Triply Positive Matrices and Quantum Measurements
Motivated by QBism

John B. DeBrota†, Christopher A. Fuchs†, and Blake C. Stacey†

†QBism Group, Physics Department, University of Massachusetts Boston,
100 Morrissey Boulevard, Boston MA 02125, USA
⋆Stellenbosch Institute for Advanced Study (STIAS), Wallenberg Research Center at Stellenbosch University,
Marais Street, Stellenbosch 7600, South Africa

February 6, 2019

Abstract

We study a class of quantum measurements that furnish probabilistic representations of finite-dimensional quantum theory. The Gram matrices associated with these Minimal Informationally Complete quantum measurements (MICs) exhibit a rich structure. They are “positive” matrices in three different senses, and conditions expressed in terms of them have shown that the Symmetric Informationally Complete measurements (SICs) are in some ways optimal among MICs. Here, we explore MICs more widely than before, comparing and contrasting SICs with other classes of MICs, and using Gram matrices to begin the process of mapping the territory of all MICs. Moreover, the Gram matrices of MICs turn out to be key tools for relating the probabilistic representations of quantum theory furnished by MICs to quasi-probabilistic representations, like Wigner functions, which have proven relevant for quantum computation. Finally, we pose a number of conjectures, leaving them open for future work.

1 Introduction

Let \( H_d \) be a \( d \)-dimensional complex Hilbert space, and let \( \{ E_i \} \) be a set of positive semidefinite operators on that space which sum to the identity:

\[
\sum_{i=1}^{N} E_i = I. \tag{1}
\]

The set \( \{ E_i \} \) is a positive-operator-valued measure (POVM), which is the mathematical representation of a measurement process in quantum theory. Each element in the set — called an effect — stands for a possible outcome of the measurement [1, §2.2.6]. A POVM is said to be informationally complete (IC) if the operators \( \{ E_i \} \) span the space of Hermitian operators on \( H_d \), and an IC POVM is said to be minimal if it contains exactly \( d^2 \) elements. For brevity, we can call a minimal informationally complete POVM a MIC.

Consider the Gram matrix of MIC elements, that is, the matrix \( G \) whose entries are given by

\[
[G]_{ij} := \text{tr} (E_i E_j). \tag{2}
\]

A matrix \( G \) of this type is “positive” in three senses of the word. First, each element is nonnegative. Second, \( G \) is positive definite, since it is the Gram matrix of a basis on the operator space. Third, it is constructed from inner products of objects that are themselves positive semidefinite.

Of particular note among MICs are those which enjoy the symmetry property

\[
|G|_{ij} = [G_{\text{SIC}}]_{ij} := \frac{1}{d^2} \frac{d \delta_{ij} + 1}{d + 1}. \tag{3}
\]

These are known as symmetric informationally complete POVMs, or SICs for short [2, 3, 4, 5]. In addition to their purely mathematical properties, SICs are of central interest to the technical side of QBism, a research...
program in the foundations of quantum mechanics [6, 7, 8, 9]. Investigations motivated by foundational concerns led to the discovery that SICs are in many ways optimal among MICs [10, 11, 12]. In this paper, we elaborate upon some of those results and explore the conceptual context of MICs more broadly.

In quantum physics, the Born Rule is a key step in the calculation of probabilities. The common way of presenting the Born Rule suggests that it fixes probabilities in terms of more fundamental quantities, namely quantum states and measurement operators. QBism, however, has promoted a change of viewpoint. From this new perspective, the Born Rule should be thought of as a consistency condition between the probabilities assigned in diverse scenarios — for instance, probabilities assigned to the outcomes of complementary experiments. The bare axioms of probability theory do not themselves impose relations between probabilities given different conditionals: In the abstract, nothing ties together \( P(E|C_1) \) and \( P(E|C_2) \). Classical intuition suggests one way to fit together probability assignments for different experiments, and quantum physics implies another. The discrepancy between these standards encapsulates how quantum theory departs from classical expectations [13]. By studying the full lay of the land of MICs, we hope to find a probabilistic representation of the Born Rule which picks out the cleanest statement of the quantum-classical divide in all its nuance.

In the next section, we introduce the fundamentals of quantum information theory and the necessary concepts from linear algebra to prove basic results that apply to all MICs. We also deduce a condition in terms of matrix rank for when a set of vectors in \( C^d \) can be fashioned into a MIC. Then, in Section 3, we show how to construct several classes of MICs explicitly, and we start the process of exploring the triply-positive matrices associated with them. We carry on with this task in Section 4, showing that in multiple ways, the SICs are optimal among MICs. To put this in a wider context, in Section 5 we investigate the triply-positive matrices of randomly-chosen MICs. The empirical eigenvalue distributions we find have intriguing features, not all of which have been explained yet. In Section 6, we relate the probabilistic representations of quantum theory furnished by MICs to quasi-probabilistic representations, such as Wigner functions, potentially relevant to the theory of quantum computation [14]. Thanks to this relation, we expect that the close study of MICs will be beneficial for understanding which resources are necessary to give quantum computation its go.

## 2 Basic Properties of MICs

We begin by briefly establishing the necessary notions from quantum information theory on which this paper is grounded. In quantum physics, each physical system is associated with a complex Hilbert space. Often, in quantum information theory, the Hilbert space of interest is taken to be finite-dimensional. We will denote the dimension throughout by \( d \). A quantum state \( \rho \) is a positive semidefinite operator of unit trace. If an experimenter ascribes the quantum state \( \rho \) to a system, then she finds her probability for the \( i^{th} \) outcome of the measurement modeled by the POVM \( \{E_i\} \) via the Hilbert–Schmidt inner product:

\[
p(E_i) = \text{tr} (\rho E_i).
\]

This formula is a standard presentation of the Born Rule. The condition that the \( \{E_i\} \) sum to the identity ensures that the resulting probabilities are properly normalized.

If the operators \( \{E_i\} \) span the space of positive semidefinite operators, then the operator \( \rho \) can be reconstructed from its inner products with them. In other words, the state \( \rho \) can be calculated from the probabilities \( \{p(E_i)\} \), meaning that the measurement is “informationally complete” and the state \( \rho \) can, in principle, be dispensed with. Any MIC can thus be considered a “Bureau of Standards” measurement, that is, a reference measurement in terms of which all states and processes can be understood [15]. Writing a quantum state \( \rho \) is often thought of as specifying the “preparation” of a system, though this terminology is overly restrictive, and the theory applies just as well to physical systems that were not processed on a laboratory workbench [16].

The extreme points in the space of quantum states are the rank-1 projection operators:

\[
\rho = |\psi\rangle\langle\psi|.
\]

These are idempotent operators; that is, they all satisfy \( \rho^2 = \rho \).

Given any MIC \( \{E_i\} \), we can always write its elements as unit-trace positive semidefinite operators with appropriate scaling factors:

\[
E_i := e_i \rho_i, \quad \text{where} \quad e_i = \text{tr} E_i.
\]
If the operators $\rho_i$ are all rank-1 projectors, we will refer to the set $\{E_i\}$ as a rank-1 MIC. We will call a MIC equal-weight when the coefficients $e_i$ are all equal. The condition that the elements sum to the identity then fixes $e_i = \frac{1}{d}$. Equal-weight MICs are of physical interest, since they represent quantum measurements that have no intrinsic bias: They map the “garbage state” $(1/d)I$ to the flat probability distribution over $d^2$ outcomes.

**Lemma 1.** If a MIC $\{E_i\}$ is equal-weight, then its $G$ matrix is proportional to a doubly stochastic matrix.

**Proof.** $G$ is always symmetric, because the trace is cyclic. We can therefore sum over either index:

$$\sum_j [G]_{ij} = \sum_j \text{tr} (E_i E_j) = \text{tr} \left( E_i \sum_j E_j \right) = \text{tr} E_i = e_i = \frac{1}{d}. \quad (7)$$

The sum over any row or column of the matrix $dG$ is therefore 1.

In order to establish some basic properties that hold for all MICs, we first recall a result of linear algebra.

**Lemma 2.** Let $A$ and $B$ be positive semidefinite operators. If $\text{tr}AB = 0$ then $AB = 0$.

**Proof.** Write $B = \sum_i b_i |i\rangle\langle i|$. Then $0 = \text{tr} AB = \sum_i b_i \langle i|A|i\rangle$. Since $A$ and $B$ are positive semidefinite, $b_i$ and $\langle i|A|i\rangle$ are nonnegative, so each term in the sum must equal zero. Thus, for each $i$, either $b_i = 0$ or $\langle i|A|i\rangle = 0$. Thus $A$ and $B$ have orthogonal supports and $AB = 0$.

Next, we recall the concept of a dual basis. Given a basis for a vector space, the dual of that basis is a set such that the inner products of a vector with the elements of the dual basis provide the coefficients in the expansion of that vector in terms of the original basis. In the familiar case when the original basis is orthonormal, the dual basis coincides with it: When we write a vector $\vec{v}$ as an expansion over the unit vectors ($\hat{x}, \hat{y}, \hat{z}$), the coefficient of $\hat{x}$ is simply the inner product of $\hat{x}$ with $\vec{v}$. One consequence of this definition is that if we expand the original basis in terms of itself, $E_i = \sum_j (\text{tr} E_i \tilde{E}_j) E_j$, linear independence of the $\{E_i\}$ implies that

$$\text{tr} E_i \tilde{E}_j = \delta_{ij}. \quad (9)$$

With the following convention, one can “vectorize” an operator:

$$\|A\| := \sum_i (A \otimes I)|i\rangle\langle i|. \quad (10)$$

The vectorized operator inner product is equal to the standard Hilbert–Schmidt inner product, so

$$[G]_{ij} = \text{tr} E_i E_j = \langle E_i\|E_j\|. \quad (11)$$

A quantity of central importance is the frame operator

$$\mathcal{F} := \sum_i \|E_i\\rangle\langle E_i\|. \quad (12)$$

The frame operator is fundamental because it allows us to move between a basis and its dual,

$$\|E_i\\rangle = \mathcal{F} \|\tilde{E}_i\\rangle. \quad (13)$$

From this and Eq. (12), we see

$$\sum_i \|\tilde{E}_i\\rangle\langle E_i\| = \sum_i \|E_i\\rangle\langle \tilde{E}_i\| = I_{d^2}. \quad (14)$$
Acting on one side by \( \|E_k\| \) and the other by \( \langle \bar{E}_i \| \) demonstrates that
\[
[G^{-1}]_{ij} = \text{tr} \bar{E}_i E_j = \langle \bar{E}_i \| \bar{E}_j \rangle.
\] (15)

It is also easy to check that
\[
F^{-1} = \sum_i \|E_i\rangle \langle \bar{E}_i\|.
\] (16)

**Lemma 3.** The frame operator and the Gram matrix of a MIC have the same spectrum.

**Proof.** To see this, form a projector out of the state \( \sum_i \|E_i\| |i\rangle \) where \(|i\rangle \) is an orthonormal basis in \( H_d^2 \) and perform partial traces over each subsystem. The results are \( G^T \) and \( F \), and so, by the Schmidt theorem, the eigenvalue spectra of \( F \) and \( G \) are equal.

