilde{\mathcal{H}}_\Delta(\mathcal{A}), \ast)$ — twisted extended $\Delta$-Hall algebra of $\mathcal{A}$ with the basis $\{[M][K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$.

2. DEFINITION OF $\Delta$-HALL ALGEBRAS

In this paper, we take the field $k = \mathbb{F}_q$, a finite field of $q$ elements. Let $\mathcal{A}$ be an essentially small hereditary abelian category, linear over $k$. Assume $\mathcal{A}$ has finite morphism and extension spaces, i.e.,

$$|\text{Hom}_\mathcal{A}(M, N)| < \infty, \quad |\text{Ext}^1_\mathcal{A}(M, N)| < \infty, \quad \forall M, N \in \mathcal{A}.$$  

Inspired by the constructions of the Ringel-Hall algebras due to Ringel [22] and the $\ast$Hall algebras due to [18, 15], we will define a new kind of algebras associated to $\mathcal{A}$ in this section, called $\Delta$-Hall algebras.

2.1. Ringel-Hall algebras and Green’s formula. First we will recall the definition of Ringel-Hall algebra of $\mathcal{A}$. Let $\text{Iso}(\mathcal{A})$ be the collection of isomorphic classes in $\mathcal{A}$. For any $M \in \mathcal{A}$, we always use the notation $a_M$ to denote the cardinality of the automorphism group $\text{Aut}(M)$ of $M$.

For any three objects $A, B, M \in \mathcal{A}$, define $\text{Ext}^1_\mathcal{A}(A, B)_M \subseteq \text{Ext}^1_\mathcal{A}(A, B)$ as the subset parameterizing extensions whose middle term is isomorphic to $M$. The Ringel-Hall algebra (or Hall algebra for short) $\mathcal{H}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the $\mathbb{Q}$-vector space with the basis $\{[M] \mid [M] \in \text{Iso}(\mathcal{A})\}$, endowed with the multiplication defined by (see [6])

$$[A] \odot [B] = \sum_{[M] \in \text{Iso}(\mathcal{A})} \frac{|\text{Ext}^1_\mathcal{A}(A, B)_M|}{|\text{Hom}_\mathcal{A}(A, B)|} \cdot [M].$$

We remark that the Ringel-Hall algebra used here is the dual version of the original one defined in [22]. Throughout the paper, when we sum for $[M]$, we take all the elements $[M] \in \text{Iso}(\mathcal{A})$ unless stated otherwise.

For any three objects $A, B, M \in \mathcal{A}$, the Hall number $F_{AB}^M$ is given by

$$F_{AB}^M := |\{X \subseteq M \mid X \cong B, M/X \cong A\}|. \quad \tag{2.1}$$

The Riedtmann-Peng formula states that

$$F_{AB}^M = \frac{|\text{Ext}^1_\mathcal{A}(A, B)_M|}{|\text{Hom}_\mathcal{A}(A, B)|} \cdot \frac{a_M}{a_AA_B}. $$
The associativity of Hall numbers tells that
\[
\sum_{[X]} F^X_{AB} F^M_{XC} = \sum_{[Y]} F^M_{AY} F^Y_{BC},
\]
which will be denoted by \( F^M_{ABC} \) for short.

Since \( \mathcal{A} \) is hereditary, the following famous Green’s formula [9, Theorem 2] holds:
\[
\sum_{[E]} F^E_{MN} F^E_{XY} \frac{1}{a_E} = \sum_{[A],[B],[C],[D]} q^{-\langle A,D \rangle} F^M_{AB} F^N_{CD} F^X_{AC} F^Y_{BD} \frac{a_A a_B a_C a_D}{a_M a_N a_X a_Y},
\]
where \( \langle A, D \rangle = \dim \text{Hom}_A(A, D) - \dim \text{Ext}_A^1(A, D) \) is the Euler form in \( \mathcal{A} \).

2.2. \( \Delta \)-Hall algebras. In order to give a realization of quantum groups, Lu-Wang [18] introduce the notion of Hall algebras. More precisely, they firstly considered the Ringel-Hall algebra \( H(C_1(\mathcal{A})) \) of the category \( C_1(\mathcal{A}) \) of 1-periodic complexes over \( \mathcal{A} \), and then used the techniques of quotient and localization successively to obtain the so-called semi-derived Hall algebra \( SDH(C_1(\mathcal{A})) \) of \( C_1(\mathcal{A}) \), and finally by using certain twisting to obtain the \( \mathfrak{t} \)-Hall algebra \( \mathfrak{t}H(\mathcal{A}) \). It turns out that the central reduction of the \( \mathfrak{t} \)-Hall algebra provides a realization of the quantum group.

Observe that the construction of \( \mathfrak{t} \)-Hall algebras \( \mathfrak{t}H(\mathcal{A}) \) is complicated, and it is difficult to obtain the basis of \( \mathfrak{t}H(\mathcal{A}) \). In this paper, inspired by Lu-Wang’s multiplication formula, we hope to define a new kind of Hall algebra \( H_{\Delta}(\mathcal{A}) \) following Ringel’s framework, called the \( \Delta \)-Hall algebra, in order to provide a new way to realize the quantum group.

Let \( v = \sqrt{q} \). For any three objects \( A, B, M \in \mathcal{A} \), we define the \( \Delta \)-Hall number as follows,
\[
\hat{F}^M_{AB} = \sum_{[L],[I],[N]} v^{\langle L,I,N \rangle} \cdot \frac{a_L a_I a_N}{a_M} \cdot F^B_{L,I} F^M_{N,L} F^A_{I,N},
\]
where
\[
\langle L, I, N \rangle = \langle L, I \rangle + \langle I, I \rangle + \langle I, N \rangle - \langle L, N \rangle.
\]
Roughly speaking, the \( \Delta \)-Hall number \( \hat{F}^M_{AB} \) calculates the number of three-cycles of the form
\[
\begin{array}{c}
I \\
\downarrow \\
A \\
\downarrow \\
N \leftarrow M \leftarrow L.
\end{array}
\]
Here, each two adjacent arrows on the same line form a short exact sequence in \( \mathcal{A} \).

We have the following associativity for the \( \Delta \)-Hall numbers, where Green’s formula (2.2) plays key roles.

**Proposition 2.1.** For any objects \( A, B, C, M \in \mathcal{A} \), the following equation holds:
\[
\sum_{[X]} \hat{F}^X_{AB} \hat{F}^M_{XC} = \sum_{[Y]} \hat{F}^M_{AY} \hat{F}^Y_{BC}.
\]
Proof. Consider the left side of (2.5), which calculates the following three-cycles of exact sequences:

\[ \sum_{[X]} F_{AB}^X F_{XC}^M = \sum_{[X]} \sum_{[L_1],[I_1],[N_1]} \sum_{[L_2],[I_2],[N_2]} \sum_{[G_1],[G_2],[G_3],[G_4]} v^{(L_1,I_1,N_1)} \cdot \frac{a_{L_1}a_{L_1}a_{N_1}}{a_X} \cdot F_{L_1I_1}^B F_{N_1L_1}^X F_{I_1N_1}^A \]

\[ = \sum_{[X]} \sum_{[L_1],[I_1],[N_1]} \sum_{[L_2],[I_2],[N_2]} \sum_{[G_1],[G_2],[G_3],[G_4]} v^{(L_1,I_1,N_1)+(L_2,I_2,N_2)} \cdot \frac{a_{L_1}a_{L_1}a_{N_1}a_{L_2}a_{L_2}a_{N_2}}{a_M} \cdot F_{L_1I_1}^B F_{I_1N_1}^A F_{L_2I_2}^C F_{N_2L_2}^M \]

\[ = \sum_{[X]} \sum_{[L_1],[I_1],[N_1]} \sum_{[L_2],[I_2],[N_2]} \sum_{[G_1],[G_2],[G_3],[G_4]} v^{(L_1,I_1,N_1)+(L_2,I_2,N_2)} \cdot \frac{a_{L_1}a_{L_1}a_{N_1}a_{L_2}a_{L_2}a_{N_2}}{a_M} \cdot F_{L_1I_1}^B F_{I_1N_1}^A F_{L_2I_2}^C F_{N_2L_2}^M \]

