Construction of time-varying ISS-Lyapunov Functions of Impulsive Systems

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Abstract: Time-varying ISS-Lyapunov functions for impulsive systems provide a necessary and sufficient condition for ISS. This property makes them a more powerful tool for stability analysis than classical candidate ISS-Lyapunov functions providing only a sufficient ISS condition. Moreover, time-varying ISS-Lyapunov functions cover systems with simultaneous instability in continuous and discrete dynamics for which candidate ISS-Lyapunov functions remain inconclusive. The present paper links these two concepts by suggesting a method of constructing time-varying ISS-Lyapunov functions from candidate ISS-Lyapunov functions, thereby effectively combining the ease of construction of candidate ISS-Lyapunov functions with the guaranteed existence of time-varying ISS-Lyapunov functions.

Keywords: Impulsive systems, input-to-state stability, ISS-Lyapunov function.

1. INTRODUCTION

Impulsive systems are a class of hybrid dynamical systems that combine continuous behavior with abrupt changes of state, often related by the flow and the jump, respectively. The flow is typically described by an ordinary differential equation or a partial differential equation. Impulsive systems find their application in various real-world applications; see Rivadeneira and Moog (2015), Antunes et al. (2013), and Guan et al. (2012).

An essential property of dynamical systems in practice is their sensitivity to external perturbations. The notion of input-to-state stability (ISS), introduced by Sontag (1989), guarantees a certain tolerance to such inputs and is therefore helpful for classifying a system’s behavior. A well-established tool for proving ISS is the ISS-Lyapunov function. The framework of ISS-Lyapunov functions was modified to the candidate ISS-Lyapunov function to cover impulsive systems (cf. Hespanha et al., 2008). See also Dashkovskiy and Mironchenko (2013) for an ISS condition for impulsive systems on an infinite-dimensional Banach space via a candidate ISS-Lyapunov function.

A candidate ISS-Lyapunov function-based analysis, however, has certain restrictions. First, it only provides a sufficient condition for ISS. Second, it offers stability conclusions for a restricted class of systems: Either the flow behavior must be stable, and the jumps may be unstable, or the jumps ought to be stable, and the flow might be unstable. These ISS results are inconclusive for impulsive systems with simultaneous instability of the continuous and discrete dynamics. ISS of this class of impulsive systems has received little attention in the literature. It was not until recently that Dashkovskiy and Slynko (2021) treated ISS of such impulsive systems using a dwell-time approach based on the higher-order derivatives of the Lyapunov function. However, the work of Dashkovskiy and Slynko (2021) also gives only a sufficient condition of ISS and does not provide a converse ISS-Lyapunov theorem. Motivated by this, we recently proposed time-varying ISS-Lyapunov functions in implication form that provide a necessary and sufficient condition for ISS of impulsive systems over infinite-dimensional Banach spaces; see Bachmann et al. (2022). They also apply to systems with simultaneous instability in continuous and discrete dynamics.

In this paper, we provide a construction method for time-varying Lyapunov functions from candidate Lyapunov functions incorporating the existing well-established candidate Lyapunov theory into our newly proposed concept of time-varying Lyapunov functions. This enables us to combine the advantages of both, namely the simplicity of construction of candidate Lyapunov functions (cf. Dashkovskiy and Mironchenko, 2013) with the guaranteed existence of a time-varying Lyapunov function.

The rest of the paper unfolds as follows. In Section 2, we provide some preliminaries, including necessary definitions and the notion of time-varying ISS-Lyapunov functions for impulsive systems. In Section 3, we provide a necessary and sufficient condition for ISS. In Section 4, we provide the main result on the construction of ISS-Lyapunov functions. We summarize our findings in Section 5.

