Mass concentration and local uniqueness of ground states for $L^2$-subcritical nonlinear Schrödinger equations

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Abstract. We consider ground states of $L^2$-subcritical nonlinear Schrödinger equation (1.1), which can be described equivalently by minimizers of the following constraint minimization problem

$$e(\rho) := \inf \left\{ E_\rho(u) : \ u \in \mathcal{H}(\mathbb{R}^d), \|u\|_2^2 = 1 \right\}.$$  

The energy functional $E_\rho(u)$ is defined by

$$E_\rho(u) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} V(x)|u|^2 \, dx - \frac{\rho^{p-1}}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx,$$

where $d \geq 1$, $\rho > 0$, $p \in (1, 1 + \frac{4}{d})$ and $0 \leq V(x) \to \infty$ as $|x| \to \infty$. We present a detailed analysis on the concentration behavior of ground states as $\rho \to \infty$, which extends the concentration results shown in [27]. Moreover, the uniqueness of nonnegative ground states is also proved when $\rho$ is large enough.

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1. Introduction

In this paper, we study the following time-independent nonlinear Schrödinger equation

$$-\Delta u + V(x)u = \mu u + \rho^{p-1}u^p, \quad u \in \mathcal{H}(\mathbb{R}^d),$$  

where $d \geq 1$, $\mu \in \mathbb{R}$, $p \in (1, 1 + \frac{4}{d})$, and $\rho > 0$ describes the strength of the attractive interactions.

The space $\mathcal{H}(\mathbb{R}^d)$ is defined as

$$\mathcal{H}(\mathbb{R}^d) := \left\{ u(x) \in H^1(\mathbb{R}^d) \ | \ \int_{\mathbb{R}^d} V(x)|u(x)|^2 \, dx < \infty \right\},$$

and the associated norm is given by $\|u\|_{\mathcal{H}} = \left\{ \int_{\mathbb{R}^d} \left( |\nabla u(x)|^2 + [1 + V(x)]|u(x)|^2 \right) \, dx \right\}^{\frac{1}{2}}$. Equation (1.1) arises in Bose–Einstein condensates (BEC) and nonlinear optics. In particular, when $p = 3$ and $d = 1$, it is the well-known time-independent Gross–Pitaevskii (GP) equation which describes the one-dimensional

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of trapping potentials. Furthermore, the local uniqueness of minimizers as $\rho \to \infty$ is also addressed. Toward this purpose, it is necessary to analyze the concentration behavior of minimizers as $\rho \to \infty$. However, the concentration results shown in [27] are not enough, and we therefore need to give a more detailed analysis on the limit behavior of minimizers for $e(\rho)$ as $\rho \to \infty$. Besides, the equivalence between ground states of (1.1) and constraint minimizers of (1.4) is also addressed.

BEC problem; see, e.g., [12,13,31] and the references therein. From the physical point of view, we assume that the trapping potential $V(x)$ satisfies

$$V(x) \in L^\infty_{loc}(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d) \text{ with } \alpha \in (0,1), \inf_{x \in \mathbb{R}^d} V(x) = 0 \text{ and } \lim_{|x| \to \infty} V(x) = \infty. \quad (1.3)$$

It is well known that a minimizer of the following minimization problem solves equation (1.1) for some suitable Lagrange multiplier $\mu$,

$$e(\rho) := \inf \left\{ E_\rho(u) : u \in \mathcal{H}(\mathbb{R}^d), \|u\|_2^2 = 1 \right\}, \quad (1.4)$$

where $E_\rho(u)$ is the Gross–Pitaevskii (GP) energy functional defined by

$$E_\rho(u) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} V(x)|u|^2 \, dx - \frac{\rho^{p-1}}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx. \quad (1.5)$$

Equivalence between ground states of equation (1.1) and constraint minimizers of (1.4) is proved in Theorem 1.1. To discuss equivalently ground states of (1.1), in this paper, we shall therefore focus on investigating (1.4), instead of (1.1). On the other hand, as for the general constraint $\|u\|_2 = N \in (0,\infty)$, one can check that this latter case can be easily reduced to (1.4), by minimizing (1.5) under the constraint $\|u\|_2 = 1$ but simply replacing $\rho$ by $\frac{\rho}{N}$.

When $p > 1 + \frac{4}{d}$, (1.4) is the so-called $L^2$-supercritical problem (also known as mass supcritical problem). Taking a suitable trial function and substituting it into (1.5), one can check that problem (1.4) admits no minimizer for any $\rho \in (0,\infty)$ under this case. For the case $p = 1 + \frac{4}{d}$, (1.4) is known as the $L^2$-critical problem (also called mass critical problem). Recently, some interesting results on this $L^2$-critical problem have been obtained by Guo and his co-authors (cf. [16]-[20]). Roughly speaking, the authors proved in [17] that there exists a finite value $\rho^*$ such that (1.4) admits minimizers if and only if $\rho < \rho^*$ (see also [1] for similar results). The threshold value $\rho^*$ is determined by $\|\omega\|_2$, where $\omega$ is the unique (up to translations) positive radially symmetric solution of the following nonlinear scalar field equation (cf. [10,22,23,28])

$$\Delta w - w + w^p = 0, \quad w \in H^1(\mathbb{R}^d). \quad (1.6)$$

The concentration behavior of minimizers as $\rho \nearrow \rho^*$ was also analyzed in [17,18,20] under different types of trapping potentials. Furthermore, the local uniqueness of minimizers as $\rho \nearrow \rho^*$ was proved in [16], where the trapping potential is a class of homogeneous functions. Besides, when $-\Delta$ is replacing by $-\varepsilon\Delta$ in (1.1), there are many works on the concentration results of the semiclassical states of equation (1.1); see e.g., [4–8,14,15,30,35] and the references therein.

As for the $L^2$-subcritical case, i.e., $1 < p < 1 + \frac{4}{d}$, problem (1.4) admits minimizers for any $\rho \in (0,\infty)$; see, e.g., [3,19,27,32,38]. Some qualitative properties of minimizers for (1.4), such as uniqueness, concentration behavior and symmetry, were also studied in [19,27,38] and the references therein. In detail, M. Maeda showed in [27] that minimizers of (1.4) are unique when $\rho$ is small enough and the minimizers must concentrate at a global minimum of $V(x)$ as $\rho \to \infty$. Further, Guo, Zeng and Zhou presented in [19] a detailed analysis on the concentration behavior of minimizers for $e(\rho)$ with $d = 2$ as $q \nearrow 3$, and more recently, Zeng has generalized these results in [38].

Motivated by the works mentioned above, in this paper we focus on proving the local uniqueness of minimizers for the $L^2$-subcritical problem (1.4) as $\rho \to \infty$. Toward this purpose, it is necessary to analyze the concentration behavior of minimizers as $\rho \to \infty$. However, the concentration results shown in [27] are not enough, and we therefore need to give a more detailed analysis on the limit behavior of minimizers for $e(\rho)$ as $\rho \to \infty$. Besides, the equivalence between ground states of (1.1) and constraint minimizers of (1.4) is also addressed.
Before stating our results, we need to introduce the following classical Gagliardo–Nirenberg-type inequality (cf. [36])

\[ C_{GN} \leq \frac{\|\nabla u\|^p}{\|u\|^p_{p+1}} \quad \text{for any } u \in H^1(\mathbb{R}^d) \setminus \{0\}, \tag{1.7} \]

where

\[ C_{GN} := \|w\|^p \left(1 - \frac{p-1}{p+1} \right) \left[ \frac{d(p-1)}{2(p+1) - d(p-1)} \right]^{\frac{2}{p-1}}, \tag{1.8} \]

and \(w\) is the unique positive solution of (1.6). The equality in (1.7) is attained at \(w\). Applying the following Pohozaev identity of (1.6) (cf. [3, Lemma 8.1.2])

\[ (d-2) \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + d \int_{\mathbb{R}^d} w^2 \, dx = \frac{2d}{p+1} \int_{\mathbb{R}^d} w^{p+1} \, dx, \tag{1.9} \]

one can deduce from (1.6) that \(w\) satisfies

\[ \int_{\mathbb{R}^d} |\nabla w|^2 \, dx = \frac{d(p-1)}{2p+1} \int_{\mathbb{R}^d} w^{p+1} \, dx = \frac{d(p-1)}{2(p+1) - d(p-1)} \int_{\mathbb{R}^d} w^2 \, dx. \tag{1.10} \]

Note also from [10, Proposition 4.1] that \(w(x)\) decays exponentially in the sense that

\[ w(x), |\nabla w(x)| = O(|x|^{-\frac{d-1}{2}} e^{-|x|}) \quad \text{as } |x| \to \infty. \tag{1.11} \]

Our first result is concerned with the equivalence between minimizers of (1.4) and ground states of (1.1). For convenience, we introduce some notations in advance. For any given \(\rho \in (0, \infty)\), the set of nontrivial weak solutions for (1.1) is defined by

\[ S_{\mu,\rho} := \left\{ u \in \mathcal{H} \setminus \{(0)\} : \langle F'_{\mu,\rho}(u), \varphi \rangle = 0, \forall \varphi \in \mathcal{H} \right\}, \]

where the energy functional \(F_{\mu,\rho}(u)\) is defined as

\[ F_{\mu,\rho}(u) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} (V(x) - \mu) |u|^2 \, dx - \frac{p-1}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx. \tag{1.12} \]

Further, the set of ground states for (1.1) is given by

\[ G_{\mu,\rho} := \left\{ u \in S_{\mu,\rho} : F_{\mu,\rho}(u) \leq F_{\mu,\rho}(v) \quad \text{for all } v \in S_{\mu,\rho} \right\}. \tag{1.13} \]

Moreover, the set of minimizers for \(e(\rho)\) is defined as

\[ M_{\rho} := \left\{ u_\rho \in \mathcal{H}(\mathbb{R}^d) : u_\rho \text{ is a minimizer of } e(\rho) \right\}. \tag{1.14} \]

Our first result is stated as the following theorem.

**Theorem 1.1.** Suppose \(V(x)\) satisfies (1.3). Then we have the following.

(i) For a.e. \(\rho \in (0, \infty)\), all minimizers of \(e(\rho)\) satisfy Eq. (1.1) with a fixed Lagrange multiplier \(\mu = \mu_\rho\).

(ii) For a.e. \(\rho \in (0, \infty)\), \(G_{\mu,\rho} = M_{\rho}\).

Theorem 1.1 indicates that, for a.e. \(\rho \in (0, \infty)\), there exists a unique \(\mu = \mu_\rho\) such that Eq. (1.1) admits ground states, which are equivalent to minimizers of \(e(\rho)\) in the sense that \(G_{\mu,\rho} = M_{\rho}\). Theorem 1.1 is largely inspired by some similar conclusions on different types of problems, such as [3, Chapter 8], [18, Theorem 1.1] and [24, Theorem 1.2]. The proof of Theorem 1.1 is given in “Appendix A.1.”

We next focus on analyzing the limit behavior of minimizers as \(\rho \to \infty\). Since \(|\nabla u| \leq |\nabla u|\) holds for a.e. \(x \in \mathbb{R}^d\), without loss of generality, we always suppose minimizers of \(e(\rho)\) are nonnegative. Motivated by [17–20], in order to analyze the blowup behavior of minimizers as \(\rho \to \infty\), some additional assumptions on \(V(x)\) are required.
Theorem 1.2. Suppose now give the following theorem on the blowup behavior of nonnegative minimizers as $\rho$ that satisfies $h$.

Definition 1.1. $h(x)$ in $\mathbb{R}^d$ is homogeneous of degree $q \in \mathbb{R}^+$ (about the origin), if there exists some $q > 0$ such that

$$h(tx) = t^q h(x) \text{ in } \mathbb{R}^d \text{ for any } t > 0.$$  

This definition indicates that if $h(x) \in C(\mathbb{R}^d)$ is homogeneous of degree $q > 0$, then

$$0 \leq h(x) \leq C|x|^q \text{ in } \mathbb{R}^d,$$

where $C := \max_{x \in \partial B_1(0)} h(x)$, because $h(\frac{x}{|x|}) \leq C$ for any $x \in \mathbb{R}^d \setminus \{0\}$. Moreover, if $h(x) \to \infty$ as $|x| \to \infty$, then 0 is the unique minimum point of $h(x)$.

