WHEN EULER (CIRCLE) MEETS PONCELET (PORISM)

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Abstract.
We describe all triangles that share the same circumcircle and Euler circle. Although this two circles do not form a poristic pair of circles, we find a poristic circle "in-between" that enable to solve this problem using Poncelet porism.

1. Introduction

Which triangles share the same circumcircle and Euler circle? When is a pair of circles the circumcircle and Euler circle of some triangle? The purpose of this paper is to give a poristic answer to these questions and to provide a functorial recipe to construct them.

In a recent paper, who in fact motivated ours, P. Pamfilos gave a full answer to these questions in the acute case (see [7] and the references therein); still the obtuse angle case was left open. Other results and related problems were studied by J. Weaver in [10] and more recently by V. Oxman, in [6]. This problem resemble the first-ever poristic pair of triangles: the i-circle and circumcircle. There are indeed many similarities and our solution consistently use Euler-Chapler formula.

Here, we give a solution that use the inverses of Euler circle, w.r. to circumcircle, as the "missing poristic hoop" between circumcircle and the Euler circle. This enabled a functorial construction of all triangles that shares the same Euler circle and circumcircle: they will be the in-touch triangles associated with this poristic pair.

Except for a few lines of algebraic computations, that use the versatility of complex numbers and complex inversion to light the burden, our solution is geometric and concise. Nevertheless, a careful reader would realize that this brevity strongly relies on Poncelet porism and on the related Euler-Chapler formulas.

2. A poristic tern

Definition 1. We say that \((C_1, C_2)\) form a pair of poristic circles (for \(n = 3\)) if there exists a some triangle \(\triangle A_1B_1C_1\), whose vertices are on \(C_1\) and whose sides (or their lines) tangents circle \(C_2\). We shall call \(C_1\) the inner circle and \(C_2\) the outer circle.

If \(A_2, B_2, C_2\) are the tangency points of the sides of \(\triangle A_1B_1C_1\) with circle \(C_2\), we say that the triangles \(\triangle A_1B_1C_1\) and \(\triangle A_2B_2C_2\) are a compatible pair of poristic triangles w.r. to this couple, and call \(\triangle A_1B_1C_1\) is the circumscribed (or outer) triangle and \(\triangle A_2B_2C_2\) is the in-touch (or inner) triangle.

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These circles are either circumcircle and inscribed circle, or circumcircle and some exinscribed circle; these circles may be secant, as well.

Note that this is an ordered pair; we shall call the first circle, the "outer" and the second circle, the "inner."

Poncelet porism still works when the two circles (or conics) intercepts; see [8]. Therefore, for any choice of an initial point $A_1$ on the outer circle $C_1$, either no tangents $t_b$ and $t_c$ from $A_1$ to $C_2$ intercept again $C_1$ or, the two intercept $C_1$ in $B_1, C_1$, and then the line $B_1 C_1$ tangents $C_2$, as well.

Unfortunately, the circumcircle $C$ and Euler circle $E$ of a triangle do not match poristically; nevertheless, there is poristic hoop that join them: $E'$, the inverses of Euler circle w.r. to circumcircle.

Refer to Fig. 1.

Lemma 1. Let $\Delta ABC$ and let $C$ its circumcircle.
i) The tangents in $A, B, C$ at $C$ determine $\triangle A'B'C'$. Then $A', B', C'$ are the precisely the inverses of the midpoints $A_m, B_m, C_m$ of the sides of the triangle w.r. to $C$.

ii) Reciprocally, if $A', B', C'$ are the inverses of the midpoints $A_m, B_m, C_m$, of the sides $BC, CA, AB$ of $\triangle ABC$ then the lines $A'B', A'C', B'C'$ tangents the circle in $C, B, A$, respectively.

