ON BODIES WITH DIRECTLY CONGRUENT PROJECTIONS AND SECTIONS

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Abstract. Let $K$ and $L$ be two convex bodies in $\mathbb{R}^4$, such that their projections onto all 3-dimensional subspaces are directly congruent. We prove that if the set of diameters of the bodies satisfy an additional condition and some projections do not have certain symmetries, then $K$ and $L$ coincide up to translation and an orthogonal transformation. We also show that an analogous statement holds for sections of star bodies, and prove the $n$-dimensional versions of these results.

1. Introduction

In this paper we address the following problems (see [Ga, Problem 3.2, page 125 and Problem 7.3, page 289]).

Problem 1. Suppose that $2 \leq k \leq n-1$ and that $K$ and $L$ are convex bodies in $\mathbb{R}^n$ such that the projection $K|H$ is congruent to $L|H$ for all $H \in \mathcal{G}(n,k)$. Is $K$ a translate of $\pm L$?

Problem 2. Suppose that $2 \leq k \leq n-1$ and that $K$ and $L$ are star bodies in $\mathbb{R}^n$ such that the section $K \cap H$ is congruent to $L \cap H$ for all $H \in \mathcal{G}(n,k)$. Is $K$ a translate of $\pm L$?

Here we say that $K|H$, the projection of $K$ onto $H$, is congruent to $L|H$ if there exists an orthogonal transformation $\varphi \in O(k, H)$ in $H$ such that $\varphi(K|H)$ is a translate of $L|H$; $\mathcal{G}(n,k)$ stands for the Grassmann manifold of all $k$-dimensional subspaces in $\mathbb{R}^n$.

If the corresponding projections are translates of each other, or if the bodies are convex and the corresponding sections are translates of each other, the answers to Problems 1 and 2 are known to be affirmative [Ga, Theorems 3.1.3 and 7.1.1], (see also [A], [R1]). Besides, for Problem 1, with $k = n-1$, Hadwiger established a more general result and showed that it is not necessary to consider projections onto all $(n-1)$-dimensional subspaces; the hypotheses need only be true for one fixed subspace $H$, together with all subspaces containing a line orthogonal to $H$. In other words, one requires only a “ground” projection on $H$ and all corresponding “side” projections.

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Moreover, Hadwiger noted that in \( \mathbb{R}^n \), \( n \geq 4 \), the ground projection might be dispensed with (see [Ha], and [Ga] pages 126–127).

If the corresponding projections (sections) of convex (star-shaped) bodies are rotations of each other, the results in the case \( k = 2 \) were obtained by the third author in [R]; see also [BM].

In the general case of rigid motions, Problems 1 and 2 are open for any \( k \) and \( n \). In the special case of direct rigid motions, i.e., when the general orthogonal group \( O(k) \) is replaced by the special orthogonal group \( SO(k) \), the problems are open as well.

Golubyatnikov [Go1] obtained several interesting results related to Problem 1 in the cases \( k = 2, 3 \) [Go1, Theorem 2.1.1, page 13; Theorem 3.2.1, page 48]. In particular, he gave an affirmative answer to Problem 1 in the case \( k = 2 \) if the projections of \( K \) and \( L \) are directly congruent and have no direct rigid motion symmetries.

If the bodies are symmetric, then the answers to Problems 1 and 2 are known to be affirmative. In the case of projections they are the consequence of the Aleksandrov Uniqueness Theorem about convex bodies, having equal volumes of projections (see [Ga] Theorem 3.3.1, page 111); in the case of sections they follow from the Generalized Funk Theorem [Ga] Theorem 7.2.6, page 281.

In this paper we follow the ideas from [Go1] and [R] to obtain several Hadwiger-type results related to both Problems 1 and 2 in the case \( k = 3 \). In order to formulate these results we introduce some notation and definitions.

Let \( n \geq 4 \) and let \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \). We will use the notation \( w^\perp \) for the \((n-1)\)-dimensional subspace of \( \mathbb{R}^n \) orthogonal to \( w \in S^{n-1} \). We denote by \( d_K(\zeta) \) a diameter of a star body \( K \), which is parallel to the direction \( \zeta \in S^{n-1} \). We will also denote by \( O = O_{\zeta} \in O(n) \) the orthogonal transformation satisfying \( O|_{\zeta^\perp} = -I|_{\zeta^\perp} \), and \( O(\zeta) = \zeta \).

Let \( D \) be a subset of \( H \in G(n,3) \), and let \( \xi \in (H \cap S^{n-1}) \). We say that \( D \) has a \( \xi \)-rotational symmetry if \( \varphi(D) = D + a \) for some vector \( a \in H \) and some rotation \( \varphi \in SO(3,H) \) by the angle \( \alpha \pi, \alpha \in \mathbb{R} \setminus \{2\mathbb{Z}\} \), satisfying \( \varphi(\xi) = \xi \). In the particular case when the angle of rotation is \( \pi \), we say that \( D \) has a \((\xi,\pi)\)-rotational symmetry.

1.1. Results about directly congruent projections. We start with the following 4-dimensional result.

**Theorem 1.** Let \( K \) and \( L \) be two convex bodies in \( \mathbb{R}^4 \) having countably many diameters. Assume that there exists a diameter \( d_K(\zeta) \), such that the "side" projections \( K|w^\perp, L|w^\perp \) onto all subspaces \( w^\perp \) containing \( \zeta \) are directly congruent. Assume also that these projections have no \( \zeta \)-rotational symmetries and no \((u,\pi)\)-rotational symmetries for any \( u \in (\zeta^\perp \cap w^\perp \cap S^3) \). Then \( K = L + b \) or \( K = OL + b \) for some \( b \in \mathbb{R}^4 \).

If, in addition, the "ground" projections \( K|\zeta^\perp, L|\zeta^\perp \), are directly congruent and do not have rigid motion symmetries, then \( K = L + b \) for some \( b \in \mathbb{R}^4 \).
We state a straight $n$-dimensional generalization of Theorem 1 as a corollary.

**Corollary 1.** Let $K$ and $L$ be two convex bodies in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$ such that the “side” projections $K|H$, $L|H$ onto all 3-dimensional subspaces $H$ containing $\zeta$ are directly congruent. Assume also that these projections have no $\zeta$-rotational symmetries and no $(u, \pi)$-rotational symmetries for any $u \in (\zeta \perp \cap H \cap S^{n-1})$. Then $K = L + b$ or $K = OL + b$ for some $b \in \mathbb{R}^n$.

If, in addition, the “ground” projections $K|G$, $L|G$ onto all 3-dimensional subspaces $G$ of $\zeta \perp$, are directly congruent and have no rigid motion symmetries, then $K = L + b$ for some $b \in \mathbb{R}^n$.

In particular, we see that if $K$ and $L$ are convex bodies in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters, and directly congruent projections onto all 3-dimensional subspaces, and if the “side” and “ground” projections related to one of the diameters satisfy the conditions of the above corollary, then $K$ and $L$ are translates of each other.

This statement was proved by Golubyatnikov [Go1, Theorem 3.2.1, page 48] under the stronger assumptions that the “side” projections have no direct rigid motion symmetries. Theorem 1 and Corollary 1 under the same stronger assumptions are implicitly contained in his proof. To weaken the symmetry conditions on the “side” projections we replace the topological argument from [Go1] with an analytic one based on ideas from [R] (compare [Go1, pages 48–52] with Proposition 1 in Section 3).

We note that the assumption about countability of the sets of the diameters of $K$ and $L$ can be weakened. Instead, one can assume, for example, that these sets are subsets of a countable union of the great circles containing $\zeta$ (see the remark after Lemma 10 in Section 4). We also note that the set of bodies considered in the above statements contains the set of all polytopes whose three dimensional projections do not have rigid motion symmetries. This set of polytopes is an everywhere dense set with respect to the Hausdorff metric in the class of all convex bodies in $\mathbb{R}^n$, $n \geq 4$. For the convenience of the reader we prove this in the Appendix.

1.2. **Results about directly congruent sections.** The analytic approach also allows to obtain results related to Problem 2 (see [Ga, pages 288-290, open problems 7.1, 7.3, and Note 7.1]).

**Theorem 2.** Let $K$ and $L$ be two star-shaped bodies with respect to the origin in $\mathbb{R}^4$, having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$ containing the origin, such that for all subspaces $w^\perp$ containing $\zeta$, the “side” sections $K \cap w^\perp$, $L \cap w^\perp$, are direct rigid motions of each other. Assume also that these sections have no $\zeta$-rotational symmetries and no $(u, \pi)$-rotational symmetries for any $u \in (\zeta^\perp \cap w^\perp \cap S^3)$. Then $K = L + b$ or $K = OL + b$ for some $b \in \mathbb{R}^4$ parallel to $\zeta$. 
As in the case of projections, we state a straight \( n \)-dimensional generalization of Theorem 2 as a corollary.

**Corollary 2.** Let \( K \) and \( L \) be star-shaped bodies with respect to the origin in \( \mathbb{R}^n \), \( n \geq 4 \), having countably many diameters. Assume that there exists a diameter \( d_K(\zeta) \) containing the origin, such that for all 3-dimensional subspaces \( H \) containing \( \zeta \), the “side” sections \( K \cap H, L \cap H \) are directly congruent. Assume also that these sections have no \( \zeta \)-rotational symmetries and no \((u, \pi)\)-rotational symmetries for any \( u \in (\zeta \perp \cap H \cap S^3) \). Then \( K = L + b \) or \( K = \partial L + b \) for some \( b \in \mathbb{R}^n \) parallel to \( \zeta \).

Applying the ideas used in this paper, one can obtain similar results related to both Problems 1 and 2 in the case \( k = 2 \), see [AC]. However, we are unaware of results related to the case \( k \geq 4 \).

The paper is organized as follows. In Section 3 we formulate and prove our main auxiliary result Proposition 1. Section 4 is devoted to the proof of Theorem 2 and Corollary 2. Theorem 2 and Corollary 2 are proved in Section 5. We prove that the set of polytopes in \( \mathbb{R}^n, n \geq 4 \), with 3-dimensional projections having no rigid motion symmetries is dense in the Hausdorff metric in the class of all convex bodies in the Appendix.

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## 2. Notation and auxiliary definitions

We will use the following standard notation. The unit sphere in \( \mathbb{R}^n \) (\( n \geq 2 \)), is \( S^{n-1} \). Given \( w \in S^{n-1} \), the hyperplane orthogonal to \( w \) and passing through the origin will be denoted by \( w^\perp = \{ x \in \mathbb{R}^n : x \cdot w = 0 \} \), where \( x \cdot w = x_1w_1 + \cdots + x_nw_n \) is the usual inner product in \( \mathbb{R}^n \). The Grassmann manifold of all \( k \)-dimensional subspaces in \( \mathbb{R}^n \) will be denoted by \( \mathcal{G}(n,k) \).

The notation \( O(k) \) and \( SO(k) \), \( 2 \leq k \leq n \) for the subgroups of the orthogonal group \( O(n) \) and the special orthogonal group \( SO(n) \) in \( \mathbb{R}^n \) is standard. If \( U \in O(n) \) is an orthogonal matrix, we will write \( U^t \) for its transpose.

Refer to Figure 1 for the next two definitions. Given \( \xi \in S^{n-1} \), the great \((n - 2)\)-dimensional sub-sphere of \( S^{n-1} \) that is perpendicular to \( \xi \) will be denoted by \( S^{n-2}(\xi) = \{ \theta \in S^{n-1} : \theta \cdot \xi = 0 \} \). Similarly, for \( \zeta \in S^3 \) and \( t \in [-1,1] \), the parallel to \( S^3(\zeta) \) at height \( t \) will be denoted by \( S^3_t(\zeta) = S^3 \cap \{ x \in \mathbb{R}^4 : x \cdot \zeta = t \} \). Observe that when \( t = 0 \), \( S^3_0(\zeta) = S^2(\zeta) \).

