Withholding Potentials, Absence of Ghosts and Relationship between Minimal Dilatonic Gravity and f(R) Theories

P. P. Fiziev
Department of Theoretical Physics, University of Sofia, Boulevard 5 James Bourchier, Sofia 1164, Bulgaria
and BLTF, JINR, Dubna, 141980 Moscow Region, Russia

We study the relation between Minimal Dilatonic Gravity (MDG) and f(R) theories of gravity and establish strict conditions for their global equivalence. Such equivalence takes place only for a certain class of cosmological potentials, dubbed here withholding potentials, since they prevent change of the sign of dilaton \( \Phi \). The withholding property ensures the attractive character of gravity, as well as absence of ghosts and a tachyon in the gravi-dilaton sector and yields certain asymptotic of the functions \( f(R) \). Large classes of withholding cosmological potentials and functions \( f(R) \) are found and described in detail. The particle content of the gravi-dilaton sector is found using perturbation theory around de Sitter vacuum of MDG. Two phenomena: scalaron waves and induction of gravitational waves by the scalaron field are discussed using the derived wave equations for MDG scalaron and graviton in the de Sitter background. Seemingly, the MDG and f(R) theories, offer a unified description of dark energy and dark matter.

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I. INTRODUCTION

Consider a scalar-tensor model of gravity with action for the gravi-dilaton sector [18–23, 56]:

\[
A_{g,\Phi} = \frac{c}{2\kappa} \int d^{4}x \sqrt{|g|} \left( \Phi R - 2U(\Phi) \right). \quad (I.1)
\]

We call this model the minimal dilatonic gravity (MDG) It corresponds to the Branse-Dicke theory with identically vanishing parameter \( \omega \).

Here \( \kappa = 8\pi G_{N}/c^{2} > 0 \) is the Einstein constant, \( \Lambda > 0 \) is the cosmological constant, and \( \Phi \in (0, \infty) \) is the dilaton field. The values \( \Phi \) must be positive since a change of the sign of \( \Phi \) entails a non-physical change of the sign of the gravitational factor \( G_{N}/\Phi \) and leads to antigravity. Besides, the value \( \Phi = 0 \) must be excluded since it leads to an infinite gravitational factor and makes the Cauchy problem not well posed [5]. The value \( \Phi = \infty \) turns off the gravity and is also physically unacceptable.

The scalar field \( \Phi \) is introduced to consider a variable gravitational factor \( G(\Phi) = G_{N}/\Phi \) instead of the Newton constant \( G_{N} \). The cosmological potential \( U(\Phi) \) is introduced to consider a variable cosmological factor instead of the cosmological constant \( \Lambda \). In general relativity (GR) with cosmological constant \( \Lambda \) we have \( \Phi \equiv 1 \) and \( U(1) \equiv 1 \). Due to its specific physical meaning, the scalar field \( \Phi \) must have quite unusual properties.

The cosmological potential \( U(\Phi) \) must be a positive single valued function of the dilaton field \( \Phi \) by astrophysical reasons. Here we shall justify additional physical requirements for the cosmological potential \( U(\Phi) \) as a necessary ingredient of a sound MDG model.

Some physical and astrophysical consequences of MDG are described in [2, 5, 8]. In [8] a theory of cosmological perturbations for general scalar-tensor theories, including MDG, was developed as a generalization of the approach of [9]. For a recent attempt to develop a theory of cosmological perturbation using action (I.1) see [10].

It was shown [2, 5] that MDG can describe simultaneously: 1) The inflation and the graceful exit to the present day accelerating de Sitter expansion of the Universe. 2) The reconstruction of the cosmological potential \( U(\Phi) \), using the scalar factor \( a(t) \) in the Friedmann-Robertson-Walker model. 3) The way to avoid any conflicts with the existing solar system and laboratory gravitational experiments using a large enough mass of the dilaton field \( m_{\Phi} \gtrsim 10^{-3} \text{eV}/c^{2} \). 4) The time of inflation as a reciprocal quantity to the mass of dilaton \( m_{\Phi} \).

In the last decade the \( f(R) \) theories of gravity with the action

\[
A_{f(R)} = \frac{c}{2\kappa} \int d^{4}x \sqrt{|g|} f(R) \quad (I.2)
\]

attracted much attention, since they seem to offer a possible explanation of the observed accelerating expansion of the Universe without introducing "dark energy" and may be also an explanation of the observational problem of missing mass without introducing "dark matter", see [11–43] and a large amount of references therein. For \( f(R) = R - 2\Lambda \) we obtain once more GR with the cosmological constant.

Some of the popular choices of the function \( f(R) \) are:
1. The Starobinsky 1980 model [11]; 2. The Carroll et al. 2004 model [12]; 3. The Capozziello et al. 2006 model [13]; 4. The Appleby et al. 2007 model [14]; 5. The Hu et al. 2007 model [15]; 6. The Starobinsky 2007 model [16].
A few of the functions $\Delta f(R) = f(R) - R$, being carefully chosen, are cosmologically viable. Their further modification is still under investigation as a promising approach to the modern astrophysical problems.

Many authors think that using the Legendre transformation of the $f(R)$-action to MDG- action and vice versa they are able to prove the equivalence of these two generalizations of GR. We show that a careful analysis leads to the opposite conclusion. In general, the two models are equivalent only locally which is not enough to ensure identical physical content.

Therefore in the present paper we study the global equivalence between MDG and the $f(R)$ theories. We derive a complete set of additional requirements and define the class of the cosmological potentials $U(\Phi)$ which indeed lead to a global equivalence and thus avoid some of the well-known problems, like physically unacceptable singularities, ghosts, etc., in the $f(R)$ theories.

Some of the above authors consider the last problems seriously already if not explicitly but implicitly. Hence, these problems need further investigation.

A part of the necessary additional requirements are known and dispersed in a large amount of the existing literature on the $f(R)$ theories, where the problem of the global equivalence with MDG has been never discussed. One of the goals of the present paper is to collect all results at one place, to represent their strict derivation and correct usage, thus making transparent the mathematical structure and the physical content of the theory.

An obvious inequality of the MDG and $f(R)$ theories lies in the fact that we do not have a physical intuition on how to choose the function $\Delta f(R)$. In contrast, our large experience in different kinds of physical theories, starting with classical mechanics, is very helpful in the construction of the cosmological potential $U(\Phi)$. 2.5.

In Section II we present the different forms of the field equations of MDG and a discussion of their specific features and peculiarities.

In Section II we consider the specific Legendre transformation between MDG and $f(R)$ theories and the conditions for the global equivalence of these two models.

In Section we introduce a new type of withholding potentials $V(\Phi)$, $U(\Phi)$ and functions $f(R)$. The main objective of this paper is to find a large enough sets of such objects to be prepared to describe the real problems.

In Section we demonstrate the solution of the ghost problem in both the MDG and $f(R)$ theories using the withholding property.