We can now proceed to prove

**Theorem 1.** No element in a MIC can be proportional to an element of the MIC’s dual basis.

**Proof.** Writing the MIC as \( \{E_i\} \) and the dual basis as \( \{\bar{E}_i\} \), we can without loss of generality pick \( E_1 \) to be the element proportional to its dual, \( \bar{E}_1 \) (noting that \( E_1 \) will necessarily be orthogonal to all the other dual elements). So, assume that \( E_1 = \alpha \bar{E}_1 \). A general operator \( A \in \mathcal{L}(H_d) \) may be written as a linear combination of dual basis elements,
\[
A = \sum_{k=1}^{d^2} (\text{tr} AE_k) \bar{E}_k,
\] (17)

so we may write the element \( E_1 \) as
\[
E_1 = \sum_{k=1}^{d^2} (\text{tr} E_1 E_k) \bar{E}_k.
\] (18)

Since \( \text{tr} E_1^2 = \alpha \text{tr} E_1 \bar{E}_1 = \alpha \), we have \( (\text{tr} E_1^2) \bar{E}_1 = \alpha (1/\alpha) E_1 = E_1 \), and it follows that
\[
E_1 = E_1 + \sum_{k \geq 2} (\text{tr} E_1 E_k) \bar{E}_k \implies 0 = \sum_{k \geq 2} (\text{tr} E_1 E_k) \bar{E}_k.
\] (19)

As the \( \bar{E}_k \) are linearly independent, we must have \( \text{tr} E_1 E_k = 0 \) for all \( k \neq 1 \). By Lemma 2, this implies that \( E_1 E_k = 0 \) for all \( k \neq 1 \). This implies that the \( d^2 - 1 \) remaining \( E_k \) are operators on a \( d - \text{rank}(E_1) \) dimensional subspace. But
\[
\dim \left[ \mathcal{L}(H_d) \right] \leq (d - 1)^2 < d^2 - 1,
\] (20)

so they cannot be linearly independent.

Theorem 1 has physical meaning. In classical probability theory, we grow accustomed to orthonormal bases. For example, imagine an object that can be in any one of \( N \) distinct configurations. When we write a probability distribution over these \( N \) alternatives, we are encoding our expectations about which of these configurations is physically present — about the “physical condition” of the object, as Einstein would say [17], or in more modern terminology, about the object’s “ontic state” [18]. We can learn everything there is to know about the object by measuring its “physical condition”, and any implementation of such an ideal measurement is represented by conditional probabilities that are 1 in a single entry and 0 elsewhere. In other words, the map from the object’s physical configuration to the reading on the measurement device is, at its most complicated, a permutation of labels. Without loss of generality, we can take the vectors that define the ideal measurement to be the vertices of the probability simplex: The measurement basis is identical with its dual, and the dual-basis elements simply label the possible “physical conditions” of the object which the measurement reads off.

In the quantum theory, by contrast, we cannot construct a MIC that has an element which is even proportional to an element in the dual. This stymies the identification of the dual-basis elements as intrinsic “physical conditions” ready for a measurement to read.
Theorem 2. No effect of a MIC can be an unscaled projector.

Proof. Suppose \( E_1 = P \) is a projection operator. Then
\[
I = P + \sum_{i \geq 2} E_i.
\]
(21)

Multiplying on the left by \( P \),
\[
P = P + \sum_{i \geq 2} PE_i.
\]
(22)

Canceling \( P \) and taking the trace implies that
\[
\sum_{i \geq 2} \text{tr} PE_i = 0.
\]
(23)

Since every element of this sum must be nonnegative, we have for all \( i \geq 2 \),
\[
\text{tr} PE_i = 0.
\]
(24)

By Lemma 2, the remaining \( d^2 - 1 \) POVM elements are operators on a \( d - \text{rank}(P) \) dimensional subspace. But as before,
\[
\text{dim} \left[ \mathcal{L} \left( \mathcal{H}_{d - \text{rank}(P)} \right) \right] \leq (d - 1)^2 < d^2 - 1,
\]
(25)

so they cannot be linearly independent.

Theorem 3. No elementwise rescaling of a proper subset of a MIC may form a POVM.

Proof. Since a MIC is a linearly independent set, the identity element is uniquely formed by the defining expression
\[
I = \sum_{i=1}^{d^2} E_i.
\]
(26)

If a linear combination of a proper subset \( S \) of the MIC elements could be made to also sum to the identity,
\[
I = \sum_{i \in S} \alpha_i E_i,
\]
(27)

then subtracting (27) from (26) implies
\[
0 = \sum_{i \in S} (1 - \alpha_i) E_i + \sum_{i \notin S} E_i
\]
(28)

which is a violation of linear independence.

Corollary 1. No two elements in a \( d = 2 \) MIC may be orthogonal under the Hilbert-Schmidt inner product.

Proof. An orthogonal pair of elements in dimension 2 may be rescaled such that they sum to the identity element. Therefore, by Theorem 3, they cannot be elements of a MIC.

As with Theorem 1, these results have physics implications. For much of the history of quantum mechanics, one type of POVM had special status: the von Neumann measurements, which consist of \( d \) elements given by the projectors onto the vectors of an orthonormal basis of \( \mathbb{C}^d \). Indeed, in older books, these are the only quantum measurements that are considered (often being defined as the eigenbases of Hermitian operators called “observables”). We can now see that, from the standpoint of informational completeness, the von Neumann measurements are rather pathological: There is no way to build a MIC by augmenting a von Neumann measurement with additional outcomes. Later, we will see how to correct for the way that the von Neumann POVMs fall short of ideal (§3).

These results prompt a question: May any two elements of a MIC in arbitrary dimension be orthogonal? In other words, can any entry in a \( G \) matrix equal zero? We answer this question in the affirmative with an explicit example of a rank-1 MIC in dimension 3 with 7 orthogonal pairs.
Example 1. When multiplied by $1/3$, the following set of rank-1 projectors form a MIC in dimension 3 with 7 orthogonal pairs.

\[
\begin{align*}
\left\{ & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\} \\
& \begin{bmatrix} 5/8 & -i/8 & -3/8 \\ -i/8 & 1/8 & 3/8 \\ -3/8 & 3/8 & 1/8 \end{bmatrix}, \begin{bmatrix} 1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/8 \end{bmatrix}, \begin{bmatrix} 1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/8 \end{bmatrix}
\end{align*}
\]

These are projectors onto the following vectors in $\mathcal{H}_d$:

\[
\left\{ (1,0,0), (0,1,0), \frac{1}{\sqrt{2}} (1,0,1), \frac{1}{\sqrt{2}} (0,1,1), \frac{1}{\sqrt{2}} (1,0,-i), \frac{1}{\sqrt{2}} (0,1,-i), \frac{1}{\sqrt{3}} (1,-i,i), \right. \\
\left. \frac{1}{\sqrt{40}} (5,-1+2i,-3+i), \frac{1}{\sqrt{24}} (1,3+2i,-3+i) \right\}.
\]

The Hilbert–Schmidt Gram matrix of the MIC elements is

\[
\begin{bmatrix}
\frac{1}{9} & 0 & \frac{1}{18} & 0 & \frac{1}{18} & 0 & \frac{1}{27} & \frac{1}{72} & \frac{1}{216} \\
\frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{27} & \frac{1}{13} & \frac{1}{13} & \frac{1}{5} \\
\frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{27} & \frac{1}{13} & \frac{1}{5} \\
0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & \frac{1}{27} & \frac{1}{13} & \frac{1}{5} \\
0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & \frac{1}{27} & \frac{1}{13} & \frac{1}{5} \\
0 & \frac{1}{9} & \frac{1}{18} & 0 & \frac{1}{18} & \frac{1}{18} & \frac{1}{27} & \frac{1}{13} & \frac{1}{5} \\
\frac{1}{18} & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
\frac{1}{18} & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
\frac{1}{18} & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
\frac{1}{18} & \frac{1}{18} & \frac{1}{18} & 0 & \frac{1}{9} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} & \frac{1}{18} \\
\end{bmatrix}
\]

The process of finding this example led us to formulate the following:

Conjecture 1. A rank-1 MIC in dimension 3 can have no more than 7 pairs of orthogonal elements.

Our next result characterizes when it is possible to build a rank-1 POVM out of a set of vectors. We will then see what additional conditions must be met in order to obtain a POVM that is a rank-1 MIC.

Theorem 4. Consider a set of $N$ normalized vectors $|\phi_i\rangle$ in $\mathcal{H}_d$ and weights $0 \leq e_i \leq 1$. Then $E_i = e_i|\phi_i\rangle\langle\phi_i|$ forms a rank-1 POVM iff the Gram matrix $|g|_{ij} = \sqrt{e_i e_j} |\phi_i\rangle\langle\phi_j|$ is a rank-$d$ projector.

Proof. Suppose $g$ is a rank-$d$ projector. This implies $N \geq d$ and

\[
\text{tr} g = d = \sum_{i=1}^{N} e_i.
\]

Then the state

\[
\sum_i \sqrt{\frac{e_i}{d}} |\phi_i\rangle |i\rangle,
\]

where $|i\rangle$ is any orthonormal basis in $\mathcal{H}_N$, is a normalized state in $\mathcal{H}_d \otimes \mathcal{H}_N$. Form the projector

\[
\sum_{i,j=1}^{N} \sqrt{\frac{e_i e_j}{d}} (|\phi_i\rangle \langle \phi_j| \otimes |i\rangle \langle j|).
\]
If we trace over the first subsystem we obtain
\[ \sum_{i,j=1}^{N} \frac{\sqrt{e_i e_j}}{d} \langle \phi_j | \phi_i \rangle |i\rangle\langle j| = \frac{1}{d} g^T. \] (35)

If we trace over the second subsystem we obtain
\[ \sum_{i=1}^{N} \frac{e_i}{d} |\phi_i\rangle\langle \phi_i| = 1 \] (36)

By the Schmidt theorem, these partial traces must have the same nonzero spectrum. This implies that
\[ \sum_{i=1}^{N} e_i |\phi_i\rangle\langle \phi_i| = 1 \] (37)
is a matrix with \( d \) eigenvalues equal to 1. As it is an operator on \( \mathcal{H}_d \), it must be the identity. Thus the \( E_i \) form a rank-1 POVM.

