Two-Photon Algebra and Integrable Hamiltonian Systems

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Abstract

The two-photon algebra $h_6$ is used to define an infinite class of $N$-particle Hamiltonian systems having $(N-2)$ additional constants of the motion in involution. By construction, all these systems are $h_6$-coalgebra invariant. As a straightforward application, a new family of (quasi)integrable $N$-dimensional potentials is derived.

1 Introduction

In a recent paper [1], a systematic construction of integrable Hamiltonians with coalgebra symmetry has been proposed. Such procedure can be applied to any Poisson coalgebra $(A, \Delta)$ with generators $X_i, \ i = 1, \ldots, l$ and Casimir element $C(X_1, \ldots, X_l)$ as follows. Let us consider the $N$-th coproduct $\Delta^{(N)}(X_i)$ of the generators and the $m$-th order $(2 \leq m \leq N)$ coproducts $\Delta^{(m)}(C)$ of the Casimir operator of the coalgebra (recall that the $m$-th coproduct is an algebra homomorphism that maps $\Delta^{(m)} : A \rightarrow A \otimes A \otimes \cdots \otimes A$).

By making use of the structural properties of the coproduct it can be proven that

$$\{\Delta^{(m)}(C), \Delta^{(N)}(X_i)\} = 0, \quad i = 1, \ldots, l.$$  \hfill (1.1)

Therefore, the $(2 \leq m \leq N)$ coproducts of the Casimir operator commute with the $N$-th order coproduct of any generator of the coalgebra. This implies that, if $H$ is an arbitrary (smooth/formal power series) function of the generators of the algebra $A$, any $N$-particle Hamiltonian defined as

$$H^{(N)} := \Delta^{(N)}(H(X_1, \ldots, X_l)) = H\left(\Delta^{(N)}(X_1), \ldots, \Delta^{(N)}(X_l)\right),$$  \hfill (1.2)

Poisson-commutes with all the $(N-1)$ functions $C^{(m)} = \Delta^{(m)}(C)$:

$$\{C^{(m)}, H^{(N)}\} = 0, \quad 2 \leq m \leq N.$$  \hfill (1.3)

Furthermore, all the $C^{(m)}$ constants of the motion are in involution

$$\{C^{(m)}, C^{(n)}\} = \{\Delta^{(m)}(C), \Delta^{(n)}(C)\} = 0, \quad \forall \ m, n = 2, \ldots, N.$$  \hfill (1.4)

So far, this formalism has been considered for $sl(2)$, $(1+1)$ Poincaré and oscillator $h_4$ Poisson coalgebras [1]–[3] under certain phase space realizations. In all these cases, the systems obtained through the coalgebra formalism turned out to be completely integrable due
Explicitly, the two-photon Lie–Poisson coalgebra \((h_6)\) is important to recall that and the Casimir \((1.7)\).

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\(\text{two-photon algebra} + \{N,A\}\)

\(\map\Delta\)

Let us start with the construction of two-particle systems. In this case, the coproduct \(\{N,A_+,A_-,B_+,B_-,M\}\) together with the Poisson brackets

\[
\begin{align*}
\{N,A_+\} &= A_+, & \{N,A_-\} &= -A_-, & \{A_-,A_+\} &= M, \\
\{N,B_+\} &= 2B_+, & \{N,B_-\} &= -2B_-, & \{B_-,B_+\} &= 4N + 2M, \\
\{A_+,B_-\} &= -2A_-, & \{A_+,B_+\} &= 0, & \{M,\cdot\} &= 0, \\
\{A_-,B_+\} &= 2A_+, & \{A_-,B_-\} &= 0
\end{align*}
\]

and the (non-deformed) two-body coproduct

\[
\Delta^{(2)}(X) = X \otimes 1 + 1 \otimes X, \quad X \in \{N,A_+,A_-,B_+,B_-,M\}.
\]

