Completely normal elements in finite abelian extensions

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Abstract

We give a completely normal element in the maximal real subfield of a cyclotomic field over the field of rational numbers, which is different from [13]. This result is a consequence of the criterion for a normal element developed in [7]. Furthermore, we find a completely normal element in certain extension of modular function fields in terms of a quotient of the modular discriminant function.

1 Introduction

Let $L/K$ be a finite Galois extension. By the normal basis theorem [14] there exists an element $x \in L$ such that $\{x^\gamma \mid \gamma \in \text{Gal}(L/K)\}$ is a $K$-basis of $L$, a so-called normal basis of $L/K$. Such an element $x$ is said to be normal in $L/K$. Moreover, if $x$ is normal in $L/F$ for every intermediate field $F$, then $x$ is said to be completely normal in $L/K$. The existence of a completely normal element was first proved by Blessenohl and Johnsen [1].

Throughout this paper we let $\zeta_\ell = e^{2\pi i/\ell}$ be a primitive $\ell$th root of unity for a positive integer $\ell$. Furthermore, we let $\mathbb{Q}(\zeta_\ell)^+$ be the maximal real subfield of the $\ell$th cyclotomic field $\mathbb{Q}(\zeta_\ell)$.

Okada [13] proved that if $k$ and $\ell (> 2)$ are positive integers with $k$ odd and $T$ is a set of representatives for which $(\mathbb{Z}/\ell\mathbb{Z})^* = T \cup (-T)$, then the numbers $(1/\pi^k)(d/dz)^k(\cot \pi z)|_{z=a/\ell}$ for $a \in T$ form a normal basis of $\mathbb{Q}(\zeta_\ell)^+ / \mathbb{Q}$, which generalized the works of Chowla [2] when $k = 1$. He utilized the partial fractional decomposition of $(d/dz)^k(\cot \pi z)$ and the Frobenius determinant relation [11] Chapter 21 Theorem 5].

2010 Mathematics Subject Classification. 11F03, 11R18, 12F05.

Key words and phrases. cyclotomic extensions, modular functions, normal bases.

This research was partially supported by Basic Science Research Program through the NRF of Korea funded by MEST (2011-0001184). The second named author was partially supported by TJ Park Postdoctoral Fellowship.
On the other hand, let \( \mathcal{N} \) be the set of positive integers which are either odd or divisible by 4. Let \( \ell \in \mathcal{N} \). Hachenberger [5] constructed an element \( w \in \mathbb{Q}(\zeta_\ell) \) which is simultaneously normal in \( \mathbb{Q}(\zeta_\ell)/\mathbb{Q}(\zeta_n) \) for each \( n \in \mathcal{N} \) dividing \( \ell \). The main tool is the notion of cyclic submodules in \( \mathbb{Q}(\zeta_\ell)/\mathbb{Q}(\zeta_n) \) [4].

Let \( K \) be an imaginary quadratic field other than \( \mathbb{Q}(\sqrt{-1}) \) and \( \mathbb{Q}(\sqrt{-3}) \). For an integer \( \ell \geq 2 \) let \( K(\ell) \) be the ray class field of \( K \) modulo \( \ell \). Recently, Jung et al [7] showed that the singular value of a Siegel function is normal in \( K(\ell)/K \). To this end, they derived a criterion to determine a normal element in a finite abelian extension of number fields from the Frobenius determinant relation.

Actually the criterion can be extended to determine a completely normal element in a finite abelian extension (Theorem 2.2). In this paper we shall give a completely normal element in \( \mathbb{Q}(\zeta_\ell)^+/\mathbb{Q} \) for an integer \( \ell \geq 5 \) by using the criterion (Theorems 3.1 and 3.2). The element is expressed in terms of the cosine function, which is simple and totally different from that of [13]. Furthermore, we shall find a completely normal element in certain extension of modular function fields in terms of a quotient of the modular discriminant function (Theorem 4.3).

## 2 A criterion for completely normal elements

Throughout this section we let \( L/K \) be a finite abelian extension of degree \( n \geq 2 \) with \( G = \text{Gal}(L/K) \). Furthermore, we let \( | \cdot | \) be a valuation on \( L \). Then \( | \cdot | \) satisfies the triangle inequality, namely, \( |x + y| \leq |x| + |y| \) for all \( x, y \in L \). It follows that

\[
|x| - |y| \leq |x + y| \quad \text{for all } x, y \in L.
\]

In particular, if \( | \cdot | \) is nonarchimedean, then \( |x + y| \leq \max\{|x|, |y|\} \), from which one can readily deduce that

\[
|x|^m - |y|^m \leq |x + y|^m \quad \text{for all } x, y \in L \text{ and any positive real number } m
\]

[6, Chapter II §1].

