Bonus Symmetries of $\mathcal{N} = 4$ Super-Yang-Mills Correlation Functions via AdS Duality

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General conjectures about the $SL(2, \mathbb{Z})$ modular transformation properties of $\mathcal{N} = 4$ super-Yang-Mills correlation functions are presented. It is shown how these modular transformation properties arise from the conjectured duality with $IIB$ string theory on $AdS_5 \times S^5$. We discuss in detail a prediction of the AdS duality: that $\mathcal{N} = 4$ field theory, in an appropriate limit, must exhibit bonus symmetries, corresponding to the enhanced symmetries of $IIB$ string theory in its supergravity limit.

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1. Introduction and summary

As with all conjectured dualities, that of \[ \mathcal{N} = 4 \] supersymmetric $SU(N)$ Yang-Mills and IIB string theory with $N$ units of $F_5$ flux, which compactifies on $AdS_5 \times S^5$, relates the weakly coupled limit of one theory to the strongly coupled limit of the dual. The string side is weakly coupled in the limit of small $g_s = 4\pi g_{YM}^2$ and large ’t Hooft coupling $\lambda \equiv g_{YM}^2 N$, where it can be approximated by semi-classical IIB supergravity. In this limit, the field theory dual is strongly coupled, as the relevant coupling is $\lambda = g_{YM}^2 N$, and perturbation theory is not valid. The mapping between weak coupling of one theory and strong coupling of the dual makes duality very powerful, but also difficult to check unless one has independent, non-perturbative information about at least one of the dual theories.

A first non-trivial check of the duality \[ \mathcal{I} \] is that both theories have the same symmetry group, $PSU(2, 2|4)$, which has bosonic subgroup $SU(2, 2) \times SU(4)_R$ and 32 supercharges. Also, both have the $SL(2, Z)$ S-duality group \[ \mathcal{I} \]. $PSU(2, 2|4)$ has short representations (to be discussed in detail in what follows), labeled by positive integers $p$, whose $SU(2, 2) \times SU(4)_R$ quantum numbers are completely fixed in terms of $p$ and thus not renormalized. In the $\mathcal{N} = 4$ gauge theory, the independent $p$ are the degrees of the Casimirs of the gauge group. In the dual IIB supergravity on $AdS_5 \times S^5$, $p$ corresponds to the $S^5$ Kaluza-Klein spherical harmonics of massless 10d supergravity fields \[ \mathcal{I} \]. The two sides, the spectrum of short representation operators in the 4d field theory, versus KK modes in the 5d AdS supergravity, agree in the large $N$ limit \[ \mathcal{I} \].

Non-renormalization theorems are known for a few $\mathcal{N} = 4$ field theory current correlation functions, which can thus be used to check the conjectured duality. More generally, the feeling is that the power of $\mathcal{N} = 4$ supersymmetry has not been fully exploited and that there are other non-renormalization theorems waiting to be discovered. Quantities for which the answer from weakly coupled gravity differs from that of weakly coupled field theory presumably do not satisfy a non-renormalization theorem (assuming the duality is correct) and the answer from weakly coupled gravity is regarded as a non-trivial prediction for strongly coupled field theory.

It sometimes happens that the weakly coupled gravity result unexpectedly agrees with that of free field theory; this can be regarded as evidence for a new non-renormalization theorem. This was the case in the results of \[ \mathcal{I} \] for three-point functions of normalized primary operators in short multiplets. This led the authors of \[ \mathcal{I} \] to conjecture that these 3-point functions are independent of the ’t Hooft coupling in the large $N$ limit and perhaps
even independent of $g_{YM}$ for arbitrary $N$. The fate of the CFT/AdS correspondence is completely independent of the fate of such a conjectured non-renormalization theorem; nevertheless, the latter is an interesting question in the field theory. Evidence for the conjectured non-renormalization of such three-point functions of primary operators was obtained in [4], where it was shown in a purely field theory analysis for small $g_{YM}$ that, for all $N$, leading order radiative corrections to all such two-point and three-point correlation functions surprisingly conspire to cancel. This possibly hints at a larger symmetry of the $\mathcal{N} = 4$ theory.

We discuss predictions for such a larger symmetry of $\mathcal{N} = 4$ field theory based on assuming the duality with IIB string theory on $AdS_5 \times S^5$. In the limit where IIB string theory is approximated by IIB supergravity, there are additional approximate symmetries: the $SL(2, \mathbb{Z})$ symmetry is enlarged to an $SL(2, \mathbb{R})$ symmetry and there is its maximal compact subgroup, $U(1)_Y$, which enters into the description of interacting IIB supergravity in terms of an $SL(2, \mathbb{R})/U(1)_Y$ coset. These enhanced approximate symmetries must then also show up in the dual $\mathcal{N} = 4$ gauge theory in the appropriate limit.

Stringy corrections to IIB supergravity, which generally violate these approximate enhanced symmetries, are suppressed when

$$\frac{\alpha'}{L^2} \sim \frac{1}{\sqrt{g_{YM}^2 N}} \ll 1. \tag{1.1}$$

Here $L$ is the size of both $AdS_5$ and $S^5$, which is related by flux quantization to the units $N$ of $F_5$ flux by

$$\frac{L^4}{\kappa_{10}} \sim N, \tag{1.2}$$

with $\kappa_{10}$ the 10d gravitational coupling.

The condition (1.1) alone is not sufficient to ensure that stringy corrections are suppressed, as D-string effects also lead to $SL(2, \mathbb{R})$ and $U(1)_Y$ violating terms; to have these effects also be suppressed, we also need

$$\frac{\tilde{\alpha}'}{L^2} \sim \sqrt{\frac{g_{YM}^2}{N}} \ll 1. \tag{1.3}$$

It is in the double limit, where both (1.1) and (1.3) are satisfied, that our bonus symmetries of $\mathcal{N} = 4$ Yang Mills theories are predicted to hold; in what follows, we will refer to this
as the “double limit.” Clearly the double limit requires large \( N \). Because the natural, dimensionless, quantum expansion parameter of the gravity dual is

\[
\hbar \sim \frac{\kappa_5^2}{L^3} \sim \frac{\kappa_{10}^2}{L^8} \sim N^{-2},
\]

where \( \kappa_5 \) is the 5d gravitational coupling, which is related to \( \kappa_{10} \) by dimensional reduction, \( \hbar \ll 1 \) and the gravity dual is semi-classical in the double limit.

It must be stressed that the larger symmetry applies only to those operators of \( \mathcal{N} = 4 \) Yang-Mills which correspond to states in supergravity. Those operators in long multiplets which correspond to stringy states, which are expected to have large anomalous dimension \( \Delta \sim (g_{YM}^2 N)^{1/4} \) in the double limit, should not be expected to respect these symmetries. We consider here only operators in the standard short multiplets of \( PSU(2, 2|4) \); these always correspond to states visible in supergravity. The bonus symmetry of the double limit should also extend to those operators in long multiplets which map to non-stringy, multi-particle supergravity states, though this will not be discussed here.

We consider, then, arbitrary correlation functions of operators \( \mathcal{O}_i(x) \) in short representations of the superconformal group:

\[
\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \rangle = f_{i_1 \ldots i_n}(x_i; N; g_{YM}, \theta_{YM}).
\]  

(1.5)

We argue that a prediction of the duality of \( \mathcal{N} = 4 \) is that, in the double limit discussed above, the leading behavior of all such correlation functions is

\[
\langle \prod_{i=1}^{n} \mathcal{O}_i(x_i) \rangle = N^2 f_{i_1 \ldots i_n}(x_i),
\]

(1.6)

where the functions are independent of \( N \) and \( g_{YM} \) and \( \theta \) to leading order. The \( N \) dependence, as will be discussed, is associated with tree-level supergravity. The reason for the \( g_{YM} \) and \( \theta \) independence of (1.6) is the \( SL(2, R) \) symmetry of supergravity: because \( SL(2, R) \) maps the gauge coupling

\[
\tau \equiv \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g_{YM}^2} \rightarrow a\tau + b \quad \frac{c\tau + d}{ct + d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, R),
\]

(1.7)

which can be used to map any \( \tau \) in the upper-half-plane to any other \( \tau \), correlation functions in this limit must be independent of \( \tau \). For arbitrary correlation functions of operators

\footnote{I am grateful to N. Seiberg for reminding me about these long multiplets.}
in short multiplets, the leading term in the double limit is thus predicted to be always completely independent of the ’t Hooft coupling $\lambda = g_{YM}^2 N$! Because $SL(2,R)$ is broken to $SL(2,Z)$ in the full string theory, correlation functions are generally expected to have non-trivial $\tau$ dependence in the terms which are sub-leading in the double limit. The normalization of the operators $O_i$, which is important in making sense of the statement (1.6), will be discussed in the next section.

It is also interesting to consider the local $U(1)_Y$, which is the maximal compact subgroup of $SL(2,R)$, and enters in the $SL(2,R)/U(1)_Y$ description of IIB supergravity, which is briefly reviewed in sect. 3. Although $U(1)_Y$ is a local symmetry, there is no corresponding gauge field and thus no corresponding conserved current in the field theory. Nevertheless, $U(1)_Y$ leads to a non-trivial $R$-type symmetry, under which the super-charges transform, of the superconformal algebra. It is non-trivial that the superconformal algebra admits such a symmetry, as will be discussed in sect. 4. The operators $O_i(x)$ in short representations of the superconformal group can all be assigned definite charges, opposite to those of the supergravity fields to which these operators couple. The $U(1)_Y$ symmetry of supergravity implies a selection rule for field theory correlation functions of operators

$$\langle \prod_{i=1}^{n} O_i^{(q_i)}(x_i) \rangle = 0 \quad \text{unless} \quad \sum_{i=1}^{n} q_i = 0,$$

where $O_i^{(q_i)}$ is a short-multiplet operator of $U(1)_Y$ charge $q_i$. As we will discuss in sect. 5, $U(1)_Y$ is not a symmetry of the field theory; nevertheless, it is predicted to yield approximate selection rules (1.8) in the double limit of (1.1) and (1.3).

The $\tau$ independence of (1.6) actually follows as a consequence of the selection rule (1.8). To see this, note that the derivative of an arbitrary $n$-point correlation function with respect to the gauge coupling $\tau$ is given by

$$\partial_\tau \langle \prod_{i} O_i(x_i) \rangle = \tau^{-1} \int d^4 z \langle O_{\tau}^{(-4)}(z) \prod_{i} O_i(x_i) \rangle,$$

where $O_{\tau}^{(-4)}$ is the exactly marginal operator, to be discussed in detail in what follows, which couples to $\tau$ in the action; it’s the on-shell $N = 4$ Lagrangian. There is a conjugate operator $O_{\tilde{\tau}}^{(4)}$, which couples to $\tilde{\tau}$, allowing us to independently vary both $g_{YM}$ and $\theta$. The $U(1)_Y$ charge of $O_{\tau}^{(-4)}$ is $-4$, as indicated by the superscript. It follows from (1.8) and (1.9) that non-zero correlation functions are independent of $\tau$, as in (1.6).
In sect. 6 we make some general conjectures about the $SL(2, Z)$ modular transformation properties of $\mathcal{N} = 4$ super-Yang-Mills correlation functions. For any gauge group, we conjecture that arbitrary correlation functions transform under $SL(2, Z)$ modular transformations as
\[
\langle \prod_i \mathcal{O}^{(q_i)}(x_i) \rangle_{\tau + \frac{c\tau + d}{c\tau + d}} = \left( \frac{c\tau + d}{c\tau + d} \right)^{q_T/4} \langle \prod_i \mathcal{O}^{(q_i)}(x_i) \rangle_{\tau},
\]
(1.10)
with $q_T = \sum_i q_i$ the net $U(1)_Y$ charge of the correlation function. (In the case of $Sp(n)$ and $SO(2n + 1)$, which are exchanged by $\tau \rightarrow -1/\tau$, the correlation functions on the two sides of (1.10) would be for these two dual groups; because we are only discussing $SU(N)$, this will not concern us here.) In the supergravity limit, where $SL(2, Z)$ is extended to $SL(2, R)$, the transformation (1.10) implies the $\tau$ independence of (1.6) and the $U(1)_Y$ selection rule (1.8).