The coproduct \(\Delta \equiv \Delta^{(2)}\) is a Poisson algebra homomorphism between \(h_6\) and \(h_6 \otimes h_6\). It is important to recall that \(h_6\) has two Casimir functions: the mass \(M\) and a fourth-order Casimir given by

\[
C_{h_6} = (MB_+ - A_+^2) (MB_- - A_-^2) - (MN - A_-A_+ + M^2/2)^2,
\]

which will play a relevant role in what follows. The one-particle phase space realization \(D\) for \(h_6\) that we shall use is given by

\[
\begin{align*}
\bar{f}_N^{(1)} &= D(N) = q_1 p_1 - \frac{1}{2} \mu_1, & \bar{f}_A_+^{(1)} &= D(A_+) = p_1, \\
\bar{f}_A_-^{(1)} &= D(A_-) = \mu_1 q_1, & \bar{f}_M^{(1)} &= D(M) = \mu_1, \\
\bar{f}_B_+^{(1)} &= D(B_+) = \frac{1}{\mu_1} p_1^2, & \bar{f}_B_-^{(1)} &= D(B_-) = \mu_1 q_1^2.
\end{align*}
\]

This phase space representation is labelled by the values of the Casimirs:

\[
\bar{f}_M^{(1)} = D(M) = \mu_1, \quad C_{h_6}^{(1)} = D(C_{h_6}) = 0.
\] (1.9)

The aim of this contribution is to present a summary of the integrability properties of the \(h_6\) systems with coalgebra symmetry obtained from \([1]\) through the realization \((1.8)\) and the Casimir \((1.7)\).

### 2 Hamiltonians with \(h_6\)-coalgebra symmetry

Let us start with the construction of two-particle systems. In this case, the coproduct map \(\Delta^{(2)}\) \((1.6)\) gives us, under a \(D \otimes D\) realization, six two-particle phase space functions:

\[
\begin{align*}
\bar{f}_N^{(2)} &= (D \otimes D)(\Delta(N)) = (q_1 p_1 - \frac{1}{2} \mu_1) + (q_2 p_2 - \frac{1}{2} \mu_2), \\
\bar{f}_A_+^{(2)} &= (D \otimes D)(\Delta(A_+)) = p_1 + p_2, \\
\bar{f}_A_-^{(2)} &= (D \otimes D)(\Delta(A_-)) = \mu_1 q_1 + \mu_2 q_2, \\
\bar{f}_B_+^{(2)} &= (D \otimes D)(\Delta(B_+)) = \frac{1}{\mu_1} p_1^2 + \frac{1}{\mu_2} p_2^2, \\
\bar{f}_B_-^{(2)} &= (D \otimes D)(\Delta(B_-)) = \mu_1 q_1^2 + \mu_2 q_2^2, \\
\bar{f}_M^{(2)} &= (D \otimes D)(\Delta(M)) = \mu_1 + \mu_2.
\end{align*}
\] (2.1)
It is straightforward to check that these functions define a two-particle phase space realization of $h_6$, provided the canonical Poisson bracket $\{q_i, p_j\} = \delta_{ij}$ is considered.

By following the construction [1], any smooth function of the $f_X^{(2)}$ functions will Poisson-commute with the $\Delta^{(2)}$ map of the Casimirs of the $h_6$ algebra. However, in this particular case this statement does not provide any dynamical information, since the two-particle integrals of motion provided by the two Casimirs $M$ and $C$ are trivial:

$$f_M^{(2)} = (D \otimes D)(\Delta(M)) = \mu_1 + \mu_2,$$

(2.2)

$$C_{h_6}^{(2)} = (D \otimes D)(\Delta(C_{h_6})) = \left( f_M^{(2)} - \left( f_{A+}^{(2)} \right)^2 \right) \left( f_M^{(2)} - \left( f_{A-}^{(2)} \right)^2 \right)$$

$$- \left( f_M^{(2)} f_{A+}^{(2)} - f_M^{(2)} f_{A-}^{(2)} + \frac{1}{2} \left( f_M^{(2)} \right)^2 \right)^2 = 0.$$  

(2.3)

