ON GENERATORS OF ARITHMETIC GROUPS OVER FUNCTION FIELDS

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Abstract. Let \( F = \mathbb{F}_q(T) \) be the field of rational functions with \( \mathbb{F}_q \)-coefficients, and \( A = \mathbb{F}_q[T] \) be the subring of polynomials. Let \( D \) be a division quaternion algebra over \( F \) which is split at \( 1/T \). Given an \( A \)-order \( \Lambda \) in \( D \), we find an explicit finite set generating \( \Lambda^\times \).

1. Introduction

Notation.
\( \mathbb{F}_q \) = the finite field with \( q \) elements;
throughout the article \( q \) is assumed to be odd.
\( A = \mathbb{F}_q[T], T \) indeterminate.
\( F = \mathbb{F}_q(T) = \text{the fraction field of } A. \)
\( |F| = \text{the set of places of } F. \)
For \( x \in |F|, F_x = \text{the completion of } F \text{ at } x. \)
\( \mathcal{O}_x = \{ z \in F_x \mid \text{ord}_x(z) \geq 0 \} = \text{the ring of integers of } F_x. \)
\( \mathbb{F}_x = \text{the residue field of } \mathcal{O}_x; \deg(x) = [F_x : \mathbb{F}_q]. \)
For \( 0 \neq f \in A, \deg(f) = \text{the degree of } f \text{ as a polynomial in } T, \) and \( \deg(0) = +\infty. \)
For \( f/g \in F, \deg(f/g) = \deg(f) - \deg(g). \)
\text{ord} = - \deg \text{ defines a valuation on } F; \text{ the corresponding place is denoted by } \infty.
\( K = \mathbb{F}_\infty. \)
\( \mathcal{O} = \mathcal{O}_\infty. \)
\( \pi = T^{-1} = \text{uniformizer at infinity.} \)

Let \( D \) be a quaternion division algebra over \( F \) such that \( D \otimes_F K \cong M_2(K) \). Let \( \Lambda \) be an \( A \)-order in \( D \), i.e., \( \Lambda \), as a subring of \( D \) containing \( A \), is also an \( A \)-module containing 4 linearly independent generators over \( F \). The subgroup \( \Gamma := \Lambda^\times \) of units of \( \Lambda \) consists of elements \( \lambda \in \Lambda \) with reduced norm \( \text{Nr}(\lambda) \in \mathbb{F}_q^\times \). It is known that \( \Gamma \) is infinite finitely generated group. The problem of finding explicit sets of generators for \( \Gamma \) naturally arises in the study of these arithmetic groups. In this paper we develop a method for finding such explicit sets.

Now we explain what we mean by “explicit” and state the main result of the paper. Let \( \mathfrak{r} \in A \) be the discriminant of \( D \) - this is the product of the monic generators of the primes in \( A \) where \( D \) ramifies; see Section 2. There exists \( \mathfrak{a} \in A \) such that \( D \) is isomorphic to the algebra over \( F \) with basis 1, i, j, \( ij \) satisfying the relations
\[
i^2 = \mathfrak{a}, \quad j^2 = \mathfrak{r}, \quad ij = -ji
\]

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(see Lemma 2.1). Next, one needs to find an $A$-basis of $\Lambda$ in terms of $\{1, i, j, ij\}$. Although our method works for general $A$-orders, to simplify the exposition, we restrict the consideration to
\[
\Lambda = A \oplus Ai \oplus Aj \oplus Aij
\]
which has the simplest presentation in terms of the basis $\{1, i, j, ij\}$. Intrinsically, this is the level-a Eichler order in $D$. In this case,
\[
\Gamma = \left\{ a + bi + cj + dij \middle| a, b, c, d \in A, a^2 - ab^2 - tc^2 + atd^2 \in F_q \times \right\}
\]
Now the problem is to find finitely many quadruples $(a_n, b_n, c_n, d_n) \in A^4$, $1 \leq n \leq N$ for some $N$, such that $\gamma_n = a_n + b_n i + c_n j + d_n ij \in \Lambda$ are in $\Gamma$ and generate $\Gamma$. Finding minimal explicit sets of generators is very difficult. In fact, for the analogous problem over $\mathbb{Q}$ only in finitely many cases an explicit minimal set of generators is known, cf. [1]. (See [8] for some examples in the function field case.) Instead of trying to find minimal sets of generators for $\Gamma$, we look for possibly larger sets which are easy to describe.

**Definition 1.1.** Put $\deg(0) = 0$ and for $\lambda = a + bi + cj + dij \in \Lambda$ define
\[
\|\lambda\| = \max(\deg(a), \deg(b), \deg(c), \deg(d)).
\]
The main result of the paper is the following:

**Theorem 1.2.** The finite subset
\[
S = \{ \gamma \in \Gamma \mid \|\gamma\| \leq 4 \deg(ar) + 6 \}
\]
generates $\Gamma$.

To prove Theorem 1.2 we use the action of $\Gamma$ on the Bruhat-Tits tree $T$ of $\mathrm{PGL}_2(K)$. The key is to quantify the discontinuity of the action of $\Gamma$ on $T$. More precisely, for a ball $T_B$ of radius $B$ around a fixed vertex and $\gamma \in \Gamma$, we show that $\gamma T_B \cap T_B = \emptyset$ once $\|\gamma\| > 2B$. To deduce Theorem 1.2 from this result, one needs strong bounds on the diameter of the quotient graph $\Gamma \setminus T$. Such a bound follows from the property of $\Gamma \setminus T$ being covered by a Ramanujan graph.

A problem of similar nature over $\mathbb{Q}$ was considered by Chalk and Kelly in [2], although the bound they obtain is exponentially worse than ours. The approach in [2] is analytic in nature and relies on the study of isometric circles of $\Gamma$ (see also [4]).

**Remark 1.3.** Theorem 1.2 raises the following interesting question: What is the minimal $\sigma$ independent of $\Gamma$ such that there exists a constant $\delta$ (also independent of $\Gamma$) with the property that the set
\[
\{ \gamma \in \Gamma \mid \|\gamma\| \leq \sigma \deg(ar) + \delta \}
\]
generates $\Gamma$? The theorem gives $\sigma \leq 4$. We have a trivial lower bound $1/4 \leq \sigma$, since according to Proposition 2.6 the cardinality of a minimal set of generators for $\Gamma$ is approximately $q^{\deg(ar)}$.

