ON WEAK REDUCING DISKS FOR THE UNKNOT
IN 3-BRIDGE POSITION

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Abstract. We show that the complex of weak reducing disks for the unknot in 3-bridge position is contractible.

1. Introduction

Let $S^3$ be decomposed into two 3-balls $V$ and $W$ with common boundary sphere $S$. Let $K$ be an unknot in $n$-bridge position with respect to $S$. That is, $V \cap K$ and $W \cap K$ are collections of $n$ boundary parallel arcs in $V$ and $W$ respectively. We call $(S^3, K) = (V, V \cap K) \cup (W, W \cap K)$ a (genus-0) $n$-bridge splitting and $S$ an $n$-bridge sphere. Each arc of $V \cap K$ and $W \cap K$ is called a bridge.

An $n$-bridge sphere is unique for every $n$ [3], but to understand the mapping class group or deep structure on topological minimality we need to study the disk complex of the bridge sphere.

The disk complex $\mathcal{D}$ for $V - K$ is a simplicial complex defined as follows.

- Vertices of $\mathcal{D}$ are isotopy classes of compressing disks for $S - K$ in $V - K$.
- A collection of $k + 1$ vertices forms a $k$-simplex if there are representatives for each that are pairwise disjoint.

Some important complexes equivalent to subcomplexes of a disk complex such as sphere complex, primitive disk complex are proved to be useful to understand the genus two Heegaard splitting of $S^3$ [5], [1]. In this paper, we consider the complex of weak reducing disks for the unknot in 3-bridge position. The complex of weak reducing disks is interesting in that it is equivalent to the complex of cancelling disks (Lemma 2.1), which is reminiscent of the primitive disk complex mentioned above. It is a natural question whether the complex is connected and contractible or not. Using a criterion in [1], we show that it is contractible.

Theorem 1.1. The complex of weak reducing disks for the unknot in 3-bridge position is contractible.

In Section 2 we consider the relationship between weak reducing disks and cancelling disks and some necessary lemmas. In Section 3 we give a proof of Theorem 1.1.

2. Weak reducing disks and cancelling disks

Let $K$ be an unknot in 3-bridge position with respect to $V \cup W$. A properly embedded disk $D$ in $V - K$ is a compressing disk if $\partial D$ is an essential simple closed curve in $S - K$. An embedded disk $\Delta$ in $V$ is a bridge disk if $\partial \Delta$ is a union of two arcs $a$ and $b$ with $a \cap b = \partial a = \partial b$,
where \( a = \Delta \cap K \) and \( b = \Delta \cap S \). A compressing disk in \( W - K \) and a bridge disk in \( W \) are defined similarly.

A compressing disk \( D \) cuts off a 3-ball \( B \), which contains one bridge, from \( V \). We can take a unique bridge disk \( \Delta \) in \( B \). Conversely, for a bridge disk \( \Delta \), the frontier of a neighborhood of \( \Delta \) in \( V \) is a compressing disk \( D \). So there is a one-to-one correspondence between the set of compressing disks \( \mathcal{C} \) and the set of bridge disks \( \mathcal{B} \). Let \( d : \mathcal{C} \rightarrow \mathcal{B} \) be the bijection defined by \( d(D) = \Delta \). A similar bijection \( \overline{d} : \overline{\mathcal{C}} \rightarrow \overline{\mathcal{B}} \) exists between the set of compressing disks \( \overline{\mathcal{C}} \) in \( W - K \) and the set of bridge disks \( \overline{\mathcal{B}} \) in \( W \).

A compressing disk \( D \) is a weak reducing disk if there is a compressing disk \( E \subset W - K \) such that \( \partial D \cap \partial E = \emptyset \). A bridge disk \( \Delta \) is a cancelling disk if there is a bridge disk \( \overline{\Delta} \subset W \) such that \( \Delta \cap \overline{\Delta} = \{ \text{one point of } K \} \). Here \((\Delta, \overline{\Delta})\) is called a cancelling pair. Figure 1 illustrates weak reducing disks and cancelling disks.

![Figure 1. Weak reducing disks and cancelling disks](image1.png)

Let \( \Delta \) be a bridge disk in, say \( V \), with \( \Delta \cap K = a \) and \( \Delta \cap S = b \). An isotopy of \( a \) to \( b \) along \( \Delta \) and further, slightly into \( W \) is called a reduction. See Figure 2. A reduction along \( \Delta \) yields a 2-bridge position of \( K \) if and only if there is a bridge disk \( \overline{\Delta} \) in \( W \) such that \((\Delta, \overline{\Delta})\) is a cancelling pair [2].

