TWO RIGIDITY RESULTS FOR STABLE MINIMAL HYPERSURFACES

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Abstract. The aim of this paper is to prove two results concerning the rigidity of complete, immersed, orientable, stable minimal hypersurfaces: we show that they are hyperplane in $\mathbb{R}^4$, while they do not exist in positively curved Riemannian $(n+1)$-manifold when $n \leq 5$; in particular, there are no stable minimal hypersurfaces in $S^{n+1}$ when $n \leq 5$. The first result was recently proved also by Chodosh and Li, and the second is a consequence of a more general result concerning minimal surfaces with finite index. Both theorems rely on a conformal method, inspired by a classical work of Fischer-Colbrie.

Key Words: stable minimal hypersurface, rigidity

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1. Introduction

It is well-known that a minimal surface $M^2 \subset \mathbb{R}^3$ is a critical point of the area functional $\mathcal{A}_t$ for all compactly supported variations, i.e. $\frac{d}{dt} |_{t=0} \mathcal{A}_t = 0$; equivalently, $M^2$ is minimal if and only if the mean curvature $H$, i.e. the (normalized) trace of the second fundamental form, is identically zero, or if and only if $M^2$ can be expressed, locally, as the graph $\Gamma(u)$ of a solution $u$ of the minimal surfaces equation

$$
(1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} + (1 + u_y^2)u_{xx} = 0.
$$

In 1914, S. Bernstein showed that an entire (i.e., defined on the whole plane $\mathbb{R}^2$) minimal graph in $\mathbb{R}^3$ is necessarily a plane; the so-called “Bernstein problem” in higher dimension can be then stated in the following way: if the graph $\Gamma(u)$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is a minimal hypersurface in $\mathbb{R}^{n+1}$, does $\Gamma(u)$ have to be necessarily a hyperplane? Many famous mathematicians worked on this problem in the Sixties, in particular Fleming [14] (who gave a new proof in the case $n = 2$), De Giorgi [10] (case $n = 3$), Almgren [1] (case $n = 4$), Simons [24] (the three remaining cases for $n \leq 7$) and, eventually, Bombieri, De Giorgi and Giusti [2], who showed that, for $n \geq 8$, there are minimal entire graphs that are not hyperplanes. We explicitly remark that a minimal graph is area-minimizing, i.e. it is not only a critical point of the area functional, but also a minimum, while this is not true for minimal hypersurfaces that are “non-graphical”, and also that area-minimizing implies stability, that is the non-negativity of the second variation for the area functional $\frac{d^2}{dt^2} |_{t=0} \mathcal{A}_t \geq 0$ for all compactly supported variations.

A natural generalization of the classical Bernstein problem, thus, is the stable Bernstein problem, that is: if $M^n \hookrightarrow \mathbb{R}^{n+1}$ is a complete, orientable, isometrically immersed, stable minimal hypersurface, does $M$ have to be necessarily a hyperplane? In the case $n = 2$ the (positive) answer was given in three different papers, which appeared between 1979 and 1981 (see do Carmo and Peng [11], Fischer-Colbrie and Schoen [16] and Pogorelov [19]).

In higher dimensions, the aforementioned result of Bombieri, De Giorgi and Giusti implies that there exist non-flat orientable, complete, stable minimal hypersurfaces in $\mathbb{R}^{n+1}$ for $n \geq 8$, while for $n \leq 5$ the stable Bernstein theorems is true with some additional assumptions (for instance, if one requires bounds on the volume growth of geodesic balls, see e.g. [21]; see also [12], [3], [4], [18] and references therein for other interesting results in the same spirit). Moreover,
by [2] and [17], we also note that there are non-flat area-minimizing (and thus minimal and stable) complete orientable hypersurfaces \( M^7 \hookrightarrow \mathbb{R}^8 \).

Up until recently, without additional hypothesis, the remaining cases \((3 \leq n \leq 6)\) were still open, even if the study of minimal (in particular stable or in general with finite index) hypersurfaces immersed into a Riemannian manifold (not only the Euclidean space, then) is a very active field and has attracted a lot of interest. Then, in 2021, Chodosh and Li \([6]\) (see also \([7]\)) showed that a complete, orientable, isometrically immersed, stable minimal immersion \( M^3 \hookrightarrow \mathbb{R}^4 \) is a hyperplane. Their proof, clever and highly non-trivial, is based on the non-parabolicity of \( M \): they perform careful estimates for the quantity

\[
F(t) = \int_{\Sigma_t} |\nabla u|^2
\]

(here \( u \) is a positive Green’s function for the Laplacian and \( \Sigma_t \) is the \( t \)-level set of \( u \)), relating it to \( \int_{\Sigma_t} |A_M|^2 \) (\( A_M \) is the second fundamental form of \( M \)).

In this paper we provide a completely different proof of Chodosh and Li result, based on a conformal deformation of the metric, a comparison result and integral estimates, and we also prove another rigidity result when the ambient space is a complete Riemannian manifold with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature. To be precise, and to fix the notation, we consider smooth, complete, connected, orientable, isometrically immersed hypersurfaces \( M^n \hookrightarrow (X^{n+1}, h) \), \( n \geq 2 \), where \((X^{n+1}, h)\) is a (complete) Riemannian manifold of dimension \( n + 1 \) endowed with metric \( h \). We denote with \( g \) the induced metric on \( M \) and with \( H \) the mean curvature of \( M \); we have that \( M \) is minimal if \( H \equiv 0 \) on \( M \). In this latter case we say that \( M \) is stable if

\[
\int_M \left[ |A|^2 + \text{Ric}_h(\nu, \nu) \right] \varphi^2 \, dV_g \leq \int_M |\nabla \varphi|^2 \, dV_g \quad \forall \varphi \in C^\infty_0(M),
\]

where \( A = A_M \) is the second fundamental form of \( M^n \), \( \nu \) is a unit normal vector to \( M \) in \( X \) and \( dV_g \) is the volume form of \( g \).

