On the binomial convolution of arithmetical functions

LÁSZLÓ TÓTH
University of Pécs, Institute of Mathematics and Informatics
Ifjúság u. 6, 7624 Pécs, Hungary
E-mail address: ltoth@ttk.pte.hu

PENTTI HAUKKANEN
Department of Mathematics and Statistics,
FI-33014 University of Tampere, Finland
E-mail address: pentti.haukkanen@uta.fi

Abstract: Let \( n = \prod_p p^{\nu_p(n)} \) denote the canonical factorization of \( n \in \mathbb{N} \). The binomial convolution of arithmetical functions \( f \) and \( g \) is defined as \( (f \circ g)(n) = \sum_{d|n} \left( \prod_p \left( \binom{\nu_p(n)}{\nu_p(d)} \right) f(d)g(n/d) \right) \), where \( \binom{\nu_p(n)}{\nu_p(d)} \) is the binomial coefficient. We provide properties of the binomial convolution. We study the \( \mathbb{C} \)-algebra \((\mathcal{A}, +, \circ, \mathbb{C})\), characterizations of completely multiplicative functions, Selberg multiplicative functions, exponential Dirichlet series, exponential generating functions and a generalized binomial convolution leading to various M"{o}bius-type inversion formulas. Throughout the paper we compare our results with those of the Dirichlet convolution \( \ast \). Our main result is that \((\mathcal{A}, +, \circ, \mathbb{C})\) is isomorphic to \((\mathcal{A}, +, \ast, \mathbb{C})\). We also obtain a “multiplicative” version of the multinomial theorem.

Key Words and Phrases: multiplicative arithmetical function, Dirichlet convolution, Dirichlet series, binomial convolution, generating function, M"{o}bius inversion, multinomial theorem

Mathematics Subject Classification: 11A25, 05Axx.

1. Introduction

Let \( \mathcal{A} \) denote the set of arithmetical functions \( f : \mathbb{N} \to \mathbb{C} \). It is well known that \( \mathcal{A} \) is a \( \mathbb{C} \)-algebra under the linear operations and the Dirichlet convolution defined by

\[
(f \ast g)(n) = \sum_{d|n} f(d)g(n/d).
\]

This \( \mathbb{C} \)-algebra is denoted as \((\mathcal{A}, +, \ast, \mathbb{C})\). It is isomorphic to the \( \mathbb{C} \)-algebra of formal Dirichlet series \( D(f, s) = \sum_{n=1}^{\infty} f(n)n^{-s} \), denoted as \((D, +, \cdot, \mathbb{C})\). Then \( D(f \ast g, s) = D(f, s)D(g, s) \) and the mapping \( f \mapsto D(f, s) \) serves as an isomorphism. Furthermore, \((\mathcal{A}, +, \ast)\) is an integral domain; it is even a unique factorization domain, cf. \([12, 19]\).

Let \( n = \prod_p p^{\nu_p(n)} \) denote the canonical factorization of \( n \in \mathbb{N} \). The binomial convolution of arithmetical functions \( f \) and \( g \) is defined as

\[
(f \circ g)(n) = \sum_{d|n} \left( \prod_p \left( \binom{\nu_p(n)}{\nu_p(d)} \right) f(d)g(n/d) \right),
\]

where \( \binom{n}{k} \) is the binomial coefficient. This convolution appears in the book by P. J. McCarthy \([12]\) p. 168, and its basic properties were investigated by P. Haukkanen \([7]\), see also \([15]\) p. 116. It was
pointed out in [2] that the binomial convolution (2) possesses properties analogous to those of the Dirichlet convolution [1]. For example, \( \{ f \in A : f(1) \neq 0 \} \) is a commutative group and the set of all multiplicative arithmetical functions \( f \) forms a subgroup of this group. It is remarkable that the binomial convolution preserves complete multiplicativity of arithmetical functions. The set of completely multiplicative functions forms a subgroup of the group of multiplicative functions under the binomial convolution. This is not the case for the Dirichlet convolution. Note also that the \( \varepsilon \)-continuous functions form a subgroup of the group of multiplicative functions under the Dirichlet convolution.

In this paper we provide further properties of the binomial convolution. We study the \( \mathbb{C} \)-algebra \((A, +, \circ, \mathbb{C})\), characterizations of completely multiplicative functions, Selberg multiplicative functions, exponential Dirichlet series, exponential generating functions and a generalized binomial convolution leading to various Möbius-type inversion formulas. Throughout the paper we compare our results with those of the Dirichlet convolution. Our main result is that \((A, +, \circ, \mathbb{C})\) is isomorphic to \((A, +, \ast, \mathbb{C})\). We also obtain a “multiplicative” version of the multinomial theorem.

2. The algebra \((A, +, \circ, \mathbb{C})\)

It is easy to see that \((A, +, \circ, \mathbb{C})\) is a \( \mathbb{C} \)-algebra. In this section we show that \((A, +, \circ, \mathbb{C})\) is isomorphic to \((A, +, \ast, \mathbb{C})\) and compare the expressions and inverses of the convolutions in these algebras.

The function \( \xi \) defined by \( \xi(n) = \prod_p \nu_p(n)! \) plays a crucial role in connections between the Dirichlet and binomial convolution. We recall that an arithmetical function \( f \) is said to be multiplicative if \( f(1) = 1 \) and \( f(mn) = f(m)f(n) \), whenever \( (m, n) = 1 \), and completely multiplicative if \( f(1) = 1 \) and \( f(mn) = f(m)f(n) \) for all \( m \) and \( n \). The function \( \xi \) is multiplicative and prime independent. Also, \( \xi(n) = 1 \) if and only if \( n \) is squarefree. The function \( \xi \) is not completely multiplicative. However, \( \xi(m)\xi(n) \mid \xi(mn) \) for all \( m, n \geq 1 \). In fact, one has \( a! b! \mid (a + b)! \) for all \( a, b \geq 1 \) and hence \( \xi(p^a)\xi(p^b) = a! b! \mid (a + b)! = \xi(p^{a+b}) \) for all prime powers \( p^a, p^b \). Therefore \( \xi(m)\xi(n) \mid \xi(mn) \) and in particular \( \xi(m)\xi(n) \leq \xi(mn) \).

\[ f + g \equiv f + g \quad \text{and} \quad sf \equiv sf \equiv s g. \]

Moreover,

\[
(f \circ g)(n) = \sum_{d|n} \left( \prod_p \frac{\nu_p(n)!}{\nu_p(d)!\nu_p(e)!} \right) f(d)g(e) = \xi(n) \sum_{d|n} f(d)g(e) = \xi(n) \left( \frac{f}{\xi} \ast \frac{g}{\xi} \right)(n),
\]

showing that

\[ f \circ g = \xi \left( \frac{f}{\xi} \ast \frac{g}{\xi} \right). \]

Thus, \( f \circ g \equiv \frac{f \circ g}{\xi} \equiv \frac{f}{\xi} \ast \frac{g}{\xi} \). This shows that \( f \rightarrow \frac{f}{\xi} \) is an algebra isomorphism.

