IMAGINARY MULTIQUADRATIC NUMBER FIELDS WITH CLASS GROUP OF EXPONENT 3 AND 5

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Abstract. In this paper we obtain a complete list of imaginary \( n \)-quadratic fields with class groups of exponent 3 and 5 under ERH for every positive integer \( n \) where an \( n \)-quadratic field is a number field of degree \( 2^n \) represented as the composite of \( n \) quadratic fields.

1. Introduction

Brown and Parry [5] determine a complete list of imaginary biquadratic fields with class number 1. Yamamura [18] gives a complete list of imaginary abelian fields with class number 1. Jung and Kwon [11] show a complete list of imaginary biquadratic fields of class number 3. Elsenhans, Klüners and Nicolae [6, Theorem 1] present a complete list of imaginary quadratic fields with class groups of exponent \( E \) for every \( E \leq 5 \) and \( E = 8 \) under the extended Riemann hypothesis (ERH). We say that \( K \) is an \( n \)-quadratic field if \( K \) is a Galois extension of \( \mathbb{Q} \) with \( \text{Gal}(K/\mathbb{Q}) \cong C_{2^n} \) where \( C_2 \) is the cyclic group of order 2. In this paper we obtain a complete list of imaginary \( n \)-quadratic fields with class groups of exponent 3 and 5, that is, isomorphic to the direct products \( C_r \mathbb{Z} \) of the cyclic group \( C_u \) of order \( u \) with \( u = 3, 5 \) and positive integers \( r \), under ERH for every positive integer \( n \). We call a 2-quadratic (resp. 3-quadratic) field as a biquadratic (resp. triquadratic) field. In this paper we show

**Theorem 1.** (1) There exist at least 163, 122, 32 and 1 imaginary biquadratic fields with class groups isomorphic to \( C_3, C_2^3, C_3^3 \) and \( C_4^3 \), respectively.
(2) There exist at least 23, 29, 7 and 1 imaginary triquadratic fields with class groups isomorphic to \( C_3, C_2^3, C_3^3 \) and \( C_4^3 \), respectively.
(3) Under ERH, there exist no imaginary biquadratic and triquadratic fields with class groups of exponent 3 other than those in (1) and (2).

**Theorem 2.** (1) There exist at least 243, 274, 54 and 1 imaginary biquadratic fields with class groups isomorphic to \( C_5, C_2^5, C_3^5 \) and \( C_4^5 \), respectively.
(2) There exist at least 18, 26, 6 and 1 imaginary triquadratic fields with class groups isomorphic to \( C_5, C_2^5, C_3^5 \) and \( C_4^5 \), respectively.
(3) Under ERH, there exist no imaginary biquadratic and triquadratic fields with class groups of exponent 5 other than those in (1) and (2).

**Remark 3.** It is known that, for every \( n \geq 4 \), there exist no imaginary \( n \)-quadratic fields with odd class numbers (cf. Theorem 5 [8, Theorem 5.2] below).

The computed tables of fields and some programs can be found on the web page https://math.uni-paderborn.de/en/ag/ca/research/exponent/.

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2. Decomposition with prime power conductor subfields

In order to study the 2-part of class groups of $n$-quadratic fields, let us introduce a symbol $[x, y]$ defined in Fröhlich’s book [5, §4]. In [5] he defines $[x, y]$ as an element in $\mathbb{Z}_\ell$ and uses it as an element in $\mathbb{F}_\ell$ to apply the theorems where $\mathbb{Z}_\ell$ is the ring of $\ell$-adic integers and $\mathbb{F}_\ell$ is the field of $\ell$ elements for a prime number $\ell$. In this paper we define the symbol $[x, y]$ as an element in $\mathbb{F}_\ell$ with $\ell = 2$ from the beginning. Let $S$ be a finite set of prime numbers, and put $S' = S \cup \{-1\}$ if $2 \in S$, and $S' = S$ otherwise. For a pair $(x, y) \in S' \times S$ with $x \neq y$ and $(x, y) \neq (-1, 2)$ we define the symbol $[x, y] \in \mathbb{F}_2$ as follows. Let $\pi$ denote the image of an $a \in \mathbb{Z}$ by the canonical map $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{F}_2$. For an odd prime number $p$ let us fix a primitive $(p-1)$-st root of unity $w_p$ in the $p$-adic number field $\mathbb{Q}_p$. For a prime number $y \neq p$ there exists a rational integer $r_{p,y}$ such that

$$w_p^{r_{p,y}} \equiv y \pmod{p}.$$  

We define $[p, y] \in \mathbb{F}_2$ by $[p, y] = \overline{r_{p,y}}$ for such an integer $r_{p,y}$. For the case $p = 2$ we put $w_2 = 5$. For an odd prime number $y$ there exist rational integers $r_{2,y}$ and $r_{-1,y}$ such that

$$w_2^{r_{2,y}}(-1)^{r_{-1,y}} \equiv y \pmod{8}.$$  

We define $[2, y]$ and $[-1, y]$ in $\mathbb{F}_2$ by $[2, y] = \overline{r_{2,y}}$ and $[-1, y] = \overline{r_{-1,y}}$ for such integers $r_{2,y}$ and $r_{-1,y}$, respectively. It is easy to see the following relation between $[x, y]$ and the Legendre symbol $\left(\frac{a}{p}\right)$.

**Lemma 4.** For an odd prime number $x \in S'$ and a prime number $y \in S$ with $x \neq y$, we have $(-1)^{[x,y]} = \left(\frac{y}{x}\right)$. For an $x \in \{-1, 2\} \subset S'$ and an odd prime number $y \in S$, we have $(-1)^{[x,y]} = \left(\frac{x}{y}\right)$.

For a number field $K$ of finite degree let $\text{Cl}(K)$ and $\text{Cl}_+(K)$ denote the class group and the narrow class group of $K$, and $h(K)$ and $h_+(K)$ the class number and the narrow class number of $K$, respectively. Note that $\text{Cl}(K) = \text{Cl}_+(K)$ and $h(K) = h_+(K)$ if $K$ is totally imaginary. For a number field $K$ let $G_+(K/\mathbb{Q})$ denote the narrow genus field of $K$ over $\mathbb{Q}$, that is, the maximal extension of $K$ which is unramified at all finite primes, and is the composite $KL$ of $K$ and some abelian extension $L$ of $\mathbb{Q}$. For an abelian field $K$ let $\nu(K)$ denote the minimal number of generators of $\text{Gal}(K/\mathbb{Q})$. Note that $\nu(K) = n$ for an $n$-quadratic field $K$.

**Theorem 5** (Fröhlich [5, Theorem 5.2]). Let $K$ be an $n$-quadratic fields, and $S$ the set of prime numbers ramifying in $K$.

