THE RIGIDITY THEOREMS OF SELF SHRINKERS VIA GAUSS MAPS

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Abstract. We study the rigidity results for self-shrinkers in Euclidean space by restriction of the image under the Gauss map. The geometric properties of the target manifolds carry into effect. In the self-shrinking hypersurface situation Theorem 3.1 and Theorem 3.2 not only improve the previous results, but also are optimal. In higher codimensional case, using geometric properties of the Grassmanian manifolds (the target manifolds of the Gauss map) we give a rigidity theorem for self-shrinking graphs.

1. Introduction

Minimal submanifolds and self-shrinkers both are special solutions to the mean curvature flow. Those two subjects share many geometric properties, as shown in \cite{3}. We continue to study rigidity properties of self-shrinkers. In the previous work we discuss the gap phenomena for squared norm of the second fundamental form for self-shrinkers \cite{6}. For submanifolds in Euclidean space we have the important Gauss map, which plays essential role in submanifold theory. In the present paper we shall study the gap phenomena of the image under the Gauss maps for self-shrinkers. In the literature \cite{3} the polynomial volume growth is an adequate assumption for the complete non-compact self-shrinkers. Ding-Xin \cite{5} showed that the properness shall guarantee the Euclidean volume growth. Afterwards, Chen-Zhou \cite{2} proved that the inverse is also true. It is unclear if there exists a complete improper self-shrinker in Euclidean space. Now, we only study properly immersed self-shrinkers. We pursue the results that a complete properly immersed self-shrinker would become an affine linear subspace or a cylinder, if its Gauss image is sufficiently restricted.

In the next section we show that the Gauss map of a self-shrinker is a weighted harmonic map, which is a conclusion of the Ruh-Vilms type result for self-shrinkers, see Theorem 2.1. We also derive a composition formula for the drift-Laplacian operator defined on self-shrinkers, which enables us to obtain some results of self-shrinkers via properties of the target manifold of the Gauss map in the subsequent sections of this paper.

In §3 we study the codimension one case. If $M$ is an entire graphic self-shrinking hypersurface in $\mathbb{R}^{n+1}$, Ecker-Huisken showed that $M$ is a hyperplane \cite{8} under
the assumption of polynomial volume growth, which was removed by Wang [16]. Namely, any entire graphic self-shrinking hypersurface in Euclidean space has to be a hyperplane. It is in sharp contrast to the case of minimal graphic hypersurfaces. For constant mean curvature surfaces in $\mathbb{R}^3$ there is the well-known results, due to Hoffman-Osserman-Schoen [11]. Their results show that a plane or a circular cylinder could be characterized by its Gauss image among other complete constant mean curvature surfaces in $\mathbb{R}^3$. In this circumstance we consider a properly immersed self-shrinking hypersurface $M$ in $\mathbb{R}^{n+1}$. Now the target manifolds of the Gauss map for a self shrinking hypersurface in $\mathbb{R}^{n+1}$ is the unit sphere. We obtain a counterpart of their results and prove that if the image under the Gauss map is contained in an open hemisphere (which includes the case of graphic self shrinking hypersurfaces in $\mathbb{R}^{n+1}$), then $M$ is a hyperplane. If the image under the Gauss map is contained in a closed hemisphere, then $M$ is a hyperplane or a cylinder over a self-shrinker of one dimension lower, see Theorem 3.1. The convex geometry of the sphere has been studied extensively by Jost-Xin-Yang [12]. Using their technique we could improve the first part of Theorem 3.1 and obtain Theorem 3.2, which is the best possible. The omitting range of the Gauss image would be the codimension one closed hemisphere $\mathbb{S}_+^{n-1}$, much smaller than the closed hemisphere $\mathbb{S}_+^{n}$ in Theorem 3.1.

In §4 we study the higher codimensional graphic situation. The target manifold of the Gauss map is the Grassmannian manifold now. To study the higher codimensional Bernstein problem, Jost-Xin-Yang obtained some interesting geometric properties of the Grassmannian manifolds and developed some skilled technique [13]. This enables us to obtain rigidity results of higher codimension for self-shrinkers. Using Theorem 3.1 in [13], Ding-Wang obtained a result for this problem [4]. Now, using the method of Theorem 3.1 in [13] we prove Proposition 4.1 to fit the present situation. Therefore, we obtain Theorem 4.1 which improves corresponding results in [4]. As for Lagrangian self-shrinkers (a special higher codimensional case), readers are referred to the papers [1],[10] and [7].

The weighted harmonic maps have already been introduced and studied. For convenience we describe its basic notion in an appendix, as the final section.

2. Gauss maps for self shrinkers

If $M$ is an oriented submanifold in $\mathbb{R}^{m+n}$, we can define the Gauss map $\gamma : M \to G_{n,m}$ that is obtained by parallel translation of $T_pM$ to the origin in the ambient space $\mathbb{R}^{m+n}$. Here $G_{n,m}$ is the Grassmannian manifolds constituting of all oriented $n$-subspaces in $\mathbb{R}^{m+n}$. It is a Riemannian symmetric space of compact type. When $m = 1$, $G_{n,1}$ becomes Euclidean sphere. The properties of the Gauss map implies the properties of the submanifolds.
Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, and \(X : M \to \mathbb{R}^{m+n}\) be an isometric immersion. Let \(\nabla\) and \(\bar{\nabla}\) be Levi-Civita connections on \(M\) and \(\mathbb{R}^{m+n}\), respectively. The second fundamental form \(B\) is defined by
\[
B_V W = (\bar{\nabla}_V W)^N = \bar{\nabla}_V W - \nabla_V W
\]
for any vector fields \(V, W\) along the submanifold \(M\), where \((\cdots)^N\) is the projection onto the normal bundle \(NM\). Similarly, \((\cdots)^T\) stands for the tangential projection. Taking the trace of \(B\) gives the mean curvature vector \(H\) of \(M\) in \(\mathbb{R}^{m+n}\), a cross-section of the normal bundle. In what follows we use \(\nabla\) for natural connections on various bundles for notational simplicity if there is no ambiguity from the context. For \(\nu \in \Gamma(NM)\) the shape operator \(A^\nu : TM \to TM\), defined by
\[
A^\nu(V) = -((\bar{\nabla}_V \nu)^T),
\]
satisfies
\[
\langle B_V W, \nu \rangle = \langle A^\nu(V), W \rangle.
\]

The second fundamental form, curvature tensors of the submanifold, curvature tensor of the normal bundle and that of the ambient manifold satisfy the Gauss equations, the Codazzi equations and the Ricci equations (see [18], for example).

We now consider the mean curvature flow for a submanifold \(M\) in \(\mathbb{R}^{m+n}\). Namely, consider a one-parameter family \(X_t = X(\cdot, t)\) of immersions \(X_t : M \to \mathbb{R}^{m+n}\) with corresponding images \(M_t = X_t(M)\) such that
\[
\begin{cases}
\frac{d}{dt}X(p, t) = H(p, t), & p \in M \\
X(p, 0) = X(p)
\end{cases}
\]
is satisfied, where \(H(p, t)\) is the mean curvature vector of \(M_t\) at \(X(p, t)\) in \(\mathbb{R}^{m+n}\).

An important class of solutions to the above mean curvature flow equations are self-similar shrinkers, whose profiles, self-shrinkers, satisfy a system of quasi-linear elliptic PDE of the second order
\[
(2.1) \quad H = -\frac{X^N}{2}.
\]

Let \(\Delta\), \(\text{div}\) and \(d\mu\) be Laplacian, divergence and volume element on \(M\) induced by the metric \(g\), respectively. Colding and Minicozzi in [3] introduced a linear operator, drift-Laplacian
\[
(2.2) \quad \mathcal{L} = \Delta - \frac{1}{2}\langle X, \nabla(\cdot) \rangle = e^{-\frac{|X|^2}{4}}\text{div}(e^{-\frac{|X|^2}{4}}\nabla(\cdot))
\]
on self-shrinkers. They showed that \(\mathcal{L}\) is self-adjoint respect to the measure \(e^{-\frac{|X|^2}{4}}d\mu\). In the present paper we carry out integrations with respect to this measure. We denote
\[
(2.3) \quad \rho := e^{-\frac{|X|^2}{4}}
\]
and the volume form \(d\mu\) might be omitted in the integrations for notational simplicity.

