A degenerate three-level laser with a parametric amplifier

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The aim of this paper is to study the squeezing and statistical properties of the light produced by a degenerate three-level laser whose cavity contains a degenerate parametric amplifier. In this quantum optical system the top and bottom levels of the three-level atoms injected into the laser cavity are coupled by the pump mode emerging from the parametric amplifier. For a linear gain coefficient of 100 and for a cavity damping constant of 0.8, the maximum intracavity squeezing is found at steady state and at threshold to be 93%.

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I. INTRODUCTION

There has been a considerable interest in the analysis of the quantum properties of the squeezed light generated by various quantum optical systems [1, 2, 3, 4, 5, 6, 7, 8, 9]. In squeezed light the fluctuations in one quadrature is below the vacuum level at the expense of enhanced fluctuations in the other quadrature, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. In addition to exhibiting a nonclassical feature, squeezed light has potential applications in precision measurements and noiseless communications [10, 11].

Some authors have studied the squeezing and statistical properties of the light produced by three-level lasers when either the atoms are initially prepared in a coherent superposition of the top and bottom levels [12, 13, 14] or when these levels are coupled by a strong coherent light [15]. These studies show that a three-level laser can under certain conditions generate squeezed light. In such a laser, three-level atoms in a cascade configuration are injected at a constant rate into the cavity coupled to a vacuum reservoir via a single-port mirror. When a three-level atom makes a transition from the top to bottom level via the intermediate level, two photons are generated. The two photons are highly correlated and this correlation is responsible for the squeezing of the light produced by a three-level laser. On the other hand, it is well known that a parametric oscillator with a typical source of squeezed light [2, 3, 4, 5, 6] is a source of squeezed light [2, 3, 4, 5, 6], with a maximum intracavity squeezing of 50%. Recently Fesseha [12] has studied a three-level laser with a parametric amplifier in which three-level atoms, initially prepared in a coherent superposition of the top and bottom levels, are injected into the cavity. He has found that the effect of the parametric amplifier is to increase the intracavity squeezing by a maximum of 50%.

In this paper we consider a degenerate three-level laser whose cavity contains a degenerate parametric amplifier (DPA) and coupled to a vacuum reservoir. The top and bottom levels of the three-level atoms injected into the cavity are coupled by the pump mode emerging from the parametric amplifier. And the three-level atoms are initially prepared in such a way that the probabilities of finding the atoms at the top and bottom levels are equal. We expect that a highly squeezed light can be generated by the quantum optical system under consideration. Thus our interest is to analyze the squeezing and statistical properties of the light generated by this system.

We obtain, applying the master equation, stochastic differential equations for the cavity mode variables associated with the normal ordering. The solutions of the resulting equations are used to determine the quadrature variance, the squeezing spectrum, and the mean photon number. Moreover, applying the same solutions, we determine the antinormally ordered characteristic function with the aid of which the Q function is obtained. Then the Q function is used to calculate the photon number distribution.

II. STOCHASTIC DIFFERENTIAL EQUATIONS

Three-level atoms in a cascade configuration are injected into the laser cavity at a constant rate $r_a$ and removed from the cavity after a certain time $\tau$. We represent the top, middle, and bottom levels of a three-level atom by $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively. We assume the transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transitions between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. We consider the case for which the cavity mode is at resonance with the two transitions $|a\rangle \rightarrow |b\rangle$ and $|b\rangle \rightarrow |c\rangle$ (see Fig. 1).

The Hamiltonian describing the coupling of levels $|a\rangle$ and $|c\rangle$ by the pump mode emerging from the parametric amplifier can be expressed as

$$\hat{H}' = \frac{i}{2} \Omega (|c\rangle\langle a| - |a\rangle\langle c|),$$

in which $\Omega = 2g'\mu$ with $g'$ and $\mu$ being respectively the coupling constant and the amplitude of the pump mode.
FIG. 1: A degenerate three-level laser with a degenerate parametric amplifier.

