Error rates in quantum circuits

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Rigorous analyses of errors in quantum circuits, or components thereof, typically appeal to the diamond distance, that is, the worst-case error instead of the average gate infidelity. This has two primary drawbacks, namely, that the diamond distance cannot be directly and efficiently estimated and the diamond distance may be unnecessarily pessimistic. In this work, we obtain upper- and lower-bounds on the diamond distance that are proportional to each other and can be efficiently estimated. We also obtain an essentially dimensional-independent upper bound on the average increase in purity due to a generalized relaxation process that is proportional to the average gate infidelity. We also show that the “average” error rate in a generic quantum circuit is proportional to the diamond distance, up to dimensional factors. Consequently, the average error in a quantum circuit can differ significantly from the infidelity, even when coherent processes make a negligible contribution to the infidelity.

Currently, great experimental effort is being exerted to control quantum systems precisely enough to allow for scalable quantum error correction and to demonstrate quantum supremacy, that is, perform some task on an experimental quantum computer that is not viable on a conventional computer. For both of these efforts, it is inadequate to completely characterize every circuit component in place, as this would be more difficult then simply simulating a quantum computer directly, thus removing any possible computational advantage. Rather, the aim of quantum characterization is to provide figures of merit and efficient methods of estimating those figures such that a sufficiently small figure of merit guarantees that fault-tolerant quantum computation is possible and/or that the total error in a circuit is sufficiently small.

There are several different metrics for quantifying the error rate due to some noise process $\mathcal{E}$, which we assume to be a completely positive and trace-preserving (CPTP) linear map $\mathcal{E} : \mathbb{H}_d \to \mathbb{H}_d$ where $\mathbb{H}_d$ is the set of $d \times d$ density matrices, that is, Hermitian and positive semi-definite matrices with unit trace. The most prominent metrics are the average gate infidelity to the identity and the diamond distance from the identity (hereafter simply the infidelity and diamond distance respectively) [1]

$$r(\mathcal{E}) = 1 - \int d\psi \Tr[\psi \mathcal{E}(\psi)]$$

$$\epsilon_\diamond(\mathcal{E}) = \max_\rho \frac{1}{2} \| (\mathcal{E} - \mathcal{I}_d) \otimes \mathcal{I}_d(\rho) \|_1,$$  \hspace{1cm} (1)

respectively, where the integral is over the set of pure states according to the unitarily-invariant Haar measure $d\psi$ and

$$\| M \|_p = \left( \Tr(M^\dagger M)^{p/2} \right)^{1/p}$$  \hspace{1cm} (2)

is the Schatten $p$-norm of $M$. The relevant metric will typically depend on the task being performed (and possibly the technique for proving robustness). Current methods of proving fault-tolerance utilize the diamond distance [2–4]. Presently, there is no known efficient protocol for directly estimating the diamond distance.

In contrast, the infidelity can be efficiently estimated using randomized benchmarking [5–9] or direct fidelity estimation [10, 11]. If the error channel is a stochastic channel (i.e., has a Kraus-operator decomposition into trace-orthogonal operators with one operator proportional to the identity), then the infidelity provides an exact estimate of the diamond distance [9]. Otherwise, the best-known bound is [12]

$$\frac{d+1}{d} r(\mathcal{E}) \leq \epsilon_\diamond(\mathcal{E}) \leq \sqrt{d(d+1)} r(\mathcal{E)},$$  \hspace{1cm} (3)
where the individual scalings with respect to $d$ and $r(\mathcal{E})$ are optimal [13]. For target (small) error rates, the upper and lower bounds differ by orders of magnitude. There is a belief that, despite the dramatically different scaling, the infidelity captures the average “computationally relevant” error [7] and that the discrepancy is simply a difference between “average” and “worst-case” performance.

In this paper, we obtain lower- and upper bounds on the diamond distance for general noise in terms of functions that can be efficiently estimated, namely, the infidelity and the unitarity [14] and differ by essentially a factor of $d^{3/2}$. The bounds here improve upon similar bounds obtained concurrently in Ref. [15] by a dimensional factor and hold for general noise rather than only for unital noise. To obtain these bounds, we also prove that a worst-case version of the infidelity is proportional to the (average) infidelity and that the relaxation rate (i.e., the average rate at which the purity of a state increases) under generalized amplitude processes is at most linear in the infidelity, and so does not introduce any significant difference between the infidelity and the diamond distance.

