Non blow-up criterion for the 3-D Magneto-hydrodynamics equations in the limiting case

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Abstract

For the 3-D incompressible Magneto-hydrodynamics equations, whether the limiting case $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ implies the regularity of weak solution is unknown. Here we prove that the solution $(u, b)$ of the 3-D Magneto-hydrodynamics equations is regular in $(0, T] \times \mathbb{R}^3$ if $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ and $b_h$ satisfies the Ladyzhenskaya-Prodi-Serrin condition, where $b_h$ is the horizontal components of the magnetic field $b$.

1 Introduction

We consider the 3-D incompressible Magneto-hydrodynamics (MHD) equations:

$$
\begin{cases}
  u_t - \nu \Delta u + u \cdot \nabla u = -\nabla \pi + b \cdot \nabla b, \\
  b_t - \eta \Delta b + u \cdot \nabla b = b \cdot \nabla u, \\
  \nabla \cdot u = \nabla \cdot b = 0.
\end{cases}
$$

(1.1)

Here $u$, $b$ describe the fluid velocity field and the magnetic field respectively, $p$ is a scalar pressure, $\nu > 0$ is the kinematic viscosity, $\eta > 0$ is the magnetic diffusivity. If $\nu = \eta = 0$, (1.1) is so called the ideal MHD equations. In the absence of the magnetic field, (1.1) becomes the incompressible Navier-Stokes equations. We take $\nu = \eta = 1$ for the simplicity of notation throughout this paper.

The global existence of weak solution and local existence of strong solution to the MHD equations (1.1) were proved by Duvaut and Lions [6]. The same as the incompressible Navier-Stokes equations, the regularity and uniqueness of weak solutions remains a challenging open problem. We refer to [18] for some mathematical questions related to the MHD equations.

It is well-known that if the weak solution of the Navier-Stokes equations satisfies the Ladyzhenskaya-Prodi-Serrin condition

$$
u \in L^q(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} \leq 1, \quad p \geq 3,$$

then it is regular on $(0, T) \times \mathbb{R}^3$. Note that the limiting case $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$ does not fall into the framework of energy method, which was proved by Escauriaza-Seregin-Šverák [7]. Wu [21] [22] extended Ladyzhenskaya-Prodi-Serrin type criterions to the MHD equations in terms of both the velocity field $u$ and the magnetic field $b$ for $p > 3$. However, some
numerical experiments seem to indicate that the velocity field should play a more important role than the magnetic field in the regularity theory of solutions to the MHD equations [15]. Recently, He-Xin [8] and Zhou [25] have presented some regularity criterions to the MHD equations in terms of the velocity field only. Chen-Miao-Zhang [3, 4] extend and improve their results as follows: if the weak solution of the MHD equations (1.1) satisfies

\[ u \in L^q(0,T; B^s_{p,\infty}) \quad \text{with} \quad \frac{2}{q} + \frac{3}{p} = 1 + s, \quad \frac{3}{1 + s} < p \leq \infty, \quad -1 < s \leq 1, \]

then it is regular on \((0,T) \times \mathbb{R}^3\). Here \(B^s_{p,\infty}\) is the Besov space. We refer to [2, 10, 11, 19, 23, 24, 26] and references therein for more relevant results. However, whether the condition on \(b\) can be removed remains unknown in the limiting case (i.e., \((p,q,s) = (3, \infty, 0)\)). The case \(u, b \in L^\infty(0,T; L^3(\mathbb{R}^3))\) was considered by Mahalov-Nicolaenko-Shikin [14], and Wang-Zhang’s recent result [20] states as follows.

**Theorem 1.1** Let \((u, b)\) be a suitable weak solution of the MHD equations (1.1). Assume that \(u \in L^\infty(-1,0; L^3(\mathbb{R}^3))\), and

\[ b \in L^\infty(-1,0; BMO^{-1}(\mathbb{R}^3)) \quad \text{and} \quad b(t) \in VMO^{-1}(\mathbb{R}^3) \quad \text{for} \quad t \in (-1, 0]. \]

Then \((u, b)\) is Hölder continuous on \(\mathbb{R}^3 \times (-1, 0]\).

Note that the inclusion relation: \(L^3(\mathbb{R}^3) \subsetneq VMO^{-1}(\mathbb{R}^3)\) [12]. Hence, this result is an improvement of [14]. It’s interesting that whether the condition on the magnetic field can be removed. In this paper, we consider the condition on the horizontal components \(b_h\) of the magnetic field, and prove that if \(b_h\) is in Ladyzhenskaya-Prodi-Serrin’s class, then the solution is regular.

Our main result is the following:

**Theorem 1.2** Let \((u, b)\) be a smooth solution of the MHD equations (1.1) in \((-1,0) \times \mathbb{R}^3\), which is also suitable. Assume that \(u \in L^\infty(-1,0; L^3(\mathbb{R}^3))\) satisfying one of the following conditions

\[ i) \nabla b_h \in L^q_x L^p_t ((-\frac{1}{2},0) \times \mathbb{R}^3), \quad \frac{3}{p} + \frac{2}{q} = 2, \quad \frac{9}{4} \leq p < 3; \]

\[ ii) \nabla b_h \in L^3_x L^2_t ((-\frac{1}{2},0) \times \mathbb{R}^3), \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 3 \leq p \leq \infty, \quad \text{and} \quad (1.2) \]

\[ b_h \in L^l_x L^1_t ((-\frac{1}{2},0) \times \mathbb{R}^3), \quad \frac{3}{l} + \frac{2}{s} = 1, \quad 9 \leq l \leq \infty. \]

Then \((u, b)\) is regular on \(\mathbb{R}^3 \times (-1, 0]\).

