DRP scheme optimization

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March 30, 2022

Abstract

A new DRP scheme is built, which enables us to minimize the error due to the finite difference approximation, by means of an equivalent matrix equation.

keywords

DRP schemes, Sylvester equation

1 Introduction: Scheme classes

We hereafter propose a method that enables us to build a DRP scheme while minimizing the error due to the finite difference approximation, by means of an equivalent matrix equation.

Consider the transport equation:

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \quad x \in [0, L], \quad t \in [0, T] \]  

(1)

with the initial condition \( u(x, t = 0) = u_0(x) \).

Proposition 1.1 A finite difference scheme for this equation can be written under the form:

\[ \alpha u_i^{n+1} + \beta u_i^n + \gamma u_i^{n-1} + \delta u_{i+1}^n + \varepsilon u_{i-1}^n + \zeta u_{i+1}^{n+1} + \eta u_{i-1}^{n+1} + \theta u_{i+1}^{n-1} + \vartheta u_{i+1}^{n+1} = 0 \]  

(2)

where:

\[ u_l^m = u(lh, m\tau) \]  

(3)

\( l \in \{i - 1, i, i + 1\}, \ m \in \{n - 1, n, n + 1\}, \ j = 0, ..., n_x, \ n = 0, ..., n_t, \ h, \ \tau \) denoting respectively the mesh size and time step (\( L = n_xh, \ T = n_t\tau \)).

The Courant-Friedrichs-Lewy number (cfl) is defined as \( \sigma = c\tau/h \).
Table 1: Numerical scheme coefficient.

| Name          | $\alpha_x + \alpha_t$ | $\beta_x + \beta_t$ | $\gamma_x + \gamma_t$ | $\delta_x + \delta_t$ | $\varepsilon_x + \varepsilon_t$ | $\zeta$ | $\eta$ | $\theta$ | $\vartheta$ |
|---------------|------------------------|----------------------|------------------------|------------------------|-----------------------------|-------|-------|--------|--------|
| Leapfrog      | $\frac{1}{\tau}$      | 0                    | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | 0     | 0     | 0      | 0      |
| Lax           | $\frac{1}{\tau}$      | 0                    | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | 0     | 0     | 0      | 0      |
| Lax-Wendroff  | $\frac{1}{\tau} + \frac{1}{\tau}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | 0     | 0     | 0      | 0      |
| Crank-Nicolson| $\frac{1}{\tau} + \frac{1}{\tau}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | $\frac{1}{\tau} - \frac{1}{h}$ | 0     | $\frac{1}{\tau}$ | $\frac{1}{\tau}$ | 0      |

A numerical scheme is specified by selecting appropriate values of the coefficients $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, $\zeta$, $\eta$, $\theta$ and $\vartheta$ in equation (2), which, for sake of usefulness, will be written as:

$$\alpha = \alpha_x + \alpha_t \quad , \quad \beta = \beta_x + \beta_t \quad , \quad \gamma = \gamma_x + \gamma_t \quad , \quad \delta = \delta_x + \delta_t \quad , \quad \varepsilon = \varepsilon_x + \varepsilon_t \quad ,$$ (4)

where the $"_x"$ denotes a dependence towards the mesh size $h$, while the $"_t"$ denotes a dependence towards the time step $\tau$.

Values corresponding to numerical schemes retained for the present works are given in Table 1.

The number of time steps will be denoted $n_t$, the number of space steps, $n_x$. In general, $n_x \gg n_t$.

In the following: the only dependence of the coefficients towards the time step $\tau$ existing only in the Crank-Nicolson scheme, we will restrain our study to the specific case:

$$\alpha_t = \gamma_t = \zeta = \eta = \theta = \vartheta = 0$$ (5)

The paper is organized as follows. The building of the DRP scheme is exposed in section 2. The equivalent matrix equation, which enables us to minimize the error due to the finite difference approximation, is presented in section 3. A numerical example is given in section 4.

