QUANTUM PRINCIPAL BUNDLES AND
CORRESPONDING GAUGE THEORIES

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Abstract. A generalization of classical gauge theory is presented, in the
framework of a noncommutative-geometric formalism of quantum principal
bundles over smooth manifolds. Quantum counterparts of classical gauge bun-
dles, and classical gauge transformations, are introduced and investigated. A
natural differential calculus on quantum gauge bundles is constructed and an-
alyzed. Kinematical and dynamical properties of corresponding gauge theories
are discussed.

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1. Introduction

The aim of this study is to present a generalization of classical gauge theory, in
which quantum groups play the role of entities describing local symmetries.
All considerations will be performed within a general conceptual framework of
non-commutative differential geometry [1].

The whole paper is based on a noncommutative-geometric theory of principal
bundles over classical smooth manifolds, possessing quantum structure groups. This
theory is presented in [2]. Here, fundamental structural elements of classical gauge
theory will be generalized and incorporated into the formalism of quantum principal
bundles.

The paper is organized as follows.
In the next section, a preparatory material is collected. As first, we fix the notation and introduce in the game relevant quantum group entities. Secondly, we present the most important ideas and results of [2], which will be used in the main considerations.

The starting point for all constructions of this paper is a quantum principal $G$-bundle $P$ over a smooth manifold $M$. Here, $M$ plays the role of space-time while $G$ is a compact matrix quantum (structure) group [4] representing “local symmetries” of the system.

In Section 3 a quantum analogue of the gauge bundle will be constructed and investigated. This quantum bundle (over $M$) will be denoted by $\mathcal{C}(P)$. Various quantum counterparts of gauge transformations are naturally associated to $\mathcal{C}(P)$. Further, a differential calculus on the bundle $\mathcal{C}(P)$ will be constructed, by combining the standard differential calculus on $M$ (based on differential forms) with an appropriate differential calculus on the quantum group $G$. This calculus on $\mathcal{C}(P)$ is relevant in situations in which quantum counterparts of gauge transformations act on entities related to differential calculus on the principal bundle $P$ (connection forms, for example).

It is important to mention that there exist two natural inequivalent ways of introducing quantum counterparts of gauge transformations. The first one is to translate into the quantum context the idea that gauge transformations are vertical automorphisms of the principal bundle $P$. This approach leads to a standard group (of gauge transformations of $P$). The same group will be obtained if we consider counterparts of sections of the bundle $\mathcal{C}(P)$. However, it turns out that such a concept of a gauge transformation does not describe gauge-like phenomena related to the quantum nature of the space $G$. Namely, because of the inherent geometrical inhomogeneity of quantum groups, every quantum principal bundle $P$ over $M$ is completely determined by its classical part $P_{cl}$ (interpretable as the set of points of $P$). The classical part is an ordinary principal $G_{cl}$-bundle over $M$, where $G_{cl}$ is a group (the classical part of $G$) interpretable as consisting of points of $G$. We shall prove that gauge transformations of $P$ are in a natural bijection with standard gauge transformations of $P_{cl}$. Further, we shall prove that the set of points of $\mathcal{C}(P)$ coincides, in a natural manner, with the standard gauge bundle $\mathcal{C}(P_{cl})$. The second approach to gauge transformations is in some sense indirect. The main idea is to construct the “action” of the bundle $\mathcal{C}(P)$ on $P$ (generalizing the classical situation). This approach does not meet geometrical obstacles. In classical geometry, the mentioned action naturally contains all the information about gauge transformations.

Section 4 is devoted to the formulation and kinematical and dynamical analysis of quantum group gauge theories, in the framework of quantum principal bundles. Gauge fields will be geometrically represented by connections on $P$. Internal degrees of freedom of such gauge fields are determined by fixing a bicovariant first-order differential $^*$-calculus [5] on the structure quantum group $G$. In this paper we shall deal with a unique differential calculus on $G$ which can be characterized as the minimal bicovariant differential calculus compatible, in appropriate sense, with the geometrical structure on the bundle $P$. If we start from this calculus on the group then it is possible to build natural differential calculi on bundles $P$ and $\mathcal{C}(P)$ which are always “locally trivialized” when bundles $P$ (and $\mathcal{C}(P)$) are locally trivialized.

Dynamical properties of the gauge theory will be determined after fixing an
appropriate lagrangian. In analogy with the classical gauge theory, we shall consider
lagrangians which are quadratic functions of the curvature form. We shall compute
the corresponding equations of motion. Symmetry properties of the introduced
lagrangian will be analyzed. We shall prove the invariance of the lagrangian under
the action of the (ordinary) group of gauge transformations of \( P \). Further, it turns
out that the lagrangian is invariant, in an appropriate sense, under the natural
action of \( C(P) \) on \( P \). This corresponds to the full gauge invariance of the lagrangian
in the classical theory.

In Section 5 everything will be illustrated on a simple but highly non-trivial ex-
ample in which \( G \) is the quantum \( SU(2) \) group. The most important observation
is that the corresponding gauge theory is essentially different from the classical
\( SU(2) \) gauge theory, and does not reduce to the classical theory when the defor-
mation parameter \( 1 - \mu \) tends to zero. This is caused by the fact that the min-
imal admissible bicovariant calculus does not respect the classical limit. Namely,
a detailed analysis [2] shows that for \( \mu \in (-1, 1) \setminus \{0\} \) the space of left-invariant
elements (playing the role of the dual space of the corresponding Lie algebra) of the
mentioned minimal calculus is infinitely dimensional, and can be naturally iden-
tified with the algebra of polynomial functions on a quantum 2-sphere. Hence, the
corresponding gauge fields possess infinitely many internal degrees of freedom, in
contrast to the classical case. Finally, in Section 6 concluding remarks are made.

The paper ends with an Appendix, in which some technical properties related
to the minimal admissible bicovariant calculus on the quantum \( SU(2) \) group are
collected.

2. Mathematical Background

Let \( G \) be a compact matrix quantum group [4]. We shall denote by \( \mathcal{A} \) the *-
algebra of “polynomial functions” on \( G \), and by \( \phi: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \), \( \epsilon: \mathcal{A} \to \mathbb{C} \) and
\( \kappa: \mathcal{A} \to \mathcal{A} \) the comultiplication, counit and the antipode respectively. The symbols
\( a^{(1)} \otimes \cdots \otimes a^{(n)} \) will be used for the result of an \( (n-1) \)-fold comultiplication
of an element \( a \in \mathcal{A} \) (so that \( \phi(a) = a^{(1)} \otimes a^{(2)} \)). Let \( G_{cl} \) be the classical part [2]
of \( G \). Explicitly, \( G_{cl} \) is consisting of *-characters (nontrivial multiplicative linear
hermitian functionals) of \( \mathcal{A} \). The Hopf algebra structure on \( \mathcal{A} \) naturally induces
the group structure on \( G_{cl} \), such that

\[
\begin{align*}
gg' &= (g \otimes g')\phi \\
g^{-1} &= g\kappa,
\end{align*}
\]

for each \( g, g' \in G_{cl} \). The counit \( \epsilon: \mathcal{A} \to \mathbb{C} \) is the neutral element of \( G_{cl} \). We shall
assume that the (complex) Lie algebra \( \text{Lie}(G_{cl}) \) is realized [2] as the space of linear
functionals \( X: \mathcal{A} \to \mathbb{C} \) satisfying

\[
X(ab) = \epsilon(a)X(b) + \epsilon(b)X(a),
\]

for each \( a, b \in \mathcal{A} \).

Let \( \Gamma \) be a first-order differential calculus over \( G \). This means [5] that \( \Gamma \) is a
bimodule over \( \mathcal{A} \) endowed with a differential \( d: \mathcal{A} \to \Gamma \) such that elements of the
form a $db$ linearly generate $\Gamma$. Let

$$\Gamma^\otimes = \sum_{k \geq 0} \Gamma^\otimes k$$

be the tensor bundle algebra \cite{5} built over $\Gamma$. Let

$$\Gamma^\wedge = \sum_{k \geq 0} \Gamma^\wedge k$$

be the universal differential envelope (\cite{2}–Appendix B) of $\Gamma$. The algebra $\Gamma^\wedge$ can be obtained from $\Gamma^\otimes$ by factorising through the ideal $S^\wedge \subseteq \Gamma^\otimes$ generated by the elements of the form

$$Q = \sum_i da_i \otimes_A db_i,$$

where $a_i, b_i \in A$ satisfy $\sum_i a_i db_i = 0$. In particular, the differential $d: \Gamma^\wedge \rightarrow \Gamma^\wedge$ extends $d: A \rightarrow \Gamma$, in a natural manner.

Let us assume that $\Gamma$ is left-covariant \cite{5} and let $\ell_\Gamma: \Gamma \rightarrow A \otimes \Gamma$ be the left action of $G$ on $\Gamma$. Let $\Gamma_{\text{inv}}$ be the space of left-invariant elements of $\Gamma$ (playing the role of the dual space of the Lie algebra of $G$) and let $\pi: A \rightarrow \Gamma_{\text{inv}}$ be the canonical projection map, given by

$$\pi(a) = \kappa(a^{(1)}) da^{(2)}.$$

This map is surjective and $R = \ker(\epsilon) \cap \ker(\pi)$ is the right $A$-ideal which canonically \cite{5} corresponds to $\Gamma$.

The space $\Gamma_{\text{inv}}$ possesses a natural right $A$-module structure, which will be denoted by $\circ$. Explicitly

$$\pi(a) \circ b = \pi[(a - \epsilon(a)1)b]$$

for each $a, b \in A$.

Let us now assume that $\Gamma$ is bicovariant, and let $\wp_\Gamma: \Gamma \rightarrow \Gamma \otimes A$ be the right action of $G$ on $\Gamma$.

The "adjoint" action $\text{ad}: A \rightarrow A \otimes A$ of $G$ on $A$ is given by

$$\text{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)}) a^{(3)}.$$

The space $\Gamma_{\text{inv}}$ is right-invariant, that is $\wp_\Gamma(\Gamma_{\text{inv}}) \subseteq \Gamma_{\text{inv}} \otimes A$. The corresponding restriction $\varpi: \Gamma_{\text{inv}} \rightarrow \Gamma_{\text{inv}} \otimes A$ is interpretable as the adjoint action of $G$ on $\Gamma_{\text{inv}}$. Explicitly $\varpi$ is characterized by

$$\varpi \pi = (\pi \otimes \text{id}) \text{ad}.$$

The actions $\ell_\Gamma$ and $\wp_\Gamma$ can be naturally extended to the grade preserving homomorphisms $\wp_\Gamma^\wedge, \otimes: \Gamma^\wedge, \otimes \rightarrow \Gamma^\wedge, \otimes \otimes A$ and $\ell_\Gamma^\wedge, \otimes: \Gamma^\wedge, \otimes \rightarrow A \otimes \Gamma^\wedge, \otimes$ (their restrictions on $A$ coincide with $\phi$).

