ON ENERGY FUNCTIONALS 
FOR SECOND-ORDER ELLIPTIC SYSTEMS 
WITH CONSTANT COEFFICIENTS

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Abstract. We consider the Dirichlet problem for second-order elliptic systems with constant coefficients. We prove that non-reducible strongly elliptic systems of this type do not admit non-negatively defined energy functionals of the form

\[ f \mapsto \int_D \Phi(u_x, v_x, u_y, v_y) \, dxdy, \]

where \( D \) is the domain where the problem we are interested in is considered, \( \Phi \) is some quadratic form in \( \mathbb{R}^4 \), and \( f = u + iv \) is a function in the complex variable. The proof is based on reducing the system under consideration to a special (canonical) form, when the differential operator defining this system is represented as a perturbation of the Laplace operator with respect to two small real parameters (the canonical parameters of the system under consideration).

Keywords: second-order elliptic system, canonical representation of second-order elliptic system, Dirichlet problem, energy functional.

1. Introduction, problem statement and some background

In the paper we are dealing with the Dirichlet problem (in its classical setting) for second-order elliptic systems in \( \mathbb{R}^2 \) with constant coefficients. For real-valued functions \( u \) defined in \( \mathbb{R}^2 \) we will denote by \( u_x, u_y, u_{xx} \) etc., their partial derivatives with respect to the corresponding variables. Furthermore, we will use the differential operators \( \partial_x = \partial / \partial x \) and \( \partial_y = \partial / \partial y \). In what follows the symbol \( M_k(\mathbb{R}) \) will state for the space of all real \( k \times k \)-matrices (here \( k > 0 \) is an integer), while the symbol \( A^t \) will stand for the transposed matrix to \( A \).

We are interested in the question whether the energy functionals of the form

\[ f \mapsto \int_D \Phi(u_x, u_y, v_x, v_y) \, dxdy, \]

do exist for systems under consideration, where \( D \) is a domain where the problem in considered, \( f = u + iv \) is some function in the complex variable, and \( \Phi \) is some non-negatively determined quadratic form in \( \mathbb{R}^4 \). The question on whether such energy functional exists is mainly motivated by the Dirichlet problem for real-valued harmonic functions, since for these functions such functional exists and has the form \( \int_D ((u_x)^2 + (u_y)^2) \, dxdy \). For general systems under consideration and, in particular, for second-order elliptic equations with constant complex coefficients the question about existence of positively determined energy functionals still open in the general case (as well a question on general solvability of the corresponding Dirichlet problem in general bounded simply connected domains).

In what follows we will identify the points \( z = (x, y) \) in the plane \( \mathbb{R}^2 \) with the complex numbers \( z = x + iy \). Moreover, we will identify the pairs of functions \( u \) and \( v \), defined in \( \mathbb{R}^2 \) and taking real values, with the complex-valued function \( f(z) = u(x, y) + iv(x, y) \), and vice-versa. Furthermore, the symbol \( f \) will mean, if necessary, the vector \((u, v)^t\).

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Let $A, B, C \in M_2(\mathbb{R})$. Define the differential operator

$$\mathcal{L} = A \partial_{xx} + 2B \partial_{xy} + C \partial_{yy},$$

where, as usual, $\partial_x f = u_x + iv_y$ and $\partial_y f = u_y + iv_x$. In other words, $\mathcal{L}f = \bar{u} + iv$, where $\bar{u}$ and $\bar{v}$ are defined as follows:

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = A \begin{pmatrix} u_{xx} \\ v_{xx} \end{pmatrix} + 2B \begin{pmatrix} u_{xy} \\ v_{xy} \end{pmatrix} + C \begin{pmatrix} u_{yy} \\ v_{yy} \end{pmatrix}.$$

We consider the homogeneous system of equations of the form

$$\mathcal{L}f = 0. \tag{3}$$

The most important particular case that we are interested in during this work is the system given by the matrices $A, B, C \in M_2^2$, where

$$M_2^2 = \left\{ A \in M_2(\mathbb{R}) : A = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \right\}.$$

Observe that the set $M_2^2$ considered with the standard operations of matrix summation and multiplications is the field isomorphic to the field $\mathbb{C}$ of complex numbers. Thus the system (3) with $A, B, C \in M_2^2$ is equivalent to one second-order equation with constant complex coefficients on the (complex-valued) function $f = u + iv$, that is to the equation of the form

$$af_{xx} + 2bf_{xy} + cf_{yy} = 0, \tag{4}$$

where $a, b, c \in \mathbb{C}$, $f_x = \partial_x f$ and $f_y = \partial_y f$ (as in the real-valued case). The systems corresponding to the equations of the form (4), are often called skew-symmetric, despite on a clear inaccuracy of this term.

Recall that an ellipticity of the system (4) means that the corresponding characteristic form

$$\mathcal{F}(\xi, \eta) = \det(A\xi^2 + 2B\xi\eta + C\eta^2) \tag{5}$$

with real $\xi$ and $\eta$ vanishes only for $\xi = \eta = 0$ (see, for instance, [1]).

The ellipticity condition for the equation (4) is equivalent to the fact that the corresponding symbol $a\xi^2 + 2b\xi\eta + c\eta^2$ with real $\xi$ and $\eta$ is also vanished only for $\xi = \eta = 0$. The latter condition is equivalent to the fact that the roots of the corresponding characteristic equation $a\lambda^2 + 2b\lambda + c = 0$ are not real.