However, this degeneracy is removed in the three particle case. The third-order co-product $\Delta^{(3)}$ of any generator $X$ reads

$$\Delta^{(3)}(X) = X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X,$$

(2.4)

so that the 3-dimensional phase space realization of $(h_6, \Delta)$ is

$$f_N^{(3)} = (q_1 p_1 + q_2 p_2 + q_3 p_3) - \frac{1}{2} (\mu_1 + \mu_2 + \mu_3), \quad f_M^{(3)} = \mu_1 + \mu_2 + \mu_3,$$

$$f_{A+}^{(3)} = p_1 + p_2 + p_3, \quad f_{A-}^{(3)} = \mu_1 q_1 + \mu_2 q_2 + \mu_3 q_3,$$

$$f_{B+}^{(3)} = \frac{1}{m!} p_1^2 + \frac{1}{m!} p_2^2 + \frac{1}{m!} p_3^2, \quad f_{B-}^{(3)} = \mu_1 q_1^2 + \mu_2 q_2^2 + \mu_3 q_3^2.$$  

(2.5)

As in the previous case, the integrals of motion coming from $M$ are trivial:

$$f_M^{(1)} = \mu_1, \quad f_M^{(2)} = \mu_1 + \mu_2, \quad f_M^{(3)} = \mu_1 + \mu_2 + \mu_3,$$

(2.6)

but we find now a first non-trivial integral of motion provided by the Casimir $C_{h_6}$:

$$C_{h_6}^{(3)} = \frac{\mu_1 + \mu_2 + \mu_3}{\mu_1 \mu_2 \mu_3} (p_1 (q_2 - q_3) \mu_2 \mu_3 + p_2 (q_3 - q_1) \mu_1 \mu_3 + p_3 (q_1 - q_2) \mu_1 \mu_2)^2.$$  

(2.7)

We stress that $C_{h_6}^{(3)}$ is, by construction, in involution with any function $H^{(3)}$ of the three-particle representation of the generators (2.5). Therefore, if $H^{(3)}$ is considered as the Hamiltonian of a three-particle system, we would need another integral of motion in involution (and functionally independent) from $C_{h_6}^{(3)}$ in order to ensure complete integrability.

The generalization to an arbitrary number of particles is straightforward. The $m$-th coproduct $\Delta^{(m)}$ is

$$\Delta^{(m)}(X) = X \otimes 1 \otimes 1 \otimes \ldots \otimes m-1 \otimes 1$$

$$+ 1 \otimes X \otimes 1 \otimes \ldots \otimes m-2 \otimes 1 + \cdots + 1 \otimes 1 \otimes \ldots m-1 \otimes 1 \otimes X.$$  

(2.8)

Hence the $m$-dimensional particle phase space realization of $(h_6, \Delta)$ turns out to be:

$$f_N^{(m)} = \sum_{i=1}^{m} (q_i p_i - \frac{1}{2} \mu_i), \quad f_M^{(m)} = \sum_{i=1}^{m} \mu_i, \quad f_{A+}^{(m)} = \sum_{i=1}^{m} p_i,$$

$$f_{A-}^{(m)} = \sum_{i=1}^{m} \mu_i q_i, \quad f_{B+}^{(m)} = \frac{1}{m!} \sum_{i=1}^{m} p_i^2, \quad f_{B-}^{(m)} = \sum_{i=1}^{m} \mu_i q_i^2.$$  

(2.9)
An $N$-dimensional Hamiltonian $H^{(N)}$ with $h_6$-coalgebra symmetry will be defined through an arbitrary smooth function $\mathcal{H}$ of the two-photon generators (2.9) for $m = N$:

$$H^{(N)} = \mathcal{H}\left(f^{(N)}_N, f^{(N)}_M, f^{(N)}_{A_{+}}, f^{(N)}_{A_{-}}, f^{(N)}_{B_{+}}, f^{(N)}_{B_{-}}\right). \quad (2.10)$$

