Asymptotic behavior of global solutions of the

$$u_t = \Delta u + u^p \quad \star \star\star$$

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Abstract

We study the asymptotic behavior of nonnegative solutions of the semilinear parabolic problem

$$\begin{cases}
    u_t = \Delta u + u^p, & x \in \mathbb{R}^N, \ t > 0 \\
    u(0) = u_0, & x \in \mathbb{R}^N, \ t = 0.
\end{cases}$$

It is known that the nonnegative solution $u(t)$ of this problem blows up in finite time for $1 < p \leq 1 + 2/N$. Moreover, if $p > 1 + 2/N$ and the norm of $u_0$ is small enough, the problem admits global solution. In this work, we use the entropy method to obtain the decay rate of the global solution $u(t)$.

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1 Introduction

We considerer the semilinear parabolic problem

\[
\begin{aligned}
    & u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0 \\
    & u(0) = u_0, \quad x \in \mathbb{R}^N, \quad t = 0,
\end{aligned}
\]  \tag{1}

where \( p > 1 \) and \( u_0 \) is nonnegative and nontrivial. The interest in this article is to study the asymptotic behavior of global in time solutions in order to obtain their decay rate. Although this matter has been treated for other authors, we do it using the entropy method. This method has been successfully applied, for instead, by J.A. Carrillo and G. Toscani [1] for the asymptotic behavior of global solutions of some Fokker-Planck type equations.

In [3], Fujita considered the evolution problem (1) and proved the existence of a critical exponent \( p^* = 1 + \frac{2}{N} \), which is called the Fujita’s exponent. This exponent satisfies

- for \( p > p^* \), if the norm of \( u_0 \) is small enough, there exists a classical global in time positive solution of (1) which decays to zero when \( t \to \infty \), in the other case the solution blows up in finite time;

- for \( 1 < p < p^* \) and any choice of the initial condition \( u_0 \), every positive solution of (1) blows up in finite time.

In the case \( p = p^* \) Hayakawa [5] proved that every positive solution of (1) has the same behavior as \( 1 < p < p^* \). This fact was proved by Kobayashi, Sirao and Tanaka [8] too.

The aim of this work is to obtain the decay rate of the global solution \( u(s) \) of the problem (1) when \( t \to \infty \). More specifically, we deduce that under certain hypotheses, the \( L^2 \)-distance of \( u(s) \) decays with the rate

\[
\|u(t)\|_{L^2} \leq C (t + 1)^{-\frac{N}{4}}, \quad t \geq t_1,
\]

for certain time \( t_1 > 0 \), as well as the \( L^q \)-distance of \( u(s) \)

\[
\|u(t)\|_{L^q} \sim t^{-\frac{1}{p-1} - \frac{2}{q} \left( \frac{N}{2} - \frac{1}{p-1} \right)}.
\]

The organization of this work is the following. The section 2 contains the notation and some previous results which are necessary for the next sections. In section 3 a
short summary about the entropy method steps is given. In the section 4 we obtain the exponential decay for the entropy production and for the entropy. The last section is devoted to deduce the asymptotic behavior of the mentioned global solution.

2 Preliminaries

The idea is to transform the equation in problem (1) in order to obtain significant information on the asymptotic behavior of the global solutions. The change of variables utilized for Kavian [6] and Kawanago [7] is here employed. That is,

\[ v(y, s) = (t + 1)^{\frac{1}{p-1}} u(x, t), \]
\[ x = (t + 1)^{1/2} y \quad \text{and} \quad t = e^s - 1. \]  

(2)

Then, \( v(y, s) \) results to be a solution of the problem

\[ \begin{cases} v_s = \Delta v + \frac{y}{2} \nabla v + \frac{v}{p-1} + v^p, & y \in \mathbb{R}^N, \quad s > 0 \\ v(y, 0) = u_0, & y \in \mathbb{R}^N. \end{cases} \]  

(3)

