Slow-string limit and “antiferromagnetic” state
in AdS/CFT

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Abstract

We discuss a slow-moving limit of a rigid circular equal-spin string solution on $R \times S^3$. We suggest that the solution with the winding number equal to the total spin approximates the quantum string state dual to the maximal-dimension “antiferromagnetic” state of the $SU(2)$ spin chain on the gauge theory side. An expansion of the string action near this solution leads to a weakly coupled system of a sine-Gordon model and a free field. We show that a similar effective Hamiltonian appears in a certain continuum limit from the half-filled Hubbard model that was recently suggested to describe the all-order dilatation operator of the dual gauge theory in the $SU(2)$ sector. We also discuss some other slow-string solutions with one spin component in $AdS_5$ and one in $S^5$.

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1 Introduction

One implication of the AdS/CFT duality is that each free string state in $AdS_5 \times S^5$ should correspond to a certain single-trace gauge invariant operator in the large $N$ maximally supersymmetric SYM gauge theory; the quantum string energy $E$ should be equal to the quantum dimension $\Delta$ of the operator.

For example, if we look at the closed $SU(2)$ sector of operators like $\text{Tr}(\Phi_{J_1}^{J_1} \Phi_{J_2}^{J_2})$ built out of two complex combinations of $N = 4$ SYM scalars and diagonalize the corresponding dilatation operator (for a review see [1]) then each of its eigenstates with given R-charges $(J_1, J_2)$ should be dual to a particular string state with the same $SO(6)$ spins $(J_1, J_2)$, and one should have also $\Delta(J_1, J_2, m; \lambda) = E(J_1, J_2, m; \sqrt{\lambda})$ [2, 3, 4]. $\lambda$ in $\Delta$ is the ‘t Hooft coupling and $\sqrt{\lambda}$ in $E$ is the string tension; $m$ in $\Delta$ labels various eigenstates with fixed $J_1$ and $J_2$ while $m$ in $E$ stands for other “hidden” quantum numbers like winding number or number of folds of the string configuration.

The leading one-loop term in the dilatation operator in this sector can be identified with the Hamiltonian of the XXX$_{1/2}$ ferromagnetic spin chain of length $J = J_1 + J_2$ [5], while higher-loop corrections add long-range and multi-spin interaction terms [6, 7, 1]. For small $\lambda$ higher loop corrections are not expected to qualitatively change the structure of the spectrum of the XXX$_{1/2}$ model (modulo possible lifting of degeneracies), i.e. they should deform the eigenvalues order by order in $\lambda$. One may then conjecture that the same should be true also in the large $\lambda$ limit, i.e. the exact spectrum should have the same qualitative structure as the Heisenberg model spectrum. This conjecture seems to be supported, for large length $J$, by the close relation between the standard one-loop Bethe ansatz and the exact asymptotic Bethe ansatz [8] (see also [9]).

The AdS/CFT duality then implies that the Heisenberg model spectrum and the corresponding part of the quantum string spectrum should have the same qualitative structure. The spectrum of the one-loop ferromagnetic Heisenberg model ($\Delta = J + E^{(1)}$, $E^{(1)} = \lambda F_1(J_1, J_2)$) starts with the ground state represented by the BPS operator $\text{Tr} \Phi_{J}^{J}$ and dual to the point-like string moving along geodesic of $S^5$. Small fluctuations near the ground state, i.e. magnons, with $E^{(1)} \sim \lambda J$, are represented, at large $J$, by the BMN [2] operators (with $J_1 \gg J_2$); extrapolated to large coupling they are dual to small strings rotating within $S^3$ part of $S^5$ with their center of mass moving along the big circle.

States with higher energy may be described as bound states of magnons or “macroscopic strings” of Bethe roots [11, 12]. In the thermodynamic limit with $J_1 \sim J_2 \gg 1$ one finds, using the Bethe ansatz, that for them $E^{(1)} \sim \frac{\lambda}{2} f(J_1, J_2)$ [12]. They may be represented by “locally-BPS” $\text{Tr}(\Phi_{J_1}^{J_1} \Phi_{J_2}^{J_2})$ -type operators with slowly changing order
of \( \Phi_1 \) and \( \Phi_2 \) clusters. The strong-coupling extrapolation of these states are “fast” \( (J_1 \sim J_2, ~ J \sim \sqrt{\lambda} \gg 1) \) semiclassical strings whose world surface is approximately null \([4, 13, 14, 15, 16, 19]\).

Going higher in the energy, the spectrum is expected to contain, for \( J \gg 1 \), some “intermediate” states with \( E(1) \sim \lambda + O(1/J) \) and, finally, the highest energy state with \( E(1) \sim \lambda J + O(1) \). In the latter case the energy density will be approximately constant at large \( J \) instead of vanishing as for the magnons or “macroscopic strings”.

Indeed, the spectrum of the ferromagnetic Heisenberg chain is isomorphic to the spectrum of the antiferromagnetic chain: the two spectra are formally related by changing the sign of the overall coefficient \( \lambda \) or the sign of the energy.\(^5\) This implies that the highest-energy state in the Heisenberg ferromagnet spectrum is the same as the Neel-type antiferromagnetic (AF) ground state of the Heisenberg antiferromagnet, i.e. it should have \( J_1 = J_2 = J/2 \) and for \( J \gg 1 \) its energy should be \( E(1) = c_1 \lambda J \), \( c_1 = \ln 2/4\pi \) \([20, 21]\). The fluctuations near the AF state will lower the energy of the ferromagnetic chain, eventually filling up the part of the spectrum from the near-AF states with \( E(1) \sim \lambda J \) to the “intermediate” states with \( E(1) \sim \lambda \).

Beyond the 1-loop order one expects to find that the energy of the AF state should be given by (assuming \( J \gg 1 \) for any fixed \( \lambda \ll 1 \))

\[
E = f(\lambda) \cdot J, \quad f(\lambda \ll 1) = 1 + c_1 \lambda + c_2 \lambda^2 + \ldots. \tag{1.1}
\]

The exact expression for the “energy density” \( f(\lambda) \) was recently found by starting with the conjectured asymptotic BDS Bethe ansatz \([8, 9, 22]\)

\[
f(\lambda) = 1 + \frac{\sqrt{\lambda}}{\pi} \int_0^\infty \frac{dk}{k} J_0 \left( \frac{\lambda \pi}{2} k \right) \frac{J_1 \left( \frac{\lambda \pi}{2} k \right)}{e^k + 1}. \tag{1.2}
\]

Here \( J_n \) are the Bessel functions. The formal extrapolation of this expression to large \( \lambda \) gives \([22]\):

\[
f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi^2} + \frac{3}{4} + \ldots. \tag{1.3}
\]

The BDS ansatz is not expected to correctly represent the quantum string spectrum at the quantitative level, but it was previously found to lead to the same qualitative results for its low-energy part.\(^6\) The same is likely to be true also for the upper part of the spectrum (in the large \( J \) limit). Indeed, the arguments in \([22]\) suggest that one

\(^5\)Changing formally the sign of \( \lambda \) in the full all-loop dilatation operator will not, of course, lead to an antiferromagnetic chain with isomorphic spectrum; for example, the BDS ansatz \([8]\) has non-trivial dependence on \( \lambda \) through the magnon dispersion relation \( e(p) = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2} - 1} \) (cf. also \([9]\)). Still, as already mentioned above, the exact spin chain Hamiltonian is expected to have the spectrum (including its higher-energy near-antiferromagnetic-state part) which is a smooth \( \lambda \)-deformation of the Heisenberg model spectrum.

\(^6\)In particular, the low-energy Landau-Lifshitz-type effective actions corresponding to the BDS ansatz and the quantum string theory appear to have the same structure \([14, 23]\).
should find the same $f(\lambda \gg 1) \sim \sqrt{\lambda}$ scaling by starting with the string Bethe ansatz of [24] (though the proportionality coefficient is likely to be different from $1/\pi^2$).

It is then natural to conjecture that the energy of the string state dual to the AF state should scale in the same way at large $\lambda$ (i.e. in the classical string limit) and large $J$, i.e. we should find

$$E(\lambda \gg 1) \sim \sqrt{\lambda} J.$$  

(1.4)

This is not the familiar scaling for semiclassical strings of the type discussed in [3, 25, 4] (for which both the energy $E$ and the spin $J$ of the classical string are proportional to the string tension, i.e. scale as $\sqrt{\lambda}$ at large $\lambda$) so one may then question if the AF state can be represented by a semiclassical string.

Since the string should carry the large angular momentum $J$ one may hope that the corresponding quantum string state may still be approximated – in the large $\lambda$ limit – by a classical string configuration.

Our aim below will be to try to identify semiclassical string states that should be dual to the upper part of the gauge-theory spin chain spectrum in the limit of large $J$ and large $\lambda$. We shall find that there are indeed classical string solutions whose energy scales as (1.4). However, the semiclassical expansion here will have an unusual form, with subleading terms in the classical energy receiving contributions from higher orders in string $\alpha' \sim 1/\sqrt{\lambda}$ expansion. Also, the classical string solution will be unstable under small fluctuations. That direct semiclassical expansion may not apply here is not surprising since it is only the true quantum string state that should be dual to the AF state on the gauge theory state. While the AdS/CFT duality implies that the quantum string theory spectrum, being equivalent to the gauge spin chain spectrum, should be bounded from the above in the compact $SU(2)$ case [22, 9], adding small fluctuations to a semiclassical string one can always increase the energy.

Our main observation is that while the lower part of the $SU(2)$ spin chain spectrum is dual to fast-moving strings (which are “locally null-geodesic” or “locally BPS”), the upper part appears to be dual to slow-moving long strings which are as far as possible from the BPS limit. While for the fast strings the time ($\tau$) evolution of the string configuration dominates over the spatial ($\sigma$) evolution, with each bit of the string having a near-null-geodesic trajectory, for the slow long string one has just the opposite: each of its bits moves very slowly. The slow motion is not in contradiction with the assumption that $J \gg 1$: the effective string rotation frequency $J = \sqrt{\lambda}$ is very small in the classical string limit $\sqrt{\lambda} \gg 1$ if we assume that $\sqrt{\lambda} \gg J$. This should be contrasted with the fast string case where $J$ was fixed in the limit of $\sqrt{\lambda} \gg 1$, i.e. the related question was already studied in [27] for 2-spin string states in $AdS_5$ dual to gauge-theory operators built out of self-dual part of gauge field strength whose 1-loop anomalous dimensions are described by antiferromagnetic $XXX_1$ spin chain [28].
\( J \) and \( \sqrt{\lambda} \) were of the same order \( [4] \).

We propose that the quantum string state representing the AF state of the gauge theory may be approximated by the simplest rotating string solution in \( S^5 \) \([29]\): a circular string moving in \( S^3 \) part of \( S^5 \) with two equal angular momenta \( J_1 = J_2 = J/2 \) and wound along a big circle. The winding number \( m \) should be equal to the total momentum \( J \), i.e. the total length of the string in \( \sigma \) direction should be proportional to \( J \).