Corollary 2.1. \((A, +, \circ)\) is an integral domain. It is even a unique factorization domain.

Equation (3) expresses the binomial convolution in terms of the Dirichlet convolution. On the other hand, we have

\[ f \ast g = \frac{f \xi \ast g\xi}{\xi} \]
Remark 2.1. The function \( f \) in Theorem 2.3.

\[ f^{-1} = \xi \left( \frac{f}{\xi} \right)^{-1 \ast} \]

and

\[ f^{-1 \ast} = \left( \frac{\xi f}{\xi} \right)^{-1} \ast. \]

Proof. We have \( f \circ f^{-1} = \delta \) and from (6) we obtain \( \xi \left( \frac{f \ast f^{-1}}{\xi} \right) = \delta \) or \( f \ast f^{-1} = \delta \). This proves Theorem 2.2.

Example 2.1. For \( f = \xi \) we have \( \xi^{-1} = \xi \ast \xi = \xi \mu = \mu \). Thus, the inverse of \( \xi \) under the binomial convolution is the Möbius function \( \mu \), see [7, p. 213].

A further result involving the binomial and Dirichlet inverses is presented below. This result involves multiplicative functions.

Theorem 2.3. If \( f \) is multiplicative and \( f(p^a) = 0 \) for all prime powers \( p^a \) with \( a \geq 2 \), then for every \( n \geq 1 \),

\[ f^{-1}(n) = (-1)^{\Omega(n)} \xi(n) \prod_p f(p)^{\nu_p(n)} = \lambda(n) \xi(n) \prod_p f(p)^{\nu_p(n)} \]

and

\[ f^{-1 \ast}(n) = (-1)^{\Omega(n)} \prod_p f(p)^{\nu_p(n)} = \lambda(n) \prod_p f(p)^{\nu_p(n)}. \]

Proof. Let \( p^a \) be a prime power with \( a \geq 1 \). Then \( 0 = (f \ast f^{-1})(p^a) = f^{-1 \ast}(p^a) + f^{-1 \ast}(p^{a-1}) f(p) \), and thus \( f^{-1 \ast}(p^a) = -f^{-1 \ast}(p^{a-1}) f(p) \). This recursion gives \( f^{-1 \ast}(p^a) = (-1)^a f(p)^{a} \). Thus (6) holds for all prime powers and therefore by multiplicativity it holds for all positive integers.

Equation (6) follows from (5) and (3).

Example 2.2. For \( f = \mu^2 \) we have \( f(p) = 1 \) for all primes \( p \) and \( f^{-1}(n) = (-1)^{\Omega(n)} \xi(n) = \lambda(n) \xi(n) \). For \( f = \mu \) we have \( \mu^{-1}(n) = (-1)^{\Omega(n)} \xi(n) \prod_p (-1)^{\nu_p(n)} = (-1)^{\Omega(n)} \xi(n) (-1)^{\Omega(n)} = \xi(n) \). This follows also from the result \( \xi^{-1} = \mu \). If \( f(p) = r \) for all primes \( p \), then \( f^{-1}(n) = (-1)^{\Omega(n)} \xi(n) \prod_p r^{\nu_p(n)} = (-r)^{\Omega(n)} \xi(n) \).

Remark 2.1. The function \( f^{-1 \ast} \) in (5) is completely multiplicative for all \( f \) satisfying the conditions in Theorem 2.3.

Equation (3) can be extended to several functions. In fact, from (3) we obtain that \( f \circ g \circ h = \xi \left( \frac{g}{{\xi}} \ast \frac{h}{{\xi}} \right) \circ h = \xi \left( \frac{f}{{\xi}} \ast \frac{g}{{\xi}} \ast \frac{h}{{\xi}} \right) \) and in general for all \( f_1, \ldots, f_k \in \mathcal{A} \),

\[ f_1 \circ \cdots \circ f_k = \xi \left( \frac{f_1}{{\xi}} \ast \cdots \ast \frac{f_k}{{\xi}} \right). \]

This means that for every \( n \in \mathbb{N} \),

\[ (f_1 \circ \cdots \circ f_k)(n) = \sum_{d_1, \ldots, d_k = n} \left( \prod_p \left( \nu_p(d_1), \ldots, \nu_p(d_k) \right) \right) f_1(d_1) \cdots f_k(d_k), \]
involving multinomial coefficients. If \( f_1, \ldots, f_k \in \mathcal{A} \) are multiplicative functions, then \( f_1 \circ \cdots \circ f_k \) is multiplicative and

\[
(11) \quad (f_1 \circ \cdots \circ f_k)(n) = \sum_{d_1 \cdots d_k = n} \left( \prod_p \left( \nu_p(d_1), \ldots, \nu_p(d_k) \right) f_1(p)^{\nu_p(d_1)} \cdots f_k(p)^{\nu_p(d_k)} \right).
\]

Equation (11) can also be extended to several functions. We do not need these details in this paper.

3. Completely multiplicative functions

In [7] the second author provides properties of completely multiplicative functions with respect to the binomial convolution. In this section we provide further properties of this kind. In fact, we derive two characterizations of completely multiplicative functions in terms the binomial convolution (Theorems 3.2 and 3.3). A large number of characterizations of completely multiplicative functions in terms the Dirichlet convolution have been published in the literature, see e.g. [1] [5] [10]. In Section 5 we find the exponential Dirichlet series of completely multiplicative functions.

We begin this section by deriving “multiplicative” versions of the multinomial and binomial theorems from (11).