**Remark 1.4.** After this article was essentially completed, Ralf Butenuth informed me that he obtained a result similar to Theorem 1.2 by a different method.
2. Arithmetic of quaternion algebras

In this section we recall some facts about quaternion algebras. The standard reference for this material is \[12\].

Let $D$ be a quaternion algebra over $F$, i.e., a 4-dimensional $F$-algebra with center $F$ which does not possess non-trivial two-sided ideals. A quaternion algebra is either a division algebra or is isomorphic to the algebra of $2 \times 2$ matrices. If $L$ is a field containing $F$, then $D \otimes_F L$ is a quaternion algebra over $L$. Let $x \in |F|$ and denote $D_x := D \otimes_F F_x$. We say that $D$ ramifies (resp. splits) at $x$ if $D_x$ is a division algebra (resp. is isomorphic to $M_2(F_x)$). Let $R \subset |F|$ be the set of places where $D$ ramifies. It is known that $R$ is a finite set of even cardinality, and conversely, for any choice of a finite set $R \subset |F|$ of even cardinality there is a unique, up to an isomorphism, quaternion algebra ramified exactly at the places in $R$. In particular, $D \cong M_2(F)$ if and only if $R = \emptyset$.

Explicitly quaternion algebras can be given as follows. For $a, b \in F^\times$, let $H(a, b)$ be the $F$-algebra with basis $1, i, j, ij$ (as an $F$-vector space), where $i, j$ satisfy

$$i^2 = a, \quad j^2 = b, \quad ij = -ji.$$  

$H(a, b)$ is a quaternion algebra, and any quaternion algebra $D$ is isomorphic to $H(a, b)$ for some $a, b \in F^\times$ (although $a$ and $b$ are not uniquely determined by $D$, e.g. $H(a, b) \cong H(b, a)$).

From now on we assume that $D$ is a division algebra (equiv. $R \neq \emptyset$). Let $L \neq F$ be a non-trivial field extension of $F$. Then $L$ embeds into $D$, i.e., there is an $F$-isomorphism of $L$ onto an $F$-subalgebra of $D$, if and only if $|L : F| = 2$ and places in $R$ do not split in $L$.

There is a canonical involution $\alpha \mapsto \alpha'$ on $D$ which is the identity on $F$ and satisfies $(\alpha \beta)' = \beta' \alpha'$. The reduced trace of $\alpha$ is $\text{Tr}(\alpha) = \alpha + \alpha'$; the reduced norm of $\alpha$ is $\text{Nr}(\alpha) = \alpha \alpha'$; the reduced characteristic polynomial of $\alpha$ is

$$f(x) = (x - \alpha)(x - \alpha') = x^2 - \text{Tr}(\alpha)x + \text{Nr}(\alpha).$$

If $\alpha \not\in F$, then the reduced trace and norm of $\alpha$ are the images of $\alpha$ under the trace and norm of the quadratic field extension $F(\alpha)/F$.

From now on we also assume that $\infty \not\in R$. For $x \in |F| - \infty$, denote by $(x)$ the prime ideal of $A$ corresponding to $x$. Let $f_x \in A$ be the monic generator of $(x)$, and let $\tau = \prod_{x \in R} f_x$. Given $a, b \in A$, with $b$ irreducible and coprime to $a$, let

$$\left( \frac{a}{b} \right) = \begin{cases} 
1, & \text{if } a \text{ is a square mod } (b) \\
-1, & \text{otherwise}
\end{cases}$$

be the Legendre symbol. Let

$$\text{Odd}(R) = \begin{cases} 
1, & \text{if } \text{deg}(x) \text{ is odd for all } x \in R; \\
0, & \text{otherwise}.
\end{cases}$$

Lemma 2.1.

1. Suppose $\text{Odd}(R) = 0$. There is a monic irreducible polynomial $a \in A$ of even degree which is coprime to $\tau$ and satisfies

$$\left( \frac{a}{f_x} \right) = -1 \text{ for all } x \in R.$$  

For such $a$, $D \cong H(a, \tau)$.

2. Suppose $\text{Odd}(R) = 1$. Let $\xi \in \mathbb{F}_q$ be a non-square. Then $D \cong H(\xi, \tau)$.
Definition 2.2. Let \( \mathcal{R} \) be a Dedekind domain with quotient field \( L \) and let \( B \) be a quaternion algebra over \( L \). For any finite dimensional \( L \)-vector space \( V \), an \( \mathcal{R} \)-lattice in \( V \) is a finitely generated \( \mathcal{R} \)-submodule \( M \) in \( V \) such that \( L \otimes_\mathcal{R} M \cong V \).

An \( \mathcal{R} \)-order in \( B \) is a subring \( \Lambda \) of \( B \), having the same unity element as \( B \), and such that \( \Lambda \) is an \( \mathcal{R} \)-lattice in \( B \). A maximal \( \mathcal{R} \)-order in \( B \) is an \( \mathcal{R} \)-order which is not contained in any other \( \mathcal{R} \)-order in \( B \).

Let \( \Lambda \) be an \( A \)-order in \( D \). It is known that \( \Lambda \) is maximal if and only if for all \( x \in R \), \( \Lambda_x := \Lambda \otimes_A \mathcal{O}_x \) is a maximal \( \mathcal{O}_x \)-order in \( D_x \) for all \( x \in |F| - R - \infty \). A maximal \( \mathcal{O}_x \)-order in \( D_x \) is unique if \( x \in R \), and it is the integral closure of \( \mathcal{O}_x \) in \( D_x \). On the other hand, for \( x \not\in R \), \( \Lambda_x \) is maximal if and only if there is an invertible element \( u \in M_2(\mathcal{O}_x) \) such that \( u\Lambda_x u^{-1} = M_2(\mathcal{O}_x) \).

Definition 2.3. Suppose \( n \in A \) is square-free and coprime to \( \mathfrak{r} \). \( \Lambda \) is an Eichler order of level-\( n \) if \( \Lambda_x \) is maximal for all \( x \in R \), and for \( x \in |F| - R - \infty \), it is isomorphic to the subring of \( M_2(\mathcal{O}_x) \) given by the matrices

\[
\left\{ \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \mid a, b, c, d \in \mathcal{O}_x \right\}.
\]

Suppose \( \Lambda = A e_1 \oplus A e_2 \oplus A e_3 \oplus A e_4 \), where \( e_1, \ldots, e_4 \) is a basis of \( D \) as an \( F \)-vector space. The discriminant of \( \Lambda \) is the ideal of \( A \) generated by \( \det(\text{Tr}(e_i e_j))_{i,j} \). It is known that the discriminant of any order is divisible by \( (t)^2 \). Moreover, the maximal orders are uniquely characterized by the fact that their discriminant is \( (t)^2 \), and the level-\( n \) Eichler orders are uniquely characterized by the fact that their
discriminants are equal to $(nr)^2$. By a theorem of Eichler, since $D$ splits at $\infty$, all maximal $A$-orders are conjugate in $D$; the same is true also for the level-$n$ Eichler orders.

