Prevalent behavior and almost sure Poincaré-Bendixson Theorem for smooth flows with invariant $k$-cones

Yi Wang, Jinxiang Yao, and Yufeng Zhang

School of Mathematical Science
University of Science and Technology of China
Hefei, Anhui, 230026, P. R. China

Abstract

We investigate the global dynamics from a measure-theoretic perspective for smooth flows with invariant cones of rank $k$. For such systems, it is shown that prevalent (or equivalently, almost all) orbits will be pseudo-ordered or convergent to equilibria. This reduces to Hirsch’s prevalent convergence Theorem if the rank $k = 1$; and implies an almost-sure Poincaré-Bendixson Theorem for the case $k = 2$. These results are then applied to obtain an almost sure Poincaré-Bendixson theorem for high-dimensional differential equations.

1 Introduction and Main Results

In this article, we are interested in the global dynamics in measure-theoretic sense for flows with invariant cones of rank $k$ (abbr. $k$-cone). Such a flow is also called monotone with respect to $k$-cone $C$ (see Section 2). Here, a $k$-cone $C \subset \mathbb{R}^n$ means a closed subset that contains a linear $k$-dimensional subspace and no linear subspaces of higher dimension.

From a measure-theoretic (or probabilistic) perspective, prevalence is a quite useful notion, introduced by Christensen [4] and Hunt et al. [17], that describes properties of interest occurs for “almost surely”. For Euclidean spaces, it is equivalent to notion of full Lebesgue measure (see Section 2). Over decades since its development, prevalence have undergone extensive investigations. We refer to [5, 17, 18, 21] (and references therein) for more details.

In dynamical systems, prevalence is parallel to another classical notion called genericity, which formulates in the topological sense the properties of interest occurs residually (i.e., on a
countable intersection of open dense subsets in a Baire space). Many examples in dynamical systems are known to be both generic in the topological sense and prevalent in measure-theoretic sense. However, on the other hand, there are also many cases properties that are generic are not prevalent, and vice-versa (see, e.g. [18,21]).

We point out that a $k$-cone is not a standard cone defined in the literature, but adopted from by Fusco-Oliva [9,10] (see also Krasnoselskii et al. [22] for a Banach space). In particular, a convex cone $K$ (defined in the standard sense) gives rise to a 1-cone, $K \cup (-K)$. Therefore, the class of flows we consider here naturally includes the classical monotone dynamical systems since the ground-breaking work of M.W. Hirsch [11–14] (see monographs or surveys [1,16,34,38,39], we call it as classical monotone systems). There is however an essential difference between a $k(\geq 2)$-cone and a convex cone $K$, due to the lack of convexity in the former. The lack of convexity requires substantially new ideas for exploring the implication of “monotonicity” for the dynamics of the considered flows. Typical examples of flows monotone with respect to $k(\geq 2)$-cones are the high-dimensional competitive systems (see, for example [2,3,15,19,20,28–30,46]); systems with quadratic cones and Lyapunov-like functions (see, for example, [31,41,42]); and monotone cyclic feedback systems with negative feedback [25–27] arising from a wide range of neural and physiological control systems, etc..

Classical monotone dynamical systems are abundant and important sources of topological genericity and prevalence. For continuous-time systems, the celebrated Hirsch’s generic convergence theorem first indicates that the generic precompact orbit approaches the set of equilibria (see Poláčik [32,33] and Smith-Thieme [40] for the improved version). Meanwhile, Enciso, Hirsch and Smith [6] investigated the prevalent behavior and proved that the set of points that converge to equilibria is prevalent. For discrete-time systems (mappings), Poláčik-Tereščák [35] first proved that the generic convergence to periodic orbits occurs provided that the mapping $F$ is of class $C^{1,\alpha}$ (see Tereščák [45] and Wang-Yao [47] for $C^1$-smooth systems). Very recently, the present authors [48] proved the set of points that converge to periodic orbits is prevalent.

An insightful observation for classical strongly monotone systems exhibits a kind of important typical nontrivial orbits, called pseudo-ordered orbits. More generally, for a $k$-cone $C$, a nontrivial orbit \( O(x) := \{\Phi_t(x) : t \geq 0\} \) is pseudo-ordered if \( O(x) \) possesses one pair of distinct points \( \Phi_t(x), \Phi_s(x) \) such that \( \Phi_t(x) - \Phi_s(x) \in C \). This is also first handled by Sanchez [36]. Very recently, Feng, Wang and Wu [8] proved that generic (open and dense) orbits in \( \mathbb{R}^n \) are either pseudo-ordered or convergent to equilibria. This covers the Hirsch’s Generic Convergence Theorem in the case \( k = 1 \). Together with their previous work [7], a generic Poincaré-Bendixson Theorem is thus obtained for the case \( k = 2 \) (see [8, Theorem B]).

In the present paper, we focus on the global behavior of a flow in strongly monotone with
respect to $k(\geq 2)$-cone $C$ from a measure-theoretic perspective. Before describing our main results and our approach, we formulate the standard assumptions:

(H) The flow $\Phi_t$ is $C^{1,\alpha}$-smooth and strongly monotone with respect to a $k$-solid cone $C$ in $\mathbb{R}^n$. Moreover, the $x$-derivative $D_x\Phi_t$ satisfies $D_x\Phi_t(C\setminus\{0\}) \subset \text{Int}C$ for $t > 0$.