**Lemma 2.1.** An element \( x \in L \) is normal in \( L/K \) if and only if

\[
\sum_{\gamma \in G} \chi(\gamma^{-1})x^\gamma \neq 0 \quad \text{for all characters } \chi \text{ on } G.
\]

**Proof.** [7, Proposition 2.3].

**Theorem 2.2.** Assume that there exists an element \( x \in L \) such that

\[
|x^\gamma/x| < 1 \quad \text{for all } \gamma \in G - \{\text{Id}\}.
\]
Let \( m \) be any positive integer such that

\[ |x^{\gamma}/x|^m \leq 1/n \quad \text{for all } \gamma \in G - \{\text{Id}\}. \tag{3} \]

Then \( x^m \) is completely normal in \( L/K \). In particular, if \( |\cdot| \) is nonarchimedean, then any positive power of \( x \) is completely normal in \( L/K \).

**Proof.** Let \( F \) be an intermediate field of \( L/K \) with \( \ell = [L:F] \) and \( H = \text{Gal}(L/F) \) \((\leq G)\). For any character \( \chi \) on \( H \) we find that

\[
|\sum_{\gamma \in H} \chi(\gamma^{-1})(x^m)^{\gamma}| \geq |x^m|(1 - \sum_{\gamma \in H - \{\text{Id}\}} |(x^m)^{\gamma}/x^m|) \quad \text{by (1)}
\]

\[
\geq |x^m|(1 - (1/n)(\ell - 1)) \quad \text{by (3)}
\]

\[
= |x^m|(n - \ell + 1)/n
\]

\[
> 0 \quad \text{because } \ell \leq n.
\]

This shows that \( x^m \) is normal in \( L/F \) by Lemma 2.1 and hence \( x^m \) is completely normal in \( L/K \). Furthermore, if \( |\cdot| \) is nonarchimedean, then we derive for any positive integer \( t \) that

\[
|\sum_{\gamma \in H} \chi(\gamma^{-1})(x^t)^{\gamma}|^m \geq |x^t|^m(1 - \sum_{\gamma \in H - \{\text{Id}\}} |(x^t)^{\gamma}/x^t|^m) \quad \text{by (2)}
\]

\[
\geq |x^t|^m(1 - (1/n)^t(\ell - 1)) \quad \text{by (3)}
\]

\[
\geq |x^t|^m(1 - (1/n)(\ell - 1))
\]

\[
= |x^t|^m(n - \ell + 1)/n
\]

\[
> 0 \quad \text{because } \ell \leq n.
\]

Hence \( x^t \) is completely normal in \( L/K \) again by Lemma 2.1. This completes the proof. \( \square \)

**Corollary 2.3.** Let \( L/K \) be an abelian extension of number fields. Assume that there exists an element \( x \in L \) such that

(i) \( x \) is an algebraic integer,

(ii) \( L = K(x) \),

(iii) \( x^{\gamma} \) are real for all \( \gamma \in G \).

Let \( a \) and \( b \) be nonzero integers such that \( 2 < |a/b| \), where \( |\cdot| \) is the usual absolute value on \( \mathbb{C} \). Then, a high power of \( ax + b \) is completely normal in \( L/K \).
Proof. Suppose that there exist distinct elements $\gamma$ and $\delta$ of $G$ such that $|ax^\gamma + b| = |ax^\delta + b|$. Since $x^\gamma$ and $x^\delta$ are real by the assumption (iii), we get $ax^\gamma + b = \pm(ax^\delta + b)$. Moreover, since $x^\gamma \neq x^\delta$ by the assumption (ii) and the fact $\gamma \neq \delta$, we obtain $ax^\gamma + b = -(ax^\delta + b)$, from which it follows that $x^\gamma + x^\delta = -2b/a$. Note that $x^\gamma + x^\delta$ is an algebraic integer by the assumption (i), but $-2b/a$ is a rational number such that $0 < |2b/a| < 1$, which yields a contradiction.

