DEFORMATIONS OF PRODUCT-QUOTIENT SURFACES
AND RECONSTRUCTION OF TODOROV SURFACES VIA
Q-GORENSTEIN SMOOTHING

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Abstract. We consider the deformation spaces of some singular product-
quotient surfaces \(X = (C_1 \times C_2)/G\), where the curves \(C_i\) have genus 3
and the group \(G\) is isomorphic to \(\mathbb{Z}_4\). As a by-product, we give a new
construction of Todorov surfaces with \(p_g = 1, q = 0\) and \(2 \leq K^2 \leq 8\) by
using Q-Gorenstein smoothings.

0. Introduction

In [To81], Todorov constructed some surfaces of general type with \(p_g = 1,
q = 0\) and \(2 \leq K^2 \leq 8\) in order to give counterexamples of the global Torelli
theorem. Todorov surfaces with \(K^2 = 8 - k\) are double covers of a Kummer
surface in \(\mathbb{P}^3\) branched over a curve \(D\), which is a complete intersection
of the Kummer surface with a smooth quadric surface containing \(k\) of its
nodes, and over the remaining \(16 - k\) nodes. Surfaces with \(K^2 = 2\), and
\(p_g = 1\) have been completely classified by Catanese and Debarre [CD89],
while some examples were constructed by Todorov. C. Rito [Rito09] gave a
detailed study of Todorov surfaces with an involution.

Recently, H. Park, J. Park and D. Shin constructed simply connected
surfaces of general type with \(p_g = 1, q = 0\) and \(2 \leq K^2 \leq 8\) by considering Q-
Gorenstein smoothings of singular K3 surfaces with special configurations of
cyclic quotient singularities, see [PPS1], [PPS2]. Their construction follows
the method used by Lee and Park in the paper [LP07], where a simply
connected surface of general type with \(p_g = q = 0\) and \(K^2 = 2\) is constructed
via the Q-Gorenstein smoothing of a singular rational surface. For more
details about these kind of techniques, over a field of any characteristic, we
refer the reader to the work of Lee and Nakayama [LN11].

Moreover, Bauer, Catanese, Grunewald and Pignatelli constructed many
interesting examples of surfaces of general type with \(p_g = 0\) by considering
the minimal desingularization of singular product-quotient surfaces,
see [BC04], [BCG08], [BCGP], [BP]. Similar methods are applied to sur-
faces of general type with \(p_g = q = 1\) by Polizzi and others, see [Pol08],
[Pol09], [CP09], [MP10]. These results motivated us to start the investiga-
tion of Q-Gorenstein smoothings of singular product-quotient surfaces.

Let us recall that a projective surface \(S\) is called a product-quotient surface
if there exists a finite group \(G\), acting faithfully on two smooth curves \(C_1\) and
\(C_2\) and diagonally on their product, so that \(S\) is isomorphic to the minimal
desingularization of $X = (C_1 \times C_2)/G$. The surface $X$ is called a singular model of a product-quotient surface, or simply a singular product-quotient surface.

This paper focuses on the case $g(C_1) = g(C_2) = 3$ and $G = \mathbb{Z}_4$. More precisely, we assume that there exist two simple $\mathbb{Z}_4$-covers $g_i : C_i \to \mathbb{P}^1$, both branched in four points. Then the singular product-quotient surface

$$X := (C_1 \times C_2)/\mathbb{Z}_4$$

contains precisely 16 cyclic quotient singularities; any of them is either of type $\frac{1}{4}(1, 1)$ or of type $\frac{1}{4}(1, 3)$. Note that $\frac{1}{4}(1, 1)$ is a rational double point, whereas $\frac{1}{4}(1, 3)$ is a singularity of class $T$, so both admit a local $\mathbb{Q}$-Gorenstein smoothing, see [KSB88] or [Man08, Sections 2-4]. The problem is to understand whether these local smoothings can be glued together in order to have a global $\mathbb{Q}$-Gorenstein smoothing of $X$. We will show that in some cases this is actually possible.

This paper is organized as follows.

In Section 1 we present some preliminaries and we set up notation and terminology. In particular, we recall the definitions of simple cyclic cover of a curve and of singular product-quotient surface and we explain how to compute their basic invariants.

In Section 2 we introduce the main objects that we want to study, namely the singular product quotient surfaces of the form $X = (C_1 \times C_2)/G$, where $g(C_1) = g(C_2) = 3$, $G = \mathbb{Z}_4$ and $C_i \to C_i/G$ is a simple cyclic cover for $i = 1, 2$.

Section 3 deals with the study of the singular product-quotient surface $Y = (C_1 \times C_2)/H$, where $H$ is the unique subgroup of $G$ isomorphic to $\mathbb{Z}_2$. By construction, $Y$ contains exactly 16 ordinary double points as singularities. By using the infinitesimal techniques introduced in [Pin81] and [Cat89], we prove that $\text{Def}(Y)$ is smooth at $Y$, of dimension 18 and $\text{ESDef}(Y)$ is smooth at $[Y]$, of dimension 8 (Proposition 3.6). Moreover, if $\mu : V \to Y$ is the minimal desingularization of $Y$, we have

$$\dim[V] \text{Def}(V) = 18, \quad h^1(\Theta_Y) = 24,$$

hence $\text{Def}(V)$ is singular at $[V]$; by [BW74] this implies that the sixteen $(-2)$ curves of $V$ do not have independent behavior in deformations.

In Section 4 we discuss three examples of singular product-quotient surface $X = (C_1 \times C_2)/G$ with different $G$-action.

• In the first example we have $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 3)$, so $X$ contains only rational double points as singularities. We prove that $\text{Def}(X)$ and $\text{ESDef}(X)$ are both smooth at $[X]$, of dimension 44 and 2, respectively (Propositions 4.4 and 4.2).

The surface $X$ satisfies $h^0(\omega_X) = 5$ and $K_X^2 = 8$; moreover it is no difficult to see that the canonical map $\phi_K : X \to \mathbb{P}^4$ is a birational morphism onto its image; by [Cat97, Proposition 6.2] it follows that the general deformation of $X$ is isomorphic to a smooth complete intersection of bidegree $(2, 4)$ in $\mathbb{P}^4$.

Moreover we have

$$\dim[S] \text{Def}(S) = 44, \quad h^1(\Theta_S) = 50,$$
hence $\text{Def}(S)$ is singular at $S$. This means that the sixteen $A_3$-cycles of $S$ do not have independent behavior in deformations.

- In the second example we have $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 1)$. We show that there exist a $\mathbb{Q}$-Gorenstein smoothing $\pi : X' \to T$ of $X$, whose base $T$ has dimension 12, such that the general fibre $X_t$ of $\pi$ is a minimal surface of general type whose invariants are

$$p_g(X_t) = 1, \quad q(X_t) = 0, \quad K_{X_t}^2 = 8.$$ 

Moreover $X_t$ is isomorphic to a Todorov surface with $K^2 = 8$ (Theorem 4.6). By a slight modification of the construction, it is possible to obtain all Todorov surfaces with $2 \leq K^2 \leq 8$.

This is related to the existence of complex structures on rational blow-downs of algebraic surfaces. More precisely, one can consider the rational blow-down $S(t)$ of $t$ of the $(-4)$-curves in $S$, where $1 \leq t \leq 16$. This means that one considers the normal connected sum of $S$ with $t$ copies of $\mathbb{P}^2$, identifying a conic in each $\mathbb{P}^2$ with a $(-4)$-curve in $S$; then $S(t)$ is a symplectic 4-manifold. On can therefore raise the following:

**Question.** Is it possible to give a complex structure on $S(t)$ for $1 \leq t \leq 16$, and to describe $S(t)$ when such a complex structure exists?

Our results answer affirmatively this question when $10 \leq t \leq 16$; in these cases, indeed, one can give a complex structure to the rational blow-down $S(t)$, which make it isomorphic to a Todorov surface with $K^2 = t - 8$.

- In the third example, we have $\text{Sing}(X) = 8 \times \frac{1}{4}(1, 1) + 8 \times \frac{1}{4}(1, 3)$. Rasdeaconu and Suvaina give an explicit construction of the minimal desingularization $S$ of $X$, see [RS06, Section 3]; in fact, they prove that $S$ is a simply connected, minimal elliptic surface with no multiple fibres.