**Remark 1.3** For simplicity, we assume that \((u, b)\) is smooth and suitable. At this moment, one can integrate by parts legitimately and the energy norms are finite, i.e.

\[ \|u\|_{L^\infty(-1,0; L^2(\mathbb{R}^3)) \cap L^2(-1,0; \dot{H}^1(\mathbb{R}^3))} + \|b\|_{L^\infty(-1,0; L^2(\mathbb{R}^3)) \cap L^2(-1,0; \dot{H}^1(\mathbb{R}^3))} < \infty. \quad (1.3) \]
To apply the theory of Escauriza-Seregin-Šverák [7], at first we prove that some necessary scaling invariant quantities of $u, b$ are bounded for compactness arguments. Then we have to estimate the vertical component $b_3$ of the magnetic field, where we use the equation of $b_3$ by the energy method technically. On the other hand, the according careful interior regular criteria are needed. In fact, let
\[
G(b_h, p, q; r) \equiv r^{1-\frac{2}{p}} \frac{2}{q} \|b_h\|_{L^q_t L^p_x(Q_r)},
\]
where $Q_r = (-r^2, 0) \times B_r$ and $B_r$ is a ball of radius $r$ centered at zero. We have the following more general interior criteria in the limiting case:

**Theorem 1.4** Let $(u, b)$ be a suitable weak solution of the MHD equations (1.1) in $(-1, 0) \times B_1$. Assume that $u \in L^\infty(-1, 0; L^3(B_1))$, and satisfies the following conditions
\[
i) \liminf_{r \to 0} G(b_h, p, q; r) = 0, \quad \sup_{0 < r < 1} G(b_h, p, q; r) < \infty, \quad \frac{3}{p} + \frac{2}{q} = 2, \quad 1 \leq p \leq \infty;
\]
\[
ii) \sup_{0 < r < 1} G(b_3, l, s; r) < \infty, \quad \frac{3}{l} + \frac{2}{s} = 2, \quad 1 \leq s \leq \infty.
\]
Then $(u, b)$ is regular on $\mathbb{R}^3 \times (-1, 0]$.

**Remark 1.5** For $u \in L^\infty_t L^3_x$, whether $b \in L^\infty_t (BMO^{-1}_x)$ or the above condition of $b$ without
\[
\liminf_{r \to 0} G(b_h, p, q; r) = 0
\]
implies the regularity of $(u, b)$ is still unknown, where standard energy methods or backward uniqueness methods seem to be out of reach.

This paper is organized as follows: In section 2, we introduce the basic notations and some known interior regular criteria. In section 3, the main line is to prove Theorem 1.2 under the assumptions of Theorem 1.4. Then Section 4 is devoted to prove Theorem 1.4.

## 2 Preliminaries

Let us first introduce the definitions of suitable weak solution.

**Definition 2.1** Let $T > 0$ and $\Omega \subset \mathbb{R}^3$. We say that $(u, b)$ is a suitable weak solution of the MHD equations (1.1) in $\Omega_T = \Omega \times (-T, 0)$ if
\[
(a) (u, b) \in L^\infty(-T, 0; L^2(\Omega)) \cap L^2(-T, 0; H^1_0(\Omega));
\]
\[
(b) (u, b, \pi) \text{ satisfies the equations (1.1) in } \mathcal{D}'(\Omega_T);
\]
\[
(c) \pi \in L^\infty_x(\Omega_T) \text{ and the following local energy inequality holds: for } a.e. \ t \in [-T, 0]
\]
\[
\int_{\Omega} (|u(t,x)|^2 + |b(t,x)|^2) \phi dx + 2 \int_{-T}^t \int_{\Omega} (|\nabla u|^2 + |\nabla b|^2) \phi dx ds
\]
\[
\leq \int_{-T}^t \int_{\Omega} \left[ (|u|^2 + |b|^2)(\Delta \phi + \partial_t \phi) + u \cdot \nabla \phi(|u|^2 + |b|^2 + 2\pi) - (b \cdot u)(b \cdot \nabla \phi) \right] dx ds,
\]
for any nonnegative $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R})$ vanishing in a neighborhood of the parabolic boundary of $\Omega_T$. 

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Remark 2.2 In general, we don’t know whether the weak solution is suitable. However, this is true if \((u, b) \in L^4(\Omega_T)\).

We define a solution \((u, b)\) to be regular at \(z_0 = (x_0, t_0)\) if \((u, b) \in L^\infty(Q_r(z_0))\) with \(Q_r(z_0) = (-r^2 + t_0, t_0) \times B_r(x_0)\), and \(B_r(x_0)\) is a ball of radius \(r\) centered at \(x_0\). We also denote \(Q_r\) by \(Q_r(0)\) and \(B_r\) by \(B_r(0)\). For a function \(u\) defined on \(Q_r(z_0)\), the mixed space-time norm \(\|u\|_{L^{p,q}(Q_r(z_0))}\) is defined by

\[
\|u\|_{L^{p,q}(Q_r(z_0))}^q := \int_{t_0}^{t_0+r^2} \left(\int_{B_r(x_0)} |u(x,t)|^p \, dx \right)^{\frac{q}{p}} \, dt.
\]

The following small energy regularity result is well-known, see [8, 14]. Similar result was proved [11] for the Navier-Stokes equations.

**Proposition 2.3** Assume that \((u, b)\) is a suitable weak solution of (1.1) in \(Q_1(z_0)\). There exists an absolute constant \(\varepsilon > 0\) such that if

\[
r^{-2} \int_{Q_r(z_0)} |u|^3 + |b|^3 + |\pi|^\frac{3}{2} \, dx \, dt \leq \varepsilon
\]

for some \(r > 0\), then \((u, b)\) is regular at the point \(z_0\).

We also need the small energy interior regularity result in terms of the velocity only proved by Zhang and the author in [19], and the according boundary interior regularity result see Kang-Kim [11].

**Proposition 2.4** Assume that \((u, b)\) is a suitable weak solution of (1.1) in \(Q_1(z_0)\). There exists an absolute constant \(\varepsilon > 0\) such that if \(u \in L^{p,q}\) near \(z_0\) and

\[
\limsup_{r \to 0^+} r^{-\left(\frac{3}{p} + \frac{2}{q} - 1\right)} \|u\|_{L^{p,q}(Q_r(z_0))} < \varepsilon,
\]

(2.1)

with \(p, q\) satisfying \(1 \leq \frac{3}{p} + \frac{2}{q} \leq 2\), \(1 \leq q \leq \infty\) and \((p, q) \neq (\infty, 1)\). Then \((u, b)\) is regular at the point \(z_0\).