Especially if \(M\) is a graph over a domain \(\Omega \subset \mathbb{R}^n\), namely,
\[
M = \{(x_1, \cdots, x_n, u^1, \cdots, u^m) : u^\alpha = u^\alpha(x_1, \cdots, x_n)\}.
\]
Then the induced metric of $M$

$$g = g_{ij} dx_i dx_j = (\delta_{ij} + u^a_i u^a_j) dx_i dx_j$$

with $u^a_i = \frac{\partial u^a}{\partial x^i}$. Let $x := (x_1, x_2, \cdots, x_n)$, $(g^{ij})$ be the inverse matrix of $(g_{ij})$ and (2.4)

$$v = \Delta u = \det g$$

be the slope of the vector-valued function $u$. Then the equation (2.1) can be written as the following elliptic system (see [4])

$$v^a = \Delta u^a = \det (g^{ij})$$

(2.5)

$$\sum_{i,j=1}^{n} g^{ij} u^a_{ij} = \frac{1}{2} (x \cdot Du^a - u^a),$$

where $u^a_{ij} := \frac{\partial^2 u^a}{\partial x^i \partial x^j}$ and $Du^a := (u^a_1, \cdots, u^a_n)$.

By a straightforward calculation,

$$\partial_i (v g^{ij}) = \frac{1}{2} v g^{kl} \partial_i g_{kl} g^{ij} - v g^{ki} \partial_i g_{kl} g^{lj}$$

(2.6)

$$= \frac{1}{2} v g^{kl} (u^a_{ki} u^a_l + u^a_k u^a_{li}) g^{ij} - v g^{ki} (u^a_{ki} u^a_l + u^a_k u^a_{li}) g^{lj}$$

$$= -v g^{ki} u^a_{ki} u^a_l g^{lj}.$$ 

Substituting (2.5) into (2.6) gives

$$\partial_i (v g^{ij}) = \frac{1}{2} v (u^a - x^i u^a_i) u^a_i g^{lj}$$

(2.7)

$$= \frac{1}{2} v u^a u^a_i g^{lj} - \frac{1}{2} v x^i (g_{il} - \delta_{il}) g^{lj}$$

$$= \frac{1}{2} v u^a u^a_i g^{lj} - \frac{1}{2} v x^i g^{lj} + \frac{1}{2} v x^i g^{lj}.$$ 

For any $C^2$-function $f$ in $M$, combining (2.7), we have

$$\mathcal{L} f = \frac{1}{v} e^{\frac{|x|^2}{4}} \frac{\partial}{\partial x_i} \left( g^{ij} v e^{-\frac{|x|^2}{4}} \frac{\partial}{\partial x_j} f \right)$$

(2.8)

$$= g^{ij} f_{ij} + \frac{1}{v} \partial_i (g^{ij} v) f_j - \frac{1}{2} g^{ij} (x_i + u^a u^a_i) f_j$$

$$= g^{ij} f_{ij} + \left( \frac{1}{2} u^a u^a_i g^{ij} - \frac{1}{2} x^j + \frac{1}{2} x^i g^{ij} \right) f_j - \frac{1}{2} g^{ij} (x_i + u^a u^a_i) f_j$$

$$= g^{ij} f_{ij} - \frac{1}{2} x^j f_j.$$ 

**Remark 2.1.** For graphic self-shrinkers in $\mathbb{R}^{m+n}$, the operator $L_g$ defined in [3] is precisely the drift-Laplace $\mathcal{L}$. Please see [7] for Lagrangian case.

Once $M$ is minimal, the Gauss map of $M$ must be a harmonic map. This is a conclusion of the well-known Ruh-Vilms theorem [14], which reveals the close relationship between Liouville type theorems for harmonic maps and Bernstein type results for minimal submanifolds. There is also a notion of $\rho$-weighted harmonic
maps. Its definition shall be given in Appendix. We have the counterpart of the Ruh-Vilms theorem.

**Theorem 2.1.** For an oriented $n$-dimensional submanifold $X : M \to \mathbb{R}^{m+n}$ its Gauss map $\gamma : M \to G_{n,m}$ is $\rho$-weighted harmonic map if and only if $H + \frac{1}{2} X N$ is a parallel vector field in the normal bundle $N M$.

**Proof.** Let $\{e_1, \ldots, e_n\}$ be a local tangent orthonormal frame field on $M$ and $\{\nu_1, \ldots, \nu_m\}$ be a local normal orthonormal frame field on $M$, and we assume $\nabla e_i = 0$ and $\nabla \nu_\alpha = 0$ at the considered point. Here and in the sequel we use summation convention and assume the range of indices.

$$1 \leq i, j, k \leq n, \quad 1 \leq \alpha, \beta \leq m.$$ Using Plücker coordinates, the Gauss map $\gamma$ could be described as $\gamma(p) = e_1 \wedge \cdots \wedge e_n$, thus

$$d\gamma(e_i) = \nabla_{e_i}(e_1 \wedge \cdots \wedge e_n) = \sum_j e_1 \wedge \cdots \wedge B_{e_i e_j} \wedge \cdots \wedge e_n$$

(2.9)

$$= \sum_j e_1 \wedge \cdots \wedge h_{\alpha,ij} \nu_\alpha \wedge \cdots \wedge e_n$$

$$= h_{\alpha,ij} e_\alpha j$$

where $\{h_{\alpha,ij} = \langle B_{e_i e_j}, \nu_\alpha \rangle : 1 \leq i, j \leq n, 1 \leq \alpha \leq m\}$ are coefficients of the second fundamental form, and $e_\alpha j$ is obtained by replacing $e_j$ by $\nu_\alpha$ in $e_1 \wedge \cdots \wedge e_n$. We note that $\{e_\alpha j : 1 \leq j \leq n, 1 \leq \alpha \leq m\}$ is an orthonormal basis of the tangent space of $G_{n,m}$ at $e_1 \wedge \cdots \wedge e_n$.

At the considered point,

$$\nabla_{e_i} e_\alpha j = \nabla_{e_i}(e_1 \wedge \cdots \wedge \nu_\alpha \wedge \cdots \wedge e_n)$$

(2.10)

$$= \sum_k e_1 \wedge \cdots \wedge \nabla_{e_i} e_k \wedge \cdots \wedge \nu_\alpha \wedge \cdots \wedge e_n$$

$$+ e_1 \wedge \cdots \wedge \nabla_{e_i} \nu_\alpha \wedge \cdots \wedge e_n$$

$$= 0.$$

Using Codazzi equations one can obtain

$$\nabla_{e_i} h_{\alpha,ij} = \nabla_{e_i} \langle B_{e_i e_j}, \nu_\alpha \rangle = \langle (\nabla_{e_i} B)_{e_i e_j}, \nu_\alpha \rangle$$

(2.11)

$$= \langle (\nabla_{e_j} B)_{e_i e_i}, \nu_\alpha \rangle = \nabla_{e_j} H^\alpha$$

with $H^\alpha := \langle H, \nu_\alpha \rangle$ the coefficients of the mean curvature vector.

Combining with (2.9)-(2.11) gives

$$\nabla_{e_i} d\gamma(e_i) = \nabla_{e_i} d\gamma(e_i) = (\nabla_{e_i} h_{\alpha,ij}) e_\alpha j + h_{\alpha,ij} \nabla_{e_i} e_\alpha j$$

(2.12)

$$= (\nabla_{e_j} H^\alpha) e_\alpha j.$$
Since $\rho = e^{-\frac{|X|^2}{4}}$, \[\nabla_{e_i}\rho = -\frac{1}{4}\rho
abla_{e_i}|X|^2 = -\frac{1}{2}\rho\langle X, \nabla_{e_i}X \rangle\] (2.13) \[= -\frac{1}{2}\rho\langle X, e_i \rangle.\]

Let $X^\alpha := \langle X, \nu_\alpha \rangle = \langle X^N, \nu_\alpha \rangle$, then \[\nabla_{e_j}X^\alpha = \langle \nabla_{e_j}X, \nu_\alpha \rangle + \langle X, \nabla_{e_j}\nu_\alpha \rangle = \langle e_j, \nu_\alpha \rangle - \langle X, h_{\alpha,ij}e_i \rangle\] (2.14) \[= -h_{\alpha,ij}\langle X, e_i \rangle.\]

In conjunction with (2.9), (2.12), (2.13) and (2.14) we have \[\tau_\rho(\gamma) := \rho^{-1}(\nabla_{e_i}(\rho d\gamma))e_i = \rho^{-1}(\nabla_{e_i}\rho)d\gamma(e_i) + \nabla_{e_i}(d\gamma)(e_i)\] (2.15) \[= \left[-\frac{1}{2}h_{\alpha,ij}\langle X, e_i \rangle + \nabla_{e_j}H^{\alpha}\right]e_{\alpha j}\] \[= \nabla_{e_j}(H^{\alpha} + \frac{1}{2}X^{\alpha})e_{\alpha j}.\]

Considering the definition of $\rho-$weighted harmonic map in the appendix, the conclusion follows.

\[\square\]

**Corollary 2.1.** If $M$ is a self-shrinker in $\mathbb{R}^{m+n}$, then its Gauss map $\gamma : M \rightarrow G_{n,m}$ is a $\rho-$weighted harmonic map.

