The decays \( \rho^- \rightarrow \eta\pi^- \) and \( \tau^- \rightarrow \eta(\eta')\pi^-\nu \) in the NJL model

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Abstract

The widths of the decays \( \rho^- \rightarrow \eta\pi^- \) and \( \tau^- \rightarrow \eta(\eta')\pi^-\nu \) are calculated in the framework of the NJL model. It is shown that these decays are defined by the \( u \) and \( d \) quark mass difference. It leads to the suppression of these decays in comparison with the main decay modes. In the process \( \rho^- \rightarrow \eta\pi^- \) the intermediate scalar \( a_0 \) state is taken into account. For the \( \tau \) decays the intermediate states with \( a_0, \rho^- (770) \) and \( \rho^- (1450) \) mesons are used. Our estimates are compared with the results obtained in other works.

Keywords: tau decays, chiral symmetry, Nambu-Jona-Lasinio model, radial excited mesons

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1 Introduction

At present, the decays \( \rho^- \rightarrow \eta\pi^- \) and \( \tau^- \rightarrow \eta(\eta')\pi^-\nu \) are not well studied in experiments [1–3]. However, recently, a number of works devoted to the investigation of these processes in the framework of the different phenomenological models were published [4–10]. On the other hand, in [11–14] it was shown that different modes of the \( \tau \) decay can be satisfactorily described in the NJL model [15–21]. In the present paper, the NJL model is used for the description of the decays \( \rho^- \rightarrow \eta\pi^- \) and \( \tau^- \rightarrow \eta(\eta')\pi^-\nu \). The probabilities of the transitions \( \pi^0 \rightarrow \eta(\eta') \) and \( \rho^- (W^-) \rightarrow a_0^- \) are calculated. These transitions are defined by the mass difference between \( u \) and \( d \) quarks and can be calculated in the framework of the NJL model without attraction of any arbitrary parameters. Our results will be compared with the estimates obtained in [7–10]. It is shown that the amplitudes with intermediate vector mesons dominate in the \( \tau^- \rightarrow \eta\pi^-\nu \) decay.

2 The decay \( \rho^- \rightarrow \eta\pi^- \)

For calculation of this decay we should first calculate two non-diagonal transitions \( \pi^0 \rightarrow \eta \) and \( \rho^- \rightarrow a_0^- \) within the NJL model. These transitions go through quark loops containing \( u \) and \( d \) quarks (see Fig. 1).

Figure 1: \( \pi^0 \rightarrow \eta(\eta') \) (left) and \( \rho^- (W^-) \rightarrow a_0^- \) (right) transitions

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The amplitude of the transition $\pi^0 \to \eta(\eta')$ has the form

$$\epsilon_{\pi\eta(\eta')} = 2g_\pi^2((2I_1(m_d) + m^2_{\eta(\eta')}I_2(m_d)) - (2I_1(m_u) + m^2_{\eta(\eta')}I_2(m_u))) \frac{\epsilon_{\eta(\eta')}}{m_{\eta}^2 - m_{\eta(\eta')}^2},$$  \hspace{1cm} (1)

where $m_\pi$, $m_\eta$, $m_{\eta'}$ are masses of $\pi$, $\eta$ and $\eta'$ mesons, respectively, given in PDG [17]; $m_u$ and $m_d$ are constituent quark masses, $m_u = 280$ MeV. Using the last experimental data for the decay $\omega \to \pi\pi$ [1] we obtain $m_d - m_u \approx 3.7$ MeV. This decay was described in detail in [17]. The $\eta - \eta'$ mixing $\epsilon_\eta = \sin \theta$ for the $\eta$ meson and $\epsilon_{\eta'} = \cos \theta$ for the $\eta'$ meson. The mixing angle $\theta \approx -54^0$ was defined in [22]. The constant $g_\pi$ and integrals $I_1(m)$, $I_2(m)$ are defined in [17].

