Global Well-Posedness for the 3D Axisymmetric Hall-MHD System with Horizontal Dissipation

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Abstract
Studied in this paper is the Cauchy problem for the 3D incompressible Hall-MHD system with horizontal dissipation. It is shown that if the initial data is axisymmetric and the swirl component of the velocity and the magnetic vorticity are trivial, such a system is globally well-posed for the large initial data. The key is to take full advantage of the structure of the Hall-MHD system in axisymmetric case to overcome the main difficulty due to the absence of vertical dissipation.

Keywords  Hall-MHD equations · Axisymmetric solutions · Global well-posedness

Mathematics Subject Classification  35Q35 · 76D03

1 Introduction

In this paper, we consider the following 3D incompressible Hall-MHD system with horizontal dissipation for velocity

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla P - \Delta_h u &= B \cdot \nabla B, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t B + u \cdot \nabla B + \nabla \times ((\nabla \times B) \times B) &= \Delta B + B \cdot \nabla u, \\
\text{div}u &= \text{div}B = 0, \\
(u,B)|_{t=0} &= (u_0,B_0), \quad x \in \mathbb{R}^3.
\end{aligned}
\] (1.1)

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Here we have written $\Delta_h := \partial_1^2 + \partial_2^2$, and the unknowns are the velocity $u = (u_1, u_2, u_3)$, the magnetic field $B = (B_1, B_2, B_3)$ and the pressure $P$. $(u_0, B_0)$ denotes the given initial velocity and initial magnetic field with $\text{div} u_0 = \text{div} B_0 = 0$.

In the 1960s, Lighthill in [19] pioneered systematic study of the Hall-MHD system. In comparison with the well-known MHD system, the system (1.1) includes the Hall term $\nabla \times (\nabla \times B) \times B$ due to the Ohm’s law. The Hall term describes deviation from charge neutrality between the electrons and the ions which plays a pivotal role in magnetic reconnection. Notice also that the dissipation for velocity only occurs in the horizontal direction in (1.1), which is very natural in the study of many phenomena in geophysical fluids. Indeed, in certain regimes and after suitable rescaling, experiments show that the vertical dissipation can be neglected as compared to the horizontal dissipation. More physical explanations about system (1.1) can be found in [2, 5, 6, 11, 15].

However, limited work has been done in mathematical theory on the Hall-MHD system. Let us recall the developments of well-posedness for the incompressible Hall-MHD system in brief. For the incompressible 3D Hall-MHD system with full dissipation for velocity, Chae-Degond-Liu in [4] proved the global existence of weak solutions and the local well-posedness of smooth solution when $(u_0, B_0) \in H^s(\mathbb{R}^3)$ with $s > \frac{5}{2}$. In [24], Wan-Zhou weakened the initial condition of local well-posedness to $(u_0, B_0) \in H^s(\mathbb{R}^3)$ with $s > \frac{3}{2}$. Dai in [7] showed the system is locally well-posed for initial data $(u_0, B_0) \in H^s \times H^{s+1-\varepsilon}(\mathbb{R}^3)$ with $s > \frac{1}{2}$ and any small enough $\varepsilon > 0$ such that $s - \varepsilon > \frac{1}{2}$. Very recently, the global well-posedness for small initial conditions $u_0$, $B_0$ and $\nabla \times B_0$ in critical spaces $\dot{B}^{\frac{3}{p} - 1}_{p, 1}$, $1 \leq p < \infty$, was obtained by Danchin-Tan in [8]. For the system (1.1), Fei-Xiang in [10] investigated the global existence of the smooth solutions for small initial data $(u_0, B_0) \in H^3(\mathbb{R}^3)$.

It is worth pointing out that all the above smooth solutions are discussed in the sense that either the solutions are locally well-posed, or the solutions are globally well-posed with small initial data. Until now, it is still not known whether or not the 3D Hall-MHD system with large initial data has a unique global smooth solution. Thus some authors are devoted to studying the solutions with some special structures. An important case is the geometric structure axisymmetric. Before preceding, let us introduce the definition of the axisymmetric vector fields.

**Definition 1.1** We say a vector field $f(x, t)$ is axisymmetric, if it can be written as

$$f(t, x) = f^r(t, r, z)e_r + f^\theta(t, r, z)e_\theta + f^z(t, r, z)e_z.$$

Here $(r, \theta, z)$ is the usual cylindrical coordinates in $\mathbb{R}^3$, that is, for any $x = (x_1, x_2, x_3)$,

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}, \quad z = x_3$$

and

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1).$$
Moreover, we call an axisymmetric vector field \( f \) is without swirl if \( f^\theta = 0 \).

The well-known work by Lei in [17] established the global well-posedness of the usual MHD system without any smallness assumptions for a class of specific axisymmetric initial data. More precisely, under the assumption that \( u_0^\theta = B_0^r = B_0^z = 0 \), he proved that there exists a unique global solution if the initial data satisfies

\[
(u_0, B_0) \in H^s(\mathbb{R}^3), \quad s \geq 2, \quad \text{and} \quad \frac{B_0^\theta}{r} \in L^\infty(\mathbb{R}^3).
\]

(1.2)

Ai and the first author in [1] weakened the initial condition (1.2) to \((u_0, B_0) \in H^1 \times H^2(\mathbb{R}^3) \) and \( \nabla \times u_0 \in L^2(\mathbb{R}^3) \). In addition, Fan–Huang–Nakamura in [9] extended the result of [17] to the usual incompressible Hall-MHD system. For the axisymmetric MHD system with horizontal dissipation and vertical magnetic diffusion, Jiu-Liu in [12] proved that there exists a unique global solution under the assumption that the initial data satisfies \( u_0^\theta = B_0^r = 0 \) and smooth enough. See [21–23], the authors also investigated the global well-posedness for 3D Boussinesq system with anisotropic dissipation corresponding to large axisymmetric data without swirl.

Inspired by the ideas in [12, 21], our main result in this paper is concerning the global existence and the uniqueness of axisymmetric smooth solution to the system (1.1) which does not have the swirl component for velocity field and magnetic vorticity. This means solution of the form:

\[
\begin{align*}
    u(t, x) &= u^r(t, r, z)e_r + u^z(t, r, z)e_z, \\
    B(t, x) &= B^\theta(t, r, z)e_\theta, \\
    P(t, x) &= P(t, r, z).
\end{align*}
\]

By direct computations, we find that

\[
\begin{align*}
    u \cdot \nabla &= u^r \partial_r + u^z \partial_z, \\
    \operatorname{div} u &= \partial_r u^r + \frac{u^r}{r} + \partial_z u^z, \\
    (B \cdot \nabla)B &= -\frac{(B^\theta)^2}{r} e_r, \\
    (B \cdot \nabla)u &= \frac{u^r}{r} B^\theta e_\theta,
\end{align*}
\]

and

\[
\nabla \times ((\nabla \times B) \times B) = -2\frac{B^\theta}{r} \partial_z B^\theta e_\theta = -\partial_z \frac{(B^\theta)^2}{r} e_\theta.
\]

Then the system (1.1) can be equivalently written in the cylindrical coordinates
where the Laplacian operator \( \Delta = \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz} \) and the horizontal Laplacian operator \( \Delta_h = \partial_{rr} + \frac{1}{r} \partial_r \).

On the other hand, we know that the vorticity \( \omega := \nabla \times u \) of the vector field \( u \) takes the form
\[
\omega = (\partial_z u^r - \partial_r u^z) e_\theta := \omega^\theta e_\theta
\]
and
\[
(\omega \cdot \nabla)u = \frac{u^r}{r} \omega^\theta e_\theta
\]
in the cylindrical coordinates. It follows from (1.3) that the quantity \( \omega^\theta \) obeys to the equation
\[
\partial_t \omega^\theta + u \cdot \nabla \omega^\theta - \left( \Delta_h - \frac{1}{r^2} \right) \omega^\theta = \frac{u^r}{r} \omega^\theta - \partial_z \left( \frac{B^\theta}{r} \right)^2.
\]

Now, let us state our main result as follows.

