ANALYSIS OF A SPLITTING SCHEME FOR DAMPED
STOCHASTIC NONLINEAR SCHRÖDINGER EQUATION WITH
MULTIPLICATIVE NOISE

JIANBO CUI † AND JIALIN HONG †

Abstract. In this paper, we investigate the damped stochastic nonlinear Schrödinger (NLS) equation with multiplicative noise and its splitting-based approximation. When the damped effect is large enough, we prove that the solutions of both the damped stochastic NLS equation and the splitting scheme are exponentially stable and possess some exponential integrability. These properties show that the strong order of the scheme is \( \frac{1}{2} \) and independent of time. Additionally, we analyze the regularity of the Kolmogorov equation with respect to the stochastic nonlinear Schrödinger equation. As a consequence, the weak order of the scheme is shown to be 1 and independent of time.

Key words. Damped stochastic nonlinear Schrödinger equation, Exponential integrability, Strong order, Weak order, Kolmogorov equation.

AMS subject classifications. 60H35, 35Q55, 60H15, 65M12.

1. Introduction. In many fields of economics and the natural sciences, stochastic partial differential equations (SPDEs) play important roles. Since many SPDEs can only be solved numerically, it is a crucial research problem to construct and study discrete numerical approximation schemes which converge with strong and weak convergence rates to the solutions of such SPDEs. For SPDEs with monotone coefficients, there exist fruitful results on strong error analysis of temporal and spatial numerical approximations (see, e.g., [2, 5, 6, 17, 20, 21]). However, there exists only a few results in the scientific literature which establish strong and weak convergence rates for a time discrete approximation scheme in the case of an SPDE with a nonglobally monotone nonlinearity (see, e.g., [11, 12, 19, 23, 24, 25]). This motives us to construct strong and weak approximations for this kind of SPDE.

The stochastic nonlinear Schrödinger (NLS) equation, as a representative SPDE, models the propagation of nonlinear dispersive waves in inhomogeneous or random media (see, e.g., [3]). In [15] and [4, 16] it was proved that the stochastic NLS equation admits a unique solution in \( H \) and \( H^1 \), respectively. Recently, [8, 11] gave the global well-posedness of the one-dimensional stochastic NLS equation in \( H^2 \). In this paper, we focus on strong and weak approximations of the following one-dimensional damped stochastic nonlinear equation with multiplicative noise:

\[
\begin{align*}
    du &= (i\Delta u + \lambda |u|^2 u - \alpha u)dt + iudW(t) \quad \text{in } \mathbb{R} \times (0, \infty); \\
    u(0) &= u_0 \quad \text{in } \mathbb{R},
\end{align*}
\]

where \( \lambda = 1 \) or \( -1 \) corresponds to focusing or defocusing cases, respectively, and \( \alpha(\cdot) \) is a real-valued function. When studying the propagation of waves over very long distance in random media, the damping term \(-\alpha u\) cannot be neglected (see e.g. [18]). The diffusion term represents the fluctuation effect of a physical process in random media, where \( W = \{W(t) : t \in [0, T]\} \) is an \( L^2(\mathbb{R}; \mathbb{R}) \)-valued \( Q \)-Wiener process.
on a stochastic basis \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\); i.e., there exists an orthonormal basis \(\{e_k\}_{k \in \mathbb{N}_+}\) of \(L^2(\mathbb{R}; \mathbb{R})\) and a sequence of mutually independent, real-valued Brownian motions \(\{\beta_k\}_{k \in \mathbb{N}_+}\) such that \(W(t) = \sum_{k \in \mathbb{N}_+} Q^2 e_k \beta_k(t), \quad t \in [0, T]\).

There have been many works concentrating on construction and analysis of numerical approximations for the stochastic NLS equation. Paper [17] studies a type of Crank–Nicolson semidiscrete schemes and shows that for stochastic NLS equation with Lipschitz coefficients, these Crank–Nicolson type schemes have strong order \(1/2\) in general and order 1 if the noise is additive and that the weak order is always 1. In order to inherit the symplectic structure of the stochastic NLS equation, [7] studies symplectic Runge–Kutta methods and obtains the convergence theorem for the Lipschitz cases. Paper [1] studies an explicit exponential scheme and shows that it preserves the trace formula for stochastic linear NLS equation with additive noise.

For a stochastic NLS equation with non-Lipschitz or nonmonotone coefficients, some papers have constructed strong numerical approximations and obtained convergence rates in a certain sense such as pathwise or in probability weaker than in strong sense (see, e.g., [8, 17, 26] and references therein). Progress has been made in [11, 12], where the authors obtained strong convergence rates of the spatial centered difference method, the spatial Galerkin method and a temporal splitting method for a conservative stochastic NLS equation.

In this article, we apply the splitting ideas in [12, 21] to approximating (1) and aim to show the strong and weak order of this splitting scheme. The key to obtaining strong and weak convergence rates of numerical schemes for SPDEs with nonmonotone coefficients is to obtain some a priori estimates and exponential integrability of exact and numerical solutions (see, e.g., [8, 17, 26] and references therein). Progress has been made in [11, 12], where the authors obtained strong convergence rates of the spatial centered difference method, the spatial Galerkin method and a temporal splitting method for a conservative stochastic NLS equation.

In this article, we apply the splitting ideas in [12, 21] to approximating (1) and aim to show the strong and weak order of this splitting scheme. The key to obtaining strong and weak convergence rates of numerical schemes for SPDEs with nonmonotone coefficients is to obtain some a priori estimates and exponential integrability of exact and numerical solutions (see, e.g., [9, 11, 12, 23, 24, 25]). On the one hand, we prove some a priori estimations of the exact solution of (1), as well as those of the numerical solution, to get the time-independent strong error estimation. As a consequence, the solution of (1) is shown to be exponentially stable. On the other hand, we show the exponential integrability properties of exact and numerical solutions by an exponential integrability lemma established in [9, Corollary 2.4]; see also [11, Lemma 3.1]. This type of exponential integrability is also useful to get the strongly continuous dependence on initial data of both exact and numerical solutions and to deduce Gaussian tail estimations of these solutions (see e.g. [9, 11, 12]). To obtain the weak convergence order of the proposed scheme, we study the regularity of the transformed Kolmogorov equation of the damped stochastic NLS equation with nonmonotone coefficient. Based on this regularity result, we prove that the weak order of the proposed scheme is first order and independent of time. To the best of our knowledge, this is the first weak convergence order result of temporal approximations for the stochastic NLS equation with nonmonotone coefficients driven by multiplicative noise.

The rest of this paper is organized as follows. In Section 2, we prove that the damped stochastic NLS equation is exponentially stable and exponentially integrable. Section 3 is devoted to obtaining some a priori estimates of the numerical solution in Sobolev norms. Then the time-independent strong error of the solutions is given. In Section 4, we study the regularity of the corresponding Kolmogorov equation with respect to the damped stochastic NLS equation. Then we show that the weak order of the scheme is first order and time-independent.

2. Some properties for damped stochastic NLS equation. We first introduce some frequently used notation and assumptions. The norm and inner product of \(H := L^2(\mathbb{R}; \mathbb{C})\) are denoted by \(\| \cdot \|\) and \(\langle u, v \rangle := \mathbb{R} \left[ \int_{\mathbb{R}} \overline{u(x)} v(x) dx \right]\), respectively.
When sup \( T \) is a fixed positive number, \( u_0 \in \mathbb{H}^s \) is a deterministic function and \( Q^\frac{1}{2} \in L^2 \) with \( s \) being a nonnegative integer, i.e.,
\[
\|Q^\frac{1}{2}\|^2_{L^2} := \sum_{k \in \mathbb{N}_+} \|Q^\frac{1}{2}e_k\|^2_{\mathbb{H}^s} < \infty,
\]
where \( \{e_k\}_{k \in \mathbb{N}_+} \) is any orthonormal basis of \( L^2(\mathbb{R}; \mathbb{R}) \) and \( \mathbb{H}^s := \mathbb{H}^s(\mathbb{R}; \mathbb{C}) \) is the usual Sobolev space. In this paper, \( a \) and \( b \) are positive numbers. We use \( C \) and \( C' \) to denote generic constants, independent of the time step size \( \tau \), which differ from one place to another. In some places of this paper, the computations are formal but could be justified rigorously by truncated techniques and approximation arguments (see e.g. [17]).

For damped stochastic NLS equations with additive noise, [18] studies the long-time behavior of its solution and obtains the ergodicity of the weakly damped NLS equation. It is natural to study the long-time behaviors of the damped stochastic NLS equation with multiplicative noise, i.e., (1). In this section, we want to investigate the mutual influence among the damping effect, the cubic nonlinearity and the noise intensity and further study the long-time behaviors. This is our other motivation for considering Eq. (1). It should be mentioned that when \( \alpha(x) = \frac{1}{2}F_Q(x) := \frac{1}{2} \sum_{k=1}^{\infty} (Q^\frac{1}{2}e_k)^2(x) \), the stochastic NLS equation (1) has the conserved quantity charge (see [16]), i.e., \( \|u(t)\|^2 = \|u_0\|^2 \), \( t < T \), a.s. Next, we mainly focus on some a priori estimates and long-time behaviors of the exact solution for (1).

**Lemma 2.1.** Let \( \|\alpha\|_{L^\infty} < \infty \), \( Q^\frac{1}{2} \in L^2 \), and \( u_0 \in \mathbb{H} \). Then \( \|u\| \) is bounded a.s. in any finite interval \( [0, T] \). Moreover, if \( \sup_{x \in \mathbb{R}}(\frac{1}{2}F_Q(x) - \alpha(x)) \leq 0 \), then the upper bound is independent of \( T \).

**Proof.** By the Itô formula, we have
\[
\frac{1}{2}\|u(t)\|^2 = \frac{1}{2}\|u_0\|^2 + \int_0^t \langle u, i\Delta u + i\lambda|u|^2u - \alpha u \rangle ds
+ \int_0^t \langle u, iudW(s) \rangle + \int_0^t \frac{1}{2} \sum_{k=1}^{\infty} (iuQ^\frac{1}{2}e_k, iuQ^\frac{1}{2}e_k) ds
= \frac{1}{2}\|u_0\|^2 + \int_0^t \|u\|^2\left(\frac{1}{2}F_Q - \alpha\right) dx dt.
\]

The Sobolev embedding theorem, \( Q^\frac{1}{2} \in L^2 \) and \( \|F_Q\|_{L^\infty} < \infty \), combined with \( \|\alpha\|_{L^\infty} < \infty \), imply that
\[
\frac{1}{2}\|u(t)\|^2 \leq \frac{1}{2}\|u_0\|^2 + \int_0^t \|u\|^2\left(\frac{1}{2}\|F_Q\|_{L^\infty} + \|\alpha\|_{L^\infty}\right) dt.
\]

Then Gronwall inequality yields that
\[
\|u(t)\|^2 \leq \exp(T\|F_Q\|_{L^\infty} + 2T\|\alpha\|_{L^\infty})\|u_0\|^2.
\]

When \( \sup_{x \in \mathbb{R}}(\frac{1}{2}F_Q(x) - \alpha(x)) \leq 0 \), a similar argument yields that
\[
\|u(t)\|^2 \leq \|u_0\|^2.
\]
COROLLARY 2.1. If in addition, \( \sup_{x \in \mathbb{R}} (\frac{1}{2} F_Q(x) - \alpha(x)) \leq -a \), then the charge is exponentially stable.

Proof. Similar to Lemma 2.1, we obtain
\[
\|u(t)\|^2 \leq \|u_0\|^2 - 2a \int_0^t \|u(s)\|^2 ds,
\]
which yields that
\[
\|u(t)\|^2 \leq \exp(-2at) \|u_0\|^2. \tag{2}
\]

When \( \alpha(x) = \frac{1}{2} F_Q(x) \), (1) becomes the stochastic NLS equation with a conserved quantity: charge. One cannot expect the following long-time behaviors of the exact solution in this conserved case. When \( \alpha(x) = a + \frac{1}{2} F_Q(x) \), (1) satisfies the condition of Corollary 2.1 and thus the charge is exponentially decaying. The above results inspire us to consider the long-time behavior of \( u \), such as its corresponding invariant measure and ergodicity. Actually, direct calculation yields that the Dirac measure at 0 is one of the invariant measures. The uniqueness of the invariant can be obtained as follows.

