Novel 16-QAM and 64-QAM Near-Complementary Sequences with Low PMEPR in OFDM Systems

Tao Jiang, Senior Member, IEEE, Chunxing Ni, and Yuance Xu

Abstract

In this paper, we firstly propose a novel construction of 16-quadrature amplitude modulation (QAM) near-complementary sequences with low peak-to-mean envelope power ratio (PMEPR) in orthogonal frequency division multiplexing (OFDM) systems. The proposed 16-QAM near-complementary sequences can be constructed by utilizing novel nonlinear offsets, where the length of the sequences is \( n = 2^m \). The family size of the newly constructed 16-QAM near-complementary sequences is \( 8 \times \left( \frac{m!}{2} \right) \times 4^{m+1} \), and the PMEPR of these sequences is proven to satisfy PMEPR \( \leq 2.4 \). Thus, the proposed construction can generate a number of 16-QAM near-complementary sequences with low PMEPR, resulting in the improvement of the code rate in OFDM systems. Furthermore, we also propose a novel construction of 64-QAM near-complementary sequences with low PMEPR, which is the first proven construction of 64-QAM near-complementary sequences. The PMEPRs of two types of the proposed 64-QAM near-complementary sequences are proven to satisfy that PMEPR \( \leq 3.62 \) or PMEPR \( \leq 2.48 \), respectively. The family size of the newly constructed 64-QAM near-complementary sequences is \( 64 \times \left( \frac{m!}{2} \right) \times 4^{m+1} \).

Index Terms — Orthogonal frequency division multiplexing (OFDM), peak-to-mean envelope power ratio (PMEPR), near-complementary sequence, quadrature amplitude modulation (QAM).

I. INTRODUCTION

As an attractive physical layer technology for wireless communications, orthogonal frequency division multiplexing (OFDM) has been applied in many wireless communication standards,
due to its considerable high spectrum efficiency, high power efficiency, multipath delay spread
tolerance, and immunity to the frequency selective fading channels [11], [2]. However, one major
drawback of OFDM is the high peak-to-mean envelope power ratio (PMEPR) of transmitted
OFDM signals. Since the high power amplifier (HPA) utilized in OFDM systems has limited lin-
ear range, the OFDM signals with high PMEPR will be seriously clipped and nonlinear distortion
will be introduced, resulting in serious degradation of the bit error rate (BER) performance [3].
Moreover, the high PMEPR leads to the out-of-band radiation, which causes serious adjacent
channel interferences [4]. Therefore, in order to significantly improve the energy efficiency and
reduce communication overhead of OFDM systems, it is necessary to reduce the PMEPR of
OFDM systems.

One promising approach for the PMEPR reduction in OFDM systems is to use Golay com-
plementary sequences as codewords [5], which can be constructed by co-sets of the classical
Reed-Muller codes [6]. The upper-bound of PMEPR for the Golay complementary sequences
can be restricted to be less than 2, which means significant PMEPR reduction for OFDM signals.
However, the Golay complementary sequences have extremely low code rate, which makes this
coding approach impractical in OFDM systems. For example, the code rate of the 16-QAM
Golay complementary sequences constructed in [10] is 2.7925, 1.8962, 0.4513 for \( n = 4, 8, 
64 \), respectively, where the code rate of code \( C \) with length \( n \) is defined as \( R(C) = \frac{\log_2 |C|}{n} \)
bits/symbol [12]. Moreover, when the length of the sequences increases, the code rate of Golay
complementary sequences turns to be extremely low. Thus, the low code rate is the bottleneck
of Golay complementary sequences. The most important objective of the coding approach is to
improve the code rate of Golay complementary sequences, which means that more and more
sequences with low PMEPR should be constructed. Therefore, researchers have proposed another
type of sequences: near-complementary sequences, which generalize the Golay complementary
sequences, in order to improve the code rate with a slight loss on the PMEPR performance [7],
[8].

As discussed above, the near-complementary sequences are expected to produce more se-
quences than Golay complementary sequences at the cost of an increase of the PMEPR bound.
Thus, near-complementary sequences can offer low PMEPR with satisfactory code rate, and they
are promising to be implemented in OFDM systems for the PMEPR reduction. The main objec-
tive of near-complementary sequences is to develop novel constructions of the near-complementary
sequences to enlarge their family size. A framework for the construction of \(2^h\)-phase shift keying (PSK) near-complementary sequences have been proposed in [8], which can achieve low PMEPR in OFDM systems. Since \(2^h\)-quadrature amplitude modulation (QAM) is more widely used in OFDM systems, different constructions of 16-QAM near-complementary sequences have been proposed in [9], [11]–[13], and these 16-QAM near-complementary sequences can be utilized to significantly reduce the PMEPR in OFDM systems.

In this paper, we propose a novel construction of 16-QAM near-complementary sequences to enlarge the family size of the 16-QAM near-complementary sequences. The family size of the newly constructed 16-QAM near-complementary sequences is \(8 \times \left(\frac{m!}{2}\right) \times 4^{m+1}\). Moreover, the proposed 16-QAM near-complementary sequences satisfy that \(\text{PMEPR} \leq 2.4\), which means a significant PMEPR reduction in OFDM systems.

Furthermore, to our best knowledge, all existing researches have not proposed a construction of 64-QAM near-complementary sequences, which has been a difficult problem for years. Therefore, we develop a novel construction of 64-QAM near-complementary sequences with a low PMEPR upper-bound in this paper, and it is the first proven construction for the 64-QAM near-complementary sequences. Two types of the proposed 64-QAM near-complementary sequences satisfy that \(\text{PMEPR}^1 \leq 3.62\) or \(\text{PMEPR}^2 \leq 2.48\), respectively. With the proposed construction, the family size of the newly constructed 64-QAM near-complementary sequences is \(64 \times \left(\frac{m!}{2}\right) \times 4^{m+1}\).

The contributions of this paper are as follows: (1) A novel construction of 16-QAM near-complementary sequences is proposed, and the family size of sequences with low PMEPR can be enlarged with the help of the proposed construction. (2) A construction of 64-QAM near-complementary sequences is proposed, which is the first proven construction for the 64-QAM near-complementary sequences.

The outline of this paper is given as follows. The definitions and notations are presented in Section II. The main results on the proposed constructions of 16-QAM and 64-QAM near complementary sequences are given in Section III. In Section IV we demonstrate the numerical performances of the proposed sequences. Finally, conclusions are drawn in Section V.

II. PRELIMINARIES

A summary of basic definitions and key notations of this paper are included in this section.
— \(q\) is an arbitrary even positive integer, \(m, n\) and \(h\) are positive integers.
— \(Z_q\) is a ring of integer modulo \(q\).
— \(Z_q^m\) is an \(m\)-dimensional vector space where each component is an element in \(Z_q\).
— \(\zeta = e^{2j\pi/q}\) is a primitive \(q\)th root of unity where \(q\) is an even positive integer, \(j = \sqrt{-1}\).
— \(\zeta = e^{j\pi/2} = j\) when \(q = 4\).
— \(\mathbb{C}\) denotes the set of complex numbers.
— \(\{\pi(0), \pi(1), \ldots, \pi(m-1)\}\) denotes an arbitrary permutation of the set \(\{0, 1, \ldots, m-1\}\).
— \(A_i\) denotes the \(i\)th element of the sequences generated from the function \(A(x)\), where \(x = (x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(m-1)})\).

A. Peak-to-Mean Envelope Power Ratio (PMEPR)

In OFDM systems, the transmitted time-domain OFDM signal of the input sequence \(A = (A_0, A_1, \ldots, A_{n-1})\) can be obtained by

\[
S_A(t) = \sum_{i=1}^{n-1} A_i e^{2j\pi w_i t}, \quad 0 \leq t < T,
\]

(1)

where \(n\) is the number of subcarriers, \(w_i\) is the frequency of the \(i\)th carrier, and \(T\) is period of the OFDM data block. To ensure orthogonality, the carrier frequencies are related by \(w_i = w_0 + iw_s (i = 0, 1, \ldots, n-1)\), where \(w_0\) is the smallest carrier frequency and \(w_s\) is the spacing of the frequencies.

**Definition 1 (Instantaneous Envelope Power):** According to (1), the instantaneous envelope power of the time-domain OFDM signal \(S_A(t)\) is defined as

\[
P_A(t) = |S_A(t)|^2 = (\sum_{i=0}^{n-1} A_i e^{2j\pi w_i t})(\sum_{k=0}^{n-1} A_k^* e^{-2j\pi w_k t})
\]

\[
= \sum_{i=0}^{n-1} \sum_{k=0}^{n-1} A_i A_k^* e^{2j\pi(i-k)w_s t}.
\]

(2)

**Definition 2 (PEP of a codeword):** The peak envelope power (PEP) of a codeword \(A\) is defined as

\[
\text{PEP}(A) = \sup_{t \in [0, T]} P_A(t),
\]

(3)

where \(\sup(\cdot)\) denotes the supremum.

**Definition 3 (PMEPR of a code):** The PMEPR of a code \(C\) is defined as the ratio of the PEP to the average mean power of code \(C\) over all OFDM signals generated from a codebook \(C\).
\[ \text{PMEPR}(C) = \max_{A \in C} \frac{\text{PEP}(A)}{P_{av}(C)}, \]  

(4)

where \( P_{av}(C) \) denotes the average mean power of the code \( C \). Moreover, \( P_{av}(C) \) is a constant for codebook \( C \).

**B. Generalized Boolean Functions**

*Definition 4 (Generalized Boolean Function):* A generalized Boolean function \( f \) is defined by a mapping \( f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q \), and it can be written in an algebraic normal form [7] as

\[
    f(x) = f(x_0, x_1, \ldots, x_{m-1}) = \sum_{p=0}^{2^m-1} c_p \prod_{l=0}^{m-1} x_l^{i_l},
\]

(5)

where sequence \( x = (x_0, x_1, \ldots, x_{m-1}) \), \( c_p \in \mathbb{Z}_q \), and \( i_l \in \mathbb{Z}_2 \) (\( l = 0, 1, \ldots, m-1 \)) can be obtained from a binary representation of \( p = \sum_{l=0}^{m-1} i_l 2^l \).

