A QUANTIZED TITS-KANTOR-KOECHER ALGEBRA

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ABSTRACT.
We propose a quantum analogue of a Tits-Kantor-Koecher algebra with a Jordan torus as an coordinated algebra by looking at the vertex operator construction over a Fock space.

Quantum toroidal algebras were first introduced by Ginzburg, Kapranov and Vasserot [GKV] in the study of the Langlands reciprocity for algebraic surfaces. These algebras are quantized analogues for toroidal Lie algebras of Moody-Rao-Yokonuma [MRY]. Representations of quantum toroidal algebras have been studied by Varagnolo-Vasserot [VV], Saito-Takemura-Uglov [STU], Saito [S], Frenkel-Jing-Wang [FJW], Takemura-Uglov[TU], Gao-Jing [GJ1,2], and among others.

The Tits-Kantor-Koecher (TKK) algebra was originally defined from Jordan algebra in constructing the finite dimensional simple Lie algebras of the exceptional types $E_6$ and $E_7$. It has also played an important role in the structure theory of newly developed extended affine Lie algebras.

A TKK algebra in the extended affine Lie algebras of type $A_1$ has been realized by gluing a Clifford module and a Heisenberg module in Tan’s paper [T]. This algebra appears as the core of extended affine Lie algebras of type $A_1$[AABGP] and has been studied by Yoshii [Y].

In this note, we shall propose a quantum analogue of the above Tits-Kantor-Koecher algebra. Our motivation comes from the vertex operator construction as was done in [GJ1, GJ2]. We hope that the quantum TKK algebra will be useful in the study of quantum toroidal algebras.

Like the quantum Kac-Moody algebra case [J2] our construction relies on an interesting combinatorial identity of Hall-Littlewood type [M]. It suggests that representations

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of our quantum TKK algebra probably will provide more generalized combinatorial identities of this type. We hope that generalization of our quantum TKK algebras can lead to further interesting combinatorial structures.

I. Tits-Kantor-Koecher construction.

Recall that a Jordan algebra \( J \) is a unital commutative algebra over \( \mathbb{F} \) satisfying
\[
(ab)a^2 = a(ba^2), \quad \text{for all } a, b \in J.
\]

Note that \( J \) may not be associative.

**Example 1.1** Let \( A \) be a unital commutative associative algebra over \( \mathbb{F} \) and \( V \) be an \( A \)-module equipped with an \( A \)-bilinear form
\[
f : V \times V \rightarrow A.
\]

Then
\[
J(A, V, f) = A \oplus V
\]
becomes a Jordan algebra over \( \mathbb{F} \) under the product
\[
(a + u)(b + v) = (ab + f(u, v)) + (av + bu)
\]
for \( a, b \in A, u, v \in V \).

Let \( J \) be a Jordan algebra. Set
\[
D_{a,b} = [L_a, L_b],
\]
for \( a, b \in J \). The \( \mathbb{F} \)-linear span \( D_{J,J} \) of all \( D_{a,b} \)'s is a Lie algebra called the inner derivation algebra of \( J \). They satisfy the following relations:
\[
D_{a,b} + D_{b,a} = 0,
\]
\[
D_{ab,c} + D_{bc,a} + D_{ca,b} = 0,
\]
\[
[D, D_{a,b}] = D_{Da,b} + D_{a,Db},
\]
for \( a, b, c \in J \) and any derivation \( D \) of \( J \).

The Tits-Kantor-Koecher algebra \( K(J) \) is defined to be a Lie algebra
\[
K(J) = (sl_2(\mathbb{F}) \otimes_{\mathbb{F}} J) \oplus D_{J,J}
\]
with Lie bracket:

\[ [A \otimes a, B \otimes b] = [A, B] \otimes ab + 2tr(AB)D_{a,b}, \]
\[ [D, A \otimes a] = A \otimes Da, \]

for \( A, B \in \mathfrak{sl}_2(\mathbb{F}) \), \( a, b \in J, D \in D_{J,J} \).

In the above example, we let \( \mathbb{F} = \mathbb{C}, A = \mathbb{C}[t^{\pm 2}, t_2^{\pm 2}], V = Aw_1 \oplus Aw_2, f(w_i, w_j) = \delta_{ij} t_i^2 \).

