Quantum state correction of relic gravitons from quantum gravity

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Abstract

The semiclassical approach to quantum gravity would yield the Schrödinger formalism for the wave function of metric perturbations or gravitons plus quantum gravity correcting terms in pure gravity; thus, in the inflationary scenario, we should expect correcting effects to the relic graviton (Zel’dovich) spectrum of the order of \((\lambda/m^2_{Pl})\). These, on the other hand, could possibly be measured in a future experiment.

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I. INTRODUCTION

In spite of the fact that there still lacks a full quantum gravity theory, a predictive semiclassical quantum cosmology has already been developed due to the selection of the initial quantum state of the Universe. The latter may be developed upon quantising canonical general relativity for the metric tensor corresponding to perturbations about FRW models [1] - [3]. In such an approximation scheme, the parameter representing the relevant energy for the initial expansion of the Universe, the effective cosmological constant, $\lambda$, is much lower than the Planck mass [4],

$$\frac{\lambda}{m_{Pl}^2} \leq 10^{-9}. \quad (1)$$

Thus, upon using a Born-Oppenhiemer approximation scheme (that takes into account the above mentioned relative small value of matter energy with respect to the Planck mass), one can obtain, for a given complex solution of Wheeler-DeWitt equation, the Schrödinger formalism for the wave function of metric perturbations [5]. Moreover, quantum gravity corrections also arise [6] and, at least in principle, they should show how effects of quantum gravity might even be measured in the context of these inflationary scenarios.

The aim of this paper is precisely to show that such effects do exist, in this case, upon introducing a shift of quantum gravity origin in the spectrum of the relic gravitons being originated during the early inflationary stages of the Universe. Relic gravitons (Zel’dovich) spectrum has, as a matter of fact, already been measured from the existing 3K cosmic background radiation anisotropy [7]; on the other hand, a certain quantum gravity modification could be possibly measured in the next future by some experimental devices [8].

II. SEMICLASSICAL WAVE FUNCTION IN DE SITTER SCENARIO

The simplest (although still realistic) model for the early stage of the Universe consists of a massive scalar field with matter potential $U(\phi)$ in a FRW spacetime described by a single degree of freedom, the scale factor, $a(t)$. After the Hartle-Hawking prescription [10], the wave
function will depend on the value \( U(\phi(\Sigma)) \) (\( \Sigma \) being the boundary of the compact euclidean 4 - sphere) of the inflaton (scalar field) potential. This is just de Sitter case with \( U(\phi) \approx \lambda \) playing the role of an effective cosmological constant for large initial values of \( \phi \); thus, in this model, we only take into account, for the wave function, an implicit dependence on the inflaton field (in classical terms: we will ignore the backreaction of matter corresponding to the kinetic energy of the field, so that matter couples directly to curvature using the effective cosmological constant).

Moreover, the general approach is only physically consistent if we obtain a description for the Schrödinger evolution of metric perturbations or gravitons, i.e., the tensor harmonics of the three sphere \( d_n \). Now, upon defining \( \alpha = \log(a) \) as the gravitational variable, the metric perturbation Hamiltonian is

\[
H_m = \sum_n \frac{-1}{2} \frac{\partial^2}{\partial d_n^2} + (n^2 - 1) e^{4\alpha} d_n^2, \tag{2}
\]

where, as a consequence of the perturbative character of the \( \{d_n\} \),

\[
d_n^2/\lambda \ll 1 \tag{3}
\]

Therefore, upon assuming \( U(\phi) \approx \lambda \), and neglecting second derivatives with respect to the matter field, Wheeler-DeWitt equation reads

\[
\left[ \frac{1}{2m_{Pl}^2} \frac{\partial^2}{\partial \alpha^2} + \frac{m_{Pl}^2}{2} (-e^{4\alpha} + \lambda e^{6\alpha} + H_m) \right] \Psi = 0 \tag{4}
\]

In the semiclassical approach we write the wave functional \( \Psi[\alpha, \{d_n\}] \) as

\[
\Psi = e^{iS} \tag{5}
\]

then we expand \( S \) in the form

\[
S = m_{Pl}^2 S_0 + S_1 + m_{Pl}^{-2} S_2 + .... \tag{6}
\]

We now insert the expansion defined by Eqs. (5) and (6) in Eq.(4) and compare expressions with the same order in \( m_{Pl} \). The highest order yields to
\[
\sum_n \left( \frac{\partial S_0}{\partial d_n} \right)^2 = 0
\]  \(7\)

Thus, \(S_0\) depends only on \(\alpha\) (the three-metric).

