Finite One-Loop Calculations in Quantum Gravity: Graviton Self-Energy, Perturbative Gauge Invariance and Slavnov–Ward Identities

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Abstract

In this paper we show that the one-loop graviton self-energy contribution is ultraviolet finite, without introducing counterterms, and cutoff-free in the framework of causal perturbation theory. In addition, it satisfies the gravitational Slavnov–Ward identities for the two-point connected Green function. The condition of perturbative gauge invariance to second order for loop graphs is proved. Corrections to the Newtonian potential are also derived.

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Contents

1 Introduction 3

2 Graviton Coupling and Perturbative Gauge Invariance 4
   2.1 $S$-Matrix Inductive Construction 4
   2.2 First Order Graviton Interaction 5
   2.3 Quantization of the Graviton Field, Perturbative Gauge Invariance and Ghost Coupling 6
   2.4 Consequences of Perturbative Gauge Invariance to Second Order for Two-Point Distributions 8

3 Two-Point Distribution for Graviton Self-Energy 11
   3.1 Inductive Construction 11
   3.2 Example of the Calculation 12
   3.3 Causal $D_2(x, y)$-Distribution for Graviton Self-Energy 13
   3.4 Singular Order in Quantum Gravity 14
   3.5 Splitting of the $D_2(x, y)$-Distribution 16
   3.6 Graviton Self-Energy Tensor from the $T_2(x, y)$-Distribution 18

4 Gravitational Slavnov–Ward Identities and Perturbative Gauge Invariance 19

5 Fixing of the Freedom in the Normalization of the Self-Energy Tensor 23
   5.1 Normalization Terms of $T_2(x, y)$ 23
   5.2 Total Graviton Propagator and Mass and Coupling Constant Normalization 24

6 Corrections to the Newtonian Potential 27

7 Perturbative Gauge Invariance to Second Order for Loop Contributions 28
   7.1 Ghost Self-Energy 29
   7.2 Perturbative Gauge Invariance to Second Order for Loop Graphs 30

Appendix 33
   Appendix 1: The $\hat{D}_{\alpha\beta\mu\nu}(p)$-Functions 33
   Appendix 2: The $I^{(\pm)}(p)_\mu\nu$-Integrals 34
   Appendix 3: The $Q^{(i)}(p)_{\alpha\beta,\mu\nu}$-Projection Operators 37
   Appendix 4: Numerical Distributions in Eq. (3.15) 39

References 40
1 Introduction

In the field-theoretical approach to general relativity, Einstein’s theory can be reduced to a theory of gravitation in flat space-time by an expansion of the gravitational Lagrangian density, the Hilbert–Einstein Lagrangian density $L_{HE}$, as an infinite series in powers of the gravitational coupling constant.

In such an expansion the inherent non-linearity of Einstein equations appears as a non-linear interaction between gravitons due to their gravitational weight. Thus, Einstein’s theory will be considered as a self-interaction of ordinary massless rank-2 symmetric tensor gauge fields in flat space-time. The Lorentz covariant quantization program for the so obtained field theory was proposed in [1], [2], [3], [4], [5], [6] and [7] (and references therein).

For an ample treatment of this subject, see [8], [9] and [10].

Although covariant Feynman rules in quantum gravity (QG) were soon derived [11], [12] which allowed a calculable perturbative expansion of QG [13], it was soon realized that in the standard perturbative framework radiative corrections within the theory were plagued by severe ultraviolet (UV) divergences [14], [15] (for a review, see [16], [17], [18]).

It turned out that pure (that is without matter fields) QG was one-loop finite due to the Gauss–Bonnett identity in four dimensions. The obtained one-loop divergences are such that they can be transformed away by a field renormalization. But two-loop calculations [19], [20] and [21] yield non-renormalizable divergences.

In the meantime, it was realized that the reason for the UV divergences lies basically in the fact that one performs mathematically ill-defined operations, when using Feynman rules for closed loop graphs, because one multiplies Feynman propagators as if they were ordinary functions.

Therefore a new strategy was developed in order to avoid the appearance of UV divergences once and for all. This was done by Epstein and Glaser in the early seventies [22], then further applied to QED by Scharf [23] and to Yang–Mills theories by Dütsch et al. [24].

In the resulting scheme, called ‘causal perturbation theory’, the central object is the $S$-matrix, whose perturbative expansion is computed taking causality as cornerstone so that all expressions are finite and well-defined. UV finiteness is then a consequence of a deeper mathematical understanding of how loop graph contributions have to be calculated.

Therefore, power-counting perturbative renormalizability no longer represents a criterion for distinguishing viable theories from ill-defined unrealistic theories.

Within the causal perturbation scheme, one-loop contributions to graviton self-energy are calculated in this paper and shown to be UV finite without the introduction of a regularization scheme and therefore cutoff-free. The obtained graviton self-energy tensor depends logarithmically on a mass scale, which breaks scale invariance, and satisfies the appropriate gravitational Slavnov–
Ward identities [13], [25], [26], only if graviton and ghost loops are added up together.

An aspect of causal theory applied to QG relies in the fact that one works with free graviton fields in a fixed gauge, therefore, for general gauge calculations we refer to [27], [28].

Although in this paper we do not present two-loop calculation, the causal method ensures us of their UV finiteness [29].

For the explicit quantization of the graviton field and the subsequent construction of the physical subspace of the graviton Fock space which contains physical graviton states and for the proof of unitarity of the $S$-matrix restricted to the physical subspace, we refer to [30], which also provides us with the basic notations and definitions.

QG coupled to photon fields and to scalar matter fields within causal perturbation theory is considered in [31] and [32], respectively.

The paper is organized as follow: in the next section, after a brief introduction to causal perturbation theory, the transition from general relativity to perturbative quantum gravity in the causal approach is grounded and the condition of perturbative gauge invariance is presented and some consequences are drawn. In Sec. 3 the inductive construction of the graviton self-energy is explicitly carried out, the issues of non-normalizability of QG and distribution splitting are touched on. In Sec. 4 the Slavnov–Ward identities for the two-point function under investigation are verified and their relation to perturbative gauge invariance is discussed. In Sec. 5 the freedom in the normalization that is inherent to the causal inductive construction is settled. Corrections to the Newtonian potential due to graviton self-energy are discussed in Sec. 6, while perturbative gauge invariance to second order in the loop graph sector is shown in Sec. 7. In the technical appendices, the formulae needed for the causal construction of the 2-point distributions and for the sum of self-energy insertions are derived.

We use the unit convention: $\hbar = c = 1$, Greek indices $\alpha, \beta, \ldots$ run from 0 to 3, whereas Latin indices $i, j, \ldots$ run from 1 to 3.

2 Graviton Coupling and Perturbative Gauge Invariance

2.1 $S$-Matrix Inductive Construction

The central object of causal perturbation theory [24], [29], [30] is the scattering matrix $S$. Being a formal power series in the coupling constant, we consider it as a sum of smeared operator-valued distributions of the following form:

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \ldots d^4 x_n T_n(x_1, \ldots, x_n) g(x_1) \cdot \ldots \cdot g(x_n),$$  (2.1)
where \( g \) is a Schwartz test function \( (g \in \mathcal{S}(\mathbb{R}^4)) \) which switches the interaction and provides a natural infrared cutoff in the long-range part of the interaction.

The \( S \)-matrix maps the asymptotically incoming free fields on the outgoing ones and it is possible to express the \( T_n \)'s by means of free fields. In causal perturbation theory interacting quantum fields do not appear.

The \( n \)-point operator-valued distribution \( T_n \) is a well-defined ‘renormalized’ time-ordered product expressed in terms of Wick monomials of free fields. \( T_n \) is constructed inductively from the first order \( T_1(x) \), which corresponds to the interaction Lagrangian in terms of free fields, and from the lower orders \( T_j \), \( j = 2, \ldots, n - 1 \) by means of Poincaré covariance and causality.

Causality leads directly to UV finite and cutoff-free \( T_n \)-distributions in every order without introducing any counterterm.

### 2.2 First Order Graviton Interaction

Following the usual approach [13], [25], we start from the Hilbert–Einstein Lagrangian (without cosmological constant)

\[
\mathcal{L}_{HE} = \frac{-2}{\kappa^2} \sqrt{-g} R ,
\]

where \( R \) is the Ricci scalar and \( \kappa^2 = 32\pi G \) with \( G \) = Newton’s constant. We use the same notations as in [30] and [33]. Expanding the Goldberg variable \( \tilde{g}^{\mu\nu} := \sqrt{-g} g^{\mu\nu} \) in an asymptotically flat geometry

\[
\tilde{g}^{\mu\nu}(x) = \eta^{\mu\nu} + \kappa h^{\mu\nu}(x) ,
\]

where \( \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \) is the flat space-time metric tensor, we find the non-terminating expansion of \( \mathcal{L}_{HE} \)

\[
\mathcal{L}_{HE} = \sum_{j=0}^{\infty} \kappa^j \mathcal{L}_{HE}^{(j)} ,
\]

where \( \mathcal{L}_{HE}^{(j)} \) represents an interaction involving \( j + 2 \) gravitons. Eq. (2.3) defines the dynamical graviton field \( h^{\mu\nu}(x) \) propagating in the flat space-time geometry.

The lowest order \( \mathcal{L}_{HE}^{(0)} \) is quadratic in \( h^{\mu\nu}(x) \) and in the Hilbert gauge \( h^{\mu\nu}(x)_{\nu} = 0 \) the graviton field \( h^{\mu\nu}(x) \) obeys the wave equation

\[
\Box h^{\mu\nu}(x) = 0 .
\]

Since the perturbative expansion for the \( S \)-matrix (2.1) is in powers of the coupling constant \( \kappa \), we consider the normally ordered product of the first order term in (2.4)

\[
T_1^h(x) = i \kappa :\mathcal{L}_{HE}^{(1)}: = i \frac{\kappa}{2} \{ + :h^{\rho\sigma} h^{\alpha\beta} h_{\alpha\rho}^{\phantom{\alpha\rho} \beta} : - \frac{1}{2} :h^{\rho\sigma} h_{\rho\sigma} : + 2 :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\beta}_{\rho\sigma} : + :h^{\rho\sigma} h_{\rho\sigma} : - 2 :h^{\rho\sigma} h^{\alpha\beta} h^{\alpha\beta}_{\rho\sigma} : \} ,
\]

(2.6)
as the cubic interaction between gravitons or first order graviton interaction. For brevity, we omit the space-time dependence of the fields, if the meaning is clear.

For convenience of notation, the trace of the graviton field is written as $h = h^\gamma_\gamma$, and all Lorentz indices of the graviton fields are written as superscripts whereas the derivatives acting on the fields are written as subscripts. All indices occurring twice are contracted by the Minkowski metric $\eta^{\mu\nu}$.

The ‘non-renormalizability problem’ of quantum gravity arises because of the presence of two derivatives on the graviton fields in (2.6) whose origin lies in the dimensionality of the coupling constant: $[\kappa] = \text{mass}^{-1}$.

### 2.3 Quantization of the Graviton Field, Perturbative Gauge Invariance and Ghost Coupling

We consider the graviton field $h^{\mu\nu}(x)$ as a free quantum tensor field which satisfies the wave equation (2.3) and quantize it by imposing the Lorentz covariant commutation rule

$$
[h^{\alpha\beta}(x), h^{\mu\nu}(y)] = -i b^{\alpha\beta\mu\nu} D_0(x - y),
$$

with

$$
b^{\alpha\beta\mu\nu} := \frac{1}{2} \left( \eta^{\alpha\mu} \eta^{\beta\nu} + \eta^{\alpha\nu} \eta^{\beta\mu} - \eta^{\alpha\beta} \eta^{\mu\nu} \right);
$$

and $D_0(x)$ is the mass-zero Jordan–Pauli causal distribution:

$$
D_0(x) = D_0^{(+)}(x) + D_0^{(-)}(x) = \frac{1}{2\pi} \delta(x^2) \text{sgn}(x^0)
$$

$$
= \frac{i}{(2\pi)^3} \int d^4 p \delta(p^2) \text{sgn}(p^0) e^{-ip\cdot x}.
$$

The gauge content of quantum gravity is formulated by means of the Lorentz invariant and time-independent gauge charge $Q$:

$$
Q := \int_{x^0=\text{const}} d^3 x h^{\alpha\beta}(x)_{\beta} \overset{\leftarrow}{\partial_0} u^\alpha(x),
$$

where $u^\mu(x)$ is a C-number vector field satisfying $\Box u^\mu(x) = 0$. The gauge charge generates the infinitesimal operator gauge transformation of the graviton field $h^{\mu\nu}(x)$:

$$
d_Q h^{\mu\nu}(x) := [Q, h^{\mu\nu}(x)] = -i b^{\mu\nu\rho\sigma} u^\rho(x)_{,\sigma}.
$$

The ten components of the symmetric rank-2 tensor $h^{\mu\nu}$ contain more than the true physical degrees of freedom of a massless spin-2 particle, this additional freedom could be suppressed by a gauge condition $h^{\mu\nu}_{,\nu} = 0$ and a trace condition
\( h = 0 \). As in gauge theories these conditions are disregarded at the beginning and considered later as conditions on the physical states. In the explicit construction of the Fock space for the physical graviton states is carried out and there it is shown that the physical subspace can be defined as \( \mathcal{F}_{\text{phys}} = \ker \{ Q, Q^\dagger \} \).

Gauge invariance of the \( S \)-matrix means
\[
\lim_{g \to 1} [Q, S(g)] = 0.
\]
This condition is fulfilled, if the perturbative gauge invariance condition for the \( n \)-point distribution
\[
d_Q T_n(x_1, \ldots, x_n) = \text{sum of divergences},
\]
holds, because divergences do not contribute in the adiabatic limit \( g \to 1 \) due to partial integration and Gauss’ theorem.

For \( n = 1 \) the above requirement is not at all trivial, because
\[
d_Q u^\nu(x) := \{ Q, u^\nu(x) \} = 0, \quad d_Q \tilde{u}^\nu(x) := \{ Q, \tilde{u}^\nu(x) \} = i \eta^{\mu\nu} D_0(x - y),
\]
whereas all other anti-commutators vanish. The ghost and anti-ghost fields undergo the infinitesimal operator gauge variations
\[
d_Q u^\alpha(x) := \{ Q, u^\alpha(x) \} = 0, \quad d_Q \tilde{u}^\alpha(x) := \{ Q, \tilde{u}^\alpha(x) \} = i h^{\alpha\beta}(x),\beta
\]
under the action of \( Q \), so that the sum of (2.6) and (2.14) preserves perturbative gauge invariance (2.13) to first order:
\[
d_Q T_1^{h+u}(x) = d_Q \left( T_1^h(x) + T_1^u(x) \right) =: \partial_\nu^\rho T_1^{\nu/1}(x).
\]
One possible form of \( T_1^{\nu/1}(x) \), the so-called \( Q \)-vertex, was derived in (33).

The definition of the \( Q \)-vertex from Eq. (2.17) allows us to give a precise prescription on how the right side of Eq. (2.13) has to be inductively constructed. We define the concept of ‘perturbative quantum operator gauge invariance’ by the equation
\[
d_Q T_n(x_1, \ldots, x_n) = \sum_{l=1}^n \frac{\partial}{\partial x^\nu_l} T_{n/l}^\nu(x_1, \ldots, x_l, \ldots, x_n),
\]
where \( T_{n/l}^\nu \) is the ‘renormalized’ time-ordered product, obtained according to the inductive causal scheme, with a \( Q \)-vertex at \( x_l \), while all other \( n - 1 \) vertices are ordinary \( T_1 \)-vertices.
2.4 Consequences of Perturbative Gauge Invariance to Second Order for Two-Point Distributions

We derive now some consequences from the condition of perturbative gauge invariance to second order for loop graphs. From the structure of \( T_1 = T_1^h + T_1^u \), it follows straightforwardly that by performing two field contractions (3.1) the resulting 2-point distribution \( T_2(x, y) \) will be of the form

\[
\begin{align*}
T_2(x, y) &= + :h^{\alpha \beta}(x)h^{\mu \nu}(y): \ t_{hh}(z)^{\alpha \beta \mu \nu} + :h^{\alpha \beta}(x),_t h^{\mu \nu}(y): \ t_{\partial hh}(z)^{\alpha \beta \mu \nu} + \\
&+ :h^{\alpha \beta}(x)h^{\mu \nu}(y),_\rho: \ t_{h\partial h}(z)^{\alpha \beta \mu \nu} + :h^{\alpha \beta}(x),_t h^{\mu \nu}(y),_\rho: \ t_{\partial h\partial h}(z)^{\alpha \beta \mu \nu} + u^{\gamma}(x)\tilde{u}^\mu(y),_\rho: \ i t_{u\partial u\partial u}(z)^{\gamma \mu} + \tilde{u}^\alpha(x),_\beta u^\delta(y): \ i t_{\partial u\partial u\partial u}(z)^{\alpha \beta \delta} + \\
&+ :u^{\gamma}(x)\tilde{u}^\mu(y),_\rho: \ i t_{u\partial u\partial u}(z)^{\gamma \mu} + \tilde{u}^\alpha(x),_\beta u^\delta(y): \ i t_{\partial u\partial u\partial u}(z)^{\alpha \beta \delta},
\end{align*}
\]

where \( z := x - y \). The subscript on the numerical \( t \)-distribution denotes the structure of the external fields attached to them. This \( T_2(x, y) \) describes the graviton self-energy and the ghost self-energy. The corresponding tensors will be given in Sec. 3.6 and in Sec. 7.1 respectively.

