ON INTEGERS WHOSE SUM IS THE REVERSE OF THEIR PRODUCT

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Abstract. We determine all pairs of positive integers \((a, b)\) such that \(a + b\) and \(a \times b\) have the same decimal digits in reverse order:

\[(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), \ldots\]

We use deterministic finite automata to describe our approach, which naturally extends to all other numerical bases. Our automata are a variation on the notion of Young graphs, which were introduced by Sloane to study “reverse multiples”.

1. Introduction

During a homeschool math lesson, the first author’s children made the curious observation that \(9 + 9 = 18\) and \(9 \times 9 = 81\) — that is, the sum and product are the reverse of each other. A short computer search revealed the more interesting examples

\[2 + 47 = 49\quad \text{and}\quad 2 \times 47 = 94\]
\[3 + 24 = 27\quad \text{and}\quad 3 \times 24 = 72.\]

Are there other examples of integer pairs \((a, b)\) for which the digits of \(a + b\) are the reverse of the digits of \(ab\)?

To formalize the problem, we say that a base-\(\beta\) representation of a positive integer is in canonical form if it has no leading zero. A pair of positive integers \(a \leq b\) will be called a reversed sum-product pair for the base \(\beta\) if the canonical representations of \(a + b\) and \(ab\) in base \(\beta\) are the reverse of each other. We insist that all numbers be written in canonical form in order to avoid examples like \(a = 15\) and \(b = 624\) in base 10, for which \(ab = 9360\) and \(a + b = 0639\). (Allowing non-canonical representations is also interesting, but we do not take up that mantle in this paper.)

Theorem 1.1. The complete list of reversed sum-product pairs in base 10 is

\[(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), (2, 49997), \ldots\]

Our proof technique is algorithmic in nature. If \(a\) is too large, then \(a + b\) will have fewer digits than \(ab\). Given any choice of small \(a\) and a guess for the first and last digit of \(b\), we give a recursive procedure for constructing more digits of \(b\). This procedure only continues indefinitely in one case — the infinite family \((2, 47), (2, 497), (2, 4997), (2, 49997), \ldots\)

Whether the sum and product of two numbers have the same digits in reverse order depends on the choice of numerical base. Our procedure for describing reversed sum-product pairs for the base 10 applies to an arbitrary base \(\beta \geq 2\). Consider base 18, where we use the digits 0, 1, 2, \ldots, 9, A, B, C, \ldots, H. The complete list of reversed sum-product pairs for the base 18 is

\[(2, 2), (H, H), (3, 37), (4, 25),\]
\[(7, 2483D8), (7, 2483D9E483D8), (7, 2483D9E483D9E483D8), \ldots\]
\[(B, 1961DC5), (B, 1961DBG461DC5), (B, 1961DBG461DBG461DC5), \ldots\]

For a given base \(\beta\) and value \(a\), the recursive procedure for constructing digits of \(b\) is best described by a deterministic finite automaton (DFA). In this paper, a DFA is a directed graph with
one vertex designated as the “initial state”, one or more vertices that are “accepting states”, and edge labels from some “alphabet”. Starting at the initial state of a DFA, we can walk through the graph while writing down the edge labels we pass. If we stop at an accepting state, then the string of labels we have written is “accepted” by the DFA. See [3, §2.2] for the formal definition of a DFA and many more details. For additional connections between automata and number theory, we recommend [1] and [4].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1}
\caption{A deterministic finite automaton for $a = 2$ for the base 10.}
\end{figure}

Returning to the issue at hand, integers $b$ that make a base-$\beta$ reversed sum-product pair with $a$ correspond to accepted strings in a particular DFA. For example, a DFA for reversed sum-product pairs for the base 10 with $a = 2$ is given in Figure 1. The initial state is $s_i$. The accepting states are $s_{1,1}$ and $s_o$, drawn with double circles. (The notation for the states will be explained when we construct the DFAs in §5.3.) As we walk through the DFA along directed edges, the edge labels describe how to build $b$ — not from left-to-right, but from out-to-in. Consider the sequence of states $s_i \rightarrow s_{1,1} \rightarrow s_{1,1}$. The state $s_{1,1}$ is accepting, so we can legally stop there. The associated sequence of edge labels is $(4, 7), (9, 9)$. The first term tells us that $b = 4 \cdot \cdots \cdot 7$; the second term gives $b = 4997$. Similarly, the sequence of states $s_i \rightarrow s_{1,1} \rightarrow s_{1,1} \rightarrow s_o$ gives rise to $b = 49997$.

Looking again at our list of reversed sum-product pairs for the base 18 in (1), the patterns become more apparent when we examine the associated DFAs. For example, Figure 2 illustrates the DFA for $a = 7$. Going around the cycle in the DFA gives an infinite family of values $b$ such that $(7, b)$ is a reversed sum-product pair for the base 18.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2}
\caption{A deterministic finite automaton for the base 18 with $a = 7$.}
\end{figure}

The upshot of our investigation for arbitrary bases is the following finiteness result — see §5.3 for a more precise statement:

**Theorem 1.2.** Fix a base $\beta \geq 2$. For each $1 \leq a < \beta$, there is a deterministic finite automaton $A_{\beta,a}$ whose accepted strings correspond to the values $b$ such that $(a, b)$ is a reversed sum-product pair for the base $\beta$. Conversely, every reversed sum-product pair arises in this way.
A closely related phenomenon can be found in “reverse multiples”: integers whose digit reversals are multiples of themselves. For example, in base 10, the only 4-digit reverse multiples are $9 \times 1089 = 9801$ and $4 \times 2178 = 8712$. Young found a construction of these numbers using special rooted trees in [7, 8], and Sloane reworked these trees into a DFA construction similar to ours [5], though the author refers to them as “Young graphs”. See [2] for yet another variation on this theme: integers $n$ for whom some nontrivial multiple permutes the digits of $n$.

