Spinor Geometry

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Abstract

It has been proposed that quantum mechanics and string theory share a common inner syntax, the relational logic of C. S. Peirce. Along this line of thought we consider the relations represented by spinors. Spinor composition leads to the emergence of Minkowski spacetime. Inversely the Minkowski spacetime is instantiated by the Weyl spinors, while the merge of two Weyl spinors gives rise to a Dirac spinor. Our analysis is applied also to the string geometry. The string constraints are represented by real spinors, which create a parametrization of the string worldsheet identical to the Enneper-Weierstrass representation of minimal surfaces. Further, a spinorial study of the $AdS_3$ spacetime reveals a Hopf fibration $AdS_3 \rightarrow AdS_2$. The conformal symmetry inherent in $AdS_3$ is pointed out. Our work indicates the hidden ties between logic-quantum mechanics-string theory-geometry and vindicates the Wheeler’s proposal of pregeometry as a large network of logical propositions.

Introduction

Discerning the foundations of a theory is not simply a curiosity. It is a quest for the internal architecture of the theory, offering a better comprehension of the entire theoretical construction and favoring the study of more complex issues. Quantum mechanics stands out as the theory of the 20th century, shaping the most diverse natural phenomena (from subatomic physics to cosmology). Yet, we are lacking a foundational principle for quantum mechanics. In another direction, the unification of quantum mechanics and
general relativity, string theory appears as the most promising example of a unified theory. In a similar vain, we are lacking the conceptual foundation of string theory.

In the present work we would like to further explore the suggestion that the relational logic of C. S. Peirce may serve as the foundation of both quantum mechanics and string theory [1]. It is most natural to invoke logic in order to build and develop the theories describing physical reality. The entire edifice of classical physics is grounded on the Aristotelian logic, as it was transformed into algebraic logic by G. Boole. The Boolean algebra is based on set theory. Set theory is the backbone of the mathematical apparatus (continuum geometry, differential equations) we employ in classical physics. Inadvertently a set-theoretic enterprise is analytic, atomistic, arithmetic. Peirce invented and elaborated a novel logical syntax, where relation is the irreducible primary datum [2]. All other terms or objects are defined in terms of relations, transformations, arrows, morphisms. The Peircean logical structure bears great resemblance to category theory [3]. A relational or categorical formulation is bound to be synthetic, holistic, geometric. There is already an important work in physics inspired by categorical notions [4,5,6]

The binary relation $R_{ij}$ between two “individual terms” $S_j$ and $S_i$ may receive multiple interpretations: as a transition from the $j$ state to the $i$ state, as a proof that the logical proposition $j$ implies logical proposition $i$. At the very core of the Peircean logical system is the composition of relations. Whenever we have relations of the form $R_{ij}$, $R_{je}$ a third transitive relation $R_{ie}$ emerges following the rule [2]

$$R_{ij}R_{ke} = \delta_{jk}R_{ie}$$

The relations generate a $W_\infty$ algebra, which is a bosonic extension of the Virasoro algebra, linked to area-preserving diffeomorphisms [1].

The simplified case of only two states ($i = 1, 2$) indicates the inherent nature of our logical system. Defining

$$R_z = \frac{1}{2}(R_{11} - R_{22})$$

and

$$R_+ = R_{12} \quad R_- = R_{21}$$

we find out that the $SU(2)$ commutation relations are satisfied [1]. Thus the underlying dynamics is similar to a “spin 1/2 particle”. Considering that the answer to a logical proposition is a “yes” or a “no” statement, analogous to a “spin up” or a “spin down” measurement, we view the spinor as the building block of our logical construction. In this paper, following the Peircean insight,
we study relations built up by spinors and the deep connection of spinors to
geometry. We find out that composition of spinors gives rise to Minkowski
spacetime. On the other hand starting from geometry, a string worldsheet
or an \(AdS_3\) spacetime, we recover the inherent spinorial structure. At the
end we comment on the hidden ties between logic-quantum mechanics-string
theory-geometry.

