Abstract

In this paper, we have analyzed the Gaussian primes and also constructed an algorithm to find the primes on the $\mathbb{R}^2$ plane for sufficient large value. If we can find the large Gaussian primes then we can calculate the moat for the higher value. We have also reduced lot of computation with this algorithm to find the Gaussian prime though their distribution on the $\mathbb{R}^2$ plane is not so regular. A moat of value $\sqrt{26}$ is already an existing result done by Genther et.al. In this note, we have constructed an algorithm using the Cramér’s bound for the prime numbers and this algorithm is focused on the higher value primes. We have also defined the notion of primality for the plane $\mathbb{R}^3$ and proposed a problem on it.

1 Introduction

It has been always an interesting question that one can walk through the real line to infinity if one takes the bounded length and steps on the primes? This is same as saying that there are arbitrarily large gaps in the primes. The proof is quick sort, a gap of size $n$ can be given by $(n + 1)! + 2, (n + 1)! + 3, \ldots, (n + 1)! + (n + 1)$ where $n$ is approaching to infinity. So the primes are getting more rare as we are walking through the real axis and going to infinity.

Now we can think about the concept of irreducibility in the plane $\mathbb{R}^2$. Well, the integers in the two dimensional plane are known as the Gaussian integers (denoted by the ring $\mathbb{Z}[i]$). So the primes in the two dimensional plane is called the Gaussian Primes. On that note, an interesting question arises that “In the complex plane, is it possible to walk to infinity in the Gaussian integers using the Gaussian primes as stepping stones and taking bounded-length steps?” This is the famous unsolved problem called “The Gaussian moat” problem posed by Basil Gordon in 1962 [1][3] at the International Congress of Mathematics in Stockholm (although it has sometimes been erroneously attributed to Paul Erdős). The question is getting more complex because of the two dimensional plane.

In 1970, Jordan and Rabung establishes that steps of lenth 4 would be required to make the journey through the Gaussian Primes. In the paper “Prime Percolation” by Ilan Vardi [14], has been given a probabilistic result with the help of the Cramér’s conjecture. He has used percolation theory which predicts that for a low enough density of random gaussian integers no walk exists.

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In the paper “A stroll through Gaussian Primes” by Genther et.al [4] has found the moat of size \sqrt{26} which covers the prime up to 10^6. They have given a computational aspect using the distance k-graph in the G-primes and with the help of mathematica software they have proved the existence of the G-prime approximately up to 10^6.

In this literature we have constructed an algorithm to find the Gaussian primes. We have analyzed the G-primes for each \( y = a \) line (for \( a \geq 1 \)) in the first octant and develop an algorithm using which we can get the large value of G-prime. Most importantly one can reduce a lot of computation with this algorithm to find the Gaussian prime though their distribution on the \( \mathbb{R}^2 \) plane is not so regular. As a new ingredient we have added that using this algorithm we can cover all the Gaussian primes inside a circle centered at origin with sufficiently large radius (say \( 10^A \) with large value of \( A \)). This is same as saying that we can get all the primes up to \( 10^A \) for sufficiently large value of \( A \) and this work has been never done before. In the section 4.1 we have shown that we can calculate the exact number of Gaussian primes of a certain area. For example, for the circle \( x^2 + y^2 = R^2 \) (\( R \approx 10^A \)) we can count the exact number of G-prime inside the circle.

We have also defined the notion of the notion of primality for the plane \( \mathbb{R}^3 \) and proposed a problem on it. We can say that it is an extension of the Gaussian Moat problem to the \( \mathbb{R}^3 \) plane.

2 Background

The Gaussian integers are the complex numbers \( z = a + bi \) where \( a, b \in \mathbb{Z}[i] \) (the ring of G-integers is denoted by \( \mathbb{Z}[i] \), consist of integers in the field \( \mathbb{Q}(i) \) and \( i = \sqrt{-1} \) [9–12]. The Gaussian integers admits a well defined notion of primality and there is a simple characterization of the Gaussian primes, so we can define Gaussian prime [13] (denoted by G-prime) as follows:

**Definition 1** Gaussian primes are Gaussian integers \( z = a + bi \) satisfying one of the following properties.

1. If both \( a \) and \( b \) are non zero then, \( a + bi \) is a Gaussian prime iff \( N(a + bi) = a^2 + b^2 \) is an ordinary prime.