Conceptually, the fundamental reason for the disconnect between the diamond distance and the infidelity is that the infidelity corresponds to an error rate for a measurement in an expected eigenbasis of the system, which is insensitive to any coherences between the expected state of the system and the orthogonal subspace (quantified by the unitarity). Such coherence is generically introduced for some input state by noise that is not completely stochastic, which can then have a significant impact on error rates for any measurement that is not in a basis containing the expected state. In general applications of quantum information, such as quantum computing, the measurement outcome is often expected to be nondeterministic even in the absence of noise, that is, measurements are often expected to be in a basis that does not contain the ideal state of the system. Consequently, the coherent contributions to the diamond and induced trace distances will generally be relevant.

We then turn to the problem of estimating the total error rate of a circuit (or of a circuit fragment consisting of the gates between two rounds of error correction). We define an average error rate and show that it behaves qualitatively like the diamond distance. In particular, primary contribution of coherent errors to the average error is due to the lowest-order terms rather than the possible worst-case alignment of rotations. We also demonstrate numerically that there are two characteristic scalings with respect to the circuit length $K$, namely, $K$ and $\sqrt{K}$ for systematic and random noise respectively.

I. THE DISCREPANCY BETWEEN THE INFIDELITY AND DIAMOND DISTANCE

We now identify the precise cause of significant discrepancies between the diamond distance and the infidelity. We first prove that while a worst-case version of the infidelity exhibits the same dimensional scaling as the diamond distance, it is nevertheless proportional to the average infidelity (Corollary 2). We then prove a dimension-independent bound on the relaxation rate due to a general non-unital process (Theorem 3). Finally, we obtain upper- and lower-bounds on the diamond distance that scale as $\sqrt{r}$ instead of $r$ if and only if there are significant coherent contributions to the noise as quantified by the unitarity [14] (Corollary 5).

The infidelity is an average over all pure states. To analyze the dependence on the input state, we define the $\psi$-infidelity and max-infidelity to be

$$r(\mathcal{E}, \psi) = 1 - \text{Tr}[\psi \mathcal{E}(\psi)],$$

$$r_{\text{max}}(\mathcal{E}) = \max_{\psi} r(\mathcal{E}, \psi)$$

respectively, where the average infidelity is $r(\mathcal{E}) = \int d\psi r(\mathcal{E}, \psi)$. In the following, the average $\psi$-infidelity over a basis (rather than the sum) is essentially independent of the dimension. We could also consider the $\psi$-infidelity of $\mathcal{E} \otimes I$ (allowing for entangled inputs), resulting in a change from $d$ to $d^2$ and again giving an average infidelity over the basis that is also essentially independent of the dimension.
**Theorem 1.** For any \( d \in \mathbb{N} \), orthonormal basis \( \{|j\rangle : j \in \mathbb{Z}_d \} \) of \( \mathbb{C}^d \), and CPTP map \( E \),

\[
\sum_{j \in \mathbb{Z}_d} r(E, |j\rangle \langle j|) \leq (d + 1) r(E).
\] (5)

**Proof.** Fix \( d \in \mathbb{N} \), \( \{|j\rangle : j \in \mathbb{Z}_d \} \) to be an arbitrary orthonormal basis of \( \mathbb{C}^d \), and define the Heisenberg-Weyl operators

\[
X_d = \sum_{j \in \mathbb{Z}_d} |j \oplus 1 \rangle \langle j|
\]

\[
Z_d = \sum_{j \in \mathbb{Z}_d} \omega_d^j |j \rangle \langle j|
\]

\[
W_{a,b} = X^a Z^b
\]

(6)

with respect to this basis, where \( \oplus \) denotes addition modulo \( d \) and \( \omega_d = \exp(2\pi i / d) \). As \( W_{a,b} |j\rangle \langle j| W_{a,b}^\dagger = |j \oplus a\rangle \langle j \oplus a| \) and the trace is linear,

\[
\sum_{j \in \mathbb{Z}_d} r(E, |j\rangle \langle j|) = \sum_{j \in \mathbb{Z}_d} r(T_W[E], |j\rangle \langle j|)
\] (7)

where

\[
T_W[E] = \frac{1}{d^2} \sum_{a,b \in \mathbb{Z}_d} W_{a,b}^\dagger E W_{a,b},
\] (8)

and \( r(E) = r[T_W(E)] \). The Heisenberg-Weyl operators satisfy

\[
\omega_d^{-bc} W_{a,b} W_{c,d} = \omega_d^{-ad} W_{c,d} W_{a,b}
\] (9)

and so \( T_W(E) \) is Weyl-covariant, that is,

\[
W_{a,b} T_W(E) = T_W(E W_{a,b}).
\] (10)

