Sarrus rules and dihedral groups

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Beginners in Linear Algebra learn to calculate the $3 \times 3$ determinant by the rule of Sarrus. For a matrix $A = (a_{ij})$, the determinant is

$$
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}
$$

and the six summands are found in the following scheme

\begin{center}
\begin{tikzpicture}
    \draw[fill=white,thick] (0,0) rectangle (3,3);
    \foreach \x in {1,...,3} {
        \draw[thick] (\x,0) -- (\x,3);
        \draw[thick] (0,\x) -- (3,\x);
        \node at (\x,\x) {$\times$};
    }
    \node at (0.5,0.5) {$1$}; \node at (1.5,0.5) {$2$}; \node at (2.5,0.5) {$3$}; \node at (1.5,1.5) {$1$}; \node at (1.5,2.5) {$2$};
    \node at (0.5,1.5) {$+$}; \node at (1.5,2.5) {$-$}; \node at (2.5,1.5) {$-$}; \node at (1.5,1.5) {$+$}; \node at (2.5,2.5) {$-$}; \node at (1.5,0.5) {$+$};
\end{tikzpicture}
\end{center}

known as “Rule of Sarrus”. The right generalization of this rule to larger square matrices is Leibniz’ explicit formula for the determinant which is quite complicated as it contains, for the $n \times n$ case, $n!$ summands where each summand is a product of $n$ entries of the matrix (see below for a formula). Hence, there seems to be a natural tendency of students to use the following generalization of Sarrus’ rule: to calculate the determinant of a $n \times n$ matrix, the $n - 1$ first columns are repeated behind the matrix. Then the sum of the product of the $n$ diagonals of this scheme from the upper left to the lower right is calculated, and the sum of the $n$ diagonals from the upper right to the lower left is taken and the latter sum is subtracted from the former sum as here:

\begin{center}
\begin{tikzpicture}
    \draw[fill=white,thick] (0,0) rectangle (4,4);
    \foreach \x in {1,...,4} {
        \draw[thick] (\x,0) -- (\x,4);
        \draw[thick] (0,\x) -- (4,\x);
        \node at (\x,\x) {$\times$};
    }
    \node at (0.5,0.5) {$1$}; \node at (1.5,0.5) {$2$}; \node at (2.5,0.5) {$3$}; \node at (3.5,0.5) {$4$}; \node at (1.5,1.5) {$1$}; \node at (1.5,2.5) {$2$}; \node at (1.5,3.5) {$3$}; \node at (2.5,1.5) {$2$}; \node at (2.5,2.5) {$3$}; \node at (2.5,3.5) {$4$}; \node at (3.5,1.5) {$3$}; \node at (3.5,2.5) {$4$}; \node at (3.5,3.5) {$1$};
    \node at (0.5,1.5) {$+$}; \node at (1.5,2.5) {$+$}; \node at (2.5,3.5) {$+$}; \node at (1.5,1.5) {$-$}; \node at (2.5,2.5) {$-$}; \node at (3.5,3.5) {$-$}; \node at (1.5,0.5) {$+$}; \node at (2.5,1.5) {$+$}; \node at (3.5,2.5) {$+$}; \node at (2.5,0.5) {$-$}; \node at (3.5,1.5) {$-$}; \node at (3.5,0.5) {$-$};
\end{tikzpicture}
\end{center}

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This amounts to the following formula:

\[
\begin{align*}
& a_{11}a_{22}a_{33}a_{44} + a_{12}a_{23}a_{34}a_{41} + a_{13}a_{24}a_{31}a_{42} + a_{14}a_{21}a_{32}a_{43} \\
& - a_{14}a_{23}a_{32}a_{41} - a_{11}a_{24}a_{33}a_{42} - a_{12}a_{21}a_{34}a_{43} - a_{13}a_{22}a_{31}a_{44}.
\end{align*}
\]

In this article we will give a generalization of this formula to the \( n \times n \) case and we will call this formula the *False Sarrus Rule*. The quantity that it calculates will be called *dihedrant* for reasons that will become clear below. Then, we will derive some properties of the determinant that are similar to certain properties of the dihedrant. Especially, we are interested in conditions that guarantee that this quantity will be equal to \( \det(A) \). Further, we shall show a scheme, which resembles Sarrus’ rule and can help to calculate the correct determinant of a \( 4 \times 4 \) matrix.

