Abstract. We compare the weight and stable rank filtrations of algebraic $K$-theory, and relate the Beilinson–Soulé vanishing conjecture to the author’s connectivity conjecture.

1. Introduction

Let $F$ be a field. For $j \geq 0$ let $K_j(F)$ be the (higher) algebraic $K$-groups of $F$ [Quillen (1970)].

There is a decreasing weight filtration

$$K_j(F) \supset F^1K_j(F) \supset \cdots \supset F^wK_j(F) \supset \cdots \supset F^jK_j(F) \supset 0$$

associated to the $\lambda$- and Adams-operations on $K_j(F)$ [Grothendieck, Quillen/Hiller (1981)].

There is also an increasing stable rank filtration

$$0 \subset F_1K_j(F) \subset \cdots \subset F_rK_j(F) \subset \cdots \subset F_jK_j(F) \subset K_j(F)$$

given by the image filtration

$$F_rK_j(F) = \text{im}(\pi_rF_jK(F) \to \pi_jK(F))$$

associated to a sequence of spectra

$$* \to F_1K(F) \to \cdots \to F_rK(F) \to \cdots \to K(F)$$

called the spectrum level rank filtration [Rognes (1992)].

Conjecture 1.1 (Beilinson–Soulé).

$$F^wK_j(F) = K_j(F)$$

for $2w \leq j + 1$.

A stronger form asserts this equality also for $2w = j + 2$ when $j > 0$. The conjecture is known with finite coefficients, so it suffices to verify it rationally, i.e., after tensoring over $\mathbb{Z}$ with $\mathbb{Q}$. We write $A_\mathbb{Q} = A \otimes_{\mathbb{Z}} \mathbb{Q}$.

Conjecture 1.2 (Connectivity).

$$F_rK_j(F) = K_j(F)$$

for $2r \geq j + 1$.

A stronger form asserts rational equality also for $2r = j$ when $j > 0$.

Conjecture 1.3 (Stable Rank).

$$F^wK_j(F)_{\mathbb{Q}} = F_rK_j(F)_{\mathbb{Q}}$$

for $w + r = j + 1$.

I will provide evidence for the connectivity and stable rank conjectures, which, if true, will imply the Beilinson–Soulé vanishing conjecture.
2. Algebraic $K$-theory

Let $\mathcal{P}(F)$ be the category of finitely generated projective $F$-modules, i.e., the category of finite-dimensional $F$-vector spaces.

The dimension $\dim V$ of an object $V$ defines an additive invariant in $K_0(F) \cong \mathbb{Z}$.

The determinant $\det A$ of an automorphism $A : V \to V$ defines an additive invariant in $K_1(F) \cong F^\times$.

The classifying space $|i\mathcal{P}(F)|$ of the subcategory $i\mathcal{P}(F)$ of isomorphisms in $\mathcal{P}(F)$ is built with one $q$-dimensional diagram $\Delta^q$ for each chain

$$V_0 \cong V_1 \cong \ldots \cong V_q$$

of $q$ composable morphisms in $i\mathcal{P}(F)$. The inclusion of the full subcategory generated by the objects $F^r$, with automorphism groups $GL_r(F)$, induces an equivalence

$$\prod_{r \geq 0} BGL_r(F) \simeq |i\mathcal{P}(F)|.$$

Direct sum of vector spaces, $(V, W) \mapsto V \oplus W$, makes $|i\mathcal{P}(F)|$ a (coherently homotopy commutative) topological monoid. To group complete $K$, i.e., a (infinite) loop space $\Omega K$, we map $K(F)_0 = |i\mathcal{P}(F)|$ to a grouplike (coherently homotopy commutative) topological monoid, i.e., a (infinite) loop space $\Omega K(F)_1 = \Omega |iS_\ast \mathcal{P}(F)|$, to be defined below. The map $K(F)_0 \to \Omega K(F)_1$ is a group completion.

**Definition 2.1.** $K_j(F) = \pi_j \Omega K(F)_1 = \pi_{j+1} K(F)_1$, where $K(F)_1 = |iS_\ast \mathcal{P}(F)|$ is given by Waldhausen’s $S_\ast$-construction.

3. The algebraic $K$-theory spectrum

Waldhausen’s $S_\ast$-construction applied to $\mathcal{P}(F)$ is a simplicial category

$$[q] \mapsto iS_q \mathcal{P}(F).$$

The category in degree $q$ has objects the sequences of injective homomorphisms

$$0 = V_0 \hookrightarrow V_1 \hookrightarrow \ldots \hookrightarrow V_q$$

in $\mathcal{P}(F)$, together with compatible choices of quotients $V_j/V_i$ for $0 \leq i \leq j \leq q$. Morphisms are vertical isomorphisms

$$0 = V_0 \cong V_1 \cong \ldots \cong V_q$$

of horizontal diagrams.

The construction can be iterated $n \geq 1$ times. We define

$$K(F)_n = |iS^{(n)}_\ast \mathcal{P}(F)|$$

as the classifying space of the simplicial category

$$[q] \mapsto iS_q^{(n)} \mathcal{P}(F)$$

with objects in degree $q$ given by $n$-dimensional cubical diagrams $[q]^n \to \mathcal{P}(F)$. In the case $n = 2$
we require to have injective homomorphisms \( V_{i-1,j} \to V_{i,j} \) and \( V_{i,j-1} \to V_{i,j} \) and injective pushout homomorphisms

\[
V_{i-1,j} \oplus V_{i-1,j-1} \to V_{i,j},
\]

for all \( 1 \leq i, j \leq q \). For higher \( n \) there are similar conditions for \( d \)-dimensional subcubes for all \( 1 \leq d \leq n \).

