Lorentz Gauge Gravity and Induced Effective Theories

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Abstract: We develop the gauge approach based on the Lorentz group to the gravity with torsion. With a Lagrangian quadratic in curvature we show that the Einstein-Hilbert action can be induced from a simple gauge model due to quantum corrections of torsion via formation of a gravito-magnetic condensate. An effective theory of cosmic knots at Planckian scale is proposed.

1. Utiyama-Kibble-Sciama gauge approach to gravity

The gauge approach to gravity based on Lorentz and Poincare group was proposed in [1] and later was developed in many studies (see refs. in [2]). The Lorentz gauge models were further studied by Carmeli [3]. The possibility of inducing the Einstein gravity via quantum corrections was considered by many physicists in various models [4]. In most of these models the Einstein-Hilbert term is induced by quantum corrections due to interaction with matter fields.

In the present article we propose a simple gauge model of quantum gravity based on the Lorentz group as a structural group. In the framework of this gauge model we demonstrate that even in a pure quantum gravity case with torsion the Einstein-Hilbert action can be induced due to the quantum dynamics of torsion via formation a non-trivial vacuum with a gravito-magnetic condensate. We develop the gauge approach to the gravity by suggesting that the torsion represents exactly the dynamical variable of quantum gravity. Moreover, we conjecture that the torsion can be confined and exists intrinsically as a quantum object, and its quantum dynamics manifests itself by inducing the Einstein-Hilbert theory as an effective theory of quantum gravity.

We start with the formalism of the Lorentz gauge model along the lines proposed in [1]. The vielbein $e_{a}^{\mu}$ is treated as a fixed background field which obtains the dynamical content after inducing the Hilbert-Einstein term in the effective theory. The covariant derivative with respect to the Lorentz structural group is defined in a standard manner

$$D_{a} = e_{a}^{\mu}(\partial_{\mu} + A_{\mu}),$$

where $A_{\mu} \equiv A_{\mu cd}^{\Omega} \Omega^{cd}$ is a general affine connection taking values in Lorentz Lie algebra. The original Lorentz gauge transformation is the following

$$\delta e_{a}^{\mu} = \Lambda_{a}^{b} e_{b}^{\mu}, \quad \delta A_{\mu} = -\partial_{\mu} A - [A_{\mu}, A],$$

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where \( \Lambda \equiv \Lambda_{cd} \Omega^{cd} \). One can split a general gauge connection \( A_{\mu cd} \) into two parts, the background (classical) part and the quantum one. In what follows we will specify the classical background as one corresponding to Riemannian space-time geometry

\[
A_{\mu c}^d = \varphi_{\mu c}^d (e) + K_{\mu c}^d,
\]

where \( K_{\mu c}^d \) is a contorsion, and \( \varphi_{\mu c}^d \) is Levi-Civita spin connection given in terms of the vielbein. In the presence of contorsion we have two types of local symmetry transformations:

(I) the classical, or background, gauge transformation

\[
\delta \varphi_{\mu} = - \partial_{\mu} \Lambda - [\varphi_{\mu}, \Lambda], \quad \delta K_{\mu} = - [K_{\mu}, \Lambda],
\]

(II) the quantum gauge transformation

\[
\delta \varphi_{\mu} = 0, \quad \delta K_{\mu} = - \hat{D}_{\mu} \Lambda - [K_{\mu}, \Lambda],
\]

where the background covariant derivative is defined with the help of Levi-Civita connection \( \hat{D}_{\mu} = \partial_{\mu} + \varphi_{\mu} \), \( \varphi_{\mu} \equiv \varphi_{\mu cd} \Omega^{cd} \).

Following the gauge principle as a guiding rule we postulate the gauge symmetry in the model under these two types of transformations. The postulate restricts strongly the admissible gauge invariants as possible candidates for the Lagrangian. For instance, all terms quadratic in torsion (like contact terms) are forbidden since they spoil the type II gauge invariance and by this the renormalizability of the theory.

Let us remind the main lines of Riemann-Cartan geometry (see, for ex., [2]). To define the derivative \( D_{\mu} \) covariant under the space-time diffeomorphisms one should include a general Christoffel symbol \( \Gamma_{\nu \rho}^{\mu} \)

\[
D_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma_{\mu \nu}^{\rho} V^{\rho}.
\]

The Christoffel symbol is related to a general Lorentz connection \( \gamma_{\mu a} \) through the equation

\[
D_{\mu} e^{\rho a} = \partial_{\mu} e^{\rho a} + \Gamma_{\mu \rho}^{\nu} e^{\nu a} - e^{\rho b} \gamma_{\mu a}^{b} = 0.
\]

This allows to convert space-time indices into Lorentz ones and vise versa by using the vielbein. The contorsion is connected with torsion as follows

\[
K_{abc} = e_a^\mu K_{\mu bc} = - \frac{1}{2} (T_{abc} - T_{bac} + T_{cab}).
\]

