Abstract. In this paper we prove local well-posedness of a space-time fractional generalization of the nonlinear Schrödinger equation with a nonlinearity of polynomial type. The linear equation coincides with a model proposed by Naber, and displays a nonlocal behavior both in space and time which accounts for long-range interactions and a so-called memory effect. Because of a loss of derivatives produced by the latter and the lack of semigroup structure of the solution operator, we employ a strategy of proof based on exploiting some smoothing effect similar to that used by Kenig, Ponce and Vega for the KdV equation.

1. Introduction

1.1. Space-time fractional NLS. In this article we study the initial value problem

$$\begin{cases} \partial_t^\beta u = (-\Delta)^{\alpha/2} u + g(u) & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u |_{t=0} = f \in H^\alpha(\mathbb{R}). \end{cases}$$

for $0 < \alpha < 2$ and $0 < \beta < 1$. We consider nonlinearities of polynomial type $g(u) = \mu |u|^{p+1} u$ for an odd integer $p$ and $\mu = \pm 1$. The operator $(-\Delta)^{\alpha/2}$ is the fractional Laplacian, given by the Fourier multiplier of symbol $|\xi|^{\alpha}$, and the operator with symbol $\partial_t^\beta$ is the Caputo fractional derivative, given by

$$\partial_t^\beta u(t, x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \partial_\tau u(\tau, x) \frac{(t-\tau)^\beta}{(t-\tau)^\beta} d\tau.$$
fact, stochastic solutions to some fractional Cauchy problems arise as scaling limits of continuous time random walks whose i.i.d. jumps are separated by i.i.d. waiting times, where the probability of waiting longer than time \( t > 0 \) decays like \( t^{-\beta} \) for large \( t \) (see [M]).

The space-fractional nonlinear equation (1.1), with \( \beta = 1 \) and \( 0 < \alpha < 1 \), was studied by Hong and Sire in [HS], where they developed a general local and global well-posedness theory for a polynomial-type nonlinearity provided the regularity \( s \) is greater or equal than the regularity invariant under the Galilean transformation, \( \frac{1}{2} - \frac{\alpha}{p-1} \), and the regularity invariant under scaling, \( s_c = \frac{1}{2} - \frac{\alpha}{p-1} \).

The combined space-time fractional linear equation with a potential has also been studied recently, see for instance [Do]. A different type of linear space-time fractional equation was proposed and studied in [ER]. However one should note that their definition of the nonlocal time derivative, although allowing them to keep some group property on the solution operator, does not agree with what Naber proposed in [Na].

Instead we propose (1.1) as a generalization of the nonlinear Schrödinger equation whose linear part agrees with Naber’s work. The coefficient \( i^\beta \), as opposed to \( i \), has been a matter of discussion. Naber argues in favor of \( i^\beta \), and among other reasons, he explains that after taking the Laplace transform in time and Fourier transform in space, the choice of \( i^\beta \) produces a movement of the pole of the solution along the imaginary axis as \( \beta \) ranges between 0 and 1. However, if one instead chooses \( i \), the pole would move to almost any desired location in the complex plane. Physically, this would mean that a small change in the order of the time derivative could change the temporal behavior from sinusoidal to growth or to decay. Moreover, we would add that if one explores the possibility of extending this work allowing exponents \( \beta \) to vary in the range \((1,2)\), the choice of \( i^\beta \) provides equations which interpolate between the Schrödinger and wave equations, allowing one to remain within the domain of dispersive equations.

1.2. Background about fractional equations. Classical equations such as the Laplace, heat, wave and even Schrödinger equations admit a generalization with a fractional Laplacian replacing the Laplacian, most commonly defined as a Fourier multiplier:

\[
(-\Delta)^s f = |\xi|^{2s} \hat{f}(\xi) \quad \text{for} \quad s > 0.
\]

We will refer to such equations as space-fractional equations.

The case of the space-fractional Laplace equation is well-known, see for instance Stein’s book [Ste] for an approach based on the use of Riesz potentials, or [Ga] for a recent survey. The case of the space-fractional porous medium equation, with a fractional Laplacian and a classical time derivative, has also been studied in a series of papers ([DP1],[DP2]).

A different approach in the generalization of such classical equations is the substitution of the time derivative for its fractional counterpart. This is a concept that mainly comes from applications, as there are evolution processes whose modelling requires to take into account the past, thus exhibiting a nonlocal behavior in time. There are different ways of making sense of such a fractional derivative in time, but one of the
most common ones is to use the Caputo derivative:
\[
\partial_t^\beta f(t) = \frac{1}{\Gamma(1 - \beta)} \int_0^t \frac{f'(\tau)}{(t - \tau)\beta} d\tau,
\]
for \(0 < \beta < 1\), where \(\Gamma\) is the Gamma function.

Such an approach has been taken by Allen, Caffarelli and Vasseur when considering a space-time fractional porous medium equation ([A1],[A2]). These types of fractional time derivatives have also been studied using probabilistic techniques, like in [M], where the authors develop stochastic solutions to Cauchy problems of type \(\partial_t^\beta u = Au\) on bounded domains via solutions to the classical Cauchy problem \(\partial_t u = Au\) together with an inverse stable subordinator of index \(\beta\).

Fractional derivatives have also been used to model some phenomena in Physics, such as non-diffusive transport in plasma turbulence ([Ne]), and in Economics, such as ruin theory of insurance companies, growth and inequality processes and high-frequency price fluctuation in financial markets (see [Sc]).

1.3. Background and properties of fractional NLS. Consider the linear case of equation (1.1), i.e. \(g = 0\). One may take the Laplace transform in time, the Fourier transform in space, and solve the resulting equation to formally find
\[
(1.2) \quad u(t, x) = \int_\mathbb{R} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=0}^\infty \frac{z^k \Gamma(\beta k + 1)}{\Gamma(\beta k)} \right] d\xi.
\]
The power series comes from the Mittag-Leffler function \(E_\beta(\xi^{\alpha} t^{\beta} i^{-\beta})\), where
\[
(1.3) \quad E_\beta(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\beta k + 1)},
\]
which is an entire function in the complex plane. More details about this derivation can be found in [Die] and Appendix A.

We can also perform some scaling analysis for equation (1.1). If \(u\) is a solution to the equation then so is
\[
u_\lambda(t, x) = \lambda^{\alpha\beta/p - 1} u(\lambda^\alpha t, \lambda^\beta x),
\]
with the obvious changes to the initial data. Then one can quickly check that the critical regularity invariant under scaling is
\[
s_c = \frac{1}{2} - \frac{\alpha}{p - 1}.
\]
Note that \(\beta\) plays no role on this formula, and so it coincides with the space-fractional case treated in [HS] for dimension 1. We will work in the subcritical regime where \(s > s_c\), and therefore we may assume that our initial data is as small as we need it to be.

An interesting feature is that even the linear equation has no conserved quantities. However, one can use the following asymptotic expansion for the Mittag-Leffler function, which may be found in Chapter 18 of [Ba],
\[
(1.4) \quad E_\beta(z) = \frac{1}{\beta} \exp(z^{\frac{1}{\beta}}) - \sum_{k=1}^{N-1} \frac{z^{-k}}{\Gamma(1 - \beta k)} + O(|z|^{-N}) \quad \text{as } |z| \to \infty,
\]
to control \( \|u(t)\|_{L^2_x} \leq 1 \) uniformly in time. In fact, one can even show that

\[
\lim_{t \to \infty} \|u(t)\|_{L^2_x} = \frac{1}{\beta} \|f\|_{L^2_x},
\]

i.e. as time passes the mass grows towards \( \frac{1}{\beta} \) times that of the initial data. Formula (1.4) is valid when \( |\arg(z)| \leq \frac{\pi}{2}\beta \) and for any integer \( N \geq 2 \). In fact, we will always choose the branch of the complex logarithm for which \( |\arg(z)| < \pi \), so that \( i^{-\beta} = e^{-i\beta \frac{\pi}{2}} \) and \( i^\beta = e^{i\beta \frac{\pi}{2}} \).

By using a fractional generalization of the Duhamel formula, we can write the solution of the nonlinear problem (1.1) as

\[
\begin{align*}
    u(t, x) &= \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) E_{\beta}(\xi^{\alpha} t^\beta \xi^{-\beta}) d\xi \\
    &+ i^{-\beta} \int_{0}^{t} \int_{\mathbb{R}} \hat{g}(\tau, \xi) (t - \tau)^{\beta-1} E_{\beta, \beta}(i^{-\beta}(t - \tau)^\beta \xi^{\alpha}) e^{ix\xi} d\tau d\xi,
\end{align*}
\]

where

\[
\hat{g}(t, \xi) = \int_{\mathbb{R}} g(u(t, x)) e^{-i\xi x} dx,
\]

and

\[
E_{\beta, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + \beta)}
\]

is known as the generalized Mittag-Leffler function, and is also known to be an entire function. As before, more details about this can be found both in Appendix A and [Die].

Because of the difficulties in dealing with this function directly, the following asymptotic formula is helpful, which may also be found in Chapter 18 of [Ba],

\[
t^{\beta-1} E_{\beta, \beta}(i^{-\beta} t^\beta \xi^{\alpha}) = \frac{1}{\beta} \left( i^{-\beta-1} \xi^{\sigma-\alpha} e^{-\xi t} \xi^\sigma \right) r^{1+\beta k-1} i^{\beta k} \\
- \sum_{k=2}^{N} \frac{\Gamma(\beta - \beta k)}{t^{1+\beta k} \xi^{\alpha}} + O \left( \frac{1}{t^{1+\beta N} \xi^{\alpha(N+1)}} \right)
\]

and is valid as \( t^{\sigma} \to \infty \), \( |\arg(z)| \leq \frac{\pi}{2}\beta \), and for any \( N \geq 2 \), where \( \sigma = \frac{\beta}{2} \). We again observe an oscillatory part and a monotonous part, which will need to be managed separately.

**Remark 1.1.** As seen in (1.2)-(1.7), the solution operator does not enjoy the usual group or even semigroup property with respect to time. This is a major obstacle that prevents us from using Strichartz estimates and other techniques that normally rely on this fact. As a result, our treatment is substantially different from the techniques used both for classical NLS and for the purely space-fractional equation in [HS].

1.4. **Statement of results.** The main theorem in this paper is the following.

**Theorem 1.1.** Consider the space-time fractional nonlinear Schrödinger initial value problem (1.1) with a nonlinearity \( g(u) = \mu |u|^{p-1} u \) for some odd integer \( p \geq 3 \), \( \mu = \pm 1 \)
and $\alpha, \beta > 0$. With $\sigma = \frac{\alpha}{\beta}$, suppose that
\begin{equation}
\alpha > \frac{\sigma + 1}{2}, \quad s \geq \frac{1}{2} - \frac{1}{2(p-1)}, \quad \text{and} \quad \delta \in \left[ s + \alpha - \frac{\sigma}{2} - \frac{1}{2(p-1)} \right].
\end{equation}
for some $s \in \mathbb{R}$. Then for every $f \in H^s(\mathbb{R})$ there exists $T = T(\|f\|_{H^s(\mathbb{R})}) > 0$ (with $T(\rho) \to \infty$ as $\rho \to 0$) and a unique solution $u(t,x)$ to the integral equation given by (1.5) satisfying
\begin{equation}
\begin{aligned}
& u \in C([0,T], H^s(\mathbb{R})), \\
& \left\| (\nabla)_{|\delta|} u \right\|_{L_x^\infty L_t^2} < \infty,
\end{aligned}
\end{equation}
and
\begin{equation}
\left\| u \right\|_{L_x^{2(p-1)} L_t^2} < \infty.
\end{equation}
Moreover, for any $T' \in (0,T)$ there exists a neighborhood $V$ of $f$ in $H^s(\mathbb{R})$ such that the map $\hat{f} \to \hat{u}$ from $V$ into the class defined by (1.9), (1.10) and (1.11) with $T'$ instead of $T$ is Lipschitz.

We now give an example of the use of this theorem for a particular choice of parameters, which we believe clarifies the exposition.

