ABSTRACT. The Multiplicity Conjecture (MC) of Huneke and Srinivasan provides upper and lower bounds for the multiplicity of a Cohen-Macaulay algebra $A$ in terms of the shifts appearing in the modules of the minimal free resolution (MFR) of $A$. All the examples studied so far have lead to conjecture (see $[HZ]$ and $[MNR2]$) that, moreover, the bounds of the MC are sharp if and only if $A$ has a pure MFR. Therefore, it seems a reasonable - and useful - idea to seek better, if possibly ad hoc, bounds for particular classes of Cohen-Macaulay algebras.

In this work we will only consider the codimension 3 case. In the first part we will stick to the bounds of the MC, and show that they hold for those algebras whose $h$-vector is that of a compressed algebra.

In the second part, we will (mainly) focus on the level case: we will construct new conjectural upper and lower bounds for the multiplicity of a codimension 3 level algebra $A$, which can be expressed exclusively in terms of the $h$-vector of $A$, and which are better than (or equal to) those provided by the MC. Also, our bounds can be sharp even when the MFR of $A$ is not pure.

Even though proving our bounds still appears too difficult a task in general, we are already able to show them for some interesting classes of codimension 3 level algebras $A$: namely, when $A$ is compressed, or when its $h$-vector $h(A)$ ends with $(\ldots, 3, 2)$. Also, we will prove our lower bound when $h(A)$ begins with $(1, 3, h_2, \ldots)$, where $h_2 \leq 4$, and our upper bound when $h(A)$ ends with $(\ldots, h_{c-1}, h_c)$, where $h_{c-1} \leq h_c + 1$.

1 Introduction

In their 1985 article $[HM]$, Huneke and Miller showed that, when a Cohen-Macaulay algebra $A$ of codimension $r$ has a pure Minimal Free Resolution (MFR), then the multi-
Multiplicity of $A$ equals the product of the different shifts appearing in its MFR divided by the factorial of $r$. The attempt to generalize this seminal result, by relating the multiplicity of any Cohen-Macaulay algebra to the shifts appearing in the modules of its MFR, led to the formulation of the so-called Multiplicity Conjecture (MC), due to Huneke and Srinivasan: namely, for any Cohen-Macaulay algebra $A$, its multiplicity times the factorial of its codimension is bounded from below (respectively, from above) by the product of the smallest (respectively, largest) shifts of the modules of the MFR of $A$.

Notice that, since the multiplicity and the MFR of a Cohen-Macaulay algebra $A$ are preserved when one considers the artinian reductions of $A$, it suffices to study the Multiplicity Conjecture for artinian algebras. We just recall here that the MC has been extended to the non-Cohen-Macaulay case by Herzog and Srinivasan, who conjectured (the stronger fact) that the upper bound of the MC actually holds for any graded algebra (whereas examples have shown that the lower bound in general does not hold when we drop the Cohen-Macaulay hypothesis). However, we will only consider the Cohen-Macaulay case in this article.

The MC has been attacked by a number of researchers over the last years, but has so far been settled only in particular cases: among them, for codimension 2 algebras (see [HS]; see also [MNR1], where the bounds of the MC are improved), Gorenstein algebras of codimension 3 (see [MNR1], where again better bounds are shown to hold), algebras with a quasi-pure resolution ([HS]), standard determinantal ideals ([Mi]), componentwise linear ideals ([Ro]), complete intersections ([HS]) and powers thereof ([GV]). We refer the reader to the recent works [Fr] and [FS] for a comprehensive history of all the main results obtained to date on the MC.

All the results achieved so far have lead to the further conjecture - due to Herzog and Zheng ([HZ]) and Migliore, Nagel and Römer ([MNR2]) - that the bounds provided by the MC are sharp if and only if the algebra we are considering has a pure MFR. Therefore, as Migliore et al. did successfully in [MNR1] for the two cases mentioned above, it seems suitable to attack the MC by seeking better - if ad hoc - bounds any time we consider particular classes of algebras (without a pure MFR). We will take this approach here in
studying codimension 3 level algebras, for which we will supply better conjectural bounds in the third section of this paper.

Throughout this work we will only consider codimension 3 algebras. In the next section, we will prove the MC (without explicit improvements) for those algebras having the same $h$-vector as that of a compressed algebra.

In the third section, we will restrict our attention to level algebras $A$, and construct ad hoc upper and lower bounds, which are better than (or, at worse, equal to) those of the MC; also, they are in general sharp, even when the MFR of $A$ is not pure. These bounds have the possibly great advantage of being entirely recovered from the $h$-vector of $A$, and therefore one does not need explicit information on the MFR’s of codimension 3 level algebras, but just on their $h$-vectors, to determine whether the bounds hold. However, since in codimension 3 even the structure of level $h$-vectors is still far from being completely understood, at this point we are able to prove our conjecture only for some interesting special classes of codimension 3 level algebras $A$: namely, when $A$ is compressed, or when its $h$-vector $h(A)$ ends with $(...,3,2)$. Furthermore, we will prove our lower bound when $h(A)$ begins with $(1,3,h_2,...)$, where $h_2 \leq 4$, and our upper bound when $h(A)$ ends with $(...,h_{c-1},h_c)$, where $h_{c-1} \leq h_c + 1$.

Finally, we will show that our bounds cannot in general be extended to the non-level case. However, even though our lower bound does not hold for all codimension 3 algebras (we will supply some non-level counterexamples), we will see that when it is verified for a given algebra $A$, then the lower bound of the MC holds for $A$ as well. We will exploit this fact to easily show the lower bound of the MC for any algebra whose $h$-vector begins with $(1,3,4,5,...)$.

