Anomalous current in diffusive ferromagnetic Josephson junctions.

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We demonstrate that in diffusive superconductor/ferromagnet/superconductor (S/F/S) junctions a finite, anomalous, Josephson current can flow even at zero phase difference between the S electrodes. The conditions for the observation of this effect are non-coplanar magnetization distribution and a broken magnetization inversion symmetry of the superconducting current. The latter symmetry is intrinsic for the widely used quasiclassical approximation and prevent previous works, based on this approximation, from obtaining the Josephson anomalous current. We show that this symmetry can be removed by introducing spin-dependent boundary conditions for the quasiclassical equations at the superconducting/ferromagnet interfaces in diffusive systems. Using this recipe we considered generic multilayer magnetic systems and determine the ideal experimental conditions in order to maximize the anomalous current.

In its minimal form the current-phase relation (CPR) characterizing the dc Josephson effect reads $I(\varphi) = I_c \sin \varphi$, where $\varphi$ is the phase difference between superconducting electrodes and $|I_c|$ is the critical current that is the maximum supercurrent that can flow through the junction. Ordinary Josephson junctions are characterized by $I_c > 0$ yielding the zero phase difference ground state $\varphi = 0$. In certain cases, however, $I_c < 0$ and the ground state corresponds to $\varphi = \pi$. Such $\pi$-junctions can be realized for example in superconductor/ferromagnet/superconductor (SFS) structure. Josephson junctions with non-equilibrium normal metal interlayer, d-wave superconductor, semi-conductor nanowire, gated carbon nanotube or multi-terminal Josephson systems. $\pi$-junctions has being suggested for building scalable superconducting digital and quantum logic.

As for $\varphi = 0, \pi$ junctions, no physical argument speaks against a CPR of the form

$$I(\varphi) = I_c \sin(\varphi + \varphi_0) \quad (1)$$

with an arbitrary phase shift $\varphi_0 \neq n\pi$ and Josephson energy $E_J = -I_c \cos(\varphi + \varphi_0)$. In such a case the ground state corresponds to $\varphi = -\varphi_0$ and a finite supercurrent at zero phase difference $I_{an} = I_c \sin \varphi_0$, termed the anomalous current. This effect, referred as the anomalous Josephson effect (AJE), takes place only in systems with a broken time reversal symmetry.

The AJE has been predicted in junctions which combine conventional superconductors with magnetism and spin-orbital interaction, between unconventional superconductors, and topologically non-trivial superconducting leads. In the presence of magnetic flux piercing the normal interlayer the superconducting proximity currents are generated which naturally leads to the phase shift of CPR. Experimentally, a $\varphi_0$-junction has been reported in nano-wire quantum dot controlled by an external magnetic field and an electrostatic gate.

Another type of systems predicted to exhibit the AJE are conventional SFS junctions with a non-homogeneous magnetization texture. In such systems the current is a functional of the magnetization distribution $M$, $I = I(\varphi, M)$. Time-inversion symmetry dictates that $I(\varphi, M) = -I(-\varphi, -M)$. If the system has an additional magnetization inversion symmetry such that

$$I(\varphi, M) = I(\varphi, -M), \quad (2)$$

then $I(\varphi, M) = -I(-\varphi, M)$ and obviously the system does not exhibit the AJE. In other words, it is necessary to break the symmetry in order to obtain the $\varphi_0$ state.

For example, for any coplanar magnetization distribution exists a global SU(2) spin rotation such that flips the direction of $M$, and the condition (2) is fulfilled. For this reason, the AJE requires a non-coplanar magnetization texture. This explains the AJE predicted for ballistic S/F/F/F/S systems with non-collinear magnetization. The anomalous current obtained in those works shows rapid oscillations as a function of the ferromagnetic thickness. These oscillations result from the Fabry-Perot interference of electronic waves reflected at the S/F and F/F interfaces.

In diffusive SFS structures, as those used in experiment, the impurity scattering randomizes directions of electron propagation and hence one expects the suppression of the rapidly oscillating anomalous current. Studies, based on quasiclassics, of the diffusive Josephson junctions through various non-coplanar structures including helix magnetic vortexes and skyrmions have shown no AJE. In contrast, in diffusive systems with half-metallic elements and in junctions...
between magnetic superconductors with spin filters\cite{41,42} a finite anomalous current has been predicted. From this apparent contradiction, the general condition for the AJE in diffusive systems still remains elusive.

In this letter we show for the first time that the AJE is a robust effect that can exists in any diffusive SFS systems with non-coplanar magnetization textures under quite general conditions. We demonstrate that the reason why anomalous currents have not been found in previous studies on diffusive SFS systems is due to the additional magnetization inversion symmetry\cite{2} that the quasiclassical approximation\cite{42,43} has with respect to the original Hamiltonian and that prevents the description of the AJE in ferromagnetic junctions. In a second part of the letter we consider a spin-filter at the S/F interfaces and demonstrate the existence of anomalous currents in diffusive SF structures. This allow us to study the AJE without having to renounce the widely used quasiclassical approximation.\cite{44,45}

We start by analyzing the inherent symmetries of the Usadel equation, which is a diffusion-like equation for the quasiclassical Green functions (GF). In the Matsubara representation it has the form\cite{41,44,45}

$$D \nabla (\tilde{g} \nabla \tilde{g}) = [\hat{\Delta} + \hat{\tau}_3 (\omega + i \hat{\sigma} h), \hat{g}],$$