Next we introduce some notations. Let \((u, b, \pi)\) be a solution of (1.1) and introduce the following scaling:

\[
u^\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad b^\lambda(x, t) = \lambda b(\lambda x, \lambda^2 t), \quad \pi^\lambda(x, t) = \lambda^2 \pi(\lambda x, \lambda^2 t),
\]

(2.2)

for any \(\lambda > 0\), then the family \((u^\lambda, b^\lambda, \pi^\lambda)\) is also a solution of (1.1). For \(z_0 = (x_0, t_0)\), we define some invariant quantities under the scaling (2.2):

\[
A(f, r) = \sup_{-r^2 \leq t < 0} r^{-1} \int_{B_r} |f(y, t)|^2 \, dy, \quad C(f, r) = r^{-2} \int_{Q_r} |f(y, s)|^3 \, dy \, ds,
\]

\[
E(f, r) = r^{-1} \int_{Q_r} |\nabla f(y, s)|^2 \, dy \, ds, \quad K(f, r) = r^{-3} \int_{Q_r} |f(y, s)|^2 \, dy \, ds,
\]

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for \( f = u, b \) and
\[
D(\pi, r) = r^{-2} \int_{Q_r} |\pi(y, s)| \frac{3}{2} dy ds,
\]
\[
\tilde{D}(\pi, r) = r^{-2} \int_{Q_r} |\pi(y, s) - (\pi)_{B_r}|^{\frac{3}{2}} dy ds, \quad (\pi)_{B_r} = \frac{1}{|B_r|} \int_{B_r} \pi(y, s) dy.
\]

Let \( A(u, b; r) = A(u, r) + A(b, r) \), and \( E(u, b; r), C(u, b; r) \) and \( K(u, b; r) \) denote similar notations. We also introduce
\[
G(f, p, q; r) = r^{1-\frac{3}{p} - \frac{2}{q}} \|f\|_{L^p; Q_r}, \quad \tilde{G}(f, p, q; r) = r^{1-\frac{3}{p} - \frac{2}{q}} \|f - (f)_{B_r}\|_{L^p; Q_r},
\]
\[
H(f, p, q; r) = r^{2-\frac{3}{p} - \frac{2}{q}} \|f\|_{L^p; Q_r}, \quad \tilde{H}(f, p, q; r) = r^{2-\frac{3}{p} - \frac{2}{q}} \|f - (f)_{B_r}\|_{L^p; Q_r}.
\]

Throughout this paper, we denote by \( C_0 \) a constant independent of \( r, \rho \) and different from line to line.

## 3 Proof of Theorem 1.2

In this section, we’ll prove Theorem 1.2 under the assumption of Theorem 1.4. Moreover, we assume that
\[
\|u\|_{L^\infty_t L^3_x((-1,0) \times R^3)} + \|b(\cdot, -\frac{1}{2})\|_{L^3_x(R^3)} \leq C_0,
\]
which is reasonable from the assumptions of Theorem 1.2 and 1.3.

First, we have the following embedding inequality and Sobolev’s interpolation inequality (for example, see [1]):

**Lemma 3.1** i) For \( 2 \leq \ell \leq 6, \ a = \frac{3}{4}(\ell - 2) \) and \( f \in H^1(R^3) \), we have
\[
\int_{R^3}|f|^\ell \leq C_0 \left( \int_{R^3} |\nabla f|^2 \right)^{a} \left( \int_{R^3} |f|^2 \right)^{\frac{\ell}{2} - a}. \tag{3.1}
\]

ii) For \( f \in L^\infty(-1, 0; L^2(R^3)) \cap L^2(-1, 0; \dot{H}^1(R^3)) \), we have
\[
\|f\|_{L^1_t L^2_x((-1,0) \times R^3)} \leq C_0 \|f\|_{L^1_t L^2_x((-1,0) \times R^3)} \leq C_0 \|f\|_{L^1_t L^2_x((-1,0) \times R^3)} \leq C_0 \|f\|_{L^1_t L^2_x((-1,0) \times R^3)}, \tag{3.2}
\]
where \( \frac{3}{4} + \frac{2}{s} = \frac{3}{2} \) with \( 2 \leq s \leq \infty \).

**Lemma 3.2** Under the assumption of Theorem 1.2, we have
\[
|b_3|^\frac{3}{2} \in L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x((-\frac{1}{2}, 0) \times R^3).
\]

**Proof.** Recall the third equation of the magnetic field:
\[
\partial_t b_3 - \Delta b_3 + u \cdot \nabla b_3 = b \cdot \nabla u_3,
\]
and multiplying \( 3|b_3|b_3 \) on both sides of it, we have
\[
\partial_t (|b_3|^3) - 3\Delta b_3 (|b_3|b_3) + u \cdot \nabla (|b_3|^3) = 3b \cdot \nabla u_3 (|b_3|b_3).
\]
Integrating on $R^3$, using integration by parts and $\nabla \cdot u = 0$ we derive that

$$
\partial_t \int_{R^3} |b_3|^3 \, dx + \frac{8}{3} \int_{R^3} |\nabla (|b_3|^\frac{2}{3})|^2 \, dx = 3 \int_{R^3} (b \cdot \nabla u_3) (|b_3|b_3) \, dx \equiv 3I. \tag{3.3}
$$

Let $\nabla_h = (\partial_1, \partial_2)^T$ and $I = I_1 + I_2$, where

$$
I_1 = \int_{R^3} (b_3 \cdot \nabla u_3) (|b_3|b_3) \, dx, \quad I_2 = \int_{R^3} (b_3 \partial_3 u_3) (|b_3|b_3) \, dx.
$$