**Proof.** i) By construction, $\triangle A'B'O$ and $\triangle A'C'O$ are two congruent right-angled triangles. Since $\triangle BOC$ is isosceles, its median $OA_m$ is also an altitude. Thus, $BA_m$ is altitude in $\triangle A'B'O$, hence $OA_m \cdot O'A' = OB^2$, and $O, A_m, A'$ are collinear, proving that $A'$ is the inverses of $A_m$ w.r. to circle $C$.

ii) If $A'$ is the inverse of $A_m$, the midpoint of $BC$, than, by the construction of the inverses, $A'B \perp OB$; similarly, $C'$ is the inverse of $C_m$, the midpoint of $BC$, than $C'B \perp OB$; hence $A', B', C'$ are collinear and $A'C'$ tangents the circle in $B$. Similarly $B'C'$, and $A'B'$ tangents the circle in $A$ and $C$, respectively. □

By construction, $A', B', C'$ are the poles of the sides of $\triangle ABC$; therefore, Lemma 1 reads as follows:

**Corollary 1.** The circumcircle of $\triangle A'B'C'$ is the inverse of Euler circle $E$ of $\triangle ABC$ w.r. to $C$.

Lemma 1 and the simple fact that inversion in circle is an involution led to the following.

**Theorem 1.** Let $\triangle ABC$ and let $C$ its circumcircle. The tangents in $A, B, C$ at $C$ determine $\triangle A'B'C'$; let $E'$ be its circumcircle. Then:

(i) $(E', C)$ is a poristic pair (for $n = 3$);

(ii) $\triangle A_1B_1C_1$ have circumcircle $C$ and Euler circle $E$ if and only if it is an in-touch triangle w.r. to $(E', C)$.

**Proof.** i) is a direct consequence of the construction and of Poncelet porism.

ii) $\Rightarrow$ is also a direct consequence of the construction and of Lemma 1.

We need to prove that if some triangle $\triangle A_1B_1C_1$ is in-touch w.r. to $(E', C)$, then its circumcircle is $C$ and its Euler circle is $E$.

Refer to figure 1 and 2; to avoid dneccessary repetition of this figure, points $A_1, B_1, C_1$ reads $A, B, C$ and so on.

By hypothesis, $\triangle A_1B_1C_1$ is in-touch w.r. to $(E', C)$; so there exists a point $A'_1 \in E'$ such that if $A'_1B'_1, A'_1C'_1$ the two tangents from $A'_1$ to circle $C$; then, by Poncelet porism, $B'_1C'_1$ also tangents $C$; and tangency point of these tangents with circle $E'$ determines the in-touch triangle.

We only have to prove that Euler triangle of $\triangle A_1B_1C_1$ is $E$. By construction, $A'_1, B_1$ and $A'_1, C_1$ are two tangents to circle $C$; therefore $\triangle A'_1B_1C_1$ is isosceles and if we denote with $A_{1m}$ the midpoint of segment $[B_1C_1]$, then $A'_1A_{1m}$ is perpendicular on $B_1C_1$; on the other side, $OA_{1m}$ is perpendicular on the cord $B_1C_1$, since $B_1, C_1$ are on the circle $C$ and $O$ is its center. Therefore, $A'_1, A_{1m}$ and $O$ are collinear. Hence, in $\triangle OB_1A'_1$ which is right-angled at $B_1$, as line $A'_1B_1$ is a tangent to $C$,

$$OB_1^2 = OA_{1m} \cdot OA'_1$$

which means that point $A_{1m}$ and $A'_1$ are inverses w.r. to circle $C$. Similarly, $B_{1m}$ and $B'_1$, and $C_{1m}$ and $C'_1$. Thus, the midpoints of the in-touch triangle $\triangle A_1B_1C_1$ are the inverses of $A'_1, B'_1, C'_1$, w.r. to $C$. These midpoints determine the Euler circle, say $E_\infty$ of the in-touch triangle $\triangle A_1B_1C_1$; but since they proved to be the inverses of three points that belong to $E'$ they are on the inverse of $E'$ w.r. to $C$, hence on $E$. □
If $\triangle ABC$ is obtuse, then its Euler circle (bordeaux) intercept its circumcircle in $P_1, P_2$; then the obtuse vertex lies necessarily on arc $P_1P_2$ and one side $B'C'$ of its outer triangle tangents the circumcircle precisely at $A$; if $L_1, L_2$ and $D_1, D_2$ are the interception of the common tangents, at the two circles, then $B', C'$ lie, on arcs $P_1L_1$ and $P_2L_2$, respectively. $A'$ lie on arc $L_1L_2$.