Therefore there exists some probability distribution \( p_{a,b} \) over \( \mathbb{Z}_d^2 \) such that [16]

\[
T_W(E)(\rho) = \sum_{a,b \in \mathbb{Z}_d} p_{a,b} W_{a,b} \rho W_{a,b}^\dagger
\] (11)

and, as the infidelity is linear and weakly unitarily invariant,

\[
r(E) = r(T_W E) = 1 - \frac{\sum_{a,b \in \mathbb{Z}_d} |\text{Tr} \sqrt{p_{a,b} W_{a,b}^\dagger}|^2 + d}{d^2 + d}
\]

\[
= \frac{d(1 - p_{0,0})}{d + 1}.
\] (12)

Substituting eq. (11) into the right-hand-side of eq. (7) and using \( p_{a,b} \geq 0 \) gives

\[
\sum_{j \in \mathbb{Z}_d} r(T_W[E], |j\rangle \langle j|) = d(1 - \sum_{b \in \mathbb{Z}_d} p_{0,b})
\]

\[
\leq (d + 1) r(E).
\] (13)

Furthermore, this bound is saturated by

\[
E'(\rho) = p\rho + (1 - p) X \rho X^\dagger.
\] (14)

\[\square\]

We now prove that the dimensional scaling from Theorem 1 is optimal. Consequently, the worst-case infidelity can exhibit the same dimensional scaling as the diamond norm but is always proportional to the average infidelity.
Corollary 2. For any CPTP map $\mathcal{E}$, the worst-case infidelity satisfies
\[ r(\mathcal{E}) \leq \max_{\psi} r(\mathcal{E}, \psi) \leq (d + 1)r(\mathcal{E}). \] (15)

Furthermore, there exist CPTP maps such that
\[ \max_{\psi} r(\mathcal{E}, \psi) \geq \frac{(d^2 + d)r(\mathcal{E})}{4(d - 1)}. \] (16)

Proof. The upper bound follows from Theorem 1 with the non-negativity of the $\psi$-infidelity. Equation (16) can be obtained by setting $\mathcal{E} = \mathcal{U}_\phi$ where $U_\phi = I_d + [(\cos \theta - 1)]|0\rangle \langle 0|$, which has average infidelity
\[ r(U_\phi) = 1 - \frac{|\text{Tr} U_\phi|^2 + d}{d^2 + d} = \frac{2(d - 1)(1 - \cos \theta)}{d^2 + d}. \] (17)

Evaluating the $\psi$-infidelity for $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$ gives
\[ \max_{\psi} r(U_\phi, \psi) \geq r(U_\phi, |+\rangle\langle +|) = \frac{1}{2}(1 - \cos \theta) \]
\[= \frac{(d^2 + d)r(U_\phi)}{4(d - 1)}. \] (18)

□

2 demonstrates that the potential discrepancy between the $O(\sqrt{r})$ scaling of the diamond distance and the average infidelity is not simply the difference between “average” and “worst-case”. The discrepancy only arises for channels that are not stochastic. The primary examples of non-stochastic processes are relaxation and coherent processes.

Relaxation to a ground state is a common physical process that cannot be described as a stochastic channel. The canonical example is the single-qubit amplitude damping channel with Kraus operators
\[ K_1 = |0\rangle \langle 0| + \sqrt{1 - \gamma}|1\rangle \langle 1|, \quad K_2 = \sqrt{\gamma}|0\rangle \langle 1|. \] (19)

For multiple qubits, the energy eigenbasis will generically be entangled relative to the computational basis. We now prove that the relaxation rate $\alpha(\mathcal{E})$ (defined below, which quantifies the average increase in purity) is proportional to the infidelity and essentially independent of the dimension. Note that the following bound coincides with [12, Prop. 12] for $d = 2$.

