On constructions and parameters of symmetric configurations $v_k$

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Abstract: The spectrum of possible parameters of symmetric configurations is investigated. We both survey known constructions and results, and propose some new construction methods. Many new parameters are obtained, in particular for cyclic symmetric configurations, which are equivalent to deficient cyclic difference sets. Both Golomb rulers and modular Golomb rulers are a key tool in our investigation. Several new upper bounds on the minimum integer $E(k)$ such that for each $v \geq E(k)$ there exists a symmetric configuration $v_k$ are obtained. Upper bounds of the same type are provided for cyclic symmetric configurations. From the standpoint of applications, it should be noted that our results extend the range of possible parameters of LDPC codes, generalized LDPC codes, and quasi-cyclic LDPC codes.

Keywords: configurations in Combinatorics; symmetric configurations; cyclic configurations; Golomb rulers; modular Golomb rulers

1 Introduction

Configurations are interesting combinatorial structures. They were defined in 1876. For an introduction to the problems connected with the configurations and their history, see [28, 29, 30] and the references therein.

Definition 1.1. [29]

(i) A configuration $(v, b, k)$ is an incidence structure of $v$ points and $b$ lines such that each line contains $k$ points, each point lies on $r$ lines, and two distinct points are connected by at most one line.
(ii) If \( v = b \) and, hence, \( r = k \), the configuration is symmetric, and it is referred to as a configuration \( v_k \).

(iii) The deficiency \( d \) of a configuration \((v_r, b_k)\) is the value \( d = v - r(k - 1) - 1 \).

A symmetric configuration \( v_k \) is cyclic if there exists a permutation of the set of its points mappings blocks to blocks, and acting regularly on both points and blocks. Equivalently, \( v_k \) is cyclic if one of its incidence matrix is circulant.

Steiner systems are configurations with \( d = 0 \) [29]. The deficiency of a symmetric configuration \( v_k \) is \( d = v - (k^2 - k + 1) \). The deficiency of \( v_k \) is zero if and only if \( v_k \) is a finite projective plane of order \( k - 1 \). In general, \( d \) indicates the number of points not joined with an arbitrary point or the number of lines parallel to an arbitrary line, see [2, 19, 24, 29].

A configuration \((v_r, b_k)\) can be treated also as a \( k \)-uniform \( r \)-regular linear hypergraph with \( v \) vertices and \( b \) hyperedges [27, 29]. Connections of configurations \((v_r, b_k)\) with numerical semigroups are noted in [10]. Some analogies between configurations \((v_r, b_k)\), regular graphs, and molecule models of chemical elements are remarked in [25]. As an example of a practical applying configurations (symmetric and nonsymmetric) we mention also the problem of user privacy for using database, see [15, 50] and the references therein.

Denote by \( M(v, k) \) an incidence matrix of a symmetric configuration \( v_k \). Any matrix \( M(v, k) \) is a \( v \times v \) 01-matrix with \( k \) units in every row and column; moreover, the \( 2 \times 2 \) matrix \( J_2 \) consisting of all units is not a submatrix of \( M(v, k) \). Therefore, \( M(v, k) \) is a \( J_2 \)-free matrix. Two incidence matrices of the same configuration may differ by a permutation on the rows and the columns.

A matrix \( M(v, k) \) can also be considered as a biadjacency matrix of a \( k \)-regular bipartite graph without multiple edges. The biadjacency matrix describes connections of two vertex subsets of the graph so that the adjacency matrix has the form

\[
\begin{bmatrix}
0_v & M(v, k) \\
M^{tr}(v, k) & 0_v
\end{bmatrix}
\]

where \( tr \) stands for transposition, and \( 0_v \) denotes the zero \( v \times v \) matrix. This graph is the Levi graph of the configuration \( v_k \) [29, Sec. 7.2]. As \( M(v, k) \) is a \( J_2 \)-free, the graph has girth at least six, i.e. it does not contain 4-cycles. Such graphs are useful for the construction of bipartite-graph codes that can be treated as low-density parity-check (LDPC) codes or generalized LDPC codes [3]–[6],[12, 14, 20, 35, 41]. If \( M(v, k) \) is circulant, then the corresponding LDPC code is quasi-cyclic; it can be encoded with the help of shift-registers with relatively small complexity, see [12, 14, 20, 35] and the references therein.

Matrices \( M(v, k) \) consisting of square circulant submatrices have a number of useful properties, e.g. they are more suitable for LDPC codes implementation. We say that a 01-matrix \( A \) is block double-circulant (BDC for short) if \( A \) consists of square circulant blocks whose weights give rise to a circulant matrix (see Definition 3.1 for details). A configuration \( v_k \) with a BDC incidence matrix \( M(v, k) \) is called a BDC symmetric configuration. Symmetric and non-symmetric configurations with incidence matrices consisting of square circulant blocks are considered, e.g. in [12]–[14],[43].
Cyclic configurations are considered, for instance, in [12]–[14], [18, 24, 38, 40]. A standard method to construct cyclic configurations (or, equivalently, circulant matrices $M_{v,k}$) is based on Golomb rulers [16, 18, 22, 24, 45]–[47].

**Definition 1.2.** [45, 18]

(i) A **Golomb ruler** $G_k$ of order $k$ is an ordered set of $k$ integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j \mid 1 \leq j < i \leq k\}$ are distinct. The **length** $L_{\text{G}}(k)$ of the ruler $G_k$ is equal to $a_k - a_1$.

(ii) A Golomb ruler $G_k$ is an **optimal Golomb ruler** $OG_k$ if no shorter Golomb ruler of the same order $k$ exists. Let $L_{\text{OG}}(k)$ and $L_{\text{G}}(k)$ be the length of an optimal ruler $OG_k$ and of the shortest known Golomb ruler $G_k$, respectively.

(iii) A $(v, k)$ **modular Golomb ruler** is an ordered set of $k$ integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all the differences $\{a_i - a_j \mid 1 \leq i, j \leq k; i \neq j\}$ are distinct and nonzero modulo $v$.

Clearly, $L_{\text{G}}(k) \geq L_{\text{OG}}(k)$ holds.

For any value $\delta \geq 0$, Golomb rulers $(a_1, a_2, \ldots, a_k)$ and $(a_1 + \delta, a_2 + \delta, \ldots, a_k + \delta)$ have the same properties. Usually, $a_1 = 0$ is assumed.

**Remark 1.3.** A $(v, k)$ modular Golomb ruler is also called a **deficient cyclic difference set** with deficiency $d = v - (k^2 - k + 1)$. For a deficient cyclic difference set the deficiency $d$ is the number of elements in $\mathbb{Z}_v \setminus \{0\}$ not represented by any difference $a_i - a_j$ [18]. Note that the expression “deficient cyclic difference set” is used in [18], whereas in [38] and [40] the expressions “difference set modulo $v$” and “deficient difference set in $\mathbb{Z}_v$” are adopted.

**Remark 1.4.** Golomb rulers and modular Golomb rulers are deeply connected with difference triangle sets and difference packings, see e.g. [37, 45, 48]. In particular, according to the notation of [45], a Golomb ruler $G_k$ is a difference triangle set $(1, k - 1)$-DΔS, whereas a $(v, k)$ modular Golomb ruler is a difference packing 1-DP$(v, k)$ [45, Prop. 19.9, Rem. 19.24]. If $a_1 = 0$ the corresponding object is said to be normalized. Note also that in [51], the expression “planar cyclic difference packing modulo $v$” is used for an object equivalent to a $(v, k)$ modular Golomb ruler.

In [16] it is proved that

$$L_{\text{OG}}(k) > k^2 - 2k\sqrt{k} + \sqrt{k} - 2.$$  

Currently, the optimal lengths $L_{\text{OG}}(k)$ are known only for orders $k \leq 25$ [16, 22, 45, 46]. So, for $k \leq 25$ we have $L_{\overline{G}}(k) = L_{\text{OG}}(k)$. The proof of the optimality of a Golomb ruler is a hard problem needing exhaustive computer search. On the other hand, for sufficiently large orders $k$, relatively short Golomb rulers are constructed and are available online, see e.g. the internet resources [16, 22, 46, 47] and the references therein. For $k \leq 150$, the
order of magnitude of the lengths $L_G(k)$ of the shortest known Golomb rulers is $ck^2$ with $c \in [0.7, 0.9]$, see [16, 18, 22, 24, 29, 45, 46]. Moreover,

$$L_{OG}(k) \leq L_G(k) < k^2$$

see [16]. Other constructions for large $k$ can be found in [17].

We say that a $0,1$-vector $u = (u_0, u_1, \ldots, u_{v-1})$ corresponds to a (modular) Golomb ruler if the increasing sequence of integers $j \in \{0, 1, \ldots, v-1\}$ such that $u_j = 1$ form a (modular) Golomb ruler.

Recall that weight of a circulant $0,1$-matrix is the number of units in each its row.

**Theorem 1.5.** [24, Sec. 4]

(i) Any Golomb ruler $G_k$ of length $L_G(k)$ is a $(v, k)$ modular Golomb ruler for all $v$ such that $v \geq 2L_G(k) + 1$.

(ii) A circulant $v \times v$ $0,1$-matrix of weight $k$ is an incidence matrix $M(v, k)$ of a cyclic symmetric configuration $v_k$ if and only if the first row of the matrix corresponds to a $(v, k)$ modular Golomb ruler.

We remark that (ii) of Theorem 1.5 is not explicitly stated in [24]. However, the assertion can be easily deduced from the results in [24].

**Corollary 1.6.** [24, Sec. 4] For all $v$ such that

$$v \geq 2L_G(k) + 1,$$  \hspace{1cm} (1.1)

there exists a cyclic symmetric configuration $v_k$.

We call the value $G(k) = 2L_G(k) + 1$ the Golomb bound.

It is well known that $v \geq k^2 - k + 1$ holds for configurations $v_k$, and that the lower bound is attained if and only if there exists a projective plane of the order $k-1$ [24, 29]. We call $P(k) = k^2 - k + 1$ the projective plane bound.

Let $v_\delta(k)$ be the smallest possible value of $v$ for which a $(v, k)$ modular Golomb ruler (or, equivalently, a cyclic symmetric configuration) exists.

Finally, we introduce other two bounds. The existence bound $E(k)$ is the integer such that for any $v \geq E(k)$, there exists a symmetric configuration $v_k$. Similarly, the cyclic existence bound $E_c(k)$ is the integer such that for any $v \geq E_c(k)$, there exists a cyclic $v_k$.

Clearly, for a fixed $k$, we have

$$k^2 - k + 1 = P(k) \leq E(k) \leq E_c(k) \leq G(k) = 2L_G(k) + 1. \hspace{1cm} (1.2)$$

The aim of this work is threefold:

- to survey the vast body of literature on constructions and parameters of symmetric configurations $v_k$;
• to describe new construction methods, paying special attention to constructions producing circulant and block double-circulant incidence matrices $M(v, k)$;

• to investigate the spectrum of possible parameters of symmetric configurations $v_k$ (with special attention to parameters of cyclic symmetric configurations) in the interval

$$k^2 - k + 1 = P(k) \leq v < G(k) = 2L_G(k) + 1. \quad (1.4)$$

Our main achievements are new constructions of BDC incidence matrices (see Theorems 3.5 and 3.11, together the examples in Section 3), improvements on the known upper bounds on $E(k)$ and $E_c(k)$, and several new parameters for cyclic and non-cyclic configurations $v_k$, see Sections 5 and 6.

From the stand point of applications, including Coding Theory, it is sometimes useful to have different matrices $M(v, k)$ for the same $v$ and $k$. This is why we attentively consider various constructions, even when they provide configurations with the same parameters.

The Extension Construction, as introduced in [3, 4], plays a key role for investigation of the spectrum of possible parameters of symmetric configurations $v_k$, $k \geq 11$, as it provides intervals of values of $v$ for a fixed $k$. To be successfully applied, the Extension Construction needs a convenient starting incidence matrix. Block double-circulant matrices turn out to be particularly useful in this context, see Corollary 4.3. In this work we use both the original starting matrices of [3, 4] and some new ones obtained by our new constructions, see Example 4.4.

We remark that new cyclic configurations provide new modular Golomb rulers, i.e. new deficient cyclic difference sets. Note also that methods considered in this work could be also used to construct non-symmetric configurations $(v, r, b_k)$.

The paper is organized as follows. In Section 2, the known constructions and parameters of configurations $v_k$ are considered. In particular, a geometrical construction from [12, 14], and the Extension Construction from [3, 4] are described. In Section 3, some new constructions of block double-circulant incidence matrices $M(v, k)$ are proposed. In Section 4, methods for constructing matrices admitting extensions are described. In Sections 5 and 6, our results on the spectra of parameters of cyclic and non-cyclic configurations are reported. An Appendix contains the proof of one of theorems from Section 3.

Some results of this work were published without proofs in [12, 13].

### 2 Some known constructions and parameters of configurations $v_k$ with $P(k) \leq v < G(k)$

The aim of this section is to provide a list of pairs $(v, k)$ for which a (cyclic) symmetric configuration $v_k$ is known to exist, see Equations (2.1)-(2.13). In most cases a brief description of the corresponding configuration is given. Infinite families of configuration $v_k$ given in this section are considered in [1–4, 8, 12, 21, 38, 40, 14–19, 24–29]; see also the references therein.
Throughout the paper, \( q \) is a prime power and \( p \) is a prime. Let \( \mathbb{F}_q \) be Galois field of \( q \) elements. Let \( \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\} \). Let \( 0_u \) be the zero \( u \times u \) matrix. Denote by \( P_u \) a permutation matrix of order \( u \).

First we recall that several pairs \((v, k-\delta)\) can be actually obtained from a given \( v_k \).

**Theorem 2.1.** [40] If a (cyclic) configuration \( v_k \) exists, then for each \( \delta \) with \( 0 \leq \delta < k \) there exists a family of (cyclic) configurations \( v_{k-\delta} \) as well.

At once we note that a cyclic configuration \( v_k \) gives a family of cyclic configurations \( v_{k-\delta} \) obtained by distmissng \( \delta \) units in the 1-st row of its incidence matrix. For the general case, Theorem 2.1 is based on the fact that an incidence ma trix \( M(v, k) \) can be represented as a sum of \( k \) permutations \( v \times v \) matrices (in many ways). This fact follows from the results of Steinits (1894) and König (1914), see e.g. [27, Sec. 5.2] and [30, Sec. 2.5].

The value \( \delta \) appearing in Equations (2.1)-(2.13) is connected with Theorem 2.1. W hen a reference is given, it usually refer to the case \( \delta = 0 \).

The families giving rise to pairs (2.1)–(2.3) below are obtained from \((v, k)\) modular Golomb rulers [16, Ch. 5],[17],[24, Sec. 5],[45, Sec. 19.3], see Theorem 1.5(ii).

\[
\begin{align*}
\text{cyclic } v_k & : \quad v = q^2 + q + 1, \quad k = q + 1 - \delta, \quad q + 1 > \delta \geq 0; \\
cyclic v_k & : \quad v = q^2 - 1, \quad k = q - \delta, \quad q > \delta \geq 0; \\
cyclic v_k & : \quad v = p^2 - p, \quad k = p - 1 - \delta, \quad p - 1 > \delta \geq 0.
\end{align*}
\]

The configurations giving rise to (2.1) use the incidence matrix of the cyclic projective plane \( P G(2, q) \) [49],[16, Sec. 5.5],[17],[45, Th. 19.15]. The family with parameters (2.2) is obtained from the cyclic starred affine plane \( A G(2, q) \) [8],[16, Sec. 5.6],[17, 18],[45, Th. 19.17], see also [14, Ex. 5] and [19] where the configurations are called anti-flags. We recall that the starred plane \( A G(2, q) \) is the affine plane without the origin and the lines through the origin. Finally, the configurations with parameters (2.3) follow from Ruzsa’s construction [44],[16, Sec. 5.4],[17],[45, Th. 19.19].