Let \( f_1, \ldots, f_k \) be completely multiplicative aritmetical functions. Then from (11) we obtain

\[
(12) \quad (f_1 \circ \cdots \circ f_k)(n) = \sum_{d_1 \cdots d_k = n} \left( \prod_p \left( \nu_p(n) \right) f_1(p)^{\nu_p(d_1)} \cdots f_k(p)^{\nu_p(d_k)} \right).
\]

On the other hand, \( f_1 \circ \cdots \circ f_k \) is also completely multiplicative and thus

\[
(13) \quad (f_1 \circ \cdots \circ f_k)(n) = \prod_p (f_1 \circ \cdots \circ f_k)(p)^{\nu_p(n)} = \prod_p (f_1(p) + \cdots + f_k(p))^{\nu_p(n)}.
\]

Now, suppose that \( f_1, \ldots, f_k \) are prime independent completely multiplicative functions, that is, \( f_1(p) = x_1, \ldots, f_r(p) = x_r \) for any prime \( p \), where \( x_1, \ldots, x_r \) are given complex numbers. Then by (12),

\[
(14) \quad (f_1 \circ \cdots \circ f_k)(n) = \sum_{d_1 \cdots d_k = n} \left( \prod_p \left( \nu_p(d_1), \ldots, \nu_p(d_k) \right) x_1^{\Omega(d_1)} \cdots x_k^{\Omega(d_k)} \right),
\]

and by (13)

\[
(15) \quad (f_1 \circ \cdots \circ f_k)(n) = (x_1 + \cdots + x_k)^{\Omega(n)}.
\]

From (14) and (15) we obtain the following “multiplicative” version of the multinomial theorem. It reduces to the usual multinomial theorem if \( n \) is a prime power.

**Theorem 3.1.** For all complex numbers \( x_1, \ldots, x_r \) and positive integers \( n \),

\[
(16) \quad \sum_{d_1 \cdots d_k = n} \left( \prod_p \left( \nu_p(d_1), \ldots, \nu_p(d_k) \right) x_1^{\Omega(d_1)} \cdots x_k^{\Omega(d_k)} \right) = (x_1 + \cdots + x_k)^{\Omega(n)}.
\]

For \( k = 2 \) we obtain the following “multiplicative” version of the binomial theorem.

**Corollary 3.1.** For all complex numbers \( x \) and \( y \) and positive integers \( n \),

\[
(17) \quad \sum_{d|n} \left( \prod_p \left( \nu_p(n/d) \right) x^d y^{n/d} \right) = (x + y)^{\Omega(n)}.
\]
And, for $x_1 = \cdots = x_k = 1$, Theorem 3.1 becomes

**Corollary 3.2.** For all positive integers $k$ and $n$,

\[
\sum_{d_1, \ldots, d_k = n} \prod_{p} \left( \nu_p(n) \right) = k^{\Omega(n)}.
\]

We next write a characterization of completely multiplicative functions in terms of binomial powers. A similar result in terms of Dirichlet powers is presented in [10].

For $k \in \mathbb{Z}, k \neq 0$ let $f^{(k)}$ denote the $k$-th power of $f \in \mathcal{A}$ under the binomial convolution, i.e. $f^{(k)} = f \circ \cdots \circ f$ ($k$ times), $f^{(-k)} = f^{-1} \circ \cdots \circ f^{-1}$ ($k$ times) for all $k > 0$.

If $f$ is completely multiplicative, then by (13) $f^{(k)}(n) = k^{\Omega(n)} f(n)$ for all $k > 0$ and $n \geq 1$. Also, $f^{(-k)}(n) = \lambda(n) f(n)$, therefore $f^{(-k)}(n) = k^{\Omega(n)} \lambda(n) f(n) = (-k)^{\Omega(n)} f(n)$, that is $f^{(k)}(n) = k^{\Omega(n)} f(n)$ for all $k \in \mathbb{Z}, k \neq 0$ and $n \geq 1$. Defining $f^{(0)} = \delta$ and $0^{\Omega(n)} = 0^1 = 1$ this holds also for $k = 0$.

The following is a sufficient condition for a multiplicative function to be completely multiplicative.

**Theorem 3.2.** Let $f$ be a multiplicative function. If there is an integer $k \in \mathbb{Z}, |k| \geq 2$ such that $f^{(k)}(n) = k^{\Omega(n)} f(n)$ for all $n \geq 1$, then $f$ is completely multiplicative.

**Proof.** Suppose that $k \geq 2$. For any prime power $n = p^\nu$ we have

\[
f^{(k)}(p^\nu) = k^\nu f(p^\nu).
\]

According to (11),

\[
f^{(k)}(p^\nu) = \sum_{\nu_1 + \cdots + \nu_k = \nu} \left( \mathbf{\nu} \right) f(p^{\nu_1}) \cdots f(p^{\nu_k}) = kf(p^\nu) + \sum_{\nu_1 + \cdots + \nu_k = \nu, \nu_1 \leq \cdots \leq \nu_k} \left( \mathbf{\nu} \right) f(p^{\nu_1}) \cdots f(p^{\nu_k}).
\]

We show by induction on $m$ that $f(p^m) = f(p)^m$. This is true for $m = 1$. Assume that it holds for any $m < \nu$. Then

\[
f^{(k)}(p^\nu) = kf(p^\nu) + \sum_{\nu_1 + \cdots + \nu_k = \nu} \left( \mathbf{\nu} \right) f(p^{\nu_1}) \cdots f(p^{\nu_k})
\]

\[
= kf(p^\nu) + \sum_{\nu_1 + \cdots + \nu_k = \nu} \left( \mathbf{\nu} \right) f(p^{\nu_1}) \cdots f(p^{\nu_k}) - k f(p^\nu) = kf(p^\nu) + k^\nu k f(p)^\nu - k f(p^\nu),
\]

by the multinomial formula. We obtain

\[
f^{(k)}(p^\nu) = kf(p^\nu) + (k^\nu - k) f(p)^\nu.
\]

By (19) and (20), $k^\nu f(p^\nu) = kf(p^\nu) + (k^\nu - k) f(p)^\nu$ or $(k^\nu - k) f(p^\nu) = (k^\nu - k) f(p)^\nu$, where $k^\nu - k \neq 0$ ($k \geq 2$). Therefore $f(p^\nu) = f(p)^\nu$.

Now, suppose that $k$ is negative. This case is reduced to what is already proved. Let $k = -j, j \geq 2$. Then for every $n \geq 1$, $f^{(-j)}(n) = (-j)^{\Omega(n)} f(n)$, which can be written as $(f^{(-1)})^{j}(n) = j^{\Omega(n)} \lambda(n) f(n)$ or $(f^{(-1)})^{j}(n) = j^{\Omega(n)} f^{-1}(n)$. Since $f^{(-1)}$ is multiplicative, it follows from the first part of the proof that $f^{(-1)}$ is completely multiplicative and we conclude that $f$ is completely multiplicative.
Remark 3.1. If $f$ is completely multiplicative, then $f^{-1} = \lambda f$. The converse is not true: If $f$ is multiplicative and $f^{-1} = \lambda f$, then $f$ need not be completely multiplicative. In fact, $f$ is given by

$$f(1) = 1,$$

$$f(p^{2n-1}) \text{ may be arbitrary, } n = 1, 2, \ldots,$$

$$f(p^{2n}) = - \sum_{k=1}^{n-1} \binom{2n}{k} (-1)^k f(p^k) f(p^{2n-k}) - \frac{1}{2} \binom{2n}{n} (-1)^n f(p^n)^2, \quad n = 1, 2, \ldots,$$

see [7, p. 215]. Note that it is well known that if $f$ is multiplicative and $f^{-1*} = \mu f$, then $f$ is completely multiplicative.