Conversely, suppose the \( E_i \) form a rank-1 POVM. This implies \( N \geq d \). The trace of the expression
\[ \sum_{i=1}^{N} e_i |\phi_i\rangle\langle \phi_i| = I \] (38)
reveals that \( \sum_i e_i = d \). Following the same argument as before with the state (33), we see that
\[ \sum_{i,j=1}^{N} \frac{\sqrt{e_i e_j}}{d} \langle \phi_j | \phi_i \rangle |i\rangle\langle j| = \frac{1}{d} g^T. \] (39)
and
\[ \sum_{i=1}^{N} \frac{e_i}{d} |\phi_i\rangle\langle \phi_i| = \frac{1}{d} I. \] (40)
must have the same nonzero spectrum. Thus, there are \( d \) nonzero eigenvalues of \( g^T \), all equal to 1. This must also be true of \( g \), so \( g \) is a rank-\( d \) projector.

What further distinguishes a rank-1 MIC? In addition to \( g \) being a rank-\( d \) projector, we need \( N = d^2 \) and \( G \) full-rank, that is, we need the elements of the POVM to be linearly independent and span operator space. What is the relationship between \( g \) and \( G \)? There may be several of interest, but certainly one worth exploring is via elementwise multiplication, sometimes called the Hadamard product [19]:
\[ g \circ g^* = G. \] (41)
For any two matrices \( A \) and \( B \), the Hadamard product satisfies the rank inequality
\[ \text{rank}(A \circ B) \leq \text{rank}(A) \text{rank}(B). \] (42)
If \( A \) and \( B \) are positive semidefinite then their Hadamard product is also positive semidefinite (this is the Schur product theorem). In this case we also have
\[ \det(A \circ B) \geq \det(A) \det(B). \] (43)
If \( N \leq d^2 \), \( g \) and \( g^* \) saturate the inequality (42) iff they are formed from a rank-1 MIC. This is not the case if \( N > d^2 \); if the POVM contains a set which spans operator space, \( G \) will be rank \( d^2 \). In the end, the rank inequality is another way of expressing what differentiates informationally complete POVMs from non-informationally complete POVMs, and then what further distinguishes minimal informationally complete POVMs from those with more than \( d^2 \) entries.

Equation (41) is a special Hadamard product because it results in an elementwise absolute value squared operation. This raises the possibility that optimality among MICs might relate to interesting properties of the Hadamard absolute value [19].
3 Explicit Constructions of MICs

The MICs that have attracted the most interest are the SICs, which in many ways are the optimal MICs [10, 11, 12, 20, 21]. SICs were studied as mathematical objects (under the name “complex equiangular lines”) before their importance for quantum information was recognized [22, 23, 24, 25]. Prior to SICs becoming a physics problem, constructions were known for dimensions \(d = 2, 3\) and 8. Exact solutions for SICs are now known in the following dimensions:

\[ d = 2, 23, 24, 28, 30, 31, 35, 37, 39, 43, 48, 53, 120, 124, 195, 323. \]  (44)

The expressions for these solutions grow complicated quickly, but there is hope that they can be substantially simplified [26]. Numerical solutions have been extracted, to high precision, in the following dimensions:

\[ d = 2, 15, 168, 172, 199, 224, 228, 255, 259, 289, 323, 327, 489, 528, 725, 844, 1155, 2208. \]  (45)

Both the numerical and the exact solutions have been found in irregular order and by various methods. Particular credit should be given to Andrew Scott for solo work [27], for collaborations with Markus Grassl [4, 28] and for contributing code used by other researchers [5].

Together, these results have created the community sentiment that SICs should exist for every finite value of \(d\). To date, however, a general proof is lacking. The current frontier of SIC research extends into algebraic number theory [29, 30, 31, 32, 33], which among other things has led to a method for uplifting numerical solutions to exact ones [34]. The topic has begun to enter the textbooks for physicists [35] and for mathematicians [36].

SICs can be considered equivalently as sets of effects, of rank-1 projectors or of vectors:

\[ E_i = \frac{1}{d} \Pi_i, \text{ where } \Pi_i = |\pi_i\rangle\langle\pi_i|. \]  (46)

It is difficult to find a meaningful visualization of structures in high-dimensional complex vector space. However, for the \(d = 2\) case, an image is available. Any quantum state for a 2-dimensional system can be written as an expansion over the Pauli matrices:

\[ \rho = \frac{1}{2} (I + x\sigma_x + y\sigma_y + z\sigma_z). \]  (47)

The coefficients \((x, y, z)\) are then the coordinates for \(\rho\) in the Bloch ball. The surface of this ball, the Bloch sphere, lives at radius 1 and is the set of pure states. In this picture, the quantum states \(\{\Pi_i\}\) comprising a SIC form a regular tetrahedron; for example,

\[ \Pi_{s,s'} = \frac{1}{2} \left( I + \frac{1}{\sqrt{3}} (s\sigma_x + s'\sigma_y + ss'\sigma_z) \right), \]  (48)

where \(s\) and \(s'\) take the values \(\pm 1\).

The matrix \(G_{\text{SIC}}\) has the spectrum

\[ \lambda(G_{\text{SIC}}) = \left( \frac{1}{d}, \frac{1}{d(d+1)}, \ldots, \frac{1}{d(d+1)} \right). \]  (49)

The flatness of this spectrum will turn out to be significant; we will investigate this point in depth in the next section.

The matrix \(G_{\text{SIC}}\) is real, symmetric and positive definite. Its inverse \(G_{\text{SIC}}^{-1}\) is also real, symmetric and positive definite, and all its off-diagonal entries are nonpositive. Thus, \(G_{\text{SIC}}^{-1}\) is a Stieltjes matrix [19]. It also belongs to the (unilluminatingly named) class of \(L\)-matrices, since while the off-diagonal entries are nonpositive, the diagonal entries are all positive [37]. The question of how widely these properties hold over the set of all MICs is largely unexplored.

It is possible to construct a MIC for any arbitrary dimension \(d\). One way to do so was found by Caves, Fuchs and Schack in the course of proving a quantum version of the de Finetti theorem [38]. (For background on this theorem, a key result in probability theory, see [39, §5.3] and [40].) We will refer to these as the
orthocross MICs. To construct an orthocross MIC in dimension $d$, first pick an orthonormal basis $\{|j\rangle\}$. This is a set of $d$ objects, and we want a set of $d^2$, so our first step is to take all possible combinations:

$$\Gamma_{jk} := |j\rangle \langle k|.$$  

(50)

The orthocross MIC will be built from a set of $d^2$ rank-1 projectors $\{\Pi_\alpha\}$, the first $d$ of which are given by

$$\Pi_\alpha = \Gamma_{\alpha\alpha}.$$  

(51)

Then, for $\alpha = d + 1, \ldots, \frac{1}{2}d(d + 1)$, we take all the quantities of the form

$$\frac{1}{2} (|j\rangle + |k\rangle) (|j\rangle + \langle k|) = \frac{1}{2} (\Gamma_{jj} + \Gamma_{kk} + \Gamma_{jk} + \Gamma_{kj}),$$  

(52)

where $j < k$. We construct the rest of the $\{\Pi_\alpha\}$ similarly, by taking all quantities of the form

$$\frac{1}{2} (|j\rangle + i|k\rangle) (|j\rangle - i\langle k|) = \frac{1}{2} (\Gamma_{jj} + \Gamma_{kk} - i\Gamma_{jk} + i\Gamma_{kj}),$$  

(53)

where again the indices satisfy $j < k$. Thus, the set $\{\Pi_\alpha\}$ contains the projectors onto the original orthonormal basis, as well as projectors built from the “cross terms”.

The operators $\{\Pi_\alpha\}$ are all positive semidefinite, and the set is linearly independent. The final step is to transform this set into a POVM, which requires making them sum to the identity. We do this by constructing

$$\Omega := \sum_{\alpha=1}^{d^2} \Pi_\alpha,$$  

(54)

which is easily shown to be a positive definite operator, and thus invertible. Multiplying both sides of the above equation by $\Omega^{-1/2}$ from the left and the right, we find that

$$I = \sum_{\alpha=1}^{d^2} \Omega^{-1/2} \Pi_\alpha \Omega^{-1/2}.$$  

(55)

The operators that appear in this sum,

$$E_\alpha := \Omega^{-1/2} \Pi_\alpha \Omega^{-1/2},$$  

(56)

inherit the rank and linear independence properties of the original projectors $\{\Pi_\alpha\}$, and by construction they sum to the identity, thereby constituting a POVM.

The operator $\Omega$ has a comparatively simple matrix representation: The elements along the diagonal are all equal to $d$, the elements above the diagonal are all equal to $\frac{1}{2}(1 + i)$, and the rest are $\frac{1}{2}(1 - i)$, as required by $\Omega = \Omega^\dagger$. The matrix $\Omega$ is not quite a circulant matrix, thanks to that change of sign, but it can be turned into one by conjugating with a diagonal unitary matrix. Consequently, the eigenvalues of $\Omega$ can be found explicitly via discrete Fourier transformation. The result is that, for $m = 0, \ldots, d - 1$,

$$\lambda_m = d + \frac{1}{2} \left( \cot \frac{\pi(4m + 1)}{4d} - 1 \right).$$  

(57)

This mathematical result has a physical implication [15].

**Theorem 5.** The probability of any outcome $E_\alpha$ of an orthocross MIC, given any quantum state $\rho$, is bounded above by

$$P(E_\alpha) \leq \left[ \frac{d}{2} - \frac{1}{2} \left( 1 + \cot \frac{3\pi}{4d} \right) \right]^{-1} < 1.$$  

(58)

**Proof.** The maximum of $\text{tr}(\rho E_\alpha)$ over all $\rho$ is bounded above by the maximum of $\text{tr}(\Pi E_\alpha)$, where $\Pi$ ranges over the rank-1 projectors. In turn, this is bounded above by the maximum eigenvalue of $E_\alpha$. We then invoke that

$$\lambda_{\text{max}}(E_\alpha) = \lambda_{\text{max}}(\Omega^{-1/2} \Pi_\alpha \Omega^{-1/2}) = \lambda_{\text{max}}(\Pi_\alpha \Omega^{-1} \Pi_\alpha) \leq \lambda_{\text{max}}(\Omega^{-1}).$$  

(59)

The desired bound then follows.
Note that all the entries in the matrix $2\Omega$ are Gaussian integers. Consequently, all the coefficients in the characteristic polynomial of $2\Omega$ will be Gaussian integers, and so the eigenvalues of $2\Omega$ will be algebraic integers. This is an example of how, in the study of MICs, number theory becomes relevant to physically meaningful quantities — in this case, a bound on the maximum probability of a reference-measurement outcome. Number theory has also turned out to be very important for SICs, in a much more sophisticated way [29, 30, 31, 32, 33].