2. If \( a = 0 \), then \( bi \) is a Gaussian prime iff \( |b| \) is an ordinary prime and \( |b| \equiv 3 \pmod{4} \).

3. If \( b = 0 \), then \( a \) is a Gaussian prime iff \( |a| \) is an ordinary prime and \( |a| \equiv 3 \pmod{4} \).

**Note:** \( |a + bi|^2 = a^2 + b^2 = p \equiv 1 \pmod{4} \) is called Pythagorean primes.

Two Gaussian integers \( v, w \) are associates if \( v = uw \) where \( u \) is a unit. In such a case \( N(v) = N(w) \). It is well known and not hard to prove that a prime \( p \equiv 1 \pmod{4} \) can be written uniquely as the sum of two squares where the primes \( p \equiv 3 \pmod{4} \) cannot be written in such fashion. Which is nothing to say that the primes of the form \( 4k + 1 \) are split into two G-prime factors where the primes of the form \( 4k + 3 \) remain prime. Now, what is about the only even prime 2? Well, the prime 2 is a special G-prime which can be written as \( 2 = 1^2 + i^2 = (1 + i)(1 - i) \) and it is the only G-prime which lives on the line \( x = y \).

Let us consider the prime 5. There are exactly eight G-primes corresponding to the prime 5 because we can write it as \( 5 = (\pm 1)^2 + (\pm 2)^2 \). So the G-primes corresponding to 5 are \( \pm 1 \pm 2i \) and \( \pm 2 \pm i \). Hence, up to associates, there are exactly two distinct G-primes corresponding to each prime \( p \equiv 1 \pmod{4} \) [6–8]. Now, let us look at the geometrical interpretation of the G-prime. If
$p$ is a prime congruent to $1$ (mod 4) consider the circle of radius $\sqrt{p}$ centered at the origin then for eight fold symmetry of the $\mathbb{R}^2$ plane, two G-primes lie in each quadrant corresponding to the prime $p$. Similarly, if a Gaussian integer $a+bi$ is composite then $\pm a \pm bi$ and $\pm b \pm ai$ are composite as well. Thus the geometric structure of the Gaussian integers has an induced eightfold symmetry. Figure 1 (taken from [4]) shows all the G-prime of the norm less than 1000.

Now the question arises that “one can walk to infinity using steps of length $k$ or less?” To prove that the answer is **No** we can try to find a moat from G-primes in the first octant (the sector $0 \leq \theta \leq \frac{\pi}{4}$) such that the width of the moat will increase as one is walking towards the infinity. So, for the eightfold symmetry we can take the first octant and through the line, $x = y$ we can cut it and one side is the reflection of the other one. If we can cut a swath from the positive $x-$axis to the line $y = x$ to get the moat, then the eightfold symmetry allows the reflection across the $y-$axis to the line $y = -x$. In this note, we will think two G-primes in the first quadrant from different octant as the reflection of one another with respect to the line $x = y$. For example, the prime 73, in the first octant there are two G-prime corresponding to 73 namely, $8+3i$ and $3+8i$. According to our consideration $3+8i$ is the reflection of $8+3i$ (as shown in figure 2).

Now let us look at the geometrical view of the G-primes. It has been conjectured that

$$\lim_{n \to \infty} \sqrt{p_{n+1}} - \sqrt{p_n} = 0$$

where $p_n$ denotes the $n$th prime. If the conjecture is true, then the circles upon which the G-primes lie become more crowded as one travels farther away from the origin in the complex plane. Thus there would be no chance of finding truly annular moat (i.e., a moat that is the region between two circles) of composite Gaussian integers. Well, analytic number theorists have been proved that the answer is **No**. So mathematically which is same as saying that

$$\lim sup p_n - p_{n-1} = \infty$$
Figure 2: Eight fold symmetry reflection of the first octant
where $p_n$ denotes the $n$-th prime number.

