TRANSMUTATION OPERATORS: CONSTRUCTION AND APPLICATIONS

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Abstract. Recent results on the construction and applications of the transmutation (transformation) operators are discussed. Three new representations for solutions of the one-dimensional Schrödinger equation are considered. Due to the fact that they are obtained with the aid of the transmutation operator all the representations possess an important for practice feature. The accuracy of the approximate solution is independent of the real part of the spectral parameter. This makes the representations especially useful in problems requiring computation of large sets of eigendata with a nondeteriorating accuracy.

Applications of the exact representations for the transmutation operators to partial differential equations are discussed as well. In particular, it is shown how the methods based on complete families of solutions can be extended onto equations with variable coefficients.

1. TRANSMUTATION OPERATORS

Transmutation operators also called transformation operators are a widely used tool in the theory of linear differential equations (see, e.g., [2], [4], [5], [18], [19], [20] and many other publications). In particular, let \( q \in C[-b,b] \) be a complex valued function. Consider the Sturm-Liouville equation

\[
Ay := y'' - q(x)y = -\omega^2 y.
\]

It is well known (see, e.g., [19]) that there exists a Volterra integral operator \( T \) called the transmutation (or transformation) operator defined on \( C[-b,b] \) by the formula

\[
T u(x) = u(x) + \int_{-x}^{x} K(x,t) u(t) dt
\]

such that for any \( u \in C^2[-b,b] \) the following equality is valid

\[
ATu = Tu''
\]

and hence any solution of (1) can be written as \( y = T[u] \) where \( u(x) = c_1 \cos \omega x + c_2 \sin \omega x \) with \( c_1 \) and \( c_2 \) being arbitrary constants.

The transmutation kernel \( K \) is a solution of a certain Goursat problem for the hyperbolic equation

\[
\left( \frac{\partial^2}{\partial x^2} - q(x) \right) K(x,t) = \frac{\partial^2}{\partial t^2} K(x,t).
\]

2. CONSTRUCTION OF THE TRANSMUTATION KERNEL AND NEW REPRESENTATIONS FOR SOLUTIONS OF THE STURM-LIOUVILLE EQUATION

In spite of fundamental importance of the transmutation kernel \( K \) in the theory of linear differential equations, besides the method of successive approximations derived directly from the Goursat problem (see, e.g., [5]) very few attempts of its practical construction have been reported. In this relation we mention the paper [3] where analytic approximation formulas for the integral kernel were obtained and the recent publications [14], [15] where another procedure of analytical approximation was proposed.

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To the difference of those previous results, in the recent paper [13] an exact representation for \( K \) in the form of a Fourier-Legendre series with explicit formulas for the coefficients was obtained. Suppose that \( q \in W^{-1}_2[-b, b] \) (that is, \( q \) can be a piecewise continuous function, may have a singularity, e.g., \( q(x) \sim c/x \), etc.). In this case \( K(x, t) \) is an \( L^2 \)-function. Under these conditions the following theorem was proved in [13].

**Theorem 1.** The kernel \( K \) has the form

\[
K(x, t) = \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x} P_n \left( \frac{t}{x} \right)
\]

where for every \( x \in [-b, b] \) the series converges with respect to \( t \) in the \( L^2 \)-norm (if \( q \in C[-b, b] \) the series converges uniformly),

\[
\beta_n(x) = \frac{2n + 1}{2} \left( \sum_{k=0}^{n} \frac{l_{k,n}}{x^k} \varphi_k(x) - 1 \right),
\]

with \( l_{k,n} \) being the coefficient at \( x^k \) of the Legendre polynomial \( P_n \), and \( \varphi_k \) being the so-called formal powers constructed as follows (see [8], [12]).

