WALDSPURGER FORMULAS IN HIGHER COHOMOLOGY

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Abstract. The classical Waldspurger formula, which computes periods of quaternionic automorphic forms in maximal torus, has been used in a wide variety of arithmetic applications, such as the Birch and Swinnerton-Dyer conjecture in rank 0 situations. This is why this formula is considered the rank 0 analogue of the celebrated Gross-Zagier formula.

On the other hand, Eichler-Shimura correspondence allows us to interpret this quaternionic automorphic form as a cocycle in higher cohomology spaces of certain arithmetic groups. In this way we can realize the corresponding automorphic representation in the etale cohomology of certain Shimura varieties. In this work we find a formula, analogous to that of Waldspurger, which relates cup-products of this cocycle and fundamental classes associated with maximal torus with special values of Rankin-Selberg L-functions.

1. Introduction

Let $F$ be any number field, and let $E/F$ be any quadratic extension of discriminant $D$. Let $B$ be a quaternion algebra over $F$ admitting an embedding $E \hookrightarrow B$. Let $\pi$ be an irreducible automorphic representation of $B^\times$ and let $\Pi$ be its Jacquet-Langlands lift to $GL_2/F$. The classical Waldspurger formula computes the period integral of an automorphic form $f \in \pi$ twisted by a Hecke character $\chi$ of $E^\times$ in terms of a critical value of the Rankin-Selberg $L$-function $L(s, \Pi, \chi)$ associated with $\Pi$. More concretely, if $d^\times t = \prod_v d^\times t_v$ is the usual Tamagawa measure of $\mathbb{A}_E/E^\times \mathbb{A}_F^\times$ and we write

$$\ell(f, \chi) = \int_{\mathbb{A}_E/E^\times \mathbb{A}_F^\times} \chi(t)f(t)d^\times t,$$

the formula reads in one of the best known forms:

$$\frac{\ell(f_1, \chi) \cdot \ell(f_2, \chi^{-1})}{(f_1, f_2)} = \frac{\Lambda_\ell(2) \cdot \Lambda(1/2, \Pi, \chi)}{2\Lambda(1, \Pi, \text{ad})} \prod_v \beta_v(f_1, f_2),$$

where $\Lambda$ stands for complete global $L$-functions, that is, those as products of local $L$-functions over all places, $\Lambda_\ell(s) = \zeta_\ell(s) \prod_v \zeta_\ell_v(s)$ is the complete $L$-series for the trivial character, $\eta_\ell$ is the quadratic character associated with $E/F$, $\beta_v(f_1, f_2, \chi_v) := \frac{L(1, \eta_\ell \chi_v) \cdot L(1/2, \Pi_v, \chi_v)}{\zeta_\ell(2) \cdot L(1/2, \Pi_v, \chi_v^2)} \int_{T(F_v)} \chi_v(t_v) \langle \pi_v(t_v) f_1, f_2 \rangle_v d^\times t_v,$

and $\langle \cdot, \cdot \rangle_v = \Pi_v \langle \cdot, \cdot \rangle_v$ is the Petersson inner product for the usual Tamagawa measure of $G(F) \backslash G(A_F)$

$$\langle f_1, f_2 \rangle = \int_{G(F) \backslash G(A_F)} f_1(g) f_2(g) d^\times g.$$

The reader will find in §3.2 more details on these classical formulas.

Waldspurger formula has many very important arithmetic applications. When $F = \mathbb{Q}$ is totally real, $E/\mathbb{Q}$ is imaginary and $B$ is definite, one can relate the corresponding period with the reduction of certain Heegner points. This has been used to relate certain Euler systems with critical values of the classical Birch and Swinnerton-Dyer conjecture.

Let $\Sigma_B$ be the set of split archimedean places of $B$, namely, the set of archimedean completions $F_v \subseteq \mathbb{C}$ where $B \otimes_F F_v = M_2(F_v)$, and write $r = \# \Sigma_B$. The \textit{(lower) Harder-Eichler-Shimura isomorphism} realizes the automorphic representation $\pi$ in the $r$-cohomology space of certain arithmetic groups attached to our quaternion algebra $B$. We proceed to give a more precise description of such cohomology spaces: We will assume in this note that $\pi$ has trivial central character and let $G$ be the algebraic group associated to $B^\times/F^\times$. Hence $\pi$ can be seen as a representation of $G(A_F)$, the adelic points of $G$. For any $k = 2 = (k_3 - 2) \in (2\mathbb{N})[E, \mathbb{Q}]$, let us consider the $\mathbb{C}$-vector spaces

$$V(k - 2) := \bigotimes_{\delta \in \mathbb{C}} \text{Sym}^{k_3 - 2}(\mathbb{C}^2).$$

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Then \( V(k - 2) \) is endowed with a natural action of \( G(F) \). For any open compact subgroup \( U \subset G(\mathbb{A}_F^\infty) \) of the finite adelic points of \( G \), we consider the group cohomology spaces

\[
H^r(G(F)_+, \mathcal{A}^\omega(V(k - 2))^U) = \bigoplus_{g \in \Gamma_g} H^r(\Gamma_g, V(k - 2)),
\]

where \( \Gamma_g = G(F)_+ \cap gUg^{-1} \) and \( G(F)_+ \) is the subgroup of elements with positive norm at all real places. The (lower) Harder-Eichler-Shimura isomorphism realizes the automorphic representation \( \pi |_{G(\mathbb{A}_F^\infty)} \) of weight \( k \) in the spaces

\[
H^r(G(F)_+, \mathcal{A}^\omega(V(k - 2))^U) = \bigcup_{U \subset G(\mathbb{A}_F^\infty)} H^r(G(F)_+, \mathcal{A}^\omega(V(k - 2))^U),
\]

where the superindex given by a fixed character \( \lambda : G(F)/G(F)_+ \to \pm 1 \) stands for the corresponding \( \lambda \)-isotypical component. In this paper (more precisely §3.1) we will recall the construction of an element

\[ \phi^\lambda \in H^r(G(F)_+, \mathcal{A}^\omega(V(k - 2))^\lambda) \]

generating \( \pi^\lambda = \pi |_{G(\mathbb{A}_F^\infty)} \).

For many arithmetic applications, it is more convenient to consider the above realization of \( \pi \) in the cohomology of the corresponding arithmetic groups, instead of thinking of it as a space of automorphic forms. Indeed, since the finite dimensional complex vector spaces \( V(k - 2) \) admit rational models, one can show that \( \pi |_{G(\mathbb{A}_F^\infty)} \) is in fact the extension of scalars of a representation defined over a number field, called the coefficient field of \( \pi \). Moreover, such rational representation can be realized in the rational analogues of the given cohomology spaces.

On the other hand, homology classes associated to quadratic extensions \( E/F \) can be defined using the group of relative units of \( E^\times \). Such fundamental classes lie in an \( r \)-th homology group \( (r = \#\Sigma_B) \) if we make the following hypothesis:

**Assumption 1.1.** The set \( \Sigma_B \) coincides with the set of archimedean places \( \sigma \) where \( E \) splits. We will write

\[
\Sigma_B = \Sigma_B^R \cup \Sigma_B^C; \quad \Sigma_B^R = \{ \sigma \in \Sigma_B; T(F_\sigma) = \mathbb{R}^\times \}, \quad \Sigma_B^C = \{ \sigma \in \Sigma_B; T(F_\sigma) = \mathbb{C}^\times \},
\]

and \( r_R = \#\Sigma_B^R, r_C = \#\Sigma_B^C \) with \( r = r_R + r_C \).

Throughout the paper the previous assumption will be fulfilled, hence we will be able to construct a fundamental class

\[ \eta \in H_r(T(F)_+, C^0(T(\mathbb{A}_F^\infty), \mathbb{Z})), \]

where \( T \) is the algebraic group associated with \( E^\times/F^\times \), \( C^0 \) denotes the subspace of locally constant functions, \( C^0 \) those with compact support, and \( T(F)_+ \) is again the subgroup of elements with positive norm at all real places.

As pointed out above, there are many arithmetic applications where both the cohomology class \( \phi \) and the fundamental class \( \eta \) play a very important role. For instance, in [5], [8], [6] and [10], classes \( \phi \) and \( \eta \) in certain precise situations are used to construct \( p \)-adic points in elliptic curves over \( F \). These points are the classical Darmon points in rank one situations, and the more recent plectic points in rank \( n \leq [F : \mathbb{Q}] \) situations. It is conjectured that they are defined over precise abelian extensions of \( E \). Much progress has been made recently towards the algebraicity of Darmon points over \( \mathbb{Q} \) in [2]. Moreover, their heights are conjecturally related to higher derivatives of the classical \( L \)-function \( L(s, \Pi, \chi) \) in the spirit of the Birch and Swinnerton-Dyer conjecture. Such conjectural formulas involving derivatives of \( L(s, \Pi, \chi) \) at \( s = 1/2 \) can be seen as generalizations of the Gross-Zagier formula for Heegner points. A different but related arithmetic application is the construction of \( p \)-adic \( L \)-functions. An anticyclotomic \( p \)-adic \( L \)-function is a \( p \)-adic avatar of the classical \( L \)-function \( L(s, \Pi, \chi) \). In [6], [10] and [11] anticyclotomic \( p \)-adic \( L \)-functions are constructed using \( \phi \) and \( \eta \). The relations between such \( p \)-adic \( L \)-functions and \( L(s, \Pi, \chi) \) are the so-called interpolation formulas. In fact, the main theorem of this note (theorem 1.2) is used in [10] and [11] to obtain precise interpolation formulas. There is a close relation between Darmon and plectic points and anticyclotomic \( p \)-adic \( L \)-functions. Indeed in [1], [6], [10] and [11] \( p \)-adic Gross-Zagier formulas are proved, relating higher derivatives of the corresponding anticyclotomic \( p \)-adic \( L \)-functions and logarithms of Darmon and plectic points.

Assume that there exists an embedding \( E \hookrightarrow \mathcal{B} \), that we will fix once for all. This provides an inclusion \( T \hookrightarrow G \) as algebraic groups. Moreover, we have a natural \( T(F) \)-equivariant pairing (see Remark 3.7 for further details)

\[
\varphi : \mathcal{A}^\omega(V(k - 2)) \times (C^0(T(\mathbb{A}_F^\infty), \mathbb{C}) \otimes V(k - 2)) \to C(T(\mathcal{A}_F), \mathbb{C}).
\]
The elements in $C^0(T(A_F), C) \otimes V(\mathbb{k} - 2)$ can be seen as a subspace of locally polynomial functions in $T(A_F)$. Since the Tamagawa measure on $T(A_F)$ provides a pairing between $C(T(A_F), C)$ and $C^0(T(A_F^n), \mathbb{Z})$, for any $T(F)$-invariant locally polynomial character $\chi \in H^0(T(F), C^0(T(A_F^n), \mathbb{C})) \otimes V(\mathbb{k} - 2)$, we can consider the cup product

$$\mathcal{P}(\phi^1, \chi) := \phi(\phi^1 \cup \chi) \cap \eta \in \mathbb{C}.$$ 

We can think $\mathcal{P}(\phi^1, \chi)$ as an analogue in higher cohomology of the period $\ell(f, \chi)$ above. In fact, if $\Sigma_F = \emptyset$ both concepts coincide. Thus, it is natural to ask if there is a formula analogous to that of (1.1) computing $\mathcal{P}(\phi^1, \chi)$, and this is precisely the main result of this paper. Furthermore, we will give a formula in an explicit form. Indeed, one of the main results of [11] where the local factors $\beta_c(f_{i,v}, f_{2,v})$ are computed at the cost of the forms $f_i$ belonging to a given one-dimensional space (see [4, Theorem 1.8]). In §3.2 we define a canonical subspace $V(\pi_\infty, \chi) \subset \pi_\infty$, where $U = \hat{O}_\Sigma$ and $O_N \subset B$ is certain admissible Eichler order of conductor $N = \text{cond}(\pi)$ such that the conductor $c_1$ of the order $O_N \cap E$ is closely related with the conductor $c$ of $\chi$ (see definition 3.5 for more details). The space $V(\pi_\infty, \chi)$ is one dimensional when the local root number $e(1/2, \pi_v, \nu(\chi)) = \eta_{\pi,v}(-1)e(B_{v})$, where $e(B_{v}) = 1$ if $B_{v}$ is a matrix algebra and $e(B_{v}) = -1$ otherwise. We will fix non-zero $\phi^1_{0,1} \in V(\pi_\infty, \chi)$ and $\phi^1_{0,2} \in V(\pi_\infty, \chi^{-1})$ in that case and our explicit Waldspurger formulas in higher cohomology will provide $\mathcal{P}(\phi^1, \chi)$ for forms $\phi^1 \in \pi_\infty$ that differ from $\phi^1_{0,2}$ at finitely many places $\Sigma$.

**Theorem 1.2.** Assume that $\chi : T(A_F)/T(F) \to \mathbb{C}^\times$ is a locally polynomial character of conductor $c$ such that

$$\chi |_{T(F_v)}(t) = \chi_0(t) \prod_{\mathfrak{p} \subset \mathcal{O}_F \subset \mathbb{C}} t^{m_{\mathfrak{p}}^c}, \quad m = (m_\mathfrak{p}) \in \mathbb{Z}^{[\mathbb{F} : \mathbb{Q}]} ,$$

for some locally constant character $\chi_0$ and some $m \in \mathbb{Z}^{[\mathbb{F} : \mathbb{Q}]}$. Given decomposable $\phi^1 = \bigotimes_{\mathfrak{p} \subset \mathcal{O}_F \subset \mathbb{C}} \phi^1_{\mathfrak{p}}$, $\phi^1 = \bigotimes_{\mathfrak{p} \subset \mathcal{O}_F \subset \mathbb{C}} \phi^1_{\mathfrak{p},v} \in \pi_\infty \subset H^1(G(F), \mathcal{A}^{\omega}(V(\mathbb{k} - 2)(\lambda)))$, we have $\mathcal{P}(\phi^1, \chi) = 0$ unless $\chi_0 = \chi$ and the root number $e(1/2, \pi_v, \nu(\chi)) = \chi_0 \eta_{\pi,v}(-1)e(B_{v})$, for all $v \nmid 1$. In that case, if $\phi^1_0$ differ from $\phi^1_{0,1}$ in a finite set of places $\Sigma$ then

$$\mathcal{P}(\phi^1_0, \chi) \cdot \mathcal{P}(\phi^1_0, \chi^{-1}) = 2^{\# S} L_{\Sigma}(1, \eta|_{\mathbb{R}}^2) \cdot L(1/2, \Pi, \chi) \cdot \langle \Phi_1, \Phi_2 \rangle \cdot \text{vol}(U_\mathbb{C}(N)) \prod_{\mathfrak{p} \subset \mathcal{O}_F} \beta_{\mathfrak{p}}(\phi^1_{\mathfrak{p},v} \phi^1_{\mathfrak{p},2,v})$$

where $S := \{ v | (N, D) ; v \parallel N \text{ then ord}_{v}(c(N)) \geq 0 \}$, $S_D := \{ v | (N, D) ; \text{ord}_{v}(c) < \text{ord}_{v}(N) \}$, $L^2$ is the L-function with the local factors at places in $S$ removed, and $L_{\Sigma}$ is the product of the local factors at $c_1$.

For this result will be found in §3.5 (theorem 3.10). In order to prove it, we will relate the value of $\mathcal{P}(\phi^1, \chi)$ to a period $\ell(f_{\phi^1}, \chi)$, for some automorphic form $f_{\phi^1} \in \pi$ associated with $\phi^1$ via the explicit description of the Harder-Eichler-Shimura morphism given in [13], and then we will apply the classical Waldspurger formula. I would like to highlight that, in order to obtain our result, one has to compute the local terms $\beta_{\mathfrak{p}}(f_{\phi^1_{\mathfrak{p},v}}, f_{\phi^1_{\mathfrak{p},2,v}})$ for all archimedean places $\mathfrak{p}$. To accomplish this, one needs the deep analysis of the archimedean representation $\pi_{\mathfrak{p}}$ performed in §4.

Finally, let me remark that our formula has been recently used in [14] to show rationality of Stark-Heegner cycles associated to Bianchi modular forms.}

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**1.1. Notation.** We write $\hat{\mathcal{O}} := \prod_{\mathfrak{p}} \mathcal{O}_\mathfrak{p}$ and $\hat{\mathcal{R}} := \mathbb{R} \otimes \hat{\mathcal{O}}$, for any ring $R$. Let $F$ be a number field with integer ring $\mathcal{O}_F$, and let $\mathcal{H}$ and $\mathcal{H}^\infty = \hat{\mathcal{O}} \otimes \mathbb{Q}$ be its ring of adeles and finite adeles, respectively. Write $\Sigma_F$ for the set of archimedean places of $F$. For any embedding $\sigma : F \to \mathbb{C}$ whose equivalence class is $\sigma \in \Sigma_F$, we will write $\sigma$. We define

$$\mathcal{Z}^\Sigma_F := \{ k = (k_{\sigma})_{\sigma \in \Sigma_F} ; k_{\sigma} = (k_{\sigma})_{\sigma \in \mathbb{Z}^{[\mathbb{F} : \mathbb{Q}]}}, \exists \mathbb{Z}^{[\mathbb{F} : \mathbb{Q}]}, \} \in \mathbb{Z}^{[\mathbb{F} : \mathbb{Q}]}.$$
Given \( k \in \mathbb{Z}^{2r} \) and \( x \in F_\infty \), we write
\[
\mu^k := \prod_{\sigma \in \Sigma_f} \prod_{\mathfrak{p} \mid \mathfrak{p}_\sigma} \vartheta(x)^{k_{\mathfrak{p}}}.
\]

Let \( B \) be a quaternion algebra over \( F \) with maximal order \( \mathcal{O}_B \). Let \( G \) be the algebraic group associated with the group of units of \( B \) modulo scalars, namely, for any \( \mathcal{O}_F \)-algebra \( R \)
\[
G(R) := (\mathcal{O}_B \otimes_{\mathcal{O}_F} R)^g / R^X.
\]
Write \( \Sigma_B \) for the set of split infinite places of \( B \).
Assume that we have a fixed embedding \( E \leftarrow B \) such that \( \mathcal{O}_{c_0} = E \cap \mathcal{O}_B \) is an order of conductor \( c_0 \). We write
\[
T(R) := (\mathcal{O}_{c_0} \otimes_{\mathcal{O}_F} R)^g / R^X.
\]
This implies that \( T \subset G \) as algebraic groups.

Write \( \Sigma_T \) for the set of split infinite places of \( E/F \). Thus,
\[
\Sigma_T = \Sigma_T^R \cup \Sigma_T^C, \quad \Sigma_T^R = \{ \sigma \mid \mathfrak{p}_\sigma \text{ is archimedean}, \quad T(F_\sigma) = \mathbb{R}^X \}, \quad \Sigma_T^C = \{ \sigma \mid \mathfrak{p}_\sigma \text{ is non-archimedean}, \quad T(F_\sigma) = \mathbb{C}^X \}.
\]
For any archimedean place \( \sigma \), write \( T(F_\sigma)_0 \subset T(F_\sigma) \) to be the intersection of all connected subgroups \( N \) for which the quotient \( T(F_\sigma)/N \) is compact. Write also \( T(F_\sigma)_+ \) for the connected component of \( 1 \) in \( T(F_\sigma) \). We can visualize \( T(F_\sigma), T(F_\sigma)_+, T(F_\sigma)_0 \) depending on the ramification type of \( \sigma \) in the following table:

| ramification | \( T(F_\sigma) \) | \( T(F_\sigma)_+ \) | \( T(F_\sigma)_0 \) | \( T(F_\sigma)/T(F_\sigma)_0 \) |
|--------------|------------------|-----------------|-----------------|------------------|
| \( \Sigma_T^R \) | \( \mathbb{R}^X \) | \( \mathbb{R}_+ \) | \( \mathbb{R}_+ \) | \( \pm 1 \) |
| \( \Sigma_T^C \) | \( \mathbb{C}^X \) | \( \mathbb{C}^X \) | \( \mathbb{R}_+ \) | \( S^1 \) |
| \( \infty \setminus \Sigma_T \) | \( \mathbb{C}^X / \mathbb{R}^X \) | \( \mathbb{C}^X / \mathbb{R}^X \) | \( 1 \) | \( \mathbb{C}^X / \mathbb{R}^X \) |

Given \( t_\infty \in T(F_\infty) = E_\infty^X / F_\infty^X \), write \( \tilde{t}_\infty \in E_\infty^X \) for a representative. If \( E_\sigma/F_\sigma \) splits, write \( \lambda(\tilde{t}_\sigma)_1, \lambda(\tilde{t}_\sigma)_2 \in E_\sigma^X \) for the two components of \( \tilde{t}_\sigma \in F_\sigma^X \times F_\sigma^X \). If \( E_\sigma/F_\sigma \) does not split, write \( \lambda(\tilde{t}_\sigma)_1, \lambda(\tilde{t}_\sigma)_2 \in E_\sigma^X \) for the image of \( \tilde{t}_\sigma \) under the two \( F_\sigma \)-isomorphisms. In this last non-split case, choose an embedding \( \vartheta : E_\sigma \to \mathbb{C} \) above \( \sigma \). For any \( m \in \mathbb{Z}^{2r} \), write
\[
\tilde{t}_\infty^m := \prod_{\sigma \mid \infty} \prod_{\mathfrak{p} \mid \mathfrak{p}_\sigma} \vartheta \left( \frac{\lambda(\tilde{t}_\sigma)_1}{\lambda(\tilde{t}_\sigma)_2} \right)^{m_{\mathfrak{p}}} \in \mathbb{C}.
\]
For any place \( v \) of \( F \), we write \( F_v \) for the completion. If \( v \mid p \), write \( O_{F,v} \) its integer ring, \( \text{ord}_v() \) its valuation, \( k_v \) its residue field, \( \omega_v \) a uniformizer, \( d_{F,v} \) its different over \( Q_v \), and \( q_v = \# k_v \).