In the general case it follows from the ellipticity condition that $\det A \neq 0$ and $\det C \neq 0$ (otherwise $\mathcal{F}(t, 0) = 0$ and $\mathcal{F}(0, t) = 0$ for $t \neq 0$, respectively). Since $\mathcal{F}(\xi, \eta) = \eta^4 \det(A\lambda^2 + 2B\lambda + C)$, where $\lambda = \xi/\eta$, the ellipticity of the system (4) is equivalent to the fact that $\det A \neq 0$ and all roots of the fourth-order equation (with real coefficients)

$$\det(A\lambda^2 + 2B\lambda + C) = 0$$

are not real. In such a case we have two pairs of complex conjugate roots that we denote by $\lambda_1$, $\bar{\lambda}_1$, $\lambda_2$ and $\bar{\lambda}_2$.

The Dirichlet problem for the system (4) is as follows: given a bounded simply connected domain $D$ in the plane and a continuous function $h$ on the boundary $\partial D$ of $D$, to find a function $f$ of class $C^2(D) \cup C(\overline{D})$ such that $\mathcal{L}f = 0$ in $D$ and $f|_{\partial D} = h$ (in view of the ellipticity of $\mathcal{L}$ it is sufficient to state the problem to find $f \in C(\overline{D})$ satisfying the equation $\mathcal{L}f = 0$ in $D$ in the distributional sense). The problem is appeared natural to describe such domains $D$, where the Dirichlet problem is solvable for every given continuous function $h$ on $\partial D$. Domains satisfying this property are called $\mathcal{L}$-regular.

In the problem on description of $\mathcal{L}$-regular domains the concept appears natural of an equivalence of systems under consideration. Two systems of the form (4) are equivalent, if they can
be reduced one to another by the use of the following *admissible transformations*: not degenerate real linear changes of variables and sought-for functions, and not degenerate real linear combinations of equations. In what follows we will call them admissible transformation of the *first*, *second* and *third* kinds, respectively. If two systems of the form (3) set by the operators $L_1$ and $L_2$ are equivalent, and if a domain $D$ is $L_1$-regular, then the domain obtained from $D$ using the corresponding linear transformation is $L_2$-regular. Such notion of system equivalence was introduced, for instance, in [2].

One of the most simple case is when the system (3) can be reduced by transformation of three kinds mentioned above to the system with upper triangular matrices $A$, $B$ and $C$. Such system is called *reducible*. This term is related with the fact that the system (3) with upper triangular matrices splits into two independent elliptic equations with constant *real* coefficients, the first one of which is homogeneous, while the second one has nonzero right-hand part. It can be shown that the first this equation is equivalent to the Laplace equation, while the second one is equivalent to the Poisson equation for harmonic functions.

For harmonic functions (that is for the system given by the Laplace operator $\Delta$, $\Delta u = u_{xx} + u_{yy}$) the result is well known that any bounded simply connected domain is $\Delta$-regular (this breakthrough result was proved by Lebesgue [3] in 1907, and one of the important ingredient of the proof was the fact that for the Laplace operator the energy functional of the form under consideration do exist). Thus, for reducible systems it is known the complete description of regular domains, as well an answer to the question on existence of energy functional of the desired form. Notice that the class of reducible systems is the almost one class, for which the complete answers to both questions under consideration were obtained (except such systems the complete answer is known for the system corresponding to the anisotropic Lame equation).

In what follows we are dealing with non-reducible systems. One of the most known and important case of such systems are skew-symmetric systems that arise from equations of the form (4).

Studying non-reducible systems in the context of the Dirichlet problem as well in the context of the problem on existence of energy functionals of the form (1) it is natural to pay attention to the notion of *strong ellipticity*.

The next definition of strong ellipticity was introduced by Vishik in [4]: the system (3) is said to be strongly elliptic, if for every $\xi, \eta \in \mathbb{R}$ the following matrix

$$A_+\xi^2 + 2B_+\xi\eta + C_+\eta^2,$$

is positively determined, where $X_+ = (X + X^t)/2$ for $X \in M_2(\mathbb{R})$. Notice that every strongly elliptic system in the sense of Vishik is elliptic, but the opposite is, in general, not the case.

In [2] the notion of strong ellipticity was introduced in a slightly different way. Namely, the system (3) is strongly elliptic, if

$$\det(\alpha A + 2\beta B + \gamma C) \neq 0$$

for all real $\alpha$, $\beta$ and $\gamma$ satisfying the condition $\beta^2 - \alpha\gamma < 0$. It can be shown that both these definitions of strong ellipticity are equivalent by modulo of system equivalence mentioned above.

For further considerations we need yet another properties of systems (3). One says that a *strongly elliptic system* of the form (3) is *symmetrizable*, if it can be reduced by admissible transformations to the system with symmetric matrices $A$, $B$ and $C$ such that the block $4 \times 4$-matrix

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}$$
is positively determined. In the opposite case the strongly elliptic system (3) under consideration is said to be non-symmetrizable.