The non-cyclic families with parameters (2.4) and (2.5) are given in [1, Constructions (i),(ii), p. 126] and [21, Constructions 3.2.3.3, Rem. 3.5], see also the references therein and [3],[4, Sec. 3],[14, Sec. 7.3].

\[
\begin{align*}
\begin{align*}
\text{cyclic } v_k & : \quad v = q^2 - qs, \quad k = q - s - \delta, \quad q > s \geq 0, \quad q - s > \delta \geq 0; \\
\text{cyclic } v_k & : \quad v = q^2 - (q - 1)s - 1, \quad k = q - s - \delta, \quad q > s \geq 0, \quad q - s > \delta \geq 0.
\end{align*}
\end{align*}
\]

In the projective plane \( P G(2, q) \) we fix a line \( \ell \) and a point \( P \) and assign an integer \( s \geq 0 \). If \( P \in \ell \) we choose \( s \) points on \( \ell \) distinct from \( P \), and \( s \) lines through \( P \) distinct from \( \ell \). If \( P \notin \ell \) we choose \( s \) arbitrary points on \( \ell \) and consider the \( s \) lines connecting \( P \) with these points. The incidence structure obtained from \( P G(2, q) \) by dismissing all the lines through the \( s+1 \) selected points and all the points lying on the \( s+1 \) selected lines provides the family of (2.4) if \( P \in \ell \) and the family with parameters (2.5) if \( P \notin \ell \). For \( s = 0 \), the construction of (2.5) is given in [40]. In [3],[4, Eqs (3.2),(3.3)], the family with parameters
(2.4) is described by using a block structure of the incidence matrix of the affine plane \( AG(2,q) \), see the Extension Construction below. Configurations \((q^2)_q\) and \((q^2 - 1)_q\) are mentioned in many papers, see e.g. [19],[24, Sec. 5].

For \( q \) a square, in [1, Conjec. 4.4, Rem. 4.5, Ex. 4.6] and [21, Construction 3.7, Th. 3.8], families of non-cyclic configuration \( v_k \) with parameters (2.6) are provided; see also [14, Ex. 8]. The configurations with parameters (2.7) belong to these families; here, \( c = q - \sqrt{q} \). Configurations with parameters (2.7) are also described in [12, Ex. 2(ii)] and [19].

\[
v_k : v = c(q + \sqrt{q} + 1), k = \sqrt{q} + c - \delta, c = 2, 3, \ldots, q - \sqrt{q}, \delta \geq 0, q \text{ square}; \quad (2.6)
\]

\[
v_k : v = q^2 - \sqrt{q}, k = q - \delta, q > \delta \geq 0, q \text{ square.} \quad (2.7)
\]

In [1, 12, 14, 19, 21], the partition \( PG(2,q) \) into Baer subplanes for \( q \) a square is used; see also Example 3.9(ii) of the present work.

In [19, Th. 1.1], a family of non-cyclic with parameters

\[
v_k : v = 2p^2, \quad k = p + s - \delta, \quad 0 < s \leq q + 1, \quad q^2 + q + 1 \leq p, \quad p + s > \delta \geq 0 \quad (2.8)
\]

is given. In [14, Sec. 6], based on the cyclic starred affine plane, a construction of non-cyclic configuration with parameters

\[
v_k : v = c(q - 1), \quad k = c - \delta, \quad c = 2, 3, \ldots, b, \quad b = q \text{ if } \delta \geq 1,
\]

\[
b = \left\lceil \frac{q}{2} \right\rceil \text{ if } \delta = 0, \quad c > \delta \geq 0, \quad (2.9)
\]

is provided.

In [12, Sec. 2],[14, Sec. 3], the following geometrical construction which uses point orbits under the action of a collineation group is described.

**Construction A.** Take any point orbit \( \mathcal{P} \) under the action of a collineation group in an affine or projective space of order \( q \). Choose an integer \( k \leq q + 1 \) such that the set \( \mathcal{L}(\mathcal{P}, k) \) of lines meeting \( \mathcal{P} \) in precisely \( k \) points is not empty. Define the following incidence structure: the points are the points of \( \mathcal{P} \), the lines are the lines of \( \mathcal{L}(\mathcal{P}, k) \), the incidence is that of the ambient space.

**Theorem 2.2.** In Construction A the number of lines of \( \mathcal{L}(\mathcal{P}, k) \) through a point of \( \mathcal{P} \) is a constant \( r_k \). The incidence structure is a configuration \((v_{r_k}, b_k)\) with \( v = |\mathcal{P}|, b = |\mathcal{L}(\mathcal{P}, k)| \).

By Definition 1.1, if \( r_k = k \) Construction A produces a symmetric configuration \( v_k \).

It is noted in [12, 14] that Construction A works for any 2-(\( v, k, 1 \)) design \( D \) and for any group of automorphisms of \( D \). The role of \( q + 1 \) is played by the size of any block in \( D \).

Families of non-cyclic configuration \( v_k \) obtained by Construction A with the following parameters are given in [14, Exs 2, 3].

\[
v_k : v = \frac{q(q - 1)}{2}, \quad k = \frac{q + 1}{2} - \delta, \quad \frac{q + 1}{2} > \delta \geq 0, \quad q \text{ odd.} \quad (2.10)
\]

\[
v_k : v = \frac{q(q + 1)}{2}, \quad k = \frac{q - 1}{2} - \delta, \quad \frac{q - 1}{2} > \delta \geq 0, \quad q \text{ odd.} \quad (2.11)
\]

\[
v_k : v = q^2 + q - q\sqrt{q}, \quad k = q - \sqrt{q}, \quad q - \sqrt{q} > \delta \geq 0, \quad q \text{ square.} \quad (2.12)
\]
In [3], a construction method for non-cyclic configuration \( v_k \) with parameters (2.13) is proposed, and called “Construction \( \theta \)-extension”. This construction is also considered in [4], where it is called CE-construction (“Cancellation+Enlargement”). The terminology we use here is “Extension Construction”.

\[
v_k : v = q^2 - qs + \theta, \quad k = q - s - \Delta, \quad q > s \geq 0, \quad q - s > \Delta \geq 0, \quad \theta = 0, 1, \ldots, q - s + 1. \quad (2.13)
\]

We first describe the Extension Construction in geometrical terms. Let \( v_k \) be a configuration \((P, L)\) with incidence matrix \( M(v, k) \). Assume that there exists a set of \( k - 1 \) pairwise disjoint lines of \( v_k \), say \( \ell_1, \ell_2, \ldots, \ell_{k-1} \), and a set of \( k - 1 \) pairwise non-collinear points, say \( P_1, P_2, \ldots, P_{k-1} \), with the property that each \( P_i \) belongs to precisely one \( \ell_{\pi(i)} \). Here \( \pi \) denotes a permutation of the indexes \( 1, 2, \ldots, k - 1 \). We define a new incidence structure \((P', L')\) as follows:

1. \( P' = P \cup \{P_{\text{new}}\} \); 
2. \( L' = L \cup \{\ell_{\text{new}}\} \); 
3. the lines incident with \( P_{\text{new}} \) are \( \ell_1, \ldots, \ell_{k-1} \) and \( \ell_{\text{new}} \); 
4. the points incident with \( \ell_{\text{new}} \) are \( P_1, \ldots, P_{k-1} \) and \( P_{\text{new}} \); 
5. \( P_i \) is not incident with \( \ell_{\pi(i)} \); 
6. for a point \( P \in P \) and a line \( \ell \in L \) we have that \( P \) is incident with \( \ell \) if and only if \( P \in \ell \) in \( v_k \), with the only \( k - 1 \) exceptions of \( P = P_i \) and \( \ell = \ell_{\pi(i)} \), \( i = 1, \ldots, k - 1 \).

It is easy to check that \((P', L')\) is a configuration \((v + 1)_k\).

It is interesting to note that this procedure can be viewed as a generalization of a classical construction by V. Martinetti for configurations \( v_3 \), going back to 1887 [39]. According to Martinetti’s construction (quoted, e.g. in [7, 9, 25],[30, Sec. 2.4, Fig. 2.4.1]) two parallel lines \( a, b \) and two noncollinear points \( A, B \) are chosen so that \( A \in a, B \in b \). Then a line \( c \) and a point \( C \) are added. The points \( A, B \) are removed from the lines \( a \) and \( b \) and are included into the new line \( c \). The new point \( C \) is included into all lines \( a, b, \) and \( c \).

Below we provide a description of the Extension Construction, as given in [3, 4].

**Definition 2.3.** [3, 4] Let \( M(v, k) \) be an incidence matrix of a symmetric configuration \( v_k \).

In \( M(v, k) \), we consider an aggregate \( \mathcal{A} \) of \( k - 1 \) rows corresponding to pairwise disjoint lines of \( v_k \) and \( k - 1 \) columns corresponding to pairwise non-collinear points of \( v_k \). The \((k - 1) \times (k - 1)\) submatrix \( C(\mathcal{A}) \) formed by the intersection of the rows and columns of \( \mathcal{A} \) is called a **critical submatrix** of \( \mathcal{A} \). The aggregate \( \mathcal{A} \) is called an **extending aggregate** (or \( E\)-aggregate) if its critical submatrix \( C(\mathcal{A}) \) is a permutation matrix \( P_{k-1} \). The matrix \( M(v, k) \) **admits an extension** if it contains at least one \( E\)-aggregate. The matrix \( M(v, k) \) **admits \( \theta \) extensions** if it contains \( \theta \) \( E\)-aggregates that do not intersect each other. We also will say that a configuration \( v_k \) **admits an extension** or **admits \( \theta \) extensions** if its incidence matrix does.
**Procedure E (Extension Procedure).** Let $M(v, k) = [m_{ij}]$ be an incidence matrix of a symmetric configuration $v_k = (\mathcal{P}, \mathcal{L})$. Assume that $M(v, k)$ admits an extension.

1. To the matrix $M(v, k)$, add a new row from below and a new column to the right. Denote the new $(v + 1) \times (v + 1)$ matrix by $B = [b_{ij}]$, and let $b_{v+1,v+1} = 1$ whereas $b_{v+1,1} = \ldots = b_{v+1,v} = 0$, $b_{1,v+1} = \ldots = b_{v,v+1} = 0$.

2. One of $E$-aggregates of $M(v, k)$, say $A$, is chosen. In the matrix $B$, we “clone” all $k-1$ units of the critical submatrix $C(A)$ writing their “projections” to the new row and column. Finally, the units cloned are changed by zeroes. In other words, let the aggregate $A$ consist of rows with indexes $i_u$, $u = 1, 2, \ldots, k - 1$, and columns with indexes $j_d$, $d = 1, 2, \ldots, k - 1$. Then the units of $C(A)$ are as follows: $m_{i_u,j_d(u)} = 1$, $u = 1, 2, \ldots, k - 1$, for some permutation $\pi$ of the indexes $1, \ldots, k - 1$. Then $B$ arising from Step 1 is changed as follows: $b_{i_u,v+1} = 1$, $b_{v+1,j_d} = 1$, $b_{i_u,j_d(u)} = 0$, $u = 1, 2, \ldots, k - 1$, $d = 1, 2, \ldots, k - 1$.

It is easily seen that $B$ is an incidence matrix for $(\mathcal{P}', \mathcal{L}')$. Therefore, the following result can be easily proved.

**Theorem 2.4.** [3, 4] Let $M(v, k)$ be an incidence matrix of a symmetric configuration $v_k$. Assume that $M(v, k)$ admits $\theta$ extensions, for some $\theta \geq 1$.

(i) $\theta$ repeated applications of Procedure E to $M(v, k)$ gives an incidence matrix $M(v+\theta, k)$ of a symmetric configuration $(v+\theta)_k$.

(ii) If $\theta \geq k - 1$, then any $k - 1$ new rows and $k - 1$ new columns obtained as a result of repeated application of Procedure E form an $E$-aggregate.

In [3, 4] the Extension Construction is applied to affine planes and provides configuration with parameters as in (2.13). Let $(x_1, x_2)$ denote coordinates for the affine plane $AG(2, q)$. The incidence structure of $AG(2, q)$ is a resolvable $2-(q^2, q, 1)$-design with $q + 1$ resolution classes. Each class contains $q$ parallel lines. The $q$ classes are lines with equation $x_2 = wx_1 + u$ where $w \in F_q$ is a constant for the given class and $u$ runs over $F_q$. One more class contains $q$ lines $x_1 = c$. This class is removed from $AG(2, q)$ in order to obtain a symmetric configuration $(q^2)_q$, whose incidence matrix $M(q^2, q)$ can be represented as a superposition of $q^2$ permutation matrices $P_q$. Each block row contains one resolution class. Each block column corresponds to $q$ points $(d, x_2)$ where $d$ is a constant for the given block column and $x_2$ runs over $F_q$. Then from $M(q^2, q)$ one removes $s$ block rows and columns. An incidence matrix $M(q^2 - qs, q - s)$ is obtained. It is a superposition of $(q - s)^2$ matrices $P_q$. Further, a $(q - s) \times (q - s)$ $0$-$1$-matrix $S_\Delta$ with $\Delta$ units in every row and column is taken. In $M(q^2 - qs, q - s)$, submatrices $P_q$ marked by units of $S_\Delta$ are changed by $0_q$. An incidence matrix $M(q^2 - qs, q - s - \Delta)$ is obtained; it admits $\theta \leq q - s$ extensions. When Procedure E is executed by $q - s$ times, Procedure E can be applied once more, according to Theorem 2.4(ii).
Some known results on existence and non-existence of sporadic symmetric configurations will be mentioned in Sections 5 and 6.

We end this section by remarking that cyclic symmetric configurations can be constructing from Sidon sets. Sidon sets are combinatorial objects equivalent to Golomb rulers.

**Definition 2.5.** [16, 42] A *Sidon k-set* (respectively, *($v, k$) modular Sidon set*) is an ordered set of $k$ integers $(a_1, a_2, \ldots, a_k)$ such that $0 \leq a_1 < a_2 < \ldots < a_k$ and all pairwise sums $\{a_i + a_j \mid 1 \leq i \leq j \leq k\}$ are different (respectively, different modulo $v$).

Sidon sets are called also *Sidon sequences*, or *$B_2$ sequence*; see [16, 42] and the references therein for more details and terminology. It should be noted that in Sidon sets we consider sums $a_i + a_j$ of not necessarily distinct elements.

The relation between Sidon sets and Golomb rulers is described in the following well-known result (for a proof see e.g. [16, Ch. 4]).

**Theorem 2.6.** A *Sidon k-set* (respectively, *($v, k$) modular Sidon set*) is a *Golomb ruler of order k* (respectively, *($v, k$) modular Golomb ruler*), and conversely.

The smallest possible value of $v$ for which a *($v, k$) modular Sidon set* exists coincides with $v_\delta(k)$. This makes our notation consistent with [23]. General bounds on $v_\delta(k)$ and precise results for smaller $k$’s can be found in [23, 31, 47, 51].

