GENERAL DECOMPOSITION OF RADIAL FUNCTIONS ON $\mathbb{R}^n$ AND APPLICATIONS TO $N$-BODY QUANTUM SYSTEMS

CHRISTIAN HAINZL$^1$ AND ROBERT SEIRINGER$^2$

Abstract. We present a generalization of the Fefferman-de la Llave decomposition of the Coulomb potential to quite arbitrary radial functions $V$ on $\mathbb{R}^n$ going to zero at infinity. This generalized decomposition can be used to extend previous results on $N$-body quantum systems with Coulomb interaction to a more general class of interactions. As an example of such an application we derive the high density asymptotics of the ground state energy of jellium with Yukawa interaction in the thermodynamic limit, using a correlation estimate by Graf and Solovej [GS].

1. Introduction

For the description of physical systems interacting with Coulomb forces, the Fefferman-de la Llave decomposition of the Coulomb potential has proved very useful. It was introduced in [FL], where it was used in the proof of stability of relativistic matter. It states that, for $x \in \mathbb{R}^3$,

$$\frac{1}{|x|} = \frac{1}{\pi} \int_0^\infty dr \frac{1}{r^5} \chi_r * \chi_r(x),$$

(1)

where $\chi_r(x) = \theta(r - |x|)$ is the characteristic function of a ball of radius $r$ centered at the origin, and $*$ denotes convolution on $\mathbb{R}^3$. Except for the constant $1/\pi$, it is easily checked that (1) holds true, since the right side is a radial, homogeneous function of order $-1$.

We are interested in a generalization of (1) to arbitrary radial functions $V$ on $\mathbb{R}^n$ going to zero at infinity, with a weight function $g(r)$ replacing $1/r^5$ in the integrand (see (3)). In Theorem 1 below we present a simple and straightforward derivation of such a decomposition, assuming $V$ to satisfy some decrease and regularity properties specified below. We give an explicit expression for the weight function $g(r)$, which turns out to be related to the $\lfloor n/2 \rfloor + 2'th derivative of $V$. Here $\lfloor \cdot \rfloor$ denotes the Gauss bracket, i.e., $\lfloor m \rfloor = \max\{n \in \mathbb{N}_0, n \leq m\}$. In particular, the case of a positive $g$ is of interest, implying positive definiteness of the function $V$.

\textit{Date:} March 28, 2022.

$^1$Marie Curie Fellow

$^2$Erwin Schrödinger Fellow. On leave from Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria
Decompositions of the type (3) have also been studied in [G1] for the case of positive and bounded functions \( V \), where the decomposition (3) is referred to as “scale mixtures of Euclid’s Hat”. The improvement of our Theorem 1 compared to [G1] lies in its concise proof and the specific, unified form of the formula for \( g \).

The decomposition we derive in the next section is of particular interest in generalizing results on \( N \)-body quantum systems so far only applicable to the Coulomb potential. For instance, it can be used in estimating the “indirect part” of the interaction energy, as was done in the Coulomb case in [LO], or to determine the validity of Hartree-Fock approximations (see [B] and [GS]). This is due to the fact that with the help of (3) the interaction energy in an \( N \)-particle quantum system can be decomposed as

\[
\sum_{1 \leq i < j \leq N} V(x_i - x_j) = \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^n} dz \int_0^\infty drg(r) X^{r,z}_i X^{r,z}_j ,
\]

where \( X^{r,z} \) denotes the one-particle multiplication operator \( \chi_r(\cdot - z) \), being actually a projection, and the subscript \( i \) means that the operator acts on the \( i \)'th component in the \( N \)-fold tensor product appropriate for \( N \)-particle systems. Working with the right side of (3) is often easier than handling the left side, using the fact that it is a superposition of products of one-particle projection operators.

In Section 3 below we will use the decomposition of the Yukawa potential to derive the high density asymptotics of the ground state energy of jellium with Yukawa interaction in the thermodynamic limit. This is a generalization of [GS], where an analogous expansion in the case of Coulomb interaction was accomplished.

Yukawa potentials are actually used in solid state physics as a model for the screened Coulomb interaction between electrons in a solid (see, e.g., [M]).

