A random attractor for stochastic porous media equations on infinite lattices

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Abstract: The paper is devoted to studying the existence of a random attractor for stochastic porous media equations on infinite lattices under some conditions.

Keywords: Random dynamical system; stochastic porous media lattice equations; random attractor

1 Introduction

In this paper, we study the following stochastic porous media lattice equations perturbed by a multiplicative noise:

\[
\frac{du_i(t)}{dt} = \Phi(u_{i-1}) - 2\Phi(u_i) + \Phi(u_{i+1}) - \lambda(u_i + u_i^{p-1}) + g_i + \alpha_i u_i \circ \frac{dw(t)}{dt}
\]  

(1.1)

with the initial data

\[ u_i(0) = u_{0,i}, \quad i \in \mathbb{Z}, \]  

(1.2)

where \( \mathbb{Z} \) denotes the integer set, \( u_i \in \mathbb{R} \), \( \lambda \) and \( \alpha_i \in \mathbb{R} \) are positive constants, \( p > 1 \), \( \Phi \) satisfies certain dissipative conditions, \( g_i \in \mathbb{R} \), \( w(t) \) is a Brownian motion (Wiener process) and \( \circ \) denotes the Stratonovich sense of the stochastic term.

System (1.1) can be regarded as the spatial discrete form on 1D infinite lattices of a type of stochastic porous media equations

\[ u_t = \Delta(\Phi(u)) - \lambda(u + |u|^{p-1}u) + g(x) + \alpha u \circ \frac{dw(t)}{dt}, \quad x \in \mathbb{R}. \]  

(1.3)

When \( \lambda = 0 \), the (similar) stochastic porous media equations have been intensively investigated in recent years, see e.g. 2 3 4 14 25 26 and references therein. The long-time behavior of stochastic porous media equations with additive white noise in terms of the existence of a random attractor has first been established in 8 which then has been extended to more generally distributed additive noise in 15 16, and linear multiplicative noise in space and time in 17.
Recently, lattice dynamical systems, which can be considered as the spatial discrete of some PDEs, have drawn much attention from mathematicians and physicists, due to the wide range of applications in various areas (see [12]). For existence and properties of different attractors for autonomous deterministic lattice dynamical systems, see e.g. [6, 24, 28, 32, 34] and e.g. [29, 33] for non-autonomous deterministic cases. For stochastic ones, stochastic lattice dynamical systems (SLDS) arise naturally while random influences or uncertainties are taken into account, these noises may play an important role as intrinsic phenomena rather than just compensation of defects in deterministic models. Since Bates et al. [5] initiated the study of SLDS, a lot of work has been done regarding the existence of global random attractors for SLDS with white noise in regular or weighted spaces of infinite sequences, see e.g. [10, 11, 21]. For lattice dynamical systems perturbed by “rough” noises, see [18, 19, 20] for more details.

Notice that there are amounts of work considered for random attractors for PDEs defined on unbounded domains (see e.g. [7, 9, 30, 31]). This introduces a major obstacle that Sobolev embeddings are not compact for these cases. In fact, the study of lattice differential systems generated by some spatially discrete PDEs can be regarded as the cases considered on unbounded domains. We take the square-summable infinite sequences space \( \ell^2 \) as the phase space, so that there is not any embedding relationships to tackle these questions. Often, lattice models are used more in a physical and numerical treatment in porous media (see e.g. [22, 23, 27]). There are few detailed analysis in the sense of infinite dynamical systems. Furthermore, we can see that all the references on the asymptotic behavior of stochastic porous media equations studied above are restricted to the bounded domains. To our knowledge, there is no result in the case of unbounded domains. Here, we set up the stochastic porous media lattice equations, and give the existence of a random attractor for the lattice model, which can be seen as a first attempt to the unbounded cases.

The paper is organized as follows. In next section, we recall some preliminaries on random dynamical systems and random attractors. In section 3 we formulate the model of stochastic porous media lattice equations and give a unique solution to system (1.1), which generates a continuous random dynamical system. We obtain the main result, that is the existence of a random attractor, in section 4.

In the sequel, we denote \( \ell^p \) the space of \( p \)-times summable infinite sequences with norm \( \| \cdot \|_p \), especially when \( p = 2 \), we denote \( \ell^2 = (\ell^2, (\cdot, \cdot), \| \cdot \|) \).
2 Preliminaries

For the reader’s convenience, we first introduce some basic concepts related to random dynamical systems and random attractors, which are taken from [1], [13] and [21]. Let \((\mathbb{H}, \| \cdot \|_\mathbb{H})\) be a separable Hilbert space and \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space.