1. If $h_+(K)$ is odd, then $K = G_+(K/\mathbb{Q})$.

2. Assume $K = G_+(K/\mathbb{Q})$. Then $h_+(K)$ is odd if and only if one of the following conditions (2.1), (2.2) and (2.3) holds:

   (2.1) $n = 1$.

   (2.2) $n = 2$, and (i) or (ii) holds:

   (i) $S = \{2\}$, that is, $K = \mathbb{Q}(\sqrt{-1}, \sqrt{-2})$.

   (ii) $S = \{p_1, p_2\}$ with distinct prime numbers $p_1 < p_2$ such that $[p_2, p_1] \neq 0$ or $a(p_2) \neq 0$ where the value $a(p)$ for $p \neq p_1$ is defined by
\[ a(p) = \begin{cases} [p_1,p] & \text{if } p_1 \not\equiv 2 \text{ or } \sqrt{2} \in K, \\ [-1,p] & \text{if } \sqrt{-1} \in K, \\ [2,p] + [-1,p] & \text{if } \sqrt{-2} \in K. \end{cases} \]

(2.3) \( n = 3 \), and (i) or (ii) holds:

(i) \( S = \{2,p_2\} \) with an odd prime number \( p_2 \) such that \( [p_2,2] \neq 0 \),
(ii) \( S = \{p_1,p_2,p_3\} \) with distinct prime numbers \( p_1 < p_2 < p_3 \) such that \( \det M \neq 0 \) where

\[ M = \begin{pmatrix} a(p_2) & 0 & [p_3,p_1] \\ 0 & [p_2,p_3] & a(p_3) \end{pmatrix} \]

and the values \( a(p) \) for \( p = p_2,p_3 \) are defined in the same way as that at (ii) of (2.2) above.

Remark 6. (1) For an \( n \)-quadratic field \( K \), the cases (A), (B), (C) and (D) at §5 in [8] mean that \( \mathbb{Q}(\zeta_8) \cap K \) are equal to \( \mathbb{Q}(\zeta_8), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-1}) \) and \( \mathbb{Q}(\sqrt{2}) \), respectively, where \( \zeta_8 \) is a primitive 8-th root of unity in \( \mathbb{C} \). In our situation, the invariant \( \lambda \) for the case (B) is equal to 1.

(2) We have \([x,y] = -[x,y]\) since \([x,y]\) is an element in \( F_2 \). For general \( \ell \), the definition of \( a(p) \) for the third case is \( a(p) = [2,p] - [-1,p] \) in Theorem [5](2.2) (ii). The same adjustment occurs at the main diagonal of \( M \) in Theorem [5](2.3) (ii).

Theorem 7 (Fröhlich [8] Theorem 2.15]). For an abelian field \( K \), the following conditions (i) and (ii) are equivalent:

(i) \( K = G_+(K/\mathbb{Q}) \).
(ii) \( K \) is the composite of fields with prime power conductors.

As in the paper [15], we say that \( K \) is a field of type \( I \) if \( K \) is an imaginary abelian field with the conditions of Theorem 7. For a field \( K \) of type \( I \) we say that \( K = K_1 \cdots K_r \) is the decomposition of \( K \) with prime power conductor fields, or simply the decomposition of \( K \) where \( K_i \) are subfields of \( K \) of prime power conductors \( p_i^{e_i} \) with distinct prime numbers \( p_i \) and positive integers \( e_i \).

As a corollary of Theorem 5 we have

Corollary 8. Let \( K \) be a field of type \( I \) with decomposition \( K = K_1K_2K_3 \). If \( K_1 \) is an imaginary quadratic field, and \( K_2 \) and \( K_3 \) are real quadratic fields, then \( 2 \mid b(K) \).

Proof. We consider the following four essential cases.

Case 1: \( K = \mathbb{Q}(\sqrt{-p_1},\sqrt{p_2},\sqrt{p_3}) \) with \( p_1 \equiv 3 \pmod{4} \) and \( p_2 \equiv p_3 \equiv 1 \pmod{4} \).
It follows from \( p_2 \equiv 1 \pmod{4} \) that \( a(p_2) = [p_1,p_2] = [p_2,p_1] \) and \([p_2,p_3] = [p_3,p_2] \).
One has \( a(p_3) = [p_1,p_3] = [p_3,p_1] \) since \( p_3 \equiv 1 \pmod{4} \). Thus, if \([p_2,p_1],[p_3,p_2] \) or \([p_3,p_1] \) is equal to 0, then the first, second or third column of \( M \) is zero vector, which implies \( \det M = 0 \). When \([p_2,p_1] = [p_3,p_2] = [p_3,p_1] = 1 \), the sum of the three columns of \( M \) is equal to zero vector, which yields \( \det M = 0 \).

Case 2: \( K = \mathbb{Q}(\sqrt{2},\sqrt{p_2},\sqrt{-p_3}) \) with \( p_1 \equiv 2 \), \( p_2 \equiv 1 \pmod{4} \) and \( p_3 \equiv 3 \pmod{4} \).
It follows from \([2,p] = [p,2] \) that \( a(p_2) = [2,p_2] = [p_2,p_1] \) and \( a(p_3) = [2,p_3] = [p_3,p_1] \). By \( p_2 \equiv 1 \pmod{4} \) one has \([p_2,p_3] = [p_3,p_2] \). In the same way as the case 1, one sees \( \det M = 0 \).

Case 3: \( K = \mathbb{Q}(\sqrt{-1},\sqrt{p_2},\sqrt{p_3}) \) with \( p_1 \equiv 2 \) and \( p_2 \equiv p_3 \equiv 1 \pmod{4} \). Then one has \( a(p_2) = a(p_3) = 0 \). Thus the first, third column of \( M \) or their sum is equal to
zero vector, which implies $\det M = 0$.

Case 4: $K = \mathbb{Q}(\sqrt{-2}, \sqrt{p_2}, \sqrt{p_3})$ with $p_1 = 2$ and $p_2 = p_3 = 3 \pmod{4}$. Then one has $a(p_2) = [2, p_2] + [-1, p_2] = [2, p_2] = [p_2, p_1]$, $[p_2, p_3] = [p_3, p_2]$ and $a(p_3) = [2, p_3] + [-1, p_3] = [2, p_3] = [p_3, p_1]$. In the same way as the case 1, one sees $\det M = 0$. \qed

**Remark 9.** Uchida shows Corollary \[ \text{[13, Lemma 1]} \] that is, if $K_1$ is an imaginary cyclic field of two power degree, and $K_2$ and $K_3$ are real quadratic fields, then $2 \mid h(K)$.