In addition, the interaction of a three-level atom with the cavity mode can be described by the Hamiltonian

\[
\hat{H}"" = ig[\hat{a}^\dagger(\langle b|a\rangle + |c\rangle\langle b\rangle) - \hat{a}(\langle a|b\rangle + |b\rangle\langle c\rangle)],
\]

(2)

where \( g \) is the coupling constant and \( \hat{a} \) is the annihilation operator for the cavity mode. Thus the Hamiltonian describing the interaction of a three-level atom with the cavity mode and with the pump mode emerging from the parametric amplifier has the form

\[
\hat{H} = ig[\hat{a}^\dagger(\langle b|a\rangle + |c\rangle\langle b\rangle) - \hat{a}(\langle a|b\rangle + |b\rangle\langle c\rangle)] + i\frac{\Omega}{2}(|c\rangle\langle a| - |a\rangle\langle c|).
\]

(3)

We take the initial state of a single three-level atom to be

\[
|\psi_A(0)\rangle = \frac{1}{\sqrt{2}}|a\rangle + \frac{1}{\sqrt{2}}|c\rangle
\]

(4)

and hence the density operator for a single atom is

\[
\hat{\rho}_A(0) = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|c\rangle\langle c| + \frac{1}{2}|a\rangle\langle c| + \frac{1}{2}|c\rangle\langle a|.
\]

(5)

It can be readily established that the equation of evolution of the density operator for the laser cavity mode, coupled to a vacuum reservoir, has in the linear and adiabatic approximation the form

\[
\frac{d}{dt}\hat{\rho} = R(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + S(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}) + U(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}) + V(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}),
\]

(6)

where

\[
R = \frac{A}{4B}\left[1 - \frac{3\beta}{2} + \beta^2\right],
\]

(7a)

\[
S = \frac{A}{4B}\left[\frac{2\kappa B}{A} + 1 + \frac{3\beta}{2} + \beta^2\right],
\]

(7b)

\[
U = \frac{A}{4B}\left[-1 + \beta + \frac{\beta^2}{2} + \frac{\beta^3}{2}\right],
\]

(7c)

\[
V = \frac{A}{4B}\left[-1 - \beta + \frac{\beta^2}{2} - \frac{\beta^3}{2}\right],
\]

(7d)

\[
B = (1 + \beta^2)(1 + \frac{\beta^2}{4}),
\]

(7e)

\[
\beta = \Omega/\gamma,
\]

(7f)

\[
A = \frac{2g^2r_0}{\gamma^2}
\]

(8)

is the linear gain coefficient, \( \kappa \) is the cavity damping constant, and \( \gamma \) is the atomic decay rate assumed to be the same for all the three levels.

Moreover, a degenerate parametric amplifier with the pump mode treated classically is describable in the interaction picture by the Hamiltonian

\[
\hat{H} = \frac{i\varepsilon}{2}(\hat{a}^2 - \hat{a}^2),
\]

(9)

in which \( \varepsilon = \lambda\mu \) with \( \lambda \) being the coupling constant. The master equation associated with this Hamiltonian has the form

\[
\frac{d}{dt}\hat{\rho} = \frac{\varepsilon}{2}(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2).
\]

(10)

Now on account of Eqs. (5) and (11), the master equation for the cavity mode of the quantum optical system under consideration can be written as

\[
\frac{d}{dt}\hat{\rho} = \frac{\varepsilon}{2}(\hat{\rho}\hat{a}^2 - \hat{a}^2\hat{\rho} + \hat{a}^2\hat{\rho} - \hat{\rho}\hat{a}^2) + R(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}) + S(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}) + U(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}) + V(\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{\rho}).
\]

(11)

We next proceed to determine, applying this master equation, the stochastic differential equations for the cavity mode variables. To this end, applying (11) one readily finds

\[
\frac{d}{dt}(\hat{a}) = (R - S)(\hat{a}) + (U - V + \varepsilon)(\hat{a}^\dagger),
\]

(12)

\[
\frac{d}{dt}(\hat{a}^2) = 2(R - S)(\hat{a})^2 + 2(U - V + \varepsilon)(\hat{a}^\dagger\hat{a}) + \varepsilon - 2V,
\]

(13)

\[
\frac{d}{dt}(\hat{a}^\dagger\hat{a}) = 2(R - S)(\hat{a}^\dagger\hat{a}) + (U - V + \varepsilon)((\hat{a}^2) + (a^2)) + 2R.
\]

