Universality of Makespan in Flowshop Scheduling Problem

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Abstract

Makespan, which is defined as the time difference between the starting time and the terminate time of a sequence of jobs or tasks, as the time to traverse a belt conveyor system, is well known as one of the most important criteria in scheduling problems. It is often used by manufacturing firms in practice in order to improve the operational efficiency with respect to the order of job processing to be performed. It is known that the performance of a machine depends on the particular timing of the job processing even if the job processing order is fixed. That is, the performance of a system with respect to flowshop processing depends on the procedure of scheduling. In this present work, we first discuss the relationship between makespan and several scheduling procedures in detail by using a small example and provide an algorithm for deriving the makespan. Using our proposed algorithm, several numerical experiments are examined so as to reveal the relationship between the typical behavior of makespan and the position of the fiducial machine, with respect to several distinguished distributions of the processing time. We also discuss the behavior of makespan by using the properties of the shape functions used in the context of percolation theory. Our contributions are firstly giving a detail discussion on the univer-

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ality of makespan in flowshop problems and obtaining several novel properties of makespan, as follows: (1) makespan possesses universality in the sense of being little affected by a change in the probability distribution of the processing time, (2) makespan can be decomposed into the sum of two shape functions, and (3) makespan is less affected by the dispatching rule than by the scheduling procedure.

*Keywords:* Flowshop Scheduling, Gantt chart, Universality, Percolation Theory, Shape Function

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1. **Introduction**

Recently, in manufacturing, the importance of producing a wide variety of products in small quantities in order to meet the variety of customer needs is increasingly being recognized [1]. Flowshop processing systems are among the most widely used manufacturing systems for such production of a wide variety of products in small quantities. Furthermore, flowshop scheduling plays one of the most important roles in product planning and, furthermore, the corresponding scheduling problem is one of the most well-known scheduling problems in production as a special case of the jobshop scheduling problem. One significant feature of flowshop scheduling is that the orders of job processing of whole machines in a processing system are consistent, and it is known that the production capacity of the processing system can be maximally desterilized with little effort compared with the jobshop scheduling approach.

In the case that the number of machines in the processing system is two, Johnson’s algorithm can be used to find the optimal scheduling of the order that jobs are processed in time $O(N \log N)$ where the number of jobs is $N$, and the optimality of the job processing order derived from Johnson’s algorithm is mathematically guaranteed if the order of all jobs processed in the processing system is same as the order for the flowshop scheduling problem [2]. This approach could be easily adapted to other special situations by French [3]. How-
ever, it is known that the optimal order of the case of three or more machines (except for certain specific cases) is difficult to solve, since the flowshop problem is NP-hard [4]. Many researchers have used meta-heuristic algorithms and dispatching rules in order to overcome this difficulty of the scheduling problem [5, 6, 7]. For instance, Nowicki and Smutnicki proposed one of the most famous meta-heuristic algorithms, the tabu search algorithm, for resolving the flowshop scheduling problem [5]. Osman and Potts also investigated applying permutation to flowshop scheduling problem using a simulated annealing algorithm and compare it with other meta-heuristic algorithms[6]. Moreover, Bierwirth and Stöppler developed a genetic algorithm so as to analyze the flowshop scheduling problem [7]. However, because of the problem being an NP-hard problem [4], the genetic algorithm was not sufficient to solve completely the flowshop scheduling problem for all cases.

In recent decades, with respect to the optimality of a processing system composed on more than three machines, the theory of constraints has drawn great attention. The theory of constraints is a management paradigm proposed by Goldratt [12]. The mechanism of management scheduling is explained in the form of a novel in his book. He also developed software called optimized production technology. The core concepts of the theory of constraints are to improve the performance of the bottleneck machine in the processing system and to subordinate the other machines to the bottleneck machine. That is, in the theory of constraints, it is necessary to implement an improvement of the performance of the bottleneck machine before anything else. Improving the bottleneck machine as required by the theory of constraints is easy to do, but this does not always improve the total throughput, as described in detail below. Since there also exist cases that the total throughput can be greatly improved by improving the performance of a non-bottleneck machine, from the unified viewpoint, we need to examine the asymptotical behavior of the production capacity of the processing system with respect to the starting point of scheduling of flowshop, both numerically and mathematically.

This paper is organized as follows; in the next section, we set up a flow-
| Fiducial machine | processing time $y_\mu$ | makespan $C_{\text{max}}$ |
|------------------|------------------------|------------------------|
| M1               | $y_\mu = 22$           | $C_{\text{max}} = 48$  |
| M2               | $y_\mu = 25$           | $C_{\text{max}} = 49$  |
| M3               | $y_\mu = 26$           | $C_{\text{max}} = 50$  |
| M4               | $y_\mu = 18$           | $C_{\text{max}} = 55$  |
| M5               | $y_\mu = 20$           | $C_{\text{max}} = 48$  |
| M6               | $y_\mu = 23$           | $C_{\text{max}} = 48$  |
| M7               | $y_\mu = 17$           | $C_{\text{max}} = 48$  |

Table 1: The relationship between the makespan and the position of the fiducial machine and processing time table is shown in table 2, where the job processing order is fixed. The Gantt charts are shown in Fig.1 to Fig.7.

shop scheduling model for simplicity of our discussion and explain briefly the relationship between the production capacity of the processing system and the machine location in the processing system which is the starting point of the scheduling with the help of a small example. Moreover, our motivation in this paper of deriving the universal properties from some examples is explained and a flowshop scheduling algorithm is proposed in order to solve systematically flowshop problems. In section 3, numerical experiments show the validity of our proposed approaches under several different distribution models, namely, for a processing time independently and identically distributed according to an exponential, uniform, or $\chi^2$ distribution. We discuss mathematically the typical behaviors of the production capacity of the processing system using the shape function developed in percolation theory in section 4 and are able to explain the results of the numerical experiments mathematically. The last section is devoted to a summary and a statement of planned future work.
2. Model Setting and Flowshop Scheduling Algorithm

