Modular Invariants, Graphs and $\alpha$-Induction for Nets of Subfactors II

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Abstract

We apply the theory of $\alpha$-induction of sectors which we elaborated in our previous paper to several nets of subfactors arising from conformal field theory. The main application are conformal embeddings and orbifold inclusions of $SU(n)$ WZW models. For the latter, we construct the extended net of factors by hand. Developing further some ideas of F. Xu, our treatment leads canonically to certain fusion graphs, and in all our examples we rediscover the graphs Di Francesco, Petkova and Zuber associated empirically to the corresponding $SU(n)$ modular invariants. We establish a connection between exponents of these graphs and the appearance of characters in the block-diagonal modular invariants, provided that the extended modular $S$-matrices diagonalize the endomorphism fusion rules of the extended theories. This is proven for many cases, and our results cover all the block-diagonal $SU(2)$ modular invariants, thus provide some explanation of the A-D-E classification.

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1 Introduction

1.1 Background

The $SU(n)_q$ subfactors of Wenzl [51] can be understood from the viewpoint of statistical mechanics [17], the IRF models of [8] or from the viewpoint of conformal field theory, irreducible highest weight positive energy representations of the loop groups of $SU(n)$ [50]. These viewpoints are also related to the study and classification of modular partition functions on a torus. The statistical mechanical models of [8] are generalizations of the Ising model. The configuration space of the Ising model, distributions of symbols $+$, $-$ on the vertices of the square lattice $\mathbb{Z}^2$, can also be thought of as distributions of edges of the Dynkin diagram $A_3$ on the edges of a square lattice, where the end vertices are labelled by $+$ and $-$. This model can be generalized by replacing $A_3$ by other graphs $\Gamma$ such as Dynkin diagrams or indeed the Weyl alcove $A^{(m)}$ of the level $k$ integrable representations of $SU(n)$, where $m = k + n$ is the altitude. The vertices of $A^{(n+k)}$ are given by weights

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\begin{align*}
\{ \Lambda = \sum_{i=1}^{n-1} m_i \Lambda_{(i)} : m_i \in \mathbb{N}_0, \sum_{i=1}^{n-1} m_i \leq k \} \text{ where the } \Lambda_{(i)} \text{ are the } n-1 \text{ weights of the fundamental representation, and the oriented edges are given by the vectors } e_i \text{ defined by } e_1 = \Lambda_{(1)}, e_i = \Lambda_{(i)} - \Lambda_{(i-1)}, i = 1, 2, \ldots, n-1, e_n = \Lambda_{(n-1)}. \text{ We can also label our states by partitions or Young tableaux } (p_j)_{j=1}^{n-1}, k \geq p_1 \geq p_2 \geq \cdots \geq p_{n-1} \geq p_n \equiv 0 \text{ obtained by the transformation } (m_i)_{i=1}^{n-1} \mapsto (p_j)_{j=1}^{n-1}, \text{ where } p_j = \sum_{i=j}^{n-1} m_i. \text{ The unoccupied state corresponds to } (0, 0, \ldots, 0) \text{ or the empty Young tableau in the two descriptions, which we often denote by } \ast \text{ or } 0.
\end{align*}

A configuration is then a distribution of the edges of \( \Gamma \) over \( \mathbb{Z}^2 \), and associated to each local configuration is a Boltzmann weight (see Figure 1) satisfying the Yang-Baxter equation of Figure 2. The justification of the term \( SU(n) \) models is as follows. By Weyl duality, the representation of the permutation group on \( \otimes M_n \) is the fixed point algebra of the product action of \( SU(n) \). Deforming this, there is a representation of the Hecke algebra in \( \otimes M_n \) whose commutant is a representation of a deformation of \( SU(n) \), the quantum group \( SU(n)_q \). The Boltzmann weights lie in this Hecke algebra representation, and at criticality reduce to the natural braid generators \( g_i \), so that the Yang-Baxter equation of Figure 2 reduces to the braid relation \( g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \). The labels of the irreducible representations of either the Hecke algebra (e.g. the permutation group when \( q = 1 \)) or the quantum group (e.g. \( SU(n) \) when \( q = 1 \)) are generically given by \( A_\ell \), a Young tableaux of at most \( n-1 \) rows. However when \( q \) is a root of unity \( e^{2\pi i / m} \) we have the further constraint of at most \( k = m - n \) columns, where \( k \) is the level, i.e. \( A(m) \) (e.g. when \( n = 2 \) the vertices of the Dynkin diagram \( A_{k+1} \)).
Figure 3: Matrices of partition functions \([T_{\xi\eta}]\) where \(\xi, \eta\) are paths on \(\Gamma\) with fixed initial vertex \(*\) and same terminal vertex generate a von Neumann algebra \(N\).

\[
\begin{array}{cccc}
\xi & \downarrow & \eta \\
\end{array}
\]

Figure 4: Embedding of \(N \subset M\) by \(T \rightarrow \text{Ad}(V)(T)\) where \(V = g_1g_2\cdots\) is the product of Boltzmann weights at criticality.

The Boltzmann weights involve paths of length two in the Bratteli diagram using the embedding graph \(\Gamma\). As we look at larger and larger partition functions (based on some fixed initial vertex \(*\)) then we can complete with respect to a natural trace and obtain a von Neumann algebra as in Figure 3.

A subfactor \(N \subset M\) can be obtained with the aid of the initial Boltzmann weights placed on the boundary as in Figure 4. For the \(SU(n)_q\) subfactors, this just amounts to \(\{g_i : i = 1, 2, 3, \ldots\}'' \subset \{g_i : i = 0, 1, 2, \ldots\}''\) because of the braid relations \(\text{Ad}(g_1g_2\cdots)(g_i) = g_{i+1}\). The center \(Z_n\) of \(SU(n)\) acts on \(A^{(m)}\) leaving the Boltzmann weights invariant and hence induces an action on \(M\), leaving \(N\) globally invariant, yielding the orbifold subfactor \(N^{Z_n} \subset M^{Z_n}\). The action of the center \(Z_n\) (corresponding to the simple currents) on \(A^{(m)}\) is as follows. We set \(A_0 = \ast\), and label the other end vertices of \(A^{(m)}\) by \(A_1 = A_0 + (m - n)e_1, A_2 = A_1 + (m - n)e_2, \ldots, A_{n-1} = A_{n-2} + (m - n)e_{n-1}\). Define a rotation symmetry of the graph \(A^{(m)}\) by \(\sigma(A_j + \sum c_r e_r) = A_{j+1} + \sum c_r e_{r+1}\) where the indices are in \(Z_n\).

Let us now turn to the loop group picture. The loop group \(LG\) is the group of smooth maps from \(S^1\) into a compact Lie group \(G\) under pointwise multiplication. We are interested in projective representations of \(LG \rtimes \text{Rot}(S^1)\), where \(\text{Rot}(S^1)\) is the rotation group, which are highest weight representations in that the generator \(L_0\) of the rotation group is bounded.
below. Such representations are called positive energy representations and are classified by irreducible representations of $G$ and a level $k$. For unitary irreducible positive energy representations, the possibilities are severely restricted. Indeed $k$ must be integral and, for a given value of the level, there are only a finite number of admissible (vacuum vector) irreducible representations of $G$. For the case of $G = SU(n)$, the admissible ones at level $k$ are the vertices of $A^{(m)}$, where $m = n + k$. Restricting to loops $L_I G$ concentrated on an interval $I \subset S^1$, $L_I G = \{ f \in LG : f(z) = e, \ z \notin I \}$, we get for each positive energy representation $\pi$ a subfactor $\pi(L_I G)'' \subset \pi(L_I G)''$ if $I^c$ is the complementary interval, of type III and of finite index — e.g. of index $4 \cos^2(\pi/(k + 2))$ in the case of the fundamental representation of $SU(2)$ and level $k$.

We next turn to the modular invariant picture. It is argued on physical grounds that the partition function $Z(\tau)$ in a conformal field theory should be invariant under re-parameterization of the torus by $SL(2, \mathbb{Z})$. In the string theory formulation, modular invariance is essentially built into the definition of the partition function (although Nahm [37] has argued the case for modular invariance in terms of the chiral algebra and its representations rather than a functional integral setting). In the transfer matrix picture of statistical mechanics we can write the partition function as an average over $e^{-\beta H}$, where $H$ is the Hamiltonian, now $L_0 + \bar{L}_0 - c/12$ (the shift by $c/24$ arising from mapping the Virasoro algebra on the plane to a cylinder). We have a momentum $P (= L_0 - \bar{L}_0)$ describing evolution along the closed string, so taking both evolutions into account, we first compute

$$Z(\tau) = \text{tr} \left( e^{-\beta H} e^{i\eta P} \right) = \text{tr} \left( e^{2\pi i\tau (L_0 - c/24)} e^{-2\pi i\bar{\tau} (\bar{L}_0 - c/24)} \right).$$

(1)

Here $2\pi i\tau = -\beta + i\eta$ parameterizes the metric of the torus, and we then have to average over $\tau$. If we choose one $\tau$ from each orbit under the action of $PSL(2, \mathbb{Z})$ and integrate we implicitly assume that $Z(\tau)$ is modular invariant.

From a Hilbert space decomposition of the loop group representation the partition function Eq. (1) decomposes as

$$Z = \sum_{\Lambda, \Lambda'} Z_{\Lambda, \Lambda'} \chi_{\Lambda} \bar{\chi}_{\Lambda'}$$

(2)

where $\chi_{\Lambda}$ is the conformal character $\text{tr} (q^{L_0 - c/24})$, $q = e^{2\pi i\tau}$ of the unitary positive energy irreducible representation $\pi_{\Lambda}, \Lambda \in A^{(m)}$, where $m = n + k$ for some fixed level $k$.

The problem then is to find or classify all expressions of the form Eq. (2) where $Z$ is $SL(2, \mathbb{Z})$ invariant, subject to the normalization $Z_{0,0} = 1$ and
$Z_{\Lambda,\Lambda'}$ is a non-negative integer. A simple argument of Gannon [21] shows that $\sum_{\Lambda,\Lambda'} Z_{\Lambda,\Lambda'} \leq 1/S_{0,0}^2$, where $S_{0,0}$ is a matrix entry of the S-matrix action of $SL(2,\mathbb{Z})$ on characters, and hence for a given $G$ at a fixed level, there are only finitely many possible modular invariants. They have been completely classified in the case $SU(2)$ by [7] and in the case $SU(3)$ by [22], and the program of Gannon to the complete classification is far advanced — see e.g. the notes of Chapter 8 of [18] for a review. The Gannon program involves identifying first a special class of modular invariants, the $ADE_7$ invariants which satisfy $Z_{0,\Lambda} \neq 0$ implies that $\Lambda$ is (the weight labelling) a simple current or equivalently $Z_{0,\Lambda} \neq 0$ implies $S_{\Lambda,0} = S_{0,0}$, and then identify what appear to be very few remaining exceptions which include those arising from conformal embeddings. The $ADE_7$ invariants include all the automorphism invariants, for which $Z_{0,\Lambda} = 0, \Lambda$. Such an invariant is basically an automorphism of the fusion ring. There is a permutation $\sigma$ of $\mathcal{A}(m)$ such that $Z_{\Lambda,\Lambda'} = \delta_{\Lambda,\sigma(\Lambda')}$.

The $ADE_7$ invariants also include the simple current invariants for which $Z_{\Lambda,\Lambda'} \neq 0$ implies $\Lambda = J \cdot \Lambda'$ for a simple current $J$. Automorphism and simple current invariants constitute the $A$ and $D$ type modular invariants. Note that there are two kinds of modular invariants:

$$\sum |\chi_i|^2, \quad \text{type I}$$
$$\sum \chi_i \bar{\chi}_{\sigma(i)}, \quad \text{type II}$$

where $\chi_i$ are (possibly extended) characters and $\sigma$ is a permutation of the (extended) fusion rules. The type II invariants where the characters are properly extended (i.e. at least one $\chi_i$ is a proper sum over two or more $\chi_\Lambda$’s) finally constitute the $E_7$ modular invariants. The type I modular invariants where the characters $\chi_i$ are proper extensions are also called block-diagonal.

In fact any $SU(n)$ block-diagonal modular invariant can be interpreted as a completely diagonal invariant of a larger theory embedding the $SU(n)$ level $k$ WZW theory. In the case of a conformal inclusion the larger theory is given in terms of a $G$ (necessarily level 1) WZW theory with $G$ a simple Lie group, and in the orbifold inclusion case the larger theory is given in terms of a simple current extension of the $SU(n)$ theory, and the $SU(n)$ theory itself can be thought of as the $\mathbb{Z}_n$ orbifold of the extended object. For $SU(2)$ and $SU(3)$ both cases actually exhaust all the block-diagonal modular invariants.\footnote{However, for $SU(n)$ with larger $n$ there appear also block-diagonal modular invariants which arise from level-rank duality but neither from conformal nor orbifold inclusions, e.g. for $SU(10)$ at level 2. This kind of invariants will not be treated in this paper.}

The modular invariants appear to be labelled naturally, in the case of
$SU(2)$ and $SU(3)$, by graphs. The $SU(2)$ modular are labelled by A-D-E Dynkin diagrams in the sense that the non-vanishing diagonal entries of the modular invariant are given by the conformal characters labelled by the Coxeter exponents of the labelling ADE graph. Recall the eigenvalues of the (adjacency matrix of the) D and E graphs constitute subsets of the vertices of the A graph with the same Coxeter number, and their labels are called Coxeter exponents. For example in case of $SU(2)$ at level 16 there are three modular invariants. In each case the diagonal part of the invariant is described by a certain subset $I = \{j\}$ of the vertices of $A_{17}$. The (adjacency matrix of the) graph of $A_{17}$ has eigenvalues \(\{2\cos((j + 1)\pi/h)\}\) where \(j = 0, 1, 2, ..., 16\) labels the vertices of $A_{17}$ and $h = 18$ is the Coxeter number of $A_{17}$. Then $I$ is the set of the Coxeter exponents, i.e. the set \(\{2\cos((j + 1)\pi/h), j \in I\}\) gives all the eigenvalues of the Dynkin diagram, $A_{17}$, $D_{10}$ or $E_{7}$. The completely diagonal invariant then corresponds to the graph $A_{17}$ itself. In this way all $SU(2)$ modular invariants are described by A-D-E graphs.

In the subfactor theory only A-D-E Dynkin diagrams with $A_{\ell+1}$, $D_{2\ell+2}$ ($\ell = 1, 2, ...$), $E_6$ and $E_8$, appear as the (dual) principal graphs (or fusion graphs) of subfactors with index less than four. In the rational conformal field theory of $SU(2)$ models described by A-D-E Dynkin diagrams, one may argue that there is a degeneracy so that only $D_{\text{even}}$, $E_6$ and $E_8$, namely the type I cases need be counted. For example in the case of $k = 16$, the modular invariant for $E_7$ reduces to that of $D_{10}$ under the simple interchange of blocks $\chi_8$ and $\chi_2 + \chi_{14}$.

A classification of $SU(3)$ modular invariants was completed by [22]. In analogy with the A-D-E classification for $SU(2)$, we label these $A$ (the completely diagonal invariants), $D$ (the simple current invariants) and the exceptional $E$ invariants. We should also throw in their conjugates $Z^c$, $(Z^c)_{\Lambda,\Lambda'} = Z_{\Lambda,\Lambda'}$ (here $\Lambda$ labels the conjugate representation) — although $D^{(6)} = D^{(6)c}$, $D^{(9)} = D^{(9)c}$, $E^{(12)} = E^{(12)c}$, $E^{(24)} = E^{(24)c}$. For $SU(2)$, the automorphism invariants are the $A$-series and the $D_{\text{odd}}$-series. For $SU(3)$ they are $A^{(m)}$ and $A^{(m)c}$ for all $m$ and $D^{(m)}$ and $D^{(m)c}$ for $m \neq 0 \mod 3$. The $ADE_{7}$ invariants for $SU(2)$ are the $A$-series, $D$-series, and the $E_7$ exceptional (hence the name $ADE_{7}$). In the $SU(3)$ case the $A$ invariants are $A^{(m)}$ and $A^{(m)c}$, the $D$ invariants are $D^{(m)}$ and $D^{(m)c}$, and the $E_7$ invariants are the two Moore and Seiberg invariants $E^{(12)}_{\text{MS}}$ and $E^{(12)c}_{\text{MS}}$. The other invariants $E^{(8)}$, $E^{(12)}$, $E^{(24)}$ correspond to conformal embeddings $SU(3)_5 \subset SU(6)_1$, $SU(3)_9 \subset (E_6)_1$, $SU(3)_{21} \subset (E_7)_1$, respectively (cf. $E_6$ and $E_8$ for $SU(2)$). The simple current invariants for $SU(2)$ are the $A$ and $D$ series, and for
\textit{SU}(3) are again the \(A\) and \(D\) series (but not their conjugations).

Di Francesco and Zuber initiated a program to associate graphs to these invariants \bibref[12, 13]. These graphs are three colourable and such that their eigenvalues ("exponents"), constituting again a subset of the set of eigenvalues of the \(A\) graph and thus being labelled by its vertices, match the non-vanishing diagonal entries of the modular invariant. They also associated graphs to several \(SU(n)\) modular invariants with higher \(n\) and their concept is quite general, however, there may be difficulties associating a graph to some invariants — unlike the \(SU(2)\) case.

We also consider in Section 5 the modular invariants of the extended \(U(1)\) current algebras as treated in \bibref[6] and the minimal model modular invariants which arise from coset theories \((SU(2)_{m-2} \otimes SU(2)_1)/SU(2)_{m-1}\) and are labelled by pairs \((G_1, G_2)\) of A-D-E graphs, associated to levels \((m-2, m-1)\).

1.2 Outline of this paper

A conformal inclusion directly provides a net of subfactors in terms of the von Neumann algebras of local loop groups in the vacuum representation of the larger theory. For the orbifold case we start with the level \(k\) vacuum representation of the loop group \(SU(n)\) and construct a net of subfactors by a DHR construction of fields implementing automorphisms, constituting a simple current extension in terms of bounded operators. The construction is possible and yields moreover a local extended net exactly at the levels where the orbifold modular invariants occur. In both cases we arrive at a net of subfactors \(N \subset M\) satisfying the necessary conditions to apply the procedure of \(\alpha\)-induction elaborated in our previous paper.

As a consequence of Wassermann’s work, to the level \(k\) positive energy representations of the loop group \(LSU(n)\) correspond (DHR superselection) sectors of local algebras \(N(I_o)\), where \(I_o \subset S^1\) is some proper interval. These sectors are labelled by admissible weights \(\Lambda\) and it is proven that their sector products obey the well-known \(SU(n)_k\) fusion rules \bibref[50], giving rise to a fusion algebra \(W = W(n, k)\). The irreducible subsectors obtained by \(\alpha\)-induction of these sectors generate sector algebra \(V\). The results of our previous paper, in particular the homomorphism property of \(\alpha\)-induction, allows to read off partially the structure of \(V\) in terms of the fusion rules in \(W\), but it does in general not determine the multiplication table completely. However, in many examples it provides enough information to resolve the puzzle.

The homomorphism property of \(\alpha\)-induction also implies that \(V\) carries a representation of \(W\) which therefore decomposes into a direct sum over the characters \(\gamma_\Lambda\) of the fusion algebra \(W\) which are labelled by admissible
weights Λ as well. The representation matrix associated to the first fundamental weight Λ(1) can be interpreted as the adjacency matrix of a graph (which is in fact the fusion graph of αΛ(1)), and its eigenvalues are the evaluation of the characters γΛ in the decomposition of this representation of W. The weights Λ labelling the characters which in fact appear this way are called exponents as can be recognized as a generalization of the Coxeter exponents in the SU(2) case.

As a consequence of ασ-reciprocity, proved in our previous paper, there is a fusion subalgebra T ⊂ V generated by the (localized) sectors of the larger theory which correspond to the blocks in the modular invariant. It is widely believed in general but proven only for several cases that their sector products coincide with the Verlinde fusion rules known in conformal field theory. Provided that this is true for the embedding theory at hand we show that the interplay of S-matrices diagonalizing the fusion rules and implementing modular transformations at the same time forces a conformal character χΛ to appear in the modular invariant if and only if Λ is an exponent.

1.3 Preliminaries

Here we briefly review our basic notation and results, however, for precise definitions and statements we refer the reader to our previous paper [4]. There we considered certain nets of subfactors N ⊂ M on the punctured circle, i.e. we were dealing with a family of subfactors N(I) ⊂ M(I) on a Hilbert space H, indexed by the set J of open intervals I on the unit circle S^1 that do neither contain nor touch a distinguished point “at infinity” z ∈ S^1. The defining representation of N possesses a subrepresentation π₀ on a distinguished subspace H₀ giving rise to another net A = {A(I) = π₀(N(I)), I ∈ J}. We assumed this net to be strongly additive (which is equivalent to strong additivity of the net N) and to satisfy Haag duality, A(I) = C₆A(I′), where C₆ denotes the C∗-algebra generated by all A(J), with intervals J ∈ J and J ⊂ I′, the (interior of the) complement of I, and also locality of the net M. Fixing an interval I₀ ∈ J we used the crucial observation in [33] that there is an endomorphism γ of the C*-algebras M into N (the C*-algebras associated to the nets are denoted by the same symbols as the nets itself, as usual) such that it restricts to a canonical endomorphism of M(I₀) into N(I₀). By θ we denote its restriction to N. We defined a map Δ₆(I₀) → End(M), λ → α₆, called α-induction, where Δ₆(I₀) is the set of transportable endomorphisms localized in I₀. Explicitly,

\[ α₆ = γ^{-1} \circ \text{Ad}(\varepsilon(λ, θ)) \circ λ \circ γ, \]
with statistics operators $\varepsilon(\lambda, \theta)$. As endomorphisms in $\Delta_N(I_o)$ leave $N(I_o)$ invariant one can consider elements of $\Delta_N(I_o)$ as elements of $\text{End}(N(I_o))$, and therefore it makes sense to define the quotient $[\Delta]_N(I_o)$ by inner equivalence in $N(I_o)$. Similarly, the endomorphisms $\alpha_\lambda$ leave $M(I_o)$ invariant, hence we can consider them also as elements of $\text{End}(M(I_o))$ and form their inner equivalence classes $[\alpha_\lambda]$ in $M(I_o)$. We derived that in terms of these equivalence classes, called sectors, $\alpha$-induction $[\lambda] \mapsto [\alpha_\lambda]$ preserves the natural additive and multiplicative structures. Crucial for our analysis is also the formula

$$\langle \alpha_\lambda, \alpha_\mu \rangle_{M(I_o)} = \langle \theta \circ \lambda, \mu \rangle_{N(I_o)}, \quad \lambda, \mu \in \Delta_N(I_o),$$

where for endomorphisms $\rho, \sigma$ of an infinite factor $M$ we denote

$$\langle \rho, \sigma \rangle_M = \dim \text{Hom}_M(\rho, \sigma) = \dim \{ t \in M : t \rho(m) = \sigma(m) t, \ m \in M \}.$$

We also have a map $\text{End}(\mathcal{M}) \to \text{End}(\mathcal{N})$, $\beta \mapsto \sigma_\beta$, called $\sigma$-restriction. Let $\Delta_M(I_o) \subset \text{End}(\mathcal{M})$ denote the set of transportable endomorphisms localized in $I_o$, and $\Delta_M^{(0)}(I_o) \subset \Delta_M(I_o)$ the subset of endomorphisms leaving $M(I)$ for any $I \in \mathcal{J}$ with $I_o \subset I$ invariant. (If the net $\mathcal{M}$ is Haag dual then $\Delta_M^{(0)}(I_o) = \Delta_M(I_o)$.) If $\beta \in \Delta_M^{(0)}(I_o)$ then $\sigma_\beta$ leaves $N(I_o)$ invariant and hence we can consider $\beta$ and $\sigma_\beta$ as elements of $\text{End}(M(I_o))$ and $\text{End}(N(I_o))$, respectively, and we derived $\alpha\sigma$-reciprocity,

$$\langle \alpha_\lambda, \beta \rangle_{M(I_o)} = \langle \lambda, \sigma_\beta \rangle_{N(I_o)}, \quad \lambda \in \Delta_N(I_o), \quad \beta \in \Delta_M^{(0)}(I_o).$$

If one starts with a certain set $\mathcal{W}$ of sectors in $[\Delta]_N(I_o)$ one obtains a set $\mathcal{V}$ of sectors of $M(I_o)$ by $\alpha$-induction, and the above results provide close connections between the algebraic structures of $\mathcal{W}$ and $\mathcal{V}$, conveniently formulated in the language of sector algebras.

## 2 Application of $\alpha$-induction to conformal inclusions

In this section we develop our first main application of $\alpha$-induction. We consider nets of subfactors which arise from conformal inclusions of $SU(n)$.