Obviously,

$$
I_1 \leq \int_{R^3} (\nabla_h \cdot b_3) u_3 (|b_3|b_3) \, dx + 2 \int_{R^3} |b_3| |\nabla_h b_3| \, dx
\leq C_0 \|u_3\|_{L^2(R^3)} \left[ \|\nabla_h \cdot b_3\|_{L^2(R^3)}^2 + \|b_3 b_h \|_{L^\infty(R^3)} \right]
\leq C_0 \left[ \|\nabla_h \cdot b_3\|_{L^2(R^3)} + \|b_3 b_h \|_{L^\infty(R^3)} \right],
$$

and the divergence-free property of $b$ implies that

$$
I_2 \leq C_0 \int_{R^3} |u_3| |\partial_3 b_3| |b_3|^2 \, dx
\leq C_0 \int_{R^3} |u_3| |\nabla_h \cdot b_3| b_3^2 \, dx \leq C_0 \|\nabla_h \cdot b_3\|_{L^2(R^3)}.
$$

From the above estimates of $I_1, I_2$ and (3.3), we derive that

$$
\partial_t \int_{R^3} |b_3|^3 \, dx + \frac{8}{3} \int_{R^3} |\nabla (|b_3|^\frac{2}{3})|^2 \, dx \leq C_0 \left[ \|\nabla_h \cdot b_3\|_{L^2(R^3)}^2 + \|b_3 b_h \|_{L^\infty(R^3)} \right], \tag{3.4}
$$

and let

$$
II_1 = \|\nabla_h \cdot b_3\|_{L^2(R^3)}^2, \quad II_2 = \|b_3 b_h \|_{L^\infty(R^3)}.
$$

**Step I: Estimate of $II_1$.** Let $\frac{3}{p} + \frac{3}{q} = 2$ with $1 \leq q \leq \infty$ and

$$
\|\nabla b_h\|_{L^p_{t}L^q_x((-\frac{1}{2},0) \times R^3)} \leq C_0.
$$

Then by Hölder inequality we have

$$
II_1 \leq \|\nabla b_h\|_{L^p_t L^q_x((-\frac{1}{2},\tau) \times R^3)} \|b_3\|_{L^\infty_x L^\frac{6p}{6p-3}\tau (\tau,\infty) \times R^3)}^2,
$$

where the end case $p = \infty$ or $p = \frac{3}{2}$ still holds for the above inequality. For any $\tau$ with $-\frac{1}{2} < \tau < 0$, integrating to time we have

$$
\int_{-\frac{1}{2}}^{\tau} II_1 \, dt \leq \|\nabla b_h\|_{L^p_t L^q_x((-\frac{1}{2},\tau) \times R^3)} \|b_3\|_{L^\infty_x L^\frac{6p}{6p-3}\tau((-\frac{1}{2},\tau) \times R^3)}^2,
$$

where $\frac{1}{q} + \frac{1}{q'} = 1.$
To apply Gronwall’s inequality, we choose $q < \infty$ and by Lemma 3.1 $q', p$ should satisfy
\[
\frac{2p - 3}{2p} + \frac{1}{q'} = 1, \quad 3 \leq 2q' \leq \infty,
\]
which yields that $1 \leq q \leq 3$ or $\frac{9}{4} \leq p \leq \infty$. Hence
\[
\|b_3\|_{L_t^{2q'} L_x^{\frac{6p}{5p - 3}}((-\frac{1}{2}, \tau) \times \mathbb{R}^3)} \leq C_0 \left[\|(|b_3|^\frac{3}{2})\|_{L_t^\infty L_x^2((-\frac{1}{2}, \tau) \times \mathbb{R}^3)} + \|\nabla (|b_3|^\frac{3}{2})\|_{L_t^1 L_x^\infty((-\frac{1}{2}, \tau) \times \mathbb{R}^3)}\right]^\frac{2}{3}. \quad (3.5)
\]

**Step II: Estimate of $II_2$.**

Let $\frac{3}{q} + \frac{2}{s} = 1$ with $2 \leq s \leq \infty$ and
\[
\|b_h\|_{L_t^s L_x^s((-\frac{1}{2}, 0) \times \mathbb{R}^3)} \leq C_0.
\]
Then by Hölder inequality we have
\[
II_2 \leq C_0 \|b_h\|_{L_t^q L_x^q(R^3)} \|\nabla (|b_3|^\frac{3}{2})\|_{L_t^2 L_x^2(R^3)} \|b_3\|_{\frac{3p}{2p - 3} L_t^{\frac{6p}{5p - 3}} L_x^{\frac{3p}{5p - 3}}(R^3)},
\]
where $p_1$ satisfies
\[
\frac{3}{2l} + \frac{3}{4} + \frac{1}{p_1} = 1, \quad 1 \leq p_1 \leq \infty.
\]
Integrating to time with the same $\tau$ as above, we have
\[
\int_{-\frac{1}{2}}^{\tau} II_2 dt \leq \|b_h\|_{L_t^q L_x^q((-\frac{1}{2}, \tau) \times \mathbb{R}^3)} \|\nabla (|b_3|^\frac{3}{2})\|_{L_t^2 L_x^2((-\frac{1}{2}, \tau) \times \mathbb{R}^3)} \|b_3\|_{\frac{3p}{2p - 3} L_t^{\frac{6p}{5p - 3}} L_x^{\frac{3p}{5p - 3}}((-\frac{1}{2}, \tau) \times \mathbb{R}^3)},
\]
where $q_1$ satisfies
\[
\frac{1}{s} + \frac{1}{2} + \frac{1}{q_1} = 1, \quad 1 \leq q_1 \leq \infty.
\]
Using $\frac{3}{q} + \frac{2}{s} = 1$, we have
\[
\frac{4}{p_1} + \frac{4}{q_1} = 1,
\]
that is $|b_3|^\frac{3}{2} \in L_t^{\frac{4q_1}{4p_1}} L_x^{\frac{4q_1}{4p_1}}$, and
\[
\frac{3}{p_1} + \frac{2}{q_1} = \frac{3}{2},
\]
which yields that by Lemma 3.1 for $2 \leq \frac{4q_1}{4p_1} \leq \infty$,
\[
\|b_3\|_{\frac{3p}{2p - 3} L_t^{\frac{6p}{5p - 3}} L_x^{\frac{3p}{5p - 3}}((-\frac{1}{2}, \tau) \times \mathbb{R}^3)} \leq C_0 \left[\|(|b_3|^\frac{3}{2})\|_{L_t^\infty L_x^2((-\frac{1}{2}, \tau) \times \mathbb{R}^3)} + \|\nabla (|b_3|^\frac{3}{2})\|_{L_t^1 L_x^\infty((-\frac{1}{2}, \tau) \times \mathbb{R}^3)}\right]^\frac{2}{3}. \quad (3.6)
\]
Then $s$ must be $2 \leq s \leq 3$ or $9 \leq l \leq \infty$.  

