New version of fractional Simpson type inequalities for twice differentiable functions

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Abstract

Simpson inequalities for differentiable convex functions and their fractional versions have been studied extensively. Simpson type inequalities for twice differentiable functions are also investigated. More precisely, Budak et al. established the first result on fractional Simpson inequality for twice differentiable functions. In the present article, we prove a new identity for twice differentiable functions. In addition to this, we establish several fractional Simpson type inequalities for functions whose second derivatives in absolute value are convex. This paper is a new version of fractional Simpson type inequalities for twice differentiable functions.

MSC: 26D07; 26D10; 26D15

Keywords: Simpson type inequalities; Convex function; Fractional integrals

1 Introduction

Simpson’s inequality plays a considerable role in several branches of mathematics. For four times continuously differentiable functions, the classical Simpson’s inequality is expressed as follows.

Theorem 1 Suppose that \( F : [\rho_1, \rho_2] \rightarrow \mathbb{R} \) is a four times continuously differentiable mapping on \( (\rho_1, \rho_2) \), and let \( \|F(4)\|_\infty = \sup_{\kappa \in (\rho_1, \rho_2)} |F^{(4)}(\kappa)| < \infty \). Then one has the inequality

\[
\left| \frac{1}{3} \left[ \frac{F(\rho_1) + F(\rho_2)}{2} + 2F\left(\frac{\rho_1 + \rho_2}{2}\right) \right] - \frac{1}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} F(\kappa) \, d\kappa \right| \leq \frac{1}{2880} \|F^{(4)}\|_\infty (\rho_2 - \rho_1)^4.
\]

Since the convex theory is an effective way to solve a large number of problems from different branches of mathematics, many authors have studied the results of Simpson type for convex mapping. To be more precise, some inequalities of Simpson type for \( s \)-convex functions are proved by using differentiable functions [4]. In the papers [34, 36], the new variants of Simpson type inequalities are established based on differentiable convex mapping. Moreover, some papers were devoted to Simpson type inequalities for various convex classes [11, 18, 27, 30, 31].
The Simpson inequalities for differentiable functions are extended to Riemann–Liouville fractional integrals in the papers [8] and [17]. Hence, several paper focused on fractional Simpson inequalities for various fractional integral operators [1–3, 7, 9, 12, 15, 19, 21, 25, 26, 28, 32, 33, 37, 39]. For further information and unexplained subjects about Simpson type inequalities, we refer the reader to [5, 10, 14, 16, 22–24, 38] and the references therein. Besides, Sarikaya et al. established several Simpson type inequalities for functions whose second derivatives are convex [35].

The purpose of this paper is to extend the results given in [35] for twice differentiable functions to Riemann–Liouville fractional integrals. The general structure of the paper consists of four chapters including an introduction. The remaining part of the paper proceeds as follows: In Sect. 2, after giving a general literature survey and the definition of Riemann–Liouville fractional integral operators, we prove an equality for twice differentiable functions. In the next section, for utilizing this equality, we establish several Simpson type inequalities for a mapping whose second derivatives are convex. In the last section, some conclusions and further directions of research are discussed.

**Definition 1** Consider $\mathcal{F} \in L_{1}[ρ_1, ρ_2]$. The Riemann–Liouville integrals $J_{ρ_1}^{α} \mathcal{F}$ and $J_{ρ_2}^{α} \mathcal{F}$ of order $α > 0$ with $ρ_1 ≥ 0$ are defined by

$$J_{ρ_1}^{α} \mathcal{F}(κ) = \frac{1}{Γ(α)} \int_{ρ_1}^{κ} (κ - τ)^{α-1} \mathcal{F}(τ) dτ, \quad κ > ρ_1$$

and

$$J_{ρ_2}^{α} \mathcal{F}(κ) = \frac{1}{Γ(α)} \int_{κ}^{ρ_2} (τ - κ)^{α-1} \mathcal{F}(τ) dτ, \quad κ < ρ_2,$$

respectively. Here, $Γ(α)$ is the gamma function and $J_{ρ_1}^{0} \mathcal{F}(κ) = J_{ρ_2}^{0} \mathcal{F}(κ) = \mathcal{F}(κ)$.

For more information and several properties of Riemann–Liouville fractional integrals, please refer to [13, 20, 29].

