This paper analyzes the diffusion behavior of the suspicious and the infected cabins in cyberspace by establishing a rumor propagation reaction–diffusion model with Allee effect and time delay. The Turing instability conditions of the system are emphatically studied. After considering the effect of time delay on the rumor propagation system, we have studied the correlation between the stability of the system under the influence of small time delay and the homogeneous system near the equilibrium point; the critical condition of the delay-induced spatial instability is given as well. Then, we prove the existence of Hopf bifurcation induced by time delay in some cases. Further considering the possibility of diffusion coefficient changing with time, the critical parameter curves of stability and instability of approximate systems are given by means of Floquet theory, and the necessary conditions of Turing instability of periodic coefficient are studied. In the numerical simulations, we find that the variation in diffusion coefficient and time delay will change the pattern type, and the periodical diffusion behavior will affect the arrangement of the crowd gathering area in the pattern.

1 Introduction

Nowadays, social network environment is characterized by fast information dissemination speed and low access cost, making it a stepping stone for false information and influencing public opinion [1]. Rumor is the main form of false information, which will cause great misdirection and harm to the public opinion environment. Therefore, it is a very important topic to study the spread of information among people and timely control the spread of rumors.

Research on information or rumor propagation started a long time ago. Daley put forward the random rumor model in 1965 [2], and Munaki established the MK model in 1973 [3]. Due to the high similarity between rumor propagation and infectious disease propagation, many scholars have tried to improve the traditional infectious disease models and established the cabin ordinary differential equation models of rumor propagation [4–6]. In Huo’s work [4], the population was divided into three categories: the ignorant population $X(t)$, the aware population $X_{in}(t)$ and spreaders $Y(t)$, as the accumulative consciousness mechanism $M(t)$ was further introduced. However, in recent years, with the in-depth development of complex network research and the high superiority of network structure in describing user structure in social networks, the establishment of rumor propagation dynamic model based on different network topologies began to rise [7–14]. According to Huo’s work, rumor propagation...
dynamics in homogeneous networks was studied, and the global stability of the internal equilibrium of the model was proved by using Lasalle’s invariant principle [12]. In the work of Wang, the process of information transmission in multi-layer social networks was studied, and the results showed that the success rate of rumor spreading was positively correlated with the node degree and the number of layers of initial nodes [7]. Based on the two-layer network, Xiao used the Markov chain method to study the coupling propagation process of multiple information, and the correlation coefficient between the networks would negatively affect the information diffusion threshold [8]. In the process of rumor propagation, the differences between different individuals or nodes are constantly concerned. For example, heterogeneous networks represent huge differences in node degree distribution, and rumor propagation based on this has been studied in some documents [11]. In the work of Yang, a connection was established between the propagation probability of rumors and node degrees, so as to take individual differences into consideration [10]. Besides ordinary differential equation models, the stochastic differential equation method after introducing randomness can also describe the rumor propagation process well [15]. Due to the high correlation between epidemic transmission and related information in practice and the similarity of transmission process, some work have tried to establish the coupling model of information transmission and disease transmission [16–18]. It reveals the mechanism of the interaction between disease and related consciousness. These works also reflect the important potential practical value of the research on the dynamics of information and rumor propagation.

Most of the above work is to establish the ordinary differential equation or difference equation model of rumor spread under different backgrounds and express its spatial relationship by using the static network structure. These works cannot well reflect the mobility of users in cyberspace and the stability of rumor propagation in space. Therefore, we would like to make more attempts to use partial differential equations to build rumor modeling and focus on studying Turing instability conditions. In 1952, Turing described a phenomenon of spatial instability [19], which became known as the Turing pattern later. Study of Turing pattern has been widely used in molecular chemistry and physics in the early days [20–25]. However, with the extensive development of biomathematics, in the predator–prey model [26–28] and parasite–host model [29], it is employed to delineate the spatial distribution instability of the population. In the subsequent population model, the reaction–diffusion process with cross-diffusion is the main cause of the pattern phenomenon [26,27,29]. After introducing the time delay in some models, it will also become one of the reasons for the appearance of pattern [30,31]. Some scholars have studied the phenomenon of speckle pattern on the network structure and obtained the corresponding conditions [32,33]. Unlike many previous assumptions of the constant diffusion coefficients, in the work of Djouda [34], the population reactivity diffusion model of mycelium and insects was established, and its diffusion coefficient had periodic characteristics. By establishing the linear approximate equation of perturbation near the equilibrium point, they try to derive the condition of pattern driven by period. The rumor propagation model established in this paper will follow this idea and improve some of the results.

The structure after this paper is as follows: In the second section, we establish a spatiotemporal model with time-delay and periodic propagation of rumor. In the third section, we will analyze the local stability of the equilibrium point in the time dimension in the case of non-diffusion. In the fourth section, the conditions of Turing instability and Hopf bifurcation in the spatiotemporal model are given. In the fifth section, extensive numerical simulations are carried out. Finally, we put forward the conclusion of our work in the sixth section.

2 Modeling

Based on previous work, this section attempts to establish a series of suspicious–infected–recovered spatiotemporal dynamics models describing rumor propagation in information cyberspace and focuses on its Turing bifurcation phenomenon. \(S\) (Suspicous) refers to people who have not been informed of rumors or doubt the authenticity of rumors, \(I\) (infected) are rumor believers, that is, people who have “infected” by rumors and begun to spread them, and \(R\) (recovered) refers to people who no longer believe in rumors or lose interest in them. The average field model of rumor propagation in the time dimension is shown as:
where $S(0) > 0$, $I(0) > 0$. The first term of the first equation represents the process that new users enter the information interaction environment. $rS(1 - \frac{S}{K})$ is a Logistic growth term, where $r$ refers to the intrinsic growth rate and $K$ is the maximum user capacity of information cyberspace. $(\frac{S}{A} - 1)$ represents the Allee effect of growth, where $A$ refers to the critical spatial carrying capacity [35–37]. The Allee effect truly reflects that when the number of users in a certain information cyberspace is low, the number of new users attracted is also relatively small. $\beta$ defines the rate of rumor spread, that is, the probability that a rumor believer communicates with a suspicious one and makes it a rumor believer in a unit time. $\mu$ refers to the rate of refuting rumors, which means that when there are two rumor believers communicating, they have the probability to check the authenticity of the rumors so that they no longer believe in rumors or lose interest in spreading.

In order to simplify the study of model (1), we make a simple linear transformation and redefine the parameters: $\beta_{new} = \frac{AK}{r} \beta$, $\mu_{new} = \frac{AK}{r} \mu$, $t_{new} = \frac{r}{AK} t$. Meanwhile, since the dynamic behavior of various groups in the model can be completely determined by the behavior of $S$ and $I$, model (1) can be simplified to the following forms:

$$\begin{align*}
\frac{dS(t)}{dt} &= rS(t)(1 - \frac{S(t)}{K})(\frac{S(t)}{A} - 1) - \beta S(t)I(t), \\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - \mu I^2(t),
\end{align*}$$

(1)

Here, for ease of writing, we have omitted the subscript “new” of the new parameters. In the following passage, when we add a new term to the model, the coefficient before the term is naturally changed to $\frac{AK}{r}$ times the original.

We extend the model to spatial latitude within a fixed bounded domain $\Omega \subset \mathbb{R}^2$. Considering the cross-diffusion behavior of users in cyberspace, the model can be written as

$$\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= D_{11}(t)\Delta S(t, x) + D_{12}(t)\Delta I(t, x) + S(t, x)(K - S(t, x)) \\
&\quad - \beta S(t, x)I(t, x), \\
\frac{\partial I(t, x)}{\partial t} &= \beta S(t, x)I(t, x) - \mu I^2(t, x),
\end{align*}$$

(2)

with homogeneous Neumann boundary conditions

$$\begin{align*}
\frac{\partial S(t, x)}{\partial n} = \frac{\partial I(t, x)}{\partial n} = 0, & \quad t \geq 0, x \in \partial \Omega,
\end{align*}$$

(3)

$D_{11}(t) > 0$, $D_{22}(t) > 0$ represent the diffusion coefficient, describing the dynamic behavior of all groups of people in cyberspace based on their own growth. $D_{12}(t)$ and $D_{21}(t)$ represent the cross-diffusion coefficient and describe the spatial trends caused by the existence of other populations. $\Delta$ is Laplace operator. It is worth noting that when $D_{11}(t)D_{22}(t) - D_{12}(t)D_{21}(t) > 0$, the flow of people in the spatial domain depends more on their own density, and in subsequent studies, we will base on this assumption. $n$ is the outward unit normal vector on boundary $\Omega$. Our study will adopt zero flux Newman boundary conditions and positive initial conditions. This condition indicates that there is no external input or internal outflow at the system boundary.

In addition, in the background of rumor propagation, time delay is always considered to be an important factor in describing the system more accurately. In the study of the stability of the reaction diffusion system, the delay is also an important reason for the phenomenon of Turing pattern [30,31]. On the basis of model (3), we consider two improved time delay as follows:

$$\begin{align*}
\frac{\partial S(t, x)}{\partial t} &= D_{11}(t)\Delta S(t, x) + D_{12}(t)\Delta I(t, x) + S(t, x)(K - S(t, x)) \\
&\quad - \beta S(t, x)I(t, x), \\
\frac{\partial I(t, x)}{\partial t} &= \beta S(t, x)I(t, x) - \mu I^2(t, x),
\end{align*}$$

(4)

with homogeneous Neumann boundary conditions

$$\begin{align*}
\frac{\partial S(t, x)}{\partial n} = \frac{\partial I(t, x)}{\partial n} = 0, & \quad t \geq 0, x \in \partial \Omega,
\end{align*}$$

(5)

and initial conditions

$$\begin{align*}
S(t, x) = S_0(t, x) \geq 0 \text{ and } \neq 0, \\
(t, x) \in [-\tau_0, 0] \times \Omega, \\
I(t, x) = I_0(t, x) \geq 0 \text{ and } \neq 0, \\
(t, x) \in [-\tau_0, 0] \times \Omega.
\end{align*}$$

(6)
where $\tau_0 = \max\{\tau_1, \tau_2\}$ and $\tau_1$, $\tau_2$ both represent a small positive time delay in the above model. Firstly, $\tau_1$ explains a time lag between the rumor reader checking the information and the refutation; secondly, $\tau_2$ reflects that the suspicious who accept and believe the rumor has a certain delay until they begin to spread.

