On a Stratification of the Moduli of K3 Surfaces

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Abstract. In this paper we give a characterization of the height of K3 surfaces in characteristic $p > 0$. This enables us to calculate the cycle classes of the loci in families of K3 surfaces where the height is at least $h$. The formulas for such loci can be seen as generalizations of the famous formula of Deuring for the number of supersingular elliptic curves in characteristic $p$. In order to describe the tangent spaces to these loci we study the first cohomology of higher closed forms.

1. Introduction

Elliptic curves in characteristic $p$ come in two sorts: ordinary and supersingular. The distinction can be expressed in terms of the formal group of an elliptic curve. Multiplication by $p$ on the formal group takes the form

$$[p](t) = at^h + \text{higher order terms}, \quad (1)$$

where $a \neq 0$ and $t$ is a local parameter. The number $h$ satisfies $1 \leq h \leq 2$ and is called the height. By definition, the elliptic curve is ordinary if $h = 1$ and supersingular if $h = 2$. There is a classical formula of Deuring for the number of supersingular elliptic curves over an algebraically closed field $k$ of characteristic $p$:

$$\sum_{E \text{ supers. } \sim \#\text{Aut}(E)^{-1} = \frac{p - 1}{24},}$$

where the sum is over supersingular elliptic curves over $k$ up to isomorphism.

If one views K3 surfaces as a generalization of elliptic curves, one can make a similar distinction of K3 surfaces in characteristic $p$ by using the formal Brauer group as Artin showed. The formal Brauer group is a 1-dimensional formal group associated to the second étale cohomology with coefficients in the multiplicative group. Multiplication by $p$ in this formal group has the form (1), but now we have $1 \leq h \leq 10$ or $h = \infty$, the latter if multiplication by $p$ is zero. The height can be used to define a stratification of the moduli spaces of K3 surfaces. A generic K3 surface will have $h = 1$; those with $h = \infty$ are most special in this respect and called supersingular.

In this paper we first express the height of a K3 surface in terms of the action of the Frobenius morphism on the second cohomology group with coefficients in the sheaf
$W(O_X)$ of Witt vectors of the structure sheaf $O_X$. The natural co-filtration $W_n(O_X)$ of $W(O_X)$ induces co-filtrations on the cohomology which correspond to approximations of the formal group. Using this characterization we can calculate the cycle classes of the strata in the moduli space where the height $\geq h$. This is done by interpreting the loci as degeneracy loci of maps between bundles. The resulting formulas can be viewed as a generalization of Deuring’s formula. Generalizations of Deuring’s formula to principally polarized abelian varieties were worked out in joint work of Ekedahl and one of us and can be found in [G]. The supersingular locus comes with a multiplicity.

In order to describe the tangent spaces to our strata we use differential forms rather than crystalline cohomology. We calculate the dimensions of cohomology groups $H^1(Z_i)$ and $H^1(B_i)$ where the sheaves $Z_i$ and $B_i$ are the sheaves of certain closed forms introduced by Illusie. We study the dimensions of the cohomology groups $H^1(Z_i)$ and $H^1(B_i)$ and of their images in $H^1(X, \Omega^1)$. We think that these spaces are quite helpful to understand the geometry of surfaces in characteristic $p$.

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1. Witt vector cohomology

Let $X$ be a non-singular complete variety defined over an algebraically closed field $k$ of characteristic $p > 0$. We denote by $W_n = W_n(O_X)$ the sheaf of Witt rings of length $n$ as defined by J.-P. Serre, cf. [S]. The sheaf $W_n(O_X)$ is a coherent sheaf of rings which comes with three operators:

i) Frobenius $F : W_n(O_X) \to W_n(O_X)$,

ii) Verschiebung $V : W_n(O_X) \to W_{n+1}(O_X)$,

iii) Restriction $R : W_{n+1}(O_X) \to W_n(O_X)$,

defined by the formulas

$F(a_0, a_1, \ldots, a_{n-1}) = (a_0^p, a_1^p, \ldots, a_{n-1}^p)$,

$V(a_0, a_1, \ldots, a_{n-1}) = (0, a_0, a_1, \ldots, a_{n-1})$,

$R(a_0, a_1, \ldots, a_n) = (a_0, a_1, \ldots, a_{n-1})$.

They satisfy the relations

$$RVF = FRV = RFV = p.$$ 

The cohomology groups $H^i(X, W_n(O_X))$ are finitely generated $W_n(k)$-modules. The projective system $\{W_n(O_X), R\}_{n=1,2,\ldots}$ induces a sequence

$$\ldots \leftarrow H^i(X, W_n(O_X)) \leftarrow R^{-1}H^i(W_{n+1}(O_X)) \leftarrow \ldots$$

so that we can define

$H^i(X, W(O_X)) = \text{proj. lim } H^i(X, W_n(O_X))$. 

This is a $W(k)$-module, but not necessarily a finitely generated $W(k)$-module, cf. Section 3. The semi-linear operators $F$ and $V$ act on it and they satisfy the relations $FV = VF = p$.

2. Formal Groups

Smooth formal Lie groups of dimension 1 over an algebraically closed field $k$ of characteristic $\neq 0$ are characterized by their height, cf. [H], [Ma]. To a smooth formal Lie group $\Phi$ of dimension one one can associate its covariant Dieudonné module $M = D(\Phi)$, a free $W(k)$-module. It possesses two operators $F$ and $V$ with the following properties: the operator $F$ is $\sigma$-linear, the operator $V$ is $\sigma^{-1}$-linear and topologically nilpotent and they satisfy $FV = VF = p$. Here $\sigma$ denotes the Frobenius map on $k$. Then $M$ is a free $W(k)$-module with the following properties:

a) $\dim(\Phi) = \dim_k(M/VM)$,

b) $\text{height}(\Phi) = \text{rank}_W(M)$.

Note that one has the equalities

$$\text{rank}_W(M) = \dim_k(M/pM) = \dim_k(M/FM) + \dim_k(M/VM).$$

3. The Formal Brauer Group of Artin-Mazur

For a proper variety $X/k$ one may consider the formal completion of the Picard group. The group of $S$-valued points of $\hat{\text{Pic}}(X)$ fits into the exact sequence

$$0 \rightarrow \hat{\text{Pic}}(S) \rightarrow H^1(X \times S, \mathbb{G}_m) \rightarrow H^1(X, \mathbb{G}_m)$$

for any local artinian scheme $S$ with residue field $k$. Here cohomology is étale cohomology. This idea of studying infinitesimal properties of cohomology was generalized to the higher cohomology groups $H^r(X, \mathbb{G}_m)$ by Artin and Mazur, cf. [A-M]. Their work leads to contravariant functors $\Phi^r: \text{Art} \rightarrow \text{Ab}$ with

$$\Phi^r(S) = \ker H^r(X \times S, \mathbb{G}_m) \rightarrow H^r(X, \mathbb{G}_m),$$

which under suitable circumstances are representable by formal Lie groups. For a K3 surface $X$ this is the case and we find for $r = 2$ the formal Brauer group $\Phi = \Phi_X = \Phi^2$. Its tangent space is

$$T_\Phi = H^2(X, O_X).$$

For a K3-surface $X$ we have two possibilities:

i) $h(\Phi) = \infty$ and $\Phi = \hat{\mathbb{G}}_a$, the formal additive group. The K3-surface is called supersingular (in the sense of Artin).

ii) $h(\Phi) < \infty$. Then $\Phi$ is a $p$-divisible formal group. Moreover, it is known that $1 \leq h(\Phi) \leq 10$. This follows from the following theorem of Artin, cf. [A]. We shall write simply $h$ for $h(\Phi)$.
(3.1) Theorem. If the formal Brauer group $\Phi_X$ of a K3 surface $X$ is $p$-divisible then it satisfies the relation $2h \leq B_2 - \rho$, where $B_2$ is the second Betti number and $\rho$ the rank of the Néron-Severi group.

For the proof one combines Theorem (0.1) of [A] with Deligne’s [D] result on lifting K3 surfaces, see also [I]. We give a proof in Section 10. This theorem implies that if $\rho = 22$ then necessarily we have $h = \infty$. If $h \neq \infty$ then it follows that $1 \leq h \leq 10$. One should view $h = 1$ as the generic case. It was conjectured by Artin that if $h = \infty$ then $\rho = 22$. This is known for elliptic K3 surfaces, see [A]. Note that a surface with $\rho = 22$ is called supersingular by Shioda, cf. [Sh].

The following result by Artin and Mazur is crucial:

(3.2) Theorem. The Dieudonné module of the formal Brauer group $\Phi_X$ is given by

$$D(\Phi_X) \cong H^2(X, W(O_X)).$$

For the proof we refer to [A-M]. The point to notice is that $D(\hat{\mathbb{G}}_a) = \mathbb{W}(k)[[T]]$.

(3.3) Remark. Note that this explains why the Witt vector cohomology is sometimes not finitely generated: if $\Phi_X \cong \hat{\mathbb{G}}_a$ then $H^2(X, W(O_X))$ is not finitely generated over $W(k)$ because $D(\hat{\mathbb{G}}_a) = \mathbb{W}(k)[[T]]$.

4. Vanishing of Cohomology

We collect a number of results on the vanishing of cohomology groups for K3 surfaces that we need in the sequel.

(4.1) Lemma. Let $X$ be a K3 surface. We have $H^1(X, W_n(O_X)) = 0$ for all $n > 0$, hence $H^1(X, W(O_X)) = 0$.

Proof. Since $X$ is a K3 surface we have by definition $H^1(X, O_X) = 0$. The lemma is deduced from this by induction on $n$. Assume that $H^1(X, W_{n-1}(O_X)) = 0$. Then the exact sequence

$$0 \rightarrow W_{n-1}(O_X) \rightarrow W_n(O_X) \rightarrow O_X \rightarrow 0$$

induces an exact sequence

$$H^1(X, W_{n-1}(O_X)) \rightarrow H^1(W_n(O_X)) \rightarrow H^1(X, O_X).$$

This implies that $H^1(X, W_n(O_X)) = 0$. □

(4.2) Lemma. For a projective surface $X$ with $H^1(X, O_X) = 0$ the induced map $R: H^2(X, W_n(O_X)) \rightarrow H^2(X, W_{n-1}(O_X))$ is surjective with kernel $\cong H^2(X, O_X)$.

Proof. This follows from the exact sequence

$$0 \rightarrow O_X \rightarrow W_n(O_X) \rightarrow W_{n-1}(O_X) \rightarrow 0$$

and the vanishing of $H^1(X, O_X)$ and of $H^3(X, O_X)$. □
\textbf{(4.3) Lemma.} In $H^2(X, W_n(O_X))$ we have
\[ RV(H^2(X, W_n(O_X))) = V(H^2(X, W_{n-1}(O_X))). \]

\textit{Proof.} The commutativity of the diagram
\[
\begin{array}{ccc}
W_n(O_X) & \xrightarrow{V} & W_{n+1}(O_X) \\
\downarrow R & & \downarrow R \\
W_{n-1}(O_X) & \xrightarrow{V} & W_n(O_X).
\end{array}
\]
gives in cohomology a commutative diagram
\[
\begin{array}{ccc}
H^2(X, W_n(O_X)) & \xrightarrow{V} & H^2(X, W_{n+1}(O_X)) \\
\downarrow R & & \downarrow R \\
H^2(X, W_{n-1}(O_X)) & \xrightarrow{V} & H^2(X, W_n(O_X)).
\end{array}
\]
The surjectivity of the left hand $R$, which follows from the preceding lemma, implies the claim. $\square$

\textbf{(4.4) Lemma.} Assume that for some $n > 0$ the map $F : H^2(X, W_n(O_X)) \rightarrow H^2(X, W_n(O_X))$ vanishes. Then for all $0 \leq i \leq n$ the map $F : H^2(X, W_i(O_X)) \rightarrow H^2(X, W_i(O_X))$ is zero. Moreover, for all $0 \leq i \leq n$ the module $H^2(X, W_i(O_X))$ is a vector space over $k$.