**Definition 2** (Formal powers \( \varphi_k \)). Let \( f \) be a solution of

\[
f'' - q(x)f = 0, \quad x \in [-b, b],
\]

\[
f(0) = 1, \quad f'(0) = 0.
\]

Then \( \{\varphi_k\}_{k=0}^{\infty} \) are defined by the equalities

\[
\varphi_k = \begin{cases} fX^{(k)}, & k \text{ odd,} \\ f\tilde{X}^{(k)}, & k \text{ even,} \end{cases}
\]

where

\[
X^{(0)} \equiv 1, \quad X^{(n)}(x) = n \int_0^x X^{(n-1)}(s) \left( f^2(s) \right)^{(1-n)} ds,
\]

and

\[
\tilde{X}^{(0)} \equiv 1, \quad \tilde{X}^{(n)}(x) = n \int_0^x \tilde{X}^{(n-1)}(s) \left( f^2(s) \right)^{(1-n-1)} ds.
\]

It is worth mentioning that \( \varphi_k \) are easily computable (at least numerically) in practice (see, e.g., [13] for additional details).

A representation for the kernel \( K \) leads to a representation for the solution of (1). Let \( u(\omega, x) \) denote the solution of (1) satisfying the initial conditions

\[
u(\omega, 0) = 1, \quad u'(\omega, 0) = i\omega.
\]

Then we have

\[
u(\omega, x) = e^{i\omega x} + \int_{-x}^{x} K(x, t)e^{i\omega t} dt.
\]

Substitution of (2) into the last integral gives us the equality [13]

\[
u(\omega, x) = e^{i\omega x} + \sum_{n=0}^{\infty} \beta_n(x) \int_{-1}^{1} P_n(y) e^{i\omega xy} dy = e^{i\omega x} + \sum_{n=0}^{\infty} i^n \beta_n(x) j_n(\omega x)
\]

where \( j_n(z) = \sqrt{\frac{2}{\pi z}} J_{n+1/2}(z) \) are spherical Bessel functions. The series converges uniformly with respect to \( x \).
Moreover, take $\omega \in \mathbb{R}$. Consider
\[
K_N(x, t) = \sum_{n=0}^{N} \beta_n(x) P_n \left( \frac{t}{x} \right)
\]
and
\[
u_N(\omega, x) = e^{i\omega x} + \sum_{n=0}^{N} \beta_n(x) j_n(\omega x), \quad x > 0.
\]
We have \[13\]
\[
|u(\omega, x) - u_N(\omega, x)| = \left| \int_{-\infty}^{\infty} (K(x, t) - K_N(x, t)) e^{i\omega t} dt \right|
\leq \|K(x, \cdot) - K_N(x, \cdot)\|_{L^2(-\infty, x)} \|e^{i\omega t}\|_{L^2(-\infty, x)}
\leq \varepsilon_N(x) \sqrt{2x}
\]
—indeedepend of $\omega$. More generally, for any $\omega \in \mathbb{C}$, $\omega \neq 0$ belonging to the strip $|\text{Im} \omega| \leq C$, $C \geq 0$,
\[
|u(\omega, x) - u_N(\omega, x)| \leq \varepsilon_N(x) \frac{\sinh(Cx)}{C}.
\]
This $\omega$-independence of the approximation accuracy was shown in \[13\] to give a very fast and efficient method for computing large sets of eigendata with a non-deteriorating accuracy. In \[17\] it was generalized onto perturbed Bessel equations, and in \[16\] onto Sturm-Liouville equations.

