ON WEAK ZARISKI DECOMPOSITIONS AND TERMINATION OF FLIPS

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Abstract. We prove that termination of lower dimensional flips for generalized klt pairs implies termination of flips for log canonical generalized pairs with a weak Zariski decomposition. Moreover, we prove that the existence of weak Zariski decompositions for pseudo-effective generalized klt pairs implies the existence of minimal models for such pairs.

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INTRODUCTION

One of the main goals of the minimal model program is to show that given a \( \mathbb{Q} \)-factorial klt pair \( (X, B) \) such that \( K_X + B \) is pseudo-effective (resp. not pseudo-effective), then there exists a finite sequence of divisorial contractions and flips

\[
X \to X_1 \to X_2 \to \ldots \to X_n
\]

such that \((X_n, B_n)\) is a minimal model (resp. there is a Mori fiber space \( X_n \to Y \) and in particular \(-(K_{X_n} + B_n)\) is ample over \( Y \)), where \( B_n \) is the strict transform of \( B \) on \( X_n \). We refer the reader to [KM98] for the details of the minimal model program. After [BCHM10], it is known that the above sequence of flips and divisorial contractions always exists and the only remaining question is whether it terminates after finitely many steps. It is well known that any such sequence can have only finitely many divisorial contractions and hence the main open question is if there are no infinite sequences of flips. A flip \( X \to X^+ \) is a small birational map of \( \mathbb{Q} \)-factorial varieties, projective over a variety \( W \) such that \( \rho(X/W) = \rho(X^+/W) = 1 \) and both \(-(K_X + B)\) and \( K_{X^+} + B^+ \) are ample over \( W \) where \( B^+ \) is the strict transform of \( B \). As a consequence of the negativity lemma, it is easy to see that flips improve certain singularity invariants known as log discrepancies. More precisely, if \( X \to X^+ \) is a flip, then we have the following inequality

\[
a_E(X, B) \leq a_E(X^+, B^+)
\]

which is strict if and only if the center of \( E \) is contained in the flipping locus i.e. the exceptional locus of the flipping contraction \( X \to W \). Shokurov has shown [Sho04] that certain natural conjectures concerning log discrepancies (namely the ascending chain condition for MLD’s and the semicontinuity for MLD’s) actually imply termination of flips. Unluckily these conjectures are very subtle and not well understood in dimension \( \geq 3 \). In [BCHM10] a different approach is introduced. Instead of...
trying to prove termination of arbitrary sequences of flips, the authors show termination of specific kinds of minimal model programs known as minimal model programs with scaling. This approach is successful whenever $K_X + B$ is big or $B$ is big or $K_X + B$ is not pseudo-effective. In particular the existence of minimal models for klt pairs of log general type follows as well as the existence of Mori fiber spaces for klt pairs $(X, B)$ such that $K_X + B$ is not pseudo-effective. This approach does not seem to shed any light on the termination of arbitrary sequences of flips.

In [Bir07], Birkar introduced a new philosophy to prove termination of flips for klt pairs such that $K_X + B$ is pseudo-effective. In this case one expects that $K_X + B \equiv G \geq 0$. Birkar shows that assuming the ascending chain condition conjecture for log canonical thresholds and the termination of flips for klt pairs of dimension $\leq d - 1$, then flips terminate for any $d$-dimensional log canonical pair $(X, B)$ such that $K_X + B \equiv G \geq 0$. The ascending chain condition conjecture for let’s was proved by Hacon, McKernan and Xu in [HMX14], and later extended to the context of generalized pairs by Birkar and Zhang in [BZ16]. In [Sho09], Shokurov shows that termination of flips with scaling holds for pseudo-effective klt fourfolds and in particular these pairs admit a minimal model and hence a Zariski decomposition. In [Mor18], the second author proves termination of pseudo-effective 4-fold flips by combining the results of [Bir07], [Sho09] and [BZ16]. Following this philosophy, in this article we prove that the existence of a weak Zariski decomposition for a generalized log canonical pair can be used to reduce termination of flips for such pair to lower dimensional terminations. More precisely, we prove the following theorems:

**Theorem 1.** Assume termination of flips for generalized klt pairs of dimension at most $n - 1$. Let $(X/Z, B + M)$ be a generalized log canonical pair of dimension $n$ admitting a weak Zariski decomposition. Then any minimal model program for $K_X + B + M/Z$ terminates.

**Theorem 2.** Assume existence of weak Zariski decompositions for pseudo-effective generalized log canonical pairs of dimension at most $n$. If $(X/Z, B + M)$ is a pseudo-effective generalized log canonical pair of dimension $n$, then any good minimal model program for $(X/Z, B + M)$ terminates.

See 1.12 for the definition of good minimal model program. In particular, any minimal model program with scaling of an ample divisor is good so we obtain the following corollary.

**Corollary 1.** Assume the existence of weak Zariski decompositions for pseudo-effective generalized log canonical pairs of dimension at most $n$. If $(X/Z, B + M)$ is a pseudo-effective log canonical pair of dimension $n$, then $(X/Z, B + M)$ has a minimal model.

Finally we remark that it is expected that the existence of a weak Zariski decompositions for pseudo-effective generalized log canonical pairs is implied by the existence of a weak Zariski decompositions for pseudo-effective log canonical pairs. We hope to adress this issue in a separate paper.

**Acknowledgement.** We would like to thank C. Birkar for useful discussions and suggestion. This paper is deeply influenced by his ideas (especially [Bir07] and [Bir12a]). After completing this paper we were informed by Jingjun Han and Zhan Li that they have obtained a result very similar to Corollary 1 [HL18].

1. **Preliminary results**

1.1. **Weak Zariski decomposition.**

**Definition 1.1.** Let $D$ be a $\mathbb{R}$-Cartier divisor on a normal variety $X/Z$. A weak Zariski decomposition for $D$ over $Z$ consists of a normal variety $X'$, a projective birational morphism $f: X' \to X$, and a numerical equivalence

$$f^*D \equiv Z P' + N'$$

such that the following properties hold
Remark 1.2. Consider an $\mathbb{R}$-Cartier divisor $D$ on a projective normal variety $X$. If there exists a projective $D$-non positive birational contraction $\pi: X \rightarrow X_1$, such that the divisorial push-forward $\pi_*D$ is a nef $\mathbb{R}$-Cartier divisor, then $D$ has a weak Zariski decomposition. Indeed, we consider a common resolution of singularities with projective birational morphisms $f: X' \rightarrow X$ and $f_1: X' \rightarrow X_1$, and we write

$$f^*D = f_1^*(\pi_*D) + E,$$

where $f_1^*(\pi_*D)$ is nef and $E$ is an effective $\mathbb{R}$-divisor. In particular, a pair $(X, \Delta)$ admitting a minimal model has a weak Zariski decomposition. Therefore, conjecturally, every pseudo-effective log canonical pair has a WZD.

Remark 1.4. In [Les14], the author constructs a psuedo-effective divisor on the blow up of $\mathbb{P}^3$ at nine very general points, which lies in the closed movable cone and has negative intersections with a set of curves whose union is Zariski dense. Hence, this pseudo-effective divisor does not admit a weak Zariski decomposition.