\textit{Proof.} The first result follows from the commutativity of the diagram
\[
\begin{array}{ccc}
H^2(X, W_n(O_X)) & \xrightarrow{R} & H^2(X, W_i(O_X)) \\
\downarrow F & & \downarrow F \\
H^2(X, W_n(O_X)) & \xrightarrow{R} & H^2(X, W_i(O_X)).
\end{array}
\]
and Lemma (4.2). The second claim follows from $p = FVR$ and $k \cong W_i(k)/pW_i(k)$. $\square$

\textbf{(4.5) Lemma.} Assume that $X$ is a K3 surface. The following two sequences are exact:
\[
0 \rightarrow H^2(X, W_{n-1}(O_X)) \xrightarrow{V} H^2(X, W_n(O_X)) \xrightarrow{R_{n-1}} H^2(X, O_X) \rightarrow 0,
\]
\[
0 \rightarrow H^2(X, W(O_X)) \xrightarrow{V} H^2(X, W(O_X)) \xrightarrow{R'} H^2(X, O_X) \rightarrow 0,
\]
where $R'$ is the map induced by $W_n(O_X) \xrightarrow{R_{n-1}} W_1(O_X)$ as $n \rightarrow \infty$.

\textit{Proof.} The first exact sequence follows from the exact sequence
\[
0 \rightarrow W_{n-1}(O_X) \xrightarrow{V} W_n(O_X) \xrightarrow{R_{n-1}} O_X \rightarrow 0
\]
and Lemma (4.2). Because the projective system $H^2(X, W_n(O_X))$ satisfies the Mittag-Leffler condition we may take the projective limit. $\square$

\textbf{5. Characterization of the Height}

Let $X$ be a K3 surface and let $\Phi_X$ be its formal Brauer group in the sense of Artin-Mazur. The isomorphism class of this formal group is determined by its height $h$. The following theorem expresses this height in terms of Witt vector cohomology.
(5.1) Theorem. The height satisfies $h(\Phi_X) \geq i + 1$ if and only if the Frobenius map $F: H^2(X, W_i(O_X)) \to H^2(X, W_i(O_X))$ is the zero map.

(5.2) Corollary. We have the following characterization of the height:

$$h(\Phi_X) = \min\{i \geq 1 : [F : H^2(W_i(O_X)) \to H^2(W_i(O_X))] \neq 0\}.$$ 

Proof of the Theorem. “⇒” In case $h(\Phi_X) = \infty$ the implication $\Leftarrow$ is immediate. So we may consider the case where the height of $\Phi_X$ is finite. Assume that the map $F: H^2(X, W_i(O_X)) \to H^2(X, W_i(O_X))$ is the zero map. We set

$$M = D(\Phi) \cong H^2(X, W(O_X)), \quad \text{the covariant Dieudonné module.}$$

Since $\dim_k(H^2(X, W(O_X))/VH^2(X, W(O_X)) = 1$ by Lemma (4.5), we have by b) in Section 2

$$\dim_k(H^2(X, W(O_X))/FH^2(X, W(O_X)) = h - 1.$$ 

The surjectivity of the projection $H^2(X, W(O_X)) \to H^2(X, W_i(O_X))$ implies the surjectivity of

$$H^2(X, W(O_X))/FH^2(X, W(O_X)) \to H^2(X, W_i(O_X))/FH^2(X, W_i(O_X)).$$

By assumption we have $H^2(X, W_i(O_X))/FH^2(X, W_i(O_X)) \cong H^2(X, W_i(O_X))$ and by Lemma (4.5) we have

$$\dim_k H^2(X, W_i(O_X)) = i,$$

i.e. we find $h - 1 \geq i$, or equivalently, $h \geq i + 1$.

Conversely, we now prove “⇐”. If $h(\Phi_X) = \infty$ then $\Phi_X = \hat{G}_a$, the formal additive group of dimension 1. So $F$ acts as zero on $D(\hat{G}_a) = D(\Phi_X) = H^2(X, W(O_X))$. As in Lemma (4.4) we conclude that $F$ acts on $H^2(X, W_i(O_X))$ as the zero map. Therefore we may assume that $h(\Phi_X) = h < \infty$. We thus assume that $h(\Phi_X) \geq i + 1$. We set

$$H = H^2(X, W(O_X))$$

and have

$$V^{h-1}H \subseteq \ldots \subseteq V^2H \subseteq VH \subseteq H.$$ 

Under projection this is mapped surjectively to

$$0 \subseteq V^{h-2}H^2(O_X) \subseteq \ldots \subseteq VH^2(W_{h-2}(O_X)) \subseteq H^2(X, W_{h-1}(O_X)).$$

All the inclusions are strict because of Lemma (4.5).

Claim. We have $V^{h-1}H^2(X, W(O_X)) = FH^2(X, W(O_X))$.

Proof of the claim. Since our modules are free over $W$ we deduce from Manin’s results [M] (but see also [H] because we use the covariant theory):

$$D(\Phi_X) \cong W[F, V]/W[F, V](F - V^{h-1}).$$
This map is formulated in an inductive way a similar characterization of the condition \( \langle F \rangle \). We now find \( FH^2(W_{h-1}(O_X)) = 0 \). By Lemma (4.4) we conclude that \( F \) acts on \( H^2(W_i(O_X)) \) for \( i \leq h - 1 \) as zero. □

**(5.3) Corollary.** The height of \( \Phi_X \) is \( \infty \) if and only if the Frobenius endomorphism \( F : H^2(X, W_{10}(O_X)) \to H^2(X, W_{10}(O_X)) \) is zero.

Proof. If the height is finite, then we know by Artin and Mazur (see (3.1)) that we have \( h \leq 10 \). □

**(5.4) Corollary.** Set \( H = H^2(X, W_{10}(O_X)) \) and consider the filtration
\[
\{0\} \subset R^0V^0H \subset R^1V^1H \subset \ldots \subset R^{h-1}V^{h-1}H \subset \ldots \subset H.
\]
If \( h \) is the height of \( \Phi_X \) then \( F(H) = R^{h-1}V^{h-1}(H) \).

Proof. The \((h-1)\)-th step \( V^{h-1}H^2(W(O_X)) \) in the filtration
\[
V^{10}H^2(W(O_X)) \subset V^9H^2(W(O_X)) \subset \ldots \subset H^2(W(O_X))
\]
maps surjectively to the corresponding step \( R^{h-1}V^{h-1}H \) of the filtration on \( H \). By our claim we have
\[
V^{h-1}H^2(W(O_X)) = FH^2(W(O_X)).
\]
This implies the assertion. □

**(5.5) Corollary.** If \( h(\Phi_X) = h < \infty \) and if \( \{\omega, V\omega, V^2\omega, \ldots, V^{h-1}\omega\} \) is a \( W \)-basis of \( H^2(X, W(O_X)) \) then \( F \) acts as zero on \( H^2(X, W_i(O_X)) \) if and only if \( F(\omega) = 0 \), with \( \omega \) the image of \( \omega \) in \( H^2(X, W_i(O_X)) \).

**(5.6) Corollary.** If \( h(\Phi_X) = h < \infty \), then \( \dim_k \ker[ F : H^2(W_i) \to H^2(W_i)] = \min\{i, h - 1\} \).

Proof. By Lemma (4.5) and Corollary (5.2), we have \( \dim_k \ker[ F : H^2(W_i) \to H^2(W_i)] = i \) if \( i \leq h - 1 \). Assume \( i \geq h \). Using the notation in Corollary 5.5, we know that \( \langle V^{i-h+1}R^{i-h+1}\omega, V^{i-h+2}R^{i-h+2}\omega, V^{i-h+3}R^{i-h+3}\omega, \ldots, V^{i-1}R^{i-1}\omega \rangle \) is a basis of \( \ker F \). □

The case \( h \geq 2 \) is characterized by the vanishing of Frobenius on \( H^2(O_X) \). We now formulate in an inductive way a similar characterization of the condition \( h \geq n + 1 \). If for a K3 surface \( X \) one assumes that \( F \) is zero on \( H^2(W_i) \) \((i = 1, \ldots n - 1)\) then one has \( FH^2(W_n) \subset V^{n-1}H^2(O_X) \) and \( F \) vanishes on \( VH^2(W_{n-1}) \). Since we have a natural \((\sigma^{-1})\)-isomorphism \( H^2(O_X) \cong V^{n-1}H^2(O_X) \), one has an induced homomorphism
\[
\phi_n : H^2(O_X) \cong H^2(W_n)/VH^2(W_{n-1}) \to V^{n-1}H^2(O_X) \cong H^2(O_X).
\]
This map is \( \sigma^n \)-linear. The following theorem is clear by the construction of \( \phi_n \).
(5.7) **Theorem.** Suppose $F$ is zero on $H^2(W_i)$ for $i = 1, \ldots, n-1$. Then $F$ vanishes on $H^2(W_n)$ if and only if $\phi_n : H^2(O_X) \to H^2(O_X)$ vanishes.

6. Closed Differential Forms

Let $F : X \to X^{(p)}$ be the relative Frobenius morphism of a K3 surface $X$. By means of the Cartier operator $C : \Omega^*_X,\text{closed} \to \Omega^*_X$ we can define sheaves $B_i\Omega^1_X$ of rings inductively by $B_0\Omega^1_X = 0$, $B_1\Omega^1_X = dO_X$ and $C^{-1}(B_i\Omega^1_X) = B_{i+1}\Omega^1_X$. Similarly, we define sheaves $Z_i\Omega^1_X$ inductively by $Z_0\Omega^1_X = \Omega^1_X$, $Z_1\Omega^1_X = \Omega^1_X,\text{closed}$, the sheaf of $d$-closed forms and by setting

$$Z_{i+1}\Omega^1_X = C^{-1}(Z_i\Omega^1_X).$$

Usually we simply write $B_i$ and $Z_i$. The sheaves $B_i$ and $Z_i$ can be viewed as locally free subsheaves of $(F^i)_*\Omega^1_X$ on $X^{(p)}$. They were introduced by Illusie in [Il] and can be used to provide de Rham-cohomology with a rich structure. The inverse Cartier operator gives rise to an isomorphism

$$C^{-i} : \Omega^1_{X^{(p)}} \cong Z_i/B_i$$

or a $\sigma^{-i}$-linear isomorphism $\Omega^1_X \cong Z_i/B_i$. Note that we have the inclusions

$$0 = B_0 \subset B_1 \subset \ldots \subset B_i \subset \ldots \subset Z_i \subset \ldots \subset Z_1 \subset Z_0 = \Omega^1_X.$$

We also have an exact sequence

$$0 \to Z_{i+1} \longrightarrow Z_i\frac{dC^i}{d\Omega^1_X} \longrightarrow 0. \quad (3)$$

(6.1) **Lemma.** If $X$ is a K3 surface $X$ we have i) $H^0(B_i) = 0$ for all $i \geq 0$; ii) the natural inclusion $B_i \to B_{i+1}$ induces an injective homomorphism $H^1(B_i) \to H^1(B_{i+1})$.

**Proof.** i) The natural injection $B_i \to \Omega^1_X$ induces an injection $H^0(B_i) \to H^0(\Omega^1_X)$ and we know $H^0(\Omega^1_X) = 0$. ii) This follows from i) and the exact sequence

$$0 \to B_i \longrightarrow B_{i+1}\frac{C^i}{B_1} \longrightarrow 0$$

with $C$ the Cartier operator. $\square$

There is a close relationship between the Witt vector cohomology and the cohomology of $B_i$ as follows. Serre introduced in [S] a map $D_i : W_i(O_X) \longrightarrow \Omega^1_X$ of sheaves in the following way:

$$D_i(a_0, a_1, \ldots, a_{i-1}) = a_0^{p^{i-1}} - a_0 + \ldots + a_{i-2}^{p^{i-2}}da_{i-2} + da_{i-1}.$$

It satisfies $D_{i+1}V = D_i$, and Serre showed that this induces an injective map of sheaves of additive groups

$$D_i : W_i(O_X)/FW_i(O_X) \longrightarrow \Omega^1_X$$

inducing an isomorphism

$$D_i : W_i(O_X)/FW_i(O_X) \cong B_i\Omega^1_X \quad (4)$$

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The exact sequence $0 \to W_i \xrightarrow{F} W_i \to W_i/FW_i \to 0$ gives rise to the exact sequence

$$0 \to H^1(W_i/FW_i) \to H^2(W_i) \xrightarrow{F} H^2(W_i) \to H^2(W_i/FW_i) \to 0 \quad (5)$$

and we thus have an isomorphism $H^1(W_i/FW_i) \cong \ker[F : H^2(W_i) \to H^2(W_i)]$. Combining the result on the dimension of the kernel of $F$ on $H^2(W_i)$ from Section 5 with (4) we get an interpretation of the height $h$ in terms of the groups $H^1(B_i)$.

