Mittag–Leffler’s function, Vekua transform and an inverse obstacle scattering problem

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Abstract
This paper studies a prototype of inverse obstacle scattering problems whose governing equation is the Helmholtz equation in two dimensions. An explicit method to extract information about the location and shape of unknown obstacles from the far-field operator with a fixed wave number is given. The method is based on an explicit construction of a modification of Mittag–Leffler’s function via the Vekua transform and the study of the asymptotic behaviour; an explicit density in the Herglotz wavefunction that approximates the modification of Mittag–Leffler’s function in the bounded domain surrounding unknown obstacles; a system of inequalities derived from Kirsch’s factorization formula of the far-field operator. Then an indicator function which can be calculated from the far-field operator acting on the density is introduced. It is shown that the asymptotic behaviour of the indicator function yields information about the visible part of the exterior of the obstacles.

1. Introduction
This paper is concerned with developing an explicit analytical method for the so-called inverse obstacle scattering problems at a fixed wave number. For the purpose we consider an inverse obstacle scattering problem in two dimensions in which the governing equation is given by the Helmholtz equation. The problem is to extract information about the location and shape of unknown sound-hard obstacles $D$ embedded in a medium with constant acoustic speed and density, from the leading term of the asymptotic expansion of the reflected wave $w$ at infinity which is caused by an incident plane wave $e^{i k \cdot d}$ for infinitely many incident directions $d \in S^1$ and a fixed wave number $k > 0$. This is a prototype of several inverse obstacle scattering problems of acoustic wave.
More precisely we assume that $D \subset \mathbb{R}^2$ is open and $\mathbb{R}^2 \setminus \overline{D}$ is connected; $\partial D$ is $C^2$. The reflected wave $w$ is the unique solution of the scattering problem:

$\Delta w + k^2 w = 0$ \quad in $\mathbb{R}^2 \setminus D$,

$\frac{\partial w}{\partial v} = \frac{\partial}{\partial v} e^{ikx} \quad \text{on } \partial D$,

$\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial w}{\partial r} - ikw \right) = 0$,

where $v$ is the unit outward normal relative to $\partial D$ and $r = |x|$. This last condition is called the Sommerfeld radiation condition.

It is well known that given $\varphi \in S^1$ the value $w(r\varphi)$ as $r \to \infty$ has the following form:

$w(r\varphi) = \frac{e^{ikr}}{\sqrt{r}} F_D(\varphi; d, k) + O\left(\frac{1}{r^{3/2}}\right)$.

The coefficient $F_D(\varphi; d, k)$ is called the far-field pattern of $w$.

The operator $F : L^2(S^1) \to L^2(S^1)$ given by the formula

$F_D g(\varphi) = \int_{S^1} F_D(\varphi; d, k) g(d) \, d\sigma(d), \quad g \in L^2(S^1),$

is called the far-field operator. It is well known that the far-field operator for a fixed $k$ uniquely determines the obstacles [20]. In this paper we consider how to extract information about the location and shape of $D$ from the far-field operator or its partial knowledge at fixed $k$.

In [10, 11] the author established the reconstruction formula of $D$ itself from the far-field operator. The formula consists of two parts: a relationship between a suitable Dirichlet-to-Neumann map on the boundary of a domain that contains $\overline{D}$ and the far-field operator; application of the probe method introduced by the author [9] to the Dirichlet-to-Neumann map.

In [25] Potthast gave a reconstruction procedure that he calls the singular sources method. The method yielded a way from the far-field operator to a scattered field outside unknown obstacles which was exerted by a point source outside the obstacles and blows up on the boundary of the obstacles.

Kirsch gave two types of reconstruction formulae of $D$ in [18, 19]. The idea behind the formulae is called the factorization method since the formulae are based on a factorization formula of the far-field operator. In particular, in [19] he made use of the quadratic form

$(F_D g, g)_{L^2(S^1)} = \int_{S^1} (F_D g)(\varphi) \overline{g}(\varphi) \, d\sigma(\varphi)$

acting on densities $g \in L^2(S^1)$ to introduce his indicator function. It is defined by

$K(x) = \inf\{|(F_D g, g)| \mid g \in L^2(S^1), (\Phi_{x}, g)_{L^2(S^1)} = 1\}$

where $\Phi_{x}(\varphi) = e^{-ikx \cdot \varphi}, \varphi \in S^1$. He established the one line formula

$D = \{x \in \mathbb{R}^2 | K(x) > 0\}$.

For applications of his method to obstacles with other boundary conditions see [5, 6].

In [12] in three dimensions the author gave an extraction formula of the convex hull of $D$ with a constraint on the Gaussian curvature of $\partial D$ from a Dirichlet-to-Neumann map calculated from the far-field operator. See also [13] for the sound-soft obstacles. It is an application of the enclosure method introduced by the author [13] and based on the asymptotic behaviour of the function

$v = e^{i(\tau \theta + \sqrt{\tau^2 + k^2} \varphi)}$
having a large parameter $\tau$ where both $\vartheta$ and $\vartheta^\perp$ are unit vectors and perpendicular to each other. This function satisfies the Helmholtz equation $\Delta v + k^2 v = 0$ in the whole space and divides the whole space into two parts: if $x \cdot \vartheta > t$, then $e^{-\tau t} |v| \to \infty$ as $\tau \to \infty$; if $x \cdot \vartheta < t$, then $e^{-\tau t} |v| \to 0$ as $\tau \to \infty$. The indicator function introduced in [12] tells us whether given $t$ the half space $x \cdot \vartheta > t$ touches unknown obstacles.

The aim of this paper is to generalize this result by introducing another indicator function which is given by the form $(F_D g, g)$ acting on explicit densities $g$ on $S^1$ and tells us whether a given cone touches unknown obstacles.

1.1. Statement of the main result and a corollary

In this paper we identify the point $\vartheta = (\vartheta_1, \vartheta_2) \in S^1$ with the complex number $\vartheta_1 + i \vartheta_2$ and denote it by the same symbol $\vartheta$. Let $B_R$ be the open disc radius $R$ centered at the origin.

**Definition 1.1.** Given $n \geq 1$, $N \geq 1$, $s > 0$ and $(y, \omega) \in B_R \times S^1$ define the indicator function

$$I^{1/n}_{y,\omega}(s)_N = (F_D g^{1/n}_{(y,\omega)}(\cdot; s,k)_N, g^{1/n}_{y,\omega}(\cdot; s,k)_N)_{L^2(S^1)}$$

where

$$g^{1/n}_{(y,\omega)}(\varphi; s,k)_N = \frac{e^{-iky \cdot \varphi}}{2\pi} \sum_{m=0}^{nN} \frac{\Gamma(m + 1)}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{\sqrt{s} \omega \varphi}{ik} \right)^m, \quad \varphi \in S^1.$$  \hfill (1.1)

Let $0 < \alpha \leq 1$. Let $C_y(\omega, \pi\alpha/2)$ denote the interior of the cone about $\omega$ of the opening angle $\pi\alpha/2$ with vertex at $y$:

$$C_y(\omega, \pi\alpha/2) = \{ x \in \mathbb{R}^2 | (x - y) \cdot \omega > |x - y| \cos(\pi\alpha/2) \}.$$  

The following theorem is the main result of this paper.

**Theorem 1.1.** Let $k^2$ be not a Neumann eigenvalue of $-\Delta$ in $D$. Assume that $D \subset B_R$. Let $\gamma_0$ be the unique positive solution of the equation $\log t + t/e = 0$. Let $\gamma$ satisfy $0 < \gamma < \gamma_0$. Let $n \geq 1$. Let $\{s(N)\}_{N=1,\ldots}$ be an arbitrary sequence of positive numbers satisfying, as $N \to \infty$,

$$(Rs(N))^\alpha = \frac{\gamma}{e} N + O(1).$$

Then, given $(y, \omega) \in B_R \times S^1$ we have

- if $C_y(\omega, \pi/2n) \cap D = \emptyset$, then $\lim_{N \to \infty} |I^{1/n}_{y,\omega}(s(N))_N| = 0$;
- if $C_y(\omega, \pi/2n) \cap D \neq \emptyset$, then $\lim_{N \to \infty} |I^{1/n}_{y,\omega}(s(N))_N| = \infty$.

