Towards a formal description of the collapse approach to the inflationary origin of the seeds of cosmic structure

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Abstract. Inflation plays a central role in our current understanding of the universe. According to the standard viewpoint, the homogeneous and isotropic mode of the inflaton field drove an early phase of nearly exponential expansion of the universe, while the quantum fluctuations (uncertainties) of the other modes gave rise to the seeds of cosmic structure. However, if we accept that the accelerated expansion led the universe into an essentially homogeneous and isotropic space-time, with the state of all the matter fields in their vacuum (except for the zero mode of the inflaton field), we can not escape the conclusion that the state of the universe as a whole would remain always homogeneous and isotropic. It was recently proposed in [A. Perez, H. Sahlmann and D. Sudarsky, “On the quantum origin of the seeds of cosmic structure”, Class. Quant. Grav. 23 (2006) 2317–2354] that a collapse (representing physics beyond the established paradigm, and presumably associated with a quantum-gravity effect à la Penrose) of the state function of the inflaton field might be the missing element, and thus would be responsible for the emergence of the primordial inhomogeneities. Here we will discuss a formalism that relies strongly on quantum field theory on curved space-times, and within which we can implement a detailed description of such a process. The picture that emerges clarifies many aspects of the problem, and is conceptually quite transparent. Nonetheless, we will find that the results lead us to argue that the resulting picture is not fully compatible with a purely geometric description of space-time.

Keywords: alternatives to inflation, physics of the early universe, quantum field theory on curved space
1 Introduction

Inflation represents an appealing framework to understand the “initial conditions” of the universe [1–7]. The early phase of accelerated expansion seems to resolve some of the classical “naturalness problems” of the big bang model, such as the flatness, the horizon, the exotic relics or the entropy ones [8–11]. Moreover, it is generally accepted that inflation also accounts for the small anisotropies observed in the cosmic microwave background (CMB), necessary for the subsequent emergence of cosmic structure [12–16]. The accelerated expansion makes it possible to “push” the short wavelength, causally connected modes of the inflaton field to the largest scales of the observable universe. If, in addition, the accelerated expansion was (nearly) de-Sitter, and the state of the quantum fields was a featureless Bunch-Davies “vacuum” (except for the zero mode of the inflaton field), a single physical scale would appear in the primordial spectrum of the quantum fluctuations (uncertainties): the Hubble parameter. Roughly speaking, this is what lies behind the appearance of a scale-free (Harrison-Zel’dovich) primordial power spectrum of density perturbations in the standard model of inflation.

However, upon a deeper examination, one finds that there is something rather strange in that picture: if the observable universe starts from the Bunch-Davies vacuum for the quantum fields in a Robertson-Walker space-time (both of which are completely homogeneous and isotropic), and considering their evolution according to the rules of standard physics, the state of the quantum fields, and the universe as a whole, will remain both homogeneous and isotropic at all times. But, how can this be compatible with the highly complex structure of the universe we are inhabiting? In fact, how can such proposal be said to explain the emergence of the seeds of cosmic structure, which early traces we study in the CMB? (See
the references [17, 18] for a full discussion of the problem and the various attempts to deal with these issues within the standard physical paradigms.) The standard and phenomenologically successful accounts on this matter rely on the identification of the quantum fluctuations of certain observables associated to a homogeneous and isotropic universe, with the averages over an ensemble of inhomogeneous universes of their analogue classical quantities: more specifically (or more crudely), on the identification of quantum uncertainties and classical perturbations. As discussed in [18], one can not avoid concluding that using standard physics there is no justification for such kind of identification. In fact, these shortcomings have started to be recognized by others in the literature: see for instance section 10.4 in [4], the end of section 8.3.3 in [5], section 10.1 in [6], section 24.2 in [7], and section 30.14 in [19].

On the other hand, it is undeniable that the standard approach has a great phenomenological success: its “predictions” match exquisitely well the most accurate observations to date. Nonetheless, as it has been argued in [17, 18], some new element must be added to the standard picture in order to have a physical description of the transition from the initial symmetric stage of the universe to the late non-symmetric one, preserving at the same time the phenomenological success of the usual approach.

In this context, it was recently proposed that a new kind of effect, which at the phenomenological level would be analogous to a “self-induced” collapse in the state function of the inflaton field, could play the required role in the early universe [17] (see also [20–25]). The point, of course, is that such kind of collapses are a serious departure from the unitary evolution of standard physics. It is worth mentioning that, in fact, conceptually similar ideas have been considered by many physicists concerned with the measurement problem in quantum mechanics (see for instance [26] and references therein), and the proposal that some quantum aspects associated to the gravitational interaction might lie at the root of this effect only serves to make the idea even more attractive [27] (see also chapter 30 in reference [19]). The suggestion that such a mechanism could play a determinant role in the breakdown of the initial symmetry of the universe seems doubly appealing to us: on the one hand, as we will see, it turns out to have the features leading to the kind of effect required for the emergence of the seeds of cosmic structure. On the other, when we take such collapse as a fundamental ingredient for the generation of primordial perturbations, these become an observationally accessible stage where such a novel aspect of physics, presumably containing some hints about the nature of gravity at the quantum level, could be investigated. Interestingly enough, some of the resulting predictions seem to differ in various aspects from the standard accounts [17, 23]. The main objective of this work is to establish a formalism that can serve as a basis to explore these ideas in a well defined and rigorous setting, which would, in the future, allow us to address questions such as the applicability of collapse theories [26, 28–33] to the inflationary universe, and to unearth the problems that might arise in each particular proposal. Our program is based on the joint use of quantum field theory in curved space-time and semiclassical gravity in self-consistent settings. However, as we will see, we will need to introduce a modification/addition to that general setting, in connection with the hypothetical collapse of the wave function. We will describe all that in detail in the main parts of the manuscript.

At this point we should note that inflation is expected to be associated with very high energy scales, well beyond the regime of the physical processes we have otherwise explored. Nonetheless, it is generally expected that such a regime is not really close to those where a quantum description of the gravitational interaction would be required (see for instance chapter 14 in reference [34]). However, the fact that we are interested in a situation where
the matter fields affecting the space-time geometry require a quantum description suggests that the appearance of some effects, ultimately traceable to a theory of quantum gravity, should not be too surprising. In fact, we should keep in mind that inflation, to the extent that it is observationally accessible through its imprints in the seeds of cosmic structure, represents the unique situation available to us which requires a treatment which simultaneously relies on both quantum theory and Einstein’s gravity. (There are several experiments which people often claim are related to the quantum/gravity interface, such as the famous neutron interference experiment \[35\] suitable for detecting the effects of the Earth’s Newtonian potential. However, a detailed analysis \[36\] indicates that these experiments are not sensitive to curvature at all, and it is curvature where gravitation truly lies, according to our post-Einsteinian views.)

The paper is organized as follows. In section 2 we review the ideas regarding the collapse of the state function, attempting to frame the proposal, even if only schematically, within our current understanding of physical theory. We also introduce some useful concepts to deal with the physics of the collapse, describing the nature of the formalism to be employed and explaining the extent to which it is taken to describe the relevant phenomena. In section 3 we study a simplified version of a wave function collapse in the early universe, describing the extent to which the transition from the symmetric to the inhomogeneous phase can be treated within the present formalism. Finally, we discuss our results in section 4. We have placed in the appendices some details of the presentation which are not essential to follow the main text.

The conventions we will be using include: \((-\), \(+\), \(+\), \(+\)) signature for the space-time metric, Wald’s convention for the Riemann tensor, and natural units with \(c = 1\). The Newton’s gravitational constant \(G\) and the reduced Planck’s constant \(\hbar\) are used to define the Planck mass \(M_p^2 \equiv \hbar^2/8\pi G\) and the Planck time \(t_p^2 = 8\pi G\hbar\). Space-time indexes are denoted by Greek letters \((\mu = 0, 1, 2, 3)\). We use Latin letters for spatial indexes \((i = 1, 2, 3)\) associated with specific hypersurfaces.

2 The nature of our effective description

Before we embark on the description and the general setting for our ideas, we would like to remind the reader that, despite the claims to the contrary, the measurement problem is still an open question in quantum mechanics \[26\]. In fact, the problem becomes even more serious in the context of quantum field theory \[37\]. Even worse, the interpretational framework of the quantum theory becomes intolerable when we want to apply it to the universe as a whole, because, in that case, we can not even rely on the practical usage of identifying an observer and/or a measuring device. Those problems have led researchers, such as J. Hartle, to argue for a modified or extended interpretational framework making the theory suitable for the cosmological applications \[38, 39\]. However, it seems that such scheme does not go far enough, lacking the kind of feature we need to address the problem that concern us here \[18\]. It is for that reason that we shall reconsider in some detail the schematic view proposed in \[17\] for dealing with this conundrum. As we have just commented in the Introduction, the proposal involves incorporating a “self induced collapse of the wave function of the matter fields”, and considering the resulting inhomogeneities and anisotropies in the energy momentum tensor generated during the collapses as the origin for the inhomogeneities and anisotropies of the space-time metric of our universe.
Now let us characterize the general nature of our approach. We should start by offering a discussion of a plausible manner whereby the collapse hypothesis might be incorporated within the general current understanding of a physical theory. However, for conciseness of the presentation this is done in the appendix \ref{sec:appendix}. Following the general arguments given there, we will take the view that the fundamental degrees of freedom for the gravitational interaction are not the ones characterizing the standard metric variables, but some other variables that are indirectly connected to those (consider, for example, the fluxes and holonomies in loop quantum gravity \cite{40}). That is, we will consider that geometry is an emergent phenomenon, and that general relativity should be regarded as some sort of hydrodynamical limit of a more fundamental theory.\footnote{Hydrodynamics is the effective theory describing the long wavelength, large frequency, low energy modes of highly coupled systems with a large number of “particles”. At those scales, only a small number of degrees of freedom are necessary to characterize the state of the system: the hydrodynamic modes. In the case of a perfect fluid, it can be described by the energy density, the entropy density and the field of velocities; all related by the Euler, the continuity and the thermodynamic equations. (Notice that it has not been necessary to introduce the concept of a fundamental constituent — the particles — for the fluid, of which, the 18th century physicist who developed the theory was naturally unaware.) This description will be appropriate to describe the fluid at length scales larger than the mean free path, and for time scales larger than the mean free time. The view we will take here is that something similar occurs with the gravitational interaction, where the characteristic physical scale should be probably determined now by the Planck mass. That is, in fact, the situation in various approaches to quantum gravity, such as the loop quantum gravity program or the poset proposal.} This point of view seems to be further supported by the analogy between the behaviour of black holes and the laws of thermodynamics \cite{41}, and by the ideas developed in \cite{42} (see also \cite{43–45}).

The regimes where the metric description for the gravitational degrees of freedom holds would correspond to the situations where the whole universe might be described by states $|\xi\rangle = |\xi\rangle_g \otimes |\xi\rangle_m$ (where the first factor corresponds to the gravitational degrees of freedom, and the second one to the matter fields) for which the gravitational sector $|\xi\rangle_g$ is such that operators corresponding to the metric variables have sharply peaked values (this would not be a strict eigenstate of the metric, where the wave function would be a delta function, but only to states where the wave function uncertainties in the geometrical quantities are rather small at the level of precision one is working).\footnote{Continuing with our hydrodynamical analogy, we might consider the very complicated wave-function characterizing the state of all the particles in a perfect fluid, but which for practical purposes can be effectively described by some classical quantities such as the energy density, the entropy density, and the field of velocities. The components of the space-time metric play now the role of the hydrodynamic modes.} Needless is to say that all our existing experience with the gravitational interaction would be, according to this view, well described by such hydrodynamical limit, and its non-geometrical behaviour would show up, generically, just in the extreme domains such as those normally associated with quantum gravity. Well outside those elusive domains, the fundamental operators for the gravitational degrees of freedom would be, for the most part, well described in terms of the metric tensor $g_{\mu\nu}$, and its dynamics can be taken as controlled by the corresponding Einstein tensor $g_{\mu\nu}(\xi)|\hat{G}|\xi\rangle = G_{\mu\nu}[g]$. At the same time, we will consider that the matter fields should be described in terms of a quantum field theory in a curved space-time. Under these conditions one can expect to recover the semiclassical description of gravitation in interaction with quantum fields as reflected in the semiclassical Einstein field equations

$$G_{\mu\nu}[g] = 8\pi G \langle \xi|\hat{T}_{\mu\nu}[g]|\xi\rangle_m .$$

Here the right hand side stands for the expectation value of the energy-momentum tensor operator in the corresponding state of the quantum fields, constructed from the field operators.
and the space-time metric $g_{\mu\nu}$. The matter fields are described in terms of operators acting on an appropriate Hilbert space $H_m$, according to the standard construction of a quantum field theory on a background space-time, with the latter satisfying the semiclassical equations given in (2.1).

Given the fact that we do not have yet a fully workable quantum theory for the gravitational interaction, it is very difficult to analyse explicitly the regime of validity of this semiclassical description. However, it seems reasonable to assume that it would be appropriate for the situations in which the matter fields $|\xi\rangle_m$ are sharply peaked around a classical field configuration $(\varphi_{i,\xi}(x), \pi_{i,\xi}(x))$, with $\varphi_{i,\xi}(x) \equiv m\langle \xi|\hat{\varphi}_i(x)|\xi\rangle_m$ and $\pi_{i,\xi}(x) \equiv m\langle \xi|\hat{\pi}_i(x)|\xi\rangle_m$, and where there are no relevant Planck scale phenomena, i.e. all curvature scalars are well below the Planck scale. Furthermore, we will be considering that the regime of validity of semiclassical gravity, with a relatively simple modification, can be extended to include the self-induced collapse (or dynamical reduction) of the wave function of matter fields. Our goal is to consider a precise formalism able to incorporate these ideas, that will allow us to explore them in a detailed manner. It is worthwhile noting here that the general formalism to be presented in the first part of the present work should be useful for the detailed studies of the dynamical wave function collapse theories (such as [26, 29–33]) in the situations where the gravitational back-reaction becomes important, as well as for certain aspects encountered in related approaches. Each one of the discontinuous modifications of the evolution equations, which are controlled by stochastic functions introduced in those schemes, would correspond to one single collapse as the one we will describe in this paper. The details of such correspondence would clearly be different for each specific case. As an example, we consider briefly the so called stochastic gravity proposal [46] in appendix B.

Next we turn to the development of the precise formalism we seek. The description of the system in the periods between collapses is considered in section 2.1. In section 2.2 we describe the proposal for incorporating the collapses within the general formalism.

### 2.1 The Semiclassical Self-consistent Configurations (SSCs)

It is clear that the semiclassical regime that we have described so far can not be but an effective description with a limited range of applicability. However, we will assume that such regime includes the cosmological setting at hand. Next, we seek to specify in some detail the nature of the formal description we will be considering, even though we do not take it to be, in any sense, fundamental. Among the advantages of having such a precise structure, is that it allows us to uncover and, in principle, to start investigating, the places where a departure from such scheme is in fact required. As we will see, this approach will allow us to discuss with precision some delicate questions, and to focus sharply on some of the less understood aspects of the collapse ideas. (Consider for comparison the formal developments in classical general relativity, which include the famous singularity theorems, which as we know indicate a limitation of the validity of that very same theory.)