We refer to Section 2 for more detailed definitions about strong monotonicity and others. Throughout the paper, we always assume (H) holds. Let

$$Q = \{x \in \mathbb{R}^n : \text{the orbit } O(x) \text{ is pseudo-ordered}\}.$$ 

**Theorem A (Prevalent Dynamics Theorem).** Let $D \subset \mathbb{R}^n$ be an open bounded set such that the orbit set $O(D)$ of $D$ is bounded. Then the set $\{x \in D : x \in Q \text{ or } \omega(x) \text{ is a singleton}\}$ is prevalent in $D$. Furthermore, the set has full Lebesgue measure in $D$.

Theorem A concludes that, for almost every point in $D$, the orbit is either pseudo-ordered or convergent to a single equilibrium. As we mentioned above, the set

$$D' \triangleq \{x \in D : x \in Q \text{ or } \omega(x) \text{ is a singleton}\}$$

is also generic (i.e., contains an open and dense subset) in $D$ (c.f. [8, Theorem A]). This entails that, for smooth flow strongly monotone with respect to $k$-cone, the fact “convergence to an equilibrium or the orbit is pseudo-ordered” is both prevalent in measure-theoretic sense and generic in the topological sense.

Needless to say, if the rank $k = 1$, Theorem A in conjunction with the Monotone Convergence Criterion (see [38]) naturally reduces to the prevalent convergence obtained by Enciso, Hirsch and Smith [6, Theorem 1] for $\mathbb{R}^n$.

Moreover, if the rank $k = 2$, we have the following:

**Theorem B (Almost Sure Poincaré-Bendixson Theorem).** Let all hypotheses in Theorem A hold. Assume also that $k = 2$ and $C$ is completed. Then, for almost every point $x \in D$, the $\omega$-limit set $\omega(x)$ containing no equilibria is a single closed orbit.

As mentioned above, Feng et.al [8] obtained the so called “Generic Poincaré-Bendixson Theorem”, that is, for generic (open and dense) points $x \in D$, the $\omega$-limit set $\omega(x)$ containing no equilibria is a single closed orbit. A drawback of topological genericity is that open dense subsets can be arbitrarily small in terms of measure. However, Theorem B here exhibits that, for smooth flow strongly monotone with respect to 2-cone, the “Poincaré-Bendixson Theorem” holds not only generically in the topological sense, but also almost-surely in measure-theoretic sense.
Compared to the classical monotone dynamical flows, the generic/prevalent behavior of the flows with invariant $k(\geq 2)$-cones is much more complicated. Due to the loss of convexity, the “order”-relation defined by $C$ (see Section 2) is not a partial order, because it is neither antisymmetric nor transitive. So, it requires substantially new techniques for exploring the implication of monotonicity for the dynamics of the considered flows.

Our approach involves the ergodic argument using the $k$-exponential separation and the associated $k$-Lyapunov exponent (that reduces to the first Lyapunov exponent if $k = 1$), which was first introduced in [8]. Plus, we present a novel interesting lemma, called the Probe Lemma (see Section 3), which plays a decisive role for the proof of the main results. Roughly speaking, this lemma indicates that one can detect the pseudo-ordered orbits surrounding the point $z \notin D'$, by tracking the information restricted on a $k$-dimensional probe $P$ passing through $z$ (see Lemma 3.1). Based on the Probe Lemma, we succeeded in proving the main results in Section 4.

In section 5, our main results will be applied to an $n$-dimensional ODE system with a quadratic cone and obtain the almost sure Poincaré-Bendixson theorem in such high-dimensional system. R. A. Smith [41, 42] has ever obtained a Poincaré-Bendixson theorem for such system under the assumption of the existence of a quadratic Lyapunov function. The connections between Smith’s results and the systems with invariant 2-cones was observed by Sanchez [36]. Feng et.al [8] proved a generic Poincaré-Bendixson Theorem even if a quadratic Lyapunov function can not be constructed. Our Theorem B here shows that for such system, the “Poincaré-Bendixson Theorem” holds not only generically, but almost-surely in measure-theoretic sense.

2 Notations and Preliminary Results

In this section, we will fix some fundamental notations. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with a norm $\| \cdot \|$ and $\Phi(t, x)$ be the flow on $\mathbb{R}^n$, which is a continuous map $\Phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ with: $\Phi_0(x) = (x)$ and $\Phi_t \circ \Phi_s(x) = \Phi_{t+s}(x)$ for $t, s \in \mathbb{R}$. We will henceforth continue to denote $\Phi(t, x)$ by $\Phi_t(x)$. A $C^{1, \alpha}$-smooth flow $\Phi_t$ on $\mathbb{R}^n$ is a flow which $\Phi_t|_{\mathbb{R} \times \mathbb{R}^n}$ is a $C^{1, \alpha}$-map (a $C^1$-map with a locally $\alpha$-Hölder derivative) with $\alpha \in (0, 1]$. We denote the derivatives of $\Phi_t$ with respect to $x$ at $(t, x)$ by $D_x \Phi_t$.

For a flow $\Phi(t, x)$ on $\mathbb{R}^n$, the orbit of $x$ is the set $O(x) = \{ \Phi_t(x) : t \in \mathbb{R} \}$. The $\omega$-limit set of $x$ is $\omega(x) = \cap_{s \geq 0} \cup_{t \geq s} \Phi_t(x)$. We say that the flow $\Phi_t$ is $\omega$-compact in a subset $X \subset \mathbb{R}^n$ if $O(x)$ is relatively compact for each $x \in X$ and $\bigcup_{x \in X} \omega(x)$ is relatively compact in $\mathbb{R}^n$. A point $x$ is an equilibrium if $O(x) = \{ x \}$. The set of equilibria is denoted by $\mathcal{E}$.