Define the set of global minimum points of $V(x)$ by

$$Z := \{ x \in \mathbb{R}^d : V(x) = 0 \} = \{ x_1, x_2, \ldots, x_m \}, \text{ where } m \geq 1. \hspace{1cm} (1.15)$$

We then assume that $V(x)$ is almost homogeneous of degree $r_i > 0$ around each $x_i$. Specifically, there exists some $V_i(x) \in C^2_{\text{loc}}(\mathbb{R}^d)$, which is homogeneous of degree $r_i > 0$ and satisfies $\lim_{|x| \to \infty} V_i(x) = +\infty$, such that

$$\lim_{x \to 0} \frac{V(x + x_i)}{V_i(x)} = 1, \quad i = 1, 2, \ldots, m. \hspace{1cm} (1.16)$$

Additionally, inspired by [14], we define $Q_i(y)$ by

$$Q_i(y) := \int_{\mathbb{R}^d} V_i(x + y) w^2 \, dx, \quad i = 1, 2, \ldots, m. \hspace{1cm} (1.17)$$

Set

$$r := \max_{1 \leq i \leq m} r_i, \quad Z := \{ x_i \in Z : r_i = r \} \subset Z, \hspace{1cm} (1.18)$$

and

$$\tilde{\lambda}_0 := \min_{i \in \Gamma} \tilde{\lambda}_i, \text{ where } \tilde{\lambda}_i := \min_{y \in \mathbb{R}^d} Q_i(y) \text{ and } \Gamma := \{ i : x_i \in Z \}. \hspace{1cm} (1.19)$$

Besides, we also introduce some useful notations,

$$\lambda := \frac{1}{2} \frac{4 - d(p - 1)}{2(p + 1) - d(p - 1)}, \hspace{1cm} (1.20)$$

$$Q(y) := \int_{\mathbb{R}^d} V_0(x + y) w^2 \, dx, \text{ where } V_0(x) := V_i(x) \text{ and } i \text{ satisfies } \tilde{\lambda}_i = \tilde{\lambda}_0, \hspace{1cm} (1.21)$$

and

$$Z_0 := \{ x_i \in Z : \tilde{\lambda}_i = \tilde{\lambda}_0 \}, \quad K_0 := \{ y : Q(y) = \tilde{\lambda}_i = \tilde{\lambda}_0 \}, \hspace{1cm} (1.22)$$

where $Z_0$ denotes the set of the flattest global minimum points of $V(x)$. Stimulated by [15, 17, 27, 34], we now give the following theorem on the blowup behavior of nonnegative minimizers as $\rho \to \infty$.

Theorem 1.2. Suppose $V(x) \in C^2(\mathbb{R}^d)$ satisfies (1.3) and (1.16), and there exists a constant $\kappa > 0$ such that

$$V(x) \leq Ce^{\kappa|x|} \text{ if } |x| \text{ is large.} \hspace{1cm} (1.23)$$

Set $a^* := \|w\|^2_2$, where $w$ is the unique positive solution of (1.6). Let $u_k$ be a nonnegative minimizer of $e(\rho_k)$, where $\rho_k \to \infty$ as $k \to \infty$. Then there exists a subsequence, still denoted by $\{u_k\}$, such that $u_k$ satisfies

$$\bar{u}_k(x) := \sqrt{a^*} \varepsilon_k \frac{d}{k} u_k(\varepsilon_k x + x_k) \to w(x) \text{ uniformly in } \mathbb{R}^d \text{ as } k \to \infty, \hspace{1cm} (1.24)$$

where

$$\varepsilon_k := \left( \frac{\rho_k}{\sqrt{a^*}} \right)^{-\frac{2(p-1)}{d(p-1)}}. \hspace{1cm} (1.25)$$
and \( x_k \) is the unique local maximum point of \( u_k \) satisfying
\[
\frac{x_k - x_0}{\varepsilon_k} \to y_0 \quad \text{for some } x_0 \in Z_0 \text{ and } y_0 \in K_0 \text{ as } k \to \infty.
\]

(1.26)

Furthermore, \( u_k \) decays exponentially in the sense that
\[
\bar{u}_k(x) \leq C e^{-\frac{|x|^2}{2}} \text{ and } |
abla \bar{u}_k| \leq C e^{-\frac{|x|^2}{2}} \quad \text{as } |x| \to \infty,
\]

(1.27)

where \( C > 0 \) is a constant independent of \( k \).

Theorem 1.2 shows that minimizers of \( e(\rho) \) must concentrate at one of the flattest global minimum points of \( V(x) \) as \( \rho \to \infty \). Some results similar to (1.24) were also obtained in [27]. In Sect. 2, we shall present a different proof for (1.24) by employing the refined energy estimates in Lemma 2.1 and blowup analysis in Lemma 2.4. Here we point out that the \( L^2 \)-subcritical nonlinearity term will lead to some difficulties on analyzing the limit behavior of minimizers, due to that the Gagliardo–Nirenberg inequality cannot be used directly. This is quite different from those obtained in [18,20], where the difficulties on analyzing the limit behavior of minimizers, due to that the Gagliardo–Nirenberg inequality cannot be used directly. Toward this aim, we have to employ the fact that \( e(\rho) \geq \bar{e}(\rho) \), where \( \bar{e}(\rho) \) is a new minimization problem defined in (A.8). Moreover, (1.26) gives the convergence rate of the unique maximum point of each minimizer as \( \rho \to \infty \), which is based on a more precise energy estimate of \( e(\rho) \). In fact, we shall show that
\[
e(\rho) = -\lambda \left( \frac{\rho}{\sqrt{a^*}} \right)^{\frac{4r(d-1)}{4-4d(p-1)}} + \bar{\lambda}_0 + o(1) \left( \frac{\rho}{\sqrt{a^*}} \right)^{-\frac{2p(p-1)}{4-4d(p-1)}} \quad \text{as } \rho \to \infty,
\]

(1.28)

where \( a^* := ||w||_2^2 \), \( r \), \( \bar{\lambda}_0 \) and \( \lambda \) are, respectively, defined by (1.18), (1.19) and (1.20).

Motivated by the uniqueness results addressed in [2,9,14,16], we finally investigate the uniqueness of nonnegative minimizers for \( e(\rho) \) as \( \rho \to \infty \). Toward this purpose, we require some additional conditions on \( V(x) \). Suppose \( V(x) \) admits a unique flattest minimum point \( x_0 \), i.e.,
\[
Z_0 \text{ contains only one element } x_0, \text{ where } Z_0 \text{ is defined in (1.22)}.
\]

(1.29)

Further, we suppose that
\[
V(x) \text{ is homogeneous of degree } r \geq 2 \text{ near } x_0.
\]

(1.30)

Moreover, we also assume that there exists a constant \( R_0 \) small enough such that
\[
\frac{\partial V(x + x_0)}{\partial x_i} = \frac{\partial V_0(x)}{\partial x_i} + W_i(x) \quad \text{and} \quad |W_i(x)| \leq C|x|^s_i \quad \text{in } B_{R_0}(0),
\]

(1.31)

where \( x_0 \in Z_0, V_0 \) is given in (1.21), and \( s_i > r - 1 \) for \( i = 1, 2, \ldots, m \). Under these assumptions, our uniqueness results can be stated as the following theorem.

**Theorem 1.3.** Suppose \( V(x) \in C^2(\mathbb{R}^d) \) satisfies (1.3), (1.16), (1.23) and (1.29)-(1.31). Moreover, we also assume that
\[
y_0 \text{ is the unique and nondegenerate critical point of } Q(y),
\]

(1.32)

where \( Q(y) \) is defined by (1.21). Then there exists a unique nonnegative minimizer for \( e(\rho) \) as \( \rho \to \infty \).

Theorem 1.3 indicates that problem (1.4) admits only one nonnegative minimizer when \( \rho \) is large enough. Together with Theorem 1.1 and the uniqueness results given in [27, Theorem 1.2], one can conclude that, for any given \( \rho \) where \( \rho \) is small enough or large enough, there exists a unique \( \mu = \mu_\rho \) such that
\[
(1.1) \text{ admits one and only one nonnegative ground state.}
\]

Theorem 1.3 is proved by establishing various types of local Pohozaev identities, which is inspired by the [2,9,14,16]. However, the proof of Theorem 1.3 requires more involved and intricate calculations,
because of the general assumptions on dimension and trapping potentials. Moreover, comparing with discussing $L^2$-critical problem, the appearance of $L^2$-subcritical term also leads to some essential differences on deriving the second Pohozaev identity.

This paper is organized as follows. Sect. 2 is concerned with proving Theorem 1.2 on the limit behavior of minimizers for $e(\rho)$ as $\rho \to \infty$. The main purpose of Sect. 3 is to prove the local uniqueness of nonnegative minimizers by deriving local Pohozaev identities. The proof of Theorem 1.1 is given in “Appendix A.1,” and we also give some useful results on $\tilde{e}(\rho)$ in “Appendix A.2.”

2. Mass concentration

In this section, we shall prove Theorem 1.2 on the limit behavior of minimizers for $e(\rho)$ as $\rho \to \infty$. We shall firstly establish the optimal energy estimates for $e(\rho)$ and then present a detailed analysis on the limit behavior of minimizers as $\rho \to \infty$.

2.1. Refined energy estimates

The main purpose of this section is to establish the refined estimates of $e(\rho)$ by the following lemma.

**Lemma 2.1.** Suppose $V(x)$ satisfies (1.3), and then, we have

$$\lim_{\rho \to \infty} \frac{e(\rho)}{\left(\frac{\rho}{\sqrt{a^*}}\right)^{\frac{4(p-1)}{4-d(p-1)}}} = -\lambda,$$

where $\lambda$ is given in (1.20), $a^* := \|w\|_2^2$ and $w$ is the unique positive solution of (1.6).

**Proof.** We start with the upper bound estimate on the energy $e(\rho)$ as $\rho \to \infty$. Suppose $\chi(x) \in C^\infty(\mathbb{R}^d)$ is a cutoff function satisfying $\chi(x) = 1$ as $|x| \leq 1$ and $\chi(x) = 0$ as $|x| \geq 2$. Choose a trial function

$$u_\tau(x) := \frac{A_\tau}{\|w\|_2^2} \tau^{\frac{d}{4}(p-1)} w(\tau x) \chi(x),$$

where $\tau = \left(\frac{\rho}{\sqrt{a^*}}\right)^{\frac{2(p-1)}{4-d(p-1)}}$, and $A_\tau$ is chosen such that $\|u_\tau\|_2^2 = 1$. Applying the identity (1.10) and the exponential decay of $w$ in (1.11), some calculations yield that

$$e(\rho) \leq E_\rho(u_\tau) \leq \frac{1}{2} \frac{d(p-1)}{2(p+1) - d(p-1)} \tau^2 - \frac{2\left(\frac{\rho}{\sqrt{a^*}}\right)^{p-1}}{2(p+1) - d(p-1)} \tau^{\frac{d}{4}(p-1)} + O(1)$$

$$= - (1 + o(1))\lambda \left(\frac{\rho}{\sqrt{a^*}}\right)^{\frac{4(p-1)}{4-d(p-1)}}$$

as $\rho \to \infty$,

where $\lambda$ is given in (1.20). This gives the upper bound of $e(\rho)$ as $\rho \to \infty$.

Next, we shall establish the lower bound estimate of $e(\rho)$ as $\rho \to \infty$ by employing the estimate of $\tilde{e}(\rho)$ given in (A.10), where $\tilde{e}(\rho)$ is a new minimization problem defined by (A.8). Let $u_\rho$ be a nonnegative minimizer of $e(\rho)$ with $\rho \to \infty$. Since $\int_{\mathbb{R}^d} V(x)u_\rho^2 \, dx \geq 0$, one can then deduce from (1.4), (A.8) and (A.10) that

$$e(\rho) \geq \tilde{e}(\rho) = -\lambda \left(\frac{\rho}{\sqrt{a^*}}\right)^{\frac{4(p-1)}{4-d(p-1)}}$$

as $\rho \to \infty$.

Combining the upper and lower bound estimates then yields (2.1), and this lemma is then proved. \qed
2.2. Blowup analysis

In this section, we shall complete the proof of Theorem 1.2. Let $u_k$ be a nonnegative minimizer of $e(\rho_k)$ with $\rho_k \to \infty$ as $k \to \infty$, and then, $u_k$ satisfies (1.1) for some suitable $\mu_k$. We firstly give the following lemma.