Let us now fix the main definitions we will need in this paper. As we said, since we are only studying the MC in the Cohen-Macaulay case, we can suppose without loss of generality that our (standard graded) algebras $A$ are artinian. We set $A = R/I$, where $R = k[x_1, ..., x_r]$, $k$ is a field of characteristic zero, $I$ is a homogeneous ideal of $R$ and the
The \( h \)-vector of \( A \) is \( h(A) = h = (h_0, h_1, \ldots, h_c) \), where \( h_i = \dim_k A_i \) and \( c \) is the last index such that \( \dim_k A_c > 0 \). Since we may suppose, without loss of generality, that \( I \) does not contain non-zero forms of degree 1, \( r = h_1 \) is defined as the codimension of \( A \). The multiplicity of \( A \) is \( e = 1 + h_1 + \ldots + h_c \), that is the dimension of \( A \) as a \( k \)-vector space. The initial degree of \( I \) is the least degree \( t \) where \( I \) is non-zero, or equivalently, the least index such that \( h_t \) is not full-dimensional (i.e. \( h_t < (r^{-1+t}) \)).

The socle of \( A \) is the annihilator of the maximal homogeneous ideal \( \mathfrak{m} = (x_1, \ldots, x_r) \subseteq A \), namely \( \text{soc}(A) = \{ a \in A \mid a\mathfrak{m} = 0 \} \). Since \( \text{soc}(A) \) is a homogeneous ideal, we define the socle-vector of \( A \) as \( s(A) = s = (s_0, s_1, \ldots, s_c) \), where \( s_i = \dim_k \text{soc}(A)_i \). Notice that \( h_0 = 1 \), \( s_0 = 0 \) and \( s_c = h_c > 0 \). The integer \( c \) is called the socle degree of \( A \) (or of \( h \)). The type of the socle-vector \( s \) (or of the algebra \( A \)) is \( \text{type}(s) = \sum_{i=0}^{c} s_i \).

If \( s = (0,0,\ldots,0,s_c) \), we say that the algebra \( A \) is level (of type \( s_c \)). In particular, if \( s_c = 1 \), \( A \) is Gorenstein. With a slight abuse of notation, we will sometimes refer to an \( h \)-vector as Gorenstein (or level) if it is the \( h \)-vector of a Gorenstein (or level) algebra.

The minimal free resolution (MFR) of an artinian algebra \( A \) is an exact sequence of \( R \)-modules of the form:

\[
0 \rightarrow F_r \rightarrow F_{r-1} \rightarrow \ldots \rightarrow F_1 \rightarrow R \rightarrow R/I \rightarrow 0,
\]

where, for \( i = 1, \ldots, r \),

\[
F_i = \bigoplus_{j=m_i}^{M_i} R^{\beta_{i,j}}(-j),
\]

and all the homomorphisms have degree 0.

The \( \beta_{i,j} \)'s are called the graded Betti numbers of \( A \).

Then \( \beta_{1,j} \) is the number of generators of degree \( j \) of \( I \). It is well-known that \( F_r = \bigoplus_{j=1}^{c} R^{e_j}(-j - r) \neq 0 \). Hence, the socle-vector may also be computed by considering the graded Betti numbers of the last module of the MFR. In particular, an artinian algebra \( A \) is level of socle degree \( c \) and type \( s_c \) if and only if \( F_r = R^{s_c}(-c - r) \). The MFR of an algebra \( A \) is called pure if it has only one different shift in each module.

Let \( h(z) = \sum_{i=0}^{c} h_i z^i \) be the Hilbert series of \( A \) (note that, since \( A \) is artinian, \( h(z) \)
here is in fact a polynomial). In particular, we have $e = h(1)$. The MFR and the Hilbert series of $A$ are related by the following well-known formula (e.g., see [FL], p. 131, point (j) for a proof):

$$h(z)(1 - z)^r = 1 + \sum_{i,j} (-1)^i \beta_{i,j} z^j.$$  \hspace{1cm} (1)

We are now ready to state the Huneke-Srinivasan Multiplicity Conjecture:

**Conjecture 1.1 (Multiplicity Conjecture).** Let $A$ be an artinian algebra of codimension $r$ and multiplicity $e$, and let, for $i = 1, 2, ..., r$, $m_i$ and $M_i$ be, respectively, the smallest and the largest shift appearing in the $i$-th module $F_i$ of the MFR of $A$. Then:

$$\frac{m_1 \cdot m_2 \cdots m_r}{r!} \leq e \leq \frac{M_1 \cdot M_2 \cdots M_r}{r!}.$$  

### 2 Compressed $h$-vectors of codimension 3

The purpose of this section is to show that all the codimension 3 algebras whose $h$-vector is the $h$-vector of a compressed algebra satisfy the Multiplicity Conjecture. The idea of a compressed algebra is a natural concept which first appeared (for the Gorenstein case) in Emsalem-Iarrobino’s 1978 seminal paper [EI], and describes those algebras having the (entry by entry) maximal $h$-vector among all the algebras with given codimension and socle-vector. Compressed algebras and their $h$-vectors were extensively studied in the Eighties by Iarrobino ([Ia]) and Fröberg-Laksov ([FL]) - who restricted their attention to the very natural case where the socle-vectors have “enough" initial entries equal to 0; see below for the exact definition -, and recently by this author in full generality - see [Za1] and [Za2], where we have defined generalized compressed algebras.

Let us now fix a codimension $r$ and a socle-vector $s = (s_0 = 0, s_1, ..., s_c)$.  

**Definition-Remark 2.1.** Following [FL], define, for $d = 0, 1, ..., c$, the integers

$$r_d = N(r, d) - N(r, 0)s_d - N(r, 1)s_{d+1} - ... - N(r, c-d)s_c,$$
where we set
\[ N(r, d) = \dim_k R_d = \binom{r - 1 + d}{d}. \]

It is easy to show (cf. [FL]) that \( r_0 < 0, r_c \geq 0 \) and \( r_{d+1} > r_d \) for every \( d \). Define \( b \), then, as the unique index such that \( 1 \leq b \leq c, r_b \geq 0 \) and \( r_{b-1} < 0 \).

