Exact Two–Point Correlation Functions of Turbulence Without Pressure in Three–Dimensions

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Abstract

We investigate exact results of isotropic turbulence in three–dimensions when the pressure gradient is negligible. We derive exact two–point correlation functions of density in three-dimensions and show that the density–density correlator behaves as $|\mathbf{x}_1 - \mathbf{x}_2|^{-\alpha_3}$, where $\alpha_3 = 2 + \frac{\sqrt{33}}{6}$. It is shown that, in three–dimensions, the energy spectrum $E(k)$ in the inertial range scales with exponent $2 - \frac{\sqrt{33}}{12} \approx 1.5212$. We also discuss the time scale for which our exact results are valid for strong 3D–turbulence in the presence of the pressure. We confirm our predictions by using the recent results of numerical calculations and experiment.

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1- Introduction

Recently, tremendous activities have started on the non-perturbative understanding of turbulence [1-12]. A statistical theory of turbulence has been put forward by Kolmogorov [13], and further developed by others [15–17]. The approach is to model turbulence using stochastic partial differential equations. The simplest approach to turbulence is the Kolmogorov’s dimensional analysis, which leads to the celebrated $k^{-5/3}$ law for the energy spectrum. This is obtained by decreeing that the energy spectrum depends neither on the wavenumber where most of the energy resides, nor on the wavenumber of viscous dissipation. Kolmogorov conjectured that the scaling exponents are universal, independent of the statistics of large–scale fluctuation and the mechanism of the viscous damping, when the Reynolds number is sufficiently large. In fact the idea of universality is based on the notion of the "inertial subrange". By inertial subrange we mean that for very large values of the Reynolds number there is a wide separation between the scale energy input $L$ and the typical viscous dissipation scale $\eta$ at which viscous friction become important and the energy is turned into heat.

However, recently it has been found that there is relation between the probability distribution function (PDF) of velocity and those of the external force [18]. This observation has been confirmed by experiments [19], and numerical simulations [20].

In this direction, Polyakov [5] has recently offered a field theoretic method to derive the probability distribution or density of states in (1+1)-dimensions in the problem of randomly driven Burgers equation [21]. In one dimension, turbulence without pressure is described by Burgers equation (see also [14] concerning the relation between Burgers equation and KPZ–equation). In the limit of high Reynold’s number, using the operator product expansion (OPE), Polyakov reduces the problem of computation of correlation functions in the inertial subrange, to the solution of a certain partial differential equation [22,23], see also [28], about generalization of Polyakov’s approach, to find the probability density and scaling exponent of the moments of "longitudinal" velocity difference in the three–dimensional strong turbulence.

In this paper we consider three–dimensional isotropic turbulence without pressure, which is described by Navier–Stokes equations, when the pressure gradient is negligible. We derive the Polyakov’s master equations in higher dimensions and solve it, in the three dimensions.
We derive the exact exponent of two-point density correlation functions and the energy spectrum exponent. We also discuss the time scale for which our exact results are valid for strong 3D–turbulence in the presence of the pressure.

2- Turbulence Without Pressure in Three–Dimensions

We consider the following quantity:

\[ e_\lambda = \rho(x, t) \exp(\lambda \cdot u(x)) \]  

where \( e_\lambda \) is the density and the velocity satisfying the Navier–Stokes equations:

\[ u_t + (u \cdot \nabla)u = \nu \nabla^2 u - \frac{\nabla p}{\rho} + f(x, t) \]  

\[ \rho_t + \partial_\alpha (\rho u_\alpha) = 0 \]  

where \( p \) and \( \nu \) are the pressure and viscosity, respectively. The stirring force \( f(x, t) \) is a Gaussian random force with the following correlation:

\[ < f_\mu(x, t)f_\nu(x', t') > = k_{\mu\nu}(x - x') \delta(t - t') \]  

We start with the situation when \( \nabla p \simeq 0 \). This mode has a characteristic time of \( T_p \simeq \frac{L}{C_s} \), where \( L \) and \( C_s \) are the large scale of the system and the velocity of sound in the turbulent system, respectively. The existence of this time scale means that for the times \( t < T_p \), we can ignore the pressure term in the Navier–Stokes equations and therefore our results are valid only for this time scale.