The following conjectures about orthocross MICs have been motivated by numerical investigations. We expect that they are “textbook exercises of the future”, in that with the proper hint, their proofs might become relatively straightforward.

**Conjecture 2.** The entries in the $G$ matrices for orthocross MICs can become arbitrarily small with increasing $d$, but no two elements of an orthocross MIC can be exactly orthogonal.

**Conjecture 3.** For any orthocross MIC, $G^{-1}$ is not a Stieltjes matrix.

**Conjecture 4.** For any orthocross MIC, the entries in $G^{-1}$ are integers or half-integers.

As mentioned above, SIC existence is an open question. It is much easier to construct a symmetric MIC when the elements are not required to be rank-1. One such class of measurements are the Wigner MICs [41]. These were defined by Appleby in terms of the Weyl–Heisenberg group, which is most conveniently written in terms of its generators. Let $\{|j\rangle : j = 0, \ldots, d-1\}$ be an orthonormal basis, and define $\omega = e^{2\pi i/d}$. Then the operator $X|j\rangle = |j + 1\rangle$, (60)

where addition is interpreted modulo $d$, effects a cyclic shift of the basis vectors. The Fourier transform of the $X$ operator is $Z|j\rangle = \omega^{|j\rangle}$, (61)

and together these operators satisfy the Weyl commutation relation $ZX = \omega XZ$. (62)

The Weyl–Heisenberg displacement operators are $D_{k,l} := (-e^{\pi i/d})^{kl} X^k Z^l$, (63)

and together they satisfy the conditions

$D_{k,l}^\dagger = D_{-k,-l}$, $D_{k,l} D_{m,n} = (-e^{\pi i/d})^{lm-kn} D_{k+m,l+n}$.

Each $D_{k,l}$ is unitary and a $d^{th}$ root of the identity. The Weyl–Heisenberg group is the set of all operators $(-e^{\pi i/d})^m D_{k,l}$ for arbitrary integers $m$, and it is projectively equivalent to $Z_d \times Z_d$.

Let the operator $B$ be constructed as

$B := \frac{1}{d+1} \sum_{k,l} D_{k,l}$, (65)

and define $B_{k,l}$ to be its conjugate under a Weyl–Heisenberg displacement operator:

$B_{k,l} := D_{k,l} B D_{k,l}^\dagger$.

The elements of the Wigner MIC have rank $(d+1)/2$, and are defined by $E_{k,l} := \frac{1}{d^2} \left(I + \frac{1}{\sqrt{d+1}} B_{k,l}\right)$, (67)

A Wigner MIC exists in any odd dimension $d$. The quantities

$W_{k,l} := (d+1) \text{tr} (E_{k,l} \rho) - \frac{1}{d}$.
are quasiprobabilities: They can be negative, but the sum over all of them is unity. The quasiprobability function \( \{W_{k,l}\} \) is known as the Wigner function of the quantum state \( \rho \). The Wigner function is the prototype for an entire genre of quasiprobability representations of quantum mechanics, a topic to which we will return in Section 6.

So far, we have not imposed any additional structure upon our Hilbert space. However, in practical applications, one might have additional structure in mind, such as a preferred factorization into a tensor product of smaller Hilbert spaces. For example, a register in a quantum computer might be a set of \( N \) physically separate qubits, yielding a joint Hilbert space of dimension \( d = 2^N \). In such a case, a natural course of action is to construct a MIC for the joint system by taking the tensor product of multiple copies of a MIC defined on the component system:

\[
E_{j_1,j_2,...,j_N} := E_{j_1} \otimes E_{j_2} \otimes \cdots \otimes E_{j_N}.
\]  

(69)

Since a collection of \( N \) qubits is a natural type of system to consider for quantum computation, we define the \( N \)-qubit tensorhedron MIC to be the tensor product of \( N \) individual qubit SICs.

**Theorem 6.** The Gram matrix of an \( N \)-qubit tensorhedron MIC is the tensor product of \( N \) copies of the Gram matrix for the qubit SIC out of which the tensorhedron is constructed.

**Proof.** Consider the two-qubit tensorhedron MIC, whose elements are given by

\[
E_{d(j-1)+j'} := \frac{1}{4} \Pi_j \otimes \Pi_{j'},
\]

(70)

with \( \{\Pi_j\} \) being a qubit SIC. The Gram matrix for the tensorhedron MIC has entries

\[
[G]_{d(j-1)+j',d(k-1)+k'} = \frac{1}{16} \text{tr} \left[ (\Pi_j \otimes \Pi_{j'})(\Pi_k \otimes \Pi_{k'}) \right].
\]

(71)

We can group together the projectors that act on the same subspace:

\[
[G]_{d(j-1)+j',d(k-1)+k'} = \frac{1}{16} \text{tr} \left( \Pi_j \Pi_k \otimes \Pi_{j'} \Pi_{k'} \right).
\]

(72)

Now, we distribute the trace over the tensor product, obtaining

\[
[G]_{d(j-1)+j',d(k-1)+k'} = \frac{1}{16} \frac{2\delta_{jk} + 1}{3} \frac{2\delta_{jj'k'} + 1}{3} = [G_{\text{SIC}}]_{jk} [G_{\text{SIC}}]_{j'k'},
\]

(73)

which is just the definition of the tensor product:

\[
G = G_{\text{SIC}} \otimes G_{\text{SIC}}.
\]

(74)

This extends in the same fashion to more qubits. \( \square \)

**Corollary 2.** The spectrum of the Gram matrix for an \( N \)-qubit tensorhedron MIC contains only the values

\[
\lambda = \frac{1}{2N} \frac{1}{3m}, \quad m = 0, \ldots, N.
\]

(75)

**Proof.** This follows readily from the linear-algebra fact that the spectrum of a tensor product is the set of products \( \{\lambda_i \mu_j\} \), where \( \{\lambda_i\} \) and \( \{\mu_j\} \) are the spectra of the factors. \( \square \)

We can also deduce properties of MICs made by taking tensor products of MICs that have orthogonal elements. Let \( \{E_j\} \) be a \( d \)-dimensional MIC with Gram matrix \( G \), and suppose that exactly \( N \) elements of \( G \) are equal to zero. The tensor products \( \{E_j \otimes E_{j'}\} \) construct a \( d^2 \)-dimensional MIC, the entries in whose Gram matrix have the form \( [G]_{jk}[G]_{j'k'} \), as above. This product will equal zero when either factor does, meaning that the Gram matrix of the tensor-product MIC will contain \( 2d^4N - N^2 \) zero-valued entries. It seems plausible that in prime dimensions, where tensor-product MICs cannot exist, the possible number of zeros is more tightly bounded, but this remains unexplored territory.
4 SICs are Optimal among MICs

What might it mean for a MIC to be the best among all MICs? In order to provide a quantitative answer to this question, we codify an ideal that a MIC should approach. In essence, we want to find a MIC that furnishes a probabilistic representation of quantum theory which looks as close to classical probability as is mathematically possible. The residuum that remains — the unavoidable discrepancy that even the most clever choice of MIC cannot eliminate — is a signal of what is truly quantum about quantum mechanics.

We set the stage with a preliminary result.

Lemma 4. Given a MIC \( \{E_i\} \) with Gramian \( G \), a quantum state is pure if and only if its probabilistic representation satisfies

\[ \sum_{ij} p(E_i)p(E_j)[G^{-1}]_{ij} = 1. \]  

(76)

Proof. Fix the Hilbert-space dimension \( d \), and let \( \{E_i\} \) be a MIC. Any MIC is an operator basis by virtue of linear independence. Denote by \( \{\tilde{E}_i\} \) the dual basis. We can expand any quantum state \( \rho \) in terms of the dual basis, and the expansion coefficients are the inner products with the elements of the original basis:

\[ \rho = \sum_i (\text{tr} E_i \rho)\tilde{E}_i. \]  

(77)

By the Born Rule, the coefficients are probabilities:

\[ \rho = \sum_i p(E_i)\tilde{E}_i. \]  

(78)

Now, recall that while \( \text{tr} \rho = 1 \) holds for any quantum state \( \text{tr} \rho^2 = 1 \) holds if and only if that operator is a “pure” state, i.e., a rank-1 projector. These operators are the extreme points of quantum state space; all other quantum states are convex combinations of them. In terms of the MIC’s dual basis, the pure-state condition is

\[ \sum_{ij} p(E_i)p(E_j)\text{tr} \tilde{E}_i\tilde{E}_j = 1. \]  

(79)

The “metric” in this quadratic form is the Gram matrix of the dual basis. Frame theory tells us that this is the inverse of the Gram matrix of the MIC itself:

\[ \sum_{ij} p(E_i)p(E_j)[G^{-1}]_{ij} = 1, \]  

(80)

as desired.

We can find an analogue for this condition in classical probability theory. The extreme points in a classical theory are the vertices of the probability simplex. The “ideal of the detached observer” (as Pauli phrased it [13]) is a measurement that reads off the system’s point in phase space, call it \( \lambda_i \), without disturbance. So, if we apply the reference measurement to each of a pair of identically prepared systems, we expect that we will obtain the same outcome both times. In other words, our “collision probability” is unity:

\[ \sum_i p(\lambda_i)^2 = 1. \]  

(81)

Now, we recall our goal: What is the unavoidable residuum that separates quantum from classical? In other words, how closely can we make the quantum condition on extreme points, Eq. (80), resemble the classical one, Eq. (81)? This depends on how close we can bring the matrix \( G^{-1} \) to the identity.

It makes sense for a reference process to be intrinsically unbiased. In this case, we have another reason to do so, because we’re trying to get as close as we can to the classical theory in terms of collision probability. So, let’s say we want the collision probability for the garbage state to be what it would be for the classical ideal, i.e., \( 1/d^2 \). We get this when the elements of the reference measurement \( \{E_i\} \) are equally weighted.

So, how close can we bring \( G^{-1} \) to the identity, by choosing an appropriate equal-weight MIC? We know the answer to this from a recent theorem [12]: The best choice is a SIC.
Theorem 7. Let $G$ be the Gram matrix of an equal-weight MIC, and let $\|\cdot\|$ be any unitarily invariant norm (i.e., any norm where $\|A\| = \|UAV\|$ for arbitrary unitaries $U$ and $V$). Then

$$\left\|I - \frac{1}{d}G^{-1}\right\| \geq \left\|I - \frac{1}{d}G_{\text{SIC}}^{-1}\right\|,$$

with equality if and only if the MIC is a SIC.

