General Mean Reflected BSDEs

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Abstract

The present paper is devoted to the study of backward stochastic differential equations with mean reflection formulated by Briand et al. [7]. We investigate the solvability of a generalized mean reflected BSDE, whose driver also depends on the distribution of the solution term \( Y \). Using a fixed-point argument, BMO martingale theory and the \( \theta \)-method, we establish the existence and uniqueness result for such BSDEs in several typical situations, including the case where the driver is quadratic with bounded or unbounded terminal condition.

Key words: mean reflection, fixed-point method, \( \theta \)-method

MSC-classification: 60H10, 60H30

1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a given complete probability space under which \( B \) is a \( d \)-dimensional standard Brownian motion. Suppose \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the corresponding natural filtration augmented by the \( \mathbb{P} \)-null sets and \( \mathcal{P} \) is the sigma algebra of progressive sets of \( \Omega \times [0, T] \). In this paper, we consider the following backward stochastic differential equation (BSDE) with mean reflection:

\[
\begin{align*}
Y_t &= \xi + \int_t^T f(s, Y_s, P_{Y_s}, Z_s) \, ds - \int_t^T Z_s \, dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
\mathbb{E}[\ell(t, Y_t)] &\geq 0, \quad \forall t \in [0, T] \quad \text{and} \quad \int_0^T \mathbb{E}[\ell(t, Y_t)] \, dK_t = 0, 
\end{align*}
\]

where \( P_{Y_t} \) is the marginal probability distribution of the process \( Y \) at time \( t \), the terminal condition \( \xi \) is a scalar-valued \( \mathcal{F}_T \)-measurable random variable, the driver \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \times \mathbb{R}^d \to \mathbb{R} \), and the running loss function \( \ell : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \) are measurable maps with respect to \( \mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathcal{P}_1(\mathbb{R})) \times \mathcal{B}(\mathbb{R}^d) \) and \( \mathcal{F}_T \times \mathcal{B}([0, T]) \times \mathcal{B}(\mathbb{R}) \) respectively. Our aim is to prove that the mean reflected BSDE \((1)\) admits a unique deterministic solution \((Y, Z, K)\), in the sense that \( K \) is a deterministic, non-decreasing, and continuous process starting from the origin.

BSDEs with mean reflection were first introduced by Briand et al. in [7] to deal with the super-hedging problem under running risk management constraints. When the driver \( f \) is independent of

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the probability distribution of \(Y_t\), the authors of [7] established the existence and uniqueness of the deterministic solution \(K\) to the mean reflected BSDE (1) based on the following representation
\[
K_t = \sup_{0 \leq s \leq T} L_s \left( \mathbb{E}_s \left[ \xi + \int_t^T f(r, Y_r, Z_r)dr \right] \right) - \sup_{t \leq s \leq T} L_s \left( \mathbb{E}_s \left[ \xi + \int_t^T f(r, Y_r, Z_r)dr \right] \right), \quad \forall t \leq T, \tag{2}
\]
where the map \(t \mapsto L_t\) is given by (4) for \(t \in [0, T]\). With the help of this representation result, they were able to construct a contraction mapping when \(f\) is uniformly Lipschitz continuous in both variables \(Y, Z\). For more details on this topic, we refer the reader to [4, 5, 10, 14] and the references therein.

In particular, combining BMO martingale theory and a fixed-point method, Hibon et al. [12] extended the results from [7] to the case with bounded terminal condition, when the driver \(f\) is allowed to have quadratic growth in the second unknown \(z\). However, in order to estimate the solution \(K\) with the representation (2), they need to assume the following additional condition on the driver:
\[
(t, y) \mapsto f(t, y, 0) \text{ is uniformly bounded,}
\]
which is not necessary for the solvability of quadratic BSDEs with bounded terminal conditions (see [6, 16, 17]).

One of our motivations is to remove this additional assumption. The key point of our fixed-point method is based on the following representation result:
\[
K_t = \sup_{0 \leq s \leq T} L_s(y_s) - \sup_{t \leq s \leq T} L_s(y_s), \quad \forall t \leq T, \tag{3}
\]
where \(y\) denotes the solution to the following BSDE
\[
y_t = \xi + \int_t^T f(s, Y_s, P_{Y_s}, z_s)ds - \int_t^T z_s dB_s.
\]
Compared with (2), the representation result (3) for the deterministic solution \(K\) does not explicitly involve the term \(Z\). We can then make use of relevant BSDE techniques to estimate the solution \(K\) and establish existence and uniqueness of the solution to the quadratic mean reflected BSDE (1) without this additional assumption.

Moreover, this method can also be used to solve BSDEs with mean reflection under weak assumptions on the data. Indeed, with the help of the corresponding BSDE theory and the representation result (3), we make a counterpart study for the case where the driver is Lipschitz and the terminal condition admits a \(p\)th-order moment. We also tackle the situation with quadratic driver and unbounded terminal condition. In the first case, we apply the representation result (3) and a linearization technique to derive a priori estimates and build a contraction mapping.

Note that the comparison theorem does not hold for mean reflected BSDEs (see [12]). Thus the monotone convergence argument is quite restrictive for quadratic BSDEs with mean reflection, which differs from the quadratic BSDEs case, see, e.g., [2, 8, 16]. Borrowing some ideas from [9, 11], we use the representation result (3) and a \(\theta\)-method to give a successive approximation procedure when the driver is quadratic and the terminal condition admits exponential moments of arbitrary order.

The main contribution herein is that we introduce a new representation result to develop the mean reflected BSDEs theory. In particular, we establish the well-posedness of equation (1) with mean reflection for several typical situations. Compared to [12], our argument also removes the additional condition in the quadratic case with bounded terminal condition.

The paper is organized as follows. In section 2, we start with the Lipschitz case to illustrate the main idea. Section 3 is devoted to the study of the quadratic case with bounded terminal condition. We remove the additional boundedness condition in Section 4 assuming convexity on the driver.
Notations.

For each Euclidian space, we denote by $\langle \cdot, \cdot \rangle$ and $| \cdot |$ its scalar product and the associated norm, respectively. Then, for each $p \geq 1$, we consider the following collections:

- $L^p$ is the collection of real-valued $\mathcal{F}_T$-measurable random variables $\xi$ satisfying
  $$\| \xi \|_{L^p} = \mathbb{E} \left[ |\xi|^p \right]^{\frac{1}{p}} < \infty;$$
- $L^\infty$ is the collection of real-valued $\mathcal{F}_T$-measurable random variables $\xi$ satisfying
  $$\| \xi \|_{L^\infty} = \text{ess sup}_{\omega \in \Omega} |\xi(\omega)| < \infty;$$
- $H^{p,d}$ is the collection of $\mathbb{R}^d$-valued $\mathcal{F}$-progressively measurable processes $(z_t)_{0 \leq t \leq T}$ satisfying
  $$\| z \|_{H^p} = \mathbb{E} \left[ \left( \int_0^T |z_t|^2 \, dt \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} < \infty;$$
- $S^p$ is the collection of real-valued $\mathcal{F}$-adapted continuous processes $(y_t)_{0 \leq t \leq T}$ satisfying
  $$\| y \|_{S^p} = \mathbb{E} \left[ \sup_{t \in [0,T]} |y(t)|^p \right]^{\frac{1}{p}} < \infty;$$
- $S^\infty$ is the collection of real-valued $\mathcal{F}$-adapted continuous processes $(y_t)_{0 \leq t \leq T}$ satisfying
  $$\| y \|_{S^\infty} = \text{ess sup}_{(t,\omega) \in [0,T] \times \Omega} |y(t,\omega)| < \infty;$$
- $P_p(\mathbb{R})$ is the collection of all probability measures over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite $p^{th}$ moment, endowed with the $p$-Wasserstein distance $W_p$;
- $L^p$ is the collection of real-valued $\mathcal{F}_T$-measurable random variables $\xi$ satisfying $\mathbb{E} \left[ e^{|\xi|} \right] < \infty$;
- $S^p$ is the collection of all stochastic processes $Y$ such that $e^Y \in S^p$;
- $L$ is the collection of all random variables $\xi \in L^p$ for any $p \geq 1$, and $H^d$ and $S$ are defined similarly;
- $\mathcal{A}$ is the collection of deterministic, non-decreasing, and continuous processes $(K_t)_{0 \leq t \leq T}$ starting from the origin, i.e. $K_0 = 0$;
- $\mathcal{T}_t$ is the collection of $[0,T]$-valued $\mathcal{F}$-stopping times $\tau$ such that $\tau \geq t \mathbb{P}$-a.s.;
- $BMO$ is the collection of $\mathbb{R}^d$-valued progressively measurable processes $(z_t)_{0 \leq t \leq T}$ such that
  $$\| z \|_{BMO} := \sup_{\tau \in \mathcal{T}_0} \text{ess sup}_{\omega \in \Omega} \mathbb{E} \left[ \int_{\tau}^T |z_s|^2 \, ds \right]^{\frac{1}{2}} < \infty.$$

We denote by $\ell_{[a,b]}$ the corresponding collections for the stochastic processes with time indexes on $[a,b]$ for $\ell = H^{p,d}, S^p, S^\infty$ and so on. For each $Z \in BMO$, we set
$$\delta(Z \cdot B)_0^T = \exp \left( \int_0^T Z_s dB_s - \frac{1}{2} \int_0^T |Z_s|^2 \, ds \right),$$
which is a martingale by [15]. Thus it follows from Girsanov’s theorem that $(B_t - \int_0^t Z_s \, ds)_{0 \leq t \leq T}$ is a Brownian motion under the equivalent probability measure $\delta(Z \cdot B)_0^T \, d\mathbb{P}$. 3


2 Lipschitz case

In this section, we study the solvability of the mean reflected BSDE \( Y \) with Lipschitz generator and \( p \)-integrable terminal condition.