Note an important result by Terence Tao [19] on the G-primes, he has showed that the G-primes $P[i] \subseteq \mathbb{Z}[i]$ contain infinitely constellations of any prescribed shape and orientation. More precisely, he has shown show that given any distinct Gaussian integers $\nu_0, \ldots, \nu_{k-1}$, there are infinitely many sets $\{a + r\nu_0, \ldots, r\nu_{k-1}\}$, with $a \in \mathbb{Z}[i]$ and $r \in \mathbb{Z}\{0\}$, all of whose elements are Gaussian primes.

3 Walking on the G-primes

Our aim in this paper is to construct an algorithm to find the distribution of G-prime in the $\mathbb{R}^2$ plane that one can walk to infinity putting the steps on the G-primes. As the analytical number theorist says that such kind of walk is not possible with the bounded length steps, our result completely agrees with this statement. In this note, we have analyzed the G-primes theoretically and used the basic facts about the prime numbers that one can walk through the G-primes more easily. In the work done by Genther et.al they have calculated a moat of size $\sqrt{26}$ by their computational method. In this paper, we have used some tools from analytic number theory to get a better way for such kind of walk on the $\mathbb{R}^2$ plane.

The main problem: In the complex plane, is it possible to “walk to infinity” in the Gaussian integers using the Gaussian primes as stepping stones and taking bounded-length steps?

In the paper, by Genther et.al they have checked the G-primes using the distance $k$-graph. Using this method they have checked the moat for the G-primes of value $10^6$. This method requires lots of counting. As we know that the primes will get rare as we increase the value. So, this method can not work for sufficiently large values of primes. We will use some tools from analytic number theory that one can get the moats of higher value.
In 1936 Swedish mathematician Harald Cramér has formulated an estimate for the size of gaps between consecutive prime numbers [20]. It states that

**Cramér’s conjecture:** The gaps between consecutive primes are always small, and the conjecture quantifies asymptotically just how small they must be, i.e.,

\[ p_{n+1} - p_n = O \left( (\log p_n)^2 \right) \]

where \( p_n \) denotes the \( n \)th prime number. Which is same as saying that

\[ \lim_{n \to \infty} \sup \frac{p_{n+1} - p_n}{(\log p_n)^2} = 1. \]

Now let us describe an overview of our work and how we have used the Cramér’s estimate.

We are going to analyse the G-prime distribution for each line \( y = a \) (for \( a \geq 1 \)) on the \( xy \) plane. For \( x = 0 \), is the real axis and by the definition of G-primes, they are the primes of the form \( 3 \mod 4 \). By the Chebyshev’s bias [21], we have an asymptotic formula for them and moat problem is focused on the G-primes lies on the quadrant. We don’t have any asymptotic formula for the primes of the form \( a^2 + b^2 \) where one of the \( a \) or \( b \) is fixed and the other one varies. Actually, if we see this problem very closely then it leads to a famous unsolved problem which is named by \( n^2 + 1 \) conjecture. That is not the focus of this paper, so let’s get back to the main topic.

In this section, we have given an overview of the constructed algorithm in this paper. We have described the eightfold symmetry (see figure 3) of the \( xy \) plane in the previous section. We will analyze the G-prime distribution only for the first octant in this paper and the result for the other octants follow from it. Now if we are standing on a G-prime on the first octant then Cramér’s conjecture is a helpful tool to find the next G-prime for that line but it will be easier if we can get the distribution of G-primes in a circle area. So we will consider a circle and inside of that circle, we will check the G-primes. It is the most important step that how we will choose the circle and it’s the radius because this choice of the circle we will be able to reduce the computation.

But to calculate the moat we need to cover all the G-primes. The main twist of our method is that we have sliced the first octant in such fashion that it will be more easy to find the G-primes. The slices are not so smooth which means we have not sliced using the straight lines. As we know the distribution of primes are very irregular as they tend to infinity and our algorithm is especially focused on the large primes. So it is more difficult to see the distribution of primes on the \( \mathbb{R}^2 \) plane for their larger values. Let us construct the paths and the lines which are slicing the first octant. After this construction, we develop the mathematics for this algorithm and also describe it’s geometrical reasoning.