**Theorem 1.2** Suppose that \( u_0 \) and \( B_0 \) are two axisymmetric divergence free vector fields with \( u_0^\theta = B_0^r = B_0^z = 0 \). Let \( (u_0, B_0) \in H^2(\mathbb{R}^3) \) and \( \frac{B_0}{r} \in L^\infty(\mathbb{R}^3) \). Then the Hall-MHD system (1.1) has a unique global solution \( (u, B) \) which satisfies
\[
(u, B) \in C([0, \infty); H^2(\mathbb{R}^3)), \quad (\nabla_h u, \nabla B) \in L^2([0, \infty); H^2(\mathbb{R}^3)).
\]

There are two main difficulties in the proof of Theorem 1.2. The first one is there is no smoothing effect on the vertical derivative due to the absence of vertical viscosity. The other is how to deal with the Hall term \( \nabla \times ((\nabla \times B) \times B) \), which is quadratic in the magnetic field and contains the second-order derivatives. To overcome these two difficulties, we need to fully use the structure of (1.1) in axisymmmetric case with \( u_0^\theta = B_0^r = B_0^z = 0 \). Now we briefly sketch the proof of Theorem 1.2. In order to absorb the magnetic stretching term \( B \cdot \nabla u \) and the vortex stretching term \( \omega \cdot \nabla u \) into the convection term, we define
\[
\Omega := \frac{\omega^\theta}{r} \quad \text{and} \quad \Pi := \frac{B^\theta}{r},
\]
which strongly relies on the geometric structure of axisymmetric flows. Then by virtue of (1.3) and (1.4), the quantity \( (\Omega, \Pi) \) verifies
We see that in $\Omega$ equation leaving only one term involving the $\Pi$ as a forcing one. Thus, we can obtain the desired control on $\Omega$ by studying the properties of $\Pi$. Moreover, using Lemma 3.3, we can obtain the estimates of $\|u\|_{L^\infty}$ and $\|ur\|_{L^\infty}$, which allows us to get the $H^1$ bound of the $u$ and $B$ (Proposition 3.6). To make up for the shortage of vertical diffusion, we deeply use anisotropic inequality to get the $H^2$ estimate of $u$ (Proposition 3.8). Finally, we need to get the Lipschitz estimates of $u$ and $B$, which play the key role in our proof. To do this, by using the axisymmetric structure and the incompressible condition, we have the following estimate

\begin{equation}
\|\nabla u\|_{L^1([0,t];L^\infty)} \leq 2\|\partial_r u\|_{L^1([0,t];L^\infty)} + 2\|\frac{u'}{r}\|_{L^1([0,t];L^\infty)} + \|\partial_z u\|_{L^1([0,t];L^\infty)} + \|\partial_z u\|_{L^1([0,t];L^\infty)}.
\end{equation}

Taking advantage of the smooth effect of the velocity in the horizontal direction and Sobolev's embedding, we obtain the estimate every term in the right side. In addition, by using the regularity theory of the parabolic equation, we get the bound of $\|\nabla B\|_{L^2(\Omega;L^\infty)}$ (Proposition 3.9). With the Lipschitz estimate of $B$ in hand, we can show the $H^2$ estimate of $B$ (Proposition 3.10).

The present paper is built up as follows. Section 2 is devoted to collecting some useful inequalities, which will be used later. With these inequalities in hand, we will show some a priori estimate in Sect. 3. Finally, we give the proof of Theorem 1.2 in Sect. 4.

We end up this section with some notations we are going to use in this context.

**Notations:** For simplicity, we denote

$$\Phi_{k,c}(t) := c \exp(\cdots \exp(ct) \cdots) \quad (k \geq 1).$$

With the denote in hand, we will frequently use the following facts

$$\int_0^t \Phi_{k,c}(\tau) \, d\tau \leq \Phi_{k,c}(t) \quad \text{and} \quad \exp \left( \int_0^t \Phi_{k,c}(\tau) \, d\tau \right) \leq \Phi_{k+1,c}(t).$$

In addition, the letter $C$ stands for some generic constant, which may vary from line to line. For $A \leq B$, we mean that there is a uniform constant $C$ such that $A \leq CB$. We always denote $\int \cdot \, dx := \int_{\mathbb{R}^3} \cdot \, dx$ and $L^p([0, t]; L^q) := L^p([0, t]; L^q(\mathbb{R}^3))$.

## 2 Preliminaries

In this section, we list some useful lemmas, which will be used in the next section.
Lemma 2.1 (Proposition 2.5 of [21]) Let $u$ be a smooth axially symmetric vector field with zero divergence and $\omega = \omega^\theta e_\theta$ be its curl. Then, the following two estimates are true:

$$\|\partial_z \left( \frac{u^r}{r} \right) \|_{L^p} \leq C \| \frac{\omega^\theta}{r} \|_{L^p}, \quad \forall p \in (1, \infty)$$

and

$$\| \frac{u^r}{r} \|_{L^{\frac{p}{p-1}}} \leq C \| \frac{\omega^\theta}{r} \|_{L^p}, \quad \forall p \in (1, 3).$$

Lemma 2.2 (Lemma 2.4 of [22]) Assume $f$ is divergence free vector field and $1 < p < \infty$. Then there exists a constant $C > 0$ depending only on the dimension $n$ such that

$$\| \nabla f \|_{L^p(\mathbb{R}^n)} \leq C \frac{p^2}{p-1} \| \nabla \times f \|_{L^p(\mathbb{R}^n)}.$$

To deal with the Hall term $\nabla \times (\nabla \times (\nabla \times B) \times B)$, we need the following inequalities.

Lemma 2.3 (Commutator Estimates [14, 16]) Let $1 < p < \infty$, $s > 0$ and $\Lambda^s := (-\Delta)^{\frac{s}{2}}$. Then there exist two constants $C_s$ such that

$$\| \Lambda^s (fg) - f \Lambda^s g \|_{L^p} \leq C \left( \| \nabla f \|_{L^{p_1}} \| \Lambda^{s-1} g \|_{L^{p_2}} + \| \Lambda^s f \|_{L^{p_3}} \| g \|_{L^{p_4}} \right)$$

and

$$\| \Lambda^s (fg) \|_{L^p} \leq C \left( \| f \|_{L^{p_1}} \| \Lambda^s g \|_{L^{p_2}} + \| \Lambda^s f \|_{L^{p_3}} \| g \|_{L^{p_4}} \right),$$

where $1 < p_2, p_3 < \infty$ satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We employ the following anisotropic inequalities to make most efficient usage of the horizontal dissipation for velocity in (1.1).

Lemma 2.4 (Lemmas F.2 and F.3 of [21]) Let $f, g, h$ be smooth functions in $\mathbb{R}^3$. Then the following two inequalities are true:

(i) \[ \int_{\mathbb{R}^3} fgh \, dx dy dz \leq C \| f \|_{L^\frac{1}{3}} \| \partial_f \|_{L^\frac{1}{3}} \| g \|_{L^\frac{1}{3}} \| \nabla_h g \|_{L^\frac{1}{3}} \| h \|_{L^\frac{1}{3}} \| \nabla_h h \|_{L^\frac{1}{3}} \frac{1}{L^\frac{1}{3}}; \]

(ii) \[ \int_{\mathbb{R}^3} fgh \, dx dy dz \leq C \| f \|_{L^\frac{1}{3}} \| \partial_f \|_{L^\frac{1}{3}} \| g \|_{L^\frac{1}{3}} \| \nabla_h g \|_{L^\frac{1}{3}} \| h \|_{L^\frac{1}{3}} \| \nabla_h h \|_{L^\frac{1}{3}} \frac{1}{L^\frac{1}{3}}; \]

where the $C_s$ are both absolute constants.
3 A priori estimates

The main goal of this section is to establish some a priori estimates needed for the proof of Theorem 1.2. Let us begin with the basic $L^2$ estimate of $(u, B)$. It should be noted that in this estimate we do not need the axisymmetric assumption.

**Proposition 3.1** Let $(u_0, B_0) \in L^2$ be two divergence free vector fields. Then any smooth solution $(u, B)$ of the system (1.1) satisfies

$$
\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 + \int_0^t \left( \|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla B(\tau)\|_{L^2}^2 \right) d\tau \\
\leq \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2.
$$

**Proof** Taking the $L^2$-inner product of the first and second equations of (1.1) with $u$ and $B$, respectively, integrating by parts and taking the divergence free property into account, summing the result together, one has

$$
\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|B\|_{L^2}^2) + \|\nabla_h u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2 = \int (B \cdot \nabla B) \cdot u \, dx + \int (B \cdot \nabla u) \cdot B \, dx \\
- \int \nabla \times ((\nabla \times B) \times B) \cdot B \, dx,
$$

(3.1)

Notice that $\text{div} u = \text{div} B = 0$ and the operator $\nabla \times$ is symmetric, we have

$$
\int (B \cdot \nabla B) \cdot u \, dx + \int (B \cdot \nabla u) \cdot B \, dx = 0
$$

and

$$
\int \nabla \times ((\nabla \times B) \times B) \cdot B \, dx = \int ((\nabla \times B) \times B) \cdot (\nabla \times B) \, dx = 0.
$$

Inserting the above fact into (3.1) and integrating on $t$, it gives

$$
\|u(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2 = \int_0^t \left( \|\nabla_h u(\tau)\|_{L^2}^2 + \|\nabla B(\tau)\|_{L^2}^2 \right) d\tau \\
= \|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2,
$$

which implies the desired result. \qed

Next, our task is to obtain the $H^1$ estimate of $(u, B)$. Let us first show some estimates for $\Pi$ and $\Omega$.