As we recalled before, stability is related to the non-negativity of the second variation or, equivalently, the non-positivity of the Jacobi operator

\[
L_M := \Delta + |A|^2 + \text{Ric}_h(\nu, \nu).
\]

The first result is thus the following:

**Theorem 1.1.** A complete, orientable, immersed, stable minimal hypersurface \( M^3 \hookrightarrow \mathbb{R}^4 \) is a hyperplane.

The second result concerns minimal hypersurfaces with finite index. We recall that a minimal immersion \( M^n \hookrightarrow (X^{n+1}, h) \) has finite index if the number of negative eigenvalues (counted with multiplicity) of the Jacobi operator \( L_M \) on every compact domain in \( M \) with Dirichlet boundary conditions is finite; in particular stability implies finite (equal zero) index. Before presenting our next result, we need to recall the notion of bi-Ricci curvature tensor introduced in \([23]\): given two orthonormal tangent vectors \( u, v \) we define

\[
\text{BRic}_h(u, v) = \text{Ric}_h(u, u) + \text{Ric}_h(v, v) - \text{Sect}_h(u, v),
\]

where \( \text{Sect}_h(u, v) \) denotes the sectional curvature of the plane spanned by \( u \) and \( v \). Our second result is the following:

**Theorem 1.2.** If \((X^{n+1}, h)\) is a complete \((n+1)\)-dimensional, \( n \leq 5 \), manifold with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature, then every complete, orientable, immersed, minimal hypersurface \( M^n \hookrightarrow (X^{n+1}, h) \) with finite index must be compact.
As a byproduct we have the

**Corollary 1.3.** If \((X^{n+1}, h)\) is a complete \((n + 1)\)-dimensional, \(n \leq 5\), manifold with non-negative sectional curvature and uniformly positive Ricci curvature, then there is no complete, orientable, immersed, stable minimal hypersurface \(M^n \hookrightarrow (X^{n+1}, h)\).

In particular, there is no complete, orientable, immersed, stable minimal hypersurface of the round spheres \(M^n \hookrightarrow (S^{n+1}, g_{\text{std}})\), provided \(n \leq 5\). In dimension \(n = 2\) this follows from a more general result proved in [22], while, in dimension \(n = 3\), it was recently proved in [8, Corollary 1.5]. We mention that Theorem 1.2 holds also for complete, orientable, immersed, stable minimal hypersurface of the cylinder \(M^n \hookrightarrow (\mathbb{R} \times S^n, g_{\text{std}})\) (observe that in this case \(\text{Sect} \geq 0\) and \(\text{BRic} \geq 1\)), provided \(n \leq 5\). As far as we know, Corollary 1.3 is new in the cases \(n = 4, 5\). We do not know if Theorem 1.2 and Corollary 1.3 hold also in dimension greater than five. We note that, in the same spirit, in [23] the authors obtained a compactness result for stable minimal hypersurfaces of dimension \(n \leq 4\) immersed in space with uniformly positive bi-Ricci curvature.

2. Proof of Theorem 1.1

In this section we give an alternative proof of [6, Theorem 1] (see Theorem 1.1). The main idea is to use a weighted volume comparison for a suitable conformal metric \(\tilde{g}\) together with a new weighted integral estimate inspired by [21].

Let \(M^n \hookrightarrow \mathbb{R}^{n+1}\) be a complete, connected, orientable, isometrically immersed, stable minimal hypersurface.

2.1. Conformal change. It is well known (see e.g. [15, Proposition 1]) that, since \(M^n \hookrightarrow \mathbb{R}^{n+1}\) is stable, then there exists a positive function \(0 < u \in C^\infty(M)\) satisfying

\[-\Delta_g u = |A|^2 u \text{ on } M.\]  

(2.1)

Following the line in [15] (see also [13]), let \(k > 0\) and consider the conformal metric

\[\tilde{g} = u^{2k} g,\]

where \(g = \iota^* h\) is the induced metric on \(M\) (and \(\iota\) denotes the inclusion). First of all we prove the following lower bound for a modified Bakry-Emery-Ricci curvature of \(\tilde{g}\). In particular, this implies the non-negativity of the 2-Bakry-Emery-Ricci curvature of \(\tilde{g}\) for a suitable \(k\).

**Lemma 2.1.** Let \(f := k(n - 2) \log u\). Then the Ricci tensor of the metric \(\tilde{g} = u^{2k} g\) satisfies

\[\text{Ric}_{\tilde{g}} + \nabla^2_{\tilde{g}} f - \frac{1 - k(n - 2)}{k(n - 2)^2} df \otimes df \geq \left( k - \frac{n - 1}{n} \right) |A|^2_{\tilde{g}} g\]

in the sense of quadratic forms. In particular, if \(n = 3\) and \(k = \frac{2}{3}\), then the 2-Bakry-Emery-Ricci tensor \(\text{Ric}^2_{\tilde{g}} f := \text{Ric}_{\tilde{g}} + \nabla^2_{\tilde{g}} f - \frac{1}{2} df \otimes df\) satisfies

\[\text{Ric}^2_{\tilde{g}} f \geq 0.\]

**Proof.** Since \(f = k(n - 2) \log u\), we have

\[df = k(n - 2) \frac{du}{u}\]

and

\[\nabla^2_{\tilde{g}} f = k(n - 2) \left( \frac{\nabla^2_{\tilde{g}} u}{u} - \frac{du \otimes du}{u^2} \right),\]
which implies
\[ \Delta_g f = k(n-2) \left( \frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} \right). \]

