Gravitational vacuum polarization III:  
Energy conditions in the (1+1) Schwarzschild spacetime  
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Building on techniques developed in a pair of earlier papers, I investigate the various point-wise and averaged energy conditions for the quantum stress-energy tensor corresponding to a conformally-coupled massless scalar field in the (1+1)-dimensional Schwarzschild spacetime. Because the stress-energy tensors are analytically known, I can get exact results for the Hartle–Hawking, Boulware, and Unruh vacua. This exactly solvable model serves as a useful sanity check on my (3+1)-dimensional investigations wherein I had to resort to a mixture of analytic approximations and numerical techniques. Key results in (1+1) dimensions are: (1) NEC is satisfied outside the event horizon for the Hartle–Hawking vacuum, and violated for the Boulware and Unruh vacua. (2) DEC is violated everywhere in the spacetime (for any quantum state, not just the standard vacuum states).

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I. INTRODUCTION

In a pair of earlier papers [1,2] I have investigated the gravitational vacuum polarization outside a Schwarzschild black hole in the Hartle–Hawking and Boulware vacuum states.

For the Hartle–Hawking vacuum in (3+1) dimensions I found that the various energy conditions were violated in a nested set of onion-like layers located between the event horizon and the unstable photon orbit [1]. Furthermore, based on the Page approximation, it seems that many of the energy conditions are also violated inside the event horizon.

For the Boulware vacuum in (3+1) dimensions I found that: (1) All standard point-wise and averaged energy conditions are violated throughout the entire region exterior to the event horizon. (2) Outside the event horizon, all standard point-wise energy conditions are violated in a maximal manner: they are violated at all points and for all null/timelike vectors. (3) Subject to caveats concerning the applicability and accuracy of the analytic approximation inside the event horizon [2].

In this paper I report a much simpler analysis which serves as a sanity check on the general formalism. I consider the (1+1)-dimensional Schwarzschild geometry

\[ ds^2 = -(1 - 2M/r)dt^2 + \frac{dr^2}{(1 - 2M/r)}. \]  

More precisely, I will work with the maximal analytic extension of this geometry, the (1+1)-dimensional Kruskal–Szekeres manifold.

This geometry is often grandiosely referred to as the (1+1)-dimensional black hole, and has many interesting features analogous to the (3+1)-dimensional black hole. One aspect that is very different from (3+1) dimensions is that the quantum stress energy tensor is explicitly calculable.

In this paper I will treat all three standard vacuum states: Hartle–Hawking, Boulware, and Unruh. I also address all point-wise energy conditions, NEC, WEC, SEC, and DEC, and finally discuss the ANEC. The analysis of this paper can also be viewed as an extension of the (1+1)-dimensional aspects of the recent papers by Ford and Roman [3,4]. The results obtained for the Hartle–Hawking and Boulware vacuum states are qualitatively similar to the (3+1)-dimensional case [1,2], and give us additional confidence in the general features deduced from the numerical analysis required in (3+1) dimensions.

II. VACUUM POLARIZATION IN (1+1) SCHWARZSCHILD SPACETIME

For a static (1+1) dimensional spacetime one knows that

\[ \langle 0 | T^\mu_\nu | 0 \rangle = \begin{bmatrix} -\rho & -f \\ f & -\tau \end{bmatrix}. \]  

Where \( \rho, \tau \) and \( f \) are functions of \( r, M \) and \( \hbar \). (Note: I set \( G \equiv 1 \), and choose to work in a local-Lorentz basis attached to the fiducial static observers [FIDOS].)

A subtlety arises when working in a local-Lorentz basis and looking at the two-index-down (or two-index-up)
versions of the stress-energy. Outside the horizon one has

\[ g^{\mu\nu}_{\text{outside}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ \end{bmatrix}. \]  

Consequently

\[ \langle 0| T^{\mu\nu} |0 \rangle_{\text{outside}} = \begin{bmatrix} +\rho & +f \\ +f & -\tau \\ \end{bmatrix}. \]  

Inside the horizon on the other hand, it is the radial direction that is timelike, so

\[ g^{\mu\nu}_{\text{inside}} = \begin{bmatrix} +1 & 0 \\ 0 & -1 \\ \end{bmatrix}. \]  

Consequently one has the potentially confusing result that

\[ \langle 0| T^{\bar{\mu}\bar{\nu}} |0 \rangle_{\text{inside}} = \begin{bmatrix} -\rho & -f \\ -f & +\tau \\ \end{bmatrix}. \]

Thus, inside the horizon, one should interpret \( \tau \) as the energy density and \( \rho \) as the tension (this tension now acting in the spacelike \( t \)-direction).

