Remarks on Murre’s conjecture on Chow groups

by

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Abstract.

For certain product varieties, Murre’s conjecture on Chow groups is investigated. In particular, it is proved that Murre’s conjecture (B) is true for two kinds of fourfolds. Precisely, if $C$ is a curve and $X$ is an elliptic modular threefold over $k$ (an algebraically closed field of characteristic 0) or an abelian variety of dimension 3, then Murre’s conjecture (B) is true for the fourfold $X \times C$.

Key Words: motivic decomposition, Chow group, curve, abelian variety, elliptic modular threefold

Mathematics Subject Classification 2010: 14C25

1. Introduction

We will work with the category $V_k$ of smooth projective varieties over a field $k$. Let $X \in V_k$ be irreducible and of dimension $d$. Let $H(X) := H^\bullet(X, \mathbb{Q}_l)$ be the $l$-adic cohomology groups over a (fixed) algebraic closure $\overline{k}$ of $k$, where $X = X \times_k \text{Spec}(\overline{k})$ and $l \neq \text{ch}(k)$ is a prime, and let $\text{cl}_X : Z^i(X) \to H^{2d}(X)$ be the cycle map associated to $H(X)$, where $Z^i(X)$ is the group of algebraic cycles of codimension $i$ of $X$. We have the well-known Künneth formula:

$$H^{2d}(X \times X) \cong \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X).$$

Let $\Delta_X \subseteq X \times X$ be the diagonal. Then $\text{cl}_{X \times X}(\Delta_X)$ has the Künneth decomposition:

$$\text{cl}_{X \times X}(\Delta_X) = \pi_0 \text{hom} + \pi_1 \text{hom} + \ldots + \pi_{2d} \text{hom},$$

where $\pi_i \text{hom} \in H^{2d-i}(X) \otimes H^i(X)$ is the $i$-th Künneth component.

Let $A^j_{\text{num}}(X)$ (resp. $A^j_{\text{rat}}(X) = CH^j(X)$) be the groups of algebraic cycles of codimension $j$ modulo the numerical equivalence (resp. rational equivalence). Grothendieck’s Lefschetz standard conjecture implies the $\pi_i \text{hom}$ are all algebraic (i.e., they are all in the image of the cycle map). Assuming additionally the conjecture that the homological equivalence coincides with the numerical equivalence ([13]), the diagonal (modulo the numerical equivalence) has a canonical decomposition into a sum of orthogonal idempotents (also called projectors)

$$\Delta_X = a_0 \text{num} + a_1 \text{num} + \ldots + a_{2d} \text{num},$$

in the correspondence ring $A^d_{\text{num}}(X \times X) \otimes \mathbb{Q}$. Then, in the category of Grothendieck motives $\mathcal{M}_k^{\text{num}}$ ([13]) (w.r.t. the numerical equivalence), the motive $h(X) \in \mathcal{M}_k^{\text{num}}$ has a canonical decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \ldots \oplus h^{2d}(X),$$

* This research is supported by National Natural Foundation of China (10871106).
where $h'(X) := h(X, \pi_i^{\text{num}}, 0) \in \mathcal{M}_k^{\text{rat}}$ (See [13] for details).

Furthermore, Murre ([15]) expected that the conjectural decomposition (1) exists even in $\text{CH}^d(X \times X; \mathbb{Q}) := \text{CH}^d(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q} := A^d_\text{rat}(X \times X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and hence in the category of Chow motives $\mathcal{M}^{\text{rat}}_k$ (w.r.t. the rational equivalence), $h(X) \in \mathcal{M}^{\text{rat}}_k$ has a decomposition as in (2). In this new setting, the decomposition is not canonical any more. However, from this conjectural decomposition, Murre ([15]) conjectured a very interesting filtration on rational Chow groups which relates the rational equivalence to the homological equivalence in finite steps as done by the conjectural Bloch-Beilinson filtration.

More precisely, as in [15], we will say that $X$ has a *Chow-Künneth decomposition* over $k$ if there exist $h_i \in \text{CH}^d(X \times X; \mathbb{Q})$, $0 \leq i \leq 2d$, satisfying

(i) $h_i$ are mutually orthogonal projectors;
(ii) $\sum_i h_i = \text{cl}_{X \times X}(\Delta_X)$;
(iii) $\text{cl}_{X \times X}(h_i) = \pi_i^{\text{hom}}$ (the $i$-th Künneth component).