(3)

where $[a, b] = (ab - ba)/2$, $\omega$ is the Matsubara frequency, $h(r)$ is the exchange field which is parallel to the local magnetization $M(r)$, $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$ is the vector of Pauli matrices in spin space $\hat{\sigma}_{1,2,3}$ and $\hat{\tau}_{1,2,3}$ are the Pauli matrices in Nambu space. The gap matrix is defined as $\hat{\Delta} = \hat{\tau}_1 \Delta e^{-i \hat{\tau}_3 \phi}$, where $\Delta$ and $\phi$ are the magnitude and phase of the order parameter. The $4 \times 4$ matrix GF in spin-Nambu space can be written in the form, which takes into account the general particle-hole symmetry of Eq. (3)

$$\tilde{g} = \begin{pmatrix} \tilde{g} & \tilde{f} \\ \tilde{f} & -\tilde{g} \end{pmatrix}$$

(4)

with $2 \times 2$ components $\tilde{g}$ and $\tilde{f}$ in the spin space and the time-reversed operation defined as $\hat{X} = \hat{\sigma}_2 \hat{X}^* \hat{\sigma}_2$. Eq. (3) is complemented by the normalization condition $\tilde{g}^2 = 1$.

We introduce the following transformation

$$\tilde{g}_{new} = \hat{\tau}_1 \tilde{g}^* \hat{\tau}_1 \hat{\sigma}_2,$$

(5)

which is a combination of two transformations $g_{new} = T \Theta \hat{g} \Theta^+ T$: the time reversal transformation, $T = i \hat{\sigma}_2 \hat{K}$, with $\hat{K}$ being the complex conjugate operation, and the transposition of the electron and hole blocks of $g$, $\Theta = \hat{\tau}_1$. Applying the transformation $\tilde{g}$ to the Usadel Eq. (3) one obtains that

$$\tilde{g}_{new}(\omega, h) = \tilde{g}(-\omega, -h).$$

(6)

On the other hand, the current is expressed as:

$$j = \frac{i}{8e} \pi T \sum_{\omega = -\infty}^{\infty} \text{Tr} \hat{\tau}_3 \tilde{g} \nabla \tilde{g},$$

(7)

where $\sigma_n = e^2 N_F D$ is the normal metal conductivity and $N_F$ is the density of states at the Fermi level. The summation is done over Matsubara frequencies $\omega = \pi T (2n + 1)$, where $n$ is the integer number and $T$ is the temperature. It follows from Eqs. (6) that the current is invariant with respect to the magnetization inversion, $j(h) = j(-h)$, as anticipated in Eq. (2). By combining this extra symmetry with the general time-reversal symmetry, $j(\varphi, h) = -j(-\varphi, -h)$ one obtains that $j(\varphi) = -j(-\varphi)$ and hence within, the quasiclassical approach, the AJE cannot take place for any spatial dependence of the exchange field $h(r)$. On the other hand, we know from previous works that anomalous current may be generated at least in ballistic SFS junctions with non-coplanar configuration of the magnetization.\cite{42,43} What is the origin of the apparent contradiction between the explicit ballistic calculations in those references and the magnetization reversal symmetry of the Usadel equation? Is the absence of AJE a specific feature of diffusive systems or is there a deeper reason for the above symmetry?

To answer these questions let us first recall the Bogoliubov-De Gennes (BdG) Hamiltonian:

$$H_{BdG} = \left( \begin{array}{cc} \xi - \hat{\sigma} h & \hat{\Delta} \\ -\hat{\Delta}^* & -\xi - \hat{\sigma} h \end{array} \right)$$

where $\xi = p^2/2m - E_F$ is the quasiparticle energy relative to the Fermi energy $E_F$. The general symmetries of the BdG Hamiltonian are well known.\cite{46} In the quasiclassical limit, which is equivalent to the Andreev approximation,\cite{47} transport properties are determined by particles living exactly at the Fermi surface. In the BdG Hamiltonian this corresponds to the $\xi = 0$ case. In this, and only in this case, the BdG Hamiltonian acquires an additional symmetry with respect to the transposition of the electron and hole blocks, namely $\hat{\tau}_1 H_{BdG}(\xi = 0, \varphi, h) \hat{\tau}_1 = H_{BdG}(\xi = 0, -\varphi, h)$. According to Eq. (5) this symmetry together with the time-reversal operation leads to the invariance of the current under magnetization inversion. Obviously, this invariance is a general feature of the quasiclassical theory, which holds true not only in the diffusive (Usadel) limit, but also for the full Eilenberger equation. In particular it explains why no AJE is obtained at the leading quasiclassical order in ballistic junctions with generic spin fields.\cite{48}

Clearly in real materials quantum effects always break this symmetry to a degree determined by the accuracy of quasiclassical approximation, which is the ratio $h/E_F$. Once this symmetry is broken the AJE may occur in any SFS system with arbitrary degree of non-magnetic disorder and non-coplanar magnetization distribution. The magnitude of the anomalous current will then be in leading order of the parameter $h/E_F$. Typical experiments on SFS junctions showing the $\pi$-junction behavior, used weak ferromagnets,\cite{49,50,51} for which $h/E_F \ll 1$. Therefore, at first glance, the AJE is hardly expected to be observed in these structures.