2 The DRP scheme

The first derivative $\frac{\partial u}{\partial x}$ is approximated at the $l^{th}$ node of the spatial mesh by:

$$\left( \frac{\partial u}{\partial x} \right)_l \simeq \beta_x u_{l+i}^n + \delta_x u_{l+i+1}^n + \varepsilon_x u_{l+i-1}^n$$ (6)

Following the method exposed by C. Tam and J. Webb in (1), the coefficients $\beta_x$, $\delta_x$, and $\varepsilon_x$ are determined requiring the Fourier Transform of the finite difference scheme (6) to be a close approximation of the partial derivative ($\frac{\partial u}{\partial x}$)$_l$.
The continuous derivative of a function $u(x)$ with respect to $x$ is approximated by a finite difference scheme as:

$$\left(\frac{\partial u}{\partial x}\right)_i \simeq \beta_x u(x + i h) + \delta_x u(x + (i + 1) h) + \varepsilon_x u(x + (i - 1) h)$$  \hspace{1cm} (7)

where $x$ is a continuous variable, and can be recovered setting $x = l h$. Applying the Fourier transform, referred to by $\hat{\cdot}$, to both sides of (7), yields:

$$j \omega \hat{u} \simeq \left\{ \beta_x e^0 + \delta_x e^{j \omega h} + \varepsilon_x e^{-j \omega h} \right\} \hat{u}$$  \hspace{1cm} (8)

$j$ denoting the complex square root of $-1$.

Comparing the two sides of (8) enables us to identify the wavenumber $\lambda$ of the finite difference scheme (6) and the quantity $\frac{1}{j} \left\{ \beta_x e^0 + \delta_x e^{j \omega h} + \varepsilon_x e^{-j \omega h} \right\}$, i.e.: The wavenumber of the finite difference scheme (6) is thus:

$$\lambda = -j \left\{ \beta_x e^0 + \delta_x e^{j \omega h} + \varepsilon_x e^{-j \omega h} \right\}$$  \hspace{1cm} (9)

To ensure that the Fourier transform of the finite difference scheme is a good approximation of the partial derivative $\left(\frac{\partial u}{\partial x}\right)_i$ over the range of waves with wavelength longer than $4 h$, the a priori unknowns coefficients $\beta_x$, $\delta_x$, and $\varepsilon_x$ must be chosen so as to minimize the integrated error:

$$E = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} |\lambda h - \lambda h|^2 d(\lambda h)$$

$$= \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} |\kappa + j h \left\{ \beta_x e^0 + \delta_x e^{j \kappa} + \varepsilon_x e^{-j \kappa} \right\} |^2 d(\kappa)$$  \hspace{1cm} (10)

The conditions that $E$ is a minimum are:

$$\frac{\partial E}{\partial \beta_x} = \frac{\partial E}{\partial \delta_x} = \frac{\partial E}{\partial \varepsilon_x} = 0$$  \hspace{1cm} (11)

and provide the following system of linear algebraic equations:

$$\begin{cases} 2 \pi h \beta_x + 4 (h \delta_x + h \varepsilon_x - 1) = 0 \\ 4 h \beta_x + \pi (2 \delta_x - 1) = 0 \\ 4 h \beta_x + 2 \pi h \varepsilon_x = 0 \end{cases}$$  \hspace{1cm} (12)

which enables us to determine the required values of $\beta_x$, $\delta_x$, and $\varepsilon_x$:

$$\begin{cases} \beta_x = \beta_{opt} x = \frac{\pi}{h(\pi^2 - 8)} \\ \delta_x = \delta_{opt} x = \frac{1}{2} - \frac{h(\pi^2 - 8)}{2} \frac{\pi}{h(\pi^2 - 8)} \\ \varepsilon_x = \varepsilon_{opt} x = -\frac{2}{\pi} \frac{\pi}{h(\pi^2 - 8)} \end{cases}$$  \hspace{1cm} (13)

