Singular matrix variate Birnbaum-Saunders distribution under elliptical models

José A. Díaz-García and Francisco J. Caro-Lopera
Faculty of Basic Sciences, Universidad de Medellín, Medellín, Colombia

ABSTRACT
This work sets the matrix variate Birnbaum–Saunders theory in the context of singular distributions and elliptical models. The termed singular matrix variate generalized Birnbaum–Saunders distribution is obtained with respect the Hausdorff measure. Several basic properties and particular cases of this distribution are also derived.

ARTICLE HISTORY
Received 8 June 2020
Accepted 26 May 2021

KEYWORDS
Singular matrix variate distributions; singular elliptical distributions; Birnbaum–Saunders distribution; Hausdorff measure

1. Introduction
The univariate Birnbaum–Saunders distribution, introduced by Birnbaum and Saunders (1969), has promoted a considerable research during the last 50 years. At first, the distribution was motivated as a lifetime model for fatigue failure caused under cyclic loading, under the assumption that the failure is due to the development and growth of a dominant crack. A more general derivation was also provided by Desmond (1985) in a context of a biological model.

However, by passing the decades, the advances in more complex scenarios were so slow. In fact, the literature of non singular matrix variate Birnbaum–Saunders is such small that in the recent review by Balakrishnan and Kundu (2018), only one of the 281 cited papers belonged to the matrix case (Caro-Lopera, Leiva, and Balakrishnan 2012). In a discussion about the addressed review, the number has increased to only three more new works, written by the same group of authors, see Caro-Lopera and Díaz-García (2016), Sánchez et al. (2015) and Díaz-García and Caro-Lopera (2018).

Now, when the research moves to the singular case, the problems are greater, because the distributions do not exist with respect the Lebesgue measure, see Khatri (1968). For a unified approach, summarizing a number of singular distributions in a wider spectra, see Khatri (1968), Uhlig (1994), Rao (2005), Díaz-García, Jáimez, and Mardia (1997); Díaz-García and González-Farías (2005a, 2005b); Díaz-García and Gutiérrez (2005, 2006a, 2006b), Zhang (2007), Bodnar and Okhrin (2008) and the references therein.
Finally, the evolution of matrix variate distributions is usually given by exploring the Gaussian case and then providing a generalization under families of distributions. The generalized theory also based on singular distribution allows the desirable unified approach for this distributional challenge.

For a contextualization of the problem, we provide some highlights of the distribution.

The original univariate random variable Birnbaum–Saunders was obtained as a function of a normal random variable. Furthermore, if \( Z \sim \mathcal{N}(0,1) \) then \( T \) is called a Birnbaum–Saunders random variable, where

\[
T = \beta \left( \frac{x}{2} Z + \sqrt{\left( \frac{x}{2} Z \right)^2 + 1} \right)^2,
\]

We shall denote this fact as \( T \sim BS(x, \beta) \), where \( x > 0 \) is the shape parameter, and \( \beta > 0 \) is both scale parameter an the median value of the distribution. Thus, the inverse relation establishes that if \( T \sim BS(x, \beta) \), then

\[
Z = \frac{1}{x} \left( \sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}} \right) \sim \mathcal{N}(0,1)
\]

Now, it said that a random variable \( Y \in \mathbb{R} \) has an elliptical distribution, with parameters \( \mu \in \mathbb{R} \) and \( \sigma^2 > 0 \), if its density can be expressed as

\[
dF_Y(y) = \frac{c}{\sigma} h \left[ \frac{(y - \mu)^2}{\sigma^2} \right] dy.
\]

Here the function \( h : \mathbb{R} \to [0, \infty) \) is termed the generator function, where \( \int_0^\infty h(u) < \infty \), and \( c \) is the normalization constant in order that \( f_Y(y) \) is a density. This aspect is denoted as \( Y \sim \mathcal{E}(\mu, \sigma^2; h) \). In particular, when \( \mu = 0 \) and \( \sigma^2 = 1 \) is termed spherical distribution and is denoted by \( \mathcal{E}(0,1; h) \). Díaz-García and Leiva-Sánchez (2005, 2006) proposed a generalization of the Birnbaum–Saunders distribution, replacing the normal distribution hypothesis in (2) by a spherical distribution, that is, they assume that \( Z \sim \mathcal{E}(0,1, h) \). Therefore, (1) defines the generalized Birnbaum–Saunders distribution, which shall be denoted by \( T \sim GBS(x, \beta; h) \).