We will break our discussion of the algorithm for describing reversed sum-product pairs into three sections, corresponding to the relative sizes of $a$, $b$, and the base $\beta$:

- (small $b$) $a \leq b < \beta$
- (large $a$) $\beta < a \leq b$
- (small $a$, large $b$) $a < \beta < b$

In Section 2, we show that there are only two reversed sum-product pairs when $b$ is small: $(2, 2)$ and $(\beta - 1, \beta - 1)$. In Section 3, we find there is no reversed sum-product pair with large $a$. For the final case, it will be useful to know that we do not need to “carry” when computing the sum of $a$ and $b$; this is proved in Section 4. We describe our recursive algorithm and the construction of DFAs in Section 5, including a careful explanation for the base 10. Python code for exploring and visualizing this construction for an arbitrary base is available at https://github.com/RationalPoint/reverse.

Next we turn to a kind of opposite problem. Instead of fixing the numerical base, we fix a positive integer $a$ and ask for which bases $\beta > a$ there exists a reversed sum-product pair containing $a$. Remarkably, this set has an enormous amount of structure. Let us say that $(2, 2)$ and $(\beta - 1, \beta - 1)$ are the uninteresting reversed sum-product pairs because they are present for all but the smallest bases $\beta$; any other pair is interesting.

**Theorem 1.3.** Fix $a \geq 2$. The set of bases $\beta$ for which there exists an interesting base-$\beta$ reversed sum-product pair $(a, b)$ is the union of a nonzero finite number of arithmetic progressions modulo $a^2 - 1$.

In particular, for a fixed $a$, the set of bases $\beta$ for which there exists an interesting base-$\beta$ reversed sum-product pair containing $a$ has positive density. We give more precise statements in Theorem 6.4 and Corollary 6.5, and we calculate this density for $a \leq 10$ at the end of Section 6.

Our investigation led to an intriguing phenomenon that we were unable to fully explain:

**Conjecture 1.4.** The only bases for which there is no interesting reversed sum-product pair $(a, b)$ are $2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 21$.

Using computer calculation and the tools in Section 6, we have verified that our conjecture holds for $\beta < 1,441,440$. We also prove that at least 99.3% of all bases admit an interesting reversed sum-product pair. This computation is explained in Section 7.

**Conventions.** Throughout this article, we assume that $a \leq b$. If an integer $n$ has base-$\beta$ expansion
\[ n = n_r \beta^r + n_{r-1} \beta^{r-1} + \cdots + n_1 \beta + n_0, \]
we will say that $n_r$ is the “first digit” of $n$ and $n_0$ is the “last digit”. If $a \leq b$ is a reversed sum-product pair for the base $\beta$, then neither one is divisible by $\beta$; indeed, the product would have trailing zeros.
2. **Small $b$**

Suppose that $a \leq b < \beta$ is a reversed sum-product pair for the base $\beta$. Then we will show that exactly one of the following is true:

- $(a, b) = (2, 2)$ and $\beta \geq 5$; or
- $(a, b) = (\beta - 1, \beta - 1)$ and $\beta \geq 3$.

Suppose first that $a + b < \beta$. Then $ab$ and $a + b$ have a single digit in base $\beta$, so $ab = a + b$. Rearranging shows

$$(a - 1)(b - 1) = 1 \implies a = b = 2.$$ 

Our hypothesis that $a + b < \beta$ now becomes $\beta > 4$.

Now suppose that $a + b > \beta$. Since $a + b < 2\beta$, we can write

$$a + b = \beta + x, \quad 1 \leq x < \beta. \quad (2)$$

Since $ab$ is the reverse of $a + b$, we have

$$ab = x\beta + 1. \quad (3)$$

Rearranging yields

$$(\beta - a)(\beta - b) = 1.$$ 

Since $1 \leq (\beta - a), (\beta - b) < \beta$, we must have $a = b = \beta - 1$. Note that this means $x = \beta - 2$, which is a valid first digit only when $\beta \geq 3$.

3. **Large $a$**

If $a$ is large, then we expect $ab$ to have more digits than $a + b$. The next lemma uses this idea to produce a coarse upper bound for $a$.

**Lemma 3.1.** Suppose that $a \leq b$ is a reversed sum-product pair with $a > \beta$. Then $a < 2\beta$ and $b < \beta(\beta + 1)$.

**Proof.** Note that $\beta(a + b)$ has one more digit than $a + b$ in base $\beta$. The fact that $a + b$ and $ab$ have the same number of digits implies that $ab < \beta(a + b)$. Solving for $b$ gives

$$b < \frac{a\beta}{a - \beta}.$$ 

The right side is a decreasing function of $a$, so it is maximized when $a = \beta + 1$, which gives the inequality $b < \beta(\beta + 1)$. Since $a \leq b < \frac{a\beta}{a - \beta}$, we can solve for $a$ to get $a < 2\beta$. \hfill \Box

We now refine the bound in the lemma and conclude there is no reversed-sum product pair with large $a$.

**Proposition 3.2.** Suppose that $a \leq b$ is a reversed sum-product pair for the base $\beta$. Then $a < \beta$.

**Proof.** Suppose for the sake of a contradiction that $a > \beta$. By Lemma 3.1, we have $a < 2\beta$ and $b < \beta(\beta + 1)$. For $\beta \leq 5$, we can examine all $a \in (\beta, 2\beta)$ and $b \in [a, \beta(\beta + 1))$ and find there is no reversed sum-product pair with these constraints.

For the remainder of the proof, we may assume that

$\beta < a < 2\beta, \quad a \leq b < \beta(\beta + 1), \quad \beta \geq 6.$

With these assumptions, we find that $a + b < 3\beta + \beta^2$, so that $a + b$ has 2 or 3 digits. We write

$$a = \beta + a_0 \quad \text{and} \quad b = b_2\beta^2 + b_1\beta + b_0, \quad (4)$$
where $0 \leq a_i, b_i < \beta$ and $a_0b_0 \neq 0$. Note that if $b_2 \neq 0$, then $ab$ has 4 digits. So we may further assume that $b_2 = 0$. As $a \leq b$, it follows that $b_1 \geq 1$.