Let \( E \) be a two or three-dimensional subspace of \( \mathbb{R}^n \). We will write \( \varphi_E \in SO(2,E) \), or \( \varphi_E \in SO(3,E) \), meaning that there exists a choice of an orthonormal basis in \( \mathbb{R}^n \) and a rotation \( \Phi \in SO(n) \), with a matrix written in this basis, such that the action of \( \Phi \) on \( E \) is the rotation \( \varphi_E \) in \( E \), and the action of \( \Phi \) on \( E^\perp \) is trivial, i.e., \( \Phi(y) = y \) for every \( y \in E^\perp \) (here \( E^\perp \) stands for the orthogonal complement of \( E \)). A similar notation will be used.
for \( \varphi_E \in O(3, E) \). We will also denote by \( O(3, S^2(w)) \), \( SO(3, S^2(w)) \), the orthogonal transformations in the 3-dimensional subspace spanned by the great subsphere \( S^2(w) \) of \( S^3 \). The restriction of a transformation \( \varphi \in O(n) \) onto the subspace of smallest dimension containing \( E \subset S^{n-1} \) will be denoted by \( \varphi|_E \). \( I \) stands for the identity transformation.

We refer to [Ga, Chapter 1] for the next definitions involving convex and star-shaped bodies. A body in \( \mathbb{R}^n \) is a compact set which is equal to the closure of its non-empty interior. A convex body is a body \( K \) such that for every pair of points in \( K \), the segment joining them is contained in \( K \). For \( x \in \mathbb{R}^n \), the support function of a convex body \( K \) is defined as \( h_K(x) = \max \{ x \cdot y : y \in K \} \) (see page 16 in [Ga]). The width function \( w_K(x) \) of \( K \) in the direction \( x \in S^{n-1} \) is defined as \( \omega_K(x) = h_K(x) + h_K(-x) \). The segment \( [z, y] \subset K \), parallel to \( \zeta \in S^{n-1} \), is called the diameter of the body \( K \) if \( |z - y| = \max_{\theta \in S^{n-1}} \omega_K(\theta) \). We will denote it by \( d_K(\zeta) \).

We say that a convex body \( K \subset \mathbb{R}^n \) has countably many diameters if the width function \( \omega_K \) reaches its maximum on a countable subset of \( S^{n-1} \).

A set \( S \subset \mathbb{R}^n \) is said to be star-shaped at a point \( p \) if the line segment from \( p \) to any point in \( S \) is contained in \( S \). Let \( x \in \mathbb{R}^n \setminus \{0\} \), and let \( K \subset \mathbb{R}^n \) be a star-shaped set with respect to the origin. The radial function \( \rho_K \) in the direction \( x \in S^{n-1} \) is defined as \( \rho_K(x) = \max \{ c : cx \in K \} \). Here the line through \( x \) and the origin is assumed to meet \( K \), (Ga page 18]). The segment \( [z, y] \subset K \), parallel to \( \zeta \in S^{n-1} \), is called the diameter of the body \( K \) if \( |z - y| = \max_{\{a, b \in K\}} |a - b| \). As in the case of a convex body, we will denote this diameter by \( d_K(\zeta) \).

Finally, we define the notion of rigid motion symmetry as will be used throughout the paper.

**Figure 1.** The great spheres \( S^2(\zeta) \) and \( S^2(\xi) \).
**Definition 1.** Let $D$ be a subset of $H \in G(n,3)$. We say that $D$ has a rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-identical orthogonal transformation $\varphi \in O(3,H)$ in $H$. Similarly, $D$ has a direct rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-trivial rotation $\varphi \in SO(3,H)$.

**Figure 2.**

**Figure 3.**

3. A result about a functional equation on $S^3$

In [R], the third author proved that if two continuous functions $F$ and $G$ on $S^2$ coincide up to rotation on each one-dimensional great circle, then either $F(x) = G(x)$ or $F(x) = G(-x)$ for every $x \in S^2$. The main result of this section is a related statement for $S^3$, which, in our opinion, has independent interest.

**Proposition 1.** Let $f$ and $g$ be two continuous functions on $S^3$. Assume that for some $\zeta \in S^3$ and for every $w \in S^2(\zeta)$ there exists a rotation $\varphi_w \in SO(3,S^2(w))$, verifying $\varphi_w(\zeta) = \zeta$, and

$$f \circ \varphi_w = g \quad \text{on} \quad S^2(w),$$
Then either \( f = g \) on \( S^3 \) or \( f(\theta) = g(O\theta) \) \( \forall \theta \in S^3 \),
where \( O \in O(4) \) is the orthogonal transformation satisfying \( O|_{S^2(\zeta)} = -I \),
and \( O(\zeta) = \zeta \).

3.1. **Auxiliary observations.** The direction \( \zeta \in S^3 \) will be fixed throughout the proof. We start with an easy observation about the geometry of the three dimensional sphere.

![Figure 4](image1)

**Figure 4.**

![Figure 5](image2)

**Figure 5.**

**Lemma 1.** Let \( \zeta \in S^3 \) and let \( \xi \in S^2(\zeta) \). Then

\[
S^3 = \bigcup_{\{w \in S^2(\xi) \cap S^2(\zeta)\}} S^2(w).
\]

**Proof.** As it is shown in Figure 4, for any \( w \in S^2(\zeta) \), the two-dimensional sphere \( S^2(w) \) can be written as the union of all one-dimensional parallels \( S^2(w) \cap S^2_\zeta \), \( t \in [-1,1] \), i.e.

\[
S^2(w) = \bigcup_{\{t \in [-1,1]\}} (S^2(w) \cap S^2_\zeta).
\]

On the other hand, we can write the two-dimensional sphere \( S^2(\zeta) \) as the union of all meridians containing a fixed direction \( \xi \in S^2(\zeta) \) (see Figure 5, left), by

\[
S^2(\zeta) = \bigcup_{\{w \in S^2(\zeta) \cap S^2(\zeta)\}} (S^2(w) \cap S^2(\zeta)),
\]
and, rescaling, the same is true for every \( S^2_t(\zeta) \), \( t \in [-1,1] \) (see Figure 5, right). Thus, we have

\[
S^2_t(\zeta) = \bigcup_{\{w \in S^2(\xi) \cap S^2(\zeta)\}} (S^2(w) \cap S^2_t(\zeta)) \quad \forall t \in [-1,1].
\]

Combining (3) and (4), we obtain

\[
S^3 = \bigcup_{\{t \in [-1,1]\}} S^2_t(\zeta) = \bigcup_{\{t \in [-1,1]\}} \bigcup_{\{w \in S^2(\xi) \cap S^2(\zeta)\}} (S^2(w) \cap S^2_t(\zeta)) = \bigcup_{\{w \in S^2(\xi) \cap S^2(\zeta)\}} \bigcup_{\{t \in [-1,1]\}} (S^2(w) \cap S^2_t(\zeta)) = \bigcup_{\{w \in S^2(\xi) \cap S^2(\zeta)\}} S^2(w).
\]

\[\square\]

\[\text{Figure 6.}\]

Let \( O \in O(4) \) be an orthogonal transformation, satisfying \( O|_{S^2(\zeta)} = -I \), and \( O(\zeta) = \zeta \). Observe that \( O|_{S^2(w)} \) commutes with every rotation \( \varphi_w \in SO(3, S^2(w)) \), such that \( \varphi_w(\zeta) = \zeta \), where \( w \in S^2(\zeta) \). It is clear that any function \( f \) on \( S^3 \) can be decomposed in the form

\[
f(\theta) = \frac{f(\theta) + f(O\theta)}{2} + \frac{f(\theta) - f(O\theta)}{2} = f_{O,e}(\theta) + f_{O,o}(\theta), \quad \theta \in S^3,
\]

where we will call \( f_{O,e}, f_{O,o} \), the even and odd parts of \( f \) with respect to \( O \). Since \( O^2 = I \), we have

\[f_{O,e}(\theta) = f_{O,e}(O\theta), \quad f_{O,o}(\theta) = -f_{O,o}(O\theta).\]

It is also clear that every \( \theta \in S^3 \) belongs to \( S^2_t(\zeta) \) for some \( t \in [-1,1] \), i.e., can be written in the form

\[
\theta = \sqrt{1-t^2}x + t\zeta, \quad \text{for some } t \in [-1,1] \text{ and } x \in S^2(\zeta) \text{ (see Figure 6)}.\]
Let \( t \in [-1, 1] \). For any function \( f \) on \( S^3 \), we can define the function \( F_t \) on \( S^2(\zeta) \),

\[
F_t(x) = F_{t,\zeta}(x) = f(\sqrt{1-t^2}x + t\zeta), \quad x \in S^2(\zeta),
\]

which is the restriction of \( f \) onto \( S^2(\zeta) \). Observe that

\[
(F_t)_e(x) = \frac{f(\sqrt{1-t^2}x + t\zeta) + f(-\sqrt{1-t^2}x + t\zeta)}{2} = \frac{f(\theta) + f(\bar{\theta})}{2},
\]

where \( \theta \) is as in (6), i.e.,

\[
(F_t)_e(x) = f_{\Theta, e}(\theta), \quad (F_t)_o(x) = f_{\Theta, o}(\theta).
\]

Note that \((F_t)_e(x) = (F_t)_e(-x)\) for every \( x \in S^2(\zeta) \).

As seen in Lemma 1, every one-dimensional great circle of \( S^2(\zeta) \) is of the form \( S^2(w) \cap S^2(\zeta) \) for some \( w \in S^2(\zeta) \). To simplify the notation, we will denote such great circles by

\[
E = E_{\zeta, w} = \{ \theta \in S^3 : \theta \cdot \zeta = \theta \cdot w = 0 \}.
\]

Since \( \varphi_w(\zeta) = \zeta \) and \( \varphi_w(S^2(w)) = S^2(w) \), we have

\[
\varphi_w(E_{\zeta, w}) = \varphi_w(S^2(w)) \cap S^2(\zeta) = S^2(w) \cap S^2(\zeta) = E_{\zeta, w}.
\]

Thus, for every \( t \in [-1, 1] \), and for the corresponding one-dimensional equator \( E = E_{\zeta, w} \) of \( S^2(\zeta) \), there is a rotation \( \phi_E \in SO(2, E) \), which is the restriction to \( E \) of the rotation \( \varphi_w \in SO(3, S^2(w)) \) given by the conditions of Proposition 1 and which satisfies

\[
F_t \circ \phi_E(x) = G_t(x) \quad \forall x \in E,
\]

(see Figure 5). Here \( G_t \) is defined from \( g \) similarly to \( F_t \) in (7).

3.2. Auxiliary Lemmata. We will use the Funk transform, \cite[Chapter III, §1]{He},

\[
Rf(w) = R_{\zeta}f(w) = \int_{S^2(w) \cap S^2(\zeta)} f(\theta)d\theta, \quad w \in S^2(\zeta).
\]

Here \( d\theta \) stands for the Lebesgue measure on the one-dimensional great circle \( E = S^2(w) \cap S^2(\zeta) \) of \( S^2(\zeta) \).

**Lemma 2.** Let \( f \) and \( g \) be as in Proposition 1. Then \( f_{\Theta, e} = g_{\Theta, e} \).

**Proof.** Let \( w \in S^2(\zeta) \), and let \( \varphi_w \in SO(3, S^2(w)) \) be such that (1) holds. Then, \( \phi_E = \varphi_w|_{S^2(w) \cap S^2(\zeta)} \in SO(2, E) \) is the corresponding rotation in \( E = S^2(w) \cap S^2(\zeta) \). By the rotation invariance of the Lebesgue measure on \( E \) and (9), we have

\[
\int_E F_t(x)dx = \int_E F_t \circ \phi_E(x)dx = \int_E G_t(x)dx, \quad \forall t \in [-1, 1],
\]

Hence, \( R_{\zeta}F_t(w) = R_{\zeta}G_t(w) \) for every \( w \in S^2(\zeta) \). Thus, \((F_t)_e(x) = (G_t)_e(x)\) for every \( x \in S^2(\zeta) \) (apply Theorem C.2.4 from \cite[page 430]{Ga} to \((F_t)_e - \)}
(G_t)_o). Using the first relation in (8), its analogue for \( g \), and (3), we obtain the desired result.

**Remark 1.** By Lemma 2, we can assume that our functions \( f \) and \( g \) are odd with respect to \( O \). In order to simplify the notation, from now on we will write \( f \) and \( g \) instead of \( f_{O,o} \) and \( g_{O,o} \). We will also write \( F_t \) for \( (F_t)_o \) and \( G_t \) for \( (G_t)_o \).