**Definition 2.4.** The order $\Lambda = A \oplus Ai \oplus Aj \oplus Aij$ in $H(a, t)$ will be called the standard order. By computing its discriminant, we see that $\Lambda$ is a level-$a$ Eichler order. In particular, $\Lambda$ is maximal if and only if $a \in \mathbb{F}_q^\times$.

Given an $A$-order $\Lambda$, the group $\Lambda^\times$ of its invertible elements consists of

$$\{\lambda \in \Lambda \mid \text{Nr}(\lambda) \in \mathbb{F}_q^\times\}.$$  

If $\lambda \in \Lambda^\times$ is a torsion element, then it is algebraic over $\mathbb{F}_q$. This easily implies that $\lambda$ is torsion if and only if $\text{Tr}(\lambda) \in \mathbb{F}_q$; such elements will be called elliptic. An element $\lambda \in \Lambda^\times$ which is not elliptic will be called hyperbolic.

**Lemma 2.5.** If $\lambda$ is hyperbolic, then its image in $\text{GL}_2(K)$ under an embedding $D \hookrightarrow \mathbb{M}_2(K)$ has two distinct $K$-rational eigenvalues.

**Proof.** The reduced characteristic polynomial of $\lambda$ is $h_\lambda := x^2 + \text{Tr}(\lambda)x + \kappa$, where $\kappa \in \mathbb{F}_q^\times$. Since $\lambda \notin F^\times$, $F(\lambda)$ is quadratic, and therefore $h_\lambda$ is irreducible. Next, since $s := \text{Tr}(\lambda) \in A$ has degree $\geq 1$, $s^2 - 4\kappa$ is a non-zero polynomial of even degree whose leading coefficient is a square. This implies that $h_\lambda$ splits over $K$ and has distinct roots.

Denote

$$g(R) = 1 + \frac{1}{q^2 - 1} \prod_{x \in R} (q^{\deg(x)} - 1) - \frac{q}{q + 1} \cdot 2^{#R - 1} \cdot \text{Odd}(R).$$

Let $\Lambda$ be the standard order in $H(a, t)$, where $a$ is a monic irreducible polynomial of even degree if $\text{Odd}(R) = 0$, and $a = \xi$ is a non-square in $\mathbb{F}_q$ if $\text{Odd}(R) = 1$, cf. Lemma 2.1. Denote $\Gamma := \Lambda^\times$. Note that $\mathbb{F}_q^\times$ is in the center of $\Gamma$.

**Proposition 2.6.**

1. $\Gamma/\mathbb{F}_q^\times$ has non-trivial torsion if and only if $\text{Odd}(R) = 1$.
2. Suppose $\text{Odd}(R) = 1$. $\Gamma$ can be generated by $g(R) + 2^{#R - 1}$ elements.
3. Suppose $\text{Odd}(R) = 0$. $\Gamma/\mathbb{F}_q^\times$ is a free group of rank

$$1 + (q^{\deg(a)} + 1)(g(R) - 1).$$

**Proof.** (1) and (2) follow from [8, Thm. 5.7]. (3) Let $\mathcal{Y}$ be a maximal order containing $\Lambda$. Denote $\Gamma' = \mathcal{Y}^\times$. By [8, Thm. 5.7], $\Gamma'/\mathbb{F}_q^\times$ is a free group of rank $g(R)$. It is not hard to show that $[\Gamma'/\mathbb{F}_q^\times : \Gamma/\mathbb{F}_q^\times] = q^{\deg(a) + 1}$, cf. [7, p. 212]. Now the claim that $\Gamma/\mathbb{F}_q^\times$ is a free group of given rank follows from Schreier’s theorem [11, p. 29].

3. **Geometry on the Bruhat-Tits tree**

We start by recalling some of the terminology from [11]. Let $\mathcal{G}$ be an (oriented) connected graph; see [11, Def. 1, p. 13]. We denote by $\text{Ver}(\mathcal{G})$ and $\text{Ed}(\mathcal{G})$ the sets of vertices and oriented edges of $\mathcal{G}$, respectively. For $e \in \text{Ed}(\mathcal{G})$, $o(e), t(e) \in \text{Ver}(\mathcal{G})$ and $\bar{e} \in \text{Ed}(\mathcal{G})$ denote its origin, terminus and inversely oriented edge. We will assume that for any $v \in \text{Ver}(\mathcal{G})$ the number of edges $e$ with $o(e) = v$ is finite; this number is the *degree* of $v$. $\mathcal{G}$ is *$m$-regular* if every vertex in $\mathcal{G}$ has degree $m$. The
distance $d(v, w)$ between $v, w \in \text{Ver}(G)$ in $G$ is the obvious combinatorial distance, i.e., the number of edges in a shortest path without backtracking connecting $v$ and $w$. The diameter $D(G)$ of a finite graph $G$ is the maximum of the distances between its vertices. A graph in which a path without backtracking connecting any two vertices $v$ and $w$ is unique is called a tree; the unique path between $v, w$ is called geodesic.

Let $\Gamma$ be a group acting on a graph $G$, i.e., $\Gamma$ acts via automorphisms. We say that $v, w \in \text{Ver}(G)$ are $\Gamma$-equivalent if there is $\gamma \in \Gamma$ such that $\gamma v = w$. $\Gamma$ acts with inversion if there is $\gamma \in \Gamma$ and $e \in \text{Ed}(G)$ such that $\gamma e = \overline{e}$. If $\Gamma$ acts without inversion, then we have a natural quotient graph $\Gamma \setminus G$ such that $\text{Ver}(\Gamma \setminus G) = \Gamma \setminus \text{Ver}(G)$ and $\text{Ed}(\Gamma \setminus G) = \Gamma \setminus \text{Ed}(G)$, cf. [11, p. 25].

