A numerical treatment for the system of singularly perturbed convection-reaction-diffusion two point boundary value problems with non-smooth data

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Abstract. Two point Singularly Perturbed Weakly Coupled System of Convection-Reaction-Diffusion Problems (WCSCRPDS), having small parameters ($\varepsilon_1, \varepsilon_2$) multiplying the highest derivative (diffusion) and the next highest derivative (convection) term with discontinuity over the source term is studied. Generally, for adequately small values of the perturbation parameters, the exact solution involves boundary and interior layers. Hence, the analysis of the problem splits into two cases, in accordance with the ratio of the convection and diffusion parameters. The continuous bounds of the solution and its derivatives are derived. A numerical method which is parameter uniform is constructed, on a suitably fitted piecewise uniform Shishkin mesh. The error estimation analysis is discussed and the numerical method applied here exhibits almost first order convergence. Numerical examples and the corresponding tables and figures demonstrate the efficiency of the numerical scheme.

1. Introduction
The Singular Perturbation Problems (SPPs) generally occur in many branches of science and engineering. For example skin layers in electrical applications, edge and shock layers in solid mechanics, boundary layers in fluid mechanics, surfaces and Stokes lines in mathematics and not the last to refer transition points in quantum mechanics. Perturbations affect these types of problems around a very confined domain where the solution undergoes a swift change which is not measurable. These narrow regions very often adjoin the interior or the boundary points of the region of interest. The only possible reason for this is the augmentation of the parameters with the highest derivatives. The formulation and analysis of best chosen numerical methods for Singularly Perturbed Differential Equations (SPDE) is a huge area of significance. Despite a large amount of work being done in the research field of SPPs for smooth [1, 2, 3, 4, 5] and non-smooth data [6, 7, 8, 9, 10, 11], there is a platform for more relevant research to explore. Recently, a numerical scheme is discussed for a Singularly Perturbed Boundary Value Problem (SPBVP), with smooth data in [12] to arrive at first order convergence. All the study discussed so far relate to SPPs in which a small parameter affects the highest derivative term. Authors in [13, 14] considered Convection-Reaction-Diffusion Problem (CRDP) with smooth data. Zahara [15] constructed a new non-polynomial cubic spline method to solve a similar type of problem in which the computational results presented confirm a higher order accuracy of order two to the nu-
merical scheme considered. Two parameter (SPBVP) with non-smooth data is considered by
the authors in [16, 17, 18]. Moreover, the solution of the linear system of SPPs for convection-
reaction-diffusion type is a wide region to examine. This type of problems can be modeled as
turbulence flow owing to the interactions of waves with the steady current[5]. Similar equations
are also found in the applications while experimenting with the diffusion process intricate by
chemical reactions in the field of electroanalytical chemistry. The diffusion coefficients of the
substances are the parameter that augment the highest derivatives. Motivated by the works of
[10, 19], in this study we have considered a numerical technique to solve a WCSCRDPs involving
discontinuous source terms in the unit interval $\bar\Upsilon = (0, 1)$. is defined as follows:
Find $\bar{u} = u_1, u_2 \in C^0(\bar{\Upsilon}) \cap C^1(\Upsilon) \cap C^2(\Upsilon^- \cup \Upsilon^+)$ such that

$$
\begin{align}
L_1 \bar{u}(s) &\equiv \varepsilon_1 u_1''(s) + \varepsilon_2 a_1(s) u_1'(s) - p_{11}(s) u_1(s) - p_{12}(s) u_2(s) = q_1(s) \quad (1) \\
L_2 \bar{u}(s) &\equiv \varepsilon_1 u_2''(s) + \varepsilon_2 a_2(s) u_2'(s) - p_{21}(s) u_1(s) - p_{22}(s) u_2(s) = q_2(s) \quad (2)
\end{align}
$$

$$
\forall x \in (\Upsilon^- \cup \Upsilon^+)
$$

$$
\begin{align}
\bar{u}_1(0) = q_1, \bar{u}_2(0) = q_2, \bar{u}_1(1) = r_1, \bar{u}_2(1) = r_2, \quad |\tilde{g}_i| \leq C.
\end{align}
$$

where $0 < \varepsilon_1 << 1, 0 < \varepsilon_2 \leq 1$, are the two perturbation parameters with

$$
a_1(s) > \alpha_1 > 0, a_2(s) > \alpha_2 > 0,
$$

$$
\begin{align}
p_{11}(s), p_{12}(s) &\geq 0, p_{11}(s) \geq |p_{12}(s)|, \quad p_{21}(s), p_{22}(s) \geq 0, p_{22}(s) \geq |p_{21}(s)|, \\
p_{11}(s) + p_{12}(s) &\geq \beta_1(s) > 0, \quad p_{21}(s) + p_{22}(s) \geq \beta_2(s) > 0.
\end{align}
$$

The coefficients $a_i(s)$ and $p_{i,j}(s)$ for $(i, j = 1, 2)$ are sufficiently smooth functions in $\bar{\Upsilon}$. The
source term functions $q_1(s)$ and $q_2(s)$ are assumed to be adequately smooth on $(\Upsilon^- \cup \Upsilon^+)$. Their
derivatives have a single jump discontinuity at $d \in \Upsilon$, denoted by $[\nu]d = \nu(d^+) - \nu(d^-)$. Eventually, this discontinuity would give rise to internal layers in the solution of $\bar{u}(s)$ of the continuous problem (1)-(3). It is comfortable to introduce the symbols $\bar{\Upsilon} = [0, 1]$, $\Upsilon^- = (0, d)$ and $\Upsilon^+ = (d, 1), d \in \Upsilon$. When $\varepsilon_2 = 1$, the problem behaves like the established diffusion-convection problem [8] and for $\varepsilon_2 = 0$, it behaves like the diffusion-reaction problem [6]. In the current
paper, the analysis of the problem (1)-(3) is split into two cases given as,

Case (i): $\sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1}$ and Case (ii): $\sqrt{\alpha \varepsilon_2} \geq \sqrt{\eta \varepsilon_1}$,

where $\alpha = \min\{\alpha_1, \alpha_2\}$ and $\eta = \min\{\eta_1, \eta_2\}$.