Perturbative gauge invariance to second order

\[
d_Q T_2(x, y) = \partial_\nu^\rho T'^{\nu}_{2/1}(x, y) + \partial_\rho^\nu T'^{\nu}_{2/2}(x, y)
\]

(2.20)

enables us to derive a set of identities for these distributions by comparing the distributions attached to same external operators on both sides of (2.20).

The left side of (2.20) is obtained by calculating the infinitesimal gauge variations of the external fields\(^1\) by means of Eqs. (2.11) and (2.16) and we isolate the terms with external operators of the type \( :u(x)h(y): \) so that

\[
\begin{align*}
d_Q T_2(x, y) &= + b^{\alpha \beta \gamma \delta} \left[ :u^{\gamma}(x),_\delta h^{\mu \nu}(y): \ t_{hh}(z)^{\alpha \beta \mu \nu} + \tilde{u}^\alpha(x),_\beta u^\delta(y): \ t_{\partial h\partial h}(z)^{\alpha \beta \mu \nu} + \\
&+ :u^{\gamma}(x),_\delta h^{\mu \nu}(y),_\rho: \ t_{h\partial h}(z)^{\alpha \beta \mu \nu} + \tilde{u}^\alpha(x),_\beta u^\delta(y),_\rho: \ t_{\partial h\partial h}(z)^{\alpha \beta \mu \nu} \right] + (2.21)
\end{align*}
\]

On the other side, \( \partial_\nu^\rho T'^{\nu}_{2/1} + \partial_\rho^\nu T'^{\nu}_{2/2} \) contains also operators of this type. Using a simplified notation which keeps track of the field type, of the derivatives and of the position of the \( \nu \)-index which forms the divergence in (2.21), then the \( Q \)-vertex of (2.17) reads (see 3.3 for the detailed form):

\[
\begin{align*}
T'^{\nu}_{1/1}(x) := & + :\partial u\partial u: + :u\partial h\partial h: + :u^\nu\partial h\partial h: + :\partial u\partial u: + \\
&+ :\partial u\partial h\partial h: + :u\partial h\partial h: + :\partial u\partial h\partial h: + :\partial u\partial u: + :u\partial u: + :\partial u\partial u: + :\tilde{u}\partial u: + :\tilde{u}\partial u: + .
\end{align*}
\]

\(^1d_Q(\tilde{u}u) = - :uQ, \tilde{u}: = - uh: \) and \( d_Q(\tilde{u}u) = :Q, \tilde{u}: = :hu: \)
while the ‘normal’ vertex reads

\[ T_1 := h \partial \theta \partial h: + \partial u \partial h u: + \partial \theta h \partial u: \quad (2.23) \]

Then, performing two contractions between \( T_{1/1}^\nu(x) \) and \( T_1(y) \), the contributions in \( T_{2/1}^\nu \) which have external operators of the type \( :u(x)h(y): \) are

\[
T_{2/1}^\nu(x, y) = + :u^\gamma(x) \partial_x h^{\mu \nu}(y) : t^\nu_{\partial u \partial h}(z)_{\partial \gamma}^{\mu \nu} + :u^\gamma(x) \partial_x h^{\mu \nu}(y) : t^\nu_{\partial u \partial h}(z)_{\partial \mu}^{\gamma \nu} + \\
+ :u^\gamma(x) h^{\mu \nu}(y) : t^\nu_{\partial_u \partial h}(z)_{\partial \gamma}^{\mu \nu} + :u^\gamma(x) h^{\mu \nu}(y) : t^\nu_{\partial_u \partial h}(z)_{\partial \mu}^{\gamma \nu} + \\
+ :u^\nu(x) h^{\mu \nu}(y) : t_{\partial u h}(z)_{\partial \mu}^{\nu \nu} + :u^\nu(x) h^{\mu \nu}(y) : t_{\partial u h}(z)_{\partial \nu}^{\nu \nu} + \\
+ :u^\nu(x) \partial_x h^{\mu \nu}(y) : t_{\partial u h}(z)_{\partial \mu}^{\nu \nu} + :u^\nu(x) \partial_x h^{\mu \nu}(y) : t_{\partial u h}(z)_{\partial \nu}^{\nu \nu} + \\
+ :u^\gamma(x, \partial_x h^{\mu \nu}(y), \partial_x) : t_{\partial \partial u h}(z)_{\partial \gamma \partial \mu}^{\nu \nu} + :u^\gamma(x, \partial_x h^{\mu \nu}(y), \partial_x) : t_{\partial \partial u h}(z)_{\partial \gamma \partial \nu}^{\nu \nu} + (2.24)
\]

One should not forget that there exist also terms with external operators of the type \( :u(x)h(y): \) coming from \( T_{2/2}^\nu(x, y) \), which is inductively constructed with \( T_1(x) \) and \( T_{1/1}^\nu(y) \). They read

\[
T_{2/2}^\nu(x, y) = + :u^\gamma(x) h^{\mu \nu}(y) : l^\nu_{\partial u \partial h}(z)_{\partial \gamma}^{\mu \nu} + :u^\gamma(x) h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \nu}^{\gamma \nu} + \\
+ :u^\gamma(x) h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \mu}^{\gamma \nu} + :u^\gamma(x) h^{\mu \nu}(y) : l^\nu_{\partial u \partial h}(z)_{\partial \mu}^{\gamma \nu} + \\
+ :u^\nu(x) h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \mu}^{\nu \nu} + :u^\gamma(x) \partial_x h^{\mu \nu}(y) : l^\nu_{\partial u \partial h}(z)_{\partial \gamma}^{\mu \nu} + \\
+ :u^\gamma(x) \partial_x h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \mu}^{\nu \nu} + :u^\gamma(x) \partial_x h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \nu}^{\nu \nu} + \\
+ :u^\gamma(x) \partial_x h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \nu}^{\nu \nu} + :u^\gamma(x) \partial_x h^{\mu \nu}(y) : l_{\partial u \partial h}(z)_{\partial \mu}^{\nu \nu} + (2.25)
\]

Here, the numerical distributions are denoted by \( l \). According to Eq. \( (2.20) \), we have to apply \( \partial_x^\nu \) to \( T_{2/1}^\nu \), \( (2.24) \), and \( \partial_x^\mu \) to \( T_{2/2}^\nu \), \( (2.25) \). After that, we gather the various terms according to their Lorentz structures given by the position of the indices and the number of derivatives acting on the external fields. We compare then the C-number distributions attached to the external operators:

\[
:u^\gamma(x) \partial_x h^{\mu \nu}(y) :, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y), :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \\
:u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y), :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \\
:u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y), :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \\
:u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y), :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \\
:u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad :u^\gamma(x) \partial_x h^{\mu \nu}(y), :u^\gamma(x) \partial_x h^{\mu \nu}(y) : \partial_x, \quad (2.26)
\]
between $d_Q T_2$ and $\partial^\nu_\alpha T_2^\nu_\mu / 1 + \partial^\nu_\alpha T_2^\nu_\mu / 2$ so that we obtain the identities

(i1), $t^{\alpha \beta \gamma \delta} t_{hh}(z)^{\alpha \beta \mu \nu} = \left\{ + \partial_\gamma^\nu t_{\partial hh}(z)^{\gamma \mu \nu} + t^\delta_\delta(z)^{\gamma \mu \nu} + \gamma^\delta t_{\partial hh}(x)^{\mu \nu} + \right. \\
+ \partial_\gamma^\nu t_{\partial hh}(z)^{\gamma \mu \nu} + \partial_\delta^\mu \partial_\delta t_{\partial hh}(z)^{\gamma \mu \nu} + \partial_\delta^\mu t_{\partial hh}(x)^{\gamma \mu \nu} \right\} ;$

(i2), $t^{\alpha \beta \gamma \delta} t_{\partial hh}(z)^{\alpha \beta \mu \nu} = \left\{ + t^\gamma_\gamma t_{\partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma t_{\partial hh}(z)^{\mu \nu} + \partial_\gamma^\nu t_{\partial hh}(z)^{\gamma \mu \nu} + \right. \\
+ \partial_\gamma^\nu t_{\partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i3), $t^{\alpha \beta \gamma \delta} t_{\partial \partial hh}(z)^{\alpha \beta \mu \nu} = \left\{ + \eta^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + t^\delta_\delta(z)^{\gamma \mu \nu} + \eta^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \right. \\
+ \partial^\gamma_t \partial_{\partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma_\delta \partial_{\partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma t_{\partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i4), $t^{\alpha \beta \gamma \delta} t_{\partial \partial hh}(z)^{\alpha \beta \mu \nu} = \left\{ + t^\gamma_\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma t_{\partial hh}(z)^{\mu \nu} + \partial_\gamma^\nu t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \right. \\
+ \partial_\gamma^\nu t_{\partial \partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i5), $t_{\mu \nu}(z)^{\gamma \mu \nu} \eta^\gamma = \left\{ + \eta^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma \partial_{\partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i6), $t_{\partial \partial \partial hh}(z)^{\gamma \mu \nu} \eta^\gamma = \left\{ + \eta^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma \partial_{\partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i7), $0 = \left\{ + \partial^\gamma t_{\partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i8), $0 = \left\{ + \partial^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \partial^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i9), $0 = \left\{ + \eta^\gamma t_{\partial \partial \partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma t_{\partial \partial \partial hh}(z)^{\gamma \mu \nu} \right\} ;$

(i10), $0 = \left\{ + \eta^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} + \eta^\gamma t_{\partial \partial hh}(z)^{\gamma \mu \nu} \right\} .

(2.27)

These identities hold among the C-number 2-point distributions constructed in second order perturbation theory. Some of them have been explicitly checked by calculations, but there is no doubt about their validity, because in Sec. 2.2 the condition (2.20) of perturbative gauge invariance to second order for loop graphs is proved.
In the case of QG coupled to photon fields \cite{31} and scalar matter fields \cite{32}, these identities are less involved and from them we can derive easily the Slavnov–Ward identities for the 2-point connected Green function with photon and matter loop, respectively. We will return on these identities and their relation to the gravitational Slavnov–Ward identities in Sec. 4.

3 Two-Point Distribution for Graviton Self-Energy

It is our aim in this section to apply the causal scheme to QG in order to calculate the 2-point distribution $T_2(x, y)$ which describes the graviton self-energy contribution. We explain step by step how $T_2(x, y)$ has to be constructed according to the general rules of the causal scheme \cite{23}. The are two important pieces in the inductive calculation that we are going to carry out: the first one is the calculation in momentum space of the product of positive/negative parts of Jordan–Pauli distributions (see App. 1 and App. 2 for the technical details) and the second one is the causal splitting procedure (see Sec. 3.5) according to the correct singular order (see Sec. 3.4).

3.1 Inductive Construction

First of all, from the commutation rules (2.7) and (2.15) we compute the contractions between two field operators:

\[
C\{ h^{\alpha\beta}(x) h^{\mu\nu}(y) \} := [ h^{\alpha\beta}(x)^{(-)}, h^{\mu\nu}(y)^{(+)} ] = -i b^{\alpha\beta\mu\nu} D_0^{(+)}(x - y),
\]

\[
C\{ u^\mu(x) \tilde{u}^\nu(y) \} := \{ u^\mu(x)^{(-)}, \tilde{u}^\nu(y)^{(+)} \} = +i \eta^\mu\nu D_0^{(+)}(x - y),
\]

\[
C\{ \tilde{u}^\mu(x) u^\nu(y) \} := \{ \tilde{u}^\mu(x)^{(-)}, u^\nu(y)^{(+)} \} = -i \eta^\mu\nu D_0^{(+)}(x - y); \tag{3.1}
\]

where $(\pm)$ refers to the positive/negative frequency part of the corresponding quantity.

The first step in the construction of $T_2(x, y)$ consists in calculating the auxiliary distributions

\[
R'_2(x, y) := -T_1^{h+u}(y) T_1^{h+u}(x), \quad A'_2(x, y) := -T_1^{h+u}(x) T_1^{h+u}(y) \tag{3.2}
\]

from these we form the causal distribution

\[
D_2(x, y) := R'_2(x, y) - A'_2(x, y) = \left[ T_1^{h+u}(x), T_1^{h+u}(y) \right]. \tag{3.3}
\]

Causal means that the numerical part of $D_2(x, y)$ has support inside the light cone. Being $T_1^{h+u}(x)$ a normally ordered product, we have to carry out all possible contractions between the two factors in (3.2) using Wick’s lemma. In this manner $D_2(x, y)$ contains tree contributions or scattering graphs (only one contraction and four external legs), loop contributions (two contractions and two external legs) and vacuum graph contributions (three contractions and no external legs). Note that, due to the presence of normal ordering, tadpole diagrams do not appear in causal perturbation theory.
3.2 Example of the Calculation

Let us illustrate how to construct $D_2(x, y)$ by explicitly working out an example. We take into account only the first term in the graviton coupling $T_1^b(x)$ so that, from the $A'_2(x, y)$-distribution, that can be decomposed as a sum of 25 different contributions $\sum_{i,j=1}^{5} A'_2(x, y)^{(i,j)}$, we pick up only the term $A'_2(x, y)^{(1,1)}$, and in addition we carry out the loop generating double-contractions only between graviton fields that carry derivatives, so that for $A'_2(x, y)^{(1,1)}$ we obtain

$$A'_2(x, y)^{(1,1)} = - \left( \frac{i\kappa}{2} \right)^2 : h^{\alpha\beta} h^{\rho\sigma} h^{\alpha\beta} \cdots h^{\mu\nu} h^{\gamma\delta} h^{\gamma\delta} : \bigg|_{2 \text{ contractions}} =$$

$$= + \frac{\kappa^2}{4} : h^{\alpha\beta}(x) h^{\mu\nu}(y) : \left[ + C \{ h^{\rho\sigma}(x),_\alpha h^{\gamma\delta}(y) \} \cdot C \{ h^{\rho\sigma}(x),_\beta h^{\gamma\delta}(y) \} \right] + \text{other contractions}.$$  

(3.4)

Using the relations in (3.1), we find

$$A'_2(x, y)^{(1,1)} = + \frac{\kappa^2}{4} : h^{\alpha\beta}(x) h^{\mu\nu}(y) : \left[ + \left( - i b^{\rho\sigma\gamma\delta} \partial_{\alpha} \partial_{\mu} D_0^{(+)}(x - y) \right) \right.$$

$$\times \left( - i b^{\rho\sigma\gamma\delta} \partial_{\beta} \partial_{\nu} D_0^{(+)}(x - y) \right) + \left( - i b^{\rho\sigma\gamma\delta} \partial_{\alpha} \partial_{\mu} D_0^{(+)}(x - y) \right) \right) + \text{other contractions}.$$  

(3.5)

Since $b^{\rho\sigma\gamma\delta} b_{\rho\sigma\gamma\delta} = 10$ and $\partial^\mu D_0^{(+)}(x - y) = -\partial^\mu D_0^{(+)}(x - y)$ we obtain

$$A'_2(x, y) = : h^{\alpha\beta}(x) h^{\mu\nu}(y) : a'_2(x - y)^{(1,1)} + \ldots,$$  

(3.6)

where

$$a'_2(x - y)^{(1,1)} = - \frac{5\kappa^2}{2} \left( D_{\alpha\mu|\beta\nu}(x - y) + D_{\alpha\nu|\beta\mu}(x - y) \right), \quad D_{\alpha\mu|\beta\nu}^{(+)}(x - y) = \partial^\alpha \partial^\mu D_0^{(+)}(x - y) \cdot \partial^\beta \partial^\nu D_0^{(+)}(x - y).$$  

(3.7)

The products between derivatives of Jordan–Pauli distributions are calculated in App. 1. Analogously, by taking into account that $D_0^{(+)}(y - x) = -D_0^{(-)}(x - y)$, we find

$$R'_2(x, y)^{(1,1)} = : h^{\alpha\beta}(x) h^{\mu\nu}(y) : r'_2(x - y)^{(1,1)} + \ldots,$$  

(3.8)

where

$$r'_2(x - y)^{(1,1)} = - \frac{5\kappa^2}{2} \left( D_{\alpha\mu|\beta\nu}^{(-)}(x - y) + D_{\alpha\nu|\beta\mu}^{(-)}(x - y) \right), \quad D_{\alpha\mu|\beta\nu}^{(-)}(x - y) = \partial^\alpha \partial^\mu D_0^{(-)}(x - y) \cdot \partial^\beta \partial^\nu D_0^{(-)}(x - y).$$  

(3.9)
Therefore, according to Eq. (3.3), the $D_2(x,y)$-distribution has the form

\[ D_2(x,y)^{(1,1)} := h^{\alpha\beta}(x)h^{\mu\nu}(y) : d_2(x-y)^{(1,1)}_{\alpha\beta\mu\nu} + \ldots , \]

\[ d_2(x-y)^{(1,1)}_{\alpha\beta\mu\nu} = r_2(x-y)^{(1,1)}_{\alpha\beta\mu\nu} - d_2'(x-y)^{(1,1)}_{\alpha\beta\mu\nu} . \quad (3.10) \]

The most important property of $D_2(x,y)$ is causality, but only the numerical distribution $d_2(x-y)$ is responsible for this property:

\[ \text{supp}(d_2(z)) \subseteq V^+(z) \cup V^-(z) , \quad \text{with } z := x - y , \quad (3.11) \]

(see below). The products of Jordan–Pauli distributions appearing in (3.10) are easily expressed in momentum space, see App. 2, so that we obtain

\[ \hat{a}_2'(p)^{(1,1)}_{\alpha\beta\mu\nu} = - \hat{P}(p)^{(4)}_{\alpha\beta\mu\nu} \Theta(p^2) \Theta(+p^0) , \]

\[ \hat{r}_2'(p)^{(1,1)}_{\alpha\beta\mu\nu} = - \hat{P}(p)^{(4)}_{\alpha\beta\mu\nu} \Theta(p^2) \Theta(-p^0) ; \]

and therefore

\[ \hat{d}_2(p)^{(1,1)}_{\alpha\beta\mu\nu} = \hat{P}(p)^{(4)}_{\alpha\beta\mu\nu} \Theta(p^2) \left[ \Theta(p^0) - \Theta(-p^0) \right] = \hat{P}(p)^{(4)}_{\alpha\beta\mu\nu} \Theta(p^2) \sgn(p^0) , \]

\[ \quad (3.13) \]

where $\hat{P}(p)^{(4)}_{\alpha\beta\mu\nu}$ is a Lorentz covariant polynomial of degree four, this degree is given by the number of derivatives on the contracted lines. Causality is evident from the scalar distribution $\hat{d}(p) := \Theta(p^2) \sgn(p^0)$. For $z^2 < 0$, we may choose a reference frame in which $z^\alpha = (0, z)$, so that

\[ d(z) = \frac{1}{(2\pi)^2} \int d^0 p \sgn(p^0) \int d^3 p \Theta(p_0^2 - p^2) e^{+i p \cdot z} = 0 , \quad (3.14) \]

because of the signum-function in $p^0$. Therefore $d(z)$ vanishes outside the light cone, see Eq. (3.11).