Let \( \varphi_{w}^{\alpha\pi} \) be the rotation of the sphere \( S^2(w) \) by the angle \( \alpha\pi \) around \( \zeta \), i.e., \( \varphi_{w}^{\alpha\pi}(\zeta) = \zeta \). By this we mean that \( \varphi_{w}^{\alpha\pi} \) is the restriction to the 3-dimensional subspace spanned by \( S^2(w) \) of a rotation \( \Phi \in \text{SO}(4) \) with the following properties: \( \Phi(\zeta) = \zeta, \Phi(w) = w \), if \( \{x,y,w,\zeta\} \) is a positively oriented orthonormal basis of \( \mathbb{R}^4 \), then for every \( v \in (\text{span}\{x,y\} \cap S^3) = S^2(w) \cap S^2(\zeta) \), the angle between the vectors \( v \) and \( \varphi_{w}^{\alpha\pi}(v) \in S^2(w) \cap S^2(\zeta) \) is \( \alpha\pi \), and if \( \alpha \) is not an integer, \( \{v, \varphi_{w}^{\alpha\pi}(v), w, \zeta\} \) form a positively oriented basis of \( \mathbb{R}^4 \).

For any \( \alpha \in \mathbb{R} \), we consider the sets \( \Xi_{\alpha} \), defined as

\[
\Xi_{\alpha} = \{ w \in S^2(\zeta) : \exists \varphi_{w}^{\alpha\pi} \in \text{SO}(3, S^2(w)) \text{ such that } f \circ \varphi_{w}^{\alpha\pi} = g \text{ on } S^2(w) \}.
\]

Observe that \( \Xi_0 = \{ w \in S^2(\zeta) : f = g \text{ on } S^2(w) \} \), and

\[
\Xi_1 = \{ w \in S^2(\zeta) : f(\theta) = g(O\theta) \quad \forall \theta \in S^2(w) \}.
\]

Our aim is to show that \( S^2(\zeta) = \Xi_0 \cup \Xi_1 \). This will be achieved in Lemma 8 if we prove the following Lemmata.

**Lemma 3.** The set \( \Xi_{\alpha} \) is closed.

**Proof.** We can assume that \( \Xi_{\alpha} \) is not empty.

Let \( (w_l)_{l=1}^{\infty} \) be a sequence of elements of \( \Xi_{\alpha} \) converging to \( w \in S^2(\zeta) \) as \( l \to \infty \), and let \( \theta \) be any point on \( S^2(w) \). Consider a sequence \( (\theta_l)_{l=1}^{\infty} \) of points \( \theta_l \in S^2(w_l) \) converging to \( \theta \) as \( l \to \infty \).

(If it is readily seen that such a sequence exists. Indeed, let \( B_{1/2}(\theta) \) be a Euclidean ball centered at \( \theta \) of radius \( 1/2 \), where \( l \in \mathbb{N} \). Since \( S^2(w_l) \) converging to \( \theta \) as \( l \to \infty \), for each \( l \in \mathbb{N} \) there exists \( m = m(l) \) such that

\[
S^2(w_m) \cap B_{1/2}(\theta) \neq \emptyset.
\]

Choose any \( \theta_l = \theta_{m(l)} \in S^2(w_{m(l)}) \) \( \cap B_{1/2}(\theta) \). Then \( \theta_l \to \theta \) as \( l \to \infty \).

By the definition of \( \Xi_{\alpha} \), we see that

\[
f \circ \varphi_{w_l}(\theta_l) = g(\theta_l) \quad \theta_l \in S^2(w_l), \quad l \in \mathbb{N}.
\]

Passing to a subsequence if necessary, we can assume that the sequence of rotations \( \varphi_{w_l}(\theta_l)_{l=1}^{\infty} \), \( \varphi_{w_l} = \varphi_{w_l}^{\alpha\pi} \in \text{SO}(3, S^2(w_l)) \), is convergent, say, to \( \varphi_w \in \text{SO}(3, S^2(w)) \). Writing out the matrices of rotations \( \varphi_{w_l}^{\alpha\pi} \) in the corresponding orthonormal bases \( \{x_l, y_l, w_l, \zeta\} \), \( x_l, y_l \in S^2(w) \cap S^2(\zeta) \), and passing to the limit as \( l \to \infty \) we see that \( \varphi_w \) is the rotation by the angle \( \alpha\pi \) and the limit of \( f \circ \varphi_{w_l}(\theta_l) = g(\theta) \). Since the choice of \( \theta \in S^2(w) \) was arbitrary, we obtain \( w \in \Xi_{\alpha} \), and the result follows. 

\( \square \)
Lemma 4. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $\Xi_\alpha \subseteq \Xi_0$.

Proof. Let $w \in \Xi_\alpha$. Following the ideas of Schneider [Sch1], we claim at first that $f^2 = g^2$ on $S^2(w)$. Indeed, since $f$ and $g$ are odd with respect to $O$, $f^2$ and $g^2$ are even with respect to $O$, and satisfy the conditions of Proposition 1. Thus, by Lemma 2 we obtain that $f^2 = g^2$ on $S^2(w)$.

Squaring (1), we have

$$f^2 \circ \varphi_w(\theta) = g^2(\theta) = f^2(\theta) \quad \forall \theta \in S^2(w).$$

Iterating for any $k \in \mathbb{Z}$,

$$f^2 \circ \varphi_w^k(\theta) = f^2 \circ \varphi_w^{k-1}(\theta) = \cdots = f^2(\theta) \quad \forall \theta \in S^2(w),$$

and using the fact that for every $\theta \in S^2(w)$, the orbit of $(\varphi_w^k(\theta))_{k \in \mathbb{Z}}$ is dense on every parallel of $S^2(w)$ orthogonal to $\zeta$, we obtain that the restrictions of $f^2$ and $g^2$ onto $S^2(w)$ are invariant under rotations leaving $\zeta$ fixed. In other words, $f^2$ and $g^2$ are constant on every parallel of $S^2(w)$ orthogonal to $\zeta$. By continuity, $f$ and $g$ must also be constant on these parallels and $f \circ \varphi_w = f$. Hence, using (1) we have $f = g$ on $S^2(w)$, and therefore $w \in \Xi_0$. Since $w$ from $\Xi_\alpha$ was chosen arbitrarily, we obtain the desired result. $\square$

In Lemma 4 we have shown that rotations whose angle is an irrational multiple of $\pi$ are not relevant under the assumptions of Proposition 1. Our next goal is to prove that rational multiples are not relevant either, excepting the rotations by the angles $0$ and $\pi$. This will be achieved in Lemma 5 by means of a topological argument, which is based on one definition and two Lemmata from [R] (see Lemmata 5 and 6 below). The argument will show that for each $t \in (-1, 1)$ and an appropriate $w \in S^2(\zeta)$, the subset of a great circle $S^2(w) \cap S^2(\zeta)$, where the functions $F_t = G_t$ are equal to each other, is open. Since such a set is closed by definition, and it is non-empty, we will conclude that $F_t$ equals $G_t$ on this large circle. Using (3) we will obtain that $f = g$ on the corresponding $S^2(w)$, which will give us the desired result.

We will reformulate the corresponding statements from [R] in a way that is more convenient for us here. Refer to Figure 7 for the next definition.

Definition 2. Let $\alpha \in (0, 1)$ and let $S_1, S_2$ be any two spherical circles in the standard metric of $S^2(\zeta)$, both of radius $\alpha \pi$. The union $l \cup m$ of two open arcs $l \subset S_1$ and $m \subset S_2$ will be called a spherical $X$-figure if the angle between arcs is in $(0, \frac{\pi}{4})$, the length of the arcs is less than $\alpha \pi$, and the arcs intersect at their centers only, $l \cap m = \{x\}$. The point $x \in S^2(\zeta)$ will be called the center of the $X$-figure.

Let $t \in (-1, 1)$, $F_t$ be a function on $S^2(\zeta)$, and $x$ be the center of a spherical $X$-figure. If for every $u \in X$ we have $F_t(u) = F_t(x)$, we will say that there exists an $X$-figure $X_{F_t(x)} \subset S^2(\zeta)$. The following result is Lemma 10 from [R] with $f = F_t$, $g = G_t$, $f_e = F_t^2$, and $S^2 = S^2(\zeta)$.

Lemma 5. Let $t \in (-1, 1)$, and let $F_t$ and $G_t$ be two continuous functions on $S^2(\zeta)$. Assume that there is an open spherical cap $U \subset \Xi_{\frac{\pi}{4}}$, with $\frac{\pi}{4} \in$
(0,1) \cap \mathbb{Q}$, such that for every $w \in U$, there exists a rotation $\phi_w = \phi_{w,\zeta}$ of the great circle $S^2(w) \cap S^2(\zeta)$ by the angle $\frac{p}{q} \pi$, verifying

$$F_t \circ \phi_w(x) = G_t(x) \quad \forall x \in S^2(w) \cap S^2(\zeta).$$

Then, for every $x \in S^2(w) \cap S^2(\zeta)$ there exists an $X$-figure $X_{F_t^2(x)} \subset S^2(\zeta)$, with one of the arcs of $X_{F_t^2(x)}$ being orthogonal to $S^2(w) \cap S^2(\zeta)$. Moreover, for every $x, y \in S^2(w) \cap S^2(\zeta)$ there exist $X$-figures $X_{F_t^2(x)}$, $X_{F_t^2(y)} \in S^2(\zeta)$, such that

$$\Theta(X_{F_t^2(x)}) = X_{F_t^2(y)},$$

where $\Theta \in SO(3, S^2(\zeta))$ is such that $\Theta(w) = w$ and $\Theta(x) = y$.

**Lemma 6.** Let $t \in (-1, 1)$, and let $F_t$, $G_t$, and $U$ be as above. Then, for every $w \in U$ there exists a constant $c$ such that $F_t^2(x) = G_t^2(x) = c$ for every $x \in S^2(w) \cap S^2(\zeta)$.

Observe that since any two great circles of $S^2(\zeta)$ intersect, the above constant is actually independent of $w \in U$.

The idea of the proof is, assuming that Lemma 6 is not true, to use Lemma 5 to show the existence of an uncountable family of disjoint spherical $X_{F_t^2(x)}$-figures, $x \in S^2(w) \cap S^2(\zeta)$ (with $F_t^2$ being constant on the corresponding figure). But this family cannot exist: if the $X$-figures are disjoint, one can find a collection of disjoint open balls, each centered at the center of the corresponding $X$-figure; we obtained the uncountable collection of disjoint open balls, which gives the desired contradiction. The exact details can be found in the proof of Lemma 5, [R], starting 17 lines from below on page 12.
Lemma 7. Let \( t \in (-1,1) \), and let \( F_t, G_t, \) and \( U \) be as above. Then \( f = g = 0 \) on \( S^2(w) \) for every \( w \in U \).

Proof. Let \( w \) be any point in \( U \), and let \( t \in (-1,1) \). By Remark 1, \( f \) and \( g \) are odd with respect to \( O \) on \( S^3 \). Using the second relation in (8), we see that \( F_t, G_t \) are odd on \( S^2(\zeta) \). By continuity, there exist \( x_1, x_2 \in S^2(w) \cap S^2(\zeta) \) such that \( F_t(x_1) = G_t(x_2) = 0 \). By Lemma 6, \( F_t^2(x) = G_t^2(x) = 0 \) for every \( x \in S^2(w) \cap S^2(\zeta) \). Using (7) and the continuity of \( f \) and \( g \), we see that the last statement is true for all \( t \in [-1,1] \). Finally, using (9) and (7) again, we conclude that \( f = g = 0 \) on \( S^2(w) \). \( \square \)

Now we are ready to prove

Lemma 8. We have \( S^2(\zeta) = \Xi_0 \cup \Xi_1 \).

Proof. Assume the contrary, the set \( A := S^2(\zeta) \setminus (\Xi_0 \cup \Xi_1) \) is not empty. By Lemma 4, \( A \cap \Xi_0 = \emptyset \), provided \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \). Hence, \( A \) may be written as

\[
A = \bigcup_{\frac{p}{q} \in \mathbb{Q}} \Xi_{\frac{p}{q}}.
\]

By Lemma 3, all \( \Xi_{\frac{p}{q}} \) are closed and \( A \) is open. Hence, by the Baire category Theorem, (cf. Lemma 8 from [R]), there exists \( \frac{p}{q} \in \mathbb{Q} \) such that \( \text{int}(\Xi_{\frac{p}{q}}) \neq \emptyset \). We can assume that there exists an open spherical cap \( U \subseteq (A \cap \Xi_{\frac{p}{q}}) \) such that for every \( w \in U \), there is a rotation \( \varphi_{\frac{p}{q}} \in SO(3, S^2(w)) \) such that

\[
f \circ \varphi_{\frac{p}{q}} = g \text{ on } S^2(w).
\]

In particular, for any \( t \in (-1,1) \), and for every large circle \( E = S^2(w) \cap S^2(\zeta) \) of \( S^2(\zeta) \) there exists a rotation \( \phi_w \in SO(2, E) \) by the angle \( \frac{p}{q} \pi \) such that (14) holds. Changing the orientation if necessary, we can assume that \( p/q \) is between 0 and 1.