## 3 Constructions of block double-circulant incidence matrices $M(v, k)$

The aim of this section is the construction of BDC incidence matrices of cyclic symmetric configurations. A method based on the Golomb ruler associated to a cyclic symmetric configuration is described in Subsection 3.2: splitting a starting modular Golomb ruler a number of quotient Golomb rulers forming a needed BDC matrix are obtained. The same method can be described in terms of the action of the automorphism group of the configuration, see Subsection 3.3. We provide two different descriptions because when dealing with a given configuration usually either one approach or the other can be more conveniently used. For example, ideas of Subsection 3.2 work better if the Golomb ruler associated to a configuration is described explicitly as a list of integers. The approach of Subsection 3.3 is useful for instance when the configuration arises from geometrical objects such as cyclic projective and affine planes. Sometimes both the approaches can be conveniently used, cf. Examples 3.7 and 3.12(i).

Throughout this section,

$$(a_1, a_2, \ldots, a_k)$$

is a *($v, k$) modular Golomb ruler*, with $v = td$ for integers $t, d$. (3.1)
3.1 BDC matrices $M(v, k)$ and families of symmetric configurations

Definition 3.1. Let $v = td$. A $v \times v$ matrix $A$ is said to be a block double-circulant matrix (or BDC matrix) if

$$A = \begin{bmatrix}
C_{0,0} & C_{0,1} & \cdots & C_{0,t-1} \\
C_{1,0} & C_{1,1} & \cdots & C_{1,t-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_{t-1,0} & C_{t-1,1} & \cdots & C_{t-1,t-1}
\end{bmatrix}, \quad (3.2)
$$

where $C_{i,j}$ is a circulant $d \times d$ 0,1-matrix for all $i, j$, and submatrices $C_{i,j}$ and $C_{l,m}$ with $j - i \equiv m - l \mod t$ have equal weights. The matrix

$$W(A) = \begin{bmatrix}
w_0 & w_1 & w_2 & w_3 & \cdots & w_{t-2} & w_{t-1} \\
w_{t-1} & w_0 & w_1 & w_2 & \cdots & w_{t-3} & w_{t-2} \\
w_{t-2} & w_{t-1} & w_0 & w_1 & \cdots & w_{t-4} & w_{t-3} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
w_1 & w_2 & w_3 & w_4 & \cdots & w_{t-1} & w_0
\end{bmatrix}, \quad (3.3)
$$

is a circulant $t \times t$ matrix whose entry in position $i, j$ is the weight of $C_{i,j}$. $W(A)$ is called the weight matrix of $A$. The vector $W(A) = (w_0, w_1, \ldots, w_{t-1})$ is called the weight vector of $A$.

We present some simple techniques for obtaining BDC matrices of symmetric configurations from a given BDC $v \times v$ matrix $A$ of (3.2) with weight matrix $W(A)$ of (3.3). We assume that $v = td$.

(i) For $h \in \{0, 1, \ldots, t-1\}$, dismiss $\delta_h \geq 0$ units in each row of every submatrix $C_{i,j}$ with $j - i \equiv h \mod t$, in such a way that the the obtained submatrix is still circulant. A BDC matrix $A'$ is then obtained; it consists of circulant matrices $C'_{i,j}$ of weight $w'_h = w_h - \delta_h$, where $j - i \equiv h \mod t$ and $h = 0, 1, \ldots, t-1$. It is an incidence BDC matrix of a configuration $v'_k$, with $W(A') = (w'_0, w'_1, \ldots, w'_{t-1})$,

$$v' = v, \quad k' = k - \sum_{h=0}^{t-1} \delta_h, \quad 0 \leq \delta_h \leq w_h, \quad w'_h = w_h - \delta_h. \quad (3.4)$$

(ii) Fix some non-negative integer $j \leq t - 1$. Let $m$ be such that $w_m \leq w_h$ for all $h \neq j$. Cyclically shift all block rows of $A$ to the left by $j$ block positions. A matrix $A^*$ with $W(A^*) = (w'_0 = w_j, w'_1 = w_{j+1}, \ldots, w'_r = w_{u+j} \mod t, \ldots, w'_{r-1} = w_{j-1})$ is obtained. By applying (i), construct a matrix $A^{**}$ with $w^{**}_0 = w^*_0 = w_j, w^{**}_h = w_m, h \geq 1$. Now remove from $A^{**}$ $t - c$ block rows and columns from the bottom and the right. In this way a $cd \times cd$ BDC matrix $A'$ is obtained, with $W(A') = (w_j, w_m, \ldots, w_m)$. It is an incidence matrix of a configuration $v'_k$, with

$$v' = cd, \quad k' = w_j + (c - 1)w_m, \quad c = 1, 2, \ldots, t. \quad (3.5)$$
(iii) Let $t$ be even. Let $A^*$ be as in (ii). Let $w_{od}, w_{ev}$ be weights such that $w_{od} \leq w_h^*$ for odd $h = 1, 3, \ldots, t - 1$, and $w_{ev} \leq w_h^*$ for even $h = 2, 4, \ldots, t - 2$. By applying (i), construct a matrix $A^{**}$ with $w_0^{**} = w_0^* = w_j^*$, $w_1^{**} = w_{od}$ for odd $h$, $w_h^{**} = w_{ev}$ for even $h \geq 2$. From $A^{**}$ remove $t - 2f$ block rows and columns from the bottom and the right. A $2fd \times 2fd$ BDC matrix $A'$ with $\overline{W}(A') = (w_j, w_{od}, w_{ev}, w_{od}, \ldots, w_{ev}, w_{od})^{f-1}$ is obtained. It is an incidence matrix of a configuration $v_{k'}$ with

\[ v' = 2fd, \quad k' = w_j + w_{od} + (f - 1)(w_{ev} + w_{od}), \quad f = 1, 2, \ldots, t/2. \tag{3.6} \]

Other methods for obtaining families of symmetric configurations from $A$ of (3.2) can be found in [14, Sec. 4].

### 3.2 Using permutations of the set of integers \(\{0, 1, \ldots, v - 1\}\)

In this subsection we show a method to obtain BDC matrices from any \((v, k)\) modular Golomb ruler with $v$ a composite integer (see Theorem 3.5 below). A key tool is the notion of quotient modular Golomb ruler, as introduced in [37] and [48, p. 3]; it should be noted that quotient rulers are used in [37, 48] with a different goal, that is, in order to obtain difference triangle sets. We will construct a permutation $\sigma$ of the set of indexes of points (and lines) of the cyclic configuration $v_h$ associated to the original modular Golomb ruler, such that the incidence matrix $A_{\sigma}$ of $v_h$ corresponding to $\sigma$ (cf. Definition 3.3) is a BDC matrix whose blocks correspond to the quotients of the original ruler.

We now sketch the construction of quotient rulers, as given in [37, 48]. For the ruler (3.1), and for any $h = 0, 1, \ldots, t - 1$, let

\[ B_h = \left\{ \frac{a_i - h}{t} : a_i \equiv h \pmod{t} \right\}, \quad w_h = |B_h|. \tag{3.7} \]

Clearly, $\sum_{h=0}^{t-1} w_h = k$ holds.

**Theorem 3.2.** [37, 48] For every $h = 0, \ldots, t - 1$, $B_h$ of (3.7) is a \((d, w_h)\) modular Golomb ruler.

**Definition 3.3.** Let \((a_1, a_2, \ldots, a_k)\) be a \((v, k)\) modular Golomb ruler. For each $u = 0, 1, \ldots, v - 1$, let

\[ L_u = \{ a_1 + u \pmod{v}, a_2 + u \pmod{v}, \ldots, a_k + u \pmod{v} \}. \tag{3.8} \]

For a permutation $\sigma$ of the set \(\{0, 1, \ldots, v - 1\}\), a $v \times v$ 01-matrix $A_{\sigma}$ is defined as follows. Let $i, j \in \{0, 1, \ldots, v - 1\}$. The element in position \((i, j)\) of $A_{\sigma}$ is 1 if and only if $\sigma(j) \in L_{\sigma(i)}$ (or, equivalently, if and only if $\sigma(j) - \sigma(i) \pmod{v} \in L_0$).

**Lemma 3.4.** For every choice of $\sigma$, the matrix $A_{\sigma}$ of Definition 3.3 is a $J_2$-free incidence matrix $M(v, k)$ of a symmetric configuration $v_k$. 

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Proof. By Theorem 1.5(ii), the matrix $A_{id}$ is the incidence matrix of a cyclic symmetric configuration $v_k$. It is easily seen that $A_{i}$ is a different incidence matrix of the same $v_k$ (points and lines are rearranged according to $\sigma$).

\[ \sigma(tad + b) = bt + a \text{ for } 0 \leq a \leq t - 1, \ 0 \leq b \leq d - 1. \quad (3.9) \]

Let $B_h$ and $w_h$ be as in (3.7), and $A_{\sigma_t}$ be as in Definition 3.3. Also, let $M_0, M_1, \ldots, M_{t-1}$ be the $d \times d$ blocks of $A_{\sigma_t}$, such that the first $d$ rows of $A_{\sigma_t}$ are a block row $[M_0M_1\ldots M_{t-1}]$. Finally, let $T_1, T_2, \ldots, T_{t-1}$ be the $d \times d$ blocks of $A_{\sigma_t}$, such that the first $d$ columns of $A_{\sigma_t}$ are a block column $[M_0T_{t-1}\ldots T_2T_1]^t$. Then

(i) Each matrix $M_h$ is a circulant $d \times d$ 01-matrix of weight $w_h$. The first row of $M_h$ corresponds to the $(d, w_h)$ modular Golomb ruler $B_h$.

(ii) Each matrix $T_h$ is a circulant $d \times d$ 01-matrix of weight $w_h$ obtained from $M_h$ by a cyclic shift of rows to the right by one position.

(iii) The matrix $A_{\sigma_t}$ is a block double-circulant incidence matrix $M(v, k)$ of a symmetric configuration $v_k$ with the following structure:

\[
A_{\sigma_t} = \begin{bmatrix}
M_0 & M_1 & M_2 & \ldots & M_{t-2} & M_{t-1} \\
T_{t-1} & M_0 & M_1 & \ldots & M_{t-3} & M_{t-2} \\
T_{t-2} & T_{t-1} & M_0 & \ldots & M_{t-4} & M_{t-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
T_2 & T_3 & T_4 & \ldots & M_0 & M_1 \\
T_1 & T_2 & T_3 & \ldots & T_{t-1} & M_0
\end{bmatrix}. \quad (3.10)
\]

Proof. (i) Let $i, j \in \{0, 1, \ldots, d - 1\}$, $h = 0, 1, \ldots, t - 1$. The position $(i, j)$ in $M_h$ is the position $(i, hd + j)$ in $A_{\sigma_t}$. The value 1 appears in this position if and only if $\sigma_t(hd + j) \in L_{\sigma_t(i)}$. By (3.9) and (3.8), $\sigma_t(hd + j) = jt + h + it$, $\sigma_t(i) = it$, and $L_{\sigma_t(i)} = \{a_1 + it \pmod{v}, \ldots, a_k + it \pmod{v}\}$. So, $\sigma_t(hd + j) \in L_{\sigma_t(i)}$ if and only if $jt \in \{a_1 - h + it \pmod{v}, \ldots, a_k - h + it \pmod{v}\}$. \hfill (3.11)

If $a_u \not\equiv h \pmod{t}$ then $t \not\mid (a_u - h + it)$ and $jt = a_u - h + it \pmod{v}$ cannot occur. Therefore, the condition (3.11) is equivalent to

\[
j \in \left\{ \frac{a_u - h}{t} + i \pmod{d} \mid a_u \equiv h \pmod{t} \right\}, \quad (3.12)
\]

which proves the assertion.
(ii) Let \( i, j \in \{0, \ldots, d-1\}, h = 1, 2, \ldots, t-1 \). The position \((i, j)\) in \(T_h\) is the position \(((t-h)d+i, j)\) in \(A_{\sigma_t}\). The value 1 appears in this position if and only if \(\sigma_t(j)\) belongs to \(L_{\sigma_t((t-h)d+i)}\). Since \(\sigma_t((t-h)d+i) = it + t - h\), we have
\[
L_{\sigma_t((t-h)d+i)} = L_{it+t-h} = \{a_1 + it + t - h \pmod{v}, \ldots, a_k + it + t - h \pmod{v}\}.
\]
Then \(\sigma_t(j) = jt\) belongs to \(L_{\sigma_t((t-h)d+i)}\) if and only if
\[
jt \in \{a_1 + it + t - h \pmod{v}, \ldots, a_k + it + t - h \pmod{v}\}.
\]
Arguing as in (i), we obtain that this condition is equivalent to
\[
j \in \left\{ \frac{a_u - h}{t} + i + 1 \pmod{d} | a_u \equiv h \pmod{t} \right\},
\]
which proves the assertion.

(iii) We need to show that for every pair \((i, j)\), \(i, j = 0, 1, \ldots, v - d - 1\), the value in position \((i, j)\) in \(A_{\sigma_t}\) is equal to that in position \((i + d, j + d)\). The value in position \((i, j)\) is equal to 1 if and only if \(\sigma_t(j) \in L_{\sigma_t(i)}\). Write \(i = i_1d + i_2, j = j_1d + j_2\), with \(0 \leq i_1, j_1 \leq t - 2, 0 \leq i_2, j_2 \leq d - 1\). Then \(\sigma_t(j) = j_2t + j_1\) and \(\sigma_t(i) = i_2t + i_1\), and hence the value in position \((i, j)\) is 1 if and only if \(j_2t + j_1 - i_2t - i_1 \pmod{v} \in L_0\). Note that \(i + d = (i_1 + 1)d + i_2\) and \(j + d = (j_1 + 1)d + j_2\). Then the value in position \((i + d, j + d)\) is 1 if and only if \(j_2t + (j_1 + 1) - i_2t - (i_1 + 1) \pmod{v} \in L_0\). Since \((j_1 + 1) - (i_1 + 1) = j_1 - i_1 \pmod{v}\) holds, the assertion is proven.

Example 3.6. Let \(p\) be a prime. Let \(g\) be a primitive element of \(F_p\). The following Ruzsa’s sequence \([44],[16, Sec. 5.4],[45, Th. 19.19]\) forms a \((p^2 - p, p - 1)\) modular Golomb ruler:
\[
e_u = pu + (p - 1)g^u \pmod{p^2 - p}, \ u = 1, 2, \ldots, p - 1, \ v = p^2 - p.
\]

(i) In \([48, Tab. 5]\), a proper divisor of \(p - 1\) is taken as \(t\) to obtain new \((d, w_h)\) modular Golomb rulers. In this case, \(d = p^{t-1}\) and \(w_h = \frac{p-1}{t}\) for every \(h\) in (3.7). The matrix \(A_{\sigma_t}\) has a weight vector \(\overline{W}(A_{\sigma_t}) = (\frac{p-1}{t}, \ldots, \frac{p-1}{t})\).