2. The Decomposition

In this section, we state and prove the decomposition (3), and comment on its implications. In the following theorem, we restrict ourselves to the case \( n \geq 2 \), and remark on the easy case \( n = 1 \) after the proof.

**Theorem 1 (Decomposition of Radial Functions).** For \( n \geq 2 \), let \( V : \mathbb{R}^n \to \mathbb{R} \) be a radial function that is \( \lceil n/2 \rceil + 2 \) times differentiable away from \( x = 0 \). For \( m \in \mathbb{N}_0 \) denote \( V^{(m)}(|x|) = \frac{d^m}{dx^m} V(x) \). Assume that \( \lim_{|x| \to \infty} |x|^m V^{(m)}(|x|) = 0 \) for all \( 0 \leq m \leq \lceil n/2 \rceil + 1 \), and let \( \chi_r(x) = \theta(r - |x|) \). Then

\[
V(x) = \int_0^\infty dr g(r) \chi_{r/2} \ast \chi_{r/2}(x) ,
\]
where

\[
g(r) = \frac{(-1)^{[n/2]} 2}{\Gamma \left( \frac{n-1}{2} \right) (\pi r^2)^{(n-1)/2}} \times \left( \int_0^\infty ds V^{(n/2)+2}(s) \left( \frac{d}{ds} \right)^{n-[n/2]} s (s^2 - r^2)^{1/2} \right) \left( \frac{n-1}{2} \right) \frac{\pi}{4} r^{n-1} \int_0^r dy (y^2 - s^2)^{1/2} \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n-1}{2} \right) \\
+ \delta_{\text{odd}} V^{(n/2)+2}(r) (2r)^{1/2} \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n-1}{2} \right)
\] (4)

and \( \delta_{\text{odd}} = 1 \) for \( n \) odd, \( \delta_{\text{odd}} = 0 \) for \( n \) even.

Note the \( r/2 \) in (3), which is chosen for convenience.

**Proof.** Elementary considerations show that

\[
\chi_{r/2} * \chi_{r/2}(x) = \frac{1}{\Gamma \left( \frac{n-1}{2} \right)^{n-1/2}} \int_0^r dy (y^2 - s^2)^{1/2} \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n-1}{2} \right)
\] (5)

for \( |x| \leq r \), and 0 otherwise. Inserting the definition (4) for \( g \), we can therefore write

\[
\left( \frac{(-1)^{[n/2]} 2}{\Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{n+1}{2} \right) 2^{n-2}} \right)^{-1} \int_0^\infty dr g(r) \chi_{r/2} * \chi_{r/2}(x) =
\int_0^\infty dr \frac{1}{n-1} \int_r^\infty ds V^{(n/2)+2}(s)
\times \left( \frac{d}{ds} \right)^{n-[n/2]} s (s^2 - r^2)^{1/2} \left( \frac{n-3}{2} \right) \int_0^r dy (y^2 - s^2)^{1/2} \left( \frac{n-1}{2} \right) \\
+ \delta_{\text{odd}} \frac{1}{r^{n-1}} \int_0^\infty dr V^{(n/2)+2}(r) r (2r)^{1/2} \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n-1}{2} \right)
\times \int_0^r dy (y^2 - r^2)^{1/2} \left( \frac{n-1}{2} \right) .
\] (6)

We now use the fact that

\[
\left( \frac{d}{ds} \right)^{n-[n/2]} \int_y^s dr \frac{1}{r^{n-1}} (y^2 - r^2)^{1/2} \left( \frac{n-3}{2} \right) s (s^2 - r^2)^{1/2} =
\int_y^s dr \frac{1}{r^{n-1}} (y^2 - r^2)^{1/2} \left( \frac{n-3}{2} \right) \left( \frac{d}{ds} \right)^{n-[n/2]} s (s^2 - r^2)^{1/2} \\
+ \delta_{\text{odd}} \frac{1}{s^{n-1}} (s^2 - y^2)^{1/2} \left( \frac{n-3}{2} \right) s (2s)^{1/2} \left( \frac{n-3}{2} \right) \Gamma \left( \frac{n-1}{2} \right)
\] (7)
and change the order of integration to get
\[
\left( \frac{(-1)^{[n/2]}}{\Gamma\left( \frac{n-1}{2}\right)\Gamma\left( \frac{n+1}{2}\right)2^{n-2}} \right)^{-1} \int_0^\infty dr g(r) \chi_{r/2} * \chi_{r/2}(x) = \\
\int_0^\infty ds V^{[n/2]+2}(s) \int_0^s dy \\
\times \left( \frac{d}{ds} \right)^{n-1-[n/2]} \int_y^s dr \frac{1}{r^{n-1}} s(s^2 - r^2)^{1/2} (n-3)(r^2 - y^2)^{1/2}(n-1) .
\]

