Hidden invexity in model predictive control

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Abstract

Non-convex optimal control problems occurring in, e.g., water or power systems, typically involve a large number of variables related through non-linear equality constraints. The ideal goal is to find a globally optimal solution, and numerical experience indicates that algorithms aiming for Karush-Kuhn-Tucker points often find (near-)optimal solutions. In our paper, we provide a theoretical underpinning for this phenomenon, showing that on a broad class of problems the objective can be shown to be an invex (invariant convex, [22, 14]) function of the control decision variables when state variables are eliminated using implicit function theory. In this way near-global optimality can be demonstrated, where the exact nature of the global optimality guarantee depends on the position of the solution within the feasible set. In a numerical example, we show how high-quality solutions are obtained for a river control problem where invexity holds.

1 Introduction

Model predictive control (MPC) is an ubiquitous technique for optimal control of systems driven by nonlinear PDEs, such as water, gas, and power systems [1, 10, 23, e.g.]. A typical objective is to steer the system into tracking target levels for the state variables, e.g., stabilizing water level around the desired level [18].

One of the key requirements for a solution is global optimality. Although locally optimal solutions are often acceptable, globally optimal ones typically yield substantially better objective values and are a more solid base for major commitments. While approaches aiming for global optimality in general nonlinear optimization have been proposed [34, 7, 30, 20, e.g.], it remains a challenge in case of tight computation time limits or large problem sizes. Therefore, common work-arounds are to use linearizations [15, 2, 17, e.g.] or convex restrictions or relaxations [28, 24, 26, e.g.] which provide tractability yet at the cost of model accuracy, or to resort to genetic algorithms [39, 31, 40, e.g.].

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It is most desirable, however, to obtain a globally optimal solution to the “most exact” nonlinear model without resorting to complex techniques. Ample numerical evidence exists that local solutions to nonlinear MPC problems are often of high quality, hardly distinguishable from true global optima [10, 20, 5, e.g.]. In this paper, we provide results that give a theoretical underpinning for this phenomenon. We do it by showing that for a broad class of MPC problems, the objective function composed with the dynamics is invex in the original sense of Hanson [22] and Craven [14]. We use invexity to prove partial global optimality guarantees for KKT points of such problems, including full global optimality for KKT points in the interior of the feasible set. We will refer to this phenomenon as hidden invexity.

The research contributions are as follows.

(a) From the optimal control angle, we show that for a large class of numerical optimal control problems, invex formulations exist that preserve the exact nonlinear dynamics. This allows to tractably determine high-quality solutions to large-scale non-convex MPC problems.

(b) From the mathematical optimization/operations research angle, we solve a difficult problem leveraging hidden invexity. Our results show that on difficult problems, certifying the hidden invexity and using standard numerical methods is a viable alternative to resorting to tools designed for general (NP-hard) optimization problems.

(c) From the nonlinear analysis angle, we show that invexity of the objective of a constrained optimization problem yields near-global optimality guarantees for KKT points with active inequality constraints, and global optimality for KKT points in the interior of the domain.

The remainder of this paper is structured as follows. Section 2 introduces the notion of regular MPC problems for which we establish our result. In Section 3 we prove the main result of the paper. Section 4 presents a numerical study for a single river reach.

2 Regular MPC problems

In this section we describe the class of optimization problems for which we demonstrate invexity. Consider the optimization problem

\[
\min_{x,u} (f \circ g)(x) \quad \text{subject to} \quad (P)
\]

\[
c(x,u) = 0
\]

\[
d(u) \leq 0,
\]

where we refer to the variables \(x \in \mathbb{R}^m\) as states and the variables \(u \in \mathbb{R}^n\) as controls, which is because the values of the controls implicitly determine the values of the states through the equality constraints \(c(x,u) = 0\). The function
$f : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective and the function $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the output function mapping states $x$ to outputs $y := g(x)$. The relationship between the controls $u$, the implicitly defined states $x$, and the output variables $y$, is illustrated in Figure 1.

![Figure 1: The relationship between the controls $u$, the implicitly defined states $x$, and the output variables $y$, for $n = 2$ and $m = 3."