Using Green’s formula, we obtain

\[ \sum_{[X]} F_{N_1L_1}^X F_{I_2N_2}^X \frac{1}{a_X} = \sum_{[G_1],[G_2],[G_3],[G_4]} q^{-(G_3,G_1)} F_{G_2G_1}^{L_1} F_{G_3G_2}^{I_2} F_{G_4G_3}^{N_3} F_{G_4G_1}^{N_2} \frac{a_{G_1}a_{G_2}a_{G_3}a_{G_4}}{a_M} \]

In the remaining of the proof, we use the simplified notation \( \sum_i A_i, \sum_j D_j \) (without confusion) to denote \( \sum_{i,j} (A_i, D_j) \) for any \( A_i, D_j \in A \). Then we get

\[ \sum_{[X]} F_{AB}^X F_{XC}^M = \sum_{[L_1],[I_1],[N_1]} \sum_{[L_2],[I_2],[N_2]} \sum_{[G_1],[G_2],[G_3],[G_4]} v^{(G_1+G_2,J_1+G_4)+(L_2,G_2+G_3,G_4+G_1)}-2(G_3,G_1) \cdot \frac{a_{L_1}a_{L_2}a_{G_1}a_{G_2}a_{G_3}a_{G_4}}{a_M} \]

\[ = \sum_{[L_1],[I_1],[N_1]} \sum_{[L_2],[I_2],[N_2]} \sum_{[G_1],[G_2],[G_3],[G_4]} v^{(L_2,G_2+G_3,G_4+G_1)}-2(G_3,G_1) \cdot \frac{a_{L_1}a_{L_2}a_{G_1}a_{G_2}a_{G_3}a_{G_4}}{a_M} \]

\[ = \sum_{[L_1],[I_1],[N_1]} \sum_{[L_2],[I_2],[N_2]} \sum_{[G_1],[G_2],[G_3],[G_4]} v^{(L_2,G_2+G_3,G_4+G_1)}-2(G_3,G_1) \cdot \frac{a_{L_1}a_{L_2}a_{G_1}a_{G_2}a_{G_3}a_{G_4}}{a_M} \]

(2.6)
On the other hand, consider the right-hand side of (2.5), which calculates the following three-cycles of exact sequences:

\[
\begin{align*}
&I_3 \\
\quad \egin{array}{c}
\not A \\
\downarrow \quad \not Y \\
N_3 \not \leftarrow \not M \not \leftarrow \not L_3
\end{array} & & I_4 \\
\quad \egin{array}{c}
B \\
\downarrow \quad \not C \\
N_4 \not \leftarrow \not Y \not \leftarrow \not L_4
\end{array}
\end{align*}
\]

We have

\[
\sum_{[Y]} \hat{F}_{AY}^M \hat{F}_{BC}^Y = \sum_{[Y]} \sum_{[L_3],[I_3],[N_3]} \nu^{(L_3,I_3,N_3)} \cdot \frac{a_{L_3}a_{I_3}a_{N_3}}{a_M} \cdot F_{L_3I_3}^Y \cdot F_{N_3L_3}^M \cdot F_{I_3N_3}^A \\
\quad \cdot \sum_{[L_4],[I_4],[N_4]} \nu^{(L_4,I_4,N_4)} \cdot \frac{a_{L_4}a_{I_4}a_{N_4}}{a_Y} \cdot F_{L_4I_4}^Y \cdot F_{N_4L_4}^M \cdot F_{I_4N_4}^B \\
= \sum_{[L_3],[I_3],[N_3],\{L_4,I_4,N_4\}} \nu^{(L_3,I_3,N_3)+(L_4,I_4,N_4)} \cdot \frac{a_{L_3}a_{I_3}a_{N_3}a_{L_4}a_{I_4}a_{N_4}}{a_M} \cdot F_{L_3N_3}^A \cdot F_{N_3L_3}^M \cdot F_{L_4I_4}^C \cdot F_{I_4N_4}^B \cdot \sum_{[Y]} F_{L_3I_3}^Y \cdot F_{N_4L_4}^Y \cdot \frac{1}{a_Y}.
\]

By using Green’s formula again, we get

\[
\sum_{[Y]} \hat{F}_{AY}^M \hat{F}_{BC}^Y = \sum_{[L_3],[I_3],[N_3],\{L_4,I_4,N_4\}} \nu^{(L_3,I_3,N_3)+(L_4,I_4,N_4)} \cdot \frac{a_{L_3}a_{I_3}a_{N_3}a_{L_4}a_{I_4}a_{N_4}}{a_M} \cdot F_{L_3N_3}^A \cdot F_{N_3L_3}^M \cdot F_{L_4I_4}^C \cdot F_{I_4N_4}^B \\
\quad \cdot \sum_{[H_1],[H_2],[H_3],[H_4]} \nu^{(H_2+H_3+H_4+H_5,N_3)+(H_1+H_2,I_4,H_3+H_4)} \phi^{-1} (H_3,H_1) \cdot F_{H_2H_3}^{L_3} \cdot F_{H_2H_4}^{L_3} \cdot F_{H_3H_4}^{N_3} \cdot F_{H_4H_1}^{I_3} \cdot \frac{a_{H_1}a_{H_2}a_{H_3}a_{H_4}}{a_{L_3}a_{I_3}a_{N_3}a_{L_4}} \\
= \sum_{[L_3],[I_3],[N_3],\{L_4,I_4,N_4\}} \nu^{(L_2+G_1,I_1+G_3,G_4)+(G_3+L_2,G_2,G_1+I_1)} \phi^{-1} (G_1,G_3) \cdot F_{H_2H_3}^{L_3} \cdot F_{H_2H_4}^{L_3} \cdot F_{H_3H_4}^{N_3} \cdot F_{H_4H_1}^{I_3} \cdot \frac{a_{G_1}a_{G_2}a_{G_3}a_{L_2}a_{G_1}a_{I_1}}{a_M} \cdot F_{H_1H_2}^{A} \cdot F_{H_1H_3}^{M} \cdot F_{H_1H_4}^{C} \cdot F_{H_2H_4}^{B} \cdot \frac{1}{a_Y}.
\]

Observe that the last summation above is taken over \([N_3],[I_4],[H_1],[H_2],[H_3],[H_4]\) for arbitrary isomorphism classes from \(A\). By replacing them by \([G_1],[G_2],[G_3],[L_2],[I_1],[G_1]\) respectively, we obtain

\[
\sum_{[Y]} \hat{F}_{AY}^M \hat{F}_{BC}^Y = \sum_{[G_1],[G_2],[G_3],[L_2],[I_1],[G_1]} \nu^{(L_2+G_1,I_1+G_3,G_4)+(G_3+L_2,G_2,G_1+I_1)} \phi^{-1} (G_1,G_3) \cdot F_{H_2H_3}^{L_3} \cdot F_{H_2H_4}^{L_3} \cdot F_{H_3H_4}^{N_3} \cdot F_{H_4H_1}^{I_3} \cdot \frac{a_{G_1}a_{G_2}a_{G_3}a_{L_2}a_{G_1}a_{I_1}}{a_M} \cdot F_{H_1H_2}^{A} \cdot F_{H_1H_3}^{M} \cdot F_{H_1H_4}^{C} \cdot F_{H_2H_4}^{B} \cdot \frac{1}{a_Y}.
\]
Comparing (2.6) and (2.7), in order to prove (2.5), it remains to show the following relation:

\[
\langle G_1 + G_2, I_1, G_3 + G_4 \rangle + \langle L_2, G_2 + G_3, G_4 + G_1 \rangle - 2\langle G_3, G_1 \rangle
\]

\[
= \langle L_2 + G_1, I_1 + G_3, G_4 \rangle + \langle G_3 + L_2, G_2, G_1 + I_1 \rangle - 2\langle G_1, G_3 \rangle,
\]

which follows easily but tediously, via expanding both sides of the equation by using (2.4). □

**Theorem 2.2.** The \(\mathbb{Q}(v)\)-vector space \(H_\Delta(A)\) with the basis \(\{ [M] \mid [M] \in \text{Iso}(A) \}\), endowed with the multiplication defined by

\[
[A] * [B] = \sum_{[M]} \widehat{F}^M_{AB} \cdot [M],
\]

forms an associative algebra with the unit \([0]\), called the \(\Delta\)-Hall algebra of \(A\).

