A MAP FROM LAWSON HOMOLOGY TO DELIGNE COHOMOLOGY

WENCHUAN HU

Abstract. A natural Abel-Jacobi type map from Lawson homology to Deligne cohomology for smooth complex projective varieties is constructed by using the Harvey-Lawson “spark” complexes. We also compare this to Abel-Jacobi type constructions by others.

Contents

1. Introduction 1
2. Sparks and differential characters 2
3. The construction of the map 4
4. Relationships to other Abel-Jacobi type constructions 8
References 10

1. Introduction

Let $X$ be a complex projective variety of dimension $n$. Let $\mathcal{C}_p(X)$ be the space of all effective algebraic $p$-cycles on $X$, i.e., an element $c \in \mathcal{C}_p(X)$ is a finite sum $c = \sum n_i V_i$, where $n_i$ is a nonnegative integer and $V_i$ is an irreducible variety of dimension $p$. Let $\mathcal{Z}_p(X)$ be the space of all algebraic $p$-cycles on $X$.

The Lawson homology $L_p H_k(X)$ of $p$-cycles is defined by

$$L_p H_k(X) := \pi_{k-2p}(\mathcal{Z}_p(X))$$

for $k \geq 2p \geq 0$, where $\mathcal{Z}_p(X)$ is provided with a natural topology so that it is an abelian topological group (cf. [F], [L1] and [L2]). For general background, the reader is referred to [L2].

Friedlander and Mazur [FM] showed that there are natural maps, called cycle class maps $\Phi_{p,k} : L_p H_k(X) \to H_k(X)$ from Lawson homology to singular homology. Denote by $L_p H_k(X)_{hom}$ the kernel of $\Phi_{p,k}$, i.e., $L_p H_k(X)_{hom} = \ker \{ \Phi_{p,k} : L_p H_k(X) \to H_k(X) \}$.

Assume that $X$ is a smooth complex manifold. Let $\Omega^k_X$ the sheaf of holomorphic $k$-form on $X$. The Deligne complex of level $p$ is the complex of sheaves

$$\mathbb{Z}_D(p) : 0 \to \mathcal{Z}(p) \to \Omega^0_X \to \Omega^1_X \to \Omega^2_X \to \cdots \to \Omega^{p-1}_X \to 0,$$

where $\mathcal{Z}(p) := (2i\pi)^p \cdot \mathbb{Z} \subset \mathbb{C}$.

The Deligne cohomology of $X$ in level $p$ is defined as the hypercohomology of this complex $H^*_p(X, \mathcal{Z}(p)) := H^*(X, \mathbb{Z}_D(p))$.

For more details on Deligne cohomology, the reader is referred to [EV].

Date: September 29, 2009.
We define a natural map from Lawson homology to the corresponding Deligne cohomology using the theory of differential characters introduced by Cheeger in [C] and the theory of D-bar sparks developed by Harvey and Lawson in [HL1] and [HL2]. This theory of differential characters has been developed in [CS], [GS], [Ha] and systematically generalized by Harvey, Lawson and Zweck in [HLZ]. This map is a generalization of the Abel-Jacobi map for Lawson homology defined by the author in [H].

The main result in this paper is the following

**Theorem 1.1.** Let $X$ be a smooth complex projective variety of dimension $n$. We have a well-defined homomorphism

$$\hat{a}_X: L_p H_{k+2p}(X) \to H_D^{2(n-p)-k}(X, \mathbb{Z}(n-p-k-1)),$$

which coincides with the generalized Abel-Jacobi map defined in [H] when $\hat{a}$ is restricted on $L_p H_k(X)_{\text{hom}}$ and the projection of the image of $\hat{a}$ under $\delta_2$ is the natural map $\Phi_{p,k}$.

If there is no confusion, $\hat{a}_X$ is simply denoted by $\hat{a}$. The notations $\hat{a}_f$ and $\delta_2$ will be defined below.

We also obtain functorial properties for $\hat{a}$, that is, it commutes with projective morphisms between smooth complex projective varieties (cf. Proposition 3.10). Moreover, our construction is also applied to higher Chow groups and we obtain a natural homomorphism from higher Chow groups to Deligne cohomology (cf. Proposition 4.4). Our maps on higher Chow groups factors those on Lawson homology (cf. Corollary 4.5). However, they seem to be different from those (called regulator maps) given by Bloch [B].

2. Sparks and differential characters

In this section we review the necessary background on materials of sparks and differential characters for our construction in section 3. For details, see [HLZ], [HL1] and [HL2]. In this section $X$ denotes a smooth manifold unless otherwise noted.