The paper is organized as follows. In Section 2 we consider the canonical forms of the system (3) and discuss the meaning of corresponding canonical parameters. In Section 3 we formulate and prove the main result of the paper (see Theorem 1) that gives a criterion in order that the system (3) admits (non-negatively determined) energy functional of the form (1). This criterion is stated in terms of canonical parameters of the system (3). In particular, it follows from Theorem 1 that for strongly elliptic equation (4) an energy functional of the desired form exists if and only if this equation has real (up to a common complex multiplier) coefficients, that is when this equation is equivalent to the Laplace one. Thus the standard proof of the fact that any bounded simply connected domain in the plane is regular with respect to the Dirichlet problem for harmonic functions cannot be generalized to the case of general strongly elliptic equations with constant complex coefficients.

2. Canonical form of second-order elliptic systems with constant coefficients

It is convenient to begin the study of our questions by reducing the system (3) to one of the canonical forms using admissible transformations, which are linear changes of variables and sought-for functions, and changes of the equations by their suitable linear combinations. Let us emphasize that most of the facts presenting in this section are not new. They are known and can be found, for example, in [2] or [5] (see also [6]). However, the complex canonical form presented at the end of this section is recently appeared in works by the authors. In what follows the symbol $z$ will mean not only the complex variable $z = x + iy$ and the corresponding point in the plane, but also the column-vector $(x, y)^t$.

The next lemma is verified by the direct differentiation.

**Lemma 1.** Let $A$, $B$ and $C$ be the matrices $A$, $B$ and $C$ that set the differential operator $\mathcal{L}$ of the form (2). Then $A$, $B$ and $C$ are changed as follows under admissible transformations:

1) Let $\zeta = Tz$, $\zeta = \xi + i\eta$, be the change of coordinates with the matrix $T \in M_2(\mathbb{R})$, det $T \neq 0$. Then

$$\mathcal{L} f = A' f_{\xi\xi} + 2B' f_{\xi\eta} + C' f_{\eta\eta},$$

where

$$A' = \begin{pmatrix} t_{11} & t_{12} \\ t_{11} \quad t_{12} \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} t_{11} \\ t_{12} \end{pmatrix} = t_{11}^2 A + 2t_{11}t_{12}B + t_{12}^2 C,$$

$$B' = \begin{pmatrix} t_{11} & t_{12} \\ t_{11} \quad t_{12} \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} t_{21} \\ t_{22} \end{pmatrix} = t_{11}t_{21}A + (t_{11}t_{22} + t_{12}t_{21})B + t_{21}t_{22}C,$$

$$C' = \begin{pmatrix} t_{21} & t_{22} \\ t_{21} \quad t_{22} \end{pmatrix} \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} t_{21} \\ t_{22} \end{pmatrix} = t_{21}^2 A + 2t_{21}t_{22}B + t_{22}^2 C,$$

and where $t_{jk}$, $j, k = 1, 2$, are the elements of $T$.

2) Let $\varphi = Qf$ be the transformation of the sought-for functions given by the matrix $Q \in M_2(\mathbb{R})$, det $Q \neq 0$. Then

$$\mathcal{L} f = A' \varphi_{xx} + 2B' \varphi_{xy} + C' \varphi_{yy},$$

where $A' = AQ$, $B' = BQ$ and $C' = CQ$.

3) A linear combination of equations of the system (3) defined by the matrix $P \in M_2(\mathbb{R})$, det $P \neq 0$, leads to the system of equations set by the operator

$$\mathcal{L}' f = A' f_{xx} + 2B' f_{xy} + C' f_{yy},$$
where \( A' = PA, B' = PB \) and \( C' = PC \).

The next lemma also can be verified by the direct computation.

**Lemma 2.** Let the elliptic system \( \mathbb{F}(\xi, \eta) \) with roots \( \lambda_1, \bar{\lambda}_1, \lambda_2, \bar{\lambda}_2 \). Then (in terms of notations introduced in Lemma \( \mathbb{F} \)) the following properties takes place.

1) The linear change of variables \( \zeta = Tz \) leads to the system with the characteristic form

\[
\det A' (\xi^2 + |\lambda_1|^2 \eta^2) (\eta^2 + |\lambda_2|^2 \xi^2),
\]

where \( A' \) is defined in the statement (1) of Lemma \( \mathbb{F} \) and \( \lambda_k = A_T(\lambda_k) \) for \( k = 1, 2 \), where

\[
A_T(\lambda) := \frac{t_{22}\lambda - t_{21}}{t_{12}\lambda + t_{11}}.
\]

2) The linear change of sought-for functions \( \varphi = Qf \) leads to the system with the characteristic form \( qF \) with \( q = \det Q \).

3) The linear combination of equations of the initial system defined by the matrix \( p \mathbb{F} \) leads to the system with the characteristic form \( pF \) with \( p = \det p \).

Using these two technical lemmata we are able to prove the first statement about canonical form of the systems under consideration.

**Proposition 1.** Every non-reducible elliptic system of the form \( \mathbb{F} \) can be reduced by admissible transformations to the system set by the operator

\[
\mathcal{L}_{k, \lambda}^1 = A\partial_{xx} + 2B\partial_{xy} + C\partial_{yy},
\]

where parameters \( \kappa \) and \( \lambda \) are such that \( \kappa \in (0, 1] \) and \( \lambda \in [-\kappa, \kappa] \setminus \{0, \kappa^2\} \), and where

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -\frac{1}{2}(1-\lambda)(1-\kappa^2/\lambda) & 1 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa^2/\lambda \end{pmatrix}.
\]

In such a case strong ellipticity of the initial system is equivalent to the fact that \( \lambda > 0 \).

