Clarifying Slow Roll Inflation and the Quantum Corrections to the Observable Power Spectra

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Abstract

Slow-roll inflation can be studied as an effective field theory. The form of the inflaton potential consistent with the data is

\[ V(\phi) = N M^4 w \left( \frac{\phi}{N M_{Pl}} \right) \]

where \(\phi\) is the inflaton field, \(M\) is the inflation energy scale, and \(N \sim 50\) is the number of efolds since the cosmologically relevant modes exited the Hubble radius until the end of inflation. The dimensionless function \(w(\chi)\) and field \(\chi\) are generically \(O(1)\). The WMAP value for the amplitude of scalar adiabatic fluctuations yields \(M \sim 0.77 \times 10^{16}\) GeV. This form of the potential encodes the slow-roll expansion as an expansion in \(1/N\). A Ginzburg-Landau (polynomial) realization of \(w(\chi)\) reveals that the Hubble parameter, inflaton mass and non-linear couplings are of the see-saw form in terms of the small ratio \(M/M_{Pl}\). The quartic coupling is \(\lambda \sim \frac{1}{N} \left( \frac{M}{M_{Pl}} \right)^4\). The smallness of the non-linear couplings is not a result of fine tuning but a natural consequence of the validity of the effective field theory and slow roll approximation. Our observations suggest that slow-roll inflation may well be described by an almost critical theory, near an infrared stable gaussian fixed point. Quantum corrections to slow roll inflation are computed and turn to be an expansion in powers \((H/M_{Pl})^2\). The corrections to the inflaton effective potential and its equation of motion are computed, as well as the quantum corrections to the observable power spectra. The near scale invariance of the fluctuations introduces a strong infrared behavior naturally regularized by the slow roll parameter \(\Delta \equiv \eta V - \epsilon V = \frac{1}{2} (n_s - 1) + r/8\). We find the effective inflaton potential during slow roll inflation including the contributions from scalar curvature and tensor perturbations as well as from light scalars and Dirac fermions coupled to the inflaton. The scalar and tensor superhorizon contributions feature infrared enhancements regulated by slow roll parameters. Fermions and gravitons do not exhibit infrared enhancement. The subhorizon part is completely specified by the trace anomaly of the fields with different spins and is solely determined by the space-time geometry. This inflationary effective potential is strikingly different from the usual Minkowski space-time result. Quantum corrections to the power spectra are expressed in terms of the CMB observables: \(n_s\), \(r\) and \(dn_s/d\ln k\). Trace anomalies (especially the graviton part) dominate these quantum corrections in a definite direction: they enhance the scalar curvature fluctuations and reduce the tensor fluctuations.

1 Inflation as an Effective Field Theory

Inflation was originally proposed to solve several outstanding problems of the standard Big Bang model \[1\] thus becoming an important paradigm in cosmology. At the same time, it provides a natural mechanism for the generation of scalar density fluctuations that seed large scale structure, thus explaining the origin of the temperature anisotropies in the cosmic microwave background (CMB), as well as that of tensor perturbations (primordial gravitational waves). A distinct aspect of inflationary perturbations...
is that these are generated by quantum fluctuations of the scalar field(s) that drive inflation. Their physical wavelengths grow faster than the Hubble radius and when they cross the horizon they freeze out and decouple. Later on, these fluctuations are amplified and grow, becoming classical and decoupling from causal microphysical processes. Upon re-entering the horizon, during the matter era, these scalar (curvature) perturbations induce temperature anisotropies imprinted on the CMB at the last scattering surface and seed the inhomogeneities which generate structure upon gravitational collapse.\cite{2, 3}. Generic inflationary models predict that these fluctuations are adiabatic with an almost scale invariant spectrum. Moreover, they are Gaussian to a very good approximation. These generic predictions are in spectacular agreement with the CMB observations as well as with a variety of large scale structure data.\cite{4}. The WMAP data\cite{4} clearly display an anti-correlation peak in the temperature-polarization (TE) angular power spectra at $l \sim 150$, providing one of the most striking confirmations of superhorizon adiabatic fluctuations as predicted by inflation.\cite{4}.

The robust predictions of inflation (value of the entropy of the universe, solution of the flatness problem, small adiabatic Gaussian density fluctuations explaining the CMB anisotropies, ...) which are common to the available inflationary scenarios, show the predictive power of the inflationary paradigm. While there is a great diversity of inflationary models, they predict fairly generic features: a gaussian, nearly scale invariant spectrum of (mostly) adiabatic scalar and tensor primordial fluctuations. More precisely, the WMAP\cite{4} data can be fit outstandingly well by simple single field slow roll models. These generic predictions of inflationary models make the inflationary paradigm robust. Whatever the microscopic model for the early universe (GUT theory) would be, it should include inflation with the generic features we know today.

Inflationary dynamics is typically studied by treating the inflaton as a homogeneous classical scalar field whose evolution is determined by its classical equation of motion, while the inflaton quantum fluctuations (around the classical value and in the Gaussian approximation) provide the seeds for the scalar density perturbations of the metric. The classical dynamics of the inflaton (a massive scalar field) coupled to a cosmological background clearly shows that inflationary behaviour is an \textit{attractor}\cite{5}. This is a generic and robust feature of inflation.

In quantum field theory, this classical inflaton corresponds to the expectation value of a quantum field operator in a translational invariant state. Important aspects of the inflationary dynamics, as resonant particle production and the nonlinear back-reaction that it generates, require a full quantum treatment of the inflaton for their consistent description. The quantum dynamics of the inflaton in a non-perturbative framework and its consequences on the CMB anisotropy spectrum were treated in refs.\cite{6, 7, 8, 9}. The quantum fluctuations are of two different kinds: (a) Large quantum fluctuations generated at the beginning of inflation through spinodal or parametric resonance depending on the inflationary scenario chosen. They have comoving wavenumbers in the range of $10^{13}$GeV $< k < 10^{15}$GeV and they become superhorizon a few efolds after the beginning of inflation. Their physical wavenumbers become subsequently very small compared with the inflaton mass. Therefore, the assembly of these modes can be treated as part of the zero mode after $5 - 10$ efolds\cite{6, 7}. That is, the use of an homogeneous classical inflaton is thus justified by the full quantum theory treatment of the inflaton. (b) Small fluctuations of high comoving wavenumbers

$$e^{N_T - 60} 10^{16} \text{GeV} < k < e^{N_T - 60} 10^{20} \text{GeV}$$

where $N_T \geq 60$ stands for the total number of efolds (see for example Ref.\cite{10}). These are the cosmologically relevant modes that exit the horizon about 50 efolds before the end of inflation and reenter later on (during the matter dominated era) being the source of primordial power for the CMB anisotropies as well as for the structure formation. While modes (b) obey linear evolution equations with great accuracy, modes (a) strongly interact with themselves calling for nonperturbative quantum field theory treatments as in refs.\cite{6, 7}. Notice that particle production is governed by linear uninstabilities (parametric or spinodal) only at the beginning of inflation. Particle production keeps strong during the nonlinear regime till particles eventually dominate the energy density and inflation stops\cite{9}. The modes (b) correspond to physical scales that were microscopic (even transplanckian) at the beginning of inflation, then after they become astronomical and produce the CMB anisotropies as well as the large scale structure of the universe.

The crucial fact is that the excitations can cross the horizon twice, coming back and bringing information from the inflationary era. We depict in fig.\cite{11} the physical wavelengths of modes (a) and (b) vs.
the logarithm of the scale factor showing that modes (b) crossed twice the horizon, modes (a) are out of the horizon still today.

Figure 1: Physical lengths $\lambda = a(t) \lambda_{\text{comoving}}$ vs. the scale factor $a(t)$ in a log-log plot. The causal horizon $d_H$ is shown for the inflationary (de Sitter), radiation dominated and matter dominated eras. The physical wavelengths ($\lambda$) for today’s Hubble radius and a typical galactic scale ($\lambda_{\text{gal}}$) are shown. One sees the modes (b) can cross the horizon twice bringing information from extremely short wavelength modes during the inflationary era. Modes (a) which have large amplitudes during inflation and dominated the energy of the universe, have not yet crossed back inside the horizon.

Recently, particle decay in a de Sitter background as well as during slow roll inflation has been studied in ref.[11] together with its implication for the decay of the density fluctuations. Quantum effects during slow roll inflation including quantum corrections to the effective inflaton potential and its equation of motion are derived in ref.[9, 12]. Recent studies of quantum corrections during inflation[9, 11, 12] revealed the robustness of classical single field slow roll inflationary models as a result of the validity of the effective field theory description. The reliability of an effective field theory of inflation hinges on a wide separation between the energy scale of inflation, determined by $H$ and that of the underlying microscopic theory which is taken to be the Planck scale $M_{\text{Pl}}$. Inflation as known today should be considered as an effective theory, that is, it is not a fundamental theory but a theory of a condensate (the inflaton field) which follows from a more fundamental one (the GUT model). The inflaton field is just an effective description while the microscopic description should come from the GUT model in the cosmological spacetime. Such derivation is not yet available.

Bosonic fields do not need to be fundamental fields, for example they may emerge as condensates of fermion-antifermion pairs $<\bar{\Psi}\Psi>$ in a grand unified theory (GUT) in the cosmological background. In order to describe the cosmological evolution is enough to consider the effective dynamics of such condensates. The relation between the low energy effective field theory of inflation and the microscopic fundamental theory is akin to the relation between the effective Ginzburg-Landau theory of superconductivity and the microscopic BCS theory, or like the relation of the $O(4)$ sigma model, an effective low energy theory of pions, photons and chiral condensates with quantum chromodynamics (QCD)[13]. The guiding principle to construct the effective theory is to include the appropriate symmetries[13]. Contrary to the sigma model where the chiral symmetry strongly constraints the model[13], only general covariance can be imposed on the inflaton model.