**Definition 2.1.** A stochastic process \(\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}\) is a continuous random dynamical system (RDS) over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) if \(\varphi\) is \((\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}^{\mathbb{H}}(\mathbb{H}))\)-measurable, and for all \(\omega \in \Omega\),

\[(i)\] the mapping \(\varphi(t, \omega) : \mathbb{H} \mapsto \mathbb{H}, x \mapsto \varphi(t, \omega)x\) is continuous for every \(t \geq 0\),

\[(ii)\] \(\varphi(0, \omega)\) is the identity on \(\mathbb{H}\),

\[(iii)\] (cocycle property) \(\varphi(s + t, \omega) = \varphi(t, \theta_s \omega) \varphi(s, \omega)\) for all \(s, t \geq 0\).

**Definition 2.2.** (i) A set-valued mapping \(\omega \mapsto B(\omega) \subset \mathbb{H}\) (we may write it as \(B(\omega)\) for short) is said to be a random set if the mapping \(\omega \mapsto \text{dist}_\mathbb{H}(x, B(\omega))\) is measurable for any \(x \in \mathbb{H}\), where \(\text{dist}_\mathbb{H}(x, D)\) is the distance in \(\mathbb{H}\) between the element \(x\) and the set \(D \subset \mathbb{H}\).

(ii) A random set \(B(\omega)\) is said to be bounded if there exist \(x_0 \in \mathbb{H}\) and a random variable \(r(\omega) > 0\) such that \(B(\omega) \subset \{x \in \mathbb{H} : \|x - x_0\|_\mathbb{H} \leq r(\omega), x_0 \in \mathbb{H}\}\) for all \(\omega \in \Omega\).

(iii) A random set \(B(\omega)\) is called a compact random set if \(B(\omega)\) is compact for all \(\omega \in \Omega\).

(iv) A random bounded set \(B(\omega) \subset \mathbb{H}\) is called tempered with respect to \((\theta_t)_{t \in \mathbb{R}}\) if for a.e. \(\omega \in \Omega\), \(\lim_{t \to +\infty} e^{-\gamma t} \text{d}(B(\theta_{-t} \omega)) = 0\) for all \(\gamma > 0\), where \(\text{d}(B) = \sup_{x \in B} \|x\|_\mathbb{H}\). A random variable \(\omega \mapsto r(\omega) \in \mathbb{R}\) is said to be tempered with respect to \((\theta_t)_{t \in \mathbb{R}}\) if for a.e. \(\omega \in \Omega\), \(\lim_{t \to +\infty} \sup_{t \in \mathbb{R}} e^{-\gamma t} r(\theta_{-t} \omega) = 0\) for all \(\gamma > 0\).

We consider a RDS \(\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}\) over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) and \(\mathcal{D}(\mathbb{H})\) the set of all tempered random sets of \(\mathbb{H}\).

**Definition 2.3.** A random set \(K\) is called an absorbing set in \(\mathcal{D}(\mathbb{H})\) if for all \(B \in \mathcal{D}(\mathbb{H})\) and a.e. \(\omega \in \Omega\) there exists \(t_B(\omega) > 0\) such that

\[\varphi(t, \theta_{-t} \omega) B(\theta_{-t} \omega) \subset K(\omega)\] for all \(t \geq t_B(\omega)\).

**Definition 2.4.** A random set \(A\) is called a global random \(\mathcal{D}(\mathbb{H})\) attractor (pullback \(\mathcal{D}(\mathbb{H})\) attractor) for \(\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}\) if the following hold:

(i) \(A\) is a random compact set, i.e. \(\omega \mapsto d(x, A(\omega))\) is measurable for every \(x \in \mathbb{H}\) and \(A(\omega)\) is compact for a.e. \(\omega \in \Omega\);

(ii) \(A\) is strictly invariant, i.e. for \(\omega \in \Omega\) and all \(t \geq 0\), \(\varphi(t, \omega) A(\omega) = A(\theta_t \omega)\);

(iii) \(A\) attracts all sets in \(\mathcal{D}(\mathbb{H})\), i.e. for all \(B \in \mathcal{D}(\mathbb{H})\) and a.e. \(\omega \in \Omega\),

\[\lim_{t \to +\infty} \text{dist}(\varphi(t, \theta_{-t} \omega) B(\theta_{-t} \omega), A(\omega)) = 0,\]

where \(\text{dist}(X, Y) = \sup_{x \in X} \inf_{y \in Y} \|x - y\|_\mathbb{H}\) is the Hausdorff semi-metric \((X \subseteq \mathbb{H}, Y \subseteq \mathbb{H})\).
Proposition 2.5. (See §3.) Let $K \in D(\mathbb{H})$ be an absorbing set for the continuous RDS $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ which is closed and which satisfies for a.e. $\omega \in \Omega$ the following asymptotic compactness condition: each sequence $x_n \in \varphi(t_n, \theta - t_n, K(\theta - t_n, \omega))$ with $t_n \to \infty$ has a convergent subsequence in $\mathbb{H}$. Then the cocycle $\varphi$ has a unique global random attractor

$$A(\omega) = \bigcap_{\tau \geq T_k(\omega)} \bigcup_{t \geq \tau} \varphi(t, \theta - t, K(\theta - t, \omega)).$$