3 Local stability analysis of non-diffusion conditions

One of the necessary conditions for Turing bifurcation is that the corresponding ordinary differential equation system is locally stable at the equilibrium point. In this section, the stability conditions of the positive equilibrium point of model (2) will be analyzed, which provides the necessary conditions for Turing instability of the reaction–diffusion rumor propagation system.

Model (2) has five equilibrium points. Among them, $P_1 = (K, 0)$, $P_2 = (A, 0)$, $P_3 = (0, 0)$ are the rumor-free equilibrium points. The rumor-spreading equilibrium points are

$$P_4 = \left(\frac{1}{2}(p - \sqrt{p^2 - 4q}), \frac{\beta}{2\mu}(p - \sqrt{p^2 - 4q})\right),$$

$$P_5 = \left(\frac{1}{2}(p + \sqrt{p^2 - 4q}), \frac{\beta}{2\mu}(p + \sqrt{p^2 - 4q})\right),$$

under the conditions that $p > 0$, $p^2 - 4q > 0$, where $p = K + A - \frac{\beta^2}{\mu}$, $q = AK$. The Jacobian matrix at the equilibrium point $P = (S^*, I^*)$ of model (2) is defined as $J(P)$:

$$J(P) = \begin{pmatrix}
S^* (\frac{2\beta^2}{\mu} - K - A) + 2AK - \beta S^* \\
S^* \frac{2\beta^2}{\mu} - \beta S^*
\end{pmatrix}.$$

The characteristic equation of model (2) can be obtained:

$\lambda^2 - \text{tr}(J)\lambda + \det(J) = 0,$

where $\text{tr}(J) = S^* (\frac{2\beta^2}{\mu} - A - K) + 2AK$, $\det(J) = -\beta S^* (S^* (\frac{2\beta^2}{\mu} - A - K) + 2AK)$. When the eigenvalues of the characteristic equation are satisfied with negative real parts, the system is locally asymptotically stable near the point $P$. Because of $\lambda_1 \lambda_2 = \det(J)$ and $\lambda_1 + \lambda_2 = \text{tr}(J)$, the judgment of the local stability of the system can be converted to discuss the positive and negative properties of determinant and trace. Further, we will analyze the sufficient conditions for the local stable state of the rumor-spreading equilibrium points based on the size relationship between $\beta$ and $\mu$.

- $\beta \geq \mu$: To make $\det(J) > 0$ in this case, just satisfy the $\text{tr}(J) < 0$. Therefore, the conditions that $P_4$ and $P_5$ are stable are $\text{tr}(J(P_4)) < 0$ and $\text{tr}(J(P_5)) < 0$. The additional constraints on the stability of the two rumor propagation equilibrium points with the above constraints satisfied are as follows:

$$\begin{align*}
\frac{2\beta^2}{\mu} &< (K + A + \beta), \\
\frac{2\beta^2}{\mu} &> (K - A - \beta)(K + A - \beta) \\
4q &< p^2 + (1 - \frac{\beta}{\mu})\beta p.
\end{align*}$$

Let $\text{tr}(J(P_4)) > 0$ and $\text{tr}(J(P_5)) < 0$, the condition that $P_4$ is instability and $P_5$ is stability is

$$\begin{align*}
\frac{2\beta^2}{\mu} &< \min\{(K + A + \beta), \\
&\frac{(K - A - \beta)(K - A + \beta)}{(K + A - \beta)}\}.
\end{align*}$$

Further, we consider the conditions that $P_5$ is stable and $P_4$ is stable. But such conditions are contradictory, and therefore, this case does not exist.

- $\beta < \mu$: To make $\text{tr}(J) > 0$ in this case, just satisfy the $\det(J) < 0$.

For $P_4$:

$$\begin{align*}
S^* (\frac{2\beta^2}{\mu} - K - A) + 2AK \\
= \frac{1}{2} \sqrt{p^2 - 4q} (p - \sqrt{p^2 - 4q}) \\
> 0 \Rightarrow \det(J(P_4)) < 0.
\end{align*}$$

For $P_5$:

$$\begin{align*}
S^* (\frac{2\beta^2}{\mu} - K - A) + 2AK \\
= \frac{1}{2} \sqrt{p^2 - 4q} (-p - \sqrt{p^2 - 4q}) \\
< 0 \Rightarrow \det(J(P_5)) > 0.
\end{align*}$$

Therefore, in this case, as long as the rumor-spreading equilibrium points exist, there is a conclusion that the $P_4$ is unstable and $P_5$ is local asymptotic stability.

To simplify the later study, we will analyze the Turing bifurcation of rumor propagation models when $\beta < \mu$ holds. Since only $P_5$ are locally asymptotically stable, the Turing pattern will only appear near this equilibrium point. We write the single stable point $P_5$ as $P = (S^*, I^*)$ to simplify the expression of following formulas.
4 Turing instability and Hopf bifurcation analysis

4.1 System of constant diffusion coefficient

Now, we consider diffusion coefficients as constants, that is, diffusion coefficients are time-independent:
\[ D_{11}(t) = d_{11}, \quad D_{12}(t) = d_{12}, \quad D_{21}(t) = d_{21}, \quad \text{and} \quad D_{22}(t) = d_{22}. \]
Model (3) of constant diffusion coefficients is simplified as follows:

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_{11} \Delta S + d_{12} \Delta I + S(K - S)(S - A) - \beta SI, \\
\frac{\partial I}{\partial t} &= d_{21} \Delta S + d_{22} \Delta I + \beta SI - \mu I^2.
\end{align*}
\]

(7)

Based on the assumption that \( \beta < \mu \), the rumor-spreading equilibrium point \( P \) is local stable in the time dimension, and the necessary condition for the occurrence of Turing bifurcation is the instability of \( P \) in the presence of diffusion in space. The perturbation \( (\delta_1, \delta_2) \) near the equilibrium point is expanded in the Fourier space:

\[
\begin{align*}
\delta_1(x, t) &= \sum_k c_k^1 e^{i\lambda t + ikx}, \\
\delta_2(x, t) &= \sum_k c_k^2 e^{i\lambda t + ikx},
\end{align*}
\]

where \( k \) and \( \lambda \) stand for wave number and wave frequency, respectively. Substituting the above equation into the linear approximation system of system (7), then the characteristic equation of system (7) is \( |H(k) - \lambda I| = 0 \), where \( H(k) = J - k^2 D \), \( D \) represents the diffusion coefficient matrix. The dispersion relation is obtained as

\[
\lambda_k^2 - \text{tr}(H(k))\lambda_k + \Delta_k = 0,
\]

where \( \text{tr}(H(k)) = \text{tr}(J) - k^2 \text{tr}(D) < 0 \), \( \Delta_k = \det(H(k)) = \eta k^4 + \theta k^2 + \Delta_0 \). Among them, \( \eta = \det(D) > 0 \), \( \theta = d_{21} d_{12} + d_{12} d_{j1} - d_{11} d_{22} - d_{22} d_{j1} \), \( j \) stands for the element of matrix \( J(P) \). If for any \( k \), the characteristic roots of system (7) all have a negative real part, then Turing pattern cannot be formed. Due to \( \text{tr}(H(k)) < 0 \), the condition of the existence of eigenvalues with positive real parts of the system is transformed into: there exist some \( k \), make \( \Delta_k < 0 \). \( \Delta_k \)'s minimum point for \( k \), namely the most dangerous modulus of the system \( k_c \), is expressed as

\[
k_c^2 = -\frac{\theta}{2\eta}.
\]

According to \( k_c \), the critical condition for the emergence of Turing bifurcation is as follows:

\[
\Delta_k = \Delta_0 - \frac{\theta^2}{4\eta} = 0.
\]

Ensure that \( k^2 \) nonnegative, \( \Delta_k < 0 \), we can give the necessary conditions for the existence of diffusion-driven Turing instability: \( \theta < 0 \) and \( 4\eta\Delta_0 < \theta^2 \).

4.2 Systems with time delays

4.2.1 Turing bifurcation

In this section, we will analyze the Turing bifurcation conditions of system (4) under \( \tau_1 = 0 \) and \( \tau_2 = 0 \), respectively, with the constant periodic coefficient. The equations are reduced to:

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_{11} \Delta S + d_{12} \Delta I + S(K - S)(S - A) - \beta SI, \\
\frac{\partial I}{\partial t} &= d_{21} \Delta S + d_{22} \Delta I + \beta SI - \mu I^2(t - \tau_1),
\end{align*}
\]

(8)

and

\[
\begin{align*}
\frac{\partial S}{\partial t} &= d_{11} \Delta S + d_{12} \Delta I + S(K - S)(S - A) - \beta S(t - \tau_2)I(t - \tau_2), \\
\frac{\partial I}{\partial t} &= d_{21} \Delta S + d_{22} \Delta I + \beta S(t - \tau_2)I(t - \tau_2) - \mu I^2,
\end{align*}
\]

(9)

where \( \tau_1 \) and \( \tau_2 \) represent the small time delays. In order to facilitate the analysis of the stability of the original system with time delay, the following theorem is given.

**Theorem 1** Set up a constant coefficient reactive diffusion system in the following form:

\[
\begin{align*}
\frac{\partial X}{\partial t} &= d_{11} \Delta X + d_{12} \Delta Y + F(X(t), Y(t)) + \varphi(X(t), Y(t)), \\
\frac{\partial Y}{\partial t} &= d_{21} \Delta X + d_{22} \Delta Y + G(X(t), Y(t)) + \psi(X(t), Y(t)),
\end{align*}
\]

(10)

where \( \varphi \) and \( \psi \) have the first partial derivatives of \( X \) and \( Y \), and the system has an equilibrium point \( P_0 \). Replace the term without delay to the term with delay: \( \varphi(X(t), Y(t)) \rightarrow \varphi(X(t - \tau), Y(t - \tau)) \), \( \psi(X(t), Y(t)) \rightarrow \psi(X(t - \tau), Y(t - \tau)) \). When \( \tau \) is very small, Taylor expansion is carried out for terms

\[ \varphi(X(t), Y(t)) \rightarrow \varphi(X(t), Y(t)) - \varphi(X(t - \tau), Y(t - \tau)) \frac{\tau}{1!} + \frac{\varphi_2(X(t), Y(t))}{2!} \tau^2 + \cdots. \]
containing \( \tau \), and the approximate system of linear terms \((X_{\text{new}}, Y_{\text{new}})^T\) is taken to satisfy the following properties near \( P_0 \):

\[
\begin{align*}
\left[ \frac{\partial X_{\text{new}}}{\partial t}, \frac{\partial Y_{\text{new}}}{\partial t} \right] &= Q \left[ \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial t} \right],
\end{align*}
\]

where \( Q = \frac{1}{(1 + \tau \frac{\partial \psi}{\partial X})(1 + \tau \frac{\partial \varphi}{\partial Y}) - \tau^2 \frac{\partial \varphi}{\partial Y} \frac{\partial \psi}{\partial X}} \left[ \begin{array}{c}
1 + \tau \frac{\partial \psi}{\partial Y} - \tau \frac{\partial \varphi}{\partial Y} \\
-\tau \frac{\partial \varphi}{\partial X} + \tau \frac{\partial \psi}{\partial X}
\end{array} \right]. \]

2 Jacobian matrix at \( P_0 \) for systems with time delay: \( J_{\text{new}}(P_0) = Q(P_0)J(P_0) \). Diffusion coefficient matrix at \( P_0 \) for delay systems: \( D_{\text{new}}(P_0) = Q(P_0)D \). Let \( H_{\text{new}}(k) = J_{\text{new}}(P_0) + k^2 D_{\text{new}}(P_0) \) and it satisfies \( H_{\text{new}}(k) = Q(P_0)H(k) \).