### 3.3 Causal $D_2(x,y)$-Distribution for Graviton Self-Energy

The total $D_2(x,y)$-distribution for the graviton self-energy through a graviton loop is obtained by calculating the 25 contributions coming from the graviton coupling $T^I_g$, not only the terms with two external graviton fields without derivatives, but also these with one or two derivatives. In addition, there are also 16 contributions coming from the ghost-graviton coupling where one performs two ghost–anti-ghost contractions. Summing graviton loop and ghost–anti-ghost loop contributions we obtain \[ D_2(x,y) = + : h^{\alpha\beta}(x)h^{\mu\nu}(y) : d^{(4)}_2(x-y)^{\mu\nu}_{\alpha\beta\mu\nu} + \]

\[ + : h^{\alpha\beta}(x)\gamma h^{\mu\nu}(y) : d^{(3\alpha)}_2(x-y)^{\gamma\mu\nu}_{\alpha\beta\mu\nu} + \]

\[ + : h^{\alpha\beta}(x)h^{\mu\nu}(y)\rho : d^{(3\beta)}_2(x-y)^{\rho\mu\nu}_{\alpha\beta\mu\nu} + \]

\[ + : h^{\alpha\beta}(x)\gamma h^{\mu\nu}(y)\rho : d^{(2)}_2(x-y)^{\rho\gamma\mu\nu}_{\alpha\beta\mu\nu} . \]

\[ \text{the notation ‘\(\cdot\)’ keeps track of the exact position of the indices} \]
The tensorial distributions have in momentum space the structure
\[ \hat{d}_2^{(i)}(p)^\dagger_{\alpha\beta\mu\nu} = \hat{P}^{(i)}(p)^\dagger_{\alpha\beta\mu\nu} \hat{d}(p), \quad i = 4, 3a, 3b, 2. \] (3.16)
The results for the tensorial distributions, being too long, are not given here. See App. 4 for the explicit form of the distributions appearing in Eq. (3.15).

In order to obtain \( T_2(x, y) \), we have to split the \( D_2 \)-distribution into a retarded part, \( R_2 \), and an advanced part, \( A_2 \), with respect to the coincidence point \( z := x - y = 0 \), so that \( \text{supp}(R_2(z)) \subseteq V^+(z) \) and \( \text{supp}(A_2(z)) \subseteq V^-(z) \). The correct treatment of this coincidence point constitutes the key to control the UV behaviour of the 2-point distribution.

This splitting procedure affects only the numerical distributions \( d_2^{(i)} \) in (3.15) and must be accomplished according to the correct singular order \( \omega(d_2^{(i)}) \) of the distribution. It describes the behaviour of \( d_2^{(i)}(z) \) near the coincidence point \( z = 0 \), or that of \( \hat{d}_2^{(i)}(p) \) for \( p \to \infty \). If \( \omega(d_2^{(i)}) < 0 \), then the splitting is trivial. On the other side, if \( \omega(d_2^{(i)}) \geq 0 \), then the splitting is non-trivial and non-unique:
\[ d_2^{(i)}(x - y)_{\alpha\beta\mu\nu}^\dagger \rightarrow r_2^{(i)}(x - y)_{\alpha\beta\mu\nu}^\dagger + \sum_{|\alpha| = 0} \{ C_{a,i} D^a \}_{\alpha\beta\mu\nu} \delta^{(4)}(x - y), \] (3.17)
and a retarded part \( r_2^{(i)}(x - y) \) is best obtained in momentum space by means of a dispersion-like integral, see Sec. 3.5, which, however, presents some difficulties in the massless case.

The \( C_{a,i} \)'s in Eq. (3.17) are undetermined but finite normalization constants and \( D^a \) is a partial differential operator acting on the local distribution \( \delta^{(4)}(x - y) \). The second term on the right side of Eq. (3.17) represents therefore a freedom in the normalization which is inherent to the causal splitting of distributions and will be discussed in Sec. 3.6 by taking physical conditions into account.

In the case of Eq. (3.15), we find from direct inspection of the distributions that
\[ \omega(d_2^{(4)}) = 4, \quad \omega(d_2^{(3a)}) = 3, \quad \omega(d_2^{(3b)}) = 3, \quad \omega(d_2^{(2)}) = 2. \] (3.18)
The singular order depends on the structure of the graph, namely on the number of derivatives acting on the contracted internal lines of the loop. For a precise formulation, see below.

The last step in the inductive construction of \( T_2(x, y) \) consists in subtracting \( R'_2(x, y) \) from \( R_2(x, y) \), see Sec. 3.3. The singular order remains unchanged after distribution splitting: \( \omega(d_2) = \omega(r_2) = \omega(t_2) \).

### 3.4 Singular Order in Quantum Gravity

Before undertaking the splitting of \( D_2(x, y) \) according to Eq. (3.17), we give the formula for the singular order of arbitrary \( n \)-point distributions in perturbative quantum gravity.
We consider in the $n$-th order of perturbation theory an arbitrary $n$-point distribution $T^G_n(x_1, \ldots, x_n)$, appearing in Eq. (2.1), as a sum of normally ordered products of free field operators multiplied by numerical distributions

$$T^G_n(x_1, \ldots, x_n) =: \prod_{j=1}^{n_h} h(x_j) \prod_{i=1}^{n_u} u(x_{m_i}) \prod_{l=1}^{n_\tilde{u}} \tilde{u}(x_{n_l}) : T^G_n(x_1, \ldots, x_n).$$  

(3.19)

This $T^G_n$ corresponds to a graph $G$ with $n_h$ external graviton lines, $n_u$ external ghost lines and $n_\tilde{u}$ external anti-ghost lines. The singular order of $G$ then reads

$$\omega(G) \leq 4 - n_h - n_u - n_\tilde{u} - d + n.$$  

(3.20)

Here $d$ is the number of derivatives on the external field operators in (3.19). The ‘≤’ means that in certain cases the singular order is lowered by peculiar conditions, e.g. by the equations of motions of the free fields.

The explicit presence of the order of perturbation theory renders the theory ‘non-normalizable’, that is the theory has a weaker predictive power but it is still well-defined in the sense of UV finiteness.

We give some hints of the inductive proof of (3.20) [23], [37]. First of all, the assumption (3.20) must be verified for $n = 1$: $\omega(T^{b+u}_1(x))$ has to be zero (because of $\omega(\delta(x)) = 0$), a result which is correctly given by (3.20) after direct inspection.

In the inductive construction of $T_n$ from the $T_m$’s, $m \leq n - 1$, we must consider tensor products of two distributions

$$T_{r,1}(x_1, \ldots, x_r) T_{s,2}(y_1, \ldots, y_s),$$  

(3.21)

with known singular order $\omega(T_{r,1}) = \omega_1 = 4 - n_{h_1} - n_{u_1} - n_{\tilde{u}_1} - d_1 + r$ and $\omega(T_{s,2}) = \omega_2 = 4 - n_{h_2} - n_{u_2} - n_{\tilde{u}_2} - d_2 + s$. According to the inductive construction, this product has to be normally ordered giving origin to all possible contraction configurations. We assume that $l$ contractions arise during this process. Taking translation invariance into account the numerical distribution of the contracted expression is of the form

$$t_1(x_1 - x_r, \ldots, x_{r-1} - x_r) \prod_{j=1}^{l} \partial^{a_j} D^{(+)}_{0}(x_{r_j} - y_{s_j}) t_2(y_1 - y_s, \ldots, y_{s-1} - y_s) =$$

$$= \tilde{t}(\xi_1, \ldots, \xi_{r-1}, \eta_1, \ldots, \eta_{s-1}, \eta),$$  

(3.22)

with $\xi_j := x_j - x_r$, $\eta_j := y_j - y_s$, $\eta := x_r - y_s$, $a_j = 0, 1, 2$ and $a = \sum_{j=1}^{l} a_j$. Then, using the distributional definition of the singular order [23], we may conclude that

$$\omega(\tilde{t}) = \omega_1 + \omega_2 + 2l - 4 + a.$$  

(3.23)
Inserting the expressions for $\omega_1$ and $\omega_2$ in Eq. (3.23), we get

$$\omega(\tilde{t}) \leq 4 - (n_{h_1} + n_{h_2} + n_{u_1} + n_{u_2} + n_{\tilde{u}_1} + n_{\tilde{u}_2} - 2l) - (d_1 + d_2 - a) + (r + s).$$

(3.24)

The first bracket represents the number $n_{h} + n_{u} + n_{\tilde{u}}$ of external fields after $l$ contractions, the second bracket gives the number $d$ of derivatives remaining on these external fields, if the $l$ contractions carry $a$ derivatives. Since $r + s = n$, Eq. (3.20) is proved.

In the usual QFT formulation, Eq. (3.20) would imply that QG is 'non-renormalizable', since $\omega(G)$ increases without bound for higher orders in the perturbative expansion. This means that there is a 'proliferation' of divergences and of counterterms to compensate them.

The situation is different in causal perturbation theory: we are facing in this case a 'non-normalizable' theory, i.e. each of its diagrams is finite due to the causal splitting method, but the number of the free, undetermined and finite normalization constants in (3.17) increases with $n$. The question is then to find enough physical conditions or requirements to fix this increasing ambiguity in the normalization.

### 3.5 Splitting of the $D_2(x, y)$-Distribution

We now carry out the splitting of the distribution $D_2(x, y)$ in Eq. (3.15).

Let us consider for example the numerical tensorial distribution $\hat{d}^{(4)}(p)_{\alpha\beta\mu\nu}$ which has singular order four from Eq. (3.18) of from Eq. (3.20). Because of the decomposition (3.16), it suffices to split the scalar distribution $\hat{d}(p) = \Theta(p^2)\text{sgn}(p^0)$ with $\omega(\hat{d}) = 0$ and then multiply the so obtained retarded part by the same tensor $\hat{P}^{(4)}(p)_{\alpha\beta\mu\nu}$ as given by Eq. (3.16).

Usually, a special retarded part in Eq. (3.17), if it exists, is given in momentum space by the dispersion integral

$$\hat{r}_0(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{(t - i0)^{\omega+1}(1 - t + i0)}, \quad p \in \mathbb{V}_+;$$

(3.25)

which is called 'central splitting solution', because the subtraction point is the origin. But this formula cannot be used directly in the case of massless theories, because the integral is divergent. In order to circumvent this deficiency, we shift the original distribution $d(x)$:

$$d_q(x) := e^{iq \cdot x} d(x), \quad \hat{d}_q(p) = \hat{d}(p + q), \quad q^2 < 0,$$

(3.26)

so that the central splitting solution $\hat{r}_0(p)$ of the shifted distribution exists.

We cannot obtain the retarded part of the original distribution simply by letting $q \to 0$, so we take advantage of the local ambiguity in the splitting procedure and consider another retarded part of $\hat{d}_q(p)$, given by $\hat{r}_q(p)$. Since
two retarded distributions differ only by local terms in configuration space, we obtain in momentum space for fixed $q$ that the difference reads

$$\hat{r}_q(p) - \hat{r}_q^0(p) = \hat{P}_q(p), \quad (3.27)$$

where $\hat{P}_q(p)$ is a $q$-dependent polynomial in $p$ of degree $\omega$. Then we construct $\hat{r}(p)$, a retarded part of the original distribution $\hat{d}(p)$, from (3.27) by taking the limit

$$\hat{r}(p) = \lim_{q \to 0} \hat{r}_q(p) = \lim_{q \to 0} [\hat{r}_q^0(p) + \hat{P}_q(p)]. \quad (3.28)$$

Here the addition of the $q$-dependent polynomial $\hat{P}_q(p)$ must be accomplished in such a way that the limit exists. This corresponds to a finite renormalization.

Using (3.25) with (3.26) in (3.28), we obtain for $p \in V^+$, with $q \to 0$ in such a way that $p - q \in V^+$, $q^2 < 0$:

$$\hat{r}(p) = \lim_{q \to 0} \left[ \frac{i}{2 \pi} \int_{-\infty}^{+\infty} dt \frac{\hat{d}_q(tp)}{(t - i0)(1 - t + i0)} + \hat{P}_q(p) \right]$$

$$= \lim_{q \to 0} \left[ \frac{i}{2 \pi} \int_{-\infty}^{+\infty} dt \frac{\hat{d}(tp + q)}{(t - i0)(1 - t + i0)} + \hat{P}_q(p) \right] \quad (3.29)$$

$$= \lim_{q \to 0} \left[ \frac{i}{2 \pi} \int_{-\infty}^{+\infty} dt \frac{dt}{t - i0} (1 - t + i0) \Theta((tp + q)^2) \sgn(tp^0 + q^0) + \hat{P}_q(p) \right].$$

The zeros of $(tp + q)^2$ are $t_{1,2} = \frac{1}{p^2}(-p \cdot q \pm \sqrt{N})$ with $N := (p \cdot q)^2 - p^2q^2$, so that the integral in (3.29) may be simplified to

$$\left\{ \frac{-i}{2 \pi} \int_{t_{1,2} < 0} dt + \frac{i}{2 \pi} \int_{t_{1,2} > 0} dt \right\} \left( \frac{1}{t - P} - \frac{1}{t - 1} - i \pi \delta(t - 1) \right) =$$

$$= \frac{i}{2 \pi} \left[ -i \pi + \log \left( \frac{p^2}{|q^2|} \right) + O(\sqrt{|q^2|}) \right], \quad (3.30)$$

and the limit reads

$$\hat{r}(p) = \lim_{q \to 0} \left[ \frac{i}{2 \pi} \left( -i \pi + \log \left( \frac{p^2}{|q^2|} \right) + O(\sqrt{|q^2|}) \right) + \hat{P}_q(p) \right]. \quad (3.31)$$

Being $\omega = 0$, we can add the polynomial $\hat{P}_q(p) = \frac{i}{2 \pi} \log \left( |q^2|/M^2 \right)$, where $M > 0$ is an arbitrary mass scale, so that we obtain a Lorentz invariant splitting solution

$$\hat{r}(p) = \frac{i}{2 \pi} \left( \log \left( \frac{p^2}{M^2} \right) - i \pi \right), \quad p \in V^+. \quad (3.32)$$

By analytic continuation in $\mathbb{R}^4 + iV^+$, we find for $p \in \mathbb{R}^4$

$$\hat{r}(p)^{an} = \frac{i}{2 \pi} \log \left( \frac{-p^2 - ip^0}{M^2} \right). \quad (3.33)$$
As pointed out at the end of Sec. 3.3, the $T_2(x, y)$-distribution is obtained from $R_2(x, y)$ by subtracting $R'_2(x, y)$. This subtraction affects only the scalar distributions. Since $\hat{r}'(p) = -\Theta(p^2)\Theta(-p^0)$, we obtain

$$\hat{t}(p) = \hat{r}(p)\alpha_{\beta\mu\nu} - \hat{r}'(p) = \frac{i}{2\pi} \left( \log \left( \frac{-p^2}{M^2} \right) - i\pi \text{sgn}(p^0) \Theta(p^2) \right) + \Theta(p^2)\Theta(-p^0)$$

(3.34)

The ambiguity in the normalization present in the splitting $D_2 \rightarrow R_2 + N_2$, Eq. (3.17), will be discussed in Sec. 5.