**Case $a + b$ has 2 digits.** From (4), we have

$$a + b = (1 + b_1)\beta + (a_0 + b_0),$$

so the first digit of $a + b$ is at least 2. Thus,

$$\frac{\beta}{2}(a + b) \geq \beta^2 > ab,$$

since $ab$ must also have 2 digits. Solving for $a$ gives

$$a < \frac{b\beta}{2b - \beta}.$$  

For $b > \beta$ the right side is a decreasing function, so it is maximized by taking $b = \beta + 1$. We then have

$$a < \frac{b\beta}{2b - \beta} \leq \beta\frac{\beta + 1}{\beta + 2} < \beta,$$

a contradiction.

**Case $a + b$ has 3 digits.** From (4), we have

$$a + b = (1 + b_1)\beta + (a_0 + b_0).$$

As we are assuming $a + b$ has 3 digits, we have two subcases to consider:

(I) $1 + b_1 \geq \beta$, or

(II) $1 + b_1 = \beta - 1$ and $a_0 + b_0 \geq \beta$.

In both (I) and (II), (4) shows that the product $ab$ satisfies

$$ab = b_1\beta^2 + (a_0b_1 + b_0)\beta + a_0b_0$$ (5)

In case (I), we must have $b_1 = \beta - 1$. Using the fact that $a_0, b_0 \geq 1$, we obtain the following estimate from (5):

$$ab \geq \beta^3 + 1.$$  

But then $ab$ has at least 4 digits, a contradiction.

In case (II), we look at the coefficient on $\beta$ in (5):

$$a_0b_1 + b_0 = a_0(\beta - 2) + b_0 \geq a_0(\beta - 2) + \beta - a_0 = a_0(\beta - 3) + \beta.$$  

This is an increasing function of $a_0$. If $a_0 \geq 2$, then this quantity is at least $2\beta$ since $\beta \geq 6$. As in case (I), we obtain the absurd conclusion that $ab$ has 4 digits. So $a_0 = 1$ and $b_0 = \beta - 1$. This completely nails down $a$ and $b$:

$$a = \beta + 1 \quad \text{and} \quad b = (\beta - 2)\beta + (\beta - 1).$$

As $a + b = \beta^2$, we do not obtain a reversed sum-product pair. This completes the proof. \qed
4. Carries

From grade school arithmetic, we know about “carrying” when computing multi-digit sums and products. Our primary goal for this section is to show that there is no carry when computing the sum \(a + b\) for a reversed sum-product pair \((a, b)\). (Typically there are carries in the product.) We begin with a careful definition of carry digits and a bound on how big they can be.

Suppose that \(a, d\) are two single-digit numbers, which means \(0 \leq a, d < \beta\). Their sum or product is at most two digits. If it is two digits, we call the leading digit the carry digit. Upon adding single-digit numbers, the resulting carry digit is at most 1:

\[
ad \leq a(\beta - 1) = (a - 1)\beta + (\beta - a).
\]

Now imagine that we are multiplying a single-digit number \(a\) by a multi-digit number \(b\). A particular digit of the product \(ab\) comes from multiplying \(a\) by a single digit of \(b\) and adding the previous carry digit. The new carry digit is still at most \(a - 1\):

\[
ad + \text{previous carry} \leq a(\beta - 1) + (a - 1) = (a - 1)\beta + (\beta - 1).
\]

Next we show that the first digit of \(a + b\) does not arise from a carry.

**Lemma 4.1.** If \((a, b)\) is a reversed sum-product pair for the base \(\beta\) with \(a < \beta < b\), then \(b\) and \(a + b\) have the same number of digits.

**Proof.** Assume the number of digits differs. Then \(a + b\) has one more digit than \(b\), and \(b = \beta^\ell - c\) for some \(\ell \geq 2\) and \(c\) with \(a > c \geq 1\). Then

\[
a + b = \beta^\ell + (a - c) \quad \text{and} \quad ab = a\beta^\ell - ac.
\]

The above expression for \(a + b\) has first digit 1, last digit \(a - c\), and all other digits 0. Therefore, the reverse is true for \(ab\):

\[
ab = (a - c)\beta^\ell + 1.
\]

Combining these two expressions for \(ab\) and rearranging, we get \(c(\beta^\ell - a) = 1\). Since \(\beta^\ell \geq \beta^2 > a + 1\), this is a contradiction.

Now we improve the preceding lemma to show that the computation of \(a + b\) involves no carry at all.

**Proposition 4.2.** If \((a, b)\) is a reversed sum-product pair for the base \(\beta\) with \(a < \beta < b\), the last digit of \(b\) is strictly smaller than \(\beta - a\).

**Proof.** Write \(b_r\) for the first digit of \(b\), and let \(d\) be the first digit of \(ab\). Since \(ab\) has the same number of digits as \(a + b\), which has the same number of digits as \(b\) (Lemma 4.1), we see that \(d = ab_r + \lambda < \beta\), where \(\lambda \leq a - 1\) is the carry from the \((r - 1)\)-st place of the product \(ab\). It follows that \(d \geq ab_r \geq a\).

Write \(b_0\) for the last digit of \(b\). Since \(a \leq b\) is a reversed sum-product pair, the last digit of \(a + b\) must be \(d \equiv a + b_0 \pmod{\beta}\). That is, \(b_0 \equiv d - a \pmod{\beta}\). Since \(d \geq a\), we conclude that \(b_0 = d - a < \beta - a\).

In particular, the above proposition shows that \(a + b\) and \(b\) have the same first digit, a fact we will capitalize on in the next section.
5. Small \(a\), Large \(b\)

Suppose that \((a, b)\) is a reversed sum-product pair for the base \(\beta\), and that \(a < \beta < b\). Write the base-\(\beta\) expansion of \(b\) as

\[ b = b_r \beta^r + \cdots + b_0. \]

We begin by determining necessary — though not sufficient — conditions on \(b_0, b_r\). From there, we will inductively determine 2 more digits (which may be the same digit if \(b\) has an odd number of digits), and so on. With careful bookkeeping, this procedure will result in only finitely many states, and we will be able to develop an algorithm for finding all valid reversed sum-product pairs.