### 4 Construction of the line \( \mathcal{L}_n \) from the path \( P_n \)

In this section, we are going to construct the line \( \mathcal{L}_n \) with respect to the path \( P_n \). Let us write the G-primes as a set up to \( 10^A \) for the sufficiently large value of \( A \).

\[
\mathcal{A} = \{ p : p = m^2 + n^2, p \leq R, m, n \neq 0 \}
\]

\[
\mathcal{A} = \{ p : n^2 - p = m^2, p \leq R, m, n \neq 0 \}
\]

\[
R - p = m^2; p \leq R, R = m^2 + p, R \approx 10^A.
\]

\[ \text{(1)} \]
where $A$ is sufficiently large and finite.

So before construct the line let us define the path $P_n$.

**Definition 2** A path $P_n$ is a set of Gaussian primes such that

$$P_n = \{p^n_{(n,a)} : \text{all the primes } p \text{ lies on the line } y = a \text{ and close from the line } L_{n-1}\},$$

where the value of $a$ varies.

**Notation:** For the $n$-th prime lies on the line $y = a$ and in the path $P_n$ (for $n \in \mathbb{N}$) is denoted by $p^n_{(n,a)}$.

Now first consider the path $P_1$. In the next two sections, we will describe more about the paths $P_1, P_2, \ldots, P_n$ in details, before that we will construct the lines first. For this section let us follow the definition 2. If we take the line $x = y$ then it is the first line, we can call it $L_0$. Now from the path $P_1$, we will construct the line $L_1$. By definition 2 we can write,

$$P_1 = \{p^1_{(n,a)} : \text{all the primes } p \text{ lies on the line } y = a \text{ and close from the line } x = y\}.$$

We know the distribution of the G-primes which are closest to the line $L_0$ up to $10^A$ from the path $P_1$. The path $P_1$ can be written as a step function. We will draw a line through the origin with the tangent $\theta_1 < \frac{\pi}{4}$ (say) and close to the path $P_1$ i.e., the line $L_1$ will pass through the farthest G-prime of the path $P_1$ from the line $L_0$ i.e., $x = y$ and which starts from the origin. So, if we see the geometrical view (see figure 4) then $L_0$ and $L_1$ has sliced the first octant up to $10^A$.

We have constructed the line $L_1$ with respect to the line $L_0$ and the path $P_1$. Similarly, we will construct the line $L_2$ with respect to the line $L_1$ and the path $P_2$ with this similar method of construction.

In such fashion we will construct the other lines $L_2, L_3, \ldots, L_n$, and the tangents are $\theta_2, \theta_3, \ldots, \theta_n$ respectively. By construction of the lines the inequality, $\theta_n < \theta_{n-1} < \ldots < \theta_1 < \frac{\pi}{4} = \theta_0$ holds.

A question can come that why we are constructing the line $L_n$ instead of using the path $P_n$ directly. Well, if we see the path then it is not so smooth, when are walking on the path $P_2$ and considering the circle then we need to compute more if we consider the path directly instead of constructing the line.

**Theorem 1** For the first path, $P_1$ the considered circle or the rings will always intersect with the line $x = y$.

**Proof:** If we see the geometrical view of the considered area of the circle then this statement is clear from the picture (see figure 7). We will describe the logic behind this and prove the statement.

Precisely, when we will get the next prime $p_{(n+1,a)}$ from the prime $p_{(n,a)}$ on the line $y = a$ using Cramér’s bound we are crossing a long distance because as we know primes will get rarer for each $y = a$ line. Similarly, if we walk through the $x = b$ line (where $p_{(n,a)} = a^2 + b^2$) then have to walk the same distance on this line too from the point $p_{(n,a)}$ as we have walked for the line $y = a$. We have considered those primes for the path $P_1$ which lie near to the line $x = y$. So after crossing this distance, we have come in the second octant.

To reach the second octant from the first octant we have crossed the $x = y$ line obviously (see figure 5). So, for the path $P_1$ the considered circle and the line $x = y$ will intersect each other. ■
Figure 4: The line $\mathcal{L}_n$ with respect to the path $P_n$
The considered circle $S_{(n,a)}$ is the path $P_1$ w.r.t. the line $L_1$.