Definition 2.1. Let $C \subset \mathbb{R}^n$ be a closed set. It is called a $k$-cone of $\mathbb{R}^n$ if it satisfies:

(i) For any $v \in C$ and $l \in \mathbb{R}$, $lv \in C$;
Moreover, the integer \( k(\geq 1) \) is called the rank of \( C \).

A \( k \)-cone \( C \subset \mathbb{R}^n \) is called \( k \)-solid if there is a \( k \)-dimensional linear subspace \( W \) such that \( W \setminus \{0\} \subset \text{Int}C \), and is called complemented if there is a \( k \)-codimensional space \( H^c \subset \mathbb{R}^n \) such that \( H^c \cap C = \{0\} \). A pair \( x, y \in \mathbb{R}^n \) are said to be ordered if \( x - y \in C \), denoted by \( x \sim y \); otherwise, \( x \) and \( y \) are called to be unordered. We also write \( x \approx y \) and call \( x \) and \( y \) are strongly ordered if \( x - y \in \text{Int}C \). Two sets \( U, V \subset \mathbb{R}^n \) are said to be strongly ordered if \( x - y \in C \) for any \( x \in U \) and \( y \in V \), denoted by \( U \approx V \).

3 The Probe Lemma

Before we focus on our main theorems, we need to present a technical lemma, called the Probe Lemma, which turns out to be very crucial for our approach. To simplify the notation, we write

\[
S = \{ x \in \mathbb{R}^n : \omega(x) \text{ is a singleton} \}.
\]

Clearly, \( S \) is a Borel set (see [6, lemma 9]).

We call a \( k \)-dim linear subspace \( P \subset \mathbb{R}^n \) a \( k \)-probe if \( P \subset \text{Int}C \) (motivated by [18]). Given any \( x \in \mathbb{R}^n \), let \( P_x = x + P \). Clearly, \( P_x = P_y \) if and only if \( x - y \in P \). Let \( B(x, \varepsilon) \) be the \( \varepsilon \)-neighborhood of \( x \). The set \( B^P(x, \varepsilon) = P_x \cap B(x, \varepsilon) \) is called the probe \( \varepsilon \)-neighborhood of \( x \).
Lemma 3.1 (Probe Lemma). Let all hypotheses in Theorem A hold. For any \( x \in D \setminus (Q \cup S) \) and any \( k \)-probe \( P \), there exists \( \varepsilon > 0 \) such that \( B^P(x, \varepsilon) \setminus \{x\} \subset Q \).

Remark 3.2. When the rank \( k = 1 \), the classical Monotone Convergence Criterion (c.f. [38, Theorem 1.2.1]) yields that \( Q \subset S \) in \( D \). Under such spacial circumstances, the Probe Lemma naturally reduces to “For \( x \in D \setminus S \), there is a neighborhood \( N \) of \( x \) such that, for any \( y \in N \) with \( y \sim x \), one has \( y \in S \).” This fact is a part of Sequential Limit Trichotomy (see, e.g. [38, Theorem 1.4.1(b) or Proposition 1.4.4(d)]) for classical monotone flows, or parallel to the Discontinuity Principle (see, e.g. [43, Proposition 3.3]) for classical monotone discrete-time systems.

Before proving the Probe Lemma, we are going to introduce some tools as \( k \)-Exponential Separation and \( k \)-Lyapunov Exponents, which we inherit from Feng, Wang and Wu [8] and play an important role in the following proof.

Let \( K \subset \mathbb{R}^n \) be an invariant compact subset. Then, \( (\Phi_t, D\Phi_t) \) naturally defines a linear skew-product flow on \( K \times \mathbb{R}^n \). Let \( \{Z_x\}_{x \in K} \) be a family of \( k \)-dimensional subspaces of \( \mathbb{R}^n \), and \( G(k, \mathbb{R}^n) \) consists of \( k \)-dim linear subspaces of \( \mathbb{R}^n \) which is a complete metric space by endowing the gap metric (see, e.g. [23, 24]). Then, \( K \times (Z_x) \) is said to be a \( k \)-dimensional continuous vector bundle if the map \( K \rightarrow G(k, \mathbb{R}^n) : x \mapsto Z_x \) is continuous. Also, we say the continuous vector bundle \( K \times (Z_x) \) is invariant with respect to \( (\Phi_t, D\Phi_t) \) if \( D\Phi_t Z_x = Z_{\Phi_t x} \) for any \( x \in K \) and \( t \geq 0 \).

We call \( (\Phi_t, D\Phi_t) \) admits a \( k \)-exponential separation along \( K \) (for short, \( k \)-exponential separation), if there are \( k \)-dimensional continuous bundle \( K \times (E_x) \) and \((n-k)\)-dimensional continuous bundle \( K \times (F_x) \) such that

(i) \( \mathbb{R}^n = E_x \oplus F_x \), for any \( x \in K \);

(ii) \( D_x \Phi_t E_x = E_{\Phi_t(x)} \), \( D_x \Phi_t F_x \subset F_{\Phi_t(x)} \) for any \( x \in K \) and \( t > 0 \);

(iii) there are constants \( M > 0 \) and \( 0 < \gamma < 1 \) such that

\[ \|D_x \Phi_t w\| \leq M \gamma^t \|D_x \Phi_t v\| \]

for all \( x \in K \), \( w \in F_x \cap S \), \( v \in E_x \cap S \) and \( t \geq 0 \), where \( S = \{v \in \mathbb{R}^n : \|v\| = 1\} \).