We denote the set of of admissible controls as $U = \{ u \in \mathbb{R}^n : d(u) \leq 0 \}$, where $d$ are the inequality constraints, and denote the set of indices of the components of $d$ with $I$.

Our goal will be to show that the objective of problem (P) is invex as a function of the controls $u$, under certain conditions. However, the equality constraints in (P) can involve nonlinear functions, making the analysis cumbersome. We shall bypass this difficulty by eliminating the constraints using implicit function theory and analyzing the problem using total gradients with respect to $u$, wherein the derivatives of the state variables $x_i$ with respect to $u$ are expressed explicitly. This step is used for the analysis, but is not required in practice.

If the Jacobian $\nabla_x c$ is invertible, then the total Jacobian of the states $x$ with respect to the controls $u$ may be expressed using the implicit function theorem as

$$D_u x = -\nabla_x^{-1} c \nabla_u c,$$

in which the prefix $\nabla_x$ denotes the matrix of partial derivatives to the components of $x$, and $D_x$ the matrix of of total derivatives with respect to $x$ of a given function.

For our reasoning to be valid, we need some assumptions. In the following, we define the regular MPC problems on which invexity can be demonstrated. After the definition, we discuss each of the conditions, most of which are typical of MPC problems.

**Definition 2.1.** Consider an optimization problem (P). Let the functions $f$, $g$, $c$ and $d$ be continuously differentiable. We say that (P) is a regular MPC (rMPC) problem if the following conditions are satisfied:

1. the set of admissible controls $U$ is bounded using the inequality constraints $d(u) \leq 0$,

where the inequality depends on the controls $u$ only and holds component-wise,
2. no explicit constraints or bounds are imposed on the states $x \in \mathbb{R}^m$, as well as linear independence constraint qualifications (LICQ):

3. the Jacobian matrix of the equality constraints $c$ with respect to the state variables $x$, i.e., $\nabla_x c(x,u)$, is square and full-rank for all $(x,u)$ such that $c(x,u) = 0$, $u \in U$,

4. the Jacobian matrix of the equality constraints $c$ with respect to the control variables $u$, i.e., $\nabla_u c(x,u)$, is full-rank for all $(x,u)$ such that $c(x,u) = 0$, $u \in U$,

5. the gradient vectors of the active inequality constraints $d_i$ at the point $u$, i.e., $\nabla_u d_i(u)$ for all $i \in I$ such that $d_i(u) = 0$, are linearly independent for all $u \in U$,

the uniqueness condition:

6. for every $u \in U$, the constraints $c(x,u) = 0$ have a unique solution $x$,

the output controllability condition:

7. the output function $g : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is such that the square matrix

$$-\nabla_x g(x) \nabla_x^{-1} c(x,u) \nabla_u c(x,u)$$

is invertible for all $(x,u)$ such that $c(x,u) = 0$, $u \in U$,

and the convexity condition:

8. the objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex.

We now discuss the meaning of each of the respective assumptions. Condition 1 states that the set of feasible controls must be described using a finite number of continuously differentiable inequality constraints. Standard convex domains such as a ball or a box satisfy this condition.

Condition 2 is necessary for the constraints $c(x,u) = 0$ to be solvable for every $u \in U$.

Condition 3 is a linear independence constraint qualification (LICQ, e.g.). It is required for the states $x_i$ to be well defined as implicit functions of $u$, and for us to be able to apply implicit function theory. Condition 3 is typically straightforward to demonstrate, if the dynamics of the underlying model are integrable in time. For the dynamics to be uniquely integrable in time, it is required that the number of states be equal to the number of equations, and furthermore that the Jacobian of the equations with respect to the states be non-singular.

Condition 4 is also a linear independence constraint qualification that essentially states that at all times, the dynamics are sensitive to all controls.

Condition 5 is the third LICQ and is satisfied automatically for standard domains such as balls and boxes.
Condition 6 states that the functions \( u \mapsto x_i \) are uniquely defined on \( U \). It is a standard MPC assumption as the required uniqueness property typically follows if the dynamics are (uniquely) integrable in time.