The inflationary scenarios are usually characterized as small and large fields scenarii. In small fields scenarios the initial classical amplitude of the inflaton is assumed small compared with $M_{\text{Pl}}$ while in large field scenarii the inflaton amplitude is initially of the order $M_{\text{Pl}}$. The first type of scenarii is usually
realized with broken symmetric potentials ($m^2 < 0$) while for the second type scenarios (‘chaotic inflation’) one can just use unbroken potentials with $m^2 > 0$.

Gravity can be treated semiclassically for inflation: the geometry is classical and the metric follows from the Einstein-Friedman equations where the r.h.s. is the expectation value of a quantum operator. Quantum gravity corrections can be neglected during inflation because the energy scale of inflation $M \sim M_{\text{GUT}} \sim 10^{-3} M_{\text{Planck}}$. That is, quantum gravity effects are at most $(M/M_{\text{Planck}})^2 \sim 10^{-6}$ and can be neglected in this context. The studies in ref. $^{[11, 12, 14]}$ reveal that quantum corrections in the effective field theory yields an expansion in $(H M_{\text{Pl}})^2$ for general inflaton potentials. This indicates that the use of the inflaton potential $V(\phi)$ from effective field theory is consistent for $(H M_{\text{Pl}})^2 \ll 1$ and hence $V(\phi) \ll M_{\text{Pl}}^4$, allowing amplitudes of the inflaton field $\phi$ well beyond $M_{\text{Pl}}$ $^{[14]}$.

1.1 Slow-roll Inflation as an expansion in $1/N_{\text{efolds}}$ and no fine tuning

In single field inflation the energy density is dominated by a homogeneous scalar condensate, the inflaton, whose dynamics can be described by an effective Lagrangian

$$\mathcal{L} = a^3(t) \left[ \frac{\dot{\phi}^2}{2} - \frac{(\nabla \phi)^2}{2 a^2(t)} - V(\phi) \right],$$

(1)

together with the Einstein-Friedman equation

$$\left[ \frac{1}{a(t)} \frac{da}{dt} \right]^2 = \frac{\rho(t)}{3 M_{\text{Pl}}^2},$$

(2)

where the energy density for an homogeneous inflaton is given by

$$\rho(t) = \frac{\dot{\phi}^2}{2} + V(\phi).$$

The inflaton potential $V(\phi)$ is a slowly varying function of $\phi$ in order to permit a slow-roll solution for the inflaton field $\phi(t)$. All this in a spatially flat FRW metric

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2 = C^2(\eta) [d\eta^2 - d\vec{x}^2],$$

(3)

where $\eta$ is the conformal time and $C(\eta) \equiv a(t(\eta))$.

The inflaton evolution equation takes the form,

$$\ddot{\phi} + 3 H \dot{\phi} + V'(\phi) = 0.$$

In the slow-roll approximation $\ddot{\phi} \ll 3 H \dot{\phi}$ and $\frac{\dot{\phi}^2}{2} \ll V(\phi)$, and the inflaton evolution equations become

$$3 H(t) \dot{\phi} + V'(\phi) = 0,$$

where $H(t) \equiv \frac{1}{a(t)} \frac{da}{dt}$ stands for the Hubble parameter. Eq. $^{[1]}$ can be integrated by quadratures with the result

$$\ln a(t) = - \frac{1}{M_{\text{Pl}}} \int_{\phi(0)}^{\phi(t)} \frac{V(\phi)}{V'(\phi)} d\phi.$$

This formula shows that the inflaton field $\phi$ scales as $M_{\text{Pl}}$ and as the square root of the number of e-folds $^{[14]}$. This suggest to introduce the dimensionless field $\chi$ and the dimensionless potential $w(\chi)$,

$$\chi \equiv \frac{\phi}{\sqrt{N} M_{\text{Pl}}}, \quad w(\chi) \equiv \frac{V(\phi)}{N M^4}$$

(5)
where $M$ stands for the scale of inflation. The dimensionless field $\chi$ is slowly varying during the stage of slow roll inflation: a large change in the field amplitude $\phi$ results in a small change in $\chi$ amplitude,

$$\Delta \chi = \frac{1}{\sqrt{N}} \frac{\Delta \phi}{M_{Pl}},$$  \hspace{1cm} (6)$$
a change in $\phi$ with $\Delta \phi \sim M_{Pl}$ results in a change $\Delta \chi \sim 1/\sqrt{N}$.

Introducing a stretched (slow) dimensionless time variable $\tau$ and a rescaled dimensionless Hubble parameter $h$ as follows

$$t = \sqrt{N} \frac{M_{Pl}}{M^2} \tau, \hspace{1cm} H = \sqrt{N} \frac{M^2}{M_{Pl}} h,$$  \hspace{1cm} (7)$$
the Einstein-Friedman equation reads

$$h^2(\tau) = \frac{1}{3} \left[ \frac{1}{2N} \left( \frac{d\chi}{d\tau} \right)^2 + w(\chi) \right],$$  \hspace{1cm} (8)$$
and the evolution equation for the inflaton field $\chi$ is given by

$$\frac{1}{N} \frac{d^2 \chi}{d\tau^2} + 3h \frac{d\chi}{d\tau} + w'(\chi) = 0.$$  \hspace{1cm} (9)$$
The slow-roll approximation follows by neglecting the $\frac{1}{N}$ terms in eqs.(8) and (9). Both $w(\chi)$ and $h(\tau)$ are of order $N^0$ for large $N$. Both equations make manifest the slow roll expansion as a systematic expansion in $1/N$.[14]

Following the spirit of the Ginsburg-Landau theory of phase transitions, the simplest choice is a quartic trinomial for the inflaton potential[15]:

$$w(\chi) = w_0 \pm \frac{1}{2} \chi^2 + \frac{G_3}{3} \chi^3 + \frac{G_4}{4} \chi^4.$$  \hspace{1cm} (10)$$
where the coefficients $w_0$, $G_3$ and $G_4$ are of order one and the signs $\pm$ correspond to large and small field inflation, respectively. Inserting eq.(10) in eq.(5) yields,

$$V(\phi) = V_0 \pm \frac{m^2}{2} \phi^2 + \frac{m g}{3} \phi^3 + \frac{\lambda}{4} \phi^4.$$  \hspace{1cm} (11)$$
where $m$, $g$ and $\lambda$ are given by the following see-saw-like relations,

$$m = \frac{M^2}{M_{Pl}} \hspace{1cm} g = \frac{G_3}{\sqrt{N}} \left( \frac{M}{M_{Pl}} \right)^2 \hspace{1cm} \lambda = \frac{G_4}{N} \left( \frac{M}{M_{Pl}} \right)^4.$$  \hspace{1cm} (12)$$
We can now input the results from WMAP[4] to constrain the scale $M$. The amplitude of adiabatic scalar perturbations in slow-roll is expressed as

$$|\Delta_{k_{ad}}^{(S)}|^2 = \frac{1}{12 \pi^2 M_{Pl}^6} \frac{V^3}{V''} = \frac{N^2}{12 \pi^2} \left( \frac{M}{M_{Pl}} \right)^4 \frac{w^3(\chi)}{w'^2(\chi)},$$  \hspace{1cm} (13)$$
Since, $w(\chi)$ and $w'(\chi)$ are of order one, we find

$$\left( \frac{M}{M_{Pl}} \right)^2 \sim \frac{2 \sqrt{3 \pi}}{N} |\Delta_{k_{ad}}^{(S)}| \simeq 1.02 \times 10^{-5}.$$  \hspace{1cm} (14)$$
where we used $N \simeq 50$ and the WMAP value for $|\Delta_{k_{ad}}^{(S)}| = (4.67 \pm 0.27) \times 10^{-5}$[4]. This fixes the scale of inflation to be

$$M \simeq 3.19 \times 10^{-3} \hspace{1cm} M_{Pl} \simeq 0.77 \times 10^{16} \text{GeV}.$$  \hspace{1cm} (15)$$
This value pinpoints the scale of the potential during inflation to be at the GUT scale suggesting a deep connection between inflation and the physics at the GUT scale in cosmological space-time.
That is, the WMAP data fix the scale of inflation $M$ for single field potentials with the form given by eq. [15]. This value for $M$ is below the WMAP upper bound on the inflation scale $3.3 \times 10^{16}$ GeV [14]. Furthermore, the Hubble scale during (slow roll) inflation and the inflaton mass near the minimum of the potential are thereby determined from eqs. [17] and [18] to be $m = \frac{H_{Pl}^2}{M} = 2.45 \times 10^{13}$ GeV, $H = \sqrt{\frac{N}{m}} h \sim 1.0 \times 10^{14}$ GeV = $4.1$ $m$ since $h = \mathcal{O}(1)$. In addition, the order of magnitude of the couplings naturally follows from eq. [12]: $g \sim 10^{-6}$, $\lambda \sim 10^{-12}$, since $M/M_{Pl} \sim 3 \times 10^{-3}$.

Since $M/M_{Pl} \sim 3 \times 10^{-3}$, these relations are a natural consequence of the validity of the effective field theory and of slow roll and relieve the fine tuning problem. We emphasize that the `see-saw-like’ form of the couplings is a natural consequence of the form of the potential eq. [15]. While the hierarchy between the Hubble parameter, the inflaton mass and the Planck scale during slow roll inflation is well known, our analysis reveals that small couplings are naturally explained in terms of powers of the ratio between the inflationary and Planck scales and integer powers of $1/\sqrt{N}$.

This is one of the main results presented in this lecture: the effective field theory and slow roll descriptions of inflation, both validated by WMAP, lead us to conclude that there is no fine tuning problem [14]. The smallness of the inflaton mass and the coupling constants in this trinomial realization of the inflationary potential is a direct consequence of the validity of both the effective field theory and the slow roll approximations through a see-saw-like mechanism.