5.1. The Recursion. To recap, we have now determined that if \((a, b)\) is a reversed sum-product pair for the base \(\beta\) with \(a < \beta < b\), then

- \(b\) and \(a + b\) and \(ab\) have the same number of digits;
- \(b\) and \(a + b\) have the same first digit; and
- the last digit of \(b\) is strictly smaller than \(\beta - a\).

Recall that we write \(b_r, b_0\) for the first and last digits of \(b\), respectively. Then we have \(0 < b_0 < \beta - a\), and the last digit of \(a + b\) is \(a + b_0\). So the first digit of \(ab\) is \(ab_r + \lambda = a + b_0\) for some \(0 \leq \lambda < a\) corresponding to the carry from the \((r - 1)\)-st place of the product. (Here, \(\lambda\) stands for “left” carry.) The first digit of \(a + b\) agrees with \(b_r\). But this is also equal to the last digit of \(ab\), so we have \(b_r \equiv ab_0 \pmod{\beta}\). Writing \(\rho\) for the carry from the units place — the “right” carry — we obtain the following constraints on the first and last digits of \(b\):

\[ a + b_0 = ab_r + \lambda \text{ for some } 0 \leq \lambda < a \]
\[ ab_0 = b_r + \rho \beta \text{ for some } 0 \leq \rho < a \]  \hfill (6)

Now suppose that we have determined the first and last \(n\) digits of \(b\) for some \(n > 0\). Write \(b = \cdots xx' \cdots yy' \cdots\), where \(x, y\) have already been determined and we would like to find \(x'\) and \(y'\). Assume further that we already know the carry into the \(x\)-column of the product \(ab\) — let us call it \(\lambda\). (This will be part of our inductive information.) The product of \(a\) with the rightmost \(n\) digits of \(b\) determines a carry out of the \(y\)-column of \(ab\) — call it \(\rho\), so that the product will have \(ay' + \rho\) (mod \(\beta\)) in the next place to the left. See Figure 3(i). Since \((a, b)\) is a reversed sum-product pair, and since the corresponding digit of \(a + b\) is \(x'\), we find that

\[ ay' + \rho = x' + \rho' \beta \text{ for some } 0 \leq \rho' < a. \]

To determine the digit of the product \(ab\) arising from multiplication by \(x'\), we need to consider the unknown carry from the middle digits — call it \(\lambda'\). See Figure 3(ii). The result is \(ax' + \lambda'\), which must agree modulo \(\beta\) with the corresponding digit of \(a + b\), namely \(y'\). That is,

\[ ax' + \lambda' = y' + \lambda \beta \text{ for some } 0 \leq \lambda' < a. \]

We combine the recursion equations for future reference:

\[ ax' + \lambda' = y' + \lambda \beta \text{ for some } 0 \leq \lambda' < a \]
\[ ay' + \rho = x' + \rho' \beta \text{ for some } 0 \leq \rho' < a \]  \hfill (7)

There are two ways for this construction to terminate: when the left and right sides may be concatenated, or when the left and right sides overlap in a digit. Suppose we know the first and last \(n\) digits of \(b\) for some \(n \geq 1\). Write \(b = \cdots xx' \cdots yy' \cdots\), where \(x, y\) have already been determined, and suppose that we know the carry \(\lambda\) into the \(x\)-column of the product \(ab\) and the carry \(\rho\) out of the \(y\)-column in the product. The left and right sides may be concatenated if the carries are compatible — i.e., if \(\lambda = \rho\). In that case, \(a\) and \(b = \cdots xy \cdots\) are a reversed sum-product pair.
To understand when the left and right sides may overlap in a digit, we must run the recursion one more step. With the setup of the previous paragraph, we solve the recursion equations (7) to obtain $x', y', \lambda', \rho'$. If $x' = y'$, then we claim that $\lambda' = \rho$ and $\lambda = \rho'$. To see it, set $y' = x'$ in (7) and subtract the two equations. We obtain

$$\lambda' - \rho = (\lambda - \rho')\beta.$$ 

Since $|\lambda' - \rho| < a < \beta$, we must have $\lambda' = \rho$ and $\lambda = \rho'$, as desired. It follows that the carries match up so that $a$ and $b = \cdots xx'y\cdots$ are a reversed sum-product pair.

To summarize, we have shown that the above procedure can terminate in two ways:

- if $\lambda = \rho$ at any step, or
- if $x' = y'$ in the recursion step.

5.2. The Case $\beta = 10$. Using the strategy from the preceding section, we complete the promised description of all reversed sum-product pairs for the base 10:

**Theorem 5.1.** If $a \leq b$ is a reversed sum-product pair for the base 10, then $(a, b)$ is among $(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), (2, 49997), \ldots$

Section 2 shows that $(2, 2)$ and $(9, 9)$ are the only instances with $b < 10$. Section 3 shows that any remaining pair must have $a < 10 < b$.

For a given $a < 10$, we begin by looking for all 4-tuples $(b_r, b_0, \lambda, \rho)$ satisfying (6) with $0 < b_0 < 10 - a$ and $0 < b_r < 10$. The result is given in Table 1.

| $a$ | $b_r$ | $b_0$ | $\lambda$ | $\rho$ |
|-----|-------|-------|-----------|-------|
| 2   | 4     | 7     | 1         | 1     |
| 3   | 2     | 4     | 1         | 1     |

Table 1. The solutions to (6) for $\beta = 10$ with $0 < b_r < 10$ and $0 < b_0 < 10 - a$.

Let us look at $a = 3$ first. Any reversed sum-product pair $(a, b)$ must have $b = 2 \cdots 4$ for some unknown (and possibly nonexistent) digits between the 2 and the 4. The equations (7) defining the recursion step have no solution, as one can check with a short calculation. Since $\lambda = \rho = 1$ in the first step, we obtain $b = 24$ as the only solution with $a = 3$.