3 The Sylvester equation

3.1 Matricial form of the finite differences problem

Theorem 3.1 The problem (2) can be written under the following matricial form:

$$M_1 U + U M_2 + L(U) = M_0$$  \hspace{1cm} (14)
where $M_1$ and $M_2$ are square matrices respectively $n_x - 1$ by $n_x - 1$, $n_t$ by $n_t$, given by:

$$M_1 = \begin{pmatrix}
\beta & \delta & 0 & \ldots & 0 \\
\varepsilon & \beta & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \beta \\
0 & \ldots & 0 & \varepsilon & \beta
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
0 & \gamma & 0 & \ldots & 0 \\
\alpha & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \gamma \\
0 & \ldots & 0 & \alpha & 0
\end{pmatrix} \quad (15)$$

the matrix $M_0$ being given by:

$$M_0 = \begin{pmatrix}
-\gamma u_0^2 - \varepsilon u_0^1 - \eta u_0^0 - \zeta u_0^2 - \delta u_2^0 & -\varepsilon u_0^3 - \eta u_0^1 - \zeta u_0^3 - \delta u_2^1 & \ldots & -\varepsilon u_0^n_t - \eta u_0^n_{t-1} \\
-\gamma u_0^{2-} - \eta u_0^{0-} - \zeta u_0^{2-} - \delta u_2^{0-} & \ldots & \ldots & \ldots \\
\vdots & \ddots & \ddots & \ddots \\
-\gamma u_0^{n_{x-1}-} - \eta u_0^{n_{x-1}-} - \zeta u_0^{n_{x-1}-} - \delta u_2^{n_{x-1}-} & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \delta u_3^{n_{x-1}} - \delta u_3^n_{x-1}
\end{pmatrix} \quad (16)$$

and where $\mathcal{L}$ is a linear matricial operator which can be written as:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \quad (17)$$

where $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ are given by:

$$\mathcal{L}_1(U) = \zeta \begin{pmatrix}
u_2^2 & u_3^3 & \ldots & u_{n_t}^{n_t} & 0 \\
u_3^2 & u_3^3 & \ldots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
u_{n_x-1}^2 & u_{n_x-1}^3 & \ldots & u_{n_x-1}^{n_x-1} & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix}, \quad \mathcal{L}_2(U) = \eta \begin{pmatrix}0 & 0 & \ldots & 0 & 0 \\
0 & u_1^1 & u_2^1 & \ldots & u_{n_t-1}^{n_t-1} \\
0 & u_1^1 & u_2^1 & \ldots & u_{n_t-1}^{n_t-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & u_{n_x-2}^1 & u_{n_x-2}^2 & \ldots & u_{n_x-2}^{n_x-1}
\end{pmatrix} \quad (18)$$

$$\mathcal{L}_3(U) = \theta \begin{pmatrix}
0 & \ldots & \ldots & \ldots & 0 \\
u_2^1 & u_3^1 & \ldots & u_{n_t}^{n_t} & 0 \\
u_2^1 & u_3^1 & \ldots & u_{n_t}^{n_t} & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
u_{n_x-2}^1 & u_{n_x-2}^3 & \ldots & u_{n_x-2}^{n_x-2} & 0
\end{pmatrix}, \quad \mathcal{L}_4(U) = \vartheta \begin{pmatrix}0 & u_1^2 & u_2^2 & \ldots & u_{n_t-1}^{n_t-1} \\
0 & u_1^2 & u_2^2 & \ldots & u_{n_t-1}^{n_t-1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & u_{n_x-1} & \ldots & u_{n_x-1} & 0 \\
0 & 0 & \ldots & 0 & 0
\end{pmatrix} \quad (19)$$

**Proposition 3.2** The second member matrix $M_0$ bears the initial conditions, given for the specific value $n = 0$, which correspond to the initialization process when computing loops, and the boundary conditions, given for the specific values $i = 0$, $i = n_x$.
Denote by $u_{\text{exact}}$ the exact solution of (1). The corresponding matrix $U_{\text{exact}}$ will be:

$$U_{\text{exact}} = [U_{\text{exact}}^n]_{1 \leq i \leq n_t, 1 \leq n \leq n_t}$$

(20)

where:

$$U_{\text{exact}}^n = U_{\text{exact}}(x_i, t_n)$$

(21)

with $x_i = i h$, $t_n = n \tau$.