A generalized Boolean function can also be represented by sequences of length \( 2^m \). The \( \mathbb{Z}_q \)-valued sequence associated with \( f \) is defined as

\[
    \psi(f) = (f_0, f_1, \ldots, f_{2^m-1}),
\]

(6)

where \( f_i = f(i_0, i_1, \ldots, i_{m-1}) \), and \( (i_0, i_1, \ldots, i_{m-1}) \) is the binary representation of \( 0 \leq i \leq 2^m - 1 \). Moreover, the sequence \( \Psi(f) \) is defined as the polyphase sequence associated with \( f \) as

\[
    \Psi(f) = (\zeta^{f_0}, \zeta^{f_1}, \ldots, \zeta^{f_{2^m-1}}),
\]

(7)

Let \( f : \mathbb{Z}_2^k \rightarrow \mathbb{Z}_q \) be a generalized Boolean function. The sequence \( \phi(f) \) of length \( (4^k + 2)/3 \) is defined as

\[
    \phi(f) = \begin{cases} 
    \zeta^{f(x_0, x_1, \ldots, x_{k-1})}, & \text{at position } \sum_{a=0}^{k-1} x_a 2^{2a}, \\
    0, & \text{otherwise}, 
\end{cases}
\]

(8)

where \( x_0, x_1, \ldots, x_{k-1} \) range over \( \mathbb{Z}_2^k \).

**C. Golay Complementary Sequences**

*Definition 5 (Polyphase Sequence):* For a complex sequence \( A = (A_0, A_1, \ldots, A_{n-1}) \), if \( A_i = \zeta^{a_i} \), and \( a_i \in \mathbb{Z}_q \) for \( i = 0, 1, \ldots, n-1 \), then \( A \) is called a polyphase sequence.
Definition 6 (Aperiodic Auto-Correlation Function): The aperiodic auto-correlation function of sequence \( A = (A_0, A_1, \ldots, A_{n-1}) \) is defined as

\[
C_A(u) = \begin{cases} 
\sum_{i=0}^{n-1-u} A_i A_i^*, & 0 \leq u < n, \\
\sum_{i=0}^{n-1} A_{i-u} A_i^*, & -n < u < 0, \\
0, & \text{otherwise.}
\end{cases}
\] (9)

Note that, \( C_A(u) \) is conjugate-symmetric, i.e., \( C_A(-u) = C_A^*(u) \), and \( C_A^*(u) \) is the complex conjugate of \( C_A(u) \).

Definition 7 (Operator \( \star \)): For two sequences \( A, B \in \mathbb{C}^n \), the operator \( \star \) is defined as

\[
A \star B = \sum_{u=1-n}^{n-1} |C_A(u) + C_B(u)|.
\] (10)

If \( A \) and \( B \) are polyphase sequences of length \( n \), we have

\[
A \star B = 2 \sum_{u=0}^{n-1} |C_A(u) + C_B(u)|
= 2n + 2 \sum_{u=1}^{n-1} |C_A(u) + C_B(u)|.
\] (11)

Definition 8 (Golay Complementary Sequence): If \( A \star B = 2n \) (i.e., \( C_A(u) + C_B(u) = 0 \) for all \( u \neq 0 \)), the pair of polyphase sequences \( (A, B) \) is called a Golay complementary pair, and the sequence \( A \) or \( B \) is called a Golay complementary sequence.

According to (2), (4) and Theorem 2 in [7], the PMEPR of \( A \) and \( B \) is at most \( A \star B \), i.e.,

\[
PMEPR(A) \leq (A \star B)/n.
\] (12)

When \( A \) is a polyphase Golay complementary sequence, then \( C_A(u) + C_B(u) = 0 \) for all \( u \neq 0 \). Substituting (11) into (12), we can obtain

\[
PMEPR(A) \leq (A \star B)/n = 2.
\] (13)

Therefore, the PMEPR of Golay complementary sequences is at most 2, which provides significant PMEPR reduction.

D. Near-Complementary Sequences

Definition 9 (Near-Complementary Sequence): We call a pair of polyphase sequences \( (A, B) \) as a near-complementary pair if \( 2n \leq A \star B \ll 2n^2 \), and sequence \( A \) or \( B \) is called as a near-complementary sequence [7].
E. Decompositions of QAM Symbols

To our best knowledge, any 16-QAM symbol can be decomposed into a pair of quadrature phase shift keying (QPSK) symbols \([10]\), because any point on the 16-QAM constellation can be written as

\[
s(u, v) = \gamma(r_1\zeta^u + r_2\zeta^v),
\]

where \(\gamma = e^{j\pi/4}, \zeta = e^{j\pi/2} = j\), \(u, v \in \mathbb{Z}_4\), \(r_1 = 2/\sqrt{5}\) and \(r_2 = 1/\sqrt{5}\) are required for the constellation to have unit average energy.

Similarly, a 64-QAM symbol could be constructed with three QPSK symbols as \([11]\)

\[
q(u, v, w) = \gamma(a_1\zeta^u + a_2\zeta^v + a_3\zeta^w),
\]

where \(u, v, w \in \mathbb{Z}_4\). To maintain its average squared magnitude, \(a_1 = \frac{4}{\sqrt{21}}, a_2 = \frac{2}{\sqrt{21}}, a_3 = \frac{1}{\sqrt{21}}\).

III. MAIN RESULTS

A. Proposed Construction of 16-QAM Near-Complementary Sequences

In this subsection, we propose a novel construction of 16-QAM near-complementary sequences in Theorem 1.

**Theorem 1**: Suppose \(m > 2\) and let \(D, E, D', E' : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_4\), be four generalized Boolean functions of variable \(x = (x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(m-1)}) \in \mathbb{Z}_2^m\), where \(\{\pi(0), \pi(1), \ldots, \pi(m-1)\}\) denotes an arbitrary permutation of the set \(\{0, 1, \ldots, m-1\}\)\. Let

\[
D(x) = 2\sum_{l=0}^{m-2} x_{\pi(l)}x_{\pi(l+1)} + \sum_{l=0}^{m-1} c_l x_{\pi(l)} + c,
\]

\[
E(x) = D(x) + 2x_{\pi(0)}x_{\pi(1)} + d_1 x_{\pi(0)} + d_2 x_{\pi(1)} + d_3,
\]

\[
D'(x) = D(x) + 2x_{\pi(m-1)},
\]

\[
E'(x) = E(x) + 2x_{\pi(m-1)},
\]

where \(c_l, c \in \mathbb{Z}_4\), and \(d_1 + 2d_3 = 2, 2d_2 = 2, d_1, d_2, d_3 \in \mathbb{Z}_4\). Define the offset \(s(x)\) as

\[
s(x) = E(x) - D(x) = 2x_{\pi(0)}x_{\pi(1)} + d_1 x_{\pi(0)} + d_2 x_{\pi(1)} + d_3.
\]

Then, the proposed 16-QAM sequence

\[
H(x) = \gamma(r_1\zeta^D(x) + r_2\zeta^{E(x)})
\]
is a near-complementary sequence and $\text{PMEPR}(H) \leq 2.4$, where $\gamma = e^{j\pi/4}$, $\zeta = e^{j\pi/2} = j$, $r_1 = 2/\sqrt{5}$ and $r_2 = 1/\sqrt{5}$.

\textbf{Proof}: See APPENDIX I.

\textbf{Corollary 1}: The family size of the construction in \textit{Theorem 1} is $8 \times (\frac{m!}{2}) \times 4^{m+1}$, and the length of the sequences is $n = 2^m$.

\textbf{Proof}: It can be seen in \textit{Theorem 1} that the number of the offsets is 8. According to [5] and \textit{Theorem 1}, we can calculate the family size of this construction to be $8 \times (\frac{m!}{2}) \times 4^{m+1}$, and the length of the sequences $n = 2^m$.

Then, we give an example of \textit{Theorem 1} as follows.

\textbf{Example 1}: Let $m = 3$, $d_1 = 0$, $d_2 = 1$ and $d_3 = 1$, $D(x) = \{0, 1, 1, 0, 1, 2, 0, 3\}$, $E(x) = \{1, 2, 3, 2, 2, 3, 0, 3\}$, and $H(x) = \{\frac{1}{\sqrt{10}}(-1+3j), \frac{1}{\sqrt{10}}(-3-j), \frac{1}{\sqrt{10}}(1-j), \frac{1}{\sqrt{10}}(-1-j), \frac{1}{\sqrt{10}}(-3-j), \frac{1}{\sqrt{10}}(1-3j), \frac{1}{\sqrt{10}}(3+3j), \frac{1}{\sqrt{10}}(3-3j)\}$. Thus, we can obtain that $\text{PMEPR}(H) \approx 2.1 \leq 2.4$.

\noindent \textbf{B. Proposed Constructions of 64-QAM Near-Complementary Sequences}

Based on the discussion in Subsection \textbf{III-A} we propose a novel construction of 64-QAM near-complementary sequences in this subsection, which is presented in \textit{Theorem 2}.