The radius of the considered circle is

$$r_{(n,a)} = O\left(\log\left(p_{(n,a)}\right)\right)^2$$

Figure 5: Intersection of the line $L_n$ and the circle $S_{(n,a)}$
**Theorem 2** For the n-th path (for all n ≥ 2) \( P_n \) the considered circle or the rings will always intersect with the line \( \mathcal{L}_{n-1} \) constructed from the path \( P_{n-1} \).

**Proof:** We will prove this theorem using the same logic of theorem 1. The line \( \mathcal{L}_1 \) has constructed by connected the farthest primes from the \( x = y \) line. So the line \( \mathcal{L}_1 \) is close from the path \( P_2 \). Precisely, when we will get the next prime \( p_{(n+1,a)} \) from the prime \( p_{(n,a)} \) on the line \( y = a \) using Cramér’s bound we are crossing a long distance because as we know primes will get rarer for each \( y = a \). Similarly, if we walk through the \( x = b \) line (where \( p_{(n,a)} = a^2 + b^2 \)) then have to walk the same distance on this line too from the point \( p_{(n,a)} \) as we have walked for the line \( y = a \). We have considered those primes for the path \( P_2 \) which lie near to the line \( \mathcal{L}_1 \). To cross this distance, we must cross the line \( \mathcal{L}_1 \) (see figure 5).

4.1 G-prime counting

In this section, we have constructed the line \( \mathcal{L}_n \) and described how it is slicing the octant. For each slice, we can count the exact number of G-prime i.e., for the n-th slice the number of G-prime is \( |P_n| \), which is clear from the definition 2. Then for the circle \( x^2 + y^2 = R^2 \) (where \( R ≈ 10^4 \)) we can count the exact number of G-prime inside it.

5 Mathematical description of the algorithm

In this section, we describe the mathematics for the analyzing process and the algorithm to find the Gaussian moat. We will also prove the result that using this algorithm we can cover all the G-prime up to the level we want to count the moat.

We consider the first octant of the \( \mathbb{R}^2 \) plane and we will analyze the G-primes for the first octant and how to calculate the moat for it. Other quadrants of the \( \mathbb{R}^2 \) plane follow similarly. For the first quadrant if we consider the function \( f \) such that \( (a, b) \downarrow (b, a) \) with \( a > b \) then for each point in the second octant can be think as a image of the point of first octant. The function \( f \) has been drawn in figure 2. Then if we calculate the moat of the first octant we will get the moat for the second octant too using this function \( f \). So after adding them, we will get the moat for the first quadrant and the moat constructing procedure for the other quadrants follows similarly.

It is quite evident that except 2 there is no other G-prime on the line \( x = y \) (see Remark 1). Let us fixed that we want to calculate the moat up to \( 10^4 \) for sufficiently large value of \( A \). So we start our walk through the G-prime from the line \( y = a \) (for some large value of \( a ∈ \mathbb{N} \) [We can start our walk from the G-prime \( 1 + i \) also]. Take the first prime on the line \( y = a \) and say it is \( p_{(1,a)} \) [Example: for the line \( y = 1 \), the first G-prime is 2, i.e., \( p_{(1,1)} = 1 + i \). We are on the first octant, so for all the pair of coordinates \( (a, b) \) we have \( a > b \). Let \( |p_{(1,a)}| = |a^2 + b_1^2| \) i.e., the value of the first prime on the line \( y = a \).

By the Cramér’s estimation we know there is a gap of size \( O \left( (\log p_{(n,a)})^2 \right) \) between the prime \( p_{(n+1,a)} \) and \( p_{(n,a)} \) for all \( n \) and \( a \). So, we take the segment of size \( O \left( (\log p_{(1,a)})^2 \right) \) on the line \( y = a \) from the point \( p_{(n,a)} \). Then consider the circle \( S_{(1,a)} \) with the radius \( r_{(1,a)} = O \left( (\log p_{(1,a)})^2 \right) \). Now, consider the first quadrant of the circle. By theorem 1 this circle \( S_{(1,a)} \) will intersect with the line \( x = y \). By Cramér’s conjecture it is possible that there is a prime at the end of the considered segment but that probability is low. The reason for this low probability is that we are
not considering all the primes. In fact, we have considered only congruent 1 modulo 4 primes and
we have fixed the line $y = a$ so primes will get more rare.