**Lemma 2.2.** Suppose $V(x)$ satisfies (1.3). Let $u_k$ be a nonnegative minimizer of $e(\rho_k)$ with $\rho_k \to \infty$ as $k \to \infty$. We then have

$$0 \leq e(\rho_k) - \bar{e}(\rho_k) \to 0 \quad \text{as} \quad k \to \infty,$$

and

$$\int_{\mathbb{R}^d} V(x)u_k^2 \, dx \to 0 \quad \text{as} \quad k \to \infty. \quad (2.4)$$

As for the proof of this lemma, one can refer to [27, Lemma 4.2] and we omit it here. □

Define

$$\hat{\varepsilon}_k := \rho_k^{\frac{2(p-1)}{4(p-1)}}$$

and

$$\hat{w}_k := \hat{\varepsilon}_k^2 u_k(\hat{\varepsilon}_k x). \quad (2.5)$$

Some calculations yield that

$$\int_{\mathbb{R}^d} |\nabla u_k|^2 \, dx = \hat{\varepsilon}_k^{-2} \int_{\mathbb{R}^d} |\nabla \hat{w}_k|^2 \, dx$$

and

$$\rho_k^{p-1} \int_{\mathbb{R}^d} u_k^{p+1} \, dx = \hat{\varepsilon}_k^{-2} \int_{\mathbb{R}^d} \hat{w}_k^{p+1} \, dx.$$

We now give the following lemma on the boundedness of $\int_{\mathbb{R}^d} |\nabla \hat{w}_k|^2 \, dx$ and $\int_{\mathbb{R}^d} \hat{w}_k^{p+1} \, dx$.

**Lemma 2.3.** Suppose $V(x)$ satisfies (1.3). Let $u_k$ be a nonnegative minimizer of $e(\rho_k)$ with $\rho_k \to \infty$ as $k \to \infty$. Then one has

$$C_1 \leq \int_{\mathbb{R}^d} |\nabla \hat{w}_k|^2 \, dx \leq C_2 \text{ and } C_1' \leq \int_{\mathbb{R}^d} \hat{w}_k^{p+1} \, dx \leq C_2', \quad (2.6)$$

where $\hat{w}_k$ is defined by (2.5), $C_1$, $C_2$, $C_1'$ and $C_2'$ are positive constants independent of $k$.

**Proof.** It follows from (1.5) and (2.1) that

$$(\sqrt{a^*})^{\frac{4(p-1)}{2(p-1)-1}} \hat{\varepsilon}_k^2 e(\rho_k) = \int_{\mathbb{R}^d} |\nabla \hat{w}_k|^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^d} \hat{w}_k^{p+1} \, dx \to -\lambda < 0 \quad \text{as} \quad k \to \infty, \quad (2.7)$$

where $\lambda$ is given in (1.20). Using the Gagliardo–Nirenberg inequality (1.7), one can then derive from (2.7) that $\int_{\mathbb{R}^d} |\nabla \hat{w}_k|^2 \, dx \leq C_2$ and $\int_{\mathbb{R}^d} \hat{w}_k^{p+1} \, dx \leq C_2'$. As for the lower bounds, from (2.7) one can deduce that $\int_{\mathbb{R}^d} \hat{w}_k^{p+1} \, dx \geq C_1'$, and using the Gagliardo–Nirenberg inequality (1.7) then yields $\int_{\mathbb{R}^d} |\nabla \hat{w}_k|^2 \, dx \geq C_1$. Hence, we complete the proof of this lemma. □

Motivated by [18, 20, 34, 37, 38], we then give the following lemma, which is a weak version of Theorem 1.2.

**Lemma 2.4.** Suppose $V(x)$ satisfies (1.3). Let $u_k$ be a nonnegative minimizer of $e(\rho_k)$ with $\rho_k \to \infty$ as $k \to \infty$. We then have the following.
(i). There exist a sequence \( \{y_k\} \subset \mathbb{R}^d \) and positive constants \( \iota \) and \( R_0 \) such that
\[
\liminf_{k \to \infty} \int_{B_{R_0}(0)} \hat{w}^{p+1}_k \, dx \geq \iota > 0. \tag{2.8}
\]
(ii). The sequence \( \{y_k\} \) satisfies that, passing to a subsequence if necessary, \( \hat{\varepsilon}_k y_k \to z_0 \) for some \( z_0 \in \mathbb{R}^d \) satisfying \( V(z_0) = 0 \).
(iii). Defining
\[
w_k := \hat{w}_k(x + y_k) = \hat{\varepsilon}_k^2 u_k(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k), \tag{2.9}
\]
and then passing to a subsequence if necessary, there holds that
\[
\lim_{k \to \infty} w_k = \left(\sqrt{a^*}\right)^{-\frac{d(p-1)}{2(p+1)}} w \left(\left(\sqrt{a^*}\right)^{-\frac{2(p-1)}{2(p+1)}} x + x_0^0\right), \tag{2.10}
\]
strongly in \( H^1(\mathbb{R}^d) \) for some \( x_0^0 \in \mathbb{R}^d \), where \( a^* := \|w\|_2^2 \) and \( w \) is the unique (up to translations) positive solution of (1.6).

Proof. (i). As for (2.8), if it is false, then for any \( R > 0 \), there exists a subsequence of \( \hat{\varepsilon}_k \) (still denoted by \( \hat{\varepsilon}_k \)) such that \( \lim_{k \to \infty} \sup_{y \in \mathbb{R}^d} \int_{B_R(y)} \hat{w}^{p+1}_k \, dx = 0 \). Applying [26, Lemma 1.1] then yields that \( \hat{w}_k \to 0 \) in \( L^{p+1}(\mathbb{R}^d) \) as \( k \to \infty \), which, however, contradicts (2.6).
(ii). Employing (2.4), this conclusion can be obtained by using the proof by contradiction. Since the proof is similar to that of [20, Lemma 2.3], we omit it here.
(iii). It follows from (1.1) and (2.5) that \( w_k \) solves
\[
-\Delta w_k + \hat{\varepsilon}_k^2 V(\hat{\varepsilon}_k x + \hat{\varepsilon}_k y_k) w_k = \hat{\varepsilon}_k^2 \mu_k w_k + u_k^p. \tag{2.11}
\]
Following (1.5) and (2.1), one can deduce that
\[
\hat{\varepsilon}_k^2 \mu_k = 2\hat{\varepsilon}_k^2 c(\rho_k) - \frac{p-1}{p+1} \int_{\mathbb{R}^d} w_k^{p+1} \, dx < 0. \tag{2.12}
\]
Using (2.6) and (2.7), one can then obtain the uniform boundedness of \( \{\hat{\varepsilon}_k^2 \mu_k\} \) as \( k \to \infty \), which indicates that passing to a subsequence if necessary, \( \hat{\varepsilon}_k^2 \mu_k \to -\beta \) for some \( \beta \in \mathbb{R}^+ \) as \( k \to \infty \). From (2.6), one can deduce that \( w_k \) is bounded uniformly in \( H^1(\mathbb{R}^d) \). Taking \( k \to \infty \), passing to a subsequence if necessary, one then has \( w_k \to w_0 \geq 0 \) in \( H^1(\mathbb{R}^d) \) for some \( w_0 \in H^1(\mathbb{R}^d) \) satisfying
\[
-\Delta w_0 + \beta w_0 = w_0^p. \tag{2.13}
\]
Applying the maximum principle, one can then conclude from (2.8) that \( w_0 > 0 \), which implies from (1.6) that \( w_0 = \beta \frac{1}{p+1} w(\beta \frac{1}{2} x + x_0^0) \) for some \( x_0^0 \in \mathbb{R}^d \), because of the uniqueness (up to translations) of positive solution of (1.6).

Here we claim that
\[
\|w_0\|_2^2 = 1 \quad \text{and} \quad \beta = \|w\|_2^{\frac{d(p-1)}{2(p+1)}}. \tag{2.14}
\]
From the following Pohozaev identity of (2.13) (cf. [3, Lemma 8.1.2]),
\[
(d-2) \int_{\mathbb{R}^d} |\nabla w_0|^2 \, dx + d\beta \int_{\mathbb{R}^d} w_0^2 \, dx = \frac{2d}{p+1} \int_{\mathbb{R}^d} w_0^{p+1} \, dx,
\]
on one can derive that
\[
\int_{\mathbb{R}^d} |\nabla w_0|^2 \, dx = \frac{p-1}{p+1} \frac{d(p-1)}{2(p+1)} \beta \int_{\mathbb{R}^d} w_0^2 \, dx. \tag{2.15}
\]
Applying the Gagliardo–Nirenberg inequality (1.7) and (2.15), some calculations yield that
\[
C_{GN} \leq \frac{\|\nabla w_0\|_2^{\frac{4}{p-1}} \|w_0\|_{p+1}^{\frac{p+1}{2}}}{\|w_0\|_{p+1}^{\frac{p+1}{2}}} = \left(\frac{p-1}{p+1}\right)^{\frac{4}{p-1}} \left(\frac{2(p+1)\beta}{2(p+1)-d(p-1)}\right)^{\frac{4}{p-1}-1} \|w_0\|_p^{\frac{p}{2}}
\]
where \(C_{GN}\) is defined in (1.8). This gives that \(\beta \geq \|w\|_2^{\frac{4(p-1)}{2(p-1)-d}}\), and further implies that \(\beta = \|w\|_2^{\frac{4(p-1)}{2(p-1)-d}}\), because \(\|w_0\|_2 = \|w\|_p^{\frac{p}{2}} \leq 1\). Moreover, one can deduce that \(\|w_0\|_2^2 = 1\).

Since \(\|w_k\|_2 = \|w_0\|_2 = 1\), passing to a subsequence if necessary, one has \(w_k \to w_0\) strongly in \(L^2(\mathbb{R}^d)\) as \(k \to \infty\). Using the interpolation inequality, one can further derive that \(w_k \to w_0\) strongly in \(L^q(\mathbb{R}^d)\) for any \(q \in [2, 2^*)\) as \(k \to \infty\). Moreover, one can conclude from (2.11) and (2.13) that (2.10) holds. \(\square\)

**Lemma 2.5.** Suppose \(V(x)\) satisfies (1.3). Let \(u_k\) be a nonnegative minimizer of \(e(\rho_k)\) with \(\rho_k \to \infty\) as \(k \to \infty\). Then \(u_k\) admits only one local maximum point \(x_k\), and passing to a subsequence if necessary, there holds that
\[
\bar{u}_k(x) := \sqrt{\alpha^*} \epsilon_k^d u_k(\epsilon_k x + x_k) \to w(x) \text{ strongly in } H^1(\mathbb{R}^d) \text{ as } k \to \infty,
\]
where \(\epsilon_k := \left(\frac{\rho_k}{\sqrt{\alpha^*}}\right)^{\frac{2(p-1)}{4(p-1)-d}}\) is defined in (1.25).

**Proof.** Applying the de Giorgi–Nash–Moser theory (cf. [21, Theorem 4.1]), one can derive from (2.10) and (2.11) that
\[
w_k(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly for large } k,
\]
which indicates that \(u_k(x)\) admits at least one global maximum point. Let \(x_k\) be a global maximum point of \(u_k(x)\) and set \(z_k := \epsilon_k y_k \to x_0\) as \(k \to \infty\). Since \(z_k := \frac{x_k - z_k}{\epsilon_k}\) is a global maximum point of \(w_k(x)\), one can thus derive from (2.8) and (2.17) that
\[
\left\{ \frac{x_k - z_k}{\epsilon_k} \right\} \text{ is bounded uniformly in } \mathbb{R}^d.
\]
Define
\[
\bar{u}_k(x) := \sqrt{\alpha^*} \epsilon_k^d u_k(\epsilon_k x + x_k)
\]
where \(\epsilon_k := \left(\frac{\rho_k}{\sqrt{\alpha^*}}\right)^{\frac{2(p-1)}{4(p-1)-d}}\) is given in (1.25). It then follows from (2.10) that, passing to a subsequence if necessary, \(\bar{u}_k(x) \to w(x + y'_0)\) for some \(y'_0 \in \mathbb{R}^d\) strongly in \(H^1(\mathbb{R}^d)\) as \(k \to \infty\). Since \(V(x) \in C^\alpha(\mathbb{R}^d)\), using the standard elliptic regular theory, we have
\[
\bar{u}_k(x) \to w(x + y'_0) \text{ in } C^2_{loc}(\mathbb{R}^d) \text{ as } k \to \infty,
\]
and one can see [20, Lemma 3.1] for a detailed proof. Note that the origin is a local maximum point of \(\bar{u}_k\) for all \(k > 0\), and it follows from (2.20) that it is also a local maximum point of \(w\). Since \(w(x)\) is radially symmetric about the origin and decreases strictly in \(|x|\) (see, e.g., [10, 23, 36]), we know that \(x = 0\) is the unique local maximum point of \(w(x)\), which thus implies from (2.20) that \(y'_0 = 0\). Hence, it follows that
\[
\bar{u}_k(x) \to w(x) \text{ strongly in } H^1(\mathbb{R}^d) \text{ as } k \to \infty.
\]
We finally prove the uniqueness of the local maximum points of \( u_k \) when \( k \) is sufficiently large. Suppose \( x_k \) is any local maximum point of \( u_k \). It is easy to know that \( \bar{u}_k \) satisfies
\[
- \Delta \bar{u}_k + \varepsilon_k^2 V(\varepsilon_k x + x_k)\bar{u}_k = \varepsilon_k^2 \mu_k \bar{u}_k + \bar{u}_k^p \quad \text{in } \mathbb{R}^d. \tag{2.22}
\]
From this, one can deduce that \( \bar{u}_k(x_k) \geq C_0 > 0 \) when \( k > 0 \) is large enough. This indicates that all local maximum points of \( \bar{u}_k \) must stay in a finite ball \( B_R(0) \) as \( k \to \infty \), where \( R > 0 \) is independent of \( k \). Employing the uniqueness of local maximum points of \( w \), one can deduce from (2.20) that the origin is the unique maximum point of \( \bar{u}_k \), i.e., \( u_k \) admits only one local maximum point \( x_k \) when \( k \) large enough. \( \square \)