**Definition 3.3** We will call error matrix the matrix defined by:

$$E = U - U_{\text{exact}}$$

(22)

Consider the matrix $F$ defined by:

$$F = M_1 U_{\text{exact}} + U_{\text{exact}} M_2 + L(U_{\text{exact}}) - M_0$$

(23)

**Proposition 3.4** The error matrix $E$ satisfies:

$$M_1 E + E M_2 + L(E) = F$$

(24)

### 3.2 The matrix equation

**Theorem 3.5** Minimizing the error due to the approximation induced by the numerical scheme is equivalent to minimizing the norm of the matrices $E$ satisfying (24).

*Note:* Since the linear matricial operator $L$ appears only in the Crank-Nicolson scheme, we will restrain our study to the case $L = 0$. The generalization to the case $L \neq 0$ can be easily deduced.

**Proposition 3.6** The problem is then the determination of the minimum norm solution of:

$$M_1 E + E M_2 = F$$

(25)

which is a specific form of the Sylvester equation:

$$AX + XB = C$$

(26)

where $A$ and $B$ are respectively $m$ by $m$ and $n$ by $n$ matrices, $C$ and $X$, $m$ by $n$ matrices.
3.3 Minimization of the error

3.3.1 Theory

Calculation yields:
\[
\begin{align*}
M_1^T M_1 &= \text{diag}\left( \begin{array}{ccc}
\beta^2 + \delta^2 & \beta \delta & \beta^2 + \delta \\
\beta \delta & \delta^2 + \epsilon^2 & \delta \epsilon \\
\beta^2 + \delta & \delta \epsilon & \delta^2 + \epsilon^2
\end{array} \right), \\
M_2^T M_2 &= \text{diag}\left( \begin{array}{ccc}
\gamma^2 & 0 \\
0 & \alpha^2
\end{array} \right)
\end{align*}
\] (27)

The singular values of \( M_1 \) are the singular values of the block matrix \( \begin{pmatrix} \beta^2 + \delta^2 & \beta \delta & \beta^2 + \delta \\ \beta \delta & \delta^2 + \epsilon^2 & \delta \epsilon \\ \beta^2 + \delta & \delta \epsilon & \delta^2 + \epsilon^2 \end{pmatrix} \), i.e.
\[
\frac{1}{2} (2\beta^2 + \delta^2 + \epsilon^2 - (\delta + \epsilon) \sqrt{4\beta^2 + \delta^2 + \epsilon^2 - 2\delta \epsilon})
\] (28)
of order \( \frac{n-1}{2} \), and
\[
\frac{1}{2} (2\beta^2 + \delta^2 + \epsilon^2 + (\delta + \epsilon) \sqrt{4\beta^2 + \delta^2 + \epsilon^2 - 2\delta \epsilon})
\] (29)
of order \( \frac{n-1}{2} \).

The singular values of \( M_2 \) are \( \alpha^2 \), of order \( \frac{n}{2} \), and \( \gamma^2 \), of order \( \frac{n}{2} \).

Consider the singular value decomposition of the matrices \( M_1 \) and \( M_2 \):
\[
U_1^T M_1 V_1 = \begin{pmatrix} \tilde{M}_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad U_2^T M_2 V_2 = \begin{pmatrix} \tilde{M}_2 & 0 \\ 0 & 0 \end{pmatrix}
\] (30)

where \( U_1, V_1, U_2, V_2 \), are orthogonal matrices. \( \tilde{M}_1, \tilde{M}_2 \) are diagonal matrices, the diagonal terms of which are respectively the nonzero eigenvalues of the symmetric matrices \( M_1^T M_1, M_2^T M_2 \).