The form of a withholding self-interaction potential $V(\phi)$ in the Einstein frame is presented in Section.

Another presented result is a correct formulation of the perturbation theory around de Sitter vacuum and clarification of the particle content of the graviti-dilaton sector in MDG (Section V C).

In the Concluding Remarks (Section VI) we outline the basic results and some open physical problems.

II. THE FIELD EQUATIONS OF MDG

The variation of the action (I.1) with respect to the dilaton field $\Phi$ gives without any restriction of the variation $\delta \Phi$ on the boundary $\partial V^3$ the algebraic relation

$$R = 2\Lambda U_\Phi(\Phi).$$

A decisive feature of MDG is the existence of such a relation, instead of a differential equation for the field $\Phi$. According to (I.1), the dilaton field $\Phi$ has no space-time properties and evolution independent of the space-time scalar curvature $R$. In particular:

i) In space-times with $R = \text{const}$ due to relation (I.1) we have $\Phi = \text{const}$ and gravitational factor $G(\Phi) = \text{const}$. This corresponds to our physical expectations for gravity, for example, in a flat space-time with $R = 0$.

ii) In the case of validity of the cosmological principle (CP) at very large scales ($\sim 10^2 \times 10^3$ Mpc) the 3D space is homogeneous and isotropic, maybe even flat. Then all quantities must depend only on the cosmic time $t$. As a result of CP, we have $R = R(t)$. Then (I.1) yields $\Phi = \Phi(t)$, i.e. the gravitational factor also obeys the CP.

In any other scalar-tensor model, or in any other field frame, instead of the algebraic equation (I.1) we will have a partial differential equation for the dilaton field $\Phi(x)$. Thus, we would lose the above attractive properties of MDG which seem to correspond to the physical reality with good precision, if one interprets the dilaton field $\Phi(x)$ as a local strength scale of gravitational interaction. For example, it would be quite strange to have some dynamics of the local intensity of gravitational interaction in an empty flat space-time.

The variation of the action (I.1) with respect to the metric field $g_{\alpha\beta}$ is more complicated. Taking into account the identity (A.3) and the four restrictions (A.4) on the surface $\partial V^3$ (see Appendix A) we obtain the vacuum equations for graviti-dilaton sector in the form

$$\Phi G_{\alpha\beta} + \Lambda U(\Phi) g_{\alpha\beta} + \nabla_\alpha \nabla_\beta \Phi - g_{\alpha\beta} \Box \Phi = 0.$$  (II.2)

The trace of the eqs. (II.2) yields the dynamical equation in vacuum:

$$\Box \Phi + \Lambda V_\Phi(\Phi) = 0.$$  (II.3)

By construction it is written in terms of the dilaton field $\Phi$. Here $V_\Phi(\Phi) = \frac{2}{3}(\Phi U_\Phi - 2U) = \frac{2}{3} \Phi^3 \frac{d\Phi}{d\phi} (e^{-2U})$ is the derivative of the dilatonic potential. For convenience, we use the normalization $V(1) = 0$. Then

$$V(\Phi) = \frac{2}{3} \int_1^\Phi (\Phi U_\Phi - 2U) d\Phi.$$  (II.4)

Independently of (II.4), we suppose that the very point $\Phi = 1$ obeys the equation $V_\Phi(\Phi) = 0$. As a result, we have $U_\Phi(1) = 2$. 34.
The other independent field equations of the system \(\text{[II.2]}\) are defined by its traceless part \[58\] in the form

\[ \Phi \hat{R}^\beta_\alpha = -\nabla_\alpha \nabla^\beta \Phi. \]  

\text{(II.5)}

Adding the standard action of the matter fields \(\Psi\), based on the minimal interaction with gravity:

\[ A_{\text{matt}} = \frac{1}{c} \int d^4x \sqrt{|g|} L_{\text{matt}}(\Psi, \nabla \Psi; g_{\alpha\beta}), \]  

\text{(II.6)}

we obtain for the gravi-dilaton sector the dynamical system (in cosmological units \(\Lambda = 1\), \(\kappa = 1\), \(c = 1\) which will (we use further on everywhere):

\[ \Box \Phi + \frac{2}{3} (\Phi U,_{\Phi} \Phi) - 2U(\Phi) = \frac{1}{3} T, \]  

\text{(II.7a)}

\[ \Phi \hat{R}^\beta_\alpha = -\nabla_\alpha \nabla^\beta \Phi - T^\beta_\alpha, \]  

\text{(II.7b)}

and Eqs. \(\text{[II.7a]}\) and \(\text{[II.12a]}\) present two equivalent forms for description of the additional scalar degree of freedom which comes into being in both the MDG and \(f(R)\) models. It is frozen in GR where the scalar curvature does not own an independent degree of freedom since \(R = T\). We call this new field degree the scalaron field [see Section \text{V C} and the corresponding spinless particle the scalaron \text{[21]}].

III. THE LEGENDRE TRANSFORM WHICH RELATES MDG AND \(f(R)\) THEORIES

Relations \(\text{[III.1]}\) and \(\text{[III.11]}\) show that the MDG and \(f(R)\) theories are related via the Legendre transform \[14, 15\]. Following the traditional notation in gravity, we obtain some particular form of the Legendre transform. For equivalence of the MDG and \(f(R)\) theories their actions \(\text{[III.1]}\) and \(\text{[III.2]}\) must give equivalent results under the corresponding variations. For this purpose, we need to satisfy the relation \(\mathcal{J}(R, \Phi) = f(R) + 2U(\Phi) - R \Phi = \partial_\alpha v^\alpha(x)\) with some vector field \(v^\alpha(x)\) which does not depend on \(R\) and \(\Phi\) \[59\]. It is enough to have \(\mathcal{J}(R, \Phi) = f(R) + 2U(\Phi) - R \Phi = \text{const}\). Then such a vector field \(v^\alpha(x)\) certainly exists.