(ii) BDC matrices such that each weight \(w_h\) is in \(\{0, 1\}\) admit an extension by Procedure E, see Section 2, and hence can be effectively used to obtain new families of configurations (cf. Section 4 and Example 4.4(ii)). There are two different possibilities to get a matrix \(A_{\sigma_t}\) with 01-weight vector from (3.14).

a) Fix \(t = p - 1, \ d = p\). Then for each \(h = 0, 1, \ldots, t - 1\) there is precisely one element \(e_u\) such that \(e_u \equiv h \pmod{t}\). We have \(e_{p-1} \equiv 0 \pmod{t}\) and \(e_u \equiv u \pmod{t}, \ u = 1, 2, \ldots, p - 2\). Therefore, \(\overline{W}(A_{\sigma_{p-1}}) = \{1, 1, \ldots, 1\}\).
b) Fix \( t = p, \) \( d = p - 1. \) In this case \( e_u \neq 0 \pmod{t} \) for all \( u. \) Also, for each \( h = 1, 2, \ldots, t - 1 \) there is precisely one element \( e_u \) such that \( e_u \equiv h \pmod{t}. \) We have \( e_u \equiv h \pmod{s} \) if and only if \(-g^u \equiv h \pmod{p}. \) Therefore, \( \mathbf{W}(A_{\sigma_p}) = (0, 1, 1, \ldots, 1). \)

Example 3.7. Consider the \((q^2 - 1, q)\) modular Golomb ruler obtained from the cyclic starred affine plane \( AG(2, q) \)[8]. Let \( q^2 - 1 = td. \) In [37], by using both counting arguments and properties of the ruler as a difference set, it is proved that if \( t \) is a divisor of \( q + 1 \) then exactly \( t - 1 \) values of \( w_h \) are equal to \( \frac{q^2 + 1}{t}, \) and there exists precisely one \( h_0 \) with \( w_{h_0} = \frac{q^2 + 1}{t} - 1. \) In [37] only proper divisor \( t \) of \( q + 1 \) are considered, as this is the relevant case in connection with difference triangle sets. Yet, the same arguments work \( t = q + 1, \) and hence on can obtain a weight vector of \( A_{\sigma_{q+1}} \) consisting of zeroes and units, and admitting an extension by Procedure E. Without loss of generality \( \mathbf{W}(A_{\sigma_{q+1}}) = (0, 1, 1, \ldots, 1) \) can be assumed. For comparison, see also Example 3.12 below.

3.3 Using subgroups of the automorphism group of a cyclic configuration

The geometrical interpretation of the procedure illustrated in Subsection 3.2 was presented in [12, 14]. Here, after summarizing some of the results from [12, 14], we apply the procedure to cyclic configurations \((q^2 - 1)_q\) associated to affine planes \( AG(2, q)\) for \( t \) a divisor of \( q - 1, \) see Theorem 3.11.

For a cyclic symmetric configuration \( v_k, \) viewed as an incidence structure \( \mathcal{I} = (\mathcal{P}, \mathcal{L}), \) let \( \sigma \) be a permutation of \( \mathcal{P} \) mapping lines to lines, and acting regularly on both \( \mathcal{P} \) and \( \mathcal{L}. \) Let \( S \) be the cyclic group generated by \( \sigma. \) Let \( \mathcal{P} = \{P_0, \ldots, P_{v-1}\} \) and \( \mathcal{L} = \{\ell_0, \ldots, \ell_{v-1}\}. \) Arrange indexes so that \( \sigma : P_i \mapsto P_{i+1} \pmod{v} \) and \( \ell_i = \sigma^i(\ell_0). \) Clearly, \( P_i = \sigma^i(P_0) \) holds.

For any divisor \( d \) of \( v, \) the group \( S \) has a unique cyclic subgroup \( \hat{S}_d \) of order \( d, \) namely the group generated by \( \sigma^t \) where \( t = v/d. \) Let \( O_0, O_1, \ldots, O_{t-1} \) (resp. \( L_0, L_1, \ldots, L_{t-1} \)) be the orbits of \( \mathcal{P} \) (resp. \( \mathcal{L} \)) under the action of \( \hat{S}_d. \) Clearly, \(|O_i| = |L_i| = d \) for any \( i. \) We arrange indexes so that \( P_0 \in O_0, O_w = \sigma^w(O_0), \ell_0 \in L_0, L_w = \sigma^w(L_0). \) For each \( i = 0, 1, \ldots, t-1, \)

\[ O_i = \{P_i, \sigma^i(P_i), \sigma^{2i}(P_i), \ldots, \sigma^{(d-1)i}(P_i)\}, \quad L_i = \{\ell_i, \sigma^i(\ell_i), \sigma^{2i}(\ell_i), \ldots, \sigma^{(d-1)i}(\ell_i)\}. \]

Equivalently, \( O_i \) (resp. \( L_i \)) consists of \( d \) points \( P_u \) (resp. \( d \) lines \( L_u \)) with \( u \) equal to \( i \) modulo \( t. \)

Let

\[ w_u = |\ell_0 \cap O_u|, \quad u = 0, 1, \ldots, t - 1. \]

(3.15)

Clearly, \( w_0 + w_2 + \ldots + w_{t-1} = k. \)

Theorem 3.8. [14] Let \( \mathcal{I} = (\mathcal{P}, \mathcal{L}) \) be a cyclic symmetric configuration \( v_k \) with \( v = td. \) Let \( d, t, \hat{S}_d, O_i, L_i \) be as above.
(i) For any $i$ and $j$, every line of the orbit $L_i$ meets the orbit $O_j$ in the same number of points $w_{j-i} \pmod{t}$ where $w_u$ is defined by (3.15).

(ii) The incidence matrix of $I$ is a block double-circulant matrix $A$ of type (3.2) where $C_{i,j}$ is a circulant $d \times d$ matrix of weight $w_{j-i} \pmod{t}$, with $w_u$ as in (3.15).

In order to use Theorem 3.8 effectively one should find intersection numbers of orbits of the cyclic subgroup $\hat{S}_q$. For cyclic projective and starred affine planes useful results on these numbers are given e.g. in [11,14] and in the references therein.

**Example 3.9.** We consider the projective plane $PG(2,q)$ as a cyclic symmetric configuration $(q^2 + q + 1)_{q+1}$ [49],[14, Sec. 5],[16, Sec. 5.5],[45, Th. 19.15]. In this case the group $S$ is a Singer group of $PG(2,q)$.

(i) Let $t = 3$, $t|(q^2 + q + 1)$, $p \equiv 2 \pmod{3}$, and let $\{i_0, i_1, i_2\} = \{0, 1, 2\}$. In [14, Prop. 4] the following is proved: $w_{i_0} = (q + 2\sqrt{q} + 1)/3$, $w_{i_1} = w_{i_2} = (q - \sqrt{q} + 1)/3$, if $q = p^{4r+2}$; $w_{i_0} = (q - 2\sqrt{q} + 1)/3$, $w_{i_1} = w_{i_2} = (q + \sqrt{q} + 1)/3$, if $q = p^{4m}$. Now we use Theorem 2.1 and (ii) of Subsection 3.1. By (3.5) with $c = 2$, we obtain families of configurations $v_k$ with parameters

$$v_k : v = \frac{2q^2 + q + 1}{3}, \quad k = \frac{2q + \sqrt{q} + 2}{3} - \delta, \quad \delta \geq 0, \quad q = p^{4m+2}, \quad p \equiv 2 \pmod{3};$$

$$v_k : v = \frac{2q^2 + q + 1}{3}, \quad k = \frac{2q - \sqrt{q} + 2}{3} - \delta, \quad \delta \geq 0, \quad q = p^{4m}, \quad p \equiv 2 \pmod{3}.$$ 

(ii) Let $q = p^{2m}$ be a square. Let $t$ be a prime divisor of $q^2 + q + 1$. Then $t$ divides either $q + \sqrt{q} + 1$ or $q - \sqrt{q} + 1$. Assume that $p \pmod{t}$ is a generator of the multiplicative group of $\mathbb{Z}_t$. By [14, Prop. 6], in this case $w_0 = (q + 1 \pm (1-t)\sqrt{q})/t$, $w_1 = w_2 = \ldots = w_{t-1} = (q + 1 \pm \sqrt{q})/t$. Now we use Theorem 2.1 and (ii) of Subsection 3.1. By (3.5), we obtain a family of configurations $v_k$ with parameters

$$v_k : v = \frac{cq^2 + q + 1}{t}, \quad k = \frac{q + 1 \pm (1-t)\sqrt{q}}{t} + (c-1)\frac{q + 1 \pm \sqrt{q}}{t} - \delta, \quad (3.16)$$

$c = 1, 2, \ldots, t$, $\delta \geq 0$, $q = p^{2m}$, $t$ prime.

The hypothesis that $p \pmod{t}$ is a generator of the multiplicative group of $\mathbb{Z}_t$ holds e.g. in the following cases: $q = 3^4$, $t = 7$; $q = 2^8$, $t = 13$; $q = 5^4$, $t = 7$; $q = 2^{12}$, $t = 19$; $q = 3^8$, $t = 7$; $q = 2^{16}$, $t = 13$; $q = 17^4$, $t = 7$; $p \equiv 2 \pmod{t}$, $t = 3$.

(iii) Let $q$ be a square. Let $v \geq 1$, $v|(q - \sqrt{q} + 1)$, and $t = \frac{1}{v}(q - \sqrt{q} + 1)$. Then $d = \frac{v(q + \sqrt{q} + 1)}{v}$ and, by [14, Prop. 7], we have $w_0 = \sqrt{q} + v$, $w_1 = w_2 = \ldots = w_{t-1} = v$. Now using (ii) of Subsection 3.1, for $v = 1$ we obtain a family of configurations with parameters (2.6). The orbits $O_0, O_1, \ldots, O_{t-1}$ are Baer subplanes. Moreover, the case $v = 1$ admits an extension by Procedure E, see Section 2; it can be effectively used for obtaining families of configurations, see Section 4 and Example 4.4(iii).
(iv) In Table 1, parameters of configurations $v'_n$ with BDC incidence matrices are given. We use both (ii) and (iii) of Subsection 3.1. The starting weights $w_i^t$ are obtained by computer forming orbits of subgroups $\tilde{S}_d$ of a Singer group of $PG(2,q)$. For $q = 81$ we use (3.16). The values $k', v'$ are calculated by (3.5),(3.6). Only cases with $v' < G(k')$ are included in the tables. Then the smallest value $k^\#$ for which $v' < G(k^\#)$ is found. As a result, each row of the table provides configurations $v'_n$ with $v' < G(n)$, $n = k^\#, k^\# + 1, \ldots, k'$, see (i) of Subsection 3.1 and (3.4).

Remark 3.10. In [43, Prop. 3, Th. 9], parity check matrices of LDPC codes based on the Hermitian curve in $PG(2,q^2)$ and consisting of square cyclic submatrices are constructed by geometrical tools that can be considered as special cases of the more general approach of Theorem 3.8. The mentioned parity check matrices are incidence matrices of non-symmetric configurations. It is possible that by dismissing some units in the matrix, BDC configurations could be obtained. This problem is not considered here, nor in [43]. It is interesting to note that the matrix of [43, Prop. 3] uses points belonging to the Hermitian curve, whereas point set of the symmetric configuration in [14, Ex. 3], whose parameters are as in (2.12), coincides with the complement of the same curve.

Throughout the rest of the section, cyclic starred affine planes $(q^2-1)Q$ are considered. In [14, 37] useful results for $t$ a divisor of $q + 1$ are obtained. Theorem 3.11 below extends our knowledge on this topic and gives new results for $t$ dividing $q - 1$. The proof is placed in Appendix; it uses orbits of cyclic subgroup.

Theorem 3.11. Let $q$ be an odd square. Consider the cyclic symmetric configuration $(q^2-q)_q$ associated to the starred affine plane of order $q$. Let $t$ be a divisor of $\sqrt{q} - 1$, and let $d = (q^2 - 1)/t$. Let $A$ be an incidence BDC matrix of this configuration as in Theorem 3.8(ii). Let $w_0, w_1, \ldots, w_{t-1}$ be weights of the circulant $d \times d$ blocks of $A$.

(i) Let $t = \sqrt{q} + 1$. Then $w_0 = 1$, $w_j = \sqrt{q} - 1$ for $j$ odd, $w_j = \sqrt{q} + 1$ for $j$ even, $j = 1, 2, \ldots, \sqrt{q}$.

(ii) Let $t = \frac{1}{2}(\sqrt{q} + 1)$, $\sqrt{q} \equiv 1$ (mod 4). Then $w_0 = \sqrt{q}$, $w_1, w_2, \ldots, w_{t-1} = 2\sqrt{q}$.

(iii) Let $t = \frac{1}{2}(\sqrt{q} + 1)$, $\sqrt{q} \equiv 3$ (mod 4). Then $w_0 = \sqrt{q} + 2$, $w_j = 2\sqrt{q} - 2$ for $j$ odd, $w_j = 2\sqrt{q} + 2$ for $j$ even, $j = 1, 2, \ldots, \frac{1}{2}(\sqrt{q} + 1) - 1$.

(iv) Let $t = \frac{1}{2}(\sqrt{q} + 1)$, $\sqrt{q} \equiv 3$ (mod 4).

- If $\frac{1}{2}(\sqrt{q} + 1)$ is odd, then $w_0 = 3\sqrt{q}$, $w_1, w_2, \ldots, w_{t-1} = 4\sqrt{q}$.
- If $\frac{1}{2}(\sqrt{q} + 1)$ is even, then $w_0 = 3\sqrt{q} + 4$, $w_j = 2\sqrt{q} - 4$ for $j$ odd, $w_j = 2\sqrt{q} + 4$ for $j$ even, $j = 1, 2, \ldots, \frac{1}{2}(\sqrt{q} + 1) - 1$.
Example 3.12. We consider the cyclic symmetric configuration \((q^2 - q)_q\) associated to the starred affine plane of order \(q\). [8],[16, Sec. 5.6],[45, Th. 19.17], see also [14, Ex. 5, Sec. 6]. In this case the group \(S\) is an affine Singer group of \(AG(2, q)\).

(i) Let \(t\) be a divisor of \(q + 1\). In [14, Prop. 10] it is proven that \(w_0 = (q + 1)/t - 1, w_1 = w_2 = \ldots = w_{t-1} = (q + 1)/t\), cf. Example 3.7 which uses results of [37] obtained by a different approach. Putting \(t = q + 1\) we obtain \(d = q - 1\) and \(w_0 = 0, w_1 = w_2 = \ldots = w_q = 1\). Now using (ii) in Subsection 3.1 one can obtain a family of configurations with parameters (2.9). Moreover, the case \(v = 1\) admits an extension by Procedure E, see Section 2; it can be effectively used for obtaining families of configurations, see Section 4 and Example 4.4(i).

(ii) Let \(q\) be an odd square. Let \(t = \sqrt{q} + 1, d = (\sqrt{q} - 1)(q + 1)\). By Theorem 3.11(i) and (iii) of Subsection 3.1 one can obtain a family of configurations with parameters

\[
v_k : v = 2f(\sqrt{q} - 1)(q + 1), \quad k = (2f - 1)\sqrt{q}, \quad f = 1, 2, \ldots, \frac{\sqrt{q} + 1}{2}, \quad q \text{ odd square.}
\]

(3.17)

By using Theorem 3.11(ii),(iii),(iv), together with (iii) of Subsection 3.1, we obtain the same parameters as in (3.17). But the structure of an incidence matrix \(M(v, k)\) is different from that arising from Theorem 3.11(i).

(iii) In Table 2, parameters of configurations \(v'_n\) with BDC incidence matrices are given. We use (iii) of Subsection 3.1. The starting weights \(w^*_i\) are obtained by computer through the constructions of the orbits of subgroups \(\hat{S}_d\) of the affine Singer group. Notations is as in Table 1.

INSERT Table 2 HERE

4 Constructing configurations \(v_k\) admitting an extension

Establishing whether a configuration admits an extension or not is not an easy task in the general case. In this section we deal with configurations admitting incidence matrices with special type, and we show that may admit several extensions.

Definition 4.1. Let \(v = td, t \geq k, d \geq k - 1\), and let \(v_k\) be a symmetric configuration. Let \(M(v, k)\) be an incidence matrix of \(v_k\), viewed as a \(t \times t\) block matrix, every block being of type \(d \times d\). We say that \(M(v, k)\) has Structure E if every \(d \times d\) block is either a permutation matrix \(P_d\) or the zero \(d \times d\) matrix \(0_d\).

Lemma 4.2. Let \(v = td, t \geq k, d \geq k - 1\), and let \(v_k\) be a symmetric configuration. Assume that \(M(v, k)\) is an incidence matrix of \(v_k\) having Structure E.