Especially, when we focus on SLDS and denote $D(\ell^2)$ the set of all tempered random sets of $\ell^2$, it yields the following result:

Proposition 2.6. (See §21.) Suppose that

(a) there exists a random bounded absorbing set $K(\omega) \in D(\ell^2)$ for the continuous RDS $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$;

(b) the RDS $\{\varphi(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ is random asymptotically null on $K(\omega)$, i.e., for any $\varepsilon > 0$, there exist $T(\varepsilon, \omega, K) > 0$ and $I_0(\varepsilon, \omega, K) \in \mathbb{N}$ such that

$$\sup_{u \in K(\omega)} \sum_{|i| > I_0(\varepsilon, \omega, K)} |(\varphi_i(t, \theta - t, u(\theta - t)))|^2 \leq \varepsilon^2, \quad \forall t \geq T(\varepsilon, \omega, K).$$

(2.1)

Then the RDS $\{\varphi(t, \omega, \cdot)\}_{t \geq 0, \omega \in \Omega}$ possesses a unique global random $D(\ell^2)$ attractor given by

$$\tilde{A}(\omega) = \bigcap_{\tau \geq T(\omega, K)} \bigcup_{t \geq \tau} \varphi(t, \theta - t, K(\theta - t, \omega)).$$

3 Stochastic porous media lattice equations

We note that system (1.1) can be interpreted as a system of integral equations

$$u_i(t) = u_i(0) + \int_0^t [\Phi(u_{i-1}(s)) - 2\Phi(u_i(s)) + \Phi(u_{i+1}(s)) - \lambda(u_i(s) + |u_i(s)|^{p-1}u_i(s))) + g_i]ds + \alpha_i \int_0^t u_i(s) \circ dw(s), \quad i \in \mathbb{Z},$$

(3.1)

where the stochastic integral is understood to be in the Stratonovich sense.

Assumptions on $\Phi$ $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, $\Phi(0) = 0$ and there exist constants $c_1, c_2 \in (0, \infty)$ such that

$$\frac{(p + 1)^2}{4} c_2 |u|^{p-1} \leq \Phi'(u) \leq c_1 (1 + |u|^{p-1}), \quad \forall u \in \mathbb{R}.$$  

(3.2)

Then (3.2) implies the following monotonicity condition:

$$(\Phi(u) - \Phi(v))(u - v) \geq k |u - v|^{p+1} - a_i, \quad \forall u, v \in \mathbb{R},$$

(3.3)
where \( k \in (0, \infty) \) and \( a_i \in [0, \infty) \) such that \((a_i)_{i \in \mathbb{Z}} \in \ell^1, p > 1 \) the same one in (1.1). Actually, these conditions are satisfied for \( \Phi(u) = |u|^p - 1 \) (see e.g. \([8, 14]\)).

For convenience, we now formulate system (3.1) as a stochastic differential equation in \( \ell^2 \). Define \( B \) and its adjoint operator \( B^* \) from \( \ell^2 \) to \( \ell^2 \) as follows. For \( u = (u_i)_{i \in \mathbb{Z}} \in \ell^2 \),

\[
(Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i.
\]

Also, define the operator \( Au \) for \( i = 0 \) and \( 1 \). We have \((Au, u) = (Bu, B^*u) = \|Bu\|^2 \leq 4\|u\|^2 \), which means that \( A \) is a bounded operator from \( \ell^2 \) to itself.

Now, system (3.1) with initial values \( u_0 = (u_0, i)_{i \in \mathbb{Z}} \) can be rewritten as the following equation in \( \ell^2 \) for \( t \geq 0 \) and \( \omega \in \Omega \),

\[
u(t) = u_0 + \int_0^t [-A(\Phi(u(s))) - \lambda(u(s) + \|u(s)\|^{p-1}u(s))] + g(s) ds
\]

\[
+ \alpha \int_0^t u(s) \circ dw(s),
\]

where \( A(\Phi(u)) = A(\Phi(u_i)_{i \in \mathbb{Z}}) = (-\Phi(u_{i-1}) + 2\Phi(u_i) - \Phi(u_{i+1}))_{i \in \mathbb{Z}} \), \( g = (g_i)_{i \in \mathbb{Z}} \in \ell^2 \) and \( \alpha = (\alpha_i)_{i \in \mathbb{Z}} \).

To prove that this stochastic equation (3.4) generates a random dynamical system, we will transform it into a random differential equation in \( \ell^2 \). First, we need to recall some properties of the Ornstein-Uhlenbeck processes.

