A MULTIPLIER INCLUSION THEOREM ON PRODUCT DOMAINS

ODYSSEAS BAKAS

Abstract. In this note it is shown that the class of all multipliers from the d-parameter Hardy space $H^1_{prod}(T^d)$ to $L^2(T^d)$ is properly contained in the class of all multipliers from $L\log^{d/2} L(T^d)$ to $L^2(T^d)$.

1. Introduction

Let $d$ be a positive integer. If $X$ is a subspace of $L^1(T^d)$, then we denote by $\mathcal{M}_{X \to L^2(T^d)}$ the class of all multipliers from $X$ to $L^2(T^d)$, namely the class $\mathcal{M}_{X \to L^2(T^d)}$ consists of all functions $m : \mathbb{Z}^d \to \mathbb{C}$ such that for every $f \in X$ one has

$$\sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d)\hat{f}(k_1, \ldots, k_d)|^2 < \infty.$$ 

In [1], it was shown that the class of all multipliers from the (real) Hardy space $H^1(T)$ to $L^2(T)$ is properly contained in the class of all multipliers from $L \log^{1/2} L(T)$ to $L^2(T)$. Our goal in this note is to extend this result to the multi-parameter setting. First of all, note that if $H^1_{prod}(T^d)$ denotes the $d$-parameter (real) Hardy space over the $d$-torus, then $L \log^d L(T^d) \subset H^1_{prod}(T^d)$ and hence, one automatically has $\mathcal{M}_{H^1_{prod}(T^d) \to L^2(T^d)} \subset \mathcal{M}_{L \log^d L(T^d) \to L^2(T^d)}$. On the other hand, by adapting the argument given in [1] to the multi-parameter case, one deduces that the best we can expect is that the multiplier inclusion (1.1) is obtained by a series of reductions. First, arguing as in [1] and by using D. Oberlin’s characterisation of the class $\mathcal{M}_{H^1_{prod}(T^d) \to L^2(T^d)}$ given in [2], it follows that the proof of (1.1) is reduced to showing the following higher-dimensional version of an inequality due to Zygmund (see Theorem 7.6 in Chapter XII of [1]), a result of independent interest. To state this version of Zygmund’s inequality on $T^d$, let $\mathcal{J}$ denote the set of all “intervals” of integers of the form $\pm \{2^n - 1, \ldots, 2^n + 1 - 2\}$, $n \in \mathbb{N}_0$; in other words, $\mathcal{J}$ consists of all the sets in $\mathbb{Z}$ of the form $\{2^k - 1, \ldots, 2^{k+1} - 2\}$, $k \in \mathbb{N}_0$ and $\{-2^l + 2, \ldots, -2^l + 1\}$, $l \in \mathbb{N}_0$.

Proposition 2. Let $\mathcal{J}$ be as above.

If $E \subset \mathbb{Z}^d$ is a non-empty set satisfying the condition

$$D_E = \sup_{I_1, \ldots, I_d \in \mathcal{J}} \# \{ E \cap (I_1 \times \cdots \times I_d) \} < \infty,$$

then

$$\mathcal{M}_{H^1_{prod}(T^d) \to L^2(T^d)} \subset \mathcal{M}_{L \log^d L(T^d) \to L^2(T^d)}.$$
then there exists a positive constant \( A_{DE} \), depending only on \( D_E \), such that

\[
\left( \sum_{(k_1, \ldots, k_d) \in E} |\tilde{f}(k_1, \ldots, k_d)|^2 \right)^{1/2} \leq A_{DE} \left[ 1 + \int_{\mathbb{T}^d} |f| \log^{d/2}(1 + |f|) \right]. \tag{1.3}
\]

In turn, (1.3) will be a corollary of a higher-dimensional extension of a result due to Seeger and Trebels [12] concerning sharp bounds of sums involving “smooth” Littlewood-Paley projections on \( \mathbb{T}^d \). To state this result, fix a Schwartz function \( \eta \) supported in \((-2, 2)\) such that \( \eta|_{[-1, 1]} = 1 \) and consider \( \phi(\xi) = \eta(\xi) - \eta(2\xi) \). For \( k \in \mathbb{N} \), set \( \phi_k(\xi) = \phi(2^{-k}\xi) \) and for \( k = 0 \), set \( \phi_0 = \eta \). One can easily see that \( \sum_{k \in \mathbb{N}_0} \phi_k(\xi) = 1 \) for every \( \xi \in \mathbb{R} \). Then, for \( k \in \mathbb{N}_0 \), the corresponding “smooth” Littlewood-Paley projection in the periodic setting is defined by

\[
\Delta_k(f)(x) = \sum_{r \in \mathbb{Z}} \phi_k(r) \hat{f}(r)e^{2\pi i rx}
\]

for any, say, trigonometric polynomial \( f \) on \( \mathbb{T} \). On the \( d \)-torus we put

\[
\Delta_{k_1, \ldots, k_d}(f)(x_1, \ldots, x_d) = \Delta_{k_1}(f)(x_1) \otimes \cdots \otimes \Delta_{k_d}(f)(x_d)
\]

\[
= \sum_{r_1, \ldots, r_d \in \mathbb{Z}} \phi_{k_1}(r_1) \cdots \phi_{k_d}(r_d) \hat{f}(r_1, \ldots, r_d)e^{2\pi i (r_1 x_1 + \cdots + r_d x_d)}
\]

initially defined over trigonometric polynomials \( f \) on \( \mathbb{T}^d \). Then, Proposition 3 is a consequence of the following result.

**Proposition 3.** There exists a constant \( C_d > 0 \), depending only on the dimension \( d \) and our choice of \( \phi \), such that the following inequality holds

\[
\|f\|_{L^p(\mathbb{T}^d)} \leq C_d p^{d/2} \left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} \|\Delta_{k_1, \ldots, k_d}(f)\|^2_{L^p(\mathbb{T}^d)} \right)^{1/2} \tag{1.4}
\]

for every trigonometric polynomial \( f \) on \( \mathbb{T}^d \) and for each \( p > 2 \).

The proof of Proposition 3 is an adaptation of the work of Seeger and Trebels [12] to the higher-dimensional setting combined with a well-known inequality on multiple martingales, see section 2.2. At this point, it should be mentioned that, in fact, we expect that

\[
\|f\|_{L^p(\mathbb{T}^d)} \leq p^{d/2} \left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} \|\Delta_{k_1, \ldots, k_d}(f)\|^2 \right)^{1/2} \left. \right|_{L^p(\mathbb{T}^d)}
\]

which, of course, implies (1.4). However, as our primary goal is to establish Theorem 1 and since (1.4) is enough for that purpose, we shall not pursue this in the present note.

The paper is organised as follows. In section 2 we give some notation and background and in section 3 we show how the proof of our multiplier inclusion theorem follows from Proposition 2. In section 4 we prove that Proposition 3 implies Proposition 2 and then, in section 5 we give a proof of Proposition 3. In the last section we briefly present some further applications of our work.

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2. Notation and background

We denote by \( \mathbb{Z} \) the set of integers, by \( \mathbb{N} \) the set of positive integers, and by \( \mathbb{N}_0 \) the set of non-negative integers.

The cardinality of a finite set \( A \) is denoted by \( \#\{A\} \).

If \( X \) and \( Y \) are positive quantities such that \( X \leq CY \), where \( C > 0 \) is a constant, then we write \( X \preceq Y \). To specify the dependence of this constant on some additional parameters \( \alpha_1, \ldots, \alpha_n \) we write \( X \preceq_{\alpha_1, \ldots, \alpha_n} Y \). If \( X \preceq Y \) and \( Y \preceq X \), we write \( X \sim Y \).