**Proof of Theorem 1.2.**

As for the exponential decay of \( u_k \) in (1.27), one can obtain it by using the comparison principle. Similar to (2.12), one can check that \( \varepsilon_k^2 \mu_k \to -1 \) as \( k \to \infty \), and then, we can derive that there exists a constant \( R > 0 \) large enough such that
\[
- \Delta \bar{u}_k + \frac{1}{2} \bar{u}_k \leq 0 \quad \text{and} \quad \bar{u}_k \leq C e^{-\frac{1}{2}R} \quad \text{for } |x| \geq R.
\]
Comparing \( \bar{u}_k \) with \( C e^{-\frac{1}{2}|x|} \) then yields \( \bar{u}_k(x) \leq C e^{-\frac{|x|}{2}} \) when \( |x| \geq R \). Furthermore, applying the standard elliptic regularity theory then yields (1.24) (see, e.g., [27, Lemma 4.9] for similar arguments).

Finally, we aim at proving (1.26). Suppose \( \bar{u}_k \) is a nonnegative minimizer of \( \bar{c}(\rho_k) \), and then, \( \bar{u}_k(x - \varepsilon_k y_0 - x_0) \) is also a nonnegative minimizer of \( \bar{c}(\rho_k) \), where \( x_0 \in Z_0, y_0 \in K_0 \) and \( z_0, k_0 \) are defined by (1.22). We then derive from (1.11), (1.16), (1.23) and (A.11) that
\[
e(\rho_k) - \bar{c}(\rho_k) \leq \int_{\mathbb{R}^d} V(x) \bar{u}_k^2(x - \varepsilon_k y_0 - x_0) \, dx
\leq \frac{1}{a^*} (1 + o(1)) \int_{B_{\frac{1}{\varepsilon_k}}(0)} \frac{V(\varepsilon_k x + \varepsilon_k y_0 + x_0)}{V_0(\varepsilon_k x + \varepsilon_k y_0)} V_0(\varepsilon_k x + \varepsilon_k y_0) w^2 \, dx \tag{2.23}
\leq \frac{1}{a^*} \varepsilon_k^r (1 + o(1)) \int_{\mathbb{R}^d} V_0(x + y_0) w^2 \, dx = \frac{1}{a^*} \varepsilon_k^r (1 + o(1)) \lambda_0,
\]
where \( \lambda_0 \) is given by (1.19). Suppose \( u_k \) is a nonnegative minimizer of \( e(\rho_k) \), and then, one can deduce from (1.16) and (1.27) that
\[
e(\rho_k) - \bar{c}(\rho_k) \geq \int_{\mathbb{R}^d} V(x) u_k^2 \, dx = \frac{1}{a^*} \int_{\mathbb{R}^d} V(\varepsilon_k x + x_k) u_k^2 \, dx
\geq \frac{1}{a^*} \varepsilon_k^r \int_{B_{\frac{1}{\varepsilon_k}}(x_i)} V_i \left( x + \frac{x_k - x_i}{\varepsilon_k} \right) u_k^2 \, dx, \tag{2.24}
\]
where \( x_i \in Z \). Comparing with the upper estimate (2.23), one can directly check that \( r_i = r \) and \( x_i = x_0 \in \bar{Z} \), where \( r \) and \( \bar{Z} \) is given by (1.18). Since \( V(x) \to \infty \) as \( |x| \to \infty \), one can further check that \( \{ \frac{x_k - x_0}{\varepsilon_k} \} \) is bounded uniformly in \( k \). More precisely, one can also verify that, passing to a subsequence if necessary,
\[
\frac{x_k - x_0}{\varepsilon_k} \to y_0,
\]
which implies that \( x_0 \in Z_0 \) and \( Z_0 \) is defined in (1.22), i.e., (1.26) holds. Moreover, we also have
\[
\lim_{k \to \infty} \frac{e(\rho_k) - e(\rho_k)}{\varepsilon_k} = \frac{1}{a^*} \bar{\lambda}_0,
\]
where \( \bar{\lambda}_0 \) is defined by (1.19). This gives (1.28), and the proof of Theorem 1.2 is thus completed. \( \square \)

3. Local uniqueness of nonnegative minimizers

In this section, we focus on the proof of local uniqueness of minimizers as \( \rho \to \infty \). Argue by contradiction. Suppose it is not true, and there exist two different nonnegative minimizers \( u_{ik} \) and \( u_{2k} \) for \( e(\rho_k) \) with \( \rho_k \to \infty \) as \( k \to \infty \). Let \( x_{1k} \) and \( x_{2k} \) denote the unique local maximum point of \( u_{1k} \) and \( u_{2k} \), respectively. Following (1.1), we have

\[
- \Delta u_{ik} + V(x)u_{ik} = \mu_{ik}u_{ik} + \rho_k^{-1}u_{ik}^p \quad \text{in} \quad \mathbb{R}^d, \quad i = 1, 2.
\]

Define
\[
\tilde{u}_{ik}(x) := \sqrt{\frac{d}{a^*}} \varepsilon_k u_{ik}(x) \quad \text{and} \quad \bar{u}_{ik}(x) := \tilde{u}_{ik}(\varepsilon_k x + x_{1k}), \quad i = 1, 2,
\]

where \( \varepsilon_k \) is given by (1.25). Since \( \lim_{k \to \infty} \frac{\varepsilon_k x_{1k}}{x_{2k}} = 0 \), by Theorem 1.2, one then has \( \bar{u}_{ik} \to w(x) \) uniformly in \( \mathbb{R}^d \) as \( k \to \infty \). One can check that \( \bar{u}_{ik} \) satisfies

\[
- \Delta \bar{u}_{ik} + \varepsilon_k^2 V(\varepsilon_k x + x_{1k}) \bar{u}_{ik} = \varepsilon_k^2 \mu_{ik} \bar{u}_{ik} + \bar{u}_{ik}^p \quad \text{in} \quad \mathbb{R}^d, \quad i = 1, 2.
\]

Since \( u_{1k} \neq u_{2k} \), define
\[
\eta_k := \frac{u_{1k} - u_{2k}}{\|u_{1k} - u_{2k}\|_{L^\infty(\mathbb{R}^d)}} \quad \text{and} \quad \bar{\eta}_k := \frac{\bar{u}_{1k} - \bar{u}_{2k}}{\|\bar{u}_{1k} - \bar{u}_{2k}\|_{L^\infty(\mathbb{R}^d)}},
\]

and then, we have \( \bar{\eta}_k = \eta_k \). Further we define
\[
\tilde{\eta}_k(x) := \bar{\eta}_k(\varepsilon_k x + x_{1k}),
\]

and thus, one can deduce from (3.3) that \( \bar{\eta}_k \) satisfies

\[
- \Delta \bar{\eta}_k + \varepsilon_k^2 V(\varepsilon_k x + x_{1k}) \bar{\eta}_k = \varepsilon_k^2 \mu_{ik} \bar{\eta}_k + \tilde{g}_k + \tilde{f}_k(x),
\]

where
\[
\tilde{g}_k(x) := \varepsilon_k^2 \frac{\mu_{ik} - \mu_{2k}}{\|\bar{u}_{1k} - \bar{u}_{2k}\|_{L^\infty(\mathbb{R}^d)}} \bar{u}_{2k} \quad \text{and} \quad \tilde{f}_k(x) := \frac{\bar{u}_{1k}^p - \bar{u}_{2k}^p}{\|\bar{u}_{1k} - \bar{u}_{2k}\|_{L^\infty(\mathbb{R}^d)}}.
\]

Now we give the following lemma on the limit of \( \bar{\eta}_k \).

**Lemma 3.1.** Suppose all the assumptions of Theorem 1.3 hold. Then passing to a subsequence if necessary, \( \bar{\eta}_k \to \bar{\eta}_0 \) in \( C_{loc}(\mathbb{R}^d) \) as \( k \to \infty \), where \( \bar{\eta}_0 \) satisfies

\[
\bar{\eta}_0(x) = b_0 \left( w + \frac{p - 1}{2} \frac{x \cdot \nabla w}{\partial x_i} \right) + \sum_{i=1}^d b_i \frac{\partial w}{\partial x_i},
\]

and \( b_0, b_1, \ldots, b_d \) are some constants.

**Proof.** Since \( \|\bar{\eta}_k\|_{\infty} \leq 1 \), the standard elliptic regularity then implies that \( \|\bar{\eta}_k\|_{C^{1,\alpha}(\mathbb{R}^d)} \leq C \), where \( C \) is a constant independent of \( k \). Therefore, passing to a subsequence if necessary, one can deduce that

\[
\bar{\eta}_k \to \bar{\eta}_0 \quad \text{in} \quad C_{loc}(\mathbb{R}^d) \quad \text{as} \quad k \to \infty
\]

for some function \( \bar{\eta}_0 \in C_{loc}(\mathbb{R}^d) \). Similar to (2.12), from (1.5) and (3.2), one can derive that

\[
\varepsilon_k^2 \mu_{ik} = 2 \varepsilon_k^2 e(\rho_k) - \frac{p - 1}{a^* (p + 1)} \int_{\mathbb{R}^d} |\bar{u}_{ik}|^{p+1} \, dx.
\]
Define
\[
\tilde{u}_{1k}^{p+1} - \bar{u}_{2k}^{p+1} = \frac{1}{dt} \int_{0}^{1} [\tilde{t}\tilde{u}_{1k} + (1 - t)\bar{u}_{2k}]^{p+1} dt
\]
\begin{equation}
(3.11)
\end{equation}
\[
= (p + 1)(\tilde{u}_{1k} - \bar{u}_{2k}) \int_{0}^{1} [\tilde{t}\tilde{u}_{1k} + (1 - t)\bar{u}_{2k}]^{p} dt
\]
\[
:= (p + 1)\tilde{C}^{p}_{k}(x)(\tilde{u}_{1k} - \bar{u}_{2k}),
\]
which implies from (1.24) that \(\tilde{C}^{p}_{k}(x) \to w^{p}(x)\) uniformly in \(\mathbb{R}^{d}\) as \(k \to \infty\). Further one can derive that
\[
\tilde{g}_{k}(x) = \frac{\mu_{1k} - \mu_{2k}}{\|\tilde{u}_{1k} - \bar{u}_{2k}\|_{L^{\infty}(\mathbb{R}^{d})}} \bar{u}_{2k} = \frac{p - 1}{a^{*}(p + 1)} \int_{\mathbb{R}^{d}} (\|\tilde{u}_{1k}\|^{p+1} - \|\bar{u}_{2k}\|^{p+1}) dx
\]
\begin{equation}
(3.12)
\end{equation}
which implies from (1.24) that
\[
\tilde{g}_{k}(x) \to -\frac{p - 1}{a^{*}} \int_{\mathbb{R}^{d}} w^{p}\tilde{\eta}_{0} dx \quad \text{uniformly in } \mathbb{R}^{d} \text{ as } k \to \infty.
\]

