EMBEDDING SEMIGROUP $C^*$-ALGEBRAS INTO INDUCTIVE LIMITS

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Abstract. The note is concerned with inductive systems of Toeplitz algebras and their $*$-homomorphisms over arbitrary partially ordered sets. The Toeplitz algebra is the reduced semigroup $C^*$-algebra for the additive semigroup of non-negative integers. It is known that every partially ordered set can be represented as the union of the family of its maximal upward directed subsets indexed by elements of some set. In our previous work we have studied a topology on this set of indexes. For every maximal upward directed subset we consider the inductive system of Toeplitz algebras that is defined by a given inductive system over an arbitrary partially ordered set and its inductive limit. Then for a base neighborhood $U_a$ of the topology on the set of indexes we construct the $C^*$-algebra $\mathcal{B}_a$ which is the direct product of those inductive limits. In this note we continue studying the connection between the properties of the topology on the set of indexes and properties of inductive limits for systems consisting of $C^*$-algebras $\mathcal{B}_a$ and their $*$-homomorphisms. It is proved that there exists an embedding of the reduced semigroup $C^*$-algebra for a semigroup in the additive group of all rational numbers into the inductive limit for the system of $C^*$-algebras $\mathcal{B}_a$.

Keywords: embedding, inductive limit, inductive system, injective $*$-homomorphism, partially ordered set, reduced semigroup $C^*$-algebra, semigroup, Toeplitz algebra, upward directed set

1. Introduction

In algebraic quantum field theory, which serves as our main motivation, one usually considers inductive systems of $C^*$-algebras and their $*$-homomorphisms. The simplest example of such a system is a net of $C^*$-algebras and their embeddings.

In [1, 2, 3] the authors study nets containing $C^*$-algebras of quantum observables for the case of curved spacetimes. The net constructed by means of the semigroup $C^*$-algebra generated by the path semigroup for a partially ordered set is treated in [4]. In [5] the authors deal with
a net consisting of \( C^* \)-algebras associated to a net of Hilbert spaces over a partially ordered set.

A part of motivation for studying inductive systems of Toeplitz algebras comes from \([6, 7, 8, 9]\). The paper \([6]\) contains results on limit automorphisms for inductive sequences of Toeplitz algebras which are closely related to the facts on the mappings of topological groups \([10, 11, 12]\). In \([7, 8]\) the authors introduce a topology associated to a partially ordered set and study its relation to properties of inductive limits arising from a system of \( C^* \)-algebras over the partially ordered set which yields that topology. The existence of an isomorphism between the inductive limit of an inductive system of Toeplitz algebras over a directed set and the reduced semigroup \( C^* \)-algebra for a semigroup in the group of rational numbers is shown in \([7, 9]\).

This note deals with inductive systems of Toeplitz algebras and their \( \ast \)-homomorphisms over arbitrary partially ordered sets. Here, by the Toeplitz algebra we mean the reduced semigroup \( C^* \)-algebra for the additive semigroup of non-negative integers. The study of such semigroup \( C^* \)-algebras goes back to L. A. Coburn \([13, 14]\), R. G. Douglas \([15]\), G. J. Murphy \([16, 17]\). There is a large literature on the subject (see, for example, \([18, 19, 20, 21, 22]\) and the references there in).

For a given inductive system of Toeplitz algebras over a partially ordered set \( K \) one can take an inductive subsystem and its inductive limit \( \mathfrak{A}^K_i \) over every maximal upward directed subset \( K_i \) in \( K \), where \( i \) runs over a set of indexes \( I \). Using the topology on \( I \) and the inductive limits \( \mathfrak{A}^K_i \), we construct new \( C^* \)-algebras \( \mathfrak{B}_a \) that are the direct products of \( C^* \)-algebras \( \mathfrak{A}^K_i \). Then we consider an inductive system consisting of \( C^* \)-algebras \( \mathfrak{B}_a \) over the upward directed set \( K_i \) and the inductive limit \( \mathfrak{B}^K_i \) of that system. We prove that the reduced semigroup \( C^* \)-algebra for a semigroup in the additive group of all rational numbers can be embedded into \( C^* \)-algebra \( \mathfrak{B}^K_i \). In other words, it is shown there exists an injective \( \ast \)-homomorphism between these \( C^* \)-algebras.