2.1. Model Setting

We consider the flowshop scheduling problem of processing \( N \) jobs on a belt conveyor and \( M \) machines configured in a line. \( x_{\mu,i} \) indicates the processing time, which includes the set-up time, of machine \( \mu \) and job \( i \), where \( \mu = 1, \cdots, M \) and \( i = 1, \cdots, N \), and the processing time depends on the workload of the particular job being processed and by the particular machine. For convenience, the processing time is independently and identically distributed according to a given probability distribution \( \Pr(x) \). As the first step of our analytical research on flowshop scheduling, we assume that this processing system has no restriction on the due dates of the jobs. Further, \( s_{\mu,i} \) and \( t_{\mu,i} \) represent the starting time and terminate time of machine \( \mu \) and job \( i \), respectively, and satisfy the following relation:

\[
t_{\mu,i} = s_{\mu,i} + x_{\mu,i}.
\]

Several different criteria for flowshop scheduling, for instance, average flow time, makespan, and average tardiness [1], have been discussed in the literature [8, 9, 10, 11], out of which the makespan under flowshop scheduling is focused on in the present paper. Here, makespan \( C_{\text{max}} \) means the total processing time of a processing system (e.g., a belt conveyor) and is defined as follows:

\[
C_{\text{max}} = t_{M,N} - s_{1,1}.
\]

That is, the total processing time of the processing system is equal to the time difference between the starting time, \( s_{1,1} \), and the terminate time, \( t_{M,N} \), of the processing system.

One of the goals of the flowshop scheduling problem is to minimize the makespan under permutations of the job processing order, given the processing times of different jobs; however, even if the processing order is fixed, as mentioned below, since the makespan depends on the position of the scheduling
fiducial machine (where herein fiducial machine means the first machine in the processing scheduling) within the processing system. Namely, before the most discussed topic in the previous works (which is permuting job processing order) is analyzed, we need to reveal in detail the relationship between makespan and the position of the scheduling fiducial machine in the processing system. Although we start off handling only one example, in tables 1 and 2, based on the results, for a fixed job processing order, the processing scheduling by starting at the center machine of this processing system gives a lower processing capacity for this processing system than does that by starting at the bottleneck machine. We need to systematically discuss this property of makespan in order to analyze the optimal processing capacity before examining permutations of job processing order using the dispatching rules which are widely discussed in previous works.

2.2. Example of Small System

We here discuss the relationship between makespan and the position of the scheduling fiducial machine in the processing system using the Gantt chart of a small example system. Let \( y_\mu \) be the sum of the necessary processing times of machine \( \mu \) if other machines do not influence machine \( \mu \), that is,

\[
y_\mu = \sum_{i=1}^{N} x_{\mu,i}. \tag{3}
\]

In accordance with common sense, the sum of processing times in practice of machine \( \mu \) in the processing system is not smaller than the sum of necessary processing times \( y_\mu \). From this, the bottleneck machine of this production system, as referred to in the literature on the theory of constraints, is determined by the following:

\[
\mu^* = \arg \max_{\mu} y_\mu. \tag{4}
\]

Typically, the bottleneck machine is uniquely determined (namely, it is the machine which maximize the sum of the necessary processing times) if the processing time table is randomly generated.
As shown in table 1, M3 is the bottleneck machine in this system. At the first step, the processing scheduling of the bottleneck machine is independently determined using the processing time table. At the next step, the processing scheduling of the nearest neighbors of the bottleneck machine, here M2 (previous machine in the processing system) and M4 (next machine), is uniquely assigned and depends only on the processing schedules of the bottleneck machine. In the same way, at the third step, the processing scheduling of M1, M5, M6, and M7 is decided in sequence. As a result, the Gantt chart of the bottleneck machine is as shown in Fig. 3. When the other machines are fiducial as the starting machine in scheduling, the Gantt charts are as shown in Figs. 1, 2, and 4 to 7. From these figures, we can see that the makespan of the bottleneck machine, here M3, is not smallest among the machines. Rather, the makespans of M1 and M7 (as well as M5 and M6), which are the first and last machines in this processing system, are minimal, whereas the makespan of M4, which is the center machine of this system, is maximal. This result conflicts with the canons of the theory of constraints, that is, the processing of the bottleneck machine is the constraint condition in this processing system and/or improvement (of the processing scheduling) of the bottleneck machine is most important. Our goal in this paper is to analyze the typical behaviors of makespan with respect to the position of the fiducial machine in the processing system.
|   | J1 | J2 | J3 | J4 | J5 | J6 | J7 | J8 | \( \mu \) | \( C_{\text{max}} \) |
|---|----|----|----|----|----|----|----|----|---------|----------------|
| M1 | 2  | 5  | 4  | 1  | 3  | 1  | 2  | 4  | 22      | 48             |
| M2 | 2  | 3  | 2  | 4  | 3  | 4  | 2  | 5  | 25      | 49             |
| M3 | 3  | 5  | 4  | 1  | 5  | 1  | 3  | 4  | 26      | 50             |
| M4 | 4  | 5  | 1  | 1  | 3  | 1  | 2  | 1  | 18      | 55             |
| M5 | 1  | 2  | 4  | 2  | 5  | 1  | 4  | 1  | 20      | 48             |
| M6 | 5  | 2  | 3  | 1  | 4  | 1  | 5  | 2  | 23      | 48             |
| M7 | 1  | 1  | 1  | 4  | 1  | 4  | 4  | 1  | 17      | 48             |

Table 2: Processing time table of \( N = 8, M = 7 \). M3 is the bottleneck machine in this system and the total necessary processing time of the bottleneck machine is 26.

Figure 1: Gantt chart for M1 as the fiducial machine. \( C_{\text{max}} = 48 \).

Figure 2: Gantt chart for M2 as the fiducial machine. \( C_{\text{max}} = 49 \).
Figure 3: Gantt chart for M3 as the fiducial machine. $C_{\text{max}} = 50$.

Figure 4: Gantt chart for M4 as the fiducial machine. $C_{\text{max}} = 55$. 

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Figure 5: Gantt chart for M5 as the fiducial machine. $C^{\text{max}} = 48$.

Figure 6: Gantt chart for M6 as the fiducial machine. $C^{\text{max}} = 48$.

Figure 7: Gantt chart for M7 as the fiducial machine. $C^{\text{max}} = 48$. 
2.3. Forward Scheduling

In the previous subsection, makespan of each machine in the processing system was discussed for an example of a small processing system. Here, the forward scheduling procedure is introduced in detail.

