ANOTHER PROOF OF TWO MODULO 3 CONGRUENCES AND ANOTHER SPT CRANK FOR THE NUMBER OF SMALLEST PARTS IN OVERPARTITIONS WITH EVEN SMALLEST PART

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Abstract. By considering the \( M_2 \)-rank of an overpartition as well as a residual crank, we give another combinatorial refinement of the congruences \( \overline{spt}_2(3n) \equiv \overline{spt}_2(3n+1) \equiv 0 \pmod{3} \). Here \( \overline{spt}_2(n) \) is the total number of occurrences of the smallest parts among the overpartitions of \( n \) where the smallest part is even and not overlined. Our proof depends on Bailey’s Lemma and the rank difference formulas of Lovejoy and Osburn for the \( M_2 \)-rank of an overpartition. This congruence, along with a modulo 5 congruence, has previously been refined using the rank of an overpartition.

1. Introduction and statement of Results

We recall a partition of a positive integer \( n \) is a non-increasing sequence of positive integers that sum to \( n \). As an example, the partitions of 4 are 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. Similar to this is the idea of an overpartition. An overpartition of \( n \) is a partition of \( n \) where the first occurrence of a part may (or may not) be overlined. For example, the overpartitions of 4 are 4, \( \overline{4} \), 3 + 1, \( \overline{3} + \overline{1} \), \( \overline{3} + 1 \), \( \overline{3} + \overline{1} \), 2 + 2, \( \overline{2} + 2 \), \( 2 + \overline{1} + 1 \), \( \overline{2} + \overline{1} + 1 \), \( \overline{2} + \overline{1} + 1 \), 1 + 1 + 1 + 1, and \( \overline{1} + 1 + 1 + 1 \).

We have a weighted count on partitions, and overpartitions, given by counting a partition by the number of times the smallest occurs. We use the convention of not counting overpartitions where the smallest part is overlined. We let \( spt(n) \) denote the total number of occurrences of the smallest parts among the partitions of \( n \). We let \( \overline{spt}(n) \) denote the total number of occurrences of the smallest parts among the overpartitions of \( n \) without smallest part overlined. The function \( spt(n) \) was introduced by Andrews in [1] and the function \( \overline{spt}(n) \) was introduced by Bringmann, Lovejoy, and Osburn in [3]. Two restrictions of \( \overline{spt}(n) \) are \( \overline{spt}_1(n) \) and \( \overline{spt}_2(n) \), where restrict to overpartitions where the smallest part is odd and even respectively. We see \( spt(4) = 10, \overline{spt}(4) = 13, \overline{spt}_1(4) = 10, \) and \( \overline{spt}_2(4) = 3. \)

Similar to the work of Andrews, Garvan, and Liang in [2] for \( spt(n) \), in [4] Garvan and the author gave combinatorial refinements of congruences satisfied by \( spt(n), \overline{spt}_1(n), \) and \( \overline{spt}_2(n) \). The idea is to introduce an extra variable into the generating function of each \( spt \)-function to get a crank type statistic. This statistic can then be shown in certain cases to equally split up the numbers \( spt(n), \overline{spt}_1(n), \) and \( \overline{spt}_2(n) \) based on the residue class of the statistic. We explain this in more detail shortly.

For \( \overline{spt}_2(n) \) we have the congruences

\[
\begin{align*}
\overline{spt}_2(3n) & \equiv 0 \pmod{3}, \\
\overline{spt}_2(3n+1) & \equiv 0 \pmod{3}, \\
\overline{spt}_2(5n+3) & \equiv 0 \pmod{5}.
\end{align*}
\]

In this paper we give another proof of the modulo 3 congruences.