PROPOSITION 2.1. Assume that \( \alpha \in \mathbb{H}^1 \), \( \sup_{x \in \mathbb{R}} (\frac{1}{2} F_Q(x) - \alpha(x)) \leq -a \), \( u_0 \in \mathbb{H}^1 \) and \( Q_\perp \in \mathcal{L}_1 \). For any \( p \geq 2 \), we have
\[
\sup_{t \in [0, \infty)} \mathbb{E} \left[ \|u(t)\|_{\mathbb{H}^1}^p \right] \leq C (1 + \|u_0\|_{\mathbb{H}^1}^p + \|u_0\|^{3p}).
\]

Proof. For simplicity, we only prove the case \( p = 2 \). One can apply the Itô formula to the appropriate power of the energy functional \( H(u) : \frac{1}{2} \|\nabla u\|^2 - \frac{1}{4} \|u\|_{L^4}^4 \), and apply the Burkholder–Davis–Gundy inequality to get the desired result for \( p > 2 \). Similar to [11], thanks to Gagliardo–Nirenberg inequality \( \|u\|_{L^4}^4 \leq 2 \|\nabla u\| \|u\|^3 \), we need only prove the uniform boundedness of the energy functional \( H(u(t)) \). The Itô formula yields that
\[
\mathbb{E}[H(u(t))] - H(u_0) = \int_0^t \mathbb{E} \left[ \langle \nabla u, \nabla (\frac{F_Q}{2} - \alpha) \rangle \right] ds + \int_0^t \mathbb{E} \left[ \langle \nabla u, u (\sum_k Q_\perp e_k \nabla Q_\perp e_k - \nabla \alpha) \rangle \right] ds
\]
\[
+ \int_0^t \frac{1}{2} \sum_k \mathbb{E} \left[ \langle u, u |\nabla Q_\perp e_k|^2 \rangle \right] ds + \int_0^t \lambda \mathbb{E} \left[ \langle |u|^2 u, u(\alpha - \frac{F_Q}{2}) \rangle \right] ds.
\]

The Hölder, Gagliardo–Nirenberg and Young inequalities and Sobolev embedding theorem imply that for \( a > \epsilon > 0 \),
\[
\mathbb{E}[H(u(t))] \leq H(u_0) - (a - \epsilon) \int_0^t \mathbb{E}[\|\nabla u\|^2] ds + C(\epsilon) \int_0^t \mathbb{E}[\|u\|^2 (\|Q_\perp\|_{\mathcal{L}_1}^8 + \|\nabla \alpha\|^4 + \|u\|^6 \|\alpha - \frac{F_Q}{2}\|^2)] ds.
\]

By the fact that \( \frac{1}{2} \|\nabla u\|^2 - \frac{1}{4} \|u\|_{L^4}^4 \leq H(u) \leq \frac{1}{2} \|\nabla u\|^2 + \frac{1}{4} \|u\|_{L^4}^4 \), and the Young
inequality, we have for small \( \eta > 0 \),
\[
\mathbb{E}[H(u(t))] \leq H(u_0) - \frac{2(\alpha - \epsilon)}{1 + \eta} \int_0^t \mathbb{E}[H(u)] ds + C(\epsilon, \eta) \int_0^t \mathbb{E} \left[ \|u\|^2 (\|Q_+^2\|^4_{L_2^2} + \|Q_+^2\|^6_{L_2^2} + \|
abla \alpha\|^4 + \|\alpha\|^6 (1 + \|\alpha - \frac{FQ}{2\|L_\infty\|}) \right] ds.
\]

The Gronwall inequality, together with the charge evolution law in Corollary 2.1, yields that
\[
\mathbb{E}[H(u(t))] \leq e^{-2(\alpha - \epsilon) t} H(u_0) + C(\epsilon, \eta) e^{-\frac{2(\alpha - \epsilon)}{1 + \eta} t} \int_0^t \left( e^{(\frac{2(\alpha - \epsilon)}{1 + \eta} - 2\|\alpha\|^2)} \|u_0\|^2 (\|Q_+^2\|^4_{L_2^2} + \|\alpha\|^4_{L_2^2}) + e^{(\frac{2(\alpha - \epsilon)}{1 + \eta} - 6\|\alpha\|^2)} \|u_0\|^6 (1 + \|\alpha - \frac{FQ}{2\|L_\infty\|}) \right) ds \\
\leq e^{-\frac{2(\alpha - \epsilon)}{1 + \eta} t} C(\epsilon, \eta, \alpha, Q) (1 + H(u_0) + \|u_0\|^6).
\]

Finally, the Gagliardo–Nirenberg and Young inequalities and the Sobolev embedding theorem imply the uniform boundedness for the \( p \)-moment of \( \|u\|_{\mathbb{H}^1} \).

Next we show that (1) admits a unique invariant measure \( \delta_0 \) and a unique stationary solution \( 0 \) in \( \mathbb{H}^1 \) similarly [14].

**Corollary 2.2.** Under the same condition as Proposition 2.1, the following statements hold:

(i) We have
\[
\lim_{t \to \infty} P_t \phi(w) = \phi(0), \quad w \in \mathbb{H}^1, \quad \phi \in C_b(\mathbb{H}^1),
\]
where \( P_t \) is the Markov semigroup associated with the solution \( u(t) \).

(ii) \( \delta_0 \) is the unique invariant measure for \( P_t \).

(iii) For any Borel probability measure \( \nu \in \mathcal{P}(\mathbb{H}^1) \), we have
\[
\lim_{t \to \infty} \int_{\mathbb{H}^1} P_t \phi(w) \nu(dw) = \phi(0).
\]

(iv) There exists \( b > 0 \) such that for any functional \( \phi \in C_b^1(\mathbb{H}^1) \), we have
\[
\left| P_t \phi(w) - \phi(0) \right| \leq C\|\phi\|_{C_b^1} e^{-bt} (1 + \|w\|_{\mathbb{H}^1}).
\]

**Proof.** We show that for any time sequence \( \{t_n\}_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} t_n = \infty \), the sequence \( \{u(t_n)\}_{n \in \mathbb{N}} \) admits a unique limit. For any \( t_n \leq t_m, \quad n \leq m \), by Minkowski and Young inequality and Sobolev embedding theorem, we have
\[
\mathbb{E}\|u(t_n) - u(t_m)\|^2_{\mathbb{H}^1} \\
\leq C \left( \mathbb{E}\|S_a(t_m - t_n) - I\|u(t_n)\|^2_{\mathbb{H}^1} + \mathbb{E}\int_{t_n}^{t_m} \|S_a(t_m - s)\lambda u(s)\|^2_{\mathbb{H}^1} ds \right)^2 \\
+ \mathbb{E}\int_{t_n}^{t_m} \|S_a(t_m - s)(-\alpha + a)u(s)\|^2_{\mathbb{H}^1} ds \right) + \mathbb{E}\left[ \|S_a(t_m - s)u(s)dW(s)\|^2_{\mathbb{H}^1} \right] \right] \\
\leq C\mathbb{E}\|u(t_n)\|_{\mathbb{H}^1} e^{-a(t_m - t_n)} (\|u(s)\|_{\mathbb{H}^1} + \|u(s)\|^3_{\mathbb{H}^1}) ds \\
+ C\int_{t_n}^{t_m} \mathbb{E}\left[ e^{-2a(t_m - s)} \|u(s)\|^2_{\mathbb{H}^1} \right] ds.
\]

This manuscript is for review purposes only.
where $S_n(t) := e^{i\Delta t-at}$. The arguments and estimate (3) in Proposition 2.1 yield that for some $b > 0$, we have

$$
\mathbb{E}[|u(t_n) - u(t_m)|^2_{H^s}] \leq C(a, \eta, \alpha, Q)e^{-bt}(1 + H(u_0) + \|u_0\|^6).
$$

This implies that $\{u(t_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence and thus $\{u(t)\}_{t \in \mathbb{R}^+}$ admits at least a strong limit. Combining those with the exponential decay estimate (3), we get 0 is the unique strong limit of $\{u(t)\}_{t \in \mathbb{R}^+}$. The strong mixing property is immediately obtained, and we finish the proof by the exponential decay estimate and strong mixing property.

To get the a priori estimates in $H^s$, we introduce the auxiliary Lyapunov functional $f(u) := \|\nabla^s u\|^2 - \lambda \langle (-\Delta)^{s-1} u, |u|^2 u \rangle$ from [11].

**Proposition 2.2.** Assume that $\alpha \in \mathbb{R}^s$, $\sup_{x \in \mathbb{R}}(\frac{1}{2}F_Q(x) - \alpha(x)) \leq -a$, $Q^\frac{3}{2} \in \mathcal{L}_2^*$ and $u_0 \in H^s$, $s \geq 2$. For any $p \geq 2$, we have

$$
\sup_{t \in [0, \infty)} \mathbb{E}\left[\|u(t)\|_{H^s}^p\right] \leq C(\alpha, Q)(1 + \|u_0\|_{H^s}^p + \|u_0\|_{H^{s-1}}^{5p}).
$$

**Proof.** We prove the uniform boundedness by induction. Assume that the $p$-moment of $\|u\|_{H^{s-1}}$ is uniformly controlled. For simplicity, we show the case $p = 2$ under the $H^s$-norm. Applying Itô formula to the functional $f(u(t))$, we can get the terms similar to those in [11]. Similar arguments yield that for $s \geq 2$,

$$
\mathbb{E}[f(u(t))] \leq f(u_0) - (a - \epsilon) \int_0^t \mathbb{E}[\|\nabla^s u\|^2]ds + C(\epsilon, \alpha, Q) \int_0^t \mathbb{E}[|u|_{H^{s-1}}^{4} + \|u\|_{H^{s-1}}^6]ds.
$$

Since $f(u) \leq \|\nabla^s u\|^2 + C\|u\|_{H^{s-1}}^4$, iterative arguments similar to those in Proposition 2.1 complete the proof.

**Remark 2.1.** Due to the particular structure of charge and energy, the exponential decay estimates in $H^s$, $s \geq 2$ can also be obtained similarly to Proposition 2.1 by iterative arguments. This show that (1) is an ergodic system and admits the unique stationary solution $0$ in $H^s$. This long-time behavior result still holds when we consider (1) in a bounded domain with homogeneous boundary condition.

Beyond these a priori estimations, we need the exponential integrability to construct numerical schemes with strong and weak convergence order similar to those in [11, 12, 23]. We also note that this type of exponential integrability has many other applications (see e.g. [9, 11, 12, 19, 23, 24, 25]).

**Proposition 2.3.** Assume $\alpha \in \mathbb{R}^2$, $\sup_{x \in \mathbb{R}}(\frac{1}{2}F_Q(x) - \alpha(x)) \leq -a$, $Q^\frac{1}{2} \in \mathcal{L}_2^*$, and $u_0 \in \mathbb{R}^1$. There exist $\beta$ and $C$ depending on $\alpha$, $Q$, and $u_0$ such that

$$
\sup_{t \in [0, \infty)} \mathbb{E}\left[\exp\left(e^{-\beta t}H(u(t))\right)\right] \leq C.
$$

**Proof.** Denote $\mu(u) = i\Delta u + i\lambda|u|^2u - au$ and $\sigma(u) = iuQ^\frac{1}{2}$. Simple calculations
yield that
\[
DH(u)\mu(u) + \frac{1}{2} \text{tr} [D^2H(u)\sigma(u)\sigma(u)^*] + \frac{1}{e^{\beta t}} \|\sigma^*(u)DH(u)\|^2
= \langle \nabla u, \nabla u(FQ_2 - \alpha) \rangle - \sum_k \langle u, \nabla u(Q^\perp e_k \nabla Q^\perp e_k - \nabla \alpha) \rangle
+ \sum_k \langle |\nabla Q^\perp e_k|^2, |u|^2 \rangle + \langle |u|^4, \alpha - \frac{FQ_2}{2} \rangle + \frac{1}{2e^{\beta t}} \sum_k \langle \nabla u, iu\nabla Q^\perp e_k \rangle^2.
\]

The Hölder, Young and Gagliardo–Nirenberg inequalities, combined with Corollary 2.1, yield that
\[
DH(u)\mu(u) + \frac{1}{2} \text{tr} [D^2H(u)\sigma(u)\sigma(u)^*] + \frac{1}{e^{\beta t}} \|\sigma^*(u)DH(u)\|^2
\leq - \left( a - \epsilon - \frac{1}{2e^{\beta t}} \|u_0\|^2 \sum_k \|\nabla Q^\perp e_k\|^2_{L^\infty} \right) \|\nabla u\|^2
+ C(\epsilon)\|u_0\|^2e^{-2at} \left( \|Q^\perp\|^4_{L^2} + \|\alpha\|^2_{L^2} + \|u_0\|^4e^{-4at}\|\alpha - \frac{FQ_2}{2}\|^2_{L^\infty} \right)
\]