With respect to machine $\mu$, after the starting and terminate times of machine $\mu - 1$ have been determined, the starting and terminate times of machine $\mu$ and job $i$, $s_{\mu,i}$ and $t_{\mu,i}$, are assigned as follows:

$$s_{\mu,i} = \max (t_{\mu-1,i}, t_{\mu-1,i}^-), \quad (5)$$
$$t_{\mu,i} = s_{\mu,i} + x_{\mu,i}, \quad (6)$$

where the starting time $s_{\mu,i}$ is decided as the larger of the terminate time of machine $\mu$ and job $i - 1$ and that of machine $\mu - 1$ and job $i$, and the terminate time $t_{\mu,i}$ is assessed using equation (1) or (6).

2.4. Backward Scheduling

In a similar way to those in previous subsections, the backward scheduling procedure is explained in detail as follows. With respect to machine $\mu$, after those of machine $\mu + 1$ have been assigned, the terminate and starting times of machine $\mu$ and job $i$, $t_{\mu,i}$ and $s_{\mu,i}$, are calculated as follows:

$$t_{\mu,i} = \min (s_{\mu,i+1}, s_{\mu+1,i}), \quad (7)$$
$$s_{\mu,i} = t_{\mu,i} - x_{\mu,i}, \quad (8)$$

where the terminate time $t_{\mu,i}$ is evaluated as the smaller of the starting time of machine $\mu$ and job $i + 1$ and that of machine $\mu + 1$ and job $i$, and the starting time $s_{\mu,i}$ is assessed using equation (1) or (8).

2.5. Algorithm for Evaluating Makespan

Summarizing our explanations in the previous subsections, the algorithm for evaluating makespan with each choice of fiducial machine (the starting machine in terms of scheduling) in the processing system is organized as follows:
Step 0: The processing time of machine $\mu$ and job $i$, $x_{\mu,i}$, is randomly assigned according to probability distribution $\Pr(x)$ to create a processing time table matrix $X = \{x_{\mu,i}\} \in \mathbb{R}^{M \times N}$. Initially, the number of the fiducial machine is $\nu = 1$.

Step 1: With respect to fiducial machine $\nu$, the starting time of job 1 is $s_{\nu,1}(\nu) = 0$ by assumption. Moreover, since the starting and terminate times of the fiducial machine do not depend on the processing operation schedules of the other machines, the starting and terminate times of fiducial machine $\nu$ and job $i$ are determined in accordance with the following relations:

\[
\begin{align*}
    t_{\nu,i}(\nu) &= s_{\nu,i}(\nu) + x_{\nu,i}, \\
    s_{\nu,i+1}(\nu) &= t_{\nu,i}(\nu).
\end{align*}
\]  

In Step 1, $s_{\nu,1}(\nu), t_{\nu,1}(\nu), \ldots, s_{\nu,N}(\nu), t_{\nu,N}(\nu)$ are uniquely determined. Note that hereafter we use $s_{\mu,i}(\nu)$ and $t_{\mu,i}(\nu)$ to denote the starting and terminate times instead of $s_{\mu,i}$ and $t_{\mu,i}$, since they strongly depend on constraint conditions influenced by fiducial machine $\nu$.

Step 2: (Forward Scheduling) With respect to machine $\mu(\nu < \mu \leq M)$ of the processing system, starting and terminate times of this machine for job $i$ are uniquely determined by the following relations:

\[
\begin{align*}
    s_{\mu,i}(\nu) &= \max (t_{\mu,i-1}(\nu), t_{\mu-1,i}(\nu)), \\
    t_{\mu,i}(\nu) &= s_{\mu,i}(\nu) + x_{\mu,i}.
\end{align*}
\]

In Step 2, $s_{\nu+1,1}(\nu), t_{\nu+1,1}(\nu), \ldots, s_{\nu+1,N}(\nu), t_{\nu+1,N}(\nu), \ldots, s_{M,1}(\nu), t_{M,1}(\nu), \ldots, s_{M,N}(\nu), t_{M,N}(\nu)$ are uniquely derived in sequence.

Step 3: (Backward Scheduling) With respect to machine $\mu(1 \leq \mu < \nu)$ of the processing system, the starting and terminate times of this machine for job $i$ are uniquely determined by the following relations:

\[
\begin{align*}
    t_{\mu,i}(\nu) &= \min (s_{\mu,i+1}(\nu), s_{\mu+1,i}(\nu)), \\
    s_{\mu,i}(\nu) &= t_{\mu,i}(\nu) - x_{\mu,i}.
\end{align*}
\]
In Step 3, $s_{1,1}(\nu), t_{1,1}(\nu), \cdots, s_{1,N}(\nu), t_{1,N}(\nu), \cdots, s_{\nu-1,1}(\nu), t_{\nu-1,1}(\nu), \cdots, s_{\nu-1,N}(\nu), t_{\nu-1,N}(\nu)$ are easily determined.

**Step 4:** We estimate the makespan of fiducial machine $\nu$ as follows:

$$C_{\text{max}}(\nu) = t_{M,N}(\nu) - s_{1,1}(\nu).$$  \hfill (15)

**Step 5:** If $\nu < M$, we replace $\nu$ by $\nu + 1$ and return to Step 1, otherwise this algorithm stops.

As mentioned above, when the processing time table $X = \{x_{\mu,i}\} \in \mathbb{R}^{M \times N}$ is given as a function of the position of the fiducial machine in the processing system, the makespan is uniquely determined, since the processing time is independently and identically distributed according to the probability distribution $\Pr(x)$. Thus, we need to average the makespan over multiple randomly generated processing time tables $X$, that is, estimate $\mathbb{E}_X[C_{\text{max}}(\nu)]$, in order to examine the typical behavior of a particular statistic measure of the processing performance of the processing system.