Now we assume $F$ to be a $C^2$-function on $G_{n,m}$, then $f = F \circ \gamma$ gives a $C^2$-function on $M$. We also choose a local orthonormal frame field $\{e_i\}$ on $M$ such that $\nabla e_i = 0$ and $\nabla \nu_\alpha = 0$ at the considered point. A straightforward calculation shows

\[\mathcal{L}f = \rho^{-1}\text{div}(\rho \nabla f) = \rho^{-1}\nabla_{e_i}(\rho df(e_i))\]
\[= \rho^{-1}\nabla_{e_i}(\rho dF(\gamma_*e_i)) = \rho^{-1}\nabla_{e_i}(dF(\rho_{\gamma_*e_i}))\]
\[= \rho^{-1}(\nabla_{e_i}(dF))(\rho_{\gamma_*e_i}) + \rho^{-1}dF\left((\nabla_{e_i}(\rho d\gamma))(e_i)\right)\]
\[= \text{Hess } F(\gamma_*e_i, \gamma_*e_i) + dF(\tau_\rho(\gamma)).\]

If $M$ is a self-shrinker, by Corollary 2.1 we have the composition formula \[\mathcal{L}f = \text{Hess } F(\gamma_*e_i, \gamma_*e_i).\] (2.16)

**Remark 2.2.** This composition formula shall play a key role in the proof of rigidity theorems for self-shrinkers. Certainly, the above formula could be obtained from the usual composition formula without the notion of $\rho-$weighted harmonic maps. In fact, an extra term in drift-Laplacian would be canceled by the tension field term.
3. Rigidity results for hypersurfaces

Let \((\cdot, \cdot)\) be the canonical Euclidean inner product on \(\mathbb{R}^{n+1}\), then for any fixed \(a \in S^n \subset \mathbb{R}^{n+1}\), \((\cdot, a)\) is obviously a smooth function on \(S^n\).

By the theory of spherical geometry, the normal geodesic \(\gamma\) starting from \(x \in S^n\) and with the initial vector \(v\) (\(|v| = 1\) and \((x, v) = 0\)) has the form
\[
\gamma(t) = \cos t \ x + \sin t \ v.
\]

Then
\[
(\gamma(t), a) = \cos t \ (x, a) + \sin t \ (v, a).
\]

Differentiating twice both sides of the above equation with respect to \(t\) implies
\[
\text{Hess}(\cdot, a)(v, v) = -(\cdot, a).
\]

In conjunction with the formula\(\text{Hess} h(v, w) = \text{Hess} h(v+w, v+w) - \text{Hess} h(v, v) - \text{Hess} h(w, w)\), it is easy to obtain
\[
(3.1) \quad \text{Hess}(\cdot, a) = -(\cdot, a) \ g_s
\]
with \(g_s\) the canonical metric on \(S^n\).

Denote
\[
(3.2) \quad F = 1 - (\cdot, a),
\]
then
\[
(3.3) \quad \text{Hess} F = (1 - F)g_s.
\]

If \(M\) is a self-shrinker in \(\mathbb{R}^{n+1}\), we put \(f = F \circ \gamma\), then combining the composition formula (2.10) with (3.3) yields
\[
(3.4) \quad \mathcal{L}f = \text{Hess} F(\gamma_* e_i, \gamma_* e_i) = (1 - f) \langle \gamma_* e_i, \gamma_* e_i \rangle = (1 - f)|B|^2.
\]

Note that \(f < 1\) \((f \leq 1)\) equals to say that the Gauss image of \(M\) is contained in the open (closed) hemisphere centered at \(a\).

(3.4) is equivalent to
\[
(3.5) \quad (1 - f)|B|^2 \rho = \text{div}(\rho \nabla f).
\]
Let $\phi$ be a smooth function on $M$ with compact supporting set. Multiplying $\phi^2 f$ with both sides of (3.5) and then integrating by parts imply
\[
\int_M \phi^2 f(1 - f)|B|^2 \rho = \int_M \phi^2 f \text{div}(\rho \nabla f)
\]
\[
= \int_M \text{div}(\phi^2 f \rho \nabla f) - \int_M \langle \nabla (\phi^2 f), \nabla f \rangle \rho
\]
\[
= -\int_M \phi^2 |\nabla f|^2 \rho - 2 \int_M \langle f \nabla \phi, \phi \nabla f \rangle \rho
\]
\[
\leq -\int_M \phi^2 |\nabla f|^2 \rho + \frac{1}{2} \int_M \phi^2 |\nabla f|^2 \rho + 2 \int_M |\nabla \phi|^2 f^2 \rho
\]
\[
= -\frac{1}{2} \int_M \phi^2 |\nabla f|^2 \rho + 2 \int_M |\nabla \phi|^2 f^2 \rho.
\]
i.e
\[
(3.6) \quad \int_M \phi^2 f(1 - f)|B|^2 \rho + \frac{1}{2} \int_M \phi^2 |\nabla f|^2 \rho \leq 2 \int_M |\nabla \phi|^2 f^2 \rho.
\]
The above 'generalized stability inequality' enables us to obtain the following rigidity theorem.

**Theorem 3.1.** Let $M$ be a complete self-shrinker hypersurface properly immersed in $\mathbb{R}^{n+1}$. If the image under the Gauss map is contained in an open hemisphere, then $M$ has to be a hyperplane. If the image under the Gauss map is contained in a closed hemisphere, then $M$ is a hyperplane or a cylinder whose cross section is an $(n - 1)$-dimensional self-shrinker in $\mathbb{R}^n$.

**Proof.** In (3.6), we put $\phi$ to be a cut-off function with $\phi \equiv 1$ on $D_R$ (the intersection of the Euclidean ball of radius $R$ and $M$), $\phi \equiv 0$ outside $D_{2R}$ and $|\nabla \phi| \leq \frac{c_0}{R}$ with a positive constant $c_0$. Noting that $0 \leq f \leq 1$ under the Gauss image assumptions, we have
\[
\frac{1}{2} \int_{D_R} |\nabla f|^2 \rho \leq \frac{1}{2} \int_M \phi^2 |\nabla f|^2 \rho \leq 2 \int_M |\nabla \phi|^2 f^2 \rho
\]
\[
\leq \frac{2c_0^2}{R^2} \int_{D_{2R}\setminus D_R} f^2 \rho \leq \frac{2c_0^2}{R^2} e^{-\frac{R^2}{4}} \text{Vol}(D_{2R}\setminus D_R).
\]
Since $M$ has Euclidean volume growth by a result in [5], letting $R \to +\infty$ we get
\[
\int_M |\nabla f|^2 \rho = 0.
\]
Hence $\nabla f \equiv 0$ and $f \equiv \text{const.}$
If \( f \equiv 0 \), then the Gauss image of \( M \) is a single point, which implies \( M \) is a hyperplane. If \( f \equiv t_0 \) with \( t_0 \in (0, 1) \), then again using (3.6) gives
\[
\begin{align*}
t_0(1-t_0) \int_{D_0} |B|^2 \rho &\leq \int_M \phi^2 f(1-f)|B|^2 \rho \\
&\leq 2 \int_M |\nabla \phi|^2 f^2 \rho \leq \frac{2c_0^2}{R^2} e^{-\frac{R_0^2}{4}} \text{Vol}(D_{2R} \setminus D_R).
\end{align*}
\]
Letting \( R \to +\infty \) forces \(|B|^2 \equiv 0\), thus \( M \) has to be a hyperplane.