1 The determinant and the “dihedrant”.

Let \( A = (a_{i,j}), i, j = 1, \ldots, n \). The *Leibniz formula* for the determinant is the following sum which runs over all permutations of the indices \( \{1, \ldots, n\} \), i.e., over the whole symmetric group \( S_n \):

\[
\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.
\]

To obtain a formula for the false Sarrus rule we note that the product that corresponds to the diagonal from the upper left to the lower right beginning with \( a_{1,k} \) is of the form

\[
\prod_{i=1}^{n} d_{i,\text{mod}(i+k-1,n)} = \prod_{i=1}^{n} d_{i,\rho_k(i)}.
\]

The respective permutation \( \rho_k \) of \( (1, \ldots, n) \) has the values

\[
\rho_k(1) = k, \ldots, \rho_k(n-k+1) = n, \rho_k(n-k+2) = 1, \ldots, \rho_k(n) = k-1.
\]

It can be visualized as the rotation of a regular \( n \)-gon by an angle \((k-1)2\pi/n\), where the vertices are numbered by \( 1, \ldots, n \).

These permutations will be called *rotations* in the following.
The product that corresponds to the diagonal from the upper right to the lower left beginning with $a_{1,k}$ is of the form
\[ \prod_{i=1}^{n} a_{i, \text{mod}(k+1-i,n)} = \prod_{i=1}^{n} a_{i, \mu_k(i)}. \]

The respective permutation $\mu_k$ of $(1, \cdots, n)$ has the values
\[ \mu_k(1) = k, \mu_k(2) = k - 1, \cdots, \mu_k(k) = 1, \mu_k(k+1) = n, \cdots, \mu_k(n) = k + 1. \]
It can be visualized as the reflection of the above regular $n$-gon reflecting it in its axis of symmetry that lies in the middle of the vertices with numbers 1 and $k$.

These permutations will be called reflections in the following.

**Exercise 1.** Visualize all rotations $\rho_k$ and reflections $\mu_k$ for for $n = 3, 4, 5$.

All involved permutations together form the *dihedral group* $D_n$ (sometimes also denoted by $D_{2n}$ because it has $2n$ elements). We assume that the reader is familiar with these groups. In the following we mainly use the fact that the composition of two rotations and the composition of two reflections is a rotation, whereas the composition of a reflection and a rotation is a reflection. With the help of rotations $\rho_k$ and reflections $\mu_k$ we write the announced formula for the False Sarrus Rule as
\[ \text{dih}(A) = \sum_{\sigma \in D_n} \text{sig}(\sigma) \prod_{i=1}^{n} a_{i, \sigma(i)}. \]

Since this quantity is built upon the dihedral group, we call it *dihedrant*. To push the similarity between the determinant and the dihedrant a little further, we define
\[ \text{sig}(\rho_k) = 1, \quad \text{sig}(\mu_k) = -1, \quad k = 1, \ldots, n, \]
and get the formula
\[ \text{dih}(A) = \sum_{\sigma \in D_n} \text{sig}(\sigma) \prod_{i=1}^{n} a_{i, \sigma(i)}. \]

The above mentioned composition rules imply that for all $\sigma, \tau \in D_n$ it holds that
\[ \text{sig}(\tau \circ \sigma) = \text{sig}(\tau) \text{sig}(\sigma). \]
Let us consider the first cases:
• The group $D_1$ (visualized as a loop) consists of one reflection and the identity map, both permuting the vertices of the 1-gon in the same way and thus, $\text{dih}(A) = 0$ for any $1 \times 1$ matrix $A$.

