On the dimension dependence of some weighted inequalities

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Dedicated with admiration to Professor Shanzhen Lu.

Abstract

In the context of radial weights we study the dimension dependence of some weighted inequalities for maximal operators. We study the growth of the $A_1$-constants for radial weights and show the equivalence between the uniform boundedness of these constants, a dimension-free weak $L^1$ estimate for the maximal operator on annuli and the condition on the weight to be decreasing and essentially constant over dyadic annuli. Each one of these conditions is shown to provide dimension-free weighted weak type $L^1$ estimates for the centred maximal Hardy-Littlewood operator acting on radial functions. Finally we show that the universal maximal operator is of restricted weak type on weighted $L^n(\mathbb{R}^n)$ with constants uniformly bounded in dimension whenever we consider an $A_1$ weight.

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1 Introduction.

In this paper we will study the dimension dependence of the bounds for some maximal operators when acting over weighted spaces. First, we consider the centered Hardy-Littlewood maximal operator over Euclidean balls. For a locally integrable $g$ on $\mathbb{R}^n$ it is defined as

$$Mg(x) = \sup_{R>0} \left( \int_{B_R(x)} |g(y)| dy \right),$$

where $B_R(x)$ is the Euclidean ball of radius $R$ centered at $x$. It is well-known since the time of Hardy and Littlewood that for each $n$, this operator is bounded on $L^p(\mathbb{R}^n)$ for $p > 1$ and weakly bounded on $L^1(\mathbb{R}^n)$. Much later, E.M. Stein raised the questions

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whether the operator norm in these inequalities could be bounded independently of the dimension and whether this uniformity in dimension could be related to an infinite-dimensional phenomenon. As far as we are concerned, very little is known about the second question. As for the first one, Stein himself showed in [40] (details in [31]) that for all $p > 1$ one has $\|M\|_{L^p(R^n) \rightarrow L^p(R^n)} \leq C_p$ with $C_p$ independent of $n$. In joint work with J.O. Strömberg [31] he also proved that $\|M\|_{L^1(R^n) \rightarrow L^{1,\infty}(R^n)} = O(n)$ as $n \to \infty$. Although this does not solve the still open problem of deciding whether these weak $L^1$ operator norms grow to infinity with the dimension or not, it is still the best known result.

This problem of uniform bounds in dimension has also been studied for maximal functions where the averages are taken over balls given by arbitrary norms in $\mathbb{R}^n$. In all cases one has $\|M\|_{L^p(R^n) \rightarrow L^p(R^n)} \leq C$ with $C$ depending only on $p$ for $p \geq 3/2$ (see [5], [6], [7] and [10]). For the balls given by the $\ell^q$ metrics in $\mathbb{R}^n$, with $1 \leq q \leq \infty$, for all $p > 1$ one has $\|M\|_{L^p(R^n) \rightarrow L^p(R^n)} \leq C_{p,q}$ with $C_{p,q}$ independent of $n$ (see [26] for the case $1 \leq q < \infty$ and [9] for $q = \infty$). As for the weak $L^1$ inequalities in this case, Stein and Strömberg proved that, in general, $\|M\|_{L^1(R^n) \rightarrow L^{1,\infty}(R^n)} = O(n \log n)$ as $n \to \infty$. It is still unknown if these operator norms remain bounded in all dimensions, except when those averages are taken over the balls of the $\ell^\infty$ metric, that is, the case of cubes with sides parallel to the coordinate axes. In this case Aldaz showed in [2] that these weak $L^1$ operator norms grow to infinity with the dimension (see also [4]).

A variation of the problem arises when the maximal operator is defined using measures different from the Lebesgue one. For a Radon measure $\mu$ on $\mathbb{R}^n$ we define the associated maximal operator

$$M_\mu g(x) = \sup_{R > 0, \mu(B_R(x)) > 0} \frac{1}{\mu(B_R(x))} \int_{B_R(x)} |g(y)| d\mu(y).$$

If $\mu$ has a radial density $w$, then $M_\mu$ and $M_\mu$ can be defined in all dimensions. We may ask then if the operator norm of $M_\mu$ in $L^p(\mu)$ is uniformly bounded in dimension. If $\mu$ is finite this is not true in general (see [1], [13], [3]). For instance, when $\mu$ is the Gaussian measure, the $L^p(\mu)$ operator norms of $M_\mu$ grow exponentially to infinity with the dimension for all $p < \infty$ (see [14]). The situation is very different when $\mu$ satisfies a doubling condition. One says that $\mu$ is uniformly strong $n$-microdoubling if there exist $K > 0$ and $N > 0$ so that for all $n \geq N$, $x \in \mathbb{R}^n$, $R > 0$ and $y \in B_R(x)$ one has

$$\mu(B_{(1+1/n)R}(x)) \leq K \mu(B_R(x)) \quad \text{and} \quad \mu(B_R(y)) \leq K \mu(B_R(x)).$$

Roughly speaking this property guarantees that small dilations and translation do not alter essentially the measure of a ball. If $\mu$ is such a measure, then one recovers the Stein and Strömberg bound $\|M_\mu\|_{L^1(\mathbb{R}^n, d\mu) \rightarrow L^{1,\infty}(\mathbb{R}^n, d\mu)} = O(n \log n)$ (see [27]). Moreover one has the uniform bound $\|M_\mu\|_{L^p(\mathbb{R}^n, d\mu) \rightarrow L^p(\mathbb{R}^n, d\mu)} \leq C_{p,\mu}$ for all $n \geq N$ and $p > 1$ (see [15]).

Still another problem is the one that considers weighted inequalities for the maximal operator. A weight $w$ is an a.e. nonnegative and locally integrable function over $R^n$. A weight is often regarded as the density of a measure over $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure. Following the usual notation we will also denote this measure by $w$, i.e. for a measurable $E$ we will write $w(E) = \int_E w$ and we will say that a function $f \in L^p(R^n, w)$ if $\int_{R^n} |f|^p w < \infty$. For $p \geq 1$ we say that a weight $w$ is in
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the class \( A_p(\mathbb{R}^n) \) if \( M \) sends \( L^p(\mathbb{R}^n, w) \) into \( L^{p,\infty}(\mathbb{R}^n, w) \) boundedly. It is well-known that for \( p > 1 \) this is equivalent to \( M : L^p(\mathbb{R}^n, w) \to L^p(\mathbb{R}^n, w) \) boundedly. These bounds have been studied extensively. For more information see [20] or [22].

In this work we will consider radial weights, so that we can define them in all dimensions. For such a weight \( w \) we will write \( w(x) = w_0(|x|) \) with \( w_0 : [0, \infty) \to [0, \infty] \). J. Duoandikoetxea and L. Vega announced in [19] the following result.