If \( f \equiv 1 \), then the Gauss image of \( M \) is contained in a subsphere of codimension 1. Without loss of generality one can assume \( \gamma(M) \subset \{ x = (x_1, \cdots, x_{n+1}) \in S^n : x_{n+1} = 0 \} \). Then for any \( p \in M, \epsilon_{n+1} \in T_p M \). (Here and in the sequel, \( \epsilon_i \) (1 \( \leq i \) \( \leq n + 1 \)) is a vector in \( \mathbb{R}^{n+1} \) whose \( i \)-th coordinate is 1 and other coordinates are all 0.) In other words, \( Y := \epsilon_{n+1} \) is a tangent vector field on \( M \). Let \( \xi : (-\varepsilon, \varepsilon) \to M \) be a curve on \( M \) satisfying the following ODE system
\[
(3.7) \quad \begin{cases} 
\dot{\xi}(t) = Y(\xi(t)), \\
\xi(0) = X(p).
\end{cases}
\]
Then \(|Y| \equiv 1\) and the completeness of \( M \) implies \( \xi \) can be infinitely extended towards both ends. Noting that \( Y \) can be viewed as a vector field on \( \mathbb{R}^{n+1} \), we put \( \zeta : \mathbb{R} \to \mathbb{R}^{n+1} \) to be a curve satisfying
\[
(3.8) \quad \begin{cases} 
\dot{\zeta}(t) = Y(\zeta(t)), \\
\zeta(0) = X(p).
\end{cases}
\]
Then obviously \( \zeta \) is the straight line going through \( X(p) \) and being perpendicular to hyperplane \( \{ X \in \mathbb{R}^{n+1} : (X, \epsilon_{n+1}) = 0 \} \). By the uniqueness of ODE system with the given initial conditions we have \( \xi(t) = \zeta(t) \). Thus \( \tilde{M} \) constitutes of straight line orthogonal to a fixed hyperplane. More precisely, if we denote \( \tilde{M} = M \cap \{ X \in \mathbb{R}^{n+1} : (X, \epsilon_{n+1}) = 0 \} \), then \( \tilde{M} \) is obviously a self-shrinker in \( \mathbb{R}^n \) and \( M = \tilde{M} \times \mathbb{R} \).

Let \((\varphi, \theta)\) be the geographic coordinate of \( S^2 \). More precisely, there is a covering mapping \( \chi : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R} \to S^2 \setminus \{ N, S \} \)
\[
(\varphi, \theta) \mapsto (\cos \varphi \cos \theta, \cos \varphi \sin \theta, \sin \varphi).
\]
Here \( N \) and \( S \) are the north pole and the south pole, \( \varphi \) and \( \theta \) are the latitude and the longitude, respectively. Note that each level set of \( \theta \) is a meridian, i.e. a half of great circle connecting the north pole and the south pole. Although \( \chi \) is not one-to-one, the restriction of \( \chi \) on \( (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\pi, \pi) \) is a bijective mapping to the open domain \( \mathbb{V} \) that is obtained by deleting the International date line from \( S^2 \).

The longitude function \( \theta \) and \( \mathbb{V} \) can be generalized to higher dimensional spheres.

Let \( p : \mathbb{R}^{n+1} \to \mathbb{R}^2 \)
\[
x = (x_1, \cdots, x_{n+1}) \mapsto (x_1, x_2)
\]
be a natural orthogonal projection, then \( p \) maps \( S^n \) onto the closed unit disk \( \bar{D} \).

Denote

\[
(3.9) \quad V = \bar{D}\backslash \{(a, 0) : -1 \leq a \leq 0\}
\]

and

\[
(3.10) \quad \mathbb{V} = p^{-1}(V) \cap S^n.
\]

Then it is easily seen that \( S^n \backslash \mathbb{V} \) is a closed hemisphere of codimension 1. So we also write \( \mathbb{V} = S^n \backslash S^n_{+1} \) in the following text. It is shown in \([9]\) \([12]\) that \( \mathbb{V} \) is a convex supporting subset in \( S^n \), i.e. any compact subset \( K \subset \mathbb{V} \) admits a strictly convex function on it.

Obviously there is a \((0, 1]\)-valued function \( r \) and a \((-\pi, \pi)\)-valued function \( \theta \) on \( \mathbb{V} \), such that

\[
(3.11) \quad p(x) = (x_1, x_2) = (r \cos \theta, r \sin \theta) \quad \text{for all } x \in \mathbb{V}.
\]

\( \{x_i : 1 \leq i \leq n+1\} \) can be viewed as coordinate functions on \( S^n \), and \( x_i = (x, \epsilon_i) \).

By (3.11),

\[
(3.12) \quad \text{Hess } x_i = -x_i \, g_s \quad \text{for every } 1 \leq i \leq n+1.
\]

From (3.11), \( r^2 = x_1^2 + x_2^2 \), hence

\[
(3.13) \quad \text{Hess } r^2 = 2x_1 \text{Hess } x_1 + 2x_2 \text{Hess } x_2 + 2dx_1 \otimes dx_1 + 2dx_2 \otimes dx_2
\]

\[
= -2x_1^2 g_s - 2x_2^2 g_s + 2(\cos \theta \, dr - r \sin \theta \, d\theta) \otimes (\cos \theta \, dr - r \sin \theta \, d\theta)
\]

\[
+ 2(\sin \theta \, dr + r \cos \theta \, d\theta) \otimes (\sin \theta \, dr + r \cos \theta \, d\theta)
\]

\[
= -2r^2 g_s + 2dr \otimes dr + 2r^2 d\theta \otimes d\theta.
\]

On the other hand,

\[
(3.14) \quad \text{Hess } r^2 = 2r \text{Hess } r + 2dr \otimes dr.
\]

(3.13) and (3.14) implies

\[
(3.15) \quad \text{Hess } r = -r \, g_s + r d\theta \otimes d\theta.
\]

Furthermore (3.12), (3.11) and (3.15) yield

\[
-x_1 \, g_s = \text{Hess } x_1
\]

\[
= \cos \theta \, \text{Hess } r - r \sin \theta \, \text{Hess } \theta - r \cos \theta \, d\theta \otimes d\theta - \sin \theta (dr \otimes d\theta + d\theta \otimes dr)
\]

\[
= -x_1 \, g_s + x_1 \, d\theta \otimes d\theta - r \sin \theta \, \text{Hess } \theta - x_1 \, d\theta \otimes d\theta - \sin \theta (dr \otimes d\theta + d\theta \otimes dr)
\]

\[
= -x_1 \, g_s - r \sin \theta \, \text{Hess } \theta - \sin \theta (dr \otimes d\theta + d\theta \otimes dr).
\]

i.e.

\[
r \sin \theta \, \text{Hess } \theta = -\sin \theta (dr \otimes d\theta + d\theta \otimes dr).
\]

Similarly computing Hess \( x_2 \) with the aid of (3.11) and (3.15) gives

\[
r \cos \theta \, \text{Hess } \theta = -\cos \theta (dr \otimes d\theta + d\theta \otimes dr).
\]
Therefore
\[(3.16)\quad \text{Hess } \theta = -r^{-1}(dr \otimes d\theta + d\theta \otimes dr).\]

It tells us \(\text{Hess } \theta(v, v) = 0\) for any vector \(v\) on \(V\) satisfying \(\theta(v) = 0\); i.e. the level sets of \(\theta\) are all totally geodesic hypersurfaces in \(S^n\). In fact, \(V = S^n \setminus S^{n-1}_+\) has so-called warped product structure, see [15].

If the Gauss image of \(M\) is contained in \(V\), using composition formula we obtain
\[
\mathcal{L}(\theta \circ \gamma) = \text{Hess } \theta(\gamma_* e_i, \gamma_* e_i) \\
= -(r \circ \gamma)^{-1}(dr \otimes d\theta + d\theta \otimes dr)(\gamma_* e_i, \gamma_* e_i) \\
= -2(r \circ \gamma)^{-1}\langle \nabla(r \circ \gamma), \nabla(\theta \circ \gamma) \rangle.
\]

In the following text, \(\theta \circ \gamma\) and \(r \circ \gamma\) are also denoted by \(\theta\) and \(r\), if there is no ambiguity from the context. Then the above equality can be rewritten as
\[(3.17)\quad \mathcal{L}(\theta) = -2r^{-1}\langle \nabla r, \nabla \theta \rangle.
\]

i.e.
\[
\rho^{-1}\text{div}(\rho \nabla \theta) = -2r^{-1}\langle \nabla r, \nabla \theta \rangle.
\]

Hence
\[(3.18)\quad \text{div}(r^2 \rho \nabla \theta) = r^2 \text{div}(\rho \nabla \theta) + \langle \nabla r^2, \rho \nabla \theta \rangle \\
= -2\langle r \nabla r, \rho \nabla \theta \rangle + \langle \nabla r^2, \rho \nabla \theta \rangle = 0.
\]

With the aid of the above formula one can improve the rigidity result in Theorem 3.1.

