Wave scattering of non-planar trifurcated waveguide by varying the incident through multiple regions

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Abstract
This article elucidates a non-planar trifurcated wave scattering problem by considering incident wave through multiple regions of the underlying waveguide. The scattered field potentials in respective duct regions are obtained via mode-matching technique (MMT) thereby dealing with the non-planar surfaces and step discontinuities positively. The scattering effects are determined by allowing the incident wave from each region explicitly and then simultaneously. The distribution of energies subject to different choices of regions for incidents are presented to analyze the effects of varying regions of incidents. It is mentioned that obtained solutions are thoroughly validated thereby displaying the scattering characteristics graphically.

Keywords
Acoustics, scattering, trifurcated, flexible waveguide, mode-matching

Introduction
Reduction of noise produced during industrial and mechanical processes has grabbed considerable attention from researchers in the field of structural acoustics, applied mathematics and engineering. Mechanical devices like combustion engines, modified silencers and fans etc. are primarily responsible for the production of noise which propagates through a network of ducts/channels to the surroundings. These channels can be modeled in terms of Helmholtz or Laplace equations. The elimination of this unwanted noise is a necessary and natural phenomenon. For this purpose, much care is taken in designing these devices. A variety of techniques have been utilized to deal with the propagation of waves which resulted in minimizing the amount of noise to a considerable level. Modern devices like heating, ventilation and air-conditioning (HVAC) systems, silencers and combustion engines etc. are designed in a way to contribute minimum noise to the surroundings. In this context the current study can be taken as an attempt to provide an effective tool to control unwanted noise.

In the absence of geometrical discontinuity, the Wiener-Hopf technique can simply be employed to obtain a solution of the boundary value problem. For example, this technique has been employed by a number of researchers.¹⁻⁴ However, with the inclusion of geometrical discontinuity the Wiener-Hopf method proves to be inappropriate for the solution of such problems. Consequently, one is forced to adopt an

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alternate method, like MMT to tackle such complex nature problems.

In case, when the boundaries of a waveguide are represented by sound-hard/soft or impedance type boundary conditions the problem is categorized as Sturm-Liouville (SL). Usual orthogonality relations can easily recover the unknown scattering coefficients. On the other hand, when the waveguide is described by high order boundary conditions (flexible surfaces) the resulting eigen system, being non-Sturm-Liouville (SL) in nature, requires specific orthogonality relation to provide a complete solution of the considered problem. Lawrie and Abrahams have developed an orthogonality relation for waveguide described by higher order boundary conditions. Jan and Porter has considered a two-dimensional uniform rectangular waveguide with a meta material cavity attached to one wall. The meta material cavity consists of micro-structures in the form of vertically arranged sound-hard channels. It has been shown that the cavity is an excellent absorber over a wide range of frequencies. The bifurcated, trifurcated and pentafurcated waveguide problems have been investigated with planar boundaries by many researchers, for example. A variety of techniques have been employed by various researchers to cope with the problems relating to the minimization of unwanted sound, for example refer to Williams et al. Wang and Huang. Hassan et al. has investigated a pentafurcated waveguide with soft-hard boundaries. Duan and Kirby has employed the finite element method to study acoustic radiations from circular, cylindrical ducts.

Keeping in view the aforementioned studies, this article is designed to analyze the effects of incident wave from different regions of the inlet duct. The article is divided in the following sections. Section 2 includes formulation of the problem in terms of MMT. Section 3 provides mathematical expressions for the unknown amplitudes when the incident wave is introduced from different regions. Power expressions and numerical results in the form of graphs are presented in section 4 and 5, respectively.

**Problem description**

Consider a two-dimensional rectangular trifurcated waveguide consisting of ducts of different materials and widths. Both the walls of the interior region, of the inlet duct, are sound-hard where as the outer boundaries of the lower and upper regions and consists of elastic plates. In terms of dimensional setting of coordinates, the trifurcated waveguide is stretched infinitely along x-axis with finite width. At the interface it is split into two semi-infinite sections. The width of the inlet and outlet ducts is 2b and 2h, respectively. The semi-infinite inlet duct contains three regions \( R_1 := \{ x < 0, \ -a < y < a \} \), \( R_2 := \{ x < 0, \ -b < y < -a \} \), \( R_3 := \{ x < 0, \ a < y < b \} \). Where as the duct on the right-hand side of the interface consists of the region \( R_4 = \{ x > 0, \ -h < y < h \} \). The upper surface and the vertical strip joining the plates with the boundaries of the outlet duct are described by the impedance boundary conditions. The geometrical configuration of the trifurcated waveguide is shown in Figure 1. An incident wave is approaching the interface \( x = 0 \) from negative x-axis which is partially reflected and partially transmitted and thus is propagating in different regions of the trifurcated waveguide. The dimensional field potentials in various regions are given as

\[
\phi(\bar{x}, \bar{y}) = \begin{cases} 
\phi_1(\bar{x}, \bar{y}), & (\bar{x}, \bar{y}) \in R_1 \\
\phi_2(\bar{x}, \bar{y}), & (\bar{x}, \bar{y}) \in R_2 \\
\phi_3(\bar{x}, \bar{y}), & (\bar{x}, \bar{y}) \in R_3 \\
\phi_4(\bar{x}, \bar{y}), & (\bar{x}, \bar{y}) \in R_4.
\end{cases}
\]