**Scheme of the proof.** Let \( \mathcal{L} \) from \( \mathbb{F} \) is set by the matrices \( A, B \) and \( C \). In order to reduce \( \mathcal{L} \) to the desired canonical form we simplify \( A, B \) and \( C \) in four steps.

**Step 1. Simplification of the characteristic form.** Since the set of characteristic roots of the initial system consists of two pairs of complex conjugate numbers (and since one may assume that \( \lambda_1 \) and \( \lambda_2 \) belongs to the upper half-plane), then there exists a Möbius (linear-fractional) transformation, which maps the upper half-plane onto itself and moving all characteristic roots of the initial system to the imaginary axis. Such transformation \( A \) may be found using the following conditions:

\[
A(\lambda_1) = \kappa i, \quad A(\lambda_2) = i,
\]

where \( \kappa \in \mathbb{R}, \kappa \neq 0 \), is an unknown parameter that needs to be determined.

In the case, where \( \lambda_1 = \lambda_2 \), the desired \( A \) is a composition of shift and dilation. In this case \( k = 1 \). For \( \lambda_1 \neq \lambda_2 \) let us observe that the points \( \lambda_1, \lambda_2, \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \) lie on some circle orthogonal to the real axis. Let \( \zeta_\ast \) and \( \zeta_{\ast\ast} \) are two points where this circle intersects the real line. The function

\[
A(\zeta) = \rho \frac{\zeta - \zeta_\ast}{\zeta - \zeta_{\ast\ast}}
\]

maps the circle under consideration onto the imaginary line, and the parameter \( \rho \) is determined by the condition \( A(\lambda_2) = i \). After that the relation \( A(\lambda_1) = \kappa i \) gives the value of \( \kappa \). If \( \kappa > 1 \), then instead of \( A \) we will use the composition of this \( A \) and dilation to \( \kappa \) times. Therefore we obtain a Möbius transformation

\[
A(\zeta) = \frac{a\zeta + b}{c\zeta + d}
\]
that possesses the properties $A(\lambda_1) = \kappa i$, $\kappa \in (0, 1]$, and $A(\lambda_2) = i$.

Apply the change of variable given by the matrix

$$T_1 = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$ 

Doing this we pass from the operator $\mathcal{L}$ to the operator $\mathcal{L}_1$ set by $A_1$, $B_1$ and $C_1$ defined in the first statement of Lemma [1]. According to Lemma [2] the new system (given by $\mathcal{L}_1$) has the characteristic form

$$\mathcal{F}_1(\xi, \eta) = \det A_1 \cdot (\xi^2 + \eta^2)(\xi^2 + \kappa^2 \eta^2).$$

Notice, that $\det A_1 \neq 0$ and $\det C_1 \neq 0$ in view of ellipticity of the initial system.

**Step 2. Diagonalization of $A_1$.** Apply the third-kind transformation (linear combination of equations) given by the matrix $A_1^{-1}$ to the system corresponding to $\mathcal{L}_1$. We obtain the system set by the operator $\mathcal{L}_2$ with the matrices $A_2 = A_1^{-1}A_1 = I$, $B_2 = A_1^{-1}B_1$ and $C_2 = A_1^{-1}C_1$, where $I$ is the identity matrix. For this system the characteristic form is $\mathcal{F}_2(\xi, \eta) = (\xi^2 + \eta^2)(\xi^2 + \kappa^2 \eta^2)$.

**Step 3. Diagonalization of $C_2$.** Let $C_3$ be the Jordan canonical form of $C_2$. We have $C_3 = PC_2P^{-1}$, where $P$ is some suitable non-degenerate matrix. Such transformation of $C_2$ to $C_3$ corresponds to consequent admissible transformations of the second and third kinds defined by $P$. Let $A_3 = I$ and $B_3 = PB_2P^{-1}$. So, we pass from the system corresponding to $\mathcal{L}_2$ to the system defined by the operator $\mathcal{L}_3$ with the matrices $A_3$, $B_3$ and $C_3$. The characteristic form does not change under such a transformation, thus for the new system we have

$$\mathcal{F}_3(\xi, \eta) = (\xi^2 + \eta^2)(\xi^2 + \kappa^2 \eta^2) = \xi^4 + (1 + \kappa^2)\xi^2 \eta^2 + \kappa^2 \eta^4. \quad (9)$$

Notice, that $C_3$ may have one of the following forms:

1) $C_3 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, where $\lambda$ and $\mu$ are real (simple) eigenvalues of the matrix $C_2$, which we assume to be such that $|\lambda| \leq |\mu|$;

2) $C_3 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, where $\lambda$ is the real eigenvalue of $C_2$ of order two;

3) $C_3 = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$, where $\lambda \pm i\mu$ are two complex conjugate eigenvalues of $C_2$.

