A chord-arc covering theorem in Hilbert space

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Abstract

We prove that there exists $M > 0$ such that for any closed rectifiable curve $\Gamma$ in Hilbert space, almost every point in $\Gamma$ is contained in a countable union of $M$ chord-arc curves whose total length is no more than $M\ell(\Gamma)$.

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1 Introduction

We say a curve $C$ in a Hilbert space $\mathcal{H}$ is $M$ chord-arc if $\ell(C_{x,y}) \leq M|x-y|$ whenever $|x-y| < |C|$, where $C_{x,y}$ is the arc joining $x$ to $y$ in $C$, $|C|$ is the diameter, and $\ell(C)$ is the one dimensional Hausdorff measure. Equivalently, $C$ is $M$ chord-arc if it is the image of a function $\psi : \partial \mathcal{D} \to \mathcal{H}$ satisfying $|\psi'| \equiv \ell(C)/2\pi$ a.e.

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\[ c_1 \leq \frac{\lvert \psi(s) - \psi(t) \rvert}{\lvert s - t \rvert} \leq c_2, \quad c_2/c_1 \leq \frac{\pi}{2} M. \] (1)

See [4] for details. We prove the following theorem:

**Theorem 1.** There is \( M > 0 \) such that if \( \Gamma \) is a rectifiable simple closed curve in a Hilbert space \( \mathcal{H} \), then there is a collection \( \{ C_j \} \) of \( M \) chord-arc curves of positive length in \( \mathcal{H} \) such that

\[ \sum \ell(C_j) \leq M \ell(\Gamma) \] (2)

and

\[ \ell(\Gamma \setminus \bigcup C_j) = 0. \] (3)

For \( \dim \mathcal{H} = 2 \), this theorem is due to Jones in [5]. It was then generalized to \( \dim \mathcal{H} = n < \infty \) by Garnett, Jones, and Marshall in [4]. First recall that a

**minimal surface with boundary** \( \Gamma \subseteq \mathbb{R}^n \) is \( F(\mathbb{D}) \), where \( F: \mathbb{D} \to \mathbb{R}^n \) is a continuous map such that

1. \( F|_{\partial \mathbb{D}} \to \Gamma \) is a homeomorphism,
2. \( F|\mathbb{D} \) is \( C^2 \),
3. \( f_j = \frac{\partial F}{\partial x_j} - i \frac{\partial F}{\partial y_j} \) is analytic,
4. \( \sum f_j^2 = 0 \).

See [7] for a reference on minimal surfaces.

**Theorem 2** (Garnett, Jones, and Marshall, 1992). There is a universal constant \( M \) such that if \( n \geq 2 \), if \( \Gamma \) is a rectifiable simple closed curve in \( \mathbb{R}^n \), and \( F(\mathbb{D}) \) is a minimal surface with boundary \( \Gamma \), then there is a locally finite partition \( \{ D_j \} \) of \( \mathbb{D} \) such that

1. \( F|_{\partial D_j} \to \Gamma \) is a homeomorphism of \( \overline{D_j} \) onto \( F(\overline{D_j}) \),
2. \( F(\partial D_j) \) is an \( M \) chord-arc curve, and
3. \( \sum \ell(F(\partial D_j)) \leq M \ell(\Gamma) \).

Hence, in the finite dimensional case, Theorem 1 holds with \( C_j = F(\partial D_j) \).

Our theorem is a generalization of this result to Hilbert space: we may enclose any rectifiable curve \( \Gamma \) inside the closure of a connected "web" of \( M \) chord-arc curves whose total length no more than \( M \ell(\Gamma) \).

Without loss of generality we may assume \( \mathcal{H} \) is separable, since \( \Gamma \) is contained in a separable subspace of \( \mathcal{H} \). Let \( \{ e_n \} \) be an orthonormal basis for \( \mathcal{H} \) and \( \mathbb{R}^n \) be the span of the first \( n \) basis vectors. Let \( P_n = P_{\mathbb{R}^n} \) be the projection onto \( \mathbb{R}^n \).

**Remark 3.** The approach we take in proving this theorem, loosely speaking, is to apply Theorem 2 to the projections \( P_n(\Gamma) \subseteq \mathbb{R}^n \) to get sequences of \( M \) chord-arc curves \( \{ C^*_j \} \), and then take a limit of these (passing to a subsequence if necessary) to produce \( M \) chord-arc curves \( \{ C_j \} \) in \( \mathcal{H} \) satisfying the conclusion of the theorem.

The first difficulty that arises in this approach is that the original statement of Theorem 2 doesn’t say (and nor does the proof immediately indicate) whether the constant \( M \) depends on the dimension \( n \), which would be necessary for any limiting argument to work. We give an outline of the proof Theorem 2 in Section 3 and verify that the constants involved are independent of \( n \).

Secondly, we may not be able to apply Theorem 2 directly to \( P_n(\Gamma) \) since it may be self intersecting. To fix this, we will introduce an extra two dimensions to \( \mathcal{H} \) and each \( \mathbb{R}^n \). This allows us to adjust the projections \( P_n(\Gamma) \) in \( \mathbb{R}^n \) into Jordan
curves $\Gamma^n$ in $\mathbb{R}^{n+2}$, and adjust them less and less as $n \to \infty$ so they converge to $\Gamma$. This is proved in Lemma 5. We can then apply Theorem 2 to the $\Gamma^n$ to obtain a collection of $M$ chord-arc curves $C^*_{\gamma}$. These in turn will converge to $M$ chord-arc curves $C_j$ in $\mathcal{H}$ satisfying (2) of the main theorem.

To prove (3) of Theorem 1 it suffices by Lemma 6 below to show that $\Gamma \subseteq \bigcup_j C^*_j$. This needs verifying (and is the main difficulty of the proof) since it may happen that the $C^*_j$ near $x$ all converge to singleton points or drift away from $x$, as seems possible if the curves $\Gamma^n$ coil up or bend more and more as $n \to \infty$. Heuristically speaking, if this were the case, the limiting curve would not be rectifiable. Hence, we remedy this issue by utilizing the rectifiability of $\Gamma$ to show that in any neighborhood of a point $x \in \Gamma$, there exists $\delta > 0$ such that for large $n$ there is $C^*_j$ intersecting that neighborhood with $|C^*_j| \geq \delta$. Thus, in the limit, we may find a $C_j$ of positive diameter near $x$ as close as we’d like, and that will prove (2).

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2 Proof of main theorem

2.1 Basic lemmas

We let $E \delta$ denotes the $\delta$-neighborhood of a set $E$, and we write $A \subseteq B$ to mean that $A \subseteq CB$ for some constant $C$, and $A \subseteq \mathbb{R}^2$ if $C$ depends on the parameter $N$. We first mention a simple yet key lemma that much of the work below depends on. This appears as lemma 7.1 in [17]:

Lemma 4. If $E$ is a minimal surface with boundary curve $C$ in $\mathbb{R}^n$, then $E \subseteq \text{co } C$, where co $C$ denotes the convex hull of $C$.

Let $\Gamma$ be as in the theorem. We will now construct the sequence of curves $\Gamma^n$ we will be working with. Define $\mathcal{H}' = \mathcal{H} \oplus \mathbb{R}^2$, the direct sum of the two Hilbert spaces $\mathcal{H}$ and $\mathbb{R}^2$, and let $\gamma : [0, 2\pi] \to \mathcal{H}'$ denote the arclength parameterization of $\Gamma$.

Lemma 5. There exist smooth Jordan curves $\Gamma^n \subseteq \mathbb{R}^n \oplus \mathbb{R}^2$ that converge to $\Gamma \subseteq \mathcal{H}$. In particular, co $\Gamma^n \to \text{co } \Gamma \subseteq \mathcal{H}$.

The second condition ensures that if the $C^n_j$ are contained in co $\Gamma^n$, every subsequential limit of $\{C^n_j\}$ will converge to a chord-arc curve properly contained in $\mathcal{H}$.

Proof. Fix an $n > 2$. Let $F : \mathbb{R}^n \to \mathbb{R}^{n+2}$ be the minimal surface function for

$$P^n \circ \gamma(t) \oplus \left( \frac{\cos t}{n}, \frac{\sin t}{n} \right),$$

which is a Jordan curve of finite length in $\mathbb{R}^{n+2}$. Pick $r \in (0, 1)$ close enough to 1 so that $\gamma^n(t) := F(re^{it})$ defines a curve such that $|\gamma^n(t) - F(e^{it})| < \frac{1}{r}$ for all $t$ and $|\ell(\gamma^n(t)) - \ell(F(e^{it}))| < \frac{1}{r}$. Let $\gamma^n$ be the arclength parameterization of $\gamma^n$.