The first result on fractional Simpson inequality for twice differentiable functions was proved by Budak et al. in [6] as follows.

**Theorem 2** Assume that the assumptions of Lemma 1 hold. Assume also that the mapping $|\mathcal{F}''|$ is convex on $[ρ_1, ρ_2]$. Then we have the following inequality:

$$\left| \frac{1}{6} \left[ \mathcal{F}(ρ_1) + 4 \mathcal{F}\left(\frac{ρ_1 + ρ_2}{2}\right) + \mathcal{F}(ρ_2) \right] \right| \leq \frac{(ρ_2 - ρ_1)^2}{6} A(α) \left( |\mathcal{F}''(ρ_1)| + |\mathcal{F}''(ρ_2)| \right), \quad 2α + 1, \quad (1.1)$$

where

$$A(α) = \frac{1}{4(α + 2)} \left( α \left( \frac{α + 1}{3} \right)^{\frac{2}{3}} + \frac{3}{α + 1} \right) - \frac{1}{8}. \quad (1.2)$$

In this paper, we prove a new version of inequality (1.1).
2 Some equalities

In this section, we give equalities on twice differentiable functions for using the main results.

**Lemma 1** If $F : [\rho_1, \rho_2] \to \mathbb{R}$ is an absolutely continuous mapping $(\rho_1, \rho_2)$ such that $F'' \in L_1([\rho_1, \rho_2])$, then the following equality

\[
\frac{1}{6} \left[ F'(\rho_1) + 4F\left(\frac{\rho_1 + \rho_2}{2}\right) + F'(\rho_2) \right] - 2(\rho_2 - \rho_1)^2 \left[ j_{\rho_2} F'\left(\frac{\rho_1 + \rho_2}{2}\right) - j_{\rho_1} F\left(\frac{\rho_1 + \rho_2}{2}\right) \right] \]

\[
= \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \int_0^1 \left( \frac{1 - 2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^\alpha \right) \left[ F''\left(\frac{1 + \tau}{2} \rho_2 + \frac{1 - \tau}{2} \rho_1\right) + F''\left(\frac{1 + \tau}{2} \rho_1 + \frac{1 - \tau}{2} \rho_2\right) \right] d\tau.
\]

is valid.

**Proof** By using integration by parts, we obtain

\[
\gamma_1 = \int_0^1 \left( \frac{1 - 2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^\alpha \right) F''\left(\frac{1 + \tau}{2} \rho_2 + \frac{1 - \tau}{2} \rho_1\right) d\tau
\]

\[
= -2 \left( \frac{1 - 2\alpha}{3(\rho_2 - \rho_1)} \right) F'\left(\frac{\rho_1 + \rho_2}{2}\right) \]

\[
= \frac{2}{(\rho_2 - \rho_1)} \int_0^1 \left( \frac{2(\alpha + 1)}{3} - \frac{(\alpha + 1)\tau^\alpha}{3} \right) F''\left(\frac{1 + \tau}{2} \rho_2 + \frac{1 - \tau}{2} \rho_1\right) d\tau
\]

\[
= -2 \left( \frac{1 - 2\alpha}{3(\rho_2 - \rho_1)} \right) F'\left(\frac{\rho_1 + \rho_2}{2}\right) + 4(\alpha + 1) \left( \frac{3(\rho_2 - \rho_1)^2}{3(\rho_2 - \rho_1)^2} \right) F(\rho_2) + \frac{8(\alpha + 1)}{3(\rho_2 - \rho_1)^2} F\left(\frac{\rho_1 + \rho_2}{2}\right)
\]

\[
= -\frac{4(\alpha + 1)}{\rho_2 - \rho_1} \int_0^1 \tau^\alpha F''\left(\frac{1 + \tau}{2} \rho_2 + \frac{1 - \tau}{2} \rho_1\right) d\tau.
\]

