1-Factor and Cycle Covers of Cubic Graphs

Eckhard Steffen

PADERBORN INSTITUTE FOR ADVANCED STUDIES IN COMPUTER SCIENCE AND ENGINEERING
PADERBORN UNIVERSITY
PADERBORN, GERMANY
E-mail: es@upb.de

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Abstract: Let $G$ be a bridgeless cubic graph. Consider a list of $k$ 1-factors of $G$. Let $E_i$ be the set of edges contained in precisely $i$ members of the $k$ 1-factors. Let $\mu_k(G)$ be the smallest $|E_0|$ over all lists of $k$ 1-factors of $G$. Any list of three 1-factors induces a core of a cubic graph. We use results on the structure of cores to prove sufficient conditions for Berge-covers and for the existence of three 1-factors with empty intersection. Furthermore, if $\mu_3(G) \neq 0$, then $2\mu_3(G)$ is an upper bound for the girth of $G$. We also prove some new upper bounds for the length of shortest cycle covers of bridgeless cubic graphs. Cubic graphs with $\mu_4(G) = 0$ have a 4-cycle cover of length $\frac{3}{2}|E(G)|$ and a 5-cycle double cover. These graphs also satisfy two conjectures of Zhang [18]. We also give a negative answer to a problem stated in [18]. © 2014 Wiley Periodical, Inc. J. Graph Theory 78: 195–206, 2015

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1. INTRODUCTION

Graphs in this article may contain multiple edges or loops. An edge is a loop if its two ends are incident to the same vertex. The degree of a vertex $v$ of a graph $G$ is the number of ends of edges that are incident to $v$. A 1-factor of a graph $G$ is a spanning 1-regular
subgraph of \( G \). Hence, a loop cannot be an edge of a 1-factor, and a bridgeless cubic graph does not contain a loop. One of the first theorems in graph theory, Petersen’s Theorem from 1891 [15], states that every bridgeless cubic graph has a 1-factor.

Let \( G \) be a cubic graph, \( k \geq 1 \), and \( S_k \) be a list of \( k \) 1-factors of \( G \). By a list we mean a collection with possible repetition. For \( i \in \{0, \ldots, k\} \) let \( E_i(S_k) \) be the set of edges that are in precisely \( i \) elements of \( S_k \). We define \( \mu_k(G) = \min\{|E_0(S_k)| : S_k \text{ is a list of } k \text{ 1-factors of } G\} \). If there is no harm of confusion we write \( E_i \) instead of \( E_i(S_k) \).

If \( \mu_3(G) = 0 \), then \( G \) is 3-edge-colorable. Cubic graphs with chromatic index 4 are the subject of many papers, since these graphs are potential counterexamples to many hard conjectures, see e.g. [3]. Cyclically 4-edge-connected cubic graphs with chromatic index 4 and girth at least 5 are also called snarks.

If \( \mu_5(G) = 0 \), then the edges of \( G \) can be covered by five 1-factors. This is conjectured to be true for all bridgeless cubic graphs by Berge (unpublished, see e.g. [5,14]).

**Conjecture 1.1** (Berge Conjecture). *Let \( G \) be a cubic graph. If \( G \) is bridgeless, then \( \mu_5(G) = 0 \).*

We say that \( G \) has a Berge-cover, if it satisfies Conjecture 1.1. Conjecture 1.1 is true if the following conjecture is true, which is attributed to Berge in [16]. This conjecture was first published in an article by Fulkerson [6].

**Conjecture 1.2** (Berge–Fulkerson Conjecture [6]). *Let \( G \) be a cubic graph. If \( G \) is bridgeless, then \( G \) has six 1-factors such that \( E_2 = E(G) \).*

We say that \( G \) has a Fulkerson-cover, if it satisfies Conjecture 1.2. Mazzuoccolo [14] proved that if all bridgeless cubic graphs have a Berge-cover, then all bridgeless cubic graphs have a Fulkerson-cover. Hence, Conjectures 1.1 and 1.2 are equivalent. Mazzuoccolo’s result does not prove the equivalence of a Berge- and a Fulkerson-cover for a single bridgeless cubic graph. It is an open question whether a given cubic graph with a Berge-cover has a Fulkerson-cover as well.

The following conjecture of Fan and Raspaud is true if Conjecture 1.2 is true. There are several other conjectures of this type involving cycle covers.

**Conjecture 1.3** ([4]). *Every bridgeless cubic graph has three 1-factors such that \( E_3 = \emptyset \).*

Eulerian graphs will be called *cycles* in the following. A *cycle cover* of a graph \( G \) is a set \( C \) of cycles such that every edge of \( G \) is contained in at least one cycle. It is a *double cycle cover* if every edge is contained in precisely two cycles, and a *k*-(double) cycle cover \( (k \geq 1) \) if \( C \) consists of at most \( k \) cycles. Circuits of length 2 are allowed, but the two edges of such a circuit must be different. The following conjecture was stated by Celmins and Preissmann independently.

**Conjecture 1.4** (see [18]). *Every bridgeless graph has a 5-cycle double cover.*

This conjecture is equivalent to its restriction to cubic graphs. In this case, a cycle is a 2-regular graph. If it is connected, then we call it *circuit*; that is, a cycle is a set of disjoint circuits. A cycle is *even* if it consists of even circuits. The length of a cycle cover \( C \) is the number of edges of \( C \). A 3-edge-colorable cubic graph \( G \) has a 2-cycle cover of length \( \frac{3}{2}|E(G)| \). Alon and Tarsi stated the following conjecture.