Defining $\gamma^n$ in this way for each $n$ gives a sequence of smooth Jordan curves of finite length in $\mathbb{R}^{n+2}$ and the $\gamma^n$ converge uniformly to an arclength parameterization $\gamma$ of $\Gamma$. Let $\Gamma^n$ be the image of $\gamma^n$.

We now show that $\lim \text{co } \Gamma^n = \text{co } \Gamma$. By definition, any point in $\text{co } \Gamma^n$ is of the form $\int \gamma^n(t) d\mu(t)$, where $\mu$ is some Borel probability measure on $[0, 2\pi)$. Since $\gamma^n \to \gamma$ uniformly, it follows that for any $\varepsilon > 0$, $|\int (\gamma^n - \gamma) d\mu| < \varepsilon$ for $n$ sufficiently large. Thus, every point in $\text{co } \Gamma^n$ is within $\varepsilon$ of a point $\int \gamma d\mu \in \text{co } \Gamma$, which proves the claim. 

\[\square\]
Lemma 6. If $C$ is a connected set in $\mathcal{H}$, if $\Gamma = \overline{C} \setminus C$, and if $\ell(\overline{C}) < \infty$, then 
\[ \ell(\Gamma \setminus C) = 0. \]

Proof. Suppose $A = \Gamma \setminus C$ has positive length. Choose $r > 0$ small such that
\[ \ell(A \cap C) < \frac{\ell(A)}{4}. \]
Choose a packing $B_j = B(x_j, r_j)$ of balls with $|B_j| < \frac{r}{2}$, so $x_j \in A$ for all $j$ and the $B_j$ are mutually disjoint. Since they form a packing, $A \subseteq \bigcup_{j} 2B_j$, and thus $\sum |2B_j| \geq \ell(A)$, where $2B_j = B(x_j, 2r_j)$. Since $C$ is dense in $A$, and $C \setminus A_r \neq \emptyset$, we may find paths $p_j : [0, 1) \to C$ such that $p_j(t) = y$ for some $y \in C \setminus A_r$ and $\lim_{t \to 1} p_j(t) = x_j$. Thus, since the $B_j \subseteq A$, they are disjoint,
\[ \ell(A_r \cap C) \geq \sum_j \ell(B_j \cap C) \geq \sum_j \ell(B_j \cap p_j([0, 1])) \geq \sum_j \frac{|B_j|}{2} \geq \frac{\ell(A)}{4}, \]
and we have a contradiction. \qed

2.2 Finding chord-arc curves close to $\Gamma$

From now on, let $F^n : \overline{\mathbb{B}} \to \mathbb{R}^n$ be the minimal surface function for $\Gamma^n$. $D^n$ the decomposition of $\mathbb{B}$ from Theorem 2, applied to $F^n$, and $C^n_\delta = F^n(\partial D^n_\delta)$. By passing to a subsequence if necessary, we may assume the $F^n(\partial \mathbb{B})$ converge to a set $S \subset \mathcal{H}$.

By the previous lemma, to prove (3) of Theorem 1 it will suffice to prove the following lemma. For a set $E$, let $N(E, \varepsilon)$ denote the covering number of $E$, defined to be minimum number of sets of diameter $\varepsilon > 0$ needed to cover the set $E$.

Lemma 7. Let $x \in \Gamma$ be a tangent point, let $r > 0$, and
\[ \mathcal{C}_n(r) = \{ j : C_n^j \cap B(x, r) \neq \emptyset \}. \]

Then there is $\delta > 0$ and $N$ such that $n > N$ implies there is $j \in \mathcal{C}_n(r)$ such that $|C_n^j| \geq \delta$.

This will show that the collection $\{C_j\}$ of chord-arc curves obtained by taking the limits of the $C_n^j$ will contain chord-arc curves of positive length as close as we’d like to any tangent point of $\Gamma$. Since the tangent points are dense in $\Gamma$, this will imply that $\Gamma$ is in the closure of the union of these curves.

We’ll give here an outline of the proof of Lemma 7. First, we will show that if we can find a sequence of points $x_n \in F^n(\mathbb{D})$ and $\rho > 0$ such that $|x_n - x| < \rho/2$ and $d(x_n, \Gamma^n) \geq \rho$ for all $n$, then we can find a subset $E \subseteq S \cap B(x, \rho/2)$ that satisfies $N(E, \delta) \geq \delta^{-2}$. Next, we will assume the conclusion of Lemma 7 is false and show that for each $\delta > 0$ we can cover $S \cap B(x, \rho/2)$ by at most $C\delta^{-1}$ balls of diameter $\sim \delta$, where $C$ is a constant depending only on $\Gamma$, and hence $N(S \cap B(x, \rho/2), \delta) \leq \delta^{-1}$. Combining the two bounds on the covering number, we get
\[ \delta^{-1} \geq N(S \cap B(x, \rho/2), \delta) \geq N(E, \delta) \geq \delta^{-2}. \]

Letting $\delta \to 0$ gives the desired contradiction. In Section 2.3, we will show that we can find the aforementioned sequence $x_n$ and follow that with the final proof of the main theorem.

Lemma 8. Suppose $E$ is a minimal surface with boundary $C$. Let $x \in E$ such that $d(x, C) > 2\rho > 0$. Then the connected component $E_\rho$ of $B(x, \rho) \cap E$ containing $x$ satisfies $N(E_\rho, \rho\varepsilon) \geq \varepsilon^{-2}$ for $\varepsilon \in (0, \rho)$, where the implied constant is independent of dimension $n$ and our choice of $E$.

Proof. Assume without loss of generality that $x = 0$. By Sard’s theorem, we may also assume that our choice of $\rho$ gives that $\partial B(0, \rho) \cap E$ is a smooth compact one dimensional manifold. Let $A_\rho$ be the connected component of this set contained
Claim: \(|A_\rho| \geq \rho/2\). If not, fix \(z \in A_\rho\). Then

\[ A_\rho \subseteq B(z, |A_\rho|) \subseteq B(z, \rho/2) \subseteq \{v : z \cdot v > 0\} =: H_z. \]

Note that \(E_\rho\) is the minimal surface for the curve \(A_\rho\) by virtue of being a minimal surface in general, so by Lemma 4 \(E_\rho\) is contained in the convex hull of \(A_\rho\) and hence is contained in \(H_z\). Thus, \(0 \in H_z^c \cap E_\rho = \emptyset\), which is a contradiction.

By Sard’s theorem, the same proof gives \(|A_r| \geq r/2\) for a.e. \(r < \rho\), where \(A_r\) is the boundary curve of the component \(E_r\) of \(B(0, r) \cap E\) containing 0.

Claim: \(N(E_\rho, \rho \delta) \gtrsim \delta^{-2}\) for \(0 < \delta < 1\), the constant independent of the minimal surface \(E\). Choose \(0 = r_0 < r_1 < \cdots < r_m = 1\) such that for \(j = 1, \ldots, m\), \(A_{\rho r_j}\) is a smooth compact one dimensional manifold bounding a surface \(E_j\) containing \(0, r_j > \frac{1}{m}\), and \(|r_j - r_{j-1}| > \frac{1}{2m}\) (that is, pick \(r_j\) sufficiently close to \(\frac{1}{m}\)). Since \(|A_{r_j}| \geq \frac{1}{2}\), we have

\[ N(A_{\rho r_j}, \rho \frac{2m}{m - 1}) \geq \frac{1}{2} \cdot \frac{\rho}{\frac{2m}{m - 1}} = \frac{m \rho r_j}{m - 1}, \]

and thus

\[ N(E_\rho, \rho \frac{2m}{m - 1}) \geq \sum_{j=1}^{m} N(A_{\rho r_j}, \rho \frac{2m}{m - 1}) \geq \sum_{j=1}^{m} \frac{m \rho r_j}{m - 1} = \sum_{j=1}^{m} \frac{j}{2} = \frac{m^2 + m}{4} \geq \left(\frac{m}{m - 1}\right)^{-2} \]

where the first inequality comes from the fact that a set of diameter \(\frac{\rho}{2m}\) can intersect at most one \(A_{\rho r_j}\), since the distance between each \(A_{\rho r_j}\) is at least \(\frac{1}{2m}\). This proves the claim.

Next, we show that if the covering numbers for a sequence of sets satisfies a uniform two dimensional growth rate, then so does the limit.

Lemma 9. Suppose \(E^n\) is a sequence of compact sets satisfying \(N(E^n, \delta) \geq C\delta^{-2}\) for \(\delta > 0\) uniformly in \(n\). Suppose the \(E^n\) converge in the Hausdorff metric to some compact set \(E\). Then \(N(E, \delta) \geq \frac{C}{4}\delta^{-2}\).