By using equation (2.2), the change of the variable $\kappa = \frac{1 + \tau}{2} \rho_2 + \frac{1 - \tau}{2} \rho_1$ for $\tau \in [0, 1]$ can be rewritten as follows:

\[
\gamma_1 = -2 \left( \frac{1 - 2\alpha}{3(\rho_2 - \rho_1)} \right) F'\left(\frac{\rho_1 + \rho_2}{2}\right) + 4(\alpha + 1) \left( \frac{3(\rho_2 - \rho_1)^2}{3(\rho_2 - \rho_1)^2} \right) F(\rho_2) + \frac{8(\alpha + 1)}{3(\rho_2 - \rho_1)^2} F\left(\frac{\rho_1 + \rho_2}{2}\right)
\]

\[
= -\frac{2^{\alpha+2}(\alpha + 1)\Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^{\alpha+2}} \int_{\rho_2} F\left(\frac{\rho_1 + \rho_2}{2}\right) d\tau.
\]

Similarly, we have

\[
\gamma_2 = \int_0^1 \left( \frac{1 - 2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^\alpha \right) F''\left(\frac{1 + \tau}{2} \rho_1 + \frac{1 - \tau}{2} \rho_2\right) d\tau
\]

\[
= 2 \left( \frac{1 - 2\alpha}{3(\rho_2 - \rho_1)} \right) F'\left(\frac{\rho_1 + \rho_2}{2}\right) + 4(\alpha + 1) \left( \frac{3(\rho_2 - \rho_1)^2}{3(\rho_2 - \rho_1)^2} \right) F(\rho_1) + \frac{8(\alpha + 1)}{3(\rho_2 - \rho_1)^2} F\left(\frac{\rho_1 + \rho_2}{2}\right)
\]

\[
= -\frac{2^{\alpha+2}(\alpha + 1)\Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^{\alpha+2}} \int_{\rho_1} F\left(\frac{\rho_1 + \rho_2}{2}\right) d\tau.
\]
From equations (2.3) and (2.4), we get

\[ \Upsilon_1 + \Upsilon_2 = \frac{4(\alpha + 1)}{3(\rho_2 - \rho_1)^2} \left[ F(\rho_1) + 4F\left(\frac{\rho_1 + \rho_2}{2}\right) + F(\rho_2) \right] \]

\[ - \frac{2^{2\alpha+1}(\alpha + 1)\Gamma(\alpha + 1)}{\rho_2 - \rho_1} \left[ \int_{\rho_1}^{\rho_2} \frac{\rho_2^\frac{\alpha}{\rho_2 + \rho_1}}{\rho_2 - \rho_1} \right] \left[ F\left(\frac{\rho_1 + \rho_2}{2}\right) + \rho_1^\frac{\alpha}{\rho_1 + \rho_2} \right]. \]  

Multiplying both sides of (2.5) by \( \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \), we obtain equation (2.1). This ends the proof of Lemma 1.

**Lemma 2** Let us consider the function \( \sigma : [0,1] \rightarrow \mathbb{R} \) by \( \sigma(\tau) = \frac{1-2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^{\alpha+1} \) with \( \alpha > 0 \).

1. If \( 0 < \alpha \leq \frac{1}{2} \), then we have
   \[ \int_0^1 |\sigma(\tau)| \, d\tau = \frac{1 - \alpha^2}{3(\alpha + 2)}. \]

2. If \( \alpha > \frac{1}{2} \), then there exists a real number \( \varsigma_\alpha \) such that \( 0 < \varsigma_\alpha < 1 \), and we have
   \[ \int_0^1 |\sigma(\tau)| \, d\tau = 2 \left( \frac{(\varsigma_\alpha)^{\alpha+2}}{\alpha + 2} - \frac{1 - 2\alpha}{3} \varsigma_\alpha + \frac{(\alpha + 1)(\varsigma_\alpha)^2}{3} \right) + \frac{1 - \alpha^2}{3(\alpha + 2)}. \]

**Proof** Let us note that \( 0 < \alpha \leq \frac{1}{2} \). Then \( \sigma(\tau) \geq 0 \) for all \( \tau \in [0,1] \). Thus, it can be easily seen that

\[ \int_0^1 |\sigma(\tau)| \, d\tau = \int_0^1 \sigma(\tau) \, d\tau = \frac{1 - \alpha^2}{3(\alpha + 2)}. \]

