Uniqueness of minimal Fourier-type extensions in $L_1$-spaces

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Abstract We give a characterization of uniqueness of finite rank Fourier-type minimal extensions in $L_1$-norm. This generalizes the main result obtained by Lewicki (Proceedings of the Fifth International Conference on Function Spaces, Lecture Notes in Pure and Applied Mathematics, vol. 213, pp. 337–345, 1998) to the case of $n$-circular sets in $\mathbb{C}^n$.

Keywords Fourier projection · Minimal extension · Uniqueness of minimal extension

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1 Introduction

We start with some notation which will be used in this paper. By $S_X(x, r)$ ($S_X$ if $x = 0$ and $r = 1$) we denote a sphere in a Banach space $X$ with the center $x$ and the radius $r$, by $\text{ext}(S_X)$ the set of extreme points of $S_X$ and the symbol $X^*$ stands for a dual

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space of $X$. An element $x \in X$ is called a norming point for $f \in X^*$ if $x \in S_X$ and $f(x) = \|x\|$. For $z \in \mathbb{C}$, $\text{sgn} z = \overline{z}/|z|$ for $z \neq 0$ and 0 for $z = 0$.

Let $Y$ be a linear subspace of $X$ and let $\mathcal{L}(X, Y)$ ($\mathcal{L}(X)$ if $X = Y$) be the space of all linear, continuous operators from $X$ into $Y$. Given $A \in \mathcal{L}(Y)$, an operator $P \in \mathcal{L}(X, Y)$ is called an extension of $A$ (or a projection in the case of $A = \text{Id}_Y$) if $P|_Y = A$. The set of all extensions of $A$ will be denoted by $\mathcal{P}_A(X, Y)$. An extension $P_0 \in \mathcal{P}_A(X, Y)$ is minimal if

$$\|P_0\| = \lambda_A(X, Y) = \inf\{|P| : P \in \mathcal{P}_A(X, Y)\},$$

(1)

We write briefly $\mathcal{P}(X, Y)$ and $\lambda(Y, X)$ instead of $\mathcal{P}_{\text{Id}_Y}(X, Y)$ and $\lambda_{\text{Id}_Y}(Y, X)$ respectively. Basic results on minimal projections and extensions can be found in [1,7,9,12,13,15–18,20,22,24]. Set

$$\mathcal{L}_Y(X, Y) = \{L \in \mathcal{L}(X, Y) : L|_Y = 0\}.$$  

(2)

Let $\pi_n$ denote the space of all trigonometric polynomials of degree $\leq n$ and let $\mathcal{C}_0(2\pi)$ be the space of all continuous, real valued, $2\pi$-periodic functions. The classical Fourier projection from $\mathcal{C}_0(2\pi)$ onto $\pi_n$ is defined by a formula

$$(F_n f)(t) = (f \ast D_n)(t) = (1/2\pi) \int_0^{2\pi} f(s) D_n(t - s) \, ds,$$  

(3)

where $D_n(t) = \sum_{j=-n}^{n} e^{ijt}$. It is well-known that $F_n$ is the unique operator of minimal norm in the space $\mathcal{P}(\mathcal{C}_0(2\pi), \pi_n)$ [10,23]. Moreover, $\|F_n\| = \lambda(\pi_n, L_p[0, 2\pi])$ for $1 \leq p \leq \infty$, which follows from Rudin Theorem [8], however, in general, it is an open question if $F_n$ is the unique minimal projection from $L_p[0, 2\pi]$ onto $\pi_n$ for $p \neq 1, 2, +\infty$. Partial results concerning subject can be found in [26] and [27].

In this paper we study the problem of the unique minimality of the Fourier-type extensions in the space $L_1$. More precisely, let $M$ be a set, $\Sigma$-$\sigma$-algebra of subsets of $M$, $\nu$ a positive measure on $\Sigma$ such that $(M, \Sigma, \nu)$ is a complete measure space. By $L_1(M, \Sigma, \nu)$ denote a space of complex-valued, $\nu$-measurable functions on $M$ satisfying a condition

$$\|f\|_1 = \int_M |f(z)| \, d\nu(z) < +\infty.$$  

To the end of this paper we assume that $(L_1(M, \Sigma, \nu))^* = L_\infty(M, \Sigma, \nu)$, which is satisfied, for example, if $\nu$ is $\sigma$-finite.

**Definition 1** It is said that $w \in V \subset L_\infty(M, \Sigma, \nu), w \neq 0$ is determined by its roots in the space $V$ if and only if for any $g \in V$ the condition $g/w \in L_\infty(M, \Sigma, \nu)$ implies that $g = cw$ for some $c \in \mathbb{C}$.

**Definition 2** A subspace $V \subset L_1(M, \Sigma, \nu)$ is called smooth if and only if each member of $V \setminus \{0\}$ is almost everywhere different from 0.
Take $V$ a smooth, finite-dimensional subspace of $L_1(M, \Sigma, \nu)$ with a basis \{${v_1, \ldots, v_k}$\} and fix an operator $A \in \mathcal{L}(V)$. Observe that any $P \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ has a form

$$Pf = \sum_{j=1}^{k} \hat{u}_j(f)v_j, \quad \hat{u}_j(v_i) = a_{j,i}, \quad i, j = 1, \ldots, k,$$

where $\{a_{i,j}\}_{i,j=1}^{k}$ is a matrix of the operator $A$ in the basis $\{v_1, \ldots, v_k\}$ and $\hat{u} \in (L_1(M, \Sigma, \nu))^*$ denotes a functional associated with $u \in L_\infty(M, \Sigma, \nu)$ by

$$\hat{u}(f) = \int_M f(z)u(z) \, d\nu(z).$$

(5)

Let $P \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ be given by (4) and $z, w \in M$. Define

$$x^P_z(w) = \sum_{j=1}^{k} u_j(z)v_j(w),$$

$$V_j(z) = \int_M v_j(w)\text{sgn}(x^P_z(w)) \, d\nu(w),$$

$$P_z = \int_M |x^P_z(w)| \, d\nu(w).$$

(6)

A map $z \to P_z$ is called the Lebesgue function of the operator $P$. It is well known that

$$\|P\| = \text{ess sup}_{z \in M} P_z \text{ (see [14, Lem. 2]).}$$

(7)

