NONREGULAR IDEALS

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Abstract. Most of the regularity properties of ideals introduced by Taylor are equivalent at successor cardinals. For \( \kappa = \mu^+ \) with \( \cf(\mu) \) uncountable, we can rid the universe of dense ideals on \( P_\kappa(\lambda) \) for while preserving nonregular ideals on the same set.

An ideal on a set \( X \) is a collection of subsets of \( X \) closed under taking subsets and pairwise unions. If \( \kappa \) is a cardinal, an ideal \( I \) is called \( \kappa \)-complete if it is also closed under unions of size less than \( \kappa \). An ideal \( I \) on \( X \) gives a notion of a “negligible” subset of \( X \), and members of \( I \) are called \( I \)-measure-zero. Subsets of \( X \) which are not in \( I \) are called \( I \)-positive, and the collection of these is typically denoted by \( I^+ \). The dual filter to \( I \), the collection of all complements of members of \( I \), constitutes the collection \( I \)-measure-one sets and will be denoted by \( I^* \). If an ideal \( I \) renders every subset of \( X \) either measure zero or measure one, then its dual filter is called an ultrafilter.

The notion of regularity of ultrafilters was introduced by Keisler \[10\] and has had many applications in set theory and model theory \[3\]. An ultrafilter \( U \) is called \((\alpha, \beta)\)-regular when there is a sequence of sets \( \langle A_i : i < \beta \rangle \subseteq U \) such that for all \( z \subseteq \beta \) of ordertype \( \alpha \), \( \bigcap_{i \in z} A_i = \emptyset \). Taylor \[12\] generalized this notion to arbitrary filters (or equivalently, ideals), defining an ideal \( I \) to be \((\alpha, \beta)\)-regular when for every sequence \( \langle A_i : i < \beta \rangle \subseteq I^+ \), there is a refinement \( \langle B_i : i < \beta \rangle \subseteq I^+ \), which means \( B_i \subseteq A_i \) for each \( i \), such that for all \( z \subseteq \beta \) of ordertype \( \alpha \), \( \bigcap_{i \in z} B_i = \emptyset \).

An ideal on a cardinal \( \kappa \) is called simply regular when it is \((\omega, \kappa)\)-regular. Taylor showed some connections between regularity properties of ideals and the structure of their associated quotient boolean algebras, most notably the following:

**Theorem 0** (Taylor). A countably complete ideal \( I \) on \( \omega_1 \) is nonregular iff there is a set \( A \in I^+ \) such that \( \mathcal{P}(A)/I \) contains a dense set of size \( \omega_1 \).

Taylor also discussed degrees of regularity indexed by three ordinals. An ideal \( I \) is said to be \((\alpha, \beta, \gamma)\)-regular when for every sequence \( \langle A_i : i < \gamma \rangle \subseteq I^+ \), there is a refinement \( \langle B_i : i < \gamma \rangle \subseteq I^+ \) such that for every \( x \subseteq \gamma \) of ordertype \( \beta \), \( |\bigcap_{i \in x} B_i| \leq \alpha \).

We note the following easy relations between the regularity properties:

1. If \( \alpha_0 < \alpha_1 \), then \((\alpha_0, \beta, \gamma)\)-regularity implies \((\alpha_1, \beta, \gamma)\)-regularity.
2. If \( \beta_0 < \beta_1 \), then \((\alpha, \beta_0, \gamma)\)-regularity implies \((\alpha, \beta_1, \gamma)\)-regularity.
3. If \( \gamma_0 < \gamma_1 \), then \((\alpha, \beta, \gamma_1)\)-regularity implies \((\alpha, \beta, \gamma_0)\)-regularity.
Taylor [12] showed that if $I$ is a $\kappa$-complete ideal on a regular cardinal $\kappa$, then $I$ is $(\omega, \kappa)$-regular iff it is $(2, \kappa)$-regular. The latter is known as the disjoint refinement property or Fodor’s property [1]. In [5], the author showed that under GCH, many more degrees of regularity are equivalent for $\kappa$-complete ideals on $\kappa$, where $\kappa$ is the successor of a regular cardinal, and this was used to examine the relationship between regularity and density of ideals on cardinals above $\omega_1$. In this paper as elsewhere, we only consider degrees of regularity of ideals on $\kappa$ for which the last index in the degree is at most $\kappa$. Under this restriction, we show that without any assumptions, there are only two possible flavors of two-variable regularity:

**Theorem 1.** Suppose $\mu$ is an infinite cardinal, $\kappa = \mu^+$, and $I$ is a $\kappa$-complete ideal on $\kappa$. Then $I$ is $(\text{cf}(\mu) + 1, \kappa)$-regular, $(1, \text{cf}(\mu), \kappa)$-regular, and $(2, \delta)$-regular when $\delta < \kappa$. If $I$ is $(\text{cf}(\mu), \kappa)$-regular, then $I$ is $(2, \kappa)$-regular. Furthermore, if $I$ is $(\alpha, \beta, \kappa)$-regular for some $\alpha, \beta < \kappa$ such that $\mu^\beta = \mu$, then $I$ is $(2, \kappa)$-regular.