The symbol $\otimes$ will be used for the graded tensor product of graded-differential algebras. The comultiplication $\phi$ admits the unique extension $\hat{\phi}: \Gamma^\wedge \rightarrow \Gamma^\wedge \otimes \Gamma^\wedge$ which is a homomorphism of graded-differential algebras \cite{2}. In particular,

$$\hat{\phi}(\xi) = \ell_\Gamma(\xi) + \wp_\Gamma(\xi)$$

for each $\xi \in \Gamma$. The antipode $\kappa$ admits the unique extension $\hat{\kappa}: \Gamma^\wedge \rightarrow \Gamma^\wedge$, which is graded-antimultiplicative and satisfies $\hat{\kappa}d = \hat{d} \hat{\kappa}$. 
Let us denote by $\Gamma^\otimes_{inv}$ and $\Gamma^\wedge_{inv}$ subalgebras of left-invariant elements of $\Gamma^\otimes$ and $\Gamma^\wedge$ respectively. We have

$$\Gamma^\otimes_{inv} = \sum_{k \geq 0} \Gamma^\otimes_k \ , \quad \Gamma^\wedge_{inv} = \sum_{k \geq 0} \Gamma^\wedge_k \ ,$$

where $\Gamma^\otimes_k$ and $\Gamma^\wedge_k$ consist of left-invariant elements from $\Gamma^\otimes$ and $\Gamma^\wedge$ respectively. The space $\Gamma^\otimes_{inv}$ is actually the tensor product of $k$-copies of $\Gamma^\otimes_{inv}$.

The space $\Gamma^\otimes_k$ is actually the tensor product of $k$-copies of $\Gamma^\otimes_{inv}$. The following natural isomorphism holds

$$\Gamma^\wedge_{inv} = \Gamma^\otimes_{inv} / S^\wedge_{inv} ,$$

where $S^\wedge_{inv}$ is the left-invariant part of $S^\wedge$. This space is an ideal in $\Gamma^\otimes_{inv}$ generated by elements of the form

$$q = \pi(a^{(1)}) \otimes \pi(a^{(2)}) ,$$

where $a \in R$.

All introduced spaces of the form $\Gamma^*_{inv}$ are right-invariant. We shall denote by $\varpi^*$ the adjoint actions of $G$ on the corresponding spaces.

The formula

$$\vartheta \circ a = \kappa(a^{(1)}) \vartheta a^{(2)}$$

defines an extension of the right $\mathcal{A}$-module structure $\circ$ from $\Gamma^\otimes_{inv}$ to $\Gamma^\wedge_{inv}$. We have

$$1 \circ a = \epsilon(a)1$$

for each $\vartheta, \eta \in \Gamma^\wedge_{inv}$ and $a \in \mathcal{A}$.

The algebra $\Gamma^\wedge_{inv} \subseteq \Gamma^\wedge$ is $d$-invariant. The differential $d: \Gamma^\wedge_{inv} \rightarrow \Gamma^\wedge_{inv}$ is explicitly determined by

$$d \pi(a) = -\pi(a^{(1)}) \pi(a^{(2)}) .$$

If $\Gamma$ is *-covariant then the *-involution *: $\Gamma \rightarrow \Gamma$ is naturally extendible from $\Gamma$ to $\Gamma^\wedge_{inv}$ (such that for each $\vartheta, \eta \in \Gamma^\wedge_{inv}$ we have $(\vartheta \eta)^* = (-1)^{\delta \delta \delta} \eta^* \vartheta^*$). Algebras $\Gamma^\wedge_{inv}, \Gamma^\otimes_{inv} \subseteq \Gamma^\wedge_{inv}$ are *-invariant. We have

$$(\vartheta \circ a)^* = \vartheta^* \circ \kappa(a)^*$$

for each $a \in \mathcal{A}$ and $\vartheta \in \Gamma^\wedge_{inv}$.

Explicitly, the *-involution on $\Gamma^\otimes_{inv}$ is determined by

$$\pi(a)^* = -\pi[\kappa(a)^*] .$$

The map $\hat{\vartheta}$, as well as the left and the right actions of $G$ on $\Gamma^\wedge_{inv}$ are *-preserving, in a natural manner.

Let $M$ be a compact smooth manifold. By definition [2] a quantum principal $G$-bundle over $M$ is a triplet $P = (\mathcal{B}, i, F)$ where $\mathcal{B}$ is a (unital) *-algebra, consisting of appropriate “functions” on $P$, while $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ and $i: S(M) \rightarrow \mathcal{B}$ are (unital) *-homomorphisms, interpretable as the dualized right action of $G$ on $P$, and the dualized projection of $P$ on $M$. Further, the bundle $P$ is locally trivial in the sense
that for each $x \in M$ there exists an open set $U \subseteq M$ such that $x \in U$, and a $*$-homomorphism $\pi_U : B \to S(U) \otimes A$ such that

\[
\pi_U (i(f)) = (f|_U) \otimes 1 \\
\pi_U (B) \supseteq S_c(U) \otimes A \\
(id \otimes \phi) \pi_U = (\pi_U \otimes id) F,
\]

and such that

\[
\pi_U (i(f)b) = 0 \implies i(f)b = 0
\]

for each $f \in S_c(U)$. Here $S$ and $S_c$ denote the corresponding $*$-algebras of complex smooth functions (with compact supports, respectively).

The homomorphism $\pi_U$ is interpretable as the dualized trivialization of $P$ over $U$. Every pair $(U, \pi_U)$, consisting of an open set $U \subseteq M$ and of a $*$-homomorphism $\pi_U : B \to S(U) \otimes A$, satisfying above conditions is called a local trivialization of $P$.

A trivialization system for $P$ is a family $\tau = \{(U, \pi_U) | U \in \mathcal{U}\}$ of local trivializations of $P$, where $\mathcal{U}$ is a finite open cover of $M$.

For each $k \in \mathbb{N}$ we shall denote by $N^k(\mathcal{U})$ the set of $k$-tuples $(U_1, \ldots, U_k) \in \mathcal{U}^k$ such that $U_1 \cap \cdots \cap U_k \neq \emptyset$.

The main structural result concerning quantum principal bundles is that there exists a natural correspondence between quantum principal $G$-bundles $P$ and classical principal $G_{cl}$-bundles $P_{cl}$ over $M$. This correspondence can be described as follows.

From a given trivialization system $\tau$ it is possible to construct the corresponding $G$-cocycle which is a system of $*$-automorphisms $\psi_{UV}$ of $S(U \cap V) \otimes A$, where $(U, V) \in N^2(\mathcal{U})$, realizing transformations between $(V, \pi_V)$ and $(U, \pi_U)$. Such systems of maps completely determine the bundle $P$.

Explicitly, let us consider the $*$-algebra

\[
\Sigma(\mathcal{U}) = \sum_{U \in \mathcal{U}}^\oplus [S(U) \otimes A].
\]

The algebra $B$ is realizable as a subalgebra of $\Sigma(\mathcal{U})$, consisting of elements $b \in \Sigma(\mathcal{U})$ satisfying

\[
(U|_{U \cap V} \otimes id)p_U(b) = \psi_U (V|_{U \cap V} \otimes id)p_V(b)
\]

for each $(U, V) \in N^2(\mathcal{U})$. Here, $p_U : \Sigma(\mathcal{U}) \to S(U) \otimes A$ are coordinate projections. In terms of this realization we have

\[
\pi_U = p_U|B,
\]

for each $U \in \mathcal{U}$.

However, it turns out that $G$-cocycles are in a natural bijection with standard $G_{cl}$-cocycles (over $\mathcal{U}$), which are systems of smooth maps $g_{UV} : U \cap V \to G_{cl}$ satisfying

\[
g_{UV}g_{VW}(x) = g_{UW}(x),
\]

for each $(U, V, W) \in N^3(\mathcal{U})$ and $x \in U \cap V \cap W$ (in particular $g_{U}^{-1} = g_{UV}$). The correspondence is established via the following formula

\[
\psi_{UV}(\varphi \otimes a) = \varphi g_{UV}(a^{(1)}) \otimes a^{(2)}.
\]
Here, maps \( g_{UV} \) are understood as \(*\)-homomorphisms \( g_{UV} : A \to S(U) \), in a natural manner. On the other hand, \( G_{cl} \)-cocycles determine, in the standard manner, principal \( G_{cl} \)-bundles \( P \) over \( M \).

The bundle \( P_{cl} \) is interpretable as the “classical part” of \( P \). The elements of \( P_{cl} \) are in a natural bijection with \(*\)-characters of \( B \). The correspondence \( P \leftrightarrow P_{cl} \) has a simple geometrical explanation. The “transition functions” \( \psi_{UV} \) are, at the geometrical level, vertical “diffeomorphisms” of \((U \cap V) \times G\). Therefore they preserve the geometrical structure of \((U \cap V) \times G\). In particular, they must preserve the classical part \((U \cap V) \times G_{cl}\) consisting of points of \((U \cap V) \times G\). Moreover, transition diffeomorphisms are completely determined by their restrictions on \((U \cap V) \times G_{cl}\), because of the right covariance. The corresponding “restrictions” are precisely transition functions for the classical bundle \( P_{cl} \).

So far about the structure of quantum principal bundles. For each (nonempty) open set \( U \subseteq M \) let \( \Omega(U) \) be the graded-differential \(*\)-algebra of differential forms on \( U \). In developing a differential calculus over quantum principal bundles it is natural to assume that the calculus is fully compatible with the geometrical structure on the bundle, such that all local trivializations of the bundle locally trivialize the calculus too (a precise formulation of this condition is given in [2]–Section 3). It turns out that this condition completely fixes the calculus on the bundle (if the calculus on the structure quantum group is fixed). However, the condition implies certain restrictions on a possible differential calculus \( \Gamma \) over \( G \).

Namely, all retrivialization maps \( \psi_{UV} \) must be extendible to differential algebra automorphisms \( \psi_{UV}^\wedge : \Omega(U \cap V) \hat{\otimes} \Gamma^\wedge \to \Omega(U \cap V) \hat{\otimes} \Gamma^\wedge \). Differential calculi \( \Gamma \) satisfying this condition are called admissible. If \( \Gamma \) is left-covariant then it is admissible if

\[
(X \otimes \text{id})\text{ad}(\hat{R}) = \{0\}
\]

for each \( X \in \text{lie}(G_{cl}) \). This fact implies that there exists the minimal admissible left-covariant calculus \( \Gamma \). This calculus is based on the right \( A \)-ideal \( \hat{R} \subseteq \ker(\epsilon) \) consisting of all elements \( a \in \ker(\epsilon) \) satisfying

\[
(X \otimes \text{id})\text{ad}(a) = 0,
\]

for each \( X \in \text{lie}(G_{cl}) \).

Moreover, we have \( \text{ad}(\hat{R}) \subseteq \hat{R} \otimes A \) and \( \kappa(\hat{R})^* = \hat{R} \), which implies [5] that \( \Gamma \) is bicovariant and \(*\)-covariant respectively.

In the following, \( \Gamma \) will be this minimal admissible (bicovariant \(*\)) calculus.