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Step III: Arguments for Theorem 1.2. From the above two estimates, we obtain that if the condition \( i \) of (1.2) holds, i.e.,

\[
\| \nabla b_h \|_{L^q_t L^p_x((-\frac{1}{2},0) \times \mathbb{R}^3)} \leq C_0, \quad \frac{9}{4} \leq p < 3,
\]

then the embedding inequality implies

\[
\| b_h \|_{L^q_t L^l_x((-\frac{1}{2},0) \times \mathbb{R}^3)} \leq C_0, \quad \frac{3}{l} + \frac{2}{q} = 1, \quad 9 \leq l < \infty,
\]

thus the estimates (3.5)-(3.6) hold, which yields that for \(-\frac{1}{2} < \tau < 0\), there holds

\[
\int_{-\frac{1}{2}}^{\tau} \int_{\mathbb{R}^3} \partial_t(|b_3|^3) dx dt + \frac{8}{3} \int_{-\frac{1}{2}}^{\tau} \int_{\mathbb{R}^3} |\nabla (|b_3|^\frac{3}{2})|^2 dx dt \leq C_0 \left[ \| (|b_3|^\frac{3}{2}) \|_{L^\infty_t L^3_x((-\frac{1}{2},\tau) \times \mathbb{R}^3)} + \| \nabla (|b_3|^\frac{3}{2}) \|_{L^2_{t,x}((-\frac{1}{2},\tau) \times \mathbb{R}^3)} \right]^\frac{2}{3}. \tag{3.7}
\]

Note that \( b(\cdot, -\frac{1}{2}) \| L^3_x(\mathbb{R}^3) \leq C_0 \). Hence, Gronwall’s inequality implies

\[
\| (|b_3|^\frac{3}{2}) \|_{L^\infty_t L^3_x((-\frac{1}{2},\tau) \times \mathbb{R}^3)} + \| \nabla (|b_3|^\frac{3}{2}) \|_{L^2_{t,x}((-\frac{1}{2},\tau) \times \mathbb{R}^3)} < \infty,
\]

and taking the supremum of \( \tau \to 0 \), we have

\[
\| (|b_3|^\frac{3}{2}) \|_{L^\infty_t L^3_x((-\frac{1}{2},0) \times \mathbb{R}^3)} + \| \nabla (|b_3|^\frac{3}{2}) \|_{L^2_{t,x}((-\frac{1}{2},0) \times \mathbb{R}^3)} \leq C_0. \tag{3.8}
\]

On the other hand, when the condition \( i \) of (1.2) holds, i.e.,

\[
\| \nabla b_h \|_{L^q_t L^p_x((-\frac{1}{2},0) \times \mathbb{R}^3)} \leq C_0, \quad 3 \leq p \leq \infty,
\]

and

\[
\| b_h \|_{L^q_t L^l_x((-\frac{1}{2},0) \times \mathbb{R}^3)} \leq C_0, \quad \frac{3}{l} + \frac{2}{s} = 1, \quad 9 \leq l \leq \infty,
\]

then we can obtain the same estimate as (3.3). This lemma is proved. \( \square \)

Proof of Theorem 1.2. Due to the assumptions on \( b_h \), we have

\[
\| b_h \|_{L^q_t L^l_x((-\frac{1}{2},0) \times \mathbb{R}^3)} \leq C_0, \quad \frac{3}{l} + \frac{2}{s} = 1, \quad 9 \leq l \leq \infty.
\]

Moreover, Lemma 3.2 yields that

\[
\| b_3 \|_{L^2_{t,x}((-\frac{1}{2},0) \times \mathbb{R}^3)} < \infty.
\]

Obviously, the conditions of Theorem 1.4 are satisfied, thus the proof is complete. \( \square \)
4 Blow-up analysis and Proof of Theorem [1.4]

We will apply Proposition 2.3 to prove the interior regularity of the solution by blow-up analysis, which was early used for the 3D Navier-Stokes equations in [13], see also [17]. Note that the velocity $u$ is in the critical class, hence backward uniqueness results in [7] are still needed. Firstly, we prove that the basic energy norms $A(u, b; r)$, $E(u, b; r)$, and $D(\pi, r)$ are uniformly bounded for all $0 < r < 1$. (see Theorem 4.1); secondly, a standard compactness argument for suitable weak solutions of the 3-D MHD equations and backward uniqueness results imply Theorem 1.4.

4.1 Bounded estimates of $A(u, b; r)$ and $E(u, b; r)$

To ensure the validity of blow-up analysis, we have to prove that $A(u, b; r)$, $E(u, b; r)$, and $D(\pi, r)$ are uniformly bounded for all $0 < r < r_1$ with some $r_1 > 0$.

**Theorem 4.1** Under the assumptions of [1.4] there exists a $r_1 > 0$ such that

$$A(u, b; r) + E(u, b; r) + D(\pi, r) < \infty, \quad 0 < r < r_1,$$

where $r_1$ depends on $C(u, b; 1)$ and $D(\pi, 1)$.

For completeness, we supply the following technical lemmas. First, we will control $A(u, b; r) + E(u, b; r)$ in terms of the other scaling invariant quantities by using the following local inequality.