From Díaz-García and Leiva-Sánchez (2005, 2006) if \( T \sim GBS(x, \beta, h) \), then

\[
dF_T(t) = \frac{t^{-3/2}(t + \beta)}{2x\sqrt{\beta}} h \left[ \frac{1}{x^2} \left( \frac{t}{\beta} + \frac{\beta}{t} - 2 \right) \right] dt, \quad t > 0.
\]

Alternatively, let \( V = \sqrt{T} \), with \( dt = 2vdv \), then under a symmetric distribution, (2) can be rewritten as

\[
Z = \frac{1}{x} \left( \frac{V}{\sqrt{\beta}} - \sqrt{\beta} \right),
\]

and its density is given by
\[ dF_V(v) = \frac{(1 + \beta v^{-2})}{\alpha \sqrt{\beta}} h \left[ \frac{1 + \beta}{\alpha^2} \left( v^2 + \beta - \frac{2}{v^2} \right) \right] dv, \quad v > 0, \quad (5) \]

which shall be termed \textit{square root generalized Birnbaum–Saunders distribution}.

The study of the Birnbaum–Saunders distribution has been very profuse in the univariate case (basic properties and estimation), but the matrix variate extensions under Gaussian or elliptical models have been elusive for decades. As we have quoted, Balakrishnan and Kundu (2018) make a detailed compilation of this distribution in the last five decades. Especially, the results in the specialized literature for these multivariate generalizations, have been proposed defining the vector and matrix variate cases, element-to-element. For the vectorial case we mention Díaz-García and Domínguez Molina (2006), Díaz-García and Domínguez Molina (2007) and for the matrix case we have Caro-Lopera, Leiva, and Balakrishnan (2012), Sánchez et al. (2015), Caro-Lopera and Díaz-García (2016). Recently, Díaz-García and Caro-Lopera (2021) closed the problem for a non singular matrix variate version in terms of a matrix transformation open a perspective to the solution in the singular case.

Thus in this paper we obtain a singular version of this matrix transformation and then we find the corresponding singular matrix variate generalized Birnbaum–Saunders distribution with respect to the Hausdorff measure (see Billingsley (1986, Section 19) and Díaz-García, Jáimez, and Mardia 1997). For this task we need consider the following aspects:

- Given a singular random matrix, if a density function with respect certain measure is obtained, we must know that both the density and the measure are nonunique. At first, the non-uniqueness of the density and its corresponding measure might seem a disadvantage, however, when obtaining the maximum likelihood estimate through different choices, each one leads to the same maximum likelihood estimate, see Khatri (1968) and Rao (2005).
- Another very important aspect warns about the combination of results under different densities and measures.
- In terms of the previous point, this is the source of explanation for a number of wrong applications of the published results, see Díaz-García (2007).
- As a consequence, when alternative approaches of literature are used in the setting of the present work, it should be noted that inconsistent algebraic, probabilistic and conceptual results are obtained. Furthermore, in those papers the approach followed in this work is validated by different authors, see Zhang (2007), Díaz-García (2007), and Bodnar and Okhrin (2008), among others.
- A consistent presentation on the theory of singular random matrix and vector distributions can be found in Khatri (1968), Uhlig (1994), Rao (2005), Díaz-García and Gutiérrez (1997, 2005, 2006a, 2006b), Díaz-García, Jáimez, and Mardia (1997), Díaz-García and González-Farías (2005a, 2005b), among others.

The above discussion is placed in the paper in two parts. Some preliminary results and new Jacobians are studied in Section 2. Then, Section 3 proposes the main result of the article. Finally, certain special cases and some basic properties are derived.
2. Preliminary results

Some preliminary results about the singular matrix variate elliptical distribution are summarized below.

Two Jacobians for matrix transformations with respect to Hausdorff measure are computed. First, some results and notations about the required matrix algebra are considered, see Díaz-García and González-Farías (2005b), Díaz-García and Gutiérrez (2005), Rao (2005), and Muirhead (2005).

