Note on (De)homogenized Gröbner Bases*

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Abstract. By employing the (de)homogenization technique in a relatively extensive setting, this note studies in detail the relation between non-homogeneous Gröbner bases and homogeneous Gröbner bases. As a consequence, a general principle of computing Gröbner bases (for an ideal and its homogenization ideal) by passing to homogenized generators is clarified systematically. The obtained results improve and strengthen the work of [LWZ], [Li1], [Li2], [Li3], and very recent [SL] concerning the same topic.

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In the computational Gröbner basis theory, though it is a well-known fact that by virtue of both the structural advantage (mainly the degree-truncated structure) and the computational advantage (mainly the degree-preserving fast ordering), most of the popularly used commutative and noncommutative Gröbner basis algorithms produce Gröbner bases by homogenizing generators first, it seems that in both the commutative and noncommutative case a general principle of computing Gröbner bases (for an ideal and its homogenization ideal) by passing to homogenized generators is still missing.

Let $K$ be a field, and let $\mathbb{N}$ be the additive monoid of nonnegative integers. Recall from [Li2] that if $R = \oplus_{p \in \mathbb{N}} R_p$ is an $\mathbb{N}$-graded $K$-algebra with an admissible system $(\mathcal{B}, \prec)$, in which $\mathcal{B}$ is a skew multiplicative $K$-basis of $R$ consisting of $\mathbb{N}$-homogeneous elements (i.e., $u, v \in \mathcal{B}$ implies that $u, v$ are homogeneous elements, $uv = 0$ or $uv = \lambda w$ for some nonzero $\lambda \in K$ and $w \in \mathcal{B}$), and $\prec$ is a monomial ordering on $\mathcal{B}$, then, theoretically every (two-sided) ideal $I$ of $R$ has a Gröbner basis $\mathcal{G}$ in the sense that if $f \in I$ and $f \neq 0$ then there is some $g \in \mathcal{G}$ such that $\text{LM}(g)|\text{LM}(f)$, where

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Let $1. Central (De)homogenized Gröbner Bases$

Considering the onto ring homomorphism $\phi: R[t] \to R$ defined by $\phi(t) = 1$, then for each $f \in R$, there exists a homogeneous element $F \in R[t]_p$, for some $p$, such that $\phi(F) = f$. More precisely, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$ and $f_p \neq 0$, then $f^* = f_p + tf_{p-1} + \cdots + t^s f_{p-s}$ is a homogeneous element of degree $p$ in $R[t]_p$ satisfying $\phi(f^*) = f$. We call the homogeneous element $f^*$ obtained this way the central homogenization of $f$ with respect to $t$ (for the reason

Algebras considered in this paper are associative algebras with multiplicative identity $1$. Unless otherwise stated, ideals considered are meant two-sided ideals. If $S$ is a nonempty subset of an algebra, then we use $\langle S \rangle$ to denote the two-sided ideal generated by $S$. Moreover, if $K$ is a field, then we write $K^* = K - \{0\}$.

1. Central (De)homogenized Gröbner Bases

Let $R = \bigoplus_{p \in \mathbb{N}} R_p$ be an arbitrary $\mathbb{N}$-graded $K$-algebra, and let $R[t]$ be the polynomial ring in the commuting variable $t$ over $R$. Then $R[t]$ has the mixed $\mathbb{N}$-gradation, that is, $R[t] = \bigoplus_{p \in \mathbb{N}} R[t]_p$ is an $\mathbb{N}$-graded algebra with the degree-$p$ homogeneous part

$$R[t]_p = \left\{ \sum_{i+j=p} F_i t^j \left| \begin{array}{c} F_i \in R_i, \ j \geq 0 \end{array} \right. \right\}, \ p \in \mathbb{N}.$$ 

Considering the onto ring homomorphism $\phi: R[t] \to R$ defined by $\phi(t) = 1$, then for each $f \in R$, there exists a homogeneous element $F \in R[t]_p$, for some $p$, such that $\phi(F) = f$. More precisely, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in R_p$, $f_{p-j} \in R_{p-j}$ and $f_p \neq 0$, then $f^* = f_p + tf_{p-1} + \cdots + t^s f_{p-s}$ is a homogeneous element of degree $p$ in $R[t]_p$ satisfying $\phi(f^*) = f$. We call the homogeneous element $f^*$ obtained this way the central homogenization of $f$ with respect to $t$ (for the reason
that \( t \) is in the center of \( R[t] \)). On the other hand, for an element \( F \in R[t] \), we write

\[ F_\ast = \phi(F) \]

and call it the **central dehomogenization** of \( F \) with respect to \( t \) (again for the reason that \( t \) is in the center of \( R[t] \)). Hence, if \( I \) is an ideal \( R \), then we write \( I^* = \{ f^* \mid f \in I \} \) and call the \( \mathbb{N} \)-graded ideal \( \langle I^* \rangle \) generated by \( I^* \) the **central homogenization ideal** of \( I \) in \( R[t] \) with respect to \( t \); and if \( J \) is an ideal of \( R[t] \), then since \( \phi \) is a ring epimorphism, \( \phi(J) \) is an ideal of \( R \), so we write \( J_\ast \) for \( \phi(J) = \{ H_\ast = \phi(H) \mid H \in J \} \) and call it the **central dehomogenization ideal** of \( J \) in \( R \) with respect to \( t \). Consequently, henceforth we will also use the notation \( (J_\ast)^* = \{ (h_\ast)^* \mid h \in J \} \).

Since each \( f \in R \) has a unique decomposition by \( \mathbb{N} \)-homogeneous elements, if \( f = f_p + f_{p-1} + \cdots + f_{p-s} \) with \( f_p \in R_p, f_{p-j} \in R_{p-j} \) and \( f_p \neq 0 \), then we call \( f_p \) the \( \mathbb{N} \)-leading homogeneous element of \( f \), denoted \( LH(f) = f_p \). Similarly, for an element \( F \in R[t] \), the \( \mathbb{N} \)-leading homogeneous element \( LH(F) \) is defined with respect to the mixed \( \mathbb{N} \)-gradation of \( R[t] \).

