A NATURAL LIE SUPERALGEBRA BUNDLE ON RANK THREE WSD MANIFOLDS

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Abstract. We determine the structure of the ∗-Lie superalgebra generated by a set of carefully chosen natural operators of an orientable WSD manifold of rank three. This Lie superalgebra is formed by global sections of a natural Lie superalgebra bundle, and turns out to be a product of sl(4, C) with the full special linear superalgebras of some graded vector spaces isotypical with respect to a natural action of so(3, R). We provide an explicit description of one of the real forms of this superalgebra, which is geometrically natural being made of so(3, R)-invariant operators which preserve the Poincaré (odd Hermitean) inner product on the bundle of forms.

1. Introduction

In the present paper we continue the search for natural (∗-)Lie superalgebra bundles on Weakly Self Dual manifolds, started in [GG]. By carefully choosing the initial geometrical operators canonically defined on any orientable Weakly Self Dual manifold of rank three, we generate a complex ∗-Lie superalgebra \( L_{3,C} \) which is invariant with respect to pointwise automorphism group \( \text{Aut}_p(X) = \text{so}(3, R) \) at any point \( p \) of the WSD structure. We describe the action of this ∗-Lie superalgebra on the differential forms of the WSD manifold, decomposing them into irreducible representations, and we also show that the algebra of all the operators of \( L_{3,C} \) which preserve the Poincaré (odd Hermitean) inner product on the bundle of forms at any point \( p \) is a real form of it.

We prove the following main theorem:

Theorem 6.4 (simplified form) On any rank three orientable WSD manifold there is a natural real Lie superalgebra bundle, based on the real Lie superalgebra

\[
\text{su}(20|20, < , >) \oplus \text{su}(36|36, < , >) \oplus \text{su}(20|20, < , >) \oplus \text{sl}(4, R)
\]

where we indicate with \( \text{su}(n|n, < , >) \) the sub Lie superalgebra of \( \text{sl}(n|n, \mathbb{C}) \) which preserves the standard odd Hermitean inner product \( < , > \) on \( \mathbb{C}^n|n \). This is not the "usual" \( \text{su}(n|n) \), as the inner product is odd; indeed, in a good basis the matrices of this Lie superalgebra have the form

\[
\begin{pmatrix}
A & B \\
-C & -A
\end{pmatrix}
\]

The motivations for this search go back to [G2], where it was shown that on any weakly self dual manifold (WSD for brevity) there is a natural \( \text{sl}(4, R) \) bundle. Afterwards in [G3] it was conjectured that there should be a "good" (super)Lie algebra bundle acting on \( \wedge^* T^*X \) for a rank three WSD manifold \( X \) (in that case degenerate of dimension 11), which should give rise, using the (generalized) lagrangian dynamics of [G1], to a natural and geometrically motivated field theory on any such WSD manifold. It was further conjectured that once quantized this field theory

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would provide a good playground where to look for a unifying theory for the various string theories. To put this last part of the conjectured construction into context, one should recall that the generalized lagrangian dynamics of \([G1]\) is also as well a generalized hamiltonian dynamics, at least for nondegenerate lagrangians, and this is clearly the starting point in the quest for a quantization procedure. Our basic motivations come therefore from mathematical physics, and from this point of view it is both reassuring and stimulating that our final result Theorem 6.4 finds a Lie superalgebra of "super-antihermitean" operators with respect to an odd nondegenerate Hermitian inner product \(< , >\).

Polysymplectic manifolds were introduced in \([G1]\) to provide a geometric approach to string theory, and WSD manifolds, which are very special polysymplectic manifolds, were introduced in \([G2]\) to build a geometric approach to Mirror Symmetry. In just a few words, if two Calabi-Yau manifolds are mirror dual, it is conjectured that there should be a family of WSD manifolds which converges (in a normalized Gromov-Hausdorff sense) to either of them on two different boundary points of the deformation space, and that for this to happen for any family of WSD manifolds it is actually necessary to have a mirror pair of Calabi-Yau manifolds (see \([G2],[G3]\) and the introduction to \([GG]\) for further details). In \([G2]\) this approach was shown to work for simpler cases, like Elliptic curves and Affine-Kähler manifolds, while in \([G3]\) it was shown that many aspects of this method do work in the case of anticanonical families in complex projective spaces, which includes the historically significant case of the quintic threefold.

Moved by these motivations, in \([GG]\) we started with a "proof of concept" computation for rank two WSD manifolds, to develop the techniques necessary to tackle the higher rank cases. Even if the rank two context was clearly less significant from a physical point of view, the result was nevertheless interesting algebraically and geometrically, as we found a natural \(\mathfrak{sl}(6,\mathbb{C})\) bundle, which is a direct generalization of the \(\mathfrak{sl}(2,\mathbb{C})\) bundle arising naturally on Kähler manifolds.

As mentioned above, here we show that indeed in the rank three nondegenerate case (and therefore also in the degenerate one) there is a very structured Lie superalgebra bundle acting as conjectured. We do not provide suggestions for the possible generalized lagrangian theory based on this bundle, even if there are some natural candidates which one should try first, from a geometrical point of view. To arrive to the final description, we faced a much more intricate situation with respect to \([GG]\), as first of all in the rank three (oriented) case one is working with superLie algebras, instead of ordinary (even) Lie algebras. To see why this happens, it is necessary to give the definition of WSD manifold: we reproduce here the rank three nondegenerate case, which is the one relevant for our present purposes (look at \([G2]\) for the general definition); also recall that we always assume the WSD manifolds to be orientable:

**Definition 1.1.** A rank three nondegenerate weakly self-dual manifold (WSD manifold for brevity) is given by a smooth manifold \(X\), together with two smooth 2-forms \(\omega_1, \omega_2\) a Riemannian metric and a third smooth 2-form \(\omega_D\) (the dualizing form) on it, which satisfy the following conditions:

1) \(d\omega_1 = d\omega_2 = d\omega_D = 0\) and the distribution \(\omega_1 + \omega_2\) is integrable.
2) For all \(p \in X\) there exist an orthonormal basis

\[
dx_1, dx_2, dx_3, dy_1^1, dy_2^1, dy_3^1, dy_1^2, dy_2^2, dy_3^2
\]

of \(T^*_p X\) such that the

\[
(\omega_1)_p = \sum_{i=1}^{m} dx_i \wedge dy_i^1, \quad (\omega_2)_p = \sum_{i=1}^{m} dx_i \wedge dy_i^2, \quad (\omega_D)_p = \sum_{i=1}^{m} dy_i^1 \wedge dy_i^2
\]
Any basis of $T_pX$ dual to a basis of 1-forms as above is said to be adapted to the structure, or standard.

Starting from this geometric data, one constructs six natural "wedge" operators.

\textbf{Definition 1.2.} For $\phi \in \Omega^1_p X$,

\[
L_0(\phi) = \omega_p \wedge \phi, \quad L_1(\phi) = -\omega_p \wedge \phi, \quad L_2(\phi) = \omega_p \wedge \phi
\]

\[
V_0(\phi) = dx_0 \wedge dx_1 \wedge dx_2 \wedge \phi, \quad V_1(\phi) = dy_0^2 \wedge dy_1^2 \wedge dy_2 \wedge \phi, \quad V_2(\phi) = dy_0^1 \wedge dy_1^1 \wedge dy_2 \wedge \phi
\]

and their (super) adjoints $\Lambda_j = L_j^\star$, $A_j = V_j^\star$ with respect to the standard Hermitean form $(\ , \ )$. Here the $V_j$ are actually wedge operators with volume forms associated to canonical oriented distributions on $T^*X$, and therefore are well defined. As can be seen by inspection, while the $L_j$ are even, the $V_j$ are odd, while in the rank two case the corresponding $V_j$ operators were also even. We call $\mathcal{L}_{3,C}$ the complex Lie superalgebra generated by these 12 operators. One of the nice aspects of this theory is that the operators above are invariant with respect to a group which is the space of sections of the $\text{SO}(3, \mathbb{R})$ bundle determined point by point by the automorphism group $\text{Aut}_p(X)$ of the WSD structure. The Lie algebra associated to this group acts on the faithful, defining representation of $\text{SO}(3, \mathbb{R})$, namely $\Lambda^* T^*X$, and by conjugation on $\mathcal{L}_{3,C}$. This action on both $\mathcal{L}_{3,C}$ (which as mentioned is $\text{SO}(3, \mathbb{R})$-invariant) and on its representation makes the representation theory that we use both richer and more complex. This is a general trend, and in the rank $k$ case the invariance group is the space of sections of a $\text{SO}(k, \mathbb{R})$ bundle. To obtain the complete description given in Theorem 6.4, in the following corollary and in the tables in appendix, we used also the $\mathfrak{sl}(4, \mathbb{C})$ structure uncovered in [G1], which is present on a WSD manifold of any rank. To obtain enough operators we needed however to use also the supercommutators $[V_j, A_j]$ to split the operators in this $\mathfrak{sl}(4, \mathbb{C})$ into homogeneous components. Another geometric ingredient that comes into play is the odd, superHermitean, nondegenerate Poincaré inner product $<\ , \ >$ between forms, given at any fixed point $p \in X$ by the formula

\[
<\alpha_p, \beta_p > = dVol_p = \alpha_p \wedge \bar{\beta}_p
\]

where $dVol_p$ is the volume form at the point $p$ given by the metric and the orientation. We show that the the operators $iL_j$, $i\Lambda_k$, $iV_j$, $A_j$ are all superantiHermitean with respect to the inner product $<\ , \ >$, and the real Lie superalgebra $\mathcal{L}_3$ which they generate is a proper real form for $\mathcal{L}_{3,C}$. Moreover, the inclusion of $\mathcal{L}_3$ inside the full $\text{SO}(3, \mathbb{R})$-invariant superantiHermitean superalgebra (which has dimension 8444) is actually almost an isomorphism, its dimension being off by a mere 48.

Coming to a more precise description of the contents of this paper, in Section 2 we introduce the algebra and its symmetries. There are many pieces of the puzzle that need to be introduced, and which will be put together in the subsequent sections. Among them, besides the geometric operators $L_j$, $\Lambda_j$, $V_j$, $A_j$ and the real form $\mathcal{L}_3$, we have the operators $J_k$ generating the $\text{SO}(3, \mathbb{R})$ symmetry, the twisted adjoint $\phi^\star$ to an operator, and the Poincaré odd Hermitean inner product $<\ , \ >$, which we show that is preserved by $\mathcal{L}_3$. There is also the Hodge star, which is ubiquitous and sends even forms to odd ones, as our manifolds are odd dimensional.