Let us observe that problem (3) has the same initial condition as problem (1). Moreover, we work with the following spaces

\[ L^r_\rho = \left\{ f / \int_{\mathbb{R}^N} |f|^r \rho \, dy < \infty \right\}, \]
\[ H^1_\rho = \left\{ f \in L^2_\rho / \nabla f \in L^2_\rho \right\}, \quad H^2_\rho = \left\{ f \in H^1_\rho / \nabla f \in H^1_\rho \right\}, \]

where \( \rho(y) = \exp(|y|^2/4) \) and \( r \geq 1 \) is a constant. Related to these spaces are

\[ (f, g)_{L^2_\rho} = \int_{\mathbb{R}^N} f g \rho \, dy, \quad \|f\|_{L^2_\rho} = (f, f)^{\frac{1}{2}_2}, \]
\[ (f, g)_{H^1_\rho} = (f, g)_{L^2_\rho} + (\nabla f, \nabla g)_{L^2_\rho}, \]
\[ \|f\|_{H^1_\rho} = \left[ \|f\|_{L^2_\rho}^2 + \|\nabla f\|_{L^2_\rho}^2 \right]^{\frac{1}{2}}. \]

Observe that the equation in (3) has to be written in the shape

\[ v_s = -L_v + \frac{v}{p-1} + v^p, \]
where $L$ is the self-adjoint operator given by

$$Lv = -\triangle v - \frac{y}{2} \nabla v,$$

defined over $D(L) := H^2_\rho$. We know that this operator satisfies

- $\lambda_1 = \frac{N}{2}$ is the least eigenvalue of $L$. Then, the following inequality holds

$$\frac{N}{2} \|v\|_{L^2_\rho} \leq \int_{\mathbb{R}^N} |\nabla v|^2 \rho \, dy,$$  \hspace{1cm} (4)

- the operator $L$ has compact inverse.

As well, it is known that $u_\infty = C(N, p) |x|^{-\frac{2}{p-1}}$ with $C(N, p) = \left[ \frac{2}{p-1} \left( N - \frac{2p}{p-1} \right) \right]^{\frac{1}{p-1}}$ is a singular equilibrium of (1), that is, $u_\infty$ is a solution of the Lane-Emden equation

$$\triangle u + u^p = 0, \quad x \in \mathbb{R}^N \quad u > 0, \quad N \geq 3,$$

which arises in astrophysics and Riemannian geometry. It is well-known that this fact is possible only for those values of $p$ that verify $p \geq \frac{N}{N-2}$, since Gidas and Spruck [4] proved that there are not stationary solutions in any other cases. In 1993, Wang [9] proved that if $N \geq 3$, $p > \frac{N}{N-2}$ and

$$0 \leq u_0(x) \leq \lambda u_\infty,$$

where $0 < \lambda < 1$, then (1) has a unique global classical solution $u$ with $0 \leq u \leq \lambda u_\infty$. It also satisfies that

$$u(x, t) \leq [(\lambda^{1-p} - 1)(p - 1) t]^{-\frac{1}{p-1}}.$$

This inequality, in terms of the problem (3), can be expressed as

$$v(y, s) \leq \frac{1}{[(\lambda^{1-p} - 1)(p - 1)]^{\frac{1}{p-1}}} \left[ \frac{e^s}{e^s - 1} \right]^{\frac{1}{p-1}}.$$  \hspace{1cm} (5)

From now on, we assume that the Wang’s theorem hypotheses are satisfied. Moreover, the framework will be the set of global solutions $v$ of (3) such that $v \in X$, with

$$X = \left\{ f \in H^1_\rho \cap L^\infty / \ f \geq 0 \ \text{and} \ \lim_{s \to \infty} \int_{\mathbb{R}^N} |\nabla f(s)|^2 \rho \, dy = 0 \right\}.$$
3 Entropy method

We study the asymptotic behavior of the global in time solutions of the problem (3). For it, we use the already mentioned entropy method. The essential application of this method will consist in the following steps.

- Define a suitable **entropy functional** $E(v(s))$ for the equation (3) and study its properties.
- Compute the **entropy production**
  \[ I(v(s)) = \frac{d}{ds} E(v(s)) . \]
- Compute the derivative of entropy production and obtain a differential equation of type
  \[ \frac{d}{ds} I(v(s)) = - C I(v(s)) - R(s) , \]
  for certain constant $C > 0$ and certain function $R(s)$.
- Prove the properties of $R(s)$ that permit to obtain an exponential decay of $I(v(s))$,
  \[ I(v(s)) \leq A e^{-Cs} . \]
- Obtain the same decay rate for $E(v(s))$ from the previous items, more specifically
  \[ E(v(s)) \leq B e^{-Cs} , \text{ for } s \geq s_1 \text{ and certain } s_1 > 0. \]
- Give a bound of $\|v\|_{L^2_\rho}$ in terms of the entropy and entropy production which permits to get conclusions on the decay of the mentioned norm.