To start the actual analysis I require explicit analytic formulae for the stress-energy tensor. By working from the analysis in Christensen and Fulling \cite{5}, or the presentation in the textbook by Birrell and Davies \cite{6}, and the analysis in Christensen and Fulling \cite{5}, it is easy to show that the stress energy tensor is given by simple rational polynomials in the variable \( z = (2M/r) \).

In the Hartle–Hawking vacuum:

\[ \begin{align*}
\rho(z) &= +p_\infty \left(1 + z + z^2 - 7z^3\right), \\
\tau(z) &= -p_\infty \left(1 + z\right) \left(1 + z^2\right).
\end{align*} \]  

Here I have defined a constant

\[ p_\infty \equiv \frac{\hbar}{6(16\pi)(2M)^2}. \]  

In the Hartle–Hawking vacuum \( p_\infty \) can be interpreted as the pressure at spatial infinity.

In the Boulware vacuum:

\[ \begin{align*}
\rho(z) &= -p_\infty z^3 \left(8 - 7z\right), \\
\tau(z) &= +p_\infty z^4 \frac{1}{1-z}.
\end{align*} \]  

In the Unruh vacuum:

\[ \begin{align*}
\rho(z) &= +p_\infty \left(\frac{1}{1-z} - 16z^3 + 14z^4\right), \\
\tau(z) &= -p_\infty \left(\frac{1}{1-z} - 2z^4\right), \\
f(z) &= +p_\infty \frac{1}{2(1-z)}.
\end{align*} \]  

To get these expressions I have written (in the notation of Christensen and Fulling) \cite{5}

\[ \begin{align*}
H_2(z) &= p_\infty (1 - z^4), \\
T(z) &= 8 \ p_\infty z^3,
\end{align*} \]  

and carried through the straightforward analysis indicated in Birrell and Davies \cite{6}. For the Boulware and Unruh vacua the resulting stress-energy tensors have been checked by taking the explicit formulae of Unruh \cite{6}, and Ford and Roman \cite{4}, and translating them into a local orthonormal basis. With these analytic formulae in hand, investigation of the energy conditions is straightforward.

I mention in passing several cautionary notes: In (1+1) dimensions one has

\[ \langle U^+ | T^{\bar{\mu}\bar{\nu}} | U^+ \rangle + \langle U^- | T^{\bar{\mu}\bar{\nu}} | U^- \rangle = \langle H | T^{\bar{\mu}\bar{\nu}} | H \rangle + \langle B | T^{\bar{\mu}\bar{\nu}} | B \rangle. \]  

Here \( U^\pm \) denote the ordinary and time-reversed Unruh vacuum states. This special relationship does not survive in (3+1) dimensions \cite{5, page 2098}. The fact that it happens to work in (1+1) dimensions is a result of the conformal flatness of (1+1)-dimensional spacetimes, which implies that all the asymptotic scattering amplitudes are unity.

A second cautionary note: In (1+1) dimensions, one has the exact result

\[ \langle H | T^{\bar{\mu}\bar{\nu}} | H \rangle - \langle B | T^{\bar{\mu}\bar{\nu}} | B \rangle = p_\infty \frac{1}{1-z} \begin{bmatrix} +1 & 0 \\ 0 & -1 \\ \end{bmatrix}. \]  

Thus the difference between the Hartle–Hawking and Boulware stress-energy is exactly a thermal distribution of massless particles at the Hawking temperature. Despite an early conjecture \cite{5, page 2101, equation (6.29)}, this special relationship does not survive in (3+1) dimensions \cite{8}. The fact that it happens to work in (1+1) dimensions is again a result of the conformal flatness of (1+1)-dimensional spacetimes.