Let \( S = k[y_1, y_2, ..., y_r] \), and consider \( S \) as a graded \( R \)-module where the action of \( x_i \) on \( S \) is partial differentiation with respect to \( y_i \). Recall that, in the theory of inverse systems (for which we refer the reader to [Ge] and [IK]), the \( R \)-submodule \( M = I^{-1} \) of \( S \), (bijectively) associated to the algebra \( R/I \) having socle-vector \( s \), is generated by \( s_i \) elements of degree \( i \), for \( i = 1, 2, ..., c \). Furthermore, the \( h \)-vector of \( R/I \) is given by the number of linearly independent partial derivatives obtained in each degree by differentiating the generators of \( M \).

The number
\[ N(r, d) - r_d = N(r, 0)s_d + N(r, 1)s_{d+1} + ... + N(r, c-d)s_c \]
is therefore an upper bound for the number of linearly independent derivatives supplied in degree \( d \) by the generators of \( M \) and, hence, is also an upper bound for the \( h \)-vector of \( R/I \). This is the reason for the introduction of the numbers \( r_d \).

**Proposition 2.2** (Fröberg-Laksov). Fix a codimension \( r \) and a socle-vector \( s = (0, s_1, ..., s_c) \). Then an upper bound for the \( h \)-vectors of all the algebras having data \( (r, s) \) is given by
\[ H = (h_0, h_1, ..., h_c), \]
where, for \( i = 0, 1, ..., c \),
\[ h_i = \min\{N(r, i) - r_i, N(r, i)\}. \]

**Proof.** See [FL], Proposition 4, i). (Fröberg and Laksov gave a direct proof of this proposition; notice that a second proof immediately follows from our comment on inverse systems and the numbers \( r_d \). The same upper bound was already supplied by Iarrobino
Theorem 2.3 (Iarrobino, Fröberg-Laksov). Let \( r \) and \( s \) be as above. If \( s_1 = \ldots = s_{b-1} = 0 \), then the upper bound \( H \) of Proposition 2.2 is actually the \( h \)-vector of “almost all” the algebras (that is, those parameterized by a suitable non-empty Zariski-open set) having data \((r, s)\).

Proof. See [Ia], Theorem II A; [FL], Proposition 4, iv) and Theorem 14. □

Definition 2.4 (Iarrobino). Fix a pair \((r, s)\) such that \( s_1 = \ldots = s_{b-1} = 0 \). An algebra having data \((r, s)\) is called compressed (with respect to the pair \((r, s)\)) if its \( h \)-vector is the upper bound \( H \) of Proposition 2.2.

Thus, Theorem 2.3 shows the existence of compressed algebras and provides an explicit description of their \( h \)-vectors. We are now going to see that the MFR’s of compressed algebras also have a very nice shape; that is, in each module (of course, except possibly for the last one, which represents the socle) they have at most two different shifts. Precisely:

Proposition 2.5 (Fröberg-Laksov). Let \( A \) be a compressed algebra (with respect to a given pair \((r, s)\) such that \( s_1 = \ldots = s_{b-1} = 0 \)). Then, for each \( i = 1, 2, \ldots, r-1 \), the \( i \)-th module \( F_i \) of the MFR of \( A \) has at most two different shifts, occurring in degrees \( t + i - 1 \) and \( t + i \).

Proof. See [FL], Proposition 16. □

Let us now restrict our attention to codimension \( r = 3 \). Next we show that the Multiplicity Conjecture holds for all compressed algebras; we will see later that this result implies the MC for any algebra whose \( h \)-vector is the same as that of a compressed algebra.
Theorem 2.6. Let $A$ be a codimension 3 compressed algebra. Then the MC holds for $A$.

Proof. With the notation above, let $h$ be the $h$-vector of a compressed algebra $A = R/I$ of codimension 3 and socle-vector $s = (0, 0, ..., 0, s_q, ..., s_c)$, where we suppose that $q$ is the smallest index such that $s_q > 0$. Hence, by definition, $q \geq b$. Notice that the initial degree $t$ of $I$ equals $b$ if $r_b > 0$, whereas $t = b + 1$ if $r_b = 0$. Also, by definition, we have

$$ N(3, c - t)s_c + N(3, c - t - 1)s_{c-1} + ... + N(3, 1)s_{t+1} + N(3, 0)s_t < N(3, t). \quad (2) $$

By Proposition 2.5, the MFR of $A$ has the form:

$$ 0 \longrightarrow F_3 = \oplus_{j=q}^{c} R^{s_j}(-(j + 3)) \longrightarrow F_2 = R^{\beta_{2,t+1}}(-(t + 1)) \oplus R^{\beta_{2,t+2}}(-(t + 2)) \longrightarrow F_1 = R^{\beta_{1,t}}(t) \oplus R^{\beta_{1,t+1}}(-(t + 1)) \longrightarrow R \longrightarrow R/I \longrightarrow 0. $$

The multiplicity of $A$ is

$$ e = 1 + N(3, 1) + ... + N(3, t - 1) + (1 + N(3, 1) + ... + N(3, c - t))s_c $$

$$ + (1 + N(3, 1) + ... + N(3, c - t - 1))s_{c-1} + ... + (1 + N(3, 1) + ... + N(3, q - t))s_q $$

$$ = N(4, t - 1) + N(4, c - t)s_c + N(4, c - t - 1)s_{c-1} + ... + N(4, q - t)s_q. \quad (3) $$

the last equality following from the combinatorial identity $\sum_{i=0}^{d} N(r, i) = N(r + 1, d)$.

In order to see which are the smaller and the larger shifts in $F_1$ and $F_2$, we need to determine when $\beta_{2,t+2}$ is positive and to distinguish the three cases $\beta_{2,t+1} - \beta_{1,t+1} < 0$, $\beta_{2,t+1} - \beta_{1,t+1} > 0$ and $\beta_{2,t+1} - \beta_{1,t+1} = 0$.