2. PRELIMINARIES AND INPUT-TO-STATE STABILITY

We denote the set of natural numbers by $\mathbb{N}$, the set of nonnegative integers by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the set of real
numbers by $\mathbb{R}$, the set of nonnegative real numbers by $\mathbb{R}_+^*$, the space of continuous functions from normed spaces $X$ to $Y$ by $C(X,Y)$, and the ball of radius $r > 0$ around 0 by $B_X(r)$. Let $I = [t_0, \infty) \subset \mathbb{R}$ and let $S = \{\tau_n\}_{n \in \mathbb{N}}$ be a set, which contains the elements of a strictly increasing sequence of impulse times $(\tau_n)_{n \in \mathbb{N}}$ in $(t_0, \infty)$ such that $\tau_n \to \infty$ for $i \to \infty$. Let $(X, \| \cdot \|_X)$ be a Banach space representing the state space. Let $PC(I, X)$ be the space of piecewise continuous functions from $I$ to $X$, which are right-continuous and the left limit exists for all times $t \in I$. Let the Banach space $(U, \| \cdot \|_U)$ represent the input space. Let furthermore $U_t$ be the space of bounded functions from $I$ to $U$ with norm $\|u\|_U := \sup_{t \in I} \{\|u(t)\|_U\}$. We denote the left limit of a function $f$ at $t$ as $f^-(t)$. We consider an impulsive system described by the inter-

gation continuous and discontinuous evolution maps: $x(t) = g_i(x^-(t), u^-(t))$, $t = \tau_i \in S$, $i \in \mathbb{N}$, (1) where $u \in U_c$ and $x : I \to X$. The closed linear operator $A : D(A) \to X$ is the infinitesimal generator of a $C_0$-

semigroup $T(t)$ on $X$, where $D(A)$ is a dense subset of $X$, $f : I \times X \times U \to X$, and $g_i : I \times X \to X$ for all $i \in \mathbb{N}$. We are interested in solutions in the mild sense, i.e., a function $x \in PC(\bar{I}, X)$ such that

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^t T(t-s)f(s, x(s), u(s)) \, ds + \sum_{i \in \mathbb{N} : \tau_i \leq t} T(t-t_i)(g_i(x^-(t_i), u^-(t_i)) - x^-(t_i))$$ (2)

holds for all $t \in [t_0, \infty)$ (cf. Ahmed, 2003). We assume that for system (1), a (forward)-unique global mild solution exists for every initial condition $x(t_0) = x_0$ and every $u \in U_c$. We denote the value of the solution trajectory at time $t$ with the initial condition $x(t_0) = x_0$ and the input $u \in U_c$ by $x(t; t_0, x_0, u)$. We shorten the notation by $x(t)$ if the parameters are clear from the context or can be chosen arbitrarily.

We next define the notion of input-to-state stability.

**Definition 1.** For a given sequence of impulse times $S$, we call system (1) input-to-state stable (ISS) if there exist functions $\beta \in K\mathcal{L}$ and $\gamma \in K\mathcal{C}$ such that for all initial values $(t_0, x_0) \in I \times X$ and every input function $u \in U_c$, the system has a global solution, which satisfies for all $t \in [t_0, \infty)$

$$\|x(t; t_0, x_0, u)\|_X \leq \beta(\|x_0\|_X, t - t_0) + \gamma(\|u\|_U).$$ (3)

We recall the notion of a candidate ISS-Lyapunov function from Dashkovskiy and Mironchenko (2013).

**Definition 2.** Let $V_{\text{cand}} : X \to \mathbb{R}_+^*$ be such that $V_{\text{cand}} \in C(X, \mathbb{R}_+^*)$. We call $V_{\text{cand}}$ a candidate ISS-Lyapunov function for system (1) if it fulfills the following conditions:

(i) There exist functions $\psi_1, \psi_2 \in K\mathcal{C}$ such that

$$\psi_1(\|x\|_X) \leq V_{\text{cand}}(x) \leq \psi_2(\|x\|_X)$$

hold true for all $x \in X$.

(ii) There exist functions $\eta \in K\mathcal{C}$ and $\alpha \in \mathcal{P}$ and a continuous function $\rho : \mathbb{R}_+^* \to \mathbb{R}$ for which $\rho(x) = 0$ $\iff x = 0$ such that for all inputs $u \in U_c$ and all solutions $x = x(t; t_0, x_0, u)$ of (1), whenever $V_{\text{cand}}(x) \geq \eta(\|u\|_U)$, the inequalities

$$\frac{d}{dt}V_{\text{cand}}(x) \leq -\rho(V_{\text{cand}}(x)), \quad t \in I \setminus S,$$

$$V_{\text{cand}}(g_i(x, u)) \leq \alpha(V_{\text{cand}}(x)), \quad t_i \in S,$$ (4)

(5)

hold true.

