ON GENERALIZATIONS OF LUKASIEWICZ RINGS

1. Preliminaries

Given any ring $R$ satisfying: for every $x \in R$, there exists $r, r' \in R$ such that $xr = r'x = x$, the lattice of ideals of $R$ form a pseudo residuated lattice $A(R) := (\text{Id}(R), \land, \lor, \bowtie, \rightarrow, \sim, \{0\}, R)$, where $I \land J = I \cap J$, $I \lor J = I + J$, $I \bowtie J := \{x \in R :Ix \subseteq J\}$, $I \rightarrow J := \{x \in R :xI \subseteq J\}$.

The authors of [1] investigated the rings $R$ for which $A(R)$ is an MV-algebra. Recall that MV-algebras, which constitute the algebraic counterpart of Lukasiewicz many value logic are equivalent to $\ell$-groups with strong units [?]. Several commutative and noncommutative generalizations of MV-algebras, among which pseudo MV-algebras and BL-algebras have been introduced and studied thoroughly (see for e.g., []).

The main goal of this work is to investigate two generalizations of Lukasiewicz rings: noncommutative Lukasiewicz rings which will be refer to as GLR, which are rings for which $A(R)$ is a pseudo MV-algebra, and BL-rings which are rings $R$ for which $A(R)$ is a BL-algebra. It turns that the class of BL-rings coincides with that of multiplication rings as studied in [5].

A (pseudo) residuated lattice is a nonempty set $L$ with five binary operations $\land, \lor, \bowtie, \rightarrow, \sim$, and two constants $0, 1$ satisfying:

L-1: $\mathbb{L}(L) := (L, \land, \lor, 0, 1)$ is a bounded lattice;

L-2: $(L, \bowtie, 1)$ is a monoid;

L-3: $x \bowtie y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \sim z$ (pseudo-Residuation);

A pseudo-RL monoid is a pseudo-residuated lattice $L$ which satisfies the following condition:

L-4: $y \bowtie (y \sim x) = x \land y = (x \rightarrow y) \bowtie x$ (pseudo-Divisibility).

A pseudo-MTL algebra is a pseudo-residuated lattice $L$ which satisfies the following condition:

L-5: $(x \rightarrow y) \lor (y \rightarrow x) = 1 = (x \sim y) \lor (y \sim x)$ (pseudo-Prelinearity);

A pseudo BL-algebra is a pseudo-MTL-algebra $L$ which satisfies the pseudo-Divisibility.

A pseudo MV-algebra is a pseudo BL-algebra $L$ which satisfies the following condition:

L-7: $\overline{\overline{x}} = x = \overline{\overline{x}}$. 

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In the literature, as for example in \cite{3}, pseudo-MV algebras are also defined as algebras $A = (A, \oplus, \odot, -, \sim, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ satisfying the following for all $x, y, z \in A$:

\begin{itemize}
  \item psMV-1 $$(x \oplus y) \oplus z = x \oplus (y \oplus z);$$
  \item psMV-2 $x \oplus 0 = 0 \oplus x = x$;
  \item psMV-3 $x \oplus 1 = 1 \oplus x = 1$;
  \item psMV-4 $1^\sim = 0, 1^\sim = 0$;
  \item psMV-5 $(x^\sim \oplus y^\sim)^\sim = (x^\sim \oplus y^\sim)^\sim$;
  \item psMV-6 $x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^\sim) \oplus y = (y \odot x^\sim) \oplus x$;
  \item psMV-7 $x \odot (x^\sim \oplus y) = (x \oplus y^\sim) \odot y$;
  \item psMV-8 $(x^\sim)^\sim = x$;
\end{itemize}

Every pseudo MV-algebra has an underline distributive lattice structure defined by $x \leq y$ if and only if $x \oplus z = y$ for some $z \in A$. Moreover, the infimum and supremum are given by

(i) $x \lor y = x \oplus (x^\sim \odot y) = y \oplus (y^\sim \odot x) = (x \odot y^\sim) \oplus y = (y \odot x^\sim) \oplus x$,

(ii) $x \land y = x \odot (x^\sim \oplus y) = y \odot (y^\sim \oplus x) = (x \oplus y^\sim) \odot y = (y \oplus x^\sim) \odot x$.

Pseudo MV-algebras have a wealth of properties that will be used repeatedly with any explicit citation.

We will need the following properties of Pseudo MV-algebras that we could not find in the literature.

**Proposition 1.1.** Let $A$ be a pseudo MV-algebra and $x, y, z \in A$.

(i) If $x \leq y$, then $(z \odot y^\sim) \oplus (y \odot x^\sim) = (y \lor z) \odot x^\sim$;

(ii) If $x \leq y, z$, then $z \oplus (x^\sim \odot y) = (z \odot x^\sim) \oplus y$.

**Proof.** (i) Suppose that $x \leq y$, then

\[
(y \lor z) \odot x^\sim = (y \odot x^\sim) \lor (z \odot x^\sim) \\
= [(z \odot x^\sim) \odot (y \odot x^\sim)] \oplus (y \odot x^\sim) \\
= [(x \odot z^\sim) \odot (y \odot x^\sim)] \oplus (y \odot x^\sim) \\
= [(y \odot x^\sim) \odot (x \odot z^\sim)] \oplus (y \odot x^\sim) \\
= [(y \lor x) \odot (y \odot x^\sim) \oplus (y \odot x^\sim) \\
= (y \lor z^\sim) \oplus (y \odot x^\sim) \quad \text{since } x \leq y \\
= (z \odot y^\sim) \oplus (y \odot x^\sim)
\]

(ii) Since $x \leq y, z$, there exists $a \in A$ such that $y = x \oplus a$ and $z \odot x^\sim = 1$.