String theory leads to higher dimension terms in the effective action which violate the $SL(2, R)$ and $U(1)_Y$ symmetries. This agrees with the fact that these are not symmetries of $\mathcal{N} = 4$ field theory for general $g_{YM}$ and $N$. The predicted form of the corresponding corrections to the $\mathcal{N} = 4$ field theory correlation functions, away from the double limit, is discussed in sect. 7. These corrections, which violate $SL(2, R)$ and $U(1)_Y$, are subleading by at least $N^{-3/2}$, for fixed $g_{YM}$, and satisfy our $SL(2, Z)$ modular transformation rule (1.10). For small $g_{YM}$, these corrections are sub-leading by at least order $(g_{YM}^2 N)^{-3/2}$.

Based on the form of the stringy violations of $U(1)_Y$ found in the $\alpha'$ expansion of IIB string theory, we conjecture that $U(1)_Y$ is actually an exact symmetry of $n \leq 4$-point functions, i.e. valid for all $g_{YM}$ and $N$. Using (1.3), this would have as a consequence the exact $SL(2, R)$ invariance of $n \leq 3$-point functions, in line with the conjecture and calculations of [4]. In sect. 8 we discuss some aspects of attempting to prove exact $U(1)_Y$ invariance of $n$-point functions with low $n$, though we only succeeded in finding a simple proof of exact $U(1)_Y$ invariance for $n = 2$-point functions. The exact $U(1)_Y$ invariance of 2-point functions implies that arbitrary $n$-point functions also respect $U(1)_Y$ in the leading Born-approximation appropriate for small $g_{YM}^2 N$.

In sect. 9 we examine $U(1)_Y$ in the context of the $\mathcal{N} = 4$ harmonic superspace formalism of [1], and find a contradiction: assuming the validity of this formalism and the classification of invariants in [8], we prove that an arbitrary $n$-point correlation function would exactly respect $U(1)_Y$, for all $g_{YM}$ and $N$, for any $n$. This result would imply that all $n$-point correlation functions of operators in short multiplets would be exactly
independent of $g_{YM}$ for all $g_{YM}$ and $N$, a result which is definitely incorrect for general $n$-point functions! As discussed further in sect. 9, this contradiction shows that the $\mathcal{N} = 4$ harmonic superspace formalism is either invalid or incomplete. This issue does not in any way affect the results or conclusions of the other sections of this paper.

The enhanced approximate $\text{SL}(2, R)$ and $U(1)_Y$ symmetries of the double limit (I.1) and (I.3) are also predicted to occur in the $\mathcal{N} = 2, 1, 0$ Yang-Mills theories associated with orbifolds of the $\mathcal{N} = 4$ theory [14,15] and with the $\mathcal{N} = 1$ theory of [16]. They should also occur for the 3d $\mathcal{N} = 0$ theory obtained from the 4d theory at finite temperature. However, there is no analog of these additional global symmetry in the case of 11d supergravity (which has no symmetries), so no such enhanced approximate symmetry is to be expected for the 3d or 6d theories associated via [1] with $M$ theory on $AdS_4 \times S^7$ or $AdS_7 \times S^4$. Therefore, the 4d $\mathcal{N} = 0$ theory obtained as in [17], from a compactification of the 6d theory which breaks supersymmetry will also not have such enhanced approximate symmetries.

The $U(1)_Y$ symmetry also entered in the discussion in a recent work on $\mathcal{N} = 6$ supergravity and $SU(2,2\vert 3)$ superconformal invariance [18], which appeared in the final stages of writing up this paper. In particular, the discussion in the last section of [18] has some overlap with the bonus symmetries discussed here.

2. The normalization of $\mathcal{N} = 4$ operators

Before discussing the enhanced symmetries of supergravity, we here consider some basic points concerning the $N$ dependence of correlation functions of operators in $\mathcal{N} = 4$

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2 In the original version of this paper, the conclusion that arbitrary $n$-point functions are not renormalized was referred to as “highly suspicious,” and it was pointed out that it could probably be disproved directly in perturbation theory by generalizing the calculations of [8] to 4-point functions. It was also pointed out that such non-renormalization would already be in conflict with the analysis of [8], where it was shown that Yang-Mills instantons do contribute to certain four and higher-point correlation functions. Subsequently it was pointed out to me by D. Freedman [10] that the four-point function of the stress tensor $T_{\mu\nu}$ must get renormalized, already in perturbation theory, because of results already appearing in [11]: the OPE of two $T_{\mu\nu}$ stress tensors contains the Konishi current, and the anomalous dimension of the Konishi current receives $g_{YM}$ quantum corrections (even in the $\mathcal{N} = 4$ theory). In addition, the first-order radiative contributions to the four-point function of the superconformal primary operator $O_2$ (to be discussed in what follows) were subsequently explicitly calculated [12,13] and were indeed found to be non-vanishing. In sum, the result we obtained via $\mathcal{N} = 4$ harmonic superspace is definitely incorrect.
SU(N) gauge theory in the large N limit. The primary operators \( \mathcal{O}_p \) of small representations of the superconformal group are Lorentz scalars, with dimension \( \Delta = p \), and in the SU(4)\(_R\) representation with Dynkin indices \((0, p, 0)\) (corresponding to a Young tableaux with \( p \) columns, each two rows deep). In terms of the SU(N) adjoint scalar \( \phi \), which is in the \( (0, 1, 0) \) (i.e. 6) representation of the SU(4)\(_R\) global symmetry, \( \mathcal{O}_p \sim [\text{Tr}_{SU(N)}(\phi^p)]_{(0, p, 0)} \); the subscript means to keep only the \((0, p, 0)\) representation, which is obtained by taking the totally symmetric, traceless product of the \( p \) \( \phi \)'s.

There is a normalization of the operators \( \mathcal{O}_p \) which is natural for the large N limit and convenient for comparing with supergravity. We start with the fields normalized so that the \( \mathcal{N} = 4 \) gauge theory lagrangian is

\[
\mathcal{L} = N \cdot \frac{1}{g_{YM}^2 N} (-\frac{1}{4} \text{Tr} F_{\mu\nu}^2 + \frac{1}{2} (D\phi)^2 + \bar{\psi} D\psi + \ldots). \tag{2.1}
\]

We then normalize the \( \mathcal{O}_p \) as

\[
\mathcal{O}_p = N(g_{YM}^2 N)^{-p/2} [\text{Tr}_{SU(N)}(\phi^p)]_{(0, p, 0)}. \tag{2.2}
\]

A virtue of this normalization can be seen in terms of the rescaled fields \( \hat{\phi} = \phi / \sqrt{g_{YM}^2 N} \), with sources introduced for the composite operators:

\[
\mathcal{L} = N \cdot (\frac{1}{2} \text{Tr}(D\hat{\phi})^2 + \ldots + \sum_p J_p \text{Tr}(\hat{\phi}^p)). \tag{2.3}
\]

The overall factor of \( N \) in (2.3) simplifies the \( N \)-counting: for arbitrary sources \( J_p \), the connected vacuum graph with Euler character \( \chi = 2 - 2g - b \) is of order

\[
S_{\text{field theory}}^{\text{eff}} [J_p] \sim N^\chi \tag{2.4}
\]

in the large \( N \) limit; see e.g. [19]. The leading contribution in the large \( N \) limit comes from planar diagrams and is of order \( N^2 \). In terms of the original fields \( \phi \) entering (2.1), the normalization of the operator coupling to the source \( J_p \) in (2.3) is that of (2.2). Thus arbitrary correlation functions of the operators normalized as in (2.2) satisfy

\[
\langle \prod_{i=1}^{n} \mathcal{O}_{\lambda_i}(x_i) \rangle = N^2 f_{\lambda_i}(x_i; \lambda \equiv g_{YM}^2 N) \tag{2.5}
\]

in the planar limit.
The factor of $g_{YM}^{-p}$ in (2.2) ensures that in (2.3) the functions $f_{p_i}(x_i; \lambda) \to f_{p_i}(x_i)$ are independent of $\lambda$ in the free-field, Born approximation appropriate for $\lambda \to 0$. As will be discussed in sect. 6, these factors of $g_{YM}^{-p}$ are also crucial for ensuring nice $SL(2, Z)$ modular transformation properties of the operators and correlation functions.

Having fixed the normalization of the primary operators $O_p$ as in (2.2), the normalization of all other operators in the short superconformal multiplet, which are descendents of $O_p$, are fixed by acting with the $Q$ and $\overline{Q}$ (the structure of the multiplet will be discussed in detail in what follows). Thus all operators in the small representation of the superconformal group labeled by $p$ have the same $N(g_{YM}^2 N)^{-p/2}$ normalization as in (2.2). The most general correlation function of all such operators then has the same $N^2$ dependence as in (2.5) in the large $N$ limit, and the same independence of $\lambda$ in the $\lambda \to 0$ limit.

According to the prescription in [3,4] for computing $\mathcal{N} = 4$ correlation functions via the duality of [1], (2.4) is understood as the supergravity or string theory effective action with the boundary condition that the fields equal the sources $J_p(x)$ on the boundary of $AdS_5$. The above normalization of the operators and sources nicely agrees with this method of computation. This is because the quantum loop expansion parameter, $\hbar$, of the supergravity or string theory dual is given by (1.4). The $g$ loop contribution to the effective action with fields set to equal the sources $J_p(x)$ on the boundary of $AdS_5$ is thus given by

$$S_{\text{gravity}}^{\text{eff}}[J_p] \sim \hbar^{1-g} \sim N^\chi,$$

with $\chi = 2 - 2g$, exactly as in (2.4); in particular, the leading, semi-classical contribution to $S_{\text{eff}}$ is $\sim N^2$. Normalizing the operators as in (2.2) corresponds to normalizing the supergravity fields, which approach the sources $J_p$ on the boundary, without any unnatural factors of $\hbar$.