(iii) There exists a function $\alpha_3 \in K\mathcal{C}$ such that for all $x \in X$, all $u \in U_c$, and all $i \in \mathbb{N}$, which satisfy $V_{\text{cand}}(x) < \eta(\|u\|_X)$, the jump inequality satisfies

$$V_{\text{cand}}(g_i(x, u)) \leq \psi_3(\|u\|_U).$$ (6)

**Remark 1.** Condition (iii) is necessary for establishing ISS, and it was not taken care of in Dashkovskiy and Mironchenko (2013). Once the state has reached the perturbation radius $\chi(\|u\|_U)$, it can escape out of it afterward if Condition (iii) is not fulfilled. Consequently, the system will not be ISS. For discrete-time systems, a similar issue was discussed by Grüne and Kellett (2014).

We now provide the notion of time-varying ISS-Lyapunov functions introduced in Bachmann et al. (2022).

**Definition 3.** Let $V : I \times X \to \mathbb{R}_+^*$ be a function such that $V \in C(I \setminus S \times X, \mathbb{R}_+^*)$ and $V \in PC(I \times X, \mathbb{R}_+^*)$ hold. We call $V$ an ISS-Lyapunov function for system (1) if it fulfills all of the following conditions:

(i) There exist functions $\alpha_1, \alpha_2 \in K\mathcal{C}$ such that

$$\alpha_1(\|x\|_X) \leq V(t, x) \leq \alpha_2(\|x\|_X)$$ (7)

holds true for all $t \in I$ and all $x \in X$.

(ii) There exist functions $\chi \in K\mathcal{C}$ and $\varphi \in \mathcal{P}$ such that for all inputs $u \in U_c$ and all solutions $x = x(t; t_0, x_0, u)$ of (1), whenever $V(t, x) \geq \chi(\|u\|_U)$, the inequalities

$$\frac{d}{dt}V(t, x) \leq -\varphi(V(t, x)), \quad t \in I \setminus S,$$

$$V(t_i, g_i(x, u)) \leq V(t_{i-}, x), \quad t_i \in S,$$ (8)

(9)

hold true.

(iii) There exists a function $\alpha_3 \in K\mathcal{C}$ such that for all $x \in X$, all $u \in U_c$, and all $i \in \mathbb{N}$, which satisfy $V(t_{i-}, x) < \chi(\|u\|_U)$, the jump inequality satisfies

$$V(t_i, g_i(x, u)) \leq \alpha_3(\|u\|_U).$$ (10)

**Remark 2.** From inequalities (8) and (9), it follows that the Lyapunov value of a trajectory of the ISS-Lyapunov function strictly falls with time when the state is outside the perturbation radius $\chi(\|u\|_U)$, whereas the candidate ISS-Lyapunov functions given in Definition 2 do not strictly fall and additional dwell-time conditions have to be imposed (see Theorems 3 and 5) to conclude ISS such that the stabilizing dynamics is dominant over the destabilizing one, which is a restriction for establishing ISS of certain classes of impulsive systems with simultaneous instability in jump and flow.

**Remark 3.** Note that inequality (8) does not necessarily imply that the flow is stable because (8) only holds for time intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}_0$. In general, $V$ is not continuous and may increase at time instants $t = t_i$, $i \in \mathbb{N}_0$. Similar arguments also hold for inequality (9).
We define a feedback system related to system (1) as follows. Let there exist a function $\eta : X \to \mathbb{R}^+_0$, which is Lipschitz-continuous on bounded subsets of $X$, and a function $\psi \in \mathcal{K}_\infty$ such that $\eta(x) \geq \psi(\|x\|_X)$ and 
$$
\dot{x}(t) = Ax(t) + f(t, x(t), d(t)\eta(x(t))) 
= Ax(t) + f(t, x(t), d(t)), \quad t \in I \setminus S, 
$$
(11) 
where $x(t) = g_i(x^-(t), d^-(t)\eta(x^-(t)))$ holds for all $d \in D$ and 
$$
0 \leq \|x(t) - x(t^-)\|_X \leq L_1^i(C, D), \quad i \in I \setminus S.
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$$
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where $x(t) = g_i(x^-(t), d^-(t)\eta(x^-(t)))$ holds for all $d \in D$ and 
$$
0 \leq \|x(t) - x(t^-)\|_X \leq L_1^i(C, D), \quad i \in I \setminus S.
$$
= F(v(t)) - F(v_0) + \sum_{k=1}^{n} F(v_k) - F(v_k) + F(v_k) - F(v_{k-1}) \\
\leq -(t - t_n) - \sum_{k=1}^{n} (t_k - t_{k-1}) = -(t - t_0). \quad (14)