**Definition 2.1.** Set $\mathcal{E}^k(X) := \text{the space of smooth differential forms $k$-form on $X$ with $C^\infty$-topology}$; $\mathcal{D}^k(X) := \{\phi \in \mathcal{E}^k(X) | \text{supp}(\phi) \text{ is compact}\}$. We say the space of currents of degree $k$ (and dimension $n-k$) on $X$, it means the topological dual space $\mathcal{D}'^k(X) \equiv \mathcal{D}^n-k(X)'$. Set $\mathcal{R}^k(X) := \text{the locally rectifiable currents of degree $k$ (dimension $n-k$) on $X$}$; $\mathcal{IF}^k(X) := \text{the locally integrally flat currents of degree $k$ on $X$}$; and $\mathcal{I}^k(X) := \text{the locally integral currents of degree $k$ on $X$}$.

The following notation was firstly given in [HLZ]:

**Definition 2.2.** The space of sparks of degree $k$ on $X$ is defined to be

$$\mathcal{S}^k(X) := \{s \in \mathcal{D}^k(X)|da = \phi - R, \text{ where } \phi \in \mathcal{E}^{k+1}(X) \text{ and } R \in \mathcal{IF}^{k+1}(X)\}.$$

**Definition 2.3.** For each integer $k$, $0 \leq k \leq n$, the de Rham-Federer characters of degree $k$ id defined to be the quotient

$$\mathcal{H}^k(X) := \mathcal{S}^k(X)/\{d\mathcal{D}^{k-1}(X) + \mathcal{IF}^k(X)\}$$
The equivalence class in \( \hat{H}^k(X) \) of a spark \( a \in S^k(X) \) will be denoted by \( \hat{a} \).
It has been proved that \( \phi \) and \( R \) in the decomposition of \( da \) above is unique (cf. [HLZ], Lemma 1.3). Moreover, there are two well-defined surjective maps:
\[
\delta_1 : \hat{H}^k(X) \to Z_0^{k+1}(X) ; \quad \delta_2 : \hat{H}^k(X) \to H^{k+1}(X, \mathbb{Z});
\]
where \( Z_0^{k+1}(X) \) denotes the lattice of smooth \( d \)-closed, degree \( k + 1 \) forms on \( X \) with integral periods.

Now we can give the definition of Riemannian Abel-Jacobi map. Let \( X \) be compact Riemannian manifold. Any current \( R \) on \( X \) has a Hodge decomposition (cf. [HP])
\[
R = H(R) + dd^*G(R) + d^*dG(R)
\]
where \( H \) is harmonic projection and \( G \) is the Green operator. Also recall that \( d \) commutes with \( G \), so that if \( R \) is a cycle, then \( dG(R) = 0 \). For \( R \in \mathcal{IF}^{k+1}(X) \), set
\[
a(R) := -d^*G(R)
\]
then we have a well-defined map
\[
\hat{a} : \mathcal{B}^{k+1}(X) \to \text{Jac}^k(X)
\]
which is called the \( k \)-th Riemannian Abel-Jacobi map.

In [HL1], the concept of homological spark complex and its associated group of homological spark classes are given. In [HL2], a generalized version of homological spark complex has been defined.

**Definition 2.4.** A homological spark complex is a triple of cochain complexes \( (F^*, E^*, I^*) \) together with morphisms
\[
\Psi : I^* \to F^* \supset E^*
\]
such that:
\[
(1) \quad \Psi(I^k) \cap E^k = \{0\} \text{ for } k > 0,
(2) \quad H^*(E) \cong H^*(F), \text{ and }
(3) \quad \Psi : I^0 \to F^0 \text{ is injective.}
\]

**Definition 2.5.** In a given spark complex \( (F^*, E^*, I^*) \) a spark of degree \( k \) is a pair
\[
(a, r) \in F^k \oplus I^{k+1}
\]
which satisfies the spark equation
\[
(1) \quad da = e - \Psi(r) \text{ for some } e \in E^{k+1}, \text{ and }
(2) \quad dr = 0.
\]
The group of sparks of degree \( k \) is denoted by \( S^k = S^k(F^*, E^*, I^*) \).

**Definition 2.6.** Two sparks \( (a, r), (a', r') \in S^k(F^*, E^*, I^*) \) are equivalent if there exists a pair
\[
(b, s) \in F^{k-1} \oplus I^k
\]
such that
(1) \( a - a' = db + \Psi(s) \), and
(2) \( r - r' = -ds \).

The set of equivalence classes is called the \textbf{group of spark classes of degree} \( k \) associated to the given spark complex and will be denoted by \( \hat{H}^k(F^*, E^*, I^*) \) or simply \( \hat{H}^k(F) \).

As usual, let \( Z^k(E) = \{ e \in E^k | de = 0 \} \) and set
\[
Z^k_1(E) := \{ e \in Z^k(E) | [e] = \Psi(\rho) \text{ for some } \rho \in H^k(I) \}
\]
where \([e]\) denotes the class of \( e \) in \( H^k(E) \) (note that \( de = 0 \)). The following lemma was proved in [HLZ].

