Exceptional point description of one-dimensional chiral topological superconductors/superfluids in BDI class

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We show that certain singularities of the Hamiltonian in the complex wave vector space can be used to identify topological quantum phase transitions for 1D chiral topological superconductors/superfluids in the BDI class. These singularities fall into the category of the so-called exceptional points (EP’s) studied in the context of non-Hermitian Hamiltonians describing open quantum systems. We also propose a generic formula in terms of the properties of the EP’s to quantify the exact number of Majorana zero modes in a particular chiral topological superconducting phase, given the values of the parameters appearing in the Hamiltonian. This formula serves as an alternative to the familiar integer ($\mathbb{Z}$) winding number invariant characterizing topological superconductor/superfluid phases in the chiral BDI class.

I. INTRODUCTION

Exceptional points (EP’s) are singular points in the parameter space of an operator at which two or more of its eigenvalues and eigenvectors coalesce\cite{10}. If the operator is the Hamiltonian itself, EP’s describe collapse of two or more energy eigenvalues at certain points in the parameter space similar to a degeneracy point, but with the important difference that the energy eigenvectors are not orthogonal to each other. The degeneracy of the eigenvalues with concomitant degeneracy of the eigenvectors gives rise to a whole host of non-trivial phenomena. For this reason, EP’s have recently attracted enormous interest in the literature\cite{31–33}. Although the concept of EP’s is known in mathematics for many years, their application in physics has been mostly limited to open quantum systems with dissipation, appropriately described by non-Hermitian Hamiltonians\cite{7–9}. Recently, EP’s have been invoked\cite{10} to describe interesting physics associated with zero energy Majorana bound states (MBSs) in one-dimensional (1D) topological superconductors/superfluids in class D, characterized by a $\mathbb{Z}_2$ invariant.

Topological superconductors\cite{11} are systems characterized by a bulk superconducting gap, and, topologically protected zero energy edge states known as MBSs (Majorana zero modes or simply ‘Majorana fermions’) described by second quantized operators satisfying the operator relation $\gamma^\dagger = \gamma$. In the context of condensed matter physics, aside from being fascinating emergent non-elementary particles (which can be identified with their own anti-particles), MBSs obey Ising type non-Abelian braiding statistics\cite{12,13} potentially useful in implementing a fault-tolerant topological quantum computer\cite{12,14}. While MBSs have not yet been conclusively found in nature, they have been theoretically shown to exist in low dimensional spinless $p$-wave superconducting systems\cite{15,16} as well as other systems involving various heterostructures with proximity-induced superconductivity which are topologically similar to them\cite{15,17}.

In particular, the spin-orbit coupled semiconductor-superconductor heterostructure scheme has motivated tremendous experimental efforts with a number of recent works claiming to have observed experimental signatures of MBSs in zero bias tunneling experiments\cite{22–25}. More recently, experiments on ferromagnetic Fe-atom chains embedded on Pb superconductor substrate, have also seen tantalizing evidence of MBSs in spatially resolved scanning tunneling microscopy measurements\cite{26}.

Recent theoretical work\cite{27,28} has established that the quadratic Hamiltonians for gapped topological insulators and topological superconductors can be classified into ten topological symmetry classes, each of which is characterized by a topological invariant. The symmetry classification is important as it provides an understanding of the effects of various perturbations on the stability of the protected surface modes, such as MBSs. The (strictly 1D) semiconductor-superconductor nanowire structure\cite{19,20} as well as the system of ferromagnetic atomic chains or nanowires deposited on Pb superconductor\cite{26,29} are in the topological class BDI. They are also known as chiral topological superconductors, described by an integer ($\mathbb{Z}$) winding number topological invariant\cite{31} that counts the number of protected zero energy Majorana modes at the individual edges. While the 1D semiconductor-superconductor nanowire structure is topologically isomorphic to the 1D spinless $p$-wave superconductor or Kitaev model\cite{32,33} (which is in the chiral BDI class in the absence of symmetry breaking perturbation\cite{34,35}), the system of ferromagnetic atomic chain or nanowire embedded on Pb superconductor is isomorphic to the doubled (or time reversal symmetric) Kitaev model\cite{32}, which is also in the chiral BDI class with a $\mathbb{Z}$ invariant. In the presence of chiral symmetry breaking terms (say, for example, stray magnetic fields and/or magnetic impurities), the symmetry classification of these systems reduces to class D, described by a $\mathbb{Z}_2$ invariant, and the number of protected MBSs at any given end reduces to zero or one.