Theorem 3. For any CPTP map $\mathcal{E} \in T_d$,
\[ \alpha(\mathcal{E}) := \|\mathcal{E}(\frac{1}{d}I_d) - \frac{1}{d}I_d\|_2 \leq \frac{\sqrt{2(d + 1)r(\mathcal{E})}}{d}. \] (20)

Proof. Consider the spectral resolution $\mathcal{E}(I_d) = \sum_{j,k} c_{j,k}|j\rangle \langle k|$ in some orthonormal basis $\{|j\rangle : j \in \mathbb{Z}_d\}$ of $\mathbb{C}^d$, where
\[ c_{j,k} = \text{Tr}|j\rangle \langle j| \mathcal{E}(|k\rangle \langle k|) \in [0,1]. \] (21)

Then
\[ \alpha(\mathcal{E})^2 = \text{Tr}(\mathcal{E}(\frac{1}{d}I_d))^2 - \frac{2}{d} \text{Tr}(\mathcal{E}(\frac{1}{d}I_d)) + \frac{1}{d} \]
\[= \frac{1}{d^2} \sum_j (\sum_k c_{j,k})^2 - \frac{1}{d} \]
\[= \frac{1}{d^2} \sum_j c_{j,j}^2 + \frac{1}{d^2} \sum_j 2c_{j,j}v_j + \frac{1}{d^2} \sum_j v_j^2 - \frac{1}{d} \]
\[= \frac{1}{d^2} \sum_j (r(\mathcal{E}, |j\rangle \langle j|) - v_j)^2 \] (22)
as the basis is orthonormal with \( v_j = \sum_{k \neq j} c_{j,k} \) and we have used \( c_{j,j} = 1 - r(\mathcal{E}, |j\rangle \langle j|) \) and

\[
\sum_j v_j = \sum_{j, k \neq j} c_{j,k} = \sum_{k, j \neq k} c_{j,k} = \sum_j r(\mathcal{E}, |j\rangle \langle j|)
\]  

(23)

for trace-preserving maps. The difference \( r(\mathcal{E}, |j\rangle \langle j|) - v_j = \text{Tr}(\mathbb{I}_d - |j\rangle \langle j|)\mathcal{E}(|j\rangle \langle j|) - \text{Tr} |j\rangle \langle j|\mathcal{E}(\mathbb{I}_d - |j\rangle \langle j|) \) is the net flow out of the state \(|j\rangle \langle j|\) and will typically be smaller than indicated by the following bound. Noting that \( v_j, r(\mathcal{E}, |j\rangle \langle j|) \geq 0 \), we have

\[
\alpha(\mathcal{E})^2 \leq \frac{1}{d^2} \sum_j [r(\mathcal{E}, |j\rangle \langle j|)^2 + v_j^2]
\]

\[
\leq \frac{1}{d^2} \left( \sum_j r(\mathcal{E}, |j\rangle \langle j|) \right)^2 + \frac{1}{d^2} \left( \sum_j v_j \right)^2
\]

\[
= \frac{2}{d^2} \left( \sum_j r(\mathcal{E}, |j\rangle \langle j|) \right)^2
\]

\[
\leq 2d^{-2}(d+1)^2 r(\mathcal{E})^2,
\]

(24)

where the final inequality follows from Theorem 1.

We now obtain lower- and upper-bounds on the diamond norm of a general linear map \( \mathcal{T} \) in terms of the purity of the Jamiołkowski-isomorphic state

\[
J(\mathcal{T}) = d^{-1} \sum_{j, k \in \mathbb{Z}_d} \mathcal{T}(|j\rangle \langle k|) \otimes |j\rangle \langle k|
\]

(25)

that differ by a factor of \( d^{3/2} \). We will then state upper and lower bounds on the diamond distance in terms of the infidelity and the unitarity [14]

\[
u(\mathcal{E}) = \frac{d}{d-1} \int d\psi \text{ Tr} \mathcal{E}(\psi - \mathbb{I}_d/d)^2,
\]

(26)

both of which can be efficiently estimated experimentally. In particular, \( u(\mathcal{E}) \in [p(\mathcal{E})^2, 1] \) with \( u - p(\mathcal{E})^2 \in O(r^2) \) giving a diamond distance that scales linearly with \( r(\mathcal{E}) \) if \( \mathcal{E} \) is stochastic where \( p(\mathcal{E}) = 1 - dr(\mathcal{E})/(d-1) \) is the randomized benchmarking decay constant.