### 3.6 Graviton Self-Energy Tensor from the $T_2(x, y)$-Distribution

Gathering all the results of the previous sections, Eqs. (3.15), (3.16) and (3.34), we find that the 2-point distribution that contributes to the graviton self-energy

$$T_2(x, y) = + : h^{\alpha\beta}(x)h^{\mu\nu}(y) : t_2^{(4)}(x - y)\alpha_{\beta\mu\nu} +$$

$$+ : h^{\alpha\beta}(x)\gamma h^{\mu\nu}(y) : t_2^{(3a)}(x - y)\alpha_{\beta\mu\nu} +$$

$$+ : h^{\alpha\beta}(x)h^{\mu\nu}(y)_{\rho} : t_2^{(3b)}(x - y)\alpha_{\beta\mu\nu} +$$

$$+ : h^{\alpha\beta}(x)\gamma h^{\mu\nu}(y)_{\rho} : t_2^{(2)}(x - y)\alpha_{\beta\mu\nu},$$

(3.35)

where the tensorial distributions have in momentum space the structure

$$\tilde{t}_2^{(i)}(p)\alpha_{\beta\mu\nu} = \tilde{P}_2^{(i)}(p)\alpha_{\beta\mu\nu} \hat{t}(p), \quad i = 4a, 3b, 2,$$

(3.36)

where $\hat{t}(p)$ is given in Eq. (3.34), and the polynomials are those of Eq. (3.16).

The $\tilde{t}_2^{(4)}$-distributions appearing in Eq. (3.33) have already been introduced in Eq. (2.19): $\tilde{t}_2^{(4)} = i\tilde{t}_{hh}, \tilde{t}_2^{(3a)} = i\tilde{t}_{h\rho h}, \tilde{t}_2^{(3b)} = i\tilde{t}_{h\rho 3h}$ and $\tilde{t}_2^{(2)} = i\tilde{t}_{h\rho 3h}$.

Since divergences in the adiabatic limit of Eq. (2.17) do not contribute, we can obtain from $T_2(x, y)$ by partial integration the graviton self-energy contribution

$$T_2(x, y)_{hSE} = : h^{\alpha\beta}(x)h^{\mu\nu}(y) : i\Pi(x - y)\alpha_{\beta\mu\nu}.$$  

(3.37)

The main result of our calculation in second order causal perturbation theory is the graviton self-energy tensor $\Pi(x - y)\alpha_{\beta\mu\nu}$ which is given by the following combination of $t_2(x - y)$-distributions

$$i\Pi(x - y)\alpha_{\beta\mu\nu} := + t_2^{(4)}(x - y)\alpha_{\beta\mu\nu} - \partial_\gamma t_2^{(3a)}(x - y)\alpha_{\beta\mu\nu} +$$

$$+ \partial_\rho t_2^{(3b)}(x - y)\alpha_{\beta\mu\nu} - \partial_\gamma \partial_\rho t_2^{(2)}(x - y)\alpha_{\beta\mu\nu},$$

(3.38)
where we have carried the derivatives acting on the external fields in Eq. (3.35) on the corresponding \( t_2(x - y) \)-distributions. In momentum space, it reads

\[
\hat{\Pi}(p)_{\alpha\beta\mu\nu} = \hat{\Pi}^{\text{grav. loop}}_{\alpha\beta\mu\nu} + \hat{\Pi}^{\text{ghost loop}}_{\alpha\beta\mu\nu} = \frac{\kappa^2 \pi}{960(2\pi)^5} \left[ -656 p^\alpha p^\beta p^\mu p^\nu - 208 p^2 (p^\alpha p^\beta \eta^{\mu\nu} + p^\mu p^\nu \eta^{\alpha\beta}) + 162 p^2 (p^\alpha p^\beta \eta^{\mu\nu} + p^\mu p^\nu \eta^{\alpha\beta} + p^\beta p^\alpha \eta^{\mu\nu} + p^\mu p^\nu \eta^{\alpha\beta}) - 162 p^4 (\eta^{\alpha\mu} \eta^{\beta\nu} + \eta^{\alpha\nu} \eta^{\beta\mu}) + 118 p^4 \eta^{\alpha\beta} \eta^{\mu\nu} \right] \log \left( \frac{-p^2 - i0}{M^2} \right). 
\]

(3.39)

Separate calculations for the graviton loop and ghost loop give the following contributions to the graviton self-energy tensor, respectively (see App. 4):

\[
\hat{\Pi}^{\text{grav. loop}}_{\alpha\beta\mu\nu} = \Xi \left[ -880, -260, +160, -170, +110 \right] \log \left( \frac{-(p^2 - i0)/M^2}{(2\pi)^5} \right),
\]

\[
\hat{\Pi}^{\text{ghost loop}}_{\alpha\beta\mu\nu} = \Xi \left[ +224, +52, +2, +8, +8 \right] \log \left( \frac{-(p^2 - i0)/M^2}{(2\pi)^5} \right); 
\]

(3.40)

where we have adopted the convention of writing only the coefficients of the tensor according to the structure given in Eq. (3.39) and \( \Xi := \kappa^2 \pi/960(2\pi)^5 \).

Our result, numerical coefficients and logarithmic dependence on \( p^2/M^2 \), agrees with the finite part of previous calculations \[25\], \[38\] obtained using ad-hoc regularization schemes. As a consequence, the absence of UV divergences means that we do not need to add counterterms \[14\], \[15\] involving four derivatives to the original Hilbert–Einstein Lagrangian in order to obtain UV finite radiative corrections to the graviton propagator. In our approach, all the expressions are cutoff-free and finite at each stage of the calculation due to the causal scheme.

4 Gravitational Slavnov–Ward Identities and Perturbative Gauge Invariance

Gravitational Slavnov–Ward Identities

The gravitational Slavnov–Ward identities (SWI) \[25\], \[26\], \[38\], \[88\] are derived in standard quantum field theory from the connected Green functions. We construct the 2-point connected Green function as

\[
\hat{G}(p)_{\alpha\beta\mu\nu} := b_{\alpha\beta\gamma\delta} \hat{D}_0^\gamma(p) \hat{\Pi}(p)^8_{\gamma\delta\rho\sigma} b_{\rho\sigma\mu\nu} \hat{D}_0^\rho(p),
\]

(4.1)

where \( \hat{D}_0^\gamma(p) = (2\pi)^{-2}(-p^2 - i0)^{-1} \). The two attached lines represent free graviton Feynman propagators. Then the gravitational SWI reads

\[
p^\alpha p^\rho \hat{G}(p)_{\alpha\beta\mu\nu} = 0.
\]

(4.2)
Since the tensorial structure of a general self-energy tensor may be characterized by the five coefficients $A, B, C, E, F$ in the standard representation $\hat{\Pi}(p) = \Xi \left[ A, B, C, E, F \right] \log \left( (-p^2 - i0)/M^2 \right)$ as in Eq. (3.39), then the SWI are equivalent to the following relations

$$\frac{A}{4} - B + E + F = 0, \quad C + E = 0.$$  

(4.3)

These relations are satisfied by the coefficients of the self-energy tensor in Eq. (3.39), only if both graviton and ghost loops are taken into account. Therefore our result satisfies the SWI.

**SWI and Other First Order Couplings**

If we had chosen another ghost coupling instead of the one in Eq. (2.14), for example $T^u_1(x) = -\frac{i\kappa}{2} \left( + : \tilde{u}^\mu h^\nu_{\rho \sigma} u^\rho_{\nu} : + 2 : \tilde{u}^\mu_{\rho} h^\nu_{\rho \sigma} u^\rho_{\nu} : \right)$, then we would have violated the SWI, because in this case the new graviton self-energy tensor through graviton and ghost loop would have had the form

$$\hat{\Pi}(p)_{\alpha\beta\mu\nu}^{\text{new}} = \Xi \left[ -836, -238, +\frac{309}{2}, -162, +118 \right] \log \left( (-p^2 - i0)/M^2 \right),$$

(4.4)

so that it would have not satisfied the SWI in Eq. (4.3).

Analogously, if we had disregarded the last two terms in the graviton coupling $T^h_i(x)$, Eq. (2.1), being a divergence due to the presence of two equal derivatives, then we would have violated the SWI, too, because in this case we would have obtained a ‘reduced’ graviton self-energy tensor through graviton loop of the form

$$\hat{\Pi}(p)_{\alpha\beta\mu\nu}^{\text{red.}} = \Xi \left[ -880, +160, -108, +58, -62 \right] \log \left( (-p^2 - i0)/M^2 \right),$$

(4.5)

so that the sum of graviton and ghost loop would have not satisfied the SWI in Eq. (4.3). This happens although the difference between the two tensors can be written as a divergence

$$\hat{\Pi}(p)^{\text{grav.\ loop}}_{\alpha\beta\mu\nu} - \hat{\Pi}(p)^{\text{red.}}_{\alpha\beta\mu\nu} = \partial_\sigma^x \Omega^\sigma(x - y) = \text{divergence},$$

(4.6)

due to the vanishing of the coefficient proportional to $p^\alpha p^\beta p^\rho p^\nu$ and therefore this difference should not be physically relevant in the adiabatic limit $g \to 1$ of $S(g)$.

**SWI and Perturbative Gauge Invariance**

The SWI for QG coupled to photon fields [21] and matter fields [22] are equivalent to the transversality of $\hat{G}(p)_2^{\alpha\beta\mu\nu}$, namely $p^\alpha \hat{G}(p)_2^{\alpha\beta\mu\nu}$. In the case of self-coupled and ghost-coupled gravitons, the complexity of the gauge structure implies that only the weaker condition (4.2) can be satisfied by the graviton self-energy tensor, which is not transverse.
Perturbative gauge invariance to second order for loop graphs can be formulated by means of the identities (i1), . . . , (i10) in Eq. (2.27). Note that these identities can be easily generalized to the n-th order for graphs with two external operators ('legs'). The identities will imply a relation among the graviton self-energy tensor

\[ \Pi(p)^{\alpha\beta\mu\nu} = t_{hh}(p)^{\alpha\beta\mu\nu} + i p_\epsilon t_{\partial\partial\epsilon}(p)^{\alpha\beta\mu\nu} - i p_\rho t_{h\partial\epsilon}(p)^{\alpha\beta\mu\nu} + p_\epsilon p_\rho t_{\partial\partial\epsilon}(p)^{\alpha\beta\mu\nu}, \tag{4.7} \]

the ghost self-energy tensors \( t_{u\partial\epsilon}(p) \) and \( t_{\partial\u\partial\epsilon}(p) \) and distributions coming from the Q-vertex, namely \( t_{u\partial\epsilon}, t_{\partial\u\partial\epsilon} \) and others with one graviton and one ghost as external legs. We work in momentum space, but we skip the hat on the \( \epsilon \) and others with one graviton and one ghost.

From the structure of the first four identities in (2.27), it follows straightforwardly that

\[ b^{\gamma\alpha\beta} \Pi(p)^{\alpha\beta\mu\nu} = \frac{1}{2} \left[ + t^\delta_{u\partial\epsilon}(p)^{\gamma|\mu\nu} + \eta^{\gamma\delta} t_{u\partial\epsilon}(p)^{|\mu\nu} - i p_\tau t^\delta_{u\partial\epsilon}(p)^{|\mu\nu} + \right. \]

\[ \left. - i \eta^{\gamma\delta} p_\tau t_{u\partial\epsilon}(p)^{|\mu\nu} + \gamma \leftrightarrow \delta \right] + \frac{1}{4} \left[ + p^2 t_{\partial\u\partial\epsilon}(p)^{|\mu\nu} + p_\epsilon p_\tau t^\lambda_{\partial\u\partial\epsilon}(p)^{|\mu\nu} \right. \]

\[ + p_\epsilon p_\tau t^\lambda_{\partial\u\partial\epsilon}(p)^{|\mu\nu} + \frac{1}{4} \left( \gamma^{|\mu} + \gamma \leftrightarrow \mu \leftrightarrow \nu \right). \tag{4.8} \]

Here we have used (i1), . . . , (i4) to express the graviton self-energy distributions by means of the Q-vertex distributions and also (i9), (i10) to simplify the expression. Symmetrization of the left side is indeed necessary, because the right side is symmetric. With (i6), we can write the second bracket as

\[ \frac{p_\epsilon p_\tau}{4} \left[ + \eta^{\nu\tau} t_{\partial\u\partial\epsilon}(p)^{|\mu\nu} \right. \]

\[ \left. + \mu \leftrightarrow \nu + \gamma \leftrightarrow \delta \right]. \tag{4.9} \]

The connected 2-point Green function has a second b-tensor attached on the right, therefore we compute

\[ b^{\gamma\alpha\beta} \Pi(p)^{\alpha\beta\mu\nu} b^{\rho\sigma} = \frac{1}{2} \left[ + t^\delta_{u\partial\epsilon}(p)^{\gamma|\rho\sigma} + 2 t^\delta_{u\partial\epsilon}(p)^{\gamma|\rho\sigma} - \frac{1}{2} t^\delta_{u\partial\epsilon}(p)^{|\rho\sigma} - \frac{1}{2} t^\delta_{u\partial\epsilon}(p)^{|\rho\sigma} + \gamma \leftrightarrow \delta + \right. \]

\[ + \frac{1}{2} \eta^{\gamma\delta} t_{u\partial\epsilon}(p)^{|\rho\sigma} + \frac{1}{2} \eta^{\gamma\delta} t_{u\partial\epsilon}(p)^{|\rho\sigma} - \frac{1}{2} \eta^{\gamma\delta} t_{u\partial\epsilon}(p)^{|\rho\sigma} + \gamma \leftrightarrow \delta + \]

\[ - i \frac{1}{2} p_\epsilon t^\delta_{u\partial\epsilon}(p)^{\gamma|\rho\sigma} - i \frac{1}{2} p_\epsilon t^\delta_{u\partial\epsilon}(p)^{\gamma|\rho\sigma} + i \frac{1}{2} p_\epsilon t^\delta_{u\partial\epsilon}(p)^{\gamma|\rho\sigma} + \eta^{\gamma\delta} p_\tau t_{u\partial\epsilon}(p)^{|\rho\sigma} + \gamma \leftrightarrow \delta + \]

\[ - i \frac{1}{2} \eta^{\gamma\delta} p_\tau t_{u\partial\epsilon}(p)^{|\rho\sigma} - i \frac{1}{2} \eta^{\gamma\delta} p_\tau t_{u\partial\epsilon}(p)^{|\rho\sigma} + i \frac{1}{2} \eta^{\gamma\delta} p_\tau t_{u\partial\epsilon}(p)^{|\rho\sigma} + \gamma \leftrightarrow \delta + \]

\[ + \frac{p_\epsilon p_\tau}{4} \left( \eta^{\nu\tau} t_{\partial\u\partial\epsilon}(p)^{|\mu\nu} + \mu \leftrightarrow \nu + \gamma \leftrightarrow \delta \right) 2 b^{\mu\nu\rho\sigma} \right]. \tag{4.10} \]
We apply now $p_{\gamma}$ to the above expression. This enables us to use other identities
((i7) and (i8)), so that after a little work we get

$$p_{\gamma} \left( b^\gamma \delta_{\alpha \beta} \Pi(p)^{\alpha \beta \mu \nu} \gamma_{\mu \nu} \right) = \frac{1}{2} \left[ + p_{\gamma} t^\delta_{\alpha \beta}(p) \gamma | \rho \sigma - \frac{1}{2} p_{\gamma} t^\delta_{\alpha \beta}(p) \gamma | \mu \eta \right]$$

$$+ p_{\delta} t_{\alpha \beta}(p) | \rho \sigma - \frac{1}{2} p_{\delta} t_{\alpha \beta}(p) | \mu \eta$$

$$- i p_{\gamma} p_{\epsilon} t^\delta_{\alpha \beta}(p) \gamma | \rho \sigma + \frac{i}{2} p_{\gamma} p_{\epsilon} t^\delta_{\alpha \beta}(p) \gamma | \mu \eta$$

$$- i p_{\delta} p_{\epsilon} t_{\alpha \beta}(p) | \rho \sigma + \frac{i}{2} p_{\delta} p_{\epsilon} t_{\alpha \beta}(p) | \mu \eta$$

$$+ \frac{p_{\epsilon} p_{\gamma} p_{\sigma}}{4} \left[ \left( \eta^{\nu \sigma} t_{d0}(p) \gamma | \mu + \mu \leftrightarrow \nu + \gamma \leftrightarrow \delta \right) 2 b^{\nu \rho \sigma} \right] +$$

$$+ \frac{i p_{\epsilon}}{2} \left[ - p_{\epsilon} t_{d0}(p) | \rho \sigma - p_{\epsilon} t_{d0}(p) | \mu \eta - p_{\lambda} t^{\lambda}_{d0}(p) | \rho \sigma$$

$$- p_{\lambda} t^{\lambda}_{d0}(p) | \mu \eta - p_{\rho} t^{\rho}_{d0}(p) | \delta \sigma$$

$$+ \eta^{\rho \sigma} \left( p_{\epsilon} t_{d0}(p) | \mu \eta + p_{\lambda} t^{\lambda}_{d0}(p) | \delta \sigma + p_{\lambda} t^{\lambda}_{d0}(p) | \mu \eta \right) \right] .$$