Upon making the decomposition \( [3] \) the curvature tensor is split into two parts

\[
R_{abcd} = \hat{R}_{abcd} + \tilde{R}_{abcd},
\]

\[
\hat{R}_{abcd} = - \hat{D}_{\mu} \varphi_{[\mu}^{d \nu]} - \varphi_{[\mu} e_{\nu]}^{d},
\]

\[
\tilde{R}_{abcd} = - \tilde{D}_{\mu} K_{[\mu}^{d \nu]} - K_{[\mu} e_{\nu]}^{d},
\]

where the underlined indices stand for indices over which the covariantization is performed, and here we introduce a short notation for the antisymmetrization over indices \([a, b] = ab - ba\).

The classical action for a pure quantum gravity in our approach contains the Maxwell type term quadratic in curvature

\[
S_{cl} = \frac{1}{4 g^2} \int \sqrt{-g} d^4x \mathbf{R}^2_{\mu \nu} = - \frac{1}{4 g^2} \int \sqrt{-g} d^4x R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma},
\]

(9)
where $\mathbf{R}_{\mu\nu} \equiv R_{\mu\nu}^{\alpha\beta} \Omega_{\alpha\beta}$, and we have written down explicitly a new gravitational gauge coupling constant $g$ corresponding to the Lorentz gauge group. For brevity of notations we will use a redefined contorsion which absorbs the coupling constant. The same Lagrangian with the general Lorentz connection constructed from $SL(2, C)$ dyads and vielbeins was considered by Carmeli [3]. It was demonstrated that the corresponding equations of motion after projection with vielbein result in Newman-Penrose form of Einstein-Hilbert equation in the vacuum. Later Martellini and Sodano considered Carmeli’s model treating the connection as an independent quantity on vielbein and proved the renormalizability of the model [5].

One should mention, since the Lorentz group is not compact the classical Lagrangian leads to the Hamiltonian which is not positively definite. We adopt the point of view that even though the classical action (9) does not lead to a positively definite Hamiltonian, nevertheless, a consistent quantum theory can be formulated. Since the canonical quantization method fails to handle our model we will apply the quantization scheme based on continual functional integration in Euclidean space-time. Within this quantization scheme the quantum theory can be constructed since in the Euclidean space-time the Lorentz group is locally isomorphic to the product of compact unitary groups $SU(2) \times SU'(2)$.

2. Effective action

The general approach to derivation of the effective theory is to integrate out all high energy (heavy mass) modes while keeping light modes (massless or light particles). Starting with the classical action (9) and imposing the gauge fixing condition $\hat{D}_\mu K^\mu = 0$ one can write down the effective action

\[
\exp \left[ i S_{\text{eff}} \right] = \int DcD\bar{c}D\mathbf{K} \exp \left\{ i \int \sqrt{-g} d^4x \text{tr} \left[ \frac{1}{4} \hat{\mathbf{R}}_{\mu\nu}^2 + \frac{1}{2} \mathbf{K}_\mu \left( g_{\mu\nu} \hat{\mathbf{D}}\hat{\mathbf{D}} - 4 \hat{\mathbf{R}}_{\mu\nu} K_\nu + \bar{c} (\hat{\mathbf{D}}\hat{\mathbf{D}}) c \right) \right] \right\},
\]

where $c, \bar{c}$ are Faddeev-Popov ghosts. The formal expression for the one-loop effective action can be written in the form

\[
S_{\text{eff}} = S_{\text{cl}} - \frac{i}{2} Tr \ln [(g_{\mu\nu} (\hat{\mathbf{D}}\hat{\mathbf{D}})_{ab})^{cd} - 2 \hat{\mathbf{R}}_{\mu\nu}^{ef} (f_{ef})_{ab}] + i Tr \ln [(\hat{\mathbf{D}}\hat{\mathbf{D}})^{cd}]_{ab},
\]

where $(f_{ef})_{ab}^{cd}$ are the structural constants of Lorentz Lie algebra. The functional determinants in (11) are not well-defined in Minkowski space-time. As is known, the adding of infinitesimal number factor $-i \epsilon$ to the bare Laplace operator in $\hat{\mathbf{D}}\hat{\mathbf{D}}$ is conditioned by the requirement of causality. The infinitesimal addition $-i \epsilon$ defines uniquely the Wick rotation from Minkowski space-time to Euclidean one. In our case we should perform the Wick rotation in the base space-time and in the tangent space-time both, so that the Lorentz group in Euclidean sector turns into the compact group $SO(4) \simeq SU(2) \times SU'(2)$. With this the functional integral becomes well-defined. Certainly, there remains a problem of analytical continuation of the final expressions from Euclidean space-time back to Minkowski space-time.