**Theorem.** We have

$$\dim H^1(B_i) = \begin{cases} \min\{i, h-1\} & \text{if } h \neq \infty, \\ i & \text{if } h = \infty. \end{cases}$$

The Verschiebung induces an exact sequence

$$0 \to W_i/FW_i \xrightarrow{V} W_i+1/FW_i+1 \to O_X/FO_X \to 0$$

and this gives rise to

$$0 \to H^1(W_i/FW_i) \xrightarrow{V} H^1(W_i+1/FW_i+1) \to H^1(\Omega^1_X) \to \ldots$$

i.e., Verschiebung induces for all $i$ an injective map. Moreover, it is surjective if and only if $h \neq \infty$ and $i \geq h - 1$.

We have a commutative diagram (with $\beta_i$ the natural map induced by $B_i \subset \Omega^1_X$)

$$\begin{array}{ccc}
H^1(W_i/FW_i) & \xrightarrow{D_i} & H^1(\Omega^1_X) \\
\Downarrow \cong & & \Downarrow = \\
H^1(B_i\Omega^1_X) & \xrightarrow{\beta_i} & H^1(\Omega^1_X)
\end{array}$$

We study the kernel of $D_i$, equivalently the kernel of the natural map $\beta_i : H^1(B_i\Omega^1_X) \to H^1(\Omega^1_X)$ in Sections 9-11.

**Lemma.** The Euler-Poincaré characteristics of $B_i$ and $Z_i$ are given by $\chi(B_i) = 0$ and $\chi(Z_i) = -20$.

**Proof.** Since the kernel and the cokernel of $F$ on $H^2(W_i)$ have the same dimension by (5) the result for $B_i$ follows from (4) and (5). The identity $\chi(B_i) + \chi(\Omega^1_X) = \chi(Z_i)$ resulting from the isomorphism $Z_i/B_i \cong \Omega^1_X$ implies the result. \( \square \)

**7. De Rham Cohomology**

The de Rham cohomology of a K3 surface is the hypercohomology of the complex $(\Omega^\bullet_X, d)$. The dimensions $h^{p,q}$ of the graded pieces are given by the Hodge diamond.

$$\begin{array}{ccc}
1 & & \\
0 & 0 & \\
1 & 20 & 1 \\
0 & 0 & \\
1 & & 
\end{array}$$

On $H^2_{dR}$ we have a perfect pairing $\langle \cdot, \cdot \rangle$ given by Poincaré duality; cf. [D].
The Hodge spectral sequence with $E_2^{i,j} = H^j(X, \Omega^i_X)$ converges to $H_{dR}^*(X)$. The second spectral sequence of hypercohomology has as $E_2^{i,j} = H^i(\mathcal{H}^j(\Omega^*_X))$ abutting to $H_{dR}^{i+j}(X/k)$. But the Cartier operator yields an isomorphism of sheaves

$$C^{-1} : \Omega_X^{i(p)} \overset{\sim}{\longrightarrow} H^i(F_*(\Omega^*_X/k)),$$

so that we can rewrite this as

$$E_2^{i,j} = H^i(X', \mathcal{H}^j(\Omega^*_X)) \cong H^i(X', \Omega^j_X) \Rightarrow H_{dR}^*(X),$$

where $X' = X^{(p)}$ is the base change of $X$ under Frobenius. We thus get two filtrations on the de Rham cohomology: the Hodge filtration

$$(0) \subset F^2 \subset F^1 \subset H_{dR}^2,$$

and the so-called conjugate filtration

$$(0) \subset G_1 \subset G_2 \subset H_{dR}^2.$$

We have rank$(F^1) = \text{rank}(G_2) = 21$, rank$(F^2) = \text{rank}(G_1) = 1$ and

$$(F^1) \perp = F^2 \quad \text{and} \quad G_1^\perp = G_2.$$

We have also

$$F^1/F^2 \cong H^1(X, \Omega^1_X), \quad G_2/G_1 \cong H^1(X, \Omega^1_X),$$

cf. [D]. Moreover, from the description with the second spectral sequence it follows that $G_1$ is the image under Frobenius of $H_{dR}^2$ and also of $H^2(X, O_X)$. The conjugate filtration is an analogue of the complex conjugate of the Hodge filtration in characteristic zero.

The relative position of these two filtrations is an interesting invariant of a K3 surface. We have the three cases

a) $F^1 \cap G_1 = \{0\}$;
b) $G_1 \subset F^1$; $G_1 \neq F^2$;
c) $G_1 = F^2$.

The first case happens if and only if $F : H^2(X, O_X) \to H_{dR}^2(X) \to H^2(X, O_X)$ is not zero, i.e. if $h = 1$. Such $X$ are called ordinary. The second case happens if $h \geq 2$, while the last case is by definition the superspecial case. In this case the two filtrations coincide. It is known that two superspecial K3 surfaces are isomorphic (as unpolarized varieties)(cf. [O]).

We have the following result of Ogus (cf. [O]) which provides us with an interpretation of $H^1(Z_1)$.

(7.1) Proposition. We have an isomorphism $F^1 \cap G_2 \cong H^1(X, Z_1)$.

Proof. The map $z_1 : H^1(Z_1) \to H_{dR}^2$ given by $\{f_{ij}\} \mapsto (0, \{f_{ij}\}, 0)$ is injective. Indeed, if $\{f_{ij}\}$ represents an element in the kernel, then it is of the form $(\delta h_{ij}, d\eta_i + \omega_j - \omega_i, d\omega_i)$ for a $h_{ij} \in C_1(O_X)$, $\omega_i \in C_0(\Omega^1_X)$. Then the $\omega_i$ are closed and $h_{ij}$ defines a cocycle. Since $H^1(O_X) = 0$ we can write $d\eta_i = \eta_j - \eta_k$ and $f_{ij}$ is a coboundary. The image is contained in $F^1$ and is orthogonal to the image $G_1$ of Frobenius. Indeed, take a class $F(a)$ and consider the cupproduct $\langle F(a), z_1(f_{ij}) \rangle$. Applying the Cartier operator we see
that it is zero. But \( C : H^1_{\text{dR}} \to H^1_{\text{dR}} \) is a bijection. Hence the image lies in \( F^1 \cap G_2 \). This implies that \( \dim H^1(Z_1) \leq 20 \) if \( X \) is not superspecial and \( \leq 21 \) for superspecial \( X \). The exact sequence

\[
0 \to H^1(B_1) \to H^1(Z_1) \xrightarrow{C} H^1(\Omega^1_X) \to H^2(B_1) \to H^2(Z_1) \to 0
\]

implies together with the value of \( h^1(B_1) = h^2(B_1) \) and \( \chi(Z_1) = -20 \) that \( h^1(Z_1) = 20 \) unless \( X \) is superspecial. But if \( X \) is superspecial then because of \( F^2 = G_1 \) the Cartier operator gives an isomorphism \( C : H^1(Z_1)/H^1(B_1) \cong H^1(\Omega^1_X) \) implying that \( h^1(Z_1) = 21 \). □

8. An Extension of de Rham Cohomology

We define an extension of de Rham cohomology by considering an enlarged complex. (It captures the \([0,1)\)-part of crystalline cohomology.) It is defined as follows. We denote by \( H^i_{\text{dRw}}(X/S) \) the cohomology of the double complex \( CW_n \) of additive groups which is defined by the commutative digram:

\[
\begin{array}{ccc}
C_2(W_n(O_{X/S})) & \xrightarrow{D_n} & C_2(\Omega^1_{X/S}) \\
\partial \uparrow & & \partial \uparrow \\
C_1(W_n(O_{X/S})) & \xrightarrow{D_n} & C_1(\Omega^2_{X/S}) \\
\partial \uparrow & & \partial \uparrow \\
C_0(W_n(O_{X/S})) & \xrightarrow{D_n} & C_0(\Omega^2_{X/S}) \\
\end{array}
\]

where \( C_i \) are the \( i \)-th Čech cochains, \( D_n \) are the maps induced by the differential of Serre given in Section 6, the differentials \( d \) are defined by the exterior differentiation of differential forms and the vertical differentials are taken in the Čech sense. As usual, we denote by \( \delta \) the differential of the single complex associated with \( CW_n \). An element of \( H^2_{\text{dRw}}(X/S) \) is represented by a triple \( (\alpha, \alpha_1, \alpha_2) \in C_2(W_n(O_{X/S})) \oplus C_1(\Omega^1_{X/S}) \oplus C_0(\Omega^2_{X/S}) \). In case \( n = 1 \), \( H^2_{\text{dRw}}(X/S) \) is nothing but the de Rham cohomology \( H^2_{\text{dR}}(X/S) \). On \( H^2_{\text{dRw}}(X/S) \) we have the Hodge filtration

\[
0 \subset F^2 \subset F^1 \subset H^2_{\text{dRw}}(X/S).
\]

Here the \( F^i \) (for \( i \neq 0 \)) is naturally isomorphic to the \( F^i \)-part in the Hodge filtration of \( H^2_{\text{dR}}(X/S) \). We have a natural isomorphism

\[
H^2_{\text{dRw}}(X/S)/F^1 \cong H^2(X, W_n(O_{X/S})).
\]

Since the Frobenius morphism \( F \) is a zero map on \( F^1 \), we have an induced homomorphism

\[
F : H^2(X, W_n(O_{X/S})) \longrightarrow H^2_{\text{dRw}}(X/S).
\]
The map $V^{n-1} : O_{X/S} \rightarrow W_n(O_{X/S})$ gives rise to a homomorphism

$$V^{n-1} : C_i(O_{X/S}) \rightarrow C_i(W_n(O_{X/S})).$$

Using this homomorphism and taking the identity mapping from $C_i(\Omega^2_{X/S})$ to $C_i(\Omega^2_{X/S})$, we have a homomorphism of complexes of additive groups $CW_1 \rightarrow CW_n$. Therefore, we have a homomorphism of additive groups:

$$V^{n-1} : H^2_{\text{dR}}(X/S) \rightarrow H^2_{\text{dR}}(X/S).$$

Let $X_0$ be a K3 surface over a field $k$ and assume now that $F : H^2(W_i(O_{X_0})) \rightarrow H^2(W_i(O_{X_0}))$ is zero for $i = 1, \ldots, n-1$. Then, by the same argument as in Section 5, we have $FH^2_{\text{dR}}(X_0) \subset V^{n-1}H^2_{\text{dR}}(X_0)$. Therefore, using the inverse of the natural isomorphism of additive groups $H^2_{\text{dR}}(X_0) \cong V^{n-1}H^2_{\text{dR}}(X_0) \subset H^2_{\text{dR}}(X_0)$, we have a homomorphism

$$\Phi_n : H^2(W_n(O_X)) \rightarrow V^{n-1}H^2_{\text{dR}}(X) \cong H^2_{\text{dR}}(X).$$

Since we have $H^2(W_n(O_X))/VH^2(W_n(O_X)) \cong H^2(O_{X_0}) \cong H^2_{\text{dR}}(X)/F^1$, and since $\Phi_n$ maps $VH^2(W_n(O_X))$ to $F^1$, the map $\Phi_n$ induces a homomorphism from $H^2(O_{X_0})$ to $H^2(O_{X_0})$. This homomorphism coincides with the map $\phi_n$ which was constructed in Section 5.

We now take a basis $\omega_0$ of $H^0(X_0, \Omega^2_{X_0})$ and take the dual basis $\zeta_0$ of $H^2(X_0, O_{X_0})$. Via the Hodge filtration of $H^2_{\text{dR}}(X_0)$, we can naturally regard $H^0(X_0, \Omega^2_{X_0})$ as a subspace of $H^2_{\text{dR}}(X_0)$. Therefore, we may assume $\omega_0$ is an element of $H^2_{\text{dR}}(X_0)$.