**1.1. Lemma** With every definition and notation made above, the following statements hold.

(i) For \( F, G \in R[t] \), \( (F + G)_\ast = F_\ast + G_\ast \). (ii) For any \( f \in R \), \( (f^*)^\ast = f \).

(iii) If \( F \in R[t]_p \) and if \( (F_\ast)_p \in R[t]_q \), then \( p \geq q \) and \( t^r(F_\ast)_p = F \) with \( r = p - q \).

(iv) If \( f, g \in R \) are such that \( fg \) has nonzero \( LH(fg) \in R_m \) and \( f^\ast g^\ast \) has nonzero \( LH(f^\ast g^\ast) \in R[t]_q \), then \( f^\ast g^\ast = t^k(fg)^\ast \) with \( k = q - m \).

(v) If \( f, g \in R \) with nonzero \( LH(f) \in R_p \) and nonzero \( LH(g) \in R_q \), then \( (f + g)^\ast = f^\ast + g^\ast \) in case \( p = q \), and \( (f + g)^\ast = f^\ast + t^\ell g^\ast \) in case \( p > q \), where \( \ell = p - q \).

(vi) If \( I \) is a two-sided ideal of \( R \), then each homogeneous element \( F \in \langle I^* \rangle \) is of the form \( t^r f^\ast \) for some \( r \in \mathbb{N} \) and \( f \in I \).

(vii) If \( J \) is a graded ideal of \( R[t] \), then for each \( h \in J_\ast \) there is some homogeneous element \( F \in J \) such that \( F_\ast = h \).

**Proof** Exercise. \( \square \)

Suppose that the \( \mathbb{N} \)-graded \( K \)-algebra \( R = \bigoplus_{p \in \mathbb{N}} R_p \) has an admissible system \( (B, \prec_{gr}) \), where \( B \) is a skew multiplicative \( K \)-basis of \( R \) consisting of \( \mathbb{N} \)-homogeneous elements, and \( \prec_{gr} \) is an \( \mathbb{N} \)-graded monomial ordering on \( B \), i.e., \( R \) has a Gröbner basis theory. Consider the mixed \( \mathbb{N} \)-gradation of \( R[t] \) and the \( K \)-basis \( B^\ast = \{ t^r w \mid w \in B, r \in \mathbb{N} \} \) of \( R[t] \). Since \( B^\ast \) is obviously a skew multiplicative \( K \)-basis for \( R[t] \), the \( \mathbb{N} \)-graded monomial ordering \( \prec_{gr} \) on \( B \) extends to a monomial ordering on \( B^\ast \), denoted \( \prec_{t-gr}, \) as follows:

\[ t^{r_1} w_1 \prec_{t-gr} t^{r_2} w_2 \text{ if and only if } w_1 \prec_{gr} w_2, \text{ or } w_1 = w_2 \text{ and } r_1 < r_2. \]

Thus \( R[t] \) holds a Gröbner basis theory with respect to the admissible system \( (B^\ast, \prec_{t-gr}) \).
As usual we call elements in $B$ and $B^*$ monomials and use $\text{LM}(\cdot)$ to denote taking the leading monomial of elements with respect to the given monomial ordering.

It follows from the definition of $\prec_{t-\text{gr}}$ that $t^r \prec_{t-\text{gr}} w$ for all integers $r > 0$ and all $w \in B - \{1\}$ (if $B$ contains the identity element 1 of $R$). Hence $\prec_{t-\text{gr}}$ is not a graded monomial ordering on $B^*$. But noticing that $\prec_{\text{gr}}$ is an $\mathbb{N}$-graded monomial ordering on $B$, under taking the $\mathbb{N}$-leading homogeneous element and central (de)homogenization the leading monomials behave harmonically as described in the next lemma.

1.2. Lemma With notation given above, the following statements hold.
(i) If $f \in R$, then $\text{LM}(f) = \text{LM}(\text{LH}(f))$ w.r.t. $\prec_{\text{gr}}$ on $B$.
(ii) If $f \in R$, then $\text{LM}(f^*) = \text{LM}(f)$ w.r.t. $\prec_{t-\text{gr}}$ on $B^*$.
(iii) If $F$ is a nonzero homogeneous element of $R[t]$, then $\text{LM}(F_s) = \text{LM}(F)_s$ w.r.t. $\prec_{\text{gr}}$ on $B$.

Proof The proof of (i) and (ii) is an easy exercise. To prove (iii), let $F \in R[t]_p$ be a nonzero homogeneous element of degree $p$, say

$$F = \lambda t^r w + \lambda_1 t^{r_1} w_1 + \cdots + \lambda_s t^{r_s} w_s,$$

where $\lambda, \lambda_i, \in K^*$, $r, r_i \in \mathbb{N}$, $w, w_i \in B$, such that $\text{LM}(F) = t^r w$. Since $B$ consists of $\mathbb{N}$-homogeneous elements and $R[t]$ has the mixed $\mathbb{N}$-gradation by the previously fixed assumption, we have $d(t^r w) = d(t^{r_i} w_i) = p$, $1 \leq i \leq s$. Thus $w = w_i$ will imply $r = r_i$ and thereby $t^r w = t^{r_i} w_i$. So we may assume that $w \neq w_i$, $1 \leq i \leq s$. Then it follows from the definition of $\prec_{t-\text{gr}}$ that $w_i \prec_{\text{gr}} w$ and $r \leq r_i$, $1 \leq i \leq s$. Therefore $\text{LM}(F_s) = w = \text{LM}(F)_s$, as desired. \[ \square \]

The next result is a generalization of ([LWZ] Theorem 2.3.2).