Assuming harmonic time dependence \( e^{-i\omega t} \) where \( \omega = ck \) is the angular velocity and \( k \) is the wave number. The boundary value problem is non-dimensionalized with respect to time scale \( \frac{k}{\omega} \) and length scale \( \frac{a}{\omega} \) using the transformation \( \frac{x}{a} = \bar{x}, \frac{y}{a} = \bar{y}, \) and \( \frac{t}{\omega} = \bar{t} \). The non-dimensional fluid potential \( \phi(x, y) \) in the trifurcated waveguide satisfies the Helmholtz’s equation,

\[
(\nabla^2 + 1)\phi(x, y) = 0.
\]

In dimensionless form, the rigid and impedance boundary conditions in the regions \( R_i(i = 1, 2, 3, 4) \) of the inlet and outlet ducts are given as

\[
\frac{\partial \phi_1}{\partial \bar{y}} = 0, \quad \bar{y} = a, -a,
\]

and

\[
p\phi_4 \pm q \frac{\partial \phi_4}{\partial \bar{y}} = 0, \quad \bar{y} = \pm \bar{h}, \quad \bar{x} > 0,
\]

respectively. In regions \( R_2 \) and \( R_3 \), the lower and upper elastic surfaces are described by the boundary conditions,

\[
\left( \frac{\partial^4}{\partial \bar{x}^4} - \mu^4 \right)\phi_{iy} \pm \alpha \phi_i = 0, \quad i = 2, 3 \quad \bar{y} = \pm \bar{b},
\]

where the dimensionless parameters \( \alpha \) and \( \mu \) are defined by

\[
\mu = \sqrt{\frac{12(1 - \nu^2)c^2\rho_p}{k_0^2h^4E}}, \quad \alpha = \frac{12(1 - \nu^2)c^2\rho_p}{k_0^2h^4E},
\]

respectively. The quantities in the above equation (6) are known as fluid loading parameter and plate wave
number, respectively, whereas $E$, $\rho_p$, $\rho_a$, and $\nu$ denote Young’s modulus, plate density, compressible fluid density, and the Poisson’s ratio, respectively. The rigid vertical strip, connecting the elastic surfaces of the inlet duct ($x < 0$) with the rigid boundaries of the outlet ducts ($x > 0$) are given by

\[
r_4 \phi_4 + s \frac{\partial \phi_4}{\partial x} = 0, \quad x = 0,
\]

for $-h \leq y \leq -b$, and $b \leq y \leq h$.

Here $p$, $q$, $r$, and $s$, appearing in equations (4) and (7), are arbitrary constants. Apart from boundary conditions, the velocity potentials propagating in various regions of the trifurcated duct are matched at the interface of waveguide. For this purpose, conditions of continuity of pressure and normal velocity at the interface are employed in the following form,

\[
\phi_4 = \begin{cases} 
0, & x = 0, \ b \leq y \leq h \\
\phi_1, & x = 0, \ -a \leq y \leq a \\
\phi_2, & x = 0, \ -b \leq y \leq -a \\
\phi_3, & x = 0, \ a \leq y \leq b \\
0, & x = 0, \ -b \leq y \leq -h,
\end{cases}
\]

and

\[
\phi_{4x} = \begin{cases} 
-\frac{\partial}{\partial x} \phi_4, & x = 0, \ b \leq y \leq h \\
\phi_{1x}, & x = 0, \ -a \leq y \leq a \\
\phi_{2x}, & x = 0, \ -b \leq y \leq -a \\
\phi_{3x}, & x = 0, \ a \leq y \leq b \\
-\frac{\partial}{\partial x} \phi_4, & x = 0, \ -b \leq y \leq -h.
\end{cases}
\]

The subscript “$x$” in equation (9) represents differentiation with respect to $x$. Furthermore, some extra conditions, termed as “edge conditions” are imposed at the joining point of the elastic plates and the vertical strip. From these conditions we know about the nature of contact of the plates with the vertical strip described by the impedance boundary condition. An appropriate list of edge conditions can be viewed in Nawaz and Lawrie and Malhiuzhinets.

Mode matching solution

Here, a detailed description of the derivation of the eigenfunction expansion in different sections of the trifurcated waveguide is provided. The section is further subdivided on the basis of the region from which the incident wave is introduced. These are discussed in detail in the following.

Incidence through region $R_1$

In this case, the plane incident wave is introduced into the trifurcated duct through the plate-bounded region $R_1$. The velocity potentials in terms of the eigenfunction expansions in various regions of the waveguide are as follows:

\[
\phi_1(x, y) = e^{ix} + \sum_{n=-\infty}^{\infty} A_n \cos[\tau_n(y + a)] e^{-in\lambda x},
\]

\[
\phi_2(x, y) = \sum_{n=-\infty}^{\infty} B_n \cosh[\beta_n(y + a)] e^{-in\lambda x},
\]
\[ \phi_3(x,y) = \sum_{n=0}^{\infty} C_n \cosh[\beta_n(y - a)] e^{-i\beta_n x}, \] (12)

and

\[ \phi_4(x,y) = \sum_{n=0}^{\infty} D_n Y_n(y) e^{i\beta_n x}, \] (13)

where

\[ Y_n(y) = \text{psin} [\gamma_n(y + h)] + q \gamma_n \cos [\gamma_n(y + h)]. \] (14)

