ERROR ANALYSIS FOR A PARABOLIC PDE MODEL PROBLEM ON A COUPLED MOVING DOMAIN IN A FULLY EULERIAN FRAMEWORK

HENRY VON WAHL† AND THOMAS RICHTER‡

Abstract. We introduce an unfitted finite element method with Lagrange-multipliers to study an Eulerian time stepping scheme for moving domain problems applied to a model problem where the domain motion is implicit to the problem. We consider a parabolic partial differential equation (PDE) in the bulk domain, and the domain motion is described by an ordinary differential equation (ODE), coupled to the bulk partial differential equation through the transfer of forces at the moving interface. The discretisation is based on an unfitted finite element discretisation on a time-independent mesh. The method-of-lines time discretisation is enabled by an implicit extension of the bulk solution through additional stabilisation, as introduced by Lehrenfeld & Olshanskii (ESAIM: M2AN, 53:585–614, 2019). The analysis of the coupled problem relies on the Lagrange-multiplier formulation, the fact that the Lagrange-multiplier solution is equal to the normal stress at the interface and that the motion of the interface is given through rigid body motion. This paper covers the complete stability analysis of the method and an error estimate in the energy norm, under an assumption on the discrete interface velocity. This includes the dynamic error in the domain motion resulting from the discretised ODE and the forces from the discretised PDE. To the best of our knowledge this is the first error analysis of this type of coupled moving domain problem in a fully Eulerian framework. Numerical examples illustrate the theoretical results.

Key words. Eulerian time stepping, coupled moving domain problems, unfitted FEM, ghost penalty

AMS subject classifications. 65M12, 65M60, 65M85

1. Introduction. Particulate flows, particle settling and in the broader sense fluid solid interactions play a major role in applications, ranging from medicine [11, 39, 12] and biology [29] to industry [3, 44].

The most well-established method to solve the resulting fluid-structure interaction problem is the so-called Arbitrary Lagrangian-Eulerian (ALE) method [10]. Here a mesh of a reference geometry is created, and the moving domain problem is solved by mapping the equations into the reference configuration. A significant burden in this approach occurs when the deformation with respect to the reference configuration becomes very large. In this case, re-meshing procedures [42] must be included, or Eulerian approaches [36, 38] need to be considered. In this paper, we shall focus on the latter approach. In particular, we shall focus on an unfitted Eulerian approach in the context of fluid-rigid body interactions. Such Eulerian approaches are based on a fixed background mesh to define a set of potential unknowns, and the geometry of the problem is described separately.

The main challenge in Eulerian approaches for time-dependent moving domain problems is the approximation of the time-derivative. Standard approximations based on finite differences are not easily applicable since the expression \( \partial_t u \approx (u^n - u^{n-1})/\Delta t \) is not well-defined if \( u^n \) and \( u^{n-1} \) live on different domains. A successful approach to deal with this challenge is a class of space-time Galerkin formulation in an Eulerian set-
This approach has been proven to work for scalar bulk problems [27, 23, 37, 50], problems on moving surfaces [33, 34] and coupled bulk-surface problems [19]. However, space-time Galerkin methods have the drawback that a higher-dimensional problem has to be solved. This problem can be circumvented by an approach using adjusted quadrature rules to reduce the space-time problem into a classical time stepping scheme [14]. However, this comes at the expense of costly computations of projections between different function spaces.

In this paper, we shall follow a different approach that recovers the use of standard time stepping schemes by using an extension of the previous solution to the domain of the next time step. This concept was first introduced in [35] for problems on moving surfaces and then for scalar bulk convection-diffusion problems in [26]. The essential idea in the latter is to apply additional stabilisation in a strip around the moving interface, such that the discrete solution $u_h^{n-1}$ is well-defined in a larger, non-physical domain $\Omega_h^{n-1} \supset \Omega_h^n$. As a result, the expression $(u_h^n - u_h^{n-1})/\Delta t$ is again well-defined on the domain $\Omega_h^n$.

This unfitted finite element method with Eulerian time stepping schemes for partial differential equation problems posed on moving domains has so far been considered for problems where the motion of the domain is a given quantity [26, 6, 49, 28, 1]. Furthermore, the method developed in these papers has been successfully applied to a fluid-structure interaction problem, where the geometry motion is part of the problem to be solved [48, 47]. However, no error analysis is available for this setting.

The main contribution of this paper is the development of an error estimate for this Eulerian time stepping scheme for a partial differential equation (PDE) in a moving domain, where the domain motion is driven by an ordinary differential equation (ODE) coupled to the PDE. To this end, we consider a set of simplified equations to analyse this kind of Eulerian time stepping with coupled domain motion. This will be a parabolic PDE in the time-dependent bulk domain, while the motion of the moving interface is driven by translational rigid body motion. These two equations are then coupled on the moving interface by the non-homogeneous Dirichlet boundary conditions and the forces acting on the moving interface. The coupling condition is the same that would be typical for standard fluid-structure interactions problems [39]. The main simplification is the restriction to a parabolic PDE model, which allows us to avoid the additional difficulties that would be involved in treating the divergence constraint. This restriction allows us a clearer presentation of the nevertheless technically complex proofs. We assume that an extension to the Stokes equations would not bring any significant surprises.

The remainder of this paper is structured as follows. In section 2, we discuss the mathematical model under consideration and show the unique solvability thereof. We then begin by a temporal semi discretisation of the problem in section 3 and show the stability of the resulting scheme under an assumption on the discrete interface velocity. Section 4 then covers the full discretisation of our problem. We introduce our CutFEM Lagrange-multiplier discretisation, then show the discrete problem’s solvability and stability of the discrete scheme. We then quantify the error in the time-dependent geometry resulting from the discretisation of the ODE governing the motion of the domain. This is then used to prove a consistency error estimate and finally an error estimate in the energy norm. In section 5, we illustrate our theoretical results with some numerical examples, including extensions to higher-order in both space and time. Finally, we give a brief summary of the results and an outlook for potential future work in section 6.
2. Mathematical Problem. Let $\Omega \subset \mathbb{R}^d$, for $d \in \{2, 3\}$, be an open bounded domain, which we denote as the background domain. We divide $\Omega$ into the $d$-dimensional open bulk domain of interest $\Omega(t)$, the $d$-dimensional complement of $\Omega$ in $\tilde{\Omega}$ denoted as $\Sigma(t) = \tilde{\Omega} \setminus \Omega(t)$ and $d - 1$-dimensional interface $\Gamma(t) = \partial \Sigma$ between the two, i.e., $\tilde{\Omega} = \Omega(t) \cup \Gamma(t) \cup \Sigma(t)$. We assume that the interface $\Gamma(t)$ can be described by a smooth level set function $\phi(t, x)$, i.e,

$$\Gamma(t) = \{ x \in \tilde{\Omega} \mid \phi(t, x) = 0 \} \quad \text{and} \quad \Omega(t) = \{ x \in \tilde{\Omega} \mid \phi(t, x) < 0 \}.$$ 

Furthermore, we denote the fixed part of the boundary of the bulk domain as $\Gamma_{\text{out}} = \partial \tilde{\Omega}$. A sketch of such a domain can be seen in Figure 2.1. Now let $[0, t_{\text{end}}]$ be a finite time interval and assume that $\Omega(t) \subset \tilde{\Omega}$ for all $t \in [0, t_{\text{end}}]$. We then define the space-time domain

$$\mathcal{Q} := \bigcup_{t \in (0, t_{\text{end}})} \Omega(t) \times \{ t \}.$$ 

Let $u$ denote the solution in the bulk domain $\Omega$, $\xi$ the velocity of the interface $\Gamma$ and $\chi$ the centre of mass of $\Sigma$ relative to the initial position. In $\mathcal{Q}$, we then consider the vector-valued parabolic model problem

\begin{align*}
(2.1a) & \quad \partial_t u - \Delta u = 0 \quad \text{in } \Omega(t) \times (0, t_{\text{end}}] \\
(2.1b) & \quad u = \xi \quad \text{in } \Gamma(t) \times (0, t_{\text{end}}] \\
(2.1c) & \quad u = 0 \quad \text{in } \Gamma_{\text{out}} \times (0, t_{\text{end}}] \\
(2.1d) & \quad u(0) = u_0 \quad \text{in } \Omega(0)
\end{align*}

where the motion of the moving interface $\Gamma(t)$ is determined by

\begin{align*}
(2.2a) & \quad \frac{d}{dt} \xi = g + F \quad \text{in } (0, t_{\text{end}}] \\
(2.2b) & \quad \frac{d}{dt} \chi = \xi \quad \text{in } (0, t_{\text{end}}] \\
(2.2c) & \quad \xi(0) = \xi_0 \\
(2.2d) & \quad \chi(0) = 0
\end{align*}

Here, $u_0$ and $\xi_0$ are given initial conditions, $g$ is a constant external force acting on $\Sigma$, and $F = - \int_{\Gamma(t)} \partial_n u \, ds$ is the force acting from $\Omega$ onto $\Sigma$, and $\Gamma$ moves with velocity $\xi$ through $\tilde{\Omega}$. The system (2.1) can be seen as a simplification of the transient Stokes equations on a moving domain [49, 6] by restricting the velocity to the space of divergence-free functions, but with the added complexity of the motion being driven by an ODE, coupled to the bulk equations through the transfer forces. We consider this model problem, since this already shows the difficulties in the error analysis of a coupled moving domain problem in a purely Eulerian framework.

The more complicated problem of coupling a rigid body to the non-linear Navier-Stokes equations has been studied extensively. In [9] the authors show the existence of weak solutions which are global in time up to collision of the rigid body with the boundary. In three spatial dimensions smallness of the data is required. Strong solutions are studied in [45, Theorem 2.2] and their unique existence, global in time up to collision, is shown in the two dimensional case. In the three dimensional case,
solutions that are global in time are shown to exist under smallness requirements on the data. For the linearised case, the authors of [30] could even show maximal regularity.

2.1. Stability estimate. We begin by showing that the system (2.1)–(2.2) satisfies a stability estimate, depending only on the problem data. Let us consider the spaces

\[ V(t) = \{ v \in H^1(\Omega(t)) \mid v|_{\Gamma_{\text{out}}} = 0 \} \quad \text{and} \quad N(t) = H^{-1/2}(\Gamma(t)). \]

Multiplying (2.1) with test-functions from the appropriate spaces and by using integration by parts we get the weak formulation: Find \((u(t), \lambda(t), \xi(t), \chi(t)) \in V(t) \times N(t) \times \mathbb{R}^d \times \mathbb{R}^d\) such that

\[
(2.3) \quad (\partial_t u, v)_{\Omega(t)} + (\nabla u, \nabla v)_{\Omega(t)} + (\lambda, v)_{\Gamma(t)} + (\mu, u - \xi)_{\Gamma(t)} + (\frac{d}{dt}\xi, \xi_1)_2 + (\frac{d}{dt}\chi, \chi_2)_2 = (g + F, \xi)_2 + (\xi, \chi)_2 \]

holds for all \((v, \mu, \xi_1, \xi_2) \in V(t) \times N(t) \times \mathbb{R}^d \times \mathbb{R}^d\). Furthermore, the solution of the Lagrange-multiplier is \(\lambda = -\partial_n u\), see [2, Theorem 3.2].

**Lemma 2.1.** For the velocity and position solution \((u, \xi, \chi)\) of (2.3), it holds that

\[
\|u(t)\|^2_{\Omega(t)} + \|\xi(t)\|^2_2 + \|\chi(t)\|^2_2 + \int_0^t \|\nabla u(s)\|^2_{\Omega(s)} \, ds \leq \exp(t) \left[ \|u_0\|^2_{\Omega(0)} + \|\xi_0\|^2_2 \right] + c_{L^2.1} \left( \exp(t) - 1 \right) \|g\|^2_2,
\]

with a constant \(c_{L^2.1} > 0\) that only depends on the domain size \(|\Omega|\) and \(|\Gamma|\).