**Conjecture 1.5** ([1]). *Every bridgeless graph \( G \) has a cycle cover of length at most \( \frac{5}{2}|E(G)| \).*
In Section 2, we introduce the core of a cubic graph. Using structural properties of cores we show that if \( \mu_3(G) \neq 0 \), then \( 2\mu_3(G) \) is an upper bound for the girth of a cubic graph \( G \). If \( G \) is a bridgeless cubic graph without nontrivial 3-edge-cut and \( \mu_3(G) \leq 4 \), then \( G \) has a Berge-cover. If \( G \) is a bridgeless cubic graph and \( \mu_3(G) \leq 6 \), then \( G \) satisfies Conjecture 1.3. If \( \mu_3(G) < \text{girth}(G) \), then \( E_3 = \emptyset \) for any three 1-factors with \( |E_0| = \mu_3(G) \). We prove some new upper bounds for the length of a shortest cycle cover of bridgeless cubic graphs. If \( G \) is triangle-free and \( \mu_3(G) \leq 5 \), then \( G \) has an even 3-cycle cover of length at most \( \frac{4}{3}|E(G)| + 2 \). With those methods, some earlier results of [4] and [11] are improved to even 3-cycle covers.

Section 3 proves that if \( \mu_4(G) \in \{0, 1, 2, 3\} \), then \( G \) has 4-cycle cover of length \( \frac{4}{3}|E(G)| + 4\mu_4(G) \). If \( \mu_4(G) = 0 \), then there is a shortest 4-cycle cover of \( G \) that is even and has edge depth at most 2, i.e. these graphs satisfy two conjectures of Zhang (Conjectures 8.11.5 and 8.11.6 in [18]). Furthermore, \( G \) has a 5-cycle double cover. We also give a negative answer to Problem 8.11.4 of [18].

It seems that snarks that have only cycle covers with more than \( \frac{4}{3}|E(G)| \) edges are rare. There are only two such graphs with at most 36 vertices [3]. One of these graphs is the Petersen graph and the other one has 34 vertices. Both graphs have a cycle cover of length \( \frac{4}{3}|E(G)| + 1 \).

The article concludes with a few remarks on hypohamiltonian snarks.

2. THREE 1-FACTORS AND THE CORE OF A CUBIC GRAPH

Let \( M \) be a graph. If \( X \subseteq E(M) \), then \( M[X] \) denotes the graph whose vertex set consists of all vertices of edges of \( X \) and whose edge set is \( X \). Let \( A, B \) be two sets, then \( A \oplus B \) denotes their symmetric difference.

Let \( G \) be a cubic graph and \( S_3 \) be a list of three 1-factors \( M_1, M_2, M_3 \) of \( G \). Let \( \mathcal{M} = E_2 \cup E_3, U = E_0 \), and \( |U| = k \). The \( k \)-core of \( G \) with respect to \( S_3 \) (or to \( M_1, M_2, M_3 \)) is the subgraph \( G_c \) of \( G \) that is induced by \( \mathcal{M} \cup U \); that is, \( G_c = G[M \cup U] \). If the value of \( k \) is irrelevant, then we say that \( G_c \) is a core of \( G \). Clearly, if \( M_1 = M_2 = M_3 \), then \( G_c = G \). A core \( G_c \) is proper if \( G_c \neq G \).

Proposition 2.1 follows from the strict version of Petersen’s Theorem [15] that for every edge \( e \) of a bridgeless cubic graph there is a 1-factor containing \( e \).

**Proposition 2.1.** Let \( G \) be a cubic graph. If \( G \) has a 1-factor, then it has a core. Furthermore, if \( G \) is bridgeless, then for every \( v \in V(G) \) there is a core \( G_c \) such that \( v \notin V(G_c) \).

**Lemma 2.2.** Let \( G_c \) be a core of a cubic graph \( G \), and \( K_c \) be a component of \( G_c \). Then \( \mathcal{M} \) is a 1-factor of \( G_c \), and

1. \( K_c \) is either an even circuit or it is a subdivision of a cubic graph \( K \), and
2. if \( K_c \) is a subdivision of a cubic graph \( K \), then \( E(K_c) \cap E_3 \) is a 1-factor of \( K \), and every edge of \( K \) is subdivided by an even number of vertices.

**Proof.** Let \( G_c \) be a core of \( G \) with respect to three 1-factors \( M_1, M_2, \) and \( M_3 \). The set \( \mathcal{M} \) is a matching in \( G \). Hence, every vertex \( v \) of \( G_c \) is incident to at most one edge of \( \mathcal{M} \). Since \( M_1, M_2, \) and \( M_3 \) are 1-factors, \( v \) cannot be incident to three edges of \( U \). Hence, \( \mathcal{M} \) is a 1-factor of \( G_c \). If \( v \) is incident to an edge of \( E_3 \), then \( d_{G_c}(v) = 3 \), and if it is incident to an edge of \( E_2 \), then \( d_{G_c}(v) = 2 \).
Let $K_c$ be a component of $G_c$. If it has no trivalent vertices, then $E(K_c) \cap M$ is a 1-factor of $K_c$ and hence, $K_c$ is an even circuit, whose edges are in $M$ and $U$, alternately. If $K_c$ contains trivalent vertices, then it is a subdivision of a cubic graph $K$.

(2) The set $E(K_c) \cap E_3$ is a matching in $K_c$ that covers all trivalent vertices of $K_c$. Since $K$ is obtained from $K_c$ by suppressing the bivalent vertices and the edges of $E(K_c) \cap E_3$ are unchanged, it follows that $E(K_c) \cap E_3$ is a 1-factor $F$ of $K$.

Furthermore, every edge $e \in E(K) - F$ corresponds to a path in $K_c$ that starts and ends with an edge of $U$. Hence, it is subdivided by an even number of vertices. \hspace{1cm} \blacksquare

**Lemma 2.3.** Let $k \geq 0$. If $G_c$ is a $k$-core of a cubic graph $G$, then $|M| = k - |E_3|$.

**Proof.** Every vertex of an edge of $E_2$ has degree 1 in $G[U]$, and every vertex of an edge of $E_3$ has degree 2 in $G[U]$. Hence, $k = \frac{1}{2}[2|E_2| + 4|E_3|] = |E_2| + 2|E_3| = |M| + |E_3|$. \hspace{1cm} \blacksquare

The **dumbbell graph** is the unique cubic graph that is obtained from $K_2$ by adding a loop to each vertex.