2.1. The obtuse case. Let us take a closer look to the obtuse triangles. In this case, some arcs of circumcircle $C$, or on $E'$, are either "unreachable" or "infertile:" either they cannot be reached by any tangent line or there is not possible to draw tangents to $C$ form that points of of $E'$.

Referring to Fig 2

Lemma 2. **The Euler circle $E$ and the circumcircle $C$ of a triangle are secant if and only if the triangle is obtuse.**

If this is the case, let the obtuse angle be in $B$ and let $P_1, P_2$ the intersection points of Euler circle and circumcircle. Then $B$ locates inside the Euler circle, on the arc of $C$ delimited by $P_1$ and $P_2$.

Proof. Refer to figures 1 and 2. Let $A_h$ and $C_h$ the foots of the altitudes from $A$ and $C$ to the sides of triangle. Since $\angle B$ is obtuse, these two points $A_h$ and $C_h$ locates outside the sides $[BC]$ and $[AC]$. Thus, $C_mC_h > AC_m$ and $A_mA_h > CA_m$. Therefore, $B$ is located on the segments $[A_mC_h]$ and on $[C_mA_h]$; since $A_m, A_h, B_m, B_h$ are on the Euler circle, by convexity, these segments thus, point $B$ itself is inside the Euler circle.

Since $B$ is on circle $C$, it therefore located on the arc $P_1P_2$ of $C$, situated inside the Euler circle $E$. \[\square\]
Corollary 2. With notations as in Lemma 2, the triangles sharing the same circumcircle and Euler circle with an obtuse \( \triangle ABC \) have necessarily one vertex on arc \( P_1P_2 \).

In the obtuse case, the circles \((E',C)\) form a porisic pair of secant circles. The intersection of tangents in \( P_1 \) and \( P_2 \) to the circumcircle \( C \) with \( E' \) give explicit restriction on "fertile" or "sterile" arcs: the arcs of \( E' \) that may or may not contain a vertex of some triangle, that generate a triangle inscribed in \( C \).

2.2. The right-angle case. Euler circle tangents internally the circumcircle iff the triangle is right-angled; in this case, all triangles sharing the circumcircle and Euler circle have a common point in vertices in \( A \), the tangency point; in this case, the polars of the vertices does not define any circle and the inverses of Euler circle w.r. to circumcircle is a line: the tangent in \( A \).

3. Pairs of compatible circumcircle-Euler circle

Till now, our pair of circles \((C,E)\), were respectively, the circumcircle and Euler circle of some pre-existent triangle.

Euler-Chapler formulas give the necessary and sufficient conditions in order that two circles with known distance between their centers and known radius be either the circumcircle and the incircle or the circumcircle and ex-inscribed circle of some triangle. Here, we formulate similar conditions for the (non poristic) pair circumcircle and Euler circle.

First, some necessary conditions.

Theorem 2. Let \( C(O,R) \) and \( E(N,r) \) the circumcircle and Euler circle of some triangle \( \triangle ABC \). Then \( r = \frac{R}{2} \). Further,

i) if \( \triangle ABC \) is acute, then \( ON < \frac{R}{2} \).

ii) if \( \triangle ABC \) is right-angled, then \( ON = \frac{R}{2} \).

iii) if \( \triangle ABC \) is obtuse then \( ON > \frac{R}{2} \).
Proof. Refer to figure 1. Euler circle is the circumcircle of $\triangle A_mB_mC_m$ which similar to $\triangle ABC$ itself, hence the radius of their circumcircle shares the same proportionality rate. Hence $r = \frac{R}{2}$.

i) If the triangle is acute, its orthocenter $H$ is inside the triangle; thus, $OH < R$. On the other hand the center of Euler circle, $N$, is the midpoint of $OH$. This implies $ON < \frac{R}{2}$, Since the radius of the Euler circle is $\frac{R}{2}$, this ensures that Euler circle is contained into the circumcircle.