Two points should be noticed here. First, from the definitions of forward scheduling and backward scheduling, because of system symmetry, $\mathbb{E}_X[C_{\text{max}}(1)] = \mathbb{E}_X[C_{\text{max}}(M)], \mathbb{E}_X[C_{\text{max}}(2)] = \mathbb{E}_X[C_{\text{max}}(M-1)],$ and $\mathbb{E}_X[C_{\text{max}}(3)] = \mathbb{E}_X[C_{\text{max}}(M-2)]$ are symmetrically and statistically satisfied, that is,

$$\mathbb{E}_X[C_{\text{max}}(\nu)] = \mathbb{E}_X[C_{\text{max}}(M + 1 - \nu)].$$  \hfill (16)

Next we define the expected terminate time function $h(\nu)$ as

$$h(\nu) = \mathbb{E}_X[t_{\nu,N}(\nu) - s_{1,1}(\nu)].$$ \hfill (17)

Then

$$C_{\text{max}}(\nu) = (t_{M,N}(\nu) - s_{\nu,1}(\nu)) - (t_{\nu,N}(\nu) - s_{\nu,1}(\nu))$$

$$+ (t_{\nu,N}(\nu) - s_{1,1}(\nu)),$$  \hfill (18)

is obtained, and the expectation of makespan can be decomposed into the sum of two terms as follows:

$$\mathbb{E}_X[C_{\text{max}}(\nu)] = h(\nu) + h(M + 1 - \nu) - NI,$$ \hfill (19)
where $l = E_X[X_{\mu,i}]$. Here we replace $L - t_{M,N}(\nu) = s'_{1,1}(M + 1 - \nu)$ and $L - s_{\nu,1}(\nu) = t'_{M+1-\nu,N}(M + 1 - \nu)$ using an appropriate constant $L$, and $E_X[t_{M,N}(\nu)-s_{\nu,1}(\nu)] = E_X[t'_{M+1-\nu,N}(M+1-\nu)-s'_{1,1}(M+1-\nu)] = h(M+1-\nu)$ is obtained by symmetry. The decomposition of $E_X[C_{\text{max}}(\mu)]$ in Eq.(19) is sketched in Fig. 8.

If $M$ and $N$ are sufficiently large, we can prove some properties of $h(\nu)$, such as concavity, regardless of the probability distribution of $x_{\mu,i}$. Furthermore, an explicit form of function $h(\nu)$ can be obtained for some classes of probability distributions. We will present these results in detail in section 4.

3. Numerical Experiments

3.1. Exponential Distribution
Table 3: Parameter estimates for fitting functions (plotted in the figures) for the case that the processing time is exponentially distributed.

| $M$ | $N$ | $A$    | $B$    | $\alpha$ | Figure |
|-----|-----|--------|--------|-----------|--------|
| 1000| 200 | 53.1185| 2333.8463| 0.50652  | Fig.9  |
| 1000| 400 | 72.8693| 2660.1637| 0.51548  | Fig.10 |
| 1000| 600 | 96.5423| 3107.2000| 0.50059  | Fig.11 |
| 1000| 800 | 106.8016| 3443.8947| 0.50941  | Fig.12 |
Figure 9: $M = 1000$ and $N = 200$, exponential distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C^{\text{max}}(\mu)$.

Figure 10: $M = 1000$ and $N = 400$, exponential distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C^{\text{max}}(\mu)$. 
Figure 11: $M = 1000$ and $N = 600$, exponential distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C^{\text{max}}(\mu)$.

Figure 12: $M = 1000$ and $N = 800$, exponential distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C^{\text{max}}(\mu)$.
In Fig. 8, an example of the expected terminate time $h(\nu)$ is depicted as a concave function. We can prove that indeed if $M$ and $N$ are sufficiently large, $h(\nu)$ will be a concave function regardless of the probability distribution of $X_{\mu,i}$. However, we will first discuss the relationship between the expectation of makespan and the position of the fiducial machine in the processing system when the processing time is independently and identically distributed according to an exponential distribution. In this discussion, the following representation of the exponential distribution is used:

\[
f(X_{\mu,i}) = \begin{cases} 
    \frac{1}{l}e^{-\frac{X_{\mu,i}}{l}} & X_{\mu,i} > 0 \\
    0 & \text{otherwise}
\end{cases},
\]

where $l$ is the scale parameter and is such that $E[X_{\mu,i}] = l$ and $\text{Var}[X_{\mu,i}] = E[X_{\mu,i}^2] - (E[X_{\mu,i}])^2 = l^2$.

Here we apply $l = 2$ in numerical experiments, the number of machines in this system is $M = 1000$, and the number of jobs processed in the processing system is $N = 200, 400, 600, \text{or} 800$. These results for the average of makespan with respect to the index of fiducial machine $\nu$ are shown in Figs. 9 to 12, respectively. The horizontal and vertical axes in these figures indicate the index of the fiducial machine $\nu$ and the average of the makespan $E[X_{\mu,i}]$ as evaluated by numerical experiments, respectively. Specifically, the symbols and error bars are results from evaluating 100 random processing time tables. As shown, the behavior of the mean makespan depends not on the position of the bottleneck machine but on the position of the fiducial machine in this system because the bottleneck machine equivalently exists in the processing system.

Moreover using equation (19), the fitting function (solid lines in these figures) is assumed as

\[
E[X[C_{\mu,i}^{\text{max}}(\nu)]] = A(\nu - 1)^\alpha + A(M - \nu)^\alpha + B,
\]

where $B = 2b + c(M - 1) - NE[X_{\mu,i}]$. This equation corresponds to the expected terminate time functions $h(\nu) = A(\nu - 1)^\alpha + b + c(\nu - 1)$ and $h(M + 1 - \nu) = A(M - \nu)^\alpha + b + c(M - \nu)$ already used. These figures show that the
results of numerical experiments and fitting functions are consistent with each other. These estimates of parameters are listed in Table 3. From this table, it is expected that $\alpha$ is constant with respect to the number of jobs in the case of this exponential distribution; that is, the assumption in equation (19) that the mean of the makespan can be expressed by two expected terminate time functions is satisfied.