**Proof.** Testing (2.3) with \((v, \mu, \xi_1, \xi_2) = (u, -\lambda, \xi, \chi)\) gives

\[
(2.4) \quad (\partial_t u, u)_{\Omega(t)} + \|\nabla u\|^2_{\Omega(t)} - (\partial_n u, \xi)_{\Gamma(t)} + (\frac{d}{dt}\xi, \xi)_2 + (\frac{d}{dt}\chi, \chi)_2 = (g + F, \xi)_2 + (\xi, \chi)_2.
\]
Using the Reynolds transport theorem for moving domains and the fact that $\xi$ is both the velocity of the moving interface and the trace of $u$ on this interface, we have

$$\mathcal{T}_1 = (\partial_t u, u)_{\Omega(t)} = \frac{1}{2} \left( \frac{d}{dt} \|u\|_{L^2(\Omega(t))}^2 - \int_{\Gamma(t)} (u \cdot \xi) \cdot n \, ds \right).$$

Now, since we have that $u|_{\Gamma} = \xi$, we may interchange $\xi$ and $u$ in the integral over the interface. Using the fact that $\xi$ is constant in space, we then find using the divergence theorem that

$$- \int_{\Gamma(t)} (u \cdot \xi) \cdot n \, ds = -\|\xi\|_{L^2(\Gamma(t))}^2 \int_{\Gamma(t)} \xi \cdot n \, ds = -\|\xi\|_{L^2(\Gamma(t))}^2 \int_{\Omega(t)} \nabla \cdot \xi \, dx = 0.$$  

As a result, we have

$$\mathcal{T}_2 = - (\partial_n u, \xi)_{\Gamma(t)} = - \int_{\Gamma(t)} \partial_n u \, ds \cdot \xi = (F, \xi)_2.$$  

This then cancels with the drag contribution on the right-hand side of (2.4) in $\mathcal{T}_5$. For the third and fourth term, we immediately have

$$\mathcal{T}_3 = \left( \frac{d}{dt} \xi, \xi \right)_2 = \frac{1}{2} \frac{d}{dt} \|\xi\|_{L^2(\Omega(t))}^2$$

and

$$\mathcal{T}_4 = \left( \frac{d}{dt} \chi, \chi \right)_2 = \frac{1}{2} \frac{d}{dt} \|\chi\|_{L^2(\Omega(t))}^2.$$  

Under our assumption that $g$ is constant in space and using that $u = \xi$ on $\Gamma(t)$, we can rewrite the first part of the fourth term as an integral over $\Gamma(t)$, i.e., $(g, \xi)_2 = \frac{1}{|\Gamma|} \int_{\Gamma} (g, \xi)_{\Gamma(t)}$. Using the trace and Poincaré estimates, we then find

$$|(g, \xi)_2| \leq |\Gamma|^{-\frac{1}{2}} \|g\|_{L^2(\Omega(t))} c_{\partial\Omega} \|u\|_{H^1(\Omega(t))} \leq |\Gamma|^{-\frac{1}{2}} \|g\|_{L^2(\Omega(t))} c_{\partial\Omega} \|\nabla u\|_{L^2(\Omega(t))},$$

where with an abuse of notation, we set $c_{\partial\Omega} = \max\{2, c_P\}$. Note that the Poincaré inequality is applicable, since $u|_{\Gamma_{out}} = 0$, c.f. [21, Remark A.37]. With a weighted Young’s inequality, we then have

$$|(g, \xi)_2| \leq \frac{c_{\partial\Omega}^2}{2 |\Gamma|} \|g\|_{L^2(\Omega(t))}^2 + \frac{1}{2} \|\nabla u\|_{L^2(\Omega(t))}^2.$$  

For the final term, we use the Cauchy-Schwarz and Young’s inequalities to estimate

$$\mathcal{T}_6 = (\xi, \chi)_2 \leq \|\xi\|_{L^2(\Omega(t))} \|\chi\|_{L^2(\Omega(t))} \leq \frac{1}{2} \|\xi\|_{L^2(\Omega(t))}^2 + \frac{1}{2} \|\chi\|_{L^2(\Omega(t))}^2.$$  

We insert (2.5), (2.6), (2.7), (2.8), (2.9) into (2.4) to get

$$\frac{d}{dt} \|u\|_{L^2(\Omega(t))}^2 + \|\nabla u\|_{L^2(\Omega(t))}^2 + \frac{d}{dt} \|\xi\|_{L^2(\Omega(t))}^2 + \frac{d}{dt} \|\chi\|_{L^2(\Omega(t))}^2 \leq \frac{c_{\partial\Omega}^2}{|\Gamma|} \|g\|_{L^2(\Omega(t))}^2 + \|\xi\|_{L^2(\Omega(t))}^2 + \|\chi\|_{L^2(\Omega(t))}^2.$$  

Using a version of Gronwall’s lemma in differential form, see [21, Lemma A.55], and setting $c_{L^2,1} := \frac{c_{\partial\Omega}^2 c_P^2}{|\Gamma|}$ proves the claim. We note that the dependence on the domain in $c_{L^2,1}$ is through the $d$-dimensional measure of $\Omega$ and $d - 1$-dimensional measure of $\Gamma$, see [15] for details. Since our problem only contains rigid body motion, this is constant and does not depend on the solution. \[\square\]
3. Discretisation in Time. As a first step, we consider the temporal semi-discretisation of (2.1)–(2.2) in an Eulerian framework. To this end, let us consider a uniform time step $\Delta t := t_{end}/N$ for some $N \in \mathbb{N}$ and denote $t^n = n\Delta t$. We define the $\delta$-neighbourhood of $\Omega(t)$ as

$$O_\delta(\Omega(t)) := \{ x \in \bar{\Omega} \mid \text{dist}(x, \Omega(t)) \leq \delta \}.$$  

As in [26, 6, 49], the Eulerian time stepping method requires $\delta$ to be sufficiently large, such that the domain $\Omega(t^n)$ is a subset of the $\delta$-neighbourhood of the the previous time step, i.e.,

$$(3.1) \quad \Omega(t^n) \subset O_\delta(\Omega(t^{n-1})), \quad \text{for } n = 1, \ldots, N.$$  

In the aforementioned literature, the motion of the interface was a known quantity, such that the relation (3.1) is guaranteed by setting $\delta$ proportional to the maximal interface normal speed and the time step. In our case, the motion of the interface is an additional unknown in the system. However, since we know by Lemma 2.1 that the interface-velocity solution is bounded, we make the following assumption:

3.1. Temporal Discretisation. To enable our Eulerian time stepping, we need a suitable extension operator. For this extension operator, we require the following family of space-time anisotropic spaces

$$\mathcal{L}^{\infty}(0, T; H^m(\Omega(t))) := \left\{ v \in L^2(Q) \mid v(\cdot, t) \in H^m(\Omega(t)) \text{ for a.e. } t \in (0, T) \text{ and } \text{ess sup}_{t \in (0, T)} \|v(\cdot, t)\|_{H^m(\Omega(t))} < \infty \right\},$$  

for $m = 0, \ldots, k + 1$. We then denote $\partial_t v = v_t$ as the weak partial derivative with respect to the time variable, if this exists as an element of the space-time space $L^2(Q)$. We now assume the existence of a spatial extension operator

$$\mathcal{E} : L^2(\Omega(t)) \to L^2(O_\delta(\Omega(t))),$$  

which fulfils the following properties:

**Assumption 3.1.** Let $v \in \mathcal{L}^{\infty}(0, T; H^{k+1}(\Omega(t))) \cap W^{2, \infty}(Q)$ and $\delta > 0$. There exist positive constants $c_{3.1a}$, $c_{3.1b}$ and $c_{3.1c}$ that are uniform in $t$ such that

$$(3.2a) \quad \|\mathcal{E}v\|_{H^k(O_\delta(\Omega(t)))} \leq c_{3.1a}\|v\|_{H^k(\Omega(t))}$$

$$(3.2b) \quad \|\nabla(\mathcal{E}v)\|_{O_\delta(\Omega(t))} \leq c_{3.1b}\|\nabla v\|_{\Omega(t)}$$

$$(3.2c) \quad \|\mathcal{E}v\|_{W^{2,\infty}(O_\delta(Q))} \leq c_{3.1c}\|v\|_{W^{2,\infty}(Q)}$$

holds. Furthermore, if for $v \in \mathcal{L}^{\infty}(0, T; H^{k+1}(\Omega(t)))$ it holds for the weak partial time-derivative that $v_t \in \mathcal{L}^{\infty}(0, T; H^k(\Omega(t)))$, then

$$(3.3) \quad \|\mathcal{E}v_t\|_{H^k(O_\delta(\Omega(t)))} \leq c_{3.1d}\left[\|v\|_{H^{k+1}(\Omega(t))} + \|v_t\|_{H^k(\Omega(t))}\right],$$

where the constant $c_{3.1d} > 0$ again only depends on the motion of the spatial domain.

Such an extension operator can be constructed explicitly from the classical linear and continuous universal extension operator for Sobolev spaces (see, e.g., [43, Section VI.3]), when the motion of the domain is described by a diffeomorphism $\Psi(t) : \Omega_0 \rightarrow \Omega(t)$, with
\( \Omega(t) \) for each \( t \in [0, T] \) from the reference domain \( \Omega_0 \) that is smooth in time. See [26] for details thereof. Although the motion of the domain is not given a priori here, we assume that the resulting motion is sufficiently smooth.

For the weak formulation of the semi-discrete problem, let us consider the spaces \( V^n := V(t^n) \) and \( N^n := N(t^n) \). The temporal semi-discrete weak formulation of our scheme then reads as follows: Given compatible initial data \((\Omega_0, u_0, \xi_0)\), i.e., \( u_0|\Gamma(0) = \xi_0 \), for \( n = 1, \ldots, N \) find \((u^n, \lambda^n, \xi^n, \chi^n) \in V^n \times N^n \times \mathbb{R}^d \times \mathbb{R}^d \) such that

\[
(3.4) \quad \left( \frac{1}{\Delta t} (u^n - \mathcal{E} u^{n-1}), v \right)_{\Omega^n} + (\nabla u^n, \nabla v)_{\Omega^n} + (\lambda^n, v)_{\Gamma^n} + (\mu, u^n - \xi^n)_{\Gamma^n} \\
+ \left( \frac{1}{\Delta t} (\xi^n - \xi^{n-1}), \zeta_1 \right)_2 + \left( \frac{1}{\Delta t} (\chi^n - \chi^{n-1}), \zeta_2 \right)_2 = (F^n + g, \zeta_1)_2 + (\xi^n, \zeta_2)_2
\]

holds for all \((v, \mu, \zeta_1, \zeta_2) \in V^n \times N^n \times \mathbb{R}^d \times \mathbb{R}^d\).

In order to specify the appropriate choice of \( \delta \) in the above method, we require the following assumption.

**Assumption 3.2.** We assume that the time step \( \Delta t > 0 \) is sufficiently small, such that there exists a constant \( c_{3.2} > 1 \) with

\[
w^{n, \Delta t}_\infty \leq c_{3.2} w^{n}_\infty,
\]

where \( w^{n, \Delta t}_\infty = \max_{i=1, \ldots, N} \| \xi^i \cdot n \|_{L^\infty(\Gamma(t))} \) is the maximal interface velocity resulting from the temporally semi-discretised scheme, \( w^n_\infty = \max_{t \in [0, t_{end}]} \| \xi(t) \cdot n \|_{L^\infty(\Gamma(t))} \) is the maximal interface velocity of the smooth problem (2.3) and \( n \) is the outward pointing unit normal vector vector on \( \Gamma(t) \).

With Assumption 3.2, we then set

\[
\delta = c_\delta \Delta t w^{n}_\infty,
\]

with \( c_\delta > c_{3.2} > 1 \), such that (3.1) is fulfilled.

**Remark 3.3.** Let us comment upon, why we consider Assumption 3.2 reasonable. The discretisations of the time derivative for both the bulk and interface velocities is a standard first-order finite difference approximation. For sufficiently small \( \Delta t \), we can reasonably expect that the discrete approximation is close to the real value, so that (3.5) can be fulfilled.

Let us briefly discuss the solvability of the system (3.4). To this end, we introduce an iteration in \( u^n_l, \lambda^n_l, \xi^n_l, \chi^n_l \). Let \( u^n_0, \lambda^n_0, \xi^n_0, \chi^n_0 = u^{n-1}, \lambda^{n-1}, \xi^{n-1}, \chi^{n-1} \), and \( \Omega_l, \Gamma_l \) denote the domains resulting from the position \( \chi^n_l \). For \( l = 1, \ldots, \), solve

\[
(3.6a) \quad \left( \frac{1}{\Delta t} (u^n_l, v)_{\Omega_{l-1}} + (\nabla u^n_l, \nabla v)_{\Omega_{l-1}} + (\lambda^n_l, v)_{\Gamma_{l-1}} + (\mu, u^n_l)_{\Gamma_{l-1}} \right) \\
= (\mu, \xi^n_{l-1})_{\Gamma_{l-1}} + \left( \frac{1}{\Delta t} (\mathcal{E} u^{n-1}_l, v)_{\Omega_{l-1}} \right) \\
(3.6b) \quad \xi^n_l = \xi^{n-1} + \Delta t \int_{\Gamma_{l-1}} -\partial_n u^n_l \, ds + g \\
(3.6c) \quad \chi^n_l = \chi^{n-1} + \Delta t \xi^n_l
\]

Therefore, the system (3.4) has a solution if the mapping \( g : (\xi^n_l, \chi^n_l) \mapsto (\xi^n_{l+1}, \chi^n_{l+1}) \)
has a fixed point. To this end, we observe that

\[
\|\chi^0_l - \chi^n_k\|_2 = \Delta t \|\xi^0_l - \xi^n_k\|_2 = \Delta t^2 \left\| \int_{\Gamma_{k-1}} -\partial_n u^n_l \, ds - \int_{\Gamma_{k-1}} -\partial_n u^n_k \, ds \right\|_2.
\]

As a result, we have that \( g \) is a contraction for sufficiently small \( \Delta t \), if the bulk solution and consequently the drag is Lipschitz continuous with respect to the interface velocity and position. This can be achieved easily by rewriting (3.6a) in a reference domain, using a smooth mapping which is a small distortion of the identity. As the solution of (3.6a) is bounded by the data, for \( \xi^n_{k-1}, \chi^n_{k-1} \) from a bounded ball around \( \xi^{n-1}, \chi^{n-1} \), we have that the drag can be bounded by a uniform constant. Then for \( \Delta t \) sufficiently small, it follows that \( \xi^n_k, \chi^n_k \) are also contained in this ball. Consequently, the Banach fixed-point theorem gives that \( g \) has a unique fixed point, so (3.4) admits a unique solution.

