No generalized statistics from dynamics in curved spacetime

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Abstract

We consider quon statistics in a dynamically evolving curved spacetime in which prior to some initial time and subsequent to some later time is flat. By considering the Bogoliubov transformations associated with gravitationally induced particle creation, we find that the consistent evolution of the generalized commutation relations from the first flat region to the second flat region can only occur if $q = \pm 1$.

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There has been an increasing interest in generalized commutation relations of the form

\[ a_i a_j^\dagger - qa_j^\dagger a_i = \delta_{ij} \tag{1} \]

where \( q \) is real, coming from two main directions. One motivation comes from the study of quantum groups (for example see Refs. [1–5]); in particular the quantum version \([\mathbb{R}]\) of SU(2). The other motivation is Greenberg’s study of particles with infinite statistics \([\mathbb{S}]\). In either case, the relations (1) can be used to quantize the simple harmonic oscillator. (The generalized commutation relations used in Refs. \([\mathbb{B}]\) can be put in the form (1) by an appropriate transformation.) As \( q \) ranges from +1 to -1, it can be seen that the relations (1) interpolate between commutation relations and anticommutation relations (appropriate for bosons and fermions respectively).

Given an operator \( a_i \) and its adjoint \( a_i^\dagger \) which satisfy (1), a vacuum state \( |0> \) can be defined in the usual way by

\[ a_i|0> = 0 \tag{2} \]

and the Fock space built up as normal by applying monomials in \( a_i^\dagger \) to \( |0> \). A detailed study of the resulting space has been given \([\mathbb{I}][\mathbb{I}]\). It was shown in these references that for \( |q| < 1 \), the Fock space is well defined with all states of positive norm. An important feature, emphasised by Greenberg \([\mathbb{S}]\) is that it is not possible to impose any relation which relates \( a_i a_j \) to \( a_j a_i \). A consequence of this is that multiparticle states have no definite symmetry properties under particle interchange. In the special case \( |q| = 1 \), it is still true that all vectors in the Fock space have non-negative norms; however, unless the vectors are totally symmetric or antisymmetric under particle interchange, they have zero norm. The allowed cases when \( |q| = 1 \) correspond to the normal Bose or Fermi statistics. If \( |q| > 1 \) then the Fock space has states of negative norm, meaning that the Hilbert space structure is lost, along with the usual probability interpretation. We will restrict ourselves to \( |q| < 1 \).
As far as we are aware, all studies of the relations (1) have been in flat spacetime. The aim of the present paper is to examine the consequences of imposing such relations in curved spacetime. In particular, we will examine Parker’s proof of the spin-statistics theorem applied to a system quantized using (1). The basic idea of the proof is to suppose that we have a time dependent spacetime which for \( t < t_1 \) is flat, and which for \( t > t_2 \) is also flat. (Here \( t_1 \) and \( t_2 \) are two arbitrary times.) The spacetime can be dynamic for \( t_1 \leq t \leq t_2 \). As a specific example we could consider a spatially flat Robertson-Walker spacetime with

\[
\text{In the in-region we may expand the field operator } \Phi(x) \text{ in terms of creation and annihilation operators as}
\]

\[
\Phi(x) = \sum_i (F_i(x)a_i + F_i^*(x)a_i^\dagger).
\]
Klein-Gordon equation. We will assume that the operators in (4) satisfy (1) with $|q| < 1$. We note that the quantum fields we consider are interacting with a gravitational field of arbitrary strength, but are otherwise free.

An expansion similar to (4) may be imposed in the out-region:

$$\Phi(x) = \sum_i (G_i(x)b_i + G^*_i(x)b^\dagger_i)$$  \hspace{1cm} (5)

with $\{G_i(x)\}$ a complete set of positive frequency solutions. The operators $b_i$ and $b^\dagger_i$ will in general differ from $a_i$ and $a^\dagger_i$ if there is particle creation due to the expansion of the universe \cite{12,13,16}. We assume that

$$b_ib^\dagger_j - q'b^\dagger_jb_i = \delta_{ij}$$  \hspace{1cm} (6)

where $|q'| < 1$ with $q'$ not necessarily equal to $q$.

Because both sets $\{F_i(x)\}$ and $\{G_i(x)\}$ are assumed to be complete, we may expand one in terms of the other:

$$G_i(x) = \sum_j (\alpha_{ij}F_j(x) + \beta_{ij}F^*_j(x))$$  \hspace{1cm} (7)

for some coefficients $\alpha_{ij}$ and $\beta_{ij}$. The expansion coefficients in (7) are called Bogoliubov coefficients. They were first used to study particle creation in the expanding universe by Parker \cite{16}. We may now substitute (7) and its complex conjugate into (5). Comparison of the result with (4) shows that

$$a_i = \sum_j (\alpha_{ji}b_j + \beta^*_{ji}b^\dagger_j).$$  \hspace{1cm} (8)

We have demanded that $a_i$ and $a^\dagger_j$ satisfy (1), whereas $b_i$ and $b^\dagger_j$ satisfy (6). However, for a spacetime whose metric is of the form given in (3), the Bogoliubov coefficients are diagonal. Therefore we restrict ourselves to the case where

$$\alpha_{ij} = \alpha_i\delta_{ij}$$  \hspace{1cm} (9)

$$\beta_{ij} = \beta_i\delta_{ij}$$  \hspace{1cm} (10)
where there is no sum over the repeated indices. Substitution of (8) and its complex conjugate into (1), and using (6) leads to

$$
\delta_{ij} = (|\alpha_i|^2 - q|\beta_i|^2)\delta_{ij} + \alpha_i\beta_j(b_ib_j - qb_jb_i) + \beta_i^*\alpha_j^*(b_i^\dagger b_j^\dagger - qb_j^\dagger b_i^\dagger) \\
+ (1 - q'q)\beta_i^*\beta_j b_i b_j + (q' - q)\alpha_i\alpha_j^* b_i^\dagger b_i.
$$

(11)

This differs considerably from the standard case since it is an operator relation. In the cases $q' = q = \pm 1$, terms in (11) which involve operators automatically vanish on account of the commutation or anticommutation of $b_k$ with $b_l$ and $b_k^\dagger$ with $b_l^\dagger$, corresponding to the requirement that the states be symmetric or antisymmetric under particle interchange. For $|q| < 1$, the requirement that (11) holds imposes extra conditions to be satisfied which are not found in the usual case.