We show similar results for $\kappa$-complete normal ideals on $\mathcal{P}_\kappa(\lambda)$. We will say a normal ideal on $Z \subseteq \mathcal{P}(\lambda)$ is simply regular when it is $(2, \lambda)$-regular.

It is easy to see that if a $\lambda$-dense normal ideal on $\mathcal{P}(\lambda)$ is nonregular, Taylor’s theorem uses a result of Baumgartner-Hajnal-Máté [1], who showed that if a countably complete ideal on $\omega_1$ is nowhere $\omega_1$-dense, then it has the disjoint refinement property. This generalizes to normal ideals $I$ on $Z \subseteq \mathcal{P}_\kappa(\lambda)$ for $\kappa$ a successor cardinal, with an additional assumption about the quotient boolean algebra $\mathcal{P}(Z)/I$ that is trivially satisfied for $\kappa = \omega_1$ (see [5]). However, it is possible to separate density and nonregularity above $\omega_1$:

**Theorem 2.** Suppose $\kappa = \mu^+$, $\omega_1 \leq \text{cf}(\mu)$, $\kappa \leq \lambda$, and there is a nonregular, $\kappa$-complete, normal ideal on $\mathcal{P}_\kappa(\lambda)$. There is a cardinal-preserving forcing extension that also has such an ideal, but in which there are no $\lambda$-dense, $\kappa$-complete, normal ideals on $\mathcal{P}_\kappa(\lambda)$.

By results in [5], the existence of a $\lambda$-dense, $\kappa$-complete, normal ideal on $\mathcal{P}_\kappa(\lambda)$, where $\kappa = \mu^+$, is consistent relative to an almost-huge cardinal, for any choice of regular $\mu$ and $\lambda$.

1. **The regularity dichotomy**

This section is devoted to a proof of Theorem 1. We will prove some more general facts about the regularity of normal ideals on $\mathcal{P}(\lambda)$ and show how they imply the desired results about $\kappa$-complete ideals on successor cardinals $\kappa$.

Our notations are mostly standard. By $\mathcal{P}_\kappa(\lambda)$ we mean $\{z \subseteq \lambda : |z| < \kappa\}$. If $x$ is a set of ordinals, then $\text{ot}(x)$ denotes its ordertype.

The following facts can be found in [7]. Recall that an ideal $I$ on $Z \subseteq \mathcal{P}(X)$ is normal when for all $x \in X$, $\hat{x} := \{z \in Z : x \in z\} \in I^*$, and for all sequences $(A_x : x \in X) \subseteq I$, the diagonal union $\nabla_{x \in X} A_x := \bigcup_{x \in X} (A_x \cap \hat{x}) \in I$. This is equivalent to the statement that for every $A \in I^+$ and every $f : A \rightarrow X$ such that $f(z) \in z$ for all $z \in A$, there is $B \in I^+$ such that $f$ is constant on $B$.

The smallest normal ideal on a set $Z$ is the nonstationary ideal on $Z$, which is the dual ideal to the club filter (closed-unbounded filter) generated by sets of the form $\{z \in Z : f[z < \omega] \subseteq z\}$, where $f$ is a function $X^{<\omega} \rightarrow X$. As the name suggests,
positive sets for the nonstationary ideal are called \emph{stationary}. Consequently, if there is a (proper) normal ideal on \( Z \subseteq \mathcal{P}(X) \), then \( Z \) is stationary.

A normal ideal \( I \) on \( Z \subseteq \mathcal{P}(X) \) is \emph{\( \delta \)-saturated} for a cardinal \( \delta \) if there is no sequence \( \langle A_\alpha : \alpha < \delta \rangle \) such that \( A_\alpha \cap A_\beta \in I \) for \( \alpha < \beta \), and simply \emph{saturated} if it is \( |X|^+ \)-saturated. If \( I \) is saturated, then \( \mathcal{P}(Z)/I \) is a complete boolean algebra, with suprema given by diagonal unions. If \( \langle A_x : x \in X \rangle \) is an antichain, then we can use normality to refine it to a pairwise disjoint sequence of \( I \)-positive sets by replacing \( A_x \) with \( A_x \cap \hat{x} \setminus \bigcup_{y \neq x} (A_y \cap \hat{y}) \).

The idea behind the following lemma is taken from \cite{1}.

\textbf{Lemma 3.} Suppose \( I \) is a normal ideal on \( \mathcal{P}(\lambda) \) and \( \delta \leq \lambda \). If there is no \( A \in \mathcal{P}(\lambda) \) such that \( I \upharpoonright A \) is \( \delta \)-saturated, then \( I \) is \( (2, \delta) \)-regular.

\textit{Proof.} Let \( \langle A_\alpha : \alpha < \delta \rangle \subseteq I^+ \), and for each \( A_\alpha \), choose a sequence of \( I \)-positive sets \( \langle B^\beta_\alpha : \beta < \delta^+ \rangle \) such that each \( B^\beta_\alpha \subseteq A_\alpha \) and \( B^\beta_\alpha \cap B^{\beta'}_\alpha \in I \) when \( \beta < \beta' < \delta^+ \). For each \( \alpha < \delta \), let \( f(\alpha) \leq \alpha \) be the minimal ordinal such that \(|\{ \beta : A_\alpha \cap B^\beta_\alpha \cap f(\alpha) \in I^+ \}| = \delta^+ \). We can find \( \xi < \delta^+ \) such that for all \( \alpha < \delta \), all \( \alpha' < f(\alpha) \), and all \( \beta \geq \xi \), \( A_\alpha \cap B^\beta_{\alpha'} \in I \).