**Proof.** We have

\[
([A] * [B]) * [C] = \sum_{[X]} \widehat{F}^X_{AB} \cdot [X] * [C]
\]

\[
= \sum_{[X]} \widehat{F}^X_{AB} \cdot \sum_{[M]} \widehat{F}^M_{XC} \cdot [M]
\]

\[
= \sum_{[M]} \sum_{[X]} \widehat{F}^X_{AB} \widehat{F}^M_{XC} \cdot [M].
\]

On the other hand, we have

\[
[A] * ([B] * [C]) = [A] * (\sum_{[Y]} \widehat{F}^Y_{BC} \cdot [Y])
\]

\[
= \sum_{[Y]} \widehat{F}^Y_{BC} \cdot [A] * [Y]
\]

\[
= \sum_{[M]} \sum_{[Y]} \widehat{F}^M_{AY} \widehat{F}^Y_{BC} \cdot [M].
\]

By Proposition 2.1, we immediately get

\[
([A] * [B]) * [C] = [A] * ([B] * [C]).
\]

Clearly, \([0]\) is the unit by definition. Then we are done. □

### 3. \(\Delta\)-Hall Algebra and Quantum Group

In this section, we establish an isomorphism between \(\Delta\)-Hall algebras and 1-periodic derived Hall algebras, and then provide a new realization of quantum groups.

#### 3.1. \(\Delta\)-Hall algebras and 1-periodic derived Hall algebras

Recall that \(A\) is an essentially small hereditary abelian category over the finite field \(k = \mathbb{F}_q\). Let \(D_1(A)\) be the derived category of 1-periodic complexes on \(A\). For any object \(M^\bullet = (M, d) \in D_1(A)\), there exists an isomorphism \((M, d) \cong (H(M^\bullet), 0)\) in \(D_1(A)\), where \(H(M^\bullet) = \text{Ker}(d)/\text{Im}(d) \in A\) is the cohomology group of \(M^\bullet\). Hence, the isoclasses \(\text{Iso}(D_1(A))\) of \(D_1(A)\) coincide with the isoclasses \(\text{Iso}(A)\) of \(A\). From now on, we identify \(\text{Iso}(D_1(A))\) and \(\text{Iso}(A)\).
For any object $M \in \mathcal{A}$, we denote $|\text{Aut}_{D_1(A)}(M)|$ by $\tilde{a}_M$. Then we have
$$\tilde{a}_M = a_M \cdot |\text{Ext}^1_A(M, M)|.$$  
For any objects $A, B, M \in \mathcal{A}$, following [27] we use the notations
$$(A, B)_M := \{ f \in \text{Hom}_{D_1(A)}(A, B)|\text{Cone}(f) \cong M\}$$
and
$$\{A, B\} := \frac{1}{|\text{Hom}_{D_1(A)}(A, B)|}.$$  
Then by [27, Corollary 2.7] we have
$$\frac{|(B, M)_A|}{\tilde{a}_B} \sqrt{\frac{\{B, M\}}{\{B, B\}}} = \frac{|(M, A)_B|}{\tilde{a}_A} \sqrt{\frac{\{A, B\}\{M, M\}}{\{A, A\}\{B, B\}}}.$$  
We denote this number by $G^M_{AB}$ in this paper, and call it the derived Hall number, which satisfies the following derived Riedtmann-Peng formula by [23, Proposition 3.3]:
$$G^M_{AB} = \frac{\tilde{a}_M \cdot |(A, B)_M|}{\tilde{a}_A \cdot \tilde{a}_B} \sqrt{\frac{\{A, B\}\{M, M\}}{\{A, A\}\{B, B\}}}.$$  

The 1-periodic derived Hall algebra $\mathcal{D}H_1(A)$ of $\mathcal{A}$ is a $\mathbb{Q}(v)$-vector space with the basis \{ $u_{[M]} | [M] \in \text{Iso}(D_1(A)) := \text{Iso}(A)$ \}, endowed with the multiplication defined by
$$u_{[A]} \ast u_{[B]} = \sum_{[M]} G^M_{AB} \cdot u_{[M]}.$$  

The following proposition indicates that the derived Hall number $G^M_{AB}$ can be calculated inside the abelian category $\mathcal{A}$.

**Proposition 3.1.** [7, Proposition 3.5] For any objects $A, B, M \in \mathcal{A}$, we have

$$G^M_{AB} = \sum_{|L|, |I|, |N|} v^{(L, I) + (I, N) - (L, N)} \cdot \frac{a_L a_I a_N}{a_A a_B} \cdot F^B_L F^M_{NL} F^A_{IN}.$$  

Comparing (3.1) with (2.3), and using (2.4), we can obtain an isomorphism from the 1-periodic derived Hall algebra $\mathcal{D}H_1(A)$ to the $\Delta$-Hall algebra $\mathcal{H}_\Delta(A)$.

**Proposition 3.2.** There is an algebra isomorphism
$$\Xi_A : \mathcal{D}H_1(A) \longrightarrow \mathcal{H}_\Delta(A).$$  

$$u_{[M]} \mapsto \frac{1}{a_M} \cdot [M]$$

**Proof.** Recall that we have the identity $\text{Iso}(D_1(A)) = \text{Iso}(A)$. For any $A, B \in \mathcal{A}$, we have
$$\Xi_A(u_{[A]} \ast u_{[B]}) = \Xi_A(\sum_{[M]} G^M_{AB} \cdot u_{[M]})$$
$$= \sum_{[M]} \sum_{|L|, |I|, |N|} v^{(L, I, N)} \cdot \frac{a_L a_I a_N}{a_A a_B} \cdot F^B_L F^M_{NL} F^A_{IN} \cdot \Xi_A(u_{[M]})$$
$$= \sum_{[M]} \sum_{|L|, |I|, |N|} v^{(L, I, N)} \cdot \frac{a_L a_I a_N}{a_A a_B a_M} \cdot F^B_L F^M_{NL} F^A_{IN} \cdot [M].$$
On the other hand,
\[ \Xi_A(u_{[A]}) \ast \Xi_A(u_{[B]}) = \frac{1}{a_A a_B} [A] \ast [B] \]
\[ = \frac{1}{a_A a_B} \sum_{[M]} \hat{F}^M_{A B} \cdot [M] \]
\[ = \frac{1}{a_A a_B} \sum_{[M]} \sum_{[L], [I], [N]} v^{(L, I, N)} \cdot \frac{a_L a_I a_N}{a_M} \cdot F^B_L F^M_{NL} F^A_{IN} \cdot [M] \]
\[ = \sum_{[M]} \sum_{[L], [I], [N]} v^{(L, I, N)} \cdot \frac{a_L a_I a_N}{a_A a_B a_M} \cdot F^B_L F^M_{NL} F^A_{IN} \cdot [M]. \]

Hence \( \Xi_A(u_{[A]}) \ast \Xi_A(u_{[B]}) = \Xi_A(u_{[A] \ast u_{[B]}}) \) and then \( \Xi_A \) is an algebra homomorphism. Since the algebras \( \mathcal{D} \mathcal{H}_1(\mathcal{A}) \) and \( \mathcal{H}_\Delta(\mathcal{A}) \) share the same basis \( \text{Iso}(\mathcal{D}_1(\mathcal{A})) = \text{Iso}(\mathcal{A}) \), we obtain that \( \Xi_A \) is bijective. Therefore, \( \Xi_A \) is an isomorphism of algebras. \( \square \)

### 3.2. New realization of \( i \)-Quantum groups of split type

Now we will recall the basic setting on (universal) quantum groups and (universal) quantum groups. Let \( Q \) be a quiver (without loops) with vertex set \( Q_0 = \mathbb{I} \). Let \( n_{ij} \) be the number of edges connecting vertices \( i \) and \( j \). The symmetric generalized Cartan matrix of the underlying graph of \( Q \) is denoted by \( C = (c_{ij})_{i,j \in \mathbb{I}}, \) where \( c_{ij} = 2 \delta_{ij} - n_{ij} \). Let \( \mathfrak{g} \) be the corresponding Kac-Moody Lie algebra of \( Q \) and \( C \).