**Definition 3.1.** The algebraic \( K \)-theory spectrum of \( \mathbb{F} \) is the spectrum

\[
\mathbf{K}(\mathbb{F}) = \{ n \mapsto K(\mathbb{F})_n = [\mathbb{S}^n_\bullet, \mathcal{P}(\mathbb{F})] \}.
\]

It is positive fibrant, in the sense that \( K(\mathbb{F})_n \to \Omega K(\mathbb{F})_{n+1} \) is an equivalence for each \( n \geq 1 \). Hence

\[
K_j(\mathbb{F}) = \pi_j K(\mathbb{F}) = \pi_{j+n} K(\mathbb{F})_n
\]

for each \( n \geq 1 \).

This construction produces a symmetric spectrum: the group \( \Sigma_n \) permutes the order of the \( n \) instances of the \( S_\bullet \)-construction.

## 4. Weight filtration

Let \( k \geq 0 \). The \( k \)-th exterior power \( V \mapsto \Lambda^k V \) induces \( \lambda \)-operations

\[
\lambda^k: K_j(\mathbb{F}) \to K_j(\mathbb{F})
\]

For \( V = L_1 \oplus \cdots \oplus L_r \) a direct sum of lines,

\[
\Lambda^k V = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq r} L_{i_1} \otimes \cdots \otimes L_{i_k}
\]

corresponds to the \( k \)-th elementary symmetric polynomial

\[
\sigma_k(x_1, \ldots, x_r) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} x_{i_1} \cdots x_{i_k}.
\]

The \( k \)-th Adams operation

\[
\psi^k: K_j(\mathbb{F}) \to K_j(\mathbb{F})
\]

is induced by \( L_1 \oplus \cdots \oplus L_r \mapsto L_1^{\otimes k} \oplus \cdots \oplus L_r^{\otimes k} \) and corresponds to the \( k \)-th power sum polynomial

\[
s_k(x_1, \ldots, x_r) = \sum_{i=1}^r x_i^k.
\]

It can thus be expressed in terms of the \( \lambda \)-operations.

The weight filtration \( \{ F^w K_j(\mathbb{F}) \}_{w \geq 0} \) on \( K_j(\mathbb{F}) \) is constructed by means of the \( \lambda^k \). The Adams operations satisfy

\[
\psi^k(x) \equiv k^w x \mod F^{w-1} K_j(\mathbb{F})
\]

for \( x \in F^w K_j(\mathbb{F}) \). Hence \( \psi^k \) acts as multiplication by \( k^w \) on \( F^w K_j(\mathbb{F})/F^{w+1} K_j(\mathbb{F}) \), for all \( k \geq 0 \).

Rationally the weight filtration splits as a direct sum of common eigenspaces for the Adams operations. Let

\[
K_j(\mathbb{F})_Q^{(w)} = \{ x \in K_j(\mathbb{F})_Q \mid \psi^k(x) = k^w x \text{ for all } k \}
\]

be the weight \( w \) rational eigenspace. Then

\[
F^w K_j(\mathbb{F})_Q = \bigoplus_{v \geq w} K_j(\mathbb{F})_Q^{(v)}
\]

is the subspace of weights \( \geq w \), and

\[
\frac{F^w K_j(\mathbb{F})_Q}{F^{w+1} K_j(\mathbb{F})_Q} \cong K_j(\mathbb{F})_Q^{(w)}.
\]

Soule proved that \( K_j(\mathbb{F})_Q^{(w)} = 0 \) for \( w < 0 \) and for \( w > j \), i.e., \( K_j(\mathbb{F})_Q \) only contains classes of weight \( 0 \leq w \leq j \).
Remark 5.2. By the Steinberg relation $\{ u, 1 - u \} \in K_2(F)$, these all map to $K_j(F)$, and land in the weight $j$ eigenspace.

5. Motivic cohomology

By analogy with the Atiyah–Hirzebruch spectral sequence from singular cohomology to topological $K$-theory for a topological space, there is a motivic spectral sequence

$$E^2_{s,t}(mot) = H^{t-s}_{mot}(F; Z(t)) \Rightarrow K_{s+t}(F).$$

It is of homological type, concentrated in the first quadrant ($s \geq 0$ and $t \geq 0$), and collapses rationally at the $E^2$-term ($d^r = 0$ after rationalization for $r \geq 2$).

| \( t \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 3 \) | \( s \) |
|---|---|---|---|---|---|
| \( 4 \) | \( H^4_{mot}(F; Z(4)) \) | \( H^3_{mot}(F; Z(4)) \) | \( H^2_{mot}(F; Z(4)) \) | \( H^1_{mot}(F; Z(4)) \) | \( \ldots \) |
| \( 3 \) | \( H^3_{mot}(F; Z(3)) \) | \( H^2_{mot}(F; Z(3)) \) | \( H^1_{mot}(F; Z(3)) \) | \( H^0_{mot}(F; Z(3)) \) | \( \ldots \) |
| \( 2 \) | \( H^2_{mot}(F; Z(2)) \) | \( H^1_{mot}(F; Z(2)) \) | \( H^0_{mot}(F; Z(2)) \) | \( H^{-1}_{mot}(F; Z(2)) \) | \( \ldots \) |
| \( 1 \) | \( H^1_{mot}(F; Z(1)) \) | \( H^0_{mot}(F; Z(1)) \) | \( H^{-1}_{mot}(F; Z(1)) \) | \( H^{-2}_{mot}(F; Z(1)) \) | \( \ldots \) |
| \( 0 \) | \( H^0_{mot}(F; Z(0)) \) | \( H^{-1}_{mot}(F; Z(0)) \) | \( H^{-2}_{mot}(F; Z(0)) \) | \( H^{-3}_{mot}(F; Z(0)) \) | \( \ldots \) |

Rationally, the motivic cohomology groups can be defined in terms of the weight filtration.