Let \( J \) be the resulting Jordan algebra. The Tits-Kantor-Koecher algebra \( K(J) \) is called a Baby TKK in \([T]\). This TKK algebra is indeed the smallest possible core of the extended affine Lie algebra which is coordinated by a Jordan torus—a nonassociative algebra.

Let \( d_1, d_2 \) be the degree derivations of \( J \). Define \( \chi : J \to \mathbb{C} \) to be the \( \mathbb{C} \)-linear function given by

\[
\chi(w_1^{n_1} w_2^{n_2}) = \begin{cases} 
1, & \text{if } n_1 = n_2 = 0, \\
0, & \text{otherwise}.
\end{cases}
\]

Define a two dimensional central extension of \( K(J) \) as follows:

\[ \hat{K}(J) = K(J) \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \]

with Lie bracket

\[
[A \otimes a, B \otimes b] = [A, B] \otimes ab + 2tr(AB)D_{a,b} \\
+ tr(AB)\chi((d_1a)b)c_1 + tr(AB)\chi((d_2a)b)c_2
\]

where \( A, B \in \mathfrak{sl}_2(\mathbb{C}) \), \( a, b \in J \), and \( c_1, c_2 \) are central elements of \( \hat{K}(J) \).

The semi-direct product of the Lie algebra \( \tilde{K}(J) \) and the two degree derivations:

\[ \tilde{K}(J) = \tilde{K}(J) \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2 \]

is an extended affine Lie algebra which is the smallest extended affine Lie algebra beyond the finite and affine types.

**Remark 1.2** Note that the Lie algebra \( \hat{K}(J) \) is generated by

\[
e_{12} \otimes w_1^m, \quad e_{21} \otimes w_1^m, \quad (e_{11} - e_{22}) \otimes w_1^m, \\
e_{12} \otimes (w_2 w_1^{2m}), \quad e_{21} \otimes (w_2 w_1^{2m}), \quad (e_{11} - e_{22}) \otimes (w_2 w_1^{2m})
\]

for \( m \in \mathbb{Z} \), where \( e_{ij} \)'s are the standard matrix units.
II. A Quantum TKK algebra.

The quantum TKK algebra \( U_q(\hat{K}(J)) \) is the unital associative algebra generated by

\[ q^{\pm c/2}, k_1^\pm, k_0^\pm, x_{1,m}^\pm, \psi_{1,m}^\pm, x_{0,2m}^\pm, \psi_{0,2m}^\pm, m \in \mathbb{Z} \]

subject to the following relations that \( q^{\pm c/2} \) is central and

\[
(2.1) \quad [h_{im}, h_{in}] = \frac{[2m]}{m} [mc] \delta_{m,-n},
\]

\[
(2.2) \quad [h_{1m}, h_{0n}] = -\frac{[m]}{m} [mc] (d^m + d^{-n}) \delta_{m,-n},
\]

\[
(2.3) \quad [h_{im}, x_{i,n}^\pm] = \pm \frac{[2m]}{m} q^{\pm |mc|/2} x_{i,m+n}^\pm,
\]

\[
(2.4) \quad [h_{im}, x_{j,n}^\pm] = \pm \frac{[m]}{m} q^{\pm |mc|/2} (d^m + d^{-m}) x_{i,m+n}^\pm,
\]

\[
(2.5) \quad x_{1,m+1}^\pm x_{1,n}^- - q^{\pm 2} x_{1,n}^- x_{1,m+1} = q^2 x_{1,m}^\pm x_{1,n+1}^- - x_{1,n+1}^- x_{1,m}^\pm,
\]

\[
(2.6) \quad x_{1,m}^\pm x_{0,n+2}^- + q^{\pm 2} x_{1,m+2}^- x_{0,n}^- + x_{0,n}^- x_{1,m+2}^- + q^{\pm 2} x_{0,n+2}^- x_{1,m}^- = 0,
\]

\[
(2.7) \quad [x_{im}^+, x_{jn}^-] = \frac{\delta_{ij}}{q - q^{-1}} (\psi_{i,m+n}^m q^{(m-n)c/2} - \psi_{i,m+n}^- q^{(n-m)c/2}),
\]