It is assumed that

$$
\begin{align}
\eta_1 &= \min_{\Upsilon \setminus \{d\}} \left\{ \frac{p_{11}(s) + p_{12}(s)}{a_1(s)} \right\}, \\
\eta_2 &= \min_{\Upsilon \setminus \{d\}} \left\{ \frac{p_{21}(s) + p_{22}(s)}{a_2(s)} \right\}
\end{align}
$$

The results in forthcoming sections show that the considered CRDP behaves more like the reaction type problem for $\sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1}$ with a layer width of $\mathcal{O}(\sqrt{\varepsilon_1})$ appearing in the neighborhood of $s = 0, s = d$ and $s = 1$. When $\sqrt{\alpha \varepsilon_2} \geq \sqrt{\eta \varepsilon_1}$, a layer width of $\mathcal{O}(\frac{\varepsilon_1}{\varepsilon_2})$ in the neighborhood of $s = 0, s = d$ and a layer width of $\mathcal{O}(\frac{\varepsilon_1}{\varepsilon_2})$ in the neighborhood of $s = d$ and $s = 1$ can be predicted. The matrix-vector form of the considered problem (1)-(3) is represented as
\[ L\bar{u} = \varepsilon_1 \bar{u}'' + \varepsilon_2 A(s)\bar{u}' - B(s)\bar{u} = \bar{g}(s), \quad x \in (\Upsilon^- \cup \Upsilon^+), \quad (4) \]

with the boundary conditions

\[ \bar{u}(0) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \bar{u}(1) = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \quad (5) \]

where

\[
L = \begin{pmatrix} L_1(s) \\ L_2(s) \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u_1(s) \\ u_2(s) \end{pmatrix}, \quad \bar{\varepsilon}_1 = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_1 \end{pmatrix}, \quad \bar{\varepsilon}_2 = \begin{pmatrix} \varepsilon_2 & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \quad A(s) = \begin{pmatrix} a_1(s) & 0 \\ 0 & a_2(s) \end{pmatrix}, \quad B(s) = \begin{pmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{pmatrix}, \quad \bar{g}(s) = \begin{pmatrix} g_1(s) \\ g_2(s) \end{pmatrix} = \begin{cases} g_{11}(s), & \text{for } s \leq d \\ g_{12}(s), & \text{for } s \geq d \end{cases} \begin{cases} g_{21}(s), & \text{for } s \leq d \\ g_{22}(s), & \text{for } s \geq d \end{cases} 
\]

The other sections of the paper are coordinated as below. In Section 2 some a priori bounds and decomposition of the problem (1)-(3) is described. Discretization of the continuous problem and the methods to be applied with the discrete bounds are described in Section 3. Decomposition and bounds for the discrete solution are conferred in Section 4. In Section 5, the convergence and the methods to be applied with the discrete bounds are described in Section 3. Decomposition of the problem (1)-(3) is described. Discretization of the continuous problem (1)-(3) mitigate the following minimum principle.

**Lemma 1.** The WCSCRDP (1)-(3) has a solution, such that \(u_1(s), u_2(s) \in C^0(\Upsilon) \cap C^1(\Upsilon) \cap C^2(\Upsilon^- \cup \Upsilon^+)\). \(\square\)

The differential operator \(L\) of the continuous problem (1)-(3) mitigate the following minimum principle.

**Lemma 2.** (Minimum principle) Let the solution \(\bar{\nu}(s) \in C^0(\Upsilon) \cap C^2(\Upsilon^- \cup \Upsilon^+)\) satisfies \(\bar{\nu}(0) \geq \bar{0}, \quad \bar{\nu}(1) \geq \bar{0}\), and \(L_1\bar{\nu}(s) \leq \bar{0}, L_2\bar{\nu}(s) \leq \bar{0} \forall s \in (\Upsilon^- \cup \Upsilon^+)\) and \(|\bar{\nu}|d \leq \bar{0}\). Then \(\bar{\nu}(s) \geq \bar{0}, \quad \forall s \in \Upsilon\). \(\square\)

The very next result that follows, minimum principle is the stability of the solution.

The Lemmas 3 and 4 can be proved following the steps and techniques adopted in \([2, 13]\).
Lemma 3. Let \( u_1(s), u_2(s) \in C^0(\bar{\Upsilon}) \cap C^1(\Upsilon) \cap C^2(\Upsilon^- \cup \Upsilon^+) \) then
\[
||u_i(s)||_{\bar{\Upsilon}} \leq C \max \{||u_i(0)||, ||u_i(1)||, ||u_1(1)||, ||u_2(1)||\} + \Sigma_i \left\{ \frac{1}{\eta_i} \{||g_i||_{(\Upsilon^- \cup \Upsilon^+)}\} \right\}, \quad s \in \bar{\Upsilon}, \ i = 1, 2. \]

Lemma 4. For all \( 0 \leq k \leq 3 \), \( j = 1, 2 \), \( u_j(s) \) be the solution of the problem (1)-(3) and the derivatives follow the following bounds.
\[
||u_j^{(k)}||_{\bar{\Upsilon}} \leq C \left(1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^k\right) \max \{||u||, ||g||\}, \quad k = 1, 2, \]
\[
||u_j^{(3)}||_{\bar{\Upsilon}} \leq C \left(1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^3\right) \max \{||u||, ||g||, ||g'||\} \]