We show that there exists a $\mathbb{Q}$-Gorenstein smoothing of $X$, although $H^2(\Theta_X) \neq 0$ and all the natural deformations of the $G$-cover $u : X \to Q$ preserve the 8 singularities of type $\frac{1}{4}(1, 1)$, see Proposition 4.8. Indeed we prove that a general surface $\bar{X}$ in the subfamily of natural deformations of the $G$-cover of $X$ can be deformed to a bidouble cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over three smooth divisors of bidegree $(2, 2)$. By taking a general deformation of these three divisors we obtain a $\mathbb{Q}$-Gorenstein smoothing of $X$ which smooths all the singularities. More generally, by using the same method one can construct surfaces of general type with $p_g = 3$, $q = 0$ and $K^2 = k$ ($2 \leq k \leq 8$) by first taking a $\mathbb{Q}$-Gorenstein smoothing of $k$ singular points of type $\frac{1}{4}(1, 1)$ of $\bar{X}$ and then the minimal resolution of the remaining $8 - k$ singular points of the same type.

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Notation and conventions.
We work over the field $\mathbb{C}$ of complex numbers.

By “surface” we mean a projective, non-singular surface $S$, and for such a surface $\omega_S = \mathcal{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S, \omega_S)$ is the geometric genus, $q(S) = h^1(S, \omega_S)$ is the irregularity and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the Euler-Poincaré characteristic.

If $X$ is any (possibly singular) projective scheme, we denote by $\mathrm{Def}(X)$ the base of the Kuranishi family of deformations of $X$ and by $\mathrm{ESDef}(X)$ the base of the equisingular deformations of $X$. The tangent spaces to $\mathrm{Def}(X)$ and $\mathrm{ESDef}(X)$ at the point $[X]$ corresponding to $X$ are given by $\text{Ext}^1(\Omega^1_X, \mathcal{O}_X)$ and $H^1(\Omega^1_Y)$, respectively.

If $L$ is a line bundle $L$ on $X$, we use the notation $L^n$ instead of $L^\otimes n$ if no confusion can arise.

If $G$ is any finite abelian group, we denote by $\hat{G}$ its dual group, namely the group of irreducible characters of $G$.

1. Preliminaries

1.1. Simple cyclic covers of curves. Let $\Gamma$ be a smooth, projective curve and $B \subset \Gamma$ an effective divisor such that $\mathcal{O}_\Gamma(B) = \mathcal{L}^n$ for some $\mathcal{L} \in \text{Pic}(\Gamma)$. Therefore there exists a $\mathbb{Z}_n$-cover $g : C \to \Gamma$, totally branched over $B$, which is called a simple cyclic cover. We identify $\mathbb{Z}_n$ with the group of $n$-th roots of unity, namely $\mathbb{Z}_n = \langle \zeta \rangle$, where $\zeta$ is a primitive $n$-th root. The dual group $\hat{\mathbb{Z}}_n$ is isomorphic to $\mathbb{Z}_n$, and it is generated by the character $\chi_1 : \mathbb{Z}_n \to \mathbb{C}$ such that $\chi_1(\zeta) = \zeta^{-1}$. We will write $\chi_j$ instead of $\chi_1^j$; then $\chi_j(\zeta) = \zeta^{-j}$.

The group $\mathbb{Z}_n$ acts naturally on $g_*\mathcal{O}_C$, so there is a canonical splitting

\begin{equation}
 g_*\mathcal{O}_C = \mathcal{O}_\Gamma \oplus \mathcal{L}^{-1} \oplus \ldots \oplus \mathcal{L}^{-(n-1)},
\end{equation}

where the summand $\mathcal{L}^{-j}$ is the eigensheaf $(g_*\mathcal{O}_C)_{\chi_j}$ corresponding to the character $\chi_j$.

Similarly, $\mathbb{Z}_n$ acts naturally on $g_*\omega_C$ and $g_*\omega_C^2$, giving the following decompositions (see [Pa91] and [Cat89 Section 2]):

\begin{align}
 g_*\omega_C &= \omega_\Gamma \oplus (\omega_\Gamma \otimes \mathcal{L}) \oplus \ldots \oplus (\omega_\Gamma \otimes \mathcal{L}^{n-1}), \\
 g_*\omega_C^2 &= (\omega_\Gamma^2(B) \otimes \mathcal{L}^{-1}) \oplus \omega_\Gamma^2(B) \oplus \ldots \oplus (\omega_\Gamma^2(B) \otimes \mathcal{L}^{n-2}).
\end{align}

In the equations (2), the eigensheaves corresponding to $\chi_j$ are $\omega_\Gamma \otimes \mathcal{L}^j$ and $\omega_\Gamma^2(B) \otimes \mathcal{L}^j$, respectively.
1.2. Cyclic quotient singularities, Hirzebruch Jung resolutions and singular product-quotient surfaces. Let $n$ and $q$ be natural numbers with $0 < q < n$, $(n, q) = 1$ and let $\zeta$ be a primitive $n$-th root of unity. Let us consider the action of the cyclic group $\mathbb{Z}_n = \langle \zeta \rangle$ on $\mathbb{C}^2$ defined by $\zeta \cdot (x, y) = (\zeta^q x, \zeta y)$. Then the analytic space $X_{n,q} = \mathbb{C}^2/\mathbb{Z}_n$ has a cyclic quotient singularity of type $\frac{n}{n}(1,q)$, and $X_{n,q} \cong X_{n',q'}$ if and only if $n = n'$ and either $q = q'$ or $qq' \equiv 1 \pmod{n}$. The exceptional divisor on the minimal resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is a Hirzebruch-Jung string, that is to say, a connected union $E = \bigcup_{i=1}^k Z_i$ of smooth rational curves $Z_1, \ldots, Z_k$ with self-intersection $\leq -2$, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all $i$, and $Z_i Z_j = 0$ if $|i - j| \geq 2$. More precisely, given the continued fraction $\frac{n}{q} = [b_1, \ldots, b_k] = b_1 - \frac{1}{b_2 - \frac{1}{\vdots - \frac{1}{b_k}}}$, the dual graph of $E$ is 

\[ \begin{array}{ccccccc}
- b_1 & - b_2 & - b_{a-1} & - b_k \\
\end{array} \]

(cf. [Lau71, Chapter II]). Notice that a rational double point of type $A_n$ corresponds to the cyclic quotient singularity $\frac{1}{n+1}(1,n)$.

**Definition 1.1.** Let $x$ be a cyclic quotient singularity of type $\frac{1}{n}(1,q)$. Then we set

$$ h_x = 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^k (b_i - 2), $$

$$ \varepsilon_x = k + 1 - \frac{1}{n}, $$

$$ B_x = 2\varepsilon_x - h_x = \frac{1}{n} (q + q') + \sum_{i=1}^k b_i, $$

where $1 \leq q' \leq n - 1$ is such that $qq' \equiv 1 \pmod{n}$.

**Definition 1.2.** [BP] We say that a projective surface $S$ is a product-quotient surface if there exists a finite group $G$ acting faithfully on two smooth projective curves $C_1$ and $C_2$ and diagonally on their product, so that $S$ is isomorphic to the minimal desingularization of $X := (C_1 \times C_2)/G$. The surface $X$ is called a singular model of a product-quotient surface, or simply a singular product-quotient surface.

From this definition it follows that a singular product quotient surface contains a finite number of cyclic quotient singularities.

**Proposition 1.3** (cf. [MP10], Section 3). Let $S$ be a product quotient surface, minimal desingularization of $X := (C_1 \times C_2)/G$. Then the invariants of $S$ are

1. $K_S^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G|} + \sum_{x \in \text{Sing } X} h_x$. 


\[ e(S) = \frac{4(g(C_1)-1)(g(C_2)-1)}{[\mathcal{E}]} + \sum_{x \in \text{Sing } X} e_x. \]

\[ q(S) = g(C_1/G) + g(C_2/G). \]

Set \( \Gamma_i := C_i/G \) and let \( g_i : C_i \to \Gamma_i \). The group \( G \) acts naturally on the sheaves \( g_i^* \mathcal{O}_{C_i}, g_i^* \omega_{C_i}, g_i^* \omega_{C_i}^2 \). Assuming that \( G \) is abelian, we can write the following generalizations of (1) and (2):

\[ g_i^* \mathcal{O}_{C_i} = \bigoplus_{\chi \in \hat{G}} (g_i^* \mathcal{O}_{C_i})^\chi, \]

\[ g_i^* \omega_{C_i} = \bigoplus_{\chi \in \hat{G}} (g_i^* \omega_{C_i})^\chi, \]

\[ g_i^* \omega_{C_i}^2 = \bigoplus_{\chi \in \hat{G}} (g_i^* \omega_{C_i}^2)^\chi, \]

where \((*)^\chi\) is the eigensheaf corresponding to the character \( \chi \in \hat{G} \).