To start, by summing according to the smallest part, we find a generating function for \( \overline{spt}_2(n) \) to be given by

\[
\sum_{n=1}^{\infty} \overline{spt}_2(n) q^n = \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1}; q)_{\infty}}{(1-q^{2n})^2(q^{2n+1}; q)_{\infty}}.
\]

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Here we use the standard product notation,

\[(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),\]  
(1.5)

\[(a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n,\]  
(1.6)

\[(a; q)_\infty = \prod_{j=0}^{\infty} (1 - aq^j),\]  
(1.7)

\[(a_1, a_2, \ldots, a_k; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty .\]  
(1.8)

In [4] we considered the two variable generalization given by

\[S_2(z, q) = \sum_{n=1}^{\infty} q^{2n} \frac{(-q^{2n+1}; q)_\infty (q^{2n+1}; q)_\infty}{(zq^{2n}; q)_\infty (z^{-1}q^{2n}; q)_\infty} .\]  
(1.9)

We note setting \(z = 1\) gives the generating function for \(spt_2(n)\). It turns out \(S_2(z, q)\) can be expressed in terms of the Dyson rank of an overpartition and a residual crank from [3]. In [6] Lovejoy and Osburn determined difference formulas for the Dyson rank of an overpartition, these formulas are essential to the proofs in [4]. In the same paper, we also used the difference formulas for the \(M_2\)-rank of a partition without repeated odd parts, these formulas were also determined by Lovejoy and Osburn in [7]. Lovejoy and Osburn also found difference formulas for the \(M_2\)-rank of an overpartition [8], but we did not use these formulas in [4].

In [5] the author gave higher order generalizations of \(spt(n)\) and \(spt_2(n)\) and noted that one could use the \(M_2\)-rank and another residual crank from [3], as used in that paper in working with \(spt_2(n)\), to explain the modulo 3 congruences for \(spt_2(n)\). Here we give the proof. In this paper we instead use

\[S(z, q) = \sum_{n=1}^{\infty} q^{2n} \frac{(-q^{2n+1}; q)_\infty (q^{2n+1}; q)_\infty}{(zq^{2n}, z^{-1}q^{2n}, q^2)_\infty (q^{2n+1}, q^2)_\infty} .\]  
(1.10)

\[= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N_S(m, n) z^m q^n .\]  
(1.11)

Again setting \(z = 1\) gives the generating function for \(spt_2(n)\). This is not the same series \(S(z, q)\) used in [4], however we do not want to overly complicate matters with additional notation.

For a positive integer \(t\) we let

\[N_S(k, t, n) = \sum_{m \equiv k \ (\text{mod} \ t)} N_S(m, n) .\]  
(1.12)

We then have

\[spt_2(n) = \sum_{m=-\infty}^{\infty} N_S(m, n) = \sum_{k=0}^{t-1} N_S(k, t, n) .\]  
(1.13)

Additionally we see if \(\zeta\) is a \(t^{th}\) root of unity, then

\[S(\zeta, q) = \sum_{n=1}^{\infty} \left( \sum_{k=0}^{t-1} N_S(k, t, n) \zeta^k \right) q^n .\]  
(1.14)

We consider when \(\zeta = \zeta_3\) is a primitive third root of unity. Here the minimal polynomial for \(\zeta_3\) is \(x^2 + x + 1\). If \(N_S(0, t, N) + N_S(1, t, N)\zeta_3 + N_S(2, t, N)\zeta_3^2 = 0\) then we must in fact have \(N_S(0, t, N) = N_S(1, t, N) = N_S(2, t, N)\). That is to say, if the coefficient of \(q^N\) in \(S(\zeta_3, q)\) is zero, then \(spt_2(N) = 3 \cdot N_S(0, t, N)\) and so \(spt_2(N) \equiv 0 \ (\text{mod} \ 3)\).