Proof. Suppose we could cover \(E\) by \(N\) balls of diameter \(\delta\) with \(N < \frac{C}{4}\delta^{-2}\). Choose \(n\) large enough so that \(E^n\) is in the \(\delta\)-neighborhood of \(E\). Then we can cover \(E^n\) by the same balls with the same centers but double the radius, hence

\[ C(2\delta)^{-2} \leq N(E^n, 2\delta) \leq N < \frac{C}{4}\delta^{-2} \]

which gives the desired contradiction.

Lemma 10. Let \(x \in \overline{S}\) and \(r > 0\). Then there exists a subset \(E \subseteq S \cap B(x, r/2)\) that satisfies \(N(E_\rho, \delta) \gtrsim \delta^{-2}\).

Proof. By Lemma 9 it suffices to find a sequence of sets \(E^n \subseteq F^n(\mathbb{D})\) with a subsequence that converges to a set \(E \subseteq S \cap B(x, r/2)\) and satisfies \(N(E^n, \delta) \gtrsim \epsilon^{-2}\) uniformly in \(n\). By Lemma 11 and compactness, it suffices to pick \(E^n\) be the connected component of \(B(\nu_n, \rho) \cap F^n(\mathbb{D})\), where \(\nu_n \in F^n(\mathbb{D})\) and \(\rho > 0\) are such that \(d(\nu_n, \Gamma^n) > \rho\) for all \(n\), which we prove exist in Lemma 11 below.
Proof of Lemma\footnote{2} Assume for the sake of a contradiction that for every $\delta > 0$ there are infinitely many $n$ such that

$$\sup \{|C^n_j| : j \in \mathcal{C}_n(r)\} < \delta. \quad (4)$$

As mentioned earlier, we will construct a cover of $S \cap B(x, r/2)$ by no more than $C\delta^{-1}$ sets of diameter $\sim \delta$, where $C$ is a constant depending only on $\Gamma$, which will give $N(S \cap B(x, r), \delta) \lesssim \delta^{-1}$. To find this cover, we first find an $F^n(\partial)$ sufficiently close to $S$ so that $S$ is in the $\partial$-neighborhood of the chord-arc domains $D^n_j \subseteq F^n(\partial)$ with diameters less than $\delta$. Next, we will group these domains into clusters whose unions will be sets of diameters at least $\delta/2$. The $\partial$-neighborhoods of each of these clusters together will constitute the $\partial$-cover we’re looking for since we may control their number via the length estimate in Theorem\footnote{2} Now we begin the proof.

Fix $\delta < r/2$. Define

$$C_n(r) = \bigcup_{j \in \mathcal{C}_n(r)} F^n(D^n_j)$$

and

$$C'_n(r) = \bigcup_{j \in \mathcal{C}_n(r)^c} F^n(D^n_j).$$

Note that, by construction, $d(x, C_n'(r)) > r$ and $F^n(\partial) = C_n(r) \cup C'_n(r)$. Then

$$F^n(\partial) \cap B(x, r) \subseteq C_n(r) \subseteq F^n(\partial)$$

since if $s \in F^n(\partial) \cap B(x, r)$ it’s covered by some set $F(D^n_j)$, and its boundary $C^n_j$ must intersect $B(x, r)$, so $j \in \mathcal{C}_n(r)$ and thus $s \in F^n(D^n_j) \subseteq C_n(r)$.

Claim: For sufficiently large $n$,

$$S \cap B(x, r/2) \subseteq C_n(r)_{\delta}.$$  

Choose $N$ such that $n \geq N$ implies $S \subseteq F^n(\partial)_{\delta}$, and let $n \geq N$. First note that

$$S \cap B(x, r/2) \subseteq (C'_n(r))_{\delta}$$

for $\delta$ smaller than $r/2$, for if a point $s \in S \cap B(x, r/2) \cap C'_n(r)_{\delta}$, then $d(s, C^n_j) < \delta$ for some $C^n_j \subseteq C'_n(r)$. Since $d(x, C'_n(r)) \geq d(x, C''_n(r)) > r$, we have

$$\frac{r}{2} > |x - s| > d(x, C''_n) - d(C''_j, s) > r - \delta > \frac{r}{2},$$

which is a contradiction. Furthermore, since $S \subseteq F^n(\partial)_{\delta} = C_n(r)_{\delta} \cup C'_n(r)_{\delta}$,

$$S \cap B(x, r/2) \subseteq F^n(\partial)_{\delta} \cap C'_n(r)_{\delta} = C_n(r)_{\delta} \cap C'_n(r)_{\delta} \subseteq C_n(r)_{\delta}$$

which proves the claim.

Fix $n \geq N$ such that \footnote{4} holds.

We will now form the clusters mentioned earlier, whose $3\delta$-neighborhoods will form the desired cover of $S \cap B(x, r)$. Let $X$ be a maximal $2\delta$-net in $C_n(r)$, and for each $z \in X$, let

$$E_z = \bigcup \{F^n(D^n_j) : j \in \mathcal{C}_n(r), d(z, C^n_j) \leq \delta/2\},$$

that is, for each $z \in X$ we form a set in $F^n(\partial)$ containing $z$ by incorporating all sets $F^n(D^n_j)$ in our cover of $F^n(\partial)$ that are close to $z$. Then the collection $\{E_z\}_{z \in X}$ satisfies the following properties:

i. $\bigcup_{z \in X} E_z \subseteq C_n(r)$. 


ii. The $E_z$ are disjoint. If $w \in E_z \cap E_{z'}$ for $z$ and $z'$ distinct, then $w \in C^n_j$ where $d(z, C^n_j) < \delta/2$ and $d(z', C^n_j) < \delta/2$, and hence $C^n_j \subseteq B(z, \frac{\delta}{2} + \delta) \cap B(z', \frac{\delta}{2} + \delta) = \emptyset$

since the $C^n_j$ for $j \in \mathcal{E}(r)$ have diameter no more than $\delta$ and $|z - z'| > 2\delta$.

iii. Each $E_z$ has diameter at least $\delta/2$ if $2\delta < |\Gamma|$. To see this, note that $B(z, \delta/2) \cap F^n(\mathbb{D})$ has diameter at least $\delta/2$ since $F^n(\mathbb{D})$ is connected, so there must be a path connecting $z$ to some point outside $B(z, \delta/2) \cap F^n(\mathbb{D})$ of diameter at least $\delta/2$. Then we observe that $E_z \supseteq B(z, \delta/2) \cap F^n(\mathbb{D})$ by construction.

iv. The collection $\{(E_z)_{z \in X}\}$ covers $C_n(r)$ since $C_n(r) \subseteq \bigcup_{x \in X} B(x, 2\delta)$ by construction, and thus $\{(E_z)_{z \in X}\}$ covers $S \cap B(x, r/2) \subseteq C_n(r)$. Thus, we have a cover of $S \cap B(x, r/2)$ by $\#X$ many sets $(E_z)_{z \in X}$ of diameter at most $2(3\delta) = 7\delta$, so $N(S \cap B(x, r/2), 7\delta) \leq \#X$, where

\[
\#X \leq \frac{2}{\delta} \sum_{z \in X} |E_z| \leq \frac{2}{\delta} \sum_{z \in X} \sum_{d(z, F^n(\mathbb{D})) < \delta/2} |F^n(\mathbb{D})| \\
\leq \frac{2}{\delta} \sum_{z \in X} \sum_{d(z, F^n(\mathbb{D})) < \delta/2} |C^n_j| \\
\leq \frac{2}{\delta} \sum_{z \in X} \sum_{d(z, F^n(\mathbb{D})) < \delta/2} \ell(C^n_j) \\
\leq \frac{2}{\delta} \sum_{j} \ell(C^n_j) \leq \frac{2}{\delta} M\ell(\Gamma^n).
\]

The first inequality holds by (iii); since $E_z = \bigcup_{d(z, F^n(\mathbb{D})) < \delta/2}$, we have the second inequality; the third is because $F^n(\mathbb{D})$ is a minimal surface containing $C^n_j$, so it’s contained in the convex hull of $C^n_j$ which has diameter $|C^n_j|$; the fifth holds because the sets $\{C^n_j : d(z, F^n(\mathbb{D})) < \delta/2\} \subseteq E_z$ are disjoint for different $z \in X$, so the terms in the sum don’t repeat; finally, the sixth inequality follows from Theorem 2. Since the $\ell(\Gamma^n) \leq \ell(\Gamma)$ for large $n$, we have that for sufficiently small $\delta$, $N(S \cap B(x, r/2), 7\delta) \leq \delta^{-1}$, where the implied constant depends only on the length of $\Gamma$. By Lemma 11, this gives a contradiction.

\[\square\]

2.3 Finding points that stay away from $\Gamma^n$

Lemma 11. For any tangent point $x \in \Gamma$, there exists a sequence $y_n \in F^n(\mathbb{D})$ and a $\rho > 0$ such that $d(y_n, \Gamma^n) > \rho$ for all $n$.