Consider \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, where \( \Omega \) is a subset of \(C_0(\mathbb{R}, \mathbb{R}) = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\} \), endowed with the compact open topology (see [11]), \( \mathcal{F} \) is the Borel \( \sigma \)-algebra and \( \mathbb{P} \) is the corresponding Wiener measure on \( \Omega \). Let \( \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R} \), then \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\) is an ergodic metric dynamical system.

To solve (3.4), we make a change of variables

\[
\nu(t) = e^{-\alpha \theta_t \omega} u(t),
\]

where \( u(t) \) is a solution of (3.4) and \( z(\theta_t \omega) = -\int_{-\infty}^0 e^{\tau} \theta_t \omega(\tau) d\tau \) is a pathwise solution of the Ornstein-Uhlenbeck equation

\[
dz + z dt = dw(t).
\]

By \([5, 10]\), we know that there exists a \( \theta_t \)-variant set \( \Omega' \subseteq \Omega \) of full \( \mathbb{P} \) measure such that \( z(\theta_t \omega) \) is continuous in \( t \) for every \( \omega \in \Omega' \), and the random variable \( |z(\omega)| \) is tempered. In addition, for every \( \omega \in \Omega' \), we have the following limits:

\[
\lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{|t|} = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t |z(\theta_s \omega)| ds = 0.
\]
Hereafter, we will write $\Omega$ as $\Omega'$ instead. Then $v(t)$ satisfies the following evolution equation with random coefficients but without white noise

$$
\frac{dv(t)}{dt} = -e^{-\alpha z(\theta t)} A(\Phi(e^{\alpha z(\theta t)} v)) + (\alpha z(\theta t) - \lambda) v
$$

$$
- \lambda e^{\alpha(p-1) z(\theta t)} |v|^{p-1} v + e^{-\alpha z(\theta t)} g,
$$

(3.7)

$$
v(0) = v_0 \in \ell^2.
$$

Now, we have the following result.

**Theorem 3.1.** Let $T > 0$ and $v_0 \in \ell^2$. Then the following two statements hold:

1. For every $\omega \in \Omega$, system (3.7) has a unique solution $v(\cdot, \omega, v_0) \in C([0, T], \ell^2)$;
2. The solution $v$ of (3.7) depends continuously on the initial data $v_0$.

**Proof.** (1) For any fixed $T > 0$ and $\omega \in \Omega$, let $u, v \in \ell^2$,

$$
\|A(\Phi(e^{\alpha z(\theta t)} u)) - A(\Phi(e^{\alpha z(\theta t)} v))\|^2
$$

$$
= \sum_{i \in \mathbb{Z}} (A(\Phi(e^{\alpha z(\theta t)} u_i)) - A(\Phi(e^{\alpha z(\theta t)} v_i)))^2
$$

$$
\leq 12 \sum_{i \in \mathbb{Z}} (\Phi(e^{\alpha z(\theta t)} u_i) - \Phi(e^{\alpha z(\theta t)} v_i))^2
$$

$$
\leq 12C_1^2 (1 + e^{\alpha(p-1) z(\theta t)} (\|u\| + \|v\|)^{p-1})^2 \|u - v\|^2,
$$

whence

$$
\|A(\Phi(e^{\alpha z(\theta t)} u)) - A(\Phi(e^{\alpha z(\theta t)} v))\|
$$

$$
\leq C(1 + e^{\alpha(p-1) z(\theta t)} (\|u\| + \|v\|)^{p-1}) \|u - v\|
$$

$$
\leq C(1 + e^{\alpha(p-1) \max_{\in [0,T]} z(\theta t)} (\|u\| + \|v\|)^{p-1}) \|u - v\|,
$$

and

$$
\| |u|^{p-1} u - |v|^{p-1} v | \leq C'(\|u\| + \|v\|)^{p-1} \|u - v\|,
$$

which implies that $A(\Phi(e^{\alpha z(\theta t)} u))$ and $|v|^{p-1} v$ are Lipschitz in bounded sets of $\ell^2$ with respect to $v$ uniformly for any $t \in [0, T]$. So by standard arguments, system (3.7) possesses a local solution $v(\cdot, \omega, v_0) \in C([0, T_{\text{max}}], \ell^2)$, where $[0, T_{\text{max}})$ is the maximal interval of existence of the solution of (3.7). Next, we need to prove that the local solution in fact a global one. From (3.7), it yields that

$$
\frac{d}{dt} \|v(t)\|^2 + 2\lambda e^{\alpha(p-1)z(\theta t)} \|v\|^{p+1}_{p+1} = 2e^{-\alpha z(\theta t)} (-A(\Phi(e^{\alpha z(\theta t)} v)), v)
$$

$$
+ 2(\alpha z(\theta t) - \lambda) \|v\|^2 + 2e^{-\alpha z(\theta t)}(g, v).
$$