For each intermediate field $F$ of $L/K$ with $[L : F] \geq 2$, the preceding argument shows that there is a unique element $\gamma_F$ of $\text{Gal}(L/F)$ and a positive integer $m_F$ such that $|(ax^\gamma + b)/(ax^{\gamma_F} + b)|^{m_F} \leq 1/[L : F]$ for all $\gamma \in \text{Gal}(L/F) - \{\gamma_F\}$. If we set $m = \max\{m_F| F \}$, then we get from Theorem 2.2 that $(ax^{\gamma_F} + b)^m$ is completely normal in $L/F$. In particular, the set $\{(ax^{\gamma_F} + b)^m| \gamma \in \text{Gal}(L/F)\}$ is a normal basis of $L$ over $F$; and hence $((ax^{\gamma_F} + b)^m)^{\gamma_F^{-1}} = (ax + b)^m$ is normal in $L/F$. This implies that $(ax + b)^m$ is completely normal in $L/K$ because $F$ is arbitrary. This completes the proof. 

Remark 2.4. If $L/K$ is an abelian extension of totally real number fields, then there always exists such an element $x \in L$ which satisfies the assumptions of Corollary 2.3.

3 Maximal real subfields of cyclotomic fields

Let $\ell$ be a positive integer. As is well-known, $\mathbb{Q}(\zeta_\ell)^+ = \mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$ and $\text{Gal}(\mathbb{Q}(\zeta_\ell)^+ / \mathbb{Q}) \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times /\{\pm 1\}$, whose actions are given as follows: if $t \in (\mathbb{Z}/\ell\mathbb{Z})^\times /\{\pm 1\}$, then $(\zeta_\ell + \zeta_\ell^{-1})^t = \zeta_\ell^t + \zeta_\ell^{-t}$. Denote the number of positive integers relatively prime to $\ell$ by $\phi(\ell)$. Then we have

$$[\mathbb{Q}(\zeta_\ell)^+ : \mathbb{Q}] = \begin{cases} 1, & \text{if } \ell = 1, 2, 3, 4, 6, \\ \phi(\ell)/2 \geq 2, & \text{otherwise} \end{cases}$$

[15] Chapter 2.

Let $| \cdot |$ denote the usual absolute value on $\mathbb{C}$.

Theorem 3.1. Let $\ell \neq 1, 2, 3, 4, 6$ be a positive integer. If $m$ is any positive integer such that

$$((\cos(4\pi/\ell) + 1)/(\cos(2\pi/\ell) + 1))^{m} \leq 2/\phi(\ell),$$

then $(\cos(2\pi/\ell) + 1)^m$ is completely normal in $\mathbb{Q}(\zeta_\ell)^+ / \mathbb{Q}$.

Proof. Let $x = (\zeta_\ell + \zeta_\ell^{-1})/2 + 1 = \cos(2\pi/\ell) + 1$. If $\gamma \in \text{Gal}(\mathbb{Q}(\zeta_\ell)^+ / \mathbb{Q}) - \{\text{Id}\}$, then $x^\gamma = (\zeta^t + \zeta^{-t})/2 + 1$ for some integer $t$ with $\gcd(\ell, t) = 1$ and $1 < t \leq [\ell/2]$, where $[\cdot]$ is the Gauss symbol. We achieve that

$$|x^\gamma/x| = |((\zeta^t + \zeta^{-t})/2 + 1)/((\zeta + \zeta^{-1})/2 + 1)|$$

$$= |(\cos(2t\pi/\ell) + 1)/(\cos(2\pi/\ell) + 1)|$$

$$\leq |(\cos(4\pi/\ell) + 1)/(\cos(2\pi/\ell) + 1)|,$$
which is less than 1. The result follows from Theorem 2.2.

**Theorem 3.2.** Let \( \ell (\geq 5) \) be an odd integer. If \( m \) is any positive integer such that
\[
(\cos(2\pi/\ell)/\cos(\pi/\ell))^m \leq 2/\phi(\ell),
\]
then \( \cos^m(\pi/\ell) \) is completely normal in \( \mathbb{Q}(\zeta_\ell)^+/\mathbb{Q} \).