Proof. This is a special case of Theorem 1 in [12].

Theorem 7 applies to all unitarily invariant norms. This includes the Frobenius norm, the trace norm, the operator norm and all the other Schatten $p$-norms, as well as the Ky Fan $k$-norms. With respect to all of these distance measures, the SICs bring $G^{-1}$ as close to the identity as possible. Theorem 7 was originally proven for foundational reasons [12], but it turns out to answer in the affirmative a conjecture regarding a practical matter of quantum computation [43, §VII.A].

So far, we have investigated the matrix $G^{-1}$. What can we say about the matrix $G$ directly? This is easiest to explore when the MIC elements are all proportional to rank-1 projection operators.

Theorem 8. No equal-weight MIC composed of rank-1 elements can be closer to an orthonormal basis than a SIC is, when measured by Frobenius distance between the MIC’s Gram matrix and the identity.

Proof. Let $\{E_i\}$ be a MIC on $d$-dimensional Hilbert space, and assume that this MIC is equal-weight, that is, $E_i = (1/d)\rho_i$ for some rank-1 projection operators $\{\rho_i\}$. Consider the quantity

$$F := \sum_{ij} (\delta_{ij} - \text{tr } E_i E_j)^2,$$

which is the squared Frobenius distance between the MIC’s Gram matrix and the identity. If the MIC could be an orthonormal basis of the operator space, then the distance $F$ would vanish. We can split $F$ into two sums, as follows:

$$F = \sum_i (1 - \text{tr } E_i^2)^2 + \sum_{i \neq j} (\text{tr } E_i E_j)^2.$$

The equal-weight and rank-1 conditions let us simplify the first sum, yielding

$$F = d^2 \left(1 - \frac{1}{d^2}\right)^2 + \sum_{i \neq j} (\text{tr } E_i E_j)^2.$$

Applying the Cauchy–Schwarz inequality to the remaining sum, we get that

$$F \geq d^2 \left(1 - \frac{1}{d^2}\right)^2 + \frac{1}{d^3 - d^2} \left(\sum_{i \neq j} \text{tr } E_i E_j\right)^2,$$

with equality iff all the terms in the sum are equal, i.e., $\text{tr } E_i E_j$ is constant for all $i \neq j$. Because the $\{E_i\}$ must sum to the identity,

$$\sum_{i \neq j} \text{tr } E_i E_j = \sum_i \text{tr } E_i (I - E_i) = d - 1.$$

Consequently,

$$F \geq \left(d - \frac{1}{d}\right)^2 + \frac{d - 1}{d^4 - d^2},$$

with equality iff the MIC is symmetric, i.e., $\text{tr } E_i E_j$ is constant for all $i \neq j$. The value of this constant is fixed by dividing its sum evenly across all contributions:

$$\text{tr } E_i E_j = \frac{1}{d^2} \frac{1}{d^2 - 1} \sum_{i \neq j} \text{tr } E_i E_j = \frac{1}{d^2} \frac{1}{d + 1}.$$
In terms of the projection operators \( \{ \rho_i \} \),
\[
\text{tr} \rho_i \rho_j = \frac{d \delta_{ij} + 1}{d+1}.
\] (90)

Thus, the Gram matrix of an equal-weight, rank-1 MIC saturates the lower bound on its squared Frobenius distance to the identity iff the MIC is a SIC.

The rank-1 condition can in fact be relaxed, at the expense of a slightly more elaborate proof. However, the equal-weight condition is not optional: The Gram matrix of a MIC can be made closer to the identity by taking the weights \( \{ e_i \} \) to be biased. \(^1\)

One property of the \( G \) matrices that we can deduce is a constraint on their eigenvalues, expressed in the language of majorization \([19, 44]\). Let \( x \) and \( y \) be two vectors of \( N \) entries each, and suppose that each element in \( x \) and in \( y \) is nonnegative. Write \( x^\downarrow \) and \( y^\downarrow \) for the vectors made by sorting \( x \) and \( y \) in nonincreasing order. Then \( x \) weakly majorizes \( y \) if
\[
\sum_{i=1}^{k} x_i^\downarrow \geq \sum_{i=1}^{k} y_i^\downarrow
\] (91)
for all \( k = 1, \ldots, N \). We write this condition as \( x \succ_w y \). If the \( k = N \) condition is satisfied with equality, then \( x \) majorizes \( y \), which we write as \( x \succ y \). Majorization is a partial order on vectors of nonnegative numbers. Heuristically speaking, \( x \) majorizes \( y \) if \( y \) is a flatter vector than \( x \).

To understand the context in which our next theorem applies, we first need a preliminary result:

**Lemma 5.** The Gram matrix \( G \) of any rank-1 MIC has a trace equal to or larger than 1.

**Proof.** By definition,
\[
\text{tr} \ G = \sum_i \text{tr} \ E_i^2 = \sum_i e_i^2 \text{tr} \rho_i^2.
\] (92)
If the MIC elements are proportional to rank-1 projectors, then \( \rho_i^2 = \rho_i \), and
\[
\text{tr} \ G = \sum_i e_i^2.
\] (93)
Applying the Cauchy–Schwarz inequality,
\[
\text{tr} \ G \geq \frac{1}{d^2} \left( \sum_i e_i \right)^2.
\] (94)
Recalling that the \( e_i \) sum to \( d \) because the \( E_i \) sum to the identity, we obtain that
\[
\text{tr} \ G \geq 1,
\] (95)
as desired.

**Theorem 9.** Let \( G \) be the Gram matrix of a MIC which satisfies \( \text{tr} \ G \geq 1 \) and let \( \lambda \) denote its spectrum. Then \( \lambda \succ_w \lambda_{\text{SIC}} \) with equality iff the MIC is a SIC.

**Proof.** We may write \( E_i = e_i \rho_i \) where \( e_i \) is a positive number and \( \rho_i \) is a quantum state. Since the frame operator has the same spectrum as the Gram matrix, we may establish a lower bound on the largest eigenvalue of the Gram matrix:
\[
\lambda_1 \geq \frac{1}{d} \langle I \| F \| I \rangle = \frac{1}{d} \sum_j (\text{tr} \ E_j)^2 = \frac{1}{d} \sum_j e_j^2 (\text{tr} \rho_j)^2 = \frac{1}{d} \sum_j e_j^2 \geq \frac{1}{d^3} \left( \sum_j e_j \right)^2 = \frac{1}{d},
\] (96)
\(^1\)An earlier paper by one of us (CAF) and a collaborator \([7]\) made the claim that the equal-weight condition could be derived by minimizing the squared Frobenius distance; this is erroneous. For the purposes of that earlier paper, it is sufficient to impose by hand the requirement that the MIC be equal-weight, since it is naturally desirable that a reference measurement have no intrinsic bias. Having made this extra proviso, the conceptual conclusions of that work are unchanged. Moreover, the same conceptual conclusions can be made much more robustly by using a refined version of the argument \([12]\).
where the second inequality has equality iff \( e_j = 1/d \) for all \( j \). The spectrum of the Gram matrix for a SIC reveals that this lower bound is tight. Now we have

\[
\lambda \succ \left( \frac{\lambda_1}{d^2 - 1}, \frac{\lambda_2}{d^2 - 1}, \ldots, \frac{\lambda_n}{d^2 - 1} \right) \succ_w \left( \frac{1}{d}, \frac{1}{d(d + 1)}, \ldots, \frac{1}{d(d + 1)} \right) = \lambda_{\text{SIC}}. \tag{97}
\]

The majorization follows because the second list is the flattest distribution which sums to \( \text{tr} G = \sum_j \lambda_j \) having largest eigenvalue \( \lambda_1 \). The weak majorization follows if the inequality

\[
\lambda_1 + n \left( \frac{\text{tr} G - \lambda_1}{d^2 - 1} \right) \geq \frac{1}{d} + n \left( \frac{1}{d(d + 1)} \right)
\]

holds when \( 1/d \leq \lambda_1 \leq \text{tr} G, 0 \leq n \leq d^2 - 1, \text{tr} G \geq 1, \) and \( d > 1 \). Equation (98) is algebraically equivalent to

\[
(d\lambda_1 - 1)(d^2 - 1 - n) + nd(\text{tr} G - 1) \geq 0,
\]

which is clearly satisfied by the conditions. When both majorizations are equalities, the state \( 1/\sqrt{d} \langle I \rangle \) is the eigenvector of the frame operator corresponding to the maximal eigenvalue of \( 1/d \). As indicated by the spectrum, in the diagonalizing basis the frame operator takes the form

\[
\mathcal{F} = \frac{1}{d(d + 1)} (I_{d^2} + \| I \rangle \langle I \|). \tag{100}
\]

We know from prior work (see [11, Corollary 1], and also Lemma 7 in the appendix) that this implies that \( dE_j \) forms a set of SIC projectors.

Specific examples of SICs have turned out to have interesting properties from a physical standpoint, that is, for purposes of quantum information and computation. For example, SICs in dimensions \( d = 2, d = 3 \) and \( d = 8 \) have been identified as optimal resources for quantum-computation protocols [14, 45, 46]. A SIC in dimension \( d = 4 \) has surprising properties with regard to quantum entanglement [47]. Similarly, SICs emerge as significant in the study of quantum cloning. Earlier work [48] identified two measures for evaluating a cloning device, the average global fidelity and the average local fidelity. These two measures imply different standards of “best”; that is, they are maximized for differing values of the inner product between pure quantum states. Using the average global fidelity, the optimum (where “two states are most quantum with respect to each other”) occurs when the inner product \( \text{tr} \rho \sigma \) is \( 1/3 \). On the other hand, using the average local fidelity, the optimum occurs when the inner product is \( 1/4 \). Therefore, a qubit SIC is a maximal set of states that pairwise optimize the average global fidelity, while a qutrit SIC does the same for average local fidelity. The results of this section indicate that it is not just specific examples of SICs that have interest for quantum information theory, but the entire class of measurement.

## 5 Computational Overview of MIC Gramians

In order to explore the realm of MICs more broadly, and to connect them with other areas of mathematical interest, it is worthwhile to generate MICs randomly and study the \( G \) matrices that result. One way to do so is to pick a random vector in a \( d \)-dimensional Hilbert space and take its orbit under the Weyl–Heisenberg group. The rank-1 projectors constructed from this orbit, rescaled by \( 1/d \), form an equal-weight MIC. (More precisely, they will do so except for initial vectors chosen from a set of measure zero.) The eigenvalue spectra of the \( G \) matrices for these Weyl–Heisenberg MICs display intriguing patterns, as shown in Figure 1.