Let us consider the odd part $\text{Cl}_n(K)$ of $\text{Cl}(K)$, that is, the subgroup of $\text{Cl}(K)$ consisting of all the classes of odd order. The following well-known theorem is useful to study $\text{Cl}_n(K)$ for an $n$-quadratic field $K$.

**Theorem 10** (Lemma Meyer (11 (1.1)))). Let $k$ be a number field with finite degree, and $K/k$ a Galois extension with $\text{Gal}(K/k) \simeq V_4$ where $V_4$ is Klein’s four-group. Let $k_1, k_2$ and $k_3$ stand for the three intermediate fields of $K/k$ with $[K : k_j] = |k_j : k| = 2$. Then we have $\text{Cl}_n(K) \simeq \text{Cl}_n(k_1) \times \text{Cl}_n(k_2) / \text{Cl}_n(k)$ where $\prod$ denotes the direct product over $j = 1, 2$ and 3.

Let $P$ denote the set consisting of all the prime numbers. We use a notation $p^*$ for $p \in P$ such that $p^* = (-1)^{(p-1)/2}p$ if $p$ is odd, and $2^* \in \{8, -4, -8\}$. Put $P^* = \{8, -4, -8\} \cup \{p^* \mid p \in P, p \neq 2\}$ and decompose it into two subsets $P^*_+ \text{ and } P^*_-$ such that $P^*_+ = \{p^* \in P^* \mid p^* > 0\}$ and $P^*_- = \{p^* \in P^* \mid p^* < 0\}$. Let $E(K)$ represent the exponent of $\text{Cl}(K)$ for a number field $K$.

**Theorem 11.** Fix an odd number $u > 0$. Let $K$ be an imaginary $n$-quadratic field $K$ such that $E(K) \mid u$. Then $K$ is one of the following forms (1), (2) and (3).

1. $K = \mathbb{Q}(\sqrt{p^*})$ with $p^* \in P^*_+$ such that $E(\mathbb{Q}(\sqrt{p^*})) = u$.
2. $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2})$ with $p_1^* \in P^*_+$ and $p_2^* \in P^*_-$ such that $\mathbb{Q}(\sqrt{p_1})$ is of the form (1) above and the following (2a) or (2b) holds:
   2a. $p_2 < 0$ and $E(\mathbb{Q}(\sqrt{p_2})) = u$, i.e., $\mathbb{Q}(\sqrt{p_2})$ is also of the form (1),
   2b. $p_2 > 0$ and $E(\mathbb{Q}(\sqrt{p_2})) = 2u$.
3. $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ with $p_1^*, p_2^* \in P^*_+$ and $p_3^* \in P^*_-$ such that three subfields $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}), \mathbb{Q}(\sqrt{p_1}, \sqrt{p_3})$ and $\mathbb{Q}(\sqrt{p_2}, \sqrt{p_3})$ are of the form (2) above.

**Proof.** If $u = 1$, then the case (1) is obvious from genus theory. Assume $u = 2$. Theorems \[ \text{[13]} \text{and } [11] \] imply that $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ for some $p_1^* \in P^*_+$ and $p_2^*, p_3^* \in P^*$. When $p_1 = p_2$, one has $p_1 = p_2 = 2$, that is, $K = \mathbb{Q}(\sqrt{-4}, \sqrt{-8})$, for which (2) holds. Under $p_1 \neq p_2$, since the extension $K / \mathbb{Q}(\sqrt{p_1})$ is ramified at $p_2$, the norm map $\text{Cl}(K) \rightarrow \text{Cl}(\mathbb{Q}(\sqrt{p_1}))$ is surjective (cf. [10] Theorem 10.1)). This means that $E(\mathbb{Q}(\sqrt{p_1})) \mid E(K)$, which implies that $\mathbb{Q}(\sqrt{p_1})$ is of the form (1). In the same way, if $p_2^* < 0$, then $\mathbb{Q}(\sqrt{p_2})$ is of the form (1). When $p_2 > 0$, put $k = \mathbb{Q}(\sqrt{p_1}, p_2^*)$. Theorem \[ \text{[11]} \] shows that $E(k) = 2u$ for some rational integer $r \geq 0$. It follows from genus theory that $2 \mid E(k)$. Suppose $4 \mid E(k)$. Then $k$ has an unramified extension $M_k$ with $\text{Gal}(M_k/k) \simeq C_4$. Note that $M_k$ is not contained in $K$ since $\text{Gal}(K/\mathbb{Q})$ has exponent 2. Thus the lift $M_kK/K$ of $M_k/k$ to $K$ is nontrivial, that is, $2 \mid h(K)$ by class field theory. It is a contradiction. Thus we have $E(k) = 2u$.

For the case $n = 3$, Theorems \[ \text{[13]} \text{and } [11] \] imply that $K = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3})$ for some $p_1^* \in P^*_+$ and $p_2^*, p_3^* \in P^*$. Due to Corollary \[ \text{[8]} \] we have $p_2 > 0$ or $p_3 < 0$, say,
$p_2^* \in P^*$. In the same way as for $n = 2$, due to the surjectivity of the norm maps, the imaginary subfields $\mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*}), \mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*})$ and $\mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*})$ have class groups of exponent dividing $u$, and thus they are of the form (2).

Let $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ denote the families of all the fields of the forms (1), (2) and (3), respectively. As a corollary of Theorem 11 we have

**Corollary 12.** If $\mathcal{K}_2$ is finite, then so is $\mathcal{K}_3$.

**Remark 13.** Our inductive method in Theorem 11 with $u = 1$ coincides with that in the proof of Proposition 5 in [15].

### 3. Finiteness of the family under ERH

**Theorem 14** (Boyd and Kisilevsky [4, Theorem 4]). Let $K$ be an imaginary quadratic field of discriminant $D < 0$. Then, under the extended Riemann hypothesis (ERH), for any $\eta > 0$, we have $E(k) > (\log |D|)/(2+\eta \log |D|)$ for sufficiently large $|D|$.

**Corollary 15.** Under ERH, the families $\mathcal{K}_1$ and $\mathcal{K}_2$ are finite for an odd $u$.

**Proof.** It follows directly from Theorem 14 that $K_3$ is finite. Assume $K \in \mathcal{K}_2$, that is, $K = \mathbb{Q}(\sqrt{p_1^*}, \sqrt{p_2^*})$ where $p_1^* \in P^*$ and $p_2^* \in P^*$ satisfy $E(K) \leq 2u$ and $E(\mathbb{Q}(\sqrt{p_1^*})), E(\mathbb{Q}(\sqrt{p_2^*})) \leq 2u$ where $c = 0$ if $p_2^* < 0$ and $c = 1$ otherwise. Let $\mathcal{F}$ denote the family consisting of all the imaginary quadratic fields $k$ with $E(k) \leq 2u$. Theorem 14 yields that $\mathcal{F}$ is finite under ERH. Since $\mathbb{Q}(\sqrt{p_1^*})$ and $\mathbb{Q}(\sqrt{p_1^*}p_2^*)$ belong to the finite family $\mathcal{F}$, the set of such pairs $(p_1^*, p_2^*)$ is also finite. Hence $\mathcal{K}_2$ is finite.