### 2.1 The general method

We first explain that conformal inclusions of $SU(n)$ give rise to quantum field theoretical nets of subfactors so that we can apply the machinery of
$\alpha$-induction developed in our previous paper. Let $H_k \subset G_1$ be a conformal inclusion of $H = SU(n)$ at level $k$ with $G$ a connected compact simple Lie group. Then there is an associated block-diagonal modular invariant of $SU(n)$,

$$
Z = \sum_{t \in T} |\chi^\text{ext}_t|^2, \quad \chi^\text{ext}_t = \sum_{\Lambda \in \mathcal{A}(n+k)} b_{t,\Lambda} \chi_\Lambda.
$$

(3)

Here $T$ denotes the labelling set of positive energy representations $(\pi^t, \mathcal{H}^t)$ of $LG$ at level 1, $\chi^\text{ext}_t$ the characters of $\mathcal{H}^t$, and $\chi_\Lambda$ the characters of the level $k$ positive energy representation spaces $\mathcal{H}_\Lambda$ of $LSU(n)$, and $(\pi_\Lambda, \mathcal{H}_\Lambda)$ appears in the decomposition of $(\pi^t, \mathcal{H}^t)$ with multiplicity $b_{t,\Lambda}$. Thus we have in terms of the positive energy representations

$$
\pi^t|_{LH} = \bigoplus_{\Lambda \in \mathcal{A}(n+k)} b_{t,\Lambda} \pi_\Lambda.
$$

(4)

Now let us define a net of subfactors $N \subset M$ on the Hilbert space $\mathcal{H} = \mathcal{H}^0$ by

$$
N(I) = \pi_0(L_I H)'', \quad M(I) = \pi_0(L_I G)'',
$$

(5)

and also the net $A$ by

$$
A(I) = \pi_0(L_I H)''
$$

(6)

for intervals $I \subset S^1$. For conformal embeddings the index of the subfactors $N(I) \subset M(I)$ is finite (see e.g. [49], [47]), and as the nets $M$ and $A$ constitute Möbius covariant precosheaves on the circle they satisfy Haag duality on the closed circle [5] and hence in particular locality. Moreover, we have a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for each $M(I)$ ($I \neq S^1$ any interval such that $\bar{I} \neq S^1$) on $\mathcal{H}$ and $N(I)$ on $\mathcal{H}_0 \subset \mathcal{H}$. The modular group of $M(I)$ associated to the state $\omega(\cdot) = \langle \Omega, \cdot \Omega \rangle$ is geometric, i.e. its action restricts to a geometric action on the loop group elements, see [50] for the case $G = SU(m)$ and [49] for the general case that $G$ is any compact simple Lie group. Hence the modular group leaves the subalgebra $N(I)$ invariant for $SU(n) \subset G$ is a subgroup. Therefore there is a normal conditional expectation $E_I$ from $M(I)$ onto $N(I)$ and preserves the state $\omega$ by Takesaki’s theorem [48]. Furthermore, $E_I$ is unique and faithful as the inclusion is also irreducible. The net $N \subset M$ is standard (by the Reeh-Schlieder theorem) and hence the Jones projection $e_N$ from $\mathcal{H}$ onto $\mathcal{H}_0$ does not depend on the interval $I$. Therefore we have $E_I(m)\Omega = E_I(m)e_N\Omega = e_Nm\Omega = e_Ne_N\Omega = e_N\Omega$ for any $I$ and $m \in M(I)$. Hence $E_I(m)\Omega = E_J(m)\Omega$ for any pair $I \subset J$ since $\Omega$ is separating for $M(J)$. We conclude that we have a faithful normal
conditional expectation $E$ from $\mathcal{M}$ onto $\mathcal{N}$ and it obviously preserves the vector state $\omega$.

Now let us remove a “point at infinity” $z \in S^1$ and take the set $\mathcal{J}_z$ as the index set of our nets $\mathcal{A}$, $\mathcal{N}$, $\mathcal{M}$. Then we are clearly dealing with directed nets. For $H = SU(n)$, Haag duality on the closed circle, $A(I) = A(I)'$, has been proven directly by Wassermann [51], as has strong additivity or “irrelevance of points” i.e. $A(I) = A(I_1) \vee A(I_2)$ if the intervals $I_1$ and $I_2$ are obtained by removing one single point from the interval $I$. Moreover, $A(I) = \bigvee_n A(J_n)$ for any sequence of increasing intervals $J_n$ tending to $I$ [49]. Both arguments imply that we have Haag duality even on the punctured circle, $A(I) = C_A(I)'$. In fact, as the proofs in [49] are formulated for any compact connected simple Lie group we similarly have Haag duality on the punctured circle for $M$, $M(I) = C_M(I)'$. (For $G = SO(m)$ (level 1) this has also been proven directly in [3].) As $\pi_0$ appears (precisely once) in $\pi^0|_{LH}$ we conclude that the net $\mathcal{N}$ has a Haag dual subrepresentation, and the corresponding net is given by $A = \{A(I) = \pi_0(N(I)), I \in \mathcal{J}_z\}$ (note that we take, by abuse of notation, the same symbol $\pi_0$ for the subrepresentation of the net $\mathcal{N}$ and for the vacuum representation of $LH$). Let us summarize the discussion in the following

**Proposition 2.1** Starting from a conformal inclusion $SU(n)_k \subset G_1$ with $G$ a compact connected simple Lie group the net $\mathcal{N} \subset \mathcal{M}$ (over the index set $\mathcal{J}_z$) defined as above is a quantum field theoretical net of subfactors of finite index where $\mathcal{M}$ is Haag dual (hence local) and $\mathcal{N}$ is strongly additive and has a Haag dual subrepresentation.

As the positive energy representations of $LH = LSU(n)$ satisfy local equivalence [51],

$$\pi_A(L_I H) \simeq \pi_0(L_I H),$$

we have by the standard arguments endomorphisms $\lambda_{0;A} \in \Delta_A(I_0)$ that correspond to $\pi_A$ for some interval $I_0 \in \mathcal{J}_z$. Wassermann has related the $LSU(n)$ fusion rules to the (relative tensor) product of bimodules, and this is equivalent to the composition of endomorphisms. Hence we have complete information about the sector products $[\lambda_{0;A}] \times [\lambda_{0;A'}]$. Equivalently, we can also take the lifted endomorphisms

$$\lambda_A = \pi_0^{-1} \circ \lambda_{0;A} \circ \pi_0 \in \Delta_N(I_0),$$

and then we clearly have the same sector product rules.
By Eq. (27) of [4] we have

$$\theta = \bigoplus_{\Lambda \in \mathcal{A}^{(n+k)}} b_{0,\Lambda} \lambda_{\Lambda}$$

where this decomposition corresponds to the the vacuum block in the modular invariant, $\chi^\text{ext}_0 = \sum_{\Lambda \in \mathcal{A}^{(n+k)}} b_{0,\Lambda} \lambda_{\Lambda}$, see Eq. (2). Our procedure is then as follows. Recall that a sector basis is a finite set of irreducible sectors with finite statistical dimension which contains the identity sector and is closed under sector products and conjugation. A sector basis canonically defines an algebra called sector algebra. (We refer again to [4] for precise definitions.)

We take the sector basis $W \equiv W(n,k) = \{\lambda_{\Lambda} \mid \Lambda \in \mathcal{A}^{(n+k)}\} \subset [\Delta]_\mathcal{A}(I_0)$ and we denote by $W \equiv W(n,k)$ the associated fusion algebra. By $\alpha$-induction (see Theorem 4.2 of [4]) we obtain a sector algebra $V \subset \text{Sect}(M(I_0))$, consisting of the distinct irreducible subsectors of the $\lambda_{\Lambda}$. (We write $\alpha_{\Lambda}$ for $\alpha_{\lambda_{\Lambda}}$.) Picking endomorphisms $\lambda_{\Lambda(p)}$, associated to the $p$-th fundamental representation, $p = 1, 2, \ldots, n - 1$ ($\Lambda(p)$ denotes the $p$-th fundamental weight) and forming $\alpha_{\Lambda(p)}$, we can compute the sector products $[\alpha_{\Lambda(p)}] \times [\alpha_{\Lambda}]$ for all $\Lambda \in \mathcal{A}^{(n+k)}$. In many cases, the homomorphism $[\alpha]$ is surjective and therefore all the fusion rules in $V$ can be read off from the fusion rules in $W$. But even for those of our examples where the homomorphism $[\alpha]$ is not surjective we can at least determine the fusion rules $[\alpha_{\Lambda(p)}] \times [\beta]$ for all $[\beta] \in V$, and thus we can draw the associated fusion graphs.

Since the positive energy representations of a loop group of any connected compact simple satisfy local equivalence [13] we have endomorphisms $\beta_t \in \Delta_{\mathcal{M}}(I_0) = \Delta_{\mathcal{M}}(I_0)$, $t \in \mathcal{T}$, corresponding to the level 1 positive energy representations of $LG$. As we know the branching rules of the decomposition of $\pi_t|_{LH}$, Eq. (1), and as $\sigma$-restriction corresponds to the restriction of representations it follows $[\sigma_{\beta_t}] = \bigoplus_{\Lambda \in \mathcal{A}^{(n+k)}} b_{\Lambda} \lambda_{\Lambda}$. As a consequence of $\alpha\sigma$-reciprocity, $\langle \alpha_{\Lambda'}, \beta_t \rangle_{M(I_0)} = \langle \lambda_{\Lambda'}, \sigma_{\beta_t} \rangle_{N(I_0)} = b_{\Lambda',\Lambda}$, $\Lambda \in \mathcal{A}^{(n+k)}$, $t \in \mathcal{T}$, we conclude (cf. Theorem 4.3 of [4]) that $\mathcal{T} \subset \mathcal{V}$ and that the associated fusion algebra $T$ must be a sector subalgebra $T \subset V$. (We identify the labelling set $\mathcal{T}$ itself with the associated sector basis: $\mathcal{T} \equiv \{[\beta_{\Lambda}]\} \subset [\Delta]_\mathcal{A}(I_0)$, $t \equiv [\beta_{\Lambda}]$.)

It is widely believed but in general not known whether the endomorphisms associated to the positive energy representations of a loop group $LG$ obey the Verlinde fusion rules of the corresponding WZW theory. However, for the level 1 theories which are relevant here, this is proven for many cases including $G = SU(m)$ as a special (and most trivial) case of Wassermann’s analysis [50] and $G = SO(m)$ as done in [3], moreover, for $G = G_2$ it follows
from our treatment of the conformal embedding $SU(2)_{28} \subset (G_2)_1$.

Now recall that Di Francesco, Petkova and Zuber (see [13, 42] or [9, 10]) associated certain graphs to modular invariants by some empirical procedure. For these graphs they constructed fusion algebras (which are possibly not uniquely determined), and they discovered for the block-diagonal modular invariants some subalgebras spanned by a subset of the vertices, called marked vertices, which obey the fusion rules of the extended theory. Indeed, in our examples we rediscover their graphs by drawing the fusion graphs of $\alpha_{\Lambda(p)}$. The elements of $T$ turn out to represent exactly the marked vertices, and our theory provides an explanation why the graph algebras (which are in fact the fusion algebras $V$) possess subalgebras corresponding to the fusion rules of the extended theory.

2.2 Example: $SU(2)_{10} \subset SO(5)_1$

We consider the conformal inclusion $SU(2)_{10} \subset SO(5)_1$. The corresponding $SU(2)$ modular invariant is the $E_6$ one,

$$Z_{E_6} = |\chi_0 + \chi_6|^2 + |\chi_4 + \chi_{10}|^2 + |\chi_3 + \chi_7|^2. \quad (7)$$

The three blocks come from the basic (0), the vector (v) and the spinor (s) representation of $LSO(5)$ at level 1. For $LSU(2)$ at level 10 there are 11 positive energy representations $\pi_j$, labelled by the (doubled, thus integer valued) spin $j = 0, 1, 2, ..., 10$. Let $\lambda_j \in \Delta_N(I_0)$ be corresponding endomorphisms. The fusion algebra $W \equiv W(2, k)$ is characterized by the fusion rules

$$[\lambda_{j_1}] \times [\lambda_{j_2}] = \bigoplus_{j=|j_1-j_2|, \ j+j_1+j_2 \ \text{even}}^{\min(j_1+j_2, 2k-(j_1+j_2))} [\lambda_j], \quad (8)$$

and here $k = 10$. From the vacuum block in Eq. (7) we read off $[\theta] = [\lambda_0] \oplus [\lambda_6]$. By Theorem 3.9 of [4] we obtain (writing $\alpha_j$ for $\alpha_{\lambda_j}$)

$$\langle \alpha_{j_1}, \alpha_{j_2} \rangle_{M(I_0)} = \langle \lambda_{j_1}, \theta \circ \lambda_{j_2} \rangle_{N(I_0)} = \langle \lambda_{j_1}, \lambda_{j_2} \rangle_{N(I_0)} + \langle \lambda_{j_1}, \lambda_6 \circ \lambda_{j_2} \rangle_{N(I_0)}.$$

We find this way

$$\langle \alpha_j, \alpha_j \rangle_{M(I_0)} = \left\{ \begin{array}{ll} 1 & \text{for } j = 0, 1, 2, 8, 9, 10 \\ 2 & \text{for } j = 3, 4, 5, 6, 7 \end{array} \right..$$

We further compute $\langle \alpha_3, \alpha_9 \rangle_{M(I_0)} = 1$, hence $[\alpha_3] = [\alpha_9] \oplus [\alpha_3]^{(1)}$ with $[\alpha_3]^{(1)}$ irreducible. As $\langle \alpha_j, \alpha_j \rangle_{M(I_0)} = 0$ for $j = 0, 1, 2, 8, 10$, there is no irreducible
\[ \alpha_0 \quad \alpha_1 \quad \alpha_2 \quad \alpha_9 \quad \alpha_{10} \]

Figure 5: E_6

\[[\alpha_3^{(1)}] \quad \bullet \quad [\alpha_0] \quad [\alpha_1] \quad [\alpha_2] \quad [\alpha_9] \quad [\alpha_{10}] \]

\[
\begin{align*}
[\alpha_3] &= [\alpha_3^{(1)}] \oplus [\alpha_9], \\
[\alpha_4] &= [\alpha_2] \oplus [\alpha_{10}], \\
[\alpha_5] &= [\alpha_1] \oplus [\alpha_9], \\
[\alpha_6] &= [\alpha_0] \oplus [\alpha_2], \\
[\alpha_7] &= [\alpha_1] \oplus [\alpha_3^{(1)}].
\end{align*}
\]

We are in the fortunate situation that we can write all elements of \( V \) as (integral) linear combinations of \([\alpha_j]\)'s, i.e. the homomorphism \([\alpha]\) is surjective. Thus we can determine their fusion rules from those of \( LSU(2) \). For instance, we compute
\[
[\alpha_3^{(1)}] \times [\alpha_1] = ([\alpha_3] \times [\alpha_1]) \oplus ([\alpha_9] \times [\alpha_1])
\]
\[
= ([\alpha_2] \oplus [\alpha_4]) \oplus ([\alpha_8] \oplus [\alpha_{10}]) = [\alpha_2].
\]

In particular, we can draw the fusion graph for \([\alpha_{\Lambda^{(1)}}] \equiv [\alpha_1]\). It is straightforward to check that this is \( E_6 \), Fig. 5.

The homomorphism \([\alpha] : W \to V\) induces an induction-restriction graph connecting \( A_{11} \) and \( E_6 \). We just draw an edge from each spin \( j \) vertex of \( A_{11} \) to the vertices of \( E_6 \) that represent the irreducible subsectors in the decomposition of \( [\alpha_j] \). For example, we draw from the spin \( j = 4 \) vertex one line to the vertex \( [\alpha_2] \) and one to \( [\alpha_{10}] \). Completing the picture we obtain a graph with two connected components \( \Gamma_1 \) and \( \Gamma_2 \) corresponding to the even and odd spins, respectively, see Figs. 6, 7. These graphs are actually well known as graphs connecting \( A_{11} \) and \( E_6 \), cf. [40, 15, 28, 24]. Indeed, one can show that \( \Gamma_1 \) is the principal graph for the inclusion \( N(\mathcal{I}_0) \subset M(\mathcal{I}_0) \), and \( \Gamma_1 \) and \( \Gamma_2 \). We plan to come back to this fact in a separate publication.
Now we turn to the discussion of the marked vertices. Let $\beta_0, \beta_v, \beta_s \in \Delta_M(I_6)$ be endomorphisms corresponding to the level 1 basic, vector and spinor representation of $\text{LSO}(5)$ (as constructed in [3]). From the blocks in Eq. (7) we can read off the decomposition of the $\sigma$-restricted endomorphisms,

$$\sigma_{\beta_0} = [\lambda_0] \oplus [\lambda_6], \quad \sigma_{\beta_v} = [\lambda_4] \oplus [\lambda_{10}], \quad \sigma_{\beta_s} = [\lambda_3] \oplus [\lambda_7].$$

By $\alpha\sigma$-reciprocity, we conclude that $[\beta_0]$ must appear in $[\alpha_0]$ and $[\beta_v]$ in $[\alpha_{10}]$ and $[\beta_s]$ in $[\alpha_3]$ and $[\alpha_7]$ with multiplicity one. Hence we conclude

$$[\alpha_0] = [\beta_0], \quad [\alpha_{10}] = [\beta_v], \quad [\alpha_3^{(1)}] = [\beta_s].$$

In Fig. 6 we encircled the marked vertices (and we will do it also in the following examples). It is easy to check that $[\alpha_0], [\alpha_{10}]$ and $[\alpha_3^{(1)}]$ indeed obey the Ising fusion rules, e.g.

$$[\alpha_3^{(1)}] \times [\alpha_3^{(1)}] = ([\alpha_3 \ominus [\alpha_9]]) \times ([\alpha_3 \ominus [\alpha_9]]) = [\alpha_0] \oplus [\alpha_{10}]$$

as it is well known for the end vertices in the graph algebra of $E_6$. This finds now an explanation by the machinery of $\alpha$-induction and $\sigma$-restriction.
Put differently, our theory proves again the result of [3], namely that the endomorphisms associated to the level 1 positive energy representations of $LSO(5)$ obey the Ising fusion rules.

2.3 More examples

(i) Example: $SU(2)_{28} \subset (G_2)_1$. The corresponding modular invariant is the $E_8$ one,

$$Z_{E_8} = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2.$$  

The two blocks come from the positive energy representations $\pi_0$ and $\pi_\phi$ of $L(G_2)$ at level 1. With $[\theta] = [\lambda_0] \oplus [\lambda_{10}] \oplus [\lambda_{18}] \oplus [\lambda_{28}]$ we can determine the structure of the induced sector algebra $V$. We omit the straightforward calculations and just present the result here. We find that the sector basis $V$ has elements, given by $[\alpha_{0}]$, $[\alpha_{1}]$, $[\alpha_{2}]$, $[\alpha_{3}]$, $[\alpha_{4}]$, $[\alpha_{5}^{(1)}]$, $[\alpha_{5}^{(2)}]$ and $[\alpha_{6}^{(1)}]$. The decompositions of the reducible $[\alpha_{j}]$’s read

$$[\alpha_{5}] = [\alpha_{5}^{(1)}] \oplus [\alpha_{5}^{(2)}], \quad [\alpha_{6}] = [\alpha_{4}] \oplus [\alpha_{6}^{(1)}],$$

$$[\alpha_{7}] = [\alpha_{3}] \oplus [\alpha_{5}^{(1)}], \quad [\alpha_{8}] = [\alpha_{2}] \oplus [\alpha_{4}],$$

$$[\alpha_{9}] = [\alpha_{1}] \oplus [\alpha_{3}] \oplus [\alpha_{5}^{(2)}], \quad [\alpha_{10}] = [\alpha_{0}] \oplus [\alpha_{2}] \oplus [\alpha_{4}],$$

$$[\alpha_{11}] = [\alpha_{1}] \oplus [\alpha_{3}] \oplus [\alpha_{5}^{(1)}], \quad [\alpha_{12}] = [\alpha_{2}] \oplus [\alpha_{4}] \oplus [\alpha_{6}^{(1)}],$$

$$[\alpha_{13}] = [\alpha_{3}] \oplus [\alpha_{5}^{(1)}] \oplus [\alpha_{5}^{(2)}], \quad [\alpha_{14}] = 2[\alpha_{4}],$$

and we have $[\alpha_{28-j}] = [\alpha_{j}]$. The fusion graph of $[\alpha_{1}]$ is in fact $E_8$, given in Fig. 3. The marked vertices are given by

$$[\alpha_{0}] = [\beta_{0}], \quad [\alpha_{6}^{(1)}] = [\beta_{\phi}],$$

and it is easy to check that they indeed obey the Lee-Yang fusion rules

$$[\alpha_{6}^{(1)}] \times [\alpha_{6}^{(1)}] = [\alpha_{0}] \oplus [\alpha_{6}^{(1)}]$$

of $(G_2)_1$, i.e. here our theory proves that the endomorphisms associated to the $(G_2)_1$ positive energy representations obey these fusion rules.

(ii) Example: $SU(2)_{4} \subset SU(3)_1$. The corresponding modular invariant is the $D_4$ one,

$$Z_{D_4} = |\chi_0 + \chi_{4}|^2 + 2|\chi_2|^2.$$
The first block comes from the vacuum representation $\pi_{(0,0)}$ and the second one from the positive energy representations $\pi_{(1,0)}$ and $\pi_{(1,1)}$ of $LSU(3)$ at level 1 which both restrict to the spin 2 representation of $LSU(2)$ at level 4. With $[\theta] = [\lambda_0] \oplus [\lambda_4]$ we find that $\mathcal{V}$ has four elements, namely $[\alpha_0]$, $[\alpha_1]$, $[\alpha_2^{(1)}]$, $[\alpha_2^{(2)}]$ where we have the decomposition $[\alpha_2] = [\alpha_2^{(1)}] \oplus [\alpha_2^{(2)}]$, and also $[\alpha_{4-j}] = [\alpha_j]$. Note that we cannot isolate $[\alpha_2^{(1)}]$ and $[\alpha_2^{(2)}]$. Thus in this case the homomorphism $[\alpha]$ is not surjective! However, since

$$[\alpha_1] \times [\alpha_2] = [\alpha_1] \oplus [\alpha_3] = 2[\alpha_1]$$

it follows that

$$[\alpha_1] \times [\alpha_2^{(i)}] = [\alpha_1], \quad i = 1, 2,$$

and we find

$$[\alpha_1] \times [\alpha_1] = [\alpha_0] \oplus [\alpha_2] = [\alpha_0] \oplus [\alpha_2^{(1)}] \oplus [\alpha_2^{(2)}],$$

hence the fusion graph of $[\alpha_1]$ is uniquely determined to be $D_4$, see Fig. 9.

The $SU(3)_1$ positive energy representations obey $\mathbb{Z}_3$ fusion rules, and from $\alpha\sigma$-reciprocity we conclude that the marked vertices are given by

$$[\alpha_0] = [\beta_{(0,0)}], \quad [\alpha_2^{(1)}] = [\beta_{(1,0)}], \quad [\alpha_2^{(2)}] = [\beta_{(1,1)}].$$

Figure 8: $E_8$

![Diagram of E8]

Figure 9: $D_4$

![Diagram of D4]
(Clearly we have the freedom to define which is $\alpha_2^{(1)}$ and which $\alpha_2^{(2)}$.)

(iii) Example: $SU(3)_3 \subset SO(8)_1$. We now turn to the treatment of the $SU(3)$ conformal embeddings. We shall label the $LSU(3)$ level $k$ positive energy representations by pairs of integers $(p, q)$, $k \geq p \geq q \geq 0$, that give the lengths of the rows of the associated Young tableaux. Thus the (first) fundamental representation has the label $(1, 0)$. We denote the endomorphism that corresponds to the positive energy representation labelled by $(p, q)$ by $\lambda_{(p, q)}$. Thus the sectors $[\lambda_{(p, q)}]$ constitute the sector basis of the fusion algebra $W(3, k)$. Recall that the fusion of the sector $[\lambda_{(1, 0)}]$ that corresponds to the fundamental representation is

$$[\lambda_{(p, q)}] \times [\lambda_{(1, 0)}] = [\lambda_{(p+1, q)}] \oplus [\lambda_{(p, q+1)}] \oplus [\lambda_{(p-1, q-1)}],$$

where it is understood that on the r.h.s. only sectors inside $A^{(k+3)}$ contribute.

Now for the conformal embedding $SU(3)_3 \subset SO(8)_1$, the corresponding modular invariant reads

$$Z_{D(6)} = |\chi_{(0,0)} + \chi_{(3,0)} + \chi_{(3,3)}|^2 + 3|\chi_{(2,1)}|^2,$$

thus we have $[\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(3,0)}] \oplus [\lambda_{(3,3)}]$. We find that $\mathcal{V}$ has six elements, $[\alpha_{(0,0)}], [\alpha_{(1,0)}], [\alpha_{(1,1)}], [\alpha_{(2,1)}], [\alpha_{(2,1)}]$, and $[\alpha_{(3,1)}]$. Here the only non-trivial decomposition is $[\alpha_{(2,1)}] = [\alpha_{(1,1)}] \oplus [\alpha_{(2,1)}] \oplus [\alpha_{(2,1)}]$, and we have $[\alpha_{(p, q)}] = [\alpha_{(3-q,p-q)}]$. The fusion graph of $[\alpha_{(1,0)}]$ is indeed $D(6)$, see Fig. Fig. 10. The marked vertices are $[\alpha_{(0,0)}], [\alpha_{(1,1)}], [\alpha_{(2,1)}]$ and $[\alpha_{(3,1)}]$ and hence obey the $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules of $SO(8)_1$.