**Proposition 4.2** Let $0 < 4r < \rho < r_0$ and $1 \leq p, q \leq \infty$. There holds

$$A(u, b; r) + E(u, b; r) \leq C_0\left(\frac{r}{\rho}\right)^2 K(u, b; \rho) + C_0\left(\frac{\rho}{r}\right)^2 \left[C(u, \rho) + C(u, \rho)^{1/3}(C(b, \rho)^{2/3} + D(\pi, \rho)^{2/3})\right].$$

**Proof.** Let $\zeta$ be a cutoff function, which vanishes outside of $Q_\rho$, and equals 1 in $Q_{\rho/2}$, and satisfies

$$|\nabla \zeta| \leq C_0\rho^{-1}, \quad |\partial_t \zeta|, |\Delta \zeta| \leq C_0\rho^{-2}.$$

Define the backward heat kernel as

$$\Gamma(x, t) = \frac{1}{4\pi(r^2 - t)^{3/2}} e^{-\frac{|x|^2}{4(r^2 - t)}}.$$

Taking the test function $\phi = \Gamma \zeta$ in the local energy inequality, and noting $(\partial_t + \Delta)\Gamma = 0$, we obtain

$$\sup_t \int_{B_\rho} (|u|^2 + |b|^2)\phi dx + \int_{Q_\rho} (|\nabla u|^2 + |\nabla b|^2)\phi dx dt$$

$$\leq \int_{Q_\rho} \left[ (|u|^2 + |b|^2)(\Delta \phi + \partial_t \phi) + u \cdot \nabla \phi(|u|^2 + |b|^2 + 2\pi - 2\pi_{B_\rho}) - (b \cdot u)(b \cdot \nabla \phi) \right] dx dt$$

$$\leq \int_{Q_\rho} \left[ (|u|^2 + |b|^2)(\Gamma \Delta \zeta + \Gamma \partial_t \zeta + 2\nabla \Gamma \cdot \nabla \zeta) + |\nabla \phi||u||u|^2 + |b|^2 + 2|\pi - \pi_{B_\rho}| \right] dx dt.$$
By some direct computations, it is easy to verify that
\[ \Gamma(x, t) \geq C_0^{-1} r^{-3} \quad \text{in } Q_r; \]
\[ |\nabla \phi| \leq |\nabla \Gamma| \zeta + |\nabla \zeta| \leq C_0 r^{-4}; \]
\[ |\Gamma \Delta \zeta| + |\Gamma \partial_t \zeta| + 2|\nabla \Gamma \cdot \nabla \zeta| \leq C_0 r^{-5}, \]
from which and Hölder inequality, it follows that
\[ f, r \leq C_0 \frac{\rho^2}{r} K(u, b; \rho) + C_0(\frac{\rho}{r})^{2-2} \int_{Q_r} (|u|^3 + |u||b| + |u||\pi - \pi_{B_r}|)dxdt \]
\[ \leq C_0(\frac{\rho}{r})^{2} K(u, b; \rho) + C_0(\frac{\rho}{r})^{2} [C(u, \rho) + C(u, \rho)^{1/3} (C(b, \rho)^{2/3} + \bar{D}(\pi, \rho)^{2/3})]. \]
The proof is complete. \(\square\)

The following is an interpolation inequality.

**Lemma 4.3** For any \(0 < r < r_0\), let \(\frac{2}{p} + \frac{2}{q} = 2\) with \(1 \leq q \leq \infty\), there holds
\[ C(f, r) \leq C_0 G(f, p, q; r) (E(f, r) + A(f, r)), \]
where \(f = u, b\).

**Proof.** Without loss of generality, we consider the estimate of \(u\). By Hölder inequality and Sobolev inequality, we get
\[ \int_{B_r} |u|^3 dx = \int_{B_r} |u|^{3+3\beta + 3-3\alpha - 3\beta} dx \]
\[ \leq \left( \int_{B_r} |u|^2 dx \right)^{3\alpha/2} \left( \int_{B_r} |u|^6 dx \right)^{\beta/2} \left( \int_{B_r} |u|^p dx \right)^{(3-3\alpha - 3\beta)/p} \]
\[ \leq C_0 \left( \int_{B_r} |u|^2 dx \right)^{3\alpha/2} \left( \int_{B_r} |\nabla u|^2 + |u|^2 dx \right)^{3\beta/2} \left( \int_{B_r} |u|^p dx \right)^{(3-3\alpha - 3\beta)/p}, \]
where \(\alpha, \beta\) are chosen so that
\[ \frac{1}{3} = \frac{\alpha}{2} + \frac{\beta}{6} + \frac{1 - \alpha - \beta}{p}, \quad \frac{3\beta}{2} + \frac{3-3\alpha - 3\beta}{q} = 1. \]
Taking \(\alpha = \frac{2p-3}{3p}\) and \(\beta = \frac{1}{p}\), we get
\[ \int_{Q_r} |u|^3 dx \leq C_0 \left( \sup_{-r^2 < t < 0} \int_{B_r} |u|^2 dx \right)^{1-\frac{3}{2p}} \left( \int_{Q_r} |\nabla u|^2 + |u|^2 dxdt \right)^{\frac{3}{2p}} \]
\[ \times \left( \int_{-r^2}^0 \left( \int_{B_r} |u|^p dx \right)^{\frac{2}{q}} dt \right)^{\frac{1}{q}}, \]
which yields that
\[ C(u, r) \leq C_0 A(u, r)^{1-\frac{3}{2p}} (E(u, r) + A(u, r))^{\frac{3}{2p}} G(u, p, q, r) \]
\[ \leq C_0 G(u, p, q, r) (E(u, r) + A(u, r)). \]
The proof is complete. \(\square\)
We present the estimate of the pressure in terms of scaling invariant quantities, see also [16].

**Lemma 4.4** Let \((u, b)\) be a suitable weak solution of (1.1) in \(Q_1\). Then there hold

\[
\tilde{D}(\pi, r) \leq C_0 \left( \left( \frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi, \rho) + \left( \frac{\rho}{r} \right)^2 C(u, b; \rho) \right),
\]

(4.2)

for any \(0 < 4r < \rho < 1\).

