A BOUQUET OF PSEUDO-ARCS

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Abstract. We prove the existence of a transcendental entire function whose Julia set is a “bouquet of pseudo-arcs”. More precisely, $J(f) \cup \{\infty\}$ is an uncountable union of pseudo-arcs, which are pairwise disjoint except at infinity.

The existence of such a function follows from a more general result of the second author, but our construction is considerably simpler and more explicit. In particular, the function we construct can be chosen to have lower order $1/2$, while the lower order in the previously known example is infinite.

1. Introduction

We consider the iteration of transcendental entire functions; i.e. of non-polynomial holomorphic self-maps of the complex plane. The subject was founded by Fatou in 1926 [Fat26]. In particular, he observed that the Julia sets of certain transcendental entire functions contained curves of escaping points, i.e., points whose orbits tend to infinity. Fatou asked whether this holds more generally. Eremenko made this question more precise in 1989, when he asked whether every escaping point of a transcendental entire function can be connected to infinity by a curve of escaping points. The latter question was answered in the negative in [RRRS11]. The counterexample belongs to a class of entire functions whose dynamics is of a particularly simple form.

1.1. Definition.
An entire function $f$ is of disjoint type if there is a bounded Jordan domain $D \subset \mathbb{C}$ such that $f(D) \subset D$, and such that

$$f : \mathbb{C} \setminus D \to \mathbb{C} \setminus D$$

is a covering map.

Equivalently, $f$ is hyperbolic with connected Fatou set; see [Rem16, Definition 1.1] and [BK07, Lemma 3.1]. We refer to [Rem16] for background on disjoint-type entire functions and their significance for wider classes of transcendental entire functions. If $f$ is a disjoint-type entire function, then its Fatou set consists of a single immediate attracting basin. In particular, the escaping set

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty\}$$

is contained in the Julia set $J(f)$. Thus the following theorem from [RRRS11] does indeed give a negative answer to Eremenko’s question.

1.2. Theorem ([RRRS11, Theorem 1.1]).
There is a disjoint-type entire function $f$ such that $J(f)$ contains no curve to infinity.
In [RRRS11] Theorem 8.4, the authors also sketch a proof of the following stronger result.

1.3. Theorem.
There is a disjoint-type entire function $f$ such that $J(f)$ contains no arc.

Every connected component $C$ of the Julia set $J(f)$ of a disjoint-type entire function $f$ is an unbounded closed connected set. Thus $\hat{C} := C \cup \{\infty\}$ is a compact connected set, which, following [Rem16], we call a Julia continuum. For the function in Theorem 1.3, each such continuum may be considered “pathological” in that it contains no arc, and it is natural to ask how complicated the topology of a Julia continuum may become. In particular, we may ask whether such a continuum may be hereditarily indecomposable, i.e., have the property that any two subcontinua are either nested or disjoint. Clearly a hereditarily indecomposable continuum contains no arcs.

A famous example of a hereditarily indecomposable continuum is the pseudo-arc (see Section 2). So we may ask, in particular, whether the pseudo-arc may arise as the Julia continuum of a disjoint-type entire function. This question was answered by the second author in [Rem16, Theorem 1.5].

1.4. Theorem.
There is a disjoint-type entire function $f$ such that every Julia continuum of $f$ is a pseudo-arc.

In fact, [Rem16, Theorem 2.7] shows that any continuum that arises as the inverse limit of a self-map of the interval below the identity (see [Rem16, Definition 2.6]) can be realised as a forward-invariant Julia continuum. Theorem 1.4 is then obtained as a corollary, using a classical theorem of Henderson [Hen64] that states that the pseudo-arc can be represented as such an inverse limit. The goal of this article is to give a simpler and more direct proof of Theorem 1.4 by directly applying some of the ideas from [Hen64]. This may also be the simplest proof of Theorem 1.3 yet. (Making the proof sketched in [RRRS11] precise would require a considerable amount of book-keeping and notation.) Since our approach is much more explicit, it offers the possibility of obtaining more detailed information about the function-theoretic behaviour of such a “pathological” function $f$. In particular, we show the following.

1.5. Theorem.
The function $f$ in Theorem 1.4 can be chosen to have lower order of growth $1/2$; that is,
\[
\liminf_{r \to \infty} \frac{\log \log M(r, f)}{\log r} = \frac{1}{2},
\]
where $M(r, f) = \max\{|f(z)| : |z| = r\}$.

In contrast, the construction in [Rem16] leads to functions of infinite lower order. Using the techniques of [Bis15a, Section 18] or [Rem16, Section 15], the function in Theorem 1.5 can also be constructed to have only two critical values and no finite asymptotic values.

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1In fact, it follows from [Rem16, Theorem 1.4] and [HO16] that any hereditarily indecomposable Julia continuum of a disjoint-type function is a pseudo-arc.
1.6. Theorem (Pseudo-arc bouquets in the Speiser class).

The function $f$ from Theorems 1.4 and 1.5 can be chosen such that $f$ has exactly two critical values and no finite asymptotic values, and such that there is $D > 1$ such that all critical points are of degree at most $D$.

Let us also note a consequence of Theorem 1.6. In polynomial dynamics, local connectivity of Julia sets is an important and much-studied property. The reason for this is that, when $J(f)$ is locally connected, the topological dynamics of the polynomial $f$ can be completely described in combinatorial terms; compare e.g. [Dou93]. Local connectivity of Julia sets has also been studied for transcendental entire functions. Of course, when $J(f) = \mathbb{C}$, the Julia set is locally connected, but this does not imply that the dynamics is simple! The following example shows that, even when $J(f) \neq \mathbb{C}$, local connectivity of the Julia set does not imply simple dynamics.

1.7. Corollary (Locally connected Julia sets with pseudo-arcs).

There exists a transcendental entire function $f$ with the following property.

(a) $J(f)$ is a Sierpiński carpet, and in particular locally connected.
(b) $J(f)$ contains an infinite collection $P_1, P_2, \ldots$ of pairwise disjoint unbounded connected invariant sets such that $P_j \cup \{\infty\}$ is a pseudo-arc for all $j$.
(c) $J(f)$ contains an uncountable collection $P$ of pairwise disjoint unbounded connected sets such that, for all $P \in P$, $P \cup \{\infty\}$ is a pseudo-arc and $f(P) \in P$.

Remark 1. This phenomenon is also remarked upon (without proof) in [BFRG15, Paragraph after Corollary 1.9] and [Rem16, Discussion after Theorem 2.11].

Remark 2. Similar phenomena are known for rational maps, where it is possible e.g. that the Julia set is a Sierpiński carpet, while the map may have a Cremer fixed point; see [Roe08]. In contrast, our example is hyperbolic (see Section 10).

Let us outline the basic idea of the various constructions mentioned above, as well as ours, and comment on the differences between them. The connected components of the set $V$ as in Definition 1.1 are called the tracts of $f$; each such tract $V$ is simply connected and mapped by $f$ as a universal covering. The proofs of the above-mentioned theorems from [RRRS11] and [Rem16] – and also our proof of Theorems 1.4 and 1.5 – proceed by first constructing a suitable simply-connected domain $T$ with

(1.1) \[ T \subset \mathbb{H} := \{x + iy: a > 0\} \]

and a conformal isomorphism

\[ F: T \to \mathbb{H}, \]

where $T$ is disjoint from its $2\pi i \mathbb{Z}$-translates. The map $F$ is constructed so that the universal covering map \[ f: \exp(T) \to \exp(\mathbb{H}) = \{z \in \mathbb{C}: |z| > 1\}; \quad f(\exp(\zeta)) = \exp(F(\zeta)) \]

and its “Julia set”

\[ J(f) := \{z \in \mathbb{C}: f^n(z) \text{ is defined and of modulus } \geq 1 \text{ for all } n \geq 0\} \]
Figure 1. The tract used in the proof of Theorem 1.2; reproduced from [RRRS11, Figure 5].

Figure 2. The tract used in the proof of Theorem 1.3; reproduced from [RRRS11, Figure 8].

have the desired properties. Then an approximation result (see e.g. Theorem 3.8 below) is applied in order to obtain a disjoint-type entire function $g$ for which $J(g)$ is homeomorphic to $J(f)$.

The tract $T$ used for the proof of Theorem 1.2 consists of a long straight half-strip, into which a countable number of “wiggles” are inserted. Here by a wiggle, we mean a winding strip that first increases to a large real part $R$, decreases back down to a (much) smaller real part $r$, and finally begins to grow again. See Figure 1. By thickening the intermediate straight pieces, the corresponding function can be made to have lower order 1/2; see [RRRS11, Proposition 8.3] and [Rem13, Theorem 1.10]. The construction in the proof of Theorem 1.3, which is only sketched in [RRRS11], is considerably more complicated. Here, the “wiggles” are each made up of further “subwiggles” (see Figure 2). This construction is iterated to a greater and greater depth the further to the right of the tract these wiggles are inserted. The proof of Theorem 1.4 in [Rem16] uses an even more elaborate construction; see [Rem16, Figure 9].
In contrast, our proof of Theorem 1.4 in this paper can be made to work using exactly the same type of tract as in Figure 1 (naturally, with different choices of where to place the “wiggles”). We use a slight modification, also used in Bis15a, Figures 42 and 43, to ensure lower order $1/2$; see Figure 3.

**Structure of the paper.** In Section 2, we collect necessary background concerning arc-like continua and pseudo-arcs. We also give a brief overview of the history of the pseudo-arc. Section 3 introduces the class of conformal isomorphisms that are used as models for our construction, and discusses their fundamental properties. In Section 4, we introduce the one-dimensional projection $\varphi$ of such a function $F$, a crucial tool for our proof that encodes the essential mapping behaviour of the inverse $F^{-1}$ via a (usually non-injective) function of one real variable. Section 5 gives a sufficient condition on the one-dimensional projection to ensure that all Julia continua are pseudo-arcs. The remainder of the paper is dedicated to constructing a conformal isomorphism with these properties. In Section 6, we consider the mapping properties of the projection $\varphi$ further.

Section 7 sets up our main construction, which is inductive. The idea is that we start with a straight half-strip, and inductively continue inserting additional wiggles into this tract, which cause more and more “crooked” mapping behaviour of the one-dimensional projection. In this way, we obtain a sequence of simply connected domains $T_n$, and associated functions $F_n: T_n \to \mathbb{H}$. A key fact in the construction is that the map $F_{n+1}$ will be close to $F_n$, as long as the new wiggle is inserted far enough to the right. Finally, the proofs of our main theorems are carried out in Section 9.

**Notation.** As usual, $\mathbb{C}$ and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the complex plane and the Riemann sphere, respectively. Recall from (1.1) that $\mathbb{H}$ is the right half-plane. Euclidean distance, diameter and length are denoted by $\text{dist}$, $\text{diam}$ and $\ell$, respectively. The open Euclidean disc with centre $z_0$ and radius $r$ is denoted

$$D(z_0, r) := \{ z \in \mathbb{C} : |z - z_0| < r \}.$$ 

The closure in $\mathbb{C}$ of a set $X \subset \mathbb{C}$ is denoted $\overline{X}$; the closure of $D(z_0, r)$ is denoted $\overline{D}(z_0, r)$.

## 2. Background from continuum theory

A *continuum* $X$ is a non-empty compact connected metric space. The continuum $X$ is *non-degenerate* if it contains more than one point. We refer to Nad92 for an introduction to continuum theory.

**2.1. Definition** (Arc-like continua).

A continuum $X$ is *arc-like* if, for every $\varepsilon > 0$, there exists a surjective continuous map $g: X \to [0, 1]$ such that

$$\max_{t \in [0, 1]} \text{diam}_X(g^{-1}(t)) \leq \varepsilon.$$ 

Such $g$ is called an $\varepsilon$-map.

**Remark.** Here $\text{diam}_X$ denotes the diameter with respect to the given metric on $X$. Since $X$ is compact, equivalent metrics on $X$ give rise to the same notion of arc-likeness, so that being arc-like is a topological property. All continua considered in this paper are subsets of the Riemann sphere, and are endowed with the usual spherical metric.
The second notion from continuum theory that we use in this paper is that of hereditarily indecomposable continua.

2.2. Definition (Hereditarily indecomposable continua).
A continuum $X$ is hereditarily indecomposable if, for any two subcontinua $X_1, X_2 \subset X$ with $X_1 \cap X_2 \neq \emptyset$, either $X_1 \subset X_2$ or $X_2 \subset X_1$.

Knaster [Kna22], in his thesis, was the first to construct an example of a hereditarily indecomposable continuum. Moise [Moi48] answered a long-standing open question of Mazurkiewicz by constructing a hereditarily indecomposable continuum that is homeomorphic to each of its non-degenerate subcontinua. Moise called this continuum a pseudo-arc. Building on Moise’s construction, Bing [Bin48] showed that Moise’s pseudo-arc $X$ is homogeneous; i.e. for any pair of points in $X$ there is a homeomorphism of $X$ to itself that maps one to the other. This answered a long-standing question of Knaster and Kuratowski.

All these examples are arc-like by construction; Moise himself remarked that his construction was very similar to Knaster’s and asked whether the two continua are homeomorphic. Bing [Bin51] gave a positive answer by showing that any two arc-like hereditarily indecomposable continua are homeomorphic. Thus we use the following definition.

2.3. Definition (Pseudo-arcs).
A pseudo-arc is a hereditarily indecomposable arc-like continuum.

We refer to [Lew99] for further background on the pseudo-arc.

3. A family of disjoint-type models

As in [RRRS11] and [Rem16], we construct our examples by approximating suitable models, constructed in logarithmic coordinates. More precisely, these models have the following general form. (See Section 7 for the specific functions used in our construction.)