Definition 5.1.

$$H^{t-s}_{mot}(F; Z(t)) \mid \mathbb{Q} = H^{t-s}_{mot}(F; \mathbb{Q}(t)) = K_{s+t}(F)^{(t)} \mathbb{Q}$$

so that

$$H^i_{mot}(F; \mathbb{Q}(w)) = K_{2w-i}(F)^{(w)} \mathbb{Q}.$$
6. Mixed Motives

It is expected that there exists a category $MM$ of mixed motives, such that the motivic cohomology groups are given by the Ext-groups

$$H^i_{mot}(F; \mathbb{Q}(w)) \cong \text{Ext}^i_{MM}(\mathbb{Q}, \mathbb{Q}(w))$$

classifying $i$-fold extensions from (the pure motive associated to) $\mathbb{Q}(w)$ to $\mathbb{Q}$ in this category.

If this is true, $H^i_{mot}(F; \mathbb{Q}(w)) = 0$ for $i < 0$, which is equivalent to $K_{2w-i}(F)_{\mathbb{Q}}(w) = 0$ for $i < 0$, hence also to $K_j(F)_{\mathbb{Q}}(w) = 0$ for $2w < j$.

Furthermore, $H^0_{mot}(F; \mathbb{Q}(w)) \cong \text{Hom}_{MM}(\mathbb{Q}, \mathbb{Q}(w)) = 0$ for $w > 0$, so $K_{2w}(F)_{\mathbb{Q}}(w) = 0$ for $w > 0$, which means that $K_j(F)_{\mathbb{Q}}(w) = 0$ for $2w < j$ when $w > 0$.

These assertions are the content of the Beilinson–Soulé vanishing conjecture.

**Conjecture 6.1** (Beilinson–Soulé). $K_j(F)_{\mathbb{Q}}(w) = 0$ for $w < j/2$ (and for $w \leq j/2$ when $j > 0$).

This is equivalent to the rational version of the conjecture as first stated.

7. Higher Chow Groups

A construction of integral motivic cohomology groups is given by Bloch’s higher Chow groups.

**Definition 7.1.** For each $q \geq 0$ let

$$\Delta^q_p = \text{Spec} \mathbb{F}[x_0, \ldots, x_q]/(x_0 + \cdots + x_q = 1)$$

be the affine $q$-simplex over $\mathbb{F}$. It is isomorphic to $\mathbb{A}^q = \text{Spec} \mathbb{F}[x_1, \ldots, x_q]$, but the $\Delta^q_p$ combine more naturally to a precosimplicial variety:

$$\Delta^q_0 \xrightarrow{d_0} \Delta^q_1 \xrightarrow{d_1} \Delta^q_2 \xrightarrow{d_2} \cdots$$

Let

$$z^p(\mathbb{F}, q) = \{ \text{codimension } p \text{ cycles } V \subset \Delta^q_p \text{ meeting each face } \Delta^q_z \to \Delta^q_y \text{ transversely} \}.$$

A cycle is an integral sum of irreducible subvarieties. Pullback of $V$ along the cofaces $d_i: \Delta^q_{p-1} \to \Delta^q_p$ defines face operators $d_i: z^p(\mathbb{F}, q) \to z^p(\mathbb{F}, q)$ that assemble to a precosimplicial abelian group

$$z^p(\mathbb{F}, 0) \xrightarrow{d_0} z^p(\mathbb{F}, 1) \xrightarrow{d_1} z^p(\mathbb{F}, 2) \xrightarrow{d_2} \cdots$$

There is an associated chain complex $(z^p(\mathbb{F}, *), \partial)$

$$0 \leftarrow z^p(\mathbb{F}, 0) \xleftarrow{\partial_1} z^p(\mathbb{F}, 1) \xleftarrow{\partial_2} z^p(\mathbb{F}, 2) \xleftarrow{\partial_3} \cdots$$

with $\partial_1 = d_0 - d_1$, $\partial_2 = d_0 - d_1 + d_2$, etc.

Bloch’s higher Chow groups are the homology groups

$$CH^p(\mathbb{F}, q) = \ker \partial_q/\text{im} \partial_{q+1} = H_q(z^p(\mathbb{F}, *), \partial)$$

of this chain complex.

(In what generality is $CH^p(\mathbb{F}, 0) = CH^p(\mathbb{F})$?)

**Definition 7.2.** Integral motivic cohomology groups can be defined as

$$H^i_{mot}(\mathbb{F}; \mathbb{Z}(w)) = CH^w(\mathbb{F}, 2w - i).$$

**Remark 7.3.** These give an integral motivic spectral sequence converging to $K_*(\mathbb{F})$.

Rationally they agree with the weight eigenspace definition of rational motivic cohomology.

There are no codimension $p$ subvarieties in $\Delta^q_p$ for $q < p$, so $z^p(\mathbb{F}, q) = 0$ and $CH^p(\mathbb{F}, q) = 0$ for $q < p$. Hence $H^i_{mot}(\mathbb{F}; \mathbb{Z}(w)) = 0$ for $2w - i < w$, i.e., for $i > w$.

This definition does not tell us whether $H^i_{mot} = 0$ for $i < 0$, or equivalently, if $CH^p(\mathbb{F}, q) = 0$ for $2p < q$. 
Theorem 7.4 (Suslin). With finite coefficients,
\[ \text{CH}^p(\mathbb{F}; q; \mathbb{Z}/m) \cong H^{2p-q}_\text{mot}(\mathbb{F}; \mathbb{Z}/m(p)) \]
when \( q \geq p \), i.e., when \( 2p-q \leq p \). In particular, \( \text{CH}^p(\mathbb{F}; q; \mathbb{Z}/m) = 0 \) for \( 2p < q \), so \( H^i_{\text{mot}}(\mathbb{F}, \mathbb{Z}/m(w)) = 0 \) for \( i < 0 \).