Recall the notation $K := F_{\infty}$ and $O := O_{\infty}$. Let $V$ be a 2-dimensional $K$-vector space. Let $\Lambda$ be an $O$-lattice in $V$. For any $x \in K^\times$, $x\Lambda$ is also a lattice. We call $\Lambda$ and $x\Lambda$ equivalent lattices. The equivalence class of $\Lambda$ is denoted by $[\Lambda]$.

Let $T$ be the graph whose vertices $\text{Ver}(T) = \{[\Lambda]\}$ are the equivalence classes of lattices in $V$, and two vertices $[\Lambda]$ and $[\Lambda']$ are adjacent if we can choose representatives $L \in [\Lambda]$ and $L' \in [\Lambda']$ such that $L' \subset L$ and $L/L' \cong F_q$. One shows that $T$ is an infinite $(q + 1)$-regular tree. This is the Bruhat-Tits tree of $\text{PGL}_2(K)$.

Fix a vertex $[\Lambda]$ of $T$. The set of vertices of $T$ at distance $n$ from $[\Lambda]$ is in natural bijection with $\mathbb{P}^1(L/\pi^n L)$. An end of $T$ is an equivalence class of half-lines, two half-lines being equivalent if they differ in a finite graph. Taking the projective limit over $n$, we get a bijection

$$\partial T := \text{set of ends of } T \cong \mathbb{P}^1(O) = \mathbb{P}^1(K),$$

which is independent of the choice of $[\Lambda]$. Given two vectors $f_1, f_2$ spanning $V$, we denote $[f_1, f_2] = [O f_1 \oplus O f_2]$. $\text{GL}_2(K)$, as the group of $K$-automorphisms of $V$, acts on the Bruhat-Tits tree $T$ via $g[f_1, f_2] = [gf_1, gf_2]$. It is easy to check that this action preserves the distance between any two vertices, and the induced action on $\partial T$ agrees with the usual action of $\text{GL}_2(K)$ on $\mathbb{P}^1(K)$ through fractional linear transformations.

Fix the standard basis $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ of $V$, and let $O := [e_1, e_2]$. Since $\text{GL}_2(K)$ acts transitively on the vertices of $T$ and the stabilizer of $O$ is $\text{GL}_2(O)K^\times$, we have a bijection

\begin{equation}
\text{GL}_2(K)/\text{GL}_2(O)K^\times \xrightarrow{\sim} \text{Ver}(T) \quad g \mapsto g \cdot O.
\end{equation}

Using the Iwasawa decomposition, one easily sees that the set of matrices

\begin{equation}
\left\{ \begin{pmatrix} \pi^k u \\ 0 \\ 1 \end{pmatrix} \mid k \in \mathbb{Z}, u \in K, u \mod \pi^k O \right\}
\end{equation}

is a set of representatives for $\text{Ver}(T)$; see [3, p. 370]. The map \eqref{eq:31} becomes

$$\begin{pmatrix} \pi^k u \\ 0 \\ 1 \end{pmatrix} \mapsto [\pi^k e_1, ue_1 + e_2].$$

Note that under this bijection the identity matrix corresponds to $O$. We say that a matrix $M \in \text{GL}_2(K)$ is in reduced form if it belongs to the set of matrices in \eqref{eq:32}.

For two matrices $M, M' \in \text{GL}_2(K)$, we write $M \sim M'$ if they represent the same vertex in $T$. 


Lemma 3.1. The distance between \( \begin{pmatrix} \pi^k & u \\ 0 & 1 \end{pmatrix} \) (in reduced form) and \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is
\[
\begin{cases} 
|k|, & \text{if } u = 0 \text{ or } \text{ord}(u) \geq 0; \\
 k - 2 \cdot \text{ord}(u), & \text{if } u \neq 0 \text{ and } \text{ord}(u) < 0.
\end{cases}
\]
(Note that for a matrix in reduced form \( k > \text{ord}(u) \).)

Proof. This is an easy calculation. \( \square \)

Given two distinct points \( P, Q \in \mathbb{P}^1(K) \), there is a unique path in \( \mathcal{T} \), without backtracking and infinite in both directions, whose ends are \( P \) and \( Q \); this is the geodesic \( \mathcal{A}(P, Q) \) connecting the two boundary points of \( \mathcal{T} \). For example, the geodesic connecting \( 0 = (0 : 1) \) and \( \infty = (1 : 0) \) is the subgraph of \( \mathcal{T} \) with vertices \( \left\{ \begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\} \).

Assume \( \gamma \in \text{GL}_2(K) \) has two distinct \( K \)-rational eigenvalues \( a \) and \( b \). The eigenvectors corresponding to \( a \) and \( b \) can be regarded as two well-defined points on \( \mathbb{P}^1(K) \) - if \( \begin{pmatrix} x \\ y \end{pmatrix} \) is an eigenvector, then the corresponding point is \( (x : y) \).

Let \( \mathcal{A}(\gamma) \) be the geodesic connecting these points. Suppose \( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \) and \( \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \) are eigenvectors corresponding to \( a \) and \( b \), respectively. Since \( \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \) maps \( \infty \) to \( (x_1 : y_1) \) and \( 0 \) to \( (x_2 : y_2) \),
\[
\text{Ver}(\mathcal{A}(\gamma)) = \left\{ \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.
\]

The distance from \( \mathcal{A}(\gamma) \) to \( O \) is the minimum of the distances from the vertices on \( \mathcal{A}(\gamma) \) to \( O \).

Lemma 3.2. The action of \( \gamma \) on \( \mathcal{T} \) induces translation on \( \mathcal{A}(\gamma) \) of amplitude \( |\text{ord}(a) - \text{ord}(b)| \).

Proof. This is easy to see after choosing the eigenvectors of \( \gamma \) as a basis of \( V \). \( \square \)

Lemma 3.3. Suppose \( x_1 y_1 \neq 0 \) and \( x_2 y_2 \neq 0 \). Denote \( x := x_1 / y_1 \) and \( y := x_2 / y_2 \). Suppose \( \text{ord}(x) \geq B \) and \( \text{ord}(y) \geq B \), or \( \text{ord}(x) \leq -B \) and \( \text{ord}(y) \leq -B \) for some \( B \geq 0 \). Then the distance from \( \mathcal{A}(\gamma) \) to \( O \) is at least \( B \).