The assumption that the winding number \( m \) should be proportional to the angular momentum \( J \) is a natural one from the spin chain/Bethe ansatz point of view: for the AF state case the excitation momenta \( p_i \) determined by the Bethe ansatz and thus the energy density \( E/J \) (with \( E = \sum_{i=1}^{J/2} [\sqrt{1 + \frac{\lambda}{\pi} \sin^2 \frac{2}{\lambda} - 1}] \)) should be constant in the large length \( J \) limit (similar limit was considered in \([32]\)). The classical energy of the circular string \( E = \sqrt{J^2 + m^2 \lambda} \) with \( m = J \) becomes \( E = \sqrt{1 + \lambda J} = (\sqrt{1 + \frac{1}{2}\lambda} + ...) J \).

Here the first term in the large \( \lambda \) expansion is indeed the same as in \([1, 4]\). Only this leading term in the large \( \lambda \) expansion of the classical energy should be trusted since the subleading terms will receive contributions from higher quantum string corrections (see below).

A qualitative reason for the existence of the “slow” string states is the compactness of \( S^5 \): in flat space the closed string needs to rotate or pulsate to balance its tension, while on a sphere it can be wrapped on a big circle and thus can be static (to embed such a state in the \( SU(2) \) sector we need still to add two angular momenta and take \( J \) large). The apparent small-fluctuation instability of the wrapped (and rotating) classical string solution \([4]\) may be interpreted as an indication that the corresponding quantum state has the maximal energy for given spins \( J_1 = J_2 \).

The rest of the paper is organized as follows. In section 2 we shall review the circular two-spin solution and consider its fast and slow limits.

In section 3 we shall discuss the effective Hamiltonian for the fluctuations around the string state corresponding to the AF state of the gauge theory chain. It is found by expanding the string Hamiltonian near the circular string solution and is expected to be related to a strong coupling limit of an effective action describing fluctuations near the AF state of the gauge-theory spin chain. In contrast to the \( \text{XXX}_{1/2} \) case this Hamiltonian need not be just that of a relativistic sigma model on \( S^2 \), but may be related to a bosonized field theory limit of a Hubbard-type model that may represent the all-loop dilatation operator of gauge theory in the \( SU(2) \) sector \([9]\).

Indeed, we will show in section 4 that a similar effective Hamiltonian appears in a continuum limit from the half-filled Hubbard model. The bosonized Hamiltonian exhibits a certain discontinuous behaviour as we move away from half-filling. This bears certain similarities with the non-closure of the \( SU(2) \) sector at large coupling for

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8 There the effective coupling \( \hat{\lambda} = \frac{1}{J^2} = \frac{1}{J} \) was fixed while \( \lambda \) was taken large. One could then expand in powers of \( \hat{\lambda} \) at each order of semiclassical expansion in \( \frac{1}{\lambda} \).
states with $J_1 \neq J_2$ \cite{10} suggesting that the Hubbard model may need to be modified to take this into account.

In section 5 we shall consider a similar slow string limit of some string solutions with one spin in $AdS_5$ and one spin in $S^5$ that are related, in particular, to the $SL(2)$ sector on the gauge side. Quantum 1-loop correction to the energy of the latter solution will be discussed in Appendix.

2 Circular rotating $J_1 = J_2$ string on $S^3$

Here we shall start with recalling the form of the simplest 2-spin solution \cite{4} for the string on $R_t \times S^3$ (in the form given in \cite{29}) and then discuss its new “slow-string” limit.

2.1 Classical string energy and its limits

Parameterizing $S^3$ by two complex coordinates $X_i$ with $|X_1|^2 + |X_2|^2 = 1$, the classical string equations in conformal gauge may be written as $\partial^2 X_i + \Lambda X_i = 0$, where $\Lambda = \partial^m X_i \partial_m X_i$. One finds the following solution

$$t = \kappa \tau \ , \quad X_1 = a e^{i w \tau + i m \sigma} \ , \quad X_2 = a e^{i w \tau - i m \sigma} \ , \quad a = \frac{1}{\sqrt{2}} ,$$

(2.1)

where $m$ is an integer winding number (we shall choose it to be positive). Note that a similar solution exists in flat space where $w = m$ ($\Lambda = 0$) and $a$ is arbitrary. The conformal gauge constraint gives $\kappa^2 = w^2 + m^2$. The corresponding $SO(4)$ spins and the energy are ($T = \frac{\sqrt{\lambda}}{2\pi}$ is the string tension)

$$J_1 = J_2 = J/2 \ , \quad J = \sqrt{\lambda} w \ , \quad (2.2)$$

$$E = \sqrt{\lambda} \kappa = \sqrt{\lambda} \sqrt{w^2 + m^2} = \sqrt{J^2 + \lambda m^2} . \quad (2.3)$$

The quadratic fluctuation spectrum near this solution was found in \cite{4, 29}. There are 4 massive $AdS_5$ fluctuations with mass $\kappa$, i.e. with the characteristic frequency $\omega_n = \sqrt{n^2 + \kappa^2} = \sqrt{n^2 + w^2 + m^2}$. In addition, there is a free massive field from $S^5$ with mass $w^2 - m^2$, i.e. with $\omega_n = \sqrt{n^2 + w^2 - m^2}$ which is real if

$$n^2 + w^2 - m^2 \geq 0 . \quad (2.4)$$

The remaining three $S^5$ bosonic fluctuations are coupled and the corresponding frequencies are given by \cite{29}

$$\omega_n^2 = n^2 + 2w^2 \pm 2\sqrt{w^4 + n^2 w^2 + m^2 n^2} . \quad (2.5)$$
Their reality condition is

\[ n \geq 2m , \tag{2.6} \]

which, if satisfied, implies also (2.4). As a result, there is always a finite number of unstable modes with \( 0 < n < 2m \), i.e. the solution is always unstable.

Returning to the classical energy, we see that it is a function of three independent parameters: \( \lambda, J, m \). Taking different limits of these parameters one finds special cases of this solution that have different physical interpretation.

Let us first consider the cases where the standard semiclassical expansion applies, which assumes that the parameters of the solution \( w, m \) are fixed while \( \lambda \) is taken to be large to suppress quantum string (inverse tension) corrections. Then \( J = \sqrt{\lambda} w \) is also large, and \( J \gg m \).

There are several possible choices of the rotation velocity \( w \) and the winding number \( m \):

(i) \( m = 0, \ w \neq 0 \): this is the point-like (BPS) case with \( E = J \).\(^9\)

(ii) \( m \ll w \): this is the “fast string” case of [4] when \( E \) has a regular expansion in the small semiclassical parameter \( m^2 w^2 = m^2 \tilde{\lambda}, \ \tilde{\lambda} \equiv \frac{\lambda}{J^2} \):

\[
E = J \sqrt{1 + m^2 \tilde{\lambda}} = J (1 + \frac{1}{2} m^2 \tilde{\lambda} + ... ) \tag{2.7}
\]

Here the time evolution is dominating over the spatial evolution: the effective string tension \( \frac{m \sqrt{\lambda}}{J} \) is small.

(iii) \( m = w \): such solution is formally the same as in flat space (the Lagrange multiplier \( \Lambda \) vanishes), but the classical energy here is still linear in \( J \):

\[
E = \sqrt{2} J \tag{2.8}
\]

(iv) \( m \gg w \): here \( \kappa \) or \( E \) can be expanded in \( \frac{w}{m} \ll 1 \) getting

\[
E = \sqrt{\lambda} m \sqrt{1 + \frac{J^2}{m^2 \lambda}} = \sqrt{\lambda} m + \frac{J^2}{2m \sqrt{\lambda}} + ... . \tag{2.9}
\]

\( m \) may be of order 1 or much bigger than 1 but is still much smaller than \( J = \sqrt{\lambda} w \) since \( \lambda \) is assumed to be taken to be large first.\(^10\)

The cases (iii) and (iv) are different from the fast-moving string case (ii) where string world-surface is nearly null. In the “slow string” case of (iv) the \( \sigma \) dependence

\(^9\)The corresponding massless geodesic runs along big circle in the “diagonal” 2-plane in \((X_1, X_2)\) space.

\(^10\)The scaling of the energy of long wound strings with winding \( E = \sqrt{\lambda} m + ... \) was observed in the uniform gauge Hamiltonian formalism in [30]; however, in contrast to [4] and the present discussion, the winding there was assumed to be in the same direction as the momentum \( J \). Similar behavior is found also in the \( su(1|1) \) sector which was analyzed in detail (for any \( J \) and \( m \)) in [31].
dominates over \( \tau \) dependence, and a reflection of that is the explicit dependence of the classical energy on the string tension \( \sqrt{\lambda} \) (in the fast string case the classical energy depends only on the square of string tension, i.e. is analytic in \( \lambda \) [4]). Such slow strings should correspond to an intermediate part of the spin chain spectrum where the energy scales as \( (J \gg 1) \)

\[
E = f(\lambda) + O\left(\frac{1}{J}\right),
\]

\[
f(\lambda \ll 1) = a_1 \lambda + a_2 \lambda^2 + \ldots, \\
f(\lambda \gg 1) = b_1 \sqrt{\lambda} + b_2 + \ldots.
\]

To sum up, fixing the spins \( J_1 = J_2 = J/2 \) we may label the string or the corresponding spin chain states by growing values of \( m \); then the energy increases

- from \( E(m = 0) = J \)
- to \( E(m \ll \frac{J}{\sqrt{\lambda}}) = J + \frac{m^2 \lambda}{J^2} + \ldots \)
- to \( E(m = \frac{J}{\sqrt{\lambda}}) = \sqrt{2}J \)
- to finally to \( E(m \gg \frac{J}{\sqrt{\lambda}}) = \sqrt{\lambda}m + \ldots \).

One may wonder what will happen if we increase \( m \) or the string length further. The spin chain correspondence suggests that the highest possible value of \( m \) should be \( J \) (which takes integer values in the quantum theory). If we assume that\(^{11}\)

\[(v) : \\
m = J \gg 1,
\]

but still \( m \ll \sqrt{\lambda} \) then \( w = \frac{J}{\sqrt{\lambda}} = \frac{m}{\sqrt{\lambda}} \ll 1 \). Hence in this case the string motion is “very slow” for large tension: the string wrapped many times on big circle is nearly static in the classical \( \sqrt{\lambda} \gg 1 \) limit. The energy (2.3) for \( m = J \) is then

\[
E = J\sqrt{1 + \lambda}, \quad \text{i.e.} \quad E(\lambda \gg 1) = \sqrt{\lambda}J + \ldots.
\]

Our main conjecture is that this special case of the circular string solution should be dual to the highest-energy antiferromagnetic state of the corresponding gauge-theory spin chain.\(^{12}\)

Like in the previous cases (i)–(iv) the solution in the case (v) is still unstable, with the number of tachyonic modes with \( n < 2m = 2J \) growing with \( J \). This instability may, however, be an artifact of the naive semiclassical expansion near the highest-energy state: our conjecture implies that there is a well-defined maximal-energy state in the discrete quantum string spectrum which in the large \( \lambda \) limit may be approximated by the above classical solution with \( m = J \).