(3.8)
By (3.3), we have

\[
(e^{-\alpha z(\theta t)} A(\Phi(e^{\alpha z(\theta t)} v)), v) = e^{-2\alpha z(\theta t)} (B(\Phi(e^{\alpha z(\theta t)} v)), B(e^{\alpha z(\theta t)} v))
\]

\[
e^{-2\alpha z(\theta t)} \sum_{i \in \mathbb{Z}} (\Phi(e^{\alpha z(\theta t)} v_{i+1}) - \Phi(e^{\alpha z(\theta t)} v_i), e^{\alpha z(\theta t)} (v_{i+1} - v_i))
\]

\[
\geq k e^{\alpha(p-1)z(\theta t)} \sum_{i \in \mathbb{Z}} |v_{i+1} - v_i|^p + e^{-2\alpha z(\theta t)} \sum_{i \in \mathbb{Z}} a_i
\]

(3.9)

which implies

\[
\frac{d}{dt} \|v(t)\|^2 + 2\lambda e^{\alpha(p-1)z(\theta t)} \|v\|^p + \lambda \|v(t)\|^2 \leq (2\alpha z(\theta t) - \lambda) \|v\|^2 + \left(\frac{8\|g\|^2}{\lambda} + 2\|a\|_1\right)e^{-2\alpha z(\theta t)}.
\]

(3.10)

Due to Gronwall lemma, for \( t > 0 \),

\[
\|v(t)\|^2 + \lambda \int_0^t e^{-\lambda s + 2\alpha \int_0^s z(\theta r)dr} \|v(s, \omega, v_0)\|^2 ds
\]

\[
+ 2\lambda \int_0^t e^{\alpha(p-1)z(\theta s) - \lambda s + 2\alpha \int_0^s z(\theta r)dr} \|v(s, \omega, v_0)\|^p ds \leq e^{-\lambda t + 2\alpha \int_0^t z(\theta r)dr} \|v_0\|^2
\]

\[
+ \left(\frac{8\|g\|^2}{\lambda} + 2\|a\|_1\right)e^{-\lambda t + 2\alpha \int_0^t z(\theta r)dr} \int_0^t e^{-2\alpha z(\theta r) + \lambda s - 2\alpha \int_0^s z(\theta r)dr} ds.
\]

(3.11)

Denote

\[
\eta(\omega) = \left(\frac{8\|g\|^2}{\lambda} + 2\|a\|_1\right) \max_{t \in [0, T]} \left(\int_0^t e^{-\lambda t + 2\alpha \int_0^s z(\theta r)dr} ds \right)
\]

and

\[
\xi(\omega) = 2\alpha \int_0^T |z(\theta \omega)| ds,
\]

then we have

\[
\|v(t)\|^2 \leq \|v_0\|^2 e^{\xi(\omega)} + \eta(\omega),
\]

which implies that the solution \( v \) is defined in any interval \( [0, T] \).

(2) Let \( u_0, v_0 \in \ell^2 \) and \( X(t) = X(t, \omega, u_0), Y(t) = Y(t, \omega, v_0) \) be two solutions of (3.7). Then, denoting \( \Delta(t) = X(t) - Y(t) \), we get

\[
\frac{d\Delta(t)}{dt} = -e^{-\alpha z(\theta t)} (A(\Phi(e^{\alpha z(\theta t)} X)) - A(\Phi(e^{\alpha z(\theta t)} Y)) + (\alpha z(\theta t) - \lambda)\Delta(t)
\]

\[
- e^{\alpha(p-1)z(\theta t)} (|X|^{p-1} X - |Y|^{p-1} Y),
\]

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and then
\[
\frac{d}{dt} \|\Delta(t)\|^2 = -2e^{-\alpha z(\theta_t\omega)}(A(\Phi(e^{\alpha z(\theta_t\omega)}X)) - A(\Phi(e^{\alpha z(\theta_t\omega)}Y)), e^{\alpha z(\theta_t\omega)}\Delta(t)) \\
+ 2(\alpha z(\theta_t\omega) - \lambda)\|\Delta(t)\|^2 - e^{\alpha(p-1)z(\theta_t\omega)}(|X|^{p-1}X - |Y|^{p-1}Y, \Delta(t))
\]
\[
\leq 2(L + \alpha z(\theta_t\omega))\|\Delta(t)\|^2 \leq \rho\|\Delta(t)\|^2,
\]
where \(\rho = 2(L + \alpha \max_{t \in [0,T]}|z(\theta_t\omega)|)\), here \(L\) denotes the Lipschitz constant of \(\Phi\) and the term \(|X|^{p-1}X - |Y|^{p-1}Y\) corresponding to a bounded set where \(X\) and \(Y\) belong to. Now by a simply computation, we have
\[
\sup_{t \in [0,T]} \|X(t) - Y(t)\|^2 \leq e^{\rho T}\|u_0 - v_0\|^2,
\]
which completes the proof.

\[\square\]

**Theorem 3.2.** System \((3.7)\) generates a continuous random dynamical system \((\varphi(t))_{t \geq 0}\) over \((\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})\), where
\[
\varphi(t, \omega, v_0) = v(t, \omega, v_0) = e^{-\alpha z(\theta_t\omega)}u(t, \omega, e^{\alpha z(\omega)}v_0)
\]
for \(v_0 \in \ell^2\), \(t \geq 0\) and for all \(\omega \in \Omega\).