### 3.2. Stability Analysis of the Semi-Discrete Scheme.

We show that the temporal semi-discretisation (3.4) results in a stable solution.

**Lemma 3.4.** Let \( \{u^m, \xi^m, \chi^m\}_{m=1}^N \) be the velocity and position solution to (3.4) with compatible initial data \( \Omega(0) \) and \( (u^0, \xi^0) \in V^0 \times \mathbb{R}^d \), and the time step \( \Delta t \) be sufficiently small. If Assumption 3.2 holds, we have for \( m = 1, \ldots, N \) the stability estimate

\[
\|u^n\|_{\Omega^n}^2 + \|\xi^m\|_2^2 + \Delta t \sum_{n=1}^{m} \frac{1}{2} \|\nabla u^n\|_{\Omega^n}^2 \leq \exp \left( t^n \frac{c_{L3.4a}}{1 - \Delta t c_{L3.4a}} \right) \left[ \|u^0\|_{\Omega^0}^2 + \|\xi^0\|_2^2 + \frac{\Delta t}{2} \|\nabla u^0\|_{\Omega^0} + c_{L3.4b}\|g\|_2^2 m \right],
\]

with constants \( c_{L3.4a}, c_{L3.4b} > 0 \) independent of the time step and the number of steps \( n \).

**Proof.** We test (3.4) with \( (v, \mu, \zeta_1, \zeta_2) = 2\Delta t (u^n, -\lambda^n, \xi^n, \chi^n) \) to obtain

\[
2(u^n - \mathcal{E} u^{n-1}, u^n)_{\Omega^n} + 2(\xi^n - \xi^{n-1}, \xi^n)_2 + 2(\chi^n - \chi^{n-1}, \chi^n)_2 \\
+ 2\Delta t \|\nabla u^n\|_{\Omega^n} + 2\Delta t (\lambda^n, \xi^n)_{\Gamma^n} = 2\Delta t (g + F^n, \xi^n)_2 + 2\Delta t (\xi^n, \chi^n)_2.
\]

For the two terms originating from the time-derivative, we have the polarisation identity \( 2(u^n - \mathcal{E} u^{n-1}, u^n)_{\Omega^n} = \|u^n\|_{\Omega^n}^2 + \|u^n - \mathcal{E} u^{n-1}\|_{\Omega^n}^2 - \|\mathcal{E} u^{n-1}\|_{\Omega^n}^2 \). For the Lagrange-multiplier, external forcing and sold velocity-position coupling terms, we have as in the proof of Lemma 2.1 above that

\[
(\lambda^n, \xi^n)_{\Gamma^n} = (F^n, \xi^n)_2
\]

and

\[
(g, \xi^n)_2 \leq \frac{c_1}{4} \|g\|_2^2 + \|\nabla u^n\|_{\Omega^n}^2, \quad (\xi^n, \chi^n)_2 \leq \frac{1}{2} \|\xi^n\|_2^2 + \frac{1}{2} \|\chi^n\|_2^2,
\]

with \( c_1 = c_1^2 / |\Gamma| \). Using these equalities and estimates, we get from (3.7) that

\[
\|u^n\|_{\Omega^n}^2 + \|\xi^n\|_2^2 + \|\chi^n\|_2^2 + \Delta t \|\nabla u^n\|_{\Omega^n}^2 \\
\leq \|\mathcal{E} u^{n-1}\|_{\Omega^n}^2 + \|\xi^{n-1}\|_2^2 + \|\chi^{n-1}\|_2^2 + \frac{c_1 \Delta t}{2} \|g\|_2^2 + 2\Delta t \|\xi^n\|_2^2 + \Delta t \|\chi^n\|_2^2.
\]
Now, for arbitrary $\varepsilon > 0$, we have from [26, Lemma 3.5] that
\[
\|E u\|_{O_1(\Omega)} \leq (1 + (1 + \varepsilon^{-1})\delta c')\|u\|_{\Omega}^2 + \delta c''\varepsilon\|\nabla u\|_{\Omega}^2.
\]
Then with $\delta$ as given in (3.5) and $\varepsilon = 1/(2c''c_3w_{n-1}^\infty)$, it follows
\[
\|E u^{n-1}\|_{O_1(\Omega_{n-1})}^2 \leq \|E u^{n-1}\|_{O_1(\Omega_{n-1})}^2 \\
\leq \left(1 + (1 + 2c''c_3w_{n-1}^\infty)c'c_3w_{n-1}^\infty\Delta t\right)\|u^{n-1}\|_{\Omega_{n-1}}^2 + \frac{\Delta t}{2}\|\nabla u^{n-1}\|_{\Omega_{n-1}}^2 \\
\leq (1 + \delta \Delta t)\|u^{n-1}\|_{\Omega_{n-1}}^2 + \frac{\Delta t}{2}\|\nabla u^{n-1}\|_{\Omega_{n-1}}^2.
\]
Applying this to (3.8) and summing this over $n = 1, \ldots, m$ leads to
\[
\|u^n\|_{O_1(\Omega)}^2 + \|\xi^n\|_{2}^2 + \|\chi^n\|_{2}^2 + \frac{\Delta t}{2}\sum_{n=1}^m\|\nabla u^n\|_{\Omega_{n-1}}^2 \leq \|u^0\|_{O_1(\Omega)}^2 + \frac{\Delta t}{2}\|\nabla u^0\|_{\Omega_{n-1}}^2 + \|\xi^0\|_{2}^2 \\
+ \delta \Delta t\sum_{n=0}^{m-1}\|u^n\|_{O_1(\Omega)}^2 + \Delta t\sum_{n=1}^m\left[\|\xi^n\|_{2}^2 + \|\chi^n\|_{2}^2\right] + \frac{c_1}{2}t^m\|g\|_{2}^2.
\]
Applying a discrete version of Gronwall's lemma, see [20, Lemma 5.1], with the choices $c_{L.3.4.a} = \max\{c_2,1\}$ and $c_{L.3.4.b} = c_1/2$ then proves the claim. \hfill $\square$

4. Discretisation in Space and Time. We now come to the full discretisation of (2.1) – (2.2). For the fully discrete method, we use an unfitted finite element method with Lagrange-multipliers to implement the Dirichlet boundary condition on the moving, unfitted interface. This will allow us to reuse aspects of the semi-discrete analysis. This unfitted FEM has its origins in [7]. The geometry is defined implicitly on a background mesh using a level set function. So-called “bad-cuts” between the mesh and the boundary $\Gamma$ given by the level set function are stabilised using ghost penalty stabilisation [5]. The ghost penalty stabilisation is also responsible for the implicit extension into a strip around the moving interface, ensuring that the solution is well-defined on domains at subsequent time steps. At each time step, the discrete domain $\Omega_h^n$ is extended by a strip of width
\[
\delta_h := c_3 h w_{n-1}^\infty\Delta t
\]
such that $\Omega_h^{n+1}$ is a subset of of the extended domain. We further assume that $c_3 h > 1$ is sufficiently small, such that
\[
O_{\delta_h} (\Omega_h^n) \subset O_{\delta} (\Omega^n).
\]

4.1. Spatial Discretisation Method. Let $\hat{T}_h$ be a simplicial, shape-regular and quasi-uniform mesh of the domain $\Omega$, where $h > 0$ is the characteristic size of the simplexes. We collect the elements that are in the extended domain as the active mesh in
\[
\mathcal{T}_{h,\delta_h} := \{T \in \hat{T}_h \mid \exists x \in T \text{ such that } \text{dist}(x, \Omega_h^n) \leq \delta_h \} \subset \hat{T}_h
\]
and denote the active domain as
\[
O_{\delta_h,\mathcal{T}} := \{x \in T \mid T \in \mathcal{T}_{h,\delta_h} \} \subset \mathbb{R}^d.
\]
We further define the cut mesh as \( \mathcal{T}_h^n := \mathcal{T}_{h,0}^n \) of all elements that contain some part of \( \Omega_h^n \) and the cut domain \( \mathcal{O}_{\mathcal{T}}^n := \mathcal{O}_{0,\mathcal{T}}^n \). Similarly, we collect the set of interface elements as

\[
\mathcal{T}_{h,\Gamma_h^n}^n := \{ T \in \mathcal{T}_h \mid \text{meas}_{d-1}(T \cap \Gamma_h^n) > 0 \}
\]

and the domain of these elements as \( \mathcal{O}_{\Gamma_h^n}^n := \{ x \in T \mid T \in \mathcal{T}_{h,\Gamma_h^n}^n \} \). For the extension ghost penalty operators, we collect the elements in the extension strip in

\[
\mathcal{T}_{h,S^n}^n := \{ T \in \mathcal{T}_{h,\delta_h} \mid \exists x \in T \text{ such that } \text{dist}(x, \Gamma_h^n) \leq \delta_h \}
\]

and the set of interior facets of this strip in

\[
\mathcal{T}_{h,\delta_h} := \{ F = \overline{T_1 \cap T_2} \mid T_1 \in \mathcal{T}_{h,\delta_h}, T_2 \in \mathcal{T}_{h,S^n} \text{ with } T_1 \neq T_2 \text{ and } \text{meas}_{d-1}(F) > 0 \}.
\]

Finally, for the analysis below, we also define the set of extension strip elements

\[
\mathcal{T}_{h,S^+}^n := \{ T \in \mathcal{T}_{h,\delta_h} \mid \exists x \in T \cap (\overline{\Omega} \setminus \Omega_h^n) \text{ such that } \text{dist}(x, \Gamma_h^n) \leq \delta_h \}.
\]

An illustration of these sets of elements and facets can be seen in Figure 4.1.

### 4.1.1. Finite Element Spaces.

On the active mesh, we consider for \( k \geq 2 \) the finite element spaces for velocity and Lagrange-multipliers

\[
\begin{align*}
V_h^n &= \{ v_h \in C(\mathcal{O}_{\delta_h,\mathcal{T}}^n) \mid v_h|_T \in \mathbb{P}^k(T) \text{ for all } T \in \mathcal{T}_{h,\delta_h}^n \}, \\
N_h^n &= \{ \mu_h \in C(\mathcal{O}_{\Gamma_h^n}^n) \mid \mu_h|_T \in \mathbb{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_{h,\Gamma_h^n}^n \}.
\end{align*}
\]

### 4.1.2. Variational Formulation.

The fully discrete variational formulation of the method then reads as follows: Given an appropriate and compatible set of initial conditions \( \Omega_h^0, u_0 \in V_h^0 \) and \( \xi_0 \in \mathbb{R}^d \), for \( n = 1, \ldots, N \) find \( (u_h^n, \lambda_h^n, \xi_h^n, \chi_h^n) \in V_h^n \times N_h^n \times \mathbb{R}^d \times \mathbb{R}^d \) such that

\[
\begin{align*}
\left( \frac{1}{\Delta t}(u_h^n - u_h^{n-1}), v_h \right)_{\Omega_h^n} + a_h^n(u_h^n, v_h) + b_h^n(\lambda_h^n, v_h) + b_h^n(\mu_h, u_h^n - \xi_h^n) \\
+ \gamma g_p h_h^n(u_h^n, v_h) - \gamma \lambda h_h^n(\lambda_h^n, \mu_h) + \left( \frac{1}{\Delta t}(\xi_h^n - \xi_h^{n-1}), \xi_1 \right)_2 + \left( \frac{1}{\Delta t}(\chi_h^n - \chi_h^{n-1}), \xi_2 \right)_2 \\
= (F_h^n + g, \xi_1)_2 + (\xi_h^n, \xi_2)_2
\end{align*}
\]
holds for all \((\mathbf{v}_h, \mathbf{u}_h, \zeta_1, \zeta_2) \in V_h^\alpha \times V_h^\mu \times \mathbb{R}^d \times \mathbb{R}^d\). The stabilisation parameters \(\gamma_{\text{gp}}, \gamma_{\lambda} > 0\) are to be specified later. The bilinear forms \(a_h^\alpha(\cdot, \cdot)\) and \(b_h^\mu(\cdot, \cdot)\) are defined by

\[
a_h^\alpha(\mathbf{u}_h, \mathbf{v}_h) := \int_{\Gamma_h^\alpha} \nabla \mathbf{u}_h : \nabla \mathbf{v}_h \, d\mathbf{x} \quad \text{and} \quad b_h^\mu(\lambda_h, \mathbf{v}_h) := \int_{\Gamma_h^\mu} \lambda_h \cdot \mathbf{v}_h \, ds,
\]

respectively. To stabilise the system (4.2) with respect to “bad-cuts”, we use the direct version of the ghost penalty stabilisation operator [37]. To define this, let \(F = T_1 \cap T_2\) be an interior facet and \(\omega_F = T_1 \cup T_2\) be the corresponding facet-patch. We then define \(\|u\| := u_1 - u_2\) with \(u_1 = \mathcal{E}^\varphi u|_{T_1}\), where \(\mathcal{E}^\varphi : \mathbb{P}^m(T) \to \mathbb{P}(\mathbb{R}^d)\) is the canonical extension of polynomials to \(\mathbb{R}^d\). The Laplace form is then stabilised with the ghost penalty form

\[
i_h^\mu(\mathbf{u}_h, \mathbf{v}_h) := \frac{1}{h^2} \sum_{F \in \mathcal{F}_h} \int_{\omega_F} \|\mathbf{u}_h\| \cdot \|\mathbf{v}_h\| \, d\mathbf{x}.
\]

Furthermore, the Lagrange-multiplier is stabilised with

\[
j_h^\lambda(\lambda_h, \mu_h) := h^2 \int_{\partial \Gamma_h^\lambda} (n \cdot \nabla \lambda_h) \cdot (n \cdot \nabla \mu_h) \, d\mathbf{x}.
\]

Here we extend the interface unit normal vector into a field in the bulk domain by \(n = -\nabla \phi / \|\nabla \phi\|_2\). On the discrete level, we define the force via the discrete Lagrange multiplier, i.e.,

\[
F_h^\alpha := \int_{\Gamma_h^\alpha} \lambda_h^\mu \, ds.
\]

### 4.1.3. Extension and Stabilisation through Ghost Penalties.

For the ghost penalty mechanism, we require the following assumption, see also [26, Assumption 5.3] and [49, subsection 5.1.3].

