Adaptive control of a class of time-varying nonlinear systems via immersion and invariance

Xu Liang\textsuperscript{1,2}*, Tingting Su\textsuperscript{1,2}, Shengda Liu\textsuperscript{3} and Guangping He\textsuperscript{2}

\textsuperscript{1}Beijing University of Technology, Beijing, China
\textsuperscript{2}North China University of Technology, Beijing, China
\textsuperscript{3}Institute of Automation, Chinese Academy of Sciences, Beijing, China

Abstract
This paper addresses the adaptive control of a class of time-varying nonlinear systems. Under the framework of Immersion and Invariance (I&I) adaptive control, a set of sufficient conditions is obtained to stabilize the concerned time-varying nonlinear systems. It is shown that the presented controller can also be utilized to complete tracking control for a class of nonholonomic constraint systems if the desired trajectories satisfy certain conditions. The effectiveness of the new adaptive controller is demonstrated by some numerical simulations on a nonholonomic mobile robot.

Keywords
Immersion and invariance, adaptive control, nonlinear systems, time-varying systems, nonholonomy

Introduction
The control issues of time-varying systems were studied by many scholars in a relatively long time.\textsuperscript{1} Time-varying systems can be roughly classified as time-varying model structure,\textsuperscript{2} time-varying model parameters,\textsuperscript{3–5} and time-varying feedback systems,\textsuperscript{6} etc. Compared with the research on the control method of time-varying linear system,\textsuperscript{7} the research on the control method of time-varying nonlinear system has been addressed relatively less. In this paper, the stabilization of a kind of nonlinear system with continuous time-varying model parameter is investigated. The purpose of the present article is to utilize the I&I adaptive control approach to obtain global result in stabilizing the concerned kind of time-varying nonlinear system. In particular, the developed new I&I...
Adaptive controller design method should be able to be used for trajectory tracking control of the first-order nonholonomic mechanical system.

Adaptive control of nonholonomic systems is a relatively active research field. This topic has been discussed in plenty of literature. Compared with the researches about the stabilization issues of various nonholonomic systems, we notice that the researches about the tracking control of arbitrary trajectory for the nonholonomic systems are relatively less and the trajectory tracking control of nonholonomic mobile robot remains an open problem. Earlier studies on adaptive tracking control of nonholonomic systems only revealed that the nonholonomic systems could track certain special trajectories, such as straight lines or circles. This situation was more systematically discussed in, which revealed that the target trajectories provided to a nonholonomic system should satisfy the nonholonomic constraints. Therefore, for the arbitrarily given trajectories, the best tracking result for a nonholonomic system is uniformly ultimately bounded or asymptotically stable. This result then has been confirmed by the researches of adaptive tracking control for specific mobile robot. In the papers and, even though the trajectory tracking control problem was investigated with regard to the nonholonomic mobile robot, which is actually a third-order nonlinear system with nonholonomic constraints, the adaptive controller presented in used output feedbacks, while used a special coordinate transformation. A similar approach used in can also be observed in. As these control methods depend on the special property of the specific robot prototype, it is not clear how these control methods could be applied to other more generally similar systems. More recently, robust adaptive tracking control of nonholonomic systems and control of stochastic nonholonomic systems also have been investigated. However, using non-smooth feedbacks or non-continuous feedbacks to increase the closed-loop system's robustness commonly causes impractical non-smooth control inputs.

Since the I&I adaptive control method was presented by Astolfi and Ortega to stabilize certain classes of nonlinear system, the method was rapidly developed by a variety of literature which were systematically organized in the monograph. In more recent years, this novel tool was also developed to stabilize some more intractable systems, such as under-actuated mechanical systems, speed observers for systems with nonholonomic constraints, and nonlinear system with time-varying parameter. In literature, a different procedure to realize the construction of I&I controller for stabilizing nonlinear system was also presented by applying the contraction-based method. It is worthy of mentioning that, besides the I&I-based controllers were mainly developed for autonomous nonlinear systems during the past time, the I&I adaptive controllers were also mostly adopted for dealing with the unknown structure parameters of a controlled system. However, there are few studies on I&I adaptive control to stabilize time-varying system.

In this article, the I&I adaptive controllers are utilized in a new way. More specifically, the introduction of the “off-the-manifold” coordinates in designing a controller is used to construct an auxiliary dynamic subsystem, such that the responses of the primary dynamic subsystem could be influenced by the auxiliary dynamic subsystem. By constructing an auxiliary “regressor”, we can show that the adaptive law can influence the developments of the states of the nonlinear systems, such that partial states of the time-
varying nonlinear system could be implicitly stabilized by following the trajectory of other states. It provides additional flexibility to design the control laws for the time-varying nonlinear systems. The originality of the presented paper is that the I&I adaptive control approach is firstly developed to stabilize a kind of nonlinear system with time-varying drift vector fields. As a result, the control law obtained here for stabilizing the type of time-varying nonlinear system can be utilized for trajectory tracking control of a kind of nonholonomic constraint system. The I&I-based adaptive control law has certain good properties, such as smooth control inputs, exponential convergence and small overshoots for the closed-loop systems, its target trajectory can be rather general, and the control parameters can be selected in a rather large range.