The same entropy functional introduced by Kavian and Kawanago ([6] and [7] respectively) will be used in the present article.

**Definition 3.1.** For every $v \in H_\rho^1 \cap L_{\rho}^{p+1}$ the **entropy functional** is defined by

\[
E(v) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2(p-1)} v^2 - \frac{1}{p+1} v^{p+1} \right] \rho \, dy .
\]
In order to obtain the decays announced above, some properties of this functional are needed. These properties are summarized in the next proposition. The first of them was proved in [6].

**Proposition 3.2.** Let \( u_0 \in H^1_\rho \cap L^\infty \), \( u_0 \geq 0 \), \( E(u_0) < \infty \) and \( v = v(y, s) \) the global solution of (3), \( v \in X \). Then

1. if there exists \( s_0 \geq 0 \) such that \( E(v(s_0)) \leq 0 \) and \( v(s_0) \neq 0 \), \( v \) blows up in finite time;

2. \( \frac{d}{ds} E(v(s)) = -I(v(s)) \) where \( I(v(s)) = \int_{\mathbb{R}^N} v^2 \rho \ dy \);

3. there exists \( M := \lim_{s \to \infty} E(v(s)) \) and, moreover, \( M = 0 \).

**Definition 3.3.** The functional \( I(v(s)) \) of the proposition 3.2 is called entropy production.

**Proof.** For the first property, the reader is referred to [6]. The second one is deduced by derivating \( E(v(s)) \) with respect to \( s \), integrating by parts and keeping in mind that \( v \) is solution of (3). In order to prove the third property, first observe that owing to the second one 2 we have that \( E(v(s)) \) is non increasing. As \( 0 \leq E(v(s)) \leq E(v(0)) < \infty \), the existence of the limit is warranted. Therefore, we must only see that the limit is equal to zero. For this issue, we observe that the first term in the expression of \( E(v(s)) \) tends to zero when \( s \to \infty \) since \( v \in X \). The second term of that expression goes to 0 too when \( s \to \infty \) due to the inequality (4). To see the behavior of the third term of the entropy functional we use the inequality (5) and obtain

\[
0 \leq \int_{\mathbb{R}^N} \frac{1}{p+1} v^{p+1} \rho \ dy \leq \left[ \frac{e^s}{e^s - 1} \right] \frac{1}{(\lambda^1 - p - 1)(p^2 - 1)} \int_{\mathbb{R}^N} v^2 \rho \ dy.
\]

The last expression can be bounded for large values of \( s \) as following

\[
\int_{\mathbb{R}^N} \frac{1}{p+1} v^{p+1} \rho \ dy \leq C \frac{1}{(\lambda^1 - p - 1)(p^2 - 1)} \int_{\mathbb{R}^N} v^2 \rho \ dy,
\]

where \( C \) is a positive constant. Taking into account the last inequality, we get that the third term of \( E(v(s)) \) tends to zero when \( s \to \infty \). \( \square \)
Now, we want to prove the decay of the entropy production and, as a result, the decay of the entropy functional. It will be made in the next section.

4 Decay of the entropy functional

The computation of $\frac{dI(v(s))}{ds}$ is needed to obtain the decay of the entropy production. We write this derivative in a convenient way using that $v$ is the solution of the equation (3). That is,

$$
\frac{d}{ds} I(v(s)) = \int_{\mathbb{R}^N} 2v_s v_{ss} \rho \, dy
= \frac{2}{p-1} I(v(s)) - 2 (Lv_s, v_s) + 2p \int_{\mathbb{R}^N} v^{p-1} v_s^2 \rho \, dy
= -2\gamma I(v(s)) - 2 R(s),
$$

(6)

where $\gamma$ is a positive constant and $R(s)$ is an appropriate function. Both $\gamma$ and $R(s)$ are defined by

$$
\gamma = \frac{N}{2} - \frac{1}{p-1} > 0,
R(s) = (Lv_s, v_s)_{L_\rho^2} - \frac{N}{2} \|v_s\|_{L_\rho^2}^2 - p \int_{\mathbb{R}^N} v^{p-1} v_s^2 \rho \, dy.
$$

(7)

We need to have more information about $R(s)$ that allows to obtain conclusions about the decay of the entropy production. This is the aim of the next three lemmas.

