ASYMPTOTICS OF G-EQUIVARIANT SZEGŐ KERNELS

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Abstract. Let \((X, T^{1,0}X)\) be a compact connected orientable CR manifold of dimension \(2n + 1\) with non-degenerate Levi curvature. Assume that \(X\) admits a connected compact Lie group \(G\) action. Under certain natural assumptions about the group \(G\) action, we define \(G\)-equivariant Szegő kernels and establish the associated Boutet de Monvel-Sjöstrand type theorems. When \(X\) admits also a transversal CR \(S^1\) action, we study the asymptotics of Fourier components of \(G\)-equivariant Szegő kernels with respect to the \(S^1\) action.

1. Introduction and statement of the main results

Let \((X, T^{1,0}X)\) be a CR manifold of dimension \(2n + 1\), \(n \geq 1\). Let \(\Box^{(q)}_b\) be the Kohn Laplacian acting on \((0, q)\) forms. The orthogonal projection \(S^{(q)}_b : L^2_{(0,q)}(X) \rightarrow \text{Ker} \, \Box^{(q)}_b\) onto \(\text{Ker} \, \Box^{(q)}_b\) is called the Szegő projection. The Szegő kernel is its distribution kernel \(S^{(q)}(x, y)\). The study of the Szegő projection and kernels is a classical subject in several complex variables and CR geometry. When \(X\) is the boundary of a strictly pseudoconvex domain, Boutet de Monvel-Sjöstrand [1] showed that \(S^{(0)}(x, y)\) is a complex Fourier integral operator.

The Boutet de Monvel-Sjöstrand theorem had a profound impact in many research areas, especially through [2]: several complex variables, symplectic and contact geometry, geometric quantization, Kähler geometry, semiclassical analysis, quantum chaos, etc. cf. [3, 4, 6, 15, 16, 17, 18, 23, 25]. Recently, Hsiao-Huang [10] obtained \(G\)-invariant Boutet de Monvel-Sjöstrand type theorems and Hsiao-Ma-Marinescu [11] established geometric quantization on CR manifolds by using \(G\)-invariant Szegő kernels asymptotic expansions.

In this paper, we study \(G\)-equivariant Szegő kernels with respect to all equivalent classes of irreducible unitary representations of \(G\). We establish \(G\)-equivariant Boutet de Monvel-Sjöstrand type theorems. When \(X\) admits also a transversal CR \(S^1\) action, we derive the asymptotic expansion of Fourier components of \(G\)-equivariant Szegő kernels with respect to the \(S^1\) action.

We now formulate the main results. We refer to Section 2 for some notations and terminology used here. Let \((X, T^{1,0}X)\) be a compact connected orientable CR manifold of dimension \(2n + 1\), \(n \geq 1\), where \(T^{1,0}X\) denotes the CR structure of \(X\). Fix a global non-vanishing real 1-form \(\omega_0 \in C^\infty(X, T^*X)\) such that \(\langle \omega_0, u \rangle = 0\), for every \(u \in T^{1,0}X \oplus T^{0,1}X\). The Levi form of \(X\) at

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Let \( HX = \{ \text{Re}\ u; u \in T^{1,0}X \} \) and let \( J : HX \to HX \) be the complex structure map given by \( J(u + ii\omega) = iu - i\omega \), for every \( u \in T^{1,0}X \). In this paper, we assume that \( X \) admits a \( d \)-dimensional connected compact Lie group action \( \xi \). For any \( \xi \in \mathfrak{g} \), we write \( \xi_X \) to denote the vector field on \( X \) induced by \( \xi \). That is, \( (\xi_X u)(x) = \left. \frac{\partial}{\partial t} \left( u(\exp(t\xi) \circ x) \right) \right|_{t=0} \), for any \( u \in C^\infty(X) \). Let \( \mathfrak{g} = \text{Span} (\xi_X; \xi \in \mathfrak{g}) \). We assume throughout that

**Assumption 1.2.** The Lie group \( G \) action is CR and preserves \( \omega_0 \) and \( J \).

We recall that the Lie group \( G \) action preserves \( \omega_0 \) and \( J \) means that \( g^* \omega_0 = \omega_0 \) on \( X \) and \( g_* J = J g_* \) on \( HX \), for every \( g \in G \), where \( g^* \) and \( g_* \) denote the pull-back map and push-forward map of \( G \), respectively. The \( G \) action is CR means that for every \( \xi_X \in \mathfrak{g} \),

\[
[\xi_X, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X).
\]

**Definition 1.3.** The moment map associated to the form \( \omega_0 \) is the map \( \mu : X \to \mathfrak{g}^* \) such that, for all \( x \in X \) and \( \xi \in \mathfrak{g} \), we have

\[
\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)).
\]

We assume also that

**Assumption 1.4.** \( 0 \) is a regular value of \( \mu \) and \( G \) acts locally free near \( \mu^{-1}(0) \).

By Assumption 1.4, \( \mu^{-1}(0) \) is a \( d \)-codimensional orbifold of \( X \). Note that if \( G \) acts freely near \( \mu^{-1}(0) \) and the Levi form is positive at \( \mu^{-1}(0) \), it is known that \( \mu^{-1}(0)/G \) is a CR manifold with natural CR structure induced by \( T^{1,0}X \) of dimension \( 2n - 2d + 1 \), (see [10]).

Let \( R = \{ R_1, R_2, \ldots \} \) be the collection of all irreducible unitary representations of \( G \), including only one representation from each equivalent class (see Section 2.4). Write

\[
R_k : G \to GL(\mathbb{C}^{d_k}), \quad d_k < \infty,
\]

\[
g \to (R_{k,j,i}(g))^{d_k}_{j,i=1},
\]

where \( d_k \) is the dimension of the representation \( R_k \). Denote by \( \chi_k(g) := \text{Tr} R_k(g) \) the trace of the matrix \( R_k(g) \) (the character of \( R_k \)). Let \( u \in \Omega^{0,0}(X) \). For every \( k = 1, 2, \ldots \), we define

\[
u_k(x) = d_k \int_G (g^* u)(x) \chi_k(g) d\mu(g),
\]
where $d\mu(g)$ is the probability Haar measure on $G$. We will show that $u = \sum_{k=0}^{\infty} u_k$. By Assumption 1.2, $u_k \in \Omega^{0,q}(X)$ if $u \in \Omega^{0,q}(X)$. Set
\[
\Omega^{0,q}(X)_k := \{u(x) \in \Omega^{0,q}(X) | u(x) = u_k(x)\}. \tag{1.4}
\]
Denote by $L^2_{(0,q)}(X)_k$ the completion of $\Omega^{0,q}(X)_k$ with respect to the inner product $(\cdot | \cdot)$.

**Definition 1.5.** The $G$-equivariant Szegő projection is the orthogonal projection
\[
S^{(q)}_k : L^2_{(0,q)}(X) \to \text{Ker} \square^{(q)}_b \cap L^2_{(0,q)}(X)_k
\]
with respect to $(\cdot | \cdot)$. Denote by $S^{(q)}_k(x,y) \in D'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ the distribution kernel of $S^{(q)}_k$.

The first main result of this work is the following

**Theorem 1.6.** With the assumptions and notations above, suppose that $\square^{(q)}_b : \text{Dom} \square^{(q)}_b \to L^2_{(0,q)}(X)$ has closed range. If $q \notin \{n_-, n_+\}$, then $S^{(q)}_k \equiv 0$ on $X$.

Suppose $q \in \{n_-, n_+\}$. Let $D$ be an open neighborhood of $X$ with $D \cap \mu^{-1}(0) = \emptyset$. Then, $S^{(q)}_k \equiv 0$ on $D$.

Let $p \in \mu^{-1}(0)$ and let $U$ be an open neighborhood of $p$ and let $x = (x_1, \ldots, x_{2n+1})$ be local coordinates defined in $U$. Let $N_p = \{g \in G : g \circ p = p\} = \{g_1 = e_0, g_2, \ldots, g_r\}$. Then, there exist continuous operators $\hat{S}_{k,-}, \hat{S}_{k,+} : \Omega^{0,q}(U) \to \Omega^{0,q}(U)$ such that
\[
S^{(q)}_k \equiv \hat{S}_{k,-} + \hat{S}_{k,+} \text{ on } U, \tag{1.5}
\]
and $\hat{S}_{k,-}(x,y), \hat{S}_{k,+}(x,y)$ satisfy
\[
\hat{S}_{k,-}(x,y) \equiv \sum_{\alpha=1}^{r} \int_{0}^{\infty} e^{\Phi_{k,-}(g_0, x, y) t} a_{k,\alpha,-}(x,y,t) dt \text{ on } U,
\]
\[
\hat{S}_{k,+}(x,y) \equiv \sum_{\alpha=1}^{r} \int_{0}^{\infty} e^{\Phi_{k,+}(g_0, x, y) t} a_{k,\alpha,+}(x,y,t) dt \text{ on } U,
\]
with
\[
a_{k,\alpha,+}(x,y,t), a_{k,\alpha,-}(x,y,t) \in S^{n-d\frac{q}{2}}_{\text{cl}}(U \times U \times \mathbb{R}^+; T^{*0,q}X \boxtimes (T^{*0,q}X)^*),
\]
\[
a_{k,\alpha,-}(x,y,t) = 0 \text{ if } q \neq n_-, \quad a_{k,\alpha,+}(x,y,t) = 0 \text{ if } q \neq n_+,
\]
\[
a^{0}_{k,\alpha,-}(x,x) \neq 0, \quad \forall x \in U, \quad a^{0}_{k,\alpha,+}(x,x) \neq 0, \quad \forall x \in U,
\]
where $a^{0}_{k,\alpha,-}(x,x)$ and $a^{0}_{k,\alpha,+}(x,x)$, $x \in \mu^{-1}(0) \cap U$, are the leading terms of the asymptotic expansion of $a_{k,\alpha,-}(x,x)$ and $a_{k,\alpha,+}(x,x)$ in $S^{n-d\frac{q}{2}}_{1,0}(U \times U \times \mathbb{R}^+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, respectively, and $\Phi_{k,-}(x,y) \in C^0(U \times U)$.

By $\Phi_{k,+}(x,y) \geq 0, \quad d_x \Phi_{k,-}(x,y) = -d_y \Phi_{k,-}(x,x) = -\omega_0(x), \quad \forall x \in U \cap \mu^{-1}(0), \tag{1.8}$

and $-\Phi_{k,+}(x,y)$ satisfies (1.8).
We refer the readers to the discussion before [2.1] and Definition [3.1] for the precise meanings of \( A \equiv B \) and the symbol space \( S^m_{÷ \mathbb{R}} \), respectively.