In this note, we identify \( \mathbb{T} \) with \([0, 1)\) in the standard way.

2.1. Product Hardy spaces and the class \( \mathcal{M}_{H^1_{\prod}(\mathbb{T}^d) \rightarrow L^2(\mathbb{T})} \). For \( 0 < r < 1 \), let \( P_t \) denote the Poisson kernel on \( \mathbb{T} \) given by \( P_t(x) = (1 - r^2)/(1 - 2r \cos x + r^2) \), \( x \in \mathbb{T} \). For \( x \in \mathbb{T} \), let \( \Gamma(x) = \{ z \in \mathbb{D} : |z - e^{2\pi i}x| \leq 2(1 - |z|) \} \), where \( \mathbb{D} \) denotes the unit disc in the complex plane. Then, the \( d \)-parameter (real) Hardy space \( H^1_{\prod}(\mathbb{T}^d) \) consists of all integrable functions \( f \) on the \( d \)-torus such that \( f^\ast \in L^1(\mathbb{T}^d) \), where for \( (x_1, \ldots, x_d) \in \mathbb{T}^d \) one has

\[
\int_{\mathbb{R}} f^\ast(x_1, \ldots, x_d) = \sup_{z \in \Gamma(x_1), \ldots, z \in \Gamma(x_d)} \left| \int f^\ast (P_{x_1} \otimes \cdots \otimes P_{x_d})(z_1, \ldots, z_d) \right|.
\]

It follows by the work of D. Oberlin \( [7] \) that \( m : \mathbb{Z}^d \rightarrow \mathbb{C} \) belongs to the class \( \mathcal{M}_{H^1_{\prod}(\mathbb{T}^d) \rightarrow L^2(\mathbb{T})} \) if and only if,

\[
\sum_{N_1, \ldots, N_d \in \mathbb{N}_0} \sum_{N_1 \leq |k_1| \leq 2N_1} \cdots \sum_{N_d \leq |k_d| \leq 2N_d} |m(k_1, \ldots, k_d)|^2 < \infty. \tag{2.1}
\]

2.2. Dyadic square functions. If \( f \in L^1(\mathbb{T}) \) and \( m \in \mathbb{N}_0 \), then the \( m \)-th conditional expectation of \( f \) is given by

\[
\mathbb{E}_m(f)(x) = 2^m \int_I f(x') dx',
\]

where \( I \) is the unique dyadic interval in \( \mathbb{T} \) of the form \( I = [s2^{-m}, (s + 1)2^{-m}) \), \( s = 0, 1, \ldots, 2^m - 1 \) such that \( x \in I \).

For \( m \in \mathbb{N} \), let \( \mathbb{D}_m = \mathbb{E}_m - \mathbb{E}_{m-1} \) denote the martingale differences acting on functions defined on \( \mathbb{T} \). For \( m = 0 \), we set \( \mathbb{D}_0 = \mathbb{E}_0 \).

For a given \( d \)-tuple \( (m_1, \ldots, m_d) \) of non-negative integers, we define

\[
\mathbb{E}_{m_1, \ldots, m_d} = \mathbb{E}_{m_1} \otimes \cdots \otimes \mathbb{E}_{m_d}
\]

and

\[
\mathbb{D}_{m_1, \ldots, m_d} = \mathbb{D}_{m_1} \otimes \cdots \otimes \mathbb{D}_{m_d} = (\mathbb{E}_{m_1} - \mathbb{E}_{m_1-1}) \otimes \cdots \otimes (\mathbb{E}_{m_d} - \mathbb{E}_{m_d-1})
\]

to be the corresponding operators acting on functions on the \( d \)-torus.

In \( [3] \), Chang, Wilson, and Wolff obtained the “good-\( \lambda \)” inequality

\[
|\{ x \in \mathbb{T} : \sup_{m \in \mathbb{N}_0} |\mathbb{E}_m f(x)| > 2\lambda \} \cap \{ x \in \mathbb{T} : (\sum_{m \in \mathbb{N}_0} |\mathbb{D}_m f(x)|^2)^{1/2} < \epsilon \lambda \} | \leq C_0 \exp\left[ -\frac{(1 - \epsilon)^2}{2\epsilon^2} \right] |\{ x \in \mathbb{T} : \sup_{m \in \mathbb{N}_0} |\mathbb{E}_m f(x)| > \lambda \}|,
\]

which holds for all \( \lambda > 0 \) and \( \epsilon > 0 \), where \( C_0 > 0 \) is an absolute constant. In particular, this estimate implies that there exists a constant \( C > 0 \) such that

\[
\| f \|_{L^p(\mathbb{T})} \leq C p^{1/2} \left( \sum_{m \in \mathbb{N}_0} |\mathbb{D}_m f(x)|^2 \right)^{1/2} \| f \|_{L^p(\mathbb{T})} \tag{2.2}
\]

for all \( p > 2 \). By using \( (2.2) \), Chang, Wilson, and Wolff obtained in \( [3] \) an inequality analogous to \( (2.2) \) involving Lusin area integrals. In \( [3] \), Pipher extended \( (2.2) \) and
its analogous version on Lusin area integrals to the two-parameter setting and in \[3\], R. Fefferman and Pipher extended the aforementioned inequality of Chang, Wilson, and Wolff involving Lusin area integrals to \(L^2\)-valued functions. The argument of R. Fefferman and Pipher can be easily adapted to obtain an \(L^2\)-valued extension of \((2.2)\), see \[3\]. By using this \(L^2\)-valued extension of \((2.2)\) together with induction on \(d\), one deduces that there exists a constant \(C_d > 0\), depending only on the dimension \(d \in \mathbb{N}\), such that

\[
\|f\|_{L^p(T^d)} \leq C_d d^{d/2} \left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} |D_{m_1, \ldots, m_d}(f)|^2 \right)^{1/2} \|f\|_{L^p(T^d)} \tag{2.3}
\]

for every \(p > 2\), see also, e.g., \[1\] Proposition 4.5 and \[3\].

2.3. Thin sets in Harmonic Analysis. Let \(G\) be a compact abelian group and let \(\Lambda\) be a non-empty set in its dual \(\widehat{G}\). In this note, we shall only consider the case \(G = \mathbb{T}^d, d \in \mathbb{N}\). A trigonometric polynomial \(f\) on \(G\) whose spectrum lies in \(\Lambda\) is said to be a \(\Lambda\)-polynomial.

Let \(p > 2\). We say that \(\Lambda \subset \widehat{G}\) is a \(\Lambda(p)\) set if there exists a constant \(A(\Lambda, p) > 0\) such that

\[
\|f\|_{L^p(G)} \leq A(\Lambda, p) \|f\|_{L^2(G)}
\]

for every \(\Lambda\)-polynomial \(f\). The smallest constant \(A(\Lambda, p)\) such that the above inequality holds is called the \(\Lambda(p)\) constant of \(\Lambda\).

A set \(\Lambda \subset \widehat{G}\) is called Sidon if there is a constant \(S_\Lambda > 0\) such that

\[
\sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)| \leq S_\Lambda \|f\|_{L^2(G)} \tag{2.4}
\]

for every \(\Lambda\)-polynomial. It follows by the work of Rudin \[11\] and Pisier \[9\] that a spectral set \(\Lambda\) is Sidon if and only if, it is a \(\Lambda\)-polynomial.