By formula (II), since $\beta_{3,t+2} = s_{t-1}$, we easily have

$$ \beta_{2,t+2} - s_{t-1} = h_{t+2} - 3h_{t+1} + 3h_t - h_{t-1} = $$

$$ N(3, c - t - 2)s_c + N(3, c - t - 3)s_{c-1} + ... + N(3, 0)s_{t+2} - 3(N(3, c - t - 1)s_c + ... + N(3, 1)s_{t+2} $$

$$ + N(3, 0)s_{t+1}) + 3(N(3, c - t)s_c + ... + N(3, 2)s_{t+2} + N(3, 1)s_{t+1} + N(3, 0)s_t) - N(3, t - 1). $$

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From the identity $3N(3, i) - 3N(3, i - 1) + N(3, i - 2) = N(3, i + 1)$, it follows that

$$
\beta_{2,t+2} - s_{t-1} = N(3, c - t - 1)s_c + N(3, c - t)s_{c-1} + \ldots + N(3, 2)s_{t+1} + N(3, 1)s_t - N(3, t - 1)
$$

$$
= -r_{t-1} - s_{t-1}.
$$

Thus, $\beta_{2,t+2} = -r_{t-1}$. Therefore, it immediately follows that $\beta_{2,t+2} = 0$ if and only if $b = t - 1$, if and only if $r_b = 0$ - that is, when $A$ is called extremely compressed.

Again by formula (1), we have:

$$
\beta_{2,t+1} - \beta_{1,t+1} = h_{t+1} - 3h_t + 3h_{t-1} - h_{t-2} =
$$

$$
N(3, c - t - 1)s_c + N(3, c - t - 2)s_{c-1} + \ldots + N(3, 0)s_{t+1} - 3(N(3, c - t)s_c +
$$

$$
N(3, c - 1 - t)s_{c-1} + \ldots + N(3, 1)s_{t+1} + N(3, 0)s_t + 3N(3, t - 1) - N(3, t - 2).
$$

Hence, from the identity $3N(3, i) - N(3, i - 1) = (i + 1)(i + 3)$, we immediately get

$$
\beta_{2,t+1} - \beta_{1,t+1} = t(t + 2) - \sum_{j=q}^{c} s_j (j - t + 1)(j - t + 3).
$$

(4)

Let us first consider the case $\beta_{2,t+2} = 0$. We have seen that this is equivalent to $r_b = 0$. In order to show the MC, we have to prove that

$$
\frac{t(t + 1)(q + 3)}{6} \leq e \leq \frac{t(t + 1)(c + 3)}{6}.
$$

(5)

The first inequality is

$$
\frac{t(t + 1)(q + 3)}{6} \leq N(4, t - 1) + N(4, c - t)s_c + N(4, c - t - 1)s_{c-1} + \ldots + N(4, q - t)s_q.
$$

If we bring the summand $N(4, t - 1)$ to the l.h.s., it is easy to see that the previous inequality is a consequence of the following:

$$
t(t + 1) \leq (c - t + 2)(c - t + 3)s_c + (c - t + 1)(c - t + 2)s_{c-1} + \ldots + (q - t + 2)(q - t + 3)s_q.
$$

(6)

But (10) means

$$
N(t - 1, 3) \leq N(3, c - t + 1)s_c + N(3, c - t)s_{c-1} + \ldots + N(3, q - t + 1)s_q,
$$

which is true (actually, an equality holds), since $t - 1 = b$ and $r_b = 0$. This proves the first inequality of (5).
In order to prove the second inequality, recall that $r_b = 0$ means

\[ N(3, t - 1) = N(3, c - t + 1)s_c + N(3, c - t)s_{c-1} + \ldots + N(3, 1)s_t + N(3, 0)s_{t-1}. \quad (7) \]

We want to show that

\[ N(4, t - 1) + N(4, c - t)s_c + N(4, c - t - 1)s_{c-1} + \ldots + N(4, 0)s_t \leq \frac{(t + 1)(c + 3)}{6}. \]

By (3), (7), and the identities $N(r, i) - N(r - 1, i) = N(r, i - 1)$ and $\sum_{i=0}^{d} N(r, i) = N(r + 1, d)$, it is easy to see that the last inequality is equivalent to

\[ N(4, t - 2) + N(4, c - t + 1)s_c + N(4, c - t)s_{c-1} + \ldots + N(4, 1)s_t \leq \frac{(t + 1)(c + 3)}{6}. \]

If we bring $N(4, t - 2)$ to the r.h.s., one moment’s thought shows that it suffices to prove that

\[ N(3, c - t + 1)s_c + N(3, c - t)s_{c-1} + \ldots + N(3, 1)s_t \leq N(3, t - 1). \]

But, since $t - 1 = b$, this is equivalent to

\[-r_b - s_b + N(3, t - 1) \leq N(3, t - 1), \]

i.e. $-r_b - s_b \leq 0$, which is true since $r_b = 0$ and $s_b \geq 0$. This completes the proof of (4).

Hence, from now on, suppose that $r_b > 0$, i.e. that $\beta_{2,t+2} > 0$. Thus, $b = t$ and $s_{t-1} = 0$. Let us first consider the case where the r.h.s. of formula (4) is lower than 0. Therefore, $\beta_{2,t+1}$ can possibly be equal to 0, whereas we must have $\beta_{1,t+1} > 0$. Hence, in order to prove the MC, it suffices to show that

\[ \frac{t(t + 2)(q + 3)}{6} \leq e \leq \frac{(t + 1)(t + 2)(c + 3)}{6}. \quad (8) \]

We will actually show (8) supposing that the r.h.s. of (4) is less than or equal to 0. As far as the first inequality of (8) is concerned, what we want to prove is that

\[ t(t + 1)(t + 2) + (c - t + 1)(c - t + 2)(c - t + 3)s_c + (c - t)(c - t + 1)(c - t + 2)s_{c-1} + \ldots + (q - t + 1)(q - t + 2)(q - t + 3)s_q \geq t(t + 2)(q + 3). \]

To this purpose, it suffices to show that

\[ (c - t + 1)(c - t + 3)s_c + \ldots + (q - t + 1)(q - t + 3)s_q \geq t(t + 2), \]
which is true under the current assumption that the r.h.s. of formula (4) is less than or equal to 0. This proves the first inequality of (8).