This is equivalent to
\[ v(t) \leq F^{-1}(F(v_0) - (t - t_0)). \quad (15) \]

Due to the case distinction in case \( v(t_n) = 0 \) for some \( t_n \in I \) mentioned above it is appropriate to choose a more relaxed estimate than (15)
\[ \tilde{\beta}(v_0, t - t_0) := F^{-1}\left(F(v_0) - (F(v_0) - m)\left(1 - e^{-\frac{t - t_0}{\alpha(t_0)}}\right)\right) \geq F^{-1}(F(v_0) - (t - t_0)) \]
for \( m > -\infty \), where \( m := F^{-1}(0) \). Otherwise, we set
\[ \tilde{\beta}(v_0, t - t_0) := F^{-1}(F(v_0) - (t - t_0)). \]

In both cases we have \( \tilde{\beta} \in KL \) and we define \( \beta \in KL \), \( \beta(r, s) := \alpha_1^{-1}\left(\alpha_2(r, s)\right) \) to obtain the desired bound
\[ \|x(t; t_0, x_0, u)\|_X \leq \beta\|x_0\|_X, t - t_0 \quad (16) \]
for \( t \in [t_0, t^*] \), \( t^* := \inf\{t \in [t_0, \infty) \mid V(t, x(t)) < \chi(\|u\|_\infty)\} \).

Step 2: Next, we show that trajectories that are in \( A_1(t) \) for some \( t \in I \), stay bounded for all times. Therefore, we apply (10) from which we can conclude that all trajectories jumping from \( A_1(t) \) for some \( t = t_i \) are bounded by
\[ A_2(t) := \{x \in X \mid V(t, x) \leq \alpha_3(\|u\|_\infty)\}. \]

It is not possible that the trajectories leave
\[ A_2(t) = A_1(t) \cup A_2(t) = \{x \in X \mid V(t, x) \leq \max\{\alpha_3(\|u\|_\infty), \chi(\|u\|_\infty)\}\} \]
neither by jump nor by flow because the boundary of \( A_1(t) \cup A_2(t) \) is in the complement of \( A_1 \). From (8), it follows that \( \frac{d}{dt}V(t, x(t)) < 0 \) holds on the boundary, and jump inequality (9) prevents the trajectories from leaving \( A_2(t) \). We define \( \gamma \in K_{\infty}, \gamma := \alpha_1^{-1} \max\{\alpha_3(\cdot), \chi(\cdot)\} \). Then \( \|x(t; t_0, x_0, u)\|_X \leq \gamma(\|u\|_\infty) \) holds for all \( t > t^* \). From this equation and (16), we can conclude
\[ \|x(t; t_0, x_0, u)\|_X \leq \beta\|x_0\|_X, t - t_0 + \gamma(\|u\|_\infty). \]

Theorem 2. Let impulsive system (1) be ISS and satisfy Assumption 1. Then, there exists an ISS-Lyapunov function for system (1).

Proof. Let system (1) be ISS. By Lemma 1 in the Appendix, it is WURS. Therefore, feedback system (11) is UGAS for all \( D \in D \) with \( \eta \) and \( \psi \) as defined in Definition 5. Proposition 1 in the Appendix then gives us that every solution of system (11) is Lipschitz continuous with respect to initial values in bounded subsets of \( X \), and locally Lipschitz continuous on the intervals \([t_i, t_{i+1}], i \in \mathbb{N}_0 \). With Proposition 1 at hand, the converse Lyapunov theorem by Karafyllis and Jiang (2011, Thm. 3.4) ensures that there exists a UGAS-Lyapunov function \( V : I \times X \to \mathbb{R}_+^n \) for feedback system (11), which is Lipschitz continuous on bounded balls in space and locally Lipschitz continuous on the intervals \([t_i, t_{i+1}], i \in \mathbb{N}_0 \). From Lemma 5 in the Appendix, it follows that there exists an ISS-Lyapunov function for system (1) as stated in the theorem.

4. MAIN RESULTS

As the main concern of this paper, we study the construction of an ISS-Lyapunov function from a candidate ISS-Lyapunov function. More precisely, we show a method for constructing an ISS-Lyapunov function \( V = V(t, x) \) as in Definition 3 from a candidate ISS-Lyapunov function \( V_{\text{cand}} \) with rates \( \varphi \) and \( \alpha \) as defined in Definition 2.