**Corollary 16.** For a fixed odd number $u > 0$, under ERH, there exist finitely many imaginary $n$-quadratic fields $K$ such that $E(K) \leq u$.

Corollaries 15 and 16 with $u = 3$ hold without ERH since we have

**Theorem 17** (Heath-Brown [9, Theorems 1 and 2]). If $E$ is equal to 5, $2^m$ or $2^m3$ with a rational integer $m \geq 0$, then there exists an ineffective constant $d_E$ such that $E(\mathbb{Q}(\sqrt{-d})) \neq E$ for every fundamental discriminant $-d$ with $d > d_E$.

**Remark 18.** Unfortunately, the constants $d_E$ in Theorem 17 are not effective. Therefore we need to apply ERH for the explicit computation of finding all such quadratic fields.

### 4. Computation for the case $u = 3$ under ERH

Let us fix $u = 3$. Let $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$ stand for the subfamilies of $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$ consisting of all the fields $K$ such that $E(K) \leq 3$, respectively. Under ERH, let $\kappa_1, \kappa_2$ and $\kappa_3$ denote the cardinalities of the finite families $\mathcal{K}_1, \mathcal{K}_2$ and $\mathcal{K}_3$, and $\mu_1, \mu_2$ and $\mu_3$ those of $\mathcal{M}_1, \mathcal{M}_2$ and $\mathcal{M}_3$, respectively.

**Theorem 19** (Baker [2], Stark [14]). Let $d > 0$ be a squarefree rational integer. Then $\mathbb{Q}(\sqrt{-d})$ has class number 1 if and only if $d$ is equal to 1, 2, 3, 7, 11, 19, 43, 67 or 163.

**Theorem 20** (Elsenhans, Klüners and Nicolae [6, Theorems 1 and 2]). The maximal absolute value $|D|$ of the discriminants $D$ of all the imaginary quadratic fields with class group of exponent 3 under the condition $|D| < 3.1 \cdot 10^{20}$ is $|D| = 4027$. Under the extended Riemann hypothesis (ERH), the maximal absolute value $|D|$ of such fields without the condition $|D| < 3.1 \cdot 10^{20}$ is equal to 4027.
By definition, one has that (2)

\[ p \text{, that is, the set consisting of all the discriminants of 26 imaginary quadratic fields} \]

Remark 22. On Corollary 21, possible failure examples contradicting ERH have \( \kappa \) respectively. In particular, we have (4)

\[ \kappa \text{, not determined yet even assuming ERH} [6]. \]

For each (Bach and Sørensen [1, Theorem 5.1 and Table 3])

Therefore we only need to check the odd part of the class group of \( \mathbb{Q} \) due to Theorem 5 for the oddness and 

Lemma 23. We have \( \mu_{2b} = 307 \) under ERH.

Next consider the set \( R_{p^r} \) for \( \mathbb{M}_{2b}. \) It is enough for computing \( R_{p^r} \) to find all the imaginary quadratic fields \( k \) with \( E(k) \mid 2u. \) However, all of such fields are not determined yet even assuming ERH [6]. For each \( p^r \in I \) with \( |p^r| \leq 4027 \) we can obtain all the imaginary quadratic fields \( k = \mathbb{Q}(\sqrt{p^r q^r}) \) such that \( q^r \in \mathbb{P}_r^* \) and 

\[ E(\mathbb{Q}(\sqrt{p^r q^r})) \mid 2u. \] Indeed, fortunately, the 2-Sylow subgroup of \( \text{Cl}(\mathbb{Q}(\sqrt{p^r q^r})) \) is small and has a tractable generator in our situation.

Lemma 24 (Boyd and Kisilevsky [4 Lemma 1]). Let \( k \) be an imaginary quadratic field of discriminant \( D. \) If \( \alpha \) is an algebraic integer with \( \alpha \notin \mathbb{Z}, \) then \( N_{k/\mathbb{Q}}(\alpha) \geq |D|/4 \) where \( N_{k/\mathbb{Q}} \) is the norm map from \( k \) to \( \mathbb{Q}. \)

Theorem 25 (Bach and Sørensen [1 Theorem 5.1 and Table 3]). Let \( k \) be a quadratic field of discriminant \( D \) such that \( |D| > e^{25} \approx 7.2 \cdot 10^{10}. \) Assume the extended Riemann hypothesis (ERH). Then there exists a prime number \( \ell \) splitting in \( k \) such that \( \ell \leq (1.881 \log(|D|) + 0.34 \cdot 2 + 5.5)^2. \)
Theorem 26 (Elsenhans, Klüners and Nicolae [6, Theorem 1]). The maximal absolute value $|D|$ of the discriminants $D$ of all the imaginary quadratic fields with class group of exponent dividing 6 under the condition $|D| < 3.1 \cdot 10^{20}$ is $|D| = 5761140$.

Lemma 27. Assume that $p^* \in I$ and $q^* \in R_{p^*}$ with $|p^*| \leq 4027$. Then, under ERH, $q^*$ is less than $5761140/|p^*|$.

Proof. Let $p^*$ be in $I$ with $|p^*| \leq 4027$, and $q^*$ in $R_{p^*}$. Then $k = \mathbb{Q}(\sqrt{p^*q^*})$ satisfies $\text{Cl}(k) \cong C_3^r \times C_2$ for some rational integer $r \geq 0$. Here the discriminant $D$ of $k$ is equal to $p^*q^*$. For finding an upper bound of $|D|$ we may assume that $|D| > e^{25} \approx 7.2 \cdot 10^{10}$. Let $p$ be unique prime ideal of $k$ above $p$. It follows from genus theory that $p$ is of order 2 in $\text{Cl}(k)$. Indeed, $p^2 = (p)$ and $p$ is not principal for $q^* \geq 5$. Theorem 26 guarantees that there exists a prime number $\ell$ splitting in $k$ such that $\ell \leq f_1(|D|)$ where $f_1(x)$ is a function defined by