A final cautionary note is that a subtlety arises when I turn to discussing the SEC: one must first decide how exactly to continue the SEC to (1+1) dimensions. In (3+1) dimensions one writes the SEC as

\[ \bar{T}_{\mu\nu} V^\mu V^\nu \geq 0? \]  

Where \( \bar{T} \) is the trace-reversed stress tensor

\[ \bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T. \]  

In (1+1) dimensions one has to decide whether to literally retain the above definition, or whether to use the (1+1) dimensional version of trace reversal

\[ \bar{T}_{\mu\nu} \equiv T_{\mu\nu} - g_{\mu\nu} T. \]  

Under the first option, working outside the horizon,
This option is uninteresting, because with this definition the SEC is identical to the NEC. To see this note that with this definition the SEC would be \((\beta \in [0,1])\)
\[
\gamma^2[(\rho - \tau)/2 + 2\beta f + \beta^2(\rho - \tau)/2] \geq 0? 
\]
This is easily rearranged to give
\[
(\rho - \tau) \pm 2f \geq 0? 
\]
Since this is to hold for all \(\beta \in [0,1]\), this implies and is implied by
\[
(\rho - \tau) \pm 2f \geq 0? 
\]
Which is exactly the NEC.

On the other hand, with the second option one has
\[
\langle 0|\Tilde{T}^{\hat{\rho}^0}|0 \rangle = \left[\begin{array}{c}
\rho - \tau/2 + f \\
\rho - \tau/2 + f
\end{array}\right]. 
\]

With this definition the SEC is
\[
\gamma^2[-\tau \pm 2f + \beta^2 f] \geq 0? 
\]
This is equivalent to
\[
-\tau \geq 0? \quad \rho - \tau \pm 2f \geq 0? 
\]
Thus this definition of SEC implies but is not implied by the NEC.

III. HARTLE–HAWKING VACUUM

A. Outside the horizon:

Outside the event horizon, the NEC reduces to the single constraint
\[
\rho(r) - \tau(r) \geq 0? 
\]
It is easy to see that
\[
\rho(r) - \tau(r) = 2\rho_\infty (1 - z) (1 + 2z + 3z). 
\]
This is explicitly positive everywhere outside the event horizon. Therefore the NEC is definitely satisfied everywhere outside the event horizon.

Outside the event horizon, the WEC reduces to the pair of constraints
\[
\rho \geq 0? \quad \rho(r) - \tau(r) \geq 0? 
\]
It is easy to see that \(\rho\) switches sign and becomes negative for \(z > 0.671907\) corresponding to \(r < 2.9776M\). (One has to numerically solve a cubic.) Therefore the WEC is definitely violated in the region \(r \in [2M, 2.9776M]\).

Outside the event horizon, the DEC reduces to the three constraints
\[
\rho(r) \geq 0? \quad \rho(r) - \tau(r) \geq 0? \quad \rho(r) + \tau(r) \geq 0? 
\]

It is easy to see that
\[
\rho(r) + \tau(r) = -8\rho_\infty z^3. 
\]
(This is in fact just the negative of the anomalous trace.)

This is explicitly negative everywhere outside the event horizon. Therefore the DEC is definitely violated everywhere outside the event horizon.

Outside the event horizon, the SEC reduces to the pair of constraints
\[
-\tau(r) \geq 0? \quad \rho(r) - \tau(r) \geq 0? 
\]
Both of these quantities are positive outside the horizon, so SEC is satisfied everywhere outside the horizon.

B. Inside the horizon:

Inside the event horizon, the radial coordinate becomes timelike, and the roles played by \(\rho(r)\) and \(\tau(r)\) are interchanged. The NEC reduces to the constraint
\[
\tau(r) - \rho(r) \geq 0? 
\]
We have already seen that \(\rho - \tau\) explicitly factorizes, goes to zero and switches sign at the event horizon. This sign switch in \(\rho - \tau\) exactly matches the definition switch in the NEC. We conclude that the NEC is definitely satisfied everywhere inside the event horizon.