Let \( v \) be an indeterminant. For \( r, m \in \mathbb{N} \), denote by
\[ [r] = \frac{v^r - v^{-r}}{v - v^{-1}}, \quad [r]! = \prod_{i=1}^{r} [i], \quad [m]_r = \frac{[m][m-1] \ldots [m-r+1]}{[r]!}. \]

The **universal quantum group** \( \hat{\mathbf{U}} := \hat{\mathbf{U}}(\mathfrak{g}) \) of \( \mathfrak{g} \) is defined to be the \( \mathbb{Q}(v) \)-algebra generated by \( E_i, F_i, \hat{K}_i, \hat{K}'_i, i \in \mathbb{I} \), where \( \hat{K}_i, \hat{K}'_i \) are invertible, subject to the following relations for \( i, j \in \mathbb{I} \):

\begin{align*}
(3.2) \quad & [E_i, F_j] = \delta_{ij} \hat{K}_i - \hat{K}'_i, \quad [\hat{K}_i, \hat{K}_j] = [\hat{K}'_i, \hat{K}'_j] = [\hat{K}_i, \hat{K}'_j] = 0, \\
(3.3) \quad & \hat{K}_i E_j = v^{c_{ij}} E_j \hat{K}_i, \quad \hat{K}_i F_j = v^{-c_{ij}} F_j \hat{K}_i, \\
(3.4) \quad & \hat{K}'_i E_j = v^{-c_{ij}} E_j \hat{K}'_i, \quad \hat{K}'_i F_j = v^{c_{ij}} F_j \hat{K}'_i,
\end{align*}

and the quantum Serre relations for \( i \neq j \in \mathbb{I} \),

\begin{align*}
(3.5) \quad & \sum_{r=0}^{1-c_{ij}} (-1)^r E_i^{(r)} E_j F_i^{(1-c_{ij}-r)} = 0, \\
(3.6) \quad & \sum_{r=0}^{1-c_{ij}} (-1)^r F_i^{(r)} F_j E_i^{(1-c_{ij}-r)} = 0.
\end{align*}

Here, \( [A, B] = AB - BA, \) and for any \( n \geq 1, i \in \mathbb{I} \),
\[ E_i^{(n)} = \frac{E_i^n}{[n]!}, \quad F_i^{(n)} = \frac{F_i^n}{[n]!}. \]

Note that \( \hat{K}_i, \hat{K}'_i \) (\( i \in \mathbb{I} \)) are central in \( \hat{\mathbf{U}} \).

The **universal \( i \)-quantum group of split type** \( \hat{\mathbf{U}}^i \) is defined to be the \( \mathbb{Q}(v) \)-subalgebra of \( \hat{\mathbf{U}} \) generated by
\[ B_i = E_i + E_i \hat{K}'_i, \quad k_i = \hat{K}_i \hat{K}'_i, \quad \forall i \in \mathbb{I}. \]
Let $\tilde{U}$ be the $\mathbb{Q}(v)$-subalgebra of $\tilde{U}$ generated by central elements $k_i$, for $i \in I$.

Analogous as for $U$, the quantum group $U := U(q)$ is defined to be the $\mathbb{Q}(v)$-algebra generated by $E_i, F_i, K_i, K_i^{-1}$, $i \in I$, subject to the relations modified from (3.2)–(3.6) with $\tilde{K}_i$ and $\tilde{K}_i'$ replaced by $K_i$ and $K_i^{-1}$, respectively.

Let $\xi = (\xi_i)_{i \in I} \in (\mathbb{Q}(v))_1^I$. The quantum group of split type $U^\prime := U^\prime_1$ is the $\mathbb{Q}(v)$-subalgebra of $U$ generated by

$$B_i = F_i + \xi_i E_i K_i^{-1}, \quad \forall i \in I.$$ 

In order to give a realization of the quantum groups via $\Delta$-Hall algebra approach, we consider the nilpotent representation category $A = \text{rep}^{nil}(kQ)$. Let $S_i$ $(i \in I)$ be the set of simple $kQ$-modules. We denote by $D\mathcal{H}_1(kQ) := D\mathcal{H}_1(A)$, $\mathcal{H}_\Delta(kQ) := \mathcal{H}_\Delta(A)$ and $\Xi_Q := \Xi_A$.

**Theorem 3.3.** Let $Q$ be an arbitrary quiver without loops. Then there exists a $\mathbb{Q}(v)$-algebra embedding

$$\Theta_Q : U^\prime_{|v = v} \longrightarrow \mathcal{H}_\Delta(kQ),$$

which sends

$$B_i \mapsto -v^{-1}[S_i]_{a_{S_i}}, \quad \text{for } i \in I.$$ 

In particular, if $Q$ is a Dynkin quiver, the embedding $\Theta_Q$ is precisely an isomorphism.

**Proof.** Recall from [7, Theorem 4.4] that there exists a $\mathbb{Q}(v)$-algebra embedding

$$\Psi_Q : U^\prime_{|v = v} \longrightarrow D\mathcal{H}_1(kQ),$$

which sends

$$B_i \mapsto -v^{-1}u[S_i], \quad \text{for } i \in I.$$ 

Let $\Theta_Q$ be the following composition

$$\Theta_Q = \Xi_Q \circ \Psi_Q : U^\prime_{|v = v} \longrightarrow D\mathcal{H}_1(kQ) \longrightarrow \mathcal{H}_\Delta(kQ).$$

Then the result follows from Proposition 3.2 immediately. \hfill \Box

4. **Extended $\Delta$-Hall algebra and universal quantum group**

In this section, we define the extended $\Delta$-Hall algebra of $A$, and prove that it is isomorphic to the $\Delta$Hall algebra of $A$, hence provide a new realization of the universal quantum group associated to $A$.

4.1. **Extended $\Delta$-Hall algebras.** Let $K_0(A)$ be the Grothendieck group of $A$. It is well-known that the Ringel-Hall algebra $\mathcal{H}(A)$ has an extended version via tensoring with the group algebra of $K_0(A)$. Similarly, the $\Delta$-Hall algebras also admit extended versions, which can be used to realize the universal quantum groups.

For technical reason we assume $K_0(A)$ is a free abelian group. Both the category of finite-dimensional representations of a quiver, and the category of coherent sheaves over a weighted projective curve, satisfy this assumption.

For any $M \in A$, the image of $M$ in $K_0(A)$ is denoted by $\widetilde{M}$. The symmetric Euler form in $A$ (or in $K_0(A)$) is determined by $(M, N) = (M, N) + (N, M)$ for any $M, N \in A$. We have the following observation.
Lemma 4.1. For any three-cycle in $\mathcal{A}$ as follows:

\[
\begin{array}{ccc}
I & \downarrow & \\
\nearrow & A & B \\
\searrow & N & \leftarrow M & \leftarrow L,
\end{array}
\]

we have

(i) \(2\hat{L} = \hat{M} + \hat{B} - \hat{A}\);
(ii) \(2\hat{I} = \hat{A} + \hat{B} - \hat{M}\);
(iii) \(2\hat{N} = \hat{M} + \hat{A} - \hat{B}\).

Consequently, \(2(N, L) = (M, M) - (A, A) - (B, B) + (A, B)\).

Proof. The three-cycle (4.1) indicates that we have the following exact sequence:

\[
0 \rightarrow N \rightarrow A \rightarrow B \rightarrow M \rightarrow N \rightarrow 0.
\]

Hence, \(2\hat{N} = \hat{M} + \hat{A} - \hat{B}\), which proves (iii). Similarly, we can obtain (i) and (ii).