In addition, if \( C \subset \mathbb{R}^n \) is a \( k \)-solid cone and

(iv) \( E_x \subset \text{Int} C \cup \{0\} \) and \( F_x \cap C = \{0\} \) for any \( x \in K \),

then \( (\Phi_t, D\Phi_t) \) is called to admit a \( k \)-exponential separation along \( K \) associated with \( C \).

It is known that \( (\Phi_t, D\Phi_t) \) admits a \( k \)-exponential separation along \( K \) associated with \( C \) if the flow \( \Phi_t \) satisfied our assumption (H) (see, e.g. [44, Corollary 2.2 or Theorem 4.1] and [29, Proposition A.1]).
Let $K \subset \mathbb{R}^n$ be an invariant compact subset for $\Phi_t$, and $(\Phi_t, D\Phi_t)$ admit a $k$-exponential separation

$$K \times \mathbb{R}^n = E \oplus F,$$

where $E = K \times (E_x)$ and $F = K \times (F_x)$. For each $x \in K$, we define the $k$-Lyapunov exponent as

$$\lambda_{kx} = \limsup_{t \to \infty} \frac{\log m(D_x \Phi_t|E_x)}{t},$$

where $m(D_x \Phi_t|E_x) = \inf_{v \in E_x \cap S} \|D_x \Phi_t v\|$ is the infimum norm of $D_x \Phi_t$ restricted to $E_x$. When $k = 1$, $\lambda_{kx}$ naturally reduces to first (or principal) Lyapunov exponent (see, e.g. [35]).

A regular point is a point $x \in K$ for which $\lambda_{kx} = \lim_{t \to \infty} \frac{\log m(D_x \Phi_t|E_x)}{t}$.

**Proposition 3.3.** For any $x \in D$, one of the alternatives must occur:

(i) $x \in Q \cup S$;

(ii) For any sequence $\{x_n\}_{n=1}^\infty \subset D \setminus \{x\}$ with $x_n \to x$ and $x_n \sim x$, there are two points $p \in \omega(x)$, $q \in \omega_c(x)$ such that $p \sim q$, where $\omega_c(x) = \cap_{n \geq 1} \cup_{m \geq n} \omega(x_m)$.

**Proof.** Our proof is motivated by the approach in [8, Section 4]. Denote by $\omega_0(x)$ the set of regular points on $\omega(x)$, i.e.,

$$\omega_0(x) = \{z \in \omega(x) : z \text{ is a regular point}\}.$$

Then one of the following alternatives must occur:

(a) $\lambda_{kz} \leq 0$ for some point $z \in \omega_0(x)$;

(b) $\lambda_{kz} > 0$ for any $z \in \omega(x)$;

(c) $\lambda_{kz} > 0$ for any $z \in \omega_0(x)$; while $\lambda_{k\tilde{z}} < 0$ for some $\tilde{z} \in \omega(x) \setminus \omega_0(x)$.

We will discuss the three cases one by one.

If (a) holds, then it follows from [8, Theorem 4.2] that $x \in Q \cup S$. Thus, item (i) holds.

If (b) holds, we choose any sequence $\{x_n\}_{n=1}^\infty \subset D \setminus \{x\}$ satisfying $x_n \to x$ and $x_n \sim x$.

Clearly, $\omega_c(x)$ is nonempty, compact and invariant. By virtue of [8, Lemma 4.4], there exists a constant $\delta > 0$ such that

$$\limsup_{t \to \infty} \|\Phi_t(x_n) - \Phi_t(x)\| \geq \delta, \quad \text{for each } n \geq 1.$$

Consequently, for each $n \geq 1$, there are $p_n \in \omega(x)$ and $q_n \in \omega(x_n)$ such that $\|p_n - q_n\| \geq \delta$ and $p_n \sim q_n$. Without loss of generality, one may assume that $p_n \to p \in \omega(x)$ and $q_n \to q \in \omega_c(x)$, as $n \to \infty$. Therefore, one has $p \sim q$ and $p \neq q$. Thus, item (ii) holds.

Finally, if (c) holds, then for any $y \in \omega(x)$, we write $v_y := \frac{d}{dt}|_{t=0} \Phi_t(y)$ and let

$$\lambda(z, v_z) = \limsup_{t \to \infty} \frac{\log \|v_{\Phi_t(z)}\|}{t}, \quad \text{for any } z \in \omega(x) \setminus \omega_0(x).$$

Fix any $z \in \omega(x) \setminus \omega_0(x)$, clearly, $\lambda(z, v_z) \leq 0$, since $\|v_{\Phi_t(z)}\|$ is bounded uniformly for $t \geq 0$.
When $\lambda(z,v_z) = 0$, we have $v_z \notin F_z$ (For otherwise, [8, Lemma 3.6(i)] implies that $\lambda(z,v_z) \leq \lambda_k + \log(\gamma) < 0$, a contradiction). Therefore, by the $k$-exponential separation, there is $T > 0$ such that $D_z \Phi_t(v_z) \in \text{IntC}$ for any $t > T$, which implies $z \in Q$. Noticing that $z \in \omega(x)$, we obtain $x \in Q$. Thus, item (i) holds.