The last integral can be evaluated to be
\[
\int_y^s dr \frac{1}{r^{n-1}} (s^2 - r^2)^{1/2}(n-3)(r^2 - y^2)^{1/2}(n-1) = \frac{\Gamma\left( \frac{n-1}{2}\right)\Gamma\left( \frac{n+1}{2}\right)2^{n-2}(s-y)^{n-1}}{s\Gamma(n)} ,
\]
and therefore
\[
\int_0^\infty dr g(r) \chi_{r/2} * \chi_{r/2}(x) = \\
\frac{(-1)^{[n/2]}}{\Gamma\left( \frac{n-1}{2}\right)\Gamma\left( \frac{n+1}{2}\right)2^{n-2}} \int_0^\infty ds V^{[n/2]+2}(s)(s - |x|)^{[n/2]+1} = V(x) ,
\]
where we integrated by parts in the last step, using the demanded decrease properties of \(|x|^m V^{(m)}(|x|)|.

EXAMPLE 1. A particular simple example which our decomposition applies to is the Coulomb potential \(V(x) = 1/|x|\). For the general case of \(n\) dimensions we compute the weight function to be
\[
g(r) = \frac{1}{2} \frac{\Gamma(n+2)}{\Gamma(n/2+1)} \frac{\pi^{1-n/2}}{r^{n+2}} .
\]
EXAMPLE 2. Another example, which will be used below, is the Yukawa potential \( Y_\mu(x) = \exp(-\mu|x|)/|x| \), with \( \mu > 0 \). In three dimensions, the corresponding weight function reads
\[
g(r) = \frac{2}{\pi} \frac{e^{-\mu r}}{r^5} \left[ 8 + 8\mu r + 4(\mu r)^2 + (\mu r)^3 \right].
\] (13)

The decomposition (3) provides conditions for positive definiteness of \( V \). By positive definiteness of a locally integrable function \( V \) we mean that
\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} V(x - y) \varphi(x) \varphi(y) dx dy \geq 0
\] for all \( \varphi \in C_0^\infty(\mathbb{R}^n) \). This is equivalent to \( V \) having a positive Fourier transform in the distributional sense. Since \( \chi_r * \chi_r(x) \) is obviously positive definite (as a function of \( x \)), any \( V \) with a positive weight function \( g(r) \) is positive definite. It is easy to see that
\[
\left( \frac{d}{ds} \right)^{n-1-\lfloor n/2 \rfloor} s(s^2 - r^2)^{1/2} (s^{-n-3}) \geq 0 \quad \text{for } s \geq r,
\] (15)

and therefore positivity of \((-1)^{\lfloor n/2 \rfloor} V^{[\lfloor n/2 \rfloor]+2} \) implies that \( V(x) \) is positive definite. This is a result due to Askey [A1], who deduces it from certain positivity properties of Bessel functions proven in [A2] and [F1]. Further extensions and refinements can be found in [G2] and references therein.

Consider now the (physically interesting) special case \( n = 3 \). Here the weight function \( g \) takes the simple form
\[
g(r) = \frac{2}{\pi r^2} \left( V''(r) - rV'''(r) \right) = -\frac{2}{\pi} \left( \frac{V''(r)}{r} \right)'.
\] (16)

Hence a monotone decrease of \( V''/r \) (instead of \( V'' \)) is sufficient for positive definiteness of \( V \). This condition for positive definiteness of a radial function on \( \mathbb{R}^3 \) has been obtained before by A. Martin [GM]. See also [HN] for an earlier reference.