Theorem 1.1 is a direct consequence of two lemmas below.

**Lemma 1.1.** There exists a positive constant $C$ such that for all $g \in L^2(S^1)$

$$C^{-1}\|Hg\|_{H^2(B_R \cup \partial D)}^2 \leq |(F_D g, g)_{L^2(S^1)}| \leq C\|Hg\|_{H^2(B_R \cup \partial D)}^2$$

where

$$Hg(x) = \int_{S^1} e^{ikx \cdot \varphi} g(\varphi) \, d\sigma(\varphi), \quad x \in \mathbb{R}^2$$

and is called the Herglotz wavefunction with the density $g$.

In the following lemma both $k$ and $R$ are arbitrary positive numbers.
Lemma 1.2. Given \((y, \omega) \in B_R \times S^1\) we have

if \(C_y(\omega, \pi/2n) \cap \overline{D} = \emptyset\), then \(\lim_{N \to \infty} \|H\bar{g}_{p(y,\omega)}^{1/n}(:s(N), k, \omega)\|_{H^1(D)} = 0\);

if \(C_y(\omega, \pi/2n) \cap D \neq \emptyset\), then \(\lim_{N \to \infty} \|H\bar{g}_{p(y,\omega)}^{1/n}(:s(N), k, \omega)\|_{D^2(\partial D)} = \infty\).

Lemma 1.1 has been pointed out in [5]. It is a corollary of a factorization formula in [18] and a coerciveness of an operator in the formula. A known proof of the coerciveness is given by a contradiction argument (cf. lemma 4.2 in [6]) and therefore not direct. It seems that at the present time, there is no direct proof of this fact.

Lemma 1.2 follows from corollary 2.1 and (3.16) in sections 2 and 3, respectively.

So from theorem 1.1 what information about unknown obstacles was extracted? To answer precisely we formulate the visible part of \(B_R \setminus \overline{D}\).

Definition 1.2. We say that a point \(y\) in \(B_R \setminus \overline{D}\) is visible if the point \(y\) can be connected with infinity by a straight line that started at \(y\) and goes to infinity without intersecting \(\overline{D}\). We denote by \(V(B_R \setminus \overline{D})\) the set of all points in \(B_R \setminus \overline{D}\) that are visible. We call this set the visible part of \(B_R \setminus \overline{D}\).

It is easy to see that the point \(y\) in \(B_R \setminus \overline{D}\) belongs to the visible part of \(B_R \setminus \overline{D}\) if and only if there exist \(n\) and \(\omega \in S^1\) such that \([C_y(\omega, \pi/2n)] \cap \overline{D} = \emptyset\) with \(\alpha = 1/n\). The set \(V(B_R \setminus \overline{D})\) is a non-empty open subset of \(B_R \setminus \overline{D}\). If \(D\) is convex, then so is \(\overline{D}\) and for any \(x \in \mathbb{R}^2\setminus \overline{D}\) one can find a direction \(\varphi \in S^1\) such that \(x \cdot \varphi > \sup_{y \in \overline{D}} y \cdot \varphi\). This yields \(V(B_R \setminus \overline{D}) = B_R \setminus \overline{D}\).

The next theorem tells us that the asymptotic behaviour of the indicator function \(I_{D}^{1/n}(s(N))\) as \(N \to \infty\) for all \(n\) and \((y, \omega) \in B_R \times S^1\) uniquely determines the visible part of \(B_R \setminus \overline{D}\) except for a thin set.

Corollary 1.1. Let \(D_1\) and \(D_2\) be two obstacles such that \(k^2\) be not a Neumann eigenvalue of \(-\Delta\) in \(D_1\) and \(\overline{D}_f \subset B_R\). Assume that for each fixed \(n\) and \((y, \omega) \in B_R \times S^1\) we have

\[
\lim_{N \to \infty} \left( (F_{D_1} - F_{D_2})g_{(y,\omega)}^{1/n}(::s(N), k), g_{(y,\omega)}^{1/n}(::s(N), k) \right)_{L^2(S^1)} = 0.
\]

Then \(V(B_R \setminus \overline{D}_1) \setminus \partial D_2 = V(B_R \setminus \overline{D}_2) \setminus \partial D_1\).

This is derived from theorem 1.1 as follows. It suffices to prove \(V(B_R \setminus \overline{D}_1) \setminus \partial D_2 \subset V(B_R \setminus \overline{D}_2)\). Let \(y \in V(B_R \setminus \overline{D}_1) \setminus \partial D_2\). Then there exist \(n\) and \(\omega \in S^1\) such that \([C_y(\omega, \pi/2n)] \cap \overline{D}_1 = \emptyset\) with \(\alpha = 1/n\). By theorem 1.1 we have

\[
\lim_{N \to \infty} \left( (F_{D_1} - F_{D_2})g_{(y,\omega)}^{1/n}(::s(N), k), g_{(y,\omega)}^{1/n}(::s(N), k) \right)_{L^2(S^1)} = 0.
\]

Since \(y\) does not belong to \(\partial D_2\), it belongs to \(D_2\) or \(B_R \setminus \overline{D}_2\). If \(y \in D_2\), then from theorem 1.1 one has

\[
\lim_{N \to \infty} \left| (F_{D_2}g_{(y,\omega)}^{1/n}(::s(N), k), g_{(y,\omega)}^{1/n}(::s(N), k) \right)_{L^2(S^1)} = \infty, \tag{1.2}
\]

A combination of this and assumption gives

\[
\lim_{N \to \infty} \left| (F_{D_2}g_{(y,\omega)}^{1/n}(::s(N), k), g_{(y,\omega)}^{1/n}(::s(N), k) \right)_{L^2(S^1)} = \infty.
\]

This is a contradiction. So \(y\) has to be in the set \(B_R \setminus \overline{D}_2\). If \([C_y(\omega, \pi/2n)] \cap D_2 \neq \emptyset\), then, from theorem 1.1, we again obtain (1.2) and the same contradiction as above. Thus, \([C_y(\omega, \pi/2n)] \cap D_2 = \emptyset\). Therefore, if one chooses a larger \(n'\) than \(n\), then one gets \([C_y(\omega, \pi/2n')] \cap \overline{D}_2 = \emptyset\) with \(\alpha' = 1/n'\). This means that \(y \in V(B_R \setminus \overline{D}_2)\).
1.2. A brief explanation of the idea

Here we give a brief explanation of the origin of the density $g^{1/n}_{(y,ω)}(\cdot; s, k)_N$.

Finding the density is closely related to Mittag–Leffler’s function $E_α(z)$ which is an entire function and defined by the formula

$$E_α(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(αm + 1)}$$

where $α$ is a parameter and satisfies $0 < α \leq 1$. The function $E_α(τ(x_1 + ix_2))$ of independent variables $x = (x_1, x_2)$ with parameter $0 < τ < \infty$ is harmonic in the whole plane. This function divides the whole plane into two parts as $τ \rightarrow \infty$: in a sector it is exponentially growing; outside the sector decaying algebraically. In [15, 17] we applied this property of the harmonic function to an inverse boundary value problem for an elliptic equation $\nabla \cdot γ\nabla u = 0$ with a discontinuous coefficient $γ$ which is a special, however, very important version of the Calderón problem [3] and a continuum model of electrical impedance tomography.

In section 2 we modify this harmonic function by using the Vekua transform [27, 28] (see also [2, 4]) which transforms the given solution of the Laplace equation in $\mathbb{R}^2$ into that of the Helmholtz equation $Δv + k^2v = 0$ in $\mathbb{R}^2$. Using the solution obtained by the transform, we define a special solution with a large parameter $s > 0$ of the Helmholtz equation which is denoted by $E_s(x; s, k, ω)$. In particular, the function $E_1(x; s, k, ω)$ is the Vekua transform of the harmonic function

$$\exp\left\{ \frac{s}{2}(ω_1 - iω_2)(x_1 + ix_2) \right\}$$

where $ω = (ω_1, ω_2) \in S^1$. We show that the function $E_s(x; s, k, ω)$ has the asymptotic behaviour as $s \rightarrow \infty$ similar to that of the original Mittag–Leffler’s function:

- if $x \in C_0(ω, πα/2)$, then $\lim_{s \rightarrow \infty} |E_s(x; s, k, ω)| = \infty$;
- if $x \in \mathbb{R}^2 \setminus C_0(ω, πα/2)$, then $\lim_{s \rightarrow \infty} |E_s(x; s, k, ω)| = 0$.