Proof. Note that \( x \neq y \), as \( \begin{pmatrix} x \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} y \\ 1 \end{pmatrix} \) are eigenvectors for \( a \) and \( b \). Without loss of generality we can assume \( \text{ord}(x) \geq \text{ord}(y) \). The geodesic \( \mathcal{A}(\gamma) \) has vertices \( \begin{pmatrix} x \pi^k & y \\ \pi^k & 1 \end{pmatrix} \), \( k \in \mathbb{Z} \).

First, consider the case when \( k > 0 \). Then \( \begin{pmatrix} 1 & 0 \\ -\pi^k & 1 \end{pmatrix} \in \text{GL}_2(O) \), so by multiplying \( \begin{pmatrix} x \pi^k \\ \pi^k \\ y \\ 1 \end{pmatrix} \) by this matrix from the right we see that
\[
\begin{pmatrix} x \pi^k \\ \pi^k \\ y \\ 1 \end{pmatrix} \sim \begin{pmatrix} (x - y) \pi^k & y \\ 0 & 1 \end{pmatrix}.
\]
On the other hand, \( \text{ord}(x - y)\pi^k \geq k + \text{ord}(y) > \text{ord}(y) \), so the resulting matrix is in reduced form. The distance from this matrix to \( O \) is

\[
\text{ord}(x - y)\pi^k \geq k + \text{ord}(y), \quad \text{if \( \text{ord}(y) \geq 0 \)},
\]

and

\[
\text{ord}(x - y)\pi^k - 2\text{ord}(y) \geq k - \text{ord}(y), \quad \text{if \( \text{ord}(y) < 0 \)}.
\]

In either case, we conclude that the distance is at least \( 1 + |\text{ord}(y)| \geq 1 + B \).

The matrix with \( k = 0 \) is adjacent to the matrix with \( k = 1 \), so from the previous paragraph we conclude that the corresponding matrix and \( O \) are at a distance at least \((1 + B) - 1 = B\).

Now suppose \( k < 0 \). Then

\[
\begin{pmatrix}
\pi^k & y \\
1 & 1
\end{pmatrix} \sim \begin{pmatrix} x & \pi^{-k} y \\
1 & \pi^{-k}
\end{pmatrix} \sim \begin{pmatrix} \pi^{-k} (y - x) & x \\
0 & 1
\end{pmatrix}.
\]

If \( \text{ord}(x) = \text{ord}(y) \), then \( \text{ord}(x - y) \geq \text{ord}(y) = \text{ord}(x) \). Hence \(-k + \text{ord}(x - y) > \text{ord}(x)\), and the above matrix is in reduced form. The distance from \( O \) is

\[-k + \text{ord}(x - y) \geq -k + \text{ord}(y) > B, \quad \text{if \( \text{ord}(y) \geq 0 \)},\]

or

\[-k + \text{ord}(x - y) - 2\text{ord}(y) \geq -k - \text{ord}(y) > B, \quad \text{if \( \text{ord}(y) < 0 \)}.\]

If \( \text{ord}(x) > \text{ord}(y) \), then \( \text{ord}(x - y) = \text{ord}(y) \). If \( \text{ord}(\pi^{-k} (x - y)) = -k + \text{ord}(y) \leq \text{ord}(x) \), then

\[
\begin{pmatrix} \pi^{-k} (y - x) & x \\
0 & 1
\end{pmatrix} \sim \begin{pmatrix} \pi^{-k} (y - x) & 0 \\
0 & 1
\end{pmatrix},
\]

so the distance is \( | -k + \text{ord}(y)| \). If \( \text{ord}(y) \geq B \), then this last quantity is obviously \(\geq 1 + B \). On the other hand, if \( \text{ord}(y) \leq -B \), then, due to the assumption of the lemma, \(-k + \text{ord}(y) \leq \text{ord}(x) \leq -B \). Thus, \(| -k + \text{ord}(y)| \geq B \). Finally, if \(-k + \text{ord}(y) > \text{ord}(x) \), then the distance is

\[-k + \text{ord}(x - y) = -k + \text{ord}(y) > B, \quad \text{if \( \text{ord}(x) \geq 0 \)},\]

or

\[-k + \text{ord}(x - y) - 2\text{ord}(x) = -k + \text{ord}(y) - 2\text{ord}(x) > -\text{ord}(x) \geq B, \quad \text{if \( \text{ord}(x) < 0 \)}.\]

\[\square\]

Remark 3.4. The inequalities \( \text{ord}(x) \gg 0, \text{ord}(y) \gg 0 \) (resp. \( \text{ord}(x) \ll 0, \text{ord}(y) \ll 0 \)) essentially mean that both \( x \) and \( y \) are in a small neighborhood of 0 (resp. \( \infty \)).

Now one can visualize the previous lemma as follows: for a sufficiently small interval on \( \mathbb{R} \) the geodesic in the hyperbolic upper half-plane \( \mathcal{H} \) connecting any two distinct points in that interval is far from a fixed point in \( \mathcal{H} \).

Notation 3.5. For \( B \geq 0 \), let \( \mathcal{T}_B \) be the finite subtree of \( \mathcal{T} \) with set of vertices

\[\text{Ver}(\mathcal{T}_B) = \{ v \in \text{Ver}(\mathcal{T}) \mid d(v, O) \leq B \} \text{.}\]

Lemma 3.6. Let \( \alpha \) be the amplitude of translation with which \( \gamma \) acts on \( A(\gamma) \). Let \( m \) be the distance from \( A(\gamma) \) to \( O \). If \( 2m + n > 2B \), then \( \gamma \mathcal{T}_B \cap \mathcal{T}_B = \emptyset \).

Proof. Let \( P \in A(\gamma) \) be the vertex closest to \( O \). Then \( m = d(P, O) \) and

\[d(O, \gamma O) = d(O, P) + d(P, \gamma P) + d(\gamma P, \gamma O) = 2m + n,\]

cf. \[\Pi\] Prop. 24 (iv), p. 63]. On the other hand, if \( \gamma \mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset \), then \( d(O, \gamma O) \leq 2B \). Thus \( 2m + n \leq 2B \). \[\square\]
Proposition 3.7. Assume $\Gamma$ acts without inversion on $\mathcal{T}$ and $\Gamma \setminus \mathcal{T}$ is finite. Let $B = D(\Gamma \setminus \mathcal{T})$. Let $S$ denote the set of $\gamma \in \Gamma$ such that $\gamma\mathcal{T}_B \cap \mathcal{T}_B \neq \emptyset$. Then $S$ generates $\Gamma$.