(14)
We note that these equations are in the normal order and the corresponding c-number equations are
\[
\frac{d}{dt} \langle \alpha \rangle = -(S - R) \langle \alpha \rangle + (U - V + \varepsilon) \langle \alpha^* \rangle,
\]
\[\text{(15)}\]
\[
\frac{d}{dt} \langle \alpha^2 \rangle = -2(S - R) \langle \alpha^2 \rangle + 2(U - V + \varepsilon) \langle \alpha^* \alpha \rangle + \varepsilon - 2V,
\]
\[\text{(16)}\]
\[
\frac{d}{dt} \langle \alpha^* \alpha \rangle = -2(S - R) \langle \alpha^* \alpha \rangle
\]
\[+ (U - V + \varepsilon) (\langle \alpha^2 \rangle + \langle \alpha^2 \rangle) + 2R.
\]
\[\text{(17)}\]
On the basis of Eq. (15), one can write the stochastic differential equation
\[
\frac{d}{dt} \alpha(t) = -(S - R) \alpha(t) + (U - V + \varepsilon) \alpha^*(t) + f(t),
\]
\[\text{(18)}\]
where \(f(t)\) is a noise force the properties of which remain to be determined. We observe that Eq. (15) and the expectation value of Eq. (18) will have the same form if
\[
\langle f(t) \rangle = 0.
\]
\[\text{(19)}\]
Moreover, it can be readily verified using (18) that
\[
\frac{d}{dt} \langle \alpha^2(t) \rangle = -2(S - R) \langle \alpha^2(t) \rangle
\]
\[+ 2(U - V + \varepsilon) \langle \alpha^*(t) \alpha(t) \rangle
\]
\[+ 2 \langle \alpha(t) f(t) \rangle,
\]
\[\text{(20)}\]
and
\[
\frac{d}{dt} \langle \alpha^*(t) \alpha(t) \rangle = -2(S - R) \langle \alpha^*(t) \alpha(t) \rangle
\]
\[+ (U - V + \varepsilon) (\langle \alpha^2(t) \rangle + \langle \alpha^2(t) \rangle)
\]
\[+ \langle \alpha(t) f^*(t) \rangle + \langle \alpha^*(t) f(t) \rangle.
\]
\[\text{(21)}\]
Comparison of Eqs. (16) and (20) as well as Eqs. (17) and (21) shows that
\[
\langle \alpha(t) f(t) \rangle = \frac{1}{2} (\varepsilon - 2V),
\]
\[\text{(22)}\]
\[
\langle \alpha(t) f^*(t) \rangle + \langle \alpha^*(t) f(t) \rangle = 2R.
\]
\[\text{(23)}\]
Furthermore, a formal solution of Eq. (18) can be written as
\[
\alpha(t) = \alpha(0) e^{-(S - R)t}
\]
\[+ \int_0^t e^{-(S - R)(t - t')} [(U - V + \varepsilon) \alpha^*(t') + f(t')] dt'.
\]
\[\text{(24)}\]
Using Eq. (24) along with (22), one easily finds
\[
\int_0^t e^{-(S - R)(t - t')} \langle f(t') f(t) \rangle dt' = \frac{1}{2} (\varepsilon - 2V).
\]
\[\text{(25)}\]
Based on this result, one can write [12,15]
\[
\langle f(t') f(t) \rangle = (\varepsilon - 2V) \delta(t - t').
\]
\[\text{(26)}\]
It can also be established in a similar manner that
\[
\langle f(t) f^*(t') \rangle = 2R \delta(t - t').
\]
\[\text{(27)}\]
We note that Eqs. (26) and (27) describe the correlation properties of the noise force \(f(t)\) associated with the normal ordering.

Now introducing a new variable defined by
\[
\alpha_{\pm}(t) = \alpha^*(t) \pm \alpha(t),
\]
\[\text{(28)}\]
we easily get with the help of (18) that
\[
\frac{d}{dt} \alpha_{\pm}(t) = -\lambda_{\mp} \alpha_{\pm}(t) + f^*(t) \pm f(t),
\]
\[\text{(29)}\]
where
\[
\lambda_{\mp} = (S - R) \mp (U - V + \varepsilon).
\]
\[\text{(30)}\]
The solution of Eq. (29) can be written as
\[
\alpha_{\pm}(t) = \alpha_{\pm}(0) e^{-\lambda_{\mp} t} + \int_0^t e^{-\lambda_{\mp} (t - t')} (f^*(t') \pm f(t')) dt'.
\]
\[\text{(31)}\]
It then follows that
\[
\alpha(t) = A(t) \alpha(0) + B(t) \alpha^*(0) + F(t),
\]
\[\text{(32a)}\]
in which
\[
A(t) = \frac{1}{2} (e^{-\lambda_{-} t} + e^{-\lambda_{+} t}),
\]
\[\text{(32b)}\]
\[
B(t) = \frac{1}{2} (e^{-\lambda_{-} t} - e^{-\lambda_{+} t}),
\]
\[\text{(32c)}\]
and
\[
F(t) = F_+(t) + F_-(t),
\]
\[\text{(33a)}\]
with
\[
F_\pm(t) = \frac{1}{2} \int_0^t e^{-\lambda_{\mp} (t - t')} (f^*(t') \pm f(t')) dt'.
\]
\[\text{(33b)}\]