• For $n = 2$, there are exactly two rotations and two reflections in $D_2$, where for both rotations there is a reflection that permutes the vertices in the same way, and hence $\text{dih}(A) = a_{11}a_{22} + a_{12}a_{21} - a_{12}a_{21} - a_{11}a_{22} = 0$ for any $2 \times 2$ matrix $A$, too.

• In the case $n = 3$ we have that $D_3$ is the full symmetric group and the rotations fulfill $\text{sig}(\rho_k) = sgn(\rho_k) = +1$ and the reflections fulfill $\text{sig}(\mu_k) = sgn(\mu_k) = -1$ and consequently, the dihedrant equals the determinant. The case $n = 3$ is a lucky coincidence and for $n > 3$ the dihedrant is only loosely related to the determinant. Nonetheless, the dihedrant has some interesting properties. Our first example for such a property says that the dihedrant also fulfills an equation similar to $\text{det}(A) = \text{det}(A^T)$.

Theorem 1. $\text{dih}(A^T) = \text{dih}(A)$.

Proof. Let $A^T = (b_{i,j})$. Since $A^T$ is the transposed of $A$, we have $b_{i,j} = a_{j,i}$. Hence,

$$\text{dih}(A^T) = \sum_{\sigma \in D_n} \text{sig}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)} = \sum_{\sigma \in D_n} \text{sig}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

$$= \sum_{\sigma \in D_n} \text{sig}(\sigma) \prod_{i=1}^{n} a_{i,\sigma^{-1}(i)}.$$

Since $D_n$ is a group and the composition rule for sig implies $\text{sig}(\sigma^{-1}) = \text{sig}(\sigma)$, the last expression equals $\text{dih}(A)$. \qed

For any matrix $A = (a_1, \ldots, a_n)$ and any permutation $\sigma \in S_n$ it holds that $\text{det}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = sgn(\sigma) \text{det}(A)$. However, the following example shows that this is not true for the dihedrant and arbitrary $\sigma \in S_n$, $n \geq 4$: Consider the identity matrix $I$ and exchange the first two columns to get the matrix $\tilde{I}$. Then we have $\text{det}(I) = -1$, but $\text{dih}(I) = 1$ and $\text{dih}(\tilde{I}) = 0$. If we restrict ourselves to $\sigma \in D_n$, however, we still get a similar property.

Theorem 2. For an $n \times n$ matrix $A = (a_1, \ldots, a_n)$ and $\sigma \in D_n$ it holds that $\text{dih}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = \text{sig}(\sigma) \text{dih}(A)$.

Proof. The proof follows immediately from

$$\text{dih}(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = \sum_{\tau \in D_n} \text{sig}(\tau) \prod_{i=1}^{n} a_{i,(\tau \circ \sigma)(i)}$$

$$= \text{sig}(\sigma) \sum_{\tau \in D_n} \text{sig}(\sigma \circ \tau) \prod_{i=1}^{n} a_{i,(\tau \circ \sigma)(i)} = \text{sig}(\sigma) \text{dih}(A).$$ \qed
Combining Theorem 1 and Theorem 2, we see that an analogous statement is valid, if one replaces columns by rows.

Now, we consider \( \det \) and \( \dih \) as maps from \((\mathbb{R}^n)^n\) to \(\mathbb{R}\), it becomes clear that both are multilinear:

**Theorem 3.** The map \( \dih \) is linear in any row and in any column of the matrix \( A \).

**Proof.** Let \( j \in \{1, \ldots, n\} \) be fixed \( \mathbf{a}_i^T = (a_{i,l}), i, l = 1, \ldots, n \), and \( \mathbf{b}_j^T = (b_{j,l}), l = 1, \ldots, n \). The assertion concerning the rows follows from the following equation

\[
\dih((\mathbf{a}_1, \ldots, \alpha \mathbf{a}_j + \beta \mathbf{b}_j, \ldots, \mathbf{a}_n)^T) = \sum_{\sigma \in D_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} (\alpha a_{j,\sigma(j)} + \beta b_{j,\sigma(j)})
\]

Now, the assertion concerning the columns follows from Theorem 1.

