Wilson loops in the adjoint representation and multiple vacua in 
two-dimensional Yang-Mills theory

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Abstract

$QCD_2$ with fermions in the adjoint representation is invariant under $SU(N)/Z_N$ and thereby is endowed with a non-trivial vacuum structure ($k$-sectors). The static potential between adjoint charges, in the limit of infinite mass, can be therefore obtained by computing Wilson loops in the pure Yang-Mills theory with the same non-trivial structure. When the (Euclidean) space-time is compactified on a sphere $S^2$, Wilson loops can be exactly expressed in terms of an infinite series of topological excitations (instantons). The presence of $k$-sectors modifies the energy spectrum of the theory and its instanton content. For the exact solution, in the limit in which the sphere is decompactified, a $k$-sector can be mimicked by the presence of $k$-fundamental charges at $\infty$, according to a Witten’s suggestion. However, this property neither holds before decompactification nor for the genuine perturbative solution which corresponds to the zero-instanton contribution on $S^2$. 

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I. INTRODUCTION

One of the fundamental problems in discussing the quantum dynamics of nonabelian gauge theories is to understand their vacuum structure. Many important aspects of four-dimensional QCD such as chiral symmetry breaking and confinement, are believed to be related to some non-trivial properties of the vacuum itself. Particularly intriguing in this context is the appearance of a topological parameter, the $\theta$ angle \cite{1}, directly related to the existence of multiple vacuum states and to quantum tunnelling between them mediated by instanton effects \cite{2}. It would be extremely interesting to understand how physical quantities could depend on $\theta$, but, unfortunately, four-dimensional QCD appears too complicated to deal with. A non-perturbative analysis is in fact mandatory, and satisfactory results concerning the $\theta$-dependence have been up to now obtained only in particular supersymmetric cases \cite{3}.

Nevertheless, a formulation of the Quantum Field Theory exists, in which, at least in principle, the exact ground state is described as a simple Fock vacuum: the Light-Front (LF) Quantum Field Theory, where the theory is quantized not on a space-like surface but on a light-like one (see \cite{4} for an extensive review). Actually, in four-dimensional gauge theories such an approach is far from being simple, due to the intricate dynamics of the so-called zero-modes that mix with the Fock vacuum and substantially spoil the kinematical character of the ground state \cite{5}. This problem does not exist in two dimensions (zero-modes are related to transverse degrees of freedom in LF quantization) and recently it has been shown \cite{6} that, at the perturbative level, LF quantization encodes a complicated instanton dynamics present in the equal-time (ET) formulation. Therefore two-dimensional gauge theories candidate themselves as the simplest models in which the influence of topological parameters on physical quantities and their interplay with the vacuum structure, as well as on their perturbative (if any) interpretation, can be probed. The best example of a theory which admits multiple vacua and shares relevant features with four-dimensional gauge theories is $QCD_2$ with adjoint fermions, as noticed many years ago by Witten \cite{7}: here a single integer
$k$ labels inequivalent vacua, taking the value $0, 1, \ldots, N - 1$ in the $SU(N)$ case. This model exhibits a rich spectrum [8], a Hagedorn transition at finite temperature [9] and interesting confining vs screening properties [10]: the presence of adjoint matter in some sense mimics transverse degrees of freedom inducing a complex behaviour. Of particular interest is the link between topological sectors, multiple vacua, instantons and condensates: the presence of a fermionic condensate, in the $SU(N)$ case, can be argued by looking at the bosonized version of the theory [11] and it is also expected from arguments of quark-hadron duality [8]. Nevertheless, only for $N = 2$ the instanton computation, relating the presence of zero-modes of the Dirac operator in a non-trivial background to the condensate, produces the desired result [12]. In the general case no satisfactory answer is known to our knowledge, since on one hand the results obtained within the “Discretized Light-Cone Quantization” (DLCQ) depend on the size of the discretization parameter [13], on the other hand the approach of [14] is essentially restricted to the $SU(2)$ case in the small volume limit.

We therefore find worthwhile examining these problems at the light of the result of [6,15], where LF quantization of Yang-Mills theory has been shown to reproduce the non-perturbative series of instantons that naturally appears in the ET formulation in a simple manner. Our present goal is to investigate carefully the vacuum properties in both quantization schemes and to understand the relation with the usual perturbative series. We limit ourselves to the case when fermionic dynamics is essentially frozen, considering infinitely massive adjoint quarks and studying the static potential between them. To this purpose we survey Wilson loops in the adjoint representation. This problem was tackled in [16] (and at finite temperature in [17], where also many finite volume results were derived): there the exact dependence of the string tension from the topological parameter labelling the vacuum was found. It was also shown that the same results can be obtained using an effective Hamiltonian in the LF theory, in the limit of infinitely massive quarks. To perform the computations the authors assumed the point of view, suggested by Witten [7], of simulating topological sectors by defining the theory with a Wilson loop at infinity in the $k$-fundamental representation. We pursue instead a different approach, closer to the instanton
interpretation of the results and suitable for a comparison with perturbative physics.

In Sect. II we present the general description of $k$-sectors in adjoint $QCD_2$, analysing the same issue both in the heat-kernel (Hamiltonian) language and in the instanton expansion: in the latter case the $k$-dependence arises as a phase factor in summing over inequivalent bundle structures in the gauge connection space. In Sect. III we explicitly compute the adjoint Wilson loop on the sphere in the $k$-sector for the $SU(N)$ theory, as a sum over a set of $N$ integers, a representation useful for further developments. We also perform the decompactification limit, showing that the final result is in full agreement with the one of [16], where the computation was done directly on the plane with a Wilson loop at infinity in the $k$-fundamental representation. In Sect. IV we study the correlation function of an adjoint and a $k$-fundamental Wilson loop first on the sphere and then in the decompactification limit, in order to mimic the inclusion of asymptotic $k$-charges, according to Witten’s suggestion. On the plane we recover the result of the previous section. Next we check in Sect. V that this result is consistent, at the fourth order in the coupling constant, with the LF perturbative computation plus the $k$-holonomy at infinity. Sect. VI is devoted to derive the instanton representation on the sphere for both cases, by performing the Poisson resummation on the exact results: we find that inequivalent gauge bundles enter the game, the relative contributions being weighted by the topological parameter. Nevertheless the instanton patterns turn out to be completely different in the two cases. In particular we notice that the zero-instanton contribution for the former case (adjoint Wilson loop in the $k$-th sector) does not depend on the topological number $k$: in the decompactification limit it coincides with the sum of the perturbative series in which the propagator is prescribed according to Wu-Mandelstam-Leibbrandt (WML) [18–20], for the adjoint Wilson loop without the presence of the $k$-holonomy at infinity. In the latter case (adjoint Wilson loop enclosed in a $k$-fundamental one) instead, string tensions do not depend on $k$ but the polynomial part does. After checking this fact by using the WML propagator in a perturbative expansion at $\mathcal{O}(g^4)$ in Sect. VII, we conclude that only for the complete theory on the plane (i.e. full-instanton resummed and then decompactified) the equivalence between $k$-sectors
and theories with \( k \)-fundamental Wilson loops at infinity holds. In Sect. VIII we draw our conclusions and discuss future developments, whereas technical details are deferred to the Appendix.

II. \( k \)-SECTORS AND INSTANTONS

It was first noticed by Witten \[7\] that two-dimensional Yang-Mills theory and two-dimensional QCD with adjoint matter do possess \( k \)-sectors. We consider \( SU(N) \) as the gauge group: since Yang-Mills fields transform in the adjoint representation, the true local symmetry is the quotient of \( SU(N) \) by its center, \( Z_N \). A standard result in homotopy theory tells us that the quotient is no longer simply connected, the first homotopy group being

\[
\Pi_1(SU(N)/Z_N) = Z_N.
\]

This result is of particular relevance for the vacuum structure of a two-dimensional gauge theory: according to the classical picture \[1\], vacuum states are related to static pure gauge configurations

\[
A_1 = ig(x)^{-1}\partial_1g(x),
\]

identified when connected by a continuous deformation. Assuming boundary conditions that allow for the compactification of the space-manifold to \( S^1 \), we see that the relevant maps \( g(x) \) to the gauge group \( G \) fall into equivalence classes labelled by \( \Pi_1(G) \). Starting from this observation, standard geometrical and field theoretical arguments lead to the conclusion that all physical states carry an irreducible representation of \( \Pi_1(G) \) \[21\]. In the case at hand we have exactly \( N \) irreducible representations for \( Z_N \), labelled by a single integer parameter \( k \), taking the values \( k = 0, 1, ..., N - 1 \) (it is obviously related to a \( N \)-th root of the identity). On a physical state \( |\psi> \) the generator \( C \) of \( Z_N \) simply acts as a phase

\[
C|\psi> = e^{2\pi i \frac{k}{N}}|\psi>;
\]

(2)
from a gauge theoretical point of view, while physical states are strictly invariant under small gauge transformations (generated by the Gauss’ law), large gauge transformations are projectively realized on them. Inequivalent quantizations, parametrized by \( k \) in the nonabelian (SU(N)) case are therefore seen to appear when the matter content singles out the effective gauge group, eventually changing the topological properties of the theory itself.

We remark that the situation is quite different in the two-dimensional abelian case: there the homotopy group is \( \Pi_1(U(1)) = \mathbb{Z} \) and the parameter labelling the irreducible representation is a real number, taking values between 0 and \( 2\pi \): it is usually called \( \theta \) in analogy with the four-dimensional case (we see that the crucial homotopy group for QCD\(_4\) is in fact \( \Pi_3(SU(N)) = \mathbb{Z} \)).

Concerning the pure SU(N) Yang-Mills theory, the explicit solution when \( k \)-states are taken into account was presented in Ref. [17]: their main result, the heat-kernel propagator on the cylinder, allows to compute partition functions and Wilson loops on any two-dimensional compact surface, therefore generalizing the well-known Migdal’s solution [22] to \( k \)-sectors.

Let us first recall the standard procedure: on a two-dimensional cylinder of length \( \tau \) and base circle \( L \), with area \( A = L\tau \), one introduces the heat kernel

\[ \mathcal{K}[A; A_2, A_1], \]

\( A_1 \) and \( A_2 \) being the potentials at the two boundaries.

It is well-known that, if we introduce the unitary matrices

\[ U_{1(2)} = \mathcal{P} \exp \left( i \int_0^L dx A_{1(2)}(x) \right), \]

\( \mathcal{P} \) denoting path-ordering as usual, the heat kernel is a class function of \( U_1 \) and \( U_2 \).

It enjoys in particular the basic sewing property

\[ \mathcal{K}[L\tau; U_2, U_1] = \int dU(u) \mathcal{K}[Lu; U_2, U(u)] \mathcal{K}[L(\tau - u); U(u), U_1]. \] (3)

Thanks to the invariance under area-preserving diffeomorphisms, knowing \( \mathcal{K}[A; U_2, U_1], \)
one can easily derive the exact partition function on the sphere \(S^2\) by setting \(U_2 = U_1 = 1\)

\[
Z(A) = \mathcal{K}[A; 1, 1].
\]

By expanding \(\mathcal{K}\) in terms of group characters \(\chi_R(U)\)

\[
\mathcal{K}[A; U_2, U_1] = \sum_R \chi_R(U_1)\chi_R^\dagger(U_2) \exp \left[ -\frac{g^2 A}{4} C_2(R) \right],
\]

\(R\) denoting an \(SU(N)\) irreducible representation and \(C_2(R)\) its quadratic Casimir, one recovers the well-known expression for the partition function

\[
Z(A) = \sum_R (d_R)^2 \exp \left[ -\frac{g^2 A}{4} C_2(R) \right],
\]

\(d_R\) being the dimension of the representation \(R\) \(22\). The extension to a general compact Riemann surface of genus \(G\) without boundaries is trivial, consisting in a change of exponent for \(d_R\) from 2 to \((2 - 2G)\).

The generalization of the above construction to \(SU(N)/Z_N\) is quite simple: following \(17\), we observe that the heat kernel in a \(k\)-sector can be obtained by projecting its final state onto \(k\)-states. This is done by summing over all transformations of \(U_2\) by the elements of the center \(Z_N\) and weighting each term in the sum by a phase factor:

\[
\mathcal{K}_k[A; U_2, U_1] = \sum_{z \in Z_N} z^k \mathcal{K}[A; zU_2, U_1],
\]

with \(z = \exp(2\pi i \frac{n}{N})\), \(n = 0, \ldots, N - 1\). The explicit form of the partition function on the sphere can be written as follows

\[
Z_k(A) = \sum_{n=0}^{N-1} e^{2\pi i \frac{n k}{N}} \sum_R \frac{\chi_R \left( e^{-2\pi i \frac{n}{N}} \mathbf{1} \right)}{d_R} d_R \exp \left[ -\frac{g^2 A}{4} C_2(R) \right].
\]

It is possible to give a beautiful interpretation of the above expression; in fact, using the Young tableau representation, one can show that

\[
\sum_{n=0}^{N-1} e^{2\pi i \frac{n k}{N}} \chi_R \left( e^{-2\pi i \frac{n}{N}} \mathbf{1} \right) = \sum_{n=0}^{N-1} \exp \left[ 2\pi i \frac{n}{N} \left( k - \sum_{\alpha=1}^{N-1} m_\alpha^{(R)} \right) \right] = \delta_{[N]}(k - m^{(R)}),
\]

where
\[ m^{(R)} = \sum_{\alpha=1}^{N-1} m_\alpha^{(R)} \] (9)

is the total number of boxes of the Young tableaux and \( \delta_{[N]} \) is the \( N \)-periodic delta function.