**Lemma 2.1.** There exist well-defined surjective homomorphisms:
\[
\delta_1 : \hat{H}^k(F) \rightarrow Z^k_1(E) \quad \text{and} \quad \delta_2 : \hat{H}^k(F) \rightarrow H^{k+1}(I)
\]
given on any representing spark \((a, r) \in S^k \) by
\[
\delta_1(a, r) = e \quad \text{and} \quad \delta_2(a, r) = [r]
\]
where \( da = e - \Psi(r) \) as in Definition 2.5

The following example is the main object which will be dealt with in the next section.

**Example 2.2.** Now let \( X \) be a smooth complex projective variety of dimension \( n \). Set
\[
F^m = D^m(X, q) := \bigoplus_{r+s=m, r<q} D^{r,s}(X) \quad \text{and} \quad \tilde{d} = \Psi \circ d
\]
where
\[
\Psi : D^m(X) \rightarrow D^m(X, q)
\]
is the projection \( \Psi(a) = a^{0,m} + a^{1,m-1} + \cdots + a^{q-1, m-q+1} \).
\[
E^m = E^m(X, q) := \bigoplus_{r+s=m, r<q} E^{r,s}(X) \quad \text{and} \quad \tilde{d} = \Psi \circ d
\]
and
\[
I^m = I^m(X)
\]

It has been shown in [HL2] that the above triple \((F^*, E^*, I^*) \) is a homological spark complex. The group of associated spark classes in degree \( m \) will be denoted by \( \hat{H}^m(X, q) \). From this homological spark complex, one has
\[
\ker(\delta_1) = H^{m+1}_D(X, \mathbb{Z}(q)).
\]

Denote by \( H^*(X, q) \) the cohomology \( H^*(E, q) \cong H^*(F, q) \) and set \( H^*_Z(X, q) := \text{Image}(\Psi : H^*(X, \mathbb{Z}) \rightarrow H^*(X, q)) \).

3. The construction of the map

In this section, \( X \) denotes a smooth projective variety. Let \( Z_p(X) \) be the space of algebraic \( p \)-cycles with a natural topology and a base point, i.e., the ‘null’ \( p \)-cycle (cf. [LL1], [L2] or [F]). Let \( \Omega^k \mathcal{Z}_p(X) \) be the \( k \)-th iterated loop space with the given base point. Explicitly, we write \( S^k \) as \( \mathbb{R}^k \cup \infty \) and we have
\[
\Omega^k Z_p(X) = \{ f : S^k \rightarrow Z_p(X) | \text{ f is continuous with } f(\infty) = 0 \}.
\]

For \( k = 0 \), an element \( f \in \Omega^0 Z_p(X) \) is the difference of two maps \( f_1, f_2 \), where \( f_i : S^0 \rightarrow \mathcal{C}_p(X) \) are maps from \( S^0 \) to the space of effective \( p \)-cycles \( \mathcal{C}_p(X) \) for \( i = 1, 2 \) (cf. [LF] or [FG]). Therefore \( \Omega^0 Z_p(X) \) can be identified with the space
containing differences of effective $p$-cycles, i.e., $\mathcal{Z}_p(X)$. The map $f$ determines an integral current $c_f = f_1(s_0) - f_2(s_0)$, where $s_0 \in S^0 - \{\infty\}$. In fact $c_f$ is a closed integral current since it is an algebraic cycle on $X$. By definition, the homology class of $c_f$ depends only on the homotopy class $[f] \in \pi_0 \mathcal{Z}_p(X) \cong L_p \mathcal{H}_{2p}(X)$ of $f$, where the isomorphism was given in [H]. Note that $c_f \in D_{p,p}(X)$.

For $k > 0$ and a continuous map $f : S^k \to \mathcal{Z}_p(X)$, we can find a map $g : S^k \to \mathcal{C}_p(X)$ such that $g$ is homotopic to $f$ (cf. [LF] or [FG]) and $g$ is piecewise linear with regard to a triangulation of $\mathcal{C}_p(X)$. Hence one can define an integral current $c_g$ over $X$. Note that this current $c_g$ is actually a cycle, i.e., $d(c_g) = 0$. Moreover, the homology class of $c_g$ depends only on the homotopy class of $g$. The detail of the construction of $c_g$ from the map $g : S^k \to \mathcal{Z}_p(X)$ can be found in section 3 of [H].

In any case, we obtain an integral cycle $c_f$ from an element $f \in \Omega^k \mathcal{Z}_p(X)$. In the following of this section we will construct an element in a suitable Deligne cohomology group from the integral cycle $c_f$.

Consider Example 2.2 with $m = 2(n - p) - k - 1, q = n - p - k - 1$. All the following argument will focus on this homological spark complex.