1.2. Haar Measures. For any number field \( F \) and any place \( v \), we choose the Haar measure \( d\xi_v^X \) for \( F_v^X \):  
\[
d\xi_v = \zeta_v(1)|x_v|_{v}^{-1}d x_v;
\]
where
\[
d x_v \text{ is } [F_v : \mathbb{R}] \text{ times the usual Lebesgue measure, } v \mid \infty;
\]
\[
d x_v \text{ is the Haar measure of } F_v \text{ such that vol}(O_{F,v}) = [d_{F,v}]^{1/2}, v \nmid \infty,
\]
where \( \zeta_v(s) = (1 - q_v^{-s})^{-1}, \) if \( v \nmid \infty, \zeta_v(s) = \pi^{-s/2} \Gamma(s/2), \) if \( F_v = \mathbb{R}, \) and \( \zeta_v(s) = 2(2\pi)^{-s} \Gamma(s), \) if \( F_v = \mathbb{C}. \) One checks easily that, if \( v \) is non-archimedean, \( \text{vol}(O_{F,v}^X) = [d_{F,v}]^{1/2} \) as well.

The product of such measures provides a Tamagawa measure \( d^X x \) on \( A_F^X / F^X \). In fact, such Haar measure satisfies
\[
\text{Res}_{s=1} \int_{x \in A_F^X / F^X, |x| \leq 1} |x|^s d^X x = \text{Res}_{s=1} \Lambda_F(s),
\]
where \( \Lambda_F(s) = \zeta_F(s) \prod_{v \mid \infty} \zeta_{F_v}(s), \) is the completed Riemann zeta function associated with \( F \). This implies that, if we choose \( d^X x \) to be the quotient measure for \( T(A_F)/T(F) = A_F^X / A_F^X E^X \), one has that \( \text{vol}(T(A_F)/T(F)) = 2L(1, \eta_T). \)

For the group \( G \) we just choose the usual Haar measure so that \( \text{vol}(G(A_F)/G(F)) = 2. \)
1.3. Finite dimensional representations. Let \( k \) be a positive even integer. Let \( \mathcal{P}(k) = \text{Sym}^k(C^2) \) be the space of polynomials in 2 variables homogeneous of degree \( k \) with \( \text{PGL}_2(\mathbb{C}) \)-action:
\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix} (X, Y) = (ad - bc)^{-\frac{k}{2}} P(aX + cY, bX + dY), \quad P \in \mathcal{P}(k).
\]
Let us denote \( \mathcal{V}(k) = \mathcal{P}(k)^\vee \) with dual \( \text{PGL}_2(\mathbb{C}) \)-action:
\[
(g \mu)(P) = \mu(g^{-1}P), \quad \mu \in \mathcal{V}(k).
\]
Notice that \( \mathcal{V}(k) \cong \mathcal{P}(k) \) by means of the isomorphism
\[
(1.4) \quad \mathcal{V}(k) \longrightarrow \mathcal{P}(k), \quad \mu \mapsto \mu((Xy - Yx)^k).
\]

1.3.1. Polynomials and torus. Recall that the fixed embedding \( \iota : E \hookrightarrow B \) provides an isomorphism \( B \otimes_E E \cong M_2(E) \) that we will fix throughout the article. Indeed, \( \iota \) implies that \( B = \mathcal{E} \oplus \mathcal{E} J \), where \( J \) normalizes \( E \) and \( J^2 \in F^x \), hence we have the corresponding embedding
\[
(1.5) \quad E \hookrightarrow M_2(E); \quad e_1 + e_2 J \mapsto \begin{pmatrix} e_1 & e_2 J \\ \bar{e}_2 & \bar{e}_1 \end{pmatrix},
\]
where \((e \mapsto \bar{e}) \in \text{Gal}(E/F)\) denotes the non-trivial automorphism. For a given \( \bar{\sigma} : F \hookrightarrow \overline{C} \), the composition \( B \hookrightarrow B \otimes_E E \cong M_2(E) \) together with the fixed extension \( \bar{\sigma}_E : E \hookrightarrow \overline{C} \) of \( \bar{\sigma} \) gives rise to an embedding \( G(F_\sigma) \hookrightarrow \text{PGL}_2(\overline{C}), \) if \( \bar{\sigma} \mid \sigma \mid \infty \). This provides an action of \( G(F_\infty) \) on \( V(k) := \mathcal{V}(k)_{\overline{C}} \) and \( \mathcal{P}(k) := \mathcal{P}(k)_{\overline{C}} \), for any \( k = (k_\sigma) \in (2\mathbb{N})^d \) indexed by the embeddings \( \bar{\sigma} \) of \( F_\infty \). The subspaces \( V(k)_{\overline{C}} \) and \( \mathcal{P}(k)_{\overline{C}} \) are fixed by the action of the subgroup \( G(F) \leq G(F_\infty) \).

The morphism \( E \hookrightarrow B \hookrightarrow B \otimes_E E \cong M_2(E) \) maps \( e \) to \((e_1, e_2)\). This implies that we have a \( T(F_\infty) \)-equivariant morphism
\[
(1.6) \quad \mathcal{P}(k) \longrightarrow C(T(F_\infty), \overline{C}); \quad \bigotimes_{\delta} P_\delta \longmapsto \left( (t_\sigma)_{\sigma \in \Sigma} \mapsto \prod_{\sigma \in \Sigma} \prod_{\bar{\sigma} | \sigma} P_\delta \begin{pmatrix} 1, \bar{\sigma}_E \left( \frac{t_\sigma}{t_\bar{\sigma}_E} \right) \bar{\sigma}_E \left( \frac{t_\sigma}{t_\bar{\sigma}_E} \right)^{-\frac{k_\bar{\sigma}}{2}} \end{pmatrix} \right).
\]

2. Fundamental classes
Let \( \Sigma \subseteq \infty \) be a set of archimedean places, write \( T(F_\Sigma)_0 := \prod_{\sigma \in \Sigma} T(F_\sigma)_0 \) and let us consider the space of continuous functions
\[
C^\Sigma(T(A_F), \overline{C}) = \{ f \in C(T(A_F), \overline{C}), \quad f \mid_{T(F_\infty)} \text{ \( \mathbb{R} \)-analytic and } T(F_\infty)_{\Sigma_0}\text{-invariant} \},
\]
with its natural \( T(F) \)-action.

For any \( \bar{\sigma} \in \Sigma_\mathbb{R} \), the space \( T(F_{\bar{\sigma}}) \) is a \( \mathbb{R} \)-Lie group of dimension 1 or 2 depending if \( \bar{\sigma} \in \Sigma_\mathbb{R}^0 \) or \( \Sigma_\mathbb{C}^0 \). In each of the cases, we can identify \( L_{\bar{\sigma}} = \text{Lie}(T(F_{\bar{\sigma}})) \) with \( \mathbb{R} \) or \( \mathbb{C} \). The element \( \delta_{\bar{\sigma}} := 1 \in L_{\bar{\sigma}} \) generates the Lie algebra of \( T(F_{\bar{\sigma}})_0 \). This claim is clear for \( \bar{\sigma} \in \Sigma_\mathbb{R}^0 \), while whether \( \bar{\sigma} \in \Sigma_\mathbb{C}^0 \) it follows from
\[
(2.7) \quad \delta_{\bar{\sigma}} f(re^{i \theta}) = \frac{d}{dt} f(re^{i \theta} e^t) \bigg|_{t=0} = r \frac{d}{dr} f(re^{i \theta}), \quad f \in C_c^\infty(T(F_{\bar{\sigma}}), \overline{C}).
\]
The analyticity condition is imposed in the definition of \( C^\Sigma(T(A_F), \overline{C}) \) so that the following \( T(F) \)-equivariant sequence is exact
\[
(2.8) \quad 0 \longrightarrow C^\Sigma(T(A_F), \overline{C}) \longrightarrow C^\Sigma(T(A_F), \overline{C}) \overset{\delta_{\bar{\sigma}}}{\longrightarrow} C^\Sigma(T(A_F), \overline{C}) \longrightarrow 0,
\]
for any \( \bar{\sigma} \in \Sigma \). Indeed, \( \delta_{\bar{\sigma}} \) is clearly surjective and its kernel are the functions that are \( T(F_{\bar{\sigma}})_0 \)-invariant.

Definition 2.1. The composition of the connection morphisms in the corresponding long exact sequences in cohomology provides a morphism
\[
\partial : \quad H^0(T(F), C^\Sigma(T(A_F), \overline{C})) \longrightarrow H^1(T(F), C^\Sigma(T(A_F), \overline{C})) \longrightarrow \cdots \longrightarrow H^{u-1}(T(F), C^{\Sigma u}(T(A_F), \overline{C})) \longrightarrow H^u(T(F), C^{\Sigma u}(T(A_F), \overline{C})),
\]
where \( u := \#\Sigma_\mathbb{R} \).
2.1. Fundamental classes. By the Dirichlet Unit Theorem the \(\mathbb{Z}\)-rank of \(T(O_F)\) is \(u\). Write

\[
T(O_F)_+ := T(O_F) \cap T(F_{\infty})_+, \quad T(F_{\infty}) := \prod_{\ell | \infty} T(F_{\ell})_+.
\]

Notice that \(T(F_{\infty})_0\) is isomorphic to \(\mathbb{R}^u\) by means of the homomorphism \(z \mapsto (\log |\sigma z|)_{\sigma \in \Sigma_F}\), moreover, under such isomorphism the image of \(T(O_F)_+\) becomes a \(\mathbb{Z}\)-lattice \(\Lambda\) of \(\mathbb{R}^u\). Thus \(T(F_{\infty})_0/\Lambda\) is a \(u\)-dimensional torus and the fundamental class \(\xi\) is a generator of \(H_u(T(F_{\infty})_0/\mathbb{Z}) \cong \mathbb{Z}\). Notice that

\[
T(F_{\infty})_+ = T(F_{\infty})_0 \times S,
\]

where \(S\) is isomorphic to a cartesian product of \(S^1\). Let \(H = T(O_F)_+ \cap S\). Since \(T(O_F)_+\) is discrete and \(S\) is compact, \(H\) is finite. It is easy to check that \(H\) is the torsion subgroup of \(T(O_F)_+\) and \(T(O_F)_+ \cong \Lambda \times H\) (see [10, Lemma 3.1]). In particular,

\[
T(F_{\infty})_+ / T(O_F)_+ \cong T(F_{\infty})_0 / \Lambda \times S / H.
\]

Since \(M := T(F_{\infty})_0 \cong \mathbb{R}^u\), the de Rham complex \(\Omega^*_M\) is a resolution for \(\mathbb{R}\). This implies that we have an edge morphism of the induced spectral sequence

\[
e : H^0(\Lambda, \Omega^*_M) \longrightarrow H^u(\Lambda, \mathbb{R}).
\]

We can identify \(c \in H_u(M / \Lambda, \mathbb{Z})\) with a group cohomology element \(c \in H_u(\Lambda, \mathbb{Z})\) by means of the relation

\[
\int \omega = e(\omega) \cap c, \quad \omega \in H^0(\Lambda, \Omega^*_M) = \Omega^*_M / \Lambda.
\]

In particular, we can think of the fundamental class as an element \(\xi \in H_u(\Lambda, \mathbb{Z})\) satisfying

\[
e(\omega) \cap \xi = \int_{T(F_{\infty})_0 / \Lambda} \omega, \quad \omega \in H^0(\Lambda, \Omega^*_M).
\]

Let us consider the compact subgroup \(U := T(O_F) \otimes \hat{\mathbb{Z}} \subset T(\mathbb{A}^\infty_F)\), and write

\[
T(F)_+ := T(F) \cap T(F_{\infty})_+, \quad \text{Cl}(T)_+ := (T(\mathbb{A}^\infty_F))/UT(F)_+.
\]

By finiteness of class groups \(\text{Cl}(T)_+\) is finite and, since \(T(O_F)_+ \cong T(F)_+ \cap U\), we have an exact sequence

\[
0 \rightarrow T(F)_+ / T(O_F)_+ \rightarrow (T(\mathbb{A}^\infty_F))/U \xrightarrow{\pi} \text{Cl}(T)_+ \rightarrow 0.
\]

Fix preimages in \(\tilde{t}_i \in T(\mathbb{A}^\infty_F)\) of all the elements \(t_i \in \text{Cl}(F)\) and consider the compact set

\[
\mathcal{F} := \bigcup_i \tilde{t}_i U \subset T(\mathbb{A}^\infty_F).
\]

By construction \(T(F_{\infty})_+ / T(O_F)_+ \times \mathcal{F}\) is a fundamental domain of \(T(\mathbb{A}^\infty_F)\) for the action of \(T(F)\). Indeed, for any \(t \in T(\mathbb{A}_F)\), there exists a unique \(t_1 \in T(F) / T(O_F)_+\) such that \(t = t_1^{-1} t \in T(F_{\infty})_+ \times \mathcal{F}\) (see [10, Lemma 3.2]). In particular,

\[
T(\mathbb{A}_F) = \bigcup_{\tau \in T(F) / T(O_F)_+} \tau (T(\mathbb{A}^\infty_F) \times \mathcal{F}).
\]

Notice that the set of continuous functions \(C(\mathcal{F}, \mathbb{Z})\) has an action of \(T(O_F)_+\) (since \(\mathcal{F}\) is \(U\)-invariant) and the characteristic function \(1_{\mathcal{F}}\) is \(T(O_F)_+\)-invariant. Let us consider the fundamental class

\[
\eta = 1_{\mathcal{F}} \cap \xi \in H_u(T(O_F)_+, C(\mathcal{F}, \mathbb{Z})); \quad 1_{\mathcal{F}} \in H^0(T(O_F)_+, C(\mathcal{F}, \mathbb{Z})),
\]

where \(\xi \in H_u(T(O_F)_+, \mathbb{Z})\) is by abuse of notation the image of \(\xi\) through the correstraction morphism.

By (2.11), the natural \(T(O_F)_+\)-equivariant embedding

\[
i : C(\mathcal{F}, \mathbb{Z}) \hookrightarrow C^0(\mathbb{A}_F, \mathbb{Z}), \quad \iota \phi(x_{\infty}, x^\infty) = 1_{T(F_{\infty})_+}(t_{\infty}) \cdot \phi(t^\infty) \cdot 1_F(t^\infty).
\]

provides an isomorphism of \(T(F)\)-modules \(\text{Ind}_{T(O_F)_+}^{T(F)}(C(\mathcal{F}, \mathbb{Z})) \cong C^0(\mathbb{A}_F, \mathbb{Z})\), where \(C^0(\mathbb{A}_F, \mathbb{Z})\) is the set of functions in \(C^0(\mathbb{A}_F, \mathbb{C})\) that are \(\mathbb{Z}\)-valued and compactly supported when restricted to \(T(\mathbb{A}_F)\) (see [10, Lemma 3.3]). Thus, by Shapiro’s Lemma one may regard

\[
\eta \in H_u(T(F), C^0(\mathbb{A}_F, \mathbb{Z})).
\]
2.2. Pairings and fundamental classes. Notice that, by the choice in §1.2, we have that \((v = 0 \mid \infty)\)

\[
d^x x_\sigma = \frac{dx_\sigma}{|x_\sigma|} = \pm \frac{dr}{r}, \quad F_\sigma = \mathbb{R}, \quad r = |x_\sigma|; \quad d^x x_\sigma = \frac{2dsdt}{\pi(s^2 + t^2)} = \frac{2drd\theta}{\pi r}, \quad F_\sigma = \mathbb{C}, \quad x_\sigma = s + it = re^{i\theta}.
\]

This implies that the restriction of \(dt^\chi\) on \(T(F_\sigma)^+\) is given by

\[
d^x t_\sigma = \begin{cases} r^{-1}dr, & T(F_\sigma)^+ = \mathbb{R}^+, \quad r \in (0, \infty); \\
\pi^{-1}2\theta \cdot r^{-1}dr, & T(F_\sigma)^+ = \mathbb{C}^\times, \quad r \in (0, \infty), \quad \theta \in [0, 2\pi] \\
\pi^{-1}2\theta, & T(F_\sigma)^+ = \mathbb{C}^\times / \mathbb{R}^\times, \quad \theta \in [0, \pi]
\end{cases}
\]

Hence, given the decomposition \(T(F_\sigma)^+ = T(F_\sigma)_0 \times T(F_\sigma)_{\sigma}/T(F_\sigma)_0\), we have that \(d^x t_\sigma = d^x t_{\sigma,0}d^x s_\sigma\), where \(d^x t_{\sigma,0} = r^{-1}dr\) if \(T(F_\sigma)_0 \neq 1\). The product of \(d^x t_0 := \prod_{v \in \infty} d^x t_{\sigma,0}\) provides a Haar measure on \(T(F_\infty)_0\). Similarly, the product \(d^x s := \prod_{v \in \infty} d^x s_\sigma\) provides a Haar measure on \(\mathcal{S}\).

On the other side, the product \(d^x t_f := \prod_{v \in \infty} d^x t_{\sigma}\) provides a Haar measure on \((A_{\sigma})^\alpha\). As seen in §1.2, we have that

\[
\text{vol}(\hat{\mathcal{O}}_F^\alpha / \hat{\mathcal{O}}_E^\alpha) = \prod_{v \in \infty} \text{vol}(\mathcal{O}_E^\alpha)\text{vol}(\mathcal{O}_F^\alpha)^{-1} = \prod_{v \in \infty} |d_{E_v}|^{1/2}/|d_{F_v}|^{1/2} = |D_E|^{1/2}/|D_F|^{1/2},
\]

where \(D_E\) and \(D_F\) are the absolute discriminants of \(E\) and \(F\).

Given \(\phi \in H^0(T(F), C^{\infty}(T(A_\sigma), \mathbb{C}))\), we can consider the complex number

\[
\partial \phi \cap \eta \in \mathbb{C},
\]

where the cap product corresponds to the \((T(F))\)-invariant pairing

\[
C^0(T(A_\sigma), \mathbb{C}) \times C_0^0(T(A_\sigma), \mathbb{Z}) \to \mathbb{C} \quad (f, \phi) = \int_{S} \int_{\mathcal{T}(A_\sigma)} f(t)\phi(s, t)f^x t^x s
\]

Notice that \((T(F))\)-invariance follows because \(d^x s\) is the Haar measure of \((A_\sigma)^\alpha\).

**Remark 2.2.** Notice that, by (2.13) and the choices for \(dt^\chi\) and \(ds^\chi\), the above pairing can be defined over \(\mathbb{Q}\), namely, it is the extension of scalars of

\[
C^0(T(A_\sigma), \mathbb{Q}) \times C_0^0(T(A_\sigma), \mathbb{Z}) \to \mathbb{Q}.
\]

**Proposition 2.3.** We have that

\[
\partial \phi \cap \eta = h \int_{(T(h)/T(F))} \phi(t)d^x t,
\]

where \(h = \#T(\mathcal{O}_F)_{+,\text{tors}}\).