Condition 7 states that different attainable states should map to different outputs. Since conditions 3 and 4 imply that the implicit function \( u_I \mapsto x \) is injective, the range of attainable state values is a subset of \( \mathbb{R}^m \) with dimension \( n \), on which invertible mappings assigning coordinates in \( \mathbb{R}^n \) arise naturally.

Another interpretation of Condition 7 is the following. If we would require the LICQ and uniqueness conditions to hold on \( \mathbb{R}^n \) (rather than on \( U \) only), condition 6 would imply that for every possible value of the output \( y \), there exists a control input \( u \in \mathbb{R}^n \) that realizes the output \( y \). It is therefore a type of output-controllability \cite{33} condition.

Condition 8 is standard and includes objectives such as \( p \)-norms raised to the \( p \)th power with \( p \geq 2 \).

Before looking at a more interesting example of rMPC problems, it is instructive to consider a few irregular problems to show that the rMPC conditions indeed eliminate some of the well-known NP-hard problems.

**Example 2.1.** Let \([0, 1] \subset U \subset \mathbb{R} \). If a problem contains a binary-restriction constraint \( u(1 - u) = 0 \), then condition 6 is not satisfied.

**Example 2.2.** Let \([-1, 1] \subset U \subset \mathbb{R} \). If a problem contains a sinusoidal constraint \( u = \sin x \), then condition 6 is, in general (i.e., barring additional structure), not satisfied.

**Example 2.3.** Let \( 0 \in U \subset \mathbb{R} \). If a problem contains a bilinear constraint of the form \( u = x_1 x_2 \), then the LICQ condition 3 is, in general (i.e., barring additional structure), not satisfied.

**Example 2.4.** Let \((0, 0) \in U \subset \mathbb{R}^2 \). If a problem contains a bilinear constraint of the form \( x = u_1 u_2 \), then the LICQ condition 4 is, in general (i.e., barring additional structure), not satisfied.

We now move to introduce examples of systems meeting the rMPC conditions. One such simple example is if the constraints are linear-affine functions satisfying the appropriate rank conditions.

**Example 2.5.** Consider an MPC problem with control vector \( u \in \mathbb{R}^n \), state vector \( x \in \mathbb{R}^m \), output vector \( y \in \mathbb{R}^n \), and trajectory tracking objective

\[
f(y) = \sum_{i=1}^{n} |y_i - y_i^t|^p
\]

with \( p \in [2, \infty) \) and linear-affine output function

\[
y = Cx + c
\]

subject to the bounds

\[-\infty < u_j^L \leq u_j \leq u_j^U < \infty \quad j \in \mathbb{N}_n,\]
and linear-affine constraints

\[ c(x, u) = Ax + Bu + b \]

with matrix \( A \) square and invertible, matrices \( B \) and \( C \) full rank and matrix \( C \) such that the square matrix \( CA^{-1}B \) is invertible. This is a regular problem.

Our next example considers trigonometric constraints, which commonly arise in control of systems with axes of rotation such as vehicles, ships, and aircraft [37, e.g.].

**Example 2.6.** An MPC problem with control vector \( u \in \mathbb{R}^2 \), state vector \( x \in \mathbb{R}^2 \), output vector \( y \in \mathbb{R}^2 \), and trajectory tracking objective

\[ f(y) = \sum_{i \in \{1, 2\}} |y_i - y'_i|^p \]

with \( p \in [2, \infty) \) and output function

\[ y = x \]

subject to the bounds

\[ 0 < u^L_1 \leq u_1 \leq u^U_1 < \infty \]
\[ 0 \leq u_2 \leq 2\pi \]

and constraints

\[ x_1 = u_1 \cos u_2, \]
\[ x_2 = u_1 \sin u_2, \]

is regular.

Since bilinear constraints are very common, we also show how many of them can meet the rMPC assumptions.