2.1. Notation

Let \( \mathcal{L}_{m,n}(q) \) be the linear space of all \( n \times m \) real matrices of rank \( q \leq \min(n,m) \) and \( \mathcal{L}_{m,n}^+(q) \) be the linear space of all \( n \times m \) real matrices of rank \( q \leq \min(n,m) \) with \( q \) distinct singular values. The set of matrices \( \mathbf{H}_1 \in \mathcal{L}_{m,n} \) such that \( \mathbf{H}_1' \mathbf{H}_1 = \mathbf{I}_m \) is a manifold denoted \( \mathcal{V}_{m,n} \), called Stiefel manifold. In particular, \( \mathcal{V}_{m,m} \) is the group of orthogonal matrices \( O(m) \). Denote by \( \mathcal{S}_m \) the homogeneous space of \( m \times m \) positive definite symmetric matrices; \( \mathcal{S}_m^+(q) \), the \( (mq - q(q - 1)/2) \)-dimensional manifold of rank \( q \) positive semidefinite \( m \times m \) symmetric matrices with \( q \) distinct positive eigenvalues. For all matrix \( \mathbf{A} \in \mathcal{L}_{m,n}(q) \) exist \( \mathbf{A}^+ \in \mathcal{L}_{n,m}(q) \) which is termed Moore–Penrose inverse. Similarly, for all matrix \( \mathbf{A} \in \mathcal{L}_{m,n}(q) \) exist \( \mathbf{A}^- \in \mathcal{L}_{m,n}(r) \), \( r \geq q \) which is termed conditional (or generalized) inverse is such that \( \mathbf{A} ^+ \mathbf{A}^- = \mathbf{A} \). The eigenvalues of \( \mathbf{A} \in \mathcal{L}_{m,m}(q) \) are the roots of the equation \( |\mathbf{A} - \lambda \mathbf{I}_m| = 0 \). The \( i \)-th eigenvalue of \( \mathbf{A} \) shall be denoted as \( \chi_i(\mathbf{A}) \). Given \( \mathbf{A} \in \mathcal{S}_m^+(q) \), there exist \( \mathbf{A}^{1/2} \in \mathcal{S}_m^+(q) \) such that \( \mathbf{A} = (\mathbf{A}^{1/2})^2 \), which is termed positive (definite) semi-definite root matrix.

2.2. Matrix variate distribution

Definition 1. It is said that \( \mathbf{Y} \in \mathcal{L}_{m,n}(q), q = \min(r,s) \), has a singular matrix variate elliptically contoured distribution if its density \( dF_Y(\mathbf{Y}) \) is given by:

\[
\frac{1}{\prod_{i=1}^r \chi_i(\Sigma)^{s/2}} \frac{1}{\prod_{j=1}^s \chi_j(\Theta)^{r/2}} h(\text{tr} \Sigma^{-1} (\mathbf{Y} - \mu)(\Theta^{-1}(\mathbf{Y} - \mu))) \, (d\mathbf{Y})
\]

where \( \mu \in \mathbb{R}^{n \times m}, \Sigma \in \mathcal{S}_m^+(r), \Theta \in \mathcal{S}_n^+(s), \) and \( (d\mathbf{Y}) \) is the Hausdorff measure. The function \( h : \mathbb{R} \rightarrow [0,\infty) \) is termed the generator function and satisfies \( \int_0^\infty \text{e}^x \, dx < \infty \). Such a distribution is denoted by \( \mathbf{Y} \sim \mathcal{E}_{n \times m}^{r,s}(\mu, \Theta \otimes \Sigma, h) \), omitting the supra-index when \( r = m \) and \( s = n \), see Díaz-García and González-Farías (2005b). Observe that if \( \mathbf{Y} \in \mathcal{L}_{m,n}(q) \), then there exist \( \mathbf{V}_1 \in \mathcal{V}_{q,m}, \mathbf{W}_1 \in \mathcal{V}_{q,m} \) and \( \mathbf{L} = \text{diag}(l_1, \ldots, l_q) \), \( l_1 > \cdots > l_q > 0 \), such that \( \mathbf{Y} = \mathbf{V}_1 \mathbf{L} \mathbf{W}_1' \), is the nonsingular part of the singular value decomposition (SVD), Rao (2005, 42, 1973). Then, the Hausdorff measure \( (d\mathbf{Y}) \) can be explicitly defined as follows

\[
(d\mathbf{Y}) = 2^{-q} \prod_{i=1}^q l_i^{m-2q} \prod_{i<j} (l_i^2 - l_j^2) (\mathbf{V}_1' d\mathbf{V}_1)(\mathbf{W}_1' d\mathbf{W}_1) \int_1^q dl_i,
\]

but quoting that the density and the measure defined in this way are not unique, see Khatri (1968) and Díaz-García and González-Farías (2005b).
When \( \mu = 0_{n \times m}, \Sigma = I_m \) and \( \Theta = I_n \), such distribution is termed \textit{matrix variate symmetric distribution} and shall be denoted as \( Y \sim \mathcal{E}_{n \times m}^s(0, I_{nm}, h) \).