### III. Quadrature Fluctuations

In this section we seek to calculate the quadrature variance and squeezing spectrum for the cavity mode under consideration.
A. Quadrature variance

The variance of the quadrature operators
\[ \Delta a_+ = \Delta a^\dagger + \Delta a \]  
(34)
and
\[ \Delta a_- = i(\Delta a^\dagger - \Delta a) \]
(35)
is expressible in terms of c-number variables associated with the normal ordering as
\[ \Delta a^2_{\pm} = 1 \pm \langle \alpha_{\pm}(t), \alpha_{\pm}(t) \rangle, \]
(36)
in which \( \alpha_{\pm}(t) \) is given by Eq. (28). Assuming the cavity mode to be initially in a vacuum state and taking into account (31) together with (19), we see that
\[ \langle \alpha_{\pm}(t) \rangle = 0. \]
(37)
In view of this result, Eq. (36) reduces to
\[ \Delta a^2_{\pm} = 1 \pm (\alpha^2_{\pm}(t)). \]
(38)
Furthermore, employing Eq. (29), one easily gets
\[
\frac{d}{dt} (\alpha^2_{\pm}(t)) = -2\lambda_{\mp} \langle \alpha^2_{\pm}(t) \rangle + 2\langle \alpha_{\pm}(t)f^*(t) \rangle \\
\quad \pm 2\langle \alpha_{\pm}(t)f(t) \rangle.
\]
(39)
With the aid of Eq. (31) along with (26) and (27), we readily obtain
\[
\frac{d}{dt} (\alpha^2_{\pm}(t)) = -2\lambda_{\mp} \langle \alpha^2_{\pm}(t) \rangle + 2(\varepsilon - 2\nu \pm 2R). \]
(40)
The steady-state solution of this equation turns out to be
\[ \langle \alpha^2_{\pm}(t) \rangle = \frac{\varepsilon - 2\nu \pm 2R}{\lambda_{\mp}}. \]
(41)
Now on account of (41) together with (30), Eq. (38) takes at steady state the form
\[
\Delta a^2_{\pm} = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(4 + \beta^2)}{(2\kappa - 4\varepsilon)(1 + \beta^2)(1 + \beta^2/4) + A(2\beta - \beta^3)} \]
(42a)
and
\[
\Delta a^2_{\mp} = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) + 3A\beta^2}{(2\kappa + 4\varepsilon)(1 + \beta^2)(1 + \beta^2/4) + A(4\beta + \beta^3)}, \]
(42b)
where we have used Eqs. (14a)–(14c).

In the absence of the parametric amplifier the quantum optical system under consideration reduces to just a degenerate three-level laser. The quadrature variance for this system has upon setting \( \beta = \varepsilon = 0 \) in (42a) and (42b) the form
\[ \Delta a^2_{\pm} = \frac{\kappa + 2A}{\kappa} \]
(43a)
and
\[ \Delta a^2_{\mp} = 1. \]
(43b)
Since neither of the quadrature variance is less than one, the light produced by the degenerate three-level laser is not in a squeezed state. We therefore observe that the particular initial preparation of the three-level atoms we have considered does not lead to the generation of squeezed light. We next consider the case in which three-level atoms are not injected into the cavity. Thus setting \( A = \beta = 0 \) (with \( \mu \neq 0 \)) in Eqs. (42a) and (42b), we have
\[ \Delta a^2_{\pm} = \frac{\kappa}{\kappa - 2\varepsilon}. \]
(44a)
and
\[ \Delta \alpha^2 = \frac{\kappa}{\kappa + 2\varepsilon}. \] (44b)

At threshold, \( \varepsilon = \kappa/2 \), the above equations reduce to
\[ \Delta \alpha^2 \rightarrow \infty \] (45a)
and
\[ \Delta \alpha^2 = \frac{1}{2}. \] (45b)

The squeezing in this case is exclusively due to the parametric amplifier.