Lastly, we compare the above flowshop scheduling results with those using two of the most discussed and applied dispatching rules, shortest processing time (SPT) and longest processing time (LPT). For the case $N = 800$ and $M = 1000$, the results of applying the two dispatching rules are given in terms of the processing times of machine 1 in Figs. 13 and 14. As shown, the behaviors of the expectation of makespan for the normal rule scheduling and those of SPT rule scheduling are statistically consistent with each other because the processing time is independently and identically distributed between jobs and machines in the case of large systems, which results in attempts at optimization being canceled out by the randomness of processing time, causing the expected effect of these dispatching rules to be negligible. In fact, $\min_{\nu}(E_X[C_{SPT}^{\max}(\nu)] - E_X[C^{\max}(\nu)]) / E_X[C^{\max}(\nu)] \approx -0.0047$ and $\max_{\nu}(E_X[C_{SPT}^{\max}(\nu)] - E_X[C^{\max}(\nu)]) / E_X[C^{\max}(\nu)] \approx 0.0012$, where $C_{SPT}^{\max}(\nu)$ and $C^{\max}(\nu)$ are the makespans under the SPT rule and the normal rule, respectively, so that the expected improvement effect is only $0.47\%$, and thus intuitively it is not cost-effective to try to obtain an improvement using the dispatching rule. In contrast, as shown in Figs. 13 and 14, the makespan under LPT scheduling which is not optimal is larger than that under normal scheduling. However, $\min_{\nu}(E_X[C_{LPT}^{\max}(\nu)] - E_X[C^{\max}(\nu)]) / E_X[C^{\max}(\nu)] \approx 0.027$ and $\max_{\nu}(E_X[C_{LPT}^{\max}(\nu)] - E_X[C^{\max}(\nu)]) / E_X[C^{\max}(\nu)] \approx 0.077$, where $C_{LPT}^{\max}(\nu)$ is the makespan under the LPT rule, so LPT is at most $7.7\%$ worse than normal scheduling.

3.2. Discrete Uniform Distribution
| $M$  | $N$  | $A$     | $B$     | $\alpha$ | Figure |
|------|------|---------|---------|----------|--------|
| 1000 | 200  | 93.7442 | 8661.74 | 0.48092  | Fig.15 |
| 1000 | 400  | 164.7893| 10218.35| 0.44421  | Fig.16 |
| 1000 | 600  | 202.0881| 11521.40| 0.44950  | Fig.17 |
| 1000 | 800  | 241.3292| 12917.00| 0.44587  | Fig.18 |

Table 4: Parameter estimates for fitting functions (plotted in the figures) in the case that processing time is distributed according to a discrete uniform distribution.
Figure 13: Makespans under normal, SPT, and LPT rules. $M = 1000$ and $N = 800$, exponential distribution with mean $t = 2$. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\max}(\mu)$.

Figure 14: Differences in makespan as plotted in Fig. 13 between normal and dispatching rules. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\max}(\mu)$. 
Figure 15: $M = 1000$ and $N = 200$, discrete uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$.

Figure 16: $M = 1000$ and $N = 400$, discrete uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$.
Figure 17: $M = 1000$ and $N = 600$, discrete uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$.

Figure 18: $M = 1000$ and $N = 800$, discrete uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$.
Table 5: Parameter estimates for fitting functions (plotted in the figures) for the case that processing time is distributed according to a continuous uniform distribution.

| $M$ | $N$ | $A$   | $B$      | $\alpha$ | Figure |
|-----|-----|-------|----------|----------|--------|
| 1000| 200 | 95.4363 | 8661.48 | 0.47832 | Fig. 19 |
| 1000| 400 | 159.8063 | 10180.61 | 0.45043 | Fig. 20 |
| 1000| 600 | 206.0768 | 11521.64 | 0.44645 | Fig. 21 |
| 1000| 800 | 246.9541 | 12975.25 | 0.44105 | Fig. 22 |

Next, we also discuss the relationship between the expectation of makespan and the position of the fiducial machine in this system when the processing time is independently and identically distributed according to a discrete uniform distribution, specifically, the following distribution:

$$
\Pr(X_{\mu,i}) = \begin{cases}
\frac{1}{13} & X_{\mu,i} = 1, 2, \cdots, 13 \\
0 & \text{otherwise}
\end{cases},
$$

which is such that $E[X_{\mu,i}] = 7$ and $\text{Var}[X_{\mu,i}] = E[X_{\mu,i}^2] - (E[X_{\mu,i}])^2 = 14$.

As in the previous subsection, the number of machines is $M = 1000$ and the number of jobs is $N = 200, 400, 600, \text{or } 800$. The results are shown in Fig. 15 to Fig. 18. As shown, the expectation of makespan under this distribution behaves similarly to the case of an exponential distribution. The estimated parameters in equation (21) for the numerical experiments are listed in table 4.

### 3.3. Continuous Uniform Distribution
Figure 19: $M = 1000$ and $N = 200$, continuous uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C^{\text{max}}(\mu)$.

Figure 20: $M = 1000$ and $N = 400$, continuous uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C^{\text{max}}(\mu)$. 
Figure 21: $M = 1000$ and $N = 600$, continuous uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{max}(\mu)$.

Figure 22: $M = 1000$ and $N = 800$, continuous uniform distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{max}(\mu)$.
Next, in order to be able to make a comparison with the previous discrete uniform case, we consider a continuous uniform distribution having the same mean and variance, that is, the density function

\[
f(X_{\mu,i}) = \begin{cases} \frac{1}{2\sqrt{42}} & |X_{\mu,i} - 7| \leq \sqrt{42} \\ 0 & \text{otherwise} \end{cases}, \quad (23)
\]

is used and is such that \(E_X[X_{\mu,i}] = 7\) and \(\text{Var}[X_{\mu,i}] = E_X[X_{\mu,i}^2] - (E_X[X_{\mu,i}])^2 = 14\).

Fig. 19 to Fig. 22 show plots of the expectations of makespan for the cases of \(N = 200, 400, 600, \) and \(800\) with \(M = 1000\). The parameter estimates are listed in table 5. Comparing table 4 and table 5, it is clear that the behaviors of makespan for the two cases are similar up to a statistical fluctuation.