**Assumption 4.1.** Our analysis requires the following further assumptions on the mesh and level set:

- **4.1.a** For every element cut by \(\Gamma_h^n\), the interface \(\Gamma_h^n\) intersects the element boundary \(\partial T\) exactly twice and each (open) edge exactly once.

- **4.1.b** For each element \(T\) intersected by \(\Gamma_h^n\), there exists a plane \(S_T\) and a piecewise smooth parametrisation \(\varphi : S_T \cap T \to \Gamma_h^n \cap T\).

- **4.1.c** We assume that for every strip element \(T \in \mathcal{T}_h^n, S^+\) there exists an uncut element \(T' \in \mathcal{T}_h^n \setminus \mathcal{T}_h^n, S^+\), which can be reached by a path which crosses a bounded number of facets \(F \in \mathcal{F}_h^n, S_h^n\). We assume that the number of facets which have to be crossed to reach \(T'\) from \(T\) is bounded by a constant \(K \lesssim (1 + \frac{h}{\Delta t})\) and that every uncut element \(T' \in \mathcal{T}_h^n \setminus \mathcal{T}_h^n, S^+\) is the end of at most \(M\) such paths, with \(M\) bounded independent of \(\Delta t\) and \(h\). In other words, each uncut elements “supports” at most \(M\) strip elements.

Since the curvature of \(\Gamma\) is bounded (and remains constant in time), the above assumption is reasonable, if the interface is sufficiently well resolved. 4.1.a and 4.1.b are necessary for a trace estimate from the interface to the entire cut element, see [18]. These assumptions therefore are standard for the analysis of CutFEM methods. Furthermore, 4.1.c is standard for unfitted moving domain discretisations. See [26] for a detailed justification thereof.
We summarise the essential stabilising property of the ghost penalty operator.

**Lemma 4.2.** With the direct ghost penalty operators, we have for all \( v_h \in V_h^n \) that
\[
\| \nabla v_h \|_{\Omega_h}^2 \simeq \| \nabla v_h \|_{\Omega_h}^2 + K^{n}(v_h, v_h).
\]

See [26, Lemma 5.5] for the proof thereof.

As seen in Lemma 4.2, the stiffness between the velocity unknown on elements \( T \in \mathcal{T}_h^\pm \setminus \mathcal{T}_h^\pm \) and \( T' \in \mathcal{T}_h^\pm, \) induced by the stabilising ghost penalty operator, depends on the inverse distance between \( T \) and \( T' \) as measured in the number of elements that need to be crossed to reach \( T \) from \( T' \). This in turn depends on the anisotropy between the spatial and the temporal discretisation, with \( K \lesssim (1 + \delta_h h) \) and \( \delta_h \lesssim \Delta t \). In the stability analysis below, we shall require that \( \gamma_{gp} \gtrsim K \), to compensate the weakening of the stabilisation for larger extension strips. As a result, we choose the ghost penalty stabilisation parameters as
\[
\gamma_{gp} = \gamma_{gp}(h, \delta_h) = \gamma_s K \quad \text{with} \quad \gamma_s > 0 \quad \text{independent of} \quad h \quad \text{and} \quad \Delta t.
\]

See also [26, section 4.4].

### 4.2. Stability Analysis

For our analysis, we consider the following mesh-dependent norms
\[
\| u_h \|_{s,n}^2 := \| \nabla u_h \|_{\Omega_h}^2 + \| h^{-1/2} u_h \|_{\Gamma_h}^2,
\]
\[
\| \lambda_h \|_{s,n}^2 := \| h^{1/2} \lambda_h \|_{\Gamma_h}^2 + \| h \mathbf{n} \cdot \nabla \lambda_h \|_{\Omega_h}^2.
\]

Note that these norms are independent of the mesh-interface cut topology, and since they are defined on the entire finite element spaces, they represent proper norms on the spaces \( V_h^n \) and \( N_h^n \), respectively. On the product space, we then take the norm
\[
\| (u_h, \lambda_h) \|_{s,n}^2 := \| u_h \|_{s,n}^2 + \| \lambda_h \|_{s,n}^2.
\]

We do not distinguish between the norms on the different spaces, since the argument makes it clear which norm is meant.

In conjunction with the stability form, we then have the following lemma

**Lemma 4.3.** For sufficiently small \( h > 0 \), it holds that
\[
\| \lambda_h \|_{\Gamma_h}^2 \lesssim \| h^{1/2} \lambda_h \|_{\Gamma_h}^2 + \| h \mathbf{n} \cdot \nabla \lambda_h \|_{\Omega_h}^2 \quad \text{for all} \quad \lambda_h \in N_h^n.
\]

See [16, Sec. 7] for the details of the proof thereof.

**Lemma 4.4.** The stabilised Laplace operator \( (a_h^n + \gamma_{gp} i_h^n)(\cdot, \cdot) \) is continuous and coercive on \( V_h^n \), i.e.,
\[
(a_h^n + \gamma_{gp} i_h^n)(u_h, v_h) \lesssim \| u_h \|_{s,n} \| v_h \|_{s,n} \quad \text{for all} \quad u_h, v_h \in V_h^n
\]
\[
(a_h^n + \gamma_{gp} i_h^n)(u_h, u_h) \gtrsim \| u_h \|_{s,n}^2 \quad \text{for all} \quad u_h \in V_h^n.
\]

See [8, Lemma 6 and Lemma 7] for a proof thereof.
LEMMA 4.5. For the discrete forms $b^n_h(\cdot, \cdot)$ and $j^n_h(\cdot, \cdot)$ in the finite element method (4.2), we have the stability estimate

$$\beta \|\lambda_h\|_{s,n} \leq \sup_{v_h \in V^n_h} \frac{b^n_h(\lambda_h, v_h)}{\|v_h\|_{s,n}} + j^n_h(\lambda_h, \lambda_h) \frac{1}{2}$$

for all $\lambda_h \in N^n_h$, with the constant $\beta > 0$ independent of $h$.

Proof. The proof follows ideas from [13, Lemma 3]. For a given $\lambda_h \in N^n_h$, let $\tilde{\lambda}_h$ be the $\mathbb{P}^{k-1}$ finite element function that is equal to $\lambda_h$ in $O^n_{\Gamma_h}$ and zero in all other degrees of freedom in $O^n_{h, \mathcal{F}}$. Then we immediately have that $h\tilde{\lambda}_h \in V^n_h$ and

$$b^n_h(\lambda_h, h\tilde{\lambda}_h) = \|h^{1/2}\lambda_h\|_{\Gamma_h}^2,$$

By the definition of the Lagrange-multiplier norm and the Cauchy-Schwarz inequality applied to the stabilising form, we therefore have

$$\|\lambda_h\|_{s,n} \leq \frac{b^n_h(\lambda_h, h\tilde{\lambda}_h)}{\|\lambda_h\|_{s,n}} + j^n_h(\lambda_h, \lambda_h) \frac{1}{2}. \tag{4.5}$$

Now, let $O^n_{\delta_h, i}$ and $O^n_{\delta_h, e}$ denote the domain of uncut elements inside and outside the physical domain, respectively, i.e., $O^n_h = O^n_{\delta_h, i} \cup O^n_{\delta_h, e}$. We then have with the observation that $\tilde{\lambda}_h$ is only non-zero on a strip of width $3h$, as well as uses of the trace and inverse estimates that

$$\|\tilde{\lambda}_h\|_{s,n}^2 \lesssim h^{-2} \|\lambda_h\|_{s,n}^2 + \|h^{-1/2}\lambda_h\|_{\Gamma_h}^2$$

$$= h^{-2}(\|\tilde{\lambda}_h\|_{s,n}^2 + h^{-1} \|\tilde{\lambda}_h\|_{s,n}^2) + \|h^{-1/2}\lambda_h\|_{\Gamma_h}^2$$

$$\lesssim h^{-1}(\|\tilde{\lambda}_h\|_{s,n}^2 + \|\lambda_h\|_{s,n}^2) + h^{-2} \|\lambda_h\|_{s,n}^2 + \|h^{-1/2}\lambda_h\|_{\Gamma_h}^2$$

$$\lesssim \|\nabla \lambda_h\|_{s,n}^2 + h^{-2} \|\lambda_h\|_{s,n}^2 + \|h^{-1/2}\lambda_h\|_{\Gamma_h}^2$$

$$\lesssim h^{-2} \|\lambda_h\|_{s,n}^2 + \|h^{-1/2}\lambda_h\|_{\Gamma_h}^2.$$

With Lemma 4.3, this gives that

$$\|h\tilde{\lambda}_h\|_{s,n} \leq c \|\lambda\|_{s,n} \tag{4.6}$$

with $c > 0$ independent of $h$ and $\lambda_h$. Inserting this estimate on the right-hand side of (4.5) gives

$$\|\lambda_h\|_{s,n} \leq \frac{b^n_h(\lambda_h, h\tilde{\lambda}_h)}{\|h\tilde{\lambda}_h\|_{s,n}} + j^n_h(\lambda_h, \lambda_h) \frac{1}{2}.$$

The claim then follows by taking the supremum over all $v_h \in V_h$. \qed

LEMMA 4.6. Let us consider the bilinear form

$$A_h^{n,*}(u_h, \lambda_h, v_h, \mu_h) := (u^n_h + \gamma_{\partial N_{h}} v^n_h)(u_h, v_h) + b^n_h(\lambda_h, v_h) + b^n_h(\lambda_h, u_h) - j^n_h(\lambda_h, \mu_h).$$
Then for all \((u_h, \lambda_h) \in V_h^\alpha \times N_h^\alpha\) there holds

\[
c_{L4.6} \| (u_h, \lambda_h) \|_{s,n} \leq \sup_{(v_h, \mu_h) \in V_h^\alpha \times N_h^\alpha} \frac{A_h^{n,*}(u_h, \lambda_h), (v_h, \mu_h))}{\| (v_h, \mu_h) \|_{s,n}},
\]

where the constant \(c_{L4.6} > 0\) is independent of the mesh size \(h\) and the mesh-interface cut position.

Proof. Let \(h\lambda_h \in V_h\) be as in the proof of Lemma 4.5. Then, using the coercivity and continuity of the stabilised Laplace operator, (4.6), Lemma 4.2 and Young’s inequality we have

\[
A_h^{n,*}(u_h^n, \lambda_h^n), (u_h^n + \alpha h\bar{\lambda}_h, -\lambda_h^n) \\
\geq \|u_h^n\|^2_{s,n} - \alpha \|u_h^n\|_{s,n}^2 h\| \bar{\lambda}_h \|_{s,n} + \alpha \|h^{1/2} \lambda_h^n\|^2_{\Gamma_h^n} + \|hn \cdot \nabla \lambda_h^n\|^2_{\Omega_h^n} \\
\geq (1 - \frac{\alpha}{2}) \|u_h^n\|^2_{s,n} + \frac{\alpha}{2} \|h^{1/2} \lambda_h^n\|^2_{\Gamma_h^n} + (1 - \frac{\alpha}{2}) \|hn \cdot \nabla \lambda_h^n\|^2_{\Omega_h^n}.
\]

For \(\alpha\) sufficiently small, it follows that

\[
A_h^{n,*}(u_h^n, \lambda_h^n), (u_h^n + \alpha h\bar{\lambda}_h, -\lambda_h^n) \geq \|u_h^n, \lambda_h^n\|^2_{s,n}.
\]

The claim then follows due to \(\|(u_h^n + \alpha h\bar{\lambda}_h, -\lambda_h^n)\|_{s,n} \leq \|(u_h^n, \lambda_h^n)\|_{s,n}\), which is a consequence of the triangle inequality and (4.6).

Corollary 4.7. The CutFEM Lagrange-Multiplier method for the stationary Poisson problem given by \(A_h^{n,*}(u_h^n, \lambda_h^n), (v_h, \mu_h) = F_h((v_h, \mu_h))\) is uniquely solvable and the condition number of the resulting stiffness matrix is bounded independent of the mesh interface cut position.