**Lemma 3** Assume that $P_0 \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ and the Lebesgue function of the operator $P_0$ is constant on $M$ (v a.a.) Let $P_1, P_2 \in \mathcal{P}_A(L_1(M, \Sigma, \nu), V)$, $\|P_1\| = \|P_2\| = \|P_0\|$ and $P_0 = (P_1 + P_2)/2$. Then the Lebesgue functions of the operators $P_j : j = 1, 2$ are constant on $M$ and for $\nu$ a.a. $z \in M$,

$$\text{sgn}(x^{P_1}_z) = \text{sgn}(x^{P_2}_z) = \text{sgn}(x^{P_0}_z).$$

**Proof** Let

$$P_j(f) = \sum_{i=1}^{k} \hat{u}_{ji}(f)v_i : j = 1, 2$$
for some $u_{ji} \in L_\infty(M, \Sigma, \nu)$: $j = 1, 2, i = 1, \ldots, k$ [see (5)]. Then

$$P_0(f) = \frac{P_1(f) + P_2(f)}{2} = \sum_{i=1}^{k} \frac{1}{2} (\hat{u}_{1i}(f) + \hat{u}_{2i}(f))v_i.$$ 

For $\nu$ a.a. $z \in M$,

$$\|P_0\| = (P_0)_z = \int_{M} \left| \sum_{i=1}^{k} \frac{1}{2} [u_{1i}(z) + u_{2i}(z)]v_i(w) \right| d\nu(w) \leq \frac{1}{2} \int_{M} \left| \sum_{i=1}^{k} u_{1i}(z)v_i(w) \right| d\nu(w) + \frac{1}{2} \int_{M} \left| \sum_{i=1}^{k} u_{2i}(z)v_i(w) \right| d\nu(w) = \frac{1}{2} (P_1)_z + \frac{1}{2} (P_2)_z \leq \frac{1}{2} (\|P_1\| + \|P_2\|) = \|P_0\|,$$

so in the above inequalities we get equalities. In particular, for $\nu$ a.a. $z, w \in M$,

$$\left| \sum_{i=1}^{k} [u_{1i}(z) + u_{2i}(z)]v_i(w) \right| = \left| \sum_{i=1}^{k} u_{1i}(z)v_i(w) \right| + \left| \sum_{i=1}^{k} u_{2i}(z)v_i(w) \right|,$$

or equivalently

$$\text{sgn}(x_z^{P_1}) = \text{sgn}(x_z^{P_2}) = \text{sgn}(x_z^{P_0})$$

and

$$(P_1)_z = (P_2)_z = \|P_0\|.$$

\[\square\]

2 Main results

Now we introduce some notation which we will use in this section. We write briefly $re^{it} \in \mathbb{C}^n$ instead of $(r_1 e^{i t_1}, \ldots, r_n e^{i t_n}) \in \mathbb{C}^n$, and put $r = (r_1, \ldots, r_n) \in [0, \infty)^n$, $t = (t_1, \ldots, t_n) \in I$, $I = [0, 2\pi]^n$. The symbol $\mathbb{T}$ stands for the unit circle in a complex plane, i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

**Definition 4** A subset $Z$ of $\mathbb{C}^n$ is called an $n$-circular set if for any $(z_1, \ldots, z_n) \in Z$ and $(\delta_1, \ldots, \delta_n) \in \mathbb{T}^n$, $(\delta_1 z_1, \ldots, \delta_n z_n)$ belongs to $Z$.

A unit ball with $p$-norm for $p \geq 1$, $\mathbb{T}^n$ or $D_1 \times \cdots \times D_n$, where $D_j \subset \mathbb{C}$ for $j = 1, \ldots, n$ denotes a classical geometric ring, are the examples of the $n$-circular sets.
Observe that any $n$-circular set $Z$ can be written in a form

$$Z = \{re^{it} : r \in W \subset [0, \infty)^n, \ t \in I\}.$$  \hfill (8)

Let $\lambda_W$ be a nonnegative measure on $W$ such that $0 < \lambda_W(W) < \infty$. For example, for $Z = \{z \in \mathbb{C} : |z| \leq 1\}$ and a Borel set $A \subset [0, 1]$, $\lambda_W(A) = \int_A r \, dr$ or for $Z = \bigcup_{j=1}^p \{z \in \mathbb{C} : |z| = r_j\}$, $\lambda_W$ is a counting measure on $W = \{r_1, \ldots, r_p\}$. Let $\lambda_I$ denote the normalized Lebesgue measure on $I$. Set

$$\mu = \lambda_W \times \lambda_I \text{ (the product measure of } \lambda_W \text{ and } \lambda_I \text{ on the set } W \times I).$$  \hfill (9)

Define a measure $\nu$ on $Z$ associated with $\mu$ by

$$\nu(A) = \mu(\{(r, t) \in W \times I : re^{it} \in A\}).$$  \hfill (10)

Throughout the remainder of this paper the symbol $L_1(Z)$ stands for the space of all $\nu$-measurable complex-valued functions on $Z$ and such that

$$\|f\|_1 = \int_Z |f(z)| \, d\nu(z) = \int_{W \times I} |f(re^{it})| \, d\mu(r, t) < \infty.$$

For $\beta \in \mathbb{N}^n, \alpha \in \mathbb{Z}^n$ define a function $e^{\beta,\alpha} \in L_1(Z)$ by

$$e^{\beta,\alpha}(re^{it}) = e^{\beta}(r)e^{\alpha}(e^{it}), \text{ where } e^{\gamma}(z) = \prod_{j=1}^n z^{\gamma_j} \text{ for } \gamma \in \mathbb{Z}^n, \ z \in \mathbb{C}^n \setminus \{0\}. \hfill (11)$$

Fix for $j = 1, \ldots, k a_j \in \mathbb{C}, \beta_j \in \mathbb{N}^n$ and $\alpha^j \in \mathbb{Z}^n$, $\alpha^i \neq \alpha^j$ for $i \neq j$. Set

$$V = \text{span}\{e^{\beta^1,\alpha^1}, \ldots, e^{\beta^k,\alpha^k}\}$$  \hfill (12)

and

$$w = \sum_{j=1}^k a_j e^{\beta^j,\alpha^j}. \hfill (13)$$

Define for $t \in I$ an operator $T_t : L_1(Z) \to L_1(Z)$ by

$$T_t(f)(ue^{is}) = f(u e^{i(s+t)}), \ s \in I.$$  \hfill (14)