Our proof of \(spt_2(3n) \equiv spt_2(3n + 1) \equiv 0 \ (\text{mod} \ 3)\) is to find the 3-dissection of \(S(\zeta_3, q)\) with the \(q^{3n}\) and \(q^{3n+1}\) terms being all zero. This is the same idea that was used in [4], we are just using \(S(z, q)\) rather than \(S_2(z, q)\).
Theorem 1.1.

\[ S(\zeta_3, q) = A_0(q^3) + q A_1(q^3) + q^2 A_2(q^3), \]  

(1.15)

where

\[ A_0(q) = 0, \]  

(1.16)

\[ A_1(q) = 0, \]  

(1.17)

\[ A_2(q) = \frac{(q^6; q^6)_\infty^4}{(q^2; q^2)_\infty^2 (q^3; q^3)_\infty^2} + \frac{2 q (-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1 - q^{6n+2}}. \]  

(1.18)

We prove these formulas by relating \( S(z, q) \) to a certain rank and crank and using dissections of these related functions. We recall the \( M_2 \)-rank of an overpartition \( \pi \) is given by

\[ M_2\text{-rank} = \left\lfloor \frac{l(\pi)}{2} \right\rfloor - \#(\pi) - \#(\pi_\circ) - \chi(\pi), \]  

(1.19)

where \( l(\pi) \) is the largest part of \( \pi \), \( \#(\pi) \) is the number of parts of \( \pi \), \( \#(\pi_\circ) \) is the number of non-overlined odd parts, and \( \chi(\pi) = 1 \) if the largest part of \( \pi \) is odd and non-overlined and otherwise \( \chi(\pi) = 0 \). The \( M_2 \)-rank was introduced by Lovejoy in [8]. We let \( \mathcal{N} \) denote the number of overpartitions of \( n \) with \( M_2 \)-rank \( m \). Lovejoy found the generating function for \( \mathcal{N} \) is given by

\[ \sum_{n=0}^{\infty} \mathcal{N}(m, n) z^n q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_{2n}}{(zq^2; q^2)_n} \frac{(q^2; q^2)_\infty}{(q^2; q^2)_\infty} \frac{(q^3; q^3)_\infty}{(q^3; q^3)_\infty} \frac{(1 - z)(1 - z^{-1})(1 - q^{3n^2+2n})}{(1 - zq^{2n})(1 - z^{-1}q^{2n})}. \]  

(1.20)

We also define

\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{N}(m, n) z^n q^n = \frac{(-q; q)_\infty (q^2; q^2)_\infty}{(zq^2; q^2)_\infty (q^3; q^3)_\infty}. \]  

(1.21)

The interpretation of this is as follows. For an overpartition \( \pi \) of \( n \) we take the crank of the partition \( \frac{\pi}{+} \) obtained by taking the subpartition \( \pi_\circ \), of the even non-overlined parts of \( \pi \), and halving each part of \( \pi_\circ \). Then \( \mathcal{M}(m, n) \) is the number of overpartitions \( \pi \) of \( n \) and such that the partition \( \frac{\pi}{+} \) has crank \( m \). However this interpretation fails when considering overpartitions whose only even non-overlined parts are a single two, as the corresponding interpretation of \( \frac{(q^3; q^3)_\infty}{(q^2; q^2)_\infty} \) as the generating function of the crank for ordinary partitions fails for the partition of 1.

Theorem 1.2.

\[ S(z, q) = \frac{1}{(1 - z)(1 - z^{-1})} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \left( \mathcal{N}(m, n) - \mathcal{M}(m, n) \right) z^n q^n. \]  

(1.22)

Using the rank difference formulas derived by Lovejoy and Osburn in [8], we have

Theorem 1.3.

\[ \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{N}(m, n) \zeta_3^m q^n = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{N}_0(q^3) + q \mathcal{N}_1(q^3) + q^2 \mathcal{N}_2(q^3), \]  

(1.23)

where

\[ \mathcal{N}_0(q) = \frac{(-q; q)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty (q^3; q^3)_\infty}, \]  

(1.24)

\[ \mathcal{N}_1(q) = \frac{2 (q^3; q^3)_\infty^2 (q^6; q^6)_\infty}{(q; q)_\infty}, \]  