**Proposition 3.2** Let $(u, B)$ be a smooth solution of the system (1.1) with $\frac{B_0}{u_0} \in L^\infty$ and $(u_0, B_0) \in H^2$ satisfying the assumptions in Theorem 1.2. Then there holds that
\[
\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}, \quad \forall \ 2 \leq p \leq \infty,
\]
\[
\int_0^t \|\nabla \Pi(\tau)\|^2_{L^2} \, d\tau \leq \|\Pi_0\|_{L^2},
\]

and

\[
\|\Omega(t)\|^2_{L^2} + \int_0^t \|\nabla \Omega(\tau)\|^2_{L^2} \, d\tau \leq \Phi_{1,\epsilon_0}(t).
\]

**Proof** We first get by multiplying the \(\Pi\) equation in (1.5) by \(|\Pi|^{p-2}\Pi\) and integrating on \(\mathbb{R}^3\) that

\[
\frac{1}{p} \frac{d}{dt} \|\Pi\|^p_{L^p} - 2\pi \int_{-\infty}^\infty \int_0^\infty \left( \frac{\partial_r^2}{r} + \frac{3}{r^2} \partial_r + \partial_z^2 \right) |\Pi|^{p-2} \Pi \, rd\tau dz
\]

\[
= \frac{2}{p+1} \int \partial_z |\Pi|^{p+1} \, dx - \int \left( u' \partial_r + u' \partial_z \right) \Pi \cdot |\Pi|^{p-2} \Pi \, dx.
\]

Using the incompressible condition \(\text{div} u = 0\) and the decay conditions

\[
\lim_{r \to \infty} \Pi(r, z, t) = \lim_{|z| \to \infty} \Pi(r, z, t) = 0,
\]

we have

\[
\frac{1}{p} \frac{d}{dt} \|\Pi\|^p_{L^p} + \frac{4(p-1)}{p^2} \int \left| \nabla |\Pi|^{\frac{p}{2}} \right|^2 \, dx = 0. \tag{3.2}
\]

Hence, integration on time implies

\[
\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}, \quad \forall \ 2 \leq p < \infty.
\]

Letting \(p \to \infty\) in above inequality, we obtain

\[
\|\Pi(t)\|_{L^\infty} \leq \|\Pi_0\|_{L^\infty}. \tag{3.3}
\]

In particular, taking \(p = 2\) in (3.2) and using Gronwall’s inequality, it infers

\[
\|\Pi(t)\|^2_{L^2} + \int_0^t \|\nabla \Pi(\tau)\|^2_{L^2} \, d\tau \leq \|\Pi_0\|^2_{L^2}. \tag{3.4}
\]

On the other hand, by taking the \(L^2\)-inner product of the \(\Omega\) equation in (1.5) with \(\Omega\), we deduce from the incompressible condition \(\text{div} u = 0\) that

\[
\frac{1}{2} \frac{d}{dt} \|\Omega\|^2_{L^2} + \|\nabla_h \Omega\|^2_{L^2} = \int \frac{2}{r} \Omega \partial_r \Omega \, dx - \int \Omega \partial_z \Pi^2 \, dx. \tag{3.5}
\]

Using the decay condition \(\lim_{r \to \infty} \Pi(r, z, t) = 0\), one has

\[
\int \frac{2}{r} \Omega \partial_r \Omega \, dx = 2\pi \int_{-\infty}^\infty \int_0^\infty \partial_r |\Omega|^2 \, dr dz = -2\pi \int_{-\infty}^\infty |\Omega(0, z, t)|^2 \, dz.
\]
The Hölder inequality and Young inequality yield

\[-\int \Omega \partial_z \Pi^2 \, dx = -2 \int \Omega \Pi \partial_z \Pi \, dx \leq 2 \| \Pi \|_{L^\infty} \| \partial_z \Pi \|_{L^2} \| \Omega \|_{L^2} \leq C \| \Pi \|_{L^\infty}^2 \| \Omega \|_{L^2}^2 + \| \partial_z \Pi \|_{L^2}^2.\]

Hence, putting all the above estimates into (3.5), we have

\[\frac{1}{2} \frac{d}{dt} \| \Omega \|_{L^2}^2 + \| \nabla_h \Omega \|_{L^2}^2 + 2\pi \int_{-\infty}^{\infty} |\Omega(0, z, t)|^2 \, dz \leq C \| \Pi \|_{L^\infty}^2 \| \Omega \|_{L^2}^2 + \| \partial_z \Pi \|_{L^2}^2.\]

An application of Gronwall’s inequality yields

\[\| \Omega(t) \|_{L^2}^2 + \int_0^t \| \nabla_h \Omega(\tau) \|_{L^2}^2 \, d\tau \leq C \left( \| \Omega_0 \|_{L^2}^2 + \int_0^t \| \partial_z \Pi(\tau) \|_{L^2}^2 \, d\tau \right) \exp \int_0^t \| \Omega(\tau) \|_{L^\infty}^2 \, d\tau \leq \Phi_{1, \alpha_0}(t),\]

where we used the estimates (3.3) and (3.4). This ends the proof of Proposition 3.2.

\[\square\]

Using the $L^p$ boundedness of Riesz operator and the Biot-Savart law, we have the following important lemma, which links the velocity to the vorticity.

**Lemma 3.3** (Proposition 3.4 of [22]) Let $u$ be a smooth axially symmetric vector field with zero divergence and $\omega = \omega^\theta e_\theta$ be its curl. Then there exist two absolute constants $C_\ast$ such that

\[\| u \|_{L^\infty} \leq C \| \omega^\theta \|_{L^\frac{3}{2}} \| \nabla_h \omega^\theta \|_{L^\frac{3}{2}}\]

and

\[\| \frac{u^\tau}{r} \|_{L^\infty} \leq C \| \frac{\omega^\theta}{r} \|_{L^\frac{3}{2}} \| \nabla_h (\frac{\omega^\theta}{r}) \|_{L^\frac{3}{2}}.\]

Thus we immediately obtain the following corollary.

**Corollary 3.4** Under the assumptions of Proposition 3.2, then there holds

\[\int_0^t \| \frac{u^\tau}{r}(\tau) \|_{L^\infty} \, d\tau \leq \Phi_{1, \alpha_0}(t).\]

**Proof** We apply the Hölder inequality and Proposition 3.2 to get
\[
\int_0^t \left\| \frac{u^r(\tau)}{r} \right\|_{L^\infty} \, d\tau \leq C \sup_{0 \leq \tau \leq t} \left\| \frac{\omega^\theta}{r}(\tau, \cdot) \right\|_{L^2} \left( \int_0^t \left\| \nabla \left( \frac{\omega^\theta}{r} \right)(\tau) \right\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t 1 \, d\tau \right)^{\frac{1}{2}} \leq \Phi_{1,c_0}(t),
\]
which gives the desired result. \(\square\)

To achieve the \(H^1\) estimate of \((u, B)\), we also need the following estimate of \(\left\| \frac{B}{u} \right\|_{L^p}\) with \(2 \leq p \leq \infty\).

