Elastic Region of Continuous Medium

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Abstract. In this paper, the elastic region model of continuous medium is established, and their constitutive relations are determined in mathematical form. The geometric properties of the deformation of the continuous medium are studied, and the spatial position and direction of each part of the continuous medium caused by the deformation are determined. We emphasize that the thermodynamic basis of the displacement concept is a finite size equilibrium region and a quasi-static process. Just from the macro point of view, the main mechanical problem of continuous medium should be how to find out its stress-strain relationship reasonably within the framework of the ancient elastic theory. This complex mesoscopic analysis will help to understand the size and constitutive relation of materials parameters, but it will not change the basic structure of elastic theory.

1. Introduction
Continuous medium theory assumes that the space occupied by a real fluid or solid can be approximated as a continuous space filled with particles. On the basis of continuum medium hypothesis, every point and every time in space have certain physical quantity. Generally speaking, these physical quantities are continuous functions of space coordinates and time. The geometric property of continuous medium deformation is to determine the change of space position and direction of various parts of the object caused by deformation and the change of distance between adjacent points, including motion, configuration, deformation gradient and strain tensor [1, 2]. However, the granular materials that are prone to plastic deformation have no tensile strength at all. Whether the concepts of displacement and strain can be applied is a basic problem that has not been completely agreed on at present [3]. Bohn and Huang Kun carefully discussed the algorithm of elastic strain in the case of uniformly deformed lattice, and thought that the long wavelength lattice wave corresponds to the classical elastic wave [4]. For non-uniform deformation and nano or disordered system, the displacement approximate formula recently proposed by goldhirsch and Goldenberg is quite different from the previous work [5]. The authors of references [6-8] suggest that the six components of stress tensor \( \sigma_{ij} \) should be regarded as independent state variables directly. Since the six unknown quantities cannot be determined by three force balance equations, three new differential and integral equations need to be established.

It is worth noting that for any system, as long as the spatial variation wavelength of the described physical phenomenon is longer than the distance between its constituent particles, there is a macro theory in the form of field variables. In other words, the concept of continuum is applicable to the sand pile on the ground where...
particles are in contact with each other, and also to the sand lifting case (just like the molecules of air are not in contact with each other, but there is still macro hydrodynamics theory) where there is no long-term contact between particles (only collision type interaction). Because the concept of strain is widely used in engineering practice, the measurement and establishment of appropriate stress-strain relationship is still one of the basic topics of soil mechanics. Considering the importance of displacement vector [9, 10], we think it is necessary to study displacement vector concept, scope of application and premise carefully. Although displacement and strain are always defined by the mass conservation principle, however, as a state variable, it is necessary for the medium to have equilibrium region and reversible process in the stress space. Since these thermodynamic concepts are generally applicable to any static macroscopic system, the displacement is still a basic variable with clear physical significance. Although the importance of elasticity has been gradually recognized, its theory and characterization are still unclear. One possible reason for this problem is that many people have been expecting to find a stress-strain theoretical relationship that is also effective when inelastic components appear in the strain. In order to establish the stress-strain theory which can describe the measured values of these inelastic experiments in a unified way, various attempts have been made, but there is still no satisfactory consensus result so far. We believe that if we focus on the study of the measured values of stress-strain close to pure elasticity, the situation is expected to be improved significantly.

2. Thermodynamic theory of displacement and strain

When apply an external force (including the volume force like gravity and the surface force on the boundary) to a medium, there will be a mass flow $V_k$ in the medium, which will cause relative displacement or deformation of different medium parts. In principle, if we know the time-space distribution $V_k(r,t)$ of the macroscopic mass flow in the deformation process, we can use the mass conservation equation

$$\frac{D\rho}{Dt} + \rho \text{div} \nu = 0$$

Its invariant form is

$$\frac{\partial \rho}{\partial t} + \text{div} (\rho \nu) = 0$$

According to Eq. (1-2), the density $\rho(r,t)$ is obtained. Then integrate the motion equation of displacement

$$\rho \frac{\partial \varepsilon_k}{\partial t} + v_k \frac{\partial \varepsilon_k}{\partial r_k} = \nu_t$$

the displacement vector field $\varepsilon_t(r,t)$ describing deformation is obtained. The strain tensor is obtained by spatial differentiation to Eq. (3)

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \varepsilon_j}{\partial r_i} + \frac{\partial \varepsilon_i}{\partial r_j} - \frac{\partial \varepsilon_k}{\partial r_i} \frac{\partial \varepsilon_k}{\partial r_j} \right)$$

Under small deformation

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial \varepsilon_j}{\partial r_i} + \frac{\partial \varepsilon_i}{\partial r_j} \right)$$