\(^{11}\)One may of course set \( m = kJ \) where \( k \) is an integer, but we expect that \( k > 1 \) cases will be equivalent to \( k = 1 \) in the exact quantum theory.

\(^{12}\)Another known solution in the \( SU(2) \) sector with \( J_1 = J_2 \) is the folded string one [13]. One may wonder if this solution also admits a limit when the number of folds \( m \) becomes large together with \( J \). The answer is no: here one cannot take the string rotation velocity to be small without having the string shrinking to a point.
The standard semiclassical expansion does not indeed directly apply in the last case (v): the classical energy depends on $\sqrt{\lambda}$ and contains subleading terms that appear also from higher orders in inverse string tension expansion (see next subsection). Still, the leading $\sqrt{\lambda}J$ term in the classical string energy (2.13) does not receive corrections from higher world-sheet loops, and this leading scaling behavior thus provides a qualitative support to our conjecture that this solution (v) is dual to the highest-energy AF state of the gauge-theory spin chain.

The fact that the proportionality coefficient $1/\pi^2$ in (1.3) as obtained in [22] by extrapolating to strong coupling the AF energy of the BDS spin chain does not match the one in (2.13) may not be considered as a contradiction. Indeed, the orders of limits taken are opposite: on the string side we first take $\lambda$ large and then $J$ large, while on the gauge side we first assume that $J$ is large and then extrapolate the perturbative in $\lambda$ result to strong coupling.

### 2.2 1-loop correction to string energy

Let us now consider the slow-string limit of the 1-loop correction to the energy of the above circular solution which was computed in [33, 34] (see also [35]). Its expansion that was discussed before was the fast string limit when $w \gg m$ (for a discussion of subtleties in this expansion see [36, 37]). Here we shall consider the opposite limit of $w \ll m$. We shall formally ignore the instability of the solution, concentrating on the real part of the 1-loop correction $E_1(m, w)$. The expansion in powers of $\frac{w}{m} = \frac{J}{\sqrt{\lambda}m}$ produces powers series in $\frac{1}{\sqrt{\lambda}}$ for fixed $J$ and $m$ (in particular, for $J = m$). As we shall see, for large $m$ the one-loop correction $E_1$ will scale linearly with $m \gg 1$ or, for $m = J$, with $J$, in agreement with the general expectation that

$$E = f(\lambda)J, \quad f(\lambda \gg 1) = a_1 \sqrt{\lambda} + a_2 + \frac{a_3}{\sqrt{\lambda}} + \ldots .$$

The expression for $E_1$ is given by the sum of the zero-mode and non-zero-mode contributions

$$E_1 = E_{\text{zero}} + E_{\text{non-zero}} , \quad E_{\text{non-zero}} = \sum_{n=1}^{\infty} S_n , \quad E_{\text{zero}} = 2 + \sqrt{1 - \frac{2m^2}{w^2 + m^2}} - 3 \sqrt{1 - \frac{m^2}{w^2 + m^2}} ,$$

$$S_n = 2 \sqrt{1 + \frac{(n + \sqrt{n^2 - 4m^2})^2}{4(w^2 + m^2)}} + 2 \sqrt{1 + \frac{n^2 - 2m^2}{w^2 + m^2}} - 4 \sqrt{1 + \frac{n^2}{w^2 + m^2}} - 8 \sqrt{1 + \frac{n^2 - m^2}{w^2 + m^2}} .$$
In contrast to the large $w$ expansion relevant for the fast string case, the small $w$ expansion of the 1-loop correction is regular: higher orders coefficients are given by convergent sums.

The leading term in the expansion in $w/m$ is found by setting $w = 0$ in the above expression (omitting the imaginary part):

$$E_1^{(0)} = 2 + \frac{1}{m} \sum_{n=1}^{\infty} \left[ \sqrt{4m^2 + (n + \sqrt{n^2 - 4m^2})^2 + 2\sqrt{n^2 - m^2} + 4\sqrt{n^2 + m^2} - 8n} \right]. \quad (2.18)$$

The expansion of this at large $m$ is subtle but numerical evaluation shows that the real part of $E_1^{(0)}$ scales linearly with $m$ at large $m$, supporting the suggested identification of the corresponding string solution with a particular $J_1 = J_2$ “intermediate” state of the spin chain.

Setting $m = J = \sqrt{\lambda} w$ in (2.16) we get

$$E_1 = E_{\text{zero}}(\lambda) + E_{\text{non-zero}}(J, \lambda), \quad (2.19)$$

$$E_{\text{zero}} = 2 + \frac{1 - \lambda}{1 + \lambda} - 3 \frac{1}{\sqrt{1 + \lambda}}, \quad E_{\text{non-zero}} = \sum_{n=1}^{\infty} S_n(J, \lambda). \quad (2.20)$$

To analyze the dependence of $E_1$ on $J$, we expand $S_n$ at large $\lambda$ for fixed $J$:

$$S_n = \frac{1}{J} (A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda^2} + ...), \quad (2.21)$$

$$A_0 = -8n + 2\sqrt{n^2 - J^2} + 4\sqrt{n^2 + J^2} + \sqrt{2n(n + \sqrt{n^2 - 4J^2})}, \quad (2.22)$$

$$A_1 = 4n - \frac{4J^2}{n} - \frac{2n^2}{\sqrt{n^2 + J^2}} + \frac{2J^2 - n}{\sqrt{n^2 - J^2}} + \frac{2J^2 - n}{\sqrt{2n(n + \sqrt{n^2 - 4J^2})}}. \quad (2.23)$$

$$A_2 = \frac{J^4}{n^3} + \frac{2J^2}{n} - 3n - \frac{n^4}{2(n^2 + J^2)^{3/2}} + \frac{4J^4 - 8J^2n^2 + 3n^4}{4(J^2 - n^2)^2} \sqrt{n^2 - J^2}$$

$$+ \frac{2n^2}{\sqrt{n^2 + J^2}} + \frac{3n^3(n + \sqrt{n^2 - 4J^2}) - 2J^4 - 2J^2n(4n + \sqrt{n^2 - 4J^2})}{2\sqrt{2}[n(n + \sqrt{n^2 - 4J^2})]^{3/2}}. \quad (2.24)$$

To study the $J$-dependence of the series we computed the sums $\sum_{n=2J} A_0$, $\sum_{n=2J} A_1$, $\sum_{n=2J} A_2$ numerically for $N = 10^5$ and $10^2 < J < 10^4$. We found that they scale as $J^2$, so that $S_n$ in (2.21) grows linearly with $J$.

Numerically evaluating the coefficients and combining $E_1$ with the classical expression (2.13) we get for the large $\lambda$ expansion of $E = E_0 + hE_1 + ...$ (ignoring $O(J^0)$ terms in $E_1$, i.e., in particular, terms coming from $E_{\text{zero}}$):

$$E = J \left[ \sqrt{\lambda} \left( 1 - 0.34 \frac{h}{\sqrt{\lambda}} \right) + \frac{1}{\sqrt{\lambda}} \left( \frac{1}{2} + 0.215 \frac{h}{\sqrt{\lambda}} \right) - \frac{1}{\lambda^{3/2}} \left( \frac{1}{8} + 0.16 \frac{h}{\sqrt{\lambda}} \right) + ... \right]. \quad (2.25)$$
Here we formally introduced the parameter $\hbar$ to distinguish between the classical and the 1-loop corrections. It is interesting to observe that while the classical part of $E/J$ contains $1/\sqrt{\lambda}$ terms in odd powers, the 1-loop corrections produce the even powers of $1/\sqrt{\lambda}$. The subleading coefficients will be further corrected by higher loop string contributions. This illustrates the point that was already mentioned above: in contrast to the usual semiclassical expansion in the $m = J$ case the string sigma model loop expansion is not equivalent to large $\lambda$ expansion.

It is interesting also to note that the leading $-0.34\hbar$ correction to the classical $\sqrt{\lambda}$ term in $E/J$ is negative, which seems consistent with the idea of interpolation from strong to weak coupling (cf. (1.1)).

Returning to the issue of instability of the solution, we expect that it is related to the fact that one tries to expand near a maximum of a potential like $\sin^2 \theta$. The exact quantization should produce a discrete set of levels in this potential with the maximal energy state being “approximated” by the above classical solution.\(^{13}\)

3. Effective action for slow-moving strings on $R \times S^3$

In the case of the lower part of the spectrum of the ferromagnetic spin chain dual to fast strings it was possible to establish a correspondence between a non-relativistic Landau-Lifshitz (LL) effective action for long-wave length excitations of the spin chain and the fast string limit of the classical string action \([16, 17, 18, 19]\). One may wonder if a similar kind of effective action correspondence exists also near the upper antiferromagnetic end of the spin chain spectrum related to a slow-string limit of the string action.

As is well known (see, e.g., \([20, 40]\)), the effective action describing near AF-state excitations of the XXX\(_{1/2}\) spin chain is a relativistic sigma model on $S^2$ (with a topological term ensuring its conformal invariance and no-gap spectrum). The exact spin chain representing gauge theory anomalous dimensions is certainly different from the XXX\(_{1/2}\) chain and the strong-coupling limit of the corresponding effective action need not be simply an $S^2$ relativistic sigma model as in the XXX\(_{1/2}\) case. The exact spin chain was suggested to be related to a version of the Hubbard model \([21]\). It is not completely clear at the moment which is the correct Hubbard-type model which should be related to string theory and which should be the corresponding near-AF state effective action for it, but one may assume that it should be qualitatively similar to that of the Hubbard model. The continuous effective action for the fluctuations near the ground AF state of the half-filled Hubbard model is a combination of the massless spinon sigma

\(^{13}\)It is possible also that the relevant quantum string state may be better represented by a pulsating string \([3, 38, 39, 19]\) (with pulsations outside of $S^3$). Given that a precise meaning of the closure of the $SU(2)$ sector is unclear on the string side (cf. \([10]\)), and, in particular, that the quantum string fluctuations “feel” all directions of $AdS_5 \times S^5$, it is possible that there exists a pulsating string solution which for large $J$ (bigger than its oscillation number) has essentially the same energy as the unstable rigid rotating string we discussed here.
model and a sine-Gordon action for massive charge excitations [11, 20, 42].

Here we shall first attempt to see what kind of effective Hamiltonian for near-AF state fluctuations may appear in the dual limit on the string side. Then in section 4 we will find that the form of this Hamiltonian and thus its spectrum is qualitatively similar to that of the Hamiltonian appearing in a scaling limit of the Hubbard model of [9].

In general, one would need to start with the full quantum string theory and integrate out all modes but the ones relevant for the description of the near-AF states of the SU(2) sector. Here we shall suppress world sheet quantum corrections by assuming that $\lambda \gg 1$, i.e. we shall consider only the classical string action. We shall follow a naive approach that essentially copies the derivation of the LL action in [16, 17, 19] but now focuses on modes close to the wrapped slow-moving string that we conjectured above to be the counterpart of the AF state. More precisely, the analogy here will be with the action of magnons as small fluctuations near the ferromagnetic state, or with the corresponding “plane wave” action on the string side.