As for the second, we want to show that

\[ t(t + 1)(t + 2) + (c - t + 1)(c - t + 2)(c - t + 3)s_c + (c - t)(c - t + 1)(c - t + 2)s_{c-1} + \ldots + (q - t + 1)(q - t + 2)(q - t + 3)s_q \leq t(t + 2)(c + 3). \]

Similarly to above, a fortiori it is enough to prove that

\[ (c - t + 1)(c - t + 2)s_c + \ldots + (q - t + 1)(q - t + 2)s_q \leq (t + 1)(t + 2), \]

i.e. that

\[ N(3, c - t)s_c + \ldots + N(3, q - t)s_q \leq N(3, t). \]

But the last inequality means exactly \( r_t \geq 0 \), and this is true since \( t = b \). This completes the proof of (8).

We now want to show the MC when the r.h.s. of formula (4) is greater than 0 (recall that we are always under the hypothesis \( \beta_{2,t+2} > 0 \)). In this case, \( \beta_{1,t+1} \) can happen to be 0, whereas we always have \( \beta_{2,t+1} > 0 \). Thus, we want to show that

\[ \frac{t(t + 1)(q + 3)}{6} \leq e \leq \frac{t(t + 2)(c + 3)}{6}. \] (9)

We omit the computations here, since they are just “symmetric” to those performed for the previous case, and again also hold when the r.h.s. of formula (4) is equal to 0. We just remark that the inequality on the r.h.s. of (4) is employed only in proving the second inequality of (8), and that, at the end, one uses the fact that \( t = b \) implies \( r_{t-1} < 0 \) to show the first inequality of (8).

Finally, it remains to prove the MC when \( \beta_{2,t+2} > 0 \) and the r.h.s. of formula (4) is equal to 0. In this case, since both \( \beta_{1,t+1} \) and \( \beta_{2,t+1} \) could be 0, we need to show that

\[ \frac{t(t + 2)(q + 3)}{6} \leq e \leq \frac{t(t + 2)(c + 3)}{6}. \] (10)

But, since we have also proven (8) and (9) when the r.h.s. of (4) is equal to 0, the two inequalities of (10) have already been shown (respectively, in proving of the first inequality of (8) and the second of (9)). This completes the proof of the theorem. ☐
The above theorem easily generalizes in the following way, giving us the main result of this section:

**Theorem 2.7.** Let $A'$ be any codimension 3 algebra whose $h$-vector is the $h$-vector of a compressed algebra. Then the MC holds for $A'$.

**Proof.** Let $A$ be a compressed algebra whose $h$-vector $h$ coincides with that of $A'$. Notice that, given $h$, the socle-vector of $A$ can be uniquely determined - one immediately computes it by induction using the definition of the numbers $r_d$; see Definition-Remark 2.1. Moreover, by the very definition of compressed algebra, we can easily see that the socle-vector of $A'$ must be (entry by entry) greater than or equal to that of $A$.

The computations made in the proof of Theorem 2.6 regarding the first two modules of the MFR of $A$ were merely numerical calculations on $h$ (and on the socle-vector of $A$, which is, as we have just said, determined by $h$); from those computations, we found out which shifts had necessarily to appear in the first two modules of the MFR of $A$, and then we used them to prove the bounds of the MC for $A$. Furthermore, since the socle-vector of $A'$ is greater than or equal to that of $A$, in last module of the MFR of $A'$ we must still have the shifts of degree $q + 3$ and $c + 3$ that we have in that of $A$.

Thus, since $A'$ has the $h$-vector of $A$, we can immediately see that the same numerical values we considered in the proof of Theorem 2.6 in bounding the multiplicity of $A$ can be considered when it comes to the algebra $A'$, and therefore we have that the proof of the previous theorem extends to $A'$. □

**Example 2.8.** Consider a codimension 3 algebra $A' = R/I'$, whose associated inverse system module $M' = (I')^{-1} \subset S$ is generated by $L_1^8, L_2^8, L_3^7, ..., L_6^7, ..., L_{12}^6, L_{13}^5, ..., L_{20}^5$, where the $L_i$’s are generic linear forms. It is easy to see that the $h$-vector of the algebra $A'$ (which has socle-vector $(0, 0, 0, 0, 8, 6, 4, 2)$) is $h = (1, 3, 6, 10, 15, 20, 12, 6, 2)$ (see [Ia], Theorem 4.8 B).
A simple calculation shows that \( h \) is also the \( h \)-vector of a compressed (level) algebra (having socle-vector \((0, 0, 0, 0, 0, 0, 0, 0, 2)\)). Therefore, it follows from Theorem 2.7, without any further computations, that \( A' \) satisfies the Multiplicity Conjecture.

3 New conjectural bounds for codimension 3 level algebras

As we said in the introduction, in general the bounds of the Multiplicity Conjecture seem too loose, although extremely difficult to prove; indeed, it was recently conjectured (see [HZ] and [MNR2]) that the only time the lower bound or the upper bound are sharp is when the algebra has a pure MFR. Therefore, we wonder if, in particular cases, we can find \textit{ad hoc} sharper bounds, which, possibly, are also easier to handle.