Inside the event horizon, the WEC reduces to the pair of constraints
\[
\tau(r) \geq 0? \quad \tau(r) - \rho(r) \geq 0? 
\]
But \(\tau\) is negative for all \(r\). Therefore the WEC is definitely violated inside the event horizon. This automatically implies that the DEC is definitely violated inside the event horizon.

Inside the event horizon, the SEC reduces to the pair of constraints
\[
-\rho(r) \geq 0? \quad \tau(r) - \rho(r) \geq 0? 
\]
Both of these quantities are positive inside the horizon, so the SEC is satisfied everywhere inside the horizon.
C. Summary:

In the Hartle–Hawking vacuum:

- NEC and SEC are satisfied throughout the spacetime.
- WEC is satisfied for \( r \in [2.9776M, \infty] \) and violated for \( r \in [0, 2.9776M] \)
- DEC is violated throughout the spacetime.

IV. BOULWARE VACUUM

A. Outside the horizon:

Outside the event horizon, the NEC reduces to the single constraint

\[
\rho(r) - \tau(r) \geq 0？
\] (38)

It is easy to see that

\[
\rho(r) - \tau(r) = -2 p_\infty \frac{4 - 3z}{1 - z}
\] (39)

This is explicitly negative everywhere outside the event horizon. Therefore the NEC, (and also WEC, DEC, and SEC) are definitely violated everywhere outside the event horizon.

B. Inside the horizon:

Inside the event horizon, we interchange \( \rho(r) \) and \( \tau(r) \). The NEC reduces to the pair of constraints

\[
\tau(r) \geq 0？ \quad \tau(r) - \rho(r) \geq 0？
\] (40)

For the Boulware vacuum, \( \tau - \rho \) is positive in the range \( z \in [0, 1] \cup [4/3, \infty] \). Thus NEC is violated in the range \( r \in [3M/2, 2M] \) and satisfied in the range \( r \in [0, 3M/2] \).

Inside the event horizon, the WEC reduces to the pair of constraints

\[
-\rho(r) \geq 0？ \quad \tau(r) - \rho(r) \geq 0？
\] (42)

For the Boulware vacuum, \( -\rho \) is positive in the range \( z \in [0, 1] \cup [8/7, \infty] \), while \( \tau - \rho \) is positive in the range \( z \in [0, 1] \cup [4/3, \infty] \). Thus SEC is again violated in the same range as the NEC.

C. Summary:

In the Boulware vacuum:

- NEC and SEC are violated for \( r \in [3M/2, \infty] \) and satisfied for \( r \in [0, 3M/2] \).
- WEC and DEC are violated throughout the spacetime.

V. UNRUH VACUUM

A. Outside the horizon:

Outside the event horizon, the presence of a flux in the Unruh vacuum implies that the NEC reduces to a pair of constraints

\[
\rho(r) - \tau(r) \pm 2f(r) \geq 0？
\] (43)

(Warning: The minus sign corresponds to an outgoing null ray, while the plus sign represents an ingoing null ray.) It is easy to see that

\[
\rho(r) - \tau(r) + 2f(r) = 2 p_\infty (1 - z)(1 + 2z + 3z^2).
\] (44)

\[
\rho(r) - \tau(r) - 2f(r) = -2 p_\infty z^3 \frac{4 - 3z}{1 - z}.
\] (45)

The second of these constraints is explicitly negative everywhere outside the event horizon. Therefore the NEC, (and also the WEC, DEC, and SEC) are definitely violated everywhere outside the event horizon.

You may wish to note that

\[
[\rho(r) - \tau(r) + 2f(r)]_U = [\rho(r) - \tau(r)]_H, \quad (46)
\]

\[
[\rho(r) - \tau(r) - 2f(r)]_U = [\rho(r) - \tau(r)]_B. \quad (47)
\]

Thus the discussion for the Hartle–Hawking and Boulware vacua can be carried over immediately to respectively the ingoing and outgoing null geodesics of the Unruh vacuum. [This result is special to (1+1) dimensions.]

A subtlety is that it is now possible to define two distinct types of NEC, a NEC\(^+\) and a NEC\(^-\), depending on whether one wishes to follow outgoing or ingoing null curves. (That is, depending on whether one is approaching Scri\(^+\) or coming in from Scri\(^-\)).