Lemma 4.1. Let $v \in H^2_\rho$ and $\Omega \subset \mathbb{R}^N$ an open set. Then

$$
\int_{\Omega} (vLv - \frac{N}{2} v^2) \rho \, dy \geq 0.
$$

Proof. We observe that, if $\chi_\Omega$ is the characteristic function of the set $\Omega$, then

$$
\int_{\Omega} vLv \rho \, dy = \int_{\mathbb{R}^N} (vLv) \chi_\Omega \rho \, dy
= \int_{\mathbb{R}^N} v \chi_\Omega(Lv) \chi_\Omega \rho \, dy.
$$
Owing to \((Lv)\mathcal{X}_\Omega = L(v\mathcal{X}_\Omega)\) almost everywhere, we have that

\[
\int\Omega vLv \rho \, dy = \int_{\mathbb{R}^N} v \mathcal{X}_\Omega L(v\mathcal{X}_\Omega) \rho \, dy \\
\geq \frac{N}{2} \int_{\mathbb{R}^N} (v \mathcal{X}_\Omega)^2 \rho \, dy \\
= \frac{N}{2} \int_{\mathbb{R}^N} v^2 \mathcal{X}_\Omega \rho \, dy \\
= \frac{N}{2} \int_{\Omega} v^2 \rho \, dy,
\]

where the inequality in (8) is a direct consequence of (4). From the last computation the lemma’s statement is established. \(\square\)

**Lemma 4.2.** Let \(f \in L^1(\mathbb{R}^N)\) such that \(\int_{\Omega} f \, dy \geq 0\) for every open set \(\Omega \subset \mathbb{R}^N\). Then, \(f \geq 0\) almost everywhere in \(\mathbb{R}^N\).

**Proof.** We suppose that \(f\) isn’t nonnegative a.e., then there exists a measurable set \(\Omega\) such that \(m(\Omega) > 0\) and \(f < 0\) in \(\Omega\). Here, \(m(\Omega)\) denotes the Lebesgue measure in \(\mathbb{R}^N\) of the set \(\Omega\). Then,

\[
\alpha = \int_{\Omega} f \, dy < 0.
\]

For each \(n \in \mathbb{N}\), there exists an open set \(G_n \subset \mathbb{R}^N\) such that \(\Omega \subset G_n\) and \(m(G_n - \Omega) < \frac{1}{n}\). We can choose those open sets \(G_n\) in such a way that the sequence \((G_n)_n\) is increasing with the inclusion. Then, we observe that

\[
\int_{G_n} f \, dy = \alpha + \int_{G_n - \Omega} f \, dy \\
= \alpha + \int_{\mathbb{R}^N} \varphi_n \, dy
\]

where \(\varphi_n = \mathcal{X}_{(G_n - \Omega)} f\). It is quite clear that the functions \(\varphi_n\) are integrable functions, they satisfy \(|\varphi_n| \leq |f|\) and, moreover, they verify that \(\varphi_n \to 0\) when \(n \to \infty\) a.e. Then, owing to dominated convergence theorem, it results that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \varphi_n \, dy = 0.
\]

Thanks to the last equality we deduce that there exists a natural \(N_0\), such that for every \(n \geq N_0\)
\[
\left| \int_{\mathbb{R}^N} \varphi_n \, dy \right| < \frac{|\alpha|}{2}. \tag{10}
\]

Using this inequality in (9), we see that it verifies
\[
\int_{G_n} f \, dy \leq \alpha + \left| \int_{\mathbb{R}^N} \varphi_n \, dy \right| < \alpha + \frac{|\alpha|}{2} = \frac{\alpha}{2},
\]
for every \( n \geq N_0 \). That is, the integral of the function \( f \) on the open sets \( G_n \), for \( n \geq N_0 \), is negative. This fact contradicts the lemma’s hypothesis. \( \square \)

The next lemma provides a bound of \( R(s) \) in terms of a new nonnegative function with exponential decay.

**Lemma 4.3.** Under the same hypotheses in the Wang’s theorem and if, moreover, \( \lambda < \left[ \frac{3p-1}{\gamma(p-1)^2} + 1 \right]^{1/p} \), then the function \( R(s) \) defined by (7) satisfies \( R(s) \geq -\frac{1}{2} K(s) \) for certain function \( K(s) \) that verifies

1. \( K(s) \geq 0 \),

2. there exists a constant \( a > 0 \) and a time \( s_1 > 0 \), which depend on \( p \), \( N \) and \( \lambda \), such that
\[
K(s) \leq K(s_1) e^{-\left(2\gamma + a\right)s}, \quad \text{for} \quad s \geq s_1, \quad \text{and}
\]

3. \( \int_0^\infty e^{2\gamma s} K(s) \, ds \leq C \), for a suitable positive constant \( C \).