Under these assumptions (and at the above specified level) we will consider that the universe can be described by what we call a Semiclassical Self-consistent Configuration (SSC). That is, a space-time geometry characterized by a classical space-time metric and a standard quantum field theory constructed on that fixed space-time background, together with a particular state in that construction such that the semiclassical Einstein equations hold. In other words, we will say that the set $\{g_{\mu\nu}(x), \hat{\varphi}(x), \hat{\pi}(x), \mathcal{H}, |\xi\rangle \in \mathcal{H}\}$ represents a SSC if and only if $\hat{\varphi}(x)$, $\hat{\pi}(x)$ and $\mathcal{H}$ correspond to a quantum field theory constructed over a
space-time with metric $g_{\mu\nu}(x)$ (as described in, say [47]), and the state $|\xi\rangle$ in $\mathcal{H}$ is such that

$$G_{\mu\nu}[g(x)] = 8\pi G \langle \xi| \hat{T}_{\mu\nu}[g(x), \hat{\varphi}(x), \hat{\pi}(x)]|\xi\rangle$$

(2.2)

for all the points $x$ in the space-time manifold. That is, we are basically relying on a strict interpretation of semiclassical gravity, considered not as a fundamental theory, but as an effective description (Note here that, as discussed in reference [48], it might even be possible to take the alternative viewpoint.) It is worth noting that an analogous approach is sometimes used in the field of quantum optics when one is not interested in some intrinsically quantum aspects of the electromagnetic field (see for instance the “self-consistent equations” in section 1.5 of reference [49]). We should keep in mind the self-referential feature of this approach, and also the fact that it is very close, in spirit, to the Schrödinger-Newton description [50–55], which seems to have a relationship with some kind of collapse-like behaviour [29, 56]. From now on we will ignore the subindexes “m” and “g”, as it would be understood that the quantum description refers only to the matter fields. For simplicity, a single scalar field $\varphi$ has been assumed, where the evident generalization is understood.

The actual construction of even one of such SSC’s is not trivial. One must somehow “guess” the appropriate space-time metric, construct the quantum theory for the matter fields “living” on that given background, and then, find an appropriate state $|\xi\rangle$ in $\mathcal{H}$ (if any) compatible with the selected space-time configuration. We should emphasize that, in general, for a given SSC, most of the states in $\mathcal{H}$ together with the original $g(x), \hat{\varphi}(x), \hat{\pi}(x)$ and $\mathcal{H}$ do not represent a valid SSC. Only for a few states, if any (besides the original one), will the equation (2.2) hold. Explicit constructions of two SSCs will be given in sections 3.1 and 3.2. The first one corresponds to a perfectly homogeneous and isotropic universe, and the second to a slightly inhomogeneous space-time, where, for simplicity, just one Fourier component is taken to be “excited”.

In order to avoid possible future misunderstandings, we should note at this point various important differences between this formalism and the one usually employed within the context of the inflationary universe. The standard approach is based on the consideration of a classical description for both, the space-time metric (taken to admit a flat Robertson-Walker description) and inflaton field (taken to be in a slow-rolling homogeneous and isotropic configuration) “backgrounds”, and the perturbations of both metric and inflaton field, which are treated at the quantum mechanical level. In contrast, in the present formalism, the split between the quantum and classical descriptions is not tied to a particular perturbative approach, but to the gravity-matter distinction. This seems to be justified, as we have argued above, not only by the lack of a workable theory of quantum gravity, but also by the conceptual problems that seem intrinsic to that program, and, in particular, to the so called “problem of time in quantum gravity”. One might be concerned with the fact that it seems always possible to move part of the degrees of freedom from the metric to the matter fields (and back) through a conformal transformation, and with the idea that changes of coordinates mix the gravity and matter field perturbations. These are common misunderstandings, and to their clarification we have devoted the appendix C.

### 2.2 Beyond a single SSC: the collapse

It should be clear that the Semiclassical Self-consistent Configurations introduced in the previous subsection are not enough in order to deal with the problem at hand, i.e. the transmutation of a symmetric universe into the actual inhomogeneous and anisotropic one. The extra element we must consider in order to represent our ideas is the quantum collapse
of the wave function of the matter fields. That is, the proposal that the normal unitary evolution characterizing the standard field theory (and then by definition the SSCs) should be supplemented by instances of quantum collapse, thought to be triggered, somehow, by the effects of the gravitational degrees of freedom which are not fully represented in the metric description (as it was indicated, the metric tensor should be regarded here as a mere effective description — at the hydrodynamical level — of the average aggregate behaviour of the true gravitational degrees of freedom). One of the main purposes of this paper is to propose a formalism that would allow us to represent and explore such idea, and to uncover its limitations.\(^3\)

We will separate the evolution of the system into the standard part and the collapse, considering the first one as described in terms of the Heisenberg picture, while the collapse will be treated in terms of the Schrödinger one. We can look at that distinction as an “interaction picture description”, where the role of the interaction is played by whatever physics lies behind the collapse process, and the rest is absorbed into the evolution of the operators, as it is done in the Heisenberg picture. Then, the space-time dependence encoded in the field operators reflects the standard unitary evolution, where the states will remain constant except when a collapse occurs, which will be characterized by a random jump of the state \(|\xi\rangle\) to one among a set of suitable related states \(\{|\zeta_1\rangle, \ldots, |\zeta_n\rangle\}\), in what we will call a “self-induced” collapse of the wave function,

\[
|\xi\rangle \rightarrow |\zeta\rangle. \tag{2.3}
\]

However, recall that, as we argued in section 2.1, any state involved in the characterization of the conditions in the universe should be always understood within a particular SSC. It is for that reason that it is more appropriate to re-express the notion depicted in (2.3) in the more precise form

\[
\text{SSC-I} \rightarrow \text{SSC-II}. \tag{2.4}
\]

In order to proceed, we need a rather precise prescription for the description of “the collapse”. Consider first, within the Hilbert space associated to the given SSC-I, that a transition \(|\xi^{(I)}\rangle \rightarrow |\zeta^{(I)}\rangle_{\text{target}}\) “is about to happen”, with both \(|\xi^{(I)}\rangle\) and \(|\zeta^{(I)}\rangle_{\text{target}}\) in \(\mathcal{H}^{(I)}\). Generically, the set \(\{g^{(I)}, \hat{\varphi}^{(I)}, \hat{\pi}^{(I)}, \mathcal{H}^{(I)}, |\zeta^{(I)}\rangle_{\text{target}}\}\) will not represent a new SSC. We will thus say that the state \(|\zeta^{(I)}\rangle_{\text{target}}\) is “not physical”. It represents a characterization (of sorts) of the state into which the collapse will take our matter fields, employing the mathematical language of the \(\mathcal{H}^{(I)}\), a language that would be inappropriate if the state of the matter fields were indeed \(|\zeta^{(I)}\rangle_{\text{target}}\), simply because in such case the space-time metric would have to be different from the one used to make the construction of that Hilbert space. In order to have a sensible picture, we need to connect this state \(|\zeta^{(I)}\rangle_{\text{target}}\) with another one \(|\zeta^{(II)}\rangle\) “living” in a new Hilbert space \(\mathcal{H}^{(II)}\) for which \(\{g^{(II)}, \hat{\varphi}^{(II)}, \hat{\pi}^{(II)}, \mathcal{H}^{(II)}, |\zeta^{(II)}\rangle\}\) is an actual SSC. We will denote the new SSC by SSC-II. Thus, first we need to determine the “target” (non-physical) state in \(\mathcal{H}^{(I)}\) to which the initial state is in a sense “tempted” to jump, and after that, we need to relate such target state with a corresponding state in the Hilbert space of a new SSC, the SSC-II. We will define these notions more precisely below. Following our previous treatments

\(^3\)Note that, despite the previously mentioned indications of collapse-like behaviour in the related Schrödinger-Newton system, we are postulating the collapse as an additional feature, because it does not seem that those previously observed features are able to account for the breaking of the initial translational and rotational symmetries of the configuration. That is, the Schrödinger-Newton system, once provided with an initial data possessing one such symmetry, would result in an equally symmetric solution, simply due to the deterministic nature of the problem and the fact that the dynamics does not break those symmetries.
on the subject (see for instance reference [17]), we will consider that the target state is chosen stochastically, guided by the quantum uncertainties of designated field operators, evaluated on the initial state $|\xi(I)\rangle$, at the collapsing time. We will maintain the essence of such prescriptions here. On the other hand, regarding the identification between the two different SSCs involved in the collapse, there seems to be in principle many natural options. The usefulness of those depends, of course, on the degree that they actually determine possible SSC’s. In this paper we will be focusing on the possibility offered by the following prescription: Consider that the collapse takes place along a Cauchy hypersurface $\Sigma_c$. A transition from the physical state $|\xi(I)\rangle$ in $H(I)$ to the physical state $|\zeta(II)\rangle$ in $H(II)$ (associated to the target non-physical state $|\zeta(I)\rangle_{target}$ in $H(I)$) will occur in a way that
\[
\langle\zeta(I)| \hat{T}_{\mu\nu}(I)[g(I) , \hat{\phi}(I) , \hat{\pi}(I)] |\zeta(I)\rangle_{target} |\Sigma_c\rangle = \langle\zeta(II)| \hat{T}_{\mu\nu}(II)[g(II) , \hat{\phi}(II) , \hat{\pi}(II)] |\zeta(II)\rangle_{\Sigma_c}, \tag{2.5}
\]
i.e. in such a way that the expectation value of the energy-momentum tensor associated to the states $|\zeta(I)\rangle_{target}$ and $|\zeta(II)\rangle$ evaluated on the Cauchy hypersurface $\Sigma_c$ coincide. Note that the left hand side in the expression above is meant to be constructed from the elements of the SSC-I (although $|\zeta(I)\rangle_{target}$ is not really the state of the SSC-I), while the right hand side correspond to quantities evaluated using the SSC-II. As we have relied for motivation of our proposal involving the collapse of the wave function on some ideas related to quantum gravity, and given that, at the classical level, the energy-momentum tensor acts as the “source of the gravitational interaction”, we find it reasonable to assume that it is precisely the expectation value of the energy-momentum tensor of the different states involved in the collapse the determining aspect for such identification.

That means that, in general, at the collapsing time we will have two different metrics $g(I)$ and $g(II)$ for a given Cauchy hypersurface. There could be some special situations for which both metrics $g(I)$ and $g(II)$ coincide over some neighbourhood of $\Sigma_c$, but in general there is no reason to expect that there might be a suitable interpolating metric description for the space-time during the process of collapse. At best we might hope to find a recipe where the induced metric on the hypersurface would be continuous at $\Sigma_c$, but its normal derivative (i.e. the extrinsic curvature), would not. Later we will find that, indeed, this would be what happens in the case of interest when we use the matching prescription given in equation (2.5). By construction, the standard unitary evolution of quantum mechanics takes place within a given SSC, thus we need the collapses in order to jump between the different SSCs, and have the possibility of describing the generation of the seeds of cosmic structure. After all, generation means that “something that did not exist at a certain time does exist at a later one”. It is worth emphasizing that the dynamical collapse of the wave function does not belong to what we could call “well established physics”, although there are several proposals formulated within the community working on foundational aspects of quantum theory (see for instance [26] and references therein) which could be connected with the general ideas outlined in this section.

We should note that the equation (2.1) would not, in general, hold through the collapses.\(^4\) At such times the excitation of the fundamental quantum gravitational degrees of freedom should be considered as “coming into play”, with the corresponding breakdown of the semiclassical approximation. In this context, it is worthwhile noting a certain similitude with the conclusions of the analysis of the resolution of the black hole singularity in loop

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\(^4\)In this work we are considering the collapse as taking place instantaneously, i.e. on a space like hypersurface $\Sigma_c$. However, it is perhaps better to think of this as an approximated description of something taking place very fast in comparison to the other time scales of the problem.
quantum gravity [57]. There, it is argued that even though at the quantum level there is no singularity and the region has a perfectly appropriate description in terms of loop variables, the metric description simply does not exist for the region that would correspond to the singularity. Getting back to our case, this possible breakdown can be represented at the formal level by the inclusion of a term $Q_{\mu\nu}$ in the semiclassical field equations, which is supposed to become nonzero only during the collapse of the quantum mechanical wave function of the matter fields:

$$G_{\mu\nu} + Q_{\mu\nu} = 8\pi G \langle \hat{T}_{\mu\nu} \rangle.$$  

(2.6)

The setting is so far general and would allow in principle to consider various situations. In this work, we want to focus on the description of the emergence of the seeds of cosmic structure in the context of a universe which was initially described by a homogeneous and isotropic state for the gravitational and matter degrees of freedom. The idea is that, at some point, the quantum state of the matter fields reaches a stage whereby the corresponding state for the gravitational degrees of freedom leads to a quantum jump of the matter field wave function. The resulting state of the matter fields needs not share the symmetries of the initial state,

$$|\text{symmetric}\rangle \to |\text{non-symmetric}\rangle,$$

(2.7)

and its connection to the gravitational degrees of freedom, which again is assumed to be accurately described by the Einstein semiclassical equations, leads to a geometry that is no longer homogeneous and isotropic. The matching at the collapsing time of the two SSCs constructed in sections 3.1 and 3.2 will be explicitly analysed in section 3.3 following the general ideas discussed here.

3 Fitting inflation into our general formal scheme

In this section we will show a detailed realization of the ideas described above. The motivation for this is twofold: on the one hand it will serve as a proof of concept, and on the other it will help us clarify the description of the cosmological situation that gave rise to the general ideas explored in [17] and subsequent works.

As we have argued, in order to have a sensible picture for the problem at hand we should determine the SSC’s suitable for describing the cosmological evolution of the universe. Our point of view is that, in principle, something like this could be done in most situations. However, it seems clear that, in practice, the complexity of the problems one is commonly interested in would make this task simply unmanageable. Nevertheless, as we are mainly interested in the study of the relatively simple case of the very early universe, we can restrict ourselves to the subset of the nearly (as characterized by the parameter $\varepsilon$ in expression (3.9)) homogeneous and isotropic SSCs. That is, even though the problem is well defined in general, we will approach its solution through a practical perturbative approach. However, we will always have under control the exact nature of the approximations we will be using, providing for the first time a clear interpretative picture which is well defined from the beginning and does not change as the discussion progresses.

We will start now to be rather specific. As it was indicated in the previous section the matter fields will be treated in the language of a quantum field theory on curved spacetimes [47, 58–60]. We will concentrate here on the case of a single scalar field, the inflaton. The other fields, and in particular all the fields of the standard model of particle physics, are assumed in their vacuum state and will be ignored in the present work. In order to
clarify our ideas, let us remind the reader that inflation is supposed to occur in a patch which emerges from the Planck era (at time $t_{IS}$, where inflation starts), corresponding to a situation where the scalar field is in a regime where the inflaton potential is sufficiently large, the kinetic term sufficiently small, and the space-time geometry sufficiently close to a homogeneous and isotropic one for inflation dynamics to take place. After the onset of such dynamics, that patch exponentially inflates, leading to a region of space-time which is very close to a homogeneous and isotropic flat Robertson-Walker universe, with the remnants of the inhomogeneities that were present at $t_{IS}$ reduced by an exponential factor in the number of e-folds, $N \equiv \ln(a/a_{IS})$, with $a$ the value for the scale factor (see expression (3.8) below). Thus, there would be at any time during inflation remnants of inhomogeneity that are at most of order $e^{-N}$. These remanent inhomogeneities are not supposed to be relevant at all in any physical consideration during much of the inflationary regime itself, and are thus said to be essentially erased by inflation. This is the situation where our analysis will be focussed on, and this is indeed necessary if we want to consider as justified any deduction that relies on the use of the vacuum state of the inflaton field to characterize the seeds of the cosmological structures we observe today. Alternatively, if one wanted to claim that such remnants from the pre-inflationary era are tied to the generation of the seeds of structure, we would have to accept that it is impossible to predict any features of such structure, because we do not know anything about the form of the spectrum that might characterize such remnants.

