New principle for scattering inside a Huygens bianisotropic medium

Akhlesh Lakhtakia

Abstract A Huygens bianisotropic medium is a linear homogeneous medium for which the Huygens principle can be formulated. When a bounded 3D scattering object composed of a linear bianisotropic medium, whether homogeneous or not, is embedded in a Huygens bianisotropic medium, the excess field phasors inside that object act as volume current densities, and the tangential components of the internal field phasors on the surface of the same object act as surface current densities, to radiate identical field phasors in the external region.

Keywords Bianisotropy · Equivalent current density · Excess field · Huygens principle · Surface integral equations · Volume integral equations

Introduction

Frequency-domain scattering by a bounded three-dimensional (3D) object in free space can be handled theoretically by analytical, semi-analytical, and numerical methods, depending on the shape and the composition of the object. Perhaps the simplest case is of a sphere composed of a homogeneous isotropic medium [1], whether the medium is dielectric-magnetic [2–4] or bianisotropic [5, 6]. Analytical solutions based on a separation-of-variables approach are available. Scattering by a spheroid composed of a homogeneous medium of one of those types has also been formulated and solved analytically [7, 8]. The same approach can be extended to radially stratified spheres [9, 10] and confocally stratified spheroids made of isotropic and bianisotropic mediums.

Except for special cases [11–13], scattering by spheres and spheroids made of homogeneous anisotropic mediums requires the use of semi-analytical [14–18] and numerical methods [18–20].

The extended boundary condition method is semi-analytical. It requires bilinear expansions of the infinite-medium dyadic Green functions for the medium of which the 3D object is composed [21]. This requirement has restricted the conventional formulation of this method to objects composed of a homogeneous medium that either are bianisotropic [22] or belong to a certain class of bianisotropic mediums [18, 23]. However, that requirement can be bypassed for a 3D object made of a homogeneous orthorhombic medium with [24, 25] or without [26] gyrotropy, because basis functions to represent the actual electric and magnetic field phasors inside the object can be synthesized from an angular spectrum of plane waves. Of course, the formulation and solution processes then become considerably toilsome.

Numerical methods can handle arbitrarily shaped 3D objects that are composed of a homogeneous/nonhomogeneous and isotropic/bianisotropic/anisotropic/bianisotropic medium. The finite-element method [27, 28] and the method of moments [29–31] partition the scattering object into several subregions. Suitable basis functions are then used to represent the electric and magnetic field phasors inside each subregion. The discrete dipole approximation partitions the scattering object into multiple subobjects. Each subobject is modeled by a set of three mutually orthogonal electric and three mutually orthogonal magnetic dipoles of strengths and phases to be determined [29, 32]. Appropriate only for homogeneous objects, the boundary element method.
partitions the object’s surface and uses subregional basis functions to determine the actual electric and magnetic field phasors on the object’s surface [33–35]. These and other numerical methods [36] are also hybridized for faster and/or higher-resolution computations [28, 37, 38]. Finally, the finite-difference time-domain method solves the differential form of the Maxwell equations directly on a spacetime grid that encompasses the scattering object [39, 40].

The absence of purely analytical methods to treat scattering by 3D objects of arbitrary shape and composition highlights the need for computationally tractable principles that the results obtained from every semi-analytical and numerical method must satisfy. One such principle is mathematical, concerned with the completeness and convergence of representations of the scattered and the internal field phasors [36, 41, 42]. The second principle is physical, that of conservation of energy in any time-invariant system [43]. This principle is often enforced through the optical theorem [44, 45]. Energy, however, is a quantity derived from electric and magnetic fields. For frequency-domain scattering problems, a universal principle involving the electric and magnetic field phasors directly is desirable.

In this paper, two new relations are obtained for 3D frequency-domain scattering problems involving:

1. the actual electric and magnetic field phasors inside the object,
2. the internal electric and magnetic field phasors on the surface of the object, and
3. the infinite-medium dyadic Green functions [46] of the linear homogeneous medium in which the object is embedded.

The external medium is assumed to extend to infinity in all directions. This medium need not be free space. Instead, it can be the most general linear, homogeneous, bianisotropic medium for which the Huygens principle has been formulated [47]. We use the phrasal adjective *Huygens bianisotropic* for such a medium. The internal medium is also linear; it can be homogeneous/nonhomogeneous and isotropic/biisotropic/anisotropic/bianisotropic. The two new relations, which can be derived from one another by using the frequency-domain Maxwell equations applied to the external medium, constitute a principle that every applicable semi-analytical and numerical method must satisfy. This new principle is independent of the principle of conservation of energy.

The plan of the paper is as follows. “Theoretical preliminaries” section sets up the 3D scattering problem by providing the constitutive relations of the external and internal mediums, the governing differential equations, and the infinite-medium dyadic Green functions of the external medium. Two volume integral equations are formulated in “Volume integral equations” section, and two surface integral equations are presented in “Surface integral equations” section. Comparisons of the volume and surface integral equations deliver two new relations in “Two new relations” section. Finally, in “New principle” section, these relations underwrite the new principle for scattering inside any Huygens bianisotropic medium.

An exp(−iωt) time-dependence is assumed with ω as the angular frequency, t the time, and i = √−1. Boldface letters denote vectors, with θ denoting the null vector and the caret (') identifying unit vectors. Dyadics are underlined twice, with δ denoting the identity dyadic and 0 the null dyadic.