The solution \( \bar{u}(s) \) is decomposed into regular \( \bar{y}(s) \) and singular \( \bar{z}(s) \) components to obtain sharper bounds in the error estimate. It is inevitable to split the analysis into two cases depending upon the ratio of \( \varepsilon_2 \) to \( \varepsilon_1 \) given by \( \sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1} \) and \( \sqrt{\alpha \varepsilon_2} \geq \sqrt{\eta \varepsilon_1} \). The solution \( \bar{u}(s) \) is decomposed as \( \bar{u}(s) = \bar{y}(s) + \bar{z}(s) + \bar{z}'(s) \), for both the cases, where \( \bar{y}(s) = (y_1(s), y_2(s))^T \) and \( \bar{z}(s) = (z_1(s), z_2(s))^T \). The regular component \( \bar{y}(s) \) is defined as the solution of the following problem,
\[
L\bar{y}(s) = \bar{y}(s), \quad s \in (\Upsilon^- \cup \Upsilon^+), \quad \bar{y}(0) = y(0), \ \bar{y}(1) = y(1), \ \bar{y}(d-) \text{ and } \bar{y}(d+) \text{ are chosen,}
\]where
\[
\bar{y}(s) = \begin{cases} 
\bar{y}^-(s), & s \in \Upsilon^-, \\
\bar{y}^+(s), & s \in \Upsilon^+.
\end{cases}
\]The singular component \( \bar{z}_l(s) \) and \( \bar{z}_r(s) \) are the solutions of
\[
L\bar{z}_l(s) = 0, \quad s \in (\Upsilon^- \cup \Upsilon^+), \ \bar{z}_l(0) = \bar{z}_l(1) = 0, \quad \text{(6)}
\]
\[
L\bar{z}_r(s) = 0, \quad s \in (\Upsilon^- \cup \Upsilon^+), \ \bar{z}_r(0) = \bar{z}_r(1) = 0, \quad \text{(7)}
\]
\[
[\bar{z}_l]'d = -[\bar{y}]d - [\bar{z}_l]'d \text{ and } [\bar{z}_r]'d = -[\bar{y}]d - [\bar{z}_r]'d. \quad \text{(8)}
\]where
\[
\bar{z}_l(s) = \begin{cases} 
\bar{z}_l^-(s), & s \in \Upsilon^-, \\
\bar{z}_l^+(s), & s \in \Upsilon^+.
\end{cases}
\]
\[
\bar{z}_r(s) = \begin{cases} 
\bar{z}_r^-(s), & s \in \Upsilon^-, \\
\bar{z}_r^+(s), & s \in \Upsilon^+.
\end{cases}
\]The sum of the regular \( \{(\bar{y}(s))\} \) and the singular \( \{(\bar{z}_l(s)), (\bar{z}_r(s))\} \) components are in \( C^1(\Upsilon) \), by (8), although they are discontinuous at \( s = d \),

The sharper bounds for \( \bar{y}(s), \bar{z}_l^-(s), \) and \( \bar{z}_r^-(s), \bar{z}_{r}^-(s) \), are defined in the succeeding Lemmas for the two ratios of the parameters.
Consider the case(i): \( \sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1} \).
The proof of Lemmas 5 and 8 follows from the principles adopted in [2, 13].

Lemma 5. For \( 0 \leq k \leq 3 \) the following bounds are satisfied by the regular component \( \bar{y}(s) \),
\[
||\bar{y}^{(k)}||_{\Upsilon \setminus \{d\}} \leq C \left(1 + \frac{1}{(\varepsilon_1)^{k-2}}\right), \quad 0 \leq k \leq 3. \quad \square
\]
Lemma 6. For $0 \leq k \leq 3$, the following bounds are satisfied by singular components $\bar{z}_l(s)$ and $\bar{z}_r(s)$,

$$
\|\bar{z}_l^{(k)}\|_{Y\setminus\{d\}} \leq \frac{C}{(\sqrt{\varepsilon_1})^k} \begin{cases}
Ce^{-\theta_1 s}, & s \in \Upsilon^-, \quad 0 \leq k \leq 3,
Ce^{-\theta_1(s-d)}, & s \in \Upsilon^+, \quad 0 \leq k \leq 3,
\end{cases}
$$

$$
\|\bar{z}_r^{(k)}\|_{Y\setminus\{d\}} \leq \frac{C}{(\sqrt{\varepsilon_1})^k} \begin{cases}
Ce^{-\theta_2(d-s)}, & s \in \Upsilon^-, \quad 0 \leq k \leq 3,
Ce^{-\theta_2(1-s)}, & s \in \Upsilon^+, \quad 0 \leq k \leq 3.
\end{cases}
$$

with

$$
\theta_1 = \frac{\sqrt{\eta}}{\sqrt{\varepsilon_1}}, \quad \theta_2 = \frac{\sqrt{\eta}}{\sqrt{\varepsilon_1}}.
$$

Consider the case(ii): $\sqrt{\alpha} \geq \sqrt{\eta}$. □

Lemma 7. The regular component $\bar{y}(s)$ satisfies the following bounds

$$
\|\bar{y}^{(k)}\|_{Y\setminus\{d\}} \leq C \left(1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^{2-k}\right), \quad 0 \leq k \leq 3. \quad \square
$$

