ABSOLUTE STABILITY WINDOW AND UPPER BOUND ON THE
MAGNETIC FIELD STRENGTH IN A STRONGLY MAGNETIZED
STRANGE QUARK STAR

A. A. ISAYEV
Kharkov Institute of Physics and Technology,
Academicheskaya Street 1, Kharkov, 61108, Ukraine
Kharkov National University, Svobody Sq., 4, Kharkov, 61022, Ukraine
Institute for the Early Universe, Ewha Woman’s University, Seoul 120-750, Korea
isayev@kipt.kharkov.ua

Received Day Month Year
Revised Day Month Year

Magnetized strange quark stars, composed of strange quark matter (SQM) and self-bound by strong interactions, can be formed if the energy per baryon of magnetized SQM is less than that of the most stable \(^{56}\text{Fe}\) nucleus under the zero external pressure and temperature. Utilizing the MIT bag model description of magnetized SQM under charge neutrality and beta equilibrium conditions, the corresponding absolute stability window in the parameter space of the theory is determined. It is shown that there exists the maximum magnetic field strength allowed by the condition of absolute stability of magnetized SQM. The value of this field, \(H \sim 3 \cdot 10^{18} \text{G}\), represents the upper bound on the magnetic field strength which can be reached in a strongly magnetized strange quark star.

Keywords: Strange quark star; strong magnetic field; absolute stability window; anisotropic pressure.

PACS numbers: 25.75.Nq, 21.65.Qr, 98.80.Jk, 95.30.Tg

1. Introduction

It was suggested some time ago that strange quark matter (SQM), composed of deconfined \(u, d\) and \(s\) quarks, can be the true ground state of matter.\(^{[1,2]}\) If this conjecture holds true, it will have important astrophysical implications. In particular, strange quark stars, composed of SQM and self-bound by strong interactions, can exist in nature.\(^{[3,4]}\) Also, if SQM is metastable at zero pressure, it can occur in the high-density core of a neutron star as a result of the deconfinement phase transition. In this case, the stability of SQM is provided by the external pressure from the outer hadronic layers. Then a relevant astrophysical object is a hybrid star having a quark core and the crust of hadronic matter.\(^{[5,6]}\)

Also, an important aspect of the problem is that compact stars can be strongly magnetized. For example, for a special class of neutron stars called magnetars and,
as basically assumed, represented by soft $\gamma$-ray repeaters (SGRs) and anomalous X-ray pulsars (AXPs), the field strength can reach values of about $10^{14-10^{15}}$ G.\cite{8,9} It was also suggested that magnetized strange quark stars can be a real source of SGRs or AXPs.\cite{10,11} Strong magnetic fields of these compact stars can give rise to the nonlinear QED effects like light bending in the vicinity of a magnetar.\cite{12} Even stronger magnetic fields up to $10^{19}$ G may potentially occur in the inner core of a neutron star.\cite{13} As a result, a pulsar can get the large kick velocity because of the asymmetric neutrino emission in direct Urca processes in the dense core of a magnetized neutron star.\cite{14,15} The origin of magnetar’s strong magnetic fields is still under discussion, and, among other possibilities, it is not excluded that this can be due to spontaneous ordering of hadron,\cite{16,17} or quark\cite{18} spins in the dense interior of a neutron star.