Given a classical string moving on $R_t \times S^3$ we are to gauge fix two coordinates (time and a spatial one) to get an action for two physical transverse degrees of freedom. In [17] this was done by fixing the momentum density corresponding to the sum of the two polar angles ($\phi_1 = \alpha + \varphi$, $\phi_2 = \alpha - \varphi$) in the two planes of $R^4$ which $S^3$ is embedded into

$$X_i = U_i e^{i\alpha}, \quad U_1 = \cos \psi e^{i\varphi}, \quad U_2 = \sin \psi e^{-i\varphi}$$

(3.1)

to be constant and equal to $\tilde{J}$. This is equivalent [19] to gauge-fixing the 2d dual coordinate $\tilde{\alpha} = \tilde{\sigma} J \sigma$.

One possible strategy is to use the same gauge also in the present case of slow strings. Then the spin $J$ will again have the interpretation of the length on the spin chain side. The difference with the fast string case is that there we had $J = J/\sqrt{\lambda}$ large so that we expanded in small $\lambda = J^2 = \frac{1}{\lambda}$. For the slow strings we may first expand in large $\lambda$, and then in large $J$, so that now we have $\sqrt{\lambda} \gg J$, or, equivalently, $J \ll 1$. In general, quantum string corrections are expected to be important (modifying subleading terms in the classical action) but we may hope that they do not change the form of the leading large $\lambda$ term in the action. Proceeding as in [19], i.e. fixing $t = \tau$, $\tilde{\alpha} = \tilde{\sigma} J \sigma$, we obtain the $R_t \times S^3$ string action in the form

$$I = \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} L, \quad L = J(C_0 - \sqrt{h}) , \quad (3.2)$$

$$h = (1 + \frac{\lambda}{J^2})(1 - |D_0 U_i|^2) + \frac{1}{4 \lambda J^2}(D_0 U_i^* D_1 U_i + c.c.)^2 , \quad (3.3)$$

$^{14}$The unit vector describing $S^2$ is related to $U_i$ by $n_i = U^\dagger \sigma_i U$, $U = (U_1, U_2)$, $\vec{n} = (\sin 2\psi \cos 2\varphi, \sin 2\psi \sin 2\varphi, \cos 2\psi)$.
where \( C_a = -iU_i^* \partial_a U_i \), \( D_a U_i = \partial_a U_i - iC_a U_i \). We expand this action at large \( \lambda \) for fixed \( J \) and fixed derivatives of the fields

\[
L = -\sqrt{\lambda}(D_1 U_i)^2(1 - |D_0 U_i|^2) + \frac{1}{4}(D_0 U_i^* D_1 U_i + \text{c.c.})^2 + O(J, \lambda^{-1/2}) .
\]  

(3.4)

Since for slow strings \( 1 \ll J \ll \sqrt{\lambda} \) we have ignored the first \( JC_0 \) term (which played the important role in the fast string case). In terms of the two angles \( \psi \) and \( \varphi \) in (3.1) we get

\[
L = -\sqrt{\lambda}(\psi'^2 + \varphi'^2 \sin^2 2\psi)(1 - \dot{\psi}^2 - \dot{\varphi}^2 \sin^2 2\psi) + (\psi' \dot{\psi} + \varphi' \dot{\varphi} \sin^2 2\psi)^2 + ... .
\]  

(3.5)

This action does admit our basic circular string configuration (2.1) as its solution for which \( \psi = \frac{\pi}{4}, \varphi = m\sigma \). We may now set \( m = J \) and expand the action near this solution

\[
\psi \to \frac{\pi}{4} + f(\tau, \sigma), \quad \varphi \to J\sigma + g(\tau, \sigma)
\]  

(3.6)

keeping all orders in the fields but dropping higher powers of their derivatives. Then the \( J \)-dependence can be absorbed into the new spatial parameter

\[
s = J\sigma
\]

and we finish with \( I = \int dt \int_0^{2\pi J} \frac{ds}{2\pi} L \), where

\[
L = \frac{1}{2} \sqrt{\lambda} \left[ - 2 \cos 2f - 2g' \cos 2f + \cos 2f \; \dot{f}^2 - \frac{f'^2 \cos 2f}{\cos 2f}
\right.
\]

\[
+ \; \dot{f}(\dot{f} g' - 2f' \dot{g}) \cos 2f + \frac{f'^2 g'}{\cos 2f} + ... ] + .... ,
\]  

(3.7)

where the prime now stands for the derivative over \( s \).

To quadratic order in fluctuations this becomes

\[
L = \frac{1}{2} \sqrt{\lambda} \left( \dot{f}^2 - f'^2 + 4f^2 + ... \right)
\]  

(3.8)

which represents the unstable mode. Its origin is similar to the tachyonic mode appearing when expanding the sine-Gordon model near the maximum of the potential.

An alternative approach to deriving the effective action is to start with the string action on \( R_t \times S^3 \) in a different – conformal – gauge

\[
L = -\frac{1}{2} \sqrt{\lambda} \left[ - (\partial t)^2 + (\partial \alpha + \cos 2\psi \partial \varphi)^2 + (\partial \psi)^2 + \sin^2 2\psi (\partial \varphi)^2 \right] .
\]  

(3.9)

For the circular string solution (2.1) we have \( t = \kappa \tau, \alpha = \sqrt{\lambda} \tau, \varphi = J\sigma \), and so for \( J \ll \sqrt{\lambda} \) one may ignore time evolution of \( \alpha \) and integrate out its spatial fluctuations.
The resulting Lagrangian for $\psi$ and $\varphi$ or their fluctuations near the wrapped string solution in (3.6) is then (here $I = \int d\tau J^0 d\sigma \frac{2\pi}{2\pi} L$)

$$L = \frac{1}{2} \sqrt{\lambda} \left( \dot{\psi}^2 - \psi'^2 + \dot{\varphi}^2 - \sin^2 2\psi \varphi'^2 + ... \right)$$

$$= \frac{1}{2} \sqrt{\lambda} \left( f'^2 - f'^2 + \dot{g}^2 - \cos^2 2f \ (J + g')^2 + ... \right). \quad (3.10)$$

For large $J$ we may replace $J + g' \to J$ and thus get a weakly-coupled combination of a sine-Gordon model for $f$ and a free homogeneous $g$ mode. In conformal gauge the action will then scale as $J^2$ but since $t = \kappa \tau \approx J\tau$ ($\kappa = \sqrt{m^2 + w^2} \approx m = J$) the target-space energy will scale linearly with $J$.

It is useful to rewrite the action for (3.10) in terms of more natural world-sheet coordinates to facilitate comparison with spin chain action in the next section, namely, in terms of the target-space time $t = J\tau + ...$ and $s = J\sigma$. The use of $s$ is natural since here the length of the wound string is large, so $J \gg 1$ corresponds to the thermodynamic limit. Then we get

$$I = \frac{\sqrt{\lambda}}{4\pi} \int dt \int_0^{2\pi J} ds \left( \dot{g}^2 + \dot{f}^2 - f'^2 - \frac{1}{2} f^2 + O \left( \frac{1}{\sqrt{\lambda}} \right) \right), \quad (3.11)$$

and it is now obvious that the action and the energy of an approximately homogeneous configurations should scale linearly with large $J$.

As stressed at the beginning, to compare to spin chain we should consider the spectrum of Hamiltonian for small fluctuations near this slow string state. The Hamiltonian corresponding to (3.11) is

$$H = \int_0^{2\pi J} ds \left[ \frac{\pi}{\sqrt{\lambda}} \Pi_g^2 + \frac{\pi}{\sqrt{\lambda}} \Pi_f^2 + \frac{\sqrt{\lambda}}{4\pi} (f'^2 + \cos^2 2f) \right]. \quad (3.12)$$

After a canonical transformation that rescales momenta and fields by $\sqrt{\lambda}$ in the opposite way we get, to quadratic order in the fluctuation field $f$ (cf. (3.8))

$$H = \frac{1}{2} \int_0^{2\pi J} ds \left( \Pi_g^2 + \Pi_f^2 + f'^2 - 4f^2 + O \left( \frac{1}{\sqrt{\lambda}} \right) \right). \quad (3.13)$$

Higher-order fluctuation terms are suppressed in the large $\lambda$ limit.

In the next section we shall see that an effective Hamiltonian similar to (3.13) appears in the relevant large $\lambda$ limit on the gauge theory spin chain side assuming it is described by the Hubbard model of [9].
4 An effective Hamiltonian for fluctuations near AF state of gauge theory spin chain described by Hubbard model

It has recently been shown [9] that the Bethe equations diagonalizing the BDS spin chain [8] are identical to those diagonalizing the infinitely long Hubbard chain with the half-filled state as the ground state. From the standpoint of the $\mathcal{N} = 4$ SYM theory the most important property of the Hubbard model is that its interactions are short-ranged. Consequently, it can be defined on a lattice of any length, providing a possible extension of the BDS chain to operators of finite length.

The relation between the Hubbard model and the AdS$_5 \times$ S$^5$ string theory is a very interesting question. In the event that (some modification of) it represents the correct extension of the BDS chain to finite length operators, the Hubbard model should also be related to the world sheet theory, perhaps in the same spirit as the Heisenberg-type chain near the ferromagnetic end of the spectrum is related to the fast string limit of the world sheet sigma model [16]. There are important differences however. The ground state of the half-filled Hubbard model is anti-ferromagnetic, in the sense of possessing Néel order. As was pointed out earlier, in the leading perturbative gauge theory limit the effective action of excitations around this state is relativistic and also strongly coupled. The lack of an expansion parameter analogous to $\lambda/J^2$ raises the question of how to compare this action to some action derived from the string world sheet. A possibility is that on the string side the relevant action may be obtained by integrating out all fields except those describing the SU(2) sector in the $\lambda \to \infty$ limit. Deriving such a quantum effective action appears to be beyond our reach at the moment.

If a version of Hubbard model does give the correct representation for the gauge theory dilatation operator it would then allow to establish a contact with the perturbative/semiclassical (i.e. large tension or large $\sqrt{\lambda}$) limit of the string world sheet theory. In the large ’t Hooft coupling limit, the effective action of small excitations around the AF ground state of the Hubbard model should be compared to the classical world sheet action expanded around the classical solution dual to this ground state. In what follows we shall compare the classical continuum limit of the standard Hubbard chain with the effective Hamiltonian (3.12) of fluctuations around the solution corresponding to the AF state. It is important to stress again that this comparison is qualitatively different from that of the ferromagnetic case coherent state continuum limit and the fast string action in [16, 17]. Rather, it should be thought of as the comparison between the spectrum of eigenvalues of the gauge theory dilatation operator close to some large anomalous dimension with the eigenvalues of the effective fluctuations Hamiltonian obtained by expanding the string effective action around a specific solution.

Also, it is clear that here we may not expect the precise match between the string
and spin chain Hamiltonians. As was found in [9], the standard Hubbard model does not resolve the “3-loop discrepancy”, i.e. it does not reproduce the precise string-theory values of subleading coefficients in the energy of fast-rotating strings in the large \( \lambda \) limit; this indicates that this model does not capture all the details of the world sheet theory. The best we may hope for is a qualitative agreement between the continuum limit of the Hubbard Hamiltonian and the slow-string effective fluctuation Hamiltonian.