**Theorem 1.1.** Let \( w_0 \) be a nonnegative function on \([0, \infty)\), so that \( w = w_0(|\cdot|) \in A_p(\mathbb{R}^N) \) with \( p < 1 \). Then for all \( n \geq N \) one has \( w \in A_p(\mathbb{R}^n) \) and, moreover,

\[ \| Mf \|_{L^p(\mathbb{R}^n, w)} \leq C \| f \|_{L^p(\mathbb{R}^n, w)}, \]

with a constant \( C \) that might depend on \( p \) and \( w_0 \) but not on \( n \).

This solves completely the problem of finding weighted \( L^p \) bounds that are uniform in dimension for \( p > 1 \). In this paper we present some partial results in the case \( p = 1 \). In the next section we present some result relating the growth of the \( A_1 \) constant of a weight with the uniformity of the weak \( L^1 \) weighted bounds for the maximal operator over radial functions. The proofs of these results are contained in Section 3. Finally, Section 4 is devoted to show some uniform bounds for the universal maximal operator and the Kakeya maximal operator over radial functions.

## 2 The maximal operator over radial functions

First, let us describe briefly the concepts and notation that we are going to deal with. We recall that \( w \in A_1(\mathbb{R}^n) \) if and only if for some \( C > 0 \) one has

\[ \frac{w(B_R(x))}{|B_R(x)|} \leq C \text{ ess inf}_{y \in B_R(x)} w(y), \quad \text{a.e. } x \in \mathbb{R}^n, \forall R > 0. \]

Equivalently, \( w \in A_1(\mathbb{R}^n) \) if

\[ Mw(x) \leq C w(x), \quad \text{a.e. } x \in \mathbb{R}^n. \]

The smallest values of \( C \) for which the previous inequalities hold will be denoted by \([w]_{A_1(\mathbb{R}^n)}\) and \([w]^*_{A_1(\mathbb{R}^n)}\) respectively. Both are usually called the \( A_1 \) constant of the weight.

For radial weights there is still another characterization of the \( A_1 \) class. We use that if \( w \) is radial, then \( Mw \) is pointwise comparable with \( Aw \), where \( A \) is the maximal operator over centered rings given by

\[ Au(x) := \sup_{0 \leq a \leq |x| \leq b} \int_{a \leq |y| \leq b} |u(y)| \, dy. \]

More precisely one has:

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1In what follows, the expression \( T : X \to Y \) will denote that the (sublinear) operator \( T \) is bounded between the spaces \( X \) and \( Y \).
Lemma 2.1. Let \( u(x) = u_0(|x|) \) be a radial function in \( L^1_{\text{loc}}(\mathbb{R}^n) \). Then there exists a constant \( K_n \) only depending on the dimension such that for all \( x \in \mathbb{R}^n \) one has

\[
\frac{1}{K_n} A u(x) \leq M u(x) \leq 2 A u(x).
\]

When acting on radial functions \( A \) can be written as a one-dimensional maximal operator. Given a weight \( v \) over \([0, \infty)\) the associated uncentered maximal operator is defined as

\[
\tilde{M}_v g(x) = \sup_{0 \leq a \leq b} \frac{1}{v([a,b])} \int_a^b |g(y)| v(y) \, dy,
\]

for \( g \in L^1_{\text{loc}}([0, \infty), v) \). If \( u(x) = u_0(|x|) \) is a radial function over \( \mathbb{R}^n \), writing \( v_n(t) = t^{n-1} \), note that \( A u(x) = \tilde{M}_{v_n} u_0(|x|) \).

As a consequence \( w(x) = w_0(|x|) \) is in \( A_1(\mathbb{R}^n) \) if and only if there exists a constant \( C > 0 \) so that for a.e. \( x \) one has

\[
A w(x) = \tilde{M}_{v_n} w_0(|x|) \leq C w_0(|x|) = C w(x).
\]

The smallest of such constants will be denoted by \((w)_{A_1(\mathbb{R}^n)} \). With the usual arguments in weight theory one can see that the previous condition is equivalent to the existence of a constant \( C > 0 \) so that for all intervals \( I \subset [0, \infty) \) one has

\[
\frac{w_0 v_n(I)}{v_n(I)} \leq C \text{ess inf}_{x \in I} w_0(x).
\]

The smallest of such constants is again \((w)_{A_1(\mathbb{R}^n)} \).

The inequality (2.1) has already appeared in the literature. In [25] T. Menárguez and the second author used the inequality \( M u(x) \leq 2 \tilde{M}_{v_n} u_0(|x|) \) to prove (2.4) below. Inequality (2.1) in the form \( c_n A u(x) \leq M u(x) \leq C_n A u(x) \), with the constants depending on the dimension, was used by Duoandikoetxea, Moyua, Oruetxebar and Seijo in [18]. From it they deduced that for any \( p \geq 1 \) the radial weights \( w \) so that \( M : L^p_{\text{rad}}(w) \to L^p_{\text{rad}}(w) \) boundedly are exactly the radial \( A_p \) weights.

First, we point out that if \( w \in A_1(\mathbb{R}^m) \) for some \( m \in \mathbb{N} \), then \( w \in A_1(\mathbb{R}^n) \) for all \( n \geq m \) and \([w]_{A_1(\mathbb{R}^n)}^* \) and \((w)_{A_1(\mathbb{R}^n)} \) grow at most linearly with \( n \).

Lemma 2.2. Let \( w_0 \) be a non-negative function over \([0, \infty)\) and set \( w(x) = w_0(|x|) \) and \( d\mu(x) = w(x) \, dx \). If for some \( m \in \mathbb{N} \) one has \( w \in A_1(\mathbb{R}^m) \) then \( w \in A_1(\mathbb{R}^n) \) for all \( n \geq m \) and \([w]_{A_1(\mathbb{R}^n)}^* \leq C_m \frac{m}{m} [w]_{A_1(\mathbb{R}^m)}^* \). The same holds for \((w)_{A_1(\mathbb{R}^n)} \).

These bounds are almost optimal, let us see this with an example. Consider \( w(x) = (1 - |x|)^{-\alpha} \) with \( 0 < \alpha < 1 \). It is easy to see that \( w \in A_1(\mathbb{R}^n) \) for all \( n \in \mathbb{N} \). Note that in \( \mathbb{R}^n \) one has

\[
\frac{w(B_t(0))}{|B_t(0)|} = \frac{1}{\omega_{n-1}/n} \int_{B_t} (1-|x|)^{-\alpha} \, dx = n \int_0^1 (1-t)^{-\alpha} t^{n-1} \, dt = n \frac{\Gamma(1-\alpha)\Gamma(n)}{\Gamma(n+1-\alpha)},
\]

(2.2)
where we have used the equality
\[
\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 (1-t)^{x-1}t^{y-1} \, dt.
\]
Since \( \Gamma \) is logarithmically convex, writing \( n + 1 - \alpha = \alpha n + (1 - \alpha)(n + 1) \) one has \( \Gamma(n + 1 - \alpha) \leq \Gamma(n)^\alpha \Gamma(n + 1)^{1-\alpha} \), which together with (2.2) says that
\[
\frac{w(B_1(0))}{|B_1(0)|} \geq \Gamma(1 - \alpha) n^\alpha \tag{2.3}
\]
But note that \( \inf_{y \in B_1(0)} w(y) = 1 \). This gives \( [w]_{A_1(\mathbb{R}^n)}^* \geq \Gamma(1 - \alpha) n^\alpha \). Since \( \alpha \) can be taken arbitrarily close to 1, this shows that the upper bound for \( [w]_{A_1(\mathbb{R}^n)}^* \) is near to be optimal. The same calculation works for \( (w)_{A_1(\mathbb{R}^n)} \).