In all these cases the eigenvalues of $C_2$ differ from zero because $C_2$ is non-degenerate. Let

$$B_3 = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}. \quad (10)$$

In the first case mentioned above we can specify $B_3$ and $C_3$ comparing (9) with the explicit expression for the characteristic form $\mathcal{F}_3$ in terms of the elements of $A_3$, $B_3$ and $C_3$:

$$\mathcal{F}_3(\xi, \eta) = \det \begin{pmatrix} \xi^2 + 2b_1\lambda\xi \eta + \lambda \eta^2 & 2b_2\lambda\xi \eta \\ 2b_3\lambda\xi \eta & \xi^2 + 2b_4\lambda\xi \eta + \lambda^2 \end{pmatrix}$$

$$= \xi^4 + 2(b_1 + b_4)\xi^3 \eta + (\lambda + \mu + 4b_1b_4 - 4b_2b_3)\xi^2 \eta^2 + 2(b_1 \mu + b_4 \lambda)\xi \eta^3 + \lambda \mu \eta^4.$$

In the case where $\lambda \neq \mu$ we have

$$A_3 = I, \quad B_3 = \begin{pmatrix} 0 & b_2 \\ b_3 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa^2 / \lambda \end{pmatrix}, \quad (10)$$

where $b_2b_3 = -\frac{1}{2}(1 - \lambda)(1 - \kappa^2 / \lambda)$, while in the opposite case $\lambda = \mu$ we have

$$A_3 = I, \quad B_3 = \begin{pmatrix} b_1 & b_2 \\ b_3 & -b_1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} \pm \kappa & 0 \\ 0 & \pm \kappa \end{pmatrix}, \quad (11)$$
where $b_1^2 + b_2 b_3 = -\frac{1}{4}(1 + \kappa)^2$.

In the second case we conclude that $B_3$ and $C_3$ are upper-triangular matrices (in this case we are arguing by the same way and comparing (9) with its explicit expression via elements of $B_3$ and $C_3$). It means that the system corresponding to $L_3$ is reducible in such a case.

Finally, in the third case we consider the system of conditions that appears after equalizing the corresponding coefficients in (9) and in explicit expression for $F_3$ via elements of $B_3$ and $C_3$. It turns out that this system is inconsistent.

Thus in the case of non-reducible systems only the first form of the Jordan canonical form of $C_2$ may take place. In this case the matrices of the corresponding differential operator can be reduced to the form (10) or (11).

**Step 4. Exclusion of superfluous parameters.** Let $A_3$, $B_3$ and $C_3$ are defined in (10), and let $P = \text{diag}(b_2, 1)$ (i.e. $P$ is the corresponding diagonal matrix). Then $A_4 = P^{-1} A_3 P$, $B_4 = P^{-1} B_3 P$ and $C_4 = P^{-1} A_3 P$ are such that

$$
A_4 = I, \quad B_4 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{4}(1 - \lambda)(1 - \kappa^2/\lambda) & 0 \end{pmatrix}, \quad C_4 = \begin{pmatrix} \lambda & 0 \\ 0 & \kappa^2/\lambda \end{pmatrix},
$$

and we can pass from the operator with $A_4$, $B_4$ and $C_3$ to the operator set by $A_4$, $B_4$ and $C_4$ using admissible transformations of the second and third kind given by $P$ and $P^{-1}$, respectively.

In the case where $A_3$, $B_3$ and $C_3$ are defined by (11), they also can be reduced to matrices of the form $A_4$, $B_4$ and $C_4$ (with $\lambda = \pm \kappa$) by conjugation with a certain suitable non-degenerate matrix for which we do not present an explicit form in order to avoid some bulk computations.

**Refinement of the set of possible values of $\lambda$.** Let us find the set of possible values of $\lambda$ for non-reducible elliptic systems whose matrices are reduced to the form (10) or (11). First of all let us recall that since $|\lambda| \leq |\mu|$ and $\lambda \mu = \kappa^2$, then $\lambda \in [-\kappa, \kappa]$. Furthermore, all matrices in (10) simultaneously become triangle if and only if $b_3 = 0$. Then, since $b_2 b_3 = -\frac{1}{4}(1 - \lambda)(1 - \kappa^2/\lambda)$ we obtain that $\lambda = \kappa^2$ (notice that the value $\lambda = 1$ is not admissible because of $|\lambda| \leq |\mu|$). Moreover, all matrices in (11) simultaneously become triangle if and only if $b_3 = 0$, which gives $b_1 = -\frac{1}{4}(1 + \kappa)^2$, i.e. $b_1 = 0$ and $\lambda = \kappa = 1$. Combining all these observations we obtain that $\lambda \in [-\kappa, \kappa] \setminus \{0, \kappa^2\}$. Moreover, it can be directly verified that the strong ellipticity property of the system under consideration is equivalent to the fact that $\lambda > 0$. Indeed, it is enough to observer that $\det(C_1 - \lambda A_1) = 0$ (since $\lambda$ is the eigenvalue of the matrix $C_2 = A_1^{-1} C_1$) and to use the formulæ for $A_1$ and $C_1$ obtained in Lemma 1.