When $\lambda(z,v_z) < 0$, we have $\nu_{\Phi_t(z)} \rightarrow 0$ as $t \rightarrow \infty$. This implies $\omega(z)$ only consists of equilibria. Thus, $\omega(z) \subset \omega_0(x)$. Hence, due to (c), one has

$$\lambda_{k\bar{z}} > 0, \text{ for any } \bar{z} \in \omega(z). \quad (3.1)$$

Now, define $C_x = \{y \in D : y \neq x \text{ and } y \sim x\}$ and choose a sequence $t_n \rightarrow \infty$ such that $\Phi_{t_n}(x) \rightarrow z$ as $n \rightarrow \infty$.

If there exists $y \in C_x$ with a subsequence $\{t_{n_i}\}_{i=1}^\infty$ of $\{t_n\}_{n=1}^\infty$, such that $\Phi_{t_{n_i}}(y) \rightarrow z$, then [7, Lemma 4.3] (or [8, Lemma 2.4]) implies that $z \in Q \cup E$. Recall that $z \notin \omega_0(x)$. Then, we have $z \in Q$. Hence, we again obtain that $x \in Q$, which means item (i) holds.

On the other hand, if for any $a \in C_x$, there is a subsequence $t_{n_j} \rightarrow \infty$ of $\{t_n\}_{n=1}^\infty$ such that $\Phi_{t_{n_j}}(a) \rightarrow z_a \neq z$ as $j \rightarrow \infty$. Clearly, $z_a$ and $z$ are ordered for any $a \in C_x$. By virtue of (3.1), we can again utilize [8, Lemma 4.4] (for $\omega(z)$) to obtain that there exists $\delta > 0$ such that

$$\limsup_{t \rightarrow \infty} \|\Phi_t(z_a) - \Phi_t(z)\| \geq \delta, \text{ for any } a \in C_x. \quad (3.2)$$

Take any sequence $\{x_k\}_{k=1}^\infty \subset D \setminus \{x\}$ with $x_k \rightarrow x$ and $x_k \sim x$. For each $k$, we utilize (3.2) (hence, for each $z_{x_k}$) to obtain that there are $p_k \in \omega(z)$ and $q_k \in \omega(z_{x_k})$ such that $\|p_k - q_k\| \geq \delta$ and $p_k \sim q_k$. Recall that $\Phi_{t_{n_j}}(x_k) \rightarrow z_{x_k}$ as $j \rightarrow \infty$, one has $z_{x_k} \in \omega(x_k)$ and hence $\omega(z_{x_k}) \subset \omega(x_k)$. Thus we obtain $q_k \in \omega(x_k)$. And also $p_k \in \omega(x)$ since $\omega(z) \subset \omega(x)$. Without loss of generality, one may assume that $p_k \rightarrow p \in \omega(x)$ and $q_k \rightarrow q \in \omega_c(x)$, as $k \rightarrow \infty$. Therefore, one has $p \sim q$ and $p \neq q$, which means item (ii) holds. Thus, we have completed the proof.

Now, we are going to prove the Lemma 3.1.

**Proof of lemma 3.1 (Probe Lemma).** We prove this lemma by contradiction. Suppose that there exists a sequence $x_n \rightarrow x$ with $x_n \neq x$, such that $x_n \in P \setminus Q$ for any $n \in \mathbb{N}$.

Then, by recalling that $x \notin Q \cup S$, it follows from Proposition 3.3 that there are $p \in \omega(x)$ and $q \in \cap_{n \geq 1} \cup_{m \geq n} \omega(x_m)$ such that $p \neq q$ and $p \sim q$. Since $\Phi$ is strongly monotone, we may assume that $p \approx q$. Let $U$ and $V$ be the neighborhood of $p$ and $q$, respectively, such that $U \approx V$. Choose two sequences $s_k \rightarrow \infty$ and $n_k \rightarrow \infty$ such that $\Phi_{s_k}(x_{n_k}) \in V$ for all $k$ sufficiently large. Let also $\tau > 0$ be such that $\Phi_{\tau}(x) \in U$. Then, $\Phi_{\tau}(x_{n_k}) \in U$; and hence, $\Phi_{\tau}(x_{n_k}) \approx \Phi_{s_k}(x_{n_k})$ for all $k$ sufficiently large. As a consequence, one has $x_{n_k} \in Q$ for all $k$ sufficiently large, a contradiction. Thus, we have completed the proof. \qed
4 Proof of the main Theorems

In this section, we will prove our main Theorems. For this purpose, besides the Lemma 3.1 (Probe Lemma), we still need the following lemma to verify the shyness of a subset \( W \subset \mathbb{R}^n \).

**Lemma 4.1.** Let \( W \subset \mathbb{R}^n \) be a Borel subset. Assume that there exists a \( k \)-probe \( P \subset \text{int} C \) such that \( W \cap P_v \) is at most countable for any \( v \in \mathbb{R}^n \). Then \( W \) is shy.