(4.11)

Using now the identity (i5), we can express the $t_{d0}$-distributions by means of the $t_{u0}$-distributions:

$$p_{\gamma} \left( b^\gamma \delta_{\alpha \beta} \Pi(p)^{\alpha \beta \mu \nu} \gamma_{\mu \nu} \right) = \frac{1}{2} \left[ + p_{\gamma} t^\delta_{\alpha \beta}(p) \gamma | \rho \sigma - \frac{1}{2} p_{\gamma} t^\delta_{\alpha \beta}(p) \gamma | \mu \eta \right]$$

$$+ p_{\delta} t_{\alpha \beta}(p) | \rho \sigma - \frac{1}{2} p_{\delta} t_{\alpha \beta}(p) | \mu \eta$$

$$- i p_{\gamma} p_{\epsilon} t^\delta_{\alpha \beta}(p) \gamma | \rho \sigma + \frac{i}{2} p_{\gamma} p_{\epsilon} t^\delta_{\alpha \beta}(p) \gamma | \mu \eta$$

$$- i p_{\delta} p_{\epsilon} t_{\alpha \beta}(p) | \rho \sigma + \frac{i}{2} p_{\delta} p_{\epsilon} t_{\alpha \beta}(p) | \mu \eta$$

$$+ \frac{p_{\epsilon} p_{\gamma} p_{\sigma}}{4} \left[ + p_{\sigma} t_{d0}(p) | \rho \sigma + p_{\rho} t_{d0}(p) | \delta \sigma$$

$$- p_{\rho} t_{d0}(p) | \delta \sigma + p_{\rho} t_{d0}(p) | \mu \eta$$

$$+ \eta^{\rho \sigma} \left( p_{\sigma} t_{d0}(p) | \mu \eta + p_{\rho} t_{d0}(p) | \delta \sigma + p_{\rho} t_{d0}(p) | \mu \eta \right) \right] .$$

(4.12)

This expression is the desired identity that involves the graviton self-energy tensor $\Pi(p)^{\alpha \beta \mu \nu}$, the ghost self-energy tensors $t_{d0}(p)$ and $t_{u0}(p)$ and the distributions coming from the $Q$-vertex. Schematically:

$$p_{\gamma} \left( b \Pi(p) b \right)^{\gamma \delta \rho \sigma} = X(t_{d0}(p), t_{u0}(p), \ldots) \delta, \rho \sigma + \left( t_{d0}(p) + t_{u0}(p) \right)^{\delta, \rho \sigma} ,$$

(4.13)

where $X^{\delta, \rho \sigma}$ represents the first eight distributions in (4.12). In QG, $(b \Pi(p) b)$ is not transversal, because the right side of (4.13) does not vanish. Note that the
remaining $Q$-vertex distributions cannot be eliminated from (4.13) as already noticed in [26], although in a very different approach to gauge invariance in QG. By multiplying (4.12) with $p_\rho$, we obtain

$$p_\rho p_\gamma (b^{\gamma \delta \alpha \beta} \Pi(p)^{\alpha \beta \mu \nu} b^{\mu \nu \rho \sigma}) = \frac{1}{2} \left[ + p_\gamma p_\rho t^\delta_{\alpha \beta} (p)^{\gamma | \rho \sigma} - \frac{1}{2} p_\gamma p_\sigma t^\delta_{\alpha \beta} (p)^{\gamma | \mu \nu} 
+ ps p_\rho t_{\alpha \beta} (p)^{| \rho \sigma} - \frac{1}{2} ps p_\sigma t_{\alpha \beta} (p)^{| \mu \nu} 
- i p_\rho p_\gamma t_{\alpha \beta} (p)^{| \rho \sigma} + \frac{i}{2} p_\sigma p_\beta t_{\alpha \beta} (p)^{| \mu \nu} 
- i ps p_\sigma t_{\alpha \beta} (p)^{| \rho \sigma} + \frac{i}{2} ps p_\beta t_{\alpha \beta} (p)^{| \mu \nu} 
+ \frac{1}{2} p_\rho p_\sigma^2 \left[ t_{\alpha \beta \delta \epsilon} (p)^{\gamma | \delta \sigma} + t_{\alpha \beta \delta \epsilon} (p)^{\gamma | \delta \sigma} - \frac{i}{2} p_\sigma p^2 \left[ t_{\alpha \beta \delta \epsilon} (p)^{\delta | \sigma} \right] \right]. \right]$$

(4.14)

This equation does not a priori imply the SWI (4.2). We can summarize Eq. (4.14) as

$$p_\rho p_\gamma (b \Pi(p) b)^{\gamma \delta \rho \sigma} = p_\rho X(t_{\alpha \beta} (p), t_{\alpha \beta} (p), ..., \delta, \rho \sigma) + p_\rho (t_{\alpha \beta \delta \epsilon} (p) + t_{\alpha \beta \delta \epsilon} (p))^{\delta, \rho \sigma}.$$  

(4.15)

With the explicit results for $t_{\alpha \beta \delta \epsilon} (p)$ and $t_{\alpha \beta \delta \epsilon} (p)$ obtained in Sec. 7.1, we find that there must be a compensation between $Q$-vertex contributions $p_\rho X^{\delta, \rho \sigma}$ and ghost self-energy contributions so that the right side of (4.15) vanishes identically, because this has already been verified by explicitly calculating the left side of (4.14). This may also explain why in the two cases with different $T^a_1$ or different $T^b_1$, investigated above, the SWI of Eq. (4.2) or Eq. (4.3) are not fulfilled: Eq. (4.15) still holds true, therefore the theory is still gauge invariant to second order, but both sides do not vanish identically.

5 Fixing of the Freedom in the Normalization of the Self-Energy Tensor

As seen in Sec. 3.4, in causal perturbation theory the problem of eliminating infinitely many UV divergent expressions is changed into the issue of fixing an increasing number of free undetermined local normalization terms that arise as a consequence of distribution splitting in each order of perturbation theory.

5.1 Normalization Terms of $T_2(x, y)$

For simplicity, we consider only the freedom in the normalization of Eq. (3.37) instead of Eq. (2.35). Since the singular order is four, local normalization terms
$N_2(x, y) \sim \delta^{(4)}(x - y)$ of singular order $0, \ldots, 4$ appear during the process of distribution splitting and in momentum space they can be written as

$$N_2(x, y)^{hSE} = h^{\alpha\beta}(x) h^{\mu\nu}(y) \cdot i N(\partial_x, \partial_y) \delta^{(4)}(x - y),$$

$$\hat{N}(p)_{\alpha\beta\mu\nu} = \left( \hat{N}^{(0)} + \hat{N}^{(2)} + \hat{N}^{(4)} \right)(p)_{\alpha\beta\mu\nu},$$

where the odd terms are excluded owing to parity. $\hat{N}(p)^{(i)}_{\alpha\beta\mu\nu}$ is a polynomial in $p$ of degree $i$ with $i = 0, 2, 4$. We assume in addition that only scalar constants should be considered, because vector-valued or tensor-valued constants may enter in conflict with Lorentz covariance. Therefore we make the following ansatz taking also the symmetries of $\Pi(x - y)_{\alpha\beta\mu\nu}$ into account

$$\hat{N}(p)^{(0)}_{\alpha\beta\mu\nu} = \Xi \left[ c_1 (\eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu}) + c_2 \eta_{\alpha\beta} \eta_{\mu\nu} \right],$$

$$\hat{N}(p)^{(2)}_{\alpha\beta\mu\nu} = \Xi \left[ c_3 (p_{\alpha} p_{\beta} \eta_{\mu\nu} + p_{\mu} p_{\nu} \eta_{\alpha\beta}) + c_4 (p_{\alpha} p_{\mu} \eta_{\beta\nu} + p_{\alpha} p_{\nu} \eta_{\beta\mu} + p_{\beta} p_{\mu} \eta_{\alpha\nu} + p_{\beta} p_{\nu} \eta_{\alpha\mu}) + c_5 p^2 (\eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu}) + c_6 p^2 \eta_{\alpha\beta} \eta_{\mu\nu} \right],$$

$$\hat{N}(p)^{(4)}_{\alpha\beta\mu\nu} = \Xi \left[ c_7, c_8, c_9, c_{10}, c_{11} \right].$$

$c_1, \ldots, c_{11}$ are undetermined real numbers. Requiring the SWI to hold, we can reduce the normalization polynomials to

$$\hat{N}(p)^{(0)}_{\alpha\beta\mu\nu} = 0,$$

$$\hat{N}(p)^{(2)}_{\alpha\beta\mu\nu} = \Xi \left[ 0, c_5 + c_6, -c_5, c_5, c_6 \right] p^{-2},$$

$$\hat{N}(p)^{(4)}_{\alpha\beta\mu\nu} = \Xi \left[ c_7, c_{10} + c_{11} + \frac{c_7}{4}, -c_{10}, c_{10}, c_{11} \right];$$

in such a way that only five undetermined coefficients remain to be fixed. The self-energy tensor supplemented by the normalization terms then reads

$$\hat{\Pi}(p)_{\alpha\beta\mu\nu} = \hat{\Pi}(p)_{\alpha\beta\mu\nu} + \hat{N}(p)^{(2)}_{\alpha\beta\mu\nu} + \hat{N}(p)^{(4)}_{\alpha\beta\mu\nu}.$$

5.2 Total Graviton Propagator and Mass and Coupling Constant Normalization

The task of eliminating the remaining freedom in Eq. (5.4) can be accomplished in our case by considering the total graviton propagator as a sum of the free graviton Feynman propagator [30]

$$\langle \Omega | T \{ h^{\alpha\beta}(x) h^{\mu\nu}(y) \} | \Omega \rangle = -i b^{\alpha\beta\mu\nu} D_0^F(x - y),$$
For this purpose we express the inverse of the total graviton propagator in the Lagrangian basis, introduced in App. 3 and find

\[ \hat{D}(p)_{\alpha\beta\mu\nu}^{\text{tot}} = \hat{D}_0^F(p) + \hat{D}_0^F(p) b^{\alpha\beta\gamma\delta} \hat{\Pi}(p)^N_{\gamma\delta\rho\sigma} b^{\rho\sigma\mu\nu} \hat{D}_0^F(p) + \]

\[ + b^{\alpha\beta\gamma\delta} \hat{D}_0^F(p) \hat{\Pi}(p)^N_{\gamma\delta\rho\sigma} b^{\rho\sigma\tau\epsilon} \hat{D}_0^F(p) \hat{\Pi}(p)^N_{\epsilon\tau\lambda\nu} b^{\lambda\kappa\mu\nu} \hat{D}_0^F(p) + \ldots \]

\[ = \hat{D}_0^F(p) \left[ b^{\alpha\beta\mu\nu} + b^{\alpha\beta\rho\sigma} \hat{\Pi}(p)^N_{\rho\sigma\gamma\delta} \hat{D}(p)^{\gamma\delta\mu\nu}_{\text{tot}} \right], \tag{5.6} \]

where \( \hat{\Pi}(p)_{\alpha\beta\mu\nu}^N := (2\pi)^4 \hat{\Pi}(p)^N_{\alpha\beta\mu\nu} \). After multiplying with \((\hat{D}_0(p))^{-1}\) and with \(b_{\alpha\beta\epsilon\tau}\), we find

\[ \left[(\hat{D}_0^F(p))^{-1} b_{\epsilon\tau} \gamma\delta - l_{\epsilon\tau\rho\sigma} \hat{\Pi}(p)^N_{\rho\sigma\gamma\delta}\right] \cdot \hat{D}(p)^{\gamma\delta\mu\nu}_{\text{tot}} = l_{\epsilon\tau\mu\nu}. \tag{5.7} \]

Since \(b_{\alpha\beta\mu\nu} b^{\mu\nu}_{\rho\sigma} = l_{\alpha\beta\rho\sigma} := (\eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu})/2\) represents the unity for rank-4 tensors, from Eq. (5.7) we can derive the form of the inverse of the total propagator:

\[ (\hat{D}(p))_{\alpha\beta\mu\nu}^{\text{tot}} = b_{\alpha\beta\mu\nu} \left(\hat{D}_0^F(p))^{-1} - \hat{\Pi}(p)^N_{\alpha\beta\mu\nu} \right. \]

\[ \left. = (2\pi)^2 \left[ b_{\alpha\beta\mu\nu} \left(-p^2 - i0\right) - (2\pi)^2 \hat{\Pi}(p)^N_{\alpha\beta\mu\nu} \right]. \tag{5.8} \]

Our aim is to impose mass and coupling constant normalization conditions on the total graviton propagator, therefore we have to invert the expression in (5.8). For this purpose we express the inverse of the total graviton propagator in the ‘projection operator’ \(Q(p)^{(i)}_{\alpha\beta\mu\nu}\) basis, introduced in App. 3 and find

\[ (\hat{D}(p))_{\alpha\beta\mu\nu}^{\text{tot}} = (2\pi)^2 \sum_{i=1}^{6} Q(p)^{(i)}_{\alpha\beta\mu\nu} \]

\[ \times \left\{ + x_i^b \left(p^2 - i0\right) - (2\pi)^2 \Xi x_i^\pi \log \left(-p^2 - i0\right)/M^2 \right\}, \tag{5.9} \]

with

\[ b_{\alpha\beta\mu\nu} = \frac{1}{6} \sum_{i=1}^{6} Q(p)^{(i)}_{\alpha\beta\mu\nu} x_i^b \left(-p^2 - i0\right), \tag{5.10} \]

\[ \hat{\Pi}(p)^N_{\alpha\beta\mu\nu} = \frac{1}{6} \sum_{i=1}^{6} Q(p)^{(i)}_{\alpha\beta\mu\nu} x_i^\pi \log \left(-p^2 - i0\right)/M^2 \right\}; \]

so that, with Eqs. (C.10), (C.11), we obtain

\[ \hat{D}(p)_{\alpha\beta\mu\nu}^{\text{tot}} = \frac{1}{(2\pi)^2} \left[ + \frac{1}{2} \left(\eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu}\right) \frac{1}{-p^2 - i0 + f_1(p^2)} + \right. \]

\[ \left. - \frac{1}{2} \eta_{\alpha\beta\mu\nu} \frac{1}{-p^2 - i0 + f_2(p^2)} \right] + \text{non-contributing terms}. \tag{5.11} \]
The non-contributing terms vanish between conserved energy-momentum matter tensors or between physical graviton states \(30\).

Mass normalization (the graviton mass remains zero after the radiative correction due to the self-energy) requires

\[
 f_j(p^2)\bigg|_{p^2=0} = 0, \quad j = 1, 2, \quad (5.12)
\]

and coupling constant normalization (the coupling constant is not changed by the radiative correction) requires

\[
 \frac{f_j(p^2)}{p^2} \bigg|_{p^2=0} = 0, \quad j = 1, 2. \quad (5.13)
\]

We work out these two conditions for the case \(j = 1\). Since \(f_1(p^2)\) has the form (see Eqs. (5.3), (5.9), (5.10), (C.8) and (C.11))

\[
 f_1(p^2) = -2 \left(2\pi\right)^2 \Xi \left(-162 p^4 \log \left(-p^2 - i0/M^2\right) + c_5 p^2 + c_{10} p^4\right), \quad (5.14)
\]

the two conditions (5.12) and (5.13) are satisfied, if we set \(c_5 = 0\). The analysis for the case \(j = 2\) is much more involved due to the complexity of Eqs. (C.8) and (C.11), although not conceptually difficult, and it yields the condition \(c_6 = 0\). The remaining normalization constants are not determined by Eqs. (5.12) and (5.13), so that we are left with

\[
 \hat{\Pi}^N(p)_{\alpha\beta\mu\nu} = \Xi \left[ -656, -208, +162, -162, +118 \right] \log \left(-p^2 - i0/M^2\right) + \hat{N}^{(4)}(p)_{\alpha\beta\mu\nu}, \quad (5.15)
\]

where \(\hat{N}^{(4)}(p)_{\alpha\beta\mu\nu}\) is a general SWI-invariant polynomial given in Eq. (5.3) with three free normalization constants. In order to simplify the above expression and to reduce further the freedom in the normalization, we rescale the constants \(c_7, c_{10}\) and \(c_{11}\) in the following way

\[
 c_{11} = +118 \log(M^2/K^2), \quad c_7 = -656 \log(M^2/L^2), \quad c_{10} = -162 \log(M^2/N^2); \quad (5.16)
\]

so that, with \((H^2)^{104} := (N^2)^{81}(L^2)^{82}/(K^2)^{59}\), we obtain

\[
 \hat{\Pi}^N(p)_{\alpha\beta\mu\nu} = \Xi \left[ -656 \log\left(-p^2 - i0/L^2\right), -208 \log\left(-p^2 - i0/H^2\right),
 +162 \log\left(-p^2 - i0/N^2\right), -162 \log\left(-p^2 - i0/N^2\right), +118 \log\left(-p^2 - i0/K^2\right) \right], \quad (5.17)
\]

with the four arbitrary positive masses \(L, H, N\) and \(K\). Now, since the splitting of the mass zero distribution \(\hat{d}(p) = \Theta(p^2)\text{sgn}(p^0)\), requires the introduction of

26
a scale invariance breaking mass, it is natural to assume this mass scale to be unique, say $M_0$, which may correspond in case to the Planck mass, so that the graviton self-energy tensor including its normalization now reads

$$\tilde{\Pi}(p)_{\alpha\beta\mu\nu}^\text{N} = \Xi \left[ -656, -208, +162, -162, +118 \right] \log \left( -\frac{p^2 - i0}{M_0^2} \right). \quad (5.18)$$

Therefore, the whole ambiguity in the normalization of $T_2(x, y)^{\text{hSE}}$ may be reduced to the single parameter $M_0$ which remains present in the theory.