In Section 3, after introducing a natural action of the permutation group $S_3$, we first prove that $\mathcal{L}_3$ commutes with $\text{SO}(3, \mathbb{R})$ and then we continue by decomposing (doing a form of plethysm for $\mathfrak{so}(3, \mathbb{R}) \otimes \mathbb{C} \cong \mathfrak{sl}(2, \mathbb{C})$) the representation $\Lambda^\star T^*X$ with respect to the action of $\mathfrak{sl}(2, \mathbb{C})$. Any isotypical $\mathfrak{sl}(2, \mathbb{C})$-module will be clearly an invariant $\mathcal{L}_{3,C}$-submodule from Schur’s lemma; even more efficiently, the highest weight vectors inside an isotypical component form an invariant $\mathcal{L}_{3,C}$-submodule.
Corollary 3.5 we make this precise, by mapping \( L_3 \) inside a direct sum of endomorphism algebras for highest weight vectors. We then describe in general terms the structure of a Lie superalgebra preserving an odd Hermitean nondegenerate inner product, like the Poincaré one that we use. In Section 4 we first build an inclusion of \( \mathfrak{sl}(4, \mathbb{R}) \) inside \( L_3 \), and then use three toral operators \( K_j \) built as supercommutators of \( V_j \) and \( A_j \) to decompose the generators of this \( \mathfrak{sl}(4, \mathbb{R}) \) into \( K_j \)-homogeneous components. In Section 5 we describe explicitly some naturally structured bases for the spaces of highest weight vectors inside the isotypical components mentioned above, and in Section 6 we restrict \( L_3 \) to these spaces of highest weight vectors, and use the bases previously built and the information accumulated up to this point to obtain the final description of the structure of \( L_3 \) and of \( L_3, \mathbb{C} \). In section 7 we show that both \( L_3 \) and \( L_3, \mathbb{C} \) are closed with respect to the "standard" adjunction operator induced by the metric, and therefore \( L_3, \mathbb{C} \) is a \( * \)-Lie superalgebra. The tables in the Appendix describe in detail the action of the \( K_j \)-homogeneous components of the generators of the \( \mathfrak{sl}(4, \mathbb{C}) \) mentioned above, in the explicitly provided bases.

2. The algebra and its invariance properties

Let us fix a point \( p \) in the WSD manifold \( X \). In accordance with the similar notation introduced in [GG] for the rank two case, the WSD structure splits the cotangent space as \( T^*_p X = W_0 \oplus W_1 \oplus W_2 \) where the \( W_j \) are three mutually orthogonal three-dimensional canonical distributions defined as:

\[
W_0 = \{ \phi \in T^*_p X \mid \phi \wedge \omega_1^3 = \phi \wedge \omega_2^3 = 0 \}
\]

\[
W_1 = \{ \phi \in T^*_p X \mid \phi \wedge \omega_1^3 = \phi \wedge \omega_D^3 = 0 \}
\]

\[
W_2 = \{ \phi \in T^*_p X \mid \phi \wedge \omega_2^3 = \phi \wedge \omega_D^3 = 0 \}
\]

**Definition 2.1.** The decomposition \( T^*_p X = W_0 \oplus W_1 \oplus W_2 \) induces corresponding decompositions for \( s \in \{0, \ldots, 9\} \):

\[
\bigwedge^s T^*_p X = \bigoplus_{p+q+r=s} \bigwedge^p W_0 \oplus \bigwedge^q W_1 \oplus \bigwedge^r W_2
\]

We say that \( \alpha \in \bigwedge^p W_0 \oplus \bigwedge^q W_1 \oplus \bigwedge^r W_2 \) has multidegree \((p, q, r)\).

The WSD structure also determines canonical pairwise linear identifications among \( W_0, W_1 \) and \( W_2 \), so that one can also write \( T^*_p X = W_0 \otimes_{\mathbb{R}} \mathbb{R}^3 \) or more simply

\[
T^*_p X = W \otimes_{\mathbb{R}} \mathbb{R}^3
\]

where \( W = W_0 \cong W_1 \cong W_2 \).

The canonical decomposition above equips \( W \) (and hence on all the \( W_j \)) with an orientation induced by the one on \( T^*_p X \). Let us now come back to the canonical operators \( L_j, V_j \) mentioned in the Introduction using the notation \( Vol(W_j) \) for the oriented volume forms of the \( W_j \):

**Definition 1.2.** For \( \phi \in \Omega^*_C X \),

\[
L_0(\phi) = \omega_D \wedge \phi, \quad L_1(\phi) = -\omega_2 \wedge \phi, \quad L_2(\phi) = \omega_1 \wedge \phi
\]

\[
V_0(\phi) = Vol(W_0) \wedge \phi, \quad V_1(\phi) = Vol(W_1) \wedge \phi, \quad V_2(\phi) = Vol(W_2) \wedge \phi
\]

We now choose a (non-canonical) orthonormal basis \( \gamma_1, \gamma_2, \gamma_3 \) for \( W_0 \), and this together with the standard identifications of the \( W_j \) determines an orthonormal basis for \( T^*_p X \), which we write as \( \{ v_{ij} = \gamma_i \otimes e_j \mid i = 1, 2, 3, \ j = 0, 1, 2 \} \). We remark
that the $v_{ij}$ are an \textit{adapted coframe} for the WSD structure, and therefore we have the explicit expressions:

\[
\begin{align*}
\omega_1 &= v_{10} \wedge v_{11} + v_{20} \wedge v_{21} + v_{30} \wedge v_{31} \\
\omega_2 &= v_{10} \wedge v_{12} + v_{20} \wedge v_{22} + v_{30} \wedge v_{32} \\
\omega_D &= v_{11} \wedge v_{12} + v_{21} \wedge v_{22} + v_{31} \wedge v_{32}
\end{align*}
\]

A different choice of the $\gamma_1, \gamma_2, \gamma_3$ would be related to the previous one by an element in $O(3, \mathbb{R})$ or, taking into account the orientability of $X$ mentioned in the Introduction, an element of $SO(3, \mathbb{R})$. The Lie algebra of the group $SO(3, \mathbb{R})$ expressing the change from one oriented adapted basis to another is generated (point by point) by the global operators $J_1, J_2, J_3$:

**Definition 2.2.** For fixed $p \in X$, the operators $J_1, J_2, J_3 \in \text{End}_\mathbb{R}(\Lambda^* T^*_p X)$ are defined in terms of the standard basis $v_{ij}$ as

\[
\begin{align*}
J_1(v_{2j}) &= v_{3j}, & J_1(v_{3j}) &= -v_{2j}, & J_1(v_{1j}) &= 0 & \text{for } j \in \{0, 1, 2\} \\
J_2(v_{3j}) &= v_{1j}, & J_2(v_{1j}) &= -v_{3j}, & J_2(v_{2j}) &= 0 & \text{for } j \in \{0, 1, 2\} \\
J_3(v_{1j}) &= v_{2j}, & J_3(v_{2j}) &= -v_{1j}, & J_3(v_{3j}) &= 0 & \text{for } j \in \{0, 1, 2\}
\end{align*}
\]

and $J_i(v \wedge w) = J_i(v) \wedge w + v \wedge J_i(w)$ for $v, w \in \Lambda^* T^*_p X$ and $i \in \{1, 2, 3\}$.

**Remark 2.3.** The operators $J_1, J_2, J_3$ generate the Lie algebra $\text{aut}_p(X)$ of all the operators preserving the WSD structure at the given point, which does not depend on the choice of a basis for $T^*_p X$. If the metric of $X$ is such that the structural forms $\omega_1, \omega_2, \omega_D$ are covariant constant, then its holonomy must lie inside the group $\text{Aut}_p(X) = \exp(\text{aut}_p(X))$.

Using the chosen (non canonical) orthonormal basis, one can define corresponding wedge and contraction operators:

**Definition 2.4.** Let $i \in \{1, 2, 3\}$ and $j \in \{0, 1, 2\}$. The operators $E_{ij}$ and $I_{ij}$ are respectively the wedge and the contraction operator with the form $v_{ij}$ on $\Lambda^* T^*_p X$ (defined using the given basis): we use the notation $\frac{\partial}{\partial v_{ij}}$ to indicate the element of $T^*_p X$ dual to $v_{ij} \in T^*_p X$:

\[
E_{ij}(\phi) = v_{ij} \wedge \phi, \quad I_{ij}(\phi) = \frac{\partial}{\partial v_{ij}} \phi
\]

**Proposition 2.5.** The operators $E_{ij}, I_{ij}$ satisfy the following relations:

\[
\begin{align*}
\forall i, j, k, l & \quad E_{ij} E_{kl} = -E_{kl} E_{ij}, & I_{ij} I_{kl} = -I_{kl} I_{ij} \\
\forall i, j & \quad E_{ij} I_{ij} + I_{ij} E_{ij} = \text{Id} \\
\forall (i, j) \neq (k, l) & \quad E_{ij} I_{kl} = -I_{kl} E_{ij} \\
\forall i, j & \quad E^*_{ij} = I_{ij}, & I^*_{ij} = E_{ij}
\end{align*}
\]

where $*$ is adjunction with respect to the metric.

**Proof** The proof is a simple direct verification, which we omit. \hfill \Box

Using the (non canonical) operators $E_{ij}$ we can obtain simple expressions for the pointwise action of the other canonical operators, the odd operators $V_j$ induced by the volume forms of the distributions $W_J$: for $\phi \in \Lambda^* T^*_p X$,

\[
V_0(\phi) = E_{10} E_{20} E_{30}(\phi), \quad V_1(\phi) = E_{11} E_{21} E_{31}(\phi), \quad V_2(\phi) = E_{12} E_{22} E_{32}(\phi)
\]

Remember however that the operators $V_j$ do not depend on the choice of a basis, as they are simply multiplication by the volume forms of the spaces $W_J$.

We use the $v_{ij}$ also as a orthonormal basis for the complexified space $T^*_p \otimes_\mathbb{R} \mathbb{C}$ (with respect to the induced hermitian inner product). We indicate with the same symbols
V_j the complexified operators acting on the spaces $\bigwedge^*_\mathbb{C} T^*_p X$. The riemannian metric induces a riemannian metric on $T^*_p X$ and on the space $\bigwedge^*_\mathbb{C} T^*_p X$.

