EXISTENCE AND STABILITY OF STANDING WAVES FOR COUPLED NONLINEAR HARTREE TYPE EQUATIONS

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Abstract. We study existence and stability of standing waves for coupled nonlinear Hartree type equations

\[-i \frac{\partial}{\partial t} \psi_j = \Delta \psi_j + \sum_{k=1}^{m} (W \ast |\psi_k|^p) |\psi_j|^{p-2} \psi_j,\]

where \(\psi_j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}\) for \(j = 1, \ldots, m\) and the potential \(W : \mathbb{R} \to [0, \infty)\) satisfies certain assumptions. Our method relies on a variational characterization of standing waves based on minimization of the energy when \(L^2\) norms of component waves are prescribed. We obtain existence and stability results for two and three-component systems and for a certain range of \(p\). In particular, our argument works in the case when \(W(x) = |x|^{-\alpha}\) for some \(\alpha > 0\).

1. Introduction

The Pekar energy functional

\[\mathcal{P}(\phi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi(x)|^2 \, dx - \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\phi(x)|^2 |\phi(y)|^2}{|x-y|} \, dxdy\]

arises from an approximation to the Hartree-Fock theory for one component plasma as discussed in Lieb’s paper [15]. Here \(\phi\) represents the wave function of the electron. For the energy functional of the electronic wave function, it is natural to impose the normalization constraint that \(\int_{\mathbb{R}^3} |\phi|^2 \, dx\) be held constant. The minimizer of the problem of minimizing \(\mathcal{P}(\phi)\) under the normalization condition solves the equation

\[-\Delta \phi + \lambda \phi = \left( \int_{\mathbb{R}^3} \frac{|\phi(y)|^2}{|x-y|} \, dy \right) \phi, \quad \int_{\mathbb{R}^3} |\phi|^2 \, dx = M > 0, \quad (1.1)\]

where \(\lambda\) is the Lagrange multiplier. Depending on the context of the application, the equation (1.1) is also called the Choquard equation or Schrödinger-Newton equation. The theory for nonlinear Choquard equation and its variants is fairly well developed in the mathematics literature by now, though there are still many interesting open questions. A complete survey of available results goes beyond the scope of this paper; we only refer the interested reader to [8, 15, 18, 20, 22]. The theory for coupled systems of such equations is much less developed, though they, too, arise as models for a variety of physical phenomena. Considered herein are the coupled systems of nonlinear Schrödinger
equations with nonlocal interaction in the form

\[- \Delta \phi_j + \lambda_j \phi_j = \sum_{k=1}^{m} (W \ast |\phi_k|^p) |\phi_j|^{p-2} \phi_j \text{ in } \mathbb{R}^N, \quad 1 \leq j \leq m, \tag{1.2}\]

where \(*\) denotes the convolution operator and \(W : \mathbb{R}^N \to [0, \infty)\) is the convolution potential satisfying certain assumptions (see below). The information about the properties of the system (1.2) does not change with the time and it is said to be in a stationary state.

By a solution of (1.2) we mean a pair consisting of a function \((\phi_1, \ldots, \phi_m)\) in the space \(Y_m = (H^1(\mathbb{R}^N))^m\) and \(\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m\) solving the system (1.2). (Here \(H^1(\mathbb{R}^N)\) denotes the \(L^2\)-based Sobolev space of complex-valued functions on \(\mathbb{R}^N\).) Solutions \((\phi_1, \ldots, \phi_m; \lambda)\) of (1.2) can be obtained as critical points of the functional

\[I(\phi_1, \ldots, \phi_m) = \frac{1}{2} \sum_{j=1}^{m} \int_{\mathbb{R}^N} |\nabla \phi_j|^2 \, dx - \frac{1}{2p} \sum_{k,j=1}^{m} \int_{\mathbb{R}^N} (W \ast |\phi_k|^p) |\phi_j|^p \, dx \tag{1.3}\]

subject to the constraints that \(\int_{\mathbb{R}^N} |\phi_j|^2 \, dx, \quad 1 \leq j \leq m,\) be held constants. In other words, the nonlocal Schrödinger system (1.2) arises as the Euler-Lagrange equations for the problem of finding

\[I^{(m)}_{M_1, \ldots, M_m} = \inf \left\{ I(\phi) : \phi = (\phi_1, \ldots, \phi_m) \in Y_m, \quad \int_{\mathbb{R}^N} |\phi_j|^2 \, dx = M_j, \quad 1 \leq j \leq m \right\}. \tag{1.4}\]

The unknown \(\lambda_j\) in the system (1.2) appear as Lagrange multipliers. Given any solution \((\phi_1, \ldots, \phi_m; \lambda)\) of (1.2), the functions \(\psi_j : \mathbb{R}^N \times (0, \infty) \to \mathbb{C}\) defined by \(\psi_j(x, t) = e^{-i\lambda_j t} \phi_j(x)\) depends on the time explicitly and the wave function \((\psi_1, \ldots, \psi_m)\) is called a standing wave for time-dependent Schrödinger system with nonlocal nonlinearities

\[-i \frac{\partial}{\partial t} \psi_j = \Delta \psi_j + \sum_{k=1}^{m} (W \ast |\psi_k|^p) |\psi_j|^{p-2} \psi_j, \quad 1 \leq j \leq m. \tag{1.5}\]

Systems of the form (1.5) are also called nonlinear Hartree like systems. Motivation for the theoretical studies of coupled nonlinear Schrödinger equations or Hartree equations comes with the recent remarkable experimental advances in multi-component Bose-Einstein condensates (3). As pointed out in (16 19), nonlinear Hartree type systems with the Coulomb potential \(W(x) = |x|^{-1}\) are also used as models to describe the interaction between electrons in the Hartree-Fock theory in Quantum Chemistry. The interaction between electrons is said to be repulsive (resp. attractive) when the sign in front of the interaction terms in the Hamiltonian is positive (resp. negative). Systems of the form considered in this paper arise as models for a variety of physical situations in which quantum particles interactive attractively. Examples include boson stars, systems of polaron in a lattice, and some Bose gases. For a discussion of how the Hartree type equation appears as a mean-field limit for many-particle boson systems, the reader may consult (13 14 25). The two-component nonlinear Hartree type systems with \(W(x) = \delta(x)\) (the delta function) has applications especially in nonlinear optics (23 24). Nonlocal nonlinearities have attracted considerable interest as means of eliminating collapse and stabilizing multidimensional solitary waves, as was shown in the context of optics (14). It appears naturally in optical systems (21) and is also known to influence the propagation
of electromagnetic waves in plasmas ([6]). In the theory of Bose-Einstein condensation, nonlocality accounts for the finite-range many-body interaction ([12]).

The purpose of this paper is twofold. First, we prove the precompactness of minimizing sequences for two-parameter variational problem $I_{M_1, M_2}^{(2)}$. As a consequence we obtain existence and stability of two-parameter family of standing waves for coupled nonlinear Hartree equations. Another purpose of this paper is to generalize the arguments to establish the precompactness of minimizing sequences for the three-parameter problem $I_{M_1, M_2, M_3}^{(3)}$. This leads to results concerning existence and stability of true three-parameter family of standing waves for coupled nonlinear Hartree equations. To our knowledge, this is the first paper which establishes existence and stability of standing waves for 3-coupled Hartree type systems under three independent normalization constraints.

The key to our analysis is the concentration compactness lemma of P. L. Lions (Lemma I.1 of [20]). For single nonlinear dispersive evolution equations in which the variational problems characterizing standing waves take the form

$$\text{minimize } A(u) = \int_{\mathbb{R}^N} A(u(x), \nabla u(x)) \, dx \quad \text{s.t.} \quad \int_{\mathbb{R}^N} |u|^2 \, dx = M > 0,$$

the concentration compactness technique is widely used for proving the relative compactness of minimizing sequences (and hence the stability of the set of minimizers provided that both the energy $A$ and the mass functional are conserved by the flow associated to the evolution equation, see [11]). Quite differently from the one-parameter case, its application for showing the relative compactness of minimizing sequences of variational problems under two or more constraint parameters, however, seems to be more complicated. In particular, putting the method into practice requires ruling out the case which Lions called dichotomy by establishing certain strict inequality for the function of constraint parameter(s). For one-parameter variational problems, as stated in Lions’ paper [20], preventing dichotomy is equivalent to verifying the strict inequality in the form

$$I_M < I_T + I_{M-T}, \forall T \in [0, M), \quad \text{(1.6)}$$

where $I_M$ denotes the infimum of $A$ over $\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = M\}$. In [11], J. Albert has illustrated the method by proving the strict inequality in a slightly different form

$$I_{M_1+M_2} < I_{M_1} + I_{M_2}, \forall M_1, M_2 > 0. \quad \text{(1.7)}$$

More recently, the method of preventing dichotomy of minimizing sequences for two-parameter variational problems was developed in [2] (see also [7]). In order to employ strategies of [2] for the problem $I_{M_1, M_2}^{(2)}$, one requires to verify the strict inequality

$$I_{M_1+T_1, M_2+T_2}^{(2)} < I_{M_1, M_2}^{(2)} + I_{T_1, T_2}^{(2)} \quad \text{(1.8)}$$

for all $M = (M_1, M_2), T = (T_1, T_2) \in \mathbb{R}_+^2 \cup \{0\}$ satisfying $M, T \neq \{0\}$ and $M + T \in \mathbb{R}_+^2$. (Here $\mathbb{R}_+$ denotes the interval $(0, \infty)$ and $\mathbb{R}^2_+ = \mathbb{R}_+ \times \mathbb{R}_+$.) While several techniques are available to prove the strict inequality for one-parameter problems, the proof of strict inequality for two-parameter problems such as (1.8), even for the most universal choice
of coupling terms, is much less understood. Furthermore, when one generalizes the strict inequality \((1.8)\) for \(m\)-parameter problem \(I^{(m)}_{M_1,\ldots,M_m}\), it takes the form
\[
I^{(m)}_{M_1+T_1,\ldots,M_m+T_m} < I^{(m)}_{M_1,\ldots,M_m} + I^{(m)}_{T_1,\ldots,T_m}
\]
and one requires to verify \((1.9)\) for all possible cases based on the values
\[M = (M_1,\ldots,M_m), T = (T_1,\ldots,T_m) \in \mathbb{R}_+^m \cup \{0\}, M, T \neq \{0\} M + T \in \mathbb{R}_+^m.\]
This makes the situation even more complicated for \(m\)-parameter problems and the problem of employing the machinery of compactness by concentration under multiple constraints remains widely open. The task of proving the strict inequalities for \(I^{(2)}_{M_1,M_2}\) and the three-parameter problem \(I^{(3)}_{M_1,M_2,M_3}\), and preventing dichotomy of minimizing sequences will occupy us through most of Sections 3 and 4.

For any \(1 \leq r < \infty\), we denote by \(L^r_w(\mathbb{R}^N)\) (the weak \(L^r\) space) the set of all measurable functions \(f: \mathbb{R}^N \to \mathbb{C}\) such that
\[
\|f\|_{L^r_w} = \sup_{M > 0} M|\{x : |f(x)| > M\}|^{1/r} < \infty.
\]
Throughout the paper, we require the power \(p\) and the convolution potential \(W \in L^r_w(\mathbb{R}^N)\) to satisfy the following assumptions

(h0) The power \(p\) satisfies
\[
2 \leq p < \frac{2r - 1}{r} + \frac{2}{N} \quad \text{with} \quad \frac{1}{r} < \frac{2}{N},
\]
(h1) The potential \(W: \mathbb{R}^N \to [0, \infty)\) is radially symmetric i.e., \(W(x) = W(|x|)\), and satisfies \(W(r) \to 0\) as \(r \to \infty\).
(h2) There exists \(\Gamma\) satisfying \(\Gamma < 2 + 2N - pN\) such that
\[
W(\theta \xi) \geq \theta^{-\Gamma} W(\xi) \quad \text{for any} \ \theta > 1.
\]
The results in this paper hold for the Coulomb type potential \(W(x) = |x|^{-\alpha}\) for some \(\alpha > 0\). Our main results are as follows:

**Theorem 1.1.** Suppose \(m = 2, 3\) and the assumptions (h0), (h1), and (h2) hold. For every \(M = (M_1,\ldots,M_m) \in \mathbb{R}_+^m\), define
\[
\Lambda^{(m)}(M) = \{\phi = (\phi_1,\ldots,\phi_m) \in Y_m : I(\phi) = I^{(m)}_M, \|\phi_j\|_{L^2}^2 = M_j, 1 \leq j \leq m\}.
\]

The following statements hold:

(a) For every \(M = (M_1,\ldots,M_m) \in \mathbb{R}_+^m\), there exists a nonempty set \(\Lambda^{(m)}(M) \subset Y_m\) such that for every \(\phi \in \Lambda^{(m)}(M)\), there exists \((\lambda_1,\ldots,\lambda_m)\) such that \(\psi_j(x,t) = e^{-i\lambda_j t}\phi_j\) is a standing wave for \((1.3)\) satisfying \(\int_{\mathbb{R}^N} |\phi_j|^2 \, dx = M_j, 1 \leq j \leq m\).

(b) For every complex-valued minimizer \(\phi\) of \(I^{(m)}_M\), there exists \(\theta_j \in \mathbb{R}\) and real-valued functions \(\tilde{\phi}_j\) such that
\[
\tilde{\phi}_j(x) > 0 \quad \text{and} \quad \phi_j(x) = e^{i\theta_j} \tilde{\phi}_j(x), \forall x \in \mathbb{R}^N, 1 \leq j \leq m.
\]
Proof. See Lieb and Loss, Analysis [17].

Under the same hypotheses as in Theorem 1.1, the set \( M \) is stable for the associated initial-value problem of (1.5), i.e., for every \( \varepsilon > 0 \), there exists \( \delta(\varepsilon) > 0 \) such that whenever \( (\psi_0^1, \ldots, \psi_0^m) \) satisfies
\[
\inf_{\phi \in \Lambda^{(m)}(M)} \|(\psi_0^1, \ldots, \psi_0^m) - \phi\|_{Y_m} \leq \delta(\varepsilon),
\]
then any solution \( \psi(\cdot, t) = (\psi_1(\cdot, t), \ldots, \psi_m(\cdot, t)) \) of (1.5) with initial datum \( \psi_j(\cdot, 0) = \psi_{0j} \) satisfies
\[
\sup_{t \geq 0} \inf_{\phi \in \Lambda^{(m)}(M)} \|\psi(t, \cdot) - \phi\|_{Y_m} < \varepsilon.
\]

2. THE VARIATIONAL PROBLEM

In this section, we prove number of lemmas which are needed in the sequel to prove our main results. Throughout this section we do not distinguish the case \( m = 2 \) and \( m = 3 \). The results of this section remain hold for an arbitrary \( m \).