\textit{Theorem 2}: Suppose $m > 2$, and denote $D, F, G, D', F', G' : Z_2^m \rightarrow Z_4$ as six generalized Boolean functions of variable $\underline{x} = (x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(m-1)}) \in Z_2^m$, where $\{\pi(0), \pi(1), \ldots, \pi(m-1)\}$ denotes an arbitrary permutation of the set $\{0,1,\ldots,m-1\}$. Let

$$D(\underline{x}) = 2 \sum_{l=0}^{m-2} x_{\pi(l)}x_{\pi(l+1)} + \sum_{l=0}^{m-1} c_l x_{\pi(l)} + c,$$

$$F(\underline{x}) = D(\underline{x}) + s^{(1)}(\underline{x}),$$

$$G(\underline{x}) = D(\underline{x}) + s^{(2)}(\underline{x}),$$

$$D'(\underline{x}) = D(\underline{x}) + 2x_{\pi(m-1)},$$

$$F'(\underline{x}) = F(\underline{x}) + 2x_{\pi(m-1)},$$

$$G'(\underline{x}) = G(\underline{x}) + 2x_{\pi(m-1)},$$

where $c_k \in Z_4$, $c \in Z_4$. Then, the 64-QAM sequence

$$J(\underline{x}) = \gamma (a_1 \zeta^{D(\underline{x})} + a_2 \zeta^{F(\underline{x})} + a_3 \zeta^{G(\underline{x})})$$

(20)
is a near-complementary sequence, where $\gamma = e^{i\pi/4}$, $\zeta = e^{j\pi/2} = j$, $a_1 = \frac{4}{\sqrt{21}}$, $a_2 = \frac{2}{\sqrt{21}}$, $a_3 = \frac{1}{\sqrt{21}}$ for each one of the following offset pairs.

**Type 1:**

$$s^{(1)}(x) = h_1x_{\pi(0)} + h_3,$$
$$s^{(2)}(x) = 2x_{\pi(0)}x_{\pi(1)} + d_1x_{\pi(0)} + d_2x_{\pi(0)} + d_3,$$

for $h_1 + 2h_3 = 0, h_1, h_3 \in Z_4$.

**Type 2:**

$$s^{(1)}(x) = 2x_{\pi(0)}x_{\pi(1)} + d_1x_{\pi(0)} + d_2x_{\pi(1)} + d_3,$$
$$s^{(2)}(x) = 2x_{\pi(0)}x_{\pi(1)} + h_1x_{\pi(0)} + h_2x_{\pi(1)} + h_3,$$

for $h_2 = d_2 + 2, h_1 + 2h_3 = 2, h_1, h_2, h_3 \in Z_4$.

Denote PMEPR$^1$ and PMEPR$^2$ as the PMEPR of 64-QAM sequences of Type 1 and Type 2, respectively, then, PMEPR$^1 \leq 3.62$ and PMEPR$^2 \leq 2.48$.

**Proof of Theorem 2:** See APPENDIX II.

**Corollary 2:** The family size of the 64-QAM near-complementary sequences in Theorem 2 is $64 \times \frac{m!}{2} \times 4^{m+1}$, where the length of the sequences is $2^m$.

**Proof:** It is obvious that 64 offsets can be constructed from Type 1 and Type 2. According to [5], the family size of the 64-QAM near-complementary sequences proposed in Theorem 2 is $64 \times \frac{m!}{2} \times 4^{m+1}$.

An example of Theorem 2 is given as follows.

**Example 2:** Let $m = 3, d_1 = 0, d_2 = 1, d_3 = 1, h_1 = 0$ and $h_3 = 0$, which are included in Type 2. Therefore, $D(x) = \{0, 1, 1, 0, 1, 2, 0, 3\}$, $E(x) = \{0, 1, 1, 0, 1, 2, 0, 3\}$, $F(x) = \{1, 2, 3, 2, 2, 3, 0, 3\}$, and $H(x) = \{\frac{1}{\sqrt{42}}(5+7j), \frac{1}{\sqrt{42}}(-7+5j), \frac{1}{\sqrt{42}}(-5+5j), \frac{1}{\sqrt{42}}(5+5j), \frac{1}{\sqrt{42}}(-7+5j), \frac{1}{\sqrt{42}}(-5-7j), \frac{1}{\sqrt{42}}(7+7j), \frac{1}{\sqrt{42}}(7-7j)\}$. Thus, we can obtain that PMEPR(H) $\approx 3.5 \leq 3.62$.

**IV. NUMERICAL RESULTS**

**A. PMEPR Reduction**

As discussed in Section III the PMEPR of the proposed 16-QAM near-complementary sequences has been proven to satisfy that PMEPR $\leq 2.4$. Moreover, the PMEPRs of the two types of the proposed 64-QAM near-complementary sequences satisfy that PMEPR$^1 \leq 3.62$ and PMEPR$^2 \leq 2.48$, respectively. In this subsection, some numerical results have been conducted.
to evaluate the PMEPR reduction performances of the proposed 16-QAM near-complementary sequences and the proposed 64-QAM near-complementary sequences. The complementary cumulative distribution function (CCDF) is employed in the simulations to measure the PMEPR reduction performances of the proposed 16-QAM and 64-QAM near-complementary sequences. The CCDF is defined as the probability that the PMEPR exceeds a given threshold $\text{PMEPR}_0$, i.e.,

$$\text{CCDF} = Pr\{\text{PMEPR} > \text{PMEPR}_0\}.$$  \hspace{1cm} (23)

Fig. 1 depicts the PMEPR reduction performance of the proposed 16-QAM near-complementary sequences with $m = 4$ and $n = 16$. The “Original” curve shows the PMEPR reduction performance of the conventional OFDM signals without PMEPR reduction. Seen from Fig. 1, the PMEPR of the proposed 16-QAM near-complementary sequences is less than or equal to 2.4, which is consistent with our proof in Section III. Therefore, the proposed 16-QAM near-complementary sequences can significantly reduce the PMEPR in OFDM systems.

Fig. 2 shows the PMEPR reduction performance of the proposed 64-QAM near-complementary sequences with $m = 4$ and $n = 16$. The “Original” curve shows the PMEPR reduction performance of the conventional OFDM signals without PMEPR reduction. Seen from Fig. 2, the proposed 64-QAM near-complementary sequences in Type 1 satisfy that $\text{PMEPR}^1 \leq 3.62$, and
the proposed 64-QAM near-complementary sequences in Type 2 satisfy that \( \text{PMEPR}^2 \leq 2.48 \).

The simulation results are consistent with the discussion in Section III. Therefore, the proposed 64-QAM near-complementary sequences can significantly reduce the PMEPR in OFDM systems.

In summary, the proposed 16-QAM and 64-QAM near-complementary sequences both offer significant PMEPR reductions in OFDM systems.

### B. Family Size

In this subsection, some numerical results have been presented in TABLE I and TABLE II to show family sizes of the proposed 16-QAM and 64-QAM near-complementary sequences. For comparison, we also present the family size of the 16-QAM near-complementary sequences proposed in [12]. Furthermore, since the proposed construction of 64-QAM near-complementary sequences is the first proven construction, we cannot compare it with any existing 64-QAM near-complementary sequences. In the numerical results, the length of the sequences is \( n = 2^m \), where \( m \) is a positive integer and \( m > 2 \). Moreover, we denote \( N^1_{16} \), \( N^2_{16} \), and \( N^1_{64} \) as the numbers of the proposed 16-QAM near-complementary sequences, the 16-QAM near-complementary sequences in [12], and the proposed 64-QAM near-complementary sequences, respectively.

As shown in TABLE II, it is obvious that the number of the proposed 16-QAM near-complementary sequences is larger than the 16-QAM near-complementary in [12]. For example, when \( m = 4 \)
TABLE I: NUMBERS OF 16-QAM SEQUENCES

| Numbers      | The length of sequences               |
|--------------|---------------------------------------|
| $N_{16}^1$   | $m = 3, n = 8$                        |
| $N_{16}^2$   | $m = 4, n = 16$                       |
|              | $n = 2^m$                            |
|              | $6144$                                |
|              | $98304$                               |
|              | $8 \times \left(\frac{m!}{2}\right) \times 4^{m+1}$ |

and $n = 16$, the numbers of the proposed 16-QAM near-complementary sequences and the 16-QAM near-complementary sequences in [12] are 98304 and 12288, respectively. Thus, the proposed construction of the 16-QAM near-complementary sequences can enlarge the family size of near-complementary sequences, resulting in more near-complementary sequences with low PMEPR. Therefore, more near-complementary sequences with low PMEPR can be utilized in OFDM systems, resulting in the improvement of the code rate in OFDM systems.

TABLE II: NUMBERS OF 64-QAM SEQUENCES

| Numbers      | The length of sequences               |
|--------------|---------------------------------------|
| $N_{64}^1$   | $m = 3, n = 8$                        |
| $N_{64}^2$   | $m = 123, n = 2^{123}$               |
|              | $n = 2^m$                            |
|              | $49152$                               |
|              | $64 \times \frac{123!}{2} \times 4^{124}$ |
|              | $64 \times \frac{m!}{2} \times 4^{m+1}$ |

We present the number of the proposed 64-QAM near-complementary sequences in TABLE II. As shown in TABLE II, the family size of the proposed 64-QAM near-complementary sequences is $64\left(\frac{m!}{2}\right)4^{m+1}$. For example, the number of the proposed 64-QAM near-complementary sequences is 49152 when $m = 3$ and $n = 8$. Therefore, the proposed construction of 64-QAM near-complementary sequences is of great value, and it can be employed to control the PMEPR in OFDM systems.

V. CONCLUSIONS

In this paper, a novel construction of 16-QAM near-complementary sequences was proposed to reduce the PMEPR in OFDM systems. The family size of the newly constructed 16-QAM near-complementary sequences is $8 \times \left(\frac{m!}{2}\right) \times 4^{m+1}$, and the PMEPR of the sequences is bounded by 2.4. Moreover, a construction of 64-QAM near-complementary sequences was also proposed in this paper, which is the first proven construction of 64-QAM near-complementary sequences. The
family size of the newly constructed 64-QAM near-complementary sequences is $64 \times \left(\frac{m!}{2} \right) \times 4^{m+1}$, and the PMEPR of the sequences is bounded by 2.48 or 3.62. Therefore, the proposed 16-QAM and 64-QAM near-complementary sequences offer significant PMEPR reduction.