**Remark 2.6.** The Riemannian metric on $X$ induces in the standard way a Hodge star operator

$$*: \bigwedge^* \mathbb{C} T^*_p X \to \bigwedge^* \mathbb{C} T^*_p X$$

which (as $\dim(X) = 9$) satisfies $*^2 = \text{Id}$.

As usual, one can define the induced Hermitean inner product $(\ ,\ )_p$ on $\bigwedge^* \mathbb{C} T^*_p X$ via the Hodge * operator

**Definition 2.7.** Given a homogeneous operator $\phi$ on $\bigwedge^* \mathbb{C} T^*_p X$, we indicate with $\phi^*$ the homogeneous operator which satisfies:

$$(\phi(\alpha), \beta)_p = (-1)^{\deg(\phi)\deg(\alpha)}(\alpha, \phi^*(\beta))_p$$

for all homogeneous $\alpha, \beta$.

**Definition 2.8.** For $j \in \{0, 1, 2\}$

$$A_j = V_j^*, \quad A_j = V_j^*$$

By construction the canonical operators $L_j, V_j, \Lambda_j, A_j$ on $\bigwedge^* \mathbb{C} T^*_p X$ are the pointwise restrictions of corresponding global operators on smooth differential forms, which we indicate with the same symbols: for $j \in \{0, 1, 2\}$,

$$L_j, V_j, \Lambda_j, A_j : \Omega^*_\mathbb{C}(X) \to \Omega^*_\mathbb{C}(X)$$

Summing up:

**Definition 2.9.** The Lie superalgebra $\mathcal{L}_3$ is the Lie subalgebra of the general linear Lie superalgebra of $\Omega^*_\mathbb{C}(X)$ generated by the operators

$$\{iL_j, iV_j, i\Lambda_j, A_j \mid \text{for } j = 0, 1, 2\}$$

The Lie superalgebra $\mathcal{L}_3, \mathbb{C}$ is $\mathcal{L}_3 \otimes \mathbb{C}$, and is in a natural way a Lie subalgebra of the general linear Lie superalgebra of $\Omega^*_\mathbb{C}(X)$.

**Definition 2.10.** For every $p \in X$ there is a natural odd non degenerate super Hermitean inner product $< \ , \>_p$ on $\bigwedge^* \mathbb{C} T^*_p X$, defined using the natural (standard) Hermitean inner product $(\ , \)_p$ and the (pointwise) volume form $\Omega$ associated to the Riemannian metric and the orientation:

$$< \alpha, \beta>_p = (\alpha \wedge \overline{\beta}, \Omega)_p$$

We list without proof the following standard facts:

**Proposition 2.11.** For every $p \in X$, the pairing $< \ , \>_p$ satisfies the following properties:

a) $< \alpha, \beta>_p = (\alpha, \beta)_p$

b) $< \alpha, \beta> = (-1)^{\deg(\alpha)\deg(\beta)} <\overline{\beta}, \alpha>_p$ (super Hermitean)

c) $\forall \alpha (\forall \beta <\alpha, \beta>_p = 0 \implies \alpha = 0)$ (non-degenerate)

d) $< \ , \>_p$ is preserved by the action of the group of orientation preserving isometries $SO(T^*_p X, g)$ on $\bigwedge^* \mathbb{C} T^*_p X$ (and in particular by our $SO(3, \mathbb{R})$).

e) $< \ , \>_p$ is preserved by the Hodge * operator.
Theorem 2.13. The algebra $\langle \cdot, \cdot \rangle$ is equivalent to the fact that the condition $\phi^* = -* \phi *$ must hold for all the elements $\phi$ of $L_3$.

Proof. For fixed $p \in X$, by definition, a homogeneous operator $\phi$ preserves $\langle \cdot, \cdot \rangle_p$ if and only if it satisfies the following equation for homogeneous $\alpha, \beta$:

$$\forall \alpha, \beta \quad \langle \phi(\alpha), \beta \rangle_p + (-1)^{\deg(\alpha)\deg(\phi)} \langle \alpha, \phi(\beta) \rangle_p = 0$$

Using the expression $\langle \alpha, \beta \rangle_p = (\alpha, *\beta)_p$, the above is equivalent to $\phi^* = -* \phi *$. It is enough to check the equation for the generators of $L_3$, as one has that if $\phi$ and $\psi$ are homogeneous and satisfy the equation, then also:

$$\langle [\phi, \psi](\alpha), \beta \rangle_p + (-1)^{\deg(\alpha)(\deg(\phi)+\deg(\psi))} \langle \alpha, [\phi, \psi](\beta) \rangle_p = 0$$

(by general reasoning in the category of super vector spaces and operators, or by direct computation). For the generators $iL_j$ and $iV_j$, which are wedge operators with homogeneous differential forms, the verification of the equation is immediate. The operators $i\Lambda_j$ by definition satisfy the equation

$$\forall \alpha, \beta \quad (iL_j(\alpha), \beta)_p = -\langle \alpha, i\Lambda_j(\beta) \rangle_p$$

Using the Hodge $*$, one can rephrase the previous equation as:

$$\forall \alpha, \beta \quad iL_j(\alpha) \wedge *\beta = -\alpha \wedge *i\Lambda_j(\beta)$$

As $L_j$ is a wedge operator with an even form, from this one obtains

$$i\Lambda_j = *iL_j *$$

As the Hodge $*$ is self-adjoint with respect to the product $\langle \cdot, \cdot \rangle_p$ and $iL_j$ is anti-(super)selfadjoint, one has

$$\langle i\Lambda_j(\alpha), \beta \rangle_p = <*iL_j *\alpha, \beta>_p = -<\alpha, *iL_j *\beta>_p = -<\alpha, i\Lambda_j(\beta)>_p$$

It remains to verify the case of $A_j$. The operators $A_j$ by definition satisfy the equation

$$\forall \alpha, \beta \quad (V_j(\alpha), \beta)_p = (-1)^{\deg(\alpha)}\langle \alpha, A_j(\beta) \rangle_p$$

Using the Hodge $*$, one can rephrase the previous equation as:

$$\forall \alpha, \beta \quad V_j(\alpha) \wedge *\beta = (-1)^{\deg(\alpha)}\alpha \wedge *A_j(\beta)$$

As $V_j$ is a wedge operator with an odd form, from this one obtains

$$A_j = *V_j *$$

As before, we start with the observation that $V_j$ is superselfadjoint and the Hodge star is selfadjoint (with respect to $\langle \cdot, \cdot \rangle_p$). Therefore

$$\langle A_j(\alpha), \beta \rangle_p = \langle *V_j *\alpha, \beta \rangle_p = (-1)^{\deg(\alpha)}\langle \alpha, A_j(\beta) \rangle_p$$

□
3. The actions of $S_3$ and $so(3, \mathbb{R})$

The canonical splitting $T^*_pX = W_0 \oplus W_1 \oplus W_2$ together with the canonical identifications $W_0 \cong W_1 \cong W_2$ induce an action of the symmetric group $S_3$ on $T^*_pX$, which propagates to $\bigwedge^* T^*_pX$ and to its $C^\infty$ sections. At every point, the action can be written explicitly in terms of the basis $\{v_{ij}\}$ from the beginning of section 2 as

$$\sigma(v_{ij}) = v_{i\sigma(j)}$$

The induced action on endomorphisms via conjugation, $\sigma(\phi) = \sigma \circ \phi \circ \sigma^{-1}$, preserves $L_3$. Indeed, one can check directly using the basis $v_{ij}$ at every point that for $\sigma \in S_3$

$$\sigma(V_j) = V_{\sigma(j)}, \quad \sigma(L_j) = \epsilon(\sigma)L_{\sigma(j)}$$

Since $S_3$ acts on $L_3$ by conjugation with (even) unitary operators, its action commutes with adjunction (the $\ast$ operator), and therefore

$$\sigma(A_j) = A_{\sigma(j)}, \quad \sigma(\Lambda_j) = \epsilon(\sigma)\Lambda_{\sigma(j)}$$

Moreover, one immediately verifies that $\sigma(J_i) = J_i$ for all $\sigma \in S_3$ and for all $i \in \{1, 2, 3\}$, which means that the action of $S_3$ commutes with that of $so(3, \mathbb{R})$.

**Theorem 3.1.** The algebra $L_3$ commutes with the action of $SO(3, \mathbb{R})$.

**Proof** It is enough to show that the generators of $L_3$ commute with the (even) generators $J_1, J_2, J_3$ of $so(3, \mathbb{R})$. As the operators $J_k$ are (real) anti-Hermitean with respect to the standard scalar product $(\ , \ )_p$, it is enough to check the commutation of them with the $L_j$ and the $V_j$ (indicating with $Vol(W_j)$ the oriented volume form of the distribution $W_j$).

$$[L_j, J_k](\alpha) = \omega_j \wedge J_k(\alpha) - J_k(\omega_j \wedge \alpha) = -J_k(\omega_j) \wedge \alpha$$

$$[iV_j, J_k](\alpha) = iVol(W_j) \wedge J_k(\alpha) - J_k(iVol(W_j)) \wedge \alpha = -J_k(iVol(W_j)) \wedge \alpha$$

As the $J_k$ are anti-Hermitean and preserve the distributions $W_j$, $J_k(Vol(W_j)) = 0$. To verify that $J_k(\omega_j) = 0$ one makes a direct computation in (orthonormal) coordinates using Definition 2.2. Using the action of $S_3$, one actually reduces to the following two verifications:

$$J_1(\omega_1) = 0, \quad J_1(\omega_2) = 0$$

$$J_1(\omega_1) = J_1(v_{10} \wedge v_{11} + v_{20} \wedge v_{21} + v_{30} \wedge v_{31}) =$$

$$= v_{30} \wedge v_{21} + v_{20} \wedge v_{31} - v_{20} \wedge v_{31} - v_{30} \wedge v_{21} = 0$$

$$J_1(\omega_2) = J_1(v_{10} \wedge v_{12} + v_{20} \wedge v_{22} + v_{30} \wedge v_{32}) =$$

$$= -v_{30} \wedge v_{12} - v_{10} \wedge v_{32} + v_{10} \wedge v_{32} + v_{30} \wedge v_{12} = 0$$

$$\square$$

**Remark 3.2.** The algebra $L_3$ (and its complexification $L_{3, \mathbb{C}}$) commutes with the complexification $so(3, \mathbb{C})$ of the Lie algebra generated by the operators $J_k$.