It must be stressed that these order of magnitude estimates follow from the statement that $w(\chi)$ and $\chi$ are of order one. They are thus independent of the details of the model. Indeed, model-dependent factors of order one appearing in the observables should allow to exclude or accept a given model by using the observational data.

The WMAP results rule out the purely quartic potential ($m = 0, g = 0$). From the point of view of an effective field theory this is not surprising: it is rather unnatural to set $m = 0$, since $m = 0$ is a particular point at which the correlation length is infinite and the theory is critical. Indeed the systematic study in ref. [15] shows that the best fit to the WMAP data requires $m^2 \neq 0$.

The general quartic Lagrangian eq. [20] describes a renormalizable theory. However, one can choose in the present context arbitrary high order polynomials for $V(\phi)$. These nonrenormalizable models are also effective theories where $M_{Pl}$ plays the rôle of UV cutoff. However, already a quartic potential is rich enough to describe the full physics and to reproduce accurately the WMAP data [15].

For a general potential $V(\phi)$,

$$V(\phi) = \sum_{n=0}^{\infty} \lambda_n \phi^n \quad \text{i.e.} \quad w(\chi) = \sum_{n=0}^{\infty} G_n \chi^n,$$  

we find from eq. [5]:

$$\lambda_n = \frac{G_n}{(N M_{Pl}^2)^{\frac{2}{N} - 1}},$$

where the dimensionless coefficients $G_n$ are of order one. We find the scaling behavior $\lambda_n \sim 1/N^{\frac{2}{N} - 1}$. Eq. [12] displays particular cases of eq. [16] for $n = 3$ and 4.

The slow-roll parameters naturally result of the order $1/N$, $1/N^2$ etc. when expressed in terms of the inflaton potential

$$V(\phi) = N M^4 w(\chi).$$

That is,

$$\epsilon_V = \frac{M_{Pl}^2}{2} \left[ \left( \frac{\phi}{\Phi_0} \right) \right]^2 = \frac{1}{2N} \left[ \frac{w'(\chi)}{w(\chi)} \right]^2, \quad \eta_V = M_{Pl}^2 \frac{V''(\Phi_0)}{V(\Phi_0)} = \frac{1}{N} \frac{w''(\chi)}{w(\chi)},$$

The spectral index $n_s$, its running and the ratio of tensor to scalar fluctuations are expressed as

$$n_s - 1 = -\frac{3}{N} \left[ \frac{w'(\chi)}{w(\chi)} \right]^2 + \frac{2}{N} \frac{w''(\chi)}{w(\chi)},$$
\[
\frac{dn_s}{d\ln k} = -\frac{2}{N^2} \frac{w'\chi}{w^2\chi} - \frac{6}{N^2} \frac{[w'\chi]^4}{w^4\chi} + \frac{8}{N^2} \frac{[w'\chi]^2}{w^3\chi},
\]
\[
r = \frac{8}{N} \left[\frac{w'\chi}{w\chi}\right]^2.
\]

In eqs. (13), (18) and (19), the field \(\chi\) is computed at horizon exiting. We choose \(N[\chi] = N = 50\). The WMAP results favoured single inflaton models and among them new and hybrid inflation emerge to be preferable than chaotic inflation\(^\text{[15]}\).

The inflationary era ends when the particles produced during inflation dominate the energy density overcoming the vacuum energy. At such stage the universe slows down its expansion to a radiation dominated regime.

### 1.2 Connection with Supersymmetry

The form of the inflaton potential
\[
V(\phi) = N M^4 w(\chi)
\]
(20)

resembles (besides the factor \(N\)) the moduli potential arising from supersymmetry breaking
\[
V_{\text{susy}}(\phi) = m^4_{\text{susy}} v \left(\frac{\phi}{M_{\text{Pl}}}\right),
\]
(21)

where \(m_{\text{susy}}\) stands for the supersymmetry breaking scale. In our context, eq. (21) indicates that \(m_{\text{susy}} \approx M \approx M_{\text{GUT}}\). That is, the susy breaking scale \(m_{\text{susy}}\) turns out to be at the GUT and inflation scales. This may be a first observational indication of the presence of supersymmetry. It should be noticed that this supersymmetry scale is unrelated to an eventual existence of supersymmetry at the TeV scale.

Notice that the invariance of the basic interactions (the lagrangian) and the invariance of the physical states (or density matrices) describing the matter are different issues. For example, no thermal state at non-zero temperature can be invariant under supersymmetry since Bose-Einstein and Fermi-Dirac distributions are different. More generally, the inflationary stage is described by a scalar condensate (the inflaton) while fermions cannot condense due to Pauli principle. This makes quite hard to uncover supersymmetric properties of the lagrangian from the physics of the early universe.

### 1.3 Conjecture: inflation is near a trivial infrared fixed point

There are several remarkable features and consistency checks of the relations\(^\text{[12]}\):

- Note the relation \(\lambda \sim g^2\). This is the correct consistency relation in a renormalizable theory because at one loop level there is a renormalization of the quartic coupling (or alternatively a contribution to the four points correlation function) of orders \(\lambda^2\), \(g^4\) and \(\lambda^3 g^2\) which are of the same order for \(\lambda \sim g^2\). Similarly, at one loop level there is a renormalization of the cubic coupling (alternatively, a contribution to the three point function) of orders \(g^3\) and \(\lambda g\) which again require \(g^2 \sim \lambda\) for consistency.

- In terms of the effective field theory ratio \((H/M_{\text{Pl}})^2\) and slow roll parameters, the dimensionless couplings are

\[
\frac{m}{H} \sim \frac{1}{N} \frac{H}{M_{\text{Pl}}}, \quad \lambda \sim \frac{1}{N^2} \left(\frac{H}{M_{\text{Pl}}}\right)^2.
\]

These relations agree with those found for the dimensionless couplings in ref.\(^\text{[11, 12]}\) once the slow roll parameters are identified with the expressions\(^\text{[18]}\) in terms of \(1/N\). The results of refs.\(^\text{[11, 12]}\) revealed that the loop expansion is indeed an expansion in the effective field theory ratio \((H/M_{\text{Pl}})^2\) and the slow roll parameters. The study in ref.\(^\text{[13]}\) allows us to go further in this direction and state that the loop expansion is a consistent double series in the effective field theory ratio \((H/M_{\text{Pl}})^2\) and in \(1/N\). Loops are either powers of \(g^2\) or of \(\lambda\) which implies that for each loop there is a factor \((H/M_{\text{Pl}})^2\). The counting of powers of \(1/N\) is more subtle: the nearly scale invariant spectrum of fluctuations leads to infrared enhancements of quantum corrections in which the small factor \(1/N\)
enters as an infrared regulator. Therefore, large denominators that feature the infrared regulator of order $1/N$ cancel out factors $1/N$ in the numerator. The final power of $1/N$ must be computed in detail in each loop contribution.

- We find the relation (12) truly remarkable. Since the scale of inflation $M$ is fixed, presumably by the underlying microscopic (GUT) theory, the scaling of $\lambda$ with the inverse of the number of e-folds strongly suggests a renormalization group explanation of the effective field theory because the number of e-folds is associated with the logarithm of the scale $N = \ln a$. A renormalization group improved scale dependent quartic coupling [16] behaves as $\lambda(K) \propto 1/\ln K$ with $K$ the scale at which the theory is studied. Since in an expanding cosmology the physical scale grows with the scale factor it is natural to expect that a renormalization group resummation yields the renormalized coupling scaling as

$$\lambda \sim 1/\ln a \sim 1/N.$$  

There are several aspects of slow roll inflation, which when considered together, lead us to conjecture that the effective field theory of inflation is an almost critical theory near but not at a trivial fixed point of the renormalization group. These aspects are the following:

- The fluctuations of the inflaton are almost massless, this is the statement that the slow roll parameter

$$\eta_V = M_{Pl}^2 \frac{V''(\phi)}{V(\phi)} \approx 3 \frac{V''(\phi)}{H^2} \ll 1. $$

The slow roll relation (15) states that the dimensionless ratio of the inflaton mass and the Hubble scale is $\sim 1/N \sim 1/\ln a$.

- The higher order couplings are suppressed by further powers of $1/N \sim 1/\ln a$ [see eq. (16)]. In the language of critical phenomena, the mass is a relevant operator in the infrared, the quartic coupling $\lambda$ is marginal, and higher order couplings are irrelevant.

These ingredients taken together strongly suggest that for large $N \sim \ln a$, the effective field theory is reaching a trivial gaussian infrared fixed point [14]. The evidence for this is manifest in that:

i) the power spectrum of scalar fluctuations is nearly scale invariant (a consequence of $\eta_V \ll 1$),

ii) the coupling constants vanish in the asymptotic limit $N \to \infty$. If the number of e-folds were infinite, the theory would be sitting at the trivial fixed point: a massless free field theory.

The physical situation in inflationary cosmology is not to be sitting exactly at the fixed point, inflation must end, and is to be followed by a radiation dominated standard big bang cosmology. Therefore, we conclude that during the stage of slow roll inflation, the theory is hovering near a trivial gaussian infrared fixed point but the presence of a small relevant operator, namely the inflaton mass which eventually becomes large at the end of slow roll, drives the theory away from criticality.

Our investigations[14] reveal that it is not the ultraviolet behavior of the renormalization group that is responsible for the near criticality of the effective field theory, but rather the infrared, superhorizon physics. That this is the case can be gleaned in eq. (12): the coefficient $(M/M_{Pl})^4$ in front of the term $1/\ln a$ cannot be obtained from the usual Minkowski-space renormalization group solution for the running coupling. Furthermore, the true trivial fixed point is obtained in the infrared limit when the scale factor $a \to \infty$, namely infinitely long physical wavelengths.