In section 3 we establish the relationship between the density $g^{1/n}_{(y,ω)}(\cdot; s, k)_N$ and the function $E_s(x - y; s, k, ω)$ for $α = 1/n$ and $y \in \overline{B}_R$:

$$H^s g^{1/n}_{(y,ω)}(\cdot; s(N), k)_N(x) \approx E_1(x - y; s(N), k, ω), \quad x \in \overline{B}_R,$$

as $N \rightarrow \infty$. Thus, one can say that $g^{1/n}_{(y,ω)}(\cdot; s, k)_N$ and $s(N)$ are chosen in such a way that the corresponding Herglotz wavefunction approximates a modification of Mittag–Leffler’s function.

It should be pointed out that the result in [16] is closely related to the construction of the density. Therein the author considered the case when $D$ is polygonal. This means that $D$ has the expression $D = D_1 \cup \cdots \cup D_m$ with $1 \leq m < \infty$ where $D_1, \ldots, D_m$ are simply connected open sets, polygons, and satisfy $\overline{D}_j \cap \overline{D}_j^c = \emptyset$ for $j \neq j'$. $D$ is not $C^2$; however, using a variational method (cf [8]), one obtains the unique solvability of the scattering problem and can formulate the far-field pattern $F_D(\cdot; d, k)$ of the reflected wave. The observation data are given by $F_D(\cdot; d, k)$ for fixed $d$ and $k$ provided we know the disc $B_R$ that contains $\overline{D}$. Using the enclosure method [14], we established a direct extraction formula of the convex hull of $D$ from the quantity

$$\int_{S^1} F_D(\cdot; d, k)g(\psi) \, d\sigma(\psi)$$

for some explicit densities $g$ independent of $D$. One of key points is the choice of the densities. Those are chosen in such a way that

$$Hg(x) \approx e^{i(τx + iτ^2s + ikω)}, \quad x \in \overline{B}_R,$$
where \( \omega^\perp = (-\omega_2, \omega_1) \). However, to obtain more than the convex hull of unknown obstacles the function on the right-hand side is not enough. In this paper we give explicitly the desired function by using the idea of the Vekua transform and Mittag–Leffler’s function.

Finally we point out that there are other approaches with a single incident plane wave: the point source method [24], the no response test [22], the range test [26] and the notion of the scattering support [21].

### 2. Modified Mittag–Leffler’s function

In this section we introduce a modification of \( E_\alpha(\tau(x_1 + ix_2)) \) with \( 0 < \tau < \infty \) that satisfies the Helmholtz equation \( \triangle u + k^2 u = 0 \) in \( \mathbb{R}^2 \) and study its asymptotic behaviour as \( \tau \to \infty \).

The Bessel function of order \( m = 0, 1, \ldots \) is given by the formula

\[
J_m(t) = \left( \frac{t^2}{4} \right)^m m! \sum_{n=0}^\infty \frac{(-1)^n}{(m+n)n!} \left( \frac{t^2}{4} \right)^n.
\]

Note that \( (2/t)^m m! J_m(t) \) can be extended as a smooth function on the whole line and in the following we will use this symbol for the extension.

**Definition 2.1.** Let \( k \geq 0 \) and \( 0 < \alpha \leq 1 \). Define

\[
E^k_\alpha(x; \tau) = \sum_{m=0}^\infty \frac{\tau^m}{\Gamma(am+1)} m! J_m(k|x|), \quad 0 < \tau < \infty.
\]

Using the well-known inequality (see Ex. 9.6, p 59 of [23])

\[
|J_m(t)| \leq \left( \frac{t^2}{4} \right)^m m!, \quad t \in \mathbb{R},
\]

one knows that \( E^k_\alpha(x; \tau) \) is well defined and satisfies \( |E^k_\alpha(x; \tau)| \leq E_\alpha(\tau|x|) \).

The idea behind definition 2.1 is the following. Let \( x = (r \cos \theta, r \sin \theta) \). One has

\[
E_\alpha(\tau(x_1 + ix_2)) = \sum_{m=0}^\infty \frac{\tau^m}{\Gamma(am+1)} e^{im\theta};
\]

\[
E^k_\alpha(x; \tau) = \sum_{m=0}^\infty \frac{\tau^m}{\Gamma(am+1)} \left\{ \left( \frac{2}{k} \right)^m m! J_m(k|x|) \right\} e^{im\theta}.
\]

Therefore, by replacing \( r^m \) in the expansion of \( E_\alpha(\tau(x_1 + ix_2)) \) with \( (2/k)^m m! J_m(k|x|) \) one obtains \( E^k_\alpha(x; \tau) \). From (2.1) one knows that the absolute value of \( (2/k)^m m! J_m(k|x|) \) is not greater than \( r^m \) and

\[
\left( \frac{2}{k} \right)^m m! J_m(k|x|) \sim r^m
\]
as \( r \to 0 \). In particular, we have \( E^k_\alpha(0; \tau) = 1 \).

Using the change of variables \( w = \sqrt{1-t} \) and the formula

\[
k|x| \int_0^1 (1-w^2)^m J_1(k|x|w) \, dw = 1 - \left( \frac{2}{k|x|} \right)^m m! J_m(k|x|)
\]

which can be easily checked by using the power series expansions of \( J_1 \) and \( J_m \) with the help of the Beta function, one can see that the function \( E^k_\alpha(x; \tau) \) has the integral representation given by the formula

\[
E^k_\alpha(x; \tau) = E_\alpha(\tau(x_1 + ix_2)) - \frac{k|x|}{2} \int_0^1 E_\alpha(\tau t(x_1 + ix_2)) J_1(k|x|\sqrt{1-t}) \frac{dt}{\sqrt{1-t}}.
\]
The integral transform
\[ u(x) = v(x) - \frac{k|x|}{2} \int_0^1 v(tx) J_1(k|x|\sqrt{1-t}) \frac{dt}{\sqrt{1-t}} \]
is called the Vekua transform of the function \(v(x)\) into the function \(u(x)\). This transforms the given solution of the Laplace equation in \(\mathbb{R}^2\) into that of the Helmholtz equation \(\Delta u + k^2 u = 0\) in \(\mathbb{R}^2\). Formula (2.2) says that \(E^k_\alpha(x; \tau)\) is the Vekua transform of \(E_\alpha(\tau(x_1 + ix_2))\) and therefore satisfies the Helmholtz equation.

In this section we show that \(E^k_\alpha(x; \tau)\) as \(\tau \to \infty\) has the almost same asymptotic behaviour as \(E_\alpha(\tau(x_1 + ix_2))\).

In this paper, for convenience we introduce
\[ \hat{J}_m(t) = \left( \frac{2}{\pi t} \right)^m m! J_m(t). \]
From (2.1) we have \(|\hat{J}_m(t)| \leq 1\). In what follows we make use of this inequality frequently.

Let \(f(z)\) be an arbitrary entire function of the independent variable \(z = x_1 + ix_2\). Let \(u(x; \tau)\) denote the Vekua transform of \(f(\tau(x_1 + ix_2))\). \(u(x; \tau)\) takes the form
\[ u(x; \tau) = f(\tau(x_1 + ix_2)) - \left(\frac{k|x|}{2}\right)^2 \int_0^1 f(\tau t(x_1 + ix_2)) \hat{J}_1(k|x|\sqrt{1-t}) \frac{dt}{\sqrt{1-t}}. \] (2.3)
In this section we write \(C(\pi \alpha/2) = C_0((1, 0), \pi \alpha/2)\).