If \( \alpha > \frac{1}{2} \), then there exists a real number \( \varsigma_\alpha \in (0,1) \) such that \( \sigma(\tau) \leq 0 \) for \( 0 \leq \tau \leq \varsigma_\alpha \) and \( \sigma(\tau) \geq 0 \) for \( \varsigma_\alpha \leq \tau \leq 1 \). Therefore, we obtain

\[ \int_0^1 |\sigma(\tau)| \, d\tau = \int_0^{\varsigma_\alpha} (-\sigma(\tau)) \, d\tau + \int_{\varsigma_\alpha}^1 \sigma(\tau) \, d\tau \]

\[ = 2 \left( \frac{(\varsigma_\alpha)^{\alpha+2}}{\alpha + 2} - \frac{1 - 2\alpha}{3} \varsigma_\alpha + \frac{(\alpha + 1)(\varsigma_\alpha)^2}{3} \right) + \frac{1 - \alpha^2}{3(\alpha + 2)}. \]

**Lemma 3** Define the function \( \sigma : [0,1] \rightarrow \mathbb{R} \) by \( \sigma(\tau) = \frac{1-2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^{\alpha+1} \) with \( \alpha > 0 \).

1. Let us consider \( 0 < \alpha \leq \frac{1}{2} \). Then we have
   \[ \int_0^1 |\sigma(\tau)| \tau \, d\tau = \frac{3 - \alpha - 2\alpha^2}{18(\alpha + 3)}. \]

2. If we take \( \alpha > \frac{1}{2} \), then there exists a real number \( \varsigma_\alpha \) so that \( 0 < \varsigma_\alpha < 1 \), and we get
   \[ \int_0^1 |\sigma(\tau)| \tau \, d\tau = 2 \left( \frac{(\varsigma_\alpha)^{\alpha+3}}{\alpha + 3} - \frac{3(1 - 2\alpha)(\varsigma_\alpha)^2 + 4(\alpha + 1)(\varsigma_\alpha)^3}{18} \right) + \frac{3 + \alpha - 2\alpha^2}{18(\alpha + 3)}. \]

**Proof** The proof can be done similar to the proof of Lemma 2.
3 New Simpson type inequalities for twice differentiable functions

In this section, we prove several Simpson type inequalities for a mapping whose second derivatives are convex.

Theorem 3 Let us note that the assumptions of Lemma 1 are valid. Let us also note that the mapping $|F''|$ is convex on $[\rho_1, \rho_2]$. Then we have the following inequality:

$$
\begin{align*}
\left| \frac{1}{6} \left[ F(\rho_1) + 4F\left(\frac{\rho_1 + \rho_2}{2}\right) + F(\rho_2) \right] 
&- \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^{\alpha}} \left[ \int_{\rho_2}^{\rho_1} F\left(\frac{\rho_1 + \rho_2}{2}\right) d\rho_1 \right] \right| \\
&\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \Omega_1(\alpha) \left[ |F''(\rho_1)| + |F''(\rho_2)| \right],
\end{align*}
$$

where $\Omega_1$ is defined by

$$
\Omega_1(\alpha) = \begin{cases} 
1 - \frac{\alpha^2}{3(\alpha^2 + 1)}, & \text{if } 0 < \alpha \leq \frac{1}{2}, \\
\frac{2(1 - 2\alpha \epsilon_{\alpha+\frac{1}{2}} + \alpha + 1)(\epsilon_{\alpha+\frac{1}{2}})^2 + \frac{1}{3} \alpha^2}{\alpha + 1}, & \text{if } \alpha > \frac{1}{2}.
\end{cases}
$$

Proof By taking modulus in Lemma 1, we have

$$
\begin{align*}
\left| \frac{1}{6} \left[ F(\rho_1) + 4F\left(\frac{\rho_1 + \rho_2}{2}\right) + F(\rho_2) \right] 
&- \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^{\alpha}} \left[ \int_{\rho_2}^{\rho_1} F\left(\frac{\rho_1 + \rho_2}{2}\right) d\rho_1 \right] \right| \\
&\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \int_0^1 \left| 1 - \frac{2\alpha}{3} + \frac{2(\alpha + 1)}{\alpha + 1} \tau - \tau^{\alpha+1} \right| d\tau \\
&\times \left[ |F''\left(\frac{1}{2} \rho_2 + \frac{1 - \tau}{2} \rho_1\right)| + |F''\left(\frac{1}{2} \rho_1 + \frac{1 - \tau}{2} \rho_2\right)| \right] d\tau.
\end{align*}
$$