There are radial weights for which the \( A_1 \) constants remain uniformly bounded in dimension. Indeed, we have the following characterization.

**Proposition 2.3.** Let \( w(x) = w_0(|x|) \) be a radial weight. The following statements are equivalent:

a) There exists \( N > 0 \) so that \( [w]_{A_1(\mathbb{R}^n)} \), \( [w]_{A_1(\mathbb{R}^n)}^* \) or \( (w)_{A_1(\mathbb{R}^n)} \) are uniformly bounded for all \( n > N \),

b) There exists a constant \( C > 0 \) so that for all \( n > N \), all \( f \in L^1_{\text{rad}}(w) \) and \( \lambda > 0 \) we have
\[
w(\{x \in \mathbb{R}^n : Af(x) > \lambda \}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|w(x) \, dx.
\]

c) \( w_0 \) is essentially constant over dyadic intervals and decreasing up to a constant. This means that there exist positive constants \( \beta \) and \( \eta \) so that
\[
\text{ess sup}_{r \in [R, 2R]} w_0(r) \leq \beta \text{ ess inf}_{r \in [R, 2R]} w_0(r), \quad \forall R > 0,
\]
\[
w_0(s) \leq \eta w_0(t), \quad \forall s \geq t \geq 0.
\]

Power weights with negative powers, that is \( w_0(t) = t^{-\alpha} \) with \( \alpha \geq 0 \), are examples of weights with these properties. It is easy to check that they satisfy condition c with \( \beta = 2^\alpha \) and \( \eta = 1 \).

Moreover, for the weights satisfying the properties in Proposition 2.3, the maximal operator \( M \) acting over radial functions is weakly bounded in weighted \( L^1 \) with constants uniformly bounded in dimension. This was already known for \( w \equiv 1 \). M.T. Menárguez and the second author proved in [25] that for all radial \( f \) over \( \mathbb{R}^n \) and \( \lambda > 0 \) one has
\[
|\{x \in \mathbb{R}^n : Mf(x) > \lambda \}| \leq \frac{4}{\lambda} ||f||_{L^1(\mathbb{R}^n, dx)}. \tag{2.4}
\]

We point out the following extension, that is a corollary of part b of Proposition 2.3 and (2.1).
**Theorem 2.4.** Let \( w(x) = w_0(|x|) \) be a radial weight. Assume that there exist \( N > 0 \) and \( C > 0 \) so that \( (w)_{A_1(\mathbb{R}^n)} < C \) for all \( n > N \). Then for all radial \( f \) over \( \mathbb{R}^n \) with \( n \geq N \) and \( \lambda > 0 \) one has

\[
 w(\{x \in \mathbb{R}^n : Mf(x) > \lambda\}) \leq \frac{4C}{\lambda} \int |f(x)|w(x) \, dx. \tag{2.5}
\]

Note that if \( w \) is a radial weight so that (2.5) holds for radial functions over \( \mathbb{R}^k \), then \( w \in A_1(\mathbb{R}^k) \). If \( w \) is decreasing up to a constant, the argument leading to (3.3) in the proof of Proposition 2.3 below, shows that \( w \) is essentially constant over dyadic intervals. Hence, by Proposition 2.3 we are in the hypothesis of Theorem 2.4 and we have (2.5) in \( \mathbb{R}^n \) for all \( n \geq k \) with a constant independent of \( n \).

We finish remarking that for the weights \( w \) characterized in Proposition 2.3 \( Mw \) and \( Aw \) are comparable with constants independent of the dimension because one always has

\[
 Aw(x) \leq (w)_{A_1(\mathbb{R}^n)}w(x) \leq (w)_{A_1(\mathbb{R}^n)}Mw(x).
\]

It is easy to find examples of radially increasing functions, for instance \( w_0(t^\alpha) \) with \( \alpha > 0 \), so that \( (w)_{A_1(\mathbb{R}^n)} = \infty \) for all \( x \). Therefore \( Aw \leq Cw \) with \( C \) independent of the dimension does not imply any of the conditions \( (1), (2), (3) \).

### 3 Proofs of the main results

We begin proving Proposition 2.3. We mention that the equivalence \( (1) \iff (3) \) already appeared in [15] for \( [w]_{A_1} \). Here we will use similar arguments. Among them, the method of differentiation through dimensions also presented in [15] that is contained in the following Lemma.

**Lemma 3.1.** Take \( w_0 \in L^1_{\text{loc}}([0, \infty), t^{N-1} \, dt) \) for some \( N \geq 1 \). Then, for almost every \( T > 0 \) and for all \( s \geq 0 \) and \( R > 0 \) so that \( s^2 + R^2 = T^2 \), if we take points \( z^n \in \mathbb{R}^n \) with \( |z^n| = s \) and we denote \( B(z^n, R) = \{ y \in \mathbb{R}^n : |z^n - y| < R \} \), the following holds

\[
 \lim_{n \to \infty} \int_{B(z^n, R)} w_0(|x|) \, dx = w_0(T).
\]

We now proceed to prove Proposition 2.3.

**Proof of Proposition 2.3.** It is easy to see that

\[
 [w]_{A_1}^* \leq [w]_{A_1}, \tag{3.1}
 [w]_{A_1}^* \leq 2(w)_{A_1}, \tag{3.2}
\]

Let us prove \( (1) \implies (3) \). Assume that for \( n \geq N \) one has \( [w]_{A_1(\mathbb{R}^n)} \leq C_* \) with \( C_* \) independent of \( n \). First, we prove that \( w_0 \) is decreasing up to a constant. Assume that
$t \geq s \geq 0$ and for each $n \geq N$ take $x_n, y_n \in \mathbb{R}^n$ so that $x_n = \alpha y_n$ and $|x_n| = s$, $|y_n| = t$. Consider the ball $B_R(x_n)$ with $R^2 = t^2 - s^2$. By hypothesis we have, in the almost everywhere sense,

$$w(B_R(x_n)) |B_R(x_n)| \leq C_* w(x_n) = C_* w_0(s).$$

In view of Lemma 3.1 we take limits as $n \rightarrow \infty$ and obtain

$$w_0(t) = w_0(\sqrt{s^2 + R^2}) \leq C_* w_0(s),$$

for almost every $s \leq t$.

