The Abundancy Index of Divisors of Odd Perfect Numbers - Part III

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Abstract

Let \( \sigma(x) \) be the sum of the divisors of \( x \). If \( N \) is odd and \( \sigma(N) = 2N \), then the odd perfect number \( N \) is said to be given in Eulerian form if \( N = q^k n^2 \) where \( q \) is prime with \( q \equiv k \equiv 1 \pmod{4} \) and \( \gcd(q, n) = 1 \). In this note, we show that \( q < n \). It then follows from the lower bound \( N > 10^{1500} \) (obtained by Ochem and Rao) that the ratio \( n^2/q \) is greater than \( 10^{375} \cdot \sqrt{10} \), which (somewhat) significantly improves on a recent result by Broughan, et. al. We end with a discussion of some remaining open problems.

Keywords: Sorli’s conjecture, odd perfect number, abundancy index

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1. Introduction

If $J$ is a positive integer, then we write $\sigma(J)$ for the sum of the divisors of $J$. A number $L$ is perfect if $\sigma(L) = 2L$.

An even perfect number $M$ is said to be given in Euclidean form if

$$M = (2^p - 1) \cdot 2^{p-1}$$

where $p$ and $2^p - 1$ are primes. We call $M_p = 2^p - 1$ the Mersenne prime factor of $M$. Currently, there are only 48 known Mersenne primes [10], which correspond to 48 even perfect numbers.

An odd perfect number $N$ is said to be given in Eulerian form if

$$N = q^k n^2$$

where $q$ is prime with $q \equiv k \equiv 1 \pmod{4}$ and $\gcd(q, n) = 1$. We call $q^k$ the Euler part of $N$ while $n^2$ is called the non-Euler part of $N$. (We will call $q$ the Euler prime factor of $N$.)

It is currently unknown whether there are infinitely many even perfect numbers, or whether any odd perfect numbers exist. It is widely believed that there is an infinite number of even perfect numbers. On the other hand, no examples for an odd perfect number have been found (despite extensive computer searches), nor has a proof for their nonexistence been established.
Ochem and Rao [13] recently proved that $N > 10^{1500}$. In a recent preprint, Nielsen [11] claims to have obtained the lower bound $\omega(N) \geq 10$ for the number of distinct prime factors of $N$, improving on his last result $\omega(N) \geq 9$ (see [12]).

Sorli conjectured in [15] that $k = \nu_q(N) = 1$ always holds. (More recently, Beasley [2] points out that Descartes was the first to conjecture $k = \nu_q(N) = 1$ “in a letter to Marin Mersenne in 1638, with Frenicle’s subsequent observation occurring in 1657.”) Dris conjectured in [8] and [7] that the components $q^k$ and $n$ are related by the inequality $q^k < n$. This conjecture was made on the basis of the result $I(q^k) < \sqrt{2} < I(n)$.

Broughan, et. al. [3] recently showed that for any odd perfect number, the ratio of the non-Euler part to the Euler part is greater than $315/2$.

We denote the abundancy index $I$ of the positive integer $x$ as $I(x) = \sigma(x)/x$.

2. Preparatory Results

The following result was communicated to the second author (via e-mail, by Pascal Ochem) in April of 2013.

**Theorem 2.1.** If $N = q^n n^2$ is an odd perfect number given in Eulerian form, then

$$I(n) > \left(\frac{8}{9}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} \approx 1.4440557.$$ 

The proof of Theorem 2.1 uses the following lemma.