**Theorem 3.2.** Let \(M^n\) be a complete self-shrinker hypersurface properly immersed in \(\mathbb{R}^{n+1}\). If the image under Gauss map is contained in \(S^n \setminus S^{n-1}_+\), then \(M\) has to be a hyperplane.

**Proof.** Let \(\phi\) be a smooth function on \(M\) with compact supporting set, multiplying \(\phi^2 \theta\) with both sides of (3.18) and then integrating by parts give

\[
0 = \int_M \phi^2 \theta \text{div}(r^2 \rho \nabla \theta) \\
= \int_M \text{div}(\phi^2 \theta r^2 \rho \nabla \theta) - \int_M \langle \nabla(\phi^2 \theta), \nabla \theta \rangle r^2 \rho \\
= -\int_M \phi^2 |\nabla \theta|^2 r^2 \rho - 2\int_M \langle \theta \nabla \phi, \phi \nabla \theta \rangle r^2 \rho \\
\leq -\frac{1}{2} \int_M \phi^2 |\nabla \theta|^2 r^2 \rho + 2\int_M |\nabla \phi|^2 \theta^2 r^2 \rho
\]

i.e.
\[
(3.19) \quad \int_M \phi^2 |\nabla \theta|^2 r^2 \rho \leq 4 \int_M |\nabla \phi|^2 \theta^2 r^2 \rho.
\]
Choosing $\phi$ to be a cut-off function, which satisfies $\phi \equiv 1$ on $D_R$, $\phi \equiv 0$ outside $D_{2R}$ and $|\nabla \phi| \leq \frac{c_0}{R}$, then combining with $\theta \in (-\pi, \pi)$ and $r \in (0, 1]$ we have

$$
\int_{D_R} |\nabla \theta|^2 r \rho \leq \int_M \phi^2 |\nabla \theta|^2 r^2 \rho \leq 4 \int_M |\nabla \phi|^2 \theta^2 r^2 \rho \leq \frac{4 \pi^2 c_0^2}{R^2} e^{-\frac{c_0^2}{4}} \text{Vol}(D_{2R}\setminus D_R).
$$

By letting $R \to +\infty$ we arrive at $|\nabla \theta| \equiv 0$. Hence $\theta \equiv \theta_0 \in (-\pi, \pi)$.

Denote $a_0 = (\cos \theta_0, \sin \theta_0, 0, \ldots, 0)$, then for arbitrary $p \in M$,

$$(\gamma(p), a_0) = r(p)(\cos^2 \theta_0 + \sin^2 \theta_0) = r(p) > 0.$$

(Note that $\gamma(p) = (r(p) \cos \theta(p), r(p) \sin \theta(p), \cdots)$.) It implies the Gauss image of $M$ is contained in an open hemisphere centered at $a_0$, and then the final conclusion immediately follows from Theorem 3.1.

\[ \square \]

**Remark 3.1.** It is shown in [12] that even we add a point to $S^n \setminus S^{n-1}_+$, it will contain a great circle. Hence the nontrivial self-shrinker $S^1 \times \mathbb{R}^{n-1} \subset \mathbb{R}^{n+1}$ whose Gauss image is just a great circle tells us that the Gauss image restriction in Theorem 3.2 is optimal.

### 4. Rigidity results in High Codimension

Via Plücker embedding, Grassmannian manifold $G_{n,m}$ can be viewed as a minimal submanifold in a higher dimensional Euclidean sphere. The restriction of the Euclidean inner product on $G_{n,m}$ is denoted by $w : G_{n,m} \times G_{n,m} \to \mathbb{R}$

$$w(P, Q) = \langle e_1 \wedge \cdots \wedge e_n, f_1 \wedge \cdots \wedge f_n \rangle = \det W. \quad (4.1)$$

Here $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_n\}$ are oriented orthonormal basis of $P$ and $Q$, respectively, and $W := (\langle e_i, f_j \rangle)$. The eigenvalues of $W^TW$ are denoted by $\mu_1^2, \cdots, \mu_n^2$, then $\mu_i$ takes value between 0 and 1. We also note that $\mu_i^2$ can be expressed as

$$\mu_i^2 = \frac{1}{1 + \lambda_i^2} \quad (4.2)$$

with $\lambda_i \in [0, +\infty]$.

The Jordan angles between $P$ and $Q$ are critical values of the angle $\theta$ between a nonzero vector in $P$ and its orthogonal projection in $Q$. A direct calculation shows there are $n$ Jordan angles $\theta_1, \cdots, \theta_n$, with

$$\theta_i = \arccos \mu_i. \quad (4.3)$$
Hence

\[(4.4) \quad |w| = \left( \det(W^T W) \right)^{\frac{1}{2}} = \prod_{i=1}^{n} \mu_i = \prod_{i=1}^{n} \cos \theta_i. \]

Fix $P_0 \in \mathbf{G}_{n,m}$ spanned by $\epsilon_1, \cdots, \epsilon_n$, which are complemented by $\epsilon_{n+1}, \cdots, \epsilon_{n+m}$, such that $\{\epsilon_1, \cdots, \epsilon_{n+m}\}$ is an orthonormal basis of $\mathbb{R}^{m+n}$. Define

\[(4.5) \quad U := \{P \in \mathbf{G}_{n,m} : w(P, P_0) > 0\}.\]

Our interested quantity will be

\[(4.6) \quad v(\cdot, P_0) := w^{-1}(\cdot, P_0) \quad \text{on } U.\]

Then it is easily-seen that

\[(4.7) \quad v(P, P_0) = \prod_{i} \sec \theta_i = \prod_{i} \mu_i^{-1} = \prod_{i} \sqrt{1 + \lambda_i^2}. \]

In this terminology, $\text{Hess}(v(\cdot, P_0))$ has been estimated in [19]. By (3.8) in [19], we have

\[(4.8) \quad \text{Hess\,}(v(\cdot, P_0)) = \sum_{j \neq \alpha} v \omega_{\alpha j}^2 + \sum_{1 \leq j \leq p} \left(1 + 2\lambda_j^2 \right) v \omega_{jj}^2 + \sum_{1 \leq j, k \leq p, j \neq k} \lambda_j \lambda_k (\omega_{jj} \otimes \omega_{kk} + \omega_{jk} \otimes \omega_{kj}) \]

\[+ \sum_{\max\{j, \alpha\} > p} v \omega_{\alpha j}^2 + \sum_{1 \leq j \leq p} \left(1 + 2\lambda_j^2 \right) v \omega_{jj}^2 + \sum_{1 \leq j, k \leq p, j \neq k} \lambda_j \lambda_k (\omega_{jj} \otimes \omega_{kk}) \]

\[+ \sum_{1 \leq j < k \leq p} \left[ (1 + \lambda_j \lambda_k) v \left( \frac{\sqrt{2}}{2} (\omega_{jk} + \omega_{kj}) \right)^2 \right] \]

\[+ (1 - \lambda_j \lambda_k) v \left( \frac{\sqrt{2}}{2} (\omega_{jk} - \omega_{kj}) \right)^2 \]

with $p := \min\{m, n\}$ and $\{\omega_{\alpha i} : 1 \leq i \leq n, 1 \leq \alpha \leq m\}$ is a dual basis of $\{\epsilon_{\alpha i} : 1 \leq i \leq n, 1 \leq \alpha \leq m\}$, namely, $\{\omega_{\alpha i} : 1 \leq i \leq n, 1 \leq \alpha \leq m\}$ is a local orthonormal coframe field on $\mathbf{G}_{n,m}$ at $P = \epsilon_1 \wedge \cdots \wedge \epsilon_n$.