On the other hand, from the standard formulas for a conformal change of the metric \( \tilde{g} = e^{2\varphi} g \), \( \varphi \in C^\infty(M) \), \( \varphi > 0 \) we get
\[ \text{Ric}_{\tilde{g}} = \text{Ric}_g - (n-2) \left( \nabla^2 \varphi - d\varphi \otimes d\varphi \right) - \left[ \Delta_g \varphi + (n-2)|\nabla \varphi|^2 \right] g \]
and
\[ \nabla^2 \tilde{f}_g = \nabla^2 f - (df \otimes d\varphi + d\varphi \otimes df) + g(\nabla f, \nabla \varphi) g. \]

Note that, in our case, \( \varphi = k \log u \); now we exploit the facts that \( u \) is a solution of equation (2.1) to write
\[ \text{Ric}_{\tilde{g}} + \nabla^2 \tilde{f}_g = \text{Ric}_g - k^2(n-2) \frac{du \otimes du}{u^2} + k|A|_g^2 g + k \frac{|\nabla u|^2}{u^2} g \]
\[ = \text{Ric}_g - \frac{dA \otimes df}{n-2} + k|A|_g^2 g + \frac{|\nabla f|^2}{k(n-2)^2} g. \]

From the Cauchy-Schwarz inequality we have
\[ |\nabla f|^2 g \geq df \otimes df, \]
thus
\[ \text{Ric}_{\tilde{g}} + \nabla^2 \tilde{f}_g \geq \text{Ric}_g + \frac{1 - k(n-2)}{k(n-2)^2} dA \otimes df + k|A|_g^2 g; \]
from Gauss equations in the minimal case we get \( \text{Ric}_g = -A^2 \); since \( A \) is traceless we have the inequality
\[ A^2 \leq \frac{n-1}{n} |A|^2 g, \]
and substituting in the previous relation we conclude
\[ \text{Ric}_{\tilde{g}} + \nabla^2 \tilde{f}_g - \frac{1 - k(n-2)}{k(n-2)^2} dA \otimes df \geq \left( k - \frac{n-1}{n} \right) |A|_g^2 g. \]

2.2. Completeness. In this subsection we are going to prove that the conformal metric \( \tilde{g} = u^{2k} g \) is complete, provided \( n = 3 \) and \( k = \frac{3}{2} \). In order to do this we follow the strategy in [15] and we use some computations in [13]. First, we recall that in the proof of [15, Theorem 1], given a reference point \( O \in M^n \), the author showed the existence of a \( \tilde{g} \)-minimizing geodesic,
\[ \gamma(s) : [0, \infty) \to M^n; \]
where \( s \) is the \( g \)-arclength and \( M^n \to \mathbb{R}^{n+1} \) is the usual complete, connected, orientable, isometrically immersed, stable minimal hypersurface. For the sake of completeness, we report the argument here. First of all, for every \( R > 0 \), we consider the geodesic ball of \( g \) centered at \( O \) of radius \( R \), \( B_R(O) \). Then, we first claim that there exists a \( \tilde{g} \)-minimizing geodesic, \( \gamma_R \), joining \( O \) to any boundary point of \( B_R(O) \). Indeed, consider \( u_R := u + \eta \), where \( \eta \) is a smooth function such that \( \eta \equiv 0 \) in \( B_R(O) \) and \( \eta \equiv 1 \) in \( M^n \setminus B_{R+1}(O) \). Since \( u_R \) is bounded below, the metric
\[ \bar{g}_R = u_R^{2(n-1)} g \]
is complete, and these geodesics exist. Therefore, for every \( R_i > 0 \), since \( \partial B_{R_i}(O) \) is compact, there exists \( x_i \in \partial B_{R_i}(O) \) so that \( x_i \) is closest (in \( \bar{g}_R \)) to \( O \). Let \( \gamma_i \) be the \( \bar{g}_R \)-minimizing geodesic
joining $O$ to $x_i$. Note that $\gamma_i \subset B_{R_i}(O)$ or another point would be closer to $O$. Since $u_{R_i} = u$ in $B_{R_i}(O)$, then $\gamma_i$ is a $\bar{g}$-minimizing geodesic. We parametrize $\gamma_i$ with respect to $g$-arclength. In particular, since $|\dot{\gamma}_i(s)|_g = 1$ for every $s$, up to subsequences, the sequence $\dot{\gamma}_i(0)$ converges to a limit vector as $R_i \to \infty$. Thus, by ODE theory and Ascoli-Arzelà, $\gamma_i$ converges on compact sets of $[0, \infty)$ to a limiting curve $\gamma$ which is a $\bar{g}$-minimizing geodesic ad is parametrized by $g$-arclength.

**Remark 2.2.** (i) We observe that the completeness of the metric $\bar{g} = u^{2k}g$ will follow if we can show that the $\bar{g}$-length of $\gamma$ is infinite, i.e.

$$\int_\gamma \bar{g} = \int_\gamma u^k \, ds = +\infty.$$ 

Indeed, by construction, the $\bar{g}$-length of every other divergent geodesic starting from $O$ (i.e. its image does not lie in any ball $B_R(O)$) must be greater or equal than the one of $\gamma$.

(ii) Note that $\gamma$ has unit speed with respect to $g$ and to $\bar{g}$, when it is parametrized by the arclength $s$ and $\bar{s}$, respectively.