### 3.4. Chi-Squared Distribution

Lastly, the relationship between the average of makespan and the position of the fiducial machine in the processing system is examined for the case that the processing time is independently and identically distributed according to the chi-squared distribution

\[
f(X_{\mu,i}) = \begin{cases} \frac{(X_{\mu,i})^{k/2 - 1} e^{-X_{\mu,i}/2}}{2^{k/2} \Gamma(k/2)} & X_{\mu,i} > 0 \\ 0 & \text{otherwise} \end{cases}, \quad (24)
\]

where \(E_X[X_{\mu,i}] = k\) and \(\text{Var}[X_{\mu,i}] = E_X[X_{\mu,i}^2] - (E_X[X_{\mu,i}])^2 = 2k\). If \(k = 7\), then the mean and variance are consistent with those for the previous uniform distributions. Fig. 23 to Fig. 26 show that the expectation of makespan is concave for this case and similar to the other distributions considered up to a statistical fluctuation. Namely, from these numerical experiments, we can assume that the asymptotical behavior of makespan is universal, that is, it does not depend on the distribution of processing time. The next section will discuss this universality mathematically.
Table 6: Parameter estimates for fitting functions (plotted in the figures) for the case that the processing time is chi-squared distributed.

| $M$ | $N$ | $A$   | $B$   | $\alpha$ | Figure |
|-----|-----|-------|-------|-----------|--------|
| 1000| 200 | 89.9326 | 8289.92 | 0.51588  | Fig.23 |
| 1000| 400 | 150.0050 | 9884.91 | 0.48407  | Fig.24 |
| 1000| 600 | 177.9403 | 11131.28 | 0.49390  | Fig.25 |
| 1000| 800 | 211.6288 | 12650.38 | 0.48702  | Fig.26 |
Figure 23: $M = 1000$ and $N = 200$, chi-squared distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$.

Figure 24: $M = 1000$ and $N = 400$, chi-squared distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$. 
Figure 25: $M = 1000$ and $N = 600$, chi-squared distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$.

Figure 26: $M = 1000$ and $N = 800$, chi-squared distribution. The horizontal axis shows the position of the fiducial machine $\mu$ and the vertical axis shows the makespan $C_{\text{max}}(\mu)$. 
4. Mathematical Discussion

In this section, we discuss the function $h$ appearing in Eq.(19) or its asymptotic function $g$, defined in detail below, for large numbers of machines and jobs. Unexpectedly, $g$ is related to a function called the shape function in percolation theory, about which much has already been revealed. We will see how percolation theorems can be used to derive scheduling theorems.

4.1. Percolation Theory and Shape Function

Let discuss the forward scheduling Eq.(5) and Eq.(6). The following argument for forward scheduling can be applied to backward scheduling, by symmetry. Set $T_{0,0} = 0$. Then for $(m, n) \in \mathbb{Z}_2$, we can show

$$T_{m,n} = \max_{\gamma \in \{(0,0) \rightarrow (m,n)\}} \sum_{(\mu,i) \in \gamma} X_{\mu,i}. \quad (25)$$

Here, $\{(0,0) \rightarrow (m,n)\}$ means the set of the shortest paths in the square grid graph connecting $(0,0)$ and $(m,n)$. This relation can be derived from Eq.(5) and Eq.(6) by induction.

If $X_{\mu,i} \in \mathbb{R}$ for $\mu, i \in \mathbb{Z}_2$ is sampled i.i.d. from a probability distribution $F$, then the model above is known as the two-dimensional last-passage directed site percolation. This name comes from the problem of finding the time of last-passage of a particle which starts from the origin and moves only in the “right” or “up” direction for a lattice with weights (time required to pass through) on its vertices (rather than its edges). Last-passage directed site percolation has been studied mainly in mathematics and physics [13, 14, 15], and it is also referred to as zero-temperature directed polymer in a random environment when used as a polymer model.

Asymptotic properties of $T_{m,n}$ have been reported (see [15]). Throughout the present paper, we assume $\mathbb{E}_X[X_{\mu,i}] < \infty$ and that $X_{\mu,i}$ is non-degenerate, i.e., $\text{Var}[X_{\mu,i}] > 0$ for any $\mu$ and $i$. There exists a function $g(\vec{v})$ on $\mathbb{R}_2$ as an almost sure asymptotic limit of $T$: for every $\vec{v} = (v_1, v_2) \in \mathbb{R}_2$,

$$\frac{1}{N} T_{\lfloor N v_1 \rfloor, \lfloor N v_2 \rfloor} \xrightarrow{a.s.} g(v_1, v_2) \quad (26)$$
as $N \to \infty$. The convergence in $L_1$ also holds:

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}_{X}[T_{\lfloor Nv_1 \rfloor, \lfloor Nv_2 \rfloor}] = g(v_1, v_2).$$

(27)

Here, $\xrightarrow{a.s.}$ means almost sure convergence (in other words, convergence with probability one) and $\lfloor \cdot \rfloor$ is the floor function. The function $g(v_1, v_2)$ is called a shape function or is sometimes known as a time constant. Note that the existence of such a function $g$ is not obvious but can be proved using Kingman’s subadditive ergodic theorem (see [15]).

### 4.2. Shape Functions and Makespan of Scheduling

Let $\lambda$ be the mean of random variable $X_{\mu,i}$ and $F(X_{\mu,i})$ be its cumulative distribution function. To make the argument simpler, we will use the normalization $\tilde{X}_{\mu,i} = X_{\mu,i} - \lambda$. The corresponding $T_{m,n}$ becomes

$$\tilde{T}_{m,n} = \max_{\gamma \in \{(0,0) \to (m,n)\}} \sum_{(\mu,i) \in \gamma} (x_{\mu,i} - \lambda) = T_{m,n} - (m + n)\lambda.$$  

(28)

Therefore, the corresponding shape function $\tilde{g}$ is represented as

$$\tilde{g}(v_1, v_2) = g(v_1, v_2) - \lambda(v_1 + v_2).$$

(29)

Next we derive makespan for the algorithm in Section 2.5. Assume $M$, the number of machines, and $\nu$, the index of the fiducial machine, are proportional to $N$ in the following sense:

$$M = \kappa N + o(N),$$

(30)

and

$$\nu = \tau M + o(M).$$

(31)

for some $\kappa \geq 0$ and $\tau \in [0, 1]$. The following theorem states that the normalized makespan is asymptotically represented by the sum of two shape functions.
Theorem 4.1. Assume Eq.(30) and Eq.(31). Then

\[
\frac{1}{N} \mathbb{E}[T(\nu)] \to g(\kappa(1-\tau), 1) + g(\kappa \tau, 1) - \lambda \\
= \tilde{g}(\kappa(1-\tau), 1) + \tilde{g}(\kappa \tau, 1) + \lambda(\kappa + 1)
\]

and

\[
\frac{1}{N} T(\nu) \to g(\kappa(1-\tau), 1) + g(\kappa \tau, 1) - \lambda \\
= \tilde{g}(\kappa(1-\tau), 1) + \tilde{g}(\kappa \tau, 1) + \lambda(\kappa + 1)
\]

as \( N \to \infty \).