The partition function therefore takes the form

\[ Z_k(A) = \sum_R (d_R)^2 \exp \left[ -\frac{g^2 A}{4} C_2(R) \right] \delta_{[N]}(k - m^{(R)}) . \] (10)

We now recall that the Casimir \( C_2(R) \) is related to the allowed energies of the system: while in the \( SU(N) \) theory all representations contribute, in \( SU(N)/Z_N \) only a particular class of Casimir invariants appear, depending on \( k \). Different \( k \)-theories have different energy spectrum. In particular they exhibit a different ground state, the \( k \)-fundamental representation. The modification naturally survives the decompactification limit, where it is the \( k \)-fundamental which dominates instead of the 1-fundamental. We remark that the above characterization of \( k \) is, in some sense, nonperturbative, since the \( k \)-dependence shows up in solving for the Hamiltonian eigenvalues.

On the other hand, it is possible to implement the topological dependence even without passing through an energy interpretation. Let us consider the familiar case of \( QCD_4 \) with gauge group \( SU(N) \). There the partition function takes contribution from four-dimensional connections belonging to disconnected sectors, as the existence of inequivalent \( SU(N) \) bundles over \( S^4 \) displays (we consider four-dimensional Euclidean space-time compactified to \( S^4 \)). Inequivalent \( SU(N) \) bundles are classified according to a single integer, \( n \), \( i.e. \) the Chern number, related to the field topological charge that is again determined by \( \Pi_3(SU(N)) = \mathbb{Z} \).

Notice that at present we are discussing genuine four-dimensional vector fields, whereas the configuration space in the Hamiltonian formalism is three-dimensional. If \( Z^{(n)} \) is the partition function in a given sector, we infer for the full answer

\[ Z(\theta) = \sum_{n=-\infty}^{\infty} e^{i\omega(n)} Z^{(n)} , \] (11)

and, requiring cluster properties, it can be easily shown that

\[ \omega(n) = 2\pi n \theta , \quad \theta \in [0, 2\pi] . \] (12)
We can proceed in perfect analogy in two dimensions. To begin with, we realize that every \( SU(N) \) bundle is trivial on \( S^2 \), since \( \Pi_1(SU(N)) = 0 \). Considering instead \( SU(N)/\mathbb{Z}_N \), we deduce from \( \Pi_1 \neq 0 \) there exists exactly \( N \) different, inequivalent classes of bundles, characterized by \( \mathbb{Z}_N \) fluxes \( n = 0, 1, \ldots, N - 1 \). The presence of the quotient reflects on a richer topological structure in the connection space, non-trivial bundles entering the game.

As in \( QCD_4 \), we have

\[
\mathcal{Z}_k = \sum_{n=0}^{N-1} e^{i\omega(n)} \mathcal{Z}^{(n)}, \tag{13}
\]

\( \mathcal{Z}^{(n)} \) pertaining to the \( n \)-th principal bundle. Imposing cluster decomposition we get

\[
\omega(n) = \frac{2\pi k}{N} n, \quad k = 0, 1, \ldots, N - 1. \tag{14}
\]

We expect perturbative physics to be related to the trivial \( n = 0 \) sector, independent of the topological parameter \( k \): as the coupling \( g^2 A \to 0 \), all \( \mathcal{Z}^{(n)} \) except \( \mathcal{Z}^{(0)} \) should be exponentially suppressed. While it is in principle possible to directly compute \( \mathcal{Z}_k \) by means of an instanton expansion, namely by finding the classical solution in any \( k \)-sector and then expanding the functional integral around it, we follow here an alternative route: we obtain a dual version of Eq. (11) via a Poisson resummation and there we identify the different instanton contributions. It turns out Eq. (10) is localized around solutions of Yang-Mills equations of \( SU(N)/\mathbb{Z}_N \), as predicted by Witten [23]: the zero-instanton sector comes only from the \( n = 0 \) part, reproducing a truly \( SU(N) \) perturbative result.

Hence, we have two complementary representations for the partition function, in which the parameter \( k \) plays different roles: the Hamiltonian heat-kernel representation, in which \( k \) selects the energies, and the instanton representation, where \( k \) weights the contribution of inequivalent bundles.

We emphasize that the parallel with the four-dimensional case, or even with the \( U(1) \) model in two dimensions is still not complete: in four dimensions, in fact, Eq. (11) can be reinterpreted as the addition of the topological action

\[
S_{top}^{(4D)} = \frac{i\theta}{16\pi^2} \int d^4x \ Tr \left[ F_{\mu\nu} \tilde{F}^{\mu\nu} \right] \tag{15}
\]
to the YM term, \( \hat{F}^{\mu
u} \) being the dual of \( F^{\mu
u} \), defined in the usual way, and
\[
\frac{1}{16\pi^2} \int d^4 x \, \text{Tr} \left[ F^{\mu\nu} \hat{F}^{\mu\nu} \right] = n \text{ for every connection } A_{\mu} \text{ in the } n\text{-th sector.}
\]
Analogously, in the two-dimensional case we have to add for \( U(1) \)
\[
\mathcal{L}_{\text{top}}^{(2D)} = \frac{i\theta}{4\pi} \int d^2 x \, F_{\mu\nu} \varepsilon^{\mu\nu}.
\] (16)

These modifications are motivated by an instanton expansion and are general, insensitive to whether the theory is defined on a compact surface or in the Euclidean space. Nevertheless, inspired by the \( U(1) \) case, Witten suggested a way to implement the phase factor (or equivalently the \( k \)-dependence) in Eq. (13) directly on the plane, using the \( SU(N) \) theory. The argument goes as follows: as pointed out long ago by Coleman \[25\], in the Schwinger model one introduces the parameter \( \theta \) as the strength of a fractional charge \( \frac{\theta}{2\pi} \) at the spatial right end of the two-dimensional world and its opposite at the left end. We observe that the inclusion of the topological term can be understood as imposing the following generalized boundary condition: we can interpret such charges in terms of a Wilson loop enclosing the world, which can be explicitly included in the action of the theory
\[
Z_\theta = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_{\mu} \, \exp \left[ - \int d^2 x \, \mathcal{L} + \frac{i\theta}{2\pi} \int_{C_{\infty}} dx_{\mu} A_{\mu} \right]
= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_{\mu} \, \exp \left[ - \int d^2 x \, \mathcal{L} + \frac{i\theta}{4\pi} \int d^2 x \, F_{\mu\nu} \varepsilon^{\mu\nu} \right].
\] (17)

In the nonabelian case the suggestion is to consider static colour charges \( T_R \) and \( T_{\bar{R}} \) at the boundary. Here the \( T \)'s are the generator of the nonabelian colour group in the representation \( R \) and its conjugate, respectively. Unlike the abelian case, there is no continuous parameter, the only choice is discrete, depending on the representation of the boundary charges. If \( \text{Tr}_R \) is the trace in the representation \( R \) of the gauge group, it holds
\[
Z_{(R)} = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A_{\mu} \, \exp \left[ - \int d^2 x \, \mathcal{L} \right] \text{Tr}_R \mathcal{P} \exp \left( i \int_{C \to \infty} dx_{\mu} A_{\mu} \right),
\] (18)
which manifests that the Wilson loop at infinity cannot be written as a local addition to the Lagrangian. Let us notice that the theory was written in presence of adjoint dynamical fermions. Witten showed that this system is stable (there are no energetically favoured pair
creations in the external background) only when \( R \) is one of the \( N \) (antisymmetric) fundamental representation of \( SU(N) \), that are exactly the \( k \)-ground states appearing in Eq. (13). We remark that \( Z(R) \) is computed with \( A_\mu \) taking values in the \( SU(N) \) algebra, and has to be used to evaluate general observables (like a Wilson loop in the adjoint representation) in the absence of dynamical fermions.

In the next section we explicitly check that this prescription is correct and that Wilson loops computed on \( S^2 \) in the \( SU(N)/Z_N \) theory match, in the decompactification limit, with the result of Ref. [14], where the above picture was adopted. However, when remaining on the sphere, \( i.e. \) before taking such a limit, we do not see any reason why such a procedure should lead to the correct \( SU(N)/Z_N \) result. A \( k \)-fundamental loop on \( S^2 \) for \( SU(N) \) cannot mimic non-trivial bundles, as it cannot obviously modify its instanton structure. Thus we do not expect that the zero-instanton result for an adjoint loop in this context will coincide with the genuine zero-instanton term for \( SU(N)/Z_N \), neither before nor after decompactification.

III. THE WILSON LOOP IN THE ADJOINT REPRESENTATION

Starting from the sewing property of the heat kernel Eq. (3), it is also easy to get an expression for a Wilson loop winding around a smooth non self-intersecting closed contour on \( S^2 \). By choosing again \( U_1 = U_2 = 1 \) and by inserting the Wilson loop expression for a contour in a given representation \( T \), we get

\[
W(A_1, A_2) = \frac{1}{Z(A)d_T} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 A_1}{4} C_2(R) - \frac{g^2 A_2}{4} C_2(S) \right] \\
\times \int dU \text{Tr}_T[U] \chi_R(U) \chi_S^\dagger(U),
\]

(19)

\( A_1 \) and \( A_2 (A_1 + A_2 = A) \) being the areas singled out by the loop.

A particularly interesting case is represented by the choice of the loop in the adjoint representation \( T \equiv \text{adj} \); in so doing invariance under the quotient group is preserved. On the other hand the loop might somehow mimic contributions from the would-be “transverse” vector degrees of freedom in higher dimensions.
The projection on a given sector $k$ can now again be realized by “twisting” $U_2 = 1$ with the center factor $z_k$

$$W_k(A_1, A_2) = \frac{1}{Z_k(N^2 - 1)} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 A_1}{4} C_2(R) - \frac{g^2 A_2}{4} C_2(S) \right]$$

$$\times \int dU \text{Tr}_{adj}[U] \chi_R(U) \chi_S^\dagger(U) \delta_{[N]} (k - m^{(S)}) \, .$$  \hspace{1cm} (20)

In the decompactification limit $A \to \infty$, keeping $A_1$ fixed, the above quantity is to be interpreted as the Wilson loop average in a $k$-vacuum, for the theory defined on the plane.