**Definition 2.1** By a deterministic solution to \( Y \), we mean a triple of progressively measurable processes \((Y, Z, K) \in S^p \times \mathcal{H}_{p,d} \times \mathcal{A}\) such that \( Y \) holds for some \( p > 1 \).

In what follows, we make use of the following conditions on the terminal condition \( \xi \), the generator \( f \) and the running loss function \( \ell \).

- **(H1)** There exists \( p > 1 \) such that \( \xi \in \mathcal{L}^p \) with \( E[\ell(T, \xi)] \geq 0 \).

- **(H2)** The process \((f(t, 0, \delta_0, 0))\) belongs to \( \mathcal{H}^{p,1} \) and there exists a constant \( \lambda > 0 \) such that for any \( t \in [0, T], y_1, y_2 \in \mathbb{R}, v_1, v_2 \in \mathcal{P}_1(\mathbb{R}) \), and \( z_1, z_2 \in \mathbb{R}^d \),

\[
|f(t, y_1, v_1, z_1) - f(t, y_2, v_2, z_2)| \leq \lambda(|y_1 - y_2| + W_1(v_1, v_2) + |z_1 - z_2|).
\]

- **(H3)** There exists a constant \( L > 0 \) such that,

1. \((t, y) \to \ell(t, y)\) is continuous,
2. \( \forall t \in [0, T], y \to \ell(t, y)\) is strictly increasing,
3. \( \forall t \in [0, T], E[\ell(t, \infty)] > 0 \),
4. \( \forall t \in [0, T], \forall y \in \mathbb{R}, |\ell(t, y)| \leq L(1 + |y|) \).

- **(H4)** There exist two constants \( \kappa > 1 \) and \( C > 0 \) such that for each \( t \in [0, T] \) and \( y_1, y_2 \in \mathbb{R} \),

\[
C|y_1 - y_2| \leq |\ell(t, y_1) - \ell(t, y_2)| \leq \kappa C|y_1 - y_2|.
\]

In order to study mean reflected BSDEs, we introduce the following map \( L_t : \mathcal{L}^1 \to \mathbb{R} \) for each \( t \in [0, T] \):

\[
L_t(\eta) = \inf\{x \geq 0 : E[\ell(t, x + \eta)] \geq 0\}, \forall \eta \in \mathcal{L}^1.
\]

When assumption \( (H3) \) is satisfied, the map \( X \mapsto L_t(X) \) is well-defined, see \[1\]. In particular, \( L_t(0) \) is continuous in \( t \). Moreover, if assumption \( (H4) \) is also fulfilled, then for each \( t \in [0, T] \),

\[
|L_t(\eta^1) - L_t(\eta^2)| \leq \kappa E[|\eta^1 - \eta^2|], \forall \eta^1, \eta^2 \in \mathcal{L}^1.
\]

**Remark 2.2** Remark that one can use the map \( X \mapsto L_t(X) \) to construct the term \( K \) via a standard BSDE involving the term \( Y \), which is crucial for solving the mean reflected BSDEs, see Lemma 2.5.

We are now ready to state the main result of this section.

**Theorem 2.3** Assume that \( (H1)-(H4) \) are fulfilled. Then the quadratic mean reflected BSDE \( Y \) admits a unique deterministic solution \((Y, Z, K) \in S^p \times \mathcal{H}_{p,d} \times \mathcal{A}\).

**Remark 2.4** Using a fixed-point method, Briand et al. \[5\] established the well-posedness of mean field BSDEs \[1\] in the case that \( p = 2 \). Note that the driver furthermore depends on the distribution of the first component \( Y \) of the solution in our framework.

In order to prove Theorem 2.3, we introduce a representation result for the solution to the problem \[4\], which plays a key role in establishing the existence and uniqueness result.
Lemma 2.5 Suppose Assumptions (H1)-(H3) hold. Let \((Y, Z, K) \in S^p \times H^{p,d} \times A\) be a deterministic solution to the BSDE with mean reflection \(\mathbb{1}\). Then, for each \(t \in [0,T]\)

\[
(Y_t, Z_t, K_t) = \left( y_t + \sup_{t \leq s \leq T} L_s(y_s), z_t, \sup_{0 \leq s \leq T} L_s(y_s) - \sup_{t \leq s \leq T} L_s(y_s) \right)
\]

where \((y, z) \in S^p \times H^{p,d}\) is the solution to the following BSDE with the driver \(f^Y(s, z) = f(s, Y_s, P_{Y_s}, z)\) on the time horizon \([0,T]\):

\[
y_t = \xi + \int_t^T f^Y(s, z_s) ds - \int_t^T z_s dB_s.
\]  

(6)

Proof. It follows from \([1, \text{Theorem 4.2}]\) that the BSDE (6) admits a unique solution \((y, z) \in S^p \times H^{p,d}\).

Since \(K\) is a deterministic process, \((Y, (K_T - K_t), Z)\) is again a \(S^p \times H^{p,d}\)-solution to the BSDE (6), which implies that

\[
(Y_t, Z_t) = (y_t + K_T - K_t, z_t), \quad \forall t \in [0,T].
\]

On the other hand, \((Y, Z, K) \in S^p \times H^{p,d} \times A\) can also be regarded as a deterministic solution to the following mean reflected BSDE with fixed generator \(f(., Y, P_Y, Z)\).

\[
\begin{aligned}
&\tilde{Y}_t = \xi + \int_t^T f(s, Y_s, P_{Y_s}, Z_s) ds - \int_t^T \tilde{z}_s dB_s + \tilde{K}_T - \tilde{K}_t, \quad 0 \leq t \leq T, \\
&E[\ell(t, \tilde{Y}_t)] \geq 0, \quad \forall t \in [0,T] \quad \text{and} \quad \int_0^T E[\ell(t, \tilde{Y}_t)] d\tilde{K}_t = 0.
\end{aligned}
\]

Note that \(y_t = E_t[\xi] + \int_t^T f(s, Y_s, P_{Y_s}, Z_s) ds\) for any \(t \in [0, T]\). Thus, recalling \([2, \text{Proposition 7}]\), we have \(K_t = \sup_{0 \leq s \leq T} L_s(y_s) - \sup_{t \leq s \leq T} L_s(y_s)\) for each \(t \in [0,T]\). This concludes the proof. \(\blacksquare\)

Next, we use a linearization technique and a fixed-point argument to get existence and uniqueness of the solution. For this purpose, we need to introduce the following solution map \(\Gamma\) defined for \(U \in S^p\) by \(\Gamma(U) = Y\) where \(Y\) is the first component of the solution \((Y, Z, K)\) to the following mean reflected BSDE with driver \(f^U\):

\[
\begin{aligned}
&Y_t = \xi + \int_t^T f^U(s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \\
&E[\ell(t, Y_t)] \geq 0, \quad \forall t \in [0,T] \quad \text{and} \quad \int_0^T E[\ell(t, Y_t)] dK_t = 0.
\end{aligned}
\]  

(7)

Lemma 2.6 Assume that (H1)-(H3) are satisfied and \(U \in S^p\). Then, the mean reflected BSDE (7) admits a unique solution \((Y, Z, K) \in S^p \times H^{p,d} \times A\).

Proof. The uniqueness is immediate from Lemma 2.5. It follows from \([1, \text{Theorem 4.2}]\) that BSDE (6) with the driver \(f(., U, P_U, z)\) has a unique solution \((y, z) \in S^p \times H^{p,d}\). Then in view of \([2, \text{Proposition 7}]\), we obtain that the following mean reflected BSDE with fixed generator \(f(., U, P_U, z)\)

\[
\begin{aligned}
&\tilde{Y}_t = \xi + \int_t^T f(s, U_s, P_{U_s}, z_s) ds - \int_t^T \tilde{z}_s dB_s + \tilde{K}_T - \tilde{K}_t, \quad 0 \leq t \leq T, \\
&E[\ell(t, \tilde{Y}_t)] \geq 0, \quad \forall t \in [0,T] \quad \text{and} \quad \int_0^T E[\ell(t, \tilde{Y}_t)] dK_t = 0
\end{aligned}
\]  

(8)

has a unique solution \((Y, Z, K) \in S^p \times H^{p,d} \times A\). In light of the representation result (Lemma 2.5), we get \(z \equiv Z\) and thus \((Y, Z, K) \in S^p \times H^{p,d} \times A\) is the solution to the mean reflected BSDE (7). \(\blacksquare\)

Let us now prove that \(\Gamma\) defines a contraction map on a small time interval \([T-h, T]\), in which \(h\) is to be determined later. Note that in the spirit of Lemma 2.6 we have \(\Gamma \left( S^p_{[T-h, T]} \right) \subset S^p_{[T-h, T]}\) for any \(h \in (0, T]\). 

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Lemma 2.7 Assume that (H1)-(H4) hold. Then there exists a constant \( \delta > 0 \) depending only on \( p, \lambda \) and \( \kappa \) such that for any \( h \in (0, \delta) \), the mean reflected BSDE \( \text{(11)} \) admits a unique solution \( (Y, Z, K) \in S^p_{[T-h, T]} \times \mathcal{H}^p_{[T-h, T]} \times \mathcal{A}^p_{[T-h, T]} \) on the time interval \([T-h, T]\).