**Lemma 2.4.** Let $k \geq 0$. If $G_c$ is a $k$-core of a connected cubic graph $G$, then

1. $k < 3$ if and only if $G$ is either the dumbbell graph or $G$ is 3-edge-colorable.
2. $|V(G_c)| = 2k - 2|E_3|$, and $|E(G_c)| = 2k - |E_3|$.
3. girth($G_c$) $\leq 2k$.
4. $G_c$ has at most $2k / \text{girth}(G_c)$ components.

**Proof.** Let $G_c$ be a $k$-core of $G$ with respect to three 1-factors $M_1, M_2, M_3$.

(1) $k < 3$. If $G_c$ has a bridge $e$, then all edges that are adjacent to $e$ are elements of $U$. Hence, $G_c$ is the dumbbell graph, and consequently $G = G_c$. If $G_c$ has no bridge, then there are $i, j$ such that $1 \leq i < j \leq 3$ and $M_i \cap M_j = \emptyset$. Hence, $M_i \cup M_j$ is an even 2-factor of $G$ and therefore, $G$ is 3-edge-colorable. The other direction is trivial.

(2) $M$ is a 1-factor of $G_c$. Hence, $|V(G_c)| = 2k - 2|E_3|$ by Lemma 2.3. Since $|U| = k$ and $M \cap U = \emptyset$ it follows that $|E(G_c)| = |M| + |U| = 2k - |E_3|$.

(3, 4) Since $G_c$ has minimum degree 2 and at most $2k$ vertices, it follows that every component contains a circuit of length at most $2k$. Since $M$ is a 1-factor of $G_c$, it follows that each circuit contains at least $\frac{1}{2} \text{girth}(G_c)$ edges of $U$. Hence, there are at most $\frac{2k}{\text{girth}(G_c)}$ pairwise disjoint circuits in $G_c$. \hspace{1cm} \blacksquare

**Corollary 2.5.** Let $G$ be a loopless cubic graph. If $\mu_3(G) \neq 0$, then $\mu_3(G) \geq 3$ and girth($G$) $\leq 2\mu_3(G)$.

**A. The Conjecture of Fan and Raspaud**

If a core $G_c$ of a cubic graph is a cycle, then we say that $G_c$ is a cyclic core. A cubic graph $G$ has three 1-factors such that $E_3 = \emptyset$ if and only if $G$ has a cyclic core. Hence, Conjecture 1.3 can be formulated as a conjecture on cores in bridgeless cubic graphs.

**Conjecture 2.6 (Conj. 1.3).** Every bridgeless cubic graph has a cyclic core.

Let $K_2^3$ be the unique cubic graph on two vertices that are connected by three edges.
Theorem 2.7. Let \( k > 0 \) and \( G_c \) be a proper \( k \)-core of a cubic graph \( G \) with \( \text{girth}(G_c) \geq k \). Then

1. \( G_c \) is a circuit of length \( 2k \), or
2. \( k \) is even, \( \text{girth}(G_c) = k \) and \( G_c \) is the disjoint union of two circuits of length \( k \), or \( G_c \) is the graph \( K_2^3 \) with two of its edges subdivided by \( k - 2 \) vertices.

In particular, if additionally \( \text{girth}(G_c) > k \) or \( k \) is odd, then \( G_c \) is a circuit.

Proof. Let \( G_c \) be proper \( k \)-core of \( G \) and \( \text{girth}(G_c) \geq k \). By Lemma 2.4, \( |E(G_c)| = 2k - |E_3| \) and hence, \( |E(G_c)| \leq 2 \text{girth}(G_c) - |E_3| \) (*). Furthermore, \( G_c \) has at most two components.

If it has two components, then each of them contains a circuit. Hence, \( |E(G_c)| \geq 2 \text{girth}(G_c) \geq 2k \). Thus, \( |E_3| = 0 \) and \( |E(G_c)| = 2k \) and \( G_c \) is the disjoint union of two circuits \( C_1', C_2' \) of length \( k \). By Lemma 2.2, \( M \cap E(C_i') \) is a 1-factor of \( C_i' \). Hence, \( k \) is even.

Now suppose that \( G_c \) is connected. If \( |E_3| = 0 \), then it is a circuit of length \( 2k \). If \( |E_3| > 0 \), then it contains at least two circuits and, by (*), any two circuits of \( G_c \) intersect. Thus, \( G_c \) is bridgeless. Let \( e \in E_3 \). Since \( G_c - E_3 \) is a circuit \( C_e \), it follows that \( e \) is a chord of \( C_e \). Thus, there are two circuits \( C_1 \) and \( C_2 \) with \( E(C_1) \cap E(C_2) = \{e\} \). Hence, \( 2 \text{girth}(G_c) - |E_3| \geq |E(G_c)| \geq |E(C_1) \cup E(C_2)| = |E(C_1)| + |E(C_2)| - 1 \geq 2 \text{girth}(G_c) - 1 \), and therefore, \( |E_3| = 1 \) and \( |E(G_c)| = 2k - 1 \). Thus, \( G_c \) is a subdivision of \( K_2^3 \), where two edges are subdivided by \( k - 2 \) vertices. Furthermore, \( k \) is even by Lemma 2.2.

Since \( \text{girth}(G_c) \geq \text{girth}(G) \), Theorem 2.7 implies the following corollary.

Corollary 2.8. Let \( G \) be a cubic graph with \( \text{girth}(G) \geq \mu_3(G) \). Then every \( \mu_3(G) \)-core is bipartite. In particular, if additionally \( \text{girth}(G) > \mu_3(G) \) or \( \mu_3(G) \) is odd, then every \( \mu_3(G) \)-core is a circuit.