3.1. Definition (Class $\mathcal{H}$).
We denote by $\mathcal{T}$ the set of all simply-connected domains

$$T \subset S := \{x + iy : x > 4, |y| < \pi\}$$

with $5 \in T$, where

(a) $T$ is unbounded, and $\partial T$ is locally connected;

(b) there is only one access to infinity in $T$; i.e., any two curves connecting the same finite endpoint to infinity in $T$ are homotopic.

If $T \in \mathcal{T}$, then there is a unique conformal isomorphism $F : T \to \mathbb{H}$ with $F(5) = 5$ that extends continuously to $\infty$ with $F(\infty) = \infty$. We denote the class of all such functions $F$ by $\mathcal{H}$, and the domain of $F \in \mathcal{H}$ by $T(F)$.

For $\nu > 0$, we also consider the subclass $\mathcal{H}_\nu$ of $F \in \mathcal{H}$ such that

$$\text{diam}(F^{-1}(\{w \in \mathbb{H} : |w| = R\})) \leq \nu$$
for all \( R > 0 \). In other words, the geodesics of \( T \) that are perpendicular to \( F^{-1}((0, \infty)) \) have uniformly bounded Euclidean diameter. We will call these the “vertical” geodesics of \( T \) associated to \( F \).

### 3.2. Remark.

Let \( T \in \mathcal{T} \); we justify the existence of the conformal isomorphism \( F \) in the definition. There is a conformal isomorphism \( F : T \to \mathbb{H} \) satisfying \( F(5) = 5 \), and this map is unique up to postcomposition by a Möbius transformation of \( \mathbb{H} \) fixing 5. By the Carathéodory-Torhorst theorem, \([\text{Pom92, Theorem 1.7}]\), \( F^{-1} : \mathbb{H} \to T \) extends continuously to \( \mathbb{H} \cup \{ \infty \} \).

Property (b) in the definition of \( T \) implies that there exists exactly one point \( \zeta \in \partial \mathbb{H} \cup \{ \infty \} \) such that \( F^{-1}(\zeta) = \infty \); by postcomposing \( F \) with a Möbius transformation, we may assume that \( \zeta = \infty \), which makes \( F \) unique. Since \( \mathbb{H} \cup \{ \infty \} \) is compact, it follows that \( F \) itself extends continuously to infinity. (However, unless \( T \) is a Jordan domain, \( F \) does not extend continuously to all finite boundary points.)

In order to estimate functions \( F \in \mathcal{H} \), we frequently use that conformal isomorphisms are isometries of the corresponding hyperbolic metrics; compare \([\text{BM07}]\) for background on plane hyperbolic geometry. The density \( \rho_{\mathbb{H}} \) of the hyperbolic metric on the right half-plane is given by

\[
\rho_{\mathbb{H}}(\zeta) = \frac{1}{\Re \zeta}.
\]

Furthermore, by monotonicity of the hyperbolic metric, the density \( \rho_{T} \) of any tract \( T \in \mathcal{T} \) is bounded below by the density \( \rho_{\tilde{S}} \) of the bi-infinite strip

\[
\tilde{S} := \{ a + ib : |b| < \pi \} \supset S \supset T.
\]

So

\[
\rho_{T}(z) \geq \rho_{\tilde{S}}(z) = 1/(2 \cos(\Im z/2)) \geq 1/2
\]

for all \( z \in T \). (In fact, \( \rho_{T}(z) \geq 1/2 \) holds for any simply-connected domain \( T \) that is disjoint from its \( 2\pi i \mathbb{Z} \)-translates, even without the assumption that \( T \subset \tilde{S} \). Compare \([\text{Rem21, Corollary 2.2}]\).) In particular, functions in \( \mathcal{H} \) uniformly expand the Euclidean metric.

### 3.3. Lemma (Expansion for \( F \in \mathcal{H} \)).

Suppose that \( F \in \mathcal{H} \). Then \( |F'(z)| \geq \Re F(z)/2 \) for all \( z \in T \). In particular,

\[
|F'(z)| \geq 2
\]

for all \( z \in T \) with \( \Re F(z) \geq 4 \).

**Proof.** (See also \([\text{Rem21, Lemma 2.1}]\).) \( F \) is an isometry between \( T \) and \( \mathbb{H} \) with their respective hyperbolic metrics, so

\[
|F'(z)| = \frac{\rho_{T}(z)}{\rho_{\mathbb{H}}(F(z))} \geq \frac{\Re F(z)}{2}.
\]

Given a function \( F \in \mathcal{H} \), we may extend \( F \) to a \( 2\pi i \)-periodic function

\[
\tilde{F} : T + 2\pi i \mathbb{Z} \to \mathbb{H}.
\]
Note that $\hat{F}^{-1}$ has countably many branches, defined by
$$\hat{F}_s^{-1}(z) := F^{-1}(z) + 2\pi is.$$  

We are interested in the set of points that remain in the half-plane under iteration of $\hat{F}$, and its decomposition into individual components according to symbolic dynamics.

3.4. Definition (Julia continua of $\hat{F}$).

Let $F \in \mathcal{H}$ and let $\hat{F}$ be the $2\pi i$-periodic extension of $F$. We define

$$J(\hat{F}) := \{ z \in \mathbb{H} : \hat{F}^n(z) \text{ is defined and belongs to } T + 2\pi i \mathbb{Z} \text{ for all } n \}.$$  

If $s = s_0 s_1 s_2 \ldots$ is a sequence of integers, we define

$$J_s(\hat{F}) := \{ z \in J(\hat{F}) : \hat{F}^n(z) \in T + 2\pi is_n \text{ for all } n \geq 0 \}.$$  

If $J_s(\hat{F}) \neq \emptyset$, we call $\hat{J}_s(\hat{F}) := J_s(\hat{F}) \cup \{ \infty \}$ a Julia continuum of $\hat{F}$.

The following is a special case of [Rem16, Proposition 7.6].

3.5. Lemma (Julia continua are arc-like).

Every Julia continuum of $\hat{F}$ is an arc-like continuum.

Proof. We can write $J_s(\hat{F})$ as a nested intersection of compact, connected sets as follows. For $n \geq j \geq 0$, define inductively $X^n_j := (S + 2\pi is_n) \cup \{ \infty \}$ and

$$X^n_{j+1} := \hat{F}_s^{-1}(X^n_j).$$

(Recall from (3.1) that $S$ is a half-strip containing $T$.) Each $X^n_j$ is a continuum, as the image of a continuum under a continuous function, and $X^n_{j+1} \subset X^n_j$. So

$$J_\infty(\hat{F}) = \bigcap_{j=0}^{\infty} X^n_j$$

is a nested intersection of continua, and thus also a continuum.

To see that $J_\infty(\hat{F})$ is arc-like, define

$$g_n : \hat{J}_\infty(\hat{F}) \to [4, \infty]; \quad z \mapsto \begin{cases} \text{Re } \hat{F}^n(z) & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

Then $g_n$ is a continuous function onto a non-degenerate compact interval. If $t$ is a finite point in the range of $g_n$, and $\zeta \in J_\infty(\hat{F})$ with $g_n(\zeta) = t$, then

$$\text{diam } g_n^{-1}(t) \leq \text{diam } \hat{F}^{-n}(\{ \text{Re } \zeta + it : (2s_n - 1)\pi \leq t \leq (2s_n + 1)\pi \}) \leq 2^{-n} \cdot 2\pi.$$ 

So the Euclidean diameter, and hence also the spherical diameter, of $g_n^{-1}(t)$ tends to zero uniformly in $t$ as $n \to \infty$, while $g_n^{-1}(\infty)$ consists of a single point. Therefore the continuum is arc-like. \[\Box\]

By expansion of $F$, and since $T$ is contained in a strip of bounded height, the real parts of points in the same Julia continuum eventually separate under iteration.
3.6. Lemma (Separation of orbits).
Let $F \in \mathcal{H}$, and let $z, w \in J(\hat{F})$ belong to the same Julia continuum. Then
\[ |\hat{F}^n(z) - \hat{F}^n(w)| \geq 2^n \cdot |z - w|. \]
In particular, $|\text{Re} \hat{F}^n(z) - \text{Re} \hat{F}^n(w)| \to \infty$.

Proof. It is enough to prove the result for $n = 1$; the general case then follows by induction. Connect $z_1 := \hat{F}(z)$ and $w_1 := \hat{F}(w)$ by a straight line segment $\gamma$. Then $(F^{-1})'(\zeta) \leq 1/2$ for all $\zeta \in \gamma$ by Lemma 3.3. Since $z$ and $w$ are in the same Julia continuum, we have $z - w = F^{-1}(z_1) - F^{-1}(w_1)$. Thus
\[ |z - w| = |F^{-1}(z_1) - F^{-1}(w_1)| \leq \ell(F^{-1}(\gamma)) \leq \frac{1}{2} |z_1 - w_1|. \]
\[ \square \]

We shall also use the following fact. If $F \in \mathcal{H}_\nu$, and we fix finitely many points in the same Julia continuum, then under a sufficiently large iterate of $\hat{F}$, all but one will eventually lie in a sector around the real axis. More precisely:

3.7. Lemma (Orbits enter a sector).
For every $\nu > 0$, there is $\delta > 0$ with the following property. Suppose that $F \in \mathcal{H}_\nu$, and $z, w \in T(F)$, with $|z - w| \geq \delta$, $|F(z)| \geq |F(w)| \geq 4$, and $|\text{Im} F(z) - \text{Im} F(w)| \leq 2\pi$. Then $\text{Re} F(z) \geq |\text{Im} F(z)| + 2\pi$.

Proof. Set $\delta := 2 \cdot (\nu + \log(2 + 3\pi/2))$ and let $z$ and $w$ be as in the statement of the lemma. Set $\tilde{z} := F^{-1}(|F(z)|)$ and $\tilde{w} := F^{-1}(|F(w)|)$. Since $F \in \mathcal{H}_\nu$, we have $|\tilde{z} - z| \leq \nu$, and likewise $|\tilde{w} - w| \leq \nu$. In particular,
\[ |\tilde{z} - \tilde{w}| \geq \delta - 2\nu, \]
and hence the hyperbolic distance in $T$ between $\tilde{z}$ and $\tilde{w}$ satisfies
\[ \tilde{\delta} := \text{dist}_T(\tilde{z}, \tilde{w}) \geq \delta - 2\nu = \log \left( 2 + \frac{3\pi}{2} \right); \]
by (3.2) and choice of $\delta$.

Since $F$ is a conformal isomorphism, the hyperbolic distance in $\mathbb{H}$ between $|F(z)|$ and $|F(w)|$ is also $\tilde{\delta}$, so
\[ |F(z)| = \exp(\tilde{\delta}) \cdot |F(w)| \geq \left( 2 + \frac{3\pi}{2} \right) \cdot |F(w)|. \]
Thus
\[ \text{Re} F(z) \geq |F(z)| - |\text{Im} F(z)| \geq (2 + 3\pi/2) \cdot |F(w)| - |\text{Im} F(z)| \]
\[ \geq 2 \cdot |\text{Im} F(w)| + 6\pi - |\text{Im} F(z)| \geq |\text{Im} F(z)| + 2\pi. \]
\[ \square \]

To recover our results in the form that they are stated in the introduction, we must pass from a conformal isomorphism $F \in \mathcal{H}$ to an entire function $f$ with the same mapping properties. This is made possible by the following theorem, which is a combination of results of Bishop [Bis15b] and the second author [Rem09], as explained in [Rem16, Theorem 3.5].
3.8. Theorem (Realisation of models).
Let $F \in \mathcal{H}$. Then there is a disjoint-type entire function $f$ such that every Julia continuum of $F$ is homeomorphic to a Julia continuum of $F$, and vice versa.

Proof. Let $\hat{F}$ be again the $2\pi i$-periodic extension of $F$. Let $\mathbb{H}_1$ denote the right half-plane $\{a + ib : a > 1\}$. The restriction of $\hat{F}$ to $\hat{F}^{-1}(\mathbb{H}_1)$ is a disjoint-type function in the class $B^p_\log$ as defined in Rem16 Definition 3.3. The claim follows from Rem16 Theorem 3.5. ■

4. The one-dimensional projection

Lemma 3.7 means that, for $F \in \mathcal{H}_\nu$, with some $\nu > 0$, the mapping properties of $F$ are essentially related to the behaviour of $F^{-1}$ on the real axis. (See Lemma 4.3.) This motivates the following definition.

4.1. Definition (One-dimensional projection).
Given $F \in \mathcal{H}$, we call

$$\varphi : [4, \infty] \to [4, \infty], \quad \varphi(t) := \text{Re } F^{-1}(t)$$

the one-dimensional projection of $F$.

4.2. Lemma (Properties of $\varphi$).
For every $F \in \mathcal{H}$, the one-dimensional projection $\varphi$ has the following properties.

(a) $|\varphi(t_1) - \varphi(t_2)| \leq \frac{|t_1 - t_2|}{2}$ when $t_1, t_2 \geq 4$.
(b) In particular, $\varphi(t) < t$ for $t \geq 5$, and $\varphi(t) < 6$ for $t \in [4, 5]$.
(c) $\varphi(t) < 5 + 2(\log t - \log 5)$ for all $t \geq 5$.
(d) In particular $\varphi(t) < t - 1$ for $t \geq 7$ and $\varphi(t) < t/2$ for $t \geq 15$.

Proof. If $t_1, t_2 \geq 4$, then by Lemma 3.3 we have $|(F^{-1})'(x)| \leq 1/2$ for $x \in [t_1, t_2]$. Hence

$$|\varphi(t_1) - \varphi(t_2)| = |\text{Re}(F^{-1}(t_1) - F^{-1}(t_2))| \leq |F^{-1}(t_1) - F^{-1}(t_2)| \leq \frac{|t_1 - t_2|}{2}.$$  

This proves (a). In particular, if $t \geq 5$, then $\varphi(t) - 5 \leq (t - 5)/2$, and therefore $\varphi(t) < t$.
Similarly, if $|t - 5| \leq 1$, then $|\varphi(t) - 5| \leq 1/2$. This proves (b).