$$f_1(x) = (1.881 \log x + 0.34 \cdot 2 + 5.5)^2$$

for $x > 0$. Let $l$ be a prime ideal of $k$ above such an $\ell$. By the assumption of the structure of $\text{Cl}(k)$, the ideal $l^3$ is of order 1 or 2 in $\text{Cl}(k)$. The group $C_3^r \times C_2$ has a unique element of order 2. Thus $l^3$ or $\mathfrak{p}l^3$ is principal. There exists an algebraic integer $\alpha$ in $k$ such that $l^3 = (\alpha)$ or $\mathfrak{p}l^3 = (\alpha)$. Here $\mathfrak{f} \mid \alpha$ and $\ell \nmid \alpha$, which means that $\alpha \notin \mathbb{Z}$. Lemma 24 implies that $N_{k/\mathbb{Q}}(\alpha) \geq |D|/4$. Thus $\ell^3$ or $\mathfrak{p}\ell^3$ is not less than $|D|/4$, in particular, $p\ell^3 \geq |D|/4$. Hence we have $\ell \geq \sqrt[3]{|D|/(4p)} \geq f_2(|D|)$ where $f_2(x)$ is a function defined by

$$f_2(x) = \sqrt{x/(4 \cdot 4027)}$$

for $x > 0$. Considering the derivatives of $\sqrt{f_1(x)}$ and $\sqrt{f_2(x)}$ and using a calculator, one can see that $\max\{|\rho | \in \mathbb{R} | f_1(\rho) \geq f_2(\rho)| \leq 2.4 \cdot 10^{15}$. It concludes $|D| \leq 2.4 \cdot 10^{15}$ since some $\ell$ exists. Since the bound $3.1 \cdot 10^{20}$ of Theorem 26 is greater than our $2.4 \cdot 10^{15}$, Theorem 26 yields $|D| \leq 5761140$ under ERH. Hence we have $q^* = |D|/|p^*| \leq 5761140/|p^*|$. $\square$

Remark 28. The method used in the proof of Lemma 27 above is a refinement of that in [6]. One has that $5761140 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 29 \cdot 43$ and $\text{Cl}((\sqrt{-5761140})) \cong C_6^r \times C_2^2$.

For each $p^* \in I$ and $q^* \in R_{p^*}$, we can check whether $E(\mathbb{Q}(\sqrt{p^*}, \sqrt{q^*})) | u$ or not, by the same way as for $p^*, q^* \in I$ due to Theorems 5 and 10.

Lemma 29. We have $\mu_{2b} = 58$ under ERH.

The table of the numbers of fields in $\mathcal{M}_{2a}$ and $\mathcal{M}_{2b}$ classified by rank is as follows. For example, the family $\mathcal{M}_{2b}$ has 12 fields $K$ such that $\text{Cl}(K) \cong C_5^r$ with $r = 2$.

| family | $r = 0$ | $r = 1$ | $r = 2$ | $r = 3$ | $r = 4$ | total |
|--------|---------|---------|---------|---------|---------|-------|
| $\mathcal{M}_{2a}$ | 32 | 133 | 110 | 31 | 1 | 307 |
| $\mathcal{M}_{2b}$ | 15 | 30 | 12 | 1 | 0 | 58 |
| total | 47 | 163 | 122 | 32 | 1 | 365 |

The maximal discriminants of $K$ in $\mathcal{M}_{2a}$ with $\text{Cl}(K) \cong C_3, C_3^2, C_3^3$ and $C_3^4$ are
20715557041 = 163^2 883^2 for $K = \mathbb{Q}(\sqrt{-163}, \sqrt{-883})$,
430862272801 = 163^2 4027^2 for $K = \mathbb{Q}(\sqrt{-163}, \sqrt{-4027})$,
13340675895121 = 907^2 4027^2 for $K = \mathbb{Q}(\sqrt{-907}, \sqrt{-4027})$ and
6704790388321 = 643^2 4027^2 for $K = \mathbb{Q}(\sqrt{-643}, \sqrt{-4027})$,
respectively. The maximal discriminants of $K$ in $\mathcal{M}_{2b}$ with $\text{Cl}(K) \simeq C_3, C_3^2$ and $C_3^3$ are

$$7767369 = 3^2 929^2$$
for $K = \mathbb{Q}(\sqrt{-3}, \sqrt{929})$, 
$$1157836729 = 7^2 4861^2$$
for $K = \mathbb{Q}(\sqrt{-7}, \sqrt{4861})$ and
$$503688249 = 3^2 7481^2$$
for $K = \mathbb{Q}(\sqrt{-3}, \sqrt{7481})$.

respectively.

Let us determine $\mathcal{M}_3$ under ERH. Using $k_2$, one can obtain $k_3$ without computation of further class groups by Theorem 9 (3). For each triquadratic field $K = \mathbb{Q}(\sqrt{P_1}, \sqrt{P_2}, \sqrt{P_3})$ in $\mathcal{K}_3$ with $p_1^*, p_2^*, p_3^* \in P^*$, we can determine whether $E(K) \mid u$ or not, using the structures of its quadratic subfields due to Theorem 5 for the oddness and Theorem 10 for the explicit odd part, in particular, $\text{Cl}_o(K)$ is isomorphic to

$$\text{Cl}_o(k_1) \times (\text{Cl}_o(k_1, k_2) / \text{Cl}_o(k_1)) \times (\text{Cl}_o(k_1, k_3) / \text{Cl}_o(k_1)) \times (\text{Cl}_o(k_1, k_2, k_3) / \text{Cl}_o(k_1))$$

or

$$\text{Cl}_o(k_1) \times (\text{Cl}_o(k_2) / \text{Cl}_o(k_1)) \times (\text{Cl}_o(k_3) / \text{Cl}_o(k_1)) \times (\text{Cl}_o(k_2, k_3) / \text{Cl}_o(k_1))$$

where $k_{j_1} = \mathbb{Q}(\sqrt{p_{j_1}}), k_{j_2j_3} = \mathbb{Q}(\sqrt{p_{j_2j_3}})$ and $k_{j_1j_2j_3} = \mathbb{Q}(\sqrt{p_{j_1j_2j_3}})$. Let $\mathcal{M}_{3a}$ and $\mathcal{M}_{3b}$ denote the subfamilies of $\mathcal{M}_3$ such that

$$\mathcal{M}_{3a} = \{ \mathbb{Q}(\sqrt{p_{j_1}}, \sqrt{p_{j_2}}, \sqrt{p_{j_3}}) \in \mathcal{M}_3 \mid p_{j_1}^*, p_{j_2}^*, p_{j_3}^* \in P^* \},$$

$$\mathcal{M}_{3b} = \{ \mathbb{Q}(\sqrt{p_{j_1}}, \sqrt{p_{j_2}}, \sqrt{p_{j_3}}) \in \mathcal{M}_3 \mid p_{j_1}^*, p_{j_2}^*, p_{j_3}^* \in P^*_+ \}$$

with cardinalities $\mu_{3a}$ and $\mu_{3b}$, respectively. The field $\mathbb{Q}(\sqrt{-4}, \sqrt{-8}, \sqrt{p^*})$ with $p^* \in P^*_-$ (resp. $p^* \in P^*_+$) is treated as a member of $\mathcal{M}_{3a}$ (resp. $\mathcal{M}_{3b}$).