Recursively choose a refinement \( \langle C_\alpha : \alpha < \delta \rangle \) of \( \langle A_\alpha : \alpha < \delta \rangle \) and an increasing sequence of ordinals \( \langle \beta_\alpha : \alpha < \delta \rangle \) as follows. Let \( C_0 = B^0_0 \) and \( \beta_0 = \xi \). Given \( \langle C_\alpha' : \alpha' < \alpha \rangle \), let \( C_\alpha \) be an \( I \)-positive set of the form \( A_\alpha \cap B^{\beta_\alpha}_{f(\alpha)} \), where \( \beta_\alpha \geq \sup_{\alpha' < \alpha} (\beta_{\alpha'} + 1) \). Note that whenever \( \alpha \neq \alpha' \) are less than \( \delta \), it is ensured that \( C_\alpha \cap C_{\alpha'} \in I \). This is because if \( f(\alpha) = f(\alpha') = \eta \), then \( B^\beta_{\eta'} \cap B^\beta_{\eta'} \in I \) by construction, and if \( f(\alpha) < f(\alpha') \), then \( B^\beta_{f(\alpha)} \cap A_{\alpha'} \in I \).

Finally, we refine \( \langle C_\alpha : \alpha < \delta \rangle \), to a pairwise disjoint sequence \( \langle D_\alpha : \alpha < \delta \rangle \) by putting \( D_\alpha = C_\alpha \cap \hat{\alpha} \setminus \bigcup_{\alpha' \neq \alpha} (C_{\alpha'} \cap \hat{\alpha'}) \).

The following lemma contains the key combinatorial idea of this section: constructing a full disjoint refinement from a collection of partial ones under certain assumptions.

\textbf{Lemma 4.} Suppose \( I \) is a normal ideal on \( \mathcal{P}(\lambda) \), \( \mu \) is a cardinal such that \( \{ \alpha : \mathop{\text{cf}}(\sup z) \geq \mu \} \subseteq \mathcal{P}(\lambda) \), and for all \( A \in \mathcal{P}(\lambda) \) and \( \delta < \lambda \), \( I \upharpoonright A \) is not \( \delta^+ \)-saturated. If \( I \) is \( (\mu, \lambda) \)-regular, then \( I \) is regular.

\textit{Proof.} Let \( \langle A_\alpha : \alpha < \lambda \rangle \subseteq I^+ \). We may assume that for all \( \alpha < \lambda \) and all \( z \in A_\alpha \), \( \mathop{\text{cf}}(\sup z) \geq \mu \) and \( \alpha \in z \). Let \( \langle B_\alpha : \alpha < \lambda \rangle \subseteq I^+ \) be such that \( B_\alpha \subseteq A_\alpha \) for all \( \alpha \), and for all \( z \), \( s(z) := \{ \alpha : z \in B_\alpha \} \) has size \( < \mu \). Note that \( s(z) \subseteq z \). For all \( z \in \bigcup_{\alpha < \lambda} B_\alpha \), let \( f(z) \in z \) be such that \( s(z) \subseteq f(z) \). By normality, for all \( \alpha \), there is an \( I \)-positive \( C_\alpha \subseteq B_\alpha \) on which \( f \) is constant. Let \( g(\alpha) \) be this constant value, and note that \( g(\alpha) > \alpha \).

For each \( \alpha < \lambda \), choose a pairwise disjoint refinement \( \langle D^\alpha_\beta : \beta < \alpha \rangle \subseteq I^+ \) of \( \langle C_\beta : \beta < \alpha \rangle \), using Lemma 3. Then let \( E_{\alpha_0} = D^\alpha_{\alpha_0}(\alpha) \). If \( g(\alpha_0) = g(\alpha_1) \), then \( E_{\alpha_0} \cap E_{\alpha_1} = \emptyset \) by construction. If \( g(\alpha_0) \neq g(\alpha_1) \), then \( E_{\alpha_0} \cap E_{\alpha_1} = \emptyset \), since for \( i < 2 \) and \( z \in E_{\alpha_i} \), \( f(z) = g(\alpha_i) \).