Now let $t \geq 5$. By (3.2), and since $F(5) = 5$, we have

$$|F^{-1}(t) - 5| \leq 2 \text{dist}_{\mathbb{T}}(F^{-1}(t), 5) = 2 \text{dist}_{\mathbb{H}}(t, 5) = 2(\log t - \log 5).$$

Hence

$$\varphi(t) \leq 5 + |F^{-1}(t) - 5| \leq 5 + 2(\log t - \log 5).$$

This proves (c), and it is easy to check numerically that (d) follows. ■

We also need the following fact about the relation between $F^{-1}(z)$ and $\varphi(\text{Re } z)$, for points $z$ that are not necessarily on the real axis.

4.3. Lemma.
Let $z \in \mathbb{H}$ with $\text{Re } z \geq 4$ and $|\text{Im } z| \leq \text{Re } z + 2\pi$. Then

$$|\text{Re } F^{-1}(z) - \varphi(\text{Re } z)| \leq 6.$$
Proof. The assumption implies that
\[\text{dist}_{\mathbb{H}}(z, \Re z) \leq \left| \frac{\Im z}{\Re z} \right| \leq 1 + \frac{2\pi}{\Re z} \leq 1 + \frac{\pi}{2} < 3.\]
By (3.2),
\[|\Re F^{-1}(z) - \varphi(\Re z)| \leq |F^{-1}(z) - F^{-1}(\Re z)| \leq 2 \text{dist}_{\mathbb{H}}(F^{-1}(z), F^{-1}(\Re z)) = 2 \text{dist}_{\mathbb{H}}(z, \Re z) < 6. \]
\[\blacksquare\]

5. A sufficient condition for pseudo-arc continua

5.1. Definition (Quadruples).
By a quadruple, we always mean a tuple of four different real numbers \(\geq 9\). We denote such a quadruple in increasing order as \(Q = (A < B < C < D)\). We also define the size of the quadruple as
\[|Q| = \min(A - 5, B - A, C - B, D - C).\]
Following [Hen64], we use the following terminology.

5.2. Definition (Crooked iterates of \(\varphi\)).
We say that \(\varphi^k\) maps a closed interval \(I\) crookedly over the quadruple \((A < B < C < D)\), if \(\varphi^k(I) \supset [A, D]\), and furthermore the convex hull of \(I \cap \varphi^{-k}(B)\) intersects the convex hull of \(I \cap \varphi^{-k}(C)\).

Remark. The final condition means that two points of \(I \cap \varphi^{-k}(B)\) surround a point of \(I \cap \varphi^{-k}(C)\), or vice versa.

The following result gives a sufficient condition for all Julia continua of a function \(F \in \mathcal{H}_\nu\) to be a pseudo-arc. Compare [Hen64, Lemma 1].

5.3. Theorem (Sufficient condition for pseudo-arcs Julia continua).
Let \(F \in \mathcal{H}_\nu\), for some \(\nu > 0\). Suppose that there exists a constant \(K > 0\) such that the following property holds for all integer quadruples \(Q = (A < B < C < D)\) with \(|Q| \geq K\).
There exists \(k_0 = k_0(Q) \in \mathbb{N}\) such that, for every \(k \geq k_0\) and every compact interval \(I \subset [6, \infty)\) with \(\varphi^k(I) \supset [A, D]\), the iterate \(\varphi^k\) maps \(I\) crookedly over \(Q\).
Then every Julia continuum of the \(2\pi i\)-periodic extension \(\hat{F}\) is a pseudo-arc.

Remark. By [Rem16, Corollary 8.7], it would be enough to prove the result for the invariant set of the conformal isomorphism \(F\), i.e. for the Julia continuum of \(\hat{F}\) at address \(s = 000\ldots\). To remain self-contained, we instead give an argument for arbitrary Julia continua, avoiding the results of [Rem16, Section 8].

Proof. Suppose, by contradiction, that \(\hat{J}_s(\hat{F})\) is a Julia continuum, and that there are subcontinua \(Z^0, Z^1 \subset \hat{J}_s(\hat{F})\) whose intersection is non-empty, but such that neither is contained in the other. Set \(C := Z^0 \cup Z^1; C\) is a continuum by assumption. We define
\[Z_n^j := \hat{F}^n(Z^j) \subset T(F) + 2\pi is_n\]
and \(C_n := Z_n^0 \cup Z_n^1\).
Let \( j \in \{0, 1\} \) and let \( z_0 \in \mathcal{Z}^j \setminus \mathcal{Z}^{1-j} \). Suppose that \( \varepsilon > 0 \) is so small that
\[
\overline{D}(z_0, \varepsilon) \cap \mathcal{Z}^{1-j} = \emptyset.
\]
By Lemma 3.6
\[
\overline{D}(z_n, 2^n \cdot \varepsilon) \cap \mathcal{Z}_n^{1-j} = \emptyset,
\]
where \( z_n = \hat{F}^n(z_0) \). If \( 2^n > 2\pi/\varepsilon \), the disc in (5.2) contains a straight line segment of length \( 4\pi \) centred at \( z_n \), which separates the strip \( S \) into two parts, one to the left and one to the right of \( z_n \). In particular, \( \mathcal{Z}^{1-j} \) is either entirely to the left or entirely to the right of \( z_n \).

**Claim 1.** For sufficiently large \( n_0 \), and \( j = 0, 1 \), there are continua \( \hat{\mathcal{Z}}_{n_0}^j \subset \mathcal{Z}_{n_0}^j \setminus \{\infty\} \) such that \( \bigcap_{j=0}^1 \hat{\mathcal{Z}}_{n_0}^j \neq \emptyset \), such that \( \hat{\mathcal{Z}}_{n_0}^j \not\subset \mathcal{Z}_{n_0}^{1-j} \) for \( j = 0, 1 \), and such that additionally
\[
\text{Re} \hat{F}^n(z) \geq 2\pi|s_{n+n_0}| + \pi
\]
for all \( n \geq 0 \) and all \( z \in \hat{\mathcal{Z}}_{n_0}^0 \cup \hat{\mathcal{Z}}_{n_0}^1 \).

**Proof.** Pick two points \( z^0, z^1 \) with \( z^j \in \mathcal{Z}^j \setminus (\mathcal{Z}^{1-j} \cup \{\infty\}) \) and let \( \varepsilon_1 \) be so small that (5.1) holds for both \( z^0 \) and \( z^1 \). Let \( \hat{\mathcal{Z}}^j \) be a connected component of \( \mathcal{Z}^j \setminus D(z^j, \varepsilon_1) \) that contains a point of \( \mathcal{Z}^0 \cap \mathcal{Z}^1 \). By the boundary bumping theorem [Nad92] Theorem 5.6], \( \hat{\mathcal{Z}}^j \) intersects \( \overline{D}(z^j, \varepsilon_1) \), and in particular \( \hat{\mathcal{Z}}^j \not\subset \mathcal{Z}^{1-j} \).

Now pick a second point \( \tilde{z}^j \in \mathcal{Z}^j \cap D(z^j, \varepsilon_1) \setminus \{z^j\} \), for \( j = 0, 1 \). By Lemmas 3.6 and 3.7 there is \( n_0 \) such that
\[
\max(\text{Re} z^j_{n_0}, \text{Re} \tilde{z}^j_{n_0}) \geq \min(|\text{Im} z^j_{n_0}|, |\text{Im} \tilde{z}^j_{n_0}|) + 2\pi \geq 2\pi|s_{n_0}| + \pi,
\]
for \( j = 0, 1 \) and all \( n \geq n_0 \). We may additionally choose \( n_0 \) so large that
\[
2^n_{n_0} > \frac{2\pi}{\varepsilon_1 - \max|z^j - \tilde{z}^j|}.
\]
If \( z \in \hat{\mathcal{Z}}^0 \cap \hat{\mathcal{Z}}^1 \), then by (5.2), for each \( n \geq n_0 \), \( \hat{F}^n(z) \) is to the right of the left-hand side of (5.4) for either \( j = 0 \) or \( j = 1 \). Similarly, \( \hat{F}^n(z) \) is to the left of \( \text{Re} z^0_{n_0} \) or of \( \text{Re} z^1_{n_0} \). Hence the continua \( \hat{\mathcal{Z}}^j_{n_0} := F^{n_0}(\hat{\mathcal{Z}}^j) \) have the desired properties. \( \square \)

For simplicity of notation, we assume in the following that \( \mathcal{Z}^j \) were chosen to satisfy (5.3) to begin with, with \( n_0 = 0 \).

**Claim 2.** Let \( z_0 \in \mathcal{C} \) and set \( z_k := \hat{F}^k(z_0) \in C_k \) for \( k \geq 0 \). If \( 0 \leq k \leq n \), then
\[
|\varphi^{n-k}(\text{Re} z_n) - \text{Re} z_k| \leq 12.
\]

**Proof.** Set \( r_k := \varphi^{n-k}(\text{Re} z_n), k = 0, \ldots, n \); we prove the claim by induction over \( n - k \); the case \( k = n \) is trivial. Suppose the claim holds for some \( k \in \{1, \ldots, n\} \). We have
\[
|r_{k-1} - \varphi(\text{Re} z_k)| = |\varphi(r_k) - \varphi(\text{Re} z_k)| \leq 6
\]
by the inductive hypothesis and Lemma 4.2(a) By (5.3) we also have
\[
|\text{Im} z_k| \leq 2\pi|s_n| + \pi \leq \text{Re} z_k,
\]
so we may apply Lemma 4.3 and obtain that
\[
|\varphi(\text{Re} z_k) - \text{Re} z_{k-1}| \leq 6.
\]


Thus $z^0, z^1$ with $z^j \in \mathcal{Z} \setminus D$ and let $\varepsilon$ be again so small that (5.1) holds for both $z^0$ and $z^1$. Define $z^j_n := \hat{F}^n(z^j)$, and fix $k$ so large that $2^k > (2\pi + 25 + 3K)/\varepsilon$. Let $j \in \{0, 1\}$ be such that $\operatorname{Re} z^j_k < \operatorname{Re} z^{-j}_k$. We define an integer quadruple $Q := (A < B < C < D)$ by setting

$$A := \lceil \operatorname{Re} z^j_k + 12 + K \rceil; \quad B := A + K; \quad D := \lceil \operatorname{Re} z^{-j}_k - 12 \rceil; \quad C := D - K.$$  

By choice of $k$, we have

$$\operatorname{Re} z^j_k + 25 + 3K \leq \hat{F}^k(z)$$

for all $z \in \mathcal{Z}^{1-j}$. Similarly, $\mathcal{Z}^j_k$ is to the left of $\operatorname{Re} z^{-j}_k - 25 - 3K$. In particular, $|Q| \geq K$, all points in $\mathcal{Z}^{1-j}_k$ have real part greater than $B + 12$, and all points in $\mathcal{Z}^j_k$ have real part less than $C - 12$.

Let $n_0$ be as in the hypothesis for the quadruple $Q$, and set $n := k + n_0$. By Claim 2, and choice of $A$ and $B$, we have

$$\varphi^{n_0}(\operatorname{Re} z^j_n) \leq \operatorname{Re} z^j_k + 12 \leq A, \quad \varphi^{n_0}(\operatorname{Re} z^{-j}_n) \geq \operatorname{Re} z^{-j}_k - 12 \geq D.$$  

So if $I$ is the interval bounded by $\operatorname{Re} z^j_k$ and $\operatorname{Re} z^{-j}_k$, then $\varphi^{n_0}(I) \supseteq [A, D]$. By hypothesis, this means that $I$ contains either two elements of $I \cap \varphi^{-n_0}(B)$ that surround an element of $I \cap \varphi^{-n_0}(C)$, or vice versa. To fix our ideas, let us suppose the former, so there are $\zeta_1 < \omega < \zeta_2$ such that $\varphi^{n_0}(\zeta_1) = \varphi^{n_0}(\zeta_2) = B$, and $\varphi^{n_0}(\omega) = C$. (The opposite case is analogous.)

Let $\tilde{\zeta}_1$ be a point in $\mathcal{C}_n$ with real part $\zeta_1$, and similarly $\tilde{\zeta}_2$. Let $z_1, z_2$ be the corresponding preimages under $\hat{F}^{n_0} : \mathcal{C}_{n_0} \to \mathcal{C}_n$. By Claim 2,

$$\operatorname{Re} z_1, \operatorname{Re} z_2 \leq B + 12.$$  

Thus $z_1, z_2 \in \mathcal{Z}^j_k$, and hence $\tilde{\zeta}_1, \tilde{\zeta}_2 \in \mathcal{Z}^j_n$. Since $\mathcal{Z}^j_n$ is a continuum, it also contains a point $\tilde{\omega}$ with real part $\omega$. The point $w \in \mathcal{Z}^j_k$ with $\hat{F}^{n_0}(w) = \omega$ satisfies $\operatorname{Re} w \geq C - 12$ by Claim 2. This is a contradiction, since $\mathcal{Z}^j_k$ is to the left of $C - 12$. The proof of the theorem is complete. \hfill \Box

6. Mapping properties of the one-dimensional projection

Throughout this section, let $F \in \mathcal{H}$, and let $\varphi$ be its one-dimensional projection. We introduce some notions and concepts that are adapted from [Hen64].

6.1. Definition (Minimal maps to an interval).

Let $I = [A, D] \subset [6, \infty)$ be a closed interval, and let $Q = (A < B < C < D)$ be a quadruple.