**Lemma 30.** We have $\mu_{3a} = 35$ and $\mu_{3b} = 42$ under ERH.

The table of the numbers of fields in $\mathcal{M}_{3a}$ and $\mathcal{M}_{3b}$ classified by rank is as follows. For example, $\mathcal{M}_{3b}$ has 12 fields $K$ such that $\text{Cl}(K) \simeq C_3^r$ with $r = 2$.

| family | $r = 0$ | $r = 1$ | $r = 2$ | $r = 3$ | $r = 4$ | total |
|--------|---------|---------|---------|---------|---------|-------|
| $\mathcal{M}_{3a}$ | 8 | 8 | 17 | 2 | 0 | 35 |
| $\mathcal{M}_{3b}$ | 9 | 15 | 12 | 5 | 1 | 42 |
| total | 17 | 23 | 29 | 7 | 1 | 77 |

The maximal discriminants of $K$ in $\mathcal{M}_{3a}$ with $\text{Cl}(K) \simeq C_3, C_3^2$ and $C_3^3$ are

$$1048870932736 = 2^4 11^2 23^3$$
for $K = \mathbb{Q}(\sqrt{-4}, \sqrt{-11}, \sqrt{-23})$
$$391872794865536 = 2^{12} 23^4 43^4$$
for $K = \mathbb{Q}(\sqrt{-8}, \sqrt{-23}, \sqrt{-43})$ and
$$13142294978742801 = 3^4 43^4 83^4$$
for $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-43}, \sqrt{-83})$,
respectively. The maximal discriminants of $K$ in $\mathcal{M}_{3b}$ with $\text{Cl}(K) \simeq C_3, C_3^2, C_3^3$ and $C_3^4$ are

$$633304165557681 = 7^4 13^4 31^4$$
for $K = \mathbb{Q}(\sqrt{-7}, \sqrt{-31}, \sqrt{13})$
$$7860782851035136 = 2^{12} 11^4 107^4$$
for $K = \mathbb{Q}(\sqrt{-11}, \sqrt{-107}, \sqrt{8})$
$$69969611497148416 = 2^{12} 19^4 107^4$$
for $K = \mathbb{Q}(\sqrt{-19}, \sqrt{-107}, \sqrt{8})$ and
$$6505835909336928256 = 2^{12} 59^4 107^4$$
for $K = \mathbb{Q}(\sqrt{-59}, \sqrt{-107}, \sqrt{8})$,
respectively.
5. Computation for the case $u = 5$ under ERH

Let us fix $u = 5$. As in the previous section for $u = 3$, let $\mathcal{M}_i$ stand for the subfamily of $K_i$ consisting of all the fields $K$ such that $E(K) \mid u$. Under ERH, let $\kappa_i$ denote the cardinality of the finite family $K_i$, and $\mu_i$ that of $\mathcal{M}_i$. We remark that the hard part is to compute the families $\mathcal{M}_1$ and $\mathcal{M}_{2a}$, i.e. the imaginary quadratic fields of exponent 5 and the imaginary quadratic fields with class group of type $C_2 \times C_5^r$ for some $r \geq 0$. We will describe the latter part in more detail in the next section. As Corollary 21 we have

**Theorem 31.** (Elsenhans, Klüners and Nicolae [6, Theorems 1 and 2]). Under ERH, the following is the list of all the squarefree rational integers $d > 0$ such that $\text{Cl}(\mathbb{Q}(\sqrt{-d})) \simeq C_5^r$ for some rational integers $r \geq 0$.

| $r$ | $d$ |
|-----|-----|
| 0   | 1, 2, 3, 7, 11, 19, 43, 67, 163 |
| 1   | 47, 79, 103, 127, 131, 179, 227, 347, 443, 523, 571, 619, 683, 691, 739, 787, 947, 1051, 1123, 1723, 1747, 1867, 2203, 2347, 2683 |
| 2   | 12451, 37363 |

Here the numbers at the right column $\sharp$ are the numbers of $d$’s at the rows, respectively. In particular, we have $\kappa_1 = \mu_1 = 36$ under ERH.

**Remark 32.** On Theorem 31 possible failure examples contradicting ERH have class group $C_5^r$ for $r \geq 3$ by Watkins’ result [17].

In the same way as for $u = 3$ in (1), we denote

$$I = \{ p^* \in P^*_+ \mid E(\mathbb{Q}(\sqrt{p^*})) \text{ divides } u \}. $$

Analogously, as in (2), (3) and (4) we define $R_{p^*}$, $\mathcal{K}_{2a}$, and $\mathcal{K}_{2b}$ and denote by $\mu_{2a}$ and $\mu_{2b}$ the cardinalities of $\mathcal{M}_{2a} = \mathcal{M}_2 \cap \mathcal{K}_{2a}$ and $\mathcal{M}_{2b} = \mathcal{M}_2 \cap \mathcal{K}_{2b}$, respectively. Like in Lemma 23 we can prove

**Lemma 33.** We have $\mu_{2a} = 537$ under ERH.

The difficult computation will be to compute the sets $R_{p^*}$. We give some details of this computation later on.

| $p^*$ | $R_{p^*}$ |
|-------|-----------|
| -3    | 5, 8, 17, 41, 53, 89, 101, 233, 569, 593, 641, 881, 1049, 1097, 1361, 1409, 1721, 1889, 11177, 38729 |
| -4    | 5, 13, 37, 181, 197, 229, 317, 373, 421, 541, 613, 709, 757, 853, 877, 1093, 1213, 22717 |
| -7    | 5, 13, 17, 61, 661, 733, 829, 1069, 9973, 12157, 21661 |
| -8    | 5, 29, 37, 61, 109, 173, 197, 269, 7829, 9461, 19301 |
| -11   | 8, 13, 17, 29, 73, 281, 569, 953 |
| -19   | 41, 193, 241, 337, 433 |
| -43   | 8, 73, 929 |
| -47   | 5, 13 |
| -103  | 37 |
| -127  | 5, 29, 109 |
| -227  | 17, 41 |
| -347  | 8 |

Here $R_{p^*} = \emptyset$ for 24 numbers $p^* \in I$ which do not appear in the above table.
Lemma 34. We have $\mu_{2b} = 82$ under ERH.