**Example 2.7.** An MPC problem with control vector \( u \in \mathbb{R}^2 \), state vectors \( x \in \mathbb{R}^2 \) and \( z \in \mathbb{R}^2 \), output vector \( y \in \mathbb{R}^2 \), and objective

\[ f(y) = \sum_{i \in \{1, 2\}} |y_i - y'_i|^p \]

with \( p \in [2, \infty) \) and output function

\[ y = z \]

subject to the bounds

\[ 0 < u^L_j \leq u_j \leq u^U_j < \infty \quad j \in \{1, 2\} \]
and constraints

\[ u_1 = x_1 z_0, \]
\[ x_1 = z_0 - z_1, \]
\[ u_2 = x_2 z_1, \]
\[ x_2 = z_1 - z_2, \]

with the fixed boundary condition \( z_0 \in \mathbb{R} \), is regular. Conditions 1–5 and 7–8 are readily verified. To verify condition 6, i.e., that for any \( u \in U \) the constraints admit a unique solution, note that the constraints may be solved in the displayed order, starting from the fixed value \( z_0 \).

Situations like Example 2.7 commonly occur when modelling the generation of a hydroelectric turbine in a power station. Instantaneous generation (\( u \)) is non-negative and bounded, and it is bilinear in flow (\( x \)) and the water level difference (\( z \)) across a dam. At the same time, an increase in flow results in a decrease of the water level difference (\( z \)). Similar reasoning applies to the power consumption of pumps.

Conditions 1–2, 5, and 7–8 may be satisfied by design. Conditions 3–4 and 6 are also satisfied by appropriate discretizations of certain hyperbolic PDEs, used to model the following examples.

**Example 2.8.** A river or canal network modelled using the Saint-Venant equations, with control authority exercised at weirs, dams, gates, and pumps [6, e.g.].

**Example 2.9.** A drinking water distribution network modelled using the Darcy-Weisbach or Hazen-Williams equations, with control authority exercised at valves and pumps [10, e.g.].

**Example 2.10.** A natural gas and/or hydrogen distribution network modelled using the isothermal Euler equations, with control authority exercised at valves and compressors [23, e.g.].

Such PDEs have a time dimension along which they may be integrated, starting from a fixed initial condition, analogous to Example 2.7. If the discretization is implicit or semi-implicit in time (in the sense of the implicit Euler method [25, e.g.] and Example 2.7), then conditions 3 and 6 are satisfied if, given values for time step \( t \), the Jacobian matrices arising when solving for time step \( t + 1 \) are multi-diagonal. Such discretizations exist for Examples 2.8–2.10. Proofs of this fact may be found in [11, 12, 38, 35, 23, e.g.].

Conditions 4 and 7 also arise naturally in the context of systems driven by hyperbolic PDEs. Since the number of output variables needs to be equal to that of the controls, for every control we can make the output function \( g \) select a state corresponding to the discretization node upon which the control variable acts, or corresponding to a spatially adjacent node if the grid is staggered [36, e.g.]. In a river control example, that would mean that for every dam in a river,
the corresponding output variable is the water level directly upstream of the dam. Such an output function \( g \) is linear and injective on the set of attainable states \( 1 \), whence it satisfies the LICQ condition \( 4 \) and the invertibility condition \( 7 \). The function of the remaining states \( x_i \) is to ensure physically accurate wave propagation, by means of a sufficiently fine spatial discretization, in between of the control nodes. Experience shows that such a setup produces appropriate control strategies, as will be illustrated with a concrete example in Section \( 4 \).

3 Hidden invexity

3.1 Introduction and the main result

In this section we present our main result that rMPC problems have hidden invexity when reduced to optimization over control variables. We begin by recalling the definition of invexity.

\textbf{Definition 3.1.} A function \( f : X \subset \mathbb{R}^n \rightarrow \mathbb{R} \) is invex on the open set \( X \) if there exists a vector function \( \eta(x_2, x_1) : X \times X \rightarrow \mathbb{R}^n \) such that

\[
   f(x_2) - f(x_1) \geq \eta^T(x_2, x_1) \nabla f(x_1)
\]

for all \( x_1, x_2 \in X \).

The name invex follows from invariant convex \( 14 \). A function is invex if and only if every stationary point is a global minimum. To see the first implication, set \( \nabla f = 0 \) in Equation \( 3 \). A concise proof of the reverse implication may be found in \( 8 \).