**Definition 3.1.** Set $a_f := -\Psi(d^*G(c_f))$. Then $(a_f, c_f)$ is called the Hodge spark of the map $f : S^k \to \mathcal{Z}_p(X)$, where $G$ is the Green’s operator. Let

$$\hat{a}_f \in \hat{H}^{2(n-p)-k-1}(X, n-p-k-1)$$

be the differential character corresponding to the Hodge spark $(a_f, c_f)$.

**Remark 3.1.** In Definition 3.1 we need to choose a Riemannian metric of $X$. However, $\hat{a}_f$ is independent of the choice of such a metric (cf. [HLZ], p. 830.).

**Lemma 3.2.** If $f \in \Omega^k \mathcal{Z}_p(X)$, then $\delta_1(\hat{a}_f) = 0 \in \hat{H}^{2(n-p)-k-1}(X, n-p-k-1)$.

**Proof.** By definition of $\delta_1$, we have $\delta_1(\hat{a}_f) = \Psi\left(\hat{H}(c_f)\right)$, where $\hat{H}(c_f)$ is the harmonic part of $c_f$. Note that $\hat{H}(c_f)$ and $c_f$ are of the same $(\ast, \ast)$-type. Moreover, from the construction of $c_f$, we have

$$c_f \in \bigoplus_{r+s=k, |r-s| \leq k} D'_{p+r,p+s}(X)$$

(cf. [H], Remark 3.3)

By the type reason, the projection of $\hat{H}(c_f)$ under $\Psi$ on

$$\bigoplus_{r+s=k, r > k+1} D'_{p+r,p+s}(X)$$

is zero. This is exactly the image of $\hat{a}_f$ under $\delta_1$. \hfill \square

Hence we get an element $\hat{a}_f \in \ker(\delta_1) = H^{2(n-p)-k}_D(X, \mathbb{Z}(n-p-k-1))$. Therefore we get a map

$$\hat{a} : \Omega^k \mathcal{Z}_p(X) \to H^{2(n-p)-k}_D(X, \mathbb{Z}(n-p-k-1))$$

which is defined by $\hat{a}(f) = \hat{a}_f$.

If the map $f$ is homotopically trivial, then we have the following:

**Lemma 3.3.** If $f : S^k \to \mathcal{Z}_p(X)$ is homotopically trivial, then $\hat{a}(f) = 0$. 
Proof. Let $F : D^{k+1} \to Z_p(X)$ be an extension of $f$, i.e., $F|_{\partial(D^{k+1})} = f$. Let $c_F$ be the current over $X$ defined by $F$. As showed in Lemma 3.2, $c_F$ is an integral current with boundary $\partial(c_F) = c_f$. By Corollary 12.11 in [HLZ], we have $\widehat{\alpha}_f = H(\Psi(c_F))$. Since

$$c_F \in \bigoplus_{r+s=k+1, |r-s| \leq k+1} D'_{p+r,p+s}(X),$$

the projection of $H(c_F)$ under $\Psi$ on

$$\bigoplus_{r+s=k, r+k+1} D'_{p+r,p+s}(X)$$

is zero. Note that $\Psi$ commutes with the Laplace operator, we have $\widehat{\alpha}_f = 0$. □

By the Lemma 3.2 and Lemma 3.3, we have a well-defined map

$$\hat{\alpha} : L_pH_{k+2p}(X) \to H^2_{D}(n-p-k)(X, \mathbb{Z}(n-p-k-1)),$$

given by

$$\hat{\alpha}([f]) = \widehat{\alpha}_f.$$

Recall that the Deligne cohomology can be written as the middle part of a short exact sequence

$$0 \to H^2_{D}(n-p-k-1)(X, n-p-k-1) \to H^2_{D}(n-p-k)(X, \mathbb{Z}(n-p-k-1)) \to \ker(\Psi) \to 0,$$

where

$$H^2_{D}(n-p-k-1)(X, n-p-k-1) = \bigg\{ \bigoplus_{r+s=2(n-p)-k-1, r \leq n-p-k-1} H^{r,s}(X) \bigg\}$$

and

$$\ker(\Psi) = H^2(n-p-k)(X, \mathbb{Z}) \cap \bigg\{ \bigoplus_{r+s=2(n-p)-k, |r-s| \leq k+2} H^{r,s}(X) \bigg\}.$$

**Proposition 3.4.** The restriction of the above map to

$$L_pH_{k+2p}(X)_{\text{hom}} := \ker\{ \Phi_{p,k+2p} : L_pH_{k+2p}(X) \to H_{k+2p}(X, \mathbb{Z}) \}$$

is the generalized Abel-Jacobi map defined in [H] if we identify the $H^{r,s}(X)$ with $\{H^{n-r,n-s}(X)\}^*$ for all $0 \leq r, s \leq n$ and $H^2(\mathbb{Z}, \mathbb{Z})$ with $H_{2n-q}(X, \mathbb{Z})$.