Equivalently, the Chow motive of $X$ has a (Chow-Künneth) decomposition

$$h(X) = h^0(X) \oplus h^1(X) \oplus \ldots \oplus h^{2d}(X),$$

where $h_i(X) := h(X, \pi_i, 0) \in \mathcal{M}_k^{\text{rat}}$.

Then, Murre proposed in [15] the following famous conjecture.

**Murre’s Conjecture**

(A): There exists a Chow-Künneth decomposition for every irreducible variety $X \in \mathcal{F}_k$ of dimension $d$.

(B): $\pi_0, \ldots, \pi_{j-1}$ and $\pi_{2j+1}, \ldots, \pi_{2d}$ act as zero on $\text{CH}^j(X; \mathbb{Q})$.

(C): Let $F^i \text{CH}^j(X; \mathbb{Q}) = \text{Ker} \pi_{2j-1} \cap \text{Ker} \pi_{2j-2} \cap \ldots \cap \text{Ker} \pi_{2j-2j+1}$. Then the filtration $F^i$ is independent of of the ambiguity in the choice of the $\pi_i$.

(D): $F^1 \text{CH}^j(X; \mathbb{Q}) = \text{CH}^j_{\text{hom}}(X; \mathbb{Q}) := \text{Ker}(\text{cl}_X)$.

It was shown by Jannsen ([9]) that Murre’s conjecture is equivalent to the Bloch-Beilinson conjecture on rational Chow groups and the two conjectural filtrations proposed respectively by Murre, and Bloch and Beilinson coincides. The main advantage of Murre’s conjecture over Bloch-Beilinson’s is that one can check the statements for specific varieties as we will do in this paper.

Until now, Murre’s conjecture is verified for only a few special varieties. It is known that (A) is true for curves, surfaces ([14]), Abelian varieties ([17][4]), Brauer-Severi varieties, some threefold ([2][3]), some special fourfold ([11]), certain modular varieties ([6][7]) and varieties whose Chow motives are finite-dimensional ([10]). As for the other parts of Murre’s conjecture, it is known that (B) and (D) are true for the product of a curve and a surface ([15]), that (B) is true for the product of two surfaces ([11]) and that some part of (D) is true for the product of two surfaces ([11][12]). Jannsen ([10]) proved that (A), (B), (C) and (D) are true for some very special higher dimensional varieties over some special ground fields, in particular, he proved that if $k$ is a rational or elliptic function field (in one variable) over a finite field $\mathbb{F}$ and $X_0$ is an arbitrary product of rational and elliptic curves over $\mathbb{F}$, then (A)-(D) hold for $X_0 \times_{\mathbb{F}} k$. Gordon and Murre ([8]) proved that (A)-(D) are true for elliptic modular threefold over a field of characteristic 0.

In this paper, we consider Murre’s conjecture for certain product varieties. We consider such a problem: if the conjecture is true for $X$, when is it also true for the product of $X$ with a curve or some other variety? In section 2, we consider the case of the product of a variety with a projective space. In section 3, we consider the case of the product of a variety with a curve. In particular, we generalize Murre’s discussion given in [16], and as consequences, we prove that if $C$ is a (smooth projective connected) curve, then Murre’s conjecture (B) is true for $X \times C$, where $X$ is an elliptic modular threefold over $k$ (an algebraically closed field of characteristic 0) or $X$ is an abelian variety of
dimension 3. This implies particularly that (B) is true for two new kinds of fourfolds other than products of two surfaces considered in [11] and [12].

2. Products with projective spaces

Fix a field $k$. Let $X$ (resp. $C$) be a smooth projective irreducible variety (resp. curve) over $k$. Let $X$ be of dimension $d$. In the following, we will always denote by $Z \in \text{CH}^j(X)$ a cycle class. In addition, we denote by $p$ with some lower indices the projection from a product variety to the corresponding factors.

In the proof of Theorem 2.3, the following lemma is crucial.

Lemma 2.1 ([5]) Let $E$ be a vector bundle of rank $r = e + 1$ on a scheme $X$ of finite type over $\text{Spec}(k)$, with the projection $\pi : E \to X$. Let $P(E)$ be the associated projective bundle, $p$ the projection from $P(E)$ to $X$, and $O_{P(E)}(1)$ the tautological line bundle on $P(E)$. Then there are canonical isomorphisms

\[
\bigoplus_{i=0}^e \text{CH}^{j-i}(X) \longrightarrow \text{CH}^j(P(E))
\]

\[
(\alpha_i) \mapsto \sum_{i=0}^e c_1(O_{P(E)}(1))^i \cap p^*\alpha_i,
\]

where $c_1(O_{P(E)}(1))$ is the first Chern class.