In any case the conditions \(\partial_\Phi \mathcal{J}(R, \Phi) = 0\) and \(\partial_R \mathcal{J}(R, \Phi) = 0\) produce relations \(\text{[III.1]}\) and \(\text{[III.11]}\), respectively, i.e. the function \(\mathcal{J}(R, \Phi)\) generates the Legendre transform. Adding the condition \(\mathcal{J}(R, \Phi) = 0\), we obtain the transformation from \(U(\Phi)\) to \(f(R)\) - \(\text{[III.1a]}\), and the inverse transformation \(f(R)\) to \(U(\Phi)\) - \(\text{[III.1b]}\), in the following parametric form:

\[ f = 2(\Phi U,_{\Phi} \Phi) - U(\Phi), \]  

\text{(III.1a)}

\[ R = 2U,_{\Phi} \Phi, \Phi \in (0, \infty), \]  

\text{(III.1b)}

The dynamical equations \(\text{[II.12]}\) of the problem look more complicated than the system \(\text{[II.7]}\). The advantage of the form \(\text{[II.12]}\) is in the absence of the dilaton field \(\Phi\) in it. If one solves this system, the field \(\Phi\) can be obtained using relation \(\text{[III.11]}\). The disadvantage of the simple form \(\text{[II.7]}\) is that it generates a wrong feeling about possible "independent dynamics" of the dilaton \(\Phi\). This is a little bit subtle circumstance. The character of Eq. \(\text{[II.7a]}\), which actually describes the dynamics of the scalar curvature \(R\), can be explained in the context of the Legendre transform. Its specific property is that the basic relations take the simplest possible form, if one uses a mixed representation in which half the variables are new and the other half is the old ones.
new ones:

\[
f_{RR} \Phi \Phi = \frac{1}{2}, \quad f_\Phi = 2 \Phi U_{\Phi \Phi} = \Phi R_\Phi, \\
U_\Phi = \frac{1}{2} R f_{RR} = \frac{1}{2} R \Phi_\Phi, \\
f_\Phi U_\Phi = f_{RR} U_\Phi = \frac{1}{2} R \Phi,
\]

(III.2)

which will be used as a dictionary for translation of the results in MDG to f(R)-results and vice versa.

All the above relations are mathematically correct and can be solved unambiguously, and the two models are physically equivalent.

The conditions (IV.2) at the ends of the interval \( \Phi \in (0, \infty) \), and \( f(R) \) in the interval \( R \in (-\infty, \infty) \).

Applying the Legendre transform to a function which is not convex, we will obtain a multi-valued new function. This means that the correspondence between MDG and f(R) theories will be only local. To what extent such a local equivalence may be acceptable from a physical point of view is a problem which we discuss in more details in Section IV.E.

Hence, there is a one-to-one global correspondence between the MDG and f(R) models if and only if the functions \( U(\Phi) \) and \( f(R) \) are convex in their physical domains. Only under this condition the second of the equations Eqs. (III.1a) and (III.1b) can be solved unambiguously, and the two models are physically equivalent.

### IV. THE WITHHOLDING PROPERTY

#### A. The withholding property in the interval \( \Phi \in (0, \infty) \)

Looking at the dynamical Eq. (II.3) for dilaton \( \Phi(x) = \Phi(R(x)) \) we see that the only way to force the dilaton to stay all the time in its physical domain

\[
\Phi \in (0, \infty),
\]

preserving each value \( \Phi > 0 \) attainable, is to impose the conditions

\[
V(0) = V(\infty) = +\infty
\]

(IV.1)
on the dilaton potential \( V(\Phi) \). Then the infinite potential barriers at the ends of the physical domain (IV.4) will confine dynamically the dilaton \( \Phi(x) \) inside this domain, if it is initially there \( \Phi = 0 \). We call this new phenomenon the withholding property of the dilatonic potential \( V(\Phi) \).

For further use we need to derive the consequences of this condition for the cosmological potential \( U(\Phi) \). For a given dilaton potential \( V(\Phi) \) under normalization \( U(1) = 1 \) the \( U(\Phi) \) gives

\[
U(\Phi) = \frac{3}{2} \Phi^2 \int_1^\Phi \Phi^{-3} V_\Phi d\Phi + \Phi^2.
\]

(IV.3)

Let us choose a behavior of the dilatonic potential \( V(\phi) \sim v \Phi^\mu \) (with some constant \( v > 0 \) ) to satisfy the condition (IV.2) at the ends of the interval \( \Phi \in (0, \infty) \). Then from Eq. (IV.3) we obtain:

- a) For \( \Phi \to 0: n < 0, and \)

\[
U(\Phi) \sim \frac{3}{2} \frac{|n|}{|n| + 3} v \Phi^{-|n| - 1}.
\]

(IV.4)

- b) For \( \Phi \to \infty: n > 0, and \)

\[
U(\Phi) \sim \begin{cases} 
\frac{\Phi^2}{v} & \text{for } n \in (0, 3), \\
\frac{3}{2} \frac{n}{n - 3} v \Phi^{n - 1} & \text{for } n > 3.
\end{cases}
\]

(IV.5)

As seen, in each case we have

\[
U(0) = U(\infty) = +\infty,
\]

(IV.6)

but the increase of \( U(\Phi) \) and \( V(\Phi) \) at the ends of the physical domain (IV.1) is, in general, not the same.

An observational astrophysical fact: the cosmological term has a definite positive sign in the observable Universe and leads to the additional requirement

\[
U(\Phi) > 0 \quad \text{for } \Phi \in (0, \infty).
\]

(IV.7)

For potentials with properties (IV.6), (IV.7) the convex condition reads

\[
U_{\Phi\Phi} > 0 \quad \text{for } \Phi \in (0, \infty).
\]

(IV.8)

It ensures the uniqueness of the Einstein vacuum. We call such \( U(\Phi) \) the withholding cosmological potentials.

It is not hard to derive the qualitative behavior of admissible functions \( f(R) \) of general type created via the Legendre transform of the withholding cosmological potential \( U(\Phi) \). Indeed, from Eqs. (IV.4) and (IV.5) we obtain the correspondence: \( \Phi \to 0 \iff R \to -\infty, \ f \to -\infty, \) and \( \Phi \to +\infty \iff R \to +\infty, \ f \to +\infty \). The translation of the properties (IV.1) and (IV.7) in the language of the \( f(R) \) models is \( f_{RR} > 0 \) and \( f_{RR} > 0 \) for all \( R \in (-\infty, \infty) \). Hence, \( f(R) \) is for sure a strictly monotonically increasing and convex function.

For cosmological potentials \( U \) with asymptotic (IV.4) and (IV.5) one obtains from (III.1):

\[
f(R) \sim \begin{cases} 
\frac{R^2}{v} & \text{for } n \in (0, 3), \\
9v \Phi(R)^2 \ln \Phi(R) & \text{for } n = 3, \\
(n - 2) \left( \frac{R}{v R} \right)^{1+n^{-1}} & \text{for } n > 3,
\end{cases}
\]

(IV.9)

where in \( \Phi(R) = \text{LambertW} \left( \exp \left( \frac{R}{v R} \right) \right) \) we use the Lambert-W-function [40], and \( \mu = \frac{n - 2}{n - 1} \in (1/2, 1) \).
The general form of the function $f(R)$, which follows the asymptotic (IV.9) and (IV.10) and is in addition a convex function, is shown in Fig. 1. We dub such functions the *withholding* $f(R)$ functions.