APPENDIX I

The proof of Theorem 1 consists of two steps: (1) To prove that the proposed 16-QAM sequence $H(x) = \gamma(r_1 \zeta^{D(x)} + r_2 \zeta^{E(x)})$ is a near-complementary sequence; (2) To prove that the PMEPR upper bound of $H(x)$ satisfies that $\text{PMEPR}(H) \leq 2.4$.

Then let us start the proof of Theorem 1 as follows.

Firstly, let $i = (i_0, i_1, ..., i_{m-1})$ be the binary representation of $i$, i.e., $i = \sum_{k=0}^{m-1} i_k 2^{m-k}$. Then, recall (16) and (17), we have

$$D(x) = 2 \sum_{l=0}^{m-2} x_{\pi(l)}x_{\pi(l+1)} + \sum_{l=0}^{m-1} c_l x_{\pi(l)} + c.$$  

$$E(x) = D(x) + 2x_{\pi(0)}x_{\pi(1)} + d_1 x_{\pi(0)} + d_2 x_{\pi(1)} + d_3; \quad (24)$$

$$D'(x) = D(x) + 2x_{\pi(m-1)};$$

$$E'(x) = E(x) + 2x_{\pi(m-1)},$$

and the offset $s(x)$ is expressed as

$$s(x) = E(x) - D(x) = 2x_{\pi(0)}x_{\pi(1)} + d_1 x_{\pi(0)} + d_2 x_{\pi(1)} + d_3. \quad (25)$$

Then, we define $H'(x)$ as follows:

$$H'(x) = \gamma(r_1 \zeta^{D'(x)} + r_2 \zeta^{E'(x)}). \quad (26)$$

Substituting (18) into (9), the aperiodic auto-correlation function of the proposed 16-QAM sequence $H(x)$ is

$$C_H(u) = \sum_{i=0}^{n-u-1} \left( r_1 \zeta^{D_i} + r_2 \zeta^{E_i} \right) \left( r_1 \zeta^{D_{i+u}} + r_2 \zeta^{E_{i+u}} \right)^*,$$

$$= \sum_{i=0}^{n-u-1} \left[ r_1^2 \zeta^{D_i-D_{i+u}} + r_2^2 \zeta^{E_i-E_{i+u}} + r_1 r_2 \zeta^{D_i-E_{i+u}+D_{i+u}-E_i} \right. \left. + r_1 r_2 \zeta^{E_i-D_{i+u}} \right]. \quad (27)$$
Similarly, substituting (26) into (9), the aperiodic auto-correlation function of $H(x)$ is

$$C_{H'}(u) = \sum_{i=0}^{n-u-1} (r_1\xi^{D_i'} + r_2\xi^{E_i'}) (r_1\xi^{D_{i+u}'} + r_2\xi^{E_{i+u}'})^*$$

$$= \sum_{i=0}^{n-u-1} \left[ r_1^2\xi^{D_i'-D_{i+u}'} + r_2^2\xi^{E_i'-E_{i+u}'} + r_1r_2\xi^{D_i'-E_{i+u}'} + r_1r_2\xi^{E_i'-D_{i+u}'} \right].$$

(28)

According to (27) and (28), we have

$$C_H(u) + C_{H'}(u) = \sum_{i=0}^{n-u-1} \left\{ r_1^2[\xi^{D_i-D_{i+u}} + \xi^{D_i'-D_{i+u}'}] + r_2^2[\xi^{E_i-E_{i+u}} + \xi^{E_i'-E_{i+u}'}] \right\} + r_1r_2[\xi^{D_i-E_{i+u}} + \xi^{D_i'-E_{i+u}'} + \xi^{E_i-D_{i+u}} + \xi^{E_i'-D_{i+u}'}].$$

(29)

With (24)~(25), we have

$$r_1r_2\xi^{D_i-E_{i+u}} + r_1r_2\xi^{D_i'-E_{i+u}'} = r_1r_2\xi^{D_i-D_{i+u}}(\xi^{s_i} + \xi^{-s_{i+u}}),$$

$$r_1r_2\xi^{D_i'-E_{i+u}'} + r_1r_2\xi^{E_i-D_{i+u}}$$

$$= r_1r_2\xi^{D_i-D_{i+u}} \times 2^{2(i-u)\pi(m-1)} \times (\xi^{s_i} + \xi^{-s_{i+u}}),$$

$$r_1r_2\xi^{D_i-E_{i+u}} + r_1r_2\xi^{E_i-D_{i+u}}$$

$$+ r_1r_2\xi^{D_i'-E_{i+u}'} + r_1r_2\xi^{E_i'-D_{i+u}'}$$

$$= r_1r_2\xi^{D_i-D_{i+u}}(\xi^{s_i} + \xi^{-s_{i+u}})$$

$$\times [1 + (-1)^{i\pi(m-1)-(i+u)\pi(m-1)}].$$

where $(i+u)\pi(0), (i+u)\pi(1), \ldots, (i+u)\pi(m-1)$ is the binary representation of $i+u$.

Therefore, (29) can be rewritten as,

$$C_H(u) + C_{H'}(u) = \sum_{i=0}^{n-u-1} \left\{ r_1^2[\xi^{D_i-D_{i+u}} + \xi^{D_i'-D_{i+u}'}] + r_2^2[\xi^{E_i-E_{i+u}} + \xi^{E_i'-E_{i+u}'}] \right\} + r_1r_2[\xi^{D_i-E_{i+u}} + \xi^{D_i'-E_{i+u}'} + \xi^{E_i-D_{i+u}} + \xi^{E_i'-D_{i+u}'}].$$

(30)

To prove Theorem 1, the following lemma is needed.

**Lemma 1:** We can obtain that

$$\sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \xi^{D_i-D_{i+u}}(\xi^{s_i} + \xi^{-s_{i+u}}) \times [1 + (-1)^{i\pi(m-1)-(i+u)\pi(m-1)}] = 0.$$
Proof: Firstly, we consider $u > 0$, let $k = i + u$ have a binary representation $k = (k_0, k_1, ..., k_{m-1})$.

Case 1.1: $i_{\pi(m-1)} \neq k_{\pi(m-1)}$. Obviously, $\zeta D_i - D_k \times [1 + (-1)^{i_{\pi(m-1)} - k_{\pi(m-1)}}] [\zeta^{-s_k} + \zeta^{s_i}] = 0$.

Case 1.2: $i_{\pi(m-1)} = k_{\pi(m-1)}$. (31) can be rewritten as $2 \times \sum_{i=1}^{n-1} \sum_{k=1}^{n-1} \zeta^{(D_i - D_k)} (\zeta^{s_i} + \zeta^{-s_k}) = 0$.

Let $v$ denote the biggest index for which $i_{\pi(v)} \neq k_{\pi(v)}$, where $1 \leq v \leq m - 2$. Let $i'$ and $k'$ be the integers whose binary representations differ from those of $i$ and $k$ only at position $\pi(v+1)$, respectively, i.e.,

$$i' = (i_0, i_1, ..., 1 - i_{\pi(v+1)}, ..., i_{\pi(m-1)}),$$

$$k' = (k_0, k_1, ..., 1 - k_{\pi(v+1)}, ..., k_{\pi(m-1)}).$$

Due to $i_{\pi(v+1)} = k_{\pi(v+1)}$, we have $k' = i' + u$. Therefore, we define an invertible mapping $(i, k) \rightarrow (i', k')$.

According to (24) and the definition of $v$, we have

$$D_i - D_k = 2 \sum_{l=0}^{m-2} [i_{\pi(l)} i_{\pi(l+1)} - k_{\pi(l)} k_{\pi(l+1)}] + \sum_{l=0}^{v} c_{\pi(l)} (i_{\pi(l)} - k_{\pi(l)}),$$

$$D_{i'} - D_{k'} = 2 \sum_{l=0}^{m-2} [i'_{\pi(l)} i'_{\pi(l+1)} - k'_{\pi(l)} k'_{\pi(l+1)}] + \sum_{l=0}^{v} c_{\pi(l)} (i'_{\pi(l)} - k'_{\pi(l)}),$$

$$(D_i - D_k) - (D_{i'} - D_{k'})$$

$$= \sum_{l=0}^{m-2} [i_{\pi(l)} i_{\pi(l+1)} - i'_{\pi(l)} i'_{\pi(l+1)}] - \sum_{l=0}^{m-2} [k_{\pi(l)} k_{\pi(l+1)} - k'_{\pi(l)} k'_{\pi(l+1)}] + \sum_{l=0}^{v} c_{\pi(l)} (i_{\pi(l)} - i'_{\pi(l)}) - \sum_{l=0}^{v} c_{\pi(l)} (k_{\pi(l)} - k'_{\pi(l)})$$

$$= 2[i_{\pi(v)} i_{\pi(v+1)} - i'_{\pi(v)} i'_{\pi(v+1)}] - 2[k_{\pi(v)} k_{\pi(v+1)} - k'_{\pi(v)} k'_{\pi(v+1)}]$$

$$= 2i_{\pi(v)} - 2k_{\pi(v)} = 2.$$

Therefore,

$$\zeta D_{i'} - D_{k'} = -\zeta D_i - D_k.$$

If $v \geq 1$, according to the definition of $v$, we have $s_i = s'_{i'}$ and $s_k = s'_{k'}$. Hence,

$$\zeta^{s_i} + \zeta^{-s_k} = \zeta^{s'_{i'}} + \zeta^{-s'_{k'}}.$$
\[ \zeta^{D_{i'}-D_{i'}} (\zeta^{s_{i'}} + \zeta^{-s_{k'}}) + \zeta^{D_i-D_k} (\zeta^{s_i} + \zeta^{-s_k}) = 0. \]