This conclusion is however not fully correct, and there
is indeed a way to enhance the anomalous Josephson currents in systems with weak ferromagnets if one introduces spin-filtering tunnel barriers at the S/F interfaces, i.e. barriers with spin-dependent transmission for up and down spins. As we show below such barriers breaks the quasiclassical symmetry, Eq. 2, and can lead to a measurable AJE in realistic SFS junctions.

Spin-filtering barriers are described by the generalized Kuprianov-Lukichev boundary conditions,\(^5\), that include spin-polarized tunneling at the SF interfaces,\(^6\)

\[
\gamma \tilde{g} \partial_n \tilde{g} = [\tilde{g} \Sigma F \tilde{g}^\dagger, \tilde{g}].
\]

(8)

Here \(\partial_n = (\mathbf{n} \cdot \nabla)\) is the normal derivative at the surface, \(\gamma = \sigma_n R\) is the parameter describing the barrier strength, \(R\) is the normal state tunneling resistance per unit area, and \(\tilde{g}\) is the Green function of the superconducting electrode. We assume that the magnetization of the barriers points in \(\mathbf{m}\) direction. The spin-polarized tunneling matrix has the form \(\Gamma = t\sigma_0 \tilde{g}_0 + u(\mathbf{m} \tilde{\sigma}) \tilde{g}_n\), with \(t = \sqrt{(1 + \sqrt{1 - P^2})/2}\), \(u = \sqrt{(1 - \sqrt{1 - P^2})/2}\) and \(P\) being the spin-filter efficiency of the barrier that ranges from 0 (no polarization) to 1 (100% filtering efficiency).

By applying the transformation \(\tilde{g} = \tilde{g}_n\) to Eq. (8) one can easily check the sign of the barrier polarization does not change and hence

\[
I(\varphi, h, P) = I(\varphi, -h, P),
\]

(9)

where \(P = P_m\). On the other hand, the time-reversal transformation flips all the magnetic moments including the exchange field and the barrier polarizations

\[
I(\varphi, h, P) = -I(-\varphi, -h, -P).
\]

(10)

Combining Eqs. (9) and (10), we see, that in principle, \(I(\varphi, h, P) \neq -I(-\varphi, h, P)\) and the zero-phase difference current at \(\varphi = 0\) is not prohibited by symmetry.

From this analysis is clear that the general features of the CPR can be deduced from the symmetry relations (9). First we consider the S/FI/F/S structure of Fig. 1a. Here FI stands for the spin-filtering barriers with magnetizations \(P_{r,l}\) and \(F\) is the mono-domain weak ferromagnet with exchange field \(h\). From previous works\(^4\), one would expect that the anomalous current is proportional to the spin chirality \(\chi = \mathbf{h} \cdot (P_r \times P_l)\). However, this term is prohibited by the symmetry \(\tilde{g}\) because of the change of sign of \(\chi\). Instead, one can construct the scalar \(I_{an} \propto (P_r, h) \chi\) which is invariant to the sign change of \(h \rightarrow -h\) and therefore is robust to the quasiclassical symmetry (9). In such a case the anomalous current is finite if all vectors are non-collinear and in addition \(h\) has a component parallel to at least one of the magnetizations \(P_{r,l}\).

To get an agreement with the results based on the Bogolubov - de Gennes calculations,\(^4\), that yield \(I_{an} \neq 0\) for any non-coplanar spin texture and has to take into account the magnetic proximity effects,\(^5\) that induces an effective exchange field \(b_r\) and \(b_l\) in the superconducting electrodes [Fig. 1b]. In this case we define the chiralities \(\chi_{l,r} = P_{l,r} \cdot (b_{l,r} \times h)\), which are invariant respect to the quasiclassical symmetry since they contain two exchange fields changing signs under the transformation \(\tilde{g}\). Thus, in this case the AJE is expected to be proportional to a linear combination of the chiralities \(\chi_{l,r}\).

A similar behavior can be obtained for the structure shown in Fig. 1c. It is a S/FI/F/F'/S junction with non-coplanar configuration of the one barrier polarization \(P_l\) and two ferromagnetic layers \(h\) and \(h_1\). In this case the chirality \((P_l \times h_l)h \neq 0\) is invariant under the symmetry \(\tilde{g}\) thus allowing for the existence of the AJE.

To quantify these effects we calculate the CPR analytically focusing on the weak proximity effect in the F layer that allows for a linearization of the Usadel equation with respect to the anomalous Green’s function.\(^6\) The latter can be written as a superposition of the scalar singlet amplitude \(f_s\) and the vector of triplet states \(\mathbf{f} = (f_x, f_y, f_z)\), \(\hat{f} = f_s \hat{\sigma}_0 + f_l \hat{\sigma} \). From Eq. (3) we get the following sys-
tem of equations for $\omega > 0$
\[
(D\nabla^2 - 2\omega)f_s - i\hbar f_i = 0 \quad (11)
\]
\[
(D\nabla^2 - 2\omega)f_i - if_s\hbar = 0 . \quad (12)
\]

supplemented by the linearized boundary condition obtained from Eq. (8) in case the possible exchange field in the superconductor is parallel to the barrier polarization $b \parallel P$.

\[
\gamma \theta_n \hat{f} = -\left[\hat{G}_s, P\sigma \cdot \hat{f}\right] - \left[\hat{G}_s, \hat{f}\right] + \sqrt{1 - P^2}\hat{F}_s, \quad (13)
\]

where $\{a, b\} = (a + b)/2$, $\hat{G}_s$ and $\hat{F}_s$ are the the normal and anomalous GF in the superconducting electrodes. The first term in the right hand side of Eq. (13) is novel as compared to the boundary conditions for non-magnetic interfaces ($P = 0$). This term provides a $\pi/2$ phase rotation of the triplet superconducting components non-collinear with the barrier polarization $P$. It is precisely this phase rotation that may lead to an effective shift of the phase difference between the Cooper pairs across the junction resulting in the AJE.