Since $R^{n-1} : H^2(W_n(O_{X_0})) \rightarrow H^2(O_{X_0})$ is surjective, there exists an element $\alpha_0 \in H^2(W_n(O_{X_0}))$ such that $R^{n-1}(\alpha_0) = \zeta_0$. Then, by Theorems 5.1 and 5.7 we have the following proposition.

**8.2 Proposition.** Suppose that for a K3 surface $X_0$ the map $F : H^2(W_i(O_{X_0})) \rightarrow H^2(W_i(O_{X_0}))$ is zero for $i = 1, \ldots, n-1$. Then, with the notation introduced above, $\nu(\Phi_{X_0}) \geq n + 1$ if and only if $\langle \Phi_n(\alpha_0), \omega_0 \rangle = 0$.

9. The Dimensions of the Spaces of Closed Forms

We study the dimensions of the spaces $H^1(X, B_n)$ and $H^1(X, Z_n)$. We also consider their images in $H^2_{\text{dR}}(X)$ and this gives a finer structure on these de Rham cohomology groups.

Let us consider the natural map $\beta_n$ induced in $H^1$ by the inclusion $B_n \subset \Omega^1_X$:

$$\beta_n : H^1(B_n) \rightarrow H^1(\Omega^1).$$

**9.1 Proposition.** If $\beta_n$ is not injective then $\beta_m$ is not injective for every $m \geq n$ and $\dim H^1(B_m) < \dim H^1(B_{m+1})$.

**Proof.** The maps $\beta_n$ are compatible with the natural maps $H^1(B_n) \rightarrow H^1(B_{n+1})$ and by (6.1) these maps $H^1(B_n) \rightarrow H^1(B_{n+1})$ are injective. If $\beta_n$ is not injective it follows that $\beta_{n+1}$ is not injective. To prove the second statement, we start with the case $n = 1$. If $\beta_1 : H^1(B_1) \rightarrow H^1(\Omega^1_X)$ is not injective then there exists a non-trivial cocycle
$f_{ij} \in C^1(O_X/F_O X)$ and a 1-cochain $\omega_i$ of 1-forms such that $df_{ij} = \omega_j - \omega_i$. Since for affine open sets $U$ the Cartier map $H^0(\Omega_{U,\text{closed}}) \to H^0(\Omega^1_X)$ is surjective we can find closed forms $\tilde{\omega}_i$ and regular functions $g_{ij}$ on $U_i \cap U_j$ such that we have a relation

$$f_{ij}^{-1}df_{ij} + dg_{ij} = \tilde{\omega}_j - \tilde{\omega}_i.$$  

(6)

Note that this implies that the map $H^1(B_2) \to H^1(Z_1)$ has a non-trivial kernel. Suppose that the left-hand-side of (6) represents an element in the image of $H^1(B_1) = H^1(dO_X) \to H^1(B_2)$. Then we find a relation $dh_{ij} = \tilde{\omega}_j - \tilde{\omega}_i$ with $\tilde{\omega}_i$ closed. Then since $C$ annihilates $dh_{ij}$ the $C\tilde{\omega}_i$ define a global 1-form and this must be zero. Hence we can write $\tilde{\omega}_i = d\phi_i$ and this shows that $dh_{ij}$ represents the zero-class in $H^1(dO_X)$, contrary to the assumption. Hence we find a non-trivial element in $H^1(B_2)$ which does not lie in the image of the natural inclusion $H^1(B_1) \to H^1(B_2)$. Carrying out this argument for all $n$ proves the claim. \qed

(9.2) Corollary. Assume that $h < \infty$. Then for all $n \geq 1$ the natural map $\beta_n : H^1(B_n) \to H^1(X, \Omega^1_X)$ is injective and the image has dimension $\min\{n, h - 1\}$.

Proof. Since by (6.2) $\dim H^1(B_n)$ stabilizes for $h \neq \infty$, non-injectivity would contradict the preceding proposition. \qed

Note that the natural map $H^1(B_n) \to H^1(\Omega^1_X)$ is not necessarily injective for $h = \infty$ because $\dim H^1(B_n) > 20$ for $n > 20$. In the case of $h \neq \infty$, we often identify $H^1(B_n)$ with the image of the natural inclusion $H^1(B_n) \to H^1(\Omega^1_X)$ in Corollary 9.2.

Let $Z_n \to \Omega^1_X$ be the natural inclusion. We have an induced map

$$z_n : H^1(Z_n) \to H^1(\Omega^1_X).$$

We would like to characterize both the image and the kernel of this map. We often write $\text{Im}(H^1(Z_n))$ for the image of $z_n$.

(9.3) Lemma. i) We have $\text{Im}(H^1(B_n)) \subseteq \text{Im}(H^1(Z_n))^\perp$, in particular, $\text{Im}(H^1(B_n)) \subseteq \text{Im}(H^1(B_n))^\perp$. ii) Assume that $h < \infty$. If $C : H^1(B_{n+1}) \to H^1(B_n)$ is surjective then we have the equality $\text{Im}(H^1(Z_n)) = \text{Im}(H^1(X, B_n))^\perp$.

Proof. We first show that $\text{Im}(H^1(B_n))$ and $\text{Im}(H^1(Z_n))$ are orthogonal. Let $\alpha \in \text{Im}(H^1(B_1))$ and $\beta \in \text{Im}(H^1(Z_1))$. Then we find an element $\alpha \wedge \beta \in H^2(\Omega^2_X) \cong k$ representing the cup product $\langle \alpha, \beta \rangle$. If we apply Cartier to $\alpha \wedge \beta$ a suitable number of times then it is zero. Now use the exact sequence

$$0 \to d\Omega^1_X \to \Omega^2_X, \text{closed} \xrightarrow{C} \Omega^2_X \to 0,$$

(7)

and the fact that $\Omega^2_X, \text{closed} = \Omega^2_X$. Then, we have from the the long exact sequence the exact sequence

$$H^2(d\Omega^1_X) \to H^2(\Omega^2_X) \xrightarrow{C} H^2(\Omega^2_X) \to 0.$$ 

The fact that $\dim H^2(\Omega^2_X) = 1$ implies that $C : H^2(\Omega^2_X) \to H^2(\Omega^2_X)$ is an isomorphism as a $p$-linear mapping. Therefore, for $x \in H^2(\Omega^2_X)$ we have $x = 0$ if and only if $C^n(x) = 0$ for some $n$. Hence, we conclude $\alpha \wedge \beta = 0$. We now prove equality by induction. For $n = 1$ we have $\text{Im}(H^1(B_1))^\perp = \text{Im}(H^1(Z_1))$ because $\text{Im}(H^1(B_1))$ is the kernel $G_1/F^2$
of the Cartier operator and \( \text{Im}(H^1(Z_1)) \) is \( F^1 \cap G_2 = F^1 \cap G_2^1 \). Suppose that we have proved that \( \text{Im}(H^1(B_i)) = \text{Im}(H^1(Z_i)) \) for \( i \leq n \). If \( \beta \in \text{Im}(H^1(Z_1)) \) is orthogonal to all \( \alpha \in \text{Im}(H^1(B_{n+1})) \) then we have \( \langle C\alpha, C\beta \rangle = 0 \) and since \( C : H^1(B_{n+1}) \to H^1(B_n) \) is surjective this implies that \( C\beta \in \text{Im}(H^1(Z_n)) \), i.e. \( \beta \in \text{Im}(H^1(Z_{n+1})) \). \( \square \)

**Lemma.** The Cartier operator \( C : H^1(B_n) \to H^1(B_{n-1}) \) is surjective for \( n \leq h - 1 \). Moreover, for \( n \leq h - 1 < \infty \) we have \( \dim \text{Im}(H^1(Z_n)) = 20 - n \).

**Proof.** Note that we know that \( h^1(B_n) = n \) for \( n \leq h - 1 \) and thus the exact sequence \( 0 \to B_1 \to B_n \to B_{n-1} \to 0 \) implies that \( C : H^1(B_n) \to H^1(B_{n-1}) \) is surjective for \( n \leq h - 1 \). The rest follows from \( (9.3) \). \( \square \)

**Corollary.** If \( h \neq \infty \) we have the following orthogonal filtration in \( H^1(\Omega^1) \):

\[
0 \subset H^1(B_1) \subset H^1(B_2) \subset \ldots \subset H^1(B_{h-1}) \subset \text{Im}(H^1(Z_{h-1})) \subset \text{Im}(H^1(Z_{h-2})) \subset \ldots \subset \text{Im}(H^1(Z_1)) \subset H^1(\Omega^1_X). \tag{8}
\]

The exact sequence (3) gives for \( i = 0 \) rise to the exact sequence

\[
0 \to H^0(d\Omega^1_X) \to H^1(Z_1) \to H^1(\Omega^1_X)_d \to H^1(d\Omega^1_X) \to H^2(Z_1) \to \ldots
\]

The natural map \( H^1(Z_1) \to H^1(\Omega^1) \) is the composition of \( H^1(Z_1) \to H^2_{\text{dR}}^1 \) and the projection \( H^2_{\text{dR}} \to F^1/F^2 \), i.e. by (7.1) it is the map \( F^1 \cap G_2 \to F^1/F_2 \). This is an isomorphism for \( h = 1 \) and it has a 1-dimensional kernel otherwise. It follows that

\[
\dim H^0(d\Omega^1) = \dim H^1(d\Omega^1) = \begin{cases} 0 & \text{if } h = 1 \\ 1 & \text{if } h \neq 1. \end{cases}
\]

From the exact sequence

\[
0 \to H^0(d\Omega^1_X) \to H^1(Z_{n+1}) \xrightarrow{\psi_{n+1}} H^1(Z_n) \to H^1(d\Omega^1_X) \to \ldots
\]

with \( \psi_{n+1} \) the map induced by inclusion we deduce that for \( h \neq 1 \)

\[
\psi_{n+1} \text{ is surjective } \iff \dim H^1(Z_{n+1}) > \dim H^1(Z_n). \tag{9}
\]

**Lemma.** For \( h \neq \infty \) we have \( \dim H^1(Z_n) = 20 \).

**Proof.** If \( h = 1 \) we have \( h^0(d\Omega^1_X) = 0 \) and \( h^1(d\Omega^1_X) = 0 \) hence all \( \psi_n \) are isomorphisms. Since we know \( h^1(Z_0) = 20 \) the result follows for \( h = 1 \). If \( h \neq 1 \) then \( H^0(d\Omega^1_X) \cong k \). For \( n \leq h - 1 \) we have \( \text{Im}(H^1(Z_n)) = 20 - n \) by Lemma 9.3 and \( \dim H^1(B_n) = \min\{n, h - 1\} \). Suppose there exists an \( n \) \((n \leq h - 1)\) such that \( \psi_n \) is surjective. Take the smallest such \( n \). Then, the image of \( z_n \) coincides with the image of \( z_{n-1} \), which contradicts \( \dim \text{Im}(H^1(Z_n)) \neq \text{Im}(H^1(Z_{n+1})) \). Hence, \( h^1(Z_n) = 20 \) for \( n \leq h - 1 \). Consider for
$n = h$ the commutative diagram of exact sequences

\[
\begin{array}{ccc}
0 & 0 & \\
\downarrow & \downarrow & \\
B_1 & = & B_1 \\
\downarrow & \downarrow & \\
0 & \rightarrow & B_n \\
\downarrow & \downarrow & \downarrow \cC \rightarrow \Omega^1_X \rightarrow 0. \\
\downarrow & \downarrow & \\
0 & \rightarrow & B_{n-1} \\
\downarrow & \downarrow & \\
0 & 0 & .
\end{array}
\]

The diagram shows that $C^h : H^1(Z_h) \rightarrow H^1(\Omega^1_X)$ factors through the image of $C^{h-1}$. This implies that $\dim H^1(Z_h) - (h-1) \leq 20 - (h-1)$. Since $h^1(Z_n) \geq 20$ for all $n \geq 1$ we get $h^1(Z_h) = 20$. We can repeat this argument for $H^1(Z_m)$ with $m \geq h$. □

10. Chern classes of line bundles and closed forms

We start with a well-known result due to Ogus [O, Cor. 1.5]. We give here the proof by Shafarevich [Sh] for the reader’s convenience.