4.1 Stable flows and unstable jumps

Let us consider system (1) with stable flows and unstable jumps. Theorem 1 of Dashkovskiy and Mironchenko (2013) provides the following sufficient stability result.

Theorem 3. Let \( V_{\text{cand}} \) be a candidate ISS-Lyapunov function for system (1) and \( \rho, \alpha \) be as in Definition 2 and \( \rho \in P \).

If for some \( \theta, \delta > 0 \) and all \( a > 0 \) it holds that
\[ \int_a^\infty \frac{\rho(s)}{\theta - \delta} \, ds \leq \theta - \delta \quad (17) \]
then (1) is ISS for all impulse time sequences \( S \) for which \( \theta \leq \inf_{i \in \mathbb{N}_1} \{t_{i+1} - t_i\} \) holds.

Now we show that ISS-Lyapunov functions given in Definition 3 can be constructed from \( V_{\text{cand}} \) given in the previous theorem.

Theorem 4. Let \( V_{\text{cand}} \) be a candidate ISS-Lyapunov function for system (1) and parameters \( \rho, \alpha, \theta, \delta \) as specified in Theorem 3, fulfill (17). Let \( \bar{F}(q) := \int_q^\infty \frac{1}{\rho(s)} \, ds \). Let \( \kappa \in K_{\infty} \cap C^1(\mathbb{R}_+^n, \mathbb{R}_+^n) \) such that \( \kappa \leq \min\{\alpha^{-1}, id\} \) and \( \kappa' \in P \) hold, then an ISS-Lyapunov function is given by
\[ V(t, x) = \max\left\{v_1(t, x), v_2(t, x)\right\} \]
where \( v_1, v_2 : I \times X \to \mathbb{R}_+^n \) are given by
\[ v_1(t, x) := \bar{F}^{-1}\left(\max\left\{\bar{F}(V_{\text{cand}}(x)) - \frac{t_i - t_{i-1}}{t_{i+1} - t_i} (\theta - \delta), \bar{F}(0)\right\}\right) \]
for \( t \in [t_i, t_{i+1}], i \in \mathbb{N}_1 \).

The role of \( v_2 \) is to prevent possible issues with finite time convergence. In that case, \( F(V_{\text{cand}}(x)) - \frac{t_i - t_{i-1}}{t_{i+1} - t_i} (\theta - \delta) \) might become smaller than zero. When \( v_1(t, x) \) is not strictly positive for \( x \neq 0 \) anymore.

Proof. Note that \( \bar{F} \) is strictly increasing and invertible (cf. \( F \) in the proof of Theorem 1).

We show that all the conditions of Definition 3 are fulfilled.

(i) By the definition of \( \kappa \), it follows that
\[ V(t, x) \geq \kappa(V_{\text{cand}}(x)) \geq \kappa(\psi(\|x\|_X)) =: \alpha_3(\|x\|_X). \]

Analogously, we obtain
\[ V(t, x) \leq \max\{V_{\text{cand}}(x), \kappa(V_{\text{cand}}(x))\} = V_{\text{cand}}(x) \geq \kappa_2(\|x\|_X) =: \alpha_2(\|x\|_X). \]

(ii) Next, we treat \( V(t, x) \geq \chi(\|u\|_\infty) := \theta(\|u\|_\infty) \). Then \( V_{\text{cand}}(x) \geq \alpha_1^{-1}(V(t, x)) \geq \eta(\|u\|_\infty) \).

We bound the Dini-derivative
\[ \frac{d}{dt} V(t, x) \]
\[ \leq \left\{ \begin{array}{ll}
\dot{v}_1(t, x), & \text{if } v_1(t, x) > v_2(t, x), \\
\dot{v}_2(t, x), & \text{if } v_1(t, x) < v_2(t, x), \\
\max\{\dot{v}_1(t, x), \dot{v}_2(t, x)\}, & \text{if } v_1(t, x) = v_2(t, x).
\end{array} \right. \quad (18) \]
For $v_1(t, x) \geq v_2(t, x) \geq 0$, we have
\begin{equation}
\begin{aligned}
\dot{v}_1(t, x) &\leq \left(\tilde{F}^{-1}\right)' \left(\tilde{F}(V_{\text{cand}}(x)) - \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \left(\theta - \delta\right)\right) \\
&\quad \times \left(\frac{V_{\text{cand}}(x)}{\rho(V_{\text{cand}}(x))} + 1 - \frac{\rho}{\theta}\right) \\
&\leq -\frac{\delta}{\theta} \rho \left(\tilde{F}(V_{\text{cand}}(x)) - \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \left(\theta - \delta\right)\right) \\
&= -\frac{\delta}{\theta} \rho(v_1(t, x)),
\end{aligned}
\end{equation}