If \( v = 0 \), we have \( i\pi(0) = i\pi(0) \neq k\pi(0) = k\pi(0) \) and \( i\pi(1) = k\pi(1) \neq i\pi'(1) = k\pi'(1). \)

(1) \( i\pi(1) = k\pi(1) = 0 \Rightarrow i\pi'(1) = k\pi'(1) = 1 \). And \( s_i = d_1i\pi(0) + d_3, s_k = d_1k\pi(0) + d_3 \), and \( s_{i'} = 2i\pi(0) + d_1i\pi(0) + d_2 + d_3, s_{k'} = 2k\pi(0) + d_1k\pi(0) + d_2 + d_3 \). According to the definition of Theorem 1, we have

\[ s_i + s_k = d_1 + 2d_3 = 2, \]
\[ s_{i'} + s_{k'} = 2 + d_1 + 2d_2 + 2d_3 = 2, \]
\[ \zeta^{s_i} + \zeta^{-s_k} = \zeta^{s_{i'}} + \zeta^{-s_{k'}} = 0, \]
\[ \zeta^{D_{i'}-D_{i'}} (\zeta^{s_{i'}} + \zeta^{-s_{k'}}) + \zeta^{D_i-D_k} (\zeta^{s_i} + \zeta^{-s_k}) = 0. \]

(2) \( i\pi(1) = k\pi(1) = 1 \Rightarrow i\pi'(1) = k\pi'(1) = 0 \). Similarly,

\[ s_i + s_k = d_1 + 2d_3 = 2, \]
\[ s_{i'} + s_{k'} = 2 + d_1 + 2d_2 + 2d_3 = 2, \]
\[ \zeta^{s_i} + \zeta^{-s_k} = \zeta^{s_{i'}} + \zeta^{-s_{k'}} = 0, \]
\[ \zeta^{D_{i'}-D_{i'}} (\zeta^{s_{i'}} + \zeta^{-s_{k'}}) + \zeta^{D_i-D_k} (\zeta^{s_i} + \zeta^{-s_k}) = 0. \]

Then, considering the situation of \( u = 0 \), (31) can be rewritten as \( 2 \times \sum_{i=0}^{n-1} 2 \times (\zeta^{s_i} + \zeta^{-s_i}) = 0. \)

Let \( i' \) be the integer whose binary representation is \( (1 - i\pi(0), i\pi(1), \ldots, i\pi(m-1)) \), now an invertible mapping \( i \rightarrow i' \) is defined, and \( \sum_{i=0}^{n-1} (\zeta^{s_i} + \zeta^{-s_i}) = \sum_{i=0}^{n-1} (\zeta^{s_{i'}} + \zeta^{-s_{i'}}) \). According to Theorem 1,

\[ s_i + s_{i'} = 2, \]

whenever \( i\pi(1) = 1 \) or \( i\pi(1) = 0 \). Therefore,

\[ \zeta^{s_i} + \zeta^{-s_i} + \zeta^{s_{i'}} + \zeta^{-s_{i'}} = 0. \]

Since we have the mappings \( (i, k) \rightarrow (i', k') \) and \( i \rightarrow i' \) is invertible, the term is equal to zero in Case 1.1, and it sums to zero in pairs in Case 1.2. Then we can have

\[ \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{(D_i-D_{i+u})} (\zeta^{s_i} + \zeta^{-s_{i+u}}) \]
\[ \times [1 + (-1)^{s_{(m-1)-(i+u)s_{(m-1)}}}] = 0. \]

Therefore, the proof of Lemma 1 is complete.

Now we are ready to give the proof of the Theorem 1.
Proof of Theorem 1: According to (30), we can obtain that

$$H \ast H' = \sum_{u=1-n}^{n-1} |C_H(u) + C_{H'}(u)|$$

$$= \sum_{u=0}^{n-1} \sum_{i=0}^{n-u-1} \{ r_1^2[\zeta^{D_i-D_{i+u}} + \zeta^{D_{i}'-D_{i+u}'}] \}$$

$$+ r_2^2[\zeta^{E_i-E_{i+u}} + \zeta^{E_{i}'-E_{i+u}'}]$$

$$+ r_1 r_2 \zeta^{D_i-D_{i+u}'} (\zeta^{s_i} + \zeta^{-s_{i+u}})$$

$$\times [1 + (-1)^{s_{(m-1)}-(i+u)s_{(m-1)}}] \}$$

(32)

$$\leq \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \{ r_1^2[\zeta^{D_i-D_{i+u}} + \zeta^{D_{i}'-D_{i+u}'}] \}$$

$$+ \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \{ r_2^2[\zeta^{E_i-E_{i+u}} + \zeta^{E_{i}'-E_{i+u}'}] \}$$

$$+ r_1 r_2 \times \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} |\zeta^{D_i-D_{i+u}} (\zeta^{s_i} + \zeta^{-s_{i+u}})$$

$$\times [1 + (-1)^{s_{(m-1)}-(i+u)s_{(m-1)}}] \}.$$

According to Lemma 1, the term $r_1 r_2 \times \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} |\zeta^{D_i-D_{i+u}} (\zeta^{s_i} + \zeta^{-s_{i+u}})$$

$$\times [1 + (-1)^{s_{(m-1)}-(i+u)s_{(m-1)}}] \} = 0,$$ thus, (32) can be rewritten as

$$H \ast H' \leq \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \{ r_1^2[\zeta^{D_i-D_{i+u}} + \zeta^{D_{i}'-D_{i+u}'}] \}$$

$$+ \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \{ r_2^2[\zeta^{E_i-E_{i+u}} + \zeta^{E_{i}'-E_{i+u}'}] \}$$

(33)

$$\equiv r_1^2 D \star D' + r_2^2 E \star E'.$$

According to [5], it is easy to verify that the sequence pair $(D, D')$ is a Golay complementary pair, and the sequence pair $(E, E')$ is a near-complementary pair. Thus, the PMEPR of sequence D is at most 2 (according to (13)), while the PMEPR of sequence E is at most 4 (Theorem 10 in [7] and Corollary 1 in [12]), i.e.,

$$D \star D' \leq 2n,$$

$$E \star E' \leq 4n.$$

(34)
Substituting (34) into (33), we can obtain that
\[
H \ast H' \leq 2r_1^2 n + 4r_2^2 n
\]
\[
= 0.8 \times 2n + 0.2 \times 4n
\]
\[
= 2.4n \ll 2n^2. \tag{35}
\]
Therefore, the proposed 16-QAM sequence \(H\) is a near-complementary sequence because \(2n \leq H \ast H' \ll 2n^2\).

Moreover, according to (12) and (35), the PMEPR of the sequence \(H\) is
\[
PMEPR(H) \leq (H \ast H')/n \leq 2.4. \tag{36}
\]
Therefore, the proof of Theorem 1 is complete.

APPENDIX II

The Proof of Theorem 2

Firstly, we recall (19) and (20), i.e.,
\[
D(x) = 2 \sum_{l=0}^{m-2} x_{\pi(l)}x_{\pi(l+1)} + \sum_{l=0}^{m-1} c_l x_{\pi(l)} + c.
\]
\[
F(x) = D(x) + s^{(1)}(x),
\]
\[
G(x) = D(x) + s^{(2)}(x),
\]
\[
D'(x) = D(x) + 2x_{\pi(m-1)},
\]
\[
F'(x) = F(x) + 2x_{\pi(m-1)},
\]
\[
G'(x) = G(x) + 2x_{\pi(m-1)},
\]
\[
J(x) = \gamma (a_1 \zeta^{D(x)} + a_2 \zeta^{F(x)} + a_3 \zeta^{G(x)}).
\]
Then, we define \(s^{(3)}(x)\) and \(J'(x)\) as follows
\[
s^{(3)}(x) = s^{(1)}(x) - s^{(2)}(x),
\]
\[
J'(x) = \gamma (a_1 \zeta^{D'(x)} + a_2 \zeta^{F'(x)} + a_3 \zeta^{G'(x)}). \tag{38}
\]
According to (9) and (37), the aperiodic auto-correlation function of $J(x)$ can be expressed as follows

$$C_J(u) = \sum_{i=1}^{n-u-1} (a_1 \zeta^D_i + a_2 \zeta^F_i + a_3 \zeta^G_i) \times (a_1 \zeta^{D_{i+u}} + a_2 \zeta^{F_{i+u}} + a_3 \zeta^{G_{i+u}})^*$$
$$= \sum_{i=0}^{n-u-1} \left[ a_1^2 \zeta^{D_{i+u} - D_{i+u}} + a_2^2 \zeta^{F_{i+u} - F_{i+u}} + a_3^2 \zeta^{G_{i+u} - G_{i+u}} \right] + a_1 a_2 (\zeta^{D_{i+u} - F_{i+u}} + \zeta^{F_{i+u} - D_{i+u}}) + a_1 a_3 (\zeta^{D_{i+u} - G_{i+u}} + \zeta^{G_{i+u} - D_{i+u}}) + a_2 a_3 (\zeta^{F_{i+u} - G_{i+u}} + \zeta^{G_{i+u} - F_{i+u}}) \tag{39}$$

Similarly, according to (9) and (38) the aperiodic auto-correlation function of $J'(x)'$ can be expressed as

$$C_{J'}(u) = \sum_{i=1}^{n-u-1} (a_1 \zeta'^D_i + a_2 \zeta'^F_i + a_3 \zeta'^G_i) \times (a_1 \zeta'^{D_{i+u}} + a_2 \zeta'^{F_{i+u}} + a_3 \zeta'^{G_{i+u}})^*$$
$$= \sum_{i=0}^{n-u-1} \left[ a_1^2 \zeta'^{D_{i+u} - D_{i+u}} + a_2^2 \zeta'^{F_{i+u} - F_{i+u}} + a_3^2 \zeta'^{G_{i+u} - G_{i+u}} \right] + a_1 a_2 (\zeta'^{D_{i+u} - F_{i+u}} + \zeta'^{F_{i+u} - D_{i+u}}) + a_1 a_3 (\zeta'^{D_{i+u} - G_{i+u}} + \zeta'^{G_{i+u} - D_{i+u}}) + a_2 a_3 (\zeta'^{F_{i+u} - G_{i+u}} + \zeta'^{G_{i+u} - F_{i+u}}) \tag{40}$$