Let \( G \) be a bridgeless cubic graph. The minimum number of odd circuits in a 2-factor of \( G \) is the oddness of \( G \), which is denoted by \( \omega(G) \). Mácajová and Škoviera proved that Conjecture 1.3 is true for bridgeless cubic graphs with oddness at most 2.

Theorem 2.9 ([12]). Let \( G \) be a bridgeless cubic graph. If \( \omega(G) \leq 2 \), then \( G \) has a cyclic core.

We will use the following proposition for the proof of the next theorem.

Proposition 2.10 ([17]). Let \( G \) be a bridgeless non-3-edge-colorable cubic graph. There is a proper 4-edge-coloring of \( G \) with a color class that contains precisely two edges if and only if \( \omega(G) = 2 \).

Theorem 2.11. Let \( G \) be a simple bridgeless cubic graph. If \( \mu_3(G) \leq 6 \), then \( G \) has a cyclic core. In particular, if \( G \) is triangle-free and \( \mu_3(G) \leq 5 \), then every \( \mu_3(G) \)-core is cyclic.

Proof. Let \( G_c \) be a core of \( G \) with respect to three 1-factors \( M_1, M_2, M_3 \), and \( |E_0| = \mu_3(G) \). If \( \mu_3(G) = 0 \), then there is nothing to prove. Since \( G \) is bridgeless, it follows by Lemma 2.4 that \( \mu_3(G) \geq 3 \). Furthermore, \( G \) has no loop and hence, if \( \mu_3(G) = 3 \), then \( G_c \) is cyclic. We will use the following claim.
Claim 2.11.1. If $C$ is a circuit whose edges are in $\mathcal{M}$ and $U$ alternately, then for every $i \in \{1, 2, 3\}$ there is an edge $e \in \mathcal{M} \cap E(C)$ such that $e \notin M_i$.

Proof. Suppose to the contrary that there is $i \in \{1, 2, 3\}$ such that $|\mathcal{M} \cap E(C)| = |M_i \cap E(C)|$. Let $i = 3$. Then $M_3' = (M_3 - E(C)) \cup (U \cap E(C))$ is a 1-factor of $G$. With $M_1 = M_1', M_2 = M_2$, we get $|E_0'| = |E(G) - \bigcup_{i=1}^3 M_i| < |E(G) - \bigcup_{i=1}^3 M_i| = \mu_3(G)$, a contradiction.

Claim 2.11.1 implies, that $G_c$ does not contain a circuit of length 4 whose edges are in $\mathcal{M}$ and $U$ alternately. Hence, every component of $G_c$ contains at least three edges of $U$. If it has two components, then $\mu_3(G) = 6$ and each component is a circuit of length 6. Hence, $G_c$ is cyclic in this case.

Thus, we now assume that $G_c$ is connected. Suppose to the contrary that there is an edge $e \in E_3$.

If no edge of $E_3$ is a bridge, then $G_c - E_3$ is a circuit. If $|E_3| \in \{2, 3\}$, then it follows that $G_c$ contains a circuit with edges in $\mathcal{M}$ and $U$ alternately and there is an $i \in \{1, 2, 3\}$ such that $\mathcal{M} \cap E(C) = M_i \cap E(C)$. Hence, we obtain a contradiction with Claim 2.11.1. Hence, $|E_3| = 1$ and $G_c$ is a subdivision of $K_3^2$ where two edges are subdivided by four vertices, that is $\mu_3(G) = 6$.

(***) Consider $(\bigcup_{i=1}^3 M_i) - E(G_c)$ as proper 3-edge-coloring $\phi$ of $G - E(G_c)$. In any case $\phi$ can be extended to a proper 4-edge-coloring of $G$ that has a color class that contains precisely two edges. Now the result follows with Proposition 2.10 and Theorem 2.9.

For the remainder of the proof we suppose that $E_3$ contains a bridge of $G_c$. If it has more than one bridge, then $G_c - E_3$ has at least three components. It is easy to see that $|U| > 6$ in this case, contradicting the fact that $\mu_3(G) \leq 6$.

We now assume that $e$ is the only bridge of $G_c$. Then $e$ connects two disjoint circuits $C_1$ and $C_2$ that form a 2-factor of $G_c$. Since $\mu_3(G) \leq 6$, it follows that $|E(C_1)| + |E(C_2)| \in \{6, 8, 10\}$. If $\mu_3(G) \leq 5$, then one of these two circuits is a triangle. Hence, if $G$ is triangle-free, then we obtain a contradiction and the statement for triangle-free graphs is proved. In the other cases we argue as in (*** that there is an appropriate proper 4-edge-coloring of $G$ such that $G$ has a cyclic core by Proposition 2.10 and Theorem 2.9. □

B. Berge–Fulkerson Conjecture

Following [16] we define a $p$-tuple edge multicoloring ($p > 1$) of a bridgeless cubic graph $G$ as a list of $3p$ 1-factors such that $E_p = E(G)$.

Theorem 2.12 ([16]). Let $G$ be a bridgeless cubic graph that has no nontrivial 3-edge-cut. If $M$ is a 1-factor of $G$, then there are an integer $p > 1$ and a $p$-tuple edge multicoloring of $G$ using $M$.

Lemma 2.13. Let $G$ be a bridgeless cubic graph that has no nontrivial 3-edge-cut, $M$ a 1-factor of $G$ and $P$ a path of length 3. If $M$ contains no edge of $P$, then there is a 1-factor $M'$ of $G$ that contains the two endedges of $P$.

Proof. Let $P$ be a path with vertex set $\{v_1, \ldots, v_4\}$ and edge set $\{v_i; v_{i+1} : 1 \leq i \leq 3\}$, and let $e = v_1v_2$ and $e' = v_3v_4$. We will show that there is a 1-factor $M'$ that contains $e$ and $e'$.

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Let $f_2, f_3$ be the edges that are adjacent to $v_2, v_3$, respectively, and which are no edges of $P$. If $f_2 = f_3$, (i.e. $v_2$ and $v_3$ are connected by two edges) then every 1-factor that contains $e$ has to contain $e'$.