(i) The matrix \(M(v, k)\) admits \(\theta(t, d, k) := t \cdot \lceil d/(k - 1) \rceil \geq t \geq k\) extensions.
(ii) Let $M(v + \theta(t, d, k), k)$ be the matrix obtained from (i) by applying $\theta(t, d, k)$ extensions. Then $M(v + \theta(t, d, k), k)$ above admits $\theta_2(t, d, k) := \lfloor \theta(t, d, k)/(k - 1) \rfloor \geq 1$ extensions.

Proof. (i) We need to provide $\theta(t, d, k)$ pairwise disjoint E-aggregates of $M(v, k)$. For each block of type $P_d$, one can easily define a set $E$ of $\lfloor d/(k - 1) \rfloor$ disjoint E-aggregates consisting of row and columns with non-trivial intersection with $P_d$. Let $B$ be the 01-matrix of type $t \times t$ such that each entry corresponds to a $d \times d$ block in $M(v, k)$: an entry is 1 if the corresponding block is of type $P_d$, 0 otherwise. Each row and each column of $B$ has weight $k$. Therefore, it is possible to obtain a permutation matrix $P_t$ by dismissing some units in $B$. The sets $E$ defined from blocks $P_d$ corresponding to units of $P_t$ are clearly disjoint, and their union gives $t \cdot \lfloor d/(k - 1) \rfloor$ not intersecting E-aggregates.

(ii) Theorem 2.4(ii) can be applied $\theta_2(t, d, k)$ times. \hfill \qed

Corollary 4.3. Let $v = td$, $t \geq k$, $d \geq k - 1$, and let $v_k$ be a symmetric configuration. Assume that an incidence matrix $A$ of $v_k$ is a BDC matrix as in (3.2) with weight vector $\overline{W}(A) = (w_0, \ldots, w_{t-1})$.

(i) If all the weights $w_u$ belong to the set $\{0, 1\}$, then $A$ admits $t + 1$ extensions.

(ii) If $\overline{W}(A) = (0, 1, 1, \ldots, 1)$, then one can obtain a family of symmetric configurations $v_k$ with parameters

$$v_k : v = cd + \theta, \ k = c - 1 - \delta, \ c = 2, 3, \ldots, t, \ \theta = 0, 1, \ldots, c + 1, \ \delta \geq 0. \quad (4.1)$$

(iii) If $\overline{W}(A) = (1, 1, \ldots, 1)$, then one can obtain a family of symmetric configurations $v_k$ with parameters

$$v_k : v = cd + \theta, \ k = c - \delta, \ c = 2, 3, \ldots, t, \ \theta = 0, 1, \ldots, c + 1, \ \delta \geq 0. \quad (4.2)$$

Proof. The matrix $A$ has clearly Structure E. Then (i) follows from Lemma 4.2, together with Theorem 2.4(i). As to (ii) and (iii), we use (ii) of Subsection 3.1. \hfill \qed

In order to obtain a configuration having Structure E from a given one, sometimes the procedures described in Subsection 3.1 are useful; see Example 4.4(iii) below.

Example 4.4. (i) We consider the starred affine plane of order $q$ as a cyclic configuration $(q^2 - 1)_q$, see Examples 3.7 and 3.12(i). Let $t = q + 1$, $d = q - 1$, $w_0 = 0$, $w_1 = w_2 = \ldots = w_q = 1$. By Corollary 4.3(ii) we obtain a family of symmetric configurations $v_k$ with parameters

$$v_k : v = c(q-1)+\theta, \ k = c-1-\delta, \ c = 2, 3, \ldots, q+1, \ \theta = 0, 1, \ldots, c+1, \ \delta \geq 0. \quad (4.3)$$
(ii) We consider Ruzsa’s configuration \((p^2 - p)_{p-1}\), see Example 3.6(ii). Put \(t = p - 1\), \(d = p\). Then \(w_0 = w_1 = \ldots = w_{p-2} = 1\). By Corollary 4.3(iii) we obtain a family with

\[
 v_k : v = cp + \theta, \ k = c - \delta, \ c = 2, 3, \ldots, p - 1, \ \theta = 0, 1, \ldots, c + 1, \ \delta \geq 0, \ p \ prime. \tag{4.4}
\]

If \(t = p, \ d = p - 1\) then \(w_0 = 0, \ w_1 = \ldots = w_{p-1} = 1\). We obtain a family with

\[
 v_k : v = c(p-1) + \theta, \ k = c - \delta, \ c = 2, 3, \ldots, p, \ \theta = 0, 1, \ldots, c+1, \ \delta \geq 0, \ p \ prime. \tag{4.5}
\]

(iii) Let \(q\) be a square. We consider \(PG(2, q)\) as a cyclic configuration \((q^2 + q + 1)_{q+1}\), see Example 3.9(iii). Let \(v = 1, \ t = q - \sqrt{q} + 1, \ d = q + \sqrt{q} + 1, \ w_0 = \sqrt{q} + 1, w_1 = w_2 = \ldots = w_{t-1} = 1\). By (i) of Subsection 3.1 we can put \(w'_0 = 1\) and obtain the weight vector \((1, 1, \ldots, 1)\). Now, by Corollary 4.3(iii), we obtain a family with

\[
 v_k : v = c(q + \sqrt{q} + 1) + \theta, \ k = c - \delta, \ c = 2, 3, \ldots, q - \sqrt{q} + 1, \ \theta = 0, 1, \ldots, c+1, \ \delta \geq 0, \ q \ square \tag{4.6}
\]

5 The spectrum of parameters of cyclic symmetric configurations

In order to widen the ranges of parameter pairs \(\{v, k\}\) for which a cyclic symmetric configuration \(v_k\) exists, we consider a number of procedures that allow to define a new modular Golomb ruler from a known one. Some methods have already been introduced in the paper, see Theorem 2.1.

Here we first recall a result from [45], which describes a method to construct different rulers with the same parameters.

**Theorem 5.1.** [45] If \((a_1, a_2, \ldots, a_k)\) is a \((v, k)\) modular Golomb ruler and \(m\) and \(b\) are integers with \(\gcd(m, v) = 1\) then \((ma_1 + b \pmod{v}, ma_2 + b \pmod{v}, \ldots, ma_k + b \pmod{v})\) is also a \((v, k)\) modular Golomb ruler.

It should be noted that a \((v, k)\) modular Golomb ruler can be a \((v + \Delta, k)\) modular Golomb ruler for some integer \(\Delta\) [24]. This property does not depend on parameters \(v\) and \(k\) only. This is why Theorem 5.1 can be useful for our purposes.

**Example 5.2.** We consider the \((31, 6)\) modular Golomb ruler

\[
(a_1, \ldots, a_6) = (0, 1, 4, 10, 12, 17)
\]

obtained from \(PG(2, 5)\), see [46]. We can apply Theorem 5.1 for \(m = 19, \ b = 0\). The \((31, 6)\) modular Golomb ruler \((ma_1 \pmod{31}, \ldots, ma_6 \pmod{31})\) is

\[
(a'_1, \ldots, a'_6) = (0, 4, 11, 13, 14, 19).
\]
Now we take $\Delta = 4$ and calculate the set of differences \( \{a'_i - a'_j \pmod{35} \mid 1 \leq i, j \leq 6; i \neq j\} \), that is \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34\} \). As the all differences are distinct and nonzero, the starting \((31, 6)\) modular Golomb ruler is also a \((35, 6)\) modular Golomb ruler.

**Example 5.3.** We take the \((57, 8)\) modular Golomb ruler \((a_1, \ldots, a_8) = (0, 4, 5, 17, 19, 25, 28, 35)\) obtained from \(PG(2,7)\). Apply Theorem 2.1 for \(\delta = 1\), and remove the integer 35. A \((57, 7)\) modular Golomb ruler \((a'_1, \ldots, a'_7) = (0, 4, 5, 17, 19, 25, 28)\) is obtained. Now we take \(\Delta = -2\). Due to Definition 2.5 and Theorem 2.6, instead of differences we calculate the set of sums \(\{a'_i + a'_j \pmod{55} \mid 1 \leq i, j \leq 7\}\), that is \(\{0, 4, 5, 8, 9, 10, 17, 19, 21, 22, 23, 24, 25, 28, 29, 30, 32, 33, 34, 36, 38, 42, 44, 45, 47, 50, 53, 56\}\). As the all sums are distinct, the \((57, 7)\) modular Golomb ruler \((0, 4, 5, 17, 19, 25, 28)\) is also a \((55, 7)\) modular Golomb ruler.

For \(k \leq 81\), we performed a computer search starting from the \((v, k)\) modular Golomb rulers corresponding to (2.1)–(2.3). For projective and affine planes, we got a concrete description of the ruler from [46]. For Ruzsa’s construction, we used (3.14) of Example 3.6. For every starting \((v, k)\) modular ruler we first considered all possible \(m\) with \(\gcd(m, v) = 1\), and applied Theorem 5.1 for \(b = 0\) to get new rulers with the same parameters \(v\) and \(k\). Then, we checked whether this ruler was also a \((v + \Delta, k)\) for some \(\Delta\).

We obtained improvements on the known results for \(k \geq 16\). For the sake of completeness, we summarize the known results about the case \(k \leq 15\) in Table 3. Table 4 lists known and new results about \(16 \leq k \leq 41\), whereas Table 5 deals with the case \(42 \leq k \leq 83\).

As to Table 3, the values of \(v_5(k)\) are taken from [23, Tab. IV], [31, Tab. 2], [47, Tab. 1a], [51]. The values of \(v\) for which cyclic symmetric configurations \(v_k\) exist (resp. do not exist) are written in normal (resp. in italic) font. Moreover, \(\overline{v}\) means that no configuration \(v_k\) exists while \(\overline{\overline{v}}\) notes that no cyclic configuration \(v_k\) exists. Data from [18, 24, 26, 29, 33, 38, 47] are listed in the 4-th column of the table. We take into account that an entry of the form “\(t+\)” in the row “\(n\)” of [47, Tab. 1] means the existence of a cyclic symmetric configurations \(v_n\) with \(v \geq t\). Also, we use the following non-existence results: \(\overline{32_6}\) [24, Th. 4.8]; \(\overline{33_6}\) [33]; \(\overline{34_6}, \overline{59_6-62_8}\) [38]; \(\overline{75_9-79_9, 81_9-84_9}\) [18]. The following theorem of [26] is taken into account.

**Theorem 5.4.** [26, Th. 2.4] There is no symmetric configuration \((k^2 - k + 2)_k\) if \(5 \leq k \leq 10\) or if neither \(k\) or \(k - 2\) is a square.

The values of \(k\) for which the spectrum of parameters of cyclic symmetric configurations \(v_k\) is completely known are indicated by a dot “•”; the corresponding values of \(E_c(k)\) are sharp and they are noted by the dot “•” too. Values of \(v\) obtained by our search lies within the range of the known parameters (see the 5-th column of the table).

**INSERT Table 3 HERE**

In Table 4, for \(16 \leq k \leq 41\), \(P(k) \leq v < G(k)\), data on the existence of cyclic symmetric configuration \(v_k\) are given. The known results from (2.1)–(2.3) are written in
normal font; the entries \( v_a, v_b, \) and \( v_c \) means, respectively, that the relations (2.1), (2.2), and (2.3) are used. The values of \( v \) obtained in this work are given in bold font; the entry \( v-w \) notes an interval of sizes from \( v \) to \( w \) without gaps. If an already known value lies within an interval \( v-w \) obtained in this work, then it is written immediately before the interval. Also, some data on the nonexistence (including those arising from Theorem 5.4) are written in italic font, in the form \( \overline{v} \) or \( \overline{v'} \). For \( k = 16 \) the value \( v_6(k) = 255 \) [47] is taken into account. The nonexistence of some projective planes by Bruck-Ryser theorem is also indicated.

\[ \text{INSERT Table 4 HERE} \]

In Table 5, for \( 42 \leq k \leq 83 \), the upper bounds on the cyclic existence bound \( E_c(k) \) obtained in this work are listed.

\[ \text{INSERT Table 5 HERE} \]

6 The spectrum of parameters of symmetric (non-necessarily cyclic) configurations

The known results regarding to parameters of symmetric configurations can be found in [1]–[5],[7]–[9],[12]–[14],[18, 19, 21, 33, 44, 45, 47],[23]–[31],[38]–[40]; see also the references therein.

The known families of configurations \( v_k \) were described in Section 2. In Table 6, for \( k \leq 37 \), \( P(k) \leq v < G(k) \), values of \( v \) for which a symmetric configuration \( v_k \) from one of the families of Section 2 exists are given. A subscript of an entry indicates that a specific (2.i) is used: more precisely \( v_a \) indicates that \( v \) is obtained from (2.1), and similarly \( v_b \rightarrow (2.2), v_c \rightarrow (2.3), v_d \rightarrow (2.4), v_e \rightarrow (2.5), v_f \rightarrow (2.6), v_g \rightarrow (2.7), v_{h} \rightarrow (2.8), v_i \rightarrow (2.9), v_{j} \rightarrow (2.10), v_k \rightarrow (2.11), v_{l} \rightarrow (2.12), v_m \rightarrow (2.13) \). An entry with more than one subscript means that the same value can be obtained from different constructions. An entry of type \( v_{\text{subscript}_1, \text{subscript}_2, \ldots} - v'_{\text{subscript}_1, \text{subscript}_2, \ldots} \) indicates that a whole interval of values from \( v \) to \( v' \) can be obtained from the constructions corresponding to the subscripts.

To save space, in Table 6 if a value belongs to an interval obtained by the Extension Construction of (2.13), then it is listed only once, even if it can be obtained from different constructions as well.

\[ \text{INSERT Table 6 HERE} \]

In Table 7, for \( P(k) \leq v < G(k) \), parameters of the symmetric configurations \( v_k \) from Sections 3 and 4 are listed. An entry of type \( v_{\text{subscript}} \) indicates that either relations (3.i),(4.j) or Tables 1, 2 are used. More precisely \( v_a \rightarrow (3.16), v_p \rightarrow (3.17), v_r \rightarrow (4.3), v_s \rightarrow (4.4), v_t \rightarrow (4.5), v_u \rightarrow (4.6), v_v \rightarrow \text{Table 1}, v_w \rightarrow \text{Table 2} \). For \( k \leq 37 \), we listed all the results we got, whereas for \( k = 38-41, 49, 56 \) we only give some illustrative examples.

\[ \text{INSERT Table 7 HERE} \]
We note that a number of parameters are new: 322_{16}, 458_{19}, 459_{19}, 482_{20}, 574_{22}, 674_{24}, 782_{26}, 1066_{27}-1072_{27}, 1104_{27}-1106_{27}, 1066_{28}-1072_{28}, 1104_{28}-1109_{28}, 1142_{28}-1146_{28}, 1104_{29}-1109_{29}, 1142_{29}-1146_{29}, 1180_{29}-1183_{29}, 1220_{29}, 1142_{30}-1146_{30}, 1180_{30}-1183_{30}, 1218_{30}-1220_{30}, 1180_{31}-1183_{31}, 1218_{31}-1220_{31}, 1256_{31}, 1257_{31}, 1218_{32}-1220_{32}, 1250_{32}-1257_{32}, 1294_{32}, 1256_{33}, 1257_{33}, 1294_{33}, 1294_{34}, 1430_{34}-1434_{34}, 1472_{35}-1475_{35}, 1514_{36}-1516_{36}, 1556_{37}, 1557_{37}; sometimes the gaps in an interval arising from (2.13) are filled.