1.2. Generalized pairs. In this subsection, we recall the language of generalized pairs.

Definition 1.5. A generalized pair is a triple $(X/Z, B + M)$, such that the following conditions hold

1. $X$ is a quasi-projective normal algebraic variety,
2. $X \rightarrow Z$ is a projective morphism of normal varieties,
3. $M$ is the push-forward of a nef $\mathbb{R}$-divisor on a higher birational model of $X$ over $Z$,
4. $B$ is an effective $\mathbb{R}$-divisor,
5. $K_X + B + M$ is an $\mathbb{R}$-Cartier divisor.

More precisely, there exists a projective birational morphism $f: X' \rightarrow X$ from a normal quasi-projective variety $X'$ and a nef $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $M'$ such that $M = f_* M'$. We can define $B'$ via the equation

$$K_{X'} + B' + M' = f^*(K_X + B + M).$$

We will say that $B$ is the boundary part and $M$ is the nef part of the generalized pair. Observe that $M'$ defines a nef $b$-Cartier $\mathbb{R}$-divisor in the sense of [Cor07, Definition 1.7.3]. We will say that this is the nef $b$-divisor associated to the generalized pair.

Definition 1.6. Given a projective birational morphism $g: X'' \rightarrow X$ which dominates $X' \rightarrow X$, we can write

$$K_{X''} + B'' + M'' = g^*(K_X + B + M),$$
where $M''$ is the pull-back of $M'$ to $X''$. Given a prime divisor $E$ on $X''$, we define the log discrepancy of $(X/Z, B + M)$ at $E$ to be

$$a_E(X/Z, B + M) = 1 - \text{coeff}_E(B'').$$

We say that $(X/Z, B + M)$ is Kawamata log terminal or klt if the log discrepancy of $(X/Z, B + M)$ at any prime divisor over $X$ is positive, and we say that $(X/Z, B + M)$ is log canonical or lc if the log discrepancy of $(X/Z, B + M)$ at any prime divisor over $X$ is non-negative.

By Hironaka’s resolution of singularities we may assume that $X''$ is smooth and $B''$ has simple normal crossing support. In this case, $(X/Z, B + M)$ is klt (resp. lc) iff $\text{coeff}(B'') < 1$ (resp. $\text{coeff}(B'') \leq 1$).

**Definition 1.7.** Let $(X, B + M)$ be a generalized pair and $(X'', B'' + M'')$ any log resolution as above. A prime divisor $E$ of $X''$ such that $\text{coeff}_E(B'') \geq 1$ is called a generalized non-klt place of the generalized pair $(X, B + M)$. Moreover, if $\text{coeff}_E(B'') = 1$ (resp. $\text{coeff}_E(B'') > 1$) then we may call it a generalized log canonical place (resp. generalized non-lc place) of the generalized pair on $X''$. The image of a generalized non-klt place (resp. generalized log canonical place) on $X$ is called a generalized non-klt center (resp. generalized log canonical center) of the generalized pair. A generalized non-klt center of a generalized pair $(X, B + M)$ is said to be minimal if it is minimal with respect to inclusion.

**Definition 1.8.** Let $(X/Z, B + M)$ be a generalized log canonical pair. A weak contraction $\phi: X \to W$ for the generalized pair is a projective birational contraction over $Z$, such that $-(K_X + B + M)$ is nef over $W$. A quasi-flip of $\phi$ is a projective birational map $\pi: X \dashrightarrow X'$ with a projective birational contraction $\phi^+: X' \to W$ over $Z$, such that the following conditions hold

1. The triple $(X', B' + M')$ is a generalized log canonical pair,
2. The $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $K_{X'} + B' + M'$ is nef over $W$,
3. The inequality $\phi_*^! B' \leq \phi_* B$ of Weil $\mathbb{R}$-divisors on $W$ holds, and
4. The nef parts $M$ and $M'$ are the trace of a common nef b-Cartier b-divisor.

As usual, the morphism $\phi$ (resp. $\phi^+$) is called the flipping contraction (resp. flipped contraction). We may call $(X/Z, B + M)$ (resp. $(X'/Z, B' + M')$) the flipping generalized pair (resp. flipped generalized pair) when the flip is clear from the context.

**Definition 1.9.** A quasi-flip $\pi$ is said to be ample if $-(K_X + B + M)$ and $K_{X'} + B' + M'$ are ample over $W$, and at most one of the morphisms $\phi$ and $\phi^+$ is the identity. Observe that if $\phi^+$ is the identity, then $\phi$ is a divisorial contraction, and vice-versa. In the above case, the quasi-flip will be called a weak divisorial contraction and weak divisorial extraction, respectively. The quasi-flip $\pi$ is said to be small if both $\phi$ and $\phi^+$ are small morphisms. A flip is an ample small quasi-flip of relative Picard rank one. A divisorial contraction (resp. divisorial extraction) is a weak divisorial contraction (resp. weak divisorial extraction) of relative Picard rank one.

**Definition 1.10.** A sequence of quasi-flips for a generalized log canonical pair $(X, B + M)$ is said to be with a common b-nef divisor if all the nef parts $M_i$ in the sequence of quasi-flips are the trace of a common b-nef b-Cartier $\mathbb{R}$-divisor. A sequence of quasi-flips for a generalized log canonical pair $(X, B + M)$ is said to be under a set satisfying the DCC if the coefficients of all the boundary parts $B_i$ in the sequence of quasi-flips belong to a fixed set satisfying the DCC. Moreover, we say that the sequence is with a fixed boundary divisor if the boundary divisor on the flipped pair is the divisorial push-forward of the boundary divisor on the flipping pair.

**Definition 1.11.** A minimal model program for $K_X + B + M$ over $Z$ is a sequence of flips and divisorial contractions for $K_X + B + M$ over $Z$. A weak minimal model program for $K_X + B + M$ over $Z$, is a sequence of ample quasi-flips for $K_X + B + M$ over $Z$. 

Definition 1.12. We say that a weak minimal model program

\[(X/Z, B + M) \to_{\pi_1} (X_1/Z, B_1 + M_1) \to_{\pi_2} (X_2/Z, B_2 + M_2) \to_{\pi_3} \cdots \to_{\pi_i} (X_i/Z, B_i + M_i) \to_{\pi_{i+1}} \cdots\]

for \((X/Z, B + M)\) is good if every irreducible component \(B_s (K_{X_i} + B_i + M_i/Z)\) is contained in the exceptional locus of \(\pi_j\) for some \(j \geq i + 1\).

Remark 1.13. An \(\mathbb{R}\)-Cartier divisor \(D\) is nef if and only if its diminished base locus \(B_s (D)\) is empty. By the negativity lemma, every irreducible component of \(B_s (K_{X_i} + B_i + M_i/Z)\) which is not contained in the exceptional locus of \(\pi_{i+1}\) is an irreducible component of the diminished base locus of the flipped generalized pair. Therefore, conjecturally every minimal model program is good. Moreover, it is known that a minimal model program with scaling of an ample divisor is good.

The following proposition is well-known to experts (see, e.g., [Sho04, Monotonicity]).

Proposition 1.14. Given a quasi-flip \(\pi: X \to X^+\) for generalized log canonical pairs \((X/Z, B + M)\) and \((X^+/Z, B^+ + M^+)\) over \(Z\), with flipping contraction \(\phi: X \to W\), and a prime divisor \(E\) over \(X\), we have that

\[a(E)(X/Z, B + M) \leq a(E)(X^+/Z, B^+ + M^+)\]

and the inequality is strict if and only if the center of \(E\) on \(X\) is contained in the flipping locus.