Proof. The proof is divided into three steps.

Step 1 (A priori estimate). The main idea is similar to \cite[Lemma 2.8]{14} and we give the sketch of the proof for readers’ convenience. Let \( U^i \in S^p_{[T-h, T]} \) for \( i = 1, 2 \). It follows from Lemma 2.5 that

\[
\Gamma(U^i)_t := y^i_t + \sup_{t \leq s \leq T} L_s(y^i_s), \quad \forall t \in [T-h, T],
\]

where \( y^i \) is the solution to the BSDE \( \text{(3)} \) with driver \( f^{U^i} \) and terminal condition \( \xi \). For each \( t \in [0, T] \), we denote

\[
\beta_t = \frac{f^{U^1}(t, z^1_t) - f^{U^1}(t, z^2_t)}{|z^1_t - z^2_t|^2} (z^1_t - z^2_t) \mathbf{1}_{\{|z^1_t - z^2_t| \neq 0\}}.
\]

Then, the pair of processes \((y^1_t - y^2_t, z^1_t - z^2_t)\) solves the following BSDE:

\[
y^1_t - y^2_t = \int_t^T \left( \beta_s (z^1_s - z^2_s) + f^{U^1}(s, z^2_s) - f^{U^2}(s, z^2_s) \right) ds - \int_t^T (z^1_s - z^2_s) dB_s.
\]

Since \( \bar{B}_t := B_t - \int_0^t \beta_s^T ds \) defines a Brownian motion under the equivalent probability measure \( \tilde{P} \) given by \( d\tilde{P} := \xi(s \cdot B_t) dP \), it follows that for every \( t \in [0, T] \),

\[
y^1_t - y^2_t = \mathbb{E}_t \left[ \xi(s \cdot B_T) \left( \int_t^T \left( f^{U^1}(s, z^2_s) - f^{U^2}(s, z^2_s) \right) ds \right) \right].
\]

Applying Hölder’s inequality yields, for any \( \mu \in (1, p) \) and any \( t \in [T-h, T] \),

\[
|y^1_t - y^2_t| \leq \exp \left( \frac{\lambda^2 h}{2(\mu - 1)} \right) \lambda h \mathbb{E}_t \left[ \left( \sup_{s \in [T-h, T]} |U^1_s - U^2_s| + \sup_{s \in [T-h, T]} \mathbb{E} \left[ |y^1_s - y^2_s| \right] \right)^\mu \right]^{\frac{1}{\mu}}.
\]

Step 2 (The contraction). Recalling \( \text{(9)} \) and \( \text{(10)} \), we have

\[
\sup_{s \in [T-h, T]} |\Gamma(U^1)_s - \Gamma(U^2)_s|^p \leq 2p^{-1} \left( \sup_{s \in [T-h, T]} |y^1_s - y^2_s|^p + \kappa^p \sup_{s \in [T-h, T]} \mathbb{E} \left[ |y^1_s - y^2_s|^p \right] \right).
\]

Recalling \( \text{(11)} \) and applying Doob’s maximal inequality, we derive

\[
\mathbb{E} \left[ \sup_{t \in [T-h, T]} |\Gamma(U^1)_t - \Gamma(U^2)_t|^p \right] \leq 2p^{-1} (1 + \kappa^p) \lambda^p h^p \exp \left( \frac{p \lambda^2 h}{2(\mu - 1)} \right) \times \left( \frac{p}{p - \mu} \right)^{\frac{1}{p}} \mathbb{E} \left[ \sup_{s \in [T-h, T]} |U^1_s - U^2_s| + \sup_{s \in [T-h, T]} \mathbb{E} \left[ |U^1_s - U^2_s|^p \right] \right]^{\frac{1}{p}}.
\]

Consequently, for any \( \mu \in (1, p) \) and \( h \in (0, \mu - 1) \), we have

\[
\mathbb{E} \left[ \sup_{t \in [T-h, T]} |\Gamma(U^1)_t - \Gamma(U^2)_t|^p \right] \leq \Lambda(\mu) \mathbb{E} \left[ \sup_{s \in [T-h, T]} |U^1_s - U^2_s|^p \right]^{\frac{1}{p}}.
\]
with
\[\Lambda(\mu) = 4(1 + \kappa)\lambda \exp \left(\frac{\lambda^2}{2}\right) \left(\frac{p}{p - \mu}\right)^{\frac{1}{p}} (\mu - 1).\]

Then we choose a small enough constant \(\mu^* \in (1, p)\) depending only on \(p, \lambda\) and \(\kappa\) such that \(\Lambda(\mu^*) < 1\) and set \(\delta := \mu^* - 1\). It follows that \(\Gamma\) is a contraction map on the time interval \([T - h, T]\) for any \(h \in (0, \delta]\).

**Step 3 (Uniqueness and existence).** The uniqueness is immediate from the fact that any solution to the mean reflected BSDE \((1)\) with bounded terminal condition is a fixed point of the map \(\Gamma\). For any \(h \in (0, \delta]\), the function \(\Gamma\) has a unique fixed point \(\tilde{Y} \in S_{[T-h,T]}^p\). Then the mean reflected BSDE \((7)\) with driver \(f^Y\) admits a unique solution \((\tilde{Y}, Z, K)^\) in \(S_{[T-h,T]}^p \times H_{[T-h,T]}^p \times A_{[T-h,T]}\). It immediately follows that \(\tilde{Y} = \Gamma(Y) = Y\), so \((Y, Z, K)^*\) is the desired solution to the mean reflected BSDE \((1)\) on the time interval \([T - h, T]\). This completes the proof.

We now prove the main result with the help of the intermediate lemmas above.

**Proof of Theorem 2.3** Note that the length of the time interval on which the map \(\Gamma\) is contractive depends only on \(p, \kappa\) and \(\lambda\). By a standard BSDE approach, we split the arbitrary time interval \([0, T]\) into a finite number of small time intervals. On each small time interval, we can then apply Lemma 2.7 to get a local solution. A global solution on the whole time interval is obtained by stitching the local ones. The global uniqueness on \([0, T]\) follows from the local uniqueness on each small time interval. The proof is complete.

**Remark 2.8** By more involved and delicate estimates, our method can still be applied to study Lipschitz mean reflected BSDEs when the driver depends on the distribution of \(\gamma\) and \(\delta\) and set \((H1')\)

\[\mu \in \mathcal{F}_{[T-h,T]}^p\]

**3 Bounded terminal condition**

In this section, we combine BMO martingale theory and a fixed-point argument in order to analyze the quadratic mean reflected BSDE \((1)\) with bounded terminal condition.

In what follows, we make use of the following conditions on the terminal condition \(\xi\) and the driver \(f\).

**(H1')** The terminal condition \(\xi \in L^\infty\) with \(\mathbb{E}[\ell(T, \xi)] \geq 0\).

**(H2')** The process \((f(t, 0, \delta_0, 0))\) is uniformly bounded and there exist two positive constants \(\beta\) and \(\gamma\) such that for any \(t \in [0, T]\), \(y_1, y_2 \in \mathbb{R}\), \(v_1, v_2 \in \mathcal{P}_1(\mathbb{R})\), and \(z_1, z_2 \in \mathbb{R}^d\),

\[|f(t, y_1, v_1, z_1) - f(t, y_2, v_2, z_2)| \leq \beta (|y_1 - y_2| + W_1(v_1, v_2)) + \gamma (1 + |z_1| + |z_2|)|z_1 - z_2|.

We are now ready to state the main result of this section.

**Theorem 3.1** Assume that \((H1'), (H2'), (H3)\) and \((H4)\) are satisfied. Then the quadratic BSDE \((1)\) with mean reflection admits a unique solution \((Y, Z, K)^* \in S^\infty \times BMO \times A\).

**Remark 3.2** In view of [13, Theorem 7.3.3] and [12, Theorem 3.1], Lemma 2.9 still holds under conditions \((H1'), (H2')\) and \((H3)\). When the driver does not depend on the distribution of \(Y\), the authors of [12] proved that quadratic mean reflected BSDEs admits a unique solution. Compared with that of [12], we apply Lemma 2.9 to remove the following additional assumption:

\[(t, y) \mapsto f(t, y, 0)\] is uniformly bounded.
As in the Lipschitz case, we will prove that the solution map $\Gamma$ defines a contraction map.

**Lemma 3.3** Assume that $(H1')$, $(H2')$ and $(H3)$ are satisfied and $U \in S^\infty$. Then, the quadratic BSDE (11) with mean reflection, with driver $f^{U}$, admits a unique solution $(Y, Z, K) \in S^\infty \times BMO \times A$.

**Proof.** It follows from [17, Theorem 7.3.3] that the quadratic BSDE (6) with the driver $f(\cdot, U, P_U, z)$ has a unique solution $(y, z) \in S^\infty \times BMO$. Then with the help of [12, Theorem 3.1], we have that the quadratic mean reflected BSDE (8) with the fixed generator $(Y, Z, K) \in S^\infty \times BMO \times A$. Recalling Remark 3.2 and Lemma 2.5, we derive that $z \equiv Z$ and $(Y, Z, K) \in S^\infty \times BMO \times A$ is the solution to the mean reflected BSDE (7). The uniqueness eventually follows from Lemma 2.5 which ends the proof. \[\square\]

We are now ready to state the proof of the main result of this section.