**Proof** The system with time delay derived from the new system is as follows:

\[
\begin{align*}
\frac{\partial X}{\partial t} &= d_{11} \Delta X + d_{12} \Delta Y + F(X(t), Y(t)) \\
&\quad + \phi(X(t-\tau), Y(t-\tau)), \\
\frac{\partial Y}{\partial t} &= d_{21} \Delta X + d_{22} \Delta Y + G(X(t), Y(t)) \\
&\quad + \psi(X(t-\tau), Y(t-\tau)).
\end{align*}
\]

Taylor expansion is performed for \( \varphi, \psi \) with \( \tau \) as the variable, and the linear term is taken as

\[
\begin{align*}
\varphi(X(t-\tau), Y(t-\tau)) &= \varphi(X(t), Y(t)) \\
-\tau \frac{\partial \varphi(X(t), Y(t))}{\partial t} + o(\tau^2), \\
\psi(X(t-\tau), Y(t-\tau)) &= \psi(X(t), Y(t)) \\
-\tau \frac{\partial \psi(X(t), Y(t))}{\partial t} + o(\tau^2).
\end{align*}
\]

Write \( M = (m_{ij})_{2 \times 2} \) and define

\[
M \Delta \frac{\partial (\varphi, \psi)}{\partial (X, Y)} = \left[ \begin{array}{c}
\tau \frac{\partial \varphi}{\partial X} \\
\tau \frac{\partial \varphi}{\partial Y} \\
\tau \frac{\partial \psi}{\partial X} \\
\tau \frac{\partial \psi}{\partial Y}
\end{array} \right].
\]

In this case, we have \( \tau \frac{\partial \varphi}{\partial t} = m_{11} \frac{\partial X}{\partial t} + m_{12} \frac{\partial Y}{\partial t} \) and \( \tau \frac{\partial \psi}{\partial t} = m_{21} \frac{\partial X}{\partial t} + m_{22} \frac{\partial Y}{\partial t} \). The linear approximation term and the upper formula are substituted into the system, and the delay approximation system is arranged as follows:

\[
\begin{align*}
\left(1 + \tau \frac{\partial \varphi}{\partial t}\right) + m_{11} \frac{\partial X}{\partial t} + m_{12} \frac{\partial Y}{\partial t} &= F(X, Y), \\
\left(1 + \tau \frac{\partial \psi}{\partial t}\right) + m_{21} \frac{\partial X}{\partial t} + m_{22} \frac{\partial Y}{\partial t} &= G(X, Y), \\
\left(1 + \tau \frac{\partial \varphi}{\partial t}\right) + \frac{\partial X_{\text{old}}}{\partial t} &= F(X, Y), \\
\left(1 + \tau \frac{\partial \psi}{\partial t}\right) + \frac{\partial Y_{\text{old}}}{\partial t} &= G(X, Y), \\
\left(1 + \tau \frac{\partial \varphi}{\partial t}\right) + m_{11} \frac{\partial X}{\partial t} + m_{12} \frac{\partial Y}{\partial t} &= \frac{\partial X_{\text{old}}}{\partial t}, \\
\left(1 + \tau \frac{\partial \psi}{\partial t}\right) + m_{21} \frac{\partial X}{\partial t} + m_{22} \frac{\partial Y}{\partial t} &= \frac{\partial Y_{\text{old}}}{\partial t}.
\end{align*}
\]

Obviously the following equation is true:

\[
\begin{align*}
\left[ \begin{array}{c}
\frac{\partial X_{\text{old}}}{\partial t} \\
\frac{\partial Y_{\text{old}}}{\partial t}
\end{array} \right] &= \left[ \begin{array}{ccc}
1 + m_{11} & m_{12} \\
m_{21} & m_{22} + 1
\end{array} \right] \frac{\partial X_{\text{new}}}{\partial t}, \\
\frac{\partial X_{\text{new}}}{\partial t} = \frac{\partial Y_{\text{new}}}{\partial t}.
\end{align*}
\]

Because \( \varphi \) and \( \psi \) have the first partial derivative of \( X \) and \( Y \) around \( P_0 \), we can assume that \( \frac{\partial \varphi}{\partial X}, \frac{\partial \varphi}{\partial Y}, \frac{\partial \psi}{\partial X}, \frac{\partial \psi}{\partial Y} \) and \( \frac{\partial \varphi}{\partial t}, \frac{\partial \psi}{\partial t} \) are bounded in a neighborhood of \( P_0 \). So when \( \tau \) is small enough, the determinant of the matrix in above equation: \( (1 + m_{11})(1 + m_{22}) - m_{12}m_{21} = m \) is always positive. Take the inverse transformation of the above equation,

\[
\begin{align*}
\left[ \begin{array}{c}
\frac{\partial X_{\text{new}}}{\partial t} \\
\frac{\partial Y_{\text{new}}}{\partial t}
\end{array} \right] &= Q \left[ \begin{array}{c}
\frac{\partial X_{\text{old}}}{\partial t} \\
\frac{\partial Y_{\text{old}}}{\partial t}
\end{array} \right],
\end{align*}
\]

where

\[
Q = \frac{1}{m} \left[ \begin{array}{cc}
1 + m_{22} & -m_{12} \\
-m_{21} & 1 + m_{11}
\end{array} \right].
\]

The Jacobian matrix of the original system (10) at \( P_0 \) is

\[
J_{\text{old}}(P_0) = \left[ \begin{array}{cccc}
F_X + \varphi_X & F_Y + \psi_Y & G_X + \varphi_X & G_Y + \psi_Y
\end{array} \right]_{P_0},
\]

and in the jacobian matrix of system (12), the elements at \( P_0 \) are shown as

\[
\begin{align*}
J_{11}^{\text{new}} &= \left[ \frac{\partial (1 + m_{22})}{\partial X} (F + \varphi) + \frac{1 + m_{22}}{m} (G + \psi) \right]_{P_0}, \\
J_{12}^{\text{new}} &= \left[ \frac{\partial (1 + m_{22})}{\partial Y} (F + \varphi) + \frac{1 + m_{22}}{m} (G + \psi) \right]_{P_0}, \\
J_{21}^{\text{new}} &= \left[ \frac{\partial (1 + m_{12})}{\partial X} (F + \varphi) + \frac{1 + m_{11}}{m} (G + \psi) \right]_{P_0}, \\
J_{22}^{\text{new}} &= \left[ \frac{\partial (1 + m_{12})}{\partial Y} (F + \varphi) + \frac{1 + m_{11}}{m} (G + \psi) \right]_{P_0}.
\end{align*}
\]
Because $P_0$ is the equilibrium point, $(\varphi + F)P_0 = 0$, $(\psi + G)P_0 = 0$. Therefore,

$$
J_{new}(P_0) = \begin{bmatrix}
(1 + m_{22})(F_X + \varphi_X) - m_{12}(G_X + \psi_X) & m \frac{m_{12}}{m} (G_Y + \psi_Y) \\
-m_{21}(F_X + \varphi_X) + (1 + m_{11})(G_X + \psi_X) & (1 + m_{22})(F_Y + \varphi_Y) - m_{12}(G_Y + \psi_Y)
\end{bmatrix}_{P_0}
$$

$$= \frac{1}{m_{21}} \begin{bmatrix}
1 + m_{22} & -m_{12} \\
-m_{21} & 1 + m_{11}
\end{bmatrix}
\begin{bmatrix}
F_X + \varphi_X & F_Y + \varphi_Y \\
G_X + \psi_X & G_Y + \psi_Y
\end{bmatrix}_{P_0}
= Q(P_0)J_{old}(P_0).
$$

From the expansion of $\left(\frac{\partial X_{new}}{\partial t}, \frac{\partial Y_{new}}{\partial t}\right)' = Q\left(\frac{\partial X_{old}}{\partial t}, \frac{\partial Y_{old}}{\partial t}\right)'$, there is obviously a conclusion: $D_{new}(P_0) = Q(P_0)D$. Further, we have

$$
H_{new}(k) = J_{new}(P_0) + k^2 D_{new}(P_0) = Q(P_0)(J(P_0) + k^2 D) = Q(P_0)H_{old}(k).
$$

(13)

This completes the proof.

The above theorem gives the relation between linear approximation system with delay term and the system without delay when $\tau$ is small, which provides a more convenient method to study the stability of equilibrium point by linearizing the time delay term. By solving the transformation matrix $Q$, key information such as jacobian matrix and characteristic equation of the new system can be obtained quickly. The Turing instability analysis of the approximate system (8), (9) is further given based on Theorem 1.