Observe that $T_t$ is an isometry and $V$ is an invariant subspace of $T_t$. One can find that

$$T_t(e^{\beta,\alpha}) = e^{\beta,\alpha} \cdot e^{\alpha}(e^{it}).$$  \hfill (15)
Now we will search for a minimal extension of an operator $R_w = F_w | V$, where $F_w \in \mathcal{L}(L_1(Z), V)$ is given by

$$
(F_w f)(re^{it}) = (f * w)(re^{it}) = \iint_{W \times I} f(u e^{is}) w(re^{i(t-s)}) d\mu(u, s).
$$

(16)

**Remark 5** Let $n = 1$, $Z = \mathbb{T}$, $V = \text{span}\{e^{-k}, \ldots, e^k\}$ and $w = \sum_{j=-k}^{k} e^j$. Then $F_w$ is a classical Fourier projection from $L_1(\mathbb{T})$ onto $V$.

**Lemma 6** The Lebesgue function of the operator $F_w$ is constant and $\|F_w\| = \|w\|_1$.

**Proof** By (13), (14) and (16),

$$
(F_w f)(re^{it}) = \iint_{W \times I} f(u e^{is}) \left( \sum_{j=1}^{k} a_j e^{\beta_j, \alpha_j} \right)(re^{i(t-s)}) d\mu(u, s)
$$

$$
= \sum_{j=1}^{k} a_j \iint_{W \times I} f(u e^{is}) e^{0, -\alpha_j}(u e^{is}) d\mu(u, s) e^{\beta_j, \alpha_j}(re^{it}).
$$

Hence

$$
F_w f = \sum_{j=1}^{k} a_j \hat{e}^{0, -\alpha_j}(f) e^{\beta_j, \alpha_j},
$$

(17)

where for $v \in L_\infty(Z)$,

$$
\hat{v}(f) = \iint_{W \times I} f(re^{it}) v(re^{it}) d\mu(r, t).
$$

(18)

Combining it with (6) and (17) we get that for $v$ a.a. $ue^{is} \in Z$,

$$
\chi_{ue^{is}} F_w (re^{it}) = \sum_{j=1}^{k} a_j e^{0, -\alpha_j}(ue^{is}) e^{\beta_j, \alpha_j}(re^{it})
$$

$$
= \sum_{j=1}^{k} a_j e^{\beta_j, \alpha_j}(re^{i(t-s)}) = w(re^{i(t-s)})
$$

(19)

and

$$
(F_w)_{ue^{is}} = \iint_{W \times I} |\chi_{ue^{is}} F_w (re^{it})| d\mu(r, t) = \iint_{W \times I} |w(re^{i(t-s)})| d\mu(r, t)
$$

$$
= \iint_{W \times I} |w(re^{it})| d\mu(r, t) = \|w\|_1.
$$

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By (7),

\[ \| F_w \| = \text{ess sup}_{ue^{it} \in \mathbb{Z}} (F_w)_{ue^{it}} = \| w \|_1. \]

Now we formulate three lemmas whose proofs go in the same manner as the proofs of Lemmas 1.3–1.5 in [19], so we omit them.

**Lemma 7** A subspace \( V \subset L_1(\mathbb{Z}) \) defined by (12) is smooth (see Definition 2).

**Lemma 8** For \( N \) a finite subset of \( \mathbb{Z}^n \) there exists a real function \( f \in L_\infty(\mathbb{Z}) \), \( f \neq 0 \) such that

\[
\int \int_{W \times I} f(re^{it})e^{\beta,\alpha}(re^{it}) d\mu(r,t) = 0 \quad \text{for} \quad \alpha \in N \quad \text{and} \quad \beta \in \mathbb{N}^n.
\]

**Lemma 9** Assume that \( g, w \in SV \), \( g/w \in L_\infty(\mathbb{Z}) \), \( \text{sgn}(g) = \text{sgn}(w) \) \( \nu \) almost everywhere in \( \mathbb{Z} \). Then for any \( \varepsilon \in \mathbb{R} \) such that \( |\varepsilon| < (\| (g - w)/w \|_\infty)^{-1} \) we have

\[
\text{sgn}(w + \varepsilon(g - w)) = \text{sgn}(w) \ \nu \text{a.e.}
\]

Now define

\[ X = \text{span}\{e^{\beta,\alpha}j : \beta \in \mathbb{N}^n, \ j = 1, \ldots, k\} \quad \text{[see (11)]}. \quad (20) \]

Observe that (12) and (20) imply that \( V \subset X \). Set

\[ \mathcal{L}_X(L_1(\mathbb{Z}), V) = \{ L \in \mathcal{L}(L_1(\mathbb{Z}), V) : L|_X = 0 \} \]

and

\[ \mathcal{P}_X(L_1(\mathbb{Z}), V) = F_w + \mathcal{L}_X(L_1(\mathbb{Z}), V) \quad \text{[see (2)]}. \]

We say that \( P_0 \in \mathcal{P}_X(L_1(\mathbb{Z}), V) \) is a minimal extension of the operator \( R_w \) in the set \( \mathcal{P}_X(L_1(\mathbb{Z}), V) \), if

\[ \| P_0 \| = \inf \{ \| P \| : P \in \mathcal{P}_X(L_1(\mathbb{Z}), V) \}. \]

It is easy to see that \( \mathcal{P}_X(L_1(\mathbb{Z}), V) \subset \mathcal{P}_{R_w}(L_1(\mathbb{Z}), V) \). Denote

\[ M^X_w \text{—the set of minimal extensions of } R_w \text{ in the space } \mathcal{P}_X(L_1(\mathbb{Z}), V). \quad (23) \]