**Proof.** Recall the definition that

$$\Phi : L_pH_{k+2p}(X)_{\text{hom}} \to \bigg\{ \bigoplus_{r+k+1, r+s=k+1} H^{r+s}(X) \bigg\}^* \bigg/ H_{2p+k+1}(X, \mathbb{Z})$$

is given by $\Phi([f]) = \Phi_f$, where $\Phi_f(\omega) = \int_\omega (H_{2p+k}(X, \mathbb{Z}))$ with $\partial(\omega) = c_f$. The Lemma 12.10 in [HLZ] implies that the two constructions coincide. □

**Remark 3.5.** It is easy to see from the definition of $\Psi_*$ that the image of $\Phi_{p,k+2p}$ is in $\ker(\Psi_*)$. Hence the natural map $\Phi_{p,k+2p} : L_pH_{k+2p}(X) \to H_{k+2p}(X)$ factors through $\hat{\alpha}$ and the map $\delta_2$ in the first $3 \times 3$ grid given in [HLZ], p.26.

**Remark 3.6.** Gillet and Soulé [GS] first showed that the Griffiths’ higher Abel-Jacobi map coincides with the Riemanian Abel-Jacobi. Harris [Ha] also discussed some related topics.

In summary, we have the following
Theorem 3.7. Let $X$ be a smooth complex projective variety of dimension $n$. We have a well-defined map

$$\hat{a} : L_p H_{k+2p}(X) \to H^{2(n-p)-k}_D(X, \mathbb{Z}(n-p-k-1)),$$

given by

$$\hat{a}([f]) = \hat{a}_f$$

which coincides with the generalized Abel-Jacobi map defined in [H] when $\hat{a}$ is restricted on $L_p H_{k+2p}(X)_{hom}$; and the projection of the image of $\hat{a}$ under $\delta_2$ is the natural map $\Phi_{p,k}$.

Proof. The first statement follows from Lemma 3.2 and 3.3 that $\hat{a}$ is well-defined. The second one follows from Proposition 3.4 and the third one follows from 3.5. □

Remark 3.8. The map $\hat{a}$ can be nontrivial even if restricted on $L_p H_{k+2p}(X)_{hom}$. This has been shown by Griffiths [G] for $k = 0$ and by the author [H] for $k > 0$.

In this section below, we will show that the homomorphism $\hat{a}$ is well-behaved with projective morphisms between smooth projective varieties.

Recall that Friedlander and Lawson have defined morphic cohomology groups $L^p H^k(X)$ for all $k \leq 2p$. Moreover, they have defined a duality between morphic cohomology groups $L^p H^k(X)$ with Lawson homology groups $L_{n-p} H_{2n-k}(X)$ for a projective variety $X$ (cf. [FL], [FL2]).

Theorem 3.9 ([FL2]). If $X$ is smooth projective of dim $X = n$, then the duality

$$D : L^p H^k(X) \to L_{n-p} H_{2n-k}(X)$$

is an isomorphism for all $k \leq 2p$.

Proposition 3.10. Let $\varphi : X \to Y$ be a projective morphism between smooth projective varieties $X$ and $Y$. Then we have the following commutative diagram

$$
\begin{array}{ccc}
L^p H^{2p-k}(X) & \xrightarrow{\bar{a}_X} & H^{2p-k}_D(X, \mathbb{Z}(p-k-1)) \\
\varphi^* & & \varphi^* \\
L^p H^{2p-k}(Y) & \xrightarrow{\bar{a}_Y} & H^{2p-k}_D(Y, \mathbb{Z}(p-k-1)).
\end{array}
$$

where $\bar{a}_X = \hat{a}_X \circ D$ and similarly for $\bar{a}_Y$.

Proof. Let $n = \dim(Y)$. Given $\alpha \in L^p H^{2p-k}(Y)$, we set

$$\beta = D(\alpha) \in L_{n-p} H_{2(n-p)+k}(Y)$$

and then choose a PL map $f : S^k \to Z_{n-p}(Y)$ such that $[f] = \beta$, where $[f]$ denotes the homotopy class of $f$. Then we have $\bar{a}_Y(\alpha) = \bar{a}_Y([f])$.