\[
+ [3]_{1} x_{i,m_1}^\pm x_{i,m_2}^\pm x_{i,m_3}^- x_{j,n}^- + [3]_{1} x_{i,m_1}^\pm x_{i,m_2}^\pm x_{i,m_3}^- x_{j,n}^\pm
\]

\[
+ [3]_{1} x_{i,m_1}^\pm x_{i,m_2}^\pm x_{i,m_3}^- x_{j,n}^\pm + x_{j,n}^\pm x_{i,m_1}^\pm x_{i,m_2}^\pm x_{i,m_3}^- + \text{Perm}\{m_1, m_2, m_3\} = 0, \quad \text{for } i \neq j,
\]

where \( d = -\sqrt{-1} \),

\[
[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [mc] = \frac{q^{mc} - q^{-mc}}{q - q^{-1}},
\]

and

\[
(2.9) \quad \sum_{n=0}^{\infty} \psi_{1,n}^\pm z^{\mp n} = k_1^\pm \exp(\pm(q - q^{-1}) \sum_{n>0} h_{1,n}^\pm z^{\mp n}),
\]

\[
(2.10) \quad \sum_{n=0}^{\infty} \psi_{0,2n}^\pm z^{\mp 2n} = k_0^\pm \exp(\pm(q - q^{-1}) \sum_{n>0} h_{0,n}^\pm z^{\mp 2n}),
\]

Write

\[
e_1(z) = \sum_{n \in \mathbb{Z}} x_{1m}^+ z^{-m}, \quad f_1(z) = \sum_{n \in \mathbb{Z}} x_{1m}^- z^{-m}
\]

\[
e_0(z) = \sum_{n \in \mathbb{Z}} x_{02m}^+ z^{-m}, \quad f_0(z) = \sum_{n \in \mathbb{Z}} x_{02m}^- z^{-m}
\]
Then the defining relations for the quantum TKK algebra can be rewritten as

\[ (2.11) \quad \psi_i^+(z) \psi_j^-(w) = \psi_j^-(w) \psi_i^+(z) q^{2c-4(wz)^2 +1} c^4(wz)^4 + w^2 c^4(wz)^2 + 1, \quad i \neq j \]

\[ (2.12) \quad \psi_i^+(z) \psi_i^-(w) = \psi_i^-(w) \psi_i^+(z) q^{-2c-2(wz)^2 - 1} c^2(wz)^2 - q^{-2} \]

\[ (2.13) \quad (z - q^2 w) e_i(z) e_i(w) = (q^2 z - w) e_i(w) e_i(z), \]

\[ (2.14) \quad [e_i(z), f_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} \{ \psi_i^+(cw) \delta(c^{-2} \frac{z}{w}) - \psi_i^-(cz) \delta(c^{2} \frac{z}{w}) \}
\]

\[ (2.15) \quad (w^2 + q^{-2} z^2) e_1(z) e_0(w) = -(z^2 + q^{-2} w^2) e_0(w) e_1(z) \]

\[ (2.16) \quad (w^2 + q^2 z^2) f_1(z) f_0(w) = -(z^2 + q^2 w^2) f_0(w) f_1(z) \]

\[ (2.17) \quad \psi_i^+(z) e_i(w) = e_i(w) \psi_i^+(z) \frac{q^{2c-2(wz)^2 +1}}{c^2(wz)^2 + q^{2}} \]

\[ (2.18) \quad \psi_i^+(z) f_i(w) = f_i(w) \psi_i^+(z) \frac{q^{2c-2(wz)^2 +1}}{c^2(wz)^2 + q^{2}} \]

\[ (2.19) \quad \psi_i^+(z) e_j(w) = e_j(w) \psi_i^+(z) \frac{q^{2c-2(wz)^2 +1}}{c^2(wz)^2 + q^{2}}, \quad i \neq j \]

\[ Sym_{z_1, z_2, z_3} \{ e_i(z_1) e_i(z_2) e_i(z_3) e_j(w) + [3] e_i(z_1) e_i(z_2) e_j(w) e_i(z_3) + \]

\[ + [3] e_i(z_1) e_j(w) e_i(z_2) e_i(z_3) + e_j(w) e_i(z_1) e_i(z_2) e_i(z_3) \} = 0, \quad \text{for} \quad a_{ij} = -2 \]

\[ Sym_{z_1, z_2, z_3} \{ f_i(z_1) f_i(z_2) f_i(z_3) f_j(w) + [3] f_i(z_1) f_i(z_2) f_j(w) f_i(z_3) + \]

\[ + [3] f_i(z_1) f_j(w) f_i(z_2) f_i(z_3) + f_j(w) f_i(z_1) f_i(z_2) f_i(z_3) \} = 0, \quad \text{for} \quad a_{ij} = -2 \]

**Remarks 2.22**

1. The subalgebra generated by $h_{1m}$, $x_{1m}^+$ or $q_{1m}^+$, $x_{1m}^-$ is isomorphic to the quantum affine algebra $U_q(\hat{s}l_2)$.