**Theoretical preliminaries**

Suppose that all space V is divided into the unbounded region \( V_e \) and the bounded region \( V_i \), the boundary of the two regions being the closed surface \( S \), as shown in Fig. 1. The external region \( V_e \) is occupied by a Huygens bianisotropic medium characterized in the frequency domain by the linear constitutive relations [21, 47]

\[
D(r) = \varepsilon \cdot E(r) + \left[ \varepsilon + (K - \Gamma) \times I \right] \cdot H(r) \quad , \quad r \in V_e ,
\]

\[
B(r) = \mu \cdot H(r) - \left[ \mu - (K + \Gamma) \times I \right] \cdot E(r) \quad , \quad r \in V_e .
\]

(1)

Here and hereafter, \( \mathbf{r} \) is the position vector, \( \mathbf{E}(\mathbf{r}) \) the electric field phasor, \( \mathbf{H}(\mathbf{r}) \) the magnetic field phasor, \( \mathbf{D}(\mathbf{r}) \) the electric displacement phasor, and \( \mathbf{B}(\mathbf{r}) \) the magnetic induction phasor. The arbitrary vectors \( K \) and \( \Gamma \) as well as the symmetric dyadics \( \varepsilon = \varepsilon^T, \mu = \mu^T \), and \( \xi = \xi^T \) are implicit functions of \( \omega \), and the superscript \( ^T \) denotes the transpose.

A linear bianisotropic medium also occupies the internal region \( V_i \), but it need not be homogeneous. Its frequency-domain constitutive relations are stated as
Equations 5 govern the electric and magnetic field phasors where the operators
\[ J'_e(r) = \frac{1}{i\omega} \cdot H(r) + \frac{1}{i\omega} \cdot E(r), \quad r \in \mathcal{V}_e \]
\[ J'_m(r) = -J_m(r), \quad r \in \mathcal{V}_m \]
(4)

Substitution of Eqs. 1 and 2 in Eqs. 4 yields
\[ \frac{\partial \mathcal{G}^{(p)}(r', r)}{\partial r} \cdot H(r) = J'_e(r), \quad r \in \mathcal{V}_e \]
\[ \frac{\partial \mathcal{G}^{(m)}(r', r)}{\partial r} \cdot E(r) = -J_m(r), \quad r \in \mathcal{V}_m \]
\[ \mathcal{G}^{(p)}(r', r) = \mathcal{G}^{(m)}(r', r) \]
(5)

where the operators
\[ \frac{\partial \mathcal{G}^{(e)}(r', r)}{\partial r} = \left\{ \nabla \times \mathbf{I} + i\omega \left[ \mathbf{\xi} + (\mathbf{K} - \mathbf{\Gamma}) \times \mathbf{\xi} \right] \right\} \]
\[ \frac{\partial \mathcal{G}^{(m)}(r', r)}{\partial r} = \left\{ \nabla \times \mathbf{I} + i\omega \left[ \mathbf{\xi} - (\mathbf{K} + \mathbf{\Gamma}) \times \mathbf{\xi} \right] \right\} \]
(6)

and the dyadics
\[ \mathcal{G}^{(p)}(r', r) = 0, \quad \mathcal{G}^{(p)}(r', r) = 0, \quad \mathcal{G}^{(p)}(r', r) = 0, \quad \mathcal{G}^{(p)}(r', r) = 0 \]
(7)

Equations 5 govern the electric and magnetic field phasors throughout \( \mathcal{V} \).

**Dyadic Green functions**

Four infinite-medium dyadic Green functions \( \mathcal{G}^{(p)}(r', r) \), \( p \in \{e, m\} \) and \( q \in \{e, m\} \), can be prescribed for the chosen external medium [46, Sec. 1.3.1]. Of these, \( \mathcal{G}^{(e)}(r, r') \) and \( \mathcal{G}^{(m)}(r, r') \) are, respectively, the solutions of the differential equations
\[ \left( \frac{\partial \mathcal{G}^{(e)}(r, r')}{\partial r} \cdot \mathbf{\xi} + \frac{\partial \mathcal{G}^{(e)}(r, r')}{\partial r} \cdot \mathbf{\xi} - \omega^2 \mathbf{\xi} \right) \cdot \mathcal{G}^{(e)}(r, r') = i\omega \mathcal{I}(r - r') \]
and
\[ \left( \frac{\partial \mathcal{G}^{(m)}(r, r')}{\partial r} \cdot \mathbf{\xi} + \frac{\partial \mathcal{G}^{(m)}(r, r')}{\partial r} \cdot \mathbf{\xi} - \omega^2 \mathbf{\xi} \right) \cdot \mathcal{G}^{(m)}(r, r') = i\omega \mathcal{I}(r - r'), \]
(8)
(9)

where \( \mathcal{I}(\cdot) \) is the Dirac delta, \( r \) serves as the field point, and \( r' \) is the source point. The remaining two dyadic Green functions can be obtained from the first two as
\[ \mathcal{G}^{(m)}(r, r') = \frac{1}{i\omega} \mathbf{\xi} \cdot \mathcal{G}^{(m)}(r, r') \]
\[ \mathcal{G}^{(m)}(r, r') = \frac{1}{i\omega} \mathbf{\xi} \cdot \mathcal{G}^{(m)}(r, r') \]
(10)
(11)