On the other hand, similar to (3.11), one can also define \(\tilde{D}^{p-1}_{k}(x)\) satisfying
\[
pD^{p-1}_{k}(x)(\tilde{u}_{1k} - \bar{u}_{2k}) := \tilde{u}_{1k}^{p} - \bar{u}_{2k}^{p},
\]
and then, \(\tilde{D}^{p-1}_{k}(x) \to w^{p-1}(x)\) uniformly in \(\mathbb{R}^{d}\) as \(k \to \infty\). Further, one has
\[
\tilde{f}_{k}(x) = \frac{\tilde{u}_{1k}^{p} - \bar{u}_{2k}^{p}}{\|\tilde{u}_{1k} - \bar{u}_{2k}\|_{L^{\infty}(\mathbb{R}^{d})}} = p\tilde{D}^{p-1}_{k}(x)\tilde{\eta}_{0},
\]
\begin{equation}
(3.15)
\end{equation}
and
\[
\tilde{f}_{k}(x) \to pw^{p-1}\tilde{\eta}_{0} \quad \text{uniformly in } \mathbb{R}^{d} \text{ as } k \to \infty.
\]

By the above results, taking \(k \to \infty\), it follows from (3.6) that \(\tilde{\eta}_{0}\) solves
\[
-\Delta\tilde{\eta}_{0} + (1 - pw^{p-1})\tilde{\eta}_{0} = -\frac{p - 1}{a^{*}} \int_{\mathbb{R}^{d}} w^{p}\tilde{\eta}_{0} dx w.
\]

Set \(\mathcal{L} := -\Delta + (1 - pw^{p-1})\) and one can check that \(\mathcal{L}(w + \frac{p - 1}{2} x \cdot \nabla w) = -(p - 1)w\). Recall from (cf. [22,29]) that
\[
\ker\mathcal{L} = \left\{ \frac{\partial w}{\partial x_{1}}, \frac{\partial w}{\partial x_{2}}, \ldots, \frac{\partial w}{\partial x_{d}} \right\},
\]
and then, one can derive that
\[
\tilde{\eta}_{0}(x) = b_{0} \left( w + \frac{p - 1}{2} x \cdot \nabla w \right) + \sum_{i=1}^{d} b_{i} \frac{\partial w}{\partial x_{i}},
\]
\begin{equation}
(3.18)
\end{equation}
where \(b_{0}, b_{1}, b_{2}, \ldots, b_{d}\) are some constants.

\[\square\]

**Lemma 3.2.** Under the assumptions of Theorem 1.3, there holds that
\[
\frac{b_{0}}{2} \int_{\mathbb{R}^{d}} \frac{\partial V_{0}(x + y_{0})}{\partial x_{j}} (x \cdot \nabla w)^{2} - \sum_{i=1}^{d} \frac{b_{i}}{2} \int_{\mathbb{R}^{d}} \frac{\partial^{2} V_{0}(x + y_{0})}{\partial x_{i} \partial x_{j}} w^{2},
\]
\begin{equation}
(3.19)
\end{equation}
where \(j = 1, 2, \ldots, d\) and \(V_{0}\) is given by (1.21).
Proof. At first, we claim that for any $\bar{x}_0$, there exist a small $\delta > 0$ and a constant $C > 0$ such that

$$\varepsilon_k^2 \int_{\partial B_{\delta}(\bar{x}_0)} |\nabla \hat{n}_k|^2 dS + \varepsilon_k^2 \int_{\partial B_{\delta}(\bar{x}_0)} V(x) \hat{n}_k^2 dS + \int_{\partial B_{\delta}(\bar{x}_0)} \hat{n}_k^2 dS \leq C\varepsilon_k^d, \quad (3.20)$$

where $\hat{n}_k$ is given by (3.4).

Following from (3.4), (3.5) and (3.6), one can deduce that $\hat{n}_k$ satisfies

$$-\varepsilon_k^2 \Delta \hat{n}_k + \varepsilon_k^2 V(x) \hat{n}_k = \varepsilon_k^2 \mu_{1k} \hat{n}_k + \hat{g}_k(x) + \hat{f}_k(x), \quad (3.21)$$

where

$$\hat{g}_k(x) := \varepsilon_k^2 \frac{\mu_{1k} - \mu_{2k}}{\|\hat{u}_{1k} - \hat{u}_{2k}\|_{L^\infty(\mathbb{R}^d)}} \hat{u}_{2k}, \quad \hat{f}_k(x) := \frac{\hat{u}_1^p - \hat{u}_2^p}{\|\hat{u}_{1k} - \hat{u}_{2k}\|_{L^\infty(\mathbb{R}^d)}}. \quad (3.22)$$

Similar to (3.12) and (3.15), one has

$$\hat{g}_k(x) = -\frac{p - 1}{a^*} \varepsilon_k^{-d} \int_{\mathbb{R}^d} \hat{C}_k^p(x) \hat{n}_k^d x \hat{u}_{2k} \quad \text{and} \quad \hat{f}_k(x) = p \hat{D}_k^{p-1}(x) \hat{n}_k, \quad (3.23)$$

where $\hat{C}_k^p(x) \epsilon_{x} + x_{1k} := \tilde{C}_k^p(x)$ and $\hat{D}_k^{p-1}(x) \epsilon_{x} + x_{1k} := \tilde{D}_k^{p-1}(x)$.

Multiplying (3.21) by $\hat{n}_k$ and integrating over $\mathbb{R}^d$ yield that

$$\varepsilon_k^2 \int_{\mathbb{R}^d} |\nabla \hat{n}_k|^2 dx + \varepsilon_k^2 \int_{\mathbb{R}^d} V(x) \hat{n}_k^2 dx - \varepsilon_k^2 \mu_{1k} \hat{n}_k dx \int_{\mathbb{R}^d} \hat{n}_k^2 dx = p \int_{\mathbb{R}^d} \hat{D}_k^{p-1}(x) \hat{n}_k^2 dx - \frac{p - 1}{a^*} \varepsilon_k^{-d} \int_{\mathbb{R}^d} \hat{C}_k^p(x) \hat{n}_k dx \int_{\mathbb{R}^d} \hat{u}_{2k} \hat{n}_k dx$$

$$= p \int_{\mathbb{R}^d} \hat{D}_k^{p-1}(x) \hat{n}_k^2 dx - \frac{p - 1}{a^*} \varepsilon_k^{-d} \int_{\mathbb{R}^d} \hat{C}_k^p(x) \hat{n}_k dx \int_{\mathbb{R}^d} \hat{u}_{2k} \hat{n}_k dx = O(\varepsilon_k^d) \quad \text{as} \quad k \to \infty,$$

where the last equality holds because $\hat{n}_k \to \hat{n}_0$, $\hat{u}_{2k} \to w$, $\tilde{C}_k^p(x) \to w^p$ and $\tilde{D}_k^{p-1}(x) \to w^{p-1}$ uniformly in $\mathbb{R}^d$ as $k \to \infty$. Applying Lemma 4.5 in [2] then yields (3.20), and this completes the proof of this claim.

Following from (3.1), one can deduce that $\hat{u}_{ik}$ solves

$$-\varepsilon_k^2 \Delta \hat{u}_{ik} + \varepsilon_k^2 V(x) \hat{u}_{ik} = \varepsilon_k^2 \mu_{ik} \hat{u}_{ik} + \hat{u}_i^p \quad \text{in} \quad \mathbb{R}^d, \quad i = 1, 2. \quad (3.24)$$

Multiplying (3.24) by $\frac{\partial \hat{u}_{ik}}{\partial x_j}$ and integrating over $B_{\delta}(x_{1k})$, where $i = 1, 2$, $j = 1, 2, \ldots, d$ and $\delta$ is given by (3.20), one can obtain the following equality,

$$-\varepsilon_k^2 \int_{B_{\delta}(x_{1k})} \Delta \hat{u}_{ik} \frac{\partial \hat{u}_{ik}}{\partial x_j} + \frac{\varepsilon_k^2}{2} \int_{B_{\delta}(x_{1k})} V(x) \frac{\partial \hat{u}_{ik}}{\partial x_j}$$

$$= -\frac{\varepsilon_k^2 \mu_{ik}}{2} \int_{B_{\delta}(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial x_j} + \frac{1}{p + 1} \int_{B_{\delta}(x_{1k})} \frac{\partial \hat{u}_i^{p+1}}{\partial x_j}. \quad (3.25)$$
Some calculations yield that

\[-\varepsilon_k^2 \int_{B_\delta(x_{1k})} \Delta \hat{u}_{ik} \frac{\partial \hat{u}_{ik}}{\partial x_j} = -\varepsilon_k^2 \sum_{l=1}^{d} \int_{B_\delta(x_{1k})} \frac{\partial^2 \hat{u}_{ik}}{\partial x_l^2} \frac{\partial \hat{u}_{ik}}{\partial x_j} \]

\[= -\varepsilon_k^2 \sum_{l=1}^{d} \left[ \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial x_l} \frac{\partial \hat{u}_{ik}}{\partial x_j} \nu_l \, dS - \int_{B_\delta(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial x_l} \frac{\partial \hat{u}_{ik}}{\partial x_l} \frac{\partial}{\partial x_j} \right] \]

\[= -\varepsilon_k^2 \sum_{l=1}^{d} \left[ \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial v} \frac{\partial \hat{u}_{ik}}{\partial x_j} \nu_l \, dS - \frac{1}{2} \int_{B_\delta(x_{1k})} \frac{\partial}{\partial x_j} \left( \frac{\partial \hat{u}_{ik}}{\partial x_l} \right)^2 \right] \]

\[= -\varepsilon_k^2 \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial v} \frac{\partial \hat{u}_{ik}}{\partial x_j} \, dS - \frac{1}{2} \int_{\partial B_\delta(x_{1k})} |\nabla \hat{u}_{ik}|^2 \nu_j \, dS, \tag{3.26}\]

and

\[\varepsilon_k^2 \int_{B_\delta(x_{1k})} V(x) \frac{\partial \hat{u}_{ik}}{\partial x_j} \, dS = \frac{\varepsilon_k^2}{2} \left[ \int_{\partial B_\delta(x_{1k})} V(x) \hat{u}_{ik}^2 \nu_j \, dS - \int_{B_\delta(x_{1k})} \frac{\partial V(x)}{\partial x_j} \hat{u}_{ik}^2 \, dS \right] \tag{3.27}\]

It then follows from (3.25)–(3.27) that

\[\frac{\varepsilon_k^2}{2} \int_{B_\delta(x_{1k})} \frac{\partial V(x)}{\partial x_j} \hat{u}_{ik}^2 \, dS = \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{1k})} V(x) \hat{u}_{ik}^2 \nu_j \, dS - \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{1k})} \frac{\partial V(x)}{\partial x_j} \hat{u}_{ik}^2 \, dS - \frac{1}{p+1} \int_{\partial B_\delta(x_{1k})} \hat{u}_{ik}^{p+1} \nu_j \, dS \]

\[-\frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial v} \frac{\partial \hat{u}_{ik}}{\partial x_j} \, dS - \frac{1}{2} \int_{\partial B_\delta(x_{1k})} |\nabla \hat{u}_{ik}|^2 \nu_j \, dS \]

Further, we have

\[\frac{\varepsilon_k^2}{2} \int_{B_\delta(x_{1k})} \frac{\partial V(x)}{\partial x_j} (\hat{u}_{1k} + \hat{u}_{2k}) \bar{\eta}_k \]

\[= \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{1k})} V(x) (\hat{u}_{1k} + \hat{u}_{2k}) \bar{\eta}_k \nu_j \, dS + \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{1k})} (\nabla \hat{u}_{1k} + \nabla \hat{u}_{2k}) \nabla \bar{\eta}_k \nu_j \, dS \]

\[-\frac{\varepsilon_k^2}{2} \mu_{1k} \int_{\partial B_\delta(x_{1k})} (\hat{u}_{1k} + \hat{u}_{2k}) \bar{\eta}_k \nu_j \, dS - \frac{1}{2} \int_{\partial B_\delta(x_{1k})} \bar{g}_k \hat{u}_{2k} \nu_j \, dS \]

\[-\frac{1}{p+1} \int_{\partial B_\delta(x_{1k})} \frac{\hat{u}_{1k}^{p+1} - \hat{u}_{2k}^{p+1}}{\||\hat{u}_{1k} - \hat{u}_{2k}|_{L^\infty(\mathbb{R}^d)}|} \nu_j \, dS \]

\[-\varepsilon_k^2 \left[ \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{u}_{1k}}{\partial v} \frac{\partial \bar{\eta}_k}{\partial x_j} \, dS + \int_{\partial B_\delta(x_{1k})} \frac{\partial \bar{\eta}_k}{\partial v} \frac{\partial \hat{u}_{2k}}{\partial x_j} \, dS \right] \]
where \( \hat{g}_k \) is defined in (3.22).