**Proposition 3.5** Let \((u, B)\) be a smooth solution of the system (1.1) with \(\frac{\omega^\theta}{r} \in L^\infty\) and \((u_0, B_0) \in H^2\) satisfying the assumptions in Theorem 1.2. Then we have
\[
\left\| B^\theta(t) \right\|_{L^p} \leq \Phi_{2,c_0}(t), \quad \forall \ 2 \leq p \leq \infty.
\]

**Proof** For any \(2 \leq p < \infty\), multiplying the \(B^\theta\) equation in (1.3) by \(|B^\theta|^{p-2}B^\theta\) and integrating on \(\mathbb{R}^3\), we get from the Hölder inequality that
\[
\frac{1}{p} \frac{d}{dt} \left\| B^\theta \right\|_{L^p}^p + \frac{4(p-1)}{p^2} \int \left| \nabla |B^\theta|^\frac{p}{2} \right|^2 \, dx + \left\| \frac{B^\theta}{r} \right\|_{L^2}^2 \leq \int \left| B^\theta \right|^p \left| \frac{u^r}{r} \right| \, dx \leq \left\| \frac{u^r}{r} \right\|_{L^\infty} \left\| B^\theta \right\|_{L^p}^p.
\]

Hence, thanks to Gronwall’s inequality and Corollary 3.4, one has
\[
\left\| B^\theta(t) \right\|_{L^p} \leq \left\| B^\theta_0 \right\|_{L^p} \exp \left( \int_0^t \left\| \frac{u^r}{r} \right\|_{L^\infty} \, d\tau \right) \leq \Phi_{2,c_0}(t). \quad (3.6)
\]

Passing \(p \to \infty\) in (3.6), we obtain the desired result. \(\square\)

Based on the estimates established above, we derive the \(H^1\) estimate of \((u, B)\).

**Proposition 3.6** Let \((u, B)\) be a smooth solution of the system (1.1) with \(\frac{\omega^\theta}{r} \in L^\infty\) and \((u_0, B_0) \in H^2\) satisfying the assumptions in Theorem 1.2. Then there holds that
\[
\left\| u(t) \right\|_{H^1}^2 + \int_0^t \left\| \nabla u(\tau) \right\|_{H^1}^2 \, d\tau \leq \Phi_{2,c_0}(t),
\]
and
\[
\left\| B(t) \right\|_{H^1}^2 + \int_0^t \left\| \nabla B(\tau) \right\|_{H^1}^2 \, d\tau \leq \Phi_{3,c_0}(t).
\]
\textbf{Proof} Firstly, from the incompressibility $\text{div}u = 0$, we get by taking the $L^2$- inner product of the Eq. (1.4) with $\omega^\theta$ that
\[
\frac{1}{2} \frac{d}{dt} \|\omega^\theta\|_{L^2}^2 + \|\nabla_h \omega^\theta\|_{L^2}^2 + \left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2 = \int \frac{u^\rho}{r} |\omega^\theta|^2 \, dx - \int \partial_z (B^\theta)^2 \frac{\omega^\theta}{r} \, dx. \tag{3.7}
\]

Thanks to Hölder’s inequality and Young’s inequality, it infers
\[
\int \frac{u^\rho}{r} |\omega^\theta|^2 \, dx \leq \|\frac{u^\rho}{r}\|_{L^\infty} \|\omega^\theta\|_{L^2}^2
\]
and
\[
- \int \partial_z (B^\theta)^2 \frac{\omega^\theta}{r} \, dx = -2 \int B^\theta \partial_z B^\theta \frac{\omega^\theta}{r} \, dx \leq 2 \|B^\theta\|_{L^\infty} \|\partial_z B^\theta\|_{L^2} \|\frac{\omega^\theta}{r}\|_{L^2}
\leq \|B^\theta\|_{L^\infty}^2 \|\partial_z B^\theta\|_{L^2}^2 + \frac{1}{2} \left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2.
\]

From this and (3.7), we obtain
\[
\frac{d}{dt} \|\omega^\theta\|_{L^2}^2 + \|\nabla_h \omega^\theta\|_{L^2}^2 + \left\| \frac{\omega^\theta}{r} \right\|_{L^2}^2 \leq \|\frac{u^\rho}{r}\|_{L^\infty} \|\omega^\theta\|_{L^2}^2 + \|B^\theta\|_{L^\infty}^2 \|\partial_z B^\theta\|_{L^2}^2.
\]

Therefore, thanks to Gronwall’s inequality, we get from Propositions 3.1, 3.5 and Corollary 3.4 that
\[
\|\omega^\theta(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \omega^\theta(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \left\| \frac{\omega^\theta}{r}(\tau) \right\|_{L^2}^2 \, d\tau \leq \left( \|\omega^\theta_0\|_{L^2}^2 + \int_0^t \|B^\theta(\tau)\|_{L^\infty}^2 \|\partial_z B^\theta(\tau)\|_{L^2}^2 \, d\tau \right) \exp \int_0^t \left\| \frac{u^\rho}{r}\right\|_{L^\infty} \, d\tau \leq \Phi_{2,c_0}(t).
\]

Notice that
\[
\|\omega\|_{L^2} = \|\omega^\theta\|_{L^2} \quad \text{and} \quad \|\nabla_h \omega\|_{L^2} = \|\nabla_h \omega^\theta\|_{L^2} + \left\| \frac{\omega^\theta}{r} \right\|_{L^2}.
\]

So we finally obtain that
\[
\|\omega(t)\|_{L^2}^2 + \int_0^t \|\nabla_h \omega(\tau)\|_{L^2}^2 \, d\tau \leq \Phi_{2,c_0}(t).
\]

Combining with Lemma 2.2 leads to the first claimed estimate.

Applying Lemma 3.3 yields
\[
\int_0^t \|u(\tau)\|_{L^\infty}^2 \, d\tau \leq C \int_0^t \|\omega^\theta(\tau)\|_{L^2} \|\nabla_h \omega^\theta(\tau)\|_{L^2} \, d\tau \leq C \sup_{0 \leq \tau \leq t} \|\omega^\theta(\tau)\|_{L^2} \left( \int_0^t \|\nabla_h \omega^\theta(\tau)\|_{L^2}^2 \, d\tau \right)^{1/2} \left( \int_0^t 1 \, d\tau \right)^{1/2} \leq \Phi_{2,c_0}(t).
\]

(3.8)

On the other hand, due to the fact that \( B = B^\theta e_\theta \), we can rewrite the \( B \) equation in (1.1) as

\[
\partial_t B + u \cdot \nabla B = \Delta B + \frac{u^r}{r} B + 2 \frac{B}{r} \partial_z B.
\]

(3.9)

Multiplying the Eq. (3.9) by \(-\Delta B\) and integrating on \( \mathbb{R}^3 \), we get

\[
\frac{1}{2} \frac{d}{dt} \|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 = \int u \cdot \nabla B \cdot \Delta B \, dx - \int \frac{u^r}{r} B \cdot \Delta B \, dx - 2 \int \frac{B}{r} \partial_z B \Delta B \, dx.
\]

(3.10)

While thanks to Hölder’s inequality and Young’s inequality again, we deduce that

\[
\int u \cdot \nabla B \cdot \Delta B \, dx \leq \|u\|_{L^\infty} \|\nabla B\|_{L^2} \|\Delta B\|_{L^2} \leq C \|u\|_{L^\infty}^2 \|\nabla B\|_{L^2}^2 + \frac{1}{4} \|\Delta B\|_{L^2}^2
\]

and

\[
\left| \int \frac{u^r}{r} B \cdot \Delta B \, dx \right| \leq \left| \frac{u^r}{r} \right|_{L^\infty} \|B\|_{L^2} \|\Delta B\|_{L^2} \leq C \left| \frac{u^r}{r} \right|_{L^\infty}^2 \|B\|_{L^2}^2 + \frac{1}{4} \|\Delta B\|_{L^2}^2.
\]

Using the interpolation estimate \( \|f\|_{L^3} \leq C \|f\|_{L^2}^{1/2} \|\nabla f\|_{L^2}^{1/2} \) gives rise to

\[
2 \int \frac{B}{r} \partial_z B \cdot \Delta B \, dx \leq 2 \left| \frac{B}{r} \right|_{L^\infty} \|\partial_z B\|_{L^3} \|\Delta B\|_{L^2} \leq C \left| \frac{B}{r} \right|_{L^\infty} \|\nabla B\|_{L^2}^{1/2} \|\Delta B\|_{L^2}^{3/2} \leq C \left| \frac{B}{r} \right|_{L^\infty}^4 \|\nabla B\|_{L^2}^2 + \frac{1}{4} \|\Delta B\|_{L^2}^2.
\]

Thus, inserting the above estimates into (3.10), we have

\[
\frac{d}{dt} \|\nabla B\|_{L^2}^2 + \|\Delta B\|_{L^2}^2 \leq C \left( \|u\|_{L^\infty}^2 + \|\frac{B}{r}\|_{L^\infty}^4 \right) \|\nabla B\|_{L^2}^2 + C \left| \frac{u^r}{r} \right|_{L^\infty}^2 \|B\|_{L^2}^2.
\]

By the Gronwall inequality and Propositions 3.1, 3.2, Corollary 3.4 and (3.8), one has
\[ \| \nabla B(t) \|_{L^2}^2 + \int_0^t \| \Delta B(\tau) \|_{L^2}^2 \, d\tau \leq C \left( \| \nabla B_0 \|_{L^2}^2 + \int_0^t \frac{\| u(\tau) \|_{L^\infty}^2 + \| B(\tau) \|_{L^4}^4}{r(\tau)} \, d\tau \right) \times \exp \int_0^t \left( \| u(\tau) \|_{L^\infty}^2 + \| B(\tau) \|_{L^4}^4 \right) \, d\tau \leq \Phi_{3, \epsilon_0}(t), \]

which implies the second desired estimate. We finish the proof of Proposition 3.6.