**Lemma 2.2.** Let $x(n) = \ln(I(n^2))/\ln(I(n))$. If $\gcd(a, b) = 1$, then

$$\min(x(a), x(b)) < x(ab) < \max(x(a), x(b)).$$

**Proof.** First, note that $x(a) \neq x(b)$ (since $\gcd(a, b) = 1$). Without loss of generality, we may assume that $x(a) < x(b)$. Thus, since

$$x(n) = \frac{\ln(I(n^2))}{\ln(I(n))},$$

we have

$$\frac{\ln(I(a^2))}{\ln(I(a))} < \frac{\ln(I(b^2))}{\ln(I(b))}.$$ 

This implies that

$$\ln(I(a^2)) \ln(I(b)) < \ln(I(b^2)) \ln(I(a)).$$

Adding $\ln(I(a^2)) \ln(I(a))$ to both sides of the last inequality, we get

$$\ln(I(a^2)) \left(\ln(I(b)) + \ln(I(a))\right) < \ln(I(a)) \left(\ln(I(b^2)) + \ln(I(a^2))\right).$$
Using the identity \( \ln(X) + \ln(Y) = \ln(XY) \), we can rewrite the last inequality as
\[
\ln(I(a^2)) \ln(I(ab)) < \ln(I(a)) \ln(I((ab)^2))
\]
since \( I(x) \) is a weakly multiplicative function of \( x \) and \( \gcd(a, b) = 1 \). It follows that
\[
\min(x(a), x(b)) = x(a) = \frac{\ln(I(a^2))}{\ln(I(a))} < \frac{\ln(I((ab)^2))}{\ln(I(ab))} = x(ab).
\]

Under the same assumption \( x(a) < x(b) \), we can show that
\[
x(ab) < x(b) = \max(x(a), x(b))
\]
by adding \( \ln(I(b^2)) \ln(I(b)) \) (instead of \( \ln(I(a^2)) \ln(I(a)) \)) to both sides of the inequality
\[
\ln(I(a^2)) \ln(I(b)) < \ln(I(b^2)) \ln(I(a)).
\]

This finishes the proof. \( \square \)

**Remark 2.3.** From Lemma 2.2 we note that \( 1 < x(n) < 2 \) follows from
\[
I(n) < I(n^2) < (I(n))^2
\]
and \( I(n^2) = (I(n))^{x(n)} \).

The trivial lower bound for \( I(n) \) is
\[
\left( \frac{8}{5} \right)^{1/2} < I(n).
\]

Note that decreasing the denominator in the exponent gives an increase in the lower bound for \( I(n) \).

**Remark 2.4.** We sketch a proof for Theorem 2.1 here, as communicated to the second author by Pascal Ochem.

Suppose that \( N = q^k n^2 \) is an odd perfect number given in Eulerian form. We want to obtain a lower bound on \( I(n) \). We know that
\[
I(n^2) = 2/I(q^k) > 2/(5/4) = 8/5.
\]

We need to improve the trivial bound \( I(n^2) < (I(n))^2 \).

Let \( x(n) \) be such that
\[
I(n^2) = \left( I(n) \right)^{x(n)}.
\]

That is, \( x(n) = \ln(I(n^2))/\ln(I(n)) \). We want an upper bound on \( x(n) \) for \( n \) odd. By Lemma 2.24 we consider the component \( r^s \) with \( r \) prime that maximizes \( x(r^s) \).

We have
\[
I(r^s) = \frac{r^{s+1} - 1}{r^s(r - 1)} = 1 + \frac{1}{r - 1} - \frac{1}{r^s(r - 1)}.
\]
Also,  
\[ I(r^{2s}) = \frac{r^{2s+1} - 1}{r^{2s}(r - 1)} = I(r^s) \left( 1 + \left( \frac{1 - r^{-s}}{r^{s+1} - 1} \right) \right). \]

So,  
\[ x(r^s) = \frac{\ln(I(r^{2s}))}{\ln(I(r^s))} = \frac{\ln(I(r^s)) + \ln(1 + \left( \frac{1 - r^{-s}}{r^{s+1} - 1} \right))}{\ln(I(r^s))}, \]

from which it follows that  
\[ x(r^s) = 1 + \frac{\ln(1 + \left( \frac{1 - r^{-s}}{r^{s+1} - 1} \right))}{\ln(1 + \frac{1}{r - 1} - \frac{1}{r^s(r - 1)}).} \]