By using the convexity of $|F''|$, we obtain

$$
\begin{align*}
\left| \frac{1}{6} \left[ F(\rho_1) + 4F\left(\frac{\rho_1 + \rho_2}{2}\right) + F(\rho_2) \right] 
&- \frac{2^{\alpha-1} \Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^{\alpha}} \left[ \int_{\rho_2}^{\rho_1} F\left(\frac{\rho_1 + \rho_2}{2}\right) d\rho_1 \right] \right| \\
&\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \int_0^1 \left| 1 - \frac{2\alpha}{3} + \frac{2(\alpha + 1)}{\alpha + 1} \tau - \tau^{\alpha+1} \right| \\
&\times \left[ \left(\frac{1}{2}\right) |F''(\rho_2)| + \left(\frac{1 - \tau}{2}\right) |F''(\rho_1)| \right] d\tau \\
&\quad + \left(\frac{1}{2}\right) |F''(\rho_1)| + \left(\frac{1 - \tau}{2}\right) |F''(\rho_2)| \right] d\tau
\end{align*}
$$
By using the convexity of $|F''|^q$, we obtain
\[
\frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \int_0^1 \left| 1 - 2\alpha + 2(\alpha + 1) - \frac{1}{3} \right| \left( F''(\rho_2) + F''(\rho_1) \right) d\tau 
\]
\[
= \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \Omega_1(\alpha) \left[ |F''(\rho_1)| + |F''(\rho_2)| \right].
\]
This completes the proof of Theorem 3. □

**Theorem 4** Let us consider that the assumptions of Lemma 1 hold. If the mapping $|F''|^q$, $q > 1$ is convex on $[\rho_1, \rho_2]$, then we have the following inequality:

\[
\left| \frac{1}{6} F(\rho_1) + 4F\left( \frac{\rho_1 + \rho_2}{2} \right) + F(\rho_2) \right|
\]
\[
- \frac{2^{q-1} \Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^q} \left[ \int_{\rho_2}^{\rho_1} F\left( \frac{\rho_1 + \rho_2}{2} \right) + \int_{\rho_1}^{\rho_2} F\left( \frac{\rho_1 + \rho_2}{2} \right) \right]
\]
\[
\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \Psi(\alpha, p) \left[ \left| F''(\rho_1) \right|^q + \left| F''(\rho_2) \right|^q \right]^{\frac{1}{q}}.
\]

Here, $\frac{1}{p} + \frac{1}{q} = 1$ and $\Psi$ is defined by

\[
\Psi(\alpha, p) = \left( \int_0^1 \left| 1 - 2\alpha + 2(\alpha + 1) - \frac{1}{3} \right| d\tau \right)^{\frac{1}{p}}.
\]

**Proof** With the help of H"{o}lder’s inequality in inequality (3.1), we get

\[
\left| \frac{1}{6} F(\rho_1) + 4F\left( \frac{\rho_1 + \rho_2}{2} \right) + F(\rho_2) \right|
\]
\[
- \frac{2^{q-1} \Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^q} \left[ \int_{\rho_2}^{\rho_1} F\left( \frac{\rho_1 + \rho_2}{2} \right) + \int_{\rho_1}^{\rho_2} F\left( \frac{\rho_1 + \rho_2}{2} \right) \right]
\]
\[
\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \left\{ \left( \int_0^1 \left| 1 - 2\alpha + 2(\alpha + 1) - \frac{1}{3} \right|^p d\tau \right)^{\frac{1}{p}} \right\}^{\frac{1}{q}}
\]
\[
\times \left( \int_0^1 \left| F''\left( \frac{1 + \tau \rho_2 + 1 - \frac{1}{3} \rho_1}{2} \right) \right|^q d\tau \right)^{\frac{1}{q}}
\]
\[
+ \left( \int_0^1 \left| 1 - 2\alpha + 2(\alpha + 1) - \frac{1}{3} \right|^p d\tau \right)^{\frac{1}{p}}
\]
\[
\times \left( \int_0^1 \left| F''\left( \frac{1 + \tau \rho_1 + 1 - \frac{1}{3} \rho_2}{2} \right) \right|^q d\tau \right)^{\frac{1}{q}} \}.
\]