The first term of equation (10) indicates the incident wave whereas the coefficients \( A_n, B_n, C_n, \) and \( D_n, \) in equations (10)–(13), are complex amplitudes of the \( n \)th reflected and transmitted fields. Furthermore, \( \nu_n, \eta_n, \) and \( \lambda_n \) are the dimensionless wave numbers in the regions \( R_i (i = 1 \text{ to } 4) \) and are given by

\[ \nu_n = \sqrt{\beta_n^2 + 1}, \quad \eta_n = \sqrt{-r_n^2 + 1}, \quad \text{and} \quad \lambda_n = \sqrt{-\gamma_n^2 + 1}, \]

respectively. The eigenvalues corresponding to the non-orthogonal eigenfunctions in the non-planar regions \( R_2 \) and \( R_3 \) satisfy a non-linear dispersion relation of the form

\[ K(\beta_n) = [\beta_n^2 + 1]^2 - \mu^4 |\beta_n \sinh[\beta_n(b - a)] - a \cosh[\beta_n(b - a)]]. \] (15)

These roots possess certain properties as listed in detail by Nawaz and Lawrie.\(^{20}\) The eigenfunctions of the elastic plate-bounded regions \( R_2 \) and \( R_3, \) being orthogonal in nature, satisfy a generalized orthogonality relation of the form,

\[ \alpha \int_a^b Y_n(\beta_n, y) Y_m(\beta_n, y) \, dy = \delta_{nm} E_n - (\beta_n^2 + \beta_n^4 + 2Y(\beta_n, -b)Y(\beta_n, -b)), \]

(16)

where the prime denotes the differentiation of the corresponding eigenfunction w.r.t. \( y. \) The quantity \( E_n \) in equation (16) can easily be obtained from the dispersion relation (15) as

\[ E_n = \frac{\alpha(b - a)}{2} + \frac{\alpha Y(\beta_m, -b)Y'(\beta_m, -b) + 2(\beta_n^2 + 1)Y'(\beta_m, -b))^2}{2\beta_n^2}. \] (17)

The unknown complex amplitudes \( A_n, B_n, C_n, \) and \( D_n \) in equations (10)–(13) can be calculated from the linear infinite coupled systems of equations arising as a result of applications of the continuity conditions of pressure and normal component of velocity at \( x = 0. \) Thus using equations (10) and (13) together with equation (8) yield

\[ A_m = -\frac{2\delta_m}{\epsilon_m} + \frac{1}{\alpha \epsilon_m} \sum_{n=0}^{\infty} D_n R_{mn}, \] (18)

where

\[ \epsilon_m = \begin{cases} 1, & m \neq 0 \\ 2, & m = 0 \end{cases} \] (19)

and

\[ R_{mn} = \int_{-a}^{a} \cos[\tau_m(y + a)]Y_m \, dy. \] (20)

To obtain a similar expression for \( B_n, \) substituting equations (11) and (13) in equation (8) and using equation (15) gives

\[ B_m = \frac{\beta_m \sinh[\beta_m(-b + a)]}{E_m} [E_1 + (\beta_m^2 + 2)E_2] \]

(21)

\[ + \frac{\alpha}{E_m} \sum_{n=0}^{\infty} D_n P_{mn}, \]

where

\[ P_{mn} = \int_{-b}^{b} \cosh[\beta_m(y + a)]Y_m \, dy, \] (22)

\[ E_1 = \sum_{n=0}^{\infty} B_n \beta_n^3 \sinh[\beta_n(-b + a)], \] (23)

and

\[ E_2 = \sum_{n=0}^{\infty} B_n \beta_n \sinh[\beta_n(-b + a)]. \] (24)

A Similar procedure is adopted to determine \( C_n \) which is given by

\[ C_m = \frac{\beta_n \sinh[\beta_n(b - a)]}{E_m} [E_3 + (\beta_n^2 + 2)E_4] \]

(25)

\[ - \frac{\alpha}{E_m} \sum_{n=0}^{\infty} D_n Q_{mn}, \]

where

\[ Q_{mn} = \int_{a}^{b} \cosh[\beta_m(y - a)]Y_m \, dy, \] (26)

\[ E_3 = \sum_{n=0}^{\infty} C_n \beta_n^3 \sinh[\beta_n(b - a)], \] (27)

and

\[ E_4 = \sum_{n=0}^{\infty} C_n \beta_n \sinh[\beta_n(b - a)]. \] (28)

Finally, the transmitted coefficients \( D_n \) can be obtained from the relation
\begin{align*}
D_m &= \frac{1}{\lambda_m G_m} \left[ \frac{\pi}{2} \sum_{n=-\infty}^{\infty} D_{nm}^2 H_{nm} + R_{nm} - \sum_{n=-\infty}^{\infty} A_n \eta_n R_{nm} \right. \\
&\quad - \sum_{n=-\infty}^{\infty} B_n v_m P_{nm} - \sum_{n=-\infty}^{\infty} C_n v_m Q_{nm} + \frac{i \pi}{2} \sum_{n=-\infty}^{\infty} D_{nm} S_{nm} \right], \\
(29) &
\end{align*}