1.3. Theorem With notions and notations as fixed before, let $I = \langle G \rangle$ be the ideal of $R$ generated by a subset $G$, and $\langle I^* \rangle$ the central homogeneization ideal of $I$ in $R[t]$ with respect to $t$. The following two statements are equivalent.
(i) $G$ is a Gröbner basis for $I$ in $R$ with respect to the admissible system $(B, \prec_{\text{gr}})$;
(ii) $G^* = \{g^* \mid g \in G\}$ is a Gröbner basis for $\langle I^* \rangle$ in $R[t]$ with respect to the admissible system $(B^*, \prec_{t-\text{gr}})$.

Proof In proving the equivalence below, without specific indication we shall use (i) and (ii) of Lemma 1.2 wherever it is needed.
(i) $\Rightarrow$ (ii) First note that $\text{LM}(G^*) \subset \text{LM}(I^*)$. We have to prove that $\text{LM}(G^*)$ generates $\langle \text{LM}(I^*) \rangle$ in order to see that $G^*$ is a Gröbner basis for $\langle I^* \rangle$. If $F \in \langle I^* \rangle$, then since $\text{LM}(F) = \text{LM}(F^*) \in \text{LM}(G^*)$. Hence $\text{LM}(F) = \text{LM}(F^*)$ and

$$F = \lambda t^{r_F} w + \lambda_1 t^{r_1} w_1 + \cdots + \lambda_s t^{r_s} w_s,$$

where $\lambda, \lambda_i, \in K^*$, $r, r_i \in \mathbb{N}$, $w, w_i \in B$, such that $\text{LM}(F) = t^{r_F} w$. Since $B$ consists of $\mathbb{N}$-homogeneous elements and $R[t]$ has the mixed $\mathbb{N}$-gradation by the previously fixed assumption, we have $d(t^{r_F} w) = d(t^{r_i} w_i) = p$, $1 \leq i \leq s$. Thus $w = w_i$ will imply $r = r_i$ and thereby $t^{r_F} w = t^{r_i} w_i$. So we may assume that $w \neq w_i$, $1 \leq i \leq s$. Then it follows from the definition of $\prec_{t-\text{gr}}$ that $w_i \prec_{\text{gr}} w$ and $r \leq r_i$, $1 \leq i \leq s$. Therefore $\text{LM}(F_s) = w = \text{LM}(F)_s$, as desired. \[ \square \]
\( \text{LM}(\text{LH}(F)) \), we may assume, without loss of generality, that \( F \) is a homogeneous element. So, by Lemma 1.1(vi) we have \( F = t^r f^* \) for some \( f \in I \). It follows from the equality \( \text{LM}(f^*) = \text{LM}(f) \) that

\[
\text{LM}(F) = t^r \text{LM}(f^*) = t^r \text{LM}(f).
\]

Since \( \mathcal{G} \) is a Gröbner basis for \( I \), \( \text{LM}(f) = \lambda v \text{LM}(g_i)w \) for some \( \lambda \in K^* \), \( g_i \in \mathcal{G} \), and \( v, w \in \mathcal{B} \). Thus,

\[
\text{LM}(F) = t^r \text{LM}(f) = \lambda t^r v \text{LM}(g_i^*)w \in \langle \text{LM}(\mathcal{G}^*) \rangle.
\]

This shows that \( \langle \text{LM}(\langle I^* \rangle) \rangle = \langle \text{LM}(\mathcal{G}^*) \rangle \), as desired.

(ii) \( \Rightarrow \) (i) Suppose \( \mathcal{G}^* \) is a Gröbner basis for the homogenization ideal \( \langle I^* \rangle \) of \( I \) in \( R[t] \). Let \( f \in I \). Then \( \text{LM}(f^*) = \lambda v \text{LM}(g_i^*)w \) for some \( \lambda \in K^* \), \( v, w \in \mathcal{B}^* \) and \( g_i^* \in \mathcal{G}^* \). Since \( \text{LM}(f) = \text{LM}(f^*) \), it follows that

\[
\text{LM}(f) = \lambda v \text{LM}(g_i)w \in \langle \text{LM}(\mathcal{G}) \rangle.
\]

This shows that \( \langle \text{LM}(I) \rangle = \langle \text{LM}(\mathcal{G}) \rangle \), i.e., \( \mathcal{G} \) is a Gröbner basis for \( I \) in \( R \). \( \square \)

We call the Gröbner basis \( \mathcal{G}^* \) obtained in Theorem 1.3 the central homogenization of \( \mathcal{G} \) in \( R[t] \) with respect to \( t \), or \( \mathcal{G}^* \) is a central homogenized Gröbner basis with respect to \( t \).

By Lemma 1.2 and Theorem 1.3, we have immediately the following corollary.

1.4. Corollary Let \( I \) be an arbitrary ideal of \( R \). With notation as before, if \( \mathcal{G} \) is a Gröbner basis of \( I \) with respect to the data \( (\mathcal{B}, \prec_{\text{gr}}) \), then, with respect to the data \( (\mathcal{B}^*, \prec_{t, \text{gr}}) \) we have

\[
\mathcal{B}^* - \langle \text{LM}(\mathcal{G}^*) \rangle = \{ t^r w \mid w \in \mathcal{B} - \langle \text{LM}(\mathcal{G}) \rangle, \ r \in \mathbb{N} \},
\]

that is, the set \( N(\langle I^* \rangle) \) of normal monomials (mod \( \langle I^* \rangle \)) in \( \mathcal{B}^* \) is determined by the set \( N(I) \) of normal monomials (mod \( I \)) in \( \mathcal{B} \). Hence, the algebra \( R[t]/\langle I^* \rangle = R[t]/\langle \mathcal{G}^* \rangle \) has the \( K \)-basis

\[
\overline{N}(\langle I^* \rangle) = \{ \overline{tw} \mid w \in N(I), \ r \in \mathbb{N} \}.
\]

\( \square \)

Theoretically we may also obtain a Gröbner basis for an ideal \( I \) of \( R \) by dehomogenizing a homogeneous Gröbner basis of the ideal \( \langle I^* \rangle \subset R[t] \). Below we give a more general approach to this assertion.