2. The deformation $U_q(\hat{K}(J))$ to $U(\hat{K}(J))$ can be achieved via

\[ x_{1m}^+ \rightarrow e_{12} \otimes w_{1m}^1, \quad x_{1m}^- \rightarrow e_{21} \otimes w_{1m}^n, \]

\[ h_{1m} \rightarrow (e_{11} - e_{22}) \otimes w_{1m}^1, \]

\[ x_{0,2m}^+ \rightarrow e_{12} \otimes (w_{2w_{1m}}^2), \quad x_{0,2m}^- \rightarrow e_{21} \otimes (w_{2w_{1m}}^2), \]

\[ h_{02m} \rightarrow (e_{11} - e_{22}) \otimes (w_{2w_{1m}}^2), \]

\[ q^{c/2} \rightarrow c_1, \quad k_0^{+} k_1^{+} \rightarrow c_2 \]
III. Vertex operator representation.

Let $P = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2$ be a rank 2 free abelian group provided with a $\mathbb{Z}$-bilinear form $\langle \cdot , \cdot \rangle$ defined by $\langle \epsilon_i , \epsilon_j \rangle = \delta_{ij}$, $1 \leq i, j \leq 2$. Let $Q = \mathbb{Z}(\epsilon_1 - \epsilon_2)$ be the rank 1 free subgroup of $P$.

Let $C[Q] = \bigoplus \mathbb{C}e^\alpha$ be the group algebra of $Q$. Also, for $\beta \in H = Q \otimes \mathbb{Z}\mathbb{C}$, define $\beta(0) \in \text{End} \mathbb{C}[Q]$ by

$$\beta(0)e^\alpha = (\beta, \alpha)e^\alpha, \text{ for } \alpha \in Q.$$ 

Next let $\epsilon_i(n)$ and $C$ be the generators of the Heisenberg algebra $\mathcal{H}$, $1 \leq i \leq 2, n \in \mathbb{Z} \setminus \{0\}$, subject to relations that $C$ is central and

$$[\epsilon_i(m), \epsilon_j(n)] = m\delta_{ij}\delta_{m+n,0}C. \quad (3.1)$$ 

Let $S(\mathcal{H}^-) = \mathbb{C}[\epsilon_i(n) : 1 \leq i \leq 2, n \in -\mathbb{Z}_+]$ denote the symmetric algebra of $\mathcal{H}^-$, which is the algebra of polynomials in infinitely many variables $\epsilon_i(n), 1 \leq i \leq 2, n \in -\mathbb{Z}_+$, where $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n > 0\}$. $S(\mathcal{H}^-)$ is an $\mathcal{H}$-module in which $C = 1$, $\epsilon_i(n)$ acts as the multiplication operator for $n \in -\mathbb{Z}_+$, and $\epsilon_i(n)$ acts as the partial differential operator for $n \in \mathbb{Z}_+$.

Set $V_Q = S(\mathcal{H}^-) \otimes \mathbb{C}[Q]$. The operator $z^\alpha \in (\text{End} \mathbb{C}[Q])[z, z^{-1}]$ is defined as

$$z^\alpha e^\beta = z^{(\alpha, \beta)}e^\beta$$

for $\alpha, \beta \in Q$.