Positivity of \( g \) might be a useful way of checking positive definiteness of a function \( V \). However, this condition is of course not necessary, as the following example shows.

EXAMPLE 3. For \( n = 3 \), \( V(x) = 1/(|x|^2 + 1) \) is positive definite, but the corresponding \( g(r) \) is not a positive function.

3. GROUND STATE ENERGY OF YUKAWA JELLIUM

We now demonstrate the usefulness of our decomposition (3) in applications to \( N \)-body quantum systems. We consider a model of electrons interacting with Yukawa potentials and with a uniform charged background. Yukawa potentials are used in solid state physics as a model for screened Coulomb potentials [M]. They were first introduced by Debye and Hückel [DH], and were later used in meson theory by Yukawa.
The system under consideration is described by the Hamiltonian
\[ H = \sum_{i=1}^{N} (-\Delta_i + V(x_i)) + \sum_{1 \leq i < j \leq N} Y_\mu(x_i - x_j), \] (17)
acting on \( H = \bigwedge_{i=1}^{N} L^2(\Lambda, dx_i; \mathbb{C}^q) \), the antisymmetric tensor product of the one-particle space \( L^2(\Lambda, dx_i; \mathbb{C}^q) \) of one electron with spin \((q-1)/2\). The particles are confined to the cube \( \Lambda = [0, L]^3 \) of side length \( L \), with volume \(|\Lambda| = L^3\). They interact via the Yukawa potential
\[ Y_\mu(x) = e^{-\mu|x|}/|x|, \] (18)
with \( \mu > 0 \). The electrons move in the potential \( V \) created by a charged background, given by
\[ V(x) = \rho \int_{\Lambda} dy Y_\mu(x - y), \] (19)
where \( \rho > 0 \) is the background density. We assume neutrality of the system, i.e., \( \rho = N/|\Lambda| \).

We have to specify boundary conditions for \( H \) on \( \Lambda \), and we take Dirichlet boundary conditions for simplicity. Denoting \( E(N, \Lambda, \mu) = \inf_{\text{spec}} H \), we define the energy density for a neutral system in the thermodynamic limit as
\[ e(\rho, \mu) = \lim_{L \to \infty} \frac{1}{|\Lambda|} \left( E(\rho|\Lambda|, \Lambda, \mu) + \frac{1}{2} \rho^2 \int_{\Lambda \times \Lambda} dx dy Y_\mu(x - y) \right). \] (20)

For convenience, we have added the self energy of the background charge. For \( \mu = 0 \) this is essential, because otherwise the limit does not exist. We take the existence of the limit (20) for granted. For the Coulomb case (i.e. \( \mu = 0 \)), existence (and independence of the boundary conditions for \( H \)) of the thermodynamic limit has been shown in [LN]. However, because of the short-range nature of the Yukawa potential, the existence of the thermodynamic limit for \( \mu > 0 \) is much easier to show.

Our main theorem concerns the behavior of \( e(\rho, \mu) \) for large \( \rho \). This is an extension of the result in [GS] for \( \mu = 0 \). We denote by \( k_F \) the Fermi momentum
\[ k_F = (6\pi^2 \rho/q)^{1/3}. \] (21)

**THEOREM 2 (Ground State Energy of Yukawa Jellium).** As \( \rho \to \infty \),
\[ \frac{e(\rho, \mu)}{\rho} = \frac{3}{5} k_F^2 - \frac{3}{4\pi} k_F J \left( \frac{\mu}{k_F} \right) \left[ 1 + o(1) \right], \] (22)
uniformly in \( \mu \) for bounded \( \mu/k_F \). Here \( J \) is the function
\[ J(\eta) = 1 - \frac{1}{6} \eta^2 + \frac{1}{2} \left( \eta^2 + \frac{1}{12} \eta^4 \right) \ln \left( 1 + \frac{4}{\eta^2} \right) - \frac{4}{3} \eta \arctan \frac{2}{\eta}. \] (23)
As in [GS], $o(1)$ is positive and smaller than $O(\rho^{-1/15+\varepsilon})$ for any $\varepsilon > 0$ and for $\mu/k_F$ fixed. The second term on the right side of (22) is entirely due to correlations in the ground state wave function and is usually referred to as “exchange energy”. The function $J(\eta)$ in Theorem 2 is in fact the integral

$$J(\eta) = \frac{4}{3} \int_0^2 \frac{dk}{k^2 + \eta^2} \left( 1 - \frac{3}{4} k + \frac{1}{16} k^3 \right).$$

