SKEW CHARACTERS AND CYCLIC SIEVING

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Abstract. In 2010, B. Rhoades proved that promotion together with the fake-degree polynomial associated with rectangular standard Young tableaux give an instance of the cyclic sieving phenomenon.

We extend this result to all skew standard Young tableaux where the fake-degree polynomial evaluates to nonnegative integers at roots of unity, albeit without being able to specify an explicit group action. Put differently, we determine in which cases a skew character of the symmetric group carries a permutation representation of the cyclic group.

We use a method proposed by N. Amini and the first author, which amounts to establishing a bound on the number of border-strip tableaux of skew shape.

Finally, we apply our results to the invariant theory of tensor powers of the adjoint representation of the general linear group. In particular, we prove the existence of a bijection between permutations and J. Stembridge's alternating tableaux, which intertwines rotation and promotion.

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1. Introduction

We determine which tensor powers of a skew character $\chi^{\lambda/\mu}$ of the symmetric group $S_n$ carry a permutation representation of the cyclic group of order $n$.

This problem can be rephrased in terms of V. Reiner, D. Stanton & D. White’s cyclic sieving phenomenon [RSW04]. Let SYT($\lambda/\mu$) be the set of standard Young tableaux of skew shape $\lambda/\mu$, and let $f^{\lambda/\mu}(q)$ be G. Lusztig’s fake degree polynomial for $\chi^{\lambda/\mu}$.

Then there exists an action $\rho$ of the cyclic group of order $n = |\lambda/\mu|$ such that

$$\left(\text{SYT}(\lambda/\mu) \times \cdots \times \text{SYT}(\lambda/\mu), \langle \rho \rangle, f^{\lambda/\mu}(q)^m\right)$$

exhibits the cyclic sieving phenomenon, if and only if $f^{\lambda/\mu}$ evaluates to nonnegative integers at $n^{th}$ root of unity. If $m$ is even this is always the case. If $m$ is odd, this is the case if and only if there exists a tiling of $\lambda/\mu$ with border-strips of size $k$ of even height for every $k | n$, see Proposition 41.

We also show that for any skew shape $\lambda/\mu$ and integer $s > 0$ there is an action $\tau$ of the cyclic group of order $s$ on stretched shapes such that

$$\left(\text{SYT}(s\lambda/s\mu), \langle \tau \rangle, f^{s\lambda/s\mu}(q)\right)$$

exhibits the cyclic sieving phenomenon, see Theorem 46.

At this point we are unable to present $\rho$ and $\tau$ explicitly for general skew shapes $\lambda/\mu$. Instead, we use a characterization of P. Alexandersson & N. Amini [AA19], which says that $f \in \mathbb{N}[q]$ is a cyclic sieving polynomial for a group action of the cyclic group of order $n$, if and only if for a primitive $n^{th}$ root of unity $\xi$ and all $k | n$ we have that $f(\xi^k) \in \mathbb{N}$ and

$$\sum_{d|k} \mu(k/d) f(\xi^d) \geq 0,$$

where $\mu$ is the number-theoretic Möbius function.

To apply this result, our main tool is a new bound for the absolute value of the skew character evaluated at a power of the long cycle. More precisely, with Theorem 28 we show that for any $k | n$

$$|f^{\lambda/\mu}(\xi^k)| \geq \sum_{d|k, d < k} |f^{\lambda/\mu}(\xi^d)|$$

provided $|f^{\lambda/\mu}(\xi^k)| \geq 2$.

To do so, we note that $|f^{\lambda/\mu}(\xi^d)| = |\chi^{\lambda/\mu}((m^d))| = |\text{BST}(\lambda/\mu, m)|$, the number of border-strip tableaux of shape $\lambda/\mu$ with strips of size $m$, extending the theorems for straight shapes by T. Springer [Spr74] and G. James & A. Kerber [JKS81]. We finally approximate the number of border-strip tableaux using a bound by S. Fomin & N. Lulov [FL97].

Our main motivation is an implication for the invariant theory of the general linear group, as we now explain. Let $\mathfrak{gl}_r$ be the adjoint representation of $\text{GL}_r$, and
consider its \( n \)th tensor power \( \mathfrak{gl}_r^\otimes n \). The symmetric group \( \mathfrak{S}_n \) acts on this space by permuting tensor positions. Thus, using Schur–Weyl duality, we can determine the subspace of \( \text{GL}_n \)-invariants of \( \mathfrak{gl}_r^\otimes n \), regarded as a representation of \( \mathfrak{S}_n \). It turns out to be isomorphic to

\[
\bigoplus_{\lambda \vdash n \atop \ell(\lambda) \leq r} S_{\lambda} \otimes S_{\lambda},
\]

where the direct sum is over all partitions of \( n \) into at most \( r \) parts, and \( S_{\lambda} \) is the irreducible representation of \( \mathfrak{S}_n \) corresponding to \( \lambda \). In particular, for \( r \geq n \), the dimension of the space of invariants equals the size of \( \mathfrak{S}_n \).

A fundamental question of invariant theory is to find an explicit basis of the space of invariants, and if possible enjoying further desirable properties. One such property is invariance under rotation of tensor positions, following G. Kuperberg’s idea of web bases \([\text{Kup96}]\).

An elegant and useful solution would be to describe a set of permutations in \( \mathfrak{S}_n \) and a bijection from these to the basis elements which intertwines rotation of permutations (that is, conjugation with the long cycle) and rotation of tensor positions. It would be even nicer if the set of permutations for the invariants of \( \mathfrak{gl}_r^\otimes n \) would be a subset of the set of permutations for the invariants of \( \mathfrak{gl}_r^\otimes (n+1) \).

Although it appears to be difficult to exhibit such an intertwining bijection explicitly, our results, combined with previous work of S. Pfannerer, M. Rubey & B. Westbury \([\text{PRW20}]\), implies that such a solution must exist, see Theorem 58.

The existence of such an intertwining bijection is closely related to the existence of a rotation invariant statistic \( st \) mapping permutations to partitions, such that \( |\{\sigma \in \mathfrak{S}_n : st(\sigma) = \lambda\}| = |\text{SYT}(\lambda) \times \text{SYT}(\lambda)| \), see Corollary 53.

1.1. Outline of the paper. In Section 2 we recall the definition of the cyclic sieving phenomenon and establish the connection with characters of cyclic group actions. In Sections 3 to 6 we generalize T. Springer’s theorem to skew shapes and show, using the Murnaghan–Nakayma rule, the abacus of G. James & A. Kerber and the Littlewood map, that the character evaluation of a skew character is, up to sign, equal to a certain number of border-strip tableaux. We stress that these identities are known for the straight shape case. However, they are somewhat underappreciated gems which deserve more attention.

In Section 7 we provide the crucial bound on the number of border-strip tableaux of given shape, building on the approximation of S. Fomin & N. Lulov. In Section 8 we use this bound and the characterization of P. Alexandersson & N. Amini to prove the existence of the group actions announced above for skew standard tableaux.

Finally in Section 9 we apply our results to permutations and the invariant theory of the adjoint representation of the general linear group.

2. Cyclic group actions and cyclic sieving

In this section we recall V. Reiner, D. Stanton & D. White’s cyclic sieving phenomenon, characters of cyclic group actions and a result of P. Alexandersson &
N. Amini characterizing characters arising from cyclic group actions. We also recall R. Brauer’s permutation lemma, which guarantees that two actions of the cyclic group which have the same character as linear representations are even isomorphic as group actions.

**Definition 1** ([RSW04]). Let $X$ be a finite set and let $\rho$ be a generator of an action of the cyclic group of order $n$ on $X$.

Given a polynomial $f(q) \in \mathbb{N}[q]$ we say that the triple $(X, \langle \rho \rangle, f(q))$ exhibits the **cyclic sieving phenomenon** if for all $d \in \mathbb{Z}$

$$\# \{x \in X : \rho^d \cdot x = x\} = f(\xi^d),$$

where $\xi$ is a primitive $n$th root of unity. In this case $f(q)$ is a **cyclic sieving polynomial** for the group action.

In particular, the cardinality of $X$ is given by $f(1)$. More generally, identifying the ring of characters of the cyclic group of order $n$ with $\mathbb{Z}[q] / (q^n - 1)$, the cyclic sieving polynomial $f(q)$ reduces to the character of the group action modulo $q^n - 1$.

The cyclic sieving phenomenon owes its name to the fact that, mysteriously often, there is a particularly nice cyclic sieving polynomial, for example a natural $q$-analogue of the counting formula for the cardinality of $X$ as a function of $n$.

Moreover, frequently the only known way to prove that a given $q$-analogue indeed is a cyclic sieving polynomial is by enumerating the number of fixed points of the group action and verifying that the evaluation of the polynomial yields the same number.

**Remark 2.** If $(X, \langle \rho \rangle, f(q))$ exhibits the cyclic sieving phenomenon, then so does $(X, \langle \rho^k \rangle, f(q))$ for any $k \in \mathbb{N}$. In this case also $(X^m, \langle \rho \rangle, f(q)^m)$ exhibits the cyclic sieving phenomenon for any $m \in \mathbb{N}$, where $\langle \rho \rangle$ acts on $X^m$ via $\rho \cdot (x_1, \ldots, x_m) = (\rho \cdot x_1, \ldots, \rho \cdot x_m)$.

Much attention has been given to prove cyclic sieving phenomena on certain sets of certain tableau objects. Most famously, B. Rhoades showed that SYT($a^b$) exhibits the cyclic sieving phenomenon with the group action of promotion and $f^\lambda(q)$ as cyclic sieving polynomial. There are now several alternative proofs of this result, notably [Pur13, PPR08] and [SW18]. For an overview of some of these approaches, see [Rhe12]

| Set                  | Group action | Statistic/$f(q)$ | Reference |
|----------------------|--------------|------------------|-----------|
| SYT($a^b$)           | Promotion    | maj              | Rho10a    |
| SYT((n − m, 1^m))    | Promotion†   | $[n-1]_q$       | BMS14     |
| SYT($\lambda$)       | Evacuation‡  | maj              | Ste96     |
| SSYT($a^b, k$)        | $k$-promotion| $q^{-\kappa(\lambda)}s_{a^b}(1, q, \ldots, q^{k-1})$ | Rho10a    |
| COF($n\lambda/n\mu$) | Cyclic shift | (variant of) maj | AU19      |

| Table 1. Summary of known cyclic sieving phenomena on tableau objects. †Note that the group action on hook shaped SYT has order $n-1$. ‡Evacuation is an involution, so the cyclic group has order two. |
It turns out that it is possible to determine whether a polynomial carries a cyclic group action of given order.

**Theorem 3** ([AA19, Thm. 2.7]). Let \( f(q) \in \mathbb{N}[q] \) and suppose that \( f(\xi^d) \in \mathbb{N} \) for all \( d \in \{1, \ldots, n\} \), where \( \xi \) is a primitive \( n \)th root of unity. Let \( X \) be any set of size \( f(1) \). Then there exists a cyclic group action \( \rho \) of order \( n \) such that \((X, \langle \rho \rangle, f(q))\) exhibits the cyclic sieving phenomenon if and only if for every \( k \mid n \),

\[
\sum_{d \mid k} \mu(k/d) f(\xi^d) \geq 0,
\]

where \( \mu \) is the number-theoretic Möbius function.

**Remark 4.** Except for its size, the nature of the set \( X \) is irrelevant in this theorem. Put differently, the theorem merely classifies the linear characters of the cyclic group of order \( n \) which are the character of a group action.