In order to ensure that $w_0$ is essentially constant over dyadic intervals, we only need to prove that for all $R > 0$, $w_0(R) \leq CW_0(2R)$ with $C$ independent of $R$. Take $n \geq N$ and consider $x \in \mathbb{R}^n$ with $|x| = 1$ and the balls $B = B_{R/2}((R/2) x)$, $B^* = B_{R/2}((3/2) x)$ and $B^{**} = B_{R/2}(5/2 x)$. Since $w \in A_1(\mathbb{R}^n)$ it is doubling and for some constant $K$ we have $w(B) \leq Kw(B^*) \leq K^2w(B^{**})$. By the decreasing property of $w_0$ that we have just proved, for all $y \in B$ we have $w_0(B) \leq C_* w(y)$ and for all $y \in B^{**}$ we have $w(y) \leq C_* w_0(2R)$. This yields

$$w_0(R) \leq C_* \frac{w(B)}{|B|} \leq C_* K^2 \frac{w(B^{**})}{|B^{**}|} \leq (C_* K)^2 w_0(2R). \tag{3.3}$$

We now proceed with $\alpha = a$. In view of (3.1) and (3.2) it is enough to see that $(w)_{A_1}$ and $[w]_{A_1}$ are uniformly bounded. To see the first, consider an interval $[a, b] \subset [0, \infty)$. If $b \leq 2a$, we have

$$\frac{w_0v_n([a, b])}{v_n([a, b])} \leq \operatorname{ess sup}_{a \leq \tau \leq b} w_0(\tau) \leq \operatorname{ess sup}_{a \leq \tau \leq 2a} \beta \operatorname{ess inf}_{a \leq \tau \leq b} w_0(\tau) \leq \beta \operatorname{ess inf}_{a \leq \tau \leq b} w_0(\tau).$$

If $b > 2a$ we use the fact that under the hypothesis that $w_0$ is essentially constant over dyadic intervals for any $R > 0$ we have

$$w_0v_n ([0, R]) = \sum_{k=1}^{\infty} \int_{2^{-k+1} R}^{2^{-k} R} w_0(t) t^{-1} dt \leq \sum_{k=1}^{\infty} \beta^k w_0(R) \frac{2^{(-k+1)n} - 2^{-kn}}{n} \leq \frac{2^n R^n}{n} w_0(R) \sum_{k=1}^{\infty} \left( \frac{\beta}{2^n} \right)^k \leq 2\beta w_0(R) v_n([0, R]),$$

the last inequality provided we take $n > N = \log_2 \beta + 1$. Using this

$$\frac{w_0v_n([a, b])}{v_n([a, b])} \leq \frac{w_0v_n([0, b])}{v_n([a, b])} \leq 2\beta w_0(b) \frac{v_n([0, b])}{v_n([a, b])},$$

and this is all we need. Note that by the decreasing property of $w_0$ we have $w_0(b) \leq \operatorname{ess inf}_{a \leq \tau \leq b} w_0(\tau)$ and

$$\frac{v_n([0, b])}{v_n([a, b])} = \frac{b^n}{b^n - a^n} \leq \frac{b^n}{b^n - (b/2)^n} \leq 2.$$

One proves the uniform boundedness of $[w]_{A_1}$ in a similar way (see 15 for the details).
To see the equivalence $[a] \iff [b]$ observe that $[b]$ is equivalent to

$$w_0 v_n \left( \{ r \geq 0 : \tilde{M}_{v_n} f_0(r) > \lambda \} \right) \leq \frac{C}{\lambda} \int_0^\infty |f_0(r)| w_0(r) v_n(r) \, dr, \quad (3.4)$$

with $C$ independent of $f_0$, $n$ and $\lambda$. For the implication $[a] \implies [b]$ assume that $(w)_{A_1}$ is uniformly bounded. Consider a radial function $f(x) = f_0(|x|)$ over $\mathbb{R}^n$ and $\lambda > 0$. The set $E_\lambda = \{ r \geq 0 : \tilde{M}_{v_n} f_0(r) > \lambda \}$ is the union of all the intervals $I \subset [0, \infty)$ verifying

$$\frac{1}{v_n(I)} \int_{I} |f_0(t)| v_n(t) \, dt > \lambda.$$

We use Young’s selection principle (see Lemma 4.2.1 in [21]) to obtain a subset $I$ of such intervals so that $E_\lambda \subset \bigcup_{I \in I} I$ and $\sum_{I \in I} \chi_I \leq 2$.

For each $I \in A$ by our assumption we have

$$w_0 v_n (I) \leq (w)_{A_1(\mathbb{R}^n)} v_n(I) \ \text{ess inf} \ w_0 \leq \frac{(w)_{A_1(\mathbb{R}^n)}}{\lambda} \int_{I} |f_0(t)| v_n(t) \, dt \ \text{ess inf} \ w_0$$

and hence,

$$w_0 v_n (E_\lambda) \leq \sum_{I \in I} w_0 v_n (I) \leq \frac{2(w)_{A_1(\mathbb{R}^n)}}{\lambda} \int_0^\infty |f_0(t)| w_0(t) v_n(t) \, dt$$

We finish with the implication $[b] \implies [a]$. Consider intervals $J \subset I \subset [0, \infty)$, take $f_0 = \chi_J$ and $\lambda = w_0 v_n(J)/v_n(I)$. Then $I \subset E_\lambda$ and by (3.4) we have

$$w_0 v_n (I) \leq w_0 v_n (E_\lambda) \leq \frac{C}{\lambda} w_0 v_n (J),$$

or equivalently

$$w_0 v_n (I) \leq \frac{w_0 v_n (J)}{v_n (I)} \leq C \frac{w_0 v_n (J)}{v_n (J)}.$$

Since $J$ is arbitrary, this implies that

$$w_0 v_n (I) \leq C \text{ess inf}_{t \in I} w_0 (t).$$

Our next goal is to show Lemma 2.2 and Theorem 1.1. The technical parts of the proofs are summarized in the following lemmas. The first one, due to Stein (see [31]), provides a method of rotations that reduces the dimension when estimating the mean values over balls.
Lemma 3.2. Let \( k < n \) be natural numbers. For each \( x = (x^1, \ldots, x^n) \in \mathbb{R}^n \) we call \( x_1 = (x^1, \ldots, x^k) \in \mathbb{R}^k \) and \( x_2 = (x^{k+1}, \ldots, x^n) \in \mathbb{R}^{n-k} \). By abuse of the language we will write \( x = (x_1, x_2) \). For any positive and measurable function \( f \) on \( \mathbb{R}^n \) one has
\[
\frac{\int_{|x| < R} f(x) \, dx}{\int_{|x| < R} \, dx} = \frac{\int_{SO(n)} \int_{|x_1| < R} f(\tau(x_1,0)) |x_1|^{-k} \, dx_1 \, d\tau}{\int_{|x_1| < R} |x_1|^{-k} \, dx_1},
\]
where \( SO(n) = \{ \tau \in \mathcal{M}_{n \times n} : \tau \tau^t = I \} \), i.e. the special orthonormal group in \( \mathbb{R}^n \) equipped with its Haar measure.