**Proof.** Fix the \( k \)-dimensional unit disc \( D \subset P \). Define the measure on \( \mathbb{R}^n \) as

\[
\mu_D(A) \triangleq m(A \cap D), \quad \text{for any } A \in \mathcal{B}(\mathbb{R}^n).
\]

Here \( m \) is the Lebesgue measure on \( P \), and \( \mathcal{B}(\mathbb{R}^n) \) consists of Borel sets of \( \mathbb{R}^n \). Clearly, \( \mu \) is a nonzero compactly supported Borel measure on \( \mathbb{R}^n \). Moreover, for any \( v \in \mathbb{R}^n \), one has

\[
(v + W) \cap D \subset (v + W) \cap P = v + (W \cap P_v).
\]

Since \( W \cap P_v \) is at most countable, we have \( m((v + W) \cap D) = 0 \). By (4.1), this entails that \( \mu_D(v + W) = 0 \). Therefore, \( W \) is shy. \( \square \)

Now we are ready to prove the main Theorems.

**Proof of Theorem A.** Let \( N = D \setminus (Q \cup S) \). Then \( N \) is a Borel set, since \( Q \) and \( S \) are both Borel sets. Fix a probe \( P \subset \text{Int} C \). In order to show that \( N \) is shy, by Lemma 4.1, it suffices to show that \( N \cap P_v \) is at most countable, for any \( v \in \mathbb{R}^n \).

To this end, fix any \( x \in N \cap P_v \), then lemma 3.1 (Probe Lemma) implies that there exists \( \varepsilon > 0 \) such that \( B^P(x, \varepsilon) \setminus \{x\} \subset Q \). Consequently, \( x \) can not be an accumulation point in \( N \cap P_v \). By the arbitrariness of \( x \) and the separability of \( P_v \), we obtain that \( N \cap P_v \) is at most countable.

Thus, we have proved that \( N \) is shy. Hence, \( Q \cup S \) is prevalent in \( D \). Finally, in \( \mathbb{R}^n \), a subset is prevalent if and only if it has full Lebesgue measure. Thus, \( Q \cup S \) is full Lebesgue measure in \( D \). We have completed the proof. \( \square \)

**Proof of Theorem B.** By virtue of Theorem A, the subset \( Q \cup S \) is full Lebesgue measure in \( D \). Recall that \( k = 2 \). Then, for any \( x \in D \cap (Q \cup S) \), the fact \( \omega(x) \cap E = \emptyset \) directly implies that \( x \in Q \). It then follows from Feng et al. [7, Theorem C] (see also Sanchez [36, Theorem 1]) that \( \omega(x) \) must be a periodic orbit. Thus, we have completed the proof. \( \square \)
5 Applications

In this section, we will apply the obtained Almost Sure Poincaré-Bendixson Theorem for high-dimensional autonomous systems to the ordinary differential equations.

Consider a system of ODEs
\[ \dot{x} = F(x), \quad x \in \mathbb{R}^n, \tag{5.1} \]
for which \( F \) is a \( C^2 \)-smooth vector field defined in \( \mathbb{R}^n \). Let \( \Phi_t \) be the flow generated by the system \((5.1)\). We call the system \((5.1)\) is dissipative if there exists a bounded set \( \Lambda \subset \mathbb{R}^n \) satisfying for each \( x \in \mathbb{R}^n \) there exists a constant \( t_0 > 0 \) such that \( \Phi_t(x) \in \Lambda \) for all \( t \geq t_0 \).

Let \( C \subset \mathbb{R}^n \) be a \( k \)-cone. We call the system \((5.1)\) is \( C \)-cooperative if the fundamental solution matrix \( X_{ij}(t) \) of the linear system
\[ \dot{X} = Q_{ij}(t)X, \quad U(0) = I \]
satisfies
\[ Q_{ij}(t)(C \setminus \{0\}) \subset \text{Int} C, \quad \text{for all} \ t > 0, \tag{5.2} \]
which \( Q_{ij}(t) = \int_0^1 DF(\tau \Phi_t(i) + (1 - \tau)\Phi_t(j))d\tau. \)

Specially, a system satisfying \((5.2)\) is called strongly cooperative if \( C \) is a classical cone.

Assume now that \( C \subset \mathbb{R}^n \) is a complemented 2-solid cone. Then, we give the following Almost Sure Poincaré-Bendixson Theorem for system \((5.1)\).

**Theorem 5.1.** Assume that system \((5.1)\) is dissipative and \( C \)-cooperative. Then, for almost every point \( x \in \mathbb{R}^n \), if the \( \omega \)-limit set \( \omega(x) \) contains no equilibrium, then \( \omega(x) \) is a periodic orbit.

**Proof.** The flow \( \Phi_t \) satisfies the assumption (H) if system \((5.1)\) is \( C^{1,\alpha} \)-smooth and \( C \)-cooperative by [36, Proposition 1].

Let \( B_i = \{ x \in \mathbb{R}^n : ||x|| < i \} \) for any integer \( i \geq 1 \). Since the system \((5.1)\) is dissipative, the orbit set \( O(B_i) \) of \( B_i \) is bounded. Let \( \mu \) be the Lebesgue measure on \( \mathbb{R}^n \). According to the Almost Sure Poincaré-Bendixson Theorem, for each \( B_i \), there is a subset \( A_i \subset B_i \) with \( \mu(A_i) = \mu(B_i) \), such that for any \( x \in A_i \) the \( \omega \)-limit set containing no equilibria is a closed orbit.

Now, let \( A = \bigcup_{i=1}^\infty A_i \). Note that \( \mathbb{R}^n = \bigcup_{i=1}^\infty B_i \). We have \( \mu(A^c) = 0 \) as \( A^c = \bigcup_{i \geq 1} (B_i \setminus A_i) \). Also, for any \( x \in A \), the \( \omega \)-limit set \( \omega(x) \) containing no equilibria is a single closed orbit. We have completed the proof.