(10.1) Proposition. The map $c_1 : NS(X)/pNS(X) \rightarrow H^2_{dR}$ is injective and factors through $F^1H^2_{dR}$.

Proof. (Shafarevich) We take an affine open covering $\{U_i\}$ of $X$. A class in $H^2_{dR}$ is represented by a triple $(a, b, c) \in C^2(O_X) \oplus C^1(\Omega^1_X) \oplus C^0(\Omega^2_X)$. The boundaries are of the form $(\delta h_{ij}, d h_{ij} + \omega_j - \omega_i, d \omega_i)$ with $(h_{ij}, \omega_i) \in C^1(O_X) \oplus C^0(\Omega^1_X)$. So if a Chern class $c_1(L)$, represented by $(0, d \log f_{ij}, 0)$, is zero in $H^2_{dR}$ then there exists $(h_{ij}, \omega_i) \in C^1(O_X) \oplus C^0(\Omega^1_X)$ with $d \omega_i = 0$ and $\delta h_{ij} = 0$ and we have $d \log f_{ij} = \omega_j - \omega_i + d h_{ij}$. By the relation $\delta(h_{ij}) = 0$ the $h_{ij}$ defines a class in $H^1(X, O_X) = 0$, so we have $h_{ij} = \eta_j - \eta_i$ with $\eta_i$ regular and we can replace $\omega_i$ by $\omega_i + d \eta_i$ and obtain a relation

$$d \log f_{ij} = \omega_j - \omega_i \quad \text{with} \quad \omega_i \quad \text{closed}. \quad (10)$$

Applying the Cartier operator we find

$$d \log f_{ij} = C \omega_j - C \omega_i. \quad (11)$$

subtracting (1) from (2) we find $C \omega_i - \omega_i = C \omega_j - \omega_j$. This defines a global 1-form which must be zero. Hence we see $C \omega_i = \omega_i$ and it follows that $\omega_i = d \log \phi_i$ (after shrinking the $U_i$ if necessary). We find

$$d \log f_{ij} = d \log \phi_j \phi_i^{-1},$$
hence
\[ f_{ij} = \phi_i \phi_j^{-1} \psi_{ij}^p. \]
for some \( \psi_{ij} \in O(U_i \cap U_j) \). Thus modulo a \( p \)-th power \( L \) is trivial. The proof also shows that the image lands in \( F^1 H^2_{dR} \). \( \square \)

(10.2) Proposition. If \( h < \infty \) then we have \( \langle c_1(NS(X)) \rangle \cap \text{Im}(H^1(B_n)) = \{0\} \) for all \( n \). Moreover, \( c_1(NS(X)) \) is orthogonal with \( \text{Im}(H^1(B_n)) \) for all \( n \).

Proof. First we show that \( c_1(NS(X)) \cap \text{Im}(H^1(B_n)) = \{0\} \) for all \( n > 0 \). If it is not, then take a minimal \( n \) such that \( \text{Im}(H^1(B_n)) \) contains a Chern class \( 0 \neq [d \log f_{ij}] \). We can write a (non-trivial) relation as
\[ d \log f_{ij} = \beta_{ij} + \omega_j - \omega_i, \] (12)
where the \( \beta_{ij} \) are forms in \( B_n \), but not in \( B_{n-1} \). Apply the inverse Cartier operator as in (9.1) to get a relation
\[ d \log f_{ij} = \tilde{\beta}_{ij} + \tilde{\omega}_j - \tilde{\omega}_i \] (13)
where the \( \tilde{\omega}_i \) are closed forms with \( C(\tilde{\omega}_i) = \omega_i \) and the \( \tilde{\beta}_{ij} \) are forms in \( B_{n+1} \) with \( C(\tilde{\beta}_{ij}) = \beta_{ij} \). Subtracting (12) from (13) shows that \( \tilde{\beta}_{ij} - \beta_{ij} \) is a boundary. Since \( \beta_{ij} \) defines a non-zero element of \( H^1(B_n) \) which is not in the image of \( H^1(B_{n-1}) \) the cocycle \( \beta_{ij} \) gives an element of \( H^1(B_{n+1}) \) not in the image of \( H^1(B_n) \). Hence the left hand side is not zero in \( H^1(B_{n+1}) \) and this shows that \( H^1(B_{n+1}) \to H^1(\Omega^1) \) is not injective.

Suppose now that \( \langle c_1(NS(X)) \rangle \cap \text{Im}(H^1(B_n)) \neq \{0\} \). Considering all \( n \) which satisfy this condition, we then have a relation with \( m \geq 2 \) minimal
\[ d \log f^{(1)}_{ij} + \sum_{\nu=2}^m a_{\nu} d \log f^{(\nu)}_{ij} = \beta_{ij} + \omega_j - \omega_i. \]
We may assume that \( m \geq 2 \) and that \( a_{\nu} \not\in \mathbb{F}_p \) for all \( \nu \geq 2 \). Then by applying \( C^{-1} \) as before we find
\[ d \log f^{(1)}_{ij} + \sum_{\nu=2}^m a_{\nu}^p d \log f^{(\nu)}_{ij} = \tilde{\beta}_{ij} + \tilde{\omega}_j - \tilde{\omega}_i, \]
where the \( \tilde{\omega}_i \) are closed and \( C\omega_i = \omega_i \). Subtracting the two relations we find a shorter relation (\( m \) smaller but with \( n \) maybe larger). This contradiction shows that \( \langle c_1(NS(X)) \rangle \cap \text{Im}(H^1(B_n)) = \{0\} \).

The orthogonality of \( \langle c_1(NS(X)) \rangle \) and \( \text{Im}(H^1(B_n)) \) follows from the fact that \( \langle c_1(NS(X)) \rangle \subset \text{Im}(H^1(Z_n)) \) and Lemma (9.3). \( \square \)

(10.3) Proposition. Suppose that \( h < \infty \). Then the Chern class map
\[ c_1 \otimes k : NS(X)/pNS(X) \otimes k \to H^1(X, \Omega^1_X) \]
is injective.

Proof. Suppose we have a relation \( \sum_{\nu=1}^r a_{\nu} c_1(L_{\nu}) = \omega_j - \omega_i \) for line bundles \( L_{\nu} \) and \( a_{\nu} \in k \). We may assume that \( a_1 = 1 \) and that the relation is the shortest possible (\( r \) minimal). Furthermore, we can assume that \( a_{\nu}/a_\mu \not\in \mathbb{F}_p \) for \( \nu \neq \mu \); otherwise we can
easily find a shorter one. Now apply the inverse Cartier operator $C^{-1}$ to the relation as we did before. We find a new relation

$$ dg_{ij} + c_1(L_1) + \sum_{\nu=2}^r a_\nu^0 c_1(L_\nu) - \tilde{\omega}_i + \tilde{\omega}_j = 0, $$

where the $g_{ij}$ are regular on $U_i \cap U_j$. If the cocycle $dg_{ij}$ defines a zero class in $H^1(X, \Omega^1_X)$, we can write $dg_{ij} = \eta_i - \eta_j$, and we can replace the relation by a shorter one by subtracting the two relations contradicting the minimality of $r$. Hence $\{dg_{ij}\}$ defines a non-zero class in $H^1(X, \Omega^1_X)$ and we find a non-zero element in $\text{Im}(H^1(B_1)) \cap \langle c_1(\text{NS}(X)) \rangle$. □

As a corollary of (6.2), (10.2) and (10.3) we now find the well-known result of Artin and Mazur on the rank $\rho$ of the Néron-Severi group:

(10.4) Corollary. For $h \neq \infty$ we have $\rho \leq 22 - 2h$.

(10.5) Remark. A line bundle $L$ defined by transition functions $f_{ij}$ defines a cocycle $d \log f_{ij}$ with values in $\mathbb{Z}_n \Omega^1_X$ for all $n \geq 0$. We thus can view the class $c_1(L)$ as a class in $H^1(\mathbb{Z}_n)$ for all $n \geq 0$ as well as in $H^2_{\text{dR}}$. If $h < \infty$ the maps

$$ c_1 \otimes k : \text{NS}(X)/p\text{NS}(X) \otimes k \rightarrow H^1(\mathbb{Z}_n) $$

are injective for all $n \geq 0$.

11. The Supersingular Case

The map $c_1 : \text{NS}/p\text{NS} \rightarrow H^2_{\text{dR}}$ is injective and factors through $H^1(Z_j)$ for all $j \geq 1$. However, the map $c_1 \otimes k : \text{NS} \otimes k \rightarrow H^2_{\text{dR}}$ is not necessarily injective. For $X$ supersingular in Shioda’s sense, i.e. $\rho = B_2 = 22$, it cannot be injective since $\dim_k H^1(Z_1) = 20$ or $21$, the latter if $X$ is superspecial.

We define for $j = 0, 1, \ldots$

$$ U_j := \ker\{c_1 \otimes k : \text{NS} \otimes k \rightarrow H^1(Z_j)\} $$

and we set

$$ \dim U_1 = \sigma_0. $$

Using the natural maps $H^1(Z_i) \rightarrow H^1(Z_{i-1})$ we have $U_{j+1} \subseteq U_j$ for $j = 0, 1, 2, \ldots$. We define two bijective operators on $\text{NS} \otimes k$

$$ \varphi = 1 \otimes F \quad \text{and} \quad \gamma = 1 \otimes F^{-1}, $$

with the Frobenius action $F : a \mapsto a^p$ on the second factor $k$.

(11.0) Remark. If we assume that $\rho = B_2 = 22$ (i.e. the truth of the Artin conjecture that $h = \infty$ implies $\rho = 22$) then one can show that the invariant $\sigma_0$ just introduced equals the Artin invariant $\sigma_0$, i.e. the intersection form on the lattice $\text{NS}(X)$ has discriminant

$$ \text{disc}(\text{NS}(X)) = -p^{2\sigma_0}. $$

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(11.1) Lemma. We have $\gamma(U_{j+1}) \subseteq U_j$; equivalently, we have $U_{j+1} \subseteq \varphi(U_j)$. Moreover, we have $U_{j+1} \subseteq U_j \cap \varphi(U_j)$.

Proof. This follows from the commutativity of the diagram

$$
\begin{array}{ccc}
NS \otimes k & \rightarrow & NS \otimes k \\
\downarrow c_1 \otimes k & & \downarrow c_1 \otimes k \\
H^1(Z_{j+1}) & \rightarrow & H^1(Z_j)
\end{array}
$$

with $C$ the Cartier operator. The second result follows from this and the inclusion $U_{j+1} \subset U_j$. □

Now choose an element $u_{\text{min}} = u^{(j)}_{\text{min}} \neq 0$ of minimal length in $U_j$ under the assumption that $U_j$ is non-zero, i.e. write $u_{\text{min}} = \sum_{i=1}^m a_i [L_i]$ and require $m \geq 2$ to be minimal. We also may assume –and we shall– that $a_1 = 1$.

(11.2) Lemma. For $j \geq 1$ we have $u_{\text{min}} \not\in \varphi(U_j)$. Similarly we have $u_{\text{min}} \not\in \gamma(U_j)$. If $X$ is not superspecial the conclusion holds also for $j = 0$.

Proof. If $u_{\text{min}} \in \varphi(U_j)$ is such a minimal element with $a_1 = 1$ then $\gamma(u_{\text{min}}) - u_{\text{min}}$ would be a shorter element or zero. If it is zero, then $u_{\text{min}} \in NS \otimes F_p \cap U_1 = \{0\}$. For $j = 0$ the argument is similar. Note that $NS \otimes F_p \cap U_0 \neq \{0\}$ if and only if $X$ is superspecial, cf. [O 1], Cor. 1.4.

(11.3) Lemma. The map $c_1 \otimes k : \varphi(U_j) \rightarrow H^1(Z_{j+1})$ factors via $H^1(B_1) \rightarrow H^1(Z_{j+1})$ and the induced map $\varphi(U_j) \rightarrow H^1(B_1)$ is surjective if $U_j \neq \{0\}$.