- For system (8), transformation matrix $Q_1$ can be figured out:

$$
Q_1 = \begin{bmatrix}
1 & 0 \\
0 & \frac{1}{1 - 2\tau_1 \mu I}
\end{bmatrix}.
$$

Since the time delay is small and the system is bounded near $P$, we may assume that: $2\mu I^* \tau_1 < 1$, which ensures that $\det(Q_1)$ is positive. According to Theorem 1, we have $J_1(P) = Q_1(P)J(P)$, $D_1(P) = Q_1(P)D$, $H_1(K) = Q_1(P)H(K)$, where $J_1(P)$ and $D_1(P)$, respectively, represent the Jacobian matrix and diffusion coefficient matrix of system (8), and $H_1(K) \triangleq J_1(P) - k^2 D_1(P)$. Further, the positive and negative properties of $J_1$ and $H_1$’s determinants and traces of the new system are analyzed in order to obtain the conditions of Turing pattern. The following conclusions can be reached:

$$
\begin{align*}
\text{tr}(J_1(P)) &= S^*(2\beta^2 \frac{1}{\mu} - K - A) + 2AK \\
&\geq \frac{\beta S^*}{1 - 2\tau_1 \mu I^*} < \text{tr}(J(P)), \\
\text{tr}(H_1(k)) &= \text{tr}(J_1(P)) - k^2(d_{11} \\
&\geq \frac{d_{22}}{1 - 2\tau_1 \mu I^*} < \text{tr}(H(k)), \\
\det(J_1(P)) &= \frac{1}{1 - 2\tau_1 \mu I^*} \det(J(P)), \\
\det(H_1(k)) &= \frac{1}{1 - 2\tau_1 \mu I^*} \det(H(k)).
\end{align*}
$$

Observe the necessary conditions for the existence of Turing bifurcation for system (6) in the previous section. Under the assumption of $\beta < \mu$, we have $\text{tr}(J(P)) < 0$, $\text{tr}(H(P)) < 0$; therefore, the new system satisfies $\text{tr}(J_1(P)) < 0$, $\text{tr}(H_1(P)) < 0$. At the same time, since $\det(J_1(P))$ has the same sign as $\det(J(P))$, so do $\det(H_1(P))$ and $\det(H(P))$, it can be found that the necessary conditions for the occurrence of Turing instability of the system given by the method in previous section are also necessary for Turing bifurcation of the system (8) to occur when the time delay is small. The conditions for the existence of diffusion-driven Turing instability of system (8) are

$$
\theta < 0, \quad 4\eta \Delta_0 < \theta^2, \quad \tau_1 < \frac{1}{2\mu I^*}.
$$

It is worth noting that although this conclusion is concise and effective, it is not universal. The main reason is that the time delay approximation matrix is a simple positive diagonal matrix, and the element in the lower right corner amplifies the negative term in the $\text{tr}(J)$, so that the trace of $J_1$ and
Fig. 1 Critical curves of Turing bifurcation and Hopf bifurcation for system (9). The green curve indicates the threshold of $\tau_2$ which induces the Turing bifurcation. The orange curve indicates the threshold of $\beta$ which induces the Hopf bifurcation. $K = 1, A = 0.1, \beta = 0.46, \mu = 0.5, d_{11} = 0.5, d_{22} = 0.5, d_{21} = 0$

$H_1$ is always negative under the original condition. In the discussion of the correlation conditions of determinant of $J_1$ and $H_1$, the conditions controlling the value of $\det(J)$ and $\det(H)$ can also control the positive and negative properties of $\det(J_1)$ and $\det(H_1)$, because they have the same sign as the original system.

For system (9), transformation matrix $Q_2$ can be figured out:

$$Q_2 = \frac{1}{1 + \tau_2 \beta S - \tau \beta I} \begin{bmatrix} 1 + \tau_2 \beta S & \tau_2 \beta S \\ -\tau_2 \beta I & 1 - \tau_2 \beta I \end{bmatrix}.$$  

In the vicinity of the equilibrium point $P$, based on the fact that $\beta S^* = \mu I^*$, and the hypothesis that $\beta < \mu$, $\det(Q_2) = \frac{1}{1 + \tau_2 \beta S - \tau_2 \beta I}$ is greater than 0. Let’s label the matrix associated with system (9) with subscript 2. Similar to the above process, we draw the following conclusions directly:

$$\begin{align*}
\text{tr}(J_2(P)) &= \frac{1}{1 + \tau_2 (\mu - \beta) I^*} \left[ \text{tr}(J(P)) + \tau_2 \beta S^* (5\beta^2 \frac{1}{\mu} - K - A) + 2AK \right], \\
\text{tr}(H_2(k)) &= \text{tr}(J_2(P)) - \frac{k^2}{1 + \tau_2 (\mu - \beta) I^*} \left[ d_{11} + d_{22} + \tau_2 I^*(\mu d_{21} + \mu d_{11} - \beta d_{12} - \beta d_{22}) \right], \\
\det(J_2(P)) &= \frac{1}{1 + \tau_2 (\mu - \beta) I^*} \det(J(P)), \\
\det(H_2(k)) &= \frac{1}{1 + \tau_2 (\mu - \beta) I^*} \det(H(k)).
\end{align*}$$

Unlike the time-delay approximation system discussed in the previous part, although the determinant of $J_2$ and $H_2$ is still the same sign as the original system, the trace of $J_2$ and $H_2$ cannot be guaranteed to be less than 0 under the constraint of existing conditions. In order to make the real part of the eigenvalue under the influence of no diffusion negative, let $\text{tr}(J_2) < 0$, the following constraints can be obtained:

$$\tau_2 < \frac{-\text{tr}(J(P))}{\beta S^* [5\beta^2 \frac{1}{\mu} - K - A] + 2AK}, \quad \text{if } \beta S^* [5\beta^2 \frac{1}{\mu} - K - A] + 2AK > 0.$$  

If $\beta S^* [5\beta^2 \frac{1}{\mu} - K - A] + 2AK \leq 0$, it is always true that $\text{tr}(H_2) < 0$. In this case, when the parameters satisfy the existence of $k$ such that $\text{tr}(H(k)) > 0$, the conclusion that $P$ is unstable under diffusion-driven can also be deduced. The instability conditions are as follows:

$$\begin{align*}
\tau_2 &> \frac{d_{11} + d_{22}}{\text{tr}(\beta d_{12} + \beta d_{22} - \mu d_{21} - \mu d_{11})}, \\
0 < &\frac{d_{12} + \beta d_{22} - \mu d_{21} - \mu d_{11}}{\text{tr}(H_2(k))}.
\end{align*}$$

When there is no Turing bifurcation in original system (3) near $P$, namely $\text{tr}(J) < 0$, $\det(J) > 0$, $\text{tr}(H) < 0$, $\det(H) > 0$, $\det(H_2) > 0$, but system (9) meets the above conditions $\text{tr}(H_2) > 0$, such instability can be called the Turing bifurcation phenomenon co-driven by diffusion and time delay. As shown in Fig. 1, the threshold curve of $\tau_2$-induced instability in the approximate system is given, which varies with $d_{12}$. This instability condition does not exist in the model analyzed in the previous part. When added the $\text{tr}(H_2) < 0$, similar to the previous model (8), the determinant is the same sign as the original model. When $\tau_2$ is small enough, the necessary condition for the existence of Turing bifurcation in the original model is also the necessary condition of model (9).
4.2.2 Hopf bifurcation

In the temporal dimension, the Hopf bifurcation of systems with time delay is one of the studied focuses. Time oscillations caused by Hopf bifurcation are often expected to be present. Further, for an appropriate simplification, we consider the analysis of the Hopf bifurcations of model (8) and model (9) on a relatively large parameter space: $-5\beta I^* < 3j_{11}(P) < -\beta I^*$, where $j_{11}(P_0)$ denotes the value of the first element of the Jacobi matrix taken at the positive equilibrium point $P$ when the time delay is 0.

**Theorem 2** Assume $-5\beta I^* < 3j_{11}(P) < -\beta I^*$, we have that, for some $k$, system (8) and system (9) occur Hopf bifurcations at $\tau_1 = \tau_1^*$ and $\tau_2 = \tau_2^*$, respectively, that is,

1. There exists $\tau_1^*$ such that system (8) is locally asymptotically stable near the equilibrium point $P$ when $\tau_1 \in (0, \tau_1^*)$, and unstable in the time dimension for some $\epsilon > 0$ when $\tau_1 \in (\tau_1^*, \tau_1^* + \epsilon)$.
2. There exists $\tau_2^*$ such that system (9) is locally asymptotically stable near the equilibrium point $P$ when $\tau_2 \in (0, \tau_2^*)$, and unstable in the time dimension for some $\epsilon > 0$ when $\tau_2 \in (\tau_2^*, \tau_2^* + \epsilon)$.

**Proof** Firstly, we provide the following symbols: $U(t) \triangleq (S(\cdot, t), I(\cdot, t))^T$, $U_1(\tau) \triangleq U(t - \tau)$ and $\psi(\tau) = (\psi_1(\tau), \psi_2(\tau))^T \triangleq U_1(\tau)$.

For system (8), in the phase space $C([-\tau_0, 0], X)$, the non-diffusive parts of the right end of the equation can be separated into linear and nonlinear parts with $\psi_1(0), \psi_2(0)$ and $\psi(\tau)$ as variables, that is,

$$\frac{dU(t)}{dt} - D\Delta U(t) = L(U_1) + N(U_1),$$

where $D$ is the diffusion coefficient matrix, linear operator $L : C([-\tau_0, 0], X) \rightarrow X$ and nonlinear operator $N : C([-\tau_0, 0], X) \rightarrow X$, are given by

$$L(\psi) = \left[ S^* \left( \frac{2\beta^2}{\mu} - K - A \right) + 2AK \right] \psi_1(0) - \beta S^* \psi_2(0) \psi_1(\tau)$$

and

$$N(\psi) = \left( A + K - 3S^* \right) \psi_1^2(0) - \beta \psi_1(0) \psi_2(0) - \mu \psi_2^2(\tau).$$

At this point, system (14) has a following characteristic equation of this form:

$$\lambda y - D\Delta y = L(e^\lambda y),$$

where $y \in \text{dom}(\Delta) \subseteq X$ and $y \neq 0$. As shown in the previous analysis of Turing stability, operator $\Delta$ has eigenvalues $-k_n^2$ in $X$, where $k_n = \frac{n\pi}{\tau}, n = 0, 1, 2, \ldots$. The eigenfunctions corresponding to the eigenvalue $k$ are $\alpha_1^k = (\cos(kx), 0)^T$ and $\alpha_2^k = (0, \cos(kx))^T$. Any element $y \in X$ can be expanded as

$$y = \sum_k Y_k^T \left( \alpha_1^k \alpha_2^k \right),$$

where $C_k = S^* \left( \frac{2\beta^2}{\mu} - K - A + 2AK - k^2d_{11}, D_k = -\beta S^* - k^2d_{12}, E_k = \beta I^* - k^2d_{21}, F_k = \beta S^* - k^2d_{22}, G_k = -\beta S^* \right.$

Therefore, the characteristic equation is

$$\lambda^2 - (C_k + F_k) \lambda + C_k F_k - E_k D_k + e^{-\lambda \tau} \left( -G_k \lambda + C_k G_k \right) = 0.$$

By substituting $\lambda = iz$ into the characteristic equation (16) and separating the real part from the imaginary part, we have

$$\begin{align*}
C_k G_k \cos(z \tau) - G_k z \sin(z \tau) &= z^2 - (C_k F_k - D_k E_k), \\
C_k G_k \sin(z \tau) + G_k z \cos(z \tau) &= - (C_k + F_k)z.
\end{align*}$$

Add the sum of the squares of the two equations, and we have

$$z^4 + z^2 \left[ 2(D_k E_k - C_k F_k) + (C_k + F_k)^2 - G_k^2 \right] + (C_k F_k - D_k E_k)^2 - C_k^2 G_k^2 = 0.$$  

At this point, the existence condition of Hopf bifurcation is transformed into the existence condition of positive solutions of the quadratic equation (17) with $z^2$ as the variable. When $k$ takes 0,
\((C_0 F_0 - D_0 E_0)^2 - C_0^2 G_0^2 = (C_0 F_0 - D_0 E_0 - C_0 G_0)\)
\((C_0 F_0 - D_0 E_0 + C_0 G_0) = (\beta S^*)^2\)
\((3C_0 + \beta I^*) (\beta I^* - C_0)\).