3. Review of the $U(1)_Y$ and $SL(2, R)$ symmetries of $IIB$ supergravity

It is perhaps useful to briefly review some textbook (see, e.g. [20]) facts about $IIB$ supergravity. Type $IIB$ supergravity in 10d has a $U(1)$ symmetry which rotates the two chiral supersymmetries, and thus is an $R$ symmetry, which we will refer to as $U(1)_Y$. Normalizing the supercharges to have $U(1)_Y$ charge $\pm 1$, the complex scalar dilaton has $Y = 4$, the complex two-form gauge field $B_{\mu\nu}$ has $Y = 2$, the complex Weyl spinor dilatino $\lambda$ has $Y = 3$, and the complex Weyl gravitino $\psi$ has $Y = 1$. The complex conjugate fields have the opposite $Y$ charges and the remaining fields, which are real, all have $Y = 0$ [21].
The entire collection of massless physical fields can be described in terms of a 10 superfield \( \Phi(x, \theta) \), where the Grassmann coordinate \( \theta \) is in the complex Weyl 16 of \( SO(9,1) \) and \( \Phi \) is subject to the constraint \( \overline{D} \Phi = 0 \) and also \( D^4 \Phi = \overline{D}^4 \Phi = 0 \).

The interacting IIB supergravity theory is formulated in terms of a \( SL(2, R)/U(1)_Y \) coset. Originally, for convenience, the coset was given in terms of \( SU(1,1) \cong SL(2, R) \) [21]; the \( SL(2, R) \) form can be found e.g. in [22] and will be briefly reviewed here. The scalars are given in terms of the “driebein” field, which is used to convert between \( SL(2, R) \) indices \( \alpha = 1, 2 \) and \( U(1) \) charges \( Y = \pm 2 \).

\[ V = (V^-_\alpha, V^+\alpha) = (-2i\tau_2)^{-1/2} \begin{pmatrix} \tau e^{-i\phi} & \tau e^{i\phi} \\ e^{-i\phi} & e^{i\phi} \end{pmatrix} \] (3.1)

this \( V \) is related to an element of \( SL(2, R) \) by a change of basis to a complex basis. \( V \) transforms under global \( SL(2, R) \) and local \( U(1)_Y \) transformations as \( V^\alpha_\pm \rightarrow e^{\pm 2i\Sigma(x)} U^\alpha_\beta \sigma V^\beta_\pm \), where \( U^\alpha_\beta \in SL(2, R) \), \( \Sigma \) is the local \( U(1) \) phase and the normalization reflects our choice of normalizing \( V^\alpha_\pm \) to have \( U(1)_Y \) charge \( \pm 2 \). The real scalar \( \phi \) in (3.1) is unphysical and can be set to zero by choice of \( U(1)_Y \) gauge. \( SL(2, R) \) transformations only preserve the gauge fixed form of \( V \) when accompanied by particular \( U(1)_Y \) transformations and the upshot is that \( \tau \) in (3.1) has the standard transformation (1.7) under \( SL(2, R) \).

The driebein \( V^\alpha_\pm \) is used to convert all other fields to be invariant under \( SL(2, R) \) but charged under \( U(1)_Y \); so the dilaton \( \tau \) will be the only field to transform under \( SL(2, R) \). In particular, the complex antisymmetric tensor \( A^\alpha_{\mu\nu} \) (with \( A^1_{\mu\nu} = A^2_{\mu\nu}^* \)), which transforms as an \( SL(2, R) \) doublet and neutral under \( U(1)_Y \), is converted to the \( SL(2, R) \) singlet \( B_{\mu\nu} = \epsilon_{\alpha\beta} V^\alpha_+ A^\beta_{\mu\nu} \), which has \( U(1)_Y \) charge 2, as in the free theory spectrum mentioned above. The \( SL(2, R) \) invariant object

\[ P_\mu = -\epsilon_{\alpha\beta} V^\alpha_+ \partial_\mu V^\beta_+ = \frac{i}{2\tau_2} \partial_\mu \tau \] (3.2)

has \( U(1)_Y \) charge 4, in line with the \( U(1)_Y \) charge of the dilaton of the free theory mentioned above. Similarly, the remaining fields and \( U(1)_Y \) charges are as mentioned above for the free theory, and are all \( SL(2, R) \) singlets.

Given the principle that supersymmetry should respect the \( SU(1,1) \cong SL(2, R) \) and \( U(1)_Y \) symmetries, with the supercharges carrying charge \( \pm 1 \) under \( U(1)_Y \), it was shown in [21] that consistency of the super-algebra completely determines (actually over determines) the form of the supersymmetry variations up to a single, real, dimensionful coupling constant \( \kappa \), which is the 10d gravitational coupling constant. Finally, requiring closure of
this super-algebra determines the interacting $IIB$ supergravity equations of motion \cite{21}, as the algebra only closes on shell. The equations of motion determined in this way will clearly also respect the $SL(2, R)$ and $U(1)_Y$ symmetries. Even in the gauge fixed form, with the unphysical degree of freedom in $V$ eliminated, the equations of motion found in \cite{21} manifestly respect a residual global $U(1)_Y$ symmetry, under which the fields have the charge assignments given above.

In converting the discussion of \cite{21} to one in which $SL(2, R)$ is used instead of $SU(1, 1)$, there is a small subtlety with regard to the $U(1)_Y$ symmetry. In the $SU(1, 1)$ formulation, the $SU(1, 1)$ invariant object (3.2) is given upon gauge fixing $U(1)_Y$ by $P_\mu = (1 - B^* B)^{-3} \partial_\mu B$ and, since $P_\mu$ is assigned $U(1)_Y$ charge 4, the complex scalar $B$ also carries $U(1)_Y$ charge 4. In the $SL(2, R)$ form (3.2), $P_\mu$ again has $U(1)_Y$ charge 4, but $\tau$ does not have a well-defined $U(1)_Y$ charge assignment because of the $\tau_2$ in (3.2). The reason is that the map between $B$ and $\tau$

$$\tau = i \frac{1 - B}{1 + B} \quad (3.3)$$

maps the origin $B = 0$, where $U(1)_Y$ is unbroken, to $\tau = i$ and a simple $U(1)$ phase for $B$ gives a more complicated transformation for $\tau$. More generally, non-zero $\langle B \rangle$ or $\langle \tau \rangle$ spontaneously break $U(1)_Y$. For our purposes, however, it is useful to note that the leading order variation $\delta \tau$ of $\tau$ around a constant $\langle \tau \rangle$ can be assigned a well-defined $U(1)_Y$ charge. As (3.2) gives $P_\mu = i \partial_\mu \delta \tau / 2 \langle \tau_2 \rangle$, we can assign $U(1)_Y$ charge 4 to $\delta \tau$ and zero to $\langle \tau_2 \rangle$. In any case, $SL(2, R)$ invariance implies that amplitudes expanded around vanishing fields and constant $\langle \tau \rangle$ will be independent of $\langle \tau \rangle$. For this reason, the spontaneous breaking of $U(1)_Y$ by $\langle \tau \rangle$ will not be relevant for our concerns.

The action which gives the equations of motion, modulo the self-duality of $F_5$ which can be imposed by hand or treated as in \cite{23}, takes the $SL(2, R)$ and $U(1)_Y$ invariant form in the Einstein frame \cite{24}:

$$S = \frac{1}{2 \kappa^2_{10}} \int d^{10}x \left[ \sqrt{-g} \left( R - \frac{\partial \tau \partial \tau}{2 \tau^2} - \frac{1}{12} G^{\mu \nu \lambda} G^*_{\mu \nu \lambda} - \frac{1}{4 \cdot 5!} F_5^2 - \frac{1}{(12)!} C_4 \wedge G \wedge G^* + \cdots \right) \right], \quad (3.4)$$

with $G = \epsilon^{\alpha \beta} V_{\alpha} A_{2, \beta} = \tau_2^{-1/2} (\tau dB_2 + dC_2)$ and $F_5 = dC_4 + 5 \epsilon^{\alpha \beta} A_{2, \alpha} \wedge dA_{2, \beta}$.

The $SL(2, R)$ and $U(1)_Y$ symmetries of $IIB$ supergravity will be respected by all tree-level amplitudes, and thus by the generating functional of these tree level amplitudes.
4. Representations of the superconformal group $PSU(2,2|4)$ and its $U(1)_Y$ automorphism

Because $F_5$ is neutral under $SL(2,R)$ and $U(1)_Y$, they will also be symmetries of the supergravity theory with $N$ units of $F_5$ flux and vacuum $AdS_5 \times S_5$. In particular, $U(1)_Y$ must act as an $R$-symmetry of the superconformal group $PSU(2,2|4)$. It is non-trivial that $PSU(2,2|4)$ indeed does admit such an outer automorphism.

In order to clarify the connection between the $U(1)_Y$ of supergravity and the superconformal group, it is useful to review a general subtlety of the supergroups $SU(M|N)$ when $M = N$; see e.g. [25] for useful facts about super matrices, groups, and algebras. Our case of interest is $M = (2,2)$ and $N = 4$; the non-compact signature of $M$ will not introduce any further subtleties. An element of the $u(M|N)$ algebra can be written as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (4.1)$$

with $A \in u(M)$ and $D \in u(N)$ bosonic and $B$ and $C$ fermionic. There is a decoupled $u(1)_D$ ideal generated by $g_D = 1_{M+N}$ and an $R$-symmetry $u(1)_Y$ generated by

$$g_Y = \frac{1}{2} \begin{pmatrix} 1_M & 0 \\ 0 & -1_N \end{pmatrix}, \quad (4.2)$$

under which $A$ and $D$ are neutral and the generators $B$ and $C$ have charge $\pm 1$. For $M \neq N$, the ideal $u(1)_D$ is eliminated by the condition $strg \equiv \text{tr} A - \text{tr} D = 0$; the resulting algebra is $su(M|N)$, which contains $u(1)_R$ generated by $g_R = g_Y + \frac{1}{2}(M+N)(N-M)^{-1}g_D$ in its bosonic subalgebra. On the other hand, for $M = N$ the condition $strg = 0$ eliminates $u(1)_Y$ (4.2) rather than $g_D = 1_{N+N}$ and thus $su(N|N) = psu(N|N) \oplus u(1)_D$ does not contain the $R$-symmetry generated by $u(1)_Y$. Although $u(1)_Y$ is not contained in $su(N|N)$ or $psu(N|N)$, it clearly acts as a consistent automorphism on them: indeed, these groups can be consistently extended to include $g_Y$ (4.2) as an additional element by simply not imposing the $strg = 0$ condition. The larger group thus obtained, which we refer to as $PU(N|N)$ in the case where the decoupled $U(1)_D$ is eliminated by hand, is $U(1)_Y \ltimes PSU(N|N)$, rather than $U(1)_Y \otimes PSU(N|N)$, since $U(1)_Y$ acts as a non-trivial $R$ symmetry on the fermionic generators.

Representations of $PSU(2,2|4)$ can be assigned definite charges under the $U(1)_Y$ automorphism group. The short representations of $PSU(2,2|4)$ were constructed by the oscillator method in [26]. The full short representation is labeled by an integer $p > 0$ and
consists of a number of particular representations of the bosonic $SU(2, 2) \times SU(4)_R$ subgroup. The motivation in [26] was to use $PSU(2, 2|4)$ representation theory to understand the spectrum of fields in 5d, $\mathcal{N} = 8$ supergravity; the same spectrum was obtained as with linearized KK reduction of $IIB$ supergravity on $S^5 \times AdS_5$ in [27], where $p$ is related to the KK spherical harmonic. The relation between these supergravity fields and operators in the 4d $\mathcal{N} = 4$ gauge theory was discussed in detail in [4] and the fact that these operators are also classified by the $PSU(2, 2|4)$ representation theory of [26] was emphasized in [28].