Assume that \( G \) acts freely on \( \mu^{-1}(0) \) for a moment. To state the formulas for \( a_{k,-}^0(x, x) \) and \( a_{k,+}^0(x, x) \), we introduce some notations. For a given point \( x_0 \in X \), let \( \{W_j\}_{j=1}^n \) be an orthonormal frame of \((T^{1,0}X, \langle \cdot | \cdot \rangle)\) near \( x_0 \), for which the Levi form is diagonal at \( x_0 \). Put

\[
\mathcal{L}_{x_0}(W_j, \overline{W}_\ell) = \mu_j(x_0) \delta_{j\ell}, \quad j, \ell = 1, \ldots, n.
\]

We will denote by

\[
\det \mathcal{L}_{x_0} = \prod_{j=1}^n \mu_j(x_0).
\]

Let \( \{T_j\}_{j=1}^n \) denote the basis of \( T^{0,1}X \), dual to \( \{W_j\}_{j=1}^n \). We assume that \( \mu_j(x_0) < 0 \) if \( 1 \leq j \leq n_\pm \) and \( \mu_j(x_0) > 0 \) if \( n_- + 1 \leq j \leq n \). Put

\[
\mathcal{N}(x_0, n_-) := \{ cT_1(x_0) \land \ldots \land T_{n_-}(x_0); \; c \in \mathbb{C} \},
\]

\[
\mathcal{N}(x_0, n_+) := \{ cT_{n_-+1}(x_0) \land \ldots \land T_n(x_0); \; c \in \mathbb{C} \}
\]

and let

\[
\tau_{x_0, n_-} = \tau_{x_0, n_-} : T^{0,q}_{x_0}X \to \mathcal{N}(x_0, n_-), \quad \tau_{x_0, n_+} = \tau_{x_0, n_+} : T^{0,q}_{x_0}X \to \mathcal{N}(x_0, n_+),
\]

be the orthogonal projections onto \( \mathcal{N}(x_0, n_-) \) and \( \mathcal{N}(x_0, n_+) \) with respect to \( \langle \cdot | \cdot \rangle \), respectively.

Fix \( x \in \mu^{-1}(0) \), consider the linear map

\[
R_x : g_x \to g_x, \quad u \to R_x u, \quad \langle R_x u | v \rangle = \langle d\omega_0(x), J u \lor v \rangle.
\]

Let \( \det R_x = \lambda_1(x) \cdots \lambda_d(x) \), where \( \lambda_j(x), \; j = 1, 2, \ldots, d \), are the eigenvalues of \( R_x \).

Fix \( x \in \mu^{-1}(0) \), put \( Y_x = \{ g \circ x; \; g \in G \} \). \( Y_x \) is a \( d \)-dimensional submanifold of \( X \). The \( G \)-invariant Hermitian metric \( \langle \cdot | \cdot \rangle \) induces a volume form \( dv_{Y_x} \) on \( Y_x \). Put

\[
V_{\text{eff}}(x) := \int_{Y_x} dv_{Y_x}.
\]

Note that the function \( V_{\text{eff}}(x) \) was already appeared in Ma-Zhang [19 (0,10)] as exactly the role in the expansion, cf. [19 (0.14)].

**Theorem 1.7.** With the notations used above, if \( G \) acts freely on \( \mu^{-1}(0) \), then for \( a_{k,-}^0(x, y) \) and \( a_{k,+}^0(x, y) \) in (1.7), we have

\[
a_{k,-}^0(x, x) = 2^{d-1} \frac{d^2}{V_{\text{eff}}(x)} \pi^{-n+\frac{3}{2}} |\det R_x|^{-\frac{1}{2}} |\det \mathcal{L}_x|^\tau_{x, n_-}, \quad \forall x \in \mu^{-1}(0)
\]

\[
a_{k,+}^0(x, x) = 2^{d-1} \frac{d^2}{V_{\text{eff}}(x)} \pi^{-n+\frac{3}{2}} |\det R_x|^{-\frac{1}{2}} |\det \mathcal{L}_x|^\tau_{x, n_+}, \quad \forall x \in \mu^{-1}(0).
\]

Assume that \( X \) admits an \( S^1 \) action \( e^{i\theta} : S^1 \times X \to X \). Let \( T \in \mathcal{C}^{\infty}(X, TX) \) be the global real vector field induced by the \( S^1 \) action given by \( (Tu)(x) = \frac{\partial}{\partial \theta} (u(e^{i\theta} \circ x)) \mid_{\theta=0}, \; u \in \mathcal{C}^{\infty}(X) \). Let the \( S^1 \) action \( e^{i\theta} \) be CR and transversal (see Definition [4.1]). We assume throughout that
Assumption 1.8.

\[ T \text{ is transversal to the space } g \text{ at every point } p \in \mu^{-1}(0), \]
\[ e^{i\theta} \circ g \circ x = g \circ e^{i\theta} \circ x, \quad \forall x \in X, \quad \forall \theta \in [0, 2\pi], \quad \forall g \in G, \]
and
\[ G \times S^1 \text{ acts locally freely near } \mu^{-1}(0). \]

Let \( u \in \Omega^{0,q}(X) \). Define
\[ Tu := \frac{\partial}{\partial \theta} ((e^{i\theta})^* u)|_{\theta=0} \in \Omega^{0,q}(X). \]

For every \( m \in \mathbb{Z} \), let
\[ \Omega^q_m(X) := \left\{ u \in \Omega^{0,q}(X); Tu = imu \right\}, \quad q = 0, 1, 2, \ldots, n, \]
\[ \Omega^q_m(X)_k := \left\{ u \in \Omega^{0,q}(X)_k; Tu = imu \right\}, \quad q = 0, 1, 2, \ldots, n. \]  

Denote by \( C^\infty_m(X) := \Omega^q_m(X), C^\infty_m(X)_k := \Omega^q_m(X)_k \). From the CR property of the \( S^1 \) action and \( (1.13) \), we have
\[ Tg^* \overline{\partial}_b = g^* T \overline{\partial}_b = \overline{\partial}_b g^* T = \overline{\partial}_b T g^* \text{ on } \Omega^{0,q}(X), \quad \forall g \in G. \]

Hence,
\[ \overline{\partial}_b : \Omega^{0,q}_m(X)_k \rightarrow \Omega^{0,q+1}_m(X)_k, \quad \forall m \in \mathbb{Z}. \]  

Assume that the Hermitian metric \( \langle \cdot | \cdot \rangle \) on \( CTX \) is \( G \times S^1 \) invariant. Then the \( L^2 \) inner product \( \langle \cdot | \cdot \rangle \) on \( \Omega^{0,q}(X) \) induced by \( \langle \cdot | \cdot \rangle \) is \( G \times S^1 \)-invariant. We then have
\[ Tg^* \overline{\partial}_b = g^* T \overline{\partial}_b = \overline{\partial}_b g^* T = \overline{\partial}_b T g^* \text{ on } \Omega^{0,q}(X), \quad \forall g \in G, \]
\[ Tg^* \square^{(q)}_b = g^* T \square^{(q)}_b = \square^{(q)}_b g^* T = \square^{(q)}_b T g^* \text{ on } \Omega^{0,q}(X), \quad \forall g \in G, \]

where \( \overline{\partial}_b \) is the \( L^2 \) adjoint of \( \overline{\partial}_b \) with respect to \( \langle \cdot | \cdot \rangle \).

Let \( L^2_{(0,q),m}(X)_k \) be the completion of \( \Omega^q_m(X)_k \) with respect to \( \langle \cdot | \cdot \rangle \). Write \( L^2_m(X)_k := L^2_{(0,0),m}(X)_k \). Put
\[ H^q_{b,m}(X)_k := (\text{Ker } \square^{(q)}_b) \cap L^2_{(0,q),m}(X)_k. \]

Since \( \square^{(q)}_b - T^2 \) is elliptic, we have for every \( m \in \mathbb{Z} \), \( H^q_{b,m}(X)_k \subset \Omega^q_m(X)_k \) and \( \dim H^q_{b,m}(X)_k < \infty \).

Definition 1.9. The \( m \)-th \( G \)-equivariant Szegö projection is the orthogonal projection
\[ S^{(q)}_{k,m} : L^2_{(0,q)}(X) \rightarrow (\text{Ker } \square^{(q)}_b) \cap L^2_{(0,q),m}(X)_k \]
with respect to \( \langle \cdot | \cdot \rangle \). Let \( S^{(q)}_{k,m}(x,y) \in C^\infty(X \times X, T^*\Omega^{0,q}X \otimes (T^*\Omega^{0,q}X)^*) \) be the distribution kernel of \( S^{(q)}_{k,m} \).

The second main result of this work is the following
Theorem 1.10. With the assumptions and notations used above, if \( q \notin n_\ast \), then, as \( m \to +\infty \), \( S_{k,m}^{(q)} = O(m^{-\infty}) \) on \( X \).

Suppose \( q = n_\ast \). Let \( D \) be an open neighborhood of \( X \) with \( D \cap \mu^{-1}(0) = \emptyset \). Then, as \( m \to +\infty \),

\[
S_{k,m}^{(q)} = O(m^{-\infty}) \quad \text{on} \ D.
\]

Let \( p \in \mu^{-1}(0) \) and let \( N_p = \{ g \in G : g \circ p = p \} = \{ g_1 = e_0, g_2, ..., g_r \} \). Let \( U \) be an open neighborhood of \( p \) and let \( x = (x_1, ..., x_{2n+1}) \) be local coordinates defined in \( U \). Then, as \( m \to +\infty \),

\[
S_{k,m}^{(q)}(x,y) = \sum_{\alpha=1}^{r} e^{im\Psi_k(g_{\alpha}o x,y)}b_{k,\alpha}(x,y,m),
\]

\[
b_{k,\alpha}(x,y,m) \in \mathcal{S}^{n-d}_{\text{loc}} (1; U \times U, T^{s_0q} X \boxtimes (T^{s_0q} X)^*) ,
\]

\[
b_{k,\alpha}(x,y,m) \sim \sum_{j=0}^{\infty} m^{n-d-j} b_{k,\alpha}^j (x,y) \quad \text{in} \quad \mathcal{S}^{n-d}_{\text{loc}} (1; U \times U, T^{s_0q} X \boxtimes (T^{s_0q} X)^*) ,
\]

\[
b_{k,\alpha}^j (x,y) \in C^\infty (U \times U, T^{s_0q} X \boxtimes (T^{s_0q} X)^*) , \quad j = 0, 1, 2, ..., \]

\[
\Psi_k(x,y) \in C^\infty (U \times U) , \quad d_x \Psi_k(x,y) = -d_y \Psi_k(x,y) = -\omega_0(x) , \quad \forall x \in \mu^{-1}(0), \quad \Psi_k(x,y) = 0
\]

if and only if \( x = y \in \mu^{-1}(0) \).

In particular, if \( G \times S^1 \) acts freely near \( \mu^{-1}(0) \), then

\[
b_{k}^0(x,y) = 2^{d-1} \frac{d^2}{V_{\text{eff}}(x)} \pi^{-n+\frac{d}{2}} |\det R_x|^\frac{1}{2} |\det L_x| \tau_{x,n_\ast} , \quad \forall x \in \mu^{-1}(0),
\]

where \( \tau_{x,n_\ast} \) is given by (1.10).