1.5. Theorem Let \( J \) be a graded ideal of \( R[t] \). If \( \mathcal{G} \) is a homogeneous Gröbner basis of \( J \) with respect to the data \( (\mathcal{B}^*, \prec_{t, \text{gr}}) \), then \( \mathcal{G}_* = \{ G_* \mid G \in \mathcal{G} \} \) is a Gröbner basis for the ideal \( J_* \) in \( R \) with respect to the data \( (\mathcal{B}, \prec_{\text{gr}}) \).

Proof If \( \mathcal{G} \) is a Gröbner basis of \( J \), then \( \mathcal{G} \) generates \( J \) and hence \( \mathcal{G}_* = \phi(\mathcal{G}) \) generates \( J_* = \phi(J) \). For a nonzero \( f \in J_* \), by Lemma 1.1(vii), there exists a homogeneous element \( H \in J \) such that
$H_*=f$. It follows from Lemma 1.2 that

\[(1)\quad \text{LM}(f) = \text{LM}(f^*) = \text{LM}((H_*)^*).\]

On the other hand, there exists some $G \in \mathcal{G}$ such that $\text{LM}(G)|\text{LM}(H)$, i.e.,

\[(2)\quad \text{LM}(H) = \lambda t^{r_1}w\text{LM}(G)t^{r_2}v\]

for some $\lambda \in K^*$, $r_1,r_2 \in \mathbb{N}$, $w,v \in B$. But by Lemma 1.1(iii) we also have $t^r(H_*) = H$ for some $r \in \mathbb{N}$, and hence

\[(3)\quad \text{LM}(H) = \text{LM}(t^r(H_*)^*) = t^r\text{LM}(H_*)^*).\]

So, $(1) + (2) + (3)$ yields

\[
\lambda t^{r_1+r_2}w\text{LM}(G) v &= \text{LM}(H) \\
&= t^r\text{LM}((H_*)^*) \\
&= t^r\text{LM}(f).
\]

Taking the central dehomogenization for the above equality, by Lemma 1.2(iii) we obtain

\[
\lambda w\text{LM}(G) v = \lambda w\text{LM}(G_*) v = \text{LM}(f).
\]

This shows that $\text{LM}(G_*)|\text{LM}(f)$. Therefore, $G_*$ is a Gröbner basis for $J_*$. \hfill \Box

We call the Gröbner basis $G_*$ obtained in Theorem 1.5 the central dehomogenization of $\mathcal{G}$ in $R$ with respect to $t$, or $G_*$ is a central dehomogenized Gröbner basis with respect to $t$.

1.6. Corollary Let $I$ be an ideal of $R$. If $\mathcal{G}$ is a homogeneous Gröbner basis of $\langle I^* \rangle$ in $R[t]$ with respect to the data $(B^*, <_{t-gr})$, then $\mathcal{G}_* = \{g_* \mid g \in \mathcal{G}\}$ is a Gröbner basis for $I$ in $R$ with respect to the data $(B, <_{gr})$. Moreover, if $I$ is generated by the subset $F$ and $F^* \subset \mathcal{G}$, then $F \subset \mathcal{G}_*$.

Proof Put $J = \langle I^* \rangle$. Then since $J_* = I$, it follows from Theorem 1.5 that if $\mathcal{G}$ is a homogeneous Gröbner basis of $J$ then $\mathcal{G}_*$ is a Gröbner basis for $I$. The second assertion of the theorem is clear by Lemma 1.1(ii). \hfill \Box

Let $S$ be a nonempty subset of $R$ and $I = \langle S \rangle$ the ideal generated by $S$. Then, with $S^* = \{f^* \mid f \in S\}$, in general $\langle S^* \rangle \subseteq \langle I^* \rangle$ in $R[t]$ (for instance, consider $S = \{y^3 - x - y, y^2 + 1\}$ in the commutative polynomial ring $K[x,y]$ and the homogenization in $K[x,y,t]$ with respect to $t$). So, from both a practical and a computational viewpoint, it is the right place to set up the procedure of getting a Gröbner basis for $I$ and hence a Gröbner basis for $\langle I^* \rangle$ by producing a homogeneous Gröbner basis of the graded ideal $\langle S^* \rangle$. 1
1.7. Proposition Let $I = \langle S \rangle$ be the ideal of $R$ as fixed above. Suppose that Gröbner bases are algorithmically computable in $R$ and hence in $R[t]$. Then a Gröbner basis for $I$ and a homogeneous Gröbner basis for $\langle I^* \rangle$ may be obtained by implementing the following procedure:

**Step 1.** Starting with the initial subset $S^* = \{ f^* \mid f \in S \}$, compute a homogeneous Gröbner basis $\mathcal{G}$ for the graded ideal $\langle S^* \rangle$ of $R[t]$.

**Step 2.** Noticing $\langle S^* \rangle_*= I$, use Theorem 1.5 and dehomogenize $\mathcal{G}$ with respect to $t$ in order to obtain the Gröbner basis $\mathcal{G}_*$ for $I$.

**Step 3.** Use Theorem 1.3 and homogenize $\mathcal{G}_*$ with respect to $t$ in order to obtain the homogeneous Gröbner basis $(\mathcal{G}_*)^*$ for the graded ideal $\langle I^* \rangle$.