In addition to finding the $SU(2, 2) \times SU(4)_R$ quantum numbers, the $U(1)_Y$ charges of the representations were also determined in [26], where it was appreciated that the 5d $\mathcal{N} = 8$ supergravity must also have the $U(1)_Y$ symmetry of the 10d $IIB$ supergravity. The $U(1)_Y$ charges of the 5d supergravity fields are simply those of the corresponding 10d $IIB$ supergravity field of which the 5d field is a $S^5$ spherical harmonic KK mode. We emphasize again that $U(1)_Y$ acts as a non-trivial $R$-symmetry on $PSU(2, 2|4)$; clearly $U(1)_Y$ of supergravity is an $R$-symmetry since the graviton is neutral and the gravitino is charged. This differs from a brief discussion in [29], where the $U(1)_Y$ of supergravity was instead identified with the decoupled, non-R-symmetry $u(1)_D$ in $su(N|N) = u(1)_D \oplus psu(N|N)$.

For convenience, we included the table of representations and $U(1)_Y$ charges determined in [26] in appendix A. We changed the normalization of the $U(1)_Y$ charges for convenience and also changed the signs to be those of the operators in the $\mathcal{N} = 4$ field theory, which are of exactly opposite $U(1)_Y$ charge from the supergravity fields to which these operators couple. Also indicated in the table is the supermultiplet form of the representations: there is a primary representation $O_p$, which has $U(1)_Y$ charge 0, and superconformal descendents obtained by acting with powers of the supercharges $Q^I_\alpha$ and $\overline{Q}_{\dot{I}, \dot{\alpha}}$ on $O_p$, with $Q$ represented by $\delta$ and $\overline{Q}$ represented by $\overline{\delta}$. When the representations $O_p$ are operators rather than fields, it should be understood that the $\delta^r \overline{\delta}^s O_p$ appearing in the table is shorthand for a nested sequence of commutators and anti-commutators with the supercharges, e.g. $\delta^2 \overline{\delta} O_p$ should be understood as $[Q, \{Q, [Q, O_p]\}]$. The supercharge descendent structure truncates at $\delta^4 \overline{\delta}^4 O_p$ rather than at $\delta^8 \overline{\delta}^8 O_p$ because it is a short rather than long $PSU(2, 2|4)$ representation. A representation $\delta^r \overline{\delta}^s O_p$ has $U(1)_Y$ charge $s - r$.

The representations with $p < 4$ truncate further. The $SU(4)_R$ quantum numbers of the representations are given by the Dynkin labels $(l_1, l_2, l_3)$ (which corresponds to a Young tableaux with $l_k$ columns of boxes which are $k$ rows deep, $k = 1, 2, 3$). Those representations which would have Dynkin index $l_2 < 0$ according to the table, of course, vanish. The $p = 0$ representation contains the identity as its only element and the $p =$
1 representation is the decoupled representation sometimes referred to as the singleton or doubleton; it is not present if the $\mathcal{N} = 4$ Yang-Mills group is simple. The $p = 2$ representation is the “massless” representation which contains, among other operators, the conserved superconformal currents.

Another interesting pair of operators in the $p = 2$ representation are

$$O^{(-4)}_\tau = \delta^4 O_2 \quad \text{and} \quad O^{(4)}_\tau = \bar{\delta}^4 O_2.$$  (4.3)

These operators are Lorentz and $SU(4)_R$ singlets, and exactly marginal as $\Delta = 4$. They are also annihilated when acted on with any more powers of $Q$ or $\bar{Q}$ since all such descendents in the short representation would have a $SU(4)_R$ Dynkin index $l_2 < 0$ for $p = 2$. In the $\mathcal{N} = 4$ gauge theory, $O^{(-4)}_\tau$ is the exactly marginal operator corresponding to changing the gauge coupling $\tau$. The corresponding field in supergravity to which $O^{(-4)}_\tau$ couples is the lowest KK mode of the dilaton, which we also denote by $\tau$, which has $U(1)_Y$ charge $+4$. $O^{(-4)}_\tau$ in the gauge theory will be discussed further in the next section.

5. $\mathcal{N} = 4$ gauge theory and the $U(1)_Y$ non-symmetry

There are some points to be made concerning how $U(1)_Y$ acts in the $\mathcal{N} = 4$ gauge theory. To illustrate a first point, it will suffice to consider Abelian $U(1)$ $\mathcal{N} = 4$ gauge theory. The fields are the gauge field $A_{\alpha \dot{\alpha}}$, scalars satisfying the reality condition $\phi^{[IJ]} \equiv (\phi_{[IJ]})^* = \frac{1}{2} \epsilon^{IJKL} \phi_{[KL]}$, and fermions $\psi_{I;\alpha}$, $\bar{\psi}^I_{\dot{\alpha}}$, where the $I$ is a fundamental $SU(4)_R$ representation index. The on-shell supersymmetry transformations are given by

$$
\delta A_{\alpha \dot{\alpha}} = \bar{\eta}^{I \dot{\gamma}} \epsilon_{\dot{\alpha} \dot{\gamma}} \psi_I \alpha + \eta_I^I \epsilon_{\alpha \dot{\beta}} \bar{\psi}^{I \dot{\beta}} \\
\delta \phi_{[IJ]} = \eta^I_I \psi_{J;\alpha} + \epsilon_{IJKL} \bar{\eta}_{KL} \bar{\psi}^{L}_{\dot{\alpha}} \\
\delta \psi_{I;\alpha} = \eta_I^I \bar{F}_{(\alpha \beta)} + \bar{\eta}^{I \dot{\gamma}} \partial_{\alpha \dot{\gamma}} \psi_{IJ} \\
\delta F_{(\alpha \beta)} = \bar{\eta}^{I \dot{\gamma}} \partial_{(\alpha \beta)} \psi_I,$$

where $\eta^I_I$ and $\bar{\eta}^{I \dot{\alpha}}$ are Grassmann parameters to keep track of the action of $Q_I^I$ and $\bar{Q}_{I \dot{\alpha}}$, there are similar transformations for $\bar{\psi}^I_{\dot{\alpha}}$ and $\bar{F}_{I \dot{\alpha}}$, and we have left out numerical constants for simplicity. (This notation differs from that of the appendix, where $\delta$ denotes acting with $Q_{I \dot{\alpha}}$ only.) There is no known off-shell formulation of $\mathcal{N} = 4$ supersymmetry at $\phi = 0$.

Note that the on-shell amplitudes in the supergravity or string theory dual apparently do provide a fully supersymmetric, off-shell formulation of the $\mathcal{N} = 4$ superconformal symmetry of the boundary field theory.
The fields transform under $U(1)_Y$ with the charges

$$
U(1)_Y \begin{pmatrix} \phi_{IJ} & \psi_{I\alpha} & \bar{\psi}^I_{\dot{\alpha}} & F_{(\alpha\beta)} & \bar{F}_{(\dot{\alpha}\dot{\beta})} \end{pmatrix}.
$$

Note that this transformation is compatible with the $\phi$ reality condition, because $\phi$ is neutral, but bizarre, because $F_{\alpha\beta}$ is not neutral. It is not the same as the $U(1)$ in $U(4) \cong U(1) \times SU(4)_R$. Assigning charges 1 to $\eta_{I}^{\alpha}$ and $\bar{\eta}^{I\dot{\alpha}}$, this transformation is respected by all of the supersymmetry variations with the exception of that of $A_{\alpha\dot{\alpha}}$, which is not a gauge invariant physical field anyway. The $U(1)_Y$ transformation is also a symmetry of the equations of motion for the physical fields. Indeed, as the super-transformations are purely on-shell, they close on the equations of motion, which must then also respect $U(1)_Y$. Note that $F_{\alpha\beta}$ involves $\vec{E} + i \vec{B}$, and thus the $U(1)_Y$ symmetry involves a continuous rotation between electric and magnetic fields - i.e. a continuous version of the discrete electric-magnetic duality transformation $S$.

While $U(1)_Y$ is a symmetry of the equations of motion of the Abelian theory, it is not a symmetry of the lagrangian:

$$
\mathcal{L} = \tau \left( - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{2} \bar{\psi}_{\dot{\alpha}} \partial^{\dot{\alpha}} \psi_{I\alpha} - \frac{i}{2} \partial_{\alpha\dot{\alpha}} \phi^{IJ} \partial^{\alpha\dot{\alpha}} \phi_{IJ} \right) + \bar{\tau} \left( - \frac{1}{4} \bar{F}_{\dot{\alpha}\dot{\beta}} \bar{F}^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} \bar{\psi}_{\dot{\alpha}} \partial^{\dot{\alpha}} \psi_{I\alpha} + \frac{i}{2} \partial_{\alpha\dot{\alpha}} \phi^{IJ} \partial^{\alpha\dot{\alpha}} \phi_{IJ} \right).
$$

It is trivially a symmetry if the equations of motion are imposed, as then the lagrangian simply vanishes. The subtlety of having to impose the equations of motion is also apparent in our identification of $O_{(4)} = \delta^4 O_2$ as the exactly marginal, supersymmetry preserving, operator corresponding to changing $\tau$. Applying $Q^4$ using (5.1) to $O_2 \sim \phi_{[IJ]} \phi_{[KL]} - (\text{trace})$ gives $\delta^4 O_2 \sim F^2$, which is zero upon imposing the equations of motion. This corresponds to varying $\tau$ in the lagrangian with the equations of motion imposed; this is trivial in the Abelian case.

In the non-Abelian case, the supersymmetry transformations are modified by replacing all $\partial_{\alpha\dot{\alpha}} \rightarrow D_{\alpha\dot{\alpha}}$ gauge covariant derivatives and there is an additional term in

$$
\delta \psi_{I\alpha} = \ldots + \eta_{I}^{\beta} \varepsilon_{\alpha\beta} [\phi_{IK}, \phi^{JK}].
$$

Assigning charges and charge +1 to $\eta_{I}^{\beta}$ as before, we see that the additional term does not respect the $U(1)_Y$ symmetry. Thus $U(1)_Y$ is not a symmetry for general $g_{YM}$ and $N$. When combined with the operation of changing the sign of all fields, a $Z_4$
subgroup of $U(1)_Y$ is preserved, but uninteresting, as it is simply the center of the $SU(4)_R$ symmetry.

In the non-Abelian case, the operator $O^{(-4)} = \delta^4 O_2$ is non-vanishing and corresponds to infinitesimally changing $\tau$ in the Lagrangian. To be precise, the change in the on-shell Lagrangian upon varying $\tau \to \tau + \delta \tau$ is given by

$$
\delta L^{\text{on-shell}} = \frac{\delta \tau}{\langle \tau_2 \rangle} O^{(-4)}_\tau + \frac{\delta \overline{\tau}}{\langle \overline{\tau}_2 \rangle} O^{(4)}_{\overline{\tau}}.
$$

The factor of $\langle \tau_2 \rangle = 4 \pi g_{YM}^{-2}$ in (5.3) is due to the normalization of $O^{(-4)}_\tau$ given by (2.2) for $p = 2$. It will be important in the next section, when we discuss modular transformation properties. The fact that the exactly marginal operator corresponding to changing $\tau$ is $\sim \delta^4 \text{Tr}(\phi^i \phi^j)_{20'}$, where $\text{Tr}(\phi^i \phi^j)_{20'}$ is the operator $O_2$, with $20'$ the $SU(4)_R$ representation with Dynkin indices $(0, 2, 0)$, was noted in [32].