This class of matrix variate distributions includes \textit{normal, contaminated normal, Pearson type II and VI, Kotz, logistic, power exponential}, and so on; these distributions have tails that are weighted more or less, and/or they have greater or smaller degree of kurtosis than the normal distribution.

In addition, note that if \( Y \sim \mathcal{E}_{n \times m}^s(\mu, \Theta \otimes \Sigma, h) \), and \( A \in \mathcal{L}_{n,a}(k) \) and \( B \in \mathcal{L}_{m,b}(t) \). Then, \( AYB \sim \mathcal{E}_{a \times b}^{x,\beta}(A\mu B', A'\Theta A \otimes B'\Sigma B, h) \), where \( x \) and \( \beta \) are the ranks of \( A'\Theta A \) and \( B'\Sigma B \), respectively; where \( x \leq \min(r, k) \) and \( \beta \leq \min(s, t) \).

### 2.3. Jacobians

**Lemma 1.** Let \( U \) and \( W \in \mathcal{L}_{m,n}(p) \), such that

\[
U = W - W^+.
\]

Then

\[
(dU) = \begin{cases} 
\prod_{i=1}^{p} (1 - d_i^{-2})^{n+m-2p} \prod_{i<j}^p \left( 1 - d_i^{-2} d_j^{-2} \right) (dW) \\
\prod_{i=1}^{p} d_i^{-2(2n+m-p)} (d_i^2 - 1)^{n+m-2p} \prod_{i<j}^p \left( d_i^2 d_j^2 - 1 \right) (dW),
\end{cases}
\]

where \( d_i^2 = \text{ch}_i(W'W), i = 1, 2, ..., p, d_1 > d_2 > \cdots > d_p > 0 \).

**Proof.** Let \( W = H_1 D Q_1' \) the nonsingular part of the singular value factorization of \( W \), where \( H_1 \in \mathcal{V}_{p,m}, D = \text{diag}(d_1, ..., d_p), d_1 > \cdots > d_p > 0 \) and \( Q_1 \in \mathcal{V}_{p,n} \), with \( d_i^2 = \text{ch}_i(W'W) \), see Muirhead (2005, Theorem A9.10, 593). By Rao (2005, Problem 28e, 76–77) is know that \( W^+ = Q_1 D^{-1} H_1' \). Then from (7)

\[
U = H_1 D Q_1' - (Q_1 D^{-1} H_1')' = H_1 (D - D^{-1}) Q_1'.
\]

By Díaz-García and Gutiérrez (2005), if we take \( g(x_i) = d_i - d_i^{-1} \) then we obtain

\[
(dU) = \prod_{i=1}^{p} \left( \frac{d_i - d_i^{-1}}{d_i} \right)^{(n+m-2p)} \prod_{i<j}^p \left( \frac{(d_i - d_i^{-1})^2 - (d_j - d_j^{-1})^2}{d_i^2 - d_j^2} \right) \\
\times \prod_{i=1}^{p} d_i (d_i - d_i^{-1}) (dW).
\]

Observe that

\[
\prod_{i=1}^{p} d_i (d_i - d_i^{-1}) = \begin{cases} 
\prod_{i=1}^{p} (1 + d_i^{-2}) \\
\prod_{i=1}^{p} d_i^{-2} \prod_{i=1}^{m} (1 + d_i^{2}),
\end{cases}
\]

(10)
\[
\prod_{i=1}^{m} \left( \frac{d_i - d_i^{-1}}{d_i} \right)^{n+m-2p} = \begin{cases} 
\prod_{i=1}^{m} (1 - d_i^{-2})^{n+m-2p} \\
\prod_{i=1}^{m} d_i^{-2(n+m-2p)} \prod_{i=1}^{m} (d_i^2 - 1)^{n+m-2p}.
\end{cases}
\] (11)