Let \( \Omega(P) \) be the graded-differential \(*\)-algebra representing differential calculus on \( P \) (constructed by combining differential forms on \( M \) with the universal envelope \( \Gamma^\wedge \) of \( \Gamma \)). Explicitly, let us consider the direct sum

\[
\Sigma^\wedge(U) = \sum_{U \in \mathcal{U}} [\Omega(U) \hat{\otimes} \Gamma^\wedge].
\]

Then \( \Omega(P) \) can be viewed as a graded-differential subalgebra consisting of elements \( w \in \Sigma^\wedge(U) \) satisfying

\[
(U|_{U \cap V} \otimes \text{id})p_U(w) = \psi_{UV}^\wedge(V|_{U \cap V} \otimes \text{id})p_V(w)
\]

for each \((U, V) \in N^2(\mathcal{U})\). Here \( p_U : \Sigma^\wedge(U) \to \Omega(U) \hat{\otimes} \Gamma^\wedge \) are corresponding coordinate projections.
As a differential algebra, $\Omega(P)$ is generated by $B = \Omega^0(P)$. For every local trivialisation $(U, \pi_U)$ of $P$ there exists the unique differential algebra homomorphism $\pi_U^\wedge: \Omega(P) \to \Omega(U) \otimes \Gamma^\wedge$ extending $\pi_U$ (in fact $\pi_U^\wedge = p_U|\Omega(P)$). The map $i: S(M) \to \mathcal{B}$ admits a natural extension $i^\wedge: \Omega(M) \to \Omega(P)$, which is interpretable as the “pull back” of differential forms on $M$ to $P$. We have

$$\pi_U^\wedge i^\wedge(w) = (w|_U) \otimes 1.$$ 

The right action $F: \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}$ is (uniquely) extendible to a differential algebra homomorphism $\hat{F}: \Omega(P) \to \Omega(P) \otimes \Gamma^\wedge$, imitating the corresponding pull back map. Let $\psi(P)$ be the space of all linear maps $\phi: \Gamma_{inv} \to \Omega(P)$ satisfying

$$(\phi \otimes \text{id})\omega = F^\wedge \phi.$$ 

Another important algebra naturally associated to $\Omega(P)$ is a graded *-subalgebra $\mathfrak{hor}(P) \subseteq \Omega(P)$ representing horizontal forms. By definition, $\mathfrak{hor}(P)$ consists of forms $w \in \Omega(P)$ with the property

$$\pi_U^\wedge w \in \Omega(U) \otimes \mathcal{A},$$

for each local trivialization $(U, \pi_U)$. Equivalently,

$$\mathfrak{hor}(P) = (\hat{F})^{-1}\{\Omega(P) \otimes \mathcal{A}\}.$$ 

The algebra $\mathfrak{hor}(P)$ is invariant under the right action of $G$, in other words

$$F^\wedge(\mathfrak{hor}(P)) \subseteq \mathfrak{hor}(P) \otimes \mathcal{A}.$$ 

Let $\psi(P)$ be the space of all linear maps $\varphi: \Gamma_{inv} \to \Omega(P)$ satisfying

$$(\varphi \otimes \text{id})\omega = F^\wedge \varphi.$$
This space is naturally graded (the grading is induced from \(\Omega(P)\)). The elements of \(\psi(P)\) are quantum counterparts of pseudotensorial forms on the bundle with coefficients in the structure group Lie algebra (relative to the adjoint representation). The space \(\psi(P)\) is closed with respect to compositions with \(d\colon \Omega(P) \to \Omega(P)\).

Let \(\tau(P) \subseteq \psi(P)\) be the subspace consisting of \(\text{hor}(P)\)-valued maps. This space is imaginable as consisting of the corresponding tensorial forms.

There exists a natural \(*\)-involution on \(\psi(P)\). It is given by
\[
\phi^*(\vartheta) = \varphi(\vartheta^*)^*.
\]
The space \(\tau(P)\) is \(*\)-invariant.

Tensorial forms possess the following local representation:
\[
\pi_U^\top \varphi(\vartheta) = (f_U \otimes \text{id}) \varphi(\vartheta),
\]
where \(f_U\colon \Gamma_{\text{inv}} \to \Omega(U)\) is a linear map.

For the purposes of this paper the most important topic of the theory of quantum principal bundles is the formalism of connections. By definition, a connection on \(P\) is every pseudotensorial hermitian 1-form \(\omega\) satisfying
\[
\pi_u \omega(\vartheta) = 1 \otimes \vartheta
\]
for each \(\vartheta \in \Gamma_{\text{inv}}\). The above formula is the quantum counterpart for the classical condition that connections map fundamental vector fields into their generators. Connections form a real affine space \(\text{con}(P)\).

In local terms, connections possess the following representation
\[
\pi_U^\top \omega(\vartheta) = (A_U \otimes \text{id}) \varphi(\vartheta) + 1_U \otimes \vartheta,
\]
where \(A_U\colon \Gamma_{\text{inv}} \to \Omega(U)\) is a 1-form valued hermitian linear map (playing the role of the corresponding gauge potential).

The curvature operator can be described as follows.

Let us fix a map \(\delta\colon \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}}\) which intertwines the corresponding adjoint actions and such that if
\[
\delta(\vartheta) = \sum_k \vartheta_1^k \otimes \vartheta_2^k
\]
then
\[
\delta(\vartheta^*) = -\sum_k (\vartheta_2^k)^* \otimes (\vartheta_1^k)^* \quad d\vartheta = \sum_k \vartheta_1^k \vartheta_2^k.
\]

Every such a map will be called an embedded differential. Further, for each pair of linear maps \(\varphi, \psi\) on \(\Gamma_{\text{inv}}\) with values in an arbitrary algebra \(\Omega\) let \(\langle \varphi, \psi \rangle\colon \Gamma_{\text{inv}} \to \Omega\) be a map given by
\[
\langle \varphi, \psi \rangle(\vartheta) = \sum_k \varphi(\vartheta_1^k) \psi(\vartheta_2^k).
\]

By construction, if \(\varphi, \psi \in \psi(P)\) then \(\langle \varphi, \psi \rangle \in \psi(P)\), too.

Finally, the curvature \(R_\omega\) of a connection \(\omega\) can be defined as
\[
R_\omega = d\omega - \langle \omega, \omega \rangle.
\]
The above formula corresponds to the structure equation in the classical theory. It turns out that $R_{\omega}$ is a tensorial 2-form. Locally, in terms of the corresponding gauge potentials we have
\[
\pi_U^\wedge R_{\omega}(\vartheta) = (F^U \otimes \text{id})\omega(\vartheta),
\]
where
\[
F^U = dA^U - \langle A^U, A^U \rangle.
\]
For each open set $U \subseteq M$ the symbol $\otimes_U$ will be used for the tensor product over $S(U)$. Similarly, the symbol $\hat{\otimes}_U$ will denote the graded tensor product of graded-differential $\ast$-algebras containing $\Omega(U)$ as their subalgebra.

3. Quantum Gauge Bundles

This section is devoted to generalizations of the most important aspects of the concept of gauge transformations, in the framework of the formalism of quantum principal bundles. The main geometrical object that will be constructed is the quantum gauge bundle, a noncommutative-geometric counterpart of the gauge bundle of the classical theory.

3.1. Classical Consideration

In order to present motivations for constructions of this section let us assume for a moment that $G$ is an ordinary compact Lie group, and let $P$ be a (classical) principal bundle over $M$.

By definition, gauge transformations of $P$ are vertical automorphisms of this bundle. In other words, gauge transformations are diffeomorphisms $\psi: P \to P$ satisfying
\[
\pi_M \psi = \pi_M \\
\psi(pg) = \psi(p)g,
\]
for each $p \in P$ and $g \in G$, where $(p, g) \mapsto pg$ is the right action of $G$ on $P$ and $\pi_M: P \to M$ is the projection map. Equivalently, gauge transformations are interpretable as (smooth) sections of the gauge bundle $\mathcal{C}(P)$, which is the bundle associated to $P$, with respect to the adjoint action of $G$ onto itself.

The equivalence between two definitions is established via the following formula
\[
\psi(p) = pf(p),
\]
where $f: P \to G$ is a smooth equivariant function in the sense that
\[
f(pg) = g^{-1}f(p)g,
\]
for each $p \in P$ and $g \in G$. Such functions are in a natural correspondence with sections of the corresponding associated bundle $\mathcal{C}(P)$.

For each $x \in M$ the fiber $G_x = \pi_M^{-1}(x)$ over $x$ (where $\pi_M: \mathcal{C}(P) \to M$ is the projection map) possesses a natural Lie group structure. The group $G_x$ is isomorphic (generally non-invariantly) to $G$. For a given $p \in \pi_M^{-1}(x) = P_x$ there exists a canonical diffeomorphism $G \leftrightarrow P_x$ defined by $g \mapsto pg$, and a group isomorphism $G \leftrightarrow G_x$ given by $g \leftrightarrow [(p, g)]$. Here, $\mathcal{C}(P)$ is understood as the orbit space of the right
action \(((p, g'), g) \mapsto (pg, g^{-1}g'g)\) of \(G\) on \(P \times G\) and \(\lfloor \rfloor\) denotes the corresponding orbit.

There exists a natural left action of \(G_x\) on \(P_x\). In terms of the above identifications this action becomes the multiplication on the left. Collecting all these fiber actions together, we obtain a smooth map
\[
\beta_{M}^*: \mathcal{C}(P) \times_M P \to P.
\]
(3.1)

With the help of \(\beta_{M}^*\) the equivalence between gauge transformations \(\psi\) and sections \(\varphi: M \to \mathcal{C}(P)\) can be described as follows
\[
\psi = \beta_{M}^*(\varphi \times_M \text{id}).
\]
(3.2)

Moreover, the correspondence \(\psi \leftrightarrow \varphi\) is an isomorphism between the group \(G\) of gauge transformations of \(P\), and the group \(\Gamma(\mathcal{C}(P))\) of smooth sections of \(\mathcal{C}(P)\).

The group structure in fibers of \(\mathcal{C}(P)\) determine the following maps of bundles
\[
\begin{align*}
\text{the fibewise multiplication} & \quad \phi_{M}^*: \mathcal{C}(P) \times_M \mathcal{C}(P) \to \mathcal{C}(P) \\
\text{the unit section} & \quad \epsilon_{M}^*: M \to \mathcal{C}(P) \\
\text{the fiberwise inverse} & \quad \kappa_{M}^*: \mathcal{C}(P) \to \mathcal{C}(P)
\end{align*}
\]
(3.3)

At the dual level of function algebras (3.1) and (3.3) are represented by the corresponding \(S(M)\)-linear *-homomorphisms
\[
\begin{align*}
\phi_{M}: S(\mathcal{C}(P)) \to S(\mathcal{C}(P)) \otimes_M S(\mathcal{C}(P)) \\
\epsilon_{M}: S(\mathcal{C}(P)) \to S(M) \\
\kappa_{M}: S(\mathcal{C}(P)) \to S(\mathcal{C}(P)) \\
\beta_{M}: S(P) \to S(\mathcal{C}(P)) \otimes_M S(\mathcal{C}(P)).
\end{align*}
\]
(3.4)

The structure of the gauge group is completely encoded in maps \(\{\phi_{M}, \kappa_{M}, \epsilon_{M}\}\).