Then, combine (39) and (40), we have

$$C_J(u) + C_{J'}(u) = \sum_{i=0}^{n-u-1} \left[ a_1^2 (\zeta^{D_{i+u} - D_{i+u}} + \zeta'^{D_{i+u} - D_{i+u}}) + a_2^2 (\zeta^{F_{i+u} - F_{i+u}} + \zeta'^{F_{i+u} - F_{i+u}}) + a_3^2 (\zeta^{G_{i+u} - G_{i+u}} + \zeta'^{G_{i+u} - G_{i+u}}) + a_1 a_2 (\zeta^{D_{i+u} - F_{i+u}} + \zeta'^{D_{i+u} - F_{i+u}}) + a_1 a_3 (\zeta^{D_{i+u} - G_{i+u}} + \zeta'^{D_{i+u} - G_{i+u}}) + a_2 a_3 (\zeta^{F_{i+u} - G_{i+u}} + \zeta'^{F_{i+u} - G_{i+u}}) \right] \tag{41}$$
The last three terms in (41) can be rewritten as follows

\[
\begin{align*}
a_1a_2(\zeta^{D_i-D_{i+u}} + \zeta^{F_i-D_{i+u}} + \zeta^{D_i'-D_{i+u}} + \zeta^{F_i'-D_{i+u}}) \\
= a_1a_2 \sum_{i=0}^{n-u-1} \zeta^{D_i-D_{i+u}} \\
\times [1 + (-1)^{(i+u)(m-1)}][\zeta^{s_i(1)} + \zeta^{s_i(1)}], \\
a_1a_3(\zeta^{D_i-G_{i+u}} + \zeta^{G_i-D_{i+u}} + \zeta^{D_i'-G_{i+u}} + \zeta^{G_i'-D_{i+u}}) \\
= a_1a_3 \sum_{i=0}^{n-u-1} \zeta^{D_i-D_{i+u}} \\
\times [1 + (-1)^{(i+u)(m-1)}][\zeta^{s_i(2)} + \zeta^{s_i(2)}], \\
a_2a_3(\zeta^{F_i-G_{i+u}} + \zeta^{G_i-F_{i+u}} + \zeta^{F_i'-G_{i+u}} + \zeta^{G_i'-F_{i+u}}) \\
= a_2a_3 \sum_{i=0}^{n-u-1} \zeta^{F_i-F_{i+u}} \\
\times [1 + (-1)^{(i+u)(m-1)}][\zeta^{s_i(3)} + \zeta^{s_i(3)}].
\end{align*}
\]

To prove Theorem 2, we should firstly prove Lemma 2 and Lemma 3.

**Lemma 2:** In Type 1, the following relationships can be obtained

\[
\begin{align*}
a_1a_2 \sum_{u=1}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{D_i-D_{i+u}} \\
\times [1 + (-1)^{(i+u)(m-1)}] \times (\zeta^{s_i(1)} + \zeta^{s_i(1)}) = 0, \\
a_1a_3 \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{D_i-D_{i+u}} \\
\times [1 + (-1)^{(i+u)(m-1)}] \times (\zeta^{s_i(2)} + \zeta^{s_i(2)}) = 0, \\
a_2a_3 \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{F_i-F_{i+u}} \\
\times [1 + (-1)^{(i+u)(m-1)}] \times (\zeta^{s_i(3)} + \zeta^{s_i(3)}) = 0.
\end{align*}
\]

**Proof:** According to Lemma 1, it is obvious that (46) is satisfied. The offset \(s^{(1)}(x)\) is in subset cases of \([10]\). Based on the proof in \([10]\) and \([13]\), we can prove that (45) is satisfied. Then we will prove (47).

Firstly, we consider \(u > 0\), let \(k = i + u\) have a binary representation \(k = (k_0, k_1, \ldots, k_{m-1})\).

**Case 2.1:** \(i_{\pi(m-1)} \neq k_{\pi(m-1)}\). Obviously, \(F_k \times [1 + (-1)^{(i+u)(m-1)}] \times (\zeta^{s_k^3} + \zeta^{s_{i}^3}) = 0.\)
**Case 2.2:** \( i_{\pi(v-1)} = k_{\pi(v-1)} \). \((47)\) can be rewritten as:

\[
\sum_{i=1}^{n-1} \sum_{i=1}^{n-1} |2 \times (F_i - F_k) (\zeta^{s_{i}^{(3)}} + \zeta^{-s_{k}^{(3)}})| = 0.
\]

Let \( v \) denote the biggest index for which \( i_{\pi(v)} \neq k_{\pi(v)} \), where \( v \leq m - 2 \). Denote \( i' \) and \( k' \) as the integers whose binary representations differ from those of \( i \) and \( k \) only at position \( \pi(v + 1) \), respectively, i.e.,

\[
i' = (i_0, i_1, ..., 1 - i_{\pi(v+1)}, ..., i_{\pi(m-1)}),
\]

\[
k' = (k_0, k_1, ..., 1 - k_{\pi(v+1)}, ..., k_{\pi(m-1)}),
\]

since \( i_{\pi(v+1)} = k_{\pi(v+1)} \), we can obtain \( k' = i' + u \). Then, we define an invertible mapping \( (i, k) \rightarrow (i', k') \).

In Type 1, we have

\[
F_i - F_k = 2 \sum_{l=0}^{m-2} [i_{\pi(l)}i_{\pi(l+1)} - k_{\pi(l)}k_{\pi(l+1)}] + \sum_{l=0}^{m} c_{\pi(l)}(i_{\pi(l)} - k_{\pi(l)}) + s_{i}^{(1)} - s_{k}^{(1)},
\]

\[
F_{i'} - F_{k'} = 2 \sum_{l=0}^{m-2} [i'_{\pi(l)}i'_{\pi(l+1)} - k'_{\pi(l)}k'_{\pi(l+1)}] + \sum_{l=0}^{m} c_{\pi(l)}(i'_{\pi(l)} - k'_{\pi(l)}) + s_{i'}^{(1)} - s_{k'}^{(1)},
\]

\[
(F_i - F_k) - (F_{i'} - F_{k'}) = 2[i_{\pi(v)}i_{\pi(v+1)} - i'_{\pi(v)}i'_{\pi(v+1)}] - 2[k_{\pi(v)}k_{\pi(v+1)} - k'_{\pi(v)}k'_{\pi(v+1)}] + s_{i}^{(1)} - s_{k}^{(1)} - s_{i'}^{(1)} + s_{k'}^{(1)} = 2 + s_{i}^{(1)} - s_{k}^{(1)} - s_{i'}^{(1)} + s_{k'}^{(1)}.
\]

According to \([10]\) and \([13]\), we have \( s_{i}^{(1)} - s_{k}^{(1)} - s_{i'}^{(1)} + s_{k'}^{(1)} = 0 \) for all \( u > 0 \). Therefore,

\[
\zeta^{F_{i'} - F_{k'}} = -\zeta^{F_i - F_k}.
\]

If \( v \geq 1 \), with the definition of \( v \), we obtain \( s_{i}^{(3)} = s_{i'}^{(3)} \) and \( s_{k}^{(3)} = s_{k'}^{(3)} \). Hence,

\[
\zeta^{s_{i}^{(3)}} + \zeta^{-s_{k}^{(3)}} = \zeta^{s_{i'}^{(3)}} + \zeta^{-s_{k'}^{(3)}},
\]

\[
\zeta^{F_{i'} - F_{k'}}(\zeta^{s_{i}^{(3)}} + \zeta^{-s_{k}^{(3)}}) + \zeta^{F_i - F_k}(\zeta^{s_{i'}^{(3)}} + \zeta^{-s_{k'}^{(3)}}) = 0.
\]

If \( v = 0 \), we have \( i'_{\pi(0)} = i_{\pi(0)} \neq k_{\pi(0)} = k'_{\pi(0)} \) and \( i_{\pi(1)} = k_{\pi(1)} \neq i'_{\pi(1)} = k'_{\pi(1)} \).
(1) \( i_{\pi(1)} = k_{\pi(1)} = 1 \Rightarrow i'_{\pi(1)} = k'_{\pi(1)} = 0 \). So 
\[ s_i^{(3)} = (h_1 - d_1)i_{\pi(0)} + h_3 - d_3, \]
\[ s_{k'}^{(3)} = (h_1 - d_1)k_{\pi(0)} + h_3 - d_3, \]
and 
\[ s_i^{(3)} = 2i_{\pi(0)}i_{\pi(1)} + (h_1 - d_1)i_{\pi(0)} - d_2 + h_3 - d_3, \]
\[ s_{k'}^{(3)} = 2k_{\pi(0)}k_{\pi(1)} + (h_1 - d_1)k_{\pi(0)} - d_2 + h_3 - d_3. \]
Moreover,
\[ s_i^{(3)} + s_{k'}^{(3)} = h_1 + 2h_3 - d_1 - 2d_3 = 2, \] (55)
\[ s_i^{(3)} + s_{k'}^{(3)} = 2 + h_1 + 2h_3 - d_1 - 2d_2 - 2d_3 = 2, \] (56)
\[ \zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}} = \zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}} = 0, \] (57)
\[ \zeta^{F_{i'}^{(3)}}F_k^{(3)}(\zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}}) = \zeta^{F_{i'}^{(3)}}F_k^{(3)}(\zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}}) = 0. \] (58)

