On Piercing Numbers of Families Satisfying the \((p, q)\)\textsubscript{r} Property

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Abstract

The Hadwiger-Debrunner number \(\text{HD}_d(p, q)\) is the minimal size of a piercing set that can always be guaranteed for a family of compact convex sets in \(\mathbb{R}^d\) that satisfies the \((p, q)\) property. Hadwiger and Debrunner showed that \(\text{HD}_d(p, q) \geq p - q + 1\) for all \(q\), and equality is attained for \(q \geq \frac{p - 1}{d} + p + 1\). Almost tight upper bounds for \(\text{HD}_d(p, q)\) for a 'sufficiently large' \(q\) were obtained recently using an enhancement of the celebrated Alon-Kleitman theorem, but no sharp upper bounds for a general \(q\) are known.

In \cite{MontejanoSoberon}, Montejano and Soberón defined a refinement of the \((p, q)\) property: \(\mathcal{F}\) satisfies the \((p, q)_r\) property if among any \(p\) elements of \(\mathcal{F}\), at least \(r\) of the \(q\)-tuples intersect. They showed that \(\text{HD}_d(p, q)_r \leq p - q + 1\) holds for all \(r > \frac{(p_r)}{(q_r)} - \frac{(p_r+1-r)}{(q_r+1-r)}\); however, this is far from being tight.

In this paper we present improved asymptotic upper bounds on \(\text{HD}_d(p, q)_r\), which hold when only a tiny portion of the \(q\)-tuples intersect. In particular, we show that for \(p, q\) sufficiently large, \(\text{HD}_d(p, q)_r \leq p - q + 1\) holds with \(r = \frac{1}{pq} (\frac{p}{q})\). Our bound misses the known lower bound for the same piercing number by a factor of less than \(pq^d\).

Our results use Kalai's Upper Bound Theorem for convex sets, along with the Hadwiger-Debrunner theorem and the recent improved upper bound on \(\text{HD}_d(p, q)\) mentioned above.

1 Introduction

Throughout this paper, \(\mathcal{F}\) denotes a finite family of compact convex sets in \(\mathbb{R}^d\), \(p, q \in \mathbb{N}\) satisfy \(p \geq q \geq d + 1\), and \(|\mathcal{F}| \geq p\). \(\mathcal{F}\) is said to satisfy the \((p, q)\) property if among any \(p\) elements of \(\mathcal{F}\) there is a \(q\)-tuple with a non-empty intersection. We say that \(\mathcal{F}\) is pierced by \(S \subset \mathbb{R}^d\) if any \(A \in \mathcal{F}\) satisfies \(A \cap S \neq \emptyset\). The smallest cardinality of a set that pierces \(\mathcal{F}\) is called the piercing number of \(\mathcal{F}\). We call \(\mathcal{F}\) \(t\)-degenerate if all elements of \(\mathcal{F}\) except at most \(t\) can be pierced by a single point. Otherwise, \(\mathcal{F}\) is called non-\(t\)-degenerate.

The classical Helly’s theorem asserts that if \(\mathcal{F}\) satisfies the \((d + 1, d + 1)\) property (namely, if any \(d + 1\) elements of \(\mathcal{F}\) have a non-empty intersection), then the piercing number of \(\mathcal{F}\) is \(1\).

In 1957, Hadwiger and Debrunner \cite{HadwigerDebrunner} considered a natural generalization of Helly’s theorem to \((p, q)\) properties. Let \(\text{HD}_d(p, q)\) be the maximum piercing number taken over all families \(\mathcal{F}\) of at least \(p\) compact convex sets in \(\mathbb{R}^d\) that satisfy the \((p, q)\) property. Is \(\text{HD}_d(p, q)\) necessarily bounded for all \(p \geq q \geq d + 1\)? (It is easy to see that \(\text{HD}_d(p, q)\) can be unbounded for \(q \leq d\).) Hadwiger and Debrunner showed that for all \(p \geq q \geq d + 1\) we have \(\text{HD}_d(p, q) \geq p - q + 1\), and that equality is attained for any \((p, q)\) such that \(q > \frac{p - 1}{d} + p + 1\) (and in particular, in \(\mathbb{R}^1\) equality is attained for all \(p \geq q \geq 2\)). In a celebrated result from 1992, Alon and Kleitman \cite{AlonKleitman} proved the Hadwiger-Debrunner conjecture, obtaining the upper bound \(\text{HD}_d(p, q) = O(p^{d^2 + d})\).

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However, as mentioned in [1], this bound is very far from being tight. The best currently known lower bound (implicitly implied by a result of Bukh et al. [3]), is $\text{HD}_d(p, q) = \Omega\left(\frac{p}{q} \log^{d-1} \frac{p}{q}\right)$.