Next we look at $a = 2$. Any reversed sum-product pair $(a, b)$ must have $b = 4 \cdots 7$. Since $\lambda = \rho = 1$, we obtain a first solution $b = 47$. The recursion equations (7) have a single solution: $(x', y', \lambda', \rho') = (9, 9, 1, 1)$. Since $x' = y' = 9$, we obtain the value $b = 497$. Since $\lambda' = \rho'$, we also obtain the value $b = 4997$. Finally, note that the recursion equations depend only on $\lambda, \rho$; it follows that $(9, 9, 1, 1)$ is the unique solution obtained by running the recursion again. We obtain the solutions 49997 and 499997 from the next step of the recursion, and so on. This completes the proof of the theorem.

Figure 3. An illustration of the carries involved in the beginning and end of the recursion step.

To understand when the left and right sides may overlap in a digit, we must run the recursion one more step. With the setup of the previous paragraph, we solve the recursion equations (7) to obtain $x', y', \lambda', \rho'$. If $x' = y'$, then we claim that $\lambda' = \rho$ and $\lambda = \rho'$. To see it, set $y' = x'$ in (7) and subtract the two equations. We obtain

$$\lambda' - \rho = (\lambda - \rho')\beta.$$ 

Since $|\lambda' - \rho| < a < \beta$, we must have $\lambda' = \rho$ and $\lambda = \rho'$, as desired. It follows that the carries match up so that $a$ and $b = \cdots xx'y\cdots$ are a reversed sum-product pair.

To summarize, we have shown that the above procedure can terminate in two ways:

- if $\lambda = \rho$ at any step, or
- if $x' = y'$ in the recursion step.

5.2. The Case $\beta = 10$. Using the strategy from the preceding section, we complete the promised description of all reversed sum-product pairs for the base 10:

**Theorem 5.1.** If $a \leq b$ is a reversed sum-product pair for the base 10, then $(a, b)$ is among $(2, 2), (9, 9), (3, 24), (2, 47), (2, 497), (2, 4997), (2, 49997), \ldots$

Section 2 shows that $(2, 2)$ and $(9, 9)$ are the only instances with $b < 10$. Section 3 shows that any remaining pair must have $a < 10 < b$.

For a given $a < 10$, we begin by looking for all 4-tuples $(b_r, b_0, \lambda, \rho)$ satisfying (6) with $0 < b_0 < 10 - a$ and $0 < b_r < 10$. The result is given in Table 1.

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Let us look at $a = 3$ first. Any reversed sum-product pair $(a, b)$ must have $b = 2 \cdots 4$ for some unknown (and possibly nonexistent) digits between the 2 and the 4. The equations (7) defining the recursion step have no solution, as one can check with a short calculation. Since $\lambda = \rho = 1$ in the first step, we obtain $b = 24$ as the only solution with $a = 3$.

Next we look at $a = 2$. Any reversed sum-product pair $(a, b)$ must have $b = 4 \cdots 7$. Since $\lambda = \rho = 1$, we obtain a first solution $b = 47$. The recursion equations (7) have a single solution: $(x', y', \lambda', \rho') = (9, 9, 1, 1)$. Since $x' = y' = 9$, we obtain the value $b = 497$. Since $\lambda' = \rho'$, we also obtain the value $b = 4997$. Finally, note that the recursion equations depend only on $\lambda, \rho$; it follows that $(9, 9, 1, 1)$ is the unique solution obtained by running the recursion again. We obtain the solutions 49997 and 499997 from the next step of the recursion, and so on. This completes the proof of the theorem.
5.3. A Deterministic Finite Automaton. Fix a base $\beta \geq 2$ and an integer $1 \leq a < \beta$. The procedure described in Section 5.1 suggests a method for constructing a DFA, which we now carry out. The allowable edge labels — i.e., the “alphabet” in DFA theory — are pairs $(x, y)$ with $0 \leq x, y < \beta$ as well as singletons $(x)$ for $0 \leq x < \beta$. We construct three types of states:

- An initial state $s_i$;
- An “odd state” $s_o$; and
- A “carry state” $s_{\lambda, \rho}$ for each pair of integers $0 \leq \lambda, \rho < a$.

The accepting states are $\{s_o\} \cup \{s_{\lambda, \lambda} : 0 \leq \lambda < a\}$. The transitions for our DFA are as follows:

- For each solution $(b_o, b_0, \lambda, \rho)$ to (6), we have a transition from the initial state $s_i$ to the state $s_{\lambda, \rho}$ with label $(b_o, b_0)$.
- For each carry state $s_{\lambda, \rho}$ and each solution $(x', y', \lambda', \rho')$ to the recursion equations (7), we have a transition from $s_{\lambda, \rho}$ to $s_{\lambda', \rho'}$ with label $(x', y')$. If $x' = y'$, we also include a transition from $s_{\lambda, \rho}$ to the odd state $s_o$ with label $(x')$.
- If $\beta \geq 5$ and $a = 2$, include a transition from $s_i$ to $s_o$ with label $(2)$.
- If $\beta \geq 3$ and $a = \beta - 1$, include a transition from $s_i$ to $s_o$ with label $(\beta - 1)$.

Write $A = A_{\beta, a}$ for the DFA thus constructed.

The definition of our DFA captures the digit construction involved in the recursion in Section 5. We formalize this in the following statement, which is a more precise version of Theorem 1.2 from the introduction.

**Theorem 5.2.** For $\beta \geq 2$ and $a < \beta$, let $A = A_{\beta, a}$ be the DFA constructed above.

- Suppose that the string $(x_1, y_1)(x_2, y_2) \cdots (x_n, y_n)$ is accepted by $A$ for some $n \geq 1$. Then $a$ and $b = x_1 x_2 \cdots x_n y_n \cdots y_2 y_1$ are a reversed sum-product pair for the base $\beta$.
- Suppose that the string $(x_1, y_1)(x_2, y_2) \cdots (x_n, y_n)(z)$ is accepted by $A$ for some $n \geq 0$. Then $a$ and $b = x_1 x_2 \cdots x_n z y_n \cdots y_2 y_1$ are a reversed sum-product pair for the base $\beta$.