ii) If the triangle is right-angled, its orthocenter coincides with its right-angle vertex, and the Euler circle of a right-angled triangle tangents internally the circumcircle and pass through its center.

iii) If the triangle is obtuse, at least one feet of its altitudes lie outside the triangle, and outside the circumcircle. Since the midpoints $A_m, B_m, C_m$ are on Euler circle, then by convexity, are contained into the circumcircle $C$. Euler circle of an obtuse triangle therefore contain both points inside and outside the circumcircle, hence in this case the two are secant. $\square$

Now we shall prove that these conditions are also sufficient.

**Theorem 3.** Let $C = C(O, R)$ and $E = E(N, \frac{R}{2})$ be two circles such that $0 \leq ON < \frac{R}{2}$. Then there exists infinitely many triangles whose circumcircle is $C(O, R)$ and whose Euler circle is $E(N, \frac{R}{2})$. Further,

i) if $ON < \frac{R}{2}$, then all these triangles are acute;

ii) if $ON = \frac{R}{2}$, then all triangles are right-angled;

iii) $ON > \frac{R}{2}$, then all triangles are obtuse.

We shall give an indirect prove: we show that if these conditions are fulfilled, then $(E', C)$ form a poristic pair of circles. To see this, we shall need the following fact.

**Lemma 3.** If $(C, E)$ verify the conditions above, then $(E', C)$ form a poristic pair for $n = 3$.

**Proof.** Two circles form a poristic pair for $n = 3$ if and only if they are either the circumcircle and inscribed circle or circumcircle and exinscribed circle. Euler- Chapler formulas guarantees that this hapens if and only if $E'$ and $C$ verify one of the following relations:

\[ (R_1 - r_1)^2 = d_1^2 + r_1^2; \quad (1) \]

\[ (R_1 + r_{1ex})^2 = d_1^2 + r_{1ex}^2 \quad (2) \]

The first relation is the necessary and sufficient condition in order to a pair of circles which have radius $R_1, r_1$ and whose centers dist $d_1$, be, respectively, the circumcircle and the inscribed circle of some triangle.

The second relation is the necessary and sufficient condition in order to a pair of circles of radius $R_1, r_{1ex}$ and whose centers dist $d_1$, be, respectively, the circumcircle and the ex-inscribed circle of some triangle.

For our proof, $R_1$ is the radius of $E'$, (as the external circle), $r_1$ the radius of $C$ (as the "inscribed circle") and $r_{1ex}$ the radius of an "ex-inscribed" circle; and $d_1$ is the distance between the centers of $E'$ and $C$.

We prove that the first case occurs when $0 \leq ON < \frac{R}{2}$ and the second, when $\frac{R}{2} < ON < \frac{R}{2}R$.

The proof now reduces to a straightforward verification. We use complex numbers.

In order to simplify the computations, assume, without lose of generality, that $C$ is the unit circle, (the circle centred in 0 and radius 1 and let $d$ be the center of a circle $E_d$ of radius $\frac{1}{2}$ (half if the radius of $C$) and whose interior intercepts $C$. 

Again, without lose of generality, we may assume that its center is a point on the real positive line; therefore $d \in [0, 3/2)$ and $d \neq 1$. We shall treat the case $d = 1$ separately; since it is the tangency case.

In order to prove the first assertion, note that the center of the given circle $E = E_d$ whose radius is half the radius of $C$, hence $\frac{1}{2}$, is located at a distance $d \in [0, \frac{1}{2})$, being internal circle. We want to prove that $(E_d', C)$ form a poristic pair of circles for $n = 3$.