![Figure 2. A reduction along \( \Delta \)](image2.png)

**Lemma 2.1.** A compressing disk \( D \) is a weak reducing disk if and only if \( \Delta \) is a cancelling disk.

**Proof.** Suppose that \( E \subset W - K \) is a compressing disk disjoint from \( D \). Let \( \Delta = d(D) \) and \( \Delta' = \overline{d}(E) \). A simultaneous reduction along \( \Delta \) and \( \Delta' \) results in a 1-string decomposition of...
K, i.e. $K = K_1 \# K_2$ for some knots $K_1$ and $K_2$. Because $K$ is an unknot, both $K_1$ and $K_2$ are unknots $[4]$. Thus we can easily see that a reduction along the single $\triangle$ results a 2-bridge position of $K$. By $[2$, Theorem 1.1$]$, there exists a bridge disks $\overline{\triangle}$ in $W$ such that $(\triangle, \overline{\triangle})$ is a cancelling pair.

Conversely, suppose that $(\triangle, \overline{\triangle})$ is a cancelling pair. A cancellation along $\triangle \cup \overline{\triangle}$ yields a 2-bridge position of $K$. Take a compressing disk $E$ in $W - K$ for the 2-bridge position. We recover the original 3-bridge position by giving a perturbation using $\triangle$ and $\overline{\triangle}$: it is done in such a way that $E$ is disjoint from $\triangle \cup \overline{\triangle}$. Hence $D = d^{-1}(\triangle)$ is a weak reducing disk. □

**Lemma 2.2.** For a weak reducing disk $D \subset V - K$, there exists a weak reducing disk $D' \subset V - K$ that is disjoint from $D$.

**Proof.** Let $\triangle = d(D)$ be the cancelling disk corresponding to $D$ and let $(\triangle, \overline{\triangle})$ be a cancelling pair. Since $(\triangle, \overline{\triangle})$ is a cancelling pair, we can take a compressing disk $D' \subset V - K$ disjoint from $\triangle \cup \overline{\triangle}$ as in the proof of Lemma 2.1. Then $D'$ is a weak reducing disk because it is disjoint from the compressing disk $d^{-1}(\triangle)$ in $W - K$, and $D'$ is disjoint from $D$ also. □

The following fact is well known, so we omit the proof here.

**Lemma 2.3.** There is a unique compressing disk (up to isotopy) for a knot in 2-bridge position.

A collection of six cancelling disks $\{\triangle_1, \triangle_2, \ldots, \triangle_6\}$ is called a complete cancelling disk system if $(\triangle_i, \triangle_{i+1})$ $(i = 1, \ldots, 5)$ and $(\triangle_6, \triangle_1)$ are cancelling pairs and $\triangle_i \cap \triangle_j = \emptyset$ for other pairs of $\{i, j\}$.

We remark that a complete cancelling disk system consisting of four cancelling disks for an unknot in 3-bridge position can be defined similarly.

**Lemma 2.4.** If $\triangle$ and $\triangle'$ are disjoint cancelling disks in $V$, then $\{\triangle, \triangle'\}$ extends to a complete cancelling disk system (Figure 3).

![Figure 3. A complete cancelling disk system extending $\{\triangle, \triangle'\}$](image)

**Proof.** Since $\triangle$ is a cancelling disk, a reduction along $\triangle$ yields an unknot in 2-bridge position. After the reduction, there exists a cancelling pair $(\triangle', \overline{\triangle})$ by uniqueness of the compressing disk and bridge disk for a knot in 2-bridge position (Lemma 2.3). By $[2$, for any bridge in $W$ a bridge disk can be chosen so that it does not intersect the rectangular region $R$ in $W$ (as in Figure 2 and Figure 3) that are created after the reduction along $\triangle$. In particular, $\overline{\triangle}$ does not intersect $R$. Then do the reverse operation of the reduction along $\triangle$. We have disjoint $\triangle$ and $(\triangle', \overline{\triangle})$.