Lemma 8. The singular components $\bar{z}_l(s)$ and $\bar{z}_r(s)$ satisfy the following bounds

$$
\|\bar{z}_l^{(k)}\|_{Y\setminus\{d\}} \leq C \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^k \begin{cases}
Ce^{-\theta_1 s}, & s \in \Upsilon^-, \quad 0 \leq k \leq 3,
Ce^{-\theta_1(s-d)}, & s \in \Upsilon^+, \quad 0 \leq k \leq 3,
\end{cases}
$$

$$
\|\bar{z}_r^{(k)}(s)\|_{Y\setminus\{d\}} \leq C \left(\frac{1}{\varepsilon_2}\right)^k \begin{cases}
Ce^{-\theta_2(d-s)}, & x \in \Upsilon^-, \quad 0 \leq k \leq 3,
Ce^{-\theta_2(1-s)}, & s \in \Upsilon^+, \quad 0 \leq k \leq 3.
\end{cases}
$$

with

$$
\theta_1 = \frac{\alpha \varepsilon_2}{2\varepsilon_1}, \quad \theta_2 = \frac{\eta}{2\varepsilon_2} \quad \square
$$

The unique solution $\bar{u}(s)$ of the continuous problem (1)-(3) is now given by

$$
\bar{u}(s) = \begin{cases}
\bar{y}^-(s) + \bar{z}_l^-(s) + \bar{z}_r^+(s), & s \in \Upsilon^-,
\bar{y}^- (d^-) + \bar{z}_l^- (d^-) + \bar{z}_r^+ (d^-) = \bar{y}^+ (d^+) + \bar{z}_l^+ (d^+) + \bar{z}_r^+ (d^+) \quad \text{at } s = d,
\bar{y}^+ (s) + \bar{z}_l^+ (s) + \bar{z}_r^+ (s), & s \in \Upsilon^+.
\end{cases}
$$

3. Discrete Problem

The continuous problem is discretized using Upwind finite difference scheme with suitable Shishkin mesh. On $\check{\Upsilon}$ a piecewise uniform mesh size over $M$ is builded by splitting the domain $\check{\Upsilon}$ into six subintervals defined as

$$
\check{\Upsilon} = [0, q_1] \cup [q_1, d - q_2] \cup [d - q_2, d] \cup [d, d + q_3] \cup [d + q_3, 1 - q_4] \cup [1 - q_4, 1].
$$
The subintervals $[0, \varrho_1]$, $[d - \varrho_2, d]$, $[d, d + \varrho_3]$ and $[1 - \varrho_4, 1]$ are scaled with a uniform mesh of $M/8$ mesh intervals, while $[\varrho_1, d - \varrho_2]$ and $[d + \varrho_3, 1 - \varrho_4]$ have a uniform mesh with $M/4$ mesh intervals. The step sizes in each subinterval is defined by $H_1 = 8\varrho_1/N$, $H_2 = 4(d - \varrho_1 - \varrho_2)/N$, $H_3 = 8\varrho_2/N$, $H_4 = 8\varrho_3/N$, $H_5 = 4(1 - \varrho_3 - \varrho_4)/N$ and $H_6 = 8\varrho_4/N$. If the discontinuous point is considered at the mesh point $x_i = x_d = M/2$ then the mesh points are represented by

$$\Upsilon^M = \{s_i : 1 \leq i \leq M/2 - 1\} \cup \{s_i : M/2 + 1 \leq i \leq M - 1\},$$

and the mesh points of the discrete domain are denoted by $\Upsilon^M = \{s_i\}^M_0 \cup \{d\}$. If $\varrho_1 = \varrho_2 = d/4$ and $\varrho_3 = \varrho_4 = (1 - d)/4$ and $\varrho_1 = \varrho_2 = \varrho_3 = \varrho_4 = 1/8$ (it is a special case of the discontinuous point at $d = 1/2$) then the mesh is uniform. The transition values in $\Upsilon$ are chosen as

$$\begin{align*}
\varrho_1 &= \min\left\{\frac{d}{4}, \frac{1}{\varrho_1} \ln M\right\}, \quad \varrho_2 = \min\left\{\frac{d}{4}, \frac{2}{\varrho_2} \ln M\right\}, \\
\varrho_3 &= \min\left\{\frac{1 - d}{4}, \frac{1}{\varrho_1} \ln M\right\}, \quad \varrho_4 = \min\left\{\frac{1 - d}{4}, \frac{2}{\varrho_2} \ln M\right\},
\end{align*}$$

(12)

where $\varrho_1, \varrho_2$, are defined in section 2.

The WCSCRDP boundary value problem (1)-(3) is discretized in the mesh domain $\Upsilon^M$ using the standard upwind finite difference method. Let us find a mesh function $\bar{U}(s_i)$, $\forall s_i \in \Upsilon^M$ such that

$$L^M \bar{U}(s_i) \equiv \varepsilon_1 \delta^2 \bar{U}(s_i) + \varepsilon_2 A(s_i) D^+ \bar{U}(s_i) + B(s_i) \bar{U}(s_i) = \bar{g}(s_i)$$

(13)

$$U(s_0) = \bar{u}(0), \ U(s_M) = \bar{u}(1),$$

(14)

$$D^- \bar{U}(s_{M/2}) = D^+ \bar{U}(s_{M/2}),$$

(15)

where the matrix $A$ and $B$ are defined in section 2. The stiffness matrix is obtained for the above discrete problem with the following operators.

$$D^+ U(s_i) = \frac{U(s_{i+1}) - U(s_i)}{h_i}, \quad D^- U(s_i) = \frac{U(s_i) - U(s_{i-1})}{h_i}, \delta^2 U(s_i) = \frac{(D^+ U(s_i) - D^- U(s_i))}{h_i},$$

where, $h_{i+1} = s_{i+1} - s_i, h_i = s_i - s_{i-1}, \bar{h}_i = \frac{h_{i+1} + h_i}{2}, i = 1, 2, ..., M$.