In what follows, for \( s > 0 \), we denote by \( \Sigma_s \) the sphere
\[
\Sigma_s = \left\{ f \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |f|^2 \, dx = s \right\}.
\]
We always denote \( m \)-tuples in \( \mathbb{R}^m_+ \) as \( M = (M_1, \ldots, M_m) \), \( T = (T_1, \ldots, T_m) \), etc. For any \( M \in \mathbb{R}^m_+ \), we write \( \Sigma^{(m)}_M = \Sigma_{M_1} \times \cdots \times \Sigma_{M_m} \). To avoid tedious expressions, we often write
\[
Q(f, g) = |f(x)|^p |g(y)|^p 
\]
for \( x, y \in \mathbb{R}^N \)
and for any \( q > 0 \), we shall denote the Coulomb-type potential by
\[
\mathbb{F}_q(f, g) = \frac{1}{q} \int_{\mathbb{R}^N \times \mathbb{R}^N} W(|x-y|)Q(f, g) \, dxdy. \tag{2.1}
\]
We will make use of the following Hardy-Littlewood-Sobolev inequality.

Lemma 2.1. For every \( f \in L^q(\mathbb{R}^N), g \in L^r_w(\mathbb{R}^N), \) and \( h \in L^t(\mathbb{R}^N) \) with \( 1 < q, r, t < \infty \) and \( \frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 2 \), there exists \( C = C(q, N, r) > 0 \) such that
\[
\left| \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x)g(x-y)h(x) \, dxdy \right| \leq C \|f\|_{L^q} \|g\|_{L^r_w} \|h\|_{L^t}.
\]
Proof. See Lieb and Loss, Analysis [17].
In what follows we use the Sobolev interpolation inequalities
\[
\left( \int_{\mathbb{R}^N} |u|^s \, dx \right)^{2/s} \leq C(s, N) \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^{\theta} \left( \int_{\mathbb{R}^N} |u|^2 \, dx \right)^{1-\theta} \tag{2.2}
\]
holds for every \( u \in H^1(\mathbb{R}^N) \) and \( s \) such that \( 2 \leq s \leq \infty \) if \( N = 1, 2 \leq s < \infty \) if \( N = 2, \) and \( 2 \leq s \leq 2N/(N-2) \) if \( N \geq 3, \) where \( C(s, N) > 0 \) and \( \theta \) satisfies
\[
\frac{N}{s} = \frac{\theta(N-2)}{2} + \frac{(1-\theta)N}{2}.
\]

The following lemma shows that \( J_m^{(m)} \) is well posed and minimizing sequences are uniformly bounded in \( Y_m. \)

**Lemma 2.2.** For \( M = (M_1, \ldots, M_m) \in \mathbb{R}^m_+, \) let \( \{(u^n_1, \ldots, u^n_m)\}_{n \geq 1} \) be any sequence in \( Y_m \) satisfying
\[
I(u^n_1, \ldots, u^n_m) \to J_m^{(m)} \quad \text{and} \quad \lim_{n \to \infty} \|u^n_j\|_{L^2}^2 = M_j, \quad 1 \leq j \leq m.
\]
Then there exists a constant \( C > 0 \) such that \( \sum_{j=1}^m \|u^n_j\|_{H^1(\mathbb{R}^N)}^2 \leq C \) for all \( n. \) Moreover, for every \( M \in \mathbb{R}^m_+, \) one has
\[
-\infty < J_M^{(m)} < 0.
\]

**Proof.** We begin with the following observation. In view of the Hardy-Littlewood-Sobolev inequality, the integral
\[
\mathbb{F}_q(f, g) = \frac{1}{q} \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)|f(x)|^p|g(y)|^p \, dxdy, \quad q > 0,
\]
is well-defined if \( |f|^p, |g|^p \in L^t(\mathbb{R}^N) \) for all \( t > 1 \) satisfying the condition
\[
\frac{1}{t} + \frac{1}{r} + \frac{1}{t} = 2, \quad \text{or} \quad t = \frac{2r}{2r-1}.
\]
By our assumption, we have that
\[
\frac{1}{tp} = \frac{2r-1}{2pr} = \frac{1}{p} - \frac{1}{2pr} > \frac{1}{p} - \frac{2N + 2 - pN}{2Np} = \frac{1}{2} - \frac{1}{Np} \geq \frac{N-2}{2N}.
\]
It follows that \( |f|^p \in L^{\frac{2r}{2r-1}}(\mathbb{R}^N) \) for every \( f \in H^1(\mathbb{R}^N). \) Using the Hardy-Littlewood-Sobolev inequality and the Gagliardo-Nirenberg inequality, we obtain that
\[
\mathbb{F}_q(u^n_j, u^n_k) = \frac{1}{q} \int_{\mathbb{R}^N} (W \ast |u^n_k|^p)|u^n_j|^p \, dx \leq C\|W\|_{L^\infty} \|u^n_j\|_{L^{2p}}^p \|u^n_k\|_{L^{\frac{2p}{2r-1}}}^p \\
\leq C\|u^n_j\|_{L^2}^{(1-\mu)p} \|\nabla u^n_j\|_{L^{2}}^\mu \|u^n_k\|_{L^2}^{(1-\mu)p} \|\nabla u^n_k\|_{L^2}^\mu \tag{2.3}
\]
where \( \mu = (Nrp - 2Nr + N)/2rp. \) To show that \( \{(u^n_1, \ldots, u^n_m)\}_{n \geq 1} \) is bounded, using the estimate (2.3) and the fact that the sequence \( \{I(u^n_1, \ldots, u^n_m)\}_{n \geq 1} \) is bounded in \( \mathbb{R}, \) we
Lemma 2.3. Define the functional \( E : H^1(\mathbb{R}^N) \to \mathbb{R} \) as follows
\[
E(h) = \frac{1}{2} \| \nabla h \|_{L^2}^2 - \frac{1}{2p} \int_{\mathbb{R}^N \times \mathbb{R}^N} W(|x-y|)Q(h, h) \, dx dy.
\]
Let \( M \in \mathbb{R}_{\geq 0}^n \) be such that \( M_1 + \ldots + M_m > 0 \) and suppose that \( \{u_n^1, \ldots, u_n^m\}\) be any minimizing sequence for \( I^{(m)}_M \). Then for each \( j \) with \( M_j > 0 \) and any number \( \Gamma > 1 \), there
obtain
\[
\frac{1}{2} \|(u_1^n, \ldots, u_m^n)\|_{L^2_m}^2 = I(u_1^n, \ldots, u_m^n) + \sum_{k,j=1}^m F_{2p}(u_k^n, u_j^n) + \frac{1}{2} \sum_{j=1}^m \|u_j^n\|_{L^2}^2
\leq C(N, r, p, M) \left( 1 + \|(u_1^n, \ldots, u_m^n)\|_{L^2_m}^{2p} \right).
\]
By the assumption (h0), we have \( 2\mu p = Nrp - 2Nr + N < 2 \). Then it follows that \( \{u_1^n, \ldots, u_m^n\}\) is bounded in \( Y_m \). The proof that \( I^{(m)}_M > -\infty \) is immediate from (2.3) and we omit the details.

We next prove that \( I^{(m)}_M < 0 \). It is enough to show that there exists \( (\tilde{u}_1, \ldots, \tilde{u}_m) \in Y_m \) such that \( \sum_{j=1}^m u_j = (M_j/M_1)^{1/2} u_1 \) for \( 2 \leq j \leq m \). Consider the functions \( u_j^\theta : \mathbb{R}^N \to \mathbb{R} \) defined by \( u_j^\theta(x) = \theta^{N/2} u_j(\theta x) \) for \( 1 \leq j \leq m \). Then one has \( \tilde{u} = (u_1^\theta, \ldots, u_m^\theta) \in Y_m \) and for every \( 0 < \theta < 1 \), we compute
\[
\int_{\mathbb{R}^N} (W * |u_k^\theta|^p |u_j^\theta|^p) \, dx = \int_{\mathbb{R}^N \times \mathbb{R}^N} \theta^{Np} W(|x-y|) |u_j(\theta x)|^p |u_k(\theta y)|^p \, dxdy
\]
\[
= \int_{\mathbb{R}^N \times \mathbb{R}^N} \theta^{Np} W(\theta^{-1} |\theta x - \theta y|) |u_j(\theta x)|^p |u_k(\theta y)|^p \, dxdy
\]
\[
\geq \int_{\mathbb{R}^N \times \mathbb{R}^N} \theta^{Np-2N+\Gamma} W(|x-y|) |u_k(y)|^p |u_j(x)|^p \, dxdy,
\]
where in the last inequality we used the assumption (h2). Using this estimate, a direct computation yields
\[
I(u_1^n, \ldots, u_m^n) \leq \frac{\theta^2}{2} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \theta^{Np-2N+\Gamma} \sum_{k,j=1}^m \int_{\mathbb{R}^N} (W * |u_k|^p) |u_j|^p \, dx
\]
\[
= \frac{\theta^2}{2} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 - \Omega \theta^{Np-2N+\Gamma} \int_{\mathbb{R}^N} (W * |u_1|^p) |u_1|^p \, dx,
\]
where the number \( \Omega = \Omega(M_1, \ldots, M_m, p) > 0 \) is given by
\[
\Omega = \frac{1}{2} + \frac{1}{p} \sum_{j=1}^m \left( M_j / M_1 \right)^{p/2} + \frac{1}{2p} \sum_{k,j=2}^m \left( M_j / M_1 \right)^{p/2} \left( M_k / M_1 \right)^{p/2} > 0.
\]
By our assumption \( Np - 2N + \Gamma < 2 \), it follows from (2.4) that \( I(u_1^\theta, \ldots, u_m^\theta) < 0 \) for sufficiently small \( \theta \) and consequently, we get \( I^{(m)}_M < 0 \). \( \square \)
exists $\delta > 0$ (independent of $n$) such that for sufficiently large $n,$
$$E \left( \Gamma^{1/2} u^n_j \right) \leq \Gamma E \left( u^n_j \right) - \delta. \quad (2.5)$$

**Proof.** We claim that for any minimizing sequence $\{(u^n_1, \ldots, u^n_m)\}_{n \geq 1}$ of the problem $I^{(m)}_M$ there exists $\delta > 0$ and $n_0 = n_0(\delta) \in \mathbb{N}$ such that
$$\int_{\mathbb{R}^N} |u^n_j|^{2pr} \, dx \geq \delta, \quad \forall n \geq n_0,$$
provided that $M_j > 0.$ To see this, suppose to the contrary that there exists some minimizing sequence $\{(\tilde{u}^n_1, \ldots, \tilde{u}^n_m)\}_{n \geq 1}$ of $I^{(m)}_M$ such that
$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} |\tilde{u}^n_j|^{2pr} \, dx = 0.$$

Using the Hardy-Littlewood-Sobolev inequality, we obtain that for any $q > 0,$
$$F_q(\tilde{u}^n_k, \tilde{u}^n_j) = \frac{1}{q} \int_{\mathbb{R}^N \times \mathbb{R}^N} W((x-y)) Q(\tilde{u}^n_k, \tilde{u}^n_j) \, dxdy \leq C \left\| \tilde{u}^n_k \right\|_{L^p} \left\| \tilde{u}^n_j \right\|_{L^{2pr}} \leq C \left\| \tilde{u}^n_k \right\|_{L^{2pr}} \left\| \tilde{u}^n_j \right\|_{L^{2pr}} \to 0$$
as $n \to \infty.$ Then it follows that
$$I^{(m)}_M = \lim_{n \to \infty} I(\tilde{u}^n_1, \ldots, \tilde{u}^n_m) \geq \liminf_{n \to \infty} \frac{1}{2} \sum_{j=1}^m \left\| \nabla \tilde{u}^n_j \right\|_{L^2} \geq 0,$$
which is a contradiction and hence the claim follows. To see (2.5), it follows from the Hardy-Littlewood-Sobolev inequality that
$$E \left( \Gamma^{1/2} u^n_j \right) = \Gamma E \left( u^n_j \right) + (\Gamma - \Gamma^p) F_{2p}(u^n_j, u^n_j) \leq \Gamma E \left( u^n_j \right) + C(\Gamma - \Gamma^p) \left\| u^n_j \right\|_{L^{2pr}}^{2p} \quad (2.6)$$
Since $\Gamma > 1,$ $p > 2,$ and $\left\| u^n_j \right\|_{L^{2pr}} \geq \delta$ for sufficiently large $n,$ the desired inequality follows from (2.6).