where we have used $t_{i+1} - t_i \geq \theta$ and the definition of $\tilde{F}$ in the first step and the inverse function theorem in the second step. Furthermore,
\begin{equation}
\begin{aligned}
\dot{v}_2(t, x) &= \kappa(V_{\text{cand}}(x)) \dot{V}_{\text{cand}}(x) \\
&\leq -\kappa(V_{\text{cand}}(x)) \rho(V_{\text{cand}}(x))
\end{aligned}
\end{equation}
holds. As $\rho, \kappa \in \mathcal{P}$ there exists such a function $\varphi \in \mathcal{P}$ that from (18), (20), and (21) follows.
\[\frac{d}{dt} V(t, x) \leq -\varphi(V(t, x))\]

Therefore, $V(t, x)$ is strictly falling. For the jumps, we find the bound $V(t_i, g_i(x, u))$
\begin{equation}
\begin{aligned}
\dot{V}(t_i, x) &\leq \max \left\{\tilde{F}^{-1} \left(\tilde{F}(\alpha(V_{\text{cand}}(x))) - (\theta - \delta)\right)\kappa(\alpha(V_{\text{cand}}(x))) \right\} \\
&\leq \max \left\{\tilde{F}^{-1}(\tilde{F}(V_{\text{cand}}(x)))\right\} V_{\text{cand}}(x) \\
&\leq \tilde{F}(V_{\text{cand}}(x)) - \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \left(\theta - \delta\right).
\end{aligned}
\end{equation}

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&\leq \max \left\{\tilde{F}^{-1}(\tilde{F}(V_{\text{cand}}(x)))\right\} V_{\text{cand}}(x) \\
&\leq \tilde{F}(V_{\text{cand}}(x)) - \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \left(\theta - \delta\right).
\end{aligned}
\end{equation}

Then, an ISS-Lyapunov function is defined by
\[V(t, x) := \tilde{F}^{-1} \left(\tilde{F}(V_{\text{cand}}(x)) - \frac{t_{i+1} - t_i}{t_{i+1} - t_i} \left(\theta + \delta\right)\right)\]
for $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}$.

The proof follows similar argumentation as the case with stable jumps and unstable flows and therefore is omitted here.

5. CONCLUSION

We proposed a method to construct time-varying ISS-Lyapunov functions from candidate ISS Lyapunov functions. By this, we could show that the new concept of time-varying ISS-Lyapunov functions integrates into the existing stability theory for impulsive systems. Even more, time-varying Lyapunov functions can serve as a new standard formulation for Lyapunov methods for impulsive systems as the existence of ISS-Lyapunov functions is guaranteed by Theorem 2. Moreover, the time-varying Lyapunov functions cover broad system classes (including systems with simultaneous instability in continuous and discrete dynamics), which is a fundamental advantage over candidate ISS-Lyapunov functions.

Appendix A. TECHNICAL RESULTS

Lemma 1. If impulsive system (1) is ISS, then it is WURS.

Proof. From the definition of ISS, it follows that there exist functions $\beta \in \mathcal{K}L$ and $\gamma \in \mathcal{K}_\infty$. We define $\alpha(s) := \beta(s, 0)$ for all $s \in \mathbb{R}^+_0$. From inequality (3), it follows that $\alpha(s) \geq s$ for all $s \in \mathbb{R}^+_0$, which means that $\alpha \in \mathcal{K}_\infty$.

We define $\sigma \in \mathcal{K}_\infty$ as $\sigma(s) := \left(\gamma^{-1} - (\frac{\alpha^{-1}}{\alpha^{-1} - (\frac{\alpha^{-1}}{\alpha^{-1} - \delta}))\right)$, and choose locally Lipschitz continuous functions $\eta : X \rightarrow \mathbb{R}^+_0$ and $\psi \in \mathcal{K}_\infty$ such that $\psi(\|x\|_X) \leq \eta(x) \leq \sigma(\|x\|_X)$.