**The emergence of Minkowski spacetime**

With \(|r_i\rangle\) standing for a state or a proposition, the suggested representa-
tion for \(R_{ij}\) is [1]

\[
R_{ij} = |r_i\rangle \langle r_j|
\]

where \(\langle r_i|\) is the dual of \(|r_i\rangle\). Consider the spinor \(|u\rangle = \left( \begin{array}{c} \xi \\ \eta \end{array} \right)\). The spinor
relationship \(R_s\) is defined by

\[
R_s = |u\rangle \langle u| = \left( \begin{array}{cc} \xi \xi^* & \xi \eta^* \\ \eta \xi^* & \eta \eta^* \end{array} \right)
\]

The relationship \(R_s\) receives the decomposition

\[
R_s = \frac{1}{2} \sum_{\mu=0}^3 X_\mu \sigma^\mu
\]

with \(\sigma_0 = 1\) and \(\sigma_i (i = 1, 2, 3)\) the Pauli matrices. We deduce that

\[
\begin{align*}
X_0 &= \xi \xi^* + \eta \eta^* = \langle u | u \rangle \\
X_1 &= \xi \eta^* + \eta \xi^* = \langle u | \sigma_1 | u \rangle \\
X_2 &= i (\xi \eta^* - \eta \xi^*) = \langle u | \sigma_2 | u \rangle \\
X_3 &= \xi \xi^* - \eta \eta^* = \langle u | \sigma_3 | u \rangle
\end{align*}
\]

\(X_\mu\) satisfies identically the equation

\[
X_0^2 = X_1^2 + X_2^2 + X_3^2
\]

and it may be viewed as a null vector belonging to Minkowski spacetime. Thus the logic-algebraic origin of Minkowski spacetime becomes manifest. It is Cartan who first indicated the profound links between spinors and geometry [7] and Penrose who introduced spinors as a powerful tool to address physics issues [8].

For a normalized spinor with

\[
|\xi|^2 + |\eta|^2 = 1.0
\]
the coordinates \( X_i \) \( (i = 1, 2, 3) \) belong to the Bloch sphere
\[
X_1^2 + X_2^2 + X_3^2 = 1.0
\] (10)
We observe a mapping from the \( S^3 \) sphere, formed by the pair of complex numbers satisfying eqn (9), to the \( S^2 \) sphere, eqn (10). It is the celebrated Hopf fibration of \( S^3 \) by great circles \( S^1 \) and base space \( S^2 \) [9,10].

A stereographic projection from the north pole to the plane \( X_3 = 0 \) will give the point \( x_1 + ix_2 \) in the complex plane with
\[
x_1 + ix_2 = \frac{X_1 + iX_2}{1 - X_3} = \frac{\xi^*}{\eta^*} \equiv \zeta^*
\] (11)
Another choice of the north pole will give another projection parametrized by \( \zeta' \), where
\[
\zeta' = \frac{a\zeta + b}{c\zeta + d}
\] (12)
with \( a, b, c, d \) complex numbers, subject to the condition \( ad - bc \neq 0 \). The Möbius transformation, eqn (12), is an automorphism which preserves the conformal structure on the Bloch sphere. Specific cases include dilation, rotation, translation, inversion.

Notice that the spinor \( |u\rangle \) connected to the vector \( \vec{X} \) belonging to a Bloch sphere, eqn. (10), creates an oriented plane normal to the vector \( \vec{X} \). The vectors \( \vec{Y} (Y_1, Y_2, Y_3), \vec{Z} (Z_1, Z_2, Z_3) \) can be constructed as mutually orthogonal, orthogonal to \( \vec{X} \) and with unit norm. They are defined by
\[
Y_1 + iZ_1 = \xi^2 - \eta^2
Y_2 + iZ_2 = i(\xi^2 + \eta^2)
Y_3 + iZ_3 = -2\xi\eta
\] (13)
All spinor bilinears lie in this transverse plane.