The uniqueness of the physical de Sitter vacuum in such MDG is still not guaranteed. Indeed, from Eq. (II.4) we obtain

$$V,\Phi = \frac{2}{3} (\Phi U,\Phi - U,\Phi), \quad V,\Phi = \frac{2}{3} \Phi U,\Phi. \quad (IV.11)$$

These relations show that in the physical domain (IV.1) the functions $U,\Phi$ and $V,\Phi$ have the same signs and zeros, but the functions $U,\Phi$ and $V,\Phi$ do not own this property. Hence, $V(\Phi)$ may have several minima in the domain (IV.1), see Fig. 2. A similar, but not limited from below potential with infinite number of minima was considered in a quite different cosmological model in [4]. Thus, we obtain a new kind of models with many locally stable dSV at the points $\Phi_k=0,1,... \geq 1$, ($\Phi_0 = 1$) which are solutions of Eq. $\Phi_k U,\Phi(\Phi_k) - 2U(\Phi_k) = 0$ and have $U,\Phi(\Phi_k) > 2U(\Phi_k)/(\Phi_k)^2 > 0$. This means that the dilaton behaves around different minima like a field with different masses $m_{\Phi_k}^2 = (2/3)(\Phi_k U,\Phi(\Phi_k) - U,\Phi(\Phi_k)) > 0$ and excludes a scalar tachyon. In the language of the $f(R)$ theories we have a series of dSV-curvature-values $R_k$ defined by Eq. $R_k f, f(R_k) + f(R_k) = 0$ with $f,R(R_k)/R_k > f,R(R_k) > 0$. In each dSV state we have different values of the gravitational factor $G_k = G_N/\Phi_k$ and of the cosmological constant we need for consistency with observations.

**B. Examples of the $f(R)$ models which are equivalent to MDG with unique dSV**

There is one more option: To postulate the uniqueness of the dSV [4]. In this case the function $V(\Phi)$ with properties (IV.2) will be convex: $V,\Phi(\Phi) > 0$ for each $\Phi \in (0, \infty)$. This ensures the stability of dSV and may exclude a scalar tachyon since $m^2 = V,\Phi(\Phi) = (2/3)(U,\Phi(1) - 2) > 0$, see Section V C. Besides, the function $(2/3)(\Phi U,\Phi(\Phi) - U,\Phi(\Phi)) = (f,R(R) - f,R(R))/3f,R(R) > 0$ is strictly positive in the whole physical domain and this may avoid the presence of a scalar tachyon in the chameleon models, see reviews in [37,43] and the references therein. In ad-
dition the condition \( f_{,RR}(R) > 0 \) entails the requirement
\[
f_{,R}(R)/f_{,RR}(R) > R, \quad R \in (-\infty, \infty). \quad (\text{IV.12})
\]
For \( R = R = 4 \) this yields the condition for stability of dSV [21]. The stability of dSV in MDG is an obvious consequence of the withholding property.

One can write down a simple example for a pair of withholding potentials with unique dSV:
\[
V(\Phi) = \frac{1}{2} p^2 \left( \Phi + \frac{1}{\Phi} - 2 \right), \quad \Phi \in (0, \infty),
\]
\[
U(\Phi) = \Phi^2 + \frac{3}{16} p^2 \left( \Phi - \frac{1}{\Phi} \right)^2, \quad (\text{IV.13})
\]
where \( p \) is a small parameter, related with the nonzero mass of the dilaton field [60] [14, 5], see Fig. 3.

The corresponding function \( f(R) \) can be written in the following parametric form:
\[
f = \frac{3}{8p^2} \left( \Phi^2 - \frac{3}{\Phi^2} + 2 \right) + 2\Phi^2,
\]
\[
R = \frac{3}{4p^2} \left( \Phi - \frac{1}{\Phi^4} \right) + 4\Phi, \quad \Phi \in (0, \infty). \quad (\text{IV.14})
\]

More general potentials with the same property were described in [3] and can be used as a basis for a very general class of withholding potentials [61]:

\[
V(\Phi) = p^2 \sum \frac{w_{\nu_+, \nu_-}}{\nu_+ + \nu_-} \left( \frac{\Phi^{\nu_+} - 1}{\nu_+} + \frac{\Phi^{-\nu_-} - 1}{\nu_-} \right), \quad \nu_\pm > 0, \quad \Phi \in (0, \infty),
\]
\[
U(\Phi) = \Phi^2 + \frac{3}{2p^2} \sum \frac{w_{\nu_+, \nu_-}}{\nu_+ + \nu_-} \left( \frac{\Phi^{\nu_+ - 1} - \Phi^2}{\nu_+ - 3} + \frac{\Phi^{-\nu_- - 1} - \Phi^2}{\nu_- + 3} \right). \quad (\text{IV.15})
\]

By construction \( V(1) = 0, \ V_{,R}(1) = 0 \). The additional requirements
\[
w_{\nu_+, \nu_-} \geq 0, \quad \sum w_{\nu_+, \nu_-} = 1 \quad (\text{IV.16})
\]
ensure that \( U(1) = 1 \) and \( V_{,RR}(1) = p^2 \), as well as that the convex condition \( U_{,RR}(\Phi) > 0 \) for \( \Phi \in (0, \infty) \) is fulfilled.

The corresponding withholding function \( f(R) \) is defined in the parametric form:
\[
f = 2\Phi^2 + \frac{3}{p^2} \sum \frac{w_{\nu_+, \nu_-}}{\nu_+ + \nu_-} \left( (\nu_+ - 2)\Phi^{\nu_+ - 1} - \Phi^2 - (\nu_- + 2)\Phi^{-\nu_- - 1} + \Phi^2 \right) / (\nu_+ - 3) + \Phi^2, \quad \nu_\pm > 0, \quad (\text{IV.17})
\]
\[
R = 4\Phi + \frac{3}{p^2} \sum \frac{w_{\nu_+, \nu_-}}{\nu_+ + \nu_-} \left( (\nu_+ - 1)\Phi^{\nu_+ - 2} - 2\Phi^2 - (\nu_- + 1)\Phi^{-\nu_- - 2} + 2\Phi^2 \right) / (\nu_+ - 3) + \Phi^2, \quad \Phi \in (0, \infty). \quad (\text{IV.18})
\]

In all of these cases the necessary and sufficient condition for global equivalence of the MDG and \( f(R) \) models are satisfied. The functions \( U(\Phi) \) and \( f(R) \) are both convex and single-valued. The proper choice of the powers \( \nu_\pm \) which enter into the sums can make the withholding barriers at the ends of the physical domain [IV.1] impenetrable not only in the classical dynamics of fields under consideration, but also at quantum level.

Much more strong withholding barriers at the ends of the physical domain [IV.1] are produced, for example,
Its qualitative behavior is the same as the one shown in Fig. 1.

From an analytical point of view, when in the formulæ 
(IV.15), (IV.17) only a finite number of terms enters, at the ends of the physical domain (IV.1) we will have: i) poles; ii) branching points – in corresponding functions; iii) log singularities – in cosmological potential $U(\Phi)$; and iv) more singular term related with the Lambert-W function – in $f(R)$; v) the ends of the physical domain are essential singular points, as in the example (IV.18), (IV.19).