We now show stability of the discrete scheme in the following fully discrete counterpart to Lemma 3.4.

Theorem 4.8. Let \(\{(u_h^m, \xi_h^m, \chi_h^m)\}_{m=1}^\infty\) be the velocity and position solution to (4.2). Then under assumptions 3.2, 3.1 and 4.1, with \(\gamma_\alpha\) sufficiently large, and \(\Delta t\) sufficiently small, we have for \(m = 1, \ldots, N\) the stability estimate

\[
\|u_h^m\|^2_{\Omega_h^n} + \|\xi_h^m\|^2_{\Omega_h^n} + \|\chi_h^m\|^2_{\Omega_h^n} + \Delta t \sum_{n=1}^m \left[ c_{T4.86} \|\nabla u_h^n\|^2_{\Omega_h^n} + \gamma \lambda_j^n(\lambda_h^n, \lambda_h^n) \right] \\
\leq \exp \left( \frac{c_{T4.86}}{1 - \Delta t c_{T4.86}} \right) \left[ \|u_h^0\|^2_{\Omega_h^n} + \|\xi_h^0\|^2_{\Omega_h^n} + \frac{c_{T4.86} \Delta t}{2} \|\nabla u_h^n\|^2_{s,n} + t m c_{T4.86} \|g\|^2_2 \right],
\]

with constants \(c_{T4.86}, c_{T4.86}, c_{T4.86} > 0\) independent of \(h, \Delta t\) and \(m = 1, \ldots, N\).

Proof. The proof follows similar lines to that of Lemma 3.4. We test (4.2) with \((v_h, \mu_h, \zeta_1, \zeta_2) = 2\Delta t(u_h^n, -\lambda_h^n, \xi_h^n, \chi_h^n)\), use BDF1 polarisation identity, the observation that \((\lambda_h^n, \xi_h^n)_{\Gamma_h^n} = (F_h^n, \xi_h^n)_{\Omega_h^n}\) due to (4.3), and the estimates (2.8) and (2.9). This leads to

\[
(4.7) \quad \|u_h^{n+1}\|^2_{\Omega_h^n} + \|\xi_h^{n+1}\|^2_{\Omega_h^n} + \|\chi_h^{n+1}\|^2_{\Omega_h^n} + \Delta t \|\nabla u_h^n\|^2_{\Omega_h^n} \\
+ 2 \Delta t \gamma g_p^n (u_h^n, u_h^n) + 2 \Delta t \gamma j^n (\lambda_h^n, \lambda_h^n) \\
\leq \|u_h^{n-1}\|^2_{\Omega_h^n} + \|\xi_h^{n-1}\|^2_{\Omega_h^n} + \|\chi_h^{n-1}\|^2_{\Omega_h^n} + \Delta t \|\xi_h^n\|^2_{\Omega_h^n} + \Delta t \|\chi_h^n\|^2_{\Omega_h^n} + \frac{c_1 \Delta t}{|\Gamma|} \|g\|^2_2.
\]
To deal with the norm of \( u_h^{n-1} \) on \( \Omega_h^n \), we recall [26, Lemma 5.7], i.e., there exists a constant \( c_{\text{LO},5.7} > 0 \), independent of \( \Delta t \) and \( h \), such that

\[
\| u_h^{n-1} \|^2_{C^0_h(\Omega_h^n)} \leq (1 + c_{\text{LO},5.7}(\varepsilon)) \| u_h^{n-1} \|^2_{\Omega_h^n} + c_{\text{LO},5.7}(\varepsilon) \Delta t \| \nabla u_h^{n-1} \|_{\Omega_h^n}
\]

with constants \( c_{\text{LO},5.7}(\varepsilon) = c_{\text{LO},5.7}c_{\delta_h}w_{\infty}^n(1 + \varepsilon^{-1}) \), \( c_{\text{LO},5.7}(\varepsilon) = c_{\text{LO},5.7}c_{\delta_h}w_{\infty}^n \varepsilon \) and \( c_{\text{LO},5.7}(\varepsilon) = c_{\text{LO},5.7}c_{\delta_h}w_{\infty}^n(\varepsilon + h^2 + h^2\varepsilon^{-1}) \). Since we have that \( \Omega_h^n \subset \Omega(\Omega_h^n-1) \), we have with the choice of \( \varepsilon \leq 1/2c_{\text{LO},5.7}c_{\delta_h}w_{\infty}^n \) that

\[
(4.8) \quad \| u_h^{n-1} \|^2_{\Omega_h^n} \leq (1 + c_2 \Delta t) \| u_h^{n-1} \|^2_{\Omega_h^n-1} + \frac{1}{2} \Delta t \| \nabla u_h^{n-1} \|^2_{\Omega_h^{n-1}} + c_3 K \Delta t \| u_h^{n-1} \|^2_{\Omega_h^{n-1}},
\]

where \( c_2, c_3 > 0 \) are independent of \( \Delta t \) and \( h \). Inserting this into (4.7), under the assumption that \( \gamma_{sp} \geq c_3 K \), summing over \( n = 1, \ldots, k \leq N \) and applying Lemma 4.2 then gives

\[
\| u_h^{n-1} \|^2_{\Omega_h^n} + \| \xi_h^{n-1} \|^2_{\Omega_h^n} + \| \chi_h^{n-1} \|^2_{\Omega_h^n} + \Delta t \sum_{n=1}^k \left[ c_4 \| \nabla u_h^{n-1} \|^2_{\Omega_h^n} + \gamma \lambda j_h^n(\lambda_h^{n-1}, \lambda_h^n) \right]
\]

\[
\leq \| u_h^{0-1} \|^2_{\Omega_h^n} + \| \xi_h^{0-1} \|^2_{\Omega_h^n} + \| \chi_h^{0-1} \|^2_{\Omega_h^n} + \frac{cL_{L2} \Delta t}{2} \| \nabla u_h^{0-1} \|^2_{\Omega_h^n} + \Delta t \sum_{n=1}^k \left[ \| \nabla u_h^{n-1} \|^2_{\Omega_h^n} + \| \xi_h^{n-1} \|^2_{\Omega_h^n} \right].
\]

The claim then follows by an application of the discrete form of Gronwall’s lemma. □

4.3. Domain Error. In this section, we shall formalise the discrepancy between the exact domain and the domain resulting from the discretised problem. In contrast to [26, 49, 28] but as in [6], we shall assume exact geometry handling, i.e., if the motion of the domain were known, we would have \( \Omega(t^n) = \Omega_h^n \). We thus choose to ignore the geometry consistency error of order \( \mathcal{O}(h^2) \) introduced by the piecewise linear level set approximation inherent in CutFEM. We do this to focus our analysis on the fact, that in the discretised setting, the motion of the domain results from a discretisation of the ODE governing the motion. As a result there is a miss-match between the motion of the domain between the smooth and discrete case, i.e., \( \Omega(t^n) \neq \Omega_h^n \), and our analysis will focus on this error source.

We note that the correct geometry order for high-order finite element spaces can be recovered in CutFEM by using, for example, the isoparametric CutFEM approach [24], which has been studied for a range of stationary problems and recently extended to moving domain problems in [28].

Now, the position of \( \Sigma(t^n) \) and \( \Sigma_h^n \) are governed by

\[
\chi(t^n) = \int_0^{t^n} \xi \, dt = \chi(t^{n-1}) + \int_{t^{n-1}}^{t^n} \xi \, dt \quad \text{and} \quad \chi_h^n = \chi^{n-1} + \Delta t \xi_h^n.
\]

The difference \( \chi(t^n) - \chi_h^n \) represent the miss-match between the domains at time \( t^n \), which results from the discretisation of the problem (2.3). To analyse this error, we
define a mapping from the discrete domain to the exact domain by \( \Phi^n : \Omega^n_h \to \Omega(t^n) \) by

\[
\Phi^n := \text{id} + (\chi(t^n) - \chi^n_h)\varphi^n,
\]

where \( \varphi^n \in C^\infty(\Omega^n_h) \), such that \( \varphi^n|_{\Gamma^n_h} = 1 \) and \( \varphi^n|_{\Omega^n_h \setminus \Gamma^n_h} = 0 \). We take this mapping to be invertible. In the following, this mapping takes a similar role as the geometry approximation mapping in, e.g., [26, 49]. Consequently, the domain error can be quantified by \( \| \text{id} - \Phi^n \| \) in an appropriate norm.

**Lemma 4.9.** Let \( \| \cdot \|_\infty \) be the \( \mathcal{L}^{\infty} \) norm on \( \Omega^n_h \). Then for the mapping \( \Phi^n : \Omega^n_h \to \Omega(t^n) \) defined in (4.9), it holds that

\[
\| \text{id} - \Phi^n \|_\infty^2 \lesssim \Delta t^2 \sum_{j=0}^n \| U^j \|_2^2 + \Delta t^3 t^n \| \partial_t \xi \|_\infty^2 =: \mathfrak{M}(U^n, \Delta t),
\]

where \( U^n := \xi(t^n) - \xi^n_h \).

**Proof.** By the definitions of \( \Phi^n \), \( \chi(t^n) \) and \( \chi^n_h \), and the fact that \( \varphi^n \) is smooth, we have that

\[
\| \text{id} - \Phi^n \|_\infty^2 \lesssim \| \chi(t^n) - \chi^n_h \|_2^2 \| \varphi^n \|_\infty^2 \lesssim \| \chi(t^n) - \chi^n_h \|_2^2 \lesssim \| \chi(t^{n-1}) - \chi^{n-1}_h \|_2^2 + \| \int_{t^{n-1}}^{t^n} \xi(t) - \xi^n_h \, dt \|_2^2.
\]

With respect to the final term, we have for \( t \in [t^{n-1}, t^n] \) that

\[
\xi(t) \leq \xi(t^n) - t \| \partial_t \xi \|_{\infty, [t^{n-1}, t^n]} \leq \xi(t^n) - \Delta t \| \partial_t \xi \|_{\infty, [t^{n-1}, t^n]}.
\]

Therefore, we have the bound

\[
\left\| \int_{t^{n-1}}^{t^n} \xi(t) - \xi^n_h \, dt \right\|_2^2 \leq \left\| \int_{t^{n-1}}^{t^n} \xi(t^n) - \Delta t \| \partial_t \xi \|_{\infty, [t^{n-1}, t^n]} - \xi^n_h \, dt \right\|_2^2 \\
\leq \Delta t^2 \left\| U^n \right\|_2^2 + \Delta t^4 \| \partial_t \xi \|_{\infty, [t^{n-1}, t^n]}^2.
\]

Iteratively repeating the above estimate for the \( \| C(t^{n-1}) - C^{n-1}_h \|_2^2 \) term then gives

\[
\| \text{id} - \Phi^n \|_\infty^2 \lesssim \Delta t^2 \sum_{j=0}^n \left\| U^j \right\|_2^2 + \Delta t^4 \| \partial_t \xi \|_{\infty, [t^{j-1}, t^j]}^2 \\
\lesssim \Delta t^2 \sum_{j=0}^n \left\| U^j \right\|_2^2 + \Delta t^3 t^n \| \partial_t \xi \|_\infty^2.
\]

**Lemma 4.10.** For \( \Phi^n \) defined in (4.9), describing the mismatch between the exact and the discrete domain at time \( t^n \), we have that

\[
\| I - D \Phi^n \|_\infty \lesssim \| \text{id} - \Phi^n \|_\infty; \quad \| 1 - \det(D \Phi^n) \|_\infty \lesssim \| \text{id} - \Phi^n \|_\infty; \quad \| \text{id} - \Phi^n \|_{\infty, \Gamma^n} \lesssim \| \text{id} - \Phi^n \|_\infty; \quad \| 1 - \det(D \Phi^n) \|_{\infty, \Gamma^n} \lesssim \| \text{id} - \Phi^n \|_\infty.
\]

**Proof.** This follows by \( \varphi^n \in C^\infty(\Omega^n) \) and the fact that the remaining components of \( \Phi^n \) are independent of space. \( \square \)
Lemma 4.11. Let $\mathcal{M}(U^n, \Delta t)$ be as defined in (4.10). Then for $u \in \mathcal{H}^3(\Omega(t^n))$, it holds that

\begin{align}
\|u \circ \Phi^n - \mathcal{E} u\|_{\Omega^n}^2 &\lesssim \|u\|_{\mathcal{H}^3(\Omega(t^n))}^2 \mathcal{M}(U^n, \Delta t), \\
\|\nabla u \circ \Phi^n - \nabla \mathcal{E} u\|_{\Omega^n}^2 &\lesssim \|u\|_{\mathcal{H}^3(\Omega(t^n))}^2 \mathcal{M}(U^n, \Delta t), \\
\|u \circ \Phi^n - \mathcal{E} u\|_{\Gamma_h^n}^2 &\lesssim \|u\|_{\mathcal{H}^3(\Omega(t^n))}^2 \mathcal{M}(U^n, \Delta t), \\
\|(\partial_n u) \circ \Phi^n - \partial_n \mathcal{E} u\|_{\Gamma_h^n}^2 &\lesssim \|u\|_{\mathcal{H}^3(\Omega(t^n))}^2 \mathcal{M}(U^n, \Delta t).
\end{align}

Proof. $\Phi^n$ maps the approximated interface location $\Gamma_h^n$ to the exact interface location $\Gamma(t^n)$, and we know that the distance between the two is given by $|\chi(t^n) - \chi_h^n|$ for which we have proven the estimate in the proof of Lemma 4.9. The proof of (4.11)–(4.14) is therefore completely analogous to that of the geometry approximation error in [17, Lemma 7.3].