It shows that the mass flow $\nu_t$ in the mass conservation equation determines not only the mass change, but also size of the deformation. When the mass flow in the process of state change is always zero, there will be no corresponding deformation. Eq. (1-5) reflect relationship among the concepts of mass change, mass flow, deformation, etc. This relationship can be applied to both solids and granular materials, even applicable for liquids. In other words, they are general kinematic relations, not including materials coefficients, not involving the specific properties of materials.
3. Results and discussion

If same medium is loaded to the same strength along two different paths, Eq. (4-5) may give different strains. Therefore, the strain defined by the above method is generally related to the loading process (stress path). There are two extreme cases. One is the ideal solid, whose deformation process is completely reversible. In other words, the ideal solid has a fully resilient elastic response to any loading or unloading process, regardless of size and mode (shear, compression or tension). Since the ideal solid remains unchanged in any cyclic loading process, the strain defined by Eq. (4-5) must only depend on the initial state and the final state, independent of the path. The elastic region of an ideal solid extends over the whole force plane. But the actual solid will yield when the stress is too large, and the elastic region is limited (Fig. 1 (a)). At every point in the elastic region, a solid can reach thermal equilibrium. On the contrary, the system in stable thermal equilibrium must respond elastically to any disturbance, the equilibrium state must be in the elastic region. So the elastic region and the heat balance region are essentially the same concept. Generally, the loading or unloading process of solid in the equilibrium region can be regarded as quasi-static (or reversible). In addition, any stress change will produce mass flow and change of strain. Therefore, when $\nu_i = 0$, it is static equilibrium

$$\sigma_{ij,j} + \rho F_i = 0,$$

Here $F$ is the volume force per unit mass. According to equilibrium thermodynamics, energy is also a state quantity. Therefore, the energy $E$ of a solid must be a single value function of strain

$$E = \varphi(e_{ij}).$$

It is noted that the momentum density is equal to the mass flow and the stress is equal to the momentum flow. Using the general law of energy-momentum conservation and the standard hydrodynamic derivation [11], the relationship between stress and energy is
\[
\sigma_{ij} = -\frac{\partial \varphi}{\partial \varepsilon_{ij}} + \sum \varepsilon_{ik} \left( \frac{\partial \varphi}{\partial \varepsilon_{kj}} \right),
\]

(8)

Here \( \varepsilon_{ik} \) is a reversible elastic strain. If the material is not particularly soft, \( [\varepsilon_{ik}]_{\approx 1} \), the last item above can be omitted. Therefore, \( \sigma_{ij} \approx -\frac{\partial \varphi}{\partial \varepsilon_{ij}} \) is not limited to the linear elastic theory. If considering Hertz contact, we will always omit higher-order correction.

From the above discussion, we can see that the constitutive relations (6-8) describing the static behavior are based on the basic concept of reversible process about thermodynamics. In the thermal dynamic equilibrium state, the free energy takes the minimum value. If there is no external force acting on the object, as \( F \) is a function of \( \varepsilon_{ij} \), then \( F \) should be minimized. Thus, the stress tensor and strain tensor are obtained

\[
(\varepsilon_{ij} = \frac{1}{K} \delta_{ij} + \frac{1}{2\mu} (\sigma_{ij} - \frac{1}{3} \delta_{ij} \sigma_{kk})),
\]

(9)

\[
(\sigma_{ij} = K \varepsilon_{ij} \delta_{ij} + 2\mu (\varepsilon_{ij} - \frac{1}{3} \delta_{ij} \varepsilon_{kk})),
\]

(10)

where \( K, \mu \) are expansion coefficients. Note that they are only applicable to equilibrium systems and corresponding equilibrium processes (i.e. inverse processes), and do not include various dissipation effects of irreversible dynamic processes.

As another extreme case, the heat equilibrium region of ideal fluid which has no resistance to both tension and shear degenerates into a straight line (Fig. 1(c)). When the medium is an ideal fluid, its constitutive relation is

\[
\sigma_{ij} = -P \delta_{ij},
\]

(11)

here \( P \) is the pressure and \( \delta_{ij} \) is the \( \delta \) tensor component. Note that the pressure in the liquid is greater than its saturation pressure). According to the general equation of continuum dynamics

\[
\rho \left( F_i - \frac{\partial \mathcal{V}}{\partial t} \right) + \sigma_{ij} = 0,
\]

(12)

obtain the Euler equation of ideal hydrodynamics

\[
\rho \left( F_i - \frac{\partial \mathcal{V}}{\partial t} \right) - P_i = 0.
\]

(13)

The fluid can not enter the heat balance state under any external force state, unless it is strictly compressed. The degradation of this equilibrium region makes the strain defined by Eq. (4-5) generally related to the path of the external force. Because there is no finite area of equilibrium state, the fluid has no strain-stress function relation like the solid, and the concept of strain has no significance to the fluid. Of course, we can introduce an effective strain concept only on the one-dimensional line in Fig. 1(c), but it is often related to the mass density \( \rho \), not a new state variable.