From now on 

$$n = 3 \quad \text{and} \quad k = \frac{2}{3}.$$ 

**Lemma 2.3.** The metric $\bar{g} = u^{\frac{4}{3}}g$ is complete.

**Proof.** We do part of the computations for every $n$. We consider the $\bar{g}$-minimizing geodesic $\gamma$ just constructed and as observed in Remark 2.2 the completeness of $\bar{g}$ is equivalent to prove that $\gamma$ has infinite $\bar{g}$ length, i.e.

$$\int_0^{+\infty} u^k(\gamma(s)) \, ds = +\infty.$$ 

Since $\gamma$ is minimizing, by the second variation formula, following the computations in the proof of Theorem 1 (with $H = 0$) in [13], we obtain

$$(n - 1) \int_0^{+\infty} (\varphi_s)^2 u^{-k} \, ds \geq \int_0^{+\infty} \varphi^2 u^{-k} \left( k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) \, ds$$

$$- k(n - 2) \int_0^{+\infty} \varphi^2 u^{-k} (\log u)_s \, ds + k \int_0^{+\infty} \varphi^2 u^{-k} |\nabla u|^2 \, ds$$

for every smooth function $\varphi$ with compact support in $(0, +\infty)$ and for every $k > 0$. Since $A$ is trace-free, we have

$$|A|^2 \geq A_{11}^2 + A_{22}^2 + \ldots + A_{nn}^2 + 2 \sum_{j=2}^n A_{1j}^2 \geq \frac{n}{n - 1} A_{11}^2 + 2 \sum_{j=2}^n A_{1j}^2,$$

thus

$$k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \geq \left( \frac{kn}{n - 1} - 1 \right) A_{11}^2 + (2k - 1) \sum_{j=2}^n A_{1j}^2.$$ 

In particular, if

$$k \geq \frac{n - 1}{n}$$ 

we have

$$\int_0^{+\infty} \varphi^2 u^{-k} \left( k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) \, ds \geq 0.$$ 

\[\text{(2.2)}\]
Using this estimate, the fact that $|\nabla u|^2 \geq (u_s)^2$ and integrating by parts, we obtain
\[
(n - 1) \int_0^{+\infty} (\varphi_s)^2 u^{-k} \, ds \geq 2k(n - 2) \int_0^{+\infty} \varphi \varphi_s u^{-k-1} u_s \, ds \\
+ k[1 - k(n - 2)] \int_0^{+\infty} \varphi^2 u^{-k-2} (u_s)^2 \, ds.
\]

Let now $\varphi = u^k \psi$, with $\psi$ smooth with compact support in $(0, +\infty)$. We have
\[
\varphi^2 u^{-k} = u^k \psi^2, \\
\varphi_s = ku^k u^{-1} u_s + u^k \psi_s, \\
(\varphi_s)^2 u^{-k} = k^2 \psi^2 u^{-2} (u_s)^2 + u^k (\psi_s)^2 + 2k \psi \psi_s u^{-1} u_s,
\]
and substituting in the previous relation we get
\[
(n - 1) \int_0^{+\infty} u^k (\psi_s)^2 \, ds \geq -2k \int_0^{+\infty} \psi \psi_s u^{-1} u_s \, ds + k(1 - k) \int_0^{+\infty} \psi^2 u^{-2} (u_s)^2 \, ds. \tag{2.3}
\]

Let
\[
I := \int_0^{+\infty} \psi \psi_s u^{-1} u_s \, ds;
\]
thus we have
\[
I = \frac{1}{k} \int_0^{+\infty} \psi \psi_s (u^k)_s \, ds = -\frac{1}{k} \int_0^{+\infty} u^k (\psi_s)^2 \, ds - \frac{1}{k} \int_0^{+\infty} \psi \psi_s u^k \, ds.
\]

Moreover, for every $t > 1$ and using Young's inequality for every $\varepsilon > 0$, we have
\[
2kI = 2ktI + 2k(1 - t)I \\
= -2t \int_0^{+\infty} u^k (\psi_s)^2 \, ds - 2t \int_0^{+\infty} \psi \psi_s u^k \, ds + 2k(1 - t) \int_0^{+\infty} \psi^2 u^{-2} (u_s)^2 \, ds \\
\leq -2t \int_0^{+\infty} u^k (\psi_s)^2 \, ds - 2t \int_0^{+\infty} \psi \psi_s u^k \, ds \\
+ k(t - 1)\varepsilon \int_0^{+\infty} \psi^2 u^{-2} (u_s)^2 \, ds + k(t - 1) \frac{k(t - 1)}{\varepsilon} \int_0^{+\infty} u^k (\psi_s)^2 \, ds.
\]

Assuming
\[
k < 1 \tag{2.4}
\]
and choosing
\[
\varepsilon := \frac{1 - k}{t - 1}
\]
we obtain
\[
2kI \leq -2t \int_0^{+\infty} \psi \psi_s u^k \, ds + k(1 - k) \int_0^{+\infty} \psi^2 u^{-2} (u_s)^2 \, ds \\
+ \left[ \frac{k(t - 1)^2}{1 - k} - 2t \right] \int_0^{+\infty} u^k (\psi_s)^2 \, ds.
\]

From (2.3) we get
\[
0 \leq \left[ \frac{k(t - 1)^2}{1 - k} - 2t + (n - 1) \right] \int_0^{+\infty} u^k (\psi_s)^2 \, ds - 2t \int_0^{+\infty} \psi \psi_s u^k \, ds
\]
for every $t > 1$ and every $k$ satisfying (2.2) and (2.4). Let
\[
P(t) := \frac{k(t - 1)^2}{1 - k} - 2t + (n - 1)
\]
and choose $k = \frac{n-1}{n}$. It is easy to see that $P(t)$ is negative for some $t > 1$ if $n = 3$: indeed

$$P(t) = (n - 1)t^2 - 2nt + 2(n - 1) = 2t^2 - 6t + 4 = -2(t - 1)(2 - t).$$

Therefore, if $n = 3$, $k = \frac{2}{3}$ and $t = \frac{3}{2}$, we deduce

$$0 \leq -\int_{0}^{+\infty} u^{\frac{2}{3}}(\psi_s)^2 ds - 6 \int_{0}^{+\infty} u^{\frac{2}{3}} \psi \psi_{ss} ds$$

for every $\psi$ smooth with compact support in $(0, +\infty)$. Now we choose $\psi = s\eta$ with $\eta$ smooth with compact support in $(0, +\infty)$: thus

$$\psi_s = \eta + s\eta_s, \quad \psi_{ss} = 2\eta_s + s\eta_{ss},$$

and we get

$$\int_{0}^{+\infty} u^{\frac{2}{3}} \eta^2 ds \leq \int_{0}^{+\infty} u^{\frac{2}{3}} (-14s\eta s \eta_{ss} - 6s^2 \eta_{ss} - s^2 (\eta_s)^2) ds.$$