Multiplying respectively \( 2^5 \) on the left side by \( U_1^T \), on the right side by \( V_2 \), yields:
\[
U_1^T M_1 E V_2 + U_1^T E M_2 V_2 = U_1^T F V_2
\] (31)

which can also be taken as:
\[
U_1^T M_1 V_1 E V_2 + U_1^T E U_2^T U_2^T M_2 V_2 = U_1^T F V_2
\] (32)

Set:
\[
T V_1 E V_2 = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}, \quad T U_1 E^T U_2 = \begin{pmatrix} \tilde{E}_{11} & \tilde{E}_{12} \\ \tilde{E}_{21} & \tilde{E}_{22} \end{pmatrix}
\] (33)

\[
T U_1 F V_2 = \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix}
\] (34)

We have thus:
\[
\begin{pmatrix} \tilde{M}_1 \tilde{E}_{11} \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{E}_{11} \tilde{M}_2 \\ \tilde{E}_{21} \tilde{M}_2 \end{pmatrix} = \begin{pmatrix} \tilde{F}_{11} & \tilde{F}_{12} \\ \tilde{F}_{21} & \tilde{F}_{22} \end{pmatrix}
\] (35)
It yields:
\[
\begin{align*}
\tilde{M}_1 \tilde{E}_{11} + \tilde{E}_{11} \tilde{M}_2 &= \tilde{F}_{11} \\
\tilde{M}_1 \tilde{E}_{12} &= \tilde{F}_{12} \\
\tilde{E}_{21} \tilde{M}_2 &= \tilde{F}_{21}
\end{align*}
\] (36)

One easily deduces:
\[
\begin{align*}
\tilde{E}_{12} &= \tilde{M}_1^{-1} \tilde{F}_{12} \\
\tilde{E}_{21} &= \tilde{F}_{21} \tilde{M}_2^{-1}
\end{align*}
\] (37)

The problem is then the determination of the \(\tilde{E}_{11}\) and \(\tilde{E}_{11}\) satisfying:
\[
\tilde{M}_1 \tilde{E}_{11} + \tilde{E}_{11} \tilde{M}_2 = \tilde{F}_{11}
\] (38)

Denote respectively by \(\tilde{e}_{ij}\), \(\tilde{e}_{ij}\) the components of the matrices \(\tilde{E}, \tilde{E}\).

The problem (38) uncouples into the independent problems:

\[
\text{minimize} \quad \sum_{i,j} \tilde{e}_{ij}^2 + \tilde{e}_{ij}^2
\] (39)

under the constraint
\[
\tilde{M}_{1ii} \tilde{e}_{ij} + \tilde{M}_{2ii} \tilde{e}_{ij} = \tilde{F}_{11ij}
\] (40)

This latter problem has the solution:
\[
\begin{align*}
\tilde{e}_{ij} &= \frac{\tilde{M}_{1ii} \tilde{F}_{11ij}}{\tilde{M}_{1ii} + \tilde{M}_{2ii}} \\
\tilde{e}_{ij} &= \frac{\tilde{M}_{2ii} \tilde{F}_{11ij}}{\tilde{M}_{1ii} + \tilde{M}_{2ii}}
\end{align*}
\] (41)

The minimum norm solution of (25) will then be obtained when the norm of the matrix \(\tilde{F}_{11}\) is minimum.

In the following, the euclidean norm will be considered.

Due to (34):
\[
\| \tilde{F}_{11} \| \leq \| \tilde{F} \| \leq \| U_1 \| \| F \| \| V_2 \| \leq \| U_1 \| \| V_2 \| \| M_1 \| U_{\text{exact}} + \| M_2 \| \| U_{\text{exact}} M_2 - M_0 \| \]
\] (42)

\(U_1\) and \(V_2\) being orthogonal matrices, respectively \(n_x - 1\) by \(n_x - 1\), \(n_t\) by \(n_t\), we have:
\[
\| U_1 \|^2 = n_x - 1 \quad , \quad \| V_2 \|^2 = n_t
\] (43)

Also:
\[
\| M_1 \|^2 = \frac{n_x - 1}{2} (2 \beta^2 + \delta^2 + \epsilon^2) \quad , \quad \| M_2 \|^2 = \frac{n_t}{2} (\alpha^2 + \gamma^2)
\] (44)

The norm of \(M_0\) is obtained thanks to relation (16).