For cohesionless granular materials, due to the easy plastic deformation and no ability to resist tensile, recently some scholars point out that displacement and strain are no longer the basic variables to describe granular materials. But they have shear resistance after being compressed. In a certain range of pressure and shear force, it can reach the thermal equilibrium state. Therefore, the particle has a finite size, and its shape is approximately the elastic region shown in Fig. 1(b). Although the granular matter is discrete in the visible range, as discussed above, as long as we pay attention to the limited scale and reasonable variability, it can still be treated as a continuous medium, because here is a larger scale to consider the statistical average of various mechanical quantities. That is to say, we can describe the granular matter in the framework of continuum theory on the macro accuracy which is far greater than the adjacent distance between particles.

Only from the shape of the equilibrium region, it can be said that the granular materials are between the solid and the fluid. The existence of a finite equilibrium region indicates that in principle it has the same...
quasi-static process as a solid, which satisfies Eq. (9-10). It is worth noting that the particles rearrange easily due to sliding in the particles, and its dissipation is much stronger than that of normal solids. Especially in the case of small bulk density or pressure, it is easy to trigger various dissipation phenomena, such as viscosity, small creep and obvious sliding. At this point, it can be very difficult to make it a quasi-static process. But what's important is that this doesn't mean that it has no equilibrium region and corresponding elastic strain concept. Another important property different from solids is that the system begins to lose its thermodynamic stability (or yield) when it is not stressed. Therefore, its zero stress point is located on the boundary of the equilibrium region, and the functions $\varepsilon_{ij}$ and $\varphi$ can not be resolved at this point. This characteristic shows that it is a pure nonlinear elastic body, and there is no linear stress-strain relationship even in the lowest approximation. Therefore, for particle mechanics, this term is very important. Without it, the stress distribution will become very similar to the results of linear elastic theory, and the unique mechanical behavior of granular materials will not be obtained.

4. Conclusion
The discussion in this paper shows that when continuous medium is in equilibrium at the beginning, under the action of external forces, the continuous medium is either in a new equilibrium state or in motion. Although the new equilibrium and motion occur after the deformation of continuous medium, the equilibrium and motion also affect the action of external forces and the response characteristics of the object. The applicable premise of elastic theory is that there are finite equilibrium regions and reversible processes in the stress space of the system. With the equilibrium region, there are displacement vectors and strains as state variables, and both stress and energy are functions of strain. The importance of the displacement and strain is that they reflect this fundamental property. Because of the heat balance, the number of macroscopic freedom degrees of the system is 3 (displacement vector), not 6 (stress tensor). This makes the static analysis only need three force balance equations plus boundary conditions. Therefore, as long as we admit the fact that continuous medium has a finite equilibrium region in the stress space, we cannot deny the applicability of elastic theory to granular materials. In this paper, we emphasize that the thermodynamic basis of the displacement concept is a finite size equilibrium region and a quasi-static process. Although the relationship between it and mesoscopic theory is not clear at present, we can see that displacement is still effective for particles only from the simple fact that dry sand piles can exist stably under the earth’s gravity.

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Reference
[1] L.D. Landau, E.M. Lifshitz: *Theory of Elasticity* (Pergamon Press, New York 1986).
[2] S. Luding: *Granular Matter* Vol. 10 (2008), P. 235
[3] P.G. de Gennes: *Rev. Mod. Phys.* Vol. 71 (1999), P. 374
[4] M. Born and K. Huang: *Dynamical Theory of Crystal Lattice* (Clarendon Press, Oxford 1954).
[5] I. Goldhirsch and C. Goldenberg: *Eur. Phys. J. E* Vol. 9 (2002), p. 245
[6] M.E. Cates, J.P. Wittmer, J.P. Bouchaud and P. Claudin: *Phys. Rev. Lett.* Vol. 81 (1998), P. 1841
[7] Q.C. Sun, F. Jin and G.D. Zhou: *Granular Matter* Vol. 15 (2013), P. 119
[8] H.A. Makse, N. Gland, D.L. Johnson and L.M. Schwartz: *Phys. Rev. E.* Vol. 70 (2004), P. 061302
[9] Y.M. Jiang and M. Liu: *Phys. Rev. Lett.* Vol. 99 (2007), P. 105501
[10] C.M. Song, P. Wang and H.A. Makse: *Nature* Vol. 453 (2008), P. 629
[11] H. Temmen, H. Pleiner, M. Liu and H.R. Brand: *Phys. Rev. Lett.* Vol. 84 (2000), P. 3228