Let \( \beta \geq -2a \). By the Gagliardo–Nirenberg and Young inequalities, we get
\[
DH(u)\mu(u) + \frac{1}{2} \text{tr} [D^2H(u)\sigma(u)\sigma(u)^*] + \frac{1}{e^{\beta t}} \|\sigma^*(u)DH(u)\|^2
\leq - \left( a - \epsilon - \frac{1}{2}\|u_0\|^2 \sum_k \|\nabla Q^\perp e_k\|^2_{L^\infty} \right) \frac{2}{1 + \eta}H(u) + C(\epsilon, \eta)\|u_0\|^2e^{-2at} \left( \|Q^\perp\|^4_{L^2} + \|\alpha\|^2_{L^2} + \|u_0\|^4e^{-4at}\|\alpha - \frac{FQ_2}{2}\|^2_{L^\infty} + 1 \right)
\]
\[
:= - \left( a - \epsilon - \frac{1}{2}\|u_0\|^2 \sum_k \|\nabla Q^\perp e_k\|^2_{L^\infty} \right) \frac{2}{1 + \eta}H(u) + V(\epsilon, \eta, t, u_0).
\]

By [11, Lemma 3.1], we need \( \beta \geq \frac{-2a + 2\epsilon + \|u_0\|^2 \sum_k \|\nabla Q^\perp e_k\|^2_{L^\infty}}{1 + \eta} \). Thus there always exist \( \epsilon \) and \( \eta \) such that \( -2a - \beta < 0 \) and
\[
\sup_{t \in [0, \infty)} \mathbb{E} \left[ \exp \left( e^{-\beta t}H(u(t)) \right) \right] \leq \mathbb{E} \left[ \exp \left( H(u_0) + \int_0^t e^{-\beta r}V(\epsilon, \eta, r, u_0)dr \right) \right] \leq C. \quad \square
\]

3. Strong convergence. We use a splitting idea similar to that in [12, 21] to discretize (1) and obtain the strong convergence rate independent of the time domain. The key tool is applying the stability in \( \mathbb{H}^2 \) and the exponential integrability of both numerical and exact solutions. The main idea is to split (1) in \( T_m = [t_m, t_{m+1}) \), \( t_m = m\tau, m \in \mathbb{Z}_M := \{0, 1, 2, \ldots, M - 1\} \), into a deterministic NLS equation with random initial datum and a linear damped SPDE:
\[
(5) \quad du^D_x(t) = (i\Delta u^D_x(t) + i\lambda |u^D_x(t)|^2 u^D_x(t)) dt,
\]
\[
(6) \quad du^S_x(t) = -\alpha u^S_x(t) dt + iu^S_x(t)dW(t).
\]
For simplicity, we denote the solution operators of (5) and (6) in $T_m$ as $\Phi^D_{m,t-t_m}$ and $\Phi^S_{m,t-t_m}$, respectively. Next we set the splitting process $u_\tau$ in $T_m$ as

$$u_\tau(t) := u^S_{\tau,m}(t) := (\Phi^S_{j,t-t_m} \Phi^D_{j,\tau}) \prod_{j=1}^{m-1} (\Phi^S_{j,\tau} \Phi^D_{j,\tau}) u_\tau(0), \quad t \in T_m,$$

and

$$u^D_{\tau}(t) := u^D_{\tau,m}(t) := \Phi^D_{j,t-t_m} \prod_{j=1}^{m-1} (\Phi^S_{j,\tau} \Phi^D_{j,\tau}) u_\tau(0), \quad t \in \{t_m \cup T_m\}/t_{m+1}.$$

For the sake of simplicity, we take the initial datum of the splitting process to be $u_\tau(0) = u_0$. Iterating previous procedures, we obtain a splitting process $u_\tau = \{u_\tau(t) : t \in [0, T]\}$, which is left-continuous with finite right-hand limits and $\mathcal{F}_t$-adapted. We note that there are some results on numerically approximating SPDEs by splitting schemes (see [10, 13, 19, 21, 26] and references therein). Since (5) has no analytic solution, we apply the Crank–Nicolson type scheme to temporally discretize (5). Based on the explicitness of the solution of (6), we get the splitting Crank–Nicolson type scheme starting from $u_0$:

$$(8) \quad \begin{cases} u^D_{m+1} = u_m + \eta \Delta t u^D_{m+\frac{1}{2}} + 1\tau \frac{|u_m|^2 + |u^D_{m+\frac{1}{2}}|^2}{2} u^D_{m+\frac{1}{2}}, \\
 u_{m+1} = \exp \left( -\alpha + \frac{F_\alpha}{2} + i(W_{m+1} - W_{m}) \right) u^D_{m+1}, \quad m \in \mathbb{Z}_M,
\end{cases}$$

with $u^D_{m+\frac{1}{2}} = \frac{1}{2}(u_m + u^D_{m+1})$. We can also get the continuous extension of $u_m$ as

$$\hat{u}_\tau(t) := \hat{u}^S_{\tau,m}(t) := (\Phi^S_{j,t-t_m} \Phi^D_{j,\tau}) \prod_{j=1}^{m-1} (\Phi^S_{j,\tau} \Phi^D_{j,\tau}) u_\tau(0), \quad t \in T_m,$$

where $\Phi^D_{j,\tau}$ is the solution operator of the Crank–Nicolson type scheme.

Throughout this paper, we do not consider the spatial discretization since our approach and proof can be extended to the study of a fully discrete scheme as in [12]. Some estimates need to be modified accordingly. However, this requires long and technical computations and would probably increase the length of our paper. For more results on the strong convergence result of spatial approximations for the stochastic NLS equation, we refer the reader to [11, 12]. However, the study of strong and weak convergence rate of numerical schemes both in time and space for a higher dimensional stochastic NLS equation requires the a priori estimates in a higher Sobolev norm and further investigation.

Next, we always assume that $\sup_{x \in \mathbb{R}}(\frac{1}{2}F_Q(x) - \alpha(x)) \leq -a$. Since (5) possesses the charge conservation law and (6) is weakly damped, it is not difficult to obtain the following results about the charge of this splitting process.

**Lemma 3.1.** Let $\alpha \in \mathbb{H}^1$, $\sup_{x \in \mathbb{R}}(\frac{1}{2}F_Q(x) - \alpha(x)) \leq -a$, $Q^\frac{1}{2} \in \mathcal{L}_2^1$, and $u_0 \in \mathbb{H}$. The splitting process $u_\tau = \{u_\tau(t) : t \in [0, T]\}$ is uniquely solvable and $\mathcal{F}_t$-measurable. Moreover, for any $t \in [0, T]$ there holds a.s. that

$$\|u_\tau(t)\|^2 \leq e^{-2at}\|u_0\|^2.$$

For $t \in T_n$, we have

$$\|u^S_{\tau,m}(t)\|^2 \leq e^{-2at}\|u_0\|^2, \quad \|u^D_{\tau,m}(t)\|^2 \leq e^{-2at}\|u_0\|^2.$$
Proposition 3.1. Assume that \( \alpha \in H^s \), \( \sup_{x \in R} (x F_Q(x) - \alpha(x)) \leq -a \), \( Q^{1/2} \in L^2 \), and \( u_0 \in H^s \), \( s \geq 1 \). Then for any \( p \geq 2 \), we have
\[
\tag{9}
\sup_{t \in [0, \infty)} \mathbb{E}\left[ \| u_r(t) \|_{H^s}^p \right] \leq C(1 + \| u_0 \|_{H^s}^p + \| u_0 \|_{L^{2p-1}}^{5p}).
\]

Proof. For simplicity, we give the proof for \( p = 2 \). The case \( p > 2 \) is made similar to the proof in [11, Theorem 2.1] by applying the Itô formula to appropriate power of the auxiliary functionals \( H \) and \( f \), and applying Burkholder–Davis–Gundy inequality. Notice that the energy evolution of splitting process (7) is same as Eq. (1) in each interval \( T_m \). The Itô formula, combined with the energy conservation law of Eq. (5), yields that
\[
\mathbb{E}[H(u^S_{r,m}(t))] - \mathbb{E}[H(u^D_{r,m}(t_m))] = \int_{t_m}^t \mathbb{E}\left[ \langle \nabla u^S_{r,m}, \nabla u^S_{r,m}(\frac{F_Q}{2} - \alpha) \rangle \right] ds + \int_{t_m}^t \sum_k \mathbb{E}\left[ \langle \nabla u^S_{r,m}, u^S_{r,m}(Q^{1/2} e_k \nabla Q^{1/2} e_k - \nabla \alpha) \rangle \right] ds \\
+ \int_{t_m}^t \frac{1}{2} \sum_k \mathbb{E}\left[ \| u^S_{r,m}, u^S_{r,m} | \nabla Q^{1/2} e_k |^2 \| \right] ds + \int_{t_m}^t \lambda \mathbb{E}\left[ \langle |u^S_{r,m}|^2 u^S_{r,m}, u^S_{r,m}(\alpha - \frac{F_Q}{2}) \rangle \right] ds.
\]

Similar to Proposition 2.1, we get
\[
\mathbb{E}[H(u^S_{r,m}(t))] \leq H(u^S_{r,m}(t_m)) - \frac{2(a - \epsilon)}{1 + \eta} \int_{t_m}^t \mathbb{E}[H(u^S_{r,m})] ds + C(\epsilon, \eta) \int_{t_m}^t \left( \| u^S_{r,m} \|_2 \| Q^{1/2} \|_{L^2} + \| Q^{1/2} \|_{L^2} + \| \nabla \alpha \|_4 + \| u^S_{r,m} \|_6 \| \alpha - \frac{F_Q}{2} \|_{L^\infty} + \| u^S_{r,m} \|_6 \right) ds.
\]

The Gronwall inequality implies
\[
\mathbb{E}[H(u^S_{r,m}(t))] \leq e^{-\frac{2(a - \epsilon)}{1 + \eta} (t - t_m) H(u^S_{r,m}(t_m)) + e^{-\frac{2(a - \epsilon)}{1 + \eta} t} C(\epsilon, \eta, \alpha, Q, \| u_0 \|)(t - t_m).
\]

Then by repeating the above procedures in each interval and combining them with discrete Gronwall inequality, we obtain
\[
\mathbb{E}[H(u_r(t))] \leq e^{-\frac{2(a - \epsilon)}{1 + \eta} H(u_0) + e^{-\frac{2(a - \epsilon)}{1 + \eta} (1 + t)} C(\epsilon, \eta, \alpha, Q) \leq H(u_0) + C(\epsilon, \eta, \alpha, Q).
\]

Then similar arguments lead to the uniform boundedness for \( p \geq 2 \).