We will consider the area which this bounded by the intersection of the first octant of the circle
$S_{(1,a)}$ and the line $x = y$ (or $L_0$) (see figure 7). Next, we will check the primes inside the considered
area of the circle $S_{(1,a)}$. If there are primes then choose the nearest prime from the point $p_{(1,a)}$
where we are standing on. Otherwise, take the ring of radius $r_{(1,a)}$ and continue the same process
until we reach the target value.

Continuing this process we have got some $G$-primes from $p_{(1,a)}$ upto $10^4$. If we join them then
we will get a path through the $G$-primes and this path is very near from the line $x = y$. So, we call
this path $P_1$ and we can say that the path $P_1$ is nothing but a set of selected $G$-primes. We can
write

$$P_1 = \{ p_{(1,a)}^1 : \text{all the primes } p \text{ lies on the line } y = a \text{ and close from the line } x = y \},$$

where the value of $a$ varies.

Our target is to calculate the Gaussian moat for which we have to cover all the $G$-primes up
to $10^4$. The path $P_1$ does not cover all the $G$-primes. To cover all of them we start the same
procedure from the point $p_{(k,a)}$ (for some $k \geq 2$) and instead of taking the line $x = y$ we will take
the line $L_1$ (as we have constructed in section 4) and obtain the path $P_2$.

So in general to get the path $P_n$ we take the prime $p_{(n',a)}$ (for some $n' \geq n$) and we will consider
the circle area with respect to the line $L_{n-1}$. We define the path $P_n$

$$P_n = \{ p_{(n,a)}^n : \text{all the primes } p \text{ lies on the line } y = a \text{ and close from the line } L_{n-1} \},$$

where the value of $a$ varies.

Continuing this process we will get the paths,

First Path = $P_1$
Second Path = $P_2$
::

$n$-th Path = $P_n$ (for some $n$)

until we cover all the $G$-primes upto $10^4$. We will calculate the moat after getting all the path
$P_1, P_2, \ldots, P_n$ upto $10^4$.

We will be able to consider all the $G$-primes using this process. Now let us state and prove this
result.

**Theorem 3** One can cover all the $g$-prime using this algorithm.

**Proof:** We have started from the prime $p_{(1,a)}$ and obtain the path $P_1$ up to $10^4$. $P_1$ is the nearest
path from the line $x = y$ by it’s definition. We will start the same procedure from the prime $p_{(2,a)}$
to obtain the path $P_2$. Continuing this process we will get a set of paths $\{ P_1, P_2, \ldots, P_n \}$ for some
value of $n$ and $P_n$ is the closest path from the real axis. Since, we have fixed our target up to $10^4$
where $A$ is sufficiently large but finite. So, the value of $n$ is also finite and the set $\{ P_1, P_2, \ldots, P_n \}$
is finite. The set of lines $L_0$ (which is actually the line $x = y$), $L_1, L_2, \ldots, L_{n-1}$ are also finite. If
we consider the circle $x^2 + y^2 = R^2$ where $R = 10^4$ then the lines $\{ L_0, L_1, \ldots, L_{n-1} \}$ have sliced


the first octant of the circle and each path \( P_1, P_2, \ldots, P_n \) has covered all the G-primes in each of the slices (see figure 4). So if we take their union i.e.,

\[
P = P_1 \cup P_2 \cup \ldots \cup P_n = \bigcup_{n} P_n,
\]

then it is straightforward that \( P \) will consider all the G-primes in the first octant of the circle \( x^2 + y^2 = R^2 \). We will continue the same process for the other octants.

Hence, we have proved that using this algorithm we can consider all the G-primes.

\[\text{\textbf{6 Geometrical interpretation of the algorithm}}\]

In this section, we describe the geometrical view of the algorithm.

We consider the \( x = y \) line and we analyze the area in the first quadrant under the \( x = y \) line (which is the first octant). Our goal is to find a path such that a person can walk over the G-primes through infinity. We need not consider the G-primes lies on the \( x \)-axis. The nearest G-prime from the origin on the first quadrant is \( 1 + i \). It is the only exception where a G-prime lies on the line \( x = y \) (see remark 3). So, let us start our walking from the G-prime \( (1, 1) \).