The other $D$-type block-diagonal modular invariants, namely $D_{2\varrho+2}$ for $SU(2)$ and $D^{(3\varrho+3)}$ for $SU(3)$, $\varrho = 2, 3, 4, ...$, do not come from conformal inclusions. This will be discussed in the following section.
(iv) Example: $SU(3)_5 \subset SU(6)_1$. The corresponding modular invariant reads

$$Z_{\mathcal{C}}(\alpha) = |\chi(0,0) + \chi(4,2)|^2 + |\chi(2,0) + \chi(5,3)|^2 + |\chi(2,2) + \chi(5,2)|^2 + |\chi(3,0) + \chi(3,3)|^2 + |\chi(3,1) + \chi(5,5)|^2 + |\chi(3,2) + \chi(5,0)|^2,$$

hence

$$[\theta] = [\lambda(0,0)] \oplus [\lambda(4,2)].$$

By computing all the numbers

$$\langle \alpha_{(p,q)}, \alpha_{(r,s)} \rangle_{M(I_\theta)} = \langle \theta \circ \lambda_{(p,q)}, \lambda_{(r,s)} \rangle_{N(I_\theta)}$$

(where we denote $\alpha_{(p,q)} = \alpha \lambda_{(p,q)}$) we find that $\mathcal{V}$ has 12 elements, and the reducible $[\alpha_{(p,q)}]$'s decompose into these irreducibles as

$$\begin{align*}
[\alpha_{(2,0)}] &= [\alpha_{(4,4)}] \oplus [\alpha_{(2,0)}], \\
[\alpha_{(2,1)}] &= [\alpha_{(5,1)}] \oplus [\alpha_{(5,4)}], \\
[\alpha_{(2,2)}] &= [\alpha_{(4,0)}] \oplus [\alpha_{(1,1)}], \\
[\alpha_{(3,0)}] &= [\alpha_{(5,4)}] \oplus [\alpha_{(3,0)}], \\
[\alpha_{(3,1)}] &= [\alpha_{(1,0)}] \oplus [\alpha_{(4,0)}] \oplus [\alpha_{(5,5)}], \\
[\alpha_{(3,2)}] &= [\alpha_{(1,1)}] \oplus [\alpha_{(4,4)}] \oplus [\alpha_{(5,0)}], \\
[\alpha_{(3,3)}] &= [\alpha_{(5,1)}] \oplus [\alpha_{(3,0)}], \\
[\alpha_{(4,1)}] &= [\alpha_{(1,1)}] \oplus [\alpha_{(4,4)}], \\
[\alpha_{(4,2)}] &= [\alpha_{(0,0)}] \oplus [\alpha_{(5,1)}] \oplus [\alpha_{(5,4)}], \\
[\alpha_{(4,3)}] &= [\alpha_{(1,0)}] \oplus [\alpha_{(4,0)}], \\
[\alpha_{(5,2)}] &= [\alpha_{(1,0)}] \oplus [\alpha_{(1,1)}] \oplus [\alpha_{(2,2)}], \\
[\alpha_{(5,3)}] &= [\alpha_{(1,1)}] \oplus [\alpha_{(2,0)}].
\end{align*}$$

We find that the homomorphism $[\alpha]$ is surjective as we can invert these formula, namely we obtain

$$\begin{align*}
[\alpha_{(2,0)}^{(1)}] &= [\alpha_{(2,0)}] \oplus [\alpha_{(4,4)}], \\
[\alpha_{(2,2)}^{(1)}] &= [\alpha_{(2,2)}] \oplus [\alpha_{(4,0)}], \\
[\alpha_{(3,0)}^{(1)}] &= \frac{1}{2} ([\alpha_{(3,0)}] \oplus [\alpha_{(3,3)}] \oplus [\alpha_{(5,1)}] \oplus [\alpha_{(5,4)}]).
\end{align*}$$

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It is then a straightforward calculation to determine the fusion rules of $V$, and the fusion graph of $[\alpha_{(1,0)}]$ is given by Figure 11.

The marked vertices are given by $[\alpha_{(0,0)}], [\alpha_{(2,2)}], [\alpha_{(5,0)}], [\alpha_{(3,0)}], [\alpha_{(5,5)}]$ and $[\alpha_{(2,0)}]$, and one may check that they in fact obey the $\mathbb{Z}_6$ fusion rules of $SU(6)_1$.

### 2.4 A non-commutative sector algebra

**Example:** $SU(4)_4 \subset SO(15)_1$. Labelling the positive energy representations of $SU(4)_4$ with partitions $(p_1, p_2, p_3) \in \mathbb{Z}^3$, $4 \geq p_1 \geq p_2 \geq p_3 \geq 0$, the corresponding modular invariant reads

$$Z = |\chi(0,0,0) + \chi(3,1,0) + \chi(3,3,2) + \chi(4,4,0)|^2$$
$$+ |\chi(2,1,1) + \chi(4,0,0) + \chi(4,3,1) + \chi(4,4,4)|^2 + 4 |\chi(3,2,1)|^2,$$

where the blocks correspond to the basic, the vector and the spinor representation of $SO(15)_1$. With

$$[\theta] = [\lambda_{(0,0,0)}] + [\lambda_{(3,1,0)}] + [\lambda_{(3,3,2)}] + [\lambda_{(4,4,0)}]$$

we can compute the table $$(\alpha(p_1, p_2, p_3), \alpha(q_1, q_2, q_3))_{M(I_0)} = \langle \theta \circ \lambda_{(p_1, p_2, p_3)}, \lambda_{(q_1, q_2, q_3)} \rangle_{N(I_0)}.$$
We find seven further elements in \( \mathcal{V} \), namely \([\alpha_{(2,1,0)}^i], [\alpha_{(2,2,0)}^i], [\alpha_{(2,2,1)}^i], \)
\( i = 1, 2 \), and \([\alpha_{(3,2,1)}^i] \), and the reducible \([\alpha_{(p_1, p_2, p_3)}^i] \) decompose in this basis as follows,

\[
\begin{align*}
[\alpha_{(2,1,0)}] &= [\alpha_{(1,1,1)}] \oplus [\alpha_{(2,1,0)}^1] \oplus [\alpha_{(2,1,0)}^2], \\
[\alpha_{(2,1,1)}] &= [\alpha_{(4,0,0)}] \oplus [\alpha_{(2,2,0)}^1] \oplus [\alpha_{(2,2,0)}^2], \\
[\alpha_{(2,2,0)}] &= [\alpha_{(2,2,0)}^1] \oplus [\alpha_{(2,2,0)}^2], \\
[\alpha_{(2,2,1)}] &= [\alpha_{(1,0,0)}] \oplus [\alpha_{(2,2,1)}^1] \oplus [\alpha_{(2,2,1)}^2], \\
[\alpha_{(3,1,0)}] &= [\alpha_{(0,0,0)}] \oplus [\alpha_{(2,2,0)}^1] \oplus [\alpha_{(2,2,0)}^2], \\
[\alpha_{(3,1,1)}] &= [\alpha_{(1,0,0)}] \oplus [\alpha_{(2,2,1)}^1] \oplus [\alpha_{(2,2,1)}^2], \\
[\alpha_{(3,2,0)}] &= [\alpha_{(1,0,0)}] \oplus [\alpha_{(2,2,1)}^1] \oplus [\alpha_{(2,2,1)}^2], \\
[\alpha_{(3,2,1)}] &= 2[\alpha_{(1,1,0)}] \oplus 2[\alpha_{(3,2,1)}^1], \\
[\alpha_{(3,2,2)}] &= [\alpha_{(1,1,1)}] \oplus [\alpha_{(2,1,0)}^1] \oplus [\alpha_{(2,1,0)}^2], \\
[\alpha_{(3,3,1)}] &= [\alpha_{(1,1,1)}] \oplus [\alpha_{(2,1,0)}^1] \oplus [\alpha_{(2,1,0)}^2], \\
[\alpha_{(3,3,2)}] &= [\alpha_{(0,0,0)}] \oplus [\alpha_{(2,2,0)}^1] \oplus [\alpha_{(2,2,0)}^2], \\
[\alpha_{(4,2,1)}] &= [\alpha_{(1,1,1)}] \oplus [\alpha_{(2,1,0)}^1] \oplus [\alpha_{(2,1,0)}^2], \\
[\alpha_{(4,2,2)}] &= [\alpha_{(2,2,0)}^1] \oplus [\alpha_{(2,2,0)}^2], \\
[\alpha_{(4,3,1)}] &= [\alpha_{(4,0,0)}] \oplus [\alpha_{(2,2,0)}^1] \oplus [\alpha_{(2,2,0)}^2], \\
[\alpha_{(4,3,2)}] &= [\alpha_{(1,0,0)}] \oplus [\alpha_{(2,2,1)}^1] \oplus [\alpha_{(2,2,1)}^2].
\end{align*}
\]

The marked vertices are \([\alpha_{(0,0,0)}], [\alpha_{(4,0,0)}] \) and \([\alpha_{(3,2,1)}]^1 \), corresponding to the basic, vector and spinor representation of \( SO(15)_1 \), respectively.
the spinor representation \( \pi_s \) of \( SO(15)_1 \) restricts to two copies of \( \pi_{(3,2,1)} \), i.e. \( b_{s,(3,2,1)} = 2 \), implies in particular that \( [\alpha_{(3,2,1)}^{(1)}] \) appears in the decomposition of \( [\alpha_{(3,2,1)}] \) with multiplicity 2 by \( \alpha\sigma\text{-reciprocity} \).

Using the \( SU(4)_4 \) fusion rules, i.e. of \( W(4,4) \), we obtain the following sector products by the homomorphism property of \( \alpha\text{-induction} \),

\[
\begin{align*}
[\alpha_{(1,0,0)}] \times [\alpha_{(1,0,0)}] &= 2[\alpha_{(1,1,0)}], \\
[\alpha_{(1,0,0)}] \times [\alpha_{(1,1,0)}] &= 2[\alpha_{(1,1,1)}] \oplus [\alpha_{(2,1,0)}^{(1)}] \oplus [\alpha_{(2,1,0)}^{(2)}], \\
[\alpha_{(1,0,0)}] \times [\alpha_{(1,1,1)}] &= [\alpha_{(0,0,0)}] \oplus [\alpha_{(4,0,0)}] \oplus [\alpha_{(2,2,0)}^{(1)}] \oplus [\alpha_{(2,2,0)}^{(2)}], \\
[\alpha_{(1,0,0)}] \times [\alpha_{(4,0,0)}] &= [\alpha_{(1,0,0)}], \\
[\alpha_{(1,0,0)}] \times [\alpha_{(3,2,1)}] &= [\alpha_{(2,1,0)}^{(1)}] \oplus [\alpha_{(2,1,0)}^{(2)}].
\end{align*}
\]

However, as the homomorphism \([\alpha] : W \to V\) is not surjective we cannot isolate \([\alpha_{(2,1,0)}^{(i)}], [\alpha_{(2,2,0)}^{(i)}] \) and \([\alpha_{(2,2,1)}^{(i)}], i = 1, 2 \). First we can only compute

\[
\begin{align*}
[\alpha_{(1,0,0)}] \times \left( [\alpha_{(2,1,0)}^{(1)}] \oplus [\alpha_{(2,1,0)}^{(2)}] \right) &= 2[\alpha_{(2,2,0)}^{(1)}] \oplus 2[\alpha_{(2,2,0)}^{(2)}], \\
[\alpha_{(1,0,0)}] \times \left( [\alpha_{(2,2,0)}^{(1)}] \oplus [\alpha_{(2,2,0)}^{(2)}] \right) &= 2[\alpha_{(1,0,0)}] \oplus 2[\alpha_{(2,2,1)}^{(1)}] \oplus 2[\alpha_{(2,2,1)}^{(2)}], \\
[\alpha_{(1,0,0)}] \times \left( [\alpha_{(2,2,1)}^{(1)}] \oplus [\alpha_{(2,2,1)}^{(2)}] \right) &= 2[\alpha_{(1,1,0)}] \oplus 2[\alpha_{(3,2,1)}].
\end{align*}
\]

Now recall that the statistical dimension of the positive energy representation of \( SU(n)_k \) labelled by a partition \( (p_1, p_2, ..., p_{n-1}) \), \( k \geq p_1 \geq p_2 \geq ... \geq p_{n-1} \geq p_n \equiv 0 \), is given by

\[
d_{(p_1, p_2, ..., p_{n-1})} = \prod_{1 \leq i < j \leq n} \frac{\sin \left( \frac{(p_j - p_i + j - i)\pi}{n + k} \right)}{\sin \left( \frac{(j - i)\pi}{n + k} \right)}.
\]

With \( n = k = 4 \) we obtain \( d_{(1,0,0)} = \sin(\pi/8)^{-1} \). Since the marked vertex \([\alpha_{(3,2,1)}^{(1)}] \) has statistical dimension (we write \( d_{(p_1, p_2, p_3)}^{(i)} \equiv d_{\alpha_{(p_1, p_2, p_3)}^{(i)}} \))

\[
d_{(3,2,1)} = \sqrt{2} \equiv 4 \sin(\pi/8) \cos(\pi/8) \text{ we obtain from Eq. [\ref{13}]} d_{(2,1,0)}^{(1)} + d_{(2,1,0)}^{(2)} = d_{(1,0,0)} d_{(3,2,1)} = 4 \cos(\pi/8). \text{ So we may and do assume without loss of generality that }} d_{(2,1,0)}^{(1)} \leq 2 \cos(\pi/8). \text{ As } 4 \cos^2(\pi/8) = 2 + \sqrt{2} < 4 \text{ it follows that } [\alpha_{(2,1,0)}^{(1)}] \text{ decomposes into at most three irreducible sectors.}
Therefore we conclude by Eq. (11) that
\[
\langle \alpha(1,0,0) \circ \alpha^{(1)}_{(2,1,0)}, \alpha(1,0,0) \circ \alpha^{(1)}_{(2,1,0)} \rangle_{M(I_0)} = \\
= \langle \alpha(1,1,1) \circ \alpha(1,0,0), \alpha^{(1)}_{(2,1,0)} \circ \alpha^{(1)}_{(2,1,0)} \rangle_{M(I_0)} \leq 3,
\]
and thus \([\alpha(1,0,0)] \times [\alpha^{(1)}_{(2,1,0)}]\) cannot contain an irreducible sector with multiplicity larger than one. But we also have
\[
\langle \alpha(1,0,0) \circ \alpha^{(1)}_{(2,1,0)}, \alpha(2,2,0) \rangle_{M(I_0)} = \langle \alpha^{(1)}_{(2,1,0)}, \alpha(1,1,1) \circ \alpha(2,2,0) \rangle_{M(I_0)} = 2
\]
since one checks \([\alpha(1,1,1)] \times [\alpha(2,2,0)] = 2[\alpha(2,1,0)]. \) It follows by comparison with Eq. (14)
\[
[\alpha(1,0,0)] \times [\alpha^{(i)}_{(2,1,0)}] = [\alpha^{(1)}_{(2,2,0)}] \oplus [\alpha^{(2)}_{(2,2,0)}], \quad i = 1, 2,
\]
and \(d^{(i)}_{(2,1,0)} = 2\cos(\pi/8), \ i = 1, 2.\)

We have \([\alpha^{(i)}_{(2,1,0)}] = [\alpha^{(2)}_{(2,2,1)}],\) and hence with a suitable choice of notation \([\alpha^{(i)}_{(2,1,0)}] = [\alpha^{(i)}_{(2,2,1)}]\) for \(i = 1, 2.\) One checks
\[
\left([\alpha^{(1)}_{(2,1,0)}] \oplus [\alpha^{(2)}_{(2,1,0)}]\right) \times [\alpha(1,1,1)] = 2[\alpha(1,1,0)] \oplus 2[\alpha^{(1)}_{(3,2,1)}],
\]
and since \(2 + \sqrt{2} = d^{(1)}_{(1,1,0)} \neq d^{(1)}_{(3,2,1)} = \sqrt{2}\) we find
\[
[\alpha^{(i)}_{(2,1,0)}] \times [\alpha(1,1,1)] = [\alpha(1,1,0)] \oplus [\alpha^{(1)}_{(3,2,1)}], \quad i = 1, 2,
\]
and conjugation yields
\[
[\alpha(1,0,0)] \times [\alpha^{(i)}_{(2,2,1)}] = [\alpha(1,1,0)] \oplus [\alpha^{(1)}_{(3,2,1)}], \quad i = 1, 2,
\]
\[
[\alpha^{(1)}_{(2,2,0)}] \oplus [\alpha^{(2)}_{(2,2,0)}] = [\alpha^{(1)}_{(2,2,0)}] \oplus [\alpha^{(1)}_{(2,2,0)}],
\]
hence conjugation of Eq. (17) yields
\[
[\alpha(1,1,1)] \times [\alpha^{(i)}_{(2,2,1)}] = [\alpha^{(1)}_{(2,2,0)}] \oplus [\alpha^{(2)}_{(2,2,0)}], \quad i = 1, 2.
\]
Thus we find for \(i, j = 1, 2,\)
\[
\langle \alpha(1,0,0) \circ \alpha^{(i)}_{(2,2,0)}, \alpha^{(j)}_{(2,2,0)} \rangle_{M(I_0)} = \langle \alpha^{(i)}_{(2,2,0)}, \alpha(1,1,1) \circ \alpha^{(j)}_{(2,2,1)} \rangle_{M(I_0)} = 1,
\]
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and similarly we obtain (by use of Eq. (11))

\[
\langle \alpha_{(1,0,0)} \circ \alpha_{(i,2,0)}, \alpha_{(1,0,0)} \rangle_{M(I_2)} = \langle \alpha_{(i,2,0)}, \alpha_{(1,1,1)} \circ \alpha_{(1,0,0)} \rangle_{M(I_2)} = 1
\]

for \(j = 1, 2\). It follows now from Eq. (15)

\[
[\alpha_{(1,0,0)}] \times [\alpha_{(i,2,0)}] = [\alpha_{(1,0,0)}] \oplus [\alpha_{(1,1,1)}] \oplus [\alpha_{(2,2,1)}] \oplus [\alpha_{(2,2,0)}], \quad i = 1, 2.
\]

We have succeeded to compute \([\alpha_{(1,0,0)}] \times [\beta]\) for each \([\beta] \in \mathcal{V}\), and thus we can draw the fusion graph given in Fig. 12.

Similarly one finds for the sector products of \([\alpha_{(1,1,0)}]\)

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(1,1,0)}] = [\alpha_{(0,0,0)}] \oplus [\alpha_{(4,0,0)}] \oplus 2[\alpha_{(1,2,0)}] \oplus 2[\alpha_{(2,2,0)}],
\]

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(1,1,1)}] = 2[\alpha_{(1,0,0)}] \oplus [\alpha_{(1,2,1)}] \oplus [\alpha_{(2,2,1)}],
\]

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(4,0,0)}] = [\alpha_{(1,1,0)}],
\]

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(3,2,1)}] = [\alpha_{(2,2,0)}] \oplus [\alpha_{(2,2,0)}],
\]

and also (we omit some details)

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(2,1,0)}] = [\alpha_{(1,0,0)}] \oplus [\alpha_{(1,2,1)}] \oplus [\alpha_{(2,2,1)}], \quad i = 1, 2,
\]

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(2,2,0)}] = 2[\alpha_{(1,1,0)}] \oplus [\alpha_{(3,2,1)}], \quad i = 1, 2,
\]

\[
[\alpha_{(1,1,0)}] \times [\alpha_{(2,2,1)}] = [\alpha_{(1,1,1)}] \oplus [\alpha_{(2,1,0)}] \oplus [\alpha_{(2,1,0)}], \quad i = 1, 2.
\]
These equations are visualized in in the (disconnected) fusion graph of $[\alpha_{(1,1,0)}]$, see Fig. 13.

We now want to show that for this example the $\alpha$-induced sector algebra is in fact non-commutative! (The appearance of a non-commutative sector structure associated to the conformal embedding $SU(4) \subset SO(15)$ was first observed by Xu [52] in his framework.) From Eqs. (18) and (19) we obtain (recall $\bar{\alpha}^{(i)}_{(2,1,0)} = [\alpha^{(i)}_{(2,2,1)}]$)

$$\langle \alpha^{(i)}_{(2,1,0)} \circ \alpha^{(j)}_{(2,1,0)}, \alpha_{(1,0,0)} \circ \alpha_{(1,0,0)} \rangle_{M(I_0)} =$$

$$= \langle \alpha^{(i)}_{(2,1,0)} \circ \alpha_{(1,1,1)}, \alpha^{(j)}_{(2,2,1)} \circ \alpha_{(1,0,0)} \rangle_{M(I_0)} = 2, \quad i, j = 1, 2,$$

but from $[\alpha_{(1,0,0)}] \times [\alpha_{(1,0,0)}] = 2[\alpha_{(1,1,0)}]$ we conclude that $[\alpha_{(1,1,0)}]$ is a subsector of $[\alpha^{(i)}_{(2,1,0)}] \times [\alpha^{(j)}_{(2,1,0)}]$, and by matching the statistical dimensions we find indeed

$$[\alpha^{(i)}_{(2,1,0)}] \times [\alpha^{(j)}_{(2,1,0)}] = [\alpha_{(1,1,0)}], \quad i, j = 1, 2,$$

and hence

$$\langle \alpha^{(i)}_{(2,1,0)} \circ \alpha^{(i)}_{(2,2,1)}, \alpha^{(i)}_{(2,2,1)} \circ \alpha^{(i)}_{(2,1,0)} \rangle_{M(I_0)} =$$

$$= \langle \alpha^{(i)}_{(2,1,0)} \circ \alpha^{(i)}_{(2,1,0)}, \alpha^{(i)}_{(2,1,0)} \circ \alpha^{(i)}_{(2,1,0)} \rangle_{M(I_0)} = 1, \quad i = 1, 2. \quad (21)$$

But $[\alpha^{(i)}_{(2,1,0)}] \times [\alpha^{(i)}_{(2,2,1)}]$ as well as $[\alpha^{(i)}_{(2,2,1)}] \times [\alpha^{(i)}_{(2,1,0)}]$ must contain the identity sector $[\alpha_{(0,0,0)}]$, and also other sectors since $\phi^{(i)}_{(2,1,0)} = \phi^{(i)}_{(2,2,1)} = 2 \cos(\pi/8) >$

Figure 13: Fusion graph of $[\alpha_{(1,1,0)}]$
1, \ i = 1, 2. Because Eq. (21) tells us that these products have only the identity sector in common we have shown
\[
[\alpha_{(2,1,0)}^{(i)}] \times [\alpha_{(2,2,1)}^{(i)}] \neq [\alpha_{(2,2,1)}^{(i)}] \times [\alpha_{(2,1,0)}^{(i)}], \quad i = 1, 2.
\]
Indeed one can compute these products as follows. Since
\[
[\alpha_{(2,1,0)}^{(i)}] \times [\alpha_{(1,0,0)}] = [\alpha_{(1,0,0)}] \times [\alpha_{(2,1,0)}^{(i)}] = [\alpha_{(2,2,0)}^{(1)}] \oplus [\alpha_{(2,2,0)}^{(2)}], \quad i = 1, 2,
\]
it follows
\[
\langle \alpha_{(2,1,0)}^{(i)} \circ \alpha_{(2,2,1)}^{(i)} \circ \alpha_{(1,1,1)} \circ \alpha_{(1,0,0)} \rangle_{M(I_0)} = \\
= \langle \alpha_{(1,0,0)} \circ \alpha_{(2,1,0)}^{(i)} \circ \alpha_{(1,0,0)} \circ \alpha_{(2,1,0)}^{(i)} \rangle_{M(I_0)} = 2, \quad i = 1, 2,
\]
and
\[
\langle \alpha_{(2,2,1)}^{(i)} \circ \alpha_{(2,1,0)}^{(i)} \circ \alpha_{(1,0,0)} \circ \alpha_{(1,1,1)} \rangle_{M(I_0)} = \\
= \langle \alpha_{(2,1,0)}^{(i)} \circ \alpha_{(1,0,0)} \circ \alpha_{(2,1,0)}^{(i)} \circ \alpha_{(1,0,0)} \rangle_{M(I_0)} = 2, \quad i = 1, 2,
\]
and from Eq. (21) we conclude that both \([\alpha_{(2,1,0)}^{(i)}] \times [\alpha_{(2,2,1)}^{(i)}]\) and \([\alpha_{(2,2,1)}^{(i)}] \times [\alpha_{(2,1,0)}^{(i)}]\) must contain, besides the identity sector, one of the sectors \([\alpha_{(4,0,0)}], \ [\alpha_{(2,2,0)}],\) and \([\alpha_{(2,2,0)}]\). Let us assume that \([\alpha_{(2,1,0)}^{(i)}] \times [\alpha_{(2,2,1)}^{(i)}]\) contains \([\alpha_{(4,0,0)}]\). Then, because of a mismatch of quantum dimensions of \(\sqrt{2}\), it contains necessarily a third sector (which is determined to be \([\alpha_{(3,2,1)}]\)).