In what follows it will be convenient to use yet another canonical representation for the system (9), which is related with the Cauchy–Riemann operator. Recall, that the Cauchy–Riemann operator is the differential operator

$$
\overline{\partial} = \frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\partial_x - i \partial_y)
$$

Together with $\overline{\partial}$ we will use the operator

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x + i \partial_y)
$$

Let us recall that the kernel of the operator $\overline{\partial}$ in some domain $D \subset \mathbb{C}$ is the space of all holomorphic functions in $D$, while the kernel of $\partial$ in $D$ is the space of all antiholomorphic functions in $D$ (of course, both kernels are considered in the space of continuous functions in $D$).

Let $L$ be the operator of the form (2) with the canonical parameters $\kappa \in (0, 1]$ and $\lambda \in [-\kappa, \kappa]$, $\lambda \neq 0, \kappa^2$. We put

$$
\tau = \frac{1 - \kappa}{1 + \kappa}, \quad \sigma = \frac{\kappa - \lambda}{\kappa + \lambda}.
$$
Let $\lambda > 0$ (this case corresponds to the case where the system (3) given by $L$ is strong elliptic). Then $|\sigma| < 1$. Define the operator

$$L_{\tau, \sigma} = (\partial \overline{\partial} + \tau \partial^2) f + \sigma (\tau \partial \overline{\partial} + \partial^2) f = 0.$$  

Let now $\lambda < 0$, that is the system (3) set by $L$ is not strongly elliptic. In such a case $|\sigma| > 1$. For $\lambda = -\kappa$ we put $\sigma = \infty$. Setting $s = \frac{1}{\sigma}$ we define $L_{\tau, \sigma}$ in this case as follows

$$L_{\tau, \sigma} = (\overline{\partial}^2 + \tau \overline{\partial}\partial) f + s (\tau \overline{\partial}^2 + \overline{\partial}\partial) f = 0.$$  

Notice that $L_{\tau, \sigma}$ for $|\sigma| < 1$ may be regarded as a perturbation of the Laplace operator by a pair of ‘small’ parameters $\tau$ and $\sigma$, while $L_{\tau, \sigma}$ for $|\sigma| > 1$ may regarded as a perturbation of the Bitsadze operator $\overline{\partial}^2$ by small parameters $\tau$ and $s = \frac{1}{\sigma}$.

**Proposition 2.** Let $L$ be the strongly elliptic operator of the form (2). Then it can be reduced by admissible transformations to the form $L_{\tau, \sigma}$ with $|\sigma| < 1$. In the case where $L$ is not strongly elliptic, it can be reduced by admissible transformations to the form $L_{\tau, \sigma}$ with $|\sigma| > 1$.

In particular, every strongly elliptic operator of the form (2) may be reduced to the form $L_{\tau, 0}$, while every operator of the form (2), which is not strongly elliptic, may be reduced to the form $L_{\tau, \infty}$.

**Proof.** First of all we observe that every non-reducible elliptic system (3) given by $L$ of the form (2), may be rewritten in the form of a single equation to the function $f = u + iv$:

$$\begin{align*}
(1 - \kappa)(\kappa + \lambda)\partial^2 f(z) + (1 + \kappa)(\kappa + \lambda)\overline{\partial} \partial f(z) + \\
+ (1 + \kappa)(\kappa - \lambda)\partial^2 \overline{\partial f(z)} + (1 - \kappa)(\kappa - \lambda)\overline{\partial} \partial \overline{f(z)} = 0,
\end{align*}$$

where $\kappa$ and $\lambda$ are defined for $L$ in Proposition 1. Indeed, let us continue transformation of matrices $A$, $B$ and $C$ from $L$, which was begun in the proof of Proposition 1.

At the first step we multiply $A_4$, $B_4$ and $C_4$ from (12) by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & \lambda/\kappa^2 \end{pmatrix}$ on the left to obtain new matrices

$$A_5 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda/\kappa^2 \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 & 1 \\ (\lambda - 1)(\lambda - \kappa^2)/(4\kappa^2) & 0 \end{pmatrix}, \quad C_5 = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}.$$  

At the second step we use the matrix $L = \begin{pmatrix} 2\kappa/(\lambda - \kappa^2) & 0 \\ 0 & 1 \end{pmatrix}$ to pass from $A_5$, $B_5$ and $C_5$ to $A_6 = LA_5L^{-1}$, $B_6 = LB_5L^{-1}$ and $C_6 = LC_5L^{-1}$. Next, multiplying $A_6$, $B_6$ and $C_6$ on the left to

$$\begin{pmatrix} \kappa/(\kappa^2 - \lambda) & 0 \\ 0 & \kappa/(1 - \lambda) \end{pmatrix},$$

we obtain the matrices

$$A_7 = \begin{pmatrix} \kappa/(\kappa^2 - \lambda) & 0 \\ 0 & \lambda/(\kappa(1 - \lambda)) \end{pmatrix},$$

$$B_7 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

$$C_7 = \begin{pmatrix} \lambda\kappa/(\kappa^2 - \lambda) & 0 \\ 0 & \kappa/(1 - \lambda) \end{pmatrix}.$$  

Finally, at the third step it remains to pass from the system of equations (on $u$ and $v$) given by $A_7$, $B_7$ and $C_7$, to a single equation on the function $f = u + iv$. To do this we need to add first equation of this system with the second one multiplied by the imaginary unit $i$, and then replace
derivatives by \(x\) and \(y\) with their representations via \(\partial\) and \(\partial\), while \(u, v\) by their representations via \(f\) and \(\bar{f}\).