We provide a special case when \( G = T^d \) on an irregular Sasakian manifold, where \( T^d \) denotes the \( d \)-dimensional torus. Recall that a CR manifold \( X \) is irregular Sasakian if it admits a CR transversal \( \mathbb{R} \)-action, which does not come from any circle action. The \( \mathbb{R} \)-action can be interpreted as a CR torus action \( T^{d+1} = T^d \times S^1 \) (cf. Herrmann-Hsiao-Li [7]). Fix \((p_1, ..., p_d) \in \mathbb{Z}^d\), we define a \( T^d \)-action as follows:

\[
T^d \times X \rightarrow X
\]

\[
((e^{i\theta_1}, ..., e^{i\theta_d}), x) \mapsto (e^{i\theta_1}, ..., e^{i\theta_d}, e^{-ip_1\theta_1} \cdots e^{-ip_d\theta_d}) \circ x.
\]

The \( T^d \)-action satisfies Assumption[18] All irreducible unitary representations of \( T^d \) are \( \{ R_{p_1, ..., p_d} : (e^{i\theta_1}, ..., e^{i\theta_d}) \mapsto e^{ip_1\theta_1} \cdots e^{ip_d\theta_d} \} \). Let \( T_j \) be the induced vector fields of the \( T^d \)-action. That is,

\[
T_j u(x) := \frac{\partial}{\partial \theta_j}|_{\theta_j = 0} u((1, ..., e^{i\theta_1}, ..., 1) \circ x), \quad \text{for} \ j = 1, ..., d \text{ and } u \in C^\infty(X).
\]

Then

\[
H_{b,mp_1, ..., mp_d,m}^q(X) := \{ u \in H_b^q(X) : Tu = imu, T_j u = imp_j u, j = 1, ..., d \}.
\]

The \( m \)-th equivariant Szegö kernel is the distributional kernel of the orthogonal projection

\[
S_{mp_1, ..., mp_d,m}^q : L^2_{(0,q)} \rightarrow H_{b,mp_1, ..., mp_d,m}^q(X).
\]

Assume that \((-p_1, ..., -p_d) \in \mathbb{Z}^d\) is a regular value of the torus invariant CR moment map

\[
\mu_0 : X \rightarrow \mathbb{R}^d, \mu_0(x) := (\omega_0(x), T_1(x), ..., \omega_0(x), T_d(x)).
\]
The CR moment map of $T^d$-action defined in (1.20) is

$$
\mu : X \to \mathbb{R}^d, \mu(x) := (\omega_0(x), -p_1 T + T_1(x)), ..., (\omega_0(x), -p_d T + T_d(x)).
$$

(1.23)

For $x \in \mu_0^{-1}(-p_1, ..., -p_d)$, $\omega_0(-p_j T + T_j) = 0$. Then 0 is the regular value of $\mu$. By Theorem [1.10] we deduce the following which covers Shen’s result [22] when $X$ is strongly pseudoconvex.

**Corollary 1.11.** Fix $(-p_1, ..., -p_d) \in \mathbb{Z}^d$, the action of $T^d$ on the irregular Sasakian manifold $X$ is defined in (1.20). With the assumptions and notations used above, If $q \notin n_-$, then, as $m \to +\infty$, $S_{mp_1, ..., mp_d,m}^{(q)} = O(m^{-\infty})$ on $X$.

Suppose $q = n_-$. Let $D$ be an open neighborhood of $X$ with $D \cap \mu^{-1}(0) = \emptyset$. Then, as $m \to +\infty$,

$$
S_{mp_1, ..., mp_d,m}^{(q)} = O(m^{-\infty}) \text{ on } D.
$$

Let $p \in \mu^{-1}(0)$ and let $N_p = \{g \in G : g \circ p = p\} = \{g_1 = e_0, g_2, ..., g_r\}$. Let $U$ be an open neighborhood of $p$ and let $x = (x_1, ..., x_{2n+1})$ be local coordinates defined in $U$. Then, as $m \to +\infty$,

$$
S_{mp_1, ..., mp_d,m}^{(q)}(x, y) \equiv \sum_{\alpha=1}^{r} e^{im\Psi(g_{\alpha} ox, y)} b_{\alpha}(x, y, m),
$$

(1.24)

$$
b_{\alpha}(x, y, m) \in S_{1U \times X, (T^{*d} q X)^*}^{n-d}(1; U \times U, T^{*d} q X \boxplus (T^{*d} q X)^*),
$$

$$
b_{\alpha}(x, y, m) \sim \sum_{j=0}^{\infty} m^{n-d-j} b_{\alpha}(x, y) \text{ in } S_{1U \times X, (T^{*d} q X)^*}^{n-d}(1; U \times U, T^{*d} q X \boxplus (T^{*d} q X)^*),
$$

$$
b_{\alpha}^{(j)}(x, y) \in C^{\infty}(U \times U, T^{*d} q X \boxplus (T^{*d} q X)^*), \quad j = 0, 1, 2, ..., \Psi(x, y) \in C^{\infty}(U \times U), \quad d_x \Psi(x, x) = - d_x \Psi(x, x) = -\omega_0(x), \text{ for every } x \in \mu^{-1}(0), \Psi(x, y) = 0 \text{ if and only if } x = y \in \mu^{-1}(0).
$$

In particular, if $T^d \times S^1$ acts freely near $\mu^{-1}(0)$, then

$$
b^{(j)}(x, x) = \frac{q^{d-1}}{V_{\text{eff}}(x)} \pi^{n-1+\frac{d}{2}} |\det R_x|^{-\frac{1}{2}} |\det \mathcal{L}_x| \tau_{x, n_-}, \quad \forall x \in \mu^{-1}(0),
$$

(1.25)

where $\tau_{x, n_-}$ is given by (1.10).

2. Preliminaries

### 2.1. Standard notations

Let $M$ be a $C^{\infty}$ paracompact manifold. We let $TM$ and $T^*M$ denote the tangent bundle of $M$ and the cotangent bundle of $M$, respectively. The complexified tangent bundle of $M$ and the complexified cotangent bundle of $M$ will be denoted by $\mathbb{C}TM$ and $\mathbb{C}T^*M$, respectively. Write $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between $TM$ and $T^*M$. We extend $\langle \cdot, \cdot \rangle$ bilinearly to $\mathbb{C}TM \times \mathbb{C}T^*M$.

Let $F$ be a $C^{\infty}$ vector bundle over $M$. The fiber of $F$ at $x \in M$ will be denoted by $F_x$. Let $E$ be a vector bundle over a $C^{\infty}$ paracompact manifold $M_1$. We write $F \boxplus E^*$ to denote the vector bundle over $M \times M_1$ with fiber over $(x, y) \in M \times M_1$ consisting of the linear maps from $E_y$ to $F_x$. Let $Y \subset M$ be an open set. From now on, the spaces of distribution sections of $F$ over $Y$ and smooth sections of $F$ over $Y$ will be denoted by $D'(Y, F)$ and $C^\infty(Y, F)$, respectively.
Let $E'(Y, F)$ be the subspace of $D'(Y, F)$ whose elements have compact support in $Y$. Put $C^\infty_c(M, F) := C^\infty(M, F) \cap E'(M, F)$.

We recall the Schwartz kernel theorem \[8, \text{Theorems 5.2.1, 5.2.6}, \ \ [16, \text{Thorem B.2.7}]. \] Let $F$ and $E$ be $C^\infty$ vector bundles over paracompact orientable $C^\infty$ manifolds $M$ and $M_1$, respectively, equipped with smooth densities of integration. If $A : C^\infty_c(M_1, E) \to D'(M, F)$ is continuous, we write $K_A(x, y)$ or $A(x, y)$ to denote the distribution kernel of $A$. The following two statements are equivalent

1. $A$ is continuous: $E'(M_1, E) \to C^\infty(M, F)$,
2. $K_A \in C^\infty(M \times M_1, F \boxtimes E^*)$.

If $A$ satisfies (1) or (2), we say that $A$ is smoothing on $M \times M_1$. Let $A, \hat{A} : C^\infty_0(M_1, E) \to D'(M, F)$ be continuous operators. We write

$$A \equiv \hat{A} \ (\text{on } M \times M_1) \quad (2.1)$$

if $A - \hat{A}$ is a smoothing operator. If $M = M_1$, we simply write “on $M$”.

Let $H(x, y) \in D'(M \times M_1, F \boxtimes E^*)$. We write $H$ to denote the unique continuous operator $C^\infty_c(M_1, E) \to D'(M, F)$ with distribution kernel $H(x, y)$. In this work, we identify $H$ with $H(x, y)$.

### 2.2. Some standard notations in semi-classical analysis.

Let $W_1$ be an open set in $\mathbb{R}^{N_1}$ and let $W_2$ be an open set in $\mathbb{R}^{N_2}$. Let $E$ and $F$ be vector bundles over $W_1$ and $W_2$, respectively. An $m$-dependent continuous operator $A_m : C^\infty_c(W_2, F) \to D'(W_1, E)$ is called $m$-negligible on $W_1 \times W_2$ if, for $m$ large enough, $A_m$ is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices $\alpha, \beta$ and any $N \in \mathbb{N}$, there exists $C_{K, \alpha, \beta, N} > 0$ such that

$$|\partial_x^\alpha \partial_y^\beta A_m(x, y)| \leq C_{K, \alpha, \beta, N} m^{-N} \text{ on } K, \ \forall m \gg 1. \quad (2.2)$$

In that case we write

$$A_m(x, y) = O(m^{-\infty}) \text{ on } W_1 \times W_2,$$

or

$$A_m = O(m^{-\infty}) \text{ on } W_1 \times W_2.$$

If $A_m, B_m : C^\infty_c(W_2, F) \to D'(W_1, E)$ are $m$-dependent continuous operators, we write $A_m = B_m + O(m^{-\infty})$ on $W_1 \times W_2$ or $A_m(x, y) = B_m(x, y) + O(m^{-\infty})$ on $W_1 \times W_2$ if $A_m - B_m = O(m^{-\infty})$ on $W_1 \times W_2$. When $W = W_1 = W_2$, we sometime write “on $W$”.

Let $X$ and $M$ be smooth manifolds and let $E$ and $F$ be vector bundles over $X$ and $M$, respectively. Let $A_m, B_m : C^\infty(M, F) \to C^\infty(X, E)$ be $m$-dependent smoothing operators. We write $A_m = B_m + O(m^{-\infty})$ on $X \times M$ if on every local coordinate patch $D$ of $X$ and local coordinate patch $D_1$ of $M$, $A_m = B_m + O(m^{-\infty})$ on $D \times D_1$. When $X = M$, we sometime write on $X$.

We recall the definition of the semi-classical symbol spaces.
Definition 2.1. Let $W$ be an open set in $\mathbb{R}^N$. Let
\[
S(1; W) := \left\{ a \in C^\infty(W) \mid \forall \alpha \in \mathbb{N}_0^N : \sup_{x \in W} |\partial^\alpha a(x)| < \infty \right\},
\]
and
\[
S^0_{\text{loc}}(1; W) := \left\{ (a(\cdot, m))_{m \in \mathbb{R}} \mid \forall \alpha \in \mathbb{N}_0^N, \forall \chi \in C^\infty_0(W) : \sup_{m \in \mathbb{R}, m \geq 1} \sup_{x \in W} |\partial^\alpha (\chi a(x, m))| < \infty \right\}.
\]
For $k \in \mathbb{R}$, let
\[
S^k_{\text{loc}}(1; W) := S^k_{\text{loc}}(1; W) = \left\{ (a(\cdot, m))_{m \in \mathbb{R}} \mid (m^{-k} a(\cdot, m)) \in S^0_{\text{loc}}(1; W) \right\}.
\]
Hence $a(\cdot, m) \in S^k_{\text{loc}}(1; W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in C^\infty_0(W)$, there exists $C_\alpha > 0$ independent of $m$, such that $|\partial^\alpha (\chi a(\cdot, m))| \leq C_\alpha m^k$ holds on $W$.