\textbf{Lemma 5.} Suppose \( I \) is a normal ideal on \( \mathcal{P}(\lambda) \), \( \mu \) is a cardinal such that \( \{ \alpha : \mathop{\text{cf}}(\sup z) = \mu \} \subseteq \mathcal{P}(\lambda) \), and \( \lambda \) is regular. Then \( I \) is \( (\mathop{\text{cf}}(\mu) + 1, \lambda) \)-regular. If the function \( z \mapsto \sup z \) is \( \leq \delta \) to one on a set in \( I^* \), then \( I \) is \( (\delta, \mathop{\text{cf}}(\mu), \lambda) \)-regular.
Proof. Let \( \langle A_i : i < \lambda \rangle \subseteq I^+ \). Let \( Z = \{ z \subseteq \lambda : \text{cf}(\sup z) = \mu \} \). For each \( z \in Z \), let \( c_z \subseteq z \) be a cofinal subset of ordertype \( \text{cf}(\mu) \). By induction, we build an increasing sequence \( \langle i < \lambda \rangle \subseteq \lambda \) and a refinement \( \langle B_i : i < \lambda \rangle \subseteq I^+ \) of \( \langle A_i : i < \lambda \rangle \) as follows. Given \( \langle i < j \rangle, \sup z > \sup_{i < j} \alpha_i \) for \( I \)-almost all \( z \in A_j \). For such \( z \), let \( \sup_{i < j} \alpha_i < \alpha_j(z) \in c_z \). Let \( B_j \subseteq A_j \) be an \( I \)-positive set on which the function \( z \mapsto \alpha_j(z) \) is constant, and let \( \alpha_j \) be this constant value. For each \( z \), let \( s(z) = \{ i < \lambda : z \in B_i \} \). Note that \( z \in B_i \) implies \( \alpha_i \in c_z \), so \( \sup(s(z)) \leq \text{cf}(\mu) \). Also, if \( \sup(s(z)) = \text{cf}(\mu) \), then \( s(z) \) is cofinal in \( c_z \) and thus in \( z \). Thus, if \( z \mapsto \sup z \) is \( \leq \delta \) to one on a set in \( I^* \), then we may take the sequence \( \langle B_i : i < \lambda \rangle \) such that \( |\{ i \in X : B_i \}| \leq \delta \) whenever \( \sup x = \mu \).

The following result was independently observed by Burke-Matsubara \( [2] \) and Foreman-Magidor \( [3] \). Its proof uses deep results of Shelah \( [11] \) and Cummings \( [4] \).

**Lemma 6.** Suppose \( I \) is a normal saturated ideal on \( \mathcal{P}(\lambda) \). Then \( \{ z : \text{cf}(\sup z) = \text{cf}(\|z\|) \} \subseteq I^* \).

The following basic fact can be proved in multiple ways, for example via Ulam matrices or via generic ultrapowers (see \( [6] \)).

**Lemma 7.** If \( \kappa \) is a successor cardinal, then no \( \kappa \)-complete ideal on \( \kappa \) is \( \kappa \)-saturated, and no \( \kappa \)-complete normal ideal on \( \mathcal{P}_\kappa(\lambda) \) is \( \lambda \)-saturated.

**Theorem 8.** Suppose \( \kappa = \mu^+ \) and \( I \) is a \( \kappa \)-complete normal ideal on \( \mathcal{P}_\kappa(\lambda) \). If \( I \) is \( (\text{cf}(\mu), \lambda) \)-regular, then \( I \) is regular. If \( \lambda \) is regular, then \( I \) is \( (\text{cf}(\mu) + 1, \lambda) \)-regular.

**Proof.** Let \( \langle A_\alpha : \alpha < \lambda \rangle \subseteq I^+ \). We first separate the saturated and non-saturated parts. We choose an initial refinement by putting \( B_\alpha = A_\alpha \) if there is no \( B \subseteq A_\alpha \) such that \( I \upharpoonright B \) is saturated, and otherwise choose \( B_\alpha \subseteq A_\alpha \) such that \( I \upharpoonright B_\alpha \) is saturated. Let \( Y_0 \) be the ordinals below \( \lambda \) falling into the first case, and \( Y_1 \) those falling into the second. Note that whenever \( \alpha \in Y_0 \) and \( \beta \in Y_1 \), we have \( B_\alpha \cap B_\beta \subseteq I^+ \).

As in the proof of Lemma \( [3] \), we may refine to a sequence \( \langle C_\alpha : \alpha < \lambda \rangle \) such that \( C_\alpha \cap C_\beta = \emptyset \) whenever at least one of \( \alpha, \beta \) is in \( Y_0 \). If we put \( C = \bigcup_{\alpha \in Y_1} C_\alpha \), then \( I \upharpoonright C \) is saturated, since if \( \langle D_\alpha : \alpha < \lambda^+ \rangle \) is an antichain in \( \mathcal{P}(C)/I \), then for some \( \beta < \lambda \), there are \( \lambda^+ \)-many \( \alpha \) such that \( C_\beta \cap D_\alpha \subseteq I^+ \).

We may assume \( \text{cf}(\sup z) = \text{cf}(\mu) \) for all \( z \in C \). Since \( I \upharpoonright A \) is not \( \lambda \)-saturated for any \( A \subseteq I^+ \), Lemma \( [1] \) implies that if \( I \) is \( (\text{cf}(\mu), \lambda) \)-regular, then there is a disjoint refinement of \( \langle C_\alpha : \alpha \in Y_1 \rangle \) into \( I \)-positive sets \( \langle D_\alpha : \alpha \in Y_1 \rangle \). Putting this together with \( \langle C_\alpha : \alpha \in Y_0 \rangle \), we have a disjoint refinement of the original sequence into \( I \)-positive sets.

If \( \lambda \) is regular, then by Lemma \( [5] \) there is a refinement \( \langle E_\alpha : \alpha \in Y_1 \rangle \subseteq I^+ \) of \( \langle C_\alpha : \alpha \in Y_1 \rangle \) such that \( \bigcap_{\alpha \in Y} E_\alpha = \emptyset \) whenever \( \sup(x) > \text{cf}(\mu) \), showing that \( I \) is \( (\text{cf}(\mu) + 1, \lambda) \)-regular.