The next order leads to the Hamilton-Jacobi equation for gravity alone:

\[
-\left( \frac{\partial S_0}{\partial \alpha} \right)^2 + \lambda e^{6\alpha} - e^{4\alpha} = 0.
\]  \(8\)

If we define a functional \(\psi_0\) according to \((S_1 \equiv \sum_n S_{1n})\)

\[
\psi_0 \equiv D(\alpha) e^{i\sum_n S_{1n}} = \Pi_n \psi_{0n}
\]  \(9\)

where,

\[
D(\alpha)^2 = \left( \frac{\partial S_0}{\partial \alpha} \right) \equiv S'_0
\]  \(10\)

then order \(m_{Pl}^0\) will imply that \(\psi_0\) is the solution of

\[
-i \frac{\partial S_0}{\partial \alpha} \frac{\partial \psi_0}{\partial \alpha} \equiv i \frac{\partial \psi_0}{\partial t} = H_m \psi_0.
\]  \(11\)

Which is the functional Schrödinger equation for matter fields propagating on a fixed curved background; \(t\) is usually called WKB time.

If we do not take into account the value of \(S_2\) in the formal expansion, we get to this order of approximation the WKB wave function given by

\[
\Psi^{(1)} = \frac{1}{D} e^{im_{Pl}^2 S_0} \psi_0.
\]  \(12\)

### III. QUANTUM GRAVITY CORRECTIONS

Upon defining \(S_2 = \sigma(\alpha) + \eta(\{d_n\}, \alpha)\), we obtain the second order correction for the wave function

\[
\Psi^{(2)} = \frac{1}{D} e^{im_{Pl}^2 S_0 + i\sigma/m_{Pl}^2} \psi_0 e^{in/m_{Pl}^2}
\]  \(13\)

where \(\sigma\) and \(\eta\), representing quantum gravity correcting terms to the WKB wave function, satisfy, after some algebra \[3\]
\[- S'_0 \eta' = \frac{i}{2} \sum_n \frac{\partial^2 \eta}{\partial d_n^2} + \frac{i}{\psi_0} \sum_n \frac{\partial \eta}{\partial d_n} \frac{\partial \psi_0}{\partial d_n} + \frac{1}{2S'_0 \psi_0} \psi'_0 S''_0 - \frac{1}{2 \psi_0} \psi''_0 \]  

Eq. (14) involves entanglement of the modes and, therefore, it is difficult to solve. Moreover, in general, field modes and gravitational degrees of freedom are also entangled so we do not expect to obtain its general solution. Nonetheless, let us define \( \eta(d_n, \alpha) \equiv \sum_n \eta_n(d_n, \alpha) \). In that case, we obtain,

\[- S'_0 \sum_n \eta'_n = \sum_n \left( \frac{i}{2} \frac{\partial^2 \eta_n}{\partial d_n^2} + \frac{i}{\psi_0} \frac{\partial \eta_n}{\partial d_n} \frac{\partial \psi_0}{\partial d_n} + \frac{1}{2S'_0 \psi_0} \psi'_0 S''_0 - \frac{1}{2 \psi_0} \psi''_0 \right) + \frac{i}{2} \sum_{l \neq k} \frac{1}{\psi_0 \psi_{0l}} \psi'_{0k} \psi'_{0l} \]  

where \( \psi_0 \) is the solution of the Schrödinger equation. The non boundary proposal \( \Psi^{(2)} \) picks up the ground state (these results are desired in the semiclassical approach to gravity \( \Psi^{(2)} \)),

\[ \psi_0(a, d_n) = N_n(a) e^{-\frac{1}{2}a^2d_n^2} \]  

and as a result of this, last term in Eq. (16) is \( O(a^4d_k^2d_l^2) \) which, after Eq. (3) should be negligible for \( a^2 \rightarrow 1/\lambda \). Thus the modes are exactly disentangled after the selection of this particular initial condition and we can finally write, for the correcting phase of a single mode wave function,

\[- S'_0 \eta'_n = \frac{i}{2} \frac{\partial^2 \eta_n}{\partial d_n^2} + \frac{i}{\psi_0} \frac{\partial \eta_n}{\partial d_n} \frac{\partial \psi_0}{\partial d_n} + \frac{1}{2S'_0 \psi_0} \psi'_0 S''_0 - \frac{1}{2 \psi_0} \psi''_0 \]  