Inversely we may attach to a null vector its spinor. Consider the null momentum
\[
P_0^2 = P_1^2 + P_2^2 + P_3^2
\] (14)
with the two solutions for \( P_0 \)
\[
P_0 = + (P_1^2 + P_2^2 + P_3^2)^{1/2}
\] (15)
\[
P_0 = - (P_1^2 + P_2^2 + P_3^2)^{1/2}
\] (16)
The spinor equations corresponding to eqns (15), (16) are the Cartan-Weyl equations [11]
\[
(\vec{P} \cdot \vec{\sigma} - P_0) |u_+\rangle = 0
\] (17)
\[
(\vec{P} \cdot \vec{\sigma} + P_0) |u_-\rangle = 0
\] (18)
where now the spinors $|u\rangle$ are determined from the components of the four-momentum $P_\mu$. The derivation suggests that we may view a spinor as the square root of a null vector. The Weyl spinors combine to provide the four-component Dirac spinor

$$|\psi_D\rangle = \begin{pmatrix} u_- \\ u_+ \end{pmatrix}$$

satisfying the massless Dirac equation [12]

$$\gamma^\mu \partial_\mu |\psi_D\rangle = 0$$

with $\sigma^\mu = (1, \vec{\sigma}), \bar{\sigma}^\mu = (1, -\vec{\sigma})$. We thus regard Dirac equation as the outcome of a logical construction.

**Spinor reconstruction of the string worldsheet**

The defining operation of relation composition, eqn (1), acquires another perspective when we represent the relation $R_{ij}$ by a double line [1]. Each state or proposition is represented by a line, with a downward (upward) arrow attached to the initial (final) state or proposition. The composition rule appears then as string joining or string splitting. Repeated application of the composition rule generates patterns derived within simplicial string theory and reminding a triangulated Riemann surface [1].

In the continuum the Polyakov string action is

$$S = -\frac{1}{2} T_0 \int d\tau d\sigma \sqrt{-h} \hbar^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$$

with $T_0$ the string tension, $h^{\alpha\beta}(\tau, \sigma)$ an auxiliary world-sheet metric and $X_\mu(\tau, \sigma)$ the parametrization of the string world-sheet [13]. Adopting a flat world-sheet metric as a gauge choice, the solution of the string equations is a sum of right-movers and left-movers

$$X^\mu(u, v) = X^\mu_L(u) + X^\mu_R(v)$$

with $u = \tau + \sigma$, $v = \tau - \sigma$. The constraints become

$$\begin{align*}
(\partial_u X_L)^2 &= 0 \\
(\partial_v X_R)^2 &= 0.
\end{align*}$$
We consider string motion in three dimensions, the lowest dimension for which string dynamics is not trivial. With \( \mu = 0,1,2 \) and defining \( P_{L_i} = \partial_u X^i_L \) the constraint (24) becomes

\[
P^2_{L_0} = P^2_{L_1} + P^2_{L_2}.
\]

(26)

This is the celebrated Pythagoras equation. Pythagoras’ equation is derived from the geometrical requirement of composing two squares into a third square. We may attempt a linear representation of the quadratic Pythagorean form by writing

\[
R_P = P_{L_0} \sigma_0 + P_{L_1} \sigma_1 + P_{L_2} \sigma_3 = \begin{pmatrix} P_{L_0} + P_{L_2} & P_{L_1} \\ P_{L_0} - P_{L_2} & 0 \end{pmatrix}
\]

(27)

Equation (26) becomes then \( \det R_P = 0 \). \( R_P \) viewed as a relationship of a real spinor \(|\phi\rangle = \begin{pmatrix} p \\ q \end{pmatrix}\) takes the form

\[
R_P = |\phi\rangle \langle \phi| = \begin{pmatrix} p^2 & pq \\ pq & q^2 \end{pmatrix}
\]

(28)

We deduce then

\[
P_{L_0} = \frac{1}{2} (p^2 + q^2)
\]

(29)

\[
P_{L_1} = pq
\]

(30)

\[
P_{L_2} = \frac{1}{2} (p^2 - q^2)
\]

(31)

The above parametrization is a zero Hopf map with \( p \) and \( q \) the coordinates of \( \mathbb{R}^2 \). We may notice also that through a physical spinorial process we managed to enact the geometrical theorem of Pythagoras. In a similar fashion we represent the right-movers by two real functions \( r \) and \( s \). Thus the entire string world-sheet is described by four independent dynamical variables \((p,q,r,s)\). Within our approach the parametrization of the string world-sheet becomes

\[
X^0 (u,v) = \frac{1}{2} \left[ \int^u (p^2 + q^2) \, du + \int^v (r^2 + s^2) \, dv \right]
\]

(32)

\[
X^1 (u,v) = \int^u pq \, du + \int^v rs \, dv
\]

(33)

\[
X^2 (u,v) = \frac{1}{2} \left[ \int^u (p^2 - q^2) \, du + \int^v (r^2 - s^2) \, dv \right]
\]

(34)
We recognise that the above parametrization is the Enneper-Weierstrass parametrization of a minimal surface [14,15].