**Proof.** We define the function \( K(s) \) as follows
\[
K(s) = 2p \int_{\mathbb{R}^N} v^{p-1} v_s^2 \rho \, dy.
\]
It is clear that \( K(s) \) is nonnegative. To see 2, we compute the derivative of the function \( K(s) \) and write it conveniently.
\[
\frac{dK(s)}{ds} = 2p \int_{\mathbb{R}^N} \left[ (p-1) v^{p-2} v_s^3 + 2 v^{p-1} v_s v_{ss} \right] \rho \, dy
\]
\[
= \left( 1 + \frac{2}{p-1} \right) K(s) + 2p (3p-1) \int_{\mathbb{R}^N} v^{2p-2} v_s^2 \rho \, dy
\]
\[
-2p (p-1) \int_{\mathbb{R}^N} v^{p-2} v_s^2 L v \rho \, dy - 4p \int_{\mathbb{R}^N} v^{p-1} v_s L v \rho \, dy. \tag{11}
\]
To get a bound of the second of the four terms of (11), we use the inequality (5) in order to obtain that

\[ 2p (3p - 1) \int_{\mathbb{R}^N} v^{2p-2} v_s^2 \rho \, dy \leq \frac{3p - 1}{(\lambda^{1-p} - 1)(p - 1)} \frac{e^s}{e^s - 1} K(s). \]  

(12)

For the bound of the third term, we apply first the lemmas 4.1 and 4.2 in order to deduce that \( vLv - \frac{N}{2} v^2 \geq 0 \) a.e., recalling that \( v^{p-3} v_s^2 \geq 0 \) we conclude

\[ \int_{\mathbb{R}^N} (vLv - \frac{N}{2} v^2) v^{p-3} v_s^2 \rho \, dy \geq 0. \]

This inequality quickly leads to a bound of the third term of (11) which is a multiple of \( K(s) \).

\[ 2p (p - 1) \int_{\mathbb{R}^N} v^{p-2} v_s^2 v^2 \rho \, dy \geq 2p (p - 1) \frac{N}{2} \int_{\mathbb{R}^N} v^{p-3} v_s^2 \rho \, dy = (p - 1) \frac{N}{2} K(s). \]  

(13)

For the last term of (11), we use first that \( v_s L v_s - \frac{N}{2} v_s^2 \geq 0 \) a.e. (it’s owing to the lemmas 4.1 and 4.2) and that also \( v^{p-1} \geq 0 \). We obtain

\[ \int_{\mathbb{R}^N} (v_s L v_s - \frac{N}{2} v_s^2) v^{p-1} \rho \, dy \geq 0. \]

This inequality permits to bound the last term of (11) by a multiple of \( K(s) \) as follows

\[ 4p \int_{\mathbb{R}^N} v^{p-1} v_s L v_s \rho \, dy \geq 2p N \int_{\mathbb{R}^N} v^{p-3} v_s^2 \rho \, dy = N K(s). \]  

(14)

Then, taking into account (12), (13) and (14), we have that

\[ \frac{dK(s)}{ds} \leq - \left( N - \frac{2}{p - 1} \right) K(s) + \left( 1 - (p - 1) \frac{N}{2} \right) K(s) + \frac{3p - 1}{(\lambda^{1-p} - 1)(p - 1)} \frac{e^s}{e^s - 1} K(s) \]

\[ = -2 \gamma K(s) - \mu(s) K(s). \]

(15)

where

\[ \mu(s) = -1 + (p - 1) \frac{N}{2} - \frac{(3p - 1)}{(\lambda^{1-p} - 1)(p - 1)} \frac{e^s}{e^s - 1} = (p - 1) \gamma - \frac{(3p - 1)}{(\lambda^{1-p} - 1)(p - 1)} f(s), \]
with \( f(s) = \frac{e^s}{e^s - 1} \). Now, we must prove that \( \mu(s) > 0 \) for \( s \geq s_1 \), where \( s_1 \) is a time which depends on \( p, N \) and \( \lambda \). It is clear that, owing to (15), it is equivalent to require that for \( f(s) < B \), where \( B \) is the constant defined by

\[
B = \frac{\gamma (p - 1)^2 (\lambda^{1-p} - 1)}{3p - 1}.
\]