The present note consists of four sections. The first two sections are Introduction and Preliminaries. and three more sections containing the results. Section 3 deals with the topology on the index set \( I \). Section 4 contains an auxiliary statement and the main results about embedding the reduced semigroup \( C^* \)-algebra.

2. Preliminaries

Let \( K \) be an upward directed set. We shall consider the category associated to the set \( K \), which is denoted by the same letter \( K \). We recall that the objects of this category are the elements of the set \( K \),
and, for any pair $a, b \in K$, the set of morphisms from $a$ to $b$ consists of the single element $(a, b)$ provided that $a \leq b$, and is the void set otherwise.

Further, we consider a covariant functor $F$ from the category $K$ into the category of unital $C^*$-algebras and their unital $*$-homomorphisms. Such a functor is called an inductive system in the category of $C^*$-algebras over the set $(K, \leq)$. It may be given by a collection $(K, \{A_a\}, \{\sigma_{ba}\})$ satisfying the properties from the definition of a functor. We shall write $F = (K, \{A_a\}, \{\sigma_{ba}\})$. Here, $\{A_a \mid a \in K\}$ is a family of unital $C^*$-algebras, and $\sigma_{ba} : A_a \to A_b$, where $a \leq b$, are unital $*$-homomorphisms of $C^*$-algebras. Recall that the equations $\sigma_{ca} = \sigma_{cb} \circ \sigma_{ba}$ hold for all elements $a, b, c \in K$ satisfying the condition $a \leq b \leq c$. Furthermore, for each element $a \in K$ the morphism $\sigma_{aa}$ is the identity mapping.

Throughout the paper, for a unital algebra $A$ its unit will be denoted by $I_A$.

We recall the definition and some facts concerning the inductive limits for inductive systems of $C^*$-algebras (see, for example, [23, Section 11.4], [24, Appendix L]).

The inductive limit of the system $F = (K, \{A_a\}, \{\sigma_{ba}\})$ is a pair $(\mathfrak{A}^K, \{\sigma^K_a\})$ where $\mathfrak{A}^K$ is a $C^*$-algebra and $\{\sigma^K_a : A_a \to \mathfrak{A}^K \mid a \in K\}$ is a family of canonical $*$-homomorphisms such that the following diagram commutes whenever $a \leq b$:

\[
\begin{array}{ccc}
A_a & \xrightarrow{\sigma_{ba}} & A_b \\
\sigma^K_a \downarrow & & \downarrow \sigma^K_b \\
\mathfrak{A}^K & & \\
\end{array}
\]

that is, the equality for mappings

\[\sigma^K_a = \sigma^K_b \circ \sigma_{ba}\] (1)

holds. We note that one has the equality

\[\mathfrak{A}^K = \bigcup_{a \in K} \sigma^K_a(A_a),\] (2)

where the bar means the closure of the set with respect to the norm topology in the $C^*$-algebra $\mathfrak{A}^K$.

The inductive limit $(\mathfrak{A}^K, \{\sigma^K_a\})$ is denoted as follows:

$$\{\mathfrak{A}^K, \{\sigma^K_a\}\} := \lim_{\longrightarrow} F.$$

The $C^*$-algebra $\mathfrak{A}^K$ itself is often called the inductive limit.
The exact construction of the inductive limit (2) and the explicit form of the canonical $\ast$-homomorphisms are less important than the following universal behavior.

