AMPLIFICATION EFFECTS ON THE TRANSMISSION AND REFLEXION PHASES IN 1D PERIODIC SYSTEMS

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Abstract

We investigate the localization observed recently for locally non-hermitian Hamiltonians by studying the effect of the amplification on the scaling behavior of the transmission and reflection phases in 1D periodic chains of δ-potentials. The amplification here is represented by an imaginary term added to the on-site potential. It is found that both phases of the transmission and reflection amplitudes are strongly affected by the amplification term. In particular, the phases in the region of amplification become independent of the length scale while they oscillate strongly near the maximum transmission (or reflection). The interference effects on the phase in passive systems are used to interpret those observed in the presence of amplification. The phases of the transmission and reflection are found to oscillate in passive systems with increasing periods in the allowed band for the transmission phase while for the reflection phase, its initial value is always less than π/2 in this band.

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1 Introduction

Recently, there was an increase of interest in non-hermitian hamiltonians and quantum phase transitions (typically localized to extended wavefunctions) in systems characterized by them. There are in general two classes of problems in this context: one in which the non-hermiticity is in the nonlocal part \[1, 2\] and the other in which it is in the local part \[3-8\]. In the first category, one considers an imaginary vector potential added to the momentum operator in the Schrödinger hamiltonian. In the second category (non-hermiticity in the local term), an imaginary term is introduced in the one-body potential. It is well-known from textbooks on quantum mechanics that depending on the sign of the imaginary term, this means the presence of a sink (absorber) or a source (amplifier) in the system. It may be noted that this second category does also have a counterpart in classical systems characterized by a Helmholtz (scalar) wave equation as well, where the practical application is in the studies of the effects of classical wave (light) localization due to backscattering in the presence of an amplifying (lasing) medium that has a complex dielectric constant with spatial disorder in its real part \[3, 6\]. There is a common thread binding both the problems though, namely that the spectrum for both becomes complex (the hamiltonian being non-hermitean or real non-symmetric), but can admit real eigenvalues as well. The common property is that the real eigenvalues represent localized states and the eigenvalues off the real lines extended states. That it is so in the first category has been shown in the recent works starting with Hatano and Nelson and followed by others \[1, 2\]. In the rest of the paper we would be concerned with non-hermitian hamiltonians of the second category only. For this category with sources at each scatterer and in the absence of impurities, it seems counter-intuitive that there are localized solutions; but it has been shown in a simple way \[5,8,9\] that the real eigenvalues are always localized. However, up to now the physical origins of this effect have not been provided. Since the localization is a consequence of the backscattering and the destructive interferences, we expect this effect to be related to the scaling behavior of the phases of the transmission and reflection amplitudes. This is the aim of this letter where we examine numerically
the effect of the amplification on the phase of the transmission and reflection amplitudes. We use for this end the Kronig-Penney model which is a continuous multiband model. We first consider a periodic passive system in order to understand the behavior of the phase for localized and extended states. This allow us to explain the phase behavior in such amplifying systems.

2 Model description

We consider a non interacting electron of energy $E$ moving through a linear chain of $\delta$-potentials strengths strength $\beta_n, n$ is the site position. In each site an imaginary term $\eta$ is included leading to a Non Hermitian Hamiltonian. The Schrödinger equation then reads

$$\left\{-\frac{d^2}{dx^2} + \sum_n (\beta_n + \eta) \delta(x - n)\right\} \Psi(x) = E\Psi(x)$$

Here $\Psi(x)$ is the single particle wavefunction at $x$, and $E$ is expressed in units of $\hbar^2/2m$ with $m$ being the free electron effective mass. For simplicity, the lattice spacing is taken to be unity in all this work. Since we are interested only in periodic systems, the potential strength $\beta_n$ is a constant $\beta_0$. The complex potential appearing in the local part of the Hamiltonian in (1) leads either to complex eigenvalues and real wavenumbers or real eigenvalues and complex vavenumbers. We consider the system Ohmically connected to ideal leads so that the second case is used since the total energy is conserved. In this case the imaginary part acts either as a sink (absorber) if $\eta < 0$ or as a source (amplifier) if $\eta > 0$. From the computational point of view it is more useful to consider the discrete version of the Schrödinger equation which is called the generalized Poincaré map and can be derived without any approximation from (1). It reads

$$\Psi_{n+1} = \left[ 2 \cos k + \frac{\sin k}{k} (\beta_0 + i\eta) \right] \Psi_n - \Psi_{n-1}$$