□

2. Non-central (De)homogenized Gröbner Bases

In a similar way, we proceed now to consider the free algebra $K\langle X \rangle = K\langle X_1, ..., X_n \rangle$ of $n$ generators as well as the free algebra $K\langle X, T \rangle = K\langle X_1, ..., X_n, T \rangle$ of $n + 1$ generators, and demonstrate how Gröbner bases in $K\langle X \rangle$ are related with homogeneous Gröbner bases in $K\langle X, T \rangle$ if the non-central (de)homogenization with respect to $T$ is employed.

Let $K\langle X \rangle$ be equipped with a fixed weight $\mathbb{N}$-gradation, say each $X_i$ has degree $n_i > 0$, $1 \leq i \leq n$. Assigning to $T$ the degree 1 in $K\langle X, T \rangle$ and using the same weight $n_i$ for each $X_i$ as in $K\langle X \rangle$, we get the weight $\mathbb{N}$-gradation of $K\langle X, T \rangle$ which extends the weight $\mathbb{N}$-gradation of $K\langle X \rangle$. Let $B$ and $\bar{B}$ denote the standard $K$-bases of $K\langle X \rangle$ and $K\langle X, T \rangle$ respectively. To be convenient we use lowercase letters $w, u, v, ...$ to denote monomials in $B$ as before, but use capitals $W, U, V, ...$ to denote monomials in $\bar{B}$.

In what follows, we fix an admissible system $(B, \prec_{gr})$ for $K\langle X \rangle$, where $\prec_{gr}$ is an $\mathbb{N}$-graded lexicographic ordering on $B$ with respect to the fixed weight $\mathbb{N}$-gradation of $K\langle X \rangle$, such that

$$X_{i_1} \prec_{gr} X_{i_2} \prec_{gr} \cdots \prec_{gr} X_{i_n}.$$  

Then it is not difficult to see that $\prec_{gr}$ can be extended to an $\mathbb{N}$-graded lexicographic ordering $\prec_{\gamma gr}$ on $\bar{B}$ with respect to the fixed weight $\mathbb{N}$-gradation of $K\langle X, T \rangle$, such that

$$T \prec_{\gamma gr} X_{i_1} \prec_{\gamma gr} X_{i_2} \prec_{\gamma gr} \cdots \prec_{\gamma gr} X_{i_n},$$

and thus we get the admissible system $(\bar{B}, \prec_{\gamma gr})$ for $K\langle X, T \rangle$. With respect to $\prec_{gr}$ and $\prec_{\gamma gr}$ we use $\text{LM}()$ to denote taking the leading monomial of elements in $K\langle X \rangle$ and $K\langle X, T \rangle$ respectively.

Consider the fixed $\mathbb{N}$-graded structures $K\langle X \rangle = \bigoplus_{p \in \mathbb{N}} K\langle X \rangle_p$, $K\langle X, T \rangle = \bigoplus_{p \in \mathbb{N}} K\langle X, T \rangle_p$, and the ring epimorphism

$$\psi : K\langle X, T \rangle \longrightarrow K\langle X \rangle$$
defined by $\psi(X_i) = X_i$ and $\psi(T) = 1$. Then each $f \in K\langle X \rangle$ is the image of some homogeneous element in $K\langle X, T \rangle$. More precisely, if $f = f_p + f_{p-1} + \cdots + f_{p-s}$ with $f_p \in K\langle X \rangle_p$, $f_{p-j} \in K\langle X \rangle_{p-j}$ and $f_p \neq 0$, then

$$\tilde{f} = f_p + T f_{p-1} + \cdots + T^s f_{p-s}$$

is a homogeneous element of degree $p$ in $K\langle X, T \rangle_p$ such that $\psi(\tilde{f}) = f$. We call the homogeneous element $\tilde{f}$ obtained this way the non-central homogenization of $f$ with respect to $T$ (for the reason that $T$ is not a commuting variable). On the other hand, for $F \in K\langle X, T \rangle$, we write

$$F_\sim = \psi(F)$$

and call $F_\sim$ the non-central dehomogenization of $F$ with respect to $T$ (again for the reason that $T$ is not a commuting variable). Furthermore, if $I = \langle S \rangle$ is the ideal of $K\langle X \rangle$ generated by a subset $S$, then we write

$$\tilde{S} = \{ \tilde{f} \mid f \in S \} \cup \{ X_i T - TX_i \mid 1 \leq i \leq n \},$$

$$\tilde{I} = \{ \tilde{f} \mid f \in I \} \cup \{ X_i T - TX_i \mid 1 \leq i \leq n \},$$

and call the graded ideal $\langle \tilde{I} \rangle$ generated by $\tilde{I}$ the non-central homogenization ideal of $I$ in $K\langle X, T \rangle$ with respect to $T$; while if $J$ is an ideal of $K\langle X, T \rangle$, then since $\psi$ is a surjective ring homomorphism, $\psi(J)$ is an ideal of $K\langle X \rangle$, so we write $J_\sim$ for $\psi(J) = \{ H_\sim \mid H \in J \}$ and call it the non-central dehomogenization ideal of $J$ in $K\langle X \rangle$ with respect to $T$. Consequently, henceforth we will also use the notation

$$(J_\sim)_\sim = \{ (h_\sim)_\sim \mid h \in J \} \cup \{ X_i T - TX_i \mid 1 \leq i \leq n \}.$$

It is straightforward to check that with respect to the data $(\tilde{B}, \prec_{\sim \psi})$, the subset $\{ X_i T - TX_i \mid 1 \leq i \leq n \}$ of $K\langle X, T \rangle$ forms a homogeneous Gröbner basis with $LM(X_i T - TX_i) = X_i T$, $1 \leq i \leq n$. In the latter discussion we will freely use this fact without extra indication.

**2.1. Lemma** With notation as fixed before, the following properties hold.

(i) If $F, G \in K\langle X, T \rangle$, then $(F + G)_\sim = F_\sim + G_\sim$, $(FG)_\sim = F_\sim G_\sim$.