Proof. Let $\varepsilon > 0$ be a small number to be determined later. Then we may choose $\delta \in (0, r/2)$ such that if $L$ is the line tangent to $\Gamma$ at $x$, we have

i. $\ell(B(x, \delta) \cap \Gamma \setminus \mathcal{L}_x) < \varepsilon \delta,$

ii. $|\ell(B(x, \delta) \cap \Gamma)| - 2\delta < 2\delta \varepsilon,$

iii. If $H_0$ is the hyperplane through $x$ perpendicular to $L$ and $H^\perp_0$ are the two components of $\mathcal{H} \setminus H_0$, then $\partial B(x, \delta) \cap \mathcal{L}_x \cap H^\perp_0 \neq \emptyset$.

Since $\Gamma^n \to \Gamma$ and $\gamma^n \to \gamma$, we know that (i), (ii), and (iii) are satisfied with $\Gamma^n$ in place of $\Gamma$ for $n$ large. These conditions say that the majority of $\Gamma^n$ inside
the ball $B(x, \delta)$ is inside the tube $L_{\epsilon\delta}$, and that the curve goes from one end of the tube to the other. More specifically, (i) implies

$$\Gamma^n \cap B(x, \delta) \setminus L_{\epsilon\delta} \subseteq (B(x, \delta) \setminus B(x, \delta - \epsilon\delta)) \cup (L_{2\epsilon\delta} \setminus L_{\epsilon\delta}).$$

(5)

Otherwise, there would be a point in $\Gamma^n \cap B(x, \delta)$ that was at least $\epsilon\delta$ away from $\partial B(x, \delta)$ and $L_{\epsilon\delta}$, and hence there would be a portion of $\Gamma^n$ in $B(x, \delta) \setminus L_{\epsilon\delta}$ of length at least $2\epsilon\delta$, which contradicts (i). Since $\Gamma^n$ is concentrated near $\partial B(x, \delta)$ and $L_{\epsilon\delta}$, our aim is to show that we can find a point on the minimal surface far from these sets in $B(x, \delta)$ and hence far away from $\Gamma^n$, as desired.

Pick a hyperplane $H \subseteq \mathcal{K}$ perpendicular to $L$ and such that the hyperplane $H' = H \cap \mathbb{R}^{n+2}$ in $\mathbb{R}^{n+2}$ (recalling that $\Gamma^n \subseteq \mathbb{R}^{n+2}$ and, for large $n$, $\mathbb{R}^{n+2} \neq H'$) such that

a. $H' \cap \Gamma^n$ is a countable union of disjoint 1-dimensional curves,

b. $H' \cap \Gamma^n$ is a finite set of points whose tangent lines are not parallel with $H$,

c. $d(x, H) < \epsilon\delta$ and $d(\partial B(x, 3\epsilon\delta) \cap H', L_{\epsilon\delta}) > 2\epsilon\delta$

We can do this as follows: define a function $H : \mathcal{K} \to \mathbb{R}$ by $u \mapsto \langle u - v, v \rangle$, where $v$ is the vector tangent to $\Gamma$ at $x$. Then $f |_{\mathbb{R}^{n+2}}$ is a continuously differentiable function. By (iii), there is an open interval containing 0 in $f(\Gamma^n) \cap L_{\epsilon\delta} \cap B(x, \delta)$ (that is, the curve $\Gamma^n$ runs from the top of the sphere to the bottom, so $f$ goes from positive to negative or negative to positive along $\Gamma^n$). By Sard’s theorem applied to both $f|_{\Gamma^n}$ and $f|_{\mathbb{R}^{n+2}}$, and using the fact that

$$\ell(P_0(z) : z \in \Gamma^n, \text{the line tangent to } \Gamma^n \text{ at } z \in P_0^{-1}(z)) = 0,$$

there exists $t$ such that by letting $H = f^{-1}(t)$, $H'$ satisfies (a), (b), and (c).

Claim: There is at most one component of $\Gamma^n \cap B(x, \delta)$ containing $B(x, 3\epsilon\delta) \cap \Gamma^n$. Suppose there were two components intersecting $B(x, 3\epsilon\delta)$. Then the endpoints of these two component curves must be contained in $\partial B(x, \delta)$, thus each of their lengths are at least $\delta - 3\epsilon\delta = \delta(1 - 3\epsilon)$. Thus $\ell(B(x, \delta) \cap \Gamma) > 2(2\delta(1 - 3\epsilon))$, which contradicts (ii) for small $\epsilon$.

Let $A$ be this component of $\Gamma^n \cap B(x, \delta)$ and let $P = \Gamma^n \cap H' \cap B(x, 3\epsilon\delta)$.

Claim: $P$ is odd. To show this, consider $[a, b] = (\gamma^n)^{-1}(A)$. Then $g = f \circ \gamma : [a, b] \to \mathbb{R}$ is a differentiable function whose derivatives on $g^{-1}(0)$ are nonzero, and $g(a)$ and $g(b)$ have opposite sign, thus $g^{-1}(0) = P$ must be odd.

By (a), $H' \cap \Gamma^n \cap B(x, \delta)$ is a countable union of 1 dimensional curves, each having either two or no endpoints points, all of which must be either in $P$, $H' \cap \Gamma^n \setminus B(x, 3\epsilon\delta)$, or $H' \cap \partial B(x, \delta)$. However, (5) implies

$$H' \cap \Gamma^n \setminus B(x, 3\epsilon\delta) \subseteq H' \cap \Gamma^n \cap \partial B(x, \delta) \cap B(x, \delta - \epsilon\delta),$$

so in fact, all boundary points are either in $P$ or $H' \cap \partial B(x, \delta) \cap B(x, \delta - \epsilon\delta)$. Since $P$ is odd, one of these curves, call it $M$, must have boundary points in both of these sets. Pick a point $y_n \in M \cap B(x, \delta - 2\epsilon\delta) \setminus B(x, 3\epsilon\delta)$. Then $y_n$ satisfies the lemma with $r = \epsilon\delta$ since $B(y_n, \epsilon\delta) \subseteq B(x, \delta - \epsilon\delta) \cap (L_{2\epsilon\delta})^c$, and (5) implies $y_n$ is at least $\delta$ away from $\Gamma^n$.

Proof of the Main Theorem. Let $\{q_k\}$ be a countable dense set in $D \setminus \bigcup_{j \neq k} \partial D^n_j$, which is dense in $\mathbb{R}$ since $\bigcup_{j \neq k} \partial D^n_j$ is a $\sigma$-finite one dimensional set by Theorem

Let $D^n_{\epsilon\delta}$ be the $D^n_j$ containing $q_k$. Let $\psi_k : \partial D \to \mathbb{R}^{n+2}$ be the arclength parametrization of $C^n_{\epsilon\delta}$ so that its derivative is $\ell(C^n_{\epsilon\delta})/2\pi$ a.e.. Then $\psi_k$ satisfies (4).

Note that $c_2 \sim \ell(C^n_{\epsilon\delta}) \leq \ell(\Gamma^n)$, which are uniformly bounded, hence $\{\psi_k^n\}_n$ is a bounded sequence of uniformly Lipschitz functions. By Arzelà-Ascoli, we
may choose a subsequence \( n_{m} \) such that \( \psi_{1,n}^{i} \) converges in \( m \). Choose a subsequence \( n_{m}^{2} \) of \( n_{m}^{1} \) such that \( \psi_{2,m}^{2} \) converges. Inductively, choose a subsequence \( n_{m}^{3} \) of \( n_{m}^{2} \) such that \( \psi_{3,m}^{3} \) converges. Finally, let \( n_{k} = n_{k}^{k} \), so that \( \psi_{k}^{m} \) converges to some \( \psi_{k} \) for each \( k \). In particular, by equation (1) \( \psi_{k} \) is \( M \) chord arc or constant.