**Remark 5.** If \((X, \langle \rho \rangle, f(q))\) exhibits the cyclic sieving phenomenon, the expression

\[
\frac{1}{k} \sum_{d \mid k} \mu(k/d) f(\xi^d)
\]

is the number of orbits of size \( k \) of the group action. Therefore, the sum

\[
\sum_{d \mid k} \mu(k/d) f(\xi^d)
\]

must be nonnegative and divisible by \( k \). The condition that the sum is divisible by \( k \) follows from the hypothesis that \( f(q) \in \mathbb{N}[q] \) and \( f(\xi^d) \in \mathbb{N} \) for all \( d \in \{1, \ldots, n\} \), see [AA19, Lem. 2.5].

**Remark 6.** It may be the case that \( f(q) \in \mathbb{N}[q] \) evaluates to nonnegative integers at \( n \)th roots of unity, but is not a cyclic sieving polynomial. As an example (see [AA19, Ex. 2.10]), take \( f(q) = q^5 + 3q^3 + q + 10 \). At \( 6 \)th roots of unity, \( f(\xi^j) \) takes nonnegative integer values. However, for \( k = 3 \) we have \( \sum_{d \mid 3} \mu(k/d) f(\xi^d) = -3 \).

We conclude this section by recalling a fact that makes cyclic groups special. In general, two non-isomorphic group actions may have the same linear character. This is not the case for group actions of a cyclic group, as R. Brauer’s permutation lemma shows:

**Theorem 7** ([Bra41, Kov82]). Two cyclic group actions are isomorphic if and only if they are isomorphic as linear representation, that is, their characters coincide.

### 3. Some properties of skew Schur functions

In this section we recall some basic properties of skew Schur functions, the Littlewood–Richardson rule and fake degree polynomials.

Let \( \text{SYT}(\lambda/\mu) \) and \( \text{SSYT}(\lambda/\mu) \) denote the set of standard and, respectively, semi-standard Young tableaux of skew shape \( \lambda/\mu \). We refer to the books by I. G. Macdonald [Mac95] and R. Stanley [Sta01] for definitions. We use English notation in all our figures.
Given a skew shape $\lambda/\mu$ with $n$ boxes, the associated skew Schur function $s_{\lambda/\mu}$ is defined as

$$s_{\lambda/\mu}(x) := \sum_{T \in SSYT(\lambda/\mu)} \prod_{\square \in \lambda/\mu} x_T(\square).$$

(2)

This generalizes the ordinary Schur function $s_{\lambda} := s_{\lambda/\emptyset}$. It is well-known that \{s_{\lambda}\}_{\lambda}$, where $\lambda$ runs over all partitions, is a basis for the ring of symmetric functions. The power sum symmetric functions indexed by partitions are defined as

$$p_{\nu}(x) := p_{\nu_1}(x)p_{\nu_2}(x) \cdots p_{\nu_\ell}(x), \quad p_j(x) := x^j_1 + x^j_2 + \cdots.$$  

(3)

The skew characters $\chi^{\lambda/\mu}(\nu)$ of the symmetric group $S_n$ are then defined via

$$s_{\lambda/\mu}(x) = \sum_{\nu} \chi^{\lambda/\mu}(\nu) p_{\nu}(x) z_{\nu},$$

(4)

where the sum is over all partitions $\nu$ of the same size as $\lambda/\mu$, $p_{\nu}$ denotes a power sum symmetric function, $z_{\nu} = \prod m_j j^{m_j}$, and $m_j$ is the number of parts in $\lambda$ equal to $j$.

We shall also make use of the Littlewood–Richardson rule, (see for example [Mac95, Sta01]):

$$s_{\lambda/\mu}(x) = \sum_{\nu} c_{\lambda,\mu,\nu}^\lambda s_{\nu}(x)$$

(5)

where $c_{\lambda,\mu,\nu}^\lambda \in \mathbb{N}$ are the Littlewood–Richardson coefficients. Combining Equation (5) and Equation (4), we obtain the Littlewood–Richardson rule for characters,

$$\chi^{\lambda/\mu} = \sum_{\nu} c_{\lambda,\mu,\nu}^\lambda \chi^{\nu}.$$  

(6)

Although it is very difficult to determine whether a given Littlewood–Richardson coefficient vanishes, the following particular case is straightforward.

**Lemma 8.** Let $\lambda/\mu$ be a skew shape with $n$ boxes. Then $c_{\mu,(n)}^{\lambda} = c_{\mu,(1^n)}^{\lambda} = 0$ if and only if the skew shape $\lambda/\mu$ has some column with at least two boxes, and some row containing at least two boxes.

**Proof.** We shall first prove the statement

$$c_{\mu,(n)}^{\lambda} = 0 \iff \lambda/\mu \text{ has some column with at least two boxes.}$$

We expand both sides of the Littlewood–Richardson rule (5) in the monomial basis. The left hand side contains the monomial $x^n_1$ if and only if there is no column of $\lambda/\mu$ with at least two boxes. Since the only semi-standard Young tableau of straight shape that contains precisely $n$ times the letter 1 has shape $(n)$, the monomial $x^n_1$ appears in the right hand side if and only if $c_{\mu,(n)}^{\lambda} \neq 0$.

The statement concerning $c_{\mu,(1^n)}^{\lambda}$ follows by applying the involution $\omega$ on both sides of the Littlewood–Richardson rule (5), which yields

$$s_{\lambda'/\mu'}(x) = \sum_{\nu} c_{\lambda',\mu',\nu}^{\lambda'} s_{\nu}(x).$$

A similar argument as in the previous paragraph now finishes the proof.

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1With the property that $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$, see [Mac95, Sta01].
Alternatively, Lemma 8 can also be proved using the fact that $c_{\mu,\nu}^\lambda$ counts the number of standard Young tableaux of shape $\lambda/\mu$ which are jeu-de-taquin equivalent to some fixed standard Young tableau of shape $\nu$.

Note that jeu-de-taquin does not decrease the number of cells in a row, nor the number of cells in a column. Thus, if $\nu = (n)$ but $\lambda/\mu$ has two cells in a column, the Littlewood–Richardson coefficient $c_{\mu,\nu}^\lambda$ must vanish. Conversely, filling a shape $\lambda/\mu$ which does not have any two cells in the same column from left to right with the numbers 1 to $n$ yields a standard Young tableau which is jeu-de-taquin equivalent to the unique tableau of shape $(n)$.

Evidently, the same argument works for $\nu = (1^n)$.

Definition 9. Given a skew standard Young tableau $T$ with $n$ boxes, we say that $j \in [n-1]$ is a descent of $T$ if $j+1$ appear in a row with index strictly higher than that of $j$. We let the major index of $T$, denoted $\text{maj}(T)$, be the sum of the descents of $T$. The fake-degree polynomial $f_{\lambda/\mu}(q)$ associated with a skew Young diagram $\lambda/\mu$ is defined as the sum

$$f_{\lambda/\mu}(q) := \sum_{T \in \text{SYT}(\lambda/\mu)} q^{\text{maj}(T)}.$$ (7)

The following lemma relates the skew Schur functions and the fake-degree polynomials.

Lemma 10 ([Sta01, Prop. 7.19.11]). Let $\lambda/\mu$ be a skew shape with $n$ boxes. Then

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \frac{f_{\lambda/\mu}(q)}{(1-q)(1-q^2) \cdots (1-q^n)}.$$ (8)

4. Skew characters and their fake degrees

A result by T. Springer [Spr74] gives an expression for the evaluation of an irreducible character of the symmetric group at a power of the long cycle $(1, \ldots, n) \in S_n$. In this section, we extend this result to skew characters.

Proposition 11. Let $\lambda/\mu$ be a skew shape with $n$ boxes and let $\xi$ be a primitive $n^{th}$ root of unity. Then, for $n = dm$,

$$\chi_{\lambda/\mu}(m^d) = f_{\lambda/\mu}(\xi^d).$$ (8)

We shall first prove a more general result, from which we deduce (8). The following proposition was first proved explicitly by B. Sagan, J. Shareshian and M. Wachs [SSW11, Prop. 3.1]. However, as they note, it is already implicit in work of J. Désarménien [Dés83]. We include yet another, different, proof.

Proposition 12. Let $F(x)$ be a homogeneous symmetric function of degree $n$, such that

$$F(x) = \sum_{\nu \vdash n} \chi_{\lambda/\mu}(\nu) \frac{p_{\mu}(x)}{z_{\nu}}.$$
Let us furthermore introduce $f^F(q)$ as

$$f^F(q) := \prod_{j=1}^{n}(1 - q^j)\, F(1, q, q^2, \ldots).$$

Suppose that $f^F(q) \in \mathbb{Z}[q]$. Then, for a primitive $n^{th}$ root of unity $\xi$ and $n = dm$, we have $\chi^F((m^d)) = f^F(\xi^d)$.

Proof. We have that $p_k(1, q, q^2, \ldots) = (1 - q^k)^{-1}$. By definition,

$$f^F(q) = \left(\prod_{j=1}^{n}(1 - q^j)^{\ell(\nu)}\right) \sum_{\nu \vdash n} \frac{\chi^F(\nu)}{z_{\nu}} \prod_{k=1}^{d} \frac{1}{1 - q^{\nu_k}}.$$

Each summand on the right hand side approaches 0 as $q \to \xi^d$ unless $\nu = (m^d)$, as the first product has a zero with multiplicity $d$ at $q = \xi^d$. By only considering the terms involving $\nu = (m^d)$ in the sum and rearranging the expression slightly, we have

$$\lim_{q \to \xi^d} f^F(q) = \lim_{q \to \xi^d} \left(\prod_{k=1}^{d} \frac{1}{1 - q^{km^d}}\right) \prod_{j=1}^{n} (1 - q^j)^{\ell(\nu)} \frac{\chi^F((m^d))}{z_{(m^d)}} \prod_{k=1}^{d} \frac{1}{1 - q^{km^d}}.$$

The last product approaches $d!$ due to l'Hospital’s rule. After taking the limit and using $z_{(m^d)} = d!m^d$, we obtain

$$f^F(\xi^d) = \left(\frac{1}{m} \prod_{j=1}^{m-1} (1 - \xi^{jd})\right)^d \chi^F((m^d)).$$

From the fact that $x^n - 1 = \prod_{j=0}^{n-1}(x - \xi^j)$ we deduce that $\prod_{j=1}^{m-1} (1 - \xi^{jd}) = m$. This shows that $f^F(\xi^d) = \chi^F((m^d))$, which proves our claim.

Proof of Proposition 11. By Lemma 10, this is the special case of Proposition 12 with $F(x) = s_{\lambda/\mu}(x)$.

5. Skew border-strip tableaux and the abacus

In this section, we recall the definition of border-strip tableaux, abaci and the Littlewood map. We end with a generalization of a theorem of G. James and A. Kerber.

A border-strip (or ribbon or skew hook) is a connected non-empty skew Young diagram containing no $2 \times 2$-square of boxes, as in Figure 1. The height $\text{height}(B)$ of a border-strip $B$ is one less than the number of rows it spans.

![Figure 1. A border-strip of height 3.](image-url)
Let \( \lambda/\mu \) be a skew shape. The size of \( \lambda/\mu \) is its number of boxes, denoted \(|\lambda/\mu|\). Suppose that \( \nu = (\nu_1, \ldots, \nu_\ell) \) is a partition of \(|\lambda/\mu|\). A border-strip tableau of shape \( \lambda/\mu \) and type \( \nu \) is a partition \( B \) of the Young diagram of \( \lambda/\mu \), into labeled border-strips \( B_1, \ldots, B_\ell \) with the following properties:

- the border-strip \( B_j \) has label \( j \) and size \( \nu_j \),
- labeling all boxes in \( B_j \) with \( j \) results in a labeling of the diagram \( \lambda/\mu \) where labels in every row and every column are weakly increasing.