We will need the following result concerning the existence of positive solutions for the functional $E(u).$

**Lemma 2.4.** Suppose that the assumptions (h0), (h1), and (h1) hold. Then for each $M > 0,$ there exists a real-valued function $\phi_M > 0$ such that
$$E(\phi_M) = \inf \left\{ E(h) : h \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |h|^2 \, dx = M \right\}.$$

**Proof.** This can be proven using the concentration compactness argument and a proof appears in [20] for the potential $W(x) = |x|^{-1},$ and in [8] for $W : \mathbb{R}^N \to [0, \infty)$ satisfying the assumptions (h0), (h1), and (h2). \qed
Lemma 2.5. For every $M_1 > 0$ and $M_2 > 0$, let $\{(u^n_1, u^n_2)\}$ be a sequence in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that $\mathcal{I}(u^n_1, u^n_2) \to J_{M_1, M_2}^{(2)}$ and $\|u^n_j\|_{L^2} \to M_j$. Then there exists $\delta_j > 0$ such that for all sufficiently large $n$,

$$E(u^n_1) - F_p(u^n_1, u^n_2) \leq -\delta_1 \text{ and } E(u^n_2) - F_p(u^n_1, u^n_2) \leq -\delta_2.$$ 

Proof. Suppose to the contrary that there exists some minimizing sequence $\{(u^n_1, u^n_2)\}_{n \geq 1}$ of $J_{M_1, M_2}^{(2)}$ such that

$$\liminf_{n \to \infty} (E(u^n_1) - F_p(u^n_1, u^n_2)) \geq 0.$$ 

Then this implies that

$$J_{M_1, M_2}^{(2)} = \lim_{n \to \infty} \mathcal{I}(u^n_1, u^n_2) \geq \liminf_{n \to \infty} \left(\frac{1}{2} \|\nabla u^n_2\|_{L^2}^2 - F_{2p}(u^n_2, u^n_2)\right).$$ 

Let $\phi_{M_2}$ be as defined in Lemma 2.4 with $M = M_2$. Then it follows from (2.7) that $J_{M_1, M_2}^{(2)} \geq E(\phi_{M_2})$. Next let $\psi \geq 0$ be an arbitrary function with compact support satisfying $\psi(0) = 1$ and $\|\psi\|_{L^2} = M_1$. For any $\theta > 0$, define $\psi_\theta(x) = \theta^{N/2} \psi(\theta x)$. Then one can show as in the proof of Lemma 2.4 that for sufficiently small $\theta$

$$E(\psi_\theta) - F_p(\psi_\theta, \phi_{M_2}) < 0.$$ 

Thus, for this choice of $\theta$, one obtains that

$$J_{M_1, M_2}^{(2)} = E(\psi_\theta) - F_p(\psi_\theta, \phi_{M_2}) + E(\phi_{M_2}) < E(\phi_{M_2}),$$

which contradicts the fact $J_{M_1, M_2}^{(2)} \geq E(\phi_{M_2})$. This proves that $E(u^n_1) - F_p(u^n_1, u^n_2)$ is negative for sufficiently large $n$. The proof that $E(u^n_2) - F_p(u^n_1, u^n_2) \leq -\delta_2$ goes through the same steps and we omit the details. \qed

Lemma 2.6. Let $N \geq 1$. Assume that $\{u_n\}_{n \geq 1}$ and $\{|
abla u_n|\}_{n \geq 1}$ are bounded in $L^2(\mathbb{R}^N)$. If for some $R > 0$,

$$\lim_{n \to \infty} \left(\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^2 \, dx\right) = 0,$$

then the sequence $\{u_n\}_{n \geq 1}$ converges to zero in $L^q(\mathbb{R}^N)$ for any $2 < q < \infty$ if $N = 1, 2$ and for every $2 < q < \tfrac{2N}{N-2}$ if $N \geq 3$.

Proof. This lemma is a special case of Lions’ concentration compactness lemma, see Lemma 1.1 of [20], but for the sake of completeness we include a proof here. Let us denote $\omega_n = \sup_{y \in \mathbb{R}^N} \|u_n\|_{L^2(B_R(y))}$. By assumption, we have that $\omega_n \to 0$ as $n \to \infty$. Using the Sobolev inequalities, we obtain

$$\|u_n\|_{L^q(B_R(y))} \leq C \|u_n\|_{L^2(B_R(y))}^{1-\lambda} \|u_n\|_{H^1(B_R(y))}^\lambda,$$

where $\lambda = N(q-2)/2q$. Thus, one has that

$$\int_{B_R(y)} |u_n|^q \, dx \leq C^q \|u_n\|_{L^2(B_R(y))}^{(1-\lambda)q} \|u_n\|_{H^1(B_R(y))}^\lambda \leq \omega_n^{(1-\lambda)q/2} \|u_n\|_{H^1(B_R(y))}^\lambda,$$
Now, if $q\lambda \geq 2$, it is obvious from (2.8) that
\[
\int_{B_R(y)} |u_n|^q \leq C (\omega_n)^{(1-\lambda)q/2} \left( \int_{B_R(y)} (|\nabla u_n|^2 + |u_n|^2) \, dx \right) \|u_n\|_{H^1}^{q\lambda - 2}
\]
\[
\leq C (\omega_n)^{(1-\lambda)q/2} \int_{B_R(y)} (|\nabla u_n|^2 + |u_n|^2) \, dx
\]
(2.9)
Consider a countable family of balls $\{B_R(z_i)\}$ which covers $\mathbb{R}^N$ in such a way that every vector in $\mathbb{R}^N$ belongs to at most $m+1$ balls. Then, summing (2.9) over the balls $\{B_R(z_i)\}$, we obtain that
\[
\int_{\mathbb{R}^N} |u_n|^q \leq (m + 1) C (\omega_n)^{(1-\lambda)q/2} \int_{\mathbb{R}^N} (|\nabla u_n|^2 + |u_n|^2) \, dx \leq C (\omega_n)^{(1-\lambda)q/2},
\]
which gives the result for $q\lambda \geq 2$, i.e., $q > 2 + \frac{4}{N}$. Next consider the case that $q < 2 + \frac{4}{N}$. Using the Hölder inequality, we have that
\[
\|u_n\|_{L^q} \leq \|u_n\|_{L^2}^{\theta} \|u_n\|_{L^{2+\frac{4}{N}}}^{(1-\theta)(2+\frac{4}{N})},
\]
where $q = 2\theta + (1-\theta)(2 + \frac{4}{N})$ for some $\theta \in (0, 1)$. Making use of the result for the case $q = 2 + \frac{4}{N}$, it follows that $\|u_n\|_{L^q} \to 0$, proving the lemma.

Given any minimizing sequence $\{(u_1^n, \ldots, u_m^n)\}_{n \geq 1}$ of $J^{(m)}_M$, we introduce the Lévy concentration function
\[
Q_n^{(m)}(R) = \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} (|u_1^n|^2 + \ldots + |u_m^n|^2) \, dx, \quad n = 1, 2, \ldots,
\]
where $B_R(x) \subset \mathbb{R}^N$ represents a ball with center at $x$ and radius $R$. Then $\{Q_n^{(m)}\}$ is a sequence of nondecreasing functions on $[0, M_1 + \ldots + M_m]$. By Helly’s selection theorem, we can assume (up to a subsequence)
\[
Z^{(m)} = \lim_{R \to \infty} \left( \lim_{n \to \infty} Q_n^{(m)}(R) \right) \in [0, M_1 + \ldots + M_m].
\]
(2.10)
The case $Z^{(m)} = 0$ is called the vanishing, $0 < Z^{(m)} < M_1 + \ldots + M_m$ is the case of dichotomy, and $Z^{(m)} = M_1 + \ldots + M_m$ is the tightness.

**Lemma 2.7.** For any minimizing sequence $\{(u_1^n, \ldots, u_m^n)\}_{n \geq 1}$ of $J^{(m)}_M$, the vanishing does not occur, that is, $Z^{(m)} > 0$.

**Proof.** If the vanishing does occur, then Lemma 2.6 implies that $\lim_{n \to \infty} \|u_j^n\|_{L^q} = 0$ for any $2 < q < \frac{2N}{N-2}$. Since $2 < \frac{2\rho}{2\rho - 1} < \frac{2N}{N-2}$, it follows from the Hardy-Littlewood-Sobolev inequality that for any $t > 0$,
\[
\mathcal{F}_t (u_k^n, u_j^n) = \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) Q(u_k^n, u_j^n) \, dx \, dy \to 0
\]
as $n \to \infty$. Consequently, we have that
\[
J^{(m)}_M = \lim_{n \to \infty} \mathcal{I}(u_1^n, \ldots, u_m^n) \geq \lim_{n \to \infty} \frac{1}{2} \sum_{j=1}^m \|\nabla u_j\|_{L^2}^2 \geq 0.
\]
In the sequel we denote \( w \subset \) type embedding, we have that for every bounded domain \( \Omega \) the solution set \( Y \) converges in \( \square \) Lemma 2.8. Suppose that \( \{(u^n_1, \ldots, u^n_m)\}_{n \geq 1} \) be any minimizing sequence for \( I^{(m)}_M \) and \( Z^{(m)} = M_1 + \ldots + M_m \). Then there exists \( \{y_n\} \subset \mathbb{R}^N \) such that the sequence \( \{(u^n_1(x + y_n), \ldots, u^n_m(x + y_n))\}_{n \geq 1} \), \( x \in \mathbb{R}^N \), converges in \( Y_m \) up to a subsequence to a function \( (\phi_1, \ldots, \phi_m) \in \Lambda^{(m)}(M) \). In particular, the solution set \( \Lambda^{(m)}(M) \) is nonempty.

Proof. We write \( \Sigma(M) = M_1 + \ldots + M_m \). Since \( Z^{(m)} = \Sigma(M) \), we can find \( \{y_n\} \subset \mathbb{R}^N \) such that if we write \( w^n_j = u^n_j(x + y_n), 1 \leq j \leq m \), then for every \( k \in \mathbb{N} \), one can find \( R_k > 0 \) satisfying for sufficiently large \( n \),

\[
\int_{B_{R_k}(0)} \sum_{j=1}^m |w^n_j|^2 \, dx > \Sigma(M) - \frac{1}{k}. \tag{2.11}
\]

In the sequel we denote \( w_n = (w^n_1, \ldots, w^n_m) \). Since \( \|w_n\|_{Y_m} \leq B \) for all \( n \), so from Rellich-type embedding, we have that for every bounded domain \( \Omega \subset \mathbb{R}^N \), the sequence \( \{w_n\} \) has some subsequence (still denoted by the same) which converges in \( (L^2(\Omega))^m \) to some function \( \phi = (\phi_1, \ldots, \phi_m) \) satisfying

\[
\int_{B_{R_k}(0)} \sum_{j=1}^m |\phi_j|^2 \, dx > \Sigma(M) - \frac{1}{k}. \tag{2.12}
\]

Using Cantor diagonalization argument and the fact \( \sum_{j=1}^m \|w^n_j\|^2_{L^2} = \Sigma(M), \forall n \), one then concludes that \( (w_n) \) converges (up to a subsequence) strongly to \( \phi \) in \( (L^2(\mathbb{R}^N))^m \) satisfying \( \sum_{j=1}^m \|\phi_j\|^2_{L^2} = \Sigma(M) \). For any \( t > 0 \), we now estimate

\[
\left| \mathcal{F}_t(w^n_k, w^n_j) - \mathcal{F}_t(\phi_k, \phi_j) \right|
\leq \frac{1}{t} \int_{\mathbb{R}^N} W(|x - y|)|w^n_k(x)|^p|w^n_j(y)|^p - |\phi_k(x)|^p|\phi_j(y)|^p \, dxdy
\leq \frac{1}{t} \int_{\mathbb{R}^N} W(|x - y|)|w^n_k(x)|^p \left| |w^n_j(y)|^p - |\phi_j(y)|^p \right| \, dxdy \tag{2.13}
+ \frac{1}{t} \int_{\mathbb{R}^N} W(|x - y|)|\phi_j(y)|^p \left| |w^n_k(x)|^p - |\phi_k(x)|^p \right| \, dxdy
\]

Using the Hardy-Littlewood-Sobolev inequality and the fact that \( \{w^n_k\}_{n \geq 1} \) is bounded in \( H^1(\mathbb{R}^N) \), we obtain that

\[
\left| \mathcal{F}_t(w^n_k, w^n_j) - \mathcal{F}_t(\phi_k, \phi_j) \right| \leq C\|W\|_{L^{\infty}_\nu} \|w^n_k\|^p_{L^{2p/\nu}_{w^n_k}} \|w^n_j\|^p_{L^{2p/\nu}_{w^n_j}} - |\phi_k|^p_{L^{2p/\nu}_{\phi_k}} + C\|W\|_{L^{\infty}_\nu} \|\phi_j\|^p_{L^{2p/\nu}_{\phi_j}} \|w^n_k\|^p_{L^{2p/\nu}_{w^n_k}} - |\phi_k|^p_{L^{2p/\nu}_{\phi_k}} \leq 2C\|w^n_k\|^p - |\phi_k|^p_{L^{2p/\nu}_{\phi_k}}
\]
Next, using the inequality, \(|a|^{p-1}a - |b|^{p-1}b| ≤ \frac{p}{2}|a - b|(\|a\|^{p-1} + \|b\|^{p-1})\), holds for any \(a, b ∈ \mathbb{R}\) and \(p ≥ 1\), and applying Holder’s inequality, we obtain that

\[
|F_t(w^n_k, w^n_j) - F_t(φ_k, φ_j)| ≤ C \left( \int_{\mathbb{R}^N} |w^n_k|^p - |φ_k|^p \, dx \right)^{\frac{2r-1}{2r}}
\]

\[
≤ C \left( \int_{\mathbb{R}^N} (|w^n_k|^{p-1} + |φ_k|^{p-1}) |w^n_k - φ_j|^{\frac{2r}{2r-1}} \, dx \right)^{\frac{2r-1}{2r}}
\]

\[
≤ C \left( \int_{\mathbb{R}^N} (|w^n_k|^{\frac{2μr}{2r-1}} + |φ_k|^{\frac{2μr}{2r-1}}) \, dx \right)^{\frac{2r}{μr}} \|w^n_k - φ_k\|_{L^{\frac{2μr}{2r-1}}}
\]

where \(ρ = \frac{2r-1}{2r}(1 - \frac{1}{p})\). Now, using the standard Interpolation inequality and the Sobolev inequality, it follows that

\[
|F_t(w^n_k, w^n_j) - F_t(φ_k, φ_j)| ≤ C\|w^n_k - φ_k\|_{L^2}^2 \|w^n_k - φ_j\|_{L^{\frac{2μr}{2r-1}}}^{1-\frac{2r}{μr}}
\]

(2.14)

where \(κ = (rN - N + 2pr - 2Npr)/(2pr - Npr)\). The right-hand side of (2.14) goes to zero since \(w^n_k → φ_k\) in \(L^2\). Thus, we have that \(\lim_{n→∞} F_t(w^n_k, w^n_j) = F_t(φ_k, φ_j)\). Furthermore, as a consequence of the weak lower semi-continuity of the norm in a Hilbert space, we can assume, by extracting another subsequence if necessary, that \(w_n → φ\) weakly in \(Y_m\), and that

\[
\|φ\|_{Y_m} = \|(φ_1, \ldots, φ_m)\|_{Y_m} ≤ \liminf_{n→∞} \|w^n_1, \ldots, w^n_m\|_{Y_m}.
\]

Then it follows that

\[
I(φ) = I(φ_1, \ldots, φ_m) ≤ \lim_{n→∞} I(w^n_1, \ldots, w^n_m) = I^{(m)}_M,
\]

and since \(w^n_j → φ_j\) in \(L^2(\mathbb{R}^N)\), we also have that \(\|φ_j\|_{L^2}^2 = \lim_{n→∞} \|w^n_j\|_{L^2}^2 = M_j\) for \(1 ≤ j ≤ m\). By the definition of the infimum \(I^{(m)}_M\), we must have \(I(φ_1, \ldots, φ_m) = I^{(m)}_M\) and \(u ∈ Σ^{(m)}_M\). Finally, the facts \(I(φ) = \lim_{n→∞} I(w^n), F_t(φ_k, φ_j) = \lim_{n→∞} F_t(w^n_k, w^n_j)\), and \(\|φ_j\|_{L^2} = \lim_{n→∞} \|w^n_j\|_{L^2}\) together imply that \(\|φ\|_{Y_m} = \lim_{n→∞} \|w^n\|_{Y_m}\), and from a standard exercise in the elementary Hilbert space theorem one then obtains that \(w_n → φ\) in \(Y_m\) norm.