**Theorem 10** The operator \( F_w \) is the unique extension of \( R_w \) belonging to the space \( \mathcal{P}_X(L_1(\mathbb{Z}), V) \) and commutative with a group \( \{ T_t : t \in I \} \).
Proof By the Stone–Weierstrass Theorem and a density of the continuous functions in the space $L_1(Z)$,

$$L_1(Z) = \text{span}\{e^{\beta \cdot \alpha} : \alpha \in \mathbb{Z}^n, \beta \in \mathbb{N}^n\}. \tag{24}$$

Let $P = \sum_{j=1}^{k} \tilde{w}_j(\cdot)e^{\beta_j \cdot \alpha_j}$ be a minimal extension of $R_w$ in the space $\mathcal{P}_X(L_1(Z), V)$ which commutes with the group of isometries $\{T_t : t \in I\}$. We show that $P(e^{\beta \cdot \alpha}) = F_w(e^{\beta \cdot \alpha})$ for $\beta \in \mathbb{N}^n, \alpha \in \mathbb{Z}^n$. If $\beta \in \mathbb{N}^n, \alpha \in \{\alpha_j : j = 1, \ldots, k\}$ the above inequality follows from the fact that $P \in \mathcal{P}_X(L_1(Z), V)$. If $\beta \in \mathbb{N}^n, \alpha \notin \{\alpha_j : j = 1, \ldots, k\}$ the condition

$$T_s \circ P(e^{\beta \cdot \alpha}) = P \circ T_s(e^{\beta \cdot \alpha}) \quad \text{for } s \in I$$

is equivalent to

$$\sum_{j=1}^{k} \tilde{w}_j(e^{\beta \cdot \alpha})e^{\beta_j \cdot \alpha_j} \cdot e^{\alpha_j(e^{is})} = \sum_{j=1}^{k} \tilde{w}_j(e^{\beta \cdot \alpha})e^{\beta_j \cdot \alpha_j} \cdot e^{\alpha(e^{is})}, \quad s \in I.$$

Hence

$$\sum_{j=1}^{k} \tilde{w}_j(e^{\beta \cdot \alpha})e^{\beta_j \cdot \alpha_j} \cdot (e^{\alpha_j(e^{is})} - e^{\alpha(e^{is})}) = 0, \quad s \in I.$$

By linear independence of the functions $\{e^{\beta_j \cdot \alpha_j}\}_{j=1}^{k}$ we get that $\tilde{w}_j(e^{\beta \cdot \alpha}) = 0$ for $j = 1, \ldots, k$. Hence $P(e^{\beta \cdot \alpha}) = 0 = F_w(e^{\beta \cdot \alpha})$ for $\alpha \notin \{\alpha^1, \ldots, \alpha^k\}, \beta \in \mathbb{N}^n$, which by (24) shows that $P = F_w$. \(\square\)

Theorem 11 $F_w$ is a minimal extension of $R_w$ in the set $\mathcal{P}_X(L_1(Z), V)$ and for any $P \in \mathcal{P}_X(L_1(Z), V)$,

$$F_w = \int_I T_s^{-1}PT_s d\lambda_I(s).$$

Proof Let $P \in M^X_w$ [see (23)]. Define

$$Q_P = \int_I T_s^{-1}PT_s d\lambda_I(s).$$

By (15) we obtain that $Q_P \in \mathcal{P}_X(L_1(Z), V)$. Properties of the Lebesgue measure imply that $Q_P$ is an operator commutative with a group of isometries $\{T_t : t \in I\}$. By Theorem 10, $Q_P = F_w$. Since $\{T_t : t \in I\}$ are the isometries, $\|F_w\| = \|Q_P\| \leq \|P\|$, which completes the proof of minimality of an operator $F_w$ in the space $\mathcal{P}_X(L_1(Z), V)$. \(\square\)
A convenient tool for studying minimality of Fourier-type extensions in the space $\mathcal{P}_A(L_1(M, \Sigma, \nu), V)$ is the following:

**Theorem 12** ([5, Cor. 1]) Let $V$ be a smooth, $k$-dimensional subspace of $L_1(M, \Sigma, \nu)$ with a basis $\{v_1, \ldots, v_k\}$. Then $P$ is a minimal extension of the operator $A$ if and only if two conditions are satisfied:

(i) the Lebesgue function of the operator $P$ is constant on $M$;
(ii) there exist a matrix $B = [B_{ij}]_{i,j=1}^k$ and a positive function $\Phi$ such that for $v = (v_1, \ldots, v_k)$,

$$\Phi(z)(V_1(z), \ldots, V_k(z)) = Bv(z) = \left[ \sum_{j=1}^k B_{ij}v_j(z) \right]_{i=1}^k. \quad (25)$$

**Theorem 13** Assume that $\# \{a_j \neq 0 : j = 1, \ldots, k\} \geq 2$ [see (13)]. Let $Z$ be an $n$-circular setsuch that $\{e^{\beta_j} | W \}_{j=1}^k$ are linearly independent functions. Then $F_w$ is not a minimal extensionof an operator $R_w$.

**Proof** Assume on the contrary that $F_w \in \mathcal{P}_{R_w}(L_1(Z), V)$ is a minimal extensionof an operator $R_w$. By Theorem 10, $F_w$ commutes with a group $G = \{T_t : t \in I\}$. Taking $\nu_1$ a Haar measure on $G$ and $\int_I T_s^{-1}BT_sd\nu_1(s)$ instead of a matrix $B$ we can assume that the matrix $B$ from Theorem 12 is commutative with $G$. It is easy to check that such a matrix is diagonal. Set $B = \text{diag}(B_1, \ldots, B_k)$. We calculate [see (6) and (19)],

$$V_j(re^{it}) = \int_{W \times I} e^{\beta_j} \alpha_j(e^{is}) \text{sgn}(x_{F_w} e^{re^{is}}) d\mu(u, s)$$

$$= \int_{W \times I} e^{\beta_j} \alpha_j(e^{is}) \text{sgn}(w(e^{i(s-t)})) d\mu(u, s)$$

$$= \int_{W \times I} e^{\beta_j} \alpha_j(e^{i(s+t)}) \text{sgn}(w(e^{is})) d\mu(u, s)$$

$$= e^{\alpha_j(e^{it})} \int_{W \times I} e^{\beta_j} \alpha_j(e^{is}) \text{sgn}(w(e^{is})) d\mu(u, s) = C_j e^{\alpha_j(e^{it})},$$