These four functions have the following symmetries with respect to the interchange of the source and field points [21]:
\[ \mathcal{G}^{(e)}(r', r) = \left\{ \mathcal{G}^{(e)}(r', r) \right\}_T \exp \left[ 2i\omega \mathbf{\Gamma} \cdot (r - r') \right] \]
\[ \mathcal{G}^{(m)}(r', r) = \left\{ \mathcal{G}^{(m)}(r', r) \right\}_T \exp \left[ 2i\omega \mathbf{\Gamma} \cdot (r - r') \right] \]
\[ \mathcal{G}^{(e)}(r', r) = -\left\{ \mathcal{G}^{(e)}(r', r) \right\}_T \exp \left[ 2i\omega \mathbf{\Gamma} \cdot (r - r') \right] \]
\[ \mathcal{G}^{(m)}(r', r) = -\left\{ \mathcal{G}^{(m)}(r', r) \right\}_T \exp \left[ 2i\omega \mathbf{\Gamma} \cdot (r - r') \right] \]
(12)

Closed-form expressions for \( \mathcal{G}^{(p)}(r, r') \), \( p \in \{e, m\} \) and \( q \in \{e, m\} \), are not available for every medium described by Eq. 1. However, \( \mathcal{G}^{(p)}(r, r') \) can always be synthesized using the spatial Fourier transform [46, Sec. 1.4.1].

**Volume integral equations**

The solutions of Eqs. 5 come in two parts. The first part is due to the source current density phasors in \( \mathcal{V}_e \), the second due to the scattering object. Accordingly, the actual field phasors everywhere are written as
\[ E(r) = E_{\text{source}}(r) + E_{\text{obj}}(r), \quad H(r) = H_{\text{source}}(r) + H_{\text{obj}}(r) \]
(13)

By definition, the field phasors
exist everywhere when \( \mathcal{V}_e \) is also occupied by the external medium. Analogously, the field phasors

\[
E_{\text{obj}}(r) = -i\omega \int_{\mathcal{V}_e} \left\{ G^{\text{ee}}(r, r') \cdot J_\text{e}(r') + G^{\text{em}}(r, r') \cdot J_\text{m}(r') \right\} d^3 r',
\]

\[
H_{\text{obj}}(r) = -i\omega \int_{\mathcal{V}_e} \left\{ G^{\text{me}}(r, r') \cdot J_\text{e}(r') + G^{\text{mm}}(r, r') \cdot J_\text{m}(r') \right\} d^3 r',
\]

come into existence only if the internal medium is different from the external medium. Thus, the excess field phasors

\[
D_{\text{ex}}(r) = \frac{\mathcal{V}_e}{\mathcal{V}_m} (r) \cdot E(r) + \frac{\mathcal{V}_e}{\mathcal{V}_m} (r) \cdot H(r)
\]

\[
B_{\text{ex}}(r) = \frac{\mathcal{V}_m}{\mathcal{V}_e} (r) \cdot H(r) + \frac{\mathcal{V}_m}{\mathcal{V}_e} (r) \cdot E(r)
\]

are equivalent to volume current densities in \( \mathcal{V}_e \).

When Eqs. 16 and 17 are substituted in Eqs. 13, the resulting equations

\[
E(r) = E_{\text{source}}(r) - i\omega \int_{\mathcal{V}_e} \left\{ G^{\text{ee}}(r, r') \cdot \left[ \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot E(r') + \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot H(r') \right] \right\} d^3 r'
\]

\[
H(r) = H_{\text{source}}(r) - i\omega \int_{\mathcal{V}_e} \left\{ G^{\text{me}}(r, r') \cdot \left[ \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot E(r') + \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot H(r') \right] \right\} d^3 r',
\]

are volume integral equations because the unknown quantities, \( E(r) \) and \( H(r) \), appear inside as well as outside the volume integrals.

**Surface integral equations**

Mathematical statements of the Huygens principle pertinent to the region \( \mathcal{V}_e \) have been derived elsewhere [21, 47] in detail. After assuming that the fields decay far away from their sources sufficiently rapidly, those statements deliver

\[
E(r) \quad 0 \quad = E_{\text{source}}(r) + \int_{S} \left\{ G^{\text{ee}}(r, r') \cdot \left[ \hat{u}(r') \times H_+(r') \right] \right\} d^2 r', \quad \left\{ r \in \mathcal{V}_e \right\}
\]

\[
H(r) \quad 0 \quad = H_{\text{source}}(r) + \int_{S} \left\{ G^{\text{me}}(r, r') \cdot \left[ \hat{u}(r') \times E_+(r') \right] \right\} d^2 r', \quad \left\{ r \in \mathcal{V}_e \right\}
\]

In these equations, \( \hat{u}(r) \) is the unit normal to \( S \) pointing into \( \mathcal{V}_e \) (see Fig. 1), whereas \( E_+(r) \) and \( H_+(r) \) are the electric and magnetic field phasors on the external side of \( S \).

When \( r \in \mathcal{V}_e \), Eqs. 21 and 22 constitute the Ewald–Oseen extinction theorem [21]. With \( r \) chosen on the exterior side of \( S \), Eqs. 21 and 22 are surface integral equations because the unknown quantities, \( E(r) \) and \( H(r) \), appear inside as well as outside the surface integrals.