Let \(q \geq 1\). A set \(\Lambda \subset \widehat{G}\) is said to be \(q\)-Rider if there is a constant \(R_{\Lambda, q} > 0\) such that

\[
\left( \sum_{\gamma \in \Lambda} |\widehat{f}(\gamma)|^q \right)^{1/q} \leq R_{\Lambda, q} \|f\|
\]

for every \(\Lambda\)-polynomial. Here, we use the notation \(\|f\| = \mathbb{E}\left[\|\sum_{\gamma \in \widehat{G}} r_\gamma \widehat{f}(\gamma)\|_{L^2(G)}\right]\), where \((r_\gamma)\) denotes the set of Rademacher functions.

It is well-known that if \(\Lambda\) is a \(\Lambda(p)\) set for all \(p > 2\) with \(\Lambda(p)\) constant growing like \(p^{1/2}\), \(k \in \mathbb{N}\), then \(\Lambda\) is a \(q\)-Rider set with \(q = 2k/(k+1)\), see \[10\] Théorème 6.3.

3. Proposition \[2\] implies Theorem \[1\]

To prove that Proposition \[2\] implies Theorem \[1\] we adapt the argument given in \[1\] to the multi-parameter setting by using the characterisation of \(\mathcal{M}_{H^1_{\text{prod}}(T^d)} \to L^2(T^d)\). To be more specific, assume that Proposition \[2\] holds and take an arbitrary \(m\) in the class \(\mathcal{M}_{H^1_{\text{prod}}(T^d)} \to L^2(T^d)\). Then, by definition, we need to show that for every \(f \in L \log^{d/2} L(T^d)\) one has

\[
\sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d) \widehat{f}(k_1, \ldots, k_d)|^2 < \infty.
\]

Towards this aim, fix an \(f \in L \log^{d/2} L(T^d)\) and note that the sum

\[
\sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d) \widehat{f}(k_1, \ldots, k_d)|^2
\]
is bounded by
\[ \sum_{I_1, \ldots, I_d \in \mathcal{J}} \max_{(k_1, \ldots, k_d) \in I_1 \times \ldots \times I_d} |\hat{f}(k_1, \ldots, k_d)|^2 \left( \sum_{k_1 \in I_1} \cdots \sum_{k_d \in I_d} |m(k_1, \ldots, k_d)|^2 \right), \]
where \( \mathcal{J} \) is as in the introduction and the statement of Proposition \( \ref{prop:multiplier-inclusion} \). Hence, by (\ref{eq:multiplier-inclusion}), it follows that
\[ \sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d)|^2 \leq m \sum_{(\tilde{k}_1, \ldots, \tilde{k}_d) \in E_f} |\hat{\tilde{f}}(\tilde{k}_1, \ldots, \tilde{k}_d)|^2, \]
where \( E_f \) is a set in \( \mathbb{Z}^d \) defined as follows. Given \( I_1, \ldots, I_d \in \mathcal{J} \), choose \( (\tilde{k}_1, \ldots, \tilde{k}_d) \)
in \( I_1 \times \ldots \times I_d \) so that
\[ |\hat{\tilde{f}}(\tilde{k}_1, \ldots, \tilde{k}_d)| = \max_{(k_1, \ldots, k_d) \in I_1 \times \ldots \times I_d} |\hat{f}(k_1, \ldots, k_d)|. \]
Then, having chosen a set of \( d \)-tuples \( (\tilde{k}_1, \ldots, \tilde{k}_d) \) as above, we define \( E_f = \{(\tilde{k}_1, \ldots, \tilde{k}_d) \in \mathbb{Z}^d : \text{for } I_1, \ldots, I_d \in \mathcal{J}, \ (\tilde{k}_1, \ldots, \tilde{k}_d) \in I_1 \times \ldots \times I_d \text{ being as above}. \}
Notice that as the choice of \( d \)-tuples \( (\tilde{k}_1, \ldots, \tilde{k}_d) \) is not necessarily unique, there might be several choices of sets \( E_f \). We just choose one of them to write
\[ \sum_{I_1, \ldots, I_d \in \mathcal{J}} \max_{(k_1, \ldots, k_d) \in I_1 \times \ldots \times I_d} |\hat{f}(k_1, \ldots, k_d)|^2 = \sum_{(\tilde{k}_1, \ldots, \tilde{k}_d) \in E_f} |\hat{\tilde{f}}(\tilde{k}_1, \ldots, \tilde{k}_d)|^2. \]
Note that any such set \( E_f \) satisfies condition (\ref{eq:multiplier-inclusion}) in Theorem \( \ref{thm:multiplier-inclusion} \) with \( D_{E_f} = 1 \). Therefore, as \( f \in L \log^{d/2} L(\mathbb{T}^d) \), it follows by (\ref{eq:multiplier-inclusion}) that
\[ \sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d)|^2 \leq \infty, \]
as desired.

3.1. Sharpness of (\ref{eq:multiplier-inclusion}). We remark that, in fact, the above argument shows that if \( m \in M_{H^1_{\text{prod}}(\mathbb{T}^d)} \to L^2(\mathbb{T}^d) \), then there is a constant \( C_m > 0 \), depending only on \( m \), such that
\[ \left( \sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d)|^2 \right)^{1/2} \leq C_m \left[ 1 + \int_{\mathbb{T}^d} |f| \log^{d/2}(1 + |f|) \right]. \]
To see that the exponent \( r = d/2 \) in \( L \log^{d/2} L(\mathbb{T}^d) \) in (\ref{eq:multiplier-inclusion}) cannot be improved, we argue as in (\ref{eq:multiplier-inclusion}). More specifically, assume that for some \( r > 0 \) every multiplier from \( H^1_{\text{prod}}(\mathbb{T}^d) \) to \( L^2(\mathbb{T}^d) \) is a multiplier from \( L^r(\mathbb{T}^d) \) to \( L^{2r}(\mathbb{T}^d) \). We shall prove that \( r \geq d/2 \). To this end, for a large positive integer \( N \), take \( f \) to be a trigonometric polynomial on \( \mathbb{T}^d \) given by \( f = V_{2N} \otimes \cdots \otimes V_{2N} \), where \( V_{2N} = 2K_{2N+1} - K_{2N-1} \) denotes the de la Vallée Poussin kernel of order \( 2N \) and \( K_n \) is the Fejér kernel on \( \mathbb{T} \) of order \( n \in \mathbb{N} \). Since \( \|K_n\|_{L^r(\mathbb{T})} = 1 \) and \( \|K_n\|_{L^{2r}(\mathbb{T})} \leq n \), we deduce that
\[ \int_{\mathbb{T}^d} |f(x_1, \ldots, x_d)| \log^r(1 + |f(x_1, \ldots, x_d)|) \, dx_1 \cdots dx_d \leq r, d \, N^r. \]
So, if we take \( M = (m(k_1, \ldots, k_d))_{k_1, \ldots, k_d \in \mathbb{Z}} \) with \( m(k_1, \ldots, k_d) = 1/\sqrt{k_1 \cdots k_d} \) for \( k_1 > 0, \ldots, k_d > 0 \) and \( m(k_1, \ldots, k_d) = 0 \) otherwise, namely when at least one of the coordinates is less or equal than 0, then \( M \in M_{H^1_{\text{prod}}(\mathbb{T}^d)} \to L^2(\mathbb{T}^d) \) and hence,
\[ \left( \sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d)|^2 \right)^{1/2} \leq C_m \]
Since

\[ \left( \sum_{(k_1, \ldots, k_d) \in \mathbb{Z}^d} |m(k_1, \ldots, k_d)\hat{f}(k_1, \ldots, k_d)|^2 \right)^{1/2} \geq \left( \sum_{1 \leq k_1, \ldots, k_d \leq 2^N} \frac{1}{k_1 \cdots k_d} \right)^{1/2} = \prod_{i=1}^d \left( \sum_{1 \leq k_i \leq 2^N} \frac{1}{k_i} \right)^{1/2} = O(N^{d/2}), \]
we see that, by choosing \( N \) to be large enough, we must have \( r \geq d/2 \).