We calculate the amplitude $I_{an}$ for the structures shown in Fig. (1) in the practically relevant regime when the coherence length in the middle ferromagnetic layer $\xi_F = \sqrt{D/\hbar}$ is much shorter than that in a normal metal $\xi_N = \sqrt{D/T}$. The analytical result can be obtained by assuming that the length $d$ of the junction is $\xi_F \ll d \ll \xi_N$. Under such conditions the Josephson current is mediated by long-range triplet superconducting correlations (LRTSC) since short-range modes decay over $\xi_F$.

For the S/F/1/F/F/S structure shown in Fig. (1), we neglect the magnetic proximity effect and assume the bulk GF in the S electrodes $G_s = \omega/\sqrt{\omega^2 + |\Delta|^2} \equiv G_0$, $\hat{F}_s = \Delta G_0/\omega \equiv F_0$. Then the anomalous current is \[\frac{eRI_{an}}{2\pi} = \chi(h\hat{P})\sqrt{(1 - P_1^2)(1 - P_2^2)} \frac{\xi_F^2 T}{\gamma^2\hbar^2d^2} \sum_{\omega > 0} \frac{F_0^2 R_0^2}{8k_\omega^2}, \quad (14)\]

where $\hat{P} = P_r + P_i$ and $k_\omega = \sqrt{\omega/D}$. As expected for this case, the anomalous current is proportional to $(h\hat{P})\chi$, where $\chi = h \cdot (P_r \times P_i)$ is the spin chirality. It is important to note that the usual contribution to the Josephson current $I_0 = I(\varphi = \pi/2)$ determined by the LRTSC is proportional to $I_0 \propto \gamma^{-4}$, and hence it dominates over the anomalous one, $I_0 \gg I_{an} \propto \gamma^{-5}$.

If we now take the inverse proximity effect into account, and assume effective exchange fields $b_r$ and $b_l$ in the superconductors [Fig. (1)], we obtain \[\frac{eRI_{an}}{2\pi} = (\chi_r - \chi_l)\sqrt{(1 - P_r^2)(1 - P_l^2)} \frac{\xi_F^2 T}{\gamma^2\hbar^2d\omega} \sum_{\omega > 0} \frac{TF_0F_0G_0}{2\sqrt{2k_\omega^2}} \quad (15)\]

where the chiralities $\chi_r,l$ are defined above and $F_0 = dF_0/\omega$. The usual current carried by the LRTSC is, to the lowest order in transparency, given by $I_0 \propto \gamma^{-2}(b_{r,\perp}b_{l,\perp})$, where $b_{r,\perp} = b_{r,\perp} - h(b_{r,\perp}/h^2)$ are the projections onto the plane perpendicular to the exchange field $h$. In contrast to the previous example, $I_0$ is given by the lower order in $\gamma^{-1}$ since the LRTSC can tunnel directly from the superconducting electrodes modified by the exchange fields $b_{r,l}$. Hence in general if $(b_{r,\perp}b_{l,\perp}) \neq 0$ the anomalous current is \[\frac{eRI_{an}}{2\pi} = \frac{eRI_{an}}{2\pi} \quad (16)\]

\]
where $\chi = (P_1 \times h_1)h \neq 0$. As in our first example, the usual component of the current is proportional to $I_0 \propto \gamma^{-4}$ and therefore $I_{an} \gg I_0$. This type of S/F/F/F/S structure provides the maximal AJE since the anomalous current is of the opposite order of the critical one $I_{an} \sim I_c$.

All previous results are strictly speaking valid in the quasiclassical limit in which $h/E_F \ll 1$. However, in the case of strong ferromagnets, $h/E_F \lesssim 1$, the difference between Fermi velocities for spin up and down electrons can be described by an effective spin-filtering effect at the S/F interfaces, and therefore they also apply for for ballistic systems and strong ferromagnets.

To summarize, the proposed mechanism for the AJE and $\varphi_0$ ground states in SFS structures is rather generic and exists in any system with non-coplanar magnetization configuration. This conclusion is in contrast to a number of previous studies which did not obtain anomalous currents in diffusive and ballistic systems in the framework of quasiclassical approximation. We clarify this apparent controversy by demonstrating that the absence of AJE within quasiclassics is due to an additional symmetry which is only exact at the Fermi level. In order to restore the symmetries of the original Hamiltonian we have considered spin-filtering boundary conditions to the Usadel equations and found analytical expressions for the anomalous current in different geometries. Our results show that in structures as those shown in Figs. (1), (c), the amplitude of the anomalous current comparable to critical one $I_{an} \sim I_c$, and therefore the AJE may be observed in such junctions.

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I. SUPPLEMENTARY MATERIAL: DERIVATION OF CURRENT-PHASE RELATIONS.