where $C_j = \int_{W \times I} e^{\beta_j} \alpha_j(e^{is}) \text{sgn}(w(e^{is})) d\mu(u, s)$. Now the condition (ii) of Theorem 12 [see (25)] is equivalent to

$$\Phi(re^{it})(C_1 e^{\alpha_1(e^{it})}, \ldots, C_k e^{\alpha_k(e^{it})}) = (B_1 e^{\beta_1} \alpha_1(e^{it}), \ldots, B_k e^{\beta_k} \alpha_k(e^{it})). \quad (26)$$
By [5, Lem. 4], dim span\{V_1, \ldots, V_k\} ≥ 2. Hence there exist \( j_1, j_2 \in \{1, \ldots, k\} \) such that \( C_{j_1} \neq 0 \) i \( C_{j_2} \neq 0 \) and

\[
\Phi(re^{it}) = \frac{B_{j_1}}{C_{j_1}} e^{\beta j_1}(r) = \frac{B_{j_2}}{C_{j_2}} e^{\beta j_2}(r) \quad \text{for } r \in W,
\]

which leads to a contradiction with a linear independence of the functions \( e^{\beta j_1}|_W \) and \( e^{\beta j_2}|_W \). By Theorem 12, \( F_w \) is not a minimal extension of \( R_w \).

\[\square\]

Remark 14 [19] In the case \( Z = \mathbb{T}^n \), the operator \( F_w \) is a minimal extension of \( R_w \) in the whole space \( \mathcal{P}_{R_w}(L_1(\mathbb{T}^n), V) \) [it is sufficient to take \( \Phi \equiv 1 \) and \( B_j = C_j \) for \( j = 1, \ldots, k \) in the equality (26)].

Theorem 15 An operator \( F_w \) is an extreme point of the set \( S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V) \) if and only if \( w/\|w\|_1 \) is a unique norming point \( g \in SV \) for a functional

\[L_1(Z) \ni h \mapsto \int \int_W \int_I h(re^{it}) sgn(w)(re^{it}) \, d\mu(r, t)\]

such that \( g/w \in L_\infty(Z) \).

Proof Assume that there exists \( g \in SV \), \( g \neq w \), \( g/w \in L_\infty(Z) \) such that \( g \) is a norming point for \( sgn(w) \) [see (18)], i.e. \( sgn(w) = sgn(g) \) \( \nu \) a.e. Define \( h = g - w \in V \). By Lemma 8, there exists a real function \( f \in L_\infty(Z) \) which is orthogonal to \( e^{\beta\alpha} \) for \( \beta \in \mathbb{N}^n \), \( \alpha \in \tilde{V} - \tilde{V} \), where \( \tilde{V} = \{\alpha : e^{\beta\alpha} \in V\} \). We can assume that \( \|f\|_{\infty} < (\|h/w\|_{\infty})^{-1} \). By Lemma 9,

\[
sgn(w)(re^{it}) = sgn\left( w \pm f(ue^{is}) \cdot h \right)(re^{it}) \quad \text{for } v \text{ a.a. } re^{it}, \, ue^{is} \in Z. \quad (27)
\]

Set \( Q_1 = F_w + L \) and \( Q_2 = F_w - L \), where

\[
(Lk)(re^{it}) = \int \int_W \int_I f(ue^{is})k(ue^{is})h(re^{i(t-s)}) \, d\mu(u, s). \quad (28)
\]

For any \( k \in L_1(Z) \) a function \( Lk \in V \) because \( h = \sum_{j=1}^k B_j e^{\beta_j,\alpha_j} \) for some \( B_j \in \mathbb{C}, \, j = 1 \ldots, k \) and

\[
Lk(re^{it}) = \int \int_W \int_I f(ue^{is})k(ue^{is}) \left( \sum_{j=1}^k B_j e^{\beta_j,\alpha_j}(re^{i(t-s)}) \right) \, d\mu(u, s)
\]

\[
= \sum_{j=1}^k B_j f(ue^{0,-\alpha_j})(k)e^{\beta_j,\alpha_j}(re^{it}).
\]
Observe that $L \neq 0$. We calculate,

$$x^L_{ueis} (re^{it}) = \sum_{j=1}^{k} B_j f (ue^{is}) e^{0,-\alpha_j} (ue^{is}) e^{\beta_j,\alpha_j} (re^{it})$$

$$= \sum_{j=1}^{k} B_j e^{\beta_j,\alpha_j} (re^{(t-s)}) f (ue^{is}) = h (re^{i(t-s)}) f (ue^{is}).$$

By the properties of $f$,

$$L(e^{\beta,\alpha}) = \sum_{j=1}^{k} B_j \int_W \int_I f (ue^{is}) e^{\beta,\alpha} (ue^{is}) d\mu(u,s) = 0 \text{ for } \alpha \in \hat{V}.$$

Hence $Q_1$ and $Q_2$ are the minimal extensions of $R_w$ and $Q_j \neq F_w : j = 1, 2$ (since $L \neq 0$). By (19) and (27)–(29) for $v$ a.a. $ue^{is} \in Z$ and $j = 1, 2$,

$$(Q_j)_{ueis} = \int_W \int_I |x_{ueis} F_w \pm L (re^{it})| d\mu(r,t) = \int_W \int_I |x_{ueis} L (re^{it})| d\mu(r,t)$$

$$\int_W \int_I |w(re^{i(t-s)}) \pm h(re^{i(t-s)}) f (ue^{is})| d\mu(r,t)$$

$$\int_W \int_I \text{sgn}(w \pm h(re^{i(t-s)}) f (ue^{is})) |(w \pm h)(re^{i(t-s)}) f (ue^{is})| d\mu(r,t)$$

$$\int_W \int_I \text{sgn}(w)(re^{i(t-s)}) (w(re^{i(t-s)}) \pm h(re^{i(t-s)}) f (ue^{is})) d\mu(r,t)$$

$$\int_W \int_I |w(re^{i(t-s)})| d\mu(r,t) \pm f (ue^{is}) \int_W \int_I \text{sgn}(w) \cdot (g - w)(re^{i(t-s)}) d\mu(r,t)$$

$$= \|w\|_1 \pm f (ue^{is}) \|g\|_1 - \|w\|_1 = \|w\|_1 = \|F_w\|.$$ 

Applying (7) we obtain that $\|Q_1\| = \|Q_2\| = \|F_w\|$. Since $F_w = (Q_1 + Q_2)/2$, $F_w$ is not an extreme point of the set $\mathcal{P}_{R_w}(L_1(Z), V) \cap S(0, \|w\|_1)$.