**Lemma 2.1.** [24, Appendix L, theorem L.1.1] Let $\mathcal{B}$ be another $C\ast$-algebra and $\psi_a : \mathcal{A}_a \rightarrow \mathcal{B}$ be a canonical $\ast$-homomorphism for each $a \in K$, and the condition analogous to (1) is satisfied, that is, $\psi_a = \psi_b \circ \sigma_{ba}$ for every $a \leq b$. Then the following commutative diagram can be filled out with precisely one $\ast$-homomorphism $\theta$ from $\mathcal{A}^K$ onto $\mathcal{B}$ that leaves the diagram commutative:

![Commutative diagram](image)

The next lemma is useful when we try to find out whether $\theta$ is injective.

**Lemma 2.2.** [24, Appendix L, lemma L.1.3] Let the commutative diagram (3) be held for all $a \leq b$. If each $\psi_a : \mathcal{A}_a \rightarrow \mathcal{B}$ is injective then $\theta$ is also injective.

Further, we recall the definition of the reduced semigroup $C\ast$-algebras for semigroups in the group of all rational numbers $\mathbb{Q}$.

Assume that $\Gamma$ is an arbitrary subgroup in $\mathbb{Q}$. Let $\Gamma^+ := \Gamma \cap [0, +\infty)$ be the positive cone in the ordered group $\Gamma$. As usual, the symbol $l^2(\Gamma^+)$ stands for the Hilbert space of all square summable complex-valued functions on the additive subgroup $\Gamma^+$. The canonical orthonormal basis in the Hilbert space $l^2(\Gamma^+)$ is denoted by $\{ e_g | g \in \Gamma^+ \}$. That is, for all elements $g, h \in \Gamma^+$, we set $e_g(h) = \delta_{g,h}$, where $\delta_{g,h} = 1$ if $g = h$, and $\delta_{g,h} = 0$ if $g \neq h$.

Let us consider the $C\ast$-algebra of all bounded linear operators $B(l^2(\Gamma^+))$ in the Hilbert space $l^2(\Gamma^+)$. For every element $g \in \Gamma^+$, we define the isometry $V_g \in B(l^2(\Gamma^+))$ by

$$V_g e_h = e_{g+h},$$

where $h \in \Gamma^+$.

We denote by $C\ast^r(\Gamma^+)$ the $C\ast$-subalgebra in the algebra $B(l^2(\Gamma^+))$ generated by the set $\{ V_g | g \in \Gamma^+ \}$. It is called the reduced semigroup $C\ast$-algebra of the semigroup $\Gamma^+$, or the Toeplitz algebra generated by $\Gamma^+$.
In the case when $\Gamma$ is the group of all integers $\mathbb{Z}$, we also denote the semigroup $C^*$-algebra $C^*_r(\mathbb{Z}^+)$ by $\mathcal{T}$ and use the symbols $T$ and $T^n$ instead of $V_1$ and $V_n$, respectively, where $n \in \mathbb{Z}^+$.

In the similar way a semigroup $C^*$-algebra can be defined for an arbitrary cancellative semigroup. As is noted in [19, Section 2], a semigroup $C^*$-algebra is a very natural object. It is generated by the left regular representation of a given semigroup.

Let $P = (p_1, p_2, p_3, ...)$ be an arbitrary sequence of prime numbers. In what follows we shall consider the reduced semigroup $C^*$-algebra $C^*_r(Q^+)$ for the semigroup $Q^+ = \{m p_1 \cdot p_2 \cdot ... \cdot p_n \mid m \in \mathbb{Z}^+, n \in \mathbb{N}\}$ of rational numbers.

It follows from Coburn’s theorem [25, Theorem 3.5.18] that for every number $n \in \mathbb{N}$, there exists a unique isometric $*$-homomorphism of $C^*$-algebras $\varphi : \mathcal{T} \rightarrow \mathcal{T}$ such that $\varphi(T) = T^n$. We note that a straightforward proof of the existence of the homomorphism $\varphi$ is given in [26, Proposition 3].