In order to prove Theorem \( [11] \) we use some results from \( [12] \) which allow a reduction to normal ideals:

**Lemma 9** (Taylor). Let \( I \) be a \( \kappa \)-complete ideal on \( \kappa \).

1. Suppose every sequence \( \langle A_i : i < \kappa \rangle \subseteq I^+ \) has a refinement \( \langle B_i : i < \kappa \rangle \subseteq I^+ \) such that \( I \upharpoonright B_i \) is \( (\alpha, \beta, \kappa) \)-regular for each \( i \). Then \( I \) is \( (\alpha, \beta, \kappa) \)-regular.

2. If \( \kappa = \mu^+ \) and \( I \) is \( \kappa^+ \)-saturated, then there is \( A \subseteq I^+ \) and a bijection \( f : \kappa \to \kappa \) such that \( \{ f[X] : X \subseteq I \upharpoonright A \} \) is a normal ideal on \( \kappa \).
Let $I$ be a $\kappa$-complete ideal on $\kappa = \mu^+$. Then by Lemmas $3$ and $7$, $I$ is $(2, \delta)$ regular for $\delta < \kappa$. For the other regularity properties, let $(A_\alpha : \alpha < \kappa) \subseteq I^+$. Let $B_\alpha \subseteq A_\alpha$, be an $I$-positive set such that $I \upharpoonright B_\alpha$ is $\kappa^+$-saturated if there is such a $B_\alpha$. In such a case, part $(2)$ of Lemma $9$ implies that we can find an $I$-positive $C_\alpha \subseteq B_\alpha$ such that $I \upharpoonright C_\alpha$ is isomorphic to a normal ideal. By Lemmas $5$ and $9$, $I \upharpoonright C_\alpha$ is $(\text{cf}(\mu) + 1, \kappa)$-regular and $(1, \text{cf}(\mu), \kappa)$-regular whenever $C_\alpha$ is defined. If $C_\alpha$ is undefined, then $I \upharpoonright A_\alpha$ is regular by Lemma $8$. Part $(1)$ of Lemma $9$ then gives that $I$ is $(\text{cf}(\mu) + 1, \kappa)$-regular and $(1, \text{cf}(\mu), \kappa)$-regular. If $I$ is $(\text{cf}(\mu), \kappa)$-regular, then so is each $I \upharpoonright C_\alpha$ when $C_\alpha$ is defined, and thus $I \upharpoonright C_\alpha$ is regular by Theorem $8$. Again by part $(1)$ of Lemma $9$ $I$ is regular in this case. This concludes the proof of the part Theorem $3$ that assumes no cardinal arithmetic.

To show that “furthermore” part of Theorem $3$ we introduce an extension of Taylor’s three-variable notion of regularity. Let us say an ideal $I$ is $(\text{cf}(\mu), \alpha, \beta)$-regular if every sequence $(\langle A_i : i < \beta \rangle \subseteq I^+$ has a refinement $(B_i : i < \beta) \subseteq I^+$ such that $\bigcap_{i \in x} B_i \in I$ whenever $\text{ot}(x) \geq \alpha$. If $I$ is a $\kappa$-complete ideal on $\kappa$, then $(I, \alpha, \beta, \kappa)$-regularity is a weakening of $(\alpha, \beta, \kappa)$-regularity for every $\alpha < \kappa$.

**Lemma 10.** Suppose $\kappa = \mu^+$ and $I$ is a $\kappa$-complete ideal on $\kappa$. If $I$ is $(I, \alpha, \beta, \kappa)$-regular, where $\mu^{\mu^X} = \mu$, then $I$ is regular.

**Proof.** Let $(\langle A_\alpha : \alpha < \kappa \rangle \subseteq I^+$, and let $(\langle B_\alpha : \alpha < \kappa \rangle \subseteq I^+$ be a refinement such that $B_\alpha \subseteq \hat{\alpha}$ for all $\alpha$, and $\bigcap\{B_\alpha : \alpha \in I\}$ whenever $\text{ot}(x) \geq \xi$. For every $\alpha < \kappa$ we can define an $I$-positive $C_\alpha \subseteq B_\alpha$ by

$$C_\alpha = B_\alpha \setminus \bigcup_{x \in [\alpha]^\xi} \bigcap_{\beta \in x} B_\beta.$$

If $x$ is a subset of $\kappa$ of ordertype $\xi + 1$, then let $\alpha = \text{max}(x)$. If $\beta \in C_\alpha$, then $\beta \notin \bigcap_{\gamma \in x'\alpha} C_\gamma$. This shows $I$ is $(\xi + 1, \kappa)$-regular and therefore regular by the first part of Theorem $3$. $\square$

2. Consistency results

This section is devoted to a proof of Theorem $2$. If $V \subseteq W$ are models of set theory and $I \subseteq V$ is an ideal, then in $W$ we can generate an ideal $\check{I}$ from $I$ by taking all sets which are covered by a set from $I$. Let us first show the preservation of nonregular ideals by forcings with a strong enough chain condition, as a consequence of Theorem $8$.