where \(H_{nm}\) and \(S_{nm}\) are vertical step discontinuities given by

\begin{align*}
H_{nm} &= \int_{-b}^{b} Y_m Y_n dy, \\
(30) &
\end{align*}

and

\begin{align*}
S_{nm} &= \int_{b}^{h} Y_m Y_n dy. \\
(31) &
\end{align*}

The unknown constants \(E_i (i = 1 - 4)\) in equations (21) and (25) actually involve dependent sums which are determined by imposing certain edge conditions at the point of contact of the planar and non-planar boundaries. This helps in determining the physical nature of the two walls of the duct. A detailed discussion about different edge conditions can be found in Nawaz and Lawrie.\(^{20}\) The clamped edge condition has been imposed in this case, which is given by

\begin{align*}
\phi_{iy}(0, b) = 0, \quad \phi_{ijy}(0, b) = 0, \quad i = 3, 6, \\
(32) &
\end{align*}

and

\begin{align*}
\phi_{iy}(0, - b) = 0, \quad \phi_{ijy}(0, - b) = 0, \quad j = 2, 5. \quad (33)
\end{align*}

The values of the unknown constants \(E_i (i = 1 - 8)\) are determined by assuming that the edges of the plate are clamped with the vertical strips. This yields

\begin{align*}
E_2 &= 0 \quad \text{and} \quad E_4 = 0. \quad (34)
\end{align*}

The multiplication of equation (21) by \(v_m \beta_m \sinh[\beta_m (- b + a)]\) and then the use of \(\phi_{2ijy}(0, - b) = 0\) results in

\begin{align*}
&\sum_{m=-\infty}^{\infty} B_m v_m \beta_m \sinh[\beta_m (- b + a)] \\
&= \sum_{m=-\infty}^{\infty} \frac{v_m \beta_m^3 \sinh[\beta_m (- b + a)]}{E_m} [E_1 + (\beta_m^2 + 2)E_2] \\
&\quad + \alpha \sum_{m=-\infty}^{\infty} \frac{v_m \beta_m \sinh[\beta_m (- b + a)]}{E_m} \sum_{n=-\infty}^{\infty} D_{nm} P_{mn}. \\
(35) &
\end{align*}

Using equations (33) and (34) reduce the above equation to

\begin{align*}
E_1 &= - \frac{\alpha}{S_1} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{D_{nm} v_m \beta_m \sinh[\beta_m (- b + a)] P_{mn}}{E_m}, \\
(36) &
\end{align*}

where

\begin{align*}
S_1 &= \sum_{m=-\infty}^{\infty} v_m \beta_m^3 \sinh[\beta_m (- b + a)] E_m. \\
(37) &
\end{align*}

In a similar manner, the unknown \(C_m\) is found by multiplying equation (25) with \(v_m \beta_m \sinh[\beta_m (b - a)]\) and using \(\phi_{3ijy}(0, b) = 0\) which gives

\begin{align*}
&\sum_{m=-\infty}^{\infty} C_m v_m \beta_m \sinh[\beta_m (b - a)] \\
&= \sum_{m=-\infty}^{\infty} \frac{v_m \beta_m^3 \sinh[\beta_m (b - a)]}{E_m} [E_3 + (\beta_m^2 + 2)E_4] \\
&\quad + \alpha \sum_{m=-\infty}^{\infty} \frac{v_m \beta_m \sinh[\beta_m (b - a)]}{E_m} \sum_{n=-\infty}^{\infty} D_{nm} Q_{mn}, \\
(38) &
\end{align*}

by virtue of equations (32) and (34) the above equation reduces to

\begin{align*}
E_3 &= - \frac{\alpha}{S_2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{D_{nm} v_m \beta_m \sinh[\beta_m (b - a)] Q_{mn}}{E_m}. \\
(39) &
\end{align*}

where

\begin{align*}
S_2 &= \sum_{m=-\infty}^{\infty} \frac{v_m \beta_m^3 \sinh^2[\beta_m (b - a)]}{E_m}. \quad (40)
\end{align*}

**Incidence through region \(R_2\)**

In order to investigate the effects of changing the region of incidence we introduce the incident wave from the region \(R_2\). As a result, the eigenfunction expansion in the corresponding region \(R_2\) takes the form

\begin{align*}
\phi_3(x, y) &= F_i \cosh[\beta_i (y + a)] e^{i \omega x} \\
&\quad + \sum_{n=-\infty}^{\infty} C_n \cosh[\beta_n (y + a)] e^{-i \omega x}, \quad (41)
\end{align*}

where the first term of equation (41) represents the incident field. The incident power is scaled at unity by taking \(F_i = \sqrt{\frac{\omega}{k_i}}\). The subscript “\(i\)” determines the mode of the incident wave, when “\(i\)” = 0 the mode is fundamental and when “\(i\)” = 1 the mode is secondary. The velocity potentials in the regions \(R_1\) and \(R_3\) remains unaltered and thus equations (18) and (25) are the same whereas equation (21) is recalculated by utilizing equations (13) and (41) together with equation (8) in the following form

\begin{align*}
B_m &= - F_i \delta_{ml} + \frac{\beta_m \sinh[\beta_m (- b + a)] E_5}{E_m} \\
&\quad + \frac{(\beta_m^2 + 2) \beta_m \sinh[\beta_m (- b + a)] E_6}{E_m} + \frac{\alpha}{E_m} \sum_{n=-\infty}^{\infty} \frac{D_{nm} P_{mn}}{E_m}, \quad (42)
\end{align*}

where
By comparing equations (33) and (43) together with (42) we get $E_0 = 0$ and

$$E_5 = \frac{2}{S_1}[F_i \nu \beta_3 \sinh[\beta_1(y - a)]] - \frac{\alpha}{S_2} \sum_{n = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} \frac{D_n \nu_m \beta_m \sinh[\beta_m(y - b + a)]P_{nm}}{E_m}.$$  