Remark 3.2. The function $\mu^{k*}(n)$ plays the role of the function $\xi^{2\Omega(n)}$ in the Dirichlet convolution. It is easy to see that $k^{2\Omega(n)} = \lambda^{k*}(n)$. Note that the functions $\lambda^{k*}(n)$ form an infinite cyclic subgroup of the group of completely multiplicative functions under the binomial convolution, while the functions $\mu^{k*}(n)$ form an infinite cyclic subgroup of the group of multiplicative functions under the Dirichlet convolution, see [4].

It is well known that distributivity over the Dirichlet convolution is a characterization of completely multiplicative functions, for details, see e.g. [3]. Similar results can also be derived for the binomial convolution. As an example of such characterizations we present the following basic result.

Theorem 3.3. Let $f$ be a multiplicative function. Then $f$ is completely multiplicative if and only if $f(g \circ h) = fg \circ fh$ for all $g, h \in A$.

Proof. The “$\Rightarrow$” direction is immediate.

We prove the “$\Leftarrow$” direction. Let $g(n) = \mu(n), b(h) = \xi(n)$. Then $\delta = fg = f(g \circ \xi) = f\mu \circ f\xi$; hence $(f\mu)^{-1} = f\xi$. On the other hand, $f\mu$ is multiplicative and $(f\mu)(p^2) = 0$ for all $a \geq 2$, and thus according to Theorem 2.3, $(f\mu)^{-1\Omega(n)} = (-1)^{2\Omega(n)} \xi(n) \prod_p (-f(p))^{\nu_p(n)} = \xi(n) \prod_p f(p)^{\nu_p(n)}$.

We obtain $f(n)^{\xi(n)} = \xi(n) \prod_p f(p)^{\nu_p(n)}$ or $f(n) = \prod_p f(p)^{\nu_p(n)}$ for all $n \in \mathbb{N}$, showing that $f$ is completely multiplicative.

Remark 3.3. For a construction which is similar to the binomial convolution of completely multiplicative arithmetical functions see [3] Section 4].

4. Selberg multiplicative functions

An arithmetical function $F$ is said to be Selberg multiplicative if for each prime $p$ there exists $f_p : \mathbb{N}_0 \to \mathbb{C}$ with $f_p(0) = 1$ for all but finitely many $p$ such that

$$F(n) = \prod_p f_p(\nu_p(n))$$

(21)

for all $n \in \mathbb{N}$. An arithmetical function $F$ is said to be semimultiplicative if

$$F(m)F(n) = F((m, n))F([m, n])$$

for all $m, n \in \mathbb{N}$, where $(m, n)$ and $[m, n]$ stand for the gcd and lcm of $m$ and $n$. It is known that an arithmetical function $F$ (not identically zero) is semimultiplicative if and only if there exists a nonzero constant $c_F$, a positive integer $a_F$ and a multiplicative function $F'$ such that

$$F(n) = c_F F'(n/a_F)$$

(22)

for all $n \in \mathbb{N}$. (We interpret that the arithmetical function $F'$ possesses the property that $F'(x) = 0$ if $x$ is not a positive integer.) We will take $a_F$ as the smallest positive integer $k$ such that $F(k) \neq 0$. Note that $c_F = F(a_F)$. Furthermore, it is known that an arithmetical function is Selberg
multiplicative if and only if it is semimultiplicative. Semimultiplicative functions \( F \) with \( F(1) \neq 0 \) are known as quasimultiplicative functions. Quasimultiplicative functions \( F \) possess the property \( F(1)F(mn) = F(m)F(n) \) whenever \( (m, n) = 1 \). Semimultiplicative functions \( F \) with \( F(1) = 1 \) are the usual multiplicative functions.

A semimultiplicative function not identically zero possesses a Selberg expansion (21) as

\[
F(n) = F(a_F) \prod_p \left( \frac{F(a_F p^{\nu_p(n)-\nu_p(a_F)})}{F(a_F)} \right).
\]

A Selberg expansion (21) of a multiplicative function is

\[
F(n) = \prod_p F(p^{\nu_p(n)}).
\]

It is known that semimultiplicative functions form a commutative semigroup with identity under the Dirichlet convolution and

\[
a_{F \ast G} = a_F a_G, \quad c_{F \ast G} = c_F c_G, \quad (F \ast G)' = F' \ast G'.
\]

Quasimultiplicative functions form a commutative group under the Dirichlet convolution. For material on Selberg multiplicative and semimultiplicative functions we refer to [5, 6, 9, 13, 14, 16, 17].

We next prove that semimultiplicative functions form a commutative semigroup with identity under the binomial convolution. In the proof we use the following result. If \( F \) and \( G \) are semimultiplicative, then \( FG \) is also semimultiplicative. In particular, if \( F \) is semimultiplicative (not identically zero) and \( f \) is multiplicative with \( f(a_F) \neq 0 \), then

\[
a_{f_F} = a_F, \quad c_{f_F} = f(a_F)c_F, \quad (f F)' = \frac{f a_F}{f(a_F)} F',
\]

where \( f_a(n) = f(an) \) for all \( n \in \mathbb{N} \).

**Theorem 4.1.** Semimultiplicative functions form a commutative semigroup with identity under the binomial convolution. Furthermore,

\[
a_{F \circ G} = a_{F \circ G}, \quad c_{F \circ G} = c_{F \circ G} \xi(a_F a_G) \xi(a_F) \xi(a_G), \quad (F \circ G)' = \frac{\xi(a_F a_G)}{\xi(a_F) \xi(a_G)} \left( \frac{\xi(a_F) F'}{\xi(a_F)} \circ \frac{\xi(a_G) G'}{\xi(a_G)} \right),
\]

where \( \xi_a(n) = \xi(an) \) for all \( n \in \mathbb{N} \).