The fact that we do not even have a truly non-perturbative method for dealing with interacting quantum fields in curved spaces leads us to set our attention on the case of a massive, non-interactive scalar field. In fact, such a model applied to the early phase of accelerated expansion has been extensively analysed in the literature, see for instance [61]. At the classical level the inflaton field satisfies the Klein-Gordon equation,

$$g^{\mu\nu}\nabla_\mu\nabla_\nu \phi - m^2 \phi = 0,$$  

(3.1)

with $g^{\mu\nu}$ the inverse of the space-time metric, and $\nabla_\mu$ the covariant derivative. Given $\phi_1(x)$ and $\phi_2(x)$ two solutions to the classical equation of motion (3.1), the symplectic product is defined by

$$(\phi_1, \phi_2)_{\text{Symp}} \equiv -i \int_{\Sigma} \left[ \phi_1 (\partial_\mu \phi_2^*) - (\partial_\mu \phi_1^*) \phi_2^* \right] d\Sigma^\mu.$$  

(3.2)

Here $d\Sigma^\mu \equiv n^\mu d\Sigma$, with $n^\mu$ a time-like, future-directed, normalized 4-vector orthogonal to the 3-dimensional Cauchy hypersurface $\Sigma$, and $d\Sigma = \sqrt{g_{\Sigma}} d^3x$ its volume element. As usual the expression (3.2) does not depend on the selected Cauchy hypersurface. In terms of the conjugate momentum associated to the inflaton field, $\pi(x) \equiv \sqrt{g_{\Sigma}} (n^\mu \partial_\mu \phi)$, the symplectic product can be re-written in the form

$$(\phi_1, \pi_1, \phi_2, \pi_2)_{\text{Symp}} \equiv -i \int_{\Sigma} [\phi_1 \pi_2^* - \pi_1 \phi_2^*] d^3x.$$  

(3.3)

At the quantum level the inflaton field and its conjugate momentum are promoted to field operators acting on a Hilbert space $\mathcal{H}$. These operators must satisfy the standard equal time commutation relations between them, which, upon an appropriate choice of space-time coordinates (to be further specified shortly), take the form,

$$[\hat{\phi}(\eta, \vec{x}), \hat{\pi}(\eta, \vec{y})] = i\hbar \delta(\vec{x} - \vec{y}), \quad [\hat{\phi}(\eta, \vec{x}), \hat{\phi}(\eta, \vec{y})] = [\hat{\pi}(\eta, \vec{x}), \hat{\pi}(\eta, \vec{y})] = 0.$$  

(3.4)
The standard way to proceed now is to decompose $\hat{\phi}(x)$ in terms of the time-independent creation and annihilation operators,

$$\hat{\phi}(x) = \sum_\alpha (\hat{a}_\alpha u_\alpha(x) + \hat{a}_\alpha^\dagger u^*_\alpha(x)),$$

(3.5)

with the functions $u_\alpha(x)$ a complete set of normal modes orthonormal with respect to the symplectic product, i.e.

$$(g^{\mu\nu} \nabla_\mu \nabla_\nu - m^2)u_\alpha = 0, \quad (u_\alpha, u_\alpha')_{\text{Symp}} = \hbar \delta_{\alpha\alpha'}.$$  

(3.6)

A similar expression to that given in (3.5) can be obtained for the conjugate momentum $\hat{\pi}(x)$, replacing the functions $u_\alpha$ by $\sqrt{g} \Sigma (n^\mu \partial_\mu u_\alpha)$. As we will be only interested in configurations corresponding to space-times very close to a flat Robertson-Walker one, we will be able to choose the labels $\alpha$ specifying the different mode solutions to be the wave vectors $\vec{k}$, despite the fact that the spatial sections are not, in general, exactly flat. Clearly, one should always keep in mind that, in general, the functions $u_{\vec{k}}(x)$ would not correspond exactly to the standard Fourier modes in flat space (the simple functional form of the modes $u_{\vec{k}}(x) \propto e^{i\vec{k} \cdot \vec{x}}$ will be appropriate only for the exactly homogeneous and isotropic case).

With all these conventions (expressions (3.5) and (3.6) above) the commutators (3.4) translate into the standard

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k'}}^\dagger] = \delta_{\vec{k}\vec{k'}}, \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k'}}] = [\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k'}}^\dagger] = 0$$

(3.7)

for the creation and annihilation operators. As usual, the vacuum is defined to be the state that is annihilated by all the $\hat{a}_{\vec{k}}$’s, (i.e. $\hat{a}_{\vec{k}}|0\rangle = 0$ for all $\vec{k}$). The Hilbert space can be constructed (à la Fock) by successive applications of creation operators on the vacuum. However, the relations (3.6) do not determine the set of mode solutions unequivocally, and the particular choice corresponds to an election of the vacuum.

This construction for the Hilbert space of the inflaton field is generic and applies for any (globally hyperbolic) space-time configuration. An exact solution will not be possible in general, but an approximate one suitable for the study of the very early universe could be obtained perturbatively. The study of the seeds of cosmic structure depends essentially on the scalar sector of the perturbations. Ignoring for simplicity the so called vector and tensor modes, we will choose a coordinate system (the so called conformal Newtonian gauge) in which the space-time metric takes the form

$$ds^2 = a^2(\eta) \left[ - (1 + 2\psi)d\eta^2 + (1 - 2\psi)\delta_{ij}dx^i dx^j \right], \quad \text{with } \psi(\eta, \vec{x}) \ll 1.$$  

(3.8)

Here $a$ is the scale factor, $\psi$ an analogue to the Newtonian potential, and $\eta$ the cosmological time in conformal coordinates. The coordinates $x^i$ label the observers co-moving with the expansion. Note that, in accordance with our approach, both $a(\eta)$ and $\psi(\eta, \vec{x})$ are (dimensionless) classical fields. Setting $\psi = 0$ we recover a spatially flat homogeneous and isotropic Robertson-Walker universe.\(^5\) In order to avoid problems in the infrared we will work with periodic boundary conditions over a box of size $L$. We can take the limit $L \to \infty$ at the end

\(^5\)Indeed, we recover that situation even if take $\psi = \psi(\eta)$, as can be seen by a simple change of the space-time coordinates. Here we will impose $\int d^3\vec{x}\psi(\eta, \vec{x}) = 0$ in order to remove those ambiguities.
of the calculations. In the present context we can describe the Newtonian potential in terms of a Fourier decomposition and write generically
\[ \psi(\eta, \vec{x}) = \varepsilon \sum_{\vec{k} \neq 0} \tilde{\psi}_\varepsilon(\eta) e^{i\vec{k} \cdot \vec{x}}, \quad (3.9) \]
where the sum is over all vectors \( \vec{k} \) with \( k_n = 2\pi j_n/L, \ j_n = 0, \pm 1, \pm 2, \ldots \) and \( n = 1, 2, 3 \).

In order to have a real value for \( \psi(\eta, \vec{x}) \) we should demand \( \tilde{\psi}_\varepsilon(\eta) = \psi^*_\varepsilon(\eta) \), and we will be assuming \( \tilde{\psi}_\varepsilon(\eta) \lesssim \mathcal{O}(1) \) and \( \varepsilon \ll 1 \), so we can guarantee \( \psi(\eta, \vec{x}) \ll 1 \). Note that as the space-independent part of the Newtonian potential can be reabsorbed in the scale factor, it does not appear in (3.9).

Recall also that we were working in co-moving coordinates, where the \( \vec{k} \)'s are fixed in time and label each particular mode (related to their physical values through \( \vec{k}/a \)).

Working up to the first order in \( \varepsilon \), the equations (3.6) simplify to
\[
(1 - 2\psi)(\ddot{u}_k + 2H\dot{u}_k) - (1 + 2\psi)\Delta u_k - 4\dot{\psi}u_k + a^2 m^2 u_k = 0, \quad (3.10a)
\]
\[
\int_{\eta = \text{const.}} \left[ u_k(\partial_\eta u^*_k) - (\partial_\eta u_k)(\psi^*_k) \right] (1 - 4\psi) dx = i\hbar a^{-2} \delta_{k\vec{k}'} \quad (3.10b)
\]
For the general case even the construction of that SSC will not be trivial, although, in principle, it would be doable. However, in order to proceed, we will consider a simple example: the transition from an exactly homogeneous and isotropic universe, \( \psi(\eta, x) = 0 \), to a situation where, as a result of one of our collapse events, a single nontrivial plane-wave is excited.

We characterize that by the wave vector \( \vec{k}_0 \), and write \( \psi(\eta, x) = \varepsilon \tilde{\psi}_{\vec{k}_0}(\eta)e^{i\vec{k}_0 \cdot \vec{x}} + \text{c.c.} \), with \( \varepsilon \ll 1 \) and c.c. denoting the complex conjugate. Next, we proceed to explicitly construct the two SSC's: the first corresponding to the homogeneous and isotropic pre-collapse situation, and the second to the post-collapse one, characterized by a SSC where there is an actual fluctuation with wave vector \( \vec{k}_0 \). Then, we will consider in some detail the character of the description of the transition between those two.

A small comment on notation is in order here. The objects that are specific to the constructions we will be discussing in sections 3.1 and 3.2 will bear an index (I or II) indicating the specific SSC construction to which they belong. In reading those two sections the reader can simply ignore that index, as they will not change within each section. However, we have left the index in place in order to avoid confusion when the two constructions are brought together in the discussion of their matching in section 3.3.

3.1 A homogeneous and isotropic SSC

At a time corresponding to few e-foldings after inflation started the relevant region of the universe is thought to be described by a homogeneous and isotropic SSC. The Newtonian potential vanishes at that time, \( \psi(\eta, \vec{x}) = 0 \). We will call it the first SSC, or simply the SSC-I. In order to carry out that construction we will take the space-time metric to be that of a Robertson-Walker universe, with a pre-established (nearly) de Sitter scale factor. The small deviation from the exact de Sitter expansion will be parametrized by \( \epsilon^{(I)} \equiv 1 - \mathcal{H}^{(I)}/\mathcal{H}^{2(I)} \), where \( \mathcal{H}^{(I)} \equiv \dot{a}^{(I)}/a^{(I)} \) is a measure of the expansion rate of the universe, related to the standard Hubble parameter by \( H^{(I)} = \mathcal{H}^{(I)}/a^{(I)} \). During slow-roll \( 0 < \epsilon^{(I)} \ll 1 \) (not to be confused with \( \varepsilon \ll 1 \)). For practical purposes and for all the situations analysed in this paper it will be sufficient to consider the scale factor \( a^{(I)}(\eta) = (-1/H_0^{(I)} \eta)^{1+\epsilon^{(I)}} \) and the Hubble
parameter in conformal coordinates $\mathcal{H}^{(I)} = -(1 + \epsilon^{(I)})/\eta$ to the first order in the slow-roll parameter (remember that $-\infty < \eta < 0$).

Next we need to find a complete set of normal modes $u_k^{(I)}(x)$ for the previously given homogeneous and isotropic space-time configuration. Naturally, given the symmetries of the spatial background, we can look for solutions of the form

$$u_k^{(I)}(x) = v_k^{(I)}(\eta) e^{ikx/L^{3/2}}.$$  \hfill (3.11)

Introducing $\psi = 0$ and the ansatz (3.11) into the equations (3.10) we obtain

$$\ddot{v}_k^{(I)} + 2\mathcal{H}^{(I)}\dot{v}_k^{(I)} + (k^2 + a^{(2)}m^2)v_k^{(I)} = 0,$$  \hfill (3.12a)

$$v_k^{(I)}\dot{v}_k^{(I)*} - \dot{v}_k^{(I)}v_k^{(I)*} = i\hbar a^{-2(I)}.$$  \hfill (3.12b)

We will find that $a^{(2)}m^2$ is first order in the slow-roll parameter, $a^{(2)}m^2 = 3\epsilon^{(I)}H^2(I)$ to the first nonvanishing order in $\epsilon^{(I)}$ (see equation (3.20) bellow), so equation (3.12a) should be considered to this same order, i.e. we should take $\mathcal{H}^{(I)} = -(1 + \epsilon^{(I)})/\eta$. For the modes with $k \neq 0$ the most general solution to the equation (3.12a) is a linear combination of the functions $\eta^{3/2+\epsilon^{(I)}}H^{(1)}_{\nu}(-k\eta)$ and $\eta^{3/2+\epsilon^{(I)}}H^{(2)}_{\nu}(-k\eta)$, with $H^{(1)}_{\nu}(-k\eta)$ and $H^{(2)}_{\nu}(-k\eta)$ the Hankel functions of first and second kind, and $\nu(I) = 3/2 + \epsilon(I) - m^2/3H_0^{(I)}$ to the first order in the slow-roll parameter. A choice of modes corresponds to an election of the vacuum. For a space-time background without a time-like killing vector field there is no preferential choice, but following the standard literature on the subject we will take the Bunch-Davis convention: i.e. use modes such that in the asymptotic past they behave as purely “positive frequency solutions”, normalized according to (3.12b). Working to the lowest non-vanishing order in the slow-roll parameters ($\nu(I) = 3/2$), we obtain

$$v_k^{(I)}(\eta) = \sqrt{\frac{\hbar}{2k}}(-H_0^{(I)}\eta)\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}.$$  \hfill (3.13)

These functions coincide with the standard mode solutions of a massless ($m = 0$) scalar field in a de Sitter ($\epsilon^{(I)} = 0$) universe. We note, however, the Hankel functions are not well behaved at the origin, and thus the zero mode is not included in (3.13). For $k = 0$ the general solution to the equation (3.12a) is a linear combination of the functions $\eta^{3+2\epsilon^{(I)}-2\nu(I)/2}$ and $\eta^{3+2\epsilon^{(I)}+2\nu(I)/2}$. The choice is arbitrary, provided it has positive symplectic norm. Here we take

$$v_0^{(I)}(\eta) = \sqrt{\frac{\hbar}{H_0^{(I)}}}\left[1 - \frac{i}{6}(-H_0^{(I)}\eta)^3\right](-H_0^{(I)}\eta)^{m^2/3H_0^{(I)}},$$  \hfill (3.14)

which has been normalized using the condition (3.12b). Note that, contrary to what we did for the $k \neq 0$ modes, we have worked now to the first order in the slow-roll parameter. We will find that this is necessary in order to accommodate a slow-rolling expectation value for the zero mode.

So far we have given a prescription to construct the mode solutions $u_k^{(I)}(x)$ for the SSC-I, and thus we have a construction of the Hilbert space and a representation of the fundamental fields as operators acting on it. However, we still need to find a state $|\xi^{(I)}\rangle$ in $\mathcal{H}^{(I)}$ such that its expectation value for the energy-momentum tensor leads to the desired nearly de Sitter, homogeneous and isotropic cosmological expansion. As expected, the state in the SSC-I
can just have the zero mode excited. The expectation value of the field operator is then homogeneous and isotropic,

$$
\phi_{\xi,0}(\eta) \equiv \langle \xi^{(I)} | \hat{\phi}(x) | \xi^{(I)} \rangle = \xi^{(I)} v_{0}(\eta)/L^{3/2} + \text{c.c.},
$$

(3.15)

with $\xi^{(I)} \equiv \langle \hat{a}_{0}^{(I)} \rangle$. The subindex “0” in $\phi_{\xi,0}(\eta)$ makes reference to this fact, whereas the super-index “(I)” indicates that the field operators involved are those of the SSC-I, and also that the expectation values are taken in the state $|\xi^{(I)}\rangle$. We should note that as the potential is quadratic in the field operator the Ehrenfest theorem guarantees the classical equation of motion for the expectation value of the inflaton field.