We let \( \text{BST}(\lambda/\mu, \nu) \) denote the set of all such border-strip tableaux. In particular, \( \text{BST}(\lambda/\mu, 1^n) \) may be identified with the set of standard Young tableaux of shape \( \lambda/\mu \). In the remainder of the paper, we shall only concern ourselves with border-strip tableaux where all strips have the same size. We let \( \text{BST}(\lambda/\mu, d) \) denote the set of border-strip tableaux where every strip have size \( d \).

The height of a border-strip tableau \( T \), or any tiling of a tableau with border-strips, is the sum of the heights of the border-strips in the partition. The content of a box is given by its column index minus its row index. Observe that in a border-strip, the lowest leftmost box has the smallest content. As a convention, the label of a strip is placed in the unique box with minimal content in the strip, as done in Figure 2.

![Figure 2](image)

**Figure 2.** Two border-strip tableaux in \( \text{BST}((9^2,6^3,4,1)/(2,1^3), 3) \). In each strip, the label has been placed in the unique box with minimal content.

In Equation (4), the skew characters \( \chi^{\lambda/\mu} \) were defined. The skew Murnaghan–Nakayama rule describes a way to compute these skew characters.

**Theorem 13** (Murnaghan–Nakayama, see [Sta01, Cor. 7.17.5]). The skew characters are given by the signed sum

\[
\chi^{\lambda/\mu}(\nu) = \sum_{B \in \text{BST}(\lambda/\mu, \nu)} (-1)^{\text{height}(B)}.
\]

**Definition 14.** Given a partition \( \lambda \) with \( \ell \) parts and an integer \( d \geq 1 \), we define an abacus with \( d \) runners as follows.

Let \( b_1 < b_2 < \cdots < b_\ell \) be the hook lengths of the boxes in the first column of \( \lambda \), see Figure 3. Then, proceeding row by row from left to right, label the positions on the runners of the abacus from 0 to \( b_\ell \), placing a bead at each of the positions labeled \( b_1, b_2, \ldots, b_\ell \).

We can now define the core and the quotient of a partition.
Definition 15. Let $\lambda$ be a partition and let $d \geq 1$ be an integer. Then the $d$-core of $\lambda$ is the partition corresponding to the $d$-abacus with all beads moved up as far as possible. The $d$-quotient of $\lambda$ is the $d$-tuple of partitions obtained by regarding each runner as an individual 1-abacus.

Next, we extend these definitions to skew shapes. To do so, we need the following observation.

Proposition 16. Let $\lambda$ be a partition. Suppose that $\mu$ is obtained from $\lambda$ by removing a border-strip of size $d$. Then the $d$-abacus corresponding to $\mu$ is obtained from the $d$-abacus corresponding to $\lambda$ by moving a bead up by one position along its runner.

Therefore, if $\lambda/\mu$ is a skew shape such that $\text{BST}(\lambda/\mu, d)$ is non-empty, the abacus corresponding to $\lambda$ can be obtained from the abacus corresponding to $\mu$ by moving down beads along their runners. Note that in this case the $d$-cores of $\lambda$ and $\mu$ coincide.

Definition 17. The $d$-quotient of a skew shape $\lambda/\mu$ is the $d$-tuple of skew shapes obtained by regarding each corresponding pair of runners in the pair of $d$-abaci for $\lambda$ and $\mu$ as a pair of 1-abaci.

Note. We use the slightly nonstandard convention fixing the number of beads to be equal to the number of parts $\ell(\lambda)$ of $\lambda$.

Finally, we recall the Littlewood map as described by, for example, G. James & A. Kerber [JK84, Ch. 2.7] or I. Pak [Pak00, fig. 2.6], where it is called the rim hook bijection.

Let $\text{SYTT}(\lambda/\mu, d)$ be the set of $d$-tuples $(T^1, T^2, \ldots, T^d)$ of tableaux with the following properties:

- the shape of $T^j$ is $\lambda^j/\mu^j$, the $j^{th}$ entry of the $d$-quotient of $\lambda/\mu$,
- for each tableau $T^j$, the box labels in rows and columns increase,
- each of the numbers $\{1, 2, \ldots, n/d\}$ appears in precisely one tableau.

Then the Littlewood map is the following bijection between $\text{BST}(\lambda/\mu, d)$ and $\text{SYTT}(\lambda/\mu, d)$. Let $B \in \text{BST}(\lambda/\mu, d)$. Let $c(x)$ denote the content of the unique box with minimal content in the $d$-strip labeled $x$, as shown in Figure 2. Then the boxes in $T^j$ are filled with the labels $x$ such that $c(x) \equiv j - \ell(\lambda) \mod d$. Furthermore, the relative position of the labels in $T^j$ is the same as in the border-strip tableau. In particular, two labeled boxes in $T^j$ differ in content by $k$ if and only if the corresponding labels differ in content by $dk$ in $B$.

The image of the two border-strip tableaux in Figure 2 are given by the triples in (10) and (11), respectively.

Example 18. The 3-cores of $\lambda = (9^2, 6^3, 4, 1)$ and $\mu = (2, 1^3)$ are both given by the partition (2). The 3-quotient of $\lambda$ is $(4, 3), (2), (2, 1^2)$ and the 3-quotient of $\mu$ is
Thus, the 3-quotient of $\lambda/\mu$ is $(4,3)/(1)$, $(2)$, $(2,1^2)$ and the two tuples

\[
\begin{array}{c|c}
2 & 5 \\
7 & 6 \\
\hline
1 & 4 \\
1 & 9
\end{array}
\quad \text{and} \quad
\begin{array}{c|c}
2 & 5 \\
6 & 7 \\
\hline
1 & 8 \\
9 & 4
\end{array}
\]  

are elements in $\text{SYTT}(\lambda/\mu, 3)$, corresponding to the two tableaux in Figure 2.

**Figure 3.** The hook lengths of the first column of the inner and the outer shape of the diagram in Figure 2, and the pair of abaci corresponding to the shape.

**Lemma 19.** Let $\lambda/\mu$ be a skew shape of size $n$ and let $k$ and $d$ be positive integers with $dk \mid n$. Suppose that $\text{BST}(\lambda/\mu, dk)$ is non-empty and let $(\nu^1/\kappa^1, \ldots, \nu^k/\kappa^k)$ be the skew $k$-quotient of $\lambda/\mu$. Then $d \mid \gcd(|\nu^1/\kappa^1|, \ldots, |\nu^k/\kappa^k|)$ and

\begin{equation}
|\text{BST}(\lambda/\mu, dk)| = \left( \frac{\sum_i |\nu^i/\kappa^i|/d}{|\nu^1/\kappa^1|/d, \ldots, |\nu^k/\kappa^k|/d} \right) \prod_{i=1}^k |\text{BST}(\nu^i/\kappa^i, d)|. 
\end{equation}

**Remark 20.** The case $d = 1$ for straight shapes is classical and can be found in [JK84, eq. (2.7.32)] or [FL97]. In this special case, the result also follows immediately from the Littlewood map.

**Proof.** We reduce the statement to the case $d = 1$. Let $\nu_i^i/\kappa_i^i, \ldots, \nu_i^d/\kappa_i^d$ be the $d$-quotient of $\nu^i/\kappa^i$, for $1 \leq i \leq k$. We first show that the collection of skew shapes $\nu_i^j/\kappa_i^j$ for $1 \leq i \leq k$ and $1 \leq j \leq d$ coincides with the collection of skew shapes in the $dk$-quotient for $\lambda/\mu$.

To do so, we show that the bead positions on the $(i + (j - 1)k)^{th}$ runner of the $dk$-abacus for $\lambda/\mu$ coincide with the bead positions on the $j^{th}$ runner of the $d$-abacus for the skew shape corresponding to the $i^{th}$ runner of the $k$-abacus for $\lambda/\mu$, for all $1 \leq i \leq k$ and $1 \leq j \leq d$. 

Suppose that the labeled beads on the abacus for $\lambda/\mu$ are $b_1 < b_2 < \cdots < b_\ell$. Recall our convention that the number of beads on the abacus equals the number of rows of the outer partition. Thus, the positions $\{\frac{b_m - (i-1)}{k} : b_m = i - 1 \ (\text{mod} \ k)\}$.

Then, on the one hand, the $j$th runner of the $d$-abacus for the skew shape corresponding to that runner has beads at positions $\left\{\frac{b_m - (i-1)k}{d} : b_m = i - 1 \ (\text{mod} \ k) \text{ and } \frac{b_m - (i-1)}{k} = j - 1 \ (\text{mod} \ d)\right\}$.

On the other hand, the $(i + (j-1)k)$th runner of the $dk$-abacus for $\lambda/\mu$ has beads at positions $\left\{\frac{b_m - (i+(j-1)k-1)}{dk} : b_m = i + (j-1)k - 1 \ (\text{mod} \ dk)\right\}$, which are indeed the same.

To conclude the argument, we compute

$$
\left(\frac{|\lambda/\mu|/dk}{|\nu^1/\kappa^1|/d, \ldots, |\nu^k/\kappa^k|/d}\right)^k \prod_{i=1}^d [\text{BST}(\nu^i/\kappa^i, d)]
$$

$$
= \left(\frac{|\lambda/\mu|/dk}{|\nu^1/\kappa^1|/d, \ldots, |\nu^k/\kappa^k|/d}\right)^k \prod_{i=1}^k \left(\frac{|\nu^i/\kappa^i|/d}{|\nu^{i,1}/\kappa^{i,1}, \ldots, |\nu^{i,d}/\kappa^{i,d}|}\right) \prod_{j=1}^d [\text{SYT}(\nu^{i,j}/\kappa^{i,j})]
$$

$$
= \left(\frac{|\lambda/\mu|/dk}{|\nu^1/\kappa^1,1, \ldots, |\nu^{k,d}/\kappa^{k,d}|}\right)^k \prod_{i=1}^k \prod_{j=1}^d [\text{SYT}(\nu^{i,j}/\kappa^{i,j})]
$$

$$
= [\text{BST}(\lambda/\mu, dk)].
$$

\[\square\]

**Example 21.** Consider the skew shape $\lambda/\mu = (9,7,4^2,3^2,1)/(3,2,1^2)$ of 24. The corresponding pair of 6-abaci is as follows. Note that the lower abacus, corresponding to the outer shape $\lambda$, is obtained from the upper abacus, corresponding to the inner shape $\mu$, by moving down beads along a runner:
On the other hand, its pair of 3-abaci is the following:

\[ \begin{array}{c}
0 & 3 & 6 \\
1 & 4 & 2 \\
0 & 3 & 6 & 9 & 12 & 15 \\
\nu & 1 & \kappa \\
(4^2)/(2) & \emptyset & (1^2) \\
\end{array} \]

Finally, as illustration of the proof, consider the pair of 2-abaci for \( \nu^1/\kappa^1 \), corresponding to the first runner in the 3-abacus above. Observe that the positions of the beads, and therefore also the corresponding skew shapes, are the same as on the first and the fourth runner of the 6-abacus:

\[ \begin{array}{c}
0 & 2 \\
0 & 2 & 4 \\
1 & 3 & 5 \\
\nu^1/\kappa^1 \\
(2) \\
\end{array} \]

As a corollary we obtain a useful characterization of shapes with precisely one border-strip tableau.

**Corollary 22.** Let \( \lambda/\mu \) be a skew shape of size \( n \) and let \( k \) be a positive integer with \( k \mid n \). Suppose that \( \text{BST}(\lambda/\mu, k) \) is non-empty.

Then \( \text{BST}(\lambda/\mu, k) \) contains precisely one element if and only if all skew shapes in the skew \( k \)-quotient of \( \lambda/\mu \) are empty, with one one exception, which is either a single row \( (n/k) \) or a single column \( (1^{n/k}) \).

In this case \( \lambda/\mu \) is a border-strip. In particular, \( \text{BST}(\lambda/\mu, dk) \) contains precisely one element for all \( d \mid \frac{n}{k} \).