**Proof.** Actually, that \(\varphi\) is a continuous random dynamical system follows from Theorem 3.1. The measurability of \(\varphi\) is indicated by the transformation in (3.5). The rest of the proof just follows from the chain rule. \[\square\]

The random dynamical system \(\varphi\) generated by \((3.7)\) is conjugated to the one generated by \((3.4)\) (see [10]). In the sequel, we will just consider the random dynamical system \(\varphi\).

### 4 Existence of a unique global random attractor

In this section, we will prove the existence of a random attractor for the SLDS generated by system \((3.7)\). Our main result is

**Theorem 4.1.** The SLDS \(\varphi\) generated by system \((3.7)\) has a unique global random attractor.

To prove Theorem 4.1 we will use Proposition 2.6. We first need to prove that there exists an absorbing set for \(\varphi\) in \(\ell^2\). Next, we will show the RDS \(\varphi\) is random asymptotically null in the sense of (2.1).
4.1 Existence of an absorbing set

We need to prove that there exists a closed random tempered set \( K \in \mathcal{D}(\ell^2) \) of absorption.

**Lemma 4.2.** There exists a closed random tempered set \( K(\omega) \in \mathcal{D}(\ell^2) \) such that for all \( B \in \mathcal{D}(\ell^2) \) and a.e. \( \omega \in \Omega \) there exists \( T_B(\omega) > 0 \) such that

\[
\varphi(t, \theta^{-t} \omega) B(\theta^{-t} \omega) \subset K(\omega) \quad \text{for all } t \geq T_B(\omega).
\]

**Proof.** Let us start with \( v(t) = \varphi(t, \omega, v_0) \). Now, by replacing \( \omega \) with \( \theta^{-t} \omega \) in (3.11), we obtain

\[
\|\varphi(t, \theta^{-t} \omega, v_0)\|^2 + \lambda \int_0^t e^{-\lambda s + 2\alpha \int_0^s z(\theta^{-s} \omega) ds} \|v(s, \theta^{-s} \omega, v_0)\|^2 ds \\
+ 2\lambda \int_0^t e^{\alpha(p-1)z(\theta^{-s} \omega) - 2\alpha \int_0^s z(\theta^{-s} \omega) ds} \|v(s, \theta^{-s} \omega, v_0)\|^{p+1} ds \\
\leq e^{-\lambda t + 2\alpha \int_0^t z(\theta^{-s} \omega) ds} \|v_0\|^2 \\
+ \left( \frac{8\|g\|^2}{\lambda} + 2\|a\|_1 \right) \int_0^t e^{-2\alpha z(\theta^{-s} \omega) + \lambda(s-t) + 2\alpha \int_0^s z(\theta^{-s} \omega) ds} \|v(s, \theta^{-s} \omega, v_0)\|^2 ds \\
\leq e^{-\lambda t + 2\alpha \int_0^t z(\theta^{-s} \omega) ds} \|v_0\|^2 \\
+ \left( \frac{8\|g\|^2}{\lambda} + 2\|a\|_1 \right) \int_0^t e^{-2\alpha z(\theta^{-s} \omega) + \lambda s + 2\alpha \int_0^s z(\theta^{-s} \omega) ds} ds.
\]

Due to (3.6), we know that \( \int_0^t e^{-2\alpha z(\theta^{-s} \omega) + \lambda s + 2\alpha \int_0^s z(\theta^{-s} \omega) ds} ds < +\infty \). Considering for any \( v_0 \in B(\theta^{-t} \omega) \), we have

\[
\|\varphi(t, \theta^{-t} \omega, v_0)\|^2 \leq e^{-\lambda t + 2\alpha \int_0^t z(\theta^{-s} \omega) ds} d(B(\theta^{-t} \omega))^2 \\
+ \left( \frac{8\|g\|^2}{\lambda} + 2\|a\|_1 \right) \int_{-\infty}^0 e^{-2\alpha z(\theta^{-s} \omega) + \lambda s + 2\alpha \int_0^s z(\theta^{-s} \omega) ds} ds.
\]