In the rest of this section, we specifically describe a quadratic cone which is 2-solid and complemented. Let \( Q \) be a constant non-singular \( n \times n \) matrix which is real symmetric. If \( Q \)
has 2 negative eigenvalues and \((n-2)\) positive eigenvalues, then \(C^-(Q) = \{ x \in \mathbb{R}^n : x^*Qx \leq 0 \}\) is a complemented 2-solid cone, which \(x^*\) denote the transpose of the vector \(x\).

In \([36,37]\), Sanchez shows a sufficient conditions for \(C^-(Q)\)-cooperative system. Let \(\lambda : \mathbb{R}^n \rightarrow \mathbb{R}\) be a continuous function such that the matrices

\[ Q \cdot DF(x) + (DF(x))^* \cdot Q + \lambda(x)Q < 0, \text{ for any } x \in \mathbb{R}^n, \quad (5.3) \]

where \((DF(x))^*\) is the transpose of the Jacobian \(DF(x)\) and “<” represents the usual order in the space of symmetric matrices. Then system \((5.1)\) is \(C^-(Q)\)-cooperative if it satisfies \((5.3)\) (see more details in Sanchez \([36, Proposition 7]\)).

Along this, we obtain the following Almost Sure Poincaré-Bendixson Theorem for high-dimensional systems \((5.1)\) with a quadratic cone:

**Corollary 5.2.** Assume that system \((5.1)\) is dissipative and satisfies \((5.3)\). Then, for almost every point \(x \in \mathbb{R}^n\), if the \(\omega\)-limit set \(\omega(x)\) contains no equilibrium, then \(\omega(x)\) is a periodic orbit.

**Remark 5.3.** Recently, Feng et.al \([8]\) proved a generic Poincaré-Bendixson Theorem for this system \((5.1)\) with the assumption \((5.3)\). We here obtain an almost-sure Poincaré-Bendixson Theorem for \(n\)-dimensional ODE system \((5.1)\) with a quadratic cone. Thus, we conclude, for \(n\)-dimensional ODE system \((5.1)\) with a quadratic cone, the “Poincaré-Bendixson Theorem” holds not only generically, but also almost-surely in measure-theoretic sense.

**References**

[1] D. Angeli and E. Sontag, Monotone control systems, IEEE Trans. Autom. Control 48 (2003), 1684-1698.

[2] S. Baigent, Geometry of carrying simplices of 3-species competitive Lotka-Volterra systems, Nonlinearity 26 (2013), 1001-1029.

[3] S. Baigent, Carrying simplices for competitive maps, Difference equations, discrete dynamical systems and applications, 3-29, Springer Proc. Math. Stat., 287, Springer, Cham, 2019.

[4] J. Christensen, On sets of Haar measure zero in abelian Polish groups, Isr. J. Math. 13 (1972), 255-260.

[5] M. Elekes and D. Nagy, Haar null and Haar meager sets: a survey and new results, Bull. London Math. Soc. 52 (2020), 561-619.

[6] G. Enciso, M. Hirsch and H. Smith, Prevalent behavior of strongly order preserving semiflows, J. Dynam. Diff. Eqns. 20 (2008), 115-132.
[7] L. Feng, Y. Wang and J. Wu, Semiflows “Monotone with Respect to High-Rank Cones” on a Banach space, SIAM J. Math. Anal. 49(2017), 142-161.

[8] L. Feng, Y. Wang and J. Wu, Generic behavior of flows strongly monotone with respect to high-rank cones, J. Diff. Eqns. 275(2021), 858-881.

[9] G. Fusco and W. Oliva, Jacobi matrices and transversality, Proc. R. Soc. Edinb., Sect. A 109(1988), 231-243.

[10] G. Fusco and W. Oliva, A Perron theorem for the existence of invariant subspaces, Ann. Mat. Pura Appl. 160(1991), 63-76.

[11] M. W. Hirsch, The dynamical systems approach to differential equations, Bull. Amer. Math. Soc. 11(1984), 1-64.

[12] M. W. Hirsch, Stability and convergence in strongly monotone dynamical systems, J. Reine Angew. Math. 383(1988), 1-53.

[13] M. W. Hirsch, Systems of differential equations which are competitive or cooperative I: Limit sets, SIAM J. Math. Anal. 13(1982), 167-179.

[14] M. W. Hirsch, Systems of differential equations which are competitive or cooperative II: Convergence almost everywhere, SIAM J. Math. Anal. 16(1985), 423-439.

[15] M. W. Hirsch, Systems of differential equations which are competitive or cooperative III: Competing species, Nonlinearity 1(1988), 51-71.

[16] M. W. Hirsch and H. Smith, Monotone dynamical systems, Handbook of Differential Equations: Ordinary Differential Equations, Vol. 2, Elsevier, Amsterdam 2005.

[17] B. Hunt, T. Sauer and J. Yorke, Prevalence: a translation-invariant “almost every” on infinite-dimensional spaces, Bull. Am. Math. Soc. 27(1992), 217-238.

[18] B. Hunt and V. Kaloshin, Prevalence, Handbook of Dynamical Systems. vol. 3, Elsevier, 2010.