Let $n \geq 0$. We say that a closed interval $J \subset [4, \infty]$ is mapped minimally over $I$ by $\varphi^n$ if $\varphi^n(J) \supseteq I$, and no smaller subinterval of $I$ has this property. In this case, we also say that $J$ is mapped minimally over $Q$.

For $n \geq 0$, we define

$$U_n(Q, \varphi) := U_n(I, \varphi) := \{J \subset [4, \infty] : J \text{ is mapped minimally to } I \text{ by } \varphi^n\}.$$
We also define
\[ \hat{\mathcal{U}}_n(Q, \varphi) := \{ J \in \mathcal{U}_n(Q, \varphi) : J \text{ is mapped crookedly over } Q \} \).

6.2. Lemma (Properties of the sets \( \mathcal{U}_n \)).

Let \( I = [A, D] \) and \( Q \) be as in Definition 6.1 and let \( n \geq 0 \).

(a) If \( \tilde{J} \) is a closed interval with \( \varphi^n(\tilde{J}) \supset I \), then \( \tilde{J} \) contains an element of \( \mathcal{U}_n(I, \varphi) \).

(b) If \( J = [a, d] \in \mathcal{U}_n(I, \varphi) \), then \( \varphi^n([a, d]) = [A, D] \) and \( [a, d] = \varphi^{-n}([A, D]) \cap \tilde{J} \).

(c) The interiors of intervals in \( \mathcal{U}_n \) are pairwise disjoint.

(d) If \( J \in \mathcal{U}_n(I, \varphi) \), then \( \varphi^n(J) = I \).

(e) If \( J \in \mathcal{U}_n(I, \varphi) \), then \( J \subset [6, \infty) \).

(f) No interval \( J \in \mathcal{U}_n(I, \varphi) \) contains an interval of \( \mathcal{U}_n(I, \varphi) \) for \( n \neq \hat{n} \).

(g) Each \( \mathcal{U}_n \) is finite.

(h) If \( J \in \mathcal{U}_n(I, \varphi) \), then \( \varphi^k(J) \in \mathcal{U}_{n-k}(I, \varphi) \) and \( J \in \mathcal{U}_k(\varphi^k(J)) \) for \( k = 0, \ldots, n \).

(i) In particular, for \( k \leq n \),
\[ \mathcal{U}_n(I, \varphi) = \bigcup_{J \in \mathcal{U}_{n-k}(I, \varphi)} \mathcal{U}_k(J, \varphi). \]

(j) If \( J \in \mathcal{U}_n(Q, \varphi) \) and \( \varphi^k(J) \in \mathcal{U}_{n-k}(Q, \varphi) \) for some \( k \leq n \), then \( J \in \hat{\mathcal{U}}_n(Q, \varphi) \).

(k) If \( \mathcal{U}_n(Q, \varphi) = \mathcal{U}_n(Q, \varphi) \), then \( \mathcal{U}_k(Q, \varphi) = \hat{\mathcal{U}}_n(Q, \varphi) \) for all \( k \geq n \).

Proof. Since we are only using a single function \( \varphi \) in this section, we suppress it from notation; i.e., we write \( \mathcal{U}_n(J) := \mathcal{U}_n(J, \varphi) \). We also simply write \( \mathcal{U}_n = \mathcal{U}_n(I) \), for \( I = [A, D] \), where this does not cause confusion.

Claim [a] is a consequence of Zorn’s lemma: consider the collection \( \mathcal{J} \) of all intervals \( J \subset \tilde{J} \) with \( \varphi^n(J) \supset I \). This collection is partially ordered by inclusion, and any descending chain has a lower bound (its intersection), which is also in \( \mathcal{J} \). Hence there is a minimal element, which is in \( \mathcal{U}_n \) by definition.

If \( J \in \mathcal{U}_n \), then \( \varphi^{-n}(A) \cap J, \varphi^{-n}(D) \cap J \neq \emptyset \). Moreover, any subinterval of \( J \) bounded by an element of \( \varphi^{-n}(A) \) and an element of \( \varphi^{-n}(D) \) also maps over \( I \). Hence the only points in \( J \) that can map to \( A \) and \( D \) are the two endpoints, proving [b].

The next two claims follow immediately from [b]. Indeed, if two intervals intersect, but are not equal, then at least one has to contain one of the endpoints of the other. This never occurs for two intervals of \( \mathcal{U}_n \), by [b], giving [c]. Similarly, if \( J = [a, d] \in \mathcal{U}_n \), then \( \varphi^n([a, d]) \) is a connected subset of \( \mathbb{R} \setminus \{ A, D \} \) that contains \( (A, D) \), and therefore \( \varphi^n([a, d]) = (A, D) \) and \( \varphi^n([a, d]) = [A, D] \), as claimed in [d].

We have \( \varphi([4, 6]) \subset [4, 6] \) by Lemma 1.2 [b]. If \( J \in \mathcal{U}_n \) for some \( n \geq 0 \), then \( \varphi^n(J) = I \subset [6, \infty) \) by [d] and therefore \( J \cap [4, 6] = \emptyset \). This proves [e]. Claim [f] follows similarly from Lemma 1.2. Indeed, recall that \( \varphi(t) < t \) for \( t \geq 6 \). If \( J \in \mathcal{U}_n \), then \( \varphi^n(J) = I \) by [d] and therefore \( \varphi^n(J) \) to the left of \( D \) for \( \hat{n} > n \) and to the right of \( A \) for \( \hat{n} < n \). Thus \( J \) does not map over \( I \) under \( \varphi^n \) for \( n \neq \hat{n} \).

Next, note that \( \lim_{n \to \infty} \varphi(t) = \infty \), and thus \( \lim_{t \to \infty} \varphi^n(t) = \infty \) for fixed \( n \). In particular, \( \varphi^{-n}(I) \supset \bigcup \mathcal{U}_n \) is bounded for every \( n \), and \( \delta := \text{dist}(\varphi^{-n}(A), \varphi^{-n}(D)) > 0 \). In conclusion, the intervals in \( J \in \mathcal{U}_n \) have diameters bounded from below by \( \delta \), their interiors are disjoint, and their union is bounded. Hence \( \mathcal{U}_n \) is finite as claimed in [g].
Next, we prove \( (h) \). Let \( J \in U_n(I) \) and consider \( J_k := \varphi^k(J) \). Then \( \varphi^{n-k}(J_k) = \varphi^n(J) = I \). Clearly \( J \) is mapped minimally to \( J_k \) by \( \varphi^k \); otherwise, there would be a proper sub-interval also mapped to \( J_k \) by \( \varphi^k \), and hence to \( I \) by \( \varphi^n \). So \( J \in U_k(J_k) \). Similarly, suppose that \( J_k \) was not mapped minimally by \( \varphi^{n-k} \). By \( (a) \) there is a proper sub-interval \( \tilde{J}_k \subset J_k \) which belongs to \( U_{n-k}(I) \). But then \( J \) contains a proper sub-interval that is mapped over \( \tilde{J}_k \), which would itself be mapped over \( I \) by \( \varphi^{n-k} \). This is a contradiction to minimality of \( J \), and proves that \( J_k \in U_{n-k}(I) \).

The inclusion “\( \subset \)” in \( (i) \) follows directly from \( (h) \). For the converse inclusion, let \( J \in U_{n-k}(I) \) and \( L \in U_k(J) \). Then \( \varphi^n(L) = I \). So \( L \) contains an element \( \tilde{L} \) of \( U_n(I) \). We have \( \varphi^k(\tilde{L}) \in U_{n-k}(I) \) by \( (h) \) and \( \varphi^k(\tilde{L}) \subset \varphi^k(L) = J \). So \( \varphi^k(\tilde{L}) = J \). But also \( \tilde{L} \in U_k(\varphi^k(\tilde{L})) = U_k(J) \). So we must have \( \tilde{L} = L \), as required.

Next suppose that \( \varphi^k(J) \) is mapped crookedly over \( Q \); say the preimages \( t_B, \tilde{t}_B \) of \( B \) surround a preimage \( t_C \) of \( C \). Then, by \( (a) \), a subinterval \( \tilde{J} \) of \( J \) is mapped minimally over \( [t_B, \tilde{t}_B] \) by \( \varphi^k \). In particular, \( \tilde{J} \) contains a preimage of \( t_C \) under \( \varphi^k \), which is an element of \( \varphi^{-n}(C) \cap J \). By \( (b) \), the subinterval \( \tilde{J} \) is bounded by a preimage of \( t_B \) and a preimage of \( \tilde{t}_B \) under \( \varphi^k \), both of which are elements of \( \varphi^{-n}(B) \). Hence \( J \) is mapped crookedly by \( f^n \), as claimed.

The final claim \( (k) \) is an immediate corollary of \( (j) \). \( \blacksquare \)

7. Tracts and Wiggles

In view of Theorem 3.8 and Lemma 6.2 \( (k) \), our goal is to construct a function \( F \) such that, for each of the countably many quadrilaterals \( Q \) in Theorem 3.8, there is a number \( n \) such that all members of \( U_n(Q, \varphi) \) are mapped crookedly by \( \varphi^n \). To do so, we begin with the strip \( S \), and inductively introduce a sequence of “wiggles” to the tract over increasing real parts; these wiggles allow us to deal with each quadrilateral \( Q \) in turn.

The desired model function \( F \) is then obtained as the limit of these maps.

More precisely, each of the functions involved in the construction is determined by specifying a tract \( T \in \mathcal{T} \), which in turn is determined by sequences \( (r_j)_{0 \leq j < N} \) and \( (R_j)_{0 \leq j < N} \) of positive real numbers, where \( N \leq \infty, r_0 > 6, r_j > R_{j-1} + 1 \) and \( R_j > r_j + 2 \) for all \( j < N \). The corresponding tract is then defined as follows (see Figure 3):

\[(7.1) \quad T := \{ x + iy : 4 < x, |y| < \pi \} \setminus \bigcup_{0 \leq j < N} (\{ r_j + iy : -\pi < y \leq \pi/3 \} \cup \{ x + \pi i/3 : r_j < x \leq R_j - 1 \} \cup \{ R_j + iy : -\pi/3 \leq y < \pi \} \cup \{ x - \pi i/3 : r_j + 1 \leq x < R_j \}). \]
Every tract \( T \) is an element of \( \mathcal{T} \) as defined in Definition 3.1. We denote the subclass of \( \mathcal{H} \) defined by the tracts as above by \( \mathcal{F} \). If \( F \in \mathcal{F} \), then we write \( T(F) \) for the tract of \( F \), as before. We also write \( r_j(F) \) and \( R_j(F) \) for the corresponding sequences as above. For inductive purposes, we use the convention that \( R_{-1}(F) = 5 \).

We also write \( N(F) = N \leq \infty \) for the number of “wiggles” occurring in the tract of \( F \), and denote by \( \mathcal{F}_N \) the subclass consisting of maps \( F \in \mathcal{F} \) with \( F(N) = N < \infty \). In the following, when \( F, \tilde{F}, F_j \) etc. are members of \( \mathcal{F} \), we always take \( \varphi, \tilde{\varphi}, \varphi_j \) etc. to denote their one-dimensional projections.

7.1. Proposition (Bounded decorations in \( \mathcal{F} \)).
There exists a universal constant \( \nu_0 > 0 \) such that \( \mathcal{F} \subset \mathcal{H}_{\nu_0} \).

**Proof.** Let \( F \in \mathcal{F} \) and \( T = T(F) \). The set \( \partial T \setminus \{4\} \) has two connected components, one containing \([4, \infty) + \pi i\) and one containing \([4, \infty) - \pi i\). We call these the upper and lower boundaries of \( T \), respectively. By the definition of \( \mathcal{F} \), every \( z \in T \) can be connected both to the upper and to the lower boundary by a straight vertical segment \( \alpha_z \subset T \) of length at most \( 2\pi \).

**Claim.** There is a universal constant \( C \) with the following property. If \( F \in \mathcal{F} \) and \( z \in T = T(F) \), there are half-open arcs \( \beta^+_z, \beta^-_z \subset T \) such that

1. Each \( \beta^+_z \) and \( \beta^-_z \) connects \( z \) to an endpoint on \( \partial T \),
2. \( \beta^+_z \) and \( \beta^-_z \) have Euclidean diameter at most \( C \);
3. \( F(\beta^+_z) \) connects \( F(z) \) to an endpoint on \( i[0, \infty) \), and \( F(\beta^-_z) \) connects \( F(z) \) to a point of \( i(-\infty, 0] \).

**Proof.** The harmonic measure in \( \mathbb{H} \) of \( i[0, \infty) \), seen from 5, is \( 1/2 \), as is that of \( i(-\infty, 0] \). (See e.g. [GM05] for the definition of harmonic measure.) Furthermore, \( \text{dist}(5, \partial T) = 1 \) by construction. It follows from [Pom92] Corollary 4.18 that there is a universal constant \( K_0 \) with the following property: there exist a hyperbolic geodesics \( \gamma^+ \) and \( \gamma^- \) of \( T \) that have diameter at most \( K_0 \), and such that \( F(\gamma^+) \) connects 5 to \( \pm i[0, \infty) \). We set \( C := K_0 + 2\pi + 1 \).

Let \( \vartheta \in \mathbb{R} \) be the unique number with \( F^{-1}(i\vartheta) = 4 \). To fix our ideas, suppose that \( \vartheta \leq 0 \); the case \( \vartheta \geq 0 \) is analogous. Let \( z \in T \). The curve \( F(\alpha_z) \) has endpoints \( i\vartheta^+ \) and \( i\vartheta^- \) with \( \vartheta^- < \vartheta < \vartheta^+ \). In particular, we can take \( \beta^- \) to be the piece of \( \alpha_z \) connecting \( z \) to the lower boundary. If \( \vartheta^+ \geq 0 \), we can do the same for the upper boundary, and are done.