Roughly speaking, it asserts that an integral mean over a ball in \( \mathbb{R}^n \) can be transformed into an integral mean over a ball in \( \mathbb{R}^k \) combined with all possible rotations in \( \mathbb{R}^n \). In [31] Lemma 3.2 was used to obtain the following pointwise controls of the maximal function.

Lemma 3.3. Let \( k < n \) be natural numbers. With the notation of the previous Lemma 3.2, if \( g \) is a function over \( \mathbb{R}^n \) we write \( g_{x_2}(x_1) := g(x_1, x_2) = g(x) \). Denoting by \( M_m \) the maximal operator in \( \mathbb{R}^m \), we have the following bounds:
\[
a) \quad M_n f(x) \leq \frac{n}{k} \int_{SO(n)} M_k [(f \circ \tau)_{\tau^{-1}(x_2)}] (\tau^{-1}(x_1)) \, d\tau,
b) \quad M_n f(x) \leq \int_{SO(n)} M_k [(f \circ \tau)_{\tau^{-1}(x_2)}] (\tau^{-1}(x_1)) \, d\tau,
\]
where \( M_k \) stands for the \( k \)-dimensional spherical maximal operator.

We will also employ a technical result for radial weights.

Lemma 3.4. Let \( w(x) = w_0(|x|) \) be a radial weight over \( \mathbb{R}^k \) so that for some \( p \geq 1 \) we have \( w \in A_p(\mathbb{R}^k) \). For each \( \rho > 0 \) consider the weights \( w_\rho(x) = w_0(\sqrt{\rho^2 + |x|^2}) \). Then one has \( w_\rho \in A_p(\mathbb{R}^k) \), and moreover there exists a constant \( C_k > 0 \) only depending on \( k \) so that for all \( \rho \geq 0 \) one has \( |w_\rho|_{A_p(\mathbb{R}^k)} \leq C_k |w|_{A_p(\mathbb{R}^k)} \). As a consequence, there exists a constant \( \tilde{C}_k > 0 \) only depending on \( k \) and on \( w \) so that for all \( \rho \geq 0 \) and \( f \in L^p(w_\rho) \) one has
\[
\| M_k f \|_{L^p(w_\rho)} \leq \tilde{C}_k \| f \|_{L^p(w_\rho)}.
\]

Now we are in conditions to prove Lemma 2.2.

Proof of Lemma 2.2. Assume that \( w(x_1) = w_0(|x_1|) \) is an \( A_1(\mathbb{R}^k) \) weight. In view of Lemma 3.3 part a) and Lemma 3.4 if \( n \geq k \) and \( x \in \mathbb{R}^n \) we have
\[
M_n w(x) \leq \frac{n}{k} \int_{SO(\mathbb{R}^n)} M_k [w_{|\tau^{-1}(x_2)|}] (\tau^{-1}(x_1)) \, d\tau
\leq C_k |w|_{A_1(\mathbb{R}^k)} \frac{n}{k} \int_{SO(\mathbb{R}^n)} w_{|\tau^{-1}(x_2)|} (\tau^{-1}(x_1)) \, d\tau = C_k |w|_{A_1(\mathbb{R}^k)} \frac{n}{k} w(x).
\]
The bound for \((w)_{A_1}\) is immediate once we observe that for any \([a, b] \subset [0, \infty)\) one has
\[
\frac{w_0v_n([a, b])}{v_n([a, b])} = \frac{n}{b^n - a^n} \int_a^b w_0(t) t^{n-1} dt \leq b^{n-k} \frac{b^k - a^k}{b^n - a^n} \frac{n}{k} \frac{k}{b^k - a^k} \int_a^b w_0(t) t^{k-1} dt \leq \frac{n}{k} \frac{w_0v_k([a, b])}{v_k([a, b])}.
\]

The second inequality uses that \(b^{n-k}(b^k - a^k) \leq b^n - a^n\), whenever \(0 \leq a \leq b\). From the previous calculation one deduces that \(M_{v_n}w_0 \leq n/k \ M_{v_k}w_0\), which implies the bound for \((w)_{A_1}\).

In order to prove Theorem 1.1 we will use Lemma 3.3, part b), where the maximal spherical operator appears. We recall that if \(n \geq 2\), for a suitable smooth function \(f\) the maximal spherical operator is defined as
\[
\mathcal{M}_n f(x) = \sup_{r > 0} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} |f(x + ry)| \, d\sigma_{n-1}(y).
\]

This operator is known to be bounded on \(L^p(\mathbb{R}^n)\) if and only if \(p > n/(n-1)\). E.M. Stein proved this in [29] in the case that \(n \geq 3\), and J. Bourgain in [8] for \(n = 2\). This allows to define the maximal spherical operator over functions in \(L^p(\mathbb{R}^n)\) with \(p > n/(n-1)\).

We say that a weight \(w\) is in the class \(W_p(\mathbb{R}^n)\) if \(\mathcal{M}_n\) is bounded on \(L^p(w)\). If \(w\) is radial we have the following relation with the \(A_p\) classes.

**Lemma 3.5.** Let \(\mu\) be a radial measure over \(\mathbb{R}^n\) with density \(w(x) = w_0(|x|)\). If for certain \(k\) one has \(w \in A_p(\mathbb{R}^k)\) then there exists \(m \geq k\) so that \(w \in W_p(\mathbb{R}^m)\). Moreover \(\|\mathcal{M}_m\|_{L^p(\mathbb{R}^m, w)} \to L^p(\mathbb{R}^m, w)\) is controlled by \(\|M_k\|_{L^p(\mathbb{R}^k, w)} \to L^p(\mathbb{R}^k, w)\).

Both, Lemmas 3.4 and 3.5 are proved below. Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1** Since \(w \in A_p(\mathbb{R}^k)\), by Lemma 3.4 for all \(a \geq 0\) one has \(w_a \in A_p(\mathbb{R}^k)\). Furthermore by Lemma 3.5 there exist \(m \geq k\) so that for all \(a \geq 0\) one has \(w_a \in W_p(\mathbb{R}^m)\). Moreover \(\|\mathcal{M}_m\|_{L^p(\mathbb{R}^m, w_a)} \to L^p(\mathbb{R}^m, w_a)\) is controlled by \(\|M_k\|_{L^p(\mathbb{R}^k, w)} \to L^p(\mathbb{R}^k, w)\).