The organization of this article is as follows. In section 2, the formulation of the control problem concerned in this paper, some basic concepts and preliminary results are presented. Section 3 is devoted to designing the adaptive controller based on I&I for a kind of time-varying nonlinear system. In section 4, the I&I adaptive tracking controller obtained in section 3 is utilized to trajectory tracking control for a nonholonomic mobile robot. Section 5 presents some numerical simulation results to demonstrate the practical effects of the controlled nonholonomic robot. Some concluding remarks are provided in section 6.

**Problem formulation and preliminaries**

This article considers the control problem of the following affine time-varying nonlinear system

\[
\dot{x} = f(x, t) + g(x)u
\]

where \(x \in \mathbb{R}^n\) represents the state, \(u \in \mathbb{R}^m\) represents the control input, \(f(x, t):\mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n\) and \(g(x):\mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}\) are smooth mappings, and \(t \geq 0\) is the time variable. The control object is to search a continuously adaptive state feedback control law in the form shown below

\[
\begin{align*}
    u &= \alpha + \varphi(x)^T \rho \\
    \alpha &= \mu(x, \hat{\rho}) \\
    \dot{\hat{\rho}} &= w(x, \hat{\rho})
\end{align*}
\]

where \(\alpha \in \mathbb{R}^m\) is defined to be a new control input, \(\rho \in \mathbb{R}^s\) represents an arbitrary but unknown nonzero constant vector, and \(\rho\) is usually associated with the structural parameters. \(\hat{\rho} \in \mathbb{R}^s\) represents the estimation of the constant vector \(\rho\), and \(\varphi(x):\mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}\) is a given smooth mapping, such that the closed-loop systems (1), (2) have a globally stable equilibrium at \((x, \hat{\rho}) = (0, \rho)\) and

\[
\lim_{t \to \infty} x(t) = 0
\]

**Remark 1.** It is different from a traditional adaptive control system, of which certain unknown structural parameters exist in the uncertain system and should be estimated by an adaptive law, so that the closed-loop system is asymptotically stable under the
condition that the estimation errors of the unknown structural parameters are bounded. However, the adaptive controller given by (2) intentionally introduces a set of unknown parameters \( \hat{\rho} \in \mathcal{R}^l \), and then designing an auxiliary dynamic system \( \hat{\rho} = w(x, \hat{\rho}) \) will provide a new pathway to synthesize a feasible control law for stabilizing the time-varying nonlinear system (1), with certain additional conditions.

To find a feasible approach to solve the control problem of system (1), the following preliminary concepts and technical tools should be mentioned for clarity.

Recall that \( \phi(x)|_{x \in (a,b)} \rightarrow \mathcal{R} \) is uniformly continuous if for any \( \varepsilon > 0 \), there exists a \( \delta(\varepsilon) > 0 \) such that if \( |x_1 - x_2| < \delta \) with \( x_1, x_2 \in (a, b) \), then \( |\phi(x_1) - \phi(x_2)| < \varepsilon \).

A bounded function \( \phi(x)|_{x \in (a,b)} \rightarrow \mathcal{R} \) indicates its \( L_\infty \)-norm, i.e. \( \|\phi(x)\|_\infty = \sup \{ \phi(x) : a < x < b \} \) exists and is finite. \( L_p \)-norm of a signal \( f(t) \), for all \( t \geq 0 \), for \( 1 < p < \infty \), is defined as \( \|f(t)\|_p = (\int_0^\infty (|f(t)|^p)dt)^{1/p} < + \infty \).

Two lemmas are provided since they will be used in the sequel of the paper.

**Lemma 1.** (Barbălat) If \( \phi: \mathcal{R}^+ \rightarrow \mathcal{R} \) is uniformly continuous, and if the limit of the integral \( \int_0^t \phi(\tau)d\tau \) exists and is finite, then \( \lim_{t \to \infty} \phi(t) = 0 \).

**Lemma 2.** For a scalar system \( \dot{x} = -kx + p(t) \), where \( k > 0 \) and \( p(t) \) is bounded and uniformly continuous function. If for any initial time \( t_0 \geq 0 \) and any initial condition \( x(t_0) \), the solution \( x(t) \) is bounded and converges to zero as \( t \to \infty \), then \( \lim_{t \to \infty} p(t) = 0 \).

**I&I adaptive control of a class of nonlinear systems**

Consider the time-varying nonlinear system (1), the assumptions are provided as follows.

(A1). The origin \( x(t) = 0 \) of the time-varying nonlinear system (1) is globally controllable for all \( t \geq 0 \).

(A2). The time-varying vector \( f(x, t) \in \mathcal{R}^n \) in system (1) is bounded and uniformly continuous, i.e. \( f(x, t) \in L_\infty(0, \infty) \), which denotes the \( L_\infty \)-norm of the vector \( f(x, t) \), and there exists a set of bounded and uniformly continuous functions \( \psi_j(x, t) : \mathcal{R}^n \times \mathcal{R} \to \mathcal{R}, j \leq r \leq m \) such that the following equation is true

\[
x^Tf(x, t) = \sum_{j=1}^r \xi_j \psi_j(x, t) \tag{4}
\]

where any variable \( \xi_j \) denotes a state variable \( x_k \) of the system (1), i.e. \( \xi_j = x_k, 1 \leq k \leq n \), and we define the variable set \( \chi_\xi = \{ \xi_1, \ldots, \xi_j, \ldots, \xi_r \} \).