Using the Hölder inequality, one can derive from (3.20) and (1.27) that

\[
\varepsilon_k^2 \int_{\partial B_k(x_{1k})} \left| \frac{\partial \hat{u}_{1k}}{\partial \nu} \frac{\partial \hat{\eta}_k}{\partial x_j} \right| dS + \varepsilon_k^2 \int_{\partial B_k(x_{1k})} \left| \frac{\partial \hat{\eta}_k}{\partial \nu} \frac{\partial \hat{u}_{2k}}{\partial x_j} \right| dS \\
\leq \varepsilon_k^2 \left( \int_{\partial B_k(x_{1k})} |\nabla \hat{\eta}_k|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_k(x_{1k})} |\nabla \hat{u}_{1k}|^2 \right)^{\frac{1}{2}} \left( \int_{\partial B_k(x_{1k})} |\nabla \hat{u}_{2k}|^2 \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon_k \frac{d-1}{2} \int_{\partial B_k(x_{1k})} |\nabla \hat{u}_{1k}|^2 dS + \int_{\partial B_k(x_{1k})} |\nabla \hat{u}_{2k}|^2 dS + \frac{1}{2} \int_{\partial B_k(x_{1k})} |\hat{\eta}_k|^2 dS
\]

\[
\leq C \int_{\mathbb{R}^d} \nabla \hat{\eta}_k \cdot \int_{\partial B_k(x_{1k})} \hat{u}_{2k} dS \]

\[
= o(1) e^{-\frac{c\varepsilon_k}{R}},
\]

and

\[
\left| \frac{1}{p+1} \int_{\partial B_k(x_{1k})} \frac{\hat{u}_{1k}^{p+1} - \hat{u}_{2k}^{p+1}}{\hat{u}_{1k} - \hat{u}_{2k}} \nu \right| \left( \mathbb{B}_{\varepsilon_k x_{1k}} \right) \leq C \int_{\partial B_k(x_{1k})} |\hat{\eta}_k|^2 dS = o(1) e^{-\frac{c\varepsilon_k}{R}}.
\]

On the other hand, let \( x_0 \) be the unique point of \( Z_0 \), where \( Z_0 \) satisfies (1.22) and (1.29). Employing (3.29)–(3.33), and applying (1.16), (1.31) and (3.9), one can derive from (3.28) that

\[
o(1) e^{-\frac{c\varepsilon_k}{R}} = \varepsilon_k^2 \int_{B_k(x_{1k})} \frac{\partial V(x)}{\partial x_j} (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k
\]

\[
= \varepsilon_k^2 \int_{B_k(x_{1k})} \frac{\partial V(x)}{\partial x_j} (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k
\]

\[
= \varepsilon_k^2 \int_{B_k(x_{1k})} \frac{\partial V(x)}{\partial x_j} (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k
\]
Proof of Theorem 1.3. At first, we claim that the coefficient $b_0$ given in (3.18) satisfies
\begin{equation}
    b_0 = 0.
    \tag{3.35}
\end{equation}
Multiplying (3.24) by $(x - x_{1k}) \cdot \nabla \hat{u}_{ik}$ and integrating over $B_{\delta}(x_{1k})$, where $i = 1, 2$ and $\delta$ is given in (3.20), one has
\begin{align*}
    -\varepsilon_k^2 & \int_{B_{\delta}(x_{1k})} \Delta \hat{u}_{ik} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}] + \varepsilon_k^2 \int_{B_{\delta}(x_{1k})} V(x) [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}^2] \\
    &= \frac{\varepsilon_k^2 \mu_{ik}}{2} \int_{B_{\delta}(x_{1k})} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}^2] + \frac{1}{p + 1} \int_{B_{\delta}(x_{1k})} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}^{p+1}].
    \tag{3.36}
\end{align*}
Some calculations yield that
\begin{align*}
    -\varepsilon_k^2 & \int_{B_{\delta}(x_{1k})} \Delta \hat{u}_{ik} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}] \\
    &= -\varepsilon_k^2 \int_{\partial B_{\delta}(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial \nu} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}] + \varepsilon_k^2 \int_{B_{\delta}(x_{1k})} \nabla \hat{u}_{ik} \cdot \nabla [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}]
\end{align*}
Moreover, one can also deduce that

\[ \frac{2 - d}{2} \varepsilon_k^2 \int_{\partial B_s(x_{1k})} \nabla \hat{u}_{ik}^2 \cdot \nu \]

where the last “=” holds due to that

\[
\varepsilon_k^2 \int_{B_s(x_{1k})} \nabla \hat{u}_{ik} \cdot \nabla [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}] \\
= \varepsilon_k^2 \sum_{j=1}^d \int_{B_s(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial x_j} \left[ \frac{\partial \hat{u}_{ik}}{\partial x_j} + (x - x_{1k}) \cdot \nabla \frac{\partial \hat{u}_{ik}}{\partial x_j} \right] \\
= \varepsilon_k^2 \sum_{j=1}^d \int_{B_s(x_{1k})} \left( \frac{\partial \hat{u}_{ik}}{\partial x_j} \right)^2 + \varepsilon_k^2 \sum_{j=1}^d \int_{B_s(x_{1k})} \left( x - x_{1k} \right) \cdot \nabla \left( \frac{\partial \hat{u}_{ik}}{\partial x_j} \right)^2 \\
= \varepsilon_k^2 \int_{B_s(x_{1k})} |\nabla \hat{u}_{ik}|^2 + \frac{\varepsilon_k^2}{2} \int_{B_s(x_{1k})} \left( x - x_{1k} \right) \cdot \nabla |\nabla \hat{u}_{ik}|^2 \\
= \frac{\varepsilon_k^2}{2} \int_{\partial B_s(x_{1k})} |\nabla \hat{u}_{ik}|^2 (x - x_{1k}) \cdot \nu + \frac{2 - d}{2} \varepsilon_k^2 \int_{B_s(x_{1k})} |\nabla \hat{u}_{ik}|^2 ,
\]

and

\[
\frac{2 - d}{2} \varepsilon_k^2 \int_{B_s(x_{1k})} |\nabla \hat{u}_{ik}|^2 \\
= \frac{2 - d}{4} \varepsilon_k^2 \int_{\partial B_s(x_{1k})} \nabla \hat{u}_{ik}^2 \cdot \nu - \frac{2 - d}{2} \varepsilon_k \int_{B_s(x_{1k})} \hat{u}_{ik} \Delta \hat{u}_{ik} \\
= \frac{2 - d}{4} \varepsilon_k \int_{\partial B_s(x_{1k})} \nabla \hat{u}_{ik}^2 \cdot \nu \\
- \frac{2 - d}{2} \varepsilon_k \int_{B_s(x_{1k})} V(x) \hat{u}_{ik}^2 - \varepsilon_k^2 \hat{u}_{ik} \int_{B_s(x_{1k})} \hat{u}_{ik}^2 - \int_{B_s(x_{1k})} \hat{u}_{ik}^{p+1} \right].
\]

Moreover, one can also deduce that

\[
\int_{B_s(x_{1k})} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}^2] = \int_{\partial B_s(x_{1k})} \hat{u}_{ik}^2 [(x - x_{1k}) \cdot \nu] - d \int_{B_s(x_{1k})} \hat{u}_{ik}^2 ,
\]

\[
\int_{B_s(x_{1k})} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}^{p+1}] = \int_{\partial B_s(x_{1k})} \hat{u}_{ik}^{p+1} [(x - x_{1k}) \cdot \nu] - d \int_{B_s(x_{1k})} \hat{u}_{ik}^{p+1} ,
\]
Following from (3.10), one has

Substituting the above results into (3.36) yields that

\[
\begin{align*}
- \frac{\varepsilon_k^2}{d} & \int_{\partial B_s(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial \nu} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}] + \frac{\varepsilon_k^2}{2} \int_{\partial B_s(x_{1k})} [(x - x_{1k}) \cdot \nu] |\nabla \hat{u}_{ik}|^2 \\
+ \frac{2 - d}{4} \varepsilon_k^2 & \int_{\partial B_s(x_{1k})} \nabla \hat{u}_{ik}^2 \cdot \nu \\
- \frac{2 - d}{2} & \left[ \frac{\varepsilon_k^2}{d} \int_{B_s(x_{1k})} V(x) \hat{u}_{ik}^2 - \frac{\varepsilon_k^2}{2} \int_{B_s(x_{1k})} \hat{u}_{ik}^2 - \int_{B_s(x_{1k})} \hat{u}_{ik}^{p+1} \right] \\
- \frac{d \varepsilon_k^2}{2} & \int_{B_s(x_{1k})} V(x) \hat{u}_{ik}^2 - \frac{\varepsilon_k^2}{2} \int_{B_s(x_{1k})} \nabla V(x) \cdot (x - x_{1k}) \hat{u}_{ik}^2 \\
+ \frac{\varepsilon_k^2}{2} & \int_{\partial B_s(x_{1k})} V(x) \hat{u}_{ik}^2 [(x - x_{1k}) \cdot \nu] \\
= & \frac{\varepsilon_k^2 \mu_{ik}}{2} \int_{\partial B_s(x_{1k})} \hat{u}_{ik}^2 [(x - x_{1k}) \cdot \nu] - \frac{d \varepsilon_k^2 \mu_{ik}}{2} \int_{B_s(x_{1k})} \hat{u}_{ik}^2 \\
+ & \frac{1}{p + 1} \int_{\partial B_s(x_{1k})} \hat{u}_{ik}^{p+1} [(x - x_{1k}) \cdot \nu] - \frac{d}{p + 1} \int_{B_s(x_{1k})} \hat{u}_{ik}^{p+1}.
\end{align*}
\]

Following from (3.10), one has

\[
\begin{align*}
\frac{\varepsilon_k^2 \mu_{ik}}{d} \int_{\mathbb{R}^d} \hat{u}_{ik}^2 = 2 a^* \frac{d \varepsilon_k^2 \mu_{ik}}{\varepsilon_k^2} \int_{\mathbb{R}^d} |\hat{u}_{ik}|^{p+1} \, dx,
\end{align*}
\]

and it then follows that

\[
\begin{align*}
- \frac{\varepsilon_k^2}{d} & \int_{\partial B_s(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial \nu} [(x - x_{1k}) \cdot \nabla \hat{u}_{ik}] + \frac{\varepsilon_k^2}{2} \int_{\partial B_s(x_{1k})} [(x - x_{1k}) \cdot \nu] |\nabla \hat{u}_{ik}|^2 \\
+ \frac{2 - d}{4} \varepsilon_k^2 & \int_{\partial B_s(x_{1k})} \nabla \hat{u}_{ik}^2 \cdot \nu \\
+ \frac{\varepsilon_k^2}{2} & \int_{\partial B_s(x_{1k})} V(x) \hat{u}_{ik}^2 [(x - x_{1k}) \cdot \nu] - \frac{\varepsilon_k^2}{2} \int_{B_s(x_{1k})} \nabla V(x) \cdot (x - x_{1k}) \hat{u}_{ik}^2 \\
- \frac{\varepsilon_k^2 \mu_{ik}}{2} & \int_{\partial B_s(x_{1k})} \hat{u}_{ik}^2 [(x - x_{1k}) \cdot \nu] - \frac{1}{p + 1} \int_{\partial B_s(x_{1k})} \hat{u}_{ik}^{p+1} [(x - x_{1k}) \cdot \nu]
\end{align*}
\]
\[ \begin{align*}
&= \varepsilon_k^2 \int_{B_\delta(x_{1k})} V(x) \hat{u}_{ik}^2 - \varepsilon_k^2 \mu_{ik} \int_{B_\delta(x_{1k})} \hat{u}_{ik}^2 - \frac{2(p+1) - d(p-1)}{2(p+1)} \int_{B_\delta(x_{1k})} \hat{u}_{ik}^{p+1} \\
&= \varepsilon_k^2 \int_{B_\delta(x_{1k})} V(x) \hat{u}_{ik}^2 + \frac{d(p-1) - 4}{2(p+1)} \int_{\mathbb{R}^d} \hat{u}_{ik}^{p+1} - 2a^2 \varepsilon_k^2 \varepsilon_k^2 e(\rho_k) \\
&+ \varepsilon_k^2 \int_{\mathbb{R}^d \setminus B_\delta(x_{1k})} \hat{u}_{ik}^2 + \frac{2(p+1) - d(p-1)}{2(p+1)} \int_{\mathbb{R}^d \setminus B_\delta(x_{1k})} \hat{u}_{ik}^{p+1}. \quad (3.38)
\end{align*} \]