**Proof of Theorem 3.1.** Let $U^i \in S^\infty$, $i = 1, 2$. It follows from Lemma 2.5 that

$$\Gamma(U^i)_t := y^i_t + \sup_{t \leq s \leq T} L_s(y^i_s), \quad \forall t \in [T - h, T],$$

(11)

where $y^i$ is the solution to the quadratic BSDE (6) with driver $f^{U^i}$ and the terminal condition $\xi$. Following the proof of Lemma 2.7 step by step (noting that $(\beta_t) \in BMO$ in this case), we conclude that for any $t \in [0, T],$

$$y^1_t - y^2_t = E\left[ \int_T^t \left( f^{U^1}(s, z^1_s) - f^{U^2}(s, z^2_s) \right) ds \right],$$

which together with Assumption $(H2')$ implies that for any $t \in [T - h, T],$

$$|y^1_t - y^2_t| \leq \beta h \| U^1 - U^2 \|_{S^{\infty}_{[T-h,T]}} + \beta h \sup_{s \in [T-h,T]} E[|U^1_s - U^2_s|].$$

In view of (11) and (5), we again derive that

$$\| \Gamma(U^1) - \Gamma(U^2) \|_{S^{\infty}_{[T-h,T]}} \leq 2(1 + \kappa) \beta h \| U^1 - U^2 \|_{S^{\infty}_{[T-h,T]}}.$$

Then we can find a small enough constant $h$ depending only on $\beta$ and $\kappa$ such that $2(1 + \kappa) \beta h < 1$. Therefore, $\Gamma$ defines a contraction map on the time interval $[T - h, T]$. Proceeding exactly as in Theorem 2.3 and Lemma 2.7 we complete the proof. \[\square\]

Note that the process $(\beta_t)$ may be unbounded in the BMO space and then the fixed-point argument fails to work when the terminal condition is unbounded. In the next section, we make use of a $\theta$-method to overcome this difficulty under the further assumption of either convexity or concavity on the generator.

### 4 Unbounded terminal condition

In this section, we investigate the solvability of the mean reflected BSDE (11) with quadratic generator $f$ and unbounded terminal value $\xi$. In what follows, we make use of the following conditions on the parameters $\xi$ and $f$.

**\((H1'')\)** The terminal condition $\xi \in L$ with $E[\ell(T, \xi)] \geq 0$.

**\((H2'')\)** There exist a positive progressively measurable process $(\alpha_t)_{0 \leq t \leq T}$ with $\int_0^T \alpha_t dt \in L$ and two positive constants $\beta$ and $\gamma$ such that
1. $\forall (t, y, v, z) \in [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R}) \times \mathbb{R}^d$, $|f(t, y, v, z)| \leq \alpha_t + \beta(|y| + W_1(v, \delta_0)) + \frac{\gamma}{2} |z|^2$.
2. $\forall t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $v_1, v_2 \in \mathcal{P}_1(\mathbb{R})$, $z \in \mathbb{R}^d$,
   \[ |f(t, y_1, v_1, z) - f(t, y_2, v_2, z)| \leq \beta(|y_1 - y_2| + W_1(v_1, v_2)) \]
3. $\forall (t, y, v) \in [0, T] \times \mathbb{R} \times \mathcal{P}_1(\mathbb{R})$, $z \to f(t, y, v, z)$ is convex or concave.

Let us now state the main result of this section.

**Theorem 4.1** Assume that (H1’), (H2’), (H3) and (H4) are fulfilled. Then the quadratic mean reflected BSDE (1) admits a unique deterministic solution $(Y, Z, K) \in \mathcal{S} \times H^d \times A$.

**Remark 4.2** Note that in view of [9, Corollary 6] and [7, Proposition 7], the representation result in Lemma 2.5 still holds under (H1’), (H2’) and (H3).

In order to prove Theorem 4.1 we need to recall some technical results on quadratic BSDEs. Consider the following standard BSDE on the time horizon $[0, T]$

$$y_t = \eta + \int_t^T g(s, z_s)ds - \int_t^T z_s dB_s.$$ (12)

The following result is important for our subsequent computations, and can be found in [11, Lemmas A3 and A4].

**Lemma 4.3** Assume that $(y, z) \in \mathcal{S}^2 \times \mathcal{H}^{2,d}$ is a solution to (12). Suppose that there is a constant $p \geq 1$ such that

$$E\left[ \exp\left\{ 2p\gamma \sup_{t \in [0, T]} |y_t| + 2p\gamma \int_0^T \alpha_t dt \right\} \right] < \infty.$$ 

Then, we have

(i) If $|g(t, z)| \leq \alpha_t + \frac{\gamma}{2} |z|^2$, then for each $t \in [0, T]$,

$$\exp\{p\gamma |y_t|\} \leq E_t\left[ \exp\left\{ p\gamma |\eta| + p\gamma \int_t^T \alpha_s ds \right\} \right].$$

(ii) If $g(t, z) \leq \alpha_t + \frac{\gamma}{2} |z|^2$, then for each $t \in [0, T]$,

$$\exp\{p\gamma (y_t)^+\} \leq E_t\left[ \exp\left\{ p\gamma |\eta| + p\gamma \int_t^T \alpha_s ds \right\} \right].$$

We are now ready to combine the $\theta$-method and the representation result to prove Theorem 4.1. In order to illustrate the main idea, we first deal with the uniqueness.

**Lemma 4.4** Assume that all the conditions of Theorem 4.1 are satisfied. Then, the quadratic mean reflected BSDE (1) has at most one deterministic solution $(Y, Z, K) \in \mathcal{S} \times \mathcal{H}^d \times A$. 

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Proof. For \( i = 1, 2 \), let \((Y^i, Z^i, K^i)\) be a deterministic \( S \times \mathcal{H}^d \times \mathcal{A}\)-solution to the quadratic mean reflected BSDE \((\mathcal{H})\). From Lemma 2.5 and Remark 4.2, we have

\[
Y^i_t := y^i_t + \sup_{t \leq s \leq T} L_s(y^i_s), \quad \forall t \in [0, T],
\]

where \((y^i, z^i) \in S \times \mathcal{H}^d\) is the solution to the following quadratic BSDE:

\[
y^i_t = \xi + \int_t^T f\left(s, Y^i_s, P_{Y^i_s}, z^i_s\right) ds - \int_t^T z^i_s dB_s.
\]

Assume without loss of generality that \( f(t, y, v, \cdot) \) is concave (see Remark 4.5). For each \( \theta \in (0, 1) \), we denote

\[
\delta_\theta \ell = \frac{\theta \ell^1 - \ell^2}{1 - \theta}, \quad \delta_\theta \bar{\ell} = \frac{\theta \ell^2 - \ell^1}{1 - \theta} \quad \text{and} \quad \delta_\theta \ell := |\delta_\theta \ell| + |\delta_\theta \bar{\ell}|
\]

for \( \ell = Y, y \) and \( z \). Then, the pair of processes \((\delta_\theta y_t, \delta_\theta z_t)\) satisfies the following BSDE:

\[
\delta_\theta y_t = -\xi + \int_t^T (\delta_\theta f(s, \delta_\theta z_s) + \delta_\theta f_0(s)) ds - \int_t^T \delta_\theta z_s dB_s,
\]

where the generator is given by

\[
\delta_\theta f_0(t) = \frac{1}{1 - \theta} \left( f(t, Y^1_s, P_{Y^1_s}, z^1_s) - f(t, Y^2_s, P_{Y^2_s}, z^2_s) \right),
\]

\[
\delta_\theta f(t, z) = \frac{1}{1 - \theta} \left( \theta f(t, Y^1_s, P_{Y^1_s}, z^1_s) - f(t, Y^2_s, P_{Y^2_s}, -(1 - \theta)z + \theta z^1_s) \right).
\]

Recalling assumptions \((H2^\circ)\), we have that

\[
\delta_\theta f_0(t) \leq \beta \left( |Y^1_s| + |\delta_\theta Y_1| + \mathbb{E}[|Y^1_s|] + |\delta_\theta Y_1| \right),
\]

\[
\delta_\theta f(t, z) \leq -f(t, Y^1_s, P_{Y^1_s}, -z) \leq \alpha_1 + \beta \left( |Y^1_s| + \mathbb{E}[|Y^1_s|] + \frac{\gamma}{2} |z|^2 \right).
\]

Set \( C_1 := \sup_{s \in [0, T]} \mathbb{E}[|Y^1_s|] + |Y^2_s| \) and

\[
\chi = \int_0^T \alpha_s ds + 2\beta C_1 T + 2\beta T \left( \sup_{s \in [0, T]} |Y^1_s| + \sup_{s \in [0, T]} |Y^2_s| \right),
\]

\[
\bar{\chi} = \int_0^T \alpha_s ds + 2\beta C_1 T + 2\beta T \left( \sup_{s \in [0, T]} |Y^1_s| + \sup_{s \in [0, T]} |Y^2_s| \right) + \sup_{s \in [0, T]} |y^1_s| + \sup_{s \in [0, T]} |y^2_s|.
\]

Using assertion (ii) of Lemma 4.3 to 4.4, we derive that for any \( p \geq 1 \),

\[
\exp \left\{ p \gamma (\delta_\theta y_t)^+ \right\} \leq \mathbb{E}_t \left[ \exp \left\{ p \gamma \left( |\xi| + \chi + \beta (T - t) \left( \sup_{s \in [t, T]} |\delta_\theta Y_s| + \sup_{s \in [t, T]} \mathbb{E}[|\delta_\theta Y_s|] \right) \right) \right\}.
\]