**Proof.** Let \( F \) and \( G \) be semimultiplicative. We show that \( F \circ G \) is also semimultiplicative. We use the formula

\[
F \circ G = \xi \left( \frac{F}{\xi} \ast \frac{G}{\xi} \right).
\]

The function \( \xi \) is a multiplicative function such that \( \xi(n) \neq 0 \) for all \( n \in \mathbb{N} \). Therefore \( F/\xi \) and \( G/\xi \) are semimultiplicative and thus \( (F/\xi) \ast (G/\xi) \) and \( \xi((F/\xi) \ast (G/\xi)) \) have the same property.

Now, since \( \xi(n) \neq 0 \) for all \( n \in \mathbb{N} \), on the basis of (23) and (24) we obtain

\[
a_{F \circ G} = a_{(F/\xi) \ast (G/\xi)} = a_{F/\xi a_G/\xi} = a_{F a_G}.
\]

Furthermore,

\[
c_{F \circ G} = (F \circ G)(a_{F \circ G}) = (F \circ G)(a_F a_G) = \xi(a_F a_G)((F/\xi) \ast (G/\xi))(a_F a_G),
\]

where, taking into account, that \( a_F := k \) and \( a_G := \ell \) are the least numbers such that \( F(k) \neq 0 \) and \( G(\ell) \neq 0 \), respectively, the second factor of the last expression is

\[
((F/\xi) \ast (G/\xi))(a_F a_G) = \frac{F(a_F)}{\xi(a_F)} \frac{G(a_G)}{\xi(a_G)} = \frac{c_{F a_G}}{\xi(a_F) \xi(a_G)}.
\]
Finally, on the basis of (23) and (24) we have

\[(F \circ G)' = \left( \xi \left( \frac{F}{\xi} \ast \frac{G}{\xi} \right) \right)' = \frac{\xi a_{FG}}{\xi(a_{FG})} \left( \frac{F}{\xi} \ast \frac{G}{\xi} \right) \]

\[= \frac{\xi a_{FG}}{\xi(a_{FG})} \left( \left( \frac{F}{\xi} \right)' \ast \left( \frac{G}{\xi} \right)' \right) = \frac{\xi a_{FG}}{\xi(a_{FG})} \left( \frac{\xi(a_{FG})}{\xi_{a_{FG}}} F' \ast \frac{\xi(a_{FG})}{\xi_{a_{FG}}} G' \right) \]

This completes the proof.

Let \( S \) denote the class of Selberg multiplicative functions \( F \) such that there exists a universal \( f : \mathbb{N}_0 \rightarrow \mathbb{C} \) with \( f(0) = 1 \) such that

\[ F(n) = \prod_p f(\nu_p(n)) \]

for all \( n \in \mathbb{N} \). The class \( S \) is exactly the class of prime independent multiplicative functions. It is known that \( S \) forms a subgroup of the commutative group of multiplicative functions under the Dirichlet convolution and for \( H = F \ast G \) we have \( h(r) = \sum_{i=0}^r f(i)g(r-i) \). In a similar way we can prove that \( S \) forms a subgroup of the commutative group of multiplicative functions under the binomial convolution and for \( H = F \circ G \) we have \( h(r) = \sum_{i=0}^r \binom{r}{i} f(i)g(r-i) \). Note that the functions \( \lambda^{ko}(n) \) form a subgroup of the group \((S, \circ)\), while the functions \( \mu^{ko}(n) \) form a subgroup of the group \((S, \ast)\).

5. Exponential Dirichlet series

For an arithmetical function \( f \) we define the (formal) exponential Dirichlet series by

\[ \overline{D}(f, s) = D \left( \frac{f}{\xi}, s \right) = \sum_{n=1}^{\infty} \frac{f(n)}{\xi(n)n^s}. \]

Then \( \overline{D}(\xi, s) = \zeta(s) \) is the Riemann zeta function (the Dirichlet series of the constant function 1) and we let \( \overline{D}(I, s) = \overline{\zeta}(s) \) denote the exponential Dirichlet series of the constant function 1. Exponential Dirichlet series has not hitherto been investigated in the literature, while the usual Dirichlet series is one of the most fundamental concepts in analytic number theory, see e.g. [2] [19]. It is evident that the exponential Dirichlet series possesses properties similar to the usual Dirichlet series.

**Theorem 5.1.** The product of two exponential Dirichlet series is the exponential Dirichlet series of the binomial convolution of the corresponding arithmetical functions, i.e.

\[ \overline{D}(f, s) \overline{D}(g, s) = \overline{D}(f \circ g, s). \]

**Proof.** According to (3),

\[ \overline{D}(f \circ g, s) = \overline{D} \left( \xi \left( \frac{f}{\xi} \ast \frac{g}{\xi} \right), s \right) = D \left( \frac{f}{\xi} \ast \frac{g}{\xi}, s \right) = D \left( \frac{f}{\xi}, s \right) D \left( \frac{g}{\xi}, s \right) = \overline{D}(f, s) \overline{D}(g, s). \]

**Remark 5.1.** The algebra \((\overline{D}, +, \cdot, \mathbb{C})\) of exponential Dirichlet series is isomorphic to the algebras given above.

Now, we consider the exponential Dirichlet series of completely multiplicative functions. Let \( \exp(t) = \sum_{k=0}^{\infty} t^k/k! \) be the (formal) exponential power series.
**Theorem 5.2.** If $f$ is completely multiplicative, then

$$
\widetilde{D}(f, s) = \exp \left( \sum_p \frac{f(p)}{p^s} \right).
$$

**Proof.** The function $\xi(n)$ is multiplicative. Therefore using the (formal) Euler product formula,

$$
\widetilde{D}(f, s) = \prod_p \sum_{\nu=0}^{\infty} \frac{1}{\nu!} (f(p)p^{-s})^\nu = \prod_p \exp(f(p)p^{-s}) = \exp \left( \sum_p \frac{f(p)}{p^s} \right).
$$

This completes the proof.

Let $\zeta_P(s) = \sum_p 1/p^s$ denote the prime zeta function and let $\tilde{\zeta}(s) = \exp(\zeta_P(s))$ (Re $s > 1$). Then, by applying Theorem 5.2 and the Glaisher formula\

$$
\zeta_P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(ns) \quad (\text{Re } s > 1),
$$

we obtain the following consequence.