We can take \( s_1 = f^{-1}(\frac{1+B}{2}) \) in order to get \( \mu(s) > 0 \) for \( s \geq s_1 \), since \( B > 1 \) due to the hypotheses of the lemma. Then, a bound of the derivative of \( K(s) \) is obtained from

\[
\frac{dK(s)}{ds} \leq -(2\gamma + a) K(s), \quad s \geq s_1,
\]

where \( a \) is a positive constant (we can take, for example, \( a = \frac{\mu(s_1)}{2} \)). Thus, from a certain \( s_1 > 0 \), the function \( K(s) \) has exponential decay rate. That is,

\[
K(s) \leq K(s_1) e^{-(2\gamma + a)s}.
\] (16)

This statement proves the second part of the lemma. The last part of the lemma can be deduced immediately from the previous one.

Applying the former lemma to (6), we obtain the mentioned decay for the entropy production as it will be proved in the next theorem.

**Theorem 4.4.** Under the hypotheses of the lemma 4.3 and if, moreover, \( I(u_0) < \infty \), then \( I(v(s)) \) has an exponential decay rate. More precisely,

\[
I(v(s)) \leq [I(u_0) + C] e^{-2\gamma s}.
\]

**Proof.** From the expression (6) and the lemma 4.3 we have

\[
\frac{d}{ds} (e^{2\gamma s} I(v(s))) = -2 e^{2\gamma s} R(s) \leq e^{2\gamma s} K(s).
\]

Integrating between 0 and \( s \) we obtain

\[
e^{2\gamma s} I(v(s)) - I(v(0)) \leq \int_0^s e^{2\gamma \sigma} K(\sigma) \, d\sigma \leq \int_0^\infty e^{2\gamma \sigma} K(\sigma) \, d\sigma \leq C
\]

\[
\therefore \quad I(v(s)) \leq [I(u_0) + C] e^{-2\gamma s}.
\]
Now, we are in a position to prove that the entropy functional decays exponentially. It will be proved in the next theorem.

**Theorem 4.5.** Under the hypotheses of the theorem 4.4, the entropy functional $E(v(s))$ has exponential decay, that is,

$$E(v(s)) \leq Ce^{-2\gamma s}, \quad \text{for every } s \geq s_1,$$

for certain $s_1 > 0$ which depends on $p, N$ and $\lambda$, and for certain positive constant $C$ that depends on $I(u_0), p, N$ and $\lambda$.

**Proof.** From part 2 of proposition 3.2 and expression (6), we have that

$$\frac{dE(v(s))}{ds} = \frac{1}{2\gamma} \frac{dI(v(s))}{ds} + \frac{1}{\gamma} R \geq \frac{1}{2\gamma} \frac{dI(v(s))}{ds} - \frac{1}{2\gamma} K(s), \quad (17)$$

where the inequality in (17) is owing to the bound of $R(s)$ from $K(s)$. Now, we integrate between $s$ and $b$ in (17) and we use part 2 of lemma 4.3 to obtain

$$E(v(b)) - E(v(s)) \geq \frac{1}{2\gamma} [I(v(b)) - I(v(s))] - \frac{1}{2\gamma} \left( \frac{K(s_1)}{2\gamma + a} e^{-(2\gamma+a)s} \right) \bigg|_s^b,$$

for every $s \geq s_1$. Taking the limit for $b \to \infty$ and using part 3 of proposition 3.2 we get

$$E(v(s)) \leq \frac{1}{2\gamma} I(v(s)) + \frac{K(s_1)}{2\gamma (2\gamma + a)} e^{-(2\gamma+a)s},$$

for every $s \geq s_1$. Therefore, owing to the theorem (4.4), the announced decay for the entropy functional takes place

$$E(v(s)) \leq Ce^{-2\gamma s}, \quad \text{for every } s \geq s_1.$$

\[\square\]

5 **Asymptotic behavior of the solution**

To finish with the application of this method, we must obtain a bound of the norm of the solution $v(s)$ in terms of the entropy $E(v(s))$ and the entropy production $I(v(s))$. For
this purpose we define the following function \( g(s) \)
\[
g(s) = \frac{1}{2} \int_{\mathbb{R}^N} v^2(s) \rho \, dy. \quad (18)
\]
The next lemma provides a bound of the function \( g(s) \) in terms of \( E(v(s)) \) and \( I(v(s)) \), that is, a bound of the norm of \( v(s) \) in the space \( L^2_{\rho} \).