Otherwise, the segment \( \alpha_z \) intersects \( \gamma \cup (4, 5) \). We can thus connect \( z \) to a point \( z' \) of \( \gamma \cup (4, 5) \), and connect \( z' \) to the endpoint of \( \gamma \) on \( \partial T \) by a subcurve of \( \gamma \cup (4, 5) \). Overall, this curve has diameter at most \( 2\pi + K_0 + 1 = C \), as desired. \( \triangle \)

To complete the proof, we proceed as in [Rem16] Proposition 7.4. By [Pom92] Lemma 4.21, there is a universal constant \( K_1 \) with the following property. Let \( x \in \mathbb{R} \), let \( \beta \) be the geodesic of \( \mathbb{H} \) connecting \( x \) to \( ix \), and let \( \beta \subset \mathbb{H} \) be any curve connecting \( ix \) to \([0, \infty) \). Then

\[
\text{diam}(F^{-1}(\beta)) \leq K_1 \cdot \text{diam}(F^{-1}(\tilde{\beta})).
\]
The diameter \( \text{diam}(\tilde{\beta}) \) can be chosen less than \( C + \varepsilon \), for any \( \varepsilon > 0 \). Indeed, we may choose \( \delta \) small enough that \( F^{-1}([0, \delta] + ix) \) has diameter less than \( \varepsilon \), and then connect \( z := F^{-1}(\delta + ix) \) to \( F^{-1}([0, \infty)) \) using the curve \( \beta^+_z \) if \( x < 0 \), and using \( \beta^-_z \) if \( x > 0 \).

So \( \text{diam}(F^{-1}(\beta)) \leq K_1 \cdot C \), and the claim follows if we take \( \nu := 2 \cdot K_1 \cdot C \).

### 7.2. Proposition (Growth of functions in \( F \))

There is a constant \( C > 1 \) with the following property. Let \( F \in \mathcal{F} \), and let \( z \in T(F) \) with \( |F(z)| \geq 4 \). If \( z \) does not belong to the two bottom thirds of any “wiggle”, i.e., if

\[
z \notin W_j(F) := \{ \zeta \in T(F) : r_j < \text{Re} \zeta < R_j \text{ and } \text{Im} \zeta < \pi/3 \}
\]

for all \( j < N(F) \), then

\[
\frac{\text{Re} z}{C} \leq \log |F(z)| \leq C \cdot \text{Re} z.
\]

If \( z \in W_j(F) \), then

\[
\frac{R_j}{C} \leq \log |F(z)| \leq C \cdot R_j.
\]

**Proof.** Note that there is an arc \( \alpha \) that connects \( 5 \) to \( \infty \) in \( T \) and remains at distance at most \( 1/2 \) from \( \partial T \), namely

\[
\alpha := \left[ 5, r_0 - \frac{1}{2} \right] \cup \bigcup_{j \geq 0} \left( \left[ r_j - \frac{1}{2}, \frac{1}{2} + 2\pi i/3 \right] \cup \left[ r_j - \frac{1}{2}, R_j - \frac{1}{2} \right] + 2\pi i/3 \right) \cup \left( \left[ r_j = \frac{1}{2}, R_j - \frac{1}{2} \right] \cup \left[ r_j + \frac{1}{2}, R_j - \frac{1}{2} \right] \right) \cup \left( \left[ r_j + \frac{1}{2}, \frac{1}{2} - 2\pi i/3 \right] \cup \left[ r_j + \frac{1}{2}, R_j + \frac{1}{2} \right] - 2\pi i/3 \right) \cup \left( \left[ R_j + \frac{1}{2}, \frac{1}{2} - 2\pi i/3 \right] \cup \left[ R_j + \frac{1}{2}, r_{j+1} - \frac{1}{2} \right] \right).
\]

(See Figure [4].) Let \( z \in \alpha \), and let \( \alpha_z \) be the part of \( \alpha \) that connects \( 5 \) to \( z \). Let us estimate the Euclidean length \( \ell(\alpha_z) \), first when \( z \notin W_j(F) \) for any \( j \). Let \( j \) be maximal such that \( \text{Re} z > R_j \). Recall that \( R_j > 7 + 3j \). We have

\[
\ell(\alpha_z) \leq \text{Re} z - 5 + \pi + \sum_{i \leq j} (2\pi + R_j - r_j - 1) \\
\leq 3 \text{Re} z + (2j + 3) \pi < 3 \text{Re} z + R_j \pi < (3 + \pi) \cdot \text{Re} z
\]

If \( z \in W_j(F) \), then similarly

\[
\ell(\alpha_z) \leq (3 + \pi) \cdot R_j.
\]
On the other hand, clearly any curve connecting 5 to a point \( z \in T \) must have Euclidean length at least \( \text{Re} z - 5 \), and at least \( R_j \) if \( z \in W_j \).

Now let \( z \) be as in the statement of the proposition, and let

\[
\gamma := F^{-1}(\{ \zeta \in \mathbb{H} : |\zeta| = |F(z)| \})
\]

be the vertical geodesic containing \( z \), and set \( \bar{z} := F^{-1}(|F(z)|) \in \gamma \). By Proposition 7.1, the Euclidean diameter of \( \gamma \) is bounded from above by \( \nu_0 \). Define

\[
R := \begin{cases} 
\text{Re} z & \text{if } z \notin W_j \text{ for all } j; \\
R_j & \text{if } z \in W_j.
\end{cases}
\]

By the above, and the standard estimates on the hyperbolic metric on \( T \), we have

\[
R - \nu_0 - 5 \leq \text{dist}_T(5, \gamma) \leq (3 + \pi) \cdot (R + \nu_0) \leq (3 + \pi + \nu_0/4) \cdot R.
\]

Moreover,

\[
\text{dist}_T(5, \gamma) = \text{dist}_T(5, \bar{z} = \text{dist}_H(5, |F(z)|)) = \log \frac{|F(z)|}{5}.
\]

When \( R > 2(\nu_0 + 5) \), say, the claim follows; we can ensure that it holds for smaller \( R \) also by choosing \( C \) sufficiently large. (Recall that \( R, |F(z)| \geq 4 \).)

7.3. Proposition (Number of intervals increases only through wiggles).

Let \( F \in \mathcal{F} \). Let \( I = [A, D] \) be an interval with \( A \geq 6 \) and \( |I| \geq \nu_0 \), where \( \nu_0 \) is the constant from Proposition 7.1. Suppose that furthermore

\[
I \notin [r_k(F) - \nu_0, R_k(F) + \nu_0]
\]

for any \( k \geq 0 \). Then

\[
\#\mathcal{U}_1(I, \varphi) = 1.
\]

Proof. The hypothesis means that there is \( \tau \in [A, D] \) such that

\[
R_k + \nu_0 < \tau < r_{k+1} - \nu_0
\]

for some \( k \).

In particular, \( T \) intersects the vertical lines at real parts \( \tau \) and \( \tau + \nu_0 \) each in exactly one vertical line segment. Let \( r \) be maximal such that the vertical geodesic

\[
\gamma = F^{-1}(\{ z \in \mathbb{H} : |z| = r \}).
\]

contains a point at real part \( \tau \). Then, by Proposition 7.1, \( \gamma \) separates all points of \( T \) at real part less than \( \tau \) from all points at real part greater than \( \tau + \nu_0 \).

If \( J \in \mathcal{U}_1(I, \varphi) \), then \( F^{-1}(J) \) connects real parts \( \tau \) and \( \tau + \nu_0 \), and must hence intersect \( \gamma \), which means that \( r \in J \). By Lemma 6.2(c), two different intervals in \( \mathcal{U}_1(I, \varphi) \) cannot both contain the same point \( r \). This proves the claim.

8. Approximation properties of the construction

As already mentioned, the positions \((r_n, R_n)\) of the “wiggles” in our example will be defined inductively. A key fact is that, if \( F \in \mathcal{F}_N \), and the next wiggle \((r_N, R_N)\) is chosen sufficiently far to the right, then the inverse of the next function \( \tilde{F} \) will be close to \( F^{-1} \), at least up until the end of the \( n \)-th “wiggle” of \( T(\tilde{F}) \). In particular, key properties of the one-dimensional projection \( \varphi \) of \( F \) will be preserved under this approximation.
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To make this statement precise, let us say that $\tilde{F}$ is $(N, \rho)$-close to a function $F \in \mathcal{F}_N$ if

(a) $N(\tilde{F}) > N$;
(b) $r_n(\tilde{F}) = r_n(F)$ and $R_n(\tilde{F}) = R_n(F)$ for $n < N$;
(c) $r_N(\tilde{F}) \geq \rho$.

We begin by noting that $(N, \rho)$-close tracts, for large $\rho$, are also close in the sense of Carathéodory kernel convergence. (See [Pom92, Section 1.4] for a discussion of kernel convergence.)

8.1. Lemma (Carathéodory convergence).
Let $F \in \mathcal{F}_N$. Suppose that $(F_j)_{j=0}^\infty$ is a sequence of functions in $\mathcal{F}$, such that $F_j$ is $(N, \rho_j)$-close to $F$, with $\rho_j \to \infty$. Then $T(F_j)$ converges to $T(F)$, with respect to any point $z_0 \in T$, in the sense of Carathéodory kernel convergence.

Proof. Let $K$ be a compact, connected set. We must show that the following are equivalent:

(i) $K \subset T(F)$;
(ii) $K \subset T(F_j)$ for all but at most finitely many $j$.

By condition (b) in the definition of $(N, \rho)$-closeness, we have $T(F_j) \subset T(F)$ and $T(F) \cap \{\text{Re } z < \rho_j\} \subset T(F_j)$ for all $j$. So if $j$ is sufficiently large that $\text{Re } z < \rho_j$ for all $z \in K$, then $K \subset T(F_j)$ if and only if $K \subset T(F)$.

8.2. Proposition (Convergence of inverse functions).
Let $F \in \mathcal{H}_\nu$, let $\nu > 0$, and suppose that $(F_j)_{j=0}^\infty$ is a sequence of functions in $\mathcal{H}_\nu$ such that $T_j(F)$ converges to $T(F)$, with respect to the point 5, in the sense of Carathéodory kernel convergence.

Then $F_j^{-1} \to F^{-1}$ locally uniformly on $\mathbb{H}$.

Proof. We write $T = T(F)$ and $T_j = F(T_j)$. Let $\psi : \mathbb{D} \to T$ and $\psi_j : \mathbb{D} \to T_j$ be conformal isomorphisms, normalised such that $\psi(5) = \psi_j(5) = 5$ and $\psi'(5), \psi_j'(5) > 0$. By Carathéodory’s kernel convergence theorem, $\psi_j \to \psi$ locally uniformly on $\mathbb{D}$. Recall that $\psi$ extends continuously to $\partial \mathbb{D}$ by the Carathéodory-Torhorst theorem, taking the value $\infty$ at exactly one point $\psi^{-1}(\infty)$ of $\partial \mathbb{D}$. We have

$$F^{-1} = \psi \circ M,$$

where $M : \mathbb{D} \to \mathbb{H}$ is the Möbius transformation with $M(5) = 0$ and $M(\infty) = \psi^{-1}(\infty)$. The analogous statements hold for $\psi_j$ and $F_j$. So it suffices to prove that $\psi_j^{-1}(\infty) \to \psi^{-1}(\infty)$.

Let $\delta > 0$. Let $R$ be so large that the set of points at real part $\geq R$ in the boundary of $\tilde{S} = \{a + ib : -\pi < b < \pi\}$ has harmonic measure at most $\delta/(2\pi)$, seen from 5. Since $T_j \subset \tilde{S}$, the set of points at real part $\geq R$ in $\partial T_j$ also has harmonic measure at most $\delta/(2\pi)$ from 5, independently of $j$. 

Choose a point \( \zeta \in T \) such that \( \text{Re} \zeta > R + \nu \) and \( |\psi^{-1}(\zeta) - \psi^{-1}(\infty)| < \delta/4 \). For sufficiently large \( j \), we have
\[
|\psi_j^{-1}(\zeta) - \psi^{-1}(\zeta)| < \frac{\delta}{4}.
\]
Now let \( \gamma_j \) be the geodesic of \( \mathbb{D} \) through \( \psi_j^{-1}(\zeta) \) which is perpendicular to the radius connecting 0 and \( \psi_j^{-1}(\infty) \). Then \( \psi_j(\gamma_j) \) is the vertical geodesic of \( T_j \) through \( \zeta \), and hence has diameter at most \( \nu \). By choice of \( \zeta \), all real parts of \( \psi_j(\gamma_j) \) are greater than \( R \), and the harmonic measure of the arc of \( \partial \mathbb{D} \) separated from 0 by \( \gamma \) is at most \( \delta/(2\pi) \). It follows that this arc has length at most \( \delta/2 \) and \( |\psi_j^{-1}(\zeta) - \psi_j^{-1}(\infty)| \leq \delta/2 \). We conclude that
\[
|\psi^{-1}(\infty) - \psi_j^{-1}(\infty)| \leq |\psi^{-1}(\infty) - \psi_j^{-1}(\zeta)| + |\psi_j^{-1}(\zeta) - \psi_j^{-1}(\infty)| < \delta.
\]
Since \( \delta > 0 \) was arbitrary, the proof is complete.

\[\hfill \square\]

8.3. Remark.

It is not sufficient to require that \( F_n \in \mathcal{H} \) (rather than \( F_n \in \mathcal{H}_\nu \)). Indeed, suppose that \( T = S = \{ x + iy : x > 4 \text{ and } |y| < \pi \} \) and
\[
T_n = \{ x + iy \in S : x < n + 6 \} \cup \{ x + iy \in S : y > \pi - 1/n \} \cup \{ x + (\pi - 1/n)i : 4 < x < 1/n \}.
\]
Then \( T_n \to T \) in kernel, and in fact the boundaries of \( T_n \) in \( \tilde{\mathbb{C}} \) converge to that of \( T \) in the Hausdorff metric. However, let \( t_n \to 5 \) be minimal with \( \text{Im} F_n^{-1}(t_n) = \pi - 1/n \); then \( \text{dist}_{T_n}(5, t_n) \to \infty \), and thus \( t_n \to \infty \). However, by the Gehring-Hayman theorem \( [\text{Pom92}, \text{Theorem 4.20}] \), \( F_n^{-1}([5, t_n]) \) remains uniformly bounded as \( n \to \infty \). So \( F_n^{-1} \) does not converge locally uniformly to \( F^{-1} \) on \( \mathbb{H} \).