**Corollary 5.1.** For $f(n) = n^r$ ($r \in \mathbb{R}$),

$$
\widetilde{D}(f, s) = \tilde{\zeta}(s-r) = \prod_{n=1}^{\infty} \zeta(n(s-r))^{\mu(n)/n} \quad (\text{Re } s > r+1).
$$

In particular,

$$
\tilde{\zeta}(s) = \prod_{n=1}^{\infty} \zeta(ns)^{\mu(n)/n} \quad (\text{Re } s > 1).
$$

**Corollary 5.2.** For $f(n) = r^{\Omega(n)}$ ($r \in \mathbb{R}$),

$$
\widetilde{D}(r^\Omega, s) = \exp(r\zeta_P(s)) \quad (\text{Re } s > 1).
$$

In particular, for the Liouville function $f(n) = \lambda(n) = (-1)^{\Omega(n)}$,

$$
\widetilde{D}(\lambda, s) = \exp(-\zeta_P(s)) \quad (\text{Re } s > 1).
$$

We now consider a binomial analog of the von Mangoldt function $\Lambda$. Let

$$
(25) \quad \widetilde{\Lambda}(n) = \begin{cases} 
\log p, & n = p \text{ prime}, \\
0, & \text{otherwise}.
\end{cases}
$$

Then $\tilde{\Lambda} \circ I = \log$, since for every $n \geq 1$,

$$
(\tilde{\Lambda} \circ I)(n) = \sum_{d|n} \left( \prod_p \begin{pmatrix} \nu_p(n) \\ \nu_p(d) \end{pmatrix} \right) \widetilde{\Lambda}(d) = \sum_p \left( \frac{\nu_p(n)}{1} \right) \log p
= \sum_p \nu_p(n) \log p = \log \prod_p p^{\nu_p(n)} = \log n.
$$
Furthermore, $\tilde{\Lambda} = \lambda \circ \log$ and we obtain the identities: for any $n$ composite number (i.e. with $\Omega(n) > 1$),

\begin{align*}
\sum_{d|n} \left( \prod_p \left( \frac{\nu_p(n)}{\nu_p(d)} \right) \right) (\log d)(-1)^{\Omega(n/d)} &= 0, \\
\sum_{d|n} \left( \prod_p \left( \frac{\nu_p(n)}{\nu_p(d)} \right) \right) (-1)^{\Omega(d)} \log d &= 0.
\end{align*}

We also have

\begin{align*}
\bar{D}(\tilde{\Lambda}, s) &= \bar{D}(\lambda, s) \bar{D}(\log, s) = -\frac{\zeta'(s)}{\zeta(s)}, \quad \text{Re } s > 1.
\end{align*}

Note that for the Chebyshev functions $\theta(x) = \sum_{p \leq x} \log p$ and $\psi(x) = \sum_{p^\nu \leq x} \log p$ we have

\begin{align*}
\theta(x) &= \sum_{n \leq x} \tilde{\Lambda}(n), \quad \psi(x) = \sum_{n \leq x} \Lambda(n).
\end{align*}

6. Exponential generating functions

The generating function or the (formal) power series of $f \in A$ is given by

\begin{align*}
P(f, z) &= \sum_{n=1}^\infty f(n)z^n.
\end{align*}

It is well known that if $f, g \in A$, then

\begin{align*}
P(f \ast g, z) &= \sum_{k=1}^\infty f(k)P(g, z^k)
\end{align*}

formally or assuming that $\sum_{m=1}^\infty \sum_{n=1}^\infty f(m)g(n)z^{mn}$ is absolutely convergent.

We define the exponential generating function (or the formal exponential power series) of $f$ by

\begin{align*}
\tilde{P}(f, z) &= \tilde{P} \left( \frac{f}{\xi}, z \right) = \sum_{n=1}^\infty \frac{f(n)}{\xi(n)} z^n.
\end{align*}

The function $\tilde{P}(f, z)$ has not hitherto been studied in the literature but it is analogous to the concept of the exponential generating function (egf) of a sequence $(a_n)_{n \geq 0}$ given by

\begin{align*}
\hat{P}(f, z) &= \sum_{n=0}^\infty \frac{a_n}{n!} z^n,
\end{align*}

used in combinatorics, see e.g. [18]. Now, $\tilde{P}(\xi, z) = P(I, z) = \exp(z)$ and we let

\begin{align*}
\Xi(z) &= \tilde{P}(I, z) = \sum_{n=1}^\infty \frac{1}{\xi(n)} z^n
\end{align*}

denote the exponential generating function of the constant function 1.

For $z = 1$,

\begin{align*}
\sum_{n=1}^\infty \frac{1}{\xi(n)} \geq \sum_p \frac{1}{\xi(p)} = \sum_p 1 = \infty,
\end{align*}
and for \( z = x \in (0, 1) \),

\[
0 < \Xi(x) = \sum_{n=1}^{\infty} \frac{1}{\xi(n)} x^n \leq \sum_{n=1}^{\infty} x^n = \frac{x}{1 - x};
\]

(35)

hence the radius of convergence of the power series \( \Xi(z) \) is \( r = 1 \).

Also \( \Xi(x) = \sum_{n=1}^{\infty} \frac{1}{\xi(n)} x^n > x + x^2 + x^3 \) for any \( 0 < x < 1 \).

The function \( \Xi(z) \) plays here the role of the exp function.

**Theorem 6.1.** If \( f, g \in A \), then

\[
\tilde{P}(f \circ g, z) = \sum_{k=1}^{\infty} \frac{f(k)}{\xi(k)} \tilde{P}(g, z^k),
\]

and the convergence is absolute assuming that \( \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{f(m)g(n)}{\xi(m)\xi(n)} z^{mn} \) is absolutely convergent.

**Proof.** According to (3) and (31),

\[
\tilde{P}(f \circ g, z) = \tilde{P} \left( \xi \left( \frac{f}{\xi} + \frac{g}{\xi} \right), z \right) = \sum_{k=1}^{\infty} \frac{f(k)}{\xi(k)} P \left( \frac{g}{\xi}, z^k \right) = \sum_{k=1}^{\infty} \frac{f(k)}{\xi(k)} \tilde{P}(g, z^k).
\]

This completes the proof.

From Corollary 3.1 we see that if \( f(n) = r^{\Omega(n)} \) and \( g(n) = s^{\Omega(n)} \), then \((f \circ g)(n) = (r + s)^{\Omega(n)}\).