C. Direct Construction of withholding functions $f(R)$.

The second relation in Eqs. (III.1a) can be solved explicitly in rare cases. Even if this is possible, as a rule, the result is complicated and not very useful for further usage. Taking into account the asymptotics (IV.2), (IV.10), and the convex condition we can construct the withholding functions $f(R)$ in a more direct way.

Consider the functions

$$F(x; \mu, \alpha) = (x^2 + 1)^{\mu/2} / (1 + \exp(-2\alpha x))$$

with the parameters $\mu, \alpha$ and construct a five parametric family of $f(R)$ functions

$$f(R; \mu_1, \mu_2; \alpha, R_1, R_2) =$$

$$= F(R/R_1; 1/\mu_1, \alpha) - F(-R/R_2; \mu_2, \alpha)$$

(IV.20)

with the parameters $\mu_{1,2} \in (1/2, 1)$, and $\alpha, R_1, R_2 \in (0, +\infty)$. By construction these functions have the asymptotic behavior (IV.9), (IV.10) and satisfy the Starobinsky condition $f(0; \mu_1, \mu_2; \alpha, R_1, R_2) = 0$. The next step is to construct the linear combinations, analogous to Eq. (IV.15).

$$f(R) = \sum_{\{\mu_1, \mu_2, \alpha, R_1, R_2\} \in CxD} W_{\{\mu_1, \mu_2, \alpha, R_1, R_2\}} f(R; \mu_1, \mu_2; \alpha, R_1, R_2),$$

(IV.21a)

$$\sum_{\{\mu_1, \mu_2, \alpha, R_1, R_2\} \in CxD} W_{\{\mu_1, \mu_2, \alpha, R_1, R_2\}} = 1, \quad W_{\{\mu_1, \mu_2, \alpha, R_1, R_2\}} \geq 0.$$  

(IV.21b)

Unfortunately, the problem of the convex property of these functions is not trivial.

Let us call the convex domain (CxD) the domain of the parameters $\mu_1, \mu_2, \alpha, R_1, R_2$ for which the functions (IV.20) are convex (62). The CxD turns to be not a simple set of points in the five dimensional parameter-space. If we perform the summation in Eq. (IV.21a) only over the values $\{\mu_1, \mu_2, \alpha, R_1, R_2\} \in CxD$, then the conditions (IV.21b) ensure the convex property of the functions $f(R)$ and these functions are withholding.

Thus, we see one more advantage of the MDG in comparison with the $f(R)$ theories: In MDG the convex domain CxD has a simple form described in the relations (IV.15) by the inequalities $\nu_{\pm} > 0$. In contrast, in the $f(R)$ theories CxD is a quite complicated set of points in the space of the parameters.

D. The withholding property in a finite interval $\Phi \in (\Phi_1, \Phi_2) \subset (0, \infty)$

Another possibility is to withhold the dilaton field in a finite interval $\Phi \in (\Phi_1, \Phi_2)$ where $0 < \Phi_1 < \Phi_2 < \infty$. We shall not discuss in detail the corresponding class of withholding potentials $U$ and $V$. The withholding potentials with unique dsV of this class can be written in the form

$$V := \sum_{\nu_1, \nu_2} w_{\nu_1, \nu_2} V_{\nu_1, \nu_2}, \quad U := \sum_{\nu_1, \nu_2} w_{\nu_1, \nu_2} U_{\nu_1, \nu_2}, \quad \nu_{1,2} \in (0, 1),$$

with coefficients $w_{\nu_1, \nu_2}$ which obey relations (IV.10). The dilatonic potentials $V_{\nu_1, \nu_2}$ have the form

$$V_{\nu_1, \nu_2}(\Phi) = \frac{c_{\nu_1, \nu_2}}{p^2} \times$$

$$\times \left( \left( \frac{1 - \Phi_1}{\Phi - \Phi_1} \right)^{\nu_1} \left( \frac{\Phi_2 - 1}{\Phi_2 - \Phi_1} \right)^{\nu_2} - 1 \right) + b_{\nu_1, \nu_2}(\Phi - 1),$$

(IV.22)
where \(b_{\nu_1,\nu_2} = \nu_1/(1 - \Phi_1) - \nu_2/(\Phi_2 - 1)\) and \(c_{\nu_1,\nu_2} = (\nu_1/(1 - \Phi_1)^2 + \nu_2/(\Phi_2 - 1)^2)^{-1}\), see an example in Fig. 4. All these potentials satisfy the requirements \(V(1) = 0, V_{,\phi}(1) = 0, \) and \(V_{,\phi\phi}(1) = 1/p^2 > 0 \) and contain enough free parameters for fitting real problems. The potential \(U(\Phi)\) can be found using Eq. (IV.3).

It can be shown that the qualitative behavior of the corresponding function \(f(R)\) is the same as shown in Fig. 1 but now the asymptotics are

\[
f(R) \sim \Phi_1 R, \quad f(R) \sim \Phi_2 R, \quad (IV.23)
\]

and corresponds to the cases with \(\mu_+ \to 1\) in (IV.9), (IV.10).

### E. \(f(R)\) models which are not equivalent to the MDG.

The \(f(R)\) models 1 - 10, listed in the Introduction, are not globally equivalent to the MDG. It is not hard to check that these models suffer from one or several of the following shortcomings:

- Negative values of \(\Phi\) are not excluded and the model, being not withholding, contains ghosts.
- The model is not withholding because the inadmissible values \(\Phi = 0\) and/or \(\Phi = \infty\) are not excluded.
- The model is not withholding since the function \(f(R)\) has no necessary asymptotic for \(R \to \pm \infty\).
- A not convex function \(f(R)\) yields multi-valued cosmological potential \(U(\Phi)\).

These shortcomings do not mean that all these models have no physical significance, but restrict the physical validity of the corresponding model. In particular, there may exists cosmologically viable solutions.

For example, the parametric representation of the multi-valued potential \(V(\Phi)\) of the astrophysically valuable Starobinsky 2007 model can be found in [18]. It was shown that this potential leads to curvature singularities in star’s models. Nevertheless, for a special initial condition \(\Phi_0\) for the dilaton in a tiny domain, the classical solution \(\Phi(t) > 0\) is positive for all time instants. Depending on the choice of the initial conditions the existing singularities in this model can be not reachable for certain class of classical solutions. An improvement of the Starobinsky 2007 model was given by Gannouji et al. [24]. This modification has a CxD for proper values of the introduced additional parameter, thus avoiding multi-valued potentials, without exclusion of the negative values of the dilaton \(\Phi\) for all solutions.

This shows that the proposed in the present paper requirement for a global equivalence of MDG and \(f(R)\) theories is acceptable but not obligate at classical level [63].

However, in the case of a pure local equivalence the \(f(R)\) theories are not physically equivalent to MDG.

Another possible scenario to avoid the negative values of \(\Phi\) only at the present epoch is to choose a proper positive value of \(\Phi_0\) and to tune the parameters of the model in a way which ensures positive \(\Phi(t) > 0\) for long enough time, i.e. to postpone the disaster for distant future (in cosmological scales) when the model has to be changed [64].

