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\title{\textit{\textbf{\textit{\textbf{\textit{j}-CATEGORIES AND \textit{j}-FUNCTORS IN REPRESENTATION THEORY}}}\
\textbf{II}}}\
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\textbf{Abstract.} This is a partial derivative of \cite{Cox94}. We give a list of examples/problems that some will find amusing.

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\section{Introduction}

In the tradition of I. M. Gelfand we will take some simple nontrivial examples and partially explore the consequences. In the tradition of Grothendieck we will categorify (this appears to be a small part of what Grothendieck had in mind - who knows what he had in mind? I certainly don’t.) If you are looking for proofs, I hate to disappoint you. As far as I can tell there are no proofs.

We use classical representation theoretic ideas found for example in \cite{Ser77} by J. P. Serre, in \cite{FH91} by W. Fulton and J. Harris, in \cite{GW09} by R. Goodman and N. Wallach and new ideas from C. Kassel and V. Drinfeld found in \cite{Kas95}. See also works by Bak-turin, Zhelobenko, Kirillov, Ibragimov, Lychagin, Komrakov, Vilekin, Vershik, Neretin and Vinberg.

I view categorification as a cheap mathematical microscope and/or telescope depending on one’s point of view.

\subsection{Clebsch-Gordan Coefficients and Clebsch-Gordan decomposition.}

The Clebsch-Gordan decomposition for $\mathfrak{sl}(2)$ is

$$F_m \otimes F_n \cong \bigoplus_{p=0}^{m-n} F_{m+n-2p}.$$ 

Thus we have an $\mathcal{F}$-category. The Clebsch-Gordan coefficients are obtained by taking a basis of $F_m$ say $v_{m}, v_{m-2}, \ldots, v_{-m}$ and equating coefficients using a non-degenerate bilinear form (see \cite{Kas95} or more precisely \cite{CE10}). This isomorphism can defined on highest weight vectors by

$$\Phi(v^{m+n-2p}_{m+n-2p}) = \sum_{k=0}^{p} (-1)^{n-p} \frac{[n-p+k]![m-k]!}{[n-p]![m]!} \cdot \Phi^{(k+p)(2+m)+p^2-k^2+n}\cdot v^{m-j}_{k} \otimes v^{n}_{n-p+k}.$$ 

Now take your favorite finite group say $D_n$ and its mutations or avatars such as $Dic_h$. The two have the same character theory. But what about their categorifications? There are a countable number of finite groups, so you have your work cut out for you. Categorify for example results in \cite{vdBC78} and \cite{Sak74}. Many of the groups often have geometric content. See for example \cite{Bre00}. Can one interpolate between categorifications? I think the answer is yes.

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2. \( \mathfrak{g} \)-CATEGORIES FROM OTHER MINDS.

2.1. The category \( \mathcal{I} \) of Enright, [Enr79]. This is the non-triangular, nonabelian but additive category whose indecomposable objects are

\[ M_n \text{ and } P_n \text{ but not } F_n. \]

Categorify \( M_n, P_n \), functors between them and study the resulting categorical structures. Only part of this work has been done. An abelian categorification of \( M_n \) appears in the work of Naisse-Vaz and an additive version starting from M. Khovanov’s work will appear hopefully some day. Then the pieces will need to be put back together.

The end result should be a categorification of Enright’s Theorem in [Enr79].

2.2. The category \( \mathcal{HT} \) of [HT92] of Howe and Tan. We might call this non-abelian categorification or non-abelian harmonic categorification? One needs to consult [HT92] for background info and notation.

Consider the representation \( \widetilde{(V_\lambda \otimes \overline{V}_\nu)} \) which has a “basis” \( v_n \otimes v_k \). Using the action of \( \mathfrak{sl}(2, \mathbb{R}) \) categorify this action. Consider the module \( U(\nu^+, \nu^-) \). Categorify its structure.

2.3. The category \( \mathcal{R} \) of Rasskazova [Ras94]. Consider the representation of Rasskazova’s \( V = V(\beta, \lambda, n) \), which has basis

\[ \{v_j^i | i = 1, \ldots, n; j \in \mathbb{Z}\}, \]

we define the homomorphism \( \varphi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V) \),

\[
\begin{align*}
\varphi(h)(v_j^i) &= h v_j^i = (2j + \beta)v_j^i \\
\varphi(e)(v_j^i) &= ev_j^i = v_{j+1}^i \\
\varphi(e)(v_j^i) &= ev_j^i = (\lambda + j\beta + j(j+1))v_{j+1}^i + v_{j+1}^{i+1} \\
\varphi(f)(v_j^i) &= fv_j^i = (\lambda + (j-1)\beta + j(j-1))v_{j-1}^i - v_{j-1}^{i+1} \\
\varphi(f)(v_j^i) &= fv_j^i = -v_{j-1}^i \\
\end{align*}
\]

Categorify these representations \( V(\beta, \lambda, n) \), functors between them and study the resulting categorical structures. I believe Rasskazova has other representations.

Categorify any functor \( F : \mathcal{HT}, \mathcal{R}, \mathcal{I} \to \mathcal{HT}, \mathcal{R}, \mathcal{I} \) etc.

3. CONCLUSION

There is no conclusion, but there are other partial derivatives of [Cox94]. My mind reels from the possibilities. What is the geometric content and structure? There is lots more to come into focus. I can see parts of images right now. I’ll write about those images later.

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