Conjecture 7.5 (Beilinson/Lichtenbaum). There are complexes (of Zariski/étale sheaves)
\[ \ldots \leftarrow \Gamma(w, \mathbb{F}^\ast)^w \leftarrow \ldots \leftarrow \Gamma(w, \mathbb{F}^0) \leftarrow \ldots \]
with cohomology calculating motivic cohomology
\[ H^i(\Gamma(w, \mathbb{F}^r)^r, \delta) \cong H^i_{\text{mot}}(\mathbb{F}, \mathbb{Z}(w)). \]

Remark 7.6. We can let \( \Gamma(0, \mathbb{F}) = \mathbb{Z} \) and \( \Gamma(1, \mathbb{F}) = \mathbb{F}^\times \) (in cohomological degree 1). Lichtenbaum has a proposed complex \( \Gamma(2, r) \). Goncharov has proposed complexes \( \Gamma_{\text{pol}}(w, \mathbb{F}) \) associated to polylogarithms, i.e., functions like
\[ L_{i_1}(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^{i_1}}. \]
Here \( L_{i_1}(z) = -\ln(1 - z) \).

8. Quillen’s rank filtration

Recall that \( K(\mathbb{F})_1 = [i \mathcal{S}_r \mathcal{P}(\mathbb{F})] \) is the classifying space of the simplicial category with \( q \)-simplices
\[ \sigma : 0 \to V_0 \to V_1 \to \ldots \to V_q. \]

Definition 8.1. Let \( F_r K(\mathbb{F}) \subset K(\mathbb{F})_1 \) be the subspace consisting of simplices where \( \dim V_q \leq r \) (so that \( \dim V_i \leq r \) for all \( i \)).
\[ \ast = F_0 K(\mathbb{F}) \subset F_1 K(\mathbb{F}) \subset F_2 K(\mathbb{F}) \subset \ldots \subset K(\mathbb{F})_1. \]

Proposition 8.2.
\[ F_r K(\mathbb{F}) \subset K(\mathbb{F})_1 \cong \Sigma^2 B(\mathbb{F}^r)_{\text{mot}(\mathbb{F})} = \text{EGL}_r(\mathbb{F})^+ \wedge_{\text{GL}_r(\mathbb{F})} \Sigma^2 B(\mathbb{F}^r) \]
is the (based) homotopy orbit space for \( \text{GL}_r(\mathbb{F}) \) acting on the double suspension of the Tits building \( B(\mathbb{F}^r) \).

Sketch proof. The (non-basepoint) \( q \)-simplices of \( F_r K(\mathbb{F}) \subset K(\mathbb{F})_1 \) are generated from the objects
\[ \sigma : 0 = V_0 \hookrightarrow V_1 \hookrightarrow \ldots \hookrightarrow V_q \]
with \( \dim V_q = r \), together with the \( \text{GL}_r(\mathbb{F}) \)-action on the latter.

This is equivalent to the \( \text{GL}_r(\mathbb{F}) \)-homotopy orbits of the subspace with \( q \)-simplices
\[ \sigma : 0 = V_0 \subset V_1 \subset \ldots \subset V_q = \mathbb{F}^r. \]
The \( i \)-th face operator deletes \( V_i \). The 0-th and \( q \)-th face operators map to the base point if \( 0 \neq V_1 \) or \( V_{q-1} \neq \mathbb{F}^r \), respectively.

This is equivalent to \( \Sigma^2 \) of the simplicial set with \( (g-2) \)-simplices the chains
\[ 0 \subset V_1 \subset \ldots \subset V_{q-1} \subset \mathbb{F}^r, \]
which is the nerve of the set of proper, nontrivial subspaces \( V \subset \mathbb{F}^r \), partially ordered by inclusion, i.e., the Tits building \( B(\mathbb{F}^r) \). An element \( A \in GL_r(\mathbb{F}) \) acts on the partially ordered set by mapping \( V \) to \( A(V) \), and has the induced action on \( B(\mathbb{F}^r) \).

Example 8.3. \( \Sigma^2 B(\mathbb{F}^1) \cong \Delta^1 / \partial \Delta^1 \cong S^1 \).

Theorem 8.4 (Solomon–Tits).
\[ B(\mathbb{F}^r) \simeq \bigvee_{\alpha} S^{r-2}. \]

Definition 8.5.
\[ \text{St}_r(\mathbb{F}) = \tilde{H}_{r-2}^{\text{mot}}(\mathbb{F}^r) \cong \tilde{H}_{r} \Sigma^2 B(\mathbb{F}^r) \cong \bigoplus_{\alpha} \mathbb{Z} \]
is the Steinberg representation of \( GL_r(\mathbb{F}) \).
Corollary 8.6. The homology
\[ \tilde{H}_*(F_r K_1(F)/F_{r-1} K_1(F)) \cong \tilde{H}_*(\Sigma^2 B(GL_r(F))) \cong H_{r+r}^p(GL_r(F); \text{St}_r(F)) \]
is concentrated in degrees \( \ast \geq r \). Hence
\[ H_{j+1}(F_r K_1(F)) \rightarrow H_{j+1} K_1(F) \]
is surjective for \( j + 1 = r \), and an isomorphism for \( j + 1 < r \). Thus
\[ F_r H_{j+1} K_1(F) = \text{im}(H_{j+1}(F_r K_1(F)) \rightarrow H_{j+1} K_1(F)) \]
is equal to \( H_{j+1} K_1(F) \) for \( r \geq j + 1 \).

This enters in the proof of the following theorem.