Observe that

\[
4(N, L) = (2\hat{N}, 2\hat{L}) = (\hat{M} + \hat{A} - \hat{B}, \hat{M} + \hat{B} - \hat{A})
\]

\[
= (\hat{M}, \hat{M}) - (\hat{A} - \hat{B}, \hat{A} - \hat{B})
\]

\[
= (\hat{M}, \hat{M}) - (\hat{A}, \hat{A}) - (\hat{B}, \hat{B}) + 2(\hat{A}, \hat{B}).
\]

By assumption, \(K_0(\mathcal{A})\) is a free abelian group. Then (4.2) follows immediately. \(\square\)

Denote by \(\tilde{\mathcal{H}}_\Delta(\mathcal{A})\) the \(\mathbb{Q}(v)\)-vector space with the basis

\[
\{[M][K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}.
\]

Proposition 4.2. The \(\mathbb{Q}(v)\)-vector space \(\tilde{\mathcal{H}}_\Delta(\mathcal{A})\) endowed with the multiplication defined by

\[
[A][K_\alpha] \ast [B][K_\beta] = \sum_{[M]} \tilde{F}^M_{AB} \cdot [M][K_\alpha \hat{\alpha} \hat{\beta} \hat{\gamma} + \alpha + \beta],
\]

forms an associative algebra with unit \([0][K_0]\), called the extended \(\Delta\)-Hall algebra of \(\mathcal{A}\).

Proof. Recall that

\[
\tilde{F}^M_{AB} = \sum_{[L],[I],[N]} v^{(L,I,N)} a^M_{\alpha \beta \gamma} a^A_M \cdot F^B_L F^M_N F^A_I.
\]

If \(\tilde{F}^M_{AB} \neq 0\), then all of \(F^B_L, F^M_N, F^A_I\) are non-zero, which follows that \(\hat{A} + \hat{B} - \hat{M} = 2\hat{I}\) by Lemma 4.1. Hence

\[
\frac{\hat{A} + \hat{B} - \hat{M}}{2} \in K_0(\mathcal{A}).
\]

So the multiplication is well-defined.

For any \([A], [B], [C] \in \text{Iso}(\mathcal{A})\) and \(\alpha, \beta, \gamma \in K_0(\mathcal{A})\), we have

\([A][K_\alpha] \ast [B][K_\beta] \ast [C][K_\gamma] = \)
Now it follows from Lemma 2.1 that

\[ ([A][K_\alpha] * ([B][K_\beta]) * [C][K_\gamma]) = [A][K_\alpha] * ([B][K_\beta] * [C][K_\gamma]) \]

Then we are done. \( \square \)

### 4.2. Extended \( \Delta \)-Hall algebras and \( \mathfrak{t} \)Hall algebras.

In this subsection we show that the extend \( \Delta \)-Hall algebra \( \widehat{H}_\Delta(\mathcal{A}) \) is isomorphic to the \( \mathfrak{t} \)Hall algebra of \( \mathcal{A} \). First let us recall the construction of the \( \mathfrak{t} \)Hall algebra due to \([15, 20, 16]\).

Denote by \( \mathcal{H}(\mathcal{C}_1(\mathcal{A})) \) the Ringel-Hall algebra for the category \( \mathcal{C}_1(\mathcal{A}) \) of 1-periodic complexes over \( \mathcal{A} \). Consider the ideal \( \mathcal{I} \) of \( \mathcal{H}(\mathcal{C}_1(\mathcal{A})) \) generated by

\[ \{ [M^\bullet] - [N^\bullet] \mid H(M^\bullet) \cong H(N^\bullet), \quad \text{Im} d_{M^\bullet} = \text{Im} d_{N^\bullet} \}. \]

Denote by

\[ \mathcal{S} := \{ a[K^\bullet] \in \mathcal{H}(\mathcal{C}_1(\mathcal{A}))/\mathcal{I} \mid a \in \mathbb{Q}(v)^\times, K^\bullet \in \mathcal{C}_1(\mathcal{A}) \text{ acyclic} \}, \]

a multiplicatively closed subset of \( \mathcal{H}(\mathcal{C}_1(\mathcal{A}))/\mathcal{I} \) with the identity \([0]\), which is right Ore and right reversible. Hence there exists the right localization of \( \mathcal{H}(\mathcal{C}_1(\mathcal{A}))/\mathcal{I} \) with respect to \( \mathcal{S} \), called the semi-derived Hall algebra of \( \mathcal{C}_1(\mathcal{A}) \) in the sense of \([15, 20]\), and will be denoted by \( \mathcal{S} \mathcal{D} \mathcal{H}(\mathcal{C}_1(\mathcal{A})) \).

The \( \mathfrak{t} \)Hall algebra \( \mathfrak{t} \widehat{H}(\mathcal{A}) \) of \( \mathcal{A} \) is defined to be the twisted semi-derived Hall algebra \( \mathcal{S} \mathcal{D} \mathcal{H}(\mathcal{C}_1(\mathcal{A})) \) via the restriction functor \( \text{res} : \mathcal{C}_1(\mathcal{A}) \to \mathcal{A} \) by forgetting differentials. That is, \( \mathfrak{t} \widehat{H}(\mathcal{A}) \) is the \( \mathbb{Q}(v) \)-algebra on the same vector space as \( \mathcal{S} \mathcal{D} \mathcal{H}(\mathcal{C}_1(\mathcal{A})) \), equipped with the following modified multiplication

\[ [M^\bullet] * [N^\bullet] = \sqrt{\text{res}(M^\bullet) \text{res}(N^\bullet)} [M^\bullet] \circ [N^\bullet]; \quad \forall M^\bullet, N^\bullet \in \mathcal{C}_1(\mathcal{A}). \]

By Theorem 4.5 and Proposition 4.10 in \([20]\), the \( \mathfrak{t} \)Hall algebra \( \mathfrak{t} \widehat{H}(\mathcal{A}) \) has a basis

\[ \{ [M] * [K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A}) \}, \]

and the elements \([K_\alpha] \ (\alpha \in K_0(\mathcal{A}))\) are central in \( \mathfrak{t} \widehat{H}(\mathcal{A}) \). In order to give an explicit description of the multiplication formula for the basis elements in \( \mathfrak{t} \widehat{H}(\mathcal{A}) \), we introduce the following notation

\[ = \sum [\hat{F}^X_{AB} \cdot [X] [K_{\frac{a+b-x}{2}+\alpha+\beta}] * [C][K_\gamma] \]

\[ = \sum [\hat{F}^X_{AB} \cdot \sum [M] [K_{\frac{a+b-x}{2}+\alpha+\beta+\gamma}] \]

\[ = \sum [\hat{F}^X_{AB} \cdot [X] [M] [K_{\frac{a+b-x}{2}+\alpha+\beta+\gamma}]] \]

\[ \text{and} \]

\[ [A][K_\alpha] * ([B][K_\beta]) * [C][K_\gamma] = [A][K_\alpha] * ([B][K_\beta] * [C][K_\gamma]). \]
for any $[A], [B] \in \text{Iso}(\mathcal{A})$:

\begin{equation}
\hat{F}^M_{AB} = \sum_{[L],[I],[N]} v^{-(A,B)} q^{(N,L)} \frac{a_L a_I a_N}{a_M} F^B_L F^M_N F^A_I.
\end{equation}

**Lemma 4.3.** For any $A, B, M \in \mathcal{A}$,

$$\hat{F}^M_{AB} = v^{(M,M) - (A,A) - (B,B)} \cdot \hat{F}^M_{AB}.$$  

**Proof.** Recall from (2.3) that

\begin{equation}
\hat{F}^M_{AB} = \sum_{[L],[I],[N]} v^{(L,I,N)} \frac{a_L a_I a_N}{a_M} F^B_L F^M_N F^A_I.
\end{equation}