(A3). The smooth mapping \( g(x) \in \mathcal{R}^{m \times n} \) satisfies the following relationship

\[
x^Tg(x) = \left[ \begin{array}{c} c_1 \xi_1 \phi_1(x) \cdots c_i \xi_i \phi_i(x) \cdots c_m \xi_m \phi_m(x) \end{array} \right] \tag{5}
\]

where \( c_i > 0 \) are constants, \( \phi_i(x) : \mathcal{R}^n \to \mathcal{R} \), for \( i = 1, \ldots, m \), are bounded continuous functions with \( 0 < |\phi_i(x)| < \Phi_0 \), any variable \( \xi_i, i \leq m \) denotes a state variable \( x_k \) of the system (1), i.e. \( \xi_i = x_k, 1 \leq k \leq n \). Define the variable set \( \chi_\xi = \{ \xi_1, \ldots, \xi_i, \ldots, \xi_m \} \), and it satisfies the relationship \( \chi_\xi \in \mathcal{R}^r \leq \chi_\xi \in \mathcal{R}^m \leq \chi \in \mathcal{R}^n \), where the set \( \chi \) is defined as \( \chi = \{ x_1, \ldots, x_n \} \).
For any variable $x_k$ of the system (1), with $1 \leq k \leq n$, which is not included in the set $\chi_{\xi}$, i.e. $x_i \in \chi_{\xi} = \{x - \xi\} \in \mathbb{R}^{n-m} \subset \chi$, there exist bounded and uniformly continuous functions $P_i(x_i, \xi, t): \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$, with $1 \leq l \leq n-m$ and $1 \leq i \leq m$, which satisfies

$$\dot{\xi}_i = -k \xi_i + P_i(x_i, \xi, t), \quad 1 \leq i \leq m$$

(6)

where $k > 0$ is a constant, and the function $P_i(x_i, \xi, t)$ can be expressed as the following form

$$P_i(x_i, \xi, t) = Q_i(x_i)p_i(\xi, t), \quad 1 \leq i \leq m$$

(7)

where $Q_i(x_i)$ is a continuous function and has a unique zero point $Q_i(x_i)|_{x_i=0} = 0$, while the function $p_i(\xi, t): \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ is also bounded and uniformly continuous but satisfies $\lim_{t \rightarrow \infty} p_i(\xi, t) \neq 0$.

**Remark 2:** The assumption (A1) is a basic condition, which just means that the nonlinear system (1) should be controllable, even though it is in general a hard problem for confirming the controllability of the rather general nonlinear systems (1). The controllability of the system (1) is usually judged by using the Chow’s theorem and its relevant corollaries. Chow’s Theorem uses the minimum distribution concept and the controllability Lie algebra condition to analyze the controllability of nonlinear systems. Chow’s theorem asserts that if the controllability Lie algebra is full rank, we can steer this system from any initial point to any final point.\(^{30-31}\)

**Remark 3:** The first part of the assumption (A2) is a classic condition for discussing the stabilization of a time-varying system. The time-varying vector $f(x, t) \in \mathbb{R}^n$ in system (1) represents the drift terms. The drift terms in a time-varying nonlinear system should be bounded and uniformly continuous, since the control inputs provided by any actual actuators are bounded and uniformly continuous. Otherwise, the time-varying nonlinear system (1) possibly loses controllability. The second part of the assumption (A2) states that the time-varying function $x^Tf(x, t)$ should be able to be reorganized into another time-varying polynomial and the new polynomial still holds linear structure. Similarly, this objective is also presented in the assumption (A3).

**Remark 4:** The assumption (A4) is provided for considering the general cases, where the nonlinear system (1) maybe a nonholonomic or under-actuated system. In particular, for the nonlinear system that could not be linearized to a controllable system by other approaches such as by the aid of differentially flat output feedbacks, or that could not be simplified to other normal forms, such as the feedback or feedforward forms. Then the assumption (A4) provides another approach to analyze the system’s stability with the help of Lemma 1 and Lemma 2.

To establish the adaptive state feedback control law (2) for system (1), following the methodology presented by Astolfi et al.\(^{20-21}\) let’s define the following “estimation
errors”
\[ z = \hat{\rho} - \rho + \beta(x) \]  (8)
where \( \beta(x):\mathbb{R}^n \to \mathbb{R}^s \) is a smooth mapping that is defined to comply with the following partial differential equation (PDE)
\[ \frac{\partial \beta(x)}{\partial x} g(x) = \gamma \varphi(x) \]  (9)
where \( \gamma > 0 \) is a given constant, then the “estimation error dynamics” of (8) can be obtained
\[ \dot{z} = \dot{\hat{\rho}} + \frac{\partial \beta(x)}{\partial x} [f(x, t) + g(x)(\alpha + \varphi(x)^T(\hat{\rho} + \beta(x) - z))] \]  (10)
\[ \dot{\hat{\rho}} = -\frac{\partial \beta(x)}{\partial x} [f(x, t) + g(x)(\alpha + \varphi(x)^T(\hat{\rho} + \beta(x)))] \]  (11)
and then the following result can be established.