We can check that  
\[ x(r^s) > x(r^t) \]
if \( s < t \) and \( r \geq 3 \). Therefore, \( x(r^s) \) is maximized for \( s = 1 \). Now,  
\[ x(r) = 1 + \frac{\ln(1 + (1/(r(r+1))))}{\ln(1 + (1/r))} = \frac{\ln(1 + (1/r) + (1/r)^2)}{\ln(1 + (1/r))} = \frac{\ln(I(r^2))/\ln(I(r))}, \]

which is maximized for \( r = 3 \). So,  
\[ x(3) = \frac{\ln(I(3^2))}{\ln(I(3))} = \frac{\ln(13)/\ln(4)}{\ln(3)} \approx 1.27823. \]

The claim in Theorem 2.1 then follows, and the proof is complete.

The argument in Remark 2.4 can be improved to account for the divisibility of \( n \) by primes \( r \) other than 3, whereby \( x(r) \leq x(5) < x(3) \). We outline an attempt on such an improvement in the following lemma.

**Lemma 2.5.** If \( N = q^k n^2 \) is an odd perfect number given in Eulerian form and \( u \) is the smallest prime factor of \( N \), then the inequality  
\[ \sqrt{3} < \left( \frac{2(u - 1)}{u} \right)^{\ln(I(u))/\ln(I(u^2))} < 2 \]

is equivalent to \( u \geq 11 \). Moreover, these inequalities imply \( I(n) > \sqrt{3} \).

**Proof.** Suppose that \( N = q^k n^2 \) is an odd perfect number given in Eulerian form, and let \( u \) be the smallest prime factor of \( N \).

Then we have \( q > u \) (since the Euler prime \( q \) is not the smallest prime factor of \( N \)), from which it follows that  
\[ I(q^k) < \frac{q}{q - 1} < \frac{u}{u - 1}. \]

This then gives  
\[ I(n^2) = \frac{2}{I(q^k)} > \frac{2(u - 1)}{u}. \]
Mimicking the proof of 2.1 since $u$ is the smallest prime factor of $N$ (and $u \neq q$), then $u \mid n$, so that $x(n) = \ln(I(n^2))/\ln(I(n))$ in this case is maximized by $x(u)$. Consequently, we have
\[
\left( \frac{I(n)}{u} \right)^{x(u)} \geq \left( \frac{I(n)}{n} \right)^{x(n)} = I(n^2) > \frac{2(u-1)}{u},
\]
from which it follows that
\[
\left( \frac{2(u-1)}{u} \right)^{\ln(I(u))/\ln(I(u^2))} = \left( \frac{2(u-1)}{u} \right)^{1/x(u)} < I(n) < 2.
\]
Thus, if
\[
\sqrt{3} < \left( \frac{2(u-1)}{u} \right)^{\ln(I(u))/\ln(I(u^2))} < 2,
\]
then $I(n) > \sqrt{3}$. Lastly, WolframAlpha (http://www.wolframalpha.com) gives the rational approximation
\[
u > 10.9334
\]
when asked to solve the inequality
\[
\sqrt{3} < \left( \frac{2(u-1)}{u} \right)^{\ln(I(u))/\ln(I(u^2))} < 2.
\]
Since we assumed that $u$ is a prime number, we can take
\[
u \geq 11.
\]
This finishes the proof. \qed

3. Main Theorems

We are now ready to prove our main results in this note.

**Theorem 3.1.** If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the inequality $q < n$ is true.

**Proof.** Let $N = q^k n^2$ be an odd perfect number given in Eulerian form, and suppose to the contrary that $n < q$. By a result in [7], this implies that
\[
k = 1.
\]
Since $k = 1$ and $n < q$, from the last remark in page 13 of [6], we have the inequality $\sigma(n) \leq q$. However, referring to the first lemma in page 6 of [5], we have the implication $k = 1 \implies q < n\sqrt{3}$, which follows from the main result in [1]. Thus, if $I(n) > \sqrt{3}$, we have a contradiction, as follows:
\[
\sqrt{3} < I(n) = \frac{\sigma(n)}{n} \leq \frac{q}{n} < \sqrt{3}.
\]
Consequently, the implication $I(n) > \sqrt{3} \implies q < n$ is true. By the contrapositive, $n < q \implies I(n) < \sqrt{3}$.