**Theorem 4.** For any linear map \( \mathcal{T} : \mathbb{C}^{d \times d} \rightarrow \mathbb{C}^{d \times d} \),

\[
\|J(\mathcal{T})\|_2 \leq \|\mathcal{T}\|_1 \leq d^{3/2} \|J(\mathcal{T})\|_2
\]

(27)

where

\[
\|\mathcal{T}\|_1 = \sup_{A \in \mathbb{C}^{d^2 \times d^2}: \|A\|_1 = 1} \|\mathcal{T} \otimes \mathbb{I}_d(A)\|_1.
\]

(28)

**Proof.** By [17, Thm. 6],

\[
\|\mathcal{T}\|_1 = \sup_{\rho, \sigma \in \mathbb{H}_d} d\|d(\mathbb{I}_d \otimes \sqrt{\rho}) J(\mathcal{T}) (\mathbb{I}_d \otimes \sqrt{\sigma})\|_1
\]

(29)
noting that our convention of $J(T)$ differs from Ref. [17] by a factor of $d$. The lower bound can be obtained by setting $\rho = \sigma = I_d/d$ and applying eq. (36). By Hölder’s inequality,

$$\|ABC\|_1 \leq \|A\|_\infty \|BC\|_1 \leq \|A\|_\infty \|B\|_2 \|C\|_2$$

(30)

for arbitrary $A, B, C \in \mathbb{C}^{m \times m}$. As $\rho, \sigma \in \mathbb{H}_d$, they are positive semi-definite and have unit trace and so $\|I_d \otimes \sqrt{\rho}\|_\infty \leq 1$ and $\|I_d \otimes \sqrt{\sigma}\|_2 \leq \sqrt{d}$ for all $\rho, \sigma \in \mathbb{H}_d$. (Note that the bound in Ref. [12] uses $\|BC\|_1 \leq \|B\|_1 \|C\|_\infty$ instead, which, together with the Fuchs-van de Graaf inequality [1], gives the bound in eq. (3).) □

**Corollary 5.** For any quantum channel $E \in \mathbb{T}_d$,

$$\frac{1}{\sqrt{2}} \|J(\Delta)\|_2 \leq \epsilon_o(E) \leq \frac{d^{3/2}}{2} \|J(\Delta)\|_2$$

(31)

where $\Delta = E - I$. In terms of the infidelity and unitarity,

$$\frac{C}{\sqrt{2}} \leq \epsilon_o(E) \leq \sqrt{\frac{d^3 C^2}{4} + \frac{(d+1)^2 r(E)^2}{2}}$$

(32)

where

$$C^2 = \frac{d^2 - 1}{d^2} (u(E) - 2p(E) + 1).$$

(33)

**Proof.** First note the factor of 1/2 from the definition of $\epsilon_o(E)$ in eq. (3) and we obtain the $\sqrt{2}$ improvement from Lemma 6 for trace-preserving maps $E$, so that $\text{Tr} \Delta(A) = 0$ for all $A$. By [14, Prop. 9],

$$\|J(\Delta)\|_2^2 = \frac{d^2 - 1}{d^2} u(\Delta) + \frac{\alpha(E)^2}{d}$$

$$= \frac{d^2 - 1}{d^2} (u(E) - 2p + 1) + \frac{\alpha(E)^2}{d}$$

(34)

where we have used $\text{Tr} \Delta(A) = 0$ for all $A$ as $E$ is a trace-preserving map, and $d\alpha(E)^2 = \|E_u\|_2^2$. The lower and upper bounds follow from the non-negative of norms and Theorem 3 respectively. □

**Lemma 6.** For any traceless Hermitian matrix $M \in \mathbb{C}^{d \times d}$,

$$\sqrt{d} \|M\|_2 \leq \|M\|_1 \leq \sqrt{d} \|M\|_2.$$

(35)

Moreover, both these bounds are saturated.

**Proof.** For any Hermitian matrix $M \in \text{GL}(d)$, writing $M = U\eta U^\dagger$ where $\eta$ is a diagonal matrix whose entries are the eigenvalues $\{\eta_j\}$ of $M$ and using the unitary invariance of the trace and Frobenius norms gives the standard bounds

$$\|M\|_2 = \sqrt{\sum_{j=1}^d \eta_j^2} \leq \|M\|_1 = \sum_{j=1}^d |\eta_j| \leq \sqrt{d \sum_{j=1}^d \eta_j^2} = \sqrt{d} \|M\|_2$$

(36)

by the Cauchy-Schwarz inequality.