**Two new relations**

Let us revert to “Volume integral equations” section and focus on \( r \in \mathcal{V}_e \). Then, the scattered field phasors can be identified as

\[
E_{\text{scat}}(r) = \int_{\mathcal{V}_e} \left\{ G^{\text{ee}}(r, r') \cdot \left[ \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot E(r') + \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot H(r') \right] \right\} d^3 r'
\]

\[
H_{\text{scat}}(r) = \int_{\mathcal{V}_e} \left\{ G^{\text{me}}(r, r') \cdot \left[ \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot E(r') + \frac{\mathcal{V}_e}{\mathcal{V}_m} (r') \cdot H(r') \right] \right\} d^3 r',
\]
\[ E_{\text{sc}}(r) = E(r) - E_{\text{source}}(r), \quad H_{\text{sc}}(r) = H(r) - H_{\text{source}}(r) \], \quad r \in V_c. \quad (23)

Furthermore, the actual field phasors at \( r \in V_i \) must be identified as the internal field phasors, i.e.,
\[ E_{\text{in}}(r) = E(r), \quad H_{\text{in}}(r) = H(r) \], \quad r \in V_i. \quad (24)

Accordingly, Eqs. 16 and 17 can be recast as
\[
\begin{align*}
E_{\text{sc}}(r) &= -i\omega \int_{V_i} \left\{ G^{ee}(r, r') \right. \\
&\quad \cdot \left[ \frac{\nu}{\omega} (r') \cdot E_{\text{in}}(r') + \frac{\nu}{\omega} (r') \cdot H_{\text{in}}(r') \right] \left. \right\} d^3r' \\
&\quad - i\omega \int_{V_i} \left\{ G^{em}(r, r') \right. \\
&\quad \cdot \left[ \frac{\nu}{\omega} (r') \cdot E_{\text{in}}(r') + \frac{\nu}{\omega} (r') \cdot H_{\text{in}}(r') \right] \left. \right\} d^3r', \quad r \in V_c,
\end{align*}
\]
and
\[
\begin{align*}
H_{\text{sc}}(r) &= -i\omega \int_{V_i} \left\{ G^{me}(r, r') \right. \\
&\quad \cdot \left[ \frac{\nu}{\omega} (r') \cdot E_{\text{in}}(r') + \frac{\nu}{\omega} (r') \cdot H_{\text{in}}(r') \right] \left. \right\} d^3r' \\
&\quad - i\omega \int_{V_i} \left\{ G^{mm}(r, r') \right. \\
&\quad \cdot \left[ \frac{\nu}{\omega} (r') \cdot E_{\text{in}}(r') + \frac{\nu}{\omega} (r') \cdot H_{\text{in}}(r') \right] \left. \right\} d^3r', \quad r \in V_c,
\end{align*}
\]
respectively.

Next, let us revert to “Surface integral equations” section and also focus on \( r \in V_c \) but with the additional stipulation that \( r \not\in S \). Equations 21, 22, and 23 then yield
\[
\begin{align*}
E_{\text{sc}}(r) &= \int_S \left\{ \frac{G^{ee}(r, r')}{\omega} \cdot \left[ \hat{n}(r') \times H_+(r') \right] \\
&\quad - G^{em}(r, r') \cdot \left[ \hat{n}(r') \times E_+ (r') \right] \right\} d^2r', \quad r \in V_c - S, \quad (27)
\end{align*}
\]
and
\[
\begin{align*}
H_{\text{sc}}(r) &= \int_S \left\{ \frac{G^{me}(r, r')}{\omega} \cdot \left[ \hat{n}(r') \times H_+(r') \right] \\
&\quad - G^{mm}(r, r') \cdot \left[ \hat{n}(r') \times E_+ (r') \right] \right\} d^2r', \quad r \in V_c - S. \quad (28)
\end{align*}
\]
With \( E_+(r) \) and \( H_+(r) \) denoting the electric and magnetic field phasors on the internal side of \( S \), the standard boundary conditions
\[
\begin{align*}
\hat{n}(r) \times E_+(r) &= \hat{n}(r) \times E_-(r), \quad r \in S, \quad (29)
\end{align*}
\]
can be restated as
\[
\begin{align*}
\hat{n}(r) \times E_+(r) &= \hat{n}(r) \times E_{\text{in}}(r), \quad r \in S. \quad (30)
\end{align*}
\]
Accordingly, Eqs. 27 and 28 can be rewritten as
\[
\begin{align*}
E_{\text{sc}}(r) &= \int_S \left\{ \frac{G^{ee}(r, r')}{\omega} \cdot \left[ \hat{n}(r') \times H_{\text{in}}(r') \right] - G^{em}(r, r') \right\} d^2r', \quad r \in V_c - S, \quad (31)
\end{align*}
\]
and
\[
\begin{align*}
H_{\text{sc}}(r) &= \int_S \left\{ \frac{G^{me}(r, r')}{\omega} \cdot \left[ \hat{n}(r') \times H_{\text{in}}(r') \right] - G^{mm}(r, r') \right\} d^2r', \quad r \in V_c - S, \quad (32)
\end{align*}
\]
respectively. According to these equations, the tangential components of the internal electric and magnetic field phasors on the object's surface are equivalent to surface current densities on \( S \).

On comparing Eqs. 25 and 31, we get
\[
\begin{align*}
\int_{V_i} \left\{ G^{ee}(r, r') \cdot \left[ \frac{\nu}{\omega} (r') \cdot E_{\text{in}}(r') + \frac{\nu}{\omega} (r') \cdot H_{\text{in}}(r') \right] \right\} d^3r' \\
&\quad + \int_{V_i} \left\{ G^{em}(r, r') \cdot \left[ \frac{\nu}{\omega} (r') \cdot E_{\text{in}}(r') + \frac{\nu}{\omega} (r') \cdot H_{\text{in}}(r') \right] \right\} d^3r' \\
&= \frac{i}{\omega} \int_S \left\{ G^{ee}(r, r') \cdot \left[ \hat{n}(r') \times H_{\text{in}}(r') \right] \\
&\quad - G^{em}(r, r') \cdot \left[ \hat{n}(r') \times E_{\text{in}}(r') \right] \right\} d^2r', \quad r \in V_c - S. \quad (33)
\end{align*}
\]
Likewise, Eqs. 26 and 32 deliver the relation
\[
\int \left\{ \frac{G_{\text{mm}}(r, r')}{V_i} \left[ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right] \right\} \, d^3r' \\
+ \int \left\{ \frac{G_{\text{mm}}^s(r, r')}{V_i} \right\} \, d^3r' \\
\frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
= \frac{i}{\omega} \int_{S} \left\{ \frac{G_{\text{mm}}(r, r')}{} \cdot [\hat{n}(r') \times H_{\text{int}}(r')] \right\} \, d^2r', \quad r \in V_e - S.
\]

