Two results in metric fixed point theory

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Abstract. We establish two fixed point theorems for certain mappings of contractive type. The first result is concerned with the case where such mappings take a nonempty, closed subset of a complete metric space $X$ into $X$, and the second with an application of the continuation method to the case where they satisfy the Leray–Schauder boundary condition in Banach spaces.

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1. Introduction

In spite of its simplicity (or perhaps because of it), the Banach fixed point theorem still seems to be the most important result in metric fixed point theory. As far as we know, the first significant generalization of Banach’s theorem was obtained in 1962 by E. Rakotch [5] who replaced Banach’s strict contractions with contractive mappings, that is, with those mappings which satisfy condition (1.1) below. Such mappings, as well as their many modifications, were studied and used by many authors. See, for instance, [4] and the references mentioned there. Recently, a renewed interest in contractive mappings has arisen. See, for example, [6, 7] where well-posedness and genericity results were established. Another important topic in fixed point theory is the search for fixed points of nonself-mappings.

In the present paper we combine these two themes by proving two fixed point theorems for contractive nonself-mappings. In Theorem 1 we present a new sufficient condition for the existence and approximation of the unique fixed point of a contractive mapping which maps a nonempty, closed subset of a complete metric space $X$ into $X$. In Theorem 2 we present an application of the continuation method to contractive mappings which satisfy the Leray–Schauder boundary condition on a subset of a Banach space with a nonempty interior.
Theorem 1. Let $K$ be a nonempty, closed subset of a complete metric space $(X, \rho)$. Assume that $T : K \to X$ satisfies

$$\rho(Tx, Ty) \leq \phi(\rho(x, y))\rho(x, y) \quad \text{for each } x, y \in K,$$

(1.1)

where $\phi : [0, \infty) \to [0, 1]$ is a decreasing function such that $\phi(t) < 1$ for all $t > 0$. Assume that $K_0 \subset K$ is a nonempty, bounded set with the following property:

(P1) For each natural number $n$, there exists $x_n \in K_0$ such that $T^i x_n$ is defined for all $i = 1, \ldots, n$.

Then

(A) the mapping $T$ has a unique fixed point $\bar{x}$ in $K$;

(B) for each $M, \epsilon > 0$, there exist $\delta > 0$ and a natural number $k$ such that for each integer $n \geq k$ and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying

$$\rho(x_0, \bar{x}) \leq M \quad \text{and} \quad \rho(x_{i+1}, Tx_i) \leq \delta, \quad i = 0, \ldots, n - 1,$$

we have

$$\rho(x_i, \bar{x}) \leq \epsilon, \quad i = k, \ldots, n.$$  

(1.2)

Let $G$ be a nonempty subset of a Banach space $(Y, \| \cdot \|)$. In [3] J. A. Gatica and W. A. Kirk proved that if $T : \overline{G} \to Y$ is a strict contraction, then $T$ must have a unique fixed point $x_1$, under the additional assumptions that the origin is in the interior $\text{Int}(G)$ of $G$ and that $T$ satisfies a certain boundary condition known as the Leray–Schauder condition:

$$Tx \neq \lambda x \quad \forall x \in \partial G, \forall \lambda > 1.$$  

(L-S)

Here $G$ is not necessarily convex or bounded. Their proof was nonconstructive. Later, M. Frigon, A. Granas and Z. E. A. Guennoun [2] and M. Frigon [1] proved that if $x_t$ is the unique fixed point of $tT$, then, in fact, the mapping $t \mapsto x_t$ is Lipschitz, so it gives a partial way to approximate $x_1$. Our second result extends these theorems to the case where $T$ merely satisfies (1.1).

Theorem 2. Let $G$ be a nonempty subset of a Banach space $Y$ with $0 \in \text{Int}(G)$. Suppose that $T : \overline{G} \to Y$ is nonexpansive and that it satisfies condition (L-S). Then for each $t \in [0, 1)$, the mapping $tT : \overline{G} \to Y$ has a unique fixed point $x_t \in \text{Int}(G)$ and the mapping $t \mapsto x_t$ is Lipschitz on $[0, b]$ for any $0 < b < 1$. If, in addition, $T$ satisfies (1.1), then it has a unique fixed point $x_1 \in \overline{G}$ and the mapping $t \mapsto x_t$ is continuous on $[0, 1]$. In particular, $x_1 = \lim_{t \to 1^-} x_t$.

Our paper is organized as follows. Part (A) of Theorem 1 is proved in Section 2 while Section 3 is devoted to the proof of Theorem 1(B). Finally, we prove Theorem 2 in Section 4.