6. Conjectures about $SL(2, Z)$ invariance and its bonus enhancement

In the duality of [1], the $SL(2, Z)$ S-duality of $\mathcal{N} = 4$ is tied to the $SL(2, Z)$ symmetry of IIB string theory, which remains a symmetry of the theory with $F_5$ flux and vacuum $AdS_5 \times S^5$ because $F_5$ is $SL(2, Z)$ invariant. In the supergravity limit, as in the theory without $F_5$ flux, the $SL(2, Z)$ symmetry is enhanced to $SL(2, R)$, with maximal compact subgroup $U(1)_Y$. Before discussing the bonus symmetry of the supergravity limit, we will discuss some general ideas and speculations for how $SL(2, Z)$ acts on correlation functions.

We expect that $SL(2, Z)$ maps any operator $O_i$ to the same $O_i$ operator in the dual gauge theory, possibly up to factors to be discussed now. The simplest realization of the $SL(2, Z)$ invariance of the $\mathcal{N} = 4$ theory with $SU(N)$ gauge group (ignoring global issues) would be that arbitrary correlation functions of operators should be modular functions of $\tau$. A more general possibility would be for correlation functions to be modular forms $F^{(w, \overline{w})}(\tau, \overline{\tau})$ of weights $(w, \overline{w})$, which transform as

$$
F^{(w, \overline{w})}(\tau, \overline{\tau}) \to (c\tau + d)^w (\overline{c\tau + d})^{\overline{w}} F^{(w, \overline{w})}(\tau, \overline{\tau}) \quad \text{under} \quad \tau \to a\tau + b \overline{c\tau + d}.
$$

One could entertain even more general possibilities, but we will not do so here.

We expect that general correlation functions transform as (6.1) and that it is possible to assign general weights $(w_i, \overline{w_i})$ to each operator $O_i$. As in (6.1), $O_i$ is mapped under
modular transformation as \( O_i \rightarrow (c\tau + d)^{w_i}(c\tau + d)^{\overline{w}_i}O_i \) and a general correlation function \( \langle \prod_i O_i(x_i) \rangle \) will have weight \((w_T, \overline{w}_T)\), with \( w_T = \sum_i w_i \) and \( \overline{w}_T = \sum_i \overline{w}_i \).

Note that the factor of \( g_{YM}^{-p} \) in (2.2) affects the weights \((w_p, \overline{w}_p)\) assigned to the operator \( O_p \). This is because \( \langle \tau_2 \rangle \equiv 4\pi g_{YM}^{-2} \) transforms as a modular form of weights \((-1, -1)\). By multiplying by powers of \( \langle \tau_2 \rangle \), it is possible to convert a modular form of weights \((-q_i/4, q_i/4)\), where \( q_i \) is the \( U(1)_Y \) charge which is assigned to the operators.

In particular, the superconformal primary operator \( O_p \) with normalization (2.2) is modular invariant. The necessity of the \( g_{YM}^{-p} \) factor in (2.2) for obtaining a modular invariant operator can be seen, for example, in the case where the gauge group is \( U(1) \) and the theory is free. Again, this factor can be understood as simply rescaling \( \varphi \) so that its kinetic term does not have the \( g_{YM}^{-2} \) factor.

To motivate the above statement about the modular weights of descendents, consider the variation (5.5) of the on-shell Lagrangian under a change \( \delta \tau \) of \( \tau \). Under a modular transformation,

\[
\frac{\delta \tau}{\tau_2} \rightarrow \left(\frac{c\tau + d}{c\tau + d}\right) \frac{\delta \tau}{\tau_2},
\]

transforms as a modular form of weight \((-1, 1)\). By assigning \( O^{(-4)} \) weight \((1, -1)\), the variation (5.5) is modular invariant. More generally, operators of \( U(1)_Y \) charge \( q_i \) should transform with weight \((-q_i/4, q_i/4)\). The supercharges \( Q^I_{\alpha} \) and \( \overline{Q}_{I,\dot{\alpha}} \) thus effectively transform as modular forms of weights \((\frac{1}{4}, -\frac{1}{4})\) and \((-\frac{1}{4}, \frac{1}{4})\), respectively. A general correlation function thus transforms under \( SL(2, Z) \) as in (1.10).

We emphasize that the above statements apply in the \( N = 4 \) gauge theory for any \( N \) and \( g_{YM} \) and are logically separate from the AdS duality.

We turn now to the AdS duality conjecture of [1] and the prescription [4] for computing general correlation functions:

\[
Z_{IIB} \left[ \Phi_i | \partial(AdS) = J_i(x) \right] = \langle e^{\sum_i \int d^4 x J_i(x) O_i(x)} \rangle_{CFT},
\]

for arbitrary source functions \( J_i(x) \). In light of the above discussion, we would like to make this prescription a bit more precise with regard to modular transformation properties and how \( \Phi_i \) is defined. First, the IIB string theory or supergravity field \( \Phi_i \) must not have any \( SL(2, R) \) or \( SL(2, Z) \) doublet indices \( \alpha \) left hanging loose: all should be soaked up with the \( V_{\pm}^{\alpha} \) (3.1). Second, appropriate factors of \( \langle \tau_2 \rangle \) should be introduced into the field \( \Phi_i \) so
that it transforms under the modular group as a form of weights \((w_i, \overline{w_i} = -w_i)\); here \(w_i = -q_i/4\), with \(q_i\) the \(U(1)_Y\) charge of \(\Phi_i\). This implies that the sources \(J_i(x)\) have modular transformation properties opposite to that of the \(\mathcal{O}_i\) discussed above. This guarantees that correlation functions computed via (3.3) will have the modular transformation properties discussed above.

As a concrete example to illustrate the factors of \(\langle \tau_2 \rangle\), consider the two point function 
\[
\langle \mathcal{O}_p^{(-4)}(x)\mathcal{O}_p^{(+4)}(y) \rangle,
\]
where \(\mathcal{O}_p^{(-4)}(x) \equiv \delta^4 \mathcal{O}_p\) and \(\mathcal{O}_p^{(+4)} \equiv \overline{\delta^4 \mathcal{O}_p}\). For \(p = 2\) these are the exactly marginal operators \(\mathcal{O}_{p=2}^{(-4)} = \mathcal{O}_\tau^{(-4)}\) and \(\mathcal{O}_{p=2}^{(+4)} = \overline{\mathcal{O}_\tau^{(4)}}\). The supergravity source for \(\mathcal{O}_{p=k+2}^{(-4)}\) is \(\delta \tau_k/\langle \tau_2 \rangle\) and the source for \(\mathcal{O}_{p=k+2}^{(+4)}\) is \(\delta \overline{\tau}_k/\langle \tau_2 \rangle\). Here \(\delta \tau_k\) is the \(k\)-th \(S^5\) spherical harmonic of the variation, \(\delta \tau\), of the 10d dilaton away from its constant expectation value \(\langle \tau \rangle\). (I hope this notation for spherical harmonics will not cause any confusion regarding \(\tau_2 \equiv \text{Im} \tau\), which is not the 2nd spherical harmonic of \(\tau\).) The reason for the factors of \(1/\langle \tau_2 \rangle\) in the source functions is, for every spherical harmonic, it is \(\delta \tau_k/\langle \tau_2 \rangle\) which transforms with weight \(\overline{\omega} = -\omega\): as seen by expanding \(\delta \tau\) in (3.2) in spherical harmonics, with \(\tau\) set to its constant expectation value, the \(\delta \tau_k/\langle \tau_2 \rangle\) all transform with weight \((-1, 1)\). So \(\delta \tau_k/\langle \tau_2 \rangle\) and \(\delta \overline{\tau}_k/\langle \tau_2 \rangle\) are the correct sources for the operators \(\delta^4 \mathcal{O}_{p=k+2}\) and \(\overline{\delta^4 \mathcal{O}_{p=k+2}}\), respectively, when these operators are properly normalized as in (2.2).

Because the total \(U(1)_Y\) charge is zero, \(\langle \mathcal{O}_p^{(-4)}(x)\mathcal{O}_p^{(+4)}(y) \rangle\) will be modular invariant. Using (3.3) with the sources as discussed above, we have

\[
\langle \mathcal{O}_p^{(-4)}(x)\mathcal{O}_p^{(+4)}(y) \rangle = \langle \tau_2 \rangle^2 \frac{\delta^2}{\delta \tau_k(x) \delta \overline{\tau}_k(y)} \mathcal{Z}_{\text{IIB}}[\delta \tau_k]. \tag{6.4}
\]

The relevant supergravity action for computing the RHS of (6.4) is simply the \(k\)-th \(S^5\) spherical harmonic of the \(S^5\) dimensional reduction of the 10d dilaton \(\tau\) kinetic term in (3.4); this yields

\[
S_{5d} = \frac{\pi^2 L^5}{2k_{10}^2} \int d^5x \sqrt{-g} \left[ -\frac{1}{2\langle \tau_2 \rangle^2} (\partial \tau_k \partial \overline{\tau}_k - k(k + 4) \tau_k \overline{\tau}_k) + \ldots \right]. \tag{6.5}
\]

As in (6.3), this gives

\[
\frac{\delta^2}{\delta \tau_k(x) \delta \overline{\tau}_k(y)} \mathcal{Z}_{\text{IIB}}[\delta \tau] \sim \frac{N^2 \langle \tau_2 \rangle^{-2}}{|x - y|^{2k + 8}}, \tag{6.6}
\]

where we used (1.4) but did not bother being careful with factors of 2 and \(\pi\). It then follows from (6.4) that

\[
\langle \mathcal{O}_p^{(-4)}(x)\mathcal{O}_p^{(+4)}(y) \rangle \sim \frac{N^2}{|x - y|^{2p + 4}}. \tag{6.7}
\]
In this limit, as well as exactly, the correlation function (6.7) is independent of $\tau$, and thus modular invariant as expected.

We now consider the enhancement of $SL(2, \mathbb{Z})$ to $SL(2, \mathbb{R})$ in the supergravity limit of $IIB$ string theory, corresponding in the $\mathcal{N} = 4$ field theory to the double limit (1.1) and (1.3). In this limit, the supergravity source fields transform under the full $SL(2, \mathbb{R})$ extension of $SL(2, \mathbb{Z})$, and thus the field theory correlation functions computed via (6.3) must also respect the enlarged $SL(2, \mathbb{R})$ symmetry. This means that, in this limit, arbitrary correlation functions must transform exactly as in (1.10), but for general $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in SL(2, \mathbb{R})$, rather than just $SL(2, \mathbb{Z})$.