**Lemma 11.** Suppose $\kappa = \mu^+$, $\lambda \geq \kappa$, and $I$ is a nonregular, $\kappa$-complete, normal ideal on $Z \subseteq P_\kappa(\lambda)$. If $\mathbb{P}$ is c.c.c., then in $V^{\mathbb{P}}$, the ideal $\check{I}$ generated by $I$ is nonregular.

**Proof.** Let $(\langle A_\alpha : \alpha < \lambda \rangle \subseteq I^+$ in $V$. If $p \Vdash \check{I}$ is regular, then there is a $\mathbb{P}$-name for a refinement $(\check{B}_\alpha : \alpha < \lambda)$ such that each $z \in Z$ is forced by $p$ to be in at most one $B_\alpha$. In $V$, for each $\alpha$ let $C_\alpha = \{z \in A_\alpha : (\exists q \leq p)q \Vdash z \in B_\alpha\}$. Since $p \Vdash \check{B}_\alpha \subseteq \check{C}_\alpha$, each $C_\alpha$ is $I$-positive. By the chain condition, for each $z$, the set $s(z) := \{\alpha : (\exists q \leq p)q \Vdash z \in B_\alpha\} = \{\alpha : z \in C_\alpha\}$ has size $< \text{cf}(\mu)$. This shows that $I$ is $(\text{cf}(\mu), \lambda)$-regular in $V$, and thus regular by Theorem $8$. $\square$

If $\check{I}$ is a $\kappa$-complete normal ideal and $\mathbb{P}$ is a $\kappa$-c.c. forcing, then it is easy to show that the ideal generated by $I$ is also $\kappa$-complete and normal in $V^{\mathbb{P}}$. If $I$ is saturated, then Foreman’s Duality Theorem $7$ allows us to say much more. This is
connected to the forcing properties of the quotient algebra and generic elementary embeddings.

The following facts can be found in \([6]\). If \( I \) is an ideal on \( Z \) and \( G \subseteq \mathcal{P}(Z)/I \) is generic, then in \( V[G] \), we can form the ultrapower embedding \( j: V \to V^Z/G \). If \( Z \subseteq \mathcal{P}(\lambda) \) and \( I \) is normal, then the pointwise image of \( \lambda \) under \( j \) is represented in the ultrapower by the identity function on \( Z \), i.e., \( [\text{id}]_G = j[\lambda] \). If \( I \) is \( \kappa \)-complete, \( \kappa = \mu^+ \), and \( Z \subseteq \mathcal{P}_\kappa(\lambda) \), then \( \kappa \) is the critical point of \( j \), and \( V^Z/G \models |j[\lambda]| < j(\kappa) \). Consequently, \( V[G] \models |\lambda| = |\mu| \). This implies that there is no condition \( A \in I^+ \) such that \( I \upharpoonright A \) is \( \lambda \)-saturated. Thus in this context, \( I \) being saturated is the same as \( \mathcal{P}(Z)/I \) having the best possible chain condition. If this occurs, then \( I \) is \( \textit{precipitous} \), meaning that whenever \( G \subseteq \mathcal{P}(Z)/I \) is generic, \( V^Z/G \) is well-founded and thus isomorphic to a transitive class \( M \subseteq V[G] \).

**Theorem 12** (Foreman \([7]\)). Suppose \( I \) is a \( \kappa \)-complete precipitous ideal on \( Z \), and \( \mathbb{P} \) is a \( \kappa \)-c.c. forcing. In \( V^\mathbb{P} \), let \( \tilde{I} \) denote the ideal generated by \( I \), and let \( j \) denote a generic ultrapower embedding obtained from forcing with \( \mathcal{P}(Z)/I \). Then there is an isomorphism

\[
\iota: B(\mathbb{P} \ast \mathcal{P}(\tilde{Z})/\tilde{I}) \cong B(\mathcal{P}(Z)/I \ast j(\mathbb{P}))
\]

given by \( \iota(p, \tilde{A}) = ||\text{id}|| \in j(\tilde{A})|| \wedge (1, j(\tilde{p})) \).

The next proposition shows the relevance of the cardinal arithmetic assumption in Lemma \([6]\). For example, we can produce a model in which CH fails and there is a nonregular ideal \( I \) on \( \omega_2 \) which is \( (I, \omega, \omega_2)-\text{regular} \).

**Proposition 13.** Suppose \( \kappa = \mu^+ \), \( \nu \leq \mu \) is such that \( \nu^{\kappa \nu} = \nu \), and \( I \) is a saturated, nonregular, \( \kappa \)-complete ideal on \( \kappa \). If \( G \subseteq \text{Add}(\nu, \kappa) \) is generic, then in \( V[G] \), \( I \) is \( (I, \nu, \kappa)-\text{regular} \).

**Proof.** Add(\( \nu, \kappa \)) is \( \nu^{\kappa \text{c.c.}} \). By Theorem \([12]\) in \( V[G] \), there is an isomorphism \( \sigma: \mathcal{P}(\kappa)/I \cong B(\mathcal{P}(\kappa)^V/I \times \text{Add}(\nu, \kappa^+)) \). If \( (A_\alpha: \alpha < \kappa) \subseteq \tilde{I}^+ \), choose for each \( \alpha \) some \( (B_\alpha, p_\alpha) \leq \sigma(A_\alpha) \). Let \( \beta < \kappa^+ \) be such that \( \text{dom} \ p_\alpha \subseteq \beta \times \nu \) for all \( \alpha \). Let \( q_\alpha = \{((\beta + \alpha, 0), 0)\} \) for \( \alpha < \kappa \), and choose \( C_\alpha \leq \sigma^{-1}(B_\alpha, p_\alpha \wedge q_\alpha) \). The intersection of any \( \nu \)-many \( C_\alpha \) is in \( I \), since there is no lower bound to \( \nu \)-many \( q_\alpha \).