Choose $\eta$ such that $\eta \equiv 1$ on $[0, R]$, $\eta \equiv 0$ on $[2R, +\infty)$ and with $|\eta_s|$ and $|\eta_{ss}|$ bounded by $C/R$ and $C/R^2$, respectively, for $R \leq s \leq 2R$ (C is a positive constant). Then

$$\int_{0}^{R} u^{\frac{2}{3}} ds \leq \int_{0}^{+\infty} u^{\frac{2}{3}} \eta^2 ds \leq C \int_{R}^{+\infty} u^{\frac{2}{3}} ds$$

for some $C > 0$ independent of $R$. We conclude that

$$\int_{0}^{+\infty} u^{\frac{2}{3}} ds = +\infty,$$

i.e. $\tilde{g} = u^{\frac{4}{3}}g$ is complete. \hfill $\square$

2.3. Weighted integral estimates. From lemma 2.1 and lemma 2.3 we have that the metric $\tilde{g} = u^{\frac{4}{3}}g$ is complete and it has non-negative 2-Bakry-Emery-Ricci curvature. Using well known comparison results (see [20]) we immediately obtain the following weighted Bishop-Gromov volume estimate for a geodesic ball $B^g_{R}(x_0)$ centered at $x_0 \in M$, of radius $R$, with respect to the metric $\tilde{g}$.

**Corollary 2.4.** Let $x_0 \in M^3$. Then, for every $R > 0$, there exists $C > 0$ such that the $f$-volume

$$\text{Vol}_{f} B^g_{R}(x_0) := \int_{B^g_{R}(x_0)} e^{-f} \, dV_{\tilde{g}} \leq CR^3,$$

where $f = \frac{2}{3} \log u$. Equivalently, in terms of $u$ and the volume form of $g$,

$$\int_{B^g_{R}(x_0)} u^{\frac{4}{3}} \, dV_{\tilde{g}} \leq CR^3.$$

The last ingredient that we need in the proof of Theorem 1.1 is the following weighted integral inequality in the spirit of [21, Theorem 1].

**Lemma 2.5.** For every $0 < \delta < \frac{4}{100}$, there exists $C > 0$ such that

$$\int_{M} |A|^5 \psi^5 \psi^{5+\delta} \, dV_{\tilde{g}} \leq C \int_{M} u^{-2 - \frac{4\delta}{3}} |\nabla \psi|^{5+\delta} \, dV_{\tilde{g}} \quad \forall \psi \in C^\infty_0(M).$$

**Proof.** Again, we do part of the computations for every $n$. From [21] we get

$$\int_{M} |A|^p \phi^2 \leq C \int_{M} |A|^{p-2} |\nabla \phi|^2 \quad \forall \phi \in C^\infty_0(M), \quad (2.5)$$

...
for every \( p \in [4, 4 + \sqrt{8/n}] \) and for some \( C = C(n, p) > 0 \). For the sake of completeness we report here the proof of (2.5). We take \( \varphi = |A|^{1+q} \psi, \ q \geq 0, \) with \( \psi \in C_0^\infty(M) \), in the stability inequality (1.1) obtaining
\[
\int_M |A|^{4+2q} \psi^2 \leq [(1 + q)^2 + \varepsilon] \int_M |A|^{2q} |\nabla A||^2 \psi^2 + \frac{1+q}{\varepsilon} \int_M |A|^{2+2q} |\nabla \psi|^2,
\]
for every \( \varepsilon > 0 \), where we used Young’s inequality. On the other hand, multiplying Simons’ inequality (see [9, Lemma 2.1] for a proof)
\[
|A|\Delta|A| + |A|^4 \geq \frac{2}{n} |\nabla |A||^2.
\]
by \( |A|^{2q} \psi^2 \) and integrating by parts, we get
\[
\left( \frac{2}{n} + 1 + 2q - \varepsilon \right) \int_M |A|^{2q} |\nabla A||^2 \psi^2 \leq \int_M |A|^{4+2q} \psi^2 + \frac{1}{\varepsilon} \int_M |A|^{2+2q} |\nabla \psi|^2
\]
for every \( \varepsilon > 0 \), where we used again Young’s inequality. Since \( q \geq 0 \), for \( \varepsilon > 0 \) sufficiently small, we obtain
\[
\left\{ 1 - [(1 + q)^2 + \varepsilon] \left( \frac{2}{n} + 1 + 2q - \varepsilon \right)^{-1} \right\} \int_M |A|^{4+2q} \psi^2 \leq C \int_M |A|^{2+2q} |\nabla \psi|^2.
\]
Let \( q := \frac{p-4}{2} \). For \( \varepsilon > 0 \) small enough, we have
\[
1 - [(1 + q)^2 + \varepsilon] \left( \frac{2}{n} + 1 + 2q - \varepsilon \right)^{-1} > 0
\]
if \( p \in [4, 4 + \sqrt{8/n}] \) and we finally obtain
\[
\int_M |A|^p \psi^2 \leq C \int_M |A|^{p-2} |\nabla \psi|^2 \quad \forall \psi \in C_0^\infty(M).
\]

Taking \( \psi = \varphi^{p/2} \), by Holder’s inequality we get (2.5).