**Definition 3.3.** Given $n \in \mathbb{Z}_{\geq 0}$, we indicate with $\rho_n$ the $2n + 1$ dimensional complex irreducible representation of the Lie algebra $so(3, \mathbb{C})$, which is a highest weight representation with highest weight equal to $2n$.

The actual representation of $so(3, \mathbb{R})$ (and $so(3, \mathbb{C})$) on $\bigwedge^* T^*_C(X)$ does depend on the choice of an orthonormal basis (indeed, two different adapted oriented orthonormal bases differ by an element in $SO(3, \mathbb{R})$). However, the decomposition in isotypical components described in the following proposition is canonical, and does not depend on any choice.
Proposition 3.4. Under the \( \mathfrak{so}(3, \mathbb{C}) \) representation induced by the operators \( J_k \), for any \( p \in X \):

a) The space \( T_C^*(X)_p \) decomposes as an \( \mathfrak{so}(3, \mathbb{C}) \) module as

\[
T_C^*(X)_p = (W_0 \otimes \mathbb{C}) \oplus (W_1 \otimes \mathbb{C}) \oplus (W_2 \otimes \mathbb{C})
\]

with \( W_j \otimes \mathbb{C} \cong \rho_1 \) as \( \mathfrak{so}(3, \mathbb{C}) \) modules.

b) More generally, Table 1 describes the decomposition of \( \bigwedge^* T_C^*(X)_p \) into isotypical components

|        | Type \( \rho_0 \) | Type \( \rho_1 \) | Type \( \rho_2 \) | Type \( \rho_3 \) |
|--------|-------------------|-------------------|-------------------|-------------------|
| \( \bigwedge^0 T_C^* X \) | \( \rho_0 \)         |                   |                   |                   |
| \( \bigwedge^1 T_C^* X \) | \( \rho_0^\oplus 3 \) | \( \rho_1^\oplus 6 \) | \( \rho_2^\oplus 3 \) |                   |
| \( \bigwedge^2 T_C^* X \) | \( \rho_0^\oplus 10 \) | \( \rho_1^\oplus 9 \) | \( \rho_2^\oplus 8 \) | \( \rho_3 \)       |
| \( \bigwedge^3 T_C^* X \) | \( \rho_0^\oplus 6 \) | \( \rho_1^\oplus 18 \) | \( \rho_2^\oplus 9 \) | \( \rho_3^\oplus 3 \) |
| \( \bigwedge^4 T_C^* X \) | \( \rho_0^\oplus 6 \) | \( \rho_1^\oplus 18 \) | \( \rho_2^\oplus 9 \) | \( \rho_3^\oplus 3 \) |
| \( \bigwedge^5 T_C^* X \) | \( \rho_0^\oplus 10 \) | \( \rho_1^\oplus 9 \) | \( \rho_2^\oplus 8 \) | \( \rho_3 \)       |
| \( \bigwedge^6 T_C^* X \) | \( \rho_0^\oplus 3 \) | \( \rho_1^\oplus 6 \) | \( \rho_2^\oplus 3 \) |                   |
| \( \bigwedge^7 T_C^* X \) | \( \rho_0^\oplus 3 \) | \( \rho_1^\oplus 6 \) | \( \rho_2^\oplus 3 \) |                   |
| \( \bigwedge^8 T_C^* X \) |                   |                   |                   |                   |
| \( \bigwedge^9 T_C^* X \) | \( \rho_0 \)         |                   |                   |                   |
| Dimension | 40              | 3 \times 72      | 5 \times 40       | 7 \times 8        |

Table 1. Isotypical components

Proof  a) The subspaces \( W_j \) are preserved by the action of \( \mathfrak{so}(3, \mathbb{R}) \), therefore there is a direct sum decomposition of \( T_C^*(X)_p \) with respect to the action of \( \mathfrak{so}(3, \mathbb{C}) \). The action of \( \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \) on \( \bigwedge^* T_C^* X \) is best explained using a Serre set of generators \( e, f, h \), defined as follows in terms of the operators of Definition 2.2:

\[
e = \imath J_1 - J_2, \quad f = \imath J_1 + J_2, \quad h = 2J_3
\]

To show the isomorphisms with \( \rho_1 \) we use a new orthonormal basis made of eigenvectors for \( h \), built starting from the basis \( \{ v_{ij} \} \):

\[
w_{1j} = v_{1j} + \imath v_{2j}, \quad w_{2j} = v_{1j} - \imath v_{2j}, \quad w_{3j} = v_{3j}
\]

One then checks that for fixed \( j \) the space \( \langle w_{1j}, w_{2j}, w_{3j} \rangle_\mathbb{C} = W_j \otimes \mathbb{C} \) isomorphic as \( \mathfrak{sl}(2, \mathbb{C}) \)-module to \( \rho_1 \), since \( w_{1j} \) is a highest weight vector of \( h \)-weight 2.
b) The proof consists now of some plethysm for $\mathfrak{sl}(2, \mathbb{C})$. We will use the well known formula

$$\rho_a \otimes \rho_b = \sum_{i=[a-b]}^{a+b} \rho_i$$

and the facts that $\bigwedge^2 \rho_1 \cong \rho_1$ and $\bigwedge^3 \rho_1 \cong \rho_0$.

The space $\bigwedge^2 T_C^* X$ is isomorphic as an $\mathfrak{sl}(2, \mathbb{C})$ module to $\bigwedge^2 (\rho_1^{\otimes 3})$, and therefore

$$\bigwedge^2 T_C^* X \cong \bigwedge^2 (\rho_1^{\otimes 3}) \cong \left[ \bigwedge^2 \rho_1 \right] \oplus (\rho_1 \otimes \rho_1) \cong \rho_1^{\otimes 3} + (\rho_0 + \rho_1 + \rho_2)^{\otimes 3}$$

The space $\bigwedge^3 T_C^* X$ is isomorphic as an $\mathfrak{sl}(2, \mathbb{C})$ module to $\bigwedge^3 (\rho_1^{\otimes 3})$, and therefore

$$\bigwedge^3 T_C^* X \cong \bigwedge^3 (\rho_1^{\otimes 3}) \cong \left[ \bigwedge^3 \rho_1 \right] \oplus (\bigwedge^2 \rho_1 \otimes \rho_1) \oplus (\bigwedge^1 \rho_1 \otimes \bigwedge^1 \rho_1) \cong \rho_1^{\otimes 3} + (\rho_0 + \rho_1 + \rho_2)^{\otimes 3}$$

The space $\bigwedge^4 T_C^* X$ is isomorphic as an $\mathfrak{sl}(2, \mathbb{C})$ module to $\bigwedge^4 (\rho_1^{\otimes 3})$, and therefore

$$\bigwedge^4 T_C^* X \cong \bigwedge^4 (\rho_1^{\otimes 3}) \cong \left[ \bigwedge^4 \rho_1 \right] \oplus (\bigwedge^3 \rho_1 \otimes \rho_1) \oplus (\bigwedge^2 \rho_1 \otimes \bigwedge^1 \rho_1) \cong \rho_1^{\otimes 3} + (\rho_0 + \rho_1 + \rho_2)^{\otimes 3}$$

As the Hodge $\ast$ commutes with the induced action of the isometries of $T_p X$, it must preserve the isotypical components with respect to $\mathfrak{so}(3, \mathbb{C})$. This implies that the computations above are enough to determine the remaining decompositions, as one has the isomorphisms of $\mathfrak{so}(3, \mathbb{C})$ modules $\ast : \bigwedge^k T_C^* X \rightarrow \bigwedge^{9-k} T_C^* X$. □

Let us call $ HW_k $ the complex vector space of $ \mathfrak{so}(3, \mathbb{C})$-highest weight vectors inside the isotypical component of type $ \rho_k $. As the algebra $ \mathcal{L}_3 $ commutes with $ \mathfrak{so}(3, \mathbb{C}) $, it preserves each isotypical component, and moreover it sends each space $ HW_k $ to itself.

**Corollary 3.5.** The algebra $ \mathcal{L}_3 $ is mapped injectively to the space

$$\text{End}_C(HW_0) \oplus \text{End}_C(HW_1) \oplus \text{End}_C(HW_2) \oplus \text{End}_C(HW_3)$$

and therefore its dimension over $ \mathbb{R} $ is at most $ 2 \times (40^2 + 72^2 + 42^2 + 8^2) = 16896 $.

**Proof** It only remains to check the dimensions of the $ HW_k $, which can be read from Table 1. □

As a consequence of this corollary, the upper bound on the dimension over $ \mathbb{R} $ of the algebra $ \mathcal{L}_3 $ has been reduced from $ 2^{19} = 524288 $ to 16896. To further reduce it, we make the following observation:

**Proposition 3.6.** The odd nondegenerate super Hermitean form $ < , >_p $ remains nondegenerate when restricted to the spaces $ HW_k $. The algebra $ \mathcal{L}_3 $ is mapped injectively to the space of operators which are super antiHermitean for this form and have supertrace zero:

$$\text{su}(HW_0^{\text{ev}} | HW_0^{\text{odd}}, < , >_p) \oplus \text{su}(HW_1^{\text{ev}} | HW_1^{\text{odd}}, < , >_p) \oplus \text{su}(HW_2^{\text{ev}} | HW_2^{\text{odd}}, < , >_p) \oplus \text{su}(HW_3^{\text{ev}} | HW_3^{\text{odd}}, < , >_p)$$
Proof Let $\alpha$ be a nonzero element of $HW_k$. As the Hodge $*$ commutes with the action of $\text{so}(3, \mathbb{C})$, $*\alpha$ is again a (nonzero) element of $HW_k$. Moreover,

$$<\alpha, *\alpha>_p = (\alpha, \alpha)_p > 0$$

The second claim is a consequence of Theorem 2.13.

The supertrace is zero because all the generators have supertrace equal to zero (the even generators $iL_j, i\Lambda_j$ are nilpotent).

As the form $<, >$ is odd and nondegenerate, the even and odd parts of the space must be in duality (like in the case of the $HW_k$). If we fix a Hermitean inner product on the odd part (and an induced compatible one on the even part), this duality can be turned into a linear isometry which we indicate with $*$, as in our case it is the Hodge $*$. As the manifold $X$ has odd dimension, the Hodge $*$ has also the property that $*^2 = Id$, so that this will hold also in all the $HW_k$. Summing up, there is an orthogonal decomposition

$$HW_k = HW_{ev}^k \oplus HW_{odd}^k$$

and the odd pairing $<,>_p$ puts in duality the odd part with the even one. The Hodge $*$ gives an isometric involution from $HW_k$ to itself which exchanges $HW_{ev}^k$ with $HW_{odd}^k$.