We end this section with the following lemma which will be used in the next section to rule out the case of dichotomy.

Lemma 2.9. For any minimizing sequence \(\{(u^n_1, \ldots, u^n_m)\}_{n≥1}\) of \(I^{(m)}_M\), let \(Z^{(m)}\) be defined by (2.10). Then there exists \(T ∈ [0, M_1] × \ldots × [0, M_m]\) such that

\[
Z^{(m)} = T_1 + \ldots + T_m \quad \text{and} \quad I^{(m)}_T + I^{(m)}_{M-T} ≤ I^{(m)}_M.
\]

(2.15)

Proof. The proof is almost same as the proof of Lemma 2.12 of [7], we only provide an outline here. Let \(ε > 0\) be arbitrary. Using the definition of \(Z^{(m)}\) and the convergence
properties of \( Q_k(R) \), there exists \( R_0(\varepsilon), k_0(\varepsilon) \) such that for all \( R \geq R_0(\varepsilon) \) and \( k \geq k_0(\varepsilon) \), we have that
\[
Z^{(m)} - \frac{3\varepsilon}{4} < Q_k(R) \leq Q_k(2R) \leq Z^{(m)} + \frac{3\varepsilon}{4}.
\] (2.16)
The inequalities (2.16) together with the definition of \( Q_k \) implies that there exists a sequence of vectors \( y_k \in \mathbb{R}^N \) such that
\[
\int_{B_R(y_k)} \sum_{j=1}^m |u_j^{(k)}|^2 \, dx > Z^{(m)} - \varepsilon, \quad \int_{B_{2R}(y_k)} \sum_{j=1}^m |u_j^{(k)}|^2 \, dx < Z^{(m)} + \varepsilon.
\] (2.17)
Let \( \phi \in C_0^\infty(\mathbb{R}^N) \) be such that \( \phi(x) \equiv 0 \) for \( |x| \geq 2 \) and \( \phi(x) \equiv 1 \) for \( |x| \leq 1 \), and take
\( \psi \in C^\infty(\mathbb{R}^N) \) such that \( \phi^2 + \psi^2 \equiv 1 \) for \( x \in \mathbb{R}^N \). For any \( R > 0 \), let \( \phi_R \) and \( \psi_R \) denote the rescaled functions \( \phi(x/R) \) and \( \psi(x/R) \) for \( x \in \mathbb{R}^N \). Let us now define
\[
u_j^{(1)} = \phi_R(x+y_k)u_j^{(k)}, \quad u_j^{(2)} = \psi_R(x+y_k)u_j^{(k)}, \quad 1 \leq j \leq m.
\]
From Lemma 2.2, the sequences \( \{u_j^{(1)}\}_{k \geq 1} \) and \( \{u_j^{(2)}\}_{k \geq 1} \), \( 1 \leq j \leq m \) are bounded in \( L^2 \).

Thus, by passing to subsequences, we may assume that there exists \( T \in [0, M_1] \times \ldots \times [0, M_m] \) such that \( \int_{\mathbb{R}^N} |u_j^{(1)}|^2 \, dx \rightarrow T_j \), whence it also follows that \( \int_{\mathbb{R}^N} |u_j^{(2)}|^2 \, dx \rightarrow M_j - T_j \).

Now we have
\[
T_1 + \ldots + T_m = \lim_{k \rightarrow \infty} \sum_{j=1}^m \int_{\mathbb{R}^N} |u_j^{(1)}|^2 \, dx = \lim_{k \rightarrow \infty} \sum_{j=1}^m \int_{\mathbb{R}^N} \phi_R^2 |u_j^{(1)}|^2 \, dx,
\]
where and in what follows we have written the rescaled functions \( \phi_R(x+y_k) \) and \( \psi_R(x+y_k) \) simply by \( \phi_R \) and \( \psi_R \) respectively. From (2.17) it follows that, for any \( k \in \mathbb{N} \),
\[
Z^{(m)} - \varepsilon < \sum_{j=1}^m \int_{\mathbb{R}^N} \phi_R^2 |u_j^{(1)}|^2 \, dx < Z^{(m)} + \varepsilon.
\]
Then it follows that
\[
| (T_1 + \ldots + T_m) - Z^{(m)} | < \varepsilon.
\]
Let us write \( U_k^{(1)} = (u_{1,k}^{(1)}, \ldots, u_{m,k}^{(1)}) \) and \( U_k^{(2)} = (u_{1,k}^{(2)}, \ldots, u_{m,k}^{(2)}) \). Then, using a standard argument, one can obtain that
\[
\mathcal{I}(U_k^{(1)}) + \mathcal{I}(U_k^{(2)}) \leq \mathcal{I}(u_1^{(1)}, \ldots, u_m^{(1)}) + C \varepsilon, \quad \forall k.
\] (2.18)

To prove (2.15), since \( \{U_k^{(1)}\}_{k \geq 1} \) and \( \{U_k^{(2)}\}_{k \geq 1} \) are bounded in \( Y_m \), so by passing to a subsequence, we may assume that \( \mathcal{I}(U_k^{(1)}) \rightarrow K_1 \) and \( \mathcal{I}(U_k^{(2)}) \rightarrow K_2 \), as \( k \rightarrow \infty \). Then, since \( \lim_{k \rightarrow \infty} \mathcal{I}(u_1^{(1)}, \ldots, u_m^{(1)}) = I_M^{(m)} \), (2.18) implies that \( K_1 + K_2 \leq I_M^{(m)} + C \varepsilon \). Taking \( \varepsilon \) sufficiently small, \( R \) sufficiently large, and making use of results from preceding paragraphs, we can find, for every \( a \in \mathbb{N} \), the sequences \( \{U_k^{(1,a)}\} \) and \( \{U_k^{(2,a)}\} \) in \( Y_m \) such that
\[
\lim_{k \rightarrow \infty} ||u_j^{(1,a)}||_L^2 = T_j(a), \quad \lim_{k \rightarrow \infty} ||u_j^{(2,a)}||_L^2 = M_j - T_j(a), \quad 1 \leq j \leq m,
\]
\[
\lim_{k \rightarrow \infty} \mathcal{I}(U_k^{(1,a)}) = \lim_{k \rightarrow \infty} \mathcal{I}(u_{1,k}^{(1,a)}, \ldots, u_{m,k}^{(1,a)}) = K_i(a), \quad i = 1, 2.
\]
where $T_j(a) \in [0, M_j]$ and $K_i(a)$ satisfy
\[
\left| \sum_{j=1}^{m} T_j(a) - Z^{(m)} \right| \leq \epsilon \quad \text{and} \quad K_1(a) + K_2(a) \leq I^{(m)}_M + \frac{1}{a}. \tag{2.19}
\]

One can further pass to a subsequence and assume that $T_j(a) \to T_j \in [0, M_j]$ and $K_i(a) \to K_i$. Furthermore, after relabeling the sequences $\{U^{(i)}_k\}$ to be the diagonal subsequences $U^{(i)}_k = U^{(i,k)}_k$, $i = 1, 2$, we can further assume that
\[
\lim_{n \to \infty} \left\| u^{(1)}_{j,k} \right\|_{L^2} = T_j, \quad \lim_{n \to \infty} \left\| u^{(2)}_{j,k} \right\|_{L^2} = M_j - T_j, \quad j = 1, \ldots, m,
\]
\[
\lim_{n \to \infty} \mathcal{I}(U^{(i)}_k) = \lim_{n \to \infty} \mathcal{I}(u^{(i)}_{1,k}, \ldots, u^{(i)}_{m,k}) = K_i, \quad i = 1, 2.
\]

Now, passing limit as $a \to \infty$ in the first inequality of (2.19), it follows that $Z^{(m)} = T_1 + \ldots + T_m$. In view of the second inequality of (2.19), the proof will be complete if we are able to deduce that $K_1 \geq I^{(m)}_T$ and $K_2 \geq I^{(m)}_T$.

To prove $K_1 \geq I^{(m)}_T$, we consider two cases, namely, $T_j > 0$ for all $1 \leq j \leq m$; and exactly $\tilde{m}$ of $T_1, \ldots, T_m$ are zero for any $1 \leq \tilde{m} \leq m - 1$. Suppose first that $T_1 > 0, \ldots, T_m > 0$. Define the numbers
\[
\beta^{(1)}_{j,k} = \frac{\sqrt{T_j}}{\left\| u^{(1)}_{j,k} \right\|_{L^2}}, \quad j = 1, \ldots, m.
\]

Then, one has that $\mathcal{I}(\beta^{(1)}_{1,k} u^{(1)}_{1,k}, \ldots, \beta^{(1)}_{m,k} u^{(1)}_{m,k}) \geq I^{(m)}_T$. Since $\beta^{(1)}_{j,k} \to 1$ as $k \to \infty$, it follows that
\[
K_1 = \lim_{k \to \infty} \mathcal{I}(\beta^{(1)}_{1,k} u^{(1)}_{1,k}, \ldots, \beta^{(1)}_{m,k} u^{(1)}_{m,k}) \geq I^{(m)}_T.
\]

Now suppose that exactly $\tilde{m}$ of $T_1, \ldots, T_m$ are zero for any $1 \leq \tilde{m} \leq m - 1$. By relabeling the indices on $T_j$'s, we may assume that $T_1 = 0, \ldots, T_{\tilde{m}} = 0$ and $T_{\tilde{m}+1} > 0, \ldots, T_m > 0$. Then, for each $j = 1, \ldots, m$, using the Hardy-Littlewood-Sobolev and Gagliardo-Nirenberg inequalities, one obtains that
\[
\int_{\mathbb{R}^N} (W \ast |u^{(1)}_{j,k}|^p)|u^{(1)}_{j,k}|^p \, dx \leq C\|W\|_{L^\infty} \|u^{(1)}_{j,k}\|^p_{L^{2p}} \|u^{(1)}_{j,k}\|^p_{L^{2p}} \leq C\|
abla u^{(1)}_{j,k}\|^p_{L^2} \|u^{(1)}_{j,k}\|^{(1-\mu)p}_{L^2} \|
abla u^{(1)}_{i,k}\|_{L^2} \|u^{(1)}_{i,k}\|^{(1-\mu)p}_{L^2} \leq C\|u^{(1)}_{i,k}\|^{(1-\mu)p}_{L^2} \to 0, \quad 1 \leq i \leq \tilde{m},
\]
as $k \to \infty$, where $\mu = N(pr - 2r + 1)/2rp$. In consequence, we obtain that
\[
K_1 = \lim_{k \to \infty} \mathcal{I}(U^{(1)}_k) = \lim_{k \to \infty} \mathcal{I}(u^{(1)}_{1,k}, \ldots, u^{(1)}_{m,k})
\]
\[
= \lim_{k \to \infty} \left( \sum_{j=1}^{m} \|\nabla u^{(1)}_{j,k}\|^2_{L^2} - \frac{1}{2p} \sum_{i,j=\tilde{m}+1}^{m} \int_{\mathbb{R}^N} (W \ast |u^{(1)}_{i,k}|^p|u^{(1)}_{j,k}|^p \, dx) \right)
\]
\[
\geq \lim_{k \to \infty} \left( \sum_{j=\tilde{m}+1}^{m} \|\nabla u^{(1)}_{j,k}\|^2_{L^2} - \frac{1}{2p} \sum_{i,j=\tilde{m}+1}^{m} \int_{\mathbb{R}^N} (W \ast |u^{(1)}_{i,k}|^p|u^{(1)}_{j,k}|^p \, dx) \right)
\]
\[
\geq I^{(m)}_{0, \ldots, 0, T_{\tilde{m}+1}, \ldots, T_m} = I^{(m)}_T.
\]
To prove that \( K_2 \geq I_{M+T}^{(m)} \), one can go through the same argument as in the proof of \( K_1 \geq I_T^{(m)} \) by treating \( M_1 - T_1, \ldots, M_m - T_m \) as \( T_1, \ldots, T_m \), respectively. 

3. The problem with two constraints

In this section, we follow the method developed in [2] to rule out the possible dichotomy of the minimizing sequences. For this purpose, we require to prove the strict subadditivity inequality for the function \( I_M^{(2)} \).

In the sequel we shall use the following notation:

\[
\langle E \rangle (f_1, f_2, \ldots, f_m) = E(f_1) + E(f_2) + \ldots + E(f_m),
\]

where the functional \( E \) is as defined in Lemma 2.3. The strict subadditivity under two constraints takes the following form:

**Lemma 3.1.** Let \( \mathbb{R}_{\geq 0} = [0, \infty) \). For any \( M, T \in \mathbb{R}_{\geq 0}^2 \) satisfying \( M, T \neq \{0\} \) and \( S = M + T \in \mathbb{R}_{\geq 0}^2 \), one has

\[
I_S^{(2)} < I_M^{(2)} + I_T^{(2)}. \tag{3.1}
\]

To prove Lemma 3.1 we use ideas from [7, 8]. Since \( M_1 + T_1 > 0 \), the following cases arise: \( M_1 > 0 \) and \( T_1 > 0 \); \( M_1 = 0 \) and \( T_1 > 0 \); or \( M_1 > 0 \) and \( T_1 = 0 \). The third case can be reduced to the second case by switching \( M_1 \) and \( T_1 \) and so we do not consider it. In the first case, since \( M_2 + T_2 > 0 \), the following cases may arise:

(a) \( M_1 > 0, T_1 > 0, M_2 > 0, \) and \( T_2 > 0 \),
(b) \( M_1 > 0, T_1 > 0, M_2 = 0, \) and \( T_2 > 0 \),
(c) \( M_1 > 0, T_1 > 0, M_2 > 0, \) and \( T_2 = 0 \).