Notice that $J(0) = 1$, whereas $J(\eta) \approx \frac{4}{9 \eta^2}$ as $\eta \to \infty$.

We conjecture that the asymptotics in (22) is uniform in $\mu$, even if $\mu/k_F$ tends to infinity. For $q = 1$, i.e., the spinless case, this is in fact easy to prove, as will be remarked after the proof of Theorem 2. Note, however, that in physical situation $\mu/k_F$ is small. The range of the potential is given by $1/\rho$, which is always bigger than the mean particle distance $\rho^{-1/3}$.

The proof of Theorem 2 follows essentially the analogous discussion of the Coulomb case in [GS]. We use freely the estimates derived there, mainly pointing out the differences to our case.

**Proof.** To obtain an upper bound to the ground state energy, we use as a trial state a Slater determinant of the $N$ lowest eigenvectors of the Laplacian on $\Lambda$, including spin. The calculation of the expectation value is essentially the same as the one done by Dirac [D] in the Coulomb case. In the thermodynamic limit the boundary conditions do not matter (see [GS] for details), and one can just consider plain waves, with momenta up to the Fermi momentum $k_F$. The function $J$ then arises from the integral

$$\frac{q}{2} \int_{\mathbb{R}^3} \frac{dk}{(2\pi)^3} \tilde{Y}_\mu(k) \int_{\mathbb{R}^3} \frac{dq}{(2\pi)^3} \theta(k_F - |q|) \theta(k_F - |k - q|) = \frac{3}{4\pi} \rho k_F J \left( \frac{\mu}{k_F} \right).$$

We are left with the lower bound. We start with the following decomposition of the Yukawa potential, proven in Section 2. If $\chi_r$ denotes the characteristic function of a ball of radius $r$ centered at the origin, then

$$Y_\mu(x - y) = \int_{\mathbb{R}^3} \int_0^\infty dz \int_{\mathbb{R}^3} dr \tilde{g}(r) \chi_r(x - z) \chi_r(y - z),$$

where

$$\tilde{g}(r) = 2g(2r) = \frac{1}{\pi} e^{-2\mu r} \left[ 1 + 2\mu r + 2(\mu r)^2 + (\mu r)^3 \right]$$

(compare with Example 2). The key to the lower bound is a correlation inequality derived in [GS, Cor. 5]. It states that for any projection $X$ and $0 \leq P \leq 1$ on the one-particle space, and for antisymmetric $\psi \in \mathcal{H}$,

$$\left\langle \psi \left| \sum_{1 \leq i < j \leq N} X_i X_j \right| \psi \right\rangle \geq \frac{1}{2} \text{Tr}[X^2] - \frac{1}{2} \text{Tr}[P X P X] - \text{const.} \text{Tr}[X(P + \gamma)] \min \left\{ 1, (\text{Tr}[X(1 - P)\gamma(1 - P)])^{1/2} \right\}.$$
Here we denote by $\gamma$ the one-particle reduced density matrix of $\psi$, with corresponding density $\rho_\gamma$. Using this, with $X = \chi_r(\cdot - z)$, and integrating over $r$ and $z$ as in (26), we get that
\[
\left\langle \psi \left| \sum_{1 \leq i < j \leq N} Y_\mu(x_i - x_j) \right| \psi \right\rangle \geq \frac{1}{2} \int_{\Lambda \times \Lambda} dxdy \rho_\gamma(x) \rho_\gamma(y) Y_\mu(x - y) - \frac{1}{2} \sum_{\sigma, \sigma'} \int_{\Lambda \times \Lambda} dxdy |P(x, \sigma; y, \sigma')|^2 Y_\mu(x - y) - \text{Error}.
\]