Consequently, for every sequence of prime numbers $P = (p_1, p_2, p_3, ...)$ one can construct the inductive sequence of Toeplitz algebras $\{\mathcal{T}_n, \varphi_{n,n+1}\}$, where $\mathcal{T}_n = \mathcal{T}$, and the bonding $*$-homomorphisms are defined as follows:

$$\varphi_{n,n+1} : \mathcal{T}_n \rightarrow \mathcal{T}_{n+1} : T \mapsto T^{p_n}, \quad n \in \mathbb{N}.$$ 

Let us denote by $\mathfrak{S}$ the inductive limit of this sequence. The following statement is proved in [6].

**Lemma 2.3.** [6, Proposition 1] There exists an isomorphism of $C^*$-algebras:

$$\mathfrak{S} \simeq C^*_r(Q^+_P).$$

For additional results in the theory of $C^*$-algebras we refer the reader, for example, to [27]; [28, Ch. 4, § 7] and [25]. Necessary facts from the theory of categories and functors are contained, for example, in [28, Ch. 0, § 2] and [29].

### 3. Topology on an index set

Throughout the next sections we shall consider an arbitrary partially ordered set $(K, \leq)$ that is not necessarily directed. Taking the family of all upward directed subsets of the set $(K, \leq)$ and using Zorn’s lemma, one can easily prove the following statement.
Lemma 3.1. Let \((K, \leq)\) be a partially ordered set. Then the following equality holds:

\[
K = \bigcup_{i \in I} K_i,
\]

where \(\{K_i | i \in I\}\) is the family of all maximal upward directed subsets of \((K, \leq)\).

We consider the topology on the index set \(I\) which was introduced in [7, 8]. For the convenience of the reader we recall the definition of this topology and its properties.

For every element \(a \in K\) we define the set

\[
U_a = \{i \in I : a \in K_i\}.
\]

The family of sets \(\{U_a | a \in K\}\) satisfies the following properties:

- If \(a, b \in K\) such that \(a \leq b\) then \(U_b \subseteq U_a\).
- The family \(\{U_a | a \in K\}\) is a base for a topology on the set \(I\).

We denote by \(\tau\) the topology generated by the base \(\{U_a | a \in K\}\).

The topological space \((I, \tau)\) is a \(T_1\)-space.

Examples of different topological spaces \((I, \tau)\) are contained in [7, 8]. In particular, \((I, \tau)\) may be: a non-Hausdorff space [8, Example 1], a locally compact space [8, Example 2], a discrete space [8, Example 3]. Here we give an example of a compact space.

Example. As the set \(K\) we consider the set of all closed arcs in the unit circle \(S^1\):

\[
K := \{A \subset S^1 \mid A = [e^{2\pi ix}, e^{2\pi iy}] \text{ or } A = S^1 \setminus (e^{2\pi ix}, e^{2\pi iy}),
\]

where \(x, y \in [0, 1)\) and \(x < y\).

A partial order on \(K\) is defined in the following way: for \(A, B \in K\) we put \(A \leq B \iff A \subset B\).

It is easily verified that the pair \((K, \leq)\) is a partially ordered set. Moreover, it is worth noting that this set is not directed. Indeed, take any \(x_1, x_2, x_3, x_4 \in [0, 1)\) such that \(x_1 < x_2 < x_3 < x_4\). Then for \(A = [e^{2\pi ix_1}, e^{2\pi ix_4}]\) and \(B = S^1 \setminus (e^{2\pi ix_2}, e^{2\pi ix_3})\) there is not \(C \in K\) such that \(A \leq C\) and \(B \leq C\).