We have the following factorization for the Lie algebra valued curvature tensor

\[
R_{\mu\nu\alpha\beta}^{cd} = -i (R_{\mu\nu}^{\alpha\beta} T^i + R_{\mu\nu}^{\alpha\beta} T^{i'}),
\]

where $T^i, T^{i'}$ are generators of the group $SU(2) \times SU(2)'$. 
The functional determinants (11) are factorized into the direct product of $SU(2)$ determinants, and the effective action takes a simple form

$$S_{\text{eff}} = S_{\text{cl}} - \frac{i}{2} Tr \ln[(g_{\mu\nu}(\hat{D}\hat{D})^{ij} - 2\hat{R}_{\mu\nu}^{ij})] - \frac{i}{2} Tr \ln[(g_{\mu\nu}(\hat{D}'\hat{D}')^{ij} - 2\hat{R}'_{\mu\nu}^{ij})]$$

$$+ i Tr \ln[(\hat{D}\hat{D})^{ij}] + i Tr \ln[(\hat{D}'\hat{D}')^{ij}).$$

(13)

where all quantities corresponding to the group $SU(2)'$ are marked with apostrophe.

Notice that the curvature squared term contains a dual tensor \( \tilde{\hat{R}}_{\mu\nu} \), for instance,

\[
(\hat{R}_{i\mu\nu})^2 = \frac{1}{8}(\hat{R}_{\mu\nu\mu\nu} + \hat{R}_{\mu\nu\mu\nu}) \equiv \frac{1}{8}(\hat{R}^2 + \hat{R} \tilde{\hat{R}}),
\]

\[
(\hat{R}'_{i\mu\nu})^2 = \frac{1}{8}(\hat{R}_{\mu\nu\mu\nu} - \hat{R}_{\mu\nu\mu\nu}) \equiv \frac{1}{8}(\hat{R}^2 - \hat{R} \tilde{\hat{R}}).
\]

(14)

For a constant background one can apply the Schwinger’s proper time method and $\zeta$-function regularization in full analogy with the case of $SU(2)$ chromodynamics (QCD) [6]. We will consider a constant homogeneous gravito-magnetic background field $H = \sqrt{\hat{R}^2_{\mu\nu\mu\nu}/2}$ which assumes that $\hat{R}_{\mu\nu\mu\nu} \tilde{\hat{R}}_{\mu\nu\mu\nu} = 0$ in an appropriate coordinate frame. For such gravito-magnetic background the final expression for the one-loop effective Lagrangian is given by

$$L_{\text{eff}} = -\frac{1}{2} H^2 - \frac{11}{48\pi^2} H^2 (\ln \frac{g H}{\bar{\mu}^2} - c),$$

(15)

$$c = 1 - \frac{1}{2} - \frac{24}{11} \zeta(-1, \frac{3}{2}) = 1.29214\ldots.$$

With a proper renormalization condition $\frac{\partial^2 V}{\partial H^2} \bigg|_{H = \bar{\mu}^2} = \frac{1}{g^2}$ one can obtain the renormalized effective potential

$$V = \frac{1}{2g^2} H^2 + \frac{11}{48\pi^2} H^2 (\ln \frac{H}{\bar{\mu}^2} - \frac{3}{2}).$$

(16)

One can check that the effective potential satisfies a renormalization group equation with the same $\beta$-function as in a pure $SU(2)$ Yang-Mills theory.

The minimum of the effective potential leads to a gravito-magnetic condensate

$$< H > = \bar{\mu}^2 \exp\left[-\frac{24\pi^2}{11g^2} + 1\right].$$

(17)

The presence of the minimum of the effective potential does not guarantee that the corresponding new vacuum is stable. The stability of the vacuum condensate even in the pure $SU(2)$ model of QCD presents a long-standing problem, and its solution have passed through several controversial results since of the first paper on that by Nielsen and Olesen [7]. Without clear evidence or at least a strong indication to the vacuum stability one can not make any serious statement based on existence of a non-trivial vacuum condensate. Recently the progress in resolving that problem in favor of stability of the magnetic vacuum has been achieved in [8]. Moreover, it has been found recently [8] that a stable classical configuration made of monopole-antimonopole strings does exist in $SU(2)$ model.
of QCD providing a strong argument that a stable magnetic vacuum can exist in QCD and, therefore, in our model of quantum gravity as well.

3. Effective induced theories

Due to two types of gauge symmetries the condensate of torsion must vanish \( \langle T_{abc} \rangle = 0 \). It is possible that there is a deep analogy with QCD, and the torsion plays a role of the off-diagonal (valence) gluon in QCD, so that one can expect that the torsion can be confined. The presence of a stable gravito-magnetic condensate generates a new scale in the theory, and one can also expect a non-vanishing vacuum averaged value for the curvature containing the torsion part

\[
\langle \tilde{R}_{abcd} \rangle = M^2 (\eta_{ac} \eta_{bd} - \eta_{ad} \eta_{bc}).
\] (18)

The sign of the number factor \( M^2 \) is chosen positive since it corresponds to the positive curvature space-time which can only be created due to quantum fluctuations.