Any stochastic matrix has an eigenvector with eigenvalue 1. Since these MICs are equal-weight, \( \text{d}G \) is always doubly stochastic, and so \( G \) will always have an eigenvalue equal to \( 1/d \). Generating large numbers of random Weyl–Heisenberg MICs, we can observe patterns in the histograms of their eigenvalues. The value \( 1/d \) appears as a peak at the upper extreme of the spectrum. Each histogram shows an exponential decay, trailing off with increasing eigenvalue, but the features in the small-eigenvalue portion appear to be dimension-dependent.

**Conjecture 5.** The plateau in the eigenvalue distribution for \( d = 3 \), seen in Figure 1(b), is related to the existence of a continuous family of unitarily inequivalent SICs in that dimension [49, 50].
6 MICs, Q-reps, and SIMs

Fix an equal-weight MIC \( \{ E_i \} \), and consider some other measurement \( \{ B_j \} \), which may or may not be a MIC. The Born Rule tells us that

\[
p(B_j) = \text{tr} (B_j \rho).
\]

(101)

Writing \( B_j = b_j \pi_j \) for some quantum state \( \pi_j \), and expanding both \( \pi_j \) and \( \rho \) in terms of the MIC’s dual basis \( \{ \tilde{E}_i \} \), we find that

\[
p(B_j) = b_j \sum_{m,n} \text{tr} (\pi_j E_m) \text{tr} (\rho E_n) \text{tr} (\tilde{E}_m \tilde{E}_n).
\]

(102)

As before, the last trace is given by the inverse of the MIC’s Gram matrix. The other two traces are the Born Rule probabilities for the events \( E_m \) and \( E_n \) given the quantum states \( \pi_j \) and \( \rho \):

\[
p(B_j) = b_j \sum_{m,n} \rho(E_m | \pi_j) \rho(E_n | \rho) (G^{-1})_{mn}.
\]

(103)

The inverse Gramian \( G^{-1} \) is playing the role of the metric in a bilinear form:

\[
p(B_j) = b_j \rho^T \pi_j G^{-1} \rho.
\]

(104)

Thanks to the informational completeness of our reference measurement \( \{ E_i \} \), we can calculate the probability for any other event in terms of probabilities, bypassing if we wish the operator \( \rho \).

For an equal-weight MIC, the matrix \( dG \) is doubly stochastic (Lemma 1). Its inverse is therefore doubly quasi-stochastic: The rows and columns each sum to 1, but the elements are not confined to the unit interval. Therefore, in the formula

\[
p(B_j) = b_j \rho^T \pi_j (dG)^{-1} \rho,
\]

(105)

we can act with \( (dG)^{-1} \) either to the left or to the right and preserve normalization. This necessarily introduces negativity somewhere: The linear map \( (dG)^{-1} \) sends some probabilities to quasi-probabilities, vectors that are properly normalized but that lie outside the probability simplex. The form of the expression (105) means that we have a certain “gauge freedom” about where the negativity can occur [51]. We can put it into the states by acting to the right, or we can put it into the effects by acting to the left. We can also “split the map down the middle” by writing

\[
(dG)^{-1} = (dG)^{-1/2}(dG)^{-1/2}
\]

(106)

and acting with one factor in either direction. This has the intriguing consequence that the final probability \( p(D_j) \) is zero exactly when the two transformed vectors are orthogonal.

This brings us into the territory of quasi-probability representations of quantum theory constructed on orthogonal operator bases [52, 53]. A Q-rep is an orthogonal Hermitian operator basis \( \{ F_i \} \) constrained to sum to the identity: \( \sum_i F_i = I \). All orthogonal bases are self-dual, meaning that each vector is proportional to its dual vector, \( F_j = \alpha_j \tilde{F}_j \). From the sum constraint, we can see that \( \text{tr} \tilde{F}_i = 1 \) for all \( i \). Define \( f_i := \text{tr} F_i \). Then \( \sum_i f_i = d \). If \( f_i = c \) for all \( i \), we say the Q-rep is equal weight. Because it is orthogonal, we have \( \text{tr} F_i F_j = c_i \delta_{ij} \). From the sum constraint,

\[
\sum_i \text{tr} F_i F_j = \text{tr} F_j \implies \sum_i c_i \delta_{ij} = f_j \implies c_j = f_j.
\]

(107)

Thus \( \text{tr} F_i F_j = f_j \delta_{ij} \), and we can see that the dual basis satisfies \( F_j = f_j \tilde{F}_j \). Contrast this with Theorem 1, which shows that such proportionality never holds for MICs.

There is a very natural way in which MICs relate to Q-reps. This connection is simplest and likely of most interest in the equal weight situation where \( f_i = 1/d \) so we will first make this special case explicit. The vectorized notation and the frame operator introduced above will turn out to be quite useful.

The definition of a Q-rep is motivated by MICs, with the positivity condition relaxed. If treated like a POVM, they can be used to define quasi-probabilities. Recall that, in the equal weight case, \( dG \) is a doubly stochastic matrix (Lemma 1). If a matrix \( A \) is column quasistochastic then \( A^{-1} \) is as well,

\[
\sum_k [A^{-1}]_{kj} = \sum_{ik} [A]_{ik} [A^{-1}]_{kj} = \sum_i \delta_{ij} = 1,
\]

(108)
and similarly if $A$ is row quasistochastic. Further, $A$ raised to any power remains row or column quasistochastic: When raised to any power, one of the eigenvalues of the result remains equal to 1 and the eigenvectors are unchanged, thus, there remains a flat right or left eigenvector with eigenvalue equal to 1 indicating that the result remains row or column quasistochastic. With these facts in hand, we can see that the following construction provides us with a Q-rep from an equal weight MIC:

$$F_i := \frac{1}{\sqrt{d}} \sum_j [G^{-1/2}]_{ij} E_j . \quad (109)$$

Summing over $i$ we get the identity because $\frac{1}{\sqrt{d}} G^{-1/2}$ is doubly quasistochastic and the $E_i$ form a POVM. Orthogonality may be checked directly:

$$\text{tr} F_i F_j = \frac{1}{d} \sum_{kl} [G^{-1/2}]_{ik} [G^{-1/2}]_{jl} \text{tr} E_k E_l = \frac{1}{d} \sum_{kl} [G^{-1/2}]_{ik} [G^{-1/2}]_{jl} = \frac{1}{d} \delta_{ij} , \quad (110)$$

where we used the fact that $G$ is symmetric. In addition to sharing the same spectrum (Lemma 3), the Gram matrix and the frame operator share the following nice relation.

**Lemma 6.** The inverse square root of the Gram matrix enjoys the following representation in terms of the frame operator:

$$[G^{-1/2}]_{ij} = \langle \tilde{E}_i \| \mathcal{F}^{-1/2} \| \tilde{E}_j \rangle . \quad (111)$$

**Proof.**

$$[G^{-1}]_{jk} = \langle \tilde{E}_j \| \tilde{E}_k \rangle = \langle \tilde{E}_j \| \mathcal{F}^{-1/2} \mathcal{F}^{-1/2} \| \tilde{E}_k \rangle = \sum_i \langle \tilde{E}_j \| \mathcal{F}^{-1/2} \| E_i \rangle \langle E_i \| \mathcal{F}^{-1/2} \| \tilde{E}_k \rangle = \sum_i \langle \tilde{E}_j \| \mathcal{F}^{-1/2} \| E_i \rangle \langle \tilde{E}_i \| \mathcal{F}^{-1/2} \| E_k \rangle \quad (112)$$

$$\implies [G^{-1/2}]_{ij} = \langle \tilde{E}_i \| \mathcal{F}^{-1/2} \| E_j \rangle .$$

Inserting this into the vectorized form of (109) and removing the resolution of identity that appears, gives us the nice identities

$$\| F_i \| = \frac{1}{\sqrt{d}} \sum_j [G^{-1/2}]_{ij} \| E_j \| = \frac{1}{\sqrt{d}} F^{-1/2} \| E_i \| . \quad (113)$$

In (109), the essential facts were that we mix together the MIC elements with a matrix which squares to $G^{-1}$ and that when scaled by $1/\sqrt{d}$ is doubly quasistochastic. It turns out that another square root of $G^{-1}$, aside from the principal square root, also satisfies these desiderata. Define the matrix

$$S := -I + \frac{2}{d^2} J , \quad (114)$$

where $J$ is the Hadamard identity, that is, the matrix of all 1s. $S$ is a real symmetric unitary, which moreover commutes with all doubly stochastic matrices, so $G^{-1/2} S$ is another square root of $G^{-1}$ preserving the properties that we need.

**Theorem 10.** For any equal weight MIC, we may construct the two Q-reps

$$\| F_1 \| := \frac{1}{\sqrt{d}} \sum_j [G^{-1/2}]_{ij} \| E_j \| \quad \text{and} \quad \| F_1^S \| := \frac{1}{\sqrt{d}} \sum_j [G^{-1/2} S]_{ij} \| E_j \| , \quad (115)$$

which we will call the principal Q-rep and the shifted Q-rep respectively.

**Corollary 3.** The relation between $\{ F_1 \}$ and $\{ F_1^S \}$ is an elementwise affine transformation:

$$\| F_1^S \| = -\| F_1 \| + \frac{2}{d^2} \| I \| . \quad (116)$$
If the MIC is a SIC, $E_i = \frac{1}{d} \Pi_i$, we have

$$\frac{1}{\sqrt{d}} [G^{-1/2}_{\text{SIC}}]_{ij} = \sqrt{d + 1} \delta_{ij} + \frac{1}{d^2} (1 - \sqrt{d + 1}),$$

and so we get the following two Q-reps:

$$F_j = \frac{1}{d} \left( \sqrt{d + 1} \right) \Pi_j + \frac{1}{d^2} \left( 1 - \sqrt{d + 1} \right) I$$ and $$F_j^S = -\frac{1}{d} \left( \sqrt{d + 1} \right) \Pi_j + \frac{1}{d^2} \left( 1 + \sqrt{d + 1} \right) I.$$

These are the two Q-reps identified by Zhu [52] for a different reason.\(^2\) Given a Q-rep, the ceiling negativity of a quantum state $\rho$ is the magnitude of the most negative entry in the quasi-probability vector that represents $\rho$. Maximizing the ceiling negativity over all quantum states yields the ceiling negativity of the Q-rep. The two Q-reps $\{F_j\}$ and $\{F_j^S\}$ provide, respectively, the lower and upper bounds on the ceiling negativity over all equal-weight Q-reps in dimension $d$. Our orthogonalization procedure, Eq. (115), sets Zhu’s result in a more broad conceptual context: Zhu’s Q-reps are the output of applying to a SIC a procedure that works for any MIC.