One has the representation of \(K\) as the union of maximal upward directed sets \(K_z\) indexed by the points of the unit circle \(S^1\), that is, \(K = \bigcup_{z \in S^1} K_z\), where \(z = e^{2\pi ix}, x \in [0, 1)\), and

\[
K_z := \{A \in K \mid A \subset S^1 \setminus \{z\}\}.
\]

A base \(\{U_A | A \in K\}\) for the topology \(\tau\) on the index set \(I = S^1\) consists of the sets

\[
U_A = \{z \in S^1 \mid A \subset K_z\} = \{z \mid z \in S^1 \setminus A\},
\]
that is, \( U_A = S^1 \setminus A \). Thus, the elements of the base for the topology \( \tau \) are all open arcs of the unit circle \( S^1 \):

\[
\{ B \subset S^1 \mid B = (e^{2\pi ix}, e^{2\pi iy}) \quad \text{or} \quad A = S^1 \setminus [e^{2\pi ix}, e^{2\pi iy}],
\]

where \( x, y \in [0, 1) \) and \( x < y \).

The topology \( \tau \) coincides with the natural topology on the unit circle \( S^1 \) that is compact.

4. MAIN RESULTS

Let \( K \) be an arbitrary partially ordered set. By Lemma 3.1 we have representation (3.1) of the set \( K \) as the union of all maximal upward directed subsets \( K_i, i \in I \).

For each index \( i \in I \) we consider an inductive system \( \mathcal{F}_i = (K_i, \{ T_a \}, \{ \text{id}_{ba} \}) \) consisting of Toeplitz algebras, that is, \( T_a = \mathcal{T} \) for all \( a \in K \), and the bonding \(*\)-homomorphisms \( \text{id}_{ba} : T_a \to T_b \), where \( a \leq b \), are the identity mappings.

Let us construct the inductive limits of the above-mentioned inductive systems:

\[
(\mathcal{T}^{K_i}, \{ \text{id}_{a}^{K_i} \}) := \lim_{\longrightarrow} \mathcal{F}_i = \lim_{\longrightarrow} (K_i, \{ T_a \}, \{ \text{id}_{ba} \}).
\]

It is clear that one has the isomorphism \( \mathcal{T}^{K_i} \simeq \mathcal{T} \) of \( C^* \)-algebras.

Further, we take any element \( a \in K \) and consider the direct product of \( C^* \)-algebras

\[
\mathfrak{B}_a := \prod_{i \in U_a} \mathcal{T}^{K_i} = \left\{ f : U_a \to \bigcup_{i \in U_a} \mathcal{T}^{K_i} : i \mapsto f(i) \in \mathcal{T}^{K_i}, \quad \|f\| = \sup_{i} \|f(i)\| < +\infty \right\}.
\]

For every pair of elements \( a, b \in K \) such that the inclusion \( U_b \subset U_a \) holds, we define the \(*\)-homomorphism \( \tau_{ba} : \mathfrak{B}_a \to \mathfrak{B}_b \) by the rule:

\[
\tau_{ba}(f)(j) = f(j),
\]

where \( f \in \mathfrak{B}_a \) and \( j \in U_b \). Obviously, one has the equality \( \tau_{ca} = \tau_{cb} \circ \tau_{ba} \) whenever \( a, b, c \in K \) and the condition \( U_c \subset U_b \subset U_a \) holds.

Therefore, for each index \( i \in I \) we can consider the inductive system \( (K_i, \{ \mathfrak{B}_a \}, \{ \tau_{ba} \}) \). Here, the bonding \(*\)-homomorphisms \( \tau_{ba} : \mathfrak{B}_a \to \mathfrak{B}_b \) are defined for all pairs of elements \( a, b \in K \) satisfying the condition \( U_b \subset U_a \), in particular, whenever \( a \leq b \). The inductive limit of this system is denoted by

\[
(\mathfrak{B}^{K_i}, \{ \tau_{a}^{K_i} \}) := \lim_{\longrightarrow} (K_i, \{ \mathfrak{B}_a \}, \{ \tau_{ba} \}).
\]
We note that the analog of equality (1) is valid, that is, \( \tau^K_i = \tau^K_i \circ \tau_{ba} \) whenever \( a \leq b \).