Similarly, we have

\[
\exp \left\{ p \gamma (\delta_\theta y_t)^+ \right\} \leq \mathbb{E}_t \left[ \exp \left\{ p \gamma \left( |\xi| + \chi + \beta (T - t) \left( \sup_{s \in [t, T]} |\delta_\theta Y_s| + \sup_{s \in [t, T]} \mathbb{E}[|\delta_\theta Y_s|] \right) \right) \right\}.
\]
In view of the fact that
\[(δθy)^− \leq (δθy)^+ + 2|y|^2 \text{ and } (δθy)^− \leq (δθy)^+ + 2|y|^2,\]
we have
\[
\exp \{pγ |δθy_t| \} \vee \exp \{pγ |δθy_t| \} \leq \exp \left\{ pγ \left( (δθy_t)^+ + (δθy_t)^+ + 2|y|^2 \right) \right\}
\leq \mathbb{E}_t \left\{ \exp \left\{ pγ \left( |ξ| + \bar{χ} + \beta(T - t) \sup_{s \in [t, T]} δθ\overline{Y}_s + \sup_{s \in [t, T]} \mathbb{E}[δθ\overline{Y}_s] \right) \right\} \right\}^2.
\]

Applying Doob’s maximal inequality and Hölder’s inequality, we get that for each \(p \geq 1 \text{ and } t \in [0, T],\)
\[
\mathbb{E} \left\{ \exp \left\{ pγ \sup_{s \in [t, T]} δθ\overline{Y}_s \right\} \right\} \leq \mathbb{E} \left\{ \exp \left\{ pγ \sup_{s \in [t, T]} |δθy_s| \right\} \right\} \exp \left\{ pγ \sup_{s \in [t, T]} |δθy_s| \right\}
\leq 4\mathbb{E} \left\{ \exp \left\{ 4pγ \left( |ξ| + \bar{χ} + \beta(T - t) \sup_{s \in [t, T]} δθ\overline{Y}_s + \sup_{s \in [t, T]} \mathbb{E}[δθ\overline{Y}_s] \right) \right\} \right\} \text{ (15)}.
\]

Set \(C_2 := \sup_{0 \leq s \leq T} |L_s(0)| + 2κ \sup_{s \in [0, T]} \mathbb{E}[|y_s^0| + |y_s^2|].\) Recalling (13) and assumption (H4), we derive that
\[
|δθY_t| \leq C_2 + |δθy_t| + κ \sup_{t \leq s \leq T} \mathbb{E}[|δθy_s|] \text{ and } |δθ\overline{Y}_t| \leq C_2 + |δθ\overline{y}_t| + κ \sup_{t \leq s \leq T} \mathbb{E}[|δθ\overline{y}_s|], \forall t \in [0, T],
\]
which together with Jensen’s inequality implies that for each \(p \geq 1 \text{ and } t \in [0, T],\)
\[
\mathbb{E} \left\{ \exp \left\{ pγ \sup_{s \in [t, T]} δθ\overline{Y}_s \right\} \right\} \leq e^{2pγC_2} \mathbb{E} \left\{ \exp \left\{ pγ \sup_{s \in [t, T]} δθ\overline{Y}_s \right\} \right\} \mathbb{E} \left\{ \exp \left\{ 2pγ \sup_{s \in [t, T]} δθ\overline{Y}_s \right\} \right\}
\leq e^{2pγC_2} \mathbb{E} \left\{ \exp \left\{ (2 + 4κ)pγ \sup_{s \in [t, T]} δθ\overline{Y}_s \right\} \right\}
\leq 4\mathbb{E} \left\{ \exp \left\{ (8 + 16κ)pγ \left( |ξ| + \bar{χ} + C_2 + \beta(T - t) \sup_{s \in [t, T]} δθ\overline{Y}_s + \sup_{s \in [t, T]} \mathbb{E}[δθ\overline{Y}_s] \right) \right\} \right\} \text{ (16)}
\leq 4\mathbb{E} \left\{ \exp \left\{ (8 + 16κ)pγ \left( |ξ| + \bar{χ} + C_2 + \beta(T - t) \sup_{s \in [t, T]} δθ\overline{Y}_s \right) \right\} \right\}
\times \mathbb{E} \left\{ \exp \left\{ (8 + 16κ)pγ \beta(T - t) \sup_{s \in [t, T]} δθ\overline{Y}_s \right\} \right\},
\]
where we have used (13) in the third inequality.

Choose a constant \(h \in (0, T] \) depending only on \(β \) and \(κ \) such that \((16 + 32κ)βh < 1.\) In the spirit of Hölder’s inequality, we derive that for any \(p \geq 1,\)
\[
\mathbb{E} \left\{ \exp \left\{ pγ \sup_{s \in [T - h, T]} δθ\overline{Y}_s \right\} \right\}
\leq 4\mathbb{E} \left\{ \exp \left\{ (16 + 32κ)pγ (|ξ| + \bar{χ} + C_2) \right\} \right\} \frac{1}{2} \mathbb{E} \left\{ \exp \left\{ (16 + 32κ)βh pγ \sup_{s \in [T - h, T]} δθ\overline{Y}_s \right\} \right\}
\leq 4\mathbb{E} \left\{ \exp \left\{ (16 + 32κ)pγ (|ξ| + \bar{χ} + C_2) \right\} \right\} \mathbb{E} \left\{ \exp \left\{ pγ \sup_{s \in [T - h, T]} δθ\overline{Y}_s \right\} \right\}^{(16 + 32κ)βh},
\]

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which together with the fact that \((16 + 32\kappa)\beta h < 1\) implies that for any \(p \geq 1\) and \(\theta \in (0, 1)\)

\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [T-h,T]} \delta_\theta Y_s \right\} \right] \leq E \left[ 4 \exp \left\{ (16 + 32\kappa)p\gamma (|\xi| + \bar{\chi} + C_2) \right\} \right]^\frac{1}{1-16+32\kappa}\beta h) < \infty.
\]

Note that \(Y^1 - Y^2 = (1 - \theta)(\delta_\theta Y + Y^1)\). It follows that

\[
E \left[ \sup_{t \in [T-h,T]} |Y^1_t - Y^2_t| \right] \leq (1 - \theta) \left( \frac{1}{\gamma} \sup_{\theta \in (0, 1)} E \left[ \exp \left\{ \gamma \sup_{s \in [T-h,T]} \delta_\theta Y_s \right\} \right] \right) + E \left[ \sup_{t \in [0,T]} |Y^1_t| \right].
\]

Letting \(\theta \to 1\) yields \(Y^1 = Y^2\). Applying Itô’s formula to \(|Y^1 - Y^2|^2\) yields \((Z^1, K^1) = (Z^2, K^2)\) on \([T - h, T]\). The uniqueness of the solution on the whole interval is inherited from the uniqueness on each small time interval. The proof is complete. ■

**Remark 4.5** In the convex case, one should use \(\ell_1 - \theta \ell_2\) and \(\ell_2 - \theta \ell_1\) instead of \(\theta \ell_1 - \ell_2\) and \(\theta \ell_2 - \ell_1\) in the definition of \(\delta_\theta \ell\) and \(\delta_\theta \ell^c\), respectively. Then the generator of BSDE \((14)\) satisfies

\[
\delta_\theta f(t, z) \leq f(t, Y^2_t, P_{Y^2_t}, z) - \alpha_t + \beta(|Y^2_t| + E[|Y^2_t|]) + \frac{\gamma}{2} z^2.
\]

One can check that \((15)\) and \((16)\) still hold in this context.

**Remark 4.6** Due to the presence of mean reflection, one cannot directly apply the \(\theta\)-method to establish the desired estimates for quadratic mean reflected BSDEs with unbounded terminal condition as in \((7)\). With the help of Lemma \((2.5)\) we could overcome this difficulty by analyzing a standard quadratic BSDE. In a similar way, \((14)\) established the well-posedness of quadratic mean-field reflected BSDEs with unbounded terminal condition via nonlinear Snell envelope representation and quadratic BSDEs techniques.

We now turn to the existence part of our result.

**Lemma 4.7** Assume that all the conditions of Theorem \((4.1)\) hold and \(U \in \mathcal{S}\). Then, the quadratic mean reflected BSDE \((7)\), with driver \(f^U\), admits a unique solution \((Y, Z, K) \in \mathcal{S} \times \mathcal{H}^d \times \mathcal{A}\).