The NEC\(^+\) condition (outgoing null curves) is violated outside the event horizon, while the NEC\(^-\) condition (ingoing null curves) is satisfied.

Notice that it does not make sense to talk about WEC\(^\pm\), DEC\(^\pm\) or SEC\(^\pm\) because the use of timelike vectors in these energy conditions does not let you make an invariant separation into classes of ingoing and outgoing geodesics.
Inside the event horizon, we again interchange $\rho(r)$ and $\tau(r)$. The NEC reduces to the pair of constraints

$$\tau(r) - \rho(r) \mp 2f(r) \geq 0? \quad (48)$$

These are identical to the two conditions outside the horizon with the signs flipped.

In particular the NEC$^-$ condition is satisfied inside the event horizon, while the NEC$^+$ condition is violated for $z \in [1, 4/3]$ and satisfied for $z \in [4/3, \infty)$, corresponding to $r \in [3M/2, 2M]$ and $r \in [0, 3M/2]$ respectively.

This is enough to tell us that WEC, DEC, and SEC are violated at least in the region $r \in [3M/2, 2M]$.

Inside the event horizon, the WEC is

$$\gamma^2(\tau \mp 2\beta f - \beta^2 \rho) \geq 0? \quad (49)$$

which reduces to the pair of constraints

$$\tau(r) \geq 0? \quad \tau(r) - \rho(r) \pm 2f(r) \geq 0? \quad (50)$$

But $\tau$ is negative inside the horizon. Therefore WEC (and also DEC) is definitely violated everywhere inside the event horizon.

Inside the event horizon the SEC is

$$\gamma^2(-\rho \mp 2\beta f + \beta^2 \tau) \geq 0? \quad (51)$$

which reduces to the triplet of constraints

$$-\rho(r) \geq 0? \quad \tau(r) - \rho(r) \pm 2f \geq 0? \quad (52)$$

For the Unruh vacuum, $\rho$ is positive in the range $z \in [1, 1.08729]$, and $r \in [0, 0.474574]$, while the second condition is always satisfied, and the third condition is violated in the range $z \in [0, 1] \cup [4/3, \infty]$. Thus SEC is again violated in the same range as the NEC.

C. Summary:

In the Unruh vacuum:

- NEC and SEC are violated for $r \in [3M/2, \infty]$ and satisfied for $r \in [0, 3M/2]$.
- NEC$^-$ is satisfied throughout the spacetime.
- NEC$^+$ is violated for $r \in [3M/2, \infty]$ and satisfied for $r \in [0, 3M/2]$.
- WEC and DEC are violated throughout the spacetime.

VI. TOTAL DEC VIOLATION

I shall now show that in any quantum state, vacuum or not, the DEC is violated throughout the spacetime. (This comes from the fact that DEC violations in 1+1 dimensions can be intimately related to the trace anomaly.)

I start from the fact that for the DEC to hold, the vector $T^\mu\nu V_\nu$ must be non-spacelike for any timelike vector $V_\nu$. This requires that for all $\beta \in [-1, 1]$ one must have

$$|f - \beta \tau| \leq |\rho + \beta f|? \quad (53)$$

Thus

$$(f - \beta \tau)^2 \leq (\rho + \beta f)^2?$$

$$\Rightarrow f^2 \pm 2|\beta|f\tau + \beta^2 \tau^2 \leq \rho^2 \pm 2|\beta|f\rho + \beta^2 f^2?$$

$$\Rightarrow f^2 + \beta^2 \tau^2 \leq \rho^2 + \beta^2 f^2?$$

$$\Rightarrow f^2 + \tau^2 \leq \rho^2 + f^2?$$

$$\Rightarrow \tau^2 \leq \rho^2?$$

$$\Rightarrow \rho \pm \tau \geq 0? \quad (54)$$

Thus a minimum condition for DEC to hold is for $\rho + \tau = -\langle T \rangle$ to be positive. But we know that $\langle T \rangle$ is given exactly by the conformal anomaly (and this is independent of the quantum state), and that in the 1+1 Schwarzschild geometry $\langle T \rangle = 8 p_{\infty} z^2$ is positive. Thus DEC is violated everywhere in the spacetime.