Applying Proposition 3.1 in [5], it is easy to show that the inverse of the map in Lemma 2.1 is the map

\[
\text{CH}^j(P(E)) \longrightarrow \bigoplus_{i=0}^e \text{CH}^{j-i}(X), \beta \mapsto (\beta_i),
\]

where $\beta_e = p_*\beta$ and for $0 \leq i \leq e - 1$,

\[
\beta_i = p_*(c_1(O_{P(E)}(1))^{e-i} \cap \beta - \sum_{t=1}^{e-i} c_1(O_{P(E)}(1))^{e+t} \cap p^*\beta_{i+t}).
\]

Lemma 2.2 ([15]) Assume that $Y_i$ ($i = 1, 2$) are smooth projective irreducible varieties over $k$. Let $Y = Y_1 \times Y_2$. If $Y_i$ ($i = 1, 2$) has a Chow-Künneth decomposition, then $Y$ has also a Chow-Künneth decomposition.

Theorem 2.3 Let $X$ be a smooth projective irreducible variety of dimension $d$ over $k$. If (A), (B) and (D) are true for $X$, then they are also true for $X \times \mathbb{P}^r$.

Proof. Let $c = c_1(O_{\mathbb{P}^r}(1))$, that is, the class of any hyperplane in $\mathbb{P}^r$. For each $0 \leq i \leq r$, set

\[
\pi_{2i} = c^{r-i} \times c^i, \quad \pi_{2i+1} = 0, \quad h^i(\mathbb{P}^r) = (\mathbb{P}^r, \pi_i).
\]

Then it is easy to see that there is the Chow-Künneth decomposition

\[
h(\mathbb{P}^r) = \bigoplus_{i=0}^{2r} h^i(\mathbb{P}^r) = \bigoplus_{t=0}^r h^{2t}(\mathbb{P}^r).
\]

Assume that $X$ has the Chow-Künneth decomposition

\[
h(X) = \bigoplus_{i=0}^{2d} h^i(X), \quad h^i(X) = (X, \pi'_i).
\]
Then, $X \times \mathbb{P}^r$ has a Chow-Künneth decomposition:

$$h(X \times \mathbb{P}^r) = \bigoplus_{m=0}^{2(d+r)} h^m(X \times \mathbb{P}^r, \pi_m),$$

where $\pi_m = \sum_{p+2q=m} \tau_*(\pi'_p \times c^{r-q} \times c^q)$.

On the other hand, from Lemma 2.1, we have the isomorphisms:

$$\phi : \text{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \rightarrow \bigoplus_{i=0}^r \text{CH}^{j-i}(X; \mathbb{Q}) \rightarrow (Z_i),$$

where $Z_i = p_{1*}([X] \times c^{r-i}) \cdot Z$ with

$$\phi^{-1}((Z_i)) = \sum_{i=0}^r ([X] \times c^i) \cdot p_{1*}Z_i = \sum_{i=0}^r Z_i \times c^i,$$

and

$$\varphi : \text{CH}^{d+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r; \mathbb{Q}) \rightarrow \bigoplus_{i=0}^r \bigoplus_{t=0}^r \text{CH}^{d+i-t}(X \times X; \mathbb{Q}),$$

where $\alpha_{it} = p_{12*}(([X] \times c^{r-t} \times c^{r-i}) \cdot \tau^* \alpha)$. In fact, we have

$$\alpha_{it} = p_{13*}(([X] \times c^{r-t} \times [X]) \cdot p_{123*}(([X \times \mathbb{P}^r \times X] \times c^{r-i}) \cdot \alpha))$$

$$= p_{13*}(([X] \times c^{r-t} \times [X] \times c^{r-i}) \cdot \alpha)$$

$$= p_{12*}(([X \times X] \times c^{r-t} \times c^{r-i}) \cdot \tau^* \alpha),$$

where $\tau$ is the isomorphism exchanging the second and the third factor of the product variety $X \times X \times \mathbb{P}^r \times \mathbb{P}^r$. Clearly, we also have $\phi^{-1}((\alpha_{it})) = \sum_{i,t} \tau_*(\alpha_{it} \times c^i \times c^i)$.