In the existing literature we can find also other choices of function \(f(R)\). For example, terms \(\sim R^{-n}(\ln R/\mu^2)^m, \sim \exp(-bR) - 1, \) or \(\sim (\exp(bR) - 1)/(\exp(bR) + \exp(bR_0))\) can be included, see the reviews [33, 40] and the references therein. In the known to us variety of such functions we were not able to find a \(f(R)\) model which is consistent with the withholding property of MDG.

### V. PRESENCE OR ABSENCE OF GHOSTS

In the present paper we call "a ghost" any physical field with a wrong sign of the kinetic term in the corresponding lagrangian. The problem of ghosts in the theories of gravity with higher derivatives has been studied in many articles, see for example, the articles [39, 32], the review articles [37, 43] and the references therein. For exclusion of ghosts in the Branse-Dicke theory with nonzero parameter \(\omega\) and massless, or very light dilaton, see [9].

#### A. The conditions for the ghost elimination in MDG

It is not hard to derive the necessary and sufficient conditions for the absence of ghosts in the MDG.

1. There is no ghost-problem for dilaton field since in the action (I.1) a kinetic term for \(\Phi\) does not exists at all.

2. The problem with the presence of ghosts in the metric field \(g_{\alpha\beta}\) is not so trivial. Generalizing the consideration of Section 93 in [6] for the MDG we see that in the action (I.1) we will have only kinetic terms of the form

\[
\frac{c}{8\kappa} \Phi g^{00}(\partial_0 g^{\alpha\beta})^2, \quad \alpha, \beta = 1, 2, 3. \quad (V.1)
\]

Since \(g^{00} > 0\), the necessary and sufficient condition for the absence of ghosts in MDG is (IV.1).

The above results show that the conditions (IV.1) and (IV.2), which define the withholding properties, ensure in a dynamical way also the absence of ghosts in MDG. Hence, the withholding property is a critical condition for the physical consistency of MDG.
B. Presence of ghosts in \( f(R) \) theories of gravity and their relation with MDG

The widespread opinion is that despite the presence of the second derivatives of the metric \( g_{\alpha\beta} \) in the \( f(R) \)-action \( (1.2) \), there is no problem with ghosts in this modification of GR (see, for example, the recent paper \cite{[29]}). As described in some detail in \cite{[30]}, the standard reason is that the \( f(R) \)-action \( (I.2) \) to the Einstein-frame-action:

\[
\ddot{\tilde{\mathcal{V}}} = \ddot{\mathcal{V}} + \frac{3}{
\begin{align}
\frac{d^4 x}{\sqrt{|g|}} \mathcal{R} +
\frac{d^4 x}{\sqrt{|\tilde{g}|}} \left( \frac{1}{2} g_{\alpha\beta} \partial_{\alpha} \varphi \partial_{\beta} \varphi - \tilde{\mathcal{V}}(\phi) \right) ,
\frac{\ddot{g}}{\tilde{g}} = \Phi g_{\alpha\beta} , \varphi = \sqrt{\frac{3}{2} \ln \Phi} ,
\frac{\ddot{\tilde{\mathcal{V}}}(\phi)}{e^{-2\sqrt{\frac{3}{2} \varphi}}} U \left( e^{V_{\phi}} \frac{\varphi}{\sqrt{3/2}} \right) ,
\end{align}
\]

which resembles the Hilbert-Einstein-action of GR, plus an additional scalar field \( \varphi \) with a normal kinetic term and self-interaction \( \tilde{\mathcal{V}}(\varphi) \). Obviously, ghosts are not present in the Einsteins-frame-action \( (V.2a) \).

However, the action \( \mathcal{V}(\phi) \) is produced by the two relations \( (V.2b) \) which both lose meaning outside the physical domain \( [1,1] \). Hence, the accurate analysis shows that the \( f(R) \) theories of gravity are not automatically free of ghosts. For this to be true certain strong additional requirements equivalent to the above constraints on the MDG are needed.

Some authors mention that for the absence of ghosts in the \( f(R) \) theories one needs the conditions

\[
f_{,R}(R) > 0 \text{ and } f_{,RR}(R) > 0 . \quad (V.3)
\]

However, the natural question: What kind of physical requirements can ensure the fulfilment of these conditions and thus indeed can exclude ghosts, remains without a definite answer. We saw that these conditions do not guarantee the withholding property of the function \( f(R) \), as well as that the withholding property is critical for exclusion of ghosts. For this purpose, relations \( (V.3) \) are not enough, and the function \( f(R) \) must have proper asymptotic behavior, say, described by Eqs. \( (IV.9) \) and \( (IV.10) \).

As a result, in Einstein-frame-action \( (V.2) \) the potential \( \tilde{\mathcal{V}}(\varphi) \) must have asymptotic

\[
\tilde{\mathcal{V}}(\varphi) \sim \begin{cases} 
\text{const} & \text{for } n \in (0, 3), \\
\varphi & \text{for } n = 3, \\
e^{(n-3)\sqrt{\frac{3}{2}} \varphi} & \text{for } n > 3.
\end{cases}
\]

A simple example which corresponds to the cosmological potential \( (IV.13) \) is

\[
\tilde{\mathcal{V}}(\varphi) = 1 + \frac{3}{16}p^2 \left( 1 - e^{-2\sqrt{\frac{3}{2} \varphi}} \right)^2 . \quad (V.6)
\]

For withholding self-interaction potentials of general form we obtain

\[
\tilde{\mathcal{V}}(\varphi) = 1 + \frac{3}{2p^2} \sum_{\nu_+, \nu_-} w_{\nu_+, \nu_-} \frac{\nu_+ + \nu_-}{\nu_+ - 3} \times \left( e^{(\nu_+ - 3)\sqrt{\frac{3}{2} \varphi}} - 1 + e^{(\nu_- + 3)\sqrt{\frac{3}{2} \varphi}} - 1 \right) . \quad (V.7)
\]

Another cosmologically valuable example is \( \tilde{\mathcal{V}}(\varphi) = V_0 \exp(-\sqrt{3} \varphi) \) (in our normalization). It was considered in \cite{[17]}. There it was shown that this very simple potential offers a good fitting of the observational date in a large domain, showing some small deviations in the CMBR angular power spectrum for small \( l \). We draw a special attention to this potential, because it is very close to the withholding property. One may hope that a small change - just to obey the withholding property, may improve the fitting of the known observational data.

C. Particle content of gravit-dilaton sector in MDG and its physics

The particle content and the ghost problem of the \( f(R) \) theories was discussed as early as in \cite{[12]}.

In the present Section we consider the perturbations around dSV in MDG. Here the propagators have a more elegant form. The physical content of the model is more transparent and simpler than in the \( f(R) \) theories which are only locally equivalent to MDG, since a lot of possible cases, considered in \cite{[12]}, are automatically excluded in MDG. In the case of a global equivalence the physical content of the two models is the same.