Consider a sequence $a_j \in S^k_{\text{loc}}(1)$, $j \in \mathbb{N}_0$, where $k_j \searrow -\infty$, and let $a \in S^k_{\text{loc}}(1)$. We say
\[
a(\cdot, m) \sim \sum_{j=0}^\infty a_j(\cdot, m) \quad \text{in} \quad S^k_{\text{loc}}(1),
\]
if, for every $\ell \in \mathbb{N}_0$, we have $a - \sum_{j=0}^\ell a_j \in S^{k_{\text{loc}}+1}_0(1)$. For a given sequence $a_j$ as above, we can always find such an asymptotic sum $a$, which is unique up to an element in $S^\infty_{\text{loc}}(1) = S^0_{\text{loc}}(1; W) := \cap_k S^k_{\text{loc}}(1)$.

Similarly, we can define $S^k_{\text{loc}}(1; Y, E)$ in the standard way, where $Y$ is a smooth manifold and $E$ is a vector bundle over $Y$.

2.3. **CR manifolds.** Let $(X, T^{1,0}X)$ be a compact, connected and orientable CR manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of $X$, that is, $T^{1,0}X$ is a subbundle of rank $n$ of the complexified tangent bundle $\mathbb{C}TX$, satisfying $T^{1,0}X \cap T^01X = \{0\}$, where $T^01X = \overline{T^{1,0}X}$, and $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$, where $\mathcal{V} = C^\infty(X, T^{1,0}X)$. There is a unique subbundle $HX$ of $TX$ such that $\mathcal{C}HX = T^{1,0}X \oplus T^01X$, i.e. $HX$ is the real part of $T^{1,0}X \oplus T^01X$. Let $J :HX \to HX$ be the complex structure map given by $J(u + i\mathcal{V}) = iu - i\mathcal{V}$, for every $u \in T^{1,0}X$. By complex linear extension of $J$ to $\mathbb{C}TX$, the i-eigenspace of $J$ is $T^{1,0}X = \{ V \in \mathcal{C}HX : JV = \sqrt{-1}V \}$. We shall also write $(X, JU, J)$ to denote a compact CR manifold.

We fix a real non-vanishing 1 form $\omega_0 \in C(X, T^*X)$ so that $\langle \omega_0(x), u \rangle = 0$, for every $u \in H_xX$, for every $x \in X$. For each $x \in X$, we define a quadratic form on $HX$ by
\[
\mathcal{L}_x(U, V) = \frac{1}{2} d\omega_0(JU, V), \forall U, V \in H_xX.
\]
We extend $\mathcal{L}$ to $\mathcal{C}HX$ by complex linear extension. Then, for $U, V \in T^{1,0}x_X$,
\[
\mathcal{L}_x(U, \overline{V}) = \frac{1}{2} d\omega_0(JU, \overline{V}) = -\frac{1}{2i} d\omega_0(U, \overline{V}).
\]
The Hermitian quadratic form $\mathcal{L}_x$ on $T^{1,0}X$ is called Levi form at $x$. Let $T \in C^\infty(X, TX)$ be the non-vanishing vector field determined by
\[
\omega_0(T) = -1,
\]
\[
d\omega_0(T, \cdot) = 0 \quad \text{on} \quad TX.
\]
Note that $X$ is a contact manifold with contact form $\omega_0$, contact plane $HX$ and $T$ is the Reeb vector field.

Fix a smooth Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ so that $T^{0,1}X$ is orthogonal to $T^{0,0}X$, $\langle u | v \rangle$ is real if $u, v$ are real tangent vectors, $\langle T | T \rangle = 1$ and $T$ is orthogonal to $T^{1,1}X \oplus T^{0,1}X$. For $u \in \mathbb{C}TX$, we write $|u|^2 := \langle u | u \rangle$. Denote by $T^{1,0}X$ and $T^{0,1}X$ the dual bundles $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with subbundles of the complexified cotangent bundle $\mathbb{C}T^*X$. Define the vector bundle of $(0, q)$-forms by $T^{0,q}X := \wedge^q T^{0,1}X$. The Hermitian metric $\langle \cdot | \cdot \rangle$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$ forms $T^{0,q}X, q = 0, 1, \cdots, n$. We shall also denote all these induced metrics by $\langle \cdot | \cdot \rangle$.

Note that we have the pointwise orthogonal decompositions:

$$\begin{align*}
\mathbb{C}T^*X &= T^{1,0}X \oplus T^{0,1}X \oplus \{ \lambda \omega_0 : \lambda \in \mathbb{C} \}, \\
\mathbb{C}TX &= T^{1,0}X \oplus T^{0,1}X \oplus \{ \lambda T : \lambda \in \mathbb{C} \}. 
\end{align*}$$

(2.6)

For $x, y \in X$, let $d(x, y)$ denote the distance between $x$ and $y$ induced by the Hermitian metric $\langle \cdot | \cdot \rangle$. Let $A$ be a subset of $X$. For every $x \in X$, let $d(x, A) := \inf \{ d(x, y) ; y \in A \}$.

Let $D$ be an open set of $X$. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{0,q}X$ over $D$ and let $\Omega_0^{0,q}(D)$ be the subspace of $\Omega^{0,q}(D)$ whose elements have compact support in $D$.

2.4. **Fourier analysis on compact Lie groups.** Let $\rho : G \to GL(\mathbb{C}^d)$ be a representation of $G$, where $d$ is the dimension of the representation $\rho$. Two representations $\rho_1$ and $\rho_2$ are equivalent if they have the same dimension and there is an invertible matrix $A$ such that $\rho_1(g) = A \rho_2(g) A^{-1}$ for all $g \in G$. Let

$$R = \{ R_1, R_2, \ldots \}$$

be the collection of all irreducible unitary representations of $G$, where each $R_k$ comes from exactly only one equivalent class. For each $R_k$, let $(R_{k,j,l})_{j,l=1}^{d_k}$ be its matrix, where $d_k$ is the dimension of $R_k$. Let $d\mu(g)$ be the probability Haar measure on $G$. Let $\langle \cdot | \cdot \rangle_G$ be the natural inner product on $C^\infty(G)$ induced by $d\mu(g)$. Let $L^2(G)$ be the completion of $C^\infty(G)$ with respect to $\langle \cdot | \cdot \rangle_G$. By the orthogonality relations for compact Lie groups and the Peter-Weyl theorem [24], we have

**Theorem 2.2.** The set $\{ \sqrt{d_k} R_{k,j,l} ; j, l = 1, \ldots, d_k, k = 1, 2, \ldots \}$ form an orthonormal basis of $L^2(G)$.

For a function $f \in C^\infty(G)$, the Fourier component of $f$ with respect to $\sqrt{d_k} R_{k,j,l}$ is

$$f_{k,j,l} := d_k R_{k,j,l}(g) \int_G f(h) \overline{R_{k,j,l}(h)} d\mu(h) \in C^\infty(G).$$

(2.7)

The smooth version of the Peter-Weyl theorem on compact Lie groups is the following [24]

**Theorem 2.3.** Let $f \in C^\infty(G)$. For every $t \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$, we have

$$\| f - \sum_{k=1}^{N} \sum_{j,l=1}^{d_k} f_{k,j,l} \|_{C^t(G)} \leq \varepsilon.$$

(2.8)
We put
\[ \chi_m(g) := \text{Tr} R_k(g) = \sum_{j=1}^{d_k} R_{k,j,j}(g). \]

**Definition 2.4.** The k-th Fourier component of \( u \in \Omega^{0,q}(X) \) is defined as
\[ u_k(x) = d_k \int_G (g^* u)(x) \chi_k(g) d\mu(g) \in \Omega^{0,q}(X). \]

We have the following theorem about Fourier components. For the readers’ convenience, we present the proof, see also [5].

**Theorem 2.5.** Let \( u \in \Omega^{0,q}(X) \). Then
\[ \lim_{N \to \infty} \sum_{k=1}^{N} u_k(x) = u(x), \forall x \in X, \]  
(2.9)
\[ \langle u_k(x) | u_t(x) \rangle = 0, \text{ if } k \neq t, \forall x \in X, \]  
(2.10)
\[ \sum_{k=1}^{N} \| u_k \|^2 \leq \| u \|^2, \forall N \in \mathbb{N}. \]  
(2.11)

**Proof.** We fix \( x \in X \) and consider a smooth function \( f : g \in G \to (g^* u)(x) \). Then
\[ f_{k,j,l}(g) = d_k R_{k,j,l}(g) \int_G (h^* u)(x) R_{k,j,l}(h) d\mu(h). \]  
(2.12)

By Theorem 2.3, for every \( \varepsilon > 0 \), there exists a \( N_0 \in \mathbb{N} \) such that for every \( N \geq N_0 \), we have
\[ \left| (g^* u)(x) - \sum_{k=1}^{N} \sum_{j,l} f_{k,j,l}(g) \right| \leq \varepsilon, \forall g \in G. \]  
(2.13)

Take \( g = e_0 \), where \( e_0 \) is the identity element of \( g \), we obtain that for every \( N \geq N_0 \),
\[ |u(x) - \sum_{k=1}^{N} \sum_{j,l} f_{k,j,l}(e_0) | \leq \varepsilon. \]  
(2.14)

Note that by (2.12),
\[ f_{k,j,l}(e_0) = d_k \delta_{j,l} \int_G (h^* u)(x) R_{k,j,l}(h) d\mu(h). \]

Then
\[ \sum_{k=1}^{N} \sum_{j,l} f_{k,j,l}(e_0) = \sum_{k=1}^{N} u_k(x). \]  
(2.15)

Hence (2.9) is true by (2.14) and (2.15).
By Theorem 2.2 and (2.12), we have

\[
\sum_{k=1}^{\infty} \sum_{j,l=1}^{d_k} \int_G |f_{k,j,l}(g)|^2 d\mu(g)
= \sum_{k=1}^{\infty} d_k \left( \int_G |(h^* u)(x) R_{k,j,l}(h)|^2 d\mu(h) \right)^2
= \int_G |(h^* u)(x)|^2 d\mu(h), \forall x \in X.
\] (2.16)

Since the metric on \( X \) is \( G \)-invariant, we have

\[
\langle p|q \rangle = \langle h^* p|h^* q \rangle, \forall p, q \in \Omega^{0,q}(X), \forall h \in G.
\]

Then for every \( k, t \in \mathbb{N} \),

\[
\langle u_k|u_t \rangle = \int_G \langle h^* u_k|h^* u_t \rangle d\mu(h).
\] (2.17)

For every \( h \in G \),

\[
h^* u_k = d_k \int_G (h^* g^* u)(x) \chi_k(g) d\mu(g)
= d_k \int_G ((g \circ h)^* u)(x) \chi_k(g) d\mu(g)
= d_k \int_G (g^* u)(x) \chi_k(g \circ h^{-1}) d\mu(g).
\] (2.18)

It is easy to see that

\[
\chi_k(g \circ h^{-1}) = \sum_{j=1}^{d_k} \sum_{l=1}^{d_k} R_{k,j,l}(g) R_{k,j,l}(h).
\] (2.19)

Hence

\[
h^* u_k = d_k \int_G (g^* u)(x) \left( \sum_{j=1}^{d_k} \sum_{l=1}^{d_k} R_{k,j,l}(g) R_{k,j,l}(h) \right) d\mu(g).
\] (2.20)

Similarly,

\[
h^* u_t = d_t \int_G (g^* u)(x) \left( \sum_{j=1}^{d_t} \sum_{l=1}^{d_t} R_{t,j,l}(g) R_{t,j,l}(h) \right) d\mu(g).
\] (2.21)

From Theorem 2.2, (2.20) and (2.21), we have

\[
\int_G \langle h^* u_k(x)|h^* u_t(x) \rangle d\mu(h) = 0, \forall k \neq t, \forall x \in X.
\] (2.22)

Then we deduce (2.10) from (2.17) and (2.22).