**Proof.** Recall that \(\eta = \xi \cap 1_{\mathcal{F}}\), hence

\[
\partial \phi \cap \eta = \partial \phi \cap 1_{\mathcal{F}} \cap \xi = \left(\int_S \int_{\mathcal{T}(A_\sigma)} 1_{\mathcal{F}}(t_f)(\partial \phi)(s, t_f)d^x t_f d^x s\right) \cap \xi = \left(\int_S \int_{\mathcal{F}} (\partial \phi)(s, t_f)d^x t_f d^x s\right) \cap \xi
\]

As above, write \(M = T(F_\infty)_0\) and recall that the function

\[
M \ni t_0 \mapsto \left(\int_S \int_{\mathcal{F}} \phi(t_{0s}, t_f)d^x t_f d^x s\right)
\]

lies in \(H^0(\Lambda, C^{\infty}(M, \mathbb{C}))\) by the \((T(F))\)-invariance of \(\phi\), and the \((T(\mathcal{O}_F))\)-invariance of \(\mathcal{F}\). Multiplying by the \(\Lambda\)-invariant differential \(d^x t_0\) of \((T(F_\infty)_0\) we obtain a differential \(\omega \in H^0(\Lambda, \Omega_M)\). It is clear by the definition of \(\partial\), \(d^x t_0\) and \(e\) that

\[
e(\omega) = \left(\int_S \int_{\mathcal{F}} (\partial \phi)(s, t_f)d^x t_f d^x s\right) \in H^1(\Lambda, \mathbb{C}).
\]

Thus, using equations (2.9) and (2.10), we deduce

\[
\partial \phi \cap \eta = e(\omega) \cap \xi = \int_{M/\Lambda} \omega = \int_{M/\Lambda} \int_S \int_{\mathcal{F}} \phi(t_{0s}, t_f)d^x t_f d^x s d^x t_0 = \#H \int_{T(F_{\infty})_{+,\text{tors}}/T(\mathcal{O}_F)} \int_{\mathcal{F}} \phi(t)d^x t.
\]

Finally, the result follows from the fact that \(\#H = \#T(\mathcal{O}_F)_{+,\text{tors}} = h\) (since \(\Lambda\) is \(\mathbb{Z}\)-free), and \(T(F_{\infty})_{+,\text{tors}} \times \mathcal{F}\) is a fundamental domain for \((T(A_\sigma)/T(F))\). \(\square\)
3. Cohomology of arithmetic groups and Waldspurger formulas

For any pair finite dimensional \(G(F)\)-representations \(M\) and \(N\) over \(\mathbb{C}\) and an open compact subgroup \(U \subset G(\mathcal{A}_F)\), we define \(\mathcal{A}^\infty(M, N)^U\) to be the set of functions

\[
\phi : G(\mathcal{A}_F)/U \longrightarrow \text{Hom}_\mathbb{C}(M, N).
\]

We write \(\mathcal{A}^\infty(N)^U := \mathcal{A}^\infty(\mathbb{C}, N)^U\). Notice that \(\mathcal{A}^\infty(M, N)^U\) has natural action of \(G(F)\):

\[
(\gamma \phi)(g) = \gamma \phi(\gamma^{-1} g); \quad \phi \in \mathcal{A}^\infty(M, N)^U, \quad \gamma \in G(F).
\]

We denote by \(\mathcal{A}^\infty(M, N)^U(\lambda)\) the twist of \(\mathcal{A}^\infty(M, N)^U\) by a character \(\lambda : G(F) \to \mathbb{C}^\times\).

3.1. Lower Eichler-Shimura for central automorphic forms.

In this section we will study some Eichler-Shimura morphisms for automorphic forms of even weight \(k\).

For any open compact subgroup \(U \subset G(\mathcal{A}_F)\), write \(\mathcal{A}(\mathbb{C})^U\) for the set of functions \(f : G(\mathcal{A}_F)/U \longrightarrow \mathbb{C}\), that is:

- \(\mathbb{C}^\infty\) when restricted to \(G(F_\infty)\).
- Right-\(K_o\)-finite, for all \(\sigma \in \Sigma_F\), where \(K_o\) is the maximal compact subgroup of \(G(F_o)\).
- Right-\(\mathbb{Z}_o\)-finite, for all \(\sigma \in \Sigma_F\), where \(\mathbb{Z}_o\) is the centre of the universal enveloping algebra of \(G(F_o)\).

Notice that automorphic forms are left-\(G(F)\)-invariant elements of \(\mathcal{A}(\mathbb{C})^U\), for some \(U \subset G(\mathcal{A}_F)\).

By a \((\mathcal{A}_o, K_o)\)-module we mean the tensor product of \((\mathcal{A}_o, K_o)\)-modules, for \(\sigma \in \Sigma_B\), and finite-dimensional \(G(F_o)\)-representations, for \(\sigma \in \Sigma_T \setminus \Sigma_B\). Given a \((\mathcal{A}_o, K_o)\)-module \(\mathcal{V}\), we define

\[
\mathcal{A}^\infty(\mathcal{V}, \mathcal{C})^U := \text{Hom}_{(\mathcal{A}_o, K_o)}(\mathcal{V}, \mathcal{A}(\mathbb{C})^U).
\]

This is consistent with notation of the beginning of the section, since by [8, Lemma 2.3], for \(V\) a finite dimensional \(G(F_o)\)-representation, \(\mathcal{A}^\infty(V, \mathcal{C})^U = \mathcal{A}^\infty(V, \mathcal{C})^U\), where \(V\) is the \((\mathcal{A}_o, K_o)\)-module associated with \(V\).

From this section onwards we will assume that the ramification sets of \(T\) and \(G\) coincide, namely, \(\Sigma_T = \Sigma_B\) and \(u = r\). For any \(\sigma \in \Sigma_B\), fix a isomorphism \(E_{\sigma} \cong F_o^2\). Recall that we already had an identification \(B \otimes F \cong M_2(E)\) such that the composition \(E \hookrightarrow B \twoheadrightarrow B \otimes F \cong M_2(E)\) is given by \(e \mapsto \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}\). Thus, both isomorphisms induce \(B_o \cong M_2(F_o)\) mapping \(E_o\) to the group of diagonal matrices. This induces an identification \(G(F_o) \cong \text{PGL}_2(F_o)\) that sends \(T(F_o)\) to the diagonal torus. Thanks to this, for any \(k \in \mathbb{Z}_o^\times := \{ (n, \sigma)_{\sigma \in \Sigma_o^c} : n_o \in \mathbb{Z}[F_o^\times, \mathcal{B}_o] \}\) with \(k_o \geq 2\) even, we can consider the \((\mathcal{A}_o, K_o)\)-module

\[
D(k) = \bigotimes_{\sigma \in \Sigma_B} D(k_{\sigma}) \otimes \bigotimes_{\sigma \in \Sigma_T \setminus \Sigma_B} V(k_{\sigma} - 2), \quad k = (k_{\sigma}),
\]

where \(V(k_{\sigma} - 2)\) and \(D(k_{\sigma})\) are the \((\mathcal{A}_o, K_o)\)-modules and finite dimensional representations described in §1.3, §4.1 and §4.2. Then an automorphic form of weight \(k\) defines an element of \(\mathcal{A}^\infty(D(k), \mathcal{C})^U\), for some \(U\), by sending a generator of \(D(k)\) to the given form. In fact, it defines a \(G(F)\)-invariant element

\[
\Phi \in H^0(G(F), \mathcal{A}^\infty(D(k), \mathcal{C})^U),
\]

and any such a \(\Phi\) defines an automorphic form of weight \(k\).

Fix a character \(\lambda : G(F_\infty)/G(F_\infty)_+ = G(F_\infty)/G(F_\infty)_+ \to \pm 1\). For any \(\sigma \in \Sigma_B\), let us consider the exact sequence of \((\mathcal{A}_o, K_o)\)-modules

\[
0 \longrightarrow D(k_{\sigma}) \overset{i_{\sigma}}{\longrightarrow} \tilde{\mathcal{B}}_{\Sigma, \sigma}^1 \overset{p}{\longrightarrow} V(k_{\sigma} - 2)(\lambda_{\sigma}) \longrightarrow 0
\]

of (4.32) and (4.40). For any \(\Sigma \subset \Sigma_B\) and \(\tau \in \Sigma\), we can also consider the intermediate \((\mathcal{A}_o, K_o)\)-modules

\[
D^1_{\Sigma}(k) := \bigotimes_{\sigma \in \Sigma} D(k_{\sigma}) \otimes \bigotimes_{\sigma \in \Sigma \setminus \Sigma^c} V(k_{\sigma} - 2)(\lambda_{\sigma}),
\]

\[
\tilde{\mathcal{B}}_{\Sigma, \tau}^1 := \bigotimes_{\sigma \in \Sigma_{\tau}} D(k_{\sigma}) \otimes \bigotimes_{\sigma \in \Sigma \setminus \Sigma_{\tau}} V(k_{\sigma} - 2)(\lambda_{\sigma}),
\]

lying in the exact sequence of \((\mathcal{A}_o, K_o)\)-modules

\[
0 \longrightarrow D^1_{\Sigma}(k) \longrightarrow \tilde{\mathcal{B}}_{\Sigma, \tau}^1 \longrightarrow D^1_{\Sigma_{\tau}}(k) \longrightarrow 0.
\]

Thus, we obtain the exact sequence of \(G(F)\)-modules

\[
0 \longrightarrow \mathcal{A}^\infty(D^1_{\Sigma_{\tau}}(k), \mathcal{C})^U \longrightarrow \mathcal{A}^\infty(\tilde{\mathcal{B}}_{\Sigma, \tau}^1, \mathcal{C})^U \longrightarrow \mathcal{A}^\infty(D^1_{\Sigma}(k), \mathcal{C})^U \longrightarrow 0.
\]
Since $D(k) = D^1_{\Sigma_0}(k)$ and $\text{Hom}(D^1_{\vartheta}(k), \mathbb{C}) = V(k - 2)(\lambda) := \bigotimes_{\vartheta \in \Sigma_0} V(k - 2)(\lambda_\vartheta)$, the connection morphisms in the corresponding long exact sequences provide the Eichler-Shimura morphism (see [13])

$$E_{\Sigma_0^r} : H^0(G(F), \mathcal{A}^\infty(D^1_{\vartheta}(k), \mathbb{C})^U) \to H^1(G(F), \mathcal{A}^\infty(D^1_{\vartheta}(k), \mathbb{C})^U) \to \cdots \to H^{r-1}(G(F), \mathcal{A}^\infty(D^1_{\vartheta}(k), \mathbb{C})^U) \to H^r(G(F), \mathcal{A}^\infty(V(k - 2))^U(\lambda)),$$

where $r = \#\Sigma_0$. By means of $E_{\Sigma_0^r}$ we realize the automorphic representation generated by $\Phi$ in the cohomology group

$$H^r(G(F), \mathcal{A}^\infty(V(k - 2))^U(\lambda)) = \lim_{\to} H^r(G(F), \mathcal{A}^\infty(V(k - 2))^U(\lambda)).$$

**Remark 3.1.** As explained in Remark 4.3, the exact sequence of (3.15) differs by a factor of $\pi$ from the one explained in [13]. Hence, each connection morphism there is given by $\pi^{-1}$ times the connection morphism here. This implies that, if $F$ is totally real, the Eichler-Shimura morphism explained in [13] is $\pi^{-r}E_{\Sigma_0^r}$.

### 3.2. Classical Waldspurger’s formulas

In this section we will recall the classical Waldspurger’s formula. We will essentially follow the reference [4], but we will use slightly different test vectors at archimedean places. Let $\pi$ be an automorphic representation of $G$ and let $\Pi$ be its Jacquet-Langlands lift to $\text{PG}L_2/F$. Given the torus $T \subset G$ considered previously and a character $\chi$ of $T(\mathbb{A}_F)/T(F)$, we define the period integral

\[
\ell(f, \chi) = \int_{T(\mathbb{A}_F)/T(F)} f(t) \cdot \chi(t) dt. 
\]

The aim of this section is to compute these periods. The first result that one can think of is the classical Waldspurger formula (see [15]) that asserts that, given decomposable $f_1 = \bigotimes_{\vartheta} f_{1, \vartheta}, f_2 = \bigotimes_{\vartheta} f_{2, \vartheta} \in \pi$,

\[
\ell(f_1, f_2) = \frac{\Lambda_f(2) \cdot \Lambda(1/2, \Pi, \chi)}{2 \Lambda(1, \Pi, \text{id})} \prod_{\vartheta} \beta_{\vartheta}(f_{1, \vartheta}, f_{2, \vartheta}),
\]

where $\Lambda$ stands for complete global $L$-functions, that is, those as products of local $L$-functions over all places,

\[
\beta_{\vartheta}(f_{1, \vartheta}, f_{2, \vartheta}) = \frac{\zeta(2 \vartheta) L(1/2, \Pi_{\vartheta}, \chi_{\vartheta})}{\zeta(2) L(1/2, \Pi, \chi)} \int_{T(F_{\vartheta})} \chi(t_\vartheta)^\vartheta \langle f_{1, \vartheta}, f_{2, \vartheta} \rangle_{\vartheta} dt_\vartheta,
\]

and $\langle \ , \ \rangle = \prod_{\vartheta} \langle \ , \ \rangle_{\vartheta}$ is the Petersson inner product

\[
\langle f_1, f_2 \rangle = \int_{G(F)\backslash G(\mathbb{A}_F)} f_1(g) f_2(g) dg.
\]

In order to get rid of the factor $2 \Lambda(1, \Pi, \text{id}) \Lambda_f(2)^{-1}$, one can use the Petersson pairing formula (see [4, proposition 2.1]) that states that, given decomposable $f_1, f_2 \in \Pi$,

\[
\langle f_1, f_2 \rangle = 2 \Lambda(1, \Pi, \text{id}) \cdot \Lambda_f(2)^{-1} \cdot \prod_{\vartheta} \alpha_{\vartheta}(W_{1, \vartheta}, W_{2, \vartheta}),
\]

where the Petersson product $\langle \ , \ \rangle$ is defined as in (3.19), the elements of the Whittaker model

\[
\prod_{\vartheta} W_{1, \vartheta} = W_1 = \int_{A_F/F} f_1 \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi(-x) dx; \quad \prod_{\vartheta} W_{1, \vartheta}^{-} = W_1^{-} = \int_{A_F/F} f_1 \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g \right) \psi(x) dx;
\]

\[
\psi = \prod_{\vartheta} \psi_{\vartheta}, \quad \psi_{\vartheta}(a) = \begin{cases} e^{2\pi i a_{\vartheta}}, & F_{\vartheta} = \mathbb{R}; \\
1, & F_{\vartheta} = \mathbb{C}; \\
e^{-2\pi i \text{Tr}_{\vartheta} (a_{\vartheta})}, & F_{\vartheta}/\mathbb{Q}_{\vartheta};
\end{cases}
\]

being $\lceil \cdot \rceil : \mathbb{Q}_{\vartheta}/\mathbb{Z}_{\vartheta} \hookrightarrow \mathbb{Q}$ is the natural inclusion, and

\[
\alpha_{\vartheta}(W_{1, \vartheta}, W_{2, \vartheta}^{-}) = \frac{\zeta(2) \cdot \langle W_{1, \vartheta}, W_{2, \vartheta}^{-} \rangle_{\vartheta}}{\zeta(2) L(1, \Pi_{\vartheta}, \chi_{\vartheta})}; \quad \langle W_{1, \vartheta}, W_{2, \vartheta}^{-} \rangle_{\vartheta} = \int_{F_{\vartheta}} W_{2, \vartheta} \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} W_{1, \vartheta}^{-} \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix}^T \, da.
\]

Combining (3.17) and (3.20), one obtains for decomposable $f_1, f_2 \in \pi$ and $f_1, f_2 \in \Pi$,

\[
\ell(f_1, f_2)^{-1} = \Lambda(1/2, \Pi, \chi) \cdot \frac{\langle f_1, f_2 \rangle_{T(\mathbb{A}_F)/T(F)}}{\prod_{\vartheta} \beta_{\vartheta}(f_{1, \vartheta}, f_{2, \vartheta}) \alpha_{\vartheta}(W_{1, \vartheta}, W_{2, \vartheta}^{-}).}
\]

Recall that, instead of working with automorphic forms, in §3.1 we consider $\Phi \in H^0(G(F), \mathcal{A}^\infty(D^1_{\vartheta}(k), \mathbb{C})^U)$, for some open compact subgroup $U \subset G(\mathbb{A}_F)$. Notice that, for any $V \in D(k)$, we can construct the automorphic
form $\phi(V)$. Hence, for any automorphic representation such that $\pi_{\infty} = \pi \mid G(F_\omega) \cong D(k)$, we can realize $\pi^\infty = \pi \mid G(\mathbb{A}_F^\infty)$ in the space $\pi^\infty \subset H^0(G(F), \mathcal{A}^\infty(D(k), \mathbb{C})) := \lim_{U} H^0(G(F), \mathcal{A}^\infty(D(k), \mathbb{C})^U)$. 

On the one hand, for a given character $\lambda : G(F_\omega) / G(F_\omega)_+ \to \pm 1$ and for any $\sigma \in \Sigma_\beta$, we have $K_\sigma$-equivariant sections of $\rho$ in (3.14) (see §4.1.1 and lemma 4.17)

$$s : V(k_\sigma - 2)(\lambda_\sigma) \to \mathcal{G}_\sigma^\lambda.$$  

On the other hand, we have the exponential map

$$(3.23) \exp_\sigma : F_\sigma \to \text{Lie}(F_\sigma) = \text{Lie}(T(F_\sigma)) \subset \mathcal{G}_\sigma.$$  

The element of the Lie algebra $\delta_\sigma^T := \exp_\sigma(1) \in \mathcal{G}_\sigma$ gives rise to a morphism

$$(3.24) \delta s : V(k_\sigma - 2)(\lambda_\sigma) \to D(k_\sigma), \quad \delta s(\mu) = \tau^{-1} s(\delta_\sigma^T(\delta(s(\mu)) - s(\delta_\sigma^T(\mu))).$$  

The tensor product of such a $\delta s$ provides

$$(3.25) \mu_{\mathbb{m}} \left( \begin{array}{cc} X & Y \\ x & y \end{array} \right)^{k_\sigma - 2} = x^{k_\sigma - m_0} y^{k_\sigma - m_0}, \quad \text{or simply} \quad \mu_{\mathbb{m}} \left( \begin{array}{cc} X & Y \\ x & y \end{array} \right)^{k_\sigma - 2} = x^{k_\sigma - m_0} y^{k_\sigma - m_0}.$$  

Hence, given $\Phi_1, \Phi_2 \in H^0(G(F), \mathcal{A}^\infty(D(k), \mathbb{C})^U)$, the formula (3.22) allows us to compute the product of periods $l(\Phi_1(\delta s(\mu_{\mathbb{m}}), \lambda); \Phi_2(\delta s(\mu_{\mathbb{m}}), \lambda))$. Nevertheless, applying directly (3.22) one obtains a formula that involves the Petersson product $(\Phi_1(\delta s(\mu_{\mathbb{m}})), \Phi_2(\delta s(\mu_{\mathbb{m}})))$ depending on $\mathbb{m}$. In order to introduce a product only depending on $\Phi_1$ and $\Phi_2$, notice that there exists a canonical $G(F_\omega)$-equivariant element

$$\bigotimes_{\sigma | \infty} \Upsilon_\sigma = \Upsilon = \left( \begin{array}{cc} x_1 & y_1 \\ x_2 & y_2 \end{array} \right)^{k_\infty - 2} \in \mathcal{P}(k - 2) \otimes \mathcal{P}(k - 2) \cong V(k - 2) \otimes V(k - 2).$$  

Thus, interpreting $\Phi_1, \Phi_2$ an element of $H^0(G(F), \mathcal{A}^\infty(D(k) \otimes D(k), \mathbb{C})^U)$, one can consider

$$(3.26) \langle \Phi_1, \Phi_2 \rangle := \int_{G(F) \backslash G(F)} \Phi_1(\delta s(\Upsilon)) \langle \sigma, \tau \rangle d^\infty \sigma.$$  

by lemmas 4.10 ad 4.25, this defines a non-zero $G(\mathbb{A}_F^\infty)$-equivariant pairing.

Regarding the automorphic representation $\Pi$ of $\text{PGL}_2$, we would like to specify a normalized newform $f$ in order to provide a more explicit formula. Write $N \subseteq O_F$ for the conductor of $\Pi$, and write $U_0(N) = \{ (a \ b) \in \mathcal{O}_F, \ c \in \mathcal{O}_F \}$.  

Let $f = \bigotimes_{\sigma} f_\sigma$ be the $U_0(0)$-invariant automorphic form of $\Pi$ normalized so that $f_\sigma$ is fixed by the diagonal matrices $(a_1 b) \in \text{GL}_2(F_\sigma), |a|_\sigma = 1$, with weight minimal for all $\sigma \in \Sigma_F$ and such that

$$\Lambda(s, \Pi) = |d_F|^s 1/2 \int_{\mathbb{A}_F^\times} f \left( \begin{array}{cc} a & \gamma \\ 1 & \delta \end{array} \right) |a|^{s - 1/2} d^\times a,$$

where $| \cdot : \mathbb{A}_F^\times \to \mathbb{R}_+$ is the standard adelic absolute value, and $d_F \subseteq O_F$ is the different of $F$. In [4], the values $(W_{\sigma_1}, W_{\sigma_2})_{\sigma}$ are computed explicitly for such a form:

**Lemma 3.2.** We have that

$$(W_{\sigma_1}, W_{\sigma_2})_{\sigma} = \begin{cases} 2(4\pi)^{-k_\sigma} \Gamma(k_\sigma) & F_\sigma = \mathbb{R}; \\
8(2\pi)^{1-k_\sigma} \Gamma \left( \frac{k_\sigma + k_{\sigma_2} - 2}{2} \right) \Gamma \left( \frac{k_{\sigma_1} + k_{\sigma_2}}{2} \right) & F_\sigma = \mathbb{C}; \sigma_1, \sigma_2 | \sigma. \end{cases}$$
Proof. If $F_o = C$, we know by [12, Theorem 6.2] that our representation $D(k_o)$ is isomorphic to the principal series $\hat{B}(\mu_1 \cdot \hat{\sigma}) = \pi(\mu_1, \mu_2)$, where

$$\mu_1(t) = \mu_2(t)(-1) = \frac{1}{2} \frac{k_o - 1}{t} \frac{t - 1}{k - 1} = \left( \frac{t}{|t|} \right) \left( \frac{t - 1}{k - 1} \right), \quad \sigma_1, \sigma_2 \mid \sigma.$$ 

In [4, before lemma 3.14], one can find a receipt to compute $\langle W_{1, o}, W_{o, -} \rangle_o$ in case $(F_o, \Pi_o) = (\mathbb{R}, D(k_o))$ and $(F_o, \Pi_o) = (\mathbb{C}, \pi(\mu_1, \mu_2))$, hence, the result follows from the given computation. $\square$

Similarly as above, it is more consistent to consider a normalized element of $H^0(\text{PGL}_2(F), \mathcal{A}^\infty(D(k), \mathbb{C}^{\text{Lin}(\mathbb{N})})$ instead of the automorphic form $\hat{f}$. Notice that $\delta_s(\mu_o) \notin D(k)$ is fixed by the diagonal matrices $\{ a_1 \} \in \text{GL}_2(F_o)$, $|a|_o = 1$, with weight minimal. Hence, we can define the normalized element

$$\Psi \in H^0(\text{PGL}_2(F), \mathcal{A}^\infty(D(k), \mathbb{C}^{\text{Lin}(\mathbb{N})}); \quad \Psi(\delta_s(\mu_o)) = \hat{f},$$

and we can consider the corresponding Petersson product $\langle \Psi, \Psi \rangle$ as in (3.26). We write $L(s, \Pi, \chi)$ for the L-function given by the product at finite places.

