A REMARK ON AN INEQUALITY FOR THE PRIME COUNTING FUNCTION

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ABSTRACT. We note that the inequalities $0.92 \frac{x}{\log(x)} < \pi(x) < 1.11 \frac{x}{\log(x)}$ do not hold for all $x \geq 30$, contrary to some references. These estimates on $\pi(x)$ came up recently in papers on algebraic number theory.

1. Chebyshev’s estimates for $\pi(x)$

Let $\pi(x)$ denote the number of primes not greater than $x$, i.e.,

$$\pi(x) = \sum_{p \leq x} 1.$$ 

One of the first works on the function $\pi(x)$ is due to Chebyshev. He proved (see [2]) in 1852 the following explicit inequalities for $\pi(x)$, holding for all $x \geq x_0$ with some $x_0$ sufficiently large:

$$c_1 \frac{x}{\log(x)} < \pi(x) < c_2 \frac{x}{\log(x)},$$

where

$$c_1 = \log\left(\frac{2^{1/2}3^{1/3}5^{1/5}}{30^{1/30}}\right) \approx 0.921292022934,$$

$$c_2 = \frac{6}{5} c_1 \approx 1.10555042752.$$

This can be found in many books on analytic number theory (see for example [1], [3], [11] and [14]). But it seems that this result is sometimes cited incorrectly: it is claimed that the estimates are valid for all $x \geq 30$. For example, in [6], page 21 we read that

$$c_1 \frac{x}{\log(x)} < \pi(x) < c_2 \frac{x}{\log(x)}, \quad \forall x \geq 30.$$

But a quick numerical computation shows that this is wrong. To give an example, take $x = 100$. Then we have $\pi(x) = 25$ and

$$c_2 \frac{x}{\log(x)} \approx 24.00672250690558538515780234 < 25.$$ 

Actually, the inequality is far from true for small $x$. We have the following result:
Theorem 1.1. Let $c_2 \approx 1.10555042752$ be Chebyshev’s constant. Then the inequality

$$\pi(x) < c_2 \frac{x}{\log(x)}$$

is true for all $x \geq 96098$. For $x = 96097$ it is false.

Proof. In [10] it is shown that

$$\pi(x) < \frac{x}{\log(x)} - 1.11, \quad x \geq 4.$$  

The RHS is less or equal to $c_2 x/\log(x)$ if and only if

$$x \geq \exp \left( \frac{1.11 \cdot c_2}{c_2 - 1} \right) \approx 112005.18.$$  

This shows the claim for $x \geq 112006$. Since $x/\log(x)$ is a monotonously increasing function it is enough to check the claimed estimate for integers $x$ in the intervall $[96098, 112006]$ by computer. For $x = 96097$ we have $\pi(96097) = 9260$ and $c_2 x/\log(x) \approx 9259.92$. \hfill \Box

The incorrect inequality was also used in a former version of Khare’s proof of Serre’s modularity conjecture for the level one case, see [8], [9]. Let $F$ be a finite field of characteristic $p$. The conjecture stated that an odd, irreducible Galois representation $\rho: \text{Gal}(\overline{Q}/Q) \rightarrow GL_2(F)$ which is unramified outside $p$ is associated to a modular form on $SL_2(\mathbb{Z})$. Khare’s proof is an elaborate induction on $p$. Starting with a $p$ for which the conjecture is known one wants to prove the conjecture for a larger prime $P$. Kahre’s arguments do only work if $P$ and $p$ are not Fermat primes, and if

$$\frac{P}{p} \leq a$$

for certain values $a > 1$, close to 1. At this point Khare used the incorrect estimate on $\pi(x)$, as explained above. Fortunately the proof easily could be repaired by using better estimates on $\pi(x)$ provided by Rosser and Schoenfeld [12], and Dusart [4].

Indeed, P. Dusart proved inequalities for $\pi(x)$ which are much better than Chebyshev’s estimates. He verifies this for smaller $x$ numerically. Nevertheless he claims in his thesis [5], that Chebyshev gave the following inequality

$$0.92 \frac{x}{\log(x)} < \pi(x) < 1.11 \frac{x}{\log(x)}, \quad x \geq 30,$$

which is equally wrong.