The following is useful for the treatment of \(E^*_\alpha(x; \tau)\) outside the cone \(C(\pi \alpha/2)\). See the appendix for the proof.

**Lemma 2.1.** One can write \(u(x; \tau)\) and the partial derivatives as
\[ \left(\frac{2}{k|x|}\right)^2 [u(x; \tau) - f(\tau(x_1 + ix_2))] = -\hat{J}_1(k|x|) \frac{1}{\tau} \int_0^\tau f(w(x_1 + ix_2)) dw + R(x; \tau) \] (2.4)
where \(R(x; \tau)\) satisfies
\[ |R(x; \tau)| \leq \frac{1}{2} \left(\frac{k|x|}{2}\right)^2 \frac{1}{\tau^2} \int_0^\tau w |f(w(x_1 + ix_2))| dw; \] (2.5)
\[ \left(\frac{2}{k|x|}\right)^2 \left\{ \frac{\partial}{\partial x_j} u(x; \tau) - \tau i^{j-1} f'(\tau(x_1 + ix_2)) \right\} = -\frac{i^{j-1} \hat{J}_1(k|x|)}{x_1 + ix_2} f(\tau(x_1 + ix_2)) \]
\[ + \left\{ -i^{j-1} \hat{J}_1(k|x|) \right\} \frac{k}{x_1 + ix_2} \hat{J}_2(k|x|) \frac{1}{\tau} \int_0^\tau f(w(x_1 + ix_2)) dw + R_j(x; \tau) \] (2.6)
where \(R_j(x; \tau)\) satisfies
\[ |R_j(x; \tau)| \leq \frac{1}{2} \left(\frac{k|x|}{2}\right)^2 \left\{ \frac{1}{\tau^2} \int_0^\tau w^2 |f'(w(x_1 + ix_2))| dw \right\} \]
\[ + \frac{2|x_j|}{|x|^2} \left( 2 + \frac{1}{3} \left(\frac{k|x|}{2}\right)^2 \right) \frac{1}{\tau^2} \int_0^\tau w |f(w(x_1 + ix_2))| dw. \] (2.7)

Let us consider the case when \(x\) is outside the cone \(C(\pi \alpha/2)\). It is known that as \(|x| \to \infty\) and \(x \in \mathbb{R}^2 \setminus C(\pi \alpha/2)\) Mittag–Leffler’s function and the partial derivatives have the asymptotic form (see [1, 7, 15])
\[ E_\alpha(x_1 + ix_2) = -\frac{1}{(x_1 + ix_2)} \frac{1}{\Gamma(1 - \alpha)} + O\left(\frac{1}{|x|^2}\right) \] (2.8)
These asymptotics are valid uniformly in the region \( \{ x \in \mathbb{R}^3 \setminus \mathcal{C}(\pi \alpha/2 + \epsilon) | R_0 < |x| \} \) for given \( \pi - \pi \alpha/2 > \epsilon > 0 \) and some \( R_0 \gg 1 \).

**Proposition 2.1.** Let \( x \in \mathbb{R}^3 \setminus \mathcal{C}(\pi \alpha/2) \). We have, as \( \tau \rightarrow \infty \),

\[
E^\alpha_0(x; \tau) = \left( \frac{k|x|}{2} \right)^2 \frac{j_1(k|x|)}{x_1 + ix_2} \frac{1}{\Gamma(1 - \alpha)} \int_0^\tau \log \frac{\tau}{\tau + i \tau} \, d\tau + O \left( \frac{1}{\tau} \right)
\]

and

\[
\frac{\partial}{\partial x_j} E^\alpha_0(x; \tau) = \frac{\partial}{\partial x_j} \left( \frac{k|x|}{2} \right)^2 \frac{j_1(k|x|)}{x_1 + ix_2} \frac{1}{\Gamma(1 - \alpha)} \int_0^\tau \log \frac{\tau}{\tau + i \tau} \, d\tau + O \left( \frac{1}{\tau} \right).
\]

These asymptotics are valid uniformly in the region \( \{ x \in \mathbb{R}^3 \setminus \mathcal{C}(\pi \alpha/2 + \epsilon) | R^{-1} < |x| < R \} \) for given \( \pi - \pi \alpha/2 > \epsilon > 0 \) and \( R > 0 \).

**Proof.** First we prove

\[
\frac{1}{\tau} \int_0^\tau E_\alpha(w(x_1 + ix_2)) \, dw = -\frac{1}{x_1 + ix_2} \frac{1}{\Gamma(1 - \alpha)} \int_0^\tau \log \frac{\tau}{\tau + i \tau} \, d\tau + O \left( \frac{1}{\tau} \right).
\]

Let \( \tau_0 > 0 \). Divide the integral

\[
\int_0^\tau E_\alpha(w(x_1 + ix_2)) \, dw = \int_0^{\tau_0} E_\alpha(w(x_1 + ix_2)) \, dw + \int_{\tau_0}^\tau E_\alpha(w(x_1 + ix_2)) \, dw, \quad \tau > \tau_0.
\]

Let \( R_0 \) be the same one as appearing before proposition 2.1. Given \( R > 0 \) choose \( \tau_0 \) in such a way that \( \tau_0 R^{-1} > R_0 \). Then for all \( w \geq \tau_0 \) and \( R^{-1} < |x| < R \) we have \( |w(x_1 + ix_2)| > R_0 \). Thus, one can use (2.8) for estimating the second integral on the right-hand side above. The first integral has a bound \( O(1) \) since \( |x| < R \) and \( \tau_0 \) is now fixed. This yields (2.12). Similarly from (2.8) and (2.9) we obtain

\[
\frac{1}{\tau^2} \int_0^\tau \left| w E_\alpha(w(x_1 + ix_2)) \right| \, dw = O \left( \frac{1}{\tau} \right);
\]

\[
\frac{1}{\tau^2} \int_0^\tau \left| w^2 E_\alpha^2(w(x_1 + ix_2)) \right| \, dw = O \left( \frac{1}{\tau} \right).
\]

Then applying lemma 2.1 to the case when \( f(z) = E_\alpha(z) \), from (2.4), (2.5), (2.12) and (2.13), we obtain (2.10). Next from (2.6)–(2.9), (2.13) and (2.14), we have

\[
\frac{\partial}{\partial x_j} E^\alpha_0(x; \tau) = \tau^{-i - 1} E^\alpha_0(\tau(x_1 + ix_2)) - \left( \frac{k|x|}{2} \right)^2 \frac{j_{i+1}(k|x|)}{x_1 + ix_2} E_\alpha(\tau(x_1 + ix_2))
\]

\[
+ \left( \frac{k|x|}{2} \right)^2 \left\{ \frac{(-1)^{i+1} j_1(k|x|)}{x_1 - ix_2} + \left( \frac{k}{2} \right)^2 x_j j_2(k|x|) \right\}
\]

\[
\times \frac{1}{\tau} \int_0^\tau E_\alpha(w(x_1 + ix_2)) \, dw + O \left( \frac{1}{\tau} \right)
\]

\[
= \left( \frac{k|x|}{2} \right)^2 \left\{ \frac{(-1)^{i+1} j_1(k|x|)}{x_1 - ix_2} + \left( \frac{k}{2} \right)^2 x_j j_2(k|x|) \right\}
\]

\[
\times \frac{1}{\tau} \int_0^\tau E_\alpha(w(x_1 + ix_2)) \, dw + O \left( \frac{1}{\tau} \right).
\]
Then from the equation
\[
\frac{\partial}{\partial x_j} \left\{ \left( \frac{k|x|}{2} \right)^2 \mathcal{J}_1(k|x|) \right\} = \left( \frac{k|x|}{2} \right)^2 \left\{ \frac{(-i)^j}{x_1 - ix_2} \mathcal{J}_1(k|x|) - \left( \frac{k}{2} \right) x_j \mathcal{J}_2(k|x|) \right\} \frac{1}{x_1 + ix_2}
\]
and (2.12) one obtains (2.11). \[\square\]