**Lemma 14.** Suppose \( I \) is a normal ideal on \( Z \subseteq \mathcal{P}(X) \). Then \( I \) is \( |X|^+ \)-saturated iff every normal \( J \supseteq I \) is equal to \( I \upharpoonright A \) for some \( A \subseteq Z \).

**Proof.** Suppose \( I \) is \( |X|^+ \)-saturated. Let \( \{A_x: x \in X\} \) be a maximal antichain in \( J \cap I^+ \). Then \( \nabla A_x \) is the \( \delta \)-largest element of \( J \cap I^+ \), so \( J = I \upharpoonright (Z \setminus \nabla A_x) \). Now suppose \( I \) is not \( |X|^+ \)-saturated, and let \( \{A_\alpha: \alpha < \delta\} \) be a maximal antichain such that \( \delta \geq |X|^+ \). Let \( J \) be the ideal generated by \( \bigcup\{\Sigma_\alpha \in Y | A_\alpha| Y \in \mathcal{P}(X)^{\langle \delta \rangle}\} \). Then \( J \) is a proper normal ideal extending \( I \). \( J \) cannot be equal \( I \upharpoonright A \) for some \( A \in I^+ \) because if this were so, there would some \( \alpha \) such that \( A \cap A_\alpha \in I^+ \). \( A \cap A_\alpha \in J \) by construction, but every \( I \)-positive subset of \( A \) is \( (I \upharpoonright A) \)-positive.

A partial order is said to be \( \kappa \)-dense if it has a dense subset of size \( \leq \kappa \). It is said to be \( \textit{nowhere} \kappa \)-dense if it is not \( \kappa \)-dense below any condition. An ideal is said to be \( \lambda \)-dense or nowhere \( \lambda \)-dense when its associated boolean algebra has these properties.
Lemma 15. Suppose $\kappa = \mu^+$, $\nu \leq \mu$ is such that $\nu^\nu = \nu$, and $Z \subseteq \mathcal{P}_\kappa(\lambda)$ is stationary. Let $\mathbb{P} = \text{Add}(\nu, \theta)$ for some $\theta \geq \kappa$. Then in $V^\mathbb{P}$, there are no normal, $\kappa$-complete, $\lambda$-dense ideals on $Z$.

Proof. Suppose $p \Vdash \dot{J}$ is a $\kappa$-complete, $\lambda^+$-saturated, normal ideal on $Z$. Let $I = \{X \subseteq Z : p \Vdash X \in \dot{J}\}$. It is easy to check that $I$ is normal and $\kappa$-complete.

The map $\sigma : \mathcal{P}(Z)/I \to \mathcal{B}(\mathbb{P} \upharpoonright p \ast \mathcal{P}(Z)/\dot{J})$ that sends $X$ to $(||X \in \dot{J}||, [X]_\dot{J})$ is an order-preserving and antichain-preserving map, so $I$ is $\lambda^+$-saturated.

Let $H$ be $\mathbb{P}$-generic over $V$ with $p \in H$. Since $\mathbb{P}$ is $\kappa$-c.c., $I$ remains normal. By Theorem 12, the map $e : q \mapsto (1, j(q))$ is a regular embedding of $\mathbb{P}$ into $\mathcal{P}(Z)/I \ast j(\mathcal{P})$. Thus in $V[H]$, $\mathcal{P}(Z)/\dot{I} \cong \mathcal{P}^V(Z)/I \ast \text{Add}(\nu, \dot{\eta})$, where $\Vdash \dot{\eta} = \text{ot}(j(\theta) \upharpoonright j(\theta))$. Since $j(\kappa) = \lambda^+$, $\Vdash \dot{\eta} \geq \lambda^+$.

$\dot{I}$ is normal and $\lambda^+$-saturated, and $\dot{I} \subseteq J$. By Lemma 14 there is $A \subseteq \dot{I}^+$ such that $J = \dot{I} \upharpoonright A$. Since $\text{Add}(\nu, \eta)$ is nowhere $\lambda$-dense, $\mathcal{P}(Z)/\dot{I}$ is nowhere $\lambda$-dense. Thus $J$ is not $\lambda$-dense.

Thus we may rid the universe of dense ideals that concentrate on $\mathcal{P}_\kappa(\lambda)^V$. This finishes the job if $\kappa = \lambda$, but not necessarily in other cases. For example, Gitik showed [3] that if $V \subseteq W$ are models of set theory, $\kappa < \lambda$ are regular in $W$, and there is a real number in $W \setminus V$, then $\mathcal{P}_\kappa(\lambda)^W \setminus \mathcal{P}_\kappa(\lambda)^V$ is stationary. In order to take care of such problems, we use some arguments of Laver and Hajnal-Juhasz that are reproduced in [6].