The diameter of $E_d$ is $|AB|$ where

$$A = d - \frac{1}{2} \quad \text{and} \quad B = d + \frac{1}{2};$$

let $d_0 = 2d \in (0, 1)$; hence $A = \frac{1}{2}(d_0 - 1) < 0$, $B = \frac{1}{2}(d_0 + 1) > 0$. Now let

$$A' = \frac{2}{d_0 - 1} \quad \text{and} \quad B' = \frac{2}{d_0 + 1}$$

be the inverses of $A, B$ w.r. to the unit circle; then the inverses of $C_d$ is the circle $C_d'$ whose diameter is $[A'B']$. Its center is

$$O' = \frac{A' + B'}{2} = \frac{2d_0}{d_0^2 - 1} \quad \text{and its radius} \quad R' = \frac{2}{d_0^2 - 1}.$$ 

Now we check that these two circles verifies the Euler-Chaplet relations.

$$(R - r)^2 = d^2 + r^2,$$ where $r = 1$, and $R = \frac{2}{d_0^2 - 1}d = \frac{2d_1}{1 - d_1}.$

This relation becomes

$$\left(\frac{2}{d_0^2 - 1} - 1\right)^2 = \frac{4d_0^2}{(d_0^2 - 1)^2} + 1,$$

$$\left(\frac{d_0^2 + 1}{d_0^2 - 1}\right)^2 = \frac{4d_0^2 + d_0^2 - 2d_0^2 + 1}{(d_0^2 - 1)^2}.$$

which is obviously verified!

The second assertion, in which the center of the given circle $E_d$ with radius $\frac{1}{2}$ is at a distance $d \in [\frac{1}{2}, \frac{3}{2})$, hence is a secant circle, proves just in the same way.

The diameter of $C_d$ is $|AB|$, where $A = d - \frac{1}{2} + B = d + \frac{1}{2};$ let $d_0 = 2d \in (1, 3)$; hence $A = \frac{1}{2}(d_0 - 1)$, $B = \frac{1}{2}(d_0 + 1)$. Now let $A' = \frac{2}{d_0 - 1}$ and $B' = \frac{2}{d_0 + 1}$ be the inverses of $A, B$ w.r. to the unit circle; then the inverses of $E_d$ is the circle $E_d'$ whose diameter is $[A'B']$. Its center is $O' = \frac{A' + B'}{2} = \frac{2d_0}{d_0^2 - 1}$ and its radius is $R' = \frac{2}{d_0^2 - 1}.$

Now we check that these two circles verifies the relation given by Euler-Chaplet theorem.

$$(R + r_{ex})^2 = d^2 + r_{ex}^2, \quad r_{ex} = 1, \quad R = R'.$$

This relation become

$$\left(\frac{2}{d_0^2 - 1} + 1\right)^2 = \frac{4d_0^2}{(d_0^2 - 1)^2} + 1,$$

$$\left(\frac{d_0^2 + 1}{d_0^2 - 1}\right)^2 = \frac{4d_0^2 + d_0^2 - 2d_0^2 + 1}{(d_0^2 - 1)^2},$$

which is obviously verified.

This expression does not make sense when $d = 1$. But this only occurs when the Euler circle tangent the circumcircle. The tangency case is straightforward, it does not require any poristic computations and we omit it.
We thus proved that any circle whose interior intercepts a given circle and whose radius is half that of the circle, is an Euler circle for some (hence infinitely many) triangles.

Finally, let us illustrate how we can generate all the triangles that shares the same circumcircle and Euler circle with a given triangle and how we can draw the triangle with a prescribed vertices.

Any poristic pair of circles generates two poristically-related families of triangles: the circumscribed triangles, whose vertices are on the outer circle and the inner (or in-touch) triangle, whose vertices are tangency point of tangents form points of outer circle, to the inner circle. Thus, any outer triangle is poristically bounded to one (and only one) inner triangle.

There are two families of poristic triangles w.r. to \((E', C)\): \(T'\), containing all triangles whose vertices are on \(E'\) and whose sides tangent the circle \(C\) (the "outer" triangles) and \(T\) the family of all triangles whose vertices are the tangency point of the sides of triangles in \(T'\), with circle \(C\), (the inner triangles, whose vertices are on \(C\) and whose Euler circle is \(E\).