Now, $z \oplus (x^\sim \odot y) = z \oplus (x^\sim \odot (x \oplus a)) = z \oplus (x^\sim \land a) = (z \oplus x^\sim) \land (z \oplus a) = z \oplus a$. 
On the other hand,
\[(z \otimes x^{-}) \oplus y = (z \otimes x^{-}) \oplus (x \oplus a) = ((z \otimes x^{-}) \oplus x) \oplus a = (z \vee x) \oplus a = z \oplus a.\]
Hence, \(z \oplus (x^{-} \otimes y) = (z \otimes x^{-}) \oplus y = z \oplus a.\)

\[\square\]

2. Semi-rings and pseudo MV-algebras

**Proposition 2.1.** Let \(A = \langle A, \oplus, \odot, -, \sim, 0, 1 \rangle\) be a pseudo MV-algebra and \(S(A) := \langle A, +, \cdot, 0, 1 \rangle\). Then \(S(A)\) is an additively idempotent semi-ring satisfying:

(i) \(x \odot y = 0\) iff \(y \leq x^{-}\) iff \(x \leq y^{-}\);
(ii) \(x \vee y = ((x^{\sim} \cdot y)^{\sim} \cdot x^{\sim})^{-} = (x^{\sim} \cdot (y \cdot x^{-})^{\sim})^{-}\);
(iii) \((y^{\sim} \cdot x^{\sim})^{-} = (y^{-} \cdot x^{-})^{\sim}\)

where \(x + y = x \vee y\), \(x \cdot y = x \odot y\), and \(x \leq y\) iff \(x^{-} \oplus y = 1\).

**Proof.** Follows easily from the main properties of pseudo MV-algebras. \(\square\)

The construction above can be reversed.

Let \(S = \langle S, +, \cdot, 0, 1 \rangle\) be an additively idempotent semi-ring, define \(x \leq y\) iff \(x + y = y\). \(S\) is called a generalized Lukasiewicz (GL) semi-ring if there exists maps \(-: S \to S\) and \(\sim: S \to S\) satisfying for all \(x, y \in S\):

(i) \(x \odot y = 0\) iff \(y \leq x^{-}\) iff \(x \leq y^{-}\);
(ii) \(x \vee y = ((x^{\sim} \cdot y)^{\sim} \cdot x^{\sim})^{-} = (x^{\sim} \cdot (y \cdot x^{-})^{\sim})^{-}\).
(iii) \((y^{\sim} \cdot x^{\sim})^{-} = (y^{-} \cdot x^{-})^{\sim}\)

**Lemma 2.2.** Let \(S = \langle S, +, \cdot, -, \sim, 0, 1 \rangle\) be a GL semi-ring and \(\leq\) the relation defined above. Then each of the following properties holds for every \(x, y \in S\).

(i) The relation \(\leq\) is an order relation on \(S\) that is compatible with + and \(\cdot\),
(ii) \(x^{\sim} \cdot x = x \cdot x^{\sim} = 0\), \(0^{\sim} = 0^{-} = 1\) and \(1^{\sim} = 1^{-} = 0\),
(iii) \(x \leq y\) implies \(y^{\sim} \leq x^{\sim}\) and \(y^{-} \leq x^{-}\),
(iv) \(x^{\sim^{-}} = x^{\sim} = x\),
(v) \(S\) is a lattice-ordered ring, where \(x \vee y = x + y\) and \(x \wedge y = (x^{-} + y^{-})^{-} = (x^{\sim} + y^{\sim})^{-}\),
(vi) \((x^{-} + y^{-})^{-} = (x^{\sim} + y^{\sim})^{-}\)

**Proof.** (i) \(\leq\) is clearly reflexive and anti-symmetric. In addition, if \(x + y = y\)
and \(y + z = z\), then \(x + z = x + y + z = y + z = z\). Thus \(\leq\) is transitive. The compatibility of \(\leq\) with + and \(\cdot\) is also easy to verify.

For the rest of the properties, note that combining (ii) and (iii), the following property holds in any GL semi-ring.

(ii)' \(x + y = ((x^{\sim} \cdot y)^{\sim} \cdot x^{-})^{-} = (x^{\sim} \cdot (y \cdot x^{-})^{\sim})^{-}\).

(ii) Since \(x^{\sim} \leq x^{-}\) and \(x^{-} \leq x^{-}\), then the result follows from (i) of the
Proposition 2.3. For every generalized Lukasiewicz semi-ring \( S = \langle S, +, \cdot, ^\sim, \sim, 0, 1 \rangle \), define \( x \oplus y = (y^\sim \cdot x^\sim)^\sim = (y^\sim \cdot x^\sim)^\sim \) and \( x \odot y = x \cdot y \).

Then, \( A(S) := \langle S, \oplus, \odot, ^\sim, \sim, 0, 1 \rangle \) is a pseudo MV-algebra.

Proof. Observe from Lemma that \( x \oplus y = (y^\sim \cdot x^\sim)^\sim = (y^\sim \cdot x^\sim)^\sim \).

psMV-1. \( x \oplus (y \oplus z) = x \oplus ((z^\sim \cdot y^\sim)^\sim) = ((z^\sim \cdot y^\sim)^\sim \cdot x^\sim)^\sim = (z^\sim \cdot (y^\sim \cdot x^\sim))^\sim = (y^\sim \cdot x^\sim)^\sim \oplus z = (x \oplus y) \oplus z \).

psMV-2, psMV-3, psMV-4: follow straight from the lemma.

psMV-5: Since \( x \oplus y = (y^\sim \cdot x^\sim)^\sim = (y^\sim \cdot x^\sim)^\sim \), it follows that \( (x^\sim \oplus y^\sim)^\sim = (x^\sim \oplus y^\sim)^\sim = y \cdot x \).

psMV-6: Note that from (ii) of the definition of GL semi-ring, \( x + y = x \bigoplus (x^\sim \odot y) = (y \odot x^\sim) \oplus x \). The equalities to the remaining expressions follow from the fact that + is commutative.

psMV-7: Note that \( (x + y)^\sim = (x^\sim \cdot y^\sim)^\sim \cdot x^\sim \) and by (ii)’ of the proof of lemma, we also have \( (x + y)^\sim = x^\sim \cdot (y \cdot x^\sim)^\sim \). Thus, \( x \odot (x^\sim \oplus y) = x \odot (y^\sim \cdot x^\sim) = (x^\sim + y^\sim)^\sim = (x^\sim + y^\sim)^\sim = (y^\sim + x^\sim)^\sim = (y \cdot x^\sim)^\sim \cdot y = (x \oplus y^\sim) \cdot y \).

psMV-8: Clear from lemma (iv).
Proposition 2.4. In a generalized Lukasiewicz semi-ring $S$, the following are equivalent:

a) $I$ is an ideal of $S$

b) $I$ is a non-empty subset, closed under $+$ and whenever $x \in I$ and $y \leq x$, then $y \in I$.