The topological phases and quantum phase transitions within the BDI class topological superconductors
are usually described in terms of the closing and re-opening of the single particle energy gap, and a winding number integer topological invariant associated with the bulk Hamiltonian with periodic boundary conditions that counts the number of protected zero energy end states localized at any given edge. In this paper, we describe the topological quantum phase transitions in the BDI class, and propose a generic formula to count the exact number of Majorana zero modes in 1D chiral topological superconductors/superfluids based on the notion of EP’s. This quantity can exactly point out which topological phase we are considering, depending on the values of the parameters. In a recent work, the notion of the EP’s was discussed in the context of Majorana zero modes in \( \mathbb{Z}_2 \) topological superconducting systems.

Corresponding to a physical 1D Hamiltonian for a chiral topological superconductor, one can construct a non-Hermitian matrix by complexifying the momentum \( k \). In the chiral basis, we first separate out each \( 2 \times 2 \) chiral block. Now, each of this chiral block can be transformed into an off-diagonal form, which is a property of a chiral topological superconducting system. The values of the complex \( k \), where any one of the two off-diagonal elements vanish, are points in the complex \( k \) space where two (or more) eigenvalues of the complexified Hamiltonian vanish, and will be an example of the EP’s. These singularities are those special points where two repelling levels are connected by a square root branch point in the complex \( k \)-plane. We will see that the two distinct EP’s collapse into one at the points of a topological quantum phase transitions in the parameter space. In other words, both the elements of the off-diagonal \( 2 \times 2 \) Hamiltonian reduce to zero at the phase transition points. In the complex \( k \)-plane, if one expands the off-diagonal Hamiltonian around a solution \( k = k_{EP} \) for an EP, the zeroth order piece is non-diagonalizable, as it consists of one of the off-diagonal elements going to zero. The leading order correction is off-diagonal with both the elements non-zero, and diagonalizable. At a topological phase transition point, the zeroth order matrix vanishes, and we are left with a completely diagonalizable piece on expanding about that value of \( k_{EP} \). Elucidating the nature of a topological quantum phase transition in terms of EP’s, we propose a generic formula in terms of the properties of the EP’s to quantify the exact number of Majorana zero modes in a given chiral topological superconducting phase, given the values of the parameters appearing in the Hamiltonian. This formula serves as an alternative to the familiar integer \((\mathbb{Z})\) winding number invariant characterizing topological superconductor/superfluid phases in the chiral BDI class.

The paper is organized as follows: In Sec. II we consider the Kitaev model of 1D spinless \( p \)-wave superconductor. Sec. III is devoted to the study of the quantum Ising chain with longer-ranged interaction, which, by a Jordan-Wigner transformation maps on to the 1D Kitaev model with longer range hopping and superconducting pair potential. We study a third model of the Majorana fermions in chiral topological ferromagnetic nanowires, with proximity-induced superconductivity in Sec. IV. All the three systems, for appropriate values of the parameters, can support chiral Majorana bound states at any given end. In Sec. V we propose a generic formula to count the total number of zero modes in a particular chiral topological phase, given the values of the parameters of the Hamiltonian. We also provide the mathematical proof of why this quantity is related to the different topological phases. Lastly, we finish with a summary and outlook in Sec. VI.