If $f_2 \neq f_3$, then $f_2, f_3 \in M$. Theorem 2.12 implies that there exist an integer $p > 1$ and a $p$-tuple edge multicoloring $\phi$ of $G$ using $M$. Let $M_1, \ldots, M_p$ be the $p$-factors of $G$ that contain $e$. If $M_i$ does not contain $e'$, then it contains $f_3$. Since $f_3 \in M$ and $M_i \neq M$, for all $i \in \{1, \ldots, p\}$, there is $j \in \{1, \ldots, p\}$ such that $f_3 \notin M_j$. Hence, $M_j$ contains $e$ and $e'$. ■

**Theorem 2.14.** Let $G$ be a bridgeless cubic graph that has no nontrivial 3-edge-cut. If $\mu_3(G) \leq 4$, then $G$ has a Berge-cover.

**Proof.** If $\mu_3(G) = 0$, then $G$ is 3-edge-colorable, and it has a Berge-cover. Let $M_1, M_2, M_3$ be three 1-factors of $G$ such that $\mu_3(G) = |\ell|$, and $G_c$ be the induced core. Then $\mu_3(G) \geq 3$, by Corollary 2.5. Since $G$ is triangle-free, it follows with Theorem 2.11 that $G_c$ is cyclic. Because of the minimality of $\mu_3(G)$ it follows that $G_c$ is connected. Hence, the edges of $\ell$ can be paired into at most two pairs (one pair and a single edge if $\mu_3(G) = 3$), such that the edges of a pair are connected by an edge of $M$. Lemma 2.13 implies that there are two 1-factors $M_4$ and $M_5$ such that $\bigcup_{i=1}^{5} M_i = E(G)$. ■

C. Short Cycle Covers

**Theorem 2.15.** Let $k, l, t$ be nonnegative integers, and $G$ be a cubic graph. If $G$ has a $k$-core that has a $l$-cycle cover $C_c$ of length $t$, then $G$ has a $(l + 2)$-cycle cover $C$ of length at most $\frac{4}{5}(|E(G)| - k) + t$. Furthermore, $C$ is even if and only if $C_c$ is even.

**Proof.** Let $G_c$ be a $k$-core of $G$ with respect to three 1-factors $M_1, M_2, M_3$. Then $\{M_1 \oplus M_2, M_1 \oplus M_3, M_2 \oplus M_3\}$ are three even cycles that together cover, and only cover, each edge of $E_1 \cup E_2$ precisely twice. Thus, there are two of these three cycles covering $E_1 \cup E_2$ with total length at most $\frac{4}{5}|E_1 \cup E_2| \leq \frac{4}{5}(|E(G)| - k)$. These two cycles together with a cycle cover of $G_c$ gives a cycle cover of $G$ with length at most $\frac{4}{5}(|E(G)| - k) + t$. The cycle cover is even if and only if the cycle cover of $G_c$ is even. ■

The following result of Alon and Tarsi [1] and Bermond et al. [2] is the best known general result on the length of cycle covers.

**Theorem 2.16 ([1,2]).** Every bridgeless graph $G$ has a 3-cycle cover of length at most $\frac{5}{5}|E(G)|$.

**Theorem 2.17.** Let $k \geq 0$, and $G$ be a cubic graph. If $G$ has a bridgeless $k$-core, then $G$ has a cycle cover of length at most $\frac{4}{5}|E(G)| + 2k$.

**Proof.** Let $G_c$ be a bridgeless $k$-core of $G$. By Theorem 2.16, $G_c$ has a cycle cover of length at most $\frac{4}{5}|E(G_c)|$. By Lemma 2.4 we have $|E(G_c)| = 2k - |E_3|$, and hence, it follows with Theorem 2.15 that $G$ has a cycle cover of length at most $\frac{4}{5}|E(G)| + 2k$. ■

We are going to prove better bounds for the length of cycle covers of a cubic graphs that have a bipartite core. First we show that bipartite cores are bridgeless.

**Theorem 2.18.** Let $G_c$ be a core of a cubic graph. If $G_c$ is bipartite, then $G_c$ is bridgeless.

**Proof.** If $G_c$ is not bridgeless, then it has a component $K_c$ that contains a bridge. Furthermore, $K_c$ has a bridge $e$ such that one component of $K_c - e$ is 2-edge-connected.
Let $K'_c$ be a 2-edge-connected component of $K_c - e$. Since $e \in E_3$ and, by Lemma 2.2, $M \cap E(K_c)$ is a 1-factor of $K_c$, it follows that $|V(K'_c)|$ is odd. Furthermore, if we remove all edges of $E_3$ from $K'_c$, then we obtain a set of circuits. Hence, $G_c$ contains an odd circuit. Therefore, it is not bipartite. ■

Theorem 2.19. Let $k \geq 0$, and $G$ be a cubic graph. If $G$ has a bipartite $k$-core $G_c$, then $G$ has an even 4-cycle cover of length at most $\frac{1}{2} |E(G)| + \frac{2}{3} k$. In particular, if $G_c$ is cyclic, then $G$ has an even 3-cycle cover of length at most $\frac{1}{4} |E(G)| + \frac{2}{3} k$.

Proof. Let $G_c$ be a bipartite $k$-core of $G$. Let $K_c$ be a component of $G_c$, and $k' = |E(K_c) \cap E_3|$. It suffices to prove that every component $K_c$ has an even cycle cover of length $2k'$. Then it follows that $G_c$ has an even cycle cover of length $2k$. Hence, $G$ has an even cycle cover of length at most $\frac{1}{4} |E(G)| + \frac{2}{3} k$ by Theorem 2.15.

Clearly, if $K_c$ is a circuit, then it has a 1-cycle cover of length $2k'$. Thus, the statement is true for $G_c$ is cyclic.