Proof. Assume that $I$ is an ideal of $S$ and let $x \in I$ and $y \in S$ with $y \leq x$. Then $x \cdot (y^\sim \cdot x)^\sim \in I$. But $x \cdot (y^\sim \cdot x)^\sim = (x^\sim + y^\sim)^\sim = y^\sim = y$. Thus, we obtain that $y \in I$.

Conversely, let $x \in I$ and $y \in S$, we have to show that $xy \in I$ and $yx \in I$. Since $1 = y + 1$, we have $x = xy + x = yx + x$. So $xy \leq x$ and $yx \leq x$ and we obtain that $xy \in I$ and $yx \in I$. \qed

Proposition 2.5. There is a natural duality between pseudo MV-algebras and generalized Lukasiewicz semi-rings.

Proof. One needs to prove that $S(A(S))$ and $S$ are equal as GL semi-rings; and $A(S(A))$ and $A$ are equal as pseudo MV-algebras. Since the underline sets remain unchanged and so do the operations: multiplication, $\cdot$, and $\sim$, one only needs to check that the additions coincide. Starting with a GL semi-ring $S = (S, +, \cdot, \sim, 0, 1)$, define $x \oplus y = (y^\sim \cdot x^\sim)^\sim$ and $x \circ y = x \cdot y$. One obtains a pseudo MV-algebra $A(S) := (S, \oplus, \circ, \sim, 0, 1)$, which has a supremum given by $x \lor y = x \oplus (x^\sim \circ y) = y \oplus (y^\sim \circ x) = (x \circ y^\sim) \oplus y = (y \circ x^\sim) \oplus x$. Now, from this pseudo MV-algebra, one constructs the GL semi-ring $S(A(S))$ whose addition is defined as the supremum. Therefore, one only needs to verify that $x \lor y = x + y$, which is clear from (ii) of the definition of a GL semi-ring. Now, starting with a pseudo MV-algebra $A = (A, \oplus, \circ, \sim, 0, 1)$, one constructs a GL semi-ring $S(A) := (A, +, \cdot, 0, 1)$, where $x + y = x \lor y$, the supremum of the pseudo MV-algebra and $x \cdot y = x \circ y$. From this GL semi-ring, one gets a pseudo MV-algebra $A(S(A))$, with $x \oplus' y = (y^\sim \circ x^\sim)^\sim = (y^\sim \circ x^\sim)^\sim$. Therefore, one needs to check that $x \oplus' y = x \oplus y$, that is $x \oplus y = (y^\sim \circ x^\sim)^\sim$, which is a known property of pseudo MV-algebras. \qed

Let $R$ be a ring which satisfies

(*) for every $x \in R$, there exist $r, r' \in R$ such that $xr = r'x = x$

Let $Sem(R) = \langle Id(R), +, \cdot, 0, R \rangle$, where $Id(R)$ denotes the set of (two-sided) ideals of $R$. Define $\sim, \sim : Id(R) \to Id(R)$ by:

$I^- = \{x \in R :Ix = 0\}$ and $I^\sim = \{x \in R :Ix = 0\}$

It is easily verified that $Sem(R) = \langle Id(R), +, \cdot, 0, R \rangle$ is a semi-ring.
**Proposition 2.6.** Given any ring $R$ and $I, J$ ideals of $R$,

(i). $I \subseteq J$ iff $I + J = J$

(ii). $I \cdot J = 0$ iff $I \subseteq I^-$

(iii). $I \subseteq J$ implies $J^- \subseteq I^-$ and $J^- \subseteq I^-$

(iv). $(I + J)^- = I^- \cap J^-$; $(I + J)^+ = I^+ \cap J^+$; $I \subseteq I^-$, $I \subseteq I^-$

(v). $I + J \subseteq ((I^- \cdot J^-) \cdot I^-)^-$ and $I + J \subseteq (I^- \cdot (J^- \cdot I^-)^-)^-$

(vi). If $R$ satisfies $(\ast)$, then $R^- = R^+$.

**Proof.** Let $I, J$ be ideals of $R$.

(i). $I \subseteq J$ iff $I + J = J + J = J$

(ii). Follows clearly from the definitions of $\cdot, \cdot, \cdot$.

(iii). Straightforward.

(iv). Since $I, J \subseteq I + J$, it follows that $Ix, Jx \subseteq (I + J)x$. Thus, $(I + J)x = 0$ implies $Ix, Jx = 0$ and $(I + J)^- \subseteq I^- \cap J^-$. In addition, if $Ix = Jx = 0$, then $(I + J)x = 0$, so $(I + J)^- \subseteq I^- \cap J^-$. Hence, $(I + J)^- = I^- \cap J^-$. Similarly, we show that $(I + J)^+ = I^+ \cap J^-$. The inclusions $I \subseteq (I^-)^-$ and $I \subseteq I^- \subseteq I^-$ follow from the definitions.