Our next step will consist in working Eqs. (10,20) out to cast them in a desirable form. In addition, we anticipate in passing that the instanton representation, which will be the subject of Sect. VI, hinges precisely on the formulae we will eventually arrive at. Taking the well-known relation

$$\text{Tr}_{adj}[U] = |\text{Tr} U|^2 - 1 \, ,$$  \hspace{1cm} (21)

into account, Eq. (20) becomes

$$\frac{1}{N^2 - 1} + W_k(A_1, A_2) = \frac{1}{Z_k(N^2 - 1)} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 A_1}{4} C_2(R) - \frac{g^2 A_2}{4} C_2(S) \right]$$

$$\times \int d\theta_1 \ldots d\theta_N \delta \left( \sum_{j=1}^N \theta_j \right) \delta_{[N]} (k - m^{(S)}) \, .$$  \hspace{1cm} (22)

We now proceed as follows. Firstly, we switch to the integers $\hat{l}_q$ so defined

$$\hat{l}_q = m_q - q + N \, , \quad q = 1, \ldots, N - 1 \, ,$$  \hspace{1cm} (23)

which satisfy the $SU(N)$ constraint $\hat{l}_1 > \hat{l}_2 > \ldots > \hat{l}_{N-1} > 0$, turning a weakly monotonous sequence into a strongly monotonous one. Secondly, with the twofold purpose of extending the range of the $\hat{l}_q$’s, $q = 1, \ldots, N - 1$, also to negative integers and of gaining the symmetry over permutations of a full set of $N$ indices in Eqs. (11,22), we introduce the obvious equality ($\hat{l}_N$ is here a dummy quantity)

$$\sqrt{\pi} = \int_0^{2\pi} d\alpha \sum_{\hat{l}_N = -\infty}^{+\infty} e^{-\left( \frac{4\pi}{N} \sum_{j=1}^{N-1} \hat{l}_j - 2\pi \hat{l}_N \right)^2} \, .$$  \hspace{1cm} (24)
Thanks to it, we extend the set of representation indices by defining

\[ l^{R,S}_q = \hat{l}^{R,S}_q + \hat{l}^{R,S}_N, \quad q = 1, \ldots, N - 1, \]
\[ l^{R,S}_N = \hat{l}^{R,S}_N, \]
\[ l^{R,S} = \sum_{i=1}^{N} l^{R,S}_i. \]  

(25)

The operations hitherto carried out, enable us to write Eqs. (10,22) explicitly in terms of the new set of indices \( l_i = (l_1, \ldots, l_N) \). By recalling the relations

\[ C_2(R) = \sum_{i=1}^{N} \left( l_i - \frac{l}{N} \right)^2 - \frac{N}{12} (N^2 - 1) \]
\[ d_R = \Delta(l_1, \ldots, l_N), \]

(26)

where \( \Delta \) is the Vandermonde determinant, we get

\[ Z_k(A) = \frac{(2\pi)^{N-1}}{N! \sqrt{\pi}} \sum_{l_i=-\infty}^{+\infty} \int_0^{2\pi} d\alpha e^{-(\alpha - \frac{2\pi}{N} l)^2} \delta_{[N]} \left( k - l + \frac{N(N-1)}{2} \right) \]
\[ \times \exp \left[ -\frac{g^2 A}{4} C_2(l_i) \right] \Delta^2(l_1, \ldots, l_N) \]

(27)

and

\[ \frac{1}{N^2 - 1} + W_k(A_1, A_2) = \frac{1}{Z_k((N^2 - 1)^{1/2})} \sum_{l_i^{R}, l_i^{S}=-\infty}^{+\infty} \frac{1}{\pi(N!)^2} \int_0^{2\pi} d\theta_1 \ldots d\theta_N \delta \left( \sum_{j=1}^{N} \theta_j \right) \]
\[ \times \int_0^{2\pi} d\alpha_1 d\alpha_2 e^{-\left( \alpha_1 - \frac{2\pi}{N} l^{R} \right)^2} e^{-\left( \alpha_2 - \frac{2\pi}{N} l^{S} \right)^2} \exp \left[ -\frac{g^2 A_1}{4} C_2(l_i^{R}) - \frac{g^2 A_2}{4} C_2(l_i^{S}) \right] \]
\[ \times \sum_{p,q=1}^{N} e^{i(\theta_p - \theta_q)} \prod_{h=1}^{N} e^{it^R_h \theta_h} \prod_{r=1}^{N} e^{-it^S_r \theta_r} \delta_{[N]} \left( k - l^{R} + \frac{N(N-1)}{2} \right) \Delta(l_1^{R}, \ldots, l_N^{R}) \Delta(l_1^{S}, \ldots, l_N^{S}). \]

(28)

It is convenient to interpret the constraint on the angles \( \theta_i \) in the equation above, as a periodic \( \delta \)-distribution

\[ \delta \left( \sum_{j=1}^{N} \theta_j \right) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in \sum_{j=1}^{N} \theta_j}, \]

(29)

the total volume being still finite and occurring also in the partition function at the denominator.
It is now natural to perform the following shift in Eq. (28)

\[ l_j^R \to l_j^R + n, \tag{30} \]

under which both \( \Delta(l_j^R) \) and \( C_2(l_j^R) \) are insensitive.

Hence, Eq. (28) reads

\[
\frac{1}{N^2 - 1} + \mathcal{W}_k(A_1, A_2) = \frac{1}{Z_k(N^2 - 1)} \sum_{l_i^R, l_i^S = -\infty}^{+\infty} \frac{1}{2\pi^2(N!)^2} \sum_{n = -\infty}^{+\infty} \int_0^{2\pi} d\theta_1 \ldots d\theta_N \times \\
\times \int_0^{2\pi} d\alpha_1 d\alpha_2 e^{-(\alpha_1 - \frac{2\pi}{N} l_j^R + 2\pi n)^2} e^{-(\alpha_2 - \frac{2\pi}{N} l_j^S)^2} \exp \left[-\frac{g^2 A_1}{4} C_2(l_j^R) - \frac{g^2 A_2}{4} C_2(l_j^S) \right] \\
\times \sum_{p,q=1}^{N} e^{i(\theta_p - \theta_q)} \prod_{h=1}^{N} e^{il_h^R \theta_h} \prod_{r=1}^{N} e^{-il_r^S \theta_r} \delta_{[N]} \left(k - l^S + \frac{N(N - 1)}{2} \right) \Delta(l_1^R, \ldots, l_N^R) \Delta(l_1^S, \ldots, l_N^S). \tag{31} \]

The next advance in the computation of \( \mathcal{W}_k(A_1, A_2) \) is reached by implementing Eq. (24) backwards and by working out the integration over the \( N \) angles \( \theta_i \), which produces

\[
(2\pi)^N N! \sum_{q_1, q_2=1}^{N} \prod_{j=1}^{N} \delta(l_j^R - l_j^S + \delta_{j,q_1} - \delta_{j,q_2}), \tag{32} \]

implying, for \( q_1, q_2 \) fixed

\[ l_j^R = l_j^S - \delta_{j,q_1} + \delta_{j,q_2}. \tag{33} \]

Although harmless as far as \( l^R \) is concerned, such a shift affects both the Casimir and the Vandermonde determinant related to the \( R \) representation. In particular, for the former we have

\[
C_2(l_j^R) = C_2(l_j^S) + 2 (l_{q_2}^S - l_{q_1}^S + 1) \quad \text{if} \quad q_1 \neq q_2
\]

\[
C_2(l_j^R) = C_2(l_j^S) \quad \text{if} \quad q_1 = q_2, \tag{34} \]

leading to the following form for \( \mathcal{W}_k \)

\[
\mathcal{W}_k(A_1, A_2) = \frac{1}{N + 1} \left\{ 1 + \frac{2}{Z_k(N - 1)} (2\pi)^{N-1} \sqrt{\pi} N! \sum_{l_i = -\infty}^{+\infty} \sum_{1=q_1<q_2}^{N} \exp \left[-\frac{g^2 A_1}{2} (l_{q_2} - l_{q_1} + 1) \right] \\
\times \int_0^{2\pi} d\alpha e^{-(\alpha - \frac{2\pi}{N} l_i^R)^2} \delta_{[N]} \left(k - l + \frac{N(N - 1)}{2} \right) \Delta(l_1, \ldots, l_N) \Delta(l_1, \ldots, l_{q_1} - 1, \ldots, l_{q_2} + 1, \ldots, l_N) \right\}, \tag{35} \]

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where also the normalization to $Z_k$ of Eq. (27) has been explicitly carried out in the first term of the summation. At this stage we possess nice formulae which enable us to deal with the decompactification limit.

The partition function (10) and the Wilson loop (20) in the limit of infinite area of the sphere are dominated by particular representations labelled by suitable indices $\{\bar{l}_i\}$. It is now easy to see that the dominant contribution is given by the following set

$$\{\bar{l}_i\} = \{0, 1, 2, \ldots, N - k - 1, N - k + 1, \ldots, N - 1, N\}, \quad (36)$$

and their permutations, which obey $\bar{l} = k + \frac{N(N-1)}{2}$ and for which the minimum value of the Casimir is reached

$$C_2(\bar{l}_i) = \frac{k(N - k)(N + 1)}{N}. \quad (37)$$

The number of distinct permutations, leaving the Casimir Eq. (37) unchanged, amounts to $\binom{N}{k}$. Evaluating $Z_k$ for $l_i = \bar{l}_i$, we obtain

$$Z_k(A \to \infty) = \frac{(2\pi)^{N-1}}{N! \sqrt{\pi}} \Delta^2(\bar{l}_1, \ldots, \bar{l}_N) \int_0^{2\pi} d\alpha \exp \left[ - \left( \alpha - \frac{2\pi}{N} k - \pi(N - 1) \right)^2 \right] \frac{g^2 A}{4} \frac{k(N - k)(N + 1)}{N}. \quad (38)$$

We observe that in the last line of Eq. (38) the exponent coincides with the Casimir of the $k$-fundamental representation, multiplied by $g^2 A/2$, and $\binom{N}{k}$ with its dimension.

We now focus on the Wilson loop in the form of Eq. (35). The decompactification limit $A \to \infty, A_1$ fixed, is performed by evaluating $W_k$ on the configurations (36). A simple form is found when the dependence on $Z_k$ is factorized out via Eq. (38)

$$W_k(A_1, A_2 \to \infty) = \frac{1}{N + 1} \left\{ 1 + \frac{1}{N - 1} \sum_{q_1 \neq q_2=1}^N \Delta(\bar{l}_1, \ldots, \bar{l}_{q_1} - 1, \ldots, \bar{l}_{q_2} + 1, \ldots, \bar{l}_N) \Delta(\bar{l}_1, \ldots, \bar{l}_N) \right\} \exp \left[ -\frac{g^2 A_1}{2} (\bar{l}_{q_2} - \bar{l}_{q_1} + 1) \right]. \quad (39)$$

This is not yet the end of the story. At this stage we have still to specify what are the shifts in $\bar{l}_j$ allowed by the Vandermonde determinants in the last term of Eq. (39) and determine
the string tensions $\sigma(q_1, q_2)$, to be read from the exponential in the same equation [4], and the relative weights they give rise to. It turns out the following four cases can occur

- $q_1 = j$ and $q_2 = j - 1$, with $j = 1, \ldots, N - k - 1, N - k + 2, \ldots, N$. As a whole, there are $N - 2$ possible swaps with vanishing string tension. Each of them contributes with the same weight; we can for instance choose $q_1 = 2$ and $q_2 = 1$

$$\frac{\Delta(\bar{l}_2, \bar{l}_1, \ldots, \bar{l}_N)}{\Delta(l_1, l_2, \ldots, l_N)} = -1$$

- $q_1 = 1$ and $q_2 = k$, with string tension $\sigma(1, k) = g^2 \frac{k}{2} (N - k)$ and weight

$$\frac{\Delta(\bar{l}_1 - 1, \ldots, \bar{l}_k + 1, \bar{l}_{k+1}, \ldots, \bar{l}_N)}{\Delta(l_1, l_2, \ldots, l_N)} = \frac{(N + 1)(N - k - 1)}{k + 1}$$

- $q_1 = k + 1$ and $q_2 = N$, with string tension $\sigma(k + 1, N) = g^2 \frac{k}{2} k$ and weight

$$\frac{\Delta(\bar{l}_1, \ldots, \bar{l}_k, \bar{l}_{k+1} - 1, \ldots, \bar{l}_N + 1)}{\Delta(l_1, l_2, \ldots, l_N)} = \frac{(N + 1)(k - 1)}{N - k + 1}$$

- $q_1 = 1$ and $q_2 = N$, with string tension $\sigma(1, N) = g^2 \frac{k}{2} (N + 1)$ and weight

$$\frac{\Delta(\bar{l}_1 - 1, \bar{l}_2, \ldots, \bar{l}_N + 1)}{\Delta(l_1, l_2, \ldots, l_N)} = \frac{kN(N + 2)(N - k)}{(k + 1)(N - k + 1)}.$$  

Finally, by substituting the previous results in Eq. (39), $W_k(A_1, A_2 \to \infty)$ becomes

$$W_k(A_1, A_2 \to \infty) = \frac{1}{N^2 - 1} \left[ 1 + \frac{kN(N + 2)(N - k)}{(k + 1)(N - k + 1)} e^{-g^2 A_1 (N + 1)} \right] + \frac{(N + 1)(N - k - 1)}{k + 1} e^{-g^2 A_1 (N - k)} + \frac{(N + 1)(k - 1)}{N - k + 1} e^{-g^2 A_1 k} \right].$$

A comment is now in order. Our result Eq. (40) coincides with Eq. (14) of Ref. [16], which was derived following an alternative route. In fact, each term in the sum (40) corresponds to an irreducible $SU(N)$ representation into which the tensor product of an adjoint representation with a $k$-fundamental one is decomposed. To realize this, notice that the overall

1We define the string tension $\sigma$ to be the exponent with changed sign divided by half the area of the loop.

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normalization is but the dimension of the adjoint, the prefactor of each term denotes the
degeneracy (dimension of the representation normalized to $\binom{N}{k}$) and the exponent is the
Casimir multiplied by $g^2A_1/2$. Nevertheless, we emphasize that such a procedure is appli-
cable only in the decompactification limit. As opposed to this, our starting point, Eq. (35),
holds in more general instances and will be unavoidable in considering the instanton repre-
sentation.