**Proof.** The uniqueness follows from Lemma \((2.5)\) and Remark \((4.2)\). In view of assumption \((H2^\prime)\), we have

\[
|f(t, U_t, P_{U_t}, z)| \leq \alpha_t + \beta(|U_t| + E[|U_t|]) + \frac{\gamma}{2} z^2.
\]  

(17)

It follows from \((6), Corollary 6\) that the BSDE \((7)\) admits a unique solution \((y, z) \in \mathcal{S} \times \mathcal{H}^d\). Then it follows from \((14), Proposition 7\) that the mean reflected BSDE \((8)\) with fixed driver \(f(\cdot, U, P_{U_t}, z)\) has a unique deterministic solution \((Y, Z, K) \in \mathcal{S} \times \mathcal{H}^d \times \mathcal{A}\). In the spirit ofLemma \((2.5)\) and Remark \((4.2)\) we conclude that \(z = Z\), which implies that \((Y, Z, K)\) is the desired solution. This completes the proof. ■

According to Lemma \((4.7)\) we recursively define a sequence of stochastic processes \((Y^{(m)})_{m=1}^\infty\) through the following quadratic BSDE with mean reflection:

\[
\begin{cases}
Y^{(m)}_t = \xi + \int_t^T f(s, Y^{(m-1)}_s, P_{Y^{(m-1)}_s}, Z^{(m-1)}_s)ds - \int_t^T Z^{(m)}_s dB_s + K^{(m)}_t - K^{(m)}_t, & 0 \leq t \leq T, \\
E[f(t, Y^{(m)}_t)] \geq 0, & \forall t \in [0, T] \text{ and } \int_0^T E[f(t, Y^{(m)}_t)]dK^{(m)}_t = 0,
\end{cases}
\]

(18)

where \(Y^{(0)} = 0\). It is obvious that \((Y^{(m)}, Z^{(m)}, K^{(m)}) \in \mathcal{S} \times \mathcal{H}^d \times \mathcal{A}\).

Next, we apply a \(\theta\)-method and BSDE techniques to prove that \((Y^{(m)}, Z^{(m)}, K^{(m)})\) defines a Cauchy sequence: the corresponding limit is the desired solution. We need the following technical results to complete the proof, whose proofs are postponed to the Appendix.
Lemma 4.8 Assume that the conditions of Theorem 4.1 are fulfilled. Then, for any \( p \geq 1 \), we have
\[
\sup_{m \geq 0} \mathbb{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0,T]} \left| Y_s^{(m)} \right| \right\} + \left( \int_0^T \left| Z_t^{(m)} \right|^2 dt \right)^p + \left| K_T^{(m)} \right| \right] < \infty.
\]

Lemma 4.9 Assume that all the conditions of Theorem 4.1 are satisfied. Then, for any \( p \geq 1 \), we have
\[
\Pi(p) := \sup_{m \to \infty} \lim_{q \to 1} \sup_{\theta \in (0,1)} \mathbb{E} \left[ \exp \left\{ p \gamma \sup_{s \in [0,T]} \left| \delta_q Y_s^{(m,q)} \right| \right\} \right] < \infty,
\]
where we use the following notations
\[
\delta_q Y^{(m,q)} = \frac{\theta Y^{(m+q)} - Y^m}{1 - \theta}, \quad \delta_q Y^{(m,q)} = \frac{\theta Y^{(m)} - Y^{(m+q)}}{1 - \theta}, \quad \text{and} \quad \delta_q Y := |\delta_q Y^{(m,q)}| + |\delta_q Y^{(m,q)}|.
\]

We are now in a position to complete the proof of the main result.

Proof of Theorem 4.1. It suffices to prove the existence. Note that for any integer \( p \geq 1 \) and for any \( \theta \in (0, 1) \),
\[
\lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^p \right] \leq 2^{p-1} (1 - \theta)^p \left( \frac{\Pi(1)p!}{\gamma^p} + \sup_{m \geq 1} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y_t^{(m)} \right|^p \right] \right),
\]
which together with Lemmas 4.8, 4.9 and the arbitrariness of \( \theta \) implies that
\[
\lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^p \right] = 0, \quad \forall p \geq 1.
\]

Applying Itô’s formula to \( Y_t^{(m+q)} - Y_t^{(m)} \)
\[
\mathbb{E} \left[ \int_0^T \left| Z_t^{(m+q)} - Z_t^{(m)} \right|^2 dt \right] \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 \right] + \sup_{t \in [0,T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right| \left| \Delta^{(m,q)} \right|
\]
\[
\leq \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 \right] + \mathbb{E} \left[ \left| \Delta^{(m,q)} \right|^2 \right] \left( \sup_{t \in [0,T]} \left| Y_t^{(m+q)} - Y_t^{(m)} \right|^2 \right)^{\frac{p}{2}}
\]
with
\[
\Delta^{(m,q)} := \int_0^T \left| f \left( t, Y_t^{(m+q-1)}, P_{Y_t^{(m+q-1)}}, Z_t^{(m+q)} \right) - f \left( t, Y_t^{(m-1)}, P_{Y_t^{(m-1)}}, Z_t^{(m)} \right) \right| dt + \left| K_T^{(m+q)} \right| + \left| K_T^{(m)} \right|.
\]

It follows from Lemma 4.8 and dominated convergence theorem that
\[
\lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \left( \int_0^T \left| Z_t^{(m+q)} - Z_t^{(m)} \right|^2 dt \right)^p \right] = 0, \quad \forall p \geq 1.
\]
Therefore, there exists a pair of processes \((Y, Z) \in \mathcal{S} \times \mathcal{H}^d\) such that
\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| Y_t^{(m)} - Y_t \right|^p + \left( \int_0^T \left| Z_t^{(m)} - Z_t \right|^2 dt \right)^p \right] = 0, \quad \forall p \geq 1. \quad (19)
\]
Set

\[ K_t = Y_t - Y_0 + \int_0^t f(s, Y_s, Z_s) \, ds - \int_0^t Z_s \, dB_s. \]

Using assumption (H2'), we obtain

\[
\lim_{m \to \infty} \mathbb{E} \left[ \int_0^T \left| \frac{dK}{dB} \right| \right] = 0,
\]

which implies that, as \( m \to \infty \),

\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| K_t - K_t^{(m)} \right| \right] = 0.
\]

In particular, we have

\[
K_t = \lim_{m \to \infty} K_t^{(m)} = \lim_{m \to \infty} \mathbb{E} \left[ K_t^{(m)} \right] = \mathbb{E} \left[ K_t \right]
\]

and then \( K \) is a deterministic, non-decreasing and continuous process. Finally, it follows from (5) that

\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| K_t - K_t^{(m)} \right| \right] = 0,
\]

which implies that, as \( m \to \infty \),

\[
\lim_{m \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left| K_t - K_t^{(m)} \right| \right] 
\]

is a deterministic solution to the quadratic mean reflected BSDE (1). The proof is complete. \( \square \)

**Appendix**

This appendix is devoted to the proofs of Lemma 4.8 and Lemma 4.9, which we give for the reader’s convenience. The main idea is the same as in Lemma 4.4 or [14, Theorem 4.1].

**A.1 Proof of Lemma 4.8**

In view of Lemma 2.5 and Remark 4.2, we have for any \( m \geq 1 \),

\[
Y_t^{(m)} := y_t^{(m)} + \sup_{t \leq s \leq T} L_s (y_s^{(m)}), \quad \forall t \in [0,T],
\]

where \( y_t^{(m)} \) is the solution to the following quadratic BSDE

\[
y_t^{(m)} = \xi + \int_t^T f \left( s, Y_s^{(m-1)}, P_{Y_s^{(m-1)}}, Z_s^{(m)} \right) \, ds - \int_t^T Z_s^{(m)} \, dB_s.
\]

Applying assertion (i) of Lemma 4.3 and (17) yields for any \( t \in [0,T] \),

\[
\exp \left\{ \alpha_t (y_t^{(m)}) \right\} \leq \mathbb{E}_t \exp \left\{ \gamma \left( \xi + \int_0^T \alpha_s ds + \beta (T - t) \left( \sup_{s \in [t,T]} |Y_s^{(m-1)}| + \sup_{s \in [t,T]} |Y_s^{(m-1)}| \right) \right) \right\}.
\]
Using Doob's maximal inequality, we get for each $m \geq 1$, $p \geq 2$ and $t \in [0, T]$,

$$
\mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [t, T]} |Y_s^{(m)}| \right\} \right] 
\leq 4 \mathbb{E} \left[ \exp \left\{ p \gamma \left( |\xi| + \int_0^T \alpha_s ds + \beta(T-t) \left( \sup_{s \in [t, T]} |Y_s^{(m-1)}| + \sup_{s \in [t, T]} \mathbb{E} \left[ |Y_s^{(m-1)}| \right] \right) \right) \right] .
$$

(23)

Recalling (20), we have

$$
|Y_t^{(m)}| \leq |y_t^{(m)}| + \sup_{0 \leq s \leq T} |L_s(0)| + \kappa \sup_{t \leq s \leq T} \mathbb{E} \left[ |y_s^{(m)}| \right] .
$$

In view of Hölder's inequality, we obtain that for any $m \geq 1$, $p \geq 2$ and $t \in [0, T],$

$$
\mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [t, T]} |Y_s^{(m)}| \right\} \right] 
\leq e^{\gamma \sup_{0 \leq s \leq T} |L_s(0)|} \mathbb{E} \left[ \exp \left\{ \gamma \sup_{s \in [t, T]} |y_s^{(m)}| \right\} \right] 
\leq e^{\gamma \sup_{0 \leq s \leq T} |L_s(0)|} \mathbb{E} \left[ \exp \left\{ (2 + 2\kappa)p\gamma \sup_{s \in [t, T]} |y_s^{(m)}| \right\} \right] 
\leq 4 \mathbb{E} \left[ \exp \left\{ (2 + 2\kappa)p\gamma |\xi| + \alpha + \beta(T-t) \sup_{s \in [t, T]} |Y_s^{(m-1)}| \right\} \right] 
\times \mathbb{E} \left[ \exp \left\{ (2 + 2\kappa)p\gamma \beta(T-t) \sup_{s \in [t, T]} |Y_s^{(m-1)}| \right\} \right] .
$$