Let $\mu$ be any non-zero complex number. Consider the valuation $\mu^\alpha$ of the operator $z^\alpha$. Namely, $\mu^\alpha$ is the operator $\mathbb{C}[Q] \to \mathbb{C}[Q]$ given by

$$\mu^\alpha e^\beta = \mu^{(\alpha, \beta)}e^\beta, \text{ for } \alpha, \beta \in Q.$$ 

Now we set $\epsilon_{i+2} = \epsilon_i, \text{ for } i \in \mathbb{Z}.$
Accordingly,

\[(\epsilon_i, \epsilon_j) = \delta_{ij} = \delta_{\overline{i}, \overline{j}}, \text{ for } \overline{i}, \overline{j} \in \mathbb{Z}/2\mathbb{Z}.\]

For \(r, i, j \in \mathbb{Z}\), we define the vertex operator \(X_{ij}(r, z)\) as follows.

\[
X_{ij}(r, z) = : \exp\left(-\sum_{n \neq 0} \frac{(\epsilon_i(n) - (-1)^{-rn}q^{i-j}|n|\epsilon_j(n))}{n} z^{-n}\right) : e^{\epsilon_i - \epsilon_j} z^{\epsilon_i - \epsilon_j} + \frac{(\epsilon_i - \epsilon_j, \epsilon_i - \epsilon_j)}{2} (-1)^{-r\epsilon_j - \frac{(\epsilon_j - \epsilon_i)}{2}} r
\]

Due to \(\epsilon_0 = \epsilon_2\) we note that \(X_{01}(r, z) = X_{21}(r, z)\). Next, for \(r, i, j \in \mathbb{Z}\), and \(i \neq j\), we define

\[
u_{ij}(r, z)
= -q^{(j-i)(\epsilon_i - \epsilon_j)} \cdot \exp\left(\sum_{n \geq 1} \frac{q^{(j-i)n} - q^{(i-j)n}}{n} (q^{j-i} n \epsilon_i(n) - (-1)^{-rn} q^{i-j} n \epsilon_j(n) z^{-n})\right)
\]

\[
u_{ij}(r, z)
= -q^{(i-j)(\epsilon_i - \epsilon_j)} \cdot \exp\left(\sum_{n \geq 1} \frac{q^{(i-j)n} - q^{(j-i)n}}{n} (q^{i-j} n \epsilon_i(-n) - (-1)^{nr} q^{i-j} n \epsilon_j(-n) z^n)\right).
\]

Write

\[
X_{ij}(r, z) = \sum_{n \in \mathbb{Z}} X_{ij}(r, n) z^{-n},
\]

\[
u_{ij}(r, z) = \sum_{n=0}^{\infty} \nu_{ij}(r, n) z^{-n},
\]

\[
u_{ij}(r, z) = \sum_{n=0}^{\infty} \nu_{ij}(r, n) z^n.
\]

We now state our main result of this note.

**Theorem 3.2.** The linear map \(\pi\) given by

\[
\pi(x_{1,m}^+) = X_{12}(0, m), \quad \pi(x_{1,m}^-) = X_{21}(0, m),
\]

\[
\pi(\psi_{1,m}^+) = u_{12}(0, m), \quad \pi(\psi_{1,m}^-) = v_{12}(0, m)
\]

\[
\pi(x_{0,2m}^+) = X_{01}(1, 2m), \quad \pi(x_{0,2m}^-) = X_{10}(-1, 2m),
\]

\[
\pi(\psi_{0,2m}^+) = u_{01}(1, 2m), \quad \pi(\psi_{0,2m}^-) = v_{01}(1, 2m),
\]

\[
\pi(k_{1}^\pm) = q^{\pm(\epsilon_1 - \epsilon_2)}, \quad \pi(k_{0}^\pm) = q^{\pm(\epsilon_2 - \epsilon_1)}, \quad \pi(q^{c/2}) = q^{1/2}
\]
gives a representation of $U_q(\hat{K}(J))$.

Proof. To prove the theorem we need to verify that the defined operators satisfy the commutation relations in the quantum TKK algebra.

The following result is from [GJ2].