Consider small perturbation of the metric \( g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu} \) around the solution \( g_{\mu\nu} \) for dSV \( (II.8) \). The linear approximation for perturbations of Eqs. \( (II.7) \) above dSV yields the particle content of the gravi-dilaton sector of MDG. The needed basic geometrical relations are given in the Appendix B. Indeed:

1. Let us introduce the scalaron \( \zeta \) as excitation of the dilaton above dSV in the following way:

\[
\zeta = \delta \Phi = \tilde{f}_{,RR} \delta R = \tilde{\Phi} \frac{\delta h}{6} \frac{d^2 h}{2} + 2U/3 \Phi \quad (V.8)
\]

Then it obeys the scalaron wave equation with a source:

\[
\left( \Box + \frac{1}{R} \right) \zeta = \delta T/3 , \quad \Leftrightarrow \\
\zeta = \frac{1}{\Box + \frac{1}{R} \Box + m^2} (\delta T/3) , \quad (V.9)
\]
where the mass of the scalaron, defined as

\[ m = \sqrt{\bar{m}^2 - \frac{1}{6} \bar{R}} \quad \text{for} \quad \bar{m}^2 \geq \frac{1}{6} \bar{R} \]  \hspace{1cm} (V.10)

is real, and \( \delta T \) is the perturbation of the trace of energy-momentum tensor of matter.

2. For the traceless part of the metric perturbation \( \delta \hat{T}_{\alpha\beta} \) (which carries the graviton) we obtain the wave equation:

\[
\left( \Box + \frac{1}{6} \bar{R} \right) \hat{T}_{\alpha\beta} = \frac{1}{\Phi} \left( -\delta \hat{T}_{\alpha\beta} - \nabla_{\alpha} \nabla_{\beta} \zeta \right) \quad \leftrightarrow \\
\hat{T}_{\alpha\beta} = \frac{1}{\Box + \frac{1}{6} \bar{R}} \frac{1}{\Phi} \left( -\delta \hat{T}_{\alpha\beta} - \nabla_{\alpha} \nabla_{\beta} \zeta \right). \]  \hspace{1cm} (V.11)

\( \delta \hat{T}_{\alpha\beta} \) being the perturbation of the traceless part of energy-momentum tensor of matter. We see that:

- The propagators of the scalaron (spin \( s = 0 \)) and graviton (spin \( s = 2 \)) resemble a presence of a ghost. Such interpretation is certainly not correct since this operator is not a propagator but just a source-operator. Its physical meaning is clear: the matter creates scalaron waves via the source operator \((\Box + \frac{1}{6} \bar{R})\) when the quadrupole moment is not zero.

Thus, the particle content of the gravi-dilaton sector of MDG and the corresponding basic physics in it is clarified.

VI. CONCLUDING REMARKS

In the present paper, we have studied carefully the relation between Minimal Dilatonic Gravity (MDG) and the \( f(R) \) theories of gravity. These models seem to offer a unified description of dark energy and dark matter. Our main result is that the two generalizations of GR are globally equivalent under certain additional assumptions.

Their physical equivalence takes place only for certain class of cosmological potentials introduced here and dubbed withholding potentials since they prevent a change of sign of dilaton potential. We shall see that this condition is met with a large stock, since from another reason physically admissible are the values \( \bar{m}^2 \gg \frac{1}{6} \bar{R} \).

- MDG predicts the existence of free spreading scalaron waves. Since the mass of the dilaton is not zero, their velocity is smaller than the velocity of light.

- The electromagnetic field and other sources with \( T = 0 \) are not able to create scalaron.

- As a result of the withholding property the graviton never becomes ghosts since \( \bar{\Phi} > 0 \).

- The scalaron is a source of gravitational waves. This phenomenon is natural for a theory (like GR) in which every physical field curves the space-time and thus becomes a source of gravitational waves.
According to Eq. (VI.11), in MDG graviton is a massless particle in the de Sitter space-time. In the corresponding conformal-invariant wave operator $\Box + \frac{1}{6} \bar{R}$ presents a constant term which resembles a mass term in a flat space-time. It corresponds to an extremely small mass, even in comparison with the electron ones $m_e$:

$$m_R = (\hbar/c)\sqrt{\bar{R}/6} = (\hbar/c)\sqrt{2\Lambda/3} \approx 1.5 \times 10^{-38} m_e.$$  

This value is consistent with the known astrophysical data about the mass of the graviton [5, 5]. Such term is needed to have a right quasiclassical limit of the theory, as well as to have a unique vacuum (up to a unitary equivalence) in the Fock space in QFT in the de Sitter space-time [49–54].

The mass of the scalaron $m$ in Eqs. (V.9), (V.10) may be arbitrary large. The only known restriction, needed to avoid conflicts with the solar system and laboratory experiments was pointed out in the Introduction. It gives

$$m \gtrsim 2 \times 10^{-9} m_e \approx 10^{29} m_R.$$  

Thus, the second condition in Eq. (V.10) is certainly fulfilled in realistic MDG models. As seen, in their gravidiaton sector we have two very different mass scales.

We recover two new phenomena in MDG:

1. Scalaron waves freely spreading in vacuum with velocity much smaller than the velocity of light because of relation (VI.2). This is consistent with the expected low velocity of dark matter particles and supports the hypothesis that the scalaron may be a good candidate for dark matter. An important problem is to find experimental methods and/or observational tools for a direct registration of scalaron waves predicted by MDG.

Our consideration shows that from a geometrical point of view the scalaron waves are freely spreading in vacuum perturbations of space-time scalar curvature $R$ (See Eq. (V.8)). Thus, their existence is an essential deviation from GR. Here the scalar curvature is rigidly related with the trace of the energy-momentum tensor of matter and vanishes identically in vacuum.

2. Induction of gravitational waves by the scalaron, according to Eq. (VI.11). Estimates of the magnitude of this effect related with astrophysical objects and with the early Universe are needed to see whether this prediction of MDG may be of interest for the future more sensitive detectors of gravitational waves.

According to the withholding condition, the dilatonic potential has at least one minimum. Hence, in MDG there exists at least one dSV. If there are several of them, as shown in Fig. 2 then the withholding condition ensures the ordering $0 < \Phi_0 < \Phi_1 < \cdots < \infty$.

Our physical normalization requires for the lowest minimum $\Phi_0 = 1$ and $U(\Phi_0) = 1$ which is a natural choice for the case of unique dSV in MDG [4, 5]. If there exist many dSV, this normalization corresponds to the hypothesis that the present state of the Universe is close to the lowest one.