The new cyclic configurations from Table 4, like 382_{17}-390_{17}, 401_{18}, 405_{18}-407_{18}, 410_{18}, 412_{18}, 413_{18}, also fill some gaps in the known range of parameters.

Parameters of the family of Example 3.9(i) are too big to be included in Table 7. For the same reason, parameters for (3.16) are only reported for \( q = 3^4 \).

Finally, in Table 8, for \( k \leq 37 \), \( P(k) \leq v < G(k) \), we summarize the data from Tables 3, 4, 6, and 7. Also, we use the following known results on existence of sporadic symmetric configurations: 45_7 [5]; 82_9 [19, Tab. 1]; 135_{12}, see [26] with reference to Mathon’s talk at the British Combinatorial Conference 1987; 34_6 [36]. The non-existence of configuration 112_{11} is proved in [34].

**Appendix: Proof of Theorem 3.11**

Let \( \xi \) be a primitive element of \( F_{q^2} \). Let \( \omega = \xi^{q+1} \), and \( \theta = \omega^{q+1} \). Identify a point \( (x, y) \in AG(2, q) \) with the element \( z = x + \omega y \in F_{q^2} \). As \( \omega^{q-1} = -1 \) it is straightforward to check that \( z^{q+1} = x^2 + \theta y^2 \).

We need to consider the orbits of \( F_{q^2}^* \) under the action of the cyclic group generated by \( \sigma^t \) where \( \sigma(\xi^i) = \xi^{i+1} \). The orbit \( O_j \) of the element \( \xi^i \) is \( \{ \xi^i, \xi^{i+t}, \xi^{i+2t}, \ldots, \xi^{i+(q^2-1)t} \} \).

Let \( \mu \) be a primitive element in \( F_q \).

(i) Assume that \( t = \sqrt{q} + 1 \). For each \( z \in O_j \) we have \( z^{q+1} = \xi^{j(q+1)}(\xi^{(q+1)(\sqrt{q}+1)})^h \) for some \( h \). Also, \( \xi^{q+1} \) is a primitive element of \( F_q \) and \( (\xi^{(q+1)(\sqrt{q}+1)})^h \in F_{\sqrt{q}}^* \). This means for each \( j = 0, \ldots, \sqrt{q} \), the orbit \( O_j \) consists precisely of the elements \( z \) such that \( z^{q+1} \in \mu^j F_{\sqrt{q}}^* \). Therefore, the following lemma holds.

**Lemma 6.1.** *The orbit \( O_j \) in \( AG(2, q) \) consists of the union of the \( \sqrt{q} - 1 \) conics with equation \( x^2 + \theta y^2 = \mu^j \alpha \) where \( \alpha \in F_{\sqrt{q}}^* \).*

The final step is to compute the sizes of the intersections \( |O_j \cap \ell| \) where \( \ell \) is any line of \( AG(2, q) \) not passing through the origin. These sizes are the integers \( u_0, \ldots, w_{\sqrt{q}} \). Choose the line \( \ell : x = 1 \).
Lemma 6.2. $w_0 = 1$.

Proof. We prove that $|\ell \cap O_0| = 1$. Note that $O_0$ consists of the conics $C_\alpha : x^2 + \theta y^2 = \alpha$. The line $x = 1$ meets the conic $C_\alpha$ in one point if $\alpha = 1$. If $\alpha \neq 1$ then the intersection is empty since $\theta$ is not a square in $F_q$ (and $\alpha$ is a square since it is an element of $F_{\sqrt{q}}$). \qed

Lemma 6.3. Let $N_j$ be the number of non-squares in the set $\mu^j F_{\sqrt{q}}^* - 1$. Then $w_j = 2N_j$.

Proof. The conic $C_\alpha : x^2 + \theta y^2 = \mu^j \alpha$ meets the line $x = 1$ in 0 or 2 points. The latter case occurs precisely when $\mu^j \alpha - 1$ is not a square. \qed

In order to compute the integers $N_j$, the following lemmas will be useful.

Lemma 6.4. The collection of sets $H_\beta = \{\mu F_{\sqrt{q}}^* - 1 \mid \beta \in F_{\sqrt{q}}\}$ coincides with $
\{\mu^j F_{\sqrt{q}}^* - 1 \mid j = 1, \ldots, \sqrt{q}\}$.

Proof. We need only need to show that sets $\frac{\mu}{1-\mu \beta} F_{\sqrt{q}}^* - 1$ are pairwise distinct. Assume on the contrary that $\frac{\mu}{1-\mu \beta} = \frac{\mu}{1-\mu \gamma}$ for $\beta, \gamma \in F_{\sqrt{q}}, \alpha \in F_{\sqrt{q}}^*$. Then $\alpha(1-\mu \beta) = 1 - \mu \gamma$ that is $\mu(\alpha \beta + \gamma) = 1 - \alpha$. If $\alpha \beta - \gamma = 0$ then $\alpha = 1$ and hence $\beta = \gamma$. If $\alpha \beta - \gamma \neq 0$ then $\mu \in F_{\sqrt{q}}$, which is a contradiction. \qed

Lemma 6.5. Fix $\beta \in F_{\sqrt{q}}$ and $j \in \{1, \ldots, \sqrt{q}\}$. Assume that the set $\frac{\mu}{1-\mu \beta} F_{\sqrt{q}}^* - 1$ coincides with $\mu^j F_{\sqrt{q}}^* - 1$ if and only if $j$ is odd.

Proof. Note that $\frac{\mu}{1-\mu \beta} F_{\sqrt{q}}^* - 1$ coincides with $\mu^j F_{\sqrt{q}}^* - 1$ if and only if $\mu^{j-1}(1-\mu \beta) \in F_{\sqrt{q}}$. Then $\mu^{j-1}(1-\mu \beta)$ is a square. Whence the assertion follows. \qed

Let $M_\beta$ be the number non-squares in the set $\frac{\mu}{1-\mu \beta} F_{\sqrt{q}}^* - 1$. By the previous lemma, the set of integers $M_\beta$ coincides with the set of integers $N_j$. Let $A = M_0 = N_1$. Next we show that every $M_\beta$ is related to $A$.

Lemma 6.6. If $1 - \mu \beta$ is a square in $F_q$ then $M_\beta = A$. If $1 - \mu \beta$ is not a square in $F_q$ then $M_\beta = \sqrt{q} - A$.

Proof. $A$ is the number of non-squares in the set $H_0 = \{\mu \alpha - 1 \mid \alpha \in F_{\sqrt{q}}^*\}$. For each $\beta \in F_{\sqrt{q}}^*$, this set coincides with $\{\mu(\alpha + \beta) - 1 \mid \alpha \in F_{\sqrt{q}}^*, \alpha \neq -\beta\} \cup \{\mu \beta - 1\}$. But since $\mu(\alpha + \beta) - 1 = \mu \alpha + \mu \beta - 1 = (1-\mu \beta)(\frac{\mu}{1-\mu \beta} \alpha - 1)$ we have that

$$H_0 = (1-\mu \beta)\{\frac{\mu}{1-\mu \beta} \alpha - 1 \mid \alpha \in F_{\sqrt{q}}^*, \alpha \neq -\beta\} \cup \{\mu \beta - 1\},$$

that is $H_0 = (1-\mu \beta)H_\beta \setminus \{-1\} \cup \{\mu \beta - 1\}$. Two cases have to be distinguished.

a) $1 - \mu \beta$ is a square. Then either $\frac{1}{\mu \beta - 1}$ and $\mu \beta - 1$ are both squares or are both non-squares. It follows that the number of non-squares in $H_0$ equals the number of non-squares in $H_\beta$. Therefore, $M_\beta = A$.

b) $1 - \mu \beta$ is not a square. As $-1$ is a square, $\mu \beta - 1$ is not a square. The number of non-squares in $H_0$ equals the number of squares in $H_\beta$ plus 1. Then $M_\beta = \sqrt{q} - A$. \qed
As a corollary to Lemmas 6.5 and 6.6, the following result is obtained.

**Lemma 6.7.** If \( j \) is odd then \( N_j = A \). If \( j \) is even then \( N_j = \sqrt{q} - A \).

By the above lemma only two possibilities occur for \( w_j \), namely \( 2A \) and \( 2(\sqrt{q} - A) \). Next we calculate with equation

\[
A \text{ integers, the assertion of Theorem 3.11(i) follows.}
\]

Then, from \( w_0 + w_1 + \ldots + w_{\sqrt{q}} = q \), we obtain

\[
q = 1 + 2u_1A + 2(\sqrt{q} - u_1)(\sqrt{q} - A) = 1 + 4u_1A + 2q - 2\sqrt{q}(u_1 + A).
\]

Hence,

\[
0 = q + 1 + 4u_1A - 2\sqrt{q}(u_1 + A) = (\sqrt{q} - 2u_1)(\sqrt{q} - 2A) + 1
\]

Since both \( \sqrt{q} - 2u_1 \) and \( \sqrt{q} - 2A \) are integers, the only possibility is that they are both equal to \( \pm 1 \). This implies that \( A = \frac{\sqrt{q} + 1}{2} \), \( u_1 = \frac{\sqrt{q} + 1}{2} \) is the only solution. So, we have proved that the integers \( w_0, \ldots, w_{\sqrt{q}} \) are such that: 1 occurs precisely once; the integer \( \sqrt{q} + 1 \) occurs \( \frac{\sqrt{q} - 1}{2} \) times; the integer \( \sqrt{q} - 1 \) occurs \( \frac{\sqrt{q} + 1}{2} \) times. Finally, since \( w_2 = w_j \) if \( j = j' \) (mod 2) and the number of odd integers in \([1, \sqrt{q}]\) is greater than that of even integers, the assertion of Theorem 3.11(i) follows.

(ii) Assume that \( t = \frac{1}{2}(\sqrt{q} + 1) \), \( \sqrt{q} \equiv 1 \) (mod 4). An orbit here is the union of two orbits \( O_j \) of case (i). More precisely, an orbit consists of the union of the \( 2(\sqrt{q} - 1) \) conics with equation

\[
x^2 + \theta y^2 = \mu^j \alpha, \quad \alpha \in F_{\sqrt{q}}^* \cup \mu(\sqrt{q}) \cup F_{\sqrt{q}}^*.
\]

Equivalently, an orbit here is the union \( O_j \cup O_{j + \frac{\sqrt{q} + 1}{2}} \) for some \( j = 0, \ldots, \frac{\sqrt{q} - 1}{2} \). Assume that \( j > 0 \). Since \( j \) and \( j + \frac{\sqrt{q} + 1}{2} \) are different modulo 2, we have that the number of points of \( \ell_i \) in this orbit is \((\sqrt{q} - 1) + (\sqrt{q} + 1) = 2\sqrt{q}\). If \( j = 0 \), since \( \frac{\sqrt{q} + 1}{2} \) is odd we have that \( \ell_1 \) meets \( O_0 \cup O_{\frac{\sqrt{q} + 1}{2}} \) in \( \sqrt{q} + 2 \) points.

(iii) Assume that \( t = \frac{1}{2}(\sqrt{q} + 1) \), \( \sqrt{q} \equiv 3 \) (mod 4). Again, an orbit here is the union \( O_j \cup O_{j + \frac{\sqrt{q} + 1}{2}} \) for some \( j = 0, \ldots, \frac{\sqrt{q} - 1}{2} \). Assume that \( j > 0 \). Since \( j \) and \( j + \frac{\sqrt{q} + 1}{2} \) are equal modulo 2, we have that the number of points of \( \ell_i \) in this orbit is \( 2\sqrt{q} - 2 \) if \( j \) is odd, \( 2\sqrt{q} + 2 \) if \( j \) is even. If \( j = 0 \), since \( \frac{\sqrt{q} + 1}{2} \) is even we have that \( \ell_1 \) meets \( O_0 \cup O_{\frac{\sqrt{q} + 1}{2}} \) in \( \sqrt{q} + 2 \) points.

(iv) Assume that \( t = \frac{1}{2}(\sqrt{q} + 1) \), \( \sqrt{q} \equiv 3 \) (mod 4). An orbit here is the union \( O_j \cup O_{j + \frac{\sqrt{q} + 1}{2}} \cup O_{j + \frac{\sqrt{q} + 1}{2}} \cup O_{j + \frac{\sqrt{q} - 3}{2}} \) for some \( j = 0, \ldots, \frac{\sqrt{q} - 3}{2} \). Then it is easy to deduce the assertion of Theorem 3.11(iv). \( \square \)

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Table 1. Parameters of configurations \( v'_n \) with BDC incidence matrices, \( v' < G(n) \), \( n = k^#, k^# + 1, \ldots, k' \), by (ii) of Subsection 3.1 from the cyclic projective plane \( PG(2, q) \).

| \( q \) | \( t \) | \( d \) | \( w'_iv'_i \) | \( c \) | \( k' \) | \( v' \) | \( G(k') \) | \( k^# \) | \( G(k^#) \) |
|---|---|---|---|---|---|---|---|---|---|
| 32 | 7 | 151 | 0, 5, 6, 5, 6 | 6 | 25 | 906 | 961 | 25 | 961 |
| 37 | 3 | 469 | 16, 1, 13 | 2 | 25 | 938 | 961 | 25 | 961 |
| 43 | 3 | 631 | 19, 3, 12 | 2 | 31 | 1262 | 1495 | 30 | 1361 |
| 49 | 3 | 817 | 21, 16, 13 | 2 | 34 | 1634 | 1877 | 33 | 1719 |
| 61 | 3 | 1261 | 25, 21, 16 | 2 | 41 | 2522 | 2611 | 40 | 2565 |
| 64 | 3 | 1387 | 27, 19, 19 | 2 | 46 | 2774 | 3407 | 42 | 2795 |
| 67 | 3 | 1519 | 28, 19, 21 | 2 | 47 | 3038 | 3609 | 44 | 3193 |
| 73 | 3 | 1801 | 28, 27, 19 | 2 | 47 | 3602 | 3609 | 47 | 3609 |
| 79 | 3 | 2107 | 31, 21, 28 | 2 | 52 | 4214 | 4541 | 51 | 4381 |
| 81 | 7 | 949 | 4, 13, 13, 13, 13, 13 | 6 | 69 | 5694 | 8291 | 58 | 5703 |
| 81 | 7 | 949 | 4, 13, 13, 13, 13, 13 | 5 | 56 | 4745 | 5451 | 54 | 4747 |
| 97 | 3 | 3169 | 39, 28, 31 | 2 | 67 | 6338 | 7639 | 62 | 6431 |
| 103 | 3 | 3571 | 39, 28, 37 | 2 | 67 | 7142 | 7639 | 65 | 7187 |
| 107 | 7 | 1651 | 24, 15, 15, 13, 13, 13 | 6 | 89 | 9906 | 13557 | 75 | 9965 |
| 107 | 7 | 1651 | 24, 15, 15, 13, 13, 13 | 5 | 76 | 8255 | 10179 | 69 | 8291 |
| 109 | 3 | 3997 | 43, 36, 31 | 2 | 74 | 7994 | 9507 | 69 | 8291 |
| 109 | 7 | 1713 | 8, 15, 15, 15, 19, 19, 19 | 6 | 83 | 10278 | 12041 | 77 | 10409 |
| 121 | 7 | 2109 | 21, 20, 13, 13, 21, 13, 21 | 6 | 86 | 12654 | 13075 | 85 | 12821 |
| 127 | 3 | 5419 | 49, 39, 36 | 2 | 85 | 10838 | 12821 | 80 | 11127 |
| 128 | 7 | 2359 | 24, 21, 21, 14, 21, 14, 14 | 6 | 94 | 14154 | 15769 | 91 | 15085 |
| 137 | 7 | 2701 | 24, 15, 15, 23, 15, 23, 23 | 6 | 99 | 16206 | 17081 | 96 | 16243 |
| 139 | 3 | 6487 | 52, 39, 49 | 2 | 91 | 12974 | 15085 | 86 | 13075 |
| 151 | 3 | 7651 | 57, 43, 52 | 2 | 100 | 15302 | 17663 | 93 | 15453 |
| 163 | 3 | 8911 | 63, 49, 52 | 2 | 112 | 17822 | 27043 | 102 | 18437 |
| 151 | 7 | 3279 | 32, 19, 19, 21, 19, 21, 21 | 6 | 127 | 19674 | 28921 | 105 | 19769 |
Table 2. Parameters of configurations $v'_n$ with BDC incidence matrices, $v' < G(n)$, $n = k^#, k^# + 1, \ldots, k'$, by (ii) and (iii) of Subsection 3.1 from the cyclic affine plane $AG(2, q)$.