Let \( C_{\psi_{k}} \) be the image of each \( \psi_{k} \). This gives us a collection of \( M \) chord-arc curves and countably many singleton points in \( \mathcal{H} \), call this set \( C \) and let \( \{ C_{j} \} \) denote those \( M \) chord-arc curves of positive length. Since \( C = \lim_{n \to \infty} \bigcup C_{j,n}^{n} \), and \( \bigcup C_{j,n}^{n} \) is connected for each \( n \), \( C \) is connected. By Theorem 3.18 of [1],

\[
\ell(C) \leq \liminf_{n \to \infty} \ell(\bigcup C_{j,n}^{n}) \leq \liminf_{n \to \infty} M\ell(\Gamma^{n}) = M\ell(\Gamma),
\]

and thus (2) of Theorem 1 is verified.

By the previous lemma, near each \( x \in \Gamma \), there is \( \delta > 0 \) such that for all sufficiently large \( \ell \), there is a \( C_{j,\ell}^{\ell} \) with diameter at least \( \delta \). Thus, in the limit, there will be a \( C_{j} \) with diameter at least \( \delta \) near \( x \). Therefore, \( \Gamma \subseteq \bigcup C_{j} \), and (3) of Theorem 1 follows from Lemma 6. By lemmas 4 and 5, \( C_{j} \subseteq \bigcap_{n=2}^{N} \bigcup_{n \geq N} \Gamma_{n} = \text{co} \Gamma \subseteq \mathcal{H} \).

\( \square \)

## 3 Proof of the Garnett-Jones-Marshall Theorem

Here we outline the proof of Theorem 2 in order to show its constants do not depend on the dimension. We also omit the details about the partition being locally finite since we do not need this condition. First, let us introduce some preliminary lemmas.

Using the properties of a minimal surface as described earlier, one can prove the following:

**Lemma 12.** If \( F(\mathbb{D}) \) is a minimal surface, then \( f_{\theta} \in H^{1} \), then

\[
\sup_{r \in (0,1)} \int_{0}^{2\pi} |f(re^{i\theta})|d\theta = \sqrt{2}\ell(\Gamma)
\]

where \(|f| = \sqrt{|f_{1}|^{2} + \cdots + |f_{m}|^{2}}\).

Now, decompose the disc into dyadic squares \( Q \). Divide each such square into \( 4^{N} \) (roughly) equal sized subsquares \( S \) which we will denote "small squares," and divide each of these into \( 4^{N'} \) "very small squares" \( S' \). Let \( Q' = \{ e^{i\theta} : re^{i\theta} \in Q \} \), i.e. \( Q' \) is the projection of \( Q \) onto \( \partial D \). Let \( Q(S) \) denote the diadic square \( Q \) with \( Q' = S' \), and \( B(S) = \{ e^{i\theta} : e^{i\theta} \in S' \text{ and } \inf_{S}|z| \leq r < \inf_{Q(S)}|z| \} \), i.e. the "tower" that includes \( S \) but not \( Q(S) \). If \( \mathcal{S} \) is some collection of small squares, note that

\[
\mathcal{S} = \mathbb{D} \setminus \bigcup_{S \in \mathcal{S}} B(S) \cup Q(S)
\]

is a chord-arc curve with constant dependent only on \( N \), and moreover, one dimensional Hausdorff measure restricted to \( \mathcal{S} \) is a Carleson measure since the length inside each dyadic cube rescaled to a unit cube is constant.

The next lemma says that if the derivative of \( F \) doesn’t vary too much from a nonzero value of \( f \) along some chord-arc curve, then the image of that curve is also chord-arc.

**Lemma 13.** Suppose \( \gamma \) is an M chord-arc curve in \( \mathbb{D} \) and that there is \( z_{0} \in \mathbb{D} \) with \(|f(z) - f(z_{0})| < \delta |f(z_{0})| \) on \( \gamma \), where \( \delta < \frac{1}{\sqrt{2M}} \). Then \( F(\gamma) \) is \( M_{1} = \left( \frac{\pi}{2} \right)^{2} \frac{1 + \sqrt{2M}}{1 - \sqrt{2M}} \) chord-arc.
The next lemma deals with images of chord arc curves near a zero of $f$.

**Lemma 14.** Suppose $f(0) = 0$ and

$$f(z) = az^m + O(z^{m+1})$$

for some $m > 0$ and $a \in \mathbb{R}^n$. Let $D_r = B(0, r)$ and $D_{j,r} = D_j \cap \{ z : 0 \in [(j-1)\pi/(m+1), j\pi/(m+1)) \}$ for $j = 1, \ldots, 2(m+1)$. If $r > 0$ is sufficiently small, then $F(D_{j,r})$ is an $M$ chord-arc curve with $M$ independent of $a$ and $M$ and

$$\sum_{j=1}^{2(m+1)} \ell(F(\partial D_{j,r})) \leq 2\ell(F(\partial D_r)).$$

The length estimates so far have not depended on dimension.

### 3.1 $|f|$ bounded above and away from zero

**Lemma 15.** Given $\eta > 0$, there is a constant $M = M(\eta)$ such that if $\eta \leq |f| \leq 1$ on a simply connected domain $\varphi$, then there is a partition $\varphi_j$ of $\varphi$ such that $F(\varphi(\varphi_j))$ is a $M$ chord-arc curve for all $j$ and

$$\sum_{j=1}^{\varphi} \ell(F(\varphi(\varphi_j))) \leq M\ell(F(\varphi(\varphi))).$$

Let $\varphi : \mathbb{D} \to \varphi$ be a conformal map, and let $G = F \circ \varphi$ and $g = (f \circ \varphi)\varphi'$. By Green’s theorem,

$$\int \int_D \Delta(|g|) \log \frac{1}{|z|} \frac{dx dy}{2\pi} = \frac{1}{2\pi} \int_{\partial D} |g| - |g(0)|,$$

and by calculus, one may show

$$\frac{|g'|^2}{|g|} \leq \Delta|g| \leq 2\frac{|g'|^2}{|g|}.$$

Thus we have the relation

$$\frac{1}{2\pi} \int_D |g| - |g(0)| = \frac{1}{2\pi} \int_D \Delta|g| \log \frac{1}{|z|} \sim \int \int_D \frac{|g'|^2}{|g|}.$$
above estimates one can show $\log \frac{\|g(z)\|}{\|g(w)\|} \lesssim \frac{|z - w|}{|Q|}$. Thus if $w = z_Q$, since $|z - z_Q| \lesssim |Q|$, 
$$|g(z)| \sim |g(z_Q)|$$
for $z$ in $(T(Q))_{|Q|/10}$ with implied constants independent of the square $Q$. Moreover, for $z$ and $w$ in this neighborhood,
$$|g(z) - g(w)| \leq |g'(c)| |z - w| \lesssim \frac{1}{1 - |c|^2} |g(c)| |z - w| \lesssim |g(c)| \frac{|z - w|}{|Q|}$$
for some point $c \in [z, w]$. Hence, if $S$ is a small square, then for $z \in S$,
$$|g(z) - g(z_S)| \lesssim |g(c)| \frac{|z - z_S|}{|Q|} \lesssim |g(z_Q)| 2^{-N} \sim |g(z_S)| 2^{-N}.$$ 

For any $\delta$, we can choose $N$ sufficiently large (and depending only on $\delta$) such that
$$|g(z) - g(z_S)| < \delta |g(z_S)| \quad (6)$$
for all $z \in S$ for any small square $S \in Q$. 

Note that the constants in this section are completely independent of dimension and estimates restricted to squares $Q$ are universal across all squares. 

We initiate a stopping time argument as follows: Let $Q$ be a dyadic square. 

a. If $\sup_{T(Q)} |g - g(z_Q)| \geq \frac{\delta}{4} g(z_Q)$, set $\mathcal{D}_Q = T(Q)$, say $\mathcal{D}_Q$ is a type 0 region. 

b. If $\sup_{T(Q)} |g - g(z_Q)| < \frac{\delta}{4} g(z_Q)$, let $\{Q_j\}$ be those dyadic squares in $Q$ such that $\sup_{T(Q_j)} |g - g(z_Q)| \geq \delta |g(z_Q)|$ and define $\mathcal{D}_Q = Q \setminus \bigcup Q_j$. Notice that $(1 - \delta/2) |g(z_Q)| < |g(z)| < (1 + \delta/2) |g(z_Q)|$ and hence $|g| \sim |g(z_Q)|$ on $\mathcal{D}_Q$.

   i. If $\ell(\partial \mathcal{D}_Q \cap \partial \mathbb{D}) \geq \frac{1}{4} \ell(\partial \mathcal{D}_Q \cap \partial \mathbb{D})$, say $\mathcal{D}_Q$ is type 1. 

   ii. If $\ell(\partial \mathcal{D}_Q \cap \partial \mathbb{D}) < \frac{1}{4} \ell(\partial \mathcal{D}_Q \cap \partial \mathbb{D})$, say $\mathcal{D}_Q$ is type 2. 