**Proof.** Let \( (\nu^1/\kappa^1, \ldots, \nu^k/\kappa^k) \) be the skew \( k \)-quotient of \( \lambda/\mu \) and suppose that \( \text{BST}(\lambda/\mu, k) \) contains precisely one element. Then, applying Lemma 19 with \( d = 1 \) we obtain that \( |\text{BST}(\lambda/\mu, k)| = 1 \) if and only if \( |\text{BST}(\nu^i/\kappa^i, 1)| = 1 \) for all \( i \) and the multinomial coefficient in Equation (12) evaluates to 1.
Since BST($\nu^i/\kappa^i, 1$) is the set of standard Young tableaux of shape $\nu^i/\kappa^i$, we have that $|\text{BST}(\nu^i/\kappa^i, 1)| = 1$ if and only if $\nu^i/\kappa^i$ is a single row or a single column. Furthermore, the multinomial coefficient equals 1 if $|\nu^i/\kappa^i| = 0$ for all but one $i$.

Finally, suppose that BST($\lambda/\mu, k$) contains precisely one element. It follows immediately that $\lambda/\mu$ is connected. Let $\nu^i/\kappa^i$ be the unique non-empty element in the quotient. Then, by definition of the Littlewood map, the contents of the boxes with minimal content in $\lambda/\mu$ are all different, and all equivalent to $i \pmod{k}$. Therefore, $\lambda/\mu$ must be a border-strip.

This in turn implies that BST($\lambda/\mu, dk$) is non-empty for all $d | n/k$, and therefore contains precisely one element. $\square$

6. Skew characters and border-strip tableaux

In this section we show that, up to sign, the evaluation of a skew character at a $d^{th}$ power of a cycle equals the number of border strip tableaux with all strips having size $d$. The non-skew case follows from a result by D. White [Whi83] and via a different technique by G. James and A. Kerber [JK84, Eq. 2.7.36]. Our proof is slightly different and uses the techniques by I. Pak [Pak00].

**Definition 23.** Let $T = (T^1, \ldots, T^d) \in \text{SYTT}(\lambda/\mu, d)$. Suppose that, after swapping entries $i$ and $i + 1$ in $T$, the resulting tuple is still an element of $\text{SYTT}(\lambda/\mu, d)$. We refer to such a transposition as a *flip* on $T$.

**Example 24.** The two flips $(6, 7)$ and $(9, 10)$ send the tableau $(10)$ to the tableau $(11)$.

**Lemma 25.** All elements of $\text{SYTT}(\lambda/\mu, d)$ are connected via a sequence of flips.

This lemma is essentially [Pak00] Thm. 3.2, although it is stated in a slightly different way. We therefore include a proof using the framework in this paper.

**Proof.** Let $(\lambda^1/\mu^1, \ldots, \lambda^d/\mu^d)$ be the $d$-quotient of $\lambda/\mu$. Let us first describe the superstandard filling $S := (S^1, \ldots, S^d) \in \text{SYTT}(\lambda/\mu, d)$. We will then show that $S$ can be obtained from any other tableau by a sequence of flips.

The boxes in $S^1$ are labeled with the numbers $1, \ldots, |\lambda^1/\mu^1|$, the boxes in $S^2$ are labeled with numbers $|\lambda^1/\mu^1| + 1, \ldots, |\lambda^1/\mu^1| + |\lambda^2/\mu^2|$, and so forth. The entries in each tableau $S^i$ are then distributed in the lexicographically smallest fashion, when reading row by row from top to bottom.

It now suffices to prove that for arbitrary $T \in \text{SYTT}(\lambda/\mu, d)$, we can obtain $S$ from $T$ by a sequence of flips. We describe a sorting algorithm which rearranges the labels, starting with 1 and then continuing with 2, 3, \ldots so that these labels agree with the corresponding entries in $S$.

Suppose that, at some point during the procedure, all boxes in $T$ with labels at most $i - 1$ agree with $S$, but the box labeled $i$ in $S$ is labeled $j > i$ in $T$. We then first flip $j$ and $j - 1$, then $j - 1$ and $j - 2$, etc. until $i + 1$ and $i$ are flipped, at which point $i$ is in the correct spot. We show inductively that all these flips are possible to perform: by construction, the boxes above and to the left of the box labeled with $j$ in $T$ must have labels smaller than $i < j - 1$. Because all tableaux in $T$ are
standard, the boxes below and to the right of the box labeled with $j - 1$ contain labels strictly larger than $j$. Performing this algorithm in order for all $i = 1, 2, \ldots$ ensures that we eventually reach $S$ from $T$. □

We shall now discuss the action of flipping $i$ and $i + 1$ on a border-strip tableau. By definition, and since the Littlewood map is bijective, the image of a flip also corresponds to a valid border-strip tableau. There are two possible cases. Either, the strips labeled $i$ and $i + 1$ are disconnected (no box in the first strip is horizontally or vertically adjacent to a box in the second), in which case the labels are just interchanged. The second case — when the strips are connected as in (13) — is more interesting.

![Diagram](13)

By analyzing the Littlewood map (see [Pak00]) one can verify that there is at least one pair of boxes, one from each strip, with the same content. Furthermore, we know that both the strips have the same length $d$. From these observations together with the properties of the Littlewood map, one can show that the “outside strip” is moved to the “inside” of the second strip, as shown in (13). See also Figure 2, where strips labeled $(6, 7)$ and $(9, 10)$ have been flipped.

We are now ready to state the main property of flips.

**Lemma 26** ([Pak00] Lem. 4.1). Suppose $B$ and $B'$ in BST($\lambda/\mu, d$) are related by a flip. Then $(−1)^{\text{height}(B)} = (−1)^{\text{height}(B')}$. 

**Proof.** Since the height is the sum of the heights of the individual strips, it suffices to consider the strips labeled $i$ and $i + 1$. If these strips are disconnected, then the flip does not alter the height of the resulting border-strip tableau. It remains to prove the assertion when the strips are connected as in (13).

The two strips form a skew Young tableau, which can be divided into three parts: a (non-empty) middle part which consists of all pairs of boxes $(b_1, b_2)$ whose contents are equal, a part to the left of this middle part and a part to the right of this middle part. A flip fixes the left and right part while it swaps the boxes in the middle part. Hence, the strips either both increase or both decrease in height by 1 and the lemma follows. □

Together with Proposition 11 we can now deduce the main result of this section.

**Corollary 27.** Let $\lambda/\mu$ be a skew shape of size $n = dm$. Then the signed sum

$$\chi^{\lambda/\mu}((m^d)) = \sum_{B \in \text{BST}(\lambda/\mu, m)} (−1)^{\text{height}(B)}$$

in the Murnaghan–Nakayama rule Theorem 13 is cancellation-free. In particular, we have that

$$f^{\lambda/\mu}(\xi^d) = \chi^{\lambda/\mu}((m^d)) = \varepsilon|\text{BST}(\lambda/\mu, m)|,$$

where $\xi$ is a primitive $n$th root of unity and $\varepsilon = (−1)^{\text{height}(B)}$ for any $B \in \text{BST}(\lambda/\mu, m)$. 
Observe that the non-skew case of the above corollary was proved already in [JK84, Thm. 2.7.27] using the abacus model.

7. Bounds on the number of border-strip tableaux

The goal of this section is to prove the following theorem.

**Theorem 28.** Let $\lambda/\mu$ be a skew shape with $n$ boxes and let $k$ be a positive integer with $k \mid n$. Suppose that $|\text{BST}(\lambda/\mu, k)| \geq 2$. Then

$$|\text{BST}(\lambda/\mu, k)| \geq \sum_{d \mid \frac{n}{k}, d > 1} |\text{BST}(\lambda/\mu, dk)|.$$ 

Additionally, the inequality holds if $n/k$ is a prime number.

**Example 29.** For $\lambda = (10,1^2) \vdash 12$ we have $|\text{BST}(\lambda, 3)| = 1$ and $|\text{BST}(\lambda, 6)| = |\text{BST}(\lambda, 12)| = 2$. By contrast, for $\lambda = (9,1^2) \vdash 12$, we have $|\text{BST}(\lambda, 1)| = 165$, $|\text{BST}(\lambda, 2)| = 5$, $|\text{BST}(\lambda, 3)| = 3$ and $|\text{BST}(\lambda, 4)| = |\text{BST}(\lambda, 6)| = |\text{BST}(\lambda, 12)| = 1$, and therefore

$$|\text{BST}(\lambda, k)| \geq \sum_{d \mid \frac{n}{k}, d > 1} |\text{BST}(\lambda, dk)|$$

for all $k$.

**Remark 30.** For $k = 1$, apart from the single row and single column partitions, there are only three shapes $\lambda/\mu$ where equality is attained: $(2^2)$, $(3^2)$ and $(2^3)$. Other than that, the minimal difference between the two sides of the inequality is attained for hooks of the form $(n-1,1)$. In this case it equals $n - \tau(n)$, where $\tau(n)$ is the number of divisors of $n$.

Our strategy is to reduce the theorem to the case of straight shapes and $k = 1$, which we prove in Section 7.2, employing a bound due to S. Fomin & N. Lulov.

In Section 7.3 we extend this to the case of skew shapes and $k = 1$, essentially using the Littlewood–Richardson rule.

Finally, in Section 7.4 we deduce the general case from the inequality with $k = 1$, using a bound on the quotient of a multinomial coefficient and a multinomial coefficient with stretched entries proved in Section 7.1 and Lemma 19 and Corollary 22.

7.1. Bounds on multinomial coefficients. We first prove two inequalities related to multinomial coefficients. For this we use the approximation due to H. Robbins.

**Proposition 31 ([Rob53]).** For any positive integer $n$,

$$n! = \sqrt{2\pi n} n^{n+1/2} e^{-n+r_n} \quad \text{for some} \quad \frac{1}{12n+1} < r_n < \frac{1}{12n}. \quad (15)$$

**Lemma 32.** For any positive integer $d$ and positive integers $m_1, \ldots, m_k$ summing to $m$,

$$\frac{\binom{dm}{m_{d_1}, \ldots, m_{d_k}}}{\binom{m}{m_1, \ldots, m_k}} \geq \frac{1}{d^{(k-1)/2}} \left( \frac{m^m}{\prod_{j=1}^k m_j^{m_j}} \right)^{d-1}. \quad (16)$$
Proof. For \(d = 1\) or \(k = 1\), the statement is trivial, so we can assume \(d > 1\) and \(k > 1\).

The left hand side of (16) is equal to
\[
\frac{(dm)! \prod_j (m_j)!}{m!},
\]
which by (15) is then larger than or equal to
\[
\frac{m^{(d-1)m}}{d(k-1)/2 \prod_i m_i^{(d-1)m_i}} \exp \left( \frac{1}{12d} - \frac{1}{12} \sum_j \frac{1}{m_j} - \frac{1}{12d} \right).
\]
It remains to show that, for \(\varepsilon = 1/12\),
\[
\frac{1}{dm} + \varepsilon + \sum_j \frac{1}{m_j} + \varepsilon \geq \frac{1}{m} + \sum_j \frac{1}{dm_j},
\]
provided \(k > 1\) and \(d > 1\). Set \(M = m \sum_j \frac{1}{m_j}\) and notice that \(M \geq k \geq 2\). Furthermore, \(m \sum_j \frac{1}{m_j + \varepsilon} \geq m \sum_j \frac{1}{m_j} = \frac{1}{1+\varepsilon} M\) and \(\frac{dm}{dm + \varepsilon} \geq \frac{1}{1+\varepsilon}\). Thus, it suffices to prove that, for \(d \geq 2\), \(M \geq 2\) and \(\varepsilon = 1/12\),
\[
\frac{1}{1+\varepsilon} (1 + dM) \geq d + M.
\]
This can be seen, for example, by replacing \(d\) with \(2 + \tilde{d}\) and \(M\) with \(2 + \tilde{M}\).