Now we are able to prove the main approximation result of this section. Note that Proposition 8.2 shows that \( \tilde{\varphi} \) is close to \( \varphi \) whenever \( \tilde{F} \) is \((N, \rho)\)-close to \( F \), with \( \rho \) sufficiently large. The following proposition sharpens this further by showing that \( \tilde{\varphi}^n(t) \) is close to \( \varphi^n(t) \), for all \( n \), provided that these values belong to a bounded subinterval of \([4, \infty)\), and furthermore \( t \) itself is not too much larger than the position of \( R_N(\tilde{F}) \). Note that the latter means that \( t \) is allowed to be large when \( \rho \) is large.

8.4. Proposition (Approximation by \((n, \rho)\)-close functions).

Suppose that \( F \in \mathcal{F}_N \). Let \( \varepsilon > 0 \) and \( \tau \geq 5 \). Then there is a number \( \rho > R_{N-1}(F) \) with the following property.

Suppose that \( \tilde{F} \in \mathcal{F}_N \) is \((N, \rho)\)-close to \( F \). If \( 4 \leq t \leq 2R_N(F) \) and \( \min(\varphi^n(t), \tilde{\varphi}^n(t)) \leq \tau \) for some \( n \geq 0 \), then
\[
(8.1) \quad |\varphi^n(t) - \tilde{\varphi}^n(t)| \leq \varepsilon.
\]

Proof. By Lemma 8.1 and Proposition 8.2, \( F_j^{-1} \to F^{-1} \) locally uniformly on \( \mathbb{H} \). In particular, \( \varphi_j \to \varphi \) uniformly on any compact subset of \((0, \infty)\). We first prove (8.1) when \( t \) belongs to a fixed interval of the real line.

Claim 1. Let \( \tilde{\tau} > 4 \) and \( \delta > 0 \). Then there is \( \rho(\tilde{\tau}, \delta) > R_{N-1}(F) \) with the following property. Suppose that \( \tilde{F} \) is \((N, \rho(\tilde{\tau}, \delta))\)-close to \( F \). If \( t_1, t_2 \in [4, \tilde{\tau}] \) with \( |t_1 - t_2| \leq \delta \), then
\[
|\varphi^n(t_1) - \tilde{\varphi}^n(t_2)| \leq \delta
\]
Claim 2. Set \( \varepsilon \) and \( \tilde{\varepsilon} \) values are comparable up to a multiplicative error of at most 6.

Proof. We prove the claim by induction; it is trivial for \( n = 0 \). Now suppose that it holds for \( n \). Let us suppose that \( \varphi^{n+1}(t) \geq C^2 \cdot R_{N-1} \); the alternative case is analogous, exchanging the roles of \( \varphi \) and \( \tilde{\varphi} \). The inductive hypothesis implies

\[
\text{dist}_H(\varphi^n(t), \tilde{\varphi}^n(t)) \leq \log \hat{C} + \frac{\pi}{2}.
\]

Therefore

\[
|\tilde{\varphi}(\varphi^n(t)) - \tilde{\varphi}^{n+1}(t)| \leq 2 \text{dist}_T(\tilde{\varphi}(\varphi^n(t)), \tilde{\varphi}^{n+1}(t)) \leq 2 \log \hat{C} + \pi < \hat{C},
\]

since \( \hat{C} \geq 6 \). Furthermore, the assumption on \( \varphi^{n+1}(t) \) means that \( \varphi^n(t) \geq \exp(C \cdot R_{N-1}) \), by Proposition 7.2. By the above,

\[
\varphi^{n+1}(t)/C^2 \leq \tilde{\varphi}(\varphi^n(t)) \leq C^2 \cdot \varphi^{n+1}(t).
\]

In particular,

\[
\varphi^{n+1}(t) \leq C^2 \tilde{\varphi}(\varphi^n(t)) \leq C^2(\varphi^n(t) + \hat{C}) \leq (C^2 + \hat{C}/4) \cdot \varphi^n(t) \leq \hat{C} \cdot \tilde{\varphi}^n(t).
\]

The opposite inequality follows analogously. \( \triangle \)
We may suppose that
\[ \tau \geq \max(4\hat{C}/\varepsilon, C^2 \cdot R_{N-1}). \]
Define \( \tilde{\tau} := \hat{C} \cdot \exp(C \cdot \exp(C \cdot \tau)) \). Then, by Proposition 7.2, \( \varphi^2(t) > \tau/\hat{C} \) when \( t > \tilde{\tau} \).

The same holds for \( \tilde{\varphi} \), assuming that we have chosen \( \rho \) larger than \( \tilde{\tau} \). Now choose
\[ \rho \geq \max(\rho(\tilde{\tau}, \delta), \rho(\tilde{\tau}, \varepsilon)), \]
where \( \rho(\tilde{\tau}, \cdot) \) is as in Claim 1, with \( \delta = \min(\hat{C}, \varepsilon/2) \). Note that \( \tilde{\tau} \) depends only on \( R_{N-1} \) and \( \tau \), and thus \( \rho \) depends only on \( F, \varepsilon \) and \( \tau \), as required.

Suppose that \( t \) is as in the statement of the proposition. It is enough to prove the claim when \( n \) is minimal with the stated property, since it then follow for larger \( n \) by Claim 1. Moreover, the claim holds for \( n < 2 \) by choice of \( \tilde{\tau} \) and Claim 1.

So now suppose that \( n \geq 2 \). We have \( \varphi^j(t), \tilde{\varphi}^j(t) > \tau \) for \( j < n \), and furthermore either \( \varphi^{n-2}(t) \leq \tilde{\tau}/\hat{C} \) or \( \tilde{\varphi}^{n-2}(t) \leq \tilde{\tau}/\hat{C} \). By Claim 2, \( \varphi^{n-2}(t) \) and \( \tilde{\varphi}^{n-2}(t) \) are comparable by a factor of \( \hat{C} \). In particular both (and their images) are less than \( \tilde{\tau} \). We now proceed similarly as in the proof of Claim 2. Firstly, again
\[ |\varphi^{n-1}(t) - \varphi(\tilde{\varphi}^{n-2}(t))| \leq 2 \log \hat{C} + \pi < \hat{C}. \]
By choice of \( \tilde{\tau} \),
\[ |\varphi(\tilde{\varphi}^{n-2}(t)) - \tilde{\varphi}^{n-1}(t)| \leq \delta, \]
and thus
\[ |\varphi^{n-1}(t) - \tilde{\varphi}^{n-1}(t)| \leq \hat{C} + \delta \leq 2\hat{C}. \]
In particular, we have
\[ \text{dist}_{\mathbb{H}}(\varphi^{n-1}(t), \tilde{\varphi}^{n-1}(t)) \leq \frac{2\hat{C}}{\tau} \leq \delta/2. \]
Arguing as above, we conclude that
\[ |\varphi^n(t) - \tilde{\varphi}^n(t)| \leq |\varphi^n(t) - \varphi(\tilde{\varphi}^{n-1}(t))| + |\varphi(\tilde{\varphi}^{n-1}(t)) - \tilde{\varphi}^{n-1}(t)| \leq \delta + \delta = \varepsilon. \]

8.5. Definition (Larger quadruplets).
If \( Q = (A < B < C < D) \) and \( \tilde{Q} = (\tilde{A} < \tilde{B} < \tilde{C} < \tilde{D}) \) are quadruples, we write \( Q < \tilde{Q} \) if
\[ \tilde{A} < A < B < \tilde{B} < \tilde{C} < C < D < \tilde{D}. \]

8.6. Proposition (U_n and approximation).
Let \( F \in \mathcal{F}_N \), and let \( Q < \tilde{Q} \) be quadruples. Let \( n_0 \geq 0 \).

Then there is \( \rho > R_{N-1}(F) \) with the following property. If \( \tilde{F} \in \mathcal{F} \) is \((N, \rho)\)-close to \( F \), then

(a) For \( n \leq n_0 \), all elements of \( \mathcal{U}_n(Q, \varphi) \) are to the left of \( \rho \).
(b) For every \( n \geq 0 \) and every \( J \in \mathcal{U}_n(Q, \varphi) \) that is to the left of \( 2R_N(\tilde{F}) \), there is \( J \in \mathcal{U}_n(Q, \varphi) \) with \( J \subset \tilde{J} \).
(c) If \( J \) as in (b) is mapped crookedly over \( Q \) by \( \varphi^n \), then \( \tilde{J} \) is mapped crookedly over \( \tilde{Q} \) by \( \tilde{\varphi}^n \).

In particular, for \( n \leq n_0 \), \( \#\mathcal{U}_n(Q, \varphi) \leq \#\mathcal{U}_n(Q, \varphi) \) and
\[ \#\mathcal{U}_n(Q, \varphi) - \#\mathcal{U}_n(Q, \varphi) \leq \#\mathcal{U}_n(Q, \varphi) - \#\mathcal{U}_n(Q, \varphi). \]
Proof. That we can ensure \([\text{a]}\) by choosing \(\rho\) sufficiently large follows from Proposition \([7.2]\).
Now let \(\varepsilon\) be the distance between the quadruples \(Q\) and \(\bar{Q}\); i.e.
\[
\varepsilon = \max(A - \bar{A}, B - \bar{B}, C - \bar{C}, D - \bar{D}).
\]
Take \(\tau > \bar{D}\), and choose \(\rho\) according to Proposition \([8.4]\). We leave it to the reader to verify that the claims follow from this choice.

\[\blacksquare\]

9. Proof of the main theorem

We are now ready to show the following result, which implies Theorem \([1.4]\).

9.1. Theorem (Model function with pseudo-arcs).
There is \(F \in \mathcal{F}\) such that every Julia continuum of the periodic extension \(\hat{F}\) is a pseudo-arc.
Moreover, \(F\) can be chosen such that
\[
\liminf_{r \to \infty} \max_{\Re \xi = r} \frac{\log \Re F(\xi)}{r} = \frac{1}{2}.
\]

We prove this theorem by inductively applying the following proposition.

9.2. Proposition (Creating crookedness over prescribed quadruples).
Let \(F \in \mathcal{F}_N\), let \(Q\) be a quadruple, and let \(\rho > 0\).
Then there is \(\bar{F} \in \mathcal{F}_N\) for some \(\bar{N} > N\), which is \((N, \rho)\)-close to \(F\), and such that \(U_n(Q, \bar{F}) = \hat{U}_n(Q, \bar{F})\) for some \(n \geq 0\).

Proof of Theorem 9.1 using Proposition 9.2. Let \((Q_k)_{k=1}^{\infty}\) be the countably many quadrilaterals appearing in the hypothesis of Theorem 5.3. We inductively construct a sequence \((F_k)_{k=0}^{\infty}\) with \(F_k \in \mathcal{F}_{N_k}\) for an increasing sequence \((N_k)\); here \(N_0 = 0\) and \(F_0\) is the unique function in \(\mathcal{F}_0\); i.e. the conformal map \(F_0 : S \to \mathbb{H}\) with \(F_0(5) = 5\) and \(F_0(\infty) = \infty\).

Along with the functions \(F_k\), we construct a sequence \(\rho_k\), in such a way that
(a) \(F_{k+1}\) is \((N_k, \rho_k)\)-close to \(F_k\);
(b) for every \(k \geq 1\), there is some \(n_k\) with the following property: if \(F \in \mathcal{F}\) is \((N_k, \rho_k)\)-close to \(F_k\), then \(U_{n_k}(Q_k, F) = \hat{U}_{n_k}(Q_k, F)\).

The number \(\rho_0\) can be chosen in an arbitrary manner.

Suppose that \(F_k\) and \(\rho_k\) have been constructed. Apply Proposition 9.2 to the function \(F_k\), the value \(\rho_k\), and a slightly smaller quadruple \(\bar{Q}_{k+1} < Q_{k+1}\), to obtain the function \(\bar{F}_{k+1} \in \mathcal{F}_{N_{k+1}}\). This function has the property that \(U_{n_k+1}(\bar{Q}_{k+1}, F_{k+1}) = \hat{U}_{n_k+1}(\bar{Q}_{k+1}, F_{k+1})\) for some \(n_{k+1} \geq 0\). By Proposition 8.6 there is \(\rho_{k+1}\) such that \(U_{n_k+1}(Q_{k+1}, F) = \hat{U}_{n_k+1}(Q_{k+1}, F)\) for every \(F\) that is \((N_{k+1}, \rho_{k+1})\)-close to \(F_{k+1}\). This completes the inductive construction.