The discrete operator $L^M$ has properties equivalent to the continuous differential operator $L$ defined in section 2.

Lemma 9. Suppose there exist a mesh function $\bar{U}(s_i)$ such that $\forall s_i \in \Upsilon^M$ which satisfies $\bar{U}(s_0) \geq 0, \bar{U}(s_M) \geq 0$, and $L^M \bar{U}(s_i) \leq 0$, $L^2 \bar{U}(s) \leq 0$ and $D^+ \bar{U}(s_{M/2}) - D^- \bar{U}(s_M/2) \leq 0$. Then $\bar{U}(s_i) \geq 0, \forall s_i \in \Upsilon$.

Lemma 10. If $\bar{U}(s_i)$ is any mesh function, then $||\bar{U}(s_i)|| \leq \left(\frac{C}{\bar{C}}\right)$, for all $s_i \in \Upsilon^M$.

4. Decomposition and Bounds for the Discrete Solution

The error to be estimated at each mesh point $s_i \in \Upsilon^M$ is represented by $||e(s_i)|| = ||\bar{U}(s_i) - \bar{u}(s_i)||$. To find the error $||e(s_i)||$, we decompose the mesh function $\bar{U}(s_i)$ of the discrete problem (13) - (15) as $\bar{U}(s_i) = \bar{Y}(s_i) + \bar{Z}_l(s_i) + \bar{Z}_r(s_i)$ in a way similar to the decomposition of continuous solutions. To obtain sharper bounds the discrete regular component $\bar{Y}(s_i)$ and singular components $\bar{Z}_l(s_i), \bar{Z}_r(s_i)$ are further decomposed as $\bar{Y}^-(s_i), \bar{Y}^+(s_i), \bar{Z}_l^-(s_i), \bar{Z}_l^+(s_i)$.
and $\bar{Z}^{-}_r(s_i)$, $\bar{Z}^+_r(s_i)$ respectively to the left and right sides of the point of discontinuity $i = M/2$
This decomposition facilitate in deriving the convergence of the nodal error $||\bar{e}(s_i)||$ in the
boundary and interior layers.

The regular discrete component $\bar{Y}(s_i)$ is defined as

$$
\bar{Y}(s_i) = \begin{cases} 
\bar{Y}^{-}(s_i), & \text{for } 1 \leq i \leq M/2 - 1, \\
\bar{Y}^{+}(s_i), & \text{for } M/2 + 1 \leq i \leq M - 1,
\end{cases}
$$

where, $\bar{Y}^{-}(s_i)$ and $\bar{Y}^{+}(s_i)$ are respectively, the solutions of the following discrete problems:

$$
\begin{align*}
L^M \bar{Y}^{-}(s_i) &= \bar{g}(s_i), & \text{for } 1 \leq i \leq M/2 - 1, & \bar{Y}^{-}(0) = y(0), \quad \bar{Y}^{-}(M/2) = y(d-), \\
L^M \bar{Y}^{+}(s_i) &= \bar{g}(s_i), & \text{for } M/2 + 1 \leq i \leq M - 1, & \bar{Y}^{+}(M/2) = y(d+), \quad \bar{Y}^{+}(1) = y(1).
\end{align*}
$$

Further the discrete singular components $\bar{Z}^{-}_l(s_i)$, $\bar{Z}^+_l(s_i)$, $\bar{Z}^{-}_r(s_i)$ and $\bar{Z}^+_r(s_i)$ are defined as

$$
\bar{Z}(s_i) = \bar{Z}_l(s_i) + \bar{Z}_r(s_i)
= \begin{cases} 
\left(\bar{Z}^{-}_l + \bar{Z}^{-}_r\right)(s_i), & \text{for } 1 \leq i \leq M/2 - 1, \\
\left(\bar{Z}^{+}_l + \bar{Z}^{+}_r\right)(s_i), & \text{for } M/2 + 1 \leq i \leq M - 1,
\end{cases}
$$

where, $\bar{Z}^{-}_l(s_i)$, $\bar{Z}^+_l(s_i)$, $\bar{Z}^{-}_r(s_i)$ and $\bar{Z}^+_r(s_i)$ are respectively the solutions of the following discrete problems:

$$
\begin{align*}
L^M \bar{Z}^{-}_l(s_i) &= 0, & \text{for } 1 \leq i \leq M/2 - 1, & \bar{Z}^{-}_l(0) = z^{-}_l(0), \quad \bar{Z}^{-}_l(M/2) = z^{-}_lM/2, \\
L^M \bar{Z}^+_l(s_i) &= 0, & \text{for } M/2 + 1 \leq i \leq M - 1, & \bar{Z}^{+}_l(M/2) = z^{+}_l(M/2), \quad \bar{Z}^{+}_l(1) = z^{+}_l(1), \\
L^M \bar{Z}^{-}_r(s_i) &= 0, & \text{for } 1 \leq i \leq M/2 - 1, & \bar{Z}^{-}_r(0) = 0, \quad \bar{Z}^{-}_r(M/2) = z^{-}_rM/2, \\
L^M \bar{Z}^+_r(s_i) &= 0, & \text{for } M/2 + 1 \leq i \leq M - 1, & \bar{Z}^{+}_rM/2 = 0, \quad \bar{Z}^{+}_r(1) = z^{+}_r(1).
\end{align*}
$$

The solution $\bar{Y}(s_i)$ of the discrete problem (13)- (15) can be now defined as

$$
\bar{U}(s_i) = \begin{cases} 
(\bar{Y}^{-} + \bar{Z}^{-}_l + \bar{Z}^{-}_r)(s_i), & \text{for } 1 \leq i \leq M/2 - 1, \\
(\bar{Y}^{-} + \bar{Z}^{-}_l + \bar{Z}^{-}_r)(s_i) = (\bar{Y}^{+} + \bar{Z}^{+}_l + \bar{Z}^{+}_r)(s_i), & \text{for } i = M/2, \\
(\bar{Y}^{+} + \bar{Z}^{+}_l + \bar{Z}^{+}_r)(s_i), & \text{for } M/2 + 1 \leq i \leq M - 1.
\end{cases}
$$