IV. TWO-LOOP CORRELATION ON THE SPHERE

As promised, in this section we consider the correlation on the sphere between two non-
intersecting (nested) loops, one in the adjoint representation and the other in the $k$-
fundamental one. We call $A_2$ the area of the annulus between the loops, $A_3$ and $A_1$ the
other two areas encircled by the loops so that $A_1 + A_2 + A_3$ equals the total area $A$ of the
sphere. We shall firstly take the decompactification limit $A_3 \to \infty$, keeping fixed the other
two areas. Eventually we shall send the $k$-fundamental loop to $\infty$, by performing the second
limit $A_2 \to \infty$ keeping $A_1$ fixed.

Our purpose in so doing is to explore to what extent this procedure reproduces the result
we have obtained in Sect. III, working with a single loop in the adjoint representation on the
sphere in a $k$-sector. As anticipated in the Introduction, we find that the Witten conjecture,
namely, in our language, that the two results have to coincide when the sphere is decom-
 pactified to the plane, is indeed verified. The $k$-selection rule on the allowed representations
can be interpreted as the presence of $k$-fundamental charges at $\infty$. However this is true only
for the exact solution, and only after decompactification of the sphere.

Following the equations given for the heat kernel in Sect. II, it is easy to write the
expression for the correlation of the two loops on the sphere we mentioned (we use the
short-hand notation $C_R \equiv C_2(R)$)

$$
W_{k,a}(A_1,A_2,A_3) = \frac{1}{Z_k \mathcal{W}_k (N^2 - 1)} \sum_{R,S,T} \exp \left[ - \frac{g^2}{4} (A_3 C_R + A_2 C_S + A_1 C_T) \right]
$$
\[ d_R d_T \int dU_1 \chi_R^\dagger(U_1) \chi_k(U_1) \chi_S(U_1) \int dU_2 \chi_S^\dagger(U_2) \chi_a(U_2) \chi_T(U_2), \quad (41) \]

where

\[ Z_k W_k = \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2}{4} (A_3 C_R + (A - A_3) C_S) \right] \int dU \chi_R^\dagger(U) \chi_k(U) \chi_S(U), \quad (42) \]

so that the natural normalizations

\[ W_{k,a}(A_1 = 0, A_2, A_3) = 1, \]
\[ W_k(A_3, A - A_3 = 0) = 1 \]

will ensue. We notice we have dropped the common factor \( \binom{N}{k} \); further irrelevant common factors will be dropped in the following.

To warm up we begin considering \( W_k \)

\[ W_k(A_3, A - A_3) = \frac{k!}{Z_k(A)} \sum_{R,S} d_R d_S \exp \left[ -\frac{g^2 A_3}{4} C_R - \frac{g^2 (A - A_3)}{4} C_S \right] \int d\theta_1 \ldots d\theta_N \delta \left( \sum_{j=1}^N \theta_j \right) \sum_{j_1 < \ldots < j_k} e^{i(\theta_{j_1} + \ldots + \theta_{j_k})} \det|e^{iR_{\theta_p}}| \det|e^{-iS_{\theta_q}}|. \quad (43) \]

We now repeat the familiar procedure, integrating over the angles and taking symmetry properties into account; we end up with the expression

\[ W_k(A_3, A - A_3) Z_k(A) = \sum_{l_i=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dl \exp \left[ i\beta \left( l - \sum_{j=1}^N l_j \right) \right] \times \]
\[ \int_0^{2\pi} d\alpha \exp \left[ -\left( \alpha - \frac{2\pi}{N} l \right)^2 \right] \exp \left[ -\frac{g^2 A_3}{4} C(l_j^R) - \frac{g^2 (A - A_3)}{4} C(l_j) \right] \Delta(l_j^R) \Delta(l_j), \quad (44) \]

where

\[ l_1^R = l_1 - 1, \ldots, l_k^R = l_k - 1, l_{k+1}^R = l_{k+1}, \ldots, l_N^R = l_N \quad (45) \]

and \( C(l_j) \) is the usual \( SU(N) \) expression of the quadratic Casimir in terms of the representation labels.

Next, we go back to Eq. (41); performing the usual harmonic analysis in terms of Young tableaux and taking symmetry properties into account, it can be written as
\[ \mathcal{W}_{k,a}(A_1, A_2, A_3) = \frac{1}{N+1} + \frac{1}{Z_k} \mathcal{W}_k (N^2 - 1) \binom{N}{k} \sum_{l_j=-\infty}^{+\infty} \sum_{q_1 \neq q_2=1}^{N} \int_0^{2\pi} d\alpha \exp \left[ - \left( \alpha - \frac{2\pi}{N} l \right)^2 \right] \]
\times \exp \left[ - \frac{g^2}{4} (A_3 C(l^R_j) + A_2 C(l_j) + A_1 C(l^T_j)) \right] \Delta(l^R_j) \Delta(l^T_j) \tag{46} \]

with the constraints

\[ l_1^R = l_1 - 1, \ldots, l_k^R = l_k - 1, \quad l_{k+1}^R = l_{k+1}, \ldots, l_N^R = l_N, \]
\[ l_1^T = l_{q_1} - 1, \quad l_2^T = l_{q_2} + 1, \quad l_j^T = l_{q_j} \]

for \( j = 3, \ldots, N \) and

\[ l = \sum_{j=1}^{N} l_j. \]

Eqs. (44) and (46) are suitable for considering the decompactification limit \( A_3 \to \infty, A - A_3 \) fixed and \( A_3 \to \infty, A_2, A_1 \) fixed, respectively.

Let us start from Eq. (44). Since the constraints (45) imply

\[ C_2(l^R) = \sum_{i=1}^{k} \left( l_i - 1 - \frac{l}{N} \right)^2 + \sum_{i=k+1}^{N} \left( l_i - \frac{l}{N} \right)^2 - \frac{k^2}{N} - \frac{N(N^2 - 1)}{2}, \tag{47} \]

it is immediately recognized that \( W_k Z_k \) is dominated by particular representations labelled by the following set of indices

\[ \{ \bar{l}_i \} = \{ 0, 1, 2, \ldots, N - k - 1, N - k + 1, \ldots, N - 1, N \}, \tag{48} \]

and their permutations, for which the minimum value of the \( R \)-representation Casimir is reached \( (C_2(l^R) = 0) \). Correspondingly we have

\[ C_2(\bar{l}) = \sum_{i=1}^{N} \left( \bar{l}_i - \frac{\bar{l}}{N} \right)^2 \frac{N(N^2 - 1)}{2} = \frac{k(N-k)(N+1)}{N}, \tag{49} \]

i.e. the Casimir of the \( k \)-fundamental representation. Notice that the dominant configurations of indices coincide with the ones we found in Sect. III, when performing the decompactification limit of the Wilson loop \( \mathcal{W}_k(A_1, A_2) \) in the \( k \)-th sector.

Proceeding further with \( \mathcal{W}_{k,a}(A_1, A_2, A_3) \), the evaluation of Eq. (46) for \( l_i = \bar{l}_i \) yields
\[ W_{k,a}(A_1, A_2, A_3 \to \infty) = \frac{1}{N+1} + \frac{1}{(N^2-1)} \sum_{q_1 \neq q_2=1}^{N} \frac{\Delta(\bar{l}_1, \ldots, \bar{l}_{q_1}-1, \ldots, \bar{l}_{q_2}+1, \ldots, \bar{l}_N)}{\Delta(l_1, \ldots, l_N)} \times \exp \left[ -\frac{g^2 A_1}{2} (\bar{l}_{q_2} - \bar{l}_{q_1} + 1) \right], \] (50)

having taken the constraints on \( \bar{t}^T \) into account. This equation exhibits the remarkable property of being independent of \( A_2 \). Indeed, it precisely coincides with Eq. (39) of Sect. III, which leads straight to Eq. (40). The independence of \( A_2 \) of Eq. (50) justifies the procedure adopted in Ref. [16], where string tensions and weights were obtained just by decomposing the direct product of an adjoint and a \( k \)-fundamental representation into its irreducible components.

At this point, we would like to stress some remarkable features of Eq. (40). If the adjoint loop is interpreted as a pair of adjoint fermions in the \( k \)-th vacuum state, we immediately see that the addition of the \( k \)-loop at infinity gives rise to up to four distinct singlet configurations: the interaction energies between adjoint charges generally depend on the vacuum \( k \) in which they are measured. Furthermore, the physics of the adjoint loops depends on \( k \mod N \), in analogy with the continuous \( \theta \)-angle of the Schwinger model, which is periodic in \( 2\pi \), and is symmetric under changing the representation of the external loop by \( k \to N - k \) (which in turn goes back to the invariance of the adjoint representation under charge conjugation). It follows that the vacua corresponding to \( k \) and \( N - k \) are degenerate in energy. Finally, surprisingly enough, a configuration presents vanishing string tension and hence the binding energy between the static adjoint charges is vanishing in the non-trivial \( k \neq 0 \) sectors of the theory.

We have thus proven Witten’s conjecture, namely that the \( SU(N)/Z_N \) theory on the plane in a \( k \)-sector is equivalent to the usual \( SU(N) \) theory in presence of a \( k \)-fundamental Wilson loop at infinity.
V. PERTURBATIVE SERIES DEFINED VIA CPV PRESCRIPTION

We now check that the expectation value of an adjoint loop enclosed in an asymptotic $k$-fundamental one on the plane, expressed by Eq. (40), is consistent with the perturbative LF computation, at least up to $O(g^4)$. Such a calculation suffices to give a flavour on how things work at higher order.

To begin with, let us briefly review the outset of quantization in the light-cone gauge $A_- = 0$ in two dimensions. If the theory is quantized on the light-front (at equal $x^+$), no dynamical degrees of freedom occur as the non-vanishing component of the vector field does not propagate

$$D^P_{++}(x) = -\frac{i}{4}|x^-| \delta(x^+), \quad x^\pm = \frac{x^0 \pm x^1}{\sqrt{2}}, \quad (51)$$

but rather gives rise to an instantaneous (in $x^+$) Coulomb-like potential. A formulation based essentially on the potential in Eq. (51) was originally proposed by G. 't Hooft in 1974 [26], to derive beautiful solutions for the $q\bar{q}$-bound state problem under the form of rising Regge trajectories.

On the other hand, when the theory is quantized at equal-times, the free propagator has the following causal expression (WML prescription) in two dimensions

$$D^{WML}_{++}(x) = \frac{1}{4\pi} \frac{x^-}{-x^+ + i\varepsilon x^-}, \quad (52)$$

first proposed by T.T. Wu [18]. In turn this propagator is nothing but the restriction in two dimensions of the expression proposed by S. Mandelstam [19] and G. Leibbrandt [20] in four dimensions and derived by means of a canonical quantization in Ref. [27].

When inserted in perturbative Wilson loop calculations, expressions (51) and (52) lead to completely different results, as first noticed in Ref. [29]. The origin of this discrepancy was

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2In dimensions higher than two, where physical degrees of freedom are switched on (transverse “gluons”), this causal prescription is the only acceptable one [28].
contributions are those with an even number of derivatives with respect to $C$ with $j$.