By Lemma 6, \( F_t^2(x) = G_t^2(x) = c \) for every \( x \in S^2(w) \cap S^2(\zeta) \), and by Lemma 7 we have \( f = g = 0 \) on \( S^2(w) \). Hence, \( w \in \Xi_0 \), which is impossible, since \( w \in A \). The result follows. \( \square \)

We need one more simple lemma.

Lemma 9. Let \( \zeta \in S^3, \xi \in S^2(\zeta) \). Assume that

\[
(S^2(\xi) \cap S^2(\zeta)) \cap \Xi_0 \cap \Xi_1 = \emptyset.
\]

Then, either

\[
(S^2(\xi) \cap S^2(\zeta)) \subset (\Xi_0 \setminus \Xi_1) \quad \text{or} \quad (S^2(\xi) \cap S^2(\zeta)) \subset (\Xi_1 \setminus \Xi_0).
\]

3438 until the end of the proof on page 3439 (use \( F_t^2 \) instead of \( f_x \) and \( S^2(w) \cap S^2(\zeta) \) instead of \( \xi^\perp \)).
Proof. By Lemma 8,
\[(16) \quad S^2(\zeta) = (\Xi_0 \setminus \Xi_1) \cup (\Xi_0 \cap \Xi_1) \cup (\Xi_1 \setminus \Xi_0).\]
By assumption, \((S^2(\xi) \cap S^2(\zeta)) \cap (\Xi_0 \cap \Xi_1) = \emptyset\). Therefore,
\[(S^2(\xi) \cap S^2(\zeta)) \subset ((\Xi_0 \setminus \Xi_1) \cup (\Xi_1 \setminus \Xi_0)).\]
If \((15)\) is not true, then
\[S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1) \neq \emptyset, \quad \text{and} \quad S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0) \neq \emptyset.\]
Take any \(w_1 \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)\) and \(w_2 \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0)\).
Rotating if necessary we can assume that
\[S^2(\xi) \cap S^2(\zeta) = \{w = w(t) \in S^3 : w(t) = (\cos t, \sin t, 0, 0), \quad t \in [0, 2\pi]\},\]
and
\[w_1 = (\cos t_1, \sin t_1, 0, 0), \quad w_2 = (\cos t_2, \sin t_2, 0, 0),\]
for some \(t_1, t_2 \in [0, 2\pi]\), \(t_1 < t_2\). Now put
\[t^* = \sup\{t \in [t_1, t_2] : w(t) \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)\}, \quad w^* = w(t^*).\]
We have two possibilities,
\[(a) \quad w^* \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1), \quad (b) \quad w^* \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0).\]
If \((a)\) is true, then \(w^* \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0)\) due to the fact that \(w(t) \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_1 \setminus \Xi_0)\) for all \(t > t^*\), and \(S^2(\xi) \cap S^2(\zeta) \cap \Xi_1\) is closed. But then,
\[(17) \quad w^* \in (\Xi_0 \setminus \Xi_1) \cap (\Xi_1 \setminus \Xi_0),\]
which is impossible since the set is empty.
If \((b)\) is true, then for every \(l \in \mathbb{N}\) there exists a \(t_l \in [t^* - \frac{1}{l}, t^*]\) such that \(w_l = w(t_l) \in S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)\), (otherwise, there would be an \(l\) such that for every \(t \in [t^* - \frac{1}{l}, t^*]\) we have \(w(t) \notin S^2(\xi) \cap S^2(\zeta) \cap (\Xi_0 \setminus \Xi_1)\), and \(t^*\) is not a supremum). Since \(w_l \to w^*\) as \(l \to \infty\) and \(S^2(\xi) \cap S^2(\zeta) \cap \Xi_0\) is closed, we again have \((17)\). Hence, \((15)\) is proved. \(\square\)

3.3. Proof of Proposition 1. By Lemma 8, we have that \(S^2(\zeta) = \Xi_0 \cup \Xi_1\).
If we assume that \(\Xi_1 = \emptyset\), then \(S^2(\zeta) = \Xi_0\), and therefore \(f(\theta) = g(\theta)\) for every \(\theta \in S^3\). On the other hand, if \(\Xi_0 = \emptyset\), we have that \(S^2(\zeta) = \Xi_1\), which means that \(f(\theta) = g(O\theta)\) for every \(\theta \in S^3\). Hence, in these two situations we obtain the desired conclusion.

Let us now assume that both \(\Xi_0, \Xi_1\) are not empty. We can also assume that \(\Xi_0 \cap \Xi_1 \neq \emptyset\). Indeed, let \(w\) be a point on the boundary of \(\Xi_0\), \((w \in \Xi_0, \text{since } \Xi_0 \text{ is closed})\). Then for every \(l \in \mathbb{N}\), the set \(B_{\frac{1}{l}}(w) \cap S^3\) contains a point \(w_l\) from \(\Xi_1\). But then \(w_l \to w\) as \(l \to \infty\), hence \(w \in \Xi_1\), and \(w \in \Xi_0 \cap \Xi_1\).
We shall consider two cases:
1) There exists \(\xi \in S^2(\zeta)\) such that \(\Xi_0 \cap \Xi_1 \cap S^2(\xi) = \emptyset\).
2) For every \(x \in S^2(\zeta)\) we have \(\Xi_0 \cap \Xi_1 \cap S^2(x) \neq \emptyset\).
Consider the first case. Using (16) and Lemma 9, we obtain (15). If the first relation in (15) holds, then, by Lemma 1, we have
\[ S_3 = \bigcup \{ w \in \Xi_0 \} S_2(w), \]
and \( f(\theta) = g(\theta) \) for every \( \theta \in S^3 \). If the second relation in (15) holds, then, using Lemma 1 again, we obtain \( S_3 = \bigcup \{ w \in \Xi_1 \} S_2(w) \), and \( f(\theta) = g(O\theta) \) for every \( \theta \in S^3 \).

Consider the second case. We claim that
\[ (18) \quad S_2(\zeta) = \bigcup \{ u \in (\Xi_0 \cap \Xi_1) \} (S_2(u) \cap S_2(\zeta)). \]
Indeed, let \( x \in S_2(\zeta) \). By the hypothesis of the second case, the set \( \Xi_0 \cap \Xi_1 \cap S_2(x) \) is non-empty. Let \( u \in \Xi_0 \cap \Xi_1 \cap S_2(x) \). Then \( x \in S_2(u) \), and hence \( x \in S_2(u) \cap S_2(\zeta) \), from which it follows that
\[ x \in \bigcup \{ u \in (\Xi_0 \cap \Xi_1) \} (S_2(u) \cap S_2(\zeta)), \]
thus proving (18).

Using (18), the fact that \( S_3 = \bigcup \{ v \in S_2(\zeta) \} S_2(v) \), and an argument similar to the one in the proof of Lemma 1, we conclude that
\[ (19) \quad S_3 = \bigcup \{ u \in (\Xi_0 \cap \Xi_1) \} S_2(u). \]

It is easy to see that if (19) holds, then \( f \) and \( g \) are zero on \( S_3 \), and we are done. Indeed, let \( \theta \in S^3 \). Then \( \theta \in S_2(w) \) for some \( w \in (\Xi_0 \cap \Xi_1) \). Using (12) we see that \( f(\theta) = g(\theta) = g(O\theta) \). Since \( g \) is odd with respect to \( O \), we have \( g(\theta) = f(\theta) = 0 \). Since \( \theta \) was arbitrary, we have proved that if (19) holds, then \( f = g = 0 \) on \( S_3 \).

Thus, in all possible cases, we have shown that if \( f \) and \( g \) are odd with respect to \( O \), then either \( f(\theta) = g(\theta) \) for every \( \theta \in S^3 \), or \( f(\theta) = g(O\theta) \) for every \( \theta \in S^3 \) (see Remark 1). Proposition 1 is proved.

### 4. Proofs of Theorem 1 and Corollary 1

The proof of Theorem 1 uses several auxiliary Lemmata, which are presented in Subsection 4.1. The main idea of the proof is more easily understood if we consider the case in which each of the bodies \( K \) and \( L \) has exactly one diameter \( d(\zeta) \) (cf. pages 51–52, our Lemmata are proven in the more general case of countably many diameters). First, we show that the diameters of \( K \) and \( L \) must be parallel (Lemma 10), and that we can translate the bodies to make the diameters coincide and be centered at the origin (Lemma 11). Next, for any 3-dimensional projection of the translated bodies \( \tilde{K} \) and \( \tilde{L} \) that contains the diameter, the direct rigid motion given by the statement of Theorem 1 must fix this diameter. There are only two
possibilities, namely, that the rigid motion is a rotation around the diameter, or a rotation around a line perpendicular to the diameter. Due to the lack of the corresponding symmetries, these cases are mutually exclusive (Lemmata 12 and 14). If one assumes that all rigid motions reflect the diameter, then it is possible to construct a continuous tangent line field on the 2-dimensional great sphere perpendicular to the diameter (Lemma 15). But this is impossible by the Theorem of Hopf (see Theorem 3). Thus, the direct rigid motions must be rotations around the diameter, which allows us to use Proposition 1 to conclude the proof.

Throughout this section, the direction \( \zeta \in S^3 \) will be fixed.

4.1. Auxiliary Lemmata. By the conditions of Theorem 1, the projections \( K|w^\perp \) and \( L|w^\perp \) are directly congruent for every \( w \in S^2(\zeta) \). Hence, for every \( w \in S^2(\zeta) \) there exists \( \chi_w \in SO(3, S^2(w)) \) and \( a_w \in w^\perp \) such that
\[
\chi_w(K|w^\perp) = L|w^\perp + a_w.
\]

We will repeatedly use the following well-known properties of the support function. For every convex body \( \tilde{K} \),
\[
(21) \quad h_{\tilde{K}|w^\perp}(x) = h_{\tilde{K}}(x) \quad \text{and} \quad h_{\chi_w(\tilde{K}|w^\perp)}(x) = h_{\tilde{K}|w^\perp}(\chi_w(x)), \quad \forall x \in w^\perp,
\]
(see, for example, \[Ga\] (0.21), (0.26), pages 17–18).

Let \( A_K \subset S^3 \) be a set of directions parallel to the diameters of \( K \). Observe that \( K \) has at most one diameter parallel to a given direction \( \theta \in S^3 \) (for, if a convex \( K \) has two parallel diameters \( d_1(\theta), d_2(\theta) \), then \( K \) contains a parallelogram \( Y \) with sides \( d_1(\theta), d_2(\theta) \), and one of the diagonals of \( Y \) is longer than \( d_1(\theta) \)).

Our first goal is to reduce matters to rotations fixing the one-dimensional subspace containing \( \zeta \). We will do this by showing that for most of the directions \( w \in S^2(\zeta) \) the projections \( K|w^\perp \) and \( L|w^\perp \) have exactly one diameter, parallel to \( \zeta \). We define
\[
(22) \quad \Omega = \{ w \in S^2(\zeta) : (A_K \cup A_L) \cap S^2(w) = \{ \pm \zeta \} \}.
\]

Lemma 10. Let \( K \) and \( L \) be as in Theorem 1 and let \( \zeta \in A_K \). Then \( \zeta \in A_L \) and \( \Omega \) is everywhere dense in \( S^2(\zeta) \). Moreover, for every \( w \in \Omega \) we have \( \chi_w(\zeta) = \pm \zeta \) and \( \omega_K(\zeta) = \omega_L(\zeta) \).

Proof. Using (21), we see that the length of diameters \( d_{K|w^\perp}(\zeta) \) and \( d_{K}(\zeta) \) is the same for every \( w \in S^2(\zeta) \). Let \( \xi \) be any element of \( A_L \), and let \( w \in S^2(\zeta) \) be such that \( S^2(w) \ni \zeta, \xi \). Since \( K|w^\perp \) and \( L|w^\perp \) are directly congruent, and the length of the diameters does not change under rigid motions, we have \( \omega_K(\zeta) = \omega_L(\zeta) \).