Take \( \varphi = u^\alpha \psi \), with \( \psi \) smooth with compact support, \( u \) the solution of (2.1) and \( \alpha < 0 \). Since, from Cauchy-Schwarz and Young’s inequalities,
\[
|\nabla (u^\alpha \psi)|^2 \leq 2\psi^2 |\nabla (u^\alpha)|^2 + 2u^{2\alpha} |\nabla \psi|^2,
\]
then (2.5) becomes
\[
\int_M |A|^{p} u^{2\alpha} \psi^2 \leq 2C \left[\int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 + \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 \right] \quad \forall \psi \in C_0^\infty(M). \tag{2.8}
\]
Now we tackle the first integral on the right-hand side of (2.8) firstly integrating by parts
\[
\int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 = -\int_M |A|^{p-2} \psi^2 u^\alpha \Delta u^\alpha - \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle - 2\int_M |A|^{p-2} u^\alpha \psi \langle \nabla u^\alpha, \nabla \psi \rangle,
\]
secondly we use the fact that
\[
\Delta u^\alpha = \alpha u^{\alpha-1} \Delta u + \alpha(\alpha - 1) u^{\alpha-2} |\nabla u|^2 \quad \text{and} \quad |\nabla u^\alpha|^2 = \alpha^2 u^{2\alpha-2} |\nabla u|^2,
\]
then with Cauchy-Schwarz and Young’s inequalities to get
\[
\int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 \leq -\alpha \int_M |A|^{p-2} u^{\alpha-1} \psi^2 \Delta u - \frac{\alpha - 1}{\alpha} \int_M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2
\]
\[
- \int M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle + \varepsilon \int M |A|^{p-2} \psi^2 |\nabla u^\alpha|^2 + \frac{1}{\varepsilon} \int M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2,
\]
for all $\varepsilon > 0$. From (2.1) we find
\[
\int_M |A|^{p-2} |\nabla u^\alpha|^2 \leq \alpha \int_M |A|^{p-2} |\nabla u^\alpha|^2 - \frac{\alpha - 1}{\alpha} \int_M |A|^{p-2} |\nabla u^\alpha|^2
\]
\[- \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle + \varepsilon \int_M |A|^{p-2} |\nabla u^\alpha|^2 + \frac{1}{\varepsilon} \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 ,
\]
i.e.
\[
\left( 1 - \varepsilon + \frac{\alpha - 1}{\alpha} \right) \int_M |A|^{p-2} |\nabla u^\alpha|^2 \leq \alpha \int_M |A|^{p-2} |\nabla u^\alpha|^2
\]
\[- \int_M u^\alpha \psi^2 \langle \nabla u^\alpha, \nabla |A|^{p-2} \rangle + \varepsilon \int_M |A|^{p-2} |\nabla u^\alpha|^2 + \frac{1}{\varepsilon} \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 ,
\]
Now, since
\[
\nabla |A|^{p-2} = (p-2)|A|^{p-3} \nabla |A| = (p-2)|A|^{\frac{p-2}{2}} |\nabla A| = (p-2)|A|^{\frac{p-4}{2}} \nabla |A|,
\]
then, from Cauchy-Schwarz and Young’s inequalities we obtain
\[
\left( 1 - \varepsilon + \frac{\alpha - 1}{\alpha} - \frac{p-2}{2t_1} \right) \int_M |A|^{p-2} |\nabla u^\alpha|^2 \leq \alpha \int_M |A|^{p-2} |\nabla u^\alpha|^2
\]
\[+ \frac{(p-2)t_1}{2} \int_M |A|^{p-4} u^{2\alpha} |\nabla |A||^2 + \frac{1}{\varepsilon} \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2 ,
\]
for every $t_1 > 0$. Now, multiplying by $|A|^{p-4} f^2$ the Simons’ inequality (2.6), integrating by parts and using Young’s inequality we obtain
\[
\int_M |A|^p f^2 \geq \left( \frac{2}{n} + p - 3 - t_2 \right) \int_M |A|^{p-4} |\nabla |A||^2 f^2 - \frac{1}{t_2} \int_M |A|^{p-2} |\nabla f|^2
\]
for every $t_2 > 0$. Choosing $f = u^\alpha \psi$ we get
\[
\int_M |A|^{p-2} |\nabla u^\alpha|^2 \geq \left( \frac{2}{n} + p - 3 - t_2 \right) \int_M |A|^{p-4} |\nabla |A||^2 u^{2\alpha} \psi^2
\]
\[- \left( \frac{1}{t_2} + \varepsilon \right) \int_M |A|^{p-2} |\nabla u^\alpha|^2 - \frac{1}{t_2} \left( 1 + \frac{1}{t_2 \varepsilon} \right) \int_M |A|^{p-2} u^{2\alpha} |\nabla \psi|^2
\]
for every $\varepsilon > 0$, since
\[
|\nabla (u^\alpha \psi)|^2 \leq (1 + t_2 \varepsilon) \psi^2 |\nabla (u^\alpha)|^2 + \left( 1 + \frac{1}{t_2 \varepsilon} \right) u^{2\alpha} |\nabla \psi|^2 .
\]
Now let $\delta > 0$. Using (2.10) in (2.9) with
\[
\alpha = -1 - \frac{\delta}{3} \leq -1
\]
we obtain
\[
\left( 1 + \frac{2 + \delta}{1 + \frac{3}{3}} - \left( 2 + \frac{\delta}{3} \right) \varepsilon - \frac{p-2}{2t_1} - \frac{1 + \frac{\delta}{3}}{t_2} \right) \int_M |A|^{p-2} |\nabla u^{-1 - \frac{\delta}{3}}|^2
\]
\[\leq \left[ \frac{1}{\varepsilon} + \frac{1 + \frac{\delta}{3}}{t_2} \left( 1 + \frac{1}{t_2 \varepsilon} \right) \right] \int_M |A|^{p-2} u^{-2 - \frac{\delta}{3}} |\nabla \psi|^2
\]
\[+ \left( \frac{(p-2)t_1}{2} - \frac{2 + \frac{\delta}{3}}{n} \right) \left( 1 + \frac{\delta}{3} \right) p + 3 + \delta + \left( 1 + \frac{\delta}{3} \right) t_2 \int_M |A|^{p-4} u^{\frac{2}{2} - \frac{\delta}{3}} |\nabla |A||^2 ,
\]
for all $\varepsilon, t_1, t_2 > 0$. Let
\[
n = 3, \quad p = 5 + \delta, \quad t_1 = \frac{2(5 + 3\delta)}{9}, \quad t_2 = 1
\]
we obtain