As it was just argued, the symmetries of the space-time background lead us to consider a state in which all the modes with $k \neq 0$ are in their vacuum state, while the zero mode is excited. Thus, we consider a state of the form

$$
|\xi^{(I)}\rangle = \mathcal{F}(\hat{a}_{0}^{(I)})|0^{(I)}\rangle,
$$

(3.16)

where $\mathcal{F}(\hat{X})$ stands for a suitable generic function acting on the operators $\hat{X}$ (as we will see it will be sufficient for our purposes in this paper to use the function that is associated with the “coherent” states, namely $\mathcal{F}(\hat{X}) \propto \exp(\hat{X})$). For those particular states with only the zero mode excited, the components 00 and $i = j$ of Einstein’s field equations simplify to

$$
3H^{2(I)} = 4\pi G \left( (\hat{\phi}^{2(I)})_{\xi,0} + a^{2(I)}m^{2}(\hat{\phi}^{2(I)})_{\xi,0} \right),
$$

(3.17a)

$$
H^{2(I)} + 2\dot{H}^{(I)} = -4\pi G \left( (\hat{\phi}^{2(I)})_{\xi,0} - a^{2(I)}m^{2}(\hat{\phi}^{2(I)})_{\xi,0} \right).
$$

(3.17b)

The two sides of the 0I and the $i \neq j$ equations vanish identically for that particular metric and state. Here $(\hat{\phi}^{2(I)})_{\xi,0} \equiv \langle \xi^{(I)} | : (\partial_{\eta} \hat{\phi}^{(I)})^{2} : | \xi^{(I)} \rangle$ and $(\hat{\phi}^{2(I)})_{\xi,0} \equiv \langle \xi^{(I)} | : (\hat{\phi}^{(I)})^{2} : | \xi^{(I)} \rangle$ are functions of $\eta$. The equations (3.17) are analogous (but not exactly equal) to those obtained in the context of a classical field theory, with the squares of the scalar field and its time derivative replaced now by the expectation value of their corresponding operators. This difference could, in principle, introduce important departures from the classical behaviour, but fortunately this problem will not affect us in the simple case treated here. As we have just mentioned, Ehrenfest’s theorem guarantees the classical equations of motion for the expectation value of the inflaton field. In general, the “classical relations” $(\hat{\phi}^{2(I)})_{\xi,0} = (\phi^{2(I)})_{\xi,0}^{2}$ and $(\phi^{2(I)})_{\xi,0} = (\phi^{2(I)})_{\xi,0}^{2}$ will not hold (that is, $\langle \hat{a}_{0}^{(I)} \rangle = \xi^{(I)}$ does not necessarily imply $\langle \hat{a}_{0}^{(I)} \rangle = \xi^{(I)}$). However, as we want to consider a state $|\xi^{(I)}\rangle$ that is sharply peaked around a classical field configuration, $\phi^{(I)}_{\xi,0}(\eta)$ and $\pi^{(I)}_{\xi,0}(\eta)$, we take it to correspond to a “highly excited coherent” state (i.e. $\langle \hat{a}_{0}^{(I)} | \xi^{(I)} \rangle = \xi^{(I)} | \xi^{(I)} \rangle$, with $\xi^{(I)} \in \mathbb{C}$), for which those classical relations do hold. In that case we recover precisely the same Friedmann equations of the standard treatments.

Requiring that the universe is characterized by a regime of slow-roll inflation with expansion rate $H^{(I)}$ and slow-roll parameter $\epsilon^{(I)} \equiv 1 - \dot{H}^{(I)}/H^{2(I)}$ implies $3(\phi^{(I)2})_{\xi,0}^{2} = \ldots$.

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6There is also an infinite contribution to the expectation value of the energy-momentum tensor coming from the vacuum of the theory, but this is an issue related to the well known cosmological constant problem and it will not be considered any further here. We could assume that the energy-momentum tensor is renormalized, say à la Haddamard, and that the cosmological constant is set to zero. In practice we will simply impose normal ordering on the energy-momentum tensor operator. : $\hat{a}_{\xi}^{\dagger} \hat{a}_{\xi}^{\dagger} := \hat{a}_{\xi}^{\dagger} \hat{a}_{\xi}$. 
\( \epsilon^{(I)} a^{2(I)} m^2 (\phi^{(I)}_{\xi,0})^2 \). This expression is obtained, working up to the first order in the slow-roll parameter \( \epsilon^{(I)} \), by combining the equations (3.17a) and (3.17b) (once we have assumed a coherent state). Solving this equation for \( \phi^{(I)}_{\xi,0}(\eta) \) we obtain

\[
\phi^{(I)}_{\xi,0}(\eta) \propto \eta \sqrt{\epsilon^{(I)} m^2 / 3 H^2_0} .
\]  

(3.18)

Comparing with expressions (3.14) and (3.15), and taking the parameter \( \xi^{(I)}_0 \) as real,

\[
\langle \xi^{(I)} | \hat{\phi}^{(I)}(x) | \xi^{(I)} \rangle = \frac{2 \xi^{(I)}_0}{L^{3/2}} \sqrt{\frac{\hbar}{H^{(I)}_0}} \left( - H^{(I)}_0 \eta \right)^{m^2 / 3 H^2_0}. 
\]  

(3.19)

Thus, we find compatibility of equations

\[
\epsilon^{(I)} = \frac{m^2}{3 H^2_0}, \quad H^{(I)}_0 = 2 \epsilon^{(I)} \eta^2 \left( \xi^{(I)}_0 \right)^2 / L^3 .
\]  

(3.20)

where \( \eta_0^2 = 8\pi G \hbar \) stands for the Planck time.\(^7\) As it is well known, in order to have \( \epsilon^{(I)} \ll 1 \) we must require \( m \ll H^{(I)}_0 \). Inflation is expected to take place at very high energies, so this requirement is not generally taken as problematic. On the other hand, given the values of the parameters \( H^{(I)}_0 \) and \( \epsilon^{(I)} \) characterizing the cosmological expansion during inflation, we can read from the second expression in (3.20) the state for the zero mode, \( \xi^{(I)}_0 \) (or more precisely, \( \xi^{(I)}_0 / L^{3/2} \)). This fixes the state for the SSC-I, as all other modes are taken to be in their vacuum state (with respect to the Bunch-Davies convention). We thus have the metric, the quantum field theory construction, and the specific state that is compatible with Einstein’s equation. This therefore completes the construction of the SSC-I. We now turn to the more complex case corresponding to a slightly inhomogeneous and anisotropic situation.

### 3.2 A SSC with the \( \vec{k}_0 \)-mode excited

Here we want to carry out the construction of a new SSC corresponding to an excitation in the Newtonian potential characterized by the wave-vector \( \vec{k}_0 \). We will denote this new SSC by SSC-II. We must consider a slight deviation from the previous homogeneous and isotropic cosmological background, characterized now by the parameters \( H^{(\text{II})}_0 \) and \( \epsilon^{(\text{II})} \) (which might, in principle, differ slightly from those corresponding to the SSC-I discussed in the previous section), and a Newtonian potential with one single term excited in the expression (3.9), described by an (in principle) arbitrary function \( \tilde{\psi}_{\vec{k}_0}(\eta) = P(\eta) \). We will latter see that the semiclassical field equations will allow us to obtain this function and the quantum state of the SSC, leading to a complete determination of this construction.

However, the first step in the building of the SSC-II will be the construction of the quantum theory for the inflaton field on the new type of space-time configuration we are considering now (with any given \( P(\eta) \)). That is, we need a complete set of normal modes \( u^{(\text{II})}_{\vec{k}}(x) \) appropriate for the new construction. As we are working up to the first order in \( \epsilon \),

\(^7\)The presence in the last expression of the factor \( L^{-3} \) might seem strange at first sight, as the size of the artificial box should have no impact on the expansion rate of the universe. However, we must recall that in our notation the expectation value of the scalar field is proportional to \( L^{-3/2} \), as can be seen in expression (3.15). Thus, what we have here is simply the fact that \( H^{(I)}_0 \) is proportional to \( \langle \hat{\phi}^2 \rangle \).
and since the Newtonian potential is given by \( \psi(\eta, \vec{x}) = \varepsilon P(\eta)e^{i\vec{k}_0 \cdot \vec{x}} + \text{c.c.} \), we can consider the ansatz:

\[
u^{(II)}(k)(x) = \varepsilon \delta v^{(II)}_k(\eta) e^{i(k-k_0) \cdot \vec{x}} / L^{3/2} + v^{(II)}_0(k) e^{i\vec{k} \cdot \vec{x}} / L^{3/2} + \varepsilon \delta v^{(II)}_k(\eta) e^{i(k+k_0) \cdot \vec{x}} / L^{3/2}.
\]

(3.21)

Introducing (3.21) into (3.10) we find that, to the zeroth order in the \( \varepsilon \), the evolution equation is given by

\[
\ddot{v}^{(II)}_k + 2H \dot{v}^{(II)}_k + (k^2 + a^2(II) m^2)v^{(II)}_k = 0,
\]

(3.22a)

with normalization condition

\[
v^{(II)}_k \dot{v}^{(II)}_k - v^{(II)*}_k \dot{v}^{(II)*}_k = i\hbar a^{-2(II)},
\]

(3.22b)

while at first order in \( \varepsilon \) the corresponding evolution equation takes the form

\[
\delta v^{(II)}_k + 2H \delta v^{(II)}_k + [(k \pm k_0)^2 + a^2(II) m^2] \delta v^{(II)}_k = F^\pm_k(\eta),
\]

(3.23a)

where,

\[
F^+_k(\eta) \equiv 4\dot{P} v^{(II)}_k - 2(2k^2 + a^2(II) m^2) P v^{(II)}_k, \\
F^-_k(\eta) \equiv 4\dot{P}^* v^{(II)}_k - 2(2k^2 + a^2(II) m^2) P^* v^{(II)}_k.
\]

(3.23b, 3.23c)

The normalization condition is given by

\[
\dot{v}^{(II)*}_k \delta v^{(II)}_k + \dot{v}^{(II)}_k \delta v^{(II)*}_k - \dot{v}^{(II)}_k \delta v^{(II)}_k - \delta v^{(II)*}_k \dot{v}^{(II)}_k = 4 \left( \dot{v}^{(II)}_k v^{(II)*}_k - v^{(II)}_k \dot{v}^{(II)*}_k \right) P.
\]

(3.23d)

The zeroth order problem coincides with the situation considered in the previous section, equations (3.12), and we can simply write:

\[
v^{(II)}_k(\eta) = \sqrt{\frac{\hbar}{2k}} (-H_0^{(II)} \eta) \left( 1 - \frac{i}{k \eta} \right) e^{-ik \eta}, \quad \text{for } k \neq 0,
\]

(3.24a)

\[
v_0^{(II)}(\eta) = \sqrt{\frac{\hbar}{H_0^{(II)}}} \left[ 1 - \frac{i}{6} (-H_0^{(II)} \eta)^3 \right] (-H_0^{(II)} \eta)^{m^2 / 3} H_0^{(II)}, \quad \text{for } k = 0.
\]

(3.24b)

Next, we consider the first order problem. The functions \( \delta v^{(II)}_k(\eta) \) satisfy the dynamical equation (3.23a), similar to that for the \( v^{(II)}_k(\eta) \), but with a source term determined by the Newtonian potential (which would be specified once we provide the function \( P(\eta) \), something that we will do shortly) and the zeroth order functions \( v^{(II)}_k(\eta) \) (given by the expressions (3.24) above). It is a second order linear equation, therefore, the solution is univocally determined in terms of the corresponding initial data: \( \delta v^{(II)}_k(\eta_c) \) and \( \delta v^{(II)}_k(\eta_c) \). (We take the initial time for the SSC-II to be the collapsing time, \( \eta_c \).) The normalization relation (3.23d) is the only constrain on the initial data. Two simple (but convenient) choices that satisfy the normalization constraints are

\[
\delta v^{(II)}_k(\eta_c) = 0, \quad \delta v^{(II)}_k(\eta_c) = 4v^{(II)}_k(\eta_c) P(\eta_c),
\]

(3.25a)

and

\[
\delta v^{(II)}_k(\eta_c) = 4v^{(II)}_k(\eta_c) P(\eta_c), \quad \delta v^{(II)}_k(\eta_c) = 0.
\]

(3.25b)
We will pick one of them, for definiteness, the first one. As we have indicated this choice completely determines the solution (assuming the function $P(\eta)$ is given). As we will see next, we will not need to calculate the functions $\delta v^{(\Pi)}_{\vec{k}}(\eta)$ explicitly in order to finish our analysis.

So far, we have provided the basic ingredients defining the construction of the quantum field theory for the SSC-II, but we still need to find a state $|\zeta^{(\Pi)}\rangle \in \mathcal{H}^{(\Pi)}$ such that its expectation value for the energy-momentum tensor leads to the desired nearly de Sitter, slightly inhomogeneous cosmological expansion, characterized by the Newtonian potential $\psi(\eta, \vec{x})$ in the space-time metric (3.8). The task of constructing that state is rather cumbersome, and the reader is encouraged to skip the details in the first reading. The important result is that this construction is carried out in general and, at the end, full compatibility is ensured by a judicious choice of the function $P(\eta)$ controlling the time dependence of the Newtonian potential.

Let us first concentrate on the state. Looking at the symmetries of the space-time background it seems natural to assume that such a state should be of the form:

$$|\zeta^{(\Pi)}\rangle = \ldots |\zeta^{(\Pi)}_{-2k_0}\rangle \otimes |\zeta^{(\Pi)}_{-k_0}\rangle \otimes |\zeta^{(\Pi)}_0\rangle \otimes |\zeta^{(\Pi)}_{k_0}\rangle \otimes |\zeta^{(\Pi)}_{2k_0}\rangle \ldots .$$  
\hspace{1cm} (3.26)

Here we are making an evident abuse of notation: the vector in Fock space is characterized by parameters indicating the specific modes that are excited (all other modes are assumed to be in the vacuum of the corresponding oscillator), and the parameters $\zeta^{(\Pi)}_{k}$ are meant to indicate exactly how the mode $\vec{k}$ has been excited. We could take these parameters to describe, for instance, a particular coherent state for each different mode, and then the state we are considering in the expression above would be precisely

$$|\zeta^{(\Pi)}\rangle = \ldots \mathcal{F}(\zeta^{(\Pi)}_{-2k_0} |\zeta^{(\Pi)}_{-k_0}\rangle \mathcal{F}(\zeta^{(\Pi)}_{-k_0} |\zeta^{(\Pi)}_0\rangle \mathcal{F}(\zeta^{(\Pi)}_0 |\zeta^{(\Pi)}_{k_0}\rangle \mathcal{F}(\zeta^{(\Pi)}_{k_0} |\zeta^{(\Pi)}_{2k_0}\rangle \ldots |\eta^{(\Pi)}\rangle ,$$  
\hspace{1cm} (3.27)

where $\mathcal{F}(\hat{X})$ again stands for the object $\mathcal{F}(\hat{X}) \propto \exp(\hat{X})$. Needless is to say that we can consider similar excitations of each mode which are not necessarily coherent, but the latter will be sufficient for our purposes here.