Let $c_f$ be the current constructed for $f$. On one hand, we have

$$\varphi^*(\bar{a}_Y(\alpha)) = \varphi^*(\bar{a}_Y \circ D(\alpha)) = \varphi^*(\bar{a}_Y([f])) = \varphi^*(-\Psi d^* G(c_f)).$$

On the other hand,

$$\begin{align*}
\bar{a}_X(\varphi^*(\alpha)) &= \hat{a}_X(D \varphi^*(\alpha)) \\
&= \hat{a}_X(\varphi^* D(\alpha)) \\
&= \hat{a}_X(\varphi^*([f])) \\
&= \hat{a}_X(\bar{a}_Y(\alpha)), \quad \text{(cf. [Pe], p.218)} \\
&= -\Psi d^* G(c_{\varphi^* f}),
\end{align*}$$

(2)
where \( \hat{\varphi}^* \) is Gysin homomorphism associated to the projective morphism \( \varphi : X \to Y \) (cf. [PC] for \( \varphi \) a regular embedding, [Pa] for any projective morphism).

Note that \( \varphi^* \) commutes with \( \Psi \), \( d^* \) and the Green operator \( G \). by comparing Equation (1) and (2), it is enough to show that

\[
\varphi^*(c_f) = c_{\hat{\varphi} \circ f}
\]

which follows from the construction of \( c_f \) and \( c_{\hat{\varphi} \circ f} \) (cf. [Hi], Section 3). This completes the proof of the proposition. \( \square \)

4. RELATIONSHIPS TO OTHER ABEL-JACOBI TYPE CONSTRUCTIONS

In this section we discuss relations between our construction to other Abel-Jacobi type constructions, e.g., the morphic Abel-Jacobi map constructed by M. Walker [Wi] and the Abel-Jacobi map for higher Chow groups by M. Kerr, J. Lewis and S. Müller-Stach [KLM].

From the construction of the homomorphism \( \hat{\alpha} \) in the last section, we see that it essentially divided into two main steps. First, for an element \( \alpha \in L_pH_{k+2p}(X) \), we can find a PL map \( f : S^k \to Z_p(X) \) such that its homotopy class \( \alpha = \hat{\alpha} \), and construct an integral cycle \( c_f \) on \( X \). Then we associate this integral cycle a Hodge spark \( \hat{\alpha} \) and show this spark is in the suitable target space, i.e., the Deligne cohomology \( H^{2(n-p)-k}_D(X, \mathbb{Z}(n-p-k-1)) \).

For the second step a Hodge decomposition is needed for currents on \( X \). Compact Riemannian manifolds are such examples.

The construction in the first step depends on what we are interested in. In section 3 we deal with those integral currents from elements of Lawson homology. It can be applied to other cases.

Applying Example 2.2 to \( m = 2(n-p) - k - 1, q = n - p - k \), we get

**Proposition 4.1.** Set

\[
\Omega^k(Z_p(X))_{htp-0} := \{ f \in \Omega^k(Z_p(X)) | f \text{ is PL and homotopically trivial} \}.
\]

Set \( b_f := -\psi_0(d^*G(c_f)) \), where \( \psi_0 \) denotes the projection corresponding to \( q = n - p - k \) in Example 2.2. Then \( (b_f, c_f) \) is the Hodge spark of the map \( f : S^k \to Z_p(X) \) and denote by \( \tilde{b}_f \in H^{2(n-p)-k-1}(X, \mathbb{Z}(n-p-k)) \) its associated differential character.

Then we have a natural homomorphism

\[
\tilde{b} : \Omega^k(Z_p(X))_{htp-0} \to H^{2(n-p)-k-1}(X, n-p-k)/H^{2(n-p)-k-1}_Z(X, n-p-k),
\]

given by

\[
\tilde{b}(f) = \tilde{b}_f.
\]

**Proof.** What we need to show is \( \delta_1(\tilde{b}_f) = 0 \in H^{2(n-p)-k-1}(X, n-p-k) \) since \( \ker(\delta_1) = H^{2(n-p)-k}_D(X, \mathbb{Z}(n-p-k)) \). By definition of \( \delta_1 \), we have \( \delta_1(\tilde{b}_f) = \psi_0(H(c_f)) \), where \( H(c_f) \) is the harmonic part of \( c_f \). From the construction of \( c_f \), we have

\[
c_f \in \bigoplus_{r+s=k, |r-s| \leq k} D'_{p+r,p+s}(X).
\]

Note that \( H(c_f) \) and \( c_f \) are of the same \((*, *)\)-type. By the type reason, the projection of \( H(c_f) \) under \( \psi_0 \) on

\[
\bigoplus_{r+s=k, r \geq k+1} D'_{p+r,p+s}(X)
\]
is zero. This is exactly the image of $\hat{b}_f$ under $\delta_1$.