| $q$ | $t$ | $d$ | $w^*_i$ | $c$ | $f$ | $k'$ | $v'$ | $G(k')$ | $k^*$ | $G(k^*)$ |
|-----|-----|-----|---------|-----|-----|------|------|---------|------|---------|
| 31  | 3   | 320 | 14,9,8  | 2   | 22  | 640  | 713  | 21      | 667  |
| 37  | 3   | 486 | 16,9,12 | 2   | 25  | 912  | 961  | 25      | 961  |
| 49  | 4   | 600 | 16,12,9,12 | 3   | 34  | 1800 | 1877 | 34      | 1877 |
| 49  | 6   | 600 | 4,9,12,8,8,8 | 5   | 36  | 2000 | 2011 | 36      | 2011 |
| 53  | 4   | 702 | 17,12,10,14 | 3   | 37  | 2106 | 2199 | 37      | 2199 |
| 67  | 3   | 1240| 25,16,20 | 2   | 41  | 2480 | 2611 | 39      | 2505 |
| 71  | 5   | 1008| 8,14,15,16,18 | 4   | 50  | 4032 | 4189 | 50      | 4189 |
| 73  | 3   | 1776| 30,21,22 | 2   | 51  | 3552 | 4381 | 47      | 3609 |
| 79  | 3   | 2080| 32,25,22 | 2   | 54  | 4160 | 4747 | 50      | 4189 |
| 79  | 6   | 1040| 8,14,13,14,18,12 | 5   | 56  | 3200 | 5451 | 56      | 5451 |
| 79  | 6   | 1040| 18,12,8,14,13,14 | 2   | 50  | 4160 | 4189 | 50      | 4189 |
| 81  | 4   | 1640| 25,20,16,20 | 3   | 57  | 4920 | 5547 | 55      | 5197 |
| 81  | 4   | 1640| 25,20,16,20 | 1   | 45  | 3280 | 3375 | 45      | 3375 |
| 81  | 8   | 820 | 16,8,8,8,9,12,8,12 | 7   | 64  | 5740 | 7055 | 59      | 5823 |
| 81  | 8   | 820 | 16,8,8,8,9,12,8,12 | 6   | 56  | 4920 | 5451 | 55      | 5187 |
| 83  | 8   | 861 | 6,10,10,10,15,11,10,11 | 7   | 66  | 6027 | 7515 | 60      | 6039 |
| 83  | 8   | 861 | 6,10,10,10,15,11,10,11 | 6   | 56  | 5166 | 5451 | 55      | 5187 |
| 89  | 4   | 1980| 26,25,18,20 | 3   | 53  | 5940 | 6431 | 60      | 6039 |
| 89  | 8   | 990 | 17,8,10,12,8,10,10,14 | 7   | 65  | 6930 | 7187 | 64      | 7055 |
| 91  | 3   | 3136| 37,34,26 | 2   | 63  | 6272 | 6783 | 62      | 6431 |
| 97  | 4   | 2352| 29,26,20,22 | 3   | 69  | 7056 | 8291 | 65      | 7187 |
| 97  | 6   | 1586| 21,20,14,16,14,12 | 5   | 69  | 7930 | 8291 | 69      | 8291 |
| 101 | 4   | 2550| 30,25,20,26 | 3   | 70  | 7650 | 8435 | 68      | 7913 |
| 101 | 4   | 2550| 30,25,20,26 | 1   | 55  | 5100 | 5197 | 55      | 5197 |
| 103 | 3   | 3336| 41,32,30 | 2   | 71  | 7072 | 8661 | 65      | 7187 |
| 103 | 6   | 1768| 24,18,14,17,14,16 | 5   | 80  | 8840 | 11127 | 72      | 8947 |
| 103 | 6   | 1768| 24,18,14,17,14,16 | 4   | 66  | 7072 | 7515 | 65      | 7187 |
| 103 | 6   | 1768| 24,18,14,17,14,16 | 2   | 70  | 7072 | 8435 | 65      | 7187 |
| 107 | 8   | 1431| 17,15,10,13,10,12,17,13 | 7   | 77  | 10017 | 10409 | 76      | 10179 |
| 107 | 8   | 1431| 17,15,10,13,10,12,17,13 | 3   | 73  | 8586 | 9027 | 71      | 8661 |
| 109 | 3   | 3960| 42,30,37 | 2   | 72  | 7920 | 8947 | 69      | 8291 |
| 109 | 4   | 2970| 32,26,22,29 | 3   | 76  | 8910 | 10179 | 72      | 8947 |
| 109 | 9   | 1320| 8,14,20,10,13,10,12,10,12 | 8   | 78  | 10560 | 10599 | 78      | 10599 |
| 113 | 4   | 3192| 32,32,24,25 | 3   | 80  | 9576 | 11127 | 75      | 9965 |
| 113 | 7   | 1824| 10,20,14,17,22,16,14 | 6   | 80  | 10944 | 11127 | 80      | 11127 |
| 121 | 4   | 3660| 36,30,25,30 | 3   | 86  | 10980 | 13075 | 80      | 11127 |
| 121 | 4   | 3660| 36,30,25,30 | 1   | 66  | 7320 | 7515 | 66      | 7515 |
| 121 | 5   | 2928| 16,24,28,25,28 | 4   | 88  | 11712 | 13491 | 83      | 12041 |
| 125 | 4   | 3906| 37,32,26,30 | 3   | 89  | 11718 | 13557 | 83      | 12041 |
| 125 | 8   | 1953| 24,16,14,15,13,16,12,15 | 7   | 96  | 13671 | 16243 | 90      | 13935 |
| 125 | 8   | 1953| 24,16,14,15,13,16,12,15 | 3   | 93  | 11718 | 15453 | 83      | 12041 |
Table 3. The existence and nonexistence of cyclic symmetric configurations $v_k$, $k \leq 15$, $P(k) \leq v_\delta(k) \leq v < G(k)$

| $k$ | $P(k)$ | $v_\delta(k)$ | $v < G(k)$ in literature | $v < G(k)$ in this work | $E_c(k)$ $\leq G(k)$ |
|-----|--------|----------------|--------------------------|--------------------------|-------------------|
| 2   | 3      | 3              |                          |                          | 3. 3             |
| 3   | 7      | 7              |                          |                          | 7. 7             |
| 4.  | 13     | 13             |                          |                          | 13. 13           |
| 5.  | 21     | 21             | 21, 22                   |                          | 23. 23           |
| 6.  | 31     | 31             | 31, 32, 33, 34          |                          | 35. 35           |
| 7.  | 43     | 48             | 48-50                    |                          | 48. 51           |
| 8.  | 57     | 57             | 57, 58, 59-62, 63-68    |                          | 63. 69           |
| 9.  | 73     | 73             | 74, 75-79, 80, 81-84, 85-88 |                          | 85. 89           |
| 10  | 91     | 91             | 91, 92, 107-110         |                          | 107. 111         |
| 11  | 111    | 120            | 120, 133, 135-144       |                          | 135. 145         |
| 12  | 133    | 133            | 133, 134, 156, 158, 159, 161-170 | 158, 162-165, 167, 169, 170 | 161. 171 |
| 13  | 157    | 168            | 168, 183, 193-212       |                          | 193. 213         |
| 14  | 183    | 183            | 183, 184, 225-254       | 226, 227, 231, 233-254  | 225. 255         |
| 15  | 211    | 255            | 255, 267-302            | 278, 282, 284, 286, 287, 290-302 | 267. 303 |
Table 4. Values of $v$ for which a cyclic symmetric configuration $v_k$ exists, $16 \leq k \leq 41$, $P(k) \leq v < G(k)$

| $k$ | $P(k)$ | $P(k) \leq v < G(k)$ | $E_{\alpha}(k)$ | $\leq G(k)$ |
|-----|--------|-----------------------|-----------------|------------|
| 16  | 241    | $241^c$, $254^c$, $255b$, $272c$, $273a$, $288b$, $307a$, $318$, $320-329$, $331-354$ | 331 | 355 |
| 17  | 273    | $273a$, $274$, $288b$, $307a$, $342c$, $343$, $353$, $360b$, $357-363$, $381a$, $365-398$ | 365 | 399 |
| 18  | 307    | $307a$, $342c$, $360b$, $381a$, $401$, $403$, $405-407$, $410$, $412-418$, $420-432$ | 420 | 433 |
| 19  | 343    | $343$, $360b$, $381a$, $455$, $457$, $464$, $467$, $468$, $470-477$, $479$, $481-492$ | 481 | 493 |
| 20  | 381    | $381a$, $382$, $503$, $506c$, $508$, $513$, $516$, $519$, $520$, $525$, $528b$, $527-530$, $532$, $553a$, $534-566$ | 534 | 567 |
| 21  | 421    | $421$, $506c$, $528b$, $553a$, $592$, $597$, $601$, $602$, $606-609$, $611$, $624b$, $651a$, $614-666$ | 614 | 667 |
| 22  | 463    | $463$, $464$, $506c$, $528b$, $553a$, $624b$, $640$, $644$, $645$, $651a$, $649-712$ | 649 | 713 |
| 23  | 507    | $507$, $508$, $528b$, $553a$, $624b$, $651a$, $683$, $696$, $698$, $699$, $702$, $707$, $709-711$, $728b$, $713-744$ | 713 | 745 |
| 24  | 553    | $553a$, $554$, $624b$, $651a$, $728b$, $739$, $757a$, $759$, $761$, $763$, $765-770$, $772$, $782a$, $840b$, $775-850$ | 775 | 851 |
| 25  | 601    | $624b$, $651a$, $728b$, $757a$, $812c$, $837$, $840b$, $842-844$, $846-854$, $856-863$, $871a$, $930c$, $960b$, $865-960$ | 865 | 961 |
| 26  | 651    | $651a$, $652$, $728b$, $757a$, $812c$, $840b$, $871a$, $900$, $905-907$, $910$, $912$, $913$, $916$, $917$, $919-921$, $924$, $925$, $929$, $930c$, $932-941$, $960b$, $943-984$ | 943 | 985 |
| 27  | 703    | $728b$, $757a$, $812c$, $840b$, $871a$, $930c$, $960b$, $971$, $975$, $977$, $978$, $987$, $991$, $993a$, $994$, $997$, $1000$, $1001$, $1003-1006$, $1008$, $1010-1015$, $1017$, $1019$, $1023a$, $1057a$, $1021-1106$ | 1021 | 1107 |
| 28  | 757    | $757a$, $758$, $812c$, $840b$, $871a$, $930c$, $960b$, $993a$, $1023b$, $1045$, $1057a$, $1063$, $1067$, $1070$, $1074$, $1075$, $1077$, $1079-1082$, $1085-1170$, $814$, $840b$, $871a$, $930c$, $960b$, $993a$, $1023a$, $1057a$, $1146$, $1151$, $1152$, $1155-1158$, $1162-1167$, $1169$, $1172$, $1173$, $1175$, $1177$, $1180-1185$, $1187-1246$ | 1187 | 1247 |
| 29  | 813    | $871a$, $872$, $930c$, $960b$, $993a$, $1023b$, $1057a$, $1198$, $1199$, $1219$, $1220$, $1224$, $1229$, $1235$, $1236$, $1238$, $1240$, $1241$, $1243$, $1248$, $1249$, $1251$, $1253$, $1255$, $1258$, $1261$, $1264-1267$, $1269-1272$, $1332c$, $1274-1360$ | 1274 | 1361 |
| 30  | 871    | $932$, $960b$, $993a$, $1023b$, $1057a$, $1324$, $1325$, $1332c$, $1341$, $1344$, $1345$, $1346$, $1348$, $1349$, $1368b$, $1407a$, $1351-1494$ | 1351 | 1495 |
| 31  | 931    | $994$, $1023b$, $1057a$, $1332c$, $1368b$, $1383$, $1393$, $1401$, $1407a$, $1409$, $1411$, $1414$, $1421$, $1424$, $1428$, $1429$, $1430$, $1432$, $1434$, $1438-1441$, $1443-1445$, $1447-1457$, $1459-1568$, $1057a$, $1058$, $1332c$, $1368b$, $1407a$, $1492$, $1506$, $1507$, $1518$, $1521$, $1529$, $1533$, $1535$, $1540$, $1542$, $1545$, $1547-1553$, $1555$, $1557-1559$, $1561-1563$, $1565-1567$, $1569-1578$, $1580-1591$, $1640c$, $1680b$, $1593-1718$ | 1593 | 1719 |
Table 4 (continue). Values of \( v \) for which a cyclic symmetric configuration \( v_k \) exists, \( 16 \leq k \leq 41 \), \( P(k) \leq v < G(k) \)

| \( k \) | \( P(k) \) | \( P(k) \leq v < G(k) \) | \( E_v(k) \leq G(k) \) |
|------|--------|-----------------|-----------------|
| 34  | 1123   | \( T \overline{1}23, \overline{T}124, 1332_c, 1368_b, 1407_a, 1640_c, 1680_b, 1699, 1723_a, 1725, 1735, 1739, 1742, 1747, 1748, 1750, 1752, 1755-1757, 1759, 1761, 1765-1770, 1772, 1773, 1777, 1779-1785, 1806_c, 1848_b, 1787-1876 \) | 1787 1877 |
| 35  | 1191   | \( 1192, 1332_c, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, 1781, 1783, 1788, 1801, 1805, 1806_c, 1807, 1810, 1812, 1814, 1817, 1823, 1825, 1826, 1829-1833, 1835-1840, 1848_b, 1842-1856, 1893_a, 1859-1974 \) | 1859 1975 |
| 36  | 1261   | \( 1332_a, 1368_a, 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1855, 1860, 1886, 1893_a, 1902, 1905, 1907, 1908, 1912, 1915-1922, 1925-1930, 1932, 1937-1939, 1941-1952, 1954, 1955, 1957-1959, 1961-2010 \) | 1961 2011 |
| 37  | 1333   | \( T \overline{3}34, 1368_b, 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 1973, 1986, 1989, 2001, 2006-2008, 2010, 2017, 2018, 2023, 2024, 2028, 2031, 2033, 2036-2039, 2041-2043, 2045, 2046-2048, 2053, 2054-2057, 2059, 2061, 2063, 2065-2074, 2076-2083, 2162_c, 2085-2198 \) | 2085 2199 |
| 38  | 1407   | \( 1407_a, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2059, 2061, 2073, 2088, 2089, 2092, 2094, 2096, 2097, 2099, 2100, 2101, 2103, 2105, 2106, 2108, 2110, 2111, 2114-2116, 2118, 2124, 2126-2129, 2136-2139, 2142-2153, 2155-2157, 2159, 2161, 2162_c, 2163, 2164, 2166-2170, 2172, 2174-2178, 2208_b, 2257_a, 2180-2292 \) | 2180 2293 |
| 39  | 1483   | \( T \overline{4}83, \overline{T}464, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, 2265, 2278, 2281, 2287, 2293, 2294, 2297, 2300, 2302, 2304, 2315, 2317, 2323, 2324, 2326, 2330, 2338, 2340-2341, 2344-2346, 2348-2350, 2352-2354, 2358, 2361-2364, 2366-2369, 2372-2383, 2385-2393, 2400_b, 2395-2401, 2451_a, 2403-2504 \) | 2403 2505 |
| 40  | 1561   | \( 1562, 1640_c, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, 2326, 2345, 2372, 2374, 2389, 2393, 2396, 2400_b, 2401, 2404, 2411, 2414, 2416, 2417, 2418, 2423, 2424, 2427, 2431, 2435, 2436, 2438, 2440-2444, 2451_a, 2449-2453, 2455, 2459-2461, 2464-2467, 2471, 2474-2480, 2482-2484, 2486, 2487, 2489-2522, 2524-2564 \) | 2524 2565 |
| 41  | 1641   | \( 1642, 1680_b, 1723_a, 1806_c, 1848_b, 1893_a, 2162_c, 2208_b, 2257_a, 2345, 2400_b, 2449, 2451_a, 2460, 2479, 2480, 2491, 2494, 2496, 2499, 2508-2511, 2513, 2516, 2518-2521, 2524, 2525, 2528-2540, 2542, 2544, 2546-2548, 2550-2553, 2555-2562, 2564-2575, 2577-2610 \) | 2577 2611 |