Here we derive analytical expressions for the anomalous and usual Josephson current components in generic trilayer SFS structures shown in Figs. 1. We use Usadel Eqs. (11,12) with boundary conditions obtained from the linearization of the Eq. (8). For the general spin structure of GF in the superconducting electrode the linearized boundary condition can be written as follows

\[ \gamma \partial_n \hat{f} = (\hat{G}_s \hat{f} + f\hat{G}_s)/2 \]  

where

\[ \hat{G}_s = t^2\hat{G}_s + u^2(\hat{m}\hat{\sigma}\hat{G}_s \hat{m}\hat{\sigma}) + 2ut\{\hat{G}_s, \hat{m}\hat{\sigma}\} \]  

\[ \hat{F}_s = t^2\hat{F}_s - u^2(\hat{m}\hat{\sigma}\hat{F}_s \hat{m}\hat{\sigma}) - 2ut\{\hat{G}_s, \hat{m}\hat{\sigma}\}. \]  

In the presence of exchange field \( \hat{b} \) in the superconducting electrode the GF are

\[ \hat{G}_s = G_0 - i(\hat{b}\hat{b})dG_0/d\omega \]  

\[ \hat{F}_s = F_0 - i(\hat{b}\hat{b})dF_0/d\omega. \]

If the exchange field is collinear with the barrier polarization \( \hat{b} \parallel \hat{m} \), the boundary condition (17) acquires the form of Eq. (13). In the right hand side of Eq. (13) the first and second terms are much smaller than the third one. Both the first and second terms are proportional to the small tunnelling parameter \( \gamma^{-1} \) but have different symmetry. The third term can be safely neglected since it has the same symmetry as the left hand side and therefore does not provide any qualitative corrections. We keep the second term which is important to obtain anomalous Josephson effect. To calculate the charge current in the ferromagnetic layer we use the expression

\[ j = \frac{2\sigma n}{e} \pi T \sum_{\omega>0} \text{Im}(f_s^* \nabla f_s - f_s^* \nabla f_s). \]  

which is obtained linearizing the general Eq. (7).

A. S/FI/F/FI/S structure

First of all we consider the simplest possible tri-layer non-coplanar structure S/FI/F/FI/S, where FI stands for the spin-filtering barriers with magnetizations \( P_{z,t} \) and F is the mono-domain weak ferromagnet with exchange field \( \mathbf{h} \). We calculate the CPR for the structure shown in Fig. 1a assuming without loss of generality that the exchange filed is \( \mathbf{h} = h \mathbf{z} \) and \( P_{z,t} \) can have arbitrary directions. Then we have the Usadel equations in components:

\[ D\nabla^2 f_s = ih_{fs} \]  

\[ D\nabla^2 f_s = ih_{fs}, \]  

\[ D\nabla^2 f_s = 2\omega f_s, \]  

\[ D\nabla^2 f_s = 2\omega f_s \] 

In Eqs. (23,24) we neglected \( \omega \) which is small compared to the exchange energy.

The boundary conditions at the left electrode \( x = -d/2 \)

\[ \gamma \partial_x f_s = -F_0\sqrt{1 - P_{z,t}^2} e^{i\varphi/2} \]  

\[ \gamma \partial_x f_s = iG_0(P_{z,t}^x f_y - P_{z,t}^y f_x) \]  

\[ \gamma \partial_x f_s = iG_0(P_{z,t}^y f_x - P_{z,t}^x f_y) \]  

\[ \gamma \partial_x f_y = iG_0(P_{z,t}^x f_x - P_{z,t}^y f_y) \]

and at the right electrode \( x = d/2 \)

\[ \gamma \partial_x f_s = F_0\sqrt{1 - P_{z,t}^2} e^{i\varphi/2} \]  

\[ \gamma \partial_x f_s = -iG_0(P_{z,t}^x f_y - P_{z,t}^y f_x) \]  

\[ \gamma \partial_x f_x = -iG_0(P_{z,t}^y f_x - P_{z,t}^x f_y) \]  

\[ \gamma \partial_x f_y = -iG_0(P_{z,t}^x f_x - P_{z,t}^y f_y) \]

Using the above boundary conditions and the general expression for current (22) we get that

\[ \frac{eRI}{2\pi} = \frac{1 - P_{z,t}^2}{\gamma} \sum_{\omega>0} F_0 \text{Im}(f_s^*(d/2)e^{i\varphi/2}). \]  

To simplify the derivation we assume that the length is \( \xi_F \ll d \ll \xi_\omega \) where \( \xi_F = \sqrt{D/h} \) and \( \xi_\omega = \sqrt{D/\omega} \) are the coherence lengths in normal and ferromagnetic regions.

The to the first order in tunnelling \( \gamma^{-1} \) we can calculate \( f_s \) and \( f_x \) near each interface independently without overlapping. For example at \( x = d/2 \) we have

\[ f_s^{(1)} = A_1 e^{k_1(x-d/2)} + A_2 e^{k_2(x-d/2)} \]  

\[ f_s^{(1)} = A_1 e^{k_1(x-d/2)} - A_2 e^{k_2(x-d/2)} \]

where \( k_{1,2}^2 = \pm ih/D \). Then we get

\[ f_s^{(1)}(d/2) = \sqrt{1 - P_{z,t}^2} (\xi_F/\sqrt{2\gamma}) F_0 e^{i\varphi/2} \]  

\[ f_s^{(1)}(d/2) = -\sqrt{1 - P_{z,t}^2} (\xi_F/\sqrt{2\gamma}) F_0 e^{i\varphi/2+\pi/2} \]

and

\[ f_s^{(1)}(-d/2) = \sqrt{1 - P_{z,t}^2} (\xi_F/\sqrt{2\gamma}) F_0 e^{-i\varphi/2} \]  