At the dual level, gauge transformations \(\psi\) can be viewed as \(S(M)\)-linear *-automorphisms \(\psi: S(P) \to S(P)\) intertwining the (dualized) right action of \(G\). Further, interpreted as sections of \(\mathcal{C}(P)\), gauge transformations become, at the dual level, \(S(M)\)-linear *-homomorphisms \(\varphi: S(\mathcal{C}(P)) \to S(M)\). In this picture, the action of \(\mathcal{G}\) on \(S(\mathcal{C}(P))\) is given by
\[
(\varphi, f) \mapsto (\varphi \otimes \text{id})\beta_{M}(f).
\]

The maps (3.4) are not suitable for considering situations in which gauge transformations act on differential forms. This can be easily “improved” by extending these maps to \(\Omega(M)\)-linear homomorphisms
\[
\begin{align*}
\tilde{\phi}_{M}: \Omega(\mathcal{C}(P)) \to \Omega(\mathcal{C}(P)) \otimes_M \Omega(\mathcal{C}(P)) \\
\tilde{\epsilon}_{M}: \Omega(\mathcal{C}(P)) \to \Omega(M) \\
\tilde{\kappa}_{M}: \Omega(\mathcal{C}(P)) \to \Omega(\mathcal{C}(P)) \\
\tilde{\beta}_{M}: \Omega(P) \to \Omega(\mathcal{C}(P)) \otimes_M \Omega(\mathcal{C}(P))
\end{align*}
\]
(3.5)

of graded-differential *-algebras. It is worth noticing that the above maps are unique, as graded-differential extensions. Actually these maps can be viewed as “pull backs” of (3.1) and (3.3).
3.2. Quantum Consideration

The presented picture admits a direct noncommutative-geometric generalization. As first, we shall construct, starting from a quantum principal bundle $P$, the corresponding quantum gauge bundle $C(P)$. Then the counterparts of maps (3.4) will be introduced and analyzed. In analogy with the classical case we shall define gauge transformations as vertical automorphisms of the bundle $P$. It turns out that such gauge transformations of $P$ are in a natural bijection with ordinary gauge transformations of the classical part $P_{cl}$ of $P$. We shall also study various equivalent interpretations of gauge transformations. Finally, a canonical differential calculus on the bundle $C(P)$ will be constructed and analyzed.

Let $G$ be a compact matrix quantum group, and let $P = (\mathcal{B}, i, F)$ be a quantum principal $G$-bundle over $\mathcal{M}$. Let us fix a trivialization system $\tau$ for $P$. For each $(U, V) \in N^2(\mathcal{U})$ let us define a linear map $\xi_{UV}: S(U \cap V) \otimes A \to S(U \cap V) \otimes A$ by the following formula

\[ \xi_{UV}(\varphi \otimes a) = \varphi g_{UV} \left[ \kappa(a^{(1)})a^{(3)} \right] \otimes a^{(2)}. \]

(3.6)

Lemma 3.1. (i) The maps $\xi_{UV}$ are $S(U \cap V)$-linear $*$-automorphisms and

\[ \xi_{UV}^{-1} = \xi_{VU}. \]

(3.7)

(ii) We have

\[ \xi_{UV}\xi_{VW}(\varphi) = \xi_{UV}(\varphi). \]

(3.8)

for each $(U, V, W) \in N^3(\mathcal{U})$ and $\varphi \in S(U \cap V \cap W) \otimes A$.

(iii) The diagrams

\[
\begin{array}{ccc}
S(U \cap V) \otimes A & \xrightarrow{id \otimes \phi} & [S(U \cap V) \otimes A] \otimes_{\mathcal{U} \cap V} [S(U \cap V) \otimes A] \\
\xi_{UV} & & \xi_{UV} \\
S(U \cap V) \otimes A & \xrightarrow{id \otimes \phi} & [S(U \cap V) \otimes A] \otimes_{\mathcal{U} \cap V} [S(U \cap V) \otimes A] \\
S(U \cap V) \otimes A & \xrightarrow{id \otimes \epsilon} & S(U \cap V) & \xrightarrow{id} & S(U \cap V) \otimes A \\
\xi_{UV} & & \xi_{UV} & & \xi_{UV} \\
S(U \cap V) \otimes A & \xrightarrow{id \otimes \phi} & [S(U \cap V) \otimes A] \otimes_{\mathcal{U} \cap V} [S(U \cap V) \otimes A] \\
\psi_{UV} & & \psi_{UV} \\
S(U \cap V) \otimes A & \xrightarrow{id \otimes \phi} & [S(U \cap V) \otimes A] \otimes_{\mathcal{U} \cap V} [S(U \cap V) \otimes A]
\end{array}
\]

are commutative.
Proof. We have

\[ \xi_{UV}\xi_{VW}(\varphi \otimes a) = \xi_{UV} \left( \varphi g_{UV} \left[ \kappa(a^{(1)})a^{(2)} \right] \otimes a^{(2)} \right) \]

\[ = \varphi g_{UV} \left[ \kappa(a^{(1)})a^{(2)} \right] g_{UV} \left[ \kappa(a^{(2)})a^{(4)} \right] \otimes a^{(3)} \]

\[ = \varphi g_{UV} \left[ \kappa(a^{(1)})a^{(3)} \right] \otimes a^{(2)} = \xi_{UW}(\varphi \otimes a), \]

for each \((U, V, W) \in N^3(\mathcal{U}), \varphi \in \Sigma(U \cap V \cap W)\) and \(a \in \mathcal{A}\). In particular, for \(W = V\) this implies that the maps \(\xi_{UV}\) are bijective and that (3.7) holds.

The maps \(\xi_{UV}\) are \(*\)-homomorphisms because of

\[ \xi_{UV}(\varphi^* \otimes a^* g_{UV}(a^{(3)*}) \otimes a^{(2)*} \]

\[ = \left[ \varphi g_{UV}(a^{(1)})g_{UV}(a^{(3)}) \right]^* \otimes a^{(2)*} = \xi_{UV}(\varphi \otimes a)^*, \]

and

\[ \xi_{UV}(\varphi \psi \otimes ab) = \varphi \psi g_{UV}(a^{(1)}b^{(1)})g_{UV}(a^{(3)}b^{(3)}) \otimes a^{(2)b^{(2)}} \]

\[ = \left[ \varphi g_{UV}(a^{(1)})g_{UV}(a^{(3)}) \otimes a^{(2)} \right] \left[ \psi g_{UV}(b^{(1)})g_{UV}(b^{(3)}) \otimes b^{(2)} \right] \]

\[ = \xi_{UV}(\varphi \otimes a)\xi_{UV}(\psi \otimes b). \]

Finally, let us check commutativity of the above diagrams. We compute

\[ (\xi_{UV} \otimes \xi_{UV})(id \otimes \phi)(\varphi \otimes a) = \varphi g_{UV} \left( \kappa(a^{(1)})a^{(2)} \kappa(a^{(4)})a^{(5)} \right) \otimes a^{(2)} \otimes a^{(5)} \]

\[ = \varphi g_{UV} \left( \kappa(a^{(1)})a^{(4)} \right) \otimes a^{(2)} \otimes a^{(3)} \]

\[ = (id \otimes \phi)\xi_{UV}(\varphi \otimes a), \]

\[ (\xi_{UV} \otimes \psi_{UV})(id \otimes \phi)(\varphi \otimes a) = \varphi g_{UV} \left( \kappa(a^{(1)})a^{(3)}g_{UV}(a^{(4)}) \otimes a^{(2)} \otimes a^{(5)} \right) \]

\[ = \varphi g_{UV}(a^{(1)}) \otimes a^{(2)} \otimes a^{(3)} \]

\[ = (id \otimes \phi)\psi_{UV}(\varphi \otimes a), \]

\[ (id \otimes \epsilon)\xi_{UV}(\varphi \otimes a) = \varphi g_{UV} \left( \kappa(a^{(1)})a^{(2)} \right) = \epsilon(a)\varphi, \]

\[ \xi_{UV}(\varphi \otimes \kappa(a)) = \varphi g_{UV} \left( \kappa^2(a^{(3)}a^{(1)}) \right) \otimes \kappa(a^{(2)}) \]

\[ = \varphi g_{UV} \left( \kappa(a^{(1)})a^{(3)} \right) \otimes \kappa(a^{(2)}) \]

\[ = (id \otimes \kappa)\xi_{UV}(\varphi \otimes a). \]

The (algebra of functions on the) quantum gauge bundle \(\mathcal{C}(P)\) can be now constructed as follows. Let \(\mathcal{D}\) be the set of elements \(q \in \Sigma(\mathcal{U})\) such that

\[ (U \mid_{U \cap V} \otimes id)p_{U}(q) = \xi_{UV}(U \mid_{U \cap V} \otimes id)p_{V}(q) \]

for each \((U, V) \in N^2(\mathcal{U}).\)

Clearly, \(\mathcal{D}\) is a \(*\)-subalgebra of \(\Sigma(\mathcal{U})\). The quantum space \(\mathcal{C}(P)\) corresponding to \(\mathcal{D}\) plays the role of the bundle associated to the principal bundle \(P\) with respect to the adoint action of \(G\) onto itself (represented by \(ad: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}\)). The fact that \(\mathcal{C}(P)\) is a bundle over \(M\) is established through the existence of a \(*\)-monomorphism.
$j_M : S(M) \to D$, playing the role of the dualized fibering of $C(P)$ over $M$. This map is defined by equalities

(3.10) \quad p_U j_M(f) = (f|U) \otimes 1.

**Definition 3.1.** The pair $C(P) = (D, j_M)$ is called the quantum gauge bundle associated to $P$.

We are going to introduce quantum counterparts of maps $\phi_M, \kappa_M, \epsilon_M$ and $\beta_M$. For each $U \in U$, let $\pi^U_\varepsilon : D \to S(U) \otimes A$ be the restriction of $p_U$ on $D$.

**Proposition 3.2.** (i) There exist the unique linear maps $\phi_M : D \to D \otimes_M D$, $\epsilon_M : D \to S(M)$, $\kappa_M : D \to D$ and $\beta_M : B \to D \otimes_M B$ such that

(3.11) \quad (\pi^U_\varepsilon \otimes \pi^U_\varepsilon) \phi_M = (id \otimes \phi) \pi^U_\varepsilon

(3.12) \quad (\pi^U_\varepsilon \otimes \pi_U) \beta_M = (id \otimes \phi) \pi^U_\varepsilon

(3.13) \quad \pi^U_\varepsilon \kappa_M = (id \otimes \kappa) \pi^U_\varepsilon

(3.14) \quad \mid_U \epsilon_M = (id \otimes \epsilon) \pi^U_\varepsilon,

for each $U \in U$. Here, $S(U) \otimes A \otimes A$ and $(S(U) \otimes A) \otimes_U (S(U) \otimes A)$ are identified, in a natural manner.

(ii) All maps are $S(M)$-linear. The maps $\phi_M, \epsilon_M$ and $\beta_M$ are $\ast$-homomorphisms while $\kappa_M$ is antimultiplicative and

(3.15) \quad \kappa_M [\kappa_M(f^\ast)^\ast] = f

for each $f \in D$.

**Proof.** The above equalities uniquely fix the values of maps $\phi_M, \epsilon_M, \kappa_M$ and $\beta_M$ because the maps $\pi_U$ and $\pi^U_\varepsilon$ distinguish points of $B$ and $D$.

Let us consider the algebra

$$\Sigma^*(U) = \sum_{U \in U}^\oplus S(U) \otimes A \otimes A.$$ 

Algebras $D \otimes_M D$ and $D \otimes_M B$ are understandable as subalgebras of $\Sigma^*(U)$. Let us consider maps $\phi_M : \Sigma(U) \to \Sigma^*(U)$, $\kappa_M : \Sigma(U) \to \Sigma(U)$ and $\epsilon_M : \Sigma(U) \to S(U)$ defined by

$$p^U_\varepsilon \phi_M = (id \otimes \phi)p_U,$$

$$p_U \kappa_M = (id \otimes \kappa)p_U,$$

$$\mid_U \epsilon_M = (id \otimes \epsilon)p_U,$$

where $p^U_\varepsilon : \Sigma^*(U) \to S(U) \otimes A \otimes A$ are coordinate projections and $S(U)$ is the direct sum of algebras $S(U)$.