By (3.4), one can deduce from (3.38) that

\[ \frac{d(p-1) - 4}{2(p+1)} \int_{\mathbb{R}^d} \| \hat{u}_{1k} - \hat{u}_{2k} \|_{L^\infty(\mathbb{R}^d)} = T_1 + T_2 + T_3 + T_4 + T_5, \quad (3.39) \]

where

\[ T_1 := -\varepsilon_k^2 \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{u}_{ik}}{\partial \nu} [(x - x_{1k}) \cdot \nabla \hat{\eta}_k] - \varepsilon_k^2 \int_{\partial B_\delta(x_{1k})} \frac{\partial \hat{\eta}_k}{\partial \nu} [(x - x_{1k}) \cdot \nabla \hat{u}_{2k}] \\
+ \varepsilon_k^2 \int_{\partial B_\delta(x_{1k})} [(x - x_{1k}) \cdot \nu] \nabla \hat{\eta}_k \cdot \nabla (\hat{u}_{1k} + \hat{u}_{2k}) \\
+ \frac{2 - d}{4} \varepsilon_k^2 \left[ \int_{\partial B_\delta(x_{1k})} \hat{\eta}_k (\nabla (\hat{u}_{1k} + \hat{u}_{2k}) \cdot \nu) + \int_{\partial B_\delta(x_{1k})} (\hat{u}_{1k} + \hat{u}_{2k}) (\nabla \hat{\eta}_k \cdot \nu) \right], \]

\[ T_2 := \frac{\varepsilon_k^2}{2} \int_{\partial B_\delta(x_{1k})} V(x) [(x - x_{1k}) \cdot \nu] (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k, \]

\[ T_3 := -\frac{\varepsilon_k^2 \mu_{ik}}{2} \int_{\partial B_\delta(x_{1k})} [(x - x_{1k}) \cdot \nu] (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k - \frac{1}{2} \int_{\partial B_\delta(x_{1k})} [(x - x_{1k}) \cdot \nu] \hat{\eta}_k \hat{u}_{2k} \\
- \varepsilon_k^2 \mu_{ik} \int_{\mathbb{R}^d \setminus B_\delta(x_{1k})} (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k - \int_{\mathbb{R}^d \setminus B_\delta(x_{1k})} \hat{\eta}_k \hat{u}_{2k}, \]

\[ T_4 := -\frac{1}{p+1} \int_{\partial B_\delta(x_{1k})} [(x - x_{1k}) \cdot \nu] \frac{\hat{u}_{ik}^{p+1} - \hat{u}_{2k}^{p+1}}{\| \hat{u}_{1k} - \hat{u}_{2k} \|_{L^\infty(\mathbb{R}^d)}} \\
- \frac{2(p+1) - d(p-1)}{2(p+1)} \int_{\mathbb{R}^d \setminus B_\delta(x_{1k})} \frac{\hat{u}_{ik}^{p+1} - \hat{u}_{2k}^{p+1}}{\| \hat{u}_{1k} - \hat{u}_{2k} \|_{L^\infty(\mathbb{R}^d)}}, \]

and

\[ T_5 := -\varepsilon_k^2 \int_{B_\delta(x_{1k})} V(x) (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k - \frac{\varepsilon_k^2}{2} \int_{B_\delta(x_{1k})} [\nabla V(x) \cdot (x - x_{1k})] (\hat{u}_{1k} + \hat{u}_{2k}) \hat{\eta}_k. \]

Similar to the estimates (3.29)-(3.33), one can deduce that

\[ |T_1|, |T_2|, |T_3|, |T_4| = o(1) e^{-\varepsilon_k^\delta}. \quad (3.40) \]
As for $T_5$, the estimate (3.34) gives that
\[
\frac{\varepsilon_k^2}{2} \int_{B_k(x_{1k})} |\nabla V(x) \cdot x_{1k}| \left(\tilde{u}_{1k} + \tilde{u}_{2k}\right) \tilde{\eta}_k = o(1)e^{-\frac{\varepsilon_k^d}{\varepsilon_k^r}},
\]
and by (1.16), one has
\[
\frac{\varepsilon_k^2}{2} \int_{B_k(x_{1k})} V(x)(\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k = \varepsilon_k^{2+d} \int_{\mathbb{R}^d} V(\varepsilon_k x + x_{1k} - x_0) V_0(\varepsilon_k x + x_{1k} - x_0)(\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k
\]
\[
= (1 + o(1))\varepsilon_k^{2+d+r} \int_{B_k(\varepsilon_k(0))} V_0(x + x_{1k} - x_0) (\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k
\]
\[
= (2 + o(1))\varepsilon_k^{2+d+r} \int_{\mathbb{R}^d} V_0(x + y_0) w\tilde{\eta}_0,
\]
where $V_0$ is given by (1.21) and $x_0 \in Z_0$ with $Z_0$ defined by (1.22). Moreover, since $\nabla V_0(x) \cdot x = r V_0(x)$, one can derive from (1.31) and (3.34) that
\[
\frac{\varepsilon_k^2}{2} \int_{B_k(x_{1k})} |\nabla V(x) \cdot x| \left(\tilde{u}_{1k} + \tilde{u}_{2k}\right) \tilde{\eta}_k
\]
\[
= \varepsilon_k^2 \int_{B_k(x_{1k})} |\nabla V(x - x_0 + x_0) \cdot (x - x_0 + x_0)| \left(\tilde{u}_{1k} + \tilde{u}_{2k}\right) \tilde{\eta}_k
\]
\[
= \varepsilon_k^2 \int_{B_k(x_{1k})} |\nabla V_0(x - x_0) + W(x - x_0)| \cdot (x - x_0)(\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k
\]
\[
+ \frac{\varepsilon_k^2}{2} \int_{B_k(x_{1k})} |\nabla V_0(x) \cdot x| (\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k
\]
\[
= \frac{\varepsilon_k^2}{2} \int_{B_k(x_{1k})} [r V_0(x - x_0) + W(x - x_0)(x - x_0)] (\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k + o(1)e^{-\frac{\varepsilon_k^d}{\varepsilon_k^r}}
\]
\[
= r(1 + o(1))\varepsilon_k^{d+r+1} \int_{B_k^0(\varepsilon_k (0))} V_0 \left(x + \frac{x_{1k} - x_0}{\varepsilon_k}\right) (\tilde{u}_{1k} + \tilde{u}_{2k})\tilde{\eta}_k + o(1)e^{-\frac{\varepsilon_k^d}{\varepsilon_k^r}}
\]
\[
= (1 + o(1))r\varepsilon_k^{2+d+r} \int_{\mathbb{R}^d} V_0(x + y_0) w\tilde{\eta}_0,
\]
where $W := (W_1, W_2, \cdots, W_d)$. It then follows from these estimates that
\[
T_5 = O(1)\varepsilon_k^{2+d+r}.
\]
(3.41)
As for the left-hand side of (3.39), one can deduce from (3.40) and (3.41) that

$$O(1)\varepsilon_k^{2+r+d} = \frac{d(p-1)-4}{2(p+1)} \int_{\mathbb{R}^d} \frac{\tilde{u}_{2k}^{p+1} - \tilde{u}_{1k}^{p+1}}{\|\tilde{u}_{1k} - \tilde{u}_{2k}\|_{L^\infty(\mathbb{R}^d)}}$$

$$= \frac{d(p-1)-4}{2} \varepsilon_k^d \int_{\mathbb{R}^d} \tilde{C}_k^p(x) \tilde{\eta}_k$$

$$= \frac{d(p-1)-4}{2} (1 + o(1)) \varepsilon_k^d \int_{\mathbb{R}^d} w^p \tilde{\eta}_0,$$

where $\tilde{C}_k^p(x)$ is defined in (3.11) and the last “=” holds because $\tilde{C}_k^p(x) \to w^p$ uniformly in $\mathbb{R}^d$ as $k \to \infty$.

Using (3.18), then (3.42) gives that

$$0 = \int_{\mathbb{R}^d} w^p \tilde{\eta}_0$$

$$= \int_{\mathbb{R}^d} w^p \left[ b_0 \left( w + \frac{p-1}{2} x \cdot \nabla w \right) + \sum_{i=1}^d b_i \frac{\partial w}{\partial x_i} \right]$$

$$= b_0 \int_{\mathbb{R}^d} w^{p+1} + \frac{b_0 p-1}{2 p+1} \int_{\mathbb{R}^d} x \cdot \nabla w^{p+1} + \sum_{i=1}^d \frac{b_i}{p+1} \int_{\mathbb{R}^d} \frac{\partial w^{p+1}}{\partial x_i}$$

$$= b_0 \int_{\mathbb{R}^d} w^{p+1} - \frac{db_0}{2 p+1} \int_{\mathbb{R}^d} w^{p+1}$$

$$= \left[ 1 - \frac{d p-1}{2 p+1} \right] b_0 \int_{\mathbb{R}^d} w^{p+1}.$$

Since $1 - \frac{d p-1}{2 p+1} \neq 0$ when $1 < p < 1 + \frac{4}{d}$, one then has $b_0 = 0$. Hence, we complete the proof of (3.35).

Further, it follows from (3.19) that

$$\sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial^2 V_0(x + y_0)}{\partial x_i \partial x_j} w^2 = \sum_{i=1}^d \frac{b_i}{\partial x_i \partial x_j} Q(y_0) = 0,$$

which implies from the nondegeneracy assumption of $Q(y_0)$ that $b_i = 0$ for $i = 1, 2, \ldots d$. From (3.18) we thus have $\tilde{\eta}_0 \equiv 0$ on $\mathbb{R}^d$.

At last, we claim that $\tilde{\eta}_0 = 0$ cannot occur. Suppose $\tilde{y}_k \in \mathbb{R}^d$ is a maximum point of $\tilde{\eta}_k$, and then $|\tilde{y}_k(\tilde{y}_k)| = \|\tilde{\eta}_k\|_{L^\infty(\mathbb{R}^d)} = 1$. It thus follows from (3.6) that $\tilde{g}_k(\tilde{y}_k) + \tilde{f}_k(\tilde{y}_k) \geq \frac{1}{2}$. One can further deduce from (3.13) and (3.16) that $w(\tilde{y}_k) \geq C_0 > 0$, which implies that $y_k$ is bounded uniformly in $k$, due to the fact that $w(x)$ decays exponentially as $|x| \to \infty$. Therefore, one can conclude from (3.9) that $\tilde{\eta}_0 \equiv 0$ on $\mathbb{R}^d$, which, however, contradicts to the fact that $\tilde{\eta}_0 \equiv 0$ on $\mathbb{R}^d$. Therefore, the proof of Theorem 1.3 is complete.

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A Appendix

A.1 Equivalence between ground states and constraint minimizers

This section is devoted to the proof of Theorem 1.1 on the equivalence between ground states of equation (1.1) and constraint minimizers of problem (1.4). At first, we give the following lemma.

**Lemma A.1.** Suppose $V(x)$ satisfies (1.3), and $u_\rho(x) \geq 0$ is a nonnegative minimizer of $e(\rho)$. For any $\rho_1, \rho_{2k} \in (0, \infty)$ satisfying $\rho_{2k} \to \rho_1$ as $k \to \infty$, passing to a subsequence if necessary, there exists $\bar{u} \in M_{\rho_1}$ such that

$$u_{\rho_{2k}} \to \bar{u} \quad \text{in} \; \mathcal{H}(\mathbb{R}^d) \quad \text{as} \; k \to \infty. \quad (A.1)$$

**Proof.** For any $\rho_1, \rho_{2k} \in (0, \infty)$, we have

$$E_\rho_1(u_{\rho_{2k}}) - E_\rho_{2k}(u_{\rho_{2k}}) \leq e(\rho_1) - e(\rho_{2k}) \leq E_\rho_1(u_{\rho_{2k}}) - E_{\rho_{2k}}(u_{\rho_{2k}}).$$

One can thus derive that

$$- \frac{\rho_{2k}^{p-1} - \rho_1^{p-1}}{p+1} \int_{\mathbb{R}^d} |u_{\rho_{2k}}|^{p+1} \, dx \leq e(\rho_{2k}) - e(\rho_1) \leq - \frac{\rho_{2k}^{p-1} - \rho_1^{p-1}}{p+1} \int_{\mathbb{R}^d} |u_{\rho_{2k}}|^{p+1} \, dx, \quad (A.2)$$

which implies that $e(\rho)$ is a decreasing function of $\rho \in (0, \infty)$ and $\lim_{k \to \infty} e(\rho_{2k}) = e(\rho_1)$, because $\|u_{\rho_{2k}}\|_{p+1}$ is bounded uniformly in $k$.