Note that Poncelet porism is still valid when the poristic circles \((E', C)\) are secant; the only restriction is on the location of the initial point \(B'\): if it lie inside the circumcircle, we cannot trace tangents from it, to \(C\). Equivalently, one side of the poristic in-touch triangle (inscribed in \(C\)) touch the arc \(C\) on a point inside \(E'\).

The two circles of the poristic pair are disjoint, if the triangle is acute and secant, if the triangle is obtuse.

A geometric recipe to construct some triangle, given its circumcircle, its Euler circle, and one of its vertices is now straightforward.

**Corollary 3.** Let \(C\) and \(E\) two non-tangent circles that are circumcircle and Euler circle for some triangle \(\triangle ABC\) and let and \(A_1\) a point on \(C\).

i) Let \(E'\) be the inverses of \(E\) w.r. to \(C\). Let \(a'\), the tangent at \(A_1\) to the circle \(C\); then:

- i) if \(a'\) does not intercept the circle \(E'\) in two distinct points there is no such triangle.
- ii) if \(a'\) is secant to \(E'\) in \(B'\) and \(C'\), then let the tangents \(b', c'\) from \(B'\)
  and \(C'\) tangent the circle \(C\) in two other points \(B, C\); these who are the vertices of the required triangle.

Note that the case i) can only occur when the two circles are secant. In this case, this happens because the point \(A_1\) lie into an unreachable arc of \(C\).

**Corollary 4.** A pair \((C, E)\) of internally tangent circles at a point \(A\) are the circumcircle and Euler circle of some triangle, if and only if the circle \(E\) pass through the center of \(C\). The triangles which have a vertex in \(A\) and whose side is a diameter of the circle passing through point \(A_1\), are inscribed in \(C\) and share the Euler circle \(C\) and is the requested triangle. If the two circles are externally tangent, there is no such a triangle.

### 4. A PORISTIC PROOF

This section reveals yet another poristical bound between Euler circle and circumcircle. This time, we shall deal with two poristic pairs: the circumcircle and the i-conic, the conic inscribed into the triangle and whose focus is in its circumcenter. The relationship between the i-conic, circumcircle and Euler circle, in a twisting of interchangeable poristic terms, spotted in a recent paper by R. Garsia and D. Reznik (see [2]), will enable a very short poristic proof of our main result.
To this end, we shall use inversive techniques and polar reciprocity; the reader not familiar with these concepts may want to see the beautiful books [1], [9]; for ad-hoc details see [4], Appendix.

Let $\triangle ABC$ and $C$ its circumcircle; let $E$ its Euler circle and $E'$ the inverse of Euler circle, w.r. to the circumcircle.

We shall denote by $\Gamma$ its i-conic, the conic tangent to the sides of triangle $\triangle ABC$ and with focus in $O$, the circumcenter.

The following lemma will prove that the poristic family of triangles associated with this pair, share the same Euler circle.

**Definition 2.** (see Appendix) The negative pedal of a curve $\gamma$ w.r. to a pedal point $D$ is the curve $\Gamma$, whose tangents are the perpendicular on $M$ to $DM$, as $M$ sweeps the curve $\gamma$.

The following fact is classic.

**Lemma 4.** The negative pedal curve $\Gamma = N(\mathcal{E})$ of a circle $\mathcal{E}$, w.r. to a pedal point $D$ is a conic $\gamma_D$, which has a focus in $D$, and whose axis coincides with the diameter of $\mathcal{E}$ through $D$.

$\Gamma$ is an ellipse (resp. hyperbola), iff $D$ is inside (resp. outside) $\mathcal{E}$.

The reader not acquainted with these concepts, may either see [5], and references therein, or Appendix of [4], for a short briefing.

This fact has an immediate consequences; see also [2], Theorem 1.
Corollary 5. The i-conic $\gamma_D$ is precisely the negative pedal curve of $\mathcal{E}_D$, w.r. to pedal point $D$.

The in-ellipse and Euler pedal circle tangents at the vertices of i-conic, have the same center and the main axis of the in-ellipse is a diameter of the Euler pedal circle.