(44)

The values of $P_{nm}$ and $S_1$ are given by (22) and (37) as in the case of incidence through region $R_1$. An expression for transmitted coefficients $D_m$ in the region $R_4$ is obtained by comparing equation (41) along with equations (11)–(13) at the interface and using (9) as

$$D_m = \frac{1}{\lambda_m G_m} \left[ i \sum_{n = -\infty}^{\infty} D_n R_{mn} + F_i \nu \nu_P ml - \sum_{n = -\infty}^{\infty} A_n \nu R_{mn} 
- \sum_{n = -\infty}^{\infty} B_n \nu \nu_P nm - \sum_{n = -\infty}^{\infty} C_n \nu \nu Q_{nm} + \frac{ir}{s} \sum_{n = -\infty}^{\infty} D_n S_{mn} \right].$$  

(45)

### Incidence through region $R_3$

When the incident wave enters the trifurcated waveguide from the left side of the interface $x = 0$ through the region $R_3$, the eigenfunction expansion in the corresponding region takes the following form

$$\phi_3(x, y) = F_3 \cosh[\beta_1(y - a)] e^{i\nu x} + \sum_{n = -\infty}^{\infty} C_n \cosh[\beta_n(y - a)] e^{-i\nu_n x}.$$  

(46)

The velocity potentials in the regions $R_1$ and $R_2$ remains unchanged and, therefore, are not listed. The unknown reflected coefficients $C_m$ can be determined by using equations (13), (46), and (8) as

$$C_m = -F_i \delta_m l + \frac{\beta_m \sinh[\beta_m(b - a)]E_i}{E_m} + \frac{(\beta_m^2 + 2) \beta_m \sinh[\beta_m(b - a)]E_8}{E_m} + \frac{\alpha}{E_m} \sum_{n = -\infty}^{\infty} D_n Q_{mn},$$  

(47)

where

$$E_7 = \phi_{633}(0, b) \quad \text{and} \quad E_8 = \phi_{60}(0, b).$$  

(48)

Comparing equations (32) and (48) yields $E_8 = 0$. To determine the value of $E_7$, we multiply equation (47) with $\nu_m \beta_m \sinh[\beta_m(b - a)]$ and then use equations (32) and (48) which results in,

$$E_7 = \frac{2}{S_2} \left[ F_i \nu \beta_3 \sinh[\beta_1(b - a)] \right] - \frac{\alpha}{S_2} \sum_{n = -\infty}^{\infty} \sum_{m = -\infty}^{\infty} \frac{D_n \nu_m \beta_m \sinh[\beta_m(b - a)]Q_{nm}}{E_m}.$$  

(49)

The value of $S_2$ is the same as in the case of incidence through the region $R_1$. In case, when the incident wave is allowed to enter the waveguide through the region $R_3$, the unknown coefficients $D_m$ can be determined by substituting equations (13) and (46) in (9) as

$$D_m = \frac{1}{\lambda_m G_m} \left[ i \sum_{n = -\infty}^{\infty} D_n R_{mn} + F_i \nu \nu_P ml - \sum_{n = -\infty}^{\infty} A_n \nu R_{mn} 
- \sum_{n = -\infty}^{\infty} B_n \nu \nu_P nm - \sum_{n = -\infty}^{\infty} C_n \nu \nu Q_{nm} + \frac{ir}{s} \sum_{n = -\infty}^{\infty} D_n S_{mn} \right].$$  

(50)

### Incidence through region $R_1$, $R_2$, and $R_3$

As in the case of incidence through regions $R_1$, $R_2$, and $R_3$ simultaneously, we utilize the equations (10), (41), and (46) with equation (9) obtained solution is

$$D_m = \frac{1}{\lambda_m G_m} \left[ i \sum_{n = -\infty}^{\infty} D_n R_{mn} + F_i \nu \nu_P ml - \sum_{n = -\infty}^{\infty} A_n \nu R_{mn} 
- \sum_{n = -\infty}^{\infty} B_n \nu \nu_P nm - \sum_{n = -\infty}^{\infty} C_n \nu \nu Q_{nm} + \frac{ir}{s} \sum_{n = -\infty}^{\infty} D_n S_{mn} \right].$$  

(51)

### Power or energy flux expression

The power fluxes in different regions of the waveguide are calculated in this section. The energy fluxes due to fluid and structure are given by

$$\frac{\partial e}{\partial t} |_{\text{fluid}} = \text{Re} \left\{ i \int \phi \left( \frac{\partial \phi}{\partial x} \right)^* dy \right\},$$  

(52)

and

$$\frac{\partial e}{\partial t} |_{\text{plate}} = \text{Re} \left\{ i \alpha \left( \frac{\partial^2 \phi}{\partial x^2} \frac{\partial \phi}{\partial y} - \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x^2} \right) \right\},$$  