Denoting

\[
R^2(\omega) = 1 + \left( \frac{8\|g\|^2}{\lambda} + 2\|a\|_1 \right) \int_{-\infty}^0 e^{-2\alpha z(\theta^{-s} \omega) + \lambda s + 2\alpha \int_0^s z(\theta^{-s} \omega) ds} ds
\]

and noticing

\[
\lim_{t \to +\infty} e^{-\lambda t + 2\alpha \int_0^t z(\theta^{-s} \omega) ds} d(B(\theta^{-t} \omega))^2 = 0,
\]

we conclude that

\[
K(\omega) = \overline{B_{\ell^2}(0, R(\omega))}
\]
is an absorbing closed random set. It remains to show that $\mathcal{K}(\omega) \in \mathcal{D}(\ell^2)$. Indeed, from Definition 2.2 (iv), for all $\gamma > 0$, we get

$$
e^{-\gamma t} R^2(\theta_{-t} \omega)
= e^{-\gamma t} + \frac{8\|g\|^2}{\lambda} + 2\|a\|_1 e^{-\gamma t} \int_{-\infty}^{0} e^{-2\alpha z(\theta_{-t} \omega) + \lambda s + 2\alpha \int_s^0 z(\theta_{-t} \omega) \, dr} \, ds
= e^{-\gamma t} + \frac{8\|g\|^2}{\lambda} + 2\|a\|_1 e^{-\gamma t} \int_{-\infty}^{-t} e^{-2\alpha z(\theta_t \omega) + \lambda(s+t) + 2\alpha \int_s^t z(\theta_{-t} \omega) \, ds} \, ds \to 0$$

as $t \to \infty$.

Thus, the proof is complete. \hfill \Box

4.2 Random asymptotic nullity

In this subsection, the property of random asymptotically null for the solution $\varphi$ of system (3.7) will be established.

Lemma 4.3. Let $v_0 \in \mathcal{K}(\omega)$ be the absorbing set given by (4.4). Then for every $\epsilon > 0$, there exist $T(\epsilon, \omega, \mathcal{K}(\omega)) > 0$ and $N(\epsilon, \omega, \mathcal{K}(\omega)) \geq 1$, such that the solution $\varphi$ of problem (3.7) is random asymptotically null, that is

$$
sup_{v \in \mathcal{K}(\omega)} \sum_{|i| > N(\epsilon, \omega, \mathcal{K}(\omega))} |\varphi_i(t, \theta_{-t} \omega, v(\theta_{-t} \omega))|^2 \leq \epsilon^2, \quad \forall t \geq T(\epsilon, \omega, \mathcal{K}(\omega)).$$

Proof. Choose a smooth cut-off function satisfying $0 \leq \rho(s) \leq 1$ for $s \in \mathbb{R}^+$ and $\rho(s) = 0$ for $0 \leq s \leq 1$, $\rho(s) = 1$ for $s \geq 2$. Suppose there exists a constant $c_0$ such that $|\rho'(s)| \leq c_0$ for $s \in \mathbb{R}^+$.

Let $N$ be a fixed integer which will be specified later, and set $x = \left( \rho \left( \frac{|i|}{N} \right) \varphi_i \right)_{i \in \mathbb{Z}}$. Then taking the inner product of (3.7) with $x$ in $\ell^2$, we obtain

$$
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2 + 2\alpha e^{\alpha(p-1)z(\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^{p+1}
= -2(e^{-\alpha z(\theta_t \omega)}(A(e^{\alpha z(\theta_t \omega)} \varphi), x)
- 2(\lambda - \alpha z(\theta_t \omega)) \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2
+ 2e^{-\alpha z(\theta_t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) g_i \varphi_i.
$$

(4.5)
We now estimate terms in (4.5) one by one. First, we have

\[
(e^{-\alpha z(\theta t \omega)} (A \Phi(e^{\alpha z(\theta t \omega)} \varphi), x) = e^{-\alpha z(\theta t \omega)} \sum_{i \in \mathbb{Z}} (\Phi(e^{\alpha z(\theta t \omega)} \varphi_{i+1}) - \Phi(e^{\alpha z(\theta t \omega)} \varphi_i))
\]

\[
\cdot (\rho \left( \frac{|i + 1|}{N} \right) \varphi_{i+1} - \rho \left( \frac{|i|}{N} \right) \varphi_i)
\]

\[
e^{-\alpha z(\theta t \omega)} \sum_{i \in \mathbb{Z}} (\Phi(e^{\alpha z(\theta t \omega)} \varphi_{i+1}) - \Phi(e^{\alpha z(\theta t \omega)} \varphi_i))
\]

\[
\cdot [(\rho \left( \frac{|i + 1|}{N} \right) - \rho \left( \frac{|i|}{N} \right) )\varphi_{i+1} + \rho \left( \frac{|i|}{N} \right) (\varphi_{i+1} - \varphi_i)].
\]