From (1.5), one can further deduce that

$$E_\rho_1(u_{\rho_{2k}}) = e(\rho_{2k}) + \frac{\rho_{2k}^{p-1} - \rho_1^{p-1}}{p+1} \int_{\mathbb{R}^d} |u_{\rho_{2k}}|^{p+1} \, dx \to e(\rho_1) \quad \text{as} \; k \to \infty. \quad (A.3)$$

Let $\{u_{\rho_{2k}}\}$ be a minimizing sequence for $e(\rho_1)$. Employing the Gagliardo–Nirenberg inequality (1.7), one can derive from (1.5) that $\{u_{\rho_{2k}}\}$ is bounded uniformly in $\mathcal{H}(\mathbb{R}^d)$ with respect to $k$. Applying the well-known compact embedding theorem (cf. [33, Theorem XIII.67]), one can deduce that passing to subsequence if necessary, $u_{\rho_{2k}} \to \bar{u}$ strongly in $L^q(\mathbb{R}^d)$ with $q \in [2, 2^*)$ for some $\bar{u} \in \mathcal{H}$. This gives the weak lower semicontinuity of $E_\rho_1(u_{\rho_{2k}})$, and implies from (A.3) that

$$e(\rho_1) = \lim_{k \to \infty} E_\rho_1(u_{\rho_{2k}}) \geq E_\rho_1(\bar{u}) \geq e(\rho_1),$$

i.e., (A.1) holds. Hence, the proof of this lemma is completed. \qed

Next, we giving the following lemma on the differentiability of $e(\rho)$.

**Lemma A.2.** Suppose $V(x)$ satisfies (1.3), and let $u_\rho(x) \geq 0$ be a nonnegative minimizer of $e(\rho)$. Then $e(\rho)$ is differentiable for a.e. $\rho \in (0, \infty)$ and

$$e'(\rho) = - \frac{p-1}{p+1} \rho^{p-2} \int_{\mathbb{R}^d} |u|^{p+1} \, dx, \quad \forall \; u \in M_\rho. \quad (A.4)$$

Since the proof of this lemma is similar to that of [18, Lemma 2.2], we omit it here. \qed

Based on the proof of lemma A.2, we remark that for any given $\rho \in (0, \infty)$, if $e(\rho)$ admits a unique nonnegative or nonpositive minimizer, then $e'(\rho)$ exists and satisfies (A.4).

**Proof of Theorem 1.1:** For any $\rho \in (0, \infty)$ and $0 \leq u_\rho \in M_\rho$, $u_\rho$ satisfies (1.1) for some Lagrange multiplier $\mu_\rho \in \mathbb{R}$. It then follows from (1.1), (1.4) and (A.4) that, for a.e. $\rho \in (0, \infty)$,

$$\mu_\rho = 2e(\rho) - \frac{p-1}{p+1} \rho^{p-1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx = 2e(\rho) + \rho e'(\rho), \quad (A.5)$$

Thus, for any $x \in \mathbb{R}^d$ and $\rho \in (0, \infty)$,

$$\mu_\rho(x) = 2e(\rho) - \frac{p-1}{p+1} \rho^{p-1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx = 2e(\rho) + \rho e'(\rho).$$
which implies that \( \mu_\rho \) depends only on the value of \( \rho \) and is independent of the choice of \( u_\rho \). This further indicates that, for a.e. \( \rho \in (0, \infty) \), all minimizers of \( e(\rho) \) satisfy equation (1.1) with the same Lagrange multiplier \( \mu_\rho \).

Taking any \( u_g \in G_{\rho,\mu} \) and setting \( \tilde{u}_g = \frac{1}{\|u_g\|_2} u_g \), one then has \( F_{\rho,\mu}(\tilde{u}_g) \geq F_{\rho,\mu}(u_g) \). Since \( u_g \) solves (1.1), one can derive from (1.12) that

\[
F_{\rho,\mu}(u_g) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \rho^{p+1} \int_{\mathbb{R}^d} |u_g|^{p+1} \, dx
\]

and

\[
F_{\rho,\mu}(\tilde{u}_g) = \left( \frac{1}{2\|u_g\|_2^2} - \frac{1}{(p+1)\|u_g\|_2^{p+1}} \right) \rho^{p+1} \int_{\mathbb{R}^d} |u_g|^{p+1} \, dx.
\]

Therefore, we have

\[
\frac{1}{2\|u_g\|_2^2} - \frac{1}{(p+1)\|u_g\|_2^{p+1}} \geq \frac{1}{2} - \frac{1}{p+1}.
\]

One can check that (A.6) holds if and only if \( \|u_g\|_2 = 1 \), i.e., \( F_{\rho,\mu}(\tilde{u}_g) = F_{\rho,\mu}(u_g) \). This implies that all ground states of equation (1.1) share the same \( L^2 \)-norm, i.e.,

for any \( u_g \in G_{\rho,\mu}, u_g \) satisfies \( \|u_g\|_2^2 = 1 \).

For any \( u_g \in G_{\rho,\mu} \) and \( u_\rho \in M_\rho \), one has

\[
E_\rho(u_g) \geq E_\rho(u_\rho) \quad \text{and} \quad F_{\rho,\mu}(u_\rho) \geq F_{\rho,\mu}(u_g).
\]

Following from (1.5) and (1.12), one has

\[
F_{\rho,\mu}(u) = E_\rho(u) - \frac{1}{2} \mu,
\]

which indicates that \( E_\rho(u_\rho) \geq E_\rho(u_g) \), i.e., \( u_g \in M_\rho \). One can further deduce from (A.5) that for a.e. \( \rho \in (0, \infty) \), there holds that \( \mu = \mu_\rho \), which implies \( F_{\rho,\mu,\rho}(u_g) \geq F_{\rho,\mu,\rho}(u_\rho) \), i.e., \( u_\rho \in G_{\rho,\mu,\rho} \). Hence, we complete the proof of Theorem 1.1. \( \square \)

A.2 Some results on the problem \( \tilde{e}(\rho) \)

In this section, we focus on studying the following minimization problem

\[
\tilde{e}(\rho) := \inf \{ \tilde{E}_\rho(u) : u \in H^1(\mathbb{R}^d), \|u\|_2 = 1 \},
\]

where \( \tilde{E}_\rho(u) \) is defined by

\[
\tilde{E}_\rho(u) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx - \frac{\rho^{p-1}}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \, dx, \quad 1 < p < 1 + \frac{4}{d}.
\]

Employing the concentration-compactness principle, one can derive that \( \tilde{e}(\rho) \) admits minimizers for any \( \rho \in (0, \infty) \); see, e.g., [3, 25, 26]. Similar to problem (1.4), without loss of generality, we restrict the minimizers of problem (A.8) to nonnegative functions. We then give our results by the following lemma.

Lemma A.3. Suppose \( \tilde{u}_\rho \) is a nonnegative minimizer of \( \tilde{e}(\rho) \). Set \( \tilde{\alpha}_\rho := \left( \frac{\rho}{\sqrt{\alpha^*}} \right)^{2(p-1)} \frac{2(p-1)}{4(p-1)} \) and \( \alpha^* := \|w\|_2^2 \), where \( w \) is the unique positive solution of (1.6). We then have

\[
\tilde{e}(\rho) = -\lambda \left( \frac{\rho}{\sqrt{\alpha^*}} \right)^{4(p-1)} \frac{4(p-1)}{4(p-1)},
\]

(A.10)
and, up to translations, $\tilde{u}_\rho$ satisfies

$$\tilde{u}_\rho := \frac{1}{\sqrt{a^*}} \hat{u}_\rho^2 w(\alpha_\rho x),$$

(A.11)

where $\lambda$ is defined in (1.20).

Proof. Suppose $\tilde{u}_\rho$ is a nonnegative minimizer of $\tilde{e}(\rho)$ and $\tilde{u}_1$ is a nonnegative minimizer of $\tilde{e}(1)$. At first, we claim that

$$\tilde{e}(\rho) = \rho^{\frac{4(\frac{p-1}{p+1})}{\frac{p}{p+1}}} \tilde{e}(1) \quad \text{and} \quad \tilde{u}_\rho = \alpha_\rho \tilde{u}_1(\alpha_\rho x),$$

(A.12)

where $\alpha_\rho := \rho^{\frac{2(p-1)}{4 - 2(p-1)}}$. In fact, setting $\tilde{v}_1 := \alpha_\rho^4 \tilde{u}_\rho(\alpha_\rho^{-1} x)$, one can check that

$$\tilde{e}(\rho) = \tilde{E}_\rho(\tilde{u}_\rho) = \rho^{\frac{4(\frac{p-1}{p+1})}{\frac{p}{p+1}}} \left[ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla \tilde{v}_1|^2 \, dx - \frac{1}{p + 1} \int_{\mathbb{R}^d} \tilde{v}_1^{p+1} \, dx \right] \geq \rho^{\frac{4(\frac{p-1}{p+1})}{\frac{p}{p+1}}} \tilde{e}(1).$$

Similarly, setting $\tilde{v}_\rho := \alpha_\rho^4 \tilde{u}_1(\alpha_\rho x)$, one can check that

$$\tilde{e}(\rho) \leq \tilde{E}_\rho(\tilde{v}_\rho) = \rho^{\frac{4(\frac{p-1}{p+1})}{\frac{p}{p+1}}} \tilde{e}(1).$$

(A.13)

The above two inequalities then give the first equality in (A.12). Furthermore, we know that $\tilde{v}_1$ is a minimizer of $\tilde{e}(1)$ and $\tilde{v}_\rho$ is a minimizer of $\tilde{e}(\rho)$, which gives the second equality in (A.12).

Next, we claim that

$$\tilde{e}(1) = -\lambda(\sqrt{a^*})^{-\frac{4(p-1)}{4 - 2(p-1)}},$$

(A.14)

where $\lambda$ is given by (1.20), and $\tilde{u}_\rho$ (up to translations) satisfies

$$\tilde{u}_1(x) = (\sqrt{a^*})^{-\frac{4}{4 - 2(p-1)}} w((\sqrt{a^*})^{-\frac{2(p-1)}{4 - 2(p-1)}} x).$$

(A.15)

Take a test function $\tilde{v}_\epsilon = \epsilon^2 \tilde{v}_0(\epsilon x)$, where $0 < \tilde{v}_0 \in H^1(\mathbb{R}^d)$ satisfies $\|\tilde{v}_0\|_2^2 = 1$ and $\epsilon > 0$ is a positive constant. One can then verify that

$$\tilde{e}(1) \leq \tilde{E}_1(\tilde{v}_\epsilon) = \frac{\epsilon^2}{2} \int_{\mathbb{R}^d} |\nabla \tilde{v}_0|^2 \, dx - \frac{\epsilon^{\frac{4}{p}(p-1)}}{p + 1} \int_{\mathbb{R}^d} |\tilde{v}_0|^{p+1} \, dx < 0, \quad \text{when } \epsilon \text{ is small enough}.$$

Let $\tilde{u}_1$ be a nonnegative minimizer of $\tilde{e}(1)$, and then, $\tilde{u}_1$ solves

$$-\Delta \tilde{u}_1 = \tilde{\mu}_1 \tilde{u}_1 + \tilde{u}_1^p,$$

(A.16)

where $\tilde{\mu}_1$ is a suitable Lagrange parameter. It follows from (A.9) and (A.16) that

$$\tilde{\mu}_1 = 2 \tilde{e}(1) - \frac{p - 1}{p + 1} \int_{\mathbb{R}^d} \tilde{u}_1^{p+1} \, dx < 0.$$  

(A.17)

Applying the maximum principle (cf. [11]) then yields that $\tilde{u}_1 > 0$, which implies that, up to translations,

$$\tilde{u}_1 = (-\tilde{\mu}_1)^{\frac{1}{p-1}} w((-\tilde{\mu}_1)^{\frac{1}{p}} x),$$

due to the fact that $w$ is the unique positive solution of (1.6). Furthermore, since $\|\tilde{u}_1\|_2^2 = 1$, one can then deduce that $\mu_1$ satisfies

$$\mu_1 = -\|w\|_2^{-\frac{4(p-1)}{4 - 2(p-1)}} = -\sqrt{a^*}^{-\frac{4(p-1)}{4 - 2(p-1)}},$$

which implies (A.15). Further, substituting (A.15) into (A.9) then yields (A.14).

Combining the above two claims then yields (A.10) and (A.11), and this completes the proof of Lemma A.3.
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