6 Corrections to the Newtonian Potential

In the last years a group of papers appeared [41], [42] and [43], based on the proposal of treating perturbative quantum gravity as a low energy effective quantum theory. In this approach, leading quantum corrections to the Newtonian potential were reliably calculated in the long range, low energy limit.

In this section we will show that causal perturbation theory for quantum gravity yields the same result for the leading corrections to the Newtonian potential for heavy spinless particles described by the scalar field $\phi$.

The Newtonian potential between two masses $m_1, m_2$

$$V(r) = -G \frac{m_1 m_2}{r} \quad (6.1)$$

can be obtained in the static non-relativistic limit of a single graviton exchange tree diagram [41], [44] to lowest order in $G$, calculated from

$$T_1^M(x) = i : L_M^{(i)}(x) := \frac{i}{2} : h^{\alpha\beta}(x) b_{\alpha\beta\mu\nu} T_{\mu\nu}^m(x) :, \quad (6.2)$$

where the conserved energy-momentum matter tensor reads

$$T_{\mu\nu}^m(x) = \varphi(x)^\mu \varphi(x)^\nu - \eta_{\mu\nu} L^0_M(x), \quad (6.3)$$

and the expansion in powers of the coupling constant of the matter Lagrangian density has the form

$$L_M = \frac{1}{2} \sqrt{-g} \left( g^{\mu\nu} \varphi_{;\mu} \varphi_{;\nu} - m^2 \varphi^2 \right) = \sum_{i=0}^{\infty} \kappa^i L_M^{(i)}. \quad (6.4)$$

From Eq. (6.2), we carry out the inductive construction of the 2-point distribution which describes the $\varphi\varphi \rightarrow \varphi\varphi$ scattering through one-graviton exchange and find

$$T_2(x, y)\bigg|_{\varphi\varphi \rightarrow \varphi\varphi} = -\frac{\kappa^2}{4} : T_m^\gamma \delta_{\alpha\beta} T_m^\rho \sigma (y) b_{\rho\sigma\mu\nu} : \left[ -i b^{\alpha\beta\mu\nu} D_0^\nu(x - y) \right]. \quad (6.5)$$
The static non-relativistic limit of the right side of (6.5) yields (6.1) using $T_{\mu\nu}^{m} \rightarrow (2\pi)\delta^{\mu0}\delta^{\nu0}m$ and the fact that the Fourier transform of the momentum transfer $p^2$ reads $(4\pi)^{-1}$. For a deeper understanding of the connection between $S$-matrix and potential, see [42]. Inserting one self-energy contribution (5.18) on the graviton contraction of Eq. (6.5), we obtain the order $\kappa^4$ correction

$$T_2(x, y)^{corr} = i \frac{\kappa^2}{4} : T_{m_1}^{\gamma\delta}(x) T_{m_2}^{\rho\sigma}(y) : \Omega(x - y)^{corr}_{\gamma\delta\rho\sigma},$$

(6.6)

where, with the self-energy tensor of (5.18), we have

$$\hat{\Omega}(p)^{corr}_{\gamma\delta\rho\sigma} = \frac{2\hat{\Pi}(p)^{\gamma\rho\sigma}}{p^4}. \quad (6.7)$$

Only the two last terms in the self-energy tensor will contribute in the following, because the others vanish when paired with conserved matter energy-momentum tensors. Therefore, even if we had not assumed a unique mass scale (as done at the end of Sec 5.2), the parameters $L, H$ and $N$ in (5.17) would not be physically observable. From (6.5) and (6.7) we obtain in the static non-relativistic limit the genuine quantum correction to the Newtonian potential (inserting back the physical constants $\hbar$ and $c$)

$$V(r) = -G \frac{m_1m_2}{r} \left( 1 + \frac{G\hbar}{c^3\pi} \frac{206}{30} \frac{1}{r^2} \right). \quad (6.8)$$

The central piece in the calculation is the distributional Fourier transform of $\log \left( p^2/M_0^2 \right)$ which yields $(-2\pi r^3)^{-1}$ and the $M_0$-dependence disappears from the non-local part of the final result, being proportional to $\delta^{(3)}(x)$. The relevant length scale appearing in (6.8) is the Planck length $l_p = \sqrt{\frac{G\hbar}{c^3}}$.

Our result agrees with the corresponding one in [42], although this represents only a partial correction to the Newtonian potential, because we have taken into account only the graviton self-energy contribution and not the complete set of diagrams of order $\kappa^4$ contributing to these corrections, as, for example, the vertex correction or the double scattering. Therefore we cannot make any statement on the absolute sign of the correction in Eq. (6.8). See also [18] for the corrections to the Newtonian potential in the framework of $R^2$-theories.

A remark about different choices of $T_1^h$ or $T_1^u$: the graviton self-energy tensor of Eq. (4.4) leads to the same corrections, whereas that of Eq. (4.3), taking also the ghost loop contribution into account, leads to different corrections.

7 Perturbative Gauge Invariance to Second Order for Loop Contributions

For the sake of completeness, before proving perturbative gauge invariance to second order in the loop graph sector, we calculate the 2-point ghost self-energy contributions.
7.1 Ghost Self-Energy

We follow the inductive causal construction described in Sec. 3 in order to construct the ghost self-energy contribution in second order perturbation theory. Starting with the ghost coupling $D_1^r(x)$ given in Eq. (2.14), we perform one graviton and one ghost–anti-ghost contraction, Eq. (3.1) in order to obtain the corresponding $D_2(x,y)$ distribution. The distribution splitting solution is the same as in Sec. 3.5, namely Eq. (3.31) with the singular order given by Eq. (3.20) in Sec. 3.4. Therefore, after all these steps, we obtain

$$T_2(x,y) = +:u^\gamma(x)\tilde{u}^\alpha(y)\gamma_\alpha\beta + :\tilde{u}^\alpha(x)\beta u^\gamma(y)\alpha_\beta\gamma + :u^\gamma(x)\beta\tilde{u}^\alpha(y)\gamma_\alpha\nu + :\tilde{u}^\alpha(x)\nu u^\gamma(y)\beta_\alpha\gamma$$  \hspace{1cm} (7.1)

The numerical distributions appearing here have already been introduced in Eq. (2.13) with the notations: $t_2^{u,(3a)} = it_{a\beta\gamma}$, $t_2^{u,(3b)} = it_{\beta\alpha\gamma}$, $t_2^{u,(2a)} = it_{a\beta\alpha\gamma}$ and $t_2^{u,(2b)} = it_{\beta\alpha\gamma\alpha\gamma}$. The results for these ghost self-energy distributions are:

$$\hat{t}_2^{u,(3a)}(p)_{(\gamma\alpha\beta)} = \hat{t}_2^{u,(3b)}(p)_{(\alpha\beta\gamma)} = \Xi \left[ -80 p_\alpha p_\beta p_\gamma + 60 p_\gamma^2 p_\alpha p_\beta - 20 p_\gamma^2 p_\alpha p_\gamma - 20 p_\gamma^2 p_\beta p_\gamma \right]$$ \hspace{1cm} (7.2)

and

$$\hat{t}_2^{u,(2a)}(p)_{(\gamma\beta\alpha)} = -\hat{t}_2^{u,(2b)}(p)_{(\alpha\beta\gamma)} = i \Xi \left[ +160 p_\gamma^2 p_\alpha p_\beta - 240 p_\gamma p_\alpha p_\beta \eta_{\alpha\beta} - 400 p_\beta^2 p_\gamma p_\alpha \eta_{\alpha\beta} \right]$$ \hspace{1cm} (7.3)

By disregarding divergence terms after partial integration, we can recast Eq. (7.1) into the form

$$T_2(x,y)_{\text{gb}} = :u^\gamma(x)\tilde{u}^\alpha(y): i \Pi_a(x-y)_{\gamma\alpha} + :\tilde{u}^\alpha(x)u^\gamma(y): i \Pi_b(x-y)_{\alpha\gamma}$$  \hspace{1cm} (7.4)

with the ghost self-energy tensors

$$i \Pi_a(x-y)_{\gamma\alpha} := + \partial^t_\beta t_2^{u,(3a)}(x-y)\gamma_\alpha\beta - \partial^t_\beta \partial^t_\nu t_2^{u,(2a)}(x-y)\nu_\alpha\beta,$$  \hspace{1cm} (7.5)

$$i \Pi_b(x-y)_{\alpha\gamma} := - \partial^t_\beta t_2^{u,(3b)}(x-y)\alpha_\gamma\beta - \partial^t_\beta \partial^t_\nu t_2^{u,(2b)}(x-y)\nu_\gamma\beta;$$

that read in momentum space

$$\Pi_a(p)_{\gamma\alpha} = \Xi \left[ 40 p_\alpha^2 p_\gamma + 20 p_\gamma^4 p_\alpha \right] \log \left( \frac{-p^2 - i0}{M^2} \right) = -\Pi_b(p)_{\alpha\gamma}. \hspace{1cm} (7.6)$$

We do not discuss here in detail the normalization of this 2-point distribution. But an investigation analogous to that of Sec. 5.2 for the sum of the series with an increasing number of ghost self-energy insertions let us assume that the normalization terms with singular order smaller than four are set equal to zero, whereas those with singular order four may be absorbed in the scale invariance breaking mass $M$.  

29
7.2 Perturbative Gauge Invariance to Second Order for Loop Graphs

The introduction of the $Q$-vertex $T^\mu_{1/1}(x)$, Eq. (2.17), enables us to formulate perturbative gauge invariance by means of Eq. (2.18). We call a theory gauge invariant to second order, if $T^\mu_{2/2}(x, y)$ satisfies

$$d_Q T^\mu_{2/2}(x, y) = \partial^\mu \mu T^\mu_{2/2}(x, y),$$

(7.7)

where $T^\mu_{2/2}(x, y)$ is the two-point ‘renormalized’ time-ordered product obtained by means of the inductive causal scheme with a $Q$-vertex at $x$ and a normal vertex at $y$.

Since $R^\mu_{2/2}(x, y)$ is trivially gauge invariant due to (3.2) and (2.17), it suffices to prove (7.7) with the retarded parts $R^\mu_{2/2}$, $R^\mu_{2/1}$ and $R^\mu_{2/2}$, instead of the corresponding $T$-distributions.

Taking Eq. (3.3) and (2.17) into account, we find that the $D^\mu_{2/2}$-distribution is trivially invariant

$$d_Q D^\mu_{2/2}(x, y) = \partial^\mu \mu D^\mu_{2/2}(x, y),$$

(7.8)

with the definitions

$$D^\mu_{2/1}(x, y) := [T^\mu_{1/1}(x), T_{1/1}(y)],$$

$$D^\mu_{2/2}(x, y) := [T_{1/1}(x), T^\mu_{1/1}(y)].$$

(7.9)

The question is whether an equation similar to (7.8) holds for the retarded parts $R^\mu_{2/2}$, $R^\mu_{2/1}$ and $R^\mu_{2/2}$ of $D^\mu_{2/2}$, $D^\mu_{2/1}$ and $D^\mu_{2/2}$, respectively. In the inductive causal construction, the splitting of distributions in $D^\mu_{2/2}$, $D^\mu_{2/1}$ and $D^\mu_{2/2}$ may give rise to local normalization terms, if the singular order is positive. We consider here only loop graphs in second order. Therefore, we must show that

$$d_Q R^\mu_{2/2}(x, y) + d_Q N^\mu_{2/2}(x, y) = \partial^\mu \mu R^\mu_{2/2}(x, y) + \partial^\mu \mu N^\mu_{2/2}(x, y)$$

(7.10)

can be satisfied by a suitable choice of the free constants in the normalization terms $N^\mu_{2/2}$, $N^\mu_{2/1}$ and $N^\mu_{2/2}$ of the general splitting solution of $D^\mu_{2/2}$, $D^\mu_{2/1}$ and $D^\mu_{2/2}$, respectively.

Since every splitting solution agrees with the original distribution on the forward light cone $V^+ \setminus \{0\}$, gauge invariance (7.10) can be violated by local terms with support $x = y$ only. Hence, the crucial point is the correct treatment of the local terms appearing in (7.10).

A careful analysis shows that that the only local terms appearing in (7.10) are those belonging to the normalizations $N^\mu_{2/2}$, $N^\mu_{2/1}$ and $N^\mu_{2/2}$ of the distribution splitting.
Let us analyze the different terms in (7.10). First of all, the normalization terms \( N_2(x, y) \, ^{\text{loops}} \) were already discussed in Sec. 5.1 and in Sec 7.1, below Eq. (7.6). They can be consistently set equal to zero.

The gauge variation of \( R_2(x, y) \, ^{\text{loops}} \) generates no local terms: the field operators in the normal products undergo the infinitesimal gauge variations (2.11) and (2.16) and the numerical distributions remain unchanged. Taking an example from Eq. (3.35)

\[
d_Q R_2(x, y) \, ^{\text{loops}} = d_Q \left( \hat{h}^{\alpha\beta}(x)h^{\mu\nu}(y): r_2^{(4)}(x - y)\alpha\beta\mu\nu + \ldots \right) =: u^\mu(x),\sigma h^{\mu\nu}(y): \left[ -i \gamma^{\alpha\beta\rho\sigma} r_2^{(4)}(x - y)\alpha\beta\rho\sigma \right] + \ldots .
\]

\( r_2^{(4)}(x - y)\alpha\beta\mu\nu \) does not contain local terms, the same holds for \( b^{\alpha\beta\rho\sigma} r_2^{(4)}(x - y)\alpha\beta\rho\sigma \).

Now we investigate the \( R_{2/1}^\mu (x, y) \, ^{\text{loops}} \)-terms. Due to the great calculation complexity, we work out only a representative example, that still contains the main features. Let us choose in \( T_{1/1}^\mu (x) \sim uhh + \tilde{u}uu \) (from [33]) the term \( -\kappa^2 \hat{u}^\gamma(x, \gamma)h(x)h(x)_\mu \): and in \( T_{1}^{h+u}(y), \) (7.13) the term \( -i \kappa^2 \gamma^{\alpha\beta}(y)h(y)_\alpha h(y)_\beta \): We construct \( D_{2/1}^\mu (x, y) \, ^{\text{loops}} \) by carrying out two graviton contractions

\[
D_{2/1}^\mu (x, y) \, ^{\text{loops}} = \frac{i \kappa^2}{16} \left( + : u^\gamma(x),\gamma h^{\alpha\beta}(y): \left( -32 \left[ + D_{\alpha\mu\beta}^{(+)\, \text{loops}} - D_{\alpha\mu\beta}^{(-)\, \text{loops}} \right](x - y) \right) + \right.
\]

\[
+ : u^\gamma(x),\gamma h(y),\alpha: \left( + 8 \left[ + D_{\alpha\mu\alpha}^{(+)\, \text{loops}} - D_{\alpha\mu\alpha}^{(-)\, \text{loops}} + D_{\alpha\mu\mu}^{(+)\, \text{loops}} - D_{\alpha\mu\mu}^{(-)\, \text{loops}} \right](x - y) \right) \big) + \ldots ,
\]

which can be written in the form

\[
D_{2/1}^\mu (x, y) \, ^{\text{loops}} = + : u^\gamma(x),\gamma h^{\alpha\beta}(y): d_{2/1}^\mu (x - y)\alpha\beta + \]

\[
+ : u^\gamma(x),\gamma h(y),\alpha: d_{2/1}^\mu (x - y)^\alpha + \ldots ,
\]

with the \( \hat{D}_{\ldots,\ldots}^{(\pm)}\) (p)-functions given in App. 1:

\[
d_{2/1}^\mu (p)\alpha\beta = -2i \kappa^2 \left[ + \hat{D}_{\alpha\mu\beta}^{(+)\, \text{loops}} - \hat{D}_{\alpha\mu\beta}^{(-)\, \text{loops}} \right](p),
\]

\[
d_{2/1}^\mu (p)^\alpha = +i \frac{\kappa^2}{2} \left[ + \hat{D}_{\alpha\mu\alpha}^{(+)\, \text{loops}} - \hat{D}_{\alpha\mu\alpha}^{(-)\, \text{loops}} + \hat{D}_{\alpha\mu\mu}^{(+)\, \text{loops}} - \hat{D}_{\alpha\mu\mu}^{(-)\, \text{loops}} \right](p).
\]