String theory appears as the most promising framework for unifying quantum mechanics and gravity, for creating quantum gravity. Our analysis points out the affinity between quantum mechanics and string theory, since both reside on the same relational principles. The known similarity of string theory to two dimensional gravity, further strengthens our confidence to string theory as a candidate theory for quantum gravity. As an extra dividend, our spinor reconstruction of the string dynamics solves effectively the constraints and allows us to use only the real, independent degrees of freedom.

**Building AdS**

Anti-de Sitter spacetime attracted enormous attention as the locus of a duality between gravity living in the interior of AdS and a conformal field theory living in the boundary of AdS [16,17,18]. It is quite natural to study the internal symmetries and architecture of AdS, in order to better discern the inherent dynamics. AdS space is the maximally symmetric solution of Einstein’s equations with an attractive cosmological constant. Anti-de Sitter spacetime is defined as a quadric surface embedded in a flat space of signature \((+ + \cdots -)\). Thus \(AdS_3\) is defined as the hypersurface

\[
    u^2 + v^2 - x_1^2 - x_2^2 = 1.0
\]

embedded in a four-dimensional flat space with the metric

\[
    ds^2 = du^2 + dv^2 - dx_1^2 - dx_2^2. \tag{36}
\]

We may proceed to a spinorial reconstruction of \(AdS_3\). Starting with a spinor

\[
    |\varphi\rangle = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},
\]

the “dual” spinor \(\tilde{\varphi} = \eta |\varphi\rangle\) where \(\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), the inner product becomes

\[
    \langle \tilde{\varphi} |\varphi\rangle = |\varphi_1|^2 - |\varphi_2|^2. \tag{37}
\]

Defining \(\varphi_1 = a + ib, \varphi_2 = c + id\), the normalization condition \(\langle \tilde{\varphi} |\varphi\rangle = 1.0\) provides

\[
    a^2 + b^2 - c^2 - d^2 = 1.0 \tag{38}
\]

The obvious identification \(a = u, b = v, c = x_1, d = x_2\) reproduces \(AdS_3\).

In analogy with the Hopf maps of a sphere to a sphere, we may look for the non-compact Hopf map of a hyperboloid. Let us define

\[
    \Sigma_1 = i\sigma_1, \quad \Sigma_2 = i\sigma_2, \quad \Sigma_3 = \sigma_3 \tag{39}
\]
We obtain then
\[ y_1 = \langle \tilde{\phi} | \Sigma_1 | \phi \rangle = i (\varphi_1^* \varphi_2 - \varphi_2^* \varphi_1) \]
\[ y_2 = \langle \tilde{\phi} | \Sigma_2 | \phi \rangle = \varphi_1^* \varphi_2 + \varphi_2^* \varphi_1 \]
\[ y_3 = \langle \tilde{\phi} | \Sigma_3 | \phi \rangle = \varphi_1^* \varphi_1 + \varphi_2^* \varphi_2 \]
\hspace{1cm} (40)

We observe that
\[ y_3^2 - y_1^2 - y_2^2 = \left[ |\varphi_1|^2 - |\varphi_2|^2 \right]^2 = 1.0 \]
\hspace{1cm} (41)

The above equation describes the hyperboloid $H^{2,0}$ and if we accept imaginary coordinates it is equivalent to $AdS_2$. Thus we observe the Hopf fibration $AdS_3 \rightarrow AdS_2$ and $AdS_3$ may be written in the form of a bundle over $AdS_2$ [19,20].