The notation \[ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \] stands for the assertion that for every $f : \alpha \times \beta \rightarrow \eta$, there is $A \in [\alpha]^{\gamma}$ and $B \in [\beta]^\delta$ such that $f$ is constant on $A \times B$. As usual with arrow notations, if ordinals on the left side are increased and ordinals on the right side are decreased, then we get a weaker statement.

Lemma 16. Suppose there is a $\lambda$-dense, $\kappa$-complete, normal ideal $I$ on $\mathcal{P}_\kappa(\lambda)$ such that every $I$-positive set has cardinality $\geq \eta$. Then for $\mu, \nu < \kappa$,

\[ \begin{pmatrix} \lambda^+ \\ \lambda < \kappa \end{pmatrix} \rightarrow \begin{pmatrix} \mu \\ \eta \end{pmatrix}^\nu. \]

Proof. Let $\theta = \lambda^{<\kappa}$, and enumerate $\mathcal{P}_\kappa(\lambda)$ as $\langle z_\alpha : \alpha < \theta \rangle$. Let $f : \lambda^+ \times \theta \rightarrow \nu$. By $\kappa$-completeness, for each $\alpha < \lambda^+$, there is $\gamma < \nu$ such that $X_\alpha := \{\beta : f(\alpha, \beta) = \gamma\} \in I^+$. By $\lambda$-density, there is a set $S \subseteq [\lambda^+]^{\lambda^+}$, a set $D \subseteq I^+$, and a $\gamma^* < \nu$ such that for all $\alpha \in S$, $D \subseteq I_X \alpha$ and $f(\alpha, \beta) = \gamma^*$ for $z_\beta \in X_\alpha$. Let $A \subseteq S$ have size $\mu$. Since $\bigcap_{\alpha \in A} X_\alpha$ is $I$-positive, there is a set $B \subseteq \theta$ of size $\geq \eta$ such that for all $\alpha \in A$ and all $\beta \in B$, $f(\alpha, \beta) = \gamma^*$.

Lemma 17. Suppose $\theta$ is regular and $\mu < \theta$ is such that $\mu^{<\mu} = \mu$. If $G \subseteq \text{Add}(\mu, \theta)$ is generic, then in $V[G]$,

\[ \begin{pmatrix} \theta^+ \\ \theta \end{pmatrix} \rightarrow \begin{pmatrix} \mu \\ \theta \end{pmatrix}^2. \]

Proof. In $V$, choose an almost-disjoint family $\{X_\alpha : \alpha < \theta^+\} \subseteq \mathcal{P}(\theta)$, and for each $\alpha$, let $\langle \gamma^*_\alpha \beta : \beta < \theta \rangle$ enumerate $X_\alpha$ in increasing order. In $V[G]$, let $f : \lambda^+ \times \theta \rightarrow 2$ be defined by $f(\alpha, \beta) = G(\gamma^*_\alpha \beta, 0)$. Let $A \subseteq \theta^+$ be a set of size $\mu$ in $V[G]$. By the chain condition, there is a $\zeta < \theta$ such that $G = G_0 \times G_1$, where $G_0$ is $\text{Add}(\mu, \zeta)$-generic, and $A \in V[G_0]$. In $V[G_0]$, let $\zeta < \delta < \theta$ be such that $\{X_\alpha \setminus \delta : \alpha \in A\}$ is pairwise
disjoint. For any \( p \in \text{Add}(\mu, \theta \setminus \zeta) \) and any \( \eta \geq \delta \), there are \( q \leq p \) and \( \alpha, \beta \in A \) such that \( q(\gamma^\alpha_0, 0) \neq q(\gamma^\beta_0, 0) \). Since \( G_1 \) is generic, we have that for all \( \eta \geq \delta \), there are \( \alpha, \beta \in A \) such that \( f(\alpha, \eta) \neq f(\beta, \eta) \). Thus there is no \( B \subseteq \theta \) of size \( \theta \) such that \( f \) is constant on \( A \times B \).

We can now prove Theorem\(^2\) Suppose that in \( V, I \) is a nonregular, \( \kappa \)-complete, normal ideal on \( P_\kappa(\lambda) \), where \( \kappa = \mu^+ \) and \( \text{cf}(\mu) \) is uncountable. Let \( \theta > \lambda^\mu \) be regular and such that \( \theta^\mu = \theta \). Let \( G \subseteq \text{Add}(\omega, \theta) \) be generic. By Lemma 11, \( \bar{I} \) is nonregular in \( V[G] \). Suppose \( Z \subseteq P_\kappa(\lambda) \) has cardinality \( < \theta \). Then there is \( \zeta < \theta \) such that \( G = G_0 \times G_1 \), where \( G_0 \) is \( \text{Add}(\omega, \zeta) \)-generic, and \( Z \in V[G_0] \). By Lemma 15, there is no \( \lambda \)-dense, \( \kappa \)-complete, normal ideal concentrating on \( Z \) in \( V[G] \). Since \( \lambda^\mu = \theta \) in \( V[G] \), Lemmas 16 and 17 imply that there is no \( \lambda \)-dense, \( \kappa \)-complete, normal ideal on \( P_\kappa(\lambda) \) for which every positive set has size \( \theta \).

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