Analogously to (2) we define a vacuum state in the out-region by

$$
b_i|0, \text{out}> = 0.
$$

(12)

( Similarly $a_i|0, \text{in}> = 0$ defines a vacuum state in the in-region. ) Taking the expectation value of (11) with $|0, \text{out}>$ leads to

$$
1 = |\alpha_i|^2 - q|\beta_i|^2.
$$

(13)

This is the obvious extension of the results of Parker [12,13,16] to general $q$ and diagonal Bogoliubov transformation. Using (13) in (11) gives

$$
0 = \alpha_i\beta_j(b_ib_j - qb_jb_i) + \beta_i^*\alpha_j^*(b_i^\dagger b_j^\dagger - qb_j^\dagger b_i^\dagger) + (1 - q'q)\beta_i^*\beta_j b_i b_j + (q' - q)\alpha_i\alpha_j^* b_i^\dagger b_i.
$$

(14)

Because the operators which appear on the RHS of (14) are independent of each other, the coefficients must vanish separately. Another way to see this is to take the expectation value of (14) in states other than the vacuum. If we define

$$
|kl, \text{out}> = b_k^\dagger b_l^\dagger |0, \text{out}>,
$$

(15)

then it is easy to show, using (6), that
\(< k, l, \text{out} | m, n, \text{out} > = \delta_{km} \delta_{ln} + q' \delta_{lm} \delta_{kn}. \quad (16)\)

It is also easy to show that the states defined in (15) are linearly independent, since if we have

\[ \sum_{k,l} c_{kl} | k, l, \text{out} > = 0 \quad (17) \]

for some coefficients \( c_{kl} \) then (16) may be used to obtain

\[ c_{mn} + q' c_{nm} = 0, \quad (18) \]

from which it follows that

\[ (1 - q'^2) c_{mn} = 0. \quad (19) \]

Since we assume \( |q'| < 1 \) the coefficients \( c_{mn} \) must vanish.

If we now operate with (14) on \( |0, \text{out} > \) and use (12) and (15) we have

\[ 0 = \beta_i^* \alpha_j^* (|ij, \text{out} > - q|ji, \text{out} >). \quad (20) \]

Using the linear independence of the states \( |ij, \text{out} > \), it then follows that

\[ \beta_i^* \alpha_j^* = 0 \quad (21) \]

The remaining relation that follows from (14) is

\[ 0 = (q' - q) \alpha_i \alpha_j^* b_j^\dagger b_i + (1 - q' q) \beta_i^* \beta_j b_i^\dagger b_j. \quad (22) \]

In particular (22) must hold for \( i = j \). Then we obtain

\[ 0 = \{(q' - q) |\alpha_i|^2 + (1 - q' q) |\beta_i|^2\} b_i^\dagger b_i. \quad (23) \]

Since \( b_i^\dagger b_i \) cannot vanish as an operator ( e.g. take the expectation value of (23) with the state \(|i, \text{out} > \) ) then

\[ 0 = (q' - q) |\alpha_i|^2 + (1 - q' q) |\beta_i|^2. \quad (24) \]
Now let us suppose that at least one of the $\alpha_i$ vanishes. Then, from (13), we cannot have $\beta_i = 0$. Thus we find $q'q = 1$ which contradicts $|q| < 1$, $|q'| < 1$. Therefore none of the $\alpha_i$ can vanish.

This in turn demands, from (21), that all of the $\beta_i$ must vanish and since $\alpha_i \neq 0$ that $q' = q$. However, the requirement that all $\beta_i = 0$ also leads to a contradiction. It implies that positive frequency solutions in the in-region remain positive frequency in the out-region even if the universe undergoes a dynamical evolution. As in the normal case, it corresponds to the absence of particle creation. However, for a spacetime of the type given in (3), the argument given in [12] holds, and shows that in general $\beta_i \neq 0$. (A number of spacetimes leading to $\beta_i \neq 0$ are known; see [17], [18] for references.)

In conclusion we have shown that the assumptions $|q| < 1$ and $|q'| < 1$ lead to an unavoidable contradiction in a dynamically evolving universe. There are only two possible alternatives. The first is to start with $q = +1$ (for bosons) or $q = -1$ (for fermions) which is already covered by Parker’s work [12], [13]. It is necessary for $q' = +1$ (for bosons) or $q' = -1$ (for fermions). The second possibility is to start with $q' = +1$ (for bosons) or $q' = -1$ (for fermions). However, since the dynamical evolution of the universe in this proof is arbitrary, one may simply reverse time in Parker’s work to see that $q = +1$ (for bosons) or $q = -1$ (for fermions). Hence the consistent evolution of the generalized commutation relations from the in-region to the out-region may only occur if $q = q' = \pm 1$. We expect that an argument similar to that given above may be presented for quons in a background electromagnetic field which allows the possibility of particle creation. In addition to problems pointed out by Greenberg [1], we expect that our paper essentially rules out a quantum field theory based on quons.

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