Now assume that $F_w$ is not an extreme point of the set $\mathcal{P}_{R_w}(L_1(Z), V) \cap S(0, \|w\|_1)$. Hence there exist $P_1, P_2 \in S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V)$ such that $P_j \neq F_w : j = 1, 2$ and $F_w = (P_1 + P_2)/2$. Define $L = (P_1 - P_2)/2$. Then $P_1 = F_w + L$ and $P_2 = F_w - L$. By Lemma 6, the Lebesgue function of the operator $F_w$ is constant.
We have \( \|F_w + L\| = \|F_w - L\| = \|w\|_1 \) and by Lemma 3, for \( \nu \) a.a. \( u e^{i s} \in Z \),
\[
(F_w + L)_{ue^{i s}} = (F_w - L)_{ue^{i s}} = \|w\|_1,
\]
\[
\text{sgn}(x_{ue^{i s}}^{F_w+L}) = \text{sgn}(x_{ue^{i s}}^{F_w-L}) = \text{sgn}(x_{ue^{i s}}^F).
\]

Let us fix \( u e^{i s} \in Z \) satisfying (30) and such that
\[
x_{ue^{i s}}^L \neq 0 \text{ and } x_{ue^{i s}}^{F_w} (re^{i(t+s)}) = w(re^{i t}) \text{ [see (19)].}
\]

Without loss of generality we can assume that \( \|w\|_1 = 1 \). Set
\[
h = T_s(x_{ue^{i s}}^L) \text{ [see (14)], } g = w + h.
\]

Since \( h \neq 0 \), \( g \neq w \). Observe that by (30)–(32) for \( \nu \) a.a. \( re^{i t} \in Z \),
\[
g(re^{i t}) = (w + h)(re^{i t}) = w(re^{i t}) + T_s(x_{ue^{i s}}^L)(re^{i t})
\]
\[
= x_{ue^{i s}}^{F_w} (re^{i(t+s)}) + x_{ue^{i s}}^L (re^{i(t+s)}) = x_{ue^{i s}}^{F_w+L} (re^{i(t+s)})
\]
and
\[
\text{sgn}(g)(re^{i t}) = \text{sgn}(x_{ue^{i s}}^{F_w+L})(re^{i(t+s)}) = \text{sgn}(x_{ue^{i s}}^{F_w})(re^{i(t+s)}) = \text{sgn}(w)(re^{i t}),
\]
\[
\text{sgn}(w - h)(re^{i t}) = \text{sgn}(x_{ue^{i s}}^{F_w-L})(re^{i(t+s)}) = \text{sgn}(x_{ue^{i s}}^{F_w})(re^{i(t+s)}) = \text{sgn}(w)(re^{i t}).
\]

Hence
\[
\text{sgn}(g) = \text{sgn}(w) = \text{sgn}(w - h) \text{ } \nu \text{ a.e.}
\]

By (30) and (33),
\[
\|g\|_1 = \iint_{W \times I} |g(re^{i t})| d\mu(r, t) = \iint_{W \times I} |x_{ue^{i s}}^{F_w+L} (re^{i(t+s)})| d\mu(r, t)
\]
\[
= \iint_{W \times I} |x_{ue^{i s}}^{F_w+L} (re^{i t})| d\mu(r, t) = (F_w + L)_{ue^{i s}} = \|w\|_1 = 1.
\]

Now we show that \( \frac{g}{w} \in L_\infty(Z) \). Assume on the contrary that \( \frac{g}{w} \notin L_\infty(Z) \). Then for any \( k \in \mathbb{N} \) there exists \( A_k \subset Z : \nu(A_k) > 0 \) and \( |g(z)/w(z)| > k \) for \( z \in A_k \). Let \( z \in A_k \) and \( a_z = \text{sgn}(w)(z) \). By (34),
\[
a_z(w(z) + h(z)) > k a_z w(z)
\]
and
\[
a_z(w(z) - h(z)) < (2 - k)|w(z)| < 0 \text{ for } k > 2.
\]
The above inequality implies that
\[ \text{sgn}(w - h)(z) \neq \text{sgn}(w)(z) \text{ for } z \in A_k, k > 2; \]
a contradiction with (34).

**Lemma 16** \( F_w \) is the unique minimal extension of \( R_w \) in the space \( P_X(L_1(Z), V) \) if and only if \( F_w \) is an extreme point of the set \( M_w^X \) [see (23)].

The proof of Lemma 16 goes in the same manner as the proof of [19, Lem. 1.2], so we omit it.

Now we can formulate a main result of this paper, a theorem, which characterize the uniqueness of minimal Fourier-type extensions in the set \( P_X(L_1(Z), V) \).

**Theorem 17** The operator \( F_w \) is the unique minimal extension of \( R_w = F_w|_V \) in the set \( P_X(L_1(Z), V) \) if and only if \( w/\|w\|_1 \) is a unique norming point \( g \in V \) for a functional
\[ L_1(Z) \ni h \mapsto \int_{W \times I} h(re^{it})\text{sgn}(w)(re^{it}) \, d\mu(r, t) \]
such that \( g/w \in L_\infty(Z) \).

**Proof** Assume that there exists \( g \in S_V, g \neq w, g/w \in L_\infty(Z) \) such that \( g \) is a norming point for \( \text{sgn}(w) \) [see (18)], i.e. \( \text{sgn}(w) = \text{sgn}(g) \) v a.e. Notice that the operator \( L \) constructed in Theorem 15 [see (28)] is actually an element of the space \( \mathcal{L}_X(L_1(Z), V) \) and extensions \( Q_1 = F_w + L \) and \( Q_2 = F_w - L \) belong to the set \( \mathcal{P}_X(L_1(Z), V) \). By the first part of the proof of Theorem 15, \( F_w \) is not an extreme point of the set \( M_w^X \) [see (23)].