Thus, the study of thermodynamic properties of cold dense matter in a strong magnetic field is the problem of a considerable interest.\cite{23,24} In particular, the pressure anisotropy, exhibited in the difference between the longitudinal and transverse (along and perpendicular to the magnetic field) pressures, becomes important for strongly magnetized matter.\cite{25,26} In this study, we consider strongly magnetized SQM within the framework of the MIT bag model,\cite{34} aiming at determining the parameter space of the theory for which magnetized SQM is absolutely stable, i.e., its energy per baryon is less than that of the most stable nucleus $^{56}$Fe under the zero external pressure and temperature. For the parameters from this absolute stability window, the formation of a strongly magnetized strange quark star is possible. Otherwise, a strongly magnetized hybrid star can be formed. In the latter case, in strong magnetic fields $H > H_{th}$ (with $10^{17} < H_{th} \lesssim 10^{18}$ G) the transverse pressure increases with the magnetic field while the longitudinal pressure decreases.\cite{33} The upper bound on the magnetic field in the quark core of a hybrid star is determined by the critical field $H_c$ beyond which the longitudinal pressure becomes negative resulting in the appearance of the longitudinal instability. There exists the upper bound on the magnetic field strength in a strongly magnetized strange quark star as well. It is related to the determination of the absolute stability window for strongly magnetized SQM and finding the maximum magnetic field strength allowed by the condition of absolute stability. Also, in this research we study how the absolute stability window and upper bound on the magnetic field strength are affected by varying the strange quark current mass $m_s$, taking into account scattering in the $m_s$ values existing in the literature.

2. Numerical results and discussion

For the details of the formalism one can address to Ref.\cite{33} We use the simplified variant of the MIT bag model, in which quarks are considered as free fermions moving inside a finite region of space called a "bag". The effects of the confinement are implemented by introducing the bag pressure $B$, providing an extra constant energy per unit volume inside the bag.
First, we will determine the absolute stability window of magnetized SQM, subject to charge neutrality and beta equilibrium conditions, at zero temperature. In the MIT bag model, the equilibrium conditions for magnetized SQM in terms of the longitudinal $p_l$ and transverse $p_t$ pressures read

$$p_l = -\Omega_H = 0,$$

$$p_t = -\Omega_H + H \frac{\partial \Omega_H}{\partial H} = 0,$$

where

$$\Omega_H = \sum_{i=u,d,s,e} \Omega_i + \frac{H^2}{8\pi} + B$$

is the total thermodynamic potential of the system including the magnetic field contribution. With account of Eq. (1) and at nonzero magnetic field, Eq. (2) is reduced to $\frac{\partial \Omega_H}{\partial H} = -\frac{\partial p_l}{\partial H} = 0$. The last equation explicitly reads

$$M - \frac{H^2}{4\pi} - \frac{\partial B}{\partial H} = 0,$$

where $M = -\sum_i (\frac{\partial \Omega_i}{\partial \mu_i})_{\mu_i}$ is the matter magnetization and we assume that the bag pressure can depend on the magnetic field. For the field-independent bag pressure, Eq. (3) would be difficult to satisfy, because it would mean that the response of the system to an external magnetic field would considerably exceed the paramagnetic response, that is hard to expect without spontaneous spin ordering in the system.

In order to be absolutely stable, the energy per baryon of magnetized SQM should be less than the energy per baryon of the most stable $^{56}$Fe nucleus under the equilibrium conditions (1), (2):

$$E_m \leq \epsilon_H(^{56}\text{Fe}),$$

where $E_m \equiv E - \frac{H^2}{8\pi}$ is the matter part of the total energy density $E$ of the system, $\varrho_B$ is the total baryon number density. From the equilibrium condition (1), one gets

$$B = -\sum_{i=u,d,s,e} \Omega_i - \frac{H^2}{8\pi}.$$

The total energy density $E$ of the system, with account of Eq. (5), is given by

$$E = \sum_{i=u,d,s,e} \mu_i \varrho_i,$$

where $\varrho_i = -(\frac{\partial \Omega_i}{\partial \mu_i})_H$ is the number density for fermions of $i$th species with the chemical potential $\mu_i$. The stability constraint (4) then reads

$$\frac{E_m}{\varrho_B} = \frac{1}{\varrho_B} \sum_{i=u,d,s,e} \mu_i \varrho_i - \frac{H^2}{8\pi \varrho_B} \leq \epsilon_H(^{56}\text{Fe}).$$
For the rough estimate, one can regard $^{56}$Fe nucleus as a system of noninteracting nucleons, and, therefore, magnetic fields $H > 10^{20}$ G are necessary in order to significantly alter its energy per nucleon. Since we will consider magnetic fields $H < 5 \times 10^{18}$ G, we use the approximation $\epsilon_H(^{56}\text{Fe}) \approx \epsilon(^{56}\text{Fe}) = 930$ MeV.