Proposition 3.3. Given decomposable $\Phi_1, \Phi_2 \in \pi^\infty \subset H^0(\text{G}(F), \mathcal{A}^\infty(D(k), \mathbb{C}))$ and a locally polynomial character $\chi$ such that $\chi \mid_{\Gamma(F_o)} = 1$, for some $\frac{k - 2}{2} \leq m \leq \frac{k - 2}{2}$ and some locally constant character $\chi_o$, we have $\ell(\Phi(\delta_s(\mu_o)), \chi) = 0$ unless $\chi_o \mid_{\Gamma(F_o)} = \lambda$ and, if this is the case,

$$\ell(\Phi_1(\delta_s(\mu_o)), \chi) \cdot \ell(\Phi_2(\delta_s(\mu_o)), \chi^{-1}) = C(k, m) \cdot L(1/2, \Pi_o) \cdot \langle \Phi_1, \Phi_2, \Psi \rangle \cdot \prod_{\varphi \in \mathcal{P}_1} \left( \prod_{\varphi \in \mathcal{P}_2} \alpha_\varphi(W_{1, \varphi}, W_{-\varphi}) \right),$$

where $\langle \cdot , \cdot \rangle$ are the inner products described in (3.26) and

$$C(k, m) = (-1)^{(\sum \#_{\mathcal{B}} - \frac{3k - 2}{2})} \cdot (32\pi)^{2\#_{\mathcal{B}}} \cdot \prod_{\varphi, \psi} \Gamma(\frac{k_o}{2} - m, \frac{k_o}{2} + m_0).$$

Proof. For the proof of this proposition we use the computations performed in §4. For all $\sigma \mid \infty$, we write $\beta_\sigma = \beta_\sigma(\Phi_1(\delta_s(\mu_o)), \Phi_2(\delta_s(\mu_o)_h)$ and $\alpha_\sigma = \alpha_\sigma(W_{1, \sigma}, W_{-\sigma})$. Since $\Phi_1(\delta_s(\mu_o))$ is proportional to $\delta_s(\mu_m)$ if $\sigma \in \Sigma$ and to $\mu_m$, otherwise, we have that

$$\left( \langle \Phi_1, \Phi_2 \rangle, \langle \delta_s(\mu_o), \Phi_2(\delta_s(\mu_o) \rangle \right) \prod_{\sigma \mid \infty} \beta_\sigma = \prod_{\sigma \mid \infty} C_\sigma L(1/2, \Pi_o, \chi).$$

where

$$C_\sigma = \begin{cases} \langle \delta_s(\Pi_o), \delta_s(\Pi_o) \rangle^{-1} \langle W_{1, o}, W_{o, -} \rangle_o L(1, \eta_o) \zeta_o(1)^{-1} \langle \delta_s(\mu_o), \delta_s(\mu_o) \rangle^{-1} \int_{\Gamma(F_o)} \chi_o(t) \pi_o(t) \mu_m \mu_m \delta_s(t) d^\infty t & \sigma \notin \Sigma; \\
\langle W_{1, o}, W_{o, -} \rangle_o L(1, \eta_o) \zeta_o(1)^{-1} \langle \delta_s(\mu_o), \delta_s(\mu_o) \rangle^{-1} \int_{\Gamma(F_o)} \chi_o(t) \pi_o(t) \delta_s(\mu_m) \mu_m d^\infty t & \sigma \in \Sigma. \\
\end{cases}$$

Notice that, in the above expression, we have freedom to choose pairings $\langle \cdot, \cdot \rangle$ except of the pairing (3.21) for the Whittaker model. Indeed, such all $G(F_o)$-invariant pairings are proportional.

If $T(F_o) = \mathbb{C}^\infty$, then $L(1, \eta_o) \zeta_o(1) = \pi^{-1}$. Moreover, we have $\chi_o = t_m, \pi_o(t) \mu_m = t_m \mu_m$, and $\text{vol}(T(F_o)) = 2$ with respect to the measure (2.12). Hence, by lemmas 3.2, 4.10, 4.11 and equation (4.35),

$$C_\sigma = 2 \langle \delta_s(\Pi_o), \delta_s(\Pi_o) \rangle^{-1} \langle W_{1, o}, W_{o, -} \rangle_o \pi^{-1} \langle \delta_s(\mu_o), \delta_s(\mu_o) \rangle^{-1} \langle \mu_m \mu_m \rangle = \frac{2 \cdot \Gamma(k_o^2 - m_0) \Gamma(k_o^2 + m_0)}{(-1)^{-4} + \frac{\#_{\mathcal{B}}}{2} \pi k_o + 1}.$$ 

If $T(F_o) = \mathbb{R}^\infty$, then $L(1, \eta_o) \zeta_o(1)^{-1} = 1$. Hence, we obtain by theorem 4.9, and lemmas 3.2 and 4.11,

$$C_\sigma = \begin{cases} \langle 4(1)^{-4} (2\pi)^{-2k_0} \Gamma(k_o^2 - m_0) \Gamma(k_o^2 + m_0), \chi_o = \lambda_o; \\
0, \chi_o \neq \lambda_o. \\
\end{cases}$$

Finally, if $T(F_o) = \mathbb{C}^\infty$ then we have as well $L(1, \eta_o) \zeta_o(1)^{-1} = 1$. Moreover, by theorem 4.24 and lemmas 4.26 and 3.2,

$$C_\sigma = \begin{cases} \frac{(-1)^{-4} + \#_{\mathcal{B}}}{2} (2\pi)^{-2k_0} \Gamma(k_o^2 - m_0) \Gamma(k_o^2 + m_0), \chi_o = \lambda_o; \\
0, \chi_o \neq \lambda_o. \\
\end{cases}$$

Hence, the result follows from (3.22) since $C(k, m) = \prod_{\sigma \mid \infty} C_\sigma$. $\square$
In [4], Cai, Shu and Tian obtain a more explicit formula by specifying concrete test vectors. Let us consider $c \subseteq O_F$ the conductor of the character $\chi$, namely, the bigger ideal such that $\chi$ is trivial on $\hat{O}_E^c / \hat{O}_E^c$. Let $O_F \subseteq E$ be the order of conductor $c$. We define $S_1 := \{ v \mid N \text{ non-split in } T; \text{ord}_c(c) < \text{ord}_c(N) \}$. Let $c_1 := \prod q \in S_1 p^{\text{ord}_p(c)}$ be the $S_1$-off part of $c$. Then, for any finite place $v$, there exists a $O_F, v$-order $O_{N, v} \subseteq B_v$ of discriminant $NO_{F, v}$ such that $O_{N, v} \cap E_v = O_{c, v}$. Such an order $O_{N, v}$ is called admissible for $(N, \chi_v)$ if at places $v \mid (N, c_1)$, the order $O_{N, v}$ is the intersection of two maximal orders $O'_{B, v}$ and $O''_{B, v}$ of $B_v$ such that

$$O'_{B, v} \cap E_v = O_{c, v}, \quad O''_{B, v} \cap E_v = \begin{cases} O_{N/v}, & \text{ if } \text{ord}_v(c/N) \geq 0 \\ O_{E, v}, & \text{ otherwise.} \end{cases}$$

**Remark 3.4.** Our concept of admissibility coincide with that of [4] because the condition (2) (see [4, definition 1.3]) does not apply in our situation because $\chi_v | f^2$ is trivial.

Let $O_N$ be an admissible $O_F$-order of $B$ for $(N, \chi)$ in the sense that, for any finite place $v$, $O_{N, v}$ is admissible for $(N, \chi_v)$. It follows that $O_N$ is an $O_F$-order of $B$ of discriminant $N$ such that $O_N \cap E = O_{c_1}$.

**Definition 3.5.** Write $U^{S_1} = \prod_{v \notin S_1} O_{N, v}^\times$. Let $V(\pi^\infty, \chi) \subset \pi^\infty$ be the subspace of elements $\otimes_v f_v \in \pi^{U, S_1}$ such that $f_v$ is a $\chi_v$-eigenform under $T(F_v)$, for all places $v \notin S_1$.

If we assume that the root number $\epsilon(1/2, \pi_{\nu, \chi_v}) = \chi_\nu(\tau_{\nu, v}(-1)\epsilon(B_v))$ for all $v \notin \infty$ then the space $V(\pi^\infty, \chi)$ is actually one-dimensional by [4, Proposition 3.7]. The following result computes the remaining local terms for such one-dimensional space of test vectors.

**Lemma 3.6.** [4, Lemma 3.13] Write $U = O_N^\times \subset G(\mathcal{A}_F^\infty)$ and let $\Phi_1 \in V(\pi^\infty, \chi)$ and $\Phi_2 \in V(\pi^\infty, \chi^{-1})$. If the root number $\epsilon(1/2, \pi_{\nu, \chi_v}) = \chi_\nu(\tau(1, 1), \epsilon(T_F))$ for all $v \notin \infty$ then

$$\prod_{v \notin \infty} \left( \beta_v(\Phi_{1, v}, \Phi_{2, v}) \cdot \alpha_v(W_{1, v}, W_{2, v}) \right) = \frac{\text{vol}(U^{\infty}(N))}{\text{vol}(U)} \left[ c_1^2 D \right]^{-\frac{3}{2} \# S_0} L(1, \eta)^2 \prod_{v \notin \infty} L(1/2, \Pi_{v, \chi_v})^{-1},$$

where $S := \{ v \mid (N, D) \; \text{if } v \parallel N \text{ then } \text{ord}_v(c/N) \geq 0 \}, S_0 := \{ v \mid (N, D); \text{ord}_v(c) < \text{ord}_v(N) \}, | \cdot |$ denotes the norm of an ideal and $L_1(1, \eta) = \prod_{\nu | 1} L(1, \eta_{\nu, v})$.

3.3. **Restriction of the Eichler-Shimura morphism.** Recall that, for a fixed character $\lambda : G(F) / G(F)_{\infty} = G(F_{\infty}) / G(F_{\infty})_{\infty} \to \pm 1$, we have the morphism $E_{S_1}$ of $\S_3.1$:

$$H^0(G(F), \mathcal{A}(D_1^2, \mathcal{A})^U) \to H^0(G(F), \mathcal{A}(V(\mathcal{A} - 2))^U(\lambda)).$$

The tensor product of the morphisms $\delta : V(k - 2) \to D(k)$ of (3.24) provides

$$\delta_S : V(k - 2) \to D_1^2(k),$$

for every $S \subseteq S_B$. Hence, we have $T(F)$-equivariant morphisms

$$\varphi_S : C^0(T(A_F), \mathcal{C}) \otimes V(k - 2) \otimes \mathcal{A}(D_1^2, \mathcal{A})^U \to C^0(T(A_F), \mathcal{C})$$

$$\varphi_S((f \otimes \mu) \otimes \Phi(z, t)) = f(z, t) \cdot \Phi(\delta_S(z^{-1} \mu))(z, t),$$

for any $z \in T(F_{\infty})$ and $t \in T(A_F^\infty)$.

**Remark 3.7.** In the special case $\Sigma = \emptyset$, the corresponding $\varphi_{\emptyset} = \varphi$ is given by

$$\varphi : C^0(T(A_F), \mathcal{C}) \otimes V(k - 2) \otimes \mathcal{A}(V(k - 2))^U(\lambda) \to C^0(T(A_F), \mathcal{C}),$$

$$\varphi((f \otimes \mu) \otimes \Phi(z, t)) = f(z, t) \cdot \lambda(z) \cdot \Phi(t)(\mu(Xy - Yx)^{k - 2})$$

for all $z \in T(F_{\infty})$, $t \in T(A_F^\infty)$.

Recall $\delta : H^0(T(F), C^0(T(A_F), \mathcal{C})) \to H^0(T(F), C^0(T(A_F), \mathcal{C}))$ from Definition 2.1.

**Proposition 3.8.** We have that

$$\delta(\varphi_{\emptyset}(h \cup \Phi)) = \varphi(h \cup ES_1(\Phi)) \in H^0(T(F), C^0(T(A_F), \mathcal{C})),$$

for all $h \in H^0(T(F), C^0(T(A_F), \mathcal{C}) \otimes V(k - 2))$ and $\Phi \in H^0(G(F), \mathcal{A}(D_1^2, \mathcal{A}))$. 

Proof. For any $\tau \in \Sigma \subseteq \Sigma_T$, we can construct the $T(F)$-equivariant morphism
\[
\tilde{\varphi}_\Sigma : \mathcal{O}(T(\mathcal{A}_F), \mathcal{C}) \otimes V(k-2) \otimes \mathcal{H}^\infty_0(\mathcal{B}_\Sigma, \mathcal{C})^U \longrightarrow C^2(T(\mathcal{A}_F), \mathcal{C})
\]
where $\tilde{\varphi}_\Sigma((f \otimes \mu) \otimes \Phi)(z, t) = f(z, t) \cdot \Phi(s(z^{-1}\mu))(z, t)$, where $s : V(k-2) \longrightarrow \mathcal{B}_\Sigma$ is defined by $s$ at $\tau$, $\delta s$ at $\Sigma \setminus \tau$ and the identity elsewhere. We claim that $\tilde{\varphi}_\Sigma$ fits in the following commutative diagram of $T(F)$-representations
\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
C^0(T(\mathcal{A}_F), \mathcal{C}) & \otimes & V(k-2) \otimes \mathcal{H}^\infty_0(\mathcal{B}_\Sigma, \mathcal{C})^U \\
\rho' \downarrow & & \downarrow \phi_{\Sigma(1)} \\
C^0(T(\mathcal{A}_F), \mathcal{C}) & \otimes & V(k-2) \otimes \mathcal{H}^\infty_0(\mathcal{B}_\Sigma, \mathcal{C})^U \\
\delta_t \downarrow & & \downarrow \phi_{\Sigma} \\
C^0(T(\mathcal{A}_F), \mathcal{C}) & \otimes & V(k-2) \otimes \mathcal{H}^\infty_0(\mathcal{B}_\Sigma, \mathcal{C})^U \\
0 & \longrightarrow & 0
\end{array}
\]
where $t$ is the natural inclusion and the left column is the exact sequence (3.15). Indeed, the upper square commutes since
\[
\tilde{\varphi}_\Sigma(\rho'(f \otimes \mu \otimes \Phi))(z, t) = f(z, t) \cdot \rho'(\Phi(s(z^{-1}\mu)))(z, t) = f(z, t) \cdot \Phi(\delta s(z^{-1}\mu))(z, t),
\]
because $s$ is a section of $\rho$. On the other side, to prove that the lower square commutes, we have
\[
\delta_t \tilde{\varphi}_\Sigma((f \otimes \mu \otimes \Phi))(z, t) = f(z, t) \cdot \delta_t (\Phi(s(z^{-1}\mu)))(z, t) = f(z, t) \cdot \frac{d}{ds}(\Phi(s(e^{-s}z^{-1}\mu))(ze^s, t))_{s=0}.
\]
If we write $F(x, y) := \Phi(s(e^{-y}z^{-1}\mu))(ze^y, t)$, we obtain
\[
\frac{\partial F}{\partial x}(0, 0) = \delta_t^x \Phi(s(z^{-1}\mu))(z, t) = \Phi(\delta_t^x \Phi(s(z^{-1}\mu)))(z, t), \quad \frac{\partial F}{\partial y}(0, 0) = -\Phi(\delta_t^y \delta_t^x \mu))(z, t).
\]
Hence,
\[
\delta_t \tilde{\varphi}_\Sigma((f \otimes \mu \otimes \Phi))(z, t) = f(z, t) \cdot \left(\frac{\partial F}{\partial x}(0, 0) + \frac{\partial F}{\partial y}(0, 0)\right) = f(z, t) \cdot \left(\Phi(\delta s(z^{-1}\mu))(z, t)\right) = \varphi_{\Sigma}(f \otimes \mu \otimes \Phi)(z, t).
\]
We deduce the result from the commutativity of the above diagram and the definitions of $\partial$ and $\text{ES}_1$. □

3.4. Higher cohomology classes and global period integrals. As in §3.2, let us consider a locally polynomial character $\chi : T(\mathcal{A}_F)/T(F) \rightarrow \mathbb{C}^\times$ of degree at most $\frac{k-2}{2}$. Namely, a character such that, when restricted to a small neighborhood of $1$ in $T(F_\infty)$, it is of the form
\[
\chi(t_\infty) = t_\infty^m, \quad m \in \mathbb{Z}^\geq, \quad \frac{2-k}{2} \leq m \leq \frac{k-2}{2}.
\]
By means of the morphisms (1.4) and (1.6), we can see $\chi$ as an element $\chi \in H^0(T(F), C^0(T(\mathcal{A}_F), \mathcal{C}) \otimes V(k-2))$. Hence, given $\Phi \in H^0(G(F), \mathcal{H}^\infty_0(D(k), \mathcal{C})^U)$, we can consider the following cup product:
\[
\varphi(\chi \cup \text{ES}_1(\Phi)) \in H^0(T(F), \mathcal{O}(T(\mathcal{A}_F), \mathcal{C})).
\]

Theorem 3.9. Let $\chi : T(\mathcal{A}_F)/T(F) \rightarrow \mathbb{C}^\times$ be a locally polynomial character such that $\chi(t_\infty) = t_\infty^m \chi_0(t_\infty)$, for some $m \in \mathbb{Z}^\geq$, with $\frac{2k}{2} \leq m \leq \frac{k-2}{2}$ and some locally constant character $\chi_0$. Then we have that, for all $\Phi \in H^0(G(F), \mathcal{H}^\infty_0(D(k), \mathcal{C})^U)$,
\[
\varphi(\chi \cup \text{ES}_1(\Phi)) \cap \eta = h \cdot \ell(\Phi(\delta s(m))), \chi) \in \mathbb{C},
\]
where $h = \#T(\mathcal{O}_F)_{t_\text{tors}}$. 
Proof. By Proposition 3.8 and Proposition 2.3,
\[ \varphi(\chi \cup \text{ES}_3 \Phi) \cap \eta = \partial(\varphi_{\Sigma}(\chi \cup \Phi)) \cap \eta = h \int_{T(A_F)/\Gamma(F)} \varphi_{\Sigma}(\chi \otimes \Phi)(\tau)d\tau. \]

One can check using (1.4) and (1.6) that \( \chi \) corresponds to \( \chi_0 \otimes \mu_m \in C^0(T(A_F), \mathbb{C}) \otimes V(k - 2) \). Moreover, \( z^{-1}\mu_m = z\tilde{\varphi}_{\mu_m} \). Hence, for all \( z \in T(F_{\infty}), t \in T(\mathbb{A}^\infty) \),
\[ \varphi_{\Sigma}(\chi \otimes \Phi)(z, t) = \chi_0(z, t) \cdot \Phi(\tilde{\varphi}(z^{-1}\mu_m))(z, t) = z^m \cdot \chi_0(z, t) \cdot \Phi(\tilde{\varphi}(\mu_m))(z, t) = \chi(z, t) \cdot \Phi(\tilde{\varphi}(\mu_m))(z, t). \]

Thus, we obtain
\[ \varphi(\chi \cup \text{ES}_3 \Phi) \cap \eta = h \int_{T(A_F)/\Gamma(F)} \chi(\tau) \cdot \Phi(\tilde{\varphi}(\mu_m))(\tau)d\tau = h \cdot \ell(\Phi(\tilde{\varphi}(\mu_m)), \chi), \]

and the result follows.