In this section we will restrict our attention to codimension 3 level algebras \( A \), and construct better candidates for bounding the multiplicity of \( A \), which - if true - are in general sharp, even when the MFR of \( A \) is not pure. Our bounds have the potentially remarkable advantage to be expressed uniquely in terms of the \( h \)-vector of \( A \), and therefore no knowledge of the MFR of \( A \) is required to prove them. Unfortunately, however, the current state of research on codimension 3 level \( h \)-vectors (e.g., see [GHMS] for a comprehensive overview up to the year 2003, but also our recent surprising results of [Za4]) does not yet provide us with a complete picture of which are actually these \( h \)-vectors and which are not, and therefore we cannot prove our conjectural bounds in general at this point.

We will be able to show them, however, in a few interesting particular cases: namely, when the codimension 3 level algebra \( A \) is compressed, or when its \( h \)-vector \( h(A) \) ends with \((..., 3, 2)\). Also, we will prove our lower bound when \( h(A) \) begins with \((1, 3, h_2, ...)\), where \( h_2 \leq 4 \), and our upper bound when \( h(A) \) ends with \((..., h_{c-1}, h_c)\), where \( h_{c-1} \leq h_c + 1 \).

Our bounds, in general, cannot be extended to the non-level case (or to codimension \( r > 3 \)). Indeed, our upper bound does not even imply that of the MC as soon as we drop the codimension 3 level hypothesis. Instead, even if our lower bound does not hold for
all algebras, we will see that when it is verified for any algebra A of codimension 3, then the lower bound of the MC also holds for A. We will exploit this fact to show the lower bound of the MC for any algebra whose h-vector begins with \(1, 3, 4, 5, \ldots\).

Let \(A = R/I\) be a codimension 3 level algebra, having \(h\)-vector \(h = (1, 3, h_2, \ldots, h_c)\) and graded Betti numbers \(\beta_{i,j}\), for \(i = 1, 2, 3\). The fact that \(A\) is level of socle degree \(c\) means the the only non-zero number \(\beta_{3,j}\) is \(\beta_{3,c+3} = h_c\). Also, recall formula (10) (for the codimension 3 case), which provides a relationship between \(h\) and the \(\beta_{i,j}\)'s:

\[
h(z)(1-z)^3 = 1 + \sum_{i,j} (-1)^i \beta_{i,j} z^j.
\]

(11)

It is easy to see, by (11), that for any integer \(n\), we have

\[
\beta_{2,n} - \beta_{3,n} - \beta_{1,n} = h_n - 3h_{n-1} + 3h_{n-2} - h_{n-3}.
\]

Set

\[
f(n) = h_n - 3h_{n-1} + 3h_{n-2} - h_{n-3}.
\]

Thus, when \(f(n) > 0\), the Betti number \(\beta_{2,n}\) has to be positive, i.e. a shift of degree \(n\) must appear in the second module of the MFR of \(A\). Conversely, since \(A\) is level, if \(f(n) < 0\) for some \(n < c + 3\) (when \(n = c + 3\) we would only be considering \(\beta_{3,c+3}\), which is positive since it indicates the socle), then \(\beta_{1,n} > 0\), i.e. there is a shift of degree \(n\) in the first module of the MFR of \(A\).

Define \(i\) as the smallest positive integer such that \(f(i) > 0\), \(j\) as the largest integer such that \(f(j) > 0\), and \(m\) as the largest integer lower than \(c + 3\) such that \(f(m) < 0\). As usual, let \(t\) be the initial degree of \(I\) (that is, the smallest integer such that \(f(t) < 0\)). Then we are ready to state our conjecture:

**Conjecture 3.1.** Let \(A\) be a codimension 3 level algebra of socle degree \(c\) and multiplicity \(e\), and let \(i, j, t\) and \(m\) be as above. Then

\[
\frac{t \cdot i \cdot (c + 3)}{6} \leq e \leq \frac{m \cdot j \cdot (c + 3)}{6}.
\]

(12)
From what we observed above in constructing the invariants $i$, $j$ and $m$, we immediately have that our Conjecture 3.1 implies the Multiplicity Conjecture. More precisely:

**Proposition 3.2.** Let $A$ be a codimension 3 level algebra satisfying the lower (respectively, upper) bound of Conjecture 3.1. Then $A$ also satisfies the lower (respectively, upper) bound of the MC.

Notice that, with Conjecture 3.1, we are believing in much more than the Multiplicity Conjecture, since the invariants we have constructed to define the bounds of do not always take into account the degrees of the shifts appearing in both of the first two modules of the MFR of $A$ - and these can happen to be many. However, all the examples we know and the computations we have performed so far seem to suggest that the bounds of might hold for all level algebras of codimension 3.

In the next example, we will show that the bounds of Conjecture 3.1 in general cannot be improved, even when the MFR of the codimension 3 level algebra $A$ is not pure.

**Example 3.3.** Let $A$ be the Gorenstein algebra associated to the inverse system cyclic module generated by $F = y_1^5 - y_1y_3^4 - y_3^2y_3^3 \in S = k[y_1, y_2, y_3]$. It is easy to see (e.g., using [CoCoA], and in particular a program on inverse systems written for us by our friend and colleague Alberto Damiano) that the $h$-vector of $A$ is $h = (1, 3, 4, 4, 3, 1)$. Hence, we can immediately check that $e = 16$, $t = m = 2$, $i = j = 6$ and $c + 3 = 8$, and therefore that the two bounds of Conjecture 3.1 are sharp for this algebra $A$.

Instead, since, as one can easily compute, in the first two modules of the MFR of $A$ there are also shifts of degree 4 (that one cannot notice just by looking at the $h$-vector, since $h_4 - 3h_3 + 3h_2 - h_1 = 3 - 12 + 12 - 3 = 0$), the bounds of the MC are not sharp: indeed, we have $2 \cdot 4 \cdot 8/6 = 10.66... < 16 < 4 \cdot 6 \cdot 8/6 = 32$.