On the other hand, when the nonlinear crystal of the parametric amplifier is removed from the cavity, the quantum optical system under consideration reduces to a coherently driven three-level laser. Hence upon setting \( \varepsilon = \lambda \mu = 0 \) (with \( \mu \neq 0 \)) in Eqs. (42a) and (42b), the quadrature variance for this system becomes
\[ \Delta \alpha^2 = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(4 + \beta^2)}{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(2\beta - \beta^3)} \] (46a)
and
\[ \Delta \alpha^2 = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) + 3A\beta^2}{2\kappa(1 + \beta^2)(1 + \beta^2/4) + A(4\beta + \beta^3)}. \] (46b)

It is interesting to consider the case for which the coherent light driving the three-level laser is sufficiently strong. Thus we note that for \( \beta = \Omega/\gamma \gg 1 \), Eq. (46a) reduces to
\[ \Delta \alpha^2 = \frac{5 + A\beta^2}{5 - A\beta^2}. \] (47)

Since \( \beta^2 \) is very large, we can drop the term \( A/\beta^2 \) in Eq. (47), so that
\[ \Delta \alpha^2 = \frac{\kappa}{\kappa - 2A/\beta}. \] (48)

Following the same procedure, we also get
\[ \Delta \alpha^2 = \frac{\kappa}{\kappa + 2A/\beta}. \] (49)

Now comparison of Eqs. (44a) and (48) as well as Eqs. (44b) and (49) shows that a degenerate three-level laser driven by a strong coherent light behaves in exactly the same manner as a degenerate parametric oscillator, which is in agreement with the assertion made in Ref. [10]. However, as can be seen clearly from Fig. 2, a relatively high degree of squeezing occurs for small values of \( \beta \).

Inspection of Eq. (40) shows that \( \lambda_+ \) is nonnegative while \( \lambda_- \) can be positive, negative, or zero. We thus note that Eq. (29) will not have a well-behaved solution if \( \lambda_- < 0 \). Now setting \( \lambda_- = 0 \) and taking into account (30), we get
\[ \varepsilon = \frac{\kappa}{2} + \frac{A(2\beta - \beta^3)}{4(1 + \beta^2)(1 + \beta^2/4)}. \] (50)

We can then interpret this as the threshold condition for the system under consideration. Therefore expressions (42a) and (42b) take at threshold the form
\[ \Delta \alpha^2 \rightarrow \infty \] (51a)
and
\[ \Delta \alpha^2 = \frac{2\kappa(1 + \beta^2)(1 + \beta^2/4) + 3A\beta^2}{4\kappa(1 + \beta^2)(1 + \beta^2/4) + 6A\beta^2}. \] (51b)

The dotted curve in Fig. 2 represents the quadrature variance for the coherently driven three-level laser and the solid curve in the same figure represents the quadrature variance for the degenerate three-level laser with the parametric amplifier at threshold. It is easy to see from Fig. 2 that the presence of the parametric amplifier leads to better squeezing. Moreover, the minimum value of the quadrature variance, described by Eq. (51b), for \( A=100 \) and \( \kappa = 0.8 \) is found to be \( \Delta \alpha^2 = 0.068 \) and occurs at \( \beta = 0.067 \). This result implies that the maximum intracavity squeezing for the above values of the linear gain coefficient and cavity damping constant is 93% below the vacuum level. In addition Fig. 3 shows that the degree of squeezing increases with the linear gain coefficient.