**Lemma 11.** The following bounds on $\bar{Z}^{-}_l(s_i)$, $\bar{Z}^+_l(s_i)$, $\bar{Z}^{-}_r(s_i)$ and $\bar{Z}^{+}_r(s_i)$ are given by

$$
\begin{align*}
|\bar{Z}^{-}_l(s_i)| &\leq C \prod_{j=1}^i \left(1 + \theta_1 h_j\right)^{-1} = \tilde{\psi}^{-}_l, \quad \tilde{\psi}^{-}_l,0 = C, \\
|\bar{Z}^+_l(s_i)| &\leq C \prod_{j=M/2+1}^i \left(1 + \theta_1 h_j\right)^{-1} = \tilde{\psi}^{+}_l, \quad \tilde{\psi}^{+}_l, M/2 = C, \\
|\bar{Z}^{-}_r(s_i)| &\leq C \prod_{j=1}^{M/2} \left(1 + \theta_2 h_j\right)^{-1} = \tilde{\psi}^{-}_r, \quad \tilde{\psi}^{-}_r, M/2 = C, \\
|\bar{Z}^{+}_r(s_i)| &\leq C \prod_{j=M/2+1}^{M} \left(1 + \theta_2 h_j\right)^{-1} = \tilde{\psi}^{+}_r, \quad \tilde{\psi}^{+}_r, M = C,
\end{align*}
$$

$\tilde{\varphi}^{-}_l = \tilde{\psi}^{-}_l \pm \bar{Z}^{-}_l(s_i)$ and $\tilde{\varphi}^{+}_r = \tilde{\psi}^{+}_r \pm \bar{Z}^{+}_r(s_i)$,
where

\[
\tilde{\psi}^{-}_{li} = \begin{cases} 
\prod_{j=1}^{i} (1 + \theta_1 h_j)^{-1}, & 1 \leq i \leq M/2, \\
1, & i = 0,
\end{cases}
\]

\[
\tilde{\psi}^{-}_{ri} = \begin{cases} 
\prod_{j=i+1}^{M/2} (1 + \theta_2 h_j)^{-1}, & 0 \leq i < M/2, \\
1, & i = M/2,
\end{cases}
\]

The values of \(\theta_1, \theta_2\) are defined in Section 3

5. Truncation Error Analysis

Lemma 12. The truncation error of the regular component satisfies the following estimate,

\[
\|\bar{Y} - \bar{y}\| \leq \left( \frac{CM^{-1}}{CM^{-1}} \right) \text{ for each mesh point } s_i \in \Omega^M.
\]

where \(\bar{Y}\) and \(\bar{y}\) are the solutions of the discrete and continuous decompositions defined in Section 4 and 2 respectively.

Lemma 13. The truncation error of the right singular component satisfies the following estimate,

\[
\|\bar{Z}_r - \bar{z}_r\| \leq \begin{cases} 
\left( \frac{CM^{-1}(\ln M)}{CM^{-1}(\ln M)} \right), & \text{if } \sqrt{\alpha \epsilon_2} \leq \sqrt{\eta \epsilon_1}, \\
\left( \frac{CM^{-1}(\ln M)^2}{CM^{-1}(\ln M)^2} \right), & \text{if } \sqrt{\alpha \epsilon_2} \geq \sqrt{\eta \epsilon_1}.
\end{cases}
\]

for each mesh point \(s_i \in \Omega^M\).

Lemma 14. The truncation error of the left singular component satisfies the following estimate,

\[
\|\bar{Z}_l - \bar{z}_l\| \leq \begin{cases} 
\left( \frac{CM^{-1} \ln M}{CM^{-1} \ln M} \right), & \text{if } \sqrt{\alpha \epsilon_2} \leq \sqrt{\eta \epsilon_1}, \\
\left( \frac{CM^{-1}(\ln M)^2}{CM^{-1}(\ln M)^2} \right), & \text{if } \sqrt{\alpha \epsilon_2} \geq \sqrt{\eta \epsilon_1}.
\end{cases}
\]

for each mesh point \(s_i \in \Omega^M\).

Lemma 15. At the discontinuity mesh point \(s_{M/2}\) the following error estimate \(\bar{e}(s_{M/2})\) is satisfied by

\[
|\{D^+ - D^-\} \bar{e}(s_{M/2})| \leq \begin{cases} 
\left( \frac{\eta \alpha \varrho}{M \epsilon_1 \epsilon_1}, \frac{\eta \alpha \varrho}{M \epsilon_1 \epsilon_1} \right)^T, & \text{if } \sqrt{\alpha \epsilon_2} \leq \sqrt{\eta \epsilon_1}, \\
\left( \frac{\alpha \varrho^2}{M \epsilon_1^2}, \frac{\alpha \varrho^2}{M \epsilon_1^2} \right)^T, & \text{if } \sqrt{\alpha \epsilon_2} \geq \sqrt{\eta \epsilon_1},
\end{cases}
\]

where \(\varrho = \min\{\varrho_2, \varrho_3\}\).