If $K_c$ is a component of $G_c$ that is not a circuit, then $E(K_c) \cap E_3 \neq \emptyset$, and by Theorem 2.18, $G_c$ is 2-edge-connected. Furthermore, $K_c$ is a subdivision of a cubic graph $H_c$. Let $E^* = E(K_c) \cap E_3$. Since the two vertices that are incident to an edge of $E^*$ have degree 3 in $K_c$, it follows that $K_c - E^*$ is a cycle. Since $K_c$ is bipartite, it is an even cycle that has a proper 2-edge-coloring. Thus, $E^*$ is a color class of a proper 3-edge-coloring $\phi$ of $K_c$. By Lemma 2.2, each edge of $H_c$ is subdivided by an even number of vertices. Hence, $\phi$ induces a proper 3-edge-coloring $\phi'$ on $H_c$, where $E^*$ is a color class. Let $C_{H_c}$ be a cycle cover of $H_c$ of length $\frac{4}{3} |E(H_c)|$, which is induced by $\phi'$ and that uses the edges of $E^*$ twice.

Now, $C_{H_c}$ induces a 2-cycle cover $C_{K_c}$ of $K_c$. The length of $C_{K_c}$ is $|E(K_c)| + |E^*|$. By Lemma 2.4 we have $|E(K_c)| = 2k' - |E^*|$. Hence, the length of $C_{K_c}$ is $2k'$.

It remains to show that $C_{K_c}$ is an even cycle. Every edge of $E^*$ is contained in precisely two cycles of $C_{K_c}$ and all the other in precisely one. Let $v$ be a vertex that is incident to two edges of $U$. Then $d_{G_c}(v) = 3$, and $v$ is incident to an edge of $E^*$. Hence, every circuit $C$ of $C_{K_c}$ does not contain any two consecutive edges of $U$. Since $M$ is a 1-factor of $G_c$, it follows that the edges of $C$ are in $M$ and $U$ alternately. Hence, $C$ has even length, and $C_{K_c}$ is an even 2-cycle cover of $K_c$. ■

Corollary 2.20. Let $G$ be a triangle-free bridgeless cubic graph. If $\mu_3(G) \leq 5$, then $G$ has an even 3-cycle cover of length at most $\frac{4}{3} |E(G)| + 2$.

Proof. If $\mu_3(G) \leq 5$, then Theorem 2.11 implies that the induced core is cyclic. If $\mu_3(G) = 5$, then $\frac{4}{3} |E(G)| + \lfloor \frac{2}{3} \mu_3(G) \rfloor$ is odd. Hence, the result follows with Theorem 2.19.

In [4] it is proved that if a cubic graph $G$ has a cyclic core, then it has a 3-cycle cover of length at most $\frac{14}{9} |E(G)|$. This result is improved to smaller than $\frac{14}{9} |E(G)|$ in [11]. We additionally deduce that there is an even 3-cycle cover of length smaller than $\frac{14}{9} |E(G)|$.

Corollary 2.21. Let $G$ be a cubic graph. If $G$ has a cyclic core, then $G$ has an even 3-cycle cover of length smaller than $\frac{14}{9} |E(G)|$.

Proof. If $G$ is 3-edge-colorable, then the statement is true. Let $G_c$ be a cyclic $k$-core of $G$. Then $G_c$ is 2-regular and it has $2k$ edges. Hence, $2k \leq \frac{2}{3} |E(G)|$ and therefore, $k \leq \frac{1}{3} |E(G)|$. If $k = \frac{1}{3} |E(G)|$, then $G_c$ is an even 2-factor of $G$ and hence, $G$ is...
3-edge-colorable. Thus, \( k < \frac{1}{3} |E(G)| \) and Theorem 2.19 implies that \( G \) has an even 3-cycle cover of length smaller than \( \frac{14}{3} |E(G)| \).

In [4] it is proved that if a cubic graph \( G \) has a Fulkerson-cover, then it has a 3-cycle cover of length at most \( \frac{22}{15} |E(G)| \). This bound is best possible for 3-cycle covers of bridgeless cubic graphs, since it is attained by the Petersen graph [11]. We additionally show that there exists a cycle cover that is even.

**Corollary 2.22.** Let \( G \) be a cubic graph which has a Fulkerson-cover.

(1) Then \( G \) has an even 3-cycle cover of length at most \( \frac{22}{15} |E(G)| \).

(2) If \( |V(G)| \not\equiv 0 \mod 10 \), then \( G \) has an even 3-cycle cover of length smaller than \( \frac{22}{15} |E(G)| \).

**Proof.**

(1) Let \( M_1, \ldots, M_6 \) be the six 1-factors of a Fulkerson-cover of \( G \). Since \( \binom{6}{2} = 15 \), there are two 1 factors, say \( M_1, M_2 \), such that \( |M_1 \cap M_2| \leq \frac{1}{15} |E(G)| \). We claim that there is \( i \in \{3, \ldots, 6\} \) such that \( |M_1 \cap M_2| + |M_i \cap M_1| + |M_2 \cap M_i| \leq \frac{1}{5} |E(G)| \). Suppose to the contrary that this is not true. Then \( \sum_{i=3}^{6} (|M_1 \cap M_2| + |M_i \cap M_1| + |M_2 \cap M_i|) > \frac{2}{5} |E(G)| \). We have \( \sum_{i=3}^{6} (|M_1 \cap M_2| + |M_i \cap M_1| + |M_2 \cap M_i|) = \frac{2}{5} |E(G)| + 2|M_1 \cap M_2| \) and hence, \( |M_1 \cap M_2| > \frac{1}{5} |E(G)| \), which contradicts our choice of \( M_1 \) and \( M_2 \).

Let \( i = 3 \) and \( |M_1 \cap M_2| + |M_1 \cap M_3| + |M_2 \cap M_3| \leq \frac{1}{2} |E(G)| \). Since every edge is contained in precisely two 1-factors, the \( k \)-core with respect to \( M_1, M_5 \) and \( M_6 \) is cyclic and \( k \leq \frac{1}{3} |E(G)| \). Theorem 2.19 implies that \( G \) has an even 3-cycle cover of length at most \( \frac{22}{15} |E(G)| \).