There is an axiomatic characterization of the determinant as the map from square matrices to numbers that is linear in every row, is invariant when adding one column to another, and gives the value one for the identity matrix (cf. the historic account [3]). Note that the dihedrant only fulfills the first and the third of these determining properties, but not the second.

2 When the False Sarrus Rule is actually right

Of course, there are matrices \( A \) for which \( \dih(A) = \det(A) \); two obvious examples are \( A = 0 \) and the identity matrix \( I \) (for which \( \dih(I) = \det(I) = 1 \)), but there are more cases.

2.1 Upper right and lower left triangular matrices

Since for upper right and lower left triangular matrices both the determinant and the dihedrant are the product of the diagonal entries, we have \( \dih(A) = \det(A) \) in this case. We can use Theorem 2 and its analogon for rows of the matrix to build more examples of matrices where \( \dih(A) = \det(A) \) and which are not easily recognized as such. One example is

\[
A = \begin{pmatrix}
1 & 0 & 0 & -1 \\
1 & -3 & 0 & -3 \\
1 & 1 & 5 & 5 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

with \( \det(A) = \dih(A) = -15 \) (Hint: Permute the last column to the first and the last row to the first.). However, there are also matrices \( A \) for which
dih(\(A\)) = \text{det}(\(A\)) \neq 0 which are not obtained by submitting indices of rows and columns of triangular matrices to rotations and reflections. One example we found by trial and error is

\[
A = \begin{pmatrix}
2 & 2 & 2 & 2 \\
1 & 2 & 1 & 1 \\
2 & 2 & 2 & 1 \\
1 & 2 & 1 & 1 \\
\end{pmatrix}
\]

with dih(\(A\)) = \text{det}(\(A\)) = 2.

### 2.2 Rank deficient matrices

Here are some classes of matrices with dih(\(A\)) = \text{det}(\(A\)) = 0:

**Theorem 4.** If rank(\(A\)) = 1, then dih(\(A\)) = 0.

**Proof.** Let \(A = (a_1, \ldots, a_n)\), and \(a_i = \alpha_i v, v \in \mathbb{R}^n, i = 1, \ldots, n\). Then, according to Theorem 3,

\[
\text{dih}(A) = \left( \prod_{i=1}^{n} \alpha_i \right) \text{dih}(v, \ldots, v).
\]

We have dih(v, \ldots, v) = 0, as all products in the False Sarrus Rule have identical values.

Now, we turn to the consideration of matrices that have the rank two. In this case all rows of the matrix are of the form \(\alpha_i a^T + \beta_i b^T, i = 1, \ldots, n\), and, by Theorem 3 it holds that

\[
\text{dih}(A) = \alpha_1 \text{dih} \left( \begin{pmatrix} a^T \\ \alpha_2 a^T + \beta_2 b^T \\ \vdots \\ \alpha_n a^T + \beta_n b^T \end{pmatrix} \right) + \beta_1 \text{dih} \left( \begin{pmatrix} b^T \\ \alpha_2 a^T + \beta_2 b^T \\ \vdots \\ \alpha_n a^T + \beta_n b^T \end{pmatrix} \right).
\]

Applying the same rule over and over again, we end up with a linear combination of dihedrants of matrices where the rows of each matrix are either \(a^T\) or \(b^T\).

**Exercise 2.** Assume that in the above matrix \(A\) we have \(a^T \neq 0, b^T \neq 0\), and \(a^T \neq b^T\). How many summands will the above mentioned linear combination have?