Next, we turn to estimate \( \mathbb{E}[\| u \|_{H^s}^p], s \geq 2 \). Similar to Proposition 2.2, we have
\[
f(u^D_{r,m}(t)) - f(u^D_{r,m}(t_m)) = - \int_{t_m}^t (\Delta)^{s-1} u^D_{r,m, i} |u^D_{r,m, i} u^D_{r,m} | dr - \lambda \int_{t_m}^t (\Delta)^{s-1} u^D_{r,m, i} |u^D_{r,m} |^2 \Delta u^D_{r,m} dr \\
- \lambda \int_{t_m}^t (\Delta)^{s-1} u^D_{r,m, i} |\nabla u^D_{r,m, i} |^2 u^D_{r,m} + 2i(\nabla u^D_{r,m, i} u^D_{r,m}) dr \\
\leq e \int_{t_m}^t f(u^D_{r,m}) ds + C(\epsilon, \alpha, Q) \int_{t_m}^t (\| u^D_{r,m} \|_{H^{s-1}}^4 + \| u^D_{r,m} \|_{H^{s-1}}^{10}) ds.
\]

By the Gronwall inequality, we obtain
\[
f(u^D_{r,m}(t_{m+1})) \leq e^{ct} f(u^D_{r,m}(t_m)) + C(\epsilon, \alpha, Q) \int_{t_m}^{t_{m+1}} (\| u^D_{r,m} \|_{H^{s-1}}^4 + \| u^D_{r,m} \|_{H^{s-1}}^{10}) ds.
\]
On the other hand, the Itô formula and the Young and Gagliardo–Nirenberg inequalities yield that
\[ E[f(u_{\tau,m}(t))] \leq E[f(u_{\tau,m}(t_m + 1))] - (a - \epsilon) \int_{t_m}^t E[\|\nabla^s u_{\tau,m}\|^2] \, ds \]
\[ + C(\epsilon, \alpha, Q) \int_{t_m}^t E[\|u_{\tau,m}\|_{L^2}^4 + \|u_{\tau,m}\|_{L^8}^{10}] \, ds. \]

Again by the Gronwall inequality, we get
\[ E[f(u_{\tau,m}(t))] \leq e^{-(a-\epsilon)(t-t_m) + \epsilon \tau} E[f(u_{\tau,m}(t_m))] \]
\[ + C(\epsilon, \alpha, Q) \int_{t_m}^t e^{-\epsilon s} E[\|u_{\tau,m}\|_{L^2}^4 + \|u_{\tau,m}\|_{L^8}^{10}] \, ds \]
\[ + C(\epsilon, \alpha, Q) e^{-(a-\epsilon)(t-t_m)} \int_{t_m}^{t+1} E[\|u_{\tau,m}\|_{L^2}^4 + \|u_{\tau,m}\|_{L^8}^{10}] \, ds. \]

Finally, the discrete Gronwall inequality, together with the induction hypothesis, leads to
\[ E[f(u_{\tau}(t))] \leq Ce^{-(a-2\epsilon)T} f(u_0) + \frac{1 - e^{-(a-2\epsilon)T}}{1 - e^{-(a-2\epsilon)\tau}} C(\epsilon, \alpha, Q, u_0) \tau \]
\[ \leq f(u_0) + C(\epsilon, \alpha, Q, u_0), \]
where we use the fact that \( \frac{1 - e^{-\alpha \tau}}{1 - e^{-\alpha \tau}} \leq 1 + \epsilon \). The relationship \( \|\nabla^s u\|^2 - C\|u\|_{L^8}^{10} \leq f(u) \leq \|\nabla^s u\|^2 + C\|u\|_{L^8}^{10} \) and induction arguments finish the proof.

We also need a priori estimation on numerical solution of the splitting Crank-Nicolson scheme (10). The detail proof for the following lemma is omitted since it is similar to the proof of Proposition 3.1.

**Lemma 3.2.** Let \( \alpha \in \mathbb{R}^1 \), \( \sup_{x \in \mathbb{R}} (\frac{1}{2} F_Q(x) - \alpha(x)) \leq -a \), \( Q^1 \in L^1_\alpha \), and \( u_0 \in \mathbb{H}^1 \). The splitting process \( u_m, m \in \mathbb{Z}_M \) is uniquely solvable and \( \mathcal{F}_{tn} \)-measurable. Moreover, it holds a.s. that
\[ \|u_m\|^2 \leq e^{-2a t_m} \|u_0\|^2. \]

For \( t \in T_m \), the energy of \( u_m \) is uniformly bounded. More precisely, for any \( p \geq 1 \), there exists \( b > 0 \) such that
\[ \sup_{m \in \mathbb{Z}_M} E[H^p(u_m)] \leq Ce^{-bt_m}(1 + H^p(u_0)). \]

**Proposition 3.2.** Assume that \( \alpha \in \mathbb{H}^2 \), \( \sup_{x \in \mathbb{R}} (\frac{1}{2} F_Q(x) - \alpha(x)) \leq -a \), \( Q^2 \in L^2_\alpha \), and \( u_0 \in \mathbb{H}^2 \). Then for any \( p \geq 2 \), there exists a constant \( C = C(\alpha, Q, u_0, p) \) such that
\[ \sup_{m \in \mathbb{Z}_M} E[\|u_m\|_{L^p}^p] \leq C. \]

**Proof.** Arguments similar to [12, Lemma 3.3], combined with the Young inequality, yield that
\[ f(u_{m+1}^D) \leq f(u_m) + \frac{C \tau}{2} (\|\Delta u_m\|^2 + \|\Delta u_{m+1}^D\|^2) \]
\[ + C(\epsilon) \tau \left( 1 + \|\nabla u_{m+1}^D\|^2 + \|\nabla u_m\|^2 \right). \]
Then we have

\[ f(u^D_{m+1}) \leq \frac{1 + \frac{C\epsilon}{2}}{1 - \frac{C\epsilon}{2}} f(u_m) + \frac{C\epsilon \tau}{1 - \frac{C\epsilon}{2}} \left( 1 + \|\nabla u^D_{m+1}\|^2 + \|\nabla u_m\|^2 \right). \]

Let \( \epsilon \tau \leq 1 \). We get

\[ f(u^D_{m+1}) \leq (1 + 2\epsilon \tau) f(u_m) + C(\epsilon) \tau \left( 1 + \|\nabla u^D_{m+1}\|^2 + \|\nabla u_m\|^2 \right). \]

Notice that \( u_m \) can be extended to a continuous process \( \tilde{u}_{\tau,m}(t) \) with \( \tilde{u}^S_{\tau,m}(t_m) = u^D_{m+1} \) in \( T_m \). The arguments in Proposition 3.1, together with Lemma 3.2 show that for some \( b_1 > 0 \),

\[
\mathbb{E}[f(\tilde{u}^S_{\tau,m}(t))] \leq e^{-(a-\epsilon)(t-t_m)} \mathbb{E}[f(\tilde{u}^D_{m+1})] + e^{-b_1 t} C(\epsilon, \alpha, Q, u_0) \tau \\
\leq e^{-(a-\epsilon)(t-t_m)}(1 + 2\epsilon \tau) \mathbb{E}[f(u_m)] + e^{-\min(b_1 a-3\epsilon t)} C(\epsilon, \alpha, Q, u_0) \tau.
\]

Using the discrete Gronwall inequality, we obtain

\[
\mathbb{E}[f(\tilde{u}_{\tau}(t))] \leq C e^{-(a-3\epsilon t) f(u_0)} + e^{-\min(b_1 a-3\epsilon t)(t+1)} C(\epsilon, \alpha, Q, u_0) \leq f(u_0) + C(\epsilon, \alpha, Q, u_0),
\]

which yields the uniform boundedness of \( f(u_m), m \in \mathbb{Z}_M \), and thus \( \|u_m\|_{\mathbb{H}^2}, m \in \mathbb{Z}_M \).

The proof of the case \( p > 2 \) is similar.

To analyze the strong and weak order of the proposed scheme, we need to show some exponential integrability of \( u_m \) and \( u_\tau \) based on [11, Lemma 3.1]. These exponential integrability properties can be used to deduce the continuous dependence on initial data of \( u_m \) and \( u_\tau \) as in [11, 23].

**Proposition 3.3.** Let \( \alpha \in \mathbb{H}^1 \), sup\( \{ 1 \} = \frac{1}{2} |x|^2 - \alpha(x) \leq -a \), \( Q^{\frac{1}{2}} \in L_2^1 \) and \( u_0 \in \mathbb{H}^1 \).

There exist \( \beta \) and \( C = C(\alpha, Q, u_0) \) such that

\[
\mathbb{E} \left[ \exp \left( e^{-\beta t^2} H(u_\tau(t)) \right) \right] \leq C,
\]

\[
\mathbb{E} \left[ \exp \left( e^{-\beta t^2} H(u_m) \right) \right] \leq C.
\]

**Proof.** We first prove the estimation (12). Since (6) has the same energy evolution as (1) and (5) possesses the energy conservation law, by Proposition 2.3 we have in \( T_m \) that there always exists \( \beta > -2a + \|u_0\|^2 \sum_{k} \|\nabla Q^{\frac{1}{2}} e_k\|^2_{\mathbb{L}^\infty} \) such that

\[
\mathbb{E} \left[ \exp \left( e^{-\beta t^2} H(u_\tau(t)) \right) \right] \leq \mathbb{E} \left[ \exp \left( e^{-\beta t^2} H(u^S_{\tau,m}(t_m)) + \int_{t_m}^t e^{-\beta s} V(\epsilon, \eta, s, u_0) ds \right) \right] \leq \mathbb{E} \left[ \exp \left( e^{-\beta t^2} H(u^D_{\tau,m}(t_m)) + \int_{t_m}^t e^{-\beta s} V(\epsilon, \eta, s, u_0) ds \right) \right],
\]

where \( V(\epsilon, \eta, s, u_0) \) is the function appearing in the proof of Proposition 2.3.

Repeating the above procedures in each interval, we deduce that

\[
\mathbb{E} \left[ \exp \left( e^{-\beta t^2} H(u_\tau(t)) \right) \right] \leq \mathbb{E} \left[ \exp \left( H(u_0) + \int_0^t e^{-\beta s} V(\epsilon, \eta, s, u_0) ds \right) \right] \leq C(\epsilon, \eta, \alpha, Q, u_0),
\]

which verifies estimation (12). Similar arguments yield estimation (13). \( \square \)
Remark 3.1. Under the condition of Proposition 3.3, by the same procedures we can obtain that

\[
E \left[ \exp \left( e^{-\beta t}H(u^D_T(t)) \right) \right] \leq C. \tag{14}
\]

Corollary 3.1. Under the condition of Proposition 3.3, there exists a constant \( C = C(\alpha, Q, u_0) \) for any \( p \geq 1 \) such that

\[
\left\| \exp \left( 2 \int_0^T \| u(s) \|_{L^\infty} \| u^D_T(s) \|_{L^\infty} \, ds \right) \right\|_{L^p(\Omega)} \leq C. \tag{15}
\]

and

\[
\left\| \exp \left( 2 \sum_{m \in Z \times \mathbb{M}} \| u^T(t_m) \|_{L^\infty} \| u^T_m \|_{L^\infty} \right) \right\|_{L^p(\Omega)} \leq C. \tag{16}
\]

Proof. By the Cauchy–Schwarz, Gagliardo–Nirenberg and Young inequalities, for \( 0 < \eta < 1 \) we have

\[
\left\| \exp \left( 2 \int_0^T \| u(s) \|_{L^\infty} \| u^D_T(s) \|_{L^\infty} \, ds \right) \right\|_{L^p(\Omega)} \leq \left\| \exp \left( \int_0^T 2e^{-at} \| u_0 \| \| \nabla u \| \, ds \right) \right\|_{L^{2p}(\Omega)} \left\| \exp \left( \int_0^T 2e^{-at} \| u_0 \| \| \nabla u^D_T \| \, ds \right) \right\|_{L^{2p}(\Omega)} \leq 2^p E \left[ \exp \left( \int_0^T \frac{4p \sqrt{2}}{\sqrt{1-\eta}} e^{-\left( a-\frac{\beta}{2} \right)^2} \| u_0 \| e^{-\frac{\beta}{2} t} \sqrt{\frac{2}{1-\eta} \| \nabla u \|} \, ds \right) \right] \cdot 2^p E \left[ \exp \left( \int_0^T \frac{4p \sqrt{2}}{\sqrt{1-\eta}} e^{-\left( a-\frac{\beta}{2} \right)^2} \| u_0 \| e^{-\frac{\beta}{2} t} \sqrt{\frac{2}{1-\eta} \| \nabla u \|} \, ds \right) \right],
\]

where \( \beta < 2a \) is as presented in Proposition 3.3. Then the Jensen, Minkovski and
Hölder inequalities yield that
\[
\left\| \exp \left( 2 \int_0^T \| u(s) \|_{L^\infty} \| u_P^2(s) \|_{L^\infty} ds \right) \right\|_{L_p(\Omega)}
\leq 2^p \sup_{t \in [0,T]} E \left[ \exp \left( \frac{4p \sqrt{2} (1 - e^{-(a - \frac{\beta}{2})T})}{\sqrt{(1 - \eta)(a - \frac{\beta}{2})}} \| u_0 \| e^{-\frac{\beta t}{2}} \sqrt{\frac{1 - \eta}{2}} \| \nabla u \| \right) \right]
\leq C(a, \beta, \eta, \| u_0 \|) \sup_{t \in [0,T]} E \left[ \exp \left( \frac{(1 - \eta)e^{-\beta t}}{2} \| \nabla u(t) \|^2 - \frac{e^{-\beta t}}{8\eta} \| u(t) \|^6 \right) \right]
\leq C(a, \beta, \eta, \| u_0 \|) \sup_{t \in [0,T]} E \left[ \exp \left( e^{-\beta t} H(u(t)) \right) \right] \sup_{t \in [0,T]} E \left[ \exp \left( e^{-\beta t} H(u_P^2(t)) \right) \right].
\]