**Corollary 6.1.** If \( f(n) = r^{\Omega(n)} \) and \( g(n) = 1 \), then \((f \circ g)(n) = (r + 1)^{\Omega(n)}\) and

\[
\sum_{n=1}^{\infty} \frac{(r+1)^{\Omega(n)}}{\xi(n)} z^n = \sum_{n=1}^{\infty} \frac{r^{\Omega(n)}}{\xi(n)} \Xi(z^n), \quad |z| < 1.
\]

(36)

In particular, if \( f(n) = (-1)^{\Omega(n)} \) and \( g(n) = 1 \), then \( f \circ g = \delta \) and

\[
\sum_{n=1}^{\infty} \frac{(-1)^{\Omega(n)}}{\xi(n)} \Xi(z^n) = z, \quad |z| < 1.
\]

(37)

If \( f(n) = 1 \) and \( g(n) = 1 \), then \((f \circ g)(n) = 2^{\Omega(n)}\) and

\[
\sum_{n=1}^{\infty} \frac{2^{\Omega(n)}}{\xi(n)} z^n = \sum_{n=1}^{\infty} \frac{1}{\xi(n)} \Xi(z^n), \quad |z| < 1.
\]

(38)

7. **A generalized binomial convolution**

Let \( \varphi : \mathbb{N} \times X \rightarrow X, \ X \subseteq \mathbb{C} \), be a function such that, using the notation \( \varphi(n, x) = \varphi_n(x) \),

(i) \( \varphi_m(\varphi_n(x)) = \varphi_{mn}(x), \ \forall m, n \in \mathbb{N}, \ \forall x \in X \),

(ii) \( \varphi_1(x) = x, \ \forall x \in X \).

With the aid of the function \( \varphi \) we define the following operation: if \( f : \mathbb{N} \rightarrow \mathbb{C} \) is an arithmetical function and \( \alpha : X \rightarrow \mathbb{C} \) is an arbitrary function, we define \( f \sqcup \varphi \alpha \) by

\[
(f \sqcup \varphi \alpha)(x) = \sum_{n=1}^{\infty} \frac{f(n)}{\xi(n)} \alpha(\varphi_n(x)), \ \forall x \in X,
\]

(39)

assuming that the series is (absolutely) convergent.
If \( \varphi_n(x) = x^n \) and \( \alpha(x) = x \), then we obtain the exponential generating function of \( f \) given in [32]. Other special cases considered in this paper are \( \varphi_n(x) = x/n, \varphi_n(x) = nx. \)

The operation \( f \circ \varphi \alpha \) defined by

\[
(f \circ \varphi \alpha)(x) = \sum_{n=1}^{\infty} f(n)\alpha(\varphi_n(x)), \ \forall \ x \in X,
\]

is investigated in the recent paper [3], where a detailed study of (40) is done, including the problem of convergence and various applications to Möbius-type inversion formulas, even in a more general algebraic context (involving arithmetical semigroups). A function \( \varphi \) satisfying conditions (i) and (ii) is called an action or a flow on \( \mathbb{N} \), the latter term being used in the theory of dynamical systems.

The following properties of \( f \square \varphi \alpha \) are similar to the properties of \( f \circ \varphi \alpha \). In what follows we write \( \square \) instead of \( \circ \) for the sake of brevity.

**Theorem 7.1.** Let \( f, g : \mathbb{N} \to \mathbb{C}, \alpha, \beta : X \to \mathbb{C} \) be arbitrary functions. Then

1) \( f \square (\alpha + \beta) = f \square \alpha + f \square \beta, \)
2) \( (f + g) \square \alpha = f \square \alpha + g \square \alpha, \)
3) \( f \square (g \square \alpha) = (f \circ g) \square \alpha, \)
4) \( \delta \square \alpha = \alpha, \)

assuming that the appropriate series converge absolutely.

**Proof.** Parts 1) and 2) are immediate by the definition. Part 4) is a consequence of (ii), since 
\( (\delta \square \alpha)(x) = \delta(1)\alpha(\varphi_1(x)) = \alpha(x), \forall x \in X. \)

Part 3) follows from the similar property \( f \circ (g \circ \varphi \alpha) = (f \circ g) \circ \varphi \alpha, \) see [3] Theorem 1] and the relation [3] between the Dirichlet convolution and the binomial convolution, but we give here a direct proof. Using (i),

\[
(f \square (g \square \alpha))(x) = \sum_{n=1}^{\infty} \frac{f(n)}{\xi(n)}(g \square \alpha)(\varphi_n(x)) = \sum_{n=1}^{\infty} \frac{f(n)}{\xi(n)} \sum_{m=1}^{\infty} \frac{g(m)}{\xi(m)} \alpha(\varphi_m(\varphi_n(x)))
\]

\[
= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{f(n)g(m)}{\xi(n)\xi(m)} \alpha(\varphi_{mn}(x)).
\]

Assuming that this series is absolutely convergent and grouping its terms according to the value \( nm = k, \)

\[
(f \square (g \square \alpha))(x) = \sum_{k=1}^{\infty} \left( \sum_{nm=k} f(n)g(m) \right) \alpha(\varphi_k(x)) = \sum_{k=1}^{\infty} \left( \frac{f}{\xi} \ast \frac{g}{\xi} \right)(k) \alpha(\varphi_k(x))
\]

\[
= \sum_{k=1}^{\infty} (f \circ g)(k) \frac{\alpha(\varphi_k(x))}{\xi(k)} = ((f \circ g) \square \alpha)(x).
\]

**Theorem 7.2.** (Möbius-type inversion) Let \( f \in \mathcal{A} \) with \( f(1) \neq 0 \) and let \( \alpha, \beta : X \to \mathbb{C} \) be arbitrary functions. Assume that

\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{f(n)f^{-1}(m)}{\xi(n)\xi(m)} \beta(\varphi_{mn}(x))
\]

is absolutely convergent. If

\[
\alpha(x) = \sum_{n=1}^{\infty} \frac{f(n)}{\xi(n)} \beta(\varphi_n(x)), \ \forall \ x \in X,
\]

then

\[
\beta(x) = \sum_{n=1}^{\infty} \frac{f^{-1}(n)}{\xi(n)} \alpha(\varphi_n(x)), \ \forall \ x \in X.
\]
The appropriate functions we have to show the series \( \sum \) then

**Theorem 7.3.** Let \( \alpha \in \mathcal{A} \) with \( f(1) \neq 0 \). If

\[
\alpha(x) = \sum_{n \leq x} \frac{f(n)}{\xi(n)} \beta(x/n), \quad \forall x \geq 1,
\]

then

\[
\beta(x) = \sum_{n \leq x} \frac{f^{-1}(n)}{\xi(n)} \alpha(x/n), \quad \forall x \geq 1.
\]

We next consider another Möbius-type inversion. In fact, we consider the case \( \varphi_n(x) = x/n \) and functions \( \alpha, \beta : (0, \infty) \to \mathbb{C} \) such that \( \alpha(x) = 0, \beta(x) = 0 \) for \( x \in (0, 1) \). Here the sums are finite and therefore we need not take care of convergence.