**Remark 4.** A similar argument shows that the Orlicz space \( L^\log d/2 L(\mathbb{T}^d) \) in (1.3) cannot be improved. Indeed, if \( E \) is a set satisfying (1.2), then by making use of the argument presented above, we see that the exponent \( r = d/2 \) in \( L^\log d/2 L(\mathbb{T}^d) \) in the right-hand side of higher-dimensional Zygmund’s inequality (1.3) is sharp.

To show that the inclusion (1.4) is proper, take \( \Lambda \) to be a Sidon set in \( \mathbb{Z} \) that cannot be written as a finite union of lacunary sequences, see [1]. Remark 2.5(3)]. Then \( M = \chi_{\Lambda \times \cdots \times \Lambda} \) belongs to the class \( \mathcal{M}_{L^\log d/2 L(\mathbb{T}^d)} \), see, e.g., [1, Proposition 4]. However, it can be easily checked that \( M = \chi_{\Lambda \times \cdots \times \Lambda} \) does not satisfy (2.4) and hence, we deduce that \( \chi_{\Lambda \times \cdots \times \Lambda} \notin \mathcal{M}_{L^\log d/2 L(\mathbb{T}^d)} \). \( \mathcal{M}_{L^\log d/2 L(\mathbb{T}^d)} \) and \( \mathcal{M}_{L^\log d/2 L(\mathbb{T}^d)} \).

4. Proposition 3 implies Proposition 2

Our goal in this section is to prove that Proposition 3 implies Proposition 2. Towards this aim, take \( E \subset \mathbb{Z}^d \) to be a set satisfying the assumption of Proposition 2 i.e. condition (1.2). Assume first that \( E \) satisfies (1.2) with \( D_E = 1 \). By duality, to prove (1.3), it suffices to show that \( E \) is a \( \Lambda(p) \) set in \( \mathbb{Z}^d \) for every \( p > 2 \) with \( \Lambda(p) \) constant growing like \( p^{d/2} \) as \( p \to \infty \). In other words, it is enough to show that for every \( E \)-polynomial \( f \) one has for every \( p > 2 \),

\[ |f|_{L^p(\mathbb{T}^d)} \leq A_E p^{d/2} |f|_{L^2(\mathbb{T}^d)}, \tag{4.1} \]

where \( A_E \) is an absolute constant, independent of \( p \) and \( f \). As we will see momentarily, if \( D_E = 1 \), then, in fact, \( A_E \) depends only on \( d \) and in particular, it can be taken to be independent of \( E \).

To prove (4.1), fix an \( E \)-polynomial \( f \) and note that for every \( (k_1, \ldots, k_d) \in \mathbb{N}_0^d \) one has by the triangle inequality

\[ \|\hat{\Delta}_{k_1, \ldots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)} \leq \sum_{(r_1, \ldots, r_d) \in E \cap (I_{k_1} \times \cdots \times I_{k_d})} |\phi_{k_1}(r_1) \cdots \phi_{k_d}(r_d)\hat{f}(r_1, \ldots, r_d)| \]

\[ \leq \phi_{(r_1, \ldots, r_d) \in E \cap (I_{k_1} \times \cdots \times I_{k_d})} |\hat{f}(r_1, \ldots, r_d)|, \]

where \( I_{k_i} \) denotes the set \( \mathbb{Z} \cap \{(-2^{k_i}+1, -2^{k_i}-1] \cup [2^{k_i}-1, 2^{k_i}+1]\}, i = 1, \ldots, d \). Observe that, thanks to condition (1.2) for \( D_E = 1 \), the sum

\[ \sum_{(r_1, \ldots, r_d) \in E \cap (I_{k_1} \times \cdots \times I_{k_d})} |\hat{f}(r_1, \ldots, r_d)| \]

consists of at most \( 6^d \) terms and hence,

\[ \|\hat{\Delta}_{k_1, \ldots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)} \leq d, \phi \sum_{(r_1, \ldots, r_d) \in E \cap (I_{k_1} \times \cdots \times I_{k_d})} |\hat{f}(r_1, \ldots, r_d)|^2 \]

and we thus deduce that

\[ \left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} \|\hat{\Delta}_{k_1, \ldots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right)^{1/2} \leq d, \phi \left( \sum_{(r_1, \ldots, r_d) \in E} |\hat{f}(r_1, \ldots, r_d)|^2 \right)^{1/2}. \tag{4.2} \]
Observe that the quantity on the right-hand side of the last inequality equals to \( \|f\|_{L^p(T^d)} \), as \( \text{supp}(f) \subset E \). Hence, (4.1) follows from (1.4) and (4.2) in the case where \( D_E = 1 \). Moreover, note that, in the case where \( D_E = 1 \), the implied constant in (4.2) depends only on the dimension \( d \) and on our choice of \( \phi \) and, in particular, it is independent of \( E \).

In the case where \( D_E > 1 \), write \( f = \sum_{i=1}^{D_E} f_i \), where \( f_i \) are trigonometric polynomials on \( T^d \) such that \( \text{supp}(f_i) \subset E_i \), where \( E = \bigcup_{i=1}^{D_E} E_i \) and \( D_{E_i} = 1 \). Then, by using the triangle inequality and the previous step we have

\[
\|f\|_{L^p(T^d)} \leq \sum_{i=1}^{D_E} \|f_i\|_{L^p(T^d)} \leq A p^{d/2} \sum_{i=1}^{D_E} \|f_i\|_{L^2(T^d)} \leq A D_E p^{d/2} \|f\|_{L^2(T^d)},
\]

since, by our construction and the \( L^2 \)-theory, \( \|f_i\|_{L^2(T^d)} \leq \|f\|_{L^2(T^d)} \) for all \( i = 1, \ldots, D_E \).

5. PROOF OF PROPOSITION 3

To prove Proposition 3, note that, as \( p > 2 \), it follows by Minkowski’s inequality that

\[
\left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} \|D_{m_1, \ldots, m_d} f\|_{L^p(T^d)}^2 \right)^{1/2} \leq \left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} \|D_{m_1, \ldots, m_d} f\|_{L^p(T^d)}^2 \right)^{1/2}.
\]

Moreover, since one trivially has

\[
\left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} \|D_{m_1, \ldots, m_d} f\|_{L^p(T^d)}^2 \right)^{1/2} \leq \left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} \|D_{m_1, \ldots, m_d} f\|_{L^2(T^d)}^2 \right)^{1/2},
\]

we deduce by (2.3) that

\[
\|f\|_{L^p(T^d)} \leq C_d p^{d/2} \left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} \|D_{m_1, \ldots, m_d} f\|_{L^2(T^d)}^2 \right)^{1/2} \quad (5.1)
\]

for all \( p > 2 \). Hence, to prove that (1.4) holds, it suffices, in view of (5.1), to show that

\[
\left( \sum_{m_1, \ldots, m_d \in \mathbb{N}_0} \|D_{m_1, \ldots, m_d} f\|_{L^2(T^d)}^2 \right)^{1/2} \leq d \left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} \|\Delta_{k_1, \ldots, k_d} f\|_{L^2(T^d)}^2 \right)^{1/2}.
\]

This last inequality follows from the next lemma which is a \( d \)-dimensional analogue of [12, Lemma 2.3].

**Lemma 5.** Let \( \delta \) be a Schwartz function that is even, supported in \((-4,4)\) and such that \( \delta |_{[-2,2]} = 1 \).