We also have from (3.9) in [19] that

\[(4.9) \quad dv(\cdot, P_0) = \sum_{1 \leq j \leq p} \lambda_j v(\cdot, P_0) \omega_{jj} \]

i.e.

\[(4.10) \quad d\log v(\cdot, P_0) = \sum_{1 \leq j \leq p} \lambda_j \omega_{jj}. \]

Combining with (4.8) and (4.10) gives

\[(4.11) \quad \text{Hess\,log\,}(v(\cdot, P_0)) = g + \sum_{1 \leq j \leq p} \lambda_j^2 \omega_{jj}^2 + \sum_{1 \leq j, k \leq p, j \neq k} \lambda_j \lambda_k (\omega_{jk} \otimes \omega_{kj}), \]

where $g$ is the metric tensor on $\mathbf{G}_{n,m}$.
Let
\begin{equation}
(4.12) \quad v := v(\cdot, P_0) \circ \gamma.
\end{equation}

By (2.9),
\[ \omega_{jk}(\gamma_*e_i) = \omega_{jk}(h_{\alpha,it}e_{\alpha t}) = h_{k,ij}. \]

Hence
\begin{equation}
(4.13) \quad |\nabla \log v|^2 = \sum_i \left[ d \log v(\cdot, P_0)(\gamma_*e_i) \right]^2
= \sum_i \left[ \sum_j \lambda_j \omega_{jj}(\gamma_*e_i) \right]^2
= \sum_i \left( \sum_j \lambda_j h_{j,ij} \right)^2.
\end{equation}

Using composition formula (2.16) yields
\begin{equation}
(4.14) \quad \mathcal{L}(\log v) = \text{Hess} \log v(\cdot, P_0)(\gamma_*e_i, \gamma_*e_i)
= |B|^2 + \sum_{i,1 \leq j \leq p} \lambda_j^2 (\omega_{jj}(\gamma_*e_i))^2 + \sum_{i,1 \leq j, k \leq p, j \neq k} \lambda_j \lambda_k \omega_{jk}(\gamma_*e_i) \omega_{kj}(\gamma_*e_i)
= |B|^2 + \sum_{i,1 \leq j \leq p} \lambda_j^2 h_{j,ij}^2 + \sum_{i,1 \leq j, k \leq p, j \neq k} \lambda_j \lambda_k h_{k,ij} h_{j,ik}.
\end{equation}

Remark 4.1. As shown in [19], if \( M \) is a graph generated by a vector-valued function \( u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m \), then the function \( v \) defined in (4.12) is just the slope of \( u \). Hence the two definitions of \( v \) given in (2.4) and (4.12) are equivalent.

Proposition 4.1. There exists a positive constant \( C_1 \). If \( M \) is a self-shrinker in \( \mathbb{R}^{n+m} \) and \( v < 3 \) on \( M \), then
\begin{equation}
(4.15) \quad \mathcal{L}(\log v) + C_1 |\nabla \log v|^2 \geq \frac{1}{2} \left( 3 - v \right) |B|^2.
\end{equation}

Proof. It suffices to prove (4.15) at any point where \( v > 1 \). For any positive constant \( C_1 > 0 \), (4.14) and (4.13) yield
\begin{equation}
(4.16) \quad \mathcal{L}(\log v) + C_1 |\nabla \log v|^2
= |B|^2 + \sum_{i,j} \lambda_j^2 h_{j,ij}^2 + 2 \sum_i \sum_{j<k} \lambda_j \lambda_k h_{k,ij} h_{j,ik} + C_1 \sum_i \left( \sum_j \lambda_j h_{j,ij} \right)^2
= \sum_{\alpha} \sum_{i,j>p} h_{\alpha,ij}^2 + \sum_{i>p} \sum_{1 \leq j < k \leq p} II_{ijk} + \sum_{1 \leq i < j < k \leq p} III_{ijk} + \sum_{1 \leq i \leq p} IV_i
\end{equation}
with
\begin{equation}
(4.17) \quad I_i = \sum_{1 \leq j \leq p} (2 + \lambda_j^2) h_{j,ij}^2 + C_1 \left( \sum_j \lambda_j h_{j,ij} \right)^2
\end{equation}
\begin{equation}
(4.18) \quad II_{ijk} = 2 h_{k,ij}^2 + 2 h_{j,ik}^2 + 2 \lambda_j \lambda_k h_{k,ij} h_{j,ik}
\end{equation}
III_{ijk} = 2h_{i,j,k}^2 + 2h_{j,k,i}^2 + 2h_{k,i,j}^2 + 2\lambda_i \lambda_j h_{i,j,k} h_{j,k,i} + 2\lambda_j \lambda_k h_{j,i,j} h_{i,j,k} + 2\lambda_k \lambda_i h_{k,i,j} h_{i,j,k}

and

IV_i = (1 + \lambda_i^2)h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i} [(2 + \lambda_j^2)h_{j,ij}^2 + h_{i,jj}^2 + 2\lambda_i \lambda_j h_{i,j,j} h_{j,j,i}] + C_1(\sum_j \lambda_j h_{j,j,i})^2.

Here we group the terms according to different types of the indices of the coefficient of the second fundamental form similarly to [13]. Note that both I_j and II_{ijk} vanish whenever p = n, and III_{ijk} vanishes whenever p \leq 2.

Obviously

\begin{equation}
I_i \geq 2 \sum_{1 \leq j \leq p} h_{j,ij}^2.
\end{equation}

As shown in (3.16) of [13],

\begin{equation}
II_{ijk} \geq (3 - v)(h_{i,ii}^2 + h_{j,ij}^2).
\end{equation}

Using Cauchy-Schwarz inequality, one can proceed as in Lemma 3.1 of [13] to get

\begin{equation}
III_{ijk} \geq (3 - v)(h_{i,ij}^2 + h_{j,ji}^2 + h_{k,ik}^2).
\end{equation}

Denote

\begin{equation}
\tau := \frac{1}{2}(v - 1),
\end{equation}

then

\begin{equation}
IV_i - \frac{1}{2}(3 - v)(h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i} (h_{j,ij}^2 + 2h_{j,ij}^2))
\end{equation}

\begin{equation}
= \sum_{1 \leq j \leq p, j \neq i} [(2\tau + \lambda_j^2)h_{j,ij}^2 + \tau h_{i,jj}^2 + 2\lambda_i \lambda_j h_{i,j,j} h_{j,j,i}] + (\tau + \lambda_i^2)h_{i,ii}^2 + C_1(\sum_j \lambda_j h_{j,j,i})^2.
\end{equation}

Completing the square yields

\begin{equation}
(2\tau + \lambda_j^2)h_{j,ij}^2 + \tau h_{i,jj}^2 + 2\lambda_i \lambda_j h_{i,j,j} h_{j,j,i}
\end{equation}

\begin{equation}
= (\tau^2 h_{i,jj} + \tau^{-1} \lambda_i \lambda_j h_{i,j,j})^2 + (2\tau + \lambda_j^2 - \tau^{-1} \lambda_i^2 \lambda_j^2)h_{j,j,i}^2
\end{equation}

\begin{equation}
\geq (2\tau + \lambda_j^2 - \tau^{-1} \lambda_i^2 \lambda_j^2)h_{j,j,i}^2.
\end{equation}

Substituting it into (4.25) implies

\begin{equation}
IV_i - \frac{1}{2}(3 - v)(h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i} (h_{j,ij}^2 + 2h_{j,ij}^2))
\end{equation}