Aside from decomposing the type 0 regions into slightly smaller regions, these three collections will give us a partition $\{\mathcal{D}_j\}$ of $\mathbb{D}$ and $\varphi(\mathcal{D}_j)$ will be the desired partition of $\mathcal{D}$, so it suffices to show that $\sum \ell(\varphi(\partial \mathcal{D}_j))$ is finite, which we verify by estimating the sums of the lengths within each type separately. We begin with the type zero regions. 

Let $\mathcal{D}_Q = T(Q)$ be a type 0 region. 

Claim: 
$$|g(z_Q)|^2 \lesssim \frac{1}{|Q|^2} \int \int_{T(Q)} |g'|^2 \quad (7)$$
with constant independent of dimension. We will reduce this to a question about normal families of analytic functions. Assume we can find a sequence of functions $g_n : (T(Q))_{|Q|/10} \to \mathbb{R}^d$ with analytic components such that $Q$ is type 0 but
$$|g_n(z_n) - g_n(z_Q)| > \frac{\delta}{4} |g_n(z_Q)|$$
for some $z_n \in T(Q)$ and
$$|g_n(z_Q)|^2 > n \frac{1}{|Q|^2} \int \int_{T(Q)} |g'|^2.$$ 

Note that we are allowing the range of $g_n$ to be in any dimension. This will ensure that the constant we get in (7) is dimensionless. Define
$$h_n = \frac{(g_n(z) - g_n(z_Q)) \cdot (g_n(z_n) - g_n(z_Q))}{|g(z_Q)|^2}.$$
Then, since $|g_n| \sim |g_n(z_Q)|$ on $Q$, 
\[
|h_n| \leq \frac{|g_n(z) - g_n(z_Q)| \cdot |g_n(z_n) - g_n(z_Q)|}{|g(z_Q)|^2} \lesssim \frac{|g_n(z_Q)|^2}{|g_n(z_Q)|^2} = 1,
\]
and 
\[
h_n(z_Q) = \frac{|g_n(z_n) - g_n(z_Q)|^2}{|g(z_Q)|^2} \geq \left(\frac{\delta}{4}\right)^2 > 0.
\]
Furthermore, 
\[
\int \int_{T(Q)} |h_n'|^2 = \int \int |g'(z)|^2 \frac{|g_n(z) - g_n(z_Q)|^2}{|g(z_Q)|^4} \lesssim \frac{|g_n(z_Q)|^2}{n} \frac{|g_n(z_n) - g_n(z_Q)|^2}{|g(z_Q)|^4} \sim \frac{1}{n}.
\]
Choose a subsequence $h_{n_k}$ that converges normally to some $h$ analytic on a neighborhood of $T(Q)$ such that $z_{n_k} \to z_0 \in T(Q)$, $h(z_0) = \lim h_{n_k}(z_{n_k}) \geq \left(\frac{\delta}{4}\right)^2 > 0$, $h_{n_k}(z_Q) = 0$, and $h' \equiv 0$ since 
\[
\int \int_{T(Q)} |h'|^2 = \lim \int \int_{T(Q)} |h'_{n_k}|^2 \lesssim \lim \frac{1}{n_k} = 0,
\]
which is a contradiction.

Now, using (6), divide each region $S_Q$ further into $4^N$ sub squares $S$ for $N$ sufficiently large such that $|g - g(z_Q)| < \delta|g(z_Q)|$ on each such $S$, with $N$ independent of $Q$ (and dimension). Then each $\partial S$ is 2 chord-arc and so Lemma [13] implies $F(\varphi(\partial S)) = G(\partial S)$ is $M$ chord-arc for some $M$ depending only on $\delta$, and moreover, 
\[
\sum_{S \subseteq S_Q} F(\varphi(\partial S)) = \sum_{S \subseteq S_Q} G(\partial S) = \int_{\partial S} |g| \lesssim \sum_{S \subseteq S_Q} |g(z_Q)| \ell(S_Q)
\]
and hence 
\[
\sum_{S \subseteq S_Q, \text{Type 0}} \ell(F(\varphi(\partial S))) \lesssim N \sum_{\text{Type 0} \ S_Q} |g(z_Q)| \ell(S_Q)
\]
where the implied constants depend on $\eta$ and $\delta$ (since $N$ depends on $\delta$, and not dimension).

For the type 1 regions $S_Q$, note that the $F(\varphi(S_Q))$ will be chord-arc by Lemma [13] so we don’t need to decompose them as in the previous case. Using the preliminary estimates and the type 1 condition on $g$, it is not difficult to show
\[ \sum_{\text{Type 1 } Q} \ell(F(\partial Q)) \lesssim \ell(F(D)) \]

where the implied constant is based on estimates already established.

Finally, let \( D \) be type 2.

Claim:

\[ |g(z_Q)|^2 \lesssim \beta \eta \int_{\partial Q} |g - g(z_Q)|^2 d\omega, \]

where \( \omega(E) = \omega(z_Q, \partial Q, E) \) is harmonic measure. We replicate the proof from \([5]\), but taking into account that \( g \) is vector valued. Let \( I_j \) be the horizontal line segments of \( D \cap \mathbb{D} \). To prove the above inequality, we first need to show that there exists \( \rho > 0 \) independent of dimension such that

\[ \ell\{ z \in I_j : |g(z) - g(z_Q)| \geq \rho |g(z_Q)| \} \geq \rho \ell(I_j) \]

for all \( j \). Assume that we may find a sequence of functions \( g_n \) satisfying all above conditions and mapping into any dimension such that

\[ \ell\{ z \in I_j : |g_n(z) - g_n(z_Q)| \geq \frac{1}{n} |g_n(z_Q)| \} < \frac{1}{n} \ell(I_j) \]

for some \( j \) (depending on \( n \)). Let \( Q_j \) be the dyadic square with top edge \( I_j \), and let \( Q'_j \) be the square of the same size “stacked” on top of it. Thus we obtain a sequence of uniformly bounded functions \( g_n \) on \( A = Q'_j \cup T(Q_j) \) such that

\[ \ell\{ z \in I_j : |g_n(z) - a_n| \geq \frac{1}{n} |a_n| \} < \frac{1}{n} \ell(I_j) \]

for some vector \( a_n \) with \( |a_n| \leq 1 \), \( |g_n - a_n| < \frac{\delta}{2} |a_n| \) on \( Q'_j \) and \( |g_n(z_n) - a_n| > \frac{\delta}{3} |a_n| \) for some \( z_n \in T(Q_j) \). By rescaling, we may assume \( Q_j = Q_1 \) for all \( n \). Define \( h_n = (g_n - a_n) \cdot (g_n(z_n) - a_n)/|a_n|^2 \). Then the \( h_n \) are uniformly bounded on a neighborhood of \( A, h_n(z_n) > \left( \frac{\delta}{2} \right)^2 \), and

\[ \ell\{ z \in I : |h_n(z)| > \frac{\delta}{2n} \} \leq \ell\{ z \in I : |g_n(z) - a_n|/|g_n(z_n) - a_n| > |a_n|^2 \frac{\delta}{2n} \} \]

\[ \leq \ell\{ z \in I : |g_n(z) - a_n| > |a_n|^2 \frac{\delta}{2n} \} \]

\[ = \ell\{ z \in I : |g_n(z) - a_n| > |a_n|^2 \frac{1}{n} \} < \frac{1}{n} \ell(I). \]

By normal families, a subsequence of these \( h_n \) converges to an analytic function \( h \) with \( h(z_0) \geq \left( \frac{\delta}{2} \right)^2 \) for some \( z_0 \in T(Q_j) \), and \( h \equiv 0 \) on \( I \), which is a contradiction. Again, by proving the statement for \( g \) with range in any dimension (as long as it satisfies all other necessary estimates), the constant \( \rho \) in the claim is independent of dimension.

