Constant Mean Curvature Spacelike Hypersurfaces in Spatially Open GRW Spacetimes

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Abstract
In this paper, we provide under certain geometric and physical assumptions new uniqueness and non-existence results for complete spacelike hypersurfaces of constant mean curvature in spatially open Generalized Robertson–Walker spacetimes. Some of our results are then applied to relevant spacetimes as the steady-state spacetime, Einstein–de Sitter spacetime, and certain radiation models.

Keywords Constant mean curvature · Spacelike hypersurface · Generalized Robertson–Walker spacetime

Mathematics Subject Classification 53C80 · 53C42 · 53C50

1 Introduction

Spacelike hypersurfaces of constant mean curvature in a spacetime are geometric objects of relevant physical and mathematical interest. They are critical points of the area functional under a suitable volume constraint [6] and, for instance, they play an important role in General Relativity as convenient initial data for the Cauchy problem [12,13] and in the proof of the positivity of the gravitational mass [27]. A summary of the many reasons justifying their physical interest can be found in [14]. As a consequence, the literature on the subject provides a number of articles dealing with
existence and uniqueness results (see for instance [3,11,19]) in a large variety of spacetimes.

From a mathematical perspective, they exhibit nice Bernstein-type properties. In fact, the only complete spacelike hypersurfaces in the $(m + 1)$-dimensional Lorentz–Minkowski spacetime $\mathbb{L}^{m+1}$ with zero mean curvature are spacelike hyperplanes. This striking result was proved by Calabi in [8] for $m \leq 4$ and later extended to arbitrary dimension by Cheng and Yau in [9]. Their approach is based on a Simons-type formula for spacelike hypersurfaces in $\mathbb{L}^{m+1}$ and the application of a generalized maximum principle due to Omori [17] and Yau [28].

In this article, we study constant mean curvature spacelike hypersurfaces in the family of cosmological models known as Generalized Robertson–Walker (GRW) spacetimes. They are warped products whose negative definite base represents a universal time and whose fiber is an arbitrary Riemannian manifold (see Sect. 2 for definitions). These spacetimes are relevant in physics and extend the classical notion of Robertson–Walker spacetime to the case when the fiber does not necessarily have constant sectional curvature [3]. Thus, GRW spacetimes are not necessarily spatially homogeneous, making them suitable cosmological models to describe the universe in a more accurate scale [22]. Furthermore, deformations of the metric on the fiber of classical Robertson–Walker spacetimes as well as GRW spacetimes with a time-dependent conformal change of metric also fit into the class of GRW spacetimes.

Classically, spacelike hypersurfaces have been studied in the class of GRW spacetimes known as spatially closed GRW spacetimes (see, for instance [3,21]). However, some observations and theoretical arguments about the total mass balance of the universe [10] suggest the convenience of using spatially open models to describe our current universe. Moreover, spatially closed models also lead to a violation of the holographic principle [5]. Due to these reasons, many authors have recently studied maximal hypersurfaces (i.e., spacelike hypersurface with zero mean curvature) in spatially open GRW spacetimes under certain other assumptions on the fiber, such as parabolicity [24] or flatness [20], and sometimes imposing some requirements, like the property of having a bounded hyperbolic angle or that of being contained in a slab [25].

The aim of this paper is to obtain new results for spacelike hypersurfaces of constant mean curvature in GRW spacetimes without any compactness assumption on the fiber and neither imposing a priori bounds on the hyperbolic angle nor the property of being contained in a slab. This will allow us to deal, respectively, with spacelike hypersurfaces that approach the null boundary as well as the future and past infinities. In order to obtain our results, we will assume a weak energy condition on the spacetime, the Null Energy Condition (NCC). Note that this energy condition is weaker than the Timelike Convergence Condition (see [18]), which is a physically reasonable assumption commonly used to obtain uniqueness results for these hypersurfaces (see, for instance, [7,16]).

Furthermore, as we point out in Remark 14, our non-existence results have a clear physical meaning. Thus, we prove that in these ambient spacetimes there are no constant mean curvature spacelike hypersurfaces where the normal observers measure a different behavior in the expansion/contraction of the universe than the comoving ones at every point.
2 Preliminaries

Let \((F, g_F)\) be an \(m (\geq 2)\)-dimensional (connected) Riemannian manifold, \(I\) an open interval in \(\mathbb{R}\) endowed with the metric \(-dt^2\), and \(\rho\) a positive smooth function defined on \(I\). Then, the product manifold \(I \times F\) endowed with the Lorentzian metric

\[
\bar{g} = -\pi_I^*(dt^2) + \rho(\pi_I)^2 \pi_F^*(g_F),
\]

where \(\pi_I\) and \(\pi_F\) denote the projections onto \(I\) and \(F\), respectively, is called a GRW spacetime with fiber \((F, g_F)\), base \((I, -dt^2)\), and warping function \(\rho\). If the fiber has constant sectional curvature, it is called a Robertson–Walker spacetime.