We must now solve the system of Eqs. (15) and (18) in order to obtain predictions from the corrected \( \Psi^{(2)} \) wave function. To this aim, it is better to consider the very early stages of the Universe where such terms should be relevant, i.e., we must restrict our calculation to the region where \( \lambda \) is really a constant; now, for \( a^2 \rightarrow 1/\lambda \), we can operate the expressions a little bit further. First, since the theory should not depend on the selection of the time
parameter, we are allowed to make our predictions for the particular semiclassical evolution parameter given by

\[ \theta(a) = \lambda D(a) = \lambda a^2 (\lambda a^2 - 1)^{1/2} \]  

(19)

thus, for \( a^2 = 1/\lambda(1 + \theta^2 + O(\theta^4)) \), we get, after Eqs. (18) and (15)

\[
\frac{i}{\lambda} \frac{\partial \eta_n}{\partial \theta} \approx -\frac{1}{2} \frac{\partial^2 \eta_n}{\partial d_n^2} - \frac{1}{\psi_{0n}} \frac{\partial \eta_n}{\partial d_n} \frac{\partial \psi_{0n}}{\partial d_n} + \frac{i}{2\theta^2 \psi_{0n}} \left\{ \frac{\partial^2 \psi_{0n}}{\partial \theta^2} - \frac{2 \partial \psi_{0n}}{\theta} \right\}
\]  

(20)

and,

\[
\frac{\partial \sigma}{\partial \theta} \approx \frac{5\lambda}{8\theta^4}
\]  

(21)

Which shows that there seems to exits an apparent divergency for \( \Psi^{(2)} \) as \( \theta \to 0 \). Moreover, from Eqs. (9) and (17), upon factorizing Van Vleck determinant, we pick up the \( n \)-mode wave function \( \psi_{0n} \) as

\[
\psi_{0n} = \theta e^{-\frac{1}{2}na^2(\theta)d_n^2}
\]  

(22)

If we now separate the factor ordering dependent part (i.e., that arising from the Van Vleck determinant) of \( \eta_n \) in the form

\[
\eta_j(\theta, d_j) = \eta_{j1}(\theta, d_j) + \eta_{j2}(\theta)\delta_{nj}
\]  

(23)

we finally obtain, replacing Eqs. (22) and (23) in Eq. (20)

\[
\frac{\partial}{\partial \theta} [\sigma(\theta) + \eta_2(\theta)] = 0
\]  

(24)

\[
\frac{i}{\lambda} \frac{\partial \eta_{n1}}{\partial \theta} = \frac{1}{2} \frac{\partial^2 \eta_{n1}}{\partial d_n^2} - d_n na^2(\theta) \frac{\partial \eta_{n1}}{\partial d_n}
\]  

(25)

Eq. (24) demostrates that, after the adiabatic approximation, the divergencies arising in \( \sigma \) and \( \eta_n \) as we approach \( \theta \to 0 \) cancel out each other exactly. This, on the other hand, is a consequence of the fact that the phase correcting terms from quantum gravity should not depend on the selection of the factor ordering \( [6] \). Therefore, in \( \Psi^{(2)} \), the only physically relevant quantum gravity correcting phase factor is \( \eta_{n1}(\theta, d_n) \).
The initial state of the Universe is taken on the three-sphere \( a^2 \rightarrow 1/\lambda \), in this case, we can obtain an exact solution of Eq. (25) upon making the obvious substitutions

\[
\eta_{n1} = e^{i\varepsilon \lambda \theta} y(d_n),
\]  

(26)

for some unknown constant \( \varepsilon \). Then we write

\[
d_n = (\lambda/n)^{1/2}x_n,
\]  

(27)

\[k = \varepsilon \lambda/n,
\]  

(28)

using these expressions, Eq. (25) transforms into

\[
\frac{d^2y}{dx_n^2} - 2x_n \frac{dy}{dx_n} + 2ky = 0,
\]  

(29)

which is the Hermite equation. The requirement of normalizability in \( d_n \) of the corrected wave function \( \psi_n = \psi_0e^{in_1} \) imposes that \( \eta_{n1} \) should only be given in terms of polynomials in \( d_n \), i.e., we select the constant \( k \) in Eq. (29)

\[k = 0, 1, 2,
\]  

(30)

or

\[
\eta_{n1}(\theta, d_n) = \frac{(2)^{1/2}g_1}{2}e^{in\theta/2}[2(n/\lambda)^{1/2}d_n\lambda] + g_2e^{in\theta}[\frac{4nd_n^2}{\lambda} - 2\lambda].
\]  