(v). Let $u \in I^-$, $v \in (I^- \cdot J^-)$ and $y \in J$, we have $uv \in I^- \cdot J$ and $0 = v(uy) = (vu)y$ and we obtain that $vu \in J^-$. So $vu \subseteq I^+ \cap J^- = (I + J)^-$. Since a typical element in $(I^+ \cdot J^-)^+ \cdot I^-$ is a sum of elements of the type $vu$, we obtain that $(I^+ \cdot J^-)^+ \cdot I^- \subseteq (I^+ + J)^-$ and conclude that $I + J \subseteq (I + J)^- \subseteq ((I^+ \cdot J^-)^+ \cdot I^-)^-$. The proof that $I + J \subseteq (I^- \cdot (J^- \cdot I^-)^-)^-$ is similar to the above.

(vi). Let $x \in R^-$, then $Rx = 0$. But, by $(\ast)$, there exists $r \in R$ such that $rx = x$. Thus $x \in Rx = 0$ and $x = 0$. □

**Definition 2.7.** A ring $R$ is called a generalized Lukasiewicz ring (GLR) if it satisfies $(\ast)$ and for all ideals $I, J$ of $R$,

( GLR-1) $I + J = ((I^- \cdot J^-) \cdot I^-)^- = (I^- \cdot (J^- \cdot I^-)^-)^-$

( GLR-2) $(J^- \cdot I^-)^- = (J^- \cdot I^-)^-$

It is clear that every Lukasiewicz ring as treated in [1] is a GLR.

**Example 2.8.** 1. Let $F$ be a field and $R = M_n(F)$ $(n \geq 1)$ be the ring of $n \times n$ matrices over $F$. Then $R$ satisfies $(\ast)$ as a unitary ring, and also GLR-1, and GLR-2 as it has only two ideals: $0, R$. Thus, $R$ is a GLR.

2. **Proposition 2.9.** A ring $R$ is a generalized Lukasiewicz ring if and only if $A(Sem(R))$ is a pseudo MV-algebra.

**Proof.** Suppose that $R$ is a generalized Lukasiewicz ring. Then $Sem(R)$ is clearly a GL semi-ring, and it follows from Proposition[2.3] that $A(Sem(R))$ is
a pseudo MV-algebra. Conversely, if $A(Sem(R))$ is a pseudo MV-algebra, it is clear that $R$ is a GLR.

From the above proposition, we have the following result.

**Proposition 2.10.** In a generalized Lukasiewicz ring $R$, we have $I^{\sim \sim} = I = I^{\sim}$.

We would like to describe the relationship between ideals of $R$ and those of $Sem(R)$, when $R$ is a GLR. For the remainder of this section, $R$ will denote a GLR and $S$ its associated semi-ring, that is $S = Sem(R)$. Note that since $R$ satisfies $(\ast)$, for every $x \in R$, $RxR := \{\sum_{i=1}^{n} r_i x_i : n \geq 1, r_i, s_i \in R\}$ is the ideal of $R$ generated by $x$.

For every ideal $I$ of $S$, we define

$$S(I) := \{J \in Id(R) : J \subseteq Rx_1R + Rx_2R + \ldots + Rx_nR, \text{ for some } x_1, \ldots, x_n \in I\}$$

Then, by Proposition 2.11, it is straightforward that $S(I)$ is an ideal of $S$ and

$$S(I) = \left\{ \sum_{i=1}^{n} J_i x_i L_i : n \geq 1, J_i, L_i \in Id(R), x_i \in I \right\}$$

Indeed, $S(I)$ is the ideal of $S$ generated by $X := \{RxR \in Id(R) : x \in I\}$. One should also observe that if $I$ is proper, so is $S(I)$. Indeed, if $R \in S(I)$, then there are $x_1, x_2, \ldots, x_n \in I$ such that $R \subseteq Rx_1R + Rx_2R + \ldots + Rx_nR \subseteq I$ and so $I = R$.

To reverse the construction above, we define $S^{-1}(I) := \{x \in R : RxR \in I\}$, for each ideal $I$ of $S$.

**Proposition 2.11.** (i) For each ideal $I$ of $S$, $S^{-1}(I)$ is an ideal of $R$;

(ii) For each ideal $I$ of $S$, $S(S^{-1}(I)) \subseteq I$;

(iii) If $I \subseteq J$, then $S^{-1}(I) \subseteq S^{-1}(J)$.

**Proof.** (i) Assume that $I$ is an ideal of $S$ and let $I = S^{-1}(I)$. Let $x, y \in I$, we have $RxR \in I$ and $RyR \in I$. Since $I \subseteq Id(S)$, we have $RxR + RyR \in I$. From the fact that $R(x + y)R \subseteq RxR + RyR$, we deduce that $R(x + y)R \in I$ and $x + y \in I$.

In addition, let $x \in I$ and $y \in R$. Since $I \subseteq Id(S)$, $RxR \in I$ and $RyR \in Id(R)$, it follows that $RxR \cdot RyR \in I$. From this and the fact that $RxR \subseteq RxR \cdot RyR$, we have $RxR \cdot RyR \in I$ as $I$ is an ideal of $S$. That is $xy \in I$. A similar argument shows that $yx \in I$. Thus, $S^{-1}(I)$ is an ideal of $R$.