The definition of $W$ will be discussed in Sect. VII. Firstly, let us start from the perturbative to the adjoint and to the $k$-fundamental representations. Notice that normalization is such that when $\Gamma_a$ is shrunk to a point, $W_{k,a} = 1$. Next we consider two light-like rectangles (see Fig. 1), one with sides $2l$, $2t$, where the adjoint representation sits, nested in a larger rectangle with sides $2L$, $2T$, where instead the $k$-fundamental sits, and choose the currents with support on the contours, so that

$$J^a(x^+, x^-) = j_1^a \delta(x^- - L) + j_2^a \delta(x^- + L) + j_3^a \delta(x^- - l) + j_4^a \delta(x^- + l).$$

With this choice the perturbative expansion Eq. (53) for $W_{k,a}$ reads

$$W_{k,a} = \frac{1}{(N^2 - 1) Z_k W_k} \left\{ \text{Tr} \left[ \mathcal{P} \exp \left[ g \oint_{\Gamma_a} \frac{t^b}{\delta J^c(x)} \, dx^+ \right] \right] \text{Tr} \left[ \mathcal{P} \exp \left[ g \oint_{\Gamma_a} \frac{T^a}{\delta J^c(x)} \, dx^+ \right] \right] \right\}$$

$$\times \exp \left[ -\frac{1}{2} \int d^2 x \, d^2 y \, J^c(x) D(x - y) J^c(y) \right] \right\}_{J=0}, \quad (53)$$

where the propagator $D(x - y)$ is defined through Eq. (51) and the matrices $T^a$, $t^b$ belong to the adjoint and to the $k$-fundamental representations, respectively. Notice that normalization is such that when $\Gamma_a$ is shrunk to a point, $W_{k,a} = 1$. Next we consider two light-like rectangles (see Fig. 1), one with sides $2l$, $2t$, where the adjoint representation sits, nested in a larger rectangle with sides $2L$, $2T$, where instead the $k$-fundamental sits, and choose the currents with support on the contours, so that

$$J^a(x^+, x^-) = j_1^a \delta(x^- - L) + j_2^a \delta(x^- + L) + j_3^a \delta(x^- - l) + j_4^a \delta(x^- + l).$$

With this choice the perturbative expansion Eq. (53) for $W_{k,a}$ reads

$$W_{k,a} = \frac{1}{(N^2 - 1) Z_k W_k} \text{Tr} \left\{ \mathcal{P} \exp \left[ g \oint_{C_2} \frac{t^b}{\delta J^a_i(x^+)} \, dx^+ \right] \mathcal{P} \exp \left[ g \oint_{C_1} \frac{T^a}{\delta j^a_i(x^+)} \, dx^+ \right] \right\}$$

$$\times \exp \left[ L \int_{-T}^{+T} dx^+ \, j^c_1(x^+) \, j^c_2(x^+) + \frac{L + l}{2} \int_{-l}^{+l} dx^+ \left( j^c_3(x^+) \, j^c_3(x^+) + j^c_4(x^+) \right) \right]$$

$$+ \frac{L - l}{2} \int_{-l}^{+l} \int_{-l}^{+l} dx^+ \, dx^+ \, j^c_1(x^+) \, j^c_4(x^+) + j^c_2(x^+) \, j^c_3(x^+) \right\}_{j_i=0} \quad (54)$$

with $C_1$, $C_2$, $C_3$, $C_4$ as in Fig. 1 and $i = 1$, $2$, $3$, $4$. Clearly, up to $\mathcal{O}(g^4)$, the only non-vanishing contributions are those with an even number of derivatives both with respect to $j_{1,2}$ and to $j_{3,4}$.
At $\mathcal{O}(g^2)$, either a factor $iL$ or $il$ is produced when a derivative acts on the first or the last term, respectively, of the last exponential in Eq. (54), which represents the unique contributions at this order, and finally $(2T)$ or $(2t)$, respectively is given by integration over the loop variable. Thus we end up with

$$\frac{i}{4}g^2 (C_k A_2 + C_A A_1)$$

(55)

where $C_k = \frac{k(N-k)(N+1)}{N}$ and $C_A = 2N$ are the quadratic Casimir of the $k$-fundamental and of the adjoint representations, respectively, and $A_2 = 4LT$, $A_1 = 4lt$ being the areas of the rectangles. One can say, equivalently, the two loops factorize at this order, so that, switching to the Euclidean space-time via a Wick rotation and taking normalization into account, $\mathcal{W}_{k,a}^{(II)}$ reads

$$\mathcal{W}_{k,a}^{(II)}(A_1, A_2) = -g^2 \frac{C_A}{4} A_1.$$  

(56)
Likewise, we can now easily derive the expression of $W_{k,a}^{(IV)}$, i.e. the adjoint loop enclosed in a $k$-fundamental one at $O(g^4)$. Very schematically, different classes of diagrams can be distinguished. Let us browse on them. Obvious contributions are the pure $k$-fundamental loop at $O(g^4)$ (which corresponds to Tr $I_{ad}$ in the inner loop) and the product of the pure adjoint by the pure $k$-fundamental loops both at $O(g^2)$, which turn out to result in

$$- \frac{g^4}{32} \left( C_k^2 A_2^2 + 2 C_A C_k A_1 A_2 \right).$$

On the other hand, it is clear they will be removed by normalizing to $Z_k W_k$. Very much in analogy with the first contribution, we have to consider the pure adjoint loop at $O(g^4)$ (which corresponds to Tr $I_k$ in the outer loop)

$$- \frac{g^4}{32} C_A^2 A_2^2.$$ 

However, the novelty in having one loop enclosed into another is supplied by graphs with propagators connecting sides of both rectangles. We naturally single out three prototypes, in which the following situations occur

1. both propagators are “long” (they travel a distance $L + l$);
2. both propagators are “short” (they travel a distance $L - l$)
3. a propagator is “short” and the other one is “long”.

Pictorially, it is straightforward to recognize that there are three diagrams falling within both the first and the second class, and among them two graphs carry a factor $1/2$, owing to the presence of integrals in the loop variables which are nested. Hence their contribution to $W_{k,a}^{(IV)}$ adds up to

$$- \frac{g^4}{4} t^2 \frac{C_A C_k}{N^2 - 1} \left( \frac{1}{2} + \frac{1}{2} + 1 \right) \left[ (L + l)^2 + (L - l)^2 \right].$$

We emphasize that the factor $\frac{C_A C_k}{N^2 - 1}$ arises from Tr $[t^a t^b]_k \cdot$ Tr $[T^a T^b]_{adj}$, properly normalized to $(N^2 - 1)\binom{N}{k}$ (the latter in $Z_k W_k$). Finally, four diagrams belong to the third class and yield
\[ g^4 t^2 \frac{C_A C_k}{N^2 - 1} (L^2 - l^2) \] (60)
in which the opposite sign with respect to Eq. (59) has to be ascribed to the “short” propagator joining sides with the same orientation (the appearance of the factor \( \frac{C_A C_k}{N^2 - 1} \) has been explained above).

When the partial results Eqs. (58-60) are summed up, we obtain for \( W_{k,a} \) in the Euclidean space-time

\[ W_{k,a}^{(IV)}(A_1, A_2) = g^4 \frac{3}{32} \left( C_A A_1^2 + 4 \frac{C_A C_k}{N^2 - 1} A_1^2 \right), \] (61)

which coincides with the \( O(g^4) \) term of Eq. (41). The perturbative LF formulation seems to capture again the exact result, even in presence of a non-trivial topology.

VI. THE INSTANTON REPRESENTATION

As first pointed out by Witten [23], one can represent the partition function \( Z_k \) in Eq. (10) and the Wilson loop \( W_k \) in Eq. (20) on the sphere \( S^2 \) as sums over instanton contributions, following the procedure proposed in [30,31] (see also [32]). By instanton we mean a non-trivial classical solution of the Yang-Mills equations on \( S^2 \), which takes the form of an Abelian Dirac monopole embedded into the non-abelian gauge group. For the general construction on any genus we refer to [24].

Any of such configurations is characterized, in the \( SU(N) \) or \( U(N) \) case, by a set of \( N \) integers \( (n_1, \ldots, n_N) \). The set \( (0, \ldots, 0) \) represents the topologically trivial solution. In Ref. [3] it has been shown, for the case of a Wilson loop in the fundamental representation of the group \( U(N) \), that it can be obtained by a \textit{bona fide} perturbative calculation [33] for the theory quantized in the light-cone gauge by means of ET canonical commutators [27,29].

From the mathematical viewpoint, the zero-instanton sector is reproduced if integration over the group manifold is replaced by integration over the tangent group algebra [34].

Here we want to generalize such results to the case of a Wilson loop in the adjoint representation for the \( SU(N) \) theory, when \( k \)-sectors are taken into account. In so doing an
intriguing interplay occurs between instantons and k-states and one may wonder to what extent perturbation theory can account for it. In Sect. IV we have shown that an adjoint loop in a k-vacuum state is equivalent to the same loop enclosed in a k-fundamental boundary one.

In the following we will carry out the instanton expansion separately for the case of $SU(N)/Z_N$ in the k-sector (i) and for the one with a k-fundamental boundary loop (ii). We will find completely different results. Let us begin with the former.

(i)

The instanton representation can be obtained by performing in Eqs. (27,35) a Poisson resummation. Starting from Eq. (27), it is convenient to introduce explicitly the constraint (25) in a factorized form

$$Z_k(A) = \frac{(2\pi)^{N-2}}{N! \sqrt{\pi}} \sum_{l_i = -\infty}^{+\infty} \int_{-\infty}^{+\infty} dl \int_0^{2\pi} d\alpha \exp\left(-\frac{g^2 A}{4} c_2(l_i)\right) \left(k - l + \frac{N(N-1)}{2}\right)$$

$$\times \int_{-\infty}^{+\infty} d\beta \exp\left[i\beta(l - \sum_{i=1}^{N} l_i)\right] \exp\left[-\frac{g^2 A}{4} C_2(l_i)\right] \Delta^2(l_1,\ldots,l_N).$$

The Poisson transformation is a kind of duality relation between two series

$$\sum_{l_i = -\infty}^{+\infty} F(l_1,\ldots,l_N) = \sum_{n_i = -\infty}^{+\infty} \tilde{F}(n_1,\ldots,n_N),$$

$$\tilde{F}(n_1,\ldots,n_N) = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N F(z_1,\ldots,z_N) \exp\left[2\pi i(z_1 n_1 + \ldots + z_N n_N)\right].$$

In order to perform the Fourier transform in (62), we remember that the transformation of a product is turned into a convolution; moreover we recall the result

$$\int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp\left[i(z_1 p_1 + \ldots + z_N p_N)\right] \Delta(\{z_i\}) \exp\left(-\frac{g^2 A}{8} \sum_{q=1}^{N} z_q^2\right) =$$

$$\left[\frac{4i}{g^2 A}\right]^{N(N-1)/2} \left[\frac{8\pi}{g^2 A}\right]^{N/2} \Delta(\{p_i\}) \exp\left(-\frac{2}{g^2 A} \sum_{q=1}^{N} p_q^2\right).$$

Taking these relations into account, Eq. (62) becomes

$$Z_k(A) = \sum_{n=0}^{N-1} \exp\left[\frac{2\pi i n k}{N}\right] Z^{(n)}(A),$$

27
where
\[
\mathcal{Z}^{(n)}(A) = (-1)^{n(N-1)} C(A, N) \sum_{n_q = -\infty}^{+\infty} \delta(n - N) \exp \left[ -\frac{4\pi^2}{g^2 A} \sum_{q=1}^{N} (n_q - \frac{n}{N})^2 \right] \zeta_n(\{n_q\}) \tag{66}
\]
with
\[
C(A, N) = \frac{(2\pi)^{2N-3}}{N!} \sqrt{\pi N} \frac{g^2}{16} N^{(N^2-1)} 2^{N+\frac{1}{2}} (g^2 A)^{-(N^2-N/2-1/2)}
\]
and
\[
\zeta_n(\{n_q\}) = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \Delta(\{\sqrt{g^2 A/2} z_q + 2\pi n_q\}) \Delta(\{\sqrt{g^2 A/2} z_q - 2\pi n_q\}). \tag{67}
\]
Eqs. (65,66) exactly provide the explicit form of the partition function in the \(k\)-sector in the instanton representation, as anticipated in Eqs. (13,14). \(\mathcal{Z}^{(n)}(A)\) is the contribution from the \(n\)-th topological sector and is localized around the classical solutions of the Yang-Mills equations in that sector. According to the localization picture, in Eq. (66) we can readily single out the contribution of the classical instanton action \(\exp \left[ -\frac{4\pi^2}{g^2 A} \sum_{q=1}^{N} (n_q - \frac{n}{N})^2 \right]\): it is remarkable that the non-trivial bundle structure \(n \neq 0\) induces a shift in the instanton numbers from integral to fractional quantities, \(n_q \to n_q - \frac{n}{N}\), while the \(\delta\)-constraint properly implements the tracelessness condition for a \(SU(N)\) matrix. Moreover, \(\zeta_n(\{n_q\})\) represents the contribution of the quantum fluctuations around the classical solutions. We notice that the coefficient \(C(A, N)\) is singular as \(\sqrt{g^2 A} \to 0\): this is expected because zero-modes appear when computing fluctuations in the instanton background, the total degree of singularity depending on the instanton numbers, as the polynomial part in Eq. (67) shows [30]. The only non-exponentially suppressed contribution, in that limit, comes from the \(n = 0\) sector, as argued by Witten in [23], and, in particular, only the fluctuations around the trivial connections survive.

The same procedure is now to be performed for \(W_k(A_1, A_2)\) starting from Eq. (32). We obtain the instanton representation
\[
W_k(A_1, A_2) = \frac{1}{N+1} + \frac{1}{\mathcal{Z}_k(A)} \sum_{n=0}^{N-1} \exp \left[ \frac{2\pi i n k}{N} \right] W^{(n)}(A_1, A_2) \tag{68}
\]
where
\[
W^{(n)} = (-1)^{n(N-1)} \frac{2C(A,N)}{N^2 - 1} \exp \left[ \frac{g^2(A_1 - A_2)^2}{8A} \right] \sum_{r<s} \sum_{n_q=-\infty}^{+\infty} \delta(n - N \sum_{q=1}^{N} n_q) \times \exp \left[ -\frac{4\pi^2}{g^2A} \sum_{q=1}^{N} (n_q - n_q/N)^2 \right] \exp \left[ 2\pi i(n_s - n_r) A_q A \right] \Omega_n(\{n_q\}) (69)
\]
and
\[
\Omega_n(\{n_q\}) = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \exp \left[ \frac{i}{2} \sqrt{\frac{g^2A}{2}} (z_r - z_s) \right] \times \Delta(\sqrt{\frac{g^2A}{2} z_1 - \tilde{n}_1}, \ldots, \sqrt{\frac{g^2A}{2} z_N - \tilde{n}_N}) \Delta(\sqrt{\frac{g^2A}{2} z_1 + \tilde{n}_1}, \ldots, \sqrt{\frac{g^2A}{2} z_N + \tilde{n}_N}), (70)
\]
with
\[
\tilde{n}_q = 2\pi n_q - (\delta_q,r - \delta_q,s) \frac{ig^2(A_1 - A_2)}{4}. (71)
\]
Following Ref. [31], in Eq. (69) one can still single out the classical instanton actions and their classical contributions to the Wilson loop (\(\exp [2\pi i(n_s - n_r) A_q A] \)).