The definition of invexity is usually stated for functions defined on open sets, whereas our goal is to optimize over a closed set \( U \). There exists a whole family of extensions of invexity to optimization problems (KT-invexity \( 29 \), HC-invexity \( 22, 14, 29 \), Type I/Type II invexity \( 21 \)). However, each of them is difficult to apply to real-world problems like ours, due to the need to find a common function \( \eta \) for the objective and the constraints. Instead, we shall stay with the standard notion of invexity of functions and, eliminating the equality constraints from the problem, show invexity of the objective in terms of the control variables.

Invexity arises naturally in the composition of convex functions with transformations that are full-rank, \( i.e. \), that have an invertible Jacobian \( 13 \). We will now show how rMPC problems fit this scheme.

Conditions 3 and 6 in Definition \( 2.1 \) facilitate the use of the implicit function theorem to express the state variables \( x \) as a function \( u \mapsto x \). Problem \( (P) \) can

\(^{1}\)Let \( t \) denote the first time step at which two control strategies differ. Since for hyperbolic PDEs, disturbances travel at finite velocity \( 16 \), the resulting change in state adjacent to a perturbed control variable at time \( t \) cannot be compensated for by \( a \) perturbations to control variables at times \( t' > t \), or \( b \) by any of the other control variables acting at another node in the spatial discretization, acting at time \( t \). Therefore different control strategies produce different output.
therefore be rewritten as:

\[
\min_{u} (f \circ g \circ x)(u) \quad \text{subject to} \quad (P^U) \\
d(u) \leq 0.
\]

In \((P^U)\), the composition \(T : U \to Y := T[U], T(u) := (g \circ x)(u)\), will be playing the role of the invertible transformation, and the composition \(f \circ T\) will be shown to be invex. This setup is illustrated in Figures 1 and 2.

Figure 2: Convex objective \(f\) composed with invertible transformation \(T\).

The key feature to deal with in our analysis is the fact that in general, the set \(Y\) is non-convex. The strength of our main result for a particular KKT point with \(y^*\) will depend on the place where point \(y^*\) is in the set \(Y\) - in the interior or on the boundary. To make this distinction rigorous and state our main result, we first recall the definition of the tangent cone [32, 19, e.g.].

**Definition 3.2.** Let \(Y \subset \mathbb{R}^n\) be a non-empty set. A vector \(d \in \mathbb{R}^n\) is tangent to \(Y\) at \(y \in Y\), if there exist sequences \(\{y^k\} \subset Y, \{t_k\} \subset \mathbb{R}^+\) such that

\[
y^k \to y, \quad t_k \to 0, \quad \frac{y^k - y}{t_k} \to d.
\]

The set of all tangent vectors at \(y \in Y\) is the tangent cone of \(Y\) at \(y\), denoted \(T_Y(y)\).

We now state our main result the proof of which is relegated to Section 3.2

**Theorem 3.1.** Consider an rMPC problem \((P^U)\). Let \((u^*, \lambda^*)\) be a KKT point of this problem. Then \(u^*\) is a global minimum of \(f \circ T\) on the set

\[
V(u^*) := \{u \in U : T(u) - T(u^*) \in T_Y(T(u^*))\}, \quad (4)
\]

where \(T_Y(T(u^*))\) denotes the tangent cone of \(Y\) at \(T(u^*)\).

It states that, in rough terms, a KKT point is a global optimum with respect to the interior of the domain and all inactive boundary segments, minus any points “hidden from view” due to local concavity of the active boundary segments. This is illustrated in Figure 3 and explored in further detail with the following corollaries and subsequent discussion.

Before going further, we note that the reverse statement of Theorem 3.1, i.e., that every minimum is a KKT point, follows from the LICQ conditions 3–5.
in Definition 2.1. The LICQ conditions form the regularity condition required for every minimum to be a KKT point [32, e.g.].

The first corollary is a direct consequence of the fact that for an interior point, \( T_y(T(u^*)) = \mathbb{R}^n \).