The table of the numbers of fields in $\mathcal{M}_{2a}$ and $\mathcal{M}_{2b}$ classified by rank is as follows. For example, the family $\mathcal{M}_{2b}$ has 18 fields $K$ such that $\text{Cl}(K) \simeq C_5^r$ with $r = 2$.

| family | $r = 0$ | $r = 1$ | $r = 2$ | $r = 3$ | $r = 4$ | total |
|--------|----------|----------|----------|----------|----------|-------|
| $\mathcal{M}_{2a}$ | 32       | 194      | 256      | 54       | 1        | 537   |
| $\mathcal{M}_{2b}$ | 15       | 49       | 18       | 0        | 0        | 82    |
| total  | 47       | 243      | 274      | 54       | 1        | 619   |

The maximal discriminants of $K$ in $\mathcal{M}_{2a}$ with $\text{Cl}(K) \simeq C_5^r, C_5^2, C_5^3$ and $C_5^4$ are

- $191256654241 = 163^22683^2$ for $K = \mathbb{Q}(\sqrt{-163}, \sqrt{-2683})$,
- $3965221594001 = 2347^22683^2$ for $K = \mathbb{Q}(\sqrt{-2347}, \sqrt{-2683})$,
- $10049045790215041 = 2683^337363^2$ for $K = \mathbb{Q}(\sqrt{-2683}, \sqrt{-37363})$ and
- $2164172685820264369 = 12451^337363^2$ for $K = \mathbb{Q}(\sqrt{-12451}, \sqrt{-37363})$,

respectively. The maximal discriminants of $K$ in $\mathcal{M}_{2b}$ with $\text{Cl}(K) \simeq C_5^r$ and $C_5^2$ are

- $109893289 = 11^2953^2$ for $K = \mathbb{Q}(\sqrt{-11}, \sqrt{953})$ and
- $23841830464 = 2^619301^2$ for $K = \mathbb{Q}(\sqrt{-8}, \sqrt{19301})$,

respectively.

Let $\mathcal{M}_{3a}$ and $\mathcal{M}_{3b}$ denote the subfamilies of $\mathcal{M}_3$ such that

- $\mathcal{M}_{3a} = \{Q(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}) \in \mathcal{M}_3 | p_1, p_2, p_3 \in P^+ \}$,
- $\mathcal{M}_{3b} = \{Q(\sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}) \in \mathcal{M}_3 | p_1, p_2 \in P^+, p_3 \in P^+ \}$

with cardinalities $\mu_{3a}$ and $\mu_{3b}$, respectively.

Lemma 35. We have $\mu_{3a} = 29$ under ERH. We have $\mu_{3b} = 39$ under ERH.

The table of the numbers of fields in $\mathcal{M}_{3a}$ and $\mathcal{M}_{3b}$ classified by rank is as follows. For example, $\mathcal{M}_{3a}$ has 12 fields $K$ such that $\text{Cl}(K) \simeq C_5^r$ with $r = 2$.

| family | $r = 0$ | $r = 1$ | $r = 2$ | $r = 3$ | $r = 4$ | total |
|--------|----------|----------|----------|----------|----------|-------|
| $\mathcal{M}_{3a}$ | 8        | 6        | 14       | 1        | 0        | 29    |
| $\mathcal{M}_{3b}$ | 9        | 12       | 12       | 5        | 1        | 39    |
| total  | 17       | 18       | 26       | 6        | 1        | 68    |

The maximal discriminants of $K$ in $\mathcal{M}_{3a}$ with $\text{Cl}(K) \simeq C_5^r, C_5^2, C_5^3$ and $C_5^4$ are

- $6722852453999296 = 2^819^267^4$ for $K = \mathbb{Q}(\sqrt{-4}, \sqrt{-19}, \sqrt{-67})$,
- $92387543897348081 = 7^443^4103^4$ for $K = \mathbb{Q}(\sqrt{-7}, \sqrt{-43}, \sqrt{-103})$ and
- $102823251817101201 = 3^447^4127^4$ for $K = \mathbb{Q}(\sqrt{-3}, \sqrt{-47}, \sqrt{-127})$,

respectively. The maximal discriminants of $K$ in $\mathcal{M}_{3b}$ with $\text{Cl}(K) \simeq C_5^r, C_5^2, C_5^3$ and $C_5^4$ are

- $136166867931136 = 2^117^461^4$ for $K = \mathbb{Q}(\sqrt{-7}, \sqrt{-8}, \sqrt{61})$,
- $324690740793595201 = 11^417^4227^4$ for $K = \mathbb{Q}(\sqrt{-11}, \sqrt{-227}, \sqrt{17})$,
- $97780726446179992321 = 19^441^4227^4$ for $K = \mathbb{Q}(\sqrt{-19}, \sqrt{-227}, \sqrt{41})$ and
- $2693876092569442561 = 11^429^4127^4$ for $K = \mathbb{Q}(\sqrt{-11}, \sqrt{-127}, \sqrt{29})$,

respectively.

6. Computation of the family $\mathcal{M}_{2b}$

In this section we describe how to compute the set $\mathcal{M}_{2b}$ for odd prime exponents $E$. In the following we assume that $p \in I$ is given with $p^* < 0$ such that the
exponent of the class group of $\mathbb{Q}(\sqrt{p^*q})$ divides $E$. We need to find a positive fundamental prime (power) discriminant $q$ such that the exponent of the class group of $L := \mathbb{Q}(\sqrt{p^*q})$ divides $2E$. Note that the 2-part of $\text{Cl}(L)$ is cyclic. Therefore we can ignore all fields $L$, where 4 divides the order of the class group.

**Lemma 36.** Let $L := \mathbb{Q}(\sqrt{p^*q})$ with $p, q$ prime, $q \equiv 1 \mathrm{mod} 4$ and $p^* < 0$. Then $4 \mid |\text{Cl}(L)|$ if and only if one of the following cases happen:

1. $p$ odd and $(\frac{q}{p}) = (\frac{p^*}{q}) = -1$,
2. $p = 2$ and $q \equiv 5 \mathrm{mod} 8$.

**Proof.** By Redei’s criterion the class group has an element of order 4, if $(\frac{q}{p}) = 1$ and $q \equiv 1 \mathrm{mod} 8$, respectively. 