$$g_\pi = \frac{m_u}{F_\pi},$$  \hspace{1cm} (2)

$$I_1(m) = -i \frac{N_c}{(2\pi)^4} \int^{2\Lambda_4} \frac{d^4k}{m^2 - k^2} = \frac{N_c}{(4\pi)^2} \left[ \Lambda_4^2 - m^2 \log \left( \frac{\Lambda_4^2}{m^2} + 1 \right) \right],$$  \hspace{1cm} (3)

$$I_2(m) = -i \frac{N_c}{(2\pi)^4} \int^{2\Lambda_4} \frac{d^4k}{m^2 - k^2} = \frac{N_c}{(4\pi)^2} \left[ \log \left( \frac{\Lambda_4^2}{m^2} + 1 \right) - \left( 1 + \frac{m^2}{\Lambda_4^2} \right)^{-1} \right],$$  \hspace{1cm} (4)

where $N_c = 3$ is a number of quark colors and $\Lambda_4 \approx 1250$ MeV is a 4-dimensionl cut-off parameter in the standard NJL model [17]. The obtained estimates coincide with those used in [9, 10]. One can see the comparison in Table 1.

| $|\epsilon_{\pi\eta}|$ [9] | $|\epsilon_{\pi\eta}'|$ [10] |
|--------------------------|--------------------------|
| $1.34 \cdot 10^{-2}$    | $1.55 \cdot 10^{-2}$    |

The transition $\rho^- \to a_0^-$ takes the form

$$\frac{\sqrt{6}}{2} (m_d - m_u)p^\mu \rho_\mu^- a_0^-, \hspace{1cm} (5)$$

where $p$ is momentum of the $\rho$ meson.

The $\rho$ meson decay width is defined by two diagrams in Figs. 2 and 3. The first diagram describes the amplitude which contains the $\pi^0 \to \eta$ transitions in the final state

$$T_1 = g_\rho \epsilon_{\pi\eta} (p_\rho^\mu - p_0^\mu) \rho_\mu^- \eta \pi^-,$$  \hspace{1cm} (6)

where $g_\rho \approx 6.14$ is defined in [17]. The second diagram describes the amplitude containing the intermediate $a_0^-$ meson

$$T_2 = 2Z g_\rho \frac{m_u(m_d - m_u)}{m_{a_0}^2 - m_\rho^2} \epsilon_{\rho\pi} p^\mu \rho_\mu^- \eta \pi^-, \hspace{1cm} (7)$$

The obtained estimates coincide with those used in [9, 10]. One can see the comparison in Table 1.

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| $|\epsilon_{\pi\eta}|$ [9] | $|\epsilon_{\pi\eta}'|$ [10] |
|--------------------------|--------------------------|
| $1.34 \cdot 10^{-2}$    | $1.55 \cdot 10^{-2}$    |
where the vertex $a_0 \to \eta \pi^-$ was defined in \[22\]

$$
\frac{4}{\sqrt{6}} Z g_{\rho a} c_\eta a_0 \eta \pi^-,
$$

$$
Z = \left(1 - 6 \frac{m_a^2}{m_{a_1}^2}\right)^{-1},
$$

and $m_{a_1} = 1230$ MeV is the mass of the $a_1$ meson [1].

Thus, for branching fractions we get

$$
B_1 = \varepsilon_{\pi \eta}^2 \frac{\lambda^{3/2}(m_{\rho}^2, m_{\eta}^2, m_{\pi}^2)}{\lambda^{3/2}(m_{\rho}^2, m_{\eta}^2, m_{\pi}^2)} = 1.78 \cdot 10^{-5},
$$

$$
B_2 = 4Z^2 \varepsilon_{\eta}^2 \left( \frac{m_u (m_d - m_u)}{m_{a_0}^2 - m_{\rho}^2}\right)^2 \frac{\lambda^{3/2}(m_{\rho}^2, m_{\eta}^2, m_{\pi}^2)}{\lambda^{3/2}(m_{\rho}^2, m_{\eta}^2, m_{\pi}^2)} = 0.33 \cdot 10^{-5},
$$

where $\lambda(s, m_{\eta(\eta')}, m_{\pi}^2) = (s - m_{\eta(\eta')}^2 - m_{\pi}^2)^2 - 4m_{\eta(\eta')}^2 m_{\pi}^2$.

We note that in these calculations we take into account only the ground state of $a_0$ because the decay with the intermediate $a_0(1450)$ is suppressed by a large mass of the radial-excited meson.