(31)

After Eq. (22), we notice that \( g_1 \neq 0 \) in Eq. (31) (i.e., a linear term in the phase of the \( n \)-mode wave function of gravitons) means that the expected initial number of gravitons is different from zero and it is given by

\[
N = \lim_{a^2 \rightarrow 1/\lambda} \left| \frac{2g_1(\lambda/\lambda)^{1/2}i}{m_{Pl}^2(2n^2)^{1/2}} \right|^2 = \left| \frac{g_1\lambda}{m_{Pl}^2} \right|^2.
\]  

(32)
IV. THE SPECTRUM OF RELIC GRAVITONS

Since the initial number of gravitons is different from zero, it will also change the measurable properties of the resulting spectrum corresponding to the statistic of gravitons in inflationary models; here, in order to calculate the expected changes, we follow the method of B. Allen [11].

In the first approximation, the graviton spectrum produced by an inflationary stage is entirely independent of the mechanism that produces the inflation. The only inputs which are needed to find the graviton spectrum are the classical metric of space-time, and the initial quantum state of the gravitational perturbations.

For convenience, one may assume that the Universe is approximately flat, so that the metric takes the form

\[ ds^2 = a^2(t)(-dt^2 + d\sigma^2) \]  

(33)

In any case, since we will only study the situation corresponding to wavelengths which are shorter than the present-day horizon scale, it could be taken as a good approximation if the Universe were either spatially open or closed.

The classical spacetime begins as de Sitter space but then undergoes an instantaneous phase transition at, say, \( t = t_1 \), after which it evolves as a radiation-dominated model until the time \( t = t_0 \). The scale factor describing this model is

\[ a(t) = \left(\frac{t}{t_1}\right)a(t_1) \text{ for } t_1 < t < t_0 \]

\[ a(t) = \left(2 - \frac{t}{t_1}\right)^{-1}a(t_1) \text{ for } t < t_1 \]

(34)

We would also require the solution of Einstein equations, imposing

\[ \frac{8\pi G\rho_0}{3} = \frac{1}{a(t_1)^2t_1^2} \equiv \lambda \]  

(35)

Let us now determine the gravitational-particle production in this spacetime. A gravitational perturbation with comoving wave number \( k \) is represented by
\[ h_{\mu\nu} = a(t)^2 \epsilon_{\mu\nu}(k) \phi(t)e^{ikx} + cc \]  

where \( \epsilon_{\mu\nu} \) is the polarization tensor. The physical frequency of the wave is \( \omega = k/a(t) \). The amplitude \( \phi \) also obeys the perturbed Einstein equations, it leads to

\[ \ddot{\phi} + (2\dot{a}/a)\dot{\phi} + k^2 \phi = 0 \]  

The choice of a solution to the equation for \( \phi \) corresponds to the choice of an initial quantum state for the gravitational field. In de Sitter stage, the solution representing a de Sitter-invariant vacuum state is

\[ \phi_v(t) = [a(t_1)/a(t)]\{1 + i(\lambda)^{1/2}(a(t)/k)\}e^{-ik(t-t_1)} \]  

On the other hand, if, after Eq. (32), the initial state is given by a small \( N \)-graviton coherent state, we must correct the vacuum state upon adding a negative frequency solution to the above expression,

\[ \phi(t) = \phi_v(t) + (ig_1\lambda/m_{Pl}^2)\phi_v^*(t) + O(\lambda/m_{Pl}^2)^2 \]  

The corresponding solution to the wave equation in the radiation stage is

\[ \phi_r(t) = (a(t_1)/a(t))\{\alpha_re^{-ik(t-t_1)} + \beta_re^{ik(t-t_1)}\} \]  

where \( \alpha_r \) and \( \beta_r \) are Bogolubov coefficients.