The central generator $M$ gives rise to $N$ trivial integrals of motion

$$f^{(m)}_M = \sum_{i=1}^{m} \mu_i, \quad m = 1, \ldots, N, \quad (2.11)$$

while the other Casimir provides $(N - 2)$ non-trivial integrals of motion $C^{(m)}_{h_6}$ ($m = 3, \ldots, N$), which are in involution and given by

$$C^{(m)}_{h_6} = \left(f^{(m)}_M f^{(m)}_B - \left(f^{(m)}_{A_{+}}\right)^2\right) \left(f^{(m)}_M f^{(m)}_B - \left(f^{(m)}_{A_{-}}\right)^2\right) - \left(f^{(m)}_M f^{(m)}_N - f^{(m)}_{A_{-}} f^{(m)}_{A_{+}} + \frac{1}{2} \left(f^{(m)}_M\right)^2\right)^2. \quad (2.12)$$

Cumbersome computations lead to the following explicit expressions for $C^{(m)}_{h_6}$ in terms of $m$-pairs of dynamical variables

$$C^{(m)}_{h_6} = \left(\sum_{s=1}^{m} \mu_s\right) \sum_{i=1}^{m} \sum_{r<s}^{m} \mu_r \mu_s (q_r - q_s)^2 + 2 \sum_{i<j}^{m} p_i p_j \sum_{k \neq i,j}^{m} \mu_k (q_k - q_i) (q_j - q_k) + \sum_{i<j<k}^{m} \frac{[p_i (q_j - q_k) \mu_j \mu_k + p_j (q_k - q_i) \mu_i \mu_k + p_k (q_i - q_j) \mu_i \mu_j]^2}{\mu_i \mu_j \mu_k}. \quad (2.13)$$

### 3 Integrability properties of $h_6$ systems

We have just shown that, given any dynamical Hamiltonian $\mathcal{H}$ defined on $h_6$, the associated $N$-particle system given by $H^{(N)} = \Delta^{(N)}(\mathcal{H})$ fulfills

$$\{C^{(m)}_{h_6}, H^{(N)}\} = 0, \quad \{C^{(m)}_{h_6}, C^{(n)}_{h_6}\} = 0, \quad m, n = 3, \ldots, N. \quad (3.1)$$

Therefore, for an arbitrary $h_6$ system there is only one integral of the motion left in order to ensure complete integrability. However, we should distinguish between two classes of $h_6$ Hamiltonians:

a) If $\mathcal{H}$ is defined on a subalgebra of $h_6$, $\mathcal{H}$ will be also in involution with the Casimir of the subalgebra. Therefore, provided the coproducts of this new Casimir are neither trivial under the realization (2.9) nor functionally dependent of the $C^{(m)}_{h_6}$ integrals, we obtain an additional set of constants of the motion. Under these conditions, these $h_6$ systems defined on subalgebras will be not only completely integrable, but superintegrable. In particular, it can be proven that this procedure gives a new algebraic construction of the superintegrability of the isotropic $N$-dimensional harmonic oscillator [9].
b) On the contrary, if $\mathcal{H}$ is not defined on a subalgebra of $h_6$, the formalism will provide only the $(N - 2)$ integrals $C^{(m)}_{h_6}$ (we could say that in this case $H^{(N)}$ is a "quasi-integrable" Hamiltonian). An interesting example is provided by the dynamical Hamiltonian given by

$$\mathcal{H} = \frac{1}{2} B_+ + \mathcal{F}(A_-) + \mathcal{G}(B_-),$$

(3.2)

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary smooth functions. This choice for $\mathcal{H}$ defines a new (and very large) family of natural Hamiltonian systems of the type

$$H^{(N)} = \frac{1}{2} \left( \sum_{i=1}^{N} \frac{1}{\mu_i} p_i^2 \right) + \mathcal{F} \left( \sum_{i=1}^{N} \mu_i q_i \right) + \mathcal{G} \left( \sum_{i=1}^{N} \mu_i q_i^2 \right),$$

(3.3)

that will always Poisson-commute with the functions $C^{(m)}_{h_6}$. Obviously, this construction does not exclude that, for a certain choice of the functions $\mathcal{F}$ and $\mathcal{G}$, more independent integrals could exist. A more extensive description of this kind of Hamiltonians will be presented elsewhere [9].

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