In our algorithm we skip the cases $q = 2$, $q^* = 8$, and $\ell = 2$ and check this cases separately. Therefore we can assume that $q = q^*$ is a prime congruent to 1 modulo 4. Let $\ell$ be the smallest split prime in $L$. Then the order of a prime ideal above $\ell$ is either $E$ or $2E$ which means that $\ell^E$ or $\ell^{2E}$ is a norm, respectively. If the order is $2E$ then $p^*\ell^E$ is a norm. Using Lemma 24 this means that we have a solution of the following norm equation:

\[
\begin{align*}
X^2 - p^*qY^2 &= 4\ell^Ep & \text{if the order is } 2E \\
X^2 - p^*qY^2 &= 4\ell^E & \text{if the order is } E.
\end{align*}
\]

Note that $p^* < 0$ and $p^* = -p$ except for $p = 2$ we have $p^* = -4$ or $p^* = -8$. The easy description of our algorithm is that it checks all $X \leq \sqrt{4\ell^Ep}$ if $\sqrt{X^2 - 4\ell^Ep}$ corresponds to a field, where $\ell$ is the smallest splitting prime. This algorithm is very similar to the one described in [6, Lemma 19, Remark 20]. In the following we would like to reduce the number of $X$ to consider. Therefore we distinguish six cases, depending on if we have an element of order $E$ or $2E$ and if $p^* = -4, -8$ or $p^*$ is odd. In these situations we can simplify the equations to the following equations (with new $X$ and $Y$). As already mentioned we assume $q \neq 2$ and $\ell \neq 2$.

**Theorem 37.** Let $\ell > 2$ be a splitting prime in $L := \mathbb{Q}(\sqrt{p^*q})$ for $q \equiv 1 \mathrm{mod} 4$ such that the ideal above $\ell$ has order $o$ in the class group of $L$. Then in each case there exists one equation with solution $(X, Y) \in \mathbb{Z}^2$ with the following property:

1. Assume $p \equiv 3 \mathrm{mod} 4, o = 2E$. Then we have $(\frac{q}{p}) = -1$ and

\[
\begin{align*}
(pX)^2 - 4\ell^Ep &= -pqY^2 & X &\equiv 1 \mathrm{mod} 2 \\
(pX)^2 - \ell^Ep &= -pqY^2 & X &\equiv 1 \mathrm{mod} 2 & \text{for } \ell \equiv 3 \mathrm{mod} 4 + \text{Test} \\
& & X &\equiv 2 \mathrm{mod} 4 & \text{for } \ell p &\equiv 7 \mathrm{mod} 8 \\
& & X &\equiv 0 \mathrm{mod} 4 & \text{for } \ell p &\equiv 3 \mathrm{mod} 8
\end{align*}
\]

2. Assume $p^* = -8$ and $o = 2E$. Then

\[
\begin{align*}
X^2 - \ell^Ep &= -pqY^2 & X &\equiv 2 \mathrm{mod} 4 & \text{for } \ell &\equiv 7 \mathrm{mod} 8 \\
& & X &\equiv 0 \mathrm{mod} 4 & \text{for } \ell &\equiv 5 \mathrm{mod} 8
\end{align*}
\]

3. Assume $p^* = -4$ and $o = 2E$. Then

\[
\begin{align*}
X^2 - 2\ell^E &= -qY^2 & X &\equiv 1 \mathrm{mod} 2 & \text{for } \ell &\equiv 3 \mathrm{mod} 4
\end{align*}
\]
Assume \( p \equiv 3 \mod 4 \), \( o = E \). Then we have \( \left( \frac{\ell}{p} \right) = 1 \) and

\[
\begin{align*}
X^2 - 4\ell^E &= -pqY^2 \quad X \equiv 1 \mod 2 \\
X^2 - \ell^E &= -pqY^2 \quad X \equiv 1 \mod 2 \text{ for } \ell \equiv 1 \mod 4 + \text{Test} \\
&\quad X \equiv 2 \mod 4 \text{ for } \ell \equiv 7 \mod 8 \\
&\quad X \equiv 0 \mod 4 \text{ for } \ell \equiv 3 \mod 8 
\end{align*}
\]

(4) Assume \( p^* = -8 \) and \( o = E \). Then

\[
\begin{align*}
X^2 - \ell^E &= -2qY^2 \quad X \equiv 1 \mod 2 \text{ for } \ell \equiv 1, 3 \mod 8 \\
&\quad (\text{Test } q \equiv 5 \mod 8)
\end{align*}
\]

(5) Assume \( p^* = -4 \) and \( o = E \). Then

\[
\begin{align*}
X^2 - \ell^E &= -qY^2 \quad X \equiv 1 \mod 2 \text{ for } \ell \equiv 1 \mod 4 + \text{Test} \\
&\quad X \equiv 2 \mod 4 \text{ for } \ell \equiv 1 \mod 8 \\
&\quad X \equiv 0 \mod 4 \text{ for } \ell \equiv 5 \mod 8 
\end{align*}
\]

(6) Assume \( p^* = -8 \) and \( o = E \). Then

\[
\begin{align*}
X^2 - 4\ell^E &= -pqY^2 \quad X \equiv 1 \mod 2 \\
X^2 - \ell^E &= -pqY^2 \quad X \equiv 1 \mod 2 \text{ for } \ell \equiv 1 \mod 4 + \text{Test} \\
&\quad X \equiv 2 \mod 4 \text{ for } \ell \equiv 7 \mod 8 \\
&\quad X \equiv 0 \mod 4 \text{ for } \ell \equiv 3 \mod 8 
\end{align*}
\]

Proof. We use (5) for the case \( 2E \). In all cases we see that \( X \) must be divisible by \( p \) (even true for \( p = 2 \)) and we can replace \( X \) by \( pX \) to get the new equation \((pX)^2 - p^*qY^2 = 4\ell^Ep\) and equivalently \(pX^2 - p^*/pqY^2 = 4\ell^E\). For odd \( p \) we arrive at the first equation and we get the identity

\[
\left( \frac{\ell}{p} \right) = \left( \frac{4\ell^E}{p} \right) = \left( \frac{p^2 + qY^2}{p} \right) = \left( \frac{q}{p} \right) = -1
\]

by Lemma 36.

In the exponent \( E \)-case we have

\[
\left( \frac{\ell}{p} \right) = \left( \frac{4\ell^E}{p} \right) = \left( \frac{X^2 + pqY^2}{p} \right) = \left( \frac{X^2}{p} \right) = 1.
\]

The proof of the odd \( p \) cases use the fact that \( -pq \equiv 5 \mod 8 \) since otherwise 2 is split or ramified. In the cases \( p^* = -4, -8 \) we see that \( X, Y \) must be divisible by 2 and we can divide the whole equation by 4. Finally we use Lemma 36 to see that \( q \equiv 5 \mod 8 \) which gives further conditions. \(\square\)

We implemented the search in the computer algebra system Hecke [7] which is based on the Julia language [3]. Our code can be downloaded from the web page https://math.uni-paderborn.de/en/ag/ca/research/exponent/. The mathematics of the code is already explained, but we used a lot of small details which makes the code efficient. We do not try to explain the code here. On the same site we put the list of computed fields. We remark that all the fields \( L = \mathbb{Q}(\sqrt[\ell]{p^*q}) \) with exponent dividing 6 or 10 can be found in the databases [12] and [13]. Our computation shows that there are no more fields with this property beyond the computed range of these databases.

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