(29)

The function $P(x, \sigma; y, \sigma')$ is the integral kernel of $P$, and $\sigma \in (1, \ldots, q)$ denotes the spin variables of the electrons. The error term is
\[
\text{Error} = \text{const.} \int_{\mathbb{R}^3} dz \int_{0}^{\infty} dr \widetilde{g}(r) \left[ \chi_r * (\rho_\gamma + \rho_P) \right](z) \min \left\{ 1, \left[ \chi_r * \rho_{Q\gamma Q} \right](z)^{1/2} \right\}.
\]

(30)

Here we introduced the operator $Q = 1 - P$; $\rho_P$ and $\rho_{Q\gamma Q}$ are the densities corresponding to $P$ and $Q\gamma Q$, respectively.

Since $\widetilde{g}(r) \leq \text{const.} r^{-5}$, we can proceed exactly as in [GS, Lem. 6] to estimate
\[
\text{Error} \leq \text{const.} \|\rho_\gamma + \rho_P\|_1^{1/6 + \varepsilon} \|\rho_\gamma + \rho_P\|_{5/3}^{5/6} \delta(\gamma, P)^{1/3 - \varepsilon}
\]

(31)

for any $0 < \varepsilon \leq 1/6$. Here $\delta(\gamma, P) = \text{Tr}[\gamma (1 - P)]$ measures the “difference” of $\gamma$ and $P$. Since $\psi$ is supposed to be the ground state, we can use the upper bound to get the $a \text{ priori}$ knowledge that $\|\rho_\gamma\|_{5/3} \leq \text{const.} N k_F^2$, see [GS, Eq. (4.11)].

Now let $P$ be the projection onto the first $N$ eigenstates of the Laplacian with periodic boundary conditions. It is shown in [GS, Eq. (4.13)] that, for any $\psi \in \mathcal{H}$,
\[
\frac{1}{N} \left\langle \psi \left| - \sum_{i=1}^{N} \Delta_i \right| \psi \right\rangle \geq k_F^2 \left( \frac{3}{5} + \text{const.} \left( N^{-1} \delta(\gamma, P) \right)^2 \right) - o(1)
\]

(32)

as $|\Lambda| \to \infty$. The second term in (29), when divided by $N$, converges in the thermodynamic limit to $\frac{3}{10} k_F J(\mu/k_F)$, as explained in the upper bound. Moreover, $\|\rho_P\|_{5/3} \leq \text{const.} N k_F^2$. Putting together (29), (31) and (32), we
therefore get that in the ground state $\psi$,
\[
\frac{1}{N} \langle \psi | H | \psi \rangle + \frac{1}{2N} \rho^2 \int_{\Lambda \times \Lambda} dxdy Y_\mu(x - y) \geq k_F^2 \left( \frac{3}{5} + \text{const.} \left( N^{-1} \delta(\gamma, P) \right)^2 \right) - \frac{3}{4\pi} k_F J(\mu/k_F) - \text{const.} \left( N^{-1} \delta(\gamma, P) \right)^{1/3 - \epsilon} \\
+ \frac{1}{2} \int_{\Lambda \times \Lambda} dxdy (\rho(\gamma(x) - \rho)(\rho(\gamma(y) - \rho) Y_\mu(x - y) - o(1) \quad (33)
\]
as $|\Lambda| \to \infty$. The last term is positive, since $Y_\mu$ is positive definite. Minimizing the right side of (33) over $\delta(\gamma, P)$ gives the desired result. \hfill \Box

**REMARK 2.** We expect the asymptotics in (22) to be uniform in $\mu/k_F$ even if this value goes to infinity. The reason is that for $\mu/k_F \gg 1$ the interaction potential is of very short range, and hence the interaction between particles of the same spin can be neglected to leading order due to the antisymmetry of the wave function. I.e., in this case one expects a contribution to $e(\rho, \mu)$ from the interaction energy of $(1 - q^{-1})2\pi \rho^2/\mu^2$. The total energy for $\mu \gg \rho^{1/3}$ should then be
\[
e(\rho, \mu) \approx \frac{3}{5} k_F^2 - \frac{4\pi}{\mu^2} \rho + (1 - q^{-1}) \frac{2\pi}{\mu^2} \rho + \frac{2\pi}{\mu^2} \rho , \quad (34)
\]
which is exactly the same as (22) as $\mu/k_F \to \infty$.