We will prove that \( \Omega \) is everywhere dense in \( S^2(\zeta) \). Suppose \( \xi \in (S^2(\zeta) \setminus \Omega) \). Then there exists \( \eta \in A_K \cup A_L \), \( \eta \not= \pm \zeta \) such that \( \eta, \zeta \in S^2(\xi) \). Hence \( \xi \in S^2(\eta) \cap S^2(\zeta) \), and
\[
(S^2(\zeta) \setminus \Omega) \subseteq \bigcup_{\{ \eta \in A_K \cup A_L, \eta \not= \pm \zeta \}} (S^2(\eta) \cap S^2(\zeta)).
\]
Since the right-hand side of the above inclusion is a countable union of one-dimensional circles, the measure of $S^2(\zeta) \setminus \Omega$ is zero. Hence, $\Omega$ is everywhere dense in $S^2(\zeta)$.

We show that $\zeta \in A_K$ implies $\zeta \in A_L$. By definition of $\Omega$, we have $A_K \cap S^2(w) = \{ \pm \zeta \}$ for every $w \in \Omega$. If $A_L \cap S^2(w) = \emptyset$, then we have $\omega_L(\theta) < \omega_K(\chi_w(\zeta))$ for every $\theta \in S^2(w)$, where $\chi_w$ is as in (20). This contradicts the fact that $K|w^\perp$ and $L|w^\perp$ are directly congruent. Thus, $A_L \cap S^2(w) = \{ \pm \zeta \}$.

Finally, assume that for some $w \in \Omega$ we have $\chi_w(\zeta) \neq \pm \zeta$. Then $\chi_w(K|w^\perp)$ has a diameter in a direction $\eta \neq \pm \zeta$. Since $\chi_w(K|w^\perp)$ and $L|w^\perp$ are translations of each other, $L|w^\perp$ must have a diameter parallel to $\eta$, which is impossible. Hence for every, $w \in \Omega$ we have $\chi_w(\zeta) = \pm \zeta$, and $\omega_K(\zeta) = \omega_L(\zeta)$. The result follows. \( \square \)

Remark 2. The previous lemma remains valid if, instead of the condition about countability of the diameters of the bodies, one assumes that, say, the sets of diameters of $K$ and $L$ are countable unions of large circles containing $\zeta$. The only fact that was used in the proof is that the set of the directions $w \in S^2(\zeta)$, such that $d_K(\zeta)$ and $d_L(\zeta)$ are the only diameters of the projections $K|w^\perp$ and $L|w^\perp$, is dense in $S^2(\zeta)$.

Our next goal is to “separate” translations from rotations. We translate the bodies $K$ and $L$ by vectors $a_K$, $a_L \in \mathbb{R}^4$, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$ such that their diameters $d_{\tilde{K}}(\zeta)$ and $d_{\tilde{L}}(\zeta)$ coincide and are centered at the origin.

Lemma 11. Let $\chi_w$ be the rotation given by (20), and let $w \in \Omega$. Then the function $\varphi_w := (\chi_w)^t$ verifies $\varphi_w(\zeta) = \pm \zeta$ and

$$h_{\tilde{K}} \circ \varphi_w(\theta) = h_{\tilde{L}}(\theta) \quad \forall \theta \in S^2(w).$$

Proof. Define $b_w = \chi_w(a_K|w^\perp) - a_L|w^\perp + a_w$, where $a_K|w^\perp$, $a_L|w^\perp$ are projections of vectors $a_K$, $a_L$, onto $w^\perp$. Then (20) holds with $\tilde{K}$ and $\tilde{L}$ instead of $K$ and $L$, and $b_w$ instead of $a_w$. We claim at first that $b_w = 0$ for all $w \in \Omega$. In other words,

$$\chi_w(\tilde{K}|w^\perp) = \tilde{L}|w^\perp.$$

Indeed, using the definition of $\tilde{K}$ and $\tilde{L}$, and Lemma 10, for every $w \in \Omega \subset S^2(\zeta)$ we have

$$d_{\tilde{K}|w^\perp}(\zeta) = d_{\tilde{K}}(\zeta) = d_{\tilde{L}|w^\perp}(\zeta)$$

and

$$\chi_w(d_{\tilde{K}}(\zeta)) = d_{\tilde{K}}(\zeta).$$

from which it follows that

$$d_{\tilde{K}|w^\perp}(\zeta) = \chi_w(d_{\tilde{K}|w^\perp}(\zeta)) = d_{\tilde{L}|w^\perp}(\zeta) + b_w = d_{\tilde{K}|w^\perp}(\zeta) + b_w.$$
Thus, $b_w = 0$ and (24) holds for every $w \in \Omega$. Then,

$$h_{\chi_w(K \delta)}(x) = h_{L \delta}(x) \quad \forall x \in \delta,$$

together with (21) gives us the desired conclusion. \hfill \Box

Consider the sets

(25) $\Xi = \{ w \in S^2(\zeta) : \mbox{ (23) holds with } \varphi_w(\zeta) = \zeta \}$

and

(26) $\Psi = \{ w \in S^2(\zeta) : \mbox{ (23) holds with } \varphi_w(\zeta) = -\zeta \}$. 

By Lemma 11 we have $\Omega \subset (\Xi \cup \Psi)$, hence $\Xi \cup \Psi \neq \emptyset$. Our final goal is to show that $S^2(\zeta) = \Xi$, which will be achieved in Lemmata 12-15. Then we will be able to invoke Proposition 1.

**Lemma 12.** The sets $\Xi$ and $\Psi$ are closed, and $\Xi \cup \Psi = S^2(\zeta)$. 

**Proof.** We prove that $\Xi$ is closed. We can assume that $\Xi$ is non-empty. Let $(w_l)_{l=1}^\infty$ be a sequence of elements of $\Xi$ converging to $w \in S^2(\zeta)$, and let $\theta$ be any point on $S^2(w)$. As in the proof of Lemma 3 we see that there exists a sequence $(\theta_l)_{l=1}^\infty$, $\theta_l \in S^2(w_l)$, converging to $\theta$ as $l \to \infty$. Then,

(27) $h_{L \theta} \circ \varphi_{w_l}(\theta_l) = h_{L \theta_l}(\theta_l)$, \quad $\varphi_{w_l}(\zeta) = \zeta$, \quad $\forall l = 1, 2, \ldots$

Using the compactness of $SO(4)$ and passing to a subsequence if necessary, we can assume that the sequence $(\varphi_{w_l})_{l=1}^\infty$ of rotations is convergent, say, to $\varphi_w$. Passing to the limit in (27) as $l \to \infty$, and using the fact that $\theta$ is an arbitrary point in $S^2(w)$, we obtain (23) with $\varphi_w(\zeta) = \zeta$. Hence, $w \in \Xi$.

The proof of the fact that $\Psi$ is closed is very similar. One has only to repeat the above arguments with $\varphi_w(\zeta) = -\zeta$ instead of the second equality in (27).

Since $\Xi \cup \Psi$ is closed, and the set $\Omega \subset (\Xi \cup \Psi)$ is everywhere dense in $S^2(\zeta)$ by Lemma 10, we conclude that $\Xi \cup \Psi = S^2(\zeta)$. \hfill \Box

**Lemma 13.** The sets $\Xi$ and $\Psi$ remain the same if, instead of the pair $h_{K \theta}$, $h_{L \theta}$ in equation (23), we take $(h_{K \theta})_{O, \theta}$, $(h_{L \theta})_{O, \theta}$. 

**Proof.** We claim at first that the conclusion of Lemma 2 and Remark 1 remain valid for $f = h_{K \theta}$ and $g = h_{L \theta}$, i.e. $h_{K \theta}$ and $h_{L \theta}$ can be assumed to be odd with respect to $O$. Indeed, let $w \in S^2(\zeta)$, and let $\varphi_w \in SO(3, S^2(w))$ be such that (23) holds. Denoting $E = S^2(w) \cap S^2(\zeta)$, if $w \in \Xi$, then, $\phi := \varphi_{w|E} \in SO(2, E)$ is the corresponding rotation in $S^2(w) \cap S^2(\zeta)$. If $w \in \Psi$, then $\psi := \varphi_{w|E} \in O(2, E)$ is the corresponding reflection with respect to $u \in S^2(w) \cap S^2(\zeta)$. By the rotation and reflection invariance of the Lebesgue measure on $E$, we see that (10) holds (with $\psi_E$ instead of $\phi_E$ if $w \in \Psi$). Thus, one can repeat the rest of the argument in the proof of Lemma 2 to see that $(h_{K \theta})_{O, \theta} = (h_{L \theta})_{O, \theta}$. The claim follows.

Since the even parts $(h_{K \theta})_{O, e}$, $(h_{L \theta})_{O, e}$ are equal, and $\varphi_w$ commutes with $O$, we conclude that (23) holds for $(h_{K \theta})_{O, \theta}$, $(h_{L \theta})_{O, \theta}$. \hfill \Box
As we did in the proof of Proposition \([\text{1}]\) for \(\alpha \in \mathbb{R}\), we consider the sets \(\Xi_\alpha\), defined by \([\text{1}]\) with \(f = h_K\) and \(g = h_L\). Reasoning as in the proof of Lemma \([\text{3}]\) it can be seen that the sets \(\Xi_\alpha\) are closed.

**Lemma 14.** We have \(S^2(\zeta) = \Xi_0 \cup \Xi_1 \cup \Psi\) and \((\Xi_0 \cup \Xi_1) \cap \Psi = \emptyset\).

**Proof.** If \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), we obtain that \(\Xi_\alpha \subset \Xi_0\) by an argument similar to the one in the proof of Lemma \([\text{4}]\) (with \(f = (h_K)_{\mathcal{O}, \alpha}\) and \(g = (h_L)_{\mathcal{O}, \alpha}\)). Also, arguing as in the proof of Lemma \([\text{8}]\) (with \(f = (h_K)_{\mathcal{O}, \alpha}\), \(g = (h_L)_{\mathcal{O}, \alpha}\), and \(A = S^2(\zeta) \setminus (\Xi_0 \cup \Xi_1 \cup \Psi)\)), we obtain that the only possible rational values for \(\alpha\) are 0 and 1.

Now we show that \((\Xi_0 \cup \Xi_1) \cap \Psi = \emptyset\). If this is not true, let \(w\) be any element of \((\Xi_0 \cup \Xi_1) \cap \Psi\). Using the definition of \(\Xi\) and \(\Psi\) together with the fact that \([\text{23}]\) and \([\text{24}]\) are equivalent, we have
\[
\chi_1(K|w^-) = \tilde{L}|w^-, \quad \chi_2(K|w^-) = \tilde{L}|w^-,
\]
where \(\chi_1, \chi_2 \in SO(3, S^2(w))\) are rotations satisfying \(\chi_1(\zeta) = \zeta, \chi_2(\zeta) = -\zeta\).

If \(w \in \Xi_0\), then \(\chi_1\) is trivial, and we have \(K|w^- = \chi_2(\tilde{K}|w^-)\). Since any 3-dimensional rotation has a one-dimensional invariant subspace, there exists \(u \in S^2(w) \cap S^2(\zeta)\) such that \(\chi_2(u) = u\). Thus, \(K\) has a \((u, \pi)\)-symmetry. Since \(K\) and \(\tilde{K}\) are translations of each other, the last equality shows that \(K|w^-\) has a \((u, \pi)\)-symmetry, which is impossible by the assumptions of Theorem \([\text{1}]\).

If \(w \in \Xi_1\), then \(\chi_1\) is the rotation of angle \(\pi\) around \(\zeta\), while \(\chi_2\) is the rotation of angle \(\pi\) around \(u \in S^2(w) \cap S^2(\zeta)\). Since \(\chi_1^{-1} = \chi_1\), it follows that \(K|w^- = \chi_1\chi_2(\tilde{K}|w^-)\). It is well known \([\text{RS}]\) that the composition of two rotations by \(\pi\) about axes that are separated by an angle \(\theta\), is a rotation by \(2\theta\) about an axis perpendicular to the axes of the given rotations. Since \(\zeta\) and \(u\) are perpendicular, we conclude that \(\chi_1\chi_2\) is a rotation by \(\pi\) around \(v \in S^2(w) \cap S^2(u) \cap S^2(\zeta)\). Hence, \(K\) has a \((v, \pi)\)-symmetry, which is impossible by the assumptions of Theorem \([\text{1}]\). Thus, \((\Xi_0 \cup \Xi_1) \cap \Psi = \emptyset\), and the Lemma is proved.