By matching the modes at \( t = t_1 \) one obtains for the negative frequency coefficient, representing particle creation

\[ \beta_r = (-ig_1\lambda/m_{Pl}^2)(1 - \frac{i}{kt_1}) + \frac{(1 + ig_1\lambda/m_{Pl}^2)}{2k^2t_1^2} \]  

On the other hand, upon taking into account that \( k = a(t_0)\omega, t_1 = 1/a(t_1)\lambda^{1/2} \) we finally obtain, for the total number of gravitons at time \( t_0 \),

\[ |\beta_r|^2 = \frac{1}{4[a(t_0)]^4} \frac{\lambda^2}{\omega^4}\epsilon(\omega) \]  

where,
\[ \epsilon(\omega) \equiv 1 - 2(\lambda/m_{Pl}^2)[2Re[g_1]\frac{a(t_1)}{a(t_0)}\frac{\omega}{\lambda^{1/2}} + Im[g_1](1 - 2[\frac{a(t_1)}{a(t_0)}]^2\omega^2)] \] (43)

Therefore, the spectrum should be given by the corresponding energy density \( d\rho_g = P(\omega)d\omega \) for a density of states \( dN = \omega^2d\omega/(2\pi^2) \), i.e.,

\[ d\rho_g = P(\omega)d\omega = 2\omega \frac{\omega^2d\omega}{2\pi^2} |\beta_r|^2 \] (44)

or,

\[ P(\omega) = \frac{1}{4\pi^2}\frac{\lambda^2}{\omega^4}[\frac{a(t_1)}{a(t_0)}]^4\epsilon(\omega). \] (45)

Here, \( \epsilon(\omega) \) differs from Zel’dovich’s spectrum due to the presence of quantum gravity effects. If we take into account some phenomenological values of \( \frac{a(t_1)}{a(t_0)} \sim 10^{28} \) (see also [11]), we conclude that such effects would already be present for those frequencies of the order of \( \omega \sim 10^{13} \) Hz.

The result in Eq. (45) may also be expressed in terms of the effective Hubble parameter during inflation, \( H = \lambda^{1/2} \), and the recombination density \( \rho_R = \rho_0[\frac{a(t_1)}{a(t_0)}]^4 \), then, Eq. (35) taken into account, we obtain the typical value

\[ \Omega_{graviton} \equiv \frac{\omega d\rho_g}{\rho_R d\omega} = \frac{2}{3\pi}(\frac{H}{m_{Pl}^2})^2 \epsilon(\omega) \] (46)

where \( \epsilon(\omega) \approx 1 + O(\lambda/m_{Pl}^2) \), represents a perturbation to the predicted Zel’dovich plateau.

**V. CONCLUDING REMARKS**

Cosmic Background Radiation is used in order to test phenomenological models of the early Universe. In these scenarios, we have just shown that, quantum gravity corrections for the spectrum of gravitons, would possibly lie on the range of frequencies technically accessible in some future experimental devices measuring CBR anisotropy [8].

Semiclassical gravity, represented by \( \Psi^{(2)} \), is, in this framework, a predictive and testable theory of initial conditions. On the other hand, we have also seen that the adiabatic approximation for the wave function of metric perturbations, leading to the ground state for the
wave function of gravitons, can be thought as being technically correct since, in this case, there would not exist formal divergencies depending on the factor ordering for the resulting quantum gravity corrections on the initial three-sphere.

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REFERENCES

[1] Hawking S. W., *The quantum state of the Universe*, Nucl. Phys., B239, 257 (1984).

[2] Halliwell J. J., and Hawking S. W., *Origin of Structure of the Universe*, Phys. Rev., D31, 1777 (1985).

[3] Wada S., *Quantum cosmological perturbations in pure gravity*, Nucl. Phys., B276, 729 (1986).

[4] Rubakov V. A., Sazhin M. V., and Veryaskin A. V., *Graviton creation in the inflationary Universe and the grand unification scale*, Phys. Lett., 115B, 189 (1983).

[5] Banks T., *Quantum gravity, the cosmological constant and all that*, Nucl. Phys., B249, 332 (1985).

[6] Kiefer C., and Singh T. P., *Quantum gravitational corrections to the functional Schrödinger equation*, Phys. Rev., D44, 1067 (1991).

[7] Steinhardt P. J., *Cosmology confronts the cosmic microwave background*, Int. J. Mod. Phys., A10, 1091 (1995).

[8] Mandolesi N., et al., *COBRAS/SAMBA: the ESA medium size mission for measurements of CBR anisotropy*, Planet. Space Sci., 43, 1459 (1995).

[9] There are also other boundary proposals leading to this, see e.g., H. D. Conradi, *Initial state in quantum cosmology*, Phys. Rev. D46, 612 (1992).

[10] Hartle J. B., and Hawking, S. W., *Wave function of the Universe*, Phys. Rev., D28, 2960 (1983).

[11] Allen B., *Stochastic gravity-wave background in inflationary-universe models*, Phys. Rev., D37, 2078 (1988).