**Theorem 1.** For the extended dynamic system \((x, z)\), which is given by (1) and (10), suppose the assumptions (A1)-(A4) hold, and there exists a smooth mapping \( \varphi(x):\mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^s \) such that the introduced vector \( \beta(x):\mathbb{R}^n \to \mathbb{R}^s \) satisfies the condition (9), then if the adaptive law is selected as (11), the control law is selected as (2), and the new inputs are chosen as
\[ \alpha_i = -\phi_i(x)^{-1}[\xi_i + \psi_i(x, t)] - \varphi_i(x)^T[\hat{\rho} + \beta(x)] \text{ for } i = 1, \ldots, m \]  (13)
where the functions \( \psi_i(x, t) \) for \( i = 1, \ldots, m \), is defined by
\[ \psi_i(x, t) := \begin{cases} \psi_j(x, t) & \text{for } \xi_i = \xi_j \\ 0 & \text{for } \xi_i \in \{ \xi - \zeta \} \end{cases} \]  (14)
with \( j = 1, \ldots, r \), then the closed-loop control systems (1), (2), (11) and (13) are uniformly globally stable at equilibrium \((x, \hat{\rho}) = (0, \rho)\) and (3) holds.

**Proof:** Define a function \( V(x, z):\mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R} \) that is given by
\[ V(x, z) = \frac{1}{2} (x^T x + \gamma^{-1} z^T z) \], then the time derivative of \( V(x, z) \) can be written as
\[ \dot{V}(x, z) = x^T (f(x, t) + g(x)u) - [\varphi(x)^T z]^T [\varphi(x)^T z] \]  (15)
Considering the assumptions (A2) and (A3), which are $x^T f(x, t) = \sum_{j=1}^{r} \zeta_j \psi_j(x, t)$, and $x^T g(x) = \begin{bmatrix} c_1 \xi_1 \phi_1(x) & \cdots & c_i \xi_i \phi_i(x) & \cdots & c_m \xi_m \phi_m(x) \end{bmatrix}$, then (15) follows that

$$
\dot{V}(x, z) = \sum_{j=1}^{r} \xi_j \psi_j(x, t) + \sum_{i=1}^{m} c_i \xi_i \phi_i(x) u_i - [\varphi(x)^T z] [\varphi(x)^T z] \tag{16}
$$

Due to the assumption (A3), that is $\zeta \in \mathcal{H}^l \subseteq \zeta \in \mathcal{H}^m \subseteq x \in \mathcal{H}^n$, the first two compound terms of the right side of the above formula can be rewritten as follows by merging similar items.

$$
\dot{V}(x, z) = \sum_{i=1}^{m} c_i \xi_i [\varphi_i(x, t) + \phi_i(x) u_i] - [\varphi(x)^T z] [\varphi(x)^T z] \tag{17}
$$

Note that $u_i = \alpha_i + \varphi_i(x)^T [\dot{\rho} + \beta(x) - z]$ for $i = 1, \ldots, m$, and selecting the new virtual inputs $\alpha_i$ as (13), and then it can be obtained that

$$
\dot{V}(x, z) = - \sum_{i=1}^{m} c_i \xi_i^2 - \sum_{i=1}^{m} c_i \xi_i \phi_i(x) \varphi_i(x)^T z - [\varphi(x)^T z] [\varphi(x)^T z] \tag{18}
$$

By applying the Young’s inequality $\left(\sqrt{\frac{1}{2a}} - \sqrt{\frac{b}{2}}\right)^2 \geq 0$ with $\forall \varepsilon > 0$ and $a, b \in \mathcal{H}$, it can be observed that the cross-coupling terms $\sum_{i=1}^{m} c_i \xi_i \phi_i(x) \varphi_i(x)^T z$ in (18) can be eliminated, and there must exist a group of constants $\tilde{c}_i > 0$ for $i = 1, \ldots, m + 1$, such that (18) follows that

$$
\dot{V}(x, z) = - \sum_{i=1}^{m} \tilde{c}_i \xi_i^2 - \tilde{c}_{m+1} [\varphi(x)^T z] [\varphi(x)^T z] \leq 0,
$$

which demonstrates that $\xi_i(t) \in L_2$ for $i = 1, \ldots, m, z_j(t) \in L_\infty$ for $j = 1, \ldots, s$, while $\varphi(x(t))^T z(t) \in L_2$. By the Lemma 1 (Barbălat), it can be concluded that $\xi_i(t)$ for $i = 1, \ldots, m$ and $\varphi(x(t))^T z(t)$ converge to zero as $t \to \infty$.