For \( k = t \), we have

\[
\int_G \langle h^* u_k(x)|h^* u_k(x) \rangle d\mu(h) = \sum_{j,l=1}^{d_k} \left| \int_G (g^* u)(x) R_{k,j,l}(g) d\mu(g) \right|^2.
\] (2.23)
With (2.16), we deduce that for every $N \in \mathbb{N}$ and every $x \in X$,
\[
\sum_{k=1}^{N} \int_{G} \langle h^* u_k(x) | h^* u_k(x) \rangle d\mu(h) \leq \int_{G} |(g^* u)(x)|^2 d\mu(g).
\] (2.24)

Then for every $N \in \mathbb{N}$,
\[
\sum_{k=1}^{N} \| u_k \|^2 = \int_{X} \left( \sum_{k=1}^{N} \int_{G} \langle h^* u_k(x) | h^* u_k(x) \rangle d\mu(h) \right) dv_X(x)
\leq \int_{X} \int_{G} |(g^* u)(x)|^2 d\mu(g) dv_X(x) = \| u \|^2.
\]

The proof is completed. \hfill \Box

We can also prove the following, see [5, Theorem 3.5].

**Theorem 2.6.** With the notations as above,
\[
\lim_{N \to \infty} \sum_{k=1}^{N} u_k(x) = u(x)
\] (2.25)
in $C^\infty$-topology.

Moreover, we have (see [5]),

**Proposition 2.7.** Let $u \in \Omega^{0,q}(X)$, then $u \in \Omega^{0,q}_k(X)$ if and only if $u = u_k$ on $X$.

### 3. G-equivariant Szegő kernel asymptotics

In this section, we establish asymptotic expansions of the $G$-equivariant Szegő kernels. We first review some known results for Szegő kernels mainly based on [9] and [13].

#### 3.1. Szegő kernel asymptotics

Fix a Hermitian metric $\langle \cdot | \cdot \rangle$ on $CT_X$ which induces a Hermitian metric on the bundles of $(0, q)$ forms $T^{*0,q} X$, $q = 0, 1, \ldots, n$. Let $D \subset X$ be an open set. Let $\Omega^{0,q}(D)$ denote the space of smooth sections of $T^{*0,q} X$ over $D$.

Let
\[
\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)
\] (3.1)
be the tangential Cauchy-Riemann operator. The natural global $L^2$ inner product $\langle \cdot | \cdot \rangle$ on $\Omega^{0,q}(X)$ induced by $dv(x)$ and $\langle \cdot | \cdot \rangle$ is given by
\[
\langle u | v \rangle := \int_{X} \langle u(x) | v(x) \rangle dv(x), \quad u, v \in \Omega^{0,q}(X).
\] (3.2)

We denote by $L^2_{(0,q)}(X)$ the completion of $\Omega^{0,q}(X)$ with respect to $\langle \cdot | \cdot \rangle$. Write $L^2(X) := L^2_{(0,0)}(X)$. We extend $\overline{\partial}_b$ to $L^2_{(0,r)}(X)$, $r = 0, 1, \ldots, n$, by
\[
\overline{\partial}_b : \text{Dom} \overline{\partial}_b \subset L^2_{(0,r)}(X) \to L^2_{(0,r+1)}(X),
\] (3.3)
where $\text{Dom} \overline{\partial}_b := \{ u \in L^2_{(0,r)}(X); \overline{\partial}_b u \in L^2_{(0,r+1)}(X) \}$ and, for any $u \in L^2_{(0,r)}(X)$, $\overline{\partial}_b u$ is defined in the sense of distributions. We also write

$$
\overline{\partial}_b^s : \text{Dom} \overline{\partial}_b^s \subset L^2_{(0,r+1)}(X) \to L^2_{(0,r)}(X)
$$

(3.4)

to denote the Hilbert adjoint of $\overline{\partial}_b$ in the $L^2$ space with respect to $(\cdot \mid \cdot)$. Let $\square^{(q)}_b$ denote the (Gaffney extension) of the Kohn Laplacian given by

$$
\text{Dom} \square^{(q)}_b = \left\{ s \in L^2_{(0,q)}(X); s \in \text{Dom} \overline{\partial}_b \cap \text{Dom} \overline{\partial}_b^s, \overline{\partial}_b s \in \text{Dom} \overline{\partial}_b, \overline{\partial}_b^s s \in \text{Dom} \overline{\partial}_b \right\},
$$

(3.5)

$$
\square^{(q)}_b s = \overline{\partial}_b \overline{\partial}_b^s s + \overline{\partial}_b \overline{\partial}_b^s s \text{ for } s \in \text{Dom} \square^{(q)}_b.
$$

By a result of Gaffney, for every $q = 0, 1, \ldots, n$, $\square^{(q)}_b$ is a positive self-adjoint operator (see [16 Proposition 3.1.2]). That is, $\square^{(q)}_b$ is self-adjoint and the spectrum of $\square^{(q)}_b$ is contained in $\mathbb{R}_+$, $q = 0, 1, \ldots, n$. Let

$$
S^{(q)} : L^2_{(0,q)}(X) \to \text{Ker} \square^{(q)}_b
$$

(3.6)

be the orthogonal projections with respect to the $L^2$ inner product $(\cdot \mid \cdot)$ and let

$$
S^{(q)}(x,y) \in D'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)
$$

(3.7)

denote the distribution kernel of $S^{(q)}$.

We recall Hörmander symbol spaces. Let $D \subset X$ be a local coordinate patch with local coordinates $x = (x_1, \ldots, x_{2n+1})$.

**Definition 3.1.** For $m \in \mathbb{R}$, $S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ is the space of all $a(x,y,t) \in C^\infty(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ such that, for all compact $K \subset D \times D$ and all $\alpha, \beta, \gamma \in \mathbb{N}_0$, $\gamma \in \mathbb{N}_0$, there is a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$
|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma a(x,y,t)| \leq C_{\alpha,\beta,\gamma}(1 + |t|)^{m-\gamma}, \ \forall (x,y,t) \in K \times \mathbb{R}_+, \ t \geq 1.
$$

Put

$$
S^{-\infty}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) := \bigcap_{m \in \mathbb{R}} S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*).
$$

Let $a_j \in S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$, $j = 0, 1, 2, \ldots$ with $m_j \to -\infty$, as $j \to \infty$. Then there exists $a \in S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ unique modulo $S^{-\infty}$, such that $a - \sum_{j=0}^{k-1} a_j \in S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$ for $k = 0, 1, 2, \ldots$.

If $a$ and $a_j$ have the properties above, we write $a \sim \sum_{j=0}^{\infty} a_j \in S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)$. We write

$$
S(x,y,t) \in S^m_{1,0}(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*)
$$

(3.8)
if \( s(x, y, t) \in \mathcal{S}_{1,0}^n(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \) and
\[
s(x, y, t) \sim \sum_{j=0}^{\infty} s_j^i(x, y)t^{m-j} \text{ in } \mathcal{S}_{1,0}^n(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),
\]
\[
s_j^i(x, y) \in C^\infty(D \times D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j \in \mathbb{N}_0.
\]

The following was proved in Theorem 4.8 in [13]

**Theorem 3.2.** Given \( q = 0, 1, 2, \ldots, n \). Assume that \( q \notin \{n_-, n_+\} \). Then, \( S^{(q)} = 0 \) on \( X \).

We have the following (see Theorem 1.2 in [9], Theorem 4.7 in [13] and see also [1] for \( q = 0 \))

**Theorem 3.3.** Let \( q = n_- \) or \( n_+ \). Suppose that \( \square_0^{(q)} \) has \( L^2 \) closed range. Then, \( S^{(q)}(x, y) \in C^\infty(X \times X \setminus \text{diag}(X \times X), T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \). Let \( D \subset X \) be any local coordinate patch with local coordinates \( x = (x_1, \ldots, x_{2n+1}) \). Then, there exist continuous operators \( S_-, S_+ : \Omega^{0,q}_0(D) \to D'(D, T^{*0,q}X) \) such that
\[
S^{(q)} = S_- + S_+ \quad \text{on } D,
\]
and \( S_-(x, y), S_+(x, y) \) satisfy
\[
S_-(x, y) \equiv \int_0^\infty e^{iy \varphi_-(x, y)t} s_-(x, y, t) dt \quad \text{on } D,
\]
\[
S_+(x, y) \equiv \int_0^\infty e^{iy \varphi_+(x, y)t} s_+(x, y, t) dt \quad \text{on } D,
\]

with
\[
s_-(x, y, t), s_+(x, y, t) \in \mathcal{S}_{1,0}^n(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),
\]
\[
s_-(x, y, t) = 0 \quad \text{if } q \neq n_-, \quad s_+(x, y, t) = 0 \quad \text{if } q \neq n_+,
\]
\[
s_-(x, y, t) \sim \sum_{j=0}^{\infty} s_j^i(x, y)t^{n-j} \text{ in } \mathcal{S}_{1,0}^n(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),
\]
\[
s_+(x, y, t) \sim \sum_{j=0}^{\infty} s_j^i(x, y)t^{n-j} \text{ in } \mathcal{S}_{1,0}^n(D \times D \times \mathbb{R}_+, T^{*0,q}X \boxtimes (T^{*0,q}X)^*),
\]
\[
s_j^i(x, y) \in C^\infty(D \times D, T^{*0,q}X \boxtimes (T^{*0,q}X)^*), \quad j = 0, 1, 2, 3, \ldots,
\]
\[
s^0_j(x, x) \neq 0, \quad \forall x \in D, \quad s^0_j(x, x) \neq 0, \quad \forall x \in D,
\]

and the phase functions \( \varphi_-, \varphi_+ \) satisfy
\[
\varphi_+(x, y), \varphi_- \in C^\infty(D \times D), \quad \text{Im } \varphi_-(x, y) \geq 0,
\]
\[
\varphi_-(x, x) = 0, \quad \varphi_-(x, y) \neq 0 \quad \text{if } x \neq y,
\]
\[
d_x \varphi_-(x, y) \big|_{x=y} = -\omega_0(x), \quad d_y \varphi_-(x, y) \big|_{x=y} = \omega_0(x),
\]
\[
\varphi_-(x, y) = -\overline{\varphi_+(y, x)},
\]
\[
-\overline{\varphi_+(x, y)} = \varphi_-(x, y).
\]
Remark 3.4. Note that for a strictly pseudoconvex CR manifold of dimension 3, \( \square^{(0)}_b \) does not have \( L^2 \) closed range in general (see [21]). Kohn [14] proved that if \( q = n_- = n_+ \) or \( |n_- - n_+| > 1 \) then \( \square^{(q)}_b \) has \( L^2 \) closed range.