**B. Squeezing spectrum**

The squeezing spectrum of a single-mode light is expressible in terms of c-number variables associated with the normal ordering as
\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm 2Re \int_0^\infty \langle \alpha_{\pm}^{\text{out}}(t), \alpha_{\pm}^{\text{out}}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau, \] (52a)
where
\[ \alpha_{\pm}^{\text{out}}(t) = \alpha_{\pm}^{\text{in}}(t) \pm \alpha_{\pm}^{\text{out}}(t). \] (52b)

For a cavity mode coupled to a vacuum reservoir, the output and intracavity variables are related by
\[ \alpha_{\pm}^{\text{out}}(t) = \sqrt{\lambda_{\pm}}(t). \] (53)

Now taking in to account Eqs. (37) and (53), the squeezing spectrum can be put in the form
\[ S_{\pm}^{\text{out}}(\omega) = 1 \pm 2Re \int_0^\infty \langle \alpha_{\pm}(t), \alpha_{\pm}(t + \tau) \rangle_{ss} e^{i\omega \tau} d\tau. \] (54)

On the other hand, the solution of Eq. (29) can also be written as
\[ \alpha_{\pm}(t + \tau) = \alpha_{\pm}(t) e^{-\lambda_{\pm}\tau} + \int_0^\tau e^{-\lambda_{\pm}(\tau - \tau')} (f^*(t + \tau') + f(t + \tau')) d\tau'. \] (55)

Upon multiplying (55) by \( \alpha_{\pm}(t) \) and taking the expectation value of the resulting equation, one readily obtains at steady state
\[ \langle \alpha_{\pm}(t)\alpha_{\pm}(t + \tau) \rangle_{ss} = \langle \alpha_{\pm}^2(t) \rangle_{ss} e^{-\lambda_{\pm}\tau}. \] (56)
FIG. 4: Plots of the squeezing spectrum [Eq. (57b)] versus $\beta$ for $\kappa = 0.8$ and $A = 25$ in the presence of the pump mode and in the absence of the nonlinear crystal (dotted curve) and [Eq. (58a)] for $\kappa = 0.8$ and $A = 25$ in the presence of the parametric amplifier (solid curve).

Now using Eqs. (41) and (56), the squeezing spectrum is found to be

$$S^\text{cont}_+(\omega) = 1 + \frac{2\kappa(\varepsilon - 2V + 2R)}{\lambda_+^2 + \omega^2}$$  \hspace{1cm} (57a)

and

$$S^\text{cont}_-(\omega) = 1 - \frac{2\kappa(\varepsilon - 2V - 2R)}{\lambda_+^2 + \omega^2}. \hspace{1cm} (57b)$$

Applying Eqs. (50), (7a)-(7e), and (30), the squeezing spectrum (57) can be put at threshold in the form

$$S^\text{cont}_+(\omega) = 1 + \frac{\kappa(\kappa + \frac{A(4 + \beta^2)}{(1 + \beta^2)(2 + \beta^2)/2})}{(\kappa + \frac{3A}{(1 + \beta^2)(2 + \beta^2)/2})^2 + \omega^2}, \hspace{1cm} (58a)$$

and

$$S^\text{cont}_-(\omega) = 1 - \frac{\kappa(\kappa + \frac{A(3 - \beta^2/2)}{(1 + \beta^2)(1 + \beta^2/2)})}{(\kappa + \frac{3A}{(1 + \beta^2)(2 + \beta^2/2)})^2 + \omega^2}. \hspace{1cm} (58b)$$

Inspection of this equation shows that there is perfect squeezing for $\beta = \omega = 0$ and for any values of $A$ and $\kappa$. The dotted curve in Fig. 4 shows that there is almost perfect squeezing at $\beta = 0.016$ and the solid curve indicates that the presence of the parametric amplifier leads to perfect squeezing at $\beta = 0$.

IV. PHOTON STATISTICS

We now proceed to calculate the mean photon number and the photon number distribution for the cavity under consideration.

![Graph](image-url)

FIG. 5: Plots of the mean photon number [Eq. (61)] at steady state versus $\beta$ for $\kappa = 0.8$, and $A = 25$ in the presence of the pump mode and in the absence of the nonlinear crystal (dotted curve) and for $\kappa = 0.8$, and $A = 25$ in the presence of the parametric amplifier with $\varepsilon = 0.3$ (solid curve).