We now focus our attention on the zero-instanton sector \((n_q = 0, \forall q)\) with the purpose of exploring its relation with possible perturbative treatments.

Taking symmetry under permutations into account, Eqs. (65-67) and (68-70) become
\[
Z_k^{(0)}(A) = C(A, N) \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \Delta^2(\sqrt{\frac{g^2A}{2} \{z_q\}}) (72)
\]
and
\[
W_k^{(0)}(A_1, A_2) = \frac{1}{N + 1} \left[ 1 + \frac{C(A, N)}{Z_k^{(0)}(A)} \exp \left[ \frac{g^2(A_1 - A_2)^2}{8A} \right] \right] \times \\
\int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \exp \left[ \frac{i}{2} \sqrt{\frac{g^2A}{2}} (z_1 - z_2) \right] \times \\
\Delta(\sqrt{\frac{g^2A}{2} z_1 + \frac{ig^2}{4}(A_1 - A_2)}, \sqrt{\frac{g^2A}{2} z_2 - \frac{ig^2}{4}(A_1 - A_2)}, \sqrt{\frac{g^2A}{2} z_3}, \ldots, \sqrt{\frac{g^2A}{2} z_N}) \times \\
\Delta(\sqrt{\frac{g^2A}{2} z_1 - \frac{ig^2}{4}(A_1 - A_2)}, \sqrt{\frac{g^2A}{2} z_2 + \frac{ig^2}{4}(A_1 - A_2)}, \sqrt{\frac{g^2A}{2} z_3}, \ldots, \sqrt{\frac{g^2A}{2} z_N}). (73)
\]
We remark that the dependence on \(k\) has completely disappeared; the label \(k\) will be thereby dropped.
We express the Vandermonde determinants in terms of Hermite polynomials and then expand them using the completely antisymmetric tensor. Afterwards, we integrate over $z_3, \ldots, z_N$, taking orthogonality into account, and are left with the expression

$$\varepsilon^{j_1 j_2 j_3 \ldots j_N} \varepsilon_{q_1 q_2 q_3 \ldots q_N} (-1)^{j_2 - q_2} \times$$

$$\int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( \frac{i \sqrt{g^2 A} z_1}{2\sqrt{2}} \right) H e_{j_1} (z_{1+}) H e_{q_1} (z_{1-}) \times$$

$$\int_{-\infty}^{+\infty} dz_2 \exp \left[ -\frac{1}{2} z_2^2 \right] \exp \left( \frac{i \sqrt{g^2 A} z_2}{2\sqrt{2}} \right) H e_{j_2} (z_{2+}) H e_{q_2} (z_{2-}),$$

(74)

where

$$z_{1,2\pm} = z_{1,2} \pm i \frac{\sqrt{2g^2}}{A} (A - 2A_1).$$

(75)

We have now the remarkable relation [34]

$$\int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( \frac{i \sqrt{g^2 A} z_1}{2\sqrt{2}} \right) H e_{q} (z_{1+}) H e_{r} (z_{1-}) =$$

$$\exp \left[ -\frac{g^2}{16A} (A - 2A_1)^2 \right] (A - A_1) \frac{g}{4\pi A_1} \Delta^2 (z_1, \ldots, z_N) \times$$

$$\int_{-\infty}^{+\infty} dz_1 \exp \left[ -\frac{1}{2} z_1^2 \right] \exp \left( igz_1 \sqrt{A_1(A - A_1)} \right) H e_{q} (z_1) H e_{r} (z_1).$$

Thanks to it, Eq. (73) takes the form

$$W^{(0)} (A_1, A_2) = \frac{1}{N + 1} + \frac{N}{Z (N + 1)} \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{j=1}^{N} z_j^2 \right] \times$$

$$\exp \left[ ig(z_1 - z_2) \sqrt{A_1(A - A_1)} \right] \Delta^2 (z_1, \ldots, z_N) =$$

$$\frac{1}{Z} \int \mathcal{D} F \exp (-\frac{1}{2} \text{Tr} F^2) \frac{1}{N^2 - 1} [\text{Tr} (\exp (igF) \mathcal{E})]^2 - 1],$$

(76)

where $Z = \int \mathcal{D} F \exp (-\frac{1}{2} \text{Tr} F^2)$.

Here $\mathcal{E} = \sqrt{\frac{A_1(A - A_1)}{2A}}$ and $\mathcal{D} F$ denotes the flat integration measure on the tangent space of constant Hermitian traceless $N \times N$ matrices. The restriction to traceless matrices only affects the overall normalization in this case. The powers appearing in Eq. (76) cancel owing to the presence of the antisymmetric tensor.
Retaining only the zero-instanton sector is therefore equivalent to integrating over the group algebra \[34\]. On the other hand, if in Eq. (76) we perform the decompactification limit \(A \to \infty\), we exactly recover the perturbative result in which we have used the WML propagator \[33,6,34\]. Memory of the \(k\)-th topological sector has been completely lost.

(ii)

We now address the issue of singling the zero-instanton (trivial) sector out for the adjoint loop enclosed in a \(k\)-fundamental one. The result we are going to obtain will keep a \(k\)-dependence and will by no means correspond to any perturbative calculation, at variance with the preceding case. One should not be too surprised by this conclusion; as a matter of fact, the instanton structures of the two cases, as long as one remains on the sphere, are completely different. Only in the decompactification limit, when all instantons are summed in both cases, the same limit ensues; but there is no reason why this miracle should occur when the two (different) zero-instanton sectors are compared.

The calculations that follow, although conceptually simple, are rather heavy and we shall try to condense them as much as possible. Starting from Eq. (44), we define

\[
\tilde{n}_1 = 2\pi n_1 - ig^2\left(2A_3 - A\right), \ldots , \tilde{n}_k = 2\pi n_k - ig^2\left(2A_3 - A\right), \tilde{n}_{k+1} = 2\pi n_{k+1}, \ldots , \tilde{n}_N = 2\pi n_N.
\]

After a long calculation we are led to the result

\[
W_k Z_k = \sum_{n_1 = -\infty}^{+\infty} \delta \left(\sum_{q=1}^{N} n_q\right) \exp \left[\frac{g^2 k^2 A_3 (A - A_3)}{4NA}\right] \exp \left[\frac{g^2 k}{16A} (2A_3 - A)^2\right] \exp \left[-\frac{4\pi^2}{g^2 A} \sum_{q=1}^{N} n_q^2\right] \exp \left(\frac{2i\pi A_3}{A} \sum_{q=1}^{k} n_q\right) \int_{-\infty}^{+\infty} dy_1 \ldots dy_N \exp \left[-\frac{1}{g^2 A} \sum_{q=1}^{N} y_q^2\right] \exp \left(-\frac{i}{2} \sum_{q=1}^{k} y_q\right) \Delta(y_j + \tilde{n}_j) \Delta(y_j - \tilde{n}_j) .
\]

We remark the quite different structure of the classical instanton action when compared to the one of the preceding section Eq. (73).

The zero-instanton sector is obtained again by choosing \(n_q = 0, \forall q\). Eq. (77) becomes
Now, rescaling the variables, expanding the Vandermonde determinants in terms of Hermite polynomials and taking Eq. (76) into account, Eq. (78) assumes the form

\[
\mathcal{W}^{(0)}_k \mathcal{Z}^{(0)}_k = \exp \left[ \frac{g^2 k^2 A_3 (A - A_3)}{4NA} \right] \exp \left[ \frac{g^2 k}{16A} (2A_3 - A)^2 \right] \times \\
\int_{-\infty}^{+\infty} dy_1 \ldots dy_N \exp \left[ -\frac{1}{g^2 A} \sum_{q=1}^{N} y_q^2 \right] \exp \left( -\frac{i}{2} \sum_{q=1}^{k} y_q \right) \times \\
\Delta(y_1 - i \frac{g^2}{4} (2A_3 - A), \ldots, y_k - i \frac{g^2}{4} (2A_3 - A), y_{k+1}, \ldots, y_N) \\
\Delta(y_1 + i \frac{g^2}{4} (2A_3 - A), \ldots, y_k + i \frac{g^2}{4} (2A_3 - A), y_{k+1}, \ldots, y_N). 
\]

(78)

F being a constant \( N \times N \) hermitian matrix and the trace being taken in the \( k \)-fundamental representation. The exponential factor is needed to turn the \( U(N) \) representation into one of \( SU(N) \). As a matter of fact, the previous equation can be written as

\[
\mathcal{W}^{(0)}_k \mathcal{Z}^{(0)}_k = \int \mathcal{D}F \exp \left( -\frac{1}{2} \text{Tr} F^2 \right) \text{Tr} \left[ \exp \left( igF \sqrt{\frac{A_3 (A - A_3)}{2A}} \right) \right]_k, 
\]

(79)

with a \textit{traceless} matrix \( F \). Again keeping the zero-instanton sector is equivalent to integrating over the group algebra.

The instanton expansion of \( \mathcal{W}_{k,a} \) is rather cumbersome and we present here just a brief account. Referring to Eq. (46), four cases are now possible:

- \( q_1 < q_2 \leq k \) with weight \( \binom{k}{2} \),

- \( k < q_1 < q_2 \) with weight \( \binom{N-k}{2} \),

- \( q_1 \leq k < q_2 \) with weight \( \frac{k(N-k)}{2} \),

- \( q_2 \leq k < q_1 \) with weight \( \frac{k(N-k)}{2} \).
Correspondingly, the second term in the r.h.s. of Eq. (46) splits into four contributions

\[
W_{k,a}(A_1, A_2, A_3) = \frac{1}{N+1} + \frac{2}{Z_k W_k (N^2 - 1)} \binom{N}{k} \sum_{l_j} \int_0^{2\pi} \! d\alpha \exp \left[ -\left(\alpha - \frac{2\pi}{N} l\right)^2 \right] \times \\
\exp \left[ -\frac{g^2 A}{4} C(l_j) - \frac{g^2}{2} A_1 \right] \exp \left[ -\frac{g^2 A_3}{4} \left(\frac{k(N - k)}{N} - 2 \sum_{j=1}^{k} (l_j - \frac{l}{N})\right) \right] \Delta(l_j^T) \times \\
\left[ \binom{k}{2} \Delta_{(1)}(l_j^T) e^{\frac{g^2 A_1}{2} (l_1 - l_2)} + \left(\frac{N - k}{2}\right) \Delta_{(2)}(l_j^T) e^{\frac{g^2 A_1}{2} (l_1 - l)} \right] + \\
\frac{k(N - k)}{2} \left( \Delta_{(3)}(l_j^T) e^{\frac{g^2 A_1}{2} (l_1 - l_N)} + \Delta_{(4)}(l_j^T) e^{-\frac{g^2 A_1}{2} (t_1 - t_N)} \right) \\
= \frac{1}{N + 1} + \frac{2}{Z_k W_k (N^2 - 1)} \binom{N}{k} \left[ W_{k,a}^{(1)} + W_{k,a}^{(2)} + W_{k,a}^{(3)} + W_{k,a}^{(4)} \right] ,
\]

where \(\Delta_{(i)}(l_j^T), i = 1, \ldots, 4,\) is the Vandermonde determinant in the four cases above and \(W_{k,a}^{(i)}(A_1, A_2, A_3)\) is conveniently defined. At this stage, each of the \(W_{k,a}^{(i)}\)'s has to undergo the same treatment of \(W_k Z_k\) (Eq. (77)). For \(W_{k,a}^{(i)}\) we define

\[
\tilde{n}_j^{(i)} = 2\pi n_j + h_j^{(i)}
\]

with \(h_j^{(i)}\) collected in Table I. After that, the Poisson transform for \(W_{k,a}^{(i)}\) reads

**TABLE I.** Values of \(h_j^{(i)}\), defined in Eq. (82). The indices \(i\) and \(j\) label the columns and the rows, respectively.