The next theorem establishes theoretically that the numerical scheme considered in this study is parameter uniform convergent to first order to the logarithmic factor \(\forall s_i \in \Omega\).
Theorem 16. Let $\bar{u}(s)$ and $\bar{U}(s_i)$ be respectively the solutions of the problems (1) and (13). Then, for adequately large $M$, we have

$$\|\bar{U} - \bar{u}\| \leq \begin{cases} 
\frac{C^M \ln M}{M^{1 - \ln M}}, & \text{if } \sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1}, \\
\frac{C^M (\ln M)^2}{M^{1 - \ln M}}, & \text{if } \sqrt{\alpha \varepsilon_2} \geq \sqrt{\eta \varepsilon_1}.
\end{cases}$$

Proof. From the results of Lemmas 4, 12, 13 and 14, it follows that

$$e(s_i) \leq \begin{cases} 
\frac{C^M \ln M}{M^{1 - \ln M}}, & \sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1}, \\
\frac{C^M (\ln M)^2}{M^{1 - \ln M}}, & \sqrt{\alpha \varepsilon_2} \geq \sqrt{\eta \varepsilon_1},
\end{cases} \quad \forall s_i \in \mathcal{Y}^M. \quad (17)$$

To prove the desired error at the point of discontinuity $x_{M/2}$:

Consider the case $\sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1}$. Let us define the discrete barrier function $\phi_j(s_i)$ for $j = 1, 2$ to be the solution of

$$\varepsilon_1 \delta^2 \phi_j(s_i) + \varepsilon_2 \alpha_j(s_i) D^+ \phi_j(s_i) - \beta_j(s_i) \phi_j(s_i) = 0,$$

$$\phi_j(s_0) = 0, \quad \phi_j(s_{M/2}) = 1 \text{ and } \phi_j(s_M) = 0.$$

We can prove that

$$D^- \phi_j(s_i) \geq 0, \quad \text{for } 1 \leq i \leq M/2 - 1 \quad \text{and}$$

$$D^+ \phi_j(s_i) \leq 0, \quad \text{for } 1 \leq i \leq M/2 + 1.$$

Note that $L^M \phi_j(s_i) \leq 0$ for all $s_i \in \mathcal{Y}^M$, using the procedure adopted from [20] we prove the following result at the point $s_{M/2} = d$,

$$D^+ \phi_j(s_{M/2}) - D^- \phi_j(s_{M/2}) = \frac{\phi_j(s_{M/2} + h_4) - 1}{h_4} - \frac{\phi_j(s_{M/2} + h_3) - 1}{h_3} \leq \frac{C^M \ln M}{\max(h_3, h_4)}.$$

Consider the barrier function for $j = 1, 2$

$$\psi_j^1(s_i) = C_3 M^{-1} \ln M + C_4 \frac{h}{\sqrt{\varepsilon_1}} \phi_j(s_i) \pm e(s_i), \quad \forall s_i \in \mathcal{Y}^M.$$

Now, $\psi_j^1(s_0) \geq 0$, $\psi_j^1(s_N) \geq 0$ and $L^M \psi_j^1(s_i) \leq 0$, $s_i \in \mathcal{Y}^M$,

$$D^+ - D^-) \psi_j^1(s_{M/2}) \leq 0, \quad i = M/2.$$

Hence applying the discrete minimum principle, we get $\psi_j^1(s_i) \geq 0 \forall s_i \in \mathcal{Y}^M$. For adequately large $M$ we derive

$$|(\bar{U} - \bar{u})(s_i)| \leq C^M \ln M, \quad \sqrt{\alpha \varepsilon_2} \leq \sqrt{\eta \varepsilon_1}. \quad (18)$$
In the second case when $\sqrt{\alpha_2 \varepsilon_2} \geq \sqrt{\eta_1}$, consider the discrete barrier function
\[ \psi^*_j(s_i) = \psi(s_i) \pm \epsilon(s_i) \text{ for } j = 1, 2 \text{ defined in the interval } (d - \varrho_2, d + \varrho_3) \]
where
\[ \psi(s_i) = C M^{-1} (\ln M)^2 + \begin{cases} C M^{-1} \varrho_2 (s_i - d - \varrho_2), & s_i \in (d - \varrho_2, d), \\ C M^{-1} \varrho_3 \varepsilon_2 (d + \varrho_3 - s_i), & s_i \in [d, d + \varrho_3). \end{cases} \]

It could be seen that $\psi_2(d - \varrho_3) > 0$, $\psi_j(d + \varrho_3) > 0$ and $L^M \psi^*_j(s_i) < 0$ and $D^+ \psi^*_j(s_i) - D^- \psi^*_j(s_i) < 0$.

Applying the discrete minimum principle to $\psi^*_j(s_i)$, we find that $\psi^*_j(s_i) \geq 0$. Hence,
\[ |(U - u)(s_i)| \leq \begin{cases} C M^{-1} \varrho_2^2 / \varepsilon_2^2 & \text{for } x_i \in (d - \varrho_2, d + \varrho_3) \\ C M^{-1} \varrho_3 \varepsilon_2^2 / \varepsilon_1^2 & \text{for } s_i \in (-\varrho_2, d + \varrho_3) \end{cases} \leq C M^{-1} (\ln M)^2. \tag{19} \]

Therefore by combining (18) and (19) we obtain the required result. \[\square\]

6. Numerical Example

In order to find the applicability of the present method, we have considered the problems of singularly perturbed two parameter BVP with discontinuous source terms.