Our basic idea in this paper is to analyze the G-prime for each \( y = a \) line. Unfortunately, we don’t have any asymptotic formula for the prime of the form \( a^2 + b^2 \) (for all \( a, b \in \mathbb{N} \)) in general. If we take \( b = 1 \), it leads to a famous problem of number theory which is known as \( n^2 + 1 \) conjecture. So, we don’t know the asymptotic behavior of the prime on the line \( y = 1 \). Actually, we are assuming that \( n^2 + 1 \) conjecture is true in this paper. Otherwise, we can not say that there exist infinitely many G-primes.

Let us get back to our main target. Now the question arises that how can we get the nearest prime on each \( y = a \) line? Well, we can reduce the problem by talking the approximation given Cramér, which is also a well-known result of number theory known as Cramér’s conjecture.

Let us observe the geometry for a point on \( xy \)-plane. For the point \( p \), there are eight possibilities where one can take another step if one is standing on that point (as shown in figure 6).

If one is putting his steps forward then we need not consider the left side points and the points below \( p \). Because those points already have been covered in the sense that we can get the nearest prime among those points with respect to \( p \). So, there are only four possibilities left. Now we will consider only those points which lie below the \( x = y \) line and our goal is to check them and find the nearest G-prime from the point \( p \). As we know primes get rare as we approach through infinity. If we see the primes on the line \( y = a \) (for \( a \in \mathbb{N} \)) then it is obvious that we will get them more rarely.

To find the larger G-primes no elementary method can work. That’s why we have used Cramér’s estimate for the gaps between two consecutive primes, which has a controllable error term. Sometimes, our method may not work for a very smaller value of G-prime. Using this method one can find the G-prime of smaller value but in that case, we will have to look the error term. This algorithm is focused on the larger values of G-primes.

It is not necessarily true that if we are standing on a prime which lies on the line \( y = a \) then the next nearest G-prime will lie on the same line. So, we will consider a circle of radius \( O((\log p_n)^2) \) (as we know the exact value of the prime where we are standing on) with center \( p \) (at the point
where we are standing on). As we have described previously that we do not need to consider the lower half portion of the circle. For the upper half portion of the circle, we only consider the first quadrant area of it. We will check the G-primes in the area bounded by $x = y$ line and the first quadrant of the circle $S_{(n,a)}$ (the shaded area as shown in figure 7).

For each $y = a$ line inside the considered area we will check the prime. For each line, we have already covered the previous prime so from that one we will take the Cramèr’s estimate for each line. In this way, we will be able to reduce lots of computation which helps us to find sufficiently large G-prime. We will take the nearest prime from the point $p$ and continue this process. If there is no prime in our considered area then again we will take a ring with outer radius $O \left((\log n)^2\right)$ and continue the same process.

At the starting point we have moved on from the $y = a$ line to $y = a + 1$ very soon and we have not covered the rest of the G-primes lie on the line $y = a$. So after computing the G-primes up to $10^A$ (for the sufficiently large value of $A$) we will get back to the starting point and take the next prime from the line $y = a$ for the same value of $a$ and continue the process to cover all the G-primes for the circle $x^2 + y^2 = R^2$.

**Remark 1** Let us consider a point $(a,a)$ on the line $x = y$, then $|a + ai|^2 = 2a$, which is not a prime. So, 2 is the only G-prime on $x = y$ line.

**Note:** For the path $P_1$ we know there is no G-prime on the line $L_0$ (i.e., $x = y$). Similarly, for the path $P_n$ ($\forall n \geq 2$), we have already counted the G-primes lie on the line $L_n$ ($\forall n \geq 1$). To construct the path $P_n$ ($\forall n \geq 2$) we need not worry about the G-primes lie on the line $L_n$ ($\forall n \geq 1$) and no need of counting.
The radius of the considered circle is
\[ r_{(n,a)} = O\left(\log(p_{(n,a)})\right)^2 \]

Figure 7: The shaded region is the considered area of the circle
7 The Main Algorithm

We have described the mathematical and geometrical interpretation of the algorithm in the previous sections. From that discussion, it is clear that one can not walk to infinity using steps of bounded length and putting the steps on the G-primes. The value of prime will increase as we are approaching infinity. We have used the Creamer’s estimate for two consecutive prime and from that result, it follows directly.