**Corollary 33.** For any integer \(d > 1\) and \(k > 1\) positive integers \(m_1, \ldots, m_k\) summing to \(m\),
\[
\frac{(dm_1, \ldots, dm_k)}{(m_1, \ldots, m_k)} \geq \prod_{j=1}^k (\tau(dm_j) - 1),
\]
where \(\tau(n)\) is the number of divisors of \(n\).

Proof. We use the inequality \(n/2 \geq \tau(n) - 1\), valid for \(n \geq 1\). By Lemma 32 it is then sufficient to show that
\[
\frac{1}{d^{(k-1)/2}} \left( \frac{m^m}{\prod_j m_j^{m_j}} \right)^{d-1} > \left( \frac{d}{2} \right)^k \prod_j m_j,
\]
or, equivalently,
\[
\left( \prod_{j=1}^k \frac{1}{m_j} \right) \left( \frac{m^m}{\prod_j m_j^{m_j}} \right)^{d-1} > \frac{1}{\sqrt{d}} \left( \frac{d}{2} \right)^k d^{k/2}.
\]
We will show the stronger inequality
\[
\left( \frac{m^m}{\prod_j m_j^{m_j+1}} \right)^{1/d} > \left( \frac{d\sqrt{d}}{2} \right)^{1/d}. \quad (*)
\]
It is not hard to see that the right hand side of this inequality, as a real function of \(d > 1\), attains its maximum between 3 and 4 and is unimodal. By direct inspection
we see that for integral \( d \) the maximum of the right hand side is attained at \( d = 3 \), where it is \( (27/4)^{1/4} \approx 1.61 \).

To find the minimum of the left hand side of (1), we consider the function
\[
h(z) = \frac{(z + \tilde{m})^{z+\tilde{m}}}{z^{z+1}}, \quad \text{where} \quad z \geq 1.
\]
For \( \tilde{m} = 1 \) we have \( h(z) = (1 + 1/z)^{z+1} \), which is strictly decreasing towards Euler’s number \( e \) as \( z \) increases. For \( \tilde{m} \geq 2 \) we show that \( h \) is strictly increasing. Indeed, the derivative of \( \ln h(z) \) equals
\[
\ln \left( 1 + \frac{\tilde{m}}{z} \right) - \frac{1}{z}.
\]
This expression is positive for \( \tilde{m} \geq 2 \) and \( z \geq 1 \), since we have
\[
\exp \left( \frac{1}{z} \right) = 1 + \frac{1}{z} \left( 1 + \sum_{k \geq 2} \frac{1}{k!} \frac{1}{z^{k-1}} \right)
\leq 1 + \frac{1}{z} \left( 1 + \sum_{k \geq 2} \frac{1}{k!} \right) = 1 + \frac{e - 1}{z} < 1 + \frac{\tilde{m}}{z}.
\]

For \( k = 2 \) and \( m_2 = 1 \), the left hand side of (1) equals \( \sqrt{h(m_1)} \) with \( \tilde{m} = 1 \). It is thus strictly larger than \( \sqrt{e} \approx 1.64 \), which in turn is larger than \( (27/4)^{1/4} \), the maximum of the right hand side of (1).

For \( k = 2 \) and \( m_2 > 1 \) and for \( k \geq 3 \) the analysis of \( h \) implies that the left hand side of (1) is strictly increasing in each of the variables \( m_i, 1 \leq i \leq k \), because it equals
\[
\left( \frac{h(m_i)}{\prod_{j \neq i} m_j^{m_j + 1}} \right)^{1/k},
\]
with \( \tilde{m} = \sum_{j \neq i} m_j \). For \( k = 2 \) and \( m_2 > 1 \), this expression is minimized at \( m_1 = 1 \), where it is larger than \( \sqrt{e} \) as shown above. For \( k > 2 \) the minimum is attained at \( m_1 = \cdots = m_k = 1 \), and is equal to \( k \). \( \square \)

7.2. **The bound for standard Young tableaux of straight shape.** The goal of this subsection is to prove the special case of Theorem 28 where \( \mu = \emptyset \) and \( k = 1 \). Note that we have the equivalence
\[
|\text{BST}(\lambda, 1)| \geq \sum_{d \mid n, d > 1} |\text{BST}(\lambda, d)| \iff \sum_{d \mid n} \frac{|\text{BST}(\lambda, d)|}{|\text{BST}(\lambda, 1)|} \leq 2. \quad (*)
\]
For the remainder of this subsection we focus on proving the latter inequality for \( \lambda \notin \{(n), (1^n)\} \). We shall first make use the following theorem by S. Fomin and N. Lulov.

**Theorem 34** [FL97]. For any partition \( \lambda \vdash n \), we have
\[
|\text{BST}(\lambda, d)| \leq Q(n, d) \cdot |\text{BST}(\lambda, 1)|^{1/d} \quad \text{where} \quad Q(n, d) := \sqrt[d]{\frac{d^n}{n! \cdots n!}}. \quad (17)
\]
We introduce the auxiliary function $B_n(x)$ as

$$B_n(x) := \sum_{d|n} Q(n, d)x^\frac{1}{d} - 1. \tag{18}$$

By plugging $x = |\text{BST}(\lambda, 1)|$ into (18), and using (17), we have that

$$B_n(|\text{BST}(\lambda, 1)|) = \sum_{d|n} Q(n, d)|\text{BST}(\lambda, 1)|^\frac{1}{d} - 1$$

$$= \sum_{d|n} Q(n, d)|\text{BST}(\lambda, 1)|^\frac{1}{d}$$

$$\geq \sum_{d|n} |\text{BST}(\lambda, d)| \frac{|\text{BST}(\lambda, 1)|}{|\text{BST}(\lambda, 1)|}.$$

Hence, if we can show that $B_n(x) \leq 2$ for suitable values of $x$ and $n$ we obtain the second inequality in (18).

**Lemma 35.** The inequality

$$\sum_{d|n} |\text{BST}(\lambda, d)| \frac{|\text{BST}(\lambda, 1)|}{|\text{BST}(\lambda, 1)|} \leq 2$$

holds for all partitions $\lambda \vdash n$ other than $(n)$ and $(1^n)$ with $n$ composite.

**Proof.** We consider several separate cases. The cases when $n$ is a prime number are trivial.

**Case** $\lambda = (n - 1, 1)$ or $\lambda = (2, 1^{n-1})$. In this case,

$$|\text{BST}(\lambda, d)| = \begin{cases} n - 1 & \text{if } d = 1 \\ 1 & \text{otherwise,} \end{cases}$$

and we observe that $n - 1 \geq \tau(n) - 1$, where $\tau(n)$ is the number of divisors of $n$.

**Case** $|\lambda| \leq 8$. The remaining 14 partitions (and their conjugates) not covered previously can be verified by hand.

**Case** $|\lambda| \geq 9$. As we noted before, it suffices to show that $B_n(|\text{BST}(\lambda, 1)|) \leq 2$. It then suffices to prove the following three properties, whenever $n \geq 9$:

- the function $x \mapsto B_n(x)$ is strictly decreasing for fixed $n$
- $B_n \left( \frac{n^2}{7} \right) \leq 2$ and
- $|\text{BST}(\lambda, 1)| \geq \frac{n^2}{7}$ for $\lambda \notin \{(n), (1^n), (n-1, 1), (2, 1^{n-2})\}$.

The first item is obvious from the definition of $B_n(x)$, as all exponents of $x$ are negative. The second item is proved later in Lemma 37. We proceed by verifying the third item. Suppose that $|\lambda| \geq 9$ and that $\lambda$ is not of the excluded shapes. We prove the statement by induction over $|\lambda|$: The computer verifies the base cases $|\lambda| \in \{9, 10\}$ easily. We now consider some exceptional shapes. If $\lambda$ is a hook of the form $(n - b, 1^b)$ with $2 \leq b \leq n - 3$, then $|\text{BST}(\lambda, 1)| = \binom{n-1}{b}$. Moreover, if $\lambda = (n - 2, 2) \text{ or } \lambda = (2^2, 1^{n-4})$ then $|\text{BST}(\lambda, 1)| = \frac{n(n-3)}{2}$. In both cases the inequality is true for $n \geq 9$. Now, if $\lambda$ is a rectangle and $n \geq 11$, we can
either remove two boxes from the last row, or the last column, to obtain \( \mu \) and \( \nu \), respectively. A simple bijective argument shows that

\[
|\text{BST}(\lambda, 1)| = |\text{BST}(\mu, 1)| + |\text{BST}(\nu, 1)|,
\]

and by induction, \( |\text{BST}(\lambda, 1)| \geq 2(n - 2)^2 / 3 \geq n^2 / 3 \) where the last inequality is true for \( n \geq 7 \). Note that \( \mu \) and \( \nu \) are not of the excluded shapes.

For the case when \( \lambda \) is not a rectangle or an exceptional shape, a similar argument works by removing two different corners of the shape \( \lambda \) (possible since the shape is not a rectangle) in order to obtain \( \mu \), \( \nu \). These smaller shapes are now either covered by the base cases or are hook shapes. They are not of the excluded shapes (this could only happen for the exceptional shapes) and we have

\[
|\text{BST}(\lambda, 1)| \geq |\text{BST}(\mu, 1)| + |\text{BST}(\nu, 1)|.
\]

A similar argument as above concludes the proof. \( \square \)

**Lemma 36.** For positive integers \( d \mid n \), we have

\[
Q(n, d) \leq \sqrt{n},
\]

where \( Q(n, d) = \left( \frac{d^n}{n!/(r(n/d))!} \right)^{1/d} \) is as in \((17)\).

**Proof.** First we prove \((19)\) for \( n \in \{1, 2\} \) by direct inspection. For \( n \geq 3 \) we use again Robbins’ approximation \((15)\) from Proposition 31. We have that

\[
Q(n, d) = \left( \frac{d^n}{n!/(r(n/d))!} \right)^{1/d} = \frac{d^{n/d}(n/d)!}{(n!)^{1/d}} = (2\pi n)^{-1/d} \left( 2\pi n^{-1/d} \right)^{1/d} \exp \left( r_{n/d} - \frac{r_n}{d} \right).
\]

By plugging this into \((19)\) and simplifying the expressions it suffices to show that

\[
\exp(2ndr_{n/d} - 2nr_n) \leq (2\pi n M_d)^n,
\]

where \( M_d := \left( \frac{d}{27} \right)^d \). By now using the approximations \( r_{n/d} < \frac{d}{12n} \) and \( r_n > 0 \) we have

\[
2ndr_{n/d} - 2nr_n \leq 2ndr_{n/d} \leq \frac{2nd^2}{12n} = \frac{d^2}{6}
\]

and thus the left hand side of \((20)\) can be bound with \( \exp(d^2/6) \).

**Claim:** For positive integers, \( n \geq 3 \), \( n \geq d \), we have

\[
\exp \left( \frac{d^2}{6} \right) \leq (2\pi n M_d)^n.
\]

**Case** \( d \geq 8 \). In this case we have \( \exp(1/6) < \frac{d}{2\pi} \), and therefore \( \exp(d/6) < M_d^d \leq M_d^n \leq (2\pi n M_d)^d \). This proves \((21)\).

**Case** \( 1 \leq d \leq 7 \). Direct inspection shows that \( 2\pi M_d \geq 2\pi M_2 = 2/\pi \), so for \( n \geq 2 \) we have \( 2\pi n M_d > 1 \) and therefore \( (2\pi n M_d)^n \) is strictly increasing in \( n \).