**Lemma 4.1.** Let \( i \in I \) be a non-isolated point with a countable neighbourhood base \( \{ U_{a_n} \mid a_n \in K_i, n \in \mathbb{N} \} \) satisfying the condition \( U_{a_1} \supset U_{a_2} \supset U_{a_3} \supset \ldots \). Let

\[
(\mathcal{B}, \{ \tau_n \}) := \lim\rightarrow (\{ \mathcal{B}_{a_n} \}, \{ \tau_{a_{n+1}a_n} \})
\]

be the inductive limit of the inductive sequence

\[
\mathcal{B}_{a_1} \xrightarrow{\tau_{a_2a_1}} \mathcal{B}_{a_2} \xrightarrow{\tau_{a_3a_2}} \mathcal{B}_{a_3} \xrightarrow{\tau_{a_4a_3}} \ldots ,
\]

where \( \tau_{a_{n+1}a_n}(f)(j) = f(j) \) for \( f \in \mathcal{B}_{a_n} \) and \( j \in U_{a_{n+1}} \). Then there exists an isomorphism of \( C^* \)-algebras

\[
(5) \quad \mathcal{B} \simeq \mathcal{B}^{K_i}.
\]

**Proof.** Using the universal property for the inductive limits (see Lemma 2.1), we have the unique \( * \)-homomorphism \( \Psi : \mathcal{B} \rightarrow \mathcal{B}^{K_i} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{B}_{a_n} & \xrightarrow{\tau_{a_{n+1}a_n}} & \mathcal{B}_{a_{n+1}} \\
\tau_n & \downarrow & \tau_{n+1} \\
\mathcal{B} & \xrightarrow{\Psi} & \mathcal{B}^{K_i}
\end{array}
\]

Take elements \( a, b \in K_i \) for which the inclusion \( U_b \subset U_a \) holds. Since the family \( \{ U_{a_n} \mid a_n \in K_i, n \in \mathbb{N} \} \) is a neighbourhood base at the point \( i \) there exists a number \( n \in \mathbb{N} \) such that we have the inclusions \( U_{a_n} \subset U_b \subset U_a \). The equality \( \tau_{a_na} = \tau_{a_n} \circ \tau_{ba} \) implies the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{B}_a & \xrightarrow{\tau_{ba}} & \mathcal{B}_b \\
\tau_n \circ \tau_{a_n} & \downarrow & \tau_n \circ \tau_{a_nb} \\
\mathcal{B} & \xrightarrow{\Psi} & \mathcal{B}^{K_i}
\end{array}
\]
Using again the universal property (Lemma 2.1), we get the unique \( \ast \)-homomorphism \( \Phi : \mathfrak{B} K_i \rightarrow \mathfrak{B} \) making the following diagram commute:

\[
\begin{array}{ccc}
\mathfrak{B}_a & \xrightarrow{\tau_a} & \mathfrak{B}_b \\
\downarrow{\tau_{ba}} & & \downarrow{\tau_{ba}} \\
\mathfrak{B}_K_i & \xrightarrow{\tau_{Ki}} & \mathfrak{B}_K_i \\
\downarrow{\tau_{Ki}} & & \downarrow{\tau_{Ki}} \\
\mathfrak{B} & \xrightarrow{\Phi} & \mathfrak{B} \\
\end{array}
\]

By the universal property of the inductive limit, making use of diagrams (6) and (7), we obtain the equalities

\[ \Psi \circ \Phi = \text{id} \quad \text{and} \quad \Phi \circ \Psi = \text{id}. \]

This means that we have isomorphism (5), as required.