Definition 1.15. Let \((X/Z, B + M)\) be a generalized pair, let \(N\) an effective divisor on \(X\) and \(P'\) a nef \(\mathbb{R}\)-Cartier divisor over \(Z\) on \(X'\), such that \(P + N\) is \(\mathbb{R}\)-Cartier, where \(P = f_* P'\). The generalized log canonical threshold of \(N + P\) with respect to the generalized pair \((X, B + M)\) is defined to be

\[\text{lct}(\pi(X/Z, B + M) | P + N) := \sup \{ \lambda | (X/Z, B + \lambda N + M + \lambda P) \text{ is generalized log canonical}\},\]

where the above generalized pair has boundary part \(B + \lambda N\) and nef part \(M + \lambda P\). If \((X, B + M)\) is generalized log canonical, then the above real number is non-negative. Observe that the above threshold may be infinite, for instance if \(P' = f^* P\) and \(N = 0\).

Remark 1.16. Given a set of positive real numbers \(\Lambda\) satisfying the DCC, we will denote by \(B(\Lambda)\) the set of generalized boundaries \(B + M\), where the coefficients of \(B\) belong to \(\Lambda\), and where we can write \(M' = \sum \lambda_i M'_i\) where \(\lambda_i \in \Lambda\) and \(M'_i\) are Cartier divisors nef over \(Z\). In [BZ16, Theorem 1.5], Birkar and Zhang prove that the set

\[\text{LCT}_n(\Lambda) = \{ \text{lct}(\pi(X/Z, B + M) | N + P) | B + M \in B(\Lambda), N + P \in B(\Lambda) \text{ and } \dim(X) = n\}\]

satisfies the ascending chain condition. Here, we assume that \(N + P\) is \(\mathbb{R}\)-Cartier so that the definition of log canonical threshold makes sense. The proof relies on [HMX14], where this result is proved in the case \(M' = N' = 0\). In [BZ16], the authors prove the statement by induction in the number of non-trivial coefficients of \(M'\) and \(N'\).

Remark 1.17. If \(M = 0\), then we will drop the word “generalized” from the definition. In this case, we are in the usual setting of log pairs as in [KM98, HK10].

1.3. Log canonical threshold with respect to weak Zariski decompositions. In this subsection, we introduce an invariant for generalized log canonical pairs admitting a weak Zariski decomposition.

Definition 1.18. Let \((X/Z, B + M)\) be a \(\mathbb{Q}\)-factorial generalized log canonical pair with a weak Zariski decomposition given by the projective birational morphism \(f: X' \to X\) over \(Z\) and the numerical equivalence \(f^*(K_X + B + M) \equiv_Z N' + P'\). We consider \(P = f_* P'\) and \(N = f_* N'\) as the nef part and boundary part of a generalized boundary, and define

\[\text{lct}_{\text{WZD}}(f, N + P)(X/Z, B + M) := \text{lct}(\pi(X/Z, B + M) | N + P).\]
We call this invariant the log canonical threshold of the generalized pair with respect to the weak Zariski decomposition or just the lct with respect to the WZD. When the weak Zariski decomposition is clear from the context, we will just write lct_{WZD} instead of lct_{WZD(f,N+P)}.

**Remark 1.19.** The generalized log canonical threshold with respect to the weak Zariski decomposition depends on the chosen WZD and not only on the given generalized pair. For instance, every effective divisor linearly equivalent to $K_X + B + M$ gives a different weak Zariski decomposition, and different choices of effective divisors give different log canonical thresholds. The above invariant is uniquely determined by the generalized pair if we choose a Nakayama decomposition (see, e.g., [Nak04]). However, the existence of a WZD is a weaker assumption (see, e.g., [Bir12a]).

**Lemma 1.20.** Let $(X/Z, B + M)$ be a $\mathbb{Q}$-factorial generalized log canonical pair with a weak Zariski decomposition. The lct with respect to the WZD is finite unless $K_X + B + M$ is nef over $Z$.

**Proof.** Without loss of generality we may assume that we have a projective birational morphism $f : X' \to X$ such that both nef b-Cartier divisors $P'$ and $M'$ descend to $X'$. If $N'$ is a non-trivial effective divisor, then the above log canonical threshold is finite, so we may assume it is trivial. Since $X$ is $\mathbb{Q}$-factorial, by the negativity lemma we can write $f^*P = P' + E$ where $E$ is an effective divisor. If $E$ is non-trivial, the above log canonical threshold is again finite. Otherwise, we have $f^*P = P'$ which implies that $P$ is a nef divisor over $Z$, and so we conclude that $K_X + B + M \equiv_Z P$ is nef over $Z$ as well. 

**Lemma 1.21.** The lct with respect to the WZD does not change if we replace $X'$ by a higher birational model.

**Proof.** The generalized log canonical threshold only depends on the nef b-Cartier divisor $P'$ and the effective divisor $N = f_* N'$, both of them are invariant by taking higher birational models of $X'$.

**Lemma 1.22.** Let $(X/Z, B + M)$ be a $\mathbb{Q}$-factorial generalized log canonical pair with a weak Zariski decomposition $f : X' \to X$ such that $f^*(K_X + B + M) \equiv_Z P' + N'$ where $P'$ is nef over $Z$ and $N' \geq 0$. If $\pi : X \dashrightarrow X_1$ is a quasi-flip that extracts no divisors, then $(X_1/Z, B_1 + M_1)$ is a $\mathbb{Q}$-factorial generalized log canonical pair with a compatible weak Zariski decomposition where $B_1 = \pi_* B$ and $M_1 = \pi_* M$.

**Proof.** We may assume that $f_1 : X' \to X_1$ is a morphism. We have

$$P' + N' \equiv_Z f^*(K_X + B + M) \equiv_Z f_1^*(K_{X_1} + B_1 + M_1) + E$$

where $E \geq 0$ is $f_1$-exceptional. Since $-(N' - E) \equiv_{X_1} P'$ is nef over $X_1$ and $f_{1,*}(N' - E) \geq 0$, it follows by the negativity lemma that $N'_1 := N' - E \geq 0$. But then $f_1^*(K_{X_1} + B_1 + M_1) \equiv_Z P' + N'_1$ is a compatible weak Zariski decomposition.

**Lemma 1.23.** Let $(X/Z, B + M)$ be a $\mathbb{Q}$-factorial generalized log canonical pair with a weak Zariski decomposition and

$$(X/Z, B + M) \xrightarrow{-\pi_1} (X_1/Z, B_1 + M_1) \xrightarrow{-\pi_2} (X_2/Z, B_2 + M_2) \xrightarrow{-\pi_3} \cdots \xrightarrow{-\pi_i} (X_i/Z, B_i + M_i) \xrightarrow{-\pi_{i+1}} \cdots$$

be a sequence of small ample quasi-flips for $K_X + B + M$ over $Z$. Then, the lct of the generalized pairs $(X_i/Z, B_i + M_i)$ with respect to the WZD induced by Lemma 1.22 forms a non-decreasing sequence of positive real numbers.