To prove Lemma \([\text{15}]\) we will need the following result of Hopf, \([\text{MH}]\), \([\text{Sa}]\).

**Theorem 3.** If a compact differentiable manifold \(M\) admits a tangent continuous line field, then the Euler characteristic of \(M\) is zero.

The idea of the proof of the following statement is taken from \([\text{Go1}]\), Lemma 3.2.1, page 48, and the third paragraph on page 51.

**Lemma 15.** We have \(S^2(\zeta) = \Xi_0 \cup \Xi_1\).

**Proof.** Since \(S^2(\zeta)\) is connected, it cannot be written as a disjoint union of two closed sets. Using the previous Lemma, either \(S^2(\zeta) = \Xi_0 \cup \Xi_1\), or \(S^2(\zeta) = \Psi\). We will prove that second case does not occur, by showing that the assumption \(S^2(\zeta) = \Psi\) leads to the existence of a continuous tangent line field on \(S^2(\zeta)\), which is impossible due to Theorem \([\text{3}]\)
Assume that $S^2(\zeta) = \Psi$. Let $A$ be the function assigning to each $w \in S^2(\zeta)$ the function $A(w) = \varphi_w$ given by (23). Recall that the rotation $\varphi_w \in SO(3, S^2(w))$ is by the angle $\pi$ around some $u \in S^2(w) \cap S^2(\zeta)$, and verifies $\varphi_w(\zeta) = -\zeta$.

We will show at first that $A$ is well-defined, i.e., $\varphi_w$ is unique. Indeed, if there were two different rotations, $\tilde{\varphi}_1 \neq \tilde{\varphi}_2$, around $u_1 \neq \pm u_2$, $u_1, u_2 \in S^2(w) \cap S^2(\zeta)$, then, using (23), we would have that

$$
(28) \quad h_{\tilde{K}} \circ \tilde{\varphi}_1(\theta) = h_{\tilde{L}}(\theta), \quad h_{\tilde{K}} \circ \tilde{\varphi}_2(\theta) = h_{\tilde{L}}(\theta) \quad \forall \theta \in S^2(w),
$$

which implies that $h_{\tilde{K}} \circ \tilde{\varphi}_1(\theta) = h_{\tilde{K}} \circ \tilde{\varphi}_2(\theta)$ for every $\theta \in S^2(w)$. Using (21) we obtain

$$
(29) \quad \chi_1(\tilde{K}|w^+) = \chi_2(\tilde{K}|w^+), \quad \text{where} \quad \chi_1 = \varphi_1^t, \quad \chi_2 = \varphi_2^t.
$$

Since $\chi_1^{-1} = \chi_1$, this implies that $\tilde{K}|w^+ = \chi_1 \chi_2(\tilde{K}|w^+)$. As in the proof of the previous lemma, $\chi_1 \chi_2$ must be the rotation by $2\theta$ around $\zeta$, where $\theta$ is the angle between $u_1$ and $u_2$. Hence, $\tilde{K}|w^+$ (and hence $K|w^+$) has a $\zeta$-rotational symmetry, which is impossible by the assumptions of Theorem 1. Therefore, the rotation $\varphi_w \in SO(3, S^2(w))$ must be unique, and the map $A$ is well-defined.

We show that $A$ is continuous. Let $(w_l)_{l=1}^\infty$ be a convergent sequence of directions from $S^2(\zeta)$, with $w_l = w$, and let $(\varphi_l)_{l=1}^\infty$ be the corresponding sequence of rotations in $S^2(w_l)$, with $\varphi_l(\zeta) = -\zeta$, for every $l \in \mathbb{N}$. First, we prove that $(\varphi_l)_{l=1}^\infty$ is convergent. Let $(\theta_l)_{l=1}^\infty$, $\theta_l \in S^2(w_l)$, be a sequence converging to any point $\theta \in S^2(w)$ as $l \to \infty$ (the existence of such a sequence can be shown as in Lemma 3). If $(\varphi_l)_{l=1}^\infty$ were not convergent, then there would exist two subsequences $(\varphi_{m_l})_{l=1}^\infty$ and $(\varphi_{j_l})_{l=1}^\infty$, with $\varphi_{m_l} \neq \varphi_{j_l}$, for every $l \in \mathbb{N}$. Using (23) on the corresponding equators $S^2(w_{m_l})$, $S^2(w_{j_l})$, we have

$$
h_{\tilde{K}} \circ \varphi_{w_{m_l}}(\theta_{m_l}) = h_{\tilde{L}}(\theta_{m_l}), \quad h_{\tilde{K}} \circ \varphi_{w_{j_l}}(\theta_{j_l}) = h_{\tilde{L}}(\theta_{j_l}).
$$

Passing to the limit in the above equalities and using the fact that $\theta$ was an arbitrary point in $S^2(w)$, we obtain (28), from which it follows as before that $K|w^+$ has a $\zeta$-rotational symmetry. This contradiction shows that the sequence $(\varphi_l)_{l=1}^\infty$ is convergent.

To show that $A$ is continuous, it remains to prove that $\lim_{l \to \infty} \varphi_l = \varphi_w$. Assume that the last equality is not true, and let $\lim_{l \to \infty} \varphi_l = \varphi_1 \neq \varphi_w$. Then we have (28) with $\varphi_2 = \varphi_w$, which is, as we have already seen, impossible. Thus, $A$ is continuous.

Consider now the map $B$ assigning to each $w \in S^2(\zeta)$ the one-dimensional invariant subspace $\mathcal{Y}(w)$ of the corresponding rotation $\varphi_w \in SO(3, S^2(w))$, $\varphi_w(\zeta) = -\zeta$. By a similar argument as the one used for $A$, the map $B$ is well-defined and continuous. Observe also that $\mathcal{Y}(w) \subset (w^+ \cap \zeta^+)$. Thus, assuming that $S^2(\zeta) = \Psi$, we have contructed a continuous tangent line field
$\mathcal{Y}(w)$ on $S^2(\zeta)$. Since the Euler characteristic of the two-dimensional sphere is 2, this contradicts Theorem 3. The proof of the Lemma is finished. \hfill $\Box$

4.2. Proof of Theorem 1. By Lemma 15, for every $w \in S^2(\zeta)$ there exists either a trivial rotation, or a rotation by the angle $\pi$, $\varphi_w^\pi \in SO(3, S^2(w))$, $\varphi_w^\pi(\zeta) = \zeta$, such that (23) holds. Applying Proposition 1 with $f = h_{\tilde{K}}$ and $g = h_{\tilde{L}}$ we obtain that either $h_{\tilde{K}} = h_{\tilde{L}}$ on $S^3$ or $h_{\tilde{K}}(\theta) = h_{\tilde{L}}(U\theta)$ for every $\theta \in S^3$, where $U \in O(4)$ is the orthogonal transformation satisfying $U|_{S^2(\zeta)} = -I$, and $U(\zeta) = \zeta$. Letting $O = U^t$, it follows from (21) that $h_{\tilde{K}}(U\theta) = h_{\tilde{K}}(\theta)$ for every $\theta \in S^3$, and therefore either $K + a_K = L + a_L$ or $K + a_K = \bar{O}L + O(a_L)$. This proves the first part of the Theorem.

Assume, in addition, that the ground projections $K|\zeta^\perp, L|\zeta^\perp$, are direct rigid motions of each other. Then, there exists $\chi_\zeta \in SO(3, S^2(\zeta))$ and $a_\zeta \in \zeta^\perp$ such that

$$
\chi_\zeta(K|\zeta^\perp) = L|\zeta^\perp + a_\zeta.
$$

If $K = \bar{O}L + b$ holds, then we have

$$
K|\zeta^\perp = (\bar{O}L)|\zeta^\perp + b|\zeta^\perp = -L|\zeta^\perp + b|\zeta^\perp.
$$

The last two equations imply that $\chi_\zeta(K|\zeta^\perp) - a_\zeta = -K|\zeta^\perp + b|\zeta^\perp$, and $K|\zeta^\perp$ has a rigid motion symmetry, which is impossible by our assumptions. Thus, we conclude that $K = L + b$ and the proof of Theorem 1 is finished. \hfill $\Box$

4.3. Proof of Corollary 1. We translate the bodies $K$ and $L$ by vectors $a_K, a_L \in \mathbb{R}^n$, to obtain $\tilde{K} = K + a_K$ and $\tilde{L} = L + a_L$ such that the origin is the center of $d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta)$. Let $J$ be an arbitrary 4-dimensional subspace of $\mathbb{R}^n$, containing $\zeta$. Observe that $\tilde{K}|J$ and $\tilde{L}|J$ satisfy the conditions of Theorem 1 with $\tilde{K}|J$ and $\tilde{L}|J$ instead of $K$ and $L$. By Theorem 1 we have $\tilde{K}|J = |J$ or $\tilde{K}|J = O_J(\tilde{L}|J)$ where $O_J \in O(4, J)$, $O_J|\zeta^\perp = -I$ and $O_J(\zeta) = \zeta$. If there existed two different 4-dimensional subspaces $J_1$ and $J_2$, such that $\tilde{K}|J_1 = \tilde{L}|J_1$ and $\tilde{K}|J_2 = O_{J_2}(\tilde{L}|J_2)$, then $\tilde{L}$ would have a 3-dimensional projection with a $\zeta$-rotational symmetry. Indeed, assume that $J_1 \cap J_2$ is a 3-dimensional subspace. Then,

$$
\tilde{L}|(J_1 \cap J_2) = (\tilde{L}|J_1)|(J_1 \cap J_2) = (\tilde{K}|J_1)|(J_1 \cap J_2) = (\tilde{K}|J_2)|(J_1 \cap J_2) = (\bar{O}_{J_2}(\tilde{L})|J_2)|(J_1 \cap J_2),
$$

and $\tilde{L}|(J_1 \cap J_2)$ has a $\zeta$-rotational symmetry, contradicting the assumptions of the Corollary. Hence, either $\tilde{K}|J = \tilde{L}|J$ for every $J$, or $\tilde{K}|J = O_J(\tilde{L}|J)$ for every $J$. If we are in the second case, let $O \in O(n)$ such that $O|_{\zeta^\perp} = -I$ and $O(\zeta) = \zeta$. Then we have that $O|J = O_J$. Since $J$ was arbitrary, the projections of $\tilde{K}$ and $\tilde{L}$ onto all two-dimensional subspaces containing $\zeta$ coincide or are reflections of each other (with respect to the line containing $\zeta$). Using Theorem 3.1.1 from [Ga, page 99] we have $\tilde{K} = \tilde{L}$ or $\tilde{K} = \bar{O}L$. Thus, $K = L + a_L - a_K$ or $K = \bar{O}L + O(a_L) - a_K$.

Now assume that the dimension of $J_1 \cap J_2$ is 2. In this case, let $\{\zeta, v_1, v_2, v_3\}$ be an orthonormal basis of $J_1$, and $\{\zeta, v'_1, v'_2, v'_3\}$ be an orthonormal basis
of $J_2$. Define $J_0$ to be the 4 dimensional subspace with basis $\{\zeta, v_1, v_2, v'_2\}$. Then, both $J_1 \cap J_0$ and $J_2 \cap J_0$ have dimension 3, and the above argument can be used. A similar argument can be used if the dimension $J_1 \cap J_2$ is 1.

Finally, assume that, in addition, the “ground” projections $K|G$, $L|G$ onto all 3-dimensional subspaces $G$ of $\zeta^\perp$, are directly congruent and have no rigid motion symmetries, then, using Theorem 1 we see that $K_1|J = L_1|J$ for an arbitrary 4-dimensional subspace $J$. Hence, the projections of $K$ and $L$ onto all two-dimensional subspaces containing $\zeta$ coincide. Using Theorem 3.1.1 from [Ga] we have $K = L$. Thus, $K + a_K = L + a_L$ and the Corollary is proved. \hfill \Box

5. Proofs of Theorem 2 and Corollary 2

The proofs are slightly different from the ones about projections. We recall that we consider star-shaped bodies with respect to the origin. The direction $\zeta \in S^3$ will be fixed throughout the proof. By the conditions of Theorem 2 the sections $K \cap w^\perp$ and $L \cap w^\perp$ are directly congruent for every $w \in S^2(\zeta)$. Hence, for every $w \in S^2(\zeta)$ there exists $\chi_w \in SO(3, S^2(w))$ and $a_w \in w^\perp$ such that

$$\chi_w(K \cap w^\perp) = (L \cap w^\perp) + a_w. \quad (30)$$

Let $l(\zeta)$ denote the one-dimensional subspace containing $\zeta$. As in Section 4 we use the notation $A_K \subset S^3$ for the set of directions that are parallel to the diameters of $K$. We consider the set $\Omega^\prime$, which is defined in the same way as $\Omega$, by (22), with $K$ being star-shaped. We will use the notation $v_K(\zeta) = \rho_K(\zeta) + \rho_K(-\zeta)$.