Now, define the map

$$\Phi : \text{CH}^{d+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r; \mathbb{Q}) \times \text{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \rightarrow \text{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}),$$

$$\Phi(\alpha, Z) := \alpha(Z) := p_{34*}(\alpha \cdot (Z \times [X \times \mathbb{P}^r])).$$

So we have the following diagram

$$\text{CH}^{d+r}(X \times \mathbb{P}^r \times X \times \mathbb{P}^r; \mathbb{Q}) \times \text{CH}^j(X \times \mathbb{P}^r; \mathbb{Q}) \rightarrow \text{CH}^j(X \times \mathbb{P}^r; \mathbb{Q})$$

Here the lower arrow is defined by the other three.

Note that if $t + i = r$, then we have

$$\tau_*(\pi'_p \times c^i \times c^q) \cdot (Z_i \times c^j) = p_{34*}(\tau_*(\pi'_p \times c^i \times c^q) \cdot (Z_i \times c^j \times [X \times \mathbb{P}^r]))$$

$$= p_{34*}((\pi'_p \times c^i \times c^q) \cdot (Z_i \times [X] \times c^j \times [\mathbb{P}^r]))$$

$$= p_{24*}((\pi'_p \cdot (Z_i \times [X])) \times c^{i+j} \times c^q))$$

$$= p_{2*}(\tau_*(\pi'_p \cdot (Z_i \times [X])) \times c^{i+j} \times c^q)$$

$$= \pi'_p(Z_i) \times c^q.$$

So we conclude that

$$\tau_*(\pi'_p \times c^i \times c^q) \cdot (Z_i \times c^j) = \begin{cases} \pi'_p(Z_i) \times c^q, & \text{if } t + i = r; \\ 0, & \text{otherwise.} \end{cases}$$
Hence, we can translate the projectors on $X \times \mathbb{P}^r$ to those on $X$ as follows.

$$
\phi(\pi_m(Z)) = \phi \cdot \Phi \cdot (\varphi \times \phi)^{-1}(\varphi(\pi_m), (Z_i)) = \phi \cdot \Phi(\pi_m, \sum_{i=0}^{r} Z_i \times c^i)
$$

$$
= \sum_{i=0}^{r} \sum_{p+2q=m} \phi(\pi_p(\pi'_p \times c^{r-q} \times c^q)) \cdot (Z_i \times c^i)
$$

$$
= \sum_{i=0}^{r} \sum_{p+2q=m, i=q} \phi(\pi_p(Z_i) \times c^q)
$$

$$
= (\pi'_m(Z_0), \pi'_{m-2}(Z_i), \ldots, \pi'_{m-2r}(Z_i))
$$

Now, we can prove that the conjectures are true for $X \times \mathbb{P}^r$.

For (B), from $0 \leq m \leq j - 1$ we have $m - 2i \leq (j - i) - 1$. If $2j + 1 \leq m \leq 2(d + r)$, then

$$
2(j - i) + 1 \leq m - 2i \iff 2j + 1 \leq m.
$$

So, by the assumptions on $X$, we see that $\pi_m(Z) = 0$, for $0 \leq m \leq j - 1$ or $2j + 1 \leq m$.

For (D), suppose that $Z \in \text{CH}^j_{\text{hom}}(X \times \mathbb{P}^r; \mathbb{Q}) = \text{Ker}(\text{cl}_{X \times \mathbb{P}^r})$. Then since $Z_i \in \text{Ker}(\text{cl}_{X}) = \text{CH}^j_{\text{hom}}(X; \mathbb{Q}) = \text{Ker}(\pi'_2(j-i))$ by assumption, we have $Z \in \text{Ker}(\pi_2)$. This completes the proof of Theorem 2.3.

\(\square\)

**Remark 2.4** (i) We expect that Theorem 2.3 is also true for non-trivial projective bundles.

(ii) For (C), we can say nothing yet since from the projectors on $X$ we can get only one but not all set of projectors on $X \times \mathbb{P}^r$.

**Corollary 2.5** Let $S_1, S_2$ be smooth projective surfaces over $k$. Then conjectures (A) and (B) are true for $S_1 \times S_2 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^n$.

**Proof:** From Lemma 2.2 and the main theorem of [14], we know that conjectures (A) and (B) are true for $S_1 \times S_2$, so the result follows from Theorem 2.3.