It is easy to see that $\phi_M(B) \subseteq D \otimes_M B$, $\phi_M(D) \subseteq D \otimes_M D$, $\kappa_M(D) \subseteq D$ and $\epsilon_M(D) \subseteq S(M)$. Let us denote by $\{\phi_M, \beta_M, \kappa_M, \epsilon_M\}$ the corresponding restrictions. By construction, (3.11)–(3.14) hold, maps $\beta_M : \phi_M$ and $\epsilon_M$ are $\ast$-homomorphisms, $\kappa_M$ is antimultiplicative and (3.15) holds. \qed
The fibers of the bundle $C(P)$ possess a natural quantum group structure. Further, the bundle $C(P)$ acts on the bundle $P$, preserving fibers and the right action. This is a geometrical background for the next proposition.

**Proposition 3.3.** The following identities hold

(3.16) $(\text{id} \otimes \phi_M)\phi_M = (\phi_M \otimes \text{id})\phi_M$

(3.17) $(\text{id} \otimes F)\beta_M = (\beta_M \otimes \text{id})F$

(3.18) $(\text{id} \otimes \beta_M)\beta_M = (\phi_M \otimes \text{id})\beta_M$

(3.19) $(\text{id} \otimes \epsilon_M) = (\epsilon_M \otimes \text{id})\phi_M = \text{id}$

(3.20) $(\epsilon_M \otimes \text{id})\beta_M = \text{id}$

(3.21) $m_M(\kappa_M \otimes \text{id})\phi_M = m_M(\text{id} \otimes \kappa_M)\phi_M = j_M\epsilon_M$

where $m_M : D \otimes_M D \rightarrow D$ is the multiplication map.

**Proof.** In terms of local trivializations, everything reduces to elementary algebraic properties of the comultiplication, the counit, and the antipode. □

We pass to the analysis of gauge transformation, in this quantum framework. In analogy with classical geometry, these transformations will be defined as vertical automorphisms of the bundle.

**Definition 3.2.** A gauge transformation of the bundle $P$ is every $S(M)$-linear $*$-automorphism $\gamma : B \rightarrow B$ such that the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{F} & B \otimes A \\
\gamma \downarrow & & \downarrow \gamma \otimes \text{id} \\
B & \xrightarrow{F} & B \otimes A
\end{array}
\]

(3.22)

is commutative.

The above diagram says that $\gamma$ intertwines the right action of $G$ on $P$, while the $S(M)$-linearity property ensures that $\gamma$ is a “vertical” automorphism of $P$. Obviously, gauge transformations form a subgroup $G \subseteq \text{Aut}(B)$.

**Proposition 3.4.** (i) The formula

(3.23) $f \leftrightarrow (f \otimes \text{id})\beta_M = \gamma$

establishes a bijection between $S(M)$-linear $*$-homomorphisms $f : D \rightarrow S(M)$ and gauge transformations $\gamma \in G$. In terms of this correspondence, the map $\epsilon_M$ corresponds to the neutral element in $G$ while the product and the inverse in the gauge group are given by

(3.24) $f\kappa_M \leftrightarrow \gamma^{-1}$

(3.25) $(f' \otimes f)\phi_M \leftrightarrow \gamma\gamma'$.

(ii) Let $\gamma$ be an arbitrary gauge transformation. Then the map $\gamma_{cl} : P_{cl} \rightarrow P_{cl}$ defined by

(3.26) $\gamma_{cl}(p) = p\gamma^{-1}$

where $m_M : D \otimes_M D \rightarrow D$ is the multiplication map.
is an ordinary gauge transformation of $P_\text{cl}$. Moreover, the above formula establishes an isomorphism between groups of gauge transformations of bundles $P$ and $P_\text{cl}$.

Proof. Identity (3.17) implies that a $S(M)$-linear homomorphism $\gamma: \mathcal{B} \to \mathcal{B}$ given by the right-hand side of (3.23) satisfies (3.22). Identity (3.20) ensures that $\epsilon_M$ corresponds to the neutral element of $\mathcal{G}$.

Let us consider an arbitrary gauge transformation $\gamma \in \mathcal{G}$. In terms of the trivialization system $\tau$ we have

$$\pi_U \gamma(b) = \sum_i \varphi_i \gamma_U(a_i^{(1)}) \otimes a_i^{(2)}$$

for each $U \in \mathcal{U}$. Here, $\pi_U(b) = \sum_i \varphi_i \otimes a_i$ and $\gamma_U: U \to G_\text{cl}$ are smooth functions uniquely determined by $\gamma$ (understood here in the “dual” manner). We have

$$\gamma_V(a^{(1)})(U \cap V) g_{VU}(a^{(2)}) = \gamma_U(a)|_{U \cap V}$$

for each $a \in \mathcal{A}$ and $(U, V) \in N^2(\mathcal{U})$.

Conversely, if *-homomorphisms $\gamma_U: \mathcal{A} \to S(U)$ are given such that equalities (3.28) hold then formula (3.27) consistently determines a gauge transformation $\gamma$.

Let us now consider a map $f: \Sigma(\mathcal{U}) \to S(\mathcal{U})$ defined by

$$f = \sum_{U \in \mathcal{U}} f_U,$$

where $f_U: S(U) \otimes \mathcal{A} \to S(U)$ are maps given by $f_U(\varphi \otimes a) = \varphi \gamma_U(a)$. It is easy to see that if $b \in \mathcal{D}$ then $f(b) \in S(M)$ (where $S(M)$ is understood as a subalgebra of $S(\mathcal{U})$). Let us pass to the corresponding restriction $f: \mathcal{D} \to S(M)$. By construction (3.23) holds (it is evident in a local trivialization). Conversely, if $f: \mathcal{D} \to S(M)$ determines a gauge transformation $\gamma$ then

$$f(b)|_U = \sum_i \varphi_i \gamma_U(a_i).$$

This easily follows from (3.23).

Let us check correspondences (3.24)–(3.25). We have

$$[(f \otimes f')\phi_M \otimes \text{id}] \beta_M = (f \otimes \gamma')\beta_M = \gamma' \gamma,$$

$$(f \kappa_M \otimes f)\phi_M = f m_M(\kappa_M \otimes \text{id}) \phi_M = \epsilon_M.$$ 

Finally, the second statement easily follows from the definition of gauge transformations, and from the local expression (3.27) for them.

A geometrical explanation of the statement (iii) is this. Gauge transformations, being diffeomorphisms of $P$ at the geometrical level, must preserve classical and quantum parts of $P$. On the other hand, because of the intertwining property, gauge transformations $\gamma$ are completely determined by their “restrictions” $\gamma_\text{cl}$ on $P_\text{cl}$, which correspond precisely to the standard gauge transformations of $P_\text{cl}$.

The quantum gauge bundle $C(P)$ is also an inherently inhomogeneous geometrical object. This is a consequence of the inhomogeneity of $G$. The classical part
of the bundle \( \mathcal{C}(P) \) (*-characters on \( D \)) is naturally identifiable with the ordinary gauge bundle of \( P_{cl} \). In other words,

\[
(\mathcal{C}(P))_{cl} = \mathcal{C}(P_{cl}).
\]

Let \( f: D \to S(M) \) be the *-epimorphism corresponding to \( \gamma \in \mathcal{G} \). This map determines a section \( f^* \) of the bundle \( (\mathcal{C}(P))_{cl} \) as follows

\[
(f^*)(x)(\varphi) = [f(\varphi)](x)
\]

where \( x \in M \) and \( \varphi \in D \). In the framework of the correspondence (3.23), the map \( f^* \) becomes the section corresponding to the gauge transformation \( \gamma_{cl} \), in the classical manner.

****

We pass to the construction and the study of differential calculus on the bundle \( \mathcal{C}(P) \). The calculus will be constructed by combining differential forms on the base manifold \( M \) with a differential calculus on the quantum group \( G \). This calculus will be based on the universal differential envelope \( \Gamma^\land \) of the minimal admissible first-order bicovariant *-calculus \( \Gamma \) over \( G \).

**Lemma 3.5.** (i) For each \((U, V) \in N^2(\mathcal{U})\) there exists the unique homomorphism \( \xi_{UV}^\land: \Omega(U \cap V) \otimes \Gamma^\land \to \Omega(U \cap V) \otimes \Gamma^\land \) of (graded) differential algebras, extending the map \( \xi_{UV} \). The map \( \xi_{UV}^\land \) is *-preserving and bijective, and

\[
(\xi_{UV}^\land)^{-1} = \xi_{UV}^\land.
\]

(ii) We have

\[
\xi_{UV}^\land \xi_{WV}^\land (\varphi) = \xi_{UV}^\land (\varphi)
\]

for each \((U, V, W) \in N^3(\mathcal{U})\) and \( \varphi \in \Omega_{\mathcal{L}}(U \cap V \cap W) \otimes \Gamma^\land \).

(iii) The diagrams

\[
\begin{array}{ccc}
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \widehat{\phi}} & [\Omega(U \cap V) \otimes \Gamma^\land] \otimes_{U \cap V} [\Omega(U \cap V) \otimes \Gamma^\land] \\
\xi_{UV}^\land & \downarrow id & \xi_{UV}^\land \\
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \widehat{\phi}} & [\Omega(U \cap V) \otimes \Gamma^\land] \otimes_{U \cap V} [\Omega(U \cap V) \otimes \Gamma^\land]
\end{array}
\]

\[
\begin{array}{ccc}
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \hat{\kappa}} & \Omega(U \cap V) \otimes \Gamma^\land \\
\xi_{UV}^\land & \downarrow id & \xi_{UV}^\land \\
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \hat{\kappa}} & \Omega(U \cap V) \otimes \Gamma^\land
\end{array}
\]

\[
\begin{array}{ccc}
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \widehat{\phi}} & [\Omega(U \cap V) \otimes \Gamma^\land] \otimes_{U \cap V} [\Omega(U \cap V) \otimes \Gamma^\land] \\
\psi_{UV}^\land & \downarrow id & \psi_{UV}^\land \\
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \widehat{\phi}} & [\Omega(U \cap V) \otimes \Gamma^\land] \otimes_{U \cap V} [\Omega(U \cap V) \otimes \Gamma^\land]
\end{array}
\]

\[
\begin{array}{ccc}
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \hat{\kappa}} & \Omega(U \cap V) \otimes \Gamma^\land \\
\psi_{UV}^\land & \downarrow id & \psi_{UV}^\land \\
\Omega(U \cap V) \otimes \Gamma^\land & \xrightarrow{id \otimes \hat{\kappa}} & \Omega(U \cap V) \otimes \Gamma^\land
\end{array}
\]
are commutative.

Proof. The uniqueness of $\xi_{UV}^\wedge$ follows from the fact that $\Omega_c(U \cap V) \otimes \Gamma^\wedge$ is generated, as a differential algebra, by $S_c(U \cap V) \otimes A$. The hermicity of $\xi_{UV}^\wedge$ follows from the fact that $\ast \xi_{UV}^\wedge \ast$ is a differential extension of the same map $\ast \xi_{UV}^\wedge \ast = \xi_{UV}^\wedge$. In a similar way it follows from Lemma 3.1 that above diagrams are commutative, and that (3.30)–(3.31) hold.

We prove the existence of $\xi_{UV}^\wedge$. The admissibility of $\Gamma$ and the universality of $\Gamma^\wedge$ imply that maps $g_{UV}$ admit the unique graded-differential ($\ast$-preserving) extensions $\hat{g}_{UV} : \Gamma^\wedge \to \Omega(U \cap V)$. 