\begin{equation}
\geq (\tau + \lambda_i^2)h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i} (2\tau + \lambda_j^2 - \tau^{-1} \lambda_i^2 \lambda_j^2)h_{j,ij}^2 + C_1(\sum_{1 \leq j \leq p} \lambda_j h_{j,j,i})^2.
\end{equation}
If there exists 2 distinct indices \( j, k \neq i \) satisfying
\[
2\tau + \lambda_j^2 - \tau^{-1}\lambda_i^2\lambda_j^2 \leq 0
\]
and
\[
2\tau + \lambda_k^2 - \tau^{-1}\lambda_i^2\lambda_k^2 \leq 0,
\]
then \( \lambda_i^2 > \tau \) and
\[
\lambda_j^2, \lambda_k^2 \geq \frac{2\tau^2}{\lambda_i^2 - \tau}.
\]
It implies
\[
(1 + \lambda_i^2)(1 + \lambda_j^2)(1 + \lambda_k^2) \geq \frac{(\lambda_i^2 + 1)(\lambda_j^2 + 2\tau^2 - \tau^2)^2}{(\lambda_i^2 - \tau)^2} \geq \frac{(2\tau + 1)^3}{\tau + 1} = \frac{2v^3}{v + 1} > v^2
\]
where the second equality holds if and only if \( \lambda_i^2 = \tau(2\tau + 3) = \frac{1}{2}(v - 1)(v + 2) \) (see (3.25) in [13]). We obtain a contradiction to (4.7). Hence one can find an index \( k \neq i \), such that
\[
2\tau + \lambda_j^2 - \tau^{-1}\lambda_i^2\lambda_j^2 > 0 \quad \text{for all} \ j \notin \{i, k\}.
\]
Denote
\[
s := \sum_{1 \leq j \leq p, j \neq k} \lambda_j h_{j,ij},
\]
then by Cauchy-Schwarz inequality,
\[
(\tau + \lambda_i^2)h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i,k} (2\tau + \lambda_j^2 - \tau^{-1}\lambda_i^2\lambda_j^2)h_{j,ij}^2 \geq \left(\frac{\lambda_i^2}{\tau + \lambda_i^2} + \sum_{1 \leq j \leq p, j \neq i,k} \frac{\lambda_j^2}{2\tau + \lambda_j^2 - \tau^{-1}\lambda_i^2\lambda_j^2}\right)^{-1}s^2. \tag{4.28}
\]
Now we denote
\[
a := \frac{\lambda_i^2}{\tau + \lambda_i^2} + \sum_{1 \leq j \leq p, j \neq i,k} \frac{\lambda_j^2}{2\tau + \lambda_j^2 - \tau^{-1}\lambda_i^2\lambda_j^2},
\]
\[
b := 2\tau + \lambda_k^2 - \tau^{-1}\lambda_i^2\lambda_k^2.
\]
We only need to deal with \( b < 0 \) case. Substituting (4.28) into (4.27) gives
\[
IV_i - \frac{1}{2}(3-v)(h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i} (h_{i,jj}^2 + 2h_{j,ij}^2)) \geq a^{-1}s^2 + bh_{k,ik}^2 + C_1(s + \lambda_kh_{k,ik})^2 = (C_1 + a^{-1})s^2 + (C_1 \lambda_k^2 + b)h_{k,ik}^2 + 2C_1\lambda_ksh_{k,ik} \tag{4.29}
\]
which is nonnegative if and only if
\[
0 \leq (C_1 + a^{-1})(C_1 \lambda_k^2 + b) - (C_1 \lambda_k)^2 = C_1(a^{-1}\lambda_k^2 + b) + a^{-1}b.
\]
Hence the remain work is to prove
\[
\frac{a^{-1}\lambda_k^2 + b}{a^{-1}b} = a + b^{-1}\lambda_k^2 = \frac{\lambda_i^2}{\tau + \lambda_i^2} + \sum_{1 \leq j \leq p, j \neq i} \frac{\lambda_j^2}{2\tau + \lambda_j^2 - \tau^{-1}\lambda_i^2\lambda_j^2}
\]
is bounded from above by \(-\delta_0\) with a positive constant \(\delta_0\); thereby \(C_1 := \delta_0^{-1}\) is the required constant.

Without loss of generality, we assume \(i = 1\) and \(k = 2\); let \(r = \lambda_1^2\) and \(t \in \mathbb{R}^+\)
satisfying
\[
(1 + r)(1 + t) = v^2.
\]
As shown in the proof of Lemma 3.2 in [13],
\[
\frac{\lambda_1^2}{\tau + \lambda_1^2} + \sum_{2 \leq j \leq p} \frac{\lambda_j^2}{2\tau + \lambda_j^2 - \tau^{-1}\lambda_1^2\lambda_j^2} \leq \frac{r}{\tau + r} + \frac{t}{2\tau + t - \tau^{-1}rt}.
\]
Hence it suffices to prove
\[
(4.30) \quad \sup_{\Omega} F(r, t) \leq -\delta_0
\]
for a positive constant \(\delta_0\) not depending on \(v \in (1, 3)\), where
\[
F(r, t) := \frac{r}{\tau + r} + \frac{t}{2\tau + t - \tau^{-1}rt}
\]
and
\[
\Omega := \{(r, t) \in \mathbb{R}^+ \times \mathbb{R}^+: (1 + r)(1 + t) = v^2, r > \tau, t \geq \frac{2\tau}{\tau^{-1}r - 1}\}.
\]

We rewrite \(F(r, t)\) as
\[
F(r, t) = (1 + r)^{-1} + (1 - \tau^{-1}r + \frac{2\tau}{t})^{-1}
\]
\[
= (1 + \frac{\tau}{r})^{-1}(2 + \tau(\frac{1}{r} + \frac{2}{t}) - \tau^{-1}r)(1 - \tau^{-1}r + \frac{2\tau}{t})^{-1}
\]
\[
:= F_1(r)^{-1}F_2(r, t)(F_2(r, t) - F_1(r))^{-1}
\]
From \(r \geq \tau\) we immediately get
\[
(4.34) \quad F_1(r) = 1 + \frac{\tau}{r} \leq 2.
\]
\((1 + r)(1 + t) = v^2\) gives \(t = \frac{v^2 - 1 - r}{1 + r}\) and moreover
\[
F_2(r, t) = 2 + \tau(\frac{1}{r} + \frac{2}{t}) - \tau^{-1}r = 2 + \tau(\frac{1}{r} + \frac{2(1 + r)}{v^2 - 1 - r}) - \tau^{-1}r
\]
\[
= \frac{r^3 + (2\tau^2 - 2\tau - v^2 + 1)r^2 + (\tau^2 + 2\tau(v^2 - 1))r + \tau^2(v^2 - 1)}{\tau r(v^2 - 1 - r)}
\]
\[
= \frac{2r^3 - (v - 1)(v + 5)r^2 + (v - 1)^2(2v + 5) + \frac{1}{2}(v - 1)^3(v + 1)}{(v - 1)r(v^2 - 1 - r)}.
\]
Let $\theta := \frac{r}{v-1}$, then $\tau < r \leq (1+r)(1+t) - 1 = v^2 - 1$ implies $\theta \in (\frac{1}{2}, v + 1]$, and

$$F_2 = \frac{2\theta^3 - (v + 5)\theta^2 + (2v + \frac{5}{2})\theta + \frac{1}{2}(v + 1)}{\theta(v + 1 - \theta)} := \frac{H_1(v, \theta)}{H_2(v, \theta)}.$$

It is easily seen that

$$H_2(v, \theta) = \theta(v + 1 - \theta) \leq \frac{1}{4}(v + 1)^2 \leq 4.$$

Denote

$$\mathcal{D} := \{(v, \theta) : v \in (1, 3), \theta \in (\frac{1}{2}, v + 1]\}.$$

Observing that $H_1(\cdot, \theta)$ is an affine linear function in $v$ for any fixed $\theta$, we have

$$\inf_{\mathcal{D}} H_1 = \min\left\{ \inf_{\theta \in [\frac{1}{2}, 2]} H_1(1, \theta), \inf_{\theta \in [2, 4]} H_1(\theta - 1, \theta), \inf_{\theta \in [\frac{3}{2}, 4]} H_1(3, \theta) \right\}.$$

A straightforward calculation shows

$$\inf_{\theta \in [\frac{1}{2}, 2]} H_1(1, \theta) = \inf_{\theta \in [\frac{1}{2}, 2]} (2\theta(\theta - \frac{3}{2})^2 + 1) = 1,$$

$$\inf_{\theta \in [2, 4]} H_1(\theta - 1, \theta) = \inf_{\theta \in [2, 4]} \theta(\theta - 1)^2 = 2,$$

$$\inf_{\theta \in [\frac{3}{2}, 4]} H_1(3, \theta) = \inf_{\theta \in [\frac{3}{2}, 4]} (2\theta(\theta - 2)^2 + \frac{1}{2}\theta + 2) > 2.$$

Therefore

$$H_1(v, \theta) \geq 1.$$

Substituting (4.37) and (4.38) into (4.36) gives

$$F_2 \geq \frac{1}{4}.$$

Since $F_2 - F_1 = 1 - \tau^{-1}r + \frac{2\tau}{l} < 0$, we have

$$|F_2 - F_1| \leq | - F_1| \leq 2.$$

Thereby one can obtain

$$|F| = |F_1|^{-1}F_2|F_2 - F_1|^{-1} \geq \frac{1}{16}$$

by substituting (4.34), (4.36) and (4.40) into (4.33); in other words, (4.30) holds true with positive constant $\delta_0 = \frac{1}{16}$ and moreover the right hand side of (4.29) is non-negative, i.e.