In order to get the \( H^2 \) estimate of \((u, B)\), we first give the following proposition.

**Proposition 3.7** Let \((u, B)\) be a smooth solution of the system (1.1) with \( \frac{\partial^6}{\partial t^6} \in L^\infty \) and \((u_0, B_0) \in H^2\) satisfying the assumptions in Theorem 1.2. Then the following estimate holds

\[ \| \nabla \Pi(t) \|_{L^2}^2 + \int_0^t \| \Delta \Pi(\tau) \|_{L^2}^2 \, d\tau \leq \Phi_{3, \epsilon_0}(t). \]

**Proof** Multiplying the \( \Pi \) equation of (1.5) by \( -\Delta \Pi \) and integrating on \( \mathbb{R}^3 \), we deduce that

\[ \frac{1}{2} \frac{d}{dt} \| \nabla \Pi \|_{L^2}^2 + \| \Delta \Pi \|_{L^2}^2 = -2 \int \frac{1}{r} \partial_r \nabla \Pi \cdot \Delta \Pi \, dx + \int u \cdot \nabla \Pi \cdot \nabla \Delta \Pi \, dx - 2 \int \nabla \Pi \cdot \Delta \Pi \, dx. \]  

(3.11)

A routine calculation gives

\[ -2 \int \frac{1}{r} \partial_r \nabla \Pi \cdot \Delta \Pi \, dx \]

\[ = -4\pi \int \int_0^\infty \partial_1 \Pi \left( \frac{\partial_1 \Pi}{r^2} + \frac{1}{r} \partial_r \Pi + \frac{1}{r^2} \partial_z \Pi \right) \, dr \, dz \]

\[ = 2\pi \int \int_0^\infty \partial_1 |\partial_1 \Pi|^2 \, dr \, dz - 4\pi \int_0^\infty \int_0^\infty \frac{1}{r} |\partial_1 \Pi|^2 \, dr \, dz + 2\pi \int \int_0^\infty \partial_1 |\partial_2 \Pi|^2 \, dr \, dz \]

\[ = -2\pi \int \int_0^\infty |\partial_1 \Pi(0, z, t)|^2 \, dr \, dz - 2 \int \frac{|\partial_r \Pi|^2}{r^2} \, dx \]

\[ - 2\pi \int \int_0^\infty |\partial_2 \Pi(0, z, t)|^2 \, dr \, dz \leq 0. \]
Thanks to Hölder’s inequality and Young’s inequality, we get
\[
\int u \cdot \nabla \Pi \cdot \Delta \Pi \, dx \leq \|u\|_{L^\infty} \|\nabla \Pi\|_{L^2} \|\Delta \Pi\|_{L^2} \leq C \|u\|_{L^\infty}^2 \|\nabla \Pi\|_{L^2}^2 + \frac{1}{4} \|\Delta \Pi\|_{L^2}^2.
\]

An application of the interpolation estimate \([f]_{L^3} \leq C [f]_{L^2}^{\frac{1}{2}} \|\nabla f\|_{L^2}^{\frac{1}{2}}\) shows that
\[
2 \int \Pi \partial_z \Pi \Delta \Pi \, dx \leq 2 \|\Pi\|_{L^6} \|\partial_z \Pi\|_{L^3} \|\Delta \Pi\|_{L^2}
\leq C \|\Pi\|_{L^6} \|\nabla \Pi\|_{L^2}^{\frac{3}{2}} \|\Delta \Pi\|_{L^2}^{\frac{3}{2}}
\leq C \|\Pi\|_{L^6}^4 \|\nabla \Pi\|_{L^2}^2 + \frac{1}{4} \|\Delta \Pi\|_{L^2}^2.
\]

Substituting the above estimates into (3.11), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \Pi\|_{L^2}^2 + \|\Delta \Pi\|_{L^2}^2 \leq C \left( \|u\|_{L^\infty}^2 + \|\Pi\|_{L^6}^4 \right) \|\nabla \Pi\|_{L^2}^2,
\]
which along with Gronwall’s inequality, Propositions 3.2 and (3.8) ensure that
\[
\|\nabla \Pi(t)\|_{L^2}^2 + \int_0^t \|\Delta \Pi(\tau)\|_{L^2}^2 \, d\tau \leq C \|\nabla \Pi_0\|_{L^2}^2 \exp \int_0^t \left( \|u(\tau)\|_{L^\infty}^2 + \|\Pi(\tau)\|_{L^6}^4 \right) \, d\tau
\leq \Phi_{3,c_0}(t).
\]
This ends the proof of Proposition 3.7. \(\square\)

The next proposition describes the \(H^2\) estimate of \(u\).

**Proposition 3.8** Let \((u, B)\) be a smooth solution of the system (1.1) with \(\frac{B_0^2}{r} \in L^\infty\) and \((u_0, B_0) \in H^2\) satisfying the assumptions in Theorem 1.2. Then we have
\[
\|\nabla \omega(t)\|_{L^2}^2 + \int_0^t \|\nabla_h^2 \omega(\tau)\|_{L^2}^2 \, d\tau + \int_0^t \|\nabla_h \partial_z \omega(\tau)\|_{L^2}^2 \, d\tau \leq \Phi_{4,c_0}(t).
\]

**Proof** Recall that the equation for vorticity \(\omega = \omega^\theta e_\theta\) satisfies
\[
\partial_t \omega - \Delta_h \omega = -u \cdot \nabla \omega + \frac{u^\theta}{r} \omega - \partial_z \left( \frac{(B^\theta)^2}{r} \right) e_\theta.
\]
Multiplying the above equation by \(-\Delta \omega\) and integrating on \(\mathbb{R}^3\), we have
In the following, we estimate 
\[
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \left( \| \nabla_h \omega \|^2_{L^2} + \| \partial_z \omega \|^2_{L^2} \right) + \left( \| \nabla_h^2 \omega \|^2_{L^2} + \| \nabla_h \partial_z \omega \|^2_{L^2} \right) \\
&= \int u \cdot \nabla \omega \cdot \Delta \omega \, dx - \int \frac{u'}{r} \omega \cdot \Delta \omega \, dx \\
&+ \int \partial_c \left( \frac{(B^\theta)^2}{r} \right) e_\theta \cdot \Delta \omega \, dx \\
&= \int \left( u' \partial_r + u' \partial_z \right) \omega \cdot \partial_c(r \partial_r, \omega) \, drdz \\
&+ \int \left( u' \partial_r + u' \partial_z \right) \omega \cdot \partial_{zz} \omega \, dx \\
&- \int \frac{u'}{r} \omega \cdot \partial_r(r \partial_r, \omega) \, drdz - \int \frac{u'}{r} \omega \cdot \partial_{zz} \omega \, dx \\
&+ 2 \int \frac{B^\theta \partial_z B^\theta}{r} e_\theta \cdot \Delta \omega \, dx + 2 \int \frac{B^\theta \partial_z B^\theta}{r} e_\theta \cdot \partial_{zz} \omega \, dx \\
&= \left( \partial_r u' \partial_z \omega \cdot \partial_r \omega + \partial_z u' \partial_z \omega \cdot \partial_z \omega \right) \, dx \\
&+ \int \left( \partial_r u' \partial_z \omega \cdot \partial_r \omega + \partial_z u' \partial_z \omega \cdot \partial_z \omega \right) \, dx \\
&+ \int \left( \partial_r u' \omega \cdot \partial_r \omega + \frac{u'}{r} \partial_r \omega \cdot \partial_z \omega \right) \, dx \\
&+ \int \left( \partial_z u' \omega \cdot \partial_z \omega + \frac{u'}{r} \partial_z \omega \cdot \partial_z \omega \right) \, dx \\
&+ 2 \int \frac{B^\theta \partial_z B^\theta}{r} e_\theta \cdot \Delta \omega \, dx + 2 \int \frac{B^\theta \partial_z B^\theta}{r} e_\theta \cdot \partial_{zz} \omega \, dx \\
&= \sum_{i=1}^6 I_i.
\end{aligned}
\]

Here we used in the cylindrical coordinates the Laplacian operator 
\( \Delta = \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz} \) and the incompressible condition \( \text{div} u = \partial_r u' + \frac{u'}{r} + \partial_z u' = 0 \).