Next consider the case when \( x \) is inside the cone \( C(\pi \alpha/2) \). From (2.2) we have the expression
\[
E^k_\alpha(x; \tau) = E_\alpha(\tau(x_1 + i x_2)) - \left( \frac{k|x|}{2} \right)^2 \int_0^1 E_\alpha(\tau t(x_1 + i x_2)) \mathcal{J}_1(k|x|\sqrt{1-t}) \, dt \tag{2.15}
\]
and a direct computation yields
\[
\frac{\partial}{\partial x_j} E^k_\alpha(x; \tau) = \tau i^{-1} \left\{ E'_\alpha(\tau(x_1 + i x_2)) - \left( \frac{k|x|}{2} \right)^2 \int_0^1 E'_\alpha(\tau t(x_1 + i x_2)) \mathcal{J}_1(k|x|\sqrt{1-t}) \, dt \right\}
\]
\[
+ \tau i^{-1} \left( \frac{k|x|}{2} \right)^2 \int_0^1 E'_\alpha(\tau(x_1 + i x_2)) (1-t) \mathcal{J}_1(k|x|\sqrt{1-t}) \, dt
\]
\[
- 2 \left( \frac{k}{2} \right) x_j \int_0^1 E_\alpha(\tau t(x_1 + i x_2)) \mathcal{J}_1(k|x|\sqrt{1-t}) \, dt
\]
\[
+ \left( \frac{k}{2} \right)^4 |x|^2 x_j \int_0^1 E_\alpha(\tau t(x_1 + i x_2)) (1-t) \mathcal{J}_2(k|x|\sqrt{1-t}) \, dt \tag{2.16}
\]

It is known that there exists a positive constant \( C \) such that, for all \( z \in C(\pi \alpha/2) \setminus \{0\} \), the estimates
\[
\left| E_\alpha(z) - \frac{1}{\alpha} e^{i z/\alpha} \right| \leq \frac{C}{(1 + |z|^2)^{1/2}} \tag{2.17}
\]
and
\[
\left| \frac{d}{dz} \left( E_\alpha(z) - \frac{1}{\alpha} e^{i z/\alpha} \right) \right| \leq \frac{C}{1 + |z|^2} \tag{2.18}
\]
are valid (see [7, 15]). Note that in (2.18) there is no restriction on \( z \) in a neighbourhood of 0. This is because of \( 0 < \alpha \leq 1 \).

**Proposition 2.2.** Given \( R > 0 \) and \( \epsilon > 0 \), let \( x \in C(\pi \alpha/2) \) satisfy \( R^{-1} \leq |x| \leq R \) and \( \text{Re} (x_1 + i x_2)^{1/\alpha} \geq \epsilon \). Then, as \( \tau \to \infty \) we have two formulae
\[
E^k_\alpha(x; \tau) = \frac{1}{\alpha} e^{i |x_1+i x_2|^{1/\alpha}} \left\{ 1 + O \left( \frac{1}{\tau^{1/\alpha}} \right) \right\} ; \tag{2.19}
\]
\[
\frac{\partial}{\partial x_j} E^k_\alpha(x; \tau) = \frac{\partial}{\partial x_j} \left\{ \frac{1}{\alpha} e^{i |x_1+i x_2|^{1/\alpha}} \right\} \left\{ 1 + O \left( \frac{1}{\tau^{1/\alpha}} \right) \right\} . \tag{2.20}
\]

**Proof.** Since (2.19) is a direct consequence of (2.15), (2.17) and the estimate
\[
\int_0^1 E_\alpha(\tau t(x_1 + i x_2)) \mathcal{J}_1(k|x|\sqrt{1-t}) \, dt = \frac{1}{\alpha} e^{i |x_1+i x_2|^{1/\alpha}} O \left( \frac{1}{\tau^{1/\alpha}} \right) \tag{2.21}
\]
first we give the proof of (2.21).
Write
\[\int_0^1 E_\alpha(\tau t(x_1 + ix_2))\hat{J}_1(k|x|\sqrt{1 - t})\,dt = \int_0^1 \frac{1}{\alpha} e^{1/\alpha(1 + \delta|x|)}\hat{J}_1(k|x|\sqrt{1 - t})\,dt + \int_0^1 \{E_\alpha(\tau t(x_1 + ix_2)) - \frac{1}{\alpha} e^{1/\alpha(1 + \delta|x|)}\} \hat{J}_1(k|x|\sqrt{1 - t})\,dt\]
\[\equiv I + II.\]

Then, from (2.17) we have
\[|II| \leq C \int_0^1 \frac{dt}{(1 + \tau^2|x|^2)^{1/2}} = C \frac{\int_0^1 |x|\,ds}{(1 + s^2)^{1/2}} = O\left(\frac{\log \tau}{\tau}\right)\] (2.22)
provided \(R^{-1} \leq |x| \leq R\).

Let \(0 < \delta < 1\). Write
\[I = \int_0^1 \frac{1}{\alpha} e^{1/\alpha(1 + \delta|x|)}\hat{J}_1(k|x|\sqrt{1 - t})\,dt + \int_0^1 \frac{1}{\alpha} e^{1/\alpha(1 + \delta|x|)}\hat{J}_1(k|x|\sqrt{1 - t})\,dt \equiv I_1 + I_2.\] (2.23)

Then, one has
\[|I_1| \leq \frac{\delta}{\alpha} e^{1/\alpha(1 + \delta)} Re(x_1 + i x_2)^1/\alpha.\] (2.24)

Write
\[I_2 = \frac{1}{\alpha} e^{1/\alpha(1 + \delta|x|)} \int_0^1 e^{-t/\alpha(1 - 1/\alpha)}(x_1 + ix_2)^1/\alpha \hat{J}_1(k|x|\sqrt{1 - t})\,dt.\]

The change of a variable \(1 - t/\alpha = s\) yields
\[\int_0^1 e^{-t/\alpha(1 - 1/\alpha)(x_1 + ix_2)^1/\alpha} \hat{J}_1(k|x|\sqrt{1 - t})\,dt \equiv \alpha \int_0^{1-\delta/\alpha} e^{-t/\alpha s(x_1 + ix_2)^1/\alpha} \hat{J}_1(k|x|\sqrt{1 - (1 - s)^\alpha}) \frac{(1 - s)^{1/\alpha}}{(1 - s)^{1/\alpha - 1}}\,ds\]

and this gives
\[\left|\int_0^1 e^{-t/\alpha(1 - 1/\alpha)(x_1 + ix_2)^1/\alpha} \hat{J}_1(k|x|\sqrt{1 - t})\,dt\right| \leq \alpha \int_0^{1-\delta/\alpha} e^{-t/\alpha s Re(x_1 + i x_2)^1/\alpha} \frac{ds}{(1 - s)^{1/\alpha - 1}} \leq \frac{\alpha}{[1 - (1 - \delta/\alpha)^{1/\alpha}]^{1 - \delta/\alpha}} \int_0^{1-\delta/\alpha} e^{-t/\alpha s Re(x_1 + i x_2)^1/\alpha} \,ds\]

Therefore, we obtain
\[|I_2| \leq \frac{\alpha}{[1 - (1 - \alpha/\alpha)^{1/\alpha}]^{1 - \alpha}} Re(x_1 + i x_2)^1/\alpha.\] (2.25)

From (2.23), (2.24) and (2.25) one concludes that
\[1/\alpha e^{-t/\alpha(x_1 + i x_2)^1/\alpha} = O\left(\frac{1}{\tau^{1/\alpha}}\right)\] (2.26)
provided \(Re(x_1 + i x_2)^1/\alpha \geq \epsilon\). A combination of (2.22) and (2.26) yields (2.21).
Next we prove for $m = 0, 1$

$$
\int_0^1 E'_\alpha(\tau t(x_1 + i x_2))(1 - t)^m \tilde{J}_1(k|x|\sqrt{1 - t}) \, dt
\quad = \frac{1}{\alpha^2} e^{i\alpha(x_1 + i x_2)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha} O\left(\frac{1}{\tau^{1/\alpha}}\right),
$$

(2.27)