Below we will first review the continuum limit and the bosonization of the Hubbard model at a general filling fraction (see, e.g., [45] for a recent thorough discussion). We shall consider the odd-length Hubbard chain to avoid complications related to the twist necessary for even lengths [9]. We shall then focus on the half-filling case and compare the result with the effective Hamiltonian of fluctuations around the slow string solution. We shall find a qualitative agreement.

### 4.1 Review of continuum limit

The Hubbard model Hamiltonian is (see, e.g., [11])

\[
H = -t \sum_{i,\alpha} \left( c_{i+1,\alpha}^\dagger c_{i,\alpha} + c_{i+1,\alpha}^\dagger c_{i,\alpha} \right) + U \sum_{i} c_{i\uparrow}^\dagger c_{i\uparrow} c_{i\downarrow}^\dagger c_{i\downarrow} \equiv H_0 + H_1 \tag{4.1}
\]

where \( c_{i\alpha}^\dagger \) and \( c_{i\alpha} \) are creation and annihilation operators of electrons of spin \( \alpha = \{\uparrow, \downarrow\} \) at site \( i \). The relation between the two parameters \( t \) and \( U \) and the 't Hooft coupling was established in [9] by comparing the ground state energy of the Hubbard model with the maximum energy state of the BDS chain:

\[
t = t_{\text{RSS}} = -\frac{1}{\sqrt{2} g}, \quad U = t U_{\text{RSS}}, \quad U_{\text{RSS}} = \frac{\sqrt{2}}{g}, \quad g^2 \equiv \frac{\lambda}{8\pi^2}. \tag{4.2}
\]

Here \( t_{\text{RSS}} \) and \( U_{\text{RSS}} \) are the \( t \) and \( U \) parameters used in [9]. In the weak gauge coupling region (where \( U \gg t \) and so the quartic term dominates over the quadratic one which is then treated as a perturbation) the effective Hamiltonian is given by a series of the form

\[
\sum_{k=0}^\infty \hat{A}_k \frac{t^{2k}}{U^{2k-1}} \tag{4.3}
\]

where \( \hat{A}_k \) are operators constructed out of \( c_{i\alpha}^\dagger \) and \( c_{i\alpha} \). The \( k = 0 \) and \( k = 1 \) terms correspond to the tree-level and one-loop dilatation operators, respectively. Let us note that the normalization in (4.2) is different from the one usually considered: here the tree-level Hamiltonian contributes \( O(1/\lambda) \) to the dimension of operators while the one-loop Hamiltonian contributes terms independent of the 't Hooft coupling. The usual extra order-\( \lambda \) factor may be restored by rescaling both \( t \) and \( U \) by \( \frac{1}{16\pi^2} = \frac{g^2}{2} \).
Indeed, in \[9\] the energy of the Hubbard model \((4.1)\) was multiplied by \(g^2\) to get the anomalous dimension. It is more natural to define the Hamiltonian so that its eigenvalues are directly related to anomalous dimensions and thus, via AdS/CFT, to string energies. To implement this, here we will adopt the following “rescaled” choice of the parameters in \((4.1)\):\(^{15}\)

\[
t = \frac{g^2}{2} t_{\text{RSS}} = -\frac{g}{2\sqrt{2}}, \quad U = t U_{\text{RSS}} = -\frac{1}{2}. \tag{4.4}
\]

The negative sign of \(t\) corrects the fact that the energy of the Hubbard model and the gauge theory anomalous dimensions have opposite signs. In relation to the world sheet theory we will choose to implement this relation by replacing \(t\) and \(U\) with \(|t|\) and \(|U|\) and reversing at the very end the sign of the time coordinate. This will ensure that the sigma model energies are identified with the negative of the Hubbard model energies.

For the comparison with the classical world sheet string theory we will be interested in the opposite limit to the one discussed in \[9\] – in the strong-coupling limit where \(\lambda \to \infty\). In this limit \(|t| \gg |U|\) and thus the Hubbard model as well as its continuum limit may be treated “semiclassically” or by expansion near the free quadratic term (the quartic term in \(H\) may be considered as a perturbation).\(^{16}\)

Our aim will be to study small fluctuations around the half-filled state. The standard procedure is to construct the operators Fourier-conjugate to \(c_{j,\alpha}\) and \(\hat{c}_{j,\alpha}^\dagger\). The operators creating the ground state fill up all momentum levels of the Fermi sea; our aim will be to find the effective action for the excitations around the Fermi level, having momenta much smaller than the Fermi momentum \(k_F\). While we are particularly interested in the half-filled state, it is possible – and, in fact, instructive – to analyze the fluctuations around the minimum energy state at a general filling fraction, i.e. for arbitrary \(J_1\) and \(J_2\) charges of the \(SU(2)\) sector. The effective Hamiltonian obtained following this procedure could then be compared to the Hamiltonian for fluctuations around a classical solution dual to the minimal energy string state with spins \(J_1\) and \(J_2\).

The annihilation operators then are

\[
c_{j,\alpha} = \sum_k e^{ikja} c_{k,\alpha} \to \sqrt{a} \left[ e^{-ik_Fja} \sum_{-k_F+\Lambda} e^{ikja} c_{k,\alpha} + e^{+ik_Fja} \sum_{k_F-\Lambda} e^{ikja} c_{k,\alpha} \right] \\
\equiv e^{-ik_Fja} L_{j,\alpha} + e^{+ik_Fja} R_{j,\alpha} \tag{4.5}
\]

where \(a\) is the lattice spacing and \(\Lambda \ll k_F\) is a cutoff enforcing that the fluctuations have momenta much smaller than \(k_F\). The Fermi level of a system of length \(J\) with \(n_c\)

\(^{15}\)Note that the Bethe ansatz (Lieb-Wu) equations for the Hubbard model that reduce to the BDS Bethe ansatz equations \[9\] depend only on the ratio \(U/t\) and thus are the same for the two choices.

\(^{16}\)Note that with the normalization \((4.1)\) it is immediately clear that in the strong-coupling limit the AF ground state energy should scale as \(t \sim g \sim \sqrt{\lambda}\), i.e. in the same way as found by extrapolating to strong coupling \((1.3)\) the perturbative expression \((1.2)\).
electrons is $k_F = \pi n_c / J$; at half-filling the number of electrons is half the number of lattice sites and, therefore, we find that $2k_F a = \pi$.

Let us then use the expansion (4.5) in the Hamiltonian (4.1) and take the continuum limit
\[
c_{j+1,a} \simeq c_{j,a} + a \partial_x c_{j,a} + \ldots \quad \text{and} \quad \sum_j \mapsto \frac{1}{a} \int_0^V dx.
\] (4.6)

By construction, the largest value of the coordinate $x$ should be $V = Ja$. One possible choice used in the near-ferromagnetic ground state case [17] is $a = 2\pi / \sqrt{\lambda}$; in that case the world-sheet coordinate had $J$-independent length while the $J$-factors combined in the scaling limit with $\sqrt{\lambda}$. Here we shall use instead $a = 1$, $V = J$; this is natural since in the thermodynamic limit $J \gg 1$ all extensive quantities describing near-AF states should scale linearly with $J$. The coordinate $x$ will then be directly related to $s$ in (3.11) up to $2\pi$ factor. For generality we shall keep the dependence on the lattice spacing $a$ explicit in what follows.

Plugging (4.5) into the quadratic and quartic terms of (4.1) leads to:

1) the quadratic Hamiltonian:
\[
H_0 = -|t| \sum_{j,\alpha} \left[ \cos k_F a (L^\dagger_{j,\alpha} L_{j,\alpha} + R^\dagger_{j,\alpha} R_{j,\alpha}) + 2i a \sin k_F a (R^\dagger_{j,\alpha} \partial_x R_{j,\alpha} - L^\dagger_{j,\alpha} L_{j,\alpha}) \right] + \ldots
\]
\[
\sim \frac{1}{a} |t| \sum_{\alpha} \int dx (L^\dagger_{\alpha} L_{\alpha} + R^\dagger_{\alpha} R_{\alpha})
\]
\[
+ 2 |t| \sin k_F a \sum_{\alpha} \int dx \left( L^\dagger_{\alpha} i\partial_x L_{\alpha} - R^\dagger_{\alpha} i\partial_x R_{\alpha} \right)
\] (4.7)

In writing the first line in the equation above we discarded summands proportional to $e^{\pm 2ik_F a}$; the reason is that, upon Fourier transforming $L$ and $R$, the sum over $j$ vanishes due to the assumption that the momenta of the excitations are much smaller than the Fermi momentum $k_F$.

2) the quartic Hamiltonian:
\[
H_1 = |U| \sum_j \left[ (L^\dagger_{j_1} L_{j_2}) (L^\dagger_{j_3} L_{j_4}) + (R^\dagger_{j_1} R_{j_2}) (R^\dagger_{j_3} R_{j_4}) + (L^\dagger_{j_1} R_{j_2} R^\dagger_{j_3} L_{j_4}) + h.c. \right] + \ldots
\]
\[
+ |U| \sum_j \left[ e^{4ik_F ja} L^\dagger_{j_1} R_{j_2} L^\dagger_{j_3} R_{j_4} + e^{-4ik_F ja} R^\dagger_{j_1} L_{j_2} R^\dagger_{j_3} L_{j_4} \right]
\] (4.8)

We have again discarded summands proportional to $e^{\pm 2ik_F a}$. Away from half-filling the second line in (4.8) is irrelevant. At half-filling we have $e^{\pm 4\pi ia} = 1$ which leads to the survival of the second line in (4.8) or in the effective action. Introducing the parameter $\zeta$ which vanishes away from half-filling and equals unity at half-filling, it follows that the continuum limit of the quartic part of the Hubbard Hamiltonian expanded around the Fermi levels is
\[
H_1 = \frac{|U|}{a} \int dx \left[ (L^\dagger_1 L_1) (L^\dagger_4 L_4) + (R^\dagger_1 R_1) (R^\dagger_4 R_4) + L^\dagger_1 R_1 R^\dagger_4 L_4 + R^\dagger_1 L_1 L^\dagger_4 R_4 \right]
\]
To summarize, the equations (4.7) and (4.9) represent the Hamiltonian of the fluctuations around the half-filled state ($\zeta = 1$) and the state at generic filling ($\zeta = 0$) of the Hubbard model. We would like to compare the large $\lambda$ limit (or linearized) spectrum of this fluctuation Hamiltonian to the spectrum of the string Hamiltonian (3.12) or (3.13). The first step is then to bosonize (4.7), (4.9).

4.2 Bosonization of the continuum-limit Hamiltonian

There are three ways to relate the above fermionic Hamiltonian to a bosonic theory. One – which we will follow here – is to directly bosonize the Hamiltonians (4.7) and (4.9). Another is to express the continuum limit of $H$ in terms of the $SU(2) \times SU(2)$ currents [41, 42]; the third possibility is to use a mean field approximation [20]. The latter two approaches yield a direct sum of the conformal $SU(2)$ level one WZW model and a massive $U(1)$ Thirring model. This representation of the scaling-limit theory is not bosonic and thus is not suitable for comparison with the slow-string actions (3.7) or (3.11). However, the WZW model is equivalent to a compact boson at self-dual radius, while the Thirring model is equivalent to a sine-Gordon model. In the end, all three approaches are equivalent, leading to the results obtained by directly bosonizing (4.7) and (4.9) as discussed below.