**Lemma 3.3.** For $r_1, r_2 \in \mathbb{Z}$ we have

$$X_{ij}(r_1, z)X_{ij}(r_2, z) =: X_{ij}(r_1, z)X_{ij}(r_2, z) = \frac{z}{w}(1 - \frac{w}{z})(1 - (-1)^{r_2-r_1} q^{-2} \frac{w}{z})(-1)^{r_1}$$

$$X_{ij}(r_1, z)X_{ji}(r_1, z) =: X_{ij}(r_1, z)X_{ji}(r_1, z) = \frac{w}{z}(1 - (-1)^{r_2} \frac{w}{qz})^{-1}(1 - (-1)^{r_1} \frac{w}{qz})^{-1}(1)^{r_1}$$

First of all we notice that the operators $X_{12}(0, z), X_{21}(0, z)$ and $u_{12}(0, z), v_{12}(0, z)$ gives a level one representation of the quantum affine algebra $U_q(sl_2)$ on the space $V$. In fact let $c = q$ we have

$$[\epsilon_1(m) - (-1)^{-rm} q^{-|m|} \epsilon_2(m), \epsilon_1(m) - (-1)^{-rm} q^{-|n|} \epsilon_2(n)] = m(1 + q^{-2|n|}) \delta_{m,-n}$$

Using the identity $e^A e^B = e^{B e^A [A,B]}$ when $[A, B]$ commutes with $A$ and $B$, one obtains immediately that

$$(z - q^2 w)X_{12}(0, z)X_{12}(0, w) = (q^2 - w)X_{12}(0, w)X_{12}(0, z)$$

$$[X_{12}(0, z), X_{12}(0, w)] = \frac{1}{q - q^{-1}}(u_{12}(cw)\delta(c^{-2} \frac{z}{w}) - v_{12}(cz)\delta(c^2 \frac{z}{x}))$$

Taking derivative on the operator $u_{ij}(r, z)$ and $v_{ij}(r, z)$, the map $\pi$ in terms of components is given by

$$h_{0m} \rightarrow (q^{|m|/2} \epsilon_1(m) - q^{-|m|/2} \epsilon_2(m)) \frac{|m|}{m}$$

$$h_{1m} \rightarrow (q^{|m|/2} \epsilon_2(m) - (-1)^{m} q^{-|m|/2} \epsilon_1(m)) d^{-m} \frac{|m|}{m}$$

It follows that

$$[\pi(h_{1m}), \pi(h_{0n})] = - \frac{|m|^2}{m} (1 + d^{2n}) d^{-n} \delta_{m,-n}$$

$$= - \frac{|m|}{m} (d^m + d^{-m}) \delta_{m,-n},$$

with $c = q$ and $C = 1$. Similarly one can check that for $i \neq j$ we have

$$[\pi(h_{im}), \pi(x_{jn}^\pm)] = \mp \frac{|m|}{m} q^{|m|/2}(d^m + d^{-m}) \pi(x_{jm+n}^\pm).$$
We now prove that \([x_{im}^+, x_{jn}^-] = 0\) for \(i \neq j\). In fact we have

\[
X_i^+(z)X_j^-(w) =: X_i^+(z)X_j^-(w) : \frac{z}{w}(1 - \frac{w}{z})(1 - p^{-1}\frac{w}{z})p
\]

\[
X_j^+(w)X_i^-(z) =: X_j^+(w)X_i^-(z) : \frac{w}{z}(1 - \frac{z}{w})(1 - p^{-1}\frac{z}{w})p
\]

It follows quickly that \([X_i^+(z), X_j^-(w)] = 0\).

Finally let’s prove the Serre relation.

The following OPE’s are direct consequences of Lemma 3.3.

\[
E_1(z)E_1(w) =: E_1(z)E_1(w) : (z - w)(z - q^{-2}w)
\]

\[
E_1(z)E_0(w) =: E_1(z)E_0(w) : \frac{zw}{z^2 + q^{-2}w^2}
\]

\[
E_1(w)E_0(z) =: E_1(w)E_0(z) : \frac{p^{-1}zw}{w^2 + q^{-2}z^2}
\]

Then we have

\[
E_1(z_1)E_1(z_2)E_1(z_3)E_0(w) =: E_1(z_1)E_1(z_2)E_1(z_3)E_0(w) :
\]

\[
\prod_{i<j} \frac{(z_i - z_j)(z_i - q^{-2}z_j)}{z_iz_j} \prod_{i=1}^3 \frac{ziwd}{z_i^2 + q^{-2}w^2}
\]

\[
E_1(z_1)E_1(z_2)E_0(w)E_1(z_3) =: E_1(z_1)E_1(z_2)E_0(w)E_1(z_3) :
\]