In the following, we estimate \( I_i \) term by term. Applying Lemmas 2.1 and 2.4, we get

\[
|I_1| \leq \| \partial_r u' \|^2_{L^2} \| \nabla_h \partial_{zz} \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \nabla_h \partial_c \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \partial_z \partial_c \omega \|_{L^2} \\
+ \| \partial_r u' \|^2_{L^2} \| \nabla_h \partial_{zz} \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \nabla_h \partial_c \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \partial_z \partial_c \omega \|_{L^2} \\
\leq \| \omega \|^2_{L^2} \| \nabla_h \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \nabla_h \partial_c \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \partial_z \partial_c \omega \|_{L^2} \\
+ \| \omega \|^2_{L^2} \| \nabla_h \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \nabla_h \partial_c \omega \|_{L^2} \| \partial_r \omega \|^2_{L^2} \| \partial_z \partial_c \omega \|_{L^2} \\
\leq \frac{1}{8} \left( \| \nabla_h^2 \omega \|_{L^2}^2 + \| \nabla_h \partial_c \omega \|_{L^2}^2 \right) + C \left( \| \omega \|_{L^2}^2 + \| \nabla_h \omega \|_{L^2}^2 \right) \left( \| \partial_r \omega \|_{L^2}^2 + \| \partial_z \omega \|_{L^2}^2 \right),
\]

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\[
|I_2| \leq \| \partial \cdot u' \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot u' \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot u' \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot u' \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot u' \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot u' \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot u' \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2}
\]
\[
+ \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2}.
\]
\[
|I_3| \leq \| \partial \cdot u' \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot u' \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot u' \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2}.
\]

and

\[
|I_4| \leq \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2} + \| \partial \cdot \omega \|_{L^2} \left( \frac{1}{2} \| \nabla_h \partial \cdot \omega \|_{L^2} \right) \| \partial \cdot \omega \|_{L^2}.
\]

Using Hölder’s inequality and Young’s inequality, we have

\[
|I_5| \leq 2 \left| \frac{B^\theta}{r} \right| \| B^\theta \|_{L^6} \| \partial \cdot B^\theta \|_{L^3} \| \Delta_h \omega \|_{L^2}
\]
\[
\leq C \| \Pi \|_{L^2} \| \nabla B \|_{L^2} \| \Delta B \|_{L^2} \| \Delta_h \omega \|_{L^2}
\]
\[
\leq C \| \Pi \|_{L^2} \left( \| \nabla B \|_{L^2}^2 + \| \Delta B \|_{L^2}^2 \right) + \frac{1}{8} \| \Delta_h \omega \|_{L^2}^2.
\]

and

\[
I_6 = -2 \int \partial_{\ell} \left( \frac{B^\theta}{r} \right) \partial_{\ell} B^\theta \cdot \partial_{\ell} \omega \ dx - 2 \int \frac{B^\theta}{r} \partial_{\ell} B^\theta \cdot \partial_{\ell} \omega \ dx
\]
\[
\leq C \| \partial_{\ell} \|_{L^2} \| \partial_{\ell} B^\theta \|_{L^3} \| \partial_{\ell} \omega \|_{L^2} + C \| \Pi \|_{L^2} \| \partial_{\ell} B^\theta \|_{L^2} \| \partial_{\ell} \omega \|_{L^2},
\]
\[
\leq C \| \partial_{\ell} \|_{L^2} \| \partial_{\ell} B^\theta \|_{L^2}^2 \| \partial_{\ell} \omega \|_{L^2} + C \| \Pi \|_{L^2} \| \Delta B \|_{L^2}^2 + C \| \partial_{\ell} \omega \|_{L^2}^2,
\]
\[
\leq C \| \partial_{\ell} \|_{L^2} \| \partial_{\ell} B^\theta \|_{L^2}^2 + C \left( \| \Pi \|_{L^2}^2 + \| \Delta B \|_{L^2}^2 \right) + C \| \partial_{\ell} \omega \|_{L^2}^2.
\]
Summing up all the estimates $I_1-I_6$ leads to
\[
\frac{d}{dt} \left( \| \nabla u \|_{L^2}^2 + \| \partial_z \omega \|_{L^2}^2 \right) + \left( \| \nabla_h^2 \omega \|_{L^2}^2 + \| \nabla_h \partial_z \omega \|_{L^2}^2 \right) \\
\leq C \left( 1 + \| \omega \|_{L^2}^2 + \| \nabla_h \omega \|_{L^2}^2 + \| \frac{\omega}{r} \|_{L^2}^2 + \frac{u'}{r} \|_{L^\infty} + \| \Delta \Pi \|_{L^2} \right) \\
\times \left( \| \nabla_h \omega \|_{L^2}^2 + \| \partial_z \omega \|_{L^2}^2 \right) \\
+ C \left( 1 + \| \Pi \|_{L^6}^2 + \| \Pi \|_{L^\infty}^2 \right) \left( \| \nabla B \|_{L^2}^2 + \| \Delta B \|_{L^2}^2 \right).
\]

The Gronwall inequality implies
\[
\| \nabla \omega(t) \|_{L^2}^2 + \int_0^t \| \nabla_h^2 \omega(\tau) \|_{L^2}^2 \, d\tau + \int_0^t \| \nabla_h \partial_z \omega(\tau) \|_{L^2}^2 \, d\tau \leq \Phi_{4,c_0}(t),
\]
where we used Propositions 3.2, 3.6, 3.7 and Corollary 3.4. This ends the proof of Proposition 3.8.

The following proposition is devoted to studying the Lipschitz estimate for $(u, B)$, which plays a key role in what follows.

**Proposition 3.9** Let $(u, B)$ be a smooth solution of the system (1.1) with $\frac{\partial u}{\partial t} \in L^\infty$ and $(u_0, B_0) \in H^2$ satisfying the assumptions in Theorem 1.2. Then
\[
\int_0^t \| \nabla u(\tau) \|_{L^\infty} \, d\tau \leq \Phi_{4,c_0}(t) \quad \text{and} \quad \int_0^t \| \nabla B(\tau) \|_{L^\infty}^2 \, d\tau \leq \Phi_{4,c_0}(t).
\]

**Proof** From the structure of axisymmetric flows and the incompressible condition, we know that $\text{div} u = \partial_r u' + \frac{u'}{r} + \partial_z u' = 0$ and $\omega = \partial_z u' - \partial_r u'$. Hence
\[
\| \nabla u \|_{L^1([0,t]; L^\infty)} \leq \| \partial_r u' \|_{L^1([0,t]; L^\infty)} + \frac{u'}{r} \|_{L^1([0,t]; L^\infty)} + \| \partial_z u' \|_{L^1([0,t]; L^\infty)} \\
+ \| \partial_r u' \|_{L^1([0,t]; L^\infty)} + \| \partial_z u' \|_{L^1([0,t]; L^\infty)} \\
\leq 2\| \partial_r u' \|_{L^1([0,t]; L^\infty)} + 2\frac{u'}{r} \|_{L^1([0,t]; L^\infty)} + \| \partial_z u' \|_{L^1([0,t]; L^\infty)} \\
+ \| \partial_z u' \|_{L^1([0,t]; L^\infty)}.
\]

We first deduce from Lemma 3.3 and Proposition 3.8 that
\[
\int_0^t \| \partial_z u'(\tau) \|_{L^\infty} \, d\tau \leq C \int_0^t \| \partial_z \omega(\tau) \|_{L^2} \| \nabla_h \partial_z \omega(\tau) \|_{L^2} \, d\tau \\
\leq C \sup_{0 \leq \tau \leq t} \| \partial_z \omega(\tau, \cdot) \|_{L^2} \left( \int_0^t \| \nabla_h \partial_z \omega(\tau) \|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t \| \omega(\tau) \|_{L^2} \right)^{\frac{3}{2}} \\
\leq \Phi_{4,c_0}(t).
\]