(2) \( i_{\pi(1)} = k_{\pi(1)} = 0 \Rightarrow i'_{\pi(1)} = k'_{\pi(1)} = 1 \). Similarly, we have
\[ \zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}} = \zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}} = 0, \] (59)
\[ \zeta^{F_{i'}^{(3)}}F_k^{(3)}(\zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}}) + \zeta^{F_{i'}^{(3)}}F_k^{(3)}(\zeta^{s_i^{(3)}} + \zeta^{-s_{k'}^{(3)}}) = 0. \] (60)

Then, we consider the situation when \( u = 0 \). The term \(|(47)|\) can be rewritten as
\[ a_2a_3 \sum_{i=0}^{n-1} \left( \zeta^{s_i^{(3)}} + \zeta^{-s_i^{(3)}} \right) = 0. \]
Let \( i' \) be the integer whose binary representation is \((1-i_{\pi(0)}, i_{\pi(1)}, \ldots, i_{\pi(m-1)})\), an invertible mapping \( i \rightarrow i' \) is defined, \( \sum_{i=0}^{n-1} (\zeta^{s_i} + \zeta^{-s_i}) = \sum_{i=0}^{n-1} (\zeta^{s_{i'}} + \zeta^{-s_{i'}}) \).
According to Type 1,
\[ s_i^{(3)} + s_{i'}^{(3)} = 2, \] (61)
whenever \( i_{\pi(1)} = 1 \) or \( i_{\pi(1)} = 0 \). Therefore,
\[ \zeta^{s_i^{(3)}} + \zeta^{-s_{i'}^{(3)}} + \zeta^{s_{i'}^{(3)}} + \zeta^{-s_i^{(3)}} = 0. \] (62)

Similar to Lemma 1, Lemma 2 is proven to be true. The proof of Lemma 2 is complete.

**Lemma 3:** In Type 2, it can be obtained that
\[ a_1a_2 \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{D_i-D_{i+u}} \]
\[ \times \left[ 1 + (-1)^{i_{\pi(m-1)}-\pi(m-1)} \right] \times \left( \zeta^{-s_{i+u}^{(1)}} + \zeta^{s_{i}^{(1)}} \right) = 0, \] (63)
\[ a_1a_3 \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{D_i-D_{i+u}} \]
\[ \times \left[ 1 + (-1)^{i_{\pi(m-1)}-\pi(m-1)} \right] \times \left( \zeta^{-s_{i+u}^{(2)}} + \zeta^{s_{i}^{(2)}} \right) = 0, \] (64)
$$a_2a_3 \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{F_i-F_{i+u}} \times [1 + (-1)^{i \pi(m-1) - (i+u) \pi(m-1)}] \times (\zeta^{-s_{i+u}} + \zeta^{s_{i}}) = 0. \quad (65)$$

**Proof:** Because the sequence pair \((D, D')\) is a Golay complementary pair, and both \((F, F')\) and \((G, G')\) are near-complementary pairs in Case 2. According to Lemma 1, we can easily verify that (63) and (64) are satisfied. Now we just need to prove (65) is true.

Firstly, we consider \(u > 0\), let \(k = i + u\) have a binary representation \(k = (k_0, k_1, ..., k_{m-1})\).

**Case 3.1:** \(i_{\pi(m-1)} \neq k_{\pi(m-1)}\). Obviously, \(\zeta^{F_i-F_k} \times [1 + (-1)^{i_{\pi(m-1)} - k_{\pi(m-1)}}] \times (\zeta^{-s_k} + \zeta^{s_i}) = 0\).

**Case 3.2:** \(i_{\pi(m-1)} = k_{\pi(m-1)}\), (65) can be rewritten as

$$a_2a_3 \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} \zeta^{F_i-F_{i+u}} \times [1 + (-1)^{i_{\pi(m-1)} - k_{\pi(m-1)}}] \times (\zeta^{-s_k} + \zeta^{s_i}) = 0. \quad (66)$$

Let \(v\) denote the biggest index for which \(i_{\pi(v)} \neq k_{\pi(v)}\), where \(v \leq m-2\). Let \(i'\) and \(k'\) denote indexes whose binary representations differ from those of \(i\) and \(k\) only at position \(\pi(v+1)\), respectively, i.e.,

\[
i' = (i_0, i_1, ..., 1 - i_{\pi(v+1)}, ..., i_{\pi(m-1)}),
\]

\[
k' = (k_0, k_1, ..., 1 - k_{\pi(v+1)}, ..., k_{\pi(m-1)}),
\]

since \(i_{\pi(v+1)} = k_{\pi(v+1)}\), we can obtain \(k' = i' + u\). Then, we define an invertible mapping \((i, k) \rightarrow (i', k')\).

In Type 2, we have

\[
F_i - F_k = 2 \sum_{l=0}^{m-2} \sum_{v} \left[ i_{\pi(l)}i_{\pi(l+1)} - k_{\pi(l)}k_{\pi(l+1)} \right] + \sum_{l=0}^{m-2} \sum_{v} c_{\pi(l)}(i_{\pi(l)} - k_{\pi(l)}) + s_{i}^{(1)} - s_{k}^{(1)}, \quad (67)
\]

\[
F_{i'} - F_{k'} = 2 \sum_{l=0}^{m-2} \sum_{v} \left[ i'_{\pi(l)}i'_{\pi(l+1)} - k'_{\pi(l)}k'_{\pi(l+1)} \right] + \sum_{l=0}^{m-2} \sum_{v} c_{\pi(l)}(i'_{\pi(l)} - k'_{\pi(l)}) + s_{i'}^{(1)} - s_{k'}^{(1)}, \quad (68)
\]

\[
(F_{i'} - F_{k'}) - (F_i - F_k) = 2[i_{\pi(v)}i_{\pi(v+1)} - i'_{\pi(v)}i'_{\pi(v+1)}] - 2[k_{\pi(v)}k_{\pi(v+1)} - k'_{\pi(v)}k'_{\pi(v+1)}] + s_{i}^{(1)} - s_{k}^{(1)} - s_{i'}^{(1)} + s_{k'}^{(1)} = 2 + s_{i}^{(1)} - s_{k}^{(1)} - s_{i'}^{(1)} + s_{k'}^{(1)}. \quad (69)
\]
If \( v \geq 1 \), with the definition of \( v \), we have \( s_i^{(1)} = s'_i \), \( s_k^{(1)} = s'_k \), \( s_i^{(3)} = s'_i \) and \( s_k^{(3)} = s'_k \).

Hence,

\[
\zeta_{F_i-F_k} = -\zeta_{F'_i-F'_k},
\]

(70)

\[
\zeta_{s_i^{(3)} + s'_i} = \zeta_{s_i^{(3)} - s'_i},
\]

(71)

\[
\zeta_{F_i-F_k} (\zeta_{s_i^{(3)} + s'_i} + \zeta_{s_i^{(3)} - s'_i}) = 0.
\]

(72)

If \( v = 0 \), we have \( \pi(0) = k'_{\pi(0)} \) and \( i_{\pi(0)} = k'_{\pi(1)} \).

(1) \( i_{\pi(1)} = k_{\pi(1)} = 1 \Rightarrow i'_{\pi(1)} = k'_{\pi(1)} = 0 \). We have

\[
s_i^{(0)} = d_1 i_{\pi(0)} + d_3, s_i^{(1)} = (d_1 - h_1) i_{\pi(0)} + d_3 - h_3,
\]

(73)

\[
s_k^{(0)} = d_1 j_{\pi(0)} + d_3, s_k^{(1)} = (d_1 - h_1) k_{\pi(0)} + d_3 - h_3,
\]

(74)

\[
s_i^{(1)} + s_k^{(1)} = 2k_{\pi(0)} + d_1 + d_2 + 2d_3,
\]

(75)

\[
s_i^{(1)} + s_k^{(1)} = 2k_{\pi(0)} + d_1 + d_2 + 2d_3,
\]

(76)

\[
s_i^{(3)} - s_k^{(3)} = 2, s_i^{(1)} + s_k^{(1)} = 2,
\]

(77)

\[
(F_i' - F_k') - (F_i - F_k) = 0.
\]

Therefore,

\[
\zeta_{F_i-F_k} = \zeta_{F'_i-F'_k},
\]

(78)

\[
\zeta_{s_i^{(3)} + s_k^{(3)}} = -(\zeta_{s_i^{(3)} - s_k^{(3)}}),
\]

(79)

\[
\zeta_{F_i'-F_k'} (\zeta_{s_i^{(3)} + s_k^{(3)}}) + \zeta_{F_i-F_k} (\zeta_{s_i^{(3)} - s_k^{(3)}}) = 0.
\]

(80)

(2) \( i_{\pi(1)} = k_{\pi(1)} = 0 \Rightarrow i'_{\pi(1)} = k'_{\pi(1)} = 1 \). Similarly, we have

\[
\zeta_{F_i-F_k} = \zeta_{F'_i-F'_k},
\]

(81)

\[
\zeta_{s_i^{(3)} + s_k^{(3)}} = -(\zeta_{s_i^{(3)} - s_k^{(3)}}),
\]

(82)

\[
\zeta_{F_i'-F_k'} (\zeta_{s_i^{(3)} + s_k^{(3)}}) + \zeta_{F_i-F_k} (\zeta_{s_i^{(3)} - s_k^{(3)}}) = 0.
\]

(83)

Then, we consider the situation when \( u = 0 \). The term \((65)\) can be rewritten as

\[
2 \times a_2 a_3 \sum_{i=0}^{n-1} \left( \zeta_{s_i^{(3)}} + \zeta_{s_i^{(3)}} \right) = 0.
\]