To obtain a sharper lower bound for traceless Hermitian matrices, let

$$\eta = \eta^+ \oplus -\eta^- = \left( \begin{array}{cc} \eta^+ & 0 \\ 0 & -\eta^- \end{array} \right)$$

(37)
FIG. 1. a) A quantum circuit $C$ with $K$ rounds of gates, where time goes from right to left. b) An implementation $\tilde{C}$ of $C$ with Markovian noise.

where $\oplus$ denotes the matrix direct sum and $\eta^\pm$ are both positive semidefinite. Then

$$\|M\|_2 = \sqrt{\sum_{j=1}^d \eta_j^2} = \sqrt{\|\eta^+\|_2^2 + \|\eta^-\|_2^2} \leq \sqrt{\|\eta^+_1\|_1^2 + \|\eta^-_1\|_1^2} = \frac{1}{\sqrt{2}} \|M\|_1,$$

(38)

where we have used $\|\eta^\pm\|_1 = \frac{1}{2} \|\eta\|_1$ which holds for traceless matrices. The above lower bound is saturated when $\eta_{11} = 1, \eta_{22} = -1$ and all other eigenvalues are zero. □

II. THE ERROR PER GATE CYCLE IN A QUANTUM CIRCUIT

We now identify the “computationally relevant” error per gate cycle under Markovian noise and prove that it is closer to the diamond distance than the infidelity (Theorem 7) and illustrate the different scalings with respect to the infidelity, the circuit length and the diamond distance.

A quantum circuit $C$ consists of some $d$-dimensional input state $\psi$ (e.g., an $n$-qubit computational basis state), $K$ rounds of gates $G_1, \ldots, G_K$, and a final measurement $M$ (e.g., of a single qubit in the computational basis) as in fig. 1 (a). Under the assumption of Markovian noise, an experimental implementation $\tilde{C}$ of $C$ consists of a preparation of $\tilde{\psi}_1$, $K$ rounds of noisy gates $\tilde{G}_1, \ldots, \tilde{G}_K$ such that $\tilde{G}_k = G_k E_k$ for some CPTP maps $\{E_k\}$, and a final measurement $\tilde{M} \approx M$ as in fig. 1 (b), where we use calligraphic font to denote CPTP maps with $U(A) = UAU^\dagger$ for any unitary operator $U$ and we denote the composition of channels by a product. For clarity of notation, we define

$$A_{b:a} = \begin{cases} A_b \ldots A_a & \text{if } b \geq a \\ \mathbb{I} & \text{otherwise.} \end{cases}$$

(39)

For decision or function problems (including, e.g., Shor’s algorithm [19]), we can coarse-grain $M$ into two elements, $M_0$ and $\mathbb{I} - M_0$, such that outcomes in $M_0$ $(\mathbb{I} - M_0)$ give a correct (incorrect) answer to the computation. The error rate of the experimental implementation $\tilde{C}$ (which does not include the error rate of the ideal circuit $C$) is then

$$\tau(C, \tilde{C}) = \frac{1}{2} \sum_j \langle \tilde{M}_j, \tilde{G}_{K:1}(\psi) \rangle - \langle M_j, G_{K:1}(\psi) \rangle$$

$$= |\langle M_0, \tilde{G}_{K:1}(\psi) \rangle - \langle M_0, G_{K:1}(\psi) \rangle|$$

$$= |\langle M_0 - M_0, \tilde{G}_{K:1}(\psi) \rangle + \sum_{k=1}^K \langle M_0, G_{K:k}(\mathbb{I} - E_k)\tilde{G}_{k-1:1}(\psi) \rangle|$$

(40)

for CPTP maps, where we have used

$$A_{b:a} - B_{b:a} = \sum_{k=a}^b A_{b,k+1}(A_k - B_k)B_{k-1:1}.$$  

(41)

The preparation of a mixed state corresponds to preparing a pure state and then applying some CPTP map, which can then be incorporated into the noise in the first gate.
As we are primarily interested in the error due to gates, we set $\tilde{M}_0 = M_0$ (although we can use the triangle inequality to separate this out as necessary).