**Theorem 4.1.** Let \( i \in I \) be a non-isolated point with a countable neighbourhood base. Then for every sequence of prime numbers \( P = (p_1, p_2, p_3, \ldots) \) there exists an injective \( \ast \)-homomorphism of \( C^\ast \)-algebras:

\[ C^\ast_r(Q^+_P) \rightarrow \mathfrak{B} K_i. \]

**Proof.** We take a countable neighbourhood base \( \{ U_{a_n} \mid a_n \in K_i, n \in \mathbb{N} \} \) at the point \( i \) satisfying the conditions

\[ U_{a_1} \supset U_{a_2} \supset U_{a_3} \supset \ldots \]

and \( a_1 \leq a_2 \leq a_3 \leq \ldots \). Consider the inductive sequence \( (\{ \mathfrak{B}_{a_n} \}, \{ \tau_{a_{n+1}a_n} \}) \), where \( \tau_{a_{n+1}a_n}(f)(j) = f(j) \) for \( f \in \mathfrak{B}_{a_n}, j \in U_{a_{n+1}} \), and its inductive limit

\[ (\mathfrak{B}, \{ \tau_n \}) := \lim \downarrow (\{ \mathfrak{B}_{a_n} \}, \{ \tau_{a_{n+1}a_n} \}). \]

For every \( n \in \mathbb{N} \) we define \( W_n := U_{a_n} \setminus U_{a_{n+1}} \) and the operator-valued function \( L_{a_n} \) in the algebra \( \mathfrak{B}_{a_n} \) as follows. For an index \( j \in U_{a_n} \) we put

\[ L_{a_n}(j) = T^{p_{n+1}p_{n+2} \ldots p_k}, \quad \text{if} \quad j \in W_k, \ k \in \mathbb{N}, \ k \geq n. \]

Recall that here \( T \) is the shift operator generating the Toeplitz algebra \( \mathcal{T} \).

Now we set

\[ L_n := \tau_n(L_{a_n}). \]

Let us show that \( L_n \) is an isometry in \( C^\ast \)-algebra \( \mathfrak{B} \). Indeed, since \( \tau_n \) is a unital \( \ast \)-homomorphism, one has the equalities

\[ L_n^* L_n = \tau_n(L_{a_n}^* L_{a_n}) = \tau_n(\mathbb{I}_{\mathfrak{B}_{a_n}}) = \mathbb{I}_{\mathfrak{B}}. \]
Further, we consider the inductive sequence of Toeplitz algebras:
\[ \{\mathcal{T}_n\}, \{\varphi_{n,n+1}\} \]
where \( \mathcal{T}_n = \mathcal{T} \) and the bonding \(*\)-homomorphisms
\( \varphi_{n,n+1} \) are defined by
\[ \varphi_{n,n+1} : \mathcal{T}_n \to \mathcal{T}_{n+1} : T \mapsto T^{p_n}, \quad n \in \mathbb{N}. \]

By Lemma 2.3, the inductive limit of this sequence is isomorphic to the
reduced semigroup \( C^*_\tau \)-algebra \( C^*_\tau(Q^+_P) \). Hence, there exist injective \(*\)-homomorphisms \( \varphi_n : \mathcal{T}_n \to C^*_\tau(Q^+_P) \) such that the equality
\[ \varphi_{n+1} \circ \varphi_{n,n+1} = \varphi_n \]
holds for every \( n \in \mathbb{N} \). This means that the diagram
\[
\begin{array}{ccc}
\mathcal{T}_n & \xrightarrow{\varphi_{n,n+1}} & \mathcal{T}_{n+1} \\
\downarrow{\varphi_n} & & \downarrow{\varphi_{n+1}} \\
C^*_\tau(Q^+_P) & & \end{array}
\]
is commutative.

It follows from Coburn’s theorem [25, Theorem 3.5.18] that for each
number \( n \in \mathbb{N} \) there is a unique isometric \(*\)-homomorphism \( \psi_n : \mathcal{T}_n \to \mathcal{B} \) such that the condition
\[ \psi_n(T) = L_n. \]
holds.