From the above estimations, Propositions 2.3 and 3.3 and Remark 3.1 yield (15). Next, we turn to the discrete case (16). Similarly, the Hölder, Gagliardo–Nirenberg and Jensen inequalities yield that
\[
\left\| \exp \left( 2\tau \sum_{m \in Z_M} \| u_r(t_m) \|_{L^\infty} \| u_m \|_{L^\infty} \right) \right\|_{L_p(\Omega)}
\leq 2^p E \left[ \exp \left( 4p \tau \sum_{m \in Z_M} e^{-(a - \frac{\beta}{2})t_m} \| u_0 \| e^{-\frac{\beta t_m}{2}} \| \nabla u_r(t_m) \| \right) \right]
\leq \sup_{m \in Z_M} E \left[ \exp \left( 4p \sqrt{2} \frac{1 + (a - \frac{\beta}{2})\tau}{\sqrt{1 - \eta(a - \frac{\beta}{2})}} \| u_0 \| e^{-\frac{\beta t_m}{2}} \sqrt{\frac{1 - \eta}{2}} \| \nabla u_r(t_m) \| \right) \right]
\leq C(a, \beta, \eta, u_0) \sup_{m \in Z_M} E \left[ \exp \left( e^{-\beta t_m} H(u_r(t_m)) \right) \right].
\]

Based on these a priori estimations and exponential integrability, we can deduce the strong convergence rate for the splitting Crank–Nicolson type scheme. We remark that when the damped assumption \( \sup_{x \in \mathbb{R}} (\frac{1}{2} F Q(x) - \alpha(x)) \leq -\alpha \) does not hold, the
strong convergence rate of the proposed scheme can also be obtained. However, we
cannot expect that the constant \( C \) in the upper estimate of the strong convergence
rate to be independent of time since the a priori estimate depends on the time interval.
A similar situation occurs when we study the weak order of the proposed scheme.

**Theorem 3.1.** Let \( \alpha \in \mathbb{H}^2 \), \( \sup_{x \in \mathbb{R}} (\frac{1}{2} F_Q(x) - \alpha(x)) \leq -\alpha \), and \( Q \frac{1}{2} \in L_2^2 \). Then for
\( p \geq 1 \), there exists a constant \( C = C(\alpha, Q, u_0, p) \) such that
\[
\mathbb{E} \left[ \sup_{m \in \mathbb{Z}_M} \| u(t_m) - u^m \|^p \right] \leq C \tau^\frac{p}{2}.
\]

**Proof.** For simplicity, we give the proof for \( p = 2 \). The proof of case \( p > 2 \) can
be similarly obtained by using a priori estimates in higher \( p \)-moments of numerical
and exact solutions in Sobolev norms. Similar to the proof in [12], we split the error
\[
\mathbb{E} \left[ \sup_{m \in \mathbb{Z}_M} \| u(t_m) - u^m \|^2 \right]
\]
as follows:
\[
\mathbb{E} \left[ \sup_{m \in \mathbb{Z}_M} \| u(t_m) - u^m \|^2 \right] 
\leq 2 \mathbb{E} \left[ \sup_{m \in \mathbb{Z}_M} \| u(t_m) - u_r(t_m) \|^2 \right] + 2 \mathbb{E} \left[ \sup_{m \in \mathbb{Z}_M} \| u_r(t_m) - u^m \|^2 \right].
\]
Denote \( e_m := u(t_m) - u_r(t_m) \), \( \tilde{e}_m := u_r(t_m) - u_m \). We first estimate the first term
\[
\mathbb{E} \left[ \sup_{m \in \mathbb{Z}_M} \| e_m \|^2 \right].
\]
By the Itô formula, the definition of \( u_r \), the Gagliardo–Nirenberg
inequality, and arguments similar to [12, Theorem 2.2], we get
\[
\| e_{m+1} \|^2 \leq \| e_m - \int_{t_m}^{t_{m+1}} i \left[ \Delta u^D_{r,m} + \lambda u^D_{r,m} u^D_{r,m} \right] dr \|^2
\]
\[
+ 2 \int_{t_m}^{t_{m+1}} \left\langle u - u^S_{r,m} , i \left[ \Delta u + \lambda |u|^2 u \right] \right\rangle ds
\]
\[
+ (1 - (a + \epsilon) \tau) \| e_m \|^2 + 2 \left\langle e_m , \int_{t_m}^{t_{m+1}} i \left[ \Delta u - \Delta u^D_{r,m} + \lambda u^2 u - \lambda u^D_{r,m} u^D_{r,m} \right] dr \right\rangle
\]
\[
+ C(\epsilon, u_0) \tau \int_{t_m}^{t_{m+1}} \left[ \left\| u^D_{r,m} \right\|^2_{L^2} + \left\| u^S_{r,m} \right\|^2_{L^2} + \left\| u_r \right\|^2_{L^2} + \left\| u \right\|^2_{L^2} \right] ds
\]
\[
+ C(\epsilon, u_0) \left( \int_{t_m}^{t_{m+1}} \| W(s) - W(t_m) \|_{L^1} \left( 1 + \| u \|^2_{L^2} \right) ds \right.
\]
\[
+ \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^{s} (u(r) - u^S_{r,m}(r)) dW(r) \right\|^2 ds + R^1_m + R^2_m + R^3_m \right).
\]
where
\[
R^1_m := \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^{r} \int_{t_m}^{r} i \left[ \Delta u + i \lambda |u|^2 u - \alpha (u - u^S_{r,m}) \right] dr_1 dW(r) \right\| \left( 1 + \| u \|^2_{L^2} \right) ds,
\]
\[
R^2_m := \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^{r} \int_{t_m}^{r} \left[ u(r_1) - u^S_{r,m}(r_1) \right] dW(r_1) dW(r) \right\| \left( 1 + \| u \|^2_{L^2} \right) ds,
\]
\[
R^3_m := \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^{r} \left( \int_{t_m}^{r} i \left[ \Delta u + \lambda |u|^2 u \right] dr_1 + \int_{t_m}^{r} i \left[ \Delta u_r + \lambda |u_r|^2 u_r \right] dr_1 \right) dW(r) \right\| \left( 1 + \| u \|^2_{L^2} \right) ds.
\]
Integrating by parts, we get
\[
2 \left\langle e_m, \int_{t_m}^{t_{m+1}} i \left[ \Delta u - \Delta u^D_{\tau,m} + \lambda |u|^2 u - \lambda |u^D_{\tau,m}|^2 u^D_{\tau,m} \right] ds \right\rangle \\
= 2 \int_{t_m}^{t_{m+1}} \left\langle \Delta e_m, i \left[ u - u^D_{\tau,m} \right] \right\rangle ds + 2\lambda \int_{t_m}^{t_{m+1}} \left\langle e_m, i \left[ |u|^2 u - |u^D_{\tau,m}|^2 u^D_{\tau,m} \right] \right\rangle ds,
\]

The Hölder inequality, cubic difference formula \(|a|^2 a - |b|^2 b = (|a|^2 + |b|^2)(a - b) + ab(\overrightarrow{a} - \overrightarrow{b})\), the Cauchy-Schwarz and Gagliardo-Nirenberg inequalities imply
\[
\left\langle e_m, \int_{t_m}^{t_{m+1}} i \left[ \Delta u - \Delta u^D_{\tau,m} + \lambda |u|^2 u - \lambda |u^D_{\tau,m}|^2 u^D_{\tau,m} \right] ds \right\rangle \\
\leq \left( \frac{\epsilon}{2} + \int_{t_m}^{t_{m+1}} \| u \|_{L^\infty} \| u^D_{\tau,m} \|_{L^\infty} ds \right) \| e_m \|^2 + C(\epsilon) \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^{s} u(r) dW(r) \right\|_{L^2}^2 ds \\
+ C(\epsilon, u_0) \int_{t_m}^{t_{m+1}} \left( \| u(s) \|_{L^2}^2 + \| u^D_{\tau,m}(s) \|_{L^2}^2 \right) \left( \tau \int_{t_m}^{s} \left( 1 + \| u(r) \|_{L^2}^2 + \| u^D_{\tau,m}(r) \|_{L^2}^2 \right) dr \\
+ \left\| \int_{t_m}^{s} u(r) dW(r) \right\|_{L^2}^2 \right) ds + C(\epsilon, u_0) \tau \int_{t_m}^{t_{m+1}} \left[ 1 + \| u^D_{\tau,m} \|_{L^2}^2 + \| u_{\tau,m} \|_{L^2}^2 + \| u \|_{L^2}^2 \right] ds.
\]

Thus we conclude that
\[
\| e_{m+1} \|^2 \leq \| e_m \|^2 + \left( -(a - 2\epsilon)\tau + 2 \int_{t_m}^{t_{m+1}} \| u(s) \|_{L^\infty} \| u^D_{\tau,m}(s) \|_{L^\infty} ds \right) \| e_m \|^2 \\
+ C(\tau) \int_{t_m}^{t_{m+1}} \left[ 1 + \| u^D_{\tau,m} \|_{L^2}^4 + \| u_{\tau,m} \|_{L^2}^4 + \| u \|_{L^2}^4 \right] \| e_m \|^2 \\
+ R^m_1 + R^m_2 + R^m_3 + C \left( \int_{t_m}^{t_{m+1}} \| W(s) - W(t_m) \|_{L^2} \left( 1 + \| u \|_{L^2}^2 \right) ds \right) \\
+ C \left( \int_{t_m}^{t_{m+1}} \left\| \int_{t_m}^{s} (u(r) - u^S_{\tau,m}(r)) dW(r) \right\|_{L^2}^2 ds \right) \\
+ C \left( \int_{t_m}^{t_{m+1}} \left( 1 + \| u(s) \|_{L^2}^2 + \| u^D_{\tau,m}(s) \|_{L^2}^2 \right) \left\| \int_{t_m}^{s} u(r) dW(r) \right\|_{L^2}^2 ds \right) \\
=: \| e_m \|^2 + \left( -(a - 2\epsilon)\tau + 2 \int_{t_m}^{t_{m+1}} \| u(s) \|_{L^\infty} \| u^D_{\tau,m}(s) \|_{L^\infty} ds \right) \| e_m \|^2 \\
+ R^m_0 + R^m_1 + R^m_2 + R^m_3 + R^m_4 + R^m_5 + R^m_6.
\]

Then repeating the above procedures yields that
\[
\| e_{m+1} \|^2 \leq \exp \left( -(a - 2\epsilon)(m + 1)\tau + \int_{t_0}^{t_{m+1}} \| u(s) \|_{L^\infty} \| u^D_{\tau,m}(s) \|_{L^\infty} ds \right) \| e_0 \|^2 \\
+ \sum_{k=1}^{m+1} \sum_{i=0}^6 \exp \left( -(a - 2\epsilon)(m + 1 - k)\tau + \int_{t_k}^{t_{m+1}} \| u(s) \|_{L^\infty} \| u^D_{\tau,m}(s) \|_{L^\infty} ds \right) R^i_k^{-1}.
\]
where terms $R^2_i$, $i = 0, \ldots, 6$, $j = 0, \ldots, M - 1$, can be controlled by Propositions 2.2, 3.1, and 3.2. We omit the detailed computations which are similar to [12, Lemma 2.4] and obtain $\|R^2_i\|_{L^2(\Omega)} \leq C\tau^2$. The exponential moment is bounded by the estimation (15) of Corollary 3.1. Thus we get for $\tau < 1$,

$$E[\|\epsilon_{m+1}\|^2] \leq C\tau^2 \frac{1 - \exp \left( - (a - 2\epsilon)T \right)}{1 - \exp \left( - (a - 2\epsilon)\tau \right)} \leq C\tau.$$ 

In fact, we can obtain the stronger result,

$$\sup_{m \in \mathbb{Z}_M} E \left[ \|\epsilon_{m+1}\|^2 \right] \leq \sum_{k=1}^{M} \sum_{i=0}^6 \left\| R_i^{k-1} \right\|_{L^2(\Omega)} \exp \left( - (a - 2\epsilon)(m + 1 - k)\tau \right) + \int_{t_k}^{t_M} \|u(s)\|_{L^\infty} \|u_{\tau,m}^D(s)\|_{L^\infty} ds \right\|_{L^2(\Omega)} \leq C\tau,$$