Expanding the original Lagrangian near the vacuum one obtains the Einstein-Hilbert Lagrangian and the cosmological constant term in lower order approximation (in units \( \hbar = c = 1 \))

\[
\mathcal{L} = -\frac{1}{4g^2}(\tilde{R}_{abcd} + \tilde{R}_{abcd})^2 = -\frac{1}{4g^2} \tilde{R}_{abcd}^2 - \frac{1}{16\pi G}(\tilde{R} + 2\lambda) + \ldots
\] (19)

where the Newton constant \( G \) and the cosmological constant \( \lambda \) are defined by only one parameter, the renormalized coupling constant \( \tilde{g} \) at some scale \( \mu^2 \) which supposed to be of order of Planckian scale \( 10^{19} \text{Gev} \).

Certainly, the assumption (18) leads straightforward to a desired induced Einstein-Hilbert term what was known very well before. The most important point is how to make foundation to that hypothesis. In our approach we put this assumption on the real ground by the explicit calculation of the effective potential and derivation of a stable classical vacuum configuration in \( SU(2) \) Yang-Mills theory [8].

Till now the numeric value of the gauge coupling \( \tilde{g} \) was not fixed, and it is a free parameter in the theory. It is possible that there are two phases corresponding to the strong and weak coupling constant. The existence of two phases in gravity was suggested in [9] in a different approach. In the strong coupling phase we can adjust the coupling constant \( \tilde{g} \) to \( \tilde{g}^2 \simeq 19 \) to obtain the value for \( G \) close to the experimental value of the Newton constant. This provides also a large value for the cosmological constant which is consistent with cosmological models containing the initial inflation at very early universe.

It is interesting to consider the possibility of existence of a weak coupling phase with \( \tilde{g}^2/4\pi < 1 \). Using the experimental data for the vacuum energy density \( \rho_v = \frac{2\lambda}{16\pi G} = 2 \cdot 10^{-47} \text{(Gev)}^4 \) one can find an appropriate value for the structure constant \( \alpha_{\tilde{g}} = \tilde{g}^2/4\pi = 0.0123 \). This value can be compared with the value \( \alpha_{SSGUT} \simeq 1/24 \) of the structure constant in supersymmetric \( SO(10) \) GUT model at unification scale \( 2 \times 10^{16} \text{Gev} \). The same order of the structure constants \( \alpha_g \) and \( \alpha_{SSGUT} \) might be a hint to the origin of the supersymmetry and its relation to quantum gravity.

Since the Lorentz group \( SO(1,3) \) contains a maximal compact subgroup \( SO(3) \simeq SU(2) \) we have the same homotopy structure as in \( SU(2) \) QCD, in particular, the Hopf mapping \( \pi_3(SO(1,3)/SO(2)) = \mathbb{Z} \). This suggests the existence of topological solitons with non-trivial Hopf numbers like the knots in Faddeev-Niemi-Skyrme model [10]. It has been
shown that the generalized Faddeev-Niemi-Skyrme model appears as an effective theory of QCD [6]. The derivation was based on Abelian decomposition of the $SU(2)$ gauge potential [11]. The essential part of this decomposition is represented by a topological triplet $\hat{n}^i$ which parameterizes the coset $S^2 \simeq SU(2)/U(1)$. We can apply the results obtained in $SU(2)$ QCD to our model in a case of gravito-magnetic background $H$. With this the effective Lagrangian corresponding to the generalized Faddeev-Niemi-Skyrme model in gravity is given by [6]

$$L_{eff} = -\frac{\mu^2}{2}(\partial_{\mu}\hat{n})^2 - \frac{1}{4}(\partial_{\mu}\hat{n} \times \partial_{\nu}\hat{n})^2 - \frac{\alpha_1}{4}(\partial_{\mu}\hat{n} \cdot \partial_{\nu}\hat{n})^2 - \frac{\alpha_2}{2}(\hat{n} \times \partial^2\hat{n})^2,$$

(20)

where $\mu, \alpha_1, \alpha_2$ are parameters proportional to vacuum averaging values of operator products of the magnetic potential $\tilde{C}_\mu(\hat{n})$. When the parameters $\alpha_{1,2}$ vanish the Lagrangian coincides with one of Faddeev-Niemi-Skyrme model [10], so that we expect the existence of cosmic knot solutions at Planckian scale with the mass of order $M_{Planck}$. Recently the knot-like cosmic strings were considered in [12].

The detailed consideration of our results will be presented elsewhere.

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