We have

$$
\int_0^1 e^{\tau i \alpha(x_1 + i x_2)/\alpha} e^{t(1-\alpha)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha} (1 - t)^m \tilde{J}_1(k|x|\sqrt{1 - t}) \, dt
\quad = e^{\tau i \alpha(x_1 + i x_2)/\alpha} e^{(1-\alpha)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha}
\times \int_0^1 e^{-\tau i \alpha(1 - t)^m(x_1 + i x_2)^{(1-\alpha)/\alpha}} (1 - t)^m \tilde{J}_1(k|x|\sqrt{1 - t}) \, dt.
$$

Since

$$
\left| \int_0^1 e^{-\tau i \alpha(1 - t)^m(x_1 + i x_2)^{(1-\alpha)/\alpha}} (1 - t)^m \tilde{J}_1(k|x|\sqrt{1 - t}) \, dt \right|
\quad \leq \int_0^1 e^{-\tau i \alpha(1 - t)^m(1-\alpha)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha} \, dt
\quad = \alpha \int_0^1 e^{-\tau i \alpha \Re(x_1 + i x_2)^{(1-\alpha)/\alpha}} ds
\quad \leq \frac{\alpha}{\tau^{1/\alpha} \Re(x_1 + i x_2)^{1/\alpha}},
$$

we obtain

$$
\int_0^1 e^{\tau i \alpha(x_1 + i x_2)/\alpha} e^{t(1-\alpha)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha} (1 - t)^m \tilde{J}_1(k|x|\sqrt{1 - t}) \, dt
\quad = e^{\tau i \alpha(x_1 + i x_2)/\alpha} e^{(1-\alpha)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha} O\left(\frac{1}{\tau^{1/\alpha}}\right).
$$

(2.28)

Now from (2.18) and (2.28) one obtains (2.27). Similarly, for $j = 1, 2$ we have

$$
\int_0^1 E_\alpha(\tau t(x_1 + i x_2))(1 - t)^m \tilde{J}_{m+1}(k|x|\sqrt{1 - t}) \, dt
\quad = \tau i^{j-1} \frac{1}{\alpha^2} e^{\tau i \alpha(x_1 + i x_2)/\alpha} e^{t(1-\alpha)/\alpha} (x_1 + i x_2)^{(1-\alpha)/\alpha} O\left(\frac{1}{\tau^{1/\alpha}}\right)
$$

(2.29)

provided $\Re(x_1 + i x_2)^{1/\alpha} \geq \epsilon$ and $R^{-1} \leq |x| \leq R$. Note that this is a ‘rough’ estimate. Now from (2.16), (2.18), (2.28) and (2.29) we obtain (2.20).

As a corollary of propositions 2.1 and 2.2 we have immediately

**Corollary 2.1.** We have for any regular $C^2$ curve $c$ with $c \subset C(\pi \alpha/2)$

$$
\lim_{t \to \infty} \| E_\alpha^t(\cdot; \tau) \|_{L^1(c)} = \infty;
$$

for any non-empty bounded open set $U$ of $\mathbb{R}^2$ with $\overline{U} \subset \mathbb{R}^2 \setminus C(\pi \alpha/2)$

$$
\lim_{t \to \infty} \| E_\alpha^t(\cdot; \tau) \|_{H^1(U)} = 0.
$$
3. Construction of the density

**Definition 3.1.** Given \( \omega = (\omega_1, \omega_2) \in S^1 \), set \( \omega^\perp = (-\omega_2, \omega_1) \). Define the function \( E_{1/n}(x; s, k, \omega) \) by the formula

\[
E_{1/n}(x; s, k, \omega) = E_{1/n}^k \left( (x \cdot \omega, x \cdot \omega^\perp); \frac{s}{2} \right), \quad s > 0.
\]

From (2.2) we already know that the function \( E_{1/n}(x; s, k, \omega) \) of \( x \in \mathbb{R}^2 \) satisfies the Helmholtz equation \( \Delta v + k^2 v = 0 \) in \( \mathbb{R}^2 \). Since \( x \cdot (\omega + i \omega^\perp) = (\omega_1 - i \omega_2)(x_1 + i x_2) \), the function \( E_{1/n}(x; s, k, \omega) \) coincides with the Vekua transform of the harmonic function \( E_{1/n}(s(\omega_1 - i \omega_2)(x_1 + i x_2)/2) \) in \( \mathbb{R}^2 \).

The aim of this section is to construct a density \( g \in L^2(S^1) \) explicitly such that

\[
Hg(x) \approx E_{1/n}(x; s, k, \omega), \quad x \in \mathbb{B}_2.
\]

The starting point is the following fact.

**Proposition 3.1 ([16]).** The Vekua transform of the harmonic function

\[
e^{ik\phi(x_1+i x_2)/2} + e^{ik\phi(x_1-i x_2)/2} - 1
\]

coincides with \( e^{ik \cdot \phi} \).

Let \( \mathcal{M} \) be a non-empty open subset of \( S^1 \). Given \( g \in L^2(S^1) \) the function

\[
\int_{\mathcal{M}} \left[ e^{ik\phi(x_1+ix_2)/2} + e^{ik\phi(x_1-ix_2)/2} - 1 \right] g(\phi) \, d\sigma(\phi)
\]

is harmonic in the whole plane. As a corollary of proposition 3.1 one knows that the Vekua transform of this harmonic function coincides with the Herglotz wavefunction \( H(\chi_{\mathcal{M}} g) \), where \( \chi_{\mathcal{M}} \) denotes the characteristic function of \( \mathcal{M} \).

Taking into account the fact mentioned above and the definition of \( E_{1/n}(x; s, k, \omega) \), it suffices to construct \( g \) in such a way that

\[
\int_{\mathcal{M}} \left[ e^{ik\phi(x_1+ix_2)/2} + e^{ik\phi(x_1-ix_2)/2} - 1 \right] g(\phi) \, d\sigma(\phi) \approx E_{1/n}(s(\omega_1 - i \omega_2)(x_1 + i x_2)/2)
\]

(3.1) where \( \omega = \omega_1 - i \omega_2 \).

Using the power series expansion of Mittag–Leffler’s function, one knows that if \( g \) satisfies the system of equations

\[
\frac{1}{\Gamma(m+1)} \left( \frac{i k}{2} \right)^m \int_{\mathcal{M}} \phi^m g(\phi) \, d\sigma(\phi) = 0, \quad m = 1, \ldots,
\]

and

\[
\frac{1}{\Gamma(m+1)} \left( \frac{i k}{2} \right)^m \int_{\mathcal{M}} \overline{\phi^m} g(\phi) \, d\sigma(\phi) = \frac{1}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s}{2} \overline{\omega} \right)^m, \quad m = 0, 1, \ldots,
\]

then \( g \) satisfies (3.1) exactly. Now consider the case when \( \mathcal{M} = S^1 \). We construct \( g \) in the form

\[
g(\phi) = \sum_{m=0}^{\infty} \beta_m \phi^m + \sum_{m=1}^{\infty} \beta_{-m} \overline{\phi}^m.
\]

Since

\[
\frac{1}{2\pi} \int_{S^1} \overline{\phi}^m g(\phi) \, d\sigma(\phi) = \beta_m, \quad \frac{1}{2\pi} \int_{S^1} \phi^m g(\phi) \, d\sigma(\phi) = \beta_{-m},
\]

we have

\[
g(\phi) = \frac{1}{2\pi} \int_{S^1} \phi^m \, d\sigma(\phi) + \frac{1}{2\pi} \int_{S^1} \overline{\phi}^m \, d\sigma(\phi) = A_{-m} \phi^m + A_m \overline{\phi}^m.
\]
from (3.2) and (3.3) we get $\beta_{-m} = 0$, $m = 1, 2, \ldots$, and

$$
\beta_m = \frac{1}{2\pi} \frac{\Gamma(m + 1)}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s \omega}{ik} \right)^m, \quad m = 0, 1, \ldots.
$$

Then $g$ becomes

$$
g(\psi) = \sum_{m=0}^{\infty} \frac{1}{2\pi} \frac{\Gamma(m + 1)}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s \omega \psi}{ik} \right)^m.
$$