Note that partial state variables of $x \in \mathcal{H}^n$ are not included in the set $\mathcal{X}$. For the state variables $x_i$ that are not included in the set $\mathcal{X}$, i.e., $x_i \in \bar{X}_i = \{\chi - \xi\}$ with $1 \leq l \leq n - m$, referring to the assumption (A4), it shows that $\dot{e}_i = -k \xi_i + P_i(x_i, \xi, t)$, with $1 \leq i \leq m$. By applying the Lemma 2, we can conclude that $\lim x_i(t) = 0$ because of $\lim \xi_i(t) = 0$ and $\lim P_i(x_i, \xi, t) = 0$ but $\lim \dot{p}_i(\xi, t) \neq 0$. Therefore, by LaSalle’s invariant principle, it can be concluded that $\lim (x(t), z(t)) = 0$ holds and the all trajectories of $(x(t), z(t))$ converge to the invariant set $\Theta = \{(x, z) \in \mathcal{H}^n \times \mathcal{H}^s : x = 0, \varphi(x)^T z = 0\}$, hence $\lim x(t) = 0$ holds, which completes the proof.

**Remark 5**: The introduction of the “regressor” $\varphi(x)$ in (2) and the smooth mapping $\beta(x)$ in (8) allows establishing of the error dynamics (10), which provides a new avenue to affect the developments of the state variables $x(t)$ of the nonlinear system (1), so that $x(t)$ could be globally asymptotically convergent for any initial values $x(0)$ and any initial time $t \geq 0$, while the zero equilibrium of $z(t)$ is uniformly stable, for any $x(t), \alpha(t)$ and for any given $\gamma > 0$, by applying the adaptive law (11).
Remark 6: Even though Theorem 1 is obtained based on several assumptions, which perhaps show certain conservatism, however, in the next section, we will demonstrate that the I&I adaptive control law given by Theorem 1 can be used for trajectory tracking control for a kind of nonholonomic constraint system. In a relatively long time, the first-order nonholonomic constraint systems, such as various kinds of mobile robots, are a class of important verification platforms of various new control laws. To the best knowledge of the authors, it may be the first time in this article that an I&I-based adaptive control law is presented for trajectory tracking control of the first-order nonholonomic constraint systems. What’s more valuable is that, compared with other control methods for the first-order nonholonomic constraint system, the I&I-based adaptive control laws demonstrate certain preferable properties, such as smooth control inputs, exponential convergence and small overshoots for the closed-loop systems, its target trajectory can be rather general, and the control parameters can be selected in a rather large range.

I&I adaptive tracking control of a nonholonomic mobile robot

In order to show the practical feasibility of the adaptive controller (2) for stabilizing a certain class of the time-varying nonlinear system (1), in this section we study the tracking control issues of a crawler-type robot system. Figure 1 illustrates the robot prototype and the corresponding parameterized model.

The velocity kinematics model of the crawler mobile robot illustrated in Figure 1 is given as

\[
\begin{align*}
\dot{x}_c &= \frac{1}{2}(v_1 + v_2) \cos \theta \\
\dot{y}_c &= \frac{1}{2}(v_1 + v_2) \sin \theta \\
\dot{\theta} &= \frac{2}{R}(v_2 - v_1)
\end{align*}
\] (19)

where \((x_c, y_c)\) denotes the Cartesian position coordinates of the robot’s mass center, \(\theta\) denotes the angular between the robot’s longitudinal axis and the x-axis of the inertial coordinate system \(o-xy\), \(v_1\) and \(v_2\) represent the velocities of two tracks of the crawler mobile robot respectively, and \(R\) denotes the distance between two tracks. Since the lateral velocity of the robot always equals zero at any time, it is well known that the nonholonomic constraint \(\dot{x} \sin \theta - \dot{y} \cos \theta = 0\) exists in the mobile robot system.

By applying the inputs transformation \(u_1 = \frac{1}{2}(v_1 + v_2)\) and \(u_2 = \frac{2}{R}(v_2 - v_1)\), the system model (19) can be changed as

\[
\begin{align*}
\dot{x} &= u_1 \cos \theta \\
\dot{y} &= u_1 \sin \theta \\
\dot{\theta} &= u_2
\end{align*}
\] (20)

Since an arbitrarily given trajectory is not necessarily feasible for a nonholonomic
system, in this paper the target trajectories are generated by a virtual mobile robot, as adopted in,\textsuperscript{6,10,14} and the target trajectories $(x_d(t), y_d(t), \theta_d(t))$ satisfy

$$
\dot{x}_d = u_{1d}(t) \cos \theta_d \\
\dot{y}_d = u_{1d}(t) \sin \theta_d \\
\dot{\theta}_d = u_{2d}(t)
$$

(21)

where $u_{1d}(t) \in \mathbb{R}$ and $u_{2d}(t) \in \mathbb{R}$ are the desired control inputs. Persistent excitation\textsuperscript{32} requires the target trajectory to be time-varying, so $u_{1d}(t)$, $u_{2d}(t)$ cannot be 0. Suppose the errors of the state variables are defined by

$$
\begin{bmatrix}
  x_e \\
y_e \\
\theta_e 
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & \sin \theta & 0 \\
  -\sin \theta & \cos \theta & 0 \\
  0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
  x_d - x \\
y_d - y \\
\theta_d - \theta 
\end{bmatrix}