Proof. If $u \in U_j$ there exist closed forms $\zeta_x \in Z_j(V_x)$ for some open covering $V_x$ such that $(c_1 \otimes k)(u)$ is a coboundary: $\zeta_\beta - \zeta_\alpha$. Now use the local surjectivity of $C$ to write

$$(c_1 \otimes k)(\varphi(u)) = \tilde{\zeta}_\beta - \tilde{\zeta}_\alpha + \phi_{\alpha \beta}$$

with $\tilde{\zeta}_x \in Z_{j+1}$, $C\tilde{\zeta}_x = \zeta_x$, $\phi_{\alpha \beta} \in B_1$ on a suitable open covering. Then this $\phi_{\alpha \beta}$ defines a cocycle, thus an element in $H^1(B_1) \subset H^1(Z_{j+1})$.

To prove the surjectivity, choose a non-zero element $u_{\text{min}} \in U_j$. Suppose that $\phi_{\alpha \beta} = \eta_\beta - \eta_\alpha$ with $\eta \in B_1$. Then $\varphi(u_{\text{min}}) \in U_j$, hence $u_{\text{min}} \in \gamma(U_j)$ which contradicts Lemma (11.2).

(11.4) Corollary. We have $U_{j+1} = U_j \cap \varphi(U_j)$ and $\dim(U_{j+1}) = \max\{\dim(U_j) - 1, 0\}$.

Proof. The kernel of $c_1 \otimes k : \varphi(U_j) \rightarrow H^1(Z_{j+1})$ equals $U_{j+1}$ by (11.1) and has codimension 1 by (11.3). Since $U_j \neq \varphi(U_j)$, and since their intersection contains $U_{j+1}$ we must have $U_{j+1} = U_j \cap \varphi(U_j)$. The statement about dimensions follows.

If we assume that $\sigma_0 \geq 1$ then we have a strictly increasing sequence

$$\{0\} = U_{\sigma_0+1} \subset U_{\sigma_0} \subset \ldots \subset U_2 \subset U_1$$

and this implies:
**Proposition.** The map $c_1 \otimes k$ factors through an injection

$$\frac{NS(X)}{p\NS(X)} \otimes k \to H^1(Z_{\sigma_0+1}).$$

We can generalize the result of Corollary (11.4).

**Lemma.** We have $\varphi^k(U_j) \cap U_j = U_{j+k}$. In particular $\varphi^{\sigma_0}(U_1) \cap U_1 = \{0\}$.

**Proof.** We prove this by induction on $k$, the case $k = 1$ was proved in (11.4). Suppose it holds for $k$. Then

$$\varphi^{k+1}(U_j) \cap U_j \subset \varphi[\varphi^k(U_{j-1}) \cap U_{j-1}] \subset \varphi(U_{j+k-1})$$

On the other hand we have

$$\varphi(U_{j+k-1}) \cap \varphi^{k+1}(U_j) \subset \varphi(U_j \cap \varphi^k(U_j)) = \varphi(U_{j+k}).$$

But by an easy induction one has

$$\varphi(U_{j+k}) \cap U_j \subset U_{j+k+1}.$$ 

In view of $\dim(\varphi^{k+1}(U_j) \cap U_j) \geq \dim(U_j) - (k + 1)$ the result follows.

**Lemma.** Suppose that $U_1 \neq \{0\}$ and let $u_{\min} \in U_1$. Then $\gamma(u_{\min}) \in U_0 \setminus U_1$. In particular, $(c_1 \otimes k)(\gamma(u_{\min})) \in H^0(\Omega^2) \subset H^1(Z_1) \subset H^1_{\text{dR}}.$

**Proof.** Since $\gamma(u_{\min})$ does not lie in $U_1$, but lies in $U_0$ we see that $(c_1 \otimes k)(\gamma(u_{\min}))$ must lie in the kernel of $H^1(Z_1) \to H^1(\Omega^1)$, which is $H^0(\Omega^2_X)$.

**Lemma.** The Chern class map $c_1 \otimes k : \varphi^m(U_j) \to H^1(Z_{j+m})$ factors through $H^1(B_m)$. For any $t \geq 1$ the natural image of $H^1(B_t)$ in $H^1(Z_{\sigma_0+1})$ is contained in the image of $\frac{NS(X)}{p\NS(X)} \otimes k$ under $c_1 \otimes k$.

**Proof.** As in the proof of (11.3) we can write $(c_1 \otimes k)(u) = \zeta_\beta - \zeta_\alpha$ with $\zeta_x \in Z_j(V_x)$. Now use the local surjectivity of $C$ to write

$$\varphi^m(u) = \tilde{\zeta}_\beta - \tilde{\zeta}_\alpha + \phi_{\alpha\beta}$$

with $\tilde{\zeta} \in Z_{j+m}$, $C^m \tilde{\zeta}_x = \zeta_x$, $\phi_{\alpha\beta} \in B_m$. Then this $\phi_{\alpha\beta}$ defines a cocycle, thus an element in $H^1(B_m) \subset H^1(Z_{j+m})$. This proves the first statement.

We prove the second statement by induction. Note that by (11.3) the image of $H^1(B_1)$ in $H^1(Z_{\sigma_0+1})$ is contained in the image of $\frac{NS(X)}{p\NS(X)} \otimes k$ under $c_1 \otimes k$. Let $\alpha$ be an element of the image of $H^1(B_t)$ and $\beta = C\alpha$ in the image of $H^1(B_{t-1})$. Then $\beta = (c_1 \otimes k)(v)$ for some $v \in NS \otimes k$. But then $\alpha - (c_1 \otimes k)(\varphi(v))$ is an element of $H^1(B_1)$. By induction this is in the image of $(c_1 \otimes k)(NS \otimes k)$. Hence $\alpha$ lies in the image of $(c_1 \otimes k)(NS \otimes k)$.

**Proposition.** Let $\sigma_0 \geq 1$. The dimension of the image of $H^1(B_{\sigma_0})$ in $H^1(Z_1)$ equals $\sigma_0$. The image in $H^1(\Omega^1_X)$ is $\sigma_0 - 1$-dimensional.

**Proof.** The first statement follows directly from (11.6) and (11.8). Arguing similarly for $U_0$ we find that $c_1 \otimes k : \varphi^{\sigma_0}(U_0) \to H^1(\Omega^1_X)$ factors through the natural map $H^1(B_{\sigma_0}) \to H^1(\Omega^1_X)$. The intersection $\varphi^{\sigma_0}(U_0) \cap U_0$ has dimension 1.
\textbf{(11.10) Theorem.} For a K3 surface $X$ with $B_2 = \rho$ and Artin invariant $\sigma_0$, we have $\dim(\operatorname{Im} H^1(Z_{\sigma_0})) = 21 - \sigma_0$ for the image in $H^1(\Omega^1_X)$ and it is generated by Chern classes.

\textit{Proof.} Since we have
\[
\langle c_1(\text{NS}(X)/p\text{NS}(X)) \rangle \subset \operatorname{Im} H^1(Z_{\sigma_0}) \subset (\operatorname{Im} H^1(B_{\sigma_0}))^\perp \subset H^1(\Omega^1_X)
\]
and $\dim \langle c_1(\text{NS}(X)/p\text{NS}(X)) \rangle = \dim (\operatorname{Im} H^1(B_{\sigma_0}))^\perp = 20 - (\sigma_0 - 1)$ by (11.9), we have
\[
\langle c_1(\text{NS}(X)/p\text{NS}(X)) \rangle = \operatorname{Im} H^1(Z_{\sigma_0}) = (\operatorname{Im} H^1(B_{\sigma_0}))^\perp
\]
and so $\dim \operatorname{Im} H^1(Z_{\sigma_0}) = 21 - \sigma_0$. \hfill \Box

Since the codimension of $\operatorname{Im} H^1(Z_{i+1})$ in $\operatorname{Im} H^1(Z_i)$ is at most one, we conclude that
\[
\operatorname{Im} H^1(Z_{\sigma_0}) = \operatorname{Im} H^1(Z_{\sigma_0-1}) \subset \operatorname{Im} H^1(Z_{\sigma_0-2}) \subset \ldots \subset \operatorname{Im} H^1(Z_1) \subset H^1(\Omega^1_X)
\]
and $\operatorname{Im} H^1(Z_n) = \operatorname{Im} H^1(Z_{\sigma_0})$ for $n \geq \sigma_0$. Here, the inclusions are strict inclusions. Moreover, we see that the injection $c_1 \otimes k : \text{NS}(X)/p\text{NS}(X) \otimes k \rightarrow H^1(Z_{\sigma_0+1})$ is an isomorphism:
\[
c_1 \otimes k : \text{NS}(X)/p\text{NS}(X) \otimes k \cong H^1(Z_{\sigma_0+1}).
\]

We now need the following lemma.

\textbf{(11.11) Lemma.} Let $X$ be a K3 surface $X$ with $B_2 = \rho$ and Artin invariant $\sigma_0$. For every $n \geq 0$ the natural map $H^1(Z_{\sigma_0+n+1}) \rightarrow H^1(Z_{\sigma_0+n})$ is surjective.

\textit{Proof.} By Theorem (11.10) the dimension of the image of $H^1(Z_{\sigma_0})$ in $H^1(\Omega^1)$ is $21 - \sigma_0$. By (14) it follows that the image of $H^1(Z_{\sigma_0-1})$ in $H^1(\Omega^1)$ has dimension at least $22 - \sigma_0 - 1$. Since the map $H^1(Z_{\sigma_0+1}) \rightarrow H^1(\Omega^1)$ factors through $H^1(Z_{\sigma_0})$ the map $H^1(Z_{\sigma_0+1}) \rightarrow H^1(Z_{\sigma_0})$ must be surjective.

We now prove that if the natural mapping $H^1(Z_{n+1}) \rightarrow H^1(Z_n)$ is surjective, then so is $H^1(Z_{m+1}) \rightarrow H^1(Z_m)$ for any $m \geq n$. Suppose that the natural homomorphism $H^1(A, Z_{n+1}) \rightarrow H^1(A, Z_n)$ is surjective. By the diagram of exact sequences
\[
\begin{array}{ccccccccc}
0 & \rightarrow & B_1 & \rightarrow & Z_{n+2} & \rightarrow & Z_{n+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & \leftarrow & \iota_{n+2} & \rightarrow & \iota_{n+1} & \\
0 & \rightarrow & B_1 & \rightarrow & Z_{n+1} & \rightarrow & Z_n & \rightarrow & 0
\end{array}
\]
we have a diagram of exact sequences
\[
\begin{array}{cccccccc}
\rightarrow & H^1(X, B_1) & \rightarrow & H^1(X, Z_{n+2}) & \rightarrow & H^1(X, Z_{n+1}) & \rightarrow & H^2(X, B_1) \\
\downarrow & \leftarrow & \iota_{n+2} & \rightarrow & \iota_{n+1} & \\
\rightarrow & H^1(X, B_1) & \rightarrow & H^1(X, Z_{n+1}) & \rightarrow & H^1(X, Z_n) & \rightarrow & H^2(X, B_1).
\end{array}
\]

From this diagram we see that the natural homomorphism $H^1(X, Z_{n+2}) \rightarrow H^1(X, Z_{n+1})$ is also surjective. So this lemma now follows by induction. \hfill \Box
(11.12) **Corollary.** Let $X$ be a K3 surface $X$ with $B_2 = \rho$ and Artin invariant $\sigma_0$. For $n \geq \sigma_0$ we have $\text{Im}(H^1(B_n)) = \text{Im}H^1(Z_n)^\perp$ and $\dim \text{Im}H^1(B_n) = \sigma_0 - 1$.

**Proof.** By the proof of (11.10), we have $\text{Im}H^1(Z_{\sigma_0})^\perp = \text{Im}H^1(B_{\sigma_0})$. Therefore, for $n \geq \sigma_0$, we have

$$\text{Im}H^1(Z_n)^\perp = \text{Im}H^1(B_{\sigma_0}) \subset \text{Im}H^1(B_n).$$

On the other hand, by the proof of (9.3), we have $\text{Im}H^1(Z_n)^\perp \supset \text{Im}H^1(B_n)$. Hence, we get the desired results. □

Since $c_1 \otimes k : NS(X)/pNS(X) \otimes k \longrightarrow H^1(Z_i)$ is injective for $i \geq \sigma_0 + 1$, we have the following proposition.