2. The main construction

Let us consider two smooth curves \( C_1, C_2 \) of genus 3, such that there are two simple \( \mathbb{Z}_4 \)-covers \( g_i : C_i \to \mathbb{P}^1 \), both branched in 4 points. In the rest of the paper we write \( G := \mathbb{Z}_4 = \langle \zeta | \zeta^4 = 1 \rangle \), where \( \zeta \) is a primitive fourth root of unity; we also denote by \( H \) the subgroup of \( G \) defined by \( H := \langle \zeta^2 \rangle \cong \mathbb{Z}_2 \).

Now set \( Z := C_1 \times C_2 \) and consider the singular product-quotient surface

\[ X := Z/G, \]

which has exactly 16 isolated singular points, corresponding to the fixed points of the \( G \)-action on \( Z \). Let \( \lambda : S \to X \) be the minimal resolution of singularities of \( X \).

The \( G \)-cover \( g_i \) factors through the double cover \( h_i : C_i \to E_i \), where \( E_i := C_i/H \). Note that \( E_i \) is an elliptic curve and that the singular product-quotient surface

\[ Y := Z/H \]

contains sixteen cyclic quotient singularities of type \( \frac{1}{2}(1, 1) \), i.e. ordinary double points, as only singularities. Let us denote by \( \mu : V \to Y \) the minimal desingularization of \( Y \). We have a commutative diagram

\[ V \xrightarrow{\mu} Y \xrightarrow{\nu} E_1 \times E_2, \]

\[ S \xrightarrow{\lambda} X \xrightarrow{u} \mathbb{P}^1 \times \mathbb{P}^1 \]

where:

- \( p : Z \to X \) and \( r : Z \to Y \) are the natural projections, so \( s : Y \to X \) is a double cover (more precisely, a \( G/H \)-cover) branched over the singular points of \( X \);
• \(g := g_1 \times g_2: Z \to \mathbb{P}^1 \times \mathbb{P}^1\) is a \(G \times G\)-cover branched on a divisor \(B \subset \mathbb{P}^1 \times \mathbb{P}^1\) of product type and of bidegree \((4, 4)\);
• \(h := h_1 \times h_2: Z \to E_1 \times E_2\) is a \(H \times H\)-cover branched on a divisor \(\Delta \subset E_1 \times E_2\) of product type and of bidegree \((4, 4)\);
• \(u: X \to \mathbb{P}^1 \times \mathbb{P}^1\) is a \(G\)-cover, whose branch locus coincides with \(B\);
• \(v: Y \to E_1 \times E_2\) is a \(H\)-cover, whose branch locus coincides with \(\Delta\);
• \(t: E_1 \times E_2 \to \mathbb{P}^1 \times \mathbb{P}^1\) is a \(G/H \times G/H\)-cover whose branch locus is \(B\) and whose ramification locus is \(\Delta\).

Let us denote by \(B_i\) the branch locus of \(g_i: C_i \to \mathbb{P}^1\) and by \(\Delta_i\) the branch locus of \(h_i: C_i \to E_i\). Both \(B_i\) and \(\Delta_i\) consist of four points; clearly \(B = B_1 \times B_2\) and \(\Delta = \Delta_1 \times \Delta_2\). From the results of Section 2, we infer that

• there is a natural action of \(G\) on the sheaves \(g_{i*}\mathcal{O}_{C_i}\), \(g_{i*}\omega_{C_i}\), \(g_{i*}\omega_{C_i}^2\),
which gives decompositions:
\[
g_{i*}\mathcal{O}_{C_i} = \mathcal{O}_{E_i} \oplus \mathcal{M}_i^{-1} \oplus \mathcal{M}_i^{-2} \oplus \mathcal{M}_i^{-3};
\]
\[
g_{i*}\omega_{C_i} = \omega_{E_i} \oplus (\omega_{E_1} \otimes \mathcal{M}_i) \oplus (\omega_{E_2} \otimes \mathcal{M}_i) \oplus (\omega_{E_1} \otimes \mathcal{M}_i);
\]
\[
g_{i*}\omega_{C_i}^2 = \omega_{E_1}^2 \otimes \mathcal{L}_i \oplus (\omega_{E_2}^2 \otimes \mathcal{M}_i) \oplus (\omega_{E_1}^2 \otimes \mathcal{M}_i) \oplus (\omega_{E_2}^2 \otimes \mathcal{M}_i)
\]
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and anti-invariant sections for the $H$-action and by $h^0(*)^+$ and $h^0(*)^-$ their dimensions. Since $Y$ has only rational double points, Künneth formula and the third equality in (11) give

$H^0(\omega_Y^2) = H^0(\omega_Y^2) = H^0(\omega_Y^2)^+ = H^0(\omega_Y^2 \boxtimes \omega_Y^2)^+$

$= (H^0(h_1^*\omega_C^2)^+ \otimes H^0(h_2^*\omega_C^2)^+) \oplus (H^0(h_1^*\omega_C^2)^- \otimes H^0(h_2^*\omega_C^2)^-)$

$\cong \mathbb{C}^{20}.$

This shows that $h^0(\omega_Y^2) = K_Y^2 + \chi(\mathcal{O}_Y)$, hence $V$ is a minimal model.

Since $Y$ is a normal surface [BW74, Proposition 1.2] gives $\mu_*\Theta_Y = \Theta_Y$. Therefore the argument in [BW74, Section 1] or [Cat89, p. 299] shows that there are two isomorphisms

(8) $H^1(\Theta_Y) \cong H^1(\Theta_Y) \oplus H^2_E(\Theta_Y), \quad H^2(\Theta_Y) \cong H^2(\Theta_Y),$ where $H^2_E(\Theta_Y)$ denotes the local cohomology with support on the exceptional divisor $E \subset V$.

By the second isomorphism in (8), we have

(9) $H^2(\Theta_Y)^* \cong H^2(\Theta_Y)^* = H^0(\Omega^2_2 \otimes \Omega^2_2)^+ = T_1 \oplus T_2 \oplus T_3 \oplus T_4,$

where

$T_1 = H^0(h_1^*\omega_C^2)^+ \otimes H^0(h_2^*\omega_C^2)^+ = H^0(\omega^2_E(\Delta_1) \otimes H^0(\omega_E^2),$

$T_2 = H^0(h_1^*\omega_C^2)^+ \otimes H^0(h_2^*\omega_C^2)^+ = H^0(\omega^2_E(\Delta_2) \otimes H^0(\omega_E^2),$

$T_3 = H^0(h_1^*\omega_C^2)^- \otimes H^0(h_2^*\omega_C^2)^- = H^0(\omega^2_E(\Delta_1) \otimes \mathcal{L}_1^{-1}) \otimes H^0(\omega_E^2 \otimes \mathcal{L}_2),$

$T_4 = H^0(h_1^*\omega_C^2)^- \otimes H^0(h_2^*\omega_C^2)^- = H^0(\omega^2_E(\Delta_2) \otimes \mathcal{L}_2^{-1}).$

Since $\dim T_i = 4$ for all $i \in \{1, 2, 3, 4\}$, we infer $h^2(\Theta_Y) = h^2(\Theta_Y) = 16$. By Riemann-Roch we have $h^1(\Theta_Y) = h^2(\Theta_Y) = 10\chi(\mathcal{O}_Y) - 2K_Y^2 = 8$, so it follows $h^1(\Theta_Y) = 24$.

**Corollary 3.2.** We have

$h^1(\Theta_Y) = 8, \quad h^2(\Theta_Y) = 16.$

**Proof.** Since $h^2(\Theta_Y) = h^2(\Theta_Y)$, the first equality follows from Proposition [34]. Furthermore, $E$ is the disjoint union of sixteen $(-2)$-curves, hence [BW74, Section 1] implies $H^2_E(\Theta_Y) \cong \mathbb{C}^{16}$. Using $h^1(\Theta_Y) = 24$ and the first isomorphism in (8) we obtain $h^1(\Theta_Y) = 8$, which completes the proof.