(53)

where the (\(\ast\)) represents the complex conjugate throughout. Using the above expressions, the incident power is found to be $\varepsilon_{inc} = 2a$. The reflected and transmitted powers in different duct sections $\varepsilon_i; i = 1 – 12$ for different incidences are determined in a similar way. In this article our focus is to analyze the effects of change on the scattering coefficients for different
incidence regions so we will calculate expressions for reflected and transmitted powers for all the cases accordingly. When the incident wave is allowed to enter the waveguide through the region $R_1$, the corresponding coefficients are given by

$$\varepsilon_1 = \frac{1}{2} Re \left\{ \sum_{n=0}^{\infty} |A_n|^2 \eta_n e_n \right\}, \quad (54)$$

$$\varepsilon_2 = \frac{1}{2a} Re \left\{ \sum_{n=0}^{\infty} |B_n|^2 \nu_n E_n \right\}, \quad (55)$$

$$\varepsilon_3 = \frac{1}{2a} Re \left\{ \sum_{n=0}^{\infty} |C_n|^2 \nu_n E_n \right\}, \quad (56)$$

and

$$\varepsilon_4 = \frac{h}{2a} Re \left\{ \sum_{n=0}^{\infty} |D_n|^2 \lambda_n G_n \right\}. \quad (57)$$

When the incident wave enters the duct through the region $R_2$, the power expressions are given by

$$\varepsilon_5 = a Re \left\{ \sum_{n=0}^{\infty} |A_n|^2 \eta_n e_n \right\}, \quad (58)$$

$$\varepsilon_6 = \frac{1}{\alpha} Re \left\{ \sum_{n=0}^{\infty} |B_n|^2 \nu_n E_n \right\}, \quad (59)$$

$$\varepsilon_7 = \frac{1}{\alpha} Re \left\{ \sum_{n=0}^{\infty} |C_n|^2 \nu_n E_n \right\}, \quad (60)$$

and

$$\varepsilon_8 = Re \left\{ \sum_{n=0}^{\infty} |D_n|^2 \lambda_n G_n \right\}. \quad (61)$$

It is to be noted that the scattering powers remain the same when the region of incidence is changed from $R_1$ to $R_2$. When the incident wave is introduced into the trifurcated waveguide through region $R_1$, $R_2$, and $R_3$ simultaneously then the powers in different regions are given by:

$$\varepsilon_9 = \frac{a}{2 + 2a} Re \left\{ \sum_{n=0}^{\infty} |A_n|^2 \eta_n e_n \right\}, \quad (62)$$

$$\varepsilon_{10} = \frac{1}{(2 + 2a)} Re \left\{ \sum_{n=0}^{\infty} |B_n|^2 \nu_n E_n \right\}, \quad (63)$$

$$\varepsilon_{11} = \frac{1}{(2 + 2a)} Re \left\{ \sum_{n=0}^{\infty} |C_n|^2 \nu_n E_n \right\}, \quad (64)$$

and

$$\varepsilon_{12} = \frac{1}{2 + 2a} Re \left\{ \sum_{n=0}^{\infty} |D_n|^2 \lambda_n G_n \right\}. \quad (65)$$

### Numerical results and discussions

Here, in this section, we will truncate the algebraic linear system of equations in (54) – (65) to N terms. This results in a system of $N + 1$ equations which are solved in terms of the unknown scattering coefficients. The numerical values of the physical constants have been taken from Kaye and Laby.$^{16}$ In this context, the speed of sound in the air is taken to be 343.5 m/sec while density of the air as 1.2 g/cm$^3$. The density of both the plates is taken to be 2700 kgm$^3$ while thickness is 0.0006 m. The dimensions of the waveguide are stated with each figure in the caption. Figures 2(a) and (b) represent the effects of increasing frequency on the power propagating in the trifurcated duct when the incident wave is introduced through region $R_1$ and the outlet duct is described by the impedance boundary condition. The only difference in the two graphs is that, in Figure 2(b), the geometrical discontinuity is removed by taking $h = \bar{h}$. It is apparent from Figure 2(a) that the entire incident power is reflected in the region $R_1$ until $f = 260$ Hz. In the rest of the regions, there are no reflections and transmissions till this frequency. After this, a gradual increase in the reflections in the region $R_2$ and a decrease in $R_1$ can be seen. The first cut-on mode occurs at a frequency of 350 Hz. There are no transmissions when $1 \leq f \leq 480$. A second cut-on mode is seen at a frequency of 480 Hz. After this, a considerable increase in transmissions occurs. Similarly, a sharp increase in the reflections in the region $R_1$ can be seen thereby dominating the magnitudes of reflections in the other two regions of the waveguide. In Figure 2(b), the removal of geometrical discontinuity reduced the transmitted power to almost zero in the entire considered frequency range. After a frequency of 260 Hz, a gradual decrease/increase in the magnitude of reflected power in the regions $R_1/R_2$ is noticed. The behavior of the reflected power in the regions $R_1$ and $R_2$ is reversed after the first cut-on mode at 600 Hz. No power is propagating in the rest of the regions. In Figure 2(c), we have assumed that the boundaries of outer region $R_4$ are rigid. Here power propagating in various regions of the waveguide is plotted against frequency, $f$, for the clamped edge condition. Maximum amount of the incident energy is reflected back in the middle region $R_1$ or transmitted to the region $R_4$. A sudden increase/decrease is noticed in the reflected and transmitted powers which reaches to a maximum/minimum at a frequency of 270 Hz (being first cut-on mode). Then a small decrease/increase can be seen in the transmitted/ reflected powers. After this the behavior of the graph remains constant. For Figure 3(a) and (b), the
configuration of the trifurcated waveguide remains the same. The reflected power in the region \( R_1 \) dominates the individual powers propagating in the other regions of the waveguide. The identical behavior of the two graphs in Figure 3 reveals that the vertical step discontinuity has no effect on the scattering coefficients. The power balance is successfully achieved in all cases. In Figures 3 and 4, the incident wave enters the trifurcated waveguide through the region \( R_2 \) which consists of an elastic plate-bounded duct. It is evident that almost all, close to 100% of the incident power is reflected back into the region \( R_2 \) in Figures 3(a) and 4(a). Whereas, the power propagating in other regions of the duct is negligible. However, for secondary mode forcing,
reflections and transmissions occur in all regions of the duct (Figure 3(b)). Reflections in the region of incidence dominates reflections in other regions of the duct except for $440 \leq f \leq 480$ in Figure 3(b). The transmitted power reaches to a maximum of 22% in this case. In Figure 4(b), more than 80% of the power propagates in regions $R_1$ and $R_2$. Also a gradual increase/decrease can be seen in the power in these regions, respectively. There is no contribution in the form of transmitted power in Figure 4(b).