For $F^B_L F^M_N F^A_I \neq 0$, we have $\hat{A} = I + \hat{N}$ and $\hat{B} = I + \hat{L}$, which implies

$$\langle B, A \rangle = \langle L, I \rangle + \langle I, I \rangle + \langle I, N \rangle + \langle L, N \rangle.$$  

It follows from (2.4) that

$$\langle B, A \rangle - 2 \langle L, N \rangle = \langle L, I, N \rangle.$$  

Now according to (4.2), we obtain

$$-\langle A, B \rangle + 2 \langle N, L \rangle = \langle M, M \rangle - \langle A, A \rangle - \langle B, B \rangle + \langle B, A \rangle - 2 \langle L, N \rangle$$  

$$= \langle M, M \rangle - \langle A, A \rangle - \langle B, B \rangle + \langle L, I \rangle.$$  

Then by comparing with (4.4) and (4.5), the result follows immediately. \hfill \square

**Proposition 4.4.** For any $A, B \in \mathcal{A}$ and $\alpha, \beta \in K_0(\mathcal{A})$, the following equation holds in $\hat{\mathcal{H}}(\mathcal{A})$:

$$([A] * [K_\alpha]) * ([B] * [K_\beta]) = \sum_{[M]} \hat{F}^M_{AB} \cdot [M] * [K_{\frac{A+B-M}{2} + \alpha + \beta}].$$

**Proof.** By [7, Proposition 3.5], in $\hat{\mathcal{H}}(\mathcal{A})$ we have

$$[A] * [B] = \sum_{[M]} \sum_{[L],[I],[N]} v^{-(A,B)} q^{(N,L)} \frac{a_L a_I a_N}{a_M} F^B_L F^M_N F^A_I \cdot [M] * [K_{\frac{A+B-M}{2}}].$$

In case $F^B_L F^M_N F^A_I \neq 0$, there is a three-cycle (4.1). Then by Lemma 4.1, we obtain $\hat{A} - \hat{N} = \frac{A + B - M}{2}$, which is independent of $[L], [I], [N]$. Hence in $\hat{\mathcal{H}}(\mathcal{A})$ we have

$$[A] * [B] = \sum_{[M]} \hat{F}^M_{AB} \cdot [M] * [K_{\frac{A+B-M}{2}}].$$

Recall that $[K_\alpha]$ ($\alpha \in K_0(\mathcal{A})$) are central in $\hat{\mathcal{H}}(\mathcal{A})$. Then we are done. \hfill \square

Now we can state the main result of this subsection.

**Proposition 4.5.** There is an algebra isomorphism

$$\hat{\Xi}_\Delta : \hat{\mathcal{H}}(\mathcal{A}) \rightarrow \hat{\mathcal{H}}_\Delta(\mathcal{A}).$$

$$[M] * [K_\alpha] \mapsto v^{-\langle M,M \rangle} [M] [K_\alpha]$$
Proof. For any \( A, B \in \mathcal{A} \) and \( \alpha, \beta \in K_0(\mathcal{A}) \), by Proposition 4.4 we have
\[
\tilde{\Xi}_A(\langle [A] * [K_\alpha] \rangle * \langle [B] * [K_\beta] \rangle) = \sum_{[M]} \hat{F}_{AB}^M \cdot \tilde{\Xi}_A(\langle [M] * [K_{\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \alpha + \beta}] \rangle)
\]
\[
= \sum_{[M]} v^{-(M,M)} \hat{F}_{AB}^M \cdot \langle [M][K_{\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \alpha + \beta}] \rangle.
\]
On the other hand,
\[
\tilde{\Xi}_A(\langle [A] * [K_\alpha] \rangle) \ast \tilde{\Xi}_A(\langle [B] * [K_\beta] \rangle) = (\langle v^{-(A,A)} [A] \rangle[K_\alpha]) \ast (\langle v^{-(B,B)} [B] \rangle[K_\beta])
\]
\[
= \langle v^{-(A,A)} \rangle \ast \langle (B,B) \rangle \sum_{[M]} \hat{F}_{AB}^M \cdot \langle [M][K_{\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \alpha + \beta}] \rangle.
\]
It follows from Lemma 4.3 that
\[
\tilde{\Xi}_A(\langle [A] * [K_\alpha] \rangle) \ast \tilde{\Xi}_A(\langle [B] * [K_\beta] \rangle) = \tilde{\Xi}_A((\langle [A] * [K_\alpha] \rangle) \ast (\langle [B] * [K_\beta] \rangle)).
\]
Hence \( \tilde{\Xi}_A \) is an algebra homomorphism. By comparing the basis of \( ^t\widetilde{\mathcal{H}}(\mathcal{A}) \) and of \( \widetilde{\mathcal{H}}_\Delta(\mathcal{A}) \), we know that \( \tilde{\Xi}_A \) is bijective, so it is an algebra isomorphism. \(\Box\)

As a consequence of Propositions 4.5 and 3.2, we obtain

**Proposition 4.6.** There exists an algebra epimorphism
\[
\tilde{\Phi}: \widetilde{\mathcal{H}}_\Delta(\mathcal{A}) \longrightarrow \mathcal{H}_\Delta(\mathcal{A}),
\]
with \( \text{Ker } \tilde{\Phi} = \langle [K_\alpha] - 1, \alpha \in K_0(\mathcal{A}) \rangle \).

**Proof.** According to [7, Theorem 3.4], there exists an algebra epimorphism
\[
\Phi: ^t\widetilde{\mathcal{H}}(\mathcal{A}) \longrightarrow \mathcal{D}\mathcal{H}_1(\mathcal{A}); \quad [M] * [K_\alpha] \longrightarrow \sqrt{\langle M, M \rangle} \cdot \tilde{a}_M \cdot u_{[M]},
\]
with \( \text{Ker } \Phi = \langle [K_\alpha] - 1, \alpha \in K_0(\mathcal{A}) \rangle \). By Propositions 4.5 and 3.2, there are isomorphisms
\[
\tilde{\Xi}_A: ^t\widetilde{\mathcal{H}}(\mathcal{A}) \rightarrow \widetilde{\mathcal{H}}_\Delta(\mathcal{A})\quad \text{and}\quad \Xi_A: \mathcal{D}\mathcal{H}_1(\mathcal{A}) \rightarrow \mathcal{H}_\Delta(\mathcal{A}).
\]
Let \( \tilde{\Phi} \) be the composition
\[
\tilde{\Phi} = \Xi_A \circ \Phi \circ \Xi_A^{-1} : \widetilde{\mathcal{H}}_\Delta(\mathcal{A}) \longrightarrow \mathcal{D}\mathcal{H}_1(\mathcal{A}) \longrightarrow \mathcal{H}_\Delta(\mathcal{A}).
\]
Therefore, \( \tilde{\Phi}(\langle [M][K_\alpha] \rangle) = [M] \) for any \( M \in \mathcal{A} \) and \( \alpha \in K_0(\mathcal{A}) \). In particular, we have the following commutative diagram:
\[
\begin{array}{ccc}
\text{Ker } \Phi & \longrightarrow & ^t\widetilde{\mathcal{H}}(\mathcal{A}) \\
\downarrow \cong & & \downarrow \Phi \\
\text{Ker } \tilde{\Phi} & \longrightarrow & \widetilde{\mathcal{H}}_\Delta(\mathcal{A})
\end{array}
\]
\[
\begin{array}{ccc}
\Xi_A & \cong & \Xi_A \\
\downarrow & & \downarrow \\
\mathcal{D}\mathcal{H}_1(\mathcal{A}) & \longrightarrow & \mathcal{H}_\Delta(\mathcal{A}).
\end{array}
\]
It follows that \( \text{Ker } \tilde{\Phi} = \Xi_A(\text{Ker } \Phi) = \langle [K_\alpha] - 1, \alpha \in K_0(\mathcal{A}) \rangle \). We are done. \(\Box\)
4.3. New realization of universal quantum groups. Let $Q$ be an arbitrary quiver without loops. The simple $kQ$-modules are denoted by $S_i$ for $i \in \mathbb{I} = Q_0$. We consider the hereditary abelian category $A = \text{rep}_{\text{nil}}(kQ)$. Denote by $^{i}\mathcal{H}(kQ) := ^{i}\mathcal{H}(A)$, $H_\Delta(kQ) := H_\Delta(A)$ and $\tilde{\Xi}_Q := \tilde{\Xi}_A$. The following result states that we can realize the universal quantum groups via extended $\Delta$-Hall algebras.