Because $SL(2, \mathbb{R})$ can be used to map any point in the upper-half plane to any other point, its modular forms are necessarily quite trivial. In particular, the only $SL(2, \mathbb{R})$ modular form which transforms as in (6.1) with weights $\overline{w} = -w$ is given by $F^{(w,-w)} = (const)\delta_{w,0}$, i.e. completely independent of $\tau$ for $w = 0$, and vanishing for $w \neq 0$. Since correlation functions have $\overline{w} = -w = qT/4$ (1.10), we find that non-zero correlation functions must respect the $qT = 0, U(1)_Y$ selection rule (1.8). This is reasonable, since supergravity respects the $U(1)_Y$ symmetry. (As mentioned in the previous section, the selection rule (1.8) is actually stronger than simple $U(1)_Y$ invariance, which would allow for non-zero net $U(1)_Y$ charge to be soaked up by powers of $\tau$; (1.8) incorporates the fact that $SL(2, \mathbb{R})$ invariance prevents this from being an option.) Further, the non-zero correlation functions with $qT = 0$ are independent of $\tau$, as stated after (1.6).

7. The breaking of $SL(2, \mathbb{R})$ and $U(1)_Y$ in string theory

The tree-level worldsheet action for the $IIB$ string theory in flat 10d spacetime\footnote{The worldsheet conformal field theory for the present case of non-zero $F_5$ flux, i.e. with a Ramond-Ramond background is not well understood. (See, however, [34] for superstring actions argued to properly describe $AdS_5 \times S^5$.) The $F_5 = 0$ worldsheet theory suffices for getting insight into some qualitative aspects, such as the breaking of $U(1)_Y$ to $Z_4$.} contains two terms, $S_1 + S_2$ discussed in detail in sect. 5.1.2 of [20]. The term $S_1$ looks well-motivated and respects the $U(1)_Y$ symmetry which rotates the two fermionic fields $\Theta$. The term $S_2$, looks less well-motivated but has to be added to $S_1$ to ensure the $\kappa$ symmetry; it is independent of the worldsheet metric and thus does not contribute to the 2d stress tensor. The effect of $S_2$ is also sub-leading to $S_1$ in the $\alpha'$ expansion. The action
S\_2 violates the U(1)\_Y symmetry, breaking it to Z\_4; the Z\_4 action involves rotating the two \( \Theta \) coordinates by \( \pi/2 \), combined with a world-sheet parity transformation \( \sigma_1 \leftrightarrow \sigma_2 \), which takes \( \epsilon^{\alpha\beta} \rightarrow -\epsilon^{\alpha\beta} \). As mentioned above, in the map to \( \mathcal{N} = 4 \) field theory, this Z\_4 corresponds to the center of the SU(4)\_R symmetry of the gauge theory and thus is not an interesting new symmetry.

As discussed e.g. in [22,35] and references cited therein, the leading \( \alpha' \) stringy correction to the spacetime effective action occurs at order \( (\alpha')^3 \) relative to the supergravity effective action and has the form (in Einstein frame)

\[
(\alpha')^3 \int d^{10}x \sqrt{-g} (f^{(12,-12)}\lambda^{16} + f^{(11,-11)}G\lambda^{14} + \ldots + f^{(4,-4)}G^8 + \ldots + f^{(0,0)}R^4) + c.c. \tag{7.1}
\]

The functions \( f^{(w,-w)}(\tau, \bar{\tau}) \) are SL(2, \( \mathbb{Z} \)) modular forms, transforming as in (6.1) with \( \bar{\tau} = -\tau \). Exact expressions for the \( f^{(w,-w)} \) are conjectured e.g. in [22,36,35], e.g.

\[
f^{(0,0)}(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^{3/2}}{|m + \tau n|^3}. \tag{7.2}
\]

The expression (7.2) is invariant under SL(2, \( \mathbb{Z} \)) modular transformations, but obviously violates SL(2, \( \mathbb{R} \)). Although \( \mathbb{R} \) is neutral under U(1)\_Y, the fact that \( \tau \) in (7.2) is charged under U(1)\_Y means that the \( R^4 \) terms in (7.1) also violates U(1)\_Y (though clearly preserves the Z\_4 since \( \tau \) has charge 4), as do the other terms in (7.1) more explicitly.

As in [37], assuming that the duality of [1,3,4] applies away from the supergravity limit, with the sub-leading stringy terms in (7.2), leads to predictions for the sub-leading corrections to the \( \mathcal{N} = 4 \) field theory correlation functions away from the double limit. Using (7.1), we find

\[
\langle \prod_{i=1}^{n} O_i^{(q_i)}(x_i) \rangle = N^2 f_{i_1 \ldots i_n}^{(0)}(x_i)\delta_{q_T,0} + N^{1/2} f_{i_1 \ldots i_n}^{(-q_T/4,q_T/4)}(\tau, \bar{\tau}) f_{i_1 \ldots i_n}^{(3)}(x_i) + \ldots, \tag{7.3}
\]

where \( q_T = \sum_i q_i \) is the total U(1)\_Y charge of the operators (which is opposite to that of the supergravity source fields). Here \( f_{i_1 \ldots i_n}^{(0)}(x_i) \) is the leading supergravity contribution and \( f_{i_1 \ldots i_n}^{(3)}(x_i) \) are the leading corrections to supergravity amplitudes coming from the additional interactions in (7.1). The relative factor of \( N^{-3/2} \) in (7.3) comes from the \( (\alpha')^3 \) in (7.1), along with (1.1). The modular forms in (7.3) are the same ones appearing in (7.1), e.g. \( f^{(12,-12)}(\tau, \bar{\tau}) \) for the 16-point function of the operator \( \delta^3 O_p \), of U(1)\_Y charge \( Y = -3 \), which is conjugate to the supergravity source \( \lambda \).
The fact that the modular forms in (7.3) have weights \((-q_T/4, q_T/4)\) is seen in (7.1): the weights of the modular forms are correlated in this way with the \(U(1)_Y\) charge of the interaction terms in (7.1). This means that the corrections in (7.3) respect the \(SL(2, Z)\) symmetry with our conjectured general modular transformation property (1.10).

The stringy correction term in (7.3) gives the leading correction, away from the double limit, which violates the approximate bonus \(SL(2, R)\) and \(U(1)_Y\) symmetries of correlation functions. It is subleading by \(N^{-3/2}\) for any fixed \(g_{YM}\). In the small \(g_{YM}\) limit, the leading contributions to the modular forms in (7.1) occur at string tree-level and are \(f(-q_T/4, q_T/4) = (\text{const}) g_{YM}^{-3/2} + \ldots\). In this limit, we see from (7.3) that the violations of the bonus symmetries are subleading by order \((g_{YM}^2 N)^{-3/2}\), as expected from (1.1) (in the limit of small \(g_{YM}\), \(D\)-string effects of size (1.3) can be ignored). There are also terms in the small \(g_{YM}\) expansion of the \(f(-q_T/4, q_T/4)\) which correspond to Yang-Mills instanton contributions to the correlation functions (7.3). It was argued in [9] and, more recently extensively analyzed and verified in [38], that \(SU(N)\) Yang-Mills instantons do lead to contributions to correlation functions precisely as expected from (7.3), with precisely the same instanton coefficients as obtained by expanding the \(f(-q_T/4, q_T/4)\). Violations of the bonus symmetries which do not get a contribution from the \((\alpha')^3\) terms in (7.3) are even more sub-leading in the \((g_{YM}^2 N)^{-1}\) expansion.

In [37] it was pointed out that the \(R^4\) term does not contribute to \(n<4\) point functions of the stress tensor because

\[
\frac{\delta^n}{\delta g^n} R^4\big|_{AdS_5 \times S^5} = 0 \quad \text{for} \quad n = 0, 1, 2, 3. \quad (7.4)
\]

Similarly, the other terms in (7.1) and low numbers of variations with respect to the fields also vanish when evaluated for the \(AdS_5 \times S^5\) vacuum. The first non-zero contribution from (7.1) is that of [37], where the \(R^4\) term contributes to the four-point function \(\langle \prod_{i=1}^4 T_{\mu_i \nu_i}(x_i) \rangle\). This leads to violation of the \(SL(2, R)\) symmetry starting at four-point functions. Using (1.9), the \(\tau\) dependence of this term also leads to violation of the \(U(1)_Y\) selection rule starting at the 5-point function \(\langle \mathcal{O}^{(-4)}_5(z) \prod_{i=1}^4 T_{\mu_i \nu_i}(x_i) \rangle\). The other \(U(1)_Y\) violating terms in (7.1) are only non-vanishing for higher \(n\)-point functions, e.g. the \(G^8\) term for \(n = 8\) point functions.

While (7.1) is just the leading string correction in the \(\alpha'\) expansion, we expect that, via the arguments of [39], all higher order \(\alpha'\) corrections to the effective supergravity action will also have the property, as in (7.4), that they vanish when evaluated for low...
numbers of variations around the $AdS_5 \times S^5$ vacuum. We thus expect that $SL(2, R)$ and $U(1)_Y$ are actually exact symmetries of $n \leq 3$ point functions for all $g_{YM}$ and $N$. The $SL(2, R)$ symmetry of $n \leq 3$ point functions is the conjecture of [4] that these correlation functions are independent of $g_{YM}$ (along with $\theta_{YM}$) for finite $N$. Descendant $n \leq 3$-point correlation functions will also be independent of $g_{YM}$ and $\theta$ for finite $N$, and respect the $U(1)_Y$ selection rule (1.8) exactly. The cancellations of radiative corrections exhibited in [6] support these conjectures.

We make a slightly stronger conjecture, which is suggested by (7.1): that the $U(1)_Y$ selection rule is an exact selection rule for all $n \leq 4$-point functions. Using (1.9), this implies that all $n \leq 3$-point correlation functions are independent of $\tau$.

The non-trivial $N$ dependence of the $n \leq 3$-point functions discussed in [4] must correspond, via (1.4), to non-trivial string loop corrections to these amplitudes. As mentioned in [4], one might expect that the scattering of three gravitons is not affected by quantum corrections. We note that this is actually completely consistent with the normalization of the 3-point function of the massless $O_2$ multiplet, which includes the conserved currents, if (1.4) is simply modified to $\bar{h} \sim (N^2 - 1)^{-1}$ for gauge group $SU(N)$ rather than $U(N)$. This can be understood simply as a one-loop string correction to the relation between $\kappa_{10}$ and $\kappa_{5}$ by $S^5$ dimensional reduction.

8. Proving exact $U(1)_Y$ invariance of $n$-point functions for low $n$.

We will now prove that all two-point functions of operators in short representations respect the $U(1)_Y$ selection rule (1.8) for all $g_{YM}, \theta_{YM},$ and $N$. Note that this selection rule is not a trivial consequence of the $SU(2, 2) \times SU(4)_R$ symmetry, as there are two-point functions which would respect these symmetries but violate $U(1)_Y$ if they were non-zero. For example, a non-zero two-point function of the operator of the form $\delta^4 O_p$, which is a Lorentz scalar and in the $(0, p - 2, 0)$ representation of $SU(4)_R$, with $U(1)_Y$ charge $-4$, with itself would respect $SU(2, 2) \times SU(4)_R$ but violate $U(1)_Y$; our argument shows that this and all other $U(1)_Y$ violating two-point functions vanish.