(2) If \( |V(G)| \not\equiv 0 \mod 10 \), then \( |E(G)| \not\equiv 0 \mod 15 \), and we deduce as above that \( G \) has cyclic \( k \)-core and \( k < \frac{1}{5} |E(G)| \). Then the statement follows with Theorem 2.19. ■

Let \( G \) be a cubic graph that has a 1-factor and consequently a core. If \( G \) has a bridge, then every core of \( G \) has a bridge. We conjecture that the opposite direction of that statement is true as well and propose two conjectures.

**Conjecture 2.23.** Every bridgeless cubic graph has a proper bridgeless core.

**Conjecture 2.24.** Every bridgeless cubic graph has a proper bipartite core.

Conjecture 1.3 implies Conjecture 2.24, which implies Conjecture 2.23, by Theorem 2.18.

### 3. FOUR 1-FACTORS

Let \( C \) be a cycle cover of a graph \( G \). For \( e \in E(G) \), let \( ced_C(e) = |\{C : C \in C \text{ and } e \in E(C)\}| \), and \( \max\{ced_C(e) : e \in E(G)\} \) be the edge-depth of \( C \), which is denoted by \( ced_C(G) \). Zhang conjectured that every bridgeless graph has a shortest cycle cover of at most four cycles (Conjecture 8.11.5 in [18]), and that every 3-edge-connected graph has a shortest cycle cover \( C \) such that \( ced_C(G) \leq 2 \) (Conjecture 8.11.6 in [18]). The next theorem shows that we get the optimal bound for the length of a cycle cover if \( \mu_4(G) = 0 \), and that these graphs have a 5-cycle double cover. It also shows that these graphs satisfy the aforementioned conjectures of Zhang as well.

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Theorem 3.1. Let $G$ be a cubic graph. If $\mu_4(G) = 0$, then

1. $G$ has an even 4-cycle cover $C$ of length $\frac{4}{3}|E(G)|$, and $\text{cyc}_C(G) \leq 2$.
2. $G$ has a 5-cycle double cover.

Proof. The statements are true for 3-edge-colorable cubic graphs. We assume that $G$ is not 3-edge-colorable. Let $M_1, \ldots, M_4$ be four 1-factors of $G$ with $\mu_4(G) = 0$. Since $G$ is cubic it follows that $E_3 = E_4 = \emptyset$, $E_2$ is a 1-factor and $E_1$ is a 2-factor of $G$. For $i \in \{1, \ldots, 4\}$ let $F_i = E_2 \oplus M_i$. Then, $F = \{F_1, F_2, F_3, F_4\}$ is an even 4-cycle cover that covers each edge of $E_1$ once and each edge of $E_2$ twice. Hence, it has length $|E_1| + 2|E_2| = \frac{4}{3}|E(G)|$. Since $E_1$ is a 2-factor it follows that $\mathcal{F} \cup E_1$ is a 5-cycle double cover of $G$.

Theorem 3.1 (2) was proved by Hou et al. [8] independently.

Following Zhang [18] we say that the Chinese postman problem is equivalent to the shortest cycle cover problem, if the shortest length of a closed trail that covers all edges of $G$ is equal to the length of a shortest cycle cover. This is certainly true for cubic graphs $G$ that have a cycle cover of length $\frac{4}{3}|E(G)|$. Zhang asked the following question (Problem 8.11.4 in [18]): Let $h \geq 5$ and $G$ be a 3-edge-connected, cyclically $h$-edge-connected graph. If the Chinese Postman problem and the shortest cycle cover problem are equivalent for $G$, does $G$ admit a nowhere-zero 4-flow? The answer to this question is negative since for $h \in \{5, 6\}$ there are cyclically $h$-edge connected snarks with $\mu_4(G) = 0$. It is known that $\mu_4(G) = 0$ if $G$ is a flower snark or a Goldberg snark, see [5].

We now study the case, when the union of four 1-factors does not cover all edges of a cubic graph $G$.

Lemma 3.2. Let $G$ be a cubic graph that has four 1-factors $M_1, \ldots, M_4$ with $|E(G) - \bigcup_{i=1}^4 M_i| = k \geq 0$. If $E_4 = \emptyset$, then $G$ has a 4-cycle cover of length $\frac{4}{3}|E(G)| + 4k$.

Proof. If $G$ is 3-edge-colorable or $k = 0$, then the statement is true by Theorem 3.1. Let $G$ be not 3-edge-colorable, and $k > 0$. Since $E_4 = \emptyset$, it follows that $G$ is bridgeless.

For $i \in \{1, \ldots, 4\}$ and $j \in \{1, 2, 3\}$ let $M^j_i = M_i \cap E_j$ and $\bar{M}^j_i = (E(G) - M_i) \cap E_j$. For $i \in \{1, \ldots, 4\}$, let $C^j_i = M^j_i \cup \bar{M}^j_i \cup \bar{M}^2_i$. Every vertex of an edge of $M^j_i$ is incident either to an edge of $\bar{M}^2_i$ and to an edge of $\bar{M}^1_i$, or to an edge of $\bar{M}^3_i$ and to an edge of $\bar{U}$.

Every vertex of an edge of $M^1_i$ is incident either to an edge of $M^1_i$ and to an edge of $\bar{M}^1_i$, or to an edge of $\bar{M}^1_i$ and to an edge of $\bar{U}$.

Every vertex of an edge of $M^3_i$ is incident to an edge of $\bar{M}^3_i$ and an edge of $\bar{U}$.

Every vertex of an edge of $\bar{U}$ is incident either to an edge of $M^1_i$ and to an edge of $\bar{M}^3_i$, or to an edge of $M^1_i$ and to an edge of $\bar{M}^3_i$, or to an edge of $M^3_i$ and to an edge of $\bar{M}^1_i$. Hence, every vertex of $G[C^j_i]$ is adjacent to precisely two edges of $C^j_i$; that is, $C^j_i$ is a cycle. Note that the edges of $M^1_i$ (and hence, the vertices as well) are not in $G[C^j_i]$.