This connection can be generalized beyond the equal weight case. For notational simplicity, we introduce the matrix $\Phi := AG^{-1}$ where $[A]_{ij} = e_i \delta_{ij}$ where $e_i := \text{tr} E_i$. Noting that the square root of $\Phi$ with all positive eigenvalues may be written as $\sqrt{\Phi} = A^{1/2} (A^{1/2} G^{-1} A^{-1/2})^{1/2} A^{-1/2}$, it is relatively straightforward to show that $f_i = e_i$. The principal Q-rep satisfies

$$\|F_i\| = \sum_j [\sqrt{\Phi}]_{ij} \|E_j\| = A^{1/2} \|E_i\|$$

where $A := \sum_i \frac{1}{d} \|E_i\|\|E_i\|$ is the frame operator for the rescaled basis $\frac{1}{\sqrt{d}} E_i$ and the shifted Q-rep satisfies a weight-dependent shifting

$$\|F_i^S\| = -\|F_i\| + \frac{2}{d} e_i \|I\|. \quad (120)$$

What about going in the other direction, from a Q-rep to a MIC? Naïvely, we could try to invert the matrix multiplication in Eq. (113) and obtain $\|E_i\|$ in terms of $\|F_i\|$. However, without an independent expression for the frame operator $F$, we lack the information to carry this out. It is nevertheless possible to construct a MIC from a Q-rep in a canonical way, following a procedure suggested by Marcus Appleby.

**Theorem 11.** Given a Q-rep $\{F_i\}$, let $f$ be the minimal (i.e., most strongly negative) eigenvalue of all the elements of the Q-rep. Then

$$H_i := \frac{F_i - f I}{1 - d^2 f}$$

defines a MIC with equal pairwise inner products $\text{tr} H_i H_j$ and constant $\text{tr} H_i^2$.

**Proof.** The essential task to be done is to remove the negativity from the Q-rep operators, while preserving the property that they sum to the identity. First, let $f$ be the minimal eigenvalue in the Q-rep $\{F_i\}$. Now define $\{H_i\}$ as stated. By inspection we see that the $\{H_i\}$ sum to the identity. Since we are subtracting the smallest eigenvalue of all of the $\{F_i\}$, we know that every $H_i$ is positive semidefinite. Thus, the $\{H_i\}$ form a MIC. The angle between them is

$$\text{tr} H_i H_j = \left( \frac{1}{1 - d^2 f} \right)^2 \text{tr} (F_i - f I)(F_j - f I) = \left( \frac{1}{1 - d^2 f} \right)^2 \left( \text{tr} F_i F_j - f \text{tr} F_i - f \text{tr} F_j + d f^2 \right)$$

$$= \left( \frac{1}{1 - d^2 f} \right)^2 \left( \frac{1}{d} \delta_{ij} - \frac{2 f}{d} + d f^2 \right) = \frac{\delta_{ij} - 2 f + d f^2}{d - 2 d^3 f + d^5 f^2},$$

so it takes one constant value when $i = j$ and another constant value when $i \neq j$. \hfill \Box

Appleby and Graydon introduced the term SIM for a symmetric measurement of arbitrary rank; a rank-1 SIM is a SIC [54, 55]. Eq. (121) yields a SIM for any Q-rep.

\(^2\)Zhu calls equal weight Q-reps “NQPRs” and prefers to report the dual basis elements. In his notation, $Q^-_j = d F_j$ and $Q^+_j = d F_j^S$. 

18
7 Conclusions

We have surveyed the domain of Minimal Informationally Complete quantum measurements. Central to understanding these objects are their Gram matrices, which are “triply positive” (having nonnegative entries, being positive definite and constructed from positive semidefinite operators). These matrices quantify how close the representation of quantum theory that a MIC furnishes is to classical probability theory. The minimal separation, we have seen, is brought about when the MIC is a SIC. The Gram matrices are also the key ingredient in converting MICs to Q-reps, thanks to the role they play in the orthogonalization procedure.

Many properties of MIC Gram matrices remain unknown. Numerical investigations have, in some cases, outstripped the proving of theorems, resulting in the conjectures we have enumerated. Another avenue for potential future exploration is the application of Shannon theory to MICs, i.e., studying the probabilistic representations of quantum states using entropic measures [56]. In the case of SICs, this has already yielded intriguing connections among information theory, group theory and geometry [45, 57, 58, 59, 60]. The analogous questions for other classes of MICs remain open for investigation.

Acknowledgments

This research was supported in part by the John E. Fetzer Memorial Trust, the John Templeton Foundation, and grants FQXi-RFP-1612 and FQXi-RFP-1811B of the Foundational Questions Institute and Fetzer Franklin Fund, a donor advised fund of Silicon Valley Community Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation. We thank Marcus Appleby, Lane Hughston, Peter Johnson, Steven van Enk and Huangjun Zhu for discussions.

A Equivalent conditions for symmetric operator bases

Here, we prove the lemma that was necessary in the final step of proving Theorem 9. This result was Corollary 1 in [11] (a follow-up to [42]), which derives it as a special case of a more general theorem about Hermitian operator bases. We now prove this special case of interest directly, following a suggestion by Zhu.

Lemma 7. Let $L_j$ for $j = 1, \ldots, d^2$ be $d^2$ Hermitian operators on $\mathcal{H}_d$, $P_s$ denote the projector into the symmetric subspace of $\mathcal{H}_d \otimes \mathcal{H}_d$, and $I$ denote the identity superoperator. The following equations are equivalent:

\begin{equation}
\sum_{j=1}^{d^2} L_j \otimes L_j = \frac{2d}{d+1} P_s, \tag{123}
\end{equation}

\begin{equation}
\sum_{j=1}^{d^2} \|L_j\| \langle L_j \| = \frac{d}{d+1} (I + \|I\| \langle I \|), \tag{124}
\end{equation}

\begin{equation}
\text{tr} L_j L_k = \frac{1}{d+1} (d \delta_{jk} + \text{tr} L_j \text{tr} L_k). \tag{125}
\end{equation}

Proof: The space $\mathcal{L}(\mathcal{H}_d) \otimes \mathcal{L}(\mathcal{H}_d)$ is isomorphic to $\mathcal{L}(\mathcal{L}(\mathcal{H}_d))$ under the map $A \otimes B \rightarrow \|A\| \langle B \|$. The equivalence of (123) and (124) follows from computing the image of $P_s = \frac{1}{2} (I + \sum |i\rangle \langle j|)$ under this map. Explicitly, for an orthonormal basis $|i\rangle$ of $\mathcal{H}_d$,

\begin{equation}
P_s = \frac{1}{2} \left( I \otimes I + \sum_{i,j=1}^{d} |i\rangle \langle j| \otimes |j\rangle \langle i| \right) \rightarrow \frac{1}{2} \left( \|I\| \langle I \| + \sum_{i,j=1}^{d} \|\langle i| \langle j|\rangle \| \langle \langle i| \langle j| \| \right) = \frac{1}{2} (I + \|I\| \langle I \|).
\end{equation}

Now applying the isomorphism to (123) immediately gives (124).
To see the equivalence of (124) and (125), first note the following identities obtained from partial tracing and tracing (123):
\[
\sum_{j=1}^{d^2} (\text{tr} L_j) L_j = \frac{d}{d+1} \text{tr}_1 \left( I \otimes I + \sum_{i,j=1}^{d} |i\rangle \langle j| \otimes |j\rangle \langle i| \right) = \frac{d}{d+1} \left( dI + \sum_{i,j,k=1}^{d} \langle k| i \rangle \langle j| k \rangle \langle j| i \rangle \right) = dI \quad (127)
\]
and
\[
\sum_{j=1}^{d^2} (\text{tr} L_j)^2 = d^2. \quad (128)
\]
Define \( L'_j = L_j - x(\text{tr} L_j)I \), where \( x \) is a parameter to be specified shortly. Then we have
\[
\sum_{j=1}^{d^2} \langle L'_j \rangle \langle L'_j \rangle = \sum_{j=1}^{d^2} \langle L_j \rangle \langle L_j \rangle - x \sum_{j=1}^{d^2} (\text{tr} L_j) (\| L_j \| \langle I \rangle \langle I \rangle + \| L_j \| \langle I \rangle \langle I \rangle) + x^2 \sum_{j=1}^{d^2} (\text{tr} L_j)^2 \langle I \rangle \langle I \rangle \]
\[
= \sum_{j=1}^{d^2} \langle L_j \rangle \langle L_j \rangle + (x^2 d^2 - 2xd) \langle I \rangle \langle I \rangle, \quad (129)
\]
where we applied (127) and (128) in the second step. From this it’s easy to see that if we choose
\[
x = \frac{1}{d} \left( 1 \pm \sqrt{\frac{1}{d+1}} \right), \quad (130)
\]
then \( x^2 d^2 - 2xd = -d/(d+1) \). Making this substitution and inserting (124) into (129), we can see that (124) is equivalent to
\[
\sum_{j=1}^{d^2} \langle L'_j \rangle \langle L'_j \rangle = \frac{d}{d+1} I \quad (131)
\]
or that
\[
L''_j \equiv \sqrt{\frac{d+1}{d}} L'_j \quad (132)
\]
satisfies
\[
\sum_{j=1}^{d^2} \langle L''_j \rangle \langle L''_j \rangle = I. \quad (133)
\]
In a finite dimensional Hilbert space, this means that \( L''_j \) forms an orthonormal basis. Thus,
\[
\delta_{jk} = \text{tr} L''_j L''_k = \frac{d+1}{d} \text{tr} L'_j L'_k = \frac{d+1}{d} \text{tr} ((L_j - x(\text{tr} L_j)I)(L_k - x(\text{tr} L_k)I))
\]
\[
= \frac{d+1}{d} \left( \text{tr} L_j L_k + (x^2 d - 2x) \text{tr} L_j L_k \right) = \frac{d+1}{d} \left( \text{tr} L_j L_k - \frac{1}{d+1}(\text{tr} L_j)(\text{tr} L_k) \right), \quad (134)
\]
which is equivalent to (125).

\[\square\]

References

[1] M. A. Nielsen and I. Chuang, Quantum Computation and Quantum Information. Cambridge University Press, tenth anniversary ed., 2010.