By using the local-to-global spectral sequence of $\mathcal{E}xt$-sheaves we obtain an exact sequence

(11) $0 \to H^1(\Theta_Y) \to \mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y) \to T^*_Y \overset{\Theta_Y}{\to} H^2(\Theta_Y),$ where $T^*_Y := H^0(\mathcal{E}xt^1(\Omega^1_Y, \mathcal{O}_Y)).$ Notice that $T^*_Y$ is a skyscraper sheaf supported on the sixteen nodes of $Y$, hence $\Theta_Y$ is a linear map

$\Theta_Y : \mathbb{C}^{16} \to \mathbb{C}^{16}.$

Thus its kernel and its cokernel have the same dimension.
Remark 3.3. The branch locus $\Delta$ of $v: Y \to E_1 \times E_2$ is a polarization of type $(4, 4)$ on the abelian surface $E_1 \times E_2$, in particular $h^0(\Delta) = 16$. Since polarized abelian surfaces form a 3-dimensional family, it follows that the deformation space $\text{Def}(Y)$ has dimension at least 18. Therefore we have
\[
\dim \text{Ext}^1(\Omega^1_Y, \mathcal{O}_Y) = \dim T_{[Y]} \text{Def}(Y) \geq \dim_{[Y]} \text{Def}(Y) \geq 18.
\]

**Proposition 3.4.** We have
\[
\dim \ker \text{ob}_Y = \dim \text{coker} \text{ob}_Y = 10.
\]

**Proof.** Notice that Remark 3.3 only gives $\dim(\ker \text{ob}_Y) \geq 10$. In order to prove equality, we apply an argument used in [Cat89, Section 2].

Let us consider the dual map $\text{ob}_Y^*: H^2(\Omega_Y^2)^* \to (T^1_Y)^*$. We set
\[
\Delta_1 = d'_1 + d'_2 + d'_3 + d'_4
\]
\[
\Delta_2 = d''_1 + d''_2 + d''_3 + d''_4
\]
and we choose local coordinates $(x, y)$ in $Z$ vanishing at $(d'_i, d''_i)$. Then the action of $H$ with respect to these coordinates is given by $(x, y) \to (-x, -y)$.

By [Cat89] we have an isomorphism $(T^1_Y)^* = (r_* \Omega^1_Z)^*/\Omega^1_Y$, therefore $\text{ob}_Y^*$ can be seen as a map
\[
\text{ob}_Y^*: H^0(\Omega^1_Z \otimes \Omega^2_Z)^+ \to (r_* \Omega^1_Z)^*/\Omega^1_Y.
\]

Near any of the ordinary double points of $Y$, the sheaf $(r_* \Omega^1_Z)^+$ is locally generated by $xdy, xdy, ydx, ydy$, whereas $\Omega^1_Y$ is locally generated by $d(x^2)$, $d(xy), d(y^2)$; then $(r_* \Omega^1_Z)^*/\Omega^1_Y$ is locally generated by $xdy - ydx$, cf. [Cat89, Lemma 2.11].

Looking at [10] and making straightforward computations, one checks that
\[
\begin{align*}
\bullet & \text{ the summand } T_1 \text{ contributes expressions of type } \alpha_1 \beta_1 xdy \otimes (dx \wedge dy); \\
\bullet & \text{ the summand } T_2 \text{ contributes expressions of type } \alpha_2 \beta_2 xdy \otimes (dx \wedge dy); \\
\bullet & \text{ the summand } T_3 \text{ contributes expressions of type } \alpha_3 \beta_3 xdy \otimes (dx \wedge dy); \\
\bullet & \text{ the summand } T_4 \text{ contributes expressions of type } \alpha_4 \beta_4 xdy \otimes (dx \wedge dy),
\end{align*}
\]
where $\alpha_i = \alpha_i(x^2)$ and $\beta_i = \beta_i(y^2)$ are pullbacks of local functions on $E_i$.

Since in the $\mathcal{O}_Y$-module $(r_* \Omega^1_Z)^*/\Omega^1_Y$ we have the relations
\[
1/2(xdy - ydx) = xdy = -ydx \text{ and } xdx = ydy = 0,
\]
it follows that the restriction of $\text{ob}_Y^*$ to the subspace $T_3 \oplus T_4$ is zero, whereas the restriction of $\text{ob}_Y^*$ to the subspace $T_1 \oplus T_2$ can be identified, up to a multiplicative constant, with the map
\[
\phi: H^0(\omega^2_{E_1}(\Delta_1)) \oplus H^0(\omega^2_{E_2}(\Delta_2)) \to \bigoplus_{i,j=1}^{4} \mathbb{C}_{ij},
\]
\[
\phi(\sigma \oplus \tau) = \bigoplus_{i,j=1}^{4} (\text{val}_{d'_i}(\sigma) - \text{val}_{d''_j}(\tau)).
\]
Here the valuation maps \( \text{val}_{d_i} \) and \( \text{val}_{d_j} \) are defined, as usual, by the short exact sequences

\[
0 \to H^0(\omega_{E_i}^2) \to H^0(\omega_{E_1}^2(\Delta_1)) \overset{\text{val}_{d_i}}{\longrightarrow} H^0(N_{\Delta_1}) \cong \bigoplus_{i=1}^4 C_i,
\]

\[
0 \to H^0(\omega_{E_2}^2) \to H^0(\omega_{E_2}^2(\Delta_2)) \overset{\text{val}_{d_j}}{\longrightarrow} H^0(N_{\Delta_2}) \cong \bigoplus_{j=1}^4 C_j.
\]

Therefore we obtain

\[
\ker \phi = \{ \sigma \oplus \tau | \text{val}_{d_i}(\sigma) = \text{val}_{d_j}(\sigma) = \text{val}_{d_i}(\sigma) = \text{val}_{d_j}(\tau) \}.
\]

As \( E_i \) is an elliptic curve, we have \( \omega_{E_i}^2 = \omega_{E_i} \) and so (12) are the standard residue sequences for meromorphic 1-forms. By the Residue Theorem we get

\[
\sum_{i=1}^4 \text{val}_{d_i}(\sigma) = \sum_{j=1}^4 \text{val}_{d_j}(\tau) = 0,
\]

hence (13) implies that \( \sigma \oplus \tau \in \ker \phi \) if and only if \( \text{val}_{d_i}(\sigma) = \text{val}_{d_j}(\tau) = 0 \) for all pairs \((i, j)\). This yields \( \ker \phi = H^0(\omega_{E_i}^2) \oplus H^0(\omega_{E_2}^2) \cong C \oplus C \).

Then \( \ker \text{ob}^* = \ker \phi \oplus T_3 \oplus T_4 \cong \mathbb{C}^{10} \), hence \( \dim \text{coker ob}_Y = 10 \) and we are done. \qed

**Corollary 3.5.** We have

\[
\dim \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y) = 18.
\]

**Proof.** Immediate from Corollary 3.2, Proposition 3.4 and exact sequence (11). \qed

**Proposition 3.6.** The following holds:

1. \( \text{Def}(Y) \) is smooth at \( [Y] \), of dimension 18;
2. \( \text{ESDef}(Y) \) is smooth at \( [Y] \), of dimension 8.

**Proof.** By Remark 3.3 and Corollary 3.5 we have

\[
18 = \dim \text{Ext}^1(\Omega_Y^1, \mathcal{O}_Y) = \dim T_{[Y]} \text{Def}(Y) \geq \dim_{[Y]} \text{Def}(Y) \geq 18,
\]

which proves (i).

On the other hand, if we move the branch loci \( B_i \subset E_i \) the curve \( \Delta \subset E_1 \times E_2 \) remains of product type, so in this way we obtain a 8-dimensional family of *equisingular* deformations of \( Y \); therefore the equisingular deformation space \( \text{ESDef}(Y) \) has dimension at least 8, and by Corollary 3.2 we have

\[
8 = \dim H^1(\Theta_Y) = \dim T_{[Y]} \text{ESDef}(Y) \geq \dim_{[Y]} \text{ESDef}(Y) \geq 8.
\]

This proves (ii). \qed

Summing up, Proposition 3.6 shows that the deformations of \( Y \) are unobstructed and that they are all obtained by deforming the pair \((A, \Delta)\), where \( A \) is an abelian surface and \( \Delta \) a polarization of type \((4, 4)\). In particular, all the deformations preserve the action of \( H \). Moreover, the equisingular deformations of \( Y \) are also unobstructed and are obtained by taking as \( A \) the product of two elliptic curves and by choosing the polarization \( \Delta \) of product type.
Remark 3.7. Since $Y$ has only rational double points, by [BW74] the dimension of $\text{Def}(Y)$ equals the dimension of $\text{Def}(V)$. Then

$$24 = h^1(\Theta_V) = \dim T_{[V]}\text{Def}(V) > \dim_{[V]}\text{Def}(V) = 18,$$

that is $\text{Def}(V)$ is singular at $[V]$. By [BW74, Theorem 3.7], this means that the sixteen $(-2)$-curves of $V$ do not have independent behavior in deformations.