**Corollary 1.1.** Consider the following nonlinear fractional Schrödinger initial value problem:
\begin{equation}
\begin{aligned}
& i \hat{\partial}_t \hat{u} = (\hat{-\Delta}_x)^{\frac{\sigma}{2}} u + \mu |u|^2 u \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\
& u |_{t=0} = f.
\end{aligned}
\end{equation}
Then for every $f \in H^s(\mathbb{R})$ there exists $T = T(\|f\|_{H^s(\mathbb{R})}) > 0$ (with $T(\rho) \to \infty$ as $\rho \to 0$) and a unique solution $u(t,x)$ to the associated integral equation given by (1.5) satisfying
\begin{equation}
\begin{aligned}
& u \in C([0,T], H^s(\mathbb{R})), \\
& \left\| (\nabla)_{|\delta|} u \right\|_{L_x^\infty L_t^2} < \infty,
\end{aligned}
\end{equation}
and
\begin{equation}
\left\| u \right\|_{L_x^{2(p-1)} L_t^2} < \infty.
\end{equation}
Moreover, for any $T' \in (0,T)$ there exists a neighborhood $V$ of $f$ in $H^s(\mathbb{R})$ such that the map $\hat{f} \to \hat{u}$ from $V$ into the class defined by (1.13), (1.14) and (1.15) with $T'$ instead of $T$ is Lipschitz.

**Remark 1.2.** To the best of our knowledge, Theorem 1.1 is the first well-posedness result for a nonlinear fractional space-time Schrödinger equation.

**Remark 1.3.** Because of (1.7), if one tries to take the $L_x^\infty L_t^2$ norm of
\begin{equation}
\int_{\mathbb{R}} \int_0^t \hat{g}(\tau, \xi) (t-\tau)^{\beta-1} E_{\beta,\beta}(i^{-\beta}(t-\tau)^{\beta}|\xi|^{\alpha}) e^{ix\xi} d\tau d\xi,
\end{equation}
we seem to lose $\sigma - \alpha$ derivatives, which is an obstacle to closing the contraction-mapping argument. In order to circumvent this problem, we exploit some smoothing
effect of the linear operator, which explains the choice of norm in (1.10). Note that this is not an issue in the space-fractional case ($\beta = 1$) because then $\sigma = \alpha$.

**Remark 1.4.** The smoothing effect mentioned above is however limited. As will be seen in Proposition 2.1, one can balance the loss of derivatives only if $\frac{n-1}{2} > \alpha$, which restricts the range of parameters to that presented in (1.8). In fact, note that this directly implies that $\alpha > 1$ and $\beta > \frac{1}{2}$, among other things.

**Remark 1.5.** If one wants to generalize this theorem to higher dimensions, a good place to start is the generalization of Proposition 2.1, which is straightforward. However, the condition needed in order to overcome the loss of derivatives in the nonlinear term becomes $\frac{n-1}{2} > \alpha$. This forces $\alpha$ to be very large and so it gives rise to a somewhat uninteresting result.

**Remark 1.6.** Another option to consider in order to avoid such loss of derivatives could be a different generalization of the NLS equation that also preserves the linear equation proposed in [Na], namely

$$i^\beta \partial_t^\alpha v = (-\Delta_x)^{\alpha/2} v + |\nabla|^{\alpha-\sigma}(|u|^{p-1}u)$$

$$v|_{t=0} = f \in H^s(\mathbb{R}^n).$$

Note that when $\beta = 1$, this problem also coincides with the space-fractional NLS. This equation could be studied with the same techniques as the one we consider, and also allows the generalization to $n$ dimensions of most of the linear estimates developed in this work, since there is no loss of derivatives we need to compensate. However, even if the equation makes sense mathematically, it is not clear what the physical interpretation of this smoother nonlinearity might be in the context of wave interactions, and thus our preference for equation (1.1).

**Remark 1.7.** It is also possible to generalize this result for even and non-integer $p$, but the exposition becomes more intricate (and slightly restricting the range of parameters might be necessary). The main idea is to use the estimates in this paper with respect to the norms (1.10) and (1.11), together with interpolation theorems to develop additional linear estimates. This strategy is based on the work of Kenig, Ponce and Vega for the KdV equation in [KPV]. Afterwards, one needs the fractional chain rule instead of Lemma 3.2, which can be found in [KPV], too.

**Remark 1.8.** Even if one could somehow control some $H^s_x$ norm of the solution globally in time, one may not easily iterate this local well-posedness result towards global well-posedness. This is precisely because of the memory effect, which also manifests itself in the lack of time-translation invariance. In other words, suppose we solve equation (1.1) for initial data $f = u(0)$ in an interval $[0,T]$ given by Theorem 1.1, and let that solution be $u(t)$. Then consider the IVP (1.1) for initial data $f = u(T)$ this time, and its solution $v$ in some small time interval. If $\beta$ were 1, we would expect $v(t) = u(t + T)$ to hold in this interval of existence, thereby extending the lifespan of our solution. However, this fails for $\beta < 1$. Instead, the right equation in the second step would be

$$i^\beta (-\tau)^{\alpha/2} v = (-\Delta_x)^{\alpha/2} v + g(v)$$

$$v|_{t=0} = u(T).$$

where $(-\tau)^{\alpha/2} v$, is a different version of the Caputo derivative:

$$-\tau^\alpha v(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\tau}^t \frac{\partial_x v(\tau,x)}{(t-\tau)^\alpha} d\tau.$$
Unfortunately, it is not clear that solving (1.16) produces an advantage over dealing with (1.1) directly, and therefore more research in this direction might be necessary.

1.5. Notation. We will denote by $A \lesssim B$ an estimate of the form $A \leq CB$ for some constant $B$ that might change from line to line. Similarly, $A \lesssim d$ $B$ means that the implicit constant $C$ depends on $d$.

We introduce the notation $\tilde{a}$ to denote the number $a - \varepsilon$ for $0 < \varepsilon \ll 1$ small enough. Similarly, we denote by $\breve{a}$ the number $a + \varepsilon$ for $0 < \varepsilon \ll 1$ small enough.

For $1 \leq p, q \leq \infty$ and $u : \mathbb{R} \times [0, T] \rightarrow \mathbb{C}$, we define

$$\|u\|_{L^p_x L^q_t} = \left( \int_0^T \left( \int_\mathbb{R} |u(t, x)|^q \, dx \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}},$$

and also

$$\|u\|_{L^q_t L^p_x} = \left( \int_0^T \left( \int_\mathbb{R} |u(t, x)|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{1}{q}},$$

with the usual modifications when $p$ or $q = \infty$. For $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{C}$, we will use the notation $L^p_x L^q_t$ and $L^q_t L^p_x$ instead, meaning $T = \infty$. We will also write $L^p_{T,x}$ in the case $p = q$.

We also use the standard notation for the spatial Fourier transform

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-ix\xi} \, dx,$$

and the fractional Laplacian will be given by the Fourier multiplier $(-\Delta_x)^{\frac{s}{2}} f(\xi) = |\nabla|^s \hat{f}(\xi)$. Similarly, $\langle \nabla \rangle^s f(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi)$. The following notation will be used for some Sobolev norms $\|f\|_{H^s(\mathbb{R})} = \|\langle \nabla \rangle^s f\|_{L^2(\mathbb{R})}$, and $\|f\|_{H^s(\mathbb{R})} = \|\langle \nabla \rangle^s f\|_{L^2(\mathbb{R})}$.

Finally, $C([0, T], H^s(\mathbb{R}))$ denotes the space of continuous functions $u$ from a time interval $[0, T]$ to $H^s(\mathbb{R})$ equipped with the norm $\max_{t \in [0, T]} \|u(t, \cdot)\|_{H^s(\mathbb{R})}$.

Additionally, we compile the following list of symbols and parameters that will be used along the paper:

- $\alpha$ - the order of the fractional Laplacian $(-\Delta_x)^{\frac{\alpha}{2}}$.
- $\beta$ - the order of the Caputo derivative in time $D_t^\beta$.
- $\sigma$ - the ratio $\frac{\alpha}{\beta}$.
- $p$ - the degree of the polynomial nonlinearity.
- $\gamma = \frac{\sigma - 1}{2}$ - the gain in the linear part thanks to the smoothing effect.
- $\tilde{\gamma} = \alpha - \frac{\sigma + 1}{2}$ - the gain in the nonlinear part thanks to the smoothing effect.

1.6. Outline. The paper is organized as follows. In Section 2, we prove linear estimates with respect to the norms given in (1.9), (1.10) and (1.11). In order to do this, we use a representation of the solution given by a Fourier multiplier for which we only have asymptotic formulas, and thus it requires a somewhat different treatment on small and large frequencies, as well as studying a remainder. In Section 3, we employ these linear
estimates to prove Corollary 1.1, which is an application of Theorem 1.1 for a special but illustrative choice of parameters, which simplifies the exposition. In Appendix A, we present additional information about Caputo fractional derivatives and explain our representation of the solutions. In Appendix B, we provide the general proof of Theorem 1.1 as stated in this introduction.

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2. Linear estimates

It will be useful to introduce the following operators. For \( f : \mathbb{R} \rightarrow \mathbb{C} \),

\[
S_t f(x) = \int_{\mathbb{R}} e^{-it|\xi|^\alpha} \chi_{|\xi|^\alpha \leq M} \hat{f}(\xi) e^{ix\xi} d\xi,
\]

(2.1)

\[
T_t f(x) = \int_{\mathbb{R}} t^{-\beta}|\xi|^{-\alpha} \chi_{|\xi|^\alpha > M} \hat{f}(\xi) e^{ix\xi} d\xi,
\]

\[
U_t f(x) = \int_{\mathbb{R}} E_\beta(i^{-\beta}t^{\beta}|\xi|^\alpha) \chi_{|\xi|^\alpha \leq M} \hat{f}(\xi) e^{ix\xi} d\xi,
\]

where \( \chi_{|\xi|^\alpha \leq M} = \chi(t, \xi) \) denotes a smooth function supported on the set \( \{ (t, \xi) \in (0, \infty) \times \mathbb{R} \mid |\xi|^\alpha \leq 2M \} \) for some large \( M > 0 \), satisfying \( \chi(t, \xi) = 1 \) if \( t|\xi|^\alpha \leq M \). Similarly, we will denote \( \chi_{|\xi|^\alpha > M} := 1 - \chi_{|\xi|^\alpha \leq M} \).

The operators \( S_t \) and \( U_t \) will capture the behavior of the solution (1.2) when \( t|\xi|^\alpha \) is small, whereas \( e^{it|\nabla|\alpha} \) and \( T_t \) will do so for large \( t|\xi|^\alpha \), based on the first and second terms of the asymptotic formula (1.4).

Similarly, the following multiplier operators will also play a role when dealing with the nonlinearity

\[
\tilde{S}_t f(x) = \int_{\mathbb{R}} |\xi|^{\sigma-\alpha} e^{-it|\xi|^\alpha} \chi_{|\xi|^\alpha \leq M} \hat{f}(\xi) e^{ix\xi} d\xi,
\]

(2.2)

\[
\tilde{T}_t f(x) = \int_{\mathbb{R}} t^{1-\beta}|\xi|^{-2\alpha} \chi_{|\xi|^\alpha > M} \hat{f}(\xi) e^{ix\xi} d\xi,
\]

\[
\tilde{U}_t f(x) = \int_{\mathbb{R}} t^{\beta-1} E_{\beta,\beta}(i^{-\beta}t^{\beta}|\xi|^\alpha) \chi_{|\xi|^\alpha \leq M} \hat{f}(\xi) e^{ix\xi} d\xi.
\]

As before, the operators \( \tilde{S}_t \) and \( \tilde{U}_t \) will capture the behavior of the second term in (1.5) when \( t|\xi|^\alpha \) is small. For large values of \( t|\xi|^\alpha \), the operators \( |\nabla|^{\sigma-\alpha} e^{it|\nabla|\alpha} \) and \( \tilde{T}_t \) will be used, based on the first and second terms in (1.7).

2.1. \( L_p^\infty L_p^2 \) estimates - smoothing effect. The following two propositions are a generalization of Theorem 3.5 in [KPV].
**Proposition 2.1.** Let $\gamma = \frac{2n}{d} \geq 0$. Then
\begin{equation}
\left\| \nabla |e^{-it\nabla}|^\gamma f \right\|_{L^p_x L^q_t} \lesssim \| f \|_{L^p_x}.
\end{equation}

**Proof.** We define
\[ F(t, x) := \int_{\mathbb{R}} |\xi|^\gamma e^{-it\xi} \hat{f}(\xi)e^{ix\xi} d\xi. \]
Then we can rewrite this as:
\[ F(t, x) = \int_0^\infty \xi^\gamma e^{-it\xi} \left( \hat{f}(\xi)e^{ix\xi} + \hat{f}(-\xi)e^{-ix\xi} \right) d\xi = \int_0^\infty \xi^\gamma e^{-it\xi} \hat{f}^*(x, \xi) d\xi = \frac{1}{\sigma} \int_0^\infty \mu^{\frac{n+1}{d} - 1} \hat{f}^*(x, \mu^\frac{1}{d}) e^{-it\mu} d\mu. \]
By the Plancherel identity,
\[ \|F(x)\|_{L^2_t}^2 = \int_0^\infty \frac{1}{\sigma} \mu^{\frac{n+1}{d} - 1} |\hat{f}^*(x, \mu^\frac{1}{d})|^2 d\mu = \int_0^\infty \frac{1}{\sigma} \xi^{2\gamma + 1 - \sigma} |\hat{f}^*(x, \xi)|^2 d\xi = \int_0^\infty \frac{1}{\sigma} |\hat{f}^*(x, \xi)|^2 d\xi, \]
Finally, since $\sup_x |\hat{f}^*(x, \xi)|^2 \leq 2|\hat{f}(\xi)|^2 + 2|\hat{f}(-\xi)|^2$ we obtain
\[ \|F\|_{L^p_x L^q_t} \lesssim \|f\|_{L^p_x}. \]

**Remark 2.1.** It is easy to further generalize this proposition to any dimension $n$. In such a case, if $\gamma = \frac{\sigma - n}{2d}$ then (2.3) holds. As we will see later, in order to gain derivatives and close the contraction-mapping principle with the techniques employed in this paper, we need $\alpha - \sigma + \gamma > 0$ to be true. This will not happen for any $n \geq 2$.