Since now \([\alpha_{(2,2,1)}^{(1)}] \times [\alpha_{(2,1,0)}^{(1)}]\) cannot contain \([\alpha_{(4,0,0)}]\) it contains either \([\alpha_{(2,2,0)}^{(1)}]\) or \([\alpha_{(2,2,0)}^{(2)}]\), and as then the quantum dimensions match this means that \([\alpha_{(2,2,1)}^{(1)}] \times [\alpha_{(2,1,0)}^{(1)}]\) decomposes into two irreducible sectors whereas \([\alpha_{(2,2,1)}^{(1)}] \times [\alpha_{(2,1,0)}^{(1)}]\) decomposes into three. However, this contradicts
\[
\langle \alpha_{(2,1,0)}^{(1)} \circ \alpha_{(2,2,1)}^{(1)} \circ \alpha_{(2,1,0)}^{(1)} \rangle_{M(I_0)} = \\
= \langle \alpha_{(2,2,1)}^{(1)} \circ \alpha_{(2,2,1)}^{(1)} \circ \alpha_{(2,1,0)}^{(1)} \rangle_{M(I_0)}.
\]

It follows, with a suitable choice of notation,
\[
[\alpha_{(2,1,0)}^{(1)}] \times [\alpha_{(2,2,1)}^{(1)}] = [\alpha_{(0,0,0)}] \oplus [\alpha_{(2,2,0)}^{(1)}], \\
[\alpha_{(2,2,1)}^{(1)}] \times [\alpha_{(2,1,0)}^{(1)}] = [\alpha_{(0,0,0)}] \oplus [\alpha_{(2,2,0)}^{(2)}].
\]
Petkova and Zuber obtained the fusion graphs of Figs. 12 and 13 in a completely different and more empirical way (Fig. A.2. in [42]). The non-commutativity of $V$ nicely explains why they could not find non-negative structure constants associated to these graphs: They were searching for a (commutative) fusion algebra.

3 The treatment of orbifold inclusions

We have seen that conformal inclusions can be described in terms of nets of subfactors. For orbifold inclusions the extended net is not a priori given. However, it is argued in [36] that orbifold type modular invariants arise from extensions of current algebras by some simple currents. The conformal dimensions of these simple currents are necessarily integers. In the following we will describe this idea in our “bounded operator framework”. The techniques we use are not essentially new. Similar and often more general statements can be found in particular in [45, 46]. However, we prefer to give a self-contained presentation and to avoid unnecessary generality if this simplifies our arguments.

Starting from the vacuum net $\mathcal{A}$ where $A(I) = \pi_0(L_I SU(n))''$ as usual we will construct a net of subfactors $\mathcal{N} \subset \mathcal{M}$ describing the analogue of the conformal inclusions now for the orbifold type modular invariants, and we will see that the extended net obtained by this construction is local exactly for the levels where the orbifold modular invariants appear.

3.1 Construction of the extended net

We call an automorphism $\hat{\sigma}_0 \in \Delta_A(I_0)$ a simple current of order $n$, if $n = 2, 3, 4, \ldots$ is the smallest positive integer such that $\hat{\sigma}_0^n$ is equivalent to the identity, i.e. $\hat{\sigma}_0^n = \text{Ad}(Y)$ for a unitary $Y \in \mathcal{B}(\mathcal{H}_0)$, and then $Y \in A(I_0)$ by Haag duality. For our construction we need an equivalent automorphism $\sigma_0$ which is periodic i.e. $\sigma_0^n$ is exactly the identity. We call $\rho_0 \in \Delta_A(I_0)$ a fixed point of the simple current $\hat{\sigma}_0$ if $[\hat{\sigma}_0 \circ \rho_0] = [\rho_0]$. The following lemma gives a sufficient criterion for the possibility of a choice of a periodic automorphism (cf. [26], Prop. 3.3, or [13], Lemmata 4.4 and 4.5).

Lemma 3.1 Let $\hat{\sigma}_0 \in \Delta_A(I_0)$ be a simple current of order $n$. If there is an irreducible fixed point $\rho_0 \in \Delta_A(I_0)$ of $\hat{\sigma}_0$ then there is a simple current $\sigma_0 \in \Delta_A(I_0)$ such that $[\sigma_0] = [\hat{\sigma}_0]$ and $\sigma_0^n = \text{id}$. 

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Then we find that it is easy to check that we define unitary field operators $U \in \mathcal{H}$ such that $\sigma_0 \circ \rho_0 = \text{Ad}(U) \circ \rho_0$. We set $\sigma_0 = \text{Ad}(U^*) \circ \hat{\sigma}_0$. Then $\sigma_0^n$ is clearly inner, namely

$$\sigma_0^n = (\text{Ad}(U^*) \circ \hat{\sigma}_0)^n = \text{Ad}(U^* \hat{\sigma}_0(U^*) \hat{\sigma}_0^2(U^*) \cdots \hat{\sigma}_0^{n-1}(U^*)) \circ \hat{\sigma}_0^n = \text{Ad}(Z),$$

where $Z = U^* \hat{\sigma}_0(U^*) \hat{\sigma}_0^2(U^*) \cdots \hat{\sigma}_0^{n-1}(U^*)Y$. (Recall $\hat{\sigma}_0^n = \text{Ad}(Y)$.) Now we have $\rho_0 = \sigma_0^n \circ \rho_0 = \text{Ad}(Z) \circ \rho_0$ and thus $Z \in \rho_0(A(I_0)) \cap A(I_0)$. Since we assumed $\rho_0$ to be irreducible it follows $Z \in \mathbb{C}1$ and hence $\sigma_0^n = \text{Ad}(Z) = \text{id}$. Q.E.D.

From now on we assume that there is an irreducible fixed point $\rho_0 \in \Delta_A(I_0)$ for the simple current $\hat{\sigma}_0 \in \Delta_A(I_0)$ of order $n$, and hence we have an equivalent periodic automorphism $\sigma_0 \in \Delta_A(I_0)$, i.e. $\sigma_0^n = \text{id}$, and also $\sigma_0 \circ \rho_0 = \rho_0$. The following construction is basically the construction of the field group and algebra as in [4]. Recall that $\mathcal{H}_0$ is the vacuum Hilbert space where $\mathcal{A}$ lives on. We set

$$\mathcal{H} = \bigoplus_{p=0}^{n-1} \mathcal{H}_0.$$

For a vector $\Psi \in \mathcal{H}$ we denote by $\Psi_p \in \mathcal{H}_0$ its $p$-th component with respect to this decomposition. We define a representation $\pi$ of $\mathcal{A}$ on $\mathcal{H}$ by $\pi(a) = \bigoplus_{p=0}^{n-1} \sigma_0^p(a)$, i.e.

$$(\pi(a)\Psi)_p = \sigma_0^p(a)\Psi_p, \quad a \in \mathcal{A}, \quad \Psi \in \mathcal{H}, \quad p = 0, 1, 2, \ldots, n - 1.$$ 

Then the net $\mathcal{N}$ is defined in terms of local algebras by

$$N(I) = \pi(A(I)), \quad I \in \mathcal{J}_z.$$ 

Pick a unitary $U_I$ such that $\sigma_{0,I} = \text{Ad}(U_I) \circ \sigma_0 \in \Delta_A(I)$ for some $I \in \mathcal{J}_z$. We define unitary field operators $f_{U_I} \in \mathfrak{B} (\mathcal{H})$ by

$$(f_{U_I} \Psi)_p = \sigma_0^{n-p-1}(U_I^*) \Psi_{p-1}, \quad \Psi \in \mathcal{H}, \quad p = 0, 1, 2, \ldots, n - 1, \quad (\text{mod } n).$$

It is easy to check that

$$(f_{U_I}^* \Psi)_p = \sigma_0^p(U_I) \Psi_{p+1}, \quad \Psi \in \mathcal{H}, \quad p = 0, 1, 2, \ldots, n - 1, \quad (\text{mod } n).$$

Then we find

$$(f_{U_I}^* \pi(a)f_{U_I} \Psi)_p = \sigma_0^p(U_I) (\pi(a)f_{U_I} \Psi)_{p+1} = \sigma_0^p(U_I) \sigma_0^{p+1}(a) (f_{U_I} \Psi)_{p+1} = \sigma_0^p(U_I) \sigma_0^{p+1}(a) \sigma_0^p(U_I^*) \Psi_p = \sigma_0^p \circ \sigma_{0,I}(a) \Psi_p,$$

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hence
\[ f_{U_I}^* \pi(a) f_{U_I} = \pi \circ \sigma_{0,I}(a), \quad a \in \mathcal{A}. \]  
(22)

We use the following notation: For \( \lambda_0 \in \text{End}(\mathcal{A}) \) we define \( \lambda \in \text{End}(\mathcal{N}) \) by \( \lambda(\pi(a)) = \pi(\lambda_0(a)), \quad a \in \mathcal{A}. \) One checks easily that \( \lambda \in \Delta_{\mathcal{N}}(I_0) \) if \( \lambda_0 \in \Delta_{\mathcal{A}}(I_0). \) Then Eq. (22) reads \( f_{U_I}^* x f_{U_I} = \sigma_I(x) \) for \( x \in \mathcal{N}, \) and in particular \( f_{U_I} \in N(I) \)'s, i.e. fields are relatively local to observables. Now we define the extended net \( \mathcal{M} \) in terms of local algebras \( \mathcal{M}(I) \) being generated by \( N(I) \) and \( f_{U_I} \),
\[ M(I) = \langle N(I), f_{U_I} \rangle, \quad I \in \mathcal{J}_z. \]

Note that we have
\[ f_{U_I} = f_1 \pi(U_I^*) \]
since
\[ (f_{U_I} \Psi)_p = \sigma_p^{p-1}(U_I^*) \Psi_{p-1} = \pi(U_I^*) \Psi_{p-1} = (f_1 \pi(U_I^*) \Psi)_p. \]
Therefore the definition of \( M(I) \) is independent on the special choice of \( U_I \) because if \( \text{Ad}(\hat{U}_I) \circ \sigma_0 \) is also localized in \( I \) then \( U_I \hat{U}_I^* \in \mathcal{A}(I) \) by Haag duality and hence \( f_{U_I} \) and \( f_{\hat{U}_I} \) differ only by an element in \( N(I) \).

Note that our construction is such that (obviously by taking \( U_{I_0} = 1 \)) we have \( M(I_0) \cong \mathcal{A}(I_0) \rtimes_{\sigma_0} \mathbb{Z}_n \). We want to show that this is similar for any \( I \in \mathcal{J}_z \).

**Lemma 3.2** For any \( I \in \mathcal{J}_z \) there is a unitary \( W \in \mathcal{A}(I) \) such that \( \tilde{\sigma}_{0,I} = \text{Ad}(W^*) \circ \sigma_{0,I} \in \Delta_{\mathcal{A}}(I) \) fulfills \( \tilde{\sigma}_{0,I}^n = \text{id} \).

**Proof.** Since the irreducible fixed point \( \rho_0 \) is transportable there is a unitary \( U_{\rho_0;I_0,I} \) such that \( \rho_{0,I} = \text{Ad}(U_{\rho_0;I_0,I}) \circ \rho_0 \in \Delta_{\mathcal{A}}(I), \) and hence
\[ \sigma_{0,I} \circ \rho_{0,I} = \text{Ad}(U_I) \circ \sigma_0 \circ \text{Ad}(U_{\rho_0;I_0,I}) \circ \rho_0 \]
\[ = \text{Ad}(U_I \sigma_0(U_{\rho_0;I_0,I})) \circ \sigma_0 \circ \rho_0 \]
\[ = \text{Ad}(U_I \sigma_0(U_{\rho_0;I_0,I})) \circ \rho_0 \]
\[ = \text{Ad}(U_I \sigma_0(U_{\rho_0;I_0,I})U_I^*) \circ \rho_{0,I}. \]
Now \( W = U_I \sigma_0(U_{\rho_0;I_0,I})U_I^* \in \mathcal{A}(I) \) by Haag duality, hence \( \rho_{0,I} \) is an irreducible fixed point for \( \sigma_{0,I}. \) Then, by the same argument as in Lemma 3.1 we find that \( \tilde{\sigma}_{0,I} = \text{Ad}(W^*) \circ \sigma_{0,I} \) fulfills \( \tilde{\sigma}_{0,I}^n = \text{id}. \) Q.E.D.

Now we take this unitary \( W \) such that \( \tilde{\sigma}_{0,I} = \text{Ad}(W^*) \circ \sigma_{0,I} \) and \( \tilde{\sigma}_{0,I}^n = \text{id}, \) and then we define \( \hat{U}_I = W^*U_I \) and set
\[ f_{\hat{U}_I} = f_{U_I} \pi(W) \in M(I). \]
Then it follows from Eq. (22) that
\[ f_{\tilde{U}_I}^* \pi(a) f_{\tilde{U}_I} = \pi \circ \tilde{\sigma}_{0,I}(a), \quad a \in A. \] (23)

**Lemma 3.3** With a suitable choice of the phase of \( W \) we have \( f_{\tilde{U}_I}^n = 1 \).

**Proof.** We have \( \sigma_0 = \text{Ad}(\tilde{U}_I) \circ \tilde{\sigma}_{0,I} \). Choose \( J \in J_z \) such that \( J \supset I \cup I_0 \).

Then for any \( a \in A(J) \) we find
\[ a = \sigma_0^n(a) = (\text{Ad}(\tilde{U}_I) \circ \tilde{\sigma}_{0,I})^n(a) = \text{Ad}(X) \circ \tilde{\sigma}_{0,I} = XaX^*, \]
where \( X = \tilde{U}_I \tilde{\sigma}_0(I) \tilde{\sigma}_{0,I}^2(\tilde{U}_I) \cdots \tilde{\sigma}_{0,I}^{n-1}(\tilde{U}_I) \), and therefore \( X \in A(J) \cap A(J) = \mathbb{C}1 \). If \( X = \xi 1, \xi \in \mathbb{C} \), the we can replace \( W \) by \( \xi^{1/n}W \) i.e. \( \tilde{U}_I \) by \( \xi^{-1/n}\tilde{U}_I \) to achieve \( X = 1 \). Now we compute
\[
\begin{align*}
f_{\tilde{U}_I}^n &= f_1 \pi(\tilde{U}_I) f_{\tilde{U}_I}^{n-1} \\
&= f_1 f_{\tilde{U}_I}^{n-1} \pi(\tilde{\sigma}_{0,I}^{n-1}(\tilde{U}_I)) \\
&= f_1 f_1 \pi(\tilde{U}_I) f_{\tilde{U}_I}^{n-2} \pi(\tilde{\sigma}_{0,I}^{n-2}(\tilde{U}_I)) \\
&= f_1^2 f_{\tilde{U}_I}^{n-2} \pi(\tilde{\sigma}_{0,I}^{n-2}(\tilde{U}_I)) \tilde{\sigma}_{0,I}^{n-1}(\tilde{U}_I)) \\
&= \cdots = f_1^n \pi(X) = 1,
\end{align*}
\]
where we used Eq. (23).

Q.E.D.

Eq. (23) holds in particular for \( a \in A(I) \), moreover, \( \tilde{\sigma}_{0,I}^p \) is outer for \( p \neq 0 \) (mod \( n \)), and \( f_{\tilde{U}_I}^n = 1 \). By the uniqueness of the crossed product we find

**Corollary 3.4** We have \( M(I) \cong A(I) \rtimes \tilde{\sigma}_{0,I} \mathbb{Z}_n \) for any \( I \in J_z \). In particular, each \( M(I) \) is a factor.

Let \( \Omega_0 \in \mathcal{H}_0 \) denote the vacuum vector. Then \( \Omega_0 \) is cyclic and separating for each local algebra \( A(I) \). Let \( \Omega \in \mathcal{H} \) denote the vector given by \( \Omega_p = \delta_{p,0} \Omega_0 \). It is clear from our construction that \( \Omega \) is cyclic and separating for each \( M(I) \), that is, our net \( \mathcal{N} \subset \mathcal{M} \) is standard.

Fixing \( U_I \) for any \( I \in J_z \) it is clear that each \( m \in M(I) \) can be uniquely written as
\[ m = \sum_{p=0}^{n-1} x_p f_{U_I}^p, \quad x_p \in N(I). \]
Then the map
\[ E_{N(I)}^{M(I)} : M(I) \to N(I), \quad m \mapsto E_{N(I)}^{M(I)}(m) = x_0, \]
is a faithful normal conditional expectation. It also satisfies \( E_{N(J)}^{M(J)}|_{M(I)} = E_{N(I)}^{M(I)} \) for \( I \subset J \) and preserves the vector state \( \omega = \langle \Omega, \cdot \Omega \rangle \). We summarize the discussion in the following

**Proposition 3.5** The net \( N \subset M \) is a standard net of subfactors with a standard conditional expectation.

Note that \( N \subset M \) is even a quantum field theoretical net of subfactors by relative locality \( M(I) \subset N(I') \). However, \( M \) will in general not be local itself. The requirement of locality of \( M \) imposes restrictions on our simple current \( \sigma_0 \).

For \( \lambda_0, \mu_0 \in \Delta_A(I_o) \) we denoted by \( \lambda \) and \( \mu \) the corresponding endomorphisms in \( \Delta_N(I_o) \). For the statistics operators we use the notation \( \epsilon^\pm(\lambda, \mu) = \pi(\epsilon^\pm(\lambda_0, \mu_0)) \) (and \( \epsilon(\lambda, \mu) = \epsilon^+(\lambda, \mu) \)) as in the previous paper \[4\]. Recall that for disjoint intervals \( I_1, I_2 \in J_z \) we write \( I_2 > I_1 \) (respectively \( I_2 < I_1 \)) if \( I_1 \) lies clockwise (respectively counter-clockwise) to \( I_2 \) relative to the point “at infinity” \( z \).

**Lemma 3.6** For \( I_1 \cap I_2 = \emptyset \) we have \( f_{U_{I_2}} f_{U_{I_1}} = \epsilon^\pm(\sigma, \sigma) f_{U_{I_1}} f_{U_{I_2}} \) with the + - sign if \( I_2 > I_1 \) and the − - sign if \( I_2 < I_1 \).

**Proof.** We compute
\[
\begin{align*}
  f_{U_{I_2}} f_{U_{I_1}} &= f_{1} \pi(U_{I_2}^* f_{1} \pi(U_{I_1}^* ) \\
    &= \pi(\sigma_0^{-1}(U_{I_2}^*)\sigma_0^{-2}(U_{I_1})) f_{1}^2 \\
    &= \pi(\sigma_0^{-1}(U_{I_2}^*)\sigma_0^{-3}(U_{I_2})\sigma_0^{-2}(U_{I_1})) f_{1} \pi(U_{I_1}) f_{1} \pi(U_{I_2}) \\
    &= \sigma^{-2} \circ \pi(\sigma_0(U_{I_2}^* U_{I_1}^*) U_{I_2} \sigma_0(U_{I_1})) f_{U_{I_1}} f_{U_{I_2}} \\
    &= \sigma^{-2} \circ \pi(\epsilon^\pm(\sigma_0, \sigma_0)) f_{U_{I_1}} f_{U_{I_2}} \\
    &= \sigma^{-2}(\epsilon^\pm(\sigma, \sigma)) f_{U_{I_1}} f_{U_{I_2}},
\end{align*}
\]
where we recognized the definition of the statistics operator in Subsection 2.3 of \[4\], and \( \epsilon^\pm(\sigma, \sigma) \) are just scalars since \( \epsilon^\pm(\sigma_0, \sigma_0) \in \sigma_0^2(A(I_o))' \cap A(A(I_o)), \) hence we can omit the symbol \( \sigma^{-2} \).

This leads us immediately to the following
Corollary 3.7 The net $\mathcal{M}$ is local if and only if $\epsilon(\sigma_0, \sigma_0) = 1$.

In Subsection 3.3 we will use Corollary 3.7 to analyze for which levels we have a local extended net if we take for $\sigma_0$ the simple current corresponding to the weight $k\Lambda_1(1)$ of the $LSU(n)$ level $k$ theory.

For completeness we also add the following

Proposition 3.8 If the net $\mathcal{M}$ is local then it is in fact Haag dual.

Proof. Let $I \in \mathcal{J}_z$ be arbitrary. We have to show that $\mathcal{C}_M(I)' = M(I)$. As $\mathcal{C}_M(I)' \supset M(I)$ follows from locality we only have to show the reverse inclusion. Thus assume $X \in \mathcal{C}_M(I)'$, and we have to show that $X \in M(I)$. Choose an interval $J \in \mathcal{J}_z$ such that $I_0 \cup I \subset J$. Then in particular $X \in \mathcal{C}_M(J')$ and therefore $X \pi(a) = \pi(a)X$ for all $a \in \mathcal{A}(J')$. This reads in matrix components (corresponding to the decomposition of $\mathcal{H}$ into $n$ copies of $\mathcal{H}_0$) $X_{p,q} \sigma_0^q(a) = \sigma_0^q(a)X_{p,q}$, $p,q \in \mathbb{Z}_n$, but $\sigma_0$ acts trivially on $\mathcal{C}(J')$ as $I_0 \subset J$. Hence $X_{p,q} \in \mathcal{C}(J') = A(J)$ by Haag duality of $\mathcal{A}$. Now choose $K \in \mathcal{J}_z$ such that $K \subset J'$. Then we have in particular $Xf_{U_K} = f_{U_K}X$. From this we obtain for the matrix components $X_{p,q+1} \sigma_0^q(U_K^*) = \sigma_0^{q-1}(U_K^*)X_{p-1,q}$, $p,q \in \mathbb{Z}_n$, and hence

$$X_{p+1,q+1} = \sigma_0^q(U_K)X_{p,q} \sigma_0^q(U_K) = \sigma_0^q(U_K) \sigma_0^{-p}(X_{p,q}) \sigma_0^{-q}(U_K) = \sigma_0^q(U_K) \cdot \sigma_{0,K} \circ \sigma_0^{-p}(X_{p,q}) \cdot \sigma_0^{-q}(U_K) = \sigma_0^q(X_{p,q}) \sigma_0^{-p}(\epsilon(\sigma_0^{-q}, \sigma_0)) = \sigma_0(X_{p,q})$$

where we used that $\sigma_0^{-p}(X_{p,q}) \in A(J)$ since $I_0 \subset J$ and $\sigma_{0,K}$ acts trivially on $A(J)$ since $K \subset J'$, and also that

$$\epsilon(\sigma_0^{-q}, \sigma_0) = \epsilon(\sigma_0, \sigma_0)\sigma_0(\epsilon(\sigma_0, \sigma_0)) \cdots \sigma_0^{q-1}(\epsilon(\sigma_0, \sigma_0)) = 1$$

since $\epsilon(\sigma_0, \sigma_0) = 1$ as $\mathcal{M}$ is local. We conclude that $X_{p+k,q+k} = \sigma_0^k(X_{p,q})$, $p,q,k \in \mathbb{Z}_n$, and by setting $\tilde{a}_p = X_{0,p} \in A(J)$ this means that $X$ can be written as $X = \sum_{p \in \mathbb{Z}_n} \pi(\tilde{a}_p) f^p_{U_1}$, i.e. $X \in M(J)$, but then we can also alternatively write $X = \sum_{p \in \mathbb{Z}_n} \pi(a_p) f^p_{U_1}$, with $a_p \in A(J)$ since also $I \subset J$. Because we assumed that $X \in \mathcal{C}_M(I')$ we must have in particular that $X \pi(b) = \pi(b)X$ whenever $b \in \mathcal{A}(I')$. Now

$$X \pi(b) = \sum_{p \in \mathbb{Z}_n} \pi(a_p) f^p_{U_1} \pi(b) = \sum_{p \in \mathbb{Z}_n} \pi(a_p) \pi(b) f^p_{U_1}$$

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by relative locality of fields and observables, and
\[ \pi(b) X = \sum_{p \in \mathbb{Z}_n} \pi(b) \pi(a_p) f^p_U, \]
hence
\[ \sum_{p \in \mathbb{Z}_n} \pi(a_p) \pi(b) f^p_U = \sum_{p \in \mathbb{Z}_n} \pi(b) \pi(a_p) f^p_U. \]
Multiplication by \( f^{-q} \) from the right and application of the conditional expectation yields \( \pi(a_q) \pi(b) = \pi(b) \pi(a_q) \) for all \( b \in \mathcal{C}_\mathcal{A}(I'), q \in \mathbb{Z}_n \). It follows \( a_q \in \mathcal{C}_\mathcal{A}(I') \), \( q \in \mathbb{Z}_n \), and therefore \( X = \sum_{p \in \mathbb{Z}_n} \pi(a_p) f^p_U \in M(I) \). Q.E.D.