II. MODEL 1: KITAEV CHAIN

Kitaev’s model of 1D \( p \)-wave superconducting quantum wire can support Majorana zero modes at the end, depending on the value of the chemical potential \( \mu \). In the Bogoliubov-de Gennes (BdG) basis, the Hamiltonian is given by:

\[
H_1(k) = (\cos(k) - \mu) \sigma_y - \Delta \sin(k) \sigma_x ,
\]

with eigenvalues

\[
E = \pm \sqrt{(\cos(k) - \mu)^2 + \Delta^2 \sin^2(k)} .
\]

Rotating the basis, we write the Hamiltonian in the following off-diagonal form:

\[
\tilde{H}_1(k) = \begin{pmatrix} 0 & A(k) \\ B(k) & 0 \end{pmatrix},
\]

\[
A(k) = (\mu - \cos(k)) + i \Delta \sin(k) ,
\]

\[
B(k) = (\mu - \cos(k)) - i \Delta \sin(k) .
\]

In the complex \( k \)-plane, the EP’s are given by

\[
k_{AK}^\pm = -i \ln \left\{ \frac{-\mu \pm \sqrt{\Delta^2 + \mu^2 - 1}}{\Delta - 1} \right\} ,
\]

\[
k_{BK}^\pm = -i \ln \left\{ \frac{\mu \pm \sqrt{\Delta^2 + \mu^2 - 1}}{\Delta + 1} \right\} ,
\]

corresponding to \( A(k_{AK}^+) = 0 \) and \( B(k_{BK}^+) = 0 \) respectively. This is in conformity with our definition of EP’s given in the introduction. Here the subscript “K” to the wave vector index \( k \) stands for Kitaev chain.

Expanding around the EP’s, we get:

\[
\tilde{H}_1(k) \approx \begin{pmatrix} 0 & A'(k_{AK}^+) \\ B'(k_{AK}^+) & 0 \end{pmatrix} (k - k_{AK}^+) ,
\]

\[
\tilde{H}_1(k) \approx \begin{pmatrix} 0 & A'(k_{BK}^+) \\ B'(k_{BK}^+) & 0 \end{pmatrix} (k - k_{BK}^+) .
\]
a) At \( \mu = -1 \), \( \tilde{H}_1(k_{AK}^+) = 0 \) and \( \tilde{H}_1(k_{AK}^-) \neq 0 \), while \( \tilde{H}_1(k_{BK}^+) = 0 \) and \( \tilde{H}_1(k_{BK}^-) \neq 0 \).

b) At \( \mu = 1 \), \( \tilde{H}_1(k_{AK}^+) \neq 0 \) and \( \tilde{H}_1(k_{AK}^-) = 0 \), while \( \tilde{H}_1(k_{BK}^+) \neq 0 \) and \( \tilde{H}_1(k_{BK}^-) = 0 \).

c) At all other values of \( \mu \), \( \tilde{H}_1(k_{AK}^+) \) and \( \tilde{H}_1(k_{BK}^+) \) are non-zero.

So, one of the two EP’s collapses at the phase transition points, irrespective of whether we consider \( A(k) = 0 \) or \( B(k) = 0 \).

\[ F_{\text{I}}. \] (Color online) (a) Purple region corresponds to \( \Im \left( k_{\tilde{A}K}^+ \right) > 0 \) and \( \Im \left( k_{\tilde{B}K}^- \right) < 0 \). Blue region corresponds to \( \Im \left( k_{\tilde{A}K}^+ \right) < 0 \) and \( \Im \left( k_{\tilde{B}K}^- \right) > 0 \). (b) Blue region corresponds to \( \Im \left( k_{\tilde{A}K}^+ \right) > 0 \) and \( \Im \left( k_{\tilde{B}K}^- \right) < 0 \). Purple region corresponds to \( \Im \left( k_{\tilde{A}K}^+ \right) < 0 \) and \( \Im \left( k_{\tilde{B}K}^- \right) > 0 \).