In Figures 5 and 6, the incident wave is introduced in the waveguide through the region $R_3$. These Figures are exactly identical to Figures 3 and 4 in terms of magnitude of the power propagating in different regions of the duct. Just like Figures 3(a) and 4(a), almost all of the incident power is reflected back into the region of incidence ($R_3$). Whereas the magnitude of reflected powers in the regions $R_3$ and $R_2$ are interchanged as were in Figures 3 and 4. In Figures 7 and 8 the incident wave is allowed to enter the trifurcated duct through the regions $R_1$, $R_2$, and $R_3$ simultaneously. The geometrical discontinuity has been removed by taking $h = b$ in Figure 7(b). Fundamental mode forcing is considered for Figures 7(a) and 8(a) whereas secondary mode is taken in Figures 8(b) and 9(b). It is evident from 7a that most of the power is reflected in the region $R_2$. The first and second cut-on modes occur at 250 Hz and 480 Hz, respectively. No power is transmitted in the frequency interval $1 \leq f \leq 480$. After the second cut-on mode, the transmitted power starts propagating and reaches to a maximum of 20%. In Figure 7(b), a sudden increase is apparent in the transmitted power after the first cut-on mode. Maximum power is transmitted for this configuration. There are no transmissions for both fundamental and secondary mode forcing in Figures 8(a) and (b). More than 80% of incident power is reflected in the regions $R_2$ and $R_3$ for fundamental mode forcing (Figure 8(a)). For secondary mode forcing (Figure 8(b)), the magnitude of reflections in the region $R_1$ dominates the powers propagating in other regions.
regions of the duct. The first cut-on occurs at a frequency of 310 Hz in this case. In Figure 8(b), no power is transmitted in the entire frequency range.

For convergence purpose, Figure 9(a) and (b) are sketched for scattered powers and their sum versus the truncation number N. It is observed that the scattering components converge appropriately whereas overall sum remains unity. This certainly confirms the accuracy of algebra together with satisfaction of power balance and all structural conditions. Figures 10 and 11 are plotted to validate the solution obtained by MMT for the trifurcated waveguide. It can be seen that the non-dimensional components of pressure and velocity excellently match in the corresponding region of the interface. Though ability of the mode-matching method to handle the underlying problem is not the primary focus, yet a low-frequency approximation is formulated to see how well mode-matching performs in low frequency regime. It is evident from Figure 12, that the results of mode-matching and low frequency approaches contrast well for fundamental mode which corresponds to the fundamental solution and, as such, can and will propagate at low frequencies.

It is worthwhile to note that power distribution pattern in case of pin-jointed edges is quite similar to that of clamped edges. Therefore, the results for pin-jointed edges are not presented. A comparative description of the power propagating in different duct regions of the trifurcated duct is presented in tabular form for different incidence regions (Table 1).

**Conclusion**

Acoustic scattering in a trifurcated waveguide where the inlet and outlet ducts were described by the elastic plate and impedance boundary conditions was investigated with the help of MMT. The incident wave was allowed to enter the waveguide through the regions $R_1$, $R_2$, and $R_3$ and finally through all the three regions simultaneously. It was observed that maximum
Figure 8. Power distribution against frequency $f$ (Hz) with incidence through region $R_1$, $R_2$ and $R_3$ and outlet region is impedance where $N = 10$, $a = 0.03$ m, $b = 0.06$ m, and $h = b$ for: (a) fundamental ($l = 0$) and (b) secondary modes ($l = 1$).

Figure 9. Power distribution against number of terms ($N$) with incidence through region $R_1$ where $f = 350$, $a = 0.03$ m, $b = 0.06$ m and $h = 0.09$ m: (a) outlet region is impedance ($p = q = r = s = 1$), and (b) outlet region is rigid.