\(\square\)

### 3. Products with curves

Let $C$ be a smooth projective curve over a field $k$ and $e \in C(k)$. It is well-known (see [18] for details) that $C$ has the Chow-K"unneth decomposition

$$
h(C) = h^0(C) \oplus h^1(C) \oplus h^2(C),
$$

where $h^i(C) = (C, \pi''_i)$ with $\pi'_0 = [e \times C], \pi''_2 = [C \times e], \pi''_1 = \Delta C - \pi'_0 - \pi''_2$.

Assume that the irreducible variety $X \in \mathcal{V}_k$ has the Chow-K"unneth decomposition

$$
h(X) = \bigoplus_{i=0}^{2d} h^i(X), \quad h^i(X) = (X, \pi'_i).
$$

Then, the product variety $X \times C$ has the Chow-K"unneth decomposition

$$
h(X \times C) = \bigoplus_{m=0}^{2(d+1)} h^m(X \times C, \pi_m),
$$

where, explicitely,

$$\pi_0 = \pi'_0 \times [e \times C],$$
\[ \pi = \pi_1 \times [e \times C] + \pi_0 \times (\Delta_C - [e \times C] - [C \times e]), \]
\[ \pi_m = \pi'_m \times [e \times C] + \pi'_{m-1} \times (\Delta_C - [e \times C] - [C \times e]) + \pi'_{m-2} \times [C \times e], \ m \geq 2. \]

Let
\[ CH^j_{\text{alg}}(X; \mathbb{Q}) := \{ Z \in CH^j(X; \mathbb{Q}) : Z \sim_{\text{alg}} 0 \}, \]
where \( Z \sim_{\text{alg}} 0 \) means that \( Z \) is algebraically equivalent to 0.

In the proof of Theorem 3.3, we need the following computations.

**Lemma 3.1** For any \( Z \in CH^j(X \times C; \mathbb{Q}) \), we have
(i) \( (\pi'_m \times [e \times C])(Z) = \pi'_m(Z(e)) \times [C] \);
(ii) \( (\pi'_m \times [C \times e])(Z) = \pi'_m(p_1Z) \times [e] \).

**Proof:** (i) We have
\[
(\pi'_m \times [e \times C])(Z) = p_{34*}(\tau_*(\pi'_m \times [e \times C])) \cdot (Z \times [X \times C]) \\
= p_{34*}(\tau_*(\pi'_m \times [e])) \cdot (Z \times [X]) \times [C] \\
= p_{34*}(\tau_*(\pi'_m \times [e])) \cdot (Z \times [X]) \times [C] \\
= \pi'_m(Z(e)) \times [C],
\]
where \( Z(e) = p_{1*}(Z \times [X]) \). Note that in the last equality, we have used the following computation.
\[
p_{34*}(\tau_*(\pi'_m \times [e])) \cdot (Z \times [X]) = p_{34*}(\tau_*(\pi'_m \times [C]) \cdot (Z \times [X]) \times [X] \times [X]) \\
= p_{34*}(\pi'_m \times [X \times e \times X]) \cdot (Z \times [X]) \\
= p_{34*}(\pi'_m \times [X \times e \times X]) \cdot (Z \times [X]) \\
= p_{2*}(\pi'_m \cdot (\pi'_m \times [X \times e \times X])) \\
= p_{2*}(\pi'_m \cdot (\pi'_m \times [X \times e \times X])) \\
= p_{2*}(\pi'_m \cdot (\pi'_m \times [X \times e \times X])) \\
= \pi'_m(Z(e)).
\]

(ii) Similar to (i), we have
\[
(\pi'_m \times [C \times e])(Z) = p_{34*}(\tau_*(\pi'_m \times [C \times e])) \cdot (Z \times [X \times C]) \\
= p_{34*}(\tau_*(\pi'_m \times [C]) \cdot (Z \times [X]) \times [e] \\
= p_{34*}(\tau_*(\pi'_m \times [C]) \cdot (Z \times [X]) \times [e] \\
= \pi'_m(p_1Z \times [e]).
\]

\[ \square \]

**Lemma 3.2** Let \( Z \in CH^j(X \times C; \mathbb{Q}) \). Then
(i) \( (\pi'_m \times \Delta_C)(Z) = p_{23*}(p_{13Z} \cdot (\pi'_m \times [C])) \);
(ii) \( (\text{id}_X \times f)^*((\pi'_m \times \Delta_C)(Z)) = (\pi'_m)_K((\text{id}_X \times f)^*Z) \), where \( K = k(C) \) is the function field of \( C \), \( f : \text{Spec}(K) \rightarrow C \) is the natural morphism and \( (\pi'_m)_K = \pi'_m \times \Delta_{\text{Spec}(K)}. \)