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Table 5. Upper bounds on the cyclic existence bound $E_c(k)$, $42 \leq k \leq 83$

| $k$ | $E_c(k)$ | $G(k)$ | $k$ | $E_c(k)$ | $G(k)$ | $k$ | $E_c(k)$ | $G(k)$ | $k$ | $E_c(k)$ | $G(k)$ |
|-----|----------|--------|-----|----------|--------|-----|----------|--------|-----|----------|--------|
| 42  | 2632     | 2795   | 55  | 4463     | 4695   | 64  | 6796     | 7055   | 75  | 9883     | 9965   |
| 43  | 2860     | 3015   | 54  | 4513     | 4747   | 65  | 6853     | 7187   | 76  | 10023    | 10179  |
| 44  | 2917     | 3193   | 55  | 5195     | 5197   | 66  | 7279     | 7515   | 77  | 10229    | 10409  |
| 45  | 3280     | 3375   | 56  | 5341     | 5451   | 67  | 7359     | 7639   | 78  | 10395    | 10599  |
| 46  | 3353     | 3407   | 57  | 5501     | 5547   | 68  | 7463     | 7913   | 79  | 10800    | 10817  |
| 47  | 3453     | 3609   | 58  | 5551     | 5703   | 69  | 8111     | 8291   | 80  | 10977    | 11127  |
| 48  | 3765     | 3775   | 59  | 5612     | 5823   | 70  | 8125     | 8435   | 81  | 11396    | 11435  |
| 49  | 3839     | 3917   | 60  | 5687     | 6039   | 71  | 8288     | 8661   | 82  | 11443    | 11629  |
| 50  | 3871     | 4189   | 61  | 5994     | 6269   | 72  | 8694     | 8947   | 83  | 11593    | 12041  |
| 51  | 4308     | 4381   | 62  | 6150     | 6431   | 73  | 8813     | 9027   |     |           |        |
| 52  | 4359     | 4541   | 63  | 6611     | 6783   | 74  | 8965     | 9507   |     |           |        |
Table 6. Values of $v$ for which a symmetric configuration $v_k$ (cyclic or non-cyclic) of families of Section 2 exists, $k \leq 37$, $P(k) \leq v < G(k)$

| $k$ | $P(k)$ | $P(k) \leq v < G(k)$ |
|-----|--------|----------------------|
| 8   | 57     | $57, 63_{a,e}, 64_{m-68}$ |
| 9   | 73     | $73_{a}, 78_{b,g}, 80_{b,e}, 81_{m-88}$ |
| 10  | 91     | $91_{a}, 98_{b, 110_{c}, 110_{d,e,i}}$ |
| 11  | 111    | $120_{b,e}, 121_{m-133}, 143_{m-144}$ |
| 12  | 133    | $133_{a}, 156_{m-170}$ |
| 13  | 157    | $168_{b,e}, 169_{m-183}, 189_{f}, 208_{m-212}$ |
| 14  | 183    | $183_{a}, 210_{f}, 224_{m-254}$ |
| 15  | 211    | $231_{f}, 240_{m-302}$ |
| 16  | 241    | $252_{f,g}, 255_{b,e}, 256_{m-321}, 323_{m-354}$ |
| 17  | 273    | $273_{a}, 288_{b,e}, 289_{m-307}, 323_{m-381}, 391_{m-398}$ |
| 18  | 307    | $307_{a}, 342_{m-381}, 403_{f}, 414_{m-432}$ |
| 19  | 343    | $360_{b,e}, 361_{m-381}, 434_{f}, 437_{m-517}, 460_{m-492}$ |
| 20  | 381    | $381_{a}, 460_{m-481}, 483_{m-566}$ |
| 21  | 421    | $483_{m-666}$ |
| 22  | 463    | $506_{m-573}, 575_{m-712}$ |
| 23  | 507    | $528_{b,e}, 529_{m-553}, 558_{f}, 575_{m-744}$ |
| 24  | 553    | $553_{a}, 589_{f}, 600_{m-673}, 675_{m-850}$ |
| 25  | 601    | $620_{f,g}, 624_{b,e}, 625_{m-651}, 675_{m-960}$ |
| 26  | 651    | $651_{a}, 702_{m-781}, 783_{m-984}$ |
| 27  | 703    | $728_{b,e}, 729_{m-757}, 783_{m-1065}, 1073_{m-1103}$ |
| 28  | 757    | $757_{a}, 812_{m-1065}, 1073_{m-1103}, 1110_{m-1141}$ |
| 29  | 813    | $840_{b,e}, 841_{m-871}, 899_{m-1057}, 1073_{m-1103}, 1110_{m-1141}$ |
| 30  | 871    | $871_{a}, 930_{m-1057}, 1110_{m-1141}, 1147_{m-1179}$ |
| 31  | 931    | $960_{b,e}, 961_{m-1057}, 1147_{m-1179}, 1184_{m-1217}$ |
| 32  | 993    | $993_{a}, 1023_{b,e}, 1024_{m-1057}, 1184_{m-1217}, 1221_{m-1255}$ |
| 33  | 1057   | $1057_{a}, 1221_{m-1255}, 1258_{m-1293}$ |
| 34  | 1123   | $1258_{m-1293}, 1295_{m-1429}, 1435_{m-1876}$ |
| 35  | 1191   | $1295_{m-1407}, 1435_{m-1471}, 1476_{m-1974}$ |
| 36  | 1261   | $1332_{m-1407}, 1476_{m-1513}, 1517_{m-2010}$ |
| 37  | 1333   | $1368_{b,e}, 1369_{m-1407}, 1517_{m-1555}, 1558_{m-2198}$ |
Table 7. Parameters of new symmetric configurations \( v_k \) (cyclic and non-cyclic) from Sections 3 and 4

| \( k \) | \( P(k) \) | \( P(k) \leq v < G(k) \) | \( G(k) \) |
|---|---|---|---|
| 8 | 57 | 63, 68, 69, 70, 71, 72, 73 | 89 |
| 9 | 73 | 80, 88, 89, 90, 91 | 89 |
| 10 | 91 | 110, 111, 112, 113, 114 | 111 |
| 11 | 111 | 120, 121, 122, 123 | 145 |
| 12 | 133 | 156, 169, 170, 171, 172, 173 | 171 |
| 13 | 157 | 168, 183, 184, 185, 186 | 213 |
| 14 | 183 | 225, 226, 227, 228, 229 | 255 |
| 15 | 211 | 240, 241, 242, 243, 244 | 303 |
| 16 | 241 | 254, 255, 256, 257, 258 | 355 |
| 17 | 273 | 288, 292, 293, 294, 295 | 399 |
| 18 | 307 | 342, 343, 344, 345, 346 | 433 |
| 19 | 343 | 360, 361, 362, 363, 364 | 493 |
| 20 | 381 | 460, 461, 462, 463, 464 | 567 |
| 21 | 421 | 483, 529, 530, 531, 532 | 667 |
| 22 | 463 | 506, 507, 508, 509, 510 | 713 |
| 23 | 507 | 528, 529, 530, 531, 532 | 745 |
| 24 | 553 | 600, 601, 602, 603, 604 | 851 |
| 25 | 601 | 624, 625, 626, 627, 628 | 961 |
| 26 | 651 | 702, 703, 704, 705, 706 | 985 |
| 27 | 703 | 999, 1027, 1028, 1029, 1030 | 1107 |
| 28 | 757 | 812, 813, 814, 815, 816 | 1171 |
| 29 | 813 | 1110, 1111, 1112, 1113, 1114 | 1247 |
| 30 | 871 | 1147, 1148, 1149, 1150, 1151 | 1361 |
Table 7 (continue). Parameters of new symmetric configurations $v_k$ (cyclic and non-cyclic) from Sections 3 and 4

| $k$ | $P(k)$ | $P(k) \leq v < G(k)$ | $G(k)$ |
|-----|--------|------------------------|--------|
| 31  | 931    | 960, -1057r, 1147s-1179s, 1152, r,t-1185r,t, 1184s-1217s, 1188, r,t-1222r,t, 1221s-1255s, 1224r,r-1494r,t, 1258s-1494s, 1262v | 1495   |
| 32  | 993    | 1023r,-1057r, 1184s-1217s, 1188, r,t-1222r,t, 1221s-1255s, 1224r,r-1568r,t, 1258s-1293s, 1295s-1568s | 1569   |
| 33  | 1057   | 1221s-1255s, 1224r-1395t, 1224r-1718r, 1258s-1293s, 1295s-1387s, 1394r-1718s, 1400r-1718t, 1634u | 1719   |
| 34  | 1123   | 1258s-1293s, 1260r-1370t, 1260r-1436r, 1295s-1369s, 1394r-1429s, 1400r-1436t, 1435s-1876s, 1440r,r-1876r,t, 1634v, 1800p,w | 1877   |
| 35  | 1191   | 1295s-1369s, 1296r-1370t, 1296r-1407r, 1435s-1471s, 1440r,r-1477r,t, 1476s-1974s, 1480r,r-1974r,t, 1800p | 1975   |
| 36  | 1261   | 1332s-1369s, 1332r-1370t, 1332r-1407r, 1476s-1513s, 1480r,r-1518r,t, 1517s-1873s, 1520r,r-2010r,t, 1880s-2010s, 2000w | 2011   |
| 37  | 1333   | 1368r,-1407r, 1517s-1555s, 1520r-1881t, 1520r-2198r, 1558s-1717s, 1886r-1928t, 1720s-1873s, 1880s-1921s, 1927s-2065s, 1932r-2022t, 2024r-2068t, 2067s-2113s, 2070r-2198t, 2106w, 2109u-2147, 2115s-2198s, 2166w-2198w | 2199   |
| 38  | 1407   | 1560r,-2292r, 2166u-2205u, 2223u-2263u, 2280u-2292u | 2293   |
| 39  | 1483   | 1600r,-2504r, 2223r-2263u, 2280r-2321u, 2337u-2379u, 2394u-2437u, 2400p, 2451w-2495u, 2480w | 2505   |
| 40  | 1561   | 1640r,-1928r, 1932r-2564r, 2280r-2321u, 2337u-2379u, 2394u-2437u, 2400p, 2451w-2495u, 2480w, 2508w-2553w, 2522v | 2565   |
| 41  | 1641   | 1680r,-1723r, 1764r,-1893r, 1932r-1975r, 1978r-2492r, 2400p, 2480w, 2611   |
| 46  | 2071   | 2494,-2610r, 2522w | 2600   |
| 49  | 2353   | 2400p, 2774v | 3407   |
| 56  | 3081   | 4745n,v, 4920w, 5200w | 5451   |
Table 8. The current known parameters of symmetric configurations $v_k$ (cyclic and non-cyclic), an integrated table

| $k$ | $P(k)$ | $P(k) \leq v < G(k)$ | $E(k) \leq G(k)$ | filling |
|-----|--------|----------------------|------------------|---------|
| 3.  | 7      | 7                    | 7                | 100%    |
| 4.  | 13     | 13                   | 13.              | 100%    |
| 5.  | 21     | 21, 22               | 23.              | 100%    |
| 6.  | 31     | 31, 32, 33, 34       | 35.              | 100%    |
| 7   | 43     | 43, 44, 45, 48-50    | 48               | 51      | 75%    |
| 8   | 57     | 57, 58, 63-68        | 63               | 69      | 67%    |
| 9   | 73     | 73, 74, 78, 80-88    | 80               | 89      | 75%    |
| 10  | 91     | 91, 92, 98, 107-110  | 107              | 111     | 35%    |
| 11  | 111    | 111, 112, 120-133, 135-144 | 135 | 145 | 76%    |
| 12  | 133    | 133, 134, 135, 156-170 | 156 | 171 | 47%    |
| 13  | 157    | 158, 168-183, 189, 193-212 | 193 | 213 | 68%    |
| 14  | 183    | 183, 184, 210, 224-254 | 224 | 255 | 47%    |
| 15  | 211    | 211, 212, 231, 240-302 | 240 | 303 | 72%    |
| 16  | 241    | 252, 255-354         | 255              | 355     | 89%    |
| 17  | 273    | 273, 274, 288-307, 323-398 | 323 | 399 | 78%    |
| 18  | 307    | 307, 342-381, 401, 403, 405-407, 410, 412-432 | 412 | 433 | 54%    |
| 19  | 343    | 344, 360-381, 434, 437-492 | 437 | 493 | 53%    |
| 20  | 381    | 381, 382, 460-566    | 460              | 567     | 59%    |
| 21  | 421    | 422, 483-666         | 483              | 667     | 75%    |
| 22  | 463    | 463, 464, 506-712    | 506              | 713     | 84%    |
| 23  | 507    | 507, 508, 528-553, 558, 575-744 | 575 | 745 | 84%    |
| 24  | 553    | 553, 554, 589, 600-850 | 600 | 851 | 85%    |
| 25  | 601    | 601, 624-651, 675-960 | 675 | 961 | 88%    |
| 26  | 651    | 651, 702-984         | 702              | 985     | 85%    |
| 27  | 703    | 728-757, 783-1106    | 783              | 1107    | 88%    |
| 28  | 757    | 757, 758, 812-1170   | 812              | 1171    | 87%    |
| 29  | 813    | 813, 840-871, 899-1057, 1073-1246 | 1073 | 1247 | 84%    |
| 30  | 871    | 871, 872, 930-1057, 1110-1360 | 1110 | 1361 | 78%    |
| 31  | 931    | 931, 932, 960-1057, 1147-1494 | 1147 | 1495 | 68%    |
| 32  | 993    | 993, 994, 1023-1057, 1184-1568 | 1184 | 1569 | 73%    |
| 33  | 1057   | 1057, 1058, 1221-1718 | 1221 | 1719 | 76%    |
| 34  | 1123   | 1123, 1124, 1258-1876 | 1258 | 1877 | 82%    |
| 35  | 1191   | 1192, 1295-1407, 1435-1974 | 1435 | 1975 | 83%    |
| 36  | 1261   | 1332-1407, 1476-2010 | 1476 | 2011 | 81%    |
| 37  | 1333   | 1334, 1368-1407, 1517-2198 | 1517 | 2199 | 84%    |