Thus, $\bigcup_{i=1}^4 C^j_i = E(G)$. Let $C^j = \{C^1_i, C^2_i, C^3_i, C^j_i\}$. Then $C^j$ is a 4-cycle cover of $G$. Each $e \in E(G)$ is either an element of $\bar{U}$ or there are $i \in \{1, \ldots, 4\}$ and $j \in \{1, 2, 3\}$ such that $e \in M^j_i$. If $e \in M^j_i$, then it is contained in precisely $j$ cycles and if $e \in \bar{U}$, then it is contained in all four cycles. Hence, the length of $C^j$ is $\frac{4}{3}|E(G)| + 4\mu_4(G)$.

Theorem 3.3. Let $G$ be a loopless cubic graph. If $\mu_4(G) \in \{0, 1, 2, 3\}$, then $G$ has a 4-cycle cover of length $\frac{4}{3}|E(G)| + 4\mu_4(G)$.
Proof. Since $\mu_4(G) \leq 3$, it follows that $E_4 = \emptyset$. The result follows with Lemma 3.2.

The bound of Corollary 3.3 is attained by the Petersen graph $P$ with $\mu_4(P) = 1$.

4. REMARK ON HYPOHAMILTONIAN SNARKS

A graph $G$ is hypohamiltonian if it is not hamiltonian but $G - v$ is hamiltonian for every vertex $v \in V(G)$. Non-3-edge-colorable, cubic hypohamiltonian graphs are cyclically 4-edge-connected and have girth at least 5, and there are cyclically 6-edge-connected hypohamiltonian snarks with girth 6, see [13]. Since hamiltonian cubic graphs are 3-edge-colorable, and $G - v$ is not 3-edge-colorable for every snark $G$, hypohamiltonian snarks could be considered as being closest to being 3-edge-colorable. Hypohamiltonian snarks have a proper 4-edge-coloring with a color class of cardinality 2. Hence, it follows with Proposition 2.10 and Theorem 2.9 that they have a cyclic core.

Corollary 2.5 implies, that if $G$ is a cubic graph and $\mu_3(G) = 3$, then $G$ has girth at most 6. It is easy to see that $\mu_3(G) = 3$, if $G$ is the Petersen graph or a flower snark, which are hypohamiltonian snarks. Jaeger and Swart [9] conjectured that (1) the girth and (2) the cyclic connectivity of a snark is at most 6. The first conjecture is disproved by Kochol [10] and the second is still open. We believe that both statements of Jaeger and Swart are true for hypohamiltonian snarks.

Conjecture 4.1. Let $G$ be a snark. If $G$ is hypohamiltonian, then $\mu_3(G) = 3$.

Hägkvist [7] proposed to prove the Fulkerson conjecture for hypohamiltonian graphs, which might be easier to prove than the general case. By Theorem 2.14, Conjecture 4.1 implies that hypohamiltonian snarks have a Berge cover, and together with Theorem 2.19 it implies that they have an even 3-cycle cover of length at most $\frac{4}{3}|E(G)| + 2$.

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REFERENCES

[1] N. Alon and M. Tarsi, Covering multigraphs by simple circuits, SIAM J Algebraic Discrete Methods 6 (1985), 345–350.
[2] J. C. Bermond, B. Jackson, and F. Jaeger, Shortest coverings of graphs with cycles, J Combin Theory Ser B 35 (1983), 297–308.
[3] G. Brinkmann, J. Goedgebeur, J. Hägglund, and K. Markström, Generation and properties of snarks, J Combin Theory Ser B 103 (2013), 468–488.
[4] G. Fan and A. Raspaud, Fulkerson’s conjecture and circuit covers, J Combin Theory Ser B 61 (1994), 133–138.
[5] J. L. Fouquet and J. M. Vanherpe, On the perfect matching index of bridgeless cubic graphs, arXiv:0904.1296v1 (2009).
[6] D. R. Fulkerson, Blocking and antiblocking pairs of polyhedra, Math Program 1 (1971), 168–194.
[7] R. Häggkvist, Problem 443. Special case of the Fulkerson Conjecture, Discrete Math 307 (2007), 650–658.
[8] X. Hou, H.-J. Lai, and C.-Q. Zhang, On matching coverings and cycle coverings, (preprint 2012).
[9] F. Jaeger and T. Swart, Conjecture 1 and 2, In: Combinatorics 79’ (M. Deza, I. G. Rosenberg, Eds.), Ann Disc Math 9 (1980), 305.
[10] M. Kochol, Snarks without small cycles, J Combin Theory Ser B 67 (1996), 34–47.
[11] E. Máčajová, A-Raspaud, M. Tarsi, and X. Zhu, Short cycle covers of graphs and nowhere-zero flows, J Graph Theory 68 (2011), 340–348.
[12] E. Máčajová and M. Škoviera, Perfect matchings with few comon edges in cubic graphs, Technical reports in Informatics TR-2009-020, Faculty of Mathematics, Physics, and Informatics, Comenius University, Bratislava, 2009.
[13] E. Máčajová and M. Škoviera, Constructing hypohamiltonian snarks with cyclic connectivity 5 and 6, Electronic J Combin 14 (2007), #R18.
[14] G. Mazzuoccolo, The equivalence of two conjectures of Berge and Fulkerson, J Graph Theory 68 (2011), 125–128.
[15] J. Petersen, Die Theorie der regulären graphs, Acta Mathematica 15 (1891), 193–220.
[16] P. D. Seymour, On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte, Proc London Math Soc 38(3) (1979), 423–460.
[17] E. Steffen, Classifications and characterizations of snarks, Discrete Math 188 (1998), 183–203.
[18] C.-Q. Zhang, Integer flows and cycle covers of graphs, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1997.