**Proof:** (i) Let \( \delta_C : C \rightarrow C \times C \) be diagonal morphism. Then, we have
\[
(\pi'_m \times \Delta_C)(Z) = p_{34*}(\tau_*(\pi'_m \times \Delta_C) \cdot (Z \times [X \times C])) \\
= p_{34*}(\tau_*(\text{id}_X \times X \times \delta_C)_*(\pi'_m \times \Delta_C) \cdot (Z \times [X \times C])) \\
= p_{34*}(\pi'_m \times \Delta_C) \cdot (\text{id}_X \times X \times \delta_C)(\pi'_m \times \Delta_C) \\
= p_{34*}(\pi'_m \times \Delta_C) \cdot (\pi'_m \times \Delta_C) \\
= p_{34*}(\pi'_m \times \Delta_C). \]

(ii) From (i) and the following diagram
\[
\begin{array}{c}
X_K \times_K X_K \\
\text{id}_X \times \text{id}_X \times f \downarrow \quad \downarrow \text{id}_X \times f \\
X \times X \times C \\
\end{array}
\]
\[
\begin{array}{c}
p_{23} \\
\end{array}
\]
\[ X \times X \rightarrow X \times C \]
Similarly, let $g$ be a morphism defined by $\eta$. Denote $X \subset \mathbb{A}_k$.

Then, for $m \geq 0$ we have

$$\eta \in \operatorname{Spec}(k) = \operatorname{Spec}(\mathbb{A}_k).$$

Assume that $m \geq 0$, and let $\eta$ be a morphism both defined by $e$. Denote $\eta_K = f_\eta(\operatorname{Spec}K)$. Then we have

$$Z_{K}(\eta_K) = p_{X_K*}(Z_{K} \cdot X \times \eta_K) = p_{X_K*}(Z_{K} \cdot (\operatorname{id}_X \times f_\eta)_*(X \times \operatorname{Spec}K))$$

$$= p_{X_K*}(\operatorname{id}_X \times f_\eta)_*(\pi_m \times [C]) = (\operatorname{id}_X \times f_\eta)_*(\pi_m \times [C])$$

Similarly, let $g_e : \operatorname{Spec}(K) \to C_K$ and $g : \operatorname{Spec}e \to C$ be the morphisms both defined by $e$. Denote $e_K = g_e(\operatorname{Spec}(K))$. Then we have $Z_{K}(e_K) = (\operatorname{id}_X \times g)^*(Z_{K})_{K} = Z(e)_K = 0$.

We claim that

$$(\pi_m^\prime \times \Delta_C)(Z) = 0, \text{ if } 1 \leq m \leq j - 1 \text{ or } 2j + 1 \leq m \leq 2(d + 1).$$

Our main theorem is the following

**Theorem 3.3** Let $k$ be an algebraically closed field, $X \in \mathcal{V}(k)$ and $C \in \mathcal{V}(k)$ an irreducible curve with the function field $K = k(C)$. Assume that (A) and (B) are true for $X$ and $X_K$, and that for any $j$, $\operatorname{CH}_j^a(X_K; \mathbb{Q}) \subseteq \operatorname{Ker}((\pi_j)_K^*).$ Then (A) and (B) are also true for $X \times C$.

**Proof:** The statement about (A) is obvious. We will consider (B) in the following. Let $Z \in \operatorname{CH}_j^a(X \times C; \mathbb{Q})$. Easy computations show that (B) is true if $Z$ is of the form $Z' \times [C]$ with $Z' \in \operatorname{CH}_j^a(X; \mathbb{Q})$. So, we can assume $Z(e) = 0$, since we have

$$Z = (Z - Z(e) \times [C]) + Z(e) \times [C],$$

$$(Z - Z(e) \times [C])(e) = Z(e) - Z(e) = 0.$$

Assume that

$$0 \leq m \leq j - 1 \text{ or } 2j + 1 \leq m \leq 2(d + 1).$$

Then, for $m \geq 1$, we have

$$0 \leq m - 1 \leq (j - 1) - 1 \text{ or } 2(j - 1) + 1 \leq m - 1,$$

and for $m \geq 2$, we have

$$0 \leq m - 2 \leq (j - 1) - 1 \text{ or } 2(j - 1) + 1 \leq m - 1.$$

From Lemma 3.1 and the assumptions on $X$, we have (note that $p_{1*}Z \in \operatorname{CH}_j^1(X; \mathbb{Q})$)