5.1. Auxiliary lemmata. Our first goal is to reduce matters to rotations leaving $l(\zeta)$ fixed. We will do this by showing that for all the directions $w \in S^2(\zeta)$, the sections $K \cap w^\perp$ and $L \cap w^\perp$ have exactly one diameter contained in $l(\zeta)$.

We will use the well-known properties of the radial function (see, for example, [Ga] (0.33), page 20)

$$\rho_{K \cap w^\perp}(\theta) = \rho_K(\theta), \quad \rho_{\chi_w(\zeta \cap w^\perp)}(\theta) = \rho_{\chi_w(\zeta \cap w^\perp)}(\chi_w^{-1}(\theta)), \quad \forall \theta \in w^\perp \cap S^3. \quad (31)$$

**Lemma 16.** Let $K$ and $L$ be as in Theorem 2. Then $L$ has a diameter $d_L(\zeta) \subset l(\zeta)$, and $\Omega^\prime$ is everywhere dense in $S^2(\zeta)$. Moreover, for every $w \in \Omega^\prime$ we have $\chi_w(\zeta) = \pm \zeta$ and $v_K(\zeta) = v_L(\zeta)$.

**Proof.** The proof that $\Omega^\prime$ is everywhere dense in $S^2(\zeta)$ is exactly as the one of Lemma 10 with $\Omega^\prime$ instead of $\Omega$.

We will show that $\zeta \in A_K$ implies $\zeta \in A_L$. By definition of $\Omega^\prime$, we have $A_K \cap S^2(w) = \{\pm \zeta\}$ for every $w \in \Omega^\prime$. If $A_L \cap S^2(w) = \emptyset$, then $v_L(\theta) < v_K(\chi_w^{-1}(\zeta))$ for every $\theta \in S^2(w)$, where $\chi_w$ is as in (30). This contradicts the fact that $K \cap w^\perp$ and $L \cap w^\perp$ are directly congruent. Thus, $A_L \cap S^2(w) = \{\pm \zeta\}$. 

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Thus, \( \rho \). Lemma 17. The result follows.

\[ r \] which is impossible by the definition of \( \Omega \).

\[ K \]

and \( \chi \)

\[ L \]

contradiction shows that there exists \( d \) and \( \tilde{w} \) of \( L \) for some \( \chi \) and \( \nu \). In other words, for all \( L \) and \( \chi \), we have \( \nu_L(\xi) = v_L(\zeta) \), since both \( d_L(\zeta), d_L(\xi) \) are subsets of \( l(\zeta) \).

The result follows.

Our next goal is to separate translations from rotations. We translate the bodies \( K \) and \( L \) by the vectors \( a_K, a_L \in \mathbb{R}^4 \), which are parallel to \( \zeta \), to obtain \( \tilde{K} = K + a_K \) and \( \tilde{L} = L + a_L \), with \( d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta) \) and the origin at the center of these diameters.

Lemma 17. For every \( w \in \Omega^r \) there exists \( \varphi_w = \chi_w^{-1} \in SO(3,S^2(w)) \), \( \varphi_w(\zeta) = \pm \zeta \), such that

\[ (32) \quad \rho_K \circ \varphi_w(\theta) = \rho_{\tilde{L}}(\theta) \quad \forall \theta \in S^2(w). \]

Proof. Define \( b_w = \chi_w(a_K) - a_L + a_w \). Then \( (30) \) holds with \( \tilde{K} \) and \( \tilde{L} \) instead of \( K \) and \( L \), and \( b_w \) instead of \( a_w \). We first claim that \( b_w = 0 \) for all \( w \in \Omega^r \). In other words, for all \( w \in \Omega^r \) we have

\[ (33) \quad \chi_w(\tilde{K} \cap w^\perp) = \tilde{L} \cap w^\perp \]

for some \( \chi_w \in SO(3,S^2(w)) \), \( \chi_w(\zeta) = \pm \zeta \). Indeed, using the definition of \( \tilde{K} \) and \( \tilde{L} \), and Lemma 16, for every \( w \in \Omega^r \subset S^2(\zeta) \) we have

\[ d_{\tilde{K} \cap w^\perp}(\zeta) = d_{\tilde{K}}(\zeta) = d_{\tilde{L}}(\zeta) = d_{\tilde{L} \cap w^\perp}(\zeta) \]

and

\[ \chi_w(d_{\tilde{K}}(\zeta)) = d_{\tilde{K}}(\zeta), \]

where \( d_{\tilde{K}}(\zeta) \) and \( d_{\tilde{L}}(\zeta) \) are contained in \( l(\zeta) \). Therefore,

\[ d_{\tilde{K} \cap w^\perp}(\zeta) = \chi_w(d_{\tilde{K} \cap w^\perp}(\zeta)) = d_{\tilde{L} \cap w^\perp}(\zeta) + b_w = d_{\tilde{K} \cap w^\perp}(\zeta) + b_w. \]

Thus, \( b_w = 0 \) and \( (33) \) holds. Then, \( \rho_{\chi_w(\tilde{K} \cap w^\perp)}(x) = \rho_{\tilde{L} \cap w^\perp}(x) \) for all \( x \in w^\perp \). In particular, we have that \( \rho_{\chi_w(\tilde{K} \cap w^\perp)}(\theta) = \rho_{\tilde{L} \cap w^\perp}(\theta) \) for all \( \theta \in S^2(w) \). We now use \( (31) \) to conclude the proof.

Consider the sets

\[ \Xi^r = \{ w \in S^2(\zeta) : \text{ (32) holds with } \varphi_w(\zeta) = \zeta \} \]

and

\[ \Psi^r = \{ w \in S^2(\zeta) : \text{ (32) holds with } \varphi_w(\zeta) = -\zeta \}. \]
5.2. Proof of Theorem 2. The first step is to show that $S^2(\zeta) = \Xi_0^r \cup \Xi_1^r$. This can be done by repeating the same reasoning as in Lemmata 12, 14 (replacing the sets $\Xi, \Xi_0^r, \Omega, \Pi, \text{with } \Xi_0^r, \Xi_1^r, \Psi, \Omega^r, \Pi^r$, the projections $K|w^+, L|w^+$ with sections $\hat{K} \cap w^+, \hat{L} \cap w^+$, and the support functions $h_{\hat{K}}, h_{\hat{L}}$ with the radial ones $\rho_{\hat{K}}, \rho_{\hat{L}}$).

We can now apply Proposition 1 with $f = \rho_{\hat{K}}, g = \rho_{\hat{L}}$, obtaining that either $\rho_{\hat{K}} = \rho_{\hat{L}}$ on $S^3$, or $\rho_{\hat{K}}(\theta) = \rho_{\hat{L}}(U\theta)$ for all $\theta \in S^3$, where $U \in O(4)$ is an orthogonal transformation, satisfying $U|_{S^2(\zeta)} = -I$ and $U(\zeta) = \zeta$. In the first case, $\hat{K} = \hat{L}$, and in the second, $\hat{K} = O\hat{L}$, where $O = U^{-1}$. Thus, $K = L + a_L - a_K$, or $K = O\Lambda L + O(a_L) - a_K$. This finishes the proof of Theorem 2.

5.3. Proof of Corollary 2. The proof is similar to the one of Corollary 1. One has only to consider the sections $\hat{K} \cap J, \hat{L} \cap J$, instead of the projections $\hat{K}|J, \hat{L}|J$, and Theorem 7.1.1 from [Ga] page 270, instead of Theorem 3.1.1 from [Ga] page 99.

6. Appendix

Let $\delta(K, P)$ be the Hausdorff distance between the convex bodies $K$ and $P$ in $\mathbb{R}^n$, $n \geq 2$, $\delta(K, P) = \max_{\theta \in S^{n-1}} |h_K(\theta) - h_P(\theta)|$.

Our goal is to prove

**Proposition 2.** Any convex body $K$ in $\mathbb{R}^n$, $n \geq 4$, can be approximated in the Hausdorff metric, by polytopes without 3-dimensional projections that have rigid motion symmetries.

Since polytopes have finitely many diameters, Proposition 2 shows that the set of bodies satisfying the conditions of Corollary 1 contains the set of polytopes which is dense in the set of all convex bodies.

Proposition 2 is not a new result (see [Go] page 48). An abstract geometric proof of this fact can be given [Pa]. However, for the convenience of the reader, we include an elementary proof. The idea is, assuming that $K$ has positive Gaussian curvature, to observe first that $K$ can be approximated by polytopes whose 3-dimensional projections have many vertices. If a polytope has a 3-dimensional projection with a rigid motion symmetry, then we use (14) to form a system of linear equations, and use the implicit function theorem to prove that these polytopes form a “manifold” of small dimension.
6.1. **Auxiliary results.** We will need the following theorem and two lemmata. Let $C^2_+ (\mathbb{R}^n)$ be the set of convex bodies in $\mathbb{R}^n$ having a positive Gaussian curvature. It is well-known, that any convex body can be approximated in the Hausdorff metric by convex bodies $K \in C^2_+ (\mathbb{R}^n)$ [Sch, pages 158-160]. Hence, we can assume that $K \in C^2_+ (\mathbb{R}^n)$.

Our first auxiliary statement is the following result of Schneider, [Sch2].

**Theorem 4.** Let $K \in C^2_+ (\mathbb{R}^n), n \geq 3$. Then, for $v \to \infty$, we have

$$\delta(K, P_v^*) \approx c_n \ v^{-\frac{n-1}{n}} \left( \int_{\partial K} \sqrt{G_K(\sigma)} \, d\sigma \right)^{\frac{n}{n-1}},$$

where $P_v^*$ is a polytope with vertices on the boundary $\partial K$, not unique in general, for which $\delta(K, P_v^*)$ equals the infimum of $\delta(K, P)$ over all convex polytopes $P$ contained in $K$ that have at most $v$ vertices, $c_n$ is a constant depending on the dimension, and $G_K(\sigma)$ is the Gaussian curvature of $K$ at $\sigma$ in $\partial K$.

We see, in particular, that the amount of vertices of $P_v^*$ gets larger, provided the Hausdorff distance between $K$ and $P_v^*$ is getting smaller. The next known statement will be used to show that the same is true for all 3-dimensional projections of $K$.

**Lemma 18.** Let $K \in C^2_+ (\mathbb{R}^n), n \geq 4$. Then $K \mid H \in C^2_+ (H)$, where $K \mid H$ is the projection of $K$ onto $H \in \mathcal{G}(n, 3)$.