Now, the maps $f_{UV} : \Gamma^\wedge \to \Omega(U \cap V) \otimes \Gamma^\wedge$ given by

$$f_{UV}(w) = \sum_i \hat{g}_{UV}(\kappa(w_i) w_i^3) \otimes w_i^2,$$

where $\sum_i w_i^1 \otimes w_i^2 \otimes w_i^3 = (\hat{\phi} \otimes \text{id}) \hat{\phi}(w) = (\text{id} \otimes \hat{\phi}) \hat{\phi}(w)$, are homomorphisms of differential $\ast$-algebras. Finally let $\xi_{UV}^\wedge$ be defined by

$$\xi_{UV}^\wedge(\alpha \otimes w) = \alpha f_{UV}(w).$$

It is evident that such defined maps are differential algebra homomorphisms extending $\xi_{UV}^\wedge$. □

Let $\Omega(\tau, C(P))$ be the set of all elements $w \in \Sigma^\wedge(\mathcal{U})$ satisfying

$$(3.32) \quad (U \mid_{U \cap V} \otimes \text{id})p_U(w) = \xi_{UV}^\wedge(V \mid_{U \cap V} \otimes \text{id})p_V(w),$$

for each $(U, V) \in N^2(\mathcal{U})$. It is clear that $\Omega(\tau, C(P))$ is a graded- differential $\ast$-subalgebra of $\Sigma^\wedge(\mathcal{U})$, and that $\Omega(\tau, C(P)) = D$. The elements of the algebra $\Omega(\tau, C(P))$ play the role of differential forms on the bundle $C(P)$. This algebra is generated by $D$, and in fact does not depend of a trivialization system $\tau$. More precisely, if $\eta$ is another trivialization system for $P$ then there exists (the unique) differential ($\ast - \ast$) isomorphism $\Omega(\tau, C(P)) \leftrightarrow \Omega(\eta, C(P))$ extending the identity map on $D$. For this reason we shall simply write $\Omega(\tau, C(P)) = \Omega(C(P))$.

**Proposition 3.6.** (i) The maps $\{\epsilon_M, j_M, \beta_M, \phi_M\}$ admit unique extensions

$$\phi_M^\wedge : \Omega(C(P)) \to \Omega(C(P)) \otimes_M \Omega(C(P))$$

$$\epsilon_M^\wedge : \Omega(C(P)) \to \Omega(M)$$

$$j_M^\wedge : \Omega(M) \to \Omega(C(P))$$

$$\beta_M^\wedge : \Omega(P) \to \Omega(C(P)) \otimes_M \Omega(P),$$

which are homomorphisms of graded-differential algebras.

(ii) The map $\kappa_M$ admits the unique extension $\kappa_M^\wedge : \Omega(C(P)) \to \Omega(C(P))$ which is graded-antimultiplicative and satisfies

$$(3.33) \quad \kappa_M^\wedge d = d\kappa_M^\wedge.$$
(iii) The following identities hold

\[(\phi^\wedge_M \otimes \text{id})\phi^\wedge_M = (\text{id} \otimes \phi^\wedge_M)\phi^\wedge_M\]
\[(\phi^\wedge_M \otimes \text{id})\beta^\wedge_M = (\text{id} \otimes \beta^\wedge_M)\beta^\wedge_M\]
\[(\text{id} \otimes \beta^\wedge_\mathcal{F})\beta^\wedge_M = (\beta^\wedge_M \otimes \text{id})\beta^\wedge_\mathcal{F}\]
\[(\text{id} \otimes \epsilon^\wedge_M)\phi^\wedge_M = (\epsilon^\wedge_M \otimes \text{id})\phi^\wedge_M = \text{id}\]
\[(\epsilon^\wedge_M \otimes \text{id})\beta^\wedge_M = \text{id}\]
\[(m^\wedge_M (\kappa^\wedge_M \otimes \text{id})\phi^\wedge_M = m^\wedge_M (\text{id} \otimes \kappa^\wedge_M)\phi^\wedge_M = j^\wedge_M \phi^\wedge_M,\]

where \(m^\wedge_M\) is the multiplication map in \(\Omega(\mathcal{C}(P))\).

(iv) We have

\[\ast \kappa^\wedge_M \ast = (\kappa^\wedge_M)^{-1},\]

while \(\{\epsilon^\wedge_M, j^\wedge_M, \beta^\wedge_M, \phi^\wedge_M\}\) are \(*\)-preserving maps.

Proof. Using the (anti)multiplicativity, the intertwining differentials properties, and the fact that all considered differential algebras are generated by corresponding zero-order subalgebras, it is easy to see that extensions of all maps in the game are, if exist, unique. The same properties, together with Proposition 3.3, imply that identities (3.34)–(3.39) hold. The statement (iv) follows from (ii) Proposition 3.2 in a similar way. Finally, existence of maps \(\epsilon^\wedge_M, j^\wedge_M, \beta^\wedge_M, \phi^\wedge_M\) and \(\kappa^\wedge_M\) can be established in a similar way as for maps \(\epsilon_M, j_M, \beta_M, \phi_M\) and \(\kappa_M\). \(\square\)

Every \(\gamma \in G\) understood as a \(*\)-homomorphism \(f: \mathcal{D} \rightarrow S(M)\) is uniquely extendible to a \(\Omega(M)\)-linear \(*\)-homomorphism \(f^\wedge: \Omega(\mathcal{C}(P)) \rightarrow \Omega(M)\) of graded-differential algebras.

The following correspondences hold

\[\gamma^{-1} \leftrightarrow f^\wedge \kappa^\wedge_M\]
\[\gamma \gamma' \leftrightarrow m^\wedge_M (f^\wedge \otimes f'^\wedge) \phi^\wedge_M.\]

4. Gauge Fields

In this section we shall present a generalization of the classical gauge theory, within the geometrical framework of quantum principal bundles. The base manifold \(M\) will play the role of space-time. The quantum group \(G\) will describe “internal symmetries” of the system. In order to simplify considerations, we shall deal only with a “pure gauge theory”.

Let us assume that \(\Gamma_{\text{inv}}\) is endowed with an \(\varpi\)-invariant scalar product \((,\)\). It means that

\[(\vartheta, \eta) \otimes 1 = \sum_{kl} (\vartheta_k, \eta_l) \otimes c_k^* d_l\]

for each \(\vartheta, \eta \in \Gamma_{\text{inv}}\), where \(\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)\) and \(\sum_l \eta_l \otimes d_l = \varpi(\eta)\).

Let us assume that \(M\) is oriented and endowed with a (pseudo)riemannian structure.

Let us denote by \(\ast\) the Hodge operation on \(\Omega(M)\). It can be (uniquely) extended to a linear map \(\ast: \mathfrak{hor}(P) \rightarrow \mathfrak{hor}(P)\) such that
\[
\star(i \wedge (\alpha) b) = i \wedge (\star(\alpha)) b,
\]
for each \( \alpha \in \Omega(M) \) and \( b \in B \).

Following the classical analogy gauge fields will be geometrically represented by connection forms \( \omega \) on the bundle \( P \).

To make possible dynamical considerations it is necessary to fix a lagrangian. Generalizing the classical situation, it is natural to consider lagrangians which are quadratic functions of the curvature \( R_\omega \). The curvature operator \( R_\omega \) depends, besides on the connection \( \omega \), also on a choice of the embedded differential map \( \delta: \Gamma_{\text{inv}} \rightarrow \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}} \). As a consequence of this, dynamical properties of the gauge theory will be essentially influenced by \( \delta \). In the classical case the curvature is \( \delta \)-independent.

Let us consider a map \( L: \text{con}(P) \rightarrow \mathfrak{so}(P) \) given by
\begin{equation}
(4.1)
L(\omega) = \sum_i R_\omega(e_i) \star [R_\omega(\bar{e}_i)],
\end{equation}
where elements \( e_i \) form an orthonormal system in \( \Gamma_{\text{inv}} \) and the bar denotes the conjugation in \( \Gamma_{\text{inv}} \). It is easy to see that \( L(\omega) \) is independent of the choice of the mentioned orthonormal system.

The map \( L \) in fact takes values from the space \( \Omega^n(M) \) (where \( n \) is the dimension of \( M \)). Indeed, in terms of local trivializations we have
\begin{equation}
(4.2)
\pi^* \left[ L(\omega) \right] = \sum_i F_U(e_i) \star [F_U(\bar{e}_i)] \otimes 1.
\end{equation}
This easily follows from the fact that \( \sum_i e_i \otimes \bar{e}_i \) is \( \mathfrak{so}^2 \)-invariant.

We shall interpret the map \( L \) as the lagrangian. In terms of the local representation, stationary points of the corresponding action functional \( S(\omega) = \int_M L(\omega) \) are given by the following equations of motion
\begin{equation}
(4.3)
d \star F_U(\bar{e}_k) + \frac{1}{2} \sum_{ij} (d_{ij}^k - d_{kj}^i) A_U(e_j) \star F_U(\bar{e}_i) = 0
\end{equation}
where numbers \( d_{ij}^k \) are determined by
\begin{equation}
(4.4)
\delta(e_k) = -\frac{1}{2} \sum_{ij} d_{ij}^k e_i \otimes e_j.
\end{equation}

The above equations correspond to the classical Yang-Mills equations of motion. The numbers \( (d_{ij}^k - d_{kj}^i)/2 \) play the role of the structure constants of (the Lie algebra of) \( G \).

If the space \( \Gamma_{\text{inv}} \) is infinite-dimensional a technical difficulty arises, related to a question of convergence of the sum in (4.1)–(4.2). In such cases, it is necessary to restrict possible values of \( \omega \) on some subspace of \( \text{con}(P) \), consisting of connections having sufficiently rapidly decreasing components, in an appropriate sense.

We pass to the study of symmetry properties of the introduced lagrangian. As first, it is easy to see that \( L(\omega) \) is invariant under gauge transformations of the bundle \( P \).
The group $G$ naturally acts on the left, via compositions, on the space $\psi(P)$ of pseudotensorial forms. The space $\tau(P)$ is invariant with respect to this action, because $\text{hor}(P)$ is $G$-invariant. The connection space is gauge invariant, too. In terms of gauge potentials the transformation of connections is

$$A^U(\vartheta) \longrightarrow \sum_k A^U(\vartheta_k)\gamma^U(c_k) + \vartheta^U(\vartheta).$$

(4.5)

Here, $\varpi(\vartheta) = \sum_k \vartheta_k \otimes c_k$, the map $\vartheta^U: \Gamma_{\text{inv}} \to \Omega^1(U)$ is given by

$$\vartheta^U\pi(a) = \gamma^U(a^{(1)})d\gamma^U(a^{(2)}),$$

while $\gamma^U: A \to S(U)$ is the map locally representing $\gamma$. Further, the transformation of the curvature is

$$F^U(\vartheta) \longrightarrow \sum_k F^U(\vartheta_k)\gamma^U(c_k).$$

(4.6)

The lagrangian (4.1) is invariant under gauge transformations of the bundle $P$. This is a simple consequence of the unitarity of the representation $\varpi$.

This invariance is a manifestation of classical symmetry properties of the lagrangian. These symmetry properties are completely expressible in terms of the classical part $P_{\text{cl}}$ of $P$.