$$(\pi_m^\prime \times [e \times C])(Z) = \pi_m^\prime(Z(e)) \times [C] = 0,$$

$$(\pi_{m-1}^\prime \times [e \times C])(Z) = \pi_{m-1}^\prime(Z(e)) \times [C] = 0,$$

$$(\pi_{m-1}^\prime \times [C \times e])(Z) = \pi_{m-1}(p_{1*}Z) \times [e] = 0,$$

$$(\pi_{m-2}^\prime \times [C \times e])(Z) = \pi_{m-2}(p_{1*}Z) \times [e] = 0.$$

So, the problem is reduced to prove

$$(\pi_{m-1}^\prime \times \Delta_C)(Z) = 0, \text{ if } 1 \leq m \leq j - 1 \text{ or } 2j + 1 \leq m \leq 2(d + 1).$$

At first, we show that $(\operatorname{id} \times f)^*Z$ is algebraically equivalent to $0$ on $X_K$. In fact, let $\eta$ be the generic point of $C$, that is, $K = k(\eta)$, and let $f_\eta : \operatorname{Spec}(K) \to C_K$ be the $K$-point defined by $\eta$. Denote $\eta_K = f_\eta(\operatorname{Spec}K)$. Then we have

$$Z_{K}(\eta_K) = p_{X_K*}(Z_{K} \cdot X \times \eta_K) = p_{X_K*}(Z_{K} \cdot (\operatorname{id}_X \times f_\eta)_*(X \times \operatorname{Spec}K))$$

$$= p_{X_K*}(\operatorname{id}_X \times f_\eta)_*(\pi_m \times [C]) = (\operatorname{id}_X \times f_\eta)_*(\pi_m \times [C])$$

Similarly, let $g_e : \operatorname{Spec}(K) \to C_K$ and $g : \operatorname{Spec}e \to C$ be the morphisms both defined by $e$. Denote $e_K = g_e(\operatorname{Spec}(K))$. Then we have $Z_{K}(e_K) = (\operatorname{id}_X \times g)^*(Z_{K})_{K} = Z(e)_K = 0$.

We claim that

$$(\pi_{m-1}^\prime)(\operatorname{id} \times f)^*Z = 0, \text{ if } 1 \leq m \leq j - 1 \text{ or } 2j + 1 \leq m \leq 2(d + 1).$$
In fact, if \( m = 2j + 1 \), since \( (\text{id} \times f)^*Z \in \text{CH}^j(X_K; \mathbb{Q}) \) is algebraically equivalent to 0, from the assumption \( \text{CH}^j_{\text{alg}}(X_K; \mathbb{Q}) \subseteq \text{Ker}((\pi'_{m-1})_K) \) we have
\[
(\pi'_{m-1})_K((\text{id} \times f)^*Z) = (\pi'_{2j})_K((\text{id} \times f)^*Z) = 0;
\]
if \( 1 \leq m \leq j - 1 \) or \( 2j + 2 \leq m \leq 2(d + 1) \), we have \( 1 \leq m - 1 \leq j - 2 \) or \( 2j + 1 \leq m - 1 \leq 2d + 1 \), so from the assumptions on \( X \) we get \((\pi'_{m-1})_K((\text{id} \times f)^*Z) = 0\) since \((\text{id} \times f)^*Z \in \text{CH}^j(X_K; \mathbb{Q})\).