For different dSV, numbered by $k = 0, 1, \ldots$, the gravitational factor has different values $G_k = G_N/\Phi_k$. The different values of the cosmological factor are $\Lambda_k = \Lambda U_k$. Hence, as a result of the withholding property, the lowest dSV is a state with minimal value of the cosmological factor and with maximal value of the gravitational factor. The scalaron has also different masses $m_k$ in the vicinity of different dSV. This new physical situation needs a more careful analysis.

A basic open problem remains a careful choice of a withholding MDG model which is able to fit known observational data in cosmology and astrophysics. The present paper is a necessary step in this direction which outlines a variety of possible future developments.

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Appendix A: Derivation of the MDG-field-equations

In this article, we use the notation written in the sign conventions of the article [21]. Then

$$\delta \left(\sqrt{|g|} R \right) = \sqrt{|g|} \left( G_{\alpha \beta} \delta g^{\alpha \beta} + \nabla_\lambda \left( \delta v^\lambda \right) \right),$$

$$G_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R. \quad (A.1)$$
The explicit form of the vector $\delta v^\lambda$ is not needed in GR. The direct calculations give

$$\delta v^\lambda = g^{\lambda\nu} \delta \partial_{\nu} \Gamma_{\mu\nu}^\mu - g^{\alpha\beta} \delta g_{\alpha\beta}^\lambda =
= \partial_{\nu} g^{\lambda\nu} + \Gamma_{\alpha\nu}^\lambda \delta g^{\alpha\beta} + \Gamma_{\nu}^\lambda \delta g^{\nu\lambda} + 2g^{\lambda\nu} \delta g_{\nu\lambda}, \quad (A.2)$$

and after some more algebra we obtain

$$\sqrt{|g|} \nabla_{\lambda} \Phi \delta v^\lambda = \sqrt{|g|} \left( \nabla_{\alpha} \nabla_{\beta} \Phi - g_{\alpha\beta} \Box \Phi \right) \delta g^{\alpha\beta} -
- \partial_{\nu} \left( \sqrt{|g|} \nabla_{\lambda} \Phi \left( \delta g^{\lambda\nu} - 2g^{\lambda\nu} \delta \left( \ln \sqrt{|g|} \right) \right) \right), \quad (A.3)$$

For derivation of the equations (1.2), the variations of the metric coefficients $\delta g^{\lambda\nu}$ and the variations of their derivatives $\delta \left( \partial_{\nu} g^{\lambda\nu} \right)$ must obey the system of four restrictions on the surface $\partial V(3)$:

$$\Phi \delta v^\lambda - \nabla_{\nu} \Phi \delta g^{\lambda\nu} + g^{\lambda\nu} \nabla_{\nu} \delta \ln |g| =
= \delta \left( \partial_{\nu} g^{\lambda\nu} \right) + \partial_{\nu} \left( \ln \Phi \sqrt{|g|} \right) \delta g^{\lambda\nu} + \Gamma_{\nu}^{\lambda\beta} \delta g^{\alpha\beta} +
+ g^{\lambda\nu} \left( \partial_{\nu} \left( \delta \ln |g| \right) - \partial_{\nu} \Phi \delta \ln |g| \right) = 0, \quad (A.4)$$

**Appendix B: Geometrical perturbations of the de Sitter space-time**

We use the general formalism for perturbations of the metric $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$ on a curved background space-time with metric $\tilde{g}_{\mu\nu}$ [3] corrected with respect to our conventions. The perturbation $h_{\mu\nu} = g_{\mu\nu} - \tilde{g}_{\mu\nu} = \delta g_{\mu\nu}$ is a small quantity of the first order. The sign $\approx$ denotes equalities valid in the linear approximation. The bar sign denotes the plane-space-time-quantities.

The tensor of the affine deformation is

$$\delta \Gamma_{\alpha\beta}^\gamma \approx H_{\alpha\beta}^\gamma = \frac{1}{2} \left( \nabla_{\alpha} h_{\beta}^\gamma + \nabla_{\beta} h_{\alpha}^\gamma - \nabla_{\gamma} h_{\alpha\beta} \right). \quad (B.1)$$

Then

$$\delta R_{\alpha\beta}^\gamma \delta \approx -2 \nabla_{[\alpha} H_{\beta]\gamma]^\delta, \quad (B.2)$$

and imposing the transversal gauge condition

$$\nabla_{\mu} h_{\alpha}^\mu - \frac{1}{3} \nabla_{\alpha} h = 0, \quad h = h_{\mu}^\mu, \quad (B.3)$$

from Eqs. (B.2) and (B.3) one obtains the general form of the variations of the Ricci tensor and scalar

$$\delta R_{\alpha\beta} = \frac{1}{2} \nabla_{\alpha} h_{\beta} - h_{\mu}^\mu R_{\mu\alpha\beta}, \quad (B.4a)$$

Taking into account the definition of the dSV in MDG it is easy to obtain the following dSV-values:

$$\Phi : V_{\phi}(\tilde{\Phi}) = 0, \quad V_{\phi}(\tilde{\Phi}) = \tilde{m}^2 > 0, \quad (B.5a)$$

$$\tilde{U} = U(\Phi), \quad \tilde{U}_\phi = 2U/\Phi, \quad (B.5b)$$

$$\tilde{R} = 4, \quad \tilde{R}_{\alpha\beta} = \tilde{g}_{\alpha\beta}, \quad \tilde{R}_{\alpha\beta\gamma} = -\frac{1}{3} \left( \tilde{g}_{\alpha\gamma} \tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta} \tilde{g}_{\beta\gamma} \right). \quad (B.5c)$$

The corresponding values for $f$, $f_{\alpha\beta}$, and $f_{\alpha\beta\gamma}$ can be obtained using Eqs. (1.1).

As a result of Eqs. (B.5) we obtain the following final form of the perturbations above dSV-space-time:

$$\delta R_{\alpha\beta} = \frac{1}{8} \nabla_{\alpha} h_{\beta} + \frac{1}{8} \nabla_{\phi} h_{\beta} + \frac{1}{8} \nabla_{\gamma} h_{\alpha\beta}, \quad (B.6a)$$

$$\delta \tilde{R}_{\alpha\beta} = \frac{1}{8} \nabla_{\alpha} h_{\beta} + \frac{1}{8} \nabla_{\phi} h_{\beta} = \delta \left( \tilde{R}_{\alpha\beta} \right) + \hat{h}_{\alpha\beta}, \quad (B.6b)$$

$$\delta \left( \tilde{R}_{\alpha\beta} \right) = \frac{1}{8} \nabla_{\alpha} h_{\beta} + \frac{1}{8} \nabla_{\phi} h_{\beta} = \frac{1}{8} \left( \nabla + \frac{1}{3} \tilde{R} \right) h_{\alpha\beta}, \quad (B.6c)$$

$$\delta \tilde{R} \approx \frac{1}{2} \nabla_{\alpha} h - h. \quad (B.6d)$$

The used barred quantities represent the corresponding values for any possible dSV. To simplify the notation, here we do not number explicitly the different dSV.