Now, by Lemma 2.5, $q > u \geq 11 \implies I(n) > \sqrt{3}$. (In other words,

$$q \geq 13 \implies I(n) > \sqrt{3},$$

since $q$ is prime with $q \equiv 1 \pmod{4}$.) But the series of inequalities

$$\sqrt{3} < I(n) < \frac{2q}{q + 1}$$

imply that $q > 3 + 2\sqrt{3} > 6$. Consequently, we have the biconditional

$$q \geq 13 \iff I(n) > \sqrt{3}.$$

Equivalently, we have

$$q = 5 \iff I(n) < \sqrt{3}.$$

But recall that we have the contrapositive $n < q \implies I(n) < \sqrt{3}$. Since $q = 5$ is equivalent to $I(n) < \sqrt{3}$, then $n < q \implies n < q = 5$.

This contradicts $10^{375} < n$, which follows from the result $q^k < n^2$ in [8] and [7], and a recent result from [13].

We therefore conclude that $q < n$, as desired. $\square$

As pointed out by Pascal Ochem to the second author in his April 2013 e-mail, a proof for the inequality $q < n$ will be considered a major breakthrough, as evidenced by the following corollary to Theorem 3.1.

**Corollary 3.2.** If $N = q^k n^2$ is an odd perfect number given in Eulerian form, then the ratio of the non-Euler part $n^2$ to the Euler prime $q$ is greater than $10^{375}. \sqrt{10}$.

**Proof.** The proof is trivial and follows from [8] and [13]. $\square$

### 4. Concluding Remarks

The inequality $I(n) < 2q/(q + 1)$ is equivalent to the inequality

$$q > \frac{I(n)}{2 - I(n)}.$$

Consequently, a (nontrivial) lower bound for $I(n)$ would correspond to a(n) (equally nontrivial) lower bound for $q$. Observe that

$$q > \frac{I(n)}{2 - I(n)} > 3$$

implies that $I(n) > 3/2$. 

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Note that Theorem 3.1 then gives
\[ n > q > \frac{I(n)}{2 - I(n)} \]
which can be transformed into (and thereby validates) the (true) inequality (\[\mathbb{3}, \mathbb{9}\])
\[ I(n) < \frac{2n}{n + 1}. \]

Comparing this analysis with the inequality \( I(n^2) = \frac{2}{I(q^k)} \leq \frac{2q}{q + 1} \), which is equivalent to the inequality
\[ q \geq \frac{I(n^2)}{2 - I(n^2)}, \]
we can see that, similarly, this last inequality (together with \( q \leq q^k < n^2 \) (\[\mathbb{8}, \mathbb{7}\])) gives (and hence validates) the (true) inequality (\[\mathbb{3}, \mathbb{9}\])
\[ I(n^2) < \frac{2n^2}{n^2 + 1}. \]

Taking off from the last theorem in page 14 of [5], we now know that we have the following series of inequalities for the indicated abundancy in dices:
\[
\begin{align*}
(q/n)I(q) & < 1 < I(q) \leq \frac{6}{5} < \sqrt{2} < I(n) < (n/q)I(n), \\
1 & < I(q) < \frac{q}{q - 1} < \sqrt{2} < I(n) < I(n^2) \leq \frac{2q}{q + 1} < \frac{2n}{n + 1} < 2,
\end{align*}
\]
and lastly,
\[ I(n^2) < (I(n))^{\ln(13/9)/\ln(4/3)} \approx (I(n))^{1.27823}. \]

Finally, note that the last dilemma in the preprint [6] was whether the implication
\[ k = 1 \implies n < q \]
is true. Since Theorem 3.1 gives \( q < n \), the question of whether Descartes’ / Frenicle’s / Sorli’s conjecture on odd perfect numbers, that \( k = 1 \), remains open.