\[
\frac{(p - 2)t_1}{2} - \frac{2 + \frac{2\delta}{3}}{n} - \left(1 + \frac{\delta}{3}\right)p + 3 + \delta + \left(1 + \frac{\delta}{3}\right)t_2 = 0
\]

and

\[
1 + \frac{2 + \frac{\delta}{3}}{1 + \frac{\delta}{3}} - \frac{p - 2}{2t_1} - \frac{1 + \frac{\delta}{3}}{t_2} = \frac{603 + 378\delta + 7\delta^2 - 12\delta^3}{180 + 16\delta + 36\delta^2}.
\]

Thus

\[
\left(\frac{603 + 378\delta + 7\delta^2 - 12\delta^3}{180 + 16\delta + 36\delta^2} - \left(\frac{2 + \frac{\delta}{3}}{3}\right)\varepsilon\right) \int_M |A|^{3+\delta}\psi^2|\nabla u|^{-\frac{\delta}{3}}|\nabla| \leq \frac{1}{\varepsilon} + \left(1 + \frac{\delta}{3}\right)\left(1 + \frac{1}{\varepsilon}\right)\int_M |A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2,
\]

for all \(\varepsilon > 0\). Choosing \(0 < \delta < 1/100\) and \(\varepsilon\) small enough we obtain

\[
\int_M |A|^{3+\delta}\psi^2|\nabla u|^{-1-\frac{\delta}{3}}|\nabla| \leq C \int_M |A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2,
\]

for some \(C > 0\). From (2.8) and Young’s inequality we get

\[
\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2 \leq C \int_M |A|^{3+\delta}u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2 \\
\leq \varepsilon' \int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2 + C \varepsilon' \int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^{5+\delta}\psi^{-(3+\delta)}
\]

for all \(\varepsilon' > 0\) and \(\psi \in C_0^\infty(M)\). Therefore

\[
\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\psi^2 \leq C \int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^{5+\delta}\psi^{-(3+\delta)} = C \int_M u^{-2-\frac{2\delta}{3}}|\nabla\psi|^2|\psi|^{5+\delta}.
\]

The conclusion now follows immediately by replacing \(\psi\) with \(\psi^{\frac{2}{5+\delta}}\).

\[\square\]

2.4. Final estimate. Combining Lemma 2.5 with Corollary 2.4 we can conclude the proof of Theorem 1.1. More precisely, let \(x_0 \in M\) and let \(\tilde{r}\) the distance function from \(x_0\) with respect to the metric \(\tilde{g} = u^\frac{4}{3}g\). We choose \(\psi := \eta(\tilde{r})\) with \(0 \leq \eta \leq 1\), \(\eta \equiv 1\) on \([0, R]\), \(\eta \equiv 0\) on \([2R, +\infty)\) and \(|\eta'| \leq C/R\) on \([R, 2R]\), for some \(C > 0\) and \(R > 0\). From Lemma 2.5, for some \(0 < \delta < 1/100\), we have

\[
\int_M |A|^{5+\delta}u^{-2-\frac{2\delta}{3}}\eta^{5+\delta} dV_g \leq C \int_M u^{-2-\frac{2\delta}{3}}|\nabla\eta|^{5+\delta}_{\tilde{g}} dV_g = C \int_M u^{-2-\frac{2\delta}{3} + \frac{2(5+\delta)}{3}}|\nabla\psi|^{5+\delta}_{\tilde{g}} dV_g \\
\leq \frac{C}{R^{5+\delta}} \int_{B^\tilde{g}_{2R}(x_0)} u^\frac{4}{3} dV_g \\
\leq \frac{C}{R^{5}}
\]

where we used the fact that \(|\nabla \tilde{r}|_{\tilde{g}} \equiv 1\) and Corollary 2.4. Since \(\delta > 0\), letting \(R \to +\infty\) we get

\[|A| \equiv 0\] on \(M^3\)

and this concludes the proof of Theorem 1.1.
3. Proof of Theorem 1.2

Proof of Theorem 1.2. Let \( (X^{n+1}, h) \) be a complete \( n \)-dimensional, \( n \leq 5 \), manifold with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature and consider an orientable, immersed, minimal hypersurface \( M^n \rightarrow (X^{n+1}, h) \) with finite index. Suppose, by contradiction, that \( M \) is non-compact. It is well known (see [15, Proposition 1]) that there exist \( 0 < u \in C^\infty(M) \) and a compact subset \( K \subset M \) such that \( u \) solves

\[
-\Delta u = [\|A\|^2 + \text{Ric}_h(\nu, \nu)] u \quad \text{on } M \setminus K.
\]