| \(h_j^{(i)}\) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | \(i\frac{g^2}{4} A_2\) | \(i\frac{g^2}{4} (A - 2A_3)\) | \(-i\frac{g^2}{4} A_2\) | \(-i\frac{g^2}{4} (A_3 - A_1)\) |
| 2 | \(i\frac{g^2}{4} (A_1 - A_3)\) | \(i\frac{g^2}{4} (A - 2A_3)\) | \(i\frac{g^2}{4} (A - 2A_3)\) | \(i\frac{g^2}{4} (A - 2A_3)\) |
| 3, \ldots, \(k\) | \(i\frac{g^2}{4} (A - 2A_3)\) | \(i\frac{g^2}{4} (A - 2A_3)\) | \(i\frac{g^2}{4} (A - 2A_3)\) | \(i\frac{g^2}{4} (A - 2A_3)\) |
| \(k + 1, \ldots, N - 2\) | 0 | 0 | 0 | 0 |
| \(N - 1\) | 0 | \(i\frac{g^2}{4} (A - 2A_1)\) | 0 | 0 |
| \(N\) | 0 | \(-i\frac{g^2}{4} (A - 2A_1)\) | \(-i\frac{g^2}{4} (A - 2A_1)\) | \(i\frac{g^2}{4} (A - 2A_1)\) |

\[
W_{k,a}^{(i)} = \sum_{n_q = -\infty}^{+\infty} \delta \left( \sum_{q=1}^{N} n_q \right) \exp \left[ -\frac{g^2 A_1}{2} \left( 1 - \frac{A_1}{A} \right) \right] \exp \left[ -\frac{g^2 A_3}{2} \left( 1 - \frac{A_3}{A} \right) \frac{k(N - k)}{2N} \right]
\]
where
\[
M^{(i)}(n_j; A_1, A_2, A_3) = \begin{cases} A_1(n_1 - n_2) & i = 1 \\ A_1(n_{N-1} - n_N) & i = 2 \\ -(A_2 + A_3)n_1 - A_1n_N & i = 3 \\ A_1(n_N - n_1) & i = 4 \end{cases}
\]
and
\[
Y^{(i)}(y_j) = \begin{cases} y_2 & i = 1 \\ y_1 - y_{N-1} + y_N & i = 2 \\ y_N & i = 3 \\ 2y_1 - y_N & i = 4 \end{cases}
\]
Choosing \( n_q = 0, \forall q \), in Eq. (83) and inserting it in (81), we obtain the zero-instanton sector
\[
\mathcal{W}^{(0)}_{k,a}(A_1, A_2, A_3) = \frac{1}{N + 1} + \frac{2}{(N^2 - 1)} \exp \left[ -\frac{g^2 A_1}{2} \left( 1 - \frac{A_1}{A} \right) \right] \frac{1}{I} \times \left[ \left( \frac{k}{2} \right) I_1 + \left( \frac{N - k}{2} \right) I_2 + \frac{k(N - k)}{2} \exp \left( \frac{g^2 A_3 A_1}{A} \right) I_3 + \frac{k(N - k)}{2} \exp \left( -\frac{g^2 A_3 A_1}{A} \right) I_4 \right],
\]
where \( I \) is the integral over \( z_j \) appearing in Eq. (79) and \( \hat{I}(A_3, A_2, A_1), i = 1, \ldots, 4 \) are explicitly given in the Appendix. We are interested in the limit \( A_3 \to \infty, A_2, A_1 \) fixed of \( \mathcal{W}^{(0)}_{k,a} \), which, far from being trivial, reads
\[
\mathcal{W}^{(0)}_{k,a}(A_1, A_2, A_3 \to \infty) = \frac{1}{N + 1} + \frac{2}{(N^2 - 1)} e^{-\frac{g^2 A_1}{2}} \times \left[ \left( \frac{k}{2} \right) I_1 + \left( \frac{N - k}{2} \right) I_2 + \frac{k(N - k)}{2} e^{\frac{g^2 A_3 A_1}{2}} I_3 + \frac{k(N - k)}{2} e^{-\frac{g^2 A_3 A_1}{2}} I_4 \right].
\]
For the sake of brevity, we defer the computation of \( I = \hat{I}(A_3 \to \infty, A - A_3) \) and \( I_i = \hat{I}_i(A_1, A_2, A_3 \to \infty) \) to the Appendix and report here just the results.
\[ I = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] H(z_1, \ldots, z_N; A_1, A_2), \]

\[ I_i = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] H_i(z_1, \ldots, z_N; A_1, A_2), \]

where

\[ H = \Delta(z_1 + 1, \ldots, z_k + 1, z_{k+1}, \ldots, z_N) \Delta(z_1 - \frac{g^2}{2} \hat{A}, \ldots, z_k - \frac{g^2}{2} \hat{A}, z_{k+1}, \ldots, z_N), \]

\[ H_1 = \Delta \left( z_1 - \frac{g^2}{2} A_2, z_2 - \frac{g^2}{2} (\hat{A} + A_1), z_3 - \frac{g^2}{2} \hat{A}, \ldots, z_k - \frac{g^2}{2} \hat{A}, z_{k+1}, \ldots, z_N \right) \times \]

\[ \Delta \left( z_1 - \frac{g^2}{2} \hat{A}, \ldots, z_k - \frac{g^2}{2} \hat{A}, z_{k+1}, \ldots, z_{N-2}, z_{N-1}, z_N + \frac{g^2}{2} A_1, z_N - \frac{g^2}{2} A_1 \right), \]

\[ H_2 = \Delta (z_1 + 1, \ldots, z_k + 1, z_{k+1}, \ldots, z_{N-2}, z_{N-1} - 1, z_N + 1) \times \]

\[ \Delta \left( z_1 - \frac{g^2}{2} \hat{A}, \ldots, z_k - \frac{g^2}{2} \hat{A}, z_{k+1}, \ldots, z_{N-2}, z_{N-1} + \frac{g^2}{2} A_1, z_N + \frac{g^2}{2} A_1 \right), \]

\[ H_3 = \Delta \left( z_1, z_2 + 1, \ldots, z_k + 1, z_{k+1}, \ldots, z_{N-1}, z_N + 1 \right) \times \]

\[ \Delta \left( z_1 - \frac{g^2}{2} A_2, z_2 - \frac{g^2}{2} (\hat{A} + A_1), z_3 - \frac{g^2}{2} \hat{A}, \ldots, z_k - \frac{g^2}{2} \hat{A}, z_{k+1}, \ldots, z_N - \frac{g^2}{2} A_1 \right), \]

\[ H_4 = \Delta (z_1 + 2, z_2 + 1, \ldots, z_k + 1, z_{k+1}, \ldots, z_{N-1}, z_N - 1) \times \]

\[ \Delta \left( z_1 - \frac{g^2}{2} (\hat{A} + A_1), z_2 - \frac{g^2}{2} \hat{A}, \ldots, z_k - \frac{g^2}{2} \hat{A}, z_{k+1}, \ldots, z_{N-1}, z_N + \frac{g^2}{2} A_1 \right). \]

We carried the calculations out for \( N = 2, 3, 4 \), with all possible value of \( k (k = 0, \ldots, N - 1) \) and found

- \( N = 2, k = 0 \)
  \[ W_{0,a}^{(0)}(A_1) = \frac{1}{3} \left[ 1 + 2 e^{-\frac{g^2 A_1}{2}} (1 - g^2 A_1) \right]; \]

- \( N = 2, k = 1 \)
  \[ W_{1,a}^{(0)}(A_1) = e^{-g^2 A_1}; \]
• $N = 3$, $k = 0$

\[
\mathcal{W}_{0,a}^{(0)}(A_1) = \frac{1}{4} \left[ 1 + 4 e^{-\frac{g^2 A_1}{2}} \left( 12 - 18g^2 A_1 + \frac{9}{2}g^4 A_1^2 - \frac{1}{2}g^6 A_1^3 \right) \right] ;
\]

• $N = 3$, $k = 1$ (or, equivalently, $k = 2$)

\[
\mathcal{W}_{1,a}^{(0)}(A_1) = \frac{1}{8} \left[ 1 - 4e^{-\frac{g^2 A_1}{2}} + (11 - 3g^2 A_1) e^{-g^2 A_1} \right] ;
\]

• $N = 4$, $k = 0$

\[
\mathcal{W}_{0,a}^{(0)}(A_1) = \frac{1}{15} \left[ 3 + \left( 12 - 24g^2 A_1 + \frac{23}{2}g^4 A_1^2 - \frac{8}{3}g^6 A_1^3 + \frac{25}{96}g^8 A_1^3 - \frac{1}{96}g^{10} A_1^5 \right) e^{-\frac{g^2 A_1}{2}} \right] ;
\]

• $N = 4$, $k = 1$ (or, equivalently, $k = 3$)

\[
\mathcal{W}_{1,a}^{(0)}(A_1) = \frac{1}{30} \left[ 4 + (-36 + 18g^2 A_1 - 3g^4 A_1^2) e^{-\frac{g^2 A_1}{2}} + (62 - 34g^2 A_1 + 3g^4 A_1^2) e^{-g^2 A_1} \right] ;
\]

• $N = 4$, $k = 2$

\[
\mathcal{W}_{2,a}^{(0)}(A_1) = \frac{1}{15} \left[ 1 + (14 - 16g^2 A_1 + 3g^4 A_1^2) e^{-g^2 A_1} \right] .
\]

We notice the following basic features:

1. for $k \neq 0$, the results are different from the single loop case;

2. for $k = 0$, the limit $A_2 \to \infty$ is immaterial, as can be understood from Eq. (89).

   Actually, the case at hand corresponds to taking the identical representation sitting in the outer loop, so that the correlator becomes insensitive to $A_2$ being finite or infinite;

3. although string tensions are independent of $k$, as Eq. (87) explicitly shows, the polynomial coefficients do depend on it.
VII. PERTURBATIVE SERIES DEFINED VIA WML PRESCRIPTION

We now check that the zero instanton contribution to the expectation value of an adjoint loop enclosed in a $k$-fundamental one on the plane, expressed by Eq. (87), is consistent, at least up to $O(g^4)$, with the perturbative computation where the propagator is prescribed according to WML.

The starting point is again the perturbative definition of $W_{k,a}$ in the light-cone gauge Eq. (53). At variance with Sect. V, the propagator $D(x-y)$ is now given by Eq. (52) 3. The choice of circles for the contours $\Gamma_k, \Gamma_a$ in the Euclidean space-time (we recall the invariance under area-preserving diffeomorphisms) will prove particularly convenient 33; the currents will be accordingly defined to have support on the contours.

The weighted basic correlator

$$\dot{x}_-(s) \dot{x}_-(s') \frac{x_+(s) - x_+(s')}{x_-(s) - x_-(s')},$$

(90)

turns out to be independent of the loop variables when describing a propagator which starts and ends on the same contour, and amounts to $2(\pi r)^2 = 2\pi A_1$ for the inner circle and to $2(\pi R)^2 = 2\pi A_2$ for the outer circle. After that, for a diagram containing only propagators of that kind, integration over the path parameters is trivial and one is left with the purely combinatorial problem of determining the group factors. At $O(g^2)$ the two loops factorize, so that we end up with

$$- \frac{g^2}{4} (C_k A_2 + C_A A_1)$$

(91)

When normalization is taken into account, $W_{k,a}^{(II)}$ reads

$$W_{k,a}^{(II)}(A_1, A_2) = - \frac{g^2 A_1}{4} C_A$$

(92)

and coincides with Eq. (56) as expected 29.

3Henceforth, for notational simplicity, we will adopt the same symbol $W_{k,a}$ for the perturbative WML two-loop correlator.
Likewise, we can now easily derive the expression of $\mathcal{W}_{k,a}^{(IV)}$, i.e. the adjoint loop enclosed in a $k$-fundamental one at $\mathcal{O}(g^4)$.

Very schematically, different classes of diagrams can be distinguished. Let us browse on them. Obvious contributions are the pure $k$-fundamental loop at $\mathcal{O}(g^4)$ (which corresponds to $\text{Tr } \mathbf{1}_{\text{adj}}$ in the inner loop) and the product of the pure adjoint by the pure $k$-fundamental loops both at $\mathcal{O}(g^2)$, which turn out to result in

$$g^4 \frac{A_1^2}{16}\left[ \frac{A_1^2}{2} \left( C_k^2 - \frac{1}{6} C_A C_k \right) + A_1 A_2 C_A C_k \right].$$  \hspace{1cm} (93)

On the other hand, it is clear they will be removed by normalizing to $\mathcal{Z}_k \mathcal{W}_k$. Very much in analogy with the first contribution, we have to consider the pure adjoint loop at $\mathcal{O}(g^4)$ (which corresponds to $\text{Tr } \mathbf{1}_k$ in the outer loop)

$$\frac{5}{192} g^4 C_A^2 A_1^2.$$  \hspace{1cm} (94)

However, the novelty in having one loop enclosed in another is supplied by graphs with propagators joining the two circles. In this case, the weighted basic correlator can be inferred from Eq. (90) and reads

$$2\pi^2 r R \frac{r e^{2\pi i s} - e^{2\pi i s'}}{e^{2\pi i s} - \frac{r}{R} e^{2\pi i s'}}.$$  \hspace{1cm} (95)

Integration over the loop variables (with $r < R$) at $\mathcal{O}(g^4)$ and insertion of the proper group factors yield the contribution of those graphs to $\mathcal{W}_{k,a}^{(IV)}$

$$\frac{g^4}{8} \frac{C_A C_k}{N^2 - 1} A_1^2.$$  \hspace{1cm} (96)

We emphasize that the factor $\frac{C_A C_k}{N^2 - 1}$ arises from $\text{Tr } [t^a t^b]_k \cdot \text{Tr } [T^a T^b]_{adj}$, properly normalized to $(N^2 - 1) \binom{N}{k}$ (the latter in $\mathcal{Z}_k \mathcal{W}_k$).