3.5. Waldspurger formulas in higher cohomology. As in previous sections, we fix \( \lambda : G(F)/G(F)_+ \rightarrow \pm 1 \). Let \( \pi \) be an automorphic representation of \( G \) of even weight \( k \) (namely, \( \pi_{\infty} \simeq D(\xi) \) and conductor \( N \). Thanks to the Eichler-Shimizu morphism \( \text{ES}_3 \) explained in §3.1, we can realize \( \pi_{\infty} = \eta \cdot |G(\mathbb{A}^\eta)| \) in the space \( H^0(G(F), \mathcal{A}^{\infty}(V(k - 2)\langle \lambda \rangle)) \). Given a locally polynomial character \( \chi \) of degree at most \( \frac{k - 2}{2} \), we can consider
\[ \mathcal{P}(\cdot, \chi) : \pi_{\infty} \subset H^0(G(F), \mathcal{A}^{\infty}(V(k - 2)\langle \lambda \rangle)) \rightarrow \mathbb{C}, \quad \mathcal{P}(\phi, \chi) := \langle \phi \chi \rangle \cap \eta \in \mathbb{C}. \]

The complex number \( \mathcal{P}(\phi, \chi) \) can be seen as an analogy in higher cohomology of the periods \( \ell(f, \chi) \) introduced in §3.2. Furthermore, we can express \( \mathcal{P}(\phi, \chi) \) in terms of certain period \( \ell(f, \chi) \) thanks to Theorem 3.9. We will use this fact together with classical Waldspurger formulas introduced in §3.2 to obtain formulas analogous to that of proposition 3.3 involving \( \mathcal{P}(\phi, \chi) \). Moreover, we will use lemma 3.6 to make those formulas explicit. Hence, for a module \( \mathcal{O}_N \subset \mathcal{B} \) admissible for \( (N, \chi) \), we recall the space \( V(\pi_{\infty}, \chi) \) of definition 3.5, we consider its realization in \( H^0(G(F), \mathcal{A}^{\infty}(V(k - 2)\langle \lambda \rangle)) \), for \( U = \mathcal{O}_N \), and we fix non-zero elements \( \phi_{\eta}^{(1)} \in V(\pi_{\infty}, \chi) \) and \( \phi_{\eta}^{(2)} \in V(\pi_{\infty}, \chi^{-1}) \). Write \( L^2(1/2, \Pi, \chi) \) for the \( L \)-function with the local factors at places of a finite set \( S \) removed. The following result is a direct consequence of proposition 3.3, lemma 3.6 and theorem 3.9:

Theorem 3.10 (Waldspurger formula in higher cohomology). Let \( \chi : T(A_F)/T(F) \rightarrow \mathbb{C}^* \) be a locally polynomial character such that \( \chi(t_{\infty}) = \frac{m}{|m|} \chi_0(t_{\infty}) \), for some \( m \in \mathbb{Z}^2 \), with \( \frac{2-k}{2} \leq m \leq \frac{k-2}{2} \) and some locally constant character \( \chi_0 \). Given decomposable \( \phi_{\eta}^{(1)} = \otimes_{\mathcal{O}_N} \phi_{\eta}^{(1), \xi}, \phi_{\eta}^{(2)} = \otimes_{\mathcal{O}_N} \phi_{\eta}^{(2), \xi} \in \pi_{\infty} \subset H^0(G(F), \mathcal{A}^{\infty}(V(k - 2)\langle \lambda \rangle)) \), we have \( \mathcal{P}(\phi_{\eta}^{(1)}, \chi) = 0 \) unless \( \chi_0 = \lambda \) and the root number \( c(1/2, \pi_{\infty}, \chi) = \chi_{\infty} \eta_{\infty}(-1)c(B, \psi) \), for all \( v \not\in S \). In that case, if \( \phi_{\eta}^{(1)} \) differ from \( \phi_{\eta}^{(1)} \) in a finite set \( \mathcal{E} \) then
\[ \mathcal{P}(\phi_{\eta}^{(1)}, \chi) \cdot \mathcal{P}(\phi_{\eta}^{(2)}, \chi^{-1}) = 2^{\#S} \cdot L^2(1/2, \Pi, \chi) \cdot \langle \phi_{\eta}^{(1)}, \phi_{\eta}^{(2)} \rangle \cdot \frac{\text{vol}(U_0(N))}{\text{vol}(U)} \cdot \prod_{\omega \in \mathcal{E}} \langle \phi_{\omega}^{(1), \xi}, \phi_{\omega}^{(2), \xi} \rangle. \]

where \( S := \{ v \mid (N, Dc); \text{if} v \not\in S \} \), \( N \) then \( \text{ord}_c(c/N) \geq 0 \}, S_D := \{ v \mid (N, D); \text{ord}_c(c) < \text{ord}_c(N) \}, \Phi_i \in H^0(G(F), \mathcal{A}^{\infty}(D(k), C)) \) are such that \( \text{ES}_i(\Phi_i) = \phi_{\eta}^{(1)} \) and
\[ C(k, m) = (-1)^{\frac{k-2}{4}} \cdot (32\pi)^c \cdot \frac{1}{n} \cdot \prod_{\gamma \in \Delta} \Gamma\left(\frac{k}{2} - m_{\gamma} + m_{\gamma}^{(1)} + m_{\gamma}^{(2)}\right) \cdot \left(\frac{1}{2\pi n}\right)^{d-r} \cdot \prod_{v_{\infty}} \prod_{E_{\infty}} \Gamma\left(\frac{k}{2} - m_{v_{\infty}} + m_{v_{\infty}}^{(1)} + m_{v_{\infty}}^{(2)}\right). \]

4. Local archimedean theory

In this section we will study the local archimedean representations generated by an automorphic form of even weight \( k \in (2N)^{[F, K]} \).

4.1. The \((G, K)\)-modules (of discrete series) of even weight \( k \): Case \( \text{PGL}_2(\mathbb{R}) \). Let us consider \( \text{PGL}_2(\mathbb{R}) \) as a real Lie group. Its maximal compact subgroup \( K \) is the image of \( O(2) = \text{SO}(2) \ltimes \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \), where
\[ \text{SO}(2) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\} \subset \text{SL}_2(\mathbb{R}). \]

Notice that \( \text{SO}(2) \simeq S^1 \). For any \( \theta \in S^1 \) we will write
\[ \kappa(\theta) := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \]
Notice that any $g \in \text{GL}_2(\mathbb{R})^+$ admits a decomposition
\begin{equation}
(4.28) \quad g = u \begin{pmatrix} y & x \\ y^{-1} & 1 \end{pmatrix} \kappa(\theta), \quad y \in \mathbb{R}_+, \ u \in \mathbb{R}_+, \ x \in \mathbb{R}, \ \theta \in S^1.
\end{equation}

For any character $\chi : \mathbb{R}_+ \to \mathbb{C}^*$, let us consider the induced representation of $\text{PGL}_2(\mathbb{R})_+$
\[\mathcal{B}(\chi) := \left\{ f : \text{GL}_2(\mathbb{R})_+ \to \mathbb{C} : f\left( \begin{pmatrix} 1 & x \\ t_2 \\ t_2 \end{pmatrix} \right) = \chi(t_1/t_2) \cdot f(g) \right\}.
\]
By the decomposition (4.28), we have a correspondence
\begin{equation}
(4.29) \quad \mathcal{B}(\chi) \rightarrow I(\chi) = \{ f : S^1 \to \mathbb{C} : f(\theta + \pi) = f(\theta) \}.
\end{equation}

For any integer $n \in \mathbb{Z}$, let us consider the character $e^{in\theta} : S^1 \to \mathbb{C}^*$. Notice that the only characters appearing in $I(\chi)$ are those of the form $e^{2in\theta}$. The corresponding Lie algebra is $\mathcal{G}_R = \text{Lie}(\text{PGL}_2(\mathbb{R})) \cong \{ g \in M_2(\mathbb{R}), \ Trg = 0 \}$.

**Definition 4.1.** Write $I(\chi, n)$ for the subspace $\mathbb{C} e^{2in\theta}$ in $I(\chi)$, then we consider
\[\tilde{I}(\chi) := \bigoplus_{n \in \mathbb{Z}} I(\chi, n) \subseteq I(\chi).
\]
Finally, we write $\tilde{\mathcal{B}}(\chi) \subseteq \mathcal{B}(\chi)$ for the $(\mathcal{G}_R, \text{SO}(2))$-module whose image under the isomorphism (4.29) is $\tilde{I}(\chi)$.

**Proposition 4.2.** Assume that $\chi(t) = \chi_k(t) := t^k$, for an even integer $k \in 2\mathbb{Z}$. We have a morphism of $\text{GL}_2(\mathbb{R})_+$-representations:
\[\rho : \mathcal{B}(\chi_k) \to V(k-2); \quad \rho(f)(P) := \int_{S^1} f(\kappa(\theta))P(-\sin \theta, \cos \theta)d\theta
\]

**Proof.** The result follows from [3, Lemma 2.6.1] since $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \mapsto t_1/t_2$ is the modular quasicharacter and the function
\[h : \text{GL}_2(\mathbb{R})_+ \to \mathbb{C}, \quad h(g) = f(g) \cdot P(c, d) \cdot \det(g)^{\frac{k-2}{2}}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]
is the induced representation by the quasicharacter.

The image under (4.29) of the subspace $I(\chi_k, n)$ is generated in $\mathcal{B}(\chi_k)$ by the function
\begin{equation}
(4.30) \quad f_n \begin{pmatrix} y & x \\ y^{-1} \end{pmatrix} \kappa(\theta) = y^k \cdot e^{2in\theta}.
\end{equation}

Notice that $\tilde{\mathcal{B}}(\chi_k)$ has a natural structure of $(\mathcal{G}_R, \text{SO}(2))$-module. To extend $\tilde{\mathcal{B}}(\chi_k)$ to a $(\mathcal{G}_R, K)$-module one has to define an action of $w = (1, -1)$. We have two possibilities providing two extensions $\mathcal{B}(\chi_k)^\pm$, by defining $w(f_n) = \pm(-1)^{\frac{k-2}{2}} f_{-n}$ (see [13, §I.3]). It turns out that $D(k) := \tilde{\mathcal{B}}(\chi_k) \cap \ker(\rho)$ is the unique sub-$(\mathcal{G}_R, K)$-module for both $\tilde{\mathcal{B}}(\chi_k)^\pm$. We have the exact sequences of $(\mathcal{G}, K)$-modules:
\[0 \to D(k) \to \tilde{\mathcal{B}}(\chi_k)^+ \xrightarrow{\rho} V(k-2) \to 0,
\]
\[0 \to D(k) \to \tilde{\mathcal{B}}(\chi_k)^- \xrightarrow{\rho} V(k-2)(-1) \to 0,
\]
where $V(k-2)(-1)$ is the representation $V(k-2)$ twisted by the character $g \mapsto \text{sign} \det(g)$. Indeed,
\[\rho(w f_n)(P) = (-1)^{\frac{k-2}{2}} \rho(\pm f_{-n})(P) = (-1)^{\frac{k-2}{2}} \int_{S^1} e^{-2in\theta}P(-\sin \theta, \cos \theta)d\theta = (-1)^{\frac{k-2}{2}} \int_{S^1} e^{2in\theta}P(\sin \theta, \cos \theta)d\theta
\]
\[= \pm \int_{S^1} e^{2in\theta}(wP)(-\sin \theta, \cos \theta)d\theta = \pm(\omega(\rho(f_n))(P)).
\]
If we write $P_m(X, Y) := (Y + iX)^m(Y - iX)^{k-2-m} \in \mathcal{P}(k-2)$, for $0 \leq m \leq k-2$, we compute
\begin{equation}
(4.31) \quad \rho(f_n)(P_m) = \int_{S^1} e^{(k-2-2m+2n)\theta}d\theta = \begin{cases} 1, & n = m - \frac{k-2}{2} \\
0, & n \neq m - \frac{k-2}{2}, \end{cases}
\end{equation}
for the good normalization of the Haar measure of $S^1$. This implies that the kernel of $\rho$ is generated by $f_n$ with $|n| \geq \frac{k}{2}$, and we deduce that
\begin{equation}
(4.32) \quad 0 \to D(k) = \bigoplus_{|n| \geq \frac{k}{2}} I(\chi_k, n) \xrightarrow{\iota_{\chi_k}} \tilde{\mathcal{B}}(\chi_k)^+ = \bigoplus_{n \in \mathbb{Z}} I(\chi_k, n) \xrightarrow{\rho} V(k-2)(\pm) \to 0,
\end{equation}
where \( \iota_k(f_n) = (\pm 1)^{\lfloor \frac{k-n}{2} \rfloor} f_n \).

**Remark 4.3.** Notice that the morphism \( \rho \) and, hence, the exact sequences (4.32) depend on the choice of the Haar measure \( d \theta \). In this note we have chosen \( d \theta \) so that \( \text{vol}(S^1) = 1 \). Nevertheless, in [13] it is chosen so that \( \text{vol}(S^1) = \pi \), so the second morphism in the exact sequence is \( \pi \cdot \rho \) there.

4.1.1. Splittings of the \((G, K)\)-module exact sequences. In this section we aim to construct sections to the exact sequences (4.32) and their dual counterparts

\[
0 \rightarrow V(k-2)(\pm) \xrightarrow{\rho^\vee} \tilde{\mathcal{B}}(\chi_{2-k})^\vee \xrightarrow{\iota_k^\vee} D(k) \rightarrow 0,
\]

where

\[
\rho^\vee(\mu) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \mu \left( \begin{array}{cc} X & Y \\ c & d \end{array} \right)^{k-2} \cdot (ad - bc)^{\frac{k-2}{2}}.
\]

Of course such exact sequences do not split as \((G, O(2))\)-modules, but they split as \((O(2))\)-modules. Hence, there exist unique \((O(2))\)-equivariant sections of \( \rho \) and \( \iota_k^\vee \), namely, \((O(2))\)-equivariant morphisms

\[
s : V(k-2)(\pm) \rightarrow \tilde{\mathcal{B}}(\chi_k)^\vee, \quad s' : D(k) \rightarrow \tilde{\mathcal{B}}(\chi_{2-k})^\vee
\]
such that \( \rho \circ s = \text{id} \) and \( s' \circ \iota_k^\vee = \text{id} \).

We aim to write down \( s \) explicitly. Notice that as \( SO(2)\)-module \( V(k-2)(\pm) \cong \sum_{\frac{k-2}{2} \leq n \leq \frac{k+2}{2}} I(\chi_k, n) \). Thus, for any \( \mu \in V(k-2)(\pm) \), we must impose

\[
s(\mu) \in \sum_{\frac{k-2}{2} \leq n \leq \frac{k+2}{2}} I(\chi_k, n) \subset \tilde{\mathcal{B}}(\chi_k)^\vee, \quad s(\mu)(\kappa(\theta)) = \sum_{\frac{k-2}{2} \leq n \leq \frac{k+2}{2}} a_n(\mu) \cdot e^{2i n \theta}.
\]

Hence, the construction of \( s(\mu) \) is equivalent to finding coefficients \( a_n(\mu) \) for \( \frac{k-2}{2} \leq n \leq \frac{k+2}{2} \). Since \( \rho \circ s = \text{id} \) we have

\[
\mu(P) = \rho(s(\mu))(P) = \int s(\mu)(\kappa(\theta))P(-\sin \theta, \cos \theta)d\theta = \sum_{\frac{k-2}{2} \leq n \leq \frac{k+2}{2}} a_n(\mu) \int e^{2in\theta} P(-\sin \theta, \cos \theta)d\theta.
\]

Evaluating at the base \( P_m(X, Y), m = 0, \ldots, k-2 \), and using orthogonality we deduce that \( a_n(\mu) = \mu\left( P_n, \frac{k-2}{2} \right) \).

We conclude that

\[
s(\mu) = \sum_{\frac{k-2}{2} \leq n \leq \frac{k+2}{2}} \mu\left( P_n, \frac{k-2}{2} \right) f_n.
\]

4.1.2. Diagonal torus and splittings. Write \( \iota : T(\mathbb{R}) \hookrightarrow \text{PGL}_2(\mathbb{R}) \) for the split diagonal torus

\[
\iota(t) = \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right).
\]

Write also

\[
\mathcal{T}_\mathbb{R} := \text{Lie}(T(\mathbb{R})) = \mathbb{R} \delta \subset G_\mathbb{R}, \quad \delta := \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right),
\]
satisfying \( \exp(t \delta) = \iota(t^\delta) \). For the section \( s : V(k-2)(\pm) \rightarrow \tilde{\mathcal{B}}(\chi_k)^\vee \) provided in previous §4.1.1, let us consider

\[
\delta s : V(k-2)(\pm) \rightarrow \iota_k D(k) \subset \mathcal{B}(\chi_k)^\vee, \quad \delta s(\mu) := \delta(s(\mu)) - s(\delta\mu).
\]

Indeed \( \delta s(\mu) \) lies in \( \iota_k D(k) \) since \( \rho \circ \delta s = 0 \) because \( \rho \) is a morphism of \((G, K)\)-modules and \( s \) is a section of \( \rho \).

**Definition 4.4.** Let \( \mu_m \in V(k-2) \) be such that

\[
\mu_m \left( \begin{array}{cc} X & Y \\ x & y \end{array} \right)^{k-2} = x^{\frac{k-2}{2} - m} y^{\frac{k-2}{2} + m}, \quad \frac{2 - k}{2} \leq m \leq \frac{k - 2}{2}.
\]

It is easy to check that \( \iota(t) \mu_m = t^{-m} \mu_m \), for all \( t \in T(\mathbb{R}) \).

**Remark 4.5.** Notice that \( \mu_m \) corresponds to \( x^{\frac{k-2}{2} - m} y^{\frac{k-2}{2} + m} \) under the isomorphism (1.4).

**Proposition 4.6.** For any \( \frac{k-2}{2} \leq m \leq \frac{k+2}{2} \), we have that

\[
\delta s(\mu_m) = \frac{(k - 1)}{2} \left( (-i)^{\frac{k-2}{2}} f_{\frac{k-2}{2}} + i^{\frac{k-2}{2}} f_{-\frac{k-2}{2}} \right).
\]
Proof. We compute on the one hand,
\[
\delta \mu_m = \frac{d}{dx} \exp(x \bar{\delta}) \mu_m \mid_{x=0} = \frac{d}{dx} \iota(e^x) \mu_m \mid_{x=0} = \frac{d}{dx} e^{-mx} \mid_{x=0} \mu_m = -m \mu_m.
\]
On the other hand, by equation (4.33) we have
\[
s(\mu_m) = \sum_{\bar{m} \leq n} \mu_m \left( P_n + z \bar{z} \right) f_n.
\]
Notice that
\[
\left| \begin{array}{cc} X & Y \\ x & y \end{array} \right|^{k-2} = 2^{2-k} \left| \begin{array}{cc} Y + iX & Y - iX \\ x - yi & -iy - x \end{array} \right|^{k-2} = 2^{2-k} \sum_{j=0}^{k-2} \binom{k-2}{j} P_j(X, Y) z^j \bar{z}^{k-2-j},
\]
where \( z = (x + yi) \) and \( \bar{z} = (x - yi) \). This implies by definition that
\[
(4.34) \quad \frac{x^{\frac{1}{2}} - y^{\frac{1}{2}} + m}{2^{2-k}} = \sum_{\bar{m} \leq n} \left( \frac{k-2}{n + k-2} \right) \mu_m \left( P_n + z \bar{z} \right) z^n \bar{z}^{k-2-n}.
\]
On the other side, we compute
\[
\frac{x^{\frac{1}{2}} - y^{\frac{1}{2}} + m}{2^{2-k}} = (z + \bar{z}) \bar{m} \cdot (z - \bar{z}) \bar{m} \cdot i \bar{m} - m \sum_{\bar{m} \leq n} (z^n \bar{z}^m).
\]
where
\[
C(n) = \sum_{j} (-1)^{n+j+m} \left( \frac{k-2}{n + k-2} \right) \mu_m \left( P_n + z \bar{z} \right) z^n \bar{z}^{k-2-n}.
\]
Comparing both expressions, we deduce
\[
\mu_m(P_n + z \bar{z}) = i^{\frac{1}{2}} - m \mu_m C(n) \left( \frac{k-2}{n + k-2} \right)^{-1} = i^{\frac{1}{2}} - m \sum_{\bar{m} \leq n} (-1)^{n+j+m} \left( \frac{k-2}{n + k-2} \right) \mu_m \left( P_n + z \bar{z} \right) z^n \bar{z}^{k-2-n}.
\]
Since \( \iota(1) \mu_m = l^{-m} \mu_m \), we have that
\[
\delta s(\mu_m) = \delta(s(\mu_m)) + ms(\mu_m) = \sum_{\bar{m} \leq n} \mu_m(P_n + z \bar{z}) \left( \delta(f_n) + m f_n \right).
\]
It is easy to check that, for any \( \left( \begin{array}{ccc} x & y \\ m & n \end{array} \right) \in \text{GL}_2(\mathbb{Z})_+ \) of determinant \( \Delta \),
\[
f_n \left( \begin{array}{ccc} x & y \\ m & n \end{array} \right) = \frac{\Delta^*(m^2 + n^2)^{\frac{n}{2}}}{(mi + n)^{2n}}.
\]
This implies that,
\[
\delta f_n(\kappa(\theta)) = \frac{d}{dt} f_n(\kappa(\theta) \cdot \exp(it \bar{\delta})) \mid_{t=0} = \frac{d}{dt} f_n(\kappa(\theta) \cdot \iota(e^t)) \mid_{t=0} = \frac{d}{dt} f_n \left( -ie^t \sin(\theta) \cos(\theta) \right) \mid_{t=0} = e^{2n i \theta} \left( \frac{k}{2} + (2n - k) \sin^2 \theta + 2ni e^{i \theta} \sin \theta \right).
\]
Hence,
\[
\delta f_n = \frac{1}{2} \left( \frac{k}{2} + n \right) f_{n+1} + \frac{1}{2} \left( \frac{k}{2} - n \right) f_{n-1}.
\]
Thus,
\[
\delta s(\mu_m) = \sum_{\bar{m} \leq n} \mu_m(P_n + z \bar{z}) \left( \frac{1}{2} \left( \frac{k}{2} + n \right) f_{n+1} + \frac{1}{2} \left( \frac{k}{2} - n \right) f_{n-1} + m f_n \right).
\]
Recall that \( \delta s(\mu_m) \in \iota_x D(k) \), hence \( \delta s(\mu_m) \in I(\chi,\frac{k}{2}) \oplus I(\chi,\frac{k}{2}) \). This implies that all terms in \( f_n \) cancel out and the only terms that survive are
\[
\delta s(\mu_m) = \frac{1}{2} \mu_m(P_{k-2})(k-1) f_{\frac{k}{2}} + \frac{1}{2} \mu_m(P_{k})(k-1) f_{\frac{k}{2}}.
\]
Thus, it provides (the unique up to constant) PGL

Hence, if we consider the function by equation (4.1).