If this conjecture[14] proves correct, it will have a major fundamental appeal as a description of inflation because field theories near a fixed point feature universal behavior independent of the underlying microscopic degrees of freedom. In fact, if the effective field theory of inflation is indeed near (not exactly) an infrared gaussian fixed point, its predictions would be nearly universal, and single field slow roll inflation describes a large universality class that features the same robust predictions. Such is the case in critical phenomena described by field theories where widely different systems feature the same behavior near a critical point.
2 Quantum corrections to the equation of motion for the inflaton and its effective potential.

We consider single field inflationary models described by a general self-interacting scalar field theory in a spatially flat Friedmann-Robertson-Walker cosmological space time with scale factor $a(t)$. In comoving coordinates the action is given by

$$S = \int d^3 x \, dt \, a^3(t) \left[ \frac{1}{2} \dot{\phi}^2 - \frac{(\nabla \phi)^2}{2a^2} - V(\phi) \right].$$

We consider a generic potential $V(\phi)$, the only requirement is that its derivatives be small in order to justify the slow roll expansion\cite{3, 17}. In order to study the corrections from the quantum fluctuations we separate the classical homogeneous expectation value of the scalar field from the quantum fluctuations by writing

$$\phi(\vec{x}, t) = \Phi_0(t) + \varphi(\vec{x}, t),$$

with

$$\Phi_0(t) = \langle \phi(\vec{x}, t) \rangle; \quad \langle \varphi(\vec{x}, t) \rangle = 0,$$

where the expectation value is in the non-equilibrium quantum state. Expanding the Lagrangian density and integrating by parts, the action becomes

$$S = \int d^3 x \, dt \, a^3(t) \mathcal{L}[\Phi_0(t), \varphi(\vec{x}, t)],$$

with

$$\mathcal{L}[\Phi_0(t), \varphi(\vec{x}, t)] = \frac{1}{2} \ddot{\Phi}_0 - V(\Phi_0) + \frac{1}{2} \dot{\varphi}^2 - \frac{(\nabla \varphi)^2}{2a^2} - \frac{1}{2} V''(\Phi_0) \varphi^2 - \varphi \left[ \ddot{\Phi}_0 + 3H \dot{\Phi}_0 + V'(\Phi_0) \right] - \frac{1}{6} V'''(\Phi_0) \varphi^3 - \frac{1}{24} V''''(\Phi_0) \varphi^4 + \text{higher orders in } \varphi.$$\hspace{1cm}(27)

We obtain the equation of motion for the homogeneous expectation value of the inflaton field by implementing the tadpole method (see\cite{11} and references therein). This method consists in requiring the condition $\langle \varphi(\vec{x}, t) \rangle = 0$ consistently in a perturbative expansion by treating the linear, cubic, quartic (and higher order) terms in the Lagrangian density eq.(27) as perturbations\cite{11}.

Our approach relies on two distinct and fundamentally different expansions: i) the effective field theory (EFT) expansion and ii) the slow-roll expansion.

Quantum corrections to the equations of motion for the inflaton and for the fluctuations are obtained by treating the second line in eq.(27), namely, the linear and the non-linear terms in $\varphi$, in perturbation theory.

The generating functional of non-equilibrium real time correlation functions requires a path integral along a complex contour in time: the forward branch corresponds to time evolution forward (+) and backward (−) in time as befits the time evolution of a density matrix. Fields along these branches are labeled $\varphi^+$ and $\varphi^−$, respectively (see refs.\cite{11} and references therein). The tadpole conditions

$$\langle \varphi^\pm(\vec{x}, t) \rangle = 0,$$

both lead to the (same) equation of motion for the expectation value $\Phi_0(t)$ by considering the linear, cubic and higher order terms in the Lagrangian density as interaction vertices. To one loop order we find

$$\ddot{\Phi}_0(t) + 3H \dot{\Phi}_0(t) + V'(\Phi_0) + g(\Phi_0) \langle |\varphi^+(\vec{x}, t)|^2 \rangle = 0.$$

The first three terms in eq.(29) are the familiar ones for the classical equation of motion of the inflaton.

The last term is the one-loop correction to the equations of motion of purely quantum mechanical origin. Another derivation of this quantum correction can be found in\cite{6, 18}. The fact that the tadpole method, which in this case results in a one-loop correction, leads to a covariantly conserved and fully renormalized energy momentum tensor has been established in the most general case in refs.\cite{6, 19, 20}.\hspace{1cm}(29)
The coupling $g$, effective ‘mass term’ $M^2$ and the quartic coupling are defined by

$$M^2 \equiv M^2(\Phi_0) = V''(\Phi_0) = 3H_0^2 \eta \nu + \mathcal{O}(\frac{1}{N}), \quad g \equiv g(\Phi_0) = \frac{1}{2} \left( \frac{V'''}{V'}(\Phi_0) \right), \quad \lambda \equiv \lambda(\Phi_0) = \frac{1}{6} \frac{V''(\Phi_0)}{V'(\Phi_0)}.$$  \hspace{1cm} (30)

The $\langle \cdot \cdot \rangle$ is computed in the free field (Gaussian) theory of the fluctuations $\varphi$ with an effective ‘mass term’ $M^2$, the quantum state will be specified below. Furthermore, it is straightforward to see that $\langle [\varphi_+^2(x,t)] \rangle = \langle [\varphi_+^2(x,t)] \rangle = \langle [\varphi_+^2(x,t)] \rangle$. In terms of the spatial Fourier transform of the fluctuation field $\varphi(x,t)$, the one-loop contribution can be written as

$$\langle [\varphi(x,t)]^2 \rangle = \int \frac{d^3 k}{(2\pi)^3} \langle [\varphi_k(t)]^2 \rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_{\varphi}(k,t),$$  \hspace{1cm} (31)

where $\varphi_k(t)$ is the spatial Fourier transform of the fluctuation field $\varphi(x,t)$ and we have introduced the power spectrum of the fluctuation

$$P_{\varphi}(k,t) = \frac{k^3}{2\pi^2} \langle [\varphi_k(t)]^2 \rangle.$$  \hspace{1cm} (32)

During slow roll inflation the scale factor is quasi de Sitter and to lowest order in slow roll it is given by:

$$C(\eta) = -\frac{1}{H_0 \eta} \frac{1}{1-\epsilon_V} = -\frac{1}{H_0 \eta}(1+\epsilon_V) + \mathcal{O}(\epsilon_V).$$  \hspace{1cm} (33)

The spatial Fourier transform of the rescaled free field Heisenberg operators $\chi(x,\eta) \equiv C(\eta)\varphi(x,t)$ obey the equation

$$\chi''_\xi(\eta) + \left[ k^2 + M^2 C(\eta) - \frac{C''(\eta)}{C(\eta)} \right] \chi_\xi(\eta) = 0.$$  \hspace{1cm} (34)

Using the slow roll expressions eqs. (30) and (33), it becomes

$$\chi''_\xi(\eta) + \left[ k^2 - \frac{\nu^2 - \frac{1}{2}}{\eta^2} \right] \chi_\xi(\eta) = 0$$  \hspace{1cm} (35)

where the index $\nu$ and the quantity $\Delta$ are given by

$$\nu = \frac{3}{2} + \epsilon_V - \eta V + \mathcal{O}(\frac{1}{N^2}), \quad \Delta = \frac{3}{2} - \nu = \eta V - \epsilon_V + \mathcal{O}(\frac{1}{N^2}).$$  \hspace{1cm} (36)

The scale invariant case $\nu = \frac{3}{2}$ corresponds to massless inflaton fluctuations in the de Sitter background, $\Delta$ measures the departure from scale invariance. In terms of the spectral index of the scalar adiabatic perturbations $n_s$ and the ratio $r$ of tensor to scalar perturbations, $\Delta$ takes the form

$$\Delta = \frac{1}{2}(n_s - 1) + \frac{r}{8}.$$  \hspace{1cm} (37)

The free Heisenberg field operators $\chi_\xi(\eta)$ are written in terms of annihilation and creation operators that act on Fock states as

$$\chi_\xi(\eta) = a_\xi S_\nu(k,\eta) + a_\xi^\dagger S_\nu^*(k,\eta)$$  \hspace{1cm} (38)

where the mode functions $S_\nu(k,\eta)$ are solutions of the eqs. (39). For Bunch-Davis boundary conditions we have

$$S_\nu(k,\eta) = \frac{1}{2} \sqrt{-\pi \eta} \ e^{i \frac{\pi}{2}(\nu+\frac{1}{2})} H^{(1)}_{\nu}(k \eta),$$  \hspace{1cm} (39)

this defines the Bunch-Davis vacuum $a_\xi^\dagger|0 >_{BD} = 0$.

There is no unique choice of an initial state, and a recent body of work has began to address this issue (see ref. [20] for a discussion and further references). A full study of the quantum loop corrections with different initial states must first elucidate the behavior of the propagators for the fluctuations in such states. Here we focus on the standard choice in the literature [3] which allows us to include the quantum corrections into the standard results in the literature. A study of quantum loop corrections with different initial states is an important aspect by itself which we postpone to later work.
The index $\nu$ in the mode functions eq. (39) depends on the expectation value of the scalar field, via the slow roll variables, hence it slowly varies in time. Therefore, it is consistent to treat this time dependence of $\nu$ as an adiabatic approximation. This is well known and standard in the slow roll expansion (33). Indeed, there are corrections to the mode functions which are higher order in slow roll. However, these mode functions enter in the propagators in loop corrections, therefore they yield higher order contributions in slow roll and we discard them consistently to lowest order in slow roll.