Our next step is to analyze the dihedrant of a matrix for which \(k\) rows are \(a^T\) and \(n-k\) rows are \(b^T\). If \(k = 0\) or \(k = n\), then those dihedrants are zero according to Theorem 4. For \(k = 1\) or \(k = n-1\) we have the following result:

**Theorem 5.** If \(n-1\) rows of the matrix \(A\) are identical, then dih(\(A\)) = 0.
Proof. Let $n - 1$ rows be equal to $b^T = (b_1, \cdots , b_n)$ and let the remaining row be $a^T = (a_1, \cdots , a_n)$. Then every product in the False Sarrus Rule is of the form $a_i \prod_{j=1, j\neq i}^n b_j$. Any of these products appears twice, once corresponding to a rotation, and once to a reflection. Hence, they cancel which implies $\text{dih}(A) = 0$. \hfill \Box

For $k = 2$ or $k = n - 2$ the same is still true:

**Theorem 6.** If $n - 2$ rows of the matrix $A$ are identical, and the remaining two rows are identical, too, then $\text{dih}(A) = 0$.

Proof. Let the $n - 2$ identical rows be equal to $b^T = (b_1, \cdots , b_n)$ and let the remaining rows be equal to $a^T = (a_1, \cdots , a_n)$. Then every product in the False Sarrus Rule is of the form $a_i a_j \prod_{l=1, l\neq i,j}^n b_l$. If a product of this form appears in the False Sarrus Rule, it appears twice. Similar to Theorem 5, one product corresponds to a rotation, the other to a reflection; again they cancel and we obtain $\text{dih}(A) = 0$. \hfill \Box

If only $n - 3$ rows are identical and the remaining three or more rows are identical, this reasoning is no longer valid. This is exemplified by the following $6 \times 6$ example

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. $$

Here $\text{dih}(A) = 1$, and $\text{rank}(A) = 2$.

Combining Theorems 5 and 6 for $n = 4$ and $n = 5$ we obtain:

**Corollary 7.** If $n = 4$ or $n = 5$, and $\text{rank}(A) \leq 2$, then $\text{dih}(A) = 0$.

As the following matrix shows, $4 \times 4$ matrices $A$ with $\text{rank}(A) = 3$ do not necessarily have dihedrant equal to zero

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Indeed, it holds that $\text{dih}(A) = -6$, but $\det(A) = 0$.

**2.3 Upper left and lower right triangular matrices**

To see when upper left or lower right triangular matrices have equal determinant and dihedrant, it is helpful to calculate the signs of rotations and reflections.
Lemma 8. Let $\rho_k$ and $\mu_k, k = 1, \ldots, n$, be as above, and

$$N(\rho_k) := (k-1)(n-k+1),$$

and

$$N(\mu_k) := \frac{(n-k-1)(n-k)}{2} + \frac{k(k-1)}{2}.$$ 

Then $\text{sgn}(\rho_k) = (-1)^{N(\rho_k)}$ and $\text{sgn}(\mu_k) = (-1)^{N(\mu_k)}$.

Proof. If we consider the values $(k, k+1, \ldots, n, 1, \ldots, k-1)$ of $\rho_k$, we see that $n - k + 1$ transpositions are needed to permute 1 to the first place. For the numbers $2, \ldots, k-1$ we also need $n - k + 1$ to gain the natural order. This implies the formula for $N(\rho_k)$.

If we consider the values $(k, k-1, \ldots, 1, n, \ldots, k+1)$ of $\mu_k$, we see that $k-1$ transpositions are needed to place 1 onto the first place, $k-2$ transpositions for the number 2, and so on. Hence, we need

$$\sum_{j=1}^{k-1} j = \frac{k(k-1)}{2}$$

transpositions for the first $k$ values and likewise $\frac{(n-k-1)(n-k)}{2}$ transpositions for the last $n - k - 1$ elements. This proves the formula for $N(\mu_k)$. \hfill $\square$

Exercise 3. Calculate a more explicit formula for the sgn of rotations and reflection, i.e. $\text{sgn}(\rho_k)$ and $\text{sgn}(\mu_k)$, or, put differently, determine for which $n$ and $k$ it holds $\text{sgn}(\rho_k) = \text{sg}(\rho_k), \text{sgn}(\mu_k) = \text{sg}(\mu_k)$. Hint: Use the remainder of $n$ and $k$ by division by two for the rotations and the remainder of $n$ and $k$ by division by four for the reflections.