To investigate the statistical description of eqs. (2) and (3), we consider the following two–point generating functional

\[ F_2(\lambda_1, \lambda_2, x_1, x_2) = < \rho(x_1)\rho(x_2) \exp(\lambda_1 \cdot u(x_1) + \lambda_2 \cdot u(x_2)) > \]  

We write the eq. (2) and (3) in two points \( x_1 \) and \( x_2 \) for \( u_1, u_2, \cdots, u_N \) and multiply the equations in \( \rho(x_2), \lambda_1 \cdot u(x_1), \lambda_2 \cdot u(x_1) \), \( \cdots, \lambda_1 \cdot \rho(x_1)\rho(x_2), \lambda_2 \cdot \rho(x_1)\rho(x_2), \cdots, \lambda_2 \cdot \rho(x_1)\rho(x_2) \), respectively.

We add the equations and multiply the result to \( \exp(\lambda_1 \cdot u(x_1) + \lambda_2 \cdot u(x_2)) \) and make averaging with respect to external random force, therefore we find:

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\[ \partial_t F_2 + \sum_{\{i=1,2\} \mu = x_1, \ldots, x_N} \frac{\partial}{\partial \lambda_i, \mu} F_2 = \sum_{\{i=1,2\} \mu = x_1, \ldots, x_N} C_{i,\mu} + D_2 \]  

(6)

where \( C_{i,\mu} \) and \( D_2 \) are,

\[ \sum_{\{i=1,2\} \mu = x_1, \ldots, x_N} C_{i,\mu} = \sum_{\{i,j=1,2\} \mu, \nu = x_1, \ldots, x_N} \lambda_{i,\mu} \lambda_{j,\nu} k_{\mu \nu}(x_i - x_j) F_2 \]  

(7)

and

\[ D_2 = \langle \nu \rho(x_1) \rho(x_2) [\lambda_1 \cdot \nabla^2 u(x_1) + \lambda_2 \cdot \nabla^2 u(x_2)] \exp (\lambda_1 \cdot u(x_1) + \lambda_2 \cdot u(x_2)) \rangle \]  

(8)

where we have used the Novikov’s theorem. \( D_2 \) is known as the anomaly term [5].

Now we consider the anomaly term in the limit of small \( \nu \) or high Reynold’s numbers. It is noted that this term can not be written in terms of \( F_2 \). To find its structure we consider the symmetries of the basic equations. The basic equations are Galilean invariant and also are invariant under the rescaling of density as \( \rho \to \alpha \rho \). In the other hand, final expression for \( D_2 \) must contain the ultraviolet finite operators \( \nabla u, \rho \) and \( e^{\lambda \cdot u} \). The only finite combination satisfying the rescaling \( \rho \to \alpha \rho \) is \( \rho e^{\lambda \cdot u} \) (see [24] for more detail). Therefore \( D_2 \) has the following form:

\[ D_2 = a F_2 \]  

(9)

where \( a \) is generally a function of \( \lambda_1 \) and \( \lambda_2 \).

Therefore in the steady state we find the following equation for \( F_2 \):

\[ \sum_{\{i=1,2\} \mu = x_1, \ldots, x_N} \frac{\partial}{\partial \lambda_i, \mu} F_2 - \sum_{\{i,j=1,2\} \mu, \nu = x_1, \ldots, x_N} \lambda_{i,\mu} \lambda_{j,\nu} k_{\mu \nu}(x_i - x_j) F_2 = a F_2. \]  

(10)

Also in this paper, we suppose that \( k_{\mu \nu} \) has the following form:

\[ k_{\mu \nu}(x_i - x_j) = k(0)[1 - \frac{|x_i - x_j|^2}{2L^2} \delta_{\mu, \nu} - \frac{(x_i - x_j)_\mu (x_i - x_j)_\nu}{L^2}] \]  

(11)

with \( k(0), L = 1 \).