3.2. \( G \)-equivariant Szegő kernel. Since \( G \) preserves \( J \) and \( \langle \cdot, \cdot \rangle \) is \( G \)-invariant, it is straightforward to see that for all \( g \in G \)

\[
g^* \overline{\partial}_b = \overline{\partial}_b g^* \quad \text{on } \Omega^{0,q}(X),
\]

\[
g^* \overline{\partial}_b^* = \overline{\partial}_b^* g^* \quad \text{on } \Omega^{0,q}(X),
\]

\[
g^* \square^{(q)}_b = \square^{(q)}_b g^* \quad \text{on } \Omega^{0,q}(X).
\]

(3.13)

Denote by \( \overline{\partial}_{b,k} \) (resp. \( \square^{(q)}_{b,k} \)) the restriction of \( \overline{\partial}_b \) (resp. \( \square^{(q)}_b \)) on \( \Omega^{0,q}(X) \). By Definition 2.4, we have

\[
\overline{\partial}_{b,k} : \Omega^{0,q}(X) \rightarrow \Omega^{0,q+1}(X),
\]

\[
\square^{(q)}_{b,k} : \Omega^{0,q}(X) \rightarrow \Omega^{0,q}(X).
\]

(3.14)

The \( G \)-equivariant Szegő projection is the orthogonal projection

\[
S^{(q)}_k : L^2(0,q)(X) \rightarrow \text{Ker } \square^{(q)}_b \bigcap L^2(0,q,k)(X)
\]

with respect to \( \langle \cdot, \cdot \rangle \). Let \( S^{(q)}_k(x,y) \in D'(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \) be its distribution kernel.

Lemma 3.5.

\[
S^{(q)}_k(x,y) = d_k \int_G S^{(q)}(g \circ x, y, \chi_k(g))d\mu(g).
\]

(3.15)

Proof. Let

\[
Q_k : L^2(0,q)(X) \rightarrow L^2(0,q,k)(X)
\]

\[
u \rightarrow u_k = d_k \int_G (g^* u)(x, y) \chi_k(g)d\mu(g).
\]

(3.16)

Then \( S^{(q)}_k = Q_k \circ S^{(q)} \). For \( u \in L^2(0,q)(X) \), we have

\[
S^{(q)}_k u = Q_k \circ S^{(q)} u
\]

\[
= Q_k \int_G S^{(q)}(x,y)u(y)dy
\]

\[
= d_k \int_G g^* \left( \int_G S^{(q)}(x,y)u(y)dy \right) \chi_k(g) d\mu(g)
\]

\[
= d_k \int_G \left( \int_G S^{(q)}(g \circ x, y)u(y)dy \right) \chi_k(g)d\mu(g)
\]

\[
= \int_G (d_k \int_G S^{(q)}(g \circ x, y) \chi_k(g) d\mu(g)) u(y)dy.
\]

Then the proof is completed. \( \square \)
Note that
\[ S_k^{(q)}(h \circ x, y) = d_k \int_G S^{(q)}(g \circ h \circ x, y) \chi_k(g) d\mu(g) \]
\[ = d_k \int_G S^{(q)}(g \circ x, y) \chi_k(g \circ h^{-1}) d\mu(g) \]
\[ = d_k \int_G S^{(q)}(g \circ x, y) \sum_{j=1}^{d_k} \sum_{l=1}^{d_k} R_{k,j,l}(g) R_{k,j,l}(h) d\mu(g). \] (3.18)

So \( S_k^{(q)}(x, y) \) is not \( G \)-invariant.

### 3.3. \( G \)-equivariant Szegő kernels near \( \mu^{-1}(0) \)

In this subsection, we will study \( G \)-equivariant Szegő kernel near \( \mu^{-1}(0) \). Let \( e_0 \in G \) be the identity element. Let \( v = (v_1, \ldots, v_d) \) be the local coordinates of \( G \) defined in a neighborhood \( V \) of \( e_0 \) with \( v(e_0) = (0, \ldots, 0) \). From now on, we will identify the element \( e \in V \) with \( v(e) \). We recall the following on group actions in local coordinates, see [10] Theorem 3.6.

**Theorem 3.6.** Let \( p \in \mu^{-1}(0) \). There exist local coordinates \( v = (v_1, \ldots, v_d) \) of \( G \) defined in a neighborhood \( V \) of \( e_0 \) with \( v(e_0) = (0, \ldots, 0) \), local coordinates \( x = (x_1, \ldots, x_{2n+1}) \) of \( X \) defined in a neighborhood \( U = U_1 \times U_2 \) of \( p \) with \( 0 \leftrightarrow p \), where \( U_1 \subset \mathbb{R}^d \) is an open set of 0 in \( \mathbb{R}^d \), \( U_2 \subset \mathbb{R}^{2n+1-d} \) is an open set of 0 in \( \mathbb{R}^{2n+1-d} \) and a smooth function \( \gamma = (\gamma_1, \ldots, \gamma_d) \in C^\infty(U_2, U_1) \) with \( \gamma(0) = 0 \in \mathbb{R}^d \) such that
\[
(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1})
\]
\[ = (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}), \]
\[
\forall(v_1, \ldots, v_d) \in V, \exists(x_{d+1}, \ldots, x_{2n+1}) \in U_2,
\]
\[
g = \text{span} \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right\},
\]
\[
\mu^{-1}(0) \cap U = \{x_{d+1} = \cdots = x_{2d} = 0\}, \] (3.20)

On \( \mu^{-1}(0) \cap U \), we have \( J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{d+j}} + a_j(x) \frac{\partial}{\partial x_{2n+1}}, \) \( j = 1, 2, \ldots, d, \)

where \( a_j(x) \) is a smooth function on \( \mu^{-1}(0) \cap U \), independent of \( x_1, \ldots, x_{2d}, x_{2n+1} \) and \( a_j(0) = 0, j = 1, \ldots, d, \)
\[
T_{p}^{1,0}X = \text{span} \{Z_1, \ldots, Z_n\},
\]
\[
Z_j = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{d+j}} \right)(p), \quad j = 1, \ldots, d,
\]
\[
Z_j = \frac{1}{2} \left( \frac{\partial}{\partial x_{2j-1}} - i \frac{\partial}{\partial x_{2j}} \right)(p), \quad j = d + 1, \ldots, n,
\]
\[
\langle Z_j | Z_l \rangle = \delta_{j,l}, \quad j, l = 1, 2, \ldots, n,
\]
\[
\mathcal{L}_p(Z_j, Z_l) = \mu_j \delta_{j,l}, \quad j, l = 1, 2, \ldots, n
\] (3.21)
and

\[ \omega_0(x) = (1 + O(|x|))dx_{2n+1} + \sum_{j=1}^{d} 4\mu_j x_{d+j} dx_j \]

\[ + \sum_{j=d+1}^{n} 2\mu_j x_{2j} dx_{2j-1} - \sum_{j=d+1}^{n} 2\mu_j x_{2j-1} dx_{2j} + \sum_{j=d+1}^{2n} b_j x_{2n+1} dx_j + O(|x|^2), \]

where \( b_{d+1} \in \mathbb{R}, \ldots, b_{2n} \in \mathbb{R} \).

The following describes the phase function in Theorem 3.3, see [10, Theorem 3.7].

**Theorem 3.7.** Let \( p \in \mu^{-1}(0) \) and take local coordinates \( x = (x_1, \ldots, x_{2n+1}) \) of \( X \) defined in an open set \( U \) of \( p \) with \( 0 \leftrightarrow p \) such that (3.20), (3.21) and (3.22) hold. Let \( \varphi_-(x, y) \in C^\infty(U \times U) \) be as in Theorem 3.3. Then,

\[ \varphi_-(x, y) = -x_{2n+1} + y_{2n+1} - 2 \sum_{j=1}^{d} \mu_j x_j x_{d+j} + 2 \sum_{j=1}^{d} \mu_j y_j y_{d+j} + i \sum_{j=1}^{n} |\mu_j| |z_j - w_j|^2 \]

\[ + \sum_{j=1}^{d} i\mu_j (\overline{x}_j w_j - \overline{z}_j \overline{w}_j) + \sum_{j=1}^{d} \left( -\frac{i}{2} b_{d+j} \right) (-z_j x_{2n+1} + w_j y_{2n+1}) \]

\[ + \sum_{j=d+1}^{n} \left( -\frac{i}{2} b_{d+j} \right) (-\overline{x}_j \overline{w}_j + \overline{z}_j \overline{w}_j) + \sum_{j=d+1}^{n} \left( \frac{1}{2} b_{2j-1} + i b_{2j} \right) (-z_j x_{2n+1} + w_j y_{2n+1}) \]

\[ + \sum_{j=d+1}^{n} \left( \frac{1}{2} b_{2j-1} - i b_{2j} \right) (-\overline{z}_j \overline{x}_j + \overline{z}_j \overline{x}_j) + (x_{2n+1} - y_{2n+1}) f(x, y) + O(|(x, y)|^3), \]

where \( z_j = x_j + ix_{d+j}, j = 1, \ldots, d; \ z_j = x_{2j-1} + ix_{2j}, j = 2d + 1, \ldots, 2n, \mu_j, j = 1, \ldots, n; \) and \( b_{d+1} \in \mathbb{R}, \ldots, b_{2n} \in \mathbb{R} \) are as in (3.22) and \( f \) is smooth and satisfies \( f(0,0) = 0, f(x,y) = f(y,x) \).

We now study \( S^q_k(x, y) \). From Theorem 3.2 and Lemma 3.5 we get

**Theorem 3.8.** Assume that \( q \notin \{n_-, n_+\} \). Then, \( S^q_k(x, y) \equiv 0 \) on \( X \).

Assume that \( q = n_- \) and \( \square^q_b \) has \( L^2 \) closed range. Fix \( p \in \mu^{-1}(0) \) and let \( v = (v_1, \ldots, v_d) \) and \( x = (x_1, \ldots, x_{2n+1}) \) be the local coordinates of \( G \) and \( X \) as in Theorem 3.6. Assume that \( d\mu = m(v) dv = m(v_1, \ldots, v_d) dv_1 \cdots dv_d \) on \( V \), where \( V \) is an open neighborhood of \( \epsilon_0 \in G \) as in Theorem 3.6. From Lemma 3.5, we have

\[ S^q_k(x, y) = d_k \int_G \chi(g) \overline{\chi_k(g) S^q_k(g \circ x, y) d\mu(g) + d_k \int_G (1 - \chi(g)) \overline{\chi_k(g) S^q_k(g \circ x, y) d\mu(g)}, \]

where \( \chi \in C_0^\infty(V), \chi = 1 \) near \( \epsilon_0 \).