4. Deformations of the singular product-quotient surface $X = Z/G$

Let us consider now the surface $X = Z/G$ defined in Section 2 and its minimal resolution of singularities $\lambda: S \to X$. We must analyze several cases, according to the type of quotient singularities that $X$ contains.

Throughout this section we set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and we denote by $\mathcal{O}_Q(a, b)$ the line bundle of bidegree $(a, b)$ on $Q$.

The following exact sequence is the analogue of (11):

$$0 \to H^1(\Theta_X) \to \text{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to T^1_X \text{ob}_X \to H^2(\Theta_X).$$

4.1. Example where $\text{Sing}(X) = 16 \times \frac{1}{4}(1, 3)$. Assume that, locally around each of the fixed points, the action of $G = \langle \zeta \mid \zeta^4 = 1 \rangle$ is given by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$. Therefore,

$$\text{Sing}(X) = 16 \times \frac{1}{4}(1, 3).$$

In this case $X$ contains only rational double points and we obtain

$$p_g(S) = 5, \quad q(S) = 0, \quad K^2_S = 8.$$

Proposition 4.1. $S$ is a minimal surface of general type.

Proof. $S$ is of general type because $p_g(S) > 0$ and $K^2_S > 0$. Since the action of $G$ is twisted on the second factor and $X$ has only rational double points, the K"unneth formula and the third equality in (6) give

$$H^0(\omega^2_S) = H^0(\omega^2_X) = H^0(\omega^2_Z)^G = H^0(\omega^2_{C_1} \boxtimes \omega^2_{C_2})^G = \bigoplus_{\chi \in \hat{G}} (H^0(g_{1, \chi} \omega^2_{C_1}) \otimes H^0(g_{2, \chi} \omega^2_{C_2})^\chi) = \mathbb{C}^{14}.$$

This shows that $h^0(\omega^2_S) = K^2_S + \chi(\mathcal{O}_S)$, hence $S$ is a minimal surface. \qed

Proposition 4.2. The following holds:

(i) $\text{ob}_X$ is surjective;

(ii) $h^1(\Theta_X) = 2, \quad h^2(\Theta_X) = 6, \quad h^1(\Theta_S) = 50, \quad h^2(\Theta_S) = 6$.

(iii) $\text{ESDef}(X)$ is smooth at $[X]$, of dimension 2.
Proof. (i) Let us consider the dual map \( \text{ob}_X^* : H^2(\Theta_X)^* \to (T_X^1)^* \). By Grothendieck duality (see [AK70, Chapter I]) and Künneth formula we obtain

\[
H^2(\Theta_X)^* = H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G
= \bigoplus_{\chi \in G} \left[ H^0(g_1, \omega_{C_1})^\chi \otimes H^0(g_2, \omega_{C_2}^2)^\chi \right]
\oplus \left( H^0(g_1, \omega_{C_1}^2)^\chi \otimes H^0(g_2, \omega_{C_2})^\chi \right)
= U_1 \oplus U_2,
\]
where

\[
U_1 = H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{M}_i^2) \otimes H^0(\omega_{\mathbb{P}^1}^2(B_2) \otimes \mathcal{M}_j^2),
U_2 = H^0(\omega_{\mathbb{P}^1}^2(B_1) \otimes \mathcal{M}_i^2) \otimes H^0(\omega_{\mathbb{P}^1} \otimes \mathcal{M}_j^2).
\]

This yields \( h^2(\Theta_X) = 6 \) and so \( h^2(\Theta_S) = 6 \). Now we set

\[
B_1 = b'_1 + b'_2 + b'_3 + b'_4,
B_2 = b''_1 + b''_2 + b''_3 + b''_4
\]
and we choose local coordinates \((x, y)\) in \( Z \) vanishing at \((b'_i, b''_j)\). As in Section 3 we can interpret \( \text{ob}_X^* \) as a map

\[
\text{ob}_X^* : H^0(\Omega_Z^1 \otimes \Omega_Z^2)^G \to (p_3^{\Omega_1})^G / \Omega_X^1,
\]
where \((p_3^{\Omega_1})^G / \Omega_X^1 \) is a skyscraper sheaf supported on the singular points of \( X \) and locally generated by \( x^i y^{i+1} dx - y^i x^{i+1} dy \), for \( i = 0, 1, 2 \), see [Cat89].

A straightforward local computation shows that the summand \( U_1 \) in (15) contributes expressions of the form \( \alpha_1 \beta_1 x dy \otimes (dx \wedge dy) \) whereas the summand \( U_2 \) contributes expressions of the form \( \alpha_2 \beta_2 y dx \otimes (dx \wedge dy) \), where \( \alpha_i = \alpha_i(x^2) \) and \( \beta_i = \beta_i(y^2) \) are pullbacks of local functions on \( \mathbb{P}^1 \). Therefore the map \( \text{ob}_X^* \) can be identified, up to a multiplicative constant, with

\[
\phi : H^0(\omega_{\mathbb{P}^1}^2(B_1) \otimes \mathcal{M}_i^2) \oplus H^0(\omega_{\mathbb{P}^1}^2(B_2) \otimes \mathcal{M}_j^2)
\to \bigoplus_{i,j=1}^4 \mathbb{C}_{ij} \subset \bigoplus_{i,j=1}^4 \mathbb{C}^{\mathbb{P}^3}_{ij} \cong (T_X^1)^*
\]

\[
\phi(\sigma \oplus \tau) = \bigoplus_{i,j=1}^4 (\text{val}_{b'_i}(\sigma) - \text{val}_{b''_j}(\tau)),
\]

where the valuation maps are defined as in Section 3. Hence we obtain

\[
\ker \phi = \{ \sigma \oplus \tau \mid \text{val}_{b'_i}(\sigma) = \text{val}_{b''_j}(\sigma) = \text{val}_{b'_i}(\sigma) = \text{val}_{b''_j}(\tau) \}.
\]

On the other hand, the valuation map \( H^0(\omega_{\mathbb{P}^1}^2(B_i) \otimes \mathcal{M}_i^2) \to H^0(N_{B_i}) \) can be identified with the residue map \( H^0(\omega_{\mathbb{P}^1}(B_i)) \to H^0(N_{B_i}) \) via the isomorphism \( H^0(\omega_{\mathbb{P}^1}(B_i) \otimes \mathcal{M}_i^2) \cong H^0(\omega_{\mathbb{P}^1}(B_i)) \). By the Residue Theorem we have

\[
\sum_{i=1}^4 \text{val}_{b'_i}(\sigma) = \sum_{j=1}^4 \text{val}_{b''_j}(\tau) = 0,
\]
so (16) implies that \( \sigma \oplus \tau \in \ker \phi \) if and only if \( \text{val}_{b'_i}(\sigma) = \text{val}_{b''_j}(\tau) = 0 \) for all pairs \((i, j)\). But there are no non-zero holomorphic 1-forms on \( \mathbb{P}^1 \),
so ker $\phi = 0$ and $\text{ob}^*_X$ is injective. Therefore the obstruction map $\text{ob}^*_X$ is surjective.

(ii) Let us denote by $F \subset S$ the exceptional divisor of $\lambda: S \to X$. Since $S$ has only rational double points, we have

$$H^1(\Theta_S) \cong H^1(\Theta_X) \oplus H^1_F(\Theta_S), \quad H^2(\Theta_S) \cong H^2(\Theta_X).$$

By Riemann-Roch theorem we obtain

$$h^1(\Theta_S) - h^2(\Theta_S) = 10 \chi(\mathcal{O}_S) - 2K_S^2 = 44,$$

then $h^1(\Theta_S) = 50$ since we have shown that $h^2(\Theta_S) = 6$, see part (i). Being $F$ the union of sixteen disjoint $A_3$-cycles, we have $H^1_F(\Theta_S) \cong \mathbb{C}^{16 \cdot 3} = \mathbb{C}^{48}$. Therefore $h^1(\Theta_X) = 2$.