A. The mean photon number

Applying Eq. (32a) and its complex conjugate, the mean photon number of the cavity mode, assumed to be initially in a vacuum state, can be written as

$$\langle \alpha^* \alpha \rangle = \langle F^*(t)F(t) \rangle. \hspace{1cm} (59)$$

On account of Eqs. (33a) and (33b) together with the correlation properties of the noise force $f(t)$, we readily obtain

$$\langle F^*(t)F(t) \rangle = \frac{2R - 2V + \varepsilon}{4\lambda_-} (1 - e^{-2\lambda_-t}) + \frac{2R + 2V - \varepsilon}{4\lambda_+} (1 - e^{-2\lambda_+t}). \hspace{1cm} (60)$$

In view of this result, the mean photon number takes the form

$$\langle \alpha^* \alpha \rangle = \frac{2\varepsilon(1 + \beta^2)(1 + \beta^2/4) + A(2 - \beta + \beta^2/2 + \beta^3/2)}{4(1 + \beta^2)(1 + \beta^2/4)(\kappa + 2\varepsilon) + 2A(2\beta - \beta^3)} \times (1 - e^{-2\lambda_-t})$$

$$- \frac{2\varepsilon(1 + \beta^2)(1 + \beta^2/4) + A(2\beta - 3\beta^2/2 + \beta^3/2)}{4(1 + \beta^2)(1 + \beta^2/4)(\kappa + 2\varepsilon) + 2A(4\beta + \beta^3)} \times (1 - e^{-2\lambda_+t}). \hspace{1cm} (61)$$

Fig. 5 clearly indicates that the parametric amplifier contributes significantly to the mean photon number for relatively small values of $\beta$. 
B. The Photon number distribution

We finally seek to calculate, employing the $Q$ function, the photon number distribution for the cavity mode. The photon number distribution for a single-mode light is expressible in terms of the $Q$ function as

$$P(n, t) = \frac{\pi}{n!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} [Q(\alpha^*, \alpha, t) e^{\alpha^* \alpha}]_{\alpha^* = \alpha = 0}. \quad (62)$$

Now using (62) and (A11), the photon number distribution for the cavity mode can be written in the form

$$P(n, t) = \frac{(c^2 - d^2)^{1/2}}{n!} \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} \times \exp[(1 - c)\alpha^* \alpha - d(\alpha^2 + \alpha^2)/2]_{\alpha^* = \alpha = 0}. \quad (63)$$

Upon expanding the exponential functions in power series, we have

$$P(n, t) = \frac{(c^2 - d^2)^{1/2}}{n!} \sum_{klm} \frac{(-1)^{l+m}(1 - c)^k d^{l+m}}{2^l m! n!} \times \frac{\partial^{2n}}{\partial \alpha^{*n} \partial \alpha^n} [\alpha^{k+2l} \alpha^{k+2l}]_{\alpha^* = \alpha = 0}, \quad (64)$$

so that on carrying out the differentiations and applying the condition $\alpha^* = \alpha = 0$, there follows

$$P(n, t) = \frac{(c^2 - d^2)^{1/2}}{n!} \sum_{klm} \frac{(-1)^{l+m}(1 - c)^k d^{l+m}(k + 2l)!}{2^l m! n! (k + 2l - n)!} \times \frac{(k + 2m)!}{(k + 2m - n)!} \delta_{k+2l,n} \delta_{k+2m,n}. \quad (65)$$

Applying the properties of the Kronecker delta symbol and the fact that a factorial is defined for nonnegative integers, we obtain

$$P(n, t) = (c^2 - d^2)^{1/2} \sum_{l=0}^{n} n! \frac{(1 - c)^{n-2l} d^{2l}}{2^{2l} l!^2 (n - 2l)!} \quad (66)$$

where $[n] = n/2$ for even $n$ and $[n] = (n - 1)/2$ for odd $n$. As can be seen from Fig. 6, the steady-state photon number distribution decreases with the photon number. Moreover, the probability of finding even number of photons is in general greater than the probability of finding odd number of photons. Although the photons are generated in pairs in this quantum optical system, there is a finite probability to find odd number of photons inside the cavity. This is because some photons leave the cavity through the port mirror. It also appears that the presence of the parametric amplifier increases the probability of finding even number of photons and decreases the probability of finding odd number of photons.