Equations 33 and 34 underwrite the novelty of this paper. Since both equations can be derived from each other because
\[
\frac{G_{\text{mm}}(r, r')}{} \cdot H_{\text{sc}}(r) + i\omega \frac{\mu}{\nu} E_{\text{sc}}(r) = 0 \\
\frac{G_{\text{mm}}^s(r, r')}{} \cdot E_{\text{sc}}(r) - i\omega \frac{\mu}{\nu} H_{\text{sc}}(r) = 0 \\
\]
they are not independent. However, one may be easier than the other to use in some situations. Also, note that Eq. 33 does not change if both sides of it are operated on from the left by \( G(r, r') \), as shown in “Appendix”. Likewise, Eq. 34 does not change if both sides of it are operated on from the left by \( G(r, r') \).

Parenthetically, Eqs. 33 and 34 arise because the scattered field phasors must be the same whether derived from surface integral equations or from volume integral equations, both types of integral equations being exact and derivable from the frequency-domain Maxwell curl postulates.

**Alternative forms**

Equations 33 and 34 can be put in alternative forms using Eqs. 12. For example, Eq. 33 may be rewritten as either
\[
\int \exp \left\{ -2i\omega \Gamma \cdot r' \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
- \int \exp \left\{ -2i\omega \Gamma \cdot r' \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
= \frac{i}{\omega} \int_{S} \left\{ \frac{G(r, r')}{} \cdot [\hat{n}(r') \times H_{\text{int}}(r')] \right\} \, d^2r', \quad r \in V_e.
\]

or
\[
\int \exp \left\{ -2i\omega \Gamma \cdot r' \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
- \int \exp \left\{ -2i\omega \Gamma \cdot r' \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \\
\left\{ \frac{G_{\text{mm}}(r, r')}{} \cdot [\hat{n}(r') \times H_{\text{int}}(r')] \right\} \, d^2r', \quad r \in V_e.
\]

**Scattering in free space**

When \( V_e \) is vacuous, \( \xi = \varepsilon_0 \frac{\omega}{c} \), \( \mu_0 = \varepsilon_0 \), and \( K = \Gamma = 0 \).

As a result,
\[
\frac{G_{\text{mm}}(r, r')}{} = i\omega \mu_0 \frac{G(r, r')}{} \\
\frac{G_{\text{mm}}^s(r, r')}{} = i\omega \frac{\mu}{\nu} \frac{G(r, r')}{} \\
\frac{G_{\text{mm}}(r, r')}{} = \frac{G_{\text{mm}}^s(r, r')}{} \\
\frac{G_{\text{mm}}^s(r, r')}{} = -\frac{G_{\text{mm}}(r, r')}{} \\
\]

where
\[
G(r, r') = \left( I + k_0^2 \nabla \nabla \right) \exp \left\{ \frac{ik_0|r - r'|}{4\pi|r - r'|} \right\}
\]
is the usual dyadic Green function for free space and \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \). Equation 33 then simplifies to
\[
\frac{i}{\omega} \mu_0 \int_{V_i} \left\{ \frac{G(r, r')}{} \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
- \nabla \times \int_{V_i} \left\{ \frac{G(r, r')}{} \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
= -\mu_0 \int_{S} \left\{ \frac{G(r, r')}{} \cdot [\hat{n}(r') \times H_{\text{int}}(r')] \right\} \, d^2r' \\
\]

or
\[
\frac{i}{\omega} \mu_0 \int_{V_i} \left\{ \frac{G(r, r')}{} \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
- \nabla \times \int_{V_i} \left\{ \frac{G(r, r')}{} \right\} \\
\left\{ \frac{\nu}{\nu_m} (r') \cdot E_{\text{int}}(r') + \frac{\nu}{\nu_m} (r') \cdot H_{\text{int}}(r') \right\} \, d^3r' \\
\frac{i}{\omega} \nabla \times \int_{S} \left\{ \frac{G(r, r')}{} \cdot [\hat{n}(r') \times E_{\text{int}}(r')] \right\} \, d^2r', \quad r \in V_e - S.
\]
and Eq. 34 to
\[
\nabla \times \int_{V_i} \left\{ G(r, r') \cdot \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \cdot \mathbf{E}_{\text{ext}}(r') + \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \cdot \mathbf{H}_{\text{ext}}(r') \right\} \, d^3r' + \frac{i \omega \varepsilon_0}{\nabla} \int_{\Sigma_{\text{ext}}} \left\{ G(r, r') \cdot \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \cdot \mathbf{E}_{\text{int}}(r') \right\} \, d^2r' \]
\[
\mathbf{E}_{\text{int}}(r') \cdot \mathbf{H}_{\text{ext}}(r') \right\} \, d^3r' \]
\[
= \int_{\Sigma_{\text{ext}}} \left\{ \nabla \times G(r, r') \cdot \left[ \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \times \mathbf{H}_{\text{int}}(r') \right] \right\} \, d^2r' + \int_{\Sigma_{\text{ext}}} \left\{ \nabla \times G(r, r') \cdot \left[ \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \times \mathbf{E}_{\text{int}}(r') \right] \right\} \, d^2r', \quad r \in \mathcal{V}_e - \mathcal{S}. \tag{41}
\]