The question is: where lies the origin for this error? Chebyshev himself proved inequalities in [2] with his constants $c_1$ and $c_2 = \frac{6}{\pi} c_1$ indeed for all $x \geq 30$, but for inequalities involving $\psi(x) = \sum_{n \leq x} \Lambda(n)$ instead of $\pi(x)$. His estimates concerning $\psi(x)$ seem to be correct for all
$x \geq 30$. For example, he shows by elementary means that, for all $x \geq 30$,

\[
\psi(x) < \frac{6}{5} c_1 x + \frac{5}{4 \log(6)} \log^2(x) + \frac{5}{4} \log(x) + 1,
\]

\[
\psi(x) > c_1 x - \frac{5}{2} \log(x) - 1.
\]

To derive from this inequalities on $\pi(x)$ for $x \geq 30$, we have to estimate

\[
\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log(x)}{\log(p)} \right\rfloor \log(p).
\]

Using the estimates $[y] \leq y < [y] + 1 \leq 2[y]$ for $y \geq 1$ we obtain

\[
\psi(x) \leq \pi(x) \log(x) \leq 2\psi(x), \quad x \geq 2.
\]

On the RHS we cannot do easily much better than $2\psi(x)$. Hence we obtain

\[
c_1 \frac{x}{\log(x)} < \pi(x) < 2c_2 \frac{x}{\log(x)}, \quad x \geq 30.
\]

On the other hand we know that

\[
\pi(x) = \frac{\psi(x)}{\log(x)} + O \left( \frac{x}{\log^2(x)} \right), \quad x \geq 2,
\]

so that we obtain, as $x$ tends to infinity,

\[
(c_1 + o(1)) \frac{x}{\log(x)} \leq \pi(x) \leq (c_2 + o(1)) \frac{x}{\log(x)}.
\]

Chebyshev used these estimates to prove Bertrand’s postulate: each interval $(n, 2n]$ for $n \geq 1$ contains at least one prime. Moreover his results were a first step towards the proof of the prime number theorem.

### 2. Other estimates for $\pi(x)$

There are many interesting inequalities on the function $\pi(x)$. Let us first consider inequalities of the form

\[
A \frac{x}{\log(x)} < \pi(x) < B \frac{x}{\log(x)}
\]
for all \( x \geq x_0 \), where \( x_0 \) depends on the constant \( A \leq 1 \) and respectively on \( B > 1 \). On the LHS we can choose \( A \) equal to 1, if \( x \geq 17 \). In fact, we have \[ \frac{x}{\log(x)} < \pi(x), \quad \forall x \geq 17. \]

Note that for \( x = 16.999 \) we have \( x/\log(x) \approx 6.0000257 \), but \( \pi(x) = 6 \). Consider the RHS of the above inequalities: if we want to hold such inequalities on \( \pi(x) \) for all \( x \geq x_0 \) with a smaller \( x_0 \), we need to enlarge the constant \( B \). Conversely, if we need this inequality for smaller \( B \), we have to enlarge \( x_0 \). The prime number theorem ensures that we can choose \( B \) as close to 1 as we want, provided \( x_0 \) is sufficiently large. The following result of Dusart \[4\] enables us to derive adjusted versions for the above inequalities:

**Theorem 2.1** (Dusart). For real \( x \) we have the following sharp bounds:

\[
\begin{align*}
\pi(x) &\geq \frac{x}{\log(x)} \left(1 + \frac{1}{\log(x)} + \frac{1.8}{\log^2(x)}\right), \quad x \geq 32299, \\
\pi(x) &\leq \frac{x}{\log(x)} \left(1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)}\right), \quad x \geq 355991.
\end{align*}
\]

One can derive, for example, the following inequalities.

\[
\begin{align*}
\pi(x) &< 1.095 \cdot \frac{x}{\log(x)}, \quad x \geq 284860, \\
\pi(x) &< 1.25506 \cdot \frac{x}{\log(x)}, \quad x \geq 17.
\end{align*}
\]

Among other inequalities on \( \pi(x) \) we mention the following ones:

\[
\frac{x}{\log(x) - m} < \pi(x) < \frac{x}{\log(x) - M}
\]

for all \( x \geq x_0 \) with real constants \( m \) and \( M \). They have been studied by various authors. A good reference is the article \[10\]. There it is shown, for example, that

\[
\begin{align*}
\pi(x) &> \frac{x}{\log(x) - \frac{26}{29}}, \quad x \geq 3299, \\
\pi(x) &< \frac{x}{\log(x) - 1.11}, \quad x \geq 4.
\end{align*}
\]

The second inequality can also be used to obtain results on our estimate \( \pi(x) < B \frac{x}{\log(x)} \), in particular for smaller \( x \), where the second inequality of Theorem 2.1 is not valid. However we have

\[
\begin{align*}
\frac{x}{\log(x)} \left(1 + \frac{1}{\log(x)} + \frac{2.51}{\log^2(x)}\right) &< \frac{x}{\log(x) - 1.11}, \quad x \geq 28516.
\end{align*}
\]
For \( x > 10^6 \) and \( a = 1.08366 \) we can use \([10]\)

\[
\pi(x) < \frac{x}{\log(x) - a}.
\]

Here the upper bound of Dusart is better only as long as \( x \geq 2846396 \).

Finally we mention the book \([13]\), providing many references on inequalities on \( \pi(x) \), and the recent article \([7]\), where lower and upper bounds for \( \pi(x) \) of the form \( \frac{n}{H_n - c} \) are discussed, where \( H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \).

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