Since the Alon-Kleitman theorem, several papers aimed at obtaining improved bounds on $\text{HD}_d(p, q)$ for various values of $(p, q, d)$. The most notable result of this kind is by Kleitman et al. [7], who showed that $\text{HD}_2(4, 3) \leq 13$ (compared to the upper bound of 345 obtained in [1]). Recently, it was shown in [6] that $\text{HD}_d(p, q) \leq p - q + 2$ for all $\varepsilon > 0$, $p \geq p_0(\varepsilon)$, and $q > p^{\frac{d-1}{d}} + \varepsilon$.

The best currently known upper bound that holds for all $q$, $\text{HD}_d(p, q) = \tilde{O}(p^{\frac{d-1}{d}})$ (also shown in [6]), is apparently far from being tight.

In an attempt to obtain improved bounds on $\text{HD}_d(p, q)$ by refining the $(p, q)$ property, Montejano and Soberón [9] introduced the following definition: A family $F$ is said to satisfy the $(p, q)_r$ property if among any $p$ elements of $F$, at least $r$ of the $q$-tuples intersect. $\text{HD}_d(p, q)_r$ is defined as the maximal piercing number taken over all families that satisfy the $(p, q)_r$ property. The main result of [9] is:

**Theorem 1.1 ([9]).** For any $d$,

$$\text{HD}_d(p, q)_r \leq p - q + 1 \quad (1)$$

holds for all $r > \left(\frac{p}{q}\right)^{(p+1-d)} \left(\frac{p}{q+1-d}\right)$.

The proof of Theorem 1.1 uses a nice geometric argument. As mentioned in [9], the upper bound of Theorem 1.1 is far from being tight. Moreover, the value of $r$ in the theorem is rather large — almost all the $\left(\frac{p}{q}\right)$ $q$-tuples are required to intersect.

In this paper we present improved upper bounds on $\text{HD}_d(p, q)_r$. For $p, q$ sufficiently large (as function of $d$), our bounds hold already when $r$ is a tiny fraction of $\left(\frac{p}{q}\right)$. Our main result is the following:

**Theorem 1.2.** $\text{HD}_d(p, q)_r$ satisfy:

1. For all $p \geq q \geq d + 1$ and $r \geq \Theta_d\left(\frac{(d+1-p)(\frac{p}{q})}{d}\right)$,

$$\text{HD}_d(p, q)_r \leq \min(p - q + 1, \frac{p}{d} > 1).$$

2. For any $\varepsilon > 0$, any $p \geq q \geq d + 1$ such that $p > p_0(\varepsilon)$ and all $r \geq \Theta_{d, \varepsilon}\left(\frac{p^{d-1} + q + 1}{(q-d)!}\right)$,

$$\text{HD}_d(p, q)_r \leq \min(p - q + 1, p - p^{\frac{d-1}{d}} + \varepsilon + 2).$$

Here, $\Theta_d(\cdot)$ hides a multiplicative factor that may depend on $d$.

The latter bound on $r$ is not far from being tight, as an explicit example presented in [9] (which we recall below) yields a lower bound of $r = \Omega\left(\frac{p^{d-1} + (q-1)!}{(q-d)!}\right)$ for the same piercing number. The upper and lower bounds differ by a multiplicative factor of $\frac{p^{d-1} + (q-1)!}{(q-d)!}$, which is smaller than $pq^{d-1}$ for all $\varepsilon \leq \frac{1}{q}$.

We note that for $p, q$ sufficiently large (as function of $d$), the condition in (1) is equivalent to $r \geq \left(\frac{p}{q}\right)^{\frac{1}{c}}$ for $c > 1$ that depends only on $d$, and the condition in (2) (for $\varepsilon = \frac{1}{q}$) is stronger than the condition $r \geq \left(\frac{p}{q}\right)^{\frac{1}{p^{d-1}}}$ stated in the abstract. This means that the assertion of Theorem 1.2 holds already when $r$ is an exponentially (in $q$) small fraction of $\left(\frac{p}{q}\right)$. 


The proof of Theorem 1.2 uses Kalai’s Upper Bound Theorem for convex sets [5], combined with the Hadwiger-Debrunner theorem and the recent improved upper bound on HD₃(p,q) obtained in [6].

In view of Theorem 1.2(1), it is natural to ask whether a smaller value of r is sufficient if we allow HD₃(p,q)ᵣ to be larger than \( \frac{p}{q} \) (but still smaller than \( p - q \)). We partially answer this question in the following generalization of Theorem 1.1.

**Theorem 1.3.** For any \( p \geq q \geq d + 1 \) and \( 0 \leq k \leq p - q - 1 \), denote by \( m₀(k) \) the smallest integer \( m \) such that \( \binom{m + 1}{2} \geq \frac{(p - q - k - 1)(p - q + k + 2)}{2} + 1 \). Let \( F \) be a non-\((p - q)\)-degenerate family of compact convex sets in \( \mathbb{R}^d \) that satisfies the \((p,q)_r\) property, with

\[
 r \geq \left( \binom{p}{q} - \binom{p - d + 1}{q - d + 1} \right) + 1 + \left( \binom{q - d - 2 + m₀(k)}{q - d + 1} \right) + \left( \binom{q - d - 1 + m₀(k)}{q - d + 1} \right).
\]

Then \( F \) can be pierced by at most \( k + 2 \) points.