Also note that
\[
(d_i - d_i^{-1})^2 - (d_j - d_j^{-1})^2 = \left( \frac{d_i^2 - 1}{d_i} \right)^2 - \left( \frac{d_j^2 - 1}{d_j} \right)^2 = \frac{d_i^2 (d_i^2 - 1)^2 - d_j^2 (d_j^2 - 1)^2}{d_i^2 d_j^2} = \frac{d_i^2 d_i^4 - 2d_i^2 d_i^2 + d_i^2 - d_i^2 d_j^4 + 2d_i^2 d_j^2 - d_j^2 d_j^2}{d_i^2 d_j^2} = \frac{(d_i^2 d_j^2 - 1)(d_i^2 - d_j^2)}{d_i^2 d_j^2}.
\]

Thus
\[
\prod_{i<j}^{p} \frac{(d_i - d_i^{-1})^2 - (d_j - d_j^{-1})^2}{d_i^2 - d_j^2} = \prod_{i<j}^{p} \frac{d_i^2 d_j^2 - 1}{d_i^2 - d_j^2} = \begin{cases} 
\prod_{i=1}^{p} d_i^{-2(p-1)} \prod_{i<j}^{p} (d_i^2 d_j^2 - 1) \\
\prod_{i<j}^{p} (1 - d_i^{-2} d_j^{-2})
\end{cases}
\] (12)

where we have used the expression
\[
\prod_{i<j}^{p} \frac{1}{d_i^2 d_j^2} = \prod_{i=1}^{p} d_i^{-2(p-1)}.
\]

Substituting (10), (11) and (12) into (9) the desired results (8) are obtained.

**Theorem 1.** Consider the follow matrix transformation
\[
Z = (V\Delta^+ - V'^+ \Delta)\Xi^+, \quad (13)
\]
where \(Z\) and \(V \in \mathcal{L}_{m,n}(p), \Delta\) and \(\Xi \in \mathcal{S}_m^+(s), s \leq m\). Then
\[
(dZ) = \frac{G(q, \theta^2)}{\prod_{i=1}^{r} \text{ch}_i(\Xi)^n \prod_{j=1}^{s} \text{ch}_j(\beta)^{n/2}} (dV) \quad (14)
\]
with
\[
G(q, \theta^2) = \begin{cases} 
\prod_{i=1}^{q} (1 - \theta_i^{-2})^{n+m-2q} (1 + \theta_i^{-2}) \prod_{i<j}^{q} (1 - \theta_i^{-2} \theta_j^{-2}) \\
\prod_{i=1}^{q} \theta_i^{-2(n+m-q)} (\theta_i^2 - 1)^{n+m-2q} (1 + \theta_i^2) \prod_{i<j}^{q} \theta_i^2 \theta_j^2 - 1.
\end{cases}
\]

Here \( \theta_i^2 = \text{ch}_i(\mathbf{V}^* \mathbf{V} \mathbf{B}^+) \), \( i = 1, \ldots, q \), where \( q \) denotes the rank of \( \mathbf{V} \mathbf{B}^+ \), \( q \leq \min(p, s) \) and \( \mathbf{B} = \Lambda^2 \).

**Proof.** Denote
\[
\mathbf{Z} = (\mathbf{V} \Lambda^+ - \mathbf{V}' \Lambda) \mathbf{E}^+ = \mathbf{U} \mathbf{E}^+,
\]
with \( \mathbf{U} = \mathbf{Y} - \mathbf{Y}'^+ \) and \( \mathbf{Y} = \mathbf{V} \Lambda^+ \). Then by Díaz-García (2007)
\[
(d\mathbf{Z}) = \prod_{i=1}^{s} \text{ch}_i(\mathbf{E}^+ \mathbf{E}_i^+)^{n/2} (d\mathbf{U}) = \frac{(d\mathbf{U})}{\prod_{i=1}^{s} \text{ch}_i(\mathbf{E})^{n}}.
\]

Similarly, consider \( \mathbf{B} = \Lambda^2 \), then
\[
(d\mathbf{Y}) = \prod_{i=1}^{s} \text{ch}_i(\mathbf{A}^+ \mathbf{A}^+)^{n/2} (d\mathbf{V}) = \prod_{i=1}^{s} \text{ch}_i((\mathbf{A}^2)^+)^{n/2} (d\mathbf{V}) = \frac{(d\mathbf{V})}{\prod_{i=1}^{s} \text{ch}_i(\