On the other hand, the lagrangian $L(\omega)$ possesses symmetry properties which are not expressible in classical terms. The appearance of these “quantum symmetries” is a purely quantum phenomena caused by the quantum nature of the space $G$. Formally, they can be described as the invariance of the lagrangian under a natural action of the quantum gauge bundle $C(P)$.

Let $\psi(P, C(P))$ be the space of linear maps $f: \Gamma_{\text{inv}} \to \Omega(C(P)) \hat{\otimes}_M \Omega(P)$ satisfying

$$(f \otimes \text{id})\varpi = (\text{id} \otimes F^\wedge)f.$$

If $\varphi \in \psi(P)$ then $\beta^\wedge_M \varphi \in \psi(P, C(P))$. Hence it is possible to introduce the map $\beta^\wedge_M: \psi(P) \to \psi(P, C(P))$ (via compositions).

Let us compute the element $\beta^\wedge_M \omega$ for $\omega \in \text{con}(P)$. Using the definition of $\beta^\wedge_M$ and the local expression for $\omega$ we obtain

$$(\pi_U^1 \otimes \pi_U)[\beta^\wedge_M(\omega)(\vartheta)] = 1_U \otimes 1 \otimes \vartheta + \sum_k \left\{ A^U(\vartheta_k) \otimes c_k^{(1)} \otimes c_k^{(2)} + 1_U \otimes \vartheta_k \otimes c_k \right\}.$$

(4.8)

Here an identification

$$[\Omega(U) \otimes \Gamma^\wedge] \hat{\otimes}_U [\Omega(U) \otimes \Gamma^\wedge] = \Omega(U) \otimes \Gamma^\wedge \otimes \Gamma^\wedge$$

is assumed. It is worth noticing that the transformation law (4.5) is contained in (4.8). Indeed, understanding gauge transformations as differential algebra homomorphisms $f^\wedge: \Omega(C(P)) \to \Omega(M)$ we obtain (4.5) by composing $\beta^\wedge_M(\omega)$ and $f^\wedge \otimes \text{id}$.
The curvature is transformed as follows
\[(\pi_U \otimes \pi_U)[\beta_M^\wedge (R_\omega)(\theta)] = \sum_k F^U(\theta_k) \otimes c_k^{(1)} \otimes c_k^{(2)}].\]

That is, in local terms we have
\[F^U \longrightarrow (F^U \otimes \text{id})\varpi.\]

The curvature operator is gauge-covariant in the sense that
\[\beta^\wedge M(R_\omega)(\varpi) = d\beta^\wedge M(\omega) - \langle \beta^\wedge M(\omega), \beta^\wedge M(\omega) \rangle.\]

A possible interpretation of the above equation (which is a trivial consequence of the fact that \(\beta^\wedge_M : \Omega(P) \rightarrow \Omega(\mathcal{C}(P)) \otimes_M \Omega(M)\) is a differential algebra homomorphism) is this. The relation between the connection \(\omega\) and its curvature is, being expressible in intrinsically geometrical terms, preserved under the action of \(\mathcal{C}(P)\).

Expression (4.6) for the curvature of the transformed connection under a gauge transformation also directly follows from (4.9).

In order to find the transformation of the local expression for the lagrangian, we should insert in (4.2) the local expression for the transformed curvature, under the action \(\beta^\wedge_M\) of \(\mathcal{C}(P)\) on \(P\). The lagrangian transforms as follows
\[\left\{ \sum_k F^U(e_k) \star F^U(\bar{e}_k) \right\} \longrightarrow \sum_{klm} F^U(e_l) \star F^U(\bar{e}_n) \otimes c_{lk}e_{nk}^*,\]

where \(\text{ad}(e_i) = \sum_j e_j \otimes c_{ji}\). On the other hand
\[\left[ \sum_k F^U(e_k) \star F^U(\bar{e}_k) \right] \otimes 1 = \sum_{klm} F^U(e_l) \star F^U(\bar{e}_n) \otimes c_{lk}e_{nk}^*,\]

because of the \(\varpi^{\otimes 2}\)-invariance of \(\sum_k e_k \otimes \bar{e}_k\).

Hence, the lagrangian is invariant with respect to the action \(\beta^\wedge_M\) of the gauge bundle \(\mathcal{C}(P)\) on \(P\).

It is important to mention that the property of “quantum gauge invariance” of the lagrangian can not be viewed as an inherent property of the local expression (4.2). Because this property essentially depends on the ordering of terms \(F^U\) and \(\star F^U\). However, in the general case, the ordering of terms \(R_\omega\) and \(\star R_\omega\) in the global representation of the lagrangian is essential, because \(\mathcal{B}\) is a noncommutative algebra.

5. Example

We shall now illustrate the presented formalism on a concrete example, assuming that \(G = SU_\mu(2)\) (with \(\mu = (-1, 1) \setminus \{0\}\)). By definition [3] this compact matrix quantum group is based on the \(2 \times 2\) matrix
\[u = \begin{pmatrix} \alpha & -\mu\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \]
where the elements $\alpha$ and $\gamma$ satisfy the following relations

\begin{equation}
\begin{aligned}
\alpha \gamma &= \mu \gamma \alpha \\
\alpha^* \alpha + \gamma^* \gamma &= 1 \\
\alpha \alpha^* + \mu^2 \gamma^* \gamma &= 1.
\end{aligned}
\end{equation}

The classical part of $G$ is isomorphic to $U(1)$. An explicit isomorphism is given by $g \leftrightarrow g(\alpha)$.

It turns out (citeD–Section 6) that the right $\mathcal{A}$-ideal $\hat{R} \subseteq \ker(\epsilon)$ determining the minimal admissible (bicovariant $\ast$-) first-order calculus $\Gamma$ over $G$ is given by

\begin{equation}
\hat{R} = (\mu^2 \alpha + \alpha^* - (1 + \mu^2)1) \ker(\epsilon).
\end{equation}

Let $X: \mathcal{A} \to \mathbb{C}$ be a generator of $\text{Lie}(G_{cl})$ specified by $X(\alpha) = -X(\alpha^*) = 1/2$ and $X(\gamma) = X(\gamma^*) = 0$.

Let $\rho: \mathcal{A} \to \mathcal{A}$ be a map given by $\rho = (X \otimes \text{id}) \text{ad}$. Let $\nu: \Gamma_{inv} \to \mathbb{C}$ and $\tilde{\rho}: \Gamma_{inv} \to \mathcal{A}$ be the maps defined by $\nu \pi = X$ and $\tilde{\rho} \pi = \rho$. Then $\tilde{\rho} = (\nu \otimes \text{id}) \varpi$, and $\tilde{\rho}$ maps isomorphically the space $\Gamma_{inv}$ onto the $\ast$-subalgebra $\mathcal{Q} \subseteq \mathcal{A}$ of left $G_{cl}$-invariant elements of $\mathcal{A}$. The subalgebra $\mathcal{Q}$ is interpretable as the algebra of polynomial functions on a quantum 2-sphere.

The adjoint action $\varpi$ is reducible. The space $\Gamma_{inv}$ is decomposable into the orthogonal sum

\begin{equation}
\Gamma_{inv} = \bigoplus_{k \geq 0} \Gamma_{inv}^k
\end{equation}

of irreducible subspaces. The subspace $\Gamma_{inv}^k$ is $(2k + 1)$-dimensional (that is, all integer-spin irreducible multiplets are in the game).

The space $\tilde{\rho}(\Gamma_{inv}^k) = \mathcal{Q}_k$ is spanned by quantum spherical harmonics $\zeta_{km}$, where $m \in \{-k, \ldots, k\}$. They constitute a standard basis for the action of $G$. Explicitly, these elements are given by

\begin{equation}
\begin{aligned}
\zeta_{km} &= (-)^m \mu^{km-m} \left[ (k - m)_\mu ! / (k + m)_\mu ! \right]^{1/2} \partial^m p_k(\gamma \gamma^*) \gamma^m \alpha^m \\
\zeta_{-m,k} &= \mu^{km} \alpha^* \gamma^m \left[ (k - m)_\mu ! / (k + m)_\mu ! \right]^{1/2} \partial^m p_k(\gamma \gamma^*).
\end{aligned}
\end{equation}

Here, $m \geq 0$ and $\partial: P(x) \to P(x)$ is a “quantum differential” (acting on the space $P(x)$ of $x$-polynomials) specified by $\partial(x^n) = n \mu x^{n-1}$. Finally, $p_k(x)$ are polynomials given by

\begin{equation}
\begin{aligned}
p_k(x) &= (-)^k c_k \partial^k \left[ x^k \prod_{j=1}^k (1 - \mu^{1-j} x) \right] \\
p_0(x) &= 1,
\end{aligned}
\end{equation}

while $c_k > 0$ and

\begin{equation}
k^{j_{\mu}!} = \prod_{j=1}^k j_{\mu} \quad j_{\mu} = \frac{1 - \mu^{2j}}{1 - \mu^2}.
\end{equation}

Let us now describe a construction of the natural embedded differential map $\delta$. We shall first construct a complement $\mathcal{L} \subseteq \ker(\epsilon)$ of the space $\hat{R}$. 
The elements $\gamma^k (k \in \mathbb{N})$ are primitive for the adjoint action of $G$ on $\ker(\epsilon)$. Let $\mathcal{L} \subseteq \ker(\epsilon)$ be the minimal $\omega$-invariant subspace containing these elements, and the ad-invariant element $\mu^2 \alpha + \alpha^* - (1 + \mu^2)1$. It turns out that the restriction $(\pi|\mathcal{L}) \colon \mathcal{L} \to \Gamma_{\text{inv}}$ is bijective. Evidently, this restriction intertwines the adjoint actions. Let $\delta \colon \Gamma_{\text{inv}} \to \Gamma_{\text{inv}} \otimes \Gamma_{\text{inv}}$ be defined by
\begin{equation}
\delta(\vartheta) = - (\pi \otimes \pi) \phi \left[ (\pi|\mathcal{L})^{-1}(\vartheta) \right].
\end{equation}
(5.7)

It is clear, by construction, that $\delta$ is an embedded differential map. Moreover,
\begin{equation}
\delta \kappa = - (\kappa \otimes \kappa) \phi \delta
\end{equation}
where the extension of the antipode $\kappa \colon \Gamma_{\text{inv}} \to \Gamma_{\text{inv}}$ is given by
\begin{equation}
\kappa \pi = - \pi \kappa^2.
\end{equation}
(5.8)

Let us compute the values of $\delta$ on the singlet and the triplet subspace of $\Gamma_{\text{inv}}$. The singlet space $\Gamma_{\text{inv}}^0$ is spanned by
\begin{equation}
\varepsilon = \pi (\mu^2 \alpha + \alpha^*),
\end{equation}
while the triplet space $\Gamma_{\text{inv}}^1$ is spanned by
\begin{equation}
\eta_+ = \pi (\gamma), \quad \eta = \pi (\alpha - \alpha^*), \quad \eta_- = \pi (\gamma^*).
\end{equation}
(5.11)

Applying the definition of $\delta$ we obtain
\begin{align*}
-\delta(\varepsilon) &= (\varepsilon \otimes \varepsilon + \mu^2 \eta \otimes \eta)/(1 + \mu^2) - \mu \eta_+ \otimes \eta_- - \mu^3 \eta_- \otimes \eta_+ \\
-\delta(\eta_+) &= ((\varepsilon - \mu^2 \eta) \otimes \eta_+ + \eta_+ \otimes (\varepsilon + \eta))/(1 + \mu^2) \\
-\delta(\eta_-) &= (\eta_- \otimes (\varepsilon - \mu^2 \eta) + (\varepsilon + \eta) \otimes \eta_+)/(1 + \mu^2) \\
-\delta(\eta) &= (\varepsilon \otimes \eta + \eta \otimes \varepsilon + (1 - \mu^2) \eta \otimes \eta)/(1 + \mu^2) + \mu (\eta_+ \otimes \eta_- - \eta_- \otimes \eta_+).
\end{align*}

****

The corresponding gauge theory based on the bundle $P$, calculus $G$ and the lagrangian $L(\omega)$ is essentially different from the classical gauge theory with $G = SU(2)$.