Now let us consider the eq.(10) in three dimensions. We change the variables as: \( x_\pm = x_1 \pm x_2, \lambda_+ = \lambda_1 + \lambda_2 \) and \( \lambda_- = \frac{\lambda_1 - \lambda_2}{2} \) and and consider the spherical coordinates, so that
$x_-(r, \theta, \phi)$ and $\lambda_- (\rho', \theta', \phi')$. Direct calculation shows that:

\[
\sum_{i=1, \mu=x,y,z}^3 \frac{\partial}{\partial \lambda_{i,\mu}} \partial_{\mu} = \sum_{\mu=x,y,z} \frac{\partial}{\partial \lambda_{-\mu}} \partial_{\mu} = \cos \gamma \partial_r \partial_{\rho'} \\
+ \frac{\sin \theta \cos \theta' \cos (\varphi - \varphi') - \cos \theta \sin \theta'}{\rho'} \partial_r \partial_{\theta'} + \frac{\sin \theta \sin (\varphi - \varphi')}{\rho' \sin \theta'} \partial_r \partial_{\phi'} \\
+ \frac{\sin \theta' \cos \cos (\varphi - \varphi') - \cos \theta' \sin \theta'}{r} \partial_{\theta'} \partial_{\phi'} + \frac{\cos \theta' \sin (\varphi - \varphi')}{{r} \rho' \sin \theta'} \partial_{\theta'} \partial_{\phi'} \\
- \frac{\cos \theta' \sin (\varphi - \varphi')}{r \rho'} \partial_{\theta'} \partial_{\phi'} + \frac{\cos (\varphi - \varphi')}{r \rho' \sin \theta'} \partial_{\theta'} \partial_{\phi'} \\
+ \frac{\cos \theta \cos \theta' \cos (\varphi - \varphi') + \sin \theta \sin \theta'}{r \rho'} \partial_{\theta'} \partial_{\phi'} \\
- \frac{\sin \theta' \sin (\varphi - \varphi')}{r \sin \theta} \partial_{\rho'} \partial_{\phi'}
\]

(12)

and

\[
\sum_{\{i,j=1,2\} \mu,\nu=x,y,z} \lambda_{i,\mu} \lambda_{j,\nu} k_{\mu,\nu}(x_i - x_j) \\
= [r^2 \rho'^2 + 2(x_- \lambda_{-x} + y_- \lambda_{-y} + z_- \lambda_{-z})^2] = r^2 \rho'^2 (1 + 2 \cos^2 \gamma)
\]

(13)

where $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\varphi - \varphi')$.

Now using the eqs. (12, 13) for isotropic turbulence we obtain:

\[
[s \partial_r \partial_{\rho'} - \frac{s(1-s^2)}{r \rho'} \partial_s^2 + \frac{1+s^2}{r \rho'} \partial_s] \\
+ \frac{1-s^2}{r \rho'} \partial_r \partial_s + \frac{1-s^2}{r} \partial_{\rho'} \partial_s - r^2 \rho'^2 (1 + 2 s^2)] F_2 = a(\rho') F_2
\]

(14)

where $\cos \gamma = s$. The $\rho'$ dependence of the $a(\rho')$ anomaly must be chosen to conform the scaling and can be different depending on the scaling properties of the force correlation functions. In general, in the case of isotropic turbulence, stirring correlation function behaves as $k_{\mu,\nu} \sim 1 - r^\eta$, where in our case we have $\eta = 2$. Therefore, $a$ must depend on $\rho'$ as follows $a(\rho') = a_0 \rho'^\sigma$, where $\sigma = \frac{2-\eta}{1+\eta}$. It is evident that for our case $a$ is independent of $\rho'$. Let us consider the universal scaling invariant solution of eq.(33) in the following form:

\[
F_2(\rho', r) = g(r) F(\rho'^\delta) \\
g(r) = r^{-\alpha_3}
\]

(15)

where $\delta = \frac{\eta+1}{3}$, and $\alpha_3$ is the exponent of two-point correlation functions of density and also using the eq.(11) we find $\delta = 1$. 

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We substitute eq.(15) in eq.(14), and find the following relation for $F(\rho'r)$:

\[
\left[-\frac{\alpha_3 s}{r} \partial_{\rho'} + \frac{s}{r} \partial_{s'} \partial_{r'} - \frac{s(1-s^2)}{\rho'} \partial_{s'}^2 + \frac{1+s^2}{\rho'} \partial_{s} \right]
\]
\[
\left[-\alpha_3 \frac{1-s^2}{\rho'} \partial_{s'} + \frac{1-s^2}{\rho'} \partial_{s} \partial_{s'} + \frac{1-s^2}{r} \partial_{s} \partial_{s'} - r^2 \rho'^2 (1+2s^2) \right] F(\rho'r) = a_0 F(\rho'r) \quad (16)
\]

Since the two point density correlators exists, in the limit of $\rho' \to 0$, $F(\rho'r)$ tends to a constant and thus we have to look for the solution of $F$ among the family of positive, finite and normalizable solution of eq.(16). In the other hands, taking the Laplace transformation of the above equation, one can show that, to consider physical solution, so that $\langle u(x_1) - u(x_2) \rangle \geq 0$, we have to consider the case $a_0 = 0$. However for different types of correlation for the stirring force, e.g. $k_{\mu,\nu} \sim 1-r^\eta$, with $\eta \neq 2$, we have to include the $a_0$ [23].