It is known that \(\beta I^* - C_0 > 0\) and by the constraints of the theorem, we have \(3C_0 + \beta I^* < 0\). When \(k \to +\infty\), it is clear that \((C_k F_k - D_k E_k)^2 - C_k^2 G_k^2 \to +\infty\). Therefore, there exists a set \(\Gamma_1 \subset [0, +\infty]\) such that when \(k \in \Gamma_1\), \((C_k F_k - D_k E_k)^2 - C_k^2 G_k^2 < 0\) and \(z\) has roots

\[z_k^\pm = \pm \sqrt{-\frac{1}{2} \left[ 2 \left( D_k E_k - C_k F_k \right) + (C_k + F_k)^2 - G_k^2 \right] + \frac{1}{2} \sqrt{\Delta_1}}.\]

where \(\Delta_1 = \left[ 2 \left( D_k E_k - C_k F_k \right) + (C_k + F_k)^2 - G_k^2 \right] - 4 \left[ (C_k F_k - D_k E_k)^2 - C_k^2 G_k^2 \right]\). From equation (16), we can obtain

\[
\cos z_k \tau = \frac{(D_k E_k - C_k F_k) G_k F_k F_k G_k Z_k^2}{(C_k G_k)^2 - G_k^2 z_k^2} = f(z_k).
\]

It can be deduced that the values that time delay takes at the Hopf bifurcation are

\[\tau_k^{(j)} = \begin{cases} \frac{1}{z_k^j} \arccos f(z_k) + \frac{2}{z_k^j} (j - 1) \pi, & \sin(z_k^j \tau) \geq 0, \\ -\frac{1}{z_k^j} \arccos f(z_k) + \frac{2}{z_k^j} j \pi, & \sin(z_k^j \tau) < 0, \end{cases} \]

\(j = 1, 2, \ldots\).

Further, we discuss the variation in the real part of the eigenvalues in the vicinity of the conjugate pure imaginary roots. Derive both sides of the characteristic equation (16) with respect to \(\tau\), and we have

\[
\frac{d\lambda}{d\tau} = \frac{\lambda e^{-\lambda \tau} (C_k G_k - G_k \lambda)}{2\lambda - C_k - F_k - \tau e^{-\lambda \tau} (C_k G_k - G_k \lambda) - G_k e^{-\lambda \tau}}.
\]

Taking the reciprocal of the upper form and separating the real part, we can get

\[
\Re \left[ \left( \frac{d\lambda}{d\tau} \right)^{-1} \right]_{\lambda = iz} = \frac{2 \left( z_k^j \right)^2 + \left[ 2 \left( D_k E_k - C_k F_k \right) + (C_k + F_k)^2 - G_k^2 \right]}{\left( z_k^j \right)^2 G_k^2 + (C_k G_k)^2} = \frac{\sqrt{\Delta_1}}{G_k^2 + (C_k G_k)^2} > 0.
\]

So far, the proof of existence of Hopf bifurcation for system (8) is complete.

For system (9), there is still the characteristic equation in the form of equation (16):

\[
\lambda^2 + a_k \lambda + b_k + e^{-\lambda \tau} (c_k \lambda + d_k) = 0, \quad (18)
\]

where

\[
b_k = \left[ S^* \left( 3 \frac{\beta^2}{\mu} - A - K \right) + 2AK - k^2 d_{11} \right] \\
\left( -2\beta S^* - k^2 d_{22} \right) - k^4 d_{12} d_{21},
\]

\[
c_k = \beta I^* - \beta S^*,
\]

\[
d_k = \left[ S^* \left( 3 \frac{\beta^2}{\mu} - A - K \right) + 2AK - k^2 d_{11} \right] \beta S^* + \left( 2\beta S^* + k^2 d_{22} \right) \beta I^* + k^2 \left( \beta I^* d_{12} - \beta S^* d_{21} \right).
\]

By substituting \(\lambda = iz\) into the equation, separating the real part and the imaginary part, we can obtain

\[
\left\{ \begin{array}{l}
\bar{c}_k \bar{z} \cos(\bar{z} \tau) + \bar{d}_k \cos(\bar{z} \tau) = \bar{z}^2 - \bar{b}_k, \\
\bar{c}_k \bar{z} \cos(\bar{z} \tau) - \bar{d}_k \sin(\bar{z} \tau) = -\bar{a}_k \bar{z}.
\end{array} \right.
\]

By substituting the below equation into the above equation and keep only the terms of \(\cos(\bar{z} \tau)\) and \(\sin(\bar{z} \tau)\), we can obtain a quadratic equation about \(\cos(\bar{z} \tau)\):

\[
0 = \left( a_k c_k d_k + b_k c_k^2 + d_k^2 \right) \cos^2(\bar{z} \tau) + \left( c_k^2 d_k + a_k^2 d_k + 2a_k c_k b_k \right) \cos(\bar{z} \tau) + a_k c_k d_k + a_k^2 b_k - d_k^2 := g_k(\cos(\bar{z} \tau)).
\]

At this point, the problem of existence of the Hopf bifurcation for system (9) can be transformed into the existence problem of the solution of \(g_k(x)\) on \([-1, 1]\). When \(k\) takes 0, we examine the values of \(g_k(x)\) taken
at 1 and −1:

\[
g_0(1) = (a_0 + c_0^2)(b_0 + d_0) = -(a_0 + c_0^2)βS^*
\]

\[
S^* \left( \frac{β^2}{μ} - A - K \right) + 2AK > 0.
\]

\[
g_0(-1) = (a_0 - c_0)^2(b_0 - d_0) = -(a_0 - c_0)^2 βS^*
\]

\[
S^* \left( 2\frac{β^2}{μ} - A - K \right) + 2AK + \frac{5}{3} βI^* < 0.
\]

Through the intermediate value theorem, when \( k = 0 \), \( g_0(x) = 0 \) must have a solution \( x_0 \in (-1, 1) \). Since \( g_k(x) \) is continuous for \( k \), there must exist a set \( T_2 \subset [0, +∞) \) such that when \( k \in T_2 \), \( g_k(x) = 0 \) have only one root \( x_k \). At this point, \( z \) has solutions:

\[
z_k = ± \frac{dk\sqrt{1 - x_k}}{ak + ckx_k}.
\]

The values of the corresponding time delay

\[
τ_k^{(j)} = \begin{cases} \frac{1}{z_k} \text{arc cos}(x_k) + \frac{2}{z_k}(j - 1)π, & \sin(x_k) ≥ 0, \\ -\frac{1}{z_k} \text{arc cos}(x_k) + \frac{2}{z_k}jπ, & \sin(x_k) < 0, \end{cases} \quad j = 1, 2, \ldots
\]

Further, similar to the previous part, we have

\[
\text{Re} \left[ \left( \frac{dλ}{dt} \right)^{-1} \right]_{λ = iz_k} = \frac{2(z_k^2 + 2bk + a_k^2 - c_k^2)}{(z_k^2)^2 + c_k^2 + d_k^2} > 0,
\]

where \( Δ_2 = (a_k^2 - c_k^2 - 2bk)^2 - 4(b_k^2 - d_k^2) \). So far, the proof of existence of Hopf bifurcation for system (9) is complete.

The Hopf bifurcation and the Turing bifurcation can jointly affect the spatiotemporal pattern of the system. In Fig. 1, we give the curve of the critical value of time delay \( τ_2^* = 0.9101 \) corresponding to \( k = 0 \) under that set of parameters.

4.3 System of period coefficient diffusion

In many reaction–diffusion systems, it is not always reasonable and effective to assume that the diffusion coefficient of the subject is constant. In fact, there are many diffusion processes in nature whose intensity is closely related to the time of diffusion. In the actual background of rumor propagation, the diffusion coefficient of the crowd in the information network space may also be considered as periodic.

We assume that the diffusion coefficient and time satisfy the following periodic relation: \( D(t) = d + d' \sin(ωt + φ) \), where \( d, d', ω \) and \( φ \) are constants. The periodic diffusion coefficient system can be written as

\[
\begin{bmatrix}
\frac{∂S}{∂t} \\
\frac{∂I}{∂t}
\end{bmatrix}
= \begin{bmatrix}
d_{11} + d'_{11} \sin(ωt + φ_{11}) \\
d_{12} + d'_{12} \sin(ωt + φ_{12})
\end{bmatrix} ΔS
+ \begin{bmatrix}
d_{21} + d'_{21} \sin(ωt + φ_{21}) \\
d_{22} + d'_{22} \sin(ωt + φ_{22})
\end{bmatrix} ΔI
\]

\[
+ \begin{bmatrix}
S(K - S)(S - A) - βSI, \\
βSI - μI^2.
\end{bmatrix}
\]

In this part, we expand the superposition perturbation near the rumor-spreading equilibrium point in the Fourier space into the system with linearization and derive the first-order approximation system on a single time dimension based on Ref. [34]. Then, the stability of the system is further analyzed by using Floquet theory [38]. The derivation process of the first-order system is highly overlapped with the derivation process of characteristic equation of the diffusion system. The first-order system is

\[
δ_t = U_ω(k, t) δ,
\]

where \( U_ω(k, t) \triangleq J - k^2D_ω(t) \), \( δ = (δ_1, δ_2)' \) is perturbation, \( k \) is wave number and \( D_ω(t) \) is the diffusion coefficient matrix with \( ω \) as the period. The stability of the system can be determined by \( U_ω(t) \).