We note that the domain error still depends on the error of the interface velocity. This is to be expected, since we will only be able to bound this error together with the entire velocity error.

4.4. Consistency Error. In this section, we analyse the consistency of our discrete formulation. To ease the upcoming notation, we shall identify $u$ and $\lambda$ with their extensions. Due to (4.1), we can define the error on the discrete domain. Therefore, let us define the bulk-velocity, interface-velocity and Lagrange-multiplier errors as

\begin{align}
E^n &:= u(t^n) - u_h^n, & \quad L^n &:= \lambda(t^n) - \lambda_h^n, \\
\mathbb{U}^n &:= \xi(t^n) - \xi_h^n, & \quad \mathbb{V}^n &:= \chi(t^n) - \chi_h^n
\end{align}

Now, to derive an error equation for our discretisation, we observe that if $(v_h, \mu_h) \in V^n_h \times N^n_h$ are suitable test-functions for the discrete problem (4.2), then they are not necessarily valid test-functions for the smooth problem (2.3). However, using the mapping $\Phi^n$, we define $v_h^\ell := v_h \circ (\Phi^n)^{-1}$ and $\mu_h^\ell := \mu_h \circ (\Phi^n)^{-1}$. Inserting these test-function into (2.3), subtracting (4.2), as well as adding and subtracting appropriate terms, we get the error equation

\begin{align}
\frac{1}{\Delta t} (E^n - E^{n-1}, v_h)_{\Omega_h^n}^2 + (\nabla E^n, \nabla v_h)_{\Omega_h^n}^2 + \gamma_{gp} (E^n, v_h)_{\Omega_h^n} + (L^n, v_h)_{\Gamma_h^n} + (\mu_h, E^n - \mathbb{U}^n)_{\Gamma_h^n} - \gamma \lambda_h^\ell (L^n, \mu_h) + \frac{1}{\Delta t} (U^n - U^{n-1}, \zeta_1^\ell)_{\Omega_h^n} + (F_h^n - F(t^n), \zeta_1^\ell)_{\Omega_h^n} + \frac{1}{\Delta t} (V^n - V^{n-1}, \zeta_2^\ell)_{\Omega_h^n} + (\mathbb{U}^n, \zeta_2^\ell)_{\Omega_h^n} = \mathcal{E}^\ell_h(v_h, \mu_h, \zeta_1, \zeta_2),
\end{align}
with the consistency error

\[
\mathcal{E}_C^n(v_h, \mu_h, \zeta_1, \zeta_2) := \frac{1}{\Delta t} \left( u(t^n) - u(t^{n-1}), v_h \right)_{\Omega_h^n} - \left( \partial_t u(t^n), v_h' \right)_{\Omega(t^n)} \\
+ \left( \nabla u(t^n), \nabla v_h \right)_{\Omega_h^n} - \left( \nabla u(t^n), \nabla v_h' \right)_{\Omega(t^n)} + \left( \lambda(t^n), v_h \right)_{\Gamma_h^n} - \left( \lambda(t^n), v_h' \right)_{\Gamma(t^n)} \\
+ \left( \mu_h, u(t^n) - \xi(t^n) \right)_{\Gamma_h^n} - \left( \mu_h^*, u(t^n) - \xi(t^n) \right)_{\Gamma(t^n)} \\
+ \frac{1}{\Delta t} \left( \xi(t^n) - \xi(t^{n-1}), \zeta_1 \right)_2 + \frac{1}{\Delta t} \left( \chi(t^n) - \chi(t^{n-1}), \zeta_2 \right)_2 \\
+ \frac{d}{dt} \left( \zeta_1, \zeta_2 \right)_2 + \frac{d}{dt} \left( \zeta_1 - \zeta_2 \right)_2 \\
+ \gamma_{pp}^n \left( u(t^n), v_h \right) - \gamma_{pp}^n \left( \lambda(t^n), \mu_h \right).
\]

Lemma 4.12 (Consistency Estimate). Let the \( u \) fulfill the regularity assumption \( u \in \mathcal{W}^{3,\infty}(Q) \cap L^\infty(0, t_{\text{end}}; \mathcal{H}^{k+1}(\Omega(t))) \), then the consistency error can be bounded by

\[
|\mathcal{E}_C^n| \lesssim (\Delta t + h^k K^{3/2} + \mathfrak{M}(U^n, \Delta t)^{3/2}) \left( \|u\|_{\mathcal{W}^{3,\infty}(Q)} + \sup_{t \in [0, t_{\text{end}}]} \|u\|_{\mathcal{H}^{k+1}(\Omega(t))} \right) \|v_h\|_{*,n} \\
+ \Delta t \sup_{t \in [0, t_{\text{end}}]} \|\xi_u\|_2 \|\zeta_1\|_2 + \Delta t \sup_{t \in [0, t_{\text{end}}]} \|\xi_i\|_2 \|\zeta_2\|_2.
\]

Proof. The proof follows similar lines as [26, Lemma 5.11].

For the time derivative term we have with a change of variable, that

\[
|\Sigma_1| = \left| - \int_{t_{n-1}}^{t_n} \frac{t-t_n}{\Delta t} u(t) \cdot v_h \, dx + \left( u(t^n), v_h^n \right)_{\Omega_h^n} - \left( u(t^n), v_h' \right)_{\Omega(t^n)} \right| \\
\leq \frac{1}{2} \Delta t \|u\|_{\mathcal{W}^{2,\infty}(Q)} \|v_h\|_{\Omega_h^n} + \|u(t^n) - u(t^n) \circ \Phi^n, v_h \|_{\Omega_h^n} \\
\lesssim (\Delta t + \mathfrak{M}(U^n, \Delta t)^{3/2}) \|u\|_{\mathcal{W}^{2,\infty}(Q)} \|v_h\|_{\Omega_h^n}.
\]

In the final step, we have used

\[
|u_t(x, t^n) - (u_t \circ \Phi^n)(x, t^n)| \leq \|\nabla u_t\|_{L^\infty(\partial_\Omega(t^n))} |x - \Phi^n(x)|
\]

and Lemma 4.9. See also [26, Lemma 5.11].

For the diffusion term \( \Sigma_2 \), it follows analogously from the differentiation chain rule, Lemma 4.10 and Lemma 4.11 that, see, e.g. [17, Lemma 7.4]

\[
|\Sigma_2| = \left| \left( \nabla u(t^n) - u(t^n) \circ \Phi^n, J(D \Phi^n)^{-1} \nabla v_h \right)_{\Omega_h^n} \\
+ \left( \nabla u(t^n), (I - J(D \Phi^n)^{-1}) \nabla v_h \right)_{\Omega_h^n} \right| \\
\leq \mathfrak{M}(U^n, \Delta t)^{3/2} \|u(t^n)\|_{\mathcal{H}^2(\Omega(t^n))} \|\nabla v_h\|_{\Omega_h^n}.
\]
For the first Lagrange-multiplier term, we similarly find by additionally using a trace and the Poincaré inequality that
\[
|\mathcal{T}_3| = |(\mathbf{F}(t^n) - \mathbf{F}(t^n) \circ \mathbf{F}_n, J\mathbf{v}_h)_{\Gamma_h} - (\mathbf{F}(t^n), (1 - J)\mathbf{v}_h)_{\Gamma_h}| \\
\lesssim \mathcal{O}(|\mathbf{F}(t^n), \Delta t|^2 \|\mathbf{u}(t^n)\|_{H^3(\Omega(t^n))} \|\nabla \mathbf{v}_h\|_{H^1}).
\]

For the boundary condition term, we first note that due to \(\xi \in \mathbb{R}^d\), we can identify the extension as the constant extension. Furthermore, we can choose the extension of the bulk velocity, such that \(\mathcal{E}\mathbf{u} = \xi\) in the \(\delta_h\)-strip around \(\Gamma(t^n)\). Therefore, we need to impose \(\delta_h\)-smoothness for the ghost penalty consistency error, with \(\mathcal{E}\mathbf{u}\) sufficiently smooth in \(\mathcal{O}_{\delta_h}(\Omega_h^n)\). As a result, we have that
\[
|\mathcal{T}_4| = |(J\mathbf{u}_h, \mathbf{u}(t^n) - \mathbf{u}_h, \xi)_{\Gamma_h} - (\mathbf{u}_h, \mathbf{u}(t^n) - \xi)_{\Gamma_h}| = 0.
\]

The interface-velocity consistency error is bounded similar to \(\mathcal{T}_1\). However, the situation is simpler here because \(\xi \in \mathbb{R}^d\) does not depend on the domain consistency. Therefore,
\[
|\mathcal{T}_5| = \left| \frac{1}{\Delta t} (\mathbf{x}(t^n) - \mathbf{x}(t^{n-1}), \mathbf{z}_1)_2 - \left( \frac{d}{dt} \xi(t^n), \mathbf{z}_1 \right)_2 \right| \lesssim \Delta t \| \frac{d}{dt} \mathbf{z}_1 \|_{\infty, [t^{n-1}, t^n]} \| \mathbf{z}_1 \|_2.
\]

Similarly for the interface position, we have \(\frac{d}{dt} \mathbf{x} = \xi\) and
\[
|\mathcal{T}_6| = \left| \frac{1}{\Delta t} (\mathbf{x}(t^n) - \mathbf{x}(t^{n-1}), \mathbf{z}_2)_2 - \left( \frac{d}{dt} \mathbf{z}_2 \right)_2 \right| \lesssim \Delta t \| \frac{d}{dt} \mathbf{z}_2 \|_{\infty, [t^{n-1}, t^n]} \| \mathbf{z}_2 \|_2.
\]

Finally, for the ghost penalty consistency error, we use for \(\mathbf{u} \in H^{k+1}(\Omega_{h_{k+1}})\) the consistency estimate, see [26, Lemma 5.8],
\[
i^k_h(\mathbf{u}, \mathbf{u}) \lesssim h^{2k} \| \mathbf{u} \|_{H^{k+1}(\Omega_{h_{k+1}})}^2.
\]

As a result, we have using the Cauchy-Schwarz inequality
\[
|\mathcal{T}_7| \lesssim i^k_h(\mathbf{u}, \mathbf{u})^{1/2} i^k_h(\mathbf{v}_h, \mathbf{v}_h)^{1/2} \lesssim h^k \| \mathbf{u} \|_{H^{k+1}(\Omega_{h_{k+1}})} i^k_h(\mathbf{v}_h, \mathbf{v}_h)^{1/2} \lesssim h^k \| \mathbf{u} \|_{H^{k+1}(\Omega(t^n))}^{1/2} i^k_h(\mathbf{v}_h, \mathbf{v}_h)^{1/2}.
\]

where we used (3.2a) in the last estimate.

To the Lagrange-multiplier stabilisation form, we use that we identify \(\mathbf{F}\) in the bulk with the function which is equal to \(\mathbf{F}\) in the interface and which is constant in the normal direction \(\mathbf{n}\). Thus the stabilisation is fully consistent and vanishes.