Finally, direct inspection now verifies that \((21)\) is already satisfied for \( n = 3 \) and each \( d \in \{1, \ldots, 7\} \). This proves the claim and concludes the proof. \( \square \)
Lemma 37. For integers \( n \geq 9 \) we have \( B_n \left( \frac{n^2}{3} \right) \leq 2 \).

Proof. We first verify the inequality for all \( n \) in the range \( 9 \leq n \leq 120 \). This can be done using a computer. For \( n \geq 121 \), we shall bound \( B_n(x) \) from above by \( g_n(x) \), defined as

\[
g_n(x) = 1 + \sqrt{x} + 2 \sum_{d|n,d>1} \frac{1}{d^{2/3}}.
\]

A simple computation shows that for \( n = 121 \), we have that \( g_n \left( \frac{n^2}{3} \right) \approx 1.999 \). Hence, if we can show that

- \( B_n(x) \leq g_n(x) \) for all \( x \geq 1 \) and
- \( n \mapsto g_n \left( \frac{n^2}{3} \right) \) is strictly decreasing,

then we are done. We proceed with the first point. Recall that the number of divisors of \( n \) is at most \( 2 \sqrt{n} \) for \( n \geq 1 \). We thus obtain

\[
B_n(x) = \sum_{d|n} Q(n,d) x^{\frac{1}{d}-1} = 1 + \sum_{d|n,d>1} Q(n,d) x^{\frac{1}{d}-1} \leq 1 + \sqrt{n} x^{\frac{1}{2}} \sum_{d|n,d>1} x^{\frac{1}{d}-1} \leq 1 + \sqrt{n} x^{\frac{1}{2}} + 2 \sum_{d|n,d>1} x^{\frac{1}{d}-1} = g_n(x).
\]

For the second point, it is enough to note that

\[
g_n(n^2/3) = 1 + \frac{\sqrt{3}}{\sqrt{n}} + \frac{2 \times 9^{1/3}}{\sqrt{n}}
\]

which is obviously decreasing in \( n \). \( \square \)

7.3. The bound for skew standard Young tableaux. We now extend the result of the previous section to skew shapes.

Lemma 38. Let \( \lambda/\mu \) be a skew shape with \( n \) boxes. Then

\[
|\text{BST}(\lambda/\mu, 1)| \geq \sum_{d|n,d>1} |\text{BST}(\lambda/\mu, d)|.
\]

if and only if \( \lambda/\mu \) is neither the partition \( (n) \) nor the partition \( (1^n) \).

Proof. We distinguish two cases:

Case 1: \( \lambda/\mu \) has at least two boxes in some row and in some column.
We prove the following sequence of inequalities:

\[ |\text{BST}(\lambda/\mu, 1)| = \sum_{\nu \vdash n} c_{\mu,\nu}^{\lambda} |\text{BST}(\nu, 1)| \]
\[ \geq \sum_{\nu \vdash n} c_{\mu,\nu}^{\lambda} \sum_{d|n,d>1} |\text{BST}(\nu, d)| \] \( (22) \)
\[ \geq \sum_{d|n,d>1} |\text{BST}(\lambda/\mu, d)|. \] \( (23) \)

We first prove inequality \( (22) \) for each summand separately. That is, we show that for all partitions \( \nu \vdash n \),

\[ c_{\mu,\nu}^{\lambda} |\text{BST}(\nu, 1)| \geq c_{\mu,\nu}^{\lambda} \sum_{d|n,d>1} |\text{BST}(\nu, d)|. \]

Since \( \lambda/\mu \) has at least two boxes in the same row and two boxes in the same column, we can apply Lemma 8. It follows that \( c_{\mu,(1^n)}^{\lambda} = c_{\mu,(n)}^{\lambda} = 0 \). Thus, the inequality holds for \( \nu = (1^n) \) and \( (n) \). For all other partitions \( \nu \vdash n \), the inequality follows from Lemma 35.

We now change the order of summation and prove inequality \( (23) \) again separately for each summand. That is, for fixed \( d > 1 \) with \( dm = n \), we show

\[ \sum_{\nu \vdash n} c_{\mu,\nu}^{\lambda} |\text{BST}(\nu, d)| \geq |\text{BST}(\lambda/\mu, d)|. \]

Indeed, we have

\[ \sum_{\nu \vdash n} c_{\mu,\nu}^{\lambda} |\text{BST}(\nu, d)| = \sum_{\nu \vdash n} c_{\mu,\nu}^{\lambda} |\chi^\nu((d^m))| \]
\[ \geq |\chi^{\lambda/\mu}((d^m))| \]
\[ = |\text{BST}(\lambda/\mu, d)|. \]

The two equalities follow from Corollary 27 whereas the inequality is obtained by taking absolute values on both sides of the Littlewood–Richardson rule for characters \( \chi \), evaluated at \( (d^m) \) and applying the triangle inequality.

**Case 2:** all columns or all rows of \( \lambda/\mu \) contain at most one box.

By symmetry we may assume that every connected component of \( \lambda/\mu \) is a single row. Let the lengths of these rows be \( n_1, n_2, \ldots, n_r \). We have \( \text{BST}(\lambda/\mu, d) = \emptyset \) unless all \( n_i \) are multiples of \( d \). In this case we find by explicit enumeration that

\[ |\text{BST}(\lambda/\mu, d)| = \binom{n/d}{n_1/d, n_2/d, \ldots, n_r/d}. \]

It then suffices to prove that

\[ \binom{n}{n_1, n_2, \ldots, n_r} \geq \sum_{d>1,d\mid \gcd(n_1, n_2, \ldots, n_r)} \binom{n/d}{n_1/d, n_2/d, \ldots, n_r/d}. \]

This inequality is an easy consequence of Corollary 33. \( \square \)
7.4. **The general case.** We are now ready to prove Theorem 28 itself.

**Proof of Theorem 28.** Let \((\nu^1/\kappa^1, \ldots, \nu^k/\kappa^k)\) be the skew \(k\)-quotient of \(\lambda/\mu\). We first establish the inequality
\[
(\tau(\nu^i/\kappa^i) - 1)|\text{BST}(\nu^i/\kappa^i, 1)| \geq \sum_{d \mid \nu^i/\kappa^i, d > 1} |\text{BST}(\nu^i/\kappa^i, d)|.
\]

If \(\nu^i/\kappa^i\) is neither the single row nor the single column partition, the bound for skew standard Young tableaux, Lemma 38, applies. Moreover, in this case \(|\nu^i/\kappa^i| \geq 3\) and therefore \(\tau(\nu^i/\kappa^i) - 1 \geq 1\). Otherwise, \(|\text{BST}(\nu^i/\kappa^i, d)| = 1\) for all \(d \mid |\nu^i/\kappa^i|\), and the inequality holds trivially.

Thus, setting \(g = \gcd(|\nu^1/\kappa^1|, \ldots, |\nu^k/\kappa^k|)\),
\[
\prod_{i=1}^{k} (\tau(\nu^i/\kappa^i) - 1)|\text{BST}(\nu^i/\kappa^i, 1)| \geq \prod_{i=1}^{k} \sum_{\nu^i/\kappa^i, d > 1} |\text{BST}(\nu^i/\kappa^i, d)|
\]
\[
\geq \sum_{g \mid d, d > 1} \prod_{i=1}^{k} |\text{BST}(\nu^i/\kappa^i, d)|
\]
\[
= \sum_{d \mid g, d > 1} \left[|\text{BST}(\lambda/\mu, dk)| \prod_{i=1}^{k} \frac{|\nu^i/\kappa^i|/d}{|\nu^1/\kappa^1|/d, \ldots, |\nu^k/\kappa^k|/d}\right].
\]

Note that, for any \(d \geq 1\), there is exactly one border-strip tableaux having empty shape: \(|\text{BST}(\emptyset, k)| = 1\). Suppose that \(g = \gcd(|\nu^1/\kappa^1|, \ldots, |\nu^k/\kappa^k|) > 1\) and there are at least two non-empty skew shapes among \(\nu^1/\kappa^1, \ldots, \nu^k/\kappa^k\). Then we can apply the above inequality and Corollary 33 and obtain
\[
|\text{BST}(\lambda/\mu, k)| = \left(\sum_{i=1}^{k} |\nu^i/\kappa^i| \prod_{i=1}^{k} |\text{BST}(\nu^i/\kappa^i, 1)|\right)^{-1}
\]
\[
\geq \left(\prod_{i=1}^{k} \frac{|\nu^i/\kappa^i|}{|\nu^1/\kappa^1|/d, \ldots, |\nu^k/\kappa^k|/d}\right)^{-1}
\]
\[
\geq \sum_{d \mid g, d > 1} |\text{BST}(\lambda/\mu, dk)|
\]
\[
= \sum_{d \mid g, d > 1} |\text{BST}(\lambda/\mu, dk)|.
\]

If \(g = \gcd(|\nu^1/\kappa^1|, \ldots, |\nu^k/\kappa^k|) = 1\), the inequality is trivially true.
If there is precisely one non-empty skew shape $\nu/\kappa$ among $\nu^1/\kappa^1, \ldots, \nu^k/\kappa^k$, we have $|\text{BST}(\lambda/\mu, dk)| = |\text{BST}(\nu/\kappa, d)|$ for all $d$ by Lemma 19.

If $\nu/\kappa$ is neither $(n/k)$ nor $(n/k)$, Lemma 38 applies and we have

$$|\text{BST}(\lambda/\mu, k)| = |\text{BST}(\nu/\kappa, 1)| \geq \sum_{d \mid n, d > 1} |\text{BST}(\nu/\kappa, d)| = \sum_{d \mid n, d > 1} |\text{BST}(\nu/\kappa, d)|.$$  

Otherwise, if $\nu/\kappa$ is either $(n/k)$ or $(n/k)$, Corollary 22 implies that there is only one element in $\text{BST}(\lambda/\mu, k)$. □

8. Cyclic sieving for skew standard tableaux

In this section we apply the bounds established in the previous section and Theorem 3 to prove the existence of several cyclic sieving phenomena for various families of skew standard Young tableaux.

Let us first put the bound from Theorem 28 into the form required to apply Theorem 3.

**Proposition 39.** Let $\lambda/\mu$ be a skew shape with $n$ boxes and let $m \in \mathbb{N}$. Then, for all $k \in \mathbb{N}$ with $k \mid n$,  

$$\sum_{d \mid \frac{n}{k}} \mu(d) |\text{BST}(\lambda/\mu, dk)|^m \geq 0,$$

or, equivalently,

$$\sum_{d \mid k} \mu(k/d) |f^{\lambda/\mu}(\xi^d)|^m \geq 0.$$

**Proof.** The equivalence of the two inequalities follows from Corollary 27 and replacing $d$ with $\frac{n}{dk}$ and $k$ with $\frac{n}{d}$. We prove the former inequality.

If $|\text{BST}(\lambda/\mu, k)| = 1$, we also have $|\text{BST}(\lambda/\mu, dk)| = 1$ for any $d \mid \frac{n}{k}$ by Corollary 22. Therefore,

$$\sum_{d \mid \frac{n}{k}} \mu(d) |\text{BST}(\lambda/\mu, dk)|^m = \sum_{d \mid \frac{n}{k}} \mu(d) = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases} \geq 0.$$  

This reasoning also covers the case $m = 0$.

Otherwise, since $\mu(1) = 1$ and $\mu(d) \geq -1$, we have  

$$\sum_{d \mid \frac{n}{k}} \mu(d) |\text{BST}(\lambda/\mu, dk)|^m \geq |\text{BST}(\lambda/\mu, k)|^m - \sum_{d \mid \frac{n}{k}, d > 1} |\text{BST}(\lambda/\mu, dk)|^m \geq 0,$$

where the final inequality follows from Theorem 28. □

**Remark 40.** One might think that $|f^{\lambda/\mu}(\xi^d)|$ could be the number of fixed points of a group action, despite the fact that $|f^{\lambda/\mu}(q)|$ is not a polynomial. However, this is not the case.