(ii) For each nonzero $f \in K\langle X \rangle$, $(\tilde{f})_\sim = f$.

(iii) Let $\mathcal{C}$ be the graded ideal of $K\langle X, T \rangle$ generated by $\{ X_i T - TX_i \mid 1 \leq i \leq n \}$. If $F \in K\langle X, T \rangle_p$, then there exists an $L \in \mathcal{C}$ and a unique homogeneous element of the form $H = \sum_i \lambda_i T^r w_i$, where $\lambda_i \in K^*$, $w_i \in B$, such that $F = L + H$; moreover there is some $r \in \mathbb{N}$ such that $T^r(H_\sim)_\sim = H$, and hence $F = L + T^r(F_\sim)_\sim$.

(iv) Let $\mathcal{C}$ be as in (iii) above. If $I$ is an ideal of $K\langle X \rangle$, $F \in \langle \tilde{I} \rangle$ is a homogeneous element, then there exist some $L \in \mathcal{C}$, $f \in I$ and $r \in \mathbb{N}$ such that $F = L + T^r \tilde{f}$.

(v) If $J$ is a graded ideal of $K\langle X, T \rangle$ and $\{ X_i T - TX_i \mid 1 \leq i \leq n \} \subset J$, then for each nonzero $h \in J_\sim$, there exists a homogeneous element $H = \sum_i \lambda_i T^r w_i \in J$, where $\lambda_i \in K^*$, $r_i \in \mathbb{N}$, and $w_i \in B$, such that for some $r \in \mathbb{N}$, $T^r (H_\sim)_\sim = H$ and $H_\sim = h$. 


Proof (i) and (ii) follow from the definitions of non-central homogenization and non-central dehomogenization directly.

(iii) Since the subset \( \{X_i T - TX_i \mid 1 \leq i \leq n\} \) is a Gröbner basis in \( K\langle X, T \rangle \) with respect to \((\tilde{B}, \prec_{\text{gr}})\), such that \( \text{LM}(X_i T - TX_i) = X_i T, 1 \leq i \leq n \), if \( F \in K\langle X, T \rangle_p \), then the division of \( F \) by this subset yields \( F = L + H \), where \( L \in \mathscr{C} \), and \( H = \sum \lambda_i T^s w_i \) is the unique remainder with \( \lambda_i \in K^* \), \( w_i \in \mathcal{B} \), in which each monomial \( T^s w_i \) is of degree \( p \). By the definition of \( \prec_{\text{gr}} \), the definitions of non-central homogenization and the definition of non-central dehomogenization, it is not difficult to see that \( H \) has the desired property.

(iv) By (iii), \( F = L + T^r (F_\sim)^\prec \) with \( L \in \mathscr{C} \) and \( r \in \mathbb{N} \). Since by (ii) we have \( F_\sim \in \langle \tilde{I} \rangle_\sim = I \), thus \( f = F_\sim \) is the desired element.

(v) Using basic properties of homogeneous element and graded ideal in a graded ring, this follows from the foregoing (iii). \( \square \)

As in the case using central (de)homogenization, before turning to deal with Gröbner bases, we are also concerned about the behavior of leading monomials under taking the \( \mathbb{N} \)-leading homogeneous element and non-central (de)homogenization. Below we use \( \text{LH}(\ ) \) to denote taking the \( \mathbb{N} \)-leading homogeneous element (i.e., the highest-degree homogeneous component) of elements in both \( K\langle X \rangle \) and \( K\langle X, T \rangle \) with respect to the fixed \( \mathbb{N} \)-gradation.

2.2. Lemma With the assumptions and notations as fixed above, the following statements hold.

(i) If \( f \in K\langle X \rangle \), then

\[ \text{LM}(f) = \text{LM}(\text{LH}(f)) \text{ w.r.t. } \prec_{\text{gr}} \text{ on } \mathcal{B}; \]

If \( F \in K\langle X, T \rangle \), then

\[ \text{LM}(F) = \text{LM}(\text{LH}(F)) \text{ w.r.t. } \prec_{\text{gr}} \text{ on } \tilde{B}. \]

(ii) For each nonzero \( f \in K\langle X \rangle \), we have

\[ \text{LM}(f) = \text{LM}(\bar{f}) \text{ w.r.t. } \prec_{\text{gr}} \text{ on } \tilde{B}. \]

(iii) If \( F \) is a homogeneous element in \( K\langle X, T \rangle \) such that \( X_i T \nmid \text{LM}(F) \) with respect to \( \prec_{\text{gr}} \) for all \( 1 \leq i \leq n \), then \( \text{LM}(F) = T^r w \) for some \( r \in \mathbb{N} \) and \( w \in \mathcal{B} \), such that

\[ \text{LM}(F_\sim) = w = \text{LM}(F)_\sim \text{ w.r.t. } \prec_{\text{gr}} \text{ on } \mathcal{B}. \]

Proof The proof of (i) and (ii) is an easy exercise. To prove (iii), let \( F \in K\langle X, T \rangle_p \) be a nonzero homogeneous element of degree \( p \). Then by the assumption \( F \) may be written as

\[ F = \lambda T^r w + \lambda_1 T^{r_1} X_j W_1 + \lambda_2 T^{r_2} X_j W_2 + \cdots + \lambda_s T^{r_s} X_j W_s, \]

where \( \lambda, \lambda_i \in K^* \), \( r, r_i \in \mathbb{N} \), \( w \in \mathcal{B} \) and \( w_i \in \tilde{B} \), such that \( \text{LM}(F) = T^r w \). Since \( \mathcal{B} \) consists of \( \mathbb{N} \)-homogeneous elements and the \( \mathbb{N} \)-gradation of \( K\langle X \rangle \) extends to give the \( \mathbb{N} \)-gradation of
Since \( G \) is a Gröbner basis of \( I \) with respect to the admissible system \((\mathcal{B}, \prec_{gr})\) of \( K(X)\); (ii) \( \tilde{G} = \{ \tilde{g} \mid g \in G \} \cup \{ X_i T - TX_i \mid 1 \leq i \leq n \} \) is a homogeneous Gröbner basis for \((\tilde{I})\) with respect to the admissible system \((\tilde{\mathcal{B}}, \prec_{\tilde{gr}})\) of \( K(X, T)\).