So we have a homomorphism

$$ \hat{b} : \Omega^k(Z_p(X))_{htp \sim 0} \to H^{2(n-p)-k}_D(X, \mathbb{Z}(n-p-k)),$$

given by

$$\hat{b}([f]) = \hat{b}_f.$$

To compete the proof of the proposition, we need to show that $\delta_2(\hat{b}_f) = 0$ since

$$\ker \delta_1 \cap \ker \delta_2 = H^{2(n-p)-k-1}(X, n-p-k)/H^{2(n-p)-k-1}_D(X, n-p-k).$$

From the definition of $\delta_2$, we have $\delta_2(\hat{b}_f) = [\Psi_0(c_f)]$, where $[c]$ denotes the homological class of the integral cycle $c$. From the proof of Lemma 3.3, $c_f$ is the boundary of an integral current $c_F$. Since the projection $\Psi_0$ commutes with the boundary map, we have $\Psi_0(c_f) = \partial(\Psi_0(c_F))$ and hence $[\Psi_0(c_f)] = 0$. That is to say, $\delta_2(\hat{b}_f) = 0$. □

Sometimes we denote by $J^k(X, p) := H^k(X, p)/H^k_D(X, p)$. In proposition 4.1 those $c_f$ is called linearly equivalent to zero if if $\hat{b}_f = 0$. Set $\Omega^k(Z_p(X))_0 := \{ f \in \Omega^k(Z_p(X))_{htp \sim 0} | \hat{b}_f = 0 \}$. Therefore, we get an injection (also called $\hat{b}$) on the quotient

$$\hat{b} : \Omega^k(Z_p(X))_{htp \sim 0} \to J^{2(n-p)-k-1}(X, n-p-k).$$

**Remark 4.2.** When $k = 0$, the construction here coincides with the Griffiths’ Abel-Jacobi map on the space of algebraic cycles $p$-cycles which are algebraic equivalent to zero. (cf. [HLZ], Example 12.16).

**Remark 4.3.** The construction here for $k = 0$ is different from the morphic Abel-Jacobi map constructed by Walker [W]. On the other hand, we require $X$ is a smooth complex projective variety but there is no such requirement for these conditions in the definition. On the other hand, at least in the case that $L_p H_{2p+1}(X)_{hom} \neq 0$, the morphic Jacobian $J^p_{mor}(X)$ defined in [W] is different from $J^{2(n-p)-1}(X, n-p)$ defined above.

For a smooth complex projective varieties $X$, S. Bloch [B] introduced higher Chow groups and constructed regulator maps

$$c_{p,l} : CH^p(X, l) \to H^{2p-l}_D(X, \mathbb{Z}(p))$$

from the higher Chow groups to Deligne cohomology. An explicit description of the restriction of $c_{p,l}$ on the homologically trivial part is given by integration currents in [KLM]. Here we descript a map from $CH^p(X, l)$ to a different Deligne cohomology by using the system method in [HLZ].

Recall that (cf. [B]) for each $m \geq 0$, let

$$\Delta[d] := \{ t \in \mathbb{C}^{d+1} | \sum_{i=0}^{m} t_i = 1 \} \cong \mathbb{C}^d.$$
$F$ of $\Delta[d]$. Using intersection and pull-back of algebraic cycles, we can define face and degeneracy relations and obtain a simplicial abelian group structure for $\cdot$. Let $|z^l(X,*)|$ be the geometric realization of $z^l(X,*)$. Then the higher Chow group is defined as

$$\text{Ch}^l(X,k) := \pi_k(|z^l(X,*)|).$$

For $\alpha \in \text{Ch}^l(X,k)$, let $f : S^k \to |z^l(X,*)|$ be a PL representative of $\alpha$. Since $S^k$ is compact, there is an integer $d \geq 0$ such that $f(S^k) \subset \Delta[d] \times z^l(X,d)$. Note that for each $s \in S^k$, $f(s)$ is a codimension-$l$ algebraic cycles in $X \times \Delta[d]$ with certain additional restriction. As in section 3, we can construct an integral current $c_f$ on $X \times \Delta[d]$.

$$\text{C}_f := \text{pr}_X^* (c_f),$$

where $\text{pr}_X$ is the projection on $X$. Note that $\text{C}_f$ is an integral current of suitable dimension since elements in $z^q(X,d)$ are required to intersect faces properly.

Again we consider Example 2.2 for $m = 2p - k - 1, q = p - k - 1$. Set $A_f := -\Psi(d^*G(C_f))$ and Let $\widetilde{A}_f \in H^{2(n-p)-k-1}(X,n-p-k-1)$ be the differential character corresponding to the Hodge spark $(A_f,C_f)$.

**Proposition 4.4.** Let $X$ be a smooth projective variety of dimension $n$. There is a homomorphism

$$\delta_1(\widetilde{A}_f) = 0 = \delta_2(\widetilde{A}_f).$$

Proof. Word for word to Lemma 3.2, we show that $\delta_1(\widetilde{A}_f) = 0 = \delta_2(\widetilde{A}_f)$. Similar to Lemma 3.3 $\widetilde{A}_f$ is well-defined on $\text{CH}^p(X,k)$. □

The combination of Theorem 3.2 and Proposition 4.4 gives us the following result.