Next, we estimate the term $E \left[ \sup_{m \in \mathbb{Z}_M} \|\tilde{\epsilon}_{m}\|^2 \right]$. Similar to the previous arguments, we get

$$\|\tilde{\epsilon}_{m+1}\|^2 \leq \|\tilde{\epsilon}_{m}\|^2 + \tau \left[ (1 + \|u_{m+1}\|^6_{L^2}) + \int_{t_m}^{t_{m+1}} \left( \|u_{\tau,m}^S(r)\|^2_{L^2} + \|u_{\tau,m}^D(r)\|^2_{L^2} + \|u_{\tau,m}^D(r)\|^4_{L^2} \right) dr \right] \|\tilde{\epsilon}_{m}\|^2 = \|\tilde{\epsilon}_{m}\|^2 + \tau \|u_{\tau,m}^D(t_m)\|_{L^\infty} \|u_{m}\|_{L^\infty} \|\tilde{\epsilon}_{m}\|^2 + \tilde{R}_{m}$$

Then taking expectations on both sides and using the H"older inequality yields that

$$E \left[ \sup_{m \in \mathbb{Z}_M} \|\tilde{\epsilon}_{m+1}\|^2 \right] \leq \sum_{k=0}^{M-1} \left[ \exp \left( - (a - 2\epsilon)(m + 1 - k)\tau \right) + \int_{t_k}^{t_{k+1}} \left( \|u(s)\|_{L^\infty} \|u_{\tau,m}^D(s)\|_{L^\infty} ds \right) \right] \|\tilde{R}_k\|_{L^2(\Omega)}.$$

Then the estimation (16) in Corollary 3.1, combined with a priori estimations in Propositions 2.2, 3.1 and 3.2, implies that

$$E \left[ \sup_{m \in \mathbb{Z}_M} \|\tilde{\epsilon}_{m}\|^2 \right] \leq C\tau.$$ 

From the estimations about $\epsilon_{m}$ and $\tilde{\epsilon}_{m}$, we obtain the strong error estimate

$$E \left[ \sup_{m \in \mathbb{Z}_M} \|u(t_m) - u_{m}\|^2 \right] \leq C\tau.$$
4. Weak convergence. In this section, we first study the regularity of the Kolmogorov equation of (1). With the help of this Kolmogorov equation, we transform the weak error into two parts, one is from the splitting approach and the other is from the deterministic Crank–Nicolson type discretization. As a consequence, the rate of weak convergence is shown to be twice that of strong convergence. This is the first result about the weak order of numerical schemes approximating the stochastic nonlinear Schrödinger equation with nonmonotone coefficients.

It is well known that \( U(t, u_0) := \mathbb{E} [\phi(u(t, u_0))] \) satisfies the following infinite-dimensional Kolmogorov equation (see e.g. [17]):

\[
\begin{aligned}
\frac{dU}{dt}(t, u) &= \frac{1}{2} \text{tr} \left( (iuQ^\perp)(iuQ^\perp)^* D^2 U(t, u) \right) + \langle i\Delta u + \alpha u, DU(t, u) \rangle, \\
U(0, u) &= \phi(u).
\end{aligned}
\]

In this section, we assume that \( \phi \in C^3_\alpha(\mathbb{H}^1) \cap C^1_\alpha(\mathbb{H}) \), \( s \geq 2, Q^\perp \in \mathcal{L}_2^s, u_0 \in \mathbb{H}^s, \alpha \in \mathbb{H}^2 \), and \( \sup_{x \in \mathbb{R}} (|F_Q(x)| - \alpha(x)) \leq -a \). To remove the infinitesimal factor, we first eliminate the unbounded Laplacian operator and consider \( V(t, v) = U(t, S(-t)v) \). Direct calculations show that \( V \) satisfies

\[
\begin{aligned}
\frac{dV}{dt}(t, v) &= \frac{1}{2} \text{tr} \left( (S(t)(\bar{S}(S(-t)v)Q^\perp))(S(t)(\bar{S}(S(-t)v)Q^\perp))^* D^2 V(t, v) \right) \\
&\quad + \langle \lambda S(t)(S(-t)v)^2(S(-t)v), DV(t, v) \rangle - \langle S(t)\alpha S(-t)v, DV(t, v) \rangle, \\
V(0, v) &= \phi(v).
\end{aligned}
\]

Now, it can be shown that the functions \( U \) and \( V \) have the same regularity as the initial data \( \phi \). Proposition 2.3 is the key to proving the following regularity result, which generalizes the case of Lipschitz drift operators in [17].

**Lemma 4.1.** The functions \( U \) and \( V \) are continuous in time with values in \( C^3(\mathbb{H}^1) \cap C^1(\mathbb{H}) \).

**Proof.** Differentiating \( U \), we obtain for \( h \in \mathbb{H} \),

\[
\langle DU(t, u_0), h \rangle = \mathbb{E} \left[ \langle D\phi(u(t, u_0)), \eta^h(t) \rangle \right],
\]

where

\[
\begin{aligned}
\eta^h(t) &= i\Delta \eta^h(t) + i\lambda \left( |u|^2 \eta^h + 2\mathbb{R}(\bar{u}\eta^h) \right) dt - \alpha \eta^h(t) dt + i\eta^h dW(t) \\
\eta^h(0) &= h.
\end{aligned}
\]

The Itô formula yields that

\[
\begin{aligned}
\frac{1}{2} \| \eta^h(t) \|^2 &= \frac{1}{2} \| h \|^2 + \int_0^t \left( \langle \eta^h, i\Delta \eta^h \rangle + \langle \eta^h, i\lambda \left( 2|u|^2 \eta^h + u^2 \bar{\eta}^h \right) \rangle - \langle \eta^h, \alpha \eta^h \rangle \right) dt \\
&\quad + \int_0^t \langle \eta^h, -i\eta^h dW(r) \rangle + \int_0^t \frac{1}{2} \text{tr} \left( (-i\eta^h Q^\perp)(-i\eta^h Q^\perp)^* \right) \| d\eta^h \|^2 dr \\
&\leq \frac{1}{2} \| h \|^2 + \int_0^t \langle \eta^h, i\lambda u^2 \bar{\eta}^h \rangle dr - a \int_0^t \| \eta^h \|^2 dr.
\end{aligned}
\]

By the Gronwall inequality, we obtain

\[
\| \eta^h(t) \|^2 \leq \exp(-2at) \exp \left( \int_0^T 2 \| u \|^2 \| \bar{\eta}^h \|^2 dr \right) \| h \|^2.
\]
Then taking expectation combined with Proposition 2.3 yields that

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \| \eta^h(t) \|^2 \right] \leq C(u_0) \| h \|^2.$$ 

Applying the Itô formula to $\| \eta^h \|^p$, $p \geq 2$, we get

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \| \eta^h(t) \|^p \right] \leq C(u_0) \| h \|^p,$$

which implies that

$$\| DU(t, u_0) \|_{L(H, \mathbb{R})} \leq C(u_0) \| \phi \|_{C^1_0(H)}.$$

Similarly,

$$D^2 U(t, u_0) \cdot (h, h) = \mathbb{E} \left[ D^2 \phi(u(t, u_0)) \cdot (\eta^h(t), \eta^h(t)) + D\phi(u(t, u_0)) \cdot \xi^h(t) \right]$$

with

$$\begin{cases}
    d\xi^h = i \Delta \xi^h \, dt + \lambda (\Re(\bar{u} \eta^h) \eta^h + 2|\eta^h|^2 u) \, dt \\
    + i \lambda (|u|^2 \xi^h + 2\Re(\bar{u} \xi^h) u) \, dt - \alpha \xi^h \, dt + i \xi^h \, dW(t), \\
    \xi^h(0) = 0.
\end{cases}$$

Again by the Itô formula, we obtain

$$\frac{1}{2} \| \xi^h(t) \|^2$$

$$= \int_0^t \left( \langle \xi^h, i \Delta \xi^h \rangle + \langle \xi^h, i \lambda (\Re(\bar{u} \eta^h) \eta^h + 2|\eta^h|^2 u) \rangle \\
    + \langle \xi^h, i \lambda (|u|^2 \xi^h + 2\Re(\bar{u} \xi^h) u) \rangle > -\langle \xi^h, \alpha \xi^h \rangle \right) \, dr$$

$$+ \int_0^t \langle \xi^h, -i \xi^h \, dW(r) \rangle + \frac{1}{2} \int_0^t \text{tr}([-i \xi^h Q^\frac{1}{2} (-i \xi^h Q^\frac{1}{2})^*)] \, dr$$

$$\leq \int_0^t \left( \langle \xi^h, i \lambda (\Re(\bar{u} \eta^h) \eta^h + 2|\eta^h|^2 u) \rangle + \langle \xi^h, i \lambda 2\Re(\bar{u} \xi^h) u \rangle \right) \, dr - a \int_0^t \| \xi^h \|^2 \, dr$$

$$\leq \int_0^t - (a - c) \| \xi^h \|^2 + 2 \| u \|^2 \| \xi^h \|^2 \, dr + \int_0^t C(\epsilon) \| u \|^2 \| \eta^h \|^2 \| \nabla \eta^h \|^2 \, dr.$$ 

Then the Gronwall inequality and the charge evolution of $u$ imply that

$$\| \xi^h(t) \|^2 \leq C \exp \left( \int_0^T 4\| u \|_L^2 \, dr \right) \int_0^T e^{-2ar} \| u_0 \|^2 \| \eta^h \|^2 \| \nabla \eta^h \|^2 \, dr.$$ 

After taking expectation, by Proposition 2.3, we have

$$\mathbb{E} \left[ \sup_{t \in [0,T]} \| \xi^h(t) \|^2 \right] \leq C(u_0) \sqrt{\sup_{t \in [0,T]} \mathbb{E} \left[ \| \eta^h(t) \|^4 \right]} \sqrt{\sup_{t \in [0,T]} \mathbb{E} \left[ \| \nabla \eta^h(t) \|^4 \right]}.$$ 

This manuscript is for review purposes only.
We need to show $\mathbb{E}[\|\nabla \eta^h(t)\|^p] < \infty$. For simplicity, we give the proof for $p = 2$. The proof of $p > 2$ is similar to the previous arguments for $p > 2$ in the a priori estimate of $u$ in the $H^1$-norm. The Itô formula, integration by parts, and the Gagliardo–Nirenberg and Young inequalities show that

\[
\frac{1}{2}\|\nabla \eta^h(t)\|^2
= \frac{1}{2}\|\nabla h\|^2 + \int_0^t \left( -\Delta \eta^h, i\Delta \eta^h + i\lambda(|u|^2 \eta^h + 2\Re(\bar{u}\eta^h)u) - \alpha \eta^h \right) \, dt
+ \int_0^t \left( -\Delta \eta^h, i\eta^h dW(r) \right) + \frac{1}{2} \int_0^t \text{tr} \left( (-i\nabla(\eta^h \nabla^\perp))(-i\nabla(\eta^h \nabla^\perp))^\ast \right) \, dt
= \frac{1}{2}\|\nabla h\|^2 + \int_0^t \lambda(\nabla \eta^h, i2\Re(\bar{u}\nabla u)\eta^h + i2\Re(\bar{u}\nabla \eta^h)u + i2\Re(\bar{u}\eta^h)\nabla u) \, dt
- \alpha \int_0^t \|\nabla \eta^h(t)\|^2 + \int_0^t \langle \nabla \eta^h, \nabla(\frac{1}{2} \nabla F_Q - \nabla \alpha) \rangle \, dt + \int_0^t \langle \nabla \eta^h, i\eta^h d\nabla W(r) \rangle
+ \int_0^t \frac{1}{2} \sum_k \langle \eta^h \nabla Q^\perp e_k, \eta^h \nabla Q^\perp e_k \rangle \, dt.
\]

The Gronwall inequality implies that for $s \in [0, t]$,

\[
\|\nabla \eta^h(t)\|^2 \leq \exp \left( -2(\alpha - \epsilon) t + \int_0^T 4\|u\|^2_{L^\infty} \, dt \right) \left( \|\nabla h\|^2 + C(\epsilon, \alpha, Q) \int_0^T (\|\Delta u\|\|\nabla u\|^2 \|u\| + 1)\|\eta^h\|^2 \, dt \right)
+ \sup_{s \in [0, t]} \int_0^s \langle \nabla \eta^h, i\eta^h d\nabla W(r) \rangle.
\]