We show that with this definition of $\eta$, the inequality
\[\gamma(||d(t)||\eta(\mathcal{F}d(t)||\mathcal{F}d(t)||) \leq \frac{1}{2} \|x_0\|_X (A.1)\]
holds for all $t \geq t_0$. We first compute
\[\gamma(||d(t)||\eta(\mathcal{F}d(t)||\mathcal{F}d(t)||) \leq \gamma(\sigma(||\mathcal{F}d(t)||_X)) \leq \frac{1}{2} \mathcal{F}d(t)||_X \]
which is valid for every $d \in D$, all $x_0 \in X$, and all $t \in I$.

By definition, $\mathcal{F}d$ is right-continuous, so for each $t_* \in I$, there exists a $\delta > 0$ such that
\[\gamma(||d(t)||\eta(\mathcal{F}d(t); t_*, x_*)||_I) \leq \frac{1}{2} \mathcal{F}d(t; t_*, x_*)||_X \leq \frac{1}{2} \|x_*\|_X \]
(A.2)

We prove by contradiction that $\mathcal{F}d = \infty$. Thus, let us assume that $\mathcal{F}d$ is not equal to infinity. Then, inequality (A.2) holds for all $t \in [t_0, T]$. By substituting inequality (A.2) into (3),
\[\|\mathcal{F}d(t)||_X \leq \beta(||x_0||_X, t - t_0) + \frac{1}{2} \|x_0\|_X \leq \frac{1}{2} \alpha(||x_0||_X)\]
(A.3) is obtained. System (1) follows the principle of causality, i.e., it only depends on inputs $u(t)$ for $t \in [t_0, T]$. Therefore, we can transform (3) into
\[\|\mathcal{F}d(t; t_*, x_*)||_X \leq \beta(||x_*||_X, \mathcal{F}d - t_*)\]
\begin{align}
+ \gamma \left( \sup_{t \in [t_0, T]} \left\{ \| d(t) \eta(\xi_d(t), t, x) \|_{\mathcal{L}} \right\} \right).
\end{align}

As system (11) is only right-continuous, we need an estimate for limits from the left, which also considers possible jumps. This estimate is given by
\[
\| \xi_d(t) \|_X = \lim_{t_+ \to t} \| \xi_d(t_+; t, \xi_d(t)) \|_X \\
\leq \lim_{t_+ \to t} \big\| \eta(\xi_d(t_+), t) - \eta(\xi_d(t), t) \big\|_{\mathcal{L}} + \left( \sup_{t \in [t_0, T]} \left\{ \| d(t) \eta(\xi_d(t), t) \|_{\mathcal{L}} \right\} \right)
\leq \alpha \left( \| \xi_d(t) \|_X + \frac{1}{\gamma} \| \xi_d(t) \|_X \right) \leq \frac{\alpha}{1 - \frac{1}{\gamma}} \| x_0 \|_X,
\]
for \( t \in [t_n, t_{t+1})\), \( t \leq \tau \) and \( n \in \mathbb{N}_0\), where we define the product \( \prod_{j=0}^{n} a_j := 1\). From this, we can conclude by extreme value theorem that
\[
\|\mathcal{P}(t) - \mathcal{P}(\tau)\|_X \leq \|x_0 - x_0\|_X \max_{t_0 \in [0, \tau]} \left\{ M \prod_{j=1}^{n} (L_{\mathcal{P}}(K(C, \tau)) M) \right\} \chi(t - t_0) \quad n \in \mathbb{N}_0 \quad t \in [t_n, t_{t+1})
\]
This gives us a Lipschitz constant for the solutions of (11) with respect to the initial conditions.

Lemma 4. Let system (11) be robustly forward complete. Let \( \mathcal{P} \) be Lipschitz continuous in the second variable on bounded subsets of \( X \) uniformly with respect to the third argument and for all \( t \in I \). Then its solutions are locally Lipschitz continuous on intervals \( t \in [t_i, t_{i+1})\), \( i \in \mathbb{N}_0\), where the Lipschitz constant depends on \( \|x_0\|_X\).

The proof follows from (Mironchenko and Wirth, 2018, Lemma 4.6).

Proposition 1. Let Assumption 1 hold, and system (1) be UGARS. Then, for any \( n \) as defined in Definition 5, the closed-loop system (11) has robustly forward complete solutions, which are Lipschitz continuous with respect to initial values in bounded subsets of \( X \) and locally Lipschitz continuous on intervals \( t \in [t_i, t_{i+1})\), \( i \in \mathbb{N}_0\), where the Lipschitz constant depends on the norm of the initial condition \( \|x_0\|_X\).