As first, gauge fields possess infinitely many internal degrees of freedom. In the classical limit $\mu \to 1$ the restriction $A^U|\Gamma_{\text{inv}}^1$ on the triplet subspace can be interpreted as a classical $SU(2)$ gauge field. Restrictions on other irreducible subspaces are classically interpretable as additional vector fields.

According to the general theory, the connection $A^U$ can be decomposed into “classical” and “purely quantum” parts
\begin{equation}
A^U = A_{cl}^U + A_{\perp}^U,
\end{equation}
where $A_{cl}^U|\ker(\nu) = 0$ and $A_{\perp}^U(\varepsilon) = 0$. The map $A_{cl}^U$ can be interpreted as a connection on the classical $U(1)$-bundle $P_{cl}$. It is important to point out that the decomposition $\Gamma_{\text{inv}} = \ker(\nu) + \mathbb{C} \varepsilon$ is incompatible with the decomposition of $\Gamma_{\text{inv}}$ into irreducible multiplets.
Let us compute the singlet and the triplet components of the curvature. Applying the definition of $\delta$ and using the local expression of the curvature we find

\[ F_U(\varepsilon) = dA_U(\varepsilon) + \mu(1 - \mu^2)A_U(\eta_+)A_U(\eta_-) \]
\[ F_U(\eta_+) = dA_U(\eta_+) + A_U(\eta_+)A_U(\eta) \]
\[ F_U(\eta_-) = dA_U(\eta_-) + A_U(\eta)A_U(\eta_-) \]
\[ F_U(\eta) = dA_U(\eta) + 2\mu A_U(\eta_+)A_U(\eta_-). \]

In general, components of the restriction $F^U|_{\Gamma^k_{\text{inv}}}$ will be expressible through fields $A_U(\vartheta)$, where $\vartheta \in \Gamma^l_{\text{inv}}$ and $1 \leq l \leq k$.

Equations of motion are mutually essentially correlated. Indeed, the equation describing the propagation of fields $A_U|_{\Gamma^k_{\text{inv}}}$ will generally contain terms of the form $A_U(\vartheta) \star F_U(\eta)$, where $\vartheta, \eta \in \Gamma^{i,j}_{\text{inv}}$ and $|i - k| \leq j \leq i + k$. This easily follows from the definition of $\delta$. It is interesting to observe that nonsinglet components are not explicitly influenced by the singlet component $A_U(\varepsilon)$. On the other hand, the singlet propagation is intertwined only with $A_U(\eta_\pm)$. Explicitly,

\[ d \star F_U(\varepsilon) = 0. \]

6. CONCLUDING REMARKS

In this study we have assumed that the higher-order differential calculus on the structure group is based on the corresponding universal envelope. All constructions can be performed also in the case when the higher-order calculus is described by the corresponding bicovariant (braided) external algebra [5].

The admissibility assumption for $\Gamma$ ensures full local trivializability of differential structures on $P$ and $C(P)$. From the “local” point of view however, the whole formalism works for an arbitrary bicovariant *-calculus $\Gamma$.

In summary, physical properties of the presented gauge theory are essentially influenced by two additional structural elements. As first, it is necessary to fix a bicovariant *-calculus $\Gamma$ over $G$. This determines kinematical degrees of freedom. Secondly, the curvature is determined only after fixing an embedded differential map $\delta$. In such a way the dynamics becomes $\delta$-dependent.

The same geometrical framework of quantum principal bundles contains logically inequivalent ways of generalizing classical gauge theory. For example, in the formulation discussed in [6] the calculus on $G$ does not figure explicitly, and the curvature is defined in a different way. In this formulation internal degrees of freedom and the curvature are determined after fixing a fundamental representation of the structure quantum group, which a priori excludes quantum phenomena appearing at the level of differential calculus.

The presented gauge theory admits a natural generalization to the completely quantum context, in which the base manifold $M$ is a noncommutative space.

APPENDIX A. THE MINIMAL ADMISSIBLE CALCULUS

In this Appendix some properties of entities associated with the minimal admissible calculus $\Gamma$ over the quantum $SU(2)$ group are collected. In particular, we shall analyze in more details the structure of the space $L$ which determines the embedded differential map.
For each integer $n \geq 1$ let $u_n$ be the $n \times n$ matrix over $A$, corresponding to the irreducible representation $[3, 4]$ of $G$, having the spin $(n - 1)/2$ and acting in $C^n$. Let $A_n$ be the lineal spanned by matrix elements of $u_n$. We have
\[ A = \sum_{n \geq 1} \oplus A_n, \]
according to the representation theory of $G$. The spaces $A_n$ are invariant under the adjoint action of $G$. They are mutually orthogonal, relative to the scalar product induced by the Haar measure $h : A \to \mathbb{C}$.

In subspaces $A_n$ the adjoint action decomposes (without degeneracy) into irreducible multiplets with spins from the set $\{0, 1, \ldots, n - 1\}$.

Lemma A.1. Let $\xi \in A_n$ be a primitive element for the $k$-spin subrepresentation of $\text{ad}|A_n$. Then,
\[ \xi = p_{kn}(\lambda)\gamma^k \]
where $\lambda = \mu \alpha + \mu^{-1}\alpha^*$ and $p_{kn}$ is a polynom of degree $n - k - 1$ with real coefficients.

Proof. From the representation theory of $G$ it follows that
\[ A_2A_n = A_nA_2 = A_{n-1} \oplus A_{n+1} \]
for each $n \geq 2$. This implies that $A_n \setminus \{0\}$ is consisting of certain polynoms of degree $n - 1$ (over generators). Further, polynoms of degree $k \leq n - 1$ form the space $\sum_{i} \oplus A_i$, where $i \leq n$. Also, from the reality of commutation relations (5.2) and the orthogonality of spaces $A_n$ it follows that we can write
\[ A_n = A^R_n \oplus iA^R_n \]
where $A^R_n$ is consisting of polynoms with real coefficients.

On the other hand, every non-zero element of the form (A.1) is primitive, and generates an irreducible $k$-spin multiplet relative to the adjoint representation. Having in mind the form of the decomposition of $\varpi|A_n$ into irreducible multiplets we conclude that (A.1) covers all primitive elements of the restriction $\varpi|A_n$. \qed

Let us assume that polynoms $p_{kn}$ are fixed. For fixed $k$, polynoms $p_{kn}$ are orthogonal, with respect to the scalar product given by
\[ (p, q) = \langle q(\lambda)\gamma^k\gamma^*p(\lambda)^* \rangle, \]
Let $j : A \to A$ be the modular automorphism [4] corresponding to the Haar measure. This map is characterized by the identity
\[ h(ba) = h(j(a)b). \]
In the case of the quantum $SU(2)$ group we have
\[ j(\gamma) = \gamma, \quad j(\alpha) = \mu^2 \alpha, \quad j(\alpha^*) = \mu^{-2}\alpha^*, \quad j(\gamma^*) = \gamma^* \]
Applying (A.3)–(A.4) we see that the scalar product defined in (A.2) can be rewritten in the form
\[(p, q) = h \left[ p^* (\lambda) q (\gamma \gamma^*)^k \right]. \tag{A.5}\]

Now, starting from (A.5), observing that the above scalar product is invariant under the replacement \(\lambda \rightarrow \alpha + \alpha^*\), and using elementary properties of polynomials it can be shown that all zeroes of \(p_{kn}\) are contained in the interval \([-2, 2]\).

We have
\[(A.6) \quad L = \mathbb{C} (\mu \lambda - (1 + \mu^2)1) \oplus \left\{ \sum_{k \geq 1} \mathcal{L}_k \right\}, \]
where \(\mathcal{L}_k \subseteq A_{k+1}\) is the \(k\)-spin irreducible subspace (for the adjoint action). Let \(L_\ast \subseteq A\) be the lineal given by
\[(A.7) \quad L_\ast = \mathbb{C} 1 \oplus \left\{ \sum_{k \geq 1} \mathcal{L}_k \right\}. \]

Let \(P(\lambda) \subseteq A\) be the subalgebra generated by \(\lambda\).

**Lemma A.2.** (i) We have
\[(A.8) \quad \pi(A_n) = \sum_{k \leq n-1} \Gamma_{inv}^k \]
for each \(n \in \mathbb{N}\).

(ii) The map \(\tilde{\Omega}: P(\lambda) \otimes L_\ast \rightarrow A\) given by
\[(A.9) \quad \tilde{\Omega}(p(\lambda) \otimes a) = p(\lambda) a \]
is bijective.

**Proof.** Let us prove that \(\tilde{\Omega}\) is bijective. As first, let us observe that the elements from \(P(\lambda)\) are ad-invariant. In particular,
\[(A.10) \quad \text{ad}(p(\lambda) a) = p(\lambda) \text{ad}(a), \]
for each \(a \in A\). According to Lemma A.1 all primitive elements for ad: \(A \rightarrow A \otimes A\) are contained in the image of \(\tilde{\Omega}\). Now (A.10) implies that \(\tilde{\Omega}\) is surjective. We prove that \(\tilde{\Omega}\) is injective. It is sufficient to check that \(\tilde{\Omega}|(P(\lambda) \otimes \mathcal{L}_k)\) is injective, for each \(k \in \mathbb{N}\). However, it follows again from Lemma A.1 and (A.10), because \(\tilde{\Omega}(p_{kn}(\lambda) \otimes \mathcal{L}_k) \subseteq A_n\) is exactly the \(k\)-spin irreducible subspace.

The following identity holds
\[(A.11) \quad \rho \tilde{\Omega} = (\epsilon \otimes (p(\mathcal{L}_\ast))). \]

The statement (i) now follows from the definition of \(\Gamma_{inv}\) and from the facts that \(\epsilon(\lambda) = \mu + \mu^{-1}\) and \(p_{kn}(\mu + \mu^{-1}) \neq 0\). \(\square\)
Using (A.6) and the definition of $\delta$, it can be shown that

\[(A.12)\]

$$\delta(\Gamma_{\text{inv}}^{n}) \subseteq \sum_{ij} \delta_{ij}(\Gamma_{\text{inv}}^{i} \otimes \Gamma_{\text{inv}}^{j})$$

for each $n \in \mathbb{N}$, where the sum is taken over pairs $(i, j)$ satisfying $|i - j| \leq n \leq i + j$.

In particular,

$$\delta(\vartheta)^{0,n} = d_{n} \varepsilon \otimes \vartheta$$

$$\delta(\vartheta)^{n,0} = d_{n} \vartheta \otimes \varepsilon,$$

for each $\vartheta \in \Gamma_{\text{inv}}^{n}$, with $d_{n} \in \mathbb{R} \setminus \{0\}$. This implies that singlet components of $\omega$ do not figure in nonsinglet components of the curvature.

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