On the other hand, we have the following well known diagram
\[
\begin{align*}
\text{CH}^{j-1}(X \times (C - U); \mathbb{Q}) & \longrightarrow \text{CH}^j(X \times C; \mathbb{Q}) \longrightarrow \text{CH}^j(X \times U; \mathbb{Q}) \longrightarrow 0 \\
\text{CH}^j(X_K; \mathbb{Q}) & \longrightarrow \lim_{\mathbb{U} \subseteq C} \text{CH}^j(X \times U; \mathbb{Q})
\end{align*}
\]
where the left vertical map is \( z \mapsto (\text{id} \times f)^*z \). So from Lemma 3.2 (ii), we have
\[
(\text{id}_X \times f)^*((\pi'_{m-1} \times \Delta_C)(Z)) = (\pi'_{m-1})_K((\text{id} \times f)^*Z) = 0,
\]
hence
\[
(\pi'_{m-1} \times \Delta_C)(Z) = \sum_i Z'_i \times a_i, \quad \text{with} \quad Z'_i \in \text{CH}^{j-1}(X; \mathbb{Q}) \quad \text{and} \quad a_i \in \text{CH}^j(C; \mathbb{Q}).
\]
In view of \((\pi'_{m-1} \times \Delta_C)^2 = \pi'_{m-1} \times \Delta_C\), we conclude that
\[
(\pi'_{m-1} \times \Delta_C)(Z) = \sum_i (\pi'_{m-1} \times \Delta_C)(Z'_i \times a_i)
= \sum_i p_{23*}[p_{13}^*(Z'_i \times a_i) \cdot (\pi'_{m-1} \times [C])]
= \sum_i \pi'_{m-1}(Z'_i) \times a_i = 0.
\]
This completes the proof of the theorem. \(\square\)

Although the theorem above is restricted, we can deduce several interesting consequences.

**Corollary 3.4** If \( k \) is an algebraically closed field of characteristic 0 and \( X \) is an elliptic modular threefold over \( k \), then (A) and (B) are true for \( X \times C \).

**Proof:** It was shown in [8] that Murre’s conjecture holds for an elliptic modular threefold over a field of characteristic 0. Obviously, conjecture (D) for \( X_K \) implies the assumption of Theorem 3.3. So, the result is an immediate consequence of Theorem 3.3. \(\square\)

**Corollary 3.5** Assume that algebraic equivalence and rational equivalence coincide on \( X \). If (A) and (B) are true for \( X \) and \( X_K \), then (A) and (B) are also true for \( X \times C \).

**Remark 3.6** Cellular varieties satisfy the first hypothesis of the corollary.

By [1] (see also [4] and [15]), for an abelian variety \( X \) of dimension \( g \) over any field \( k \), we have the following decomposition
\[
\text{CH}^j(X; \mathbb{Q}) = \bigoplus_{s=j-g} \text{CH}^s(X),
\]
where
\[
\text{CH}^s(X) := \{ \alpha \in \text{CH}^j(X; \mathbb{Q}) | n^* \alpha = n^{2j-s} \alpha, \forall n \in \mathbb{Z} \}.\]
Corollary 3.7 Let $X$ be an abelian variety of dimension at most 4 over an algebraically closed field $k$. Assume that for any $j$, $\text{CH}_j^0(X_K) \cap \text{CH}_j^\text{alg}(X_K; \mathbb{Q}) = 0$. Then (A) and (B) are true for $X \times C$.

Proof: It follows from [1] that conjecture (B) is true for an abelian variety of dimension at most 4, equivalently, Beauville’s vanishing conjecture holds: $\text{CH}_j^s(X) = 0$ if $s < 0$. By assumption and the fact that the algebraic equivalence is adequate, we have

$$\text{CH}_j^\text{alg}(X_K; \mathbb{Q}) = \bigoplus_{s=1}^j (\text{CH}_s^j(X_K) \cap \text{CH}_s^\text{alg}(X_K; \mathbb{Q})).$$

On the other hand, by Lemma 2.5.1 of [15], we see that for any $j$,

$$\text{Ker}((\pi_{2j})_K) = \bigoplus_{s=1}^j \text{CH}_s^j(X_K).$$

Then we can apply Theorem 3.3 to end the proof.

Remark 3.8 The assumption of Corollary 3.7 is a consequence of a conjecture of Beauville: the restricted cycle map $c_0 : \text{CH}_j^0(X) \rightarrow H^{2j}(X)$ is injective for any $j$.

Corollary 3.9 Let $X$ be an abelian variety of dimension 3 over an algebraically closed field $k$. Then (A) and (B) are true for $X \times C$.

Proof: By [1], we know that the restricted cycle map

$$c_0 : \text{CH}_j^0(X_K) \rightarrow H^{2j}(X_K)$$

is injective for $j = 0, 1, g - 1, g$ if $X$ is an abelian variety of dimension $g$. So for any $j$,

$$\text{CH}_j^0(X_K) \cap \text{CH}_j^\text{alg}(X_K; \mathbb{Q}) \subseteq \text{CH}_j^0(X_K) \cap \text{CH}_j^\text{hom}(X_K; \mathbb{Q}) = 0.$$

Hence the result follows from Corollary 3.7.
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