By using the convexity of $|F''|^q$, we obtain

\[
\left| \frac{1}{6} F(\rho_1) + 4F\left( \frac{\rho_1 + \rho_2}{2} \right) + F(\rho_2) \right|
\]
\[
- \frac{2^{q-1} \Gamma(\alpha + 1)}{(\rho_2 - \rho_1)^q} \left[ \int_{\rho_2}^{\rho_1} F\left( \frac{\rho_1 + \rho_2}{2} \right) + \int_{\rho_1}^{\rho_2} F\left( \frac{\rho_1 + \rho_2}{2} \right) \right]
\]
\[
\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \left( \int_0^1 \left| 1 - 2\alpha + 2(\alpha + 1) - \frac{1}{3} \right|^p d\tau \right)^{\frac{1}{p}}
\]
\[\begin{align*}
\times \left[ \left( \int_0^1 \left( \frac{1 + \tau}{2} \right) |F''(\rho_2)|^q \right. \right. & \left. \left. + \left( \frac{1 - \tau}{2} \right) |F''(\rho_1)|^q \right) d\tau \right]^{\frac{1}{q}} \\
& \hspace{1em} + \left( \int_0^1 \left( \frac{1 + \tau}{2} \right) |F''(\rho_1)|^q + \left( \frac{1 - \tau}{2} \right) |F''(\rho_2)|^q \right) d\tau \right]^{\frac{1}{q}} \\
= & \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \left( \int_0^1 \left[ 1 - 2\alpha \frac{1}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^{\alpha + 1} \right] d\tau \right)^{\frac{1}{q}} \\
& \times \left[ \left( \frac{3}{4} |F''(\rho_2)|^q + |F''(\rho_1)|^q \right) \right. \left. \right]^{\frac{1}{q}} + \left( \frac{3}{4} |F''(\rho_2)|^q + 3 |F''(\rho_1)|^q \right) \right]^{\frac{1}{q}}. 
\end{align*}\]

This finishes the proof of Theorem 4.

**Theorem 5** Suppose that the assumptions of Lemma 1 hold. If the mapping \(|F''|^q\), \(q \geq 1\) is convex on \([\rho_1, \rho_2]\), then we have the following inequality:

\[
\left[ \left( \frac{1}{6} |F(\rho_1) + 4F\left( \frac{\rho_1 + \rho_2}{2} \right) + F(\rho_2) \right) \right. \\
\left. \frac{- 2^{\alpha - 1} \Gamma(\alpha + 1)}{\rho_2 - \rho_1} \left[ J_{\rho_2, \tau}^\alpha F\left( \frac{\rho_1 + \rho_2}{2} \right) + J_{\rho_1, \tau}^\alpha F\left( \frac{\rho_1 + \rho_2}{2} \right) \right] \right] \\
\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \left( \int_0^1 \left[ 1 - 2\alpha \frac{1}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^{\alpha + 1} \right] d\tau \right)^{\frac{1}{q}} \\
\times \left[ \left( \Omega_1(\alpha) + \Omega_2(\alpha) \right) |F''(\rho_2)|^q + \left( \Omega_1(\alpha) - \Omega_2(\alpha) \right) |F''(\rho_1)|^q \right] \right]^{\frac{1}{q}},
\]

where \(\Omega_1\) is defined as in Theorem 3 and \(\Omega_2\) is defined by

\[
\Omega_2(\alpha) = \begin{cases} \\
\frac{3 - 2\alpha}{18(\alpha + 1)} & \text{if } 0 < \alpha \leq \frac{1}{2}, \\
2\left( \frac{\alpha^2}{\alpha + 1} - \frac{3(1 - 2\alpha)(\alpha^2 + 4\alpha + 1)}{18(\alpha + 1)} \right) + \frac{3 - 2\alpha}{18(\alpha + 1)} & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