Using the rather standard bosonization formulae ($\gamma$ is for the time being an arbitrary constant)

\[
\begin{align*}
: L^\dagger_\alpha L_\alpha : &= \gamma^2 \partial_x \Phi_{L\alpha} , \\
: R^\dagger_\alpha R_\alpha : &= \gamma^2 \partial_x \Phi_{R\alpha} ,
\end{align*}
\]

(4.10)

translated into the Hamiltonian formalism\textsuperscript{17} we are quickly led to the following bosonic Hamiltonian

\[
H = 2 \gamma^2 |t| \sum_{\alpha=\uparrow,\downarrow} \int dx \left[ (\partial_x \Phi_{L\alpha})^2 + (\partial_x \Phi_{R\alpha})^2 \right] + \frac{\gamma^4 |U|}{a} \int dx \left[ (\partial_x \Phi_{L\uparrow} + \partial_x \Phi_{R\uparrow})(\partial_x \Phi_{L\downarrow} + \partial_x \Phi_{R\downarrow}) + 2 \cos(\Phi_{L\uparrow} - \Phi_{L\downarrow} - \Phi_{R\downarrow} + \Phi_{R\uparrow}) \right] + \zeta \frac{\gamma^4 |U|}{a} \int dx \left[ 2 \cos(\Phi_{L\uparrow} + \Phi_{L\downarrow} + \Phi_{R\uparrow} + \Phi_{R\downarrow}) \right].
\]

(4.11)

The commutation relations of the original creation and annihilation operators imply certain commutation relations between the fields

\[
\begin{align*}
\phi_\alpha &\equiv \Phi_{L\alpha} + \Phi_{R\alpha} , \\
\theta_\alpha &\equiv \Phi_{L\alpha} - \Phi_{R\alpha} .
\end{align*}
\]

(4.12)

\textsuperscript{17}The bosonization formulae used here apply to a theory defined on the plane $\mathbb{R}^2$, with $z = r e^{i \theta}$. For the purpose of comparison with string theory we need to pass to $\mathbb{R} \times S^1$, where the time-like direction is related to $r$ as $r = e^{i \tau} = e^{i \sigma}$. Note that as usual translations (but not rescalings) of the (euclidean) time coordinate correspond to dilatations on $\mathbb{R}^2$ (under which the operators in (4.10) have appropriate 2d quantum dimensions).
In particular, it turns out that $\partial_x \theta_\alpha$ can be interpreted as the momentum conjugate to $\phi_\alpha$, implying that the Hamiltonian simplifies to

$$H = \gamma^2 |t| \int dx \left[ (\Pi_\uparrow^2 + \Pi_\downarrow^2) + (\partial_x \phi_\uparrow)^2 + (\partial_x \phi_\downarrow)^2 \right]$$

Furthermore, this Hamiltonian can be rewritten as a sum of two decoupled Hamiltonians by introducing $\phi_c = \frac{1}{\sqrt{2}} (\phi_\uparrow + \phi_\downarrow)$, $\phi_s = \frac{1}{\sqrt{2}} (\phi_\uparrow - \phi_\downarrow)$. (4.14)

We then get:

$$H = H_s + H_c$$

with

$$\frac{1}{\gamma^2 |t|} H_s = \int dx \left[ \Pi_s^2 + (1 - \frac{\gamma^2 U}{4at})(\partial_x \phi_s)^2 + 2 \frac{\gamma^2 U}{at} \cos(\sqrt{2} \phi_s) \right]$$

$$\frac{1}{\gamma^2 |t|} H_c = \int dx \left[ \Pi_c^2 + (1 + \frac{\gamma^2 U}{4at})(\partial_x \phi_c)^2 + 2 \zeta \frac{\gamma^2 U}{at} \cos(\sqrt{2} \phi_c) \right]$$

Thus apparently we end up with two sine-Gordon theories.

As is well known, the continuum limit of the Heisenberg XXX$_{1/2}$ chain near the antiferromagnetic state is described by a relativistic 2-d theory [20, 42]. The excitations of the Heisenberg chain span only a subset of the excitations of the Hubbard model, namely (up to duality transformations), only those in which all sites are either empty or doubly-occupied. Taken separately and after appropriate redefinitions of the space-like coordinate, each of the two Hamiltonians (4.16) and (4.17) can be interpreted as describing a relativistic theory. However, if they are combined together, the relativistic theory interpretation is not possible because the speeds of light for the two types of decoupled excitations are different:

$$v_s = \sqrt{1 - \frac{\gamma^2 U}{4at}} = \sqrt{1 - \frac{\gamma^2}{2\sqrt{2} a g}} , \quad v_c = \sqrt{1 + \frac{\gamma^2 U}{4at}} = \sqrt{1 + \frac{\gamma^2}{2\sqrt{2} a g}} .$$

---

18The $c,s$ notation is used to emphasize the important well-known fact that the excitations rearranging the spin and the charge distributions are decoupled in the Hubbard model [41].

19We recall that, since we replaced $t$ and $U$ by $|t|$ and $|U|$, to match the energy of the bosonized continuum Hubbard model with that of a world-sheet theory we need to reverse the sign of the time coordinate. This transformation has no effect at the level of the Hamiltonian or the Lagrangian, since they have an even number of time derivatives.
It is important to emphasize that in constructing this continuum limit we have assumed that the Hubbard coupling constant $U$ is small compared to $t$, i.e. $g$ should be large enough. This is reflected in the above expressions (4.18) in that the positivity of the Hamiltonian (4.15) implies that we are not allowed to take the 't Hooft coupling or $g^2$ to be arbitrarily small. In other words, as expected from the analysis of the discrete Hamiltonian, recovering the perturbative region of the gauge theory dilatation operator requires quantum treatment of the Hubbard model of [9].

The bosonized Hamiltonians (4.16) and (4.17) are, however, not the end of the story. Their sum, while looking similar to the effective Hamiltonian (3.12) of fluctuations around the slow-string solution, is qualitatively different from (3.12): both fields appear to be interacting at half-filling ($\zeta = 1$), while one of the fields of the slow-string action (3.11) or in (3.12) is free in the large $J$ limit.

To find a way to match (4.15) and (3.12), (3.13) let us analyze (4.16) and (4.17) separately. Through a canonical transformation the speed of light factor can be moved into the argument of the cosine potential. Then, in the free theory approximation (which is valid as $\lambda$ is assumed to be large), the dimensions of the operators representing the potential terms

$$O_{s,c} = \cos \frac{\sqrt{2} \phi_{s,c}}{(1 \mp \frac{\gamma^2}{2\sqrt{2}a})^{1/4}}$$

are

$$d_{O_{s,c}} = \frac{2}{\sqrt{1 \mp \frac{\gamma^2}{2\sqrt{2}a}}}$$

This means that the interaction term is an irrelevant operator in $H_s$ but relevant one in $H_c$ (with $\zeta = 1$). From the standpoint of the world sheet infrared physics we can therefore replace $H_s$ by a free (gapless) Hamiltonian.

As a result, the effective Hamiltonian for small fluctuations around the half-filled state of the Hubbard model is

$$H = \gamma^2 |t| \int_0^{a_f} dx \left[ \Pi^2_s + (1 + \frac{\gamma^2 U}{4at})(\partial_x \phi_s)^2 + 2 \frac{\gamma^2 U}{at} \cos(\sqrt{2}\phi_s)ight.\left. + \Pi^2_s + (1 - \frac{\gamma^2 U}{4at})(\partial_x \phi_s)^2 \right]$$

Next, let us choose the free parameter $\gamma$ such that the second velocity is zero, $v_s = 0$, i.e.

$$\gamma^2 = \frac{4at}{U}$$

<sup>20</sup>The most naive suggestion – to depart from the half-filling – is of course not an option, since the slow string action (3.11) was derived by assuming that we are expanding the string action around the solution dual to the half-filled state.
Introducing the rescaled fields

\[ f = \frac{\phi_c}{2\sqrt{2}}, \quad g = \frac{\phi_s}{2\sqrt{2}}, \]  

(4.23)

we are then led to

\[ H = \gamma^2|t| \int_0^a J dx \left( \Pi_g^2 + \Pi_f^2 + 16 (\partial_x f)^2 + 16 \cos^2 2f \right). \]  

(4.24)

The identifications (4.2) combined with the choice of \( \gamma \) in (4.22) lead to

\[ \gamma^2 |t| = \frac{4a|t|^2}{|U|} = ag^2. \]  

(4.25)

Moreover, choosing, as discussed above, the lattice spacing to be \( a = 1 \), we conclude that the Hamiltonian (4.24) has essentially the same structure as (3.12) apart from order \( \lambda \) factors.

We then find the following effective Hamiltonian for the linearized fluctuations around the half-filled state:

\[ H = g^2 \int_0^J dx \left( \Pi_g^2 + \Pi_f^2 + 16 \left( (\partial_x f)^2 - 4f^2 \right) + \ldots \right). \]  

(4.26)

The relative coefficients here can be adjusted further by canonically rescaling the momenta and the fields.

There are quite obvious similarities between this Hamiltonian (4.26) and the Hamiltonian of the fluctuations around the slow string solution (3.12): both describe a massive and a massless field and the ratio between the mass and the mode number of the massive field is also the same in the two Hamiltonians. As already mentioned, the target-space time coordinate \( t \) in (3.11) should be identified (due to our choice of sign for the couplings of the Hubbard model) with the sign-reversed time coordinate conjugate to Hubbard’s Hamiltonian to ensure that the string energies match anomalous dimensions on the spin chain side. The spatial coordinates \( x \) in (4.26) and \( s \) in (3.11) are essentially the same, modulo the \( 2\pi \) factor.

The coefficients in the two Hamiltonians, however, appear to be different: \( H \) in (4.26) has an extra overall factor of \( g^2 = \frac{\lambda}{8\pi^2} \) while it is absent in (3.13).\(^{21}\) One may say this is hardly unexpected, given, in particular, the \( \frac{1}{2\pi^2} \) mismatch between the AF ground state energy of Hubbard model (1.3) and the slow-string energy in (2.13), to leading order in \( \sqrt{\lambda} \).

\(^{21}\)It is curious that this is the same rescaling coefficient that we needed to introduce to make the Hubbard model energies match the spin chain and thus string theory ones; had we used the unrescaled choice of Hubbard model parameters (4.2) we would not get that overall factor in (4.26). At the moment we do not understand if this is a coincidence or an indication that we have missed a compensating \( g^2 \) factor at some other step.
To appreciate additional subtleties that one may need to overcome on the way to better understanding the correspondence between the near-AF state spin chain described by Hubbard model and the slow-string limit on the string side it is instructive to consider the continuum limit for fluctuations around the minimal energy state at some arbitrary filling fraction. On the one hand, this limit should correspond to an effective Hamiltonian for fluctuation around the semiclassical solution dual to the \((J_1, J_2)\) operator of maximal anomalous dimension.\(^{22}\) It is reasonable to expect that at least one of the two fields of appearing in the slow-string effective Hamiltonian will be interacting in general and massive at the quadratic level. On the other hand, as we have seen earlier in this section, away from the half-filling we are to set \(\zeta = 0\). Then the interaction term in the Hamiltonian \(4.17\) vanishes and the continuum limit as constructed above yields a free theory. It appears, therefore, that the qualitative agreement that we have described above is restricted to the half-filled Hubbard model.