**Corollary 1.** Consider an rMPC problem \((P_U)\). Consider a KKT point \((u^*, \lambda^*)\) such that \( u^* \in \text{int} U \). Then \( u^* \) is a global minimum of \( f \circ T \) on \( U \).

In other words, \( f \circ T \) is invex on the interior of \( U \). The second corollary follows from the fact that for a convex set \( X \), membership \( x, y \in X \) implies that \( y - x \in T_X(x) \).

**Corollary 2.** Consider an rMPC problem \((P_U)\). Consider a KKT point \((u^*, \lambda^*)\) and a set \( W \subset U, u^* \in W \), such that \( T(W) \) is convex. Then \( u^* \) is a global minimum of \( f \circ T \) on the set \( W \).

We will now explain the meaning of these results. For this, it is instructive to first recall the reference situation: general nonlinear programming. A KKT point of a nonlinear optimization problem need not be a local minimum; it may also be a local maximum, or a saddle point. Furthermore, in case that a KKT point is a local minimum, it is only guaranteed to be minimal within an arbitrarily small neighbourhood of itself. Numerically, generic nonlinear optimization problems are hard: Local search methods may converge to KKT points that are local maxima or saddle points.

For an rMPC problem, Theorem 3.1 provides a stronger characterization of KKT points. First of all, it states that a KKT point is a local minimum within \( V(u^*) \). This is important from a numerical point of view. Secondly, it states that a local minimum \( u^* \) is a global minimum within the set \( V(u^*) \). If \( u^* \) is an interior point, we have that \( V(u^*) = U \), whence it is a global optimum. If \( u^* \) lies on the boundary, its objective value is no greater than the objective values for all points that lie on rays emanating from \( u^* \) in the directions of the tangent vectors. Both cases are illustrated in Figure 3.

![Figure 3: The highlighted areas illustrate sets \( T[V(u^*)] \) within which a solution \( y^* = T(u^*) \) is provably globally optimal, for an interior (left) and a boundary solution (right). The sets are shown in the output space \( Y = T[U] \) to highlight the role of the tangent cones.](image)

As the result relies on the behaviour of the composition of a convex objective function with an invertible transformation, we will say that an rMPC problem has **hidden invexity**. If, furthermore, the set \( Y \) is convex, then by Corollary 2 the rMPC problem has hidden **convexity** [9, 27, e.g.].
3.2 Proof of the main result

Consider the transformation $T = g \circ x$. By condition 7 of Definition 2.1, $D_u T$ is invertible whence, by the inverse function theorem, $T$ itself is invertible. The transformation and its use within the optimization problem is illustrated in Figure [2].

We will first show that a point $(u^*, \lambda^*)$ is a KKT point of the optimization problem

$$\min_u (f \circ T)(u) \quad \text{subject to} \quad d(u) \leq 0,$$

if and only if $(T(u^*), \lambda^*)$ is a KKT point of the optimization problem

$$\min_y f(y) \quad \text{subject to} \quad (d \circ T^{-1})(y) \leq 0,$$

Afterwards, we will analyze the global optimality structure of the KKT points. Let

$$L^U(u, \lambda) := (f \circ T)(u) + \lambda^T d(u)$$

denote the Lagrangian of problem $\mathcal{P}^U$, and let

$$L^Y(y, \lambda) := f(y) + \lambda^T (d \circ T^{-1})(y)$$

denote the Lagrangian of problem $\mathcal{P}^Y$. We will use the standard definition of KKT points following [32]. KKT points of $\mathcal{P}^U$ are stationary points of the Lagrangian $L^Y$ and therefore satisfy

$$0 = D_y L^Y = \nabla_y f + \lambda^T \nabla d D_u^{-1} T = \nabla_y f [D_u T D_u^{-1} T] + \lambda^T \nabla d D_u^{-1} T = [\nabla_y f D_u T + \lambda^T \nabla d] D_u^{-1} T = D_u L^U D_u^{-1} T.$$

Since $D_u T$ is invertible, a point $(u^*, \lambda^*)$ is a stationary point of $L^U$ if and only if $(T(u^*), \lambda^*)$ is a stationary point of $L^Y$. Similar reasoning applies to the primal and dual feasibility conditions ($d(u^*) \leq 0$ and $\lambda^* \geq 0$) as well as to the complementarity condition $(\lambda^* d_i(u^*) = 0$ whenever $d_i(u^*) = 0$). This completes the first part of the proof.