(22)

then the errors dynamics of the state variables can be expressed as

$$
\begin{bmatrix}
  \dot{x}_e \\
\dot{y}_e \\
\dot{\theta}_e 
\end{bmatrix} =
\begin{bmatrix}
  u_{1d}(t) \cos \theta_e \\
  u_{1d}(t) \sin \theta_e \\
  u_{2d}(t) 
\end{bmatrix} +
\begin{bmatrix}
  -1 & y_e & 0 \\
  0 & -x_e & 0 \\
  0 & 0 & -1 
\end{bmatrix}
\begin{bmatrix}
  u_1 \\
  u_2 
\end{bmatrix}

(23)

which can be expressed in the form of (1), indicating that the system is a time-varying nonlinear system. Now we consider the global tracking control issue of the time-varying nonlinear system (23). Using the adaptive controller design method presented in section 3, an adaptive controller for stabilizing the system (23) can be obtained.
**Theorem 2.** For the time-varying nonlinear system (23) with $u_{td}(t) \neq 0$, select the following virtual inputs

$$
\alpha_1 = k_x x_e + u_{td} \cos \theta_e - \varphi_1(\hat{\rho}_1 + \beta_1)
$$

$$
\alpha_2 = k_\theta \theta_e + u_{td} \int_0^1 \cos (s \theta_e) ds + u_{2d} - \varphi_2(\hat{\rho}_2 + \beta_2)
$$

(24)

with the parameter adaptive laws

$$
\dot{\hat{\rho}}_1 = -\frac{\partial \beta_1}{\partial x_e} [y_e u_2 + u_{td} \cos \theta_e - \alpha_1 - \varphi_1(\hat{\rho}_1 + \beta_1)] - \frac{\partial \beta_1}{\partial \theta_e} (u_{2d} - u_2)
$$

$$
\dot{\hat{\rho}}_2 = -\frac{\partial \beta_2}{\partial \theta_e} [u_{2d} - \alpha_2 - \varphi_2(\hat{\rho}_2 + \beta_2)]
$$

(25)

and the functions $\beta_1$ and $\beta_2$ are defined by

$$
\beta_1(x_e, \theta_e) = -\gamma_1 \int_0^{x_e} \varphi_1(s, \theta_e) ds
$$

$$
\beta_2(\theta_e) = -\gamma_2 \int_0^{\theta_e} \varphi_2(s) ds
$$

(26)

Where $\gamma_1, \gamma_2 > 0$ are constants, while the functions $\varphi_1$ and $\varphi_2$ are selected as

$$
\varphi_1(x_e, \theta_e) = x_e \theta_e
$$

$$
\varphi_2(\theta_e) = \theta_e
$$

(27)

then the controller

$$
u_1 = \alpha_1 + \varphi_1(x_e, \theta_e) \rho_1
$$

$$
u_2 = \alpha_2 + \varphi_2(\theta_e) \rho_2
$$

(28)

where $\rho_1$ and $\rho_2$ are two arbitrary constants, renders the system (23) uniformly globally stabilized to the origin $(x_e, y_e, \theta_e) = (0, 0, 0)$.

**Proof.** Define the “estimation errors”

$$
z_1 = \dot{\hat{\rho}}_1 - \rho_1 + \beta_1(x_e, \theta_e)
$$

$$
z_2 = \dot{\hat{\rho}}_2 - \rho_2 + \beta_2(\theta_e)
$$

(29)

where functions $\beta_1$ and $\beta_2$ are defined by (26), then the errors dynamics of (29) are given
by
\[ \dot{z}_1 = \dot{\hat{\rho}}_1 + \frac{\partial \beta_1}{\partial x_e} [y_e u_2 + u_{1d} \cos \theta_e - \alpha_1 - \varphi_1 (\dot{\hat{\rho}}_1 + \beta_1)] + \frac{\partial \beta_1}{\partial x_e} \varphi_1 z_1 + \frac{\partial \beta_1}{\partial \theta_e} (u_{2d} - u_2) \]
\[ \dot{z}_2 = \dot{\hat{\rho}}_2 + \frac{\partial \beta_2}{\partial \theta_e} [u_{2d} - \alpha_2 - \varphi_2 (\dot{\hat{\rho}}_2 + \beta_2)] + \frac{\partial \beta_2}{\partial \theta_e} \varphi_2 z_2 \]

Select the adaptive laws as
\[ \dot{\hat{\rho}}_1 = -\frac{\partial \beta_1}{\partial x_e} [y_e u_2 + u_{1d} \cos \theta_e - \alpha_1 - \varphi_1 (\dot{\hat{\rho}}_1 + \beta_1)] - \frac{\partial \beta_1}{\partial \theta_e} (u_{2d} - u_2) \]
\[ \dot{\hat{\rho}}_2 = -\frac{\partial \beta_2}{\partial \theta_e} [u_{2d} - \alpha_2 - \varphi_2 (\dot{\hat{\rho}}_2 + \beta_2)] \]

then (30) follows that
\[ \dot{z}_1 = -\gamma_1 \varphi_1^2 z_1 \]
\[ \dot{z}_2 = -\gamma_2 \varphi_2^2 z_2 \]