(1.25)

\[ \mathcal{N}_2(q) = \frac{4 (q^2; q^2)_\infty^3 (q^3; q^3)_\infty^2}{(q^2; q^2)_\infty (q^3; q^3)_\infty} + \frac{6 q (-q^3; q^3)_\infty}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+6n}}{1 - q^{6n+2}}. \]  

(1.26)
Theorem 1.4.
\[
\frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \frac{(q^2; q^2)_{\infty}}{(\zeta q^2, \zeta^3 q^2; q^2)_{\infty}} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M^2(m, n) \zeta^n q^n = M^2_0(q^3) + qM^2_1(q^3) + q^2M^2_2(q^3),
\]
where
\[
M^2_0(q) = \frac{(-q; q)_{\infty} (q^3; q^3)^2}{(q; q)_{\infty} (-q^3; q^3)^2},
\]
\[
M^2_1(q) = 2 \frac{(q^3; q^3)_{\infty} (q^6; q^6)_{\infty}}{(q; q)_{\infty}},
\]
\[
M^2_2(q) = \frac{(q^6; q^6)_{\infty}}{(q^2; q^2)_{\infty} (q^3; q^3)^2}.\]

We see Theorem 1.4 follows from Theorems 1.2 and 1.3 noting \(\frac{1}{(1-q)(1-\zeta^{-1})} = \frac{1}{3}\). We give the proofs of Theorems 1.2 and 1.3 in the next section. In Section 3 we give brief combinatorial interpretations of the coefficients \(M_2(m, n)\), in particular they are non-negative, and in Section 4 we end with a few remarks.

2. The Proofs

Proof of Theorem 1.3 We recall a pair of sequences \((\alpha_n, \beta_n)\), is a Bailey pair relative to \((a, q)\) if
\[
\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}.
\]

A limiting case of Bailey’s Lemma gives for a Bailey pair \((\alpha_n, \beta_n)\) that
\[
\sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_{\infty} \left( \frac{aq}{\rho_1 \rho_2} \right)^n \beta_n = \frac{(aq/\rho_1, aq/\rho_2; q)_{\infty}}{(aq/\rho_1, aq/\rho_2; q)_{\infty}} \sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_{\infty} \left( \frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n.
\]

As in [5] the Bailey pair connecting the \(M_2\)-rank of an overpartition and the residual crank is
\[
\alpha_n = \begin{cases} 
1 & n = 0 \\
(-1)^n 2q^n & n \geq 1
\end{cases},
\]
\[
\beta_n = \frac{(q; q^2)_{2n}}{(q^2; q^2)_{2n}}.
\]

This is a Bailey pair with respect to \((1, q^2)\).

We note that
\[
\frac{(q^2; q^2)_{\infty}}{(z, z^{-1}; q^2)_{\infty} (q; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}}{(1-z)(1-z^{-1}) (q; q^2)_{\infty}} = \frac{(-q; q)_{\infty}}{(1-z)(1-z^{-1}) (q; q^2)_{\infty}}.
\]

With this Bailey pair we have
\[
S(z, q) = \sum_{n=1}^{\infty} \frac{q^{2n} (-q^{2n+1}; q)_{\infty} (q^{2n+1}; q)_{\infty}}{(z q^{2n}, z^{-1} q^{2n}; q^2)_{\infty} (q^{2n+1}; q^2)_{\infty}}
\]
\[
= \sum_{n=1}^{\infty} \frac{q^{2n} (q^2; q^2)_{\infty}}{(z, z^{-1}; q^2)_{\infty} (q; q^2)_{\infty}} \sum_{n=1}^{\infty} q^{2n} (z, z^{-1}; q^2)_{n} \beta_n
\]
\[
= \frac{(q^2; q^2)_{\infty}}{(z, z^{-1}; q^2)_{\infty} (q; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{2n} (z, z^{-1}; q^2)_{n} \alpha_n
\]
\[
= \frac{(-q; q)_{\infty}}{(1-z)(1-z^{-1}) (q; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{2n} (z, z^{-1}; q^2)_{n} \alpha_n
\]
\[
\quad - \frac{(q^2; q^2)_{\infty}}{(z, z^{-1}; q^2)_{\infty} (q; q^2)_{\infty}}.
\]
For \( \pi \), \( \#(\pi) \) the number of parts in \( \pi \), \#(\pi_e) the number of even parts in \( \pi \), and \( |\pi| \) the sum of the parts of \( \pi \). For \( \pi = (\pi_1, \pi_2, \pi_3, \pi_4) \in \mathcal{P} \), we define the weight \( \omega(\pi) = (-1)^{\#(\pi) - 1} \), the crank(\( \pi \)) = \#(\pi_2^3) - \#(\pi_3), \) and the norm \( |\pi| = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4| \). We say \( \pi \) is a vector partition of \( n \) if \( |\pi| = n \).