For example, consider $\lambda = (2, 1)$. Then $f^\lambda(q) = q + q^2$ and, for a 3rd root of unity $\xi$ we have $|f^\lambda(\xi^3)| = |\text{BST}(\lambda, 1)| = 2$ and $|f^\lambda(\xi)| = |\text{BST}(\lambda, 3)| = 1$, which
is incompatible with the possible orbit sizes of a group action on a set with two elements. Indeed, for \( k = 3 \) we obtain
\[
\frac{1}{3} \sum_{d|k} \mu(k/d) |f^\lambda(\xi^d)| = \frac{1}{3}(-1 + 2),
\]
which, by Remark 5, would have to be an integer.

Taking into account the previous remark, it makes sense to look for shapes \( \lambda/\mu \) such that the character \( f_{\lambda/\mu} \) evaluated at roots of unity is nonnegative.

**Proposition 41.** Let \( \lambda/\mu \) be a skew shape with \( n \) boxes and let \( m \in \mathbb{N} \). Then there is a group action \( \rho \) such that
\[
\left( \text{SYT}(\lambda/\mu) \times \cdots \times \text{SYT}(\lambda/\mu), \langle \rho \rangle, f_{\lambda/\mu}(q)^m \right)
\]
exhibits the cyclic sieving phenomenon if and only if \( m \) is even, or \( m \) is odd and for each \( k \in \mathbb{N} \) with \( k \mid n \) there is a tiling of \( \lambda/\mu \) of even height with strips of size \( k \).

**Remark 42.** The case \( m = 2 \) of this proposition does not extend to squares of arbitrary representations of the symmetric group. For example, consider the representation with character \( \chi^{(4)} + \chi^{(2,1^2)} \). Its fake degree polynomial is \( f(q) = 1 + q^4 + q^5 + q^6 \). Then we obtain, for a primitive fourth root of unity \( \xi \), that \( f(\xi)^2 = 4 \) and \( f(\xi^2)^2 = 0 \). This violates the condition in Theorem 3 for \( k = 2 \), because \( \mu(2)f(\xi)^2 + \mu(1)f(\xi^2)^2 = -4 \).

**Proof.** Let \( \xi \) be an \( n \)th primitive root of unity. Then Proposition 5 together with Theorem 3 ensures the existence of \( \rho \), provided \( f_{\lambda/\mu}(\xi^d)^m \) is nonnegative for all \( d \mid n \). Conversely, nonnegativity is a necessary condition because, given a group action \( \rho \), the number of fixed points of \( \rho^d \) equals \( f_{\lambda/\mu}(\xi^d)^m \).

It remains to consider the case of odd \( m \). By Corollary 27, \( f_{\lambda/\mu}(\xi^d) \), with \( d = \frac{n}{m} \), is nonnegative if and only if there is a tiling of \( \lambda/\mu \) of even height with strips of size \( k \).

**Corollary 43.** Let \( \lambda = (a, 1^{n-a}) \) be a hook-shaped partition of \( n \). Then there is a group action \( \rho \) such that \( (\text{SYT}(\lambda), \langle \rho \rangle, f^\lambda(q)) \) exhibits the cyclic sieving phenomenon if and only if \( n \) and \( a \) are odd and \( a - 1 \mod m \) is even for \( m \mid n \), \( 1 \leq m < a \).

**Proof.** Suppose that \( n \), and \( a \) are odd, and \( m \mid n \). In particular, \( m \) is odd, too. Note that there is a unique tiling of a hook with border-strips of size \( m \). We have to show that the height of this tiling is even if and only if \( a - 1 \mod m \) is even.

Recall that the height of a tile is one less than the number of rows it spans. If, and only if \( a - 1 \mod m \) is even, the height of the tile covering the top left corner of the shape must be even: this tile must cover an odd number of boxes in the first row and, since its size \( m \) is odd, an even number of boxes in the first column. Since the height of all other tiles is evidently even, too, so is the total height.

If the parity of \( n \) and \( a \) is different, then the tiling with a single strip of size \( n \) has height \( n - a \), which is odd. If both \( n \) and \( a \) are even, the tiling with two strips...
of size \( n/2 \) has odd height: if \( a \leq n/2 \), the height is \( n - a - 1 \), otherwise the height is \( n - 1 \).

**Remark 44.** According to the previous theorem, for \( \lambda = (3,1^{n-3}) \) a group action of order \( n \) with character \( f^\lambda(q) = q^{(n-2)(n-3)/2} \binom{n-1}{2} \binom{n-2}{2} \) exists for all odd \( n > 3 \). In this case, there should be one singleton orbit and \((n - 3)/2\) orbits of size \( n \). Indeed, an appropriate group action can be constructed as follows:

Identify a tableau with the two entries \( x < y \) different from 1 in the first row.

Note that \( y - x \in \{1, 2, \ldots, n - 2\} \), and only the pair \( (2, n) \) has difference \( n - 2 \).

We let the generator of the group action \( \eta \) act as follows:

\[
\eta(x, y) := \begin{cases} 
(2, n) & \text{if } x = 2, \; y = n, \\
(x + 2, y + 2) & \text{if } 2 \leq x < y \leq n - 2, \\
(2, x + 1) & \text{if } y = n - 1, \\
(3, x + 1) & \text{if } x > 2, \; y = n.
\end{cases}
\]

We then note that if \((u, v) = \eta(x, y)\), then \( v - u \in \{y - x, (n - 2) - (y - x)\} \). This explains why there are \((n - 3)/2\) orbits of length \( n \). We leave the remaining details to the reader.

**Remark 45.** It turns out that one can determine the number of border strips \( \lambda/\mu \) of size \( n \) which carry a group action of order \( n \) and character \( f^{\lambda/\mu}(q) \). This will appear in a separate note [Pfa20].

A different way to ensure positivity of the character \( f^{\lambda/\mu} \) is to decrease the order of the cyclic group as in Remark 2.

**Theorem 46.** Let \( \lambda/\mu \) be a skew shape such that every row contains a multiple of \( m \) boxes. Then there is a cyclic group action \( \rho \) of order \( m \) such that

\[
\left( \text{SYT}(\lambda/\mu), \langle \rho \rangle, f^{\lambda/\mu}(q) \right)
\]

exhibits the cyclic sieving phenomenon.

**Proof.** By Theorem 3 it suffices to show that for a primitive \( m^{th} \) root of unity \( \zeta \) and every \( k \mid m \)

\[
\sum_{d \mid k} \mu(k/d) f^{\lambda/\mu}(\zeta^d) \geq 0.
\]

Let \( |\lambda/\mu| = dm \). By Proposition 39 we have

\[
\sum_{d \mid k} \mu(k/d) |f^{\lambda/\mu}(\zeta^d)| \geq 0
\]

for an \( n^{th} \) root of unity \( \xi \) and every \( k \mid dm \). Let \( \zeta = \xi^{\frac{m}{d}} \). Then, by Corollary 27

\[
f^{\lambda/\mu}(\zeta^d) = f^{\lambda/\mu}(\xi^{d \frac{m}{d}}) = (-1)^{\text{height}(B)} |\text{BST} \left( \lambda/\mu, \frac{m}{d} \right)|.
\]

Since the length of every row of \( \lambda/\mu \) is a multiple of \( m \), there is a filling with border-strips of size \( \frac{m}{d} \mid m \), where every strip has height 0. \( \square \)
We remark that stretching shapes seems to be a fruitful way to construct cyclic sieving phenomena, as was previously shown with fillings related to Macdonald polynomials by P. Alexandersson & J. Uhlin \cite{AU19}. Possibly an adaptation of the approach presented here can be used to settle the following conjecture:

**Conjecture 47** (\cite[Conj. 3.4]{AA19}). There is an action $\beta$ on the set of semi-standard Young tableaux $\text{SSYT}(m\lambda/m\mu, k)$ of order $m$ such that

$$\left(\text{SSYT}(m\lambda/m\mu, k), \langle \beta \rangle, s_{m\lambda/m\mu}(1, q, q^2, \ldots, q^{k-1})\right)$$

events the cyclic sieving phenomenon.

For some shapes $\lambda/\mu$, the tiling may have odd height, but one can multiply $f_{\lambda/\mu}(q)$ with $q^{n/2}$, provided that the size $n$ of $\lambda/\mu$ is even, to obtain positivity at roots of unity. An important example is the case of rectangular shapes. In this case, B. Rhoades proved that promotion, together with a natural $q$-analogue of the hook length formula exhibits the cyclic sieving phenomenon. The following result is much weaker, because it only establishes the existence of a group action, but it is also much easier to prove, and illustrates the method.

**Theorem 48** (\cite{Rho10a}). Let $\lambda = a^b$ be a rectangular diagram with $n = ab$ boxes, and set $\kappa(\lambda) := \sum_j \binom{\lambda_j}{2}$. Then there is a group action $\partial$ of order $n$ such that

$$\left(\text{SYT}(\lambda), \langle \partial \rangle, q^{-\kappa(\lambda)} f^\lambda(q)\right)$$

events the cyclic sieving phenomenon.

**Proof.** It is a well-known result by R. Stanley \cite[Cor. 7.21.5]{Sta01}, that

$$q^{-\kappa(\lambda)} f^\lambda(q) = \frac{[n]_q!}{\prod_{\square \in \lambda} (h(\square))_q}$$

where $h(\square)$ is the hook-value of $\square$. In particular, $q^{-\kappa(\lambda)} f^\lambda(q)$ is a polynomial. We must check that this is nonnegative whenever $q$ is an $n$\textsuperscript{th} root of unity. Suppose that $m \mid n$, $n = dm$ and let $\xi$ be a primitive $n$\textsuperscript{th} root of unity. Corollary 27 implies that $f^\lambda(\xi^d)$ is non-zero only if and only if $\text{BST}(a^b, m)$ is non-empty. Using the abacus, one can show that $m \mid a$ or $m \mid b$ if and only if the $m$-core is empty, which, for straight shapes, is equivalent to $|\text{BST}(a^b, m)| > 0$. From here, it is a straightforward exercise to show that $\kappa(\lambda) = ba(a-1)/2$ and that $\xi^{-d \cdot ba(a-1)/2} f^\lambda(\xi^d)$ is nonnegative for all $d \mid n$.

Finally, Proposition 39 and Theorem 3 gives the result. \qed

9. Permutations and invariants of the adjoint representation of $GL_n$

In this section we apply our results to study the space of invariants of tensor powers of the adjoint representation $gl_n$ of the general linear group $GL_n$.

**Definition 49.** The rotation $\text{rot} \sigma$ of a permutation $\sigma \in S_n$ is the permutation obtained by conjugating with the long cycle $(1, \ldots, n)$.

**Remark 50.** Equivalently, if $M_\sigma$ is the permutation matrix corresponding to $\sigma$, then $M_{\text{rot} \sigma}$ is obtained by removing the first column of $M_\sigma$ and appending it on the right, and then removing the first column and appending it at the bottom.
Yet equivalently, let $D_{\sigma}$ be the chord diagram associated with $\sigma$, that is, the directed graph with vertices $\{1, \ldots, n\}$ arranged counterclockwise on a circle, and arcs $(i, \sigma(i))$. Then $D_{\text{rot } \sigma}$ is the chord diagram obtained by rotating the graph clockwise. See Figure 4 for an illustration.

$$\begin{bmatrix} 5,4,1,2,3 \end{bmatrix} \rightarrow \begin{bmatrix} 3,5,1,2,4 \end{bmatrix}$$

The following theorem makes the character of rotation explicit.