**Proof.** Note that \(\pi\) satisfies the following equation in distribution sense:

\[-\Delta \pi = \partial_i \partial_j (\hat{u}_i \hat{u}_j - \hat{b}_i \hat{b}_j),\]

where \(\hat{u} = u - (u)_{B_\rho}\) and \(\hat{b} = b - (b)_{B_\rho}\). Let \(\zeta\) be a cut-off function, which equals 1 in \(Q_{\rho/2}\) and vanishes outside of \(Q_\rho\). Set \(\pi = \pi_1 + \pi_2\) with

\[\pi_1 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} [\partial_i \partial_j (((\hat{u}_i \hat{u}_j - \hat{b}_i \hat{b}_j)\zeta^2)],\]

and \(\pi_2\) is harmonic in \(Q_{\rho/2}\).

Due to the Calderon-Zygmund inequality, we have

\[
\int_{B_{\rho}} |\pi_1|^\frac{3}{2} dx \leq C_0 \int_{B_{\rho}} |\hat{u}|^3 + |\hat{b}|^3 dx.
\]

Since \(\pi_2\) is harmonic in \(Q_{\rho/2}\), we have

\[
\int_{B_r} |\pi_2 - (\pi_2)_{B_r}|^{\frac{3}{2}} dx \leq C_0 r^{3+\frac{3}{2}} \sup_{B_{\rho/4}} |\nabla \pi_2|^{\frac{3}{2}}
\]

\[
\leq C_0 \left( \frac{r}{\rho} \right)^{3+\frac{3}{2}} \int_{B_{\rho/2}} |\pi_2 - (\pi_2)_{B_{\rho/2}}|^{\frac{3}{2}} dx.
\]

Hence we infer that

\[
\tilde{D}(\pi, r) \leq \tilde{D}(\pi_1, r) + \tilde{D}(\pi_2, r)
\]

\[
\leq C_0 \left( \left( \frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi, \rho) + \left( \frac{\rho}{r} \right)^2 C(u, b; \rho) \right).
\]

The proof is complete. \(\square\)

**Proof of Theorem 4.1** Without loss of generality, we assume that

\[
\sup_{0<r<1} G(b_h, p, q; r) + \sup_{0<r<1} G(b_3, l, s; r) \leq C_0,
\]

for some \((p, q)\) and \((l, s)\) satisfying

\[
\frac{3}{p} + \frac{3}{q} = 2, \quad \frac{3}{l} + \frac{3}{s} = 2.
\]

Then, by Lemma 4.3 we have

\[
C(b, \rho) \leq C_0 [C(b_h, \rho) + C(b_3, \rho)] \leq C_0 [A(b, \rho) + E(b, \rho)].
\]
Thus, by the local energy inequality and $C(u, \rho) \leq C_0$, we have

$$A(u, b; r) + E(u, b; r) \leq C_0 \left( \frac{r}{\rho} \right)^2 A(u, b; \rho) + C_0 \left( \frac{\rho}{r} \right)^2 \left[ 1 + (C(b, \rho) + A(b, \rho))^2/3 + \tilde{D}(\pi, \rho)^2/3 \right]$$

$$\leq C_0 \left( \frac{r}{\rho} \right)^2 [A(u, b; \rho) + E(u, b; \rho)] + C_0 \left( \frac{\rho}{r} \right)^2 \left[ (\frac{\rho}{r})^8 + \tilde{D}(\pi, \rho)^2/3 \right] .$$

Take $0 < 8r \leq \rho < r_0$ and set

$$F(r) = A(u, b; r) + E(u, b; r) + \varepsilon^{-1/2} \tilde{D}(\pi, r)^2/3 .$$

Due to Lemma 4.4, we get

$$\tilde{D}(\pi, r) \leq C_0 \left( \frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi, \rho) + C_0 \left( \frac{\rho}{r} \right)^2 (1 + A(b, \rho) + E(b, \rho))$$

Hence,

$$F(r) \leq C_0 \left[ \left( \frac{r}{\rho} \right)^2 + \left( \frac{\rho}{r} \right)^2 \varepsilon^{1/2} + \left( \frac{r}{\rho} \right)^{5/3} \right] F(\rho) + C_0 \varepsilon^{-3/2} \left( \frac{\rho}{r} \right)^{10} ,$$

and choosing $\theta = \left( \frac{r}{\rho} \right)$ and $\varepsilon$ sufficiently small, we get

$$F(\theta \rho) \leq \frac{1}{2} F(\rho) + C_0 \varepsilon^{-3/2} \theta^{-10} ,$$

which yields that there exists a $r_1 > 0$ such that

$$\sup_{0 < r < r_1} F(r) < C_1 ,$$

where $C_1$ depends on $C_0$, $C(u, b; 1)$, $\tilde{D}(\pi, 1)$. The proof is complete. \(\Box\)

### 4.2 Proof of Theorem 1.4

The proof of Theorem 1.4 is based on the blow-up analysis and unique continuation theorem, for example see [17], [7].

We assume that $\|u\|_{L^\infty_t L^3_x(Q_1)} \leq C_0$, and

i) $\lim \inf_{r \to 0} G(b_h, p, q; r) = 0$, \linebreak $\lim \sup_{0 < r < 1} G(b_h, p, q; r) \leq C_0$, \linebreak $\frac{3}{p} + \frac{2}{q} = 2$, \linebreak $1 \leq p \leq \infty$;

ii) $\sup_{0 < r < 1} G(b_3, l, s; r) \leq C_0$, \linebreak $\frac{3}{l} + \frac{2}{s} = 2$, \linebreak $1 \leq s \leq \infty$.

Then, by the local energy inequality in Proposition 4.2 and Theorem 4.1 we have

$$A(u, b; r) + E(u, b; r) + \tilde{D}(\pi, r) \leq C_1 ,$$

for all $0 < r < 1$.\[12]
where \( C_1 > 0 \) may depend on \( C(u, b; 1) \) and \( D(\pi, 1) \). Moreover, suppose that \( C(u, b; 1) + D(\pi, 1) \leq C_0 \), which is reasonable by the definition of suitable weak solutions.