(iii) The cover $u: X \to Q$ is a simple $G$-cover branched on the divisor $B = B_1 \times B_2$, which has bidegree $(4, 4)$. By varying the branch loci $B_i \subset \mathbb{P}^1$ we obtain a $2$-dimensional family of equisingular deformations of $X$. Then

$$2 = \dim H^1(\Theta_X) = \dim T_{[X]} \text{ESDef}(X) \geq \dim_{[X]} \text{ESDef}(X) \geq 2,$$

which implies the claim. □

**Proposition 4.3.** The general deformation of the surface $X$ is a canonically embedded, smooth complete intersection $S_{2,4}$ of type $(2, 4)$ in $\mathbb{P}^4$.

**Proof.** By [Cat97, Proposition 6.2] it is sufficient to check that the canonical map $\phi_K: X \to \mathbb{P}^4$ is a birational morphism onto its image. Since $X$ has only Rational Double Points and $u: X \to Q$ is a simple $G$-cover, Hurwitz formula yields $K_X = u^*\mathcal{O}_Q(1, 1)$; but $|\mathcal{O}_Q(1, 1)|$ is base-point free, so $|K_X|$ is also base-point free and $\phi_K$ is a morphism.

It remains to show that $\phi_K$ separates two general points $x, y$ on $X$. The decomposition of $u_*\omega_X$ with respect to the $G$-action is

$$u_*\omega_X = \omega_Q \oplus (\omega_Q \otimes L) \oplus (\omega_Q \otimes L^2) \oplus (\omega_Q \otimes L^3),$$

where $L = \mathcal{O}_Q(1, 1)$ and $\omega_Q \otimes L^i$ is the eigensheaf corresponding to the character $\chi_i$. Therefore we obtain

$$H^0(u_*\omega_X) = H^0(\omega_Q \otimes L^2) \oplus H^0(\omega_Q \otimes L^3).$$

Now let $\{\tau\}$ be a basis of $H^0(\omega_Q \otimes L^2) = H^0(\mathcal{O}_Q)$ and let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a basis of $H^0(\omega_Q \otimes L^3) = H^0(\mathcal{O}_Q(1, 1))$. The four sections $\{\sigma_i\}$ provide an embedding $Q \to \mathbb{P}^3$, hence $\phi_K$ separates pairs of points which belong to the same fibre of $u: X \to Q$. Now let $x, y$ be two points in the same (general) fibre of $u$. Then there exists $1 \leq a \leq 3$ such that $y = \zeta^a \cdot x$. Then

$$\sigma_i(y) = \zeta^a \sigma_i(x), \quad \tau(y) = \zeta^2 \tau(x),$$

that is

$$\phi_K(y) = [\sigma_1(y): \sigma_2(y): \sigma_3(y): \sigma_4(y): \tau(y)] = [\sigma_1(x): \sigma_2(x): \sigma_3(x): \sigma_4(x): \zeta^a \tau(x)] 
\neq [\sigma_1(x): \sigma_2(x): \sigma_3(x): \sigma_4(x): \tau(x)] = \phi_K(x).$$

Therefore $\phi_K$ also separates general pairs of points lying in the same fibre of $u: X \to Q$ and we are done. □

Now we can prove the following
Proposition 4.4. Def(\(X\)) is smooth at \([X]\), of dimension 44.

Proof. By using Proposition 4.2 and exact sequence 4.4 we obtain
\begin{align}
\dim T_{[X]}\text{Def}(X) = \dim \text{Ext}^1(\Omega_\chi^1, \mathcal{O}_X) = 44.
\end{align}

On the other hand, by [Se06, Chapter 3] one knows that Def(\(S_{2,4}\)) is smooth, of dimension
\begin{align}
h^0(N_{S_{2,4}/\mathbb{P}^4}) - \dim \text{Aut}(\mathbb{P}^4) = h^0(\mathcal{O}_{S_{2,4}}(2)) + h^0(\mathcal{O}_{S_{2,4}}(4)) - 24 = 44.
\end{align}
Equality (17) and Proposition 4.3 yield
\begin{align}
44 = \dim T_{[S]}\text{Def}(\mathcal{S}) \geq \dim_{[S]}\text{Def}(\mathcal{S}) = 44,
\end{align}
so we are done. \(\square\)

Remark 4.5. Since \(X\) has only rational double points, by [BW74] the dimension of Def(\(X\)) equals the dimension of Def(\(S\)). So we infer
\begin{align}
50 = h^1(\Theta_{S}) = \dim T_{[S]}\text{Def}(\mathcal{S}) > \dim_{[S]}\text{Def}(\mathcal{S}) = 44,
\end{align}
that is Def(\(S\)) is singular at \([S]\). By [BW74, Theorem 3.7], this means that the sixteen \(A_3\)-cycles of \(S\) do not have independent behavior in deformations.

Proposition 4.3 in particular shows that the general deformation of \(X\) does not preserve the \(G\)-action. Now we want to consider some particular deformations that preserve the quadruple cover \(u: X \to Q\). According to [Pa91] we call them natural deformations, and we freely follow the notation of that paper everywhere. The building data of any totally ramified \(G\)-cover \(u: X \to Q\) are
\begin{align}
4L_{X_1} = 3D_{G,X_1} + D_{G,X_1},
2L_{X_2} = D_{G,X_1} + D_{G,X_3},
4L_{X_3} = D_{G,X_3} + 3D_{G,X_1},
\end{align}
see [Pa91, Proposition 2.1]. The \(G\)-cover \(u: X \to Q\) defines a natural embedding \(i\) of \(X\) into the total space of the vector bundle \(W = \bigoplus_{\chi \in \mathfrak{G}(\chi^0)} V(L^-_\chi)\). If \(w_\chi\) is a local coordinate on \(V(L^-_\chi)\) on an open set \(U\) and \(\sigma_{G,\psi}\) is a local equation for \(D_{G,\psi}\) on \(U\), then \(i(X)\) is defined by the equations
\begin{align}
w_\chi w_{\chi'} = \left(\prod_{\psi \in \{X_1, X_3\}} (\sigma_{G,\psi})^{G,\psi}_{\chi,\chi'}\right) w_{\chi\chi'}
\end{align}
and the covering map is given by the composition \(\pi \circ i\), where \(\pi: W \to Q\) is the projection. Moreover, the integers \(\epsilon_{G,\psi,\chi}^{G,\psi,\chi'}\) can be easily computed by using [Pa91, p. 196]:
\begin{align}
\epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0,
\epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1,
\epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 0,
\epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1, \quad \epsilon_{G,\psi,\chi}^{G,\psi,\chi'} = 1.
\end{align}
Let us consider now a collection of sections
\begin{align}
\{r_{G,\psi,\chi} \in H^0(\mathcal{O}_Q(D_{G,\psi}) \otimes L^-_\chi)\}_{\psi \in \{X_1, X_3\}, \chi \in \mathfrak{G}, \psi}.
\end{align}
where

\[ S_{G,\chi_1} := \{ \chi_0, \chi_1, \chi_2 \}, \quad S_{G,\chi_3} := \{ \chi_0, \chi_2, \chi_3 \}. \]

Let \( h_{G,\psi,\chi} \) be a local representative of \( r_{G,\psi,\chi} \) on the open set \( U \) and define

\[ \tau_{G,\psi} := \sum_{\psi \in \{ \chi_1, \chi_3 \}} \sum_{\chi \in S_{G,\psi}} h_{G,\psi,\chi} w_\chi. \]

Then the natural deformation of the \( G \)-cover \( u : X \to Q \), associated to the collection of sections \( \{ r_{G,\psi,\chi} \} \), is the subvariety \( X' \) of \( W \) locally defined by

\[ w_\chi w_{\chi'} = \left( \prod_{\psi \in \{ \chi_1, \chi_3 \}} (\tau_{G,\psi}^{G,\psi})_{\chi,\chi'} \right) w_{\chi'} , \]

together with the map \( u' : X' \to Q \) obtained by restricting the projection \( \pi : W \to Q \) to \( X' \).