To reveal the dynamics properties of the closed-loop system (23), (28), consider the function
\[ \dot{V}(x, y, \theta, z_1, z_2) = x_e (y_e u_2 + u_{1d} \cos \theta_e - u_1) + y_e (-x_e u_2 + u_{1d} \sin \theta_e) + \theta_e (u_{2d} - u_2) - \gamma_1 \varphi_1^2 z_1^2 - \gamma_2 \varphi_2^2 z_2^2 \]

Note that the identity \( \sin (\theta_e) = \theta_e \int_0^1 \cos (s \theta_e) ds \) holds, then (33) can be rewritten as
\[ \dot{V}(x, y, \theta, z_1, z_2) = x_e (u_{1d} \cos \theta_e - \alpha_1 - \varphi_1 (\dot{\hat{\rho}}_1 + \beta_1)) + \theta_e \left( u_{1d} y_e \int_0^1 \cos (s \theta_e) ds + u_{2d} - \alpha_2 - \varphi_2 (\dot{\hat{\rho}}_2 + \beta_2) \right) \]
\[ - \gamma_1 \varphi_1^2 z_1^2 - \gamma_2 \varphi_2^2 z_2^2 + x_e \varphi_1 z_1 + \theta_e \varphi_2 z_2 \]

By applying the virtual inputs (24) and considering the following Young’s inequalities
\[ x_e \varphi_1 z_1 \leq \frac{1}{2 \varepsilon_1} x_e^2 + \frac{\varepsilon_1}{2} \varphi_1^2 z_1^2 \]
\[ \theta_e \varphi_2 z_2 \leq \frac{1}{2 \varepsilon_2} \theta_e^2 + \frac{\varepsilon_2}{2} \varphi_2^2 z_2^2 \]
where $\varepsilon_1, \varepsilon_2 > 0$ are two constants, it is easy to show that

$$
\dot{V}(x, y, \theta, z_1, z_2) \leq -\left(k_x - \frac{1}{2\varepsilon_1}\right)x_e^2 - \left(k_\theta - \frac{1}{2\varepsilon_2}\right)\theta_e^2
$$

$$
- \left(\gamma_1 - \frac{\varepsilon_1}{2}\right)\varphi_1^2z_1^2 - \left(\gamma_2 - \frac{\varepsilon_2}{2}\right)\varphi_2^2z_2^2
$$

(36)

thus we can conclude that $\dot{V}(x, y, \theta, z_1, z_2) \leq 0$ if the feedback parameters $(k_x, k_\theta, \gamma_1, \gamma_2)$ are properly selected. From (36), it can be observed that $x_e(t) \in L_2, \theta_e(t) \in L_2$ and $\varphi_iz_i \in L_2$ for $i = 1, 2$. By the Lemma 1 (Barbâlat), one has

$$
\lim_{t \to \infty} |x_e(t)| + |\theta_e(t)| + |\varphi_1z_1(t)| + |\varphi_2z_2(t)| = 0
$$

(37)

To observe the convergence property of the state variable $y_e(t)$, refer to the close-loop dynamics of the variable $\theta_e(t)$, which is given by

$$
\dot{\theta}_e = -k_\theta \theta_e - u_{1d}y_e \int_{0}^{1} \cos(s\theta_e)ds + \varphi_2z_2
$$

(38)

then based on Lemma 2, we can get $\lim_{t \to \infty} y_e(t) = 0$, owing to $\lim_{t \to \infty} \theta_e(t) = 0$ and $u_{1d}(t) \neq 0$ (given condition) and $\lim_{t \to \infty} \int_{0}^{1} \cos(s\theta_e(t))ds = \lim_{t \to \infty} \frac{\sin(\theta_e(t))}{\theta_e(t)} = 1$. This completes the proof.

Remark 7. It is worthy noting that the trajectory tracking controller given by Theorem 2 does not depend on the chained normal form transformation approach, which is an important result presented by Murray et al.\textsuperscript{30} for understanding the controllability of the nonholonomic constraint systems, and has been used in many papers\textsuperscript{11,12,30,33} for different purposes. However, in this paper the errors definition (22) is used for designing the trajectory tracking controller (28), such that the nonlinear coordinate $\theta_e$ is successfully decoupled from the control vector field $g(x)$ in (23) by applying the errors definition (22), and the drift vector field $f(x, t)$ in (23) could be limited to be bounded and uniformly continuous while does not conflict with the practical applications of a mobile robot.

Remark 8. The close-loop dynamics of the variable $\theta_e(t)$, which is given by (38), directly demonstrate that the assumption (A4) is useful for applying the Theorem 1 to deal with the trajectory tracking control issues of the first order nonholonomic constraint systems.