Using the results of App. 1 and App. 2, we obtain

\[
d_{2/1}^\mu (p)\alpha\beta = -\frac{\kappa^2 \pi}{24(2\pi)^4} \left[ 2p_\alpha p_\beta p^\mu - p^2 p_\alpha \eta_\beta^\mu + p^2 p_\beta \eta_\alpha^\mu + p^2 p^\rho \eta_\alpha_\beta \right] \Theta(p^2) \text{sgn}(p^0),
\]

\[
d_{2/1}^\mu (p)^\alpha = \frac{+i \kappa^2 \pi}{8(2\pi)^4} \left[ p^\mu p^\alpha \right] \Theta(p^2) \text{sgn}(p^0).
\]
In order to obtain $R_{2/1}^{\mu\text{loops}}$, we split the distributions in (7.15). With (3.33) and assuming that the splitting of the massless distribution $\hat{d}(p)$ generates only one mass scale $M$, we find

$$r_{2/1}^{\mu}(p)_{\alpha\beta} = \frac{-i\kappa^2\pi}{24(2\pi)^5} \left[ \frac{\log \left( \frac{-p^2 - i\rho^0}{M^2} \right)}{8(2\pi)^5} \right] ,$$

so that

$$R_{2/1}^{\mu}(x, y)_{\text{loops}} = + : u^\gamma(x)\gamma h^{\alpha\beta}(y) : r_{2/1}^{\mu}(x - y)_{\alpha\beta} + 2\partial_\mu = 0,$$

The local normalization terms of the splitting are included in $N_{2/1}^{\mu}(x, y)_{\text{loops}}$ and reads:

$$N_{2/1}^{\mu}(x, y)_{\text{loops}} = + : u^\gamma(x)\gamma h^{\alpha\beta}(y) : n_{2/1}^{\mu}(x - y)_{\alpha\beta} + 2\partial_\mu = 0.$$
We turn now to the local terms in $\partial^\mu N^{\mu}_{2/1}(x,y)^{\text{loops}}$. Since $\omega(\hat{d}^{\mu\nu}_{2/1}) = 3$ and $\omega(\hat{d}^{\mu\nu}_{2/1}) = 2$, these terms have the general form

$$
\hat{n}^{\mu}_{2/1}(p)^{\alpha\beta} = +a_1 p^\mu \eta^{\alpha\beta} + a_2 p^\alpha p^\beta + a_3 p^\beta \eta^{\alpha\beta} + a_4 p^\mu p^\nu p^\beta + a_5 p^\mu \eta^{\alpha\beta} + a_6 p^\alpha \eta^{\beta\mu} + a_7 p^\beta \eta^{\alpha\mu} ,
$$

(7.21)

where $a_1, \ldots, a_{10}$ are unknown parameters. Multiplying (7.21) with $p_\mu$ leads to

$$
p_\mu \hat{n}^{\mu}_{2/1}(p)^{\alpha\beta} = +a_1 p^2 \eta^{\alpha\beta} + (a_2 + a_3) p^\alpha p^\beta + (a_4 + a_6 + a_7) p^2 p^\alpha p^\beta + a_5 p^\mu \eta^{\alpha\beta} ,
$$

(7.22)

$$
p_\mu \hat{n}^{\mu}_{2/1}(p)^{\alpha\beta} = +a_8 p^\alpha + (a_9 + a_{10}) p^2 p^\alpha .
$$

(7.23)

Therefore, the only local terms in (7.10) are those coming from (7.22). In our simplified example, perturbative gauge invariance then requires

$$
a_1 = a_2 + a_3 = a_4 + a_6 + a_7 = a_5 = a_8 = a_9 + a_{10} = 0 ,
$$

(7.24)

Obviously, these conditions are fulfilled by choosing $a_1 = \ldots = a_{10} = 0$. One may convince oneself that the above example can be generalized to all second order loop graph contributions, because these follow the same pattern, although much more involved conditions as (7.23) would appear. The important point is that $\partial^\mu R^{\mu}_{2/1}(x,y)^{\text{loops}} + \partial^\mu N^{\mu}_{2/1}(x,y)^{\text{loops}}$ does not generate local terms. This is in contrast to the tree graph sector, investigated in [33].

Let us add a final remark: if we do not fix the normalization $N_2(x,y)^{\text{loops}}$ of $R_2(x,y)^{\text{loops}}$ (as done in Sec. [3]), then the condition of perturbative gauge invariance to second order forces us to choose the normalization constants $c_i$ of $N_2(x,y)^{\text{loops}}$ and $a_j$ of $N^{\mu}_{2/1+2}(x,y)^{\text{loops}}$ in such a way that

$$
d_Q N_2(x,y)^{\text{loops}} = +\partial^\mu R^{\mu}_{2/1}(x,y)^{\text{loops}} + \partial^\mu N^{\mu}_{2/2}(x,y)^{\text{loops}}
$$

(7.24)

holds. Since (7.10) always holds among non-local terms by construction, a trivial solution $c_i = a_j = 0 \forall i, j$ always exists. This concludes the proof of perturbative gauge invariance to second order for loop graphs.

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**Appendix 1: The $\hat{D}^{\pm}_{\alpha\beta\mu\nu}(p)$-Functions**

The product of two $D_0^{\pm}$-distributions is well-defined in momentum space, because the intersection of the supports of the two $\hat{D}^{\pm}_0(p)$ is a compact set. The
product \( D^{(\pm)}_\downarrow(x) := D_0^{(\pm)}(x) \cdot D_0^{(\pm)}(x) \) goes over into a convolution of the Fourier transforms \( \hat{D}_0^{(\pm)}(p) \) using Eq. (2.9):

\[
\hat{D}_\downarrow^{(\pm)}(p) = \frac{1}{(2\pi)^2} \int d^4x \, D_\downarrow^{(\pm)}(x) \cdot D_0^{(\pm)}(x) e^{ip \cdot x}.
\]

Further calculations:

\[
\hat{D}_\downarrow^{(\pm)}(p) = -\frac{1}{(2\pi)^4} \int \frac{d^4p_1 \, d^4p_2 \, \delta(p_1^0) \, \Theta(\pm p_1^0) \, \delta(p_2^0) \, \Theta(\pm p_2^0) \cdot \int d^4x \, e^{-ix \cdot (p_1 + p_2 - p)}}{p_1^0 \cdot p_2^0}.
\]

Therefore we see that we have to deal with integrals of the type

\[
D^{(\pm)}_\alpha\beta(x) := \partial^\mu_\alpha D_0^{(\pm)}(x) \cdot \partial^\nu_\beta D_0^{(\pm)}(x),
\]

and so on if a different combination of derivatives acts on the distributions. Following the same calculation as in (A.1), we obtain in momentum space:

\[
\hat{D}_\alpha\beta^{(\pm)}(p) = \frac{1}{(2\pi)^4} \int d^4k \, \delta((p - k)^2) \, \Theta(\pm (p^0 - k^0)) \, \delta(k^2) \, \Theta(\pm k^0) \times [p_\alpha k_\beta - k_\alpha k_\beta].
\]

\[
\hat{D}_\alpha\beta\mu\nu^{(\pm)}(p) = -\frac{1}{(2\pi)^4} \int d^4k \, \delta((p - k)^2) \, \Theta(\pm (p^0 - k^0)) \, \delta(k^2) \, \Theta(\pm k^0) \times \left[ + p_\alpha p_\beta k_\mu k_\nu - p_\alpha k_\beta k_\mu k_\nu - p_\beta k_\alpha k_\mu k_\nu + k_\alpha k_\beta k_\mu k_\nu \right].
\]

Therefore we see that we have to deal with integrals of the type

\[
I^{(\pm)}_{\alpha/\alpha\beta/\alpha\beta\mu/\alpha\beta\mu}(p) := \int d^4k \, \delta((p - k)^2) \, \Theta(\pm (p^0 - k^0)) \, \delta(k^2) \, \Theta(\pm k^0) \times [1, k_\alpha, k_\alpha k_\beta, k_\alpha k_\beta k_\mu, k_\alpha k_\beta k_\mu k_\nu],
\]

which are calculated in App. 2. For the two examples in Eq. (A.3), we find the relations

\[
\hat{D}_\downarrow^{(\pm)}(p) = -\frac{1}{(2\pi)^4} I^{(\pm)}(p),
\]

\[
\hat{D}_{\alpha\beta}^{(\pm)}(p) = \frac{1}{(2\pi)^4} \left[ + p_\alpha I^{(\pm)}(p)_\beta - I^{(\pm)}(p)_{\alpha\beta} \right],
\]

\[
\hat{D}_{\alpha\beta\mu\nu}^{(\pm)}(p) = -\frac{1}{(2\pi)^4} \left[ + p_\alpha p_\beta I^{(\pm)}(p)_{\mu\nu} - p_\alpha I^{(\pm)}(p)_{\beta\mu\nu} - p_\beta I^{(\pm)}(p)_{\alpha\mu\nu} + I^{(\pm)}(p)_{\alpha\beta\mu\nu} \right].
\]
Appendix 2: The $I^{(\pm)}(p)$-Integrals

For the case of $I^{(+)}(p)$, Eq. (A.4), the momenta $k$ and $p$ are restricted to the space-time regions $\{k^2 = 0, k_0 > 0\}$ and $\{(p - k)^2 = 0, p_0 - k_0 > 0\}$, due to the $\delta$- and $\Theta$-distributions in the integrand. Then $p - k$ and $k$ are time-like and therefore $p$ is time-like. We choose a Lorentz reference frame with $p_\alpha = (p_0, 0), p_0 > 0$, then

$$I^{(+)}(p_0) = \int d^4k \delta(p_0^2 - 2p_0k_0) \Theta(p_0 - k_0) \Theta(k_0) \frac{\delta(k_0 - |k|) + \delta(k_0 + |k|)}{2E_k}$$

(B.1)

with $E_k = k_0 = |k|$, so that

$$I^{(+)}(p_0) = 4\pi \int_{-\infty}^{+\infty} dk^0 \int_0^{+\infty} d|k| |k|^2 \delta(2p_0(p_0 \frac{p_0}{2} - k_0)) \Theta(p_0 - k_0) \Theta(k_0) \frac{\delta(k_0 - |k|)}{2|k|}$$

$$= \frac{\pi}{p_0} \int_{-\infty}^{+\infty} dk^0 \int_0^{+\infty} d|k| |k| \delta(\frac{p_0}{2} - k_0) \Theta(p_0 - k_0) \delta(k_0 - |k|)$$

$$= \frac{\pi}{p_0} \int_0^{p_0} d|k| |k| \delta(|k| - \frac{p_0}{2}) = \frac{\pi}{2}.$$

(B.2)

$I^{(-)}(p)$ can be calculated analogously and the result in an arbitrary Lorentz reference frame is

$$I^{(\pm)}(p) = \frac{\pi}{2} \Theta(p^2) \Theta(\pm p_0).$$

(B.3)

Computing $I^{(\pm)}(p)_\alpha$ for $p_\alpha = (p_0, 0), p_0 > 0$, we have a non-vanishing contribution only for $\alpha = 0$. We obtain for $I^{(\pm)}(p_0)$ an additional factor $k_0$ in the integrand of (B.2), which is set equal to $|k|$, because of the distribution $\delta(k_0 - |k|)$ and finally is set equal to $p_0/2$, because of the distribution $\delta(|k| - \frac{p_0}{2})$. This leads to

$$I^{(\pm)}(p)_\alpha = \frac{\pi}{4} p_\alpha \Theta(p^2) \Theta(\pm p_0),$$

(B.4)

in an arbitrary Lorentz reference frame.

For $I^{(\pm)}(p)_{\alpha\beta}$, we consider the covariant decomposition

$$I^{(\pm)}(p)_{\alpha\beta} = A^{(\pm)}(p^2) p_\alpha p_\beta + B^{(\pm)}(p^2) \eta_{\alpha\beta}.$$  

(B.5)

It follows from $I^{(\pm)}(p)_\alpha = 0$ (because of the factor $k^2\delta(k^2)$ in the integrand of Eq. (A.4)), that

$$B^{(\pm)}(p^2) = \frac{-p^2}{4} A^{(\pm)}(p^2).$$

(B.6)
Then

\[ I^{(\pm)}(p)_{\alpha\beta} p^\alpha p^\beta = \frac{3}{4} A^{(\pm)}(p^2) p^4. \]  

(B.7)

Calculating \( I^{(\pm)}(p)_{\alpha\beta} p^\alpha p^\beta \) for \( p_\alpha = (p_0, \mathbf{0}) \), \( p_0 > 0 \), through the integral definition (A.4), an additional factor \((p_0 k_0)^2\) appears in the integrand, therefore we obtain in an arbitrary Lorentz frame

\[ I^{(\pm)}(p)_{\alpha\beta} p^\alpha p^\beta = \frac{3}{8} p^2 \Theta(p^2) \Theta(\pm p^0). \]  

(B.8)

Comparing (B.7) with (B.8) we find

\[ I^{(\pm)}(p)_{\alpha\beta} p^\alpha p^\beta = \frac{3}{8} p^2 \Theta(p^2) \Theta(\pm p^0). \]  

(B.9)

For \( I^{(\pm)}(p)_{\alpha\beta\mu} \), if we calculate \( I^{(\pm)}(p)_{\alpha\beta\mu} p^\alpha p^\beta p^\mu \) for \( p_\alpha = (p_0, \mathbf{0}) \), \( p_0 > 0 \), we get a factor \((p_0 k_0)^3\) in the integrand, so that in an arbitrary Lorentz frame we have

\[ I^{(\pm)}(p)_{\alpha\beta\mu} p^\alpha p^\beta p^\mu = \frac{3}{16} p^6 \Theta(p^2) \Theta(\pm p^0). \]  

(B.10)

The covariant decomposition of \( I^{(\pm)}(p)_{\alpha\beta\mu} \) reads

\[ I^{(\pm)}(p)_{\alpha\beta\mu} = C^{(\pm)}(p^2) p^\alpha p^\beta p^\mu + D^{(\pm)}(p^2) \left(p_\alpha \eta_{\beta\mu} + p_\beta \eta_{\alpha\mu} + p_\mu \eta_{\alpha\beta}\right). \]  

(B.11)

Since \( I^{(\pm)}(p)_{\alpha\beta} = 0 \), we obtain \( D^{(\pm)}(p^2) = \frac{-p^2}{4} C^{(\pm)}(p^2) \). On the other side, contracting the covariant decomposition of \( I^{(\pm)}(p)_{\alpha\beta\mu} \) with \( p^\alpha p^\beta p^\mu \), we find

\[ I^{(\pm)}(p)_{\alpha\beta\mu} p^\alpha p^\beta p^\mu = \frac{3}{2} C^{(\pm)}(p^2). \]  

(B.12)

Comparing (B.12) with (B.10) we conclude that \( C^{(\pm)}(p^2) = \frac{3}{2} \Theta(p^2) \Theta(\pm p^0) \) and

\[ I^{(\pm)}(p)_{\alpha\beta\mu} = \frac{3}{8} \left(p_\alpha p_\beta p_\mu - \frac{p^2}{6} \left(p_\alpha \eta_{\beta\mu} + p_\beta \eta_{\alpha\mu} + p_\mu \eta_{\alpha\beta}\right)\right) \Theta(p^2) \Theta(\pm p^0). \]  

(B.13)

We repeat this calculation scheme also for \( I^{(\pm)}(p)_{\alpha\beta\mu\nu} \), which has the covariant decomposition

\[ I^{(\pm)}(p)_{\alpha\beta\mu\nu} = E^{(\pm)}(p^2) p_\alpha p_\beta p_\mu p_\nu + F^{(\pm)}(p^2) \left(p_\alpha p_\beta \eta_{\mu\nu} + p_\alpha p_\mu \eta_{\beta\nu} + p_\alpha p_\nu \eta_{\beta\mu} + p_\beta p_\mu \eta_{\alpha\nu} + p_\beta p_\nu \eta_{\alpha\mu} + p_\mu p_\nu \eta_{\alpha\beta}\right) + C^{(\pm)}(p^2) \left(p_\alpha p_\beta p_\mu \eta_{\nu\alpha} + p_\alpha p_\mu \eta_{\beta\nu} + p_\beta p_\nu \eta_{\alpha\mu}\right) + G^{(\pm)}(p^2) \left(p_\alpha p_\mu \eta_{\beta\nu} + p_\beta p_\nu \eta_{\alpha\mu}\right). \]  

(B.14)
From $I^{(\pm)}(p)^\alpha_{\alpha\mu\nu} = 0$ and $I^{(\pm)}(p)^{\alpha}_{\alpha\mu\nu} = 0$, we obtain

\[
\begin{align*}
F^{(\pm)}(p^2) &= -\frac{p^2}{8} E^{(\pm)}(p^2) , \\
G^{(\pm)}(p^2) &= \frac{p^2}{2} F^{(\pm)}(p^2) = \frac{p^2}{32} E^{(\pm)}(p^2) .
\end{align*}
\]  (B.15)