Next, we turn to bound the quantity $\| \partial_r u' \|_{L^1([0,t]; L^\infty)}$ and $\| \partial_z u' \|_{L^1([0,t]; L^\infty)}$. Notice that for a divergence free vector function $f$, we have

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\[
f \cdot \nabla f = (\nabla \times f) \times f + \frac{1}{2} \nabla |f|^2.
\]

Hence, acting the operator $\nabla \times$ to the first equation of (1.1), we get from

\[
\nabla \times ((\nabla \times B) \times B) = -\partial_z \frac{(B^\theta)^2}{r} e_\theta
\]

that

\[
\partial_t (\nabla \times u) - \Delta_h (\nabla \times u) = -\nabla \times ((\nabla \times u) \times u) - \partial_z \frac{(B^\theta)^2}{r} e_\theta. \tag{3.12}
\]

By virtue of Proposition 3.8 and Sobolev’s embedding, we obtain

\[
\|u\|_{L^\infty([0,\tau];L^m)} \leq \|u\|_{L^\infty([0,\tau];L^2)} + \|\nabla \times u\|_{L^\infty([0,\tau];L^6)} \leq \Phi_{4,c_0}(t), \tag{3.13}
\]

which along with the Hölder inequality gives rise to

\[
\| (\nabla \times u) \times u \|_{L^1([0,\tau];L^6)} \leq \|u\|_{L^\infty([0,\tau];L^m)} \| \nabla \times u \|_{L^\infty([0,\tau];L^6)} \|
\]

\[
\| (\nabla \times u) \times u \|_{L^1([0,\tau];L^6)} \leq \Phi_{4,c_0}(t).
\]

Applying Propositions 3.2 and 3.5, one deduces that

\[
\left\| \frac{B^\theta}{r} \right\|_{L^1([0,\tau];L^6)} \leq \left\| B^\theta \right\|_{L^\infty([0,\tau];L^m)} \left\| \frac{B^\theta}{r} \right\|_{L^\infty([0,\tau];L^6)} \leq \Phi_{2,c_0}(t).
\]

Then by using the regular estimates of the velocity in horizontal direction for (3.12), see Lemma 3.4 in [21] for details, gives that

\[
\left\| \nabla \times u \right\|_{L^1([0,\tau];L^6)} \leq \left\| \nabla \omega \right\|_{L^2} + \left\| (\nabla \times u) \times u \right\|_{L^1([0,\tau];L^6)} + \left\| \frac{B^\theta}{r} B^\theta \right\|_{L^1([0,\tau];L^6)} \leq \Phi_{4,c_0}(t).
\]

Thus, the Sobolev embedding implies that

\[
\left\| \nabla u \right\|_{L^1([0,\tau];L^\infty)} \leq \Phi_{4,c_0}(t),
\]

which ensures that

\[
\left\| \partial_t u \right\|_{L^1([0,\tau];L^\infty)} + \left\| \partial_t \hat{u} \right\|_{L^1([0,\tau];L^\infty)} \leq \left\| \nabla u \right\|_{L^1([0,\tau];L^\infty)} \leq \Phi_{4,c_0}(t).
\]

Collecting these estimates with Corollary 3.4 yields that

\[
\left\| \nabla u \right\|_{L^1([0,\tau];L^\infty)} \leq \Phi_{4,c_0}(t).
\]

On the other hand, applying the operator $\nabla \times$ to the $B$ Eq. (3.9), we see

\[
\partial_t (\nabla \times B) - \Delta (\nabla \times B) = -\nabla \times (u \cdot \nabla B) + \nabla \times \left( \frac{u'}{r} B \right) + \nabla \times \left( \frac{2}{r} \frac{\partial_z B}{r} \right).
\]

Then we can write it as
\[ \nabla \times B = e^{t \Delta} (\nabla \times B_0) - \int_0^t e^{(t-\tau) \Delta} (\nabla \times (u \cdot \nabla B))(\tau) \, d\tau \\
+ \int_0^t e^{(t-\tau) \Delta} \left( \nabla \times \left( \frac{\nabla r}{r} B \right) \right)(\tau) \, d\tau + \int_0^t e^{(t-\tau) \Delta} \left( \nabla \times \left( 2 \frac{B}{r} \partial_z B \right) \right)(\tau) \, d\tau. \]

By making use of the \( L^p((0, t]; L^q) \) (\( 1 < p, q < +\infty \)) estimates for the parabolic equation of singular integral and potentials, see \([18, 25]\) for details, we deduce that

\[ \| \nabla \nabla \times B \|_{L^2([0, t]; L^p)} \leq \| B_0 \|_{H^2} + \| u \cdot \nabla B \|_{L^2([0, t]; L^p)} + \| \frac{\nabla r}{r} B \|_{L^2([0, t]; L^p)} + \| B \|_{L^2([0, t]; L^p)} \| \frac{\nabla r}{r} \|_{L^2([0, t]; L^q)} \| B \|_{L^2([0, t]; L^q)} \| \frac{\nabla r}{r} \|_{L^2([0, t]; L^q)} \| B \|_{L^2([0, t]; L^q)} \| \frac{\nabla r}{r} \|_{L^2([0, t]; L^q)} \| B \|_{L^2([0, t]; L^q)} \]

where we used Propositions 3.2, 3.6, Lemmas 2.1 and (3.13). Then the Sobolev embedding shows that

\[ \| \nabla B \|_{L^2([0, t]; L^\infty)} \leq \Phi_{4, c_0}(t). \]

This completes the proof of Proposition 3.9. \( \square \)

To the end, we show the \( H^2 \) estimate of \( B \).

**Proposition 3.10** Let \( (u, B) \) be a smooth solution of the system (1.1) with \( \frac{B_0}{r} \) \( \in L^\infty \) and \( (u_0, B_0) \) \( \in H^2 \) satisfying the assumptions in Theorem 1.2. Then

\[ \| \nabla^2 B(t) \|_{L^2}^2 + \int_0^t \| \nabla^3 B(\tau) \|_{L^2}^2 \, d\tau \leq \Phi_{4, c_0}(t). \]

**Proof** Notice that

\[ \nabla \times (u \times B) = (B \cdot \nabla) u - (u \cdot \nabla) B + u \text{div}B - B \text{div}u. \]

Hence, the second equation of (1.1) can be rewritten as

\[ \partial_t B - \nabla \times (u \times B) + \nabla \times ((\nabla \times B) \times B) = \Delta B. \]

Applying the operator \( \nabla^2 \) to the above equation implies

\[ \partial_t \nabla^2 B - \nabla^2 \Delta B - \nabla^2 \nabla \times (u \times B) + \nabla^2 \nabla \times ((\nabla \times B) \times B) = 0. \quad (3.14) \]

Taking the \( L^2 \)-inner product of the Eq. (3.14) with \( \nabla^2 B \), we get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^2 B \|^2_{L^2} + \| \nabla^3 B \|^2_{L^2} = \int \nabla^2 (u \times B) \cdot \nabla^2 (\nabla \times B) \, dx - \int \nabla^2 ((\nabla \times B) \times B) \cdot \nabla^2 (\nabla \times B) \, dx.
\] (3.15)

For the first term in the right side of (3.15), using Hölder’s inequality and Lemma 2.3, we obtain

\[
\left| \int \nabla^2 (u \times B) \cdot \nabla^2 (\nabla \times B) \, dx \right| \leq \| \nabla^2 (u \times B) \|_{L^2} \| \nabla^2 \nabla \times B \|_{L^2}
\leq (\| u \|_{L^\infty} \| \nabla^2 B \|_{L^2} + \| \nabla^2 u \|_{L^2} \| B \|_{L^\infty}) \| \nabla^3 B \|_{L^2}
\leq C \|u\|_{L^\infty}^2 \| \nabla^2 B \|^2_{L^2} + C \| \nabla^2 u \|^2_{L^2} \| B \|^2_{L^\infty} + \frac{1}{4} \| \nabla^3 B \|^2_{L^2}.
\]

As for the second term in the right side of (3.15), we get from Lemma 2.3 again that

\[
- \int \nabla^2 ((\nabla \times B) \times B) \cdot \nabla^2 (\nabla \times B) \, dx
= - \int \left[ \nabla^2 ((\nabla \times B) \times B) - \left( \nabla^2 (\nabla \times B) \right) \times B \right] \cdot \nabla^2 (\nabla \times B) \, dx
\leq \| \nabla^2 ((\nabla \times B) \times B) - \left( \nabla^2 (\nabla \times B) \right) \times B \|_{L^2} \| \nabla^2 (\nabla \times B) \|_{L^2}
\leq C \| \nabla B \|_{L^\infty} \| \nabla \nabla \times B \|_{L^2} + \| \nabla^2 B \|_{L^2} \| \nabla \times B \|_{L^\infty}) \| \nabla^3 B \|_{L^2}
\leq C \| \nabla B \|^2_{L^\infty} \| \nabla^2 B \|^2_{L^2} + \frac{1}{4} \| \nabla^3 B \|^2_{L^2}.
\]