Notice that, by definition,

Remark 4.7. Notice that the isomorphism (1.4) gives rise to a different GL(2)-invariant pairing

Given \( \mu_m, \mu_{m'} \) as in Definition 4.4, we have that

By Schur’s Lemma both pairings \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_V \) must coincide up to constant. To compute such a constant let us consider \( (y^{k-2})^\vee \in V(k-2)(\pm) \), the element of the dual basis of \( \{ x^i y^{k-2-i} \} \), corresponding to \( y^{k-2} \). It is clear that

On the other side,

This implies that both pairings coincide, and relations (4.35) also hold for \( \langle \cdot, \cdot \rangle_V \).

Once constructed the morphism \( \delta s : V(k-2)(\pm) \to \iota_4 D(k) \). We aim to compute

where \( \chi \) is a locally constant character and \( d \chi^\vee \) is the measure of (2.12).

Notice that, by definition, \( \delta s(\mu_m) := \delta(s(\mu_m)) - s(\delta \mu_m) \). Moreover,

Hence, if we consider the function

Using the above computations

\[
\mu_m(P_0) = i^{\frac{k-2}{2} + m} \sum_j (-1)^{\frac{k-2}{2} + j + m} \binom{\frac{k-2}{2} - m}{j} \binom{\frac{k-2}{2} + m + j}{j} = i^{\frac{k-2}{2} + m},
\]

\[
\mu_m(P_{k-2}) = i^{\frac{k-2}{2} + m} \sum_j (-1)^{\frac{k-2}{2} + j + m} \binom{\frac{k-2}{2} - m}{j} \binom{\frac{k-2}{2} + m + j}{j} = i^{\frac{k-2}{2} - m},
\]

and the result follows. □

4.1.3. Inner products and pairings. Let us consider the PGL(2,\( \mathbb{R} \))-invariant pairing

It satisfies the property

Thus, it provides (the unique up to constant) PGL-invariant bilinear inner products on \( D(k) \) and \( V_{k-2}(\pm) \):

\[
\langle \cdot, \cdot \rangle : D(k) \times D(k) \to \mathbb{C} \quad \langle f, h \rangle := \int_{S^1} f(x(\theta)) \cdot h(x(\theta)) d\theta.
\]

\[
\langle \cdot, \cdot \rangle_V : V(k-2)(\pm) \times V(k-2)(\pm) \to \mathbb{C} \quad \langle \mu_1, \mu_2 \rangle_V := \langle s(\mu_1), \rho(\mu_2) \rangle_B.
\]

\[
\langle \mu_1, \mu_2 \rangle' = \mu_2 \mu_1 \left( \begin{array}{c} X_1 \\ X_2 \\ Y_1 \\ Y_2 \\ k-2 \end{array} \right).
\]

By Schur’s Lemma both pairings \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_V \) must coincide up to constant. To compute such a constant let us consider \( (y^{k-2})^\vee \in V(k-2)(\pm) \), the element of the dual basis of \( \{ x^i y^{k-2-i} \} \), corresponding to \( y^{k-2} \). It is clear that

\[
\langle \mu_{\frac{k-2}{2}}, (y^{k-2})^\vee \rangle = (y^{k-2})^\vee \mu_{\frac{k-2}{2}} \left( \begin{array}{c} x \\ y \\ k-2 \end{array} \right) = (y^{k-2})^\vee \left( y^{k-2} \right) = 1.
\]

On the other side,

\[
\langle \mu_{\frac{k-2}{2}}, (y^{k-2})^\vee \rangle_V = \int_{S^1} s \left( \mu_{\frac{k-2}{2}} \right) (x(\theta)) \cdot \rho(\mu_{\frac{k-2}{2}})(x(\theta)) d\theta
\]

\[
= \sum_{\frac{k-2}{2} \leq n \leq \frac{k-2}{2}} \mu_{\frac{k-2}{2}} \left( P_n+\frac{k-2}{2} \right) \int_{S^1} e^{2in\theta} \cdot (y^{k-2})^\vee \left( \begin{array}{c} x \\ -\sin \theta \\ \cos \theta \end{array} \right) d\theta
\]

\[
= \sum_{\frac{k-2}{2} \leq n \leq \frac{k-2}{2}} \mu_{\frac{k-2}{2}} \left( P_n+\frac{k-2}{2} \right) (2i)^{2k} \left( \begin{array}{c} k-2 \\ \frac{k-2}{2} - n \end{array} \right) (-1)^{\frac{k-2}{2} - n} = 1,
\]

by equation (4.34). This implies that both pairings coincide, and relations (4.35) also hold for \( \langle \cdot, \cdot \rangle_V \).
we compute
\[
\frac{d}{dt} F(t) = m t^m \langle t(t) s(\mu_m), s' \delta s(\mu_{-m}) \rangle_B + t^m \frac{d}{dx} \langle t(t e^x) s(\mu_m), s' \delta s(\mu_{-m}) \rangle_B \bigg|_{x=0} \\
= t^m \langle -\langle t(t) s(\delta \mu_m), s' \delta s(\mu_{-m}) \rangle_B + \langle t(t) \delta s(\mu_m)), s' \delta s(\mu_{-m}) \rangle_B \rangle_B \\
= t^m \langle t(t) \delta s(\mu_m), \delta s(\mu_{-m}) \rangle_B.
\]

**Proposition 4.8.** We have that
\[
F(\infty) - F(0^+) = (-1)^m 2^{k-1} (k-1) \left( \frac{k-2}{2} - m \right)^{-1}.
\]

where \( F(\ast) := \lim_{t \to \ast} F(t). \)

**Proof.** Notice that, since \( w(f_{\frac{k}{2}}) = \pm (-1)^{\frac{k-2}{2}} f_{\frac{k}{2}}, \) we have that \( s'(f_{\frac{k}{2}}) = C f_{\frac{k}{2}} \) and \( s'(f_{-\frac{k}{2}}) = C f_{-\frac{k}{2}} \) for some constant \( C. \) If we normalize so that \( \langle f_{\frac{k}{2}}, f_{-\frac{k}{2}} \rangle_B = 1, \) we deduce that \( C = 1. \) By Proposition 4.6, this implies that
\[
s' \delta s(\mu_{-m}) = \frac{(k-1)}{2} \left( (-i) \frac{k-2}{2} s_{\frac{k}{2}} + i \frac{k-2}{2} s_{-\frac{k}{2}} \right),
\]
where now \( f_{\pm \frac{k}{2}} \) are the corresponding functions in \( B(\chi_{2-k}). \) In fact, for any \( f_n \) in \( B(\chi_{2-k}), \)
\[
f_n \begin{pmatrix} x \\ r \\ s \\ s \\ \end{pmatrix} = \Delta \frac{s_{\frac{n}{2}} (s-r)^n s_{\frac{k}{2}}}{(s+r)^{n+1}}, \quad \Delta = (xs - ry) > 0,
\]
This implies that, if \( t > 0, \)
\[
t(l^{-1}) s' \delta s(\mu_{-m}(\chi(\theta))) = s' \delta s(\mu_{-m}) \left( \frac{t^{-1} \cos \theta}{t^{-1} \sin \theta} \sin \theta \cos \cos \theta \cos \theta \right) = \\
= \frac{(k-1)}{2} \left( (-i) \frac{k-2}{2} s_{\frac{k}{2}} + i \frac{k-2}{2} s_{-\frac{k}{2}} \right) \sum_{0 \leq j \leq \frac{k}{2}} \sum_{\frac{k}{2} + m \equiv j \mod 2} \binom{k}{j} \binom{1}{j} \frac{\cos \theta^{k-j} \sin \theta^j}{(\cos^2 \theta + t^{-1} \sin^2 \theta)}.
\]
Hence we have that \( F(t) = t^m \langle s(\mu_m), t(l^{-1}) s' \delta s(\mu_{-m}) \rangle_B \) is given by
\[
F(t) = (-1)^{\frac{k}{2}} (it) \frac{k-2}{2} + m (k-1) \sum_{0 \leq j \leq \frac{k-2}{2}} \binom{k}{j} \sum_{\frac{k-2}{2} + m \equiv j \mod 2} \int_{S^1} s(\mu_m)(\chi(\theta)) \left( \frac{(it)^{\frac{k-j}{2} \cos \theta}}{(\sin^2 \theta + t^{-1} \cos \theta)} \right) d\theta.
\]

If \( t \to 0^+ \), we have that \( (\sin^2 \theta + t \cos \theta)^{-1} = \sin^{-2} \theta \sum_{s \geq 0} (it \cos \theta^2 \sin \theta^{2s}), \) hence
\[
F(t) = (-1)^{\frac{k}{2}} (it) \frac{k-2}{2} + m (k-1) \sum_{0 \leq j \leq \frac{k-2}{2}} \binom{k}{j} \sum_{s \geq 0} C(j + 2s) (it)^{2s}
\]
where
\[
C(n) := \int_{S^1} s(\mu_m)(\chi(\theta)) (- \sin \theta)^{k-2-n} \cos \theta^n d\theta.
\]
When \( 0 \leq n \leq k-2 \), we have
\[
C(n) = \int_{S^1} s(\mu_m)(\chi(\theta)) \rho^n \left( \mu_{-\frac{k-2}{2}} \right) (\chi(\theta)) d\theta = \left( \mu_m, \mu_{\frac{k-2}{2}} \right)^n.
\]
Hence, by Remark 4.7 and equation (4.35), we have that \( C(n) = 0 \) except when \( -m = n - \frac{k-2}{2} \) where \( C(\frac{k-2}{2} - m) = (-1)^m \frac{1}{2} \left( \frac{k-2}{2} - m \right)^{-1} \). This implies that
\[
F(0^+) = (-1)^{m-1} (k-1) \left( \frac{k-2}{2} - m \right)^{-1} \sum_{s \geq 0} \binom{k-2}{2} \sum_{m=2s} \frac{1}{k-2 - m - 2s}.
\]

If \( t \to \infty \), we have that \( (\sin^2 \theta + t \cos \theta^{2s})^{-1} = (t \cos \theta)^{-2} \sum_{s \geq 0} (it \cos \theta^{2s})(\sin \theta)^{2s}, \) hence
\[
F(t) = -(-1)^{\frac{k}{2}} (it) \frac{k-2}{2} + m (k-1) \sum_{0 \leq j \leq \frac{k-2}{2}} \binom{k}{j} \sum_{s \geq 0} C(j - 2 - 2s)(it)^{-2-2s}.
\]
Using the same calculations we obtain
\[ F(\infty) = (-1)^m(k-1)\left(\frac{k-2}{k-2-m}\right)^{-1} \sum_{s \geq 0} \left(\frac{k-2}{k-2-m} + 2s + 2\right). \]

We conclude that
\[ F(\infty) - F(0^+) = \frac{(-1)^m(k-1)}{\left(\frac{k-2}{k-2-m}\right)} \sum_{s \geq 0} \left(\frac{k}{k-2-m} + 2s\right) = \frac{(-1)^m(k-1)}{\left(\frac{k-2}{k-2-m}\right)^{-1}} \left(1 + \frac{(1-1)^k}{(-1)^{k-m}}\right) = \frac{2^{k-1}(-1)^m(k-1)}{\left(\frac{k-2}{k-2-m}\right)}, \]
and the result follows. \( \square \)

We write \( \tilde{B}(\chi_k)^{\#}, \) for \( \epsilon = \pm 1. \) The following result computes the local integral we are looking for:

**Theorem 4.9.** Given a locally constant character \( \chi : T(\mathbb{R}) \to \pm 1, \) we consider
\[ I(\chi, m) := \int_{T(\mathbb{R})} \chi(t)(t(1)\delta s(\mu_m), \delta s(\mu_{-m}))t^md^x t, \]
where \( d^x t \) is the Haar measure \((2.12).\) Then
\[ I(\chi, m) = \begin{cases} (-1)^m2^k(k-1)\left(\frac{k-2}{k-2-m}\right)^{-1}, & \chi(1) = \epsilon, \\ 0, & \chi(-1) = \epsilon. \end{cases} \]

**Proof.** By the explicit description of \( \delta s(\mu_m) \) given in Proposition 4.6, we compute
\[ w\delta s(\mu_m) = \epsilon(-1)^m\delta s(\mu_m), \quad w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]
This implies that, for \( t < 0, \)
\[ \chi(t)(t(1)\delta s(\mu_m), \delta s(\mu_{-m}))t^m = \chi(-1)(t(-1)w\delta s(\mu_m), \delta s(\mu_{-m}))t^m = \chi(-1)e(t(-1)\delta s(\mu_m), \delta s(\mu_{-m}))(-t)^m. \]

Since \( d^x t = |t|^{-1}dt, \) we obtain
\[ I(\chi, m) = (1 + \chi(-1)e) \int_0^\infty t^m(t(1)\delta s(\mu_m), I_T\delta s(\mu_m))dt. \]
We have seen that \( t\frac{dF(t)}{dt} = t^m(t(1)\delta s(\mu_m), I_T\delta s(\mu_m)). \) Thus, by the above calculations,
\[ I(\chi, m) = (1 + \chi(-1)e) \int_0^\infty t \frac{dF(t)}{dt} dt = (1 + \chi(-1)e)(F(\infty) - F(0^+)) \]
hence the result follows from Proposition 4.8. \( \square \)

4.1.4. Pairings of canonical sections. Let us consider the canonical PGL\(_2(\mathbb{R})\)-invariant element
\[ \mathcal{Y} = t^{-1} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \in V(k-2) \otimes V(k-2), \]
where \( t : V(k-2) \otimes V(k-2) \rightarrow \mathcal{P}(k-2) \otimes \mathcal{P}(k-2) \) is the isomorphism of \((1.4).\) Since we have the morphism \( \delta s : V(k-2) \rightarrow D(k), \) we can consider \( \delta s(\mathcal{Y}) \in D(k) \otimes D(k) \) and apply the inner product \( \langle \cdot, \cdot \rangle \) defined in the beginning of \S 4.1.3. Similarly, we can apply the pairing \( \langle \cdot, \cdot \rangle \) of remark 4.7 to \( \mathcal{Y}. \)

**Lemma 4.10.** We have that
\[ \langle \delta s(\mathcal{Y}) \rangle = (k-1) \cdot 2^{k-3}. \]

**Proof.** On the one side, we have by remark 4.5 and proposition 4.6
\[ \langle \delta s(\mathcal{Y}) \rangle = \sum_{j=0}^{k-2} \binom{k-2}{j}(1)\binom{k-2-j}{1} \left(\delta s\left(\mu_{\frac{k-2-j}{2}}\right), \delta s\left(\mu_{\frac{k-2-j}{2}}\right)\right) \]
\[ = \frac{(k-1)^2}{4} \sum_{j=0}^{k-2} \binom{k-2}{j}(1)\binom{k-2-j}{1} \left(\delta s\left(\mu_{\frac{k-2-j}{2}}\right), \delta s\left(\mu_{\frac{k-2-j}{2}}\right)\right) \]
\[ = \frac{(k-1)^2}{2} \sum_{j=0}^{k-2} \binom{k-2}{j} = (k-1)^2 \cdot 2^{k-3}, \]
4. The \((G, K)\)-modules of even weight \(\mathfrak{k}\): Case \(\text{PGL}_2(\mathbb{C})\). Let us consider \(\text{PGL}_2(\mathbb{C})\) as a real Lie group. Its maximal compact subgroup \(K\) is the image of

\[
\text{SU}(2) := \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\} \subset \text{SL}_2(\mathbb{C}).
\]

Notice that \(\text{SU}(2) = S^3 := \{(a, b \in \mathbb{C}^2); |a|^2 + |b|^2 = 1\}. \) For any \((a, b) \in S^3\) we will write

\[
\kappa(a, b) := \begin{pmatrix} a \\ -\overline{b} \end{pmatrix}.
\]

Similarly as in \((4.28)\), any \(g \in \text{GL}_2(\mathbb{C})\) admits a decomposition

\[
g = u \begin{pmatrix} r & x \\ -\overline{x} & r^{-1} \end{pmatrix} \kappa(a, b), \quad u \in \mathbb{C}^\times, x \in \mathbb{C}, r \in \mathbb{R}^\times, (a, b) \in S^3.
\]

4.2.1. \(\text{Haar measure of } SU(2).\) Notice that \(S^3\) is the boundary of \(D \subset \mathbb{C}^2, \) where

\[
D = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 \leq 1\}.
\]

Moreover, if we consider \(\mathbb{C}^2\) with the standard right action of \(\text{GL}_2(\mathbb{C})\), then \(D\) and \(S^3\) are clearly \(SU(2)\)-invariants. The measure on \(D\) given by \(dz_1dz_2d\overline{z}_1d\overline{z}_2\) is \(SU(2)\)-invariant. Indeed, the action of \((\kappa, b)\) provides a change of variables given by the same matrix \(\kappa(a, b)\), thus the action on the differential \(dz_1dz_2\) (and \(d\overline{z}_1d\overline{z}_2\)) is given by the determinant of \(\kappa(a, b)\), which is 1. If we perform the change of variables

\[
z_1 = R \cos \theta e^{i\alpha}, \quad z_2 = R \sin \theta e^{i\beta}, \quad \overline{z}_1 = R \cos \theta e^{-i\alpha}, \quad \overline{z}_2 = R \sin \theta e^{-i\beta},
\]

where \(\alpha, \beta \in [0, 2\pi], R \in [0, 1]\) and \(\theta \in [0, \pi/2]\), the corresponding Jacobian determinant is

\[
\det \begin{pmatrix} \cos \theta e^{i\alpha} & -R \sin \theta e^{i\alpha} \\ \sin \theta e^{i\beta} & R \cos \theta e^{i\beta} \\ \cos \theta e^{-i\alpha} & -R \sin \theta e^{-i\alpha} \\ \sin \theta e^{-i\beta} & R \cos \theta e^{-i\beta} \end{pmatrix} = -R^3 \sin 2\theta.
\]

Notice that any \(f \in C^\infty(S^3, \mathbb{C})\) can be seen as a function of \(D\) that is constant in \(R\), therefore

\[
\int_D f(z_1, z_2)dz_1dz_2d\overline{z}_1d\overline{z}_2 = -\int_0^1 \int_{S^3} \int_0^{\pi/2} \sin 2\theta \cdot f\left(\cos \theta e^{i\alpha}, \sin \theta e^{i\beta}\right)d\theta d\alpha R^3 dR
\]

\[
= -\frac{1}{4} \int_{S^3} \int_0^{\pi/2} \sin 2\theta \cdot f\left(\cos \theta e^{i\alpha}, \sin \theta e^{i\beta}\right)d\theta d\alpha.
\]

From the \(SU(2)\)-invariance of \(dz_1dz_2d\overline{z}_1d\overline{z}_2\), we deduce that

\[
\int_{S^3} f(a, b)d(a, b) := \int_{S^3} \int_0^{\pi/2} \sin 2\theta \cdot f\left(\cos \theta e^{i\alpha}, \sin \theta e^{i\beta}\right)d\theta d\alpha d\beta
\]

is a Haar measure for \(SU(2).\)
Lemma 4.12. We have that
\[
I_{n_1, n_2, m_1, m_2} := \int_{S^3} a^{n_1} b^{m_1} \tilde{a}^{n_2} \tilde{b}^{m_2} d(a, b) = \begin{cases} \frac{1}{n_1 + m_1} (\frac{n_1 + m_1}{n_1})^{-1}, & n_1 = n_2, \ m_1 = m_2, \\ 0, & \text{otherwise.} \end{cases}
\]

Proof. By the above calculations
\[
I_{n_1, n_2, m_1, m_2} = 2 \int_{S^1} \int_{S^1} \int_0^{\pi/2} \sin \theta^{1 + m_1 + m_2} \cos \theta^{1 + n_1 + n_2} (n_1 - n_2) \tilde{a}^{n_1 - m_2} \tilde{b} d\theta d\tilde{a} d\tilde{b},
\]
hence clearly \(I_{n_1, n_2, m_1, m_2} = 0\) unless \(m_1 = m_2\) and \(n_1 = n_2\). If this is the case, the integral can be solved by parts using a simple induction:
\[
I_{n_1, n_2, m_1, m_2} = 2 \int_0^{\pi/2} \sin \theta^{1 + 2m_1} \cos \theta^{1 + 2n_1} d\theta = \frac{1}{n_1 + m_1 + 1} \left( \frac{n_1 + m_1}{n_1} \right)^{-1},
\]
hence the result follows. \(\square\)

4.2.2. Quotients of Principal series. Let us consider the finite dimensional \(\mathbb{C}\)-representation of \(\text{SU}(2)\):
\[
\mathcal{P}(n) := \text{Sym}^n(\mathbb{C}^2) = \{P \in \mathbb{C}[x, y], P(ax, ay) = a^n P(x, y)\}, \quad V(n) = \mathcal{P}(n)^\vee,
\]
with action
\[
(\kappa(a, b) P)(x, y) := P((x, y)\kappa(a, b)) = P(ax - \tilde{b}x, bx + \tilde{a}y), \quad (a, b) \in S^3.
\]
Notice that by (1.4) we have \(\mathcal{P}(n) = V(n)\). By [9], these are all the irreducible representations of \(\text{SU}(2)\). Thus the irreducible representations of the compact subgroup of \(\text{PGL}_2(\mathbb{C})\) are the representations \(V(2n)\), where \(n \in \mathbb{N}\).