According to Floquet’s theory, there must be a solution that satisfies the following form:

\[
δ(t + \frac{2π}{ω}) = ε \cdot δ(t),
\]

where \( ε \) can be obtained by calculating the characteristic roots of \( E \), which refers to the transformation matrix of the fundamental solution matrix \( Φ(t) \) of system (20) satisfying \( Φ(t + \frac{2π}{ω}) = E \cdot Φ(t) \).

By observing the form of the solution, it is obvious that when \( |ε| > 1 \), the system has an unbounded solution, when \( |ε| < 1 \), the system has a stable solution, and when \( |ε| = 1 \), the system has a periodic solution. Further, when \( ε = 1 \), \( δ(t + \frac{2π}{ω}) = δ(t) \), the minimum period of the periodic solution is \( \frac{2π}{ω} \), and the minimum period is \( \frac{4π}{ω} \), when \( ε = -1 \), \( δ(t + \frac{4π}{ω}) = -δ(t + \frac{4π}{ω}) = δ(t) \). It is effective to calculate the multipliers in judging the stability of the solution. However, in most
cases, it is difficult to calculate the fundamental solution matrix of nonautonomous differential equations, so is this system. So it is almost impossible to figure out the characteristic multiplier by solving the fundamental solution matrix of system (20). Therefore, we need a new method to estimate the range of $\varepsilon$.

The multiplicators $\varepsilon_1$ and $\varepsilon_2$ can be expressed as the roots of

$$
\varepsilon^2 - u(k, \omega)\varepsilon + c = 0,
$$

where the form of $u(k, \omega)$ is difficult to determine, which is related to $U_\omega(k, t)$. But fortunately, $c$ can be given by the following formula:

$$
c = \varepsilon_1\varepsilon_2 = \exp\left(\int_0^{2\pi} tr(U_\omega(k, t))\,dt\right).
$$

Meanwhile, $0 < c < 1$, with $\int_0^{2\pi} tr(U_\omega(k, t))\,dt = \frac{2\pi}{\omega} \left(tr(J) - k^2(d_{11} + d_{22})\right) < 0$. Multiplicators can be represented by $u$ and $c$: $\varepsilon_1 = \frac{1}{2}(u + \sqrt{u^2 - 4c})$, $\varepsilon_2 = \frac{1}{2}(u - \sqrt{u^2 - 4c})$. We can do the following analysis for different values of $u$:

1. $|u| > 1 + c$. In this case, $u^2 - 4c > 0$, and $\varepsilon_1, \varepsilon_2$ are both negative real numbers with $\varepsilon_2 < -1 < -c < \varepsilon_1 < 0$, or $0 < \varepsilon_2 < c < 1 < \varepsilon_1$. Equation (20) has an unbounded solution.

2. $|u| = 1 + c$. In this case, $u^2 - 4c > 0$. When $u < 0$, $\varepsilon_2 = -1$, and system (20) has a solution with period $\frac{4\pi}{\omega}$. When $u > 0$, there is a solution with period $\frac{2\pi}{\omega}$ for $\varepsilon_1 = 1$.

3. $2\sqrt{c} \leq |u| < 1 + c$. In this case, $u^2 - 4c \geq 0$, $\varepsilon_1$ and $\varepsilon_2$ both are real numbers with the same sign whose modulus is less than 1. The solution to system (20) is stable.

4. $|u| < 2\sqrt{c}$. In this case, $u^2 - 4c < 0$, and $\varepsilon_1, \varepsilon_2$ are conjugate with non-zero imaginary parts. System (20) has the general solution in the following form:

$$
\delta(t) = c_1 \exp\left(\frac{2\pi i c_1 t}{\omega}\right) \left( p_{11}(t) \atop p_{21}(t) \right) + c_2 \exp\left(\frac{2\pi i c_2 t}{\omega}\right) \left( p_{12}(t) \atop p_{22}(t) \right),
$$

where $|\varepsilon_1| < 1$, $|\varepsilon_2| < 1$, $c_1, c_2$ are arbitrary constants, $p_{11}(t)$, $p_{12}(t)$, $p_{21}(t)$ and $p_{22}(t)$ are periodic function. The solution is oscillating, but not generally periodic.

The region where the above parameters make the solutions all bounded with no period is defined as the stable parametric region, the region where the system has unbounded solutions is defined as the unstable parametric region, and the region where the equation has periodic solutions is defined as the periodic region. Figure 2a intuitively shows various regional attributes of $u - c$ in the above 4 cases.

It can be found that the periodic region divides the parameter stable and unstable regions of system (20). In order to obtain the instability of system (20), we consider using the numerical solution method to solve the condition that the parameters satisfy when the periodic solution exists.

Fourier expansion of $\delta$ with period of $\frac{4\pi}{\omega}$ can be obtained by

$$
\delta(t) = \frac{\delta_1}{\delta_2} = \sum_{n=0}^{\pm \infty} \left( A_n e^{i\omega_0 t + \frac{in\omega}{2}} + B_n e^{i\omega_0 t - \frac{in\omega}{2}} \right).
$$

Let $a_0 + \frac{in\omega}{2} := a_n$, $b_0 + \frac{in\omega}{2} := b_n$. Insert expansion into system (20), make $\sin(x) = \frac{1}{2i} (e^{ix} - e^{-ix})$, and we can obtain

$$
\sum_{n=-\infty}^{\infty} e^{i\omega_0 t} \left[ (a_n - j_{11} + k^2 d_{11}) A_n + \frac{1}{2i} d_{11}^2 e^{i\phi_{11}} A_{n-2} - \frac{1}{2i} d_{11}^2 e^{-i\phi_{11}} A_{n+2} \right] = 0,
$$

$$
\sum_{n=-\infty}^{\infty} e^{i\omega_0 t} \left[ (b_n - j_{22} + k^2 d_{22}) B_n + \frac{1}{2i} d_{22}^2 e^{i\phi_{22}} B_{n-2} - \frac{1}{2i} d_{22}^2 e^{-i\phi_{22}} B_{n+2} \right] = 0,
$$

where $A_n, B_n$ are periodic function. The periodic region divides the parameter stable and unstable regions of system (20).
The formula is true for any $t$, and let $t = 0$, then
\[
(a_n - j_{11} + k^2d_{11})A_n + \frac{1}{2i}d_1'k^2 e^{\phi_{11}}A_{n-2} - \frac{1}{2i}d_1'^2 k^2 e^{-\phi_{11}}A_{n+2} + (-j_{12} + k^2d_{12})B_n + \frac{1}{2i}d_2'k^2 e^{\phi_{12}}B_{n-2} - \frac{1}{2i}d_2'^2 k^2 e^{-\phi_{12}}B_{n+2} = 0
\]
\[
(-j_{21} + k^2d_{21})A_n + \frac{1}{2i}d_1'^2 k^2 e^{\phi_{21}}A_{n-2} - \frac{1}{2i}d_2'^2 k^2 e^{-\phi_{21}}A_{n+2} + (b_n - j_{22} + k^2d_{22})B_n + \frac{1}{2i}d_2'^2 k^2 e^{\phi_{22}}B_{n-2} - \frac{1}{2i}d_2'^2 k^2 e^{-\phi_{22}}B_{n+2} = 0
\]
\[n = (\cdots - 1, 0, 1 \cdots)
\]

The formula is an infinite set of homogeneous linear equations.

**Remark 1** When system (20) has a solution with a period of $\frac{2\pi}{\omega}$, it is required that $A_n$ and $B_n$ have solutions that are not all zero when $n$ only takes odd numbers. This part is consistent with the document [34, 39–41].

**Remark 2** Similar to above, when system (20) has a solution with a period of $\frac{4\pi}{\omega}$, it is required that $A_n$ and $B_n$ have solutions that are not all zero. Given the fact that this situation includes the solution with a period of $\frac{2\pi}{\omega}$, to get solutions periodic of period $\frac{4\pi}{\omega}$ instead of $\frac{2\pi}{\omega}$, conditions that there are nonzero solutions for some $A_n$ and $B_n$ must be satisfied when $n$ takes even numbers. Therefore, we only need to consider whether the corresponding $A_n$ and $B_n$ are all nil when $n$ is only even to determine the existence condition of the solution with a minimum period of $\frac{4\pi}{\omega}$.

Since it is a homogeneous system of equations, the condition for the above two cases is that the infinite determinant formed by the coefficients of the corresponding $A_n$ and $B_n$, known as a Hill determinant, is nil. For the convergence of the Hill determinant, each equation is divided by $(1 + n^2)$. For simplicity, make the following definition:

\[
y_n^{(11)} = \frac{a_n - j_{11} + k^2d_{11}}{1 + n^2}, \quad y_n^{(12)} = \frac{-j_{12} + k^2d_{12}}{1 + n^2}, \quad y_n^{(21)} = \frac{-j_{21} + k^2d_{21}}{1 + n^2}, \quad y_n^{(22)} = \frac{b_n - j_{22} + k^2d_{22}}{1 + n^2}
\]
\[
\xi_n^{(11)} = \frac{d_1'k^2 e^{\phi_{11}}}{2i(1 + n^2)}, \quad \xi_n^{(12)} = \frac{d_1'^2 k^2 e^{\phi_{12}}}{2i(1 + n^2)}, \quad \xi_n^{(21)} = \frac{d_2'k^2 e^{\phi_{21}}}{2i(1 + n^2)}, \quad \xi_n^{(22)} = \frac{d_2'^2 k^2 e^{\phi_{22}}}{2i(1 + n^2)}
\]

The 6-order determinant of Hill determinant when $n$ takes odd number (system has a periodic solution of $\frac{2\pi}{\omega}$) is given by

\[
H_{6 \times 6}^{\text{odd}} = \begin{pmatrix}
\gamma_2^{(11)} & \gamma_2^{(12)} & \xi_2^{(11)} & \xi_2^{(12)} & 0 & 0 \\
\gamma_2^{(21)} & \gamma_2^{(22)} & \xi_2^{(21)} & \xi_2^{(22)} & 0 & 0 \\
\xi_0^{(11)} & \xi_0^{(12)} & \gamma_0 & \gamma_0 & \xi_0^{(11)} & \xi_0^{(12)} \\
\xi_0^{(21)} & \xi_0^{(22)} & \gamma_0 & \gamma_0 & \xi_0^{(21)} & \xi_0^{(22)} \\
0 & 0 & \xi_2^{(11)} & \xi_2^{(12)} & \gamma_2 & \gamma_2 \\
0 & 0 & \xi_2^{(21)} & \xi_2^{(22)} & \gamma_2 & \gamma_2
\end{pmatrix},
\]

and the 8-order determinant of Hill determinant when $n$ takes even number (system has a periodic solution of $\frac{4\pi}{\omega}$) is given by
Fig. 2 Panel (b) shows the state of system (20) under different values of \( u \) and \( c \). Panel (b) shows the state of system (20) under different values of \( k \) and \( \omega \). Panels (c)–(f) show some special solution curves whose parameters correspond to the four points