The standard arguments above can be summarized in the explicit description of all the maps which preserve the odd form $<,>_p$:

Proposition 3.7. Let $\phi : HW_k \rightarrow HW_k$ be a linear morphism whose homogeneous components preserve the odd pairing $<,>_p$ (i.e. $\phi$ preserves $<,>_p$). With respect to the above mentioned decomposition and isomorphism

$$HW_k = HW_{ev}^k \oplus HW_{odd}^k$$

$\phi$ must satisfy $\phi^* = -* \phi *$ and therefore it has the following form (indicating with $\phi_j^*$ the adjoint with respect to the Hermitean inner product)

$$\phi = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & -\phi_1^* \end{pmatrix}$$

where $\phi_1 : HW_{ev}^k \rightarrow HW_{ev}^k$, $\phi_2 : HW_{odd}^k \rightarrow HW_{ev}^k$, $\phi_3 : HW_{ev}^k \rightarrow HW_{odd}^k$.

Proof The condition follows immediately from $\phi^* = -* \phi *$ applied to the "matrix" of components of the operator $\phi$.

From the previous propositions and the fact that the supertrace of the elements of $L_3$ is zero one has

Corollary 3.8. The algebra $L_3$ has dimension at most

$$4 \times 20^2 + 4 \times 36^2 + 4 \times 20^2 + 4 \times 4^2 - 4 = 8444$$

In the remainder of the paper we will show that this estimate is almost sharp, being off by 48, and the previous propositions will be turned into a precise description of the algebra $L_3$ (see Theorem 6.4).

4. Some even operators and global relations

In this section we study some even operators obtained as superbrackets of the odd generators, the $K_{lm}$, and study how they relate to the even algebra generated by the $iL_j, i\Lambda_j$.

Definition 4.1. For $j \in \{0, 1, 2\}$

$$H_j = [i\Lambda_j, iL_j]$$
Proposition 4.2. The operators \( \{ iL_0, i\Lambda_0, iL_1, i\Lambda_1, iL_2, i\Lambda_2 \} \) generate the real Lie algebra \( \mathfrak{sl}(4, \mathbb{R}) \), and the \( \{ H_0, H_1, H_2 \} \) span a Cartan subalgebra.

Proof Referring to [GG], [G2], one sees that there is a Serre basis for the complexified algebra, given by the operators

\[
\begin{align*}
e_1 &= [L_0, \Lambda_1] & \text{f}_1 &= [L_1, \Lambda_0] & h_1 &= [e_1, f_1] \\
e_2 &= [L_1, \Lambda_2] & \text{f}_2 &= [L_2, \Lambda_1] & h_2 &= [e_2, f_2] \\
e_3 &= L_2 & \text{f}_3 &= \Lambda_2 & h_3 &= [e_3, f_3]
\end{align*}
\]

which is associated to the Dynkin diagram \( A_3 \). From this it is immediate to check that the real Lie algebra generated by \( H_j, iL_j, i\Lambda_j \) is a split form, isomorphic to \( \mathfrak{sl}(4, \mathbb{R}) \). This proof carries over also in arbitrary rank (of the WSD structure). □

In the rank 2 case (see [GG]) one verified that the brackets of the form \([V_j, A_m]\) were already contained in the Cartan subalgebra spanned by the \( H_k \). Contrary to that situation, in rank 3 the algebra is richer, and we get genuinely new operators:

Definition 4.3. For \( l, m \in \{0, 1, 2\} \)

\[ K_{lm} = iV_l A_m + iA_m V_l = [iV_l, A_m] \]

Proposition 4.4. For \( j, l, m \in \{0, 1, 2\} \) one has

\[ [H_j, K_{lm}] = (-3\delta_{jl} + 3\delta_{jm}) K_{lm} \]

In particular, \([H_j, K_{mm}] = 0 \) for all \( j, m \in \{0, 1, 2\} \).

Proof This follows using (super)Jacoby from the relations

\[ [H_j, V_m] = 3(1 - \delta_{jm}) V_m, \quad [H_j, A_m] = -3(1 - \delta_{jm}) A_m \]

which can be computed using the expressions in terms of the \( E_{ij} \), as in [GG]. □

Definition 4.5. We say that \( \alpha \in \bigwedge^* T_{p}^* X \) has \( Kw \) weight equal to \((z_0, z_1, z_2)\) if

\[ K_{mm}(\alpha) = z_m \alpha \]

for every \( m \in \{0, 1, 2\} \)

The \( H_j, K_{mm} \) form an abelian subalgebra from the previous proposition. However, although the generators of \( \mathcal{L}_3 \) are already eigenvectors for the adjoint action of the \( H_j \), they are not so for the adjoint action of the \( K_{mm} \). Nevertheless, the adjoint action of the \( K_{mm} \) can be diagonalized (over the real numbers) on the Lie subalgebra of \( \mathcal{L}_3 \) generated by the \( iL_j \) and the \( i\Lambda_k \):

Proposition 4.6. If a form \( \alpha \in \bigwedge^* T_{p}^* X \) is multidegree homogeneous of multidegree \((a, b, c)\) then is also \( Kw \) homogeneous, with \( Kw \)-weight:

\[ K\text{w}(\alpha) = i(-1)^{\text{deg}(\alpha)} (\delta_{a0} - \delta_{a3}, \delta_{b0} - \delta_{b3}, \delta_{c0} - \delta_{c3}) \]

As a consequence, the operators \( K_{mm} \) can be simultaneously diagonalized over \( \mathbb{C} \), with eigenvalues 0 and \( \pm i \).

Proof It is enough to prove the proposition for \( \alpha = v_{i_1j_1} \wedge \cdots \wedge v_{i_kj_k} \), a monomial in the forms \( v_{ij} \), which constitute as a basis for \( \bigwedge^* T_{p}^* X \). Such a monomial of multidegree \((a, b, c)\) can be written as \( m_0 \wedge m_1 \wedge m_2 \), where \( m_0 \) has multidegree \((a, 0, 0)\), \( m_1 \) has multidegree \((0, b, 0)\) and \( m_2 \) has multidegree \((0, 0, c)\). The action of \( iV_m \) and of \( A_m \) on such a monomial is easily described:

\[
\begin{align*}
iV_0 (m_0 \wedge m_1 \wedge m_2) &= i\delta_{a0} Vol(W_0) \wedge m_1 \wedge m_2 \\
A_0 (m_0 \wedge m_1 \wedge m_2) &= \delta_{a3}(-1)^{b+c} m_1 \wedge m_2 \\
iV_1 (m_0 \wedge m_1 \wedge m_2) &= i\delta_{a0} Vol(W_1) \wedge m_0 \wedge m_2
\end{align*}
\]
\[ A_1 (m_0 \wedge m_1 \wedge m_2) = \delta_{b3}(-1)^c m_0 \wedge m_2 \]
\[ tV_2 (m_0 \wedge m_1 \wedge m_2) = t\delta_{c0} Vol(W_2) \wedge m_0 \wedge m_1 \]
\[ A_2 (m_0 \wedge m_1 \wedge m_2) = \delta_{c3} m_0 \wedge m_1 \]

From these explicit expressions it is straightforward to verify that in this basis the \( K_{mm} = tV_m A_m + tA_m V_m \) are diagonal and their eigenvalues are 0 and \( \pm t \). More precisely, on a monomial \( m \) of multidegree \((a, b, c)\) one has therefore that

\[ K_{00}(m) = t(-1)^{deg(m)}(\delta_{a0} - \delta_{a3})m \]
\[ K_{11}(m) = t(-1)^{deg(m)}(\delta_{b0} - \delta_{b3})m \]
\[ K_{22}(m) = t(-1)^{deg(m)}(\delta_{c0} - \delta_{c3})m \]

As a consequence a monomial of multidegree \((a, b, c)\) will have \( Kw \) weight

\[ Kw(m) = t(-1)^{deg(m)}(\delta_{a0} - \delta_{a3}, \delta_{b0} - \delta_{b3}, \delta_{c0} - \delta_{c3}) \]

\[ \square \]

**Proposition 4.7.** The algebra \( L_3 \) contains the \( Kw \)-homogeneous components of the operators \( tL_j, t\Lambda_k \) and the \( t\Lambda_k \). The decomposition of the \( tL_j \) into \( Kw \)-homogeneous components is as follows:

\[ tL_0 = tL_0^{(0,0,0)} + tL_0^{(0,-1,0)} + tL_0^{(0,0,-1)} + tL_0^{(0,-1,-1)} \]
\[ tL_1 = tL_1^{(0,0,0)} + tL_1^{(-1,0,0)} + tL_1^{(0,0,-1)} + tL_1^{(-1,0,-1)} \]
\[ tL_2 = tL_2^{(0,0,0)} + tL_2^{(-1,0,0)} + tL_2^{(0,-1,0)} + tL_2^{(-1,-1,0)} \]

The corresponding decomposition for the \( t\Lambda_j \) has the same form, with opposite eigenvalues.

**Proof** The previous proposition allows us to identify the components \( Kw \)-homogeneous of any given operator which is multidegree homogeneous. In particular this applies to the \( tL_j \), which turn out to have only nonzero \( Kw \)-homogeneous components with eigenvalues 0, \(-t\) and have therefore the decomposition as in the statement of the proposition.

The \( t\Lambda_j \) have opposite multidegree and therefore opposite \( Kw \) degree with respect to the \( tL_j \).

If we indicate with \( p(x) \) the real polynomial \( x(x^2 + 1) \), one has that the operators \( p(ad(K_{mm})) \) vanish on the linear span of the \( tL_j \). From the decomposition \( 1 = (x^2 + 1) - x^2 \) one obtains that for all \( j, m \)

\[ (ad(K_{mm})^2 + Id) (tL_j) - ad(K_{mm})(tL_j) = tL_j \]

and moreover the first summand must be in the Kernel of \( ad(K_{mm}) \) and the second must be in the Kernel of \( ad(K_{mm})^2 + 1 \). The component of \( tL_j \) in the Kernel of \( ad(K_{mm})^2 + 1 \) is the sum of an eigenvector for \( ad(K_{mm}) \) with eigenvalue \( t \) and one with eigenvalue \(-t\). The first of these two however was shown to be zero before. It follows that iterating this procedure for the subsequent values 0, 1, 2 of \( m \) one obtains all the possibly nonzero \( Kw \)-homogeneous components of the \( tL_j \), which therefore lie inside \( L_3 \). The same reasoning applies to the \( t\Lambda_j \), whose \( Kw \)-homogeneous components are therefore in \( L_3 \).
In this section we find geometrically meaningful bases for the highest weight vector spaces \( HW_k \). These bases will allow us to study the algebra \( \mathcal{L}_3 \) more in detail, when in the next section we will restrict the newly generated operators to the subrepresentations \( HW_k \).