From these estimates and (3.15), we see

\[
\frac{d}{dt} \| \nabla^2 B \|^2_{L^2} + \| \nabla^3 B \|^2_{L^2} \leq C \left( \| u \|^2_{L^\infty} + \| \nabla B \|^2_{L^\infty} \right) \| \nabla^2 B \|^2_{L^2}
+ C \| \nabla^2 u \|^2_{L^2} \| B \|^2_{L^\infty}.
\]

Therefore, thanks to the Gronwall inequality, we deduce from Propositions 3.8 and 3.9 that

\[
\begin{align*}
\| \nabla^2 B(t) \|^2_{L^2} + \int_0^t \| \nabla^3 B(\tau) \|^2_{L^2} \, d\tau & \leq C \left( \| \nabla^2 B_0 \|^2_{L^2} + \int_0^t \| \nabla^2 u(\tau) \|^2_{L^2} \| B(\tau) \|^2_{L^\infty} \, d\tau \right) \\
& \times \exp \int_0^t \left( \| u(\tau) \|^2_{L^\infty} + \| \nabla B(\tau) \|^2_{L^\infty} \right) \, d\tau \\
& \leq \Phi_{5,c_0}(t).
\end{align*}
\]

The proof is completed. \(\square\)
4 Proof of Theorem 1.2

Now, we are in a position to complete the proof of Theorem 1.2.

Proof The existence part can be obtained by the classical Friedrichs method (see [3] for more details): For \( n \geq 1 \), the cut-off operator \( J_n \) is defined as

\[
\widehat{J_n f}(\xi) = \chi_{[0,n]}(|\xi|) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^3.
\]

We consider the following truncated system in the space \( L^2_n := \{ f \in L^2(\mathbb{R}^3) | \text{supp } f \subset B(0,n) \} \):

\[
\begin{aligned}
\partial_t u_n - \Delta u_n + \mathcal{P} J_n (u_n \cdot \nabla u_n) &= \mathcal{P} J_n (B_n \cdot \nabla B_n), \\
\partial_t B_n - \Delta B_n + \mathcal{P} J_n (u_n \cdot \nabla B_n) &= \mathcal{P} J_n (B_n \cdot \nabla u_n), \\
\text{div } u_n &= \text{div } B_n = 0, \\
u_n(x,0) &= J_n u_0, \quad B_n(x,0) = J_n B_0.
\end{aligned}
\]

(4.1)

Here \((u_0, B_0)\) is a divergence free axisymmetric vector field. Then \((u_n(x,0), B_n(x,0))\) is also axisymmetric due to the radial property of the function \( \chi \). Since the operators \( J_n \) and \( \mathcal{P} J_n \) are the orthogonal projectors for the \( L^2 \)- inner product, the above formal calculations remain unchanged.

Based on Propositions 3.1, 3.8, 3.9 and 3.10, we get by using standard arguments that the system (4.1) has a unique global solution \((u_n, B_n)_{n \in \mathbb{N}}\) such that

\[
u_n \in C([0, \infty); H^2(\mathbb{R}^3)) \cap L^1([0, \infty); W^{1,\infty}(\mathbb{R}^3))
\]

and

\[
B_n \in C([0, \infty); H^2(\mathbb{R}^3)) \cap L^2([0, \infty); W^{1,\infty}(\mathbb{R}^3)).
\]

The control is uniform with respect to the parameter \( n \). Furthermore, by using a standard compactness argument, we obtain that the approximate solutions \((u_n, B_n)_{n \in \mathbb{N}}\) converges to some \((u, B)\) which satisfies our initial data. This completes the proof of the existence part. We omit here the details, see for example [13, 22, 24].

We next prove the uniqueness part. Let \((u^1, B^1)\) and \((u^2, B^2)\) be two solutions of the system (1.1) with the same initial data such that

\[
u^i \in C([0, \infty); L^2(\mathbb{R}^3)) \cap L^1([0, \infty); W^{1,\infty}(\mathbb{R}^3))
\]

and

\[
B^i \in C([0, \infty); L^2(\mathbb{R}^3)) \cap L^2([0, \infty); W^{1,\infty}(\mathbb{R}^3)).
\]

We denote \( \delta u := u^1 - u^2, \delta B := B^1 - B^2 \) and \( \delta P := P^1 - P^2 \). Then \((\delta u, \delta B)\) satisfies
\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t \delta u + u^2 \cdot \nabla \delta u + \delta u \cdot \nabla u^1 + \nabla \delta P - \Delta_h \delta u = B^2 \cdot \nabla \delta B + \delta B \cdot \nabla B^1,
\partial_t \delta B - \Delta \delta B + u^2 \cdot \nabla \delta B - B^2 \cdot \nabla \delta u + \delta u \cdot \nabla B^1 - \delta B \cdot \nabla u^1
\end{array} \right.
\end{align*}
\]

Taking the \(L^2\)-inner product of the \(\delta u\) and \(\delta B\) equations of (4.2) with \(\delta u\) and \(\delta B\), respectively, we get

\[
\frac{1}{2} \frac{d}{dt} (\|\delta u\|_{L^2}^2 + \|\delta B\|_{L^2}^2) + \|\nabla \delta u\|_{L^2}^2 + \|\nabla \delta B\|_{L^2}^2
\]

\[
= - \int \delta u \cdot \nabla u^1 \cdot \delta u \, dx + \int \delta B \cdot \nabla B^1 \cdot \delta u \, dx - \int \delta u \cdot \nabla B^1 \cdot \delta B \, dx - \int \nabla \times ((\nabla \times B^2) \times \delta B) \cdot \delta B \, dx
\]

which we used the facts that

\[
\int \nabla \times ((\nabla \times \delta B) \times B^1) \cdot \delta B \, dx = 0,
\]

and due to \(\text{div} u^i = \text{div} B^i = 0\), we have

\[
\int B^2 \cdot \nabla \delta B \cdot \delta u \, dx + \int B^2 \cdot \nabla \delta u \cdot \delta B \, dx = 0,
\]

\[
\int u^2 \cdot \nabla \delta u \cdot \delta u \, dx = 0, \quad \int u^2 \cdot \nabla \delta B \cdot \delta B \, dx = 0.
\]

Using Hölder’s inequality and Young’s inequality, one has

\[
\left| \int \delta u \cdot \nabla u^1 \cdot \delta u \, dx + \int \delta B \cdot \nabla u^1 \cdot \delta B \, dx \right| \leq \|\nabla u^1\|_{L^\infty} (\|\delta u\|_{L^2}^2 + \|\delta B\|_{L^2}^2),
\]

\[
\left| \int \delta B \cdot \nabla B^1 \cdot \delta u \, dx + \int \delta u \cdot \nabla B^1 \cdot \delta B \, dx \right| \leq \|\nabla B^1\|_{L^\infty} (\|\delta u\|_{L^2}^2 + \|\delta B\|_{L^2}^2),
\]

and

\[
\left| \int \nabla \times ((\nabla \times B^2) \times \delta B) \cdot \delta B \, dx \right| = \left| \int ((\nabla \times B^2) \times \delta B) \cdot (\nabla \times \delta B) \, dx \right| \leq \|\nabla \times B^2\|_{L^\infty} \|\delta B\|_{L^2} \|\nabla \times \delta B\|_{L^2} \leq \frac{1}{2} \|\nabla \delta B\|_{L^2}^2 + C \|\nabla \delta B\|_{L^\infty}^2 \|\delta B\|_{L^2}^2.
\]

Therefore, substituting the above estimates into (4.3), we obtain
which along with Gronwall’s inequality applied implies that $\delta u = 0$ and $\delta B = 0$. This ends the proof of Theorem 1.2.

\[ \Box \]

5 Conclusion

We consider global axisymmetric smooth solutions for the 3D incompressible Hall-MHD system with horizontal dissipation. We prove that if the initial data is axisymmetric and the swirl component of the velocity and the magnetic vorticity are trivial, such a system is globally well-posed for the large initial data.

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Declarations

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