- Panel (b): 
  - Point a: \( (k = 1, \omega = 0.5) \)
  - Point b: \( (k = 0.8, \omega = 1.53) \)
  - Point c: \( (k = 4, \omega = 0.05) \)
  - Point d: \( (k = 1.5, \omega = 3) \)

- In panels (b)-(f), the other parameters are set as follows: \( K = 1, A = 0.1, \beta = 0.46, \mu = 0.5, d_{11} = d_{11}' = 0.5, d_{22} = d_{22}' = 0.5, d_{12} = d_{12}' = 0.1, d_{21} = d_{21}' = 0.965, \phi_{11} = 0, \phi_{22} = \frac{4\pi}{15}, \phi_{12} = \frac{\pi}{6}, \phi_{21} = \frac{\pi}{5}, \theta_{1} = 0, \theta_{2} = 0 \)
Numerical solution methods such as Newton’s method can be used to draw parametric feasible curves of the above two determinants, respectively, equal to zero.

As shown in Fig. 2b, after selecting a set of parameters, the boundary curves of stability and instability of the system based on the changes of \( k \) and \( \omega \), namely the periodic solution curve, are given. Further, the value of \( k \) and \( \omega \) marked with red letters is selected in panel (b), and the corresponding variation curves of system disturbance are given by using the four-five-order Range–Kutta method algorithm in panel (c)–(f) of Fig. 2. In Fig. 2c, we give a set of initial values satisfying the perturbation instability and obtain the solution curve of the system. In Fig. 2d and e, the parameters fall on the periodic curves, and the disturbance (initial value is set as \((0.1,-0.1)\)) presents a vibration phenomenon with a period of \( \frac{2\pi}{\omega} \) and \( \frac{4\pi}{\omega} \). In Fig. 2f, the parameter is in the stable region, and the solution curve with the initial value of \((0.1, -0.1)\) finally converges to zero.

In order to appear Turing instability, we require the system to be stable when \( k = 0 \) and unstable when \( k > 0 \), so we can try to give the expression of the necessary conditions for the existence of Turing pattern in the periodic diffusion coefficient system analogously [34]: After a set of parameters is selected, an image similar to that shown in Fig. 2b can be obtained, and \( \omega \) can be fixed at the same time. If Turing instability exists in the system, it must satisfy the conditions that when \( k = 0 \), the system falls on the stable region and there is \( k > 0 \) makes the system is in the unstable region. Since \( \omega \) satisfies the above necessary conditions in the range shown in Fig. 2b, we can try to obtain Turing pattern of the periodic system under such parameters. It is worth noting that if \( d' = 0 \), that is, when there is no periodic term, with the same rest of parameters, the pattern phenomenon of the constant diffusion coefficient system can still be derived. However, in the following numerical simulation, we give an example that Turing pattern exists with period terms but not with \( d' = 0 \).

5 Numerical simulation

In this section, we will perform extensive numerical simulations of the above model to observe their patterns of spatiotemporal oscillations. All systems are defined on in \( \Omega \times T \), where \( T = [0, +\infty) \), \( \Omega = [0, 100] \times [0, 100] \) represents a two-dimensional rectangular space or \( \Omega = [0, 100] \) represents a one-dimensional space. In this section, the numerical simulation of process adopting zero flux boundary conditions, numerical integration employs forward Euler integral method, and central difference scheme is used for the numerical approximation of Laplacian. In the process of iterative, the time step \( \Delta t \) takes 0.01, and the space step \( \Delta h \) takes 1.

In addition, the number of iterations of pattern formations given is not always the same, as detailed after each figure, and the last one of the multiple patterns given by the same system at different points in time evolution can always represent the stable state of the system, that is, the patterns after that will not change. By observing the pattern types of the system under different parameters, it can be found that the pattern types of \( S \) and \( I \) are always consistent. Therefore, in the following simulation results, we will only give the diagrams of \( S \)'s density distribution.
Fig. 3  In the case that \( d_{11} = 0.1, d_{22} = 2, d_{12} = d_{21} = 0 \), the evolution process of Turing pattern for system (3) is shown. In panel (a), \( t = 500 \). In panel (b), \( t = 1000 \). In panel (c), \( t = 1500 \). The other parameters are set as follows: \( K = 1 \), \( A = 0.1 \), \( \beta = 0.46 \), \( \mu = 0.5 \).

Fig. 4  In the case that \( d_{11} = 0.1, d_{22} = 2, d_{12} = 0.2, d_{21} = 0.4 \), the evolution process of Turing pattern for system (3) is shown. In panel (a), \( t = 100 \). In panel (b), \( t = 200 \). In panel (c), \( t = 300 \). The other parameters are set as follows: \( K = 1 \), \( A = 0.1 \), \( \beta = 0.46 \), \( \mu = 0.5 \).

Fig. 5  In the case that \( d_{11} = 0.5, d_{22} = 0.5, d_{12} = 0.1, d_{21} = 0.965 \), the evolution process of Turing pattern for system (2) is shown. In panel (a), \( t = 3000 \). In panel (b), \( t = 6000 \). In panel (c), \( t = 9000 \). The other parameters are set as follows: \( K = 1 \), \( A = 0.1 \), \( \beta = 0.46 \), \( \mu = 0.5 \).
Pattern formation in a reaction–diffusion rumor propagation system

5.1 Constant diffusion coefficient

In this part, we simulate the formation of the pattern for system (3) and study the effect of diffusion coefficients on the system.

Firstly, we examine different types of Turing patterns in $\Omega = [0, 100] \times [0, 100]$. As shown in Fig. 3, there is no cross-diffusion ($d_{12} = d_{21} = 0$). When $t = 500$, as shown in Fig. 3a the system mainly shows strips in space. With the advance of time, cold spots begin to appear, and at this time, cold spots and strips compete with each other, as shown in Fig. 3b. Finally, the spatial distribution of the system evolves into an irregular cold spots with red (high density) as the background, that is, some independent circular regions with low density appear in the space.

When the system is affected by cross-diffusion ($d_{12} = 0.2, d_{21} = 0.4$), the formation time of the stable spot pattern of the system is significantly reduced. When $t = 100$, as shown in Fig. 4a, the competition between spots and strips also appears in the pattern, but with the evolution of time, the pattern finally shows the coexistence of cold spots and cold strips. At this time, in the distribution of $S$, some independent regions with low density also appear in the space, but at this time, the shapes of these regions are greatly different.

In Figs. 3 and 4, the diffusion coefficient $d_{22}$ is much higher than $d_{11}$. In Fig. 5, $d_{11}$ and $d_{22}$ are taken as 0.5, at which time the pattern type changes significantly. In Fig. 5a, the competition between hot spots and hot bars appears, and finally, the red spots are stable on the blue background, and the distribution of the points is very regular. At this time, a high-density circular independent region appears in the background of low density, and the distribution of $S$ in space appears aggregation phenomenon.

In order to illustrate the evolution of the system in time and space simultaneously, we further evaluate the system numerically in one-dimensional space under the three different set of diffusion coefficients in Fig. 6, which vertical coordinate represents space and the horizontal coordinate represents time. Figure 6a–c corresponds to the parameters of Figs. 3, 4 and 5, respectively.

From the time dimension, the system appears no oscillations. Compared to the system on the two-dimensional space, the convergence of the system on the one-dimensional space is similar overall. In the spatial dimension, the oscillations of the system are similar at the three sets of coefficients, all of which show spaced arrangement of high-density regions and low-density regions. However, it is noteworthy that the cold color stripes in Fig. 6a are wider and more uniform than those in Fig. 6b when the system is stable, which matches the characteristics of the patterns presented by the system in two dimensions. Based on the previous observation that the distribution of spot patterns is most regular and uniform in Fig. 5, we observe Fig. 6c and also find that the distribution of its cold color stripes is indeed the most regular and uniform.

5.2 Pattern driven by time delay

As analyzed in the previous section, time delays may affect both the temporal and spatial stability of the
system, thus triggering more complex spatio-temporal oscillations. In order to better observe the dual effect of time delay on the system from the time dimension and the space dimension, in this section, we focus on the numerical evaluation of the system (8) and system (9) in the one-dimensional space.

For system (8), we fix the following parameters: $K = 1$, $A = 0.1$, $\beta = 0.4$, $\mu = 0.49$, $d_{11} = 0.05$, $d_{12} = 0.11$, $d_{21} = 0.9$. At this point, $\tau_1^* = 2.0686$. Three sets of parameters are selected for simulation: (1) $\tau_1 = 1.2$, $d_{22} = 2$; (2) $\tau_1 = 2.07$, $d_{22} = 3$; (3) $\tau_1 = 2.1$, $d_{22} = 2$.

Under the first set of parameters, the system is expected to exhibit Turing instability and Hopf stability theoretically. As shown in Fig. 7a, the system is stable in time without oscillations. Spatially, the pattern is mainly presented as an inhomogeneous distribution of smaller low-density regions against a high-density background. Figure 7d shows the 3-D view of the spatial pattern for system (8) on two-dimensional space at $t = 1000$ under the same parameters, and it can be seen that the Turing pattern is dominated by cold spots. These results are consistent with theoretical expectations. The second set of parameters satisfies the conditions of Turing stability and Hopf instability. As shown in Fig. 7b, the system exhibits periodic solutions in the time dimension and tends to be homogeneous in space, which coincides with the spatial pattern of the system in two dimensions shown in Fig. 7e. The third set of parameters satisfies both instability conditions. Figure 7c shows a very interesting spatio-temporal pattern, which exhibits the instability in two dimensions. To the naked eye, the part that exhibits spatial vibrations and the part that exhibits temporal vibrations appear to be split. In fact, all points in space are undergoing periodic variations, but it is also true that there is a large interval $[60, 85]$, where the system is homogeneous.