Computing $I^{(\pm)}(p)_{\alpha\beta\mu\nu} p^\alpha p^\beta p^\mu p^\nu$, for $p_\alpha = (p_0, \mathbf{0})$, $p_0 > 0$, we get a factor $(p_0k_0)^4$ in the integrand, so that in an arbitrary Lorentz frame we have

\[
I^{(\pm)}(p)_{\alpha\beta\mu\nu} p^\alpha p^\beta p^\mu p^\nu = \frac{\pi}{32} p^8 \Theta(p^2) \Theta(\pm p^0) .
\]  (B.16)

On the other side, contracting the covariant decomposition of $I^{(\pm)}(p)_{\alpha\beta\mu\nu}$ with $p^\alpha p^\beta p^\mu p^\nu$ and using (B.15), we obtain

\[
I^{(\pm)}(p)_{\alpha\beta\mu\nu} p^\alpha p^\beta p^\mu p^\nu = \frac{5}{16} p^8 E^{(\pm)}(p^2) .
\]  (B.17)

Comparing (B.17) with (B.16), we find $E^{(\pm)}(p^2) = \frac{\pi p^2}{16} \Theta(p^2) \Theta(\pm p^0)$, that implies with (B.15) $F^{(\pm)}(p^2) = -\frac{\pi p^2}{16} \Theta(p^2) \Theta(\pm p^0)$ and $G^{(\pm)}(p^2) = \frac{\pi p^2}{48} \Theta(p^2) \Theta(\pm p^0)$ so that

\[
I^{(\pm)}(p)_{\alpha\beta\mu\nu} = \frac{\pi}{10} \left( + p_\alpha p_\beta p_\mu p_\nu - \frac{p^2}{8} \left( + p_\alpha p_\beta \eta_{\mu\nu} + p_\alpha p_\mu \eta_{\beta\nu} + \\
+ p_\alpha p_\nu \eta_{\beta\mu} + p_\beta p_\mu \eta_{\alpha\nu} + p_\beta p_\nu \eta_{\alpha\mu} + p_\mu p_\nu \eta_{\alpha\beta} \right) + \\
+ \frac{p^4}{48} \left( + \eta_{\alpha\mu} \eta_{\beta\nu} + \eta_{\alpha\nu} \eta_{\beta\mu} + \eta_{\alpha\beta} \eta_{\mu\nu} \right) \right) \Theta(p^2) \Theta(\pm p^0) .
\]  (B.18)

**Appendix 3: The $Q^{(i)}(p)_{\alpha\beta,\mu\nu}$-Projection Operators**

The aim of this Appendix is to find a representation basis for rank-4 tensors, which allows to compute the inverse of the total graviton propagator \([5,8]\) in Sec. 5. Let us define as in \([17]\)

\[
d_{\mu\nu} := \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} , \quad e_{\mu\nu} := \frac{k_\mu k_\nu}{k^2} ;
\]  (C.1)

with

\[
d_{\mu\nu} d_{\nu\rho} = d_{\mu\rho} , \quad d_{\mu\nu} e_{\nu\rho} = 0 , \quad e_{\mu\nu} e_{\nu\rho} = e_{\mu\rho} .
\]  (C.2)

Then the so-called ‘projection’ operators are defined by

\[
Q^{(1)}(k)_{\alpha\beta,\mu\nu} = \frac{1}{2} (d_{\alpha\mu} e_{\beta\nu} + d_{\alpha\nu} e_{\beta\mu} + d_{\beta\mu} e_{\alpha\nu} + d_{\beta\nu} e_{\alpha\mu}) ,
\]

\[
Q^{(2)}(k)_{\alpha\beta,\mu\nu} = \frac{1}{2} (d_{\alpha\mu} d_{\beta\nu} + d_{\alpha\nu} d_{\beta\mu} - \frac{2}{3} d_{\alpha\beta} d_{\mu\nu}) ,
\]

\[
Q^{(3)}(k)_{\alpha\beta,\mu\nu} = \frac{1}{3} (d_{\alpha\beta} d_{\mu\nu}) , \quad Q^{(4)}(k)_{\alpha\beta,\mu\nu} = (e_{\alpha\beta} e_{\mu\nu}) ;
\]  (C.3)
and the so-called ‘transfer’ operators are defined by
\[ Q(k)^{(5)}_{\alpha\beta,\mu\nu} = \frac{1}{\sqrt{3}} (d_{\alpha\beta} e_{\mu\nu}) , \quad Q(k)^{(6)}_{\alpha\beta,\mu\nu} = \frac{1}{\sqrt{3}} (e_{\alpha\beta} d_{\mu\nu}) ; \]  
(C.4)

with the relations
\[ Q(k)^{(j)}_{\alpha\beta,\mu\nu} Q(k)^{(j)}_{\mu\nu,\rho\sigma} = \begin{cases} 1, & \text{if } j = 1, 2, 3, 4; \\ 0, & \text{if } j = 5, 6. \end{cases} \]  
(C.5)

These and other relations can be easily calculated using (C.2). We consider a rank-4 tensor in the standard basis as in Eq. (3.39) or (3.40) by giving its five coefficients and disregarding the logarithmic dependence on \( k^2/M^2 \):
\[ T(k)_{\alpha\beta\mu\nu} = [A, B, C, E, F](k)_{\alpha\beta\mu\nu} . \]  
(C.6)

We rescale it by dividing it by \( k^4 \), so that we can express the obtained ‘rescaled’ tensor \( \tilde{T}(k)_{\alpha\beta\mu\nu} \) in the projector basis given by (C.3) and (C.4):
\[ \tilde{T}(k)_{\alpha\beta\mu\nu} = \sum_{j=1}^{6} x_j Q(k)^{(j)}_{\alpha\beta,\mu\nu} . \]  
(C.7)

Comparing (C.6) with (C.7), we find the relations between the \( x_j \) coefficients and the \( A, \ldots, F \) coefficients:
\[ x_1 = 2(C + E), \quad x_2 = 2E, \quad x_3 = 2E + 3F, \]
\[ x_4 = A + 2B + 4C + 2E + F, \quad x_5 = x_6 = \sqrt{3}(B + F) . \]  
(C.8)

The inverse of \( \tilde{T}(k)_{\alpha\beta\mu\nu} \) in Eq. (C.7) satisfies
\[ (\tilde{T}(k))^{-1}_{\alpha\beta\mu\nu} (\tilde{T}(k)^{-1})^{\mu\nu}_{\rho\sigma} = l_{\alpha\beta,\rho,\sigma} , \]  
(C.9)

being \( l_{\alpha\beta,\rho,\sigma} = (\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu})/2 \) the unity for rank-4 tensors and it is given by
\[ (\tilde{T}(k)^{-1})_{\alpha\beta\mu\nu} = \left[ + x_1^{-1} Q^{(1)} + x_2^{-1} Q^{(2)} + \frac{x_4}{\Delta} Q^{(3)} + \frac{x_3}{\Delta} Q^{(4)} - \frac{x_5}{\Delta} Q^{(5)} - \frac{x_6}{\Delta} Q^{(6)} \right](k)_{\alpha\beta\mu\nu} \]
\[ =: \sum_{j=1}^{6} y_j Q(k)^{(j)}_{\alpha\beta,\mu\nu} , \]  
(C.10)

where \( \Delta := x_3 \cdot x_4 - x_5 \cdot x_6 \). The proof of Eq. (C.10) simply consists in carrying out the product in (C.9) and using the relations of Eq. (C.5). Returning back to the standard representation and multiplying with \( k^4 \), we obtain
\[ (T(k)^{-1})_{\alpha\beta\mu\nu} = \left[ -2y_1 + \frac{2y_2}{3} + \frac{y_3}{3} + y_4 - \frac{2y_5}{\sqrt{3}} + \frac{y_5}{\sqrt{3}} - \frac{y_3}{3} + \frac{y_2}{3} \right] (k)_{\alpha\beta\mu\nu} . \]  
(C.11)
Using the definitions of the \( y_j \)'s as functions of the \( x_i \)'s from (C.10) and the inverses of the relations in (C.8), we can then find the coefficients of the inverse \((\hat{T}(k)^{-1})_{\alpha\beta\mu\nu}\) in terms of the original coefficients \( A, \ldots, F\).

**Appendix 4: Numerical Distributions in Eq. (3.15)**

In this appendix we present explicitly the numerical distributions appearing in Eq. (3.15) or in Eq. (3.16). We separate the various contributions according to their singular order and to the presence of graviton or ghost loops.

For the graviton loop, the distribution with singular order four reads

\[
\tilde{d}^{(4)}_{2}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2 \pi}{960(2\pi)^4} \left[ -80, +60, +10, -70, 0 \right] \Theta(p^2) \text{sgn}(p^0). \tag{D.1}
\]

The distribution with singular order two can be written as

\[
\tilde{d}^{(2)}_{2}(p)_{\gamma|\rho} = \frac{\kappa^2 \pi}{960(2\pi)^4} \left[ -10 \hat{P}_1(p)_{\gamma|\rho} - 10 p^2 \hat{P}_{II}(p)_{\gamma|\rho} \right] \Theta(p^2) \text{sgn}(p^0), \tag{D.2}
\]

with

\[
\hat{P}_1(p)_{\gamma|\rho} = \{ p_\alpha p_\beta \left( 8 \eta_\mu^\rho \eta_\nu^\gamma + 8 \eta_\mu^\gamma \eta_\nu^\rho \right) + (\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu) \}
\]

\[
+ \{ p_\alpha p_\mu \left( 16 \beta_\nu^\gamma \eta_\rho^\beta + 12 \beta_\nu^\beta \eta_\rho^\gamma + 10 \eta_\rho^\beta \eta_\rho^\gamma \right) + (\alpha \leftrightarrow \beta) + (\mu \leftrightarrow \nu) 
\]

\[
+ (\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu) \}
\]

\[
+ p^\rho p^\gamma \{ -34 \eta_\alpha^\beta \eta_\mu^\rho - 60 \eta_\alpha^\beta \eta_\nu^\rho + 60 \eta_\alpha^\nu \eta_\beta^\mu \}
\]

\[
+ \{ 4 p_\alpha p_\gamma \left( -\eta_\beta^\mu \eta_\rho^\nu - \eta_\beta^\nu \eta_\rho^\mu + \eta_\gamma^\rho \eta_\mu^\nu \right) + (\alpha \leftrightarrow \beta) 
\]

\[
+ (\alpha \leftrightarrow \mu, \gamma \leftrightarrow \rho, \beta \leftrightarrow \nu) + (\alpha \leftrightarrow \nu, \gamma \leftrightarrow \rho, \beta \leftrightarrow \mu) \}
\]

\[
+ \{ p_\alpha p_\rho \left( -30 \eta_\beta^\gamma \eta_\mu^\nu - 30 \eta_\beta^\mu \eta_\rho^\nu + 18 \eta_\beta^\gamma \eta_\mu^\rho \right) + (\alpha \leftrightarrow \beta) 
\]

\[
+ (\alpha \leftrightarrow \mu, \gamma \leftrightarrow \rho, \beta \leftrightarrow \nu) + (\alpha \leftrightarrow \nu, \gamma \leftrightarrow \rho, \beta \leftrightarrow \mu) \}, \tag{D.3}
\]

and

\[
\hat{P}_{II}(p)_{\gamma|\rho} = \{ \eta_\alpha^\beta \eta_\mu^\nu \eta_\gamma^\rho - 8 \gamma^\rho \{ \eta_\alpha^\mu \eta_\beta^\nu + (\mu \leftrightarrow \nu) \}
\]

\[
- 3 \{ \eta_\alpha^\mu \eta_\beta^\rho + (\mu \leftrightarrow \nu) \} - 3 \{ \eta_\mu^\rho \eta_\alpha^\gamma \eta_\beta^\alpha + (\alpha \leftrightarrow \beta) \}
\]

\[
+ \{ 2 \eta_\alpha^\gamma \eta_\beta^\mu \eta_\beta^\rho + 15 \eta_\beta^\mu \eta_\gamma^\rho + (\mu \leftrightarrow \nu) + (\alpha \leftrightarrow \beta) + (\alpha \leftrightarrow \beta, \mu \leftrightarrow \nu) \}. \tag{D.4}
\]

This distribution yields after distribution splitting and after having multiplied it with \( p_\gamma p_\rho \) the following contribution to the graviton self-energy through graviton loop:

\[
p_\gamma p_\rho \tilde{d}^{(2)}_{2}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2 \pi}{960(2\pi)^4} \left[ -1200, -380, +450, -520, +330 \right] \tilde{t}(p). \tag{D.5}
\]
The distribution with singular order three attached to \( h^{\alpha\beta}(x)\gamma \, h^{\mu\nu}(y) \): reads
\[
\delta_2^{(3\alpha)}(p)_{\alpha\beta\mu\nu} = \frac{i\kappa^2\pi}{960(2\pi)^4} \left[ \hat{Q}_I(p)_{\alpha\beta\mu\nu} + p^2 \hat{Q}_{II}(p)_{\alpha\beta\mu\nu} \right] \Theta(p^2) \text{sgn}(p^0),
\]
with
\[
\hat{Q}_I(p)_{\alpha\beta\mu\nu} = -120 p_\mu p_\nu \{ p_\alpha \eta_\beta \gamma + (\alpha \leftrightarrow \beta) \}
+ 20 \{ \eta_\alpha p_\mu p_\nu \gamma + \eta_\mu p_\alpha p_\beta p_\nu + \eta_\nu p_\alpha p_\beta p_\mu \},
\]
and
\[
\hat{Q}_{II}(p)_{\alpha\beta\mu\nu} = p \gamma \left\{ +110 \eta_\mu \eta_\alpha \eta_\beta - 210 \eta_\mu \eta_\nu \eta_\beta - 210 \eta_\mu \eta_\nu \eta_\alpha \right\}
+ \{ p_\alpha (-20 \eta_\mu \eta_\beta \gamma + 130 \eta_\mu \eta_\nu \gamma + 130 \eta_\gamma \eta_\beta \nu) + (\alpha \leftrightarrow \beta) \}
+ \{ p_\mu (-20 \eta_\alpha \eta_\beta \gamma + 20 \eta_\alpha \eta_\nu \beta \gamma + 20 \eta_\gamma \eta_\beta \nu) + (\mu \leftrightarrow \nu) \}.
\]
This distribution yields after distribution splitting and after having multiplied it with \( ip_\gamma \) the following contribution to the graviton self-energy through graviton loop:
\[
ip_\gamma \delta_2^{(3\alpha)}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2\pi}{960(2\pi)^4} \left[ +200, +30, -150, +210, -110 \right] \hat{\ell}(p).
\]
The distribution \( d_2^{(3\alpha)}(p)_{\alpha\beta\mu\nu} \), with singular order three attached to the operator \( h^{\alpha\beta}(x)\gamma \, h^{\mu\nu}(y) \), is the same to the previous one with a global sign change and the replacements \( \mu \leftrightarrow \alpha, \nu \leftrightarrow \beta \) and \( \rho \leftrightarrow \gamma \). After distribution splitting and multiplication with \( -ip_\rho \), it gives the same contribution to the graviton self-energy through graviton loop as the previous one.

For the ghost loop we have: with singular order four
\[
\delta_2^{(4)}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2\pi}{960(2\pi)^4} \left[ +64, -8, +12, +8, +8 \right] \Theta(p^2) \text{sgn}(p^0),
\]
after symmetrization in \((\alpha\beta) \leftrightarrow (\mu\nu)\); with singular order three we have
\[
d_2^{(3\alpha)}(p)_{\alpha\beta\mu\nu} = \frac{i\kappa^2\pi}{960(2\pi)^4} \left[ +20 p_\alpha p_\beta \{ p_\mu \eta_\nu \rho + (\mu \leftrightarrow \nu) \}
+ 20 p^2 \{ p_\alpha \eta_\mu \eta_\nu \beta \rho + (\alpha \leftrightarrow \beta) \}
+ 10 p^2 \{ p_\mu (\eta_\alpha \eta_\nu \rho - \eta_\beta \eta_\mu \eta_\nu \rho) + (\mu \leftrightarrow \nu) \} \right] \Theta(p^2) \text{sgn}(p^0),
\]
whereas the distribution \( d_2^{(3\alpha)}(p)_{\alpha\beta\mu\nu} \), with singular order three attached to the operator \( h^{\alpha\beta}(x)\gamma \, h^{\mu\nu}(y) \), is the same to the previous one with a global sign change and the replacements \( \mu \leftrightarrow \alpha, \nu \leftrightarrow \beta \) and \( \rho \leftrightarrow \gamma \). With singular order two we obtain
\[
d_2^{(2)}(p)_{\alpha\beta\mu\nu} = \frac{\kappa^2\pi}{960(2\pi)^4} \left[ + \eta_\beta \rho \eta_\nu \gamma (80 p_\alpha p_\mu + 40 p^2 \eta_\alpha \mu) \right] \Theta(p^2) \text{sgn}(p^0),
\]
which must be symmetrized in \((\alpha \leftrightarrow \beta)\) and \((\mu \leftrightarrow \nu)\) and then one applies \( p_\gamma p_\rho \).
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