**Proof.** Since $\pi_i$ is a small ample quasi-flip over $Z$ we know that the generalized log canonical pair $(X_i/Z, B_i + M_i)$ is not nef over $Z$. Hence, by Lemma 1.20, we conclude that the lct with respect to any WZD of $K_{X_i} + B_i + M_i$ over $Z$ is finite. It suffices to prove the statement for a single small ample quasi-flip.
\( \pi: X \rightarrow X^+ \) over \( Z \), of the \( \mathbb{Q} \)-factorial generalized log canonical pair \((X/Z, B + M)\). We will denote by \((X^+/Z, B^+ + M^+)\) the flipped generalized log canonical pair. Consider two projective birational morphisms \( f: X' \rightarrow X \) and \( f^+: X' \rightarrow X^+ \) over \( Z \), such that both nef b-Cartier divisors \( P' \) and \( M' \) descend on \( X' \). We will denote by \( f^+(K_X + B + M) \equiv_Z P' + N' \) the induced weak Zariski decomposition for \( K_X + B + M \) on \( X' \). By the negativity lemma we have
\[
f^+(K_X^+ + B^+ + M^+) \equiv_Z P' + N'^+.
\]
where \( N' \geq N'^+ \geq 0 \). Hence, we have an induced Zariski decomposition for \( K_X^+ + B^+ + M^+/Z \) and we will denote
\[
P^+ = f^+ \cdot P' \quad \text{and} \quad N^+ = f^+ \cdot N'^+.
\]
Without loss of generality we may assume that \( \lambda \) is a log resolution of both generalized pairs. By Lemma 1.21, this assumption does not change the lct with respect to the WZD. Therefore, for every \( \lambda > 0 \) we have that
\[
f^+(K_X^+ + B^+ + M^+ + \lambda(P^+ + N^+)) \leq f^+(K_X + B + M + \lambda(P + N)),
\]
concluding the inequality between log canonical thresholds. \( \square \)

**Corollary 1.24.** The lct with respect to the WZD of a small ample quasi-flip \((X/Z, B + M) \rightarrow (X^+/Z, B^+ + M^+)\) strictly increases if and only if the flipping locus contains all the generalized log canonical centers of \((X/Z, B + M + \lambda(P + N))\) where \( \lambda \) is the log canonical threshold of the generalized pair \((X/Z, B + M)\) with respect to the WZD.

### 1.4. Generalized divisorially log terminal modifications.

In this subsection, we recall the proof of existence of \( \mathbb{Q} \)-factorial dlt modifications for generalized log canonical pairs (see, e.g., [BH14, Remark 4.5] and [Bir17, 2.6.(2)]). In [AH12] and [KK10, Theorem 3.1], there is a proof of existence of dlt modifications for pairs.

**Definition 1.25.** We say that the pair \((X/Z, B)\) is divisorially log terminal or dlt if the coefficients of \( B \) are less than or equal to one, and there is a log resolution \( g: X'' \rightarrow X \) over \( Z \), such that \( a_E(X/Z, B) > 0 \) for all \( g \)-exceptional prime divisors \( E \) on \( X'' \). We say that \((X/Z, B + M)\) is generalized divisorially log terminal or generalized dlt if \((X/Z, B)\) is dlt and if every generalized non-klt center of \((X/Z, B + M)\) is a non-klt center of \((X/Z, B)\).

**Lemma 1.26.** Let \((Y/Z, B_Y + M_Y)\) be a generalized pair such that the coefficients of \( B_Y \) are in \([0, 1]\). The generalized pair is dlt if and only if there exists a closed subset \( Y_0 \subseteq Y \) such that \( Y \setminus Y_0 \) is smooth, \( B_Y|_{Y \setminus Y_0} \) is a divisor with simple normal crossing support, and for every prime divisor \( E \) with center in \( Y_0 \) we have that \( a_E(Y/Z, B_Y + M_Y) > 0 \).

**Proof.** The proof is similar to the the analogous statement for dlt pairs, see eg. [KM98, Prop. 2.40]. \( \square \)

**Definition 1.27.** Let \((X/Z, B + M)\) be a generalized log canonical pair. Let \( h: Y \rightarrow X \) be a projective birational morphism of normal varieties over \( Z \). We may assume that the given projective birational morphism \( f: X' \rightarrow X \) factors through \( h \). Then, we define \( B_Y \) and \( M_Y \) to be the push-forwards of \( B' \) and \( M' \) on \( Y \), respectively. Thus, we can write
\[
K_Y + B_Y + M_Y = h^*(K_X + B + M).
\]
If \( B_Y \geq 0 \), \((Y/Z, B_Y + M_Y)\) is a \( \mathbb{Q} \)-factorial generalized dlt pair, and every \( h \)-exceptional prime divisor \( E \) has log discrepancy zero with respect to the generalized pair \((X/Z, B + M)\), then we say that \((Y/Z, B_Y + M_Y)\) is a \( \mathbb{Q} \)-factorial dlt modification of \((X/Z, B + M)\). Here, we consider \((Y/Z, B_Y + M_Y)\) as a generalized pair with nef b-Cartier divisor \( M' \).
Proposition 1.28. A generalized log canonical pair \((X/Z, B + M)\) has a \(\mathbb{Q}\)-factorial dlt modification \((Y/Z, B_Y + M_Y)\).

Proof. Assume that the morphism \(f : X' \rightarrow X\) over \(Z\) gives a log resolution of the generalized pair. We may assume that \(f\) is obtained by blowing up loci of codimension at least two, so that there exists an \(f\)-exceptional divisor \(C \geq 0\) such that \(-C\) is \(f\)-ample.

We define \(\Delta = [B]\) and \(T = B - \Delta\), and as usual we write

\[
K_{X'} + B' + M' = f^*(K_X + B + M),
\]

hence we can write \(B' = \Delta' + E^+ + E^0 - E^-\), where \(\Delta' = f_*^{-1}\Delta\), \(E^+\) is supported on the sum of the divisors with generalized log discrepancy zero, \(E^0\) is supported on the sum of the \(f\)-exceptional divisors with generalized log discrepancy in \((0, 1]\), and \(E^-\) is supported on the sum of the \(f\)-exceptional divisors with generalized log discrepancy \(\geq 1\). We may assume that the support of \(E^0\) contains the \(f\)-exceptional divisors with generalized log discrepancy equal to 1.

We consider a sufficiently ample divisor \(H\) on \(X\). For every \(\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}_{>0}\) we have

\[
E^+ + (1 + \epsilon_3)E^0 + \epsilon_2(-C + f^*H) + M' = (1 - \epsilon_1\epsilon_2)E^+ + (1 + \epsilon_3)E^0 + \epsilon_2(\epsilon_1E^+ - C + f^*H) + M'.
\]

We can choose \(\epsilon_1\) sufficiently small such that both \(-\epsilon_2(-C + f^*H) + M'\) and \(\epsilon_2(\epsilon_1E^+ - C + f^*H) + M\) are ample, so they are \(\mathbb{Q}\)-linearly equivalent to effective divisors \(H_1(\epsilon_2)\) and \(H_2(\epsilon_2)\) with coefficients in \((0, 1)\) such that \(B' + H_1(\epsilon_2) + H_2(\epsilon_2)\) has simple normal crossing support. If \(\epsilon_3\) is small enough, the pair

\[
(X, \Delta' + (1 - \epsilon_1\epsilon_2)E^+ + (1 + \epsilon_3)E^0 + H_1(\epsilon_2))
\]

is klt, so by [BCHM10] we can run a minimal model program \(\pi : X' \rightarrow Y\), of the above pair with respect to \(X\) which terminates with a minimal model \(h : Y \rightarrow X\). The above minimal model program is also a minimal model program for the pair

\[
(X, \Delta' + E^+ + (1 + \epsilon_3)E^0 + H_1(\epsilon_2)),
\]

so the minimal model is dlt.