The following cases remain to be considered:

- \( q = q^k < n \)
- \( q < q^k < n \)
- \( q < n < q^k \)

We refer the interested reader to [4] and [14] for possible avenues of further research and investigations into these remaining problems.
5. Acknowledgements

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References

[1] P. Acquaah, S. Konyagin, On prime factors of odd perfect numbers, *Int. J. Number Theory*, **08** (2012), 1537, doi [http://dx.doi.org/10.1142/S1793042112500935](http://dx.doi.org/10.1142/S1793042112500935).

[2] B. D. Beasley, Euler and the ongoing search for odd perfect numbers, ACMS 19th Biennial Conference Proceedings, Bethel University, May 29 to Jun. 1, 2013, [http://godandmath.files.wordpress.com/2013/07/acms-2013-proceedings.pdf](http://godandmath.files.wordpress.com/2013/07/acms-2013-proceedings.pdf), pages 21-31.

[3] K. A. Broughan, D. Delbourgo and Q. Zhou, Improving the Chen and Chen result for odd perfect numbers, *Integers*, **13** (2013), Article #A39, [http://www.emis.de/journals/INTEGRERS/papers/n39/n39.pdf](http://www.emis.de/journals/INTEGRERS/papers/n39/n39.pdf), ISSN 1867-0652.

[4] J. A. B. Dris, Euclid-Euler heuristics for (odd) perfect numbers, (Oct. 2013), [http://arxiv.org/abs/1310.5616](http://arxiv.org/abs/1310.5616).

[5] J. A. B. Dris, The abundancy index of divisors of odd perfect numbers - Part II, (Nov. 2013), submitted to Journal of Number Theory, preprint [http://arxiv.org/abs/1309.0906](http://arxiv.org/abs/1309.0906).

[6] J. A. B. Dris, New results for Sorli’s conjecture on odd perfect numbers, (Jul. 2013), submitted to Integers: The Electronic Journal of Combinatorial Number Theory, preprint [http://arxiv.org/abs/1302.5991](http://arxiv.org/abs/1302.5991).

[7] J. A. B. Dris, The abundancy index of divisors of odd perfect numbers, *J. Integer Seq.*, **15** (Sep. 2012), Article 12.4.4, [https://cs.uwaterloo.ca/journals/JIS/VOL15/Dris/dris8.html](https://cs.uwaterloo.ca/journals/JIS/VOL15/Dris/dris8.html), ISSN 1530-7638.

[8] J. A. B. Dris, Solving the Odd Perfect Number Problem: Some Old and New Approaches, M. Sc. thesis, De La Salle University, Manila, Philippines, 2008, [http://arxiv.org/abs/1204.1450](http://arxiv.org/abs/1204.1450).
[9] K. A. P. Dagal, J. A. Dris, A criterion for almost perfect numbers in terms of the abundancy index, (Aug. 2013), preprint, http://arxiv.org/abs/1308.6767.

[10] Various contributors, Great Internet Mersenne Prime Search, http://www.mersenne.org/, last accessed: 11/03/2013.

[11] P. P. Nielsen, Odd perfect numbers, Diophantine equations, and upper bounds, preprint, http://math.byu.edu/~pace/BestBound_web.pdf.

[12] P. P. Nielsen, Odd perfect numbers have at least nine distinct prime factors, Math. Comp., 76 (2007), 2109-2126, doi: http://dx.doi.org/10.1090/S0025-5718-07-01990-4.

[13] P. Ochem, M. Rao, Odd perfect numbers are greater than $10^{1500}$, Math. Comp., 81 (2012), 1869-1877, doi: http://dx.doi.org/10.1090/S0025-5718-2012-02563-4.

[14] T. D. Noe, Online Encyclopedia of Integer Sequences, sequence A228059, http://oeis.org/A228059, last accessed: 11/03/2013.

[15] R. M. Sorli, Algorithms in the Study of Multiperfect and Odd Perfect Numbers, Ph. D. Thesis, University of Technology, Sydney, 2003, http://epress.lib.uts.edu.au/research/handle/10453/20034.