We may seek a transformation of the spinor $|\varphi\rangle$, $|\varphi'\rangle = R |\varphi\rangle$, leaving the form of eqn (37) invariant
\[ \langle \tilde{\varphi}' | \varphi' \rangle = \langle \tilde{\varphi} | \varphi \rangle . \]
\hspace{1cm} (42)

The transformation $R$ takes the form
\[ R = \begin{pmatrix} \cosh \omega e^{i\theta} & \sinh \omega e^{-i\chi} \\ \sinh \omega e^{i\chi} & \cosh \omega e^{-i\theta} \end{pmatrix} . \]
\hspace{1cm} (43)

Two successive $R$ transformations generate a third transformation. Let us represent the transformation by $r (\alpha, \beta)$, where $\alpha = \cosh \omega e^{i\theta}$ and $\beta = \sinh \omega e^{-i\chi}$. Then with $r_1 (\alpha, \beta)$ and $r_2 (\gamma, \delta)$ the successive transformations, we find
\[ r_1 r_2 = r_3 (\alpha \gamma + \beta \delta^*, \alpha \delta + \beta \gamma^*) . \]
\hspace{1cm} (44)

The inverse of a transformation $r (\alpha, \beta)$ is the transformation $r^{-1} (\alpha^*, -\beta)$.

Writing $\zeta = \frac{\alpha}{\varphi_2}$ the transformation becomes
\[ \zeta' = \frac{\alpha \zeta + \beta}{\beta^* \zeta + \alpha^*} . \]
\hspace{1cm} (45)

Equation (45) is the M"obius transformation which preserves the conformal structure of $AdS_3$.

**In lieu of conclusions**

We suggested relational logic as the appropriate framework to comprehend relational systems, notably quantum mechanics and string theory. A logical proposition receiving a “yes” or a “no” answer, may be represented by
a spinor. A composition of logic spinors, resembling to a spinor network, would be equivalent to a logical proof. In the present work we considered the simplest spinor compositions and how they lead to the representation of Minkowski spacetime, $AdS_3$ spacetime and the string world-sheet. Geometry and spacetime, rather than abstract mathematical constructions, emerge as the outcome of a logical process. Our demarche is reminiscent of the Wheeler’s pioneering idea of pregeometry [21,22]. Wheeler pointed out that the genuine explanations about the nature of something do not come about by explicating a concept in terms of similar ones, but by reducing it to a different, more basic kind of object. Citing Peirce, Wheeler considers that law must have come into being and that physics must be built from a foundation that has no physics [23]. Taking account of the most challenging “experimental” fact, that the universe is comprehensible, we should build the universe on this very demand for comprehensibility. Within this perspective, pregeometry (the stage preceding geometry) is based on the calculus of propositions [23]. We consider our approach as a concrete realization of Wheeler’s vision, bringing together logic, geometry, string theory, quantum mechanics. The common thread of these theories is their shared inner syntax, the relational logic of Peirce.

The holographic principle, developed by ’t Hooft [24] and Susskind [25] after an analysis of the black hole dynamics, dictates that the degrees of freedom contained in a volume $V$ are encoded on the surface bounding $V$ like a holographic image. A detailed study case is given in [26]. An oscillating string, encircling a black hole is contracting towards the horizon. String fluctuations are developed in both the radial and angular directions. To a static asymptotic observer the radial fluctuations are suppressed by the Lorentz-contraction and thus the string appears eventually to cover the whole black hole horizon [26]. The identification of the horizon with the string world-sheet gives credit to the notion that the stringy degrees of freedom are registered on the entire area of the event horizon. The two-dimensional horizon may be seen as a spherical holographic screen and an infinitesimal pixel on such a screen may be represented by a null vector $X_\mu$, equ. (8) with $X_0 = 1$. The same spinor giving rise to $\vec{X}$, creates also the plane transverse to $\vec{X}$, equ. (13). The spinor holographic construction can be extended to $d$ dimensions [27] and it appears that pure spinors are best suited for an analysis of the AdS/CFT holography [28].

We studied relations built up by spinors and leading to geometrical patterns. We considered relations generated by a single spinor. It is very interesting to consider relations built by two spinors, since such a construction will lead to a better comprehension of the quantum entanglement. We may
even go further and consider, following Wheeler’s advice, the statistical analysis of a great number of logical-spinor-propositions. If this approach makes sense, the entire universe will look like a theorem. Work along these lines is in progress.

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