$$\frac{1}{2}(3 - v)\left(h_{i,ii}^2 + \sum_{1 \leq j \leq p, j \neq i} (h_{i,jj}^2 + 2h_{j,jj}^2)\right).$$

Finally (4.15) is followed from (4.16), (4.21), (4.22), (4.23) and (4.42). □

**Remark 4.2.** From the above proof, one can take $C_1 = 16$ in (4.15).
Let \( h := h(\log v) \), then
\[
\mathcal{L}h = \Delta h - \frac{1}{2}(X, \nabla h) = h''|\nabla \log v|^2 + h'\Delta \log v - \frac{1}{2}h'(X, \nabla \log v)
\]
\[
= h'\mathcal{L}(\log v) + h''|\nabla \log v|^2
\]
where \( h' \) (\( h'' \)) is the first (second) derivation of \( h \) with respect to \( \log v \). Now we choose
\begin{equation}
(4.43) \quad h := v^{C_1} = \exp(C_1 \log v),
\end{equation}
then by Proposition 4.1,
\begin{equation}
(4.44) \quad \mathcal{L}h = C_1 h(\mathcal{L}(\log v) + C_1 |\nabla \log v|^2) \geq \frac{1}{2}C_1 h(3 - v)|B|^2.
\end{equation}
Then one can proceed as in §3 to obtain

**Theorem 4.1.** Let \( u = (u^1, \ldots, u^m) : \mathbb{R}^n \to \mathbb{R}^m \) be a smooth vector-valued function, such that \( M = \text{graph } u \) is a self-shrinker. If the slope of \( u \)
\begin{equation}
(4.45) \quad \Delta_u := \det \left( \delta_{ij} + \sum_{\alpha} u^\alpha_i u^\alpha_j \right)^{\frac{1}{2}} < 3,
\end{equation}
then \( u \) has to be linear and \( M \) has to be a linear subspace.

**Proof.** Denote \( F : \mathbb{R}^n \to M \)
\[
x = (x_1, \ldots, x_n) \mapsto (x, u(x))
\]
then \( F \) is a diffeomorphism and \( M \) can be viewed as \( \mathbb{R}^n \) equipped with metric \( g = g_{ij}dx_idx_j \), where \( g_{ij} = \delta_{ij} + u^\alpha_i u^\alpha_j \). Obviously the eigenvalues of \( (g_{ij}) \) are bounded from below by 1. Moreover \( \Delta_u = \det(g_{ij})^{\frac{1}{2}} < 3 \) implies
\begin{equation}
(4.46) \quad |\xi|^2 \leq \xi_i g_{ij} \xi_j < 9|\xi|^2 \quad \forall \xi \in \mathbb{R}^n.
\end{equation}
Hence \( M \) is a simple Riemannian manifold with Euclidean volume growth.

\((4.45)\) equals to say \( v = \Delta_u \) is a smooth function on \( M \) taking values in \([1, 3)\). Let \( \phi \) be a smooth function on \( M \) with compact supporting set, multiplying \( \phi^2 h \) with both sides of \((4.44)\) and integrating by parts yield
\[
\frac{1}{2}C_1 \int_M \phi^2 h^2 (3 - v)|B|^2 \rho \leq \int_M \phi^2 h \text{ div}(\rho \nabla h)
\]
\[
= -\int_M \langle \nabla(\phi^2 h), \nabla h \rangle \rho
\]
\[
= -\int_M \phi^2 |\nabla h|^2 \rho - 2 \int_M \langle h \nabla \phi, \phi \nabla h \rangle \rho
\]
\[
\leq -\frac{1}{2} \int_M \phi^2 |\nabla h|^2 \rho + 2 \int_M |\nabla \phi|^2 h^2 \rho,
\]
i.e.
\begin{equation}
(4.47) \quad C_1 \int_M \phi^2 h^2 (3 - v)|B|^2 \rho + \int_M \phi^2 |\nabla h|^2 \rho \leq 4 \int_M |\nabla \phi|^2 h^2 \rho.
\end{equation}
Now we choose \( \phi \) to be a cut-off function which satisfies \( \phi \equiv 1 \) on \( D_R \), \( \phi \equiv 0 \) outside \( D_{2R} \) and \( |\nabla \phi| \leq \frac{\alpha}{R} \), then

\[
\int_{D_R} |\nabla h|^2 \rho \leq \int_M \phi^2 |\nabla h|^2 \rho \leq 4 \int_M |\nabla \phi|^2 h^2 \rho = 4 \int_{D_{2R}\setminus D_R} |\nabla \phi|^2 h^2 \rho \leq \frac{4c_0^2}{R^2} 3^{2C_1} e^{-\frac{4R^2}{3}} \text{Vol}(D_{2R}\setminus D_R).
\]

(4.48)

Letting \( R \to +\infty \) forces \( |\nabla h|^2 \equiv 0 \), furthermore \( v \equiv \text{const.} \). Assume \( v \equiv v_0 < 3 \) and denote \( h_0 := v_0^{C_1} \). Again using (4.47) gives

\[
C_1 h_0^2 (3 - v_0) \int_{D_R} |B|^2 \rho \leq C_1 \int_M \phi^2 h^2 (3 - v) |B|^2 \rho \leq 4 \int_M |\nabla \phi|^2 h^2 \rho \leq \frac{4c_0^2}{R^2} 3^{2C_1} e^{-\frac{4R^2}{3}} \text{Vol}(D_{2R}\setminus D_R).
\]

(4.49)

Let \( R \to +\infty \), we have \( |B|^2 \equiv 0 \). Hence \( M \) has to be a linear subspace.

\[\Box\]

5. Appendix

Let \( f : (M^m, g) \to (N^n, h) \) be a smooth map and let \( w \) be a given smooth positive function in \( M \). Let \( \{e_i\}_{i=1}^m \) be a local orthonormal frame field in \( M \), \( \nabla \) be the Levi-Civita connection of \( M \) and \( \langle \cdot, \cdot \rangle \) be the inner product of \( (N, h) \). Then we can define \( w\)-weighted energy density of the given map \( f \) by

\[ e_w(f) = \frac{1}{2} \langle f_* e_i, f_* e_i \rangle w. \]

Let \( \Omega \) be an arbitrary domain in \( M \) such that \( \overline{\Omega} \) is compact. The integral of the \( w\)-weighted energy density over \( \Omega \) yields the \( w\)-weighted energy of the map \( f \):

\[ E_{w, \Omega}(f) = \int_{\Omega} e_w(f) = \frac{1}{2} \int_{\Omega} \langle f_* e_i, f_* e_i \rangle w. \]

Now let’s deduce the first variational formula for weighted map.

Let \( f_t \) be a 1-parameter family of maps from \( M \) to \( N \) with \( f_0 = f \) and \( \frac{df_t}{dt} \big|_{t=0} \) having compact support in \( \Omega \). Let \( \{e_i\} \) be a normal coordinate frame of \( M \) at \( p \in M \),
then by (1.2.12) of [17],
\[
\frac{d}{dt}E_{w, \Omega}(f_t) = \frac{1}{2} \int_{\Omega} \langle f_t e_i, f_t e_i \rangle w
\]
\[
= \int_{\Omega} \left\langle \nabla \frac{df_t}{dt}, f_t e_i \right\rangle w = \int_{\Omega} \left\langle \nabla e_i \frac{df_t}{dt}, f_t e_i \right\rangle w
\]
\[
= \int_{\Omega} \left\langle \nabla e_i \left( \left( \frac{df_t}{dt}, w f_t e_j \right) e_j \right), e_i \right\rangle - \int_{\Omega} \left\langle \frac{df_t}{dt}, \nabla e_i (w f_t e_i) \right\rangle
\]
\[
= \int_{\Omega} \left( \frac{df_t}{dt} w f_t e_j \right) + \int_{\Omega} \left( \frac{df_t}{dt}, \nabla e_i (w f_t e_i) \right)
\]
\[
= - \int_{\Omega} \left( \frac{df_t}{dt}, \tau_w(f_t) \right) w.
\]
(5.1)

Here, \( \tau_w(f) = w^{-1} \nabla e_i (w f_t e_i) = w^{-1} (w f_t^a), \frac{\partial}{\partial y^a} \), and \( df = \frac{\partial f^a}{\partial x^i} dx_i \otimes \frac{\partial}{\partial y^a} \) is a section of the bundle \( T^*M \otimes f^{-1} TN \).

We call \( f \) is a \( w \)-weighted harmonic map if \( \tau_w(f) = 0 \).

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