Fig. 1(a) and 1(b) show the dependence of the signs of \( \Im \left( k_{\tilde{A}K,\tilde{B}K}^\pm \right) \) as functions of \((\mu, \Delta)\). At \( \mu = -1 \), we find that \( \Im \left( k_{\tilde{A}K,\tilde{B}K}^+ \right) \) change signs. For instance, \( \Im \left( k_{\tilde{A}K}^+ \right) < 0 \) for \( \mu < -1 \), and \( \Im \left( k_{\tilde{A}K}^+ \right) > 0 \) for \( \mu > -1 \). This means that \( \Im \left( k_{\tilde{A}K}^+ \right) = 0 \) at \( \mu = -1 \), which in turn implies that \( E(k) = 0 \) has a solution for a real value of \( k \) (as \( E^2(k) = A(k) B(k) \)).

FIG. 2. (Color online) \( f^{A,B}(\mu, \Delta) \) reproduces the topological phases of the Hamiltonian in Eq. (1) in the \((\mu, \Delta)\)-plane.

Since \( E(k) = 0 \) for a real \( k \) signals a topological quantum phase transition (gap-closing) in this system, we find that in the EP-description, the topological phase transition is marked by the imaginary component of one of the EP’s going through zero. We find similar behaviour at \( \mu = 1 \). These observations help us define the functions:

\[
\begin{align*}
f^{A,B}(\mu, \Delta) &= \frac{1}{2} \left[ | \text{sgn} \left\{ \Im \left( k_{\tilde{A}K,\tilde{B}K}^+ (\mu, \Delta) \right) \right\} | - \text{sgn} \left\{ \Im \left( k_{\tilde{A}K,\tilde{B}K}^+ (\mu_0, \Delta) \right) \right\} \right] + \text{sgn} \left\{ \Im \left( k_{\tilde{A}K,\tilde{B}K}^+ (\mu, \Delta) \right) \right\} - \text{sgn} \left\{ \Im \left( k_{\tilde{A}K,\tilde{B}K}^+ (\mu_0, \Delta) \right) \right\} \right],
\end{align*}
\]

which capture the number of chiral Majorana zero modes in a given phase. Here, \( \mu_0 \) is any value of \( \mu \) where we have a non-topological or zero Majorana mode phase, i.e. \( \mu_0 \notin (-1, 1) \). Fig. 2 shows the contourplot for \( f^{A,B}(\mu, \Delta) \) in the \((\mu, \Delta)\)-plane. In practice, one can work with either \( f^A \) or \( f^B \) to identify the topological phase.

These EP’s were studied in an earlier work\(^{10} \) in the context of topological phases for the Kitaev chain. The authors proposed that the function

\[
\begin{align*}
\frac{1}{2} \left[ | \text{sgn} \left\{ \Im \left( k_{\tilde{A}K}^+ (\mu, \Delta) \right) \right\} \right] + \text{sgn} \left\{ \Im \left( k_{\tilde{A}K}^+ (\mu_0, \Delta) \right) \right\} \right]
\]

\((7)\)

gives the number of Majorana zero mode(s) in the various phases. However, we find that this definition works only for the \( 0 \to 1 \) Majorana fermion topological phase transition in the Kitaev model, and furthermore, when one chooses to work with the \( E \)’s corresponding \( A(k) = 0 \) (and not for \( B(k) = 0 \)).