In the second case, since \( M_1 + M_2 > 0, T_1 + T_2 > 0, \) and \( M_2 + T_2 > 0 \), the following cases may arise:

(a) \( M_1 = 0, T_1 > 0, M_2 > 0, \) and \( T_2 > 0 \),
(b) \( M_1 = 0, T_1 > 0, M_2 = 0, \) and \( T_2 > 0 \),
(c) \( M_1 = 0, T_1 > 0, M_2 > 0, \) and \( T_2 = 0 \).

In order to prove Lemma 3.1, it suffices to consider the cases (a), (b), and (c). All other cases can be reduced to one of these cases by switching roles of \( M_j \)'s and \( T_j \)'s. We consider these three cases in the next three lemmas.

The first lemma concerns the case (a).

**Lemma 3.2.** Let \( M = (M_1, M_2) \in \mathbb{R}_{\geq 0}^2 \) and \( T = (T_1, T_2) \in \mathbb{R}_{\geq 0}^2 \). Then one has

\[
I_M^{(2)}(M + T) < I_M^{(2)} + I_T^{(2)}. \tag{3.2}
\]

**Proof.** We follow the ideas from [7, 8]. Let \( \{ (u_1^n, u_2^n) \} \) and \( \{ (v_1^n, v_2^n) \} \) be any sequences in \( Y_2 \) satisfying

\[
\lim_{n \to \infty} \| u_1^n \|^2_{L^2} = M_j, \quad \lim_{n \to \infty} \| v_1^n \|^2_{L^2} = T_j, \quad j = 1, 2,
\]

\[
\lim_{n \to \infty} \mathcal{I}(u_1^n, u_2^n) = I_M^{(2)}(M), \quad \lim_{n \to \infty} \mathcal{I}(v_1^n, v_2^n) = I_T^{(2)}(T).
\]
By passing to a subsequence if necessary, we may assume that the following values exist.

\[ A_1 = \frac{1}{M_1} \lim_{n \to \infty} (E(u_1^n) - \mathbb{F}_p(u_1^n, u_2^n)), \quad B_1 = \frac{1}{M_2} \lim_{n \to \infty} E(u_2^n), \]

\[ A_2 = \frac{1}{T_1} \lim_{n \to \infty} (E(v_1^n) - \mathbb{F}_p(v_1^n, v_2^n)), \quad B_2 = \frac{1}{T_2} \lim_{n \to \infty} E(v_2^n). \]

To prove (3.2), we consider three cases: \( A_1 < A_2 \); \( A_1 > A_2 \); and \( A_1 = A_2 \). Assume first that \( A_1 < A_2 \). Without loss of generality, we may assume that \( u_j^n \) and \( v_j^n \) are non-negative and by density argument, we may also assume that \( u_j^n \) and \( v_j^n \) have compact supports.

Let \( \tilde{v}_2^n = v_2^n(-x_n \rho) \), where \( \rho \) is some unit vector in \( \mathbb{R}^N \) and \( x_n \) is chosen such that \( x_n \to 0 \) as \( n \to \infty \), and \( \tilde{v}_2^n \) and \( u_2^n \) have disjoint supports. Define \((f_1^n, f_2^n)\) as follows: \( f_1^n = \ell^{1/2} u_1^n \) and \( f_2^n = u_2^n + \tilde{v}_2^n \), where \( \ell = (M_1 + T_1)/M_1 \). Then we have that

\[ I_{M+T}^{(2)} \leq \lim_{n \to \infty} \mathcal{I}(f_1^n, f_2^n) \]

\[ = \lim_{n \to \infty} \left( E(f_1^n) + \langle E \rangle(u_2^n, \tilde{v}_2^n) - \mathbb{F}_p(f_1^n, f_2^n) \right) \]

\[ \leq \lim_{n \to \infty} \left( E(f_1^n) + \langle E \rangle(u_2^n, \tilde{v}_2^n) - \mathbb{F}_p(f_1^n, u_2^n) \right). \]

Since \( \ell > 1 \) and \( p \geq 2 \), we have that \( \ell^{p/2} \geq \ell \). Then it follows that

\[ \mathbb{F}_p(\ell^{1/2} f, \ell^{1/2} f) = \ell^{p/2} \mathbb{F}_p(f, f) \geq \ell \mathbb{F}_p(f, f), \]

\[ \mathbb{F}_p(\ell^{1/2} f, g) = \ell^{p/2} \mathbb{F}_p(f, g) \geq \ell \mathbb{F}_p(f, g). \]

Making use of these observations, we obtain that

\[ E(f_1^n) - \mathbb{F}_p(f_1^n, u_2^n) = \ell \| \nabla u_1^n \|_{L^2}^2 - \mathbb{F}_p(\ell^{1/2} u_1^n, \ell^{1/2} u_1^n) - \mathbb{F}_p(\ell^{1/2} u_1^n, u_2^n) \]

\[ \leq \ell \left( \| \nabla u_1^n \|_{L^2}^2 - \mathbb{F}_p(u_1^n, u_1^n) - \mathbb{F}_p(u_1^n, u_2^n) \right) \]

\[ = E(u_1^n) - \mathbb{F}_p(u_1^n, u_2^n) + \frac{T_1}{M_1} (E(v_1^n) - \mathbb{F}_p(u_1^n, u_2^n)) \]

Using (3.3), (3.5), and the fact \( A_1 < A_2 \), it follows that

\[ I_{M+T}^{(2)} \leq I_{M}^{(2)} + \lim_{n \to \infty} E(\tilde{v}_2^n) + \frac{T_1}{M_1} (A_1 M_1) \]

\[ < I_{M}^{(2)} + \lim_{n \to \infty} E(\tilde{v}_2^n) + T_1 A_2 \]

\[ = I_{M}^{(2)} + \lim_{n \to \infty} \mathcal{I}(v_1^n, \tilde{v}_2^n) = I_{M}^{(2)} + I_T^{(2)}. \]

The proof in the case \( A_1 > A_2 \) goes through unchanged after swapping the indices and so we do not repeat here. Next suppose that \( A_1 = A_2 \). We consider two subcases: \( B_1 \leq B_2 \) and \( B_1 \geq B_2 \). Suppose first that \( A_1 = A_2 \) and \( B_1 \leq B_2 \). Let \( \ell \) be defined as above and \( s = (M_2 + T_2)/M_2 \). Then we have that

\[ I_{M+T}^{(2)} \leq \lim_{n \to \infty} \mathcal{I}(\ell^{1/2} u_1^n, s^{1/2} u_2^n) \]

\[ = \lim_{n \to \infty} \left( E(\ell^{1/2} u_1^n) + E(s^{1/2} u_2^n) - \mathbb{F}_p(\ell^{1/2} u_1^n, s^{1/2} u_2^n) \right). \]
Since $\ell > 1$, $s > 1$, and $p \geq 2$, we have that $\ell^{p/2} \geq \ell$ and $s^{p/2} \geq s > 1$. It follows that

$$F_2^p(\ell^{1/2}u_1^n, \ell^{1/2}u_1^n) = \ell^p F_2^p(u_1^n, u_1^n) \geq \ell F_2^p(u_1^n, u_1^n),$$

$$F_p(\ell^{1/2}u_1^n, s^{1/2}u_2^n) = \ell^{p/2} s^{p/2} F_p(u_1^n, u_2^n) \geq \ell F_p(u_1^n, u_2^n).$$

Using these observations, a similar argument as in (3.5) yields

$$E(\ell^{1/2}u_1^n) - F_p(\ell^{1/2}u_1^n, s^{1/2}u_2^n) \leq \ell (E(u_1^n) - F_p(u_1^n, u_2^n))$$

$$= E(u_1^n) - F_p(u_1^n, u_2^n) + \frac{T_1}{M_1} (E(u_1^n) - F_p(u_1^n, u_2^n)) \quad (3.7)$$

Using Lemma 2.3, there exists $\delta > 0$ such that for sufficiently large $n$,

$$E(s^{1/2}u_2^n) \leq s E(u_2^n) - \delta = E(u_2^n) + \frac{T_2}{M_2} E(u_2^n) - \delta. \quad (3.8)$$

Inserting (3.7) and (3.8) into (3.6) and using the assumptions $A_1 = A_2$ and $B_1 \leq B_2$, we obtain that

$$I_{M+T}^{(2)} \leq \lim_{n \to \infty} \left( I(u_1^n, u_2^n) + \frac{T_2}{M_2} E(u_2^n) + \frac{T_1}{M_1} (E(u_1^n) - F_p(u_1^n, u_2^n)) \right) - \delta$$

$$= I_M^{(2)} + \frac{T_2}{M_2} (M_2 B_1) + \frac{T_1}{M_1} (T_1 A_1) - \delta$$

$$\leq I_M^{(2)} + T_2 B_2 + T_1 A_2 - \delta$$

$$= I_M^{(2)} + \lim_{n \to \infty} I(v_1^n, v_2^n) - \delta = I_M^{(2)} + I_T^{(2)} - \delta,$$

which gives the desired strict inequality. The proof in the case $A_1 = A_2$ and $B_1 \geq B_2$ follows a similar argument and we do not repeat here. \(\Box\)

The following lemma establishes (3.1) in the case ($b_1$).

**Lemma 3.3.** For any $T \in \mathbb{R}_+^2$ and $M = (0, M_2)$ with $M_2 > 0$, one has

$$I_{M+T}^{(2)} < I_M^{(2)} + I_T^{(2)}.$$

**Proof.** Let $\{(0, u_2^n)\}$ and $\{(v_1^n, v_2^n)\}$ be any sequences in $Y_2$ satisfying

$$\lim_{n \to \infty} \|u_2^n\|_{L^2}^2 = M_2, \quad \lim_{n \to \infty} \|v_j^n\|_{L^2}^2 = T_j,$$

$$\lim_{n \to \infty} I(0, u_2^n) = I_M^{(2)}, \quad \text{and} \quad \lim_{n \to \infty} I(v_1^n, v_2^n) = I_T^{(2)}.$$

As in the previous lemma, after passing to a subsequence if necessary, we consider the following values

$$D_1 = \frac{1}{M_2} \lim_{n \to \infty} (E(u_2^n) - F_p(v_1^n, u_2^n)), \quad D_2 = \frac{1}{T_2} \lim_{n \to \infty} (E(v_2^n) - F_p(v_1^n, v_2^n)).$$
We consider three cases: \( D_1 < D_2 \), \( D_1 > D_2 \), and \( D_1 = D_2 \). Assume first that \( D_1 < D_2 \). Let \( s = (M_2 + T_2)/M_2 \). Since \( s > 1 \) and \( p \geq 2 \), it follows that
\[
I^{(2)}_{\Delta M + T} \leq \lim_{n \to \infty} T(v^n_1, s^{1/2}u^n_2) \\
= \lim_{n \to \infty} \left( E(v^n_1) + s\|\nabla u^n_2\|_2^2 - s^p \mathbb{E}_{2p} (u^n_2, u^n_2) - s^{p/2} \mathbb{F}_p (v^n_1, u^n_2) \right) \\
\leq \lim_{n \to \infty} (E(v^n_1) + sE(u^n_2) - s\mathbb{F}_p (v^n_1, u^n_2)).
\] (3.9)

Since \( E(f) - \mathbb{F}_p (f, g) \leq T(0, f) \) and \( D_1 < D_2 \), it follows from (3.9) that
\[
I^{(2)}_{\Delta M + T} \leq I^{(2)}_M + \lim_{n \to \infty} \left( E(v^n_1) + \frac{T_2}{M_2} (E(u^n_2) - \mathbb{F}_p (v^n_1, u^n_2)) \right) \\
= I^{(2)}_M + \lim_{n \to \infty} E(v^n_1) + \frac{T_2}{M_2} (M_2 D_1) \\
< I^{(2)}_M + \lim_{n \to \infty} E(v^n_1) + T_2 D_2 \\
= I^{(2)}_M + \lim_{n \to \infty} T(v^n_1, v^n_2) = I^{(2)}_M + I^{(2)}_T,
\] (3.10)

which is the desired strict inequality. The proof in the case \( D_1 > D_2 \) follows the same steps and we omit the details. Now consider the case that \( D_1 = D_2 \). Let \( f^n_2 = s^{1/2}u^n_2 \), where \( s \) is defined as above. Then, using Lemma 2.3, there exists a number \( \delta > 0 \) such that for sufficiently large \( n \),
\[
E(f^n_2) = E(s^{1/2}u^n_2) \leq sE(u^n_2) - \delta.
\] (3.11)

Since \( s > 1 \) and \( p \geq 2 \), we have that \( s^{p/2} \geq s \). Then it is easy to see that \( \mathbb{F}_p (f, s^{1/2}g) \geq s\mathbb{F}_p (f, g) \). Using this observation and (3.11), we obtain that
\[
I^{(2)}_{\Delta M + T} \leq \lim_{n \to \infty} T(v^n_1, s^{1/2}u^n_2) \\
= \lim_{n \to \infty} \left( E(v^n_1) + E(s^{1/2}u^n_2) - \mathbb{F}_p (v^n_1, s^{1/2}u^n_2) \right) \\
\leq \lim_{n \to \infty} (E(v^n_1) + sE(u^n_2) - s\mathbb{F}_p (v^n_1, u^n_2)) - \delta.
\] (3.12)

Once we have obtained (3.12), the desired strict inequality follows using the same lines as in (3.10). \( \square \)

To complete the proof of Lemma 3.1, it only remains to establish (3.1) in the case \((b_2)\). This will be done in the next lemma.

\[ \textbf{Lemma 3.4.} \text{ For any } M \in \{0\} \times \mathbb{R}_+ \text{ and } T \in \mathbb{R}_+ \times \{0\}, \text{ one has } \]
\[
I^{(2)}_{\Delta M + T} < I^{(2)}_M + I^{(2)}_T.
\]

\[ \text{Proof.} \text{ Using Lemma 2.4, let } \phi_{M_2} > 0 \text{ and } \phi_{T_1} > 0 \text{ be such that } \]
\[
E(\phi_{M_2}) = \inf \left\{ E(f) : f \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |f|^2 \, dx = M_2 \right\},
\]
\[
E(\phi_{T_1}) = \inf \left\{ E(f) : f \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |f|^2 \, dx = T_1 \right\}.
\]
Then it is obvious that $F_p(\phi_{M_2}, \phi_{T_1}) > 0$. Thus, it follows that
\[
I_{M+T}^{(2)} \leq T(\phi_{M_2}, \phi_{T_1}) = E(\phi_{M_2}) + E(\phi_{T_1}) - F_p(\phi_{M_2}, \phi_{T_1})
\]
\[
= I_M^{(2)} + I_T^{(2)} - F_p(\phi_{M_2}, \phi_{T_1}) < I_M^{(2)} + I_T^{(2)},
\]
which is the desired strict inequality. \hfill \Box

We are now able to rule out the case $0 < Z^{(2)} < M_1 + M_2$.