For a non-trivial circuit, that is, $G_{k:1} \neq I_d$, $M_k \approx \tilde{M}_k$ and $\psi_k$ will be in different bases that are fixed relative to each other and so the terms in the summation will typically be close to

$$t_{\text{avg},m}(\mathcal{E}) = E_{\psi,M}|\text{Tr}_{M=\text{m,}M^2=M}|\langle M, \mathcal{E}(\psi) - \psi \rangle|,$$

(42)

where $m = \text{Tr} M_0$. We now prove that this quantity is proportional to the diamond distance, up to a dimensional factor and a factor that depends on the rank of $M_0$. For many problems, such as Shor’s factoring algorithm, and for the measurements between rounds of error correction, $m \in O(d^a)$ and so the dimensional scaling between the following upper and lower bounds is primarily a consequence of the corresponding dimensional factors in Corollary 5.

**Theorem 7.** For any quantum channel $\mathcal{E}$ and $m \in \mathbb{N},$

$$\frac{2m(m + 1)\epsilon_\circ(\mathcal{E})}{d^3(d + 1)^2} \leq t_{\text{avg},m}(\mathcal{E}) \leq 2\epsilon_\circ(\mathcal{E})$$

(43)

where $\Delta = \mathcal{E} - I_d$.

**Proof.** The upper bound is trivial from the definition of $\epsilon_\circ(\mathcal{E})$ and Corollary 5 (although note the factor of 2). For the lower bound, let

$$t(M, \mathcal{E}, \psi) = |\langle M, \Delta(\psi) \rangle|,$$

(44)

so that $t_{\text{avg},m}(\mathcal{E}) = E_{\psi,M}|\text{Tr}_{M=\text{m,}M^2=M}t(M, \mathcal{E}, \psi)$). The absolute value in eq. (44) makes evaluating the mean difficult. To circumvent this, we use the identity $a^2 = a \otimes a$ for $a \in \mathbb{R}^d$ and the distributivity of the tensor product to obtain

$$V = E_{\psi,M}|\text{Tr}_{M=\text{m,}M^2=M}t(M, \mathcal{E}, \psi)^2|$$

$$= \int_{S_d} d\psi \int_{U(d)} dU \text{Tr} \left[ U(M^{(m)}) \otimes^2 \Delta^{(2)}(\phi^{(2)}) \right].$$

(45)

To evaluate the integrals, let $S = \sum_{i,j} |ij\rangle\langle ji|$ be the two-qudit swap gate and let $\pi_{a/s} = (I_d^2 \pm S_d)/2$ be the projectors onto the symmetric and antisymmetric subspaces of $\mathbb{C}^{d^2}$ respectively. For any Hermitian matrix $M$,

$$\int dU U^{\otimes^2}(M) = \frac{\text{Tr} \pi_s M}{\text{Tr} \pi_s} \pi_s + \frac{\text{Tr} \pi_a M}{\text{Tr} \pi_a} \pi_a$$

(46)

by Schur-Weyl duality and so, with $\text{Tr} S(A \otimes B) = \text{Tr} AB$,

$$\int d\psi \phi^{\otimes^2} = \int dU U(\phi^{\otimes^2})$$

$$= \frac{I_d^2 + S}{d^2 + d}$$

$$\int_{U(d)} dU U(M^{(m)})^{\otimes^2} = \frac{2m^2 + 2m}{d^2 + d} \pi_s + \frac{2m^2 - 2m}{d^2 - d} \pi_a$$

(47)
where $\phi$ is an arbitrary pure state. Substituting this into eq. (45) gives

$$V = \frac{m^2 + m}{(d^2 + d)^2} \text{Tr} \left[ (I_{d^2} + S)\Delta \otimes^2 (I_{d^2} + S) \right]$$

$$= \frac{m^2 + m}{(d^2 + d)^2} \text{Tr} \left[ S\Delta \otimes^2 (I_{d^2} + S) \right]$$

$$= \frac{m^2 + m}{(d+1)^2} \left( \|J(\Delta)\|^2_2 + \|\Delta(I_d)\|^2_2 \right)$$

$$\geq \frac{4m(m + 1)\epsilon_\diamond(\mathcal{E})^2}{d^3(d+1)^2} \quad \text{(48)}$$

where we have used $\pi_a \Delta \otimes (\pi_s) = 0$, [14, Prop. 3], the non-negativity of norms, and Corollary 5. Now note that

$$\max t(M, \mathcal{E}, \psi) \leq 2\epsilon_\diamond(\mathcal{E}), \quad \text{(49)}$$

and so, with Corollary 5 and the non-negativity of norms,

$$t_{\text{avg}}(\mathcal{E}) \geq \frac{E_{\psi,M}[t(M, \mathcal{E}, \psi)^2]}{\max t(M, \mathcal{E}, \psi)}$$