For system (9), we fix the following parameters: $K = 1$, $A = 0.1$, $\beta = 0.46$, $\mu = 0.5$, $d_{11} = 0.5$, $d_{22} = 0.5$, $d_{21} = 0$. At this point, $\tau_2^* = 0.9101$. 

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**Fig. 7** Numerical simulation results of system (8) under different parameters. $K = 1$, $A = 0.1$, $\beta = 0.4$, $\mu = 0.49$, $d_{11} = 0.05$, $d_{12} = 0.11$, $d_{21} = 0.9$. Panel (a–c) show the spatio-temporal patterns of the system in a one-dimensional space. Panel (d–f) show the 3D view of the spatial patterns for the corresponding system in two dimensions at a certain moment. In panel (a) and (d), $\tau_1 = 1.2$, $d_{22} = 2$, $t = 1000$. In panel (b) and (e), $\tau_1 = 2.07$, $d_{22} = 3$, $t = 200$. In panel (c) and (f), $\tau_1 = 2.1$, $d_{22} = 2$, $t = 1000$. 

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Fig. 8 Numerical simulation results of system (9) under different parameters. $K = 1, A = 0.1, \beta = 0.46, \mu = 0.5, d_{11} = 0.5, d_{22} = 0.5, d_{21} = 0$. Panel (a-c) show the spatio-temporal patterns of the system in a one-dimensional space. Panel (d-f) show the 3D view of the spatial patterns for the corresponding system in two dimensions at a certain moment. In panel (a) and (d), $\tau_2 = 0.7, d_{12} = 11, t = 1000$. In panel (b) and (e), $\tau_2 = 0.92, d_{12} = 11, t = 500$. In panel (e) and (f), $\tau_2 = 0.7, d_{12} = 2, t = 1000$

Three sets of parameters are selected for simulation: (1) $\tau_2 = 0.7, d_{12} = 11$; (2) $\tau_2 = 0.92, d_{12} = 11$; (3) $\tau_2 = 0.7, d_{12} = 2$.

Referring to the results in Fig. 2, the first set of parameters falls below the hopf bifurcation curve and to the right of the Turing bifurcation curve. Figure 8a shows significant heterogeneity in spatial distribution, which is consistent with the theoretical results. However, contrary to theoretical expectations, the system also exhibits periodicity in time. This is mainly due to the fact that the $\tau_2^*$ we give is based on $k = 0$, which is not exact, because we often do not know the exact value of the eigenvalue of the Laplace operator $-k^2$. At this point, the diffusion coefficient $d_{12}$ is large, and the actual $\tau_2^*$ is smaller than the one we gave by 0.9101. When we make the diffusion coefficients all 0, i.e., excluding the effect of diffusion, the system is locally asymptotically stable in time at the equilibrium point. The second set of parameters falls on the hopf bifurcation curve and the Turing stability region. The results presented in Fig. 8b and e match the theoretical expectations. The third set of parameters satisfies the conditions for both types of instability. Figure 8c shows the vibration of the system in time at this point. It looks more complex compared to Fig. 7c, and at each moment, the space exhibits heterogeneous characteristics. As shown in Fig. 8f, the two-dimensional spatial pattern of the system is mainly presented as a hot stripe pattern.

5.3 Pattern driven by period

In this section, we try to examine the effect of periodic diffusion on the system. First we simulate the system under the same three sets of parameters as shown in Fig. 6. We fix the other parameters: $d_{11}' = 0.5d_{11}$, $d_{22}' = 0.5d_{22}$, $d_{12}' = 0.5d_{12}$, $d_{21}' = 0.5d_{21}$, $\omega = 0.5$, $\phi_{11} = 0$, $\phi_{22} = \frac{3\pi}{2}$, $\phi_{12} = \frac{\pi}{2}$, $\phi_{21} = \frac{\pi}{2}$. 
Fig. 9 Numerical simulation results of system (19) under different parameters. $K = 1$, $A = 0.1$, $\beta = 0.46$, $\mu = 0.5$, $d_{11} = 0.5d_{11}$, $d_{22} = 0.5d_{22}$, $d'_{12} = 0.5d_{12}$, $d'_{21} = 0.5d_{21}$, $\omega = 0.5$, $\phi_{11} = 0$, $\phi_{22} = \frac{2\pi}{7}$, $\phi_{12} = \frac{2\pi}{7}$, $\phi_{21} = \frac{2\pi}{7}$. Panels (a-c) show the spatio-temporal patterns of the system on a one-dimensional space. Panels (d-f) show the spatial patterns for the corresponding system in two dimensions at a certain moment. In panel (a) and (d), $d_{11} = 0.1$, $d_{22} = 2$, $d_{12} = 0$, $d_{21} = 0$, $t = 16000$. In panels (b) and (e), $d_{11} = 0.1$, $d_{22} = 2$, $d_{12} = 0.2$, $d_{21} = 0.4$, $t = 600$. In panel (c) and (f), $d_{11} = 0.5$, $d_{22} = 0.5$, $d'_{12} = 0.1$, $d'_{21} = 0.965$, $t = 2000$.

Fig. 10 Periodium-induced pattern versus periodium-free condition. $K = 1$, $A = 0.1$, $\beta = 0.46$, $\mu = 0.5$, $d'_{11} = 1$, $d'_{22} = 1$, $d'_{12} = 0.2$, $d'_{21} = 1$, $d_{11} = 0.5$, $d_{22} = 0.5$, $d_{12} = 0.1$, $d_{21} = 0.5$, $\phi_{11} = 0$, $\phi_{22} = \frac{2\pi}{7}$, $\phi_{12} = \frac{2\pi}{7}$, $\phi_{21} = \frac{2\pi}{7}$, $\theta_1 = 0$, $\theta_2 = 0$. Panel (a) shows the state system (19) is in for different values of $\omega$. Panel (b) shows the spatio-temporal patterns of the system on a one-dimensional space at $\omega = 2.8$. Panel (c) shows the spatio-temporal patterns of the system without periodic diffusion.

Figure 9a–c shows the spatio-temporal patterns of the system in a one-dimensional space. Figure 9d–f shows the spatial patterns for the corresponding system in two dimensions at a certain moment. From the results, the main impact of periodic diffusion is concentrated in the time dimension. Due to the effect of the periodicity coefficient, the system does not eventually converge to a constant solution, but similar to
the system with time delay, the system exhibits regular periodic variations in the time dimension. The periodic diffusion does not have a large impact on the spatial patterns of the system. However, it is worth noting that in Fig. 9f and c, the highly ordered distribution of hot spots is destroyed, and hot spots at the junction of irregular arrangements cannot reach the highest density of other hot spots even after a long time of evolution.

Further, we give an example of periodic diffusion-induced Turing instability under a certain set of parameters: \( K = 1, A = 0.1, \beta = 0.46, \mu = 0.5, d'_{11} = 1, d'_{22} = 1, d'_{21} = 1, d_{11} = 0.5, d_{22} = 0.5, d_{12} = 0.1, d_{21} = 0.5, \phi_{11} = 0, \phi_{22} = \frac{4\pi}{3}, \phi_{12} = \frac{\pi}{3}, \phi_{21} = \frac{\pi}{6}, \theta_1 = 0, \theta_2 = 0. \) Figure 10a illustrates the variation in the system state with \( \phi \) and \( k \). Taking \( \omega = 2.8 \), we give the spatio-temporal pattern of the system in the one-dimensional space as shown in Fig. 10b. It is easy to see that the system shows significant spatial heterogeneity, i.e., some low-density gaps appear on a high-density background. To show that this spatial heterogeneity is due to the periodic diffusion, we also give the spatio-temporal pattern of the corresponding system in the absence of periodic diffusion as shown in Fig. 10c. It is clear that spatial heterogeneity vanishes in this case.

Further, we try to examine the type of Turing pattern of the system under the influence of periodic diffusion. As the space defined by the system varies, the eigenvalue \(-k^2\) corresponding to the Laplace operator changes, which makes it impossible to produce Turing instability for the system which is defined on a two-dimensional space by using exactly the same parameters as shown in Fig. 10b. So we retake \( \omega \) as 5.2 and obtain the evolution of the Turing pattern as shown in Fig. 11. At this point, the system mainly presents a very regular stripe pattern.

6 Conclusion

Based on the characteristics of rumor propagation in cyberspace, this paper establishes a reaction–diffusion model for \( S \) and \( I \), studies and analyzes the diffusion behavior of various groups in cyberspace, and presents the characteristics of spatial instability under some special parameters. In the third section, Theorem 1 is used to solve the problem of local stability at the equilibrium point of a system with a small time delay. In the analysis of the influence of time delay on the system, we find that the smaller time delay generally does not change the positivity of the determinant of the Jacobian matrix and the \( H \) matrix of the original system, but it can affect the positivity of the \( \text{tr}(H) \), thus causing instability that does not exist in the original system in space. At the same time, the existence of Hopf bifurcation in certain parameter domains is theoretically demonstrated. It is shown that time delay can trigger both Hopf bifurcation and Turing bifurcation and thus have more complex effects on the spatiotemporal patterns of the system.

For the reaction–diffusion system with periodic coefficients, we can employ Floquet’s theory to analyze the stability of a first-order approximate linear system with \( k \) as the parameter, and determine whether the system satisfies the necessary condition of Turing
instability by solving the parametric numerical solution satisfying the existence of periodic solution.

In the numerical simulations, we have compared the types of patterns with and without cross-diffusion in case of $d_{22} > d_{11}$ and found that when $d_{12} = d_{21} = 0$, the cold spot pattern is finally stable; when $d_{12} = 0.2$, $d_{21} = 0.4$, the cold spot-strip pattern is finally stable. When $d_{11} = d_{22} = 0.5$, the hot spot pattern is finally stable. Thus, we show the time-delay-induced patterns, which is shown that the time delay can cause both the Hopf bifurcation and the Turing bifurcation, which have double effects on the space–time pattern of the system. The change of time delay will also lead to the change of Turing pattern type. Furthermore, we give the period-induced patterns, which proves the correctness that periodicity will lead to spatial instability that does not exist in the original system in the previous analysis.

These conclusions are helpful for us to better explore the spatial distribution status of all kinds of people in the network environment and provide help for the control and management of the network information environment. At the same time, the above research can also help to better understand the causes of spatial pattern formation and enrich the research paradigm of reaction–diffusion systems. In the future, we will continue to discuss the dynamics and control methods of spatiotemporal network rumor propagation [42–46].

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Data availability All data during this research are included in this published article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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