To begin with, we describe the space \( HW_0 \), which is made of highest weight vectors for \( \mathcal{L}_3 \). It turns out that it is possible to use the operators \( tL_j \) applied to \( 1 \in \Lambda^* T^* X_p \) and the \( t \Lambda_k \) applied to the volume form \( Vol(W_0 \oplus W_1 \oplus W_2) = Vol(W_0) \wedge Vol(W_1) \wedge Vol(W_2) \) to build a canonical of \( HW_0 \).

**Proposition 5.1.** The elements of the form \( p(tL_0, tL_1, tL_2) \cdot 1 \), where \( p \) is a monomial in three variables of degree less than or equal to 3, form a basis for \( HW_0^{ev} \).

The elements of the form \( p(t \Lambda_0, t \Lambda_1, t \Lambda_2) \cdot Vol(W_0 \oplus W_1 \oplus W_2) \), where \( p \) is a monomial in three variables of degree less than or equal to 3, form a basis for \( HW_0^{odd} \), corresponding to the even one by way of the Hodge \( \ast \) operator.

**Proof** The dimension of \( HW_0^{ev} \) (and \( HW_0^{odd} \)) is 20 (from table 1), and the Hodge \( \ast \) sends an element of the form \( p(tL_0, tL_1, tL_2) \cdot 1 \) precisely to \( p(t \Lambda_0, t \Lambda_1, t \Lambda_2) \cdot Vol(W_0 \oplus W_1 \oplus W_2) \) (from the proof of Theorem 2.13, which shows that \( t \Lambda_j = \ast tL_j \ast \) and the fact that \( 1 = Vol(W_0 \oplus W_1 \oplus W_2) \)). It is therefore enough to prove that the 20 elements of the form \( p(tL_0, tL_1, tL_2) \cdot 1 \) obtained by varying \( p \) among the monomials of degree less than or equal to three are all independent. To do that, we first observe that all these elements have different multidegree (as the \( tL_j \) are multidegree homogeneous, with independent multidegree vectors). To conclude it is therefore enough to show that all the elements above are different from zero. Of course it is enough to check the non-vanishing when \( p \) has degree three, and using the symmetric group \( S_3 \) we are reduced to prove that \(-tL_0^2 L_1 \cdot 1 \neq 0, -tL_0^3 \cdot 1 \neq 0 \) and \(-tL_0 L_1 L_2 \cdot 1 \neq 0 \). As the above are all wedge operators, this is equivalent to showing that the following three forms are different from zero (at all points):

\[
\omega_D \wedge \omega_D \wedge \omega_2, \quad \omega_D \wedge \omega_D \wedge \omega_D, \quad \omega_D \wedge \omega_1 \wedge \omega_2
\]

This can be easily checked using an adapted coframe like the one given by the \( \{v_{ij}\} \), or one can look at Theorem 1.6 of [G1]. \( \square \)

To describe a basis for \( HW_1 \) we can use the same reasoning as before, applying operators of the form \( p(tL_0, tL_1, tL_2) \) to the three forms \( w_{10}, w_{11}, w_{12} \) defined in the proof of Proposition 3.4, which are a basis for the space \( HW_1 \cap \bigwedge^1 T_p X \).

**Proposition 5.2.** The 12 elements

\[
w_{10}, \ tL_0 w_{10}, \ tL_1 w_{10}, \ tL_2 w_{10}, \ L_0^2 w_{10}, \ L_1 L_0 w_{10}, \ L_2 L_0 w_{10} \\
L_1^2 w_{10}, \ L_2 L_1 w_{10}, \ L_2^2 w_{10}, \ tL_0^3 w_{10}, \ tL_2 L_1 L_0 w_{10}
\]

together with the other 24 elements in their \( S_3 \) orbit, constitute a basis for \( HW_1^{odd} \).

The image of this basis under the Hodge \( \ast \) constitutes a basis for \( HW_1^{ev} \).

**Proof** A simple computation in coordinates shows that with respect to the \( sl(4, \mathbb{R}) \) action the \( (h_1, h_2, h_3) \)-weight of \( w_{10} \) is \((-1, 0, -2)\), it is a lowest weight vector and therefore the \( sl(4, \mathbb{R}) \) module that it generates is irreducible of dimension 36, and hence its complexification plus its image under the Hodge \( \ast \) coincides with \( HW_1 \). Moreover, out of the ordered linear basis

\[
\Lambda_0, \Lambda_1, \Lambda_2, [L_1, \Lambda_0], [L_2, \Lambda_2], [L_2, \Lambda_0], [L_2, \Lambda_1], H_0, H_1, H_2, [L_0, \Lambda_2], [L_0, \Lambda_1], L_0, L_1, L_2
\]

only the last 8 elements do not annihilate \( w_{10} \). Using the Poincaré-Birkhoff-Witt theorem to write a linear basis for \( U(sl(4, \mathbb{R})) \), one sees that a set of generators for this module is made out of polynomials in the \( L_j \) applied to the vectors \( w_{10}, w_{11} \) and
By considering the known \((h_1, h_2, h_2)-weight\) decomposition of this \(\mathfrak{sl}(4, \mathbb{R})\) module, and the fact that the operators \(L_j\) have independent \((h_1, h_2, h_2)-weight\), one finds that the only thing to check to prove linear independence of the elements listed in the proposition is that the following elements (and hence also the elements in their \(S_3\) orbits) are different from zero:

\[
iL_0^2w_{10}, \ iL_0L_1L_2w_{10}, \ L_1^2w_{10}
\]

The non-vanishing of the first one is immediate, as \(\omega_L^3\) is a volume form for the distribution \(W_1 \oplus W_2\). The third one can be shown to be nonzero by considering that the \(H_1\)-weight of \(w_{10}\) is \(-2\) and therefore the \(\mathfrak{sl}(2, \mathbb{C}) \cong <L_1, \Lambda_1, H_1>\) module that it generates must have dimension 3. The last non-vanishing is a simple computation in coordinates (or can be similarly proved using a repeated application of \(\mathfrak{sl}(2, \mathbb{C})\) actions).

To describe a basis for \(HW_2\) we can again apply the same reasoning as before, acting with operators of the form \(p(iL_0, iL_1, iL_2)\) on the three forms \(w_{10} \wedge w_{11}, w_{10} \wedge w_{12}\) and \(w_{11} \wedge w_{12}\), which are a basis for the space \(HW_2 \cap \bigwedge_{\mathbb{C},p}^2 T^*_p X\).

**Proposition 5.3.** The 20 elements

\[
\begin{align*}
w_{10} \wedge w_{11} & \quad \quad iL_0w_{10} \wedge w_{11} \quad iL_1w_{10} \wedge w_{11} \quad iL_2w_{10} \wedge w_{11} \\
L_0^2w_{10} \wedge w_{11} & \quad iL_0L_2w_{10} \wedge w_{11} \quad L_1^2w_{10} \wedge w_{11} \\
w_{10} \wedge w_{12} & \quad iL_0w_{10} \wedge w_{12} \quad iL_1w_{10} \wedge w_{12} \quad iL_2w_{10} \wedge w_{12} \\
L_0^2w_{10} \wedge w_{12} & \quad L_1^2w_{10} \wedge w_{12} \\
w_{11} \wedge w_{12} & \quad iL_0w_{11} \wedge w_{12} \quad iL_1w_{11} \wedge w_{12} \quad iL_2w_{11} \wedge w_{12} \\
L_0^2w_{11} \wedge w_{12} & \quad L_1L_2w_{11} \wedge w_{12} \quad L_2^2w_{11} \wedge w_{12}
\end{align*}
\]

constitute a basis for \(HW_2^{\mathfrak{g}}\). They together with their image under the Hodge * constitute a basis for \(HW_2\).

**Proof** A simple computation in coordinates shows that \(w_{10} \wedge w_{11}\) is a lowest weight vector of \((h_1, h_2, h_2)-weight\) equal to \((0, -1, -1)\) with respect to the \(\mathfrak{sl}(4, \mathbb{R})\) action, and therefore the \(\mathfrak{sl}(4, \mathbb{R})\) module that it generates is irreducible of dimension 20, and hence its complexification plus its image under the Hodge * coincides with \(HW_2\). Moreover, out of the ordered linear basis

\[
\Lambda_0, \Lambda_1, \Lambda_2, [L_1, \Lambda_0], [L_1, \Lambda_2], [L_2, \Lambda_0], [L_2, \Lambda_1], H_0, H_1, H_2, [L_0, \Lambda_0], [L_0, \Lambda_1], L_0, L_1, L_2
\]

only the last 8 elements do not annihilate \(w_{10} \wedge w_{11}\). Using the Poincaré-Birkhoff-Witt theorem to write a linear basis for \(\mathcal{U}(\mathfrak{sl}(4, \mathbb{R}))\), one sees that a set of generators for this module is made out of polynomials in the \(L_j\) applied to the vectors \(w_{10} \wedge w_{11}, w_{10} \wedge w_{12}\) and \(w_{11} \wedge w_{12}\).

The non vanishing of \(L_0^2w_{12}\) was proven in the previous proposition, and from this one gets immediately the non-vanishing of \(L_0^2w_{10} \wedge w_{12}\) and of all the other 5 elements in its \(S_3\)-orbit. Looking at the multidegree, one sees that all these nonzero elements are actually independent.