Define \( \psi(\xi) = \delta(\xi) - \delta(2\xi) \). For \( k \in \mathbb{N} \), put \( \psi_k(\xi) = \psi(2^{-k}\xi) \) and for \( k = 0 \), put \( \psi_0 = \delta \). Consider the operator

\[
\Psi_k(f)(x) = \sum_{r \in \mathbb{Z}} \psi_k(r) \hat{f}(r) e^{i2\pi rx}
\]

acting on functions defined over the torus. For \( k_1, \cdots, k_d \in \mathbb{N}_0 \) we use the notation \( \Psi_{k_1, \cdots, k_d} = \Psi_{k_1} \otimes \cdots \otimes \Psi_{k_d} \).

There exists a constant \( C_d > 0 \), depending only on the dimension \( d \) and on \( \psi \), such that for all \( d \)-tuples of non-negative integers \((m_1, \cdots, m_d)\) and \((k_1, \cdots, k_d)\) one has

\[
\|E_{m_1, \cdots, m_d} \Psi_{k_1, \cdots, k_d} \|_{L^\infty(T^d) \to L^\infty(T^d)} \leq C_d \prod_{j \in A} 2^{m_j - k_j}, \quad (5.2)
\]
where \( A = \{ j \in \{1, \cdots, d\} : m_j < k_j \} \) and

\[
\|D_{m_1, \cdots, m_d} \Psi_{k_1, \cdots, k_d}\|_{L^\infty(\mathbb{T}^d) \to L^\infty(\mathbb{T}^d)} \leq C_d \prod_{j=1}^d 2^{-|k_j-m_j|} \quad (5.3)
\]

In (5.2) we make the convention that if \( A = \emptyset \), then \( \prod_{j \in A} 2^{m_j-k_j} = 1 \).

The proof of Lemma 5 will be given in the next subsection. By using the above lemma and in particular estimate (5.3), one can easily complete the proof of Proposition 2. More precisely, we consider a trigonometric polynomial \( f \) on \( \mathbb{T}^d \) and write \( f = \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \Delta_{k_1, \cdots, k_d}(f) \). For fixed \( \eta \) (and \( \phi \)), if \( \psi \) is as in the statement of Lemma 5 then \( \psi \phi = \phi \) and hence, \( \Psi_{k_1, \cdots, k_d} \Delta_{k_1, \cdots, k_d} = \Delta_{k_1, \cdots, k_d} \). So, by using (5.3), we obtain

\[
\|D_{m_1, \cdots, m_d}(f)\|_{L^\infty(\mathbb{T}^d)} \leq \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \|D_{m_1, \cdots, m_d} \Delta_{m_1, \cdots, m_d}(f)\|_{L^\infty(\mathbb{T}^d)}
\]

\[
\leq \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \|D_{m_1, \cdots, m_d} \Psi_{k_1, \cdots, k_d}\|_{L^\infty(\mathbb{T}^d) \to L^\infty(\mathbb{T}^d)} \|\Delta_{k_1, \cdots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)}
\]

\[
\leq d \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \left( \prod_{j=1}^d 2^{-|m_j-k_j|} \right) \|\Delta_{k_1, \cdots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)}
\]

and it thus follows that

\[
\left( \sum_{m_1, \cdots, m_d \in \mathbb{Z}} \|D_{m_1, \cdots, m_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right)^{1/2} \leq d
\]

\[
\left[ \sum_{m_1, \cdots, m_d \in \mathbb{Z}} \left( \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \left( \prod_{j=1}^d 2^{-|m_j-k_j|} \right) \|\Delta_{k_1, \cdots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right) \right]^{1/2},
\]

where the implied constant depends only on the dimension \( d \). Hence, by Minkowski’s integral inequality,

\[
\left[ \sum_{m_1, \cdots, m_d \in \mathbb{Z}} \left( \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \left( \prod_{j=1}^d 2^{-|m_j-k_j|} \right) \|\Delta_{k_1, \cdots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right) \right]^{1/2} \leq
\]

\[
\sum_{m_1, \cdots, m_d \in \mathbb{Z}} \left( \prod_{j=1}^d 2^{-|m_j|} \right) \left( \sum_{k_1 \geq -m_1} \cdots \sum_{k_d \geq -m_d} \|\Delta_{k_1+m_1, \cdots, k_d+m_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right)^{1/2}
\]

Since we have

\[
\sum_{m_1, \cdots, m_d \in \mathbb{Z}} \left( \prod_{j=1}^d 2^{-|m_j|} \right) \left( \sum_{k_1 \geq -m_1} \cdots \sum_{k_d \geq -m_d} \|\Delta_{k_1+m_1, \cdots, k_d+m_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right)^{1/2} \leq
\]

\[
\left( \sum_{k_1, \cdots, k_d \in \mathbb{Z}} \|\Delta_{k_1, \cdots, k_d}(f)\|_{L^\infty(\mathbb{T}^d)}^2 \right)^{1/2},
\]

the proof of Proposition 3 will be complete once we prove Lemma 5. This will be done in the following subsection.

5.1 Proof of Lemma 5. The proof of this Lemma is a straightforward adaptation of [12, Lemma 2.3] to the multi-parameter setting. For the sake of simplicity, we shall only present the proof of the two-dimensional case. A similar argument establishes the higher-dimensional case.
Let \( \psi \) be as in the statement of Lemma 5. Following (12), we use the notation 
\[ \psi^{(s)}(\xi) = (i2\pi \xi)^s \psi(\xi), \quad s \in \{-1, 0, 1\} \]
and for \( k \in \mathbb{N}_0 \) we put 
\[ \Psi_k^{(s)}(f)(x) = \sum_{r \in \mathbb{Z}} \psi^{(s)}(2^{-k}r)\hat{f}(r)e^{2\pi i r x}. \]
For \( s = 0 \) we write \( \psi^{(0)} = \psi \) and \( \Psi_k^{(0)} = \Psi_k \). Notice that we may write 
\[ \Psi_k^{(s)}(f)(x) = K_k^{(s)} \ast f(x), \]
where \( K_k^{(s)}(x) = \sum_{r \in \mathbb{Z}} \psi^{(s)}(2^{-k}r)e^{2\pi i r x}. \) Our assumption on the support of \( \psi \)
implies that \( K_k^{(s)} \) is in fact a trigonometric polynomial on \( \mathbb{T} \). By using the Poisson
summation formula, see, e.g., Corollary 2.6 in Chapter VII of [13], it is straightforward to see that 
\[ \|K_k^{(s)}\|_{L^1(\mathbb{T})} \lesssim \psi. \]
Therefore, it follows that 
\[ \|\Psi_k^{(s_1)} \ominus \Psi_k^{(s_2)}\|_{L^\infty(\mathbb{T}^2) \to L^\infty(\mathbb{T}^2)} = \|K_k^{(s_1)} \ominus K_k^{(s_2)}\|_{L^1(\mathbb{T})} \lesssim \psi \]
for all \( s_1, s_2 \in \{-1, 0, 1\} \) and we thus deduce that 
\[ \sum_{s_1, s_2 \in \{-1, 0, 1\}} \|\Psi_k^{(s_1)} \ominus \Psi_k^{(s_2)}(f)\|_{L^\infty(\mathbb{T}^2)} \lesssim \|f\|_{L^\infty(\mathbb{T}^2)} \tag{5.4} \]
for all \( k_1, k_2 \in \mathbb{N}_0 \), where the summation is taken with respect to all possible choices of
\( s_1, s_2 \in \{-1, 0, 1\} \).

5.1.1. Proof of condition (5.2) (for \( n = 2 \)). We shall consider two cases; \( A = \emptyset \)
and \( A \neq \emptyset \).