For any character \(\chi : \mathbb{C}^\times \rightarrow \mathbb{C}^\times\), let us consider the induced \(\text{PGL}_2(\mathbb{C})\)-representation
\[
\mathcal{B}(\chi) := \left\{ f : \text{GL}_2(\mathbb{C}) \rightarrow \mathbb{C} : f \left( \begin{pmatrix} t_1 & x \\ t_2 & y \end{pmatrix} \right) g = \chi(t_1/t_2) \cdot f(g) \right\}.
\]
By equation (4.36) we have a correspondence
\[
\mathcal{B}(\chi) \overset{\sim}{\rightarrow} \left\{ f : S^3 \rightarrow \mathbb{C} : f(e^{i\theta} a, e^{i\theta} b) = \chi(e^{2i\theta}) \cdot f(a, b) \right\}.
\]

Let us consider the embedding \(S^1 \hookrightarrow S^3, e^{i\theta} \mapsto (e^{i\theta}, 0)\). By (4.37), the \(\text{SU}(2)\)-representation \(\mathcal{B}(\chi)\) is induced by the character \(\chi\) of \(S^1\). By Frobenius reciprocity,
\[
\text{Hom}_{\text{SU}(2)}(V(2n), \mathcal{B}(\chi)) \overset{\sim}{\rightarrow} \mathcal{P}(2n)(\chi) := \{ P \in \mathcal{P}(2n), \kappa(e^{2i\theta}, 0) P = \chi(e^{-2i\theta}) \cdot P \}
\]
If we write \(\chi(e^{i\theta}) = e^{i\lambda\theta}\), we can easily compute the subspace \(\mathcal{P}(2n)(\chi)\):
\[
\mathcal{P}(2n)(\chi) = \{ P \in \mathcal{P}(2n), P(e^{i\theta} x, e^{-i\theta} y) = e^{-2i\lambda\theta} \cdot P(x, y) \} = \mathbb{C} x^{-\lambda} y^{n+\lambda}.
\]
This implies that
\[
\text{Hom}_{\text{SU}(2)}(V(2n), \mathcal{B}(\chi)) = \left\{ \begin{array}{ll} \mathbb{C}, & |\lambda| \leq n \\ 0, & \text{otherwise.} \end{array} \right\}
\]

Definition 4.13. Assuming that \(\chi(e^{i\theta}) = e^{i\lambda\theta}\), we will fix \(\varphi_n \in \text{Hom}_{\text{SU}(2)}(V(2n), \mathcal{B}(\chi))\) for all \(n \geq |\lambda|\) to be the following morphism:
\[
\varphi_n(\mu)(a, b) := \mu \left( \begin{array}{cc} \alpha & \beta \\ \tilde{b} & \tilde{a} \end{array} \right) \begin{cases} x \ y \\ n+\lambda \\ \tilde{a} \ x \ y \ n-\lambda \end{cases}.
\]

Let us consider the Lie algebra \(\mathcal{G}_C = \text{Lie}(\text{PGL}_2(\mathbb{C})) \cong \{ g \in M_2(\mathbb{C}), \quad \text{Tr} g = 0 \}\).

Definition 4.14. Write \(\mathcal{B}(\chi, n)\) for the image of \(V(2n)\) through \(\varphi_n\). Hence the subspace
\[
\hat{\mathcal{B}}(\chi) := \bigoplus_{n \geq |\lambda|} \mathcal{B}(\chi, n) \subseteq \mathcal{B}(\chi)
\]
is a natural \((\mathcal{G}_C, K)\)-module.
Proposition 4.15. Let $\Sigma$ be the set of $\mathbb{R}$-isomorphisms $\sigma : \mathbb{C} \rightarrow \mathbb{C}$, and let $k = (k_0)_{0 \in \Sigma} \in (2\mathbb{N})^2_+$. Assume that $\chi_k(t) = \prod_{0 \in \Sigma} \sigma(t)^{\frac{1}{k_0}}$. We have a morphism of $\text{GL}_2(\mathbb{C})$-representations

$$\rho : \mathcal{B}(\chi_{k}) \rightarrow V(k - 2) = \bigotimes_{0 \in \Sigma} V(k_0 - 2);$$

where $\text{GL}_2(\mathbb{C})$ acts on $V(k_0 - 2)$ by means of $\sigma$.

Proof. Let us consider $\delta$ the modular quasicharacter

$$\delta : P := \left\{ \left( \begin{array}{cc} r & x \\ r^{-1} & 1 \end{array} \right), r \in \mathbb{R}_{>0}, x \in \mathbb{C} \right\} \rightarrow \mathbb{R}, \quad \delta \left( \begin{array}{cc} r & x \\ r^{-1} & 1 \end{array} \right) = r^4.$$ 

Thus, the function $h : \text{GL}_2(\mathbb{C}) \rightarrow \mathbb{C}$

$$h(g) = f(g) \left( \prod_{0 \in \Sigma} \mathcal{P}_{0}(\sigma(c), \sigma(d)) \sigma(\det(g))^{\frac{a+b}{2m}} \right), \quad g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),$$

is in the induced representation by $\delta$. Since $\sigma(a, b)$ is the Haar measure of SU(2), the result follows from [3, Lemma 2.6.1].

Write $D(k) := \mathcal{B}(\chi_k) \cap \ker(\rho)$. It is the unique sub-$\langle \mathcal{G}_c, \mathbb{K} \rangle$-module of $\mathcal{B}(\chi_k)$ and, by definition, we have the exact sequence of $\langle \mathcal{G}_c, \mathbb{K} \rangle$-modules:

$$(4.40) \quad 0 \rightarrow D(k) \xrightarrow{i} \mathcal{B}(\chi_k) \xrightarrow{\rho} V(k - 2) \rightarrow 0.$$ 

Notice that, in this case, $\lambda = \frac{k_0 - k}{2}$, where $c$ is complex conjugation. If we write

$$\chi^m y^{2n-m} \in \bigotimes_{0 \in \Sigma} \mathcal{P}_{0}(k_{-2}) := \bigotimes_{0 \in \Sigma} \mathcal{P}(k_{0} - 2), \quad m = (m_0) \leq k_{-2},$$

and we consider $\chi^{2n} \in V(2n)$, the element of the dual basis of $\{\chi^m y^{2n-m}\}_m$ corresponding to $\chi^{2n}$, one obtains that

$$\rho(\varphi_\sigma((\chi^{2n})^\vee))(\chi^m y^{2n-m}) = \int_{\mathbb{R}} \mathcal{P}_{0}(\sigma(c), \sigma(d)) \sigma(\det(g))^{\frac{a+b}{2m}} d(a, b)$$

Thus by Lemma 4.12,

$$(4.41) \quad \rho(\varphi_\sigma((\chi^{2n})^\vee))(\chi^m y^{2n-m}) = \begin{cases} \frac{(-1)^m a + m b}{n + \frac{n m a - m b}{2}} (\chi^{k_0 - k_{-2}})^{-1}, \quad n = m - d - c - \frac{k_0 - k_{-2}}{2}, \\ 0, \quad n \neq m - d - c - \frac{k_0 - k_{-2}}{2}, \end{cases}$$

Since $\kappa(a, b)(\chi^{2n})^\vee$ generates $V(2n)$ and $0 \leq m_0 \leq k_0 - 2$, we deduce that

$$0 \rightarrow D(k) = \bigoplus_{n \geq \frac{k_0 - k_{-2}}{2}} \mathcal{B}(\chi_k, n) \rightarrow \mathcal{B}(\chi_k) = \bigoplus_{n \geq \frac{k_0 - k_{-2}}{2}} \mathcal{B}(\chi_k, n) \rightarrow V(k - 2) \rightarrow 0.$$ 

4.2.3. Splittings of the $\langle \mathcal{G}, \mathbb{K} \rangle$-module exact sequences. As in §4.1.1, we can construct SU(2)-equivariant sections of the exact sequence constructed previously and its dual counterpart. Indeed, dual to the exact sequence (4.40) we have

$$0 \rightarrow V(k - 2) \xrightarrow{\rho^\vee} \mathcal{B}(\chi_{2-k}) \xrightarrow{i^\vee} D(k) \rightarrow 0,$$

where $\chi_{2-k}(t) = \prod_{0 \in \Sigma} \sigma(t)^{\frac{1}{k_0}}$ and

$$\rho^\vee(\mu) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \mu \left( \begin{array}{cc} X & Y \\ c & d \end{array} \right)^{\frac{k - 2}{2}}, \quad (ad - bc)^{\frac{k - 2}{2}}.$$ 

Here we mean by $(ad - bc)^{\frac{k - 2}{2}} := \prod_{0 \in \Sigma} \sigma(ad - bc)^{\frac{1}{k_0}}$ and

$$\left( \begin{array}{cc} X & Y \\ c & d \end{array} \right)^{\frac{k - 2}{2}} := \bigotimes_{0 \in \Sigma} \left( \begin{array}{cc} X_\sigma & Y_\sigma \\ \sigma(c) & \sigma(d) \end{array} \right)^{\frac{k - 2}{2}} \in \mathcal{P}(k - 2).$$
**Definition 4.16.** Write $M := \frac{\Sigma (k_n - 2)}{2}$ and $\lambda := \frac{k_n - k_c}{2}$. Notice that as a $SU(2)$-representation $V(k - 2) \cong \bigoplus_{|\lambda| \leq n \leq M} V(2n)$. If we assume that indeed $|\lambda| \leq n \leq M$, the integer numbers $r_1 := n + \lambda$, $r_2 := n - \lambda$ and $r_3 := M - n$ are positive. In this situation we say that the even integers $k_n - 2$, $k_c - 2$ and $2n$ are balanced. Hence we can consider the $SU(2)$-equivariant morphism

$$t_n : V(k - 2) = V(k_n - 2) \otimes V(k_c - 2) \longrightarrow \mathcal{P}(2n) \cong V(2n), \quad t_n(\mu_i \otimes \mu_c) = \mu_i \mu_c(\Delta_n),$$

where $\Delta_n$ is as described: The unicity of $\Delta_n$ and the result follows.

The polynomial $\Delta$ is called the Clebsch-Gordan element (see [7, §3] for other situations where such element naturally appears). This morphism is unique up to constant.

**Lemma 4.17.** There exist unique $SU(2)$-equivariant sections of $\rho$ and $\iota'$. Namely, $SU(2)$-equivariant morphisms

$$s : V(k - 2) \longrightarrow \mathcal{B}(\chi), \quad s' : D(k) \longrightarrow \mathcal{B}(\chi_{2-\lambda})$$

such that $\rho \circ s = id$ and $s' \circ \iota' = id$. More precisely,

$$s(\mu) = \sum_{|\lambda| \leq n \leq M} (2n + 1) \cdot \frac{2n}{n + \lambda} \cdot \varphi_n \circ t_n(\mu).$$

**Proof.** The decomposition of $\mathcal{B}(\chi)$ as a direct sum of irreducible $SU(2)$-representations and the resultant decomposition of the subrepresentations $V(k - 2)$ and $D(k)$ imply the existence of $s$ and $s'$. Let us show that $s$ is as described: The unicity of $\varphi_n$ and $t_n$ implies the existence of $C_n \in \mathbb{C}$ such that

$$s = \sum_{|\lambda| \leq n \leq M} C_n \cdot \varphi_n \circ t_n.$$

Let us fix $n$ in the range $[|\lambda|, M]$. Assume that $\mu \in V(k - 2)$ is such that $t_n(\mu) = (x^{2n})^\lor$ and $t_n(\mu) = 0$ for $n' \neq n$. Hence by relation (4.41), for any $m$ with $m_i - m_{-c} - \lambda = n$, we have

$$\mu(x^{2i}y^{k_c - 2}z^{-2m}) = \rho \circ s(\mu)(x^{2i}y^{k_c - 2}z^{-2m}) = C_n \cdot \rho \circ \varphi_n((x^{2n})^\lor)(x^{2i}y^{k_c - 2}z^{-2m}) = C_n \frac{(-1)^{m_i + m_{-c}}}{n + M + 1} \cdot \frac{n + M}{m_i + 1} \cdot \frac{1}{(m_i + 1)!}.$$

and $\mu(x^{2i}y^{k_c - 2}z^{-2m}) = 0$ if $m_i - m_{-c} - \lambda \neq n$. We compute that

$$t_n(\mu) = \mu \left( \sum_{i,j,s} r_1 i - j - s \right) \left( \frac{r_2}{r_3} \right) \frac{r_3}{r_1} \frac{x^{i+j}y^{r_1+r_s+i+j}x^{r_2-j}y^{r_3+j}y^{2n-i}x^{i+j}y^{2n-i}(-1)^{i+j+s}}{r_2 \cdot r_3 \cdot r_1 \cdot r_{-3} \cdot r_{-2} \cdot r_0} \cdot \frac{n + M}{r_2 \cdot r_3 \cdot r_1 \cdot r_{-3} \cdot r_{-2} \cdot r_0}.$$
Definition 4.18. For any \(\frac{2-k}{k+2} \leq m = (m_0, m_1) \leq \frac{k-2}{2}\), let \(\mu_m \in V(k-2)\) be such that
\[
\mu_m \left( \begin{array}{ccc}
X & Y & k-2 \\
x & y & \end{array} \right) = \begin{vmatrix}
X & Y & k-2 \\
x & y & \end{vmatrix} = \prod_{o} X_o Y_o^{k-o-2}.
\]
It can be checked analogously as in §4.1.2 that \(\iota(t)\mu_m = t^{-m}\mu_m\).

Remark 4.19. Similarly as in remark 4.5, \(\mu_m\) corresponds to \(x^{k-m} y^{k+m}\) under the isomorphism \(V(k-2) \cong \mathcal{P}(k-2)\) induced by (1.4).

Proposition 4.20. For any \(\mu \in V(k-2)\) we have that
\[
\delta s(\mu) = -2 \left( \frac{2M}{k_{id} - 2} \right) \varphi_{M+1}(t_M(\mu)^*),
\]
where \(t_M(\mu)^* \in V(2M + 2)\) is given by \(t_M(\mu)^*(P) := t_M(\mu) \left( \frac{\partial P}{\partial x y} \right)\).

Proof. Notice that, for any \((\alpha, \beta) \in \text{GL}(2, \mathbb{C})\) of determinant \(\Delta\),
\[
\varphi_n(\mu)(\alpha, \beta) :=\begin{vmatrix}
\mu(P_n) & \\
(r^2 + |s|^2)M+n+2 &
\end{vmatrix}
\]
Hence,
\[
\delta \varphi_n(\mu)(\alpha, \beta) = \frac{d}{dr} \varphi_n(\mu)\left(\kappa(\alpha, \beta)i(c')\right) \left|_{r=0} \right. = \frac{d}{dr} \left. \left( \frac{c'(M+2)\mu\left( (a y - e' \beta x)^{n+1} (a y - e' \bar{\beta} y - \bar{a} x)^{n-1} \right)}{(1 - e' \bar{\beta} y - a x)^{n+2}} \right) \right|_{r=0}
\]
where in the last equality we have used the identity \((\bar{\beta} y + \bar{a} x)\beta = y - (a y - \beta x)\bar{a}\). If we use \((\bar{\beta} y + \bar{a} x)\alpha = x + (a y - \beta x)\bar{a}\) as well, we obtain
\[
\frac{\partial P_n}{\partial x}(-\bar{\beta}, \bar{a}) = \left( (\Delta - n)(a y - \beta x)\bar{a} - (n + \lambda)(a y - \beta x)\bar{a} \right) P_{n-1}(-\bar{\beta}, \bar{a}) = \left( (\Delta - n)(n + \lambda)(a y - \beta x)\bar{a} - (n + \lambda)(a y - \beta x)\bar{a} \right) \nu_{n-1}(-\bar{\beta}, \bar{a})
\]
Thus,
\[
\frac{\partial^2 P_{n+1}(-\bar{\beta}, \bar{a})}{\partial x \partial y} = \left( (-n + \lambda)(n + \lambda)(n + 2n + 1)(n + 1)(a y - \beta x)\bar{a} - (n + \lambda)(n + 2n + 1)(a y - \beta x)\bar{a} \right) \nu_{n-1}(-\bar{\beta}, \bar{a})
\]
We deduce
\[
\delta \varphi_n(\mu)(\alpha, \beta) = \frac{(\lambda + 1)(n + 2n + 1)(n + 1)(n + 2n + 1)(n + 1)(a y - \beta x)\bar{a} - (n + \lambda)(n + 2n + 1)(a y - \beta x)\bar{a}}{n} \mu_{n-1}(-\bar{\beta}, \bar{a})
\]
Therefore, \(\delta \varphi_n(\mu) = \varphi_n(\mu_0)^* + \varphi_n(\mu_1)^* + \varphi_n(\mu_{-1})^*\), for \(\mu_0 \in V(2n), \mu_1 \in V(2n + 2)\) and \(\mu_{-1} \in V(2n - 2)\), where
\[
\mu_0(P) := \frac{\lambda (M + 1)}{2n(n + 1)} \left( \frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} \right), \quad \mu_1(P) := \frac{(M + n + 2)}{2(n + 1)(n + 2)} \left( \frac{\partial P}{\partial x} \right),
\]
\[
\mu_{-1}(P) = \frac{(n + \lambda)(n + 2n + 1)}{n} \left( \frac{\partial P}{\partial x} \right).
\]
Recall that \( \rho(\delta s(\mu)) = 0 \), hence \((1 - \rho)\delta s(\mu) = \delta s(\mu)\). But \((1 - \rho)(V(2n)) = 0 \) for all \( |\lambda| \leq n \leq M \). We conclude

\[
\delta s(\mu) = \sum_{|\lambda| \leq n \leq M} (2n + 1) \left( \frac{2n}{n + \lambda} \right) \\
	imes \left( \varphi_n(t_n(\mu_\lambda)) + \varphi_{n+1}(t_{n+1}(\mu_\lambda)) + \varphi_{n-1}(t_{n-1}(\mu_\lambda)) - \varphi_n(t_n(\delta \mu_\lambda)) \right)
\]

and the result follows. \(\square\)

4.2.5. Inner products and pairings. Let us consider the \( \text{PGL}_2(\mathbb{C}) \)-invariant pairing

\[
\langle \cdot, \cdot \rangle_B : \mathcal{B}(\chi_1) \times \mathcal{B}(\chi_2) \to \mathbb{C}; \quad (f, h)_B := \int_{\Gamma(\mathfrak{g})} f(\chi(\alpha, \beta)) \cdot h(\chi(\alpha, \beta)) d\alpha(\beta).
\]

It satisfies the property

\[
\langle if, \rho(\cdot)\mu \rangle_B = 0, \quad \text{for all } f \in D(\mathfrak{k}), \mu \in V(\mathfrak{k} - 2).
\]

Again it provides (the unique up to constant) \( \text{PGL}_2(\mathbb{C}) \)-invariant bilinear inner products on \( D(\mathfrak{k}) \) and \( V(\mathfrak{k} - 2) \):

\[
\langle \cdot, \cdot \rangle : D(\mathfrak{k}) \times D(\mathfrak{k}) \to \mathbb{C}; \quad (f, h) := \langle if, \rho(\cdot)\mu \rangle_B,
\]

\[
\langle \cdot, \cdot \rangle : V(\mathfrak{k} - 2) \times V(\mathfrak{k} - 2) \to \mathbb{C}; \quad (\mu_1, \mu_2) := \langle s(\mu_1), \rho(\cdot)\mu_2 \rangle_B.
\]

As in previous sections, we aim to compute as well

\[
I(\chi; x) := \int_{\Gamma(\mathfrak{g})} \chi(t) (t(\delta s(\mu_m), \delta s(\mu_{-m}))) t^m d^s t,
\]

for a locally constant character \( \chi \) and the Haar measure \( d^s t \) of (2.12). Since \( (t(\delta s(\mu_m), \delta s(\mu_{-m}))) t^m d^s t = m \cdot \delta s(\mu_m) \),

where \( m := m_{id} + m_c \). Hence, let us consider again the function

\[
F(t) = t(t(\delta s(\mu_m), \delta s(\mu_{-m}))) t^m d^s t.B
\]