**III. MODEL 2: ISING CHAIN WITH LONGER-RANGED INTERACTIONS**

The fermionized version of the 1D transverse field Ising model, generalized to include longer-ranged spin-spin interactions, can support 0, 1 or 2 Majorana mode(s) at
The eigenvalues of the Hamiltonian in Eq. (8) are given by
\[ E_I(k) = \pm \sqrt{\xi^2(k) + \Delta_I^2(k)}. \]

Since the Hamiltonian is chiral ($\{H_2(k), \sigma_x\} = 0$), by rotating the basis, we can rewrite it in the following off-diagonal form:
\[
\hat{H}_2(k) = \begin{pmatrix}
0 & A_I(k) \\
B_I(k) & 0
\end{pmatrix},
\]
\[
A_I(k) = \xi(k) + i \Delta_I(k),
\]
\[
B_I(k) = \xi(k) - i \Delta_I(k).
\]

In the complex $k$-plane, the $EP$'s are given by
\[
k_{AI}^\pm = -i \ln \left\{ -\lambda_1 \pm \sqrt{\lambda_1^2 + 4 \lambda_2} \right\},
\]
\[
k_{BI}^\pm = -i \ln \left\{ -\lambda_1 \pm \sqrt{\lambda_1^2 + 4 \lambda_2} \right\},
\]

where $\lambda_1$ denotes the magnitudes of the nearest neighbour hopping and superconducting gap, and $\lambda_2$ denotes the amplitudes of the next nearest neighbour hopping and superconducting gap. It has been shown$^{35}$ that the presence of $\lambda_2$ gives rise to two Majorana zero modes coexisting at the same end of the 1D chain, for certain values of the parameters $\lambda_1$ and $\lambda_2$. In this system, multiple zero energy Majorana modes can coexist (and do not mix and split to finite energies), because the Hamiltonian is in the chiral BDI class with an integer invariant$^{31}$. The eigenvalues of the Hamiltonian in Eq. (8) are given by

\[ E_I(k) = \pm \sqrt{\xi^2(k) + \Delta_I^2(k)}. \]
superconductor with a single spatial channel (i.e. no transverse hopping). In the momentum space, the BdG Hamiltonian becomes $H = \sum_k \Psi_k^\dagger H_3(k) \Psi_k$, where

$$H_3(k) = (-2t \cos(k) - \mu) \sigma_0 \tau_z + [\Delta_x, \sigma_0 + \Delta_p \sin(k)] d \cdot \sigma] \tau_x + V \cdot \sigma \tau_0.$$ (14)

Here $k \equiv k_x$ is the one-dimensional crystal momentum, $\Psi_k = (c_{k\uparrow}, c_{k\downarrow}, c_{-k\uparrow}^\dagger, c_{-k\downarrow}^\dagger)^T$ is the four-component Nambu spinor which acts on the particle-hole ($\tau$) and spin ($\sigma$) spaces, and $V$ is the Zeeman field which can be induced by ferromagnetism. Also, $\Delta_x$ and $\Delta_p$ are proximity-induced $s$-wave and $p$-wave superconducting pairing potentials respectively, with $d$ determining the relative magnitudes of the components of the $p$-wave superconducting order parameter $\Delta_{\alpha\beta}$ ($\alpha, \beta = \uparrow, \downarrow$). In our calculations, we use $d = (1, 0, 0)$ and $V = (0, V, 0)$. Furthermore, we set $\Delta_x = 0$, thus only considering $p$-wave pairing, which does not change the chiral BDI class of the Hamiltonian.

The eigenvalues of the Hamiltonian are given by:

$$E_1(k) = \pm \sqrt{(2t \cos(k) + \mu - V)^2 + \Delta_p^2 \sin^2(k)},$$ (15)

$$E_2(k) = \pm \sqrt{(2t \cos(k) + \mu + V)^2 + \Delta_p^2 \sin^2(k)},$$ (16)

which correspond to bands of opposite chirality. We change the basis to transform the Hamiltonian in Eq. (14) to the form:

$$\tilde{H}_3(k) = \begin{pmatrix} h_1(k) & 0 \\ 0 & h_2(k) \end{pmatrix},$$

where

$$h_1(k) = \begin{pmatrix} 0 & A_r(k) \\ B_r(k) & 0 \end{pmatrix},$$

$$A_r(k) = 2t \cos(k) + \mu - V + i \Delta_p \sin(k),$$

$$B_r(k) = 2t \cos(k) + \mu - V - i \Delta_p \sin(k),$$

$$A_2(k) = 2t \cos(k) + \mu + V + i \Delta_p \sin(k),$$

$$B_2(k) = 2t \cos(k) + \mu + V - i \Delta_p \sin(k).$$ (17)

The upper block $h_1(k)$ and the lower block $h_2(k)$ correspond to the opposite chirality bands with eigenvalues $E_1(k)$ and $E_2(k)$ respectively.