Observe that the strict transform of the \(\mathbb{Q}\)-divisor

\[
f^*(K_X + B + M) - (K_{X'} + \Delta' + E^+ + (1 + \epsilon_3)E^0 + H_1(\epsilon_2)) \sim_{f, \mathbb{Q}} -\epsilon_3E^0 + \epsilon_2C - E^-
\]

on \(Y\) is \(h\)-anti-nef and its push-forward on \(X\) is trivial. By the negativity lemma we conclude that the push-forward on \(Y\) of the above divisor must be effective. Then, if we take \(0 < \epsilon_2 \ll \epsilon_3 \ll 1\), the irreducible divisors on the support of \(E^0\) ad \(E^-\) are contracted in the minimal model program \(\pi : X \rightarrow Y\). Thus, the generalized pair \((Y, B_Y + M_Y)\) is generalized dlt, where \(M_Y = \pi_*M'\) and \(B_Y\) is the strict transform of \(\Delta' + E^+ + E^-\) on \(Y\).

The following lemma is proved in a more general setting in [BZ16, Section 4].

Lemma 1.29. Let \((Y/Z, B_Y + M_Y)\) be a \(\mathbb{Q}\)-factorial generalized dlt pair. Let \(A\) be a general effective ample divisor on \(Y\) over \(Z\), then we can run a minimal model program for the generalized pair with scaling of \(A\) over \(Z\).

1.5. Generalized dlt adjunction. In this subsection, we recall the construction and properties of generalized divisorial adjunction in [BZ16] and introduce a generalized dlt adjunction formula.

Definition 1.30. Let \((X/Z, B + M)\) be a generalized log canonical pair, assume that \(S\) is the normalization of a component of \([B]\) and \(S'\) its birational transform on \(X'\). Replacing the morphism \(f : X' \rightarrow X\) with a higher birational model, we may assume that \(f\) is a log resolution for the generalized log canonical pair \((X, B + M)\). Then, we can write

\[
K_{X'} + B' + M' = f^*(K_X + B + M),
\]
and
\[ K_{S'} + B_{S'} + M_{S'} = (K_X' + B' + M')|_{S'}, \]
where \( B_{S'} = (B - S')|_{S'} \) and \( M_{S'} = M|_{S'} \). We have an induced morphism \( f_S: S' \to S \) and we let \( f_{S*}(B_{S'}) = B_S \) and \( f_{S*}(M_{S'}) = M_S \). Hence, we can consider the pair \((S/Z, B_S + M_S)\) as a generalized pair with b-nef b-Cartier divisor \( M_{S'} \).

**Lemma 1.31.** The divisor \( B_S \) is effective. The generalized pair \((S/Z, B_S + M_S)\) is generalized log canonical.

*Proof.* This is proved in [BZ16, Remark 4.8]. □

**Proposition 1.32.** Let \( d \) be a natural number and a set of nonegative real numbers \( \Lambda \) satisfying the DCC. There is a set of nonegative real numbers \( \Omega \) satisfying the DCC, which only depends on \( d \) and \( \Lambda \), such that if

1. \((X/Z, B + M)\) is generalized log canonical of dimension \( d \),
2. the coefficients of \( B \) belong to \( \Lambda \),
3. we can write \( M' = \sum \mu_iM'_i \), where \( M'_i \) are Cartier divisors and \( \mu_i \in \Lambda \), and
4. the generalized pair \((S/Z, B_S + M_S)\) is constructed as in Definition 1.30,

then the coefficients of \( B_S \) belong to \( \Omega \).

*Proof.* This is proved in [BZ16, Proposition 4.9]. □

**Lemma 1.33.** Let \( \Lambda \) be a set of nonegative real numbers satisfying the DCC condition and \( d \in \mathbb{Z}_{\geq 1} \), then there is a set of nonegative real numbers \( \Theta \) satisfying the DCC condition, which only depends on \( d \) and \( \Lambda \), such that if

1. \((Y/Z, B_Y + M_Y)\) is a generalized dlt pair of dimension \( d \),
2. the coefficients of \( B_Y \) belong to \( \Lambda \),
3. we can write \( M' = \sum \mu_iM'_i \), where \( M'_i \) are Cartier divisors and \( \mu_i \in \Lambda \), and
4. \( V \) is a generalized log canonical center of \((Y/Z, B_Y + M_Y)\),

then we can write an adjunction formula
\[ (K_Y + B_Y + M_Y)|_V = K_V + B_V + M_V, \]
where \((V/Z, B_V + M_V)\) is a generalized dlt pair, the coefficients of \( B_V \) belong to \( \Theta \) and we can write \( M'_V = \sum \mu_iM'_{i,V} \), where \( M'_{i,V} \) are Cartier divisors and \( \mu_i \in \Lambda \).

*Proof.* We proceed by induction on the codimension of the log canonical center. If the log canonical center has codimension one, then this is Lemma 1.32. If the log canonical center \( V \) has higher codimension, by Lemma 1.26 we know that \( V \) is contained in some divisor \( S \) which appears with coefficient one in \( B \). Therefore, by Lemma 1.32 we can do a divisorial generalized adjunction to \( S \). By [HK10, Theorem 3.24], the generalized pair \((S/Z, B_S + M_S)\) is dlt and \( V \) is a non-klt center of such generalized pair. Hence, by the induction hypothesis on the codimension, we can write an adjunction formula
\[ (K_S + B_S + M_S)|_V = K_V + B_V + M_V, \]
which induces an adjunction formula for \((Y/Z, B_Y + M_Y)\). □

**Remark 1.34.** Observe that the set \( \Theta \) of Lemma 1.33 is
\[ \Omega(\Omega(\ldots(\Omega(\Lambda, d), d - 1), \ldots, 2), 1), \]
where \( \Omega \) is the set of Lemma 1.32.

**Corollary 1.35.** If \( V \) is a minimal non-klt center of the generalized dlt pair \((Y, B_Y + M_Y)\), then the induced generalized pair \((V, B_V + M_V)\) is generalized klt.
1.6. **Generalized klt closure.** In this subsection, we prove that generalized klt closures exist for certain generalized pairs.

**Definition 1.36.** The **generalized klt locus** of a generalized pair \((X/Z, B + M)\) is the complement of the non-klt centers on \(X\).

**Definition 1.37.** Let \((X/Z, B + M)\) be a generalized log canonical pair. We say that a generalized klt pair \((Y/Z, B_Y + M_Y)\) is a **generalized klt closure** of \((X/Z, B + M)\) if the following conditions hold:

- there is a projective birational morphism \(h: Y \to X\),
- \(h\) is an isomorphism over the generalized klt locus of \((X/Z, B + M)\),
- the restrictions of \(B_Y\) and \(B\) coincide via this isomorphism, and
- \(M_Y\) is the trace on \(Y\) of the b-nef b-Cartier divisor \(M'\).

**Proposition 1.38.** Every \(\mathbb{Q}\)-factorial generalized log canonical pair admits a generalized klt closure. Moreover, we may assume that the coefficients of the boundary part on the generalized klt closure are the same as those of the starting generalized pair.

**Proof.** Let \((X/Z, B + M)\) be a generalized log canonical pair. By Proposition 1.28, we can construct a dlt modification \((Y/Z, B_Y + M_Y)\) of \((X/Z, B + M)\) and then reduce the coefficients of the extracted divisors to zero. It suffices to prove that the dlt modification is an isomorphism on the generalized klt locus of \((X/Z, B + M)\). Observe that the projective birational morphism \(h: Y \to X\) does not extract divisors over the generalized klt locus of \((X/Z, B + M)\), therefore over such locus we have a small morphism between \(\mathbb{Q}\)-factorial varieties, which is an isomorphism. \(\square\)

2. **Weak Zariski decompositions and termination of flips**

2.1. **WZD and termination of flips.** In this subsection, we prove Theorem 1.