This results in:
\[
\| \tilde{F}_{11} \| \leq \sqrt{n_t (n_x - 1)} \left\{ \| U_{\text{exact}} \| \left( \sqrt{\frac{n_x - 1}{2}} \sqrt{2 \beta^2 + \delta^2 + \epsilon^2} + \sqrt{\frac{n_t}{2}} \sqrt{\alpha^2 + \gamma^2} \right) + \| M_0 \| \right\}
\] (45)
∥\tilde{F}_{11}∥ can be minimized through the minimization of the second factor of the right-side member of (45), which is function of the scheme parameters.

∥U_{\text{exact}}∥ is a constant. The quantities \sqrt{2n-1}\beta^2 + \delta^2 + \varepsilon^2, \sqrt{\alpha^2 + \gamma^2} and ∥M_0∥ being strictly positive, minimizing the second factor of the right-side member of (45) can be obtained through the minimization of the following functions:

\begin{align*}
   f_1(\beta, \delta, \varepsilon) &= \sqrt{2\beta^2 + \delta^2 + \varepsilon^2} \\
   f_2(\alpha, \gamma) &= \sqrt{\alpha^2 + \gamma^2} \\
   f_3(\alpha, \beta, \gamma, \delta, \varepsilon) &= ∥M_0∥
\end{align*}

(46)

4 Numerical example: a new DRP scheme

Consider the scheme (2) where the values of \beta_x, \delta_x, and \varepsilon_x are given by (13).

Let, in a first time, the values of the coefficients \alpha, \beta_t, \gamma, \delta_t, and \varepsilon_t remain unknown, and advect a sinusoidal signal

\begin{align*}
   u = \cos \left[k(x - ct)\right]
\end{align*}

(47)

through this scheme, with Dirichlet boundary conditions. (c is taken equal to 1, and \( k = \pi \)).

Calculation yields then:

\begin{align*}
   f_1(\beta_x^{\text{opt}} + \delta_t, \varepsilon_t) &= \sqrt{2\left(\beta_t + \frac{\pi}{2n(\pi^2 - 8)}\right)^2 + \left(\delta_t + \frac{\pi}{2n}\right)^2 + \left(\varepsilon_t - \frac{\pi}{2n}\right)^2} \\
   f_2(\alpha, \gamma) &= \sqrt{\alpha^2 + \gamma^2} \\
   f_3(\alpha, \beta_t, \gamma, \delta_t, \varepsilon_t) &= \sqrt{3\gamma^2 + 3\left(\delta_t + \frac{\pi}{2n}\right)^2 + \left(\gamma - \varepsilon_t - \frac{\pi}{2n}\right)^2 + 3\left(\varepsilon_t + \frac{\pi}{2n}\right)^2}
\end{align*}

(48)

Minimum values for \( f_1 \) and \( f_3 \) can thus be obtained choosing negative values for \( \beta_t \), while choosing positive ones for \( \delta_t \) and \( \varepsilon_t \), the absolute values of which are respectively close to those of \( \beta_x, \delta_x \) and \( \varepsilon_x \). \( f_2 \) is minimized choosing \( \gamma = 0 \).

In the following, we have choosen to set:

\begin{align*}
   \begin{cases}
   \beta_t &= -0.9 \beta_x^{\text{opt}} \\
   \delta_t &= -0.9 \delta_x^{\text{opt}} \\
   \varepsilon_t &= -0.9 \varepsilon_x^{\text{opt}}
   \end{cases}
\end{align*}

(49)

and \( \alpha = 10 \).

The value of the \( L_2 \) norm of the error, for:

i. case 1: our new scheme, with \( \text{cfl} = 0.9 \);

ii. case 2: the Lax scheme, with \( \text{cfl} = 0.9 \);

is displayed in Figure 1. The error curve corresponding to the first case is the minimal one.
5 Conclusion

The above results open new ways for the building of DRP schemes. It seems that the research on this problem has not been performed before as far as our knowledge goes. In the near future, we are going to extend the techniques described herein to nonlinear schemes, in conjunction with other innovative methods as the Lie group theory.

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