\[
\sum_{n=0}^{\infty} q^{2n}(1 - z)(1 - z^{-1})\alpha_n = \frac{(-q; q)_\infty}{(1 - z)(1 - z^{-1}) (q; q)_\infty} \sum_{n=0}^{\infty} q^{2n}(1 - z)(1 - z^{-1})\alpha_n - \frac{(-q; q)_\infty}{(q; q)_\infty} (q^2; q^2)_\infty.
\]

By equations (1.20) and (1.21) we then have

\[
S(z, q) = \frac{1}{(1 - z)(1 - z^{-1})} \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\mathcal{N}(m, n) - \mathcal{M}(m, n)) z^n q^m.
\]

Proof of Theorem 1.4. We begin by noting

\[
\frac{(-q; q)_\infty}{(q; q)_\infty} \frac{(q^2; q^2)_\infty}{(q; q)_\infty} = \frac{\left(\frac{q^2}{q}\right)_\infty^2}{(q; q)_\infty^2}.
\]

By Gauss we have

\[
\frac{(q; q^2)_\infty}{(q; q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{n(n+1)/2}.
\]

By the Jacobi Triple Product Identity we then have

\[
\frac{\left(\frac{q^2}{q}\right)_\infty^2}{(q; q^2)_\infty} = \frac{1}{2} \sum_{k=0}^{2} \sum_{n=-\infty}^{\infty} q^{(3n+k)(3n+k+1)/2}
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{(9n^2+3n)/2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{(9n^2+9n)/2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} q^{(9n^2+15n)/2}
\]

\[
= \frac{1}{2} (-q^6, -q^3, q^3; q^9)_\infty + \frac{1}{2} q (-1, -q^3, q^3; q^9)_\infty + \frac{1}{2} (-q^{-3}, -q^{12}, q^3; q^9)_\infty
\]

\[
= (-q^6, -q^3, q^3; q^9)_\infty + q (-q^9, -q^9, q^3; q^9)_\infty.
\]

Using the above to expand \( \frac{(q^2; q^2)_\infty^2}{(q; q^2)_\infty} \) and dividing by \( \left(\frac{q^2}{q}\right)_\infty^2 \) then gives

\[
\mathcal{M}(q) = \frac{(-q^2, q^3, q^3; q^3)_\infty^2}{(q^2; q^2)_\infty^2},
\]

\[
\mathcal{M}(q) = 2 \frac{(-q^2, -q^3, q^3, q^3; q^3)_\infty}{(q^2; q^2)_\infty},
\]

\[
\mathcal{M}(q) = \frac{(-q^3, -q^3, q^3, q^3)_\infty}{(q^2; q^2)_\infty}.
\]