By (3.2) and (3.3), we obtain

\[
e^{-\alpha z(\theta t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) (\Phi(e^{\alpha z(\theta t \omega)} \varphi_{i+1}) - \Phi(e^{\alpha z(\theta t \omega)} \varphi_i)) (\varphi_{i+1} - \varphi_i)
\]

\[
\geq k e^{\alpha (p-1) z(\theta t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_{i+1} - \varphi_i|^{p+1} - e^{-2\alpha z(\theta t \omega)} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) a_i
\]

\[
\geq -e^{-2\alpha z(\theta t \omega)} \sum_{|i| \geq N} a_i
\]

and

\[
| \sum_{i \in \mathbb{Z}} (\rho \left( \frac{|i + 1|}{N} \right) - \rho \left( \frac{|i|}{N} \right) ) (\Phi(e^{\alpha z(\theta t \omega)} \varphi_{i+1}) - \Phi(e^{\alpha z(\theta t \omega)} \varphi_i)) | \varphi_{i+1} |
\]

\[
\leq \frac{c_0}{N} \sum_{i \in \mathbb{Z}} |\Phi(e^{\alpha z(\theta t \omega)} \varphi_{i+1}) - \Phi(e^{\alpha z(\theta t \omega)} \varphi_i)| |\varphi_{i+1}|
\]

\[
\leq \frac{2c_0 c_1}{N} (e^{\alpha z(\theta t \omega)} ||\varphi||^2 + e^{\alpha p z(\theta t \omega)} ||\varphi||_{p+1}^{p+1}).
\]

For the last term in (4.5),

\[
\sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) g_i \varphi_i \leq \frac{\lambda}{2} \sum_{|i| \geq N} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2 + \frac{1}{2\lambda} \sum_{|i| \geq N} |g_i|^2.
\]

Combining with (4.6), (4.7) and (4.8), we get

\[
\frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2 + (\lambda - 2\alpha z(\theta t \omega)) \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i|^2
\]

\[
\leq \frac{4c_0 c_1}{N} (||\varphi||^2 + e^{\alpha (p-1) z(\theta t \omega)} ||\varphi||_{p+1}^{p+1}) + \sum_{|i| \geq N} (2|a_i| + \frac{1}{\lambda} |g_i|^2) e^{-2\alpha z(\theta t \omega)}.
\]
By using Gronwall’s inequality for \( t > 0 \) and substituting \( \theta - t \omega \) for \( \omega \), it follows that

\[
\sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i(t, \theta - t \omega, v_0(\theta - t \omega))|^2 \\
\leq e^{-\lambda t + 2\alpha \int_0^t z(\theta - t \omega) ds} \| v_0(\theta - t \omega) \|^2 \\
+ \frac{4c_0c_1}{N} \int_0^t e^{-\lambda s + 2\alpha \int_0^s z(s, \theta - s \omega) ds} \| \varphi(s, \theta - t \omega, v_0) \|^2 ds \\
+ \frac{4c_0c_1}{N} \int_0^t e^{\alpha(p-1)z(\theta - t \omega) - \lambda s + 2\alpha \int_0^s z(s, \theta - s \omega) ds} \| \varphi(s, \theta - t \omega, v_0) \|_{p+1}^2 ds \\
+ \sum_{|i| \geq N} (2|a_i| + \frac{1}{\lambda} |g_i|^2) \int_0^t e^{-\lambda(t-s) + 2\alpha \int_0^s z(\theta - t \omega) ds - 2\alpha z(\theta - t \omega) dr} ds.
\] (4.9)

By Lemma 4.2 and (4.1), there exists \( T_1 = T_1(\epsilon, \omega, K(\omega)) > 0 \) such that for \( t \geq T_1 \),

\[
4c_0c_1 \int_0^t e^{-\lambda s + 2\alpha \int_0^s z(s, \theta - s \omega) ds} \| \varphi(s, \theta - t \omega, v_0) \|^2 ds \leq \frac{4c_0c_1}{\lambda N} R^2(\omega)
\]

and

\[
\frac{4c_0c_1}{N} \int_0^t e^{\alpha(p-1)z(\theta - t \omega) - \lambda s + 2\alpha \int_0^s z(s, \theta - s \omega) ds} \| \varphi(s, \theta - t \omega, v_0) \|_{p+1}^2 ds \\
\leq \frac{2c_0c_1}{\lambda N} R^2(\omega),
\]

where \( R(\omega) \) is given by (4.2). Since \( a \in \ell^1 \) and \( g \in \ell^2 \), by using (3.6), there exist \( \tilde{T}(\epsilon, \omega, K(\omega)) > T_1 \) and \( \tilde{N}(\epsilon, \omega, K(\omega)) \geq 1 \) such that for \( t > \tilde{T}(\epsilon, \omega, K(\omega)) \),

\[
\sum_{|i| > \tilde{N}(\epsilon, \omega, K(\omega))} |\varphi_i(t, \theta - t \omega, v_0(\theta - t \omega))|^2 \leq \sum_{i \in \mathbb{Z}} \rho \left( \frac{|i|}{N} \right) |\varphi_i(t, \theta - t \omega, v_0(\theta - t \omega))|^2 \\
\leq e^2.
\]

The proof is complete.

Thus, we have proved Theorem 4.1.

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