**Proof** By applying the power-mean inequality in (3.1), we get

\[
\left[ \left( \frac{1}{6} |F(\rho_1) + 4F\left( \frac{\rho_1 + \rho_2}{2} \right) + F(\rho_2) \right) \right. \\
\left. \frac{- 2^{\alpha - 1} \Gamma(\alpha + 1)}{\rho_2 - \rho_1} \left[ J_{\rho_2, \tau}^\alpha F\left( \frac{\rho_1 + \rho_2}{2} \right) + J_{\rho_1, \tau}^\alpha F\left( \frac{\rho_1 + \rho_2}{2} \right) \right] \right] \\
\leq \frac{(\rho_2 - \rho_1)^2}{8(\alpha + 1)} \left( \int_0^1 \left[ 1 - 2\alpha \frac{1}{3} + \frac{2(\alpha + 1)}{3} \tau - \tau^{\alpha + 1} \right] d\tau \right)^{\frac{1}{q}} \\
\times \left[ \left( \Omega_1(\alpha) + \Omega_2(\alpha) \right) |F''(\rho_2)|^q + \left( \Omega_1(\alpha) - \Omega_2(\alpha) \right) |F''(\rho_1)|^q \right] \right]^{\frac{1}{q}}, \tag{3.2}
\]
Since $|F''|^q$ is convex, we obtain
\[
\int_0^1 \left| \frac{1 - 2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau \right|^{\alpha+1} \left| F'' \left( \frac{1 + \tau}{2} \rho_1 + \frac{1 - \tau}{2} \rho_2 \right) \right|^q d\tau \\
\leq \int_0^1 \left| \frac{1 - 2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau \right|^{\alpha+1} \left[ \frac{1 + \tau}{2} |F''(\rho_2)|^q + \frac{1 - \tau}{2} |F''(\rho_1)|^q \right] d\tau \\
= \frac{(\Omega_1(\alpha) + \Omega_2(\alpha))|F''(\rho_2)|^q + (\Omega_1(\alpha) - \Omega_2(\alpha))|F''(\rho_1)|^q}{2},
\]
and similarly
\[
\int_0^1 \left| \frac{1 - 2\alpha}{3} + \frac{2(\alpha + 1)}{3} \tau \right|^{\alpha+1} \left| F'' \left( \frac{1 + \tau}{2} \rho_1 + \frac{1 - \tau}{2} \rho_2 \right) \right|^q d\tau \\
\leq \frac{(\Omega_1(\alpha) + \Omega_2(\alpha))|F''(\rho_1)|^q + (\Omega_1(\alpha) - \Omega_2(\alpha))|F''(\rho_2)|^q}{2}.
\]

Then we obtain the desired result Theorem 5. □

4 Special cases
In this section, we present special cases of the main findings in the paper.

Remark 1 If we choose $\alpha = 1$ in Theorem 3, then $\varsigma_\alpha = \frac{1}{3}$, and we have the inequality
\[
\left| \frac{1}{6} \int \left[ F(\rho_1) + 4F \left( \frac{1 + \rho_2}{2} \right) + F(\rho_2) \right] - \frac{1}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} F(\kappa) d\kappa \right| \\
\leq \frac{(\rho_2 - \rho_1)^2}{16} \left[ |F''(\rho_1)| + |F''(\rho_2)| \right],
\]
which is proved by Sarikaya et al. in [35].

Corollary 1 In Theorem 4, if we assign $\alpha = 1$, then $\varsigma_\alpha = \frac{1}{3}$, and the following inequality
\[
\left| \frac{1}{6} \int \left[ F(\rho_1) + 4F \left( \frac{1 + \rho_2}{2} \right) + F(\rho_2) \right] - \frac{1}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} F(\kappa) d\kappa \right| \\
\leq \frac{(\rho_2 - \rho_1)^2}{16} \Psi(1,p) \left[ |F''(\rho_1)|^q + |F''(\rho_2)|^q \right]^{\frac{1}{q}}
\]

is valid.

Remark 2 If we take $\alpha = 1$ in Theorem 5, then Theorem 5 reduces to [35, Theorem 2.5].

5 Conclusion
In the present article, the fractional version of Simpson type inequality for twice differentiable functions are established. Moreover, we show that our results generalize the inequalities obtained by Sarikaya et al. [35]. This work is a new version of fractional Simpson type inequalities for twice differentiable functions. In the future studies, authors can try to generalize our results by utilizing a different kind of convex function classes or another type of fractional integral operators. In addition to this, the authors can give some applications of special cases with the help of our results.
Acknowledgements
The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding
There is no funding.

Availability of data and materials
Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 28 June 2021 Accepted: 7 September 2021 Published online: 18 October 2021

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