[2] G. Zauner, Quantendesigns. Grundzüge einer nichtkommutativen Designtheorie. PhD thesis, University of Vienna, 1999. http://www.gerhardzauner.at/qdmye.html. Published in English translation: G. Zauner, “Quantum designs: foundations of a noncommutative design theory,” Int. J. Quantum Inf. 9 (2011) 445–508.
[3] J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, “Symmetric informationally complete quantum measurements,” *J. Math. Phys.* **45** (2004) 2171–2180.

[4] A. J. Scott and M. Grassl, “Symmetric informationally complete positive-operator-valued measures: A new computer study,” *J. Math. Phys.* **51** (2010) 042203.

[5] C. A. Fuchs, M. C. Hoang, and B. C. Stacey, “The SIC question: History and state of play,” *Axioms* **6** (2017) 21, arXiv:1703.07901.

[6] C. A. Fuchs, N. D. Mermin, and R. Schack, “An introduction to QBism with an application to the locality of quantum mechanics,” *Am. J. Phys.* **82** (2014) no. 8, 749–54, arXiv:1311.5253.

[7] C. A. Fuchs and R. Schack, “Quantum-Bayesian coherence,” *Rev. Mod. Phys.* **85** (2013) 1693–1715, arXiv:1301.3274.

[8] R. Healey, “Quantum-Bayesian and pragmatist views of quantum theory,” in *The Stanford Encyclopedia of Philosophy*, E. N. Zalta, ed. Metaphysics Lab, Stanford University, 2016. https://plato.stanford.edu/archives/spr2017/entries/quantum-bayesian/.

[9] C. A. Fuchs and B. C. Stacey, “QBism: Quantum theory as a hero’s handbook,” arXiv:1612.07308.

[10] D. M. Appleby, H. B. Dang, and C. A. Fuchs, “Symmetric informationally-complete quantum states as analogues to orthonormal bases and minimum-uncertainty states,” *Entropy* **16** (2014) 1484–1492.

[11] D. M. Appleby, C. A. Fuchs, and H. Zhu, “Group theoretic, Lie algebraic and Jordan algebraic formulations of the SIC existence problem,” *Quantum Inf. Comput.* **15** (2015) 61–94, arXiv:1312.0555.

[12] J. B. DeBrota, C. A. Fuchs, and B. C. Stacey, “Symmetric informationally complete measurements identify the essential difference between classical and quantum,” arXiv:1805.08721.

[13] C. A. Fuchs, “Notwithstanding Bohr, the Reasons for QBism,” *Mind and Matter* **15** (2017) 245–300, arXiv:1705.03483.

[14] V. Veitch, S. A. H. Mousavian, D. Gottesman, and J. Emerson, “The resource theory of stabilizer computation,” *New J. Phys.* **16** (2014) 013009, arXiv:1307.7171.

[15] C. A. Fuchs, “Quantum mechanics as quantum information (and only a little more),” arXiv:quant-ph/0205039.

[16] C. A. Fuchs, *Coming of Age with Quantum Information: Notes on a Paulian Idea*. Cambridge University Press, 2011.

[17] B. C. Stacey, “Misreading EPR: Variations on an Incorrect Theme,” arXiv:1809.01751.

[18] R. W. Spekkens, “Evidence for the epistemic view of quantum states: A toy theory,” *Phys. Rev. A* **75** no. 3, 032110, arXiv:quant-ph/0401052.

[19] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1994.

[20] C. A. Fuchs and M. Sasaki, “Squeezing quantum information through a classical channel: Measuring the quantumness of a set of quantum states,” *Quantum Inf. Comput.* **3** (2003) 377–404, arXiv:quant-ph/0302092.

[21] A. J. Scott, “Tight informationally complete quantum measurements,” *J. Phys. A* **39** (2006) 13507–13530, arXiv:quant-ph/0604049.

[22] P. Delsarte, J. M. Goethels, and J. J. Seidel, “Bounds for systems of lines and Jacobi polynomials,” *Philips Research Reports* **30** (1975) 91–105.

[23] S. G. Hoggar, “Two quaternionic 4-polytopes,” in *The Geometric Vein: The Coxeter Festschrift*, C. Davis, B. Grünbaum, and F. A. Sherk, eds. Springer, 1981.
[24] H. S. M. Coxeter, *Regular Complex Polytopes*. Cambridge University Press, second ed., 1991.

[25] S. G. Hoggar, “64 lines from a quaternionic polytope,” *Geometriae Dedicata* 69 (1998) 287–289.

[26] M. Appleby and I. Bengtsson, “Simplified exact SICs,” arXiv:1811.00947.

[27] A. J. Scott, “SICs: Extending the list of solutions,” arXiv:1703.03993.

[28] M. Grassl and A. J. Scott, “Fibonacci–Lucas SIC-POVMs,” *Journal of Mathematical Physics* 58 (2017) 122201, arXiv:1707.02944.

[29] D. M. Appleby, H. Yadsan-Appleby, and G. Zauner, “Galois automorphisms of a symmetric measurement,” *Quantum Inf. Comput.* 13 (2013) 672–720, arXiv:1209.1813.

[30] M. Appleby, S. Flammia, G. McConnell, and J. Yard, “Generating ray class fields of real quadratic fields via complex equiangular lines,” arXiv:1604.06098.

[31] I. Bengtsson, “The number behind the simplest SIC-POVM,” *Found. Phys.* 47 (2017) 1031–41, arXiv:1611.09087.

[32] M. Appleby, S. Flammia, G. McConnell, and J. Yard, “SICs and algebraic number theory,” *Found. Phys.* 47 (2017) 1042–59, arXiv:1701.05200.

[33] G. Kopp, “SIC-POVMs and the Stark conjectures,” arXiv:1807.05877.

[34] M. Appleby, T.-Y. Chien, S. Flammia, and S. Waldron, “Constructing exact symmetric informationally complete measurements from numerical solutions,” arXiv:1703.05981.

[35] I. Bengtsson and K. Życzkowski, “Discrete structures in Hilbert space,” in *Geometry of Quantum States: An Introduction to Quantum Entanglement*. Cambridge University Press, second ed., 2017, arXiv:1701.07902.

[36] S. Waldron, *An Introduction to Finite Tight Frames*. Springer, 2018. https://www.math.auckland.ac.nz/~waldron/Preprints/Frame-book/frame-book.html.

[37] D. M. Young, *Iterative Solution of Large Linear Systems*. Elsevier, 1971.

[38] C. M. Caves, C. A. Fuchs, and R. Schack, “Unknown quantum states: The quantum de Finetti representation,” *J. Math. Phys.* 43 (2002) no. 9, 4537–59, arXiv:quant-ph/0104088.

[39] B. C. Stacey, *Multiscale Structure in Eco-Evolutionary Dynamics*. PhD thesis, Brandeis University, 2015. arXiv:1509.02958.

[40] P. Diaconis and B. Skyrms, *Ten Great Ideas About Chance*. Princeton University Press, 2017.

[41] D. M. Appleby, “Symmetric informationally complete measurements of arbitrary rank,” *Opt. Spect.* 103 (2007) 416–428, arXiv:quant-ph/0611260.

[42] D. M. Appleby, S. T. Flammia, and C. A. Fuchs, “The Lie algebraic significance of symmetric informationally complete measurements,” *J. Math. Phys.* 52 (2011) 022202, arXiv:1001.0004.

[43] A. Veitia and S. J. van Enk, “Testing the context-independence of quantum gates,” arXiv:1810.05945.

[44] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*. Springer, 2011.

[45] B. C. Stacey, “SIC-POVMs and compatibility among quantum states,” *Mathematics* 4 (2016) no. 2, 36, arXiv:1404.3774.

[46] M. Howard and E. T. Campbell, “Application of a resource theory for magic states to fault-tolerant quantum computing,” *Phys. Rev. Lett.* 118 (2017) no. 9, 090501, arXiv:1609.07488.
[47] H. Zhu, Y. S. Teo, and B. G. Englert, “Two-qubit symmetric informationally complete positive operator valued measures,” *Phys. Rev. A* **82** (2010) 042308, arXiv:1008.1138.

[48] C. A. Fuchs, “Just two nonorthogonal quantum states,” arXiv:quant-ph/9810032.

[49] G. N. M. Tabia and D. M. Appleby, “Exploring the geometry of qutrit state space using symmetric informationally complete probabilities,” *Phys. Rev. A* **88** (2013) no. 1, 012131.

[50] L. P. Hughston and S. M. Salamon, “Surveying points in the complex projective plane,” *Advances in Mathematics* **286** (2016) 1017–1052, arXiv:1410.5862.

[51] B. C. Stacey, “Is the SIC outcome there when nobody looks?,” arXiv:1807.07194.

[52] H. Zhu, “Quasiprobability representations of quantum mechanics with minimal negativity,” *Phys. Rev. Lett.* **117** (2016) no. 12, 120404, arXiv:1604.06974.

[53] J. B. DeBrota and C. A. Fuchs, “Negativity bounds for Weyl–Heisenberg quasiprobability representations,” *Found. Phys.* **47** (2017) 1009–30, arXiv:1703.08272.

[54] M. A. Graydon and D. M. Appleby, “Quantum conical designs,” *J. Phys. A* **49** (2016) 085301, arXiv:1507.05323.

[55] M. A. Graydon and D. M. Appleby, “Entanglement and designs,” *J. Phys. A* **49** (2016) 33LT02, arXiv:1507.07881.

[56] J. R. Buck, S. J. van Enk, and C. A. Fuchs, “Experimental proposal for achieving superadditive communication capacities with a binary quantum alphabet,” *Physical Review A* **61** (2000) 032309, arXiv:quant-ph/9903039.

[57] B. C. Stacey, “Geometric and information-theoretic properties of the Hoggar lines,” arXiv:1609.03075.

[58] W. Slomczyński and A. Szymusiak, “Highly symmetric POVMs and their informational power,” *Quant. Info. Proc.* **15** (2014) no. 1, 505–606, arXiv:1402.0375.

[59] A. Szymusiak, “Maximally informative ensembles for SIC-POVMs in dimension 3,” *J. Phys. A* **47** (2014) 445301, arXiv:1405.0052.

[60] A. Szymusiak and W. Slomczyński, “Informational power of the Hoggar symmetric informationally complete positive operator-valued measure,” *Phys. Rev. A* **94** (2015) 012122, arXiv:1512.01735.
Figure 1: Empirical histograms of eigenvalues for random Weyl–Heisenberg MICs in dimensions $d = 2$ through $d = 8$. The peak at eigenvalue $1/d$ is straightforward to explain, the behavior at low eigenvalues less so. Note in particular the broad plateau for $d = 3$ and the apparent small-eigenvalue peaks in $d \geq 5$. 