Using Stirling’s formula and D’Alembert’s ratio test, we see that (3.4) is always divergent when $n > 1$ except for the case when $n = 1$ and $s < k$. So we consider a truncation of (3.4):

$$
g_N(\psi; s, k, \omega) = \frac{1}{2\pi} \sum_{m=0}^{nN} \frac{\Gamma(m + 1)}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s \omega \psi}{ik} \right)^m
$$

where $N = 1, \ldots$. Then one obtains

$$
\int_{S^1} \left| e^{i\phi(x_1 + ix_2)/2} + e^{i\phi(x_1 - ix_2)/2} - 1 \right| g_N(\psi; s, k, \omega) \, d\sigma(\psi)
= E_{1/\nu} \left( \frac{s \omega}{2} (x_1 + ix_2) \right) - \sum_{m=nN}^{N} \frac{1}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s \omega}{k} \right)^m (x_1 + ix_2)^m.
$$

This shows that $g_N(\cdot; s, k, \omega)$ satisfies (3.1) in this sense. Taking the Vekua transform of both sides of (3.6) we obtain the equation

$$
H g_N(\cdot; s, k, \omega)(x) = E_{1/\nu}(x; s, k, \omega) = \sum_{m=nN}^{N} \frac{\Gamma(m + 1)}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s \omega}{k} \right)^m J_m(kr) e^{im\theta}
$$

where $x = (r \cos \theta, r \sin \theta)$. Note that this can be checked also directly and the equation $H g_N(\cdot; s, k, \omega)(0) = E_{1/\nu}(0; s, k, \omega) = 1$ holds.

For our purpose we have to consider how to choose $s$ depending on $N$. One answer to this question is the following and it is the main result of this section.

**Theorem 3.1.** Let $\gamma_0$ be the unique positive solution of the equation $\log t + t/e = 0$. Let $\gamma$ satisfy $0 < \gamma < \gamma_0$. Let $R$ be an arbitrary fixed positive number. Let $\{s(N)\}_{N=1}^\infty$ be an arbitrary sequence of positive numbers satisfying, as $N \to \infty$,

$$
(Rs(N))^n = \frac{\nu}{e} N + O(1).
$$

Then we have, as $N \to \infty$,

$$
\sup_{|x| \leq 2R} |H g_N(\cdot; s(N), k, \omega)(x) - E_{1/\nu}(x; s(N), k, \omega)|
+ \sup_{|x| \leq 2R} | \nabla \{H g_N(\cdot; s(N), k, \omega)(x) - E_{1/\nu}(x; s(N), k, \omega)\}|
= O \left( N^{3/2} e^{N(\zeta + \log \gamma)} \right) = O(N^{-\infty}).
$$

**Proof.** Set

$$
S_N(x; s) = \sum_{m=nN}^{N} \frac{\Gamma(m + 1)}{\Gamma \left( \frac{m}{n} + 1 \right)} \left( \frac{s \omega}{k} \right)^m J_m(kr) e^{im\theta}, \quad E_{1/\nu}^N(z) = \sum_{m=0}^{nN} \frac{z^m}{\Gamma \left( \frac{m}{n} + 1 \right)}.
$$

Then from (2.1) we have, for all $x$ with $|x| \leq 2R$,

$$
|S_N(x; s)| \leq \sum_{m=nN}^{N} \frac{1}{\Gamma \left( \frac{m}{n} + 1 \right)} (Rs)^m = \left| E_{1/\nu}(z) - E_{1/\nu}^N(z) \right|_{z=Rs}.
$$
Moreover, using the recurrence relation
\[ J_{m+1}(kr) = \frac{m}{kr} J_m(kr) - J_m'(kr), \quad J_{m-1}(kr) = \frac{m}{kr} J_{m-1}(kr) + J_m'(kr) \]
one has the formulae
\[ e^{i\theta} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \right) J_m(kr) e^{im\theta} = -kJ_{m+1}(kr) e^{i(m+1)\theta}; \]
\[ e^{-i\theta} \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \right) J_m(kr) e^{im\theta} = kJ_{m-1}(kr) e^{i(m-1)\theta}. \]

Then the formulae
\[
\frac{\partial}{\partial x_1} = \frac{e^{i\theta}}{2} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \frac{e^{-i\theta}}{2} \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \right) \\
\frac{\partial}{\partial x_2} = -\frac{i}{2} e^{i\theta} \left( \frac{\partial}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} \right) + \frac{i}{2} e^{-i\theta} \left( \frac{\partial}{\partial r} - \frac{1}{r} \frac{\partial}{\partial \theta} \right)
\]
give the estimate
\[
\left| \frac{\partial}{\partial x_1} S_N(x; s) \right| + \left| \frac{\partial}{\partial x_2} S_N(x; s) \right| \leq s \frac{d}{dz} \left| E_{1/n}(z) - E_{1/n}^N(z) \right| \bigg|_{z=R}\nn + k^2 R \left| E_{1/n}(z) - E_{1/n}^N(z) \right| \bigg|_{z=R}, \quad (3.10)
\]
From the proof of proposition 3.2 in [17] one has,
\[
\left| E_{1/n}(z) - E_{1/n}^N(z) \right| \leq \sum_{l=1}^{n} \frac{|z|^{l(N+1)}}{\Gamma(N+1 + \frac{1}{n})} e^{\text{Re}z} \quad (3.11)
\]
and
\[
\left| \frac{d}{dz} \left| E_{1/n}(z) - E_{1/n}^N(z) \right| \right| \leq n|z|^{n-1} \sum_{l=1}^{n} \frac{|z|^{l(N-1)+l}}{\Gamma(N+\frac{1}{n})} e^{\text{Re}z} \quad (3.12)
\]
Note that these are sharper than (3.5) and (3.6) of proposition 3.2 in [17]. Consider the case when \( z = Rs(N) \). Then we get, as \( N \to \infty \),
\[
\sum_{l=1}^{n} \frac{|z|^{l(N+1)}}{\Gamma(N+1 + \frac{1}{n})} e^{\text{Re}z} = O \left( \frac{(Rs(N))^{n(N+1)} e^{(Rs(N))^n}}{\Gamma(N+1 + \frac{1}{n})} \right) ;
\]
\[
s(N)|z|^{n-1} \sum_{l=1}^{n} \frac{|z|^{l(N-1)+l}}{\Gamma(N+\frac{1}{n})} e^{\text{Re}z} = O \left( \frac{(Rs(N))^{n(N+1)} e^{(Rs(N))^n}}{\Gamma(N+\frac{1}{n})} \right) .
\]
Since
\[
\frac{1}{\Gamma(N+\frac{1}{n})} = O \left( \frac{1}{(N-1)!} \right),
\]
from (3.9), (3.10) and (3.11) to (3.13) we obtain
\[
|S_N(x; s(N))| + \left| \frac{\partial}{\partial x_1} S_N(x; s(N)) \right| + \left| \frac{\partial}{\partial x_2} S_N(x; s(N)) \right| \\
= O \left( \frac{N^2(Rs(N))^{n(N-1)} e^{(Rs(N))^n}}{(N-1)!} \right). \quad (3.14)
\]
Using the Stirling formula, we have
\[
\frac{\xi(N)^{N-1} e^{\xi(N)}}{(N-1)!} = O(N^{-1/2} e^{N(\xi+\log\gamma)}) \quad (3.15)
\]
where \( \{\xi(N)\}_{N=1}^{\infty} \) is an arbitrary sequence of positive numbers satisfying, as \( N \to \infty \),

\[
\xi(N) = (\gamma/e)N + O(1) \quad \text{and} \quad 0 < \gamma < \gamma_0.
\]

Now the conclusion follows from (3.7), (3.14) and (3.15). \( \square \)

From (3.5) we know that \( g^{1/n}_{(y, \omega)}(\cdot; s, k)_N \) given by (1.1) has the expression

\[
g^{1/n}_{(y, \omega)}(\phi; s, k)_N = e^{-iky}g_N(\phi; s, k, \omega), \quad \phi \in S^1.
\]

Then from definition 3.1, (3.8) and the equation \( Hg^{1/n}_{(y, \omega)}(\cdot; s, k)_N(x) = Hg_N(\cdot; s, k, \omega)(x - y) \), we immediately obtain