However these products easily reduce to those in the statement of the Theorem 1.4. □
By writing the summands of $S(z, q)$ as

$$q^{2n} \left( q^{2n+1} ; q^2 \right) \infty \cdot \frac{1}{\left( z q^{2n} ; q^2 \right) \infty \left( q^{2n+1} ; q^2 \right) \infty} \cdot \frac{1}{\left( z^{-1} q^{2n} ; q^2 \right) \infty \left( q^{2n+1} ; q^2 \right) \infty} \cdot \frac{1}{\left( z^{-1} q^{2n} ; q^2 \right) \infty \left( q^{2n+1} ; q^2 \right) \infty}.$$

we see $N_S(m, n)$ is the number of vector partitions $(\pi^1, \pi^2, \pi^3, \pi^4)$ from $\mathbb{V}$ of $n$ with the additional constraints that $\pi^1$ is non-empty, $s(\pi^1) \leq s(\pi^2)$, $s(\pi^1) \leq s(\pi^3)$, $s(\pi^1) < s(\pi^4)$, and $s(\pi^1)$ is even, but counted with the weight $\omega$.

However, this interpretation hides the fact that each $N_S(m, n)$ is non-negative, as we are counting with a weight that may be negative. We can also interpret $N_S(m, n)$ in terms of partition pairs. This interpretation makes the non-negativity clear. Using the $q$-binomial theorem we have

$$\sum_{n=1}^{\infty} \frac{q^{2n} (q^{4n+2} ; q^2)^{\infty}}{(z q^{2n} ; q^2)^{\infty} (q^{2n+1} ; q^2)^{\infty}} \cdot \frac{1}{(q^2 ; q^2)^{\infty}} \cdot \frac{1}{(q^2 q^2)^{\infty}} \cdot \frac{1}{(q^2 q^2)^{\infty}}.$$

We can now give the partition pair interpretation. We let $PP_2$ denote the set of partition pairs $(\pi^1, \pi^2)$ such that $\pi^1$ is non-empty, $s(\pi^1)$ is even, $s(\pi^1) \leq s(\pi^2)$, and the even parts of $\pi^2$ are at most $2s(\pi^1)$. For such a partition pair we let $k(\pi^1, \pi^2)$ denote the number of even parts of $\pi^1$ that are either the smallest part or are larger than $s(\pi_1) + 2 \#(\pi_2)$. We note when $\pi^2$ contains no even parts that $k(\pi_1, \pi_2)$ reduces to $\#(\pi_1)$.

We define a crank on the elements of $PP_2$ by

$$c(\pi_1, \pi_2) = k(\pi^1, \pi^2) - \#(\pi_2^2) - 1.$$

We claim $N_S(m, n)$ is also the number of partitions pairs of $n$ from $PP_2$ with $c(\pi_1, \pi_2) = m$.

For this we note the first series in (6.2) gives the cases when $\pi^2$ has no even parts. The second series in (6.2) gives the cases when $\pi^2$ has even parts, since $q^{2n} (q^2 ; q^2)_{2n+k}$ is the generating function for partitions into even parts with exactly $k$ parts and each part between $2n$ and $4n$ (inclusive).

It may be possible to define a bijection from these partition pairs to marked overpartitions with smallest part even, and through that determine a crank defined on marked overpartitions. However, we do not pursue that here.

4. Remarks

While $S(3, q)$ can be used to prove $\overline{spt}_2 (3n) \equiv 0 \pmod{3}$ and $\overline{spt}_2 (3n + 1) \equiv 0 \pmod{3}$, we cannot use $S(3, q)$ to prove $\overline{spt}_2 (5n + 3) \equiv 0 \pmod{3}$. In particular we find the coefficient of $q^8$ in $S(3, q)$ to be $z^3 + z^2 + 3z + 5 + 3z^{-1} + z^{-2} + z^{-3}$. That is to say, $N_S(0, 5, 8) = 5$, $N_S(1, 5, 8) = 3$, $N_S(2, 5, 8) = 2$, $N_S(3, 5, 8) = 2$, and $N_S(4, 5, 8) = 3$. 

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However \( \text{spt}_2(5n + 3) \equiv 0 \pmod{5} \) does follow by considering \( \text{spt}_2(\zeta_5, q) \). This can be compared with the rank of a partition explaining the congruences for \( p(5n + 4) \) and \( p(7n + 5) \) but not \( p(11n + 6) \), whereas the crank of an partition does explain all three.

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