In any GRW spacetime \(M = I \times \rho F\), the coordinate vector field \(\partial_t := \partial/\partial t\) is (unitary) timelike, and hence \(M\) is time-orientable. On the other hand, if we consider the timelike vector field \(K := \rho(\pi_I) \partial_t\), from the relation between the Levi-Civita connection of \(M\) and those of the base and the fiber [18, Cor. 7.35], it follows that

\[
\nabla_K K = \rho'(\pi_I) X,
\]

for any \(X \in \mathfrak{X}(M)\), where \(\nabla\) is the Levi-Civita connection of the Lorentzian metric (1). Thus, \(K\) is a closed conformal vector field.

From (2) we easily see that the divergence on \(\bar{M}\) of the reference frame \(\partial_t\) satisfies \(\text{div}(\partial_t) = m \rho'(t) / \rho(t)\). Therefore, the observers in \(\partial_t\) are spreading out (resp. coming together) if \(\rho' > 0\) (resp. \(\rho' < 0\)).

Given an \(m\)-dimensional manifold \(M\), an immersion \(\psi : M \rightarrow \bar{M}\) is said to be spacelike if the Lorentzian metric (1) induces, via \(\psi\), a Riemannian metric \(g\) on \(M\). In this case, \(M\) is called a spacelike hypersurface. We will denote by \(\tau := \pi_I \circ \psi\) the restriction of \(\pi_I\) along \(\psi\).

The time-orientation of \(\bar{M}\) allows to take, for each spacelike hypersurface \(M\) in \(\bar{M}\), a unique unitary timelike vector field \(N \in \mathfrak{X}^+(M)\) globally defined on \(M\) with the same time-orientation of \(\partial_t\), i.e., such that \(\bar{g}(N, \partial_t) \leq -1\). Note that \(\bar{g}(N, \partial_t) = -1\) at a point \(p \in M\) if and only if \(N = \partial_t\) at \(p\). We will denote by \(A\) the shape operator associated with \(N\). Then, the mean curvature function associated with \(N\) is given by \(H := -(1/m)\text{trace}(A)\). As it is well known, the mean curvature is constant if and only if the spacelike hypersurface is, locally, a critical point of the \(m\)-dimensional area functional for compactly supported normal variations, under certain constraints of the volume. When the mean curvature vanishes identically, the spacelike hypersurface is called a maximal hypersurface.

For a spacelike hypersurface \(\psi : M \rightarrow \bar{M}\) with Gauss map \(N\), the hyperbolic angle \(\varphi\), at any point of \(M\), between the unit timelike vectors \(N\) and \(\partial_t\), is given by \(\cosh \varphi = -\bar{g}(N, \partial_t)\). For simplicity, throughout this paper, we will refer to \(\varphi\) as the hyperbolic angle function on \(M\).

In any GRW spacetime \(\bar{M} = I \times \rho F\), there is a remarkable family of spacelike hypersurfaces, namely, its spacelike slices \(\{t_0\} \times F, t_0 \in I\). It can be easily seen that a spacelike hypersurface in \(\bar{M}\) is a (piece of) spacelike hypersurface if and only if the function \(\tau\) is constant. Furthermore, a spacelike hypersurface in \(\bar{M}\) is a (piece of) spacelike
slice if and only if the hyperbolic angle \( \phi \) vanishes identically. The shape operator of the spacelike slice \( \tau = t_0 \) is given by \( A = -\rho'(t_0)/\rho(t_0)I \), where \( I \) denotes the identity transformation, and therefore its (constant) mean curvature \( H \) is given by \( \rho'(t_0)/\rho(t_0) \). Thus, a spacelike slice is maximal if and only if \( \rho'(t_0) = 0 \) (and hence, totally geodesic).

3 The Set Up

Let \( \psi : M \to \bar{M} \) be an \( m \)-dimensional spacelike hypersurface immersed in a GRW spacetime \( \bar{M} = I \times \rho F \). Denoting by

\[
\partial^T_t := \partial_t + \bar{g}(N, \partial_t)N,
\]

the tangential component of \( \partial_t \) along \( \psi \), then it is easy to check that the gradient of \( \tau \) on \( M \) is

\[
\nabla \tau = -\partial^T_t,
\]

and so

\[
|\nabla \tau|^2 = g(\nabla \tau, \nabla \tau) = \sinh^2 \phi.
\]

Moreover, since the tangential component of \( K \) along \( \psi \) is given by \( K^T = K + \bar{g}(K, N)N \), a direct computation from (2) gives

\[
\nabla \bar{g}(K, N) = -AK^T,
\]

where we have used (3), and also the fact that

\[
\nabla \cosh \phi = A\partial^T_t + \frac{\rho'(\tau)}{\rho(\tau)} \cosh \phi \partial^T_t.
\]