**Corollary 3.1.** Let \( \{s(N)\}_{N=1}^{\infty} \) be the same as in theorem 3.1. Then for any fixed \((y, \omega) \in BR \times S^1\) we have, as \( N \to \infty \),

\[
\sup_{|x| \leq R} \left| Hg^{1/n}_{(y, \omega)}(\cdot; s(N), k)_N(x) - E_{1/n}(x - y; s(N), k, \omega) \right| + \sup_{|x| \leq R} \left| \nabla[Hg^{1/n}_{(y, \omega)}(\cdot; s(N), k)_N(x) - E_{1/n}(x - y; s(N), k, \omega)] \right| = O(N^{-\infty}). \tag{3.16}
\]

4. Remarks

**Remark 4.1.** It should be pointed out that the density (1.1) satisfies

\[
(\Phi_x, g^{1/n}_{(x, \omega)}(\cdot; s, k)_N)_{L^2(S^1)} = 1.
\]

Therefore, we obtain the relationship between our indicator function and Kirsch’s one:

\[
K(x) \leq \left| I^{1/n}_{(x, \omega)}(s(N))_N \right|.
\]

This together with theorem 1.1 explains why \( K(x) = 0 \) in a case that the point \( x \) can be connected with infinity by a straight line without intersecting \( \bar{D} \).

**Remark 4.2.** Kirsch’s formula needs to construct numerically a sequence \( \{g_n\} \) of \( L^2(S^1) \) functions with \( (\Phi_x, g_n)_{L^2(S^1)} = 1 \) such that \( |F_{Dg_n}(g_n)| \to K(x) \) for each \( x \in BR \).

However, we do not know its profile at all. Moreover, for \( x \in D \) his formula gives just \( 0 < K(x) < \infty \). Our ‘indicator function’ \( I^{1/n}_{(x, \omega)}(s(N))_N \) can be easily computed once we have accurate far-field pattern or its Fourier coefficients. It emphasizes when \( x \in D \) as \( \lim_{N \to \infty} |I^{1/n}_{(x, \omega)}(s(N))_N| = \infty \). However, theoretically our method yields only a limited information, that is, the visible part of \( BR \setminus \bar{D} \). It would be interesting to construct and test an algorithm based on theorem 1.1. This belongs to our future research plan.

**Remark 4.3.** Theorem 1.1 does not cover the ‘critical’ case when both \( C_y(\omega, \pi/2n) \cap \bar{D} \neq \emptyset \) and \( C_y(\omega, \pi/2n) \cap D = \emptyset \) are satisfied. This is due to the lack of a necessary uniform estimate of the function \( E_{\alpha}(x; s, k, \omega) \). At the present time we do not know what one can say about the behaviour of the indicator function as \( N \to \infty \) in this case. Note that the results in [15] and theorem 1.1 in [17] completely cover this type of case.

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Appendix. Proof of lemma 2.1

Write
\[ u(x; \tau) - f(\tau(x_1 + ix_2)) = -\left(\frac{k|\mathbf{x}|}{2}\right)^2 \hat{J}_1(k|\mathbf{x}|) \int_0^1 f(\tau t(x_1 + ix_2)) \, dt + \left(\frac{k|\mathbf{x}|}{2}\right)^2 R(x; \tau) \]
where
\[ R(x; \tau) = \int_0^1 f(\tau t(x_1 + ix_2)) [\hat{J}_1(k|\mathbf{x}|) - \hat{J}_1(k|x|\sqrt{1-t})] \, dt. \]

Using the expression
\[ \hat{J}_m(k|\mathbf{x}|\sqrt{1-t}) = m! \sum_{n=0}^{\infty} \frac{(-1)^n}{(m+n)!n!} \left(\frac{k|x|}{2}\right)^n (1-t)^n, \] one obtains
\[ \frac{d}{dt}[\hat{J}_1(k|x|\sqrt{1-t})] = \frac{1}{2} \left(\frac{k|x|}{2}\right)^2 \hat{J}_2(k|x|\sqrt{1-t}). \]

Then the mean value theorem yields
\[ |\hat{J}_1(k|x|\sqrt{1-t}) - \hat{J}_1(k|x|)| \leq \frac{1}{2} \left(\frac{k|x|}{2}\right)^2 t. \]

From this one obtains
\[ |R(x; \tau)| \leq \frac{1}{2} \left(\frac{k|x|}{2}\right)^2 \int_0^1 |f(\tau t(x_1 + ix_2))| t \, dt. \]

Now (2.4) and (2.5) are clear.

Next from (A.1) we have
\[ \frac{\partial}{\partial x_j} \{\hat{J}_1(k|x|\sqrt{1-t})\} = -\left(\frac{k}{2}\right)^2 x_j (1-t) \hat{J}_2(k|x|\sqrt{1-t}) \]
and this yields
\[ \frac{\partial}{\partial x_j} [u(x; \tau) - f(\tau(x_1 + ix_2))] = -\left(\frac{k}{2}\right)^2 2 x_j \int_0^1 f(\tau t(x_1 + ix_2)) \hat{J}_1(k|x|\sqrt{1-t}) \, dt \]
\[ - \left(\frac{k|x|}{2}\right) \tau i^{j-1} \int_0^1 t f'(\tau t(x_1 + ix_2)) \hat{J}_1(k|x|\sqrt{1-t}) \, dt \]
\[ + \left(\frac{k|x|}{2}\right)^2 \left(\frac{k}{2}\right)^2 x_j \int_0^1 f(\tau t(x_1 + ix_2))(1-t) \hat{J}_2(k|x|\sqrt{1-t}) \, dt \]
\[ = \left(\frac{k|x|}{2}\right)^2 \{A_j(x; \tau) + R_j(x; \tau)\} \] (A.2)
where
\[ A_j(x; \tau) = -\left(\frac{2x_j}{|\mathbf{x}|}\right) \hat{J}_1(k|\mathbf{x}|) \int_0^1 f(\tau t(x_1 + ix_2)) \, dt - \tau i^{j-1} \hat{J}_1(k|x|) \int_0^1 t f'(\tau t(x_1 + ix_2)) \, dt \]
\[ + \left(\frac{k}{2}\right)^2 x_j \hat{J}_2(k|x|) \int_0^1 f(\tau t(x_1 + ix_2)) \, dt \]
and

\[ R_j(x; \tau) = -\frac{2x_j}{|x|^2} \int_0^1 \left\{ f(\tau t(x_1 + ix_2)) \left[ \hat{J}_1(k|x|\sqrt{1-t}) - \hat{J}_1(k|\tau x|) \right] - \tau t^{\frac{1}{2}} \int_0^1 f'(\tau t(x_1 + ix_2)) \left[ \hat{J}_1(k|x|\sqrt{1-t}) - \hat{J}_1(k|\tau x|) \right] dt \right\} \frac{1}{\tau} \int_0^{\tau} f(\tau t(x_1 + ix_2)) dt \ \text{d}w. \]  

The change of variables and integration by parts yields

\[ A_j(x; \tau) = -\frac{i^{\frac{1}{2}}}{x_1 + ix_2} f(\tau(x_1 + ix_2)) \]

\[ + \left\{ -\frac{i^{\frac{1}{2}}}{x_1 - ix_2} \hat{J}_2(k|x|) + \left( \frac{k}{2} \right)^2 x_j \hat{J}_2(k|x|) \right\} \frac{1}{\tau} \int_0^{\tau} f(\tau(x_1 + ix_2)) dt. \]  

Since

\[ \frac{d}{dt} \hat{J}_2(k|x|((1-t)\sqrt{1-t}) = \frac{1}{3} \left( \frac{k|x|}{2} \right)^2 \hat{J}_3(k|x|((1-t), \]

one knows that

\[ |(1-t)\hat{J}_2(k|x|((1-t) - \hat{J}_2(k|x|)| \leq \left\{ \frac{1}{3} \left( \frac{k|x|}{2} \right)^2 + 1 \right\} t. \]

Using this together with (A.2), (A.3) and (A.4), we obtain (2.6) and (2.7).

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