### 3.2 Endomorphisms of the extended net

We have lifted endomorphisms \( \lambda_0 \) of \( \mathcal{A} \) to endomorphisms \( \lambda \) of \( \mathcal{N} \) by \( \lambda \circ \pi = \pi \circ \lambda_0 \). Next we consider the \( \alpha \)-induced endomorphisms \( \alpha \lambda \in \text{End}(\mathcal{M}) \). In the following we assume that \( \mathcal{M} \) is Haag dual, i.e. that \( \epsilon(\sigma_0, \sigma_0) = 1 \). For notation we refer again to our previous paper \([4]\) and to Subsection 1.3.

**Lemma 3.9** For \( \lambda_0 \in \Delta_\mathcal{A}(I_\circ) \) we have \( \alpha \lambda(f_1) = f_1 \epsilon(\lambda, \sigma) \).

**Proof.** By applying \( \gamma \) to \( f_1^* x f_1 = \sigma(x), x \in \mathcal{N} \), we find \( \gamma(f_1) \in \text{Hom}_\mathcal{N}(\theta \circ \sigma, \theta) \). Hence by the BFE, Eq. (22) of \([4]\), we obtain
\[ \gamma(f_1) \theta(\epsilon(\lambda, \sigma)) \epsilon(\lambda, \theta) = \epsilon(\lambda, \theta) \cdot \lambda \circ \gamma(f_1), \]
and therefore
\[ \alpha \lambda(f_1) \equiv \gamma^{-1} \circ \text{Ad}(\epsilon(\lambda, \theta)) \circ \lambda \circ \gamma(f_1) = f_1 \epsilon(\lambda, \sigma), \]
proving the lemma. Q.E.D.

Now we ask when \( \alpha \lambda \) is localized. For the sake of simplicity, we restrict the discussion to irreducible \( \lambda_0 \). Define the **monodromy** by \( Y(\lambda_0, \sigma_0) = \epsilon(\lambda_0, \sigma_0) \epsilon(\sigma_0, \lambda_0) \). Note that for irreducible \( \lambda_0 \) the monodromy is a scalar as \( Y(\lambda_0, \sigma_0) \in \sigma_0 \circ \lambda_0(A(I_\circ))' \cap A(I_\circ) = \mathbb{C} \mathbf{1} \), i.e. \( Y(\lambda_0, \sigma_0) = \omega \mathbf{1} \), \( \omega \in \mathbb{C} \). Therefore we have \( \epsilon(\lambda_0, \sigma_0) = Y(\lambda_0, \sigma_0) \epsilon(\sigma_0, \lambda_0)^* = \omega e^{-}(\lambda_0, \sigma_0) \).

**Lemma 3.10** For \( \lambda_0 \in \Delta_\mathcal{A}(I_\circ) \) irreducible \( \alpha \lambda \) is localized in \( I_\circ \) if and only if the monodromy \( Y(\lambda_0, \sigma_0) \) is trivial, i.e. \( \omega = 1 \).
Proof. It is clear that \( \alpha_\lambda(x) \equiv \lambda(x) = x \) for any \( x \in N(I) \) with \( I \cap I_0 = \emptyset \) since \( \lambda_0 \in \Delta_A(I_0) \). Thus we have to check whether \( \alpha_\lambda(f_{U_I}) = f_{U_I} \) whenever \( I \cap I_0 = \emptyset \). By definition

\[
\alpha_\lambda(f_{U_I}) = \alpha_\lambda(f_\lambda(\pi(U_I^*)) = f_\lambda(\pi(\epsilon(\lambda_0, \sigma_0)\lambda_0(U_I^*)))
\]

\[
= f_{U_I}\pi(\epsilon(\lambda_0, \sigma_0)\lambda_0(U_I^*)).
\]

For \( I \in J_\pm \) such that \( I \cap I_0 = \emptyset \) we distinguish two cases.

Case 1: \( I > I_0 \). We can choose an interval \( I_+ \in J_+ \) such that \( I_+ > I_0 \) and \( I_+ > I \). Since \( I > I_0 \) we can choose some \( J_+ \in J_+ \) such that \( J_+ \supset I \) but \( J_+ \cap I_0 = \emptyset \). For any unitary \( U_{\sigma_0,+} \) such that \( \sigma_{0,+} = \text{Ad}(U_{\sigma_0,+}) \circ \sigma_0 \in \Delta_A(I_+) \) the statistics operator can be written as \( \epsilon(\lambda_0, \sigma_0) = U_{\sigma_0,+}^*\lambda_0(U_{\sigma_0,+}) \). Since \( \sigma_{0,I} = \text{Ad}(U_I) \circ \sigma_0 \) we have \( \sigma_{0,+} = \text{Ad}(V_+) \circ \sigma_{0,I} \) with \( V_+ = U_{\sigma_0,+}U_I^* \), and hence \( V_+ \in A(J_+) \) by Haag duality. Then

\[
U_I\epsilon(\lambda_0, \sigma_0)\lambda_0(U_I^*) = V_+^*\lambda_0(V_+) = V_+^*V_+ = 1
\]

since \( J_+ \cap I_0 = \emptyset \). Hence \( \alpha_\lambda(f_{U_I}) = f_{U_I} \) for \( I > I_0 \).

Case 2: \( I < I_0 \). Recall \( \epsilon(\lambda_0, \sigma_0) = \omega \epsilon^-(\lambda_0, \sigma_0) \), hence

\[
U_I\epsilon(\lambda_0, \sigma_0)\lambda_0(U_I^*) = \omega U_I\epsilon^-(\lambda_0, \sigma_0)\lambda_0(U_I^*).
\]

We can choose an interval \( I_- \in J_- \) such that \( I_- < I_0 \) and \( I_- < I \). Since \( I < I_0 \) we can choose some \( J_- \in J_- \) such that \( J_- \supset I \) but \( J_- \cap I_0 = \emptyset \). For any unitary \( U_{\sigma_0,-} \) such that \( \sigma_{0,-} = \text{Ad}(U_{\sigma_0,-}) \circ \sigma_0 \in \Delta_A(I_-) \) the statistics operator can be written as \( \epsilon^-(\lambda_0, \sigma_0) = U_{\sigma_0,-}^*\lambda_0(U_{\sigma_0,-}) \). Then \( \sigma_{0,-} = \text{Ad}(V_-) \circ \sigma_{0,I} \) with \( V_- = U_{\sigma_0,-}U_I^* \in A(J_-) \), and

\[
U_I\epsilon^-(\lambda_0, \sigma_0)\lambda_0(U_I^*) = V_-^*\lambda_0(V_-) = V_-^*V_- = 1,
\]

and hence \( \alpha_\lambda(f_{U_I}) = \omega f_{U_I} \) for \( I < I_0 \). The statement follows. Q.E.D.

The next step is the transportability. Note that \( \Delta_{A_0}^0(I_0) = \Delta_{M}(I_0) \) since \( M \) is Haag dual. We have the following (cf. Prop. 5.2 in [16])

**Lemma 3.11** For \( \lambda_0 \in \Delta_A(I_0) \) irreducible we have \( \alpha_\lambda \in \Delta_M(I_0) \) if and only if the monodromy \( Y(\lambda_0, \sigma_0) \) is trivial, i.e. \( \omega = 1 \).

**Proof.** After Lemma 3.10 all we have to show is that \( \alpha_\lambda \) is transportable if \( \omega = 1 \). Since \( \lambda_0 \in \Delta_A(I_0) \) there is for any \( J \in J_\pm \) a unitary \( U \equiv U_{\lambda_0,I_0,J} \) such that \( \lambda_0 \in \Delta_A(J) \). Define \( \tilde{\alpha}_\lambda = \text{Ad}(\pi(U)) \circ \alpha_\lambda \). It is clear that \( \tilde{\alpha}_\lambda(x) = x \) whenever \( x \in N(I) \) with \( I \cap J = \emptyset \). We show that also \( \tilde{\alpha}_\lambda(f_{U_I}) = f_{U_I} \) in that case. We again distinguish two cases.
Case 1: $I > J$. We choose $I_+ \in \mathcal{J}^z$ such that $I_+ > I_0$ and $I_+ > J$. Since $I > J$ there is a $K_+ \in \mathcal{J}^z$ such that $K_+ \supset I_+ \cup I$ but $K_+ \cap J = \emptyset$. As before, we choose a unitary $U_{\sigma_0,+}$ such that $\sigma_{0,+} = \text{Ad}(U_{\sigma_0,+}) \circ \sigma_0 \in \Delta_A(I_+)$. Then $\sigma_{0,+} = \text{Ad}(V_+) \circ \sigma_0$ and therefore $V_+ = U_{\sigma_0,+}U_+^* \in A(K_+)$, hence $\lambda_0(V_+) = V_+$. Since $I_+ > I_0$ and $I_+ > J$ we also have $\sigma_{0,+}(U) = U$. Now we compute

$$
\tilde{\alpha}_\lambda(f_{U_1}) = \text{Ad}(\pi(U)) \circ \alpha_\lambda(f_{U_1}) = \pi(U) f_{U_1} \pi(U f(U_0,\sigma_0)\lambda_0(U_1^*)U^*) = f_{U_1} \sigma_0(f(U_0,\sigma_0)\lambda_0(U_1^*)U^*) = f_{U_1} \sigma_0(U_0,\sigma_0)\lambda_0(U_1^*)U^*) = f_{U_1} \pi(U_1\sigma_0(U)U_{\sigma_0,+}\lambda_0(U_{\sigma_0,+})U^*\lambda_0(U_1^*)) = f_{U_1} \pi(U_1U_{\sigma_0,+}\sigma_0(U)U_{\sigma_0,+}\lambda_0(U_{\sigma_0,+})U^*) = f_{U_1} \pi(V_+\lambda_0(V_+)) = f_{U_1}.
$$

Case 2: $I < J$. We choose $I_- \in \mathcal{J}^z$ such that $I_- < I_0$ and $I_- < J$. Since $I < J$ there is a $K_- \in \mathcal{J}^z$ such that $K_- \supset I_- \cup I$ but $K_- \cap J = \emptyset$. Let $\sigma_{0,-} = \text{Ad}(U_{\sigma_0,-}) \circ \sigma_0 \in \Delta_A(I_-)$. Then $V_- = U_{\sigma_0,+}U_+^* \in A(K_-)$, hence $\lambda_0(V_-) = V_-$. Since $I_- < I_0$ and $I_- < J$ we also have $\sigma_{0,-}(U) = U$. If $\omega = 1$ then $\epsilon(\lambda_0,\sigma_0) = \epsilon^-(\lambda_0,\sigma_0)$, so we can compute analogously

$$
\tilde{\alpha}_\lambda(f_{U_1}) = f_{U_1} \pi(U_1\sigma_0(U)\epsilon(\lambda_0,\sigma_0)U^*\lambda_0(U_1^*)) = f_{U_1} \pi(U_1\sigma_0(U)U_{\sigma_0,-}\lambda_0(U_{\sigma_0,-})U^*\lambda_0(U_1^*)) = f_{U_1} \pi(U_1U_{\sigma_0,-}\sigma_0(U)U_{\sigma_0,-}\lambda_0(U_{\sigma_0,-})\lambda_0(U_1^*)) = f_{U_1} \pi(V_-\lambda_0(V_-)) = f_{U_1}.
$$

We have shown that $\tilde{\alpha}_\lambda$ is localized in $J$. Since $J \in \mathcal{J}^z$ was arbitrary it follows $\alpha_\lambda \in \Delta_M(I_0)$. Q.E.D.

Our construction of the net $\mathcal{M}$ is such that (Proposition 2.10 and the discussion in Subsection 2.4 in [1])

$$
[\theta] = \bigoplus_{p=0}^{n-1} [\sigma^p] \cdot \quad (24)
$$

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Lemma 3.12 For any $\lambda_0 \in \Delta_A(I_o)$ we have

$$\text{Hom}_{M(I_o)}(\alpha_\lambda, \alpha_\lambda) = \left\{ t = \sum_{p=0}^{n-1} \pi(T_p) f_1^p, \ T_p \in \text{Hom}_{A(I_o)}(\sigma_0^{-p} \circ \lambda_0, \lambda_0) \right\}.$$ (25)

Proof. Suppose $t \in \text{Hom}_{M(I_o)}(\alpha_\lambda, \alpha_\lambda)$. We can write $t = \sum_{p=0}^{n-1} \pi(T_p) f_1^p$ with $T_p \in A(I_o)$. Now from $t \cdot \alpha_\lambda \circ \pi(a) = \alpha_\lambda \circ \pi(a) \cdot t$ for all $a \in A(I_o)$ we obtain

$$\sum_{p=0}^{n-1} \pi(T_p) f_1^p \cdot \pi \circ \lambda_0(a) = \sum_{p=0}^{n-1} \pi(T_p \cdot \sigma_0^{-p} \circ \lambda_0(a)) f_1^p = \sum_{p=0}^{n-1} \pi(\lambda_0(a)T_p)f_1^p.$$

It follows $T_p \in \text{Hom}_{A(I_o)}(\sigma_0^{-p} \circ \lambda_0, \lambda_0)$. It remains to be shown that then $t\alpha_\lambda(f_1) = \alpha_\lambda(f_1)t$. From the BFE, Eq. (11) in [4], we obtain

$$\sigma_0(T_p) \epsilon(\sigma_0^{-p}, \sigma_0) \sigma_0^{-p}(\epsilon(\lambda_0, \sigma_0)) = \epsilon(\lambda_0, \sigma_0) T_p.$$

But

$$\epsilon(\sigma_0^{-p}, \sigma_0) \equiv \epsilon(\sigma_0^{n-p}, \sigma_0) = \epsilon(\sigma_0, \sigma_0) \sigma_0(\epsilon(\sigma_0, \sigma_0)) \cdots \sigma_0^{n-p-1}(\epsilon(\sigma_0, \sigma_0)) = 1$$

as $\epsilon(\sigma_0, \sigma_0) = 1$. Hence we find

$$\sigma_0(T_p) \sigma_0^{-p}(\epsilon(\lambda_0, \sigma_0)) = \epsilon(\lambda_0, \sigma_0) T_p.$$ 

Now we compute

$$t \alpha_\lambda(f_1) = \sum_{p=0}^{n-1} \pi(T_p) f_1^{p+1} \epsilon(\lambda, \sigma)$$

$$= \sum_{p=0}^{n-1} f_1 \cdot \pi \circ \sigma_0(T_p) \cdot f_1^p \epsilon(\lambda, \sigma)$$

$$= \sum_{p=0}^{n-1} f_1 \cdot \pi \circ \sigma_0(T_p) \cdot \sigma^{-p}(\epsilon(\lambda, \sigma)) f_1^p$$

$$= \sum_{p=0}^{n-1} f_1 \pi(\sigma_0(T_p)\sigma_0^{-p}(\epsilon(\lambda_0, \sigma_0))) f_1^p$$

$$= \sum_{p=0}^{n-1} f_1 \pi(\epsilon(\lambda_0, \sigma_0)T_p) f_1^p$$

$$= \sum_{p=0}^{n-1} f_1 \epsilon(\lambda, \sigma) \pi(T_p) f_1^p$$

$$= \alpha_\lambda(f_1) t,$$

thus we have indeed Eq. (25). Q.E.D.
For the fixed point $\rho_0$ of $\sigma_0$ we have $\text{Hom}_{A(I_0)}(\sigma_0 \circ \rho_0, \rho_0) = \mathbb{C}1$. By Lemma 3.12 it is not hard to see that then $\text{Hom}_{M(I_0)}(\alpha_0, \alpha_0)$ is an $n$-dimensional commutative algebra, i.e. $\text{Hom}_{M(I_0)}(\alpha_0, \alpha_0) \cong \mathbb{C} \oplus \mathbb{C} \oplus \ldots \oplus \mathbb{C}$, and therefore $[\alpha_0]$ decomposes in $n$ distinct irreducible sectors. Since $\sigma_{\alpha_0} = \gamma \circ \alpha_0|N = \theta \circ \rho$ and thus $[\sigma_{\alpha_0}] = [\theta \circ \rho] = n[\rho]$ we arrive at the following.

**Corollary 3.13** We have $[\alpha_0] = \bigoplus_{p=0}^{n-1} [\delta_p]$ with $[\delta_p]$ distinct and irreducible. Moreover, $[\sigma_{\delta_p}] = [\rho]$ for all $p = 0, 1, \ldots, n - 1$.

### 3.3 Spin and statistics

We found that the extended net is local (and even Haag dual) if and only if $\epsilon(\sigma_0, \sigma_0) = 1$. In fact $\epsilon(\sigma_0, \sigma_0)$ can be computed by the spin and statistics connection. In the following the conformal dimensions $h_\Lambda$, which are by definition the lowest eigenvalues of the rotation generator $L_0$ in the positive energy representations $(\pi_\Lambda, \mathcal{H}_\Lambda)$, $\Lambda \in \mathcal{A}^{(n+k)}$, will play an important role. They are given by

$$h_\Lambda = \frac{(\Lambda|\Lambda + 2\rho)}{2(k+n)},$$

where $\rho = \sum_{i=1}^{n-1} \Lambda(i)$ and $(\cdot|\cdot)$ is the symmetric bilinear form. Recalling that $(\Lambda(i)|\Lambda(j)) = i(n - j)/n$ for $1 \leq i, j \leq n - 1$ one may obtain for $\Lambda = \sum_{i=1}^{n-1} m_i \Lambda(i)$

$$h_\Lambda = \sum_{1 \leq i \leq j \leq n-1} m_i m_j \frac{i(n-j)}{n(k+n)} - \sum_{i=1}^{n-1} m_i^2 \frac{i(n-i)}{2(n(k+n))} + \sum_{i=1}^{n-1} m_i \frac{i(n-i)}{2(k+n)},$$

(26)

where we used the Dynkin labelling, i.e. $m_i \in \mathbb{N}_0$ and $\sum_{i=1}^{n-1} m_i \leq k$. Now let $\lambda_{0,\Lambda} \in \Delta_A(I_0)$ denote the endomorphisms corresponding to the positive energy representations $(\pi_\Lambda, \mathcal{H}_\Lambda)$, $\Lambda \in \mathcal{A}^{(n+k)}$. Then $\sigma_0 = \lambda_{0,k\Lambda(1)}$ is a simple current of order $n$, and its fusion rules correspond to the $\mathbb{Z}_n$-rotation of $\mathcal{A}^{(n+k)}$. It has a fixed point if $k$ is a multiple of $n$, namely $\rho_0 = \lambda_{0,k\Lambda_R}$, where $\Lambda_R = \frac{k}{n} \Lambda(1) + \frac{k}{n} \Lambda(2) + \ldots + \frac{k}{n} \Lambda(n-1)$. Therefore we first require $k \in n\mathbb{N}$ so that we can construct the extended net $\mathcal{M}$ by means of $\sigma_0$ as explained in the previous subsections. Then we ask when $\mathcal{M}$ is local.

**Proposition 3.14** The net $\mathcal{M}$ is local if and only if $k \in 2n\mathbb{N}$ if $n$ is even and $k \in n\mathbb{N}$ if $n$ is odd.

**Proof.** By Corollary 3.13 the net $\mathcal{M}$ is local if and only if $\epsilon(\sigma_0, \sigma_0) = 1$. Since $\sigma_0$ is an automorphism we have $\epsilon(\sigma_0, \sigma_0) = \kappa_{\sigma_0}1$, where $\kappa_{\sigma_0} \in \mathbb{C}$
is the statistical phase. By the conformal spin and statistics theorem [25] we have
\[ \kappa_{\sigma_0} = e^{2\pi i h_{\sigma_0}} \] where \( h_{\sigma_0} \) is the infimum of the spectrum of the rotation generator \( L_0 \) in the representation \( \pi_0 \circ \sigma_0 \). But this is the conformal dimension, \( h_{\sigma_0} = h_{k\Lambda(1)} \), and by Eq. (26)
\[ h_{k\Lambda(1)} = k \frac{n-1}{2n}. \]
Therefore \( \epsilon(\sigma_0, \sigma_0) = 1 \) if and only if \( k(n-1)/2n \in \mathbb{N} \), the statement follows.
Q.E.D.

The next step is to ask for which \( \Lambda \in A^{(n+k)} \) we have \( \alpha_{\Lambda} \equiv \alpha_{\lambda_0} \in \Delta_M(I_0) \). For \( \Lambda = m_1 \Lambda(1) + m_2 \Lambda(2) + \ldots + m_{n-1} \Lambda(n-1) \) we denote \( |\Lambda| = \sum_{i=1}^{n-1} i m_i \). Recall that the \( \mathbb{Z}_n \)-rotation \( \sigma \) on \( A^{(n+k)} \) is defined by \( \sigma(\Lambda) = (k - m_1 - \ldots - m_{n-1}) \Lambda(1) + m_1 \Lambda(2) + m_2 \Lambda(3) + \ldots + m_{n-2} \Lambda(n-1) \).

**Proposition 3.15** We have \( \alpha_{\Lambda} \in \Delta_M(I_0) \) if and only if \( |\Lambda| \in n\mathbb{Z} \).

**Proof.** By Lemma 3.11 we have \( \alpha_{\Lambda} \in \Delta_M(I_0) \) if and only if \( Y(\lambda_{0;\Lambda}, \sigma_0) = 1 \).
By Lemma 3.3 of [19] we have for any \( T \in \text{Hom}_{A(I_0)}(\lambda_{0;\sigma(\Lambda)}, \sigma_0 \circ \lambda_{0;\Lambda}) \)
\[ Y(\lambda_{0;\Lambda}, \sigma_0)T = \frac{K_{\lambda_{0;\sigma(\Lambda)}}}{K_{\sigma_0} K_{\lambda_{0;\Lambda}}} T, \]
where the \( \kappa \)'s are statistical phases. Since \( [\lambda_{0;\sigma(\Lambda)}] = [\sigma_0 \circ \lambda_{0;\Lambda}] \) and since \( \lambda_{0;\Lambda} \) is irreducible we can take \( T \) unitary, hence
\[ Y(\lambda_{0;\Lambda}, \sigma_0) = \frac{K_{\lambda_{0;\sigma(\Lambda)}}}{K_{\sigma_0} K_{\lambda_{0;\Lambda}}} 1. \]
Using again the conformal spin and statistics theorem we find
\[ \frac{K_{\lambda_{0;\sigma(\Lambda)}}}{K_{\sigma_0} K_{\lambda_{0;\Lambda}}} = e^{2\pi i (h_{\lambda_{0;\sigma(\Lambda)}} - h_{\sigma_0} - h_{\lambda_{0;\Lambda}})} \equiv e^{2\pi i (h_{\sigma(\Lambda)} - h_{k\Lambda(1)} - h_{\Lambda})}. \]
Now by Lemma 2.7 of [32] we have
\[ h_{\sigma(\Lambda)} - h_{\Lambda} = \frac{1}{n} \left( \frac{(n-1)k}{2} - |\Lambda| \right), \]
hence \( h_{\sigma(\Lambda)} - h_{k\Lambda(1)} - h_{\Lambda} = -|\Lambda|/n \). Therefore \( Y(\lambda_{0;\Lambda}, \sigma_0) = 1 \) if and only if \( |\Lambda| \in n\mathbb{Z} \).
Q.E.D.
Remark. If we label the positive energy representations of $LSU(n)$ at level $k$ by partitions (or Young tableaux) $(p_1, p_2, \ldots, p_{n-1})$ with $p_i = \sum_{j=i}^{n-1} m_j$ then Proposition 3.15 reads $\alpha(p_1, \ldots, p_{n-1}) \in \Delta_M(I_0)$ if and only if $\sum_{i=1}^{n-1} p_i \in n\mathbb{Z}$.

By Proposition 3.15 it should be clear that for the orbifold modular invariants the sectors corresponding to the marked vertices are (the irreducible subsectors of) $\alpha(p_1, \ldots, p_{n-1})$ with $\sum_{i=1}^{n-1} p_i \in n\mathbb{Z}$, as these $\alpha$-induced endomorphisms are localized and transportable endomorphisms of the extended net $\mathcal{M}$. Moreover, as we will see by the treatment of the examples, their $\sigma$-restriction corresponds to the block structure of the corresponding orbifold modular invariants. The $SU(n)_k$ sectors that do not appear in the blocks of the modular invariant can be identified as “twisted sectors” if we consider the $SU(n)_k$ theory as the $\mathbb{Z}_n$ orbifold of the extended theory. In fact, $\alpha$-induction of these sectors does not provide localized sectors; here we only obtain “solitonic” localization of the $\alpha$-induced endomorphisms.