(ii) Let $J \in S(S^{-1}(I))$, then $J \subseteq Rx_1R + Rx_2R + \ldots + Rx_nR$, for some $x_1, \ldots, x_n \in S^{-1}(I)$. But $x_i \in S^{-1}(I)$ means that $Rx_iR \in I$ and then, $J \subseteq$
\[ Rx_1 R + Rx_2 R + \ldots + Rx_n R \in I. \] Hence \( J \in I. \)

(iii) Clear. \( \square \)

**Proposition 2.12.** For every ideal \( I \) of \( R, \) \( I = S^{-1}(S(I)). \)

*Proof.* Assume that \( I \) is an ideal of \( R \) and \( x \in I. \) It is clear that \( RxR \in S(I) \) and then \( x \in S^{-1}(S(I)). \) Conversely, let \( x \in S^{-1}(S(I)). \) Thus \( RxR \subseteq Rx_1 R + Rx_2 R + \ldots + Rx_n R, \) for some \( x_1, \ldots, x_n \in I. \) In particular, since each \( Rx_i R \subseteq I, \) then \( x \in RxR \subseteq I, \) and \( x \in I. \) Thus, \( I = S^{-1}(S(I)). \) \( \square \)

For the next result, \( \text{FG}(R) \) denotes the set of finitely generated ideals of \( R. \)

**Proposition 2.13.** For each ideal \( I \) of \( S, \)

(i) \( I \cap \text{FG}(R) \subseteq S(S^{-1}(I)); \)

(ii) If every ideal of \( I \) is finitely generated, then \( I = S(S^{-1}(I)). \)

*Proof.* (i) Suppose \( J \in I \) and \( J = Rx_1 R + Rx_2 R + \ldots + Rx_n R, \) for some \( x_1, \ldots, x_n \in J. \) Since \( Rx_i R \subseteq J \) for all \( i, \) and \( J \in I, \) which is an ideal of \( S, \) then \( Rx_i R \in I \) for all \( i. \) That is \( x_i \in S^{-1}(I) \) for all \( i, \) and \( J \in S(S^{-1}(I)). \) Hence, \( I \cap \text{FG}(R) \subseteq S(S^{-1}(I)) \) as needed.

(ii) By assumption, \( I \subseteq \text{FG}(R), \) hence \( I = I \cap \text{FG}(R) \subseteq S(S^{-1}(I)). \) The equality is obtained by combining the above with Proposition 2.12(ii). \( \square \)

Note all ideals of Noetherian rings are finitely generated. Therefore, if \( R \) is Noetherian, then \( I = S(S^{-1}(I)). \) Thus, there is a one-to-one correspondence between the ideals of \( R \) of those of \( S. \)

### 3. THE CATEGORY OF GLRs

It turns out that the category of GLRs is closed under several important algebraic constructions. We start with finite direct products.

**Proposition 3.1.** Any finite direct product of GLRs is a GLR. Conversely, if a product of rings is a GLR, then so is each factor.

*Proof.* Let \( R = \prod_{i=1}^n R_i \) denote the finite product of the rings \( R_i. \) Since the operations of \( R \) are component-wise, then \( R \) satisfies \((\ast)\) if and only if each \( R_i \) does.

Suppose that each \( R_i \) is a GLR and let \( I \) be an ideal of \( R, \) then \( I = \prod_{i=1}^n I_i, \) where \( I_i \subseteq R_i. \) Note that if \( J = \prod_{i=1}^n J_i, \) with \( J_i \subseteq R_i, \) then \( I + J = \prod_{i=1}^n (I_i + J_i). \)

\[ I^\sim = \prod_{i=1}^n I_i^\sim, \quad I^\sim = \prod_{i=1}^n I_i, \quad I \cdot J = \prod_{i=1}^n (I_i \cdot J_i). \] From these identities, and the fact that each \( R_i \) satisfies \((\text{GL-1})\) and \((\text{GL-2}),\) it follows that \( R \) satisfies \((\text{GL-1})\) and \((\text{GL-2}). \) Thus, \( R \) is a GLR.
Conversely, suppose that $R := \prod_{i=1}^n R_i$ is a GLR. Let $I_k, J_k$ be ideals of $R_k$. Then $I = I_k \times \prod_{i \neq k} R_i$ and $J = J_k \times \prod_{i \neq k} R_i$ are ideals of $R$. Note that $R_i^\sim = 0_i = R_i^-$, where $0_i$ is the zero ideal of $R_i$. We have $I^\sim = I_k^\sim \times \prod_{i \neq k} 0_i, I^- = I_k^\sim \times \prod_{i \neq k} 0_i, J^\sim = J_k^\sim \times \prod_{i \neq k} 0_i, J^- = J_k^\sim \times \prod_{i \neq k} 0_i$. Now $I + J = (I_k + J_k) \times \prod_{i \neq k} R_i$ and $J^\sim \cdot I^\sim = (J_k^\sim \cdot I_k^\sim) \times \prod_{i \neq k} 0_i$. So, $(J^\sim \cdot I^\sim)^\sim = ((J_k^\sim \cdot I_k^\sim))^- \times \prod_{i \neq k} R_i$.

Similarly, we show that $(J^- \cdot I^-)^\sim = (J_k^- \cdot I_k^-)^\sim \times \prod_{i \neq k} R_i$. From this and the fact that $R$ is a GLR, it follows that $(J_k^- \cdot I_k^-)^\sim \times \prod_{i \neq k} R_i = (J_k^- \cdot I_k^-)^\sim \times \prod_{i \neq k} R_i$. Thus, $(J_k^\sim \cdot I_k^\sim)^\sim = (J_k^- \cdot I_k^-)^\sim$.

On the other hand, $I^\sim \cdot J = (I_k^\sim \cdot J_k) \times \prod_{i \neq k} 0_i$ and $(I^\sim \cdot J)^\sim = (I_k^\sim \cdot J_k)^\sim \times \prod_{i \neq k} R_i$. So, $(I^\sim \cdot J)^\sim \cdot I^\sim = (I_k^\sim \cdot J_k)^\sim \cdot I_k^\sim \times \prod_{i \neq k} 0_i$ and $(I_k + J_k) \times \prod_{i \neq k} R_i = I + J = ((I^\sim \cdot J)^\sim \cdot I^\sim)^\sim = ((I_k^\sim \cdot J_k)^\sim \cdot I_k^\sim)^\sim \times \prod_{i \neq k} R_i$.