**Theorem 4.7.** Let $Q$ be an arbitrary quiver without loops. There exists a $Q(v)$-algebra embedding

$$\tilde{\Theta}_Q : \tilde{U}_v \rightarrow H_\Delta(kQ),$$

which sends

$$B_i \mapsto -\frac{1}{q-1} \cdot v^{-(S_i, S_i)} \cdot [S_i], \quad k_i \mapsto -q^{-1}[K_i], \quad \text{for } i \in \mathbb{I}.$$

In particular, if $Q$ is a Dynkin quiver, the embedding $\tilde{\Theta}_Q$ is precisely an isomorphism.

**Proof.** Recall from [20, 18, 17] that the $i$Hall algebra $^{i}\mathcal{H}(kQ)$ of $\text{rep}_{\text{nil}}(kQ)$ provides a realization of the universal quantum group $U^i$ associated to $Q$. Namely, there is an embedding

$$\tilde{\psi}_Q : \tilde{U}_v \rightarrow ^{i}\mathcal{H}(kQ),$$

which sends

$$B_i \mapsto -\frac{1}{q-1}[S_i], \quad k_i \mapsto -q^{-1}[K_i], \quad \text{for } i \in \mathbb{I}.$$

In particular, if $Q$ is a Dynkin quiver, the embedding $\tilde{\psi}_Q$ is precisely an isomorphism.

On the other hand, by Proposition 4.5, there exists an algebra isomorphism

$$\tilde{\Xi}_Q : ^{i}\mathcal{H}(kQ) \rightarrow H_\Delta(kQ),$$

which sends

$$[M] \ast [K_\alpha] \mapsto v^{-(M,M)}[M][K_\alpha]; \quad \forall M \in A, \quad \alpha \in K_0(A).$$

Set $\tilde{\Theta}_Q = \tilde{\Xi}_Q \circ \tilde{\psi}_Q$, then we are done. \qed

5. Twisting on extended $\Delta$-Hall algebras

In this section, we consider the twisting on the extended $\Delta$-Hall algebra $H_\Delta(A)$, in order to recover the semi-derived Hall algebra $SDH(C_1(A))$ of $C_1(A)$. Moreover, by extending $K_0(A)$ to $K_0(A) \otimes \mathbb{Z}\frac{1}{2}\mathbb{Z}$, we establish an algebra isomorphism between the extended version of the $\Delta$-Hall algebra and the derived Hall algebra of $A$.

5.1. Twisted extended $\Delta$-Hall algebras and semi-derived Hall algebras. Let $\varphi : K_0(A) \times K_0(A) \rightarrow \mathbb{Q}(v)$ be a multiplicative bilinear form on $K_0(A)$ in the following sense:

$$\varphi(\alpha_1 + \alpha_2, \beta) = \varphi(\alpha_1, \beta) \cdot \varphi(\alpha_2, \beta) \quad \text{and} \quad \varphi(\alpha, \beta_1 + \beta_2) = \varphi(\alpha, \beta_1) \cdot \varphi(\alpha, \beta_2)$$

for $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in K_0(A)$.

**Proposition 5.1.** For any multiplicative bilinear form $\varphi : K_0(A) \times K_0(A) \rightarrow \mathbb{Q}(v)$, the following multiplication

$$[A][K_\alpha] \ast [B][K_\beta] = \varphi(\hat{A} + 2\alpha, \hat{B} + 2\beta) \cdot [A][K_\alpha] \ast [B][K_\beta]$$

defines an associative algebra structure on $H_\Delta(A)$, called the twisted extended $\Delta$-Hall algebra of $A$ via $\varphi$, and denoted by $\varphi H_\Delta(A)$. 

Proposition 5.2. Let \( \varphi : K_0(A) \times K_0(A) \rightarrow \mathbb{Q}(v) \) be a multiplicative bilinear form given by \( \varphi(\alpha, \beta) = v^{-(\alpha, \beta)} \). Then there is an algebra isomorphism:

\[
\varphi \Xi_A : SDH(C_1(A)) \rightarrow \tilde{H}_\Delta(A).
\]

\[
[M] \circ [K_\alpha] \mapsto v^{-(M,M)}[M][K_\alpha]
\]

Proof. Recall that the basis of \( SDH(C_1(A)) \) is given by

\[
\left\{ [M] \circ [K_\alpha] \mid [M] \in \text{Iso}(A), \alpha \in K_0(A) \right\},
\]

and according to (4.3) and Proposition 4.4, its multiplication is given by

\[
([A] \circ [K_\alpha]) \circ ([B] \circ [K_\beta]) = v^{-(\hat{A} + 2\alpha, \hat{B} + 2\beta)} \sum_{[M]} F_{AB}^M \cdot [M] \circ [K_{\hat{A} + \hat{B} + 2\alpha + 2\beta}].
\]

On the other hand, the basis of \( \varphi \tilde{H}_\Delta(A) \) is given by

\[
\left\{ [M][K_\alpha] \mid [M] \in \text{Iso}(A), \alpha \in K_0(A) \right\},
\]

and according to Proposition 5.1, its multiplication is given by

\[
[A][K_\alpha] \ast [B][K_\beta] = v^{-(\hat{A} + 2\alpha, \hat{B} + 2\beta)} \sum_{[M]} F_{AB}^M \cdot [M][K_{\hat{A} + \hat{B} + 2\alpha + 2\beta}].
\]

Therefore, by Lemma 4.3,

\[
\varphi \Xi_A(([A] \circ [K_\alpha]) \circ ([B] \circ [K_\beta])) = v^{-(\hat{A} + 2\alpha, \hat{B} + 2\beta) - (M,M)} \sum_{[M]} F_{AB}^M \cdot [M][K_{\hat{A} + \hat{B} + 2\alpha + 2\beta}]
\]

\[
= v^{-(\hat{A} + 2\alpha, \hat{B} + 2\beta) - (A,B)} \sum_{[M]} F_{AB}^M \cdot [M][K_{\hat{A} + \hat{B} + 2\alpha + 2\beta}]
\]

Hence, by the associativity of \( \tilde{H}_\Delta(A) \) (see Proposition 4.2) we obtain

\[
([A][K_\alpha] \ast [B][K_\beta]) \ast [C][K_\gamma] = [A][K_\alpha] \ast ([B][K_\beta] \ast [C][K_\gamma]).
\]

Then we are done. \( \square \)

As an application, we find that the semi-derived Hall algebra \( SDH(C_1(A)) \) of \( C_1(A) \), can be obtained from the extended \( \Delta \)-Hall algebra by suitable twisting.