Consider the correlation functions in Euclidean space, with radial ordering from the origin (an arbitrary point). We then have vacuum states $|0\rangle$ and $\langle 0|$, which are annihilated by all supercharges, and correlation functions are to be understood as: $\langle 0| \prod_i O_i(x_i)|0\rangle$. For arbitrary operators $A$ and $B$,

$$\langle 0|[Q, A(x)]_\pm B(y)|0\rangle = \pm \langle 0|A(x)QB(y)|0\rangle = \pm \langle 0|A(x)[Q, B(y)]_\pm |0\rangle,$$

(8.1)
since $Q$ annihilates $\langle 0 |$ and $| 0 \rangle$; the same identity holds with $Q$ replaced by $\overline{Q}$. By repeating the operation (8.1), an arbitrary two-point function, of any operators in the table in appendix A, is equal to a two-point function of the form

$$\langle O_p(x) [D^{(n,\overline{\eta})}O_q](y) \rangle,$$

(8.2)

where $O_p(x)$ is the superconformal primary scalar operator with dimension $\Delta = p$ and $SU(4)_R$ representation $(0, p, 0)$ and $[D^{(n,\overline{\eta})}O_q](y)$ is a superconformal descendent obtained by acting with $n Q$ and $\overline{\eta} \overline{Q}$ operators on $O_q(y)$. The two-point function violates $U(1)_Y$ if it is non-zero for $n \neq \overline{n}$.

For two-point functions, there is an essential difference between whether the superconformal descendent $[D^{(n,\overline{\eta})}O_q]$ is a primary field or a descendent under the conformal group $SU(2, 2)$. The superconformal descendents in the table in appendix A are all primary under the conformal group $SU(2, 2)$. Each has an infinite tower of conformal descendents obtained by acting with $P_\mu$, corresponding to taking $x_\mu$ derivatives of the operator. As is well known, using the Ward identities of the $SU(2, 2)$ conformal group, it can be shown that the two-point function of two conformal primary operators can be non-zero only if their conformal dimensions are equal. Thus, if $[D^{(n,\overline{\eta})}O_q]$ is a $SU(2, 2)$ conformal primary operator, the two-point function (8.2) can be non-zero only if it has dimension $\Delta = p$ and in an $SU(4)_R$ representation which includes a singlet in its product with the $(0, p, 0)$ representation of $O_p$. The only operator in the table in appendix A which has these properties is the superconformal primary operator $O_p$ itself, i.e. $n = \overline{n} = 0$. The two-point function of $O_p$ with itself of course respects $U(1)_Y$, as $O_p$ is neutral.

Thus any two-point function involving a superconformal descendant which could potentially violate the $U(1)_Y$ selection rule will vanish unless, upon using (8.1) to write it in the form (8.2), the operator $[D^{(n,\overline{\eta})}O_q]$ is not a $SU(2, 2)$ conformal primary operator. This can happen because, using the supersymmetry algebra, a $Q$ and $\overline{Q}$ anti-commutator is replaced with $P_\mu$. The $P_\mu$ can be replaced with $\partial_{y_\mu}$ acting on the two-point function for the remaining operators, which is again of the form (8.2) but with an operator of the form $[D^{(n-1,\overline{\eta}-1)}O_q](y)$, since a $Q$ and $\overline{Q}$ were traded for the $\partial_{y_\mu}$. Repeating the above argument, the two-point function on which $\partial_{y_\mu}$ acts can also only be non-zero if $(n - 1, \overline{\eta} - 1) = 0$ and $q = p$ or if $[D^{(n-1,n-1)}O_q]$ is a $SU(2, 2)$ descendent, $[D^{(n-1,n-1)}O_q] = [P_\mu', [D^{(n-2,n-2)}O_q]]$. Continuing this argument, the only non-zero two-point functions have in (8.2) $n = \overline{n}$ and $q = p$, with $[D^{(n,n)}O_q]$ a $SU(2, 2)$ descendent of $O_p$. 

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Thus all non-zero two-point functions respect the $U(1)_Y$ selection rule and can be written as $y_\mu$ derivatives of the two-point function of superconformal primary operators $\langle O_p(x)O_p(y) \rangle$. For example, $\langle T_{\mu\nu}(x)T_{\rho\sigma}(y) \rangle$ and $\langle O^{(-4)}_\tau(x)O^{(4)}_\tau(y) \rangle$ can each be written as particular combinations of four $\partial_y$ derivatives acting on $\langle O_2(x)O_2(y) \rangle$, while $\langle O^{(-4)}_\tau O^{(-4)}_\tau \rangle = 0$. All two-point functions of superconformal descendents and, in particular, their normalization, are fixed by the primary $\langle O_p(x)O_p(y) \rangle$ correlation functions. This analysis, again, is valid for all $g_{YM}$, $\theta_{YM}$, and $N$.

We note that, because all two-point functions exactly respect the $U(1)_Y$ selection rule, the Born-approximation calculation of an arbitrary $n$-point function, where the $n$-point function is broken up into products of two-point functions, will also respect $U(1)_Y$. This approximation gives the leading contribution to the correlation function in the small $g_{YM}^2 N$ limit. Thus arbitrary $n$-point correlation functions will also respect the $U(1)_Y$ selection rule in the small $g_{YM}^2 N$ limit:

$$\langle \prod_{i=1}^{n} O_i^{(q_i)}(x_i) \rangle = F_{i_1...i_n}(x_i; N)\delta_{q_T,0} + \text{order } (g_{YM}^2 N),$$

(8.3)

with $q_T = \sum_{i=1}^{n} q_i$ and $F_{i_1...i_n}(x_i; N)$ independent of $g_{YM}$ and $\theta_{YM}$. This is valid for arbitrary $N$ and, in the limit of large $N$,

$$F_{i_1...i_n}(x_i; N) \approx N^2 H_{i_1...i_n}(x_i),$$

(8.4)

where the functions $H_{i_1...i_n}$ could generally differ from those of (1.6), which described the large $g_{YM}^2 N$ limit, as arbitrary correlation functions generally depend on $g_{YM}^2 N$. It would be interesting to check the field theory prediction (8.3) against IIB string theory and the duality of [1,3,4]: stringy violations of $U(1)_Y$ must also vanish in the small $\lambda = g_{YM}^2 N$ limit. Checking this in string theory would require better understanding of the worldsheet CFT with non-zero $F_5$ flux.

Manipulations of the type used above do not seem as useful for higher $n$-point functions. Although we expect that the $U(1)_Y$ symmetry is an exact symmetry for three-point functions and possibly also four-point functions, we have here succeeded only in proving it for two-point functions. In the next section, we discuss a formalism which should just be a convenient way to re-package the superconformal Ward identities. As we will discuss, however, this formalism is extremely powerful – perhaps too powerful!
9. Harmonic superspace and the $U(1)_Y$ symmetry

The $\mathcal{N} = 4$ gauge superfield $W$, as well as the operators in small representations, obey constraints which imply that they only depend on half of the coordinates of a would-be superspace. This is seen in the table, in that the small representations truncate at $\delta^{i\dot{r}\dot{s}}O_p$ rather than $\delta^{i\dot{r}}\delta^{s}O_p$. It is impossible to implement this constraint in superspace in which $SU(4)_R$ is manifest. Introducing Grassmann coordinates $\Theta^I$ and $\bar{\Theta}^{I\dot{\alpha}}$ conjugate to $Q^I$ and $\bar{Q}^{I\dot{\alpha}}$, the gauge superfield should depend on two of the four possible $\Theta^I$ coordinates and two of the four possible $\bar{\Theta}^{I\dot{\alpha}}$ coordinates. Thus at most a $SU(2) \times SU(2)$ subgroup of $SU(4)_R$ can be made manifest. Basically, we decompose the supersymmetries under $SU(4)_R \rightarrow SU(2) \times SU(2) \times U(1)$ as

$4 \rightarrow (2,1)_1 \oplus (1,2)_{-1}$ and $\bar{4} \rightarrow (\bar{2},1)_{-1} \oplus (1,\bar{2})_1$

and only keep fermionic coordinates for $(2,1)_1$ and $(1,\bar{2})_1$.

The result, then, is $\mathcal{N} = 4$ harmonic (or “analytic”) superspace, involving coordinates

$$X = \left( x_{\alpha\dot{\alpha}}, \lambda_{\alpha a'}, \pi_{a\dot{\alpha}}, y_{aa'} \right), \quad (9.1)$$

where $\lambda_{\alpha a'}$ and $\pi_{a\dot{\alpha}}$ are the fermionic coordinates, with $a = 1, 2$ and $a' = 1, 2$ labels for the $SU(2) \times SU(2)' \subset SU(4)_R$ and $y_{aa'}$ a bosonic coordinate living on the Grassmanian coset space $SU(4)/S(U(2) \times U(2)')$. Because this coset space is compact, the power of $y_{aa'}$ is constrained. The $y_{aa'}$ coordinates allow the $SU(2) \times SU(2)'$ indices to be converted back to $SU(4)_R$ indices at the end of the day. This formalism has been discussed in detail in the series of papers [7,8].

This superspace formalism is a remarkably powerful technology: it allows the $\mathcal{N} = 4$ gauge multiplet to be packaged into a single superfield $W(X)$, and the entire collection of small $\mathcal{N} = 4$ representation operators appearing in the table of appendix $A$ to be neatly packaged into the single super-space operator $A_p(X) = \text{Tr}_{SU(N)}W(X)^p$. General correlation functions of small representation operators then take the form

$$\langle \prod_{i=1}^n A_{p_i}(X_i) \rangle = f_{p_1,\ldots,p_n}(X_1,\ldots,X_n; N; \tau). \quad (9.2)$$

A key dynamical assumption [7] is that the function $f_{p_i}(X_i)$ remains analytic, with only positive powers of $y_{aa'}$, in the quantum theory. In the last reference of [7], this principle was explicitly checked for a particular correlation function, at the two loop level, in the analogous $\mathcal{N} = 2$ harmonic superspace formalism; all intermediate non-harmonic analytic terms were found to cancel upon adding contributions from all diagrams.
The function in (9.2) is then constrained by superconformal invariance; we now summarize the results of [7,8]. Two-point functions and three-point functions are argued to be completely fixed to be

\[
\langle A_p(X_1)A_q(X_2) \rangle = c_p \delta_{p,q} g_{12}^p, \quad (9.3)
\]

\[
\langle A_{p_1}A_{p_2}A_{p_3} \rangle = c_{p_1,p_2,p_3} (g_{12})^{1/2} (p_1+p_2-p_3) (g_{23})^{1/2} (p_2+p_3-p_1) (g_{13})^{1/2} (p_1+p_3-p_2), \quad (9.4)
\]

where \(c_p\) and \(c_{p_1,p_2,p_3}\) are (a priori, possibly \(\tau\) dependent) constants and

\[
g_{ij} \equiv (\text{sdet} X_{ij})^{-1} = \hat{g}_{ij}, \quad (9.5)
\]

\[
(y_{ij})_{aa'} = (y_{ij})_{aa'} - \frac{(\pi_{ij})_{\dot{a}\dot{a'}} (x_{ij})^{\dot{a}\dot{a'}} (\lambda_{ij})_{aa'}}{(x_{ij})^2}, \quad (9.6)
\]

and \(X_{ij} \equiv X_i - X_j\) (i.e. \(y_{ij} = y_i - y_j\) etc.). \(n\)-point functions with \(n \geq 4\) again involve the \(g_{ij}\) (9.5), though now there can also be undetermined functions of superconformal invariants:

\[
\langle \prod_{i=1}^{n} A_{p_i}(X_i) \rangle = \prod_{i<j}^{n} \frac{g_{ij}^{(p_i+p_j-\pi_{ij})/(n-2)}}{F_{p_i}(I)}, \quad (9.7)
\]

where \(p_T = \sum_{i=1}^{n} p_i\) and \(I\) are all possible superconformal invariants. The possible superconformal invariants were classified in [8] and found to be of two types. The first are super-cross-ratios of the \(g_{ij}\):

\[
\frac{g_{ij}g_{kl}}{g_{ik}g_{jl}}. \quad (9.8)
\]

The second type of superconformal invariants involve super-traces \(str N_i^p\), with \(p = 1, \ldots, 4\), of quantities \(N_i\) defined in [8], the simplest example, for four points, being

\[
str N = str (X_{12}^{-1}X_{23}X_{34}^{-1}X_{41}). \quad (9.9)
\]

As remarked in [7,8], the condition that there be no \(y_{ij}\) singularities puts constraints on the dependence of this second class of invariants; these aspects will not be relevant for the point we are making here.