V. CONCLUSION

In this paper we have considered a degenerate three-level laser whose cavity contains a degenerate parametric amplifier, with the top and bottom levels of the three-level atoms coupled by the pump mode emerging from the parametric amplifier. We have obtained using the master equation stochastic differential equations. Applying the solutions of the resulting equations, we have calculated the quadrature variance and squeezing spectrum. Moreover, using the same solutions, we have determined the mean photon number and the photon number distribution. We have found that the light generated by the quantum optical system is in a squeezed state, with the maximum intracavity squeezing attainable being 93%. The parametric amplifier increases the squeezing significantly over and above the squeezing achievable due to the coupling of the top and bottom levels by the pump mode. In addition, we have seen that there is perfect squeezing of the output light for $\beta = \omega = 0$ and for any values of $A$ and $\kappa$. We have also found that the presence of the parametric amplifier leads to a significant increase in the mean photon number for small values of $\beta$. The plots of the photon number distribution show that the probability of finding odd number of photons is in general less than the probability of finding even number of photons. Moreover, the same plots indicate that the probability of finding even number of photons is greater in the presence of the parametric amplifier than in the absence of the nonlinear crystal. However, the opposite of this assertion holds for the probability of finding odd number of photons.
APPENDIX A: THE Q FUNCTION

We evaluate the $Q$ function applying the relation

$$Q(\alpha^*, \alpha, t) = \frac{1}{\pi^2} \int dz^2 \phi(z^*, z, t) \exp(\alpha^* - z^* \alpha), \quad (A1)$$

where

$$\phi(z^*, z, t) = \text{Tr}(\hat{\rho} e^{-z^* \hat{a}(t)} e^{z \hat{a}^\dagger(t)}) \quad (A2)$$

is the antinormally ordered characteristic function defined in the Hiesenberg picture. Using the identity

$$e^{\hat{A}\hat{B}} = e^{\hat{B}} e^{\hat{A}[\hat{A}, \hat{B}]}, \quad (A3)$$

denotes associated with the normal ordering as this function can be written in terms of c-number variables associated with the normal ordering as

$$\phi(z^*, z, t) = e^{-z^* z} \exp\left(\frac{1}{2}(z^* \alpha^* + \alpha^* z^* - 2z^* z \alpha^*)\right). \quad (A4)$$

Since $e^{\hat{A}\hat{B}}$ is a linear differential equation, we see that $\alpha(t)$ is a Gaussian variable $[15]$. In addition, on account of Eqs. (32a), (33a), and (33b), we find mean value of $\alpha(t)$ to be zero. Hence $\alpha(t)$ is a Gaussian variable with vanishing mean. One can then express $\langle \alpha^2 \rangle$ in the form

$$\phi(z^*, z, t) = e^{-z^* z} \exp\left(\frac{1}{2}(z^* \alpha^* + \alpha^* z^* - 2z^* z \alpha^*)\right). \quad (A5)$$

Now employing Eqs. (32a), (33a), and (33b) along with the correlation properties of the noise force $f(t)$, one can establish that

$$\langle \alpha^2 \rangle = \langle \alpha^* \rangle = \frac{2R - 2V + \varepsilon}{4\lambda_-} (1 - e^{-2\lambda_+ t})$$

$$- \frac{2R + 2V - \varepsilon}{4\lambda_+} (1 - e^{-2\lambda_- t}), \quad (A6)$$

$$\langle \alpha^* \alpha \rangle = \frac{2R - 2V + \varepsilon}{4\lambda_-} (1 - e^{-2\lambda_- t})$$

$$+ \frac{2R + 2V - \varepsilon}{4\lambda_+} (1 - e^{-2\lambda_+ t}). \quad (A7)$$

Therefore, with the aid of Eqs. (A6) and (A7), Eq. (A5) can be put in the form

$$\phi(z^*, z, t) = \exp(-az^* z - b(z^2 + z^* z)/2), \quad (A8)$$

in which

$$a = 1 + \frac{2R - 2V + \varepsilon}{4\lambda_-} (1 - e^{-2\lambda_- t})$$

$$b = \frac{2R - 2V + \varepsilon}{4\lambda_-} (1 - e^{-2\lambda_- t})$$

$$- \frac{2R + 2V - \varepsilon}{4\lambda_+} (1 - e^{-2\lambda_+ t}). \quad (A9)$$

Upon substituting (A8) into (A1) and performing the integration, the $Q$ function is found to be

$$Q(\alpha^*, \alpha, t) = \frac{(e^2 - d^2)^{1/2}}{\pi} \exp\left[-c \alpha^* \alpha - \frac{d}{2} (\alpha^* + \alpha)^2\right], \quad (A11)$$

where

$$c = \frac{a}{a^2 - b^2}, \quad (A12)$$

$$d = \frac{b}{a^2 - b^2}. \quad (A13)$$

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