**Theorem 8.7** (Quillen). Let \( \mathcal{O}_F \) be the ring of integers in a number field \( F \). For each \( j \geq 0 \) the group \( K_j(\mathcal{O}_F) \) is finitely generated.

The connectivity conjecture asserts a stronger convergence result, namely \( F_r K_j(F) = K_j(F) \) for \( 2r \geq j + 1 \), but for the more powerful stable rank filtration.

9. **The spectrum level rank filtration**

Also recall that \( K(F) = \{ n \mapsto K(F)_n = [iS^m_i \mathcal{P}(F)] \} \) where \( iS^m_i \mathcal{P}(F) \) has \( q \)-simplices the category with objects
\[ \sigma : [q]^n \rightarrow \mathcal{P}(F) \]
\[ (i_1, \ldots, i_n) \mapsto V_{i_1, \ldots, i_n} \]

plus choices of subquotients, subject to lists of conditions.

**Definition 9.1** (Rognes (1992)). Let \( F_r K(F)_n \subset K(F)_n \) be the subspace where \( \text{dim} V_{i_1, \ldots, i_n} \leq r \) (so that \( \text{dim} V_{i_1, \ldots, i_n} \leq r \) for all \( (i_1, \ldots, i_n) \)). Let
\[ F_r K(F)_n = \{ n \mapsto F_r K(F)_n \} \]
be the associated (pre-)spectrum. The sequence
\[ * \mapsto F_1 K(F) \mapsto \ldots \mapsto F_{r-1} K(F) \mapsto F_r K(F) \mapsto \ldots \mapsto K(F) \]
is the spectrum level rank filtration.

Recall that \( \pi_j X = \text{colim}_n \pi_{j+n} X_n \) for a prespectrum \( X = \{ n \mapsto X_n \} \).

**Definition 9.2.** Let
\[ F_r K_j(F) = \text{im}(\pi_j F_r K(F) \rightarrow \pi_j K(F)) \]
so that
\[ 0 \subset F_1 K_j(F) \subset \cdots \subset F_r K_j(F) \subset \cdots \subset K_j(F) \].

This is the stable rank filtration.

**Proposition 9.3.**
\[ F_r K(F)/F_{r-1} K(F) \simeq D(GL_r(F))_{hGL_r(F)} = EGL_r(F) \wedge_{GL_r(F)} D(F) \]
is the homotopy orbit spectrum for \( GL_r(F) \) acting on the stable building \( D(F) \).

**Sketch proof.** At level \( n \), \( F_r K(F)_n/F_{r-1} K(F)_n \) realizes a simplicial category with \( q \)-simplices diagrams
\[ \sigma : (i_1, \ldots, i_n) \mapsto V_{i_1, \ldots, i_n} \]
with \( \text{dim} V_{i_1, \ldots, i_n} = r \). It is equivalent to the subcategory where \( V_{i_1, \ldots, i_n} = F^r \) and each \( V_{i_1, \ldots, i_n} \) is a subspace of \( F^r \), with morphisms given by the \( GL_r(F) \)-action on \( F^r \) and its subspaces. \( \Box \)
Definition 9.4. We define $D(F^r) = \{n \mapsto D(F^r)_n\}$ by letting $D(F^r)_n$ be a simplicial set with $q$-simplices diagrams $\sigma: [q]^n \to \text{Sub}(\mathbb{F}^r) \subset \mathcal{P}(\mathbb{F})$ consisting of subspaces $V_{i_1},...,i_n$ of $\mathbb{F}^r$ and inclusions between these. The case $n = 2$ appears as follows:

$$
\begin{array}{cccc}
0 & = & 0 & = \ldots = 0 \\
\| & \mid & \mid & \\
0 & \subset & V_{1,1} & \subset \ldots \subset V_{1,q} \\
\| & \mid & \mid & \\
\vdots & \vdots & \ddots & \vdots \\
\| & \mid & \mid & \\
0 & \subset & V_{q,1} & \subset \ldots \subset V_{q,q}
\end{array}
$$

with $\sigma: (i,j) \mapsto V_{i,j}$. In general we require (0) that $V_{1,...,i_n} = 0$ if some $i_s = 0$, and $V_{q,...,q} = \mathbb{F}^r$, (1) that $V_{i_1,...,i_{s-1},...,i_n} \subset V_{i_1,...,i_s,...,i_n}$ is an inclusion, (2) that the pushout morphism

$$
V_{i_1,...,i_{s-1},...,i_n} \oplus V_{i_1,...,i_s,...,i_{n-1}} \to V_{i_1,...,i_n}
$$

is injective, etc. (to $(n)$). We call these the lattice conditions.

Example 9.5. $D(F^1) \cong S$ (the sphere spectrum), so $F_1K(F) \simeq S_{hGL_1(F)} = \Sigma^\infty(BF^\infty)_+$. Rationally, $\pi_jF_1K(F) \cong \pi_j^r(BF^\infty)$ is isomorphic to $H_j(BF^\infty) = H_j^{BP}(F^\infty)$, which is also rationally isomorphic to $\Lambda^jF^\infty$. Hence $F_1K_j(F) \subset K_j(F)$ agrees rationally with the image of Milnor $K$-theory:

$$
F_1K_j(F)_q = K^j_1(F)_q
$$

as subgroups of $K_j(F)_q$.

10. THE COMPONENT FILTRATION

To analyze the stable building $D(F^r)$ we associate some invariants to the simplices $\sigma: [q]^n \to \text{Sub}(\mathbb{F}^r)$.