Case 1: \( A = \emptyset \). In this case we have \( m_1 \geq k_1 \) and \( m_2 \geq k_2 \) and (5.2) easily follows
from (5.3), 
\[ \|E_{m_1, m_2}(\Psi_{k_1, k_2})\|_{L^\infty(\mathbb{T}^2) \to L^\infty(\mathbb{T}^2)} \lesssim 1. \]

Case 2: \( A \neq \emptyset \). First, consider the subcase where \( m_1 < k_1 \) and \( m_2 < k_2 \). For
\( (x_1, x_2) \in \mathbb{T}^2 \), we denote by \( I_j = \left[s_j 2^{-m_j}, (s_j + 1)2^{-m_j}\right], s_j \in \{0, 1, \ldots, 2^{m_j} - 1\} \),
the unique dyadic interval in \( \mathbb{T} \) of length \( 2^{-m_j} \) containing \( x_j \) \( (j = 1, 2) \). If we write 
\( I_j = \left[a_j, b_j\right], \) i.e. \( a_j = s_j 2^{-m_j}, b_j = (s_j + 1)2^{-m_j} \), then we have 
\[ E_{m_1, m_2}(\Psi_{k_1, k_2}(f))(x_1, x_2) = \]
\[ 2^{m_1 + m_2} \int_{I_1} \int_{I_2} \sum_{r_1, r_2 \in \mathbb{Z}} \psi^{(2^{-k_1}r_1)}(2^{-k_2}r_2)\hat{f}(r_1, r_2)e^{2\pi i (r_1 x_1 + r_2 x_2)} dr_1 dr_2 = \]
\[ 2^{m_1 + m_2} \sum_{r_1, r_2 \in \mathbb{Z}} \psi^{(2^{-k_1}r_1)}(2^{-k_2}r_2)\hat{f}(r_1, r_2)\left[\frac{e^{2\pi i r_1 a_1} - e^{2\pi i r_1 b_1}}{i2\pi r_1} - \frac{e^{2\pi i r_2 a_2} - e^{2\pi i r_2 b_2}}{i2\pi r_2}\right]. \]

Hence, one can write 
\[ E_{m_1, m_2}(\Psi_{k_1, k_2}(f))(x_1, x_2) = \]
\[ 2^{-k_1 + m_1} 2^{-k_2 + m_2} \left[\psi^{(-1)} \otimes \psi^{(-1)}(f)(b_1, b_2) - \psi^{(-1)} \otimes \psi^{(-1)}(f)(b_1, a_2)\right] \]
\[ - \psi^{(-1)} \otimes \psi^{(-1)}(f)(a_1, b_2) + \psi^{(-1)} \otimes \psi^{(-1)}(f)(a_1, a_2)\]
and thus, by (5.4), we obtain the desired estimate,
\[ \|E_{m_1, m_2}(\Psi_{k_1, k_2})\|_{L^\infty(\mathbb{T}^2) \to L^\infty(\mathbb{T}^2)} \lesssim 2^{m_1 - k_1} 2^{m_2 - k_2}. \]

Next, consider the subcase where \( m_1 < k_1 \) but \( m_2 \geq k_2 \). In this case, for \( (x_1, x_2) \in \]
\( I_1 \times I_2 = [a_1, b_1] \times [a_2, b_2], \) \( I_1, I_2 \) being as in the previous subcase, we have 
\[ E_{m_1, m_2}(\Psi_{k_1, k_2}(f))(x_1, x_2) = \]
\[ 2^{m_1 - k_1} 2^{m_2} \int_{I_2} \left[ \sum_{r_1, r_2 \in \mathbb{Z}} \psi^{(2^{-k_1}r_1)}(2^{-k_2}r_2)\hat{f}(r_1, r_2)e^{2\pi i r_2 x_2'} \left[\frac{e^{2\pi i r_1 b_1} - e^{2\pi i r_1 a_1}}{i2\pi r_1}\right] \right] dr_2', \]
and so, \( \mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) \) can be written as
\[
2^{m_1-k_1}2^{m_2}[\int_{I_2} \psi_{k_1}^{(1)} \otimes \psi_{k_2}(f)(b_1,x_2') - \psi_{k_1}^{(-1)} \otimes \psi_{k_2}(f)(a_1,x_2')dx_2'].
\]
Since the length of \( I_2 \) is equal to \( 2^{-m_2} \), we get
\[
\|\mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^\infty(T^2)} \leq 2^{m_1-k_1} \cdot 2^{m_2} \cdot 2^{2m_2} \cdot 2\|\psi_{k_1}^{(1)} \otimes \psi_{k_2}(f)\|_{L^\infty(T^2)}
\]
and hence, by using (5.3), we have
\[
\|\mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}]\|_{L^\infty(T^2) \to L^\infty(T^2)} \leq 2^{m_1-k_1}.
\]
The subcase where \( m_1 \geq k_1 \) and \( m_2 < k_2 \) is symmetric to the previous one. Therefore, (5.2) is completely shown in the two-dimensional case.

5.1.2. Proof of condition (5.3) (for \( n = 2 \)). We shall consider two cases: \( A = \{1,2\} \) and \( \{1,2\} \backslash A \neq \emptyset \).

Case 1: \( A = \{1,2\} \). In this case we have \( m_1 < k_1 \) and \( m_2 < k_2 \) and (5.3) follows easily from (5.2). Indeed, observe that
\[
\|D_{m_1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^\infty(T^2)} \leq \|\mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^\infty(T^2)} + \|\mathbb{E}_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^\infty(T^2)}
\]
\[
+ \|\mathbb{E}_{m_1,m_2-1}[\Psi_{k_1,k_2}(f)]\|_{L^\infty(T^2)} + \|\mathbb{E}_{m_1-1,m_2-1}[\Psi_{k_1,k_2}(f)]\|_{L^\infty(T^2)}
\]
\[
\leq 2^{m_1-k_1} \cdot 2^{m_2-k_2}\|f\|_{L^\infty(T^2)}
\]
by (5.2), as \( m_1 - 1 < m_1 < k_1 \) and \( m_2 - 1 < m_2 < k_2 \).