Of course this result generalizes to any (1+1)-dimensional spacetime: We now know that the DEC must be violated at least on those regions where

$$\langle T \rangle \equiv \frac{1}{768\pi} R > 0. \quad (55)$$

That is: one needs a negative Ricci scalar to even have a hope of satisfying the DEC.

VII. ANEC VIOLATION?

Outside the event horizon we may certainly write the ANEC integral as $\frac{8}{768\pi} R$. (See also \(3, page 133, equations (12.59)–(12.63)).)

$$I_\gamma \equiv \int_\gamma T^\mu_\nu k^\mu k^\nu \ d\lambda,$$

$$= \int_\gamma (\rho - \tau \pm 2f) \xi^2 \ d\lambda,$$

$$= \int_\gamma (\rho - \tau \pm 2f) \ dt,$$

$$= \int_0^\infty (\rho - \tau \pm 2f) \frac{dr}{2M (1 - 2M/r)}. \quad (56)$$

This appears to weight the region near the event horizon very heavily—because of the explicit pole at $r = 2M$. However, the integrand $(\rho - \tau \pm 2f)$ often has a zero at
the event horizon. (This occurs in the Hartle–Hawking state, and for the ingoing null geodesics in the Unruh state).

Inside the event horizon there are additional sign-flips:

\[ I_\gamma = \int \gamma T_{\mu \nu} k^\mu k^\nu \, d\lambda, \]
\[ = \int (\tau - \rho \mp 2f) \xi^2 \, d\lambda, \]
\[ = \int (\tau - \rho \mp 2f) \, dt, \]
\[ = - \int (\tau - \rho \mp 2f) \, dr. \]
\[ = \int_0^{2M} (\rho - \tau \pm 2f) \frac{1}{1 - 2M/r} \, dr. \quad (57) \]

To see where the extra minus sign comes, assume we are looking at an outgoing null geodesic (one that approaches Scri\(^+\)), such a null geodesic starts off from the past singularity and must first pass through the past event horizon H\(^-\). Inside the past event horizon, if you want an outgoing null geodesic (increasing \( r \)) to be travelling forward in \( t \) one must take \( dr = |1 - 2M/r|dt \). Reversing the argument, the same result holds for incoming null geodesics, ones that start out from Scri\(^-\), cross the future event horizon H\(^+\), and terminate on the future singularity. (The present discussion does not address null geodesics that pass through the bifurcation two-point, see Ford and Roman [4].)

Combining these results, one may formally write

\[ I_\gamma = \int_0^\infty (\rho - \tau \pm 2f) \frac{1}{1 - 2M/r} \, dr \]
\[ = 2M \int_0^\infty (\rho - \tau \pm 2f) \frac{1}{z^2(1 - z)} \, dz. \quad (58) \]

The integral can potentially have divergences at \( r = 0 \), \( r = 2M \), and \( r = \infty \).

For the Hartle–Hawking vacuum everything is well-defined at the event horizon. It is most sensible to integrate outward or inward from \( r = 2M \), to consider

\[ I[z] = 2M \int_z^1 \frac{(\rho - \tau)}{z^2(1 - z)} \, d\bar{z} \]
\[ = 4Mp_\infty ((1 - z)(1 + 3z)/z - 2\ln(z)) \quad (59) \]

Note that for \( z < 1 \) (that is, \( r > 2M \)) this is explicitly positive (as it should be). There is of course an infra-red divergence as \( r \to \infty \). For \( z > 1 \) one should switch the integration limits, and again get a positive result. There is now an ultraviolet singularity as \( z \to \infty (r \to 0) \).

The total ANEC integral, from \( r = \infty \) to \( r = 0 \), is positive infinity.

For the Boulware vacuum there is a singularity at the horizon, so it is more sensible to integrate inward from spatial infinity and keep \( z < 1 \):

\[ I[z] = 2M \int_0^z \frac{(\rho - \tau)}{z^2(1 - z)} \, d\bar{z} \]
\[ = 4Mp_\infty \{z(2 - 3z)/(1 - z) + 2\ln(1 - z)\} \quad (60) \]

Although the polynomial piece here is (for \( z < 2/3 \)) positive it is easy enough to check that the logarithm is negative and dominant. If one tries to push this integral past the event horizon one picks up a negative infinity.