The expectation value of the field operator in such a state is given by

$$\phi^{(\Pi)}_{\zeta}(x) = \phi^{(\Pi)}_{\zeta,0}(\eta) + \left( \delta \phi^{(\Pi)}_{\zeta,2k_0}(\eta)e^{ik_0\vec{x} - c.c.} \right) + \left( \delta \phi^{(\Pi)}_{\zeta,2k_0}(\eta)e^{2k_0\vec{x} - c.c.} \right) + \ldots .$$  
\hspace{1cm} (3.28)

Contrary to what happens for the SSC-I, expression (3.15), the excitation in each mode leads now to space-dependencies with three characteristic wavelengths. For instance, even if just the $k = 0$ mode is excited, the field expectation value will include terms with the characteristic behaviour $e^{\pm ik_0\vec{x}}$. In fact, for a general state of the form (3.26) we have

\begin{align}  
\mathcal{L}^{3/2} \phi^{(\Pi)}_{\zeta,0}(\eta) &= \zeta^{(\Pi)}_0 v^{(\Pi)0}_0(\eta) + \varepsilon \left[ \zeta^{(\Pi)}_{-k_0} \delta v^{(\Pi)+}_{-k_0}(\eta) + \zeta^{(\Pi)}_k \delta v^{(\Pi)-}_{k_0}(\eta) \right] + c.c., \hspace{1cm} (3.29a) \\
\mathcal{L}^{3/2} \phi^{(\Pi)}_{\zeta,2k_0}(\eta) &= \zeta^{(\Pi)}_{2k_0} v^{(\Pi)0}_{2k_0}(\eta) + \zeta^{(\Pi)} v^{(\Pi)0}_{-2k_0}(\eta) + \varepsilon \left[ \zeta^{(\Pi)}_{k_0} \delta v^{(\Pi)+}_{k_0}(\eta) + \zeta^{(\Pi)}_{-k_0} \delta v^{(\Pi)-}_{k_0}(\eta) \right] + \zeta^{(\Pi)}_{2k_0} \delta v^{(\Pi)+}_{2k_0}(\eta) + \zeta^{(\Pi)}_{-2k_0} \delta v^{(\Pi)-}_{2k_0}(\eta), \hspace{1cm} (3.29b) \\
\mathcal{L}^{3/2} \phi^{(\Pi)}_{\zeta,2k_0}(\eta) &= \zeta^{(\Pi)}_{-2k_0} v^{(\Pi)0}_{-2k_0}(\eta) + \zeta^{(\Pi)} v^{(\Pi)0}_{2k_0}(\eta) + \varepsilon \left[ \zeta^{(\Pi)}_{k_0} \delta v^{(\Pi)}_{k_0}(\eta) + \zeta^{(\Pi)}_{-k_0} \delta v^{(\Pi)}_{-k_0}(\eta) \right] + \zeta^{(\Pi)}_{2k_0} \delta v^{(\Pi)+}_{2k_0}(\eta) + \zeta^{(\Pi)}_{-2k_0} \delta v^{(\Pi)-}_{2k_0}(\eta), \hspace{1cm} (3.29c)
\end{align}
with similar expressions for the other $\delta \phi^{(II)}_{\xi, n k_0} (\eta)$ (for positive integers $n$). We are considering a coherent state for all the different modes. However, all that is required for the validity of the expressions above is that $\zeta^{(II)}_{\pm n k_0}$ is necessary to ensure compatibility of our state ansatz with Einstein’s equations. In principle, the details behind such relations could be used to determine the exact values of the undetermined parameters, and it is easy to see that $|\zeta^{(II)}_{\pm n k_0}| \sim \varepsilon^n |\zeta^{(II)}_0|$. Recalling that we are interested here just on a first order in $\varepsilon$ calculation, it is clear that all the terms in $\varepsilon \zeta_{\pm k_0}$ and $\zeta_{\pm n k_0}$ (with $n \geq 2$) in the expressions (3.29) can be disregarded. Thus, in practice and to the order we are working here, we can simply write $|\zeta^{(II)}_0| \pm |\zeta^{(II)}_{-k_0}| \otimes |\zeta^{(II)}_{k_0}|$. In particular that implies $|\delta \phi^{(II)}_{\xi, 0} | \sim \varepsilon^{2} \zeta^{(II)}_{-k_0}$, and, from now on, we will write

$$\phi^{(II)}_{\xi} (x) = \phi^{(II)}_{\xi, 0} (\eta) + \varepsilon \left( \delta \phi^{(II)}_{\xi, k_0} (\eta) e^{i k_0 \cdot x} + \text{c.c.} \right),$$

(3.30)

where we have used the notation $\varepsilon \delta \phi^{(II)}_{\xi, k_0} \equiv \delta \phi^{(II)}_{\xi, k_0}$. Thus we have, up to order $\varepsilon$,

$$L^{3/2} \phi^{(II)}_{\xi, 0} (\eta) = \zeta^{(II)}_{0} (\eta) + \varepsilon \left( \delta \phi^{(II)}_{\xi, k_0} (\eta) e^{i k_0 \cdot x} + \text{c.c.} \right),$$

(3.31a)

$$L^{3/2} \varepsilon \delta \phi^{(II)}_{\xi, k_0} (\eta) = \zeta^{(II)}_{k_0} (\eta) \varepsilon v^{(II)}_{k_0} (\eta) + \zeta^{(II)}_{-k_0} (\eta) v^{(II)}_{-k_0} (\eta) + \varepsilon \left[ \zeta^{(II)}_{0} (\eta) (\delta v^{(II)}_{0} (\eta) + \zeta^{(II)}_{0} (\eta) (\delta v^{(II)}_{0} (\eta) - \eta v^{(II)}_{0} (\eta)) \right].$$

(3.31b)

The conditions above are necessary in order to ensure that there are no terms in $e^{\pm i n k_0 \cdot x}$ (with $n \geq 2$) appearing in the expectation value of the energy-momentum tensor. That, in turn, is necessary to ensure compatibility of our state ansatz with Einstein’s equations, and the assumption that those terms do not appear to the order $\varepsilon$ in the expression for the Newtonian potential of the SSC-II.

To the first order in $\varepsilon$, 0, $\eta$ and $i = j$ components of Einstein’s field equations take the form:

$$3 \mathcal{H}^{(II)} - 2 \varepsilon \left[ (k_0^2 P + 3 \mathcal{H}^{(II)} \tilde{P}) e^{i k_0 \cdot x} + \text{c.c.} \right]$$

(3.32a)

$$= 4 \pi G \left\{ \left( \delta \phi^{(II)}_{\xi, 0} \right)^2 + a^{2} (\phi^{(II)}_{\xi, 0} \right)^2 m^2 \right\}$$

$$+ 2 \varepsilon \left[ \left( \phi^{(II)}_{\xi, 0} \right)^2 \delta \phi^{(II)}_{\xi, 0} + a^{2} (\phi^{(II)}_{\xi, 0} \right)^2 m^2 \right\} e^{i k_0 \cdot x} + \text{c.c.} \right\},$$

$$\varepsilon \left[ \left( \hat{\mathcal{P}} + \mathcal{H}^{(II)} P \right) e^{i k_0 \cdot x} + \text{c.c.} \right]$$

(3.32b)

$$= 4 \pi G \left\{ \varepsilon \left( \left( \phi^{(II)}_{\xi, 0} \delta \phi^{(II)}_{\xi, 0} \right) e^{i k_0 \cdot x} + \text{c.c.} \right),$$

$$- \mathcal{H}^{(II)} - 2 \varepsilon \left[ \left( \hat{\mathcal{P}} + 3 \mathcal{H}^{(II)} \tilde{P} + 2 \left( \tilde{P} + \mathcal{H}^{(II)} \tilde{P} \right) \right) e^{i k_0 \cdot x} + \text{c.c.} \right]$$

(3.32c)

$$= 4 \pi G \left\{ \left( \phi^{(II)}_{\xi, 0} \right)^2 + a^{2} (\phi^{(II)}_{\xi, 0} \right)^2 m^2 \right\}$$

$$+ 2 \varepsilon \left[ \left( \phi^{(II)}_{\xi, 0} \delta \phi^{(II)}_{\xi, 0} - a^{2} (\phi^{(II)}_{\xi, 0} \right)^2 m^2 \right\} e^{i k_0 \cdot x} + \text{c.c.} \right\}. $$

Here (3.32a) and (3.32b) are two constraints due to the diffeomorphism invariance of the theory, and (3.32c) a dynamical equation. The $i \neq j$ equations vanish to this order.
Let us concentrate first on the space-independent part of the SSC-II. Integrating the 00 and the $i = j$ components of Einstein equations over a spatial hypersurface of constant $\eta$ we obtain

$$3H^2(\Pi) = 4\pi G \left( (\phi_{0,\zeta}^{(\Pi)})^2 + a^2(\Pi)m^2(\phi_{0,\zeta}^{(\Pi)})^2 \right), \quad (3.33a)$$

$$H^2(\Pi) + 2\dot{H}(\Pi) = -4\pi G \left( (\phi_{0,\zeta}^{(\Pi)})^2 - a^2(\Pi)m^2(\phi_{0,\zeta}^{(\Pi)})^2 \right). \quad (3.33b)$$

Again, these are the standard Friedmann equations. The spatial integral of the equations 0i and $i \neq j$ vanishes (to the first order in $\varepsilon$). The equations (3.33) are analogous to those obtained in section 3.1 for the SSC-I, and the state of the zero mode get fixed like in the previous case (see expression (3.20) and recall that, to the order we are working here, the terms in $\varepsilon \zeta_k$ and $\varepsilon \zeta_k^2$ must be neglected in (3.29a), i.e. see expression (3.31a) above). Note that, in contrast with the homogeneous and isotropic case corresponding to the SSC-I, in the present case the system (3.33) represents just the first contribution in a series expansion, and although here we will limit ourselves to the first order, it is clear however that, in principle, we could extend the analysis to any desired order.

Introducing the equations (3.33) into the space-dependent first order system (3.32) we obtain

$$k_0^2P + 3H(\Pi)\dot{P} = -4\pi G \left( \dot{\phi}_{0,\zeta}^{(\Pi)} \delta_{\zeta,k_0}^{(\Pi)} + a^2(\Pi)m^2\phi_{0,\zeta}^{(\Pi)} \delta_{\zeta,k_0}^{(\Pi)} + a^2(\Pi)m^2(\phi_{0,\zeta}^{(\Pi)})^2P \right), \quad (3.34a)$$

$$\dot{P} + H(\Pi)P = 4\pi G \left( \dot{\phi}_{0,\zeta}^{(\Pi)} \delta_{\zeta,k_0}^{(\Pi)} \right), \quad (3.34b)$$

$$\ddot{P} + 3H(\Pi)\dot{P} + 2(H(\Pi) + 2\dot{H}(\Pi))P = 4\pi G \left( \dot{\phi}_{0,\zeta}^{(\Pi)} \delta_{\zeta,k_0}^{(\Pi)} - a^2(\Pi)m^2\phi_{0,\zeta}^{(\Pi)} \delta_{\zeta,k_0}^{(\Pi)} + a^2(\Pi)m^2(\phi_{0,\zeta}^{(\Pi)})^2P - 2(\phi_{0,\zeta}^{(\Pi)})^2P \right). \quad (3.34c)$$

The key result, and the aspect that enables us to carry out the construction in a complete manner is the following fact: the equations (3.34) can be combined into a single dynamical equation for the Newtonian potential, which is independent of the matter fields first order quantities $\delta_{\zeta,k_0}^{(\Pi)}$ and $\delta_{\zeta,k_0}^{(\Pi)}$:

$$\ddot{P} + 2 \left[ 3H(\Pi) + \frac{a^2(\Pi)m^2\phi_{0,\zeta}^{(\Pi)}}{\dot{\phi}_{0,\zeta}^{(\Pi)}} \right] \dot{P} + \left[ k_0^2 + 2H(\Pi) + 4\dot{H}(\Pi) + 8\pi G(\phi_{0,\zeta}^{(\Pi)})^2 + 2\frac{a^2(\Pi)m^2\phi_{0,\zeta}^{(\Pi)}}{\dot{\phi}_{0,\zeta}^{(\Pi)}}H(\Pi) \right]P = 0. \quad (3.35)$$

In fact, we can now use Friedmann equations to write by expressing $a^{(\Pi)}m\phi_{0,\zeta}^{(\Pi)}/\dot{\phi}_{0,\zeta}^{(\Pi)}$ as $-\sqrt{2H(\Pi) + \dot{H}(\Pi)}/(\dot{H}(\Pi) - H(\Pi))^{1/2}$ (we have taken the negative sign because during slow-roll $\dot{\phi}_{0,\zeta}^{(\Pi)}$ and $\phi_{0,\zeta}^{(\Pi)}$ must have opposite signs). We can go further by using the definition of the slow-roll parameter, $\dot{H}(\Pi)/H(\Pi) = 1 - \epsilon(\Pi)$, and express the above equation in the simpler looking way

$$\ddot{P} + 2 \left[ 3H(\Pi) - A(\Pi) \right] \dot{P} + \left[ k_0^2 + 2(3 - \epsilon(\Pi))H(\Pi) - 2A(\Pi)H(\Pi) \right]P = 0. \quad (3.36)$$

Here we have used that $\left[ (2H(\Pi) + \dot{H}(\Pi))/(\dot{H}(\Pi) - H(\Pi)) \right]^{1/2} = [(3 - \epsilon(\Pi))/\epsilon(\Pi)]^{1/2}$, and defined $A(\Pi) \equiv \sqrt{3/\epsilon(\Pi)}m a^{(\Pi)}(1 - \epsilon(\Pi)/6)$, expression valid up to the first order in the slow-roll parameter. One might be concerned about the $1/\sqrt{\epsilon(\Pi)}$, which would be extremely large during a phase of slow-roll inflation. However, we should note that this factor appears
multiplied by the scalar field mass \( m \), and, just as in the case of the SSC-I (remember expression (3.20) in section 3.1), we would have \( \mathcal{H}^{(\II)}_0 = m(1 + r\epsilon^{(\II)})/\sqrt{3\epsilon^{(\II)}} \). Although large, this is simply the natural inflationary scale. (Note that we have kept the higher order correction term \( r \) that was not explicit in equation (3.20) to be consistent with the expansion being first order in \( \epsilon^{(\II)} \).) Making use of the expression for the scale factor \( a^{(\II)} = \mathcal{H}^{(\II)}/H^{(\II)} \), we find that the equation (3.36) becomes simply

\[
\ddot{P} + \epsilon^{(\II)}(1 + 6r)\mathcal{H}^{(\II)} \dot{P} + \left[k_0^2 - \epsilon^{(\II)}(1 - 6r)\mathcal{H}^{(\II)}_0\right]P = 0.
\]

The general solution to the equation (3.36) depends only on the zero mode (the space-independent part of the universe) and the initial conditions for the Newtonian potential, \( P_c \equiv P(\eta_c) \) and \( \dot{P}_c \equiv \dot{P}(\eta_c) \),

\[
P(\eta) = C_1 \eta \sqrt{3(1 + (6 + 1)\epsilon^{(\II)})}J_\alpha(-k\eta) + C_2 \eta \sqrt{3(1 + (6 + 1)\epsilon^{(\II)})}Y_\alpha(-k\eta),
\]

where \( J_\alpha(-k\eta) \) and \( Y_\alpha(-k\eta) \) are the Bessel functions of first and second kind, \( \alpha = 1 + 3(1 - 2r)\epsilon^{(\II)}/2 \) to the first order in \( \epsilon^{(\II)} \), and \( C_1 \) and \( C_2 \) two constants that will be determined by the initial conditions. We will not be making use of this explicit solution in the rest of the analysis. However, it is worth noting that this represents a damped oscillation of the Newtonian potential. Once we have an expression for the function \( P(\eta) \), equation (3.38), the mode solutions determining the quantum field theory construction become fully determined as well, as we already noted below equation (3.25b).

Regarding the initial values for the function \( P(\eta) \), we note that the problem has a fundamental symmetry \( \phi \to -\phi \), and, for definiteness, we will be assuming from now on that \( \phi^{(\II)}_{\zeta,0} > 0 \). Making use of the two constraints (3.34a) and (3.34b), we can express the initial values that would determine the specific solution \( P(\eta) \) in the form,\(^8\)

\[
\begin{pmatrix}
P \\
\dot{P}
\end{pmatrix} = \sqrt{4\pi G\epsilon^{(\II)}}\mathcal{H}^{(\II)}\begin{pmatrix}3\mathcal{H}^{(\II)} - \mathcal{A}^{(\II)}
-k_0^2 - (3 - \epsilon^{(\II)})\mathcal{H}^{(\II)}_0 + \mathcal{A}^{(\II)}\mathcal{H}^{(\II)} - \mathcal{H}^{(\II)}_0
\end{pmatrix}\begin{pmatrix}\delta_{\phi^{-\gamma}}^{(\II)}_{\zeta,k_0} \\
\delta_{\phi^{\gamma}}^{(\II)}_{\zeta,k_0}
\end{pmatrix}.
\]

The equations at the SSC-II side of collapsing time, so the system (3.39) can be used in order to infer the initial conditions for the Newtonian potential in terms of the characteristics of the collapse, \( \delta_{\phi^{-\gamma}}^{(\II)}(\eta_c) \) and \( \delta_{\phi^{\gamma}}^{(\II)}(\eta_c) \). Given the values for \( P_c \) and \( \dot{P}_c \) at the collapsing time \( \eta_c \) (or equivalently, \( \delta_{\phi^{-\gamma}}^{(\II)}(\eta_c) \) and \( \delta_{\phi^{\gamma}}^{(\II)}(\eta_c) \)), we have thus a completely determined space-time metric, and, as discussed in connection with (3.25b), this then determines the set of normal modes for the SSC-II. We can also use the values for \( \delta_{\phi^{-\gamma}}^{(\II)}(\eta_c) \) and \( \delta_{\phi^{\gamma}}^{(\II)}(\eta_c) \), together with the expression (3.29b) and the identities (3.25a), to determine the value of the parameters \( \zeta^{(\II)}_{k_0} \) and \( \zeta^{(\II)}_{-k_0} \) (we have two equations for two unknowns leading to definite values for the latter). A simple and naturally expected conclusion can be seen: the homogeneous and isotropic universe determines the state of the zero mode \( |\zeta^{(\II)}_{k_0}\rangle \) (or vice versa), whereas the values for \( P_c \) and \( \dot{P}_c \) at the collapse time help determine the quantum state for the modes \( \pm k_0 \) (i.e. the “mode states” \( |\zeta^{(\II)}_{k_0}\rangle \) and \( |\zeta^{(\II)}_{-k_0}\rangle \)).