The proof is in Appendix A.

From this theorem, the asymptotic values of the normalized makespan are obtained as

\[
\eta(\kappa) := g(\kappa(1-\tau), 1) + g(\kappa \tau, 1) - \lambda
\]

and we can prove the following property of \( \eta(\kappa) \).

Theorem 4.2. Assume the function \( \eta(\kappa) \) on \([0, 1]\) is a concave function which attains its maximum at the midpoint \( \tau = 1/2 \) and its minimum at the endpoints \( \tau = 0 \) and 1.

The proof is in Appendix B.

4.3. Makespan for the Exponential or Geometric Distribution

For only a few types of distributions \( F \) are the shape functions \( g \) known. The following theorem is based on the results proved in [14].

Theorem 4.3. If \( F \) is the cumulative distribution function of an exponential distribution or a geometric distribution with mean \( \lambda \) and standard deviation \( \sigma \), then

\[
g(\xi, 1) = \lambda(1 + \xi) + 2\sigma \sqrt{\xi}
\]

and

\[
\tilde{g}(\xi, 1) = 2\sigma \sqrt{\xi}.
\]
In Appendix C, we note which results in [14] correspond to Theorem 4.3.

From Theorem 4.1 and Theorem 4.3, the normalized makespan for the exponential distribution and the geometric distribution is obtained as follows.

**Corollary 4.4.** Assume Eq.(30) and Eq.(31). If $F$ is the exponential distribution or geometric distribution with mean $\lambda$ and standard deviation $\sigma$, then

$$
\frac{1}{N} T(\nu) \xrightarrow{\text{a.s.}} 2\sigma \left( \sqrt{\kappa(1-\tau)} + \sqrt{\kappa \tau} \right) + \lambda(\kappa + 1)
$$

(39)

and

$$
\frac{1}{N} \mathbb{E}_\lambda [T(\nu)] \rightarrow 2\sigma \left( \sqrt{\kappa(1-\tau)} + \sqrt{\kappa \tau} \right) + \lambda(\kappa + 1)
$$

(40)

as $N \to \infty$.

In [15], they noted that the exponential or geometric distribution $F$ "is essentially the only nontrivial case (whether directed or undirected, first- or last-passage) where the form of the shape function $g$ above is known." As far as we know, the corresponding makespan function is also unknown except for the case of an exponential or geometric distribution. Remark that, on the other hand, they proved that $g(\epsilon, 1)$ behaves as $\sigma \sqrt{\epsilon} + \lambda(1 + \epsilon)$ for small $\epsilon > 0$ for a general distribution $F$. Therefore, the normalized makespan function $\frac{1}{N} T(\nu)$ behaves asymptotically parabolic for sufficiently small $\nu/N$.

5. Conclusions and Future Work

In this paper, we have discussed the relationship between the asymptotical behavior of makespan and the position of the fiducial machine in a processing system both numerically and mathematically. When a job processing order is fixed, the first and last machines in this processing system minimize the makespan and the center machine in the processing system maximizes the makespan instead of the bottleneck machine. Regarding the dispatching rules discussed in several works, shortest processing time and longest processing time, the makespan of the job processing order without a dispatching rule is hardly
distinguishable from the makespan of the job processing order based on the shortest processing time rule, while it is less than the makespan of the job processing order based on the longest processing time rule. Namely, the dispatching rules do not work well for the flowshop scheduling problem. Also, the makespan is strongly influenced by the position of the fiducial machine in the processing system. Moreover, numerical findings are supported by the property of shape functions discussed in percolation theory. In addition, when the processing time is independently and identically distributed according to an exponential or geometric distribution, it is well known that the shape function has a term proportional to the square of the position of the fiducial machine, although the shape functions in the cases that the processing time is independently and identically distributed according to a discrete uniform distribution, continuous uniform distribution, or chi-squared distribution might have a term proportional to the square of the position, consistent with the numerical results of the expected terminate time function.

In the future work, because our results here regarding the shortest processing time and longest processing time as the dispatching rules show that not all dispatching rules consistently work well with the flowshop scheduling problem, we need to compare other dispatching rules, both numerically and mathematically. Next, we considered here only i.i.d. processing times in order to simplify our discussion; however, since processing times in practice are weakly or strongly correlated with each other, the makespan in the case of correlated processing times needs to be examined in detail.

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Appendix A. Proof of Theorem 4.1

Any shape function $g$ satisfies the following properties (listed in [15]):

(P1) $g(0, 0) = 0,$

(P2) $g(v_1, v_2) = g(v_2, v_1)$ for $v_1, v_2 \in \mathbb{R}_{\geq 0},$

(P3) $g(\alpha \vec{v}) = \alpha g(\vec{v})$ for $\alpha \geq 0, \vec{v} \in \mathbb{R}^2_{\geq 0},$

(P4) $g(\vec{v}) + g(\vec{w}) \geq g(\vec{v} + \vec{w})$ for $\vec{v}, \vec{w} \in \mathbb{R}^2_{\geq 0}.$

By Eq.(26), $\tilde{g}(v_1, v_2)$ also satisfies properties (P1)-(P4). Note that by the law of large numbers,

$$\frac{1}{N} T_{[N\nu],0} = \frac{1}{N} \sum_{\mu=0}^{[N\nu]} X_{\mu,0} \xrightarrow{a.s.} v\lambda. \quad (A.1)$$

By comparing this and Eq.(26), we obtain the properties

(P5) $g(\nu, 0) = g(0, \nu) = v\lambda$ for $\nu \geq 0$ and

(P5)$'$ $\tilde{g}(\nu, 0) = \tilde{g}(0, \nu) = 0$ for $\nu \geq 0.$

Normalizing the makespan $T(\nu)$ by the number of jobs $N$,

$$\frac{1}{N} T(\nu) = \frac{1}{N} \{t_{M,N}(\nu) - s_{1,1}(\nu)\} = \frac{1}{N} \{t_{M,N}(\nu) - s_{1,1}(\nu)\} - \frac{1}{N} \{t_{\nu,N}(\nu) - s_{1,1}(\nu)\} + \frac{1}{N} \{t_{\nu,N}(\nu) - s_{1,1}(\nu)\}. \quad (A.2)$$

36
The first and the third terms follow the same distributions as $T_{M-\nu,N-1}$ and $T_{\nu-1,N-1}$, respectively, where $T_{M-\nu,N-1}$ and $T_{\nu-1,N-1}$ are two independent random variables generated by Eq.(25).