Proof. Let $\bar{O} \in \Gamma \setminus \mathcal{T}$ be the image of $O$. Let $\bar{\nu} \in \text{Ver}(\Gamma \setminus \mathcal{T})$. Consider a path $P$ of shortest length connecting $\bar{O}$ and $\bar{\nu}$. Obviously $P$ is a subtree of $\Gamma \setminus \mathcal{T}$, hence by [11, Prop. 14, p. 25] lifts to $\mathcal{T}$. This implies that there is a vertex $v$ in $\mathcal{T}$ which maps to $\bar{\nu}$ and $d(O, v) = d(\bar{O}, \bar{\nu})$. In particular, $v \in \mathcal{T}_B$, so $\mathcal{T}_B$ surjects onto $\Gamma \setminus \mathcal{T}$ under the quotient map $\mathcal{T} \to \Gamma \setminus \mathcal{T}$.

Let $\text{real}(\mathcal{T})$ be the realization of $\mathcal{T}$; see [11, p. 14]. Recall that $\text{real}(\mathcal{T})$ is a CW-complex where each edge of $\mathcal{T}$ is homeomorphic to the interval $[0,1] \subset \mathbb{R}$. Let $U$ be the open subset of $\text{real}(\mathcal{T})$ consisting of points at distance $< 1/3$ from $\text{real}(\mathcal{T}_B)$. Then $\gamma U \cap U \neq \emptyset$ if and only if $\gamma\mathcal{T}_B \cap \mathcal{T}_B$, and $U \to \text{real}(\Gamma \setminus \mathcal{T})$ is surjective. The claim of the proposition now follows from [11 (1), p. 30].

Let $\Gamma$ be as in Proposition 2.6. $\Gamma$ acts naturally on $\mathcal{T}$ (see Section 3). The action is without inversion and the quotient graph $\Gamma \setminus \mathcal{T}$ is finite; see [8, Lem. 5.1].

Lemma 3.8. Let $V := \#\text{Ver}(\Gamma \setminus \mathcal{T})$.

1. If $\text{Odd}(R) = 1$, then
   \[
   V = \frac{2}{(q - 1)(q^2 - 1)} \prod_{x \in R} (q^{\text{deg}(x)} - 1) + \frac{q}{q + 1} 2^{\#R - 1}.
   \]

2. If $\text{Odd}(R) = 0$, then
   \[
   V = \frac{2(q^{\text{deg}(a)} + 1)}{(q - 1)(q^2 - 1)} \prod_{x \in R} (q^{\text{deg}(x)} - 1).
   \]

Proof. (1) This follows from [8, Thm. 5.5]. (2) Since $\mathbb{F}_q^\times \lessdot \Gamma$ acts trivially on $\mathcal{T}$ and $\Gamma/\mathbb{F}_q^\times$ is a free group, $\Gamma \setminus \mathcal{T}$ is $(q + 1)$-regular. Thus, if we denote by $E$ the number of (non-oriented) edges of $\Gamma \setminus \mathcal{T}$, then $E = (q + 1)V/2$. On the other hand, by [11, Thm. 4’, p. 27], the rank of $\Gamma/\mathbb{F}_q^\times$ is equal to $E + 1 - V$. Now the expression for $V$ follows from Proposition 2.6. \hfill $\square$

Lemma 3.9. Let $\mathcal{G}$ be an $m$-regular Ramanujan graph on $n$ vertices. Then
   \[
   D(\mathcal{G}) \leq 2\log_{m-1}(n) + \log_{m-1}(4).
   \]

Proof. For the definition of Ramanujan graphs see [5]. The claim of the lemma is part of Proposition 7.3.11 in loc. cit. \hfill $\square$

Proposition 3.10. $D(\Gamma \setminus \mathcal{T}) \leq 2\text{deg}(ar) + 3$.

Proof. Let $I = (T) \triangleleft A$ be the ideal generated by $T$. Let $\Upsilon$ be a maximal order containing $\Lambda$. Denote by $\Gamma(T)$ the principal level-$I$ congruence subgroup of $\Upsilon$, cf. [6]. Let $\Gamma' := \Gamma \cap \Gamma(I)$. $\Gamma \setminus \mathcal{T}$ is naturally a quotient of $\Gamma' \setminus \mathcal{T}$, so
   \[
   D(\Gamma \setminus \mathcal{T}) \leq D(\Gamma' \setminus \mathcal{T}).
   \]
Moreover, since $[\Gamma : \Gamma'] < q^4$, we have $V' < q^4V$, where $V' := \#\text{Ver}(\Gamma' \setminus \mathcal{T})$.

By [6, Thm. 1.2], $\Gamma' \setminus \mathcal{T}$ is a $(q + 1)$-regular Ramanujan graph, so the previous paragraph and Lemma 3.9 give the bound
   \[
   D(\Gamma \setminus \mathcal{T}) \leq 2\log_q(q^4V) + 1.
   \]
Now the proposition follows from the formulae in Lemma [3.8] (Note that \( \deg(r) = \sum_{x \in R} \deg(x) \), and \( \deg(a) = 0 \) when \( \text{Odd}(R) = 1 \).)

4. Proof of Theorem [1,2]

Let \((a, b) = (a', r)\) (resp. \((a, b) = (r, \xi)\)) if \( \text{Odd}(R) = 0 \) (resp. \( \text{Odd}(R) = 1 \)), with \(a', \xi\) chosen as in Lemma [1.1]. Let \( \Lambda \) be the standard order in \( H(a, b) \) and \( \Gamma := \Lambda^x \).

By Proposition [2.0] this is a finitely generated group, and we would like to find an explicit set of generators. We start by embedding \( H(a, b) \) into \( M_2(K) \). The map

\[ i \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \]

gives an embedding \( H(a, b) \to M_2(K) \). Indeed, since \( a \in A \) is monic and has even degree, the equation \( x^2 = a \) has a solution in \( K \). Thus, the above matrices are indeed in \( M_2(K) \). It remains to observe that the given matrices satisfy the same relations as \( i \) and \( j \). Under this embedding \( \Gamma \) is the subgroup of \( \text{GL}_2(K) \) consisting of matrices

\[ \Gamma = \left\{ \begin{pmatrix} a + b\sqrt{a} & c + d\sqrt{a} \\ b(c - d\sqrt{a}) & a - b\sqrt{a} \end{pmatrix} \mid a, b, c, d \in A \right\}. \]

**Proposition 4.1.** Fix some \( B \geq 0 \). If \( \gamma \in \Gamma \) is hyperbolic and satisfies \( \|\gamma\| > 2B \), then \( \gamma \mathcal{T}_B \cap \mathcal{T}_B = \emptyset \).