From the linear estimate in Proposition 2.1 we obtain the following:

**Proposition 2.2.** For $\gamma = \frac{2n}{d} \geq 0$ and a fixed time $T > 0$, we have
\[ \left\| \nabla |e^{-it\nabla}|^\gamma G(t', x) \right\|_{L^p_x L^q_t} \lesssim \| G \|_{L^p_x L^q_t}. \]

**Proof.** The dual of the estimate (2.3) is
\begin{equation}
\left\| \nabla |e^{it\nabla}|^\gamma g(t', x) \right\|_{L^p_x} \lesssim \| g \|_{L^p_x}. 
\end{equation}
From this estimate, consider the function $\hat{g}(t', x) := \chi_{[t,T]}(t') \hat{g}(t', x)$, which is clearly dominated by $g$ and therefore is also in $L^1_x L^2_t$. We substitute this in (2.4) and obtain
\[ \left\| \nabla |e^{it\nabla}|^\gamma g(t', x) \right\|_{L^p_x} \lesssim \| g \|_{L^1_x L^2([t,T])}. \]
Since the operator $e^{-it\nabla}|^\gamma$ is unitary,
\[ \left\| e^{-it\nabla}|^\gamma \right\|_{L^p_x} \lesssim \| g \|_{L^1_x L^2([t,T])}. \]
Proof.
By the Minkowski inequality,

\[ | \langle \chi_t \rangle \langle \gamma \rangle \int_0^T e^{-i(t-t')} \nabla |^\sigma g(t', x) \, dt' \rangle |_{L^2} \lesssim \|g\|_{L^1 L^2} . \]

Finally, consider

\[
\left\| \nabla \gamma \left( \int_0^T e^{-i(t-t')} \nabla |^\sigma G(t', x) \, dt' \right) \right\|_{L^2} = \sup_{\|g\|_{L^1 L^2}} \left| \int_0^T \int_R \left( \nabla \gamma \left( \int_0^T e^{-i(t-t')} \nabla |^\sigma G(t', x) \, dt' \right) \right) \, dx \, dt \right|
\]

\[
= \sup_{\|g\|_{L^1 L^2}} \left| \int_0^T \int_R G(t', x) \left( \nabla \gamma \left( \int_0^T e^{-i(t-t')} \nabla |^\sigma g(t', x) \, dt' \right) \right) \, dx \, dt \right|.
\]

Now we finish by using the Hölder inequality together with (2.5). \( \Box \)

**Proposition 2.3.** For any \( 0 \leq \gamma' < \gamma = \frac{2-\beta}{2} \), the operator \( T_t \) defined in (2.1) satisfies

\[ \left\| \nabla |^\gamma T_t f \right\|_{L^2} \lesssim T^{2-\gamma'} \|f\|_{L^2} . \]

**Proof.** By the Minkowski inequality,

\[
\left\| \nabla |^\gamma T_t f \right\|_{L^2} \lesssim \int_R \left\| t^{-\beta} |^\gamma |^\sigma \chi_{(t|^\sigma| > M)} \hat{f}(\xi) \right\|_{L^2} d\xi
\]

\[
= \int_R \left\| \xi |^\gamma |^\sigma \hat{f}(\xi) \chi_{(|\xi|^\sigma > \frac{M}{\xi}} \right\| \left( \int_\frac{M}{\xi}^T t^{-2\beta} dt \right)^{\frac{1}{2}} d\xi
\]

\[
= \int_R \left\| \xi |^\gamma |^\sigma \hat{f}(\xi) \chi_{(|\xi|^\sigma > \frac{M}{\xi}} \right\| \left( \frac{T^{1-2\beta}}{1-2\beta} - \frac{1}{1-2\beta} \left( \frac{M}{\xi} \right)^{1-2\beta} \right)^{\frac{1}{2}} d\xi.
\]

We shall see later that the assumption \( \beta > \frac{1}{2} \) will happen naturally, so now observe that because of the characteristic function we have the following bound

\[
\left\| \nabla |^\gamma T_t f \right\|_{L^2} \lesssim M, \beta \int_R \left\| \xi |^\gamma |^\sigma \hat{f}(\xi) \chi_{(|\xi|^\sigma > \frac{M}{\xi}} \right\| |^\sigma |^{\alpha-\beta} d\xi = \int_R \left\| \xi |^\gamma |^\sigma \hat{f}(\xi) \chi_{(|\xi|^\sigma > \frac{M}{\xi}} \right\| \frac{1}{2} \left( \frac{M}{\xi} \right)^{1-2\beta} d\xi.
\]

The condition \( 2\gamma' - \sigma < -1 \) is necessary to ensure integrability in the previous inequality. \( \Box \)

Finally we look at what happens for small frequencies.

**Proposition 2.4.** For any \( 0 \leq \gamma' < \gamma = \frac{2-\beta}{2} \), the operator \( U_t \) defined in (2.1) satisfies

\[ \left\| \nabla |^\gamma U_t f \right\|_{L^2} \lesssim T^{2-\gamma'} \|f\|_{L^2} . \]
Proposition 2.5. For any $0 \leq \gamma < \gamma = \frac{\alpha + 1}{2}$, the operator $S_t$ defined in (2.1) satisfies

$$\left\| \nabla^\gamma S_t f \right\|_{L^\gamma L^2_t} \lesssim T^{\frac{2-\gamma}{2}} \left\| f \right\|_{L^2}.$$

Proof. Let us remember the definition of this operator

$$U_t f(x) = \int_\mathbb{R} E_\beta(i^{-\beta} t^\beta |\xi|^\alpha) \chi_{\{|t|/|\xi| \leq M\}} \hat{f}(\xi) e^{ix\xi} d\xi.$$

Note that the series defining the function $E_\beta$ is absolutely convergent in the support of $\chi_{\{|t|/|\xi| \leq M\}}$, see (1.3). Using this fact together with the Minkowski inequality we find

$$\left\| \nabla^\gamma U_t f \right\|_{L^\gamma L^2_t} \lesssim \int_\mathbb{R} |\hat{f}(\xi)| |\xi|^{\gamma} \left( \int_0^{\min\left(T, \frac{\gamma}{\gamma - \sigma}\right)} d\xi \right)^{\frac{1}{2}} d\xi \lesssim \int_\mathbb{R} |\hat{f}(\xi)| |\xi|^{\gamma - \sigma} \chi_{\{|\xi| > M\}} d\xi + T^{\frac{1}{2}} \int_\mathbb{R} |\hat{f}(\xi)| |\xi|^{\gamma} \chi_{\{|\xi| \leq M\}} d\xi \lesssim \|f\|_{L^2} \left( \int_\mathbb{R} |\xi|^{2\gamma - \sigma} \chi_{\{|\xi| > M\}} d\xi \right)^{\frac{1}{2}} + \|f\|_{L^2} T^{\frac{1}{2}} \left( \int_\mathbb{R} |\xi|^{2\gamma} \chi_{\{|\xi| \leq M\}} d\xi \right)^{\frac{1}{2}},$$

Again, one sees that the condition $2\gamma - \sigma < -1$ is required to ensure integrability, and the exponent in $T$ coincides for both integrals. \hfill \Box

The same argument proves the following result:

**Proposition 2.5.** For any $0 \leq \gamma' < \gamma = \frac{\alpha + 1}{2}$, the operator $S_t$ defined in (2.1) satisfies

$$\left\| \nabla^\gamma S_t f \right\|_{L^\gamma L^2_t} \lesssim T^{\frac{2-\gamma'}{2}} \left\| f \right\|_{L^2}.$$

We now rewrite Propositions 2.1 and 2.2 in a clearer way for the purpose of estimating what will be the nonlinear part of the equation.

**Proposition 2.6.** Let $\tilde{\gamma} = \alpha - \frac{\sigma + 1}{2}$. Then

$$\left\| \nabla^\tilde{\gamma} |\nabla|^{\sigma-\alpha} e^{-it|\nabla|^\sigma} f \right\|_{L^\gamma L^2_t} \lesssim \|f\|_{L^2},$$

$$\left\| \nabla^\tilde{\gamma} \int_0^t |\nabla|^{\sigma-\alpha} e^{-i(t-t')|\nabla|^\sigma} G(t', x) dt' \right\|_{L^\gamma L^2_t} \lesssim \|G\|_{L^\gamma L^2_t}.$$

Note that the only way to gain derivatives is if $\tilde{\gamma} > 0$, which in particular implies that $\alpha > 1$ and $\beta > \frac{1}{2}$.

Similarly, we need the analogous of this result for the other multiplier operators taking part in the nonlinear piece.

**Proposition 2.7.** For any $0 \leq \tilde{\gamma}' < \tilde{\gamma} = \alpha - \frac{\sigma + 1}{2}$ the operator $\tilde{T}_t$ defined in (2.2) satisfies

$$\left\| \nabla^\tilde{\gamma}' \int_0^t \tilde{T}_{t-t'} G(t', x) dt' \right\|_{L^\gamma L^2_t} \lesssim T^{\frac{\tilde{\gamma} - \tilde{\gamma}'}{2}} \|G\|_{L^\gamma L^2_t}.$$
Proof. By the Minkowski inequality,

\begin{equation}
\left\| \nabla \gamma' \int_0^t \tilde{T}_{t-u}G(t', x) \, dt' \right\|_{L^\infty_x L^2_t} \leq \int_0^T \int_{\mathbb{R}} |\xi|^\gamma - 2\alpha |\tilde{G}(t', \xi)| \left( \frac{\chi_{\{(t-t') \xi > M\}}}{(t-t')^{1+\beta}} \right) \frac{d\xi}{(t-t')^{1+2\beta}} \frac{1}{M^{1+2\beta}} \, dt' \leq \int_0^T \int_{\mathbb{R}} |\xi|^\gamma - \alpha + \frac{\gamma}{2} |\tilde{G}(t', \xi)| \chi_{\{(t-t') \xi > M\}} \frac{d\xi}{(t-t')^{1+2\beta}} \leq \|G\|_{L^2_t} \left( \int_{\mathbb{R}} |\xi|^\gamma - \alpha + \frac{\gamma}{2} \chi_{\{(t-t') \xi > M\}} \frac{d\xi}{(t-t')^{1+2\beta}} \right)^{\frac{1}{2}} \leq \left( T - t' \right)^{\frac{\gamma - \alpha}{\gamma}} \|G\|_{L^1_t L^2_x}.
\end{equation}

The proofs of the following results are analogous to the ones before and so we will omit them.