Definition 10.1. The rank jump at $\vec{p} = (i_1,\ldots,i_n) \in [q]^n$ is the dimension of the cokernel of the $n$-cube pushout morphism to $V_{i_1,...,i_n}$, i.e., the alternating sum

$$
\sum_{\epsilon_1,...,\epsilon_n \in \{0,1\}} (-1)^{i_1+\cdots+i_n} \dim V_{i_1-\epsilon_1,...,i_n-\epsilon_n}.
$$

It is non-negative by the lattice conditions, and the sum over all $\vec{p}$ of the rank jumps is $r = \dim V_{q,...,q}$. Hence there are $r$ distinguished points $\vec{p}_1,...,\vec{p}_r \in [q]^n$, counted with multiplicities, where the rank jumps are positive.

(The ordering of $\vec{p}_1,...,\vec{p}_r$ is not well-defined.)

A preordering is a reflexive and transitive relation. It amounts to a small category with at most one morphism from $i$ to $j$ for each pair of objects $(i,j)$.

Definition 10.2. The $r$ distinguished points $\vec{p}_1,...,\vec{p}_r$ inherit a preordering from the product partial ordering on $[q]^n$. Let the path component count of $\sigma$, denoted $c(\sigma)$, be the number of path components of (the classifying space of the category associated to) this preordering. Clearly $1 \leq c(\sigma) \leq r$.

Face operators in $D(F^r)_n$ may merge distinguished points, which in turn may reduce the path component count.

Definition 10.3. Let $F_cD(F^r)_n \subset D(F^r)_n$ be the simplicial subset consisting of simplices $\sigma$ with path component count $c(\sigma) \leq c$. Let $F_cD(F^r) = \{n \mapsto F_cD(F^r)_n\}$ be the associated (pre-)spectrum. The sequence

$$
\ast \mapsto F_1D(F^r) \mapsto \ldots \mapsto F_{c-1}D(F^r) \mapsto F_cD(F^r) \mapsto \ldots \mapsto F_D(F^r) = D(F^r)
$$

is the component filtration of the stable building $D(F^r)$.

Example 10.4. $F_1D(F^r) \simeq \Sigma^\infty \Sigma B(F^r) \simeq V_nS^{r-1}$.
Theorem 10.5.

\[ F_\beta D(\mathcal{F}^r)/F_{c-1}D(\mathcal{F}^r) \cong \bigvee_{\beta} S^{r+c-2} \]

for \(1 \leq c \leq r\).

Sketch proof. There is a finer filtration of \(D(\mathcal{F}^r)\) (than the component filtration) given by restricting the (isomorphism classes of) preorders on \(\{1, \ldots, r\}\) given by setting \(s \leq t\) if \(\vec{p}_s \leq \vec{p}_t\). The filtration subquotients of this preorder filtration can be completely analyzed, in terms of configuration spaces and smash products of Tits buildings. The preorders that are not componentwise (pre-)linear contribute stably trivial filtration subquotients. The stable homology of configuration spaces contributes Lie representations, and the smash products of Tits buildings contribute tensor products of Steinberg representations. See [Rognes (1992)] for details.

\[ \square \]

Hence \(H_*D(\mathcal{F}^r)\) is the homology of a free chain complex

\[ 0 \to Z_{2r-2} \to \cdots \to Z_{r-1} \to 0, \]

with

\[ Z_{r+c-1} = H_{r+c-2}(F_c D(\mathcal{F}^r)/F_{c-1} D(\mathcal{F}^r)) \cong \bigoplus_{\beta} \mathbb{Z} \]

for \(1 \leq c \leq r\). In particular,

\[ Z_{2r-2} = \mathbb{Z}[GL_r(\mathcal{F})/T_r] \otimes_{\Sigma_r} \text{Lie}_r^* \]

and \(Z_{r-1} = \text{St}_r(\mathcal{F})\). Here \(T_r \subset GL_r(\mathcal{F})\) is the diagonal torus, and \(\text{Lie}_r^*\) is the dual of the Lie representation of the symmetric group \(\Sigma_r\). The group \(GL_r(\mathcal{F})\) acts naturally on this complex.

Corollary 10.6. \(H_*D(\mathcal{F}^r)\) is concentrated in the range \(r-1 \leq * \leq 2r-2\).

11. The connectivity conjecture

In [Rognes (1992)] we made the following conjecture.

Conjecture 11.1 (Connectivity). \(H_*D(\mathcal{F}^r)\) is concentrated in degree \((2r-2)\).

Equivalently, the complex

\[ 0 \to H_{2r-2} D(\mathcal{F}^r) \to Z_{2r-2} \to \cdots \to Z_{r-1} \to 0 \]

is exact, \(D(\mathcal{F}^r)\) is \((2r-3)\)-connected, and \(D(\mathcal{F}^r) \cong \bigvee_{\gamma} S^{2r-2}\).

Theorem 11.2 (Rognes). The connectivity conjecture is true for \(r = 1, 2\) and \(3\).

Definition 11.3. Let

\[ \Delta_r(\mathcal{F}) = H_{2r-2} D(\mathcal{F}^r) \cong \bigoplus_{\gamma} \mathbb{Z} \]

be the stable Steinberg representation of \(GL_r(\mathcal{F})\).

Example 11.4. \(\Delta_1(\mathcal{F}) = \mathbb{Z}\) and \(\Delta_2(\mathcal{F})\) is \(H_1\) of the complete graph on the set \(P^1(\mathcal{F})\) of lines \(L \subset \mathbb{F}^2\).

Corollary 11.5. If the connectivity conjecture holds, then

\[ H_* (F_r K(\mathcal{F})/F_{r-1} K(\mathcal{F})) \cong H_* (D(\mathcal{F}^r)_{hGL_r(\mathcal{F})}) \cong H_{*-2r+2}^{op}(GL_r(\mathcal{F}); \Delta_r(\mathcal{F})) \]

is concentrated in degrees \(* \geq 2r - 2\). Then \(F_r K(\mathcal{F}) \to K(\mathcal{F})\) is \((2r-1)\)-connected, so

\[ F_r K_j(\mathcal{F}) = \text{im}(\pi_j F_r K(\mathcal{F}) \to \pi_j K(\mathcal{F})) \]

is equal to \(K_j(\mathcal{F})\) for \(j \leq 2r-1\), or equivalently, for \(2r \geq j + 1\).