Figure 10. Validation of solution for non-dimensional real and imaginary pressures at interface, when $N = 80$, $f = 350$, $a = 0.03$ m, $b = 0.06$ m, and $h = 0.09$ m.
Figure 11. Validation of solution for non-dimensional real and imaginary velocities at interface, when $N = 80$, $f = 350$, $a = 0.03$ m, $b = 0.06$ m, and $h = 0.09$ m.

Table 1. Discontinuous trifurcated waveguide, when $a = 0.03$ m, $b = 0.06$ m, $h = 0.09$ m, $N = 10$ by varying frequency $f$(Hz) taking incidence through different regions.

| Incidence through Regions $l$ | Frequency $f$(Hz) | Energy (Impedence) $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ | $\varepsilon_5$ |
|-------------------------------|-------------------|-----------------------------------|----------------|----------------|----------------|----------------|
| $R_1$ when $l = 0$            | 150               | 0.972523                          | 0.0137385      | 0.0137385      | 0              | 1              |
|                               | 300               | 0.589621                          | 0.205189       | 0.205189       | 0              | 1              |
|                               | 450               | 0.00599807                        | 0.497001       | 0.497001       | 0              | 1              |
|                               | 600               | 0.207076                          | 0.157852       | 0.157852       | 0.47722        | 1              |
|                               | 750               | 0.272076                          | 0.12533        | 0.12533        | 0.477263       | 1              |
| $R_2$ when $l = 0$            | 150               | 0.0137385                         | 0.985318       | 0.000943481    | 0              | 1              |
|                               | 300               | 0.0389598                         | 0.952994       | 0.00804667     | 0              | 1              |
|                               | 450               | 0.0234477                         | 0.953651       | 0.0139018      | 0              | 1              |
|                               | 600               | 0.00628924                        | 0.980609       | 0.00411627     | 0.00898552     | 1              |
|                               | 750               | 0.00354141                        | 0.986708       | 0.00305154     | 0.0069871      | 1              |
| $R_3$ when $l = 1$            | 150               | 0.0949995                         | 0.0649445      | 0.00651838     | 0              | 0.166462       |
|                               | 300               | 0.166229                          | 0.799469       | 0.0343019      | 0              | 1              |
|                               | 450               | 0.464553                          | 0.336234       | 0.199213       | 0              | 1              |
|                               | 600               | 0.151563                          | 0.529398       | 0.0996349      | 0.219404       | 1              |
|                               | 750               | 0.121788                          | 0.536295       | 0.105618       | 0.236298       | 1              |
| $R_3$ when $l = 0$            | 150               | 0.0137385                         | 0.000943481    | 0.985318       | 0              | 1              |
|                               | 300               | 0.0389598                         | 0.00804667     | 0.952994       | 0              | 1              |
|                               | 450               | 0.0324477                         | 0.0139018      | 0.953651       | 0              | 1              |
|                               | 600               | 0.00628924                        | 0.00411627     | 0.980609       | 0.00898552     | 1              |
|                               | 750               | 0.00354141                        | 0.00305154     | 0.986708       | 0.0069871      | 1              |
| $R_3$ when $l = 1$            | 150               | 0.0949995                         | 0.0651838      | 0.0649445      | 0              | 0.166462       |
|                               | 300               | 0.166229                          | 0.0343019      | 0.799469       | 0              | 1              |
|                               | 450               | 0.464553                          | 0.199213       | 0.336234       | 0              | 1              |
|                               | 600               | 0.151563                          | 0.0996349      | 0.529398       | 0.219404       | 1              |
|                               | 750               | 0.121788                          | 0.374457       | 0.374457       | 0.205013       | 1              |
| $R_1$, $R_2$ and $R_3$ when $l = 0$ | 150               | 0.169953                          | 0.415024       | 0.415024       | 0              | 1              |
|                               | 300               | 0.279331                          | 0.360334       | 0.360334       | 0              | 1              |
|                               | 450               | 0.0679673                         | 0.466016       | 0.466016       | 0              | 1              |
|                               | 600               | 0.0175697                         | 0.394637       | 0.394637       | 0.193157       | 1              |
|                               | 750               | 0.0460716                         | 0.374457       | 0.374457       | 0.205013       | 1              |
| $R_1$, $R_2$ and $R_3$ when $l = 1$ | 150               | 0.116239                          | 0.0642087      | 0.0642087      | 0              | 0.244657       |
|                               | 300               | 0.462714                          | 0.268643       | 0.268643       | 0              | 1              |
|                               | 450               | 0.802988                          | 0.0985058      | 0.0985058      | 0              | 1              |
|                               | 600               | 0.0821629                         | 0.0375498      | 0.0375498      | 0.842738       | 1              |
|                               | 750               | 0.0286028                         | 0.0334003      | 0.0334003      | 0.904597       | 1              |
incident power is reflected back in the region of incidence. The reflected power is close to 100\% when fundamental mode forcing is considered for a geometrically discontinuous \((h \neq b)\) structure (Figures 3–6). A considerable amount of the energy is transmitted when the incident wave enters the waveguide through all the three regions \(R_1, R_2,\) and \(R_3\) simultaneously. Figures 4, 6, and 8 reveal the fact that almost entire of the incident acoustic energy is propagating in the inlet ducts \(R_1,\) \(R_2,\) and \(R_3\) in the form of reflections for the entire frequency range. Therefore, it can be concluded from these numerical results that the proposed model can be used as a perfect broadband absorber of the acoustic energy.

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