If we additionally suppose that \( V_i \) is occupied by a homogeneous, isotropic dielectric medium, then \( \varepsilon_0 \left( r' \right) = \varepsilon_0 \), \( \mu_0 \left( r' \right) = \mu_0 \), \( \varepsilon_\infty \left( r' \right) = \varepsilon_\infty \), \( \varepsilon_{\text{mm}} \left( r' \right) = \varepsilon_{\text{mm}} \), and \( \mu_{\text{ee}} \left( r' \right) = \mu_{\text{ee}} \). Then, Eq. 40 simplifies to
\[
\omega^2 \mu_{\text{ee}} \varepsilon_\infty \int_{\mathcal{V}_i} \left\{ \nabla \times G(r, r') \cdot \mathbf{E}_{\text{int}}(r') \right\} \, d^3r' = \int_{\Sigma_{\text{ext}}} \left\{ \nabla \times G(r, r') \cdot \left[ \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \times \mathbf{H}_{\text{int}}(r') \right] \right\} \, d^2r'
\]
\[
+ \varepsilon_0 \int_{\Sigma_{\text{ext}}} \left\{ \nabla \times G(r, r') \cdot \left[ \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \times \mathbf{E}_{\text{int}}(r') \right] \right\} \, d^2r', \quad r \in \mathcal{V}_e - \mathcal{S}. \tag{42}
\]

and Eq. 41 to
\[
\varepsilon_\infty \nabla \times \int_{\mathcal{V}_i} \left\{ G(r, r') \cdot \mathbf{E}_{\text{ext}}(r') \right\} \, d^3r' = \int_{\Sigma_{\text{ext}}} \left\{ \nabla \times G(r, r') \cdot \left[ \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \times \mathbf{H}_{\text{int}}(r') \right] \right\} \, d^2r' + \int_{\Sigma_{\text{ext}}} \left\{ \nabla \times G(r, r') \cdot \left[ \frac{\nabla e^{-i\omega \tau}}{\omega} \left( r' \right) \times \mathbf{E}_{\text{int}}(r') \right] \right\} \, d^2r', \quad r \in \mathcal{V}_e - \mathcal{S}. \tag{43}
\]

Equation 40 can also be derived [48] using the second vector-dyadic Green theorem [49, p. 300]. It is a simple matter to apply Eqs. 42 and 43 to multipolar scattering by an isotropic dielectric sphere and thus validate the analytical Lorenz–Mie theory [2–4].

**New principle**

Equations 33 and 34 allow us to enunciate a new principle for scattering inside a Huygens bianisotropic medium. The left sides of both equations contain the excess field phasors (\( D_{\text{ex}} \) and \( B_{\text{ex}} \)) inside the scattering object, as defined in Eqs. (18). The right sides of both equations contain the tangential components of the internal field phasors (\( E_{\text{int}} \) and \( H_{\text{int}} \)) on the surface of the same object. The new principle may be enunciated as follows:

The excess field phasors inside a bounded 3D scattering object act as volume current densities, and the tangential components of the internal field phasors on the surface of the same object act as surface current densities, to radiate identical field phasors in the external region, provided the external medium is a Huygens bianisotropic medium.

The commonest Huygens bianisotropic medium is free space. Thus, this new principle certainly applies to scattering in free space, and it can be useful in checking the results of semi-analytical and numerical methods for solving scattering problems in free space [36], so long as the scattering object is composed of a linear medium.

In closing, the new principle is exact, just like the celebrated optical theorem [44, 45] is for the scattering of a plane wave by an object composed of a linear medium and surrounded by free space. Unlike the optical theorem, however, the new principle applies not only to incident plane waves but to practically any incident time-harmonic field. Next, the optical theorem applies to power density, but the new principle to the field phasors. Also, whereas the optical theorem has only been extended to the external medium being an isotropic chiral medium [6, Sec. 5-2.7], “Two new relations” section shows that the new principle applies when the external medium is far more general than an isotropic chiral medium. Thus, the new principle has a huge scope in computational electromagnetics for validation of scattering results provided by diverse semi-analytical and numerical methods.

**Acknowledgements** The author thanks the Charles Godfrey Binder Endowment at Penn State for ongoing support of his research activities.

**Declarations**

**Competing interests** The author has no competing interests to declare that are relevant to the content of this paper.

**Appendix**

The integrands on both sides of Eq. 33 are of the form
\[ G^{\text{ee}}(r', r') \cdot X(r') + G^{\text{em}}(r', r') \cdot Y(r') \]

with \( r \) not lying in the integration domain. Therefore,

\[ \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \cdot G^{\text{ee}}(r', r') \cdot X(r') = \left[ \omega^2 \varepsilon \cdot G^{\text{ee}}(r', r') + i \omega \delta(r - r') \right] \cdot X(r') = \omega^2 \varepsilon \cdot G^{\text{ee}}(r', r') \cdot X(r') \]

follows after using Eq. 8, and

\[ \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \cdot G^{\text{em}}(r', r') \cdot Y(r') = -\frac{1}{i \omega} \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \cdot \varepsilon^{-1} \cdot \frac{\mathbf{Q}}{i} \cdot G^{\text{em}}(r', r') \cdot Y(r') = -\frac{1}{i \omega} \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \cdot \left[ \omega^2 \mu \cdot G^{\text{em}}(r', r') + i \omega \delta(r - r') \right] \cdot Y(r') = i \omega \mathbf{Q} \cdot G^{\text{em}}(r', r') \cdot Y(r') = \omega^2 \varepsilon \cdot G^{\text{em}}(r', r') \cdot Y(r') \]

follows using Eqs. 9 and 10. Accordingly,

\[ \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \cdot \left[ G^{\text{ee}}(r', r') \cdot X(r') + G^{\text{em}}(r', r') \cdot Y(r') \right] = \omega^2 \varepsilon \cdot \left[ G^{\text{ee}}(r', r') \cdot X(r') + G^{\text{em}}(r', r') \cdot Y(r') \right] \]

so that Eq. 33 does not change if both sides of it are operated on from the left by \( \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \). In the same way, Eq. 34 does not change if both sides of it are operated on from the left by \( \left( \frac{\mathbf{Q} \cdot \mu^{-1}}{-i} \cdot \frac{\mathbf{Q}}{i} \right) \).