**Proof.** To simplify the notation, we put \( \deg(0) = 0 \). The image of \( \gamma \) in \( \text{GL}_2(K) \) is the matrix

\[ \begin{pmatrix} a + b\sqrt{a} & c + d\sqrt{a} \\ b(c - d\sqrt{a}) & a - b\sqrt{a} \end{pmatrix}. \]

Let \( \alpha, \beta \in K \) be the eigenvalues of \( \gamma \). We know that \( \text{Nr}(\gamma) = \alpha \beta =: \kappa \in \mathbb{F}_q^\times \) and \( \text{Tr}(\gamma) = \alpha + \beta = 2a \notin \mathbb{F}_q \). Thus, \( \beta = \kappa \alpha^{-1} \) and \( \text{ord}(\beta) = -\text{ord}(\alpha) \). Without loss of generality, we assume \( \text{ord}(\alpha) \geq 0 \), so \( \text{ord}(\beta) \leq 0 \). Since \( \deg(a) \geq 1 \), \( \text{ord}(\beta) = \text{ord}(\alpha) \leq -1 \). Using Lemma [3.2] we conclude that \( \gamma \) acts on \( \mathcal{A}(\gamma) \) by translations with amplitude \( 2 \deg(a) \). Let \( m \) be the distance from \( \mathcal{A}(\gamma) \) to \( O \). Lemma [3.6] implies that if

\[ m + \deg(a) > B \]

then \( \gamma \mathcal{T}_B \cap \mathcal{T}_B = \emptyset \). In particular, if \( \deg(a) > B \), then \( \gamma \mathcal{T}_B \cap \mathcal{T}_B = \emptyset \).

Suppose \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is an eigenvector. Then \( b(c - d\sqrt{a}) = 0 \), which forces \( c = d = 0 \).

Thus, we must have \( a^2 - b^2a \in \mathbb{F}_q^\times \). If \( \deg(b) > \deg(a) - \deg(a)/2 \), then this is not possible. It is easy to see that we reach the same conclusion in the case when \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) is an eigenvector.

Assume \( \deg(b) > \deg(a) - \deg(a)/2 \). Denote \( s := \deg(b) + \deg(a)/2 - \deg(a) > 0 \).

From the previous paragraph we know that there are eigenvectors for \( \alpha \) and \( \beta \) of the form \( \begin{pmatrix} x \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} y \\ 1 \end{pmatrix} \), with \( x \neq 0, y \neq 0 \). Now

\[ x(a + b\sqrt{a}) + (c + d\sqrt{a}) = \alpha x \]
\[ xb(c - d\sqrt{a}) + (a - b\sqrt{a}) = \alpha. \]
From this we get

\[ x = \frac{c + d\sqrt{a}}{a + b\sqrt{a} - \alpha} = \frac{\alpha - a + b\sqrt{a}}{b(c - d\sqrt{a})} \]  

Similarly,

\[ y = -\frac{c + d\sqrt{a}}{a + b\sqrt{a} - \kappa\alpha^{-1}} = \frac{-\alpha - a + b\sqrt{a}}{b(c - d\sqrt{a})}. \]

Hence

\[ x = \frac{a + b\sqrt{a} - \kappa\alpha^{-1}}{a + b\sqrt{a} - \alpha} \]

and

\[ x - y = \frac{(c + d\sqrt{a})(\kappa\alpha^{-1} - \alpha)}{(a + b\sqrt{a} - \alpha)(a + b\sqrt{a} - \kappa\alpha^{-1})} = x \]  

From our assumption deg(b) > deg(a) - deg(a)/2 and (4.4), we get

\[ \text{ord}(x) = \text{ord}(y) =: n. \]

Similarly from (4.5), we get

\[ \text{ord}(x - y) = n + s. \]

By (3.3), the geodesic \( \mathcal{A}(\gamma) \) has vertices

\[ \text{Ver}(\mathcal{A}(\gamma)) = \left\{ \frac{x\pi^k}{\pi^k} \begin{pmatrix} y \\ 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}. \]

If \( k \geq 0 \), then

\[ \begin{pmatrix} x\pi^k \\ \pi^k \end{pmatrix} \sim \begin{pmatrix} (x - y)\pi^k \\ 0 \end{pmatrix} \sim \begin{pmatrix} \pi^{k+n+s} \\ 0 \end{pmatrix}. \]

Since \( y \neq 0 \) and \( k + n + s > n = \text{ord}(y) \), the last matrix is in reduced form. By Lemma 3.1, the distance of this matrix from \( O \) is \( k + |n| + s \geq s \). If \( k < 0 \), then

\[ \begin{pmatrix} x\pi^k \\ \pi^k \end{pmatrix} \sim \begin{pmatrix} x \\ y\pi^{-k} \end{pmatrix} \sim \begin{pmatrix} \pi^{-k} \\ \pi^{-k} \end{pmatrix}. \]

A similar calculation shows that the distance from this matrix to \( O \) is again greater or equal to \( s \). Now

\[ s + \text{deg}(a) = \text{deg}(b) + \text{deg}(a)/2. \]

If \( \text{deg}(b) > B - \text{deg}(a)/2 \), then this last quantity is greater than \( B \). Therefore, by (4.1), \( \gamma T_B \cap T_B = \emptyset \). Note that if \( \text{deg}(a) \leq B \), then \( \text{deg}(b) > B - \text{deg}(a)/2 \) implies \( \text{deg}(b) \geq \text{deg}(a) \). Overall, we have shown that if \( \text{deg}(a) > B \) or \( \text{deg}(b) > B - \text{deg}(a)/2 \), then \( \gamma T_B \cap T_B = \emptyset \).