**Corollary 4.5.** The range of $\delta_1(\widetilde{A}_f)$ constructed here is different from the regulator map constructed by Bloch. From the construction, we have a commutative diagram:

$$\begin{align*}
\text{Ch}^p(X,k) \xrightarrow{FG} L_{n-p}H_{2(n-p)+k}(X) \xrightarrow{\delta_1} H^{2p-k}_D(X,\mathbb{Z}(p-k-1)))
\end{align*}$$

where $FG$ is the Friedlander-Gabber homomorphism defined in [FG], Section 4.

**Acknowledgement**

I would like to express my gratitude to my former advisor, Blaine Lawson, for all his help. I also thank the referee for detailed suggestions and corrections on the first version of the paper.

**References**

[B] S. Bloch, *Algebraic cycles and higher K-theory*. Adv. in Math. 61 (1986), no. 3, 267–304.

[C] J. Cheeger, *Multiplication of differential characters*. Symposia Mathematica, Vol. XI (Convegno di Geometria, INDAM, Rome, 1972), pp. 441–445. Academic Press, London, 1973.

[CS] J. Cheeger; J. Simons *Differential characters and geometric invariants*. Geometry and topology (College Park, Md., 1983/84), 50–80, Lecture Notes in Math., 1167, Springer, Berlin, 1985.
A MAP FROM LA WSON HOMOLOGY TO DELIGNE COHOMOLOGY

H. Esnault; E. Viehweg, Deligne-Bellinson cohomology. Bellinson’s conjectures on special values of L-functions, 43–91, Perspect. Math., 4, Academic Press, Boston, MA, 1988.

E. Friedlander, Algebraic cycles, Chow varieties, and Lawson homology. Compositio Math. 77 (1991), no. 1, 55–93.

E. Friedlander and O. Gabber, Cycle spaces and intersection theory. Topological methods in modern mathematics (Stony Brook, NY, 1991), 325–370, Publish or Perish, Houston, TX, 1993.

E. Friedlander and B. Lawson, A theory of algebraic cocycles. Ann. of Math. (2) 136 (1992), no. 2, 361–428.

E. Friedlander and B. Lawson, Duality relating spaces of algebraic cocycles and cycles. (English summary) Topology 36 (1997), no. 2, 533–565.

E. Friedlander and B. Mazur, Filtrations on the homology of algebraic varieties. With an appendix by Daniel Quillen. Mem. Amer. Math. Soc. 110 (1994), no. 529, x+110 pp.

H. Gillet and C. Soulé, Arithmetic Chow groups and differential characters. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), 29–68, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, Kluwer Acad. Publ., Dordrecht, 1989.

P. Griffiths, On the periods of certain rational integrals I, II. Ann. of Math. (2) 90 (1969), 460-495; ibid. (2) 90(1969) 496–541.

B. Harris, Differential characters and the Abel-Jacobi map. Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), 69–86, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 279, Kluwer Acad. Publ., Dordrecht, 1989.

R. Harvey and B. Lawson, From Sparks to Grundies - Differential Characters. (English summary) Comm. Anal. Geom. 14 (2006), no. 1, 25–58.

R. Harvey and B. Lawson, D-bar sparks. (English summary) Proc. Lond. Math. Soc. (3) 97 (2008), no. 1, 1–30.

R. Harvey; B. Lawson and J. Zweck, The de Rham-Fedder theory of differential characters and character duality. Amer. J. Math. 125 (2003), no. 4, 791–847.

R. Harvey; J. Polking Fundamental solutions in complex analysis. I and II. The Cauchy-Riemann operator. Duke Math. J. 46 (1979), no. 2, 253–340.

W. Hu, Generalized Abel-Jacobi map on Lawson Homology. arXiv: math.AG/0608294. To appear in Amer. J. Math.

B. Lawson, Algebraic cycles and homotopy theory. Ann. of Math. 129(1989), 253-291.

B. Lawson, Spaces of algebraic cycles. pp. 137-213 in Surveys in Differential Geometry, 1995 vol.2, International Press, 1995.

P. Lima-Filho, Completions and fibrations for topological monoids. Trans. Amer. Math. Soc. 340 (1993), no. 1, 127–147.

M. Kerr, J. Lewis and S. Müller-Stach, The Abel-Jacobi map for higher Chow groups. (English summary) Compos. Math. 142 (2006), no. 2, 374–396.

C. Peters, Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups. Math. Z. 234 (2000), no. 2, 209–223.

M. Walker The morphic Abel-Jacobi map. (English summary) Compos. Math. 143 (2007), no. 4, 909–944.

Institute for Advanced Study, Einstein Drive, Princeton, N.J. 08540, U.S.A.

E-mail address: wenchuan@math.ias.edu