Case 2: \( \{1,2\} \backslash A \neq \emptyset \). Assume first that \( A = \emptyset \), that is \( m_1 \geq k_1 \) and \( m_2 \geq k_2 \). By using the definition of \( D_{m_1,m_2} \), we write
\[
D_{m_1,m_2}[\Psi_{k_1,k_2}(f)] = \mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}(f)] - \mathbb{E}_{m_1,m_2-1}[\Psi_{k_1,k_2}(f)]
\]
\[
= \mathbb{E}_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)] - \mathbb{E}_{m_1-1,m_2-1}[\Psi_{k_1,k_2}(f)].
\]
Take \( (x_1,x_2) \in T^2 \) and for \( j = 1,2 \) let \( I_j \) be the dyadic interval in \( T \) of length \( 2^{-m_j} \) containing \( x_j \). Let \( \tilde{I}_j \) denote the dyadic interval of length \( 2^{-m_j+1} \) such that \( x_j \in \tilde{I}_j \). Note that since \( I_1 \) and \( \tilde{I}_2 \) are dyadic intervals with non-empty intersection and \( |I_j| = 2|\tilde{I}_j| \), one has \( I_j \subset \tilde{I}_j \), \( j = 1,2 \). Since
\[
\mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) - \mathbb{E}_{m_1,m_2-1}[\Psi_{k_1,k_2}(f)](x_1,x_2) =
\]
\[
2^{-m_1} \int_{\tilde{I}_1} \left( \mathbb{E}_{m_1\tilde{I}_1}[\Psi_{k_1,k_2}(f)](x_1',x_2) - \mathbb{E}_{m_2-1}[\Psi_{k_1,k_2}(f)](x_1',x_2) \right) dx_1',
\]
by using the mean value theorem for integrals it follows that there exists an \( x_1^{(\alpha)} \in I_1 \) such that
\[
\mathbb{E}_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) - \mathbb{E}_{m_1,m_2-1}[\Psi_{k_1,k_2}(f)](x_1,x_2) =
\]
\[
\mathbb{E}_{m_2}[\Psi_{k_1,k_2}(f)](x_1^{(\alpha)},x_2) - \mathbb{E}_{m_2-1}[\Psi_{k_1,k_2}(f)](x_1^{(\alpha)},x_2).
\]
A similar analysis on \( \mathbb{E}_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)] - \mathbb{E}_{m_1-1,m_2-1}[\Psi_{k_1,k_2}(f)] \) shows that there exists an \( x_1^{(\beta)} \in \tilde{I}_1 \) such that
\[
\mathbb{E}_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) - \mathbb{E}_{m_1-1,m_2-1}[\Psi_{k_1,k_2}(f)](x_1,x_2) =
\]
\[
\mathbb{E}_{m_2}[\Psi_{k_1,k_2}(f)](x_1^{(\beta)},x_2) - \mathbb{E}_{m_2-1}[\Psi_{k_1,k_2}(f)](x_1^{(\beta)},x_2).
\]
Therefore,
\[
D_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = \mathbb{E}_{m_2}[\Psi_{k_1,k_2}(f)](x_1^{(\alpha)},x_2) - \mathbb{E}_{m_2-1}[\Psi_{k_1,k_2}(f)](x_1^{(\alpha)},x_2)
\]
\[
- \mathbb{E}_{m_2}[\Psi_{k_1,k_2}(f)](x_1^{(\beta)},x_2) + \mathbb{E}_{m_2-1}[\Psi_{k_1,k_2}(f)](x_1^{(\beta)},x_2).
\]
If we assume, without loss of generality, that $x_1^{(α)} < x_1^{(β)}$, then by the mean value theorem,
\[ D_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = (x_1^{(β)} - x_1^{(α)})\partial_{x_1} \{ E_{m_2} \Psi_{k_1,k_2}(f) - E_{m_1-1} \Psi_{k_1,k_2}(f) \}(x_1^{(γ)},x_2) \]
for some $x_1^{(γ)} \in (x_1^{(α)},x_1^{(β)})$. One can easily see that
\[ \partial_{x_1} \{ E_{m_2} \Psi_{k_1,k_2}(f) - E_{m_1-1} \Psi_{k_1,k_2}(f) \} = 2^{k_1} (E_{m_2} \Psi_{k_1}^{(1)} \Psi_{k_2}(f) - E_{m_1-1} \Psi_{k_1}^{(1)} \Psi_{k_2}(f)) \]
and so,
\[ D_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = 2^{k_1} (E_{m_2} \Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2) - E_{m_1-1} \Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2)) \]

Hence, by using the definition of $E_{m_1}$ and $E_{m_2-1}$, it follows by the mean value theorem for integrals that there are $x_2^{(α)} \in I_2$ and $x_2^{(β)} \in \tilde{I_2}$ such that
\[ D_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = (x_2^{(β)} - x_2^{(α)})2^{k_2} (\Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2^{(β)}) - \Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2^{(α)})) \]
Without loss of generality we may assume that $x_2^{(α)} < x_2^{(β)}$. Hence, by applying the mean value theorem, we deduce that
\[ D_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = (x_2^{(β)} - x_2^{(α)})2^{k_2} (\Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2^{(β)}) - \Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2^{(α)})) \]
for some $x_2^{(γ)} \in (x_2^{(α)},x_2^{(β)})$. Since $|x_2^{(β)} - x_2^{(α)}| < 2^{-m_2+1}$, we obtain
\[ \|D_{m_1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^∞(\mathbb{T}^2) \to L^∞(\mathbb{T}^2)} \leq 2^{k_1-m_1}2^{k_2-m_2}\|f\|_{L^∞(\mathbb{T}^2)}, \]
as desired.

It only remains to consider the subcase where $m_1 \geq k_1$ and $m_2 < k_2$, the other one ($m_1 < k_1$ and $m_2 \geq k_2$) being symmetric. We need to show that
\[ \|D_{m_1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^∞(\mathbb{T}^2) \to L^∞(\mathbb{T}^2)} \leq 2^{k_1-m_1}2^{k_2-m_2}. \]
To this end, write $D_{m_1,m_2}[\Psi_{k_1,k_2}(f)]$ as
\[ E_{m_1,m_2}[\Psi_{k_1,k_2}(f)] - E_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)] - (E_{m_1,m_2-1}[\Psi_{k_1,k_2}(f)] - E_{m_1-1,m_2-1}[\Psi_{k_1,k_2}(f)]) \]
and handle each of these two terms separately. Take $(x_1,x_2) \in \mathbb{T}^2$ and, for $j = 1, 2$, consider the dyadic intervals $I_j = [a_j, b_j]$ and $\tilde{I_j}$ as above. For the first term, by applying the mean value theorem for integrals, we see that there are $x_1^{(α)} \in I_1$ and $x_1^{(β)} \in \tilde{I_1}$ such that
\[ E_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) - E_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = E_{m_2} \Psi_{k_1,k_2}(f)(x_1^{(α)},x_2) - E_{m_2} \Psi_{k_1,k_2}(f)(x_1^{(β)},x_2). \]

Hence, if we assume that $x_1^{(α)} < x_1^{(β)}$, then by the mean value theorem there is an $x_1^{(γ)} \in (x_1^{(α)},x_1^{(β)})$ such that
\[ E_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) - E_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = (x_1^{(β)} - x_1^{(α)})\partial_{x_1} E_{m_2} \Psi_{k_1,k_2}(f)(x_1^{(γ)},x_2) \]
\[ = (x_1^{(β)} - x_1^{(α)})2^{k_1} E_{m_2} \Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2), \]
Now, by considering the definition of $E_{m_2}$, an explicit calculation shows that
\[ E_{m_1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) - E_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)](x_1,x_2) = (x_1^{(β)} - x_1^{(α)})2^{k_1}2^{m_2-2}k_2[\Psi_{k_1}^{(1)} \Psi_{k_2}(f)(x_1^{(γ)},x_2) - \Psi_{k_1}^{(1)} \Psi_{k_2}^{(-1)}(f)(x_1^{(γ)},x_2)], \]
where $I_2 = [a_2, b_2]$ as above. Since $x_1^{(α)}, x_1^{(β)} \in \tilde{I_1}$, the last expression gives
\[ \|E_{m_1,m_2}[\Psi_{k_1,k_2}(f)] - E_{m_1-1,m_2}[\Psi_{k_1,k_2}(f)]\|_{L^∞(\mathbb{T}^2)} \leq 2^{k_1-m_1}2^{m_2-2}k_2\|f\|_{L^∞(\mathbb{T}^2)}. \]
A similar argument shows that the second term also satisfies
\[ \|E_{m_1,m_2-1}[\Psi_{k_1,k_2}(f)] - E_{m_1-1,m_2-1}[\Psi_{k_1,k_2}(f)]\|_{L^p(T^2)} \lesssim 2^{k_1-m_1}2^{m_2-k_2} \|f\|_{L^p(T^2)} \]
and we thus deduce that
\[ \|D_{m_1,m_2}[\Psi_{k_1,k_2}]\|_{L^p(T^2)} \lesssim 2^{k_1-m_1}2^{m_2-k_2} \]
Hence, the proof of (5.3) for \( n = 2 \) is complete.