This means that there exists \(C_m > 0\) so that for all \(a \geq 0\) and \(g \in L^p(w_a)\) one has
\[
\|\mathcal{M}_m g\|_{L^p(\mathbb{R}^m, w_a)} \leq C_m \|f\|_{L^p(\mathbb{R}^m, w_a)}.
\]

Given \(f \in L^p(\mathbb{R}^n, w)\) with \(n > m\), by Lemma 3.3 part b) and Minkowski inequality we have
\[
\|M_n f\|_{L^p(\mathbb{R}^n, w)} \leq \int_{S^O(n)} \left( \int_{\mathbb{R}^n} |\mathcal{M}_m((f \circ \tau)(\tau^{-1}(x))_2)(\tau^{-1}(x))_1| \, w(x) \, dx \right)^{1/p} \, dt.
\]
Applying obvious changes of integration variables and Lemma 3.4, the previous is bounded by

\[
\int_{SO(n)} \left( \int_{\mathbb{R}^n} |M_m[(f \circ \tau)_{y_2}](y_1)|^p w(y) \, dy \right)^{1/p} \, d\tau
\]

\[= \int_{SO(n)} \left( \int_{\mathbb{R}^{n-k}} \int_{\mathbb{R}^k} |M_m[(f \circ \tau)_{y_2}](y_1)|^p w_{[y_2]}(y_1) \, dy_1 \, dy_2 \right)^{1/p} \, d\tau
\]

\[\leq \int_{SO(n)} \left( \int_{\mathbb{R}^{n-k}} C_m \int_{\mathbb{R}^k} |(f \circ \tau)_{y_2}(y_1)|^p w_{[y_2]}(y_1) \, dy_1 \, dy_2 \right)^{1/p} \, d\tau
\]

\[= C_m \|f\|_{L^p(w)}.
\]

We finish justifying Lemmas 3.4 and 3.5.

**Proof of Lemma 3.4** If \(w \in A_p(\mathbb{R}^k)\) there exist \(u, v \in A_1(\mathbb{R}^k)\) so that \(w = uv^{1-p}\) and \([w]_{A_p(\mathbb{R}^k)} \leq [u]_{A_1(\mathbb{R}^k)}[v]_{A_1(\mathbb{R}^k)}^{p-1}\). Moreover if \(w\) is radial, \(u\) and \(v\) can be chosen to be radial by their construction (see [20]). Therefore it is enough to prove the result in the case \(p = 1\).

Assuming \(w \in A_1(\mathbb{R}^k)\), we are going to show that there exist a constant \(C > 0\), so that for all \(x \in \mathbb{R}^k\) and \(\rho \geq 0\) one has \(Mw_\rho(x) \leq Cw_\rho(x)\). As we observed after Lemma 2.1 if \(u(x) = u_0(x)\) is a radial and locally integrable function we have \(Mu(x) \leq Cu(x)\) a.e. if and only if \(M_{v,2}u_0(|x|) \leq C'w_0(|x|)\) a.e. By hypothesis this last condition is true for \(w_0\). Now we check it for \(w_0(\sqrt{\rho^2 + (\cdot)^2})\). We take \(0 \leq a \leq |x| \leq b\). With the change of variables \(s^2 = \rho^2 + t^2\) we obtain

\[
\frac{k}{b^k - a^k} \int_a^b w_0(\sqrt{\rho^2 + t^2}) t^{k-1} \, dt
\]

\[= \frac{k}{b^k - a^k} \int_{\sqrt{\rho^2 + a^2}}^{\sqrt{\rho^2 + b^2}} w_0(s) (s^2 - \rho^2)^{k/2 - 1} \, ds
\]

\[\leq \frac{k^{\frac{2}{k}}}{(\rho^2 + b^2)^{k/2} - (\rho^2 + a^2)^{k/2}} \int_{\sqrt{\rho^2 + a^2}}^{\sqrt{\rho^2 + b^2}} w_0(s) s^{k-1} \, ds
\]

\[\leq \frac{k}{2} M_{v,2} w_0(\sqrt{\rho^2 + |x|^2}) \leq \frac{k}{2} (w)_{A_1(\mathbb{R}^k)} w_0(\sqrt{\rho^2 + |x|^2}).
\]

The only step that is not immediate is the first inequality. Clearly it would follow from

\[
\left[ (\rho^2 + b^2)^{k/2} - (\rho^2 + a^2)^{k/2} \right] \left( \frac{s^2 - \rho^2}{s^2} \right)^{k/2 - 1} \leq \frac{k}{2} (b^k - a^k).
\]

Observe that the left hand side of this inequality is increasing in \(s\) for \(s \geq \rho\). Thus, it is enough to check the case \(s^2 = \rho^2 + b^2\), that is

\[
\left[ (\rho^2 + b^2)^{k/2} - (\rho^2 + a^2)^{k/2} \right] \left( \frac{b^2}{\rho^2 + b^2} \right)^{k/2 - 1} \leq \frac{k}{2} (b^k - a^k).
\] (3.5)
Applying the Mean Value Theorem to the function \( \phi(t) = (\rho^2 + t)^{n/2} \) yields

\[
(\rho^2 + b^2)^{k/2} - (\rho^2 + a^2)^{k/2} \leq \frac{k}{2}(\rho^2 + b^2)^{k/2-1}(b^2 - a^2).
\]

This, together with the observation that \( b^{k-2}(b^2 - a^2) \leq b^k - a^k \), proves \( \text{(3.5)} \). \( \square \)

**Proof of Lemma 3.5.** We assume \( w \in A_p(\mathbb{R}^k) \). By the Reverse Hölder property there exists \( s < 1 \) so that \( w^{1/s} \) is also an \( A_p(\mathbb{R}^k) \) weight. Observe that \( w^{1/s} \in A_p(\mathbb{R}^m) \) for all \( m \geq k \). This is an easy consequence of the factorization and Lemma 2.2.

Let us use the notation \( \mathcal{M}_m f(x) = \sup_{t > 0} S_t f(x) \), where

\[
S_t f(x) = \frac{1}{\omega_{m-1}} \int_{\mathbb{R}^{m-1}} |f(x + ty)| \, d\sigma_{m-1}(y).
\]

Following J.L. Rubio de Francia in [28] we perform a dyadic decomposition of the spherical maximal function. Let \( \psi : [0, \infty) \to [0, \infty) \) be a smooth function supported in \([1/2, 2]\), so that for all \( r > 0 \)

\[
\sum_{j = -\infty}^{\infty} \psi(2^{-j} r) = 1.
\]

Let \( \phi(r) = \sum_{j = -\infty}^{1} \psi(2^{-j} r) \). We can define the operators \( S_t^j \) and \( B_t \) by \( (S_t^j f)(\xi) = (S_t f)(\xi) \psi(2^{-j} |\xi|) \) and \( (B_t f)(\xi) = (S_t f)(\xi) \phi(2^{-j} |\xi|) \). It is obvious that

\[
\mathcal{M}_m f(x) = \sup_{t > 0} \sum_{j = 0}^{\infty} B_t f(x) + S_t^j f(x) \leq M_m f(x) + \sum_{j = 0}^{\infty} \sup_{t > 0} |S_t^j f(x)|. \tag{3.6}
\]

It is known that if \( p < m/(m - 1) \) then

\[
\left\| \sup_{t > 0} |S_t^j f| \right\|_{L^p(\mathbb{R}^m, dx)} \leq 2^j (1 - m/p)^{j} \|f\|_{L^p(\mathbb{R}^m, dx)},
\]

and that \( S_t^j f(x) \leq C 2^j M_m f(x) \) (see [28] and [19]). Then since \( w^{1/s} \in A_p(\mathbb{R}^m) \), one has

\[
\left\| \sup_{t > 0} |S_t^j f| \right\|_{L^p(w^{1/s})} \leq C 2^j \|f\|_{L^p(w^{1/s})}.
\]