The passage from the equation (15) to the form \(L\tau,\sigma f = 0\) can be checked by the direct computation.

Notice that the operator \(L\tau,\sigma\) is more convenient to represent in the following form. Define the differential operator \(\partial_{\tau} = \partial + \tau\partial\) and the linear operator \(B_{\alpha,\beta} = \alpha I + \beta C\), where \(\alpha, \beta \in \mathbb{R}\), while \(I\) and \(C\) are the identity operator and the complex conjugation operator, respectively. Then

\[
L\tau,\sigma = \begin{cases} \partial B_{1,\sigma} & \text{for } |\sigma| < 1, \\ \partial B_{1,s} & \text{for } |\sigma| > 1, \end{cases}
\]

where, as previously, \(s = 1/\sigma\). Moreover, the equation \(L\tau,\sigma f = 0\) with \(\sigma \neq \infty\) may be written as a system with the operator \(L\) given by the matrices

\[
A = (1 + \tau) \begin{pmatrix} 1 + \sigma & 0 \\ 0 & 1 - \sigma \end{pmatrix},
\]

\[
B = \begin{pmatrix} 0 & \tau - \sigma \\ -(\tau + \sigma) & 0 \end{pmatrix},
\]

\[
C = (1 - \tau) \begin{pmatrix} 1 - \sigma & 0 \\ 0 & 1 + \sigma \end{pmatrix}.
\]

3. Energy functional for the system (3)

For a function \(f = u + iv\) in the complex variable \(z = x + iy\) we put

\[
\nabla f = \begin{pmatrix} u_x & v_x & u_y & v_y \end{pmatrix}^t.
\]

Moreover we will identify \(f\) with the vector \((u, v)^t\), and it will be convenient to use the notation \(f_x = (u_x, v_x)^t\) and \(f_y = (u_y, v_y)^t\). For vectors \(a, b \in \mathbb{R}^m, m \geq 1\), the symbol \((a, b)\) means, as usual, their scalar product (in an appropriate space \(\mathbb{R}^m\)). The symbol \(m_2()\) stands for the two-dimensional Lebesgue measure (i.e. area) in \(\mathbb{R}^2\).

In this section we study the question on what conditions on the operator \(L\) of the form (2) ensure that the system (3) with this operator admits a non-negatively determined energy functional of the form (1). We start with several auxiliary statements. Let \(E \in M_4(\mathbb{R})\) be a symmetric matrix, i.e. \(E = E^t\), and let

\[
E = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix},
\]

where \(K, L, M \in M_2(\mathbb{R})\) are such that \(K = K^t\) and \(M = M^t\).

**Lemma 3.** Let \(D\) be a Jordan domain in \(\mathbb{C}\) with the boundary \(\Gamma\), let \(h \in C(\Gamma)\), and \(E \in M_4(\mathbb{R})\) be a symmetric matrix. Then, the system of Euler–Lagrange equations to the functional

\[
\mathcal{E} f := \frac{1}{2} \int_D (E \nabla f, \nabla f) \, dm_2,
\]

defined on the class of functions

\[
\mathcal{F}(D, h) = \{ f \in C^2(D) \cap C^1(\overline{D}) : f|_{\Gamma} = h \},
\]

has the form (3) with the matrices \(A = K, B = (L + L^t)/2\) and \(C = M\), where \(K, L, M\) are defined according to (18).
Proof. Write the functional under consideration in terms of matrices $K$, $L$ and $M$:

$$
E f = \frac{1}{2} \int_D \left( (Kf_x, f_x) + 2(Lf_x, f_y) + (Mf_y, f_y) \right) \, dm_2.
$$

A variation of this function may be computed directly:

$$
\delta E f = \int_D \left( (Kf_x, \delta f_x) + (L^t f_y, \delta f_x) + (L f_x, \delta f_y) + (M f_y, \delta f_y) \right) \, dm_2,
$$

where $\delta f_x = (\delta u_x, \delta v_x)^t$ and $\delta f_y = (\delta u_y, \delta v_y)^t$ are variations of $f_x$ and $f_y$, respectively. The latter expression can be reduced to the form

$$
\delta E f = \int_D \left[ \partial_x ((Kf_x, \delta f) + (L^t f_y, \delta f)) + \partial_y ((L f_x, \delta f) + (M f_y, \delta f)) \right] \, dm_2
$$

$$
- \int_D \left[ (Kf_{xx}, \delta f) + (L + L^t) f_{xy}, \delta f) + (M f_{yy}, \delta f) \right] \, dm_2,
$$

where $\delta f = (\delta u, \delta v)$ is the variation of $f$, and $\partial_x$ and $\partial_y$ are operator of partial differentiation by $x$ and $y$, respectively. The integral in this expression is a divergence of some vector and, hence, it is equal to the following integral

$$
\int_D ((Kf_x, \delta f) + (L^t f_y, \delta f)) \, dy - ((L f_x, \delta f) + (M f_y, \delta f)) \, dx,
$$

which vanishes because the variations of $u$ and $v$ on $\Gamma$ equal zero (since these functions take on $\Gamma$ the prescribed values). The equality to zero of the second integral from the expression for $\delta E f$ leads to the system of the form (19) generated by the matrices $A = K$, $B = (L + L^t)/2$ and $C = M$.