[19] J. Jiang and L. Niu, On the equivalent classification of three-dimensional competitive Leslie/Gower models via the boundary dynamics on the carrying simplex, J. Math. Biol. 74(2017), 1223-1261.

[20] J. Jiang, J. Mierczyński and Y. Wang, Smoothness of the carrying simplex for discrete-time competitive dynamical systems: a characterization of neat embedding, J. Diff. Eqns. 246(2009), 1623-1672.
[21] V. Kaloshin, Some prevalent properties of smooth dynamical systems, Proc. Steklov Inst. Math. 213 (1997), 123-151.

[22] M. Krasnosel’skii, E. Lifshits and A. Sobolev, Positive linear systems, the method of positive operators, Heldermann Verlag, Berlin, 1989.

[23] Z. Lian and K. Lu, Lyapunov exponents and invariant manifolds for random dynamical systems on a Banach space, Mem. Amer. Math. Soc. 206 (2010), no. 967.

[24] Z. Lian and Y. Wang, K-dimensional invariant cones of random dynamical system in $\mathbb{R}^n$ with applications, J. Diff. Eqns. 259 (2015), 2807-2832.

[25] J. Mallet-Paret and R. Nussbaum, Tensor products, positive linear operators, and delay-differential equations, J. Dynam. Diff. Eqns. 25 (2013), 843-905.

[26] J. Mallet-Paret and G. Sell, The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay, J. Diff. Eqns. 125 (1996), 441-489.

[27] J. Mallet-Paret and H. Smith, The Poincaré-Bendixson theorem for monotone cyclic feedback systems, J. Dynam. Diff. Eqns. 2 (1990), 367-421.

[28] J. Mierczyński, The $C^1$ property of convex carrying simplices for competitive maps, Ergod. Theory Dyn. Syst. 40 (2020), 1335-1350.

[29] J. Mierczyński, The $C^1$ property of convex carrying simplices for a class of competitive system of ODEs, J. Diff. Eqns. 111 (1994), 385-409.

[30] J. Mierczyński, L. Niu and A. Ruiz-Herrera, Linearization and invariant manifolds on the carrying simplex for competitive maps, J. Diff. Eqns. 267 (2019), 7385-7410.

[31] R. Ortega and L. Sanchez, Abstract competitive systems and orbital stability in $\mathbb{R}^3$, Proc. Amer. Math. Soc. 128 (2000), 2911-2919.

[32] P. Poláčik, Convergence in smooth strongly monotone ows defined by semilinear parabolic equations, J. Diff. Eqns. 79 (1989), 89-110.

[33] P. Poláčik, Generic properties of strongly monotone semiflows defined by ordinary and parabolic differential equations, Qualitative theory of differential equations (Szeged 1988) 519-530, Colloq. Math. Soc. János Bolyai, 53, North-Holland, Amsterdam, 1990.

[34] P. Poláčik, Parabolic equations: asymptotic behavior and dynamics on invariant manifolds, Handbook on Dynamical Systems, Vol. 2, Elsevier, Amsterdam, 2002, 835-883.
[35] P. Poláčik and I. Tereščák, Convergence to cycles as a typical asymptotic behavior in smooth strongly monotone discrete-time dynamical systems, Arch. Ration. Mech. Anal. 116 (1991), 339-360.

[36] L. Sanchez, Cones of rank 2 and the Poincaré-Bendixson property for a new class of monotone systems, J. Diff. Eqns. 216 (2009), 1170-1190.

[37] L. Sanchez, Existence of periodic orbits for high-dimensional autonomous systems, J. Math. Anal. Appl. 363 (2010), 409-418.

[38] H. Smith, Monotone Dynamical Systems: an introduction to the theory of competitive and cooperative systems, Math. Surveys and Monographs, Vol. 41, Amer. Math. Soc., Providence, Rhode Island, 1995.

[39] H. Smith, Monotone Dynamical Systems: Reflections on new advances and applications, Discrete Contin. Dyn. Syst. 37 (2017), 485-504.

[40] H. Smith and H. Thieme, Convergence for strongly order-preserving semiflows, SIAM J. Math. Anal. 22 (1991), 1081-1101.

[41] R. Smith, Existence of periodic orbits of autonomous ordinary differential equations, Proc. of Royal Soc. of Edinburgh A 85 (1980), 153-172.

[42] R. Smith, Orbital stability for ordinary differential equations, J. Diff. Eqns. 69 (1987), 265-287.

[43] P. Takáč, Domains of attraction of generic ω-limit sets for strongly monotone discrete-time semigroups, J. Reine Angew. Math. 423 (1992), 101-173.

[44] I. Tereščák, Dynamical systems with discrete Lyapunov functionals, Ph.D. thesis, Comenius University, Bratislava, 1994.

[45] I. Tereščák, Dynamics of $C^1$ smooth strongly monotone discrete-time dynamical systems, preprint, Comenius University, Bratislava, 1994.

[46] Y. Wang and J. Jiang, The general properties of discrete-time competitive dynamical systems, J. Diff. Eqns. 176 (2001), 470-493.

[47] Y. Wang and J. Yao, Dynamics alternatives and generic convergence for $C^1$-smooth strongly monotone discrete dynamical systems, J. Diff. Eqns. 269 (2020), 9804-9818.

[48] Y. Wang, J. Yao and Y. Zhang, Prevalent Behavior of Smooth Strongly Monotone Discrete-Time Dynamical Systems, Proc. Amer. Math. Soc., in press.