Now assume that \( w/\|w\|_1 \) is a unique norming point \( g \in V \) for a functional
\[ L_1(Z) \ni h \mapsto \int_{W \times I} h(re^{it})\text{sgn}(w)(re^{it}) \, d\mu(r, t) \]
such that \( g/w \in L_\infty(Z) \). By Theorem 15, \( F_w \) is not an extreme point of the set \( S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V) \). Since \( M_w^X \subset S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V) \), we get that
\[ \text{ext} \left[ S(0, \|w\|_1) \cap \mathcal{P}_{R_w}(L_1(Z), V) \right] \subset \text{ext} M_w^X \]
and \( F_w \) is not an extreme point of the set \( M_w^X \). By Lemma 16 the proof is complete. \( \square \)

Directly from Theorem 17, reasoning as in the proof of [19, Cor. 1.9], we get the following result:

**Corollary 18** Let \( w \in V, w \neq 0 \) is determined by its roots in \( V \) (Definition 1). Then the operator \( F_w \) is the unique minimal extension of \( R_w \) in the set \( \mathcal{P}_X(L_1(Z), V) \).
The next theorem shows how large the set of minimal extensions can be. We present it without proof. The reader interested in the method of proof is referred to [19, Th. 1.7].

**Theorem 19** Let

\[ S_w = \text{span}\{ P - F_w : P \in M^X_w \} \].

If the operator \( F_w \) is not a unique minimal extension of the operator \( R_w \) in the set \( \mathcal{P}_X(L_1(Z), V) \), then \( \dim(S_w) = \infty \).

**Theorem 20** (Daugavet [11]) Let \( K \) be a compact set without isolated points. If \( L : \mathcal{C}_{\mathbb{K}}(K) \to \mathcal{C}_{\mathbb{K}}(K) \) (\( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \)) is a compact operator, then

\[ \| Id + L \| = 1 + \| L \|. \]

Denote

\[ (\mathcal{P}_X(L_1(Z), V))^* = \{ P^* : P \in \mathcal{P}_X(L_1(Z), V) \}. \]

We say that an operator \( P_0^* \in (\mathcal{P}_X(L_1(Z), V))^* \) is an element of best approximation to \( A : L_\infty(Z) \to L_\infty(Z) \) in the set \( (\mathcal{P}_X(L_1(Z), V))^* \) if

\[ \| A - P_0^* \| = \inf\{ \| A - P^* \| : P^* \in (\mathcal{P}_X(L_1(Z), V))^* \}. \]

In the same manner as in [20, Th.1.9] we can prove:

**Theorem 21** Let \( Z \) be a compact \( n \)-circular set. An identity operator \( Id : L_\infty(Z) \to L_\infty(Z) \) has the unique element of best approximation in \((\mathcal{P}_X(L_1(Z), V))^* \) if and only if \( w/\| w \|_1 \) is a unique norming point \( g \in V \) for a functional

\[ L_1(Z) \ni h \mapsto \int \int_{W \times I} h(re^{it}) \text{sgn}(w)(re^{it}) d\mu(r, t) \]

such that \( g/w \in L_\infty(Z) \).

### 3 Applications

Now we show some applications of Theorem 17.

**Theorem 22** [25, Th.14.3.3] Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n, n > 1 \). If \( f \) and \( g \) are holomorphic functions on \( \Omega \), continuous in \( \overline{\Omega} \) and such that

\[ |f(z)| \leq |g(z)| \text{ for } z \in \partial\Omega, \]

then

\[ |f(z)| \leq |g(z)| \text{ for } z \in \Omega. \]
Directly from Theorem 17 and Theorem 22 we get the following two examples:

**Example 23** (Uniqueness) Let $D$ be a bounded $n$-circular domain in $\mathbb{C}^n$, $n \geq 2$ and let $Z = \partial D$. Assume that $V$ is the space of algebraic polynomials of $n$ complex variables of degree $\leq k$ and fix $w \in V$. We prove that the operator $F_w$ is the unique minimal extension of $R_w$ in the set $\mathcal{P}_{X}(L_1(Z), V)$. Indeed, by Theorem 17 it is sufficient to show that if $g \in V$ satisfies the conditions $\text{sgn}(g)(z) = \text{sgn}(w)(z)$ for $z \in Z$, $\|g|Z\|_1 = 1$ and $g/w \in L_\infty(Z)$, then $g|z = w|z/\|w|z\|_1$. Take $g$ as we mentioned above. Define $F_1 = g + iw, F_2 = g - iw$. Observe that $F_1$ and $F_2$ are holomorphic functions on $D$ and continuous in $\bar{D}$ By assumptions, $g(z)w(z) \in \mathbb{R}$ for $z \in Z$ and

$$|F_1(z)| = |F_2(z)| = \sqrt{|g(z)|^2 + |w(z)|^2}, z \in Z.$$  

Applying twice Theorem 22, we get that $|F_1(z)| = |F_2(z)|$ for $z \in D$. Put $G = D \setminus \{F_2 = 0\}$ and $h(z) = F_1(z)/F_2(z)$ for $z \in G$. Note that $h$ is holomorphic in the domain $G$ and $|h| = 1$ on $G$. Since nonconstant holomorphic functions are open mappings, $h|G = c$ for some $c \in \mathbb{C}$. A condition

$$\|g|Z\|_1 = \\left\| \frac{w|Z}{\|w|Z\|_1} \right\|_1 = 1$$

implies that $g|Z = w|Z/\|w|Z\|_1$.

Reasoning as in Example 23 we get:

**Example 24** (Uniqueness) Let $Z$ be an $n$-circular domain, $n \geq 2$. Assume that $V$ is a space of algebraic polynomials of $n$ complex variables of degree $\leq k$ and fix $w \in V$. Then the operator $F_w$ is the unique minimal extension of $R_w$ in the set $\mathcal{P}_{X}(L_1(Z), V)$.

Now we give an example of $n$-circular set $Z$, a smooth space $V$ and $w \in V$, for which the operator $F_w$ is not a unique minimal extension of $R_w$ in the set $\mathcal{P}_{X}(L_1(Z), V)$.