We will now analyze the KKT points. For this, we will use some machinery related to tangent cones, for which we first recall a few additional definitions. Relevant references are [32, 19, 3].

Definition 3.3. The set

$$\mathcal{A}(y^*) := \{ i \in \mathcal{I} : (d_i \circ T^{-1})(y^*) = 0 \}$$

is the active set for the problem $\mathcal{P}^Y$ at the point $y^* \in Y$. 

\[11\]
**Definition 3.4.** The set
\[ F(y^*) := \{ t \in \mathbb{R}^n : t^T D_y (d_i \circ T^{-1})(y^*) \leq 0 \; \forall i \in \mathcal{A}(y^*) \}, \]
is the set of linearized feasible directions for the problem \( \mathcal{P}^* \) at the point \( y^* \in Y \).

**Definition 3.5.** The cone
\[ K^o := \{ y \in \mathbb{R}^n : y^T x \leq 0 \; \forall x \in K \} \]
is the polar cone of the cone \( K \).

Let \((u^*, \lambda^*)\) be a KKT point of \( \mathcal{P}^* \). Our aim is to show that the point \( u^* \) is global minimum of \( f \) on the set \( V(u^*) \) as defined in Equation (4). For this, it is convenient to reason about \( y^* = T(u^*) \) and problem \( \mathcal{P}^T \). As LICQ holds for the constraint function \( d \circ T^{-1} \), we have \( F(y^*) = T_Y(y^*) \), see \([32]\), for proof of this fact. Note that, since \((y^*, \lambda^*)\) is a KKT point,

\[ -\nabla_y f(y^*) = \lambda^T D_y (d \circ T^{-1})(y^*). \]

Following the definition of the set of linearized feasible directions \( F(y^*) \), for all \( t \in F(y^*) = T_Y(y^*) \) we have that \( t^T D_y (d_i \circ T^{-1})(y^*) \leq 0 \) for all \( i \in \mathcal{A}(y^*) \). Because of this and the facts that \( \lambda_i^* \geq 0 \) for all \( i \in \mathcal{A}(y^*) \) and \( \lambda_i^* = 0 \) for all \( i \in I \setminus \mathcal{A}(y^*) \), it holds that \( -t^T \nabla_y f(y^*) \leq 0 \) for all \( t \in F(y^*) = T_Y(y^*) \). Therefore \( -\nabla_y f \in (T_Y(y^*))^o \), the polar cone of the tangent cone.

Since \( T[V(u^*)] \subset Y \), it follows directly from Definition 3.2 that \( T_{T[V(u^*)]}(y^*) \subset T_Y(y^*) \). The inclusion reverses when taking polar cones, so that

\[ -\nabla_y f(y^*) \in (T_Y(y^*))^o \subset (T_{T[V(u^*)]}(y^*))^o. \]

In other words, for every tangent vector \( t \in T_{T[V(u^*)]}(y^*) \), we have \( t^T \nabla_y f(y^*) \geq 0 \). By convexity of \( f \), for every \( y \in T[V(u^*)] \),

\[ f(y) - f(y^*) \geq (y - y^*)^T \nabla_y f(y^*) \geq 0. \]

The second inequality follows from the fact that \( y - y^* \in T_Y(y^*) \) by construction of the set \( V(u^*) \). We conclude that \( y^* \) is a global minimum of \( f \) on \( T[V(u^*)] \), whence \( u^* \) is a global minimum of \( f \circ T^{-1} \) on \( V(u^*) \).

4 Numerical experiment

In this section, we describe a numerical experiment revolving around an MPC problem for the one-dimensional shallow water equations. These equations are also known as the Saint-Venant equations, and form a nonlinear hyperbolic PDE \([41]\), for example.