**Lemma 2.1.** Let \((Y/Z, B_Y + M_Y)\) be a \(\mathbb{Q}\)-factorial dlt pair and

\[(Y/Z, B_Y + M_Y) \xrightarrow{\pi_1} (Y_1/Z, B_{Y_1} + M_{Y_1}) \xrightarrow{\pi_2} (Y_2/Z, B_{Y_2} + M_{Y_2}) \xrightarrow{\pi_3} \cdots\]

be a minimal model program which is an isomorphism at the generic point of a log canonical center \(V\) of \((Y/Z, B_Y + M_Y)\). Then, the induced sequence of birational maps (see §1.5)

\[(V/Z, B_V + M_V) \xrightarrow{\pi_1} (V_1/Z, B_{V_1} + M_{V_1}) \xrightarrow{\pi_2} (V_2/Z, B_{V_2} + M_{V_2}) \xrightarrow{\pi_3} \cdots\]

is a sequence of ample quasi-flips or identities for the generalized dlt pair \((V, B_V + M_V)\).

**Proof.** This is proved in [Mor18, Proposition 4.3] for the divisorial generalized adjunction. The general case follows by induction on the codimension of the log canonical center. \(\square\)

**Lemma 2.2.** A sequence of ample quasi-flips with a common b-nef divisor and under a DCC set for a generalized klt pair terminates in codimension one.

**Proof.** This is proved in [Mor18, Lemma 4.26]. \(\square\)

**Corollary 2.3.** If \(V\) is a minimal non-klt center of \((Y, B_Y + M_Y)\) not contained in any of the flipping loci, then the sequence of birational transformations (2.1) is eventually a sequence of isomorphisms and small ample quasi-flips with a fixed boundary divisor and a common b-nef divisor.

**Lemma 2.4.** A small ample quasi-flip for a \(\mathbb{Q}\)-factorial generalized klt pair with a fixed boundary divisor can be factored in a sequence of flips.
Proof. Suppose that \( \pi : X \to X^+ \) is a small ample quasi-flip so that we have generalized klt pairs \((X, B + M)\) and \((X^+, B^+ + M^+)\) and projective morphisms \(\phi : X \to W\) and \(\phi^+ : X^+ \to W\) over \(Z\) such that \(- (K_X + B + M)\) and \(K_{X^+} + B^+ + M^+\) are ample over \(W\) and \(B^+ = \pi_* B\). We now run a \(K_X + B + M\) mmp with scaling over \(W\) which terminates by [BZ16, Lemma 4.4]. The output of this minimal model program is a good minimal model \((X', B' + M')\) for \(K_X + B + M\) over \(W\), it has a projective birational morphism \(\pi' : X' \to X^+\) and \(\rho\). Since flips do not change the relative Picard rank over \(W\) we conclude that both varieties \(X'\) and \(X\) have the same Picard rank over \(W\), which means that \(\pi'\) is a small morphism between \(\mathbb{Q}\)-factorial varieties, so it must be an isomorphism.

The following lemma is a version of Fujino’s special termination for dlt pairs in the context of generalized pairs (see, e.g., [Fuj07]).

Lemma 2.5. With the notation of Lemma 2.1. Assume that a minimal model program for the generalized \(\mathbb{Q}\)-factorial dlt pair \((Y/Z, B_Y + M_Y)\) is infinite. Then, this minimal model program is eventually disjoint from the generalized non-klt locus of \((Y/Z, B_Y + M_Y)\) or it induces an infinite sequence of flips for a generalized klt pair of dimension at most \(n - 1\).

Moreover, if the minimal model program for \((Y/Z, B_Y + M_Y)\) is good, then we obtain an induced infinite good minimal model program for a generalized dlt pair.

Proof. Assume that the flipping loci of the minimal model program for \((Y/Z, B_Y + M_Y)\) intersect the generalized non-klt locus infinitely many times. Then there exists a generalized log canonical center which is not contained in any exceptional locus of the minimal model program and intersects the flipping loci infinitely many times. Let \(V\) be a generalized log canonical center which is minimal with the above condition. By the minimality assumption, eventually the flipping loci only intersects the klt locus of \((V/Z, B_V + M_V)\). Since the generalized pair \((V/Z, B_V + M_V)\) is generalized dlt by Lemma 1.33, then by Proposition 1.38 it has a generalized klt closure \((V'/Z, B_{V'} + M_{V'})\) and \(h : V' \to V\). By Lemma 2.1, Corollary 2.3, and Lemma 2.4, we obtain an induced infinite sequence of flips for the generalized klt pair \((V'/Z, B_{V'} + M_{V'})\).

For the second claim, since the pair \((V/Z, B_V + M_V)\) is generalized dlt, from the inclusion

\[
\text{Bs}_-(K_Y + B_Y + M_Y/Z) \cap V \supseteq \text{Bs}_-(K_V + B_V + M_V/Z),
\]

it follows that the induced minimal model program for \((V/Z, B_V + M_V)\) is good.

Proof of Theorem 1. Assume termination of flips for generalized klt pairs of dimension at most \(n - 1\). Let \((X/Z, B + M)\) be a generalized log canonical pair of dimension \(n\) admitting a weak Zariski decomposition. We proceed by contradiction. Let

\[
(X/Z, B + M) \xrightarrow{\pi_1} (X_1/Z, B_1 + M_1) \xrightarrow{\pi_2} (X_2/Z, B_2 + M_2) \xrightarrow{\pi_3} \cdots \xrightarrow{\pi_i} (X_i/Z, B_i + M_i) \xrightarrow{\pi_{i+1}} \cdots
\]

be an infinite minimal model program for \((X/Z, B + M)\).

Step 1. We reduce to the \(\mathbb{Q}\)-factorial dlt case.

Consider the ample quasi-flip \(\pi_1 : X \to X_1\) with flipping contraction \(\phi : X \to W\). By Lemma 1.28, we have a \(\mathbb{Q}\)-factorial dlt modification

\[
\rho : (Y/Z, B_Y + M_Y) \to (X/Z, B + M).
\]

By Lemma 1.29, we can run a minimal model program for the \(\mathbb{Q}\)-factorial generalized dlt pair \((Y/Z, B_Y + M_Y)\) with scaling of a general ample divisor over \(W\). By Lemma 2.5 and the induction hypothesis, we may assume that the sequence of flips is eventually disjoint from the generalized non-klt locus of the generalized pair. However, in this case we obtain a minimal model program with scaling for a quasi-projective generalized klt
pair which is big over the base. This terminates by [BZ16, Lemma 4.4]. Thus, the above minimal model program terminates with a minimal model \((Y_1/Z, B_{Y_1} + M_{Y_1})\) over \(W\) which is a generalized dlt modification of \((X_1/Z, B_1 + M_1)\) and \((X_1/Z, B_1 + M_1)\) is its generalized log canonical model over \(W\).