By cancelling along $(\triangle', \overline{\triangle})$ as in Figure 3 we get a 2-bridge unknot. Take a complete cancelling disk system containing $\triangle$ for the 2-bridge position. Then do the reverse isotopy of
the cancellation, i.e. a perturbation, to get the cancelling pair \((\Delta', \overline{\Delta'})\) back additionally. We obtained a complete cancelling disk system containing \(\{\Delta, \Delta', \overline{\Delta'}\}\) for the original 3-bridge position.

\[\text{\(\square\)}\]

Figure 4. A cancellation along \((\Delta', \overline{\Delta'})\)

Let \(\Delta_x, \Delta_y, \Delta_z\) be three disjoint bridge disks in \(W\). Let \(x = \Delta_x \cap S, y = \Delta_y \cap S,\) and \(z = \Delta_z \cap S\). By removing a small open neighborhood \(\eta(K)\) of \(K\) from \(W\), we obtain a genus three handlebody. In order to prove the main theorem in the coming section, we would like to express a simple closed curve \(\gamma\) in \(S - K\) as a word in terms of the three generators of \(\pi_1(W - \eta(K))\). An oriented loop in \(S - K\) passing one of the simple arcs \(x, y, z\) once becomes a representative of the corresponding generator of \(\pi_1(W - \eta(K)) \cong \pi_1(W - K)\). Therefore, the simple closed curve \(\gamma\) can be represented by a word \(w\) in \(x, y, z\).

Lemma 2.5. If an essential simple closed curve \(\gamma\) in \(S - K\) bounds a disk in \(W - K\), then the word \(w\) of \(\gamma\) is reduced to an empty word.

Proof. Suppose that \(w\) is not reduced to an empty word. Then \(w\) represents a non-trivial element in the free group \(\pi_1(W - K)\). It contradicts that \(\gamma\) bounds a disk in \(W - K\).

Lemma 2.6. If an essential simple closed curve \(\delta\) contains three subarcs connecting \(x\) to \(y\), \(y\) to \(z\), and \(z\) to \(x\), then there is no compressing disk in \(W - K\) that is disjoint from \(\delta\).

Proof. Let \(E\) be a compressing disk in \(W - K\). If \(E\) is disjoint from \(\Delta_x \cup \Delta_y \cup \Delta_z\), then \(\partial E\) is a simple closed curve that does not bound a disk in \(P = S - (x \cup y \cup z)\), which is homeomorphic to a 3-punctured sphere. Then the three subarcs of \(\delta\) is an obstruction for \(\partial E\) to be disjoint from \(\delta\). See Figure 5. Suppose that \(E\) intersects \(\Delta_x \cup \Delta_y \cup \Delta_z\). We may assume that \(E \cap (\Delta_x \cup \Delta_y \cup \Delta_z)\) consists of arc components. Consider an outermost disk \(C\) in \(E\) cut off by an outermost arc of \(E \cap (\Delta_x \cup \Delta_y \cup \Delta_z)\). Then \(C \cap P\) is an essential arc in \(P\) such that the two endpoints of \(C \cap P\) are on the same component of \(\partial P\). Again, the three subarcs of \(\delta\) is an obstruction to \(\partial E\). \(\square\)
Figure 5. An obstruction to a weak reducing disk in $W - K$

3. Proof of Theorem 1.1

3.1. Setting. Let $D$ and $F$ be weak reducing disks in $V - K$ such that $D \cap F \neq \emptyset$. We assume that $|D \cap F|$ is minimal up to isotopy. By [1, Theorem 4.2], it is enough to show that one of the disks obtained by surgery of $D$ along an outermost disk in $F$ cut off by an outermost arc of $D \cap F$ is a weak reducing disk. Let $\Delta = d(D)$. By an isotopy of $D$, we assume that $\partial D$ equals the boundary of a small neighborhood of $\Delta \cap S$ in $S$. Let $\alpha_o$ be an outermost arc of $D \cap F$ in $F$ and let $C$ be the corresponding outermost disk in $F$ cut off by $\alpha_o$. Let $\alpha = C \cap S$.

First we consider the special case that there exists another weak reducing disk $D'$ in $V - K$ disjoint from $D$ and $C$. Let $\Delta' = d(D')$. We isotope $D'$ so that $\partial D'$ equals the boundary of a small neighborhood of $\Delta' \cap S$ in $S$. Let $P$ and $P'$ be the disks that $\partial D$ and $\partial D'$ bound, containing $\Delta \cap S$ and $\Delta' \cap S$ respectively. The arc $\alpha$ cuts off an annulus $A$ from $S - (P \cup P')$. Let $\beta$ be an essential arc of $A$ disjoint from $\alpha$. We give $\beta$ an orientation from $\partial D'$ to $\partial D$. We isotope $\alpha$ so that $\alpha$ equals the frontier of a small neighborhood of $\beta \cup P'$ in $\text{cl}(S - P)$. See Figure 6.