**Theorem 51** ([BRS08, Rho10b, RW14]).

$$\left( S_n, \langle \text{rot} \rangle, \sum_{\lambda \vdash n} f^\lambda(q)^2 \right)$$

exhibits the cyclic sieving phenomenon.

**Proof.** Consider the adjoint representation of $S_n$, that is, $S_n$ acting on itself by conjugation, or relabelling. It is well-known (see, e.g., [Sta01, Ex. 7.71a]) that the character of this representation equals $\sum_{\lambda \vdash n} \chi^\lambda \bar{\chi}^\lambda$. Since the restriction of the adjoint representation to the cyclic group generated by the long cycle $(1, \ldots, n)$ is precisely the action rot, the result follows from Proposition 11. \qed

**Definition 52.** Recall that the Robinson–Schensted correspondence provides a bijection

$$S_n \leftrightarrow \{(P, Q) \in \text{SYT}(\lambda) \times \text{SYT}(\lambda) : \lambda \vdash n\}.$$

The shape $\text{sh}(|)$ of a permutation $\sigma$ is the common shape of the standard Young tableaux $P$ and $Q$ corresponding to $\sigma$ under the Robinson–Schensted correspondence. We let $R_\lambda$ denote the set of permutations of shape $\lambda$.

We are now ready to prove the first major result of this section.
Corollary 53. Let $P_n$ be the set of partitions of $n$. Then there exists a map $st : S_n \to P_n$ which is invariant under rotation and equidistributed with the Robinson–Schensted shape. That is,

$$\text{st} \circ \text{rot} = \text{st} \quad \text{and} \quad \sum_{\sigma \in S_n} s_{\text{st}(\sigma)}(x) = \sum_{\sigma \in S_n} s_{\text{sh}(\sigma)}(x).$$

Moreover, with $\mathcal{S}_n^\lambda := \{ \pi \in S_n \mid \text{st}(\sigma) = \lambda \}$, the triple

$$(\mathcal{S}_n^\lambda, \langle \text{rot} \rangle, f^\lambda(q)^2)$$

exhibits the cyclic sieving phenomenon.

Remark 54. We stress that we are unable to present such a statistic explicitly.

Note that the distribution of the Robinson–Schensted shape over all permutations is essentially the Plancherel measure. Remarkably,

$$\sum_{\sigma \in S_n} s_{\text{sh}(\sigma)}(x)/f_{\text{sh}(\sigma)} = p_1^n(x).$$

Proof. By Proposition 41 there exists an action of the cyclic group of order $n$ on $R_\lambda$ with character $(f^\lambda(q))^2$. Let $\rho$ be the direct sum over all $\lambda \in P_n$ of these group actions. Thus, $(S_n, \langle \rho \rangle, \sum_{\lambda \vdash n} f^\lambda(q)^2)$ exhibits the cyclic sieving phenomenon. Since $\rho$ acts on each $R_\lambda$ separately, we have

$$\text{sh}(\rho \cdot \sigma) = \text{sh}(\sigma). \tag{26}$$

By Theorem 7 and Theorem 51 the action of $\rho$ and rotation are isomorphic. Therefore we have a bijection $\phi : S_n \to S_n$ with

$$\phi(\text{rot } \sigma) = \rho \cdot \phi(\sigma). \tag{27}$$

Defining $\text{st}(\sigma) := \text{sh}(\phi(\sigma))$, we obtain

$$\text{st}(\text{rot } \sigma) = \text{sh}(\phi(\text{rot } \sigma)) \overset{\tag{27}}{=} \text{sh}(\rho \cdot \phi(\sigma)) \overset{\tag{26}}{=} \text{sh}(\phi(\sigma)) = \text{st}(\sigma).$$

Finally we have

$$\mathcal{S}_n^\lambda = \phi^{-1}(R_\lambda),$$

yielding the last statement. \hfill \square

Remark 55. It is natural to ask whether for $\lambda \vdash n$ we have

$$\# \{ \sigma \in R_\lambda : \text{rot}^d(\sigma) = \sigma \} = f^\lambda(\xi^d)^2,$$

for all $d \in \mathbb{N}$ and a primitive $n^{th}$ root of unity $\xi$. In this case, the subset cyclic sieving technique of P. Alexandersson, S. Linusson & S. Potka [ALP19, Prop. 29] would imply the non-skew, $m = 2$ case of Proposition 11. However, this fails already for $\lambda = (2, 1)$; we have that $R_\lambda = \{132, 213, 231, 312\}$ and rot fixes 231 and 312, but $f^\lambda(q)^2 = q^2(1 + q)^2$ evaluates to 1 at $q = \exp(2\pi i/3)$.

We now turn to the connection with the invariants of tensor powers of the adjoint representation of $\text{GL}_r$, which is the original motivation for this article.
Let $V$ be an $r$-dimensional complex vector space and let $\mathfrak{gl}_r = \text{End}(V)$ be the adjoint representation $\text{GL}_r \to \text{End}(\mathfrak{gl}_r)$, $A \mapsto TAT^{-1}$. Recall that the space of $\text{GL}_r$-invariants of the $n$th tensor power of $\mathfrak{gl}_r$ is, as a representation of the symmetric group $S_n$, 

$$(\mathfrak{gl}_r^\otimes n)_{\text{GL}_r} = \text{Hom}_{\text{GL}_r} \left( \mathfrak{gl}_r^\otimes n, \mathbb{C} \right) \cong \text{Hom}_{\text{GL}_r} \left( (V \otimes V^*)^\otimes n, \mathbb{C} \right) \cong \text{End}_{\text{GL}_r}(V^\otimes n).$$

A basis for this space can be indexed by J. Stembridge’s alternating tableaux:

**Definition 56** ([Ste87]). A **staircase** is a dominant weight of $\text{GL}_r$, that is, a vector in $\mathbb{Z}^r$ with weakly decreasing entries. A $\mathfrak{gl}_r$-**alternating tableau** $\mathcal{A}$ of length $n$ (and weight zero) is a sequence of staircases 

$$\mathcal{A} = (\emptyset = \mu^0, \mu^1, \ldots, \mu^{2n} = \emptyset)$$

such that

- for even $i$, $\mu^{i+1}$ is obtained from $\mu^i$ by adding 1 to an entry, and
- for odd $i$, $\mu^{i+1}$ is obtained from $\mu^i$ by subtracting 1 from an entry.

The set of $\mathfrak{gl}_r$-alternating tableaux of length $n$ is denoted by $\mathcal{A}_n^{(r)}$.

B. Westbury defined a natural action, **promotion**, of the cyclic group of order $n$ on the set of so called invariant words of any finite crystal, in particular alternating tableaux of length $n$, generalizing Schützenberger’s promotion on rectangular standard Young tableaux. We refrain from giving a definition here and refer to S. Pfannerer, M. Rubey & B. Westbury [PRW20] instead.

For our purposes, it is enough to relate promotion to an action on the $\text{GL}_r$-invariants of the $n$th tensor power of $\mathfrak{gl}_r$. To do so, note that the symmetric group $S_n$ acts on $\mathfrak{gl}_r^\otimes n$ by permuting tensor positions, and therefore also on the space of invariants. It turns out that the action of the long cycle $(1, \ldots, n) \in S_n$ plays a special role:

**Theorem 57** ([Wes16, Sec. 6.3]). There is a basis of $(\mathfrak{gl}_r^\otimes n)^{\text{GL}_r}$ which is preserved by the action of the long cycle. Moreover, this action is isomorphic to the action of promotion on the set of alternating tableaux.

Note that B. Westbury’s theorem only asserts the existence of the basis, no explicit construction is known. The main result of this section is the following refinement of his assertion.

**Theorem 58.** Let $\mathfrak{S}_n^{(r)} := \{ \pi \in S_n \mid \ell(\text{st}(\pi)) \leq r \} = \bigcup_{\ell(\lambda) \leq r} \mathfrak{S}_n^{\lambda}$. Then there exists a bijection 

$$\mathcal{P} : \mathcal{A}_n^{(r)} \rightarrow \mathfrak{S}_n^{(r)} \quad \text{with} \quad \mathcal{P} \circ \text{pr} = \text{rot} \circ \mathcal{P}.$$ 

for $1 \leq r \leq n$.

**Remark 59.** In particular, there is an injection $\iota : \mathcal{A}_n^{(r)} \rightarrow \mathcal{A}_n^{(r+1)}$ such that $\text{pr} \circ \iota(\mathcal{A}) = \iota(\text{pr} \mathcal{A})$. This answers a question posed by S. Pfannerer, M. Rubey & B. Westbury [PRW20, rmk. 3.9].

Let us remark that for large dimension such a bijection is known:
Theorem 60 ([PRW20]). For \( r \geq n \), there is an (explicit) bijection
\[
\mathcal{P} : \mathcal{A}_n^{(r)} \rightarrow \mathfrak{S}_n \quad \text{with} \quad \mathcal{P} \circ \mathcal{P} = \text{rot} \circ \mathcal{P}.
\]

As a first step, we compute a decomposition of the space of invariants as a direct sum of tensor squares of Specht modules.

Lemma 61 ([RW14]). Let \( \mathfrak{gl}_r \) be the adjoint representation of \( \text{GL}_r \), and, given a partition \( \lambda \vdash n \), let \( S_{\lambda} \) be the corresponding irreducible representation of the symmetric group. Then there is an isomorphism
\[
(\mathfrak{gl}_r \otimes^n)_{\text{GL}_r} \cong \bigoplus_{\ell(\lambda) \leq r} S_{\lambda} \otimes S_{\lambda}.
\]

Proof. Let \( V \) be the vector representation of \( \text{GL}_r \). Schur–Weyl duality asserts that there is an isomorphism
\[
V^r \cong \bigoplus_{\ell(\lambda) \leq r} V_{\lambda} \otimes S_{\lambda},
\]
where \( V_{\lambda} \) is an irreducible representation of \( \text{GL}_r \).

Recall that, by Schur’s lemma, \( \text{Hom}_{\text{GL}_r}(V_{\lambda}, V_{\mu}) \) contains only the zero map if \( \lambda \neq \mu \), and all scalar multiples of the identity otherwise. Thus,
\[
(\mathfrak{gl}_r \otimes^n)_{\text{GL}_r} \cong \text{End}_{\text{GL}_r}(V^\otimes^n)
\cong \text{End}_{\text{GL}_r}( \bigoplus_{\ell(\lambda) \leq r} V_{\lambda} \otimes S_{\lambda})
\cong \bigoplus_{\ell(\lambda) \leq r} \text{End}(S_{\lambda})
\cong \bigoplus_{\ell(\lambda) \leq r} S_{\lambda} \otimes S_{\lambda}.
\]

Proof of Theorem 58. By Proposition [11] and Corollary [53] we obtain that the character of \( \mathfrak{S}_n^{\lambda} \) equals the character of \( \bigoplus_{\ell(\lambda) \leq r} S_{\lambda} \otimes S_{\lambda} \downarrow_{\langle (1,\ldots,n) \rangle} \). Summing over all partitions \( \lambda \) of length at most \( r \), we obtain the character of
\[
(\mathfrak{gl}_r \otimes^n)_{\text{GL}_r} \downarrow_{\langle (1,\ldots,n) \rangle} \cong \bigoplus_{\ell(\lambda) \leq r} S_{\lambda} \otimes S_{\lambda} \downarrow_{\langle (1,\ldots,n) \rangle},
\]
by Lemma [61]. Therefore, by Brauer’s permutation lemma (Theorem 7) and Westbury’s Theorem [57], the cyclic group actions
\[
(\mathcal{P}, \mathcal{A}_n^{(r)}) \cong (\text{rot}, \mathfrak{S}_n^{(r)})
\]
are isomorphic. \( \square \)
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