Note that in the case \( k = p - q - 1 \), Theorem 1.3 reduces to Theorem 1.1. The proof of Theorem 1.3 uses a bootstrapping technique based on the technique pioneered by Montejano and Soberón in [9]. The added value of Theorem 1.3 over Theorem 1.2 is demonstrated well by its simplicity. We also show that the technique of Montejano and Soberón can be used to obtain an alternative proof of the Hadwiger-Debrunner theorem, which may be of independent interest due to its simplicity.

## 2 Proof of Theorem 1.2

As mentioned already, in the proof of Theorem 1.2, we use Kalai’s Upper Bound Theorem for convex sets [5], the Hadwiger-Debrunner theorem [4], and the recent upper bound on HD₃(p,q) obtained in [6]. We state these results first.

**Theorem 2.1 (5).** Let \( F \) be a family of \( p \) convex sets in \( \mathbb{R}^d \). Denote by \( f_{q-1} \) the number of \( q \)-tuples of sets in \( F \) whose intersection is non-empty. If \( f_{d+s} = 0 \) for some \( s \geq 0 \) then for any \( q > 0 \),

\[
f_{q-1} \leq \sum_{i=0}^{d} \binom{s}{q-i} \binom{p-s}{i}.
\]

**Theorem 2.2 (4).** For \( p \geq q \geq d + 1 \) such that \( q > \frac{d+1}{d}p + 1 \),

\[
\text{HD}_p(q,p) = p - q + 1.
\]

**Theorem 2.3 (6).** Let \( \varepsilon > 0 \). There exists \( p₀(\varepsilon,d) \) such that for any \( p \geq q \geq d + 1 \) with \( p \geq p₀ \) and \( q \geq p \frac{d+1}{d} + \varepsilon \), we have

\[
\text{HD}_q(p,q) \leq p - q + 2.
\]

The intuition behind the proof is simple. Theorems 2.2 and 2.3 yield a strong bound on the piercing number for a family that satisfies the \((p,q)\) property with a ‘large’ \( q \). In order to apply them, we need to ‘enlarge’ \( q \), and this is done using Theorem 2.1. Specifically, if some family \( F′ \) of \( p \) convex sets contains ‘many’ intersecting \( q \)-tuples, Theorem 2.1 allows to deduce that it also contains an intersecting \((q+k)\)-tuple, for an appropriate value of \( k \). This implies that if a family \( F \) satisfies the \((p,q)_r\) property, then it must satisfy the \((p,q+k)\) property, for \( k = k(r) \). Applying this with a sufficiently large \( r \), we replace \( q \) with a sufficiently large \( q+k \), and then apply an improved bound on the piercing number that follows from Theorem 2.2 or Theorem 2.3.
2.1 Proof of Theorem 1.2(1)

We need the following lemma:

**Lemma 2.4.** Let \( p \geq q \geq d + 1 \), and let \( 1 \leq f(p) \leq \frac{p}{d} - 1 \). If

\[
r \geq r_0 := \sum_{i=0}^{d} \left( \frac{p - f(p) - d}{q - i} \right) \left( \frac{f(p) + d}{i} \right) + 1,
\]

then \( \text{HD}_d(p, q)_r \leq f(p) \).

**Proof.** Let \( F \) be a family of compact convex sets in \( \mathbb{R}^d \) that satisfies the \((p, q)_r\) property for some \( r \geq r_0 \). Put \( k = p - q - f(p) + 1 \). Note that \( \frac{d-1}{d} p - q + 2 \leq k \leq p - q \). By the choice of \( r_0 \), Theorem 2.1 implies that \( F \) satisfies the \((p, q + k)_r\) property. As \( q + k \geq \frac{d-1}{d} p + 2 \), Theorem 2.2 implies that the piercing number of \( F \) is at most \( p - (q + k) + 1 = f(p) \), as asserted. \( \square \)