It follows that $I_k + J_k = ((I_k^\sim \cdot J_k^\sim \cdot I_k^\sim)^\sim \cdot I_k^\sim)^\sim$. A similar argument shows that $I_k + J_k = (I_k^\sim \cdot (J_k \cdot I_k^\sim))^\sim$. Therefore, $R_k$ is a GLR as needed. □

**Remark 3.2.** An infinite (direct) product of GLRs needs not be a GLR.

Indeed, consider $R = \prod_{i=1}^\infty F$, where $F$ is a field. We claim that $R$ is not a GLR. To see this, consider $I = \{(x_n) : x_{2n} = 0\}, J = \oplus_{i=1}^\infty F$ and $K = \{(x_n) : x_{2n+1} = 0\}$, which are all ideals of $R$. One can verify that

$$I^- = I^\sim = K, (I + J)^\sim = 0, (I^\sim \cdot J)^\sim \cdot I^\sim = K$$

Thus, $(I + J)^\sim \neq (I^\sim \cdot J)^\sim \cdot I^\sim$ and GLR-1 fails.

Let $R$ be a GLR, $A(Sem(R))$ be the pseudo MV-algebra associated to $R$. For simplicity, $A(Sem(R))$ will be denoted throughout the rest of the paper by $A(R)$. Recall that $A(R) = (\text{Id}(R), \oplus, \odot, \neg, \neg, 0, 1)$ is the pseudo MV-algebra, where:

$$I \oplus J = (J^\sim \cdot I^\sim)^\sim = (J^- \cdot I^-)^\sim,$$

$$I \odot J = I \cdot J = \{\sum a_ib_i : a_i \in I, b_i \in J\},$$

$$I^\sim = \{x \in R : Ix = 0\}, I^- = \{x \in R : xI = 0\},$$

$$0 = \{0\}, 1 = R.$$

We know that $A(R)$ has an underline distributive lattice $(\text{Id}(R), \lor, \land, 0, 1)$, where:

$$I \lor J = ((I^\sim \cdot J)^\sim \cdot I^\sim)^\sim = (I^\sim \cdot (J \lor I^-)^\sim)^\sim = I + J, I \land J = I \cap J.$$ Indeed, $(\text{Id}(R), \lor, \land, 0, 1)$ is a complete lattice.

Since the identity $\oplus$ distributes over $\lor$ in any pseudo MV-algebra, then in $A(R)$ the following identity holds.

$$I \oplus (J + K) = I \oplus J + I \oplus K$$

Our next task is to show that GLRs are closed under epimorphic images.

Given an ideal $I$ of $R$, we consider the ring $R/I$. Then ideals of $R/I$ are of
the form $J/I := \{x/I : x \in J\}$ where $J$ is an ideal of $R$ such that $I \subseteq J$. For ideals $J, K$ of $R$ such that $I \subseteq J$ and $I \subseteq K$, we have $J/I + K/I = (J + K)/I$ and $(J/I) \cdot (K/I) = (J \cdot K)/I$.

**Proposition 3.3.** Let $R$ be a GLR, and $I, J$ be ideals of $R$ such that $I \subseteq J$. Then:

(i) $I \subseteq (I^\sim \cdot J)^\sim$ and $I \subseteq (J \cdot I^\sim)^\sim$.

(ii) $(J/I)^\sim = (I^\sim \cdot J)^\sim/I$ and $(J/I)^\sim = (J \cdot I^\sim)^\sim/I$.

**Proof.**

(i) We have $I^\sim \cdot J \subseteq I^\sim$ and then $I = I^\sim \subseteq (I^\sim \cdot J)^\sim$.

Similarly, $(J \cdot I^\sim)^\sim \subseteq I^\sim$ and then $I = I^\sim \subseteq (J \cdot I^\sim)^\sim$.

(ii) $(J/I)^\sim = \{x/I \in R/I : (y/I)(x/I) = 0 \text{ for all } y \in J\}$ = \{x/I \in R/I : yx \in I \text{ for all } y \in J\} = \{x/I \in R/I : Jx \subseteq I\} = \{x/I \in R/I : (I^\sim \cdot Jx) \subseteq I^\sim \cdot I = 0\} = \{x/I \in R/I : I^\sim \cdot Jx = 0\} = \{x/I \in R/I : x \in (I^\sim \cdot J)^\sim\} = (I^\sim \cdot J)^\sim/I.

A similar argument shows that $(J/I)^\sim = (J \cdot I^\sim)^\sim/I$. □

**Proposition 3.4.** (i) For all $I, J, K \in \text{Id}(R)$, $I \cap (J + K) = I \cap J + I \cap K$.

(ii) If $\{J_i\}_i$ is a family of ideals of $R$, then $I + \bigcap_i J_i = \bigcap_i (I + J_i)$.

**Proof.**

(i) Since $\langle \text{Id}(R), \wedge, \vee, 0, R \rangle$ is a distributive lattice and $\wedge = \cap, \vee = +$, we have $I \cap (J + K) = I \wedge (J \vee K) = (I \wedge J) \vee (I \wedge K) = (I \cap J) + (I \cap K)$.

(ii) Let $\{J_i\}_i$ be a family of ideals of $R$, since $A(R) = \langle \text{Id}(R), \oplus, \circ, ^\sim, ^\wedge, 0, 1 \rangle$ is a complete pseudo MV-algebra, we have $I + \bigcap_i J_i = I \vee \bigcap_i J_i = \bigcap_i (I \vee J_i) = \bigcap_i (I + J_i)$. □

**Proposition 3.5.** The quotient of a GLR by a proper ideal is again a GLR.

**Proof.** Let $I$ be a proper ideal of a GLR $R$.

It is clear that $R/I$ satisfies $(\ast)$ since $R$ does.