**Proposition 2.8.** For any \(0 \leq \gamma' < \gamma = \alpha - \frac{\alpha + 1}{2}\) the operator \(\tilde{U}_t\) defined in (2.2) satisfies

\[ \left\| \nabla \gamma' \int_0^t \tilde{U}_{t-u}G(t', x) \, dt' \right\|_{L^\infty_x L^2_t} \leq T^{\frac{\gamma - \alpha}{\gamma}} \|G\|_{L^1_t L^2_x}. \]

**Proposition 2.9.** For any \(0 \leq \gamma' < \gamma = \alpha - \frac{\alpha + 1}{2}\) the operator \(\tilde{S}_t\) defined in (2.2) satisfies

\[ \left\| \nabla \gamma' \int_0^t \tilde{S}_{t-u}G(t', x) \, dt' \right\|_{L^\infty_x L^2_t} \leq T^{\frac{\gamma - \alpha}{\gamma}} \|G\|_{L^1_t L^2_x}. \]

Now it is time to put all these results together:

**Theorem 2.1** (Norm homogeneous version). Let \(0 \leq \gamma' < \gamma = \frac{\alpha - 1}{2}\), and \(0 \leq \gamma' < \gamma = \frac{\alpha - 1}{2}\). Then

\[ \left\| \nabla \gamma' e^{-it\nabla} f \right\|_{L^\infty_x L^2_t} \leq \|f\|_{L^2_x}, \]

\[ \left\| \nabla \gamma' S_t f \right\|_{L^\infty_x L^2_t} \leq T^{\frac{\gamma - \alpha'}{\gamma}} \|f\|_{L^2_x}, \]

\[ \left\| \nabla \gamma' T_t f \right\|_{L^\infty_x L^2_t} \leq T^{\frac{\gamma - \alpha'}{\gamma}} \|f\|_{L^2_x}, \]

\[ \left\| \nabla \gamma' U_t f \right\|_{L^\infty_x L^2_t} \leq T^{\frac{\gamma - \alpha'}{\gamma}} \|f\|_{L^2_x}. \]
Additionally,
\[
\left\| \nabla |_{t=0}^t |\nabla|^{\alpha-a} e^{-i(t-t')}|\nabla|^{a} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq \| G \|_{L^1_t L^2_x},
\]
\[
\left\| \nabla |_{t=0}^t \hat{S}_{t-t'} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq T^{\frac{2-a'}{2}} \| G \|_{L^1_t L^2_x},
\]
\[
\left\| \nabla |_{t=0}^t \hat{T}_{t-t'} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq T^{\frac{2-a'}{2}} \| G \|_{L^1_t L^2_x},
\]
\[
\left\| \nabla |_{t=0}^t \hat{U}_{t-t'} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq T^{\frac{2-a'}{2}} \| G \|_{L^1_t L^2_x}.
\]

**Theorem 2.2** (Norm nonhomogeneous version). Let 0 < γ' < γ = \(\frac{\sigma-1}{2}\), and 0 < \(\hat{\gamma}'\) < \(\hat{\gamma} = \alpha - \frac{\gamma+1}{2}\). Then
\[
\left\| \langle \nabla \rangle^{\gamma} e^{-i|\nabla|^a} f \right\|_{L^2_t L^2_x} \leq (1 + T^{\frac{\hat{\gamma}}{2}}) \| f \|_{L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\gamma'} S_t f \right\|_{L^2_t L^2_x} \leq T^{\frac{\gamma-a'}{2}}(1 + T^{\frac{\hat{\gamma}}{2}}) \| f \|_{L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\gamma'} T_t f \right\|_{L^2_t L^2_x} \leq T^{\frac{\gamma-a'}{2}}(1 + T^{\frac{\hat{\gamma}}{2}}) \| f \|_{L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\gamma'} U_t f \right\|_{L^2_t L^2_x} \leq T^{\frac{\gamma-a'}{2}}(1 + T^{\frac{\hat{\gamma}}{2}}) \| f \|_{L^2_x}.
\]

Moreover,
\[
\left\| \langle \nabla \rangle^{\hat{\gamma}} \int_0^t \nabla |_{t=0}^t |\nabla|^{\alpha-a} e^{-i(t-t')}|\nabla|^{a} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq (T^{\frac{2-a'}{2}} + T^{\beta-\frac{\hat{\gamma}}{2}}) \| G \|_{L^1_t L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\hat{\gamma}} \int_0^t \hat{S}_{t-t'} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq (T^{\frac{2-a'}{2}} + T^{\beta-\frac{\hat{\gamma}}{2}}) \| G \|_{L^1_t L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\hat{\gamma}} \int_0^t \hat{T}_{t-t'} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq (T^{\frac{2-a'}{2}} + T^{\beta-\frac{\hat{\gamma}}{2}}) \| G \|_{L^1_t L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\hat{\gamma}} \int_0^t \hat{U}_{t-t'} G(t', x) dt' \right\|_{L^2_t L^2_x} \leq (T^{\frac{2-a'}{2}} + T^{\beta-\frac{\hat{\gamma}}{2}}) \| G \|_{L^1_t L^2_x}.
\]

**Proof.** We write \( f = P_{\leq 1} f + P_{>1} f \) where \((P_{\leq 1} f)^\wedge = \phi(\xi) \hat{f}(\xi)\) for \(\phi \in C^\infty_0(\mathbb{R})\) and \(\text{supp} (\phi) \subset B(0, 1)\), and \((P_{>1} f)^\wedge = (1 - \phi(\xi)) \hat{f}(\xi)\).

Take for example
\[
\left\| \langle \nabla \rangle^{\gamma} e^{-i|\nabla|^a} P_{\leq 1} f \right\|_{L^2_t L^2_x} \leq \left\| \langle \nabla \rangle^{\gamma} e^{-i|\nabla|^a} P_{\leq 1} f \right\|_{L^2_t L^2_x} + \left\| \langle \nabla \rangle^{\gamma} e^{-i|\nabla|^a} P_{>1} f \right\|_{L^2_t L^2_x}.
\]

Let \( \hat{f} = \langle \nabla \rangle^{\gamma} P_{>1} f \). Then by Proposition 2.1 and the fact that these operators commute,
\[
\left\| \langle \nabla \rangle^{\gamma} e^{-i|\nabla|^a} P_{>1} f \right\|_{L^2_t L^2_x} = \left\| e^{-i|\nabla|^a} \hat{f} \right\|_{L^2_t L^2_x} \leq \left\| \langle \nabla \rangle^{\gamma} \hat{f} \right\|_{L^2_t L^2_x} \leq \| f \|_{L^2_x},
\]
\[
\left\| \langle \nabla \rangle^{\gamma} P_{>1} f \right\|_{L^2_t L^2_x} \leq \| f \|_{L^2_x}.
\]
which follows from the fact that $|\nabla|^{-\gamma} \langle \nabla \rangle^{P_{>1}}$ is a bounded operator from $L^2_x$ to $L^2_x$.

On the other hand, by the Minkowski inequality
\[
\left\| \langle \nabla \rangle^{\gamma} e^{-i\xi |\nabla|^\sigma} P_{\leq 1} f \right\|_{L^2_x L^2_T} \leq \left\| \int_{\mathbb{R}} e^{-i\xi |\xi|^\sigma} e^{i\xi \phi(\xi)} \langle \xi \rangle^{\gamma} \hat{f}(\xi) d\xi \right\|_{L^2_x L^2_T} \lesssim T^{\frac{1}{2}} \left\| \hat{f} \right\|_{L^2_x} \lesssim T^{\frac{1}{2}} \| f \|_{L^2_x}.
\]

The idea for the others will be the same, since all $S_t$, $T_t$ and $U_t$ composed with $P_{\leq 1}$ are bounded.

Now let’s look at the estimates that we will use on the nonlinear part. Take for instance
\[
\left\| \langle \nabla \rangle^{\gamma'} \int_0^t \hat{T}_{t-t'} G(t', x) dt' \right\|_{L^2_x L^2_T} \lesssim (T^{\frac{\gamma'}{2}} + T^{\beta+\frac{1}{2}}) \| G \|_{L^1_t L^2_x},
\]
and let’s prove it is true for $P_{\leq 1} G$ and $P_{>1} G$. Let $\hat{G} = \langle \nabla \rangle^{\gamma'} P_{>1} G$, then using Proposition 2.7:
\[
\left\| \int_0^t \hat{T}_{t-t'} \hat{G}(t', x) dt' \right\|_{L^2_x L^2_T} \lesssim T^{\frac{\gamma'}{2}} \left\| \langle \nabla \rangle^{-\gamma'} \hat{G} \right\|_{L^1_t L^2_x} = T^{\frac{\gamma}{2}} \left\| \langle \nabla \rangle^{-\gamma} \langle \nabla \rangle^{\gamma'} P_{>1} G \right\|_{L^1_t L^2_x} \lesssim T^{\frac{\gamma'}{2}} \| G \|_{L^1_t L^2_x},
\]
where the last inequality follows from the fact that the operator $\langle \nabla \rangle^{-\gamma} \langle \nabla \rangle^{\gamma'} P_{>1}$ is bounded from $L^2_x$ to $L^2_x$. This argument also works for $\hat{S}_t$ and $\hat{U}_t$ when looking at the piece where $P_{>1}$.

Regarding the other piece, the idea is to go back to the proof of the results and prove them for $P_{\leq 1}$ this time. We always assume that $M \gg T$ in our setting. For instance, in the case of $\hat{T}_t$, we go back to the following inequality in (2.6):
\[
\left\| \langle \nabla \rangle^{\gamma'} \int_0^t \hat{T}_{t-t'} P_{\leq 1} G(t', x) dt' \right\|_{L^2_x L^2_T} \lesssim \int_0^T \int_{\mathbb{R}} \langle \xi \rangle^{\gamma'} |\xi|^{-2\alpha} |\hat{G}(t', \xi)| \chi_{|\xi| \leq 1} \chi_{|(t-t')| |\xi|^\sigma > M} \left( \frac{|\xi|^\sigma + 2\alpha}{M^{1+2\beta}} - \frac{1}{(T-t')^{1+2\beta}} \right)^{\frac{1}{2}} d\xi dt'.
\]

Then $\chi_{|\xi| \leq 1} \chi_{|(t-t')| |\xi|^\sigma > M} = 0$ because $M \gg T$. 

\[\square\]

2.2. $L^2_x L^2_x$ estimates.

**Proposition 2.10.** Using the definitions for $S_t$, $T_t$ and $U_t$ given in (2.1), we have
\[
\left\| e^{-i\xi |\nabla|^\sigma} f \right\|_{L^2_x L^2_T} \lesssim \| f \|_{L^2_x},
\]
\[
\| S_t f \|_{L^2_x L^2_T} \lesssim \| f \|_{L^2_x},
\]
\[
\| T_t f \|_{L^2_x L^2_T} \lesssim \| f \|_{L^2_x},
\]
\[
\| U_t f \|_{L^2_x L^2_T} \lesssim \| f \|_{L^2_x}.
\]
Proposition 2.11. The following estimate holds:
\[
\left\| \int_0^t \left| \nabla^{\sigma - \alpha} e^{-i(t-t')|\nabla|^\sigma} G(t', x) \right| dt' \right\|_{L^p_T L^q_x} \leq \left\| |\nabla|^{\sigma - \alpha} G \right\|_{L^1_T L^2_x}.
\]

Proposition 2.12. For \( \tilde{S}_t, \tilde{T}_t \) and \( \tilde{U}_t \) as defined in (2.2) we have
\[
\left\| \int_0^t \tilde{S}_{t-t'} G(t', x) dt' \right\|_{L^p_T L^q_x} \leq T^{\beta - \frac{1}{2}} \left\| G \right\|_{L^p_T L^q_x},
\]
\[
\left\| \int_0^t \tilde{T}_{t-t'} G(t', x) dt' \right\|_{L^p_T L^q_x} \leq T^{\beta - \frac{1}{2}} \left\| G \right\|_{L^p_T L^q_x},
\]
\[
\left\| \int_0^t \tilde{U}_{t-t'} G(t', x) dt' \right\|_{L^p_T L^q_x} \leq T^{\beta - \frac{1}{2}} \left\| G \right\|_{L^p_T L^q_x}.
\]

Proof. For \( \tilde{T}_t \), we use the Minkowski inequality, the Plancherel theorem, then the cut-off and finally the Cauchy-Schwartz inequality:
\[
\left\| \int_0^t \tilde{T}_{t-t'} G(t', x) dt' \right\|_{L^p_T L^q_x} \leq \left\| \int_0^t \left( t-t' \right)^{-1-\beta} \left| \xi \right|^{-2\alpha} \left| \hat{G}(t', \xi) \right| \chi_{\{(t-t')|\xi|>M\}} \right\|_{L^p_t L^q_x} dt'
\]
\[
\leq \sup_{0 \leq t \leq T} \int_0^t (t-t')^{-1+\beta} \left\| G(t') \right\|_{L^q_x} dt'
\]
\[
\leq \left\| G \right\|_{L^p_T L^q_x} \sup_{0 \leq t \leq T} \left( \int_0^t (t-t')^{-2+2\beta} dt' \right)^{\frac{1}{2}}.
\]

The proofs for the corresponding inequalities involving \( \tilde{S}_t \) and \( \tilde{U}_t \) are analogous.

2.3. \( L^p_T L^q_x \) estimates - maximal function. Now we are interested in studying what happens with these operators in spaces like \( L^p_T L^q_x \) for \( p \geq 2 \). For the oscillatory part, there are global estimates that we may use, see Lemma 3.29 in [KPV], and also Theorem 1 in [P].