When the partial results Eqs. (94,96) are summed up, we obtain for $\mathcal{W}_{k,a}^{(IV)}$

$$\mathcal{W}_{k,a}^{(IV)}(A_2, A_1) = \frac{g^4 A_1^2}{8} \left( \frac{5}{24} C_A^2 + \frac{C_A C_k}{N^2 - 1} \right).$$  \hspace{1cm} (97)

At this point a comment is in order. The coincidence of the coefficients of $g^4 A_1^2 C_A C_k$ in Eqs. (61) and (97) should not be too surprising: in fact, they correspond to graphs connecting
the inner adjoint loop to the outer \( k \)-fundamental one, in the LF and ET formulation, respectively, and at this order those graphs are in some sense “abelian-like”, since no crossing takes place (equivalently, the traces in the adjoint and in the \( k \)-fundamental representation are trivial). In fact, the different behaviour of CPV and WML prescription in such diagrams is expected to arise only at \( \mathcal{O}(g^6) \), when at least two propagators out of three, or more, joining the two loops, cross. The appearance of a dependence on \( A_2 \) is also to be ascribed to graphs with crossing propagators.

As we have already announced in the previous section, we were not successful in treating the limit \( A_2 \to \infty \) in Eqs. (87-89) for all values of \( N \) and \( k \). Nevertheless, we argue that in such a limit the zero-instanton contribution of the two-loop correlator has to coincide with the perturbative series defined above. Eqs. (88,89) suggest for \( W_{k,a}^{(0)}(A_1, A_2) \) the following form

\[
W_{k,a}^{(0)}(A_1, \hat{A}) = 1 - \frac{C_{A}}{4} g^2 A_1 + \sum_{n=2} (g^2 A_1)^n R_n(g^2 \hat{A}),
\]

(98)

where \( R_n(g^2 \hat{A}) = \frac{Q_n(g^2 \hat{A})}{P_n(g^2 \hat{A})} \), both \( Q_n \) and \( P_n \) being polynomials of degree \( k(N-k) \). In Eq. (98) the dependence on \( A_2 \) has been conveniently rearranged into \( \hat{A} \), which is the natural variable in the normalization factor \( I \) (see (79)). One should be careful not to regard Eq. (98) as an expansion in \( g^2 \), as the rational functions \( R_n \) produce a series in \( g^2 \) of their own. Nonetheless, such a form emphasizes the finiteness of the limit \( \hat{A} \to \infty, A_1 \) fixed. Notice that the contributions \( \mathcal{O}(g^6) \) contain the whole dependence on \( A_2 \), which is naturally expected to appear in the perturbative expansion, as remarked beforehand. Moreover, Eq. (98) points out one has simply to set \( \hat{A} = 0 \) to recover the contribution \( \mathcal{O}(g^4) \). This we did for a limited sample of values of \( N \) \((N = 2, 3, 4\) with all possible values of \( k \)) and found the terms up to \( \mathcal{O}(g^4) \) in the expansion of Eq. (87) are reproduced by Eqs. (92,97).

We stress the only case in which the WML perturbative expansion turns out to be independent of \( A_2 \) is \( k = 0 \); in fact, it coincides with the expansion inferred from the exact results for the two-loop correlator reported in Sect. VI. This is not at all surprising, since Eq. (97) and the expression of \( C_k \) appearing in Sect. V point out that, at least \( \mathcal{O}(g^4) \), the
correlator reduces, for \( k = 0 \), to a single adjoint loop of area \( A_1 \). Nonetheless, the argument can be pushed further and it is easy to verify it holds at any order.

**VIII. CONCLUSIONS**

The main goal of this paper was the study of vacua for \( SU(N)/Z_N \) gauge theories in two dimensions, trying to generalize previously obtained results [6,15] concerning a Wilson loop in the fundamental representation, to the adjoint case.

The motivation was twofold: on one hand matter in the adjoint representation (infinitely heavy fermions in our case) can mimic the effects of “transverse” degrees of freedom, which are obviously lacking in our two-dimensional world; on the other hand, as long as only adjoint representations are involved, the true symmetry group becomes \( SU(N)/Z_N \), those representations being insensitive to the group center. As a result, the topological properties of the theory are modified.

This feature induces indeed a much richer topological structure: many inequivalent vacua occur, which are the non-abelian counterpart of the familiar \( \theta \)-vacua of the Schwinger model (or of \( QCD_4 \)).

If the theory is considered on a sphere \( S^2 \) with area \( A \) (which is eventually to be decompactified sending \( A \to \infty \)), in the \( SU(N) \) case already an infinite set of topological excitations (instantons) are present [23]. They are responsible for the big difference we found between an ET and a LF description of the theory: confinement at large \( N \), which can be easily obtained in the LF formulation, can be only recovered in the ET scenario if those instantons are fully taken into account. This situation strengthens the belief that the LF vacuum provides indeed a simpler picture, at least in two dimensions.

In the \( SU(N)/Z_N \) case, the presence of a non-trivial bundle structure leads to inequivalent \( k \)-vacua and deeply modifies the instanton pattern on \( S^2 \). Nevertheless, we found that a LF vacuum is again closer to the exact solution, which can be reached by summing a suitable perturbative series. Moreover we have shown that different \( k \)-sectors of the theory can be
interpreted as due to the presence of $k$-charges at $\infty$ in the form of a boundary Wilson loop in the $k$-fundamental representation, as conjectured by Witten long ago [7].

However, this property holds only for the exact solution and in the decompactification limit; it is not shared by the zero-instanton contribution which corresponds to the perturbative ET result. We find remarkable that a simple, yet deep feature of the theory emerges only after all nonperturbative effects are taken into account.

At this stage, we think we have set a solid ground for the most interesting future development, namely the introduction of dynamical fermions, with a particular focus on the generation and on the properties of a chiral condensate. This problem has already been tackled in the recent literature [35], without reaching so far firm conclusions in the non-supersymmetric case.

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IX. APPENDIX

We report here the integrals $\hat{I}_i(A_1, A_2, A_3)$, $i = 1, \ldots, 4$, making their appearance in Eq. (86)

$$\hat{I}_1 = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \times (99)$$

$$\Delta \left( z_1 - ig \frac{A_2}{\sqrt{2A}}, z_2 + ig \frac{A + A_3 - A_1}{\sqrt{2A}}, z_3 + ig \frac{A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A_3 - A_3}{\sqrt{2A}}, z_{k+1} \ldots, z_N \right) \times$$

$$\Delta \left( z_1 + ig \frac{A_2}{\sqrt{2A}}, z_2 + ig \frac{A - A_3 + A_1}{\sqrt{2A}}, z_3 + ig \frac{A - A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1} \ldots, z_N \right),$$
\[ \hat{I}_2 = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \times \Delta \left( z_1 + ig \frac{A - A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1}, \ldots, z_N - ig \frac{A_1}{\sqrt{2A}}, z_N \right) \times \Delta \left( z_1 + ig \frac{A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1}, \ldots, z_N - ig \frac{A_1}{\sqrt{2A}} \right), \]

\[ \hat{I}_3 = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \times \Delta \left( z_1 + ig \frac{A_2}{\sqrt{2A}}, z_2 + ig \frac{A - A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1}, \ldots, z_N + ig \frac{A - A_1}{\sqrt{2A}} \right) \times \Delta \left( z_1 - ig \frac{A_2}{\sqrt{2A}}, z_2 + ig \frac{A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1}, \ldots, z_N + ig \frac{A - A_1}{\sqrt{2A}} \right), \]

\[ \hat{I}_4 = \int_{-\infty}^{+\infty} dz_1 \ldots dz_N \exp \left[ -\frac{1}{2} \sum_{q=1}^{N} z_q^2 \right] \times \Delta \left( z_1 + ig \frac{A + A_3 - A_1}{\sqrt{2A}}, z_2 + ig \frac{A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1}, \ldots, z_N - ig \frac{A - A_1}{\sqrt{2A}} \right) \times \Delta \left( z_1 + ig \frac{A + A_3 - A_1}{\sqrt{2A}}, z_2 + ig \frac{A - A_3}{\sqrt{2A}}, \ldots, z_k + ig \frac{A - A_3}{\sqrt{2A}}, z_{k+1}, \ldots, z_N - ig \frac{A - A_1}{\sqrt{2A}} \right). \]

As announced in Sect. VI, we want to show how the limit \( A_3 \to \infty \) can be performed so as to obtain \( I(A_1, A_2), \hat{I}_1(A_1, A_2) \). We address \( \hat{I}_1 \) as an example; the other cases can be dealt with the same technology. The product of the two Vandermonde in the integral over \( z_1, \ldots, z_N \), with Gaussian measure, can be rewritten in terms of generalized Laguerre polynomials as follows [30]

\[
(2\pi)^{N/2} \prod_{\alpha=k+1}^{N} (n_\alpha - 1)! \right|_{j_1 \ldots j_k j_{k+1} \ldots j_N} (j_1 - 1)! \left( ig \frac{A_2}{\sqrt{2A}} \right)^{l_1 - j_1} L_{j_1 - 1}^{(l_1 - j_1)} \left( -\frac{g^2 A_2^2}{2A} \right) \left\{ (j_2 - 1)! \left( ig \frac{A - A_3 + A_1}{\sqrt{2A}} \right)^{l_2 - j_2} L_{j_2 - 1}^{(l_2 - j_2)} \left( g^2 A_2^2 - (A_3 - A_1)^2 \right) \right\} \prod_{q=3}^{k} (j_q - 1)! \left( ig \frac{A - A_3}{\sqrt{2A}} \right)^{l_q - j_q} L_{j_q - 1}^{(l_q - j_q)} \left( g^2 A_2^2 (A - A_3) \right) \right\}.
\]

The next step consists in factorizing all possibly divergent terms out of Eq. (100), which, in the limit \( A_3 \to \infty (A \to \infty) \), amount to \( (ig \sqrt{A}/2) \sum_{q=1}^{k} (j_q - l_q) \). Now, recalling that \( \sum_{q=1}^{k} j_q = \sum_{q=1}^{k} l_q \), we are left with the finite expression

\[
\sum_{q=1}^{k} j_q = \sum_{q=1}^{k} l_q,
\]
\[(2\pi)^{N/2} \prod_{\alpha=k+1}^{N} (n_\alpha - 1)! \varepsilon^{j_1 \cdots j_k j_{k+1} \cdots j_N} \varepsilon^{l_1 \cdots l_k l_{k+1} \cdots l_N} \]

\[
\left\{ (j_1 - 1)! \left( -\frac{g^2 A_2}{2} \right)^{l_1 - j_1} L^{(l_1 - j_1)}_{j_1 - 1} (0) \right\}
\]
\[
\left\{ (j_2 - 1)! \left( -\frac{g^2 2A_1 + A_2}{2} \right)^{l_2 - j_2} L^{(l_2 - j_2)}_{j_2 - 1} \left( g^2 (2A_1 + A_2) \right) \right\}
\]
\[
\prod_{q=3}^{k} \left\{ (j_q - 1)! \left( -\frac{g^2 A_1 + A_2}{2} \right)^{l_q - j_q} L^{(l_q - j_q)}_{j_q - 1} \left( g^2 A_1 + A_2 \right) \right\},
\]

which can be recombined into

\[
\Delta \left( z_1, z_2 + 2, z_3 + 1, \ldots, z_k + 1, z_{k+1}, \ldots, z_N \right) \times
\]
\[
\Delta \left( z_1 - \frac{g^2}{2} A_2, z_2 - \frac{g^2}{2} (A_2 + 2A_1), z_3 - \frac{g^2}{2} (A_2 + A_1), \ldots, z_k - \frac{g^2}{2} (A_2 + A_1), z_{k+1}, \ldots, z_N \right).
\]

This is precisely the integrand \( H_1 \) in Eqs. (88, 89).
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