Now let \(F \in \mathcal{F}\) be the limit of the functions \(F_k\); that is, \(F\) has \(N(F) = \infty\) and is defined by the sequence \((r_k(F_{k+1}), R_k(F_{k+1}))_{k=0}^{\infty}\). Then \(F\) is \((N_k, \rho_k)\)-close to \(F_k\) for every \(k \geq 1\). By Lemma 6.2 \([\text{b}]\) and property \([\text{b}]\) of the inductive construction, we have \(U_n(Q_k, F) = \hat{U}_n(Q_k, F)\) for all \(n \geq n_k\). Hence \(F\) satisfies the hypotheses of Theorem 5.3 and every Julia continuum of \(\hat{F}\) is a pseudo-arc.
We claim that, if each $\rho_k$ is chosen sufficiently large, depending on $R_{N_k-1} := R_{N_k-1}(F) = R_{N_k-1}(F_k)$, then $F$ additionally satisfies (9.1). Indeed, let $a \in T(F)$ be such that $\text{Re } a = R_{N_k-1} + 1$ and such that $|F(a)|$ is maximal. Also let $\zeta \in T(F)$ with $\text{Re } \zeta = \rho_k - \nu_0$; we may assume that $\rho_k$ is chosen so large that $\text{Re } \zeta > \text{Re } a + 2\nu_0$, and in particular $|F(\zeta)| > |F(a)|$.

Let $\gamma_a$ and $\gamma_\zeta$ be the “vertical” geodesics through $a$ and $\zeta$, respectively, and let $Q \subset T$ be the quadrilateral bounded by $\gamma_a$ on the left, $\gamma_\zeta$ on the right, and pieces of the upper and lower boundaries of $T$. Then log $F$ maps $Q$ conformally to the rectangle

$$R := \{x + iy : \log|F(a)| < x < \log|F(\zeta)| \text{ and } |y| < \pi/2\}.$$ 

The conformal modulus of this rectangle is

$$\text{mod } (R) = \frac{1}{\pi} \cdot \log \frac{|F(\zeta)|}{|F(a)|}.$$ 

On the other hand,

$$Q \subset \{x + iy : 0 < x < \text{Re } \zeta + \nu_0 \text{ and } |y| < \pi\},$$ 

and hence $\text{mod } (Q) \leq (\text{Re } \zeta + \nu_0)/(2\pi)$. Hence

$$\log|F(\zeta)| \leq \frac{\text{Re } \zeta}{2} + \frac{\nu_0}{2} + \log|F(a)| \leq \frac{\text{Re } \zeta}{2} + \frac{\nu_0}{2} + C \cdot (R_{N_k-1} + 1),$$

where $C$ is the constant from Proposition 7.2. If $\rho_k$ is chosen so large that

$$\frac{\nu_0}{2} + C \cdot (R_{N_k-1} + 1) \leq \frac{\rho_k - \nu_0}{n} = \frac{\text{Re } \zeta}{n},$$

then

$$\log|F(\zeta)| \leq \text{Re } \zeta \cdot \left(\frac{1}{2} + \frac{1}{n}\right),$$

and (9.1) follows. \[\square\]

To prove Proposition 9.2, we consider a quadrilateral $Q$, and proceed to inductively introduce wiggles over the intervals in $\hat{U}_n(Q, \varphi)$. More precisely, we show the following.

**9.3. Proposition** (Increasing the number of intervals mapped crookedly). Let $F \in F_N$, let $Q < \tilde{Q}$ be quadruples, and let $\rho > 0$. Let $m$ be the integer such that

$$\#U_n(Q, \varphi) - \#\hat{U}_n(Q, \varphi) = m$$

for sufficiently large $n$.

Suppose that $m > 0$. Then there is $\tilde{F} \in F_{N+1}$, which is $(N, \rho)$-close to $F$, such that

$$\#U_n(\tilde{Q}, \tilde{\varphi}) - \#\hat{U}_n(\tilde{Q}, \tilde{\varphi}) \leq m - 1$$

for all sufficiently large $n$.

**Proof.** Let $n_0$ be so large that all elements of $U_{n_0}(Q, \varphi)$ are to the right of the last wiggle of $F$. Thus $\#U_n(Q, \varphi) = \#U_{n_0}(Q, \varphi)$ for all $n \geq n_0$ by Proposition 7.3. In particular, a number $m$ as in the statement of the proposition does indeed exist. By assumption, if $n_0$ is chosen sufficiently large, then

$$\#U_n(Q, \varphi) = \#\hat{U}_{n_0}(Q, \varphi) = \#U_{n_0}(Q, \varphi) - m$$
for \( n \geq n_0 \).

Choose \( \rho_1 > \rho \) so large that Proposition \[8.6\] applies, and let \( \rho_2 > \rho_1 \) be so large that Proposition \[8.4\] applies with \( \tau = \rho_1 \) and \( \varepsilon \) the distance between \( Q \) and \( \hat{Q} \) (as in the proof of Proposition \[8.6\]). Let \( n_1 \) be sufficiently large that all elements of \( \mathcal{U}_{n_1}(Q, \varphi) \) are to the right of \( \rho_2 + \nu_0 + 1 \). Let \([\hat{A}, \hat{D}]\) be the right-most element of this set that is not in \( \hat{\mathcal{U}}_{n_1}(Q, \varphi) \), and choose two points \( \hat{B} < \hat{C} \) that map to the elements \( B \) and \( C \) of the quadruple \( Q \) (not necessarily in that order). By Lemma \[4.2\] we have

\[
\min(\hat{B} - \hat{A}, \hat{D} - \hat{C}) \geq 2^{n_1} \cdot |Q| > 2\nu_0 + 2
\]

if \( n_1 \) was chosen sufficiently large. Let \( \hat{Q} = (\hat{A} < \hat{B} < \hat{C} < \hat{D}) \) be the resulting quadruple.

Define \( \hat{F} \in \mathcal{F}_{n+1} \) which is \((N, \rho_2)\)-close to \( F \) by setting \( r_N(\hat{F}) := \hat{B} - \nu_0 \) and \( R_N(\hat{F}) := \hat{C} + \nu_0 \).

**Claim.** The following hold.

1. \( \#\mathcal{U}_n(\hat{Q}, \hat{\varphi}) \leq \mathcal{U}_{n_0}(Q, \varphi) \) for \( n \geq n_0 \).
2. Every element of \( \mathcal{U}_{n_0}(\hat{Q}, \hat{\varphi}) \) that contains an element of \( \hat{\mathcal{U}}_{n_0}(Q, \varphi) \) is in \( \hat{\mathcal{U}}_{n_0}(\hat{Q}, \hat{\varphi}) \).
3. We have \( \mathcal{U}_1(\hat{Q}, \hat{\varphi}) = \mathcal{U}_1(\hat{Q}, \hat{\varphi}) \).

**Proof.** We have \( \#\mathcal{U}_{n_0}(\hat{Q}, \hat{\varphi}) \leq \#\mathcal{U}_{n_0}(Q, \varphi) \) by Proposition \[8.6\] and every interval of the former contains an interval of the latter. Part (2) also follows from Proposition \[8.6\].

Moreover, for \( n \geq n_0 \), no interval of \( \mathcal{U}_n(Q, \varphi) \) is contained in \([r_N(\hat{F}) - \nu_0, R_N(\hat{F}) + \nu_0]\). Indeed, otherwise this interval would be contained in \([\hat{A}, \hat{D}]\) which contradicts Proposition \[8.6\]. By Proposition \[8.6\] also no interval of \( \hat{\mathcal{U}}_n(\hat{Q}, \hat{\varphi}) \) is contained in \([r_N(\hat{F}) - \nu_0, R_N(\hat{F}) + \nu_0]\). So by Proposition \[7.3\] we have

\[
\#\mathcal{U}_n(\hat{Q}, \hat{\varphi}) \leq \#\mathcal{U}_{n_0}(Q, \varphi) \leq \mathcal{U}_{n_0}(Q, \varphi)
\]

for \( n \geq n_0 \).

Finally, let \( J \in \mathcal{U}_1(\hat{Q}, \hat{\varphi}) \). Then \( F^{-1}(J) \) is an arc that connects a point at real part \( \hat{A} < r_N(\hat{F}) \) to a point at real part \( \hat{D} > R_N(\hat{F}) \). By the shape of the tract (see Figure 3), along this curve the real parts must first grow to at least \( R_N(\hat{F}) - 1 \), decrease again at least to \( r_N(\hat{F}) + 1 \), before they can reach \( \hat{D} \). It follows that \( \hat{\varphi} \) does indeed map \( J \) crookedly to \( \hat{Q} \).

\[\triangle\]

Now we can conclude the proof. Let \( n > n_1 \), and set \( k := n - n_0 \). By Proposition \[8.6\] for every \( J \in \mathcal{U}_n(\hat{Q}, \hat{\varphi}) \), there is \( I \in \mathcal{U}_{n_0}(Q, \varphi) \) such that \( I \subset \hat{\varphi}^k(J) \). Moreover, by the claim, there is at most one such \( J \) for every \( I \).

(a) If \( I \in \hat{\mathcal{U}}_{n_0}(Q, \varphi) \), then by the claim, \( \hat{\varphi}^k(J) \in \hat{\mathcal{U}}_{n_0}(\hat{Q}, \hat{\varphi}) \), and thus \( J \in \hat{\mathcal{U}}_{n_0}(\hat{Q}, \hat{\varphi}) \).

(b) If \( I = \varphi^{n_1-n_0}([\hat{A}, \hat{D}]) \), then \( \varphi^{n_1-n_0} \supset [\hat{A}, \hat{D}] \). By the claim, \( \varphi^{n_1-n_0} \) maps \( I \) crookedly over \( \hat{Q} \). By choice of \( \rho \), and Proposition \[8.4\] one of \( \varphi^{n_1}(\hat{B}) \) and \( \varphi^{n_1}(\hat{C}) \) is between \( \hat{A} \) and \( \hat{B} \), and the other between \( \hat{C} \) and \( \hat{D} \). Thus \( \varphi^n \) maps \( I \) crookedly over \( \hat{Q} \).

In conclusion, at least one more of the intervals in \( \mathcal{U}_n(\hat{Q}, \hat{\varphi}) \) is mapped crookedly than was the case for \( \mathcal{U}_n(Q, \varphi) \), and the proof is complete.

**Proof of Proposition \[9.2\]** Let \( S_0 \) be a quadruple with \( S_0 < Q \). Let \( m \) be the integer such that

\[
\#\mathcal{U}_n(S_0, \varphi) - \#\hat{\mathcal{U}}_n(S_0, \varphi) = m
\]
for all sufficiently large $n$, as in Proposition 9.3.
Let $S_0 < S_1 < \cdots < S_m = Q$ be a sequence of quadruples. Inductively apply Proposition 9.3 to obtain a sequence of functions $F_0 = F, F_1, \ldots, F_k \equiv \hat{F}$, with $k \leq m$ and $F_j \in \tilde{F}_{N+j}$, such that

$$m_j := \lim_{n \to \infty} \# U_n(S_j, \varphi_j) - \# \hat{U}_n(S_j, \varphi_j)$$

is strictly decreasing, and $m_k = 0$. Then $\tilde{F}$ is the desired function. ■

Proof of Theorem 1.4. Let $F$ be as in Theorem 9.1, and $\hat{F}$ its $2\pi i$-periodic extension. By Theorem 3.8, there is a disjoint-type entire function for which every Julia continuum is homeomorphic to a Julia continuum of $\hat{F}$, and hence also a pseudo-arc. ■

Proof of Theorem 1.5. The theorem also follows from the construction in the next section, which uses Bishop’s quasiconformal folding. Here, we indicate how to prove the theorem without recourse to quasiconformal folding. Firstly, note that it follows from the proof of Theorem 3.8 that the entire function $g$ constructed has finite lower order. Indeed, the function $g$ is quasiconformally equivalent near infinity to the universal covering $f$ defined by

$$f(\exp(\zeta)) = \exp(F(\zeta)),$$

and the latter has finite lower order. (Compare [ERG15, Proof of Proposition 2.3(b)].)

However, with this argument we cannot directly show that the lower order can be to be $1/2$. To do so, we modify the proof by changing the functions in the class $\mathcal{F}$. Instead of considering conformal isomorphisms $T \to \mathbb{H}$, where $T$ is a tract as in Figure 3, we instead consider conformal isomorphisms

$$F: T \to H := \{x + iy: x > -14 \log |y|\},$$

again normalised so that $F(5) = 5$ and $F(\infty) = \infty$. This does not affect the proof of the theorem. Indeed, we only used the range of the function $F$ in certain estimates of the hyperbolic metric, which hold equally for $F$ with range $H$ as above.

Then we apply Theorems 1.7, 1.8 and 1.9 of [Rem13] to the function $F$, to obtain an entire function whose Julia continua are all pseudo-arcs, and which has order $1/2$ by (9.1).

10. Finitely many singular values

In this section, we indicate how to modify the construction to prove Theorem 1.6. We follow similar lines as [Bis15a, Section 18] and [Rem16, Section 15], and assume that the reader is familiar with Bishop’s technique of quasiconformal folding from [Bis15a].

Proof of Theorem 1.6. Let $F: T \to \mathbb{H}$ be the function $F \in \mathcal{F}$ from Theorem 9.1. We modify $F$ to a function $\tilde{F}: \tilde{T} \to \mathbb{H}$, where

$$\tilde{T} = T \cup \{x + iy: x \leq 4 \text{ and } |y| < \pi\},$$

and $\tilde{F}$ is chosen such that $F(5) = 5$, and $F(z) \to \infty$ as $\Re z \to \infty$ in $\tilde{T}$.

We do not have $\tilde{T} \subset \mathbb{H}$, so $\tilde{F}$ is not a function in $\mathcal{H}$. However, $\tilde{F} \in \mathcal{F}$, and hence Lemma 3.3 also holds for $\tilde{F}$. It follows that Lemma 4.2 also holds for the one-dimensional projection $\tilde{\varphi}$ of $\tilde{F}$. Moreover, the maps $\varphi$ and $\tilde{\varphi}$ behave similarly:
Claim. There exists constants $C, X > 0$ with the following property. If $x \geq 9$ and $n \geq 1$ with $\tilde{\varphi}^n(x) \geq X$, then

$$|\tilde{\varphi}^n(x) - \varphi^n(x)| \leq C.$$

Proof. Let $\varphi: \tilde{T} \to T$ be a quasiconformal homeomorphism such that $\varphi(z) = z$ when $\text{Re } z \geq 5$; such a map is easy to construct. Then we have

$$\psi \circ \tilde{F} = F \circ \varphi,$$

where $\psi$ is a quasiconformal self-map of the upper half-plane. The dilatation of $\psi$ is supported on a bounded set, and thus $\psi(z) \approx c \cdot z$ as $z \to \infty$, for some $c > 0$.