(11.13) **Proposition.** For a K3 surface $X$ with $B_2 = \rho$ the following four conditions are equivalent.

(i) The natural map $H^1(Z_i) \rightarrow H^1(Z_{i-1})$ is surjective.

(ii) The Cartier operator $C : H^1(Z_i) \rightarrow H^1(Z_{i-1})$ is surjective.

(iii) $\dim H^1(Z_{10}) \geq 31 - i$.

(iv) $\sigma_0 \leq i$.

12. **The Kodaira-Spencer Map**

Let $X_0$ be a K3 surface, and let $\pi : X \longrightarrow S$ be the versal formal $k$-deformation of $X_0$. Then, as is well-known (cf. [D]), we have $S = \text{Spf}[k[[t_1, \ldots, t_{20}]]$ with variables $t_1, \ldots, t_{20}$. We denote by $\nabla$ the Gauss-Manin connection of $H^2_{dR}(X/S)$:

$$\nabla : H^2_{dR}(X/S) \longrightarrow \Omega_{S/k}^1 \otimes H^2_{dR}(X/S).$$

We take a basis $\omega$ of $H^0(X, \Omega^2_{X/S})$. Then, $\nabla$ composed with cup product with $\omega$ gives an isomorphism:

$$\rho_\omega : H^1(X, \Omega^1_{X/S}) \sim \Omega_{S/k}^1.$$ 

We denote by $m$ the maximal ideal of the closed point of $S$. By evaluating $\rho_\omega$ at zero we have an isomorphism:

$$\rho_{\omega,0} : H^1(X_0, \Omega^1_{X_0/k}) \sim m/m^2.$$ 

(12.1) **Remark.** Ogus gave an explicit expression of the isomorphism $\rho_\omega$ as follows. For an element $\alpha \in H^1(X, \Omega^1_{X/S})$ we choose a lifting $\alpha' \in F^1H^2_{dR}(X/S)$ of $\alpha$. Since $\langle \alpha', \omega \rangle = 0$, we have

$$\rho_\omega(\alpha) = \langle \nabla \alpha', \omega \rangle = -\langle \alpha', \nabla \omega \rangle.$$

For details, see the paper by Deligne/Illusie [D], cf. also Ogus [O].

13. **Horizontality**

We consider the moduli space $M = M_{2d}$ of K3 surfaces with a polarization of degree $2d$ in characteristic $p$. Let $(X, D)$ be a polarized K3 surface with a polarization of degree $2d$. The existence of this moduli spaces follows from work of Gieseker. We view these
moduli spaces as algebraic stacks. If the Chern class \(c_1(D)\) is not zero in the de Rham cohomology of \(X\) then the moduli space is formally smooth at \([(X, D)]\).

We shall assume for simplicity that the degree \(2d\) of the polarization is prime to \(p\). Let furthermore \(\pi : \mathcal{X} \to M_{2d}\) be the universal family of polarized K3 surfaces over \(k\). We set
\[
M^{(h)} := \{ s \in M : h(X_s) \geq h \}.
\]
Then, by Artin [A], \(M^{(h)}\) is an algebraic subvariety of codimension \(\leq h - 1\) in \(M\) for \(h = 1, \ldots, 10\). We shall show that their codimension is \(h - 1\).

The direct image sheaves \(R^2\pi_* O_{\mathcal{X}}\) are coherent sheaves of rings, but not coherent \(O_M\)-modules. If there would exist a suitable Grothendieck group of such objects we could calculate Chern classes by using Theorem 5.1. Since we do not know how to do this we resort to a different method to calculate cycle classes of loci of given height.

Let \(X_0\) be a K3 surface, and assume that the height of the formal Brauer group \(\Phi_{X_0}\) is greater than or equal to \(h\), i.e., \(X_0\) corresponds to a point in \(M^{(h)}\). Then, the Frobenius morphism is zero on \(H^2(X, W_i(O_{X/S}))\) for \(i = 1, \ldots, h - 1\). We let \(S\) be a formal neighborhood of \(M^{(h)}\) at the point, and we also denote by \(\nabla\) the Gauss-Manin connection of \(H^2_{dR}(X/S)\). We consider the Hodge filtration \(0 \subset F^2 \subset F^1 \subset H^2_{dR}(X/S)\), and construct, in the same way as in Section 8, a homomorphism
\[
\Phi_h : H^2(W_h(O_X)) \to H^2_{dR}(X).
\]
We take a basis \(\omega\) of \(H^0(O^2_{X/S})\) and take the dual basis \(\zeta\) of \(H^2(O_{X/S})\). We take a lifting \(\tilde{\zeta} \in H^2_{dR}(X/S)\) of \(\zeta\). Then we have \(\langle \tilde{\zeta}, \omega \rangle = 1\). Since \(R^{n-1} : H^2(W_n(O_{X/S})) \to H^2(O_{X/S})\) is surjective, we take an element \(\alpha \in H^2(W_h(O_{X/S}))\) such that \(R^{h-1}(\alpha) = \zeta\). We set
\[
g_h = \langle \Phi_h(\alpha), \omega \rangle.
\]
Since \(\Phi_h(\alpha) - g_h \tilde{\zeta}\) is orthogonal to \(\omega\), it follows that \(\Phi_h(\alpha) - g_h \tilde{\zeta}\) is contained in the \(F^1\)-step of the Hodge filtration. Therefore, using the natural isomorphism \(H^2_{dR}/F^1 \cong H^2(O_X)\), we conclude that
\[
\phi_h(\zeta) = g_h \zeta \quad \text{in } H^2(O_X),
\]
where \(\phi_h\) was defined in section 5. This means that the equation \(g_h = 0\) gives the scheme theoretic locus of zero of \(\phi_h\), and by Theorem 8.2, the support of the locus in \(M^{(h)}\) coincides with \(M^{(h+1)}\).

(13.1) Proposition. Under the notation and assumptions made above, the image \(\text{Im } \Phi_h\) is horizontal with respect to the Gauss-Manin connection.

Proof. It suffices to prove \(\nabla(\Phi_h(\alpha)) = 0\). The element \(\alpha\) is represented by a cocycle \(\alpha_{ijk} = (\alpha_{ijk}^{(0)}, \ldots, \alpha_{ijk}^{(h-1)})\) with respect to a suitable affine open covering \(\{U_i\}\) of \(X/S\). Since the Frobenius morphism is zero on \(H^2(W_{h-1}(O_{X/S}))\), there exist a cochain \(\gamma_{ij} \in \Gamma(U_i \cap U_j, W_{h-1}(O_{X/S}))\) such that \(FR(\alpha) = \partial \{ \gamma_{ij} \} = \{ \gamma_{jk} - \gamma_{ik} + \gamma_{ij} \} (\in C_2(W_{h-1}))\). Hence we have
\[
F(\alpha) - \partial(\{ \gamma_{ij}, 0 \}) = \{(0, \ldots, 0, g_{ijk})\}.
\]
Put $\tilde{\gamma}_{ij} = (\gamma_{ij}, 0)$, an element in $\Gamma(U_i \cap U_j, W_h(O_X/S))$. Then $\phi_h(\zeta) = \{g_{ijk}\}$ and

$$\Phi_h(\alpha) = (g_{ijk}, -D_h(\tilde{\gamma}_{ij}), 0) \in C_2(O_X/S) \oplus C_1(\Omega_X^1/S) \oplus C_0(\Omega_X^2/S).$$

We write this as

$$\Phi_h(\alpha) = \{(g_{ijk}, b_{ij}, 0)\}$$

We have to calculate $\nabla(\Phi_h(\alpha))$. We use the explicit description of the Gauss-Manin connection. Katz and Oda define in [K-O] two operators

$$L_S : C_q(\Omega^p) \to C_q(\Omega^{p+1}), \quad L_S((\beta)(i_0, \ldots, i_q)) = d^i_S(\beta(i_0, \ldots, i_q))$$

and

$$\lambda : C_q(\Omega^p) \to C_{q+1}(\Omega^p), \quad \lambda((i_0, \ldots, i_{q+1})) = (-1)^p(I^0 - I^i)(\beta(i_1, \ldots, i_{q+1})).$$

Here we follow the notation of loc. cit. The (substitution) operator $I^0$ is given by $\sum_{p=1}^\ell \text{sub}(dx_i \mapsto d^{ij}_{S,q})$ and is zero for $p = 0$. In our case this gives $L_S(g_{ijk}) = d^i_S(g_{ijk}) \in C_2(\Omega^1)$, $\lambda(g_{ijk}) = 0$, and $L_S(b_{ij}) = d^i_S(b_{ij}) \in C_1(\Omega^2)$, $\lambda(b_{ij}) = -(I^i - I^j)(b_{jk}) \in C_2(\Omega^1)$. So we find

$$\nabla(\Phi_h(\alpha)) = d^i_S b_{ij} + d^i_S g_{ijk} - I^i b_{jk} + I^j b_{jk}. \quad (16)$$

Here the first term lies in $C_1(\Omega^2)$.

Using $d^i_S$ instead of $d$ we can make an operator $D^i_{h,S}$ similar to the operator $D_h$ defined by Serre. It is zero on the image of Frobenius and so the relation (15) gives

$$-D^i_{h,S}(\partial\{\gamma_{ij}\}) = d^i_S(g_{ijk}).$$

This says $d^i_S(g_{ijk}) = I^i(b_{jk} - b_{ik} + b_{ij})$. Collecting the terms we get

$$\nabla(\Phi_h(\alpha)) = d^i_S b_{ij} + I^i(-b_{ik} + b_{ij}) + I^j b_{jk}.$$ 

Put $c_{ij} = -D_h(\gamma_{ij})$. Now note that we have

$$d^i_S(D_h(\gamma_{ij})) = d(D^i_{h,S}\gamma_{ij}).$$

Therefore the right hand side of (16) is a boundary in the total complex. We conclude $\nabla \Phi_h(\alpha) = 0$ in $\Omega^{1}_{S/k} \otimes H^2_{DR}(X/S)$.

14. The Tangent Spaces to the Stratification

We denote by $D_0$ the polarization class of $X_0$ of degree $2d$ and we shall assume that is not a $p$-th power. Let $M^{(h)}$ be the closed locus of the moduli space $M = M_{2d}$ of polarized K3 surfaces given by the condition height $\geq h$ for $h = 1, \ldots, 10$ and $h = \infty$. We now determine the tangent space of $M^{(h)}$ at the point $x_0 = (X_0, D_0)$. We denote by $\text{Im} \ H^1(X_0, Z_0)$ the image of $H^1(X_0, \Omega_{X_0}^1)$ in $H^1(X_0, \Omega_{X_0}^1)$ induced by the natural inclusion $Z_0 \Omega_{X_0}^1 \to \Omega_{X_0}^1$. 

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Suppose that \((X_0, D_0)\) represents a point \(x_0\) of \(M^{(h)} - M^{(\infty)}\). Then for \(1 \leq h \leq 10\) the tangent space of \(M^{(h)}\) at \(x_0\) is in a natural way isomorphic to 
\[
\text{Im } H^1(X_0, Z_{h-1}) \cap c_1(D_0)^\perp.
\]

Proof. Note that by (9.2) the map \(H^1(X_0, B_{h-1}) \to H^1(X_0, \Omega^1_{X_0})\) is injective. Since we have \(H^1(X_0, B_{h-1}) \subset c_1(D_0)^\perp\), by Corollary (10.2) and Lemma (9.3), it suffices to prove that \(\langle H^1(X_0, B_{h-1}), c_1(D_0) \rangle\) is the normal space of \(M^{(h)}\) at \(x_0\). We will show this by induction. Note that we know \(\dim H^1(X_0, B_\ell) = \ell\) for \(\ell = 0, \ldots, h - 1\).