It would be interesting to understand if considering an effective action including other degrees of freedom would yield a better match away from the half-filling or, if possible, find a modification of the Hubbard Hamiltonian which does not affect the weak ‘t Hooft coupling limit, preserves integrability and accounts for the additional interaction in the strong ‘t Hooft coupling limit.

There is an intriguing similarity between the discontinuous behaviour of the effective Hamiltonian of the Hubbard model \(^{23}\) and that of the \(SU(2)\) sector of gauge theory at strong coupling, i.e. from the point of view of the world sheet theory. As it has been discussed in \([10]\), while the excitations around the \(J_1 \neq J_2\) states mix with other “non-SU(2)” world sheet excitations, they could be decoupled if \(J_1 = J_2\). It is tempting to conjecture that the differences between the Hubbard model and the slow string effective Hamiltonian away from half-filling can be corrected by additional interaction terms in the Hubbard Hamiltonian which account for mixing with other gauge theory operators.

5 Some “slow” string solutions with spins in \(AdS_5\) and \(S^5\)

The general case of noncompact sectors is different: there is apparently no bound on the quantum string energy. One may relate this to the fact that the string wrapped on a circle in \(S^3\) part of \(AdS_5\) can not be static and in any case can have any radius (and thus any energy). It is still useful to study “slow-string” limits of solutions that carry one spin \((S)\) in \(AdS_5\) and one spin \((J)\) in \(S^5\) as they may have some interpretation in the \(SL(2)\) sector of gauge theory. In particular, we shall find that there is again a case

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\(^{22}\)For \(J_i \ll \sqrt{\lambda}\) and before expanding to quadratic order, this should be a “slow string” Hamiltonian, of the same type as \(3.7\) or \(6.11\).

\(^{23}\)This discontinuity is an example \([10]\) of the so-called Mott-Hubbard metal-insulator phase transition \([46, 47]\).
in which the classical string energy scales as $E \sim \sqrt{\lambda} J + \ldots$.

Below we shall discuss limits of the circular $(S, J)$ solution of [29] and also consider a “flat-space like” solution which may be viewed as a special case of the more general $(S, J_1, J_2)$ circular solution in [29].

5.1 Circular solution in $AdS_3 \times S^1$

Let us review the form of the solution of [29] describing a string which has a rigid circle form in $AdS_5$ and in $S^5$ and each circle rotates “along itself”. In terms of complex combinations of global embedding coordinates ($Y_i$ in $AdS_5$ and $X_i$ in $S^5$) one has:

\begin{equation}
Y_0 = r_0 e^{i \kappa \tau}, \quad Y_1 = r_1 e^{i \omega \tau + i k \sigma}, \quad X_1 = e^{i \omega \tau - i m \sigma}, \quad Y_2 = X_2 = X_3 = 0,
\end{equation}

\begin{equation}
r_0 \equiv \cosh \rho_0, \quad r_1 \equiv \sinh \rho_0, \quad r_0^2 - r_1^2 = 1,
\end{equation}

where $\rho_0 =$const, and $k$ and $m$ are positive integer winding numbers. The charges are

\begin{equation}
E \equiv \sqrt{\lambda} \mathcal{E} = \sqrt{\lambda} \kappa r_0^2, \quad S \equiv \sqrt{\lambda} S = \sqrt{\lambda} \omega r_1^2, \quad J \equiv \sqrt{\lambda} J = \sqrt{\lambda} w.
\end{equation}

The equations of motion imply $\omega^2 = k^2 + \kappa^2$ and the conformal gauge constraints give

\begin{equation}
2\kappa \mathcal{E} - \kappa^2 = 2S \sqrt{k^2 + \kappa^2} + J^2 + m^2
\end{equation}

\begin{equation}
kS = mJ.
\end{equation}

The energy is thus a function of the three parameters, e.g., $E = \sqrt{\lambda} \mathcal{E}(\frac{J}{\sqrt{\lambda}}, \frac{S}{\sqrt{\lambda}}, m)$.

Let consider the special case of

\begin{equation}
S = J, \quad i.e. \quad k = m
\end{equation}

Then the independent parameters are $J = w$ and $m$ and

\begin{equation}
2\kappa \mathcal{E} = (w + \sqrt{m^2 + \kappa^2})^2, \quad \frac{\mathcal{E}}{\kappa} = \frac{w + \sqrt{m^2 + \kappa^2}}{\sqrt{m^2 + \kappa^2}}
\end{equation}

Solving for $\kappa$ we obtain

\begin{equation}
\kappa_{\pm} = \sqrt{\frac{w^2}{2} + \frac{w^2 + 2m^2 \pm w\sqrt{w^2 + 8m^2}}{2}}
\end{equation}

Note that the minus sign solution can exist only if $m \geq w$. The energy is

\begin{equation}
\mathcal{E}_{\pm} = \left( w + \sqrt{m^2 + \frac{1}{2} \left( w^2 + 2m^2 \pm w\sqrt{w^2 + 8m^2} \right)} \right)^2 \frac{\sqrt{2w^2 + 4m^2 \pm 2w\sqrt{w^2 + 8m^2}}}{\sqrt{2w^2 + 4m^2 \pm 2w\sqrt{w^2 + 8m^2}}}
\end{equation}
Like in the case of the $S^5$ solution with $J_1 = J_2$ energy of the $SL(2)$ sector solution with $S = J$ thus has an explicit analytic form.

Small fluctuations near this solution were discussed in [26]. There are 4 real massive fields from $S^5$ with mass $\sqrt{w^2 - m^2}$, i.e. $\omega_n = \sqrt{n^2 + w^2 - m^2}$. This frequency is real if $n^2 + w^2 - m^2 \geq 0$. From $AdS_5$ there are also 2 free massive real fields with mass $\kappa$.

Remaining fluctuations are coupled and the corresponding characteristic equation is

$$\left(\omega_n^2 - n^2\right)^2 + 4r_1^2\kappa^2\omega_n^2 - 4(1 + r_1^2)(\sqrt{m^2 + \kappa^2}\omega_n + nm)^2 = 0 \quad (5.9)$$

where $r_1^2 = w/\sqrt{m^2 + \kappa^2}$. The stability condition is the reality of the solutions of this quartic equation.

In the standard semiclassical expansion one assumes that $m, w$ are fixed while $\lambda$ is large. As in the above $S^5$ solution case we can now consider particular limits of the parameters:

(i) $w \gg m$: this is the “fast string” case [29]. Only the solution with plus sign is possible. The energy has a regular expansion in $\frac{m^2}{w^2} = \frac{m^2}{\lambda}$,

$$E = J\left(2 + \frac{m^2}{J^2} - \frac{5}{4}\frac{m^4\lambda^2}{J^4} + \ldots\right). \quad (5.10)$$

As was shown in [29], this solution is stable for large $w$.

(ii) $w = m$: this a “flat-space” type solution. We get

$$E = \frac{3\sqrt{3}}{2} J. \quad (5.11)$$

Similar “flat-space”-type solution will be discussed in the next subsection. As follows from (5.9), this solution may be unstable for certain values of $w$.

(iii) $w \ll m$: this is a slow-moving string: the $\tau$ part of the solution is much smaller then the winding $\sigma$ part. [25] There are two possible cases for the two signs in (5.7). We will concentrate only on the solution with plus sign, as the other one can be treated similarly. Here we can expand $E = \sqrt{\lambda}m \ F(\frac{m}{\lambda})$ in $w/m$, i.e.

$$E = \sqrt{\lambda}m + \sqrt{2}J + \frac{J^2}{4m\sqrt{\lambda}} - \frac{J^3}{8\sqrt{2}m^2\lambda} \ldots. \quad (5.12)$$

Here $m \ll J = \sqrt{\lambda}w$ since $\lambda$ is taken large first. As in the $SU(2)$ case this solution always has unstable modes: the condition of reality of characteristic frequencies of $S^5$.

---

24 The case when $k = m = 0$ is again the BPS one, $E - S = J$, when string world surfaces degenerates to a massless geodesic. Here the string trajectory is a massive geodesic in $AdS_5$ and a big circle in $S^5$; to make canonical identification between the string and gauge states/energies one is to apply an $AdS_3$ transformation to transform the $AdS_5$ geodesic to rest frame, $t = \tau$.

25 A different large winding limit of the $S \neq J$ solution was considered in [31].
fluctuations is \( n \geq m \). The fluctuations in other directions are non-tachyonic: expanding the solutions of the quartic equation \( \text{(5.9)} \) at large \( m \) we find that all frequencies are real in this limit.

Like in the \( SU(2) \) case, we can then increase \( m \) further, but here one is not expecting the upper bound on the string energy so there should be no obvious choice for the maximal \( m \). Still, let us formally consider again the case of \( m = J \) (which corresponds to \( w = \frac{m}{\sqrt{\lambda}} \to 0 \) in the large \( \lambda \) limit). Although it is not clear which state on the gauge spin chain side should correspond to the \( m = J \) string state, let us discuss this case by analogy with the \( SU(2) \) case. Setting \( m = J \) in \( \text{(5.8)} \) one obtains (we choose plus sign)

\[
E = J \frac{(2 + \sqrt{2 + 8\lambda + 2\sqrt{1 + 8\lambda}})^2}{4\sqrt{2 + 4\lambda + 2\sqrt{1 + 8\lambda}}},
\]

i.e. at large \( \lambda \)

\[
E = \sqrt{\lambda} J \left( 1 + \frac{\sqrt{2}}{\sqrt{\lambda}} + \frac{1}{4\lambda} + \ldots \right).
\]

As in the \( SU(2) \) case we computed the 1-loop correction to the energy for the case of \( m = J \) and found that it depends linearly on large \( J \) (details are given in Appendix). This suggests that in general for large \( J \) one should have

\[
E = f(\lambda) J;
\]

this relation may then be extrapolated to weak coupling and should correspond to the anomalous dimension of a particular state in the spectrum of the \( SL(2) \) spin chain.