On the other hand, if we represent by \( \nabla \) the Levi-Civita connection of the metric \( g \), then the Gauss and Weingarten formulas for the immersion \( \psi \) are, respectively, given by

\[
\nabla_X Y = \nabla_X Y - g(AX, Y)N,
\]

and

\[
AX = -\nabla_X N,
\]

where \( X, Y \in \mathcal{X}(M) \). Then, taking the tangential component in (2) and using (7) and (8), we obtain

\[
\nabla_X K^T = -\rho(\tau)\bar{g}(N, \partial_t)AX + \rho'(\tau)X,
\]

where \( X \in \mathcal{X}(M) \) and \( \rho'(\tau) := \rho' \circ \tau \). Next, from (3) we get

\[
\nabla_X \partial^T_t = \frac{\rho'(\tau)}{\rho(\tau)} g(X, \partial^T_t) \partial^T_t + \cosh \phi AX + \frac{\rho'(\tau)}{\rho(\tau)} X.
\]
We now choose a local orthonormal reference frame \( \{ E_1, \ldots, E_m \} \) on \( TM \) to calculate the Laplacian of \( \cosh \varphi \) as follows

\[
\Delta \cosh \varphi = \sum_{i=1}^{m} g(\nabla_{E_i} (A \partial_i^T), E_i) + \sum_{i=1}^{m} g \left( \nabla_{E_i} \left( \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \partial_i^T \right), E_i \right),
\]

where we have made use of (6). After several computations, we rewrite (11) in the form

\[
\Delta \cosh \varphi = \sum_{i=1}^{m} g(\nabla_{E_i} (A \partial_i^T), E_i) - \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sum_{i=1}^{m} g(E_i, \partial_i^T) g(\partial_i^T, E_i)
\]

\[+ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sum_{i=1}^{m} g(E_i, \partial_i^T) g(\partial_i^T, E_i)
\]

\[+ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sum_{i=1}^{m} g(E_i, \partial_i^T) g(\partial_i^T, E_i)
\]

\[+ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sum_{i=1}^{m} g(\nabla_{E_i} \partial_i^T, E_i).
\]

Taking into account (4) and \((\nabla X A)Y = \nabla X (AY) - A(\nabla X Y)\) for all \( X, Y \in \mathfrak{X}(M) \), from (12), we have

\[
\Delta \cosh \varphi = \sum_{i=1}^{m} g((\nabla_{E_i} A) \partial_i^T, E_i) + \sum_{i=1}^{m} g((\nabla_{E_i} \partial_i^T) A E_i) - \frac{\rho''(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi
\]

\[+ 2 \frac{\rho'(\tau)^2}{\rho(\tau)^2} \cosh \varphi \sinh^2 \varphi + \frac{\rho'(\tau)}{\rho(\tau)} g(A \partial_i^T, \partial_i^T)
\]

\[+ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sum_{i=1}^{m} g(\nabla_{E_i} \partial_i^T, E_i).
\]

Furthermore, Codazzi equation \( \bar{g}(\bar{R}(X, Y) N, Z) = \bar{g}((\nabla Y A_N) X, Z) - \bar{g}((\nabla X A_N) Y, Z) \) yields

\[
\Delta \cosh \varphi = \sum_{i=1}^{m} \bar{g}(\bar{R}(\partial_i^T, E_i) N, E_i) + \sum_{i=1}^{m} \bar{g}((\nabla_{\partial_i^T} A) E_i, E_i) + \sum_{i=1}^{m} \bar{g}(\nabla_{E_i} \partial_i^T, A E_i)
\]

\[- \frac{\rho''(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi + 2 \frac{\rho'(\tau)^2}{\rho(\tau)^2} \cosh \varphi \sinh^2 \varphi + \frac{\rho'(\tau)}{\rho(\tau)} g(A \partial_i^T, \partial_i^T)
\]

\[+ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sum_{i=1}^{m} g(\nabla_{E_i} \partial_i^T, E_i).
\]
Using (10) into (14), we then obtain

\[
\Delta \cosh \varphi = \sum_{i=1}^{m} g(R(\partial_t^T , E_i)N, E_i) + \sum_{i=1}^{m} g((\nabla_{\partial_t^T} A)E_i, E_i) + \frac{\rho'(\tau)}{\rho(\tau)} g(A\partial_t^T , \partial_t^T ) \\
+ \cosh \varphi \ \text{trace}(A^2) - m \frac{\rho'(\tau)}{\rho(\tau)} H - \frac{\rho''(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi \\
+ 2 \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi + \frac{\rho'(\tau)}{\rho(\tau)} g(A\partial_t^T , \partial_t^T ) + \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi \\
- m \frac{\rho'(\tau)}{\rho(\tau)} H \cosh^2 \varphi + m \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi.
\]

(15)

Since covariant derivations commute with contractions, choosing our local base in \( T_p M \) satisfying \( (\nabla_E E_i)_p = 0 \), we obtain

\[
\Delta \cosh \varphi = -\overline{\text{Ric}}(\partial_t^T , N) - m g(H, \partial_t^T ) + 2 \frac{\rho'(\tau)}{\rho(\tau)} g(A\partial_t^T , \partial_t^T ) \\
+ \cosh \varphi \ \text{trace}(A^2) - m \frac{\rho'(\tau)}{\rho(\tau)} H (\cosh^2 \varphi + 1) - \frac{\rho''(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi \\
+ 3 \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi + m \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi.
\]

(16)

Decomposing \( N \) as \( N = N_F - \overline{g}(N, \partial_t)\partial_t \), where \( N_F \) denotes the projection of \( N \) on the fiber \( F \), we know from [18, Cor. 7.43] that

\[
\overline{\text{Ric}}(\partial_t, \partial_t) = -m \frac{\rho''(\tau)}{\rho(\tau)}.
\]