Example 1.

\[ -\varepsilon_1 \ddot{u}''(s) - \varepsilon_2 A(s) \dot{u}'(s) + B(s) \bar{u}(s) = \bar{g}(s), \quad s \in \Upsilon^- \cup \Upsilon^+, \]
\[ \bar{u}(0) = (1, 1)^T, \quad \bar{u}(1) = (0, 0)^T, \]

where
\[ A(s) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right) \quad \text{for } 0 < s < 1 \quad \text{B}(s) = \left( \begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right), \]
\[ g_1(s) = \begin{cases} 0.5, & \text{for } 0 < s < 0.5 \\ -0.6, & \text{for } 0.5 < s < 1 \end{cases} \quad \text{and } g_2(s) = \begin{cases} 0.2, & \text{for } 0 < s < 0.5 \\ -2 + s, & \text{for } 0.5 < s < 1 \end{cases} \]

Since the exact solutions are not available for the considered problem, the nodal errors and the order of convergence will be estimated using the double mesh principle [2]. The error in the double mesh differences are defined as
\[ E^M_{(\varepsilon_1, \varepsilon_2)} = \max_{s_i \in \Upsilon^M} |\bar{U}^M(s_i) - \bar{U}^{2M}(s_i)|, \quad \text{and } E^M_{\varepsilon_1, \varepsilon_2} = \max_{s_i \in \Upsilon^M} E^M_{(\varepsilon_1, \varepsilon_2)}, \]

where $\bar{U}^M(s_i)$ and $\bar{U}^{2M}(s_i)$ respectively denote the numerical solutions obtained using $M$ and $2M$ mesh intervals. The parameter-robust orders of convergence are calculated from the formula
\[ R^M = \log_2 \left( \frac{E^M_{\varepsilon_1, \varepsilon_2}}{E^{2M}_{\varepsilon_1, \varepsilon_2}} \right). \]

The Table 1 and Table 2 displayed here give a clear notion on the maximum pointwise error estimates ($E^M$) and the corresponding orders of convergence ($R^M$) for $u_1$ and $u_2$ of Example 1 for different values of $\varepsilon_1$. Figures 1 and 2 represent the plot of (a) numerical solutions and (b) the errors for $\varepsilon_1 = 2^{-12}$ and $\varepsilon_2 = 2^{-20}$ when $N = 256$ for the Example 1.
Table 1: Maximum point-wise errors ($E^M$) and order of convergence ($R^M$) for $u_1$ of Example 1 when $\varepsilon_2 = 2^{-10}$.

| $\varepsilon_1$ | 64   | 128  | 256  | 512  | 1024 | 2048 |
|----------------|------|------|------|------|------|------|
| $2^0$          | 7.086e-004 | 3.522e-004 | 1.756e-004 | 8.766e-005 | 4.380e-005 | 2.189e-005 |
| $2^{-2}$       | 1.270e-003 | 6.245e-004 | 3.097e-004 | 1.542e-004 | 7.694e-005 | 3.843e-005 |
| $2^{-4}$       | 2.092e-003 | 1.049e-003 | 5.255e-004 | 2.629e-004 | 1.315e-004 | 6.577e-005 |
| $2^{-6}$       | 5.170e-003 | 2.569e-003 | 1.280e-003 | 6.391e-004 | 3.193e-004 | 1.596e-004 |
| $2^{-8}$       | 6.498e-003 | 3.212e-003 | 1.598e-003 | 7.974e-004 | 3.983e-004 | 1.900e-004 |
| $2^{-10}$      | 6.498e-003 | 3.212e-003 | 1.598e-003 | 7.974e-004 | 3.983e-004 | 1.900e-004 |

$E^M$ 1.568e-001 1.021e-001 6.240e-002 3.649e-002 2.066e-002 1.145e-002

$R^M$ 0.6189 0.7104 0.7740 0.8207 0.8515 —

Table 2: Maximum point-wise errors ($E^M$) and order of convergence ($R^M$) for $u_2$ of Example 1 when $\varepsilon_2 = 2^{-10}$.

| $\varepsilon_1$ | 64   | 128  | 256  | 512  | 1024 | 2048 |
|----------------|------|------|------|------|------|------|
| $2^0$          | 3.291e-004 | 1.664e-004 | 8.366e-005 | 4.194e-005 | 2.100e-005 | 1.051e-005 |
| $2^{-2}$       | 1.264e-003 | 6.372e-004 | 3.199e-004 | 1.603e-004 | 8.023e-005 | 4.013e-005 |
| $2^{-4}$       | 4.160e-003 | 2.087e-003 | 1.045e-003 | 5.228e-004 | 2.615e-004 | 1.308e-004 |
| $2^{-6}$       | 7.675e-003 | 3.825e-003 | 1.909e-003 | 9.539e-004 | 4.768e-004 | 2.383e-004 |
| $2^{-8}$       | 9.087e-003 | 4.510e-003 | 2.248e-003 | 1.122e-003 | 5.608e-004 | 2.803e-004 |
| $2^{-10}$      | 9.087e-003 | 4.510e-003 | 2.248e-003 | 1.122e-003 | 5.608e-004 | 2.803e-004 |

$E^M$ 1.568e-001 1.021e-001 6.240e-002 3.649e-002 2.066e-002 1.145e-002

$R^M$ 0.6189 0.7104 0.7740 0.8207 0.8515 —

7. Conclusion

Two point singularly perturbed weakly coupled system of convection-reaction-diffusion problems having two small parameters ($\varepsilon_1, \varepsilon_2$) multiplying the diffusion and convection term with
discontinuity over source term is studied. The solutions to these type of problems exhibit interior and boundary layers. An upwind difference scheme is applied to discretize the problem (1) - (3) by building a piecewise uniform mesh. The estimated analysis shows that the numerical scheme considered here converges to almost first order. Tables and figures illustrated in Section 6 test problem validate the theoretical estimates.

Figure 1: (a) Numerical Solution
Plot of Numerical Solution and Error when $\varepsilon_1 = 2^{-12}$, $\varepsilon_2 = 2^{-20}$ with $M = 256$ for Example 1

Figure 2: (b) Error

Acknowledgment
The authors thank the anonymous reviewers for their valuable suggestions to improve the paper

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