### 5.2 “Flat-space” type \((S, J)\) solution in \( AdS_3 \times S^2 \)

Let us now consider another example of an \((S, J)\) “flat-space” solution which may be viewed as a special case of the rational \((S, J_1, J_2)\) solution in \(29\): it admits a special \( J_2 = 0 \) limit when the \( S^5 \) part of the solution is left (or right) moving. Here the string is wrapping a circle of \( S^5 \) which is not the maximal radius one. Explicitly (cf. \( \text{(5.1)} \))

\[
Y_0 = r_0 e^{i\kappa \tau}, \quad Y_1 = r_1 e^{i\omega \tau + ik\sigma}, \quad X_1 = \cos \psi_0 e^{im(\tau - \sigma)}, \quad X_2 = \sin \psi_0
\]

where the coordinates \( \rho_0, \psi_0 \) specifying the position of the circular string are constant. The only non-zero elements of the rotation generators are \( S_{50} = E, S_{12} = S, J_{12} = J \), and now \( J = \sqrt{\lambda} m \cos^2 \psi_0 \). Again we have \( \omega^2 = k^2 + \kappa^2 \) and \( 2\kappa E - \kappa^2 = 2S\sqrt{k^2 + \kappa^2} + 2mJ, kS = mJ \) or

\[
2\kappa E - \kappa^2 = 2S\sqrt{k^2 + \kappa^2} + 2kS.
\]

The absence of an upper bound on the string energy is consistent with gauge-theory expectations in the \( SL(2) \) sector: for fixed \( J \), i.e. fixed length of the chain, the spin-chain energy can be arbitrarily large because the spin \( S \) can be arbitrarily large. We are grateful to K. Zarembo for this remark.

One may wonder whether other such solutions exist. One can show that a similar solution of the form \( Y_0 = \cosh \rho_0 e^{i\kappa \tau}, Y_1 = \sinh \rho_0 e^{ik(\tau - \sigma)}, X_1 = e^{i\omega \tau + im\sigma} \) does not exist. Also, a solution in \( AdS_5 \) of the form \( Y_0 = \cosh \rho_0 e^{i\kappa \tau}, Y_1 = \sqrt{2} \sinh \rho_0 e^{i\omega \tau + ik\sigma}, Y_2 = \sqrt{2} \sinh \rho_0 e^{im(\tau - \sigma)} \) does not exist.
A useful relation following from $r_0^2 - r_1^2 = 1$ is $E = \frac{S}{\sqrt{k^2 + \kappa^2}} = 1$ and the non-trivial solutions for $\kappa$

$$\kappa_{\pm} = 2^{-1/2} \sqrt{4kS - k^2 \pm k^{3/2} \sqrt{k + 8S}}. \quad (5.17)$$

Note that $\cos^2 \psi_0 = \frac{kS}{m^2}$. Therefore, a large $S$ limit with $k, m$ held fixed is not well defined. Instead, a useful limit to consider is large $S$ and large $m$ with $m/S$ fixed, e.g., equal to 1. In this limit the string is located near $\psi_0 \to \frac{\pi}{2}$ and $\rho_0 \to \infty$. The solution $\kappa_+$ has a regular expansion at small $S$, which is the flat space limit. For the physical $\kappa_+$ solution, the energy $E = E(S, k)$ becomes

$$E = \frac{1}{2} \sqrt{\frac{-k + 4S + \sqrt{k(k + 8S)}}{k + 4S + \sqrt{k(k + 8S)}}} \left(2S + \sqrt{2k + 4S + \sqrt{k(k + 8S)}}\right) \quad (5.18)$$

Its large $S$ expansion is

$$E = S + \sqrt{2k\sqrt{S}} + \frac{k}{4} - \frac{k^{3/2}}{8\sqrt{2}\sqrt{S}} + \frac{k^2}{32S} + O(1/S^{3/2}) \quad (5.19)$$

Let us now consider two "slow" limits with small $S \to 0$. The first limit is $S \to 0$ and $m \to 0$ or $J \to 0$ with $k$ finite. In this case the string shrinks to a point in both $AdS_5$ and $S^5$ and we get the usual flat-space scaling

$$E = 2\sqrt{k\sqrt{S}} + \frac{S^{3/2}}{\lambda\sqrt{k}} - \frac{5S^{5/2}}{4\lambda^2 k^{3/2}} + \ldots \quad (5.20)$$

Another limit is $S \to 0$ and $k \to \infty$ with $m, J$ kept finite. Now the string shrinks to a point in $AdS_5$ but it remains macroscopic in $S^5$. The energy in this limit is the same as in (5.20). The same result (5.20) is found also in the special case when $S = J$, i.e. when $m = k$. Thus in contrast with a similar limit in the case of the previous $(S, J)$ solution, here we obtain the flat-space behaviour of the energy instead of the $\sqrt{\lambda S}$ behaviour.

This solution is stable for sufficiently large $S$. One can compare its energy $E = E(k, S)$ with the energy of the rational solution in the $SL(2)$ sector $E = E(k, m, S)$ reviewed in the previous subsection. Numerical analysis shows that the energy of this new solution has less energy than of the old one. Since the later is known to be stable for large $S$, we conclude that the new solution presented here should also be stable for large $S$. This is confirmed by direct analysis of its fluctuation spectrum which follows the discussion in [29].
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Appendix: 1-loop correction to the energy of \((S, J)\) solution in the slow-string limit

Below we shall consider the case of \(m = J\); the discussion for the general case of \(m \gg w\) is similar.

The 1-loop correction to the energy \(E_1\) was found in [26]. It can be written as the sum of the contributions of the zero and non-zero modes

\[
E_1 = E_1^{(0)} + \bar{E}_1, \quad E_1^{(0)} = \frac{1}{2\kappa}(4\nu + 2\kappa + \omega_0 - 8\tilde{w}_0), \quad (A.1)
\]

\[
\bar{E}_1 = \frac{1}{\kappa} \sum_{n=1}^{\infty} \left( 4\sqrt{n^2 + \nu^2} + 2\sqrt{n^2 + \kappa^2} + \frac{1}{2} \sum_{I=1}^{4} \text{sign}(C_{I,B}^{(n)})\omega_{I,n} \right. \\
\left. - 4\sqrt{(n+c)^2 + a^2 + (n-c)^2 + a^2} \right),
\]

where \(\nu^2 = w^2 - m^2\) and and for \(m = J\)

\[
\omega_0 = 2\sqrt{\kappa^2 + J^2(1 + r_1^2)}, \quad \tilde{\omega}_0 = \sqrt{c^2 + a^2}, \quad r_1^2 = \frac{J}{\sqrt{\lambda}\sqrt{J^2 + \kappa^2}},
\]

\[
a^2 = \frac{1}{2}\left(\kappa^2 + \frac{J^2}{\lambda} - J^2\right), \quad c = \frac{1}{2}\kappa \left[ 1 + \frac{2J^2(1 + r_1^2)}{\kappa^2 - \frac{r_1^4}{\lambda} + J^2} \right] \sqrt{\frac{\kappa^2 - \frac{J^2}{\lambda} + J^2 - 2J^2r_1^2}{2(J^2 + \kappa^2)}},
\]

The sign functions are

\[
C_p^B = \frac{1}{2m_{11}(\omega_{p,0})\omega_{p,0} \prod_{q \neq p}(\omega_{p,0}^2 - \omega_{q,0}^2)}, \quad C_{I,B}^{(n)} = \frac{1}{m_{11}(\omega_{I,n}) \prod_{J \neq I}(\omega_{I,n} - \omega_{J,n})},
\]

where \(\omega_{I,n}\) are the bosonic frequencies for \(n \neq 0\), \(\omega_{p,0}\) are bosonic frequencies for \(n = 0\), and the relevant part of the minor \(m_{11}\) for computing the signs of \(C_{I,B}^{(n)}\). \(C_p^B\) is \(m_{11} \sim (\omega^2 - n^2)\). As in the SU(2) case discussed in section 2.2 we may expand \(\bar{E}_1\) at large \(\lambda\) for fixed \(J\) and then take \(J\) large. Again we can do this expansion inside the sum over \(n\). Expanding the zero-mode part we get (omitting the imaginary part)

\[
E_1^{(0)} = 1 - \sqrt{2} - \frac{5}{4\sqrt{\lambda}} + \frac{27}{32\sqrt{2}\lambda} + \ldots
\]

(A.6)
The non-zero mode bosonic frequencies from the quartic characteristic equation have the following large $\lambda$ expansions

$$\omega_{I=1,2,n} = -\sqrt{2}J \mp \sqrt{2J^2 - 2nJ + n^2} - \frac{J}{\sqrt{\lambda}} \frac{6\sqrt{2}J^2 + 2\sqrt{2}(n - 3J) \mp 2(n - 2J)\sqrt{2J^2 - 22Jn + n^2}}{4\sqrt{2}J^2 + 2(2 - n - 2J)(\sqrt{2n} \mp 2J + 2Jn + n^2)} + O\left(\frac{1}{\lambda}\right) \quad (A.7)$$

$$\omega_{I=3,4,n} = \sqrt{2}J \mp \sqrt{2J^2 + 2nJ + n^2} + \frac{J}{\sqrt{\lambda}} \frac{6\sqrt{2}J^2 + 2\sqrt{2}(n + 3J) \mp 2(n + 2J)\sqrt{2J^2 + 22Jn + n^2}}{4\sqrt{2}J^2 + 2(n + 2J)(\sqrt{2n} \mp 2J^2 + 2Jn + n^2)} + O\left(\frac{1}{\lambda}\right) \quad (A.8)$$

We see that in the large $\lambda$ limit these frequencies are real, so the only unstable modes with $n \leq J$ come from $S^5$ fluctuations. One way to obtain the real 1-loop correction to energy is to omit the unstable modes, i.e. to take the sum over $n$ starting with $n = J$. The sign functions can also be computed in the large $\lambda$ limit and are found to be $\text{sign}(C_{1,B}) = \text{sign}(C_{3,B}) = -1$ and $\text{sign}(C_{2,B}) = \text{sign}(C_{4,B}) = +1$. Then we get

$$\bar{E}_1 = \sum_{n=1}^{\infty} S_n(J,\lambda), \quad S_n = B_0 + \frac{B_1}{\sqrt{\lambda}} + \frac{B_2}{\lambda} + ... , \quad (A.9)$$

where

$$B_0 = \frac{1}{J} \left[ 4\sqrt{n^2 - J^2} + 2\sqrt{n^2 + J^2} + \sqrt{n^2 - 2nJ + 2J^2} + \sqrt{n^2 + 2Jn + 2J^2} \right. \left. - 2\sqrt{2}J - \sqrt{2n} - 2\sqrt{2}J + \sqrt{2n} \right]. \quad (A.10)$$

$B_1$, $B_2$ have complicated form which we will not write down, but we used them to evaluate the series numerically and plot $\bar{E}_1$ against $J$. Taking the sums over $n$ from $J$ to $N = 10^5$ we plotted $B_0$, $B_1$ and $B_2$ for $J = 10^2, ..., 10^4$. As in the $SU(2)$ case, we found linear dependence with $J$. Combining together the classical energy and the 1-loop correction we got

$$E = J \left[ \sqrt{\lambda} \left( 1 + (\sqrt{2} - \frac{h}{4}) \frac{1}{\sqrt{\lambda}} \right) + \frac{1}{\sqrt{\lambda}} \left( 1 - \frac{h}{1.75} \right) \right] \left( 7.5 - \frac{h}{8\sqrt{2}} \right) + O(\lambda^{-3/2}) . \quad (A.11)$$

Here as in (2.25) we introduced $\hbar$ to indicate the 1-loop contributions.

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