Proceeding analogously with the other steps of the minimal model program, we obtain an infinite minimal model program for \(Q\)-factorial generalized dlt pairs

\[
(Y/Z, B_Y + M_Y) = (Y_i/Z, B_{Y_i} + M_{Y_i}) \to (Y_{i+1}/Z, B_{Y_{i+1}} + M_{Y_{i+1}}) \to \ldots
\]

we denote by \(P_{Y_i}\) and \(N_{Y_i}\) the push-forward of the nef part and effective part of the weak Zariski decomposition induced by Lemma 1.22 on each generalized pair \((Y_i/Z, B_{Y_i} + M_{Y_i})\). Moreover, we denote by \(\lambda_i\) the log canonical threshold of the \(Q\)-factorial generalized dlt pair \((Y_i/Z, B_{Y_i} + M_{Y_i})\) with respect to \(P_{Y_i} + N_{Y_i}\).

Recall from Lemma 1.23 that the \(\lambda_i\) form a non-decreasing sequence of non-negative real numbers.

**Step 2.** We may assume that the non-decreasing sequence \(\lambda_i\) is eventually constant and equal to a nonnegative real number \(\lambda > 0\) and the set of non-klt centers of the \(Q\)-factorial generalized log canonical pairs \((Y_i/Z, B_{Y_i} + M_{Y_i} + \lambda(P_{Y_i} + N_{Y_i}))\) are birational for all \(i \geq 0\).

By the ACC for generalized log canonical thresholds [BZ16, Theorem 1.5] we conclude that after finitely many steps the sequence \(\lambda_i\) must stabilize to a nonnegative real number \(\lambda\). Moreover, by the monotonicity property of generalized log discrepancies (Proposition 1.14), we conclude that after finitely many steps of the minimal model program, the generic point of any of the (finitely many) generalized non-klt centers is not contained in the flipping locus. By applying Step 1 again, we may assume that the generalized pairs \((Y_i/Z, B_{Y_i} + M_{Y_i} + \lambda(P_{Y_i} + M_{Y_i}))\) are indeed \(Q\)-factorial generalized dlt.

**Step 3.** We may assume that there exists \(\lambda' > \lambda\) and divisors \(0 \leq B_Y \leq B_{Y_i}\) and \(0 \leq N_Y \leq N_{Y_i}\) such that the following conditions hold

1. \((Y_i, B_{Y_i} + \lambda'N_{Y_i} + M_{Y_i} + \lambda'P_{Y_i})\) is generalized \(Q\)-factorial dlt,
2. there exists an open subset \(U_i \subset Y_i\) containing all the flipping loci such that

\[
(B_{Y_i} + \lambda'N_{Y_i} + M_{Y_i} + \lambda'P_{Y_i})|_{U_i} = (B_{Y_i} + \lambda'N_{Y_i} + M_{Y_i} + \lambda'P_{Y_i})|_{U_i},
\]

and

3. there is a stratum of \((B_{Y_i} + \lambda'N_{Y_i})^{=1}\) that is not contained in any flipping locus but intersects infinitely many flipping loci.

In particular, each birational map \(\pi_i : Y_i \to Y_{i+1}\) is a \((K_{Y_i} + B_{Y_i} + M_{Y_i} + \lambda'(N_{Y_i} + P_{Y_i}))\)-flip.

Suppose that we are given \(\lambda' > \lambda\) and divisors \(0 \leq B_{Y_i} \leq B_{Y_i}\) and \(0 \leq N_{Y_i} \leq N_{Y_i}\) as above satisfying (1) and (2). Notice that this is the case for \(B_{Y_i} = B_{Y_i}\) and \(U_i = Y_i\). Suppose that (3) is not satisfied, then the flipping loci are eventually disjoint from the non-klt locus i.e. from the support of \((B_{Y_i} + \lambda'N_{Y_i})^{=1}\). Let

\[
B_{Y_i}' = B_{Y_i} - B_{Y_i} \wedge (B_{Y_i} + \lambda'N_{Y_i})^{=1} + \text{Supp}(B_{Y_i} + \lambda'N_{Y_i})^{=1},
\]

\[
N_{Y_i}' = N_{Y_i} - N_{Y_i} \wedge (B_{Y_i} + \lambda'N_{Y_i})^{=1},
\]

and set

\[
\lambda'' := \text{lct}(Y_i/Z, B_{Y_i}' + M_{Y_i} | N_{Y_i}' + P_{Y_i}) > \lambda'.
\]

Replace \(U_i\) by \(U_i \setminus (B_{Y_i}' + \lambda'N_{Y_i})^{=1}\). Notice that as the flipping loci are disjoint from \((B_{Y_i} + \lambda'N_{Y_i})^{=1}\) they are contained in \(U_i\). Passing to appropriate dlt models as in Step 1, we may assume that we have a sequence of the mmp for dlt pairs \((Y_i, B_{Y_i}' + M_{Y_i} + \lambda''(N_{Y_i}' + P_{Y_i}))\).
We may now replace $B^X_i$ by $B'^X_i$ and $N^X_i$ by $N'^X_i$. Notice that by the acc for generalized lct’s (cf. [BZ16, Theorem 1.5]), we may repeat this procedure at most finitely many times. Therefore condition (3) is eventually satisfied.

**Step 4.** We prove that a minimal model program as in Step 3 terminates.

Observe that we have a minimal model program for a $\mathbb{Q}$-factorial generalized dlt pair and a log canonical center which is intersected non-trivially by infinitely many flips. By Lemma 2.5, we obtain an induced sequence of flips for a generalized klt pair of dimension at most $n - 1$, leading to a contradiction. \(\square\)

**Proof of Theorem 2.** We assume the existence of weak Zariski decompositions for pseudo-effective generalized log canonical pairs of dimension at most $n$. Let $(X/Z, B + M)$ be a pseudo-effective generalized log canonical pair of dimension $n$. We proceed by contradiction. Let

$$(X/Z, B + M) \dashrightarrow (X/Z, B_1 + M_1) \dashrightarrow (X/Z, B_2 + M_2) \dashrightarrow \cdots \dashrightarrow (X/Z, B_i + M_i) \dashrightarrow \cdots$$

be an infinite good minimal model program for $(X/Z, B + M)$.

**Step 1.** We reduce to the $\mathbb{Q}$-factorial dlt case.

Consider the ample quasi-flip $\pi_1: X \dashrightarrow X_1$ with flipping contraction $\phi: X \rightarrow W$. By Lemma 1.28, we have a $\mathbb{Q}$-factorial dlt modification

$$\rho: (Y/Z, B_Y + M_Y) \rightarrow (X/Z, B + M).$$

By Lemma 1.29, we can run a minimal model program for the $\mathbb{Q}$-factorial generalized dlt pair $(Y/Z, B_Y + M_Y)$ with scaling of a general ample divisor over $W$. By Lemma 2.5 and the induction hypothesis, we may assume that the sequence of flips is eventually disjoint from the generalized non-klt locus of the generalized pair. However, in this case we obtain a minimal model program for a quasi-projective generalized klt pair which is big over the base, this terminates by [BZ16, Lemma 4.4]. Thus, the above minimal model program terminates with a minimal model $(Y_1/Z, B_{Y_1} + M_{Y_1})$ over $W$ which is a generalized dlt modification of $(X_1/Z, B_1 + M_1)$ and $(X_1/Z, B_1 + M_1)$ is its generalized log canonical model over $W$.