\(^8\)In order to study what happens when \( k_0^2 - \epsilon^{(\II)}\mathcal{H}^{(\II)}_0 = 0 \) (or close to that point) one would need to include the next order in the series expansion. However, we are not going to analyse this here.
It is worth commenting at this point that the seemingly “strange” connection that we have found between the excitation of the mode $\vec{k}_0$ and that of its higher harmonics ($\vec{k} = n\vec{k}_0$, with $n$ natural), is intimately connected with the nonlinearity of the entire proposal (which, as we have explained, is the general relativistic version of what occurs in the Schrödinger-Newton system), and is very similar, at the mathematical level, to the effect known as “parametric resonance” that occurs in quantum optics with non-linear media [62, 63].

This finalizes the construction of the SSC-II. We have seen that it is fully determined once given the values $\delta \hat{\phi}^{(II)}_{\zeta, \vec{k}_0}(\eta)$ and $\delta \hat{\pi}^{(II)}_{\zeta, \vec{k}_0}(\eta)$. Next we need to study the possibility of matching this SSC to the SSC-I on the hypersurface corresponding to the collapse time (see appendix C for a discussion of the aspect of this matching that is connected to the gauge issues).

### 3.3 The matching at the collapse time

As we have discussed before, when trying to describe the emergence of the seeds of structure in our universe, we need to consider the transition from a homogeneous and isotropic SSC to another one lacking such symmetries. Here we will give an explicit construction for such a transition using the general ideas developed in section 2.2. As we have already stressed, we are interested in discussing the formalism, rather than the completely realistic and evidently quite complicated case. We will only consider the transition from the SSC-I with $\psi(\eta, \vec{x}) = 0$ discussed in section 3.1, to a situation where a single nontrivial mode $\vec{k}_0$ in the Newtonian potential is excited. That is, the SSC-II with $\hat{\psi}_{\vec{k}}(\eta) = P(\eta)\delta_{\vec{k}_0}$ described in We note that there are two general issues to be treated in this context: a) the actual matching conditions between the SSCI and the SSCII, and b) the characterization of the target state in the Hilbert space of the SSC-I, which in turn will be used to characterize the state of the SSC-II.

As indicated before we will be considering that at time $\eta_c$ the mode $\vec{k}_0$ of the state $|\xi^{(I)}(\eta_c)|$ undergoes a collapse. (Remember that we are attempting to formalize the description of that novel — and evidently unknown — aspect of physics we have argued should be taking place in the early universe, described in this paper as the collapse of the wave function). Following the ideas developed in section 2.2, we assume that, first, the characteristics of the collapse are determined within the initial SSC. We will assume that such characterization is encoded in a state within the initial Hilbert space. We will often refer to such state (using a very loose, but heuristically helpful language) as the state the system “is tempted to jump into” or the “target state”. In our case that will be a state belonging to the Hilbert space $\mathcal{H}^{(I)}$ corresponding to the homogeneous and isotropic SSC-I, but will not be the state that makes up the SSC-I. In accordance with the comments above, we will assume that such state corresponds to a tendency of excitation in the $\vec{k}_0$ mode, related in some way to the collapse process that we are about to describe,

$$|\xi^{(I)}\rangle = |\xi^{(I)}_0\rangle \rightarrow |\xi^{(I)}\rangle_{\text{target}} = |\xi^{(I)}_{-\vec{k}_0}\rangle \otimes |\xi^{(I)}_0\rangle \otimes |\xi^{(I)}_{\vec{k}_0}\rangle,$$

(3.40)

with all the other modes remaining in their previous state. We will consider how this particular target state $|\xi^{(I)}\rangle_{\text{target}}$ in $\mathcal{H}^{(I)}$ is chosen later on. However, as we have anticipated, what we need to face is the fact that the target state in $\mathcal{H}^{(I)}$, together with $g^{(I)}_{\mu\nu}(x)$, $\hat{\phi}^{(I)}(x)$ and $\hat{\pi}^{(I)}(x)$, can not represent a new SSC. Thus, we need a new SSC corresponding to a specific version of the SSC-II, $\{g^{(II)}_{\mu\nu}(x), \hat{\phi}^{(II)}(x), \hat{\pi}^{(II)}(x), \mathcal{H}^{(II)}, |\xi^{(II)}\rangle\}$, such that the state $|\xi^{(II)}\rangle$ in $\mathcal{H}^{(II)}$ is in some way related to the state $|\xi^{(I)}\rangle_{\text{target}}$ in $\mathcal{H}^{(I)}$. In the simple case
considered here we can focus on the operator $\dot{\phi}^{(I)}(\eta)$, with $\dot{\phi}^{(I)}(x) = \sum_k \dot{\phi}^{(I)}_k(\eta) e^{ikx}$, and write $\phi^{(I)}_{\zeta,0}(\eta_c) \equiv \text{target}(\zeta^{(I)}|\phi^{(I)}_0(\eta_c))|\zeta^{(I)}\rangle_{\text{target}}$ and $\varepsilon\delta\phi^{(I)}_{\zeta,0}(\eta_c) \equiv \text{target}(\zeta^{(I)}|\phi^{(I)}_0(\eta_c))|\zeta^{(I)}\rangle_{\text{target}}$. Even though, as we have stressed, the target state is an element of $\mathcal{H}^{(I)}$ but is not part of any SSC. Here, the subindex “r” in $\zeta$ refers to the fact that the quantity corresponds to the expectation value in such a target state.

As it was anticipated in section 2.2, we will consider that the identification of the SSC-I and the SSC-II is guided by the expectation value of the energy-momentum tensor, expression (2.5). We will see this in detail in the following and, in particular, we will see how that helps determining the state corresponding to the SSC-II in terms of the target state. To the zeroth order in $\varepsilon$ the requirement (2.5) gives:

\begin{equation}
(\phi^{(I)}_{\zeta,0})^2 + a^{(I)^2}m^2(\phi^{(I)}_{\zeta,0})^2 = (\dot{\phi}^{(I)}_{\zeta,0})^2 + a^{(I)^2}m^2(\phi^{(I)}_{\zeta,0})^2, \quad (3.41a)
\end{equation}

\begin{equation}
(\phi^{(I)}_{\zeta,0})^2 - a^{(I)^2}m^2(\phi^{(I)}_{\zeta,0})^2 = (\dot{\phi}^{(I)}_{\zeta,0})^2 - a^{(I)^2}m^2(\phi^{(I)}_{\zeta,0})^2. \quad (3.41b)
\end{equation}

From these equations it is easy to conclude that $(\phi^{(I)}_{\zeta,0})^2 = (\dot{\phi}^{(I)}_{\zeta,0})^2$ and $a^{(I)^2}(\phi^{(I)}_{\zeta,0})^2 = a^{(I)^2}(\phi^{(I)}_{\zeta,0})^2$. We will assume that there is no jump in the scale factor, and also that there is no jump in sign in the respective field expectation values. Under these assumptions we are led to $\phi^{(I)}_{\zeta,0} = \dot{\phi}^{(I)}_{\zeta,0}$ and $\phi^{(I)}_{\zeta,0} = \dot{\phi}^{(I)}_{\zeta,0}$.

Now, let us proceed to discuss the matching to the first order in $\varepsilon$. At that order, expression (2.5) gives:

\begin{equation}
\dot{\phi}^{(I)}_{\zeta,0} \dot{\phi}^{(I)}_{\zeta,0} + a^{(I)^2}m^2\phi^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0} = \dot{\phi}^{(I)}_{\zeta,0} \dot{\phi}^{(I)}_{\zeta,0} + a^{(I)^2}m^2(\phi^{(I)}_{\zeta,0} \phi^{(I)}_{\zeta,0} + (\phi^{(I)}_{\zeta,0})^2) P, \quad (3.42a)
\end{equation}

\begin{equation}
\dot{\phi}^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0} = \dot{\phi}^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0}, \quad (3.42b)
\end{equation}

\begin{equation}
\dot{\phi}^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0} = a^{(I)^2}m^2\phi^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0} = \dot{\phi}^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0} - a^{(I)^2}m^2(\phi^{(I)}_{\zeta,0} \delta\phi^{(I)}_{\zeta,0} + (\phi^{(I)}_{\zeta,0})^2) P - 2(\phi^{(I)}_{\zeta,0})^2 P. \quad (3.42c)
\end{equation}

First, we can use expression (3.42b) together with our previous results to conclude that $\delta\phi^{(I)}_{\zeta,0} = \delta\phi^{(I)}_{\zeta,0}$. Next, subtracting equation (3.42c) from (3.42a) and using again the previous results we find that $(\phi^{(I)}_{\zeta,0})^2 P = 0$. Thus, as we will assume that after the collapse the universe remains in a slow-roll expansion, $\dot{\phi}^{(I)}_{\zeta,0} \neq 0$, we can conclude that $P = 0$. This might seem problematic, but let us recall that the matching conditions are supposed to hold only at the time of collapse, $\eta = \eta_c$, thus $P(\eta_c) = 0$, but $P$ at later times need not vanish. On the other hand, as we will see, this will drastically simplify our analysis. Finally, using these results and adding equations (3.42c) and (3.42a) we find that $\delta\phi^{(I)}_{\zeta,0} = \delta\phi^{(I)}_{\zeta,0}$.

It is worth mentioning that equations (3.41a) and (3.41b) have been obtained from $(T^{(I)}_{00})_{\zeta,0} = (T^{(I)}_{00})_{\zeta,0}$ and $(T^{(I)}_{ii})_{\zeta,0} = (T^{(I)}_{ii})_{\zeta,0}$, whereas (3.42a), (3.42b) and (3.42c) from $(\delta T^{(I)}_{00})_{\zeta,0} = (\delta T^{(I)}_{00})_{\zeta,0}$, $(\delta T^{(I)}_{ii})_{\zeta,0} = (\delta T^{(I)}_{ii})_{\zeta,0}$ and $(\delta T^{(I)}_{ii})_{\zeta,0} = (\delta T^{(I)}_{ii})_{\zeta,0}$. For a scalar field, and up to the first order in $\varepsilon$, the other components of the energy-momentum tensor do not contain additional information. Of course the identities (3.41) and (3.42) are only valid at the collapsing time. What is more, the target state only plays a role at that particular time.
Recapitulating, we will ask that at the matching \( a^{(II)}(\eta_c) = a^{(I)}(\eta_c) \), and found

\[
\begin{align*}
\phi^{(II)}_{\zeta,0}(\eta_c) &= \phi^{(I)}_{\zeta,0}(\eta_c), \\
\dot{\phi}^{(II)}_{\zeta,0}(\eta_c) &= \dot{\phi}^{(I)}_{\zeta,0}(\eta_c), \\
\delta\phi^{(II)}_{\zeta,k_0}(\eta_c) &= \delta\phi^{(I)}_{\zeta,k_0}(\eta_c), \\
\delta\dot{\phi}^{(II)}_{\zeta,k_0}(\eta_c) &= \delta\dot{\phi}^{(I)}_{\zeta,k_0}(\eta_c),
\end{align*}
\] (3.43a)

and \( P(\eta_c) = 0 \). We can use now expressions (3.33) to conclude that \( H^{(I)} = H^{(II)} \) and \( \epsilon^{(I)} = \epsilon^{(III)} \), i.e. we have continuity for the space-independent part of the space-time background (the spatial metric and extrinsic curvature).

Within each SSC, the expectation values of field and momentum operators satisfy the Ehrenfest theorem (recall that within each SSC there is nothing exotic going on, and each mode of the field is essentially a harmonic oscillator). This can be used to compute the quantities \( \phi_{\zeta,0}(\eta_c) \) and \( \delta\phi_{\zeta,k_0}(\eta_c) \) (see the expression (3.29) above and remember that, to the order we are working here, the parameters \( \epsilon_{\pm k_0} \) and \( \zeta_{\pm n k_0} \) (with \( n \geq 2 \)) can be disregarded, as we did for instance in equation (3.31)). The values for \( \phi_{\zeta,0}^{(II)}(\eta_c) \) and \( \delta\phi_{\zeta,k_0}^{(II)}(\eta_c) \) can be obtained from the time derivatives of the expressions given in (3.29). However, note that \( \phi_{G,0}^{(I)}(\eta_c) \) and \( \delta\phi_{G,k_0}^{(I)}(\eta_c) \) correspond to expectation values in a target state, which, as we have been emphasising, despite being an element of the Hilbert space of the SSC-I, is not the state characterizing the SSC-I. Thus, in general, such quantities can exhibit spatial dependences.

In fact, these are given by the expressions

\[
\begin{align*}
L^{3/2}\phi^{(I)}_{G,0}(\eta_c) &= \xi^{(I)}_0 u^{(I)}_0(\eta_c) + \text{c.c.}, \\
L^{3/2}\delta\phi^{(I)}_{G,k_0}(\eta_c) &= \zeta^{(I)}_{k_0} u^{(I)}_{k_0}(\eta_c) + \zeta^{(I)*}_{k_0} u^{(I)*}_{k_0}(\eta_c),
\end{align*}
\] (3.44a)

and \( \phi^{(I)}_{G,0}(\eta_c) \) and \( \delta\phi^{(I)}_{G,k_0}(\eta_c) \) by their time derivatives. Comparing the identities (3.43) with the expressions (3.31) and (3.44), we arrive to \( \xi^{(II)}_0 = \xi^{(I)}_0, \zeta^{(II)}_0 = \zeta^{(I)}_0 \) and \( \zeta^{(II)}_{k_0} = \zeta^{(I)}_{k_0} \) (recall that, according to the choice given in (3.25a), for the case \( P(\eta_c) = 0 \) we have \( \delta\phi_{k}^{(II)}(\eta_c) = \delta\phi_{k}^{(I)}(\eta_c) = 0 \)). That is, using the mode solutions chosen in sections 3.1 and 3.2, the parameters \( \zeta^{(II)}_k \) characterizing the state of the inflaton field in the SSC-II can be directly read from the parameters \( \zeta^{(I)}_k \) characterizing the target state in \( \mathcal{H}^{(I)} \). Thus, once the target state is determined, the complete SSC-II gets fixed, because as we showed in section 3.2 everything is determined there once we specify the SSC state.

From \( P(\eta_c) = 0 \) and the equation (3.39) we find that, at the time of collapse, we should have

\[
(3\mathcal{H}^{(II)} - \mathcal{A}^{(II)}) \delta\phi^{(II)}_{\zeta,k_0}(\eta_c) + \delta\dot{\phi}^{(II)}_{\zeta,k_0}(\eta_c) = 0.
\] (3.45)

Combining (3.39) and (3.45) we obtain the explicit expression for \( \dot{P}(\eta_c) \), which shows that, even though the Newtonian potential is continuous at \( \eta_c \), its time derivative is not,

\[
\dot{P}(\eta_c) = \sqrt{4\pi G c^{(II)} \mathcal{H}^{(II)} \delta\phi^{(II)}_{\zeta,k_0}(\eta_c)}.
\] (3.46)

This jump in the time derivative of the Newtonian potential at the collapsing time gives rise to the primordial perturbation in the \( k_0 \) mode. That is, the spatial metric is continuous.
at the transition, but its time derivative is not, i.e. at the collapsing time we will have a continuous but non-smooth description for the space-time manifold.