Let \( k > 0 \) and consider the conformal metric

\[
\tilde{g} = u^{2k} g.
\]

where \( g \) is the induced metric on \( M \). Let \( s \) be the arc length with respect to the metric \( g \). Following the construction in [15, Theorem 1], we can construct a minimizing geodesic \( \tilde{\gamma}(s) : [0, +\infty) \rightarrow M \setminus K \) in the metric \( \tilde{g} \) which has infinite length in the metric \( g \). Now we can argue exactly as in the proof of estimate (6) in [13], using \( H \equiv 0 \), obtaining

\[
(n - 1) \int_0^a (\varphi_s)^2 \, ds \geq k(n - 3) \int_0^a \varphi_s \frac{u_a}{u} \, ds + k \frac{[4 - k(n - 1)]}{4} \int_0^a \varphi^2 \left( \frac{u_a}{u} \right)^2 \, ds
\]

\[
+ \int_0^a \varphi^2 \left( k\text{Ric}_h(\nu, \nu) + \sum_{j=2}^n R_{1j1j}^h \right) \, ds
\]

\[
+ \int_0^a \varphi^2 \left( k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) \, ds,
\]

for every smooth function \( \varphi \) such that \( \varphi(0) = \varphi(a) = 0 \) and for every \( k > 0 \). Arguing as in [23, Section 2] we have the following identity

\[
k\text{Ric}_h(\nu, \nu) + \sum_{j=2}^n R_{1j1j}^h = k\text{BRic}_h(\nu, \nu) - k\text{Ric}_h(e_1, e_1) + kR_{11}^h + \sum_{j=2}^n R_{1j1j}^h
\]

\[
= k\text{BRic}_h(e_1, \nu) + (1 - k) \sum_{j=2}^n R_{1j1j}^h.
\]

Since \( (N^{n+1}, h) \) is with non-negative sectional curvature and either uniformly positive bi-Ricci curvature or uniformly positive Ricci curvature, we have \( R_{1j1j}^h \geq 0 \) for every \( j = 2, \ldots, n \) and either

\[
\text{BRic}_h(e_1, \nu) \geq R_0 \quad \text{or} \quad \text{Ric}_h(\nu, \nu) \geq R_0
\]

for some \( R_0 > 0 \). Therefore, if \( k \leq 1 \), we get

\[
(n - 1) \int_0^a (\varphi_s)^2 \, ds \geq k(n - 3) \int_0^a \varphi_s \frac{u_a}{u} \, ds + k \frac{[4 - k(n - 1)]}{4} \int_0^a \varphi^2 \left( \frac{u_a}{u} \right)^2 \, ds
\]

\[
+ \int_0^a \varphi^2 \left( kR_0 + k|A|^2 - A_{11}^2 - \sum_{j=2}^n A_{1j}^2 \right) \, ds.
\]

Since \( A \) is trace-free, we have

\[
|A|^2 \geq A_{11}^2 + A_{22}^2 + \ldots + A_{nn}^2 + 2 \sum_{j=2}^n A_{1j}^2 \geq \frac{n}{n - 1} A_{11}^2 + 2 \sum_{j=2}^n A_{1j}^2,
\]
thus
\[ k|A|^2 - A_{11}^2 - \sum_{j=2}^{n} A_{1j}^2 \geq \left( \frac{kn}{n-1} - 1 \right) A_{11}^2 + (2k-1) \sum_{j=2}^{n} A_{1j}^2. \]
Choose
\[ k = \frac{n-1}{n} \leq 1. \]

We get
\[ \int_{0}^{a} (\varphi_s)^2 \, ds \geq \frac{n-3}{n} \int_{0}^{a} \varphi \varphi_s \frac{u_s}{u} \, ds + \frac{6n-n^2-1}{4n^2} \int_{0}^{a} \varphi^2 \left( \frac{u_s}{u} \right)^2 \, ds + \frac{R_0}{n} \int_{0}^{a} \varphi^2 \, ds. \]
If \( n \leq 5 \), we have
\[ \frac{6n-n^2-1}{4n^2} \geq \delta_0 > 0. \]
Moreover, there exists \( C > 0 \), such that
\[ \frac{n-3}{n} \varphi \varphi_s \frac{u_s}{u} \geq -\delta_0 \varphi^2 \left( \frac{u_s}{u} \right)^2 - C(\varphi_s)^2. \]
Therefore, there exists \( C > 0 \), such that
\[ C \int_{0}^{a} (\varphi_s)^2 \, ds \geq \frac{R_0}{n} \int_{0}^{a} \varphi^2 \, ds \]
for every smooth function \( \varphi \) such that \( \varphi(0) = \varphi(a) = 0 \). Integrating by parts we obtain
\[ \int_{0}^{a} (\varphi \varphi_s s + CR_0 \varphi^2) \, ds \leq 0. \]
Choosing \( \varphi(s) = \sin(\pi s a^{-1}) \), \( s \in [0,a] \) one has
\[ \left( CR_0 - \frac{\pi^2}{a^2} \right) \int_{0}^{a} \sin^2(\pi s a^{-1}) \, ds \leq 0 \]
i.e.
\[ a^2 \leq \frac{\pi^2}{CR_0}. \]
We conclude that the length (in the metric \( g \)) of the geodesic \( \tilde{\gamma}(s) \) is finite and this gives a contradiction. Therefore \((M^n, g)\) must be compact and this concludes the proof of Theorem 1.2. \( \square \)

**Proof of Corollary 1.3.** If \( M \) is stable, by Theorem 1.2 it must be compact. Moreover there exists \( u > 0 \) satisfying
\[ -\Delta u = \left[ |A|^2 + \text{Ric}_h(\nu, \nu) \right] u \quad \text{on } M. \]
Integrating over \( M \) we get a contradiction, since \( \text{Ric}_h > 0 \) on \( M \). Equivalently, one can use \( f \equiv 1 \) in the stability inequality (1.1) to get a contradiction. \( \square \)

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