### 4.5. Error estimate in the energy norm.
We consider stable interpolation operators \(I^n, I^k\) for the bulk velocity and the Lagrange-multiplier spaces \(V_h, N^n\), respectively. For \(k_s = 1, \ldots, k\), \(\mathbf{u} \in H^{k_s+1}(\Omega(t^n))\) and \(\mathbf{F} \in H^{k_s-1/2}(\Gamma(t^n))\), it then holds that
\[
\| \mathbf{u} - I^n \mathbf{u} \|_{\Omega_{h_{k_s}}} + h \| \nabla(\mathbf{u} - I^n \mathbf{u}) \|_{\Omega_{h_{k_s}}} + h^{1/2} \| \mathbf{u} - I^n \mathbf{u} \|_{\Gamma_h} \lesssim h^{k_s+1} \| \mathbf{u} \|_{H^{k_s+1}(\Omega(t^n))}.
\]
Applying this split in (4.16) yields
\[ \frac{1}{\Delta t} (e_h^n - e_h^{n-1}, v_h) + (\nabla e_h^n, \nabla v_h)_{\Omega_h^n} + \gamma_{gp} i_h^n (e_h^n, v_h) + (I_h^n, v_h)_{\Gamma_h^n} \]
\[ + (\mu_h, e_h^n - U_h^n)_{\Gamma_h^n} - \gamma_\lambda \lambda_h^n (\nabla e_h^n, \nabla v_h)_{\Omega_h^n} + \frac{1}{\Delta t} (e_h^n - e_h^{n-1})_\Omega + \frac{1}{\Delta t} (e_h^n - e_h^{n-1})_\Gamma + \gamma_{gp} i_h^n (e_h^n, v_h) + (I_h^n, v_h)_{\Gamma_h^n} \]
\[ + \frac{1}{\Delta t} (\gamma^n - \gamma^{n-1}, \zeta_2)_2 - (U_h^n, \zeta_2)_2 = E^n(v_h, \mu_h, \zeta_1, \zeta_2) + E^n_1(v_h, \mu_h), \]
for all \((v_h, \mu_h, \zeta_1, \zeta_2) \in V_h^n \times N_h^n \times \mathbb{R}^d \times \mathbb{R}^d\), with the interpolation term
\[ E^n_1(v_h, \mu_h) = \frac{1}{\Delta t} (\eta^n - \eta^{n-1}, \nabla v_h)_{\Omega_h^n} - (\nabla \eta^n, \nabla v_h)_{\Omega_h^n} + \gamma_{gp} i_h^n (\eta^n, v_h) - (\nu^n, v_h)_{\Gamma_h^n} - (\mu_h, \eta^n)_{\Gamma_h^n} + \gamma_\lambda \lambda_h^n (\eta^n, \mu_h). \]

**Lemma 4.13 (Interpolation estimate).** Let \( u \in L^\infty(0, t_{\text{end}}, \mathcal{H}^{k+1}(\Omega(t))) \) and \( u_t \in L^\infty(0, t_{\text{end}}, \mathcal{H}^k(\Omega(t))) \). Then the interpolation error can be bounded by
\[ |E^n_1(v_h, \mu_h)| \lesssim h^k K^2 \sup_{t \in [0, t_{\text{end}}]} \left( \|u\|_{\mathcal{H}^{k+1}(\Omega(t))} + \|u_t\|_{\mathcal{H}^k(\Omega(t))} \right) \|v_h\|_{s,n} \]
\[ + h^k \sup_{t \in [0, t_{\text{end}}]} \|u_t\|_{\mathcal{H}^{k+1}(\Omega(t))} \|\mu_h\|_{s,n}. \]

**Proof.** The bound
\[ |E^n_1(v_h, \mu_h)| \lesssim h^k K^2 \sup_{t \in [0, t_{\text{end}}]} \left( \|u\|_{\mathcal{H}^{k+1}(\Omega(t))} + \|u_t\|_{\mathcal{H}^k(\Omega(t))} \right) \|v_h\|_{s,n} \]
is shown in [26, Lemma 5.12]. We therefore only need to deal with the boundary terms.
Using the Cauchy-Schwarz inequality, and since the Lagrange-multiplier is the normal derivative of the velocity, we find with the stability of the extension that
\[
|\Sigma_{12}| \leq h^{\frac{3}{2}} \|\theta^n\|_{\Gamma_h^*} h^\frac{3}{2} \|v_h\|_{\Gamma_h^*} \lesssim h^k \|\partial_n u\|_{H^{k-1/2}(\Gamma(t^n))} h^\frac{3}{2} \|v_h\|_{\Gamma_h^*} \\
\lesssim h^k \|u\|_{H^{k+1}(\Omega(t^n))} \|v_h\|_{*,n}.
\]
Similarly, we have
\[
|\Sigma_{13}| \leq h^\frac{3}{2} \|\eta^n\|_{\Gamma_h^*} h^\frac{3}{2} \|\mu_h\|_{\Gamma_h^*} \lesssim h^k \|u\|_{H^{k+1}(\Omega(t^n))} h^\frac{3}{2} \|\mu_h\|_{\Gamma_h^*} \\
\lesssim h^k \|u\|_{H^{k+1}(\Omega(t^n))} \|\mu_h\|_{*,*}. 
\]
For the Lagrange-multiplier stabilising form, we again use the Cauchy-Schwarz inequality and the interpolation estimate (4.19). This results in
\[
|\Sigma_{14}| \leq j_h^n(\theta^n, \eta^n) \frac{1}{2} j_h^n(\mu_h, \mu_h) \frac{1}{2} \lesssim h^k \|u\|_{H^{k+1}(\Omega(t^n))} \|hn \cdot \nabla \mu_h\|_{\Omega_h^*}. 
\]
The claim then follows by the triangle inequality and summing up the above estimates. 

**Theorem 4.14 (Energy error estimate).** Let \((u_m^n, \xi_m^n)\) be the solution to the discrete problem (4.2). We assume that assumptions 3.2, 3.1 and 4.1 hold, assume \(\gamma_s\) in (4.4) is sufficiently large, the time step \(\Delta t\) is sufficiently small and the exact solution fulfills the regularity \(u \in W^{3,\infty}(Q) \cap \mathcal{L}^\infty(0, t_{\text{end}}; H^{k+1}(\Omega(t)))\), \(u_t \in \mathcal{L}^\infty(0, t_{\text{end}}; H^k(\Omega(t)))\) and \(\xi_{tt} \in \mathcal{L}^\infty(0, t_{\text{end}}; \mathbb{R}^d)\). Then for \(m = 1, \ldots, N\), and the errors defined in (4.15), the following error estimate holds:

\[
\|E_m\|_{\Omega_h^*}^2 + \|U_m\|_{L^2}^2 + \|V_m\|_{L^2}^2 + \frac{\Delta t c_{4.14a}}{2} \sum_{n=1}^m \left[ \|E_n\|_{*,m}^2 + j_h^n(\xi, \xi) \right] \\
\lesssim \exp \left(\frac{c_{4.14b}}{1 - \Delta t c_{4.14a}} - t_m^m \right) (\Delta t^2 + h^{2k}K + h^{2k-1} + h^{2k} \Delta t^{-1}) R(u, \xi),
\]

with \(R(u, \xi) := \|u\|_{W^{3,\infty}(Q)} + \sup_{t \in [0, t_{\text{end}}]} \left( \|u_t\|_{H^k(\Omega(t))}^2 + \|u_{tt}\|_{H^{k-1}(\Omega(t))}^2 + \|\xi_{tt}\|_{L^2}^2 + \|u\|_{H^{k+1}(\Omega(t))} + \|\xi\|_{H^{k+1}(\Omega(t))} \right)\), and constants \(c_{4.14a}, c_{4.14b} > 0\) independent of \(n, \Delta t\) and the mesh-interface cut topology.

**Proof.** Testing (4.20) with \((v_h, \mu_h, \zeta_1, \zeta_2) = 2\Delta t(e_h^n, -1^n, U^n, V^n)\) gives

\[
(4.21) \quad \|e_h^n\|_{\Omega_h^*}^2 + \|e_h^n - e_h^{n-1}\|_{\Omega_h^*}^2 - \|e_h^{n-1}\|_{\Omega_h^*}^2 + 2\Delta t \|\nabla e_h^n\|_{\Omega_h^*}^2 + 2\Delta t \gamma_{gp} j_h^n(e_h^n, e_h^n) \\
+ 2\Delta t (I_h^n, U^n)_{\Gamma_h^*} + 2\Delta t \gamma_{gp} j_h^n(I_h^n, I_h^n) + \|U^n\|_{L^2}^2 + \|U^n - U^{n-1}\|_{L^2}^2 - \|U^{n-1}\|_{L^2}^2 \\
+ 2\Delta t (F_h^n - F(t^n), U^n)_{L^2} + \|V^n\|_{L^2}^2 + \|V^n - V^{n-1}\|_{L^2}^2 - \|V^{n-1}\|_{L^2}^2 - 2\Delta t (U^n, V^n)_2 \\
= 2\Delta t (e_h^n, -1^n, U^n, V^n) + e_h^n(e_h^n, -1^n)_{L^2}(U^n, V^n). 
\]

Now, by definition, we have that \(F_h^n = \int_{\Gamma_h^*} \lambda_h ds\) and \(F(t^n) = \int_{\Gamma(t^n)} \lambda(t^n) ds\). Since \(U^n \in \mathbb{R}^d\) is constant in space, we have
\[
(F_h^n - F(t^n), U^n)_2 = (\lambda_h^n, U^n)_{\Gamma_h^*} - (\lambda(t^n), U^n)_{\Gamma(t^n)}. 
\]
As a result, the boundary integrals involving \( \lambda_h \) on the left-hand side of (4.21) vanish, and we have an additional mixed consistency/interpolation error term

\[ e^n(U^n) := (\lambda(t^n), U^n)_{\Gamma(t^n)} - (\lambda^n, U^n)_{\Gamma_h^n} \]

on the right-hand side. This leads to the error equation

\[
\begin{align*}
\|e^n_h\|_{\Omega_h^n}^2 + \|e^n_h - e^{n-1}_h\|_{\Omega_h^n}^2 - \|e^{n-1}_h\|_{\Omega_h^n}^2 + 2\Delta t\|\nabla e^n_h\|_{\Omega_h^n}^2 & \\
& + 2\Delta t\gamma_p\delta(t^n_h, e^n_h, e^n_h) + 2\Delta t\gamma_\lambda j^n_h(I^n_h, I^n_h) \\
& + \|u^n\|_{\Omega}^2 - \|u^{n-1}\|_{\Omega}^2 - \|u^{n-1}\|_{\Omega}^2 + \|\nabla u^n\|_{\Omega}^2 + \|\nabla u^{n-1}\|_{\Omega}^2 - \|\nabla u^{n-1}\|_{\Omega}^2
\end{align*}
\]

For the second term, we estimate using the Cauchy-Schwarz inequality, the fact that

\[\|e^n_h - e^{n-1}_h\|_{\Omega_h^n} \lesssim h^{k-\frac{1}{2}}\|u\|_{\mathcal{H}^{k+1}(\Omega(t^n))}\|\Gamma\|^{\frac{1}{2}}\|U^n\|_2.\]

Using these estimates, together with Lemma 4.2, Lemma 4.12 and Lemma 4.13 in (4.22) gives

\[
\begin{align*}
\|e^n_h\|_{\Omega_h^n}^2 + \|e^n_h - e^{n-1}_h\|_{\Omega_h^n}^2 + 2c_1\Delta t\|e^n_h\|_{\Omega_h^n}^2 + 2\gamma_\lambda j^n_h(I^n_h, I^n_h) + \|u^n\|_{\Omega}^2 + \|\nabla u^n\|_{\Omega}^2 & \\
\leq \|e^{n-1}_h\|_{\Omega_h^n}^2 + \|u^{n-1}\|_{\Omega}^2 + \Delta t\|U^n\|_{\Omega}^2 + \Delta t\|\nabla u^n\|_{\Omega}^2 + 2c\Delta t \xi_{15}
\end{align*}
\]

with

\[\xi_{15} := (\Delta t + h^kK^{\frac{1}{2}} + M^{\frac{1}{2}})R_1(u)\|e^n_h\|_{\Omega_h^n}^2 + h^kR_2(u)\|I^n_h\|_{\Omega_h^n}^2 + (\Delta t + h^k\delta^{\frac{1}{2}} + M^{\frac{1}{2}})R_3(u, \xi)\|u^n\|_{\Omega}^2.\]
Here we have abbreviated $\mathfrak{M} = \mathfrak{M}(U^n, \Delta t)$ and the higher-order residual terms are

$$R_1(u) := \|u\|_{\mathcal{W}^{k,\infty}(Q)} + \sup_{t \in [0, \text{end}]} \left( \|u\|_{\mathcal{H}^{k+1}(\Omega(t))} + \|u_t\|_{\mathcal{H}^{k}(\Omega(t))} \right),$$

$$R_2(u) := \sup_{t \in [0, \text{end}]} \|u\|_{\mathcal{H}^{k+1}(\Omega(t))},$$

$$R_3(u, \xi) := \sup_{t \in [0, \text{end}]} \left( \|\xi_{tt}\|_2 + \|\xi_t\|_2 + |\Gamma|^{\frac{1}{2}} [\|u\|_{\mathcal{H}^3(\Omega(t))} + \|u\|_{\mathcal{H}^{k+1}(\Omega(t))}] \right).$$

As in (4.8) we estimate the first term on the right-hand side of (4.23) by

$$\|e^{n-1}_h\|^2_{\Omega^n} \leq (1 + c' \Delta t)\|e^{n-1}_h\|^2_{\Omega^n} + \frac{1}{2} c_1 \Delta t \|e^{n-1}_h\|^2_{s,n,1}.$$

For the Lagrange-multiplier, we test (4.20) with $(\mathbf{v}_h, \mu_h, \zeta_1, \zeta_2) = (v_h, 0, 0, 0)$ and use Lemma 4.5 to get the estimate

$$\|l^n_{\mu,n}\|_{s,n} \leq \frac{1}{\Delta t} \|e^n_h - e^{n-1}_h\|^2_{\Omega^n} + \|e^n_h\|_{s,n} + (\Delta t + \Delta t^2 K + \mathfrak{M}) R_1(u) + j^n_k(l^n_{\mu,n}, l^n_{\mu,n})^\frac{1}{2}.$$

Then, we can estimate using the weighted Young's inequality

$$2c\Delta t \Xi_{15} \leq \frac{1}{2} \|e^n_h - e^{n-1}_h\|^2_{\Omega^n} + c_1 \Delta t \|e^n_h\|^2_{s,n} + \Delta t \gamma \lambda \gamma \lambda_h(\lambda^n_h, \lambda^n_h) + \Delta t \|U^n\|^2_2$$

$$+ \hat{c}(2 + \Delta t R^2_2) \mathfrak{M} + c'' \Delta t^2 R(u, \xi) (\|e^n_h\|^2_{s,n} + \|U^n\|^2_2)$$

$$+ c_4 \Delta t (\Delta t^2 + h^{2k} K + h^{2k-1} + h^{2k} \Delta t^{-1}) R(u, \xi).$$