\[ f_s^{(1)}(-d/2) = -\sqrt{1 - P_{z,t}^2} (\xi_F/\sqrt{2\gamma}) F_0 e^{-i\varphi/2+\pi/2}. \]
To the next order in $\gamma^{-1}$ we find corrections to $f_s$ using the boundary conditions (31-32). The amplitudes $f_{x,y}$ change negligibly small and therefore can be calculated integrating the Eqs. (25, 26) and using the boundary conditions (33-34):

\begin{align*}
 f_x - i\beta \tilde{P}_x f_y &= -i\beta \tilde{P}_y f_z \\
 f_y + i\beta \tilde{P}_y f_x &= i\beta \tilde{P}_x f_z 
\end{align*}

where

\begin{align*}
 \beta &= G_0 \xi_F^2 / (2\gamma d) \\
 \tilde{P}_y f_z &= P_y f_z (d/2) + P_y^l f_z (-d/2) \\
 \tilde{P}_x &= P_x^l + P_x^r.
\end{align*}

Hence we obtain

\begin{align*}
 f_x &= -i\beta \tilde{P}_y f_z - \beta^2 \tilde{P}_x f_z \\
 f_y &= i\beta \tilde{P}_x f_z - \beta^2 \tilde{P}_y f_z.
\end{align*}

For the anomalous current we need the second terms in Eqs. (47, 48) so that

\begin{equation}
 P_x^r f_y - P_y^r f_x = \beta^2 \tilde{P}_x (P_y^r P_y^l - P_x^r P_x^l) f_z^{(0)} (-d/2) + \text{other terms}.
\end{equation}

Now we can insert Eq. (49) to the boundary conditions (33-34) to find the corrections to the component $f_s (d/2)$ needed to calculate the current (35). We search the correction $\tilde{f}_x, \tilde{f}_z$ in the form

\begin{align*}
 \gamma \partial_x \tilde{f}_x &= 0 \\
 \gamma \partial_z \tilde{f}_z &= -i G_0 \beta^2 (P_y^r P_x^l - P_x^r P_y^l) f_z^{(1)} (-d/2)
\end{align*}

which yields

\begin{align*}
 \tilde{A}_{2+} &= -\tilde{A}_{1+} + k_1 / k_2 \\
 \tilde{A}_{1+} &= -i G_0 \beta^2 k_1 (P_y^r P_x^l - P_x^r P_y^l) f_z^{(1)} (-d/2).
\end{align*}

Therefore $\tilde{f}_s (d/2) = (1 - k_1 / k_2) \tilde{A}_{1+} = (1 - i) \tilde{A}_{1+}$. Substituting Eq. (53) we obtain

\begin{equation}
 \tilde{f}_s (d/2) = - (1 + i) G_0 \beta^2 k_1 (P_y^r P_x^l - P_x^r P_y^l) f_z^{(1)} (-d/2) = \frac{\beta^2 \xi_F^2}{2\gamma^2} G_0 F_0 e^{-i\varphi/2}.
\end{equation}

Finally, substituting this expression to the Eq. (55) for the current we obtain the anomalous current amplitude

\begin{equation}
 \frac{e R I_{an}}{2\pi} = P_x (P_y^r P_y^l - P_y^l P_y^r) \sqrt{1 - P_y^l P_y^r} \times
\end{equation}

\begin{equation}
 \frac{\xi_F^2}{8\gamma^2} T \sum_{\omega > 0} \frac{F_0 G_0^2}{k_\omega^2}
\end{equation}

We can write the amplitude of the current in coordinate-independent form

\begin{equation}
 h^2 \tilde{P}_x (P_y^r P_y^l - P_y^l P_y^r) = (h \tilde{P}) \chi
\end{equation}

where $\chi = h (P_r \times P_l)$ and $\tilde{P} = P_r + P_l$.

\begin{equation}
 \frac{e R I_{an}}{2\pi} = \chi (h \tilde{P}) \sqrt{1 - P_l^2} \sqrt{1 - P_r^2} \times
\end{equation}

\begin{equation}
 \frac{\xi_F^2}{8\gamma^2} T \sum_{\omega > 0} \frac{F_0^2 G_0^2}{k_\omega^2}
\end{equation}

B. S/FI/FI/FI/S structure with exchange field in superconducting electrodes

Next let us consider the same S/FI/FI/FI/S trilayer system but take into account the induced exchange field in superconducting electrodes $b_{r,l}$ shown in Fig. 1b. In this case one can compose the chirality as follows $\chi_l = P_r \cdot (b_r \times h)$ or $\chi_r = P_l \cdot (b_l \times h)$ which are both robust against the quasiclassical symmetry since both $h$ and $b_{r,l}$ change sign.