**Proof of Theorem 1.2(1).** First, note that if \( p - q + 1 \leq \frac{p}{d} - 1 \), then \( q > \frac{d+1}{d} p + 2 \), and thus Theorem 2.2 implies \( \text{HD}_d(p, q)_r \leq p - q + 1 \) even for \( r = 1 \). Hence, we may assume \( \frac{p}{d} - 1 \leq p - q + 1 \). Substituting \( f(p) = \frac{p}{d} - 1 \) into Lemma 2.4, we get \( \text{HD}_d(p, q)_r \leq f(p) = \frac{p}{d} - 1 \) for all

\[
r \geq \sum_{i=0}^{d} \left( \frac{p - \left( \frac{p}{d} - 1 \right) - d}{q - i} \right) \left( \frac{\left( \frac{p}{d} - 1 \right) + d}{i} \right) + 1
\]

\[
= \left( \frac{\frac{d+1}{d} p + 1 - d}{q} \right) + \left( \frac{\frac{d+1}{d} p + 1 - d}{q - 1} \right) \cdot \left( \frac{p}{d} + d - 1 \right) + \ldots + \left( \frac{\frac{d+1}{d} p + 1 - d}{q - d} \right) \cdot \left( \frac{\frac{p}{d} + d - 1}{d} \right)
\]

\[
= O_d \left( \left( \frac{\frac{d+1}{d} p}{q - d} \right) \left( \frac{\frac{p}{d}}{d} \right) \right),
\]

as asserted. \( \square \)

**Remark 2.5.** Note that Lemma 2.4 actually supplies a sequence of upper bounds on \( r \), which correspond to any desired piercing number between 1 and \( \frac{p}{d} - 1 \). Piercing numbers larger than \( \frac{p}{d} - 1 \) are treated in Section 5.

2.2 Proof of Theorem 1.2(2)

The proof of Theorem 1.2(2) is similar to the proof of Theorem 1.2(1), with Theorem 2.3 replacing Theorem 2.2.

**Proof of Theorem 1.2(2).** Let \( \varepsilon > 0 \), and let \( p > p_0(\varepsilon) \) where \( p_0(\varepsilon) \) is chosen to satisfy the hypothesis of Theorem 2.3.

First, consider the case \( p - q + 1 < p - p^{\frac{d-1}{d}+\varepsilon} + 1 \). Recall that by assumption, \( F \) satisfies the \((p, q)_r\) property with

\[
r \geq \Theta_d(\varepsilon) \left( \frac{p^{\left( \frac{d-1}{d}+\varepsilon \right) + 1}}{(q - d)!} \right),
\]

and thus, also with

\[
r = \sum_{i=0}^{d} \left( \frac{q - d}{q - i} \right) \left( \frac{p - q + d}{i} \right) + 1
\]

(actually, this is assured by taking the implicit factor in \( \Theta_d(\varepsilon) \) to be sufficiently large). By Theorem 2.1, the latter implies that \( F \) satisfies the \((p, q + 1)_r\) property. As in this case, \( q > p^{\frac{d-1}{d}+\varepsilon} \), Theorem 2.3 implies that \( F \) can be pierced with at most \( p - (q + 1) + 2 = p - q + 1 \) points, as asserted. Hence, we may assume \( p - p^{\frac{d-1}{d}+\varepsilon} + 1 \leq p - q + 1 \).
Let \( k = p^{\frac{d-1}{d} + \varepsilon} - q \). Since by assumption, \( \mathcal{F} \) satisfies the \((p, q)_r\) property with \( r \) that satisfies (3), in particular \( \mathcal{F} \) satisfies the \((p, q)_r\) property with
\[
r = \sum_{i=0}^{d} \left( p^{\frac{d-1}{d} + \varepsilon} - d - 1 \right) \left( p - p^{\frac{d-1}{d} + \varepsilon} + d + 1 \right) + 1
= \sum_{i=0}^{d} \left( q + k - d - 1 \right) \left( p - q - k + d + 1 \right) + 1.
\]

By Theorem 2.1, this implies that \( \mathcal{F} \) satisfies the \((p, q + k)\) property. As \( q + k = p^{\frac{d-1}{d} + \varepsilon} \), Theorem 2.3 implies that \( \mathcal{F} \) can be pierced by at most \( p - p^{\frac{d-1}{d} + \varepsilon} + 2 \) points. This completes the proof.

**Remark 2.6.** As in Section 2.1, a similar argument (using Theorem 2.3 instead of Theorem 2.2) shows that for any \( p > p_0(\varepsilon) \) and any \( 1 \leq f(p) \leq p^{\frac{d-1}{d} + \varepsilon} + 2 \), we have \( \text{HD}(p, q)_r \leq f(p) \) for all
\[
r > \sum_{i=0}^{d} \left( p - f(p) + 1 - d \right) \left( f(p) - 1 + d \right).
\]

The upper bound on \( r \) asserted in Theorem 1.2(2) is not far from being optimal, as demonstrated by the following example (presented in [9]).

**Example.** Let \( \mathcal{F} \) be a family composed of \( p - p^{\frac{d-1}{d} + \varepsilon} + 3 \) pairwise disjoint sets and \( p^{\frac{d-1}{d} + \varepsilon} - 3 \) copies of a convex set that contains all of them. An easy computation shows that \( \mathcal{F} \) satisfies the \((p, q)_r\) property for
\[
r = \left( p^{\frac{d-1}{d} + \varepsilon} - 3 \right) + (p - p^{\frac{d-1}{d} + \varepsilon} + 3) \cdot \left( p^{\frac{d-1}{d} + \varepsilon} - 3 \right) = \Theta \left( \frac{p^{\frac{d-1}{d} + \varepsilon}(q-1)+1}{(q-1)!} \right),
\]
while it clearly cannot be pierced by \( p - p^{\frac{d-1}{d} + \varepsilon} + 2 \) points.