For $SU(2)$ the positive energy representations are labelled by the spin $j \equiv m_1 = 0, 1, 2, \ldots, k$. Then Eq. (26) reduces to

$$h_j = \frac{j(j+2)}{4k+8}.$$ 

First we find by Proposition 3.14 that we can construct the local extended net for $k = 4q$, $q \in \mathbb{N}$, since then $h_k = q \in \mathbb{Z}$. The rotation $\sigma$ is now the flip $\sigma(j) = k-j$. Hence

$$h_{\sigma(j)} - h_j = \frac{(k-j)(k-j+2)}{4k+8} - \frac{j(j+2)}{4k+8} = \frac{k-2j}{4} = \frac{q-j}{2},$$

i.e. $\alpha_j \in \Delta_M(I_0)$ if and only if $j \in 2\mathbb{Z}$.

3.4 Examples

We now consider some examples for the application of $\alpha$-induction to the extended net coming from an orbifold block-diagonal modular invariant. The simplest case is the $SU(2)$ $D_4$ modular invariant but we have already discussed this case as the extended net here coincides with the net associated to the $SU(3)_1$ theory. The $D_{2q+2}$ modular invariants with $q > 1$ do not come from conformal inclusions. They appear at level $k = 4q$ and can be written as

$$Z_{D_{2q+2}} = \frac{1}{2} \sum_{k \geq j \geq 0 \atop j \in 2\mathbb{Z}} |\chi_j + \chi_{k-j}|^2.$$
Let us first illustrate this at the next case in the D-series, namely $D_6$. The $D_6$ invariant appears at level 8, thus we start with the fusion algebra $W(2,8)$. The simple current is given by $[\lambda_8]$ and indeed $h_8 = 2$. Eq. (24) now reads $[\theta] = [\lambda_0] \oplus [\lambda_8]$ and from this we get immediately that $[\alpha_4]$ decomposes into two irreducible sectors, say $[\alpha_4(1)] \oplus [\alpha_4(2)]$. All other $[\alpha_j]$ are irreducible and $[\alpha_8 - j] = [\alpha_j]$. The fusion rules involving $[\alpha_j]$, $j = 0, 1, 2, 3$, can be read off from those of $[\lambda_j]$ by the homomorphism property of $\alpha$-induction, so one only has to find the fusion rules involving $[\alpha_4(i)]$, $i = 1, 2$. One checks that the following fusion rules,

\[
\begin{align*}
[\alpha_1] \times [\alpha_4(i)] &= [\alpha_3], \quad i = 1, 2, \\
[\alpha_2] \times [\alpha_4(i)] &= [\alpha_2] \oplus [\alpha_4^{(i+1)}], \quad i = 1, 2 \text{ (mod 2)}, \\
[\alpha_3] \times [\alpha_4(i)] &= [\alpha_1] \oplus [\alpha_3], \quad i = 1, 2, \\
[\alpha_4(i)] \times [\alpha_4(i)] &= [\alpha_0] \oplus [\alpha_4(i)], \quad i = 1, 2, \\
[\alpha_4^{(1) i}] \times [\alpha_4^{(2) i}] &= [\alpha_2],
\end{align*}
\]

determine a well-defined fusion algebra with unit $[\alpha_0]$. The fusion graph of $[\alpha_1]$ is easily seen to be $D_6$, see Fig. 14.

For arbitrary $\varrho = 1, 2, 3, ..., k = 4\varrho$, the fusion algebra can be characterized as follows. We have $2\varrho + 2$ irreducible sectors $[\alpha_j]$, $j = 0, 1, 2, ..., 2\varrho - 1$, and $[\alpha_{2\varrho}]$ and $[\alpha_2^{(1)}]$. The fusion rules are given from those in $W(2,4\varrho)$, see Eq. (3), i.e.

\[
[\alpha_1] \times [\alpha_j] = \bigoplus_{j = |j - j_2|, j + j_1 + j_2 \text{ even}}^{\min(j_1 + j_2, 2k - (j_1 + j_2))} [\alpha_j],
\]

for $j = 0, 1, 2, ..., 2\varrho$, where we identify $[\alpha_{k-j}] = [\alpha_j]$ and $[\alpha_{2\varrho}] = [\alpha_{2\varrho}^{(1)}] \oplus [\alpha_{2\varrho}^{(2)}]$ on the r.h.s. Thus associativity, the homomorphism property of $[\alpha]$ and
compatibility with \((\alpha_j, \alpha_{j'})_{M(I_k)} = (\theta \circ \lambda_j, \lambda_{j'})_{N(I_k)}\) where \([\theta] = [\lambda_0] \oplus [\lambda_k]\) are automatically guaranteed, and the fusion graph of \([\alpha_1]\) is already determined to be \(D_{2\theta+2}\). We only have to specify the fusion rules involving the isolated \([\alpha_{2\theta}^{(i)}], i = 1, 2\). But it is shown in [27] that the fusion graph \(D_{2\theta+2}\) of \([\alpha_1]\) already determines all the (endomorphism) fusion rules; they are given by

\[
[\alpha_j] \times [\alpha_{2\theta}^{(i)}] = \begin{cases} 
[\alpha_{2\theta-j}] \oplus [\alpha_{2\theta-j+2}] \oplus \ldots \oplus [\alpha_{2\theta-3}] \oplus [\alpha_{2\theta-1}], & j \in 2\mathbb{Z} + 1 \\
[\alpha_{2\theta-j}] \oplus [\alpha_{2\theta-j+2}] \oplus \ldots \oplus [\alpha_{2\theta-2}] \oplus [\alpha_{2\theta}], & j \in 4\mathbb{Z} \\
[\alpha_{2\theta-j}] \oplus [\alpha_{2\theta-j+2}] \oplus \ldots \oplus [\alpha_{2\theta-2}] \oplus [\alpha_{2\theta}^{(i+1)}], & j \in 4\mathbb{Z} + 2 
\end{cases}
\]

for \(0 < j < 2\theta\) and \(i = 1, 2 \pmod{2}\). Of course \([\alpha_0] \times [\alpha_{2\theta}] = [\alpha_{2\theta}]\), and

\[
[\alpha_{2\theta}^{(i)}] \times [\alpha_{2\theta}^{(i)}] = \begin{cases} 
[\alpha_0] \oplus [\alpha_4] \oplus \ldots \oplus [\alpha_{2\theta-4}] \oplus [\alpha_{2\theta}], & \varrho = 2, 4, 6, \ldots \\
[\alpha_2] \oplus [\alpha_6] \oplus \ldots \oplus [\alpha_{2\theta-4}] \oplus [\alpha_{2\theta}^{(i+1)}], & \varrho = 1, 3, 5, \ldots 
\end{cases}
\]

\[
[\alpha_{2\theta}^{(i)}] \times [\alpha_{2\theta}^{(i+1)}] = \begin{cases} 
[\alpha_2] \oplus [\alpha_6] \oplus \ldots \oplus [\alpha_{2\theta-6}] \oplus [\alpha_{2\theta-2}], & \varrho = 2, 4, 6, \ldots \\
[\alpha_0] \oplus [\alpha_4] \oplus \ldots \oplus [\alpha_{2\theta-6}] \oplus [\alpha_{2\theta-2}], & \varrho = 1, 3, 5, \ldots 
\end{cases}
\]

for \(i = 1, 2 \pmod{2}\).

Next we consider the \(D^{(3\varrho+3)}\), \(\varrho \in \mathbb{N}\), (block-diagonal) modular invariant that appears at level \(k = 3\varrho\),

\[
Z_{D^{(3\varrho+3)}} = \frac{1}{3} \sum_{k \geq p \geq q \geq 0 \atop p+q \in 3\mathbb{Z}} |X(p,q) + X_\sigma(p,q) + X_{\sigma^2(p,q)}|^2,
\]

where \(\sigma\) is the \(\mathbb{Z}_3\) rotation of the \(A^{(k+3)}\) graph,

\[
\sigma(p,q) = (k-q, p-q), \quad \sigma^2(p,q) = (k-p, q, k-p).
\]

This is an orbifold invariant and it can be treated completely analogously to the \(D_{\text{even}}\) invariants of \(SU(2)\). The vacuum block gives us the \([\theta]\),

\[
[\theta] = [\lambda_{(0,0)}] \oplus [\lambda_{(k,0)}] \oplus [\lambda_{(k,k)}].
\]

Using the \(LSU(3)\) fusion rules at level \(k\), in particular those for the simple currents

\[
[\lambda_{(p,q)}] \times [\lambda_{(k,0)}] = [\lambda_{\sigma(p,q)}], \quad [\lambda_{(p,q)}] \times [\lambda_{(k,k)}] = [\lambda_{\sigma^2(p,q)}],
\]

we find

\[
\langle [\alpha_{(p,q)}], [\alpha_{(r,s)}] \rangle = \delta_{(p,q),(r,s)} + \delta_{\sigma(p,q),(r,s)} + \delta_{\sigma^2(p,q),(r,s)} = \delta_{p,r} \delta_{q,s} + \delta_{k-q,r} \delta_{p-q,s} + \delta_{k-p+r} \delta_{k-p,s}.
\]

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Hence we have identifications

\[ [\alpha_{(p,q)}] = [\alpha_{\sigma(p,q)}] = [\alpha_{\sigma^2(p,q)}] \]

and all \([\alpha_{(p,q)}]\) are irreducible apart from the fixed point \((p,q) = (2\varrho,\varrho)\),
where \(\langle [\alpha_{(2\varrho,\varrho)}], [\alpha_{(2\varrho,\varrho)}]\rangle = 3\), so that it decomposes into three irreducible
sectors as follows,

\[ [\alpha_{(2\varrho,\varrho)}] = [\alpha_{(1)}] \oplus [\alpha_{(2)}] \oplus [\alpha_{(3)}]. \]

One easily checks that the fusion graphs of \([\alpha_{(1,0)}]\) are the orbifold graphs \(D^{(k+3)}\) which were first discovered in [33] in the context of statistical mechanical models and then in [17] in the subfactor context.

4 Graphs and intertwining matrices

In this section we define several fusion matrices and study their properties. Using some ideas of Xu [52], we establish identities between these matrices which allow to identify them with certain matrices considered by Di Francesco and Zuber.

4.1 Some matrices and their properties

Let again \(W \equiv W(n,k) = \{[\lambda_\Lambda], \Lambda \in A^{(n+k)}\}\) be the canonical sector basis
for \(SU(n)_k\). Recall that the structure constants of \(W\) can be written as

\[ N^\Lambda_{\Lambda',\Lambda''} = \langle \lambda_\Lambda \circ \lambda_{\Lambda'}, \lambda_{\Lambda''} \rangle_{N(I_\Lambda)}, \quad \Lambda, \Lambda', \Lambda'' \in A^{(n+k)}, \]

and this defines matrices \(N_\Lambda\) by \((N_\Lambda)_{\Lambda',\Lambda''} = N^\Lambda_{\Lambda',\Lambda''}\). We set \(A_p = N_{\Lambda(p)}\),
for the fundamental weights \(\Lambda(p)\), \(p = 1, 2, ..., n - 1\). Note that \(A_1\) is the
adjacency matrix of the first fusion graph of the fundamental representation, i.e. of \(A^{(n+k)}\) considered as a graph. For either a conformal inclusion or an orbifold inclusion as discussed above let \(V\) denote the sector algebra with basis \(V\) obtained by \(\alpha\)-induction. We denote \(\alpha_\Lambda \equiv \alpha_{\lambda_\Lambda}\). First of all we claim

Lemma 4.1 For either a conformal or an orbifold inclusion, \(\alpha_{\Lambda(1)}\) is always irreducible.

Proof. Irreducibility of \(\alpha_{\Lambda(1)}\) means that \(\langle \alpha_{\Lambda(1)}, \alpha_{\Lambda(1)} \rangle_{M(I_\Lambda)} = 1\). We have

\[ \langle \alpha_{\Lambda(1)}, \alpha_{\Lambda(1)} \rangle_{M(I_\Lambda)} = \langle \theta \circ \lambda_{\Lambda(1)}, \lambda_{\Lambda(1)} \rangle_{N(I_\Lambda)} = \langle \theta, \lambda_{\Lambda(1)} \circ \lambda_{\Lambda(n-1)} \rangle_{N(I_\Lambda)} = 1 + \langle \theta, \lambda_{\Lambda(1)} + \Lambda_{(n-1)} \rangle_{N(I_\Lambda)}, \]
as \( [\Lambda_{(1)}] = [\lambda_{\Lambda_{(n-1)}}] \) and \( [\lambda_{\Lambda_{(1)}}] \times [\lambda_{\Lambda_{(n-1)}}] = [\lambda_0] \oplus [\Lambda_{\Lambda_{(1)}+\Lambda_{(n-1)}}] \) and since \([\lambda_0] = [id]\) appears in the decomposition of \([\theta]\) precisely once. Using the formula for the conformal dimension, Eq. (26), one checks that \( h_{\Lambda_{(1)}+\Lambda_{(n-1)}} = n/(k+n) \notin \mathbb{Z} \). However, all subsectors of \([\theta]\) must have integer conformal dimension (and this corresponds to T-invariance in the modular invariant picture): For the conformal inclusion case, the decomposition of \([\theta]\) corresponds to the decomposition of the restricted vacuum representation. In the orbifold inclusion case we have \([\theta] = \bigoplus_{p=0}^{n-1} [\sigma^p], [\sigma^p] = [\lambda_{k\Lambda_{(p)}}], \) and \( h_{k\Lambda_{(p)}} = kp(n-p)/2n \in \mathbb{Z} \) as \( k \in 2n\mathbb{N} \) if \( n \) is even and \( k \in n\mathbb{N} \) if \( n \) is odd. We conclude that \( \langle \theta, \lambda_{\Lambda_{(1)}+\Lambda_{(n-1)}} \rangle N(I_o) = 0 \), proving irreducibility of \( \alpha_{\Lambda_{(1)}} \).

Q.E.D.

We define the following collection of non-negative integers,

\[
V^b_{\Lambda,a} = \langle \beta_a \circ \alpha_{\Lambda}, \beta_b \rangle_{M(I_o)}, \quad \Lambda \in \mathcal{A}^{(n+k)}, \quad a, b \in \mathcal{V},
\]

where \( \beta_a \) are representative endomorphisms of \( a \equiv [\beta_a] \) (and we will use the label \( 0 \) for the identity sector of \( M(I_o) \) as well). This defines square matrices \( V_{\Lambda}, \Lambda \in \mathcal{A}^{(n+k)} \), by \((V_{\Lambda})_{a,b} = V^b_{\Lambda,a}\), as well as rectangular matrices \( V(a), a \in \mathcal{V} \), by \((V(a))_{\Lambda,b} = V^b_{\Lambda,a} \). Also, we set \( G_p = V_{\Lambda_{(p)}}, p = 1, 2, \ldots, n-1 \). Hence \( G_p \) is the adjacency matrix of the fusion graph of \( \alpha_{\Lambda_{(p)}} \).

**Lemma 4.2** The matrices \( V_{\Lambda} \) and \( V(a) \) have the following properties.

1. \( V_0 = 1_d \),

2. \( A_p V(a) = V(a) G_p, a \in \mathcal{V}, p = 1, 2, \ldots, n-1 \),

3. \( V_{\Lambda} V_{\Lambda'} = \sum_{\Lambda''} N_{\Lambda,\Lambda'}^{\Lambda''} \cdot V_{\Lambda''} \).

**Proof.** Ad 1. We obviously have \( V^b_{0,a} = \delta_{a,b} \) as \( \mathcal{V} \) is a sector basis.

Ad 2. We compute

\[
(A_p V(a))_{\Lambda,b} = \sum_{\Lambda' \in \mathcal{A}^{(n+k)}} (A_p)_{\Lambda,\Lambda'} V^b_{\Lambda',a} \\
= \sum_{\Lambda' \in \mathcal{A}^{(n+k)}} \langle \Lambda_{\Lambda'}, \lambda_{\Lambda_{(p)}}, \lambda_{\Lambda'} \rangle_{N(I_o)} \langle \beta_a \circ \alpha_{\Lambda'}, \beta_b \rangle_{M(I_o)} \\
= \langle \beta_a \circ \alpha_{\Lambda_{(p)}}, \beta_b \rangle_{M(I_o)} \\
= \sum_{c \in \mathcal{V}} \langle \beta_a \circ \alpha_{\Lambda}, \beta_{c} \rangle_{M(I_o)} \langle \beta_{c} \circ \alpha_{\Lambda_{(p)}}, \beta_b \rangle_{M(I_o)} \\
= \sum_{c \in \mathcal{V}} V_{\Lambda,a}^c (G_p)_{c,b} \\
= (V(a) G_p)_{\Lambda,b},
\]
where we used the fact that the \([\lambda]_\Lambda\)'s and \([\beta]_a\)'s constitute sector bases, and in the third equality we used the additive homomorphism property of \(\alpha\)-induction.

Ad 3. We compute

\[
(V_\Lambda V_{\Lambda'})_{a,b} = \sum_{c \in V} V^c_{\Lambda;a} V^b_{\Lambda';c} = \sum_{c \in V} \langle \beta_a \circ \alpha_\Lambda, \beta_c \rangle_{M(I_c)} \langle \beta_c \circ \alpha_{\Lambda'}, \beta_b \rangle_{M(I_c)} = \langle \beta_a \circ \alpha_\Lambda \circ \alpha_{\Lambda'}, \beta_b \rangle_{M(I_c)} = \sum_{\Lambda'' \in A(n+k)} N^{\Lambda''}_{\Lambda,\Lambda'} \langle \beta_a \circ \alpha_{\Lambda''}, \beta_b \rangle_{M(I_c)} = \sum_{\Lambda'' \in A(n+k)} N^{\Lambda''}_{\Lambda,\Lambda'} (V_{\Lambda''})_{a,b},
\]

where we again used the homomorphism property of \(\alpha\)-induction. Q.E.D.

By some abuse of notation we also denote the sector product matrices associated to \(V\) by \(N_a\), i.e. \((N_a)_{b,c} = N^c_{b,a}\) with

\[
N^c_{b,a} = \langle \beta_b \circ \beta_a, \beta_c \rangle_{M(I_c)}, \quad a, b, c \in V.
\]

Analogous to the commutative case \([30]\), these matrices realize the “regular” representation of the sector algebra \(\hat{V}\); we have

\[
N_a N_b = \sum_{c \in V} N^c_{a,b} \cdot N_c, \quad a, b \in V,
\]

since

\[
(N_a N_b)_{d,e} = \sum_{f \in V} N^f_{d,a} N^e_{f,b} = \sum_{f \in V} \langle \beta_d \circ \beta_a, \beta_f \rangle_{M(I_c)} \langle \beta_f \circ \beta_b, \beta_e \rangle_{M(I_c)} = \langle \beta_d \circ \beta_a \circ \beta_b, \beta_e \rangle_{M(I_c)} = \sum_{c \in V} N^c_{a,b} \langle \beta_d \circ \beta_c, \beta_e \rangle_{M(I_c)} = \sum_{c \in V} N^c_{a,b} (N_c)_{d,e},
\]

where we used that \(V\) is a sector basis.

Note that Lemma 4.2 (3.) reflects basically the homomorphism property of \(\alpha\)-induction. Using the decomposition of \([\alpha_\Lambda\] one can similarly derive \(V_\Lambda = \sum_{a \in V} V^a_{\Lambda;0} \cdot N_a\) for \(\Lambda \in A^{(n+k)}\). The following lemma reflects the commutativity of \([\alpha_\Lambda\] with each \([\beta]_a\), proven in Proposition 3.16 of [4].

**Lemma 4.3** We have \(V_\Lambda N_a = N_a V_\Lambda\) for any \(\Lambda \in A^{(n+k)}\) and \(a \in V\).
Proof. We compute

\[(V_{\Lambda N} a)_{b,c} = \sum_{d \in V} V_{\Lambda, d}^d N_{d,a}^c = \sum_{d \in V} \langle \beta_b \circ \Lambda \circ \beta_d \rangle_{M(I_o)} \langle \beta_d \circ \beta_a \circ \beta_c \rangle_{M(I_o)} \]

\[= \langle \beta_b \circ \beta_a \circ \beta_c \rangle_{M(I_o)} \]

\[= \sum_{d \in V} \langle \beta_b \circ \beta_a \circ \beta_d \rangle_{M(I_o)} \langle \beta_d \circ \alpha \Lambda \circ \beta_c \rangle_{M(I_o)} \]

\[= \sum_{d \in V} N_{b,a}^d \cdot V_{\Lambda, a}^c = (N_{a} V_{\Lambda})_{b,c} \]

where we used Proposition 3.16 of \[4\]. Q.E.D.

4.2 Modular invariants and exponents of graphs

Let us briefly recall some facts about fusion algebras (see e.g. \[30\]). If \(W\) is a fusion algebra with sector basis \(W = \{w_0, w_1, ..., w_{d-1}\}\) and structure constants \(N_{i,j}^k\) then the matrices \(N_i\) defined by \((N_i)_{j,k} = N_{i,j}^k\) form the regular representation of \(W\), and since they constitute a family of normal, commuting matrices they can be simultaneously diagonalized by a unitary matrix \(S\). Then the diagonal matrices \(S^* N_i S\) form a direct sum over all the irreducible (one-dimensional) representations of \(W\) i.e. over its characters. These representations \(\rho_j\) are labelled by \(j = 0, 1, ..., d - 1\) and are given by

\[\rho_j(w_i) = \frac{S_{i,j}}{S_{0,j}}, \quad i = 0, 1, 2, ..., d - 1,\]

where \(S_{i,j}\) are the matrix elements of \(S\).

Now let us start with a conformal or orbifold inclusion of \(SU(n)\) at level \(k\), and let \(V\) again denote the sector basis obtained by \(\alpha\)-induction from the sector basis \(W = W(n,k)\) corresponding to the positive energy representations of \(SU(n)_k\). Recall that \(T \subset V\) are the sectors corresponding to the marked vertices, generating a commutative sector subalgebra \(T \subset V\) by Theorem 4.3 of \[4\]; for details see also Subsection 2.1. Note that \(N_{b,a}^c = N_{c,a}^b\), thus \(N_{\pi}\) is the transpose matrix of \(N_a\). Since \(T\) is closed under conjugation and by Lemma 4.3, the matrices \(N_t, t \in T,\) and \(V_{\Lambda, \Lambda} \in A(n+k)\), form a family of normal, commuting matrices and hence can be simultaneously diagonalized in a suitable orthonormal basis that we denote by \(\{\psi^i, i = 1, 2, ..., D\}\); here \(D = |V|\). As the matrices \(V_{\Lambda}\) constitute a representation of the fusion algebra \(W \equiv W(n,k)\) by Lemma 4.2 they decompose in the one-dimensional irreducible representations \(\gamma_{\Phi}\) of \(W\), which are labelled by weights \(\Phi \in A(n+k)\) and are given by

\[\gamma_{\Phi}(\Lambda) = \frac{S_{\Lambda,\Phi}}{S_{0,\Phi}}, \quad \Lambda \in A(n+k),\]
where $S_{\Lambda,\Phi}$ denote the entries of the matrix $S$ that diagonalizes the fusion rules of the endomorphisms associated to the $\text{LSU}(n)$ level $k$ theory. Due to Wassermann’s result [50] these endomorphisms obey the fusion rules given by the Verlinde formula in terms of the modular S-matrix $S$, therefore the modular S-matrix $S$ diagonalizes the endomorphism fusion rules, i.e. we have indeed $S = S$.

We conclude that we have a map $\Phi : \{1, 2, ..., D\} \to A^{(n+k)}$, $i \mapsto \Phi(i)$, such that

$$V_{\Lambda} = \sum_{i=1}^{D} \gamma_{\Phi(i)}(\Lambda) \langle \psi^i | \psi^i \rangle, \quad \Lambda \in A^{(n+k)},$$

i.e. in components

$$V_{\Lambda}^{b} = \sum_{i=1}^{D} \frac{S_{\Lambda,\Phi(i)}^{a}}{S_{0,\Phi(i)}^{a}} \psi_{a}^{i} (\psi_{b}^{i})^{*}, \quad \Lambda \in A^{(n+k)}, \quad a, b \in V.$$

The image of $\Phi$ is the set of weights $\Omega \in A^{(n+k)}$ such that $\gamma_{\Omega}$ appears in the $V_{\Lambda}$’s. Since in particular $G_p = V_{\Lambda(p)}$, $p = 1, 2, ..., n - 1$, we call these weights $\Omega$ *exponents* and denote the set of exponents $\text{Exp} = \text{Im} \Phi$. In other words, $\text{Exp}$ labels the joint spectrum of the matrices $V_{\Lambda}$. Similarly, as the $N_t$’s with $t \in T$ give a representation of the fusion algebra $T$ of the extended theory by Theorem 4.3 of [4] we have a map $s : \{1, 2, ..., D\} \to T$, $i \mapsto s(i)$, such that

$$N_{t} = \sum_{i=1}^{D} \eta_{s(i)}(t) \langle \psi_{a}^{i} | \psi_{c}^{i} \rangle, \quad t \in T,$$

where

$$\eta_{s}(t) = \frac{S_{t,s}^{\text{ext}}}{S_{0,s}^{\text{ext}}}, \quad t \in T,$$

are the one-dimensional representations of $T$ and $S_{t,s}^{\text{ext}}$ denote the entries of a matrix $S^{\text{ext}}$ that diagonalizes the (endomorphism!) fusion rules of $T$. It is widely believed for general conformal field theories (and even conjectured e.g. in [20], Conjecture 4.48) that endomorphisms representing the sectors of a conformal field theory obey the Verlinde fusion rules given in terms of the modular S-matrix, i.e. that we can choose $S^{\text{ext}} = S^{\text{ext}}$, where $S^{\text{ext}}$ is the S-matrix coming from the modular transformation of the extended characters. However, a proof exists only for several particular cases, see below.