Coming back to our particular case, we have

\[ D_{G,\chi_1} \in |O_Q(4, 4)|, \quad D_{G,\chi_3} = 0, \]

\[ L_{\chi_1} \cong O_Q(1, 1), \quad L_{\chi_2} \cong O_Q(2, 2), \quad L_{\chi_3} \cong O_Q(3, 3), \]

and \( B = D_{G,\chi_1} \). Since \( D_{G,\chi_3} = 0 \), the natural deformations of \( X \) are parameterized by the vector space

\[ \bigoplus_{\chi \in S_{G,\chi_1}} H^0(O_Q(D_{G,\chi_1}) \otimes L^{-1}_\chi) \]

\[ = H^0(O_Q(4, 4)) \oplus H^0(O_Q(3, 3)) \oplus H^0(O_Q(2, 2)) \cong \mathbb{C}^{50}. \]

4.2. **Example where Sing(\( X \)) = 16 \times \frac{1}{4}(1, 1).** Assume that, locally around each of the fixed points, the action of \( G = \langle \zeta \mid \zeta^4 = 1 \rangle \) is given by \( \zeta \cdot (x, y) = (\zeta x, \zeta y) \). In this case,

\[ \text{Sing}(X) = 16 \times \frac{1}{4}(1, 1). \]

By using Proposition 1.3, we obtain

\[ p_g(S) = 1, \quad q(S) = 0, \quad K_S^2 = -8, \]

hence \( S \) is not a minimal model.

**Theorem 4.6.** The following holds:

(i) \( h^2(\Theta_X) = 14; \)

(ii) all natural deformations of \( u : X \to Q \) preserve the 16 points of type \( \frac{1}{4}(1, 1); \)

(iii) there exists a 12-dimensional family of \( Q \)-Gorenstein deformations of \( X \), smoothing all the singularities. The general element \( X_t \) of this deformation is a smooth, minimal surface of general type with \( p_g(X_t) = 1, q(X_t) = 0 \) and \( K_{X_t}^2 = 8; \)

(iv) \( X_t \) is isomorphic to a Todorov surface with \( K^2 = 8 \).
Proof. (i) By using Grothendieck duality and Künneth formula as in Proposition 4.2 we obtain
\[ H^2(\Theta_X) = H^0(\Omega^1_X \otimes \Omega^2_Z)^G \]
\[ = \bigoplus_{\chi \in \mathcal{G}} \left( [H^0(g_1, \omega_{C_1})_X \otimes H^0(g_2, \omega_{C_2})_X]^{\chi^{-1}} \right) \]
\[ \oplus (H^0(g_1, \omega_{C_1})_X \otimes H^0(g_2, \omega_{C_2})_X) \]
\[ = (H^0(\mathcal{O}_{\mathbb{P}^1}) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(2))) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \]
\[ \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(2)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1))), \]
which yields \( h^2(\Theta_X) = 14 \).

(ii) The \( G \)-cover \( u: X \to Q \) is determined by the building data \[19\], with
\[ D_{G,X_1} \in |\mathcal{O}_Q(4, 0)|, \quad D_{G,X_2} \in |\mathcal{O}_Q(0, 4)|, \]
\[ L_{X_1} \cong \mathcal{O}_Q(1, 3), \quad L_{X_2} \cong \mathcal{O}_Q(2, 2), \quad L_{X_3} \cong \mathcal{O}_Q(3, 1). \]
The natural deformations of \( u \) are parameterized by the vector space
\[ \bigoplus_{\psi \in \{x_1, x_2\}} \left( \bigoplus_{\chi \in \mathcal{S}_{G,\psi}} H^0(\mathcal{O}_Q(\mathcal{D}_{G,\psi}) \otimes L_\chi^{-1}) \right) \]
\[ = H^0(\mathcal{O}_Q(4, 0)) \oplus H^0(\mathcal{O}_Q(0, 4)). \]
Therefore they form a family of dimension 10, which is exactly the one obtained by keeping the branch divisor \( B \subset Q \) of product type. In particular, all the natural deformations preserve the sixteen singular points of \( X \).

(iii) For simplicity, set \( w_i = w_{x_i} \) and \( \tau_{G,x_i} = h_iw_i \). Writing \( w_0 = 1 \), the local equations defining the family of natural deformations of \( u: X \to Q \) are the following:
\[ w_1^2 = h_3w_2, \quad w_1w_2 = h_3w_3, \quad w_1w_3 = h_1h_3, \]
\[ w_2^2 = h_1h_3, \quad w_2w_3 = h_1w_1, \quad w_3^2 = h_1w_2. \]
Relations \[24\] can be written in determinantal form in two different ways, namely
\[ (a) \quad \text{rank} \begin{pmatrix} w_2 & w_3 & w_1 & h_1 \\ w_1 & w_2 & h_3 & w_3 \end{pmatrix} \leq 1, \]
\[ (b) \quad \text{rank} \begin{pmatrix} h_3 & w_1 & w_2 \\ w_1 & w_2 & w_3 \\ w_2 & w_3 & h_1 \end{pmatrix} \leq 1. \]
In the sequel we will only consider the determinantal representation \( b \).
We can deform it by using the parameter \( s \in H^0(L_{X_2}) = \mathbb{C}^9 \), i.e.
\[ \text{rank} \begin{pmatrix} h_3 & w_1 & w_2 \\ w_1 & w_2 + s & w_3 \\ w_2 & w_3 & h_1 \end{pmatrix} \leq 1. \]
It is no difficult to check that for general \( s \neq 0 \) one obtains a smooth surface, hence \[25\] provides a smoothing \( \pi: \mathcal{X} \to T \) of \( X \). This is actually a \( \mathbb{Q} \)-Gorenstein smoothing of \( X \), since it is the globalization of the local \( \mathbb{Q} \)-Gorenstein smoothing of the quotient singularity \( 3(1, 1) \), see \[\text{Man08}\].
Chapter 4]. Therefore the general fibre $X_t$ of $\pi$ is a surface of general type whose invariants are

$$p_g(X_t) = 1, \quad q(X_t) = 0, \quad K_{X_t}^2 = 8.$$ 

The canonical divisor $K_X$ is big and nef (since $4K_X = u^*O_Q(4, 4)$), so $K_X$ is big and nef too, as $X_t$ is obtained by a $\mathbb{Q}$-Gorenstein smoothing of $X$. This shows that $X_t$ is a minimal model.

In order to give a more concrete description of $X_t$, let us look again at the double cover $v: Y \to E_1 \times E_2$ constructed in Section 3. By Proposition 3.6 we know that $\text{Def}(Y)$ is smooth at $[Y]$ of dimension 18; moreover the general deformation $Y_t$ of $Y$ is a double cover $v_t: Y_t \to A_t$ of an abelian variety $A_t$, branched on a smooth divisor $\Xi$ which is a polarization of type $(4, 4)$. Let us compute the dimension of the subspace of $\text{Def}(Y)$ consisting of surfaces for which it is possible to lift the natural involution $\iota_t: A_t \to A_t$ to an involution $\tilde{\iota}_t: Y_t \to Y_t$ such that $Y_t/\tilde{\iota}_t$ is smooth. By [BL04, Corollary 4.7.6], the divisor $\Xi$ does not contain any of the 16 fixed points of $\iota_t$. If we write locally the equation of the double cover $v_t: Y_t \to A_t$ as $z^2 = f(x, y)$ so that $\iota_t$ is given by $(x, y) \to (-x, -y)$, we see that $\iota_t$ lifts to $Y_t$ if and only if the branch locus $f(x, y) = 0$ is $\iota_t$-invariant; moreover in this case there is a unique lifting such that the quotient is smooth; it is locally given by $(x, y, z) \to (-x, -y, -z)$. By [BL04, Corollary 4.6.6], the divisors in $|\Xi|$ which are invariant under $\iota_t$ form a family of dimension $\frac{3}{2}h^0(\mathcal{O}_A(\Xi)) + 2 - 1 = 9$ and so, taking into account the three moduli of abelian surfaces, we obtain a 12-dimensional family $\{Y_t\}$ of deformations of $Y$ which admit a lifting of $\iota_t$.

One can further check that the lifted involution $\tilde{\iota}$ is fixed-point free and that the family $\{X_t\}$ constructed before can be obtained as $X_t = Y_t/\tilde{\iota}_t$.

(iv) Let us consider the Kummer surface $\text{Kum}(A_t) := A_t/\iota_t$. By (iii) a general fibre $X_t$ of the $\mathbb{Q}$-Gorenstein smoothing of $X$ is a double cover of $\text{Kum}(A_t)$ branched over the 16 nodes of $\text{Kum}(A_t)$ and the image $D$ of the curve $\Xi$.

On the other hand, $\text{Kum}(A_t)$ can be embedded in $\mathbb{P}^3$ as a quartic surface with 16 nodes and via this embedding the curve $D$ is obtained by intersecting $\text{Kum}(A_t)$ with a smooth quadric surface $\Phi$ which does not contain any of the nodes.