If $a \leq b$ is a reversed sum-product pair for the base $\beta$, then $b$ can be constructed from a string accepted by $A$ in one of these two ways.

A priori, the number of states in $A_{\beta, a}$ is $a^2 + 2$. In practice, many of these states are unreachable by a path beginning at the initial state. To avoid these superfluous states, we will always “lazy construct” the DFA: beginning with solutions to (6), only construct states as needed to satisfy the recursion. Taking this approach does not affect the set of strings accepted by the DFA. For example, if (6) has no solution, then only the initial state needs to be constructed. See Figure 4 for the end result in base 10.

After constructing the DFA, we may find that there are states $s$ that do not participate in any accepted string — i.e., no path from the initial state to an accepting state passes through $s$. We can “trim” all such $s$ from the DFA. As an extreme example, the DFA $A_{150,31}$ has 13 states accessible from the initial state, but it has no accepting state. Consequently, we can trim all states but the initial one. Said another way, there is no base-150 reversed sum-product pair of the form $(31, b)$.

When constructing examples of these DFAs, one begins to marvel at how complicated they can be. One way to measure this complexity is by the growth of the number of strings accepted by $A_{\beta, a}$. Recall that the set of strings accepted by a DFA is a regular language. Passing to a more general setting for a moment, let $\Sigma$ be an alphabet and $R \subset \Sigma^*$ any regular language. Define the growth function

$$p_R(n) = |R \cap \Sigma^n| = \text{“number of strings in } R \text{ of length } n\text{”}.$$ 

In [6], it is shown that $p_R(n) = O(n^k)$ for some $k \geq 0$ if and only if $R$ can be represented as a finite union of regular expressions of the form $xy_1^* z_1 \cdots y_k^* y_k$, where $x, y_i, z_i$ are fixed strings in $\Sigma^*$. 

[9]
In essence, this means there are only finitely many repeated patterns that can occur in the strings of $R$.

Figures 1 and 2 are examples of DFAs with $p_R(n) = O(1)$. We also have an example with exponential growth. Use the digits $0, 1, 2, \ldots, 0, A, B, C, \ldots, Q$ to represent integers in base 27. Running the DFA construction in Section 5.3, one verifies that a state diagram for $A_{27, A}$ is given by Figure 5. If $R$ is the regular language associated with this DFA, then a straightforward inductive argument gives the number of accepted strings of a given length:

$$p_R(4 + 5n) = 2^n$$

for each $n \geq 0$.

In particular, $p_R(n) \neq O(n^k)$ for any $k \geq 0$.

**Question 5.3.** Is there a combinatorial description of the set of bases $\beta$ and integers $a < \beta$ such that $A_{\beta, a}$ has bounded growth function: $p_R(n) = O(1)$?

To close this section, we give an easy criterion for showing that $A_{\beta, a}$ has no accepting state.
Lemma 6.2. Let one base in a congruence class, then it participates for all larger bases in the same class. We will take the corresponding solutions to the recursion equations \((a, b)\) and \((\beta - 1, \beta - 1)\). By setting \(T = a\) if \(\beta = a\) unless \(\beta = \beta - 1\). Then \((\beta - 1, \beta - 1)\) is a reversed sum-product pair for the base \(\beta\). Propagation in a reversed

\[
\begin{align*}
\text{Example 6.1.} & \text{ For a given } a \geq 2, \text{ we claim there are infinitely many bases } \beta \text{ for which } a \text{ participates. To see it, let } T \text{ be a variable. Define } \\
b & = (T + 1)\beta + (aT + 1) & \beta & = (a^2 - 1)T + a - 1.
\end{align*}
\]

Then \((a, b)\) is a formal reversed sum-product pair for the base \(\beta\) in the sense that

\[
a + b = (T + 1)\beta + (aT + a + 1) \quad \text{and} \quad ab = (aT + a + 1)\beta + (T + 1).
\]

By setting \(T\) equal to a positive integer, we get a reversed sum-product pair in the usual sense unless \(a = 2\) and \(T = 1\) or 2.

In the above example, we produced a set of bases \(\beta\) for which \(a\) participates; note that they all lie in the same congruence class modulo \(a^2 - 1\). This is a general phenomenon: if \(a\) participates for one base in a congruence class, then it participates for all larger bases in the same class.

**Lemma 6.2.** Let \(a \geq 2\) be an integer, and suppose that \(a\) participates for the base \(\beta > a\). Then \(a\) also participates for the base \(\hat{\beta} = \beta + a^2 - 1\).

**Proof.** Let \(b = b_\beta \beta^r + \cdots + b_0\) be such that \((a, b)\) is a reversed sum-product pair for the base \(\beta\). We will take the corresponding solutions to the recursion equations (6) and (7) and construct new solutions for the base \(\hat{\beta}\).

The quadruple \((b_\beta, b_0, \lambda, \rho)\) satisfies the equations (6). Define

\[
\begin{align*}
\hat{b}_r & = b_r + \rho \\
\hat{b}_0 & = b_0 + a\rho.
\end{align*}
\]

Since \(\rho < a\), we see that \(\hat{b}_r < \hat{\beta}\) and \(\hat{b}_0 < \hat{\beta} - a\). One verifies immediately that the quadruple \((\hat{b}_r, \hat{b}_0, \lambda, \rho)\) satisfies the equations (6) with \(\hat{\beta}\) in place of \(\beta\). Note that the carries \(\lambda, \rho\) did not change.