Having described this powerful formalism, it must be mentioned that its applicability is considered suspicious by some physicists. (See, for example, in the discussion of descendent correlation functions in [8].) A reason for concern is that there is no known off-shell superspace for \(\mathcal{N} = 4\) supersymmetry [31]; the present formalism is purely on-shell. The danger, then, is that it is incapable of reproducing the off-shell contributions to correlation functions in intermediate channels.
We will argue that assuming applicability of this formalism leads to an incorrect result: all correlation functions of operators in short representations of the superconformal group would exactly respect the $U(1)_Y$ selection rule! If correct, this would imply, as a consequence of (1.9), that all correlation functions of operators in short multiplets are completely independent of $g_{YM}$. However, as discussed in footnote 2, this latter result has been shown to be incorrect, as $n \geq 4$-point functions are definitely renormalized.

To see the above result about $U(1)_Y$, note that $U(1)_Y$ charge in this formalism is carried by $\lambda_{\alpha a'}$, which has charge +1, and $\pi_{\dot{a} \alpha}$, which has charge −1. The bosonic coordinates $x_{\alpha \dot{\alpha}}$ and $y_{aa'}$ are, of course, neutral under $U(1)_Y$. In order to have a correlation function which does not respect the $U(1)_Y$ symmetry, the RHS of (9.2) would have to contain a function $f_{p_i}(X_i; N; \tau)$ which is not invariant under the $U(1)_Y$ transformation

$$\lambda_i \rightarrow C \lambda_i$$

and

$$\pi_{i \dot{a}} \rightarrow C^{-1} \pi_{i \dot{a}},$$

(9.10)

for an arbitrary phase $C$ (which could just as well be an arbitrary complex number, corresponding to $U(1)_Y$ complexified). This transformation can be represented on the $X_i$ coordinates (9.1) as

$$X_i \rightarrow T^{-1} X_i T, \quad \text{with} \quad T = \begin{pmatrix} \sqrt{C} & 0 \\ 0 & 1 \end{pmatrix} \in \text{GL}(2|2).$$

(9.11)

Since $\text{sdet} T = C$, this $T$ is not in $SL(2|2)$ for a non-trivial $U(1)_Y$ transformation.

It is easily seen from (9.3) and (9.9) that the $g_{ij}$ are invariant under the $U(1)_Y$ transformation (9.10) or (9.11). Upon expanding out both sides of (9.3) and (9.4) in components, it then follows that all two point and three point functions of operators with non-zero total $U(1)_Y$ charge necessarily vanish. These results are plausible and in line with our conjecture, and with the descendent 3-point function calculation in [6], which had non-zero net $U(1)_Y$ charge and was found to vanish to leading and next-to-leading order in a small coupling expansion.

Moving on to four and higher point functions, the $g_{ij}$ terms in (9.7), again, respect the $U(1)_Y$ selection rule. Thus the only way there could be terms on the right side of (9.7) with non-zero $U(1)_Y$ charge is if some of the superconformal invariants $I$ carry non-zero $U(1)_Y$ charge. It is clear that all invariants of the first type (9.8) respect $U(1)_Y$, since the $g_{ij}$ all respect $U(1)_Y$. Further, the invariants of the second type also respect $U(1)_Y$. Clearly (9.9) is invariant under (9.11). Indeed, the transformation (9.10) is achieved in terms of the $u_i = (1, X_i)$ coordinates of [8] by $u_i \rightarrow T^{-1} u_i g_T$, with $g_T = \text{diag}(T, T)$, with
$T$ given by (9.11). $g_T$ is in $GL(4|4)$ rather than $SL(4|4)$, but the basic superconformal ingredients $K_i$ and $L_i$ defined in eqns. (27) and (28) of [8] are clearly invariant under $u_i \rightarrow u_ig_T$ anyway. The final invariants, by construction, must also be invariant under the $u_i \rightarrow T^{-1}u_i$ transformation needed to take $u_i$ back to the form $(1,X')$. Thus all invariants constructed in [8] respect the $U(1)_Y$ symmetry.

We thus obtain a result which is incorrect: that, for all $g_{YN}$, and $N$, all $n$-point correlation functions of short representation operators exactly obey the exact $U(1)_Y$ selection rule (1.8), which would imply their non-renormalization. This is contrary to the results of [9,10,12,13,38], where it was explicitly shown that various $n \geq 4$ point functions do, in fact, get renormalized. Again, we have conjectured that the $U(1)_Y$ selection rule actually is exact for $n \leq 4$ point functions, which would imply non-renormalization only for $n \leq 3$ point functions.

There are two options at this juncture:

(1) The $\mathcal{N} = 4$ harmonic superspace formalism is inherently problematic. Again, this might have been expected as it is a purely on-shell formalism.

(2) The $\mathcal{N} = 4$ harmonic superspace formalism can be salvaged by finding some new superconformal invariants, which violate $U(1)_Y$, which have been overlooked in the classification of [8]. This would allow the above incorrect conclusions about the exact $U(1)_Y$ selection rule to be evaded.

Option (1) would be unfortunate.

It would be nicest if option (2) is correct and that, in line with our conjecture, there is (at least one) as-yet missing superconformal invariant, which violates $U(1)_Y$, and which can only be written down for $n > 4$ point functions. However, I have not yet succeeded in constructing such an invariant. Again, this issue in no way affects the results and conjectures of the previous sections.

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Appendix A. Table of the spectrum of short multiplets

| form  | $SO(4)$   | $\Delta$ | $SU(4)_R$     | $U(1)_Y$ |
|-------|------------|----------|----------------|----------|
| $\mathcal{O}_p$ | (0, 0) | $p$ | (0, $p$, 0) | 0 |
| $\delta \mathcal{O}_p$ | ($\frac{1}{2}$, 0) | $p + \frac{1}{2}$ | (0, $p - 1$, 1) | $-1$ |
| $\overline{\mathcal{O}}_p$ | (0, $\frac{1}{2}$) | $p + \frac{1}{2}$ | (1, $p - 1$, 0) | 1 |
| $\delta^2 \mathcal{O}_p$ | (1, 0) | $p + 1$ | (0, $p - 1$, 0) | $-2$ |
| $\overline{\delta^2 \mathcal{O}}_p$ | (0, 1) | $p + 1$ | (0, $p - 1$, 0) | 2 |
| $\delta^3 \mathcal{O}_p$ | (0, 0) | $p + 1$ | (0, $p - 2$, 2) | $-2$ |
| $\overline{\delta^3 \mathcal{O}}_p$ | (0, 0) | $p + 1$ | (2, $p - 2$, 0) | 2 |
| $\delta^4 \mathcal{O}_p$ | (0, 0) | $p + 1$ | (2, $p - 2$, 0) | 2 |
| $\overline{\delta^3 \mathcal{O}}_p$ | (1, $\frac{1}{2}$) | $p + \frac{3}{2}$ | (1, $p - 1$, 2) | 0 |
| $\delta^4 \mathcal{O}_p$ | ($\frac{1}{2}$, 0) | $p + \frac{3}{2}$ | (0, $p - 1$, 0) | $-3$ |
| $\overline{\delta^4 \mathcal{O}}_p$ | (0, 1) | $p + 2$ | (0, $p - 2$, 0) | $-4$ |
| $\delta^5 \mathcal{O}_p$ | (1, 1) | $p + 2$ | (0, $p - 2$, 0) | 0 |
| $\overline{\delta^5 \mathcal{O}}_p$ | (1, 1) | $p + 2$ | (0, $p - 2$, 0) | 0 |
| $\delta^6 \mathcal{O}_p$ | ($\frac{1}{2}$, 0) | $p + \frac{5}{2}$ | (2, $p - 3$, 1) | 1 |
| $\overline{\delta^6 \mathcal{O}}_p$ | (0, $\frac{1}{2}$) | $p + \frac{5}{2}$ | (1, $p - 3$, 1) | $-1$ |
| $\delta^7 \mathcal{O}_p$ | ($\frac{1}{2}$, $\frac{1}{2}$) | $p + 2$ | (1, $p - 3$, 1) | $-1$ |
| $\overline{\delta^7 \mathcal{O}}_p$ | ($\frac{1}{2}$, $\frac{1}{2}$) | $p + 2$ | (1, $p - 3$, 1) | 1 |
| $\delta^8 \mathcal{O}_p$ | ($\frac{1}{2}$, 1) | $p + \frac{5}{2}$ | (0, $p - 3$, 1) | 1 |
| $\overline{\delta^8 \mathcal{O}}_p$ | ($\frac{1}{2}$, 1) | $p + \frac{5}{2}$ | (0, $p - 3$, 1) | $-1$ |
| $\delta^9 \mathcal{O}_p$ | (1, 0) | $p + 3$ | (0, $p - 3$, 0) | 2 |
| $\overline{\delta^9 \mathcal{O}}_p$ | (1, 0) | $p + 3$ | (0, $p - 3$, 0) | $-2$ |
| $\delta^{10} \mathcal{O}_p$ | (0, 1) | $p + 3$ | (0, $p - 3$, 0) | 2 |
| $\overline{\delta^{10} \mathcal{O}}_p$ | (0, 1) | $p + 3$ | (0, $p - 3$, 0) | $-2$ |
| $\delta^{11} \mathcal{O}_p$ | (0, 0) | $p + 3$ | (0, $p - 4$, 0) | 2 |
| $\overline{\delta^{11} \mathcal{O}}_p$ | (0, 0) | $p + 3$ | (0, $p - 4$, 0) | $-2$ |
| $\delta^{12} \mathcal{O}_p$ | (0, 0) | $p + 3$ | (0, $p - 4$, 0) | 2 |
| $\overline{\delta^{12} \mathcal{O}}_p$ | (1, $\frac{1}{2}$) | $p + \frac{5}{2}$ | (1, $p - 4$, 1) | $-1$ |
| $\delta^{13} \mathcal{O}_p$ | (0, 0) | $p + 4$ | (0, $p - 4$, 0) | 0 |
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