Another representation for the kernel $K$ and as a corollary for the solutions of \(1\) was obtained in \[9\]. Consider the following extension of the transmutation kernel $\tilde{K}$,
\[
\tilde{K}(x, t) = \begin{cases} K(x, t), & -x \leq t \leq x, \\ 0, & -\infty < t < -x. \end{cases}
\]
Then
\[
u(x, x) = e^{i\omega x} + \int_{-\infty}^{\infty} \tilde{K}(x, y) e^{i\omega y} dy = e^{i\omega x} \left( 1 + \int_{0}^{\infty} \tilde{K}(x, x-t) e^{-i\omega t} dt \right).
\]
Consider
\[
\tilde{K}(x, x-t) = k(x, t)e^{-t}.
\]
The function $k(x, \cdot)$ then belongs to the space $L^2(0, \infty; e^{-t})$ equipped with the scalar product $\langle u, v \rangle := \int_{0}^{\infty} u(t) \overline{v}(t) e^{-t} dt$. Thus, $k(x, \cdot)$ admits a Fourier-Laguerre expansion convergent in the corresponding norm,
\[
k(x, t) = \sum_{n=0}^{\infty} a_n(x) L_n(t).
\]
The kernel has the form \[9\]
\[
\tilde{K}(x, y) = \sum_{n=0}^{\infty} a_n(x) L_n(x-y)e^{-x-y},
\]
with the coefficients $a_n$ defined by
\[
a_n(x) = \sum_{j=0}^{n} (-1)^j \left( \varphi_j(x) - x^j \right) \sum_{k=j}^{n} (-1)^k \frac{n!}{(n-k)!k!(k-j)!j!} x^{k-j}.
\]
The solution $u(\omega, x)$ has the form \[9\]
\[
u(\omega, x) = e^{i\omega x} \left( 1 + \sum_{n=0}^{\infty} a_n(x) \frac{(i\omega)^n}{(1+i\omega)^{n+1}} \right).
\]
The following estimate is valid for any $\omega \in \mathbb{R}$,
\begin{equation}
|u(\omega, x) - u_N(\omega, x)| \leq \varepsilon_N(x),
\end{equation}
where
\begin{equation*}
u_N(\omega, x) := e^{i\omega x} \left(1 + \sum_{n=0}^{N} a_n(x) \frac{(i\omega)^n}{(1+i\omega)^{n+1}}\right),
\end{equation*}
and $\varepsilon_N(x)$ is a nonnegative function independent of $\omega$ and such that $\varepsilon_N(x) \to 0$ for all $x \in [-b, b]$ when $N \to \infty$. More generally,
\begin{equation*}
|u(\omega, x) - u_N(\omega, x)| \leq \varepsilon_N(x) e^{-\frac{\text{Im}\omega x}{\sqrt{1-2\text{Im}\omega}}}, \quad \text{when } \text{Im}\omega < 1/2.
\end{equation*}

Consideration of another extension of the transmutation kernel defined by
\begin{equation*}
\tilde{K}(x, y) := \begin{cases} 
K(x, y) & \text{when } x \in [-b, b] \text{ and } y \in [-x, x] \\
0 & \text{otherwise},
\end{cases}
\end{equation*}
leads to the following series expansion
\begin{equation*}
\tilde{K}(x, y) = \sum_{n=0}^{\infty} c_n(x) H_n(y) e^{-y^2}
\end{equation*}
where $H_n$ stands for an Hermite polynomial of order $n$ and the coefficients $c_n$ are to be found. Note that
\begin{equation*}
\int_{-\infty}^{\infty} \tilde{K}(x, y) H_n(y) dy = \sqrt{\pi n!} 2^n c_n(x).
\end{equation*}
Hence
\begin{equation}
\begin{split}
c_n(x) = & \frac{1}{\sqrt{\pi n!} 2^n} \int_{-x}^{x} K(x, y) H_n(y) dy = \frac{1}{\sqrt{\pi n!} 2^n} \sum_{k=0}^{n} h_{k,n} \left(\varphi_k(x) - x^k\right)
\end{split}
\end{equation}
where $h_{k,n}$ denotes the coefficient of $x^k$ from the Hermite polynomial $H_n(x)$.