6. Some Further Remarks and Applications

6.1. Applications in thin sets. Proposition \( \Delta \) gives examples of \( \Lambda(p) \) sets in \( \mathbb{Z}^d \) whose corresponding \( \Lambda(p) \) constant grows like \( p^{d/2} \) as \( p \to \infty \) and they cannot be written as products of Sidon sets. Moreover, those sets, namely the class of the sets \( E \subset \mathbb{Z}^d \) that cannot be written as \( d \)-fold products of sets in \( \mathbb{Z} \) and satisfy the condition \( \text{sup}_{i_1,\ldots,i_d \in \mathcal{J}} \# \{ E \cap (I_1 \times \cdots \times I_d) \} < \infty \), are examples of \( 2d/(d+1) \)-Rider sets in \( \mathbb{Z}^d \) that cannot be written as products of Sidon sets in \( \mathbb{Z} \).

Note that if \( \Lambda_1, \ldots, \Lambda_d \) are lacunary sequences in \( \mathbb{Z} \), then \( \Lambda_1 \times \cdots \times \Lambda_d \) satisfies (1.2) and we thus recover the well-known fact that \( \Lambda \) whose constant grows like \( p^{d/2} \) as \( p \to \infty \). However, Proposition \( \Delta \) cannot handle spectral sets of the form \( \Lambda_1 \times \cdots \times \Lambda_d \), where \( \Lambda_j \) is a Sidon set that is not a finite union of lacunary sequences \( (j = 1, \ldots, d) \).

6.2. A version of (1.4) for “rough” projections. For \( k \in \mathbb{N} \) consider the classical Littlewood-Paley projections
\[ \Delta_k(f)(x) = \sum_{n=-2^k+1}^{2^k-1} \hat{f}(n) e^{i2\pi nx} + \sum_{n=-2^k+1}^{-2^k-1} \hat{f}(n) e^{i2\pi nx}. \]
For \( k = 0 \), set \( \Delta_0(f)(x) = \hat{f}(0) \). For \( k_1, \ldots, k_d \in \mathbb{N}_0 \) we write
\[ \Delta_{k_1,\ldots,k_d} = \Delta_{k_1} \otimes \cdots \otimes \Delta_{k_d}. \]
Since for every trigonometric polynomial \( f \) on the \( d \)-torus we may write \( f = \sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \Delta_{m_1,\ldots,m_d}(f) \), we have
\[ \tilde{\Delta}_{k_1,\ldots,k_d}(f) = \sum_{m_1,\ldots,m_d \in \mathbb{N}_0} \tilde{\Delta}_{k_1,\ldots,k_d} \Delta_{m_1,\ldots,m_d}(f). \]
Observe that \( \tilde{\Delta}_{k_1,\ldots,k_d} \Delta_{m_1,\ldots,m_d} = 0 \) whenever there exists an index \( j_0 \in \{1, \ldots, d\} \) such that \( |k_j - m_j| > 1 \). We thus deduce that
\[ \|\tilde{\Delta}_{k_1,\ldots,k_d}(f)\|_{L^p(T^d)} \lesssim \sum_{|k_j - m_j| \leq 1 \text{ for all } j \in \{1, \ldots, d\}} \|\tilde{\Delta}_{k_1,\ldots,k_d} \Delta_{m_1,\ldots,m_d}(f)\|_{L^p(T^d)} \lesssim d \sum_{|k_j - m_j| \leq 1 \text{ for all } j \in \{1, \ldots, d\}} \|\Delta_{m_1,\ldots,m_d}(f)\|_{L^p(T^d)}. \]
Therefore,
\[ \left( \sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \|\tilde{\Delta}_{k_1,\ldots,k_d}(f)\|^2_{L^p(T^d)} \right)^{1/2} \lesssim d \left( \sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \|\Delta_{k_1,\ldots,k_d}(f)\|^2_{L^p(T^d)} \right)^{1/2} \]
and hence, it follows by (1.4) that for every trigonometric polynomial \( f \) on \( T^d \) one has
\[ \|f\|_{L^p(T^d)} \lesssim d \frac{p^{d/2}}{d} \left( \sum_{k_1,\ldots,k_d \in \mathbb{N}_0} \|\Delta_{k_1,\ldots,k_d}(f)\|^2_{L^p(T^d)} \right)^{1/2} \] (6.1)
for every $p > 2$. Estimate (6.1) is a multi-parameter version of an inequality due to C. Moore [6]. In particular, we obtain the following multi-parameter extension of [6, Theorem, p.30].

**Corollary 6.** There exist positive constants $c_1(\mathcal{d})$ and $c_2(\mathcal{d})$, depending only on the dimension $\mathcal{d}$, such that whenever

$$
\sum_{k_1, \ldots, k_d \in \mathbb{N}_0} \| \Delta_{k_1, \ldots, k_d}(f) \|_{L^p(\mathbb{T}^d)}^2 < \infty
$$

one has

$$
\int_{\mathbb{T}^d} \exp \left\{ c_1(\mathcal{d}) \left( \frac{|f(x_1, \ldots, x_d)|}{\left( \sum_{k_1, \ldots, k_d \in \mathbb{N}_0} \| \Delta_{k_1, \ldots, k_d}(f) \|_{L^p(\mathbb{T}^d)}^2 \right)^{1/2}} \right)^{2/d} \right\} dx_1 \cdots dx_d < c_2(\mathcal{d}).
$$

**References**

[1] Odysseas Bakas. Variants of the Inequalities of Paley and Zygmuend. *arXiv preprint arXiv:1702.07049*, 2017.

[2] Dmitriy Bilyk. Roth’s orthogonal function method in discrepancy theory and some new connections. In *A panorama of discrepancy theory*, volume 2107 of Lecture Notes in Math., pages 71–158. Springer, Cham, 2014.

[3] S.-Y. A. Chang, J. M. Wilson, and T. H. Wolff. Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.*, 60 (2):217–246, 1985.

[4] Ciprian Demeter and Francesco Di Plinio. Logarithmic $L^p$ bounds for maximal directional singular integrals in the plane. *J. Geom. Anal.*, 24(1):375–416, 2014.

[5] R. Fefferman and J. Pipher. Multiparameter operators and sharp weighted inequalities. *Amer. J. Math.*, 119(2):337–369, 1997.

[6] Charles Nelson Moore. *Some Applications Of Cauchy Integrals On Curves*. 1986. Thesis (Ph.D.)–University of California, Los Angeles.

[7] Daniel M. Oberlin. Two multiplier theorems for $H^1(U^2)$. *Proc. Edinburgh Math. Soc. (2)*, 22(1):43–47, 1979.

[8] Jill Pipher. Bounded double square functions. *Ann. Inst. Fourier (Grenoble)*, 36(2):69–82, 1986.

[9] Gilles Pisier. Ensembles de Sidon et processus gaussiens. *C. R. Acad. Sci. Paris Sér. A-B*, 286(15):A671–A674, 1978.

[10] Gilles Pisier. Sur l’espace de Banach des séries de Fourier aléatoires presque sûrement continues. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 17–18, 33, École Polytech., Palaiseau, 1978.

[11] Walter Rudin. Trigonometric series with gaps. *J. Math. Mech.*, 9:203–227, 1960.

[12] Andreas Seeger and Walter Trebels. Low regularity classes and entropy numbers. *Arch. Math. (Basel)*, 92(2):147–157, 2009.

[13] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.

[14] A. Zygmund. *Trigonometric series. Vol. I, II*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition, 2002.

Room 4606, James Clerk Maxwell Building, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH9 3FD.

E-mail address: o.bakas@sms.ed.ac.uk