Proof. The solutions of system (1) exist for every \( t \geq 0\). This implies that the solutions of system (11) also exist for every \( t \geq t_0\). Let \( \eta \) be the function constructed in Definition 5. According to Definition 5, system (11) is UGARS. Therefore, for all \( C > 0 \) and all \( t > t_0\)
\[
\sup_{t_0 \in \mathbb{B}_X(c), \in \mathbb{D} \in \mathbb{D}, \in [0, \tau]} \|\mathcal{P}(t; t_0, x_0)\|_X \leq \beta(C, 0)
\]
exists, and system (11) is robustly forward complete. We apply Lemma 2 from the Appendix to obtain that \( \mathcal{P} \) and \( \mathcal{Q} \) are Lipschitz continuous in space on bounded subsets of \( X \) and uniformly with respect to \( t \) and \( d \). Therefore, the preconditions of Lemma 3 in the Appendix are fulfilled, i.e., system (11) has a solution, which is Lipschitz continuous with respect to the initial values in bounded subsets of \( X \). From Lemma 4 in the Appendix, it follows that the trajectories are locally Lipschitz continuous on intervals \( t \in [t_i, t_{i+1})\), \( i \in \mathbb{N}_0\), as claimed in the Proposition.

Lemma 5. Let there exist a UGARS-Lyapunov function \( V \) for system (11) as given in Definition 6, which is Lipschitz continuous on bounded balls in space and locally Lipschitz continuous on the intervals \( [t_i, t_{i+1})\), \( i \in \mathbb{N}_0\). Then, there exists an ISS-Lyapunov function of the form given in Definition 3, which is Lipschitz continuous on bounded subsets of \( X \), and locally Lipschitz continuous on \( I \setminus S \) such that it is right continuous and the left limit exists.

Proof. The UGARS-Lyapunov function \( V \) as given in Definition 6 immediately implies that the inequalities (8)–(9) hold for all \( x \in X \) and \( u \in B_U(\eta(x))\). As a consequence of Definition 5, (8)–(9) hold for all \( x \in X \) and \( u \in B_U(\psi(x))\). As \( \psi \) is invertible, we can set \( \chi := \alpha_1 \circ \psi^{-1}\). By this, \( V \) satisfies (8)–(9) without additional restrictions.

It remains to show that the third property of Definition 3 is fulfilled. Let \( V(t_i, x) < \chi(\|x\|_U) \) hold, it follows from (7) that
\[
\|x\|_X \leq \min\{\alpha_i^{-1} \circ \chi\}(\|u\|_U) = \kappa(\|u\|_U).
\]
where we define \( \kappa \in \mathcal{K}_\infty\), \( \kappa(s) = (\alpha_i^{-1} \circ \chi)(s)\). For \( \|x\|_X \leq C \) and \( \|u\|_U \leq D \), we have
\[
\|g_i(x, u)\|_X \leq \|g_i(x, u) - g_i(0, u)\|_X + \|g_i(0, u)\|_X
\]
\[
\leq L_1^i(C, D)\|x\|_X + L_2^i(C, D)\|u\|_U
\]
\[
\leq L_1^i(C, D)\kappa(\|u\|_U) + L_2^i(C, D)\|u\|_U = \alpha_3(\|u\|_U).
\]
Hence, the second inequality applies Assumption 1 and the last equality follows from the fact that \( g_i(0, u) = 0 \) by (11) being UGARS. Note that \( D \) is linear in \( \|u\|_U \) and \( C \) is linear in \( \|x\|_X \). Therefore, by (A.7), \( L_1^i \) and \( L_2^i \) are weakly growing functions in \( \|u\|_U \). It can be seen that \( \alpha_3(s) := L_1^i(k(s), s) \kappa(s) + L_2^i(k(s), s) s \) belongs to class \( \mathcal{K}_\infty \). By defining the \( \mathcal{K}_\infty \)-function \( \alpha_3 := \alpha_2 \circ \alpha_3 \) we conclude
\[
V(t_i, g_i(x, u)) \leq \alpha_2(\|g_i(x, u)\|_X) \leq \alpha_3(\|u\|_U).
\]
We used (7) in the first inequality and (A.8) in the second one. So, \( V \) is an ISS-Lyapunov function for system (1).

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