By \( \text{SU}(2) \)-invariance, \( s'(\varphi_n(\mu)) = C_n \varphi_n(\mu) \) for all \( \mu \in V(2n) \). Let us normalize \( s' \) so that \( C_{M+1} = 1 \). Thus by Proposition 4.20, we have

\[
s' \delta s(\mu_{-m}) = -2 \left( \frac{2M}{k_{id} - 2} \right) \varphi_{M+1}(t_M(\mu_{-m})).
\]

Hence,

\[
s' \delta s(\mu_{-m}) \begin{pmatrix} R & S \\ R & S \end{pmatrix} = -2 \left( \frac{2M}{k_{id} - 2} \right) \varphi_{M+1}(t_M(\mu_{-m})) \begin{pmatrix} R & S \\ R & S \end{pmatrix} \Delta^2, \quad \Delta := \det \left( \begin{pmatrix} S & -R \\ x & y \end{pmatrix} \right)^{k_{id}-1}, \quad \Delta^2 = \det \left( \begin{pmatrix} S & -R \\ x & y \end{pmatrix} \right)^{k_{id}-1}.
\]

Remark 4.21. The isomorphism \( \iota : V(2n) \to \mathcal{P}(2n) \) of (1.4) provides a \( \text{GL}_2(\mathbb{C}) \)-equivariant pairing

\[
\langle \cdot, \cdot \rangle_{\mathcal{P}} : \mathcal{P}(2n) \times \mathcal{P}(2n) \to \mathbb{C},
\]

where \( (P_1, P_2)_{\mathcal{P}} = \iota^{-1}(P_1)(P_2) \). In the usual basis

\[
\begin{pmatrix} x^i y^{2n-i} \\ x^i y^{2n-i} \end{pmatrix}_{\mathcal{P}} = \begin{cases} (-1)^{i(2n-1)} & i = 2n - j \\
0 & i \neq 2n - j \end{cases}
\]

Remark 4.22. Similarly as in previous remark 4.21, the isomorphism \( \iota : V(k-2) \to \mathcal{P}(k-2) \) provides a \( \text{GL}_2(\mathbb{C}) \)-equivariant pairing

\[
\langle \cdot, \cdot \rangle : V(k - 2) \times V(k - 2) \to \mathbb{C},
\]

such that, for \( \mu_m, \mu_{m'} \) as in Definition 4.18,

\[
\langle \mu_m, \mu_{m'} \rangle = \begin{cases} (-1)^{M+m} \begin{pmatrix} k_{id} - 2 \\ k_{id} - m_{id} \end{pmatrix}^{-1} \begin{pmatrix} k_{id} - 2 \\ m_{id} - m_{id} \end{pmatrix}^{-1}, & m = -m' \\
0, & m \neq -m' \end{cases}
\]

(4.44)
On the other side, we have defined another pairing \( \langle \ , \ \rangle_V \) on \( V(k-2) \). By Schur’s lemma both pairings must be the same up to constant. To compute such a constant we can consider the elements \( \mu_1 := \mu_i \), where \( \mu_i := (\frac{2i-1}{2}, \frac{2i+1}{2}) \), and \( \mu_2 \in V(k-2) \) to be such that \( t_n(\mu_2) = 0 \) for \( n \neq M \) and \( t_M(\mu_2) = (x^{2M})^\vee \). In this situation

\[
\langle \mu_2, \mu_1 \rangle_V = \mu_2 \mu_1 \left( \begin{array}{cc}
X' & Y' \\
X & Y
\end{array} \right)^{\frac{k-2}{2}} = \rho \cdot s(\mu_2)(X_{id}^{k-2}y_c^{k-2})
\]

where \( \rho = (2M + 1) \left( \begin{array}{cc}
M & \lambda \\
M + \lambda
\end{array} \right) \). On the other side, we compute

\[
\left( \begin{array}{cc}
\alpha & \beta \\
\alpha & \beta
\end{array} \right)^{M+\lambda} \left( \begin{array}{cc}
\alpha & \beta \\
\alpha & \beta
\end{array} \right)^{M-\lambda} = (-\vec{a})^{k-2} \alpha^{k-2} d(\alpha, \beta)
\]

Thus both pairings coincide and \( \langle \ , \ \rangle_V \) by (4.41). On the other side, we compute

\[
\langle \mu_2, \mu_1 \rangle_V = \int_{S^3} s(\mu_2)(\kappa(\alpha, \beta)) \cdot \mu_1 \left( \begin{array}{cc}
X' & Y' \\
X & Y
\end{array} \right)^{\frac{k-2}{2}} d(\alpha, \beta)
\]

by Lemma 4.12. Thus both pairings coincide and (4.44) also holds for \( \langle \ , \ \rangle_V \).

**Proposition 4.23.** We have that

\[
F(0) - F(0) = (-1)^{n+M} 2(2M + 1)(2M + 2) \left( \begin{array}{cc}
M & \lambda \\
M + \lambda
\end{array} \right)^{-1} \left( \begin{array}{cc}
k_{id} - 2 \\
k_c - 2
\end{array} \right)^{-1}.
\]

where again \( F(\ast) = \lim_{t \to \ast} F(t) \).

**Proof.** By remark 4.21 we have that

\[
t_M(\mu_i)(P) = \langle \mu_i(\Delta_M), P \rangle_p = (-1)^{\frac{2i-1}{2} - m_i} \langle y^{M-m_i+m_i} x^M+m_i-m_i, P \rangle_p,
\]

for all \( P \in \mathcal{P}(2M) \). Hence again by remark 4.21, if we write \( \bar{m} := m_i - m_c \),

\[
t_M(\mu_i)^\prime \left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right)^{k-1} \left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right)^{k^\prime - 1} = t_M(\mu_i)^\prime \left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right)^{2M+2} \left( \begin{array}{cc}
k_{id} - 2 \\
k_c - 2
\end{array} \right)^{\frac{k-3}{2}} \left( \begin{array}{cc}
k_{id} - 1 \\
k_{id} - 1
\end{array} \right)^{M+\bar{m} - i} A^{i+1} B^{i-2} C^{M+i} D^{M+i} +
\]

\[
\sum_i (AD + BC)(k_i - 1)(k_{id} - 1) \left( \begin{array}{cc}
k_{id} - 2 \\
k_c - 2
\end{array} \right) \left( \begin{array}{cc}
k_{id} - 1 \\
k_{id} - 1
\end{array} \right) A^{i+1} B^{i-2} C^{M+i} D^{M+i} +
\]

Thus, by equation (4.43), the function \( F(t) = t_M(\mu_i), t(t^{-1})^2 s(\mu_i) \) is given by

\[
F(t) = -2 \left( \begin{array}{cc}
2M \\
k_{id} - 2
\end{array} \right) \int_{S^3} s(\mu_i)(\kappa(\alpha, \beta)) \left( \begin{array}{cc}
\alpha & \beta \\
\alpha & \beta
\end{array} \right)^{k-1} \left( \begin{array}{cc}
\alpha & \beta \\
\alpha & \beta
\end{array} \right)^{k^\prime - 1} \left( \begin{array}{cc}
\alpha & \beta \\
\alpha & \beta
\end{array} \right)^{M+\bar{m} - i} d(\alpha, \beta)
\]

by Lemma 4.12. Thus both pairings coincide and (4.44) also holds for \( \langle \ , \ \rangle_V \).
where $k_M := \frac{2(-1)^{k-2}m_d(2M+1)(2M+2)(2M+3)}{(M+m+1)}$.

If $r \ll 0$, we use \( \left( r^2 |\alpha|^2 + |\beta|^2 \right)^{-1} = |\beta|^{-2} \sum_{j \geq 0} (-r^2 |\alpha|^2 |\beta|^{-2})^j \) to obtain

\[
F(t) = k_M \sum_i \binom{k_c - 1}{i} \binom{k_d - 1}{i} \sum_{j \geq 0} r^{2m_c - k_c + 2j} p(i + j)
\]

\[
= k_M \sum_{N \geq 0} p(N) \cdot r^{2m_c - k_c + 2N + 2} \sum_{i \leq N} \binom{k_c - 1}{i} \binom{k_d - 1}{i}
\]

where

\[
p(N) := \int s(\mu_m)(k(\alpha, \beta)) \cdot \alpha^N(-\beta)^{k_c - 2 - N} \sum_{\lambda \geq 0} (N + \lambda - \tilde{m})(-\tilde{\beta})^{M - \tilde{m} - N} d(\alpha, \beta)
\]

By definition, for any \( m' = (m'_d, m'_c) \),

\[
\rho^\gamma(\mu_m)(k(\alpha, \beta)) = \mu_m \left( \begin{array}{c|c}
X & Y \\
\tilde{\alpha} & \tilde{\beta}
\end{array} \right) \left( \begin{array}{c}
k_c - 2 \\
-\tilde{\beta}
\end{array} \right) = (-\tilde{\beta})^{-m'_d} \tilde{\alpha}^{-m'_c} \cdot m(-\beta)^{k_c - 2 - m'_c} \cdot \alpha^{-m'_d}.
\]

Hence, if we assume \( N \leq \frac{k_c - 2}{2} - m_c \) then we have that

\[
2 - \frac{k}{2} \leq m_N := \left( N - \frac{k_c - 2}{2} - \tilde{m}, N - \frac{k_c - 2}{2} \right) \leq \frac{k - 2}{2}.
\]

Moreover, by remark 4.22,

\[
p(N) := (-1)^{N + \lambda + \tilde{m}} \langle s(\mu_m), \rho^\gamma(\mu_m) \rangle_\beta = (-1)^{N + m} \langle \mu_m, \mu_m \rangle_\nu.
\]

Hence, we conclude that

\[
F(0) = k_M(-1)^{m + m_d} \langle \mu_m, \mu_m \rangle_\nu \sum_{i \geq \frac{k_c - 2}{2} - m_c} \binom{k_c - 1}{i} \binom{k_d - 1}{i}
\]

If $r > 0$, we use \( \left( r^2 |\alpha|^2 + |\beta|^2 \right)^{-1} = r^{-2} |\alpha|^{-2} \sum_{j \geq 0} (-r^2 |\beta|^2 |\alpha|^{-2})^j \) to obtain

\[
F(t) = -k_M \sum_i \binom{k_c - 1}{i} \binom{k_d - 1}{i} \sum_{j \geq 0} r^{2m_c - k_c + 2j} p(i - j - 1)
\]

\[
= -k_M \sum_{N \leq k - 2} p(N) \cdot r^{2m_c - k_c + 2N + 2} \sum_{i \geq N + 1} \binom{k_c - 1}{i} \binom{k_d - 1}{i}
\]

Again for \( N \geq \frac{k_c - 2}{2} - m_c \), we check that \( \frac{2 - k}{2} \leq \tilde{m}_N \leq \frac{k_c - 2}{2} \). Similarly as above we obtain

\[
F(\infty) = -k_M(-1)^{m + m_d} \langle \mu_m, \mu_m \rangle_\nu \sum_{i \geq \frac{k_c - 2}{2} - m_c} \binom{k_c - 1}{i} \binom{k_d - 1}{i}
\]

We conclude again by remark 4.22 that

\[
F(\infty) - F(0) = k_M(-1)^{m + m_d} \langle \mu_m, \mu_m \rangle_\nu \sum_i \binom{k_c - 1}{i} \binom{k_d - 1}{i}
\]

\[
= k_M(-1)^{m + m_d} \langle \mu_m, \mu_m \rangle_\nu \frac{2M + 2}{M - \tilde{m} + 1} = (-1)^m \langle \mu_m, \mu_m \rangle_\nu \frac{2M + 2}{M - \tilde{m} + 1}
\]

and the result follows.

\[\square\]

**Theorem 4.24.** Given a locally constant character \( \chi : T(\mathbb{C}) \to \mathbb{C}^\times \), we denote

\[
I(\chi, m) := \int_{T(\mathbb{C})} \langle \chi(t), (\delta s(\mu_m), \delta s(\mu_m)) \rangle \ell^m d^\times t,
\]

where \( d^\times t \) is the Haar measure (2.12). Then

\[
I(\chi, m) = \begin{cases} (-1)^{M + m} s(2M + 1)(2M + 2) \frac{2M}{k_d - 2} \left( \frac{k_d - 2}{k_d - 2 - m_d} \right)^{-1} \left( \frac{k_d - 2}{k_d - 2 - m_d} \right)^{-1}, & \chi = 1, \\
0, & \chi \neq 1. \end{cases}
\]
Proof. Using polar coordinates $t = re^{i\theta}$, the function $F(t)$ satisfies
\[
\frac{r}{dr}F(t) = m \left[ i(t) (s_\mu(t s_\mu))_B + \frac{d}{dx}i(t x^\ast) (s_\mu(t s_\mu))_B \right]_{x=0} = 1 \left[ -i(t s_\mu), s' \delta s_\mu \right]_B + \left[ i(t s_\mu), s' \delta s_\mu \right]_B = t^{-m} (i(t) \delta s_\mu, s' \delta s_\mu).
\]
Hence,
\[
l(\chi, \mu) = \frac{2}{\pi} \int_0^{2\pi} \chi(\theta) \int_0^\infty \left( \frac{\partial}{\partial r} F(t) \right) dr d\theta = \frac{2}{\pi} \int_0^{2\pi} \chi(\theta) \int_0^\infty \frac{r}{dr} F(t) dr d\theta = \left\{ \begin{array}{ll} 4F(\infty) - F(0), & \chi = 1, \\ 0, & \chi \neq 1, \end{array} \right.
\]
and the result follows from proposition 4.23. \hfill \square

4.2.6. Pairings of canonical sections. Let us consider the canonical $GL_2(\mathbb{C})$-invariant element
\[
\Upsilon = \iota^{-1} \left| \begin{array}{cc} X_1 & Y_1 \\ X_2 & Y_2 \end{array} \right|^{k_2} \in V(k - 2) \otimes V(k - 2),
\]
where $\iota : V(k - 2) \otimes V(k - 2) \to \mathcal{P}(k - 2) \otimes \mathcal{P}(k - 2)$ is the isomorphism of (1.4) and $x_2 = (x_{11}, x_{12}, x_{21}, x_{22})$. Since we have the morphism $\delta s : V(k - 2) \to D(k)$, we can consider $\delta s(\Upsilon) \in D(k) \otimes D(k)$ and apply the inner product $\langle \cdot, \cdot \rangle$ defined in the beginning of §4.2.5.

**Lemma 4.25.** We have that
\[
\langle \delta s(\Upsilon) \rangle = \frac{2}{3} \left( \begin{array}{c} 2M \\ k_{id} - 2 \end{array} \right) (2M + 1)^2 (2M + 2)^2 (2M + 1)^2.
\]

**Proof.** By proposition 4.20 and equation (4.43),
\[
\langle \delta s(\Upsilon) \rangle = 4 \left( \begin{array}{c} 2M \\ k_{id} - 2 \end{array} \right) \langle \mathcal{P}(1)(m(\Upsilon^\ast)) \rangle_B,
\]
where $\mathcal{P}(1) : V(2M + 2) \otimes V(2M + 2) \to \mathcal{B}(\chi_2) \times \mathcal{B}(\chi_2^\ast)$. In order to control $m(\Upsilon^\ast)$ we compute its image through the isomorphism $\iota$ of (1.4):
\[
m(\Upsilon^\ast)(x_1, y_1, x_2, y_2) := m(\Upsilon^\ast) \left( \begin{array}{cc} X_1 \\ X_2 \end{array} \right), X_1 \begin{array}{c} Y_1 \\ Y_2 \end{array} \begin{array}{c} 2M + 2 \\ 2M \end{array} \begin{array}{c} X_2 \\ X_1 \end{array} \begin{array}{c} 2M + 2 \\ 2M \end{array},
\]
where $\mathbb{z}_1 = (x_1, y_1), \mathbb{w}_1 = (y_1, -x_1), \mathbb{z}_2 = (x_2, y_2)$ and $\mathbb{w}_2 = (y_2, -x_2)$. To compute $\langle \mathcal{P}(1)(m(\Upsilon^\ast)) \rangle_B$, notice that
\[
\langle \mathcal{P}(1)(\cdot), \mathcal{P}(1)(\cdot) \rangle_B \in \text{Hom}_{SU(2)}(V(2M + 2) \otimes V(2M + 2), \mathbb{C}) = \text{Hom}_{SU(2)}(V(2M + 2), V(2M + 2)) = \mathbb{C},
\]
by Schur lemma, but remark 4.21 provides a rather simpler $SU(2)$-equivariant pairing $\langle \cdot, \cdot \rangle$. Thus, both pairings must coincide up to constant. To find such a constant we use $\mu_1 = (y_{2M + 1})^\ast, \mu_2 = (x_{2M + 1})^\ast \in V(2M + 2)$ corresponding to $x_2^{2M + 2}, y_2^{2M + 2} \in \mathcal{P}(2M + 2)$, respectively. We compute
\[
\langle x_2^{2M + 2}, y_2^{2M + 2} \rangle = 1,
\]
\[
\langle \mathcal{P}(1)(\cdot), \mathcal{P}(1)(\cdot) \rangle = \int_{s^3} \mu_1 \left( \begin{array}{c} x \\ y \end{array} \right) \left( \begin{array}{c} 0 \\ a \end{array} \right) d(a, b) = -\frac{1}{2M + 3} \left( \begin{array}{c} 2M + 2 \\ k_{id} - 1 \end{array} \right).
Hence, we deduce
\begin{equation}
(4.45) \quad \langle \varphi_{M+1}(\cdot), \varphi_{M+1}(\cdot) \rangle_B = \frac{-1}{2M + 3} \left( \frac{2M + 2}{k_{id} - 1} \right)^{-1} \langle \cdot, \cdot \rangle_p.
\end{equation}
This implies that
\[
\frac{\langle \varphi_{M+1}(t_{M}(Y)^\prime) \rangle}{(2M + 2)^2(2M + 1)^2} = \frac{-1}{2M + 3} \left( \frac{2M + 2}{k_{id} - 1} \right)^{-1} \langle t_{M}(Y)^\prime \rangle_p
\]
\[
= \frac{1}{2M + 3} \left( \frac{2M + 2}{k_{id} - 1} \right)^{-1} \sum_{i=0}^{2M} \left( \frac{2M}{i} \right) \langle x_1^{i+1} y_1^{M-i+1}, (x_2)^{2M-i+1} y_2^{i+1} \rangle_p
\]
\[
= \frac{1}{2M + 3} \left( \frac{2M + 2}{k_{id} - 1} \right)^{-1} \sum_{i=0}^{2M} \left( \frac{2M}{i} \right) \left( \frac{2M + 2}{i + 1} \right)^{-1} = \frac{1}{6} \left( \frac{2M + 2}{k_{id} - 1} \right)^{-1}.
\]
Hence, the result follows.

Another interesting element to consider is $\delta s(\mu_\omega)$, where $\Omega = (0,0)$. Since $s$ is SU(2)-equivariant, $\delta$ is in the Lie algebra of the diagonal torus $T(\mathbb{C})$ and $\mu_\omega$ is invariant under the action of $T(\mathbb{C})$, we deduce that $\delta s(\mu_\omega)$ generates the subspace of $D(k)$ invariant under $\kappa(e^{i\theta},0) \in$ SU(2) with minimal weight.

**Lemma 4.26.** We have that
\[
\langle \delta s(\mu_\omega), \delta s(\mu_\omega) \rangle = 4(-1)^M(2M + 2)^2(2M + 1)^2(2M + 3)^{-1} \left( \frac{2M + 2}{k_{id} - 1} \right)^{-1} \left( \frac{2M}{k_{id} - 1} \right)^{-2} \left( \frac{2M + 2}{M + 1} \right)^{-1}
\]

**Proof.** By proposition 4.20 and equation (4.45),
\[
\langle \delta s(\mu_\omega), \delta s(\mu_\omega) \rangle = 4 \left( \frac{2M}{k_{id} - 2} \right)^2 \langle \varphi_{M+1}(t_{M}(\mu_\omega)^\prime), \varphi_{M+1}(t_{M}(\mu_\omega)^\prime) \rangle_B = -\frac{4(-k_{id}^{-2})^{-1} (2M)^2}{2M + 3} \langle t_{M}(\mu_\omega)^\prime, t_{M}(\mu_\omega)^\prime \rangle_p.
\]
Moreover,
\[
\langle t_{M}(\mu_\omega)^\prime \rangle^\prime = t_{M}(\mu_\omega)^\prime ((Xy - Yx)^{2M+2}) = (2M + 2)(2M + 1)t_{M}(\mu_\omega) (-xy(Xy - Yx)^{2M})
\]
\[
= -xy(2M + 2)(2M + 1)t_{M}(\mu_\omega) ((Xy - Yx)^{2M})
\]
\[
= -xy(2M + 2)(2M + 1)\mu_\omega \left[ \begin{array}{cc} X_{id} & Y_{id} \\ x & y \end{array} \right] k_{id}^{-2} \left[ \begin{array}{cc} X_{id} & Y_{id} \\ x & y \end{array} \right] k_{id}^{-2}
\]
\[
= (-1)^{M+1}(2M + 2)(2M + 1)x^{M+1}y^{M+1}.
\]
Thus,
\[
\langle t_{M}(\mu_\omega)^\prime, t_{M}(\mu_\omega)^\prime \rangle_p = (-1)^{M+1}(2M + 2)^2(2M + 1)^2 \left( \frac{2M + 2}{M + 1} \right)^{-1},
\]
and the result follows.

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