Now suppose that the quadruple \((\hat{x}', \hat{y}', \lambda', \rho')\) satisfies (7). Define

\[
\begin{align*}
\hat{x}' & = x' + \lambda a + \rho' \\
\hat{y}' & = y' + \rho' a + \lambda.
\end{align*}
\]

Again, we see that \(\hat{x}', \hat{y}' < \hat{\beta}\) and the quadruple \((\hat{x}', \hat{y}', \lambda', \rho')\) satisfies (7) with \(\hat{\beta}\) in place of \(\beta\).
To complete the proof, we will show that the termination conditions agree; that is, we get an integer \( \hat{b} \) with the same number of digits as \( b \), and such that \((a, \hat{b})\) is a reversed sum-product pair for the base \( \hat{\beta} \). If the recursion for \( b \) terminates because \( \lambda = \rho \), then clearly the same is true for \( \hat{b} \) since we used all of the same carries. If instead the recursion terminates because \( x' = y' \) at some stage, then we claim that \( \hat{x}' = \hat{y}' \). Indeed, we saw in §5.1 that \( x' = y' \) implies that \( \lambda = \rho' \) and \( \lambda' = \rho \). It follows that
\[
\hat{x}' = x' + \lambda a + \rho' = y' + \rho' a + \lambda = \hat{y}'.
\]
It follows that the recursion terminates for \( \hat{b} \), as desired. \( \square \)

There is at least one congruence class that contains no base for which \( a \) participates.

**Lemma 6.3.** Let \((a, b)\) be a reversed sum-product pair for the base \( \beta \) with \( b > \beta \). Then \( \beta \not\equiv 0 \pmod{a^2 - 1} \).

**Proof.** We can write \( b = b_r \beta^r + \cdots + b_0 \), with \( r \geq 1 \). Consequently, \( b_r \) and \( b_0 \) must satisfy the equations (6) for some choice of \( \lambda, \rho \). Eliminating \( b_0 \) from (6) shows that
\[
(a^2 - 1) b_r = a^2 - a\lambda + \rho \beta.
\]
If \( \beta \equiv 0 \pmod{a^2 - 1} \), then reducing this equation modulo \( a^2 - 1 \) yields \( a^2 \equiv a\lambda \pmod{a^2 - 1} \). As \( a \) is coprime to \( a^2 - 1 \), we conclude that \( \lambda \equiv a \pmod{a^2 - 1} \). Since \( a < a^2 - 1 \), we must have \( \lambda = a \), a contradiction. \( \square \)

For integers \( q, r \), let us agree to write \( qN + r \) for the arithmetic progression \( \{qn + r : n = 0, 1, 2, \ldots \} \).

**Theorem 6.4.** For \( a \geq 2 \), there exists a nonempty set of arithmetic progressions
\[
(a^2 - 1)N + v_1, (a^2 - 1)N + v_2, \ldots, (a^2 - 1)N + v_\ell
\]
such that
\begin{itemize}
  \item \( 0 < v_i < a^2 - 1 \) for all \( i \);
  \item If \( a \) participates for \( \beta \), then \( \beta \in (a^2 - 1)N + v_i \) for some \( i \); and
  \item For each \( i \) and each sufficiently large \( \beta \in (a^2 - 1)N + v_i \), \( a \) participates for \( \beta \).
\end{itemize}

**Proof.** Fix \( a \geq 2 \) and let \( B \) be the set of all \( \beta \geq 2 \) for which \( a \) participates. Define \( v_1, \ldots, v_\ell \) to be the set of integers in the interval \([0, a^2 - 1]\) that are congruent to some element of \( B \). Example 6.1 shows that the set \( \{v_1, \ldots, v_\ell\} \) is nonempty. Lemma 6.3 shows that no \( v_i = 0 \). The final statement is immediate from Lemma 6.2. \( \square \)

We can now address the question, “How big is the set of bases for which a given \( a \) participates in a reversed sum-product pair?” To that end, define the limit
\[
\Omega(a) = \lim_{N \to \infty} \frac{1}{N} |\{2 \leq \beta \leq N : a \text{ participates for } \beta\}|.
\]

**Corollary 6.5.** For each \( a \geq 2 \), the limit defining \( \Omega(a) \) exists and is of the form \( \ell/(a^2 - 1) \) for some integer \( 1 \leq \ell < a^2 - 1 \) that depends on \( a \). In particular, \( 0 < \Omega(a) < 1 \).
Proof. Let \( v_1, \ldots, v_\ell \) be as in the theorem. Then \( 1 \leq \ell < a^2 - 1 \). For \( N \) sufficiently large, we have

\[
|\{2 \leq \beta \leq N : a \text{ participates for } \beta\}| = \left| \bigcup_{i=1}^\ell \left\{ (a^2 - 1)n + v_i : 1 \leq n \leq \frac{N}{a^2 - 1} \right\} \right| + O(1)
\]

\[
= \sum_{i=1}^\ell \frac{N}{a^2 - 1} + O(1) = \frac{\ell N}{a^2 - 1} + O(1).
\]

Dividing by \( N \) and passing to the limit gives the result. \( \square \)

Example 6.6. We claim that \( \Omega(2) = \frac{1}{3} \). The theorem shows that all bases \( \beta \) for which \( a = 2 \) participates lies in the arithmetic progressions \( 3\mathbb{Z}+1 \) or \( 3\mathbb{Z}+2 \). We will now argue that if \( \beta \in 3\mathbb{Z}+2 \), then there is no reversed sum-product pair for the base \( \beta \) other than the uninteresting pair \((2,2)\).

Write \( \beta = 3n + 2 \). Solving the recursion equations (6) for \( b_r \), shows that \( 3b_r = 4 - 2\lambda + \rho \beta \). Reducing modulo 3 and simplifying gives \( \rho \equiv \lambda + 1 \pmod{3} \). The only solution to this congruence with \( 0 \leq \lambda, \rho < 2 \) is \( \lambda = 0 \) and \( \rho = 1 \). As \( \lambda \neq \rho \), we must continue the recursion. A similar analysis applied to (7) shows that \( \rho' = 0 \) and \( \lambda' = 1 \). But then the equations (7) become

\[
2x' + 1 = y' \\
2y' + 1 = x',
\]

which has \( x' = y' = -1 \) as their unique solution. These are not valid digits in any base. This contradiction shows that the only reversed sum-product pair for a base \( \beta \in 3\mathbb{Z}+2 \) is \((2,2)\).