In the presence of effective exchange fields $b_r$ and $b_l$ GF in the superconducting electrodes are given by Eqs. (20, 21) with $b = b_r (b_l)$ for right (left) electrodes. Substituting these expressions into boundary conditions Eq. (13) we obtain at the left electrode $x = -d/2$

\begin{equation}
 \gamma \partial_x f_s = F_0 \sqrt{1 - P_l^2} e^{-i\varphi/2}
\end{equation}

\begin{equation}
 \gamma \partial_z f_s = i G_0 (P_x f_y - P_y f_x) + i \sqrt{1 - P_l^2} b_{l-x} F_0 e^{-i\varphi/2}
\end{equation}

\begin{equation}
 \gamma \partial_z f_y = i G_0 (P_y f_x - P_x f_y) + i \sqrt{1 - P_l^2} b_{l-y} F_0 e^{-i\varphi/2}
\end{equation}

\begin{equation}
 \gamma \partial_z f_y = - i G_0 (P_y f_z - P_z f_y) + i \sqrt{1 - P_l^2} b_{l-y} F_0 e^{-i\varphi/2}
\end{equation}

and at the right electrode $x = d/2$

\begin{equation}
 \gamma \partial_x f_s = F_0 \sqrt{1 - P_r^2} e^{i\varphi/2}
\end{equation}

\begin{equation}
 \gamma \partial_x f_z = i G_0 (P_r f_y - P_y f_x) - i \sqrt{1 - P_r^2} b_{r-x} F_0 e^{i\varphi/2}
\end{equation}

\begin{equation}
 \gamma \partial_x f_y = - i G_0 (P_y f_x - P_x f_y) - i \sqrt{1 - P_r^2} b_{r-x} F_0 e^{i\varphi/2}
\end{equation}

\begin{equation}
 \gamma \partial_x f_y = i G_0 (P_r f_z - P_z f_y) - i \sqrt{1 - P_r^2} b_{r-y} F_0 e^{i\varphi/2}
\end{equation}
Using the general solution (36, 37) we obtain
\[
\frac{eRI}{2\pi} = \sqrt{1 - \frac{P_2}{\gamma}^2} \times (66)
\]
the component at the boundary \(e^{i\gamma/2}/2\) by their averages given by
\[
(2dk_c^2)f_i = \partial_x f_i(d/2) - \partial_x f_i(-d/2). (67)
\]
Substituting the boundary conditions (60, 61, 64, 65) to the Eq. (67) and neglecting the terms of the order \(\gamma^{-3}\) we obtain
\[
i\mathbf{b}_r \cdot \mathbf{f}_t^e = \frac{G_0}{2d\gamma k_0^2} (P_{rx} b_{ry} - P_{ry} b_{rx}) f_t^e(d/2) + \frac{G_0}{2d\gamma k_0^2} (P_{rbx} b_{ry} - P_{ry} b_{rx}) f_t^e(-d/2) - \frac{F_0'}{2d\gamma k_0^2} \sqrt{1 - P_1^2} (P_{rbx} b_{ly} + P_{ry} b_{rx}) e^{i\gamma/2} - \frac{F_0'}{2d\gamma k_0^2} \sqrt{1 - P_1^2} (P_{rbx} b_{ly} + P_{ry} b_{rx})^{-i\gamma/2}
\]
Thus the second term in the current Eq. (66) is given by
\[
\text{Im}(i\mathbf{b}_r \cdot \mathbf{f}_t^e e^{i\gamma/2}) = \frac{G_0}{2d\gamma k_0^2} (P_{rbx} b_{ly} + P_{ry} b_{rx}) \sin \varphi + \frac{F_0'}{2d\gamma k_0^2} \sqrt{1 - P_1^2} (P_{rbx} b_{ly} + P_{ry} b_{rx}) \cos \varphi. (68)
\]
To find the contribution of the first term in the current Eq. (66) we need to calculate the generation of singlet component at the boundary \(x = d/2\) by the long-range triplet ones \(f_t^s\). To find this we take into account only the first (red) term in the l.h.s. of the boundary conditions (63)
\[
\partial_x \tilde{f}_s = 0
\]
\[
\gamma \partial_x \tilde{f}_s = -i \frac{F_0'}{2d\gamma k_0^2} (P_{rbx} b_{ry} - P_{ry} b_{rx}) (70)
\]
Using the general solution (36, 37) we obtain
\[
f_s(0) = -\frac{\xi}{\gamma} G_o (P_{rx} \tilde{f}_y - P_{ry} \tilde{f}_x)
\]
Therefore we get
\[
\text{Im}(e^{i\gamma/2} f_s^e) = -\sqrt{1 - P_1^2} \frac{F_0' G_0 \xi \sqrt{2\gamma}}{2\sqrt{2d\gamma k_0^2}} (P_{rbx} b_{ry} - P_{ry} b_{rx}) \cos \varphi (72)
\]
The anomalous current is given by Eq. (73) and second term in (68)
\[
e_{an} \frac{RI}{2\pi} = \frac{(\chi_l - \chi_r) \sqrt{1 - \frac{P_2}{\gamma}^2} \sqrt{1 - \frac{P_1}{\gamma}^2} \xi F}{2\sqrt{2d\gamma k_0^2}} \sum_{\omega > 0} TF_0'dF_0G_0 k_0^2, (74)
\]
where the chiralities are given by \(\chi_l = P_1 \cdot (b_r \times h)\) and \(\chi_r = P_r \cdot (b_l \times h)\). The usual current is given by
\[
e_{an} \frac{RI}{2\pi} = - (b_{r,i} b_{l,i}) \sqrt{(1 - P_1^2)(1 - P_2^2)} \times \frac{1}{2\gamma^2 d} \sum_{\omega > 0} TF_0'^2 k_N^2. (75)
\]
It is is proportional to the product of the components \(b_{r,i} b_{l,i}\) perpendicular to the exchange field in the ferromagnetic interlayer \(h\).