Proposition 2.13. Assume that \( \sigma > 1 \) and \( f \in \mathcal{S}(\mathbb{R}) \), the Schwartz space. The inequality
\[
\left\| e^{-i|\nabla|^\sigma f} \right\|_{L^p L^q} \leq \left\| |\nabla|^\sigma f \right\|_{L^p_x}
\]
holds if and only if \( p = \frac{3}{\sigma-2} \) and \( \frac{1}{4} \leq s < \frac{1}{2} \).

The other operators can be treated as follows.

Proposition 2.14. For \( p > 2 \) and \( T_t \) as defined in (2.1), we have
\[
\left\| T_t f \right\|_{L^p_T L^q_x} \leq \left\| |\nabla|^s f \right\|_{L^p_x},
\]
where \( s = \frac{1}{2} - \frac{1}{p} \).
Proof. We define
\[ \mathcal{T}_t(x) := \int_{\mathbb{R}} t^{-\beta} |\xi|^{-\alpha} \chi_{\{|\xi| \geq M\}} e^{ix\xi} d\xi, \]
which is well-defined because we are working under the assumption that \( \alpha > 1 \). Consider
\[ |\nabla|^{-s} \mathcal{T}_t(x) = \int_{\mathbb{R}} t^{-\beta} |\xi|^{-s-\alpha} \chi_{\{|\xi| \geq M\}} e^{ix\xi} d\xi, \]
Our goal is to prove that
\[ (2.7) \quad \left| |\nabla|^{-s} \mathcal{T}_t(x) \right| \lesssim_s |x|^{s-1} \text{ for all } x \in \mathbb{R}, \]
and for any \( s \in [0, 1] \) independently of time. By using \( |\nabla|^{-s} \mathcal{T}_t(x) = t^{\frac{s+\alpha}{\alpha}} (|\nabla|^{-s} \mathcal{T}_1)(t^{-\frac{1}{2}} x) \)
(which follows by rescaling), we can rewrite inequality (2.7) as
\[ \left| t^{-\frac{s+1}{s}} (|\nabla|^{-s} \mathcal{T}_1)(t^{-\frac{1}{2}} x) \right| \lesssim_s |x|^{s-1}, \]
\[ (2.8) \quad \left| (|\nabla|^{-s} \mathcal{T}_1)(t^{-\frac{1}{2}} x) \right| \lesssim_s t^{\frac{1-s}{s}} |x|^{s-1} = |t^{-\frac{1}{2}} x|^{s-1}, \]
\[ \left| |\nabla|^{-s} \mathcal{T}_1(x) \right| \lesssim_s |x|^{s-1}. \]
And therefore, it is enough to prove (2.7) in the case \( t = 1 \), namely
\[ |\nabla|^{-s} \mathcal{T}_1(x) = \int_{\mathbb{R}} |\xi|^{-s-\alpha} \chi_{\{|\xi| \geq M\}} e^{ix\xi} d\xi. \]
By absolute integrability we have \( \left| |\nabla|^{-s} \mathcal{T}_1(x) \right| \lesssim 1 \), which we will use when \( |x| \) is small. For \( |x| > 2 \), we integrate by parts:
\[ |\nabla|^{-s} \mathcal{T}_1(x) = \int_{\mathbb{R}} |\xi|^{-s-\alpha} \chi_{\{|\xi| \geq M\}} \frac{1}{ix} d\xi e^{ix\xi} d\xi = - \int_{\mathbb{R}} \frac{1}{ix} d\xi (|\xi|^{-s-\alpha} \chi_{\{|\xi| \geq M\}}) e^{ix\xi} d\xi. \]
Once again, the derivative of the function is absolutely integrable and so \( \left| |\nabla|^{-s} \mathcal{T}_1(x) \right| \lesssim_s |x|^{-1}. \) Combining the \( O(1) \) and the \( O(|x|^{-1}) \) bounds, (2.8) follows. At this stage, while we could use the full decay \( O(|x|^{-1}) \) to complete the proof of this proposition, we prefer to follow a strategy of proof that also works for the other operators where this decay is not available. Finally, we use the Hardy-Littlewood-Sobolev inequality to prove the final bound
\[ (2.9) \quad \left\| |\nabla|^{-s} \mathcal{T}_t f \right\|_{L^p_x L^\infty_y} = \left\| (|\nabla|^{-s} \mathcal{T}_1) * f \right\|_{L^p_x L^\infty_y} \leq \sup_t \left\| |\nabla|^{-s} \mathcal{T}_t \right\|_{L^p_x} \left\| f \right\|_{L^p_y} \lesssim_s \left\| f \right\|_{L^2_y}. \]
The assumption \( s = \frac{1}{2} - \frac{1}{p} > 0 \) is a necessary hypothesis to use the Hardy-Littlewood-Sobolev inequality. \( \square \)

**Proposition 2.15.** For \( p > 2 \) and \( U_t \) as defined in (2.1), we have
\[ \left\| U_t f \right\|_{L^2_x L^p_y} \lesssim \left\| |\nabla|^s f \right\|_{L^2_y}, \]
where \( s = \frac{1}{2} - \frac{1}{p} \) (and for \( p = \infty \) we have \( s = \frac{1}{2} \)).
Proof. We define
\[ \mathcal{U}_t(x) = \int_{\mathbb{R}} \chi(t|\xi| \leq M) e^{i\beta t |\xi|^{\alpha}} e^{ix \xi} d\xi = \int_{\mathbb{R}} \chi(t|\xi| \leq M) \left( \sum_{k=0}^{\infty} \frac{i^{-\beta k} \xi^{1-\beta k}}{\Gamma(\beta k + 1)} \right) e^{ix \xi} d\xi. \]
By rescaling, it suffices to prove
\[(2.10) \quad ||\nabla|^{-s} \mathcal{U}_1(x)|| \leq_s |x|^{s-1}, \]
for \(s \in [0,1]\), and then use the Hardy-Littlewood-Sobolev inequality as in (2.9).

In order to prove (2.10) we can be very direct and use integration by parts together with the fact that this integral is bounded for every \(x\) (remember that the series converges absolutely in the support of \(\chi\)).
\[
|\nabla|^{-s} \mathcal{U}_1(x) = \int_{\mathbb{R}} |\xi|^{-s} \chi(|\xi| \leq M) \left( \sum_{k=0}^{\infty} \frac{i^{-\beta k} \xi^{1-\beta k}}{\Gamma(\beta k + 1)} \right) e^{ix \xi} d\xi \\
= \int_{|\xi| \leq |x|^{-1}} |\xi|^{-s} \chi(|\xi| \leq M) \left( \sum_{k=0}^{\infty} \frac{i^{-\beta k} \xi^{1-\beta k}}{\Gamma(\beta k + 1)} \right) e^{ix \xi} d\xi \\
+ \int_{|\xi| > |x|^{-1}} |\xi|^{-s} \chi(|\xi| \leq M) \left( \sum_{k=0}^{\infty} \frac{i^{-\beta k} \xi^{1-\beta k}}{\Gamma(\beta k + 1)} \right) e^{ix \xi} d\xi = I_1 + I_2.
\]
Therefore:
\[
|I_1(x)| \leq \int_{|\xi| \leq |x|^{-1}} |\xi|^{-s} d\xi \sim |x|^{s-1},
\]
\[
|I_2(x)| \leq \left| \int_{|\xi| > |x|^{-1}} \frac{1}{x} \frac{d}{d\xi} \left[ |\xi|^{-s} \chi(|\xi| \leq M) \left( \sum_{k=0}^{\infty} \frac{i^{-\beta k} \xi^{1-\beta k}}{\Gamma(\beta k + 1)} \right) \right] e^{ix \xi} d\xi \right| \\
\leq \frac{1}{|x|} \int_{|\xi| > |x|^{-1}} \left| \frac{d}{d\xi} \left[ |\xi|^{-s} \chi(|\xi| \leq M) \left( \sum_{k=0}^{\infty} \frac{i^{-\beta k} \xi^{1-\beta k}}{\Gamma(\beta k + 1)} \right) \right] \right| d\xi \leq |x|^{s-1}.
\]
The last inequality follows from the fact that the series is absolutely convergent in any compact interval in \(\xi\), as well as its derivatives, and so the leading behaviour comes from \(\frac{d}{d\xi} |\xi|^{-s} \sim |\xi|^{-s-1}\) which integrates to around \(|x|^s\). \(\square\)

One may prove the following using similar ideas.

**Proposition 2.16.** For \(p > 2\) and \(S_t\) as defined in (2.1), we have
\[
\|S_t f\|_{L_p^p L_T^\infty} \leq \|\nabla|^s f\|_{L_p^2},
\]
where \(s = \frac{1}{2} - \frac{1}{p}\) (and for \(p = \infty\) we have \(s = \frac{1}{2}\)).

Now let us study what will constitute the nonlinear terms under the same norm.

**Proposition 2.17.** For \(p \geq 4\) and \(G(t, \cdot) \in \mathcal{S}(\mathbb{R})\) pointwise for each \(t \in [0, T]\), we have
\[
\left\| \int_0^t |\nabla|^{\sigma - \alpha} e^{-i(t-t')|\nabla|^\sigma} G(t', x) dt' \right\|_{L_p^p L_T^\infty} \leq \|\nabla|^{\sigma - \alpha + s} G\|_{L_p^1 L_T^2},
\]
where \(s = \frac{1}{2} - \frac{1}{p}\).
Proof. Let \( f_t(x) := e^{it|\nabla|^\sigma|\nabla|^{\sigma-\alpha} G(t', x)} \), then by applying Proposition 2.13 and the fact that the operator is unitary we have
\[
\left\| \int_0^t |\nabla|^{\sigma-\alpha} e^{-i(t-t')|\nabla|^\sigma} G(t', x) \, dt' \right\|_{L_x^p L_T^\infty} \leq \int_0^T \left\| |\nabla|^{\sigma-\alpha} e^{-i(t-t')|\nabla|^\sigma} G(t', x) \right\|_{L_x^p L_T^\infty([0,T])} \, dt' \\
= \int_0^T \left\| e^{-it|\nabla|^\sigma} f_t(x) \right\|_{L_x^p L_T^\infty([0,T])} \, dt' \\
\lesssim \int_0^T \left\| |\nabla|^\sigma f_t(x) \right\|_{L_x^p} \, dt' \\
= \int_0^T \left\| e^{-it|\nabla|^\sigma} |\nabla|^{\sigma-\alpha} G(t', x) \right\|_{L_x^p} \, dt' \\
= \int_0^T \left\| |\nabla|^{\sigma-\alpha} G(t', x) \right\|_{L_x^p} \, dt'.
\]

**Proposition 2.18.** For \( p > 2 \) and \( \check{S}_t \) as defined in (2.2), we have
\[
\left\| \int_0^t \check{S}_{t-t'} G(t', x) \, dt' \right\|_{L_x^p L_T^\infty} \lesssim \left\| |\nabla|^{\sigma-\alpha} G \right\|_{L_x^p L_T^\infty},
\]
for \( s = \frac{1}{2} - \frac{1}{p} \).

Proof. Note that \( |\nabla|^{-s-\sigma+\alpha} \check{S}_t = |\nabla|^{-s} S_t(x) \), and therefore we may use Proposition 2.16 for the operator \( S_t \)
\[
\left\| \int_0^t |\nabla|^{-s-\sigma+\alpha} \check{S}_{t-t'} G(t', x) \, dt' \right\|_{L_x^p L_T^\infty} = \left\| \int_0^t |\nabla|^{-s} S_{t-t'} G(t', x) \, dt' \right\|_{L_x^p L_T^\infty} \\
\leq \int_0^T \left\| |\nabla|^{-s} S_{t-t'} G(t', x) \right\|_{L_x^p L_T^\infty([0,T])} \, dt' \\
\lesssim \int_0^T \left\| G(t', x) \right\|_{L_x^p} \, dt'.
\]

**Remark 2.2.** We can also prove the bound
\[
\left\| \int_0^t \check{S}_{t-t'} G(t', x) \, dt' \right\|_{L_x^p L_T^\infty} \lesssim T^{\beta-\frac{1}{2}} \left\| |\nabla|^\sigma G \right\|_{L_x^p L_T^\infty}.
\]
The main idea is to write
\[
|\nabla|^{-s} \check{S}_t(x) := \int_{\mathbb{R}} |\xi|^{-s+\sigma-\alpha} \chi_{|\xi|^\sigma \leq M} e^{i|\xi|^\sigma + ix\xi} \, d\xi \\
= \xi^{-\beta-1} \int_{\mathbb{R}} |\xi|^{-s} \chi_{|\xi|^\sigma \leq M} (t^{1-\beta} |\xi|^{\sigma-\alpha}) e^{i|\xi|^\sigma + ix\xi} \, d\xi,
\]
and then we treat the piece in parentheses as a bounded function and proceed as with the proof for \( \check{U}_t \) below. The difference is that this second proof only works for finite \( T \), whereas the one for Proposition 2.18 works for \( T = \infty \) as well.
Proposition 2.19. For $p > 2$ and $\tilde{T}_t$ as defined in (2.2), we have
\[
\left\| \int_0^t \tilde{T}_{t-t'} G(t', x) \, dt' \right\|_{L^p_t L^2_x} \lesssim \| \nabla |^{\sigma-a+s} G \|_{L^1_t L^2_x},
\]
for $s = \frac{1}{2} - \frac{1}{p}$. Moreover,
\[
\left\| \int_0^t \tilde{T}_{t-t'} G(t', x) \, dt' \right\|_{L^p_t L^2_x} \lesssim T^{\beta-\frac{1}{2}} \| \nabla |^\sigma G \|_{L^2_{t,x}}.
\]

Proof. Both proofs are analogous to those of Proposition 2.18. □

Proposition 2.20. For $p > 2$ and $\tilde{U}_t$ as defined in (2.2), we have
\[
\left\| \int_0^t \tilde{U}_{t-t'} G(t', x) \, dt' \right\|_{L^p_t L^2_x} \lesssim T^{\beta-\frac{1}{2}} \| \nabla |^\sigma G \|_{L^2_{t,x}},
\]
for $s = \frac{1}{2} - \frac{1}{p}$.