Lemma 5. Let $\triangle ABC$ and $O$ its circumcenter. The i-conic $\Gamma$ is the negative pedal of Euler circle $\mathcal{E}$, w.r. to pedal point $O$.

Proof. Refer to figure 4.

Let $C_m$ be the midpoint of $AB$; then $OC_m \perp AB$; since $C_m$ is on $\mathcal{E}$, then, by the definition of a negative pedal curve, the line $AB$ tangents $\Gamma$; the same for the other two sides. Therefore, the negative pedal curve of Euler circle is precisely the i-conic of $\triangle ABC$. □

Corollary 6. The triangles that share the same circumcircle and Euler circle, are precisely those poristically inscribed into $\mathcal{C}$ and circumscribed to $\Gamma$. 
Now we shall formulate the result in terms of a pair of circles and provide a poristic proof of our main result.

**Theorem 4.** (see Pamfilos [7], Theorem 5) The triangles that shares the same Euler circle $\mathcal{E}$ and circumcircle $\mathcal{C}$ are those whose vertices are tangency points (with circle $\mathcal{C}$) of the sides of triangles inscribed into $\mathcal{E}'$ and circumscribed to $\mathcal{C}$.

**Proof.** Refer to figure 5

Perform a dual transform w.r. to $\mathcal{C}$. The vertices $A, B, C$ converts into their polars, which are the tangents $a', b', c'$ that determines $\triangle A'B'C'$. By this polarity, the circumcircle $\mathcal{C}$ does not change, while the i-conic $\Gamma$, whose focus is $O$, the inversion center, converts into the circumcircle of the polar triangle $\triangle A'B'C'$. By 1 this is the inverse of Euler circle, $\mathcal{E}'$. Therefore, $\triangle ABC$ is inscribed in $\mathcal{C}$ and circumscribed to the i-conic $\Gamma$ if and only if $\triangle A'B'C'$ is inscribed into the symmetric of Euler circle w.r. to the circumcircle and circumscribed to $\mathcal{C}$.

We thus poristically generate two families of triangles, one circumscribed, other in-touch, the latter being precisely the family of all triangles sharing the same Euler circle and circumcircle.

The obtuse-angle case does not need a special treat as very few change. The i-conic will be a hyperbola and the poristic triangles are again the in-touch triangles with respect to the dual system. Some arc of the circumcircle $\mathcal{C}$ is not reachable. Nevertheless, this does not alter the result, nor the construction. The description of the unreachable arcs is similar to those we provide in the previous section, so we omit it.

Despite circumcircle and Euler circle are not poristic pair, a providential $i-conic$ and its negative pedal curve put the things into the right perspective and enable to give a poristic solution to a non-poristic problem.

**References**

[1] Akopyan, A., Zaslavsky, A., *Geometry of Conics*, 2 Amer. Math. Soc., 2007.
[2] Garsia, R., Reznik, D. *Any triangle can be 3-periodic*, preprint.
[3] Gheorghe, L.G. *Apollonius problem: in pursuit of a natural solution*, Int. J. of Geometry, 9 (2020), 29-51.
[4] Gheorghe, L.G., Reznik, D *A special conic associated with Reuleaux negative pedal curves*, Int. J. of Geometry, Vol. 10 (2021), No 2, 33-49. (to appear)
[5] Glaeser, G., Stachel, H., Odehnal, B., *The universe of Conics*, Springer Specktrum, Springer-Verlang Berlin Heidelberg 2016.
[6] Oxman, V. *On the existence of triangles with given circumcircle, incircle and one additional element*, Forum Geometricorum, 5 (2005) 165-171.
[7] Pamfilos, P *Triangles sharing their Euler circle and circumcircle*, Int. J. of Geometry, 9 (2020), No 1, 5-24.
[8] Poncelet, J.V., *Traité des propriétés projectives des figures*, Bachelier, 1822.
[9] Salmon, G., *A treatise on conic sections*, Longman, Brown, Green, Longraus, 1855.
[10] Weaver, J. *A system of triangles related to a poristic system*, Amer. Math. Month, 31 (1924), 337-340.