Thus, in order to find the upper bound $B_u$ on the bag pressure $B$ for magnetized SQM to be absolutely stable, it is necessary, first, to find the fermion species chemical potentials $\mu_i$ ($i = u, d, s, e$) from the constraint (7), taken with the equality sign, and charge neutrality and chemical equilibrium conditions:

$$2\varrho_u - \varrho_d - \varrho_s - 3\varrho_e = 0,$$

$$\mu_d = \mu_u + \mu_e,$$

$$\mu_d = \mu_s.$$

Then the upper bound $B_u$ on the bag pressure from the absolute stability window can be found from Eq. (5). Note that, as a solution of Eq. (5), the function $B_u(H)$ will identically satisfy to Eq. (2).

At $H = 0$, in turn, the lower bound $B_l$ on the bag pressure from the absolute stability window can be established from the experimental observation that two-flavor quark matter, consisting of $u$ and $d$ quarks, is less stable compared to $^{56}$Fe nucleus at zero external pressure and temperature. Obviously, that requirement is preserved at not too strong magnetic fields, at least, up to $H \sim 10^{17}$ G, when the impact of a magnetic field on quark matter properties is insignificant.

Before presenting the absolute stability window in the plane "magnetic field strength – bag pressure", let us also mention that there is quite a noticeable gap between the values of the strange quark current mass $m_s$ used in many researches on SQM, and that given by PDG. For example, in Refs. [13, 25, 33, 35] it was used the value $m_s = 150$ MeV while the recent PDG edition points out the value $m_s = 95 \pm 5$ MeV. In view of that, we present the upper bound on $B$ from the absolute stability window for a few values of $m_s$ in the range from 90 MeV to 150 MeV in order to study quantitatively the effect of varying $m_s$. For $u$ and $d$ quarks we use the current masses $m_u = m_d = 5$ MeV.

Figure 1 shows the upper $B_u$ and lower $B_l$ bounds on the bag pressure $B$ from the absolute stability window as functions of the magnetic field strength. It is seen that...
Fig. 1. The absolute stability window in the plane "magnetic field strength - bag pressure" for magnetized SQM at zero temperature. The upper bound $B_u$ on the bag pressure $B$ is calculated for varying $s$ quark current mass $m_s$. Under increasing the $s$ quark current mass $m_s$, the upper bound $B_u$ decreases. For example, the maximum value of $B_u$, corresponding to $H = 0$ (which is practically indistinguishable from the value of $B_u$ at $H = 10^{16}$ G) is $B_{u \text{max}} \approx 84.4$ MeV/fm$^3$ for $m_s = 90$ MeV, $B_{u \text{max}} \approx 79.9$ MeV/fm$^3$ for $m_s = 120$ MeV, and $B_{u \text{max}} \approx 74.9$ MeV/fm$^3$ for $m_s = 150$ MeV. The upper bound stays, first, practically constant and then, beginning from the magnetic field strength $H$ somewhat smaller than $10^{18}$ G, decreases with $H$ and becomes, hence, more restrictive. The upper bound $B_u$ vanishes at $H_{u \text{max}} \approx 3.3 \times 10^{18}$ G for $m_s = 90$ MeV, at $H_{u \text{max}} \approx 3.2 \times 10^{18}$ G for $m_s = 120$ MeV and at $H_{u \text{max}} \approx 3.1 \times 10^{18}$ G for $m_s = 150$ MeV. In a stronger magnetic field, in order to satisfy the constraints (1), (2), (7)-(10), the bag pressure had to become negative, contrary to the constraint $B > 0$. This means, that, under such magnetic fields, the equilibrium conditions (1), (2), (8)-(10) become incompatible with the stability condition (7) for the positively defined bag pressure. In other words, under the equilibrium conditions and in magnetic fields $H > H_{u \text{max}}$, magnetized SQM becomes less stable than the $^{56}$Fe nucleus.