Choose a constant $h \in (0, T]$ depending only on $\beta$ and $\kappa$ such that

$$
(32 + 64\kappa)\beta h < 1.
$$

(24)

In view of Hölder's inequality, we obtain that for any $p \geq 2$,

$$
\mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [T-h, T]} |Y_s^{(m)}| \right\} \right] 
\leq 4 \mathbb{E} \left[ \exp \left\{ (4 + 4\kappa)p\gamma |\xi| + \bar{\alpha} \right\} \right] \mathbb{E} \left[ \exp \left\{ (4 + 4\kappa)\beta hp\gamma \sup_{s \in [T-h, T]} |Y_s^{(m-1)}| \right\} \right] 
\leq 4 \mathbb{E} \left[ \exp \left\{ (8 + 8\kappa)p\gamma |\xi| \right\} \right] \mathbb{E} \left[ \exp \left\{ (8 + 8\kappa)p\gamma \bar{\alpha} \right\} \right] \mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [T-h, T]} |Y_s^{(m-1)}| \right\} \right] .
$$

(25)

Define $\rho = \frac{1}{1 - (4 + 4\kappa)\beta h}$ and

$$
\mu := \begin{cases} 
\frac{T}{n}, & \text{if } \frac{T}{n} \text{ is an integer;} \\
T + 1, & \text{otherwise.}
\end{cases}
$$

If $\mu = 1$, it follows from (25) that for each $p \geq 2$ and $m \geq 1,$

$$
\mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [0, T]} |Y_s^{(m)}| \right\} \right] 
\leq 4 \mathbb{E} \left[ \exp \left\{ (8 + 8\kappa)p\gamma |\xi| \right\} \right] \mathbb{E} \left[ \exp \left\{ (8 + 8\kappa)p\gamma \bar{\alpha} \right\} \right] \mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [0, T]} |Y_s^{(m-1)}| \right\} \right] .
$$
Iterating the above procedure \( m \) times yields, given the definition of \( \rho \),

\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T]} |Y_s^{(m)}| \right\} \right] \leq 4^p E \left[ \exp \left\{ (8 + 8\kappa)p\gamma |\xi| \right\} \right] \frac{\hat{\theta}}{\gamma}^{\frac{p}{2}} E \left[ \exp \left\{ (8 + 8\kappa)p\gamma \tilde{\alpha} \right\} \right] \frac{\hat{\theta}}{\gamma},
\]

(26)

which is uniformly bounded with respect to \( m \) thanks to assumptions (H1") and (H2"). If \( \mu = 2 \), proceeding identically as above, we have for any \( p \geq 2 \),

\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [T-h,T]} |Y_s^{(m)}| \right\} \right] \leq 4^p E \left[ \exp \left\{ (8 + 8\kappa)p\gamma |\xi| \right\} \right] \frac{\hat{\theta}}{\gamma}^{\frac{p}{2}} E \left[ \exp \left\{ (8 + 8\kappa)p\gamma \tilde{\alpha} \right\} \right] \frac{\hat{\theta}}{\gamma}.
\]

(27)

We then consider the following quadratic reflected BSDE on time interval \([0,T-h]\):

\[
\left\{ \begin{array}{l}
Y_t^{(m)} = Y_{T-h}^{(m)} + \int_t^{T-h} f(s, Y_s^{(m-1)}, P_s^{(m-1)}, Z_s^{(m)}) \, ds - \int_t^{T-h} Z_s^{(m)} \, dB_s + K_{T-h}^{(m)} - K_t^{(m)}, \\
E[\ell(t, Y_t^{(m)})] \geq 0, \quad \forall t \in [0,T-h] \quad \text{and} \quad \int_0^{T-h} E[\ell(t, Y_t^{(m)})] \, dK_t^{(m)} = 0.
\end{array} \right.
\]

According to the derivation of (26), we deduce that

\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T-h]} |Y_s^{(m)}| \right\} \right] \leq 4^p E \left[ \exp \left\{ 2p\gamma \sup_{s \in [0,T-h]} |Y_s^{(m)}| \right\} \right] \frac{\hat{\theta}}{\gamma}^{\frac{p}{2}} E \left[ \exp \left\{ 2p\gamma \sup_{s \in [T-h,T]} |Y_s^{(m)}| \right\} \right] \frac{\hat{\theta}}{\gamma},
\]

(28)

where we used (27) in the last inequality. Putting the above inequalities together and applying Hölder’s inequality again yields for any \( p \geq 2 \),

\[
E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T]} |Y_s^{(m)}| \right\} \right] \leq 4^p E \left[ \exp \left\{ 2p\gamma \sup_{s \in [0,T-h]} |Y_s^{(m)}| \right\} \right] \frac{\hat{\theta}}{\gamma}^{\frac{p}{2}} E \left[ \exp \left\{ 2p\gamma \sup_{s \in [T-h,T]} |Y_s^{(m)}| \right\} \right] \frac{\hat{\theta}}{\gamma} \\cdot \frac{\hat{\theta}}{\gamma}^{\frac{p}{2}},
\]

(28)

which is also uniformly bounded with respect to \( m \).

Iterating the above procedure \( \mu \) times in the general case, we get

\[
\sup_{m \geq 0} E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T]} |Y_s^{(m)}| \right\} \right] < \infty, \quad \forall p \geq 1,
\]

which together with (25) implies that

\[
\sup_{m \geq 0} E \left[ \exp \left\{ p\gamma \sup_{s \in [0,T]} |y_s^{(m)}| \right\} \right] < \infty, \quad \forall p \geq 1.
\]

(29)

It follows from Lemma 2.5 and assumption (H4) that

\[
\sup_{m \geq 0} K_T^{(m)} \leq \sup_{0 \leq s \leq T} |L_s(0)| + \sup_{m \geq 0} E \left[ \sup_{s \in [0,T]} |y_s^{(m)}| \right] < \infty.
\]

Finally, noting \( Z^{(m)} = z^{(m)} \) and applying 4. Corollary 4 to the quadratic BSDE (21) leads to

\[
\sup_{m \geq 0} E \left[ \left( \int_0^T |Z_t^{(m)}|^2 \, dt \right)^p \right] < \infty, \quad \forall p \geq 1,
\]

which ends the proof.

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\section*{A.2 Proof of Lemma 4.9}

Without loss of generality, assume $f(t, y, v, \cdot)$ is concave, since the other case can be proved by a similar analysis, as discussed in Remark \ref{rem:concave}. For each fixed $m, q \geq 1$ and $\theta \in (0,1)$, we can define similarly $\delta_y \ell^{(m, q)}$, $\delta_y \ell^{(m, q)}$ and $\delta_y \ell^{(m, q)}$ for $y, z$. Then, the pair of processes $(\delta_y \ell^{(m, q)}, \delta_y z^{(m, q)})$ satisfies the following BSDE:

\[
\delta_y y_t^{(m, q)} = -\xi + \int_t^T \left( \delta_y f^{(m, q)}(s, \delta_y z_s^{(m, q)}) + \delta_y f^{(m, q)}(s) \right) \, ds - \int_t^T \delta_y z_s^{(m, q)} \, dB_s, \tag{30}
\]

where the generator is given by

\[
\delta_y f_0^{(m, q)}(t) = \frac{1}{1-\theta} \left( f \left( t, Y_t^{(m+q-1)}, P_{Y_t^{(m+q-1)}}, \gamma_t^{(m)} \right) - f \left( t, Y_t^{(m-1)}, P_{Y_t^{(m-1)}}, \gamma_t^{(m)} \right) \right),
\]

\[
\delta_y f^{(m, q)}(t, z) = \frac{1}{1-\theta} \left( t f \left( t, Y_t^{(m+q-1)}, P_{Y_t^{(m+q-1)}}, \gamma_t^{(m)} \right) - f \left( t, Y_t^{(m+q-1)}, P_{Y_t^{(m+q-1)}}, -(1-\theta)z + \delta_2^{(m+q)} \right) \right).
\]

From assumption (H2'), we get

\[
\delta_y f_0^{(m, q)}(t) \leq \beta \left( |Y_t^{(m+q-1)}| + |\delta_y Y_t^{(m-1, q)}| + \mathbb{E} \left[ |Y_t^{(m+q-1)}| + |\delta_y Y_t^{(m-1, q)}| \right] \right),
\]

\[
\delta_y f^{(m, q)}(t, z) \leq -f \left( t, Y_t^{(m+q-1)}, P_{Y_t^{(m+q-1)}}, -(1-\theta)z + \delta_2^{(m+q)} \right) \leq \alpha_1 + \beta \left( |Y_t^{(m+q-1)}| + \mathbb{E} \left[ |Y_t^{(m+q-1)}| \right] \right) + \frac{\gamma}{2} z^2.
\]

Set $C_3 := 2 \sup_m \mathbb{E} \left[ \sup_{s \in [0, T]} |Y_s^{(m)}| \right] < \infty$ (see Lemma \ref{lem:uniform_bound}) and for any $m, q \geq 1$, denote

\[
\zeta^{(m, q)} = |\xi| + \beta T C_3 + \int_0^T \alpha_s \, ds + \beta T \left( \sup_{s \in [0, T]} |Y_s^{(m-1)}| + \sup_{s \in [0, T]} |Y_s^{(m+q-1)}| \right),
\]

\[
\chi^{(m, q)} = 2 \beta T C_3 + \int_0^T \alpha_s \, ds + \beta T \left( \sup_{s \in [0, T]} |Y_s^{(m+q-1)}| + \sup_{s \in [0, T]} |Y_s^{(m-1)}| \right).
\]