GLR-1: Let $J, K$ be ideals of $R$ both containing $I$. We need to prove the following two identities:

$$(1) \quad ((J/I)^\sim \cdot K/I)^\sim \cdot (J/I)^\sim = (J/I + K/I)^\sim$$

$$(2) \quad (J/I)^\sim \cdot (K/I \cdot (J/I)^\sim)^\sim = (J/I + K/I)^\sim$$

To prove (1), note that by Proposition 3.3, (1) is equivalent to

$$(1') \quad [((J \cdot I^\sim)^\sim \cdot K \cdot I^\sim)^\sim \cdot (J \cdot I^\sim)^\sim]/I = ((J + K) \cdot I^\sim)^\sim/I$$
But,
\[
(J \cdot I^-) \sim \cdot (K \cdot I^-) \sim \cdot (J \cdot I^-) \sim = [(J \cdot I^-) \oplus ((J \cdot I^-) \sim \cdot K \cdot I^-)] \sim \\
= (J \cdot I^-) \lor (K \cdot I^-) \\
= (J + K) \cdot I^-
\]

To prove (2), note that by Proposition 3.3, (2) is equivalent to
\[
(2') \quad [(J \cdot I^-) \sim \cdot (K \cdot (I^- \cdot J^-) \sim \cdot I^-) \sim] / I = ((J + K) \cdot I^-) \sim / I
\]

But,
\[
(J \cdot I^-) \sim \cdot (K \cdot (I^- \cdot J^-) \sim \cdot I^-) \sim = (J \cdot I^-) \sim \cdot (K \cdot (J^- \oplus I) \cdot I^-) \sim \\
= (J \cdot I^-) \sim \cdot (K \cdot (J^- \cap I^-)) \sim \\
= (J \cdot I^-) \sim \cdot (K \cdot J^-) \sim \quad \text{since } I \subseteq J \\
= [(K \cdot J^-) \oplus (J \cdot I^-)] \sim \\
= [(J \lor K) \cdot I^-] \sim \quad \text{(Prop. 3.3)} \\
= [(J + K) \cdot I^-] \sim
\]

This completes the proof of GLR-1.

GLR-2: Let \( J, K \) be ideals of \( R \) both containing \( I \). We need to show that
\[
((J/I)^- \cdot (K/I)^-) \sim = ((J/I)^- \cdot (K/I)^-) \sim , \quad \text{or equivalently by Proposition 3.3 that}
\]
\[
((I^- \cdot J^-) \sim \cdot (I^- \cdot K^-) \sim \cdot I^-) \sim / I = (I^- \cdot (J \cdot I^-) \sim \cdot (K \cdot I^-) \sim) / I
\]

We have
\[
((I^- \cdot J^-) \sim \cdot (I^- \cdot K^-) \sim \cdot I^-) \sim = I \oplus (I^- \cdot K \oplus I^- \cdot J) \\
= (I \oplus I^- \cdot K) \oplus I^- \cdot J \\
= (I + K) \oplus I^- \cdot J \\
= K \oplus I^- \cdot J
\]

Similarly,
\[
(I^- \cdot (J \cdot I^-) \sim \cdot (K \cdot I^-) \sim) \sim = (K \cdot I^- \oplus J \cdot I^-) \oplus I \\
= K \cdot I^- \oplus (J \cdot I^- \oplus I) \\
= K \cdot I^- \oplus (I + J) \\
= K \cdot I^- \oplus J \\
= K \oplus I^- \oplus J
\]
Now, the conclusion follows from Proposition 1.1(ii).

Thus, \( R/I \) is a GLR. \( \square \)

The following result provides examples of non-unitary GLRs.

**Proposition 3.6.** The direct sum of GLRs is again a GLR.

**Proof.** Let \( (R_\lambda)_{\lambda \in \Lambda} \) be a family of GLRs, and let \( R := \bigoplus_{\lambda \in \Lambda} R_\lambda \). For each \( \lambda \in \Lambda \), let \( p_\lambda \) denotes the natural projection from \( \prod_{\lambda \in \Lambda} R_\lambda \) onto \( R_\lambda \). For every subset \( S \) of \( \prod_{\lambda \in \Lambda} R_\lambda \), and \( \bigoplus_{\lambda \in \Lambda} R_\lambda \), \( p_\lambda(S) \) will be denoted by \( S_\lambda \).

As in the proof of the commutative case [1, Prop. 3.12], one shows that every ideal \( I \) of \( R \) is of the form \( I = \bigoplus_{\lambda \in \Lambda} I_\lambda \), where \( I_\lambda \subseteq R_\lambda \). Moreover, one verifies that \( I^- = \bigoplus_{\lambda \in \Lambda} I^-_\lambda \) and \( I^- = \bigoplus_{\lambda \in \Lambda} I^-_\lambda \).

Since each \( R_\lambda \) satisfies \((\star)\), it follows that \( R \) does as well. It remains to show that \( R \) satisfies GLR-1 and GLR-2.

Let \( I, J \) be two ideals of \( R \).

GLR-1: This is easily adaptable from the proof from [1, Prop. 3.12].

GLR-2: Observe that \( R_\mu \cdot R_\lambda = 0 \) whenever \( \mu \neq \lambda \). Hence,

\[
(J^- \cdot I^-)^- = \bigoplus_{\mu, \lambda \in \Lambda} J^-_\mu \cdot I^-_\lambda = \bigoplus_{\mu, \lambda \in \Lambda} (J^-_\mu \cdot I^-_\lambda)^-
\].

A similar calculation shows that \((J^- \cdot I^-)^- = \bigoplus_{\lambda \in \Lambda} (J^-_\lambda \cdot I^-_\lambda)^- \). Therefore, since \((J^- \cdot I^-)^- = (J^- \cdot I^-)^- \) for all \( \lambda \), then we obtained \((J^- \cdot I^-)^- = (J^- \cdot I^-)^- \).

Thus, \( R \) is a GLR as claimed. \( \square \)

4. **Subrings of generalized Łukasiewicz rings**

Let \( R \) be a GLR and \( M \) a subring (with or without a unity). For every ideal \( I \) of \( R \), let

\[
I^- M = \{ x \in M : Ix = 0 \} \text{ and } I^- M = \{ x \in M : xI = 0 \}
\]

Observe that \( I^- M = M \cap I^- \) and \( I^- M = M \cap I^- \).