Proof. In this case we need to be slightly more careful. Based on the definition of $\tilde{U}_t$ in (2.2) we let
\[
|\nabla|^{-s} \mathcal{U}_t(x) := \int_\mathbb{R} t^{\beta-1} |\xi|^{-s} \chi\{t|\xi| \leq M\} \left( \sum_{k=0}^\infty t^{\beta_k} \frac{\xi^{ak}}{(\beta k + \beta)} \right) e^{ix\xi} \, d\xi.
\]

As with $|\nabla|^{-s} U_t$, one proves that $|\nabla|^{-s} \mathcal{U}_t(x) \lesssim t^{\beta-1} |x|^{s-1}$, where the implicit constant is independent of time. Then
\[
\sup_{t \in [0,T]} \left| \int_0^t |\nabla|^{-s} \mathcal{U}_{t-t'} * G_{t'} \, dt' \right| \leq \sup_{t \in [0,T]} \left| \int_0^t |\nabla|^{-s} \mathcal{U}_{t-t'} | * |G_{t'}| \, dt' \right|
\]
\[
\lesssim \sup_{t \in [0,T]} \left| \int_0^t (t - t')^{\beta-1} |x|^{s-1} |G_{t'}(x)| \, dt' \right|
\]
\[
= |x|^{s-1} \left( \sup_{t \in [0,T]} \left| \int_0^t (t - t')^{\beta-1} |G_{t'}(x)| \, dt' \right| \right)
\]
\[
\lesssim |x|^{s-1} \left( \sup_{t \in [0,T]} \left( \int_0^t (t - t')^{2(\beta-1)} \, dt' \right)^{\frac{1}{2}} \| G(t', x) \|_{L^2_{t,x}(0,T)} \right)
\]
\[
= |x|^{s-1} \left( T^{\beta-\frac{1}{2}} \| G(t', x) \|_{L^2_{t,x}} \right),
\]
and we finish by using the Hardy-Littlewood-Sobolev inequality. □

We sum up these results in the following:
Moreover, small enough. Similarly, we denote by \( a \) problem (1.1). Namely, consider a cubic nonlinearity \( \| \cdot \| \) values for the parameters \( (3.1) \)

Based on the integral equation (1.5), we define the following operator

\[ \Phi(v) := \int_{\mathbb{R}} \tilde{f}(\xi) E_{\beta}(\xi^{\alpha} t^{\beta} \xi^{\gamma} \xi^{\gamma}) e^{ix \cdot \xi} \, d\xi \tag{3.1} \]

where

\[ \tilde{g}(\tau, \xi) = \int_{\mathbb{R}} |v(\tau, x)|^2 v(\tau, x) e^{-ix \cdot \xi} \, dx . \]

We remind the reader of the notation \( a - \varepsilon \) to denote the number \( a - \varepsilon \) for \( 0 < \varepsilon \ll 1 \) small enough. Similarly, we denote by \( a + \varepsilon \) the number \( a + \varepsilon \) for \( 0 < \varepsilon \ll 1 \) small enough.
Now we define the norms
\[
\eta_1(v) = \left\| \langle \nabla \rangle^{\frac{2}{3}} v \right\|_{L_T^\infty L_x^2},
\eta_2(v) = \left\| \langle \nabla \rangle^{\frac{1}{3}} v \right\|_{L_T^\infty L_x^2},
\eta_3(v) = \| v \|_{L_T^4 L_x^2},
\]
and let \( \Lambda_T := \max_{j=1,2,3} \eta_j \). Then consider the space
\[
X_T := \{ v \in C([0,T], H^{\frac{1}{2}}(\mathbb{R})) \mid \Lambda_T(v) < \infty \}.
\]
Our goal is to show that for small enough \( T \), there exists a ball \( B_R \subset X_T \) such that \( \Phi : B_R \rightarrow B_R \) is a contraction, and then apply the contraction mapping theorem.

Before we do that, we prove a useful lemma.

**Lemma 3.1.** For \( j = 1,2,3 \) the following inequality holds
\[
\eta_j(\Phi(v)) \leq \eta_j(e^{-it|\xi|^\sigma} f) + \eta_j(S t f) + \eta_j(T t^\dagger f) + \eta_j(U_t f)
\]
\[
+ \eta_j \left( \int_0^t |\nabla|^{\sigma-\alpha} e^{-i(t-t')|\nabla|^\sigma} g(t',\cdot) \, dt' \right) + \eta_j \left( \int_0^t \tilde{S}_{t-t'} g(t',\cdot) \, dt' \right)
\]
\[
+ \eta_j \left( \int_0^t \tilde{T}_{t-t'} g(t',\cdot) \, dt' \right) + \eta_j \left( \int_0^t \tilde{U}_{t-t'} g(t',\cdot) \, dt' \right)
\]
where \( T_t^\dagger \) and \( \tilde{T}_t^\dagger \) satisfy the same estimates as \( T_t \) and \( \tilde{T}_t \), respectively.

**Proof.** Let \( u \) be the linear part of \( \Phi(v) \), and remember that \( \chi := \chi_{\{t|\xi|^\sigma \leq M\}} \) is a smooth function such that \( \chi = 1 \) if \( t|\xi|^\sigma \leq M \) and \( \chi = 0 \) if \( t|\xi|^\sigma > 2M \). Let \( \tilde{P}_\chi u = \tilde{\chi} u \). Then we have
\[
u = P_\chi u + P_1 - \chi u = U_t f + P_1 - \chi u
\]
\[
u = U_t f + e^{-it|\xi|^\sigma} f - P_\chi e^{-it|\xi|^\sigma} f + \left( P_1 - \chi u - P_1 - \chi e^{-it|\xi|^\sigma} f \right)
\]
\[
u = U_t f + e^{-it|\xi|^\sigma} f - S_t f + P_1 - \chi \left( u - e^{-it|\xi|^\sigma} f \right)
\]
Now by (1.4), the Fourier multiplier corresponding to \( P_1 - \chi \left( u - e^{-it|\xi|^\sigma} f \right) \) may be controlled as follows for large enough \( M \)
\[
(1 - \chi(t,\xi)) \left| E_\beta(\xi^\alpha t^\beta) - e^{-it|\xi|^\sigma} \right| \lesssim (1 - \chi(t,\xi)) \frac{1}{t^\beta|\xi|^\alpha}.
\]
As a result, one can check that \( P_1 - \chi \left( u - e^{-it|\xi|^\sigma} f \right) \) satisfies the same estimates as \( T_t \), since in every proof involving \( T_t \) the absolute value was taken at some stage. The idea for the nonlinear part of \( \Phi(v) \) is similar.

Let us start by taking the first norm of (3.1). We will use a combination of Lemma 3.1 and Theorem 2.2 to control the two terms that form \( \Phi(v) \). Let us remind the reader of the estimates proved in Theorem 2.2 for the particular case of our parameters.
Note the gain in derivatives in this norm, almost 1 with Lemma 3.1 yield Lemma 3.2.

Now we look at the second norm of (3.1). Propositions 2.10, 2.11 and 2.12 combined These estimates above, together with Lemma 3.1 allow us to control (3.1) in terms as follows:

These estimates above, together with Lemma 3.1 allow us to control (3.1) in terms of \( \eta_1 \) as follows:

Note the gain in derivatives in this norm, almost \( \frac{1}{2} \) for the linear part and almost \( \frac{1}{2} \) in the nonlinear part.}

Now we look at the second norm of (3.1). Propositions 2.10, 2.11 and 2.12 combined with Lemma 3.1 yield

Note that in this case we clearly lose \( \frac{1}{4} \) derivatives in the nonlinear part.

Finally, we take the third norm, which we control thanks to Theorem 2.3 and Lemma 3.1.

Let us highlight once again the loss of \( \frac{1}{4} \) derivatives in the linear part and \( \frac{1}{2} \) in the nonlinear part.

Therefore, we can simultaneously estimate (3.2), (3.3) and (3.4) by controlling the quantity \( \| \langle \nabla \rangle^q |v|^2 v \|_{L^2_T,x} \) in terms of \( \eta_1, \eta_2 \) and \( \eta_3 \). To this end, we introduce the following:

**Lemma 3.2.** When \( p \) is an odd integer, we have that

\[
\| \langle \nabla \rangle^s |v|^{p-1} v \|_{L^2_T,x} \leq \| \langle \nabla \rangle^s v \|_{L^2_T} \| v \|_{L^2_T}^{p-1} \| v \|_{L^2_T}^{p-1} .
\]
Proof. When \( p = 2k + 1 \) is an odd integer, we have \( |u|^{p-1}u = u^{k+1}v^k \). Then by the Plancherel theorem,
\[
\|\langle \nabla \rangle^s(|v|^{p-1}v)\|_{L^2_T} \leq \|\langle \nabla \rangle^s|v|^{p-1}\|_{L^2_T} + \|\langle \nabla \rangle^s(|v|^{p-1})\|_{L^2_T}.
\]
Now we iterate the argument for the last term to argue that the derivative can only hit \( v \) or \( \bar{v} \). In other words,
\[
\|\langle \nabla \rangle^s(|v|^{p-1}v)\|_{L^2_T} \leq \|\langle \nabla \rangle^s|v|^{p-1}\|_{L^2_T} + \|\langle \nabla \rangle^s|v|^{k+1}v^{k-1}\|_{L^2_T}.
\]
After taking the \( L^2_T \) norm, and interchanging the order of integration in \( T \) and \( x \), we can use the Hölder inequality to control each term in terms of \( \|\langle \nabla \rangle^s|v|\|_{L^2_T} \|v|^{p-1}\|_{L^2_T}^2 \).

Using Lemma 3.2 for the case \( p = 3 \), we find that
\[
\|\langle \nabla \rangle^{\frac{1}{2}}(|v|^2v)\|_{L^2_T} \leq \eta_1(v)\eta_3(v)^2.
\]
By assuming that \( T \leq 1 \), (3.2), (3.3) and (3.4) can be rewritten as
\[
\Lambda_T(\Phi(v)) \leq \|f\|_{H^{1/4}_x} + T^{0+}\eta_1(v)\eta_3(v)^2 \leq \|f\|_{H^{1/4}_x} + T^{0+}\Lambda_T(v)^3.
\]

Pick \( B_R = \{v \in X_T | \Lambda_T(v) < R\} \) and \( f \) such that \( \|f\|_{H^{1/4}_x} \leq \frac{R}{2} \). Then we have that \( \Phi : B_R \rightarrow B_R \) as long as \( T^{0+}R^3 \leq \frac{R}{2} \), which happens for \( T \) small enough. We still must prove that \( \Phi(v) \) is actually in \( C([0,T],H^{1/4}_x(\mathbb{R})) \), but we do that for the general case in Lemma 3.3, which can be found in Appendix B below.