Finally, we must give a prescription for the election of the target state $|\zeta^{(I)}\rangle_{\text{target}}$ involved in the collapse, expression (3.40). As it was anticipated in section 2.2, and as it has been usual in our previous treatments of the subject, we will consider that the target state is chosen stochastically, guided by the quantum uncertainties, at the time of collapse, of some field operators evaluated in the pre-collapse state $|\xi^{(I)}\rangle$. These operators have been usually taken to be the corresponding modes of the inflaton field or their conjugate momenta, or some combination thereof, and sometimes even both. We had rather large freedom in what we chose in that regard. However, in the present analysis, we find that the field and momentum combination thereof, and sometimes even both. We had rather large freedom in what we chose in that regard. However, in the present analysis, we find that the field and momentum can not be assumed to change their expectation value during the collapse in an arbitrary way: the condition (2.5) imposes $P_\epsilon = 0$, and then the relation (3.45) above. Here, we will consider that the collapse is guided by the quantum uncertainties associated to the modes of the field operator, and that the immediate post-collapse expectation value of momentum operator is such that the condition $P_\epsilon = 0$ is satisfied.

We need to clarify one more point before analysing the determination of the target state. As discussed in [17], in order for the collapse to resemble as much as possible the implementation of the reduction postulate (which is connected with Hermitian observables), we decompose the operators $\hat{\phi}_k^{(I)}(\eta)$ into its real and imaginary parts, $\hat{\phi}_k^{(I)}(\eta) = \hat{\phi}_k^{(I)R}(\eta) + i\hat{\phi}_k^{(I)I}(\eta)$, and focus the collapse on those. The operators we must consider are then

$$\hat{\phi}_k^{(I)R,I}(\eta) = \frac{1}{\sqrt{2}} \left( u^{(I)}_k(\eta) \hat{a}^{(I)R,I}_k + u^{(I)}_k(\eta) \hat{a}^{(I)R,I}_k\right),$$

(3.47)

and

$$\hat{a}^{(I)R}_k = \frac{1}{\sqrt{2}} \left( \hat{a}^{(I)} + \hat{a}^{(I)}_k \right), \quad \hat{a}^{(I)I}_k = \frac{-i}{\sqrt{2}} \left( \hat{a}^{(I)}_k - \hat{a}^{(I)}_k \right).$$

(3.48)

With these definitions $\hat{\phi}_k^{(I)R,I}(\eta)$ are Hermitian operators (i.e. $\hat{\phi}_k^{(I)R,I}(\eta) = \hat{\phi}_k^{(I)R,I}(\eta)$), but the commutation relations between $\hat{a}^{(I)R}_k$ and $\hat{a}^{(I)I}_k$ are non-standard,

$$\left[ \hat{a}^{(I)R}_k, \hat{a}^{(I)R}_k \right] = \left( \delta_{k,k'} + \delta_{k',k} \right), \quad \left[ \hat{a}^{(I)I}_k, \hat{a}^{(I)I}_k \right] = \left( \delta_{k,k'} - \delta_{k',k} \right),$$

(3.49)

with all the other commutators vanishing.

Now we turn to specifying the target state in SSC-I, and thus the relevant state of SSC-II. As discussed in [17], we will be assuming, in a loose analogy with standard quantum mechanics, that the collapse is somehow similar to an imprecise measurement of the operators $\hat{\phi}_k^{(I)R,I}(\eta)$, and that the final results will be guided by

$$\varepsilon \delta_{\phi_\zeta^{(I)R,I}}(\eta_c) = x^{R,I}_{k_0} \sqrt{\langle 0^{(I)}_{k_0} | \left[ \Delta \hat{\phi}_k^{(I)}(\eta_c) \right]^2 | 0^{(I)}_{k_0} \rangle} = x^{R,I}_{k_0} \sqrt{\frac{1}{2} \varepsilon \delta_{\phi_\zeta^{(I)R,I}}(\eta_c)},$$

(3.50)

with $x^{R,I}_{k_0}$ taken to be two independent random variables distributed according to a Gaussian function centred at zero with unit-spread. The expression (3.50), together with the relation (3.45), determines the values for $\varepsilon \delta_{\phi_\zeta^{(I)R,I}}(\eta_c)$ and $\varepsilon \delta_{\phi_\zeta^{(I)R,I}}(\eta_c)$ at the collapsing time (in
terms of the random variables \( x^{R, I}_{k_0} \), and then the state \( |\zeta^{(I)}_{\text{target}}\rangle \), and thus \( |\zeta^{(III)}\rangle \)

\[
\begin{align*}
\zeta^{(II)}_{k_0} &= \zeta^{(I)}_{k_0} = -i\hbar^{-1}a^2(\eta_c)\varepsilon\left[ v^{(I)}(\eta_c)\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) - v^{(I)}(\eta_c)\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) \right], \\
\zeta^{(II)}_{-k_0} &= \zeta^{(I)}_{-k_0} = -i\hbar^{-1}a^2(\eta_c)\varepsilon\left[ v^{(I)}(\eta_c)\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) - v^{(I)}(\eta_c)\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) \right].
\end{align*}
\]

That is, the state of the inflaton field in the SSC-II will be given by \( |\zeta^{(III)}\rangle = |\zeta^{(II)}_{-k_0}\rangle \otimes |\zeta^{(II)}_{k_0}\rangle \otimes |\zeta^{(II)}_{0}\rangle \), with \( \zeta^{(II)}_{0} = |\zeta^{(I)}_{0}\rangle \), and \( \zeta^{(II)}_{k_0} \) and \( \zeta^{(II)}_{-k_0} \) determined in terms of the values for \( \varepsilon\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) \) and \( \varepsilon\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) \) at the collapsing time, expressions (3.51). Regarding the metric tensor, its space-independent part will not be affected by the collapse, \( H^{(I)} = H^{(II)} \) and \( \epsilon^{(I)} = \epsilon^{(II)} \).

On the other hand, the Newtonian potential will be given by the solution of the differential equation expression (3.35), with initial conditions \( P_c = 0 \) and \( \dot{P}_c \) determined in terms of the values for \( \varepsilon\delta\phi^{(I)}_{\zeta, k_0} (\eta_c) \) and \( \varepsilon\dot{\phi}^{(I)}_{\zeta, k_0} (\eta_c) \) by (3.39) (see also the expression (3.46) above).

One should avoid being deceived by the close relationship between the target state \( |\zeta^{(I)}_{\text{target}}\rangle \) and the state \( |\zeta^{(III)}\rangle \). While, as we have seen, their corresponding expectation values for the basic field operators at the collapse time are the same, their subsequent evolution will in general deviate from one another. In particular, the Newtonian potential will become non-vanishing, and thus the modes of the SSC-II construction will involve non-vanishing \( \delta v_{\zeta} \)'s (see equation (3.21) above). These aspects might become relevant in a detailed analysis of the resulting primordial spectrum, something that lies well beyond the scope of the present manuscript.

We emphasize that we have found that, although the Newtonian potential is continuous at the collapse time, its time derivative is not. This means we do not really have a true space-time description of the process. In the next section we will briefly discuss our views on this problematic aspect of our results, and in future works we hope to investigate ways in which this aspect of our formalism might be improved.

4 Discussion

We have considered the generic joint description of gravitation in interaction with a quantum field, to the extent that this can be done without a fully workable theory of quantum gravity. Our proposal is formalized in terms of what we call a Semiclassical Self-consistent Configuration (SSC), which is nothing but a combination of quantum field theory on a background space-time, with the requirement that the state of the matter fields and the space-time geometry be connected through the semiclassical Einstein equations. We have applied this approach to a simple inflationary cosmological model, describing both a perfectly homogeneous and isotropic configuration, and a situation that deviates from the former one by a slight excitation of a particular inhomogeneous and anisotropic perturbation. We have considered in detail a proposal for describing the “space-time” where such a perturbation actually emerges from the initial homogeneous and isotropic configuration as the result of the collapse of the wave function of the inflaton field. To our knowledge, this represents the first time such a detailed description of the process of emergence of structure is ever presented. We believe that the treatment developed in this paper can be useful in uncovering how the collapse of a wave function can be made compatible with a fundamental theory of gravity. If and when we would be in possession of a fully workable and satisfactory quantum
theory of gravitation, and are able to describe in detail its semiclassical regime, we should be able to explore the exact behaviour of the gravitational degrees of freedom on the collapse hypersurface. However, even in the absence of such a theory, a study of these issues could produce interesting insights into some of the features that such theory should contain. We have argued that the general formalism developed in this work should be useful in detailed studies of the various theories involving wave function collapse (such as \[26, 28–33\]), and in particular in their applications to situations where the gravitational back-reaction becomes important, as well as in proposals such as the stochastic gravity of Hu and Verdaguer \[46\] (see the appendix B for more details).

Regarding the inflationary regime (which provided the motivation for the development of this formalism), it is clear that what we have done here is just a starting point, as we have limited ourselves to the study of a single collapse. In order to analyse the problem of the emergence of the seeds of cosmic structure in a complete fashion we would need to consider multiplicity of collapses, occurring in multiple times and involving all the different modes, as we have done, schematically, in previous works \[17, 20–22\]. However, in contrast with what was done there, the study can now, in principle, be carried out using a well defined and precise formalism as presented here. That formalism would allow us to consider issues such as the degree to which “energy conservation” is violated during a collapse, and possibly to analyse its effects on the evolution of the universe during and immediately after inflation. Moreover, as discussed in section 3.2, the formalism suggests the existence of correlations between the excitation level of the modes \(\vec{k}\) and that of its higher harmonics, a feature that is reminiscent of the so called “parametric resonances” occurring in quantum optics with nonlinear materials \[62, 63\].

It is worthwhile to contrast the formalism we have developed in this work with the standard treatment of inflation, which is, at the fundamental level, essentially perturbative. Such a treatment is based on the separation of a background (involving both the space-time metric and the inflaton field), which is described at the classical level, reserving the quantum treatment just for the linear perturbations. The formalism we have considered here is, in principle, amenable to a non-perturbative treatment, even though in practice we are often impeded from carrying that out simply because of the usual limitations that prevent a general non-perturbative treatment of a quantum field theory. Nonetheless, if one manages to overcome this limitation on the quantum field side, for instance through treatments based on lattice approaches, or simply by considering some quantum field solvable model, the scheme could be appropriate to the consistent inclusion of gravity. One obvious advantage of such a setup is that the path to considering higher order perturbations (i.e. anything beyond first order perturbations) is clear and well defined from the beginning. On the other hand, it is quite evident that this cannot be considered as a fully satisfactory description of nature because of all the well known arguments indicating that we need a quantum theory of gravitation, see for instance references \[48, 64, 65\]. Nevertheless, it seems reasonable to assume that a theory like this can be suitable for a description of a situation where the measures and estimates (i.e. classical and quantum mechanical) of the space-time curvature are well below the Planck regime, and where the matter fields energy-momentum tensor have uncertainties that are “not too large” (i.e. see Penrose’s arguments). We will be working under the assumption that this would be the case for most of the inflationary regime we are interested in, and for the post-inflationary cosmological regimes that follow it.

However, as we have previously argued, this cannot be the full story if we want to be able to account for the transition from the completely homogeneous and isotropic universe
which is usually associated with the early and mid stages of inflation (and that we have considered in section 3.1 within a very simple model described by the SSC-I), to the late situation where inhomogeneities and anisotropies are present (and which was described in a simplified manner involving just one excited mode by the SSC-II of section 3.2). Accounting for this phenomena requires a departure from the established unitary evolution in the form of some sort of “collapse of the wave function” (as considered by Diosi [29–31], Ghirardi, Rimini and Weber [32], or Pearle [33]; see also [26]), a feature that might presumably find its full justification in a deeper theory of quantum gravity, as it has been previously discussed by R. Penrose [19, 27]. In the appendix A we have presented a speculation of how something like that might be tied to the resolution of the problem of time in quantum gravity, based on the findings of [86] that the usage of a physical clock can naturally introduce effective deviations from unitary evolution.

Nonetheless, regardless of such speculations, the issue we need to face is how to modify the SSC formalism to include such a transition. Here, we have considered an attempt to do so, which involves the selection of a state within the Hilbert space of the SSC-I (called the “target state”) to which the system “wants to jump” (to which it would jump if the result was also a SSC), and then finding a new SSC for which the associated state had, on the collapse hypersurface (a space-like hypersurface taken for simplicity to coincide with a homogeneous and isotropic one of the cosmological model described by the SSC-I), the same energy-momentum tensor as the target state. We can argue that this “matching recipe” is more or less natural, although clearly has several aspects that can be considered rather ad hoc. One can certainly consider other possible recipes. However, as it turns out a posteriori, this option has some nice features, in the sense of limiting the degree of arbitrariness on the specification of both the target state (through the requirement that the condition (3.45) is satisfied) and the SSC-II (in the sense of fixing the expectation values of the basic field operators, and the values for the Newtonian potential and its first time derivative at the collapsing time, see equation (3.43)). The resulting “space-time” turns out to be described by a continuous but not smooth metric (there is a jump in the extrinsic curvature on the matching hypersurface), and, as such, the result is not truly a space-time. In fact, we already knew that a jump associated with a quantum collapse would imply that since, at the corresponding space-time points, Einstein’s semiclassical equations would not hold, simply because of the fact that for any smooth space-time the Bianchi identity implies the vanishing of $\nabla_{\mu}G^{\mu\nu}$, while the expectation value of the energy momentum tensor would quite generally not be divergence-less during the jump of the quantum state [17]. This is certainly an unappealing aspect of the formalism, and this is one of the reasons for which we can not take this as a complete and satisfactory description of the problem at hand.

There are some other aspects of the proposal that seem unsatisfactory, with one of the most problematic being the issue of general covariance. It is evident that the association of a collapse with a particular space-like hypersurface brings up the very issue that is usually cause of grave concern regarding the compatibility with special relativity of any theory involving an instantaneous reduction of the wave function. At this point, we should mention a related (but different) issue of gauge dependence (or independence) of the proposal. As this issue has led to considerable confusion we have devoted the appendix C and turn the reader to it for a careful discussion. Turning back to the character of the collapse hypersurface, the explicit analysis we have presented here ascribes to such particular space-like hypersurface a very particular physical role: it separates the space-time region which is perfectly homogeneous and isotropic from the one where a particular kind of anisotropy has set in. It is then clear
that the selection of such hypersurface is not a simple gauge choice, but it corresponds to part of the characterization of the proposal regarding the collapse process itself.

On the other hand, the evident tension between the existence of space-like hypersurfaces with particular physical properties and relativistic ideas is clearly a very serious issue, as it seems to imply the physical breakdown of cherished and well established principles, and as such, the problem can certainly be grave. Of course, this was not unexpected, as one of the most troublesome aspects of the notion “collapse of the wave function” is precisely that it seems intrinsically associated with a global and instantaneous process, and at this point we can only hope that the idea might be eventually reconciled with the principles of relativity. This issue is in a sense close to the core of the famous EPR gedankenexperiment, and of course the modern developments including its experimental realization. The problem is thus not hopeless by any means, but, of course, the discussion of alternatives is well beyond the scope of the present paper. It is nevertheless worth mentioning that our view in this regard is that the novel physical process is associated and triggered by global conditions, and as such it might be both instantaneous and covariantly defined, but of course it would need to have features that prevent it from being used (and here the important word is used, which implies the possibility of external manipulation by conscious beings) to send information faster than light. It is worth noting that there are various works suggesting that non-locality might play an even more fundamental role in a complete theory than that it plays in standard quantum theory (see for instance [69–72]). We should, however, keep in mind that this is meant only as an effective description of limited validity, and that a truly developed theory of quantum gravity can naturally be expected to be needed in order to obtain something completely satisfactory. It is perhaps also worth pointing out that the issue of compatibility with Lorentz invariance is a difficulty that seems to emerge often in connection with bringing together the quantum theory and gravitation, such as in loop quantum gravity [73–77] (similar problems appear also in the so called Liouville approach to string theory [78, 79]), and that in those cases, it often provides important constraints that have not been dealt with in full yet [80, 81].