Recall that \( A(R) = \langle \text{Id}(R), \oplus, \odot, -, \sim, 0, 1 \rangle \) is a pseudo MV-algebra with underline lattice \( \langle \text{Id}(R), +, \cap, 0, 1 \rangle \).

**Proposition 4.1.** Let \( R \) be a GLR and \( M \) an ideal of \( R \) such that \( M \cap M^- = 0 \) and \( M \cap M^- = 0 \). Then \( M \) is a GLR.
Proof. First, observe that since $M \cap M^– = 0$ and $M \cap M^– = 0$, then $M + M^– = M \vee M^– = M \oplus M^– = R$ and $M + M^– = R$. It follows that any ideal of $M$ is an ideal of $R$, i.e., $\text{Id}(M) \subseteq \text{Id}(R)$.

Let $x \in M$, then as $R$ satisfies $(\star)$, there are $r, r' \in R$ such that $xr = r'x = x$. Since $M + M^– = R$ and $M + M^– = R$, there are $m_1, m_2 \in M$, $x_1 \in M^–, x_2 \in M^–$ such that $m_1 + x_1 = r$ and $m_2 + x_2 = r'$. It follows that $xm_1 = m_2x = x$.

It remains to prove GLR-1 and GLR-2.

GLR-1: Let $I, J$ be ideals of $M$.

$((I^\sim \cdot J)^\sim \cdot (I^\sim \cdot J^\sim)^\sim)^\sim = (((I^\sim \cap M) \cdot J)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((((I^\sim \cap M) \cdot J)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((I^\sim \cap M)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((((I^\sim \cap M)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((I^\sim \cap M)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (I + M^–) \cap (I + M^–) \cap M$,

$= I + (I + M^–) \cap M$,

$= I + J$.

A similar calculation shows that $(I^\sim \cdot J \cdot (I^\sim M)^\sim)^\sim = I + J$.

GLR-2: Let $I, J$ be ideals of $M$, then $(I^\sim \cdot J^\sim)^\sim = ((I^\sim \cap M) \cdot (J^\sim \cap M)) \cap M$,

$= (((((I^\sim \cap M) \cdot J)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((((I^\sim \cap M)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((((I^\sim \cap M)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (((((I^\sim \cap M)^\sim \cap (I^\sim \cap M)) \cap (M \cdot (I^\sim \cap M)) \cap M$,

$= (I + M^–) \cap (I + M^–) \cap M$,

$= I + J$.

A similar computation shows that $(I^\sim \cdot J^\sim \cdot (J^\sim M)^\sim)^\sim = J \oplus I$.

Thus, $(I^\sim \cdot J^\sim \cdot (J^\sim M)^\sim)^\sim = (I^\sim \cdot J^\sim \cdot (J^\sim M)^\sim)^\sim = J \oplus I$, and GLR-2 is proved.

Hence, $M$ is a GLR as claimed. 

One should observe that since $\text{Id}(M) \subseteq \text{Id}(R)$, the preceding proof (GLR-2) shows that in $A(M), I \oplus_M J = I \oplus J$.

5. BL-rings

In the previous sections, we treated a noncommutative generalization of Lukasiewicz rings. In this section we introduce a commutative generalization of Lukasiewicz rings.
Definition 5.1. A commutative ring $R$ is called a BL-ring if for all ideals $I, J$ of $R$, BLR-1: $I \cap J = I \cdot (I \to J)$ and
BLR-2: $(I \to J) + (J \to I) = R$

Note that BLR-1 is equivalent to $I \cap J \subseteq I \cdot (I \to J)$ since the inclusion $I \cdot (I \to J) \subseteq I \cap J$ holds in any ring.

Example 5.2. 1. $\mathbb{Z}$ is a BL-ring.
2. Every DVR is a BL-ring. Indeed, the ideals of a DVR form a chain, and the axioms BLR-1 and BLR-2 are straightforward.
3. Every Łukasiewicz ring is a BL-ring. Indeed, if $R$ is a Łukasiewicz ring, then $A(R)$ is an MV-algebra. Thus $A(R)$ is a BL-algebra, and the axioms BLR-1 and BLR-2 follow.

Recall [5] that a commutative ring is called a multiplication ring if every ideals $I, J$ of $R$ such that $I \subseteq J$, there exists an ideal $K$ of $R$ such that $I = J \cdot K$.

Theorem 5.3. A commutative ring is a BL-ring if and only if it is a multiplication ring.

Proof. Suppose that $R$ is a BL-ring and let $I, J$ be ideals of $R$ such that $I \subseteq J$. Then by BLR-1, $I = I \cap J = I \cdot (I \to J)$. Take $K = I \to J$.

Conversely, suppose that $R$ is a multiplication ring.

BLR-1: Let $I, J$ be ideals of $R$. Then since $I \cap J \subseteq I$, there exists an ideal $K$ of $R$ such that $I \cap J = I \cdot K$. Hence, $I \cdot K \subseteq J$ and it follows that $K \subseteq I \to J$. Thus, $I \cap J \subseteq I \cdot (I \to J)$. As observed above, the inclusion $I \cdot (I \to J) \subseteq I \cap J$ holds in any ring.

BLR-2: Need to prove or disprove. $\square$

Corollary 5.4. 1. Every BL-ring satisfies

$(\star')$ for every $x \in R$, there exist $e = e^2 \in R$ such that $xe = x$

2. A commutative ring is a BL-ring if and only if $L(R) := (\text{Id}(R), \wedge, \vee, \ominus, \to, \sim, \{0\}, R)$ is a BL-algebra.

Proof. 1. BLR-1 implies that $R$ is a multiplication ring, and it is known that every multiplication ring satisfies $(\star')$ [5, Cor. 7].

2. Follows from 1. and the axioms BLR-1/2. $\square$

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