Now we prove that \( \Phi \) is a contraction. By using the same ideas as in (3.2), (3.3) and (3.4) together with \( T \leq 1 \) we quickly find
\[
\Lambda_T(\Phi(v) - \Phi(u)) \leq T^{0+}\|\langle \nabla \rangle^{\frac{1}{2}}(|v|^2v - |u|^2u)\|_{L^2_T,x}.
\]
In order to deal with this last term, we use the following estimate based on the fundamental theorem of calculus:
\[
\|v|^2v - |u|^2u\|_{L^2_{T,H^{1/2}_x}} \leq \int_0^1 \|v + \lambda(u - v)|^2(u - v)\|_{L^2_{T,H^{1/2}_x}} d\lambda.
\]
Then a combination of Lemma 3.2 and the Hölder inequality yields
\[
\Lambda_T(\Phi(v) - \Phi(u)) \leq T^{0+}R^2\Lambda_T(v - u),
\]
so that by making \( T \) smaller (if necessary), we can arrange \( CT^{0+}R^2 < 1 \). Therefore \( \Phi \) is a contraction on the ball \( B_R = \{v \mid \Lambda_T(v) < R \sim \|f\|_{H^{1/4}_x}\} \), which shows that \( \Phi \) has the desired contraction property.
Consider the linear space-time fractional Schrödinger equation (1.1) with \( g = 0 \). By taking the Fourier transform in space and the Laplace transform in time, we obtain
\[
i^{\beta} s^{\beta} \hat{u}(s, \xi) - i^{\beta} s^{\beta-1} \hat{f}(\xi) = \xi^\alpha \hat{u}(s, \xi),
\]
and thus
\[
\hat{u}(s, \xi) = \frac{i^{\beta} s^{\beta-1}}{i^{\beta}s^{\beta} - \xi^\alpha} \hat{f}(\xi),
\]
which we then invert to find:
\[
u(t, x) = \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=0}^{\infty} \frac{i^{\beta} k |\xi|^{\alpha k - \beta k}}{\Gamma(\beta k + 1)} \right] d\xi,
\]
which was already given in (1.2).

For completeness, we now present the main ideas on how this function formally solves equation (1.1) in the case \( g = 0 \). The full details can be found in [Die].

With \( u \) as in (1.2), we formally have
\[
\partial_t^\beta u(t, x) = \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=1}^{\infty} \frac{\beta k |\xi|^{\alpha k - \beta k}}{\Gamma(\beta k + 1)} \right] d\xi,
\]
and therefore,
\[
\partial_t^\beta u(t, x) = \frac{1}{\Gamma(1 - \beta)} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=1}^{\infty} \frac{\beta k |\xi|^{\alpha k - \beta k}}{\Gamma(\beta k + 1)} \right] \int_0^t \frac{\tau^{\beta k - 1}}{(t - \tau)^{\beta}} d\tau d\xi.
\]

One can easily check that the \( \tau \)-integral is essentially a Beta function:
\[
\int_0^t \frac{\tau^{\beta k - 1}}{(t - \tau)^{\beta}} d\tau = \int_0^1 \frac{\tau^{\beta k - 1}}{(1 - \tau)^{\beta}} t d\tau = t^{\beta(k - 1)} \frac{\Gamma(\beta k) \Gamma(1 - \beta)}{\Gamma(\beta k - 1)}.
\]

Consequently, and after using the identity \( \Gamma(z + 1) = z \Gamma(z) \),
\[
\partial_t^\beta u(t, x) = \frac{1}{\Gamma(1 - \beta)} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=1}^{\infty} \frac{\beta k |\xi|^{\alpha k - \beta k} \Gamma(\beta k) \Gamma(1 - \beta)}{\Gamma(\beta k + 1)\Gamma(\beta k - 1)} \right] d\xi
\]
\[
= \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=1}^{\infty} \frac{\beta k |\xi|^{\alpha k - \beta k} \Gamma(\beta k + 1)}{\Gamma(\beta k + 1)\Gamma(\beta k - 1)} \right] d\xi
\]
\[
= \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=0}^{\infty} \frac{\beta k |\xi|^{\alpha (k + 1) - \beta k}}{\Gamma(\beta k + 1)} \right] d\xi = I.
\]

On the other hand, we consider the other term involved in equation (1.1), which formally gives
\[
(-\Delta_x)^{\alpha/2} u = \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) \left[ \sum_{k=0}^{\infty} \frac{\beta k |\xi|^{\alpha (k + 1) - \beta k}}{\Gamma(\beta k + 1)} \right] d\xi = II.
\]
Therefore \( I \) and \( II \) coincide after multiplying the former by the \( i^{\beta} \) factor.
Now consider the nonlinear space-time fractional Schrödinger equation, as defined in (1.1). Once again, we simply provide a brief exposition, since the full details may be found in [Die]. A representation for the solution to the nonlinear equation was given in (1.5), where the Fourier transform of the inhomogeneous part is precisely

\[
(3.6) \quad h(t, \xi) = i^{-\beta} \int_0^t \hat{g}(\tau, \xi) (t - \tau)^{\beta - 1} E_{\beta, \beta} (i^{-\beta}(t - \tau)^{\beta}|\xi|^\alpha) \, d\tau 
\]

\[
= \sum_{k=0}^{\infty} i^{-\beta (k + 1)}|\xi|^\alpha \, (J_{\beta k + \beta} \hat{g})(t, \xi),
\]

where

\[
(J_{\nu} \hat{g})(t, \xi) := \frac{1}{\Gamma(\nu)} \int_0^t \hat{g}(\tau, \xi) (t - \tau)^{\nu - 1} \, d\tau,
\]

which is based on the definition of $E_{\beta, \beta}$ in (1.6). One can check that $J_{\nu_1} \cdot J_{\nu_2} = J_{\nu_1 + \nu_2}$ for any $\nu_1, \nu_2 > 0$.

All we need to do now is to show that $h$ is a particular solution of the following Cauchy problem:

\[
(3.6) \quad \begin{cases}
    i^\beta \partial_t^\beta \hat{u} = |\xi|^\alpha \hat{u} + \hat{g}, \\
    \hat{u} \mid_{t=0} = 0.
\end{cases}
\]

We have the following formal equalities:

\[
I = |\xi|^\alpha h(t, \xi) = \sum_{k=0}^{\infty} i^{-\beta (k + 1)}|\xi|^{\alpha (k + 1)} J_{\beta k + \beta} \hat{g}(t) 
\]

\[
II = i^\beta \partial_t^\beta h(t, \xi) = \sum_{k=0}^{\infty} i^{-\beta k}|\xi|^\alpha \partial_t^\beta J_{\beta k + \beta} \hat{g}(t).
\]

Note that $\partial_t^\beta = D_t^\beta (\text{id} - \text{ev}_0)$, where id is the identity, $\text{ev}_0 f = f(0)$ and $D_t^\beta f := \frac{1}{\Gamma(1 - \beta)} \frac{d}{dt} \int_0^t f(s) (t - s)^{-\beta} \, ds = \frac{d}{dt} J_{1 - \beta} f$.

The proof of this fact follows directly from integration by parts. Then one shows the following identity

\[
\partial_t^\beta J_{\beta} = D_t^\beta (J_{\beta} - \text{ev}_0 J_{\beta}) = D_t^\beta J_{\beta} = \frac{d}{dt} J_{1 - \beta} J_{\beta} = \frac{d}{dt} J_1 = \text{id},
\]

by the fundamental theorem of calculus, having also used the fact that $\text{ev}_0 J_{\beta} = 0$.

With all this in mind, let us show that $h$ in (3.5) formally satisfies the equation in (3.6):

\[
II - I = \sum_{k=0}^{\infty} i^{-\beta k}|\xi|^\alpha \partial_t^\beta J_{\beta k} \hat{g} - \sum_{k=0}^{\infty} i^{-\beta (k + 1)}|\xi|^{\alpha (k + 1)} J_{\beta k + \beta} \hat{g} 
\]

\[
= \sum_{k=0}^{\infty} i^{-\beta k}|\xi|^\alpha J_{\beta k} \hat{g} - \sum_{k=0}^{\infty} i^{-\beta (k + 1)}|\xi|^{\alpha (k + 1)} J_{\beta k + \beta} \hat{g} = \hat{g} = \hat{g}.
\]

It is also obvious that $h \mid_{t=0} = 0$. 

Appendix B: General proof of Theorem 1.1

In this section, we provide the proof of Theorem 1.1 for general values of the parameters involved. As we did before, we define the following operator based on the integral equation given in (1.5),

$$
\Phi(v)(t, x) = \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi) E_\beta(|\xi|^\alpha t^\beta i^{-\beta}) \, d\xi 
+ i^{-\beta} \int_{\mathbb{R}} \hat{g}(\tau, \xi) (t - \tau)^{\beta - 1} E_{\beta,\beta}(i^{-\beta}(t - \tau)^{\beta} |\xi|^{\alpha}) e^{ix\xi} \, d\xi \, d\tau,
$$

(3.7)

where

$$
\hat{g}(\tau, \xi) = \int_{\mathbb{R}} |v(\tau, x)|^{p-1} v(\tau, x) e^{-ix\xi} \, dx.
$$

Now we define

$$
\eta_1(v) = \left\| \langle \nabla \rangle^{\delta} v \right\|_{L^q_T L^2_x},
$$

$$
\eta_2(v) = \left\| \langle \nabla \rangle^{\delta - \gamma'} v \right\|_{L^q_T L^2_x},
$$

$$
\eta_3(v) = \left\| v \right\|_{L^2_T L^{p-1}_x},
$$

for some $s$ and some $\delta$ to be chosen later, and let $\Lambda_T := \max_{j=1,2,3} \eta_j$. Then consider the space $X_T := \{ v \in C([0, T], H^s(\mathbb{R})) \mid \Lambda_T(v) < \infty \}$. Our goal is to show that for small enough $T$, there exists a ball $B_R \subset X_T$ such that $\Phi : B_R \to B_R$ is a contraction, and then apply the contraction mapping principle.

Let us start by taking the first norm of (3.7), where we will use Theorem 2.2 and Lemma 3.1.

$$
\eta_1(\Phi(v)) \lesssim (1 + T^{\frac{s}{2}}) (1 + T^{\frac{s-\delta}{\beta}}) \left\| \langle \nabla \rangle^{\delta - \gamma'} f \right\|_{L^q_T L^2_x}
+ (T^{\frac{s-\delta}{\beta}} + T^{\beta - 1}) \left\| \langle \nabla \rangle^{\delta - \gamma'} (|v|^{p-1} v) \right\|_{L^q_T L^2_x}.
$$

(3.8)

Now we look at the second norm of (3.1). Propositions 2.10, 2.11 and 2.12 combined with Lemma 3.1 yield

$$
\eta_2(\Phi(v)) \lesssim \left\| \langle \nabla \rangle^{s} f \right\|_{L^q_T L^2_x} + T^{\beta - \frac{1}{2}} (1 + T^{1-\beta}) \left\| \langle \nabla \rangle^{s+\sigma-\alpha} (|v|^{p-1} v) \right\|_{L^q_T L^2_{T,x}}.
$$

(3.9)

In order to be able to eventually control the nonlinear part with $\eta_1$, we require $s + \sigma - \alpha \leq \delta$, as well as $s \geq \delta - \gamma'$ to control the linear part too. These two conditions on $s, \delta$ are perfectly compatible, and one may easily check that they are equivalent to $\alpha > \frac{s+1}{2}$ as stated in the hypothesis of Theorem 1.1, see (1.8).

Finally, we take the third norm, which we control thanks to Theorem 2.3 and Lemma 3.1.

$$
\eta_3(\Phi(v)) \lesssim \left\| \langle \nabla \rangle^{s} f \right\|_{L^q_T L^2_x} + T^{\beta - \frac{1}{2}} \left\| \langle \nabla \rangle^{s+\sigma-\alpha} (|v|^{p-1} v) \right\|_{L^q_T L^2_{T,x}},
$$

(3.10)

for $s \geq \frac{1}{2} - \frac{1}{p(p-1)}$ and $p \geq 3$. 

Therefore, we can simultaneously estimate (3.8), (3.9) and (3.10) by controlling the quantity \(|\langle \nabla \rangle^{s+\sigma-\alpha} (|v|^{p-1}v)\|_{L^2_{T,x}}\|\) in terms of \(\eta_1, \eta_2\) and \(\eta_3\). Using Lemma 3.2, we have

\[
(3.11) \quad \|\langle \nabla \rangle^{s+\sigma-\alpha} (|v|^{p-1}v)\|_{L^2_{T,x}} \lesssim \eta_1(v) \eta_3(v)^{p-1}.
\]

By assuming that \(T \leq 1\), (3.8), (3.9) and (3.10) can be rewritten as

\[
\Lambda_T(\Phi(v)) \lesssim \|f\|_{H^\varepsilon_x} + T^\varepsilon \eta_1(v) \eta_3(v)^{p-1} \lesssim \|f\|_{H^\varepsilon_x} + T^\varepsilon \Lambda(v)^p,
\]

for some \(\varepsilon \leq \frac{3-s}{\sigma}\).