**Lemma 3.5.** Suppose that $\{u^n_1, u^n_2\}_{n \geq 1} \subset Y_2$ be any minimizing sequence of $I_M^{(2)}$ and $Z^{(2)}$ be defined by (2.10) with $m = 2$. Then, one has
\[
Z^{(2)} = M_1 + M_2.
\]

**Proof.** Since the case $Z^{(2)} = 0$ has been ruled out, we show that $Z^{(2)} \notin (0, M_1 + M_2)$. Suppose that $Z^{(2)} \in (0, M_1 + M_2)$ holds. Let $T$ be defined as in Lemma 2.9 and define $S = (S_1, S_2)$ by $S_j = M_j - T_j$, $j = 1, 2$. Then, we have that $S + T \in \mathbb{R}^2_+$. We also have $T_1 + T_2 = Z^{(2)} > 0$ and
\[
S_1 + S_2 = M_1 + M_2 - (T_1 + T_2) = M_1 + M_2 - Z^{(2)} > 0.
\]
Applying Lemma 3.1, we then have
\[
I_T^{(2)} + I_S^{(2)} > I_{S+T}^{(2)}.
\]
This is same as $I_T^{(2)} + I_S^{(2)} > I_M$, contradicting the result of Lemma 2.9. This proves that $Z^{(2)} \notin (0, M_1 + M_2)$ and we must have $Z^{(2)} = M_1 + M_2$. \hfill \Box

**Lemma 3.6.** For every $M \in \mathbb{R}^2_+$, the set $\Lambda^{(2)}(M)$ is nonempty. Moreover, the following statements hold.

(i) For every $(\phi_1, \phi_2) \in \Lambda^{(2)}(M)$, there exists $\lambda_1$ and $\lambda_2$ such that
\[
(\psi_1(x, t), \psi_2(x, t)) = (e^{-i\lambda_1 t} \phi_1(x), e^{-i\lambda_2 t} \phi_2(x)) \tag{3.13}
\]
is a standing-wave solution of (1.5) with $m = 2$.

(ii) The Lagrange multipliers $\lambda_1$ and $\lambda_2$ satisfy $\lambda_j > 0$.

(iii) For every $(\phi_1, \phi_2) \in \Lambda^{(2)}(M)$ there exists $\theta_j \in \mathbb{R}$ and real-valued functions $\phi_{M_1}$ and $\phi_{M_2}$ such that
\[
\phi_{M_j}(x) > 0 \quad \text{and} \quad \phi_j(x) = e^{i\theta_j} \phi_{M_j}(x), \ x \in \mathbb{R}^N.
\]

**Proof.** Let $(\phi_1, \phi_2) \in \Lambda^{(2)}(M)$. Then the Lagrange multiplier principle implies that each function $(\phi_1, \phi_2)$ satisfies Euler-Lagrange equations
\[
-\Delta \phi_j + \lambda_j \phi_j = \sum_{k=1}^2 (W \ast |\phi_k|^p)|\phi_j|^p \phi_j, \ 1 \leq j \leq 2, \tag{3.14}
\]
where $\lambda_1$ and $\lambda_2$ are Lagrange multipliers. Consequently the function $(\psi_1, \psi_2)$ defined by (3.13) is a standing wave for (1.5) with $m = 2$. Multiplying the first equation by $\overline{\phi_1}$ and
the section equation by \( \overline{\phi} \), and integrating by parts, we get
\[
- \lambda_j \| \phi_j \|_{L^2}^2 = \| \nabla \phi_j \|_{L^2}^2 - \sum_{k,j=1}^2 \int_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y)Q(\phi_k, \phi_j) \, dxdy
\]
\[
= \| \nabla \phi_j \|_{L^2}^2 - 2p(\mathbb{F}_{2p}(\phi_1, \phi_1) + \mathbb{F}_{2p}(\phi_2, \phi_2) + \mathbb{F}_p(\phi_1, \phi_2)).
\]
Applying Lemma 2.5 with \((u^n_1, u^n_2) = (\phi_1, \phi_2)\), it follows that there exists \( \delta_j > 0 \) such that
\[
\| \nabla \phi_j \|_{L^2}^2 - 2p(\mathbb{F}_{2p}(\phi_1, \phi_1) + \mathbb{F}_{2p}(\phi_2, \phi_2) + \mathbb{F}_p(\phi_1, \phi_2)) < 0.
\]
Since \( 2p > 2 \), it follows that the right-hand side of (3.15) is negative. Then it follows that \( \lambda_j \) must be positive.

Next, let \((\phi_1, \phi_2) \in \Lambda^{(2)}(M)\) be a complex-valued minimizer of \( I^{(2)}_{M_1, M_2} \). Using the fact that \( u \in H^1(\mathbb{R}^N) \Rightarrow |u| \in H^1(\mathbb{R}^N), \| \nabla |u| \|_{L^2} \leq \| \nabla u \|_{L^2} \), it follows that \((|\phi_1|, |\phi_2|) \in \Lambda^{(2)}(M)\) as well. By the strong maximum principle, we infer that
\[
|\phi_1| > 0 \text{ and } |\phi_2| > 0.
\]
We have that
\[
I(|\phi_1|, |\phi_2|) - I(\phi_1, \phi_2) = \frac{1}{2} \sum_{j=1}^2 \| \nabla |\phi_j| \|_{L^2}^2 - \frac{1}{2} \sum_{j=1}^2 \| \nabla \phi_j \|_{L^2}^2.
\]
Since both \((\phi_1, \phi_2)\) and \((|\phi_1|, |\phi_2|)\) belong to \( \Lambda^{(2)}(M)\), the only possibility (3.16) can happen is that
\[
\int_{\mathbb{R}^N} |\nabla \phi_j|^2 \, dx = \int_{\mathbb{R}^N} |\nabla |\phi_j||^2 \, dx, \ j = 1, 2.
\]
Once we have (3.17), a number of techniques are available to prove item (iii) of Lemma 3.6 (see for example, Theorem 5 of [5]).

4. THE PROBLEM WITH THREE CONSTRAINTS

In this section we prove the strict subadditivity inequality for \( I^{(3)}_M \) and rule out the possible dichotomy of the minimizing sequences. Throughout this section we shall use the following notation:
\[
Q(f, g, h) = E(g) - \mathbb{F}_p(f, g) - \mathbb{F}_p(g, h),
\]
\[
\mathcal{D}(f, g, h) = Q(f, g, h) - \mathbb{F}_p(f, h),
\]
where the functional \( E \) is as defined in Lemma 2.3. With these definitions, we can write
\[
I(f, g, h) = Q(f, g, h) + \langle E \rangle(f, h) - \mathbb{F}_p(f, h)
\]
\[
= \mathcal{D}(f, g, h) + \langle E \rangle(f, h)
\]
\[
= Q(g, f, h) + \langle E \rangle(g, h) - \mathbb{F}_p(g, h).
\]
The strict subadditivity condition for the function \( I^{(3)}_M \) takes the following form
Lemma 4.1. Let $\mathbb{R}_{\geq 0} = [0, \infty)$. For any $M, T \in \mathbb{R}^3_{\geq 0}$ satisfying $M, T \neq \{0\}$ and $S = M + T \in \mathbb{R}^3$, one has

$$I_S^{(3)} < I_M^{(3)} + I_T^{(3)}.$$  \hspace{1cm} (4.2)

**Proof of Lemma 4.1.** We use ideas from [8, 9, 10]. Since $M_1 + T_1 > 0$, we have the following possibilities: $M_1 > 0$ and $T_1 > 0$; or $M_1 = 0$ and $T_1 > 0$; or $M_1 > 0$ and $T_1 = 0$. The third case $M_1 > 0$ and $T_1 = 0$ can be reduced to the second case by switching $M_1$ and $T_1$ and so we do not consider it.

In the case when $M_1 > 0$ and $T_1 > 0$, the following situations may arise:

(A1) $M_1 > 0$, $T_1 > 0$, $T_2 > 0$, $M_2 = 0$, $T_3 = 0$, $M_3 > 0$,

(A2) $M_1 > 0$, $T_1 > 0$, $T_2 > 0$, $M_2 = 0$, $T_3 > 0$, $M_3 = 0$,

(A3) $M_1 > 0$, $T_1 > 0$, $T_2 > 0$, $M_2 = 0$, $T_3 > 0$, $M_3 > 0$,

(A4) $M_1 > 0$, $T_1 > 0$, $T_2 = 0$, $M_2 > 0$, $T_3 = 0$, $M_3 > 0$,

(A5) $M_1 > 0$, $T_1 > 0$, $T_2 = 0$, $M_2 > 0$, $T_3 > 0$, $M_3 = 0$,

(A6) $M_1 > 0$, $T_1 > 0$, $T_2 = 0$, $M_2 > 0$, $T_3 > 0$, $M_3 > 0$,

(A7) $M_1 > 0$, $T_1 > 0$, $T_2 > 0$, $M_2 > 0$, $T_3 = 0$, $M_3 > 0$,

(A8) $M_1 > 0$, $T_1 > 0$, $T_2 > 0$, $M_2 > 0$, $T_3 > 0$, $M_3 = 0$,

(A9) $M_1 > 0$, $T_1 > 0$, $T_2 > 0$, $M_2 > 0$, $T_3 > 0$, $M_3 > 0$.

Similarly, in the second case, i.e., when $T_1 > 0$ and $M_1 = 0$, one has to consider the following cases:

(B1) $T_1 > 0$, $M_1 = 0$, $T_2 = 0$, $M_2 > 0$, $T_3 = 0$, $M_3 > 0$,

(B2) $T_1 > 0$, $M_1 = 0$, $T_2 = 0$, $M_2 > 0$, $T_3 > 0$, $M_3 = 0$,

(B3) $T_1 > 0$, $M_1 = 0$, $T_2 = 0$, $M_2 > 0$, $T_3 > 0$, $M_3 > 0$,

(B4) $T_1 > 0$, $M_1 = 0$, $T_2 > 0$, $M_2 = 0$, $T_3 = 0$, $M_3 > 0$,

(B5) $T_1 > 0$, $M_1 = 0$, $T_2 > 0$, $M_2 = 0$, $T_3 > 0$, $M_3 > 0$,

(B6) $T_1 > 0$, $M_1 = 0$, $T_2 > 0$, $M_2 > 0$, $T_3 = 0$, $M_3 > 0$,

(B7) $T_1 > 0$, $M_1 = 0$, $T_2 > 0$, $M_2 > 0$, $T_3 > 0$, $M_3 = 0$,

(B8) $T_1 > 0$, $M_1 = 0$, $T_2 > 0$, $M_2 > 0$, $T_3 > 0$, $M_3 > 0$.

To prove the lemma, it suffices to consider the cases (A9), (A3), (B3), (B5), and (B2); since otherwise we can switch the role of the parameters and reduce to one of these cases. We consider each of these cases separately in the next five lemmas.

Before we begin, we make the following observation. For any $\ell > 1$, define $u_\ell = \ell^{1/2} u_2$ and let $U = (u_1, u_\ell, u_3)$. Then we have that

$$Q(U) = \frac{\ell}{2} \|\nabla u_2\|_{L^2} - \ell p \mathbb{F}_2(u_2, u_2) - \ell^{p/2} \mathbb{F}_p(u_1, u_2) - \ell^{p/2} \mathbb{F}_p(u_2, u_3)$$

$$\leq \ell \left( \|\nabla u_2\|_{L^2} - \mathbb{F}_2(u_2, u_2) - \mathbb{F}_p(u_1, u_2) - \mathbb{F}_p(u_2, u_3) \right)$$

$$= \ell \left( E(u_2) - \mathbb{F}_p(u_1, u_2) - \mathbb{F}_p(u_2, u_3) \right) = \ell Q(u_1, u_2, u_3).$$  \hspace{1cm} (4.3)

The following lemma establishes (4.2) in the case (A9).