Inserting these estimates in (4.23) then gives

$$\|e^n_h\|^2_{\Omega^n} + \frac{1}{2} \|e^n_h - e^{n-1}_h\|^2_{\Omega^n} + c_1 \Delta t \|e^n_h\|^2_{s,n} + \Delta t \gamma \lambda \gamma \lambda_h(\lambda^n_h, \lambda^n_h) + \|U^n\|^2_2 + \|V^n\|^2_2$$

$$\leq (1 + c' \Delta t)\|e^{n-1}_h\|^2_{\Omega^n} + \frac{c_1 \Delta t}{2} \|e^{n-1}_h\|^2_{s,n-1} + \|U^{n-1}\|^2_2 + \|V^{n-1}\|^2_2$$

$$+ 2\Delta t \|U^n\|^2_2 + \Delta t \|V^n\|^2_2 + c'' \Delta t^2 R(u, \xi) (\|e^n_h\|^2_{s,n} + \|U^n\|^2_2)$$

$$+ c_4 \Delta t (\Delta t^2 + h^{2k} K + h^{2k-1} + h^{2k} \Delta t^{-1}) R(u, \xi).$$

Now, the goal is to sum this over $n = 1, \ldots, m$ in order to use a discrete Gronwall lemma. To this end, we first note that

$$\sum_{n=1}^m \mathfrak{M}(U^n, \Delta t) = \sum_{n=1}^m \left[ \Delta t^2 \sum_{j=0}^n \|U^j\|^2_2 + \Delta t^3 t^n \|\partial_t \xi_t\|^2_\infty \right]$$

$$\leq t^m \Delta t \sum_{n=1}^m \|U^n\|^2_2 + \Delta t^2 (t^m)^2 \|\xi_t\|^2_\infty.$$
Summing (4.24) over \( n = 1, \ldots, m \), and using that \( e_h^0 = I_h^0 = U^0 = \psi^0 = 0 \), then gives

\[
\left\| e_h^m \right\|_{\Omega_h}^2 + \left\| U^m \right\|_2^2 + \left\| V^m \right\|_2^2 + \Delta t \left( \frac{c_1}{2} - c'' \Delta t R(u, \xi) \right) \sum_{n=1}^{m} \left\| e_h^n \right\|_{s,n}^2 + \Delta t \sum_{n=1}^{m} \gamma \lambda_h^n (I_h^n, I_h^n) \\
\leq c' \Delta t \sum_{n=1}^{m-1} \left\| e_h^n \right\|_{\Omega_h}^2 + \Delta t \left[ 2 + t^n c''(2 + \Delta t R_2^2) + c'' \Delta t R(u, \xi) \right] \sum_{i=1}^{m} \left\| U^n \right\|_2^2 \\
+ \Delta t \sum_{n=1}^{m} \left\| V^n \right\|_2^2 + c_4 (\Delta t^2 + h^{2k} K + h^{2k-1} + h^{2k} \Delta t^{-1}) R(u, \xi),
\]

with \( c_4 > 0 \) independent of \( \Delta t \) and \( m \). We now take \( \Delta t \) to be sufficiently small, such that \( (c_1/2 - c'' \Delta t R(u, \xi)) \geq c_1/4 \), and \( \Delta t (2 + t^n c''(2 + \Delta t R_2^2) + c'' \Delta t R(u, \xi)) =: \Delta t c_{T4.14b} \leq 1 \). Under this time step restriction, we can then apply a discrete Gronwall’s Lemma [20, Lemma 5.1] to get the estimate

\[
\left\| e_h^m \right\|_{\Omega_h}^2 + \left\| U^m \right\|_2^2 + \left\| V^m \right\|_2^2 + \Delta t c_{T4.14a} \sum_{n=1}^{m} \left[ \left\| e_h^n \right\|_{s,n}^2 + \lambda_h^n (\lambda_h^n, \lambda_h^n) \right] \\
\lesssim \exp \left( \frac{c_{T4.14b}}{1 - \Delta t c_{T4.14b}} \right) (\Delta t^2 + h^{2k} K + h^{2k-1} + h^{2k} \Delta t^{-1}) R(u, \xi),
\]

with \( c_{T4.14a} := \max\{c_1/4, \gamma \lambda \} \). The claim then follows by the triangle inequality and the optimal interpolation properties.

We note that the \( h^{2k}/\Delta t \) scaling also appears in the error estimate in [49] for the transient Stokes problem on a moving domain, also as a result of the use of an inf-sup result, in the latter case for the pressure.

5. Numerical Examples. We have implemented the method using NGSxFem [25], an add-on to NGSolve/netgen [41, 40] for unfitted finite element discretisations. The reproduction source code can be found in the archive [46].

5.1. Set-up. We consider the background domain \( \bar{\Omega} = (0, 1)^2 \), and the initial domain of interest is given by \( \Omega(0) = \Omega \setminus \{ x \in \Omega \mid (x_1 - 0.5)^2 + (x_2 - 0.8)^2 \leq 0.1^2 \} \). The external force acting on \( \Sigma \) is given by \( g = (0, -1)^T \). At \( t = 0 \), the system is at rest, i.e., \( u = 0 \) and \( \xi_0 = 0 \). The system is considered until \( t = t_{end} = 1.5 \).

As we do not have an analytical solution for this problem, we shall compare our results against a reference solution. As quantities of interest for comparison with this reference simulation, we consider the position and velocity of the moving interface. The error is then measured in the discrete space-time norm

\[
\left\| \xi^i_{ref} - \xi^i_h \right\|_{L^2(\mathbb{R}^d)}^2 := \sum_{i=1}^{N} \Delta t \left\| \xi^i_{ref} - \xi^i_h \right\|_2^2.
\]

Remark 5.1 (Reference Simulation). To compute a reference simulation of the above set-up, we consider a fitted ALE discretisation. Here we use \( P^4 \) elements together with BDF2 time stepping. As the motion of the domain is purely translational, a simple analytical form of the ALE mapping can be given, see, e.g., [48]. The PDE/ODE system is solved using a partitioned approach as in the Eulerian setting, c.f. Remark 5.2. An illustration of the solution can be seen in Figure 5.1.
Figure 5.1. Solution magnitude at $t = 0, 0.5, 1, 1.5$. Computed using the ALE method with $h_{\text{max}} = 0.1$, $k = 3$ and $\Delta t = \frac{1}{200}$.

Remark 5.2. To solve the coupled PDE/ODE system, we use a partitioned approach with a relaxation in the update of the interface velocity for stability of the scheme. To increase the convergence of the relaxation scheme to update the interface velocity, we use Aitken’s $\Delta^2$-method [22] to determine a good value for the relaxation parameter. In practice, we then require three iterations between the ODE and PDE until the relative velocity update is less than $10^{-8}$.

5.2. Convergence Study. We consider a series of shape-regular and quasi uniform meshes of the background domain $\tilde{\Omega}$, with the mesh sizes $h_{\text{max}} = h_0 \cdot 2^{-L_x}$. The initial mesh size is taken to be $h_0 = 0.1$. On these meshes we consider $P^2/P^1$ elements for the velocity and Lagrange-multiplier spaces. Similarly for the time step, we consider a series of uniformly refined time steps of $\Delta t = \Delta t_0 \cdot 2^{-L_t}$, with the initial time step $\Delta t_0 = 1/50$. We then consider the spatial convergence using the smallest time step and the temporal convergence on the finest mesh. The results for the full study over each mesh/time step combination can be found in the archive [46].

To define the extension strip, we set $w_n = |\xi_h \cdot n|$ in each time step, i.e., the explicit normal interface velocity. To ensure that the strip is wide enough to allow for acceleration of the interface, we set $c_h = 2$. Furthermore, the ghost penalty parameter is set to $\gamma_s = 0.1$ and the Lagrange-multiplier stabilisation parameter is set to $\gamma = 0.01$.

BDF1. We consider the BDF1 time-discretisation for the bulk and interface velocities as analysed above. The integration over the level set domains is then performed using standard CutFEM, i.e., using a $P^1$ approximation of the level set to explicitly construct the unfitted quadrature rules. As a result, we can only expect spatial convergence of order two, due to the geometry error of order $O(h^2)$. However, due to the first order time-discretisation, we expect the temporal error to dominate most situations.

The errors resulting from the convergence study for the interface velocity and position can be seen in Figure 5.2. Here we see the expected linear convergence with respect to the time step. Concerning the spatial convergence, we see the expected second order convergence until the temporal discretisation error starts to dominate.

BDF2 and isoparametric mapping. As an extension, we present some numerical results, based on a BDF2 discretisation of the time-derivatives. This is enabled, by making the extension strip sufficiently large, such that both $\Omega_h^0, \Omega_h^{n-1} \subset \mathcal{O}_{\xi_h}(\Omega_h^{n-2})$ and $\Omega_h^n \subset \mathcal{O}_{\xi_h}(\Omega_h^{n-1})$. We achieve this by setting $c_h = 4$. Furthermore, to increase the geometry approximation properties of the CutFEM method, we apply the isopara-
metric mapping approach introduced in [24] to increase the geometry approximation to $O(h^{k+1})$. For details of the higher-order discretisation in space and time, applied to a moving domain problem with prescribed motion, we refer to [28].

The results over the same series of meshes and time steps as considered before, can be seen in Figure 5.3. Here, we see second-order convergence with respect to the time step over the entire series of considered time steps. With respect to the spatial discretisation, we observe fourth-order convergence until the temporal error begins to dominate. This suggests that the error in the velocity and position of $\Sigma$ is dominated by the error in the force acting from $\Omega$ onto $\Gamma$, because the force is obtained with the accuracy of order $O(h^{2k})$, when computing this from the Lagrange multiplier. The proof of this follows the same lines as for the Babuška-Miller trick, see for example [4]. Furthermore, since the temporal error appears to remain dominant on finer meshes, we only see higher-order convergence over the first meshes.

5.3. Lagrange-Multipliers vs. Nitsche. Our analysis relies heavily on the Lagrange-multiplier formulation to enforce the boundary condition on the moving interface since testing with appropriate functions then lead to some terms appearing twice with differing signs and thus vanishing. However, CutFEM using Nitsche’s method [32] to implement the boundary condition is much more commonly used, see
amongst others [1, 6, 31, 49]. An advantage of the Nitsche approach is that we do not need to discretise the Lagrange-multiplier space, the resulting systems will be smaller on the same mesh. On the other hand, we need to choose the stabilisation parameter.

To investigate whether there is a significant numerical difference between the two methods, we consider the BDF2 implementation together with the isoparametric mapping and implement the boundary condition on the moving interface using Nitsche’s method. We use the symmetric version of Nitsche’s method and the penalty parameter is chosen as $40k^2/h$.

The results can be seen in Figure 5.4. The results are very similar compared to the Lagrange-multiplier results above. We keep the second-order convergence in time and the higher-order convergence in space, before the temporal error begins to dominate.

6. Conclusions and Outlook. In this work, we have studied an Eulerian, unfitted finite element method for a model moving domain problem, consisting of a parabolic PDE in the bulk together with translational coupled rigid body motion determining the motion of the domain. The analysis of the method relied on the Lagrange multiplier formulation we considered to implement the implicit non-homogeneous Dirichlet boundary condition on the moving interface. We showed stability in the temporal semi-discrete case in Lemma 3.4 and the stability of the fully discrete scheme in Theorem 4.8. For the error analysis, we treated the domain error resulting from the discretised scheme similarly as the geometry approximation error is dealt with in unfitted finite element analysis for problems with stationary domains or domains with known motion. With this, we proved an optimal-in-time error estimate for the bulk- and interface-velocity in the energy norm with Theorem 4.14. This estimate takes a similar form as the case for prescribed motion, with the key differences being a higher regularity assumption on the exact solution and a stronger time step restriction, resulting from the use of a more general form of Gronwall’s lemma. We illustrated our theoretical results with numerical convergence studies in both space and time. For the time-derivative approximation, we observed optimal order convergence for both the BDF1 and BDF2 time-derivative approximation. Concerning the mesh size, we observed the expected second-order convergence in the case of a piece-wise linear level set approximation, while we observed higher-order convergence when an isoparametric CutFEM approach was used for improved geometry approximation.
Furthermore, we observed that in practice, there is no noticeable difference in the accuracy of the analysed Lagrange-multiplier formulation and the more commonly used Nitsche formulation.

An important open issue for upcoming work is to lift Assumption 3.2, stating the stability of the discrete interface velocity. This assumption enters the definition of the $\delta$-neighbourhood used for the ghost penalty stabilisation and is therefore central in the analysis. Further, we consider the following extensions to be of interest for future research. To extend the presented analysis of the method to a more general setting, the equations governing the solid’s motion should include rotational motion. This requires additional work since the rotational component of the interface velocity $\mathbf{\omega} \times \mathbf{r}$ is no longer independent of space. Furthermore, the ODE governing the rotational motion is not linear for general shapes. Secondly, the bulk equations should be generalised to full fluid equations. Finally, the geometry approximation error inherent in CutFEM could be included in further analysis.

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