By interpolation with change of measure (see [32]), one has

\[
\left\| \sup_{t > 0} |S_t^j f| \right\|_{L^p(\mathbb{R}^m, w)} \leq C 2^{j(1-s)j(1-m/p')} \|f\|_{L^p(\mathbb{R}^m, w)}.
\]

If \( m > p'/(1 - s) \) the exponent in this bound is negative and then we can sum in \( j \) to obtain

\[
\|\mathcal{M}_m f\|_{L^p(\mathbb{R}^m, w)} \leq \|M_m f\|_{L^p(\mathbb{R}^m, w)} + \sum_{j = 0}^{\infty} \left\| \sup_{t > 0} |S_t^j f| \right\|_{L^p(\mathbb{R}^m, w)} \leq C \|f\|_{L^p(\mathbb{R}^m, dx)} + C \sum_{j = 0}^{\infty} 2^{j(1-s)j(1-m/p')} \|f\|_{L^p(\mathbb{R}^m, w)} \leq C \|f\|_{L^p(\mathbb{R}^m, w)}.
\]

Here \( C \) may depend on \( m, p \) and \( w \) but is independent of \( f \). \( C \) is indeed controlled by the operator norm of \( M_m \) in \( L^p(\mathbb{R}^m, w) \). \( \square \)
4 Kakeya maximal operator

Fixed $N > 0$, we denote by $\mathcal{R}_N$ the family of all parallelepipeds in $\mathbb{R}^n$ with edge lengths $h \times h \times \cdots \times h \times Nh$, where $h > 0$ is arbitrary. The Kakeya maximal operator of eccentricity $N$ is defined as

$$ \mathcal{K}_N f(x) = \sup_{x \in R \in \mathcal{R}_N} \frac{1}{R} \int_R |f(y)| \, dy. $$

It is easy to prove that $\mathcal{K}_N f(x) \leq N^{(n-1)} M f(x)$ where $M f$ is here the usual maximal function over all rotated cubes. One just has to replace $R \in \mathcal{R}_N$ by the smallest cube that contains $R$. Then $\mathcal{K}_N$ is weakly bounded on $L^1(\mathbb{R}^n)$ with a constant growing with $N$ at most at the rate $N^{n-1}$. By interpolation with the $L^\infty$ case the operator norm on $L^p(\mathbb{R}^n)$ grows at most like $N^{(n-1)/p}$ for $1 < p < \infty$. However, it is conjectured that for $p = n$ it grows no faster than $C \varepsilon N^\varepsilon$ for each $\varepsilon > 0$. A. Córdoba proved in [12] that the conjecture is true in the case $n = 2$. In higher dimensions A. Carbery, E. Hernández and the second author showed in [11] that the conjecture holds when restricting the action of $\mathcal{K}_N$ to radial functions. Alternative proofs and extensions are due to J. Duoandikoetxea, V. Naibo and O. Oruetxebarria [17] and J. Duoandikoetxea, A. Moyua and O. Oruetxebarria [16].

In this last three papers the result is obtained as a corollary of boundedness results for the universal maximal operator. This is defined as

$$ \mathcal{K} f(x) = \sup_{u \in S^{n-1}} \sup_{a \leq s \leq b} \frac{1}{b-a} \int_a^b |f(x + su)| \, ds. $$

$\mathcal{K}$ is related to the Kakeya maximal operator in the sense that it can be regarded as its extremal case, where the eccentricity $N$ is infinity and rectangles become segments. Moreover $\mathcal{K}$ majorizes all the $\mathcal{K}_N$ but turns out to be unbounded on every $L^p$, except for $p = \infty$ (see [23]). In spite of this, [11] established that $\mathcal{K} : L^{n,1}_\text{rad}(\mathbb{R}^n) \to L^{n,\infty}_\text{rad}(\mathbb{R}^n)$.

The basic idea to give an alternative proof of this last bound in [17] is that for $f = \chi_A$, the characteristic function of a radial set, we have $\mathcal{K} f(x) \leq C_n (Af(x))^{1/n}$. This was further refined in [16] to obtain that for a radial $f$ the bound $\mathcal{K} f(x) \leq C_q (M_{wq} f_0(|x|))^{1/q}$ for any $q > 2$, and for $q \geq 2$ if $f$ is the characteristic function of a radial set. These last constants $C_q$ are independent of the dimension, although the weighted inequalities obtained from the were not. Here we prove

**Lemma 4.1.** Let $E$ be a radial subset of $\mathbb{R}^n$ and $f = \chi_E$. Then for all $k \geq 2$ one has the pointwise inequality

$$ \mathcal{K} f(x) \leq 2(\tilde{M}_{v_k} f_0(|x|))^{1/k}. $$

The constant 2 in this inequality is sharp.

As a consequence, via Proposition [23] we obtain

**Theorem 4.2.** Let $f$ be a radial function over $\mathbb{R}^n$, with $n \geq 2$, and let $w$ be a radial weight in $A_1(\mathbb{R}^n)$, then

$$ \|\mathcal{K} f\|_{L^{n,\infty}(\mathbb{R}^n, w)} \leq \frac{2n}{n-1} \left[2(w)_{A_1(\mathbb{R}^n)}\right]^{1/n} \|f\|_{L^{n,1}(\mathbb{R}^n, w)}. $$
Observe that in view of Lemma 2.2 the previous implies a bound that is uniform in dimension. As a consequence one also has such a bound for the Kakeya maximal operator $K_N$. We remark that the only original results claimed in this section are the sharp bound in Lemma 4.1 and the uniformity in the bound of Theorem 4.2.

Assuming Lemma 4.1 for the moment, we provide a proof of the above theorem.

**Proof of Theorem 4.2.** By density, we just need to prove the result for a simple function of the form
\[
f(x) = \sum_{j=1}^{J} c_j \chi_{E_j}(x),
\]
where $E_1 \supset \ldots \supset E_J$ are radial sets and $c_1, \ldots, c_J$ are positive reals. If $E$ is a radial set, by Lemma 4.1 for $k = n$ and following the argument in the proof of Proposition 2.3 one has
\[
w(\{ x \in \mathbb{R}^n : K \chi_{E}(x) > \lambda \}) \leq \left( \frac{2}{\lambda} \right)^n 2(w)_{A_1(\mathbb{R}^n)} w(E).
\]
Hence $\|K \chi_{E}\|_{L^{n,\infty}(\mathbb{R}^n, w)} \leq 2 \left( \frac{2}{\lambda} \right)^n 2(w)_{A_1(\mathbb{R}^n)} w(E)$. For a general $f$ we use the standard procedure:
\[
\|Kf\|_{L^{n,\infty}(\mathbb{R}^n, w)} \leq \sum_{j=1}^{J} c_j \|K \chi_{E_j}\|_{L^{n,\infty}(\mathbb{R}^n, w)} \leq \frac{n}{n-1} \sum_{j=1}^{J} c_j \|K \chi_{E_j}\|_{L^{n,\infty}(\mathbb{R}^n, w)} \leq \frac{2n}{n-1} \left( \frac{2}{\lambda} \right)^n \sum_{j=1}^{J} c_j w(E_j)^{1/n} = \frac{2n}{n-1} \left( \frac{2}{\lambda} \right)^n \|f\|_{L^{n,1}(\mathbb{R}^n, w)}.
\]