The behavior of the lower bound $B_l$ with the magnetic field is similar to that of the upper bound $B_u$. At $H = 0$, the lower bound has its maximum value $B_{l \text{max}} \approx 56.5$ MeV/fm$^3$ (which almost coincides with the value of $B_l$ at $H = 10^{16}$ G). It stays practically constant till magnetic fields somewhat smaller than $10^{18}$ G, beyond which $B_l$ decreases with $H$ and becomes, hence, less restrictive. The lower bound $B_l$ vanishes at $H_{l \text{0}} \approx 2.7 \times 10^{18}$ G. Under the equilibrium conditions and in the fields $H > H_{l \text{0}}$, the lower bound $B_l$ would be negative. Because of the positiveness of the bag pressure, the inequality $B > B_l$ would be fulfilled always in the fields $H > H_{l \text{0}}$. Thus, in order magnetized SQM would be absolutely stable, the magnetic field strength should satisfy the constraint $H < H_{u \text{max}}$. In fact, the value $H_{u \text{max}}$
represents the upper bound on the magnetic field strength which can be reached in a magnetized strange quark star.

Note that the absolute stability window of magnetized SQM in the MIT bag model was studied earlier in Ref. 35. However, in that study the pure magnetic field contribution \( H^2 / 4 \pi \) (the Maxwell term) to the longitudinal \( p_l \) and transverse \( p_t \) pressures in Eqs. (1), (2) was missed. Because the Maxwell term enters with different sign to the pressures \( p_l \) and \( p_t \), it cannot be excluded by the redefinition of the bag pressure. This term becomes important just in the range of strong magnetic fields \( H > H_{th} \) with \( 10^{17} < H_{th} < 10^{18} \) G. By this reason, in Ref. 35 the bag pressure from the absolute stability window was obtained as only weakly field dependent, and no any constraint on the magnetic field strength was set from the requirement of absolute stability of magnetized SQM.

It is worthy to note at this point that in the case of magnetized hybrid stars the stability of the quark core is provided by the gravitational pressure from the outer hadronic layers, and the bag pressure can be field-independent in the metastable state. Then, without evidently counting the gravitational pressure in the equilibrium conditions, the longitudinal \( p_l \) and transverse \( p_t \) pressures in the quark core would be positive. Under strong magnetic fields, the longitudinal pressure \( p_l \) decreases with the magnetic field while the transverse pressure \( p_t \) increases. There exists the critical magnetic field beyond which the longitudinal pressure \( p_l \) becomes negative resulting in the appearance of the longitudinal instability in magnetized quark matter. In the astrophysical context, this would mean the occurrence of the gravitational collapse of a hybrid star along the magnetic field, if the magnetic field strength exceeds the critical one. The magnitude of the critical field for the appearance of the
longitudinal instability represents the upper bound on the magnetic field strength in the interior of a hybrid star. Thus, in both instances of a strongly magnetized strange quark star and a strongly magnetized hybrid star there exists the upper bound on the magnetic field strength, related to the issue of the star’s stability, but the concrete mechanism responsible for the appearance of the instability is different in each case.