Thus we have in components

$$N_{a,t}^{c} = \sum_{i=1}^{D} \frac{S_{t,s(i)}^{\text{ext}}}{S_{0,s(i)}^{\text{ext}}} \psi_{a}^{i} (\psi_{c}^{i})^{*}, \quad t \in T, \quad a, c \in V.$$
For \( \Lambda \in \mathcal{A}^{(n+k)} \) and \( t \in \mathcal{T} \) we define \( \text{Eig}(\Lambda, t) \) to be the space spanned by those \( \psi^i \) which correspond simultaneous eigenvalues \( \gamma_\Lambda(\Lambda') \) of \( V_{\Lambda'} \) and \( \eta_t(\Lambda') \) of \( N_t \) for all \( t' \in \mathcal{T}, \Lambda' \in \mathcal{A}^{(n+k)} \), i.e.

\[
\text{Eig}(\Lambda, t) = \text{span}\{ \psi^i : i \in \Phi^{-1}(\Lambda) \cap s^{-1}(t) \},
\]

so in particular \( \text{Eig}(\Lambda, t) = 0 \) iff \( \Lambda \notin \text{Exp} \). So far the vectors \( \psi^i \) are fixed up to unitary transformations in each \( \text{Eig}(\Lambda, t) \).

**Lemma 4.4** We have \( \psi^i_t = \sum_{c \in \mathcal{V}} N^c_{0,t} \psi^i_c \) for any \( t \in \mathcal{T} \) and \( i = 1, 2, \ldots, D \).

**Proof.** Clearly we have \( N^c_{0,t} = \delta_{c,t} \). Hence

\[
\psi^i_t = \sum_{c \in \mathcal{V}} N^c_{0,t} \psi^i_c = \sum_{c \in \mathcal{V}} \sum_{j=1}^D \frac{S^t_{l,s(i)}}{S^t_{0,s(i)}} \psi^j_c (\psi^j_c)^* \psi^i_c = \sum_{j=1}^D \frac{S^t_{l,s(i)}}{S^t_{0,s(i)}} \delta_{k,j} = \frac{S^t_{l,s(i)}}{S^t_{0,s(i)}} \psi^i_t
\]

by orthonormality of the \( \psi^j_t \)'s. \( \quad \) Q.E.D.

Let \( \psi_0 = (\psi^i_0)_{i=1}^D \) denote the dual vector of 0-components, and we set

\[
\|\psi_0\|_{\Lambda,t} = \sqrt{\sum_{i \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)} |\psi^i_0|^2}.
\]

**Lemma 4.5** If \( \text{Eig}(\Lambda, t) \neq 0 \) for some \( \Lambda \in \mathcal{A}^{(n+k)} \) and \( t \in \mathcal{T} \) then \( \|\psi_0\|_{\Lambda,t} \neq 0 \).

**Proof.** Since \( N_a N_b = \sum_{c \in \mathcal{V}} N^c_{a,b} \cdot N_c \) and \( N^c_{a,t} = N^c_{0,a} \) for \( t \in \mathcal{T} \) by Theorem 4.3 of \( \prod (a, b, c) \in \mathcal{V} \) we have \( N_a N_t = N_t N_a \) for any \( a \in \mathcal{V} \). Hence we find for \( i \in \Phi^{-1}(\Lambda) \cap s^{-1}(t) \)

\[
V_{\Omega} N_a \psi^i = \gamma_\Lambda(\Omega) N_a \psi^i, \quad N_u N_a \psi^i = \eta_t(u) N_a \psi^i, \quad \Omega \in \mathcal{A}^{(n+k)}, \quad u \in \mathcal{T},
\]

i.e. \( N_a \psi^i \in \text{Eig}(\Lambda, t) \). In other words, the matrices \( N_a \) are block-diagonal in the basis \( \psi^i \) corresponding to the decomposition in \( \text{Eig}(\Lambda, t) \). It follows that there are matrices, the “blocks” \( B_a \equiv B_a(\Lambda, t) \), \( (B_a)_{i,j} = B^i_{a,j} \in \mathbb{C}, \ i, j \in \Phi^{-1}(\Lambda) \cap s^{-1}(t) \) such that \( N_a \psi^i = \sum_{j \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)} B^i_{a,j} \psi^j \), hence in particular for the 0-components

\[
(N_a \psi^i)_0 = \sum_{j \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)} B^i_{a,j} \psi^j_0.
\]

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Since \((N_a \psi_i)^0 = \sum_{c \in V} N_{a,0}^c \psi_c^i = \psi_a^i\) we have for any \(i \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)\) and any \(a \in V\)
\[
\psi_a^i = \sum_{j \in h^{-1} \cap s^{-1}(t)} B_{a,j}^i \psi_0^j.
\]

It follows if \(\psi_0^j = 0\) for all \(j \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)\) then \(\psi_a^i = 0\) for all \(i \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)\) and \(a \in V\), i.e. \(\text{Eig}(\Lambda, t) = 0\).

We set \(D_{\Lambda, t} = \dim \text{Eig}(\Lambda, t) \equiv |\Phi^{-1}(\Lambda) \cap s^{-1}(t)|.\) Our vectors \(\psi^i\) are fixed up to unitary transformations \((\phi, \psi) \rightarrow \phi \cdot \psi^i, \psi^i \rightarrow \sum_{j \in h^{-1} \cap s^{-1}(t)} u_{i,j} \psi^j\), with unitary matrices \(u = (u_{i,j})_{i,j \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)}\). Thus we have in particular \(\psi_0^i \rightarrow \sum_{j \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)} u_{i,j} \psi_0^j\). This is a rotation in \(C^{D_{\Lambda, t}}\) on the sphere of radius \(\|\psi_0\|_{\Lambda, t}\). As we have shown that \(\|\psi_0\|_{\Lambda, t} \neq 0\) if \(\text{Eig}(\Lambda, t) \neq 0\) we arrive at the following

**Corollary 4.6** There is a choice of eigenvectors \(\psi^i\) such that \(\psi_0^i \neq 0\) for all \(i = 1, 2, \ldots, D\), e.g. \(\psi_0^i = D_{\Lambda, t}^{-1/2} \|\psi_0\|_{\Lambda, t} > 0\) whenever \(i \in \Phi^{-1}(\Lambda) \cap s^{-1}(t)\).

As we now can divide by \(\psi_0^i\) we obtain immediately from Lemma 4.4 the following

**Corollary 4.7** For such a choice we have for any \(t \in T\) and any \(b, c \in V\)
\[
N_{a,t}^c = \sum_{i=1}^{D} \frac{\psi_a^i \psi_i^c (\psi_i^c)^*}{\psi_0^i}.
\]

Let \(\chi_t^{\text{ext}}, t \in T\), denote the characters of the extended theory and \(\chi_\Lambda, \Lambda \in \mathcal{A}^{(n+k)}\), those of \(\text{SU}(n)_k\). Further, let \(b_{t, \Lambda}\) denote the branching coefficients, defined by \(\chi_t^{\text{ext}} = \sum_{\Lambda \in \mathcal{A}^{(n+k)}} b_{t, \Lambda} \chi_\Lambda.\) Let also \(S^{\text{ext}}\) be the modular \(S\)-matrix of the extended theory, i.e. (in the notation of [27])
\[
\chi_t^{\text{ext}} \left( \frac{1}{t}, \frac{z}{t}, \frac{(z|z)}{t} \right) = \sum_{v \in T} S^{\text{ext}}_{t,v} \chi_t^{\text{ext}}(t, z, u).
\]

**Lemma 4.8** For any \(\Lambda \in \mathcal{A}^{(n+k)}\) and \(u \in T\) we have
\[
\sum_{v \in \mathcal{T}} S^{\text{ext}}_{u,v} b_{v,\Lambda} = \sum_{\Omega \in \mathcal{A}^{(n+k)}} b_{u,\Omega} S_{\Omega,\Lambda}.
\]
Proof. This is essentially the computation in [27], p. 268, here for the special case that the branching functions are constants. By taking the S-transformation on both sides of \( \chi_{\text{ext}}^u = \sum_{\Omega \in \mathcal{A}^{(n+k)}} b_{u,\Omega} \chi_{\Omega} \) we obtain

\[
\sum_{v \in \mathcal{T}} S_{u,v}^{\text{ext}} \chi_{v}^{\text{ext}} = \sum_{v \in \mathcal{T}} \sum_{\Lambda \in \mathcal{A}^{(n+k)}} S_{u,v}^{\text{ext}} b_{v,\Lambda} \chi_{\Lambda} = \sum_{\Lambda, \Omega \in \mathcal{A}^{(n+k)}} b_{u,\Omega} S_{\Omega,\Lambda} \chi_{\Lambda} .
\]

Since the full (not the Virasoro specialized!) characters \( \chi_{\Lambda} \) are linearly independent functions the coefficients must coincide, so we are done. Q.E.D.

Note that \( V_{\Lambda,0} = \langle \alpha_{\Lambda}, \beta_t \rangle_{M(I_0)} = \langle \lambda_{\Lambda}, \sigma_{\beta_t} \rangle_{N(I_0)} \) by \( \alpha \sigma \)-reciprocity, thus we find for the branching coefficients \( b_{t,\Lambda} = V_{t,\Lambda;0} \). Let \( \tilde{N}_t \) denote the restriction of the matrices \( N_t \) to \( \mathcal{T} \), i.e. \( (\tilde{N}_t)_{u,v} = N_{u,t}^{v} \), \( t, u, v \in \mathcal{T} \).

**Lemma 4.9** Provided that \( S^{\text{ext}} \) diagonalizes the fusion matrices \( \tilde{N}_t \), \( t \in \mathcal{T} \), i.e. \( S^{\text{ext}} = \sum_{\Omega \in \mathcal{A}^{(n+k)}} (n+\kappa) b_{u,\Omega} \chi_{\Omega} \), we have for \( \Lambda \in \mathcal{A}^{(n+k)} \) and \( t \in \mathcal{T} \)

\[
b_{t,\Lambda} = \frac{\| \psi_0 \|_{\Lambda,t}^2}{S_{0,t}^{\text{ext}} S_{0,\Lambda}} .
\]

**Proof.** Exploiting \( S_{u,t}^{\text{ext}} = S_{u,t}^{\text{ext}} \), multiplying Eq. (28) by \( (S_{u,t}^{\text{ext}})^* \) and summing over \( u \in \mathcal{T} \) yields

\[
b_{t,\Lambda} = \sum_{u \in \mathcal{T}} \sum_{\Omega \in \mathcal{A}^{(n+k)}} (S_{u,t}^{\text{ext}})^* b_{u,\Omega} S_{\Omega,\Lambda}
= \sum_{u \in \mathcal{T}} \sum_{\Omega \in \mathcal{A}^{(n+k)}} (S_{u,t}^{\text{ext}})^* V_{\Omega,0}^{\mu} S_{\Omega,\Lambda}
= \sum_{u \in \mathcal{T}} \sum_{\Omega \in \mathcal{A}^{(n+k)}} \sum_{i=1}^{D} (S_{u,t}^{\text{ext}})^* \delta_{\lambda,\phi(i)} S_{\Omega,\Lambda}^{\mu} \psi_{\Omega(0)}^{*} \psi_{\Omega}^{(i)} S_{\Omega,\Lambda}
= \sum_{u \in \mathcal{T}} \sum_{i=1}^{D} (S_{u,t}^{\text{ext}})^* \psi_{\Omega(0)}^{*} \psi_{\Omega}^{(i)} S_{\Omega,\Lambda}^{\mu} S_{\Omega,\Lambda}^{\mu} \delta_{\lambda,\phi(i)}
= \sum_{i=1}^{D} \delta_{t,s(i)} \delta_{\lambda,\phi(i)} \| \psi_0 \|_{s(i)}^2 S_{0,\Lambda}^{\mu} S_{0,\Lambda}^{\mu}
= \| \psi_0 \|_{s(i)}^2 S_{0,\Lambda}^{\mu} S_{0,\Lambda}^{\mu} ,
\]

where we used \( b_{u,\Omega} = V_{\Omega,0}^{u} = V_{\Omega,0}^{\mu} \) and Lemma 4.4. Q.E.D.

Recall that the mass matrix of the modular invariant is given by \( Z_{\Lambda,N'} = \sum_{t \in \mathcal{T}} b_{t,\Lambda} b_{t,N'} \). We can now summarize Lemmata 4.3 and 4.4 in the following
Theorem 4.10 Provided that $S^{\text{ext}}$ diagonalizes the fusion matrices $\tilde{N}_t$, $t \in T$, i.e. $S^{\text{ext}} = S^{\text{ext}}$, we have $b_{t,A} \neq 0$ if and only if $\text{Eig}(\Lambda, t) \neq 0$. In particular $Z_{\Lambda, \Lambda} \neq 0$ if and only if $\Lambda \in \text{Exp}$.

Actually we would like to prove a stronger statement than Theorem 4.10, namely $b_{t,A} = \sqrt{D_{\Lambda,t}}$ because this equality holds in all our examples we have investigated so far. Let us explain why it holds in our examples. Let

$$\text{tr} Z = \sum_{\Lambda \in A^{(n+k)}} Z_{\Lambda, \Lambda} = \sum_{t \in T} \sum_{\Lambda \in A^{(n+k)}} b_{t, \Lambda}^2.$$ 

We clearly have $\sum_{t \in T} \sum_{\Lambda \in A^{(n+k)}} D_{\Lambda,t} = D = |V|$, and it is a simple observation that $\text{tr} Z = D$ in all our examples, hence

$$\sum_{t \in T} \sum_{\Lambda \in A^{(n+k)}} b_{t, \Lambda}^2 = \sum_{t \in T} \sum_{\Lambda \in A^{(n+k)}} D_{\Lambda,t}.$$ 

Thus, if all $b_{t,A} \in \{0, 1\}$ then our derived equivalence of $b_{t,A} \neq 0$ and $D_{\Lambda,t} \neq 0$ in Theorem 4.10 implies $b_{t,A} = \sqrt{D_{\Lambda,t}}$ for all $t \in T$, $\Lambda \in A^{(n+k)}$. The only case of our examples where some $b_{t,A} > 1$ appears is the conformal embedding $SU(4)_4 \subset SO(15)_1$ where the spinor $(s)$ representation of $SO(15)_1$ restricts to two copies of $\pi(3,2,1)$, i.e. $b_{s,(3,2,1)} = 2$. Because of Theorem 4.10 we have $D_{\Lambda,t} \geq b_{t, \Lambda}^2$ for all pairs $(t, \Lambda) \neq (s, (3,2,1))$. However, Petkov and Zuber [12] found a multiplicity 4 of the exponent $(3,2,1)$ in the graphs of Figs. 12 and 13, i.e. $\sum_{t \in T} D_{(3,2,1),t} = 4$. But since $b_{t,(3,2,1)} = 0$ for $t \neq s$ implies $D_{(3,2,1),t} = 0$ for $t \neq s$ it follows $D_{(3,2,1),s} = 4$, and hence indeed $b_{t,A} = \sqrt{D_{\Lambda,t}}$ for all $t \in T$, $\Lambda \in A^{(4+4)}$ because $\text{tr} Z = D = 12$. Nevertheless we have not succeeded in proving this equality for the general case.

4.3 Discussion and consequences

Let us now summarize some of the results of this section. To each block-diagonal modular invariant of $SU(n)$, coming either from a conformal inclusion or being of $\mathbb{Z}_n$-orbifold type, we have a net of subfactors such that we can apply $\alpha$-induction. By doing this, we obtain in particular a set of $n - 1$ normal, mutually commuting matrices $G_p$, $p = 1, 2, \ldots, n - 1$, which can be interpreted as adjacency matrices of fusion graphs, namely those of $[\alpha_{\Lambda(p)}]$ in the sector algebra $V$. Since $[\pi_{\Lambda(p)}] = [\alpha_{\Lambda(n-p)}]$ we find $G_p^t = G_{n-p}$. The matrices $G_p$ can be simultaneously diagonalized in an orthonormal basis $\{\psi^i, \ i = 1, 2, \ldots, |V|\}$, and the eigenvalues of $G_p$ are given by $S_{\Lambda(p), \Phi}/S_{0, \Phi}$, $\Phi \in \text{Exp}$, where $\text{Exp}$ is a subset of $A^{(n+k)}$. 

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Recall that one can define a $\mathbb{Z}_n$-valued colouring $\tau$ on the vertices of $A^{(n+k)}$ (which we can identify with the elements of $W$) by $\tau(\Lambda) = |\Lambda| \mod n$. Then one has $\tau(0) = 0$ and $N_{\Lambda,\Lambda'}^{\Lambda''} = \langle \lambda_{\Lambda} \circ \lambda_{\Lambda'}, \lambda_{\Lambda''} \rangle_{N(I_t)} = 0$ if $\tau(\Lambda) + \tau(\Lambda') \neq \tau(\Lambda'')$. If $[\theta]$ decomposes only into sectors $[\lambda_{\Lambda}]$ of colour zero then the elements of $V$ inherit the colouring from $A^{(n+k)}$: The colour of $[\beta] \in V$ is set to be $\tau(\Lambda)$ if $[\beta]$ appears in $[\alpha_{\Lambda}]$. This is then well defined because $\langle \alpha_{\Lambda}, \alpha_{\Lambda'} \rangle_{M(I_t)} = \langle \theta \circ \lambda_{\Lambda}, \lambda_{\Lambda'} \rangle_{N(I_t)} = 0$ if $\tau(\Lambda) \neq \tau(\Lambda')$. That $[\theta]$ decomposes only into sectors of colour zero is true for all orbifold inclusions and also all conformal embeddings considered here. Therefore the matrices $G_p$ satisfy all the axioms postulated in [42]. However, as already mentioned there, there are also counter examples e.g. the conformal embedding $SU(9)_1 \subset (E_8)_1$ where $[\theta]$ has also constituents of non-zero colour.

The sector algebra $V$ possesses a subalgebra given by the fusion algebra of the sectors of the extended net which restrict to the relevant sectors of the net of the $SU(n)$ theory. If the corresponding fusion matrices, coming from these sector products, are diagonalized by the modular $S$-matrix $S_{\text{ext}}$ of the extended characters then the non-zero diagonal entries $Z_{\Lambda,\Lambda'}$ of the modular invariant are precisely those with $\Lambda \in \text{Exp}$. We conjecture that the modular $S$-matrix $S_{\text{ext}}$ always diagonalizes the extended (endomorphism) fusion rules, but let us point out the cases where it has already been proven.

First we consider the modular invariants coming from conformal inclusions. It follows from Wassermann’s results [50] that in particular the endomorphisms of any $LSU(m)$ level 1 theory satisfy the $(\mathbb{Z}_m)$ fusion rules of the Verlinde formula, thus we have $S_{\text{ext}} = S_{\text{ext}}$ for all conformal inclusions $SU(n) \subset G$ with $G = SU(m)$ for some $m$. This covers the infinite series of inclusions

$$SU(n)_{n-2} \subset SU(n(n-1)/2)_1 \quad \text{and} \quad SU(n)_{n+2} \subset SU(n(n+1)/2)_1.$$ 

By the result of [3], the endomorphisms of the $LSO(m)$ level 1 theories satisfy the well-known $SO(m)_1$ fusion rules, hence $S_{\text{ext}} = S_{\text{ext}}$ also for $G = SO(m)$. This covers the infinite series of inclusions

$$SU(n)_n \subset SO(n^2 - 1)_1,$$

and also $SU(2)_{10} \subset SO(5)_1$. Moreover, we have seen from the treatment of the $E_8$ modular invariant of $SU(2)$ (at level $k = 28$) that the endomorphisms of the $L(G_2)$ level 1 theory obey the Lee-Yang fusion rules, thus we have $S_{\text{ext}} = S_{\text{ext}}$ also for the conformal inclusion $SU(2)_{28} \subset (G_2)_1$.

Now let us turn to the orbifold modular invariants. Unfortunately, our results are only complete for $SU(2)$. We have seen that the fusion algebra
V for the $D_{2\theta+2}$ modular invariants is completely determined although the homomorphism $[\alpha]$ is not surjective. The modular S-matrices $S^{\text{ext}}$ of the extended characters are known and their Verlinde fusion rules are given in \cite{1}. They coincide exactly with the fusion rules of the sectors $[\alpha_j]$, $j = 0, 2, 4, \ldots, 2\theta - 2$, and $[\alpha_{2\theta}]$, the “marked vertices”, which we gave in Subsection 3.4. Thus we have $S^{\text{ext}} = S^{\text{ext}}$ also for these cases. Summarizing we found that $S^{\text{ext}} = S^{\text{ext}}$ holds for all block-diagonal modular invariants of $SU(2)$, hence its diagonal entries are labelled by some subset $\text{Exp} \subseteq A_{k+1}$. As we have seen for the (non-trivial) block-diagonal modular invariants that $G_1 = N_{[\alpha_1]}$ is the adjacency matrix of the Coxeter graphs $E_6$, $E_8$ or $D_{\text{even}}$ (in fact since they are fusion graphs of norm $d_{[\alpha_1]}^2 = d_1^2 = 4 \cos^2(\pi/(k + 2))$ they can only be these graphs), the set $\text{Exp}$ is necessarily given by the Coxeter exponents of these graphs. Thus our theory explains in particular why the spins of the diagonal entries of the non-trivial block-diagonal modular invariants are given by the Coxeter exponents of the graphs $E_6$, $E_8$ and $D_{2\theta+2}$, $\theta \in \mathbb{N}$.

5 Other applications

We shall also discuss some other examples for the application of $\alpha$-induction which may be of some interest of their own.

5.1 Inclusions of extended $U(1)$ theories

Let $A_N$, $N = 1, 2, 3, \ldots$, denote the extension of the $U(1)$ current algebra discussed in \cite{1}. It has $2N$ sectors constituting $\mathbb{Z}_{2N}$ fusion rules. The characters are given by

$$K^{(N)}_a(q) = \frac{1}{\eta} \sum_{m \in \mathbb{Z}} q^{(a+2mN)^2/4N}, \quad a \in \mathbb{Z}_{2N},$$

where $\eta$ is Dedekind’s function. The modular invariant partition functions of these theories have been classified \cite{11}. For each factorization $N = \ell^2 p p'$, $\ell, p, p' \in \mathbb{N}$, $p$ and $p'$ coprime, associate $r, r' \in \mathbb{Z}$ such that $r' p' - r p = 1$. Define $s = r' p' + r p$. Then

$$Z^{(N)}(\ell, s) = \sum_{a, b \in \mathbb{Z}_{2N}} Z^{(N)}_{a, b}(\ell, s) K^{(N)}_a K^{(N)}_b$$

with

$$Z^{(N)}_{a, b}(\ell, s) = \begin{cases} \sum_{c \in \mathbb{Z}_d} \delta_{a, ab+2cN/\ell} & \text{if } \ell | a \text{ and } \ell | b \\ 0 & \text{otherwise} \end{cases}$$
exhaust all modular invariants. Note that $\ell = 1$, $p = N$, $p' = 1$, implying $r = 0$, $r' = 1$, $s = 1$, gives the diagonal modular invariant $Z(N)(1, 1)$. Now choose an $N$ such that $\ell \neq 1$ so that $\mathcal{A}_N$ is a non-maximal $U(1)$-extension in the terminology of [4]. Choose $p = N/\ell^2$, $p' = 1$ implying $r = 0$, $r' = 1$ and $s = 1$. The corresponding partition function reads

$$Z(N)(\ell, 1) = \sum_{a \in \mathbb{Z}_{2p}} \left| \sum_{c \in \mathbb{Z}_\ell} K^{(N)}_{a + 2cN/\ell} \right|^2. \quad (30)$$

But

$$\sum_{c \in \mathbb{Z}_\ell} K^{(\ell^2p)}_{a + 2cN/\ell}(q) = \eta^{-1} \sum_{c \in \mathbb{Z}_\ell} \sum_{m \in \mathbb{Z}} q^{(a\ell + 2c\ell^2p + 2m\ell^2p)^2/4\ell^2p} = \eta^{-1} \sum_{m \in \mathbb{Z}} q^{(a + 2mp)^2/4p} = K^{(p)}_a(q),$$

hence

$$Z^{(\ell^2p)}(\ell, 1) = \sum_{a \in \mathbb{Z}_{2p}} |K^{(p)}_a|^2 = Z^{(p)}(1, 1).$$

Indeed the inclusion $\mathcal{A}_{N=\ell^2p} \subset \mathcal{A}_p$ is of $\mathbb{Z}_\ell$ type. Note that $Z^{(\ell^2p)}(\ell, 1)$ is block-diagonal, and we can take the net of inclusions of local algebras $\mathcal{A}_N(I) \subset \mathcal{A}_p(I)$, with $I \in \mathcal{J}_z$, as our net of subfactors $\mathcal{N} \subset \mathcal{M}$.