Let \( l' \) be the integer whose binary representation is \((i_{\pi(0)}, 1 - \ldots, 1)\).
\[\begin{align*}
&\text{According to } i_{\pi(1)}, \ldots, i_{\pi(m-1)}, \text{ an invertible mapping } i \to i' \text{ is defined, }
&\quad \sum_{i=0}^{n-1} (\zeta_s^{(3)} + \zeta_{-s'}^{(3)}) = \sum_{i=0}^{n-1} (\zeta_s^{(3)} + \zeta_{-s'}^{(3)}). \\
&\text{According to Type 2, we have}
&\quad s_i^{(3)} - s_i'^{(3)} = 2, \quad (84)
&\text{whenever } i_{\pi(1)} = 1 \text{ or } i_{\pi(1)} = 0. \text{ Therefore,}
&\quad \zeta_s^{(3)} + \zeta_{-s'}^{(3)} + \zeta_{s'}^{(3)} + \zeta_{-s}^{(3)} = 0. \quad (85)
&\text{Obviously, similar to Lemma 2, Lemma 3 is proven to be true. The proof of Lemma 3 is complete.}
&\text{Now, the proof of Theorem 2 is given as follows.}
&\text{Proof of Theorem 2: According to [5], Theorem 10 in [7] and Corollary 1 in [12], for Type 1,}
&\text{both the sequence pairs } (D, D') \text{ and } (F, F') \text{ are Golay complementary pairs, while } (G, G')
&\text{is a near-complementary pair. For Type 2, both } (F, F') \text{ and } (G, G') \text{ are near-complementary pairs,}
&\text{while the sequence pair } (D, D') \text{ is a Golay complementary pair.}
&\text{Type 1: By } D \star D' \leq 2n, \ F \star F' \leq 2n, \ G \star G' \leq 4n, \text{ Lemma 2 and [14], we have}
&\quad J \star J' \\
&\quad = \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} [a_i^2 (\zeta_{D_i-D_{i+u}} + \zeta_{D_i'-D_{i+u}}) \\
&\quad + a_i^2 (\zeta_{F_i-F_{i+u}} + \zeta_{F_i'-F_{i+u}}) \\
&\quad + a_i^2 (\zeta_{G_i-G_{i+u}} + \zeta_{G_i'-G_{i+u}}) \\
&\quad + a_1a_2 (\zeta_{D_i-F_{i+u}} + \zeta_{D_i'-F_{i+u}} + \zeta_{F_i'-D_{i+u}} + \zeta_{F_i-D_{i+u}}) \\
&\quad + a_1a_3 (\zeta_{D_i-G_{i+u}} + \zeta_{G_i-D_{i+u}} + \zeta_{D_i'-G_{i+u}} + \zeta_{G_i'-D_{i+u}}) \\
&\quad + a_2a_3 (\zeta_{F_i-G_{i+u}} + \zeta_{G_i-F_{i+u}} + \zeta_{F_i'-G_{i+u}} + \zeta_{G_i'-F_{i+u}}) ] \\
&\quad \leq \sum_{u=1-n}^{n-1} \sum_{i=0}^{n-u-1} a_i^2 (\zeta_{D_i-D_{i+u}} + \zeta_{D_i'-D_{i+u}}) \quad (86)
\end{align*}\]
\[ + \sum_{u=1}^{n-1} \left| \sum_{i=0}^{n-u-1} a_2^2 \left( \zeta F_i - F_{i+u} + \zeta F'_i - F'_{i+u} \right) \right| \\
+ \sum_{u=1}^{n-1} \left| \sum_{i=0}^{n-u-1} a_3^2 \left( \zeta G_i - G_{i+u} + \zeta G'_i - G'_{i+u} \right) \right| \\
+ a_1 a_2 \times 4n + 2 \times \sum_{u=1}^{n-1} \left| \sum_{i=0}^{n-u-1} a_2 \times \zeta D_i - D_{i+u} \right| \\
\times [1 + (-1)^{i \pi (m-1) - (i+u) \pi (m-1)}] \times \left( \zeta^{-s_1}_{i+u} + \zeta^{s_1}_{i+u} \right) \\
+ \sum_{u=1}^{n-1} \left| \sum_{i=0}^{n-u-1} a_3 \times \zeta F_i - F_{i+u} \right| \\
\times [1 + (-1)^{i \pi (m-1) - (i+u) \pi (m-1)}] \times \left( \zeta^{-s_2}_{i+u} + \zeta^{s_2}_{i+u} \right) \\
+ \sum_{u=1}^{n-1} \left| \sum_{i=0}^{n-u-1} a_3 \times \zeta F'_i - F'_{i+u} \right| \\
\times [1 + (-1)^{i \pi (m-1) - (i+u) \pi (m-1)}] \times \left( \zeta^{-s_3}_{i+u} + \zeta^{s_3}_{i+u} \right) \\
+ \left| \zeta F'_i - G'_{i+u} + \zeta G'_i - F'_{i+u} \right| \right| \\
\leq a_1^2 \times 2n + a_2^2 \times 2n + a_3^2 \times 4n + a_1 a_2 \times 4n \approx 3.62n.
\]

**Type 2:** By \( D \ast D' \leq 2n, F \ast F' \leq 4n, G \ast G' \leq 4n, \) Lemma 3 and [14], we have

\[ J \ast J' = \sum_{u=1}^{n-1} \left| \sum_{i=0}^{n-u-1} \left[ a_2^2 \left( \zeta D_i - D_{i+u} + \zeta D'_i - D'_{i+u} \right) \\
+ a_2^2 \left( \zeta F_i - F_{i+u} + \zeta F'_i - F'_{i+u} \right) \\
+ a_3^2 \left( \zeta G_i - G_{i+u} + \zeta G'_i - G'_{i+u} \right) \\
+ a_1 a_2 \left( \zeta D_i - F_{i+u} + \zeta F_i - D_{i+u} + \zeta D'_i - F'_{i+u} + \zeta F'_i - D'_{i+u} \right) \\
+ a_1 a_3 \left( \zeta D_i - G_{i+u} + \zeta G_i - D_{i+u} + \zeta D'_i - G'_{i+u} + \zeta G'_i - D'_{i+u} \right) \\
+ a_2 a_3 \left( \zeta F_i - G_{i+u} + \zeta G_i - F_{i+u} + \zeta F'_i - G'_{i+u} + \zeta G'_i - F'_{i+u} \right) \right] \right| \]

\[ \leq a_2^2 \times 2n + a_2^2 \times 4n + a_3^2 \times 4n \approx 2.48n. \]

In summary, the proposed 64-QAM sequence \( H \) is a near-complementary sequence because \( 2n \leq H \ast H' \ll 2n^2 \). Moreover, according to (12), (87) and (87), the PMPERs of the \( H \) for **Type 1** and **Type 2** satisfy that \( \text{PMEPR}_1 \leq 3.62 \) and \( \text{PMEPR}_2 \leq 2.48 \), respectively. Then, the proof of **Theorem 2** is complete.

**REFERENCES**

[1] M. Sharif, V. Tarokh, B. Hassibi, “Peak power reduction of OFDM signals with sign adjustment,” *IEEE Transactions on Communications*, vol. 57, no. 7, pp. 2160-2166, Jul. 2009.

[2] T. Jiang and Y. Wu. “An overview: peak-to-average power ratio reduction techniques for OFDM signals,” *IEEE Transactions on Broadcasting*, vol. 54, no. 2, pp. 257-268, Jun. 2008.
[3] T. Jiang, C. Li and C. Ni, “Effect of PAPR reduction on spectrum and energy efficiencies in OFDM systems with Class-A HPA over AWGN channel,” IEEE Transactions on Broadcasting, vol. 59, no. 3, pp. 513-519, Sep. 2013.
[4] E. Hong, H. Kim, K. Yang, and D. Har, “Pilot-aided side information detection in SLM-based OFDM systems,” IEEE Transactions on Wireless Communications, vol. 12, no. 7, pp. 3140-3147, Jul. 2013.
[5] J. Davis and J. Jedwab, “Peak-to-mean power control for OFDM, Golay complementary sequences, and Reed-Muller codes,” IEEE Transactions on Information Theory, vol. 45, no. 7, pp. 2397-2417, Nov. 1999.
[6] K. Paterson and A. Jones, “Efficient decoding algorithms for generalized Reed-Muller codes,” IEEE Transactions on Communications, vol. 48, pp. 1272-1285, Aug. 2000.
[7] K. Schmidt, “On cosets of the generalized first-order Reed-Muller code with low PMEPR,” IEEE Transactions on Information Theory, vol. 52, no. 7, pp. 3220-3232, Jul. 2006.
[8] N. Yu and G. Gong, “Near-complementary sequences with low PMEPR for peak power control in multicarrier communications,” IEEE Transactions on Information Theory, vol. 57, no. 1, pp. 505-513, Jan. 2011.
[9] B. Tarokh and H. Sadjadpour, “Construction of OFDM M-QAM sequences with low peak-to-average power ratio,” IEEE Transactions on Communications, vol. 51, pp. 25-28, Jan. 2003
[10] C. Chong, R. Venkataramani, and V. Tarokh, “A new construction of 16-QAM Golay complementary sequences,” IEEE Transactions on Information Theory, vol. 49, no. 11, pp. 2953-2959, Nov. 2003.
[11] H. Lee and S. Golomb, “A new construction of 64-QAM Golay complementary sequences,” IEEE Transactions on Information Theory, vol. 52, no. 4, pp. 1663-1770, Apr. 2006.
[12] H. Lee and S. Golomb, “A new construction of 16-QAM near complementary sequences,” IEEE Transactions on Information Theory, vol. 56, no. 11, pp. 5772-5779, Nov. 2010.
[13] Y. Li, “Comments on “A new construction of 16-QAM Golay complementary sequence” and extension for 64-QAM Golay sequences,” IEEE Transactions on Information Theory, vol. 54, no. 7, pp. 3246-3251, Jul. 2008.
[14] Y. Li, “A construction of general QAM Golay complementary sequences,” IEEE Transactions on Information Theory, vol. 56, no. 11, pp. 5765-5771, Nov. 2010.