**Theorem 7.4.** Let \( f, g, h \in \mathcal{A} \) such that

\[
\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{h(k)h^{-1}(m)}{\xi(k)\xi(m)} g(kmn)
\]

is absolutely convergent for all \( n \geq 1 \). If

\[
f(n) = \sum_{m=1}^{\infty} \frac{h(m)}{\xi(m)} g(mn), \quad \forall n \geq 1,
\]

then

\[
g(n) = \sum_{m=1}^{\infty} \frac{h^{-1}(m)}{\xi(m)} f(mn), \quad \forall n \geq 1.
\]

**Corollary 7.1.** Let \( f, g \in \mathcal{A} \). Then the following two statements are equivalent:

A) \( f(n) = \sum_{m=1}^{\infty} \frac{1}{\xi(m)} g(mn) \), for every \( n \geq 1 \), and \( \sum_{n=1}^{\infty} n^\varepsilon |g(n)| < \infty \), for an \( \varepsilon > 0 \).

B) \( g(n) = \sum_{m=1}^{\infty} \frac{(-1)^{\Omega(m)}}{\xi(m)} f(mn) \), for every \( n \geq 1 \), and \( \sum_{n=1}^{\infty} n^\varepsilon |f(n)| < \infty \), for an \( \varepsilon > 0 \).

**Proof.** We show that A) implies B). Assume that A) holds. In order to apply Theorem 7.4 for the appropriate functions we have to show the series \( \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} g(kmn)/\xi(k)\xi(m) \) is absolutely convergent for every \( n \geq 1 \). We use that for any \( \varepsilon > 0 \) there is a constant \( C = C(\varepsilon) \) such that

\[
\sum_{j=1}^{\tau(n)} j^\varepsilon 
\]

Thus, for all \( K, M \geq 1 \), by grouping the terms according to the value \( km = \ell \),

\[
\sum_{k=1}^{K} \sum_{m=1}^{M} \frac{1}{\xi(k)\xi(m)} |g(kmn)| \leq \sum_{\ell=1}^{KM} |g(n\ell)| \sum_{km=\ell}^{KM} \frac{1}{\xi(k)\xi(m)} \leq \sum_{\ell=1}^{KM} |g(n\ell)| \tau(\ell) \leq C \sum_{\ell=1}^{KM} |g(n\ell)|(|\ell|^\varepsilon) \leq Cn^{-\varepsilon} \sum_{j=1}^{\infty} |g(j)||j|^\varepsilon < \infty.
\]
This shows that the series $\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |g(kmn)|/(\xi(k)\xi(m))$ is absolutely convergent for every $n \geq 1$.

Now, applying Theorem 7.4 for $h(n) = 1$, $h^{-1}(n) = (-1)^{\Omega(n)}$ we obtain the first part of B). By similar arguments as above, for all $N \geq 1$,

$$\sum_{n=1}^{N} n^{\varepsilon/2} |f(n)| \leq \sum_{n=1}^{N} n^{\varepsilon/2} \sum_{m=1}^{\infty} |g(mn)| \leq \sum_{\ell=1}^{\infty} |g(\ell)| \sum_{n=1}^{\infty} n^{\varepsilon/2} \leq \sum_{\ell=1}^{\infty} |g(\ell)| \ell^{\varepsilon/2} \tau(\ell) \leq C' \sum_{\ell=1}^{\infty} |g(\ell)| \ell^{\varepsilon} < \infty,$$

where $C' = C(\varepsilon/2)$, proving the second part of B).

**Remark 7.1.** A survey of various Möbius-type functions has been presented in [15, Chapter 2].

**Acknowledgement.** We thank Saku Sairanen for providing a proof of the positive part of Theorem 3.2. We also thank the referee for very careful reading of the manuscript and useful comments.

**References**

[1] T. M. Apostol, *Some properties of completely multiplicative arithmetical functions*, Amer. Math. Monthly **78** (1971), 266–271.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[3] M. Benito, L. M. Navas, J. L. Varona, *Möbius inversion formulas for flows of arithmetic semigroups*, J. Number Theory **128** (2008), 390–412.

[4] T. C. Brown, L. C. Hsu, J. Wang, P. J.-S. Shiue, *On a certain kind of generalized number-theoretical Möbius function*, Math. Sci. **25** (2000), 72–77.

[5] P. Bundschuh, L. C. Hsu, P. J.-S. Shiue, *Generalized Möbius inversion–theoretical and computational aspects*, Fibonacci Quart. **44** (2006), 109–116.

[6] P. Haukkanen, *Classical arithmetical identities involving a generalization of Ramanujan’s sum*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. Dissertationes **68** (1988), 1–69.

[7] P. Haukkanen, *On a binomial convolution of arithmetical functions*, Nieuw Arch. Wisk. (IV), **14** (1996), 209–216.

[8] P. Haukkanen, *On characterizations of completely multiplicative arithmetical functions*, in: Number theory, Turku, de Gruyter, 2001, pp. 115–123.

[9] T.-X. He, L. C. Hsu, P. J.-S. Shiue, *On generalized Möbius inversion formulas*, Bull. Austral. Math. Soc. **73** (2006), 79–88.

[10] V. Laohakosol, N. Pabhapote, N. Wechwiriyakul, *Characterizing completely multiplicative functions by generalized Möbius functions*, Int. J. Math. Math. Sci. **29** (2002), 633–639.

[11] J. H. Loxton, J. W. Sanders, *On an inversion theorem of Möbius*, J. Austral Math. Soc. (Ser. A), **30** (1980), 15–32.

[12] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer, 1986.

[13] D. Rearick, *Semi-multiplicative functions*, Duke Math. J. **33** (1966), 49–53.

[14] D. Rearick, *Correlation of semi-multiplicative functions*, Duke Math. J. **33** (1966), 623–627.

[15] J. Sándor, B. Crstici, *Handbook of Number Theory II*, Kluwer Academic Publishers, 2004.
[16] A. Selberg, *Remarks on multiplicative functions*, in: Number Theory Day (Proc. Conf. Rockefeller Univ., New York, 1976), pp. 232–241, Springer, 1977.

[17] R. Sivaramakrishnan, *Classical Theory of Arithmetic Functions*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 126. Marcel Dekker, Inc., 1989.

[18] R. P. Stanley, *Enumerative combinatorics*, Vol. 2, Cambridge Studies in Advanced Mathematics, 62, Cambridge Univ. Press, 1999.

[19] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, 1995.