A similar example, with \( p - f(p) + 2 \) instead of \( p^{\frac{d-1}{d} + \varepsilon} \), shows that the upper bound on \( r \) asserted in Remark 2.6 is also near tight.

Finally, we note that in dimension 1, the exact relation between the \((p, q)_r\) property and the piercing number can be obtained easily using the Upper Bound Theorem.

**Proposition 2.7.** For \( p \geq q \geq 2 \), let \( \mathcal{F} \) be a family of segments on the real line that satisfies the \((p, q)_r\) property. If
\[
r \geq \left( p - k - 2 \right) + (k + 2) \left( p - k - 2 \right) + 1,
\]
then \( \mathcal{F} \) can be pierced by \( k + 1 \) points. Conversely, there exists a family \( \mathcal{F}_0 \) that satisfies the \((p, q)_r\) property with \( r = \left( p - k - 2 \right) + (k + 2) \left( p - k - 2 \right) \) and cannot be pierced by \( k + 1 \) points.

**Proof.** By Theorem 2.1, if \( \mathcal{F} \) satisfies the \((p, q)_r\) property with \( r \) that satisfies (4), then \( \mathcal{F} \) satisfies the \((p, p - k)\) property. By Theorem 2.2 this implies that \( \mathcal{F} \) can be pierced by \( k + 1 \) points.

For the other direction, let \( \mathcal{F}_0 \) be a family that consists of \( k + 2 \) distinct single-point sets, and \( p - k - 2 \) copies of a segment that contains all the points. A straightforward computation shows that \( \mathcal{F}_0 \) satisfies the \((p, q)_r\) property with \( r = \left( p - k - 2 \right) + (k + 2) \left( p - k - 2 \right) \), but cannot be pierced by \( k + 1 \) points.

Proposition 2.7 will be useful for us in the next section.
3 Proof of Theorem 1.3

In the proof of Theorem 1.3 we use a bootstrapping based on the technique presented by Montejano and Soberón [9]. First we state a lemma of [9] on which we base our argument.

3.1 The technique of [9] and an alternative proof of the Hadwiger–Debrunner theorem

Lemma 3.1. For any family $F$ of convex sets in $\mathbb{R}^d$, there exist $A_1,A_2,\ldots,A_d \in F$ and a line $\ell$ such that if $C \in F$ intersects $\cap_{i \neq j} A_i$ for all $1 \leq j \leq d$ then $C \cap \ell \neq \emptyset$.

Since our argument is partially based on the proof of Lemma 3.1 presented in [9], we recall the proof below. In the general case of families in $\mathbb{R}^d$, the proof of [9] uses topological techniques. As we do not use these parts of the proof of [9], we present the proof in the case of $\mathbb{R}^2$ where the topological tools are not needed, and refer the reader to [2, Theorem 2.62] for sketch of the proof in the general case. For sake of clarity, we formulate explicitly the $d=2$ case of Lemma 3.1 whose proof we present.

Lemma (Lemma 3.1 for $d=2$). For any family $F$ of convex sets in $\mathbb{R}^2$, there exist $A,B \in F$ and a line $\ell$ such that if $C \in F$ satisfies $A \cap C \neq \emptyset$ and $B \cap C \neq \emptyset$ then $C \cap \ell \neq \emptyset$.

Proof. If some $A,B \in F$ satisfy $A \cap B = \emptyset$ then the assertion clearly holds with $A,B$ and any line $\ell$ that separates between $A$ and $B$. Thus, we assume that $A \cap B \neq \emptyset$ for all $A,B \in F$.

For any pair $A,B \in F$ such that $A \cap B \neq \emptyset$, let $\text{lexmax}(A,B)$ denote the lexicographic maximum of $A \cap B$. Let $x_0 = \text{lexmin}\{\text{lexmax}(A,B) : A,B \in F, A \cap B \neq \emptyset\}$ (i.e., the lexicographic minimum amongst $\text{lexmax}(A,B)$), and let $A,B \in F$ be such that $x_0 = \text{lexmax}(A,B)$. Denote $H = \{x \in \mathbb{R}^2 : x \geq_{\text{lex}} x_0\}$, and let $A' = A \cap H, B' = B \cap H$. As $A',B'$ are convex sets and $A' \cap B' = \{x_0\}$, there exists a line $\ell$ with $x_0 \in \ell$ that separates between $A' \setminus \{x_0\}$ and $B' \setminus \{x_0\}$. We claim that the assertion holds with $A,B,\ell$. To see this, we consider two cases:

1. $C \cap (A \cap B) \neq \emptyset$. We claim that $x_0 \in C$, and thus $C \cap \ell \neq \emptyset$. Assume to the contrary $x_0 \notin C$. Note that for any family of convex sets $C_1,C_2,\ldots,C_m \subset \mathbb{R}^2$ such that $\cap_{i=1}^m C_i \neq \emptyset$, there exist $1 \leq k < l \leq m$ such that $\text{lexmax}(\cap_{i=1}^m C_i) = \text{lexmax}(C_k \cap C_l)$. (This is a straightforward application of Helly’s theorem; see [8, Lemma 8.1.2].) In the case $\{C_1,\ldots,C_m\} = \{A,B,C\}$, by assumption $x_1 := \text{lexmax}(A \cap B \cap C) \neq \text{lexmax}(A \cap B)$, and thus w.l.o.g. $x_1 = \text{lexmax}(A \cap C)$. It follows that $x_1 \in A \cap B$, and thus $x_1 <_{\text{lex}} x_0 = \text{lexmax}(A \cap B)$. A contradiction to the definition of $x_0$. Hence, $x_0 \in C$, as asserted.

2. $C \cap (A \cap B) = \emptyset$. As $\text{lexmax}(A \cap C) >_{\text{lex}} x_0$, we have $(A' \setminus \{x_0\}) \cap C \neq \emptyset$. Similarly, we have $(B' \setminus \{x_0\}) \cap C \neq \emptyset$. As $\ell$ separates between $A' \setminus \{x_0\}$ and $B' \setminus \{x_0\}$, this implies $C \cap \ell \neq \emptyset$, as asserted.

\[\blacksquare\]

Remark 3.2. When the proof of Case (1) of Lemma 3.1 is applied for a general $d$ (as was done in [9] and as we do below), we define $x_0 = \text{lexmin}\{\text{lexmax}(\cap_{i=1}^d A_i) : A_1,\ldots,A_d \in F, \cap_{i=1}^d A_i \neq \emptyset\}$ (where the system of coordinates is chosen such that all lexicographic maxima/minima are defined uniquely). We also replace $A \cap B$ by $\cap_{i=1}^d A_i$, and replace ‘each of $A$ and $B$’ by ‘each of $\cap_{i \neq j} A_i$, $1 \leq j \leq d$’.

The argument used in Case (1) above can be used to obtain a simple proof of the Hadwiger–Debrunner theorem (Theorem 2.2 above), as follows:

Alternative proof of Theorem 2.2. Let $F$ be a family of at least $p$ compact convex sets in $\mathbb{R}^d$ that satisfies the $(p,q)$ property, and let $x_0, A_1, A_2, \ldots, A_d$ be chosen as in the proof of Lemma 3.1 and in Remark 3.2.

Consider the family $G = \{C \in F : x_0 \notin C\}$. We consider two cases:
• \(|G| \geq p - d\). We claim that in this case, \(G\) satisfies the \((p - d, q - d + 1)\) property. Indeed, let \(C_1, C_2, \ldots, C_{p-d} \in G\), and consider the family \(\{C_1, \ldots, C_{p-d}, A_1, A_2, \ldots, A_d\}\). By assumption, it contains an intersecting \(q\)-tuple. This \(q\)-tuple cannot contain all of \(A_1, A_2, \ldots, A_d\), as by the argument of Case (1) above, each of \(C_1, \ldots, C_{p-d}\) is disjoint with \(\cap_{i=1}^d A_i\). Thus, \(\{C_1, C_2, \ldots, C_{p-d}\}\) contains an intersecting \((q - d + 1)\)-tuple.

• \(|G| = p - d - t\) for \(t \geq 0\). By the same reasoning as in the previous case, \(G\) contains an intersecting \((q - d - t + 1)\)-tuple that can be pierced by a single point, and thus, it can be trivially pierced by \((p - d - t) - (q - d - t + 1) + 1 = p - q\) points.

As \(F \setminus G\) is pierced by \(x_0\), combining the two cases we get \(\text{HD}(p, q) \leq \max(\text{HD}(p - 2, q - 1) + 1, p - q + 1)\). Since \(\text{HD}(d + 1, d + 1) = 1\) by Helly’s theorem, it follows by induction that if \(q > \frac{d}{d-1}p + 1\) then \(\text{HD}(p, q) \leq p - q + 1\).

3.2 The bootstrapping technique

In [9], the authors show that if \(F\) satisfies the \((p, q)_r\) property with \(r > \binom{p}{q} - \binom{p+1-d}{q+1-d}\) then (in the notations of Lemma 3.3) the family \(G' = \{C \cap \ell : C \in F, x_0 \notin C\}\) satisfies the \((p - d, q - d + 1)\) property, and thus, by Theorem 2.2 in dimension 1, \(F\) can be pierced by \(p - q + 1\) points. In our bootstrapping argument, we show instead that the family \(G'\) satisfies the \((p - q + 1, 2)_{r'}\) property for a sufficiently large \(r'\), and then an improved piercing number for \(F\) can be derived from Proposition 2.7. We will use the following.