In the complex $k$-plane, the EP’s for the upper block are given by

$$k_{A,1}^\pm = -i \ln \left\{ \frac{V - \mu \pm \sqrt{(V - \mu)^2 + \Delta_p^2 - 4t^2}}{2t + \Delta_p} \right\},$$ (18)

$$k_{B,1}^\pm = -i \ln \left\{ \frac{V - \mu \pm \sqrt{(V - \mu)^2 + \Delta_p^2 - 4t^2}}{2t - \Delta_p} \right\},$$ (19)

corresponding to $A_1(k_{A,1}^\pm) = 0$ and $B_1(k_{B,1}^\pm) = 0$ respectively. Similarly, the EP’s for the lower block are given by

$$k_{A,2}^\pm = -i \ln \left\{ \frac{V + \mu \pm \sqrt{(V + \mu)^2 + \Delta_p^2 - 4t^2}}{2t + \Delta_p} \right\},$$ (20)

$$k_{B,2}^\pm = -i \ln \left\{ \frac{V + \mu \pm \sqrt{(V + \mu)^2 + \Delta_p^2 - 4t^2}}{2t - \Delta_p} \right\},$$ (21)

corresponding to $A_2(k_{A,2}^\pm) = 0$ and $B_2(k_{B,2}^\pm) = 0$ respectively.
a) At \( \mu = -0.5 t \), \( h_1(k_{A,1}^+) = 0 \) and \( h_2(k_{A,1}^-) \neq 0 \), while \( h_1(k_{B,1}^+) \neq 0 \) and \( h_1(k_{B,1}^-) = 0 \).

b) At \( \mu = 3.5 t \), \( h_1(k_{A,1}^-) \neq 0 \) and \( h_1(k_{A,1}^+) = 0 \), while \( h_1(k_{B,1}^-) = 0 \) and \( h_1(k_{B,1}^+) \neq 0 \).

c) At \( \mu = -3.5 t \), \( h_2(k_{A,2}^+) = 0 \) and \( h_2(k_{A,2}^-) \neq 0 \), while \( h_2(k_{B,2}^-) \neq 0 \) and \( h_2(k_{B,2}^+) = 0 \).

d) At \( \mu = 0.5 t \), \( h_2(k_{A,2}^+) \neq 0 \) and \( h_2(k_{A,2}^-) = 0 \), while \( h_2(k_{B,2}^+) = 0 \) and \( h_2(k_{B,2}^-) \neq 0 \).

e) At all other values of \( \mu \), \( h_r(k_{A,r}^+) \) and \( h_r(k_{B,r}^-) \) are non-zero.

So, one of the two \( EP \)'s collapses at each phase transition point, irrespective of whether we consider \( A_r(k) = 0 \) or \( B_r(k) = 0 \).