Note that, as soon as we drop the hypothesis that $A$ be level of codimension 3, we
are no longer able to recover, from the three invariants $i$, $j$ and $m$ defined above, the same information on the shifts of the MFR of $A$ (or therefore on the MC). In fact, if $r > 3$, from (11) we obtain, in place of $f$, a formula with at least two positive and two negative terms, and therefore the sign of that formula no longer implies the existence of a particular shift in the MFR of $A$.

When $r = 3$ but we drop the level hypothesis, the same argument holds for the integer $m$. Indeed, the inequality $f(n) < 0$ forces $\beta_{3,n} + \beta_{1,n} > 0$, but clearly this guarantees neither the positivity of $\beta_{3,n}$ nor that of $\beta_{1,n}$.

We just remark here that Francisco ([Fr]), who already studied some cases of the Multiplicity Conjecture by looking at possible numerical cancelations among the Betti numbers, suggested an approach (from which he obtained some interesting results) to perform cancelations also when - like in the above cases - there could be more than one way to make them (basically, his choice was to give priority to the rightmost cancelation). However, Francisco’s technique and results (which go in a different direction) will not be employed nor further discussed in this paper.

Instead, for any codimension 3 algebra $A$, we can see by the same reasoning as above that the invariant $i$ always implies the existence of a shift in the second module of the MFR of $A$. Hence we have:

**Proposition 3.4.** Let $A$ be any codimension 3 algebra satisfying the lower bound of (12). Then $A$ also satisfies the lower bound of the MC.

However, the lower bound we supplied in Conjecture 3.1 does not hold for all codimension 3 algebras, as the following example shows:

**Example 3.5.** Let $A$ be a codimension 3 compressed algebra having socle-vector $(0, 0, 0, 0, 0, 0, 1, 0, 1)$. Then the $h$-vector of $A$ is

$$h = (1, 3, 6, 10, 15, 21, 13, 7, 3, 1).$$

We have $e = 80$, $t = 6$, $i = 7$ and $c + 3 = 12$; therefore, the lower bound of (12) does
not hold for \( A \), since
\[
\frac{t \cdot i \cdot (c + 3)}{6} = 84 > 80 = e.
\]

Notice that \( A \) is actually an \textit{extremely} compressed algebra. In general, for all codimension 3 extremely compressed non-level algebras, it can be shown that neither bound of the MC is sharp (just by relaxing the inequalities “\( \leq \)” into strict inequalities “\( < \)” in our proof of Theorem 2.6 when \( q < c \); this result is also consistent with the above-mentioned improvement of the multiplicity conjecture due to Herzog-Zheng and Migliore-Nagel-Römer). Thus, since for these algebras the product \( t \cdot i \cdot (c + 3)/6 \) coincides with the upper bound of the MC and is therefore larger that \( e \), we have that for all codimension 3 extremely compressed non-level algebras the lower bound of Conjecture 3.1 does not hold.

There are several cases, however, when the lower bound of Conjecture 3.1 holds for a non-level algebra \( A \). For instance, we can show very easily:

**Proposition 3.6.** Let \( A \) be any codimension 3 algebra having an \( h \)-vector which begins with \((1, 3, 4, 5, ...)\). Then the lower bound of the MC holds for \( A \).

**Proof.** Let \( h(A) = (1, h_1, ..., h_c) \). By Proposition 3.4 it suffices to show that the first inequality of (12) holds for the algebra \( A \) - namely, with the above definitions, that
\[
t \cdot i \cdot (c + 3)/6 \leq e.
\]

It is immediate to check that \( t = 2 \) and \( i = 3 \). Hence what we want to prove is that \( e \geq c + 3 \). In fact, for any \( c \geq 3 \), we clearly have that \( e \geq 1 + 3 + 4 + 5 + (c - 3) = c + 10 \), and the result follows. \( \square \)

Let us now come back to the codimension 3 level case. As we said, we are not yet able to prove Conjecture 3.1, and \textit{a fortiori} the Multiplicity Conjecture, in full generality, but there are a few interesting cases where we can be successful. Namely, we have:

**Theorem 3.7.** Let \( A \) be a codimension 3 level algebra, and let \( h = (1, 3, h_2, ..., h_{c-1}, h_c) \)
be its $h$-vector. Then:

In the following cases Conjecture 3.1 holds for $A$:

i). $A$ is compressed.

ii). $h_{c-1} = 3$ and $h_c = 2$.

In the following case the lower bound of Conjecture 3.1 holds for $A$:

iii). $h_2 \leq 4$.

In the following case the upper bound of Conjecture 3.1 holds for $A$:

iv). $h_{c-1} \leq h_c + 1$.

Proof. i). This case is already implicitly shown by the argument of Theorem 2.6 (if we let $q = c$, i.e. we consider that $A$ be level): indeed, the bounds of Conjecture 3.1 are exactly those we have shown in that proof, since the shifts we have considered in the first two modules of the MFR of $A$ in order to prove the MC were always those whose existence was forced by inequalities on the Betti numbers, and we have used exactly the same inequalities to construct the invariants $i$, $j$ and $m$. This proves the theorem when $A$ is compressed.

ii). In [Za3], Theorem 2.9, we characterized the level $h$-vectors of the form $(1, 3, ..., 3, 2)$ as those which can be expressed as the sum of $(0, 1, 1, ..., 1)$ and a codimension 2 Gorenstein $h$-vector. Since the latter $h$-vectors are of the form $(1, 2, ..., p - 2, p - 1, p - 1, ..., p - 1, p - 2, ..., 2, 1)$ for some integer $p \geq 3$ (this fact is easy to see, and was first noticed by Macaulay in [Ma]), we have that the codimension 3 level $h$-vectors we are considering here are of the form either $h = (1, 3, 3, ..., 3, 3, 2)$ or $h = (1, 3, 4, ..., p, ..., p, p - 1, ..., 4, 3, 2)$.