**Definition 3.3.** Let \(F\) be a family of compact convex sets in \(\mathbb{R}^d\), \(|F| \geq p\), and let \(\ell\) be a line. \(F\) is said to satisfy the \((p, q)_{r}\) property through \(\ell\) if any \(p\)-tuple of sets in \(F\) contains at least \(q\) \(r\)-tuples that intersect on \(\ell\).

**Lemma 3.4.** If a family \(F\) satisfies the \((p, 2)_{r_0}\) property through \(\ell\) where \(r_0 = \binom{p-k-2}{2} + (k + 2)\binom{p-k-2}{1} + 1\), then \(F\) can be pierced by \(k + 1\) points.

**Proof.** Let \(H = \{C \in F : C \cap \ell = \emptyset\}\), and denote \(h = |H|\). The family \(F' = \{C \cap \ell : C \in F \setminus H\}\) clearly satisfies the \((p - h, 2)_{r_0}\) property. As

\[
\binom{p-k-2}{2} + (k + 2)\binom{p-k-2}{1} + 1 \geq \binom{p-h}{2} - (k - h - 2)
\]

\[
+ (k - h + 2)\binom{p-h}{1} - (k - h - 2) + 1,
\]

it follows that \(F'\) is a family of segments on \(\ell\) that satisfies the \((p - h, 2)_{r'}\) property with \(r' = \binom{p-h}{2} - (k - h - 2) + ((k - h) + 2)\binom{p-h}{1} - (k - h - 2) + 1\). Thus, by Proposition 2.7, \(F'\) can be pierced by \(k - h + 1\) points, and thus, \(F\) can be pierced by \(k + 1\) points, as asserted.

Now we are ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let \(F\) be a family that satisfies the assumption of the theorem, and let \(x_0, A_1, A_2, \ldots, A_d, \ell\) be chosen as in the proof of Lemma 3.3, i.e., \(\{A_i\}_{i=1}^d\) is the \(d\)-tuple in which the lexmin(lexmax(\(\cdot\))) is attained. Denote \(F' = \{C \in F : x_0 \notin C\}\). We want to show that \(F'\) satisfies the \((p - q + 1, 2)_{r_0}\) property through \(\ell\), where \(r_0 = \binom{p-q+1-k-2}{2} + (k + 2)\binom{p-q+1-k-2}{1} + 1\). By Lemma 3.4, this would imply that \(F'\) can be pierced by \(k + 1\) points, and thus, \(F\) can be pierced by \(k + 2\) points, as asserted.

By the choice of \(m_0(k)\), it is sufficient to show that \(F'\) satisfies the \((p - q + 1, 2)_{r'}\) property through \(\ell\), where \(r' = \binom{m_0(k)+1}{2}\). Furthermore, by Lemma 3.3, it is sufficient to show that among any \(p - q + 1\) elements of \(F\) there exist at least \(\binom{m_0(k)+1}{2}\) distinct pairs of elements that intersect \(\cap_{i \neq j} A_i\) for all \(1 \leq j \leq d\).
As $\mathcal{F}$ is non-$(p-q)$-degenerate, we have $|\mathcal{F}'| \geq p - q + 1$. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_{p-q+1}\} \subset \mathcal{F}'$, and let $\mathcal{D} = \{D_1, D_2, \ldots, D_{q-d-1}\} \subset \mathcal{F}$ such that $\mathcal{D} \cap (\mathcal{C} \cup \{A_1, A_2, \ldots, A_d\}) = \emptyset$. We have $|\mathcal{C} \cup \mathcal{D} \cup \{A_1, A_2, \ldots, A_d\}| = p$, and thus, the family $\mathcal{C} \cup \mathcal{D} \cup \{A_1, A_2, \ldots, A_d\}$ contains at least $r$ intersecting $q$-tuples.

Note that $q$-tuples of elements of $\mathcal{C} \cup \mathcal{D} \cup \{A_1, A_2, \ldots, A_d\}$ can be divided into three groups:

1. $q$-tuples that contain less than $d - 1$ of the sets $A_1, \ldots, A_d$.
2. $q$-tuples that contain exactly $d - 1$ of the sets $A_1, \ldots, A_d$.
3. $q$-tuples that contain all the sets $A_1, \ldots, A_d$.