Fig. 6(a) and 6(b) show the dependence of the signs of \( \Im \left( k_{A,r}^+ \right) \) as functions of \( \mu/t \). Generalising the functions defined in Eqs. 7 and 13 to the present case of a \( 4 \times 4 \) Hamiltonian, we define:

\[
\begin{align*}
\sum_{\mu} f_{A/B}^{A/B} (\mu) & = \frac{1}{2} \sum_{r=1,2} \left[ \left| \text{sgn} \left\{ \Im \left( k_{A/r}^+ (\mu) \right) \right\} - \text{sgn} \left\{ \Im \left( k_{A/r}^- (\mu) \right) \right\} \right| \\
& + \left| \text{sgn} \left\{ \Im \left( k_{B/r}^- (\mu) \right) \right\} - \text{sgn} \left\{ \Im \left( k_{B/r}^+ (\mu) \right) \right\} \right| \right],
\end{align*}
\]

which capture the number of chiral Majorana zero modes in a given phase. Here, \( \mu_0 \) is any value of \( \mu \) where we have a non-topological phase. Fig. 6(c) shows the plot for \( f_{A/B}^{A/B} (\mu) \).

**V. GENERIC FORMULA**

A generic 1D \( 2 \times 2 \) chiral Hamiltonian, having two distinct \( EP \)'s in the complex \( k \)-plane, can support up to two Majorana zero modes at each end of the fermionic chain. Let \( k = k_{EP}^+ \) be the solutions of the pair of \( EP \)'s, where any one of the two elements the BdG Hamiltonian vanish (after it has been rotated into the off-diagonal form and \( k \) has been promoted to a complex number). From the study of our models, we observe the following:

a) \( \text{sgn} \left\{ \Im \left( k_{EP}^+ (\{p_i\}) \right) \right\} \) for a phase with \( n = 0, 2 \) zero modes. Further, these signs are opposite to each other for the \( n = 0 \) and \( n = 2 \) cases.

b) \( \text{sgn} \left\{ \Im \left( k_{EP}^+ (\{p_i\}) \right) \right\} \) and \( \text{sgn} \left\{ \Im \left( k_{EP}^- (\{p_i\}) \right) \right\} \) have opposite signs for a phase with \( n = 1 \) zero mode.

Hence, we propose the following generic formula:

\[
\begin{align*}
f_{k_{EP}^+ (\{p_i\})} & = \frac{1}{2} \left[ \left| \text{sgn} \left\{ \Im \left( k_{EP}^+ (\{p_i\}) \right) \right\} - \text{sgn} \left\{ \Im \left( k_{EP}^- (\{p_i\}) \right) \right\} \right| \\
& + \left| \text{sgn} \left\{ \Im \left( k_{EP}^- (\{p_i\}) \right) \right\} - \text{sgn} \left\{ \Im \left( k_{EP}^+ (\{p_i\}) \right) \right\} \right| \right],
\end{align*}
\]

where \( \{p_i\} \) is the set of parameters appearing in the expressions for \( k_{EP}^\pm \), and \( \{p_i^0\} \) are their values at any point in the non-topological phase.
We note that we can apply this formula for several nanowires coupled by a transverse hopping term, i.e. for a chiral topological superconductor in the BDI class with arbitrary integer topological invariant $Z$. For each pair of energy eigenvalues $\pm |E_n(k)|$, we will get a pair of $E P$’s, indexed by $n \in \{1, 2, \cdots, N\}$. The total number of zero modes will be given by the sum of the functions $f_{EP,k}$ (defined in Eq. [23]) evaluated for each pair of $E P$’s. In particular, if we have a $2N \times 2N$ Hamiltonian, writing it in the block-diagonal form and unitarily rotating each $2 \times 2$ block to an off-diagonal form, we define $k = k_{EP,n}^\pm$ to be the $E P$’s corresponding to the $N$ blocks. The total count of the zero modes for the full Hamiltonian will be given by

$$\sum_{n=1}^{N} f_{k_{EP,n}} = \frac{1}{2} \sum_{n=1}^{N} \left[ \text{sgn} \left( \Im \left( k_{EP,n}^\pm \right) \right) \right] - \text{sgn} \left( \Im \left( k_{EP,n}^\mp \right) \right).$$