**Theorem 5.1.** Under the hypotheses of theorem (4.4) and if, moreover, \( p > \tilde{p} \), where \( \tilde{p} \) is defined by
\[
\tilde{p} = \begin{cases} 
\frac{N}{N-2}, & N = 3, \\
1 + \frac{4}{N}, & N \geq 4,
\end{cases}
\]
then
1. \( \left( \frac{p-1}{2} N - 2 \right) g(s) \leq \frac{1}{2} I(v(s)) + (p+1) E(v(s)) \) and
2. \( g(s) \leq C e^{-2\gamma s} \) for all \( s \geq s_1 \), where \( s_1 \) is a positive number which depends on \( p, N \) and \( \lambda \), and \( C \) is certain positive constant which depends on \( I(u_0), p, N \) and \( \lambda \).

**Proof.** We observe that the derivative of the function \( g(s) \) satisfies
\[
g'(s) = \int_{\mathbb{R}^N} vv_s \rho \, dy \tag{19}
\]
\[
\leq \int_{\mathbb{R}^N} \left[ \frac{1}{2} v^2 + \frac{1}{2} v^2_s \right] \rho \, dy = g(s) + \frac{1}{2} I(v(s)). \tag{20}
\]
On the other hand, we can obtain another expression for \( g'(s) \) replacing \( v_s \) in (19) according to the problem (3) and using the definitions of \( E(v(s)) \), \( L \) and \( g(s) \), that is,
\[
g'(s) = -(p+1) E(v(s)) + \frac{p-1}{2} (Lv, v) - g(s). \tag{21}
\]
Owing to (4), (20) and (21), we have that
\[
\left( \frac{p-1}{2} N - 2 \right) g(s) \leq \frac{1}{2} I(v(s)) + (p+1) E(v(s)).
\]
As \( p > \frac{4}{N} + 1 \) and using the bounds for the entropy and entropy production in theorems 4.4 and 4.5 respectively, we get
\[
g(s) \leq C e^{-2\gamma s} \quad \text{para} \ s \geq s_1, \tag{22}
\]
where \( C \) is a positive constant which depends on \( N, p, \lambda \) and \( I(v(0)) \).

\[\square\]
Thanks to the definition (18), we can deduce the decay of the norm of \( v(s) \) in the space \( L^2_\rho \) which is the aim in the following theorem.

**Theorem 5.2.** Under the hypotheses of the theorem 5.1, then
\[
\| v(s) \|_{L^2_\rho} \leq C e^{-\gamma s}, \quad s \geq s_1,
\]
for \( s_1 > 0 \) which depends on \( p, N \) and \( \lambda \), and for certain positive constant \( C \) which depends on \( I(u_0) \), \( p, N \) and \( \lambda \).

Now, we can use again the change of variables (2) in order to obtain the decay of the solution \( u(x, t) \) of (1), getting the next result.

**Theorem 5.3.** Under the hypotheses of the theorem 5.1, then
\[
\| u(t) \|_{L^2} \leq C (t + 1)^{-\frac{N}{4}}, \quad t \geq t_1,
\]
for \( t_1 > 0 \) which depends on \( p, N \) and \( \lambda \), and for certain positive constant \( C \) which depends on \( I(u_0) \), \( p, N \) and \( \lambda \).

**Remark 5.4.** Let us observe that, because of the Wang’s theorem and the theorem 5.3, we obtain the decay of the norm \( \| u(t) \|_{L^q} \) for \( q \geq 2 \). This is,
\[
\| u(t) \|_{L^q} \sim t^{-\frac{1}{p-1} - \frac{2}{q}(\frac{N}{4} - \frac{1}{p-1})}, \quad (22)
\]
for \( t \geq t_1 \).

**Remark 5.5.** Notice that, although the decay of the norm \( \| u \|_{L^q} \sim t^{-\frac{N}{2}(1-\frac{1}{q})} \) obtained by Kawanago in [7] is better than (22) in the case \( q > 2 \) and the same for \( q = 2 \), the result obtained in [7] is true for \( \frac{N+2}{N} < p < \frac{N+2}{N-2} \) and the result obtained in this work corresponds to the range \( p > \tilde{p} \).

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