References

1. M. Kerker (ed.), Selected Papers on Light Scattering, Part 1 (SPIE, Bellingham, 1988)
2. L.V. Lorenz, Lysvevægelsen i og uden for en af plane lysbølger belyst kugle. K. Dan. Vidensk. Selsk. Forh. 6(6), 1–62 (1890)
3. G. Mie, Beiträge zur Optik trüber Medien, speziell kolloidaler Metalllösungen. Ann. Phys. Lpz. 25(3), 377–445 (1908)
4. C.F. Bohren, D.R. Huffman, Absorption and Scattering of Light by Small Particles (Wiley, New York, 1983)
5. C.F. Bohren, Light scattering by an optically active sphere. Chem. Phys. Lett. 29(3), 458–462 (1974)
6. A. Lakhtakia, Beltrami Fields in Chiral Media (World Scientific, Singapore, 1994)
7. S. Asano, G. Yamamoto, Light scattering by a spheroidal particle. Appl. Opt. 14(1), 29–49 (1975)
8. M.F.R. Cooray, I.R. Ciric, Wave scattering by a chiral spheroid. J. Opt. Soc. Am. A 10(6), 1197–1203 (1993)
9. R. Bhandari, Scattering coefficients for a multilayered sphere: analytic expressions and algorithms. Appl. Opt. 24(13), 1960–1967 (1985)
10. D.L. Jaggard, J.C. Liu, The matrix Riccati equation for scattering from stratified chiral spheres. IEEE Trans. Antennas Propag. 47(7), 1201–1207 (1999)
11. J.C. Monzon, Three-dimensional field expansion in the most general rotationally symmetric anisotropic medium: application to scattering by a sphere. IEEE Trans. Antennas Propag. 37(6), 728–735 (1989)
12. C.-W. Qui, L.-W. Li, T.-S. Yeo, S. Zouhdi, Scattering by rotationally symmetric anisotropic spheres: potential formulation and parametric studies. Phys. Rev. E 75(2), 026609 (2007)
13. A.D.U. Jafri, A. Lakhtakia, Scattering of an electromagnetic plane wave by a homogeneous sphere made of an orthorhombic dielectric–magnetic medium. J. Opt. Soc. Am. A 31(1), 89–100 (2014), erratum: 31(12), 2630 (2014)
14. A.D. Kiselev, V.Y. Reshetnyak, T.J. Sluckin, Light scattering by optically anisotropic scatterers: T-matrix theory for radial and uniform anisotropies. Phys. Rev. E 68(5), 056609 (2003)
15. J.L.-W. Li, W.-L. Ong, K.H.R. Zheng, Anisotropic scattering effects of a gyrotropic sphere characterized using the T-matrix method. Phys. Rev. E 85(3), 036601 (2012)
16. A. Novitsky, A.S. Salahi, A.V. Lavrenchenko, Spherically symmetric inhomogeneous bianisotropic media: wave propagation and light scattering. Phys. Rev. A 95(5), 053818 (2017)
17. H.M. Alkhoori, A. Lakhtakia, J.K. Breakall, C.F. Bohren, Scattering by a three-dimensional object composed of the simplest Lorentz-nonreciprocal medium. J. Opt. Soc. Am. A 35(12), 2026–2034 (2018)
18. H.M. Alkhoori, A. Lakhtakia, J.K. Breakall, C.F. Bohren, Plane-wave scattering by an ellipsoid composed of an orthorhombic dielectric–magnetic medium with arbitrarily oriented constitutive principal axes. J. Opt. Soc. Am. A 36(5), F60–F71 (2019)
19. V.V. Varadan, A. Lakhtakia, V.K. Varadan, Scattering by anisotropic sphere. IEEE Trans. Antennas Propag. 37(6), 800–802 (1989)
20. M. Sadati, J.A. Martinez-Gonzalez, Y. Zhou, N. Taheri Qazvini, K. Kurtenbach, X. Li, E. Bukusoglu, R. Zhang, N.L. Abbott, J.P. Hernandez-Ortiz, J.J. de Pablo, Prolate and oblate chiral liquid crystal spheroids. Sci. Adv. 6(18), eaba6728 (2020)
21. A. Lakhtakia, The Ewald–Oseen extinction theorem and the extended boundary condition method, Chap. 19, in The World of Applied Electromagnetics. ed. by A. Lakhtakia, C.M. Furse (Springer, Cham, 2018)
22. A. Lakhtakia, V.K. Varadan, V.V. Varadan, Scattering and absorption characteristics of lossy dielectric, chiral, nonspherical objects. Appl. Opt. 24(23), 4146–4154 (1985)
23. A. Lakhtakia, T.G. Mackay, Vector spherical wavefunctions for orthorhombic dielectric–magnetic medium with gyrotropic-like magnetoelectric properties. J. Opt. (India) 44(4), 201–213 (2012)
24. Y.-L. Geng, Analytical solution of electromagnetic scattering by a general gyrotropic sphere. IET Microw. Antennas Propag. 11(5), 1244–1250 (2012)
25. G.P. Zouros, G.D. Kolezas, N. Stefanou, T. Wriedt, EBCM for electromagnetic modeling of gyrotropic BoRs. IEEE Trans. Antennas Propag. 69(9), 6134–6139 (2021)
26. V. Schmidt, T. Wriedt, T-matrix method for biaxial anisotropic particles. J. Quant. Spectrosc. Radiat. Transf. 110(14–16), 1392–1397 (2009)
27. S. Ishii, S.-I. Inoue, A. Otomo, Electric and magnetic resonances in strongly anisotropic particles. J. Opt. Soc. Am. B 31(2), 212–218 (2014)
28. X. Yang, M. Jian, L. Shen, P.-H. Jia, Z. Rong, Y. Chen, L. Lei, J. Hu, A flexible FEM-BEM-DDM for EM scattering by multi-scale anisotropic objects. IEEE Trans. Antennas Propag. 69(12), 8562–8573 (2021)