Pick \(B_R = \{v \in X_T \mid \Lambda_T(v) < R\}\) and \(f\) such that \(\|f\|_{H^\varepsilon_x} \lesssim \frac{R}{T}\). Then we have that \(\Phi : B_R \to B_R\) as long as \(CT^\varepsilon R^p < \frac{R}{T}\), which happens for \(T\) small enough. We still must prove that \(\Phi(v)\) is actually in \(C^1([0,T], H^s(\mathbb{R}))\), but we do that in Lemma 3.3 below.

Now we prove that \(\Phi\) is a contraction. By using the same ideas as in (3.8), (3.9) and (3.10) together with \(T \leq 1\) we quickly find

\[
\Lambda_T(\Phi(v) - \Phi(u)) \lesssim T^\varepsilon \|\langle \nabla \rangle^{s+\sigma-\alpha} (|v|^{p-1}v - |u|^{p-1}u)\|_{L^2_{T,x}}
\]

for some \(\varepsilon \leq \frac{3-s}{\sigma}\). Now we adapt an idea found in the proof of Theorem 1.2 in [HS], which is based on the fundamental theorem of calculus:

\[
\|v|^{p-1}v - |u|^{p-1}u\|_{L^2_{T}H^{2+s-\alpha}} \leq \left\| \int_0^1 p|v + \lambda(u - v)|^{p-1}(u - v) \, d\lambda \right\|_{L^2_{T}H^{s-\alpha}} \leq p \int_0^1 \|v + \lambda(u - v)|^{p-1}(u - v)\|_{L^2_{T}H^{2+s-\alpha}} \, d\lambda.
\]

Suppose \(p = 2k + 1\). By Lemma 3.2 we have

\[
\|\langle \nabla \rangle^{s+\sigma-\alpha} (v + \lambda(u - v)|^{p-1}(u - v))\|_{L^2_{T,x}} = \|\langle \nabla \rangle^{s+\sigma-\alpha} (v + \lambda(u - v)|^{k}(v + \lambda(u - v)|^{k-1}(u - v))\|_{L^2_{T,x}} + \|\langle \nabla \rangle^{s+\sigma-\alpha} (v + \lambda(u - v)|^{k}(v + \lambda(u - v)|^{k-1}(u - v))\|_{L^2_{T,x}} + \|\langle \nabla \rangle^{s+\sigma-\alpha} (v + \lambda(u - v)|^{k}(v + \lambda(u - v)|^{k-1}(u - v))\|_{L^2_{T,x}} = A + B + C.
\]

The terms \(A\) and \(B\) are bounded in the same way, so we check only one. \(C\) is the easiest and follows from the Hölder inequality.

\[
C \leq \|\langle \nabla \rangle^{s+\sigma-\alpha} (u - v)\|_{L^2_{T}L^p_x} \|\|v + \lambda(u - v)|^{k}(v + \lambda(u - v)|^{k}\|_{L^2_{T}L^p_x} = \eta_1(u - v) \|v + \lambda(u - v)|^{p-1}\|_{L^2_{T}L^p_x} = \eta_1(u - v) \|v + \lambda(u - v)|^{p-1}\|_{L^2_{T}L^p_x} = \eta_1(u - v) \eta_3(v + \lambda(u - v)|^{p-1} \leq \eta_1(u - v) \eta_3(v)^{p-1} \eta_3(u)^{p-1} \leq 2R^{p-1} \eta_1(u - v).
\]
Now let’s deal with $A$, which also follows from the Hölder inequality.

\[
A = \left\| \langle \nabla \rangle^{s+\sigma-\alpha} (v + \lambda(u - v)) \left( v + \lambda(u - v) \right)^{k-1} [\bar{v} + \lambda(\bar{u} - \bar{v})] \right\|_{L^r_T}^k
\]

\[
\leq \left\| v - u \right\|_{L^r_T} \left\| \langle \nabla \rangle^{s+\sigma-\alpha} (v + \lambda(u - v)) \right\|_{L^r_T} \left\| (v + \lambda(u - v))^{k-1} [\bar{v} + \lambda(\bar{u} - \bar{v})] \right\|_{L^r_T},
\]

where $\frac{1}{r} = \frac{1}{2(p-1)} + \frac{1}{2}$. Then one uses the pointwise bound $\left\| (v + \lambda(u - v))^{k-1} \right\|_{L^r_T} \leq |u|^{k-1} + |v|^{k-1}$.

\[
A \leq \eta_3(u - v) \left\| \lambda \langle \nabla \rangle^{s+\sigma-\alpha} u + (1 - \lambda) \langle \nabla \rangle^{s+\sigma-\alpha} v \right\|_{L^r_T} \left\| v \right\|^{2k-1}_{L^r_T} + \left\| u \right\|^{2k-1}_{L^r_T},
\]

\[
\leq \eta_3(u - v) \left( \left\| \langle \nabla \rangle^{s+\sigma-\alpha} u \right\|_{L^r_T} + \left\| \langle \nabla \rangle^{s+\sigma-\alpha} v \right\|_{L^r_T} \right) \left( \left\| v \right\|^{2k-1}_{L^r_T} + \left\| u \right\|^{2k-1}_{L^r_T} \right)
\]

\[
\leq \eta_3(u - v) \left( \eta_1(u) + \eta_1(v) \right) (\eta_3(u)^{p-2} + \eta_3(v)^{p-2}) \leq 4R^{p-1} \eta_3(u - v),
\]

after checking that $(2k-1)r = 2(p-1)$.

Putting everything together, we obtain

\[
\Lambda_T(\Phi(v) - \Phi(u)) \leq T^s R^{p-1} \Lambda_T(v - u),
\]

so that by making $T$ smaller (if necessary), we can arrange $T^s R^{p-1} < 1$. Therefore $\Phi$ is a contraction on the ball $B_R = \{ v \mid \Lambda_T(v) < R \sim \|f\|_{H^s_T} \}$ with $s = \frac{1}{2} - \frac{1}{2(p-1)}$.

Finally, we give the proof that $\Phi(v)$ is continuous for completeness.

**Lemma 3.3.** $\Phi(v) \in C([0, T], H^s(\mathbb{R}))$ whenever $v \in X_T$.

**Proof.** Let $\text{lin}_v \Phi$ be the linear part of $\Phi(v)$. It is then easy to show that $\text{lin}_v \Phi \in C([0, T], H^s(\mathbb{R}))$. Firstly, we use the Plancherel theorem to write

\[
\left\| \text{lin}_v \Phi \right\|_{H^s_T} = \left\| \hat{f}^2 \langle \xi \rangle^{2s} |E_\beta(i^{-\beta} t \xi^\alpha)|^2 \right\|_{L^1_\xi}^{\frac{1}{2}}.
\]

Then we use the Dominated convergence theorem, together with the fact that

\[
|\hat{f}^2 \langle \xi \rangle^{2s} |E_\beta(i^{-\beta} t \xi^\alpha)|^2 \lesssim_M |\hat{f}^2 \langle \xi \rangle^{2s} |E_\beta(i^{-\beta} t \xi^\alpha)|
\]

uniformly in $t$, and also the fact that $E_\beta(i^{-\beta} t \xi^\alpha)$ is continuous in $t$.

Therefore, we only need to prove that $\text{non}_1 \Phi$, the nonlinear part of $\Phi(v)$, also lives in $C([0, T], H^s(\mathbb{R}))$. Suppose $0 \leq t_2 < t_1 \leq T$, and consider

\[
\left\| \text{non}_1 \Phi(v) - \text{non}_1 \Phi(u) \right\|_{H^s_T}^2 = \int_0^{t_2} \int_0^{t_1} \hat{g}(t_1 - \tau, \xi) \tau^{\beta-1} E_{\beta, \beta}(i^{-\beta} \tau^\beta |\xi^\alpha|) d\tau d\xi
\]

\[
- \int_0^{t_2} \int_0^{t_1} \hat{g}(t_2 - \tau, \xi) \tau^{\beta-1} E_{\beta, \beta}(i^{-\beta} \tau^\beta |\xi^\alpha|) d\tau d\xi
\]

\[
+ \int_0^{t_2} \int_0^{t_1} \hat{g}(t_1 - \tau, \xi) \tau^{\beta-1} E_{\beta, \beta}(i^{-\beta} \tau^\beta |\xi^\alpha|) d\tau d\xi
\]

\[
\lesssim I + II,
\]
where

\[ I = \int_{\mathbb{R}} \left| \int_{t_2}^{t_1} \hat{g}(t_1 - \tau, \xi) \tau^{\beta-1} E_{\beta, \beta}(i^{-\beta} \tau^{\beta} | \xi |^\alpha) \, d\tau \right|^2 \langle \xi \rangle^{2s} \, d\xi, \]

\[ II = \int_{\mathbb{R}} \left| \int_{0}^{t_2} [\hat{g}(t_1 - \tau, \xi) - \hat{g}(t_2 - \tau, \xi)] \tau^{\beta-1} E_{\beta, \beta}(i^{-\beta} \tau^{\beta} | \xi |^\alpha) \, d\tau \right|^2 \langle \xi \rangle^{2s} \, d\xi. \]

The first term is easier to deal with. By Lemma 3.1, the multiplier \( \tau^{\beta-1} E_{\beta, \beta}(i^{-\beta} \tau^{\beta} | \xi |^\alpha) \) can be controlled by those associated to \( |\nabla|^{\alpha-\sigma} e^{-i\tau |\nabla|^\sigma}, \hat{T}_\tau, \hat{S}_\tau \) and \( \hat{U}_\tau \), and so we should consider each case separately. We treat one as an example: \( \int_{\mathbb{R}} \left| \int_{t_2}^{t_1} \hat{g}(t_1 - \tau, \xi) |\xi|^{\sigma-\alpha} e^{-i\tau |\xi|^\sigma} \, d\tau \right|^2 \langle \xi \rangle^{2s} \, d\xi \)

\[ \leq \|g(t_1 - \cdot)\|_{L^2((t_2, t_1); H^{s+\sigma-\alpha}_x)}^2 |t_1 - t_2| \leq \langle \nabla \rangle^{s+\sigma-\alpha} g \|_{L^2_{t,x}}^2 |t_1 - t_2| \to 0 \]

as \( t_2 \to t_1 \), where we used the Cauchy-Schwartz inequality and the fact that \( \|\langle \nabla \rangle^{s+\sigma-\alpha} g\|_{L^2_{t,x}} \) is finite as long as \( v \in X_T \), which was shown in (3.11). The other cases are treated analogously.

Now we focus on \( II \). By Lemma 3.1, one can control this quantity by the multipliers associated to \( |\nabla|^{\alpha-\sigma} e^{-i\tau |\nabla|^\sigma}, \hat{T}_\tau, \hat{S}_\tau \) and \( \hat{U}_\tau \). Once again, we treat one case as an example:

\[ \int_{\mathbb{R}} \left| \int_{t_2}^{t_1} \hat{g}(t_1 - \tau, \xi) - \hat{g}(t_2 - \tau, \xi) \right|^{\alpha-\sigma} e^{-i\tau |\xi|^\sigma} \, d\tau \right|^2 \langle \xi \rangle^{2s} \, d\xi \]

\[ \leq t_2 \int_{\mathbb{R}} \left| \int_{0}^{t_2} \hat{g}(t_1 - \tau, \xi) - \hat{g}(t_2 - \tau, \xi) \, d\tau \right|^2 \langle \xi \rangle^{2(s+\sigma-\alpha)} \, d\xi \]

Now we take \( t_2 \to t_1 \) and use the Dominated convergence theorem, since

\[ \langle \xi \rangle^{2(s+\sigma-\alpha)} \int_{0}^{T} \chi_{\{0 \leq \tau \leq t_2\}} |\hat{g}(t_1 - t_2 + \tau, \xi) - \hat{g}(\tau, \xi)|^2 \, d\tau \leq 2 \langle \xi \rangle^{2(s+\sigma-\alpha)} \int_{0}^{T} |\hat{g}(\tau, \xi)|^2 \, d\tau \in L^1 \]

independently of \( t_2 \), as was shown in (3.11). Of course, we also need to show that

\[ \int_{0}^{T} \chi_{\{0 \leq \tau \leq t_2\}} |\hat{g}(t_1 - t_2 + \tau, \xi) - \hat{g}(\tau, \xi)|^2 \, d\tau \]

is continuous on \( t_2 \). But this follows from the translation continuity of \( L^p \) norms, together with the fact that \( \hat{g}(\cdot, \xi) \in L^2([0, T]) \) for a.e. \( \xi \).

\[ \square \]

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