For the spinless case, i.e., $q = 1$, the uniformity is in fact easy to prove. Neglecting the positive interaction energy we get a lower bound
\[
\frac{e(\rho, \mu)}{\rho} \geq \frac{3}{5} k_F^2 - \frac{2\pi}{\mu^2} \rho , \quad (35)
\]
which agrees with (22) as $\mu/k_F \to \infty$. We note that the second term is now not the exchange energy, but the difference of the energy of the electrons in the background and the self energy of the background.

**Acknowledgments.** We thank Volker Bach for encouraging the present study, and Michael Loss for providing the references [A1, A2, F1]. C.H. was supported by a Marie Curie Fellowship of the European Community programme “Improving Human Research Potential and the Socio-economic Knowledge Base” under contract number HPMFCT-2000-00660. R.S. was supported by the Austrian Science Fund, and acknowledges warm hospitality at the Mathematical Institute, LMU München, where part of this work was done.

**References**

[A1] R. Askey, *Refinements of Abel Summability for Jacobi Series*, in: Harmonic Analysis on Homogeneous Spaces, Proc. Symp. in Pure Math. 26, 335–338 (1973).

[A2] R. Askey, *Summability of Jacobi Series*, Trans. Amer. Math. Soc. 179, 71–84 (1973).
[B] V. Bach, *Error bound for the Hartree-Fock energy of atoms and molecules*, Commun. Math. Phys. **147**, 527–548 (1992).

[D] P.A.M. Dirac, *Note on exchange phenomena in the Thomas-Fermi atom*, Proc. Cambridge Phil. Soc. **26**, 376–385 (1931).

[DH] P. Debye, E. Hückel, *Zur Theorie der Elektrolyte*, Phys. Z. **24**, 185–206 (1923).

[FI] J.L. Fields, M.E. Ismail, *On the Positivity of some \(1_F^2\)’s*, SIAM J. Math. Anal. **6**, 551–559 (1975).

[FL] C.L. Fefferman, R. de la Llave, *Relativistic stability of matter I*, Revista Matematica Iberoamericana **2**, 119–161 (1986).

[G1] T. Gneiting, *Radial Positive Definite Functions Generated by Euclid’s Hat*, J. Multivariate Anal. **69**, 88–119 (1999).

[G2] T. Gneiting, *Criteria of Pólya Type for Radial Positive Definite Functions*, Proc. Amer. Math. Soc. **129**, 2309–2318 (2001).

[GM] H. Grosse, A. Martin, *Particle Physics and the Schrödinger Equation*, Cambridge University Press, p. 137 (1997).

[GS] G.M. Graf, J.P. Solovej, *A correlation estimate with applications to quantum systems with coulomb interaction*, Rev. Math. Phys. **6**, 977–997 (1994).

[HN] J.M. Hammersley, J.A. Nelder, *Sampling from an isotropic Gaussian process*, Proc. Cambridge Phil. Soc. **51**, 652–662 (1955).

[HS] C. Hainzl, R. Seiringer, *Bounds on One-Dimensional Exchange Energies with Application to Lowest Landau Band Quantum Mechanics*, Lett. Math. Phys. **55**, 133–142 (2001).

[LN] E.H. Lieb, H. Narnhofer, *The Thermodynamic Limit for Jellium*, J. Stat. Phys. **12**, 291–310 (1975). Errata J. Stat. Phys. **14**, 465 (1976).

[LO] E.H. Lieb, S. Oxford, *Improved Lower Bound on the Indirect Coulomb Energy*, Int. J. Quant. Chem. **19**, 427–439 (1981).

[M] G.D. Mahan, *Many-Particle Physics*, 2nd ed., Plenum Press, New York (1990).

[P] G. Pólya, *Remarks on Characteristic Functions*, in: Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, Univ. of CA press, 115–123 (1949).