With this choice and to lowest order in slow roll, the power spectrum eq. (32) is given by

$$\mathcal{P}_\phi(k, t) = \frac{H^2}{8\pi} (-k\eta)^3 |H_\nu^{(1)}(-k\eta)|^2. \quad (40)$$

For large momenta $|k\eta| \gg 1$ the mode functions behave just like free field modes in Minkowski space-time, namely

$$S_\nu(k, \eta) = 1 + \frac{1}{\sqrt{2k}} e^{-ik\eta} \quad (41)$$

Therefore, the quantum correction to the equation of motion for the inflaton eqs. (29) and (31) determined by the momentum integral of $\mathcal{P}_\phi(k, t)$ features both quadratic and logarithmic divergences. Since the field theory inflationary dynamics is an effective field theory valid below a comoving cutoff $\Lambda$ of the order of the Planck scale, the one loop correction (31) becomes

$$\int_0^\Lambda \frac{dk}{k} \mathcal{P}_\phi(k, t) = \frac{H^2}{8\pi} \int_0^{\Lambda_p} \frac{dz}{z} z^3 \left|H_\nu^{(1)}(z)\right|^2, \quad (42)$$

where $\Lambda_p(\eta)$ is the ratio of the cutoff in physical coordinates to the scale of inflation, namely

$$\Lambda_p(\eta) = \frac{\Lambda}{H C(\eta)} = -\Lambda \eta. \quad (43)$$

The integration variable $z = -k\eta$ has a simple interpretation at leading order in slow roll

$$z \equiv -k\eta = \frac{k}{H_0 C(\eta)} = \frac{k_p(\eta)}{H_0}, \quad (44)$$

where $k_p(\eta) = k/C(\eta)$ is the wavevector in physical coordinates. If the spectrum of scalar fluctuations were strictly scale invariant, (namely for massless inflaton fluctuations in de Sitter space-time), then the index would be $\nu = 3/2$ and the integrand in (42) given by

$$z^3 \left|H_\nu^{(1)}(z)\right|^2 = \frac{2}{\pi} \left[1 + z^2\right]. \quad (45)$$

In this strictly scale invariant case, the integral of the power spectrum also features an infrared logarithmic divergence. While the ultraviolet divergences are absorbed by the renormalization counterterms in the effective field theory, this is not possible for the infrared divergence. Obviously, the origin of this infrared behavior is the exact scale invariance of superhorizon fluctuations. However, during slow roll inflation there are small corrections to scale invariance, in particular the index $\nu$ is slightly different from 3/2 and this slight departure introduces a natural infrared regularization. In ref. [11] we have introduced an expansion in the parameter $\Delta = 3/2 - \nu = \eta_V - \epsilon_V + \mathcal{O}(\epsilon_V^2, \eta_V^2, \epsilon_V \eta_V)$ which is small during slow roll and we showed in [12] that the infrared divergences featured by the quantum correction manifest as simple poles in $\Delta$.

The quantum correction to the equation of motion for the inflaton by isolating the pole in $\Delta$ as well as the leading logarithmic divergences were computed in ref. [12] with the result

$$\frac{1}{2} \langle |\varphi(\vec{x}, t)|^2 \rangle = \left(\frac{H_0}{4\pi}\right)^2 \left[\Lambda_p^2 + \ln \Lambda_p^2 + \frac{1}{\Delta} + 2\gamma - 4 + \mathcal{O}(\Delta)\right], \quad (46)$$

where $\gamma$ is the Euler-Mascheroni constant. While the quadratic and logarithmic ultraviolet divergences are regularization scheme dependent, the pole in $\Delta$ arises from the infrared behavior and is independent of the regularization scheme. In particular this pole coincides with that found in the expression for $\langle \phi^2(\vec{x}, t) \rangle$ in ref. [21]. The ultraviolet divergences, in whichever renormalization scheme, require that the effective field theory be defined to contain renormalization counterterms in the bare effective lagrangian, so that these counterterms will systematically cancel the divergences encountered in the calculation of quantum corrections in the (EFT) and slow roll approximations.
2.1 Renormalized effective field theory: renormalization counterterms

The renormalized effective field theory is obtained by writing the potential $V[\phi]$ in the Lagrangian density eq. (29) in the following form

$$V(\phi) = V_R(\phi) + \delta V_R(\phi, \Lambda),$$

(47)

where $V_R(\phi)$ is the renormalized classical inflaton potential and $\delta V_R(\phi, \Lambda)$ includes the renormalization counterterms which are found systematically in a slow roll expansion by canceling the ultraviolet divergences. In this manner, the equations of motion and correlation functions in this effective field theory are cutoff independent. We find from eqs. (29) and (46)

$$\ddot{\Phi}_0(t) + 3H\dot{\Phi}_0(t) + V'(\Phi_0) + V''(\Phi_0) \left(\frac{H_0}{4\pi}\right)^2 \left[\Lambda_\phi^2 + \ln \Lambda_\phi^2 + \frac{1}{\Delta} + 2\gamma - 4 + O(\Delta)\right] = 0. $$

(48)

From this equation it becomes clear that the one-loop ultraviolet divergences can be canceled by choosing appropriate counterterms [12] leading to the final form of the renormalized inflaton equation of motion to leading order in the slow roll expansion

$$\ddot{\Phi}_0(t) + 3H_0\dot{\Phi}_0(t) + V'_R(\Phi_0) + \left(\frac{H_0}{4\pi}\right)^2 \frac{V''_R(\Phi_0)}{\Delta} = 0. $$

(49)

Although the quantum correction is of order $V''_R(\Phi_0)$, (second order in slow roll), the strong infrared divergence arising from the quasi scale invariance of inflationary fluctuations brings about a denominator which is of first order in slow roll. Hence the lowest order quantum correction in the slow roll expansion, is actually of the same order as $V'_R(\Phi_0)$. To highlight this observation, it proves convenient to write eq. (49) in terms of the EFT and slow roll parameters,

$$\ddot{\Phi}_0(t) + 3H_0\dot{\Phi}_0(t) + V'_R(\Phi_0) \left[1 + \left(\frac{H_0}{2\pi M_{Pl}}\right)^2 \frac{\xi_V}{2\epsilon_V \Delta}\right] = 0. $$

(50)

Since $\xi_V \sim \epsilon_V^2$ and $\Delta \sim \epsilon_V$ the leading quantum corrections are of zeroth order in slow roll. This is a consequence of the infrared enhancement resulting from the nearly scale invariance of the power spectrum of scalar fluctuations. The quantum correction is suppressed by an EFT factor $H^2/M^2_{Pl} \ll 1$.

Restoring the dependence of $\Delta$ on $\Phi_0$ through the definitions (18) and (36) we find the equation of motion for the inflaton field to leading order in slow roll and in $(H/M_{Pl})^2$,

$$\ddot{\Phi}_0(t) + 3H_0\dot{\Phi}_0(t) + V'_R(\Phi_0) + \frac{1}{24(\pi M_{Pl})^2} \frac{V^3_R(\Phi_0) V''_R(\Phi_0)}{2 V_R(\Phi_0) V'_R(\Phi_0) - V''_R(\Phi_0)} = 0. $$

(51)

2.2 Quantum corrections to the Friedmann equation: the effective potential

The zero temperature effective potential in Minkowski space-time is often used to describe the scalar field dynamics during inflation [3] [22]. However, as we see below the resulting effective potential [see eq. (58)] is remarkably different from the Minkowski one [see Appendix A]. The focus of this Section is to derive the effective potential for slow-roll inflation.

Since the fluctuations of the inflaton field are quantized, the interpretation of the ‘scalar condensate’ $\Phi_0$ is that of the expectation value of the full quantum field $\phi$ in a homogeneous coherent quantum state. Consistently with this, the Friedmann equation must necessarily be understood in terms of the expectation value of the field energy momentum tensor, namely

$$H^2 = \frac{1}{3 M^2_{Pl}} \left\{ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left(\frac{\nabla \phi}{a(t)}\right)^2 + V[\phi]\right\}. $$

(52)

Separating the homogeneous condensate from the fluctuations as in eq. (28) and imposing the tadpole equation (25), the Friedmann equation becomes

$$H^2 = \frac{1}{3 M^2_{Pl}} \left[\frac{1}{2} \dot{\Phi}_0^2 + V_R(\Phi_0) + \delta V_R(\Phi_0)\right] + \frac{1}{3 M^2_{Pl}} \left\{ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \left(\frac{\nabla \phi}{a(t)}\right)^2 + \frac{1}{2} V''(\Phi_0) \varphi^2 + \cdots \right\}. $$

(53)
suggests the identification of the effective potential is strikingly different from eq. (58) valid for slow roll inflation. Where the derivative of \( V \) corrections to the energy momentum tensor by integrating the fluctuations eq. (46). Calculating the expectation value in eq. (53) in free field theory corresponds to obtaining the corrections to the energy momentum tensor up to one loop \([12]\). The first two terms of the expectation value in eq. (53) do not feature infrared divergences for \( \nu = 3/2 \) because of the two extra powers of the loop momentum in the integral. These contributions are given by

\[
\frac{1}{2} \langle \varphi^2 \rangle = \frac{H_0^4}{16 \pi} \int_0^{\Lambda_p} \frac{dz}{z} \left[ \frac{d}{dz} \left[ z^2 H^{(1)}_\nu(z) \right] \right]^2 = \frac{H_0^4 \Lambda_p^4}{32 \pi^2} + O(H_0^4 \Delta), \tag{54}
\]

\[
\frac{1}{2} \left( \frac{\nabla \varphi}{a(t)} \right)^2 = \frac{H_0^4}{16 \pi} \int_0^{\Lambda_p} \frac{dz}{z} z^5 \left| H^{(1)}_\nu(z) \right|^2 = \frac{H_0^4 \Lambda_p^4}{32 \pi^2} + \frac{H_0^4 \Lambda_p^2}{16 \pi^2} + O(H_0^4 \Delta). \tag{55}
\]

The counterterms cancel the ultraviolet divergences arising from the third term in the angular brackets in eq. \([12]\). Finally, the fully renormalized Friedmann equation to one loop and to lowest order in the slow roll expansion is \([12]\)

\[
H^2 = \frac{1}{3 M_{Pl}^2} \left[ \frac{1}{2} \Phi_0^2 + V_R(\Phi_0) + \left( \frac{H_0}{4 \pi} \right)^2 \frac{V''_{\nu}(\Phi_0)}{\Delta} + \text{higher orders in slow roll} \right] \equiv H_0^2 + \delta H^2, \tag{56}
\]

where \( H_0 \) is the Hubble parameter in absence of quantum fluctuations:

\[
H_0^2 = \frac{V_R(\Phi_0)}{M_{Pl}^2} \left[ 1 + \frac{\epsilon_{\nu}}{3} + O(\epsilon_{\nu}^2, \epsilon_{\nu} \eta_{\nu}) \right].
\]

Using the lowest order slow roll relation eq. \([59]\), the last term in eq. \([56]\) can be written as follows

\[
\frac{\delta H^2}{H_0^2} = \left( \frac{H_0}{4 \pi M_{Pl}} \right)^2 \eta_{\nu} \frac{\eta_{\nu}}{\Delta}. \tag{57}
\]

This equation defines the back-reaction correction to the scale factor arising from the quantum fluctuations of the inflaton.