The next lemma may be proved by direct differentiation.

**Lemma 4.** Let $D$ be a Jordan domain with the boundary $\Gamma$ and let $h \in C(\Gamma)$ be a given function. Moreover, let $E \in M_4(\mathbb{R})$ be a symmetric matrix and a functional $E$ on the set $F(D, h)$ is defined by (19). Then the following properties take place.

1) Let $z \mapsto \zeta = \xi + i\eta$ be the linear non-degenerate change of variables in $\mathbb{R}^2$, defined by the matrix $T \in M_2(\mathbb{R})$. Then

$$
E f = \frac{1}{2} \int_{D_0} (E_0 \nabla \zeta f, \nabla \zeta f) \, dm_2(\zeta),
$$

where $D_0$ is the image of $D$ under the mapping $z \mapsto \zeta$, while $\nabla \zeta f = (u_\xi, v_\xi, u_\eta, v_\eta)^t$, and where components $K_0$, $L_0$ and $M_0$ of $E_0$ are determined from the equality

$$
\begin{pmatrix}
K_0 & L_0 \\
L_0^t & M_0
\end{pmatrix} = T \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} T^t.
$$

(21)

2) Let $f = u + iv$ and $f_1 = u_1 + iv_1$ are connected by some non-degenerate linear transformation $f_1 = Q f$, defined by $Q \in M_2(\mathbb{R})$. Then

$$
E f = \frac{1}{2} \int_D (E_1 \nabla f_1, \nabla f_1) \, dm_2,
$$

where $E_1 \in M_4(\mathbb{R})$ is such that their components $K_1$, $L_1$ and $M_1$ of $E_1$ are determined from the equality

$$
\begin{pmatrix}
K_1 & L_1 \\
L_1^t & M_1
\end{pmatrix} = Q^t \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} Q.
$$

(22)

In the case where $E$ is non-negatively (resp., positively) determined, that both matrices $E_0$ and $E_1$ also are non-negatively (resp., positively) determined.
The main result of this paper is the following statement:

**Theorem 1.** A non-reducible elliptic system of the form (13) is the system of Euler–Lagrange equations for some functional of the form (19) with non-negatively determined matrix $E \in M_4(\mathbb{R})$ if and only if this system is strongly elliptic and its canonical parameters $\tau$ and $\sigma$ are such that $\sigma > \tau$.

**Proof.** Suppose that the system (3) is the system of Euler–Lagrange equations for some functional (19). Then, according to Lemma 3, this system has symmetric matrices, or it can be reduced to the system with symmetric matrices by suitable linear combinations of equations. The system with symmetric matrices we reduce to the system generated by matrices of the form (16). It can be done using linear combinations of equations which do not change the energy functional and by linear changes of variables and sought-for functions. The latter transformation, according to Lemma 4, preserves the property of energy functional to be non-negatively determined. Now we multiply all matrices of the obtained system on the left by the matrix

$$
\begin{pmatrix}
\sigma + \tau & 0 \\
0 & \sigma - \tau
\end{pmatrix}
$$

and arrive at the system with symmetric matrices

$$
A = (1 + \tau) \begin{pmatrix}
(1 + \sigma)(\sigma + \tau) & 0 \\
0 & (1 - \sigma)(\sigma - \tau)
\end{pmatrix},
$$

$$
B = (\tau^2 - \sigma^2) \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

$$
C = (1 - \tau) \begin{pmatrix}
(1 - \sigma)(\sigma + \tau) & 0 \\
0 & (1 + \sigma)(\sigma - \tau)
\end{pmatrix}.
$$

The system generated by $A$, $B$ and $C$ is also the system of Euler–Lagrange equations for the functional of the form (19) with non-negatively determined matrix $E$ of the form (18). Moreover, according to Lemma 3 we have $K = A$, $L + L^t = 2B$ and $M = C$ for components of $E$. Using the Silvester criterion we conclude, that if $E$ is non-negatively determined, then $K$ needs to be non-negatively determined, which yields $\sigma > \tau$ (the case $\sigma = \tau$ is already excluded since the initial system is non-reducible).

In order to prove sufficiency of our conditions let us present, for $0 \leq \tau < \sigma < 1$, some non-negatively determined matrix $E$ of the form (18) constructed by $K = A$, $M = C$ and

$$
L = \begin{pmatrix}
0 & (1 + \sigma)(\tau^2 - \sigma^2) \\
(1 - \sigma)(\tau^2 - \sigma^2) & 0
\end{pmatrix},
$$

so that $L$ possesses the condition $L + L^t = 2B$, where $A$, $B$ and $C$ are taken from (23). 

In particular, it follows from Theorem 1 that for systems determined by operators $L_{\tau,0}$ for $\tau > 0$ (that is for equations of the form (1) different from the Laplace equation, which corresponds to a reducible system) do not exist non-negatively determined energy functional of the form (19). This circumstance shows that it is not possible to generalize directly the Lebesgue theorem stated above to strongly elliptic equations of the form (1). Thus, for proving an analogue of the Lebesgue theorem for such equations (and, in particular, for solving Problem 4.2 from [7]) some essentially different techniques and ideas are needed.
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