Then taking expectation, combined with Corollary 2.1, Propositions 2.2 and 2.3, the estimation (18), and the Burkholder–Davis–Gundy, Hölder, and Young inequalities, leads that for $\frac{1}{p} + \frac{1}{q} = 1$, $1 < q < 2$,

\[
\mathbb{E}[\|\nabla \eta^h(t)\|^2] 
\leq e^{-2(\alpha - \epsilon)t} \left\| \exp \left( \int_0^T 4\|u\|^2_{L^\infty} \, dt \right) \right\|_{L^p(\Omega)} \left\| \sup_{s \in [0, t]} \int_0^s \langle \nabla \eta^h, i\eta^h d\nabla W(r) \rangle \right\|_{L^q(\Omega)}
\|\nabla h\|^2 + C(\epsilon, \alpha, Q) \int_0^T (\|\Delta u\|\|\nabla u\|^2 \|u\| + 1)\|\eta^h\|^2 \, dt \right)
\leq e^{-2(\alpha - \epsilon)t} C(p, \alpha, Q, u_0) \left( \sqrt{\mathbb{E} \left( \int_0^T \|\nabla \eta^h\|^2 \, dt \right)^2} \sup_{s \in [0, T]} \|\eta^h(s)\|^q + \|\nabla h\|^2 + \|h\|^2 \right)
\leq e^{-2(\alpha - \epsilon)t} C(p, \alpha, Q, u_0) \left( \int_0^t e \mathbb{E} \left[ \|\nabla \eta^h\|^2 \right] \, dt + C(\epsilon) (\|\nabla h\|^2 + \|h\|^2) \right).
\]

Applying again the Gronwall inequality, we get

\[
\mathbb{E}[\|\nabla \eta^h(t)\|^2] \leq C(\epsilon, \alpha, Q, u_0) (\|h\|^2 + \|\nabla h\|^2).
\]

Similar arguments yield that for any $p \geq 2$,

\[
\|\nabla \eta^h(t)\|_{L^p(\mathbb{H})} \leq C(u_0) (\|h\| + \|\nabla h\|).
\]
Then we conclude that
\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \xi^h(t) \|^p \right] \leq C(u_0) \| h \|^p \| \nabla h \|^p,
\]
which implies that
\[
\| D^2 U(t, u_0) \|_{\mathcal{L}(\mathbb{H}^1 \times \mathbb{H}^1 ; \mathbb{R})} \leq C(u_0) \max(\| \phi \|_{C^2_0(\mathbb{H}^1)} , \| \phi \|_{C^4_0(\mathbb{H}^1)}).
\]
For the function \( V(t, v) = U(t, S(-t)v) \), we have
\[
\langle DV(t, u_0), h \rangle_{\mathbb{H}} = \mathbb{E} \left[ \langle D\phi(u(t, S(-t)u_0)), \eta^h \rangle \right],
\]
and
\[
DV(t, u_0) \cdot (h, h) = \mathbb{E} \left[ D^2\phi(u(t, S(-t)u_0)) \cdot (\eta^h(t), \eta^h(t)) + D\phi(u(t, S(t)u_0)) \cdot \xi^h(t) \right].
\]
The unitarity of \( S(t) \), i.e., \( \| S(t)u_0 \|_{\mathbb{H}^s} = \| u_0 \|_{\mathbb{H}^s} \), \( s \in \mathbb{N} \), combined with previous arguments finishes the proof. Similar arguments yield that \( U \) and \( V \) belong to \( C^4(\mathbb{H}^1) \).

**Remark 4.1.** The above procedures imply the global existence of variational solutions of stochastic NLS equations, which in turn gives the theoretical support to why the phase flow, in any finite time, preserves the symplectic structure when \( \alpha = \frac{1}{2} F_Q \) and the conformal symplectic structure when \( \alpha = a + \frac{1}{2} F_Q \) (see, e.g., [7, 22]).

Based on the estimations in Lemma 4.1 and the corresponding Kolmogorov equation, we have the following weak convergence result.

**Theorem 4.1.** Assume that \( \alpha \in \mathbb{H}^4 \), \( \| Q^+ \|_{\mathcal{L}_2^{1}} < \infty \), and \( u_0 \in \mathbb{H}^4 \). For any \( \phi \in C^3_0(\mathbb{H}^1) \cap C^4_0(\mathbb{H}) \), there exists a positive constant \( C = C(\alpha, Q, u_0, \phi) \) such that
\[
\| \mathbb{E} [\phi(u(T))] - \mathbb{E} [\phi(u_M)] \| \leq C \tau.
\]

We aim to give the representation formula of the weak error and split \( \mathbb{E} [\phi(u(T))] - \mathbb{E} [\phi(u_M)] \) as follows:
\[
\mathbb{E} [\phi(u(T))] - \mathbb{E} [\phi(u_M)] = \mathbb{E} [\phi(u(T))] - \mathbb{E} [\phi(u_\tau(T))] + \mathbb{E} [\phi(u_\tau(T))] - \mathbb{E} [\phi(u_M)].
\]
The following lemmas show that the estimate (19) holds.

**Lemma 4.2.** Assume that \( \alpha \in \mathbb{H}^2 \), \( \| Q^+ \|_{\mathcal{L}_2^{1}} < \infty \) and \( u_0 \in \mathbb{H}^2 \). For any \( \phi \in C^3_0(\mathbb{H}^1) \cap C^4_0(\mathbb{H}) \), there exists a positive constant \( C = C(\alpha, Q, u_0, \phi) \) such that
\[
\| \mathbb{E} [\phi(u(T))] - \mathbb{E} [\phi(u_\tau(T))] \| \leq C \tau.
\]

**Proof.** First we split the error by the local arguments as follows:
\[
\mathbb{E} [\phi(u_\tau(T))] - \mathbb{E} [\phi(u(T))] = \sum_{k=0}^{M-1} \left( \mathbb{E} [V(T - t_{k+1}, v_\tau(t_{k+1}))] - \mathbb{E} [V(T - t_k, v_\tau(t_k))] \right),
\]
where \( v_\tau(t) = S(T - t)u_\tau(t) \). The definition of \( u_\tau \) yields that
\[
S(T - t_{k+1})u_\tau(t_{k+1}) = S(T - t_k)u_\tau(t_k) + \int_{t_k}^{t_{k+1}} S(T - t) i u_\tau^D(t) |u_\tau^D(t)|^2 u_\tau^D(t) dt \\
+ \int_{t_k}^{t_{k+1}} S(T - t) i u_\tau^S(t) dW(t) - \int_{t_k}^{t_{k+1}} S(T - t) a u_\tau^S(t) dt.
\]
With the help of the Kolmogorov equation (17), the mean value theorem, and the Itô formula, we get

\[
V(T-t_{k+1}, v_r(t_{k+1})) - V(T-t_k, v_r(t_k)) \\
= - \int_{t_k}^{t_{k+1}} \frac{dV}{dt} \, dt + \int_{t_k}^{t_{k+1}} \frac{1}{2} \text{tr} \left[ (S(T-t)(i(S(-T+t)v_r)Q^\frac{1}{2})) (S(T-t)(i(S(-T+t)v_r)Q^\frac{1}{2}))^* D^2V(T-t, v_r) \right] \, dt \\
+ \int_{t_k}^{t_{k+1}} \left\langle \lambda i S(T-t)(|u_r^D|^2 u_r^D), \int_0^1 DV(T-t_k, v_r(t_k)) + \theta \int_{t_k}^{t_{k+1}} S(T-t) i |u_r^D(s)|^2 u_r^D(s) \, ds \right\rangle \, dt \\
+ \int_{t_k}^{t_{k+1}} \left\langle S(T-t) i u_r^D(t) dW(t), DV(T-t, v_r) \right\rangle \\
= \int_{t_k}^{t_{k+1}} \left\langle \lambda i S(T-t)(|u_r^D|^2 u_r^D), \int_0^1 DV(T-t_k, v_r(t_k)) + \theta \int_{t_k}^{t_{k+1}} S(T-t) i |u_r^D(s)|^2 u_r^D(s) \, ds \right\rangle \, dt \\
- \int_{t_k}^{t_{k+1}} \left\langle \lambda i S(T-t)(|u_r|^2 u_r) - i S(T-t)(|u_r^D|^2 u_r^D), DV(T-t, v_r(t_k)) \right\rangle \, dt \\
- \int_{t_k}^{t_{k+1}} \left\langle \lambda i S(T-t)(|u_r|^2 u_r), DV(T-t, v_r(t_k)) - DV(T-t, v_r(t_k)) \right\rangle \, dt
\]

Then taking expectation shows that

\[
E\left[ V(T-t_{k+1}, v_r(t_{k+1})) - V(T-t_k, v_r(t_k)) \right] \\
= E\left[ \int_{t_k}^{t_{k+1}} \left\langle S(T-t) i |u_r^D|^2 u_r^D, \int_0^1 DV(T-t_k, v_r(t_k)) + \theta \int_{t_k}^{t_{k+1}} S(T-t) i |u_r^D(s)|^2 u_r^D(s) \, ds \right\rangle \, dt \\
- \int_{t_k}^{t_{k+1}} \left\langle \lambda i S(T-t)(|u_r|^2 u_r) - i S(T-t)(|u_r^D|^2 u_r^D), DV(T-t, v_r(t_k)) \right\rangle \, dt \\
- \int_{t_k}^{t_{k+1}} \left\langle \lambda i S(T-t)(|u_r|^2 u_r), DV(T-t, v_r(t_k)) - DV(T-t, v_r(t_k)) \right\rangle \, dt \right] := E[\mathcal{W}_1 + \mathcal{W}_2 + \mathcal{W}_3].
\]

Since $u_r$ and $u_r^D$ are both predictable, combining them with the continuity of $D^2V$ and $D^3V$ and the expansion of $DV$, we get for some $b_1 > 0$,

\[
E[\mathcal{W}_1] \leq C(u_0)^2 \tau^2 \sup_{t \in [0,T]} \sqrt{E[||u_r^D(t)||_{H_1}^4]} \\
+ E\left[ \int_{t_k}^{t_{k+1}} \left\langle S(T-t) i |u_r|^2 u_r^D, DV(T-t_k, v_r(t_k)) - DV(T-t, v_r(t_k)) \right\rangle \, dt \right] \\
\leq C(u_0)^2 \tau^2 \sup_{t \in [0,T]} \sqrt{\left( E[||u_r^D(t)||_{H_1}^4] + E[||u_r^D(t)||_{H_1}^2] \right)} \leq Ce^{-b_1 t_k} \tau^2.
\]
The cubic difference formula yields that
\[
W_2 = -\int_{t_k}^{t_{k+1}} \left\langle \lambda \mathcal{S}(T - t) \left( |u_{\tau}|^2 + |u_{\tau}^D|^2 \right)(u_{\tau} - u_{\tau}^D), DV(T - t, v_{\tau}(t_k)) \right\rangle dt \\
- \int_{t_k}^{t_{k+1}} \left\langle \lambda \mathcal{S}(T - t) \left( u_{\tau} u_{\tau}^D (\overline{u_{\tau}} - u_{\tau}^D) \right), DV(T - t, v_{\tau}(t_k)) \right\rangle dt := W_{21} + W_{22}.
\]
The estimations of \(W_{21}\) and \(W_{22}\) are similar, we only give the estimate of first term. The expressions of \(u_{\tau}^D\) and \(u_{\tau}\) yield that
\[
W_{21} = \int_{t_k}^{t_{k+1}} \left\langle \lambda \mathcal{S}(T - t) \left( |u_{\tau}|^2 + |u_{\tau}^D|^2 \right)(u_{\tau} - u_{\tau}^D), DV(T - t, v_{\tau}(t_k)) \right\rangle dt \\
+ \int_{t_k}^{t_{k+1}} \left\langle \lambda \mathcal{S}(T - t) \left( |u_{\tau}|^2 + |u_{\tau}^D|^2 \right)(u_{\tau}^D(t_k) - u_{\tau}(t)), DV(T - t, v_{\tau}(t_k)) \right\rangle dt \\
= \int_{t_k}^{t_{k+1}} \left\langle \lambda \mathcal{S}(T - t) \left( |u_{\tau}|^2 + |u_{\tau}^D|^2 \right) \left[ (S(t_{k+1} - t_k) - I)u_{\tau}(t_k) + \int_{t_k}^{t_{k+1}} S(t - s)u_{\tau}^D(s)ds \right] \right\rangle dt \\
- \int_{t_k}^{t_{k+1}} \left\langle \lambda \mathcal{S}(T - t) \left( |u_{\tau}|^2 + |u_{\tau}^D|^2 \right) \right\rangle dt \\
\left\langle \left[ (S(t_{k+1} - t_k) - I)u_{\tau}(t_k) + \int_{t_k}^{t_{k+1}} S(t_{k+1} - s)u_{\tau}^D(s)ds \right] \right\rangle dt.
\]
Then the independence of the Wiener process and the property of the stochastic integral, together with the property of \(S(t)\), the boundedness of \(DV\), and the Gagliardo–Nirenberg inequality, imply that
\[
\mathbb{E}[W_{21}] \\
\leq C(u_0)^2 \sup_{t \in [t_k, t_{k+1}]} \sqrt{\mathbb{E} \left[ \left( |u_{\tau}(t)|^2 + |u_{\tau}^D(t)|^2 \right) \left( 1 + \left| u_{\tau}(t) \right|_{H^2}^4 + \left| u_{\tau}^D(t) \right|_{H^2}^4 + \left| u_{\tau}^D(t) \right|_{H^2}^4 + \left| u_{\tau}^D(t) \right|_{H^2}^4 \right) \right]} \\
\leq C e^{-\alpha t_k \tau^2}.
\]
Thus we can obtain \(\mathbb{E}[W_2] \leq C e^{-\alpha t_k \tau^2}\).