Hence, if $c$ as usual denotes the socle degree of $h(A)$, $c \geq 2$, it is easy to compute that $h$ stabilizes at $p$ exactly $c - 2(p - 2) + 1 = c - 2p + 5$ times. In particular, $3 \leq p \leq (c + 4)/2$.

The multiplicity of $A$ is

$$e = (1 + 3 + 4 + ... + p - 1) + p(c - 2p + 5) + (p - 1 + p - 2 + ... + 3 + 2) =$$

$$= \frac{(p - 1)p}{2} - 2 + p(c - 2p + 5) + \frac{(p - 1)p}{2} - 1 = pc - 2p^2 + 5p + p^2 - p - 3 = pc - (p - 3)(p - 1).$$

Let $p = 3$, i.e. $h = (1, 3, 3, ..., 3, 2)$. Then straightforward computations show that $t = 2$, $i = 3$, $j = c + 2$, $m = c$ for $c \neq 3$ and $m = 2$ for $c = 3$. Checking the bounds of
Conjecture 3.1, i.e. proving that, under the current conditions,
\[ \frac{t \cdot i \cdot (c + 3)}{6} \leq pc - (p - 3)(p - 1) \leq \frac{m \cdot j \cdot (c + 3)}{6}, \]
is an easy exercise which is left to the reader.

Let \( p = 4 \). We have \( t = 2, i = 4 \) for \( c \neq 5 \) and \( i = 5 \) for \( c = 5, j = c + 2, m = c + 1. \)
Again, it is trivial to check the bounds of Conjecture 3.1 (just notice that here \( c \geq 4, \) since \( p = 4). \)

Finally, let \( p \geq 5 \). Thus, \( t = 2, i = 3, j = c + 2, m = c + 1. \) We want to show that
\[ \frac{2 \cdot 3 \cdot (c + 3)}{6} \leq pc - (p - 3)(p - 1) \leq \frac{(c + 1)(c + 2)(c + 3)}{6}. \] (13)

The first inequality of (13) is equivalent to \( c + 3 \leq pc - (p - 3)(p - 1), \) i.e. to \( (p-1)(c-p+3) \geq 3, \) and this is true since \( p-1 \geq 4 \) and \( c-p+3 \geq (2p-4)-p+3 = p-1 \geq 4. \)

As for the second inequality of (13), it is clearly enough to show that \( pc \leq (c + 1)(c + 2)(c + 3)/6, \) or therefore that \( p \leq (c + 2)(c + 3)/6. \) But the latter inequality is easily verified, since \( p \leq (c + 4)/2 \) and \( c \geq 6 \) for \( p \geq 5. \)

This completes the proof of (13), and that of case ii) of the theorem.

iii). Using the tables for the possible codimension 3 level \( h \)-vectors of low socle degree (see [GHMS], Appendix F), and employing a few standard considerations, one can determine the form of the level \( h \)-vectors beginning with \((1, 3, h_2, \ldots), \) for \( h_2 \leq 4. \) Precisely:

For \( h_2 = 1, \) \( h \) can only be \((1, 3, 1). \)

For \( h_2 = 2, \) the only possibility for \( h \) is \((1, 3, 2) \) (since \( h_3 \) must be lower than 3 by Macaulay’s theorem, cannot be 2 otherwise would force a one-dimensional socle in degree 1 (e.g., see [Za5], Theorem 3.5), and cannot be 1 because of the symmetry of Gorenstein \( h \)-vectors. Thus, \( h_3 = 0). \)

For \( h_2 = 3, \) by considerations similar to those we have made for the previous case, we see that \( h \) must necessarily be of the form \((1, 3, 3, \ldots, 3, h_c), \) where \( h_c \) equals 1, 2 or 3.

Let \( h_2 = 4. \) Then, likewise, we can show that all the possibilities for \( h \) are: \((1, 3, 4, 5, \ldots), \) \((1, 3, 4, 4, 4, u, \ldots) \) (where \( u \leq 4), \) \((1, 3, 4, 4, 3, 2), \) \((1, 3, 4, 4, 3, 1), \) \((1, 3, 4, 4, 3), \) \((1, 3, 4, 4, 2), \) \((1, 3, 4, 4), \) \((1, 3, 4, 3, 2), \) \((1, 3, 4, 3, 1), \) \((1, 3, 4, 3), \) \((1, 3, 4, 2), \) \((1, 3, 4). \)

In all of the cases listed above it is easy to check that the lower bound of Conjecture
3.1 is verified, so we will avoid the computations here. This completes the proof of this case.

iv). From the hypothesis $h_{c-1} \leq h_c + 1$, it clearly follows that $-h_{c-1} > -3h_c$, and therefore $j = c + 2$. First suppose that $h_{c-1} \leq h_c$. Then, since $-h_{c-2} < 0 \leq 3(h_c - h_{c-1})$, we have $m = c + 1$. Hence the upper bound of Conjecture 3.1 is $(c + 1)(c + 2)(c + 3)/6$. We have

$$e \leq 1 + 3 + ... + N(3, c) = N(4, c) = (c + 1)(c + 2)(c + 3)/6,$$

as we wanted to show.

Now let $h_{c-1} = h_c + 1$. It is immediate to check, by definition of $m$, that this time we have $m = c + 1$ if and only if $h_{c-2} > 3$. So, when $h_{c-2} > 3$, we are done by reasoning as above. Instead, by using the same standard techniques we have employed in showing point iii) of this theorem, we can see that all the codimension 3 level $h$-vectors such that $h_{c-1} = h_c + 1$ and $h_{c-2} \leq 3$ are: $(1, 3, 3, ..., 3, 2)$ (for any $c \geq 2$), $(1, 3, 4, 3)$, $(1, 3, 5, 4)$ and $(1, 3, 6, 5)$. It is easy to check the upper bound of Conjecture 3.1 for these $h$-vectors (for the first $h$-vector see the case $p = 3$ of point ii) above), so we will leave the computations as an exercise for the reader. This completes the proof of this case and that of the theorem. □

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