Hence the hyperbolic distance between $x$ and $\psi(x)$ is uniformly bounded (for $x \geq 5$, say), and therefore the Euclidean distance between $F^{-1}(x)$ and $F^{-1}(\psi(x))$ is uniformly bounded by a constant $\delta$. Recall that $F^{-1}(\psi(x)) = \tilde{F}^{-1}(x)$, as long as this point as real part at least 5.

Set $X := 9 + \delta$ and $C := 2\delta$. Recall that $\tilde{\varphi}^n(x) \geq 9 + \delta$ implies $\tilde{\varphi}^k(x) \geq 9 + \delta$ for $k = 0, \ldots, n$, by Lemma 4.2.

The claim now follows by induction. If it holds for $n$, then we have

$$|\tilde{\varphi}^{n+1}(x) - \varphi^{n+1}(x)| \leq |\tilde{\varphi}^{n+1}(x) - \varphi(\tilde{\varphi}^n(x))| + |\varphi(\tilde{\varphi}^n(x)) - \varphi^{n+1}(x)|
\leq |\tilde{F}^{-1}(\tilde{\varphi}^n(x)) - F^{-1}(\tilde{\varphi}^n(x))| + \delta \leq 2\delta = C,$$

as desired. $\triangle$

Now consider the half-plane $H = \{a + ib: a > 4\}$ and the $2\pi i$-periodic extension $G$ of the restriction

$$\tilde{F}: \tilde{F}^{-1}(H) \to H.$$

We have $\overline{G^{-1}(H)} \subset H$, at least if $r_1(F)$ was chosen sufficiently large. Indeed, on the initial part of the tract, the function $\tilde{F}$ is close to the conformal isomorphism

$$F_S: \tilde{S} = \{a + ib: |b| < \pi\} \to \mathbb{H}; \quad z \mapsto 5 \exp\left(\frac{z - 5}{2}\right),$$

and for $z \in H$ we have

$$\text{Re } F_S^{-1}(z) = 5 + 2 \cdot \log \frac{|z|}{5} \geq 5 + 2 \log \frac{4}{5} = 5 - \log \frac{25}{16} > 4.$$

So the map $G$ is a disjoint-type function in the class $\mathcal{B}^p_{\log}$ as defined in [Rem16, Definition 3.3], and it makes sense to consider its Julia continua, exactly as we did for the function $F$. That is, a Julia continuum of $G$ is a connected component of the set of points that remain in $H$ under iteration of $\tilde{F}$.

It follows from the claim above that $\tilde{\varphi}$ satisfies the hypotheses of Theorem 5.3, and the proof of that theorem applies equally to $G$. So all of the Julia continua of $G$ are pseudo-arcs.

Now we apply quasiconformal folding to the function $\tilde{F}$. To do so, introduce vertices on $\tilde{T}$ so that $G = \exp(\tilde{T})$ is a bounded-geometry tree in the sense of [Bis15a]. These vertices should be placed such that the two intervals on $\partial \mathbb{H}$ corresponding to an edge have length at least $\pi$ under $\tilde{F}$. We may use standard estimates of harmonic measure to deduce that this can be achieved while letting the length of any edge at real parts $\geq R$ tend to zero like $\exp(c \cdot R)$, for some $c > 0$. 

The quasiconformal folding theorem [Bis15a, Theorem 1.1] now yields an entire function \( f \), having at most two critical values and no asymptotic values, and a \( K \)-quasiconformal map \( \varphi \) such that 
\[
f \circ \varphi = \cosh \circ \tilde{F}
\]
on the complement of the set 
\[
G(r_0) := \bigcup_e \{ z \in \mathbb{C} : \text{dist}(z,e) < r_0 \text{diam}(e) \},
\]
where the union is over all edges of \( G \), and \( r_0 \) is a universal constant. In particular, the dilatation of the map \( \varphi \) is supported on the set \( G(r_0) \). Precomposing \( \varphi \) with a suitable linear map, we may choose \( f \) to have disjoint type.

Our assumption on the lengths of edges on \( \partial \tilde{T} \) implies that the cylindrical area of \( G(r_0) \) is finite, and therefore by the Teichmüller-Wittich-Belinski-Lehto Theorem, the map \( \varphi \) is asymptotically conformal at infinity. It follows that the map \( f \) has lower order \( 1/2 \).

It remains to show that the Julia continua of \( f \) are all pseudo-arcs. Suppose, by contradiction, that \( \tilde{K} \) was a decomposable subcontinuum of \( J(f) \). Then, as in the proof of Theorem 3.8, we can find a subcontinuum \( \tilde{K} \subset K \), for some \( n \geq 0 \), which is also decomposable, and which furthermore has the following properties:

(a) the iterates of \( f \) tend to infinity uniformly on \( \tilde{K} \);
(b) for sufficiently large \( n \), the sets \( f^n(\tilde{K}) \) are disjoint from \( G(r_0) \).

As noted in [Rem16, Section 15], any such subcontinuum of \( J(f) \) is homeomorphic to a subcontinuum of \( J(G) \). Therefore \( \tilde{K} \) is a pseudo-arc, which is a contradiction to the assumption that \( \tilde{K} \) is decomposable.

\[\blacksquare\]

**Proof of Corollary 1.7.** Let \( g \) be the function from Theorem 1.6; we may assume that its critical values are 0 and 1. Since \( g \) has no asymptotic values, the set \( T := g^{-1}([0,1]) \) has the structure of an infinite tree, with vertices at the preimages of 0 and 1.

The function \( g \) has infinitely many critical points over both 0 and 1. This follows immediately from the quasiconformal folding technique, but we may also see this directly: Every vertex of degree 1 is a leaf of the tree \( T \); since \( T \) is connected, it follows that every element of \( g^{-1}(0) \) is adjacent to at least one element of \( g^{-1}(1) \) of degree \( \geq 2 \), and vice versa.

In particular, we may pick two critical points \( c_0 \) and \( c_1 \) with \( g(c_0) = 0 \) and \( g(c_1) = 1 \) such that \( c_0 \) and \( c_1 \) are not adjacent in \( T \). Define 
\[
f(z) := g(z \cdot c_1 + (1-z) \cdot c_0).
\]

Then 0 and 1 are super-attracting fixed points of \( f \). Since these are the only critical values of \( f \), it follows that \( f \) is hyperbolic in the sense of [BFRG15, Definition 1.1]. By [BFRG15, Corollary 1.9], the Julia set \( J(f) \) is locally connected. Moreover, every component of \( F(f) = \mathbb{C} \setminus J(f) \) is a Jordan curve [BFRG15, Theorem 1.4]

To show that \( J(f) \) is a Sierpiński carpet, we must show that different components of \( F(f) = \mathbb{C} \setminus J(f) \) have pairwise disjoint closures. Let \( U_0 \) and \( U_1 \) be the components containing 0 and 1, respectively. Recall that \( U_0 \) is a Jordan domain, and that \( \overline{U_0} \) contains no critical points other than 0. So we can find a Jordan domain \( \tilde{U}_0 \supset \overline{U_0} \) whose closure
also contains no other critical points. Every connected component of $f^{-1}(\tilde{U}_0)$ is mapped to $\tilde{U}_0$ as a finite-degree proper map with a single critical point, and in particular contains a unique connected component of $f^{-1}(U_0)$. Therefore different connected components of $f^{-1}(U_0)$ have disjoint closures. The same argument works for components of $f^{-1}(U_1)$.

Next we show that $\partial U_0$ and $\partial U_1$ are disjoint. Suppose, by contradiction, that there was a common point $z_0$. There can be at most one such point. Indeed, otherwise there would be a Jordan curve in $\overline{U}_1 \cup \overline{U}_0$ that is contained in $K := \overline{U}_1 \cup \overline{U}_2$. Since $K$ is bounded and forward-invariant, the interior region of this curve would be contained in the Fatou set, which contradicts the fact that 0 and 1 belong to different Fatou components.

We may find invariant curves $\gamma_0$ and $\gamma_1$ such that $\gamma_j$ connects $z_0$ to $j$ in $U_j$. By considering a homotopy between $\gamma_0 \cup \gamma_1$ and $[0,1]$, we see that 0 and 1 are adjacent in the graph $f^{-1}([0,1])$. This contradicts the choice of $c_0$ and $c_1$, and the definition of $f$. So $\partial U_0$ and $\partial U_1$ are disjoint. This implies that different connected components of the backwards orbits of $U_0$ and $U_1$ must also be disjoint.

The existence of invariant pseudo-arcs in $J(f)$ follows from [Rem09]. Indeed, the main result of that paper shows the existence of a homeomorphism that conjugates the disjoint-type map $g$ to $f$ on the set $J_{\geq R}(g)$ of points whose orbit under $g$ remains outside a certain disc $D(0,R)$ at all times. This set contains infinitely many different invariant Julia continua of $g$, which are all pseudo-arcs by our main theorem.

Similarly, the collection of all Julia continua of $g$ that are completely contained in $J_{\geq R}(g)$ give rise to the uncountable collection of pseudo-arcs whose existence is asserted in the final part of the corollary.

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\section*{References}

[BFRG15] Walter Bergweiler, Núria Fagella, and Lasse Rempe-Gillen, \textit{Hyperbolic entire functions with bounded Fatou components}, Comment. Math. Helv. \textbf{90} (2015), no. 4, 799–829.

[Bin48] R. H. Bing, \textit{A homogeneous indecomposable plane continuum}, Duke Math. J. \textbf{15} (1948), 729–742.

[Bin51] R. H. Bing, \textit{Concerning hereditarily indecomposable continua.}, Pacific Journal of Mathematics \textbf{1} (1951), no. 1, 43 – 51.

[Bis15a] Christopher J. Bishop, \textit{Constructing entire functions by quasiconformal folding}, Acta Math. \textbf{214} (2015), no. 1, 1–60.

[Bis15b] \textit{Models for the Eremenko-Lyubich class}, J. Lond. Math. Soc. (2) \textbf{92} (2015), no. 1, 202–221.

[BK07] Krzysztof Barański and Bogusława Karpińska, \textit{Coding trees and boundaries of attracting basins for some entire maps}, Nonlinearity \textbf{20} (2007), no. 2, 391–415.

[BM07] A. F. Beardon and D. Minda, \textit{The hyperbolic metric and geometric function theory}, Quasiconformal mappings and their applications, Narosa, New Delhi, 2007, pp. 9–56. MR 2492498 (2011c:30108)

[Dou93] Adrien Douady, \textit{Descriptions of compact sets in C}, Topological methods in modern mathematics (Stony Brook, NY, 1991), Publish or Perish, Houston, TX, 1993, pp. 429–465.

[ERG15] Adam Epstein and Lasse Rempe-Gillen, \textit{On invariance of order and the area property for finite-type entire functions}, Ann. Acad. Sci. Fenn. Math. \textbf{40} (2015), no. 2, 573–599.

[Fat26] Pierre Fatou, \textit{Sur l’itération des fonctions transcendantes entières}, Acta Math. \textbf{47} (1926), 337–370.

[GM05] John B. Garnett and Donald E. Marshall, \textit{Harmonic measure}, New Mathematical Monographs, vol. 2, Cambridge University Press, Cambridge, 2005.
[Hen64] George W. Henderson, *The pseudo-arc as an inverse limit with one binding map*, Duke Mathematical Journal 31 (1964), no. 3, 421–425.

[HO16] Logan C. Hoehn and Lex G. Oversteegen, *A complete classification of homogeneous plane continua*, Acta Math. 216 (2016), no. 2, 177–216.

[Kna22] B. Knaster, *Un continu dont tout sous-continu est indécomposable*, Fundam. Math. 3 (1922), 247–286 (French).

[Lew99] Wayne Lewis, *The pseudo-arc*, Bol. Soc. Mat. Mexicana (3) 5 (1999), no. 1, 25–77. MR 1692467 (2000f:54029)

[Moi48] Edwin E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Trans. Amer. Math. Soc. 63 (1948), 581–594.

[Nad92] Sam B. Nadler, Jr., *Continuum theory. An introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York, 1992.

[Pom92] Christian Pommerenke, *Boundary behaviour of conformal maps*, Grundlehren der Mathematischen Wissenschaften, vol. 299, Springer-Verlag, Berlin, 1992.

[Rem09] Lasse Rempe, *Rigidity of escaping dynamics for transcendental entire functions*, Acta Math. 203 (2009), no. 2, 235–267.

[Rem13] Lasse Rempe-Gillen, *Hyperbolic entire functions with full hyperbolic dimension and approximation by Eremenko-Lyubich functions*, Proc. London Math. Soc. 108 (2013), no. 5, 1193–1225.

[Rem16] Lasse Rempe, *Arc-like continua, Julia sets of entire functions, and Eremenko’s Conjecture*, Preprint, 2016, arXiv:1610.06278.

[Rem21] Lasse Rempe, *The Eremenko-Lyubich constant*, Preprint, 2021.

[Roe08] P. Roesch, *On local connectivity for the Julia set of rational maps: Newton’s famous example*, Ann. of Math. (2) 168 (2008), no. 1, 127–174.

[RRRS11] Günter Rottenfußer, Johannes Rückert, Lasse Rempe, and Dierk Schleicher, *Dynamic rays of bounded-type entire functions*, Ann. of Math. (2) 173 (2011), no. 1, 77–125.

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