Now, we propose the following Ansatz for $F(\rho'r)$, with $z = \rho'r$:

\[
F(z, s) = e^{z^\gamma f(s)} \quad (17)
\]

Using the eq.(16), we find $\gamma = 3/2$ and $f(s)$ satisfy the following equations:

\[
\begin{cases}
\frac{9}{4} s f(s)^2 + 3 f(s) f'(s)(1-s^2) + f'(s)^2 (1-3s) = (1+2s^2) \\
-s(1-s^2) f''(s) + [(4+\alpha_3) - (2+\alpha_3) s^2] f'(s) + \left(\frac{9}{4} + \frac{3}{2} \alpha_3\right) s f(s) = 0
\end{cases} \quad (18)
\]

also from eq.(18-a), one can derive the following initial conditions for $f(s)$,

\[
f(1) = \frac{2}{\sqrt{3}} \quad f'(1) = \frac{\sqrt{3} + \sqrt{11}}{4} \quad (19)
\]

It is interesting to note that the equation for $f(s)$ (i.e. eq.(18-a)) is the same as equation which is found in the instanton approach [8].

The function $f(s)$ has the following expansion around $s = 1$:

\[
f(s) = \frac{2}{\sqrt{3}} + \frac{\sqrt{3} + \sqrt{11}}{4} (s-1) + \frac{5\sqrt{33} - 61}{32(3\sqrt{3} - 2\sqrt{11})} (s-1)^2 + \cdots \quad (20)
\]

Now using the boundary conditions on $f(s)$ (i.e. eq(19)), and positivity of the probability distribution function we find:

\[
\alpha_3 = \frac{12 + \sqrt{33}}{6} \approx 2.9574 \quad (21)
\]
Noting the fact that $\rho'$ has dimension $-1$, we can find the following scaling relation for the density of the energy $\epsilon(x)$,

$$\epsilon(\alpha x) = \alpha^\Delta \epsilon(x)$$  \hspace{1cm} (22)

where $\Delta = 1 - \frac{\sqrt{33}}{12}$, and therefore we can determine the behavior of the energy spectrum exactly as:

$$E(k) \sim k^{-\beta}, \quad \beta = 2 - \frac{\sqrt{33}}{12} \simeq 1.52128$$  \hspace{1cm} (23)

This behavior of energy spectrum is known as the non–Kolmogorov power laws which has been observed experimentally [25,27] and also in the numerical simulations [25,26].

The numerical calculations have been done in [25,26], where they have used the Wiener–Hermit expansion. They have shown that the energy spectrum behaves as $E(k) \sim k^{-1.521}$ for systems without boundaries (i.e. free turbulence) and also for a finite system this spectrum is not stable. In [25,26] it has been shown that in the inertial subrange for a finite system energy spectrum starts with slope $-1.521$, and after a moderate time which is less than the characteristic time $T_c \simeq \frac{L}{u_{rms}}$ (where $L$ and $u_{rms}$ are the large scale of the system and rms value of initial velocity fluctuation, respectively) the equilibrium is attained and has transformed to $-5/3$. In other words the $-5/3$’s law is the stable algebraic spectrum for the Navier–Stokes equations after a time of order $T_c$.

The experimental results (reported by Wissler [25,27]) shows that the non–Kolmogorov spectrum has been observed also experimentally only for moderate times less than $T_c$. It is noted that in general for a turbulent flow $u_{rms} \leq C_s$ and therefore the pressure time scale has the property that $T_p \leq T_c$.

Finally we can derive the PDF for the velocity difference and show that it tails as $e^{-\alpha u^3}$ in the limit $|u| \to +\infty$, which is in agreement with other approaches [18] for three–dimensional turbulence.

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