**Proof.** Let \( x = -(\zeta^{(\ell-1)/2} + \zeta^{-(\ell-1)/2})/2 \). Since \( 1 \cdot \ell + (-2) \cdot (\ell - 1)/2 = 1 \), we get \( \gcd(\ell, (\ell - 1)/2) = 1 \), which implies that \( x \) is a conjugate of \( -(\zeta + \zeta^{-1})/2 \). Hence, if \( \gamma \in \text{Gal}(\mathbb{Q}(\zeta_\ell)^+/\mathbb{Q}) - \{\text{Id}\} \), then \( x^\gamma = -(\zeta^t + \zeta^{-t})/2 \) for some integer \( t \) with \( \gcd(\ell, t) = 1 \) and \( 1 \leq t < (\ell - 1)/2 \). We find that
\[
|x^\gamma/x| = \left|\frac{-(\zeta^t + \zeta^{-t})/2}{-(\zeta^{(\ell-1)/2} + \zeta^{-(\ell-1)/2})/2}\right|
= \left|\frac{-\cos(2t\pi/\ell) - \cos(\pi/\ell)}{-\cos(2\pi/\ell)}\right|
= \left|\frac{\cos(2t\pi/\ell)}{\cos(\pi/\ell)}\right|
\leq \frac{\cos(2\pi/\ell)}{\cos(\pi/\ell)},
\]
which is less than 1. We obtain the assertion by Theorem 2.2.

**Lemma 3.3.** Let \( L_1 \) and \( L_2 \) be finite Galois extensions of a number field \( K \) such that \( L_1 \cap L_2 = K \). If \( x_k \in L_k \) is normal in \( L_k/K \) \((k = 1, 2)\), then \( x_1x_2 \) is normal in \( L_1L_2/K \).

**Proof.** [8, p.227].

**Lemma 3.4.** Let \( t = 4 \) or an odd prime \( p \) such that \( p \equiv 3 \pmod{4} \). Then \( \mathbb{Q}(\zeta_t) \) contains a unique quadratic extension of \( \mathbb{Q} \), namely \( \mathbb{Q}(\sqrt{-t}) \).

**Proof.** [6, Theorem 11.1].

**Theorem 3.5.** Let \( t = 4 \) or an odd prime \( p \) such that \( p \equiv 3 \pmod{4} \). Let \( \ell (\neq 1, 2, 3, 4, 6) \) be a positive integer. If \( m \) is any positive integer such that
\[
((\cos(4\pi/t\ell) + 1)/\cos(2\pi/t\ell) + 1))^m \leq 2/\phi(t\ell),
\]
then \( (\sqrt{-t} + 1)(\cos(2\pi/t\ell) + 1)^m \) is normal in \( \mathbb{Q}(\zeta_{t\ell})/\mathbb{Q} \).

**Proof.** One can readily show that \( \sqrt{-t} + 1 \) is normal in \( \mathbb{Q}(\sqrt{-t})/\mathbb{Q} \). And, if \( m \) is any positive integer which satisfies the condition (4), then \( (\cos(2\pi/t\ell) + 1)^m \) is normal in \( \mathbb{Q}(\zeta_{t\ell})^+/\mathbb{Q} \) by Theorem 3.1. On the other hand, since \( \mathbb{Q}(\sqrt{-t}) \) is an imaginary quadratic field contained in \( \mathbb{Q}(\zeta_t) \subset \mathbb{Q}(\zeta_{t\ell}) \) by Lemma 3.4, we have \( \mathbb{Q}(\sqrt{-t}) \cap \mathbb{Q}(\zeta_{t\ell})^+ = \mathbb{Q} \) and \( \mathbb{Q}(\sqrt{-t})\mathbb{Q}(\zeta_{t\ell})^+ = \mathbb{Q}(\zeta_{t\ell}) \). Now, the result follows from Lemma 3.3.
4 Fields of modular functions

Let $\mathbb{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ be the complex upper half-plane. For a positive integer $N$ we consider the group

\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}, \]

which acts on $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ as fractional linear transformations. Then a (meromorphic) modular function for $\Gamma_0(m)$ is a $\mathbb{C}$-valued function on $\mathbb{H}$, except for isolated singularities, which satisfies the following three conditions:

(i) $f(\tau)$ is meromorphic on $\mathbb{H}$,

(ii) $f(\tau)$ is invariant under $\Gamma_0(N)$,

(iii) $f(\tau)$ is meromorphic at the cusps $\mathbb{Q} \cup \{\infty\}$.

We denote the field of all modular functions for $\Gamma_0(N)$ by $\mathbb{C}(X_0(N))$. As is well-known, $\mathbb{C}(X_0(N))$ is a Galois extension of $\mathbb{C}(X_0(1))$ whose Galois group is isomorphic to the quotient group $\Gamma_0(1)/\Gamma_0(N)$ [11, Chapter 6].