Among the remaining elements associated to polynomials homogeneous of degree 2, there are \(L_0L_1(w_{10} \wedge w_{11})\), \(L_1L_2(w_{11} \wedge w_{12})\) and \(L_2L_0(w_{12} \wedge w_{10})\), which form an \(S_3\)-orbit and have all the same multidegree \((2, 2, 2)\). We notice that there is the following relation (verifiable by hand in coordinates) among them:

\[
(1) \quad L_0L_1(w_{10} \wedge w_{11}) + L_1L_2(w_{11} \wedge w_{12}) + L_2L_0(w_{12} \wedge w_{10}) = 0
\]

The same computation also shows as a byproduct that any two of them are independent, and therefore the space that the three of them span has exactly dimension two. Taking into account also the three generators (corresponding to polynomials of degree zero) we have up to now \(8 + 3 = 11\) independent vectors in \(HW_2^{\mathfrak{g}}\). To finish the proof it is enough to show that the nine elements corresponding to polynomials
of degree one are all independent, as \( \text{HW}_3^v \) has dimension 20. Using multigree, the action of \( S_3 \) and the non-vanishing information on the elements of degree two, we are left with the following two verifications:

\[
iL_2(w_{10} \wedge w_{11}) \neq 0, \quad \text{dim} \left( \langle iL_0(w_{10} \wedge w_{11}) \rangle \right) = 2
\]

Both these verifications are straightforward, as the forms involved have degree 4. The first one can also be verified observing that the Lie algebra \( \text{sl}(2, \mathbb{C}) \equiv \langle L_2, A_2, H_2 \rangle \) acts on \( w_{10} \wedge w_{11} \) generating an irreducible module of dimension 2, as this element has \( H_2 \)-weight equal to \(-1\).

It remains to be described a basis for \( \text{HW}_3 \). We can again apply the same reasoning as before, acting with operators of the form \( p(iL_0, iL_1, iL_2) \) on the form \( w_{10} \wedge w_{11} \wedge w_{12} \), which generates \( \text{HW}_3 \cap \Lambda^3 \mathbb{C}T_p^X \).

**Proposition 5.4.** The elements

\[
w_{10} \wedge w_{11} \wedge w_{12}, \quad iL_0(w_{10} \wedge w_{11} \wedge w_{12}), \quad iL_1(w_{10} \wedge w_{11} \wedge w_{12}), \quad iL_2(w_{10} \wedge w_{11} \wedge w_{12})
\]

constitute a basis for \( \text{HW}_3^\text{odd} \). They together with their image under the Hodge * constitute a basis for \( \text{HW}_3 \).

**Proof** A simple computation in coordinates shows that \( w_{10} \wedge w_{11} \wedge w_{12} \) is a lowest weight vector of \( (h_1, h_2, h_2) \)-weight equal to \((0, 0, -1)\) with respect to the \( \text{sl}(4, \mathbb{R}) \) action, and therefore the \( \text{sl}(4, \mathbb{R}) \) module that it generates is irreducible of dimension 4, and hence its complexification plus its image under the Hodge * coincides with \( \text{HW}_3 \). To show that the four elements listed are independent, it is enough to show that they are different from zero, as they have all different multidegrees. Using the action of \( S^3 \) it is enough to show that \( iL_2(w_{10} \wedge w_{11} \wedge w_{12}) \neq 0 \). This follows by observing that the Lie algebra \( \text{sl}(2, \mathbb{C}) \equiv \langle L_2, A_2, H_2 \rangle \) acts on \( w_{10} \wedge w_{11} \wedge w_{12} \) generating an irreducible module of dimension 2, as this element has \( H_2 \)-weight equal to \(-1\).

6. **Restriction of \( \mathcal{L}_3 \) to the isotypical components**

Using the bases constructed in the previous section, we will now provide an explicit description for the action of \( \mathcal{L}_3 \) on the spaces \( \text{HW}_k \).

**Theorem 6.1.** For every point \( p \in X \), the operators \( V_0, V_1, V_2, A_0, A_1, A_2 \) are zero when restricted to the complex vector space \( \text{HW}_3 \) of \( \text{so}(3, \mathbb{C}) \) highest weight vectors inside the isotypical component of type \( \rho_3 \) of \( \Lambda^3 \mathbb{C}T_p^X \). Moreover, in the basis of Proposition 5.4 one has that the operators \( iL_0, iL_1, iL_2 \) acting on \( \text{HW}_3 \) have matrices:

\[
M_0(iL_0) = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad M_0(iL_1) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad M_0(iL_2) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

The restriction of \( \mathcal{L}_3 \) to \( \text{HW}_3 \) is isomorphic \( \text{sl}(4, \mathbb{R}) \).

**Proof** The vanishing of the operators \( V_0, V_1, V_2, A_0, A_1, A_2 \) on the basis elements is immediate for reasons of multidegree. The matrices for \( iL_0, iL_1, iL_2 \) in terms of the given basis are straightforward to compute. The matrices of the \( iL_3 \) acting on \( \text{HW}_3 \) are the transpose of those of the corresponding \( L_3 \) up to a nonzero scalar because \( iL_j = -(iL_j)^\ast \) for \( j = 0, 1, 2 \) and the basis is orthogonal with respect to the inner product for reasons of multidegree. The restriction of \( \mathcal{L}_3 \) to \( \text{HW}_3 \) is therefore isomorphic \( \text{sl}(4, \mathbb{R}) \). From Proposition 3.7 one has that also the restriction of \( \mathcal{L}_3 \) to \( \text{HW}_3 \) is isomorphic.
to $\mathfrak{sl}(4,\mathbb{R})$ (as the map sending an operator $\phi$ to $-\phi^*$ is a Lie algebra isomorphism).

Lemma 6.2. For all $p \in X$ and for $j \in \{0, 1, 2\}$ the restriction of the Lie algebra $\mathcal{L}_3^{ev}$ to the complex vector space $HW_j^{ev}$ of even $\mathfrak{so}(3,\mathbb{C})$ highest weight vectors inside the isotypical component of type $\rho_j$ of $\bigwedge^*_p T_p X$ includes the full complex special linear algebra: specifically it includes $\mathfrak{sl}(20,\mathbb{C})$ when $j = 0$ (respectively $\mathfrak{sl}(36,\mathbb{C})$ when $j = 1$ and $\mathfrak{sl}(20,\mathbb{C})$ when $j = 2$).

Proof In the following proof we obtain an explicit presentation of the restrictions of the algebra $\mathcal{L}_3^{ev}$ to the $HW_j^{ev}$. This is done making a “root search” using as starting point the $Kw$-homogeneous components of the restrictions of the generators $iL_0, iL_1, iL_2$, which lie in $\mathcal{L}_3$ from Proposition 4.7. We list in the Appendix (Tables 2, 3, 4) the matrices of these components, which turn out to be simple, having a few nonzero entries.

To build the tables we proceed as follows: given a $Kw$-homogeneous component $iL_j^{(u,v,z)}$ of an operator and a basis element $v$, we know that the multidegree of the form $iL_j^{(u,v,z)}(w)$ would be the sum of the multidegrees of $iL_j^{(u,v,z)}$ and of $v$. Using this information and Proposition 4.6 we obtain vanishing of many candidate entries. When the $Kw$-degrees match, we check whether $iL_j(v)$ is another basis element: in this case we get non-vanishing. This is sufficient, except in a specific case in $HW_2$ when we have to use relation 1 from the proof of Proposition 5.3 to express $iL_j^{(u,v,z)}(w)$ in terms of the basis. In the other cases the non-vanishing is verified directly, but this verification turns out to be irrelevant for the root search; this fact was signalled by putting the symbol $\times$ in the table (see the Appendix).

The completed root search proves that the restriction of $\mathcal{L}_3$ to $HW_j$ (for $j = 0, 1, 2$) includes a split real form for the full complex special linear algebra. As the Cartan contains the purely imaginary elements $K_i$, all the weight spaces which have $Kw$-weight different from $(0,0,0)$ must have real dimension two, and therefore the restriction must include the full complex special linear algebra.

Theorem 6.3. For all $p \in X$ and for $j \in \{0, 1, 2\}$ the restriction of the Lie superalgebra algebra $\mathcal{L}_3$ to the complex vector space $HW_j^{ev}$ of $\mathfrak{so}(3,\mathbb{C})$ highest weight vectors inside the isotypical component of type $\rho_j$ of $\bigwedge^*_p T_p X$ is the Lie superalgebra $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)$ of all the operators which preserve the odd nondegenerate super Hermitian inner product $< , >$.p.

Proof From Propositions 3.6 and 3.7, we know how the action of $\mathcal{L}_3^{ev}$ on the $HW_j^{odd}$ is determined by the action on $HW_j^{ev}$: in particular from the previous lemma we know that $\mathcal{L}_3^{ev}$ is mapped isomorphically onto the even part of $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)$. To conclude the proof, it remains to be proven that $\mathcal{L}_3|HW_j$ contains all the odd part of $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)$ and also the elements

$$\begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix}$$

Once obtained all the odd elements of $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)$, the one above can be obtained simply via the superbracket

$$\begin{pmatrix} iI & 0 \\ 0 & iI \end{pmatrix} = \frac{1}{2} \left[ \begin{pmatrix} 0 & I \\ iI & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ iI & 0 \end{pmatrix} \right]$$

We first observe that the odd operators of $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)$ which send even forms into odd ones and the operators which send odd forms into even ones are two modules of the same dimension for the adjoint action of the Lie algebra $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)^{ev}$ on $\mathfrak{su}(HW_j^{ev}|HW_j^{odd},< , >)^{odd}$, which for simplicity
we indicate in this proof respectively with $An_j$ and $He_j$ (as, from Proposition 3.7, they are made respectively of matrices which have anti-hermitean and hermitean blocks with respect to any real orthonormal basis). These two modules have a decomposition (as real vector spaces and also as $sl(n_j, \mathbb{R})$-modules, where $n_0 = 20, n_1 = 36$ and $n_2 = 20$)

$$He_j = Re(He_j) \oplus \mathbb{R}Im(He_j), \quad An_j = Re(An_j) \oplus \mathbb{R}Im(An_j)$$

Given an element of $L^u_{sl}(1,2)$ with components $\phi$ and $-\phi^*$ which act on the even and on the odd parts of $HW_j$, its adjoint action on the odd part of the algebra is as follows:

$$\begin{pmatrix} \phi & 0 \\ 0 & -\phi^* \end{pmatrix} \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} = \begin{pmatrix} -\phi^* A - A\phi & \phi B + B\phi^* \\ -\phi^* A - A\phi & 0 \end{pmatrix}$$

from which it follows easily that considering the action of the Lie subalgebra $sl(n_j, \mathbb{R}) \subset L^u_{sl}(1,2)$ given by real operators $\phi$, the four modules indicated above are all irreducible, of dimensions respectively $n_j(n_j + 1)/2, n_j(n_j - 1)/2, n_j(n_j - 1)/2, n_j(n_j + 1)/2$. As all the $v_j$ have nonzero components exactly in $iIm(He_j)$ and in $iIm(An_j)$ simply by taking into account their multidegree (and the fact that they are purely imaginary), by acting on them with the algebra $sl(n_j, \mathbb{R})$ we can generate all $iIm(He_j)$ and $iIm(An_j)$ (which are not isomorphic having different dimension). Similarly, by acting with the algebra $sl(n_j, \mathbb{R})$ on the operators $A_j$ we generate all $Re(He_j)$ and $Re(An_j)$.