Assume first \( G \) is globally free on \( \mu^{-1}(0) \), if \( U \) and \( V \) are small, there is a constant \( c > 0 \) such that

\[ d(g \circ x, y) \geq c, \ \forall x, y \in U, g \in \text{Supp}(1 - \chi), \]

\[ \text{(3.25)} \]
where $U$ is an open set of $p \in \mu^{-1}(0)$ as in Theorem 3.6. From now on, we take $U$ and $V$ small enough so that (3.25) holds. By Theorem 3.3, $S^{(q)}(x, y)$ is smoothing away from diagonal. From this observation and (3.25), we have

$$S_k^{(q)}(x, y) \equiv d_k \int_G \chi(g)\chi_k(g)S^{(q)}(g \circ x, y)d\mu(g) \equiv 0 \text{ on } U \text{ and hence }$$

$$(3.26)$$

From Theorem 3.3 and (3.26), we have

$$S_k^{(q)}(x, y) \equiv \hat{S}_{k,-}(x, y) + \hat{S}_{k,+}(x, y) \text{ on } U,$$

$$(3.27)$$

More precisely,

$$\hat{S}_{k,-}(x, y) = d_k \int_G \chi(g)\chi_k(g)S_-(g \circ x, y)d\mu(g),$$

$$\hat{S}_{k,+}(x, y) = d_k \int_G \chi(g)\chi_k(g)S_+(g \circ x, y)d\mu(g).$$

By using stationary phase formula of Melin-Sj¨ ostrand [20], it follows from the arguments in [10, Section 3.3] that

$$\hat{S}_{k,-}(x, y) \equiv d_k \int_0^\infty \int_V e^{i\phi_{k,-}(x,y)t} \chi(v)\chi_k(v)s_-(v \circ x, y, t)m(v)dvdt. \quad (3.28)$$

where $a_{k,-}(x, y, t) \sim \sum_{j=0}^{\infty} t^{n-\frac{d}{2}-j} a_{j, k,-}(x, y)$ in $S^{n-\frac{d}{2}}(U \times U \times R_+ \times T^{s0,q}X \otimes (T^{s0,q}X)^*)$, $a_{j, k,-}(x, y) \in C^\infty(U \times U, T^{s0,q}X \otimes (T^{s0,q}X)^*)$, $j = 0, 1, \ldots ,$

In this work, $G$ acts locally free on $\mu^{-1}(0)$ under Assumption [14]. Let $N_p = \{g \in G : g \circ p = p\} = \{g_1 = e_0, g_2, \ldots , g_r\}$. Similarly to (3.25), we can choose $U$ and $V$ to be small such that the subsets $\{g_j V\}_{\alpha = 1}^{r}$ are mutually disjoint and there is a constant $c > 0$ satisfying

$$d(h \circ x, y) \geq c, \quad \forall x, y \in U, h \in \operatorname{Supp}(1 - \sum_{\alpha = 1}^{r} \chi(g_\alpha^{-1})). \quad (3.30)$$

Then on $U$

$$S_k^{(q)}(x, y) \equiv d_k \sum_{\alpha = 1}^{r} \int_G \chi(g_\alpha^{-1})\chi_k(g)S^{(q)}(g \circ x, y)d\mu(g) = d_k \sum_{\alpha = 1}^{r} \int_G \chi(g)\chi_k(gg_\alpha)S^{(q)}(gg_\alpha \circ x, y)d\mu(g). \quad (3.31)$$

$$\hat{S}_{k,-}(x, y) \equiv d_k \sum_{\alpha = 1}^{r} \int_0^\infty \int_V e^{i\phi_{k,\alpha}(x,y)t} \chi(v)\chi_k(vv_\alpha)\chi_{k,-}(vv_\alpha \circ x, y, t)m(v)dvdt, \quad (3.32)$$
where \( v_\alpha \) is the coordinate of \( g_\alpha \). By using stationary phase formula of Melin-Sjöstrand [20], it follows from the above argument that

\[
\hat{S}_{k,-}(x,y) \equiv \sum_{\alpha=1}^r \int e^{i\Phi_{k,-}(v_\alpha \cdot x,y)} a_{k,\alpha,-}(x,y,t)dt \quad \text{on } U, \quad (3.33)
\]

where \( a_{k,\alpha,-}(x,y,t) \sim \sum_{j=0}^\infty t^{n-j} a_{k,\alpha,-}^j(x,y) \) in \( S_{1,0}^{n-\frac{d}{2}}(U \times U \times \mathbb{R}^+, T^{0,q}X \otimes (T^{0,q}X)^*) \),

\[
a_{k,\alpha,-}^j(x,y) \in C^\infty(U \times U, T^{0,q}X \otimes (T^{0,q}X)^*), \quad j = 0, 1, \ldots,
\]

In particular, it follows from the arguments in [10, Subsection 3.3] with minor modification, if \( G \) acts freely on \( \mu^{-1}(0) \), then for \( a_{k,-}^0(x,y) \) and \( a_{k,+}^0(x,y) \) in (3.34), we have

\[
a_{k,-}^0(x,x) = 2t^{d-1} \frac{d^2}{V_{\text{eff}}(x)} \pi^{-n-\frac{d}{2}} |\det R_x|^{-\frac{1}{2}} |\det L_x| |\tau_{x,n-}|, \quad \forall x \in \mu^{-1}(0)
\]

\[
a_{k,+}^0(x,x) = 2t^{d-1} \frac{d^2}{V_{\text{eff}}(x)} \pi^{-n-\frac{d}{2}} |\det R_x|^{-\frac{1}{2}} |\det L_x| |\tau_{x,n+}|, \quad \forall x \in \mu^{-1}(0).
\]

3.4. \( G \)-equivariant Szegö kernel asymptotics away \( \mu^{-1}(0) \). The goal of this section is to prove the following

**Theorem 3.9.** Let \( D \) be an open neighborhood of \( X \) with \( D \cap \mu^{-1}(0) = \emptyset \). Then,

\[
S_k^{(q)} \equiv 0 \quad \text{on } D.
\]

We first need

**Lemma 3.10.** Let \( p \notin \mu^{-1}(0) \). Then, there are open neighborhoods \( U \) of \( p \) and \( V \) of \( e \in G \) such that for any \( \chi \in C_0^\infty(V) \), we have for every \( k \),

\[
\int_G S_k^{(q)}(x, g \circ y) \chi(g) \overline{\chi_k(g)} d\mu(g) \equiv 0 \quad \text{on } U. \quad (3.34)
\]

The proof of the above lemma follows from [10, Lemma 3.14] by adding \( \chi_k(g) \).

**Lemma 3.11.** Let \( p \notin \mu^{-1}(0) \) and let \( h \in G \). We can find open neighborhoods \( U \) of \( p \) and \( V \) of \( h \) such that for every \( \chi \in C_0^\infty(V) \), we have for every \( k \),

\[
\int_G S_k^{(q)}(g \circ x, y) \chi(g) \overline{\chi_k(g)} d\mu(g) \equiv 0 \quad \text{on } U.
\]

**Proof.** Let \( U \) and \( V \) be open sets as in Lemma 3.10. Let \( \hat{V} = hV \). Then, \( \hat{V} \) is an open set of \( G \). Let \( \hat{\chi} \in C_0^\infty(\hat{V}) \). We have

\[
\int_G S_k^{(q)}(g \circ x, y) \hat{\chi}(g) \overline{\chi_k(g)} d\mu(g) = \int_G S_k^{(q)}(h \circ g \circ x, y) \hat{\chi}(h \circ g) \overline{\chi_k(h \circ g)} d\mu(g)
\]

\[
= \int_G S_k^{(q)}(h \circ g \circ x, y) \chi(g) d\mu(g), \quad (3.35)
\]
where \( \chi(g) := \bar{\chi}(h \circ g) \chi_k(h \circ g) \in C^\infty_0(V) \). From (3.35) and Lemma 3.10 we deduce that
\[
\int_G S^{(q)}(g \circ x, y) \overline{\chi_k(g)} d\mu(g) \equiv 0 \quad \text{on } U.
\]
The lemma follows.

Proof of Theorem 3.29

Fix \( p \in D \). We need to show that \( S^{(q)} \) is smoothing near \( p \). Let \( h \in G \). By Lemma 3.11 we can find open sets \( U_h \) of \( p \) and \( V_h \) of \( h \) such that for every \( \chi \in C^\infty_0(V_h) \), we have
\[
\int_G S^{(q)}(g \circ x, y) \chi_k(g) d\mu(g) \equiv 0 \quad \text{on } U_h.
\]
Since \( G \) is compact, we can find open sets \( U_{h_j} \) and \( V_{h_j} \), \( j = 1, \ldots, N \), such that \( G = \bigcup_{j=1}^N V_{h_j} \). Let \( U = D \cap \left( \bigcap_{j=1}^N U_{h_j} \right) \) and let \( \bar{\chi}_j \in C^\infty_0(V_{h_j}) \), \( j = 1, \ldots, N \), with \( \sum_{j=1}^N \bar{\chi}_j = 1 \) on \( G \). From (3.36), we have
\[
S^{(q)}(x, y) = d_k \int G S^{(q)}(g \circ x, y) \overline{\chi_k(g)} d\mu(g)
\]
\[
= d_k \sum_{j=1}^N \int G S^{(q)}(g \circ x, y) \overline{\chi_j(g)} d\mu(g) \equiv 0 \quad \text{on } U.
\]
The theorem follows.

From Section 3.3 and Section 3.4 we get Theorem 1.6.

4. \( G \)-equivariant Szegö kernel asymptotics on CR manifolds with \( S^1 \) action

Let \( X \) admit an \( S^1 \) action \( e^{i\theta} \): \( S^1 \times X \to X \). Let \( T \in C^\infty(X, TX) \) be the global real vector field induced by the \( S^1 \) action given by \( (Tu)(x) = \frac{\partial}{\partial \theta} (u(e^{i\theta} \circ x)) \big|_{\theta=0} \), \( u \in C^\infty(X) \).

Definition 4.1. The \( S^1 \) action \( e^{i\theta} \) is CR if \( [T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X) \) and the \( S^1 \) action is transversal if for each \( x \in X \), \( CT(x) \oplus T^{1,0}_x X \oplus T^{0,1}_x X = CT_x X \). Moreover, the \( S^1 \) action is locally free if \( T \neq 0 \) everywhere.

Note that transversality implies local freeness. Let \((X, T^{1,0}X)\) be a compact connected CR manifold with a transversal CR \( S^1 \) action \( e^{i\theta} \) and \( T \) be the global vector field induced by the \( S^1 \) action. Let \( \omega_0 \in C^\infty(X, T^*X) \) be the global real one form determined by \( \langle \omega_0, u \rangle = 0 \), for every \( u \in T^{1,0}X \oplus T^{0,1}X \), and \( \langle \omega_0, T \rangle = -1 \). Note that \( \omega_0 \) and \( T \) satisfy (2.5). Recall that we work with Assumption 1.8

Assume that the Hermitian metric \( \langle \cdot | \cdot \rangle \) on \( CTX \) is \( G \times S^1 \) invariant. Then the \( L^2 \) inner product \( \langle \cdot | \cdot \rangle \) on \( \Omega^{0,q}(X) \) induced by \( \langle \cdot | \cdot \rangle \) is \( G \times S^1 \)-invariant. We then have
\[
Tg^* \overline{\bar{\omega}}_b = g^* T \overline{\omega}_b = \overline{\bar{\omega}}_b g^* T = \overline{\bar{\omega}}_b T g^* \quad \text{on } \Omega^{0,q}(X), \quad \forall g \in G,
\]
\[
Tg^* \Box_b^{(q)} = g^* T \Box_b^{(q)} = \Box_b^{(q)} g^* T = \Box_b^{(q)} T g^* \quad \text{on } \Omega^{0,q}(X), \quad \forall g \in G.
\]
Let \( L^2_{(0,q),m}(X) \) be the completion of \( \Omega^0_{m,q}(X) \) with respect to \( (\cdot, \cdot) \). We write \( L^2_{m}(X)_k := L^2_{(0,0),m}(X)_k \). Put
\[
H^q_{b,m}(X)_k := (\text{Ker} \square^{(q)}_b) \cap L^2_{(0,q),m}(X)_k.
\]
The \( m \)-th \( G \)-equivariant Szegö projection is the orthogonal projection
\[
S^{(q)}_{k,m} : L^2_{(0,q)}(X) \to (\text{Ker} \square^{(q)}_b) \cap L^2_{(0,q),m}(X)_k
\]
with respect to \( (\cdot, \cdot) \). Let \( S^{(q)}_{k,m}(x, y) \in C^\infty(X \times X, T^{*0,q}X \boxtimes (T^{*0,q}X)^*) \) be the distribution kernel of \( S^{(q)}_{k,m} \). Then
\[
S^{(q)}_{k,m}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S^{(q)}_k(x, e^{i\theta} \circ y) e^{im\theta} d\theta. \quad (4.1)
\]
The goal of this section is to study the asymptotics of \( S^{(q)}_{k,m} \) as \( m \to +\infty \).