Proceeding analogously with the other steps of the minimal model program, we obtain an infinite minimal model program for $\mathbb{Q}$-factorial generalized dlt pairs

$$(Y/Z, B_Y + M_Y) \dashrightarrow (Y_1/Z, B_{Y_1} + M_{Y_1}) \dashrightarrow (Y_2/Z, B_{Y_2} + M_{Y_2}) \dashrightarrow \cdots \dashrightarrow (Y_i/Z, B_{Y_i} + M_{Y_i}) \dashrightarrow \cdots$$

Observe that this minimal model program is good since we have the equality

$$\text{Bs}_-(K_Y + B_Y + M_Y/Z) = \rho^{-1}(\text{Bs}_-(K_X + B + M/Z)).$$

We denote by $P_{Y_i}$ and $N_{Y_i}$ the push-forward of the nef part and effective part of the weak Zariski decomposition induced by Lemma 1.22 on each generalized pair $(Y_i/Z, B_{Y_i} + M_{Y_i})$. Moreover, we denote by $\lambda_i$ the log canonical threshold of the $\mathbb{Q}$-factorial generalized dlt pair $(Y_i/Z, B_{Y_i} + M_{Y_i})$ with respect to $P_{Y_i} + N_{Y_i}$. Recall from Lemma 1.23 that the $\lambda_i$ form a non-decreasing sequence of non-negative real numbers.

**Step 2.** We may assume that the non-decreasing sequence $\lambda_i$ is eventually constant and equal to a nonnegative real number $\lambda > 0$ and the set of non-klt centers of the $\mathbb{Q}$-factorial generalized log canonical pairs $(Y_i/Z, B_{Y_i} + M_{Y_i} + \lambda(P_{Y_i} + N_{Y_i}))$ are birational for all $i \gg 0$. 

In particular, each birational map \( \pi \) contradicts the following conditions hold:

1. \((Y_i, B_{Y_i}' + \lambda' N_{Y_i} + M_{Y_i} + \lambda' P_{Y_i})\) is a generalized \( \mathbb{Q} \)-factorial dlt,
2. there exists an open subset \( U_i \subset Y_i \), containing all the flipping loci such that
\[
(B_{Y_i}' + \lambda' N_{Y_i} + M_{Y_i} + \lambda' P_{Y_i})|_{U_i} = (B_{Y_i} + \lambda' N_{Y_i} + M_{Y_i} + \lambda' P_{Y_i})|_{U_i},
\]
and
3. there is a stratum of \((B_{Y_i}' + \lambda' N_{Y_i}')^{-1}\) that is not contained in any flipping locus but intersects infinitely many flipping loci.

In particular, each birational map \( \pi_i : Y_i \rightarrow Y_{i+1} \) is a \((K_{Y_i} + B_{Y_i} + M_{Y_i} + \lambda(N_{Y_i} + P_{Y_i}))\)-flip.

Similar to the proof of Step 3 of Theorem 1.

**Step 4.** We prove that a minimal model program as in Step 3 terminates.

Observe that we have a good minimal model program for a \( \mathbb{Q} \)-factorial generalized dlt pair and a log canonical center which is intersected non-trivially by infinitely many flips. By Lemma 2.5, we obtain an infinite good minimal model program for a generalized klt pair of dimension at most \( n - 1 \), leading to a contradiction.

\[\square\]

**References**

[AH12] Valery Alexeev and Christopher D. Hacon, *Non-rational centers of log canonical singularities*, J. Algebra 369 (2012), 1–15.

[Bir07] Caucher Birkar, *Ascending chain condition for log canonical thresholds and termination of flips*, Duke Math. J. 136 (2007), no. 1, 173–180. MR2271298

[Bir10] ______, *On existence of log minimal models*, Compos. Math. 146 (2010), no. 4, 919–928, DOI 10.1112/S0010437X09004564. MR2660678

[Bir11] ______, *On existence of log minimal models II*, J. Reine Angew. Math. 658 (2011), 99–113, DOI 10.1515/CRELLE.2011.002. MR2831514

[Bir12a] ______, *On existence of log minimal models and weak Zariski decompositions*, Math. Ann. 354 (2012), no. 2, 787–799, DOI 10.1007/s00208-011-0756-y. MR2965261

[Bir12b] ______, *Existence of log canonical flips and a special LMMPP*, Publ. Math. Inst. Hautes Études Sci. 115 (2012), 325–368, DOI 10.1007/s10240-012-0039-5. MR2929730

[Bir17] ______, *Anti-pluricanonical systems on Fano varieties*, 2017. https://arxiv.org/abs/1603.05765.

[BH14] Caucher Birkar and Zhengyu Hu, *Polarized pairs, log minimal models, and Zariski decompositions*, Nagoya Math. J. 215 (2014), 203–224, DOI 10.1215/00277630-2781096. MR3263528

[BCHM10] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. 23 (2010), no. 2, 405–468. MR2601039

[BZ16] Caucher Birkar and De-Qi Zhang, *Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs*, Publ. Math. Inst. Hautes Études Sci. 123 (2016), 283–331. MR3502099

[Cor07] Alessio Corti (ed.), *Flips for 3-folds and 4-folds*, Oxford Lecture Series in Mathematics and its Applications, vol. 35, Oxford University Press, Oxford, 2007.

[Fuj79] Takao Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), no. 3, 106–110. MR531454

[Fuj86] ______, *Zariski decomposition and canonical rings of elliptic threefolds*, J. Math. Soc. Japan 38 (1986), no. 1, 19–37, DOI 10.2969/jmsj/03810019. MR816221

[Fuj07] Osamu Fujino, *Special termination and reduction to pl flips*, Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, pp. 63–75. MR2359942
WEAK ZARISKI DECOMPOSITIONS AND TERMINATION OF FLIPS

Christopher D. Hacon, James McKernan, and Chenyang Xu, ACC for log canonical thresholds, Ann. of Math. (2) 180 (2014), no. 2, 523–571. MR3224718

Christopher D. Hacon and Sándor J. Kovács, Classification of higher dimensional algebraic varieties, Oberwolfach Seminars, vol. 41, Birkhäuser Verlag, Basel, 2010. MR2675555

Jingjun Han and Zhan Li, Weak Zariski decompositions and log minimal models for generalized polarized pairs, Preprint (2018).

János Kollár and Sándor J. Kovács, Log canonical singularities are Du Bois, J. Amer. Math. Soc. 23 (2010), no. 3, 791–813. MR2629988

János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original. MR1658959

John Lesieutre, The diminished base locus is not always closed, Compos. Math. 150 (2014), no. 10, 1729–1741, DOI 10.1112/S0010437X14007544. MR3269465

Joaquín Moraga, Termination of pseudo-effective 4-fold flips, 2018. https://arxiv.org/abs/1802.10202.

Noboru Nakayama, Zariski-decomposition and abundance, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004. MR2104208

Yu. G. Prokhorov, On the Zariski decomposition problem, Tr. Mat. Inst. Steklova 240 (2003), no. Biratsion. Geom. Lineín. Sist. Konechno Porozhdennye Algebry, 43–72 (Russian, with Russian summary); English transl., Proc. Steklov Inst. Math. 1(240) (2003), 37–65. MR1993748

V. V. Shokurov, Letters of a bi-rationalist. V. Minimal log discrepancies and termination of log flips, Tr. Mat. Inst. Steklova 246 (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 328–351 (Russian, with Russian summary); English transl., Proc. Steklov Inst. Math. 3(246) (2004), 315–336. MR2101303

V. V. Shokurov, Letters of a bi-rationalist. VII. Ordered termination., Tr. Mat. Inst. Steklova 264 (2009), no. Mnogomernaya Algebraicheskaya Geometriya, 184–208 (Russian, with Russian summary).2590847

Oscar Zariski, The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. (2) 76 (1962), 560–615, DOI 10.2307/1970376. MR0141668

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