The boundedness of \(D^2V\), the continuity of \(u_{\tau}\) and \(v_{\tau}\) in the local interval, the property of the stochastic integral, and Lemma 3.1 and 3.2 imply for \(b_2 > 0\) that
\[
\mathbb{E}[W_3] \leq C \mathcal{E} \left[ \left( \int_{t_k}^{t_{k+1}} \left( \left| u_{\tau}(t) \right|_{H^2}^4 + \left| u_{\tau}(t_{k+1}) \right|_{H^2}^4 \right) \left| u_{\tau}(t_k) - u_{\tau}(t) \right|_{H^1} \left| v_{\tau}(t) - v_{\tau}(t_k) \right|_{H^1} dt \right)^2 \right] \\
- \mathcal{E} \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t} D^2V(T - s, v_{\tau}(s))dv_{\tau}(s)ds, i\lambda S(T - t)\left( |u_{\tau}(t_{k+1})|^2 u_{\tau}(t_k) \right) \right] dt \\
+ C \mathcal{E} \left[ \int_{t_k}^{t_{k+1}} \left\| \int_{0}^{1} D^2V(T - t, v_{\tau}(t) + \theta \int_{t_k}^{t_{k+1}} S(T - t) |u_{\tau}^D(s)|^2 u_{\tau}^D(s)ds) d\theta \right\| \right] \\
\times \left\| u_{\tau}(t_{k+1}) \right\|_{H^2}^3 \left\| \int_{t_k}^{t_{k+1}} S(T - t) |u_{\tau}^D(s)|^2 u_{\tau}^D(s)ds \right\|_{H^1} dt \\
\leq C e^{-b_2 t_k \tau^2}.
\]
The estimations of $\mathcal{W}_i$, $i = 1, 2, 3$, yield that

$$
\mathbb{E} [\phi(u(T))] - \mathbb{E} [\phi(u_\tau(T))] \leq C \sum_{k=0}^{M-1} e^{-\min(a,b_1,b_2)t_k} \tau^2 \leq C \tau.
$$

Next, we deal with the term $\mathbb{E} [\phi(u_\tau(T))] - \mathbb{E} [\phi(u_M)]$.

**Lemma 4.3.** Assume that $\alpha \in \mathbb{H}^4$, $\|Q^4\|_{C^1_\alpha} < \infty$, and $u_0 \in \mathbb{H}^4$. For any $\phi \in C^3_b(\mathbb{H}^3) \cap C^1_b(\mathbb{H})$, there exists a positive constant $C = C(\alpha, Q, u_0, \phi)$ such that

$$
|\mathbb{E} [\phi(u_\tau(T))] - \mathbb{E} [\phi(u_M)]| \leq C \tau.
$$

**Proof.** By the damped effect, we obtain for any $v, w \in \mathbb{H}$,

$$
\|\Phi^S_k(v - w)\| \leq C e^{a \tau} \|v - w\|.
$$

Then the total error are divided as follows:

$$
|\phi(u_\tau(T)) - \phi(u_M)| \leq C \|u_\tau(T) - u_M\| \\
\leq C \sum_{k=0}^{M-1} \prod_{j=1}^{M-k-1} (\Phi^S_{M-j} \Phi^D_{M-j}) (\Phi^S_k - \Phi^D_k) \prod_{l=0}^{k-1} (\Phi^S_{k-1-l} \Phi^D_{k-1-l}) u_0 \\
\leq C \sum_{k=0}^{M-1} \prod_{j=1}^{M-k-1} (\Phi^S_{M-j} \Phi^D_{M-j}) (\Phi^S_k - \Phi^D_k) u_\tau(T)_{k}.
$$

By the stability of $u_\tau^D$ in $\mathbb{H}^4$, we have

$$
\left\| (\Phi^D_k - \Phi^D_k) u_\tau^D (t_k) \right\| \leq \left\| (S(t_{k+1} - t_k) - S_\tau) u_\tau^D (t_k) \right\| \\
+ \int_{t_k}^{t_{k+1}} \left| S(t_{k+1} - s) u_\tau^D(s)^2 u_\tau^D(s) - \tau T_{\tau}, \frac{|\tilde{u}_\tau^D(t_k)|^2 + |\tilde{u}_{\tau+1}^D|^2}{2} \right| dt \\
\leq C \tau^2 \|u_\tau^D(t_k)\|_{\mathbb{H}^4} + C \tau \left( \int_{t_k}^{t_{k+1}} \|u_\tau^D(s)\|^p_{\mathbb{H}^4} ds + \tau \|u_\tau^D(t_k)\|^p_{\mathbb{H}^4} + \tau \|\tilde{u}_{\tau+1}^D\|^2_{\mathbb{H}^4} \right),
$$

where $S_\tau t = \frac{1 + \Delta \tau}{1 - \frac{1}{2} \Delta \tau}$, $T_\tau t = \frac{1}{1 - \frac{1}{2} \Delta \tau}$. After taking expectation, we see that charge evolutions and the continuous dependence on initial data of $\Phi^D_k, \Phi^D_k, \Phi^S_k$, $k \in Z_M$, together with the uniform boundedness of $\tilde{u}_k$ and $u_\tau^D$ in Proposition 3.1, and Lemma 3.2, imply that

$$
\mathbb{E} [\phi(u_\tau(T)) - \phi(u_M)] \leq C \sum_{k=0}^{M-1} e^{-a(T-t_{k+1}) \tau^2} \leq C \tau.
$$

**Remark 4.2.** Since we discretize the semigroup $S(\tau)$ by $S_\tau$, we need the same high regularity requirement on $u_0$ as in [17] to get a weak order result. This approach to analyze weak order of numerical scheme is also available for the conservative stochastic NLS equation ($\alpha = \frac{1}{2} F_Q$) and other cases, such as $\|\alpha\|_{\mathbb{H}^4} < \infty$ and for more general test functions with polynomial growths, i.e., $\phi \in C^3_p(\mathbb{H}^1) \cap C^1_p(\mathbb{H})$.

**References**

This manuscript is for review purposes only.
[1] R. Anton and D. Cohen, Exponential integrators for stochastic Schrödinger equations driven by ito noise, J. Comput. Math., to appear.
[2] R. Anton, D. Cohen, S. Larsson, and X. Wang, Full discretization of semilinear stochastic wave equations driven by multiplicative noise, SIAM J. Numer. Anal. 54 (2016), no. 2, 1093–1119. MR 3484300
[3] O. Bang, P. L. Christiansen, F. If, K. O. Rasmussen, and Y. B. Gaididei, White noise in the two-dimensional nonlinear Schrödinger equation, Appl. Anal. 57 (1995), no. 1-2, 3–15. MR 1382938
[4] V. Barbu, M. Röckner, and D. Zhang, Stochastic nonlinear Schrödinger equations, Nonlinear Anal. 136 (2016), 168–194. MR 3474409
[5] S. Becker, A. Jentzen, and P. E. Kloeden, An exponential Wagner-Platen type scheme for SPDEs, SIAM J. Numer. Anal. 54 (2016), no. 4, 2389–2426. MR 3544742
[6] Y. Cao, J. Hong, and Z. Liu, Approximating stochastic evolution equations with additive white and rough noises, SIAM J. Numer. Anal. 55 (2017), no. 4, 1958–1981. MR 3608750
[7] C. Chen and J. Hong, Symplectic Runge-Kutta Semidiscretization for Stochastic Schrödinger Equation, SIAM J. Numer. Anal. 54 (2016), no. 4, 2569–2593. MR 3542010
[8] C. Chen, J. Hong, and A. Prohl, Convergence of a θ-scheme to solve the stochastic nonlinear Schrödinger equation with Stratonovich noise, Stoch. Partial Differ. Equ. Anal. Comput. 4 (2016), no. 2, 274–318. MR 3498984
[9] S. Cox, M. Hutzenthaler, and A. Jentzen, Local lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations, arXiv:1309.5595.
[10] S. Cox and J. van Neerven, Convergence rates of the splitting scheme for parabolic linear stochastic Cauchy problems, SIAM J. Numer. Anal. 48 (2010), no. 2, 428–451. MR 2646103
[11] J. Cui, J. Hong, and Z. Liu, Strong convergence rate of finite difference approximations for stochastic cubic Schrödinger equations, J. Differential Equations 259 (2010), no. 7, 3087–3113. MR 3670334
[12] J. Cui, J. Hong, Z. Liu, and W. Zhou, Strong convergence rate of splitting schemes for stochastic nonlinear Schrödinger equations, arXiv:1701.05680.
[13] Stochastic symplectic and multi-symplectic methods for nonlinear Schrödinger equation with white noise dispersion, J. Comput. Phys. 342 (2017), 267–285. MR 3649275
[14] G. Da Prato, An introduction to infinite-dimensional analysis, Universitext, Springer-Verlag, Berlin, 2006, Revised and extended from the 2001 original by Da Prato. MR 2244975
[15] A. de Bouard and A. Debussche, A stochastic nonlinear Schrödinger equation with multiplicative noise, Comm. Math. Phys. 205 (1999), no. 1, 161–181. MR 1706888
[16] The stochastic nonlinear Schrödinger equation in H1, Stochastic Anal. Appl. 21 (2003), no. 1, 97–126. MR 1954077
[17] Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation, Appl. Math. Optim. 54 (2006), no. 3, 369–399. MR 2268663
[18] A. Debussche and C. Odasso, Ergodicity for a weakly damped stochastic non-linear Schrödinger equation, J. Evol. Equ. 5 (2005), no. 3, 317–356. MR 2174876
[19] P. Dörsek, Semigroup splitting and cubature approximations for the stochastic Navier-Stokes equations, SIAM J. Numer. Anal. 50 (2012), no. 2, 729–746. MR 2914284
[20] I. Gyöngy, Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I, Potential Anal. 9 (1998), no. 1, 1–25. MR 1644183
[21] I. Gyöngy and N. Krylov, On the splitting-up method and stochastic partial differential equations, Ann. Probab. 31 (2003), no. 2, 564–591. MR 1964941
[22] J. Hong, X. Wang, and L. Zhang, Numerical analysis on ergodic limit of approximations for stochastic NLS equation via multi-symplectic scheme, SIAM J. Numer. Anal. 55 (2017), no. 1, 305–327. MR 3605750
[23] M. Hutzenthaler and A. Jentzen, On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients, arXiv:1401.0265.
[24] M. Hutzenthaler, A. Jentzen, and X. Wang, Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations, Math. Comp 87 (2018), no. 311, 1353–1413.
[25] A. Jentzen and P. Pusnik, Exponential moments for numerical approximations of stochastic partial differential equations, arXiv:1608.07031.
[26] J. Liu, Order of convergence of splitting schemes for both deterministic and stochastic nonlinear Schrödinger equations, SIAM J. Numer. Anal. 51 (2013), no. 4, 1911–1932. MR 3072234