From Theorem 3.3 and by using integration by parts several times, we get

**Theorem 4.2.** Let \( D \subset X \) be an open set with \( D \cap \mu^{-1}(0) = \emptyset \). Then,
\[
S^{(q)}_{k,m} = O(m^{-\infty}) \quad \text{on} \quad D.
\]

We now study \( S^{(q)}_{k,m} \) near \( \mu^{-1}(0) \). We can repeat the proof of Theorem 3.6 with minor change and get

**Theorem 4.3.** Let \( p \in \mu^{-1}(0) \). There exist local coordinates \( v = (v_1, \ldots, v_d) \) of \( G \) defined in a neighborhood \( V \) of \( e_0 \) with \( v(e_0) = (0, \ldots, 0) \), local coordinates \( x = (x_1, \ldots, x_{2n+1}) \) of \( X \) defined in a neighborhood \( U = U_1 \times (\hat{U}_2 \times [-2\delta, 2\delta]) \) of \( p \) with \( 0 \leftrightarrow p \), where \( U_1 \subset \mathbb{R}^d \) is an open set of \( 0 \in \mathbb{R}^d \), \( \hat{U}_2 \subset \mathbb{R}^{2n-d} \) is an open set of \( 0 \in \mathbb{R}^{2n-d} \), \( \delta > 0 \), and a smooth function \( \gamma = (\gamma_1, \ldots, \gamma_d) \in C^\infty(\hat{U}_2 \times [-2\delta, 2\delta], U_1) \) with \( \gamma(0) = 0 \in \mathbb{R}^d \) such that
\[
(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}) = (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n+1}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n+1}), x_{d+1}, \ldots, x_{2n+1}), \quad (4.2)
\]
\forall (v_1, \ldots, v_d) \in V, \quad \forall (x_{d+1}, \ldots, x_{2n+1}) \in \hat{U}_2 \times [-2\delta, 2\delta],
\]
\[
T = -\frac{\partial}{\partial x_{2n+1}},
\]
\[
\mathfrak{g} = \text{span} \left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right\}, \quad (4.3)
\]
\[
\mu^{-1}(0) \cap U = \{x_{d+1} = \cdots = x_{2d} = 0\},
\]
On \( \mu^{-1}(0) \cap U \), we have \( J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{d+j}} + a_j(x)\frac{\partial}{\partial x_{2n+1}} \), \( j = 1, 2, \ldots, d, \)
where \( a_j(x) \) is a smooth function on \( \mu^{-1}(0) \cap U \), independent of \( x_1, \ldots, x_{2d}, x_{2n} \) and \( a_j(0) = 0 \), \( j = 1, \ldots, d \),

\[
T_p^{1, 0} X = \text{span} \{Z_1, \ldots, Z_n\},
\]

\[
Z_j = \frac{1}{2}(\partial_{x_j} - i \partial_{x_{d+j}})(p), \quad j = 1, \ldots, d,
\]

\[
Z_j = \frac{1}{2}(\partial_{x_{2j-1}} - i \partial_{x_{2j}})(p), \quad j = d + 1, \ldots, n,
\]

\[
\langle Z_j, Z_k \rangle = \delta_{j, k}, \quad j, k = 1, 2, \ldots, n,
\]

\[
E_p(Z_j, Z_k) = \mu_j \delta_{j, k}, \quad j, k = 1, 2, \ldots, n
\]

and

\[
\omega_0(x) = (1 + O(|x|))dx_{2n+1} + \sum_{j=1}^d 4\mu_j x_{d+j}dx_j
\]

\[
+ \sum_{j=d+1}^n 2\mu_j x_{2j}dx_{2j-1} - \sum_{j=d+1}^n 2\mu_j x_{2j-1}dx_{2j} + O(|x|^2).
\]

From Theorem 3.8 we get

**Theorem 4.4.** Assume that \( q \notin \{n_-, n_+\} \). Then, \( S_{k, n}^{(q)} = O(m^{-\infty}) \) on \( X \).

**Proof of Theorem 4.4.** It suffices to show the cases when \( q = n_- \) and \( q = n_+ \neq n_- \). Assume that \( q = n_- \). It is well-known \[13\] Theorem 1.12 that when \( X \) admits a transversal \( S^1 \) action, then \( \Box_b^{(q)} \) has \( L^2 \) closed range. Fix \( p \in \mu^{-1}(0) \). Let \( N_p = \{ g \in G : g \circ p = p \} = \{ g_1 = e_0, g_2, \ldots, g_r \} \). Let \( v = (v_1, \ldots, v_d) \) and \( x = (x_1, \ldots, x_{2n+1}) \) be the local coordinates of \( G \) and \( X \) as in Theorem 4.3 and let \( U \) and \( V \) be open sets as in Theorem 4.3. We take \( U \) small enough so that there is a constant \( c > 0 \) such that

\[
d(e^{i\theta} \circ g \circ x, y) \geq c, \quad \forall (x, y) \in U \times U, \quad \forall g \in G, \theta \in [-\pi, -\delta] \bigcup [\delta, \pi],
\]

where \( \delta > 0 \) is as in Theorem 4.3. We repeat the same procedure in \[11\] Section 4] as follows.

\[
S_{k, n}^{(q)}(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_k^{(q)}(x, e^{i\theta} \circ y)e^{im\theta}d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n+1} + imy_{2n+1}} S_k^{(q)}(\hat{x}, e^{i\theta} \circ \hat{y})e^{im\theta}d\theta
\]

\[
= I + II,
\]

\[
I = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n+1} + imy_{2n+1}} \chi(\theta)S_k^{(q)}(\hat{x}, e^{i\theta} \circ \hat{y})e^{im\theta}d\theta,
\]

\[
II = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx_{2n+1} + imy_{2n+1}} (1 - \chi(\theta))S_k^{(q)}(\hat{x}, e^{i\theta} \circ \hat{y})e^{im\theta}d\theta,
\]

where \( \hat{x} = (x_1, \ldots, x_{2n}, 0) \in U \), \( \hat{y} = (y_1, \ldots, y_{2n}, 0) \in U \), \( \chi \in C_0^\infty([-2\delta, 2\delta]) \), \( \chi = 1 \) on \([-\delta, \delta]\). It is easy to check that

\[
II = O(m^{-\infty}).
\]
For $I$, we have

$$I = I_0 + I_1,$$

$$I_0 = \frac{1}{2\pi} \sum_{\alpha = 1}^{r} \int_{0}^{\infty} \int_{-\pi}^{\pi} e^{-imx_{2n+1} + imy_{2n+1}} \chi(\theta) e^{i(-\theta + \hat{\Phi}_{k,+}(g_{\alpha} \circ \hat{x}, \hat{y})) t + im \theta} \, dt \, d\theta,$$

$$I_1 = \frac{1}{2\pi} \sum_{\alpha = 1}^{r} \int_{0}^{\infty} \int_{-\pi}^{\pi} e^{-imx_{2n+1} + imy_{2n+1}} \chi(\theta) e^{i(\theta - \hat{\Phi}_{k,+}(g_{\alpha} \circ \hat{x}, \hat{y})) t + im \theta} a_{k,\alpha,-}(\hat{x}, (\hat{y}, -\theta), t) \, dt \, d\theta. \tag{4.9}$$

From $\frac{\partial}{\partial \theta} \left( i(\theta + \hat{\Phi}_{k,+}(g_{\alpha} \circ \hat{x}, \hat{y})) t + im \theta \right) \neq 0$, we can integrate by parts with respect to $\theta$ several times and deduce that

$$I_1 = O(m^{-\infty}). \tag{4.10}$$

For $I_0$, we have

$$I_0 = \frac{1}{2\pi} \sum_{\alpha = 1}^{r} \int_{0}^{\infty} \int_{-\pi}^{\pi} e^{-imx_{2n+1} + imy_{2n+1}} \chi(\theta) e^{im(\theta t + \hat{\Phi}_{k,-}(g_{\alpha} \circ \hat{x}, \hat{y})) t + \theta} a_{k,\alpha,0}(\hat{x}, (\hat{y}, -\theta), mt) \, dt \, d\theta. \tag{4.11}$$

We apply the complex stationary phase formula of Melin-Sjöstrand \cite{20}, Theorem 2.3 to carry the $dt \, d\theta$ integration in (4.11). The calculation is similar as in the proof of Theorem 3.17 in [12]. Then

$$I_0 \equiv \sum_{\alpha = 1}^{r} e^{im\Psi_{k}(g_{\alpha} \circ \hat{x}, \hat{y}, x, y, m)} b_{k,\alpha}(x, y, m),$$

$$\Psi_{k}(x, y) = \hat{\Phi}_{k,-}(\hat{x}, \hat{y}) - x_{2n+1} + y_{2n+1},$$

$$b_{k,\alpha}(x, y, m) \in S_{\text{loc}}^{n-\frac{d}{2}}(1; U \times U, T^{*0,q} X \boxtimes (T^{*0,q} X)^*),$$

$$b_{k,\alpha}(x, y, m) \sim \sum_{j=0}^{\infty} m^{\frac{n-d}{2} - j} b^j_{k,\alpha}(x, y) \in S_{\text{loc}}^{n-\frac{d}{2}}(1; U \times U, T^{*0,q} X \boxtimes (T^{*0,q} X)^*),$$

$$b^j_{k,\alpha}(x, y) \in C^\infty(U \times U, T^{*0,q} X \boxtimes (T^{*0,q} X)^*), \quad j = 0, 1, 2, \ldots, \tag{4.12}$$

Assume that $q = n_+ \neq n_-$. If $m \to -\infty$, then the expansion for $S_{k,m}^{(q)}(x, y)$ as $m \to -\infty$ is similar to $q = n_-$ case. When $m \to +\infty$, we can repeat the method above with minor change and deduce that $S_{k,m}^{(q)}(x, y) = O(m^{-\infty})$ on $X$. In particular, it follows from the argument in [10] Section 4] with minor modification, if $G \times S^1$ acts freely near $\mu^{-1}(0)$, then

$$b^0_{k}(x, x) = 2^{d-1} \frac{d^2}{\text{Ve}_{\text{eff}}(x)} \pi^{-n-1+\frac{d}{2}} |\det R_x|^{-\frac{1}{2}} |\det \mathcal{L}_x|_{\tau_{x,n_-}} \chi(\text{eff}) \in \mu^{-1}(0), \quad \forall x \in \mu^{-1}(0),$$

where $\tau_{x,n_-}$ is given by (1.10). The proof is completed. \hfill $\square$

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