**Proof** In proving the equivalence below, without specific indication we shall use (i) and (ii) of Lemma 2.2 wherever it is needed.

(i) \( \Rightarrow \) (ii) Suppose that \( G \) is a Gröbner basis of \( I \) with respect to the data \((\mathcal{B}, \prec_{gr})\). Let \( F \in (\tilde{I}) \).

Thus we may assume that \( F \) is a nonzero homogeneous element. We want to show that there is some \( D \in \tilde{G} \) such that \( \text{LM}(D) \mid \text{LM}(F) \), and hence \( G \) is a Gröbner basis.

Note that \( \{ X_i T - TX_i \mid 1 \leq i \leq n \} \subseteq \tilde{G} \) with \( \text{LM}(X_i T - TX_i) = X_i T \). If \( X_i T \mid \text{LM}(F) \) for some \( X_i T \), then we are done. Otherwise, \( X_i T \not\mid \text{LM}(F) \) for all \( 1 \leq i \leq n \). Thus, by Lemma 2.2(iii), \( \text{LM}(F) = T^r w \) for some \( r \in \mathbb{N} \) and \( w \in \mathcal{B} \) and

\[
\text{LM}(F_{\sim}) = w = \text{LM}(F)_{\sim}.
\]

On the other hand, by Lemma 2.1(iv) we have \( F = L + T^q \tilde{f} \), where \( L \) is an element in the ideal \( \mathcal{G} \) generated by \( \{ X_i T - TX_i \mid 1 \leq i \leq n \} \) in \( K(X, T) \), \( q \in \mathbb{N} \), and \( f \in I \). It turns out that

\[
F_{\sim} = (\tilde{f})_{\sim} = f \quad \text{and hence} \quad \text{LM}(F_{\sim}) = \text{LM}(f).
\]

Since \( G \) is a Gröbner basis for \( I \), there is some \( g \in G \) such that \( \text{LM}(g) \mid \text{LM}(f) \), i.e., there are \( u, v \in \mathcal{B} \) such that

\[
\text{LM}(f) = u \text{LM}(g) v = u \text{LM}(\tilde{g}) v.
\]

Combining (1), (2), and (3) above, we have

\[
w = \text{LM}(F_{\sim}) = \text{LM}(f) = u \text{LM}(\tilde{g}) v.
\]

Therefore, \( \text{LM}(\tilde{g}) | T^r w \), i.e., \( \text{LM}(\tilde{g}) \text{LM}(F) \), as desired.
(ii) ⇒ (i) Suppose that \( \tilde{G} \) is a Gröbner basis of the graded ideal \( \langle \tilde{I} \rangle \) in \( K(X,T) \). If \( f \in I \), then since \( \tilde{f} \in \tilde{I} \), there is some \( H \in \tilde{G} \) such that \( \text{LM}(H)\text{LM}(\tilde{f}) \). Note that \( \text{LM}(\tilde{f}) = \text{LM}(f) \) and thus \( T \not| \text{LM}(\tilde{f}) \). Hence \( H = \tilde{g} \) for some \( g \in G \), and there are \( w,v \in B \) such that

\[
\text{LM}(f) = \text{LM}(\tilde{f}) = w\text{LM}(\tilde{g})v = w\text{LM}(g)v.
\]

This shows that \( G \) is a Gröbner basis for \( I \) in \( R \).

We call the Gröbner basis \( \tilde{G} \) obtained in Theorem 2.3 the non-central homogenization of \( G \) in \( K(X,T) \) with respect to \( T \), or \( \tilde{G} \) is a non-central homogenized Gröbner basis with respect to \( T \).

By Lemma 2.1 and Theorem 2.3, the following Corollary is straightforward.

**2.4. Corollary** Let \( I \) be an arbitrary ideal of \( K(X) \). With notation as before, if \( G \) is a Gröbner basis of \( I \) with respect to the data \( (B,\prec_{gr}) \), then, with respect to the data \( (\tilde{B},\prec_{\tau_{gr}}) \) we have

\[
\tilde{B} - \langle \text{LM}(\tilde{G}) \rangle = \{ T^r w \mid w \in B - \langle \text{LM}(\tilde{G}) \rangle, \ r \in \mathbb{N} \},
\]

that is, the set \( N(\langle \tilde{I} \rangle) \) of normal monomials \( (\text{mod} \ (\tilde{I})) \) in \( \tilde{B} \) is determined by the set \( N(I) \) of normal monomials \( (\text{mod} \ I) \) in \( B \). Hence, the algebra \( K(X,T)/\langle \tilde{I} \rangle = K(X,T)/\langle \tilde{G} \rangle \) has the \( K \)-basis

\[
N(\langle \tilde{I} \rangle) = \{ T^r w \mid w \in N(I), \ r \in \mathbb{N} \}.
\]

As with the central (de)homogenization with respect to the commuting variable \( t \) in section 1, theoretically we may also obtain a Gröbner basis for an ideal \( I \) of \( K(X) \) by dehomogenizing a homogeneous Gröbner basis of the ideal \( \langle \tilde{I} \rangle \subset K(X,T) \). Below we give a more general approach to this assertion.

**2.5. Theorem** Let \( J \) be a graded ideal of \( K(X,T) \), and suppose that \( \{ X_iT - TX_i \mid 1 \leq i \leq n \} \subset J \). If \( G \) is a homogeneous Gröbner basis of \( J \) with respect to the data \( (\tilde{B},\prec_{\tau_{gr}}) \), then \( \tilde{G}_\sim = \{ G_\sim \mid G \in G \} \) is a Gröbner basis for the ideal \( J_\sim \) in \( K(X) \) with respect to the data \( (B,\prec_{gr}) \).