$$\geq \frac{2m(m + 1)\epsilon_\diamond(\mathcal{E})}{d^3(d+1)^2}. \quad \text{(50)}$$

\[ \square \]

gives a lower bound on the typical size of the individual terms in eq. (40), which scales as $\sqrt{r}$ for general noise processes. However, these individual terms can have arbitrary signs and so can combine in several ways. In particular, there will be two characteristic scalings with respect to $K$, namely, $\tau(\mathcal{C}, \hat{\mathcal{C}}) \in O(K)$ when all the signs are the same and $\tau(\mathcal{C}, \hat{\mathcal{C}}) \in O(\sqrt{K})$ when the signs are essentially uniformly random.

Almost all the signs will be the same in at least two regimes as illustrated in fig. 2 (a), namely, for systematic stochastic and unitary noise. However, for general stochastic and unitary noise, the signs will be uniformly random and so the summation in eq. (40) will look more like a one-dimensional random walk with typical step size proportional to $t_{\text{avg},m}(\mathcal{E})$ and so the total error will scale as $\sqrt{K}t_{\text{avg},m}(\mathcal{E})$ as illustrated in fig. 2 (b).

\[ \text{III. CONCLUSION} \]

We have obtained improved bounds on the diamond distance in terms of the infidelity and the unitarity [14], which can both be efficiently estimated. When noise is approximately stochastic, the improved bound scales as $O(r)$. If the unitarity indicates that the noise contains coherent errors, then the improved upper and lower bounds both scale as $O(\sqrt{r})$.

We have also shown that the diamond distance and infidelity are not “worst-case” and “average” error rates in the same sense by constructing worst-case and average versions of the infidelity and diamond distance respectively and showing that they are proportional to the standard versions up to dimensional factors. An important semantic consequence is that referring to the diamond distance as the worst-case error rate and the infidelity as the average error rate is misleading because they quantify error rates in fundamentally different ways.

We then provided analytic and numerical arguments demonstrating that the total error rate of a circuit (or of a circuit fragment consisting of the gates between two rounds of error correction) behaves qualitatively
FIG. 2. (Color online) Average error \( \tau(\hat{C}, \hat{C}) \) from eq. (40) averaged over 100 random three-qubit circuits of each length \( K \) under unitary (blue diamonds) and stochastic (red squares) noise. The ideal circuits \( \hat{C} \) consist of preparations in the state \( |\psi\rangle^\otimes 3 \), an \( X \)-basis measurement on the final qubit and \( K \) gates drawn uniformly from the set of all combinations of: a) controlled-phase, and \( R_{\pi/8} = |0\rangle\langle 0| + \exp(i\pi/4)|1\rangle\langle 1| \) gates; and b) controlled-phase, Hadamard and \( R_{\pi/8} \) gates acting disjointly on three qubits. The systematic unitary noise is \( U = \cos(\theta)I_8 + i\sin(\theta)ZIZ \) with \( \theta = 0.001 \) and the systematic stochastic noise is \( \mathcal{E}(\rho) = 0.999\rho + 0.001ZIZ\rho ZIZ \). The random noise is fixed and gate-independent for each circuit. The random stochastic noise is generated by randomly generating a Pauli channel with infidelity 2/300 and the random unitary noise is generated by conjugating \( U = \cos(\theta)I_8 + i\sin(\theta)ZIZ \) by a Haar-random unitary with \( \theta = 0.03 \).

more like the diamond distance instead of the infidelity. The primary contribution of coherent errors to the average error is not a possible worst-case alignment of rotations giving an infidelity that scales quadratically with the number of gates. Rather, the primary contribution is due to the lowest order terms behaving like the diamond distance as the effective preparations and measurements (when propagated through the circuit to immediately precede and succeed a lowest-order error) are not in the same basis unless the circuit is trivial. As noted in Ref. [20], the \( \sqrt{r} \) scaling can be dominant even when any coherent noise processes make a negligible contribution (e.g., 1%) to the infidelity.

We also demonstrated numerically that there are two characteristic scalings with the circuit length \( K \), namely, \( K \) and \( \sqrt{K} \) for systematic and random noise respectively. In particular, the average error does not demonstrate \( K^2 \) scaling even for systematic coherent errors as would be expected if the average error was directly proportional to the infidelity.

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