We claim that the diagram
\[
\begin{array}{ccc}
\mathcal{T}_n & \xrightarrow{\varphi_{n,n+1}} & \mathcal{T}_{n+1} \\
\downarrow{\psi_n} & & \downarrow{\psi_{n+1}} \\
\mathcal{B} & & \end{array}
\]
is commutative, that is, the following equality for \(*\)-homomorphisms
holds:
\[ \psi_{n+1} \circ \varphi_{n,n+1} = \psi_n. \]

Really, to prove equality (12) it is enough to show that for the homomorphisms \( \psi_{n+1} \circ \varphi_{n,n+1} \) and \( \psi_n \) their values at the generating element \( T \) for the Toeplitz algebra \( \mathcal{T}_n \) are the same. Making use of (10), (11) and (9), we obtain the equalities
\[ (\psi_{n+1} \circ \varphi_{n,n+1})(T) = (L_{n+1})^{p_n} = \tau_{n+1}(L_{p_n}). \]

On the other hand, using (11), (9) and the definition of the inductive
limit, we get
\[ \psi_n(T) = \tau_n(L_{a_n}) = (\tau_{n+1} \circ \tau_{a_{n+1}}a_n)(L_{a_n}). \]
Now we shall show that in the algebra $B_{a^{n+1}}$ the following equality holds:

\[(15) \quad L_{a^{n+1}}^p = \tau_{a^{n+1}a_n}(L_{a_n}).\]

Indeed, firstly, by (8), we can write

\[L_{a^{n+1}}^p = (T_{p_{n+1}^{\cdots p_k}})^{p_n} = T_{p_n^{p_{n+1}^{\cdots p_k}}}, \quad \text{if } j \in W_k, \quad k \in \mathbb{N}, \quad k \geq n+1.\]

Secondly, we have the equalities

\[\tau_{a^{n+1}}a_n(L_{a_n})(j) = L_{a_n}(j), \quad \text{if } j \in U_{a^{n+1}}\]

and

\[L_{a_n}(j) = T_{p_n^{p_{n+1}^{\cdots p_k}}}, \quad \text{if } j \in W_k, \quad k \in \mathbb{N}, \quad k \geq n.\]

This means that for every index $j \in U_{a^{n+1}}$ we get the equality $(L_{a^{n+1}}^p)(j) = (\tau_{a^{n+1}a_n}(L_{a_n}))(j)$. Consequently, equality (15) is true. Thus, the expressions on the right-hand sides in (13) and (14) are equal. Therefore, equality (12) is valid, as claimed.

By the universal property for the inductive limits (see Lemma 2.1), there exists a unique $\ast$-homomorphism $\theta : C^*(Q^+_P) \to \mathcal{B}$, such that the following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{T}_n & \xymatrix{ \varphi^p_{n,n+1} \ar[rr] & & \mathcal{T}_{n+1} } \\
\downarrow \psi_n & & \downarrow \psi_{n+1} \\
\mathcal{C}^*_r(Q^+_P) & \xymatrix{ \theta \ar@{.>}[ur] \ar@{.>}[dr] & } & \mathcal{B}
\end{array}
\]

Since all the mappings $\psi_n : \mathcal{T}_n \to \mathcal{B}$ are injective $\ast$-homomorphisms, the $\ast$-homomorphism $\theta : C^*_r(Q^+_P) \to \mathcal{B}$ is also injective (see Lemma 2.2).

Finally, to complete the proof of the theorem we use Lemma 4.1 which states that the $C^*$-algebras $\mathcal{B}$ and $\mathcal{B}^{K_i}$ are isomorphic.

□

Let us consider a sequence of prime numbers $P$ such that each prime number from $\mathbb{N}$ is equal to infinitely many terms of $P$, for example, $P = (2, 3, 5, 7, 11, 13, 17, 19, \ldots)$. It is straightforward to check that the following equality holds for semigroups of rational numbers:

\[Q^+_P = Q^+ := Q \cap [0, +\infty).\]

As a consequence of Theorem 4.1, we obtain
Theorem 4.2. Let \( i \in I \) be a non-isolated point with a countable neighbourhood base. There exists an injective \(*\)-homomorphism of \( C^*\)-algebras:

\[
C^*_r(Q^+) \to \mathfrak{B}^{K_i}.
\]

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