29. A. Lakhtakia, Strong and weak forms of the method of moments and the coupled dipole method for scattering of time-harmonic electromagnetic fields. Int. J. Mod. Phys. C 3(3), 583–603 (1992). Errata: 4(3), 721–722 (1993)

30. C. Mei, M. Hasanovic, J.K. Lee, E. Arvas, Comprehensive solution to scattering by bianisotropic objects of arbitrary shape. Prog. Electromagn. Res. B 42, 335–362 (2012)

31. M. Maddah-Ali, S.H.H. Sadeghi, M. Dehmollaian, A method of moments for analysis of electromagnetic scattering from inhomogeneous bianisotropic bodies of revolution. IEEE Trans. Antennas Propag. 66(6), 2976–2986 (2018)

32. R. Alcaraz de la Osa, P. Albella, J.M. Saiz, F. Gonzalez, F. Moreno, Extended discrete dipole approximation and its application to bianisotropic media. Opt. Express 18(23), 23865–23871 (2010)

33. P. Ylä-Oijala, M. Taskinen, S. Järvenpää, Surface integral equation formulations for solving electromagnetic scattering problems with iterative methods. Radio Sci. 40(6), RS6002 (2005)

34. Q. Sun, E. Klaseboer, A.J. Yuffa, D.Y.C. Chan, Field-only surface integral equations: scattering from a dielectric body. J. Opt. Soc. Am. A 37(2), 284–293 (2020)

35. Z. Cui, S. Guo, J. Wang, F. Wu, Y. Han, Light scattering of Laguerre–Gaussian vortex beams by arbitrarily shaped chiral particles. J. Opt. Soc. Am. A 38(8), 1214–1223 (2021)

36. F.M. Kahnert, Numerical methods in electromagnetic scattering theory. J. Quant. Spectrosc. Radian. Transf. 79–80, 775–824 (2003)

37. J. Zhu, M.M. Li, Z.H. Fan, R.S. Chen, Analysis of EM scattering from 3D bi-anisotropic objects above a lossy half space using FE-BI with UV method. Appl. Comput. Electromagn. Soc. J. 28(10), 917–923 (2013)

38. J. Liu, Z. Li, J. Su, J. Song, On the volume-surface integral equation for scattering from arbitrary shaped composite PEC and inhomogeneous bi-isotropic objects. IEEE Access 7, 85594–85603 (2019)

39. K.P. Prokopidis, D.C. Zografopoulos, E.E. Kriezis, Rigorous broadband investigation of liquid-crystal plasmonic structures using finite-difference time-domain dispersive-anisotropic models. J. Opt. Soc. Am. B 30(10), 2722–2730 (2013)

40. A. Gansen, M. El Hachemi, S. Belouettar, O. Hassan, K. Morgan, EM modelling of arbitrary shaped anisotropic dielectric objects using an efficient 3D leapfrog scheme on unstructured meshes. Comput. Mech. 58(3), 441–455 (2016)

41. H. Massoudi, C.H. Durney, M.F. Iskander, Limitations of the cubical block model of man in calculating SAR distributions. IEEE Trans. Microw. Theory Tech. 32(8), 746–752 (1984)

42. K. Aydin, A. Hizal, On the completeness of the spherical vector wave functions. J. Math. Anal. Appl. 117(2), 428–440 (1986)

43. E. Noether, Invariant variation problems. Transp. Theory Stat. Phys. 1(3), 186–207 (1971)

44. A.T. de Hoop, On the plane-wave extinction cross-section of an obstacle. Appl. Sci. Res. B 7, 463–472 (1959)

45. R.G. Newton, Optical theorem and beyond. Am. J. Phys. 44(7), 639–642 (1976)

46. M. Faryad, A. Lakhtakia, Infinite-Space Dyadic Green Functions in Electromagnetism (Morgan & Claypool, San Rafael, 2018)

47. M. Faryad, A. Lakhtakia, On the Huygens principle for bianisotropic mediums with symmetric permittivity and permeability dyadics. Phys. Lett. A 381(7), 742–746 (2017). Erratum: 381(25-26), 2136 (2017)

48. M.J. Berg, C.M. Sorensen, A. Chakrabarti, J. Quant. Spectrosc. Radiat. Transf. 112(7), 1170–1181 (2011)

49. C.-T. Tai, Dyadic Green Functions in Electromagnetic Theory, 2nd edn. (IEEE, New York, 1994)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.