Hence, while the ratio \( \eta_{\nu} / \Delta \) is of order zero in slow roll, the one loop correction to the Friedmann equation is of the order \( H_0^2 / M_{Pl}^2 \ll 1 \) consistently with the EFT expansion. The Friedmann equation suggests the identification of the effective potential

\[
V_{\nu \text{ eff}}(\Phi_0) = V_R(\Phi_0) + \left( \frac{H_0}{4 \pi} \right)^2 \frac{V''_{\nu}(\Phi_0)}{\Delta} + \text{higher orders in slow roll} = \tag{58}
\]

\[
V_R(\Phi_0) \left[ 1 + \left( \frac{H_0}{4 \pi M_{Pl}} \right)^2 \frac{\eta_{\nu}}{\eta_{\nu} - \epsilon_{\nu}} + \text{higher orders in slow roll} \right]. \tag{59}
\]

We see that the equation of motion for the inflaton eq. \([59]\) takes the natural form

\[
\ddot{\Phi}_0(t) + 3 H_0 \dot{\Phi}_0(t) + \frac{\partial V_{\nu \text{ eff}}}{\partial \Phi_0}(\Phi_0) = 0.
\]

where the derivative of \( V_{\nu \text{ eff}} \) with respect to \( \Phi_0 \) is taken at fixed Hubble and slow roll parameters. That is, \( H_0 \) and \( \Delta \) must be considered in the present context as gravitational degrees of freedom and not as matter (inflaton) degrees of freedom.

Eqs. \([59]\) and \([58]\) make manifest the nature of the effective field theory expansion in terms of the ratio \( (H_0 / M_{Pl})^2 \). The coefficients of the powers of this ratio are obtained in the slow roll expansion. To leading order, these coefficients are of \( O(N^0) \) because of the infrared enhancement manifest in the poles in \( \Delta \), a consequence of the nearly scale invariant power spectrum of scalar perturbations.

A noteworthy result is the rather different form of the effective potential eq. \([58]\) as compared to the result in Minkowski space time at zero temperature. In the appendix we show explicitly that the same definition of the effective potential as the expectation value of \( T_{\nu} \) in Minkowski space-time at zero temperature is strikingly different from eq. \([58]\) valid for slow roll inflation.
3 Quantum Corrections to the Scalar and Tensor Power. Scalar, Tensor, Fermion and Light Scalar Contributions.

During slow-roll inflation many quantum fields may be coupled to the inflaton (besides itself) and can contribute to the quantum corrections to the equations of motion and to the inflaton effective potential. The scalar curvature and tensor fluctuations are certainly there. We also consider light fermions and scalars coupled to the inflaton. We take the fermions to be Dirac fields with a generic Yukawa-type coupling but it is straightforward to generalize to Weyl or Majorana fermions. The Lagrangian density is taken to be

$$\mathcal{L} = \sqrt{-g} \left\{ \frac{1}{2} \dot{\phi}^2 - \left( \frac{\sqrt{-g} \sigma}{2a} \right)^2 - V(\phi) + \frac{1}{2} \sigma^2 - \left( \frac{\sqrt{-g} \sigma}{2a} \right)^2 - \frac{1}{2} m_{\sigma}^2 \sigma^2 - G(\phi) \sigma^2 + \overline{\Psi} \left[ i \gamma^\mu D_\mu \Psi - m_f - Y(\phi) \right] \Psi \right\} \tag{60}$$

where $G(\Phi)$ and $Y(\Phi)$ are generic interaction terms between the inflaton and the scalar and fermionic fields. For simplicity we consider one bosonic and one fermionic Dirac field. The $\gamma^\mu$ are the curved space-time Dirac matrices and $D_\mu$ the fermionic covariant derivative.$^{[9, 23]}$ We will consider that the light scalar field $\sigma$ has vanishing expectation value at all times, therefore inflationary dynamics is driven by one single scalar field, the inflaton $\phi$.

We consider the contributions from the quadratic fluctuations to the energy momentum tensor. There are four distinct contributions: i) scalar metric (density) perturbations, ii) tensor (gravitational waves) perturbations, iii) fluctuations of the light bosonic scalar field $\sigma$, iv) fluctuations of the light fermionic field $\Psi$.

Fluctuations in the metric are studied as usual$^{[8, 24]}$. Writing the metric as

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta^s g_{\mu\nu} + \delta^t g_{\mu\nu}$$

where $g^0_{\mu\nu}$ is the spatially flat FRW background metric eq. (3), $\delta^s g_{\mu\nu}$ correspond to the scalar and tensor perturbations respectively, and we neglect vector perturbations. In longitudinal gauge

$$\delta^s g_{00} = C^2(\eta) \ 2 \ \phi \ \ , \ \ \delta^s g_{ij} = C^2(\eta) \ 2 \ \psi \ \delta_{ij} \ \ , \ \ \delta^t g_{ij} = -C^2(\eta) \ h_{ij}$$

where $h_{ij}$ is transverse and traceless and we neglect vector modes since they are not generated in single field inflation$^{[8, 24]}$.

Expanding up to quadratic order in the scalar fields, fermionic fields and metric perturbations the part of the Lagrangian density that is quadratic in these fields is given by

$$\mathcal{L}_Q = \mathcal{L}_s[\delta \varphi^{s_i}, \phi^{s_i}, \psi^{s_i}] + \mathcal{L}_t[h] + \mathcal{L}_\sigma[\sigma] + \mathcal{L}_\Psi[\overline{\Psi}, \Psi] \ ,$$

where

$$\mathcal{L}_t[h] = \frac{M_p^2}{8} C^{s}(\eta) \ \partial_\alpha h^i_\beta \ \partial_\beta h^i_\alpha \ \eta^{\alpha\beta} \ ,$$

$$\mathcal{L}_\sigma[\sigma] = C^4(\eta) \ \left\{ \frac{1}{2} \left( \sigma^C \right)^2 - \frac{1}{2} \left( \overline{\nabla} \sigma \right)^2 - \frac{1}{2} M^2_{\sigma}[\Phi_0] \ \sigma^2 \right\} \ ,$$

$$\mathcal{L}_\Psi[\overline{\Psi}, \Psi] = \overline{\Psi} \left[ i \gamma^\mu D_\mu \Psi - M_\Psi[\Phi_0] \right] \Psi \ ,$$

here the prime stands for derivatives with respect to conformal time and the labels (gi) refer to gauge invariant quantities$^{[24]}$. The explicit expression for $\mathcal{L}[\delta \varphi^{s_i}, \phi^{s_i}, \psi^{s_i}]$ is given in eq. (10.68) in ref.$^{[24]}$. The effective masses for the bosonic and fermionic fields are given by

$$M^2_{\sigma}[\Phi_0] = m_{\sigma}^2 + G(\Phi_0) \ \ , \ \ M_{\Psi}[\Phi_0] = m_f + Y(\Phi_0) \ . \tag{62}$$

We focus on the study of the quantum corrections to the Friedmann equation, for the case in which both the scalar and fermionic fields are light in the sense that during slow roll inflation,

$$M_{\sigma}[\Phi_0], \ M_{\Psi}[\Phi_0] \ll H_0 \ . \tag{63}$$
at least during the cosmologically relevant stage corresponding to the 60 or so e-folds before the end of inflation.

From the quadratic Lagrangian given above the quadratic quantum fluctuations to the energy momentum tensor can be extracted.

The effective potential is identified with \( \langle T^0_0 \rangle \) in a spatially translational invariant state in which the expectation value of the inflaton field is \( \Phi_0 \). During slow roll inflation the expectation value \( \Phi_0 \) evolves very slowly in time, the slow roll approximation is indeed an adiabatic approximation, which justifies treating \( \Phi_0 \) as a constant in order to obtain the effective potential. The time variation of \( \Phi_0 \) only contributes to higher order corrections in slow-roll. The energy momentum tensor is computed in the FRW inflationary background determined by the \textit{classical} inflationary potential \( V(\Phi_0) \), and the slow roll parameters are also explicit functions of \( \Phi_0 \). Therefore the energy momentum tensor depends \textit{implicitly} on \( \Phi_0 \) through the background and \textit{explicitly} on the masses for the light bosonic and fermionic fields given above.