**Example 25** (Nonuniqueness) Let $V$ be a space of algebraic polynomials of $n$ complex variables of degree $\leq k$. Set $Z = \mathbb{T}^n$. For $s \in \mathbb{T}$ define

$$h(s) = s^2 + l(1 + b)s + b,$$

$$k(s) = s^2 + m(1 + b)s + b,$$

where $l, m \in (0, \infty) \setminus \{1\}$, $m \neq l$, $|b| = 1$, $b \neq -1$ are such that polynomials $h$ and $k$ have all roots outside of $\mathbb{T}$ and

$$ml(1 + \text{Re}b) + (m + l)\text{Re}((1 + \overline{b})s) \geq 0 \text{ for } s \in \mathbb{T}.$$  

Observe that by our assumptions,

$$h(s)k(s) = (s^2 + b + l(1 + b)s)(\overline{s}^2 + \overline{b} + m(1 + b)\overline{s})$$

$$= |s^2 + b|^2 + m(1 + \overline{b})s + m(1 + b)\overline{s} + l(1 + b)\overline{s} + l(1 + \overline{b})s + ml|1 + b|^2$$

$$= |s^2 + b|^2 + 2ml(1 + \text{Re}b) + 2(m + l)\text{Re}((1 + \overline{b})s) \geq 0.$$
Consider any polynomial $l$ of $n$ variables of degree $k - 2$. Put

$$w(s, t) = h(s)l(s, t), \quad g(s, t) = k(s)l(s, t) \quad \text{for} \quad s \in T, \quad t \in T^{n-1}.$$ 

Then $g, w \in V, g/w \in L_\infty(Z)$ and $w(s, t)g(s, t) = h(s)k(s)|l(s, t)|^2 \geq 0$ for $(s, t) \in T^n$. Hence $\text{sgn}(w) = \text{sgn}(g)$ and by Theorem 17, $F_w$ is not a minimal extension of $R_w$ in the set $M^X_w$. In this case the assertion of Theorem 19 is fulfilled.

Other examples in which there is more than one minimal extension of $R_w$ can be found in [19].

Notice that methods of proofs used in this paper can be applied not only in the case of $n$-circular sets, but also to $Z = [0, 2\pi]^n \times T^n$. More precisely, let $L_1([0, 2\pi]^n \times T^n)$ denote the space of complex-valued functions, Lebesgue measurable on $[0, 2\pi]^n \times T^n$ and such that

$$\|f\|_1 = (1/2\pi)^{2n}\int_{I \times I} |f(u, e^{is})| duds = \int_{I \times I} |f(u, e^{is})| d\mu(u, s) < \infty,$$

where

$$\mu = \lambda_I \times \lambda_I, \quad \lambda_I — a \text{ normalized Lebesgue measure on } I = [0, 2\pi]^n. \quad (35)$$

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_n$ and $r = (r_1, \ldots, r_n) \in [0, 2\pi]^n$ put

$$G^\alpha = \{ f : f(r) = f_1(r_1) \cdot \ldots \cdot f_n(r_n), \quad f_j = \cos(\alpha_j \cdot \cdot) \text{ or } f_j = \sin(\alpha_j \cdot) : j = 1, \ldots, n \}$$

(36)

Let

$$G^\alpha = \{ G^\alpha_j : j = 1, \ldots, 2^n \}.$$

Let $\alpha \in \mathbb{Z}_n, \beta \in \mathbb{Z}_n$ and $j \in \{1, \ldots, 2^n\}$. Define a function

$$G_{j}^{\alpha,\beta} : L_1([0, 2\pi]^n \times T^n) \ni (r, e^{it}) \mapsto G_{j}^{\alpha}(r)e^{\beta}(e^{it}) \in \mathbb{C}, \quad (37)$$

where $e^{\beta}$ is given by the formula (11). Let

$$V = \text{span}\{ G_{j}^{\alpha,\beta} : p = 1, \ldots, k, \quad j = 1, \ldots, 2^n \},$$

$$w = \sum_{p=1}^{k} b_{p, j} G_{j}^{\alpha,\beta}, \quad (38)$$
for some $\alpha^p \in \mathbb{Z}^n$, $\beta^p \in \mathbb{Z}^n$ such that $(\alpha^p, \beta^p) \neq (\alpha^m, \beta^m)$ for $p \neq m$ and $b_{p,j} \in \mathbb{C}$. Set

$$(F_w f)(r, e^{it}) = (f * w)(r, e^{it}) = \int_{I \times I} f(u, e^{is})w(r - u, e^{i(t-s)}) d\mu(u, s).$$ (39)

Applying a group of isometries $\tilde{G} = \{ T_{u,s} f(r, e^{it}) = f(r + u, e^{i(s+t)}) \}_{u,s \in [0,2\pi]^n}$ instead of $G = \{ T_{t} f \}_{t \in [0,2\pi]^n}$ [see (14)] and reasoning in the same manner as in the case of $n$-circular sets, we can obtain the following:

**Theorem 26** The operator $F_w$ is the unique minimal extension of $R_w$ in the set $\mathcal{P}_{R_w} (L_1([0, 2\pi]^n \times \mathbb{T}^n), V)$ if and only if $w/\|w\|_1$ is a unique norming point $g \in V$ for a functional

$$L_1([0, 2\pi]^n \times \mathbb{T}^n) \ni h \mapsto \int_{I \times I} h(r, e^{it})\text{sgn}(w)(r, e^{it}) d\mu(r, t)$$

such that $g/w \in L_\infty([0, 2\pi]^n \times \mathbb{T}^n)$.

Here we have the uniqueness in the whole space $\mathcal{P}_{R_w} (L_1([0, 2\pi]^n \times \mathbb{T}^n), V)$ because the group $\tilde{G}$ is so big that $F_w$ is the unique operator in $\mathcal{P}_{R_w} (L_1([0, 2\pi]^n \times \mathbb{T}^n), V)$ commuting with $\tilde{G}$. As a consequence, $F_w$ has a minimal norm in the space $\mathcal{P}_{R_w} (L_1(Z), V)$, which is not true in the case of an arbitrary $n$-circular set.

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