This leads to another representation for the solution of (1),
\begin{equation*}
\begin{split}
u(\omega, x) = & e^{i\omega x} + \int_{-\infty}^{\infty} \tilde{K}(x, y) e^{i\omega y} dy \\
= & e^{i\omega x} + \sum_{n=0}^{\infty} c_n(x) \int_{-\infty}^{\infty} H_n(y) e^{i\omega y} e^{-y^2} dy \\
= & e^{i\omega x} + \sqrt{\pi} e^{-\frac{\omega^2}{4}} \sum_{n=0}^{\infty} c_n(x) (i\omega)^n.
\end{split}
\end{equation*}
Consider the partial sum
\begin{equation*}
u_N(\omega, x) = e^{i\omega x} + \sqrt{\pi} e^{-\frac{\omega^2}{4}} \sum_{n=0}^{N} c_n(x) (i\omega)^n.
\end{equation*}
Then it is easy to see that
\begin{equation*}
|u(\omega, x) - u_N(\omega, x)| \leq \pi \frac{1}{4} e \frac{(\text{Im}\omega)^2}{2} \varepsilon_N(x)
\end{equation*}
which means that the truncation error is uniformly bounded in any strip $|\text{Im}\omega| \leq C$.\vspace{1cm}
3. Applications to PDEs

Exact representations of the transmutation kernel lead to numerous applications for partial differential equations admitting certain symmetry. In particular, let us consider the possibility to obtain complete systems of solutions. For example, application of the transmutation operator $T$ to a complete system of harmonic functions leads to a complete system of solutions of the equation

$$(\Delta - q(x)) u(x, y) = 0.$$  

Indeed,

$$(\Delta - q(x)) T = T \Delta$$

whenever the domain of interest is such that the integration in $T$ is well defined.

**Example** Harmonic polynomials (Re $z^n$ and Im $z^n$) can be written in the form

$$p_0(x, y) = 1,$$

$$p_{2m+1}(x, y) = \text{Re } z^{m+1} = \sum_{\text{even } k=0}^{m} (-1)^{\frac{k}{2}} \binom{m+1}{k} x^{m+1-k} y^k, \quad m \geq 0,$$

$$p_{2m}(x, y) = \text{Re } (iz^m) = \sum_{\text{odd } k=1}^{m} (-1)^{\frac{k+1}{2}} \binom{m}{k} x^{m-k} y^k, \quad m \geq 1.$$

Since

$$T : x^k \mapsto \varphi_k(x),$$

the following functions are the images of $p_m$ under the action of $T$ and represent a complete system of solutions of (8)

$$u_0(x, y) = f(x),$$

$$u_{2m+1}(x, y) = \sum_{\text{even } k=0}^{m+1} (-1)^{\frac{k}{2}} \binom{m+1}{k} \varphi_{m+1-k}(x) y^k, \quad m \geq 0,$$

$$u_{2m}(x, y) = \sum_{\text{odd } k=1}^{m} (-1)^{\frac{k+1}{2}} \binom{m}{k} \varphi_{m-k}(x) y^k, \quad m \geq 1.$$

Other complete systems of solutions can be obtained.

**Example** The method of fundamental solutions (discrete sources) (see, e.g., [1], [6]) can be extended onto equations with variable coefficients. Consider the fundamental solution for the Laplace operator on the plane

$$\log |x + i y - (\eta + i \xi)| = \log |x - \tilde{Z}|.$$  

Application of $T$ leads to the following integrals

$$T [\log |x - \tilde{Z}|] = \log |x - \tilde{Z}| + \sum_{n=0}^{\infty} \frac{\beta_n(x)}{x} \int_{-x}^{x} P_n \left( \frac{t}{x} \right) \log |t - \tilde{Z}| \, dt.$$  

Their calculation gives us the image of the fundamental solution in the form

$$T [\log |x - \tilde{Z}|] = \log |x - \tilde{Z}| + \beta_0(x) \text{Re} \left( \log ((Z + x) (Z - x)) + 2Q_1 \left( \frac{Z}{x} \right) \right)$$

$$+ 2 \sum_{n=1}^{\infty} \frac{\beta_n(x)}{2n+1} \text{Re} \left( Q_{n+1} \left( \frac{Z}{x} \right) - Q_{n-1} \left( \frac{Z}{x} \right) \right)$$

where $Q_n$ are Legendre functions of the second kind.

Similar considerations can be applied to systems arising in hypercomplex analysis (see, e.g., [7]) and may lead to extensions of well known methods based on monogenic polynomials or other
complete systems of solutions onto systems with variable coefficients [10], [11] important, e.g., in electromagnetic theory and quantum physics.

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