Numerical simulations

Numerical simulations are presented to validate the effectiveness of I&I adaptive tracking controller for the crawler-type mobile robot. In the simulations, the desired trajectory is given by (39), while the control parameters and the initial conditions are shown in...
Table 1. The control parameters and initial conditions used in the numerical simulations.

| Parameters                      | Symbols | Values | Physical Units |
|--------------------------------|---------|--------|----------------|
| Coefficients of the introduced functions $\beta_1$ and $\beta_2$ | $\gamma_1, \gamma_2$ | 1, 1 | / |
| Feedback parameters            | $k_x, k_\theta$ | 1, 1 | / |
| Unknown parameters             | $\rho_1, \rho_2$ | 1, 1 | / |
| Initial position errors        | $x_e, y_e, \theta_e$ | 1, $-1, \frac{\pi}{2}$ | m, rad |
| Distance between two tracks    | $R$     | 0.2    | m |

Figure 2. Position responses of the closed-loop system.
Figure 3. Position errors responses of the closed-loop system.

Figure 4. Control inputs the closed-loop system.

Figure 5. Adaptive parameters responses of the closed-loop system.
their target trajectories. For clarity, we also draw Figure 6(b) to show $x, y, \theta$ more clearly. Figure 7 also presents both global and local representations of the position errors. By comparing Figure 2 with Figure 6, Figure 3 with Figure 7, we can get that the root-mean-square error (RMSE) of $x_e, y_e, \theta_e$ under the proposed controller in Figure 3 is 0.1148, 0.1681 and 0.2708 respectively, while the RMSE of $x_e, y_e, \theta_e$ under the comparison controller in Figure 7 is 17.9779, 1.4718 and 0.2818 respectively. Then it can be concluded that the method proposed in this paper achieve better trajectory tracking of the target trajectory (39) than the comparison method. It is worthy to mention that, the target trajectory (39) is rather general. The controller proposed in this paper is different from
many existing papers, where certain special trajectories were commonly utilized to validate their control strategies. In this paper, we can show that the adaptive law can influence the developments of the states of the nonlinear systems by constructing an auxiliary “regressor”, such that partial states of the time-varying nonlinear system could be implicitly stabilized by following the trajectory of other states. In addition, many other numerical simulations show that the control parameters \((k_x, k_\theta), (\gamma_1, \gamma_2)\) and \((\rho_1, \rho_2)\) can be selected in a rather large range. In particular, benefitting from the adaptive mechanism, the control law (28), (24) can undergo large changes on unknown parameters \((\rho_1, \rho_2)\) for a set of given feedback gains \((k_x, k_\theta)\) and \((\gamma_1, \gamma_2)\). All these simulation results verify the effectiveness of the controller proposed in this paper.

## Conclusions

I&I-based adaptive control issues of a class of time-varying nonlinear system are investigated in this paper. It is shown that, under certain additional conditions, the I&I-based adaptive control method can be utilized to stabilize a class of time-varying nonlinear system. More interesting, it is also shown the presented adaptive controller can be used for trajectory tracking control of a class of nonholonomic constraint system.

It is different from the traditionally standard adaptive control, where the adaptive law is generally used to dominate the estimation errors of the unknown structure parameters of a controlled plant. However, the I&I-based adaptive control method proposed in this paper is used to increase the design flexibility of the control inputs. By constructing an auxiliary “regressor”, we have shown that the adaptive law can influence the developments of the state variables of the nonlinear systems, such that partial state variables of a time-varying nonlinear system could be implicitly stabilized by following the trajectory of other state variables. Our future work will focus on the I&I-based adaptive control method of a class of nonholonomic constraint system with disturbances or unmodelled terms.

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## Declaration of conflicting interests

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ORCID iD
Xu Liang https://orcid.org/0000-0002-9963-3662

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Author biographies

Xu Liang received the B.S. degree in automation from Central South University, Changsha, China, in 2013, and the Ph.D. degree in control theory and control engineering from Institute of Automation, Chinese Academy of Sciences, Beijing, China, in 2020. He is currently with the North China University of Technology, Beijing, China. His research interests include human-machine interface, interaction control, and intelligent systems.

Tingting Su received the B.S. degree in automation from Central South University, Changsha, China, in 2013, and the Ph.D. degree in control theory and control engineering from the Institute of Automation, Chinese Academy of Sciences, Beijing, China, in 2018. She is currently with North China University of Technology, Beijing, China. Her current research interests include trajectory planning, robotics, and intelligent control systems.

Shengda Liu received the B.S. degree with the major of mathematics from Harbin Normal University, Harbin, China, in 2012, and the M.S. and Ph.D. degrees with the major of applied mathematics from Guizhou University, Guiyang, China, in 2016 and 2019, respectively. He is currently a post-doctoral fellow at the State Key Laboratory of Management and Control for Complex Systems, Institute of Automation, Chinese Academy of Sciences. His current research interests include iterative learning control, fractional order equation and control theory, human-robot interaction and its application in rehabilitation robots.

Guangping He received the B.S., M.S., and Ph.D. degrees in mechanical and electrical engineering from the Beihang University of China, Beijing, China, in 1994, 1997, and 2002, respectively. From 1997 to 2002, he was with Launch Vehicle Technology Academy of China, Beijing. From 2007 to 2008, he was a post-doctoral fellow with the Department of Mechanics and Engineering Science, Peking University, Beijing, China. Since 2002, he has been with North China University of Technology, Beijing, China. His current research interests include dynamics and control of robots and micro-electromechanical devices.