By results of Jerison and Kenig (see \([6]\) or page 247 of \([3]\)), we know that there exists \( \alpha, C \), constants depending only on the chord arc constant \( M \) such that for all arcs \( I \subseteq \partial Q \), and \( E \subseteq I \),

\[ C^{-1} \left( \frac{\ell(E)}{\ell(I)} \right)^{\frac{\alpha}{2}} \leq \frac{\omega(E)}{\omega(I)} \leq C \left( \frac{\ell(E)}{\ell(I)} \right)^{\alpha}. \]

Let \( E_j = \{ z \in I_j : |g(z) - g(z_Q)| \geq \rho |g(z_Q)| \} \). Using the fact just mentioned with \( E = \bigcup E_j \) and \( I = \partial D \) (so \( \omega(I) = 1 \)), the type 2 estimate, and the
fact that \((\bigcup I_j) = \partial D \setminus (\partial D_Q \cap \partial D)\), we get
\[
\int_{\partial D_Q} |g - g(z_Q)|^2 \, d\omega \geq \sum_{E_j} \int_{E_j} |g - g(z_Q)|^2 \, d\omega \geq \rho^2 |g(z_Q)|^2 \omega(\bigcup E_j)
\]
\[
\geq \rho^2 |g(z_Q)|^2 \left( \frac{\ell(\bigcup E_j)}{\ell(\partial D_Q)} \right)^\alpha
\]
\[
\geq \rho^2 |g(z_Q)|^2 \left( \frac{\sum \rho(I_j)}{\ell(\partial D_Q)} \right)^\alpha
\]
\[
= \rho^2 |g(z_Q)|^2 \left( \frac{\rho \ell(\partial D_Q) - \ell(\partial D_Q \cap \partial D)}{\ell(\partial D_Q)} \right)^\alpha
\]
\[
\geq \rho^2 |g(z_Q)|^2 \left( \frac{\rho \ell(\partial D_Q) - \ell(\partial D_Q \cap \partial D)}{\ell(\partial D_Q)} \right)^\alpha
\]
\[
\geq |g(z_Q)|^2
\]
which thus proves the claim, where the implied constant is universal.

From here on, we can proceed as in [5] using the previous estimates to show that
\[
\sum_{\text{Type 2 } Q} \ell(F(\partial D_Q)) \lesssim \ell(F(\partial D)).
\]

Combining these three estimates for each type together, we are done.

### 3.2 \(|f| \leq 1\)

Now we remove the condition that \(|f| \) be bounded away from zero. For a starting region \(\mathcal{D}\), we decompose it into regions where \(|f| \) is small and where \(|f| \) is large. The places where \(|f| \) is large we can decompose using the work of the previous section, and then repeat this decomposition on the regions where \(|f| \) is small, continuing this process indefinitely. This involves instilling some regularity into our decomposition so that we may repeat it at each level.

**Lemma 16.** There is a constant \(M\) such that if \(|f| \leq 1\) on a simply connected domain \(D \subseteq \mathbb{D}\), then there is a partition \(\mathcal{D}_j\) of \(\mathcal{D}\) such that each \(F(\partial D_j)\) is \(M\) chord-arc and
\[
\sum_{\text{Type 2 } Q} \ell(F(\partial D_Q)) \leq M \ell(\partial D).
\]

#### 3.2.1 The Stopping time process

Let \(\varphi : \mathbb{D} \to \mathcal{D}\) be a conformal map. Let \(Q\) be a dyadic square, and \(\alpha/2 > \varepsilon, N, N' > 0\) be constants to be decided later.

**Case 1:** \(\sup_{T(Q)} |f \circ \varphi| \leq \frac{\alpha}{4}\). Define descending squares to be the squares \(Q_j \subseteq Q\) such that \(\sup_{T(Q_j)} |f \circ \varphi| \geq \alpha\) and let \(\mathcal{Q}(Q) = Q \setminus \bigcup_{j \in \mathcal{E}(Q)} Q_j\) be a bad region of type 1.

**Case 2:** \(\sup_{T(Q)} |f \circ \varphi| > \frac{\alpha}{4}\). Let \(\mathcal{J}(Q)\) be the set of small squares \(S\) such that
\[
\inf_{S} |f \circ \varphi| \leq \varepsilon,
\]
and such that \(B(S)\) is maximal. Define the descendent squares in this case to be \(\{Q(S) : S \in \mathcal{J}(Q)\}\) and the components \(G_j\) of
\[
Q \setminus \bigcup_{S \in \mathcal{E}(Q)} B(S) \cup Q(S)
\]
to be good regions of the first kind. For $S \in \mathcal{S}(Q)$, let $\mathcal{S}'(S)$ be the set of very small squares in $B(S)$ that contain a zero of $f \circ \varphi$ or are adjacent to a very small square that does, call these very small squares bad regions of the second kind. Let the very small squares in $B(S)$ not in $\mathcal{S}'(S)$ be good regions of the second kind.

### 3.2.2 Bounding $|f \circ \varphi|$ on good and bad regions

By Schwarz lemma, we may choose $N$ large enough such that $|(f \circ \varphi) \cdot u| \leq 2\varepsilon < \alpha$ on all bad regions of type 2. In particular, we pick $N$ so that this holds for any analytic function $h$ defined on the disk such that $\inf_S |h| \leq \varepsilon$ for all $S \in \mathcal{S}(Q)$. By taking $u$ to range over all unit vectors, we hence know $|f \circ \varphi| < \alpha$ on all bad regions of type 2, and of course the same holds on bad regions of type 1 by definition.

We’d like to obtain a lower bound for $|f \circ \varphi|$ on the good regions. By definition, we know $|f \circ \varphi| \geq \varepsilon$ on good regions of first type, so it suffices to figure out a lower bound for it on the second type region, a lower bound independent of dimension. Let $S \in \mathcal{S}(Q)$ and suppose $\inf_S |z| = \inf_Q |z|$. Then $|f \circ \varphi(z')| \geq \varepsilon$ for some $z'$ on the top edge of $B(S)$. The reason for this is that, if this weren’t the case, then $\inf_{E_0} |f \circ \varphi| < \varepsilon$, where $E_0$ is the small square immediately above $S$, and hence $S$ would not be maximal. Choose a unit vector $u$ such that $f \circ \varphi(z') \cdot u \geq \varepsilon$ and define $g = (f \circ \varphi) \cdot u$.

**Claim:** There exists $\eta = \eta(N, N', \varepsilon) > 0$ such that, for any analytic function $g$ on $(B(S))_{\ell(\partial S)/8}$ satisfying $|g| \leq 1$, $g(z') \geq \varepsilon$ for some $z'$ on the top of $B(S)$, then $|g| \geq \eta$ on the set obtained from $B(S)$ by excluding the $2^{-N} \ell(\partial S)/4$-neighborhood of $Z(S)$, the set zeros in $(B(S))_{\ell(\partial S)/8}$.

Notice that the good regions of second type are contained in this set and that this result will give a lower bound for $f \circ \varphi$ on these regions by Cauchy-Schwarz that independent of dimension (since this bound holds for general analytic functions). Thus there are no dimensional concerns here and we refer the reader to [H] for the details. We will mention, however, that the proof requires showing that for such a function $g$ satisfying the condition of the claim that

$$\# Z(S) \lesssim 2^N \log \frac{1}{\varepsilon}$$

which also gives an upper bound on the number of second type bad regions in $B(S)$, which we will need later.

A similar universal bound for $f \circ \varphi$ on good regions of the second type will hold if $\inf_S |z| = \inf_Q |z|$ since $S$ and $Q$ are comparable and we can find a point on the top edge of $T(Q)$ such that $|f \circ \varphi| \geq \alpha/2 > \varepsilon$. Here are the facts we have obtained:

- a. $|f \circ \varphi| \leq \alpha$ on all bad regions,
- b. $|f \circ \varphi| \geq \min\{\eta, \varepsilon\}$ on all good regions, $\eta$ universal and only dependent on $\varepsilon, N'$, and $N$.
- c. There are at most a constant times $2^N \log \frac{1}{\varepsilon}$ bad regions in any $B(S)$ for $S \in \mathcal{S}(Q)$.

Notice so far that the only dependency is $N$ on $\alpha$, since we choose it sufficiently large to apply Schwarz lemma.

### 3.2.3 Carleson estimates for boundaries of good and bad regions

Now we’d like to show that length on the boundaries of good and bad regions are Carleson measures. We first prove the following:
Lemma 17. For $\alpha > 0$, there exist $N$ and $\varepsilon_0 > 0$ such that for any case 2 dyadic square $Q$,
$$\sum \{ \ell(\partial Q_j) : Q_j \text{ a descendent of } Q \} \leq \frac{\ell(\partial Q)}{100}.$$  

Proof. By Schwarz lemma, we may pick $N$ such that if $|h| \leq 1$ is any analytic function on $D$ satisfying $\inf_S |h| \leq \varepsilon < \varepsilon_0$, then $\sup_S |h| \leq 2\varepsilon_0$. Letting $h = f \circ \varphi \cdot u$ and letting $u$ vary over all unit vectors, the same holds for $|f \circ \varphi|$. Since $Q$ is a case 2 square, we may choose $u$ a unit vector such that $g = f \circ \varphi \cdot u > \alpha/2$ at some point in $T(Q)$.