**Lemma 4.2.** For any $M, T \in \mathbb{R}^3_{\geq 0}$, one has $I^{(3)}_{M+T} < I^{(3)}_M + I^{(3)}_T$.

**Proof.** For every $M, T \in \mathbb{R}^3_{\geq 0}$, let $\{(u^n_1, u^n_2, u^n_3)\}_{n \geq 1}$ and $\{(v^n_1, v^n_2, v^n_3)\}_{n \geq 1}$ be minimizing sequences for $I^{(3)}_M$ and $I^{(3)}_T$, respectively. Without loss of generality, we may assume that
numbers

Substituting (4.6) into (4.5) and taking into account the observation (4.1), it follows that
\[ L \text{ switching indices and so we omit the details. Assume now that } \]

Define the pair of numbers \((L_1, L_2) \in \mathbb{R}^2\) as follows

\[ L_1 = T_1 \lim_{n \to \infty} Q(u^n_2, u^n_1, u^n_3) \quad \text{and} \quad L_2 = M_1 \lim_{n \to \infty} Q(v^n_2, v^n_1, v^n_3). \]

Then, the following situations may occur: \(L_1 < L_2\) or \(L_1 > L_2\), or \(L_1 = L_2\). Assume first that \(L_1 < L_2\). Define

\[
\overline{v}_2^n(\cdot) = v_2^n(\cdot + x_n \rho), \quad f_2^n = u_2^n + \overline{v}_2^n, \\
\overline{v}_3^n(\cdot) = v_3^n(\cdot + x_n \rho), \quad f_3^n = u_3^n + \overline{v}_3^n,
\]

where \(\rho\) is a unit vector in \(\mathbb{R}^N\) and \(x_n\) is such that \(x_n \to 0\) as \(n \to \infty\); \(\overline{v}_2^n\) and \(u^n_2\) have disjoint supports; and \(\overline{v}_3^n\) and \(u^n_3\) have disjoint supports. Let \(\ell = 1 + \frac{T_1}{M_1}\) and take the function \(f_1^n = \ell^{1/2} u^n_1\). Then \((f_1^n, f_2^n, f_3^n) \in \Sigma^{(3)}\) and we have

\[
I^{(3)}_{M+T} \leq \lim_{n \to \infty} T(f_1^n, f_2^n, f_3^n) \\
= \lim_{n \to \infty} (\langle E(f_1^n, f_2^n, f_3^n) - F_p(f_1^n, f_2^n) - F_p(f_2^n, f_3^n) - F_p(f_1^n, f_3^n) \rangle) \\
\leq \lim_{n \to \infty} (D(u_2^n, \ell^{1/2} u_1^n, u_3^n) + \langle E(u_2^n, u_3^n, \overline{v}_2^n, \overline{v}_3^n) \rangle - F_p(\overline{v}_2^n, \overline{v}_3^n)).
\]

Since \(\ell > 1\), using (4.3), it follows that

\[
D(u_2^n, \ell^{1/2} u_1^n, u_3^n) = Q(u_2^n, \ell^{1/2} u_1^n, u_3^n) - F_p(u_2^n, u_3^n) \\
\leq \ell Q(u_2^n, u_1^n, u_3^n) - F_p(u_2^n, u_3^n) \\
= Q(u_2^n, u_1^n, u_3^n) + \frac{T_1}{M_1} Q(u_2^n, u_1^n, u_3^n) - F_p(u_2^n, u_3^n).
\]

Substituting (4.6) into (4.5) and taking into account the observation (4.1), it follows that

\[
I^{(3)}_{M+T} \leq \lim_{n \to \infty} T(u_1^n, u_2^n, u_3^n) + \frac{T_1}{M_1} L_1 + \lim_{n \to \infty} (\langle E(\overline{v}_2^n, \overline{v}_3^n) \rangle - F_p(\overline{v}_2^n, \overline{v}_3^n)) \\
< I^{(3)}_M + \frac{L_2}{M_1} + \lim_{n \to \infty} (\langle E(\overline{v}_2^n, \overline{v}_3^n) \rangle - F_p(\overline{v}_2^n, \overline{v}_3^n)) \\
= I^{(3)}_M + \lim_{n \to \infty} T(u_1^n, u_2^n, u_3^n) = I^{(3)}_M + I^{(3)}_T,
\]

which is the desired strict inequality. The same argument applies in the case \(L_1 > L_2\) by switching indices and so we omit the details. Assume now that \(L_1 = L_2\) and consider the numbers

\[
\Pi_1 = \frac{1}{M_2} \lim_{n \to \infty} \left( E(u_2^n) - \frac{1}{p} \int_{\mathbb{R}^N} (W * |u_2^n|^p)|u_2^n|^p \, dx \right), \\
\Pi_2 = \frac{1}{T_2} \lim_{n \to \infty} \left( E(v_3^n) - \frac{1}{p} \int_{\mathbb{R}^N} (W * |v_3^n|^p)|v_3^n|^p \, dx \right).
\]

We split the proof into two subcases: \(\Pi_1 \leq \Pi_2\) and \(\Pi_1 \geq \Pi_2\). Since the proofs in both subcases are similar, we only consider \(L_1 = L_2\) and \(\Pi_1 \leq \Pi_2\). Let \(F_n = (f_1^n, f_2^n, f_3^n)\), where
\( f_1^n = \ell^{1/2}u^n_1 \) with \( \ell \) is defined as above, \( f_2^n = s^{1/2}u^n_2 \) with \( s = 1 + T_2/M_2 \), and \( f_3^n \) is defined as in \( (4.4) \). Since \( s > 1 \), using Lemma 2.3, there exists \( \delta > 0 \) such that

\[
E(f_2^n) = E(s^{1/2}u_2^n) \leq sE(u_2^n) - \delta
\]

for all sufficiently large \( n \). Since \( p \geq 2 \), we have that \( s^{p/2} \geq s > 1 \). Using this fact, it is easy to check that \( \mathbb{F}_p(s^{1/2}f, g) \geq s\mathbb{F}_p(f, g) \) and \( \mathcal{Q}(s^{1/2}f, g, h) \leq \mathcal{Q}(f, g, h) \). Making use of these observations, \( (4.3) \), \( (4.8) \), and taking into account the definitions \( \ell = (M_1 + T_1)/M_1 \) and \( s = (M_2 + T_2)/M_2 \), we obtain

\[
\mathcal{I}(F_n) \leq \mathcal{D}(f_2^n, f_1^n, u_3^n) + \langle E \rangle(f_2^n, u_3^n, \tilde{v}_3^n)
\]
\[
\leq \langle E \rangle(f_2^n, u_3^n, \tilde{v}_3^n) + \ell \mathcal{Q}(f_2^n, u_1^n, u_3^n) - \mathcal{F}_p(s^{1/2}u_2^n, u_3^n)
\]
\[
\leq \langle E \rangle(f_2^n, u_3^n, \tilde{v}_3^n) + \ell \mathcal{Q}(u_2^n, u_1^n, u_3^n) - s\mathcal{F}_p(u_2^n, u_3^n)
\]
\[
\leq \mathcal{I}(u_1^n, u_2^n, u_3^n) + \frac{T_1}{M_1} \mathcal{Q}(u_2^n, u_1^n, u_3^n) + E(\tilde{v}_3^n)
\]
\[
+ \frac{T_2}{M_2} (E(u_2^n) - \mathcal{F}_p(u_2^n, u_3^n)) - \delta
\]

Using this last estimate and making use of the assumptions \( L_1 = L_2 \) and \( \Pi_1 \leq \Pi_2 \), we obtain

\[
I_M^{(3)} + I_T^{(3)} \leq \lim_{n \to \infty} \mathcal{I}(F_n) \leq I_M^{(3)} + \frac{T_1}{M_1} I_T^{(3)} + \lim_{n \to \infty} E(v_3^n) + \frac{T_2}{M_2} (M_2 \Pi_1) - \delta
\]
\[
\leq I_M^{(3)} + \frac{L_2}{M_1} I_T^{(3)} + \lim_{n \to \infty} E(v_3^n) + T_2 \Pi_2 - \delta = I_M^{(3)} + I_T^{(3)} - \delta,
\]

which gives the desired strict inequality. \( \square \)

The next lemma establishes \( (4.2) \) in the case \( (A_3) \).

**Lemma 4.3.** For any \( T \in \mathbb{R}_+ \) and \( M \in \mathbb{R}_+ \times \{0\} \times \mathbb{R}_+ \), one has

\[
I_S^{(3)} < I_M^{(3)} + I_T^{(3)}, \quad S = M + T.
\]

**Proof.** Let \( \{(u_1^n, 0, u_3^n)\}_{n \geq 1} \) and \( \{(v_1^n, v_2^n, v_3^n)\}_{n \geq 1} \) be minimizing sequences for \( I_{M_1,0,M_3}^{(3)} \) and \( I_{T_1,T_2,T_3}^{(3)} \), respectively. Define the real numbers

\[
G_1 = T_1 \lim_{n \to \infty} \mathcal{Q}(v_2^n, v_1^n, u_3^n) \quad \text{and} \quad G_2 = M_1 \lim_{n \to \infty} \mathcal{Q}(v_2^n, v_1^n, v_3^n).
\]

Assume first that \( G_1 < G_2 \). Define \( f_3^n \) as follows

\[
\tilde{v}_3(\cdot) = v_3^n(\cdot + x_n \rho), \quad f_3^n = u_3^n + \tilde{v}_3^n,
\]

where \( \rho \) is a unit vector in \( \mathbb{R}^N \) ; and \( x_n \) is chosen such that \( x_n \to 0 \) as \( n \to \infty \), and \( u_3^n \) and \( \tilde{v}_3^n \) have disjoint supports. Take \( f_1^n = \ell^{1/2}u_1^n \) and \( f_2^n = v_2^n \), where \( \ell = 1 + \frac{T_1}{M_1} \). Let us write \( F_n = (f_1^n, f_2^n, f_3^n) \). Using the same argument as in \( (4.5) \) and \( (4.6) \), we can obtain

\[
\mathcal{I}(F_n) \leq \mathcal{I}(u_1^n, 0, u_3^n) + \langle E \rangle(v_2^n, \tilde{v}_3^n) + \frac{T_1}{M_1} \mathcal{Q}(v_2^n, u_1^n, u_3^n) - \mathcal{F}_p(v_2^n, \tilde{v}_3^n).
\]
Using this estimate and the assumption \( G_1 < G_2 \), it follows that
\[
I_S^{(3)} \leq \lim_{n \to \infty} \mathcal{I}(f_1^n, f_2^n, f_3^n)
\]
\[
\leq I_M^{(3)} + \frac{T_1 G_1}{M_1 T_1} + \lim_{n \to \infty} \langle (E)(v_2^n, \bar{v}_3^n) - \mathbb{E}_p(v_2^n, \bar{v}_3^n) \rangle
\]
\[
< I_M^{(3)} + \frac{G_2}{M_1} + \lim_{n \to \infty} \langle (E)(v_2^n, \bar{v}_3^n) - \mathbb{E}_p(v_2^n, \bar{v}_3^n) \rangle
\]
\[
= I_M^{(3)} + \lim_{n \to \infty} \mathcal{I}(v_1^n, v_2^n, v_3^n) = I_M^{(3)} + I_T^{(3)},
\]
which is the desired strict inequality. The proof for the case \( G_1 > G_2 \) goes through unchanged and we do not repeat here. Assume now that \( G_1 = G_2 \). As in the previous case, we consider the numbers
\[
\Gamma_1 = \frac{1}{M_3} \lim_{n \to \infty} \left( E(u_3^n) - \frac{1}{p} \int_{\mathbb{R}^N} (W * |u_3^n|^p)|v_2^n|^p \, dx \right),
\]
\[
\Gamma_2 = \frac{1}{T_3} \lim_{n \to \infty} \left( E(v_3^n) - \frac{1}{p} \int_{\mathbb{R}^N} (W * |v_3^n|^p)|v_2^n|^p \, dx \right)
\]
and split the proof into two subcases: \( \Gamma_1 \leq \Gamma_2 \) and \( \Gamma_1 \geq \Gamma_2 \). Consider the case that \( G_1 = G_2 \) and \( \Gamma_1 \leq \Gamma_2 \). Take the functions
\[
f_1^n = \ell^{1/2} u_1^n, \quad f_2^n = v_2^n, \quad \text{and} \quad f_3^n = t^{1/2} u_3^n,
\]
where \( \ell \) is defined as above and \( t = (M_3 + T_3)/M_3 \). Since \( p \geq 2 \) and \( t > 1 \), we have that \( t^{p/2} \geq t \). Then it is straightforward to see that \( \mathbb{E}_p(f, t^{1/2}g) \geq t \mathbb{E}_p(f, g) \) and \( \mathcal{Q}(f, g, t^{1/2}h) \geq \mathcal{Q}(f, g, h) \). Using these observations and (4.13), it follows that
\[
\mathcal{I}(F_n) = \mathcal{Q}(v_2^n, \ell^{1/2} u_1^n, t^{1/2} u_3^n) + \langle (E)(f_2^n, f_3^n) - t \mathbb{E}_p(f_2^n, f_3^n) \rangle
\]
\[
\leq \ell \mathcal{Q}(v_2^n, u_1^n, t^{1/2} u_3^n) + \langle (E)(f_2^n, f_3^n) - t \mathbb{E}_p(v_2^n, u_3^n) \rangle
\]
\[
\leq \ell \mathcal{Q}(v_2^n, u_1^n, u_3^n) + \langle (E)(f_2^n, f_3^n) - t \mathbb{E}_p(v_2^n, u_3^n) \rangle
\]
Since \( t > 1 \), by an application of Lemma 2.3 there exists \( \delta > 0 \) such that for all sufficiently large \( n \), we have
\[
E(f_3^n) - t \mathbb{E}_p(u_2^n, v_3^n) \leq t (E(u_3^n) - \mathbb{E}_p(v_2^n, u_3^n)) - \delta.
\]
Using the definitions of \( \ell \) and \( t \), it follows from (4.12) and (4.13) that
\[
\mathcal{I}(F_n) \leq \mathcal{I}(u_1^n, 0, u_3^n) + E(v_2^n) + \frac{T_1}{M_1} \mathcal{Q}(v_2^n, u_1^n, u_3^n) + \frac{T_3}{M_3} (E(u_3^n) - \mathbb{E}_p(v_2^n, u_3^n)) - \delta
\]
Using the estimate above and the assumptions \( G_1 = G_2 \), \( \Gamma_1 \leq \Gamma_2 \), it then follows that
\[
I_S^{(3)} \leq \lim_{n \to \infty} \mathcal{I}(f_1^n, f_2^n, f_3^n)
\]
\[
\leq I_M^{(3)} + \lim_{n \to \infty} E(v_2^n) + \frac{T_1 G_1}{M_1 T_1} + \frac{T_3}{M_3} (M_3 \Gamma_1) - \delta
\]
\[
\leq I_M^{(3)} + \lim_{n \to \infty} E(v_2^n) + \frac{G_2}{M_1} + T_3 \Gamma_2 - \delta
\]
\[
= I_M^{(3)} + \lim_{n \to \infty} \mathcal{I}(v_1^n, v_2^n, v_3^n) - \delta = I_M^{(3)} + I_T^{(3)} - \delta.
\]
which gives the desired strict inequality. The proof in the case \( G_1 = G_2 \) and \( \Gamma_1 \geq \Gamma_2 \) is similar and we omit it.

The next lemma establishes (4.2) in the case \((B_3)\).