Our estimates coincide with one taken in [7]. These estimates do not contradict known experimental limits [1, 2].

### 3 The decay $\tau^- \to \eta(\eta') \pi^− \nu$

The description of the decay $\tau \to \pi \pi \nu$ was obtained in [13] with satisfactory agreement with current experimental data.
We use the amplitude from [13] with the $\pi^0 \to \eta(\eta')$ transitions [1] in the final states (see Fig. 4)

$$T_V = \epsilon_{\pi(\eta')(\eta')} m_\rho^2 \left( 1 - \frac{r \sqrt{\Gamma_\rho(p^2)}}{m_\rho^2} \right) BW_\rho(p^2) + \beta_\rho \frac{p^2}{m_\rho^2} BW_\rho(p^2) \left( p^\mu - p_\eta(\eta') \right) l^\mu \pi^- \eta(\eta'),$$  \hspace{1cm} (12)

where the Breit-Wigner relation $BW_\rho(p^2)$ and $\beta_\rho$ parameter were defined in [13]. For the processes with the intermediate vector meson we get contributions to branching fractions

$$B_V(\tau \to \eta \pi \nu) = 4.35 \cdot 10^{-6}, \hspace{1cm} (13)$$

$$B_V(\tau \to \eta' \pi \nu) = 1.11 \cdot 10^{-8}. \hspace{1cm} (14)$$

The $W^- \to a_0^-$ transition takes the form

$$\frac{\sqrt{3}}{4 g_\rho} g_{EW} |V_{ud}| (m_d - m_u) p^\mu W^-_{a_0^-} a_0^-,$$

where $g_{EW}$ is the electroweak constant.

$\tau^-$

$W^-$

$\eta(\eta')$

$\nu$

$\pi^-$

Figure 5: The scalar contribution to $\tau$ decay.

The amplitude with the intermediate scalar meson (see Fig. 5) takes the form

$$T_S = 2 Z m_u (m_d - m_u) \epsilon_{\eta(\eta')(\eta')} \left( BW_{a_0}(p^2) + \beta_{a_0 \eta(\eta') \pi} BW_{a_0}(p^2) \right) p^\mu l^\mu \pi^- \eta(\eta'),$$  \hspace{1cm} (16)

where $BW_{a_0}(a_0')(p^2)$ is the Breit–Wigner formula for the $a_0(a_0')$ meson with $m_{a_0} = 980$ MeV, $m_{a_0'} = 1474$ MeV, $\Gamma_{a_0}(m_{a_0}) = 265$ MeV taken from PDG [1] and $\Gamma_{a_0}(m_{a_0}) = 100$ MeV calculated from [9] which coincides with the upper PDG limit [1]. For the estimation of the contribution of the radial-excited $a_0'(1450)$ to the $\tau$ decays we should use the extended NJL model [22–25]. The amplitudes $A_{a_0' \to \eta(\eta') \pi}$ of the $a_0' \to \eta(\eta') \pi$ decays can be found in [23]. The transition $W^- \to a_0' (1450)$ takes the form

$$C_{W,a_0'} = \frac{\sqrt{3}}{4 g_\rho} g_{EW} |V_{ud}| (m_d - m_u) \left( \cos(\phi + \phi_0) \sin(2\phi_0) + \frac{\cos(\phi - \phi_0)}{\sin(2\phi_0)} \right) \frac{p^\mu W^-_{a_0'}}{m_{a_0'}},$$  \hspace{1cm} (17)

where $\phi_0 = 65.5^\circ$ and $\phi = 72.0^\circ$ are the mixing angles, and $\Gamma = 0.54$. Thus, we get the $\beta_{a_0 \eta(\eta') \pi}$ parameter

$$\beta_{a_0 \eta(\eta') \pi} = e^{i \pi} C_{W,a_0'} \frac{\sqrt{6} A_{a_0' \to \eta(\eta') \pi}}{m_{a_0'}},$$  \hspace{1cm} (18)

where phase factor $e^{i \pi}$ are taken similarly [13]. The values $\beta_{a_0 \eta \pi} = -0.24$ and $\beta_{a_0 \eta' \pi} = -0.26$ do not contradict with ones given in [5,10]. The contributions to the branching fractions from the amplitude [13] are