(17)

and

\[
\overline{\text{Ric}}(N_F, N_F) = \text{Ric}^F(N_F, N_F) + \sinh^2 \varphi \left( \frac{\rho''(\tau)}{\rho(\tau)} + (m - 1) \frac{\rho'(\tau)}{\rho(\tau)} \right),
\]

(18)

where \( \text{Ric}^F \) is the Ricci tensor of the fiber \( F \). Therefore, from (17) and (18), we deduce

\[
\overline{\text{Ric}}(\partial_t^T , N) = -\cosh \varphi \left\{ \text{Ric}^F(N_F, N_F) + (m - 1)(\log \rho)''(\tau) \sinh^2 \varphi \right\}.
\]

(19)
We now insert (19) into (15) to obtain

\[ \Delta \cosh \varphi = \cosh \varphi \left\{ \text{Ric}^F(N_F, N_F) - (m - 1)(\log \rho)^\prime\prime(\tau) \sinh^2 \varphi \right\} - m \ g(\nabla H, \partial^T_i) \]
\[ + 2 \ \frac{\rho'(\tau)}{\rho(\tau)} g(A\partial^T_i, \partial^T_i) + \cosh \varphi \ \text{trace}(A^2) - m \ \frac{\rho'(\tau)}{\rho(\tau)} H(\cosh^2 \varphi + 1) \]
\[ - \frac{\rho''(\tau)}{\rho(\tau)} \cosh \varphi \sinh^2 \varphi + 3 \ \frac{\rho'(\tau)}{\rho(\tau)^2} \cosh \varphi \sinh^2 \varphi + m \ \frac{\rho'(\tau)}{\rho(\tau)^2} \cosh \varphi. \]  

(20)

In the next step, we compute \(|\text{Hess}(\tau)|^2\) using (10), having

\[ |\text{Hess}(\tau)|^2 = \sum_{i=1}^{m} g(\nabla E_i, \partial^T_i, \nabla E_i, \partial^T_i) = \frac{\rho'(\tau)^2}{\rho(\tau)^2} \sinh^4 \varphi + \cosh^2 \varphi \ \text{trace}(A^2) \]
\[ + m \ \frac{\rho'(\tau)^2}{\rho(\tau)^2} + 2 \ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \ g(A\partial^T_i, \partial^T_i) + 2 \ \frac{\rho'(\tau)^2}{\rho(\tau)^2} \sinh^2 \varphi \]
\[ - 2 \ m \ \frac{\rho'(\tau)}{\rho(\tau)} H \cosh \varphi. \]  

(21)

We observe that (21) can be written as

\[ |\text{Hess}(\tau)|^2 = \frac{\rho'(\tau)^2}{\rho(\tau)^2} \left( m - 1 + \cosh^4 \varphi \right) + \cosh^2 \varphi \ \text{trace}(A^2) \]
\[ + 2 \ \frac{\rho'(\tau)}{\rho(\tau)} \cosh \varphi \ g(A\partial^T_i, \partial^T_i) - 2 \ m \ \frac{\rho'(\tau)}{\rho(\tau)} H \cosh \varphi. \]  

(22)

Using (20) and (22), we have

\[ \cosh \varphi \ \Delta \cosh \varphi = \cosh^2 \varphi \left\{ \text{Ric}^F(N_F, N_F) - (m - 1)(\log \rho)^\prime\prime(\tau) \sinh^2 \varphi \right\} \]
\[ - m \ \cosh \varphi \ g(\nabla H, \partial^T_i) + |\text{Hess}(\tau)|^2 \]
\[ - \frac{\rho'(\tau)^2}{\rho(\tau)^2} \left( m - 1 + \cosh^4 \varphi \right) + 2 \ m \ \frac{\rho'(\tau)}{\rho(\tau)} H \cosh \varphi \]
\[ - m \ \frac{\rho'(\tau)}{\rho(\tau)} H \cosh \varphi(\cosh^2 \varphi + 1) - \ \frac{\rho''(\tau)}{\rho(\tau)} \cosh^2 \varphi \sinh^2 \varphi \]
\[ + 3 \ \frac{\rho'(\tau)^2}{\rho(\tau)^2} \cosh^2 \varphi \sinh^2 \varphi + m \ \frac{\rho'(\tau)^2}{\rho(\tau)^2} \cosh^2 \varphi. \]  

(23)
Since $|\text{Hess}(\tau)|^2 \geq 0$, after some computations, (23) leads to the differential inequality

$$\cosh \varphi \Delta \cosh \varphi \geq \cosh^2 \varphi \left\{ \text{Ric}^F(N_F, N_F) - m(\log \rho)^{\prime\prime}(\tau) \sinh^2 \varphi \right\}$$

$$- m \cosh \varphi \g(\nabla H, \partial^T) - m \frac{\rho'(\tau)}{\rho(\tau)} H \cosh \varphi \sinh^2 \varphi$$

$$+ \frac{\rho'(\tau)^2}{\rho(\tau)^2} \sinh^2 \varphi (m + \sinh^2 \varphi). \quad (24)$$