**Proof.** Let $x$ be any point on the boundary of $K$. Changing the coordinates if necessary we can assume that $x$ is the origin and the tangent hyperplane to $K$ at $x$ is the $(x_1, \ldots, x_{n-1})$-hyperplane. Using the Taylor decomposition of the boundary of $K$ near the origin we have

$$x_n = f(x_1, \ldots, x_{n-1}) = k_1 x_1^2 + \cdots + k_{n-1} x_{n-1}^2 + o(x),$$

where $k_j > 0, j = 1, \ldots, n-1$, are the main curvatures of the boundary at $x$, and $\frac{o(x)}{|x|} \to 0$ as $|x| \to 0$. Consider the ball $B$,

$$B = \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 + (x_n - \frac{1}{k})^2 = \frac{1}{k^2} \}, \quad k = \min_{j=1,\ldots,n-1} k_j.$$

Since the main curvatures are the reciprocals of the main radii of curvature we see that in a small enough neighborhood $W$ of the origin, $K \cap W$ is contained in $B$. Let $u \in S^{n-1}$ be such that $u_n = 0$, i.e., $u$ is the unit vector contained in the $(x_1, \ldots, x_{n-1})$-hyperplane, and let $H_u \in \mathcal{G}(n, 3)$ be contained in the $(x_1, \ldots, x_{n-1})$-hyperplane, and orthogonal to $u$. Observe that the boundary of the projection $(K \cap W)|H_u$ is contained in the 3-dimensional ball of radius $\frac{1}{k}$, which is the projection of $B$. Since the main curvatures of the boundary of $(K \cap W)|H_u$ are the reciprocals of the radii of curvature, we see that the main curvatures of $(K \cap W)|H_u$ at the origin are positive. Since $x$ was an arbitrary point on the boundary of $K$, the result follows. \qed
To formulate our last auxiliary lemma, we recall the definition of the Hausdorff dimension [WikiH]. Given any subset $E$ of $\mathbb{R}^n$ and $\alpha \geq 0$, the exterior $\alpha$-dimensional Hausdorff measure of $E$ is defined by

$$m^*_\alpha(E) = \lim_{\delta \to 0^+} \inf H^\alpha_\delta(E),$$

and $diam(S) = \sup_{x,y \in S} |x - y|$ stands for the diameter of $S$. The Hausdorff dimension of $E$ is

$$\dim_H(E) = \inf\{\alpha > 0 : m^*_\alpha(E) = 0\}.$$

Lemma 19. Let $M$ be a smooth manifold of dimension $k$ in $\mathbb{R}^m$, $m \geq 3$, $k \leq m - 2$, and let $M|H$ be the orthogonal projection of $M$ onto a $l$-dimensional subspace $H$, $k < l \leq m - 1$. Then the Hausdorff dimension of $M|H$ does not exceed the dimension of $M$.

Proof. Let $\delta > 0$ and let $\bigcup_{k=1}^\infty F_k$, $diam(F_k) \leq \delta$, be a covering of $M$. Since $\bigcup_{k=1}^\infty (F_k|H)$ is a covering of $M|H$, and $diam(F_k|H) \leq diam(F_k) \leq \delta$, we see that

$$\sum_{k=1}^\infty (diam(F_k|H))^\alpha \leq \sum_{k=1}^\infty (diam(F_k))^\alpha,$$

and $m^*_\alpha(M|H) \leq m^*_\alpha(M)$. The result follows. \qed

6.2. Proof of Proposition 2. To prove the proposition it is enough to show that each $P^*_v$, having sufficiently many vertices, can be approximated by polytopes without any 3-dimensional projection rigid motion symmetries. We will do this by proving that the set of polytopes having $v$ vertices with 3-dimensional projection rigid motion symmetries is a nowhere dense set contained in the set of all polytopes having $v$ vertices.

Define $\mathcal{P}_v$ to be the set of polytopes in $\mathbb{R}^n$, $n \geq 4$, with $v$ vertices $p_1, p_2, \ldots, p_v$. We see that $\mathcal{P}_v$ can be parametrized by points from $\mathbb{R}^{nv}$, with $p_j = (p_{1j}, \ldots, p_{nj}) \in \mathbb{R}^n$, $j = 1, \ldots, v$, and we can identify $\mathcal{P}_v$ with an open domain in $\mathbb{R}^{nv}$.

We denote by $\Pi_v$ the set of polytopes in $\mathcal{P}_v$ that have a 3-dimensional projection with rigid motion symmetries. Our goal is to show that $\Pi_v$ is nowhere dense in $\mathcal{P}_v$, provided that $v$ is large enough. We can partition $\Pi_v$ into equivalence classes such that two polytopes are in the same class if there is a rigid motion in $\mathbb{R}^n$ taking one to the other. Letting $H_0$ be the $(x_1, x_2, x_3)$-plane in $\mathbb{R}^n$, each equivalence class can be represented by a polytope whose projection on $H_0$ has rigid motion symmetries. Let us define $Q_v$ to be the set of these representatives, i.e.,

$$Q_v = \{Q \in \mathcal{P}_v : \exists \varphi_{H_0} \in O(3, H_0), \varphi_{H_0} \neq I, \exists a_{H_0} \in \mathbb{R}^3 \text{ such that} \}.$$
Theorem 4, we can assume that polytopes $P$.

Lemma 20.

All that remains is to find the dimension of $Q$. Consider the set $\mathcal{M} = \mathcal{M}(\mathcal{Q}_v)$ of all triples

$$(Q, \varphi_{H_0}, a_{H_0}) \in \mathbb{R}^{nv} \times O(3, H_0) \times \mathbb{R}^3,$$

satisfying (34).

Let $H \in G(n, 3)$. Since for every $\theta \in H \cap S^{n-1}$, we have $h_K[H] = h_K(\theta) \setminus \text{Gal}(0.21)$, page 17, $K[H]$ can be approximated in the Hausdorff metric by polytopes $P^*_t[H]$. Let $V_H$ be the number of vertices of $P^*_t[H]$. Using Lemma 18 and Theorem 4, we can assume that

$$t_0 := \min_H V_H > 5 + \frac{n(n + 1)}{2}.$$  

**Lemma 20.** The set $\mathcal{M}$ is manifold in $\mathbb{R}^{nv+6}$ with dimension at most $(nv + 5 - t_0)$, provided that $v$ is so large that $t_0 > 5 + \frac{n(n + 1)}{2}$.

**Proof.** Let $Q$ be a polytope in $\mathcal{Q}_v$ and consider its projection $Q[H_0]$, which is also a polytope with $t$ vertices, where $t \geq t_0$. We will write the assumption that $Q[H_0]$ has rigid motion symmetries as a system of linear equations that equal zero precisely at the vertices of $Q[H_0]$, and explicitly compute the determinant of its Jacobian matrix to show that it is nonzero. The implicit function theorem [Wiki] will allow us to obtain the result.

Since any rigid motion maps a vertex into a vertex, an equation, similar to (34), can be written for the corresponding vertices $q_i[H_0]$ of $Q[H_0]$,

$$q_i[H_0] = \varphi_{H_0}(q_{j(i)}[H_0]) + a_{H_0},$$

where $\varphi_{H_0}$ is a nonidentical orthogonal transformation whose $3 \times 3$ matrix has coordinates $(a_{l,m})_{l,m=1,2,3}$, and $j$ is a permutation on the set \{1, \ldots, t\}, which indicates that the $j(i)$-th vertex gets mapped to the $i$-th vertex. As it is well known, a permutation can be written as a product of cycles. We will consider two cases: cycles of length one, and cycles of length greater than one.

Assume that the vertex $q_i[H_0]$ is mapped to itself, i.e., $q_i[H_0] = \varphi_{H_0}(q_i[H_0]) + a_{H_0}$. Since $\varphi_{H_0}$ is not the identity, given a basis $e_1, e_2, e_3$ of $H_0$, there exists $r \in \{1, 2, 3\}$ such that $\varphi_{H_0}(e_r) \neq e_r$. For this $r$, consider the function $F_{ri} : \mathbb{R}^{nv} \times O(3, H_0) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ defined by

$$F_{ri}(x_{11}, \ldots, x_{nv}, \varphi_{H_0}, a_{H_0}) = ((x_{1i}, x_{2i}, x_{3i}) - \varphi_{H_0}(x_{1i}, x_{2i}, x_{3i}) - a_{H_0})_r = x_{ri} - o_{r1}x_{1i} - o_{r2}x_{2i} - o_{r3}x_{3i} - (a_{H_0})_r.$$

(34) $$\varphi_{H_0}(Q[H_0]) + a_{H_0} = Q[H_0].$$

Observe that every $P \in \Pi_v$ can be written as $P = \phi(Q) + b$ for some $\phi \in O(n), Q \in \mathcal{Q}_v, b \in \mathbb{R}^n$, and hence can be represented as the triple $(Q, \phi, b) \in \mathcal{Q}_v \times O(n) \times \mathbb{R}^n$. Thus,

$$\dim(\Pi_v) \leq \dim(\mathcal{Q}_v) + \dim(O(n)) + n = \dim(\mathcal{Q}_v) + \frac{n(n + 1)}{2}.$$
Since the right hand side depends only on the variables \( x_{1i}, x_{2i}, x_{3i} \), we see that \( \frac{\partial F_{rs}}{\partial x_{rs}} = 0 \) for all \( s \neq i \) and all \( k \), while \( \frac{\partial F_{rs}}{\partial x_{ri}} \neq 0 \) because \( \varphi_{H_0}(e_r) \neq e_r \). Thus, this cycle forms a \((1 \times 1)\)-Jacobian block whose entry is not 0.

Next, suppose that the cycle is of length \( k \) and permutes the vertices \( q_{i1}, q_{i2}, \ldots, q_{ik} \) (for \( \ell < k \), \( q_{i\ell + 1} \) gets mapped to \( q_{i\ell} \) and \( q_{i1} \) is mapped back to \( q_{ik} \)). Consider the system of \( 3(k - 1) \) functions \( F_{rs} : \mathbb{R}^{nv} \times O(3, H_0) \times \mathbb{R}^3 \to \mathbb{R} \) defined by

\[
F_{rs}(x_{11}, \ldots, x_{nv}, \varphi_{H_0}, a_{H_0}) = ((x_{1s}, x_{2s}, x_{3s}) - \varphi_{H_0}(x_{1j(s)}, x_{2j(s)}, x_{3j(s)}) - a_{H_0})_r
\]

for \( r = 1, 2, 3 \) and for \( s = i_1, i_2, \ldots, i_{k-1} \).

We will order the variables in such a way that the Jacobian block corresponding to this cycle will be upper triangular. We note that for \( r = 1, 2, 3 \), and \( s = i_1, \ldots, i_{k-1} \), \( F_{rs} \) depends on the variables \( x_{rs} \) and \( x_{kj(s)} \) for \( k = 1, 2, 3 \). Thus, \( \frac{\partial F_{rs}}{\partial x_{rs}} = 0 \) for \( k \neq r \), and \( \frac{\partial F_{rs}}{\partial x_{kl}} = 0 \) for all \( l \neq s \), \( l \neq j(s) \) and all \( k \). Order the Jacobian block as follows, \( x_{11i}, x_{21i}, x_{31i}, x_{1i2}, \ldots, x_{3i_{k-1}} \). Since \( \frac{\partial F_{rs}}{\partial x_{rs}} = 1 \), the diagonal entries are all 1. In addition, the variables \( x_{kj(s)} \) occur after \( x_{rs} \), so the Jacobian block is upper triangular. Therefore, the determinant of this block is equal to 1. Thus, the Jacobian of the system of equations is a block diagonal matrix with nonzero determinant.

We observe that the number of equations in our system depends on the decomposition of the permutation \( j \) into cycles. Each 1-cycle gives us one equation, while each cycle of length \( k > 1 \) contributes \( 3(k - 1) \) equations to the system. Hence, the smallest possible number of equations in our system is \( 3 + (t - 2) \), which occurs if the decomposition of the permutation \( j \) into cycles contains only one two-cycle and all the rest are one-cycles. By the implicit function theorem, we can express at least \( t + 1 \) variables \( x_{rs} \) as functions of the coordinates of \( \varphi_{H_0}, a_{H_0} \) and at most \( nv - (t + 1) \) other variables. Since \( t \geq t_0 \), this shows that the dimension of the manifold \( M \) in \( \mathbb{R}^{nv+6} \) is at most \( (nv + \dim(O(3)) + \dim(H_0) - (t_0 + 1)) = nv + 5 - t_0 \). □

We are now ready to prove our goal.

**Lemma 21.** The set \( \Pi_v \) is nowhere dense in \( \mathcal{P}_v \).

**Proof.** By definition, \( Q_v \) is equal to the projection of \( M \) onto \( \mathbb{R}^{nv} \) and by Lemmata 19 and 20

\[
\dim(Q_v) = \dim(M|\mathbb{R}^{nv}) \leq \dim(M) \leq nv + 5 - t_0.
\]

Hence, using (35), we have \( \dim(\Pi_v) \leq nv + 5 - t_0 + \frac{n(n+1)}{2} \). Finally, (36) yields \( \dim(\Pi_v) < \dim(\mathcal{P}_v) = nv \).

To complete the proof of Proposition 2, we use Theorem 4 to approximate \( K \in C^2(\mathbb{R}^n) \) in the Hausdorff metric, by polytopes \( P_v^* \) with \( v \) so large that \( t_0 > 5 + \frac{n(n+1)}{2} \). By Lemma 21, we can approximate \( P_v^* \) by polytopes without 3-dimensional projections that have rigid motion symmetries. □
DIRECTLY CONGRUENT PROJECTIONS

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