A similar strategy to the one in Example 6.6 allows us to compute the ratio \( \Omega(a) \) for any \( a \). Table 2 gives the first few values. (Given \( a \), the procedure for determining which arithmetic progressions actually contain bases \( \beta \) for which \( a \) participates is implemented in the function \texttt{construct.generic.autowma} in our Python module.)

| \( a \) | \( \Omega(a) \) | \( a \) | \( \Omega(a) \) |
|---|---|---|---|
| 2 | \( \frac{1}{3} \approx 0.333 \) | 7 | \( \frac{4}{48} \approx 0.083 \) |
| 3 | \( \frac{1}{8} = 0.125 \) | 8 | \( \frac{22}{63} \approx 0.349 \) |
| 4 | \( \frac{4}{15} \approx 0.267 \) | 9 | \( \frac{12}{80} = 0.15 \) |
| 5 | \( \frac{3}{24} = 0.125 \) | 10 | \( \frac{26}{99} \approx 0.263 \) |
| 6 | \( \frac{13}{35} \approx 0.371 \) |

Table 2. The value of the ratio \( \Omega(a) \) for \( a \leq 10 \).

7. Existence of Interesting Pairs

Recall that a reversed sum-product pair \((a,b)\) for the base \( \beta \) is deemed to be interesting if it is not one of the pairs \((2,2)\) or \((\beta-1,\beta-1)\). For a given base \( \beta \), do we expect to find any interesting reversed sum-product pair at all? The following propositions provide support for Conjecture 1.4 in the introduction.

Proposition 7.1. Among all bases \( \beta \geq 2 \), at least 99.3\% of them admit an interesting reversed sum product-pair.

Proposition 7.2. The only bases \( \beta < 1,441,440 \) for which there is no interesting reversed sum-product pair are

\[2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 21.\]
We describe a sieving procedure that will allow us to prove both propositions simultaneously:

1. Set $B = 1, 441, 440$, and consider an array of integers from 0 to $B - 1$.
2. For each $a < 100$ such that $a^2 - 1$ divides $B$, do the following:
   (a) Compute $v_1, \ldots, v_k$ as described by Theorem 6.4.
   (b) For each arithmetic progression $(a^2 - 1)N + v_i$, let $T_i \geq 0$ be minimal such that $\beta = (a^2 - 1)T_i + v_i$ admits an interesting reversed sum-product pair.
   (c) For each $t \geq T_i$ such that $\beta = (a^2 - 1)t + v_i$ lies in the interval $[0, B)$, cross $\beta$ off of our array.
3. For each $\beta \in [0, B)$ that we have not crossed off yet, construct the DFAs $A_{\beta, a}$ for $a \geq 2$. If we find one that gives rise to an interesting reversed sum-product pair, cross $\beta$ off of our array.

Step (2a) can be accomplished by a “generic” version of the construction in §5.3. Set $\beta = (a^2 - 1)T + v$ for some fixed $0 < v < a^2 - 1$. We solve the recursion equations (6) for integers $0 \leq \lambda, \rho < a$ and with $b_0, b_r$ being linear polynomials in $T$. Then we solve (7) for $0 \leq \lambda', \rho' < a$ and with $x', y'$ being linear polynomials in $T$. This is implemented in our Python module in the function \texttt{construct\_generic\_automata}.

To determine $T_i$ as in Step (2b), we can specialize a generic DFA to $T = 0, 1, \ldots$ until we find a valid DFA $A((a^2 - 1)T + v, a)$. At the end of Step (2), we find that 1, 431, 542 of the elements of our array have been crossed off. Since $a^2 - 1$ divides $B$ for each $a$ used in the computation, Lemma 6.2 tells us that all sufficiently large integers $\beta$ in each of these residue classes admit an interesting reversed sum-product pair. That is, this holds for at least $\frac{1431542}{5354288800} \approx 99.31\%$ of all bases, which proves Proposition 7.1. This part of the computation took approximately 5.5 minutes on a Xeon(R) E5-2699 processor (2.30GHz with 500GB memory).

The construction of the DFAs $A_{\beta, a}$ as in Step (3) is implemented in our Python module in the function \texttt{construct\_automata}. Applying it, we cross off all remaining entries in the array except for

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 21.$$ 

Since $\beta = 0, 1$ are not valid numerical bases, we may drop these from consideration, thus proving Proposition 7.2. This step required an additional 1.5 hours to complete on the same processor as above.

\textbf{Remark 7.3.} Trying Step (2) again with the larger parameter $B = 53, 542, 288, 800$ took 42 hours and gives the improved lower bound of $\frac{1431542}{1441440} \approx 99.92\%$ in Proposition 7.1. We did not extend the computation in Step (3) to this larger value of $B$.

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\textbf{References}

[1] Jean-Paul Allouche and Jeffrey Shallit. \textit{Automatic sequences}. Cambridge University Press, Cambridge, 2003.
[2] Benjamin V. Holt. On permutiples having a fixed set of digits. \textit{Integers}, 17:Paper No. A20, 21, 2017.
[3] John E. Hopcroft and Jeffrey D. Ullman. \textit{Introduction to automata theory, languages, and computation}. Addison-Wesley Publishing Co., Reading, Mass., 1979. Addison-Wesley Series in Computer Science.
[4] Aayush Rajasekaran, Jeffrey Shallit, and Tim Smith. Additive number theory via automata theory. \textit{Theory Comput. Syst.}, 64(3):542–567, 2020.
[5] N. J. A. Sloane. 2178 and all that. \textit{Fibonacci Quart.}, 52(2):99–120, 2014.
[6] Andrew Szilard, Sheng Yu, Kaizhong Zhang, and Jeffrey Shallit. Characterizing regular languages with polynomial densities. In Mathematical foundations of computer science 1992 (Prague, 1992), volume 629 of Lecture Notes in Comput. Sci., pages 494–503. Springer, Berlin, 1992.

[7] Anne Ludington Young. k-reverse multiples. Fibonacci Quart., 30(2):126–132, 1992.

[8] Anne Ludington Young. Trees for k-reverse multiples. Fibonacci Quart., 30(2):166–174, 1992.

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