\textbf{Theorem 6.4.} The restrictions to the $HW_j$ provide an isomorphism of Lie superalgebras

$$L_3 = su(HW^u_0|HW^o_0,<,>,p) \oplus su(HW^u_1|HW^o_1,<,>,p) \oplus$$

$$\oplus su(HW^u_2|HW^o_2,<,>,p) \oplus su(HW^u_3|HW^o_3,<,>,p,R)^{ev} \cong$$

$$\cong su(20|20,<,>) \oplus su(36|36,<,>) \oplus su(20|20,<,>) \oplus sl(4,R)$$

\textbf{Proof} The only thing remaining to be proven is that the restriction map of $L_3$ to on $HW_0 \oplus HW_2$ is surjective onto

$$su(HW^u_0|HW^o_0,<,>) \oplus su(HW^u_2|HW^o_2,<,>)$$

To do this, it is enough to find an even operator which restricts to zero on one module and is nonzero once restricted to the other. Ad example of these is given by the $K_{ij}$, which are nonzero on $HW_0$ and are zero on $HW_2$ for reasons of multidegree.

\hfill \Box

\textbf{Remark 6.5.} The (real) dimension of $L_3$ is 8396. This makes precise the observations at the end of Section 3.

\textbf{Corollary 6.6.} The complexification $L_{3,C}$ of $L_3$ is isomorphic via the restriction maps to the complex Lie superalgebra

$$sl(20|20) \oplus sl(36|36) \oplus sl(20|20) \oplus sl(4,C)$$

\section{Final remarks}

We have at this point enough information to prove that our complexified algebra $L_{3,C}$ is indeed a $*$-Lie superalgebra, with respect to the standard adjunction operator $*$ associated to the natural superHilbert space structure on $\Lambda^*_T X$. For convenience of the reader we reproduce here the definition of superHilbert space and $*$ operator (induced by the superadjunction $\dagger$), taken from [CCTV] and [V]:
Definition 7.1 ([CCTV]). A super Hilbert space is a super vector space \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \) over \( \mathbb{C} \) with a scalar product \( (\cdot, \cdot) \) such that \( \mathcal{H} \) is a Hilbert space under \( (\cdot, \cdot) \), and \( \mathcal{H}_i (i = 0, 1) \) are mutually orthogonal closed linear subspaces. If we define

\[
\langle x, y \rangle = \begin{cases} 
0 & \text{if } x \text{ and } y \text{ are of opposite parity} \\
(x, y) & \text{if } x \text{ and } y \text{ are even} \\
i(x, y) & \text{if } x \text{ and } y \text{ are odd}
\end{cases}
\]

then \( \langle x, y \rangle \) is an even super Hermitean form with

\[
\langle y, x \rangle = (-1)^{\text{deg}(x) \text{deg}(y)} \langle x, y \rangle, \quad \langle x, x \rangle > 0 \text{ if } x \neq 0 \text{ even}, \quad i^{-1} \langle x, x \rangle > 0 \text{ if } x \neq 0 \text{ odd}.
\]

If \( T(\mathcal{H} \to \mathcal{H}) \) is a bounded linear operator, we denote by \( T^* \) its Hilbert space adjoint and by \( T^\dagger \) its super adjoint given by \( \langle Tx, y \rangle = (-1)^{\text{deg}(T) \text{deg}(x)} \langle x, T^\dagger y \rangle \).

We remark that our spaces of sections form a preHilbert space, and would need to be completed to become a Hilbert space proper.

We proved in Theorem 6.4 that our real Lie superalgebra \( \mathcal{L}_3 \) is the full real Lie superalgebra of all the odd-unitary operators with respect to the odd super Hermitean inner product \( \langle \cdot, \cdot \rangle \) for all the \( \text{so}(3, \mathbb{R}) \)-isotypical components, save for the smallest one where we obtain only the real and even part of it. From this it follows that to prove that \( \mathcal{L}_3 \) is closed with respect to \( \dagger \) it is enough to show that an element of the form

\[
\begin{pmatrix} A & B \\ C & -B^T \end{pmatrix}
\]

is sent to an element of the same form by \( \dagger \). Notice that we are working on a fixed isotypical component, and this is allowed because the different components are orthogonal with respect to the inner product \( (\cdot, \cdot) \). It is now easy to compute:

\[
\begin{pmatrix} A & B \\ C & -B^T \end{pmatrix}^\dagger = \begin{pmatrix} iA & iC \\ -iB & A \end{pmatrix}
\]

and therefore if \( \phi \in \mathcal{L}_3 \) then \( \phi^\dagger \in \mathcal{L}_3 \). This implies that also \( \mathcal{L}_{3,C} \) is closed with respect to \( \dagger \), as was to be proven.

References

[B] V. Batyrev, Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3 (1994), 493-535

[BMP] U. Bruzzo, G. Marelli, F. Pioli A Fourier transform for sheaves on real tori Part II. Relative theory I. of Geometry and Phy. 41 (2002) 312-329

[CDGP] P. Candelas, X.C. De la Ossa, P.S. Green, L. Parkes, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, Nucl. Phys. B359 (1991), p 21-74

[CCTV] C. Carmeli, G. Cassinelli, A. Toigo, V.S. Varadarajan, Unitary Representations of Super Lie Groups and Applications to the Classification and Multiplet Structure of Super Particles, Commun. Math. Phys. 263, 217-258 (2006)

[GG] G. Gaiffi, M. Grassi A geometric realization of \( \mathfrak{sl}(6, \mathbb{C}) \)

[G1] M. Grassi, Polysymplectic spaces, s-Kähler manifolds and lagrangian fibrations, math.DG/0006154 (2000)

[G2] M. Grassi, Mirror symmetry and self-dual manifolds, math.DG/0202016 (2002)

[G3] M. Grassi, Self-dual manifolds and mirror symmetry for the quintic threefold, Asian J. Math 9 (2005) 79-102

[GP] B.R. Greene, M.R. Plesser, Duality in Calabi-Yau moduli space, Nucl. Phys. B338 (1990), 15-37

[GVW] B. R. Greene, C. Vafa, N. P. Warner, Calabi-Yau manifolds and renormalization group flows, Nucl. Phys. B324 (1989), 371-390

[Gr] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser P.M. 152, Boston 1999

[GW] M. Gross, P.M.H. Wilson, Large Complex Structure limits of K3 surfaces, math.DG/0008018 (2001)
Appendix A. Tables

The notation in the tables is explained by the following example: an entry of the form $L_{0}^{(0,0,-)}$ indicates that the matrix of the $Kw$-homogeneous component $tL_{0}$ of the operator $tL_{0}$ of $Kw$-degree $(0,0,-i)$ has a nonzero entry in that position. The basis elements have been indicated as follows: $(a,b,c)$ indicates the form $(i)^{a+b+c}L_{a}^{0}L_{b}^{1}L_{c}^{2}$ (Table 2), $(a,b,c)j$ indicates the form $(i)^{a+b+c}L_{a}^{0}L_{b}^{1}L_{c}^{2}w_{1j}$ (Tables 4, 5) and $(a,b,c)jk$ indicates the form $(i)^{a+b+c}L_{a}^{0}L_{b}^{1}L_{c}^{2}w_{1j} \wedge w_{1k}$ (Table 3). When an entry is followed by the symbol $\times$ it means that the component is present with a nonzero coefficient, that this fact was proven by a direct “brute force” computation and that it is in any case irrelevant to know whether this coefficient is zero or not.
| (0,0,0) | (1,0,0) | (1,0,1) | (0,0,1) | (2,0,0) | (1,1,0) | (1,0,1) | (0,1,1) | (0,0,2) | (3,0,0) | (2,1,0) | (2,0,1) | (1,2,0) | (1,1,1) | (1,0,2) | (0,3,0) | (0,2,1) | (0,1,2) | (0,0,3) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| (1,0,0) |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| (0,1,0) |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| (0,0,1) |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| (1,1,0) |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |        |        |
| (0,1,1) |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |        |
| (0,0,2) |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |        |
| (1,0,1) |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |        |
| (0,1,0) |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |        |
| (0,0,1) |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |        |
| (1,0,0) |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |        |
| (1,0,1) |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |        |
| (1,1,0) |        |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |        |
| (1,2,0) |        |        |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |        |
| (1,1,1) |        |        |        |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |        |
| (1,0,2) |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |        |
| (0,3,0) |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |        |
| (0,2,1) |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        | $L_{1}^{(0,0,0)}$ |
| (0,1,2) |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| (0,0,3) |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |        |

Table 2. Table for $HW_0$
Table 3. Table for $HW_2$
| 0.0.0.00 | 0.1.0.00 | 1.0.0.00 | 0.0.0.10 | 0.2.0.00 | 1.1.0.00 | 1.0.1.00 | 0.1.1.00 | 0.2.0.10 | 1.1.1.00 | 1.0.1.10 | 0.2.0.11 | 1.1.1.10 | 1.0.1.11 | 0.2.0.12 | 1.1.1.12 | 1.0.1.12 | 0.2.0.13 | 1.1.1.13 | 1.0.1.13 | 0.2.0.14 | 1.1.1.14 | 1.0.1.14 |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| \( L_{0}^{(-,-,-)} \) | \( L_{0}^{(-,0,-)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-,-)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) | \( L_{0}^{(-,-0)} \) |

Table 4. Table for \( HW_1 \), first part
| Column 1 | Column 2 | Column 3 | Column 4 | Column 5 | Column 6 | Column 7 | Column 8 | Column 9 | Column 10 | Column 11 | Column 12 | Column 13 | Column 14 |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.0.0.0 | 0.2.0.1 | 0.1.1.1 | 0.0.0.2 | 0.1.1.1 | 0.0.0.1 | 0.2.0.1 | 0.1.1.1 | 0.0.0.1 | 0.2.0.1 |
| 1.0.0.1 | 0.2.0.1 | 0.1.1.1 | 0.0.0.2 | 0.1.1.1 | 0.0.0.1 | 0.2.0.1 | 0.1.1.1 | 0.0.0.1 | 0.2.0.1 |

**Table 5.** Table for HW₁, second part