We can write the effective potential as

\[
V_{\text{eff}}(\Phi_0) = V(\Phi_0) + \delta V(\Phi_0),
\]

where

\[
\delta V(\Phi_0) = \langle T^0_0(\Phi_0) \rangle_s + \langle T^0_0(\Phi_0) \rangle_t + \langle T^0_0(\Phi_0) \rangle_\sigma + \langle T^0_0(\Phi_0) \rangle_\Psi
\]

(\( s, t, \sigma, \Psi \)) correspond to the energy momentum tensors of the quadratic fluctuations of the scalar metric tensor (gravitational waves), light boson field \( \sigma \) and light fermion field \( \Psi \) fluctuations respectively. Since these are the expectation values of a quadratic energy momentum tensor, \( \delta V(\Phi_0) \) corresponds to the \textit{one loop correction} to the effective potential.

### 3.1 Light scalar fields

During slow roll inflation the effective mass of the \( \sigma \) field is given by eq. (62), just as for the inflaton fluctuation in sec. 2. It is convenient to introduce a parameter \( \eta_\sigma \) defined to be

\[
\eta_\sigma = \frac{M_\sigma^2[\Phi_0]}{3 H_0^2}.
\]

Hence, the sigma field contributions to the inflaton equations of motion and inflaton effective potential can be obtained from sec. 2 just replacing the slow roll parameter \( \epsilon_V \) by \( \eta_\sigma \). In particular, infrared divergences are now regulated by the parameter \( \Delta_\sigma \equiv \eta_\sigma - \epsilon_V \).

To leading order in the slow roll expansion and in \( \eta_\sigma \ll 1 \), the infrared contribution is given by\[2,\]

\[
\frac{M_\sigma^2[\Phi_0]}{2} \langle \sigma^2(\vec{x}, t) \rangle = \frac{3 H_0^4}{(4\pi)^2} \frac{\eta_\sigma}{\eta_\sigma - \epsilon_V} + \text{subleading in slow roll}.
\]

The fully renormalized contribution from the sigma field to \( T^0_0 \) to leading order in slow roll takes the form\[9,\]

\[
\langle T^0_0 \rangle_\sigma = \frac{3 H_0^4}{(4\pi)^2} \frac{\eta_\sigma}{\Delta_\sigma} + \frac{1}{2} \left\langle \sigma^2 + \left( \frac{\nabla \sigma}{C(\eta)} \right)^2 \right\rangle_{\text{ren}}
\]

In calculating here the second term we can set to zero the slow roll parameters \( \epsilon_V, \eta_\sigma \) as well as the mass of the light scalar, namely \( \eta_\sigma = 0 \). Hence, to leading order, the second term is identified with the 00 component of the renormalized energy momentum tensor for a free massless minimally coupled scalar field in exact de Sitter space time. Therefore we can extract this term from refs.\[21,23,\]

\[
\langle T_{\mu\nu} \rangle_{\text{ren}} = \frac{g_{\mu\nu}}{(4\pi)^2} \left\{ m_\sigma^2 H_0^2 \left[ 1 - \frac{m_\sigma^2}{2 H_0^2} \right] - \psi \left( \frac{3}{2} + \nu \right) \right. - \psi \left( \frac{3}{2} - \nu \right) + \ln \frac{m_\sigma^2}{H_0^2} \left. + \frac{2}{3} m_\sigma^2 H_0^2 - \frac{29}{30} H_0^4 \right\},
\]

where \( \nu \equiv \sqrt{\frac{9}{4} - \frac{m_\sigma^2}{H_0^2}} \) and \( \psi(z) \) stands for the digamma function. This expression corrects a factor of two in ref.\[23,\] In eq. (6.177) in\[23,\] the D’Alambertian acting on \( G^1(x, x') \) was neglected. However, in
computing this term, the D’Alambertian must be calculated \textit{before} taking the coincidence limit. Using
the equation of motion yields the extra factor 2 and the expression eq.\eqref{68}. The pole at \( \nu = 3/2 \) manifest in eq.\eqref{68} coincides with the pole in eq.\eqref{66} using that \( m^2 = 3 H^2 \eta_\sigma \) [eq.\eqref{67}]. This pole originates in the term \( m^2 < \sigma^2 > \), which features an infrared divergence in the limit \( \nu_\sigma = 3/2 \). All the terms with space-time derivatives are infrared finite in this limit. Therefore, we can extract from eq.\eqref{68} the renormalized expectation value in the limit \( H_0 \gg m_\sigma \),
\[
\langle T^0_0 \rangle_\sigma = \frac{H_0^4}{(4 \pi)^2} \left[ \frac{3 \eta_\sigma}{\eta_\sigma - \epsilon_V} - \frac{29}{30} + \mathcal{O}(\epsilon_V, \eta_\sigma, \eta_V) \right] \tag{69}
\]
The second term is completely determined by the \textit{trace anomaly} of the minimally coupled scalar fields\cite{21, 23, 25}.

3.2 Quantum Corrections to the Inflaton potential from the Scalar metric perturbations

The gauge invariant energy momentum tensor for quadratic scalar metric fluctuations has been obtained
in ref.\cite{26}. In longitudinal gauge and in cosmic time it is given by
\[
\langle T^\mu_\nu \rangle = \frac{M_{Pl}^2}{12} \left[ 12 H_0 \langle \phi \phi \rangle - 3 \langle (\dot{\phi})^2 \rangle + \frac{9}{C^2(\eta)} \langle (\nabla \phi)^2 \rangle \right]
\]
\[
+ \frac{1}{2} \langle (\delta \varphi)^2 \rangle + \frac{\langle (\nabla \delta \varphi)^2 \rangle}{2 C^2(\eta)} + \frac{V''(\Phi_0)}{2} \langle (\delta \varphi)^2 \rangle + 2 V'(\Phi_0) \langle \delta \varphi \rangle \tag{70}
\]
where the dots stand for derivatives with respect to cosmic time. In longitudinal gauge, the equations of
motion in cosmic time for the Fourier modes are\cite{21, 24}
\[
\ddot{\phi}_k + \left( H_0 - 2 \frac{\Phi_0}{\Phi_0} \right) \dot{\phi}_k + \left[ 2 \left( H_0 - 2 H_0 \frac{\Phi_0}{\Phi_0} \right) + \frac{k^2}{C^2(\eta)} \right] \phi_k = 0
\]
\[
\ddot{\delta \varphi}_k + 3 H \dot{\delta \varphi}_k + \left[ \frac{k^2}{C^2(\eta)} \right] \delta \varphi_k + 2 V'(\Phi_0) \phi_k - 4 \Phi_0 \dot{\Phi}_0 \dot{\phi}_k = 0 \ , \tag{71}
\]
with the constraint equation
\[
\dot{\phi}_k + H_0 \phi_k = \frac{1}{2M_{Pl}} \delta \varphi_k \Phi_0 \ . \tag{72}
\]
We split the contributions to the energy momentum tensor as those from superhorizon modes, which
yield the infrared enhancement, and the subhorizon modes for which we can set all slow roll parameters
to zero. Just as discussed above for the case of the \( \sigma \) field, since spatio-temporal derivatives bring higher
powers of the momenta, we can neglect all derivative terms for the contribution from the superhorizon
modes and set\cite{26}
\[
\langle T^0_0 \rangle_{IR} \approx \frac{1}{2} V''(\Phi_0) \langle (\delta \varphi(\vec{x}, t))^2 \rangle + 2 V'(\Phi_0) \langle \delta \varphi(\vec{x}, t) \delta \varphi(\vec{x}, t) \rangle \ . \tag{73}
\]
The analysis of the solution of eq.\eqref{71} for superhorizon wavelengths in ref.\cite{24} shows that in exact
de Sitter space time \( \dot{\phi}_k \sim \text{constant} \), hence it follows that during quasi-de Sitter slow roll inflation for
superhorizon modes
\[
\dot{\phi}_k \sim (\text{slow roll}) \times H_0 \phi_k \tag{74}
\]
Therefore, for superhorizon modes, the constraint equation \eqref{72} yields
\[
\dot{\phi}_k = - \frac{V'(\Phi_0)}{2V(\Phi_0)} \delta \varphi_k + \text{higher orders in slow roll} \ . \tag{75}
\]
Inserting this relation in eq.\eqref{71} and consistently neglecting the term \( \dot{\phi}_k \) according to eq.\eqref{74}, we find the
following equation of motion for the gauge invariant scalar field fluctuation in longitudinal gauge
\[
\ddot{\delta \varphi}_k + 3 H_0 \dot{\delta \varphi}_k + \left[ \frac{k^2}{C^2(\eta)} + 3 H_0^2 \eta_\delta \right] \delta \varphi_k = 0 \ , \tag{76}
\]
where we have used the definition of the slow roll parameters $\epsilon_V$: $\eta_V$ given in eq.(18), and introduced $\eta_s \equiv \eta_V - 2 \epsilon_V$. This is the equation of motion for a minimally coupled scalar field with mass squared $3 H_0^2 \eta_s$ and we can use the results obtained in the case of the scalar field $\sigma$.

Repeating the analysis presented in the case of the scalar field $\sigma$ above, we finally find $\langle T_{00}^0 \rangle_{IR} = 3 H_0^4 \eta_V - 4 \epsilon_V \eta_V - 3 \epsilon_V + \text{subleading in slow roll}$ (77)

It can be shown that the contribution from subhorizon modes to $\langle T_{00}^s \rangle$ is given by $\langle T_{00}^s \rangle_{UV} \simeq 1 \frac{1}{2} \langle (\dot{\delta} \phi)^2 \rangle + \frac{1}{2} \langle (\nabla \delta \phi)^2 \rangle r_{en}$ (78)

where we have also neglected the term with $V' \Phi_0 \sim 3 H_0^2 \eta_V$ since $k^2/a^2 \gg H_0^2$ for subhorizon modes.