The Saint-Venant equations describe levels and flows in rivers and canals. They are given by the momentum equation

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \frac{\partial H}{\partial x} + gQ \frac{Q}{AR^2} = 0, \]
Table 1: Parameters for the example problem.

| Parameter | Value | Description |
|-----------|-------|-------------|
| $T$       | 72    | Index of final time step |
| $\Delta t$ | 600 s | Time step size |
| $H^b_i$   | $(-4.90, -4.92, \ldots, -5.10)$ m | Bottom level |
| $l$       | 10000 m | Total channel length |
| $A_i(H_i)$ | $50 \cdot (H - H^b_i)$ m$^2$ | Channel cross section function |
| $P_i(H_i)$ | $50 + 2 \cdot (H - H^b_i)$ m | Channel wetted perimeter function |
| $C_i$     | $(40, 40, \ldots, 40)$ m$^{0.5}$/s | Chézy friction coefficient |
| $H_{i}(t_0)$ | $(0.000, -0.025, \ldots, -0.222)$ m | Initial water levels at $H$ nodes |
| $Q_{i}(t_0)$ | $(100, 100, \ldots, 100)$ m$^3$/s | Initial discharge at $Q$ nodes |

with longitudinal coordinate $x$, time $t$, discharge $Q$, water level $H$, cross section $A$, hydraulic radius $R := A/P$, wetted perimeter $P$, Chézy friction coefficient $C$, gravitational constant $g$, and by the mass balance (or continuity) equation

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0.$$ 

For our experiment, we consider a discretization that is semi-implicit in time, following \cite{12}, and staggered in space. The semi-implicit discretization ensures that conditions 3–4 and 6 in Definition 2.1 are met. The proof of this statement is given, in a different wording, in \cite{11, 12, e.g.}. We use the experimental setting from the draft \cite{4}, from which the following description and Table \text{1} are adapted.

We consider a single river reach with 10 uniformly spaced water level nodes and rectangular cross section, an upstream inflow boundary condition provided with a fixed time series, as well as a controllable downstream release boundary condition. The grid is illustrated in Figure \text{4}, and the hydraulic parameters and initial conditions are summarized in Table \text{1}. The model starts from steady state: the initial flow rate is uniform and the water level decreases linearly along the length of the channel.

Our objective is to keep the water level at the $H$ node upstream of the gate at $0$ m above datum:

$$f = \sum_{j=1}^{T} |H_{10}(t_j)|^2,$$

i.e., $f = \|y\|_2^2$ and $y = (H_{10}(t_j))_{j=1, \ldots, T}$ in concordance with conditions 7 and 8.
of Definition 2.1 subject to the bound on the control variables

\[ 100 \text{ m}^3/\text{s} \leq Q_{10}(t_j) \leq 200 \text{ m}^3/\text{s}, \]

in concordance with conditions 1–2 and 5 of Definition 2.1.

To give a physical context for this problem, suppose this model represents a channel downstream of a reservoir and upstream of an adjustable gate with limited capacity. The gate is trying to dampen the sudden pulse of water shown in Figure 5a released by the reservoir.

A solution to the optimization problem was obtained using the interior point solver IPOPT \cite{42} and is plotted in Figure 5. By releasing water in anticipation of the inflow using the decision variable $Q_{10}$, the optimization is able to reduce water level fluctuations and keep the water levels close to the target level.

Since some of the bounds on the control variables are active, the near-global optimality guarantee of Theorem 3.1 applies. There is also ample numerical evidence that solutions of this type are globally optimal or very close to it. In \cite{5}, the performance of an interior point-type method (IPM) for a large class of water problems is benchmarked against a so-called reduced genetic algorithm (RGA, \cite{40}). The IPM search finds qualitatively consistent solutions that always obtains better objective function values than the RGA. This benchmark includes problems with multiple river reaches, multiple spatial control points, and both coarser and finer discretizations of the shallow water equations in time and space.

Similar results are reported in \cite{20} for drinking water distribution networks, where local search using IPOPT finds solutions with objective values within a
relative distance of $10^{-3}$ of those found using the global solver COUENNE – in a fraction of the computation time.

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