Inside the event horizon one has singularities at both \( r = 0 \) and \( r = 2M \). It is perhaps most instructive to fix one end of the ANEC integral at \( z = 4/3 \), the boundary between the NEC satisfying and NEC violating region. In that case

\[ I[z] = 2M \int_{4/3}^z \frac{(\rho - \tau)}{z^2(1 - \bar{z})} \, d\bar{z} \]
\[ = 4Mp_\infty \{(z - 2)(3z - 4)/(z - 1) + 2\ln(3z - 1)\} \quad (61) \]

For \( z > 4/3 \): Although the polynomial piece here changes sign at \( z = 2 \), it is easy enough to check that the logarithm is positive and always sufficient to make the net integral positive. Consequently, integrating inward from \( r = 3M/2 \) toward \( r = 0 \) gives a positive contribution to the ANEC.

For \( z < 4/3 \): The polynomial piece is positive and the logarithm is negative and sub-dominant. Thus, after switching the limits of integration, one sees that integrating inward from just inside \( r = 2M \) to \( r = 3M/2 \) gives a negative contribution to the ANEC.

If one insists on integrating all the way from \( r = \infty \) to \( r = 0 \) one picks up two negative infinities and one positive infinity—for a net result that is at best purely formal. (With suitable regulator, one might argue that the overall integral can be taken to be negative infinity.) This is not in conflict with the ANEC theorems of Yurtsever [10], and Wald and Yurtsever [11], because those theorems were derived in asymptotically flat (1+1)-dimensional spacetimes with technical assumptions about the inextendible nature of the null geodesics. The singularity present in the Schwarzschild geometry prevents us from applying these theorems to the present case.

For ingoing null geodesics in the Unruh vacuum, the discussion is identical to that for the Hartle–Hawking vacuum, while for outgoing null geodesics in the in the Unruh vacuum, the discussion is identical to that for the Boulware vacuum.

\[ \text{VIII. DISCUSSION} \]

In a pair of companion papers [1,2] I have studied the gravitational vacuum polarization in (3+1) dimensions, discussing both the Hartle–Hawking and Boulware vacuum states. In the (3+1) Hartle–Hawking vacuum state I discovered a complicated layering of energy-condition violations confined to the region between the unstable
photon orbit and the event horizon. In the (3+1) Boul-
ware vacuum I found that all point-wise energy and aver-
eged conditions are violated throughout the entire region
exterior to the event horizon, and that the point-wise en-
ergy conditions seemed to be violated inside the event
horizon.

In this paper I have looked at the analytically more
tractable model of (1+1)-dimensional Schwarzschild
spacetime, mainly as a sanity check on the (3+1)-
dimensional calculations. Happily, the basic flavour of
the (3+1)-dimensional results follows through: Many of
the energy conditions are violated, the precise locations
and manner of violation being influenced by the partic-
ular vacuum state being considered. The results of this
paper are also compatible with, and extensions of, ear-
lier (1+1)–dimensional investigations by Ford and Ro-
mans.

The ANEC is satisfied for the Hartle–Hawking vac-
um, but is ill-defined (and arguably negative) for the
Boulware vacuum. (Certainly integrating from infinity
down to the event horizon gives a finite and negative
ANEC integral.)

Overall the situation is this: energy condition viola-
tions are ubiquitous and are particularly prevalent in the
Boulware vacuum in the region outside the event horizon,
where every point-wise energy condition is violated, and
all one-sided integrated energy conditions are violated.
(The same comment applies to outgoing null curves in
the Unruh vacuum.)

Continuing the discussion back to (3+1) dimensions:
These ubiquitous violations of the energy conditions
have immediate and significant impact on issues such as
whether or not it is possible, even in principle, to
generalise the classical singularity theorems, classical
positive mass theorems, and classical laws of black
hole dynamics to semiclassical quantum gravity. It
seems that the standard energy conditions may not be
the right tools for the job. Something along the lines of
the Ford–Roman “quantum inequalities” might be more
useful.

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