Therefore, to leading order in slow roll we find the renormalized expectation value of $T_{00}$ is given by

$$\langle T_{00} \rangle_{ren} \simeq \frac{3 H_0^4}{(4 \pi)^2} \eta_V - 4 \epsilon_V \eta_V - 3 \epsilon_V + \frac{1}{2} \left( \frac{\dot{\phi}^2 + (\nabla \delta \phi)^2}{C(\eta)} \right)^2_{ren}$$ (79)

To obtain the renormalized expectation value in eq.(79) one can set all slow roll parameters to zero to leading order and simply consider a massless scalar field minimally coupled in de Sitter space time and borrow the result from eq.(69). We find

$$\langle T_{00} \rangle_{ren} = \frac{3 H_0^4}{(4 \pi)^2} \frac{\eta_V - 4 \epsilon_V \eta_V - 3 \epsilon_V}{30} + \mathcal{O}(\epsilon_V, \eta_\sigma, \eta_V)$$ (80)

The last term in eq. (80) is completely determined by the trace anomaly of a minimally coupled scalar field in de Sitter space time $^{21, 23, 25}$

### 3.3 Quantum Corrections to the Inflaton potential from the Tensor perturbations

Tensor perturbations correspond to massless fields with two physical polarizations. The energy momentum tensor for gravitons only depends on derivatives of the field $h_{ij}$ therefore its expectation value in the Bunch Davies (BD) vacuum does not feature infrared singularities in the limit $\epsilon_V \rightarrow 0$. The absence of infrared singularities in the limit of exact de Sitter space time, entails that we can extract the leading contribution to the effective potential from tensor perturbations by evaluating the expectation value of $T_{00}$ in the BD vacuum in exact de Sitter space time, namely by setting all slow roll parameters to zero. This will yield the leading order in the slow roll expansion.

Because de Sitter space time is maximally symmetric, the expectation value of the energy momentum tensor is given by $^{23}$

$$\langle T_{\alpha\beta} \rangle_{BD} = \frac{g_{\alpha\beta}}{4} \langle T_{\alpha} \rangle_{BD}$$ (81)

and $T_\alpha$ is a space-time constant, therefore the energy momentum tensor is manifestly covariantly conserved. A large body of work has been devoted to study the trace anomaly in de Sitter space time implementing a variety of powerful covariant regularization methods that preserve the symmetry $^{25, 23, 21}$ yielding a renormalized value of $\langle T_{\mu\nu} \rangle_{BD}$ given by eq. (81). Therefore, the full energy momentum tensor is completely determined by the trace anomaly. The contribution to the trace anomaly from gravitons has been given in refs. $^{21, 23, 25}$, it is

$$\langle T_{\alpha} \rangle_t = - \frac{717}{80} \pi^2 H_0^4$$ and $\langle T_0^0 \rangle_t = - \frac{717}{320} \pi^2 H_0^4$. (82)
3.4 Summary of Quantum Corrections to the Observable Spectra

In summary, we find that the effective potential at one-loop is given by,

\[
\delta V(\Phi_0) = \frac{H_0^4}{(4\pi)^2} \left[ \frac{\eta_V - 4\epsilon_V}{\eta_V - 3\epsilon_V} + \frac{3\eta_\sigma}{\eta_\sigma - \epsilon_V} + T_\Phi + T_s + T_t + T_\Psi + \mathcal{O}(\epsilon_V, \eta_V, \eta_\sigma, \mathcal{M}^2) \right],
\]

where \((s, t, \sigma, \Psi)\) stand for the contributions of the scalar metric, tensor fluctuations, light boson field \(\sigma\) and light fermion field \(\Psi\), respectively. We have

\[
T_\Phi = T_s = -\frac{29}{30}, \quad T_t = -\frac{717}{5}, \quad T_\Psi = \frac{11}{60} \quad (83)
\]

The terms that feature the ratios of combinations of slow roll parameters arise from the infrared or superhorizon contribution from the scalar density perturbations and scalar fields \(\sigma\) respectively. The terms \(T_{s,t,\Psi}\) are completely determined by the trace anomalies of scalar, graviton and fermion fields respectively. Writing \(H_0^2 = V(\Phi_0) H_0^2/[3 M_{Pl}^2]\) we can finally write the effective potential to leading order in slow roll

\[
V_{\text{eff}}(\Phi_0) = V(\Phi_0) \left[ 1 + \frac{H_0^2}{3 (4\pi)^2 M_{Pl}^2} \left( \frac{\eta_V - 4\epsilon_V}{\eta_V - 3\epsilon_V} + \frac{3\eta_\sigma}{\eta_\sigma - \epsilon_V} - \frac{2903}{20} \right) \right] \quad (84)
\]

There are several remarkable aspects of this result:

i) the infrared enhancement as a result of the near scale invariance of scalar field fluctuations, both from scalar density perturbations as well as from a light scalar field, yield corrections of zeroth order in slow roll. This is a consequence of the fact that during slow roll the particular combination \(\Delta_\sigma = \eta_\sigma - \epsilon_V\) of slow roll parameters yield a natural infrared cutoff.

ii) the final one loop contribution to the effective potential displays the effective field theory dimensionless parameter \(H_0^2/M_{Pl}^2\).

iii) the last term is completely determined by the trace anomaly, a purely geometric result of the short distance properties of the theory.

The quantum corrections to the effective potential lead to quantum corrections to the amplitude of scalar and tensor fluctuations with the result [2]

\[
|\Delta^{(S)}_{k,\text{eff}}| = |\Delta^{(S)}_k| = \left\{ 1 + \frac{2}{3} \left( \frac{H_0}{4\pi M_{Pl}} \right)^2 \left[ 1 + \frac{3}{8} \frac{r (n_s - 1) + \frac{dn_s}{d\ln k}}{(n_s - 1)^2} + \frac{2903}{40} \right] \right\}
\]

\[
|\Delta^{(T)}_{k,\text{eff}}| = |\Delta^{(T)}_k| = \left\{ 1 - \frac{1}{3} \left( \frac{H_0}{4\pi M_{Pl}} \right)^2 \left[ 1 - \frac{1}{8} \frac{r (n_s - 1) + \frac{dn_s}{d\ln k}}{(n_s - 1)^2} + \frac{2903}{20} \right] \right\}
\]

\[
r_{\text{eff}} \equiv \frac{|\Delta^{(T)}_{k,\text{eff}}|}{|\Delta^{(S)}_{k,\text{eff}}|} = r \left\{ 1 - \frac{1}{3} \left( \frac{H_0}{4\pi M_{Pl}} \right)^2 \left[ 1 + \frac{3}{8} \frac{r (n_s - 1) + \frac{dn_s}{d\ln k}}{(n_s - 1)^2} + \frac{8709}{20} \right] \right\}. \quad (85)
\]

The quantum corrections turn out to enhance the scalar curvature fluctuations and to reduce the tensor fluctuations as well as their ratio \(r\).

Acknowledgment: H J de V thanks the organizers of JGRG15 for their kind hospitality in Tokyo.

A One loop effective potential and equations of motion in Minkowski space-time: a comparison

In this appendix we establish contact with familiar effective potential both at the level of the equation of motion for the expectation value of the scalar field, as well as the expectation value of \(T_{00}\).

In Minkowski space time the spatial Fourier transform of the field operator is given by

\[
\varphi_k(t) = \frac{1}{\sqrt{2\omega_k}} \left[ a_k e^{-i\omega_k t} + a_k^+ e^{i\omega_k t} \right], \quad (86)
\]
where the vacuum state is annihilated by $a_k$ and the frequency is given by

$$\omega_k = \sqrt{k^2 + V''(\Phi_0)}. \quad (87)$$

The one-loop contribution to the equation of motion (87) is given by

$$\frac{V'''(\Phi_0)}{2} \left(|\phi(\vec{x}, t)|^2\right) = \frac{V'''(\Phi_0)}{8 \pi^2} \int_0^\Lambda k^2 \omega_k \, dk = \frac{d}{d\Phi_0} \left[ \frac{1}{4\pi^2} \int_0^\Lambda k^2 \omega_k \, dk \right]. \quad (88)$$

The expectation value of $T_{00} = \mathcal{H}$ (Hamiltonian density) in Minkowski space time is given up to one loop by the following expression

$$\langle T_{00} \rangle = \left\langle \frac{1}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right\rangle = \frac{1}{2} \Phi_0^2 + V(\Phi_0) + \left\langle \frac{1}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} V''(\Phi_0) \varphi^2 + \cdots \right\rangle. \quad (89)$$

The expectation value of the fluctuation contribution is given by

$$\left\langle \frac{1}{2} \phi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} V''(\Phi_0) \varphi^2 + \cdots \right\rangle = \frac{1}{4\pi^2} \int_0^\Lambda k^2 \omega_k \, dk = \frac{\Lambda^4}{16\pi^2} + \frac{V''(\Phi_0) \Lambda^2}{16\pi^2} - \frac{[V''(\Phi_0)]^2}{64\pi^2} \ln \frac{4\Lambda^2}{V''(\Phi_0)}. \quad (90)$$

Renormalization proceeds as usual by writing the bare Lagrangian in terms of the renormalized potential and counterterms. Choosing the counterterms to cancel the quartic, quadratic and logarithmic ultraviolet divergences, we obtain the familiar renormalized one loop effective potential

$$V_{\text{eff}}(\Phi_0) = V_R(\Phi_0) + \frac{[V''(\Phi_0)]^2}{64\pi^2} \ln \frac{V''(\Phi_0)}{M^2}, \quad (91)$$

where $M$ is a renormalization scale. Furthermore from eq. (88) it is clear that the equation of motion for the expectation value is given by

$$\ddot{\Phi}_0 + V'_{\text{eff}}(\Phi_0) = 0. \quad (92)$$

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