Figure 2 shows the total baryon number density, normalized on the nuclear saturation density \( \rho_0 = 0.16 \text{ fm}^{-3} \), as a function of the magnetic field strength for magnetized SQM and magnetized two-flavor quark matter, calculated under the respective equilibrium conditions for \( E_m/\rho_0 = 930 \text{ MeV} \). In fact, the corresponding lines \( \rho_B^u(H) \) and \( \rho_B^l(H) \) represent the upper and lower bounds on the total baryon number density to ensure the absolute stability of magnetized SQM, assuming that, under the equilibrium conditions, two-flavor quark matter remains to be less stable than \( ^{56}\text{Fe} \) nucleus in strong magnetic fields. On the right, the absolute stability window is bound by the straight line \( H = H_{u,max} \). The corresponding maximum values \( \rho_B^{u,max} \) and \( \rho_B^{l,max} \) of the upper and lower bounds on the total baryon density are \( \rho_B^{u,max} \approx 5.6\rho_0 \) and \( \rho_B^{l,max} \approx 4.4\rho_0 \) at \( m_s = 90 \text{ MeV} \), \( \rho_B^{u,max} \approx 5.4\rho_0 \) and \( \rho_B^{l,max} \approx 4.3\rho_0 \) at \( m_s = 120 \text{ MeV} \), \( \rho_B^{u,max} \approx 5.2\rho_0 \) and \( \rho_B^{l,max} \approx 4.1\rho_0 \) at \( m_s = 150 \text{ MeV} \). It is seen that the upper \( \rho_B^{u} \) and lower \( \rho_B^{l} \) bounds for the allowable total baryon number density \( \rho_B \) from the absolute stability window stay practically constant till the magnetic field strength somewhat smaller than \( 10^{18} \text{ G} \), and then increase till the corresponding maximum value. Also, the increase of the current mass \( m_s \) leads to the decrease of the upper bound on \( \rho_B \), e.g., at \( H = 0 \) (giving nearly the same results as at \( H = 10^{16} \text{ G} \)), \( \rho_B^{l,min} \approx 2.4\rho_0 \) for \( m_s = 90 \text{ MeV} \), \( \rho_B^{l,min} \approx 2.3\rho_0 \) for \( m_s = 120 \text{ MeV} \), \( \rho_B^{l,min} \approx 2.2\rho_0 \) for \( m_s = 150 \text{ MeV} \). The lower bound on \( \rho_B \) at \( H = 0 \) is \( \rho_B^{l,min} \approx 1.5\rho_0 \). Note that magnetic fields \( H > 10^{18} \text{ G} \) strongly affect the upper and lower bounds on the baryon number density from the absolute stability window, unlike to the results of Ref. [35] where this impact was found to be modest.

In conclusion, we have considered magnetized strange quark stars, composed of SQM, within the framework of the MIT bag model aiming at determining the absolute stability window in the parameter space of the theory for which magnetized strange quark stars can be formed. To that end, we have determined the domain of magnetic field strengths, bag pressures, and total baryon number densities, for which the energy per baryon of magnetized SQM is less than that of the most stable \( ^{56}\text{Fe} \) nucleus under the zero external pressure conditions (1), (2) and vanishing temperature. In fact, this requirement sets the upper bound on the parameters from the absolute stability window. The lower bound on the parameters from the absolute stability window is determined from the constraint that magnetized two-flavor quark matter under equilibrium conditions (1), (2) and zero temperature should be less stable than the most stable \( ^{56}\text{Fe} \) nucleus. This constraint is extended from weak terrestrial magnetic fields, where it has direct experimental confirmation, to possible strong magnetar interior magnetic fields \( H > 10^{17} \text{ G} \), where such confirmation is
An important feature of our consideration is that, unlike to some of the previous studies, in the zero pressure conditions [1, 2] the pure magnetic field contribution (the Maxwell term) to the transverse and longitudinal pressures has been taken into account. It has been shown that there exists the magnetic field strength at which the upper bound on the bag pressure from the absolute stability window vanishes. In fact, the value of this field, $H_{u \text{max}} \sim 3 \cdot 10^{18}$ G, represents the upper bound on the magnetic field strength, which can be reached in a strongly magnetized strange quark star. Also, we have studied how the absolute stability window and upper bound on the magnetic field strength are affected by varying the strange quark current mass $m_s$. It has been clarified that the increase of the current mass $m_s$ leads to the decrease of the upper bound on the bag pressure and total baryon number density from the absolute stability window, and to the decrease of the upper bound on the magnetic field strength in magnetized strange quark stars as well.

Note that obtained here an estimate of the upper bound on the magnetic field strength in magnetized strange quark stars may be further improved by utilizing a more elaborated version of the MIT bag model, e.g., with taking into account the effects of short-range quark interactions or color superconductivity. Nevertheless, the given analysis provides, definitely, the correct order of magnitude of the upper bound on the magnetic field strength in strange quark stars, and is acceptable for getting the rough estimate of this quantity.

Acknowledgment

The author would like to thank Ewha Womans University for hospitality and support during his stay at Seoul.

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