Applying assertion (ii) of Lemma \ref{lem:uniform_bound} to (30) yields for any $p \geq 1,$

\[
\exp \left\{ p \gamma \left( \delta_y y_t^{(m, q)} \right) \right\} \leq \mathbb{E}_t \exp \left\{ p \gamma \left( \zeta^{(m, q)} + \beta (T-t) \left( \sup_{s \in [t, T]} |\delta_y Y_s^{(m-1, q)}| + \sup_{s \in [t, T]} \mathbb{E} |\delta_y Y_s^{(m-1, q)}| \right) \right) \right\}
\]

and in a similar way, we also have

\[
\exp \left\{ p \gamma \left( \delta_y y_t^{(m, q)} \right) \right\} \leq \mathbb{E}_t \left[ \exp \left\{ p \gamma \left( \zeta^{(m, q)} + \beta (T-t) \left( \sup_{s \in [t, T]} |\delta_y Y_s^{(m-1, q)}| + \sup_{s \in [t, T]} \mathbb{E} |\delta_y Y_s^{(m-1, q)}| \right) \right) \right] \right\}
\]

According to the fact that

\[
(\delta_y y^{(m, q)})^{-} \leq (\delta_y y^{(m, q)})^{+} + 2|y^{(m)}| \text{ and } (\delta_y y^{(m, q)})^{-} \leq (\delta_y y^{(m, q)})^{+} + 2|y^{(m+q)}|,
\]
we derive, using Hölder’s inequality and (22), that
\[
\exp \left\{ p\gamma |\delta y_t^{(m,q)}| \right\} \wedge \exp \left\{ p\gamma |\delta y_t^{(m,q)}| \right\} \\
\leq \exp \left\{ p\gamma \left( \left( \delta y_t^{(m,q)} \right)^+ + \left( \delta y_t^{(m,q)} \right)^+ + 2 |y_t^{(m)}| + 2 |y_t^{(m+q)}| \right) \right\} \\
\leq \mathbb{E}_t \left[ \exp \left\{ p\gamma \left( |x| + \chi^{(m,q)} + \beta (T-t) \left( \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} + \sup_{s \in [t,T]} \mathbb{E} \left[ \delta y_s^{(m-1,q)} \right] \right) \right) \right\} \right]^2 \\
\times \exp \left\{ 2p\gamma \left( |y_t^{(m)}| + |y_t^{(m+q)}| \right) \right\} \\
\leq \mathbb{E}_t \left[ \exp \left\{ p\gamma \left( |x| + \chi^{(m,q)} + \beta (T-t) \left( \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} + \sup_{s \in [t,T]} \mathbb{E} \left[ \delta y_s^{(m-1,q)} \right] \right) \right) \right\} \right]^2 \\
\times \mathbb{E}_t \left[ \exp \left\{ 4p\gamma \zeta^{(m,q)} \right\} \right].
\]
In view of Doob’s maximal inequality and Hölder’s inequality, we obtain that for all \( p > 1 \) and \( t \in [0,T] \)
\[
\mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [t,T]} \delta y_s^{(m,q)} \right\} \right] \\
\leq 4\mathbb{E} \left[ \exp \left\{ 8p\gamma \left( |x| + \chi^{(m,q)} + \beta (T-t) \left( \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} + \sup_{s \in [t,T]} \mathbb{E} \left[ \delta y_s^{(m-1,q)} \right] \right) \right) \right\} \right]^\frac{1}{2} \\
\times \mathbb{E} \left[ \exp \left\{ 16p\gamma \zeta^{(m,q)} \right\} \right]^\frac{1}{2} \\
\leq 4\mathbb{E} \left[ \exp \left\{ 8p\gamma \left( |x| + \chi^{(m,q)} + \beta (T-t) \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} \right) \right\} \right]^\frac{1}{2} \\
\times \mathbb{E} \left[ \exp \left\{ 8\beta (T-t)p\gamma \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} \right\} \right]^\frac{1}{2} \\
\times \mathbb{E} \left[ \exp \left\{ 16p\gamma \zeta^{(m,q)} \right\} \right]^\frac{1}{2}.
\]
Set \( C_4 := \sup_{0 \leq s \leq T} |L_s(0)| + 2\kappa \sup_{m \in [0,T]} \mathbb{E} \left[ \sup_{s \in [0,T]} |y_s^{(m)}| \right] < \infty \) (see (20)). Recalling (20) and assumption (H4),
\[
\delta y_t^{(m,q)} \leq \delta y_t^{(m,q)} + 2\kappa \sup_{t \leq s \leq T} \mathbb{E} \left[ \delta y_s^{(m,q)} \right] + 2C_4,
\]
which together with Jensen’s inequality implies that for each \( p \geq 1 \) and \( t \in [0,T] \)
\[
\mathbb{E} \left[ \exp \left\{ p\gamma \sup_{s \in [t,T]} \delta y_s^{(m,q)} \right\} \right] \leq e^{2p\gamma C_4} \mathbb{E} \left[ \exp \left\{ (2+4\kappa)p\gamma \sup_{s \in [t,T]} \delta y_s^{(m,q)} \right\} \right]^\frac{1}{2} \\
\leq 4\mathbb{E} \left[ \exp \left\{ (16 + 32\kappa)p\gamma \left( |x| + \chi^{(m,q)} + C_4 + \beta (T-t) \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} \right) \right\} \right]^\frac{1}{2} \\
\times \mathbb{E} \left[ \exp \left\{ (16 + 32\kappa)\beta (T-t)p\gamma \sup_{s \in [t,T]} \delta y_s^{(m-1,q)} \right\} \right]^\frac{1}{2} \mathbb{E} \left[ \exp \left\{ (32 + 64\kappa)p\gamma \zeta^{(m,q)} \right\} \right]^\frac{1}{2}.
\]
Choosing \( h \) as in [24], we have

\[
\mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [T-h, T]} \delta_\theta Y_s^{(m,q)} \right\} \right] \\
\leq 4\mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma |\xi| \right\} \right] \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma (\chi^{(m,q)} + C_4) \right\} \right]^{\frac{1}{2}} \\
\times \mathbb{E} \left[ \exp \left\{ (32 + 64\kappa) p_\gamma \zeta^{(m,q)} \right\} \right] \sup_{s \in [T-h,T]} \mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [T-h,T]} \delta_\theta Y_s^{(m-1,q)} \right\} \right]^{\frac{16+32\kappa}{(16+32\kappa)\beta h}}.
\]

Set \( \tilde{\rho} = \frac{1}{1-(16+32\kappa)\beta h} \). If \( \mu = 1 \), it follows from (31) that for each \( p \geq 1 \) and \( m, q \geq 1 \),

\[
\mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [0,T]} \delta_\theta Y_s^{(m,q)} \right\} \right] \\
\leq 4^{\tilde{\rho}} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma |\xi| \right\} \right] \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma (\chi^{(m,q)} + C_4) \right\} \right]^{\frac{1}{2}} \\
\times \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (32 + 64\kappa) p_\gamma \zeta^{(m,q)} \right\} \right]^{\frac{1}{2}} \mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [0,T]} \delta_\theta Y_s^{(1,q)} \right\} \right]^{\frac{16+32\kappa}{(16+32\kappa)\beta h}}.
\]

The result from Lemma 4.8 insures that for any \( \theta \in (0,1) \)

\[
\lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [0,T]} \delta_\theta Y_s^{(1,q)} \right\} \right]^{(16+32\kappa)\beta h} = 1,
\]

which implies that

\[
\sup_{\theta \in (0,1)} \lim_{m \to \infty} \sup_{q \geq 1} \mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [0,T]} \delta_\theta Y_s^{(m,q)} \right\} \right] \\
\leq 4^{\tilde{\rho}} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma |\xi| \right\} \right] \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma (\chi^{(m,q)} + C_4) \right\} \right]^{\frac{1}{2}} \\
\times \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (32 + 64\kappa) p_\gamma \zeta^{(m,q)} \right\} \right]^{\frac{1}{2}} < \infty.
\]

If \( \mu = 2 \), in view of the derivation of (28), we conclude that for any \( p \geq 1 \),

\[
\mathbb{E} \left[ \exp \left\{ p_\gamma \sup_{s \in [0,T]} \delta_\theta Y_s^{(m,q)} \right\} \right] \\
\leq 4^{\tilde{\rho} + \frac{\tilde{\rho}^2}{m}} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa)^2 p_\gamma |\xi| \right\} \right] \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa)^2 p_\gamma (\chi^{(m,q)} + C_4) \right\} \right]^{\frac{1}{2}} \\
\times \sup_{m,q \geq 1} \mathbb{E} \left[ \exp \left\{ (32 + 64\kappa)(64 + 128\kappa)^2 p_\gamma \zeta^{(m,q)} \right\} \right]^{\frac{1}{2}} \\
\times \mathbb{E} \left[ \exp \left\{ (64 + 128\kappa) p_\gamma \sup_{s \in [0,T]} \delta_\theta Y_s^{(1,q)} \right\} \right]^{\frac{1}{2}} \mathbb{E} \left[ \exp \left\{ (16+32\kappa)\beta h \right\} \right]^{m-1},
\]

which also implies the desired assertion when \( \mu = 2 \). Iterating the above procedure \( \mu \) times in the general case, we complete the proof.
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