Remark 11.6. For \(r \geq 2\), if \(H_*^{op}(GL_r(\mathcal{F}); \Delta_r(\mathcal{F}))\) is torsion, hence rationally trivial, then \(F_r K_j(\mathcal{F})_Q = K_j(\mathcal{F})_Q\) also for \(j = 2r\), i.e., for \(2r \geq j\).
12. The stable rank conjecture

Applying homology to the sequence of homotopy cofiber sequences

* \longrightarrow F_1 K(\mathbb{F}) \longrightarrow F_2 K(\mathbb{F}) \longrightarrow \ldots \longrightarrow F_r K(\mathbb{F}) \longrightarrow \ldots \longrightarrow K(\mathbb{F})

\xrightarrow{\approx} \Sigma^\infty B^{\mathbb{F}}_+ \xrightarrow{D(\mathbb{F})_{hGL_2(\mathbb{F})}} D(\mathbb{F})_{hGL_r(\mathbb{F})}

with \( F_s K(\mathbb{F}) \) in filtration \( s = r - 1 \) we obtain the homological rank spectral sequence

\[ E_{s,t}^1(rk) = H_{s+t}(D(\mathbb{F}^{s+1})_{hGL_{s+1}(\mathbb{F}))} \longrightarrow H_{s+t}(K(\mathbb{F})) \]

It is of homological type, concentrated in the first quadrant (\( s \geq 0 \) and \( t \geq 0 \)). Assuming the connectivity conjecture, the \( E^1 \)-term can be rewritten as

\[ E_{s,t}^1(rk) = H_{s-t}(GL_{s+1}(\mathbb{F}); \Delta_{s+1}(\mathbb{F})) , \]

hence is in fact concentrated in the wedge \( s \geq 0 \) and \( t \geq s \). The Hurewicz homomorphism \( K_{s+t}(\mathbb{F}) = \pi_{s+t} K(\mathbb{F}) \to H_{s+t}(K(\mathbb{F})) \) is a rational equivalence, so after rationalization the rank spectral sequence converges to \( K_{s+t}(\mathbb{F})_{\mathbb{Q}} \).

\[
\begin{array}{cccccc}
& & & & & \\
& \vdots & & & & \\
& & & & & \\
4 & H_{4}^{op}(\mathbb{F}^\times) & H_{3}^{op} (GL_{2}\mathbb{F}; \Delta_{2}\mathbb{F}) & H_{2}^{op} (GL_{3}\mathbb{F}; \Delta_{3}\mathbb{F}) & H_{1}^{op} (GL_{4}\mathbb{F}; \Delta_{4}\mathbb{F}) & \ldots \\
3 & H_{3}^{op} (\mathbb{F}^\times) & H_{2}^{op} (GL_{2}\mathbb{F}; \Delta_{2}\mathbb{F}) & H_{1}^{op} (GL_{3}\mathbb{F}; \Delta_{3}\mathbb{F}) & \Delta_{4}(\mathbb{F})_{GL_{4}\mathbb{F}} & \ldots \\
2 & H_{2}^{op} (\mathbb{F}^\times) & H_{1}^{op} (GL_{2}\mathbb{F}; \Delta_{2}\mathbb{F}) & \Delta_{3}(\mathbb{F})_{GL_{3}\mathbb{F}} & 0 & \ldots \\
1 & \mathbb{F}^\times & \delta^i \Delta_{2}(\mathbb{F})_{GL_{2}\mathbb{F}} & 0 & 0 & \ldots \\
0 & \mathbb{Z} & 0 & 0 & 0 & \ldots \\
E_{s,t}^1(rk) & 0 & 1 & 2 & 3 & s \\
\end{array}
\]

Example 12.1. \( E_{0,t}^1(rk) = H_t (B^{\mathbb{F}}) = H_{t}^{op}(\mathbb{F}^\times) \) is rationally isomorphic to \( A^t \mathbb{F}^\times \).

The \( E^1 \)-term suggests the following definition of the motivic complexes sought by Beilinson and Lichtenbaum.

Definition 12.2. For each \( w \geq 0 \) define the rank complex \( (\Gamma_{rk}(w, \mathbb{F}), \delta) \) by

\[ \Gamma_{rk}(\mathbb{F})^i = E_{w-i,w}^1(rk) \]

and \( \delta^i = d_{w-i,w}: \Gamma_{rk}(w, \mathbb{F})^i \to \Gamma_{rk}(w, \mathbb{F})^{i+1} \).

By definition, \( \Gamma_{rk}(w, \mathbb{F})^i = 0 \) for \( i > w \). If the connectivity conjecture holds, then

\[ \Gamma_{rk}(w, \mathbb{F})^i \cong H_{i}^{op} (GL_{w-i+1}(\mathbb{F}); \Delta_{w-i+1}(\mathbb{F})) \]

is nonzero only for \( 0 \leq i \leq w \).

Definition 12.3. Let the rank cohomology \( H^i_{rk}(\mathbb{F}; \mathbb{Z}(w)) \) be the cohomology of this cochain complex:

\[ H^i_{rk}(\mathbb{F}; \mathbb{Z}(w)) = \frac{\ker \delta^i}{\text{im} \delta^{i-1}} = H^i(\Gamma_{rk}(w, \mathbb{F}^*), \delta) . \]
These groups give the $E^2$-term of the homological rank spectral sequence

$$E^2_{s,t}(rk) = H^{s+t}_k(F, \mathbb{Z}(t)) \Rightarrow H_{s+t}K(F).$$

If the connectivity conjecture holds, then this spectral sequence is concentrated in the region $0 \leq s \leq t$ (with $s < t$ for $t > 0$ if $\Delta_s(F)/GL_s(F)$ is torsion).