In this section, we will write the algorithm.

First, we write the algorithm for the path $P_1$ then we generalize it for any path $P_n$.

Algorithm 1: Algorithm for the path $P_1$

1. Select the starting point $p_{(1,a)}$ (say) (where $a \in \mathbb{N}$).
2. Let $|p_{(1,a)}| = |a^2 + b^2|$. Take the length $O\left((\log p_{(1,a)})^2\right)$.
3. Consider the circle $S_{(1,a)}$ with the radius $r_{(1,a)} = O\left((\log p_{(1,a)})^2\right)$.
4. Consider the area of the first quadrant of the circle.
5. Select the considered area below the line $y = x$.
6. Check the primes inside the selected area.
7. if there are primes then
   8. select the nearest prime from the point $p_{(1,a)}$;
   9. else
   10. Take the ring with the outer radius $r_{(1,a)}$ and check the primes inside it;
11. end
12. Choose the nearest prime.
13. Continue the process.

Algorithm 2: Algorithm for the path $P_n$, for all $n \geq 2$

1. Select the starting point $p_{(n,a)}$ (say) (where $a \in \mathbb{N}$).
2. Let $|p_{(n,a)}| = |a^2 + b^2'|$. Take the length $O\left((\log p_{(n,a)})^2\right)$.
3. Consider the circle $S_{(n,a)}$ with the radius $r_{(n,a)} = O\left((\log p_{(n,a)})^2\right)$.
4. Consider the area of the first quadrant of the circle.
5. Select the considered area below the straight line $L_{n-1}$.
6. Check the primes inside the selected area.
7. if there are primes then
   8. select the nearest prime from the point $p_{(n,a)}$;
   9. else
   10. Take the ring with the outer radius $r_{(n,a)}$ and check the primes inside it;
11. end
12. Choose the nearest prime.
13. Continue the process.
8 Related Questions

Walking through the Gaussian prime is basically to find the prime distribution in the plane $\mathbb{R}^2$. So, we can extend this search for higher dimensions.

One of the famous result in mathematics is the Legendre’s three-square theorem \[15\] which states that,

**Legendre’s three-square theorem:** Any natural number can be represented as the sum of three squares of integers

$$n = x^2 + y^2 + z^2,$$

if and only if $n$ is not of the form $n = 4^a(8b + 7)$ for integers $a$ and $b$.

So, we can define the notion of primality for $\mathbb{R}^3$, if $a > 1$ then $4^a(8b + 7)$ is not prime. Our concern is only about those primes which are of the form $(8b + 7)$.

Take the set $\mathbb{P} - \{8b + 7\}$ where $\mathbb{P}$ is the set of all primes. Then the question arises that what is the distribution of the prime numbers in the plane $\mathbb{R}^3$? If we re-write the statement formally then it says,

**Problem 1** *In the plane $\mathbb{R}^3$, is it possible to walk to infinity in the integer lattices talking the steps on the primes with the bounded-length?*

Another important question arises with this problem that why we are not extending this G-prime problem for $\mathbb{R}^n$ for all $n \geq 4$. Well, by the four-square theorem \[16\] \[18\] we know every natural number can be written as the sum of four squares. So we can write any prime number as the sum of four square then the searching them may not be so interesting.

9 Conclusion

In this paper, we have analyzed the distribution of G-prime and constructed an algorithm which makes easy to walk on the G-primes. This method for Gaussian prime searching is better than the other because it can avoid computations despite having the irregular distribution of G-primes. It is clear from this algorithm that one can not walk to infinity using steps of bounded length and putting the steps on the G-primes. We have proved that this algorithm is covering all the G-primes. We have also shown that it is easy to find the exact number of G-primes for a certain area using this algorithm. As a new concept, we have defined the notion of primality for the $\mathbb{R}^3$ plane and proposed the problem which is an extension of the Gaussian moat problem.

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