where $\Psi_n$ is the value of the wavefunction at site $n$ and $k = \sqrt{E}$. This representation relates the values of the wavefunction at three successive discrete locations along the x-
axis without restriction on the potential shape at those points and is very suitable for numerical computations. The solution of equation (2) is done iteratively by taking for our initial conditions the following values at sites 1 and 2: \( \Psi_1 = \exp(-ik) \) and \( \Psi_2 = \exp(-2ik) \). We consider here an electron having a wave number \( k_F \) (at Fermi energy) incident at site \( N + 3 \) from the right (by taking the chain length \( L = N \), i.e. \( N + 1 \) scatterers). The transmission and reflection amplitudes \( (t \text{ and } r) \) can then be expressed as

\[
t = \frac{-2i \exp(-ik(N + 3)) \sin k}{\Psi_{N+3} \exp(-ik) - \Psi_{N+2}},
\]

and

\[
r = \frac{\exp(-2ik(N + 3)) (\Psi_{N+2} - \exp(ik) \Psi_{N+3})}{\Psi_{N+3} \exp(-ik) - \Psi_{N+2}},
\]

where the terms \( \exp(-ik(N + 3)) \) and \( \exp(-2ik(N + 3)) \) appearing respectively in the transmission and reflection amplitudes originate from the fact that the electron is incident at site \( N + 3 \) with an incident phase \( -k(N + 3) \). Therefore, these fictitious phases are to be discarded. Note here that the wave number \( k \) appearing in the last expressions is that of the free electron moving in the leads and is different from that inside the system (which is complex). From Eqs. (3 and 4) the phases of the transmission and reflection amplitudes depend only on the values of the wavefunction at the end sites, \( \Psi_{N+2}, \Psi_{N+3} \) which are evaluated from the iterative equation (2). The phases of the transmission and reflection amplitudes \( (\Phi_t \text{ and } \Phi_r) \) are then the arguments of \( t \) and \( r \) respectively. These phases vary obviously between 0 and \( 2\pi \).

3 Results and discussion

As discussed below, the observed asymptotic localization in amplifying periodic systems [9] should come from the phase interferences and the backscattering. Indeed, the maximum transmission length \( (L_{max}) \) in this case can be seen as the characteristic length separating the region where the amplification dominates from that where the interfer-
ences and backscattering dominate (below $L_{\text{max}}$). Let us first consider the effect on the transmission and reflection phases.

In order to understand the phase behavior in the case of constructive and destructive interferences, we start examining its scaling in a passive periodic system. We fix in this case $\beta_0 = 8$ which, from Eqs. (1) and (2) leads us to the energy spectrum shown in Fig.1. In this spectrum, we choose the energies $E = 1$, $E = 3$ and $E = 5$ to scan the phase scaling either in the gap and the allowed band (Figs.2). The transmission phase in Fig.2a oscillates around $\pi$ with decreasing periods for energies away from the allowed band while they increase inside this band. Therefore a higher frequency oscillating phase means a localization. In Fig.2b, the initial reflection phase seems to be always between $\pi/2$ and $3\pi/2$ for energies in the gap which corresponds to localized states for such finite systems.

Let us now examine the phase scaling for amplifying systems $\eta > 0$ (see Figs.3). For simplicity we consider that the on-site potential is purely imaginary (i.e., $\beta = 0$). We see in particular in these figures that both the reflection and amplification phases remain constant in the region where the transmission coefficient grows. It is important to notice that the reflection phase is greater than $\pi/2$ which indicates that there are destructive interferences in the region of growing transmission but they seem to not affect it. In the region of maximum transmission (and reflection) both phases oscillate and the transport properties of the system seems to become sensitive to them.

4 Conclusion

We used in this letter the effect of the amplification on the scaling behavior of both transmission and reflection phases in order to interpret the recently observed effect on the coefficients. The main results show a constant phase in the growth region while it starts oscillating near the maximum transmission and reflection. However, the amplification effect has been studied here only in the allowed band of the corresponding passive periodic system (since $\beta_0 = 0$ when the amplification $\eta$ is applied, all the spectrum of
the passive system is Bloch like). Therefore, it is interesting to examine this effect in the gap of the corresponding passive system. In this case the transmission coefficient is exponentially decaying (the system being finite) and the Lyapunov exponent should be affected differently by the amplification.

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Figure Captions

**Fig.1** Transmission coefficient (in a log scale) versus energy for $\beta_0 = 8$ and $\eta = 0$ (passive system).

**Fig.2** Variation of the reflection and transmission phase with the length scale for $\eta = 0$, $\beta_0 = 8$ and different energies 1, 3 and 5. a) $\Phi_t$, b) $\Phi_{hi}$.

**Fig.3** variations of the reflexion and transmission phases and the transmission coefficient with the length scale $L$ for $\beta = 0$, $\eta = 0.05$ and 0.1 and the energy $E = 1$. a) phase of the transmission, b) phase of the reflection, c) transmission coefficient.
FIGURE 1

Transmission coefficient vs. Energy
FIGURE 2

Phase

Length scale L

a) E=1 (Gap)
E=3 (Gap); V_0=8; η=0
E=6 (Band)

b)