Suppose that the statement of the theorem is false. Then there exist a series of suitable weak solutions \((v^k, b^k, \pi^k)\) and \( r_k \downarrow 0 \) such that

\[
A(v^k, b^k; r) + E(v^k, b^k; r) + \tilde{D}(\pi^k, r) \leq C_1, \quad \text{for all } 0 < r < 1, \tag{4.3}
\]

and

\[
G(\tilde{b}^k, p, q; r_k) \to 0, \quad \text{as } r_k \to 0.
\]

Moreover, \((0, 0)\) is a singular point of \((v^k, b^k, \pi^k)\).

We denote

\[
u^k(y, s) = r_k v^k(r_k y, r_k^2 s), \quad b^k(y, s) = r_k b^k(r_k y, r_k^2 s), \quad \pi^k(y, s) = r_k^2 \pi^k(r_k y, r_k^2 s),
\]

where \((y, s) \in B_{1/k} \times \left(-\frac{1}{r_k}, 0\right)\). Then it follows from (4.3) that

\[
A(u^k, b^k; r) + E(u^k, b^k; r) + \tilde{D}(\pi^k, r) \leq C_1, \quad \text{for all } 0 < r < 1,
\]

\[
G(b^k_h, p, q; 1) \to 0, \quad \text{as } r_k \to 0.
\]

\[
\|u^k\|_{L^\infty_\omega L^2_{3/2}(B_{1/k} \times \left(-\frac{a}{r_k}, 0\right))} \leq C_0. \tag{4.4}
\]

For any \( a, T > 0 \), choose sufficiently large \( k \) such that

\[
\|u^k\|_{L^\infty_\omega L^2((-T, 0) \times B_a)} + \|b^k\|_{L^\infty_\omega L^2((-T, 0) \times B_a)} + \|\nabla u^k\|_{L^2((-T, 0) \times B_a)} + \|\nabla b^k\|_{L^2((-T, 0) \times B_a)} \leq c(a, T). \tag{4.5}
\]

Hence, \( u^k : \nabla u^k, u^k : \nabla b^k, b^k : \nabla u^k, b^k : \nabla b^k \in L^{\frac{3}{2}} L^{\frac{9}{2}}(Q_a) \). This gives by the linear Stokes theory [7] that

\[
|\partial_t u^k| + |\Delta u^k| + |\partial_t b^k| + |\Delta b^k| + |\nabla p^k| \in L^{\frac{3}{2}} L^{\frac{9}{2}}(Q_{3a/4}).
\]

Then Lions-Aubin’s lemma ensures that there exists \((u, b, \pi)\) such that for any \( a, T > 0 \) (up to subsequence),

\[
\begin{align*}
\lim_{k \to +\infty} u^k &\to u, \quad \lim_{k \to +\infty} b^k \to b, \quad \text{in } L^3((-T, 0) \times B_a), \\
\lim_{k \to +\infty} u^k &\to u, \quad \lim_{k \to +\infty} b^k \to b, \quad \text{in } C([-T, 0]; L^{9/8}(B_a)), \\
\lim_{k \to +\infty} \pi^k &\to \pi \quad \text{in } L^\frac{3}{2}((-T, 0) \times B_a), \\
\lim_{k \to +\infty} \|u\|_{L^\infty_\omega L^2((-T, 0) \times \mathbb{R}^3)} &\leq C_0, \\
\lim_{k \to +\infty} b^k_h \to b_h &\to 0, \quad \text{in } L^q((-T, 0); L^p(B_a)),
\end{align*}
\]

as \( k \to +\infty \).

Hence \( \partial_3 b = 0 \) and \( b \cdot \nabla b = 0 \) due to the velocity field equations, and we get

\[
u_t - \Delta u + u \cdot \nabla u = -\nabla \pi, \quad \nabla \cdot u = 0. \tag{4.6}
\]
Using the property of weak convergence, by (4.4) we have

\[ C(u,1) + \tilde{D}(\pi,1) \leq C_0, \]

and due to \( u \in L^{\infty,3}_{t,x} \), the well-known result in [7] yields that \( \|u\|_{L^\infty(Q_{1/2})} \leq C_0 \). Applying the interior regularity criteria in Proposition 2.4, we obtain that \( \|b\|_{L^\infty(Q_{1/2})} \leq C_0 \).

Since \((0,0)\) is a singular point of \((v^k, \bar{b}^k, \bar{\pi}^k)\), by small regularity results in Proposition 2.3 we have

\[ \varepsilon < C(v^k, r) + C(\bar{b}^k, r) + \tilde{D}(\bar{\pi}^k, r), \]

for any \(0 < r < 1\). Thus,

\[ \varepsilon < C(u^k, r) + C(b^k, r) + \tilde{D}(\pi^k, r), \]

for any \(0 < r < 1\).

Take the supremum limit as \(k \to \infty\), we have

\[ \varepsilon < C_0 r^3 + \tilde{D}(\pi^k, r), \]

for any \(0 < r < 1\).

By the pressure estimate in Lemma 4.4 and \[4.3\], we have

\[ \tilde{D}(\pi^k, r) \leq C_0 \left( \frac{r}{\rho} \right)^{5/2} \tilde{D}(\pi^k, \rho) + \left( \frac{\rho}{r} \right)^2 C(u^k, b^k; \rho) \leq C_0 \left( \frac{r}{\rho} \right)^{5/2} + C_0 \left( \frac{\rho}{r} \right)^2 C(u^k, b^k; \rho), \]

for any \(0 < r < \rho < 1\). Choose \( \rho = \sqrt{r} \), then

\[ \limsup_{k \to \infty} \tilde{D}(\pi^k, r) \leq C_0 \sqrt{r}, \]

for any \(0 < r < 1\).

Hence, we have

\[ \varepsilon < C_0 r^3 + C_0 \sqrt{r}, \]

for any \(0 < r < 1\). Obviously, it’s a contradiction. The proof is complete. \( \square \)

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