**Proof of Lemma 4.1.** We need some notation. Given $w, z \in \mathbb{R}^n$ we denote by $S_{w,z}$ the segment whose extremal points are $w, z$. We may assume that $|w| \leq |z|$ and will call $y$ to the point in $S_{w,z}$ which is closest to the origin. Consider a radial set $A \subset \mathbb{R}^n$, we define its radial projection over $S_{w,z}$ as $A_0 = \{ |y| \leq |z| : \exists x \in A \text{ with } |x| = t \}$. Denoting by $|E|_k$ the $k$-dimensional Hausdorff measure of a set $E$ it is enough to prove that one has
\[
\frac{|S_{w,z} \cap A|_1}{|z - w|} \leq 2 \left( \frac{k}{|z|^k - |y|^k} \int_{|y|}^{|z|} \chi_{A_0}(s)s^{k-1} ds \right)^{1/k}.
\]

We proceed in several steps, first we show how to reduce to the case $y = w$. Define $z'$ as the point aligned with $w$ and $z$ so that $|z' - y| = |z - y|$. Note that
\[
\frac{|S_{w,z} \cap A|_1}{|z - w|} \leq \frac{|S_{z',z} \cap A|_1}{|z - y|} = 2 \frac{|S_{y,z} \cap A|_1}{|z - y|}.
\]
For the second step we assume $y = w$. We call $L := |z - y|$ and $\ell := |S_{y,z} \cap A|_1$. Consider the point $u \in S_{y,z}$ so that $\ell := |u - y|$. Defining $A^* = \{x \in \mathbb{R}^n : |y| \leq |x| \leq |u|\}$, we are done if we show that we have

$$\frac{|S_{y,z} \cap A|_1}{L} = \frac{|S_{y,z} \cap A^*|_1}{L} \leq \left( \frac{k}{|z|^k - |y|^k} \int_{|y|}^{|z|} \chi_A(s)s^{k-1} ds \right)^{1/k} \leq \left( \frac{k}{|z|^k - |y|^k} \int_{|y|}^{|z|} \chi_A(s)s^{k-1} ds \right)^{1/k} \cdot \tag{4.1}$$

The equality in (4.1) is a trivial consequence of the definition of $A^*$. Now we prove the first inequality in (4.1) in the case $k = 2$. Denoting by $\gamma$ the angle determined by $S_{0,y}$ and $S_{y,z}$ at $y$, the inequality can be rewritten as

$$\frac{\ell}{L} \leq \left( \frac{|u|^2 - |y|^2}{|z|^2 - |y|^2} \right)^{1/2} = \left( \frac{\ell^2 - 2\ell|y| \cos \gamma}{L^2 - 2L|y| \cos \gamma} \right)^{1/2},$$

where the last expression comes from the cosine law. This is equivalent to

$$\frac{\ell}{L} \leq \frac{\ell - 2|y| \cos \gamma}{L - 2|y| \cos \gamma},$$

which is obviously true since $\ell \leq L$ and $\cos \gamma \leq 0$. For $k \geq 2$ it is enough to show that

$$\left( \frac{|z|^k - |y|^k}{|u|^k - |y|^k} \right)^{1/k} \geq \left( \frac{|z|^2 - |y|^2}{|u|^2 - |y|^2} \right)^{1/2}.$$

Dividing by $|y|$ and renaming $\alpha = (|z|/|y|)^2$ and $\beta = (|u|/|y|)^2$ the previous inequality becomes

$$\left( \frac{\alpha^{k/2} - 1}{\beta^{k/2} - 1} \right)^{1/k} \geq \left( \frac{\alpha - 1}{\beta - 1} \right)^{1/2},$$

or equivalently

$$(\alpha - 1)^{k/2}(\alpha^{k/2} - 1) \geq (\beta - 1)^{k/2}(\beta^{k/2} - 1),$$

which is true for $\alpha > \beta \geq 1$ since $s \mapsto (s - 1)^{k/2}(s^{k/2} - 1)$ is clearly an increasing function for $s \geq 1$.

As for the second inequality in (4.1), let us define $T = \{|v - y| : v \in A \cap S_{y,z}\}$. Note that $|T| = \ell$ and that $T = \{s \geq 0 : s - 2|y| \cos \gamma \in A\}$. Therefore, by the change of variables $t = (s^2 + |y|^2 - 2s|y| \cos \gamma)^{1/2}$ one has

$$\int_{A_0} t^{k-1} dt = \int_{|y|}^{\ell} (s^2 + |y|^2 - 2s|y| \cos \gamma)^{(k-2)/2}(s - 2|y| \cos \gamma) ds$$

$$\geq \int_{0}^{\ell} (s^2 + |y|^2 - 2s|y| \cos \gamma)^{(k-2)/2}(s - 2|y| \cos \gamma) ds$$

$$= \int_{|y|}^{\ell} t^{k-1} dt = \int_{A_0^*} t^{k-1} dt.$$

To get the above inequality we have used that the integrated function is increasing with $s$. \qed
Remark. The constant 2 in this Lemma is optimal. To see this assume that \( C > 0 \) is a constant such that for all \( z \in \mathbb{R}^n \) and all radial set \( A \subset \mathbb{R}^n \) one has

\[
K\chi_A(z) \leq C \tilde{M}_{v^2} \chi_{A_0}(|z|)^{1/2}, \tag{4.2}
\]

Consider the segment \( S_{w,z} \) and define \( y \) as the point in \( S_{w,z} \) that is closest to the origin. Assume that \(|w| > |y|\) take \( A = \{x \in \mathbb{R}^n : |y| \leq |x| \leq |w|\} \). By orthogonality

\[
\tilde{M}_{v^2} \chi_{A_0}(|z|) = \sup_{|y| \leq t \leq |w|} \frac{|w|^2 - t^2}{|z|^2 - t^2} = \frac{|w|^2 - |y|^2}{|z|^2 - |y|^2} = \frac{|w - y|^2}{|z - y|^2} = \frac{(\ell/2)^2}{(L - \ell/2)^2}
\]

Calling as before \( L = |z - w| \) and \( \ell = 2|w - y| \) we have \( K\chi_A(z) \geq \ell/L \). Then inequality (4.2) implies

\[
C \geq 2 \frac{L - \ell/2}{L}.
\]

Since, choosing \( w \) appropriately, \( \ell \) can be taken as small as wanted, necessarily we must have \( C \geq 2 \).

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