Suppose \(h = 1\). Then, we have \(H^1(X_0, B_0) = 0\), and by the general theory of moduli spaces the tangent space of \(M^{(1)} = M\) at \(x_0\) is given by \(c_1(D_0)^\perp \subset H^1(X_0, \Omega^1_{X_0})\).

Now, we assume that the statement holds until \(h\). We use the notation above. Then, by (8.2) \(M^{(h+1)}\) is defined by \(g_h = \langle \Phi_h(\alpha), \omega \rangle = 0\) in \(M^{(h)}\). Using Proposition (13.1), we have
\[
dg_h = \langle \nabla \Phi_h(\alpha), \omega \rangle + \langle \Phi_h(\alpha), \nabla \omega \rangle = \langle \Phi_h(\alpha), \nabla \omega \rangle.
\]

We denote by \(m\) (resp. \(m_0\)) the maximal ideal which corresponds to the point \(x_0\) in the versal formal moduli space around \(x_0\) (resp. in the formal moduli around \(x_0\) in \(M^{(h)}\)). Then, under the natural homomorphism
\[
H^1(X_0, \Omega^1_{X_0}) \cong m/m^2 \longrightarrow m_0/m_0^2
\]
\(-\Phi_h(\alpha)(0)\) corresponds to the cotangent vector \(g_h\) by the argument of Ogus [O]. The kernel of this homomorphism is isomorphic to \(\langle H^1(X_0, B_{h-1}), c_1(D_0) \rangle\) by induction. We have
\[
-\Phi_h(\alpha)(0) = -\{D_h(\bar{\gamma}_{ij})\}
= -\{\sum_{m=0}^{h-1}(\gamma_{ij}^{(m)})^{p^h-m-1} d \log \gamma_{ij}^{(m)}\}.
\]

and \(D_h : H^2(W_h(O_{X_0})/FW_h(O_{X_0}) \to H^1(X_0, \Omega^1_{X_0})\) is injective by Corollary (9.2). Since \(\Phi(\alpha)(0)\) lies in \(H^1(X_0, B_h)\) but not in \(H^1(X_0, B_{h-1})\), we conclude that \(g_h \notin m_0^2\).

By induction we thus see that the tangent space to \(M^{(h+1)}\) can be identified with \(H^1(X_0, Z_h) \cap c_1(D_0)^\perp\). □

This argument does not work for \(h = \infty\), but can be made to work for the supersingular points for which the subspace \(\langle \text{Im}(H^1(B_h)), c_1(D) \rangle\) of \(H^1(\Omega^1)\) has dimension \(h\). In Section 12 we gave conditions for this. Under the assumption that \(\rho = B_2\) this is the case if the Artin invariant \(\sigma_0\) of a supersingular K3 surface satisfies \(\sigma_0 > h\). We thus find:

(14.2) Theorem. For \(h = 1, \ldots, 10\) the open stratum \(M^{(h)}\) if not empty is purely of dimension \((20 - h)\) and nonsingular at any point of the stratum \(M^{(h)}\) where the subspace \(\langle \text{Im}(H^1(B_{h-1})), c_1(D_0) \rangle\) of \(H^1(\Omega^1_{X_0})\) has dimension \(h\). In particular, it is non-singular at non-supersingular points and assuming the Artin conjecture at all supersingular points with Artin invariant \(\sigma_0 \geq h\) and \(c_1(D_0) \notin \text{Im}(H^1(B_h))\).

We refer here to a forthcoming preprint of Ogus for a description of the singularities of the strata. Ogus proved in [O, Prop. 2.6] that for \(p \neq 2\) the stratum \(M^{(2)}\) has a quadratic singularity at the superspecial points. A variation of his argument there
Hence the locus is reduced for $h^0 = h - 1$ the singular locus has multiplicity 2. In particular the stratum $M^{(11)}$ has multiplicity 2 at points with $h_0 = 10$, cf. his forthcoming preprint and the discussion in the next section.

15. The Loci of K3 Surfaces of Given Height

We now come to the description of the cycle classes of the strata defined by the height. Let $M^{(h)}$ be the closed stratum of the moduli space $M = M_{2d}$ where the height of the formal group $\Phi_X$ is at least $h$ with the convention that $M^{(11)} = M^{(\infty)}$. For simplicity we shall assume that $p$ does not divide $2d$. By our characterization of $h$ these strata can be given a natural scheme structure and these are reduced for $h \neq \infty$ by our results in Section 14. It is known by Artin that the strata $M^{(h)}$ for $h = 1, \ldots, 11$ have codimension $\leq h - 1$ in $M_{2d}$; see [A].

Define a line bundle $V$ on $M$ by $V = \pi^*(\Omega^2_X/M)$ and let the first Chern class be $v$.

(15.1) **Theorem.** Let $M = M_{2d}$ be the moduli stack of polarized K3 surfaces over $k$ with a polarization of degree $2d$ prime to $p$. Then for $h = 1, \ldots, 10, 11$ the scheme-theoretic locus $M^{(h)}$ of surfaces with height $\geq h$, if not empty, is of codimension $h - 1$ and for $h \neq 11$ it is a local complete intersection. The class of $M^{(h)}$ in the Chow group $CH^{h-1}_Q(M)$ is given by

$$(p - 1)(p^2 - 1) \ldots (p^{h-1} - 1)v^{h-1}.$$

**Proof.** We prove this by induction. Let $M$ be the moduli space of polarized K3 surfaces of degree $2d$ as above. We know that the generic K3 surface has height 1, and so for $h = 1$ the formula is correct. The codimension of $M^{(h)}$ is $\leq h - 1$ for $1 \leq h \leq 10$ as follows from (5.7). For $h = 2$ the locus $M^{(2)}$ is the non-ordinary locus. This locus is characterized by the fact that the Frobenius map $H^2(X, O_X) \to H^2(X, O_X)$ vanishes. This is a $p$-linear map and the corresponding $O_M$-linear map is $(R^2\pi_*O_X)^{(p)} \to R^2\pi_*O_X$ with associate cycle class $(p - 1)v$. Locally, at a point of $M^{(2)}$ an equation is given by $g_1 = 0$, see the proof of (14.1) and $dg_1 \neq 0$. So if $M^{(2)}$ is not empty then it is purely 18-dimensional.

Suppose now that the class of $M^{(h-1)}$ is given by the class in the formula. By Proposition (5.7) the locus in $M^{(h-1)}$ where the height increases is given by the vanishing of the map $\phi_{h-1} : (R^2\pi_*O_X)^{(p^{h-1})} \to R^2\pi_*O_X$, equivalently, by the vanishing of a section of $V^{p^{h-1}-1}$. By (14.1) it follows that for a local equation $g_h = 0$ we have $dg_h \neq 0$. Hence the locus is reduced for $h \neq \infty$ and the class on $M^{(h-1)}$ is given by $(p^{h-1} - 1)v$.

Let $j_h : M^{(h)} \to M^{(h-1)}$ and $j : M^{(h-1)} \to M$ be the natural inclusions. Then the class of the locus $M^{(h)}$ in $CH^{h-1}_Q(M)$ is given by

$$j_*j^* M^{(h)} = j_* ([M^{(h-1)}] \cdot j^*_{h-1}(p^{h-1} - 1)v) = (p^{h-1} - 1)v \cdot (j_* [M^{(h-1)}])$$

by the projection formula. □

The locus $M^{(11)}$ comes with a multiplicity in the formula because of (14.2). For $p \neq 2$ the multiplicity is 2. It makes sense to call the reduced locus $M^{(11)}_{\text{red}}$ the supersingular locus.
(15.2) Remark. In [G] a formula for the class of the supersingular locus on the moduli space of principally polarized abelian surfaces was given. Comparison with Kummer surfaces shows that this is compatible with multiplicity 2 along the supersingular locus, cf. [G-K].

We shall now assume that the line bundle $V = \pi^*(\Omega^2_{X/M})$ is ample on the moduli space. It is known by the theory of Baily and Borel (see [B-B]) that $V$ is ample on the moduli spaces in characteristic 0; indeed, modular forms of sufficiently high weight define an embedding.

(15.3) Theorem. Suppose that the class $v$ is ample. Let $X \to S$ with $S$ complete be a proper smooth family of polarized K3 surfaces with constant $h \neq \infty$. Then this family is isotrivial.

Proof. It follows from the preceding theorem that the strata $S^{(h)} - S^{(h+1)}$ where the height is constant are quasi-affine for $h = 1, \ldots, 10$.

We do not know whether the class $v$ is ample on the moduli spaces $M_{2d}$, but we expect it to be so.

Suppose that there exists a good Baily-Borel compactification. By this we mean that there exists a projective variety (stack) $\overline{M}_{2d}$ containing $M_{2d}$ such that $\overline{M}_{2d} - M_{2d}$ is 1-dimensional and consists of a configuration of elliptic modular curves. This is the case in characteristic zero, cf. Kondo [Ko]. Then it follows from our theorem that a family of K3 surfaces with $h \geq 3$ does not degenerate. Indeed, it follows from our formula that a class of the form $v^m$ with $m \geq 3$ has zero intersection with the ‘boundary components’. This implies that for each boundary component the locus with $h \geq 3$ either has empty intersection with this boundary component or contains it. The boundary components form a connected set and the generic point of each component corresponds to a degenerate K3 surface corresponding to an ordinary elliptic curve. For the degenerate surfaces the height is 1 or 2. Compare the discussion in [R-Z-Sh].

16. An Extension for Other Varieties

Though the theorem in Section 5 was formulated for K3 surfaces it holds for a more general class of surfaces.

(16.1) Theorem. Suppose that $X$ is a smooth algebraic surface such that

i) $\text{Pic}^0(X)$ is reduced,

ii) $\dim H^2(X, O_X) = 1$.

Then $\Phi^2$ is represented by a formal group of dimension 1 and its height satisfies $h(\Phi_X) \geq i + 1$ if and only if the Frobenius map $F$ on $H^2(X, W_i(O_X))$ is the zero map.

(16.2) Corollary. For such a surface we have the following characterization of the height:

$$h(\Phi_X) = \min\{i \geq 1 : [F : H^2(W_i(O_X)) \to H^2(W_i(O_X))]) \neq 0\}.$$ 

Proof. The proof is analogous to the proof given for K3 surfaces. Instead of the vanishing of $H^1(X, O_X)$ one uses the vanishing of the Bockstein operators. Recall
that $H^1(X, W_n(O_X))$ is the subgroup of $k[\epsilon]/(\epsilon^{n+1})$-valued points of the connected component of the Picard scheme $P$ at the origin, cf. [Mu]. A $k[\epsilon]/\epsilon^2$-valued point (tangent vector) is tangent to $P_{\text{red}}$ at the origin if and only if it can be lifted to $k[\epsilon]/(\epsilon^n)$-valued point for all $n$. That is, these correspond precisely to the elements of $H^1(X, O_X)$ that can be lifted to $H^1(X, W_n(O_X))$ for all $n$. So if $P = P_{\text{red}}$ then all elements of $H^1(X, O_X)$ can be lifted and this implies the analogues of Lemmas (4.2) and (4.5) that we need.

16.3) Example. 1) An abelian surface satisfies the assumptions. 2) A surface of general type with $H^1(O_X) = 0$ and $p_g = 1$. Examples of such surfaces are surfaces with $K^2 = p_g = 1$. These have $h^1(X, O_X) = 0$ and are resolutions of surfaces of type (6,6) in weighted projective space $\mathbb{P}(1, 2, 2, 3, 3)$, cf. [C].

A nonsingular complete algebraic variety $X$ of dimension $n$ is called a Calabi-Yau variety if the canonical invertible sheaf $\omega_X$ is trivial and $H^i(X, O_X) = 0$ for $1 \leq i \leq n - 1$. By a criterion of Artin-Mazur [A-M], the Artin-Mazur formal group $\Phi^n$ is pro-representable by a one-dimensional formal Lie group for such a variety. In the same way as in Section 5, we have also a characterization of the height of the formal group $\Phi^n$.

(16.4) Proposition. For a Calabi-Yau variety $X$ of dimension $n$ we have the following characterization of the height:

$$h(\Phi^n_X) = \min\{i \geq 1 : [F : H^n(W_i(O_X)) \rightarrow H^n(W_i(O_X))] \neq 0\}.$$ 

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