**Conjecture 12.4** (Stable Rank). The motivic spectral sequence and the stable rank spectral sequence are rationally isomorphic, starting from the $E^2$-terms:

$$E^2_{s,t}(mot)_\mathbb{Q} = H^{s+t}_k(F, \mathbb{Z}(t))_\mathbb{Q} \Rightarrow K_{s+t}(F)_\mathbb{Q}$$

$$E^2_{s,t}(rk)_\mathbb{Q} = H^{s+t}_k(F, \mathbb{Z}(t))_\mathbb{Q} \Rightarrow H_{s+t}K(F)_\mathbb{Q}$$

**Theorem 12.5.** The stable rank conjecture holds for $s = 0$. More precisely,

$$E^2_{0,j}(mot) \Rightarrow E^0_{0,j}(mot)$$

$$E^2_{0,j}(rk) \Rightarrow E^0_{0,j}(rk)$$

consists of rational isomorphisms for all $\rho \geq 2$.

**Sketch proof.** Consider the diagram

$$\Lambda^jF^x \to K^M_j(F) \to K_j(F)$$

$$H^g_pF^x \to E^2_{0,j}(rk) \to H_jK(F).$$

The connectivity and stable rank conjectures together imply (the rational form of) the Beilinson–Soulé vanishing conjecture.

An advantage of the stable rank point of view is that $D(F^r)$ is described only in terms of linear subspaces $V \subset F^r$, as opposed to general subvarieties of $\Delta_F^r$.

13. **THE COMMON BASIS COMPLEX**

By covering the stable building $D(F^r)$ by the $GL_r(F)$-translates of a stable apartment $A(r)$, we obtain the following elementary description of the stable building.

**Definition 13.1.** Let the common basis complex $D'(F^r)$ be the simplicial complex with vertices the proper, nontrivial subspaces $0 \not\subset V \subset F^r$, such that a set $\{V_0, \ldots, V_p\}$ of vertices spans a $p$-simplex if and only if these vector spaces admit a *common basis*, i.e., there exists a basis $B = \{b_1, \ldots, b_r\}$ for $F^r$ such that for each $i = 0, \ldots, p$ there is a subset of $B$ that is a basis for $V_i$.

**Theorem 13.2.** $\Sigma^\infty D'(F^r) \simeq D(F^r)$.

**Sketch proof.** The stable apartment $A(r)$ the a (pre-)spectrum with $n$-th space $A(r)_n$ a simplicial set with $q$-simplices diagrams $\sigma: [q]^n \to \text{Sub}(\{1, \ldots, r\})$ consisting of subsets of $\{1, \ldots, r\}$ and inclusions between these. We know that $A(r)_n \simeq S^n$, so $A(1) \simeq S$ and $A(r) \simeq S^r$ for $r \geq 2$. (This, incidentally, gives a proof of the Barratt–Pridey–Quillen theorem.)

The free $F$-vector space functor $\text{Sub}(\{1, \ldots, r\}) \to \text{Sub}(F^r)$ induces an embedding $A(r) \to D(F^r)$, and the translates $\{gA(r) | g \in GL_r(F)\}$ cover $D(F^r)$. A $(p + 1)$-fold intersection

$$g_0A(r) \cap \cdots \cap g_pA(r)$$

is isomorphic to $S$ if there is a proper, nontrivial subspace $V \subset F^r$ such that for each $0 \leq s \leq p$ there is a basis for $V$ given by a subset of the columns of $g_s \in GL_r(F)$. Otherwise, the intersection is (stably) contractible. Hence $D(F^r) \simeq \Sigma^\infty D'(F^r)$, where $D'(F^r)$ is the simplicial complex with vertices the elements $g$ of $GL_r(F)$, such that $\{g_0, \ldots, g_p\}$ span a $p$-simplex if and only if there is a $0 \not\subset V \subset F^r$ such that for each $0 \leq s \leq p$ a subset of the columns of $g_s$ is a basis for $V$. 

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For each $0 \subseteq V \subseteq \mathbb{F}^r$ the set of $g \in GL_r(\mathbb{F})$ such that a subset of the columns of $g$ is a basis for $V$ span a contractible subspace $C(V) \subset D^r(\mathbb{F}^r)$. A $p$-fold intersection

$$C(V_0) \cap \cdots \cap C(V_p)$$

is contractible if there exists a single $g \in GL_r(\mathbb{F})$ such that for each $0 \leq t \leq p$ a subset of the columns of $g$ is a basis for $V_t$. In other words, the intersection is contractible if $\{V_0, \ldots, V_p\}$ admit a common basis. Otherwise the intersection is empty. This proves that $D^r(\mathbb{F}^r) \simeq D'(\mathbb{F}^r)$. \hfill \Box

**Conjecture 13.3** (Connectivity). $\tilde{H}_* D'(\mathbb{F}^r)$ is concentrated in degree $(2r - 3)$.

**Example 13.4.** For $r = 2$, $D'(\mathbb{F}^2)$ is the complete graph on the set $\mathbb{P}^1(\mathbb{F})$ of lines $L \subset \mathbb{F}^2$. It is connected, hence its homology $\tilde{H}_* D'(\mathbb{F}^2)$ is concentrated in degree 1. Thus $\Delta_2(\mathbb{F})$ is the homology of the complete graph on $\mathbb{P}^1(\mathbb{F})$, as previously claimed.

**References**

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