We observe that none of the intersecting $q$-tuples belong to the third group, as by the proof of Lemma 3.1 above (specifically, by Lemma 8.1.2 in [8] that applies for a general $d$), all elements of $\mathcal{C}$ are disjoint with $\cap_{i=1}^{d} A_i$, and $\mathcal{D}$ contains only $q - d - 1$ elements. This implies that the total number of intersecting $q$-tuples is at most $\binom{p}{r} - \binom{p-d}{q-d}$. Furthermore, since $r$ satisfies (2), the number of non-intersecting $q$-tuples in groups 1 and 2 is at most

$$\binom{p}{q} - \binom{p-d}{q-d} - \binom{p-d+1}{q-d+1} + 1 + \binom{q-d-2+m_0(k)}{q-d} + \binom{q-d-1+m_0(k)}{q-d+1}$$

$$\left(\frac{p-d}{q-d+1} - \left(\frac{p-d}{q-d+1} + 1\right) + \left(\frac{q-d-2+m_0(k)}{q-d}\right) + \left(\frac{q-d-1+m_0(k)}{q-d+1}\right) + 1\right).$$

(5)

For each $\{S_1, S_2, \ldots, S_{q-d+1}\} \subset \mathcal{C} \cup \mathcal{D}$, we define a $d$-tuple

$$P_{\{S_1, \ldots, S_{q-d+1}\}} = \{\{S_1, \ldots, S_{q-d+1}, A_2, A_3, \ldots, A_d\}, \{S_1, \ldots, S_{q-d+1}, A_1, A_3, \ldots, A_d\}, \ldots, \{S_1, \ldots, S_{q-d+1}, A_1, A_2, \ldots, A_{d-1}\}\}.$$

Denote by $P'$ the set of all $\{S_1, S_2, \ldots, S_{q-d+1}\} \subset \mathcal{C} \cup \mathcal{D}$ for which all $d$ elements of $P_{\{S_1, \ldots, S_{q-d+1}\}}$ are intersecting. We claim that

$$|P'| \geq \frac{q-d-2+m_0(k)}{q-d} + \frac{q-d-1+m_0(k)}{q-d+1} + 1. \tag{6}$$

Indeed, note that the $q$-tuples in group 2 are naturally divided into $d$ classes according to the set $A_i$ they miss. Each class consists of $\binom{p-d}{q-d+1}$ $q$-tuples. It is clear that for a given number of intersecting $q$-tuples, $|P'|$ is minimized when all non-intersecting $q$-tuples of group 2 belong to the same class. In that case, $|P'|$ equals to the number of remaining elements in that class, and thus by Equation (6),

$$|P'| \geq \left(\frac{p-d}{q-d+1} - \left(\frac{p-d}{q-d+1} + 1\right) + \left(\frac{q-d-2+m_0(k)}{q-d}\right) + \left(\frac{q-d-1+m_0(k)}{q-d+1}\right) + 1\right),$$

meaning that (6) holds.

By the definition of $P'$, each element in $P'$ contains $q-d+1$ sets that intersect $\cap_{i \neq j} A_i$, for all $1 \leq j \leq d$. As $|\mathcal{D}| = q-d-1$, at least two of these sets belong to $\mathcal{C}$. Hence, each element of $P'$ contains at least one pair of elements in $\mathcal{C}$ that intersect $\cap_{i \neq j} A_i$, for all $1 \leq j \leq d$. Recall that we want to prove that there are at least $\binom{m_0(k)+1}{2}$ such pairs.

It is easy to see that for a given number of elements in $P'$, the number of distinct pairs $(C, C') \in \mathcal{C}$ contained in elements of $P'$ is minimized when these elements are 'packed together'. In particular, the maximal possible number of elements in $P'$ such that the number of distinct
pairs is smaller than \( \binom{m_0(k)+1}{2} \) is attained when we take some \( C_1, C_2, \ldots, C_{m_0(k)+1} \in \mathcal{C} \), and define

\[
P'' = \{ S \subset \mathcal{C} \cup \mathcal{D} : (|S| = q - d + 1) \wedge (S \cap \mathcal{C} \subset \{ C_1, C_2, \ldots, C_{m_0(k)+1} \}) \wedge \{ C_{m_0(k)}, C_{m_0(k)+1} \} \not\subset S \}.
\]  

(7)

In this case, we have \( |P''| = (q-d-2+m_0(k)) + \frac{q-d-1+m_0(k)}{q-d+1} \). Indeed, since \( |\mathcal{D}| = q - d - 1 \), then among the \( (q - d + 1) \)-tuples \( S \) for which \( S \cap \mathcal{C} \subset \{ C_1, C_2, \ldots, C_{m_0(k)+1} \} \), there are \( \frac{(q-d-1+m_0(k))}{q-d+1} \) that do not include \( C_{m_0(k)+1} \), and \( \frac{(q-d-2+m_0(k))}{q-d} \) that include \( C_{m_0(k)+1} \) and miss \( C_{m_0(k)} \). Therefore, Equation (6) implies that \( \mathcal{C} \) must contain at least \( \binom{m_0(k)+1}{2} \) distinct pairs that intersect \( \cap_{i \neq j} A_i \) for all \( 1 \leq j \leq d \), and thus, by Lemma 3.1 \( \mathcal{F}' \) satisfies the \( (p-q+1, 2) \binom{m_0(k)+1}{2} \) property through \( \ell \). This completes the proof. \( \square \)

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