\[
\prod_{i<j} \frac{(z_i - z_j)(z_i - q^{-2}z_j)}{z_iz_j} \frac{p^{-1}z_1z_2z_3w^3d^3}{(z_1^2 + q^{-2}w^2)(z_1^2 + q^{-2}w^2)(w^2 + q^{-2}z_3^2)}
\]

\[
E_1(z_1)E_0(w)E_1(z_2)E_1(z_3) =: E_1(z_1)E_0(w)E_1(z_2)E_1(z_3) :
\]

\[
\prod_{i<j} \frac{(z_i - z_j)(z_i - q^{-2}z_j)}{z_iz_j} \frac{p^{-2}z_1z_2z_3w^3d^3}{(z_1^2 + q^{-2}w^2)(w^2 + q^{-2}z_3^2)(w^2 + q^{-2}z_3^2)}
\]

\[
E_0(w)E_1(z_1)E_1(z_2)E_1(z_3) =: E_0(w)E_1(z_1)E_1(z_2)E_1(z_3) :
\]

\[
\prod_{i<j} \frac{(z_i - z_j)(z_i - q^{-2}z_j)}{z_iz_j} \frac{p^{-3}z_1z_2z_3w^3d^3}{(w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2)(w^2 + q^{-2}z_3^2)}
\]
Therefore we have

\[ E_1(z_1)E_1(z_2)E_1(z_3)E_0(w) + [3]E_1(z_1)E_1(z_2)E_0(w)E_1(z_3) + \]
\[ + [3]E_1(z_1)E_0(w)E_1(z_2)E_1(z_3) + E_0(w)E_1(z_1)E_1(z_2)E_1(z_3) \]
\[ =: E_1(z_1)E_1(z_2)E_1(z_3)E_0(w) \cdot \prod_{i<j} \frac{(z_i - z_j)(z_i - q^{-2}z_j)}{z_i z_j} (z_1 z_2 z_3)(wd)^3 \]
\[ \cdot \left\{ \frac{1}{(z_1^2 + q^{-2}w^2)(z_1^2 + q^{-2}w^2)(z_1^2 + q^{-2}w^2)} \right\} + [3] \]
\[ + \right\{ \frac{1}{(z_1^2 + q^{-2}w^2)(w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2) - [3](w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2)} - [3](w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2) - [3](w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2) \}
\[ \cdot \right\{ \prod_{i<j} \frac{z_i - q^{-2}z_j}{z_i - z_j} \right\} \]

Thus the Serre relation holds if the following combinatorial identity is truth.

**Lemma 3.4.** Let \( \mathfrak{S}_3 \) act on \( z_1, z_2, z_3 \) via \( \sigma z_i = z_{\sigma(i)} \). Then

\[
\sum_{\sigma \in \mathfrak{S}_3} \sigma \left[ (w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2)(w^2 + q^{-2}z_3^2) - [3](w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2)(w^2 + q^{-2}z_3^2) - [3](w^2 + q^{-2}z_1^2)(w^2 + q^{-2}z_2^2)(w^2 + q^{-2}z_3^2) \right] = 0.
\]

Proof of the Lemma. Considering the left-hand side as a polynomial in \( w \), we extract the constant term.

\[
\sum_{\sigma \in \mathfrak{S}_3} (q^{-6} - [3]q^{-4} + [3]q^{-2} - 1)(z_1 z_2 z_3)^2 \cdot \prod_{i<j} \frac{z_i - q^{-2}z_j}{z_i - z_j} = 0.
\]

Similarly the highest coefficient of \( w^6 \) is seen to be zero.
The coefficient of $w^2$ and $w^4$ are essentially the same up to swapping of $z_i$ with $z_i^{-1}$. Thus the identity (3.5) in Lemma 3.4 boils down to the truth of the following identity.

\[(3.6) \quad \sum_{\sigma \in S_3} \sigma.\{q^{-3}z_1^2 - (q + q^{-1})z_2^2 + q^3z_3^2\} \prod_{i < j} \frac{z_i - q^{-2}z_j}{z_i - z_j} = 0,\]

where the left-hand side times $q^{-5} - q^{-1}$ is the coefficient of $w^4$ of the polynomial in Eq. (3.5).

The identity (3.6) is easily proved by comparing coefficients of $z_i$ or direct verification. Hence Lemma 3.4 is proved. Similarly one can prove the Serre relations for the $F_i(z)$'s, and Theorem 3.2 is proved.

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