Let us assume now that the ambient spacetime $\overline{M}$ satisfies the Null Convergence Condition (NCC). It is well known that a spacetime obeys the NCC if and only if its Ricci tensor satisfies $\text{Ric}(z, z) \geq 0$ for all lightlike vectors $z$. In particular, in an $(m + 1)$-dimensional GRW spacetime, NCC is equivalent to

$$\text{Ric}^F(X_F, X_F) - m (\rho \rho'' - \rho^2) g_F(X_F, X_F) \geq 0, \quad (25)$$

for all $X_F$ tangent to the fiber $F$. This energy condition is a mathematical way to express that gravity, on average, attracts [18] and it is automatically satisfied by any spacetime that obeys the Einstein field equations with a physically reasonable stress–energy tensor. Therefore, since $\frac{1}{2} \Delta \sinh^2 \varphi = \cosh \varphi \Delta \cosh \varphi + |\nabla \cosh \varphi|^2$, from (24) and (25), we obtain

**Lemma 1** Let $\psi : M \rightarrow \overline{M}$ be a spacelike hypersurface of constant mean curvature immersed in a GRW spacetime $\overline{M} = I \times_{\rho} F$ that obeys the NCC. Then, the hyperbolic angle of $M$ satisfies

$$\frac{1}{2} \Delta \sinh^2 \varphi \geq -m \frac{\rho'(\tau)}{\rho(\tau)} H \cosh \varphi \sinh^2 \varphi + \frac{\rho'(\tau)^2}{\rho(\tau)^2} \sinh^2 \varphi (m + \sinh^2 \varphi). \quad (26)$$

**4 Main Results**

As a first consequence of Lemma 1, we obtain the following non-existence result

**Theorem 2** Let $\overline{M} = I \times_{\rho} F$ be a GRW spacetime obeying the NCC. Then there are no complete spacelike hypersurfaces $\psi : M \rightarrow \overline{M}$ with constant mean curvature $H$ satisfying

$$H \rho'(\tau) \leq 0 \quad \inf_M \frac{\rho'(\tau)^2}{\rho(\tau)^2} > 0,$$

and the volume growth condition

$$\liminf_{r \rightarrow +\infty} \frac{\log(\text{Vol}(B_r))}{r^2} < +\infty. \quad (27)$$
**Proof** By contradiction, let us suppose the existence of such a spacelike hypersurface. From Lemma 1 and the assumption $H \rho'(\tau) \leq 0$, we deduce that the hyperbolic angle of $M$ satisfies
\[
\frac{1}{2} \Delta \sinh^2 \varphi \geq \frac{\rho'(\tau)^2}{\rho(\tau)^2} \sinh^2 \varphi (m + \sinh^2 \varphi). \tag{28}
\]
Completeness and the assumption on the volume growth of geodesic balls enable us to apply [2, Thm. 4.2] to the above differential inequality to infer that $\sinh 2 \varphi$ is bounded from above. Next, thanks to the assumptions $\inf_M \rho'(\tau)^2 / \rho(\tau)^2 > 0$ and (27), we can use [2, Thm. 4.1] to deduce that the hyperbolic angle identically vanishes on $M$. It follows that $\psi(M)$ is a spacelike slice $\{t_0\} \times F$ with mean curvature $H = \rho'(t_0) / \rho(t_0)$. However, by assumption $H \rho'(\tau) \leq 0$, so we immediately deduce that $\rho'(t_0) = 0$. This contradicts the assumption $\inf_M \rho'(\tau)^2 / \rho(\tau)^2 > 0$. \hfill \square

**Remark 3** The conclusions of Theorem 2 still hold substituting (27) by the integral condition
\[
\liminf_{r \to +\infty} \frac{1}{r^2} \int_{B_r} |\sinh \varphi|^p < +\infty \tag{29}
\]
for some $p > 0$ (see [2, Rem. 4.2]).

The above technique can also be used to give a bound for the hyperbolic angle in the next theorem

**Theorem 4** Let $\overline{M} = \mathcal{I} \times \rho F$ be a GRW spacetime obeying the NCC and let $\psi : M \to \overline{M}$ be a complete spacelike hypersurface with constant mean curvature $H$ satisfying the volume growth condition (27). If $0 < H \leq \rho'(\tau) / \rho(\tau)$ (resp., $\rho'(\tau) / \rho(\tau) \leq H < 0$) on $M$, then the hyperbolic angle of the immersion $\varphi$ satisfies
\[
\sinh^2 \varphi \leq m(m - 2).
\]

**Proof** By means of Lemma 1 and the condition $0 < H \leq \rho'(\tau) / \rho(\tau)$ (resp., $\rho'(\tau) / \rho(\tau) \leq H < 0$), we obtain the differential inequality
\[
\frac{1}{2} \Delta \sinh^2 \varphi \geq \frac{\rho'(\tau)^2}{\rho(\tau)^2} \sinh^2 \varphi \left( m - m \sqrt{1 + \sinh^2 \varphi + \sinh^2 \varphi} \right), \tag{30}
\]
where $\inf_M \rho'(\tau)^2 / \rho(\tau)^2 > 0$ because $H > 0$. From (27) and [2, Thm. 4.2], we thus obtain that $\sinh^2 \varphi$ is bounded from above. Again by $\inf_M \rho'(\tau)^2 / \rho(\tau)^2 > 0$ and (27), we can apply [2, Thm. 4.1] to obtain
\[
sup_M (\sinh^2 \varphi) \left( m - m \sqrt{1 + \sup_M (\sinh^2 \varphi) + \sup_M (\sinh^2 \varphi)} \right) \leq 0,
\]
that gives the desired conclusion. \hfill \square
In particular, for spacelike surfaces immersed in a 3-dimensional GRW spacetime, we have