We now prove why one or both of $\Im \left( k_{EP}^\pm \right)$ change signs at a topological phase transition point, characterised by a set of parameters $\{p_i = p_i^1\}$. We consider a 1D $2 \times 2$ chiral system with eigenvalues of the form

$$E(k) = \pm \sqrt{a^2(k, \{p_i\}) + b^2(k, \{p_i\})},$$

where the parameters $\{p_i\}$ are real, and the functions $(a, b)$ are real for real $k$. We can rotate the basis of such a Hamiltonian to the off-diagonal form:

$$H_{od}(k) = \begin{pmatrix} 0 & a(k, \{p_i\}) - i b(k, \{p_i\}) \\ a(k, \{p_i\}) + i b(k, \{p_i\}) & 0 \end{pmatrix},$$

by a unitary transformation, and consider complex $k$. Let us consider the $E P$’s at $k = k_{EP}$ corresponding to $a(k_{EP}, \{p_i\}) = i b(k_{EP}, \{p_i\})$. Let

$$k_{EP} = x(\{p_i\}) + iy(\{p_i\}),$$

where $(x, y)$ are the real and imaginary parts of $k_{EP}$ as functions of $\{p_i\}$. Then we must have

$$a(k_{EP}, \{p_i\}) = f_{a,\text{even}}(x, y) + i f_{a,\text{odd}}(x, y),$$

$$b(k_{EP}, \{p_i\}) = f_{b,\text{even}}(x, y) + i f_{b,\text{odd}}(x, y),$$

such that $f_{a,\text{even}}(x, y)$ and $f_{a,\text{odd}}(x, y)$ are even and odd functions of $y$ respectively. Needless to add that they must be real functions too. Now the constraint $a(k_{EP}, \{p_i\}) = i b(k_{EP}, \{p_i\})$ translates into the equations

$$f_{a,\text{even}}(x, y) = -f_{b,\text{odd}}(x, y),$$

$$f_{a,\text{odd}}(x, y) = f_{b,\text{even}}(x, y),$$

whose solutions give $(x, y)$ as functions of $\{p_i\}$. At a point $\{p_i = p_i^1\}$ in parameter space, where both $a \pm ib$ vanish (i.e. $H_{od}(k_{EP}, \{p_i = p_i^1\}) = 0$), we must have

$$a(\{p_i^1\}) = b(\{p_i^1\}) = 0,$$

$$f_{a,\text{odd}}(x, y)|_{\{p_i = p_i^1\}} = f_{b,\text{even}}(x, y)|_{\{p_i = p_i^1\}} = 0,$$

$$\Rightarrow y(\{p_i^1\}) = 0.$$  

As one of the solutions for $y(\{p_i^1\})$. This is clearly a topological phase transition point, as $E(k)$ vanishes for a real value of $k$. For a point $\{p_i, p_i^1, p_i^2\}$ close to $\{p_i^1, p_i^1\}$, we have the expansion

$$y(\{p_i, p_i^1, p_i^2\}) = \begin{cases} |p_i - p_i^1| \partial_{p_i} y(\{p_i^1\}) & \text{if } p_i > p_i^1, \\ -|p_i - p_i^1| \partial_{p_i} y(\{p_i^1\}) & \text{if } p_i < p_i^1, \end{cases}$$

for the solution $y(\{p_i^1\}) = 0$. Hence, this solution for $\Im \left( k_{EP}^\pm \right)$ undergoes a sign change on crossing the phase transition point.

VI. CONCLUSION

We have derived a generic formula for counting the number of Majorana zero modes for a 1D chiral topological superconductor/superfluid. For the case when each $2 \times 2$ block of the corresponding BdG Hamiltonian can support upto two Majorana zero modes, we can write down the solutions, $k_{EP,n}^\pm$, for a pair of exceptional points in the complex momentum space, in terms of the parameters of the Hamiltonian. Our formula is based on the evolution of these $E P$’s in the complex $k$-plane as functions of the parameters. The count of the MBSs is encoded in the signs of $\Im \left( k_{EP}^\pm \right)$.

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