For a pair $(r_1, r_2) \in \mathbb{Q}^2 - \mathbb{Z}^2$, the Siegel function $g(r_1, r_2)(\tau)$ on $\mathbb{H}$ is defined by the following infinite product

\[ g(r_1, r_2)(\tau) = -q^{(1/2)B_2(r_1)}e^{\pi i r_2(r_1 - 1)}(1 - q^{r_1}e^{2\pi i r_2}) \prod_{n=1}^{\infty} (1 - q^{n+r_1}e^{2\pi i r_2})(1 - q^{n-r_1}e^{-2\pi i r_2}), \]

where $q = e^{2\pi i r}$ and $B_2(X) = X^2 - X + 1/6$ is the second Bernoulli polynomial.

For $X \in \mathbb{R}$ we let $\langle X \rangle$ be the fractional part of $X$ in the interval $[0, 1)$.

**Lemma 4.1.** Let $N \geq 2$ be an integer and $(r_1, r_2) \in (1/N)\mathbb{Z}^2 - \mathbb{Z}^2$.

(i) $g_{(r_1, r_2)}(\tau)^{12N}$ is determined by $\pm(r_1, r_2) \pmod{\mathbb{Z}^2}$.

(ii) If $(a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})$, then

\[ g_{(r_1, r_2)}(\tau)^{12N} \circ \alpha = g_{(r_1, r_2)}(\tau)^{12N} = g_{(r_1a + r_2c, r_1b + r_2d)}(\tau)^{12N}. \]

(iii) $\text{ord}_q g_{(r_1, r_2)}(\tau) = (1/2)B_2(\langle r_1 \rangle)$.

**Proof.** [10] Chapter 2 §1. \qed

Let

\[ \Delta(\tau) = (2\pi)^{12}q^2 \prod_{n=1}^{\infty} (1 - q^n)^{24} \quad (\tau \in \mathbb{H}) \]

be the modular discriminant function.
Lemma 4.2. We have the relation

\[
\Delta(\tau)/\Delta(N\tau) = N^{12} \prod_{k=1}^{N} g_{(0,k/N)}(\tau)^{-12},
\]

which is a modular function for \( \Gamma_0(N) \).

Proof. [9, Proposition 5.1]. \( \square \)

Theorem 4.3. Let \( N (\geq 2) \) be an integer. Let \( L = \mathbb{C}(X_0(N)) \) and \( K \) be the subfield of \( L \) fixed by \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) in \( \Gamma_0(1)/\Gamma_0(N) \). Then, any positive power of \( \Delta(\tau)/\Delta(N\tau) \) is completely normal in \( L/K \).

Proof. By Galois theory we have

\[
\text{Gal}(L/K) \simeq \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \text{ in } \Gamma_0(1)/\Gamma_0(N)
\]

\[
= \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t = 0, 1, \ldots, N - 1 \right\} \text{ in } \Gamma_0(1)/\Gamma_0(N).
\]

Consider the nonarchimedean valuation \( | \cdot | \) on \( L \) defined by

\[
| \cdot | : L \to \mathbb{R}_{\geq 0}, \quad \alpha \mapsto |\alpha| = \exp(-\text{ord}_q \alpha).
\]

Let \( x = \Delta(\tau)/\Delta(N\tau) \). For any \( \gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in \text{Gal}(L/K) - \{ \text{Id} \} \) we find that

\[
|x^\gamma/x|^N = \left| (N^{12N} \prod_{k=1}^{N-1} g_{(0,k/N)}(\tau)^{-12N})^N/N^{12N} \prod_{k=1}^{N-1} g_{(0,k/N)}(\tau)^{-12N} \right|^N \text{ by Lemma 4.2}
\]

\[
= \left| \prod_{k=1}^{N-1} g_{(kt,N,k/N)}(\tau)^{-12N} / N^{12N} \prod_{k=1}^{N-1} g_{(0,k/N)}(\tau)^{-12N} \right|^N \text{ by Lemma 4.1(i) and (ii)}
\]

\[
= \exp \left( - \sum_{k=1}^{N-1} (1/2)B_2(\langle kt/N \rangle) \cdot (-12N) + \sum_{k=1}^{N-1} (1/2)B_2(0) \cdot (-12N) \right)
\]

by Lemma 4.1(iii)

\[
= \exp \left( 6N \sum_{k=1}^{N-1} (B_2(\langle kt/N \rangle) - B_2(0)) \right)
\]

< 1 because \( B_2(X) \) has its maximum at \( X = 0 \) in the interval \( [0,1) \),

from which it follows that \( |x^\gamma/x| < 1 \). Therefore \( x \) is completely normal in \( L/K \) by Theorem 2.2. \( \square \)
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