In general if \(b_r\) and \(b_l\) are non-zero \(I_{an} \ll I_0\). However if either \(b_r = 0\) or \(b_l = 0\) the usual component of Josephson current is absent \(I_0 = 0\). In this case we obtain the giant anomalous Josephson effect when the CPR is given by \(I = I_{an} \sin \varphi\) and the current is maximal at zero phase difference.

Physically the situation equivalent to the case when the exchange field in one of the superconducting electrodes is absent can be realized in the setup with non-homogeneous non-collinear magnetization in the metallic layer shown in Fig. 1c.

C. S/FI/F/F/S structure with non-collinear exchange field

We consider the non-coplanar tri-layered structure shown in Fig. 1c consisting of spin filter and two metallic ferromagnets. The boundary conditions at the left electrode \(x = -d/2\)
\[
\gamma \partial_x f_s = -F_0 \sqrt{1 - P_1^2} e^{i\gamma/2} (77)
\]
\[
\gamma \partial_x f_s = iF_0 (P^l_{yx} f_y - P^l_{xfy}) (78)
\]
\[
\gamma \partial_x f_s = iG_0 (P^l_{yx} f_y - P^l_{xfy}) (79)
\]
and at the right electrode \(x = d/2 + d_1\)
\[
\gamma \partial_x f_s = F_0 e^{i\gamma/2} (81)
\]
\[
\gamma \partial_x f_s = 0. (82)
\]
To obtain the boundary conditions at \(x = d/2\) we can integrate through the layer \(d/2 < x < d/2 + d_1\) to obtain the effective boundary conditions at \(x = d/2\) which read as
\[
\gamma \partial_x f_s = F_0 e^{i\gamma/2} - i \frac{d_1}{D} (h_1 f_s) (83)
\]
\[
\gamma \partial_x f_t = -i \frac{d_1}{D} h_1 f_s (84)
\]
and at $x = -d/2$

$$\gamma \partial_x f_s = -F_0 \sqrt{1 - P^2_t} e^{-i\varphi/2}$$

$$\gamma \partial_x f_z = 0$$

$$\gamma \partial_x f_x = iG_0(P^t_y f_z - P^t_z f_y) + G_0 f_x$$

$$\gamma \partial_x f_y = iG_0(P^t_z f_x - P^t_x f_z) + G_0 f_y$$

These boundary conditions are qualitatively similar to (81,84).

Boundary conditions (81) yield the current given by

$$\frac{eRI}{2\pi} = T \sum_{\omega > 0} \frac{F_0}{\gamma} \text{Im}[e^{i\varphi/2} f_s^*]$$

where $f_s^* = f_s^*(d/2)$. To find the current we need to determine corrections $f_x$ with the help of boundary conditions (83) due to the triplet components generated at the $x = -d/2$ boundary. In this way we search the corrections to the short-range solution $\tilde{f}, \tilde{f}$ in the form (86) with the amplitudes determined by the boundary condition (83,84). Thus we obtain

$$\tilde{f}_s = -i\frac{d_1 \xi_F}{\sqrt{2D}} (h_1 f_t)$$

The components $f_t$ to be substituted in Eq.(90) can be found using the equation (67)

$$i\hbar h f_t^* = -(h_{1x}^2 + h_{1y}^2) \frac{d_1}{Dk_F^2} f_s^*(d/2)$$

$$+ \beta(h_{1y} P^l_x - h_{1x} P^l_y)f_t^*(-d/2)$$

$$- i\beta^2 P^t_x (h_{1z} P^l_x + h_{1y} P^l_y) + i\beta^2 P^t_x (h_{1z} P^l_x + h_{1y} P^l_y) f_t^*(-d/2),$$

where $\beta$ is given by (44). Using Eqs.(38,41) we obtain

$$\text{Im}[e^{i\varphi/2} f_s^*] = \frac{d_1 \xi_F}{\sqrt{2D}} \text{Im}[ie^{i\varphi/2} (h_1 f_t^*)].$$

Thus the anomalous and usual parts of the current (89) are given by

$$\frac{eRI_{an}}{2\pi} = \sqrt{1 - P^2_t} (h_{1y} P^l_x - h_{1x} P^l_y) \frac{d_1 T}{2\gamma^2 h} \sum_{\omega > 0} \beta F^2_0$$

$$\frac{eR_0}{2\pi} = \sqrt{1 - P^2_t} (h_{1z} P^l_x + h_{1y} P^l_y) \frac{d_1 T}{8\gamma^4 d^2 h} \sum_{\omega > 0} \beta^2 F^2_0$$

These expressions can be rewritten in the coordinate-independent form

$$\frac{eRI_{an}}{2\pi} = \chi \sqrt{1 - P^2_t} \frac{d_1 T}{4\gamma^3 d} \sum_{\omega > 0} \frac{F^2_0 G_0}{k^2_\omega}$$

$$\frac{eR_0}{2\pi} = \sqrt{1 - P^2_t} (P_1 h_{1\perp}) \frac{d_1 T}{8\gamma^4 d^2 h} \sum_{\omega > 0} \frac{F^2_0 G^2_0}{k^4_\omega}$$

where the chirality is given by $\chi = (P_1 \times h_1) h$ and $h_{1\perp} = h_1 - h(hh_1)/\hbar^2$ is the perpendicular component of the exchange field $h_1$. The usual current is given by the higher order corrections in the tunnel barrier transparency $I_0 \propto \gamma^{-3}$ than the anomalous one $I_{an} \propto \gamma^{-3}$. Therefore in the tunnelling limit $I_{an} \gg I_0$. 