Let us denote by $\lambda^{(N)}_a$ the endomorphisms (which are in fact the automorphisms constructed in [4]) corresponding to the sectors labelled by $a \in \mathbb{Z}_{2N}$. The $\mathbb{Z}_{2N}$ fusion rules then just read

$$[\lambda^{(N)}_a] \times [\lambda^{(N)}_b] = [\lambda^{(N)}_{a+b}], \quad a, b \in \mathbb{Z}_{2N}.$$ 

Thus the associated fusion algebra $W^{(N)}$ is the group algebra of $\mathbb{Z}_{2N}$ (with $\mathbb{Z}_{2N}$ as sector basis). Now we want to apply the machinery of $\alpha$-induction. For a non-maximal $\mathcal{A}_N$, $N = \ell^2p$, we start with the block-diagonal partition function in Eq. (30) and read off the $[\theta]$ from the vacuum block,

$$[\theta] = \bigoplus_{c \in \mathbb{Z}_\ell} [\lambda^{(N)}_{2c\ell p}]. \quad (31)$$

It is easy to see that the formula $\langle \alpha^{(N)}_a, \alpha^{(N)}_b \rangle_{M(I_0)} = \langle \theta \circ \alpha^{(N)}_a, \alpha^{(N)}_b \rangle_{N(I_0)}$ (we denote $\alpha^{(N)}_a \equiv \alpha^{(N)}_{\lambda^{(N)}_a}$) and the homomorphism property of $[\alpha]$ determine the induced sector algebra $V$ to be the group algebra of $\mathbb{Z}_{2\ell p} = \mathbb{Z}_{2N}/\mathbb{Z}_\ell$. Now $\mathbb{Z}_{2\ell p}$ has the subgroup $\mathbb{Z}_{2p}$, describing the fusion rules of $\mathcal{A}_p$, and this corresponds to the marked vertices.

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As an illustration we discuss the $\mathbb{Z}_2$ inclusion $\mathcal{A}_4 \subset \mathcal{A}_1$. The $\mathcal{A}_1$ theory has two sectors and it is known to be precisely the $\mathfrak{su}(2)_1$ theory. The $\mathcal{A}_4$ theory has eight sectors, and their conformal weights $\Delta_\nu$ are given by $\Delta_a = a^2/16$, $a = 0, \pm 1, \pm 2, \pm 3, 4$. Indeed the series of $\mathbb{Z}_2$ orbifold inclusions $SO(2n)_2 \subset SU(2n)_1$ gives the inclusion $\mathcal{A}_4 \subset \mathcal{A}_1$ when $n = 1$. The sectors of $\mathcal{A}_4$ labelled by $a = 0, \pm 2, 4$ are the basic ($v$), the spinor ($s,c$) and the vector ($v$) modules, the sectors labelled by $\pm 1$ and $\pm 3$ are the twisted sectors $\sigma, \tau$ and and $\sigma', \tau'$, respectively, in the terminology of $SO(2n)_2$. The modular invariant $Z^{(4)}(2,1)$ of $\mathcal{A}_4$ reads

$$Z^{(4)}(2,1) = |K_0^{(4)} + K_4^{(4)}|^2 + |K_2^{(4)} + K_{-2}^{(4)}|^2 = |K_0^{(1)}|^2 + |K_1^{(1)}|^2 = Z^{(1)}(1,1).$$

From $[\theta] = [\lambda_0^{(4)}] \oplus [\lambda_4^{(4)}]$ we obtain that $V$ is the group algebra of $\mathbb{Z}_4$. The sectors $[\lambda_0^{(4)}]$, $[\lambda_4^{(4)}]$ and $[\lambda_{2}^{(4)}]$, $[\lambda_{-2}^{(4)}]$, obtained from $[\lambda_1^{(1)}]$ and $[\lambda_{0}^{(1)}]$ by $\sigma$-restriction, yield irreducible sectors $[\alpha_0^{(4)}] = [\alpha_4^{(4)}]$ and $[\alpha_2^{(4)}] = [\alpha_{-2}^{(4)}]$, respectively, and constitute the $\mathbb{Z}_2 \subset \mathbb{Z}_4$ subgroup. In the fusion graph of $[\alpha_1^{(4)}]$, being the $\mathbb{Z}_4$ graph $\mathcal{A}_4^{(1)}$, these sectors represent the marked vertices.

The sectors $[\lambda_{1\pm 3}^{(4)}]$ are not obtained by $\sigma$-restriction of any $\mathcal{A}_1$ sectors. Note that these are precisely the twisted sectors. Correspondingly, $[\alpha_1^{(4)}] = [\alpha_1^{(4)}]$ and $[\alpha_3^{(4)}] = [\alpha_{-3}^{(4)}]$ yield the elements of the sector basis $V$ of $V$ which are not represented by marked vertices.

These observations generalize as follows to the block-diagonal modular invariant $Z^{(N)}(\ell,1)$ for $N = \ell^2 p$. We have seen that we then can consider $\mathcal{A}_N$ as the $\mathbb{Z}_\ell$ orbifold theory of $\mathcal{A}_p$. Reading off the $[\theta]$, Eq. (37), from the vacuum block we obtain that $V$ is the group algebra of $\mathbb{Z}_2\ell p$. The irreducible sectors $[\alpha_a^{(N)}]$ with $a$ a multiple of $\ell$, i.e. $a \in \mathbb{Z}_{2p} \subset \mathbb{Z}_{2\ell p}$, are represented as marked vertices in the fusion graph of $[\alpha_1^{(N)}]$. Correspondingly, the sectors $[\lambda_a^{(N)}]$, $a \in \mathbb{Z}_{2p}$, are obtained by $\sigma$-restriction of the sector $[\lambda_0^{(p)}]$ of $\mathcal{A}_p$. Considering $\mathcal{A}_N$ as the $\mathbb{Z}_\ell$ orbifold of $\mathcal{A}_p$, we can interpret the other sectors $[\lambda_b^{(N)}]$, $b \notin \mathbb{Z}_{2p}$, as twisted sectors.

### 5.2 Minimal models

We shall also briefly discuss the treatment of the minimal models here. The minimal models are described by the positive energy representations of the diffeomorphism group of the circle $\text{Diff}(S^1)$, or, on the level of Lie algebras, by the unitary highest weight modules of the Virasoro algebra $\mathfrak{Vir}(c)$, where
the central charge \( c \equiv c(m) \) is given by
\[
c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, 5, \ldots
\]
These models arise as coset theories \[^2\]
\[
\frac{SU(2)_{m-2} \otimes SU(2)_1}{SU(2)_{m-1}}.
\]
The modules \( H_{p,q} \), appearing at fixed \( m \), are labelled by pairs of integers \( p = 0, 1, 2, \ldots, m - 2 \) and \( q = 0, 1, 2, \ldots, m - 1 \), the conformal grid. We have a double counting, \( H_{p,q} = H_{m-p-2,m-q-1} \).

In the setting of local von Neumann algebras the minimal models have been treated in \[^3\] quite analogously to the treatment of the loop groups \( SU(n) \) by Wassermann. Here we are dealing with a net \( \mathcal{N} \) of local von Neumann algebras \( \mathcal{N}(I) = \pi_0(\text{Diff}(I(S^1)))' \), where \( \pi_0 \) is the vacuum representation of \( \text{Diff}(S^1) \) and \( \text{Diff}(I(S^1)) \) is the subgroup of diiffeomorphisms concentrated on an interval \( I \subset S^1 \). Analogously to the arguments for \( SU(n) \) we have Haag duality in the vacuum representation and the positive energy representations correspond localized, transportable endomorphisms. The well known fusion rules are proven in the bimodule setting in \[^3\] and hence they give the correct fusion rules for the corresponding sectors, explicitly,
\[
[\lambda_{p,q}] \times [\lambda_{p',q'}] = \bigoplus_{r=|p-p'|}^{\min(p+p',2m-p-p'-4)} \bigoplus_{s=|q-q'|}^{\min(q+q',2m-q-q'-2)} [\lambda_{r,s}]
\]
where \( \lambda_{p,q} \in \Delta_N(I_0) \) denote the endomorphisms associated to \( H_{p,q} \). This determines the fusion algebra \( W_{\text{Vir}(c(m))} \).

The modular invariants of the minimal models are classified, and are labelled by pairs \((G_1, G_2)\) of ADE-graphs (with Coxeter numbers \( m - 2 \) and \( m - 1 \)) \[^7\]. If we write the \( SU(2) \) modular invariants appearing at level \( k \) and labelled by ADE-graphs \( \mathcal{G} \) as
\[
Z_{\mathcal{G}} = \sum_{j=0}^{k} Z_{j,j'}^{(k)}(\mathcal{G}) \chi_j \bar{\chi}_{j'}
\]
then the \((G_1, G_2)\) modular invariants of the minimal model with \( c = c(m) \) is given by
\[
Z_{G_1,G_2} = \frac{1}{2} \sum_{p,p'=0}^{m-2} \sum_{q,q'=0}^{m-1} Z_{p,p'}^{(m-2)}(G_1) Z_{q,q'}^{(m-1)}(G_2) \chi_{p,q} \bar{\chi}_{p',q'}
\]

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Table 1: The sets $\Gamma_k(\mathcal{G})$

| Level | Graph $\mathcal{G}$ | $\Gamma_k(\mathcal{G})$ |
|-------|---------------------|-----------------------|
| $k = 1, 2, 3, \ldots$ | $A_{k+1}$ | $\{0\}$ |
| $k = 4^q, \ q = 1, 2, 3, \ldots$ | $D_{2q+2}$ | $\{0, 4^q\}$ |
| $k = 10$ | $E_6$ | $\{0, 6\}$ |
| $k = 28$ | $E_8$ | $\{0, 10, 18, 28\}$ |

where $\chi_{p,q}$ denotes the character of $H_{p,q}$. The prefactor $1/2$ is due to the double counting. Since either $m - 2$ or $m - 1$ is odd either $\mathcal{G}_1$ or $\mathcal{G}_2$ is necessarily an A-graph.

We would like to apply the procedure of $\alpha$-induction. Although these are block diagonal modular invariants we do not always know what the net $\mathcal{M}$ is. Recall that for $SU(2)$ the $E_6$ and $E_8$ modular invariants come from the conformal embedding $SU(2) \subset G$ where $G = SO(5)$ or $G_2$, respectively. So it is natural to ask whether there is an extension of $\mathfrak{Vir}$ for the $(\mathcal{G}_1, \mathcal{G}_2)$ modular invariants where $\mathcal{G}_1$ or $\mathcal{G}_2$ is $E_6$ or $E_8$. For the $(E_6, A_{12})$- and $(E_8, A_{30})$-invariants the natural candidate is the coset

$$\frac{G_1 \otimes SU(2)_1}{SU(2)_{m-1}}$$

where $G_1 = SO(5)_1$ and $m = 12$, or $G_1 = (G_2)_1$, $m = 30$, respectively. However, for the $(A_{10}, E_6)$ and $(A_{28}, E_8)$ modular invariants there is no such natural candidate. For any block diagonal modular invariant of the minimal models we proceed by assuming that there is a net $\mathcal{M}$ such that the net of subfactors $\mathcal{N} \subset \mathcal{M}$ has the correct properties, in particular, that the blocks correspond to $\sigma$-restriction of representations of the net $\mathcal{M}$. Then we may go on as follows: Let $\Gamma_k(\mathcal{G})$ denote the set of integers $j$ with $[\lambda_j]$ appears in the $[\theta]$ we associated to the $SU(2)$ $\mathcal{G}$ modular invariant, see Table 1. For the $(\mathcal{G}_1, \mathcal{G}_2)$ modular invariant of the minimal model with $c = c(m)$ define $[\theta] \in [\Delta]_{\mathcal{N}(I_0)}$ by

$$[\theta] = \bigoplus_{p \in \Gamma_{m-2}(\mathcal{G}_1)} \bigoplus_{q \in \Gamma_{m-1}(\mathcal{G}_2)} [\lambda_{p,q}],$$

so that $[\theta]$ precisely correspond to the vacuum block in $\mathcal{Z}_{\mathcal{G}_1, \mathcal{G}_2}$. (Note that one of the summations is always trivial as either $\mathcal{G}_1$ or $\mathcal{G}_2$ is an A-graph.)
Then we determine the induced fusion algebra \( V \) by \( \langle \alpha_{p,q}, \alpha_{p'q'} \rangle_{M(I_0)} = \langle \theta \circ \lambda_{p,q}, \lambda_{p'q'} \rangle_{N(I_0)} \) where we denote \( \alpha_{p,q} \equiv \alpha_{\lambda_{p,q}} \).

We have to choose an analogue of the fundamental representation of \( LSU(n) \) for the minimal models. It is instructive to discuss briefly the fusion graphs of \( [\alpha_{\lambda}] \) for the choices \( \lambda = \lambda_{0,1}, \lambda_{1,0}, \lambda_{1,1}. \) First one checks that then \( [\alpha_{\lambda}] \) is irreducible, \( \langle \alpha_{\lambda}, \alpha_{\lambda} \rangle_{M(I_0)} = 1, \) in all cases. Now \( [\alpha_{0,1}] \) (\( [\alpha_{1,0}] \)) generates the fusion subalgebra corresponding to the first column (row) of the conformal grid, being isomorphic to \( W(2, m-1) \) (\( W(2, m-2) \)). Thus the fusion graph of \( [\alpha_{0,1}] \) (\( [\alpha_{1,0}] \)) is not connected; the identity component is just \( G_2 \) (\( G_1 \)). Now consider \( [\lambda_{1,1}] \) that is \( [\lambda_{1,1}] = [\lambda_{0,1}] \times [\lambda_{1,0}] \). The fusion graph of \( [\alpha_{1,1}] \) is somehow a combination of the graphs \( G_1 \) and \( G_2 \). As an illustration, we give the result for the \( (A_4, D_4) \) modular invariant (\( m = 5, c = 4/5 \))

\[
Z_{(A_4, D_4)} = \frac{1}{2} \left( \sum_{p=0}^{3} |\chi_{p,0} + \chi_{p,4}|^2 + 2 |\chi_{p,2}|^2 \right)
\]

Then \( [\theta] = [\lambda_{0,0}] \oplus [\lambda_{0,4}] \), and we find eight distinct irreducible sectors, \( [\alpha_{0,0}], [\alpha_{1,1}], [\alpha_{2,0}], [\alpha_{3,1}], [\alpha_{0,2}] \pm, [\alpha_{2,2}] \pm \). The fusion graph of \( [\alpha_{1,1}] \) is given in Fig. 15.

### 6 Outlook

We have applied the procedure of \( \alpha \)-induction and \( \sigma \)-restriction of sectors to chiral conformal field theory models, in particular to the \( SU(n)_k \) WZW
models. Looking at the block-diagonal modular invariants arising from conformal or orbifold inclusions of $SU(n)$, we have seen that their classification by certain fusion graphs — in particular the A-D-E classification in the $SU(2)$ case — can be understood by the induction-restriction machinery of the relevant sectors. However, many questions remain unanswered. The induction turns out to be non-surjective in several cases; this is apparently closely related to multiplicities in the mass matrix $Z$, but a good understanding of this non-surjectivity (which can even lead to non-commutativity of the induced sector algebra) is still missing. It might be possible to extract more information about the structure of the induced fusion algebra from the $SU(n)_k$ data than our results in [4] like the main reducibility formula or $\alpha \sigma$-reciprocity provide. In fact, the observation $\text{tr} \, Z = D \equiv |V|$ is still awaiting a good explanation.

It will certainly be worth looking also at the block-diagonal $SU(n)$ modular invariants that come neither from conformal nor from orbifold embeddings. Moreover, it is not clear at the moment how to incorporate the type II modular invariants in our framework. Another challenging question, suggested by the treatment of $SU(2)$, concerns a better understanding of the relation between the appearance of modular invariants of $SU(n)$ WZW models and the existence of sub-(equivalent)-paragroups of the paragroups arising from the relevant $A$-type subfactors ([29, 39]). Of course it will also be interesting to construct the associated fusion graphs also for modular invariants of other Lie groups, e.g. $Sp(n)$.

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References

[1] Bauer, E., Gepner, D.: Fusion Rules for Extended Current Algebras.
   Mod. Phys. Lett. A 11 (1996) 1929-1946
[2] Bisch, D., Jones, V.F.R.: Algebras Associated to Intermediate Subfactors. Invent. Math. 128 (1997) 89-157

[3] Böckenhauer, J.: An Algebraic Formulation of Level One Wess-Zumino-Witten Models. Rev. Math. Phys. 8 (1996) 925-947

[4] Böckenhauer, J., Evans, D.E.: Modular Invariants, Graphs and α-Induction for Nets of Subfactors I. Preprint, hep-th/9801171, to appear in Commun. Math. Phys.

[5] Brunetti, R., Guido, D., Longo, R.: Modular Structure and Duality in Conformal Quantum Field Theory. Commun. Math. Phys. 156 (1993) 201-219

[6] Buchholz, D., Mack, G., Todorov, I.: The Current Algebra on the Circle as a Germ of Local Field Theories. Nucl. Phys. B (Proc. Suppl.) 5B (1988) 20-56

[7] Cappelli, A., Itzykson, C., Zuber, J.-B.: The A-D-E Classification of Minimal and $A^{(1)}_1$ Conformal Invariant Theories. Commun. Math. Phys. 113 (1987) 1-26

[8] Date, E., Jimbo, M., Miwa, T., and Okado, M.: Solvable lattice models. Theta functions — Bowdoin 1987, Part 1, Proceedings of Symposia in Pure Mathematics Vol. 49, American Mathematical Society, Providence, R.I. (1987) 295–332

[9] Di Francesco, P: Integrable Lattice Models, Graphs and Modular Invariant Conformal Field Theories. Int. J. Mod. Phys. A 7 (1992) 407-500

[10] Di Francesco, P., Mathieu, P., Sénéchal, D.: Conformal Field Theory. Springer-Verlag, New York 1996

[11] Di Francesco, P., Saleur, H., Zuber, J.-B.: Modular Invariance in Non-Minimal Two-Dimensional Conformal Field Theories. Nucl. Phys. B 285 (1987) 454-480

[12] Di Francesco, P., Zuber, J.-B.: $SU(N)$ Lattice Integrable Models Associated with Graphs. Nucl. Phys. B 338 (1990) 602-646

[13] Di Francesco, P., Zuber, J.-B.: $SU(N)$ Lattice Integrable Models and Modular Invariants. In: Randjbar, S. et al (eds.): Recent
Developments in Conformal Field Theories. World Scientific 1990, 179-215

[14] Doplicher, S., Haag, R., Roberts, J.E.: Fields, Observables and Gauge Transformations II. Commun. Math. Phys. 15 (1969) 173-200

[15] Evans, D.E., Gould, J. D.: Dimension Groups and Embeddings of Graph Algebras. Intern. J. Math. 5 (1994) 291-327

[16] Evans, D.E., Kawahigashi, Y.: The $E_7$ Commuting Squares Produce $D_{10}$ as Principal Graph. Publ. RIMS, Kyoto Univ. 30 (1994) 151-166

[17] Evans, D.E., Kawahigashi, Y.: Orbifold subfactors from Hecke algebras. Commun. Math. Phys. 165 (1994) 445-484

[18] Evans, D.E., Kawahigashi, Y.: Quantum Symmetries on Operator Algebras. Oxford Univ. Press, to appear

[19] Fredenhagen, K., Rehren, K.-H., Schroer, B.: Superselection Sectors with Braid Group Statistics and Exchange Algebras II. Rev. Math. Phys. Special Issue (1992) 113-157

[20] Fröhlich, J., Gabbiani, F.: Operator Algebras and Conformal Field Theory. Commun. Math. Phys. 155 (1993) 569-640

[21] Gannon, T.: WZW Commutants, Lattices and Level-One Partition Functions. Nucl. Phys. B 396 (1993) 708-736

[22] Gannon, T.: The Classification of Affine $SU(3)$ modular invariants. Commun. Math. Phys. 161 (1994) 233-264

[23] Goddard, P., Kent, A., Olive, D.: Virasoro Algebras and Coset Space Models. Phys. Lett. B 152 (1985) 88-93

[24] Goodman, F.M., de la Harpe, P., Jones, V.F.R.: Coxeter Graphs and Towers of Algebras. Springer-Verlag, New York 1989

[25] Guido, D., Longo, R.: The Conformal Spin and Statistics Theorem. Commun. Math. Phys. 181 (1996) 11-35

[26] Izumi, M.: Application of Fusion Rules to Classification of Subfactors. Publ. RIMS, Kyoto Univ. 27 (1991) 953-994
[27] Kac, V.G.: *Infinite Dimensional Lie Algebras*. 3rd edition, Cambridge University Press 1990

[28] Kawahigashi, Y.: *Classification of Paragroup Actions on Subfactors*. Publ. RIMS, Kyoto Univ. **31** (1995) 481-517

[29] Kawahigashi, Y.: *Quantum Galois Correspondence for Subfactors*. In preparation.

[30] Kawai, T.: *On the Structure of Fusion Algebras*. Phys. Lett. **B 217** (1989) 247-251

[31] Klümper, A., Pearce, P.A.: *Conformal Weights of RSOS Lattice Models and their Fusion Hierarchies*. Physica **A 183** (1992) 304-350

[32] Kohno, T., Takata, T.: *Symmetry of Witten’s 3-Manifold Invariants for sl(n,C)*. Journal of Knot Theory and Its Ramifications **2** (1993) 149-169

[33] Kostov, I.K.: *Free Field Presentation of the An Coset Models on the Torus*. Nucl. Phys. **B 300** (1988) 559-587

[34] Loke, T.: *Operator Algebras and Conformal Field Theory of the Discrete Series Representations of Diff(S^1)*. Dissertation Cambridge (1994)

[35] Longo, R., Rehren, K.-H.: *Nets of Subfactors*. Rev. Math. Phys. **7** (1995) 567-597

[36] Moore, G., Seiberg, N.: *Naturality in Conformal Field Theory*. Nucl. Phys. **B 313** (1989) 16-40

[37] Nahm, W.: *A Proof of Modular Invariance*. Int. J. Mod. Phys. **A 6** (1991) 2837-2845

[38] O’Brian, L., Pearce, P.A.: *Lattice Realizations of Unitary Minimal Modular Invariant Partition Functions*. J. Phys. **A 28** (1995) 4891-4906

[39] Ocneanu, A.: *Paths on Coxeter Diagrams: From Platonic Solids and Singularities to Minimal Models and Subfactors*. Lectures given at the Fields Institute (1995), notes recorded by S. Goto.

[40] Pasquier, V.: *Operator Content of the ADE Lattice Models*. J. Phys. **A 20** (1987) 5707-5717
[41] PEARCE, P.: Recent Progress in Solving A-D-E Lattice Models. Physica A 205 (1994) 15-30

[42] PETKOVA, V.B., ZUBER, J.-B.: From CFT to Graphs. Nucl. Phys. B 463 (1996) 161-193

[43] PETKOVA, V.B., ZUBER, J.-B.: Conformal Field Theory and Graphs. In: Proceedings Goslar 1996 “Group 21”

[44] PRESSLEY, A., SEGAL, G.: Loop Groups. Oxford University Press 1986

[45] REHREN, K.-H.: Space-Time Fields and Exchange Fields. Commun. Math. Phys. 132 (1990) 461-483

[46] REHREN, K.-H.: Markov Traces as Characters for Local Algebras. Nucl. Phys. B (Proc. Suppl.) 18B (1990) 259-268

[47] REHREN, K.-H.: Subfactors and Coset Models. In: DOEBNER, H.-D. ET AL (EDS.): Generalized Symmetries in Physics. World Scientific 1994, 338-356

[48] TAKESAKI, M.: Conditional Expectations in von Neumann Algebras. J. Funct. Anal. 9 (1972) 306-321

[49] WASSERMANN, A.: Subfactors Arising from Positive Energy Representations of Some Infinite Dimensional Groups. Unpublished notes 1990

[50] WASSERMANN, A.: Operator Algebras and Conformal Field Theory III. To appear in Invent. Math.

[51] WENZL, H.: Hecke algebras of type An and subfactors. Invent. Math. 92 (1988) 345-383

[52] XU, F.: New Braided Endomorphisms from Conformal Inclusions. Commun. Math. Phys. 192 (1998) 349-403