**Corollary 5** Let $\overline{M} = I \times F$ be a GRW spacetime of dimension 3 obeying the NCC and let $\psi : M \rightarrow \overline{M}$ be a complete spacelike hypersurface with constant mean curvature $H$ satisfying (27). If $0 < H \leq \frac{\rho'(\tau)}{\rho(\tau)}$ (resp., $\frac{\rho'(\tau)}{\rho(\tau)} \leq H < 0$), then $M$ is a spacelike slice.

For the next result, we shall use a different technique. Note that in Remark 7 after the proof, we give a sufficient condition for the existence of a function $\zeta$ satisfying (i), (ii), and (iii).

**Theorem 6** Let $\overline{M} = I \times F$ be a GRW spacetime obeying the NCC and let $\psi : M \rightarrow \overline{M}$ be a complete spacelike hypersurface with constant mean curvature. Assume the existence of $\zeta \in C^0(M \setminus B_{R_0}) \cap \text{Lip loc}(M \setminus B_{R_0})$ satisfying

\[
\begin{align*}
(i) \quad & \Delta \zeta \leq \Lambda_0 + \Lambda_1 \zeta \\
(ii) \quad & |\nabla \zeta| \leq \Lambda_2 \\
(iii) \quad & \zeta(x) \rightarrow +\infty \text{ as } x \rightarrow \infty
\end{align*}
\]

on $M \setminus B_{R_0}$ for some positive constants $\Lambda_0, \Lambda_1, \Lambda_2$. Furthermore, suppose that $\rho'(\tau)H \leq 0$ on $M$ and that $\sinh \varphi \in L^2(M)$. Then,

\[\rho'(\tau) \sinh^2 \varphi \equiv 0 \text{ on } M.\]

**Proof** Fix $R > T > R_0 + 2$ and let $\alpha \in C^1([0, R^+]) \cap C^2([0, R])$ be such that

\[\alpha(t) \geq 0 \text{ on } \mathbb{R}^+, \quad \alpha(t) = 1 \text{ on } [0, T], \quad \alpha(t) = 0 \text{ on } [R, +\infty)\]

and

\[\alpha'(t) \leq \frac{P}{R - T}, \quad |\alpha''(t)| \leq \frac{P}{(R - T)^2}\]  

on $[0, R]$ for some constant $P > 0$ independent of $R$ and $T$. Next, we define the cut-off function $\phi$ by setting

\[\phi = \begin{cases} 
\alpha(\zeta(x)) & \text{on } M \setminus D_{R_0} \\
1 & \text{on } D_{R_0},
\end{cases}\]

where $D_T = \{x \in M : \zeta(x) < T\}$. We then have

\[\begin{cases} 
\phi(x) \geq 0 \text{ on } M, \quad \phi(x) = 1 \text{ on } D_T, \quad \phi(x) = 0 \text{ on } M \setminus D_R \\
\nabla \phi = 0 \text{ on } \partial D_R \cup D_T \cup (M \setminus D_R), \quad \nabla \phi \leq \frac{P}{R - T} \text{ on } \overline{D_R \setminus D_T}\n\end{cases}\]  

Finally, from our assumptions on $\zeta$ and (31), we deduce

\[\Delta \phi = 0 \text{ on } D_T \cup (M \setminus D_R), \quad \Delta \phi \leq C \frac{R}{R - T} + \frac{D}{R - T} + \frac{E}{(R - T)^2} \text{ on } \overline{D_R \setminus D_T}\]

\[\square \text{ Springer}\]
for some positive constants $C, D, E$ independent of $R$ and $T$. Next we consider any $C^2$ function $u \geq 0$ on $M$. By the divergence theorem and the properties of $\phi$, we have

$$0 = \int_{D_R} \phi \Delta u \, dV + \int_{D_R \setminus D_T} g(\nabla \phi, \nabla u) \, dV,$$

(34)

with $dV$ the volume element of $M$. Observing that

$$\text{div}(u \nabla \phi) = u \Delta \phi + g(\nabla u, \nabla \phi)$$

and that $\nabla \phi = 0$ on $\partial D_R \cup \partial D_T$, a further application of the divergence theorem yields

$$0 = \int_{D_R \setminus D_T} u \Delta \phi \, dV + \int_{D_R \setminus D_T} g(\nabla u, \nabla \phi) \, dV.$$

(35)

Inserting (35) into (34), we deduce

$$\int_{D_R} \phi \Delta u \, dV = \int_{D_R \setminus D_T} u \Delta \phi \, dV.$$

(36)