In the case of an upper left or lower right triangular matrix with non-zero entries on the anti-diagonal, the False Sarrus Rule only has one non-zero summand (the product of the entries of the anti-diagonal) and we get

$$\text{dih}(A) = -\prod_{i=1}^{n} a_{i, (n-i+1)}.$$ 

For the determinant of such a matrix there is also only one non-zero summand in Leibniz’ formula (the same product as for the dihedrant) which is the one that corresponds to the reflection $\mu_n$. By the second part of Lemma 8, the sign of this summand is

$$\text{sgn}(\mu_n) = \begin{cases} 1, & \text{for } \frac{n(n-1)}{2} \text{ even,} \\ -1, & \text{for } \frac{n(n-1)}{2} \text{ odd,} \end{cases}$$

and thus,

$$\text{det}(A) = \begin{cases} -\prod_{i=1}^{n} a_{i, (n-i+1)}, & \text{if } \text{mod}(n, 4) = 2 \text{ or } \text{mod}(n, 4) = 3 \\ \prod_{i=1}^{n} a_{i, (n-i+1)}, & \text{if } \text{mod}(n, 4) = 0 \text{ or } \text{mod}(n, 4) = 1. \end{cases}$$

We conclude
**Theorem 9.** Let $A$ be an upper left or lower right triangular $n \times n$ matrix. Then it holds that $\text{dih}(A) = \det(A)$ if $\mod(n, 4) = 2$ or $\mod(n, 4) = 3$.

**Exercise 4.** Work out under which conditions there holds $\text{dih}(A) = \det(A)$ where $A$ is an $n \times n$ matrix with the following pattern of non-zero entries:

\[
\begin{pmatrix}
* & * & 0 \\
\cdot & \cdot & \cdot \\
0 & \cdot & * \\
* & * & *
\end{pmatrix}
\]

### 3 A modified Sarrus’ rule for $4 \times 4$ matrices.

The most obvious hint that the dihedrant fails to be equal to the determinant in general is that it has way too few summands. This is due to the fact that the dihedral group $D_n$ only has $2n$ elements, while the symmetric group $S_n$ has $n!$. For the case $n = 4$ we need $4! = 24$ summands and the False Sarrus Rule only provides $8$ of them (and half of them even have the wrong sign). Here is a scheme that provides all $24$ summands with the correct signs:

![Scheme for 4x4 matrices](image)

We do not show a similar rule for the cases $n \geq 5$, since one needs $(n-1)!/2$ permutations of columns to get the needed total of $n!$ permutations. We are by far not the first, to propose such an approach to memorize determinants. The oldest reference we could track down is [1] (Russian original [2]) and a recent reference is [4].

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**References**

[1] S. Arschon. Verallgemeinerte Sarrussche Regel. *Rec. Math. Moscow*, 42:121–128, 1935.

[2] С. Аршон. Обобщенное правило Саррюса. *Матем. сб.*, 42(1):121–128, 1935.
[3] Eberhard Knobloch. From Gauss to Weierstrass: determinant theory and its historical evaluations. In *The intersection of history and mathematics*, volume 15 of *Sci. Networks Hist. Stud.*, pages 51–66. Birkhäuser, Basel, 1994.

[4] M. G. Sobamowo. On the extension of Sarrus’ rule to $n \times n$ ($n > 3$) matrices: Development of new method for the computation of the determinant of $4 \times 4$ matrix. *International Journal of Engineering Mathematics*, 2016:article ID 9382739, 2016.