Using Lemma 2.4, extend $\{\Delta, \Delta'\}$ to a complete cancelling disk system $\{\Delta, \Delta', \Delta_b, \Delta_x, \Delta_y, \Delta_z\}$, where $\Delta, \Delta', \Delta_b$ are in $V$ and $\Delta_x, \Delta_y, \Delta_z$ are in $W$. Let $b = \Delta_b \cap S$, $x = \Delta_x \cap S$, $y = \Delta_y \cap S$, $z = \Delta_z \cap S$. We assume that the arcs $x, y, z, b$ satisfy the following (Figure 7).

- $x$ intersects only a point of $\partial D$.
- $y$ intersects only a point of $\partial D'$.
- $z$ intersects a point of $\partial D$ and a point of $\partial D'$.
- $b$ intersects none of $\partial D$ and $\partial D'$.

We shrink each of $P$ and $P'$ to a point. Then $\beta$ can be regarded as a properly embedded arc in a 4-punctured sphere. An isotopy class of a properly embedded arc with different endpoints in a 4-punctured sphere is completely determined by its slope. A slope $s \in \mathbb{Q} \cup \{\infty\}$ is expressed as a fraction $\frac{p}{q}$ of relatively prime integers $p$ and $q$. The oriented arc $\beta$ starts at the lower left vertex and ends at the upper left vertex. Note that for the slope $s = \frac{p}{q}$ of $\beta$, $p$ is odd and $q$ is even. See Figure 8 for an example of $s = \frac{3}{8}$. Without loss of generality, we may assume that $s$ is positive.

Each time $\beta$ passes through an arc, it is given a generator among $x^\pm 1, y^\pm 1, z^\pm 1, b^\pm 1$. So the arc $\beta$ can be written as a reduced word $w$ in $x, y, z, b$. We divide cases according to the slope.
s = \frac{p}{q} and investigate the word w. We show that either a surgery of D along the outermost disk C yields a weak reducing disk, or actually F is not a weak reducing disk.

When \( s = \infty = \frac{1}{0} \), \( \beta \) is equal to \( z \). In this case, the disk obtained by a surgery of D along C is a weak reducing disk. If \( s > 1 \), then \( \alpha \) contains all three types of subarcs connecting \( x \) to \( y \), \( y \) to \( z \), and \( z \) to \( x \). So a compressing disk in \( W - K \) disjoint from F cannot exist by Lemma 2.6. See Figure 9. The slope \( s \) cannot be \( 1 = \frac{1}{1} \). Hence we assume that \( 0 < s < 1 \).

If \( \frac{1}{2} < s < 1 \), then \( w \) begins with \( bx^{-1} \), so \( \alpha \) contains all three types of subarcs as above. Suppose that \( s = \frac{1}{2k} \) for some natural number \( k \). Then we can see that a surgery of D along C yields a weak reducing disk. See Figure 10 for an example of \( k = 2 \). Thus from now on, for \( s = \frac{p}{q} \) with \( p \) odd and \( q \) even, we assume that \( \frac{2p}{q} < 1 \) and \( p > 1 \). For such a slope \( s \), there exists a natural number \( k \) satisfying one of the following.

Case (a). \( \frac{2kp}{q} < 1 < \frac{(2k+1)p}{q} \)
Case (b). \(\frac{(2k+1)p}{q} < 1 < \frac{(2k+2)p}{q}\)

For each case, there are two subcases according to when a multiple of \(\frac{p}{q}\) exceeds 2. We list the beginning subword of \(w\) in each subcase together.

Subcase 1. \(\frac{4kp}{q} < 2 < \frac{(4k+1)p}{q}\), \(w\) begins with \(w_1 = (bz^{-1})^k x (b^{-1}z)^k y^{-1}b\).

Subcase 2. \(\frac{(4k+1)p}{q} < 2 < \frac{(4k+2)p}{q}\), \(w\) begins with \(w_2 = (bz^{-1})^k x (b^{-1}z)^k b^{-1}y\).

Subcase 3. \(\frac{(4k+2)p}{q} < 2 < \frac{(4k+3)p}{q}\), \(w\) begins with \(w_3 = (bz^{-1})^k bx^{-1}z (b^{-1}z)^k y^{-1}b\).