$$B_S(\tau \to \eta \pi \nu) = 0.37 \cdot 10^{-6}, \hspace{1cm} (19)$$

$$B_S(\tau \to \eta' \pi \nu) = 2.63 \cdot 10^{-8}. \hspace{1cm} (20)$$

\footnote{We neglect the $p^2$ dependence for a rough estimate.}
The expression for the total width is

\[
\Gamma = \frac{G_F^2 |V_{ud}|^2}{384\pi m_\tau^2} \int_{m_v^2}^{m_s^2} \frac{ds}{s^3} \lambda^{1/2}(s, m_{\eta(\eta')}^2, m_s^2)(m_s^2 - s)^2 
\times \left( |T_V|^2(2s + m_s^2)\lambda(s, m_{\eta(\eta')}^2, m_s^2) + |T_S|^2 3m_s^2(m_{\eta(\eta')}^2 - m_s^2)^2 \right). \tag{21}
\]

Note that there is no interference between the vector and scalar intermediate state contributions. Thus, for branchings we get

\[
B(\tau^- \to \eta\pi^- \nu) = 4.72 \cdot 10^{-6}, \tag{22}
\]

\[
B(\tau^- \to \eta'\pi^- \nu) = 3.74 \cdot 10^{-8}. \tag{23}
\]

Let us note that our estimations for scalar contributions are much less than ones in previous works.

4 Conclusions

Our calculations are in qualitative agreement with the previous theoretical estimates obtained in \[7–10\]. However, the NJL model allows us to describe the transitions \(\pi^0 \to \eta(\eta')\) and \(\rho^- (W^-) \to a_0\) using the same methods. As the result, we can compare the contribution of amplitudes with intermediate scalar and vector mesons from uniform positions. These calculations show that in the decays \(\rho^- \to \eta\pi^-\) and \(\tau^- \to \eta\pi^-\nu\) the scalar meson plays an insignificant role. However, in the decay \(\tau^- \to \eta'\pi^-\nu\) the processes with intermediate \(a_0\) and \(a_0'\) make contributions comparable with the contributions of intermediate vector mesons.

It is worth noticing that the width of the decay \(\tau^- \to a_0\nu\) calculated in the NJL model is close to the values obtained in \[8\]

\[
\Gamma = \frac{G_F^2 |V_{ud}|^2 m_\tau^2}{16\pi} \left( \frac{\sqrt{6}}{2} \frac{m_d - m_u}{g_\rho} \right)^2 \left( 1 - \frac{m_{a_0}^2}{m_\tau^2} \right)^2, \tag{24}
\]

\[
B(\tau^- \to a_0\nu) = 3.28 \cdot 10^{-6}. \tag{25}
\]

This confirms the relevance of our expression for the vertex \(\tau a_0\nu\) used in \[13\]. For the vertex \(a_0 \to \eta\pi\) the expression was used \[9\]. We get the amplitude \[16\] by the matching these expressions through propagator of scalar \(a_0\) meson. This contradicts the VDM-like ansatz for the intermediate-resonance used in \[3,9,10\]

\[
\frac{e_{\tau\eta}^2 M_R^2}{M_R^2 - p^2 - iM_R\Gamma_R(p^2)} \tag{26}
\]

On the other hand, if we use this ansatz for a vector to scalar transition taken in \[5,9,10\] and calculate \(\rho^- \to \eta\pi^-\) with this ansatz then we get by an order of magnitude

\[
B \sim e_{\tau\eta}^2 \left( \frac{m_{a_0}^2}{m_{a_0'}^2 - m_\rho^2} \right)^2 \frac{\lambda^{3/2}(m_{a_0'}^2, m_{a_0'}^2, m_\rho^2)}{\lambda^{3/2}(m_{a_0'}^2, m_{a_0'}^2, m_\rho^2)} \sim 10^{-3}. \tag{27}
\]

This estimate for the branching fraction is close to the current experimental limit \[1,2\] and can be tested in the near future at the high-luminosity \(e^+e^-\) colliders in Novosibirsk and Beijing, for example. Therefore, the problem of relevancy of vector–scalar transition representation can be clarified.

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