Then by Eq.(26),

$$\frac{1}{N}(T_{M-\nu,N-1} + T_{\nu-1,N-1}) \xrightarrow{a.s.} g(\kappa(1-\tau), 1) + g(\kappa\tau, 1).$$

(A.3)

The second term of the right-hand side of Eq.(A.2) converges to $-\lambda$ since

$$\frac{1}{N}\{t_{\nu,N}(\nu) - s_{\nu,1}(\nu)\} = \frac{1}{N} \sum_{i=1}^{N} X_{\nu,i} \xrightarrow{a.s.} \lambda$$

(A.4)

by the strong law of large numbers. Thus, Eq.(38) and Eq.(33) of Theorem 4.1 hold. Similarly but by using Eq.(27) instead of Eq.(26), Eq.(34) and Eq.(35) can be proved.

**Appendix B. Proof of Theorem 4.2**

It is sufficient to prove that $\tilde{g}(\xi, 1)$ is a strictly increasing concave function of $\xi \geq 0$. By properties (P3) and (P4), $g(\xi, 1)$ is concave and so is $\tilde{g}(\xi, 1)$. Because of (P5)' and the concavity, $\tilde{g}(\vec{v}) > 0$ for $\vec{v} \in \mathbb{R}^2_{>0}$ or $\tilde{g}(\vec{v}) \equiv 0$ for $\vec{v} \in \mathbb{R}^2_{\geq 0}$.

Next assume that $X_{\mu,i}$ are non-degenerate random variables and $E[X_{\mu,i}] = 0$. Then,

$$E_X[T_{1,1}] = E_X[\max(X_{1,2}, X_{2,1})]$$

$$= E_X[X_{2,1} + \max(0, X_{1,2} - X_{2,1})]$$

$$> 0.$$  

(B.1)

Since $E_X[T_{N,N}] \geq NE_X[T_{1,1}]$,

$$\tilde{g}(1, 1) = \lim_{N \to \infty} \frac{1}{N}E_X[T_{N,N}] \geq E_X[T_{1,1}] > 0.$$  

(B.2)

Thus, $\tilde{g}(\vec{v}) \neq 0$, which implies $\tilde{g}(\vec{v}) > 0$ for $\vec{v} \in \mathbb{R}^2_{>0}$.

For $0 \leq \xi < 1$ and $0 < \epsilon \leq 1 - \xi$, the point

$$\vec{p} = \left(\frac{1 + \xi}{1 + \xi + \epsilon}(\xi + \epsilon), \frac{1 + \xi}{1 + \xi + \epsilon}\right)$$

(B.3)
is an internal division of the line segment joining points \((\xi, 1)\) and \((1, \xi)\). Thus by the concavity and property (P3) of \(\tilde{g}\),

\[
\frac{1 + \xi}{1 + \xi + \epsilon} \tilde{g}(\xi + \epsilon, 1) = \tilde{g}\left(\frac{1 + \xi}{1 + \xi + \epsilon}(\xi + \epsilon), \frac{1 + \xi}{1 + \xi + \epsilon}\right) \geq \tilde{g}(\xi, 1) \quad (= \tilde{g}(1, \xi)).
\]

(B.4)

Therefore, \(\tilde{g}(\xi + \epsilon, 1) > \tilde{g}(\xi, 1)\) and \(\tilde{g}(\xi, 1)\) is a strictly increasing function of \(\xi \geq 0\). \(\Box\)

Note that \(g\) is continuous on \(\mathbb{R}_{\geq 0}^2\) since it is concave, and, therefore, \(h_\kappa\) is continuous for \(\tau > 0\). In [15], they proved that \(g\) is continuous on \(\mathbb{R}_{\geq 0}^2\), the whole domain including the boundary, if distribution \(F\) satisfies

\[
\int_0^\infty (1 - F(s))^{1/2} ds < \infty.
\]

(B.5)

Remark that condition (Eq.(B.5)) is satisfied for almost all common distributions since it corresponds to the density functions decaying faster than in cubic order.

### Appendix C. Derivation of Theorem 4.3

Theorem 4.3 is included in the results proved in [14]. Here, we will note which results in [14] correspond to Theorem 4.3.

From Theorem 1.1 of [14], the geometric distribution \(P(X_{\mu,i} = k) = (1-q)^k q\) for \(k \in \mathbb{Z}_{\geq 0}\), whose mean is \(E_X[X_{\mu,i}] = \frac{q}{1-q}\) and variance is \(V[X_{\mu,i}] = \frac{q}{(1-q)^2}\), induces the shape function

\[
g(\xi, 1) = \frac{(1 + \sqrt{q\xi})^2}{1 - q} - 1 = \frac{q}{1-q} (1 + \xi) + 2 \sqrt{\frac{q}{(1-q)^2}} \xi.
\]

(C.1)

This coincides with Eq.(37).

From Theorem 1.6 of [14], the exponential distribution whose mean is one (and, therefore, variance is one) induces the shape function

\[
g(\xi, 1) = \left(1 + \sqrt{\xi}\right)^2 = 1 + \xi + 2 \sqrt{\xi}.
\]

(C.2)
Since the exponential distribution with parameter $\lambda$, whose mean is $\lambda$ and variance is $\lambda^2$, is obtained by multiplying $X_{\mu,i}$ by $\lambda$ and , the shape function becomes

$$g(\xi, 1) = \lambda(1 + \xi) + 2\lambda\sqrt{\xi}.$$  \hspace{1cm} (C.3)

This also coincides with Eq.(37).

Remark that in Theorem 1.1 and Theorem 1.6 of [14], $\xi \geq 1$ is assumed but statements of these theorems can be generalized to $\xi > 0$ by setting $\xi := 1/\xi$. The theorem is easy to prove for $\xi = 0$. Eq.(38) for $\hat{g}$ is a direct consequence of Eq.(37).

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