This shows that $X_t$ belongs precisely to the family of surfaces with $p_g = 1, q = 0$ and $K^2 = 8$ constructed by Todorov in [To81].

**Remark 4.7.** Let us fix the abelian surface $A$ and the embedding $\text{Kum}(A) \hookrightarrow \mathbb{P}^3$. Then the choice of the deformation parameter $s \in H^0(L_{\chi_{\Upsilon}})$ corresponds to the choice of the quadric surface $\Phi \in |\mathcal{O}_{\mathbb{P}^3}(2)|$. By [To81, Lemma 2.1] there is a quadric surface $\Phi_k$ in $\mathbb{P}^3$ which contains exactly $k$ ($1 \leq k \leq 6$) of the nodes of $\text{Kum}(A)$ that are general position. This means that the pullback in $A$ of the curve $D_k := \text{Kum}(A) \cap \Phi_k$ is a polarization of type $(4, 4)$ which contains exactly $k$ of the fixed points of $\iota: A \to A$.

Therefore arguments similar to those used in the proof of Theorem 4.6 part (ii) show that there exists a partial $\mathbb{Q}$-Gorenstein smoothing of $X$, whose general fibre $X_t$ is isomorphic to the double cover of $\text{Kum}(A)$ branched over the curve $D_k$ and the remaining $16 - k$ nodes of $\text{Kum}(A)$. The surface $X_t$ is not smooth, since it contains exactly $k$ singular points of type $\frac{1}{4}(1, 1)$. 


Its minimal resolution of singularities is a Todorov surface with $K^2 = 8 - k$ ($1 \leq k \leq 6$).

4.3. **Example where Sing** $(X) = 8 \times \frac{1}{4}(1, 3) + 8 \times \frac{1}{4}(1, 1)$. We can also twist the action of $G$ on $Z$ in such a way that

$$\text{Sing}(X) = 8 \times \frac{1}{4}(1, 1) + 8 \times \frac{1}{4}(1, 3).$$

By using Proposition 4.8, we obtain

$$p_g(S) = 3, \quad q(S) = 0, \quad K_S^2 = 0,$$

hence $S$ is not a minimal model.

Rasdeaconu and Suvaina give an explicit construction of $S$ in [RS06, Section 3], showing that it is a simply connected, minimal, elliptic surface with no multiple fibers. One can also prove that $H^2(\Theta_X) \neq 0$, see [LP11, Section 3].

**Proposition 4.8.** The following holds:

(i) all natural deformations of $X$ preserve the $8$ points of type $\frac{1}{4}(1, 1)$;

(ii) there exists a family of $\mathbb{Q}$-Gorenstein deformations of $X$, smoothing all the singularities. The general element of this family is a smooth, minimal surface of general type with $p_g = 3, q = 0$ and $K^2 = 8$.

**Proof.** (i) The abelian $G$-cover $u: X \to Q$ is determined by the building data $[I3]$, with

$$D_{G,X}, D_{G,X}, \in |O_Q(2, 2)|,$$

$$L_{X_1}, L_{X_2}, L_{X_3} \cong O_Q(2, 2).$$

The same argument of Theorem 4.6, part (ii) shows that the natural deformations of $X$ are parameterized by the vector space

$$H^0(O_Q(2, 2)) \oplus H^0(O_Q(2, 2))$$

$$\oplus H^0(O_Q) \oplus H^0(O_Q) \oplus H^0(O_Q) \oplus H^0(O_Q).$$

Writing $w_i := w_{X_i}$ we have

$$h_1 = g_1 + c_1 w_1 + c_2 w_2, \quad h_3 = g_3 + d_2 w_2 + d_3 w_3,$$

where $g_i$ a local equations of $D_{G,X}$, and $c_i, d_i \in \mathbb{C}$. Therefore the equations of the natural deformations of $X$ are

$$w_1^2 = (g_3 + d_2 w_2 + d_3 w_3)w_2,$$

$$w_1 w_2 = (g_3 + d_2 w_2 + d_3 w_3)w_3,$$

$$w_1 w_3 = (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3),$$

$$w_2^2 = (g_1 + c_1 w_1 + c_2 w_2)(g_3 + d_2 w_2 + d_3 w_3),$$

$$w_2 w_3 = (g_1 + c_1 w_1 + c_2 w_2)w_1,$$

$$w_3^2 = (g_1 + c_1 w_1 + c_2 w_2)w_2.$$

For a general choice of the parameters the morphism $\bar{u}: \bar{X} \to Q$ is not a Galois cover and an easy computation shows that its branch locus is of the form

$$D_X = D_1 + \ldots + D_6$$
where the $D_i$ belong to the pencil generated by $D_{G,X_1}$ and $D_{G,X_2}$. Then the singular locus of $D_X$ is given by the 8 points $D_{G,X_1} \cap D_{G,X_3}$ and $\text{Sing}(X)$ consists of the 8 points of type $\frac{1}{3}(1,1)$ locally defined by setting
\[ g_1 = g_3 = w_1 = w_2 = w_3 = 0 \]
in $(26)$.

(ii) We note that the set of natural deformations $\tilde{X}$ of $X$ which keep the $G$-action is parameterized by the vector space $H^0(O_Q(2,2)) \oplus H^0(O_Q(2,2))$. In fact, the action of the generator $i = \sqrt{-1}$ of $G$ must be given by
\[ w_1 \mapsto -iw_1, \quad w_2 \mapsto -w_2, \quad w_3 \mapsto iw_3 \]
and substituting in $(26)$ we obtain $c_1 = c_2 = d_1 = d_3 = 0$.

The $G$-cover $X \to Q$ factors into two double covers
\[ \widetilde{X} \to K \overset{p}{\to} Q \]
where $K$ is a $K3$ surface with 8 ordinary double points and $p: K \to Q$ is a double cover branched over $D_{G,X_1} + D_{G,X_3}$. Let $D_{G,X_2}$ be a general member in the pencil induced by $D_{G,X_1}$ and $D_{G,X_3}$. Let $D_{G,X_2} = p^*D_{G,X_2}$ and $2D_{G,X_1} = p^*D_{G,X_1}$ for $i = 1, 3$. Since $D_{G,X_3}$ is linearly equivalent to $D_{G,X_1}$ for $i = 1, 3$ and a $K3$ surface is simply connected, $D_{G,X_2}$ is linearly equivalent to $D_{G,X_1} + D_{G,X_3}$. Note that both these curves have exactly 8 nodes. The double cover $\tilde{X}$ of $K$ branched over $D_{G,X_2}$ is deformation equivalent to $X$, and $\tilde{X}$ can be realized as the bidouble cover of $Q$ branched over $D_{G,X_1}$, $D_{G,X_3}$ and $D_{G,X_2}$. Therefore if one deforms $D_{G,X_2}$ to a general divisor of bidegree $(2,2)$ we have a $\mathbb{Q}$-Gorenstein smoothing of $\tilde{X}$ which smoothes all the singularities. Since $\tilde{X}$ is a deformation of $X$ and $\tilde{X}$ is deformation equivalent to $\tilde{X}$, we have a smooth projective surface in the deformation space of $X$ which is a $\mathbb{Q}$-Gorenstein one. Finally, we note that each deformation is a $\mathbb{Q}$-Gorenstein one. In fact, $\tilde{X}$ and $X$ are double covers of the $K3$ surface $K$ branched over $D_{G,X_2}$ and $D_{G,X_1} + D_{G,X_3}$, respectively. Let $X \to \Delta$ be a family of double covers of $K$ obtained deforming the branch locus from $D_{G,X_1} + D_{G,X_3}$ to $D_{G,X_2}$. By using the canonical divisor formula for a double cover, it is not hard to see that $K_X$ is a $\mathbb{Q}$-Cartier divisor. Therefore the transitive property of $\mathbb{Q}$-Gorenstein deformations implies that $X$ has a $\mathbb{Q}$-Gorenstein smoothing. \qed

Remark 4.9. By applying arguments similar to those used in Remark 4.7 and in Lecture 10 Section 2, one can construct surfaces of general type with $p_g = 3$, $q = 0$ and $K^2 = k$ ($2 \leq k \leq 8$) by first taking a $\mathbb{Q}$-Gorenstein smoothing of $k$ singular points of type $\frac{1}{3}(1,1)$ of $\tilde{X}$ and then the minimal resolution of the remaining $8 - k$ singular points of the same type.

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DEFORMATIONS OF PRODUCT-QUOTIENT SURFACES

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