**Proof** If \( G \) is a Gröbner basis of \( J \), then \( G \) generates \( J \) and hence \( G_\sim = \phi(G) \) generates \( J_\sim = \phi(J) \). We show next that for each nonzero \( h \in J_\sim \), there is some \( G_\sim \in \tilde{G}_\sim \) such that \( \text{LM}(G_\sim)|\text{LM}(h) \), and hence \( \tilde{G}_\sim \) is a Gröbner basis for \( J_\sim \).

Since \( \{ X_iT - TX_i \mid 1 \leq i \leq n \} \subset J \), by Lemma 2.1(v) there exists a homogeneous element \( H \in J \) and some \( r \in \mathbb{N} \) such that \( T^r(H_\sim) = H \) and \( H_\sim = h \). It follows that

\[
(1) \quad \text{LM}(H) = T^r\text{LM}((H_\sim) = T^r\text{LM}(\tilde{h}) = T^r\text{LM}(h).
\]
On the other hand, there is some $G \in \mathcal{G}$ such that $\text{LM}(G)|\text{LM}(H)$, i.e., there are $W, V \in \mathcal{B}$ such that

$$\text{LM}(H) = W\text{LM}(G)V. \tag{2}$$

But by the above (1) we must have $\text{LM}(G) = T^q w$ for some $q \in \mathbb{N}$ and $w \in \mathcal{B}$. Thus, by Lemma 2.2(iii),

$$\text{LM}(G_\sim) = w = \text{LM}(G) \text{ w.r.t. } \prec_{\text{gr}} \text{ on } \mathcal{B}. \tag{3}$$

Combining (1), (2), and (3) above, we then obtain

$$\text{LM}(h) = \text{LM}(H_\sim) = (W\text{LM}(G)V)_\sim = W_\sim \text{LM}(G)_\sim V_\sim = W_\sim \text{LM}(G_\sim)V_\sim.$$

This shows that $\text{LM}(G_\sim)|\text{LM}(h)$, as expected. \hfill $\square$

We call the Gröbner basis $\mathcal{G}_\sim$ obtained in Theorem 2.5 the non-central dehomogenization of $\mathcal{G}$ in $K\langle X \rangle$ with respect to $T$, or $\mathcal{G}_\sim$ is a non-central dehomogenized Gröbner basis with respect to $T$.

2.6. Corollary Let $I$ be an ideal of $K\langle X \rangle$. If $\mathcal{G}$ is a homogeneous Gröbner basis of $\langle \mathcal{G} \rangle$ in $K\langle X, T \rangle$ with respect to the data $(\mathcal{B}, \prec_{\text{gr}})$, then $\mathcal{G}_\sim = \{g_\sim \mid g \in \mathcal{G}\}$ is a Gröbner basis for $I$ in $K\langle X \rangle$ with respect to the data $(B, \prec_{\text{gr}})$. Moreover, if $I$ is generated by the subset $F$ and $\bar{F} \subset \mathcal{G}$, then $F \subset \mathcal{G}_\sim$.

Proof Put $J = \langle \mathcal{G} \rangle$. Then since $J_\sim = I$, it follows from Theorem 2.5 that if $\mathcal{G}$ is a homogeneous Gröbner basis of $J$ then $\mathcal{G}_\sim$ is a Gröbner basis for $I$. The second assertion of the theorem is clear by Lemma 2.1(ii). \hfill $\square$

Let $S$ be a nonempty subset of $K\langle X \rangle$ and $I = \langle S \rangle$ the ideal generated by $S$. Then, with $\bar{S} = \{\bar{f} \mid f \in S\} \cup \{X_iT - TX_i \mid 1 \leq i \leq n\}$, in general $\langle \bar{S} \rangle \subset \langle \mathcal{G} \rangle$ in $K\langle X, T \rangle$ (for instance, consider $S = \{Y^3 - XY - X - Y, Y^2 - X + 3\}$ in the free algebra $K\langle X, Y \rangle$ and the homogenization in $K\langle X, Y, T \rangle$ with respect to $T$). Again, as we did in the case dealing with (de)homogenized Gröbner bases with respect to the commuting variable $t$, we take this place to set up the procedure of getting a Gröbner basis for $I$ and hence a Gröbner basis for $\langle \mathcal{G} \rangle$ by producing a homogeneous Gröbner basis of the graded ideal $\langle \bar{S} \rangle$.

2.7. Proposition Let $I = \langle S \rangle$ be the ideal of $K\langle X \rangle$ as fixed above. Suppose the ground field $K$ is computable. Then a Gröbner basis for $I$ and a homogeneous Gröbner basis for $\langle \mathcal{G} \rangle$ may be obtained by implementing the following procedure:
Step 1. Starting with the initial subset 
\[ \tilde{S} = \{ \tilde{f} \mid f \in S \} \cup \{ X_i T - T X_i \mid 1 \leq i \leq n \}, \]
compute a homogeneous Gröbner basis \( \mathcal{G} \) for the graded ideal \( \langle \tilde{S} \rangle \) of \( K\langle X, T \rangle \).

Step 2. Noticing \( \langle \tilde{S} \rangle \simeq I \), use Theorem 2.5 and dehomogenize \( \mathcal{G} \) with respect to \( T \) in order to obtain the Gröbner basis \( \mathcal{G} \simeq \) for \( I \).

Step 3. Use Theorem 2.3 and homogenize \( \mathcal{G}_\simeq \) with respect to \( T \) in order to obtain the homogeneous Gröbner basis \( (\mathcal{G}_\simeq)^\simeq \) for the graded ideal \( \langle \tilde{I} \rangle \).

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