**Lemma 4.4.** For any \( M \in \{0\} \times \mathbb{R}^2_+ \) and \( T \in \mathbb{R} \times \{0\} \times \mathbb{R}_+ \), one has

\[
I^{(3)}_{M+T} < I^{(3)}_M + I^{(3)}_T.
\]

**Proof.** Let \( \{(0, u^n_2, u^n_3)\}_{n \geq 1} \) and \( \{(v^n_1, 0, v^n_3)\}_{n \geq 1} \) be minimizing sequences for \( I^{(3)}_M \) and \( I^{(3)}_T \) respectively. Let \((C_1, C_2) \in \mathbb{R}^2\) be defined by

\[
C_1 = T_3 \lim_{n \to \infty} Q(v^n_1, u^n_3, u^n_2) \quad \text{and} \quad C_2 = M_3 \lim_{n \to \infty} Q(v^n_1, v^n_3, u^n_2).
\]

We consider two cases: \( C_1 \leq C_2 \) and \( C_1 \geq C_2 \). Suppose first that \( C_1 \leq C_2 \). Define \( F_n = (f^n_1, f^n_2, f^n_3) \) as follows

\[
f^n_1 = v^n_1, \quad f^n_2 = u^n_2, \quad \text{and} \quad f^n_3 = t^{1/2} u^n_3, \quad (4.15)
\]

where \( t \) is defined as in the previous case. Using Lemma 2.3, there exists \( \delta > 0 \) such that \( E(f^n_3) \leq tE(u^n_3) - \delta \) for sufficiently large \( n \). Then, by a direct computation and using the fact \( t^{p/2} \geq t \), we obtain that

\[
\mathcal{I}(F_n) = \langle E(F_n) \rangle - t^{p/2} F_p(u^n_3, v^n_1) - t^{p/2} F_p(u^n_3, u^n_2) - F_p(v^n_1, u^n_2) \\
\leq \langle E(F_n) \rangle - tF_p(u^n_3, v^n_1) - tF_p(u^n_3, u^n_2) - F_p(v^n_1, u^n_2) \\
\leq \langle E(v^n_1, u^n_2) \rangle + t Q(v^n_1, u^n_3, u^n_2) - F_p(v^n_1, u^n_2) - \delta \quad (4.16)
\]

Since \( \lim_{n \to \infty} Q(v^n_1, u^n_3, u^n_2) = C_1/T_3 \), \( \lim_{n \to \infty} \mathcal{I}(0, u^n_2, u^n_3) = I^{(3)}_M \), and \( F_p(v^n_1, u^n_2) \geq 0 \), it follows from (4.16) that

\[
I^{(3)}_{M+T} \leq \lim_{n \to \infty} \mathcal{I}(f^n_1, f^n_2, f^n_3) \\
\leq I^{(3)}_M + \lim_{n \to \infty} E(v^n_1) + \frac{T_3 C_1}{M_3} - \delta \\
\leq I^{(3)}_M + \lim_{n \to \infty} E(v^n_1) + \frac{C_2}{M_3} - \delta \\
\leq I^{(3)}_M + \lim_{n \to \infty} \mathcal{I}(v^n_1, 0, v^n_3) - \delta = I^{(3)}_M + I^{(3)}_T - \delta,
\]

which gives the desired strict inequality. The proof in the case \( C_1 \geq C_2 \) is similar and we do not repeat here.

The following lemma establishes (4.2) in the case \((B_5)\).

**Lemma 4.5.** For any \( M \in \{0\} \times \mathbb{R}_+ \) and \( T \in \mathbb{R}^3_+ \), one has

\[
I^{(3)}_{M+T} < I^{(3)}_M + I^{(3)}_T.
\]
Proof. Let \( \{(0, 0, u^n_3)\}_{n \geq 1} \) and \( \{(v^n_1, v^n_2, v^n_3)\}_{n \geq 1} \) be minimizing sequences for \( I_{0,0,M_3}^{(3)} \) and \( I_T^{(3)} \) respectively. Let \( (D_1, D_2) \in \mathbb{R}^2 \) be defined as

\[
D_1 = T_3 \lim_{n \to \infty} Q(v^n_1, u^n_3, v^n_2) \quad \text{and} \quad D_2 = M_3 \lim_{n \to \infty} Q(v^n_1, v^n_3, v^n_2).
\]

As before, we divide the proof into two cases \( D_1 \leq D_2 \) and \( D_1 \geq D_2 \). In the first case \( D_1 \leq D_2 \), define \( F_n = (f_1^n, f_2^n, f_3^n) \in Y_3 \) as follows

\[
f_1^n = v_1^n, \quad f_2^n = v_2^n, \quad \text{and} \quad f_3^n = t^{1/2}u_3^n,
\]

where \( t \) is given by \( t = (M_3 + T_3)/M_3 \). Using Lemma 2.3, there exists a number \( \delta > 0 \) such that \( E(f_3^n) \leq tE(u_3^n) - \delta \) for sufficiently large \( n \). Then, as in the previous case, it follows that

\[
\mathcal{I}(F_n) = \langle E \rangle(F_n) - t^{\alpha/2}F_p(u_3^n, v_1^n) - t^{\beta/2}F_p(u_3^n, v_2^n) - F_p(v_2^n, v_1^n)
\]

\[
\leq \langle E \rangle(v_1^n, v_2^n) + tQ(v_1^n, u_3^n, v_2^n) - F_p(v_2^n, v_1^n) - \delta
\]

\[
\leq \mathcal{I}(0, 0, u_3^n) + \langle E \rangle(v_1^n, v_2^n) + \frac{T_3}{M_3}Q(v_1^n, u_3^n, v_2^n) - \delta.
\]

Since \( \mathcal{I}(0, 0, u_3^n) \to I_{\phi}^{(3)} \) and \( D_1 \leq D_2 \), it follows from the above estimate that

\[
I_{M+T}^{(3)} \leq \lim_{n \to \infty} \mathcal{I}(f_1^n, f_2^n, f_3^n)
\]

\[
\leq I_M^{(3)} + \lim_{n \to \infty} \langle E \rangle(v_1^n, v_2^n) + \frac{T_3}{M_3} \frac{D_1}{T_3} - \delta
\]

\[
\leq I_M^{(3)} + \lim_{n \to \infty} \langle E \rangle(v_1^n, v_2^n) + \frac{D_2}{M_3} - \delta
\]

\[
= I_M^{(3)} + \lim_{n \to \infty} \mathcal{I}(v_1^n, v_2^n, v_3^n) - \delta = I_M^{(3)} + I_T^{(3)} - \delta,
\]

which gives the desired strict inequality. The case \( D_1 \geq D_2 \) uses the same argument and we do not repeat here. \qed

**Lemma 4.6.** For any \( M \in \{0\} \times \mathbb{R}_+ \times \{0\} \) and \( T \in \mathbb{R}_+ \times \{0\} \times \mathbb{R}_+ \), one has

\[
I_{M+T}^{(3)} < I_M^{(3)} + I_T^{(3)}.
\]

**Proof.** Using Lemma 2.4, let \( \phi_{M_2} > 0 \) be such that

\[
E(\phi_{M_2}) = \inf \{ E(f) : f \in H^1(\mathbb{R}^N) \text{ and } \|f\|_{L^2}^2 = M_2 \}.
\]

Lemma 3.6 implies that there exist functions \( \phi_T > 0 \) and \( \phi_{T_3} > 0 \) such that

\[
\mathcal{I}(\phi_T, \phi_{T_3}) = \inf \{ I(f,g) : f, g \in H^1(\mathbb{R}^N) \text{ and } \|f\|_{L^2}^2 = T_1, \|g\|_{L^2}^2 = T_3 \}.
\]

Clearly, we have that \( F_p(\phi_{M_2}, \phi_T) > 0 \) and \( F_p(\phi_{M_2}, \phi_{T_3}) > 0 \). Then we obtain

\[
I_{M+T}^{(3)} \leq \mathcal{I}(\phi_{M_2}, \phi_T, \phi_{T_3})
\]

\[
= E(\phi_{M_2}) + \mathcal{I}(\phi_T, \phi_{T_3}) - F_p(\phi_{M_2}, \phi_T) - F_p(\phi_{M_2}, \phi_{T_3})
\]

\[
= I_M^{(3)} + I_T^{(3)} - F_p(\phi_{M_2}, \phi_T) - F_p(\phi_{M_2}, \phi_{T_3}) < I_M^{(3)} + I_T^{(3)},
\]

which is the desired strict inequality. \qed
We have now completed the proof of Lemma 4.7. The next lemma rules out the case of dichotomy.

**Lemma 4.8.** Suppose that \( \{(u_1^n, u_2^n, u_3^n)\}_{n \geq 1} \subset Y_3 \) be any minimizing sequence of \( I_M^{(3)} \) and \( Z^{(3)} \) be defined by (2.10) with \( m = 3 \). Then, one has
\[
Z^{(3)} = M_1 + M_2 + M_3.
\]

**Proof.** The proof goes through unchanged as in the proof of Lemma 3.5 and we do not repeat here. □

**Lemma 4.9.** For every \( M \in \mathbb{R}^3_+ \), the set \( \Lambda^{(3)}(M) \) is nonempty. Moreover, the following statements hold.

(i) For every \( (\phi_1, \phi_2, \phi_3) \in \Lambda^{(3)}(M) \), there exists \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) such that
\[
(\psi_1(x, t), \psi_2(x, t), \psi_3(x, t)) = (e^{-i\lambda_1 t} \phi_1(x), e^{-i\lambda_2 t} \phi_2(x), e^{-i\lambda_3 t} \phi_3(x))
\]
is a standing-wave solution of (1.5) with \( m = 3 \).

(ii) The Lagrange multipliers \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) satisfy \( \lambda_j > 0 \).

(iii) For every \( (\phi_1, \phi_2, \phi_3) \in \Lambda^{(3)}(M) \) there exists \( \theta_j > 0 \) and real-valued functions \( \phi_{M_1}, \phi_{M_2}, \) and \( \phi_{M_3} \) such that
\[
\phi_{M_j}(x) > 0 \quad \text{and} \quad \phi_j(x) = e^{i\theta_j} \phi_{M_j}(x), \ x \in \mathbb{R}^N.
\]

**Proof.** The proof uses the same argument as in the proof of Lemma 3.6 and we omit the details. □

## 5. Proof of main results

We are now prepared to obtain our main results.

**Proof of Theorem 1.1.** The proof follows from Lemmas 4.8 and 3.6.

**Proof of Theorem 1.2.** Once we have obtained the relative compactness of minimizing sequences, the proof of stability result uses a classical argument (11) which we repeat here for the sake of completeness. Suppose that \( \Lambda^{(m)}(M) \) is not stable. Then there exist a number \( \epsilon > 0 \), a sequence of times \( t_n \), and a sequence \( \{\psi_n(x, 0)\} = \{(\psi_1^n(x, 0), ..., \psi_m^n(x, 0))\} \) in \( Y_m \) such that for all \( n \),
\[
\inf\{\|\psi_1^n(x, 0), ..., \psi_m^n(x, 0)\| - \phi\|_{Y_m} : \phi \in \Lambda^{(m)}(M)\} < \frac{1}{n}; \quad (5.1)
\]
and
\[
\inf\{\|\psi_1^n(\cdot, t_n), ..., \psi_m^n(\cdot, t_n)\| - \phi\|_{Y_m} : \phi \in \Lambda^{(m)}(M)\} \geq \epsilon, \quad (5.2)
\]
for all \( n \), where \( (\psi_1^n(x, t), ..., \psi_m^n(x, t)) \) solves (1.5) with initial data \( \psi_n(x, 0) \). Since \( \psi_n(x, 0) \) converges to an element in \( \Lambda^{(m)}(M) \) in \( Y_m \) norm, and since for \( \phi \in \Lambda^{(m)}(M) \), we have
\[
\|\phi_j\|_{L^2}^2 = M_j, \ 1 \leq j \leq m, \ \text{and} \ I(\phi) = I^{(m)}_M, \ \text{we therefore have}
\]
\[
\lim_{n \to \infty} \|\psi_{j,n}(x, 0)\|_{L^2}^2 = M_j, \ 1 \leq j \leq m, \ \text{and} \ \lim_{n \to \infty} I(\psi_n(x, 0)) = I^{(m)}_M.
\]
Let us denote \( \psi_{j,n}(\cdot, t_n) \) by \( U_{1,n}^j \) for \( 1 \leq j \leq m \). We now choose \( \{\alpha_j^n\} \subset \mathbb{R}^N \) such that
\[
\|\alpha_j^n \psi_{j,n}(x, 0)\|_{L^2}^2 = M_j, \ 1 \leq j \leq m
\]
for all \( n \). Thus \( \alpha^n_j \to 1 \) for each \( 1 \leq j \leq m \). Hence the sequence \( (f^n_1, \ldots, f^n_m) \) defined as \( f^n_j = \alpha^n_j U^n_j \) satisfies \( \|f^n_j\|_{L^2} = M_j \) and
\[
\lim_{n \to \infty} I(f^n_1, \ldots, f^n_m) = \lim_{n \to \infty} I(\psi_n(\cdot, t^n)) = \lim_{n \to \infty} I(\psi_n(x, 0)) = I^{(m)}_M.
\]
Therefore \( \{(f^n_1, \ldots, f^n_m)\} \) is a minimizing sequence for \( I^{(m)}_M \). From Theorem 1.1, it follows that for all \( n \) sufficiently large, there exists \( \phi_n \in \Lambda^{(m)}(M) \) such that
\[
\|(f^n_1, \ldots, f^n_m) - \phi_n\|_{Y^m} \leq \epsilon/2.
\]
But then we have
\[
\begin{align*}
\epsilon \leq & \|\psi_n(\cdot, t^n) - \phi_n\|_{Y^m} \\
\leq & \|\psi_n(\cdot, t^n) - (f^n_1, \ldots, f^n_m)\|_{Y^m} + \|(f^n_1, \ldots, f^n_m) - \phi_n\|_{Y^m} \\
\leq & |1 - \alpha^n_1| \cdot \|U^n_1\|_{H^1} + \ldots + |1 - \alpha^n_m| \cdot \|U^n_m\|_{H^1} + \frac{\epsilon}{2}
\end{align*}
\]
and by taking \( n \to \infty \), we obtain that \( \epsilon \leq \epsilon/2 \), a contradiction, and we conclude that \( \Lambda^{(m)}(M) \) must in fact be stable.

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