We set $T = \frac{R}{2}$ and using (33), we finally obtain

$$\int_{D_{R/2}} \Delta u \, dV \leq G \int_{D_{R/2} \setminus D_{R/2}} u \, dV$$

(37)

for some constant $G > 0$ sufficiently large and independent of $R$. Now, let $u = \sinh^2 \varphi$ and use $\rho'(\tau)H \leq 0$ to deduce from (26) of Lemma 1

$$\frac{1}{2} \Delta u \geq \frac{\rho'(\tau)^2}{\rho(\tau)^2} u (m + u).$$

(38)

Inserting (38) into (37), we get

$$\int_{D_{R/2}} \frac{\rho'(\tau)^2}{\rho(\tau)^2} u (m + u) \, dV \leq \frac{G}{2} \int_{D_{R/2} \setminus D_{R/2}} u \, dV.$$

(39)

Since $u \in L^1(M)$ and $\zeta(x) \rightarrow +\infty$ as $x \rightarrow \infty$, letting $R \rightarrow +\infty$ in (39), we conclude that

$$\rho'(\tau) \sinh^2 \varphi \equiv 0 \text{ on } M.$$

\[\square\]

**Remark 7** Let $r(x) = \text{dist}(x, o)$ for some fixed origin $o$ in the complete manifold $M$. Observe that, by the Laplacian comparison theorem, the condition

$$\text{Ric}(\nabla r, \nabla r) \geq -(m - 1)B^2(1 + r^2) \text{ on } M,$$

(40)
for some constant $B > 0$ implies

$$\Delta r \leq \Lambda_0 + \Lambda_1 r \text{ on } M \setminus B_1$$  \hspace{1cm} (41)$$

for some positive constants $\Lambda_0, \Lambda_1$, and obviously by Gauss lemma $|\nabla r| = 1$ (see for instance [4]). This observation yields the following

**Theorem 8** Let $\overline{M} = I \times_\rho F$ be a GRW spacetime with fiber $F$ of non-negative sectional curvature and warping function $\rho$ satisfying $(\log \rho)'' \leq 0$. Let $\psi : M \rightarrow \overline{M}$ be a complete spacelike hypersurface of constant mean curvature $H$ and suppose that $\rho'(\tau)H \leq 0$ on $M$ and $\sinh \varphi \in L^2(M)$. Then,

$$\rho'(\tau) \sinh^2 \varphi \equiv 0 \text{ on } M.$$

**Proof** Using [1, Lemma 13] and the assumptions of the theorem, we deduce that the Ricci curvature of $M$ is bounded from below. Moreover, from our assumptions, the spacetime obeys the NCC. Therefore, we see from Remark 7 that we can use Theorem 6 choosing as $\zeta$ the distance function $r(\chi)$ on $M$ to prove the desired conclusion. \(\Box\)

**Remark 9** Note that the existence of $\zeta$ satisfying conditions (i) and (iii) in Theorem 6 implies the validity of the weak maximum principle for the Laplacian on the complete manifold $M$ [2, Thm. 3.1]. Observe that the conclusion of Theorem 6 can also be expressed in the form

$$\text{supp } \varphi \subseteq \{ \tau \in I : \rho'(\tau) = 0 \}.$$

(42)

A different integrability request on $\sinh^2 \varphi$ enables us to obtain conclusion (42) avoiding the assumptions on the function $\zeta$ of Theorem 6. Indeed, this is the content of the next result.

**Theorem 10** Let $\overline{M} = I \times_\rho F$ be a GRW spacetime obeying the NCC and let $\psi : M \rightarrow \overline{M}$ be a complete spacelike hypersurface of constant mean curvature $H$. If $H \rho'(\tau) \leq 0$ and $\sinh^2 \varphi \in L^q(M)$ for some $q > 2$, then

$$\text{supp } \varphi \subseteq \{ \tau \in I : \rho'(\tau) = 0 \}.$$

**Proof** From Lemma 1, (26) holds on $M$ and, setting $u = \sinh^2 \varphi$, we have

$$u \Delta u - 2 \frac{\rho'(\tau)^2}{\rho(\tau)^2} u^2 - 2 \frac{\rho'(\tau)^2}{\rho(\tau)^2} u^3 \geq 0.$$  \hspace{1cm} (43)$$

Consider the operator

$$L = \Delta - m q \frac{\rho'(\tau)^2}{\rho(\tau)^2},$$

and observe that any positive constant $\kappa$ solves $L \kappa \leq 0$ on $M$. Using now [15, Thm. 3.3] with the choices (in the notation of Theorem 3.3) $H = \frac{1}{2} q$, $\beta = \frac{1}{2} q - 1$, $K = 0$, the result follows.
\( p = 2 \) and taking into account that the assumption \( u \in L^q(M) \) for some \( q > 2 \) implies
\[
\frac{1}{\int_{\partial B_r} u^2} \notin L^1(\partial \omega),
\]
(see [23]) we get that (42) has no non-negative \( C^2 \)-solutions \( u \) on \( M \) satisfying
\[
\text{supp } u \cap \left\{ \tau \in I : 2 \frac{\rho' \tau^2}{\rho \tau^2} > 0 \right\} = \emptyset.
\]
\( \Box \)

As a consequence of Theorem 10 we have

**Corollary 11** Let \( \overline{M} = I \times_\rho F \) be a GRW spacetime obeying the NCC. Then, there are no complete spacelike hypersurfaces \( M \) of constant mean curvature in \( \overline{M} \) such that \( H \rho' \leq 0 \), \( \sinh^2 \varphi \in L^q(M) \) for some \( q > 2 \) and \( \rho' \neq 0 \) on \( M \).