Now assume \( \text{deg}(a) \leq B \) and \( \text{deg}(b) \leq B - \text{deg}(a)/2 \), but \( \text{deg}(c) > 2B \) or \( \text{deg}(d) > 2B \). From our earlier discussion we know that \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) or \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) are not eigenvectors (since otherwise \( c = d = 0 \)). Now, either \( \text{ord}(c + d\sqrt{a}) < -2B \) or \( \text{ord}(c - d\sqrt{a}) < -2B \) (and only one of the inequalities holds due to \( \text{Nr}(\gamma) \in F_d^\times \)). First, assume \( \text{ord}(c + d\sqrt{a}) < -2B \). Since \( \text{ord}(a + b\sqrt{a} - \alpha) \geq -B \) and \( \text{ord}(a + b\sqrt{a} - \kappa\alpha^{-1}) \geq -B \), from the first equality in (4.2) and (4.3) we get \( \text{ord}(x) < -B \) and \( \text{ord}(y) < -B \). Next, assume \( \text{ord}(c - d\sqrt{a}) < -2B \). Since \( \text{ord}(a - a + b\sqrt{a}) \geq -B \) and \( \text{ord}(\kappa\alpha^{-1} - a + b\sqrt{a}) \geq -B \), from the second equality in (4.2) and (4.3) we get...
get \( \text{ord}(x) > B \) and \( \text{ord}(y) > B \). In either case, Lemma \( \ref{lem:3.3} \) implies that the distance from \( A(\gamma) \) to \( O \) is greater than \( B \). As before, from \( \ref{prop:4.1} \) we conclude \( \gamma T_B \cap T_B = \emptyset \).

**Proposition 4.2.** If \( \gamma \in \Gamma \) is elliptic and satisfies \( \| \gamma \| > B \), then \( \gamma T_B \cap T_B = \emptyset \).

**Proof.** To simplify the notation, we again put \( \text{deg}(0) = 0 \). If \( \gamma \) is elliptic and satisfies the inequality of the proposition, then obviously \( \gamma \notin \mathbb{F}_q^\times \). By Proposition \( \ref{prop:2.6} \), the existence of such \( \gamma \) is possible if and only if \( \text{Odd}(\mathcal{R}) = 1 \). Hence, we can assume that

\[
\gamma = \begin{pmatrix}
a + b \sqrt{r} & c + d \sqrt{r} \\
\xi(c - d \sqrt{r}) & a - b \sqrt{r}
\end{pmatrix},
\]

Since \( \gamma \) is elliptic, \( a \in \mathbb{F}_q \). Assume \( \text{deg}(c) > B \). Consider \( \tau := \gamma \cdot j \in \Gamma, \)

\[
\tau = \begin{pmatrix}
\xi(c + d \sqrt{r}) & a + b \sqrt{r} \\
\xi(a - b \sqrt{r}) & \xi(c - d \sqrt{r})
\end{pmatrix}.
\]

Note that \( \tau \) is hyperbolic since \( \text{Tr}(\tau) = 2 \xi c \notin \mathbb{F}_q \). Suppose \( \gamma T_B \cap T_B \neq \emptyset \). Since \( j \) fixes \( O \), we get

\[
d(O, \tau O) = d(O, \gamma O) \leq 2B.
\]

On the other hand, \( \tau \) acts on its geodesic by translation with amplitude \( 2 \text{deg}(c) > 2B \), so \( d(O, \tau O) > 2B \) - a contradiction. Thus, from now on we can assume \( \text{deg}(c) \leq B \). If \( \text{deg}(b) > B \), then \( \text{deg}(a^2 - b^2 \tau) > 2B \). Since

\[
(a^2 - b^2 \tau) - \xi(c^2 - d^2 \tau) \in \mathbb{F}_q^\times,
\]

we must have \( \text{deg}(b) = \text{deg}(d) =: s \). In this case we have

\[
\gamma = \begin{pmatrix}
z_1 \pi^{-s} & z_2 \pi^{-s} \\
z_3 \pi^{-s} & z_4 \pi^{-s}
\end{pmatrix} \sim \begin{pmatrix}
z_1 & z_2 \\
z_3 & z_4
\end{pmatrix},
\]

where \( z_1, \ldots, z_4 \in \mathcal{O}^\times \). Moreover, since \( \text{det}(\gamma) \in \mathbb{F}_q^\times \),

\[
z_1 z_4 - z_2 z_3 \in \pi^{2s} \mathcal{O}^\times.
\]

This implies

\[
\begin{pmatrix}
z_1 & z_2 \\
z_3 & z_4
\end{pmatrix} \sim \begin{pmatrix}
\pi^{2s} & 0 \\
0 & 1
\end{pmatrix}.
\]

The distance from this matrix to \( O \) is obviously \( 2s \). On the other hand, the matrix corresponding to the vertex \( \gamma O \) under \( \ref{eq:3.2} \) is \( \gamma \), so \( d(O, \gamma O) = 2s > 2B \). The previous argument works also under the assumption \( \text{deg}(d) > B \), and leads to the same conclusion. \( \square \)

**Proof of Theorem \( \ref{thm:1.2} \).** Let \( B = 2 \text{deg}(ab) + 3 \). By Proposition \( \ref{prop:3.10} \), \( D(\Gamma \setminus T) \leq B \). By Proposition \( \ref{prop:4.1} \) and \( \ref{prop:4.2} \) the set of elements \( \gamma \in \Gamma \) such that \( \gamma T_B \cap T_B \neq \emptyset \) is contained in \( S \). Now the theorem follows from Proposition \( \ref{prop:3.7} \). \( \square \)

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References

[1] M. Alsina and P. Bayer, Quaternion orders, quadratic forms and Shimura curves, Amer. Math. Soc., 2004.
[2] J. Chalk and B. Kelly, Generating sets for Fuchsian groups, Proc. Roy. Soc. Edinburgh Sect. A 72 (1975), 317–326.
[3] E.-U. Gekeler, Improper Eisenstein series on Bruhat-Tits trees, manuscripta math. 86 (1995), 367–391.
[4] S. Johansson, On fundamental domains of arithmetic Fuchsian groups, Math. Comp. 69 (2000), 339–349.
[5] A. Lubotzky, Discrete groups, expanding graphs and invariant measures, Birkhäuser, 1994.
[6] A. Lubotzky, B. Samuels, and U. Vishne, Ramanujan complexes of type $\tilde{A}_d$, Israel J. Math. 149 (2005), 267–299.
[7] T. Miyake, Modular forms, Springer-Verlag, 1989.
[8] M. Papikian, Local diophantine properties of modular curves of $D$-elliptic sheaves, submitted for publication, available at [http://www.math.psu.edu/papikian]
[9] M. Rosen, Number theory in function fields, Springer, 2002.
[10] J.-P. Serre, Corps locaux, Hermann, 1968.
[11] J.-P. Serre, Trees, Springer Monographs in Math., 2003.
[12] M.-F. Vignéras, Arithmétiques des algèbres de quaternions, LNM 800, 1980.

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