Proposition 5.2. Let \( \varphi : K_0(A) \times K_0(A) \rightarrow \mathbb{Q}(v) \) be a multiplicative bilinear form given by \( \varphi(\alpha, \beta) = v^{-(\alpha, \beta)} \). Then there is an algebra isomorphism:

\[
\varphi \Xi_A : SDH(C_1(A)) \rightarrow \tilde{H}_\Delta(A).
\]

\[
[M] \circ [K_\alpha] \mapsto v^{-(M,M)}[M][K_\alpha]
\]

Proof. Recall that the basis of \( SDH(C_1(A)) \) is given by

\[
\left\{ [M] \circ [K_\alpha] \mid [M] \in \text{Iso}(A), \alpha \in K_0(A) \right\},
\]

and according to (4.3) and Proposition 4.4, its multiplication is given by

\[
([A] \circ [K_\alpha]) \circ ([B] \circ [K_\beta]) = v^{-(\hat{A} + 2\alpha, \hat{B} + 2\beta)} \sum_{[M]} F_{AB}^M \cdot [M] \circ [K_{\hat{A} + \hat{B} + 2\alpha + 2\beta}].
\]

On the other hand, the basis of \( \varphi \tilde{H}_\Delta(A) \) is given by

\[
\left\{ [M][K_\alpha] \mid [M] \in \text{Iso}(A), \alpha \in K_0(A) \right\},
\]

and according to Proposition 5.1, its multiplication is given by

\[
[A][K_\alpha] \ast [B][K_\beta] = v^{-(\hat{A} + 2\alpha, \hat{B} + 2\beta)} \sum_{[M]} F_{AB}^M \cdot [M][K_{\hat{A} + \hat{B} + 2\alpha + 2\beta}].
\]

Therefore, by Lemma 4.3,
\[ v^{-\langle A, A \rangle - \langle B, B \rangle} ([A][K_\alpha] \ast [B][K_\beta]) = \varphi \Xi_A([A] \circ [K_\alpha]) \ast \varphi \Xi_A([B] \circ [K_\beta]). \]

Hence, \( \varphi \Xi_A \) is an algebra homomorphism. Obviously, it is an isomorphism of \( \mathbb{Q}(v) \)-vector spaces. Then we are done. \( \square \)

5.2. Extended \( \Delta \)-Hall algebras and extended derived Hall algebras. In this subsection, we consider a new kind of extension of the \( \Delta \)-Hall algebra \( \mathcal{H}_\Delta(\mathcal{A}) \), with the aim to establish an algebraic isomorphism with certain extended version of the derived Hall algebra \( \mathcal{DH}_1(\mathcal{A}) \).

Recall that the Grothendieck group \( K_0(\mathcal{A}) \) is free. Consider the \( \mathbb{Z} \)-module
\[ \frac{1}{2}K_0(\mathcal{A}) := K_0(\mathcal{A}) \otimes_{\mathbb{Z}} \frac{1}{2} \mathbb{Z} = \{ \frac{\beta}{2} | \beta \in K_0(\mathcal{A}) \}. \]

Clearly, \( K_0(\mathcal{A}) \) is a subgroup of \( \frac{1}{2}K_0(\mathcal{A}) \).

Denote by \( \tilde{\mathcal{H}}_\Delta(\mathcal{A}) \) the \( \mathbb{Q}(v) \)-vector space with the basis
\[ \left\{ [M][K_\alpha] | [M] \in \text{Iso}(\mathcal{A}), \alpha \in \frac{1}{2}K_0(\mathcal{A}) \right\}. \]

Similar as Proposition 4.2, the following result holds.

**Proposition 5.3.** The \( \mathbb{Q}(v) \)-vector space \( \tilde{\mathcal{H}}_\Delta(\mathcal{A}) \) endowed with the multiplication defined by
\[ [A][K_\alpha] \ast [B][K_\beta] = \sum_{[M]} \tilde{F}_{AB}^M \cdot [M][K_{\frac{1}{2} \alpha + \frac{1}{2} \beta}] \]
forms an associative algebra with unit \( [0][K_0] \).

Consider the tensor product space \( \mathcal{H}_\Delta(\mathcal{A}) \otimes_{\mathbb{Q}} \mathbb{Q}(\frac{1}{2}K_0(\mathcal{A})) \), equipped with the standard multiplication, i.e., \((x \otimes u) \ast (y \otimes w) = (x \ast y) \otimes (u \ast w)\). It turns out that the tensor product algebra is isomorphic to the extended \( \Delta \)-Hall algebra \( \tilde{\mathcal{H}}_\Delta(\mathcal{A}) \).

**Proposition 5.4.** There is an algebra isomorphism
\[ \Phi_\otimes : \tilde{\mathcal{H}}_\Delta(\mathcal{A}) \longrightarrow \mathcal{H}_\Delta(\mathcal{A}) \otimes_{\mathbb{Q}} \mathbb{Q}(\frac{1}{2}K_0(\mathcal{A})). \]

**Proof.** For any \([A], [B] \in \text{Iso}(\mathcal{A})\) and \( \alpha, \beta \in \frac{1}{2}K_0(\mathcal{A}) \), we have
\[
\Phi_\otimes([A][K_\alpha] \ast [B][K_\beta]) = \Phi_\otimes \left( \sum_{[M]} \tilde{F}_{AB}^M \cdot [M][K_{\frac{1}{2} \alpha + \frac{1}{2} \beta}] \right)
= \sum_{[M]} \tilde{F}_{AB}^M \cdot [M] \otimes [K_{\frac{1}{2} \alpha + \frac{1}{2} \beta}]
= \sum_{[M]} \tilde{F}_{AB}^M \cdot [M] \otimes [K_{\frac{1}{2} \alpha + \frac{1}{2} \beta}],
\]
and
\[ \Phi_\otimes([A][K_\alpha]) \ast \Phi_\otimes([B][K_\beta]) \]
that there are isomorphisms $\Xi$ and $\hat{\Psi}$.

It follows that $\Phi$ is a natural subalgebra of $\tilde{\mathcal{H}}(A)$.

Recall from Propositions 3.2 and 5.4, we obtain

**Corollary 5.5.** There is an algebra isomorphism

$$
\Psi: \tilde{\mathcal{H}}(A) \rightarrow \mathcal{DH}(A) \otimes \mathbb{Q}(\frac{1}{2}K_0(A)).
$$

$$
[M][K_\alpha] \mapsto a_M \cdot u_M \otimes [K_{\frac{1}{2}} + \alpha].
$$

**Proof.** Recall from Propositions 3.2 and 5.4 that there are isomorphisms $\Xi$ and $\Phi$ respectively.

Consider the composition $\Psi := (\Xi^{-1} \otimes \text{id}) \circ \Phi$ as follows:

$$
\tilde{\mathcal{H}}(A) \rightarrow \mathcal{H}(A) \otimes \mathbb{Q}(\frac{1}{2}K_0(A)) \rightarrow \mathcal{DH}(A) \otimes \mathbb{Q}(\frac{1}{2}K_0(A)).
$$

Then we are done. □

**Remark 5.6.** (1) By definition, $\tilde{\mathcal{H}}(A)$ is a natural subalgebra of $\tilde{\mathcal{H}}(A)$. Therefore, under the isomorphisms $\Xi_A: \mathcal{H}(A) \rightarrow \mathcal{H}(A)$ and $\Phi$, the iHall algebra $\mathcal{H}(A)$ is a subalgebra of $\mathcal{DH}(A) \otimes \mathbb{Q}(\frac{1}{2}K_0(A))$.

(2) The inverse of $\Psi$ is given by:

$$
\Psi^{-1}: \mathcal{DH}(A) \otimes \mathbb{Q}(\frac{1}{2}K_0(A)) \rightarrow \tilde{\mathcal{H}}(A).
$$

$$
u_M \otimes [K_\alpha] \mapsto \frac{1}{a_M} \cdot [M][K_{\frac{1}{2}} + \alpha].
$$

Define the degree function on $\tilde{\mathcal{H}}(A)$ by

$$
\text{deg}([M][K_\alpha]) = \hat{M} + 2\alpha; \quad \forall M \in A, \alpha \in K_0(A).
$$

Then the image of $\mathcal{DH}(A) \otimes 1$ under $\Psi^{-1}$ has the expression

$$
\mathbb{Q}(\nu)\{[M][K_{\frac{1}{2}}] \mid [M] \in \text{Iso}(A)\},
$$

which is actually the subspace of $\tilde{\mathcal{H}}(A)$ with degree 0.

Therefore, the 1-periodic derived Hall algebra $\mathcal{DH}(A)$ can be embedded into $\tilde{\mathcal{H}}(A)$, which is an extended version of $\tilde{\mathcal{H}}(A)$, hence embedded into an extended version of the twisted semi-derived Hall algebra. In general, for t-periodic cases with $t \geq 1$, the embeddings from the derived Hall algebras to certain extended version of twisted semi-derived Hall algebras also hold, see Lin-Peng [13, Theorem 5.6].
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