In what follows, we prove a result that leads to interesting consequences for certain well-known spacetimes.

**Theorem 12** Let \( \overline{M} = I \times_\rho F \) be a GRW spacetime whose warping function satisfies \( \log \rho'' \leq 0 \) and whose fiber \( F \) has non-negative sectional curvature. Then, there are no complete spacelike hypersurfaces of constant mean curvature in \( \overline{M} \) satisfying \( H \rho' \leq 0 \) and \( \inf_M \frac{\rho' \tau^2}{\rho \tau^2} > 0 \).

**Proof** Let us suppose the existence of such a spacelike hypersurface \( \psi : M \rightarrow \overline{M} \). From Lemma 1 and our assumptions, that imply the NCC, the hyperbolic angle of the immersion verifies
\[
\frac{1}{2} \Delta \sinh^2 \varphi \geq \inf_M \left( \frac{\rho' \tau^2}{\rho \tau^2} \right) \sinh^4 \varphi. \quad (44)
\]
Moreover, from [1, Lemma 13] the Ricci curvature of \( M \) is bounded from below. Since \( M \) is complete and \( \inf_M \frac{\rho' \tau^2}{\rho \tau^2} > 0 \), we can use [20, Lemma 2] (this is a result obtained in [16]) on (44) to conclude that the hyperbolic angle identically vanishes on \( M \). Now, from the assumption \( H \rho' \leq 0 \), we deduce that \( M \) is a totally geodesic spacelike slice, contradicting \( \inf_M \frac{\rho' \tau^2}{\rho \tau^2} > 0 \). \( \Box \)

**Remark 13** Note that we can also guarantee the boundedness of the Ricci curvature of the spacelike hypersurface \( M \) if the fiber has Ricci curvature bounded from below (not necessarily by zero) and the hyperbolic angle of \( M \) is bounded. Indeed, the aim of this bound on the Ricci curvature of \( M \) is to guarantee that the Omori–Yau maximum principle for the Laplacian holds on \( M \). Even more, we can substitute this assumption by requiring a controlled decay of the Ricci curvature [2].

**Remark 14** We give a physical interpretation to the assumptions in our theorems referring to [26]. In order to do so, at each point \( p \in M \), we define in a neighborhood \( U \) of
in the spacetime a unitary future-pointing timelike vector field $\tilde{N}$ such that $\tilde{N} = N$ on $U \cap M$, being $N$ the unit normal vector field to $M$. If we compute the divergence in $\bar{M}$ of $\tilde{N}$ at $p \in M$, we get

$$\overline{\text{div}}(\tilde{N})_p = mH(p),$$

where $H$ is the mean curvature function of $M$. Since the integral curves of this vector field $\tilde{N}$ are known as the normal observers, if the mean curvature $H(p)$ is positive (resp. negative) at some point $p \in M$, these normal observers will measure that they are spreading out (resp. coming together).

Moreover, in a GRW spacetime $\bar{M} = I \times \rho F$ there is a distinguished family of observers known as the comoving observers, which are defined as the integral curves of the vector field $\partial_t$. Since the divergence of this vector field in $\bar{M}$ is

$$\overline{\text{div}}(\partial_t) = m \frac{\rho^\prime}{\rho},$$

we obtain that the comoving observers will measure that the spacetime is expanding or contracting depending on the sign of $\rho^\prime$. Thus, our results in Theorem 2, Corollary 5, and Theorem 12 imply that, under certain assumptions, there are no complete spacelike hypersurfaces of constant mean curvature in these ambient spacetimes where the normal observers measure that the universe is non-expanding, whereas the comoving ones measure that it is non-contracting at some point $p \in M$ or vice versa.

From Theorem 12, we obtain the following non-existence results, which extend [20, Corollaries 5, 6, 7] to the case of constant mean curvature spacelike hypersurfaces.

**Corollary 15** There are no complete spacelike hypersurfaces of non-positive constant mean curvature in the $(m + 1)$-dimensional steady-state spacetime $\mathbb{R} \times \rho F$.

**Corollary 16** There are no complete spacelike hypersurfaces of non-positive constant mean curvature bounded away from future infinity in the $(m + 1)$-dimensional Einstein–de Sitter spacetime $\mathbb{R}^+ \times \rho^{2/3} \mathbb{R}^m$.

**Corollary 17** There are no complete spacelike hypersurfaces of non-positive constant mean curvature bounded away from future infinity in the $(m + 1)$-dimensional Robertson–Walker radiation model $\mathbb{R}^+ \times (2at)^{1/2} \mathbb{R}^m$, where $a > 0$.

**Proof** We recall that for a spacelike hypersurface in a GRW spacetime, being bounded away from future infinity analytically means that $\sup_M \tau < +\infty$. \qed

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