Perhaps, a simple analogy would be helpful in order to convey what is precisely what we have in mind. Let us imagine for a moment that we do not have a mathematical description of curved surfaces (say, for concreteness, 2-surfaces embedded in 3-dimensional Euclidean space), and that we only know how to characterize planes. Let us assume that we want to describe a certain smooth rock. Clearly we would not be able to do that unless the rock was completely flat. However, we can obtain what would be for most purposes a reasonable description of the rock by wishing a large number of tangent planes and indicating that the rock is what lies within the volume that the planes define (adding all other relevant information about where certain planes would end, etc). It is clear that at the intersection of the planes we would have certain singular behaviour (at those points we would not have a unique normal characterizing the rock), but we should not be surprised by that. We know that although the sharp vertexes associated with the intersections are not to be taken seriously, many other features (like for instance the volume of the rock) can be reasonably expected

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9We should mention here the ideas of Rovelli regarding a relational wave function [66], the proposals designed to make the collapse models applicable to field theories, which assume that the collapse is triggered at a single event and then propagates on the past light cone [67], or the analysis in [68].

10Relativity requires the laws of physics to be the same in any reference frame, but that of course does not prevent the existence of special frames associated with a particular state of a physical system. Say a body determines a spacial frame in which it is at rest.
to be well characterized with our rough description. Moreover, one might even hope that perhaps such limited description could be the starting point motivating the development of the differential geometry of smooth surfaces. It is our hope that by attempting to push the well developed theories of physics (quantum field theory on curved space-time and classical general relativity) to the limit in an attempt to describe the situations of interest in inflationary cosmology (which is, as we already noted, the only regime where both quantum theory and gravitation come together in dealing with situations that are observationally accessible to us) can serve not only to push further our understanding of the specific situation, but it would be also useful in some manner to continuing investigations on the quantum gravity realm. It seems clear to us that if the collapse of the wave function is a required modification of the quantum theory brought by the quantum gravitational effects as it is suggested by Doisi and Penrose, this would be the case.

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A The collapse within our current understanding of physical theory

Let us try to frame our proposal, even if only schematically, within the general current understanding of physical theory. The basic idea underlying our current considerations, and which was initially proposed in [18], is to connect the problem at hand to that encountered when trying to write a theory of quantum gravity through the canonical quantization procedure. As it is well known, when following approaches of this kind, such as the old Wheeler de Witt proposal [64], or its more modern incarnation in the form of loop quantum gravity [40], one ends up with an atemporal theory. This is known as “the problem of time in quantum gravity” [84]. That is, in both schemes one starts with a formulation in which the basic canonical variables describe the geometry of a 3-spatial hypersurface $\Sigma$, and characterize the embedding of this 3-surface in a 4-dimensional space-time. From those quantities one is led to the identification of a set of canonical variables, which we will denote here generically by $(G, \Pi)$.

(In the Wheeler de Witt case this stands for the spatial metric $h_{\mu\nu}$ and a certain function of the extrinsic curvature $K_{\mu\nu}$, i.e. $(h_{\mu\nu}, K_{\mu\nu})$, while in loop quantum gravity these will be the densitized triad $E^i_\mu$ and connection $A^i_\mu$ variables, i.e. $(E^i_\mu, A^i_\mu)$.) The problem is that time, or its general relativistic counterpart (a time function usually specified by the lapse function and the shift vector) simply disappears from the theory, given that the Hamiltonian vanishes when acting on the physical states (those satisfying the diffeomorphism and Hamiltonian constraints).

The problem is then how would one recover a space-time description of our world, clearly an essential element one would need in order to be able to connect the theory with observations. One of the most favoured approaches towards addressing this problem is to consider, simultaneously with the geometry, some matter fields, which we will describe here schematically by a collection of ordered pairs of canonically variables,

$$\{(\varphi_1, \pi_1), \ldots, (\varphi_n, \pi_n)\},$$

(A.1)
and to identify an appropriate variable (or combination of variables) in the joint matter gravity theory that could act as a physical clock $T(\varphi, \pi, G, \Pi)$. The next step consists of characterizing the state for the remaining variables in terms of the correlations of their values with those of the physical clock. That is, one starts from the wave function for the configuration variables of the theory $\Phi(\varphi_1, \ldots, \varphi_n, G)$, which must satisfy the so called Hamiltonian and momentum constraints $H_\mu \Phi(\varphi_1, \ldots, \varphi_n, G) = 0$. Next, one needs to obtain an effective wave function $\Psi$ for the remaining variables by projecting $\Phi$ into the subspace where the operator $T(\varphi, \pi, G, \Pi)$ takes a certain range of values. That is, let us denote by $P_{T,[t,t+\delta t]}$ the projector operator onto the subspace corresponding to the region between $t$ and $t + \delta t$ of the spectrum of the operator $T$. One then attempts to recover a Schrödinger-like evolution equation by studying the dependence of $\Psi(t) \equiv P_{T,[t,t+\delta t]} \Phi$ on the parameter $t$. (In the general relativistic setting this would be a global time function $T$ defined on the reconstructed approximate space-time.) After obtaining, by the above procedure, a wave function associated with the spectrum of the operator $T$ we might use it to compute the expectation values of the 3-dimensional geometrical operators (say the triad $E^\mu_i(x)$ and connection $A^\mu_i(x)$ variables of Ashtekar, or some appropriate smoothing thereof) for the wave function $\Psi(t)$. Such collection of quantities could be seen as providing the geometrical descriptions of the “average” space-time in terms of the $3+1$ decomposition. In other words, one would have constructed a space-time where the slicing would correspond to the hypersurfaces on which the geometrical quantities are given by the expectations of the projected wave functions $\Psi(t)$, and thus one would be able to characterize the space-time and its slicing in terms of the lapse and shift functions.

The precise realization of this procedure depends strongly on the situation and specific theory for the matter fields one is considering, and such study is quite beyond the scope of the present paper, among other reasons because we do not have at this point a satisfactory and workable theory of quantum gravity. On the other hand, several works along these lines exist in the literature [85]. The point we want to make here, however, is that in such a setting, the standard Schrödinger equation emerges only as an effective description, and it is only approximately valid. Under these circumstances, small modifications to that equation would not be unexpected. In fact, in a recent analysis [86] of a quantum mechanical system, it was found that describing it in terms of the time measured by a physical clock, rather than an idealized one, implied modifications representing departures from the quantum mechanical unitary evolution. We consider that this could be the grounds where a modification of the Schrödinger evolution, involving something akin to a collapse of the wave function, might find its ultimate explanation. There are, indeed, several proposals for such a modification centering on the analysis of standard laboratory situations [26, 28–33]. Moreover, it should be noted that with a paradigm where the quantum jumps occur generically and spontaneously, rather than being thought as triggered by the decisions of observers to measure particular quantities, one might avoid the kind of problem discussed in [37] (see also the idea of the objective wave function reduction developed by R. Penrose in chapter 29 of reference [19]).

### B Connection with other approaches

Let us compare the general formalism introduced in section 2 with the stochastic gravity proposal by Hu and Verdaguer [46]. The idea behind that approach to semiclassical gravity is to attempt to take into account the “fluctuating part of the energy momentum tensor” of the (quantum) mater fields through the introduction of a stochastic field $\chi_{\mu\nu}(x)$. The
The proposal then assumes that the ensemble statistics of the stochastic term are characterized by a certain measure of the uncertainties of the energy momentum tensor. Let us show that one of the instantaneous collapses, occurring say at \( t = t_c \), could be seen as corresponding in the above scheme to a particular contribution to the stochastic field at that particular time, the time of collapse. As usual in quantum field theory we use the Heisemberg picture. However we assume that the state of the field is not constant in time, but that as a result of the collapse process (which we will be considering to occur instantaneously), it "jumps". Thus, the state of the field is described by \[ |\psi(t)\rangle = \theta(t - t_c)|\xi\rangle + \theta(t - t_c)|\zeta\rangle, \] where \( \theta(\cdot) \) is the step function (it is 0 when the argument is negative and 1 when it is positive).

Einstein equations would then be given by

\[
G_{\mu\nu}(x) = 8\pi G \left( \langle \hat{T}_{\mu\nu}(x) \rangle + \chi_{\mu\nu}(x) \right),
\]

where the stochastic term reflecting the "jump" takes the form \( \chi_{\mu\nu} \equiv \theta(t - t_c)(\langle \xi|\hat{T}_{\mu\nu}|\xi\rangle - \langle \zeta|\hat{T}_{\mu\nu}|\zeta\rangle) \). As we have indicated the exact relationship between the formalism developed in this work and the different approaches is outside of the scope of the present paper, but the above considerations indicate the existence of a relatively close connection. In future works we will explore such connections more closely in order to, on the one hand, use the formalism to better understand those proposals, and on the other hand to consider the possibility of incorporating the inflationary issue that has motivated this and previous works within the context of such theories and proposals.

C Gauge conditions, change of variables and all that

In mathematical physics space-time is characterized by a differential manifold \( M \) with a type \((0,2)\) tensor field \( g \) defined on it. This characterization is independent of the coordinates. On the same manifold we can have other fields (of scalar, vector or tensor nature), denoted generically by \( \psi \) and representing matter. Again such characterization is independent of the choice of coordinates. Modification of coordinate choices can never mix up different fields. When we choose some specific coordinates, \((x^\mu)\), the metric can be written in their components, using for instance the basis of one forms naturally associated with that coordinate chart, \( g = g_{\mu\nu}dx^\mu dx^\nu \). This is a tensor of type \((0,2)\) built with tensor products of two one-forms. A change of coordinates will change the components \( g_{\mu\nu} \). However, the metric as a mathematical object will not change.

Now let us see when and how the issue of "gauge" appears, and how it often leads to confusion. Consider a situation where we have two space-times with metric and matter fields defined on them, \((M,g,\psi)\) and \((\tilde{M},\tilde{g},\tilde{\psi})\), and assume we want to compare one with the other. For that we need to use some diffeomorphism (which will only exist when the two differential manifolds are diffeomorphic), \( F : M \rightarrow \tilde{M} \), mapping one manifold to the other. Then one might want to consider the differences in metric and fields by looking at, say, \( \delta g \equiv \tilde{g} - F^*(g) \) and \( \delta \psi \equiv \tilde{\psi} - F^*(\psi) \). When doing this, the result will evidently depend on the choice of \( F \). This is what is often done when considering perturbations from, say, a homogeneous and isotropic space-time to one that deviates from the former by a small amount. This is the setting often used to considerations involving inflationary cosmology.
There \((M, g, \psi)\) is taken as the “homogeneous and isotropic background”, and \((\tilde{M}, \tilde{g}, \tilde{\psi})\) the situation representing somehow our universe. This is not an issue of coordinates, but it can be confused with one. The fact is, however, that the issue is customarily solved by choosing a “gauge” that effectively uses the symmetries of the first space-time to determine (to the desired perturbative order) the diffeomorphism \(F\). In an alternative approach, which is often employed in cosmology, one considers suitable combinations of certain components of \(\delta g\) and \(\delta \psi\) (associated with suitable coordinate choices), looking for combinations that are invariant under “small changes of \(F\)”: these are the so called “gauge invariant perturbations”.

The approach based on the use of gauge invariant quantities can be quite useful in some calculations, but often can make things a bit more difficult when discussing interpretational aspects. We can see this by noting that what is observed using our satellites is often described using coordinates, such as the angular coordinates on the celestial sphere to characterize the CMB, just to give one example. Moreover, the fact that in our treatment the matter fields and the space-time metric are so clearly distinct forces us to avoid a gauge invariant treatment and work with the approach based on fixing the gauge. In this paper we have chosen to work in the so called “Newtonian” (sometimes also known as “longitudinal” or “conformal-Newtonian”) gauge, introduced for instance in chapter 9.2 of reference [5].

In the particular situation we want to consider in this paper we have a scalar field living on a “space-time” which is the result of gluing together two pieces. The first piece is the region characterized by \(\eta < \eta_c\) of the SSC-I, i.e. a perfectly homogeneous and isotropic space-time with the scalar field in a state where only the zero mode (which is also homogeneous and isotropic) is excited. This is described in detail in section 3.1. The second piece is the region \(\eta > \eta_c\) of the construction corresponding to a slightly inhomogeneous and anisotropic space-time, with a characteristic wave vector \(\vec{k}_0\) and a scalar field in a state where there is a nontrivial excitation not only of the zero mode, but also of several other modes \(\vec{k}_0, 2\vec{k}_0, 3\vec{k}_0, \ldots\). This is described in detail in section 3.2. The two pieces are glued together at the hypersurface \(\Sigma_c\), corresponding to the regions with coordinates \(\eta = \eta_c\) in each of the two pieces. The matching of the two pieces makes up a space-time describing the emergence (in the traditional sense of the word: i.e. something that was not there “at a given time” is there “at a latter time”) of perturbations. This is described in section 3.3. Regarding the coordinates, we consider the whole manifold to be covered by a single coordinate chart, \((\eta, x^1, x^2, x^3)\). The construction we have given describes something that is almost a space-time (i.e. is a space-time except for the fact that the extrinsic curvature is not continuous in the hypersurface \(\Sigma_c\)), and is, in the mathematical sense, analogous to the formalism employed in considering infinitely thin matter shells (see for instance [82]), and are thus not realistic in the very same sense. We expect that in a more realistic description the shells would have a finite thickness, and that the collapse hypersurface would perhaps have some small (but finite) temporal extent. The similarity breaks down in the fact that, in the case of the thin shells, we have a workable and well defined theory capable of treating the problem to any desired degree of accuracy. In the situation at hand, the collapse is expected to be described by some theory which does not exit yet, and as we have argued, such theory would probably trace its origin to the quantum gravity regime. Moreover, as we have discussed in section 2, we would expect that the characterization of any collapse theory in terms of space-time language would only be achievable once the fundamental degrees of freedom for the gravitational interaction have been given an approximate classical description. We note here that if we wanted to change coordinates, the hypersurface \(\Sigma_c\) would in general be described in the new choice by some complicated function characterizing for instance its “time coordinate” in terms of the other
three coordinates. Thus our “almost” space-time would be the same but described in a more complicated form. The construction has been carried out in a specific gauge, but the result is gauge independent. This is essentially the same when making some analysis in general relativity using some quite specific coordinates, say studying the perihelion of Mercury using Schwarzschild coordinates, one needs not worry that the result is coordinate dependent.

There is, on the other hand, a very different issue that might be confused with that of gauge freedom. It is connected to the question: what is the physics that triggers the collapse, and how does that mechanism determine that the surface where it should occur is the hypersurface $\eta = \eta_c$? We of course do not know the answer, simply because we do not have a well developed and workable theory of quantum gravity, and much less a theory of collapse. The present work is only the first step in the development of a well defined formalism that we hope could be useful in obtaining well defined but parametrized predictions that might be compared with observations (as done for intense in [20]), and as a result would help us learn something about the physics of collapse. The only thing we can say at this point is that, if something like the value of the uncertainties in the quantum state of the matter fields is connected with triggering of collapse (as it would be for instance in a theory of collapse based on Penrose’s ideas), then the fact that the region described by the SSC-I is completely homogeneous leads us to conclude that these local uncertainties would reach the same level exactly at the same value of $\eta$, and thus it would be natural to expect that the collapse would be associated with the corresponding hypersurface. But this is of course just “educated” speculation in the absence of a detailed theory of collapse.

Another source of confusion comes from the use of conformal transformations and the subsequent mixing of variables. One might be concerned with the fact that it seems always possible to move part of the degrees of freedom from the metric to the matter fields (and back) through a conformal transformation, and therefore the split between what ‘should’ and ‘should not’ be treated at the quantum level would be completely artificial, and thus intrinsically arbitrary [83]. (Given a space-time metric $g$, one can introduce a new metric $\tilde{g}$ and a new scalar field $\chi$ related to the former metric by $g = \chi^2 \tilde{g}$, and then regard $\chi$ as a matter field. That is, one might be in doubt as to which of the two metrics one should describe at classical level.) We do not share such point of view, simply because it is based on a classical treatment, where indeed such conformal transformations are well defined and meaningful. Our view is that at a truly fundamental — and thus quantum — level, the space-time degrees of freedom are of a different nature than those of the matter degrees of freedom, and that at such level there would be no ambiguity whatsoever. Of course, at the practical level we must work without a satisfactory theory of quantum gravity, and the ambiguity would have to be resolved in some other way. We will take the view that the resolution comes simply from considering the physical space-time metric to be the one for which the corresponding geodesics are associated with the paths of the free particles, i.e. the metric to which the other fields are coupled in the minimal way.

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