We need the following theorem from [2]:

Theorem 18. Let $f$ be a bounded analytic function on the upper half plane, $|f| \leq 1$. For all $\beta \in (0, 1)$ and $\varepsilon' \in (0, 1)$, there exists $A = A(\beta, \varepsilon) \in (0, 1)$ such that for any dyadic square $Q$, if
$$\sup_{T(Q)} |f| \geq \beta,$$  
then
$$\ell(E^*_\alpha) = \ell(\{ z \in Q : |f(z)| \leq A \}^*) < \varepsilon' \ell(\partial Q).$$  

Applying this theorem with $\beta = \alpha/2$, $\varepsilon' = \frac{1}{100}$, there exists $2\varepsilon_0 = A$ such that $\varepsilon < \varepsilon_0$ implies
$$\ell(\{ z \in Q : |g| \leq 2\varepsilon_0 \})^* \leq \frac{\ell(\partial Q)}{100}.$$  

Since for any case 2 square $Q$, the good regions obtained are $\sim 2^{-N-N'}$ chord-arc, the sum of their lengths are $\lesssim_{N,N'} \ell(\partial Q)$, and so it isn’t difficult to show that
$$\sum \ell(G_i \cap Q) \lesssim \ell(\partial Q).$$  

For the bad regions, we will need a better estimate that is independent of $N$,
$$\sum \ell(\partial B_j \cap Q') \lesssim \ell(\partial Q')$$  
where the implied constant is universal. To see this, notice that all case 2 squares are case 1 descendents, and since bad regions of the first type are all 6 chord-arc,

$$\sum_{B_j} \ell(\partial B_j \cap Q')$$
$$= \sum \{ \ell(\partial B_j) : B_j \text{ case 1 bad region of } Q' \text{ or a descending } Q \}$$
$$\lesssim \sum \{ \ell(\partial Q) : Q \text{ descendant of } Q' \} \lesssim \ell(\partial Q').$$
and, if \(C(Q)\) is the set containing \(Q\) and its descending squares,

\[
\sum_{B_j} \ell(B_j \cap Q') = \sum_{Q \in C(Q')} \sum_{S \in \mathcal{S}(Q)} \sum_{B_j \subseteq B(S)} \ell(\partial B_j)
\]

\[
\lesssim \sum_{Q \in C(Q')} \sum_{S \in \mathcal{S}(Q)} \left(2^N \log \frac{1}{\varepsilon}\right)(2^{-N'} \ell(\partial S))
\]

\[
\leq \sum_{Q \in C(Q')} \sum_{S \in \mathcal{S}(Q)} \ell(\partial S) \lesssim \ell(\partial Q')
\]

where the last inequality follows if we fix \(N'\) large enough.

By Carleson’s theorem,

\[
\sum_{G_i} \int_{\varphi(G_i)} |f| = \sum_{G_i} \int_{\partial G_i} |f \circ \varphi| \cdot |\varphi'| \lesssim_N \int_{\partial B} |f \circ \varphi| \cdot |\varphi'| = \int_{\partial B} |f|
\]

and

\[
\sum_{B_i} \int_{\partial B_i} |\varphi'| \leq C \int_{\partial B} |\varphi'| = C \ell(\partial \mathcal{G}).
\]

where \(C\) is a universal constant.

### 3.2.4 Inductive process

Now we repeat this process starting with each bad region \(B_j\) independently instead of \(\mathcal{G}\) and \(\sum_{B_j} \ell(\partial B_j)\). Inductively, we choose a conformal map \(\varphi_w : \mathbb{D} \to B_w\), and decompose \(\mathbb{D}\) into good and bad regions \(G_{w_j}\) and \(B_{w_j}\), where \(w\) is some word of length at least \(1\). Define a map \(T_w\) inductively by \(T_{w_j} = T_w \circ \varphi_{w_j}\) and \(T_j = \varphi \circ \varphi_j\). Notice that \(|f \circ T_w| \leq \alpha^{|w|}\) on \(\mathbb{D}\) (i.e. \(|f| \leq \alpha^{|w|}\) on the region \(T_w(\mathbb{D}) \subseteq \mathcal{G}\)). Using this, a change of variables and Carleson’s theorem one may get

\[
\sum_{|w| = n} \sum_{j} \int_{\partial T_{w_j}(\partial G_{w_j})} |f| \lesssim_N \alpha^n C^n \ell(\partial \mathcal{G}).
\]

Since \(T_{w_j}(\partial \mathbb{D}) = T_w(\partial B_w)\), by a change of variables and Carleson’s theorem,

\[
\sum_{|w| = n+1} \int_{\partial B} |T_w| \leq C \sum_{|w| = n} \int_{\partial B} |T_w| \leq \cdots \leq C^{n+1} \ell(\partial \mathcal{G})
\]

by induction. Fix \(\alpha < \frac{1}{4}\), which now fixes \(N\). Summing over all good regions and letting \(w'\) be the word \(w\) shortened one letter, we get that

\[
\sum_{w} \int_{\partial T_{w'}(G_w)} |f| = \sum_{n} \sum_{|w| = n} \int_{\partial T_{w'}(\partial G_w)} |f| \lesssim \sum_{n} (C\alpha)^n \ell(\partial \mathcal{G}) \lesssim \ell(\partial \mathcal{G}).
\]

If there is only one zero \(\xi\) of \(f \circ T_{w'}\) in \(B_w\), choose \(\varphi_w : \mathbb{D} \to B_w\) such that \(\varphi_w(0) = \xi\) and apply Lemma \([\text{[E]}]\) and the fact that, for any \(r < 1\),

\[
\int_{|z| = r} |f \circ T_w| \cdot |T_w| \leq \int_{\partial B} |f \circ T_w| \cdot |T_w| = \int_{B_w} |f \circ T_{w'}| \cdot |T_{w'}|
\]

so the regions obtained by the lemma will at most double the estimate we obtain at each stage, in which case we redefine the starting regions for the stopping time argument to not include the ball of radius \(r\). Performing this operation in the exception that there is only one zero prevents the process from continuing indefinitely around a zero of \(f \circ \varphi\).
3.3 $|f|$ not bounded above

Suppose now that $|f|$ is not bounded above by 1. Choose $r_0$ such that the convex hull $C_\theta$ of $e^{i\theta}$ and $B(0, r_0)$ contains $T(Q)$ for all $Q$ with $e^{i\theta} \in Q^\ast$. Define

$$f^\ast(\theta) = \sup\{|f(z)| : z \in C_\theta\}.$$ 

Since $|f|$ has a harmonic extension $h$ to $\mathbb{D}$ and $|f|$ is subharmonic on $\mathbb{D}$, we know

$$f^\ast(\theta) \leq h^\ast(\theta) \lesssim \mathcal{M}(h)(\theta) = \mathcal{M}(f)(\theta)$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. The Hardy-Littlewood inequality in 1 dimension gives

$$\int f^\ast(\theta) \lesssim \int |f| = \sqrt{2}\ell(\Gamma)$$

where the implied constant is independent of dimension.

For any dyadic square $Q$, let $m = m(Q)$ be such that $2^{m-1} \leq \sup_{T(Q)} |f| < 2^m$. Let $Q_j$ be the descendent squares such that $\sup_{T(Q_j)} |f| \geq 2^m$, and define $D_Q = Q \setminus \bigcup Q_j$. If $e^{i\theta} \in Q^\ast$, then $f^\ast(\theta) \geq 2^{m-1}$ and $\ell(D_Q) \lesssim \ell(\partial Q^\ast)$.

Repeat this process on the descending squares. Let $D^m$ be the union of all regions $D_Q$ with $m(Q) = m$. By Lemma 13 we can decompose $D^m$ into subregions $D_{m,j}$ such that

$$\sum_j \ell(F(\partial D^m_{m,j})) \lesssim 2^m \ell(\partial D^m) \lesssim \sum_{m(Q) = m} 2^m \ell(\partial D_Q) \lesssim \sum_{m(Q) = m} 2^m \ell(\partial Q^\ast)$$

$$\leq 2^m \ell\{\theta : f^\ast(\theta) > 2^{m-1}\}$$

since $f^\ast \geq 2^{m-1}$ on squares $Q$ with $m(Q) = m$. Therefore,

$$\sum_{m,j} \ell(F(\partial D^m_{m,j})) \lesssim \sum_m 2^m \ell\{\theta : f^\ast(\theta) > 2^{m-1}\} \sim \int_{\partial D} f^\ast \lesssim \ell(\Gamma).$$

Once again, since the only new constant introduced is from the Hardy-Littlewood maximal inequality, all constants are independent of dimension.

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