Subcase 4. \(\frac{(4k+3)p}{q} < 2 < \frac{(4k+4)p}{q}\), \(w\) begins with \(w_4 = (bz^{-1})^k bx^{-1}z (b^{-1}z)^k b^{-1}y\).
In each subcase, let $p_i$ be the intersection point of $\beta \cap (x \cup y \cup z \cup b)$ corresponding to the last generator of $w_i$. Let $s_i$ be the short arc of $(x \cup y \cup z \cup b) - \alpha$ containing $p_i$. Let $R_i$ be the disk region of $S - (\alpha \cup s_i)$ which contains $P'$.

Let $G$ be a compressing disk in $W - K$ disjoint from $F$. Let $w$ be the word of $\partial G$ in $x, y, z$. By lemma 2.5, $w$ is freely reduced to the empty word. In any case, since $\alpha$ contains subarcs connecting $x$ to $z$ and $y$ to $z$, a cancellation of two adjacent generators of $w$ is $zz^{-1}$ or $z^{-1}z$.

The cancellation and the subsequent cancellations take place in one of the following three ways.

(i) in the interior of $R_i$.

(ii) when $\partial G$ passes through $s_i$.

(iii) in the complementary region of $R_i$.

However, the arc $\alpha$ is an obstruction to $\partial G$, and we will show that actually $G$ cannot exist in any case.

In the interior of $R_i$, only a $k$-times nested cancellation of $zz^{-1}$ occurs in $w$. See Figure 11 for an example of $k = 2$. The left of Figure 11 is Case (a) and the right is Case (b). Subwords of $w$ in the left and right are $z(z(z z^{-1}z^{-1}x)z^{-1}x^{-1}$ and $z(z(z z^{-1}z^{-1}z z^{-1} z^{-1}) z^{-1}x^{-1}$ respectively. The thick solid blue curve represents the first cancellation $zz^{-1}$ and the two thin solid blue curves represent the subsequent cancellation $z(z z^{-1}z^{-1}$. The dotted curve represents that a cancellation does not occur any more.

When $\partial G$ passes through $s_i$, a cancellation does not occur as can be seen in Figure 12. The left of Figure 12 is when $w_i$ ends with $y^{-1}b$ (Subcases 1 and 3) and the right is when $w_i$ ends with $b^{-1}y$ (Subcases 2 and 4). Subwords of $w$ depicted in Figure 12 are $zy^{-1}z^{-1}$ and $zyz^{-1}$, hence a cancellation does not occur.

Now consider a cancellation in the complementary region of $R_i$. See Figure 13–16 for an example of $k = 2$. (For simplicity, we draw $\beta$ instead of $\alpha$.) In Subcase 1, the subword of $w$ is $z^{-1}(z^{-1}z z^{-1}y^{-1}$. So only a 2-times nested cancellation of $z^{-1}z$ occurs. The subwords of $w$ in Subcases 2, 3, 4 are $z^{-1}(z^{-1}z z y^\pm 1$, $z^{-1}(z^{-1}z^{-1}z z y^\pm 1$, $z^{-1}(z^{-1}z^{-1}z z y^\pm 1$, $z^{-1}(z^{-1} z z y^\pm 1$. 
respective. In general, only a $k$-times nested cancellation of $z^{-1}z$ occurs in $\overline{w}$. So we conclude that $G$ cannot exist.

Finally we show that there exists a weak reducing disk $D'$ in $V - K$ disjoint from $D$ and $C$. Choose a weak reducing disk $D'$ disjoint from $D$ using Lemma 2.2. If $C$ is disjoint from $D'$, then we are done. Suppose that $C \cap D' \neq \emptyset$. Consider an outermost arc $\gamma$ of $C \cap D'$ in $C$ such that the outermost disk $C'$ in $C$ cut off by $\gamma$ is disjoint from $D$. One of the disks obtained by surgery of $D'$ along $C'$, denoted by $D''$, is isotopic to neither $D$ nor $D'$. (The other is isotopic to $D$.) By the argument of the preceding special case, $D''$ is a weak reducing disk. Note that $|C \cap D''| < |C \cap D'|$. By repeating the argument with $D''$ instead of $D'$, we get the desired weak reducing disk disjoint from $D$ and $C$. 
Figure 13. A cancellation in the complementary region of $R_i$ — Subcase 1

Figure 14. A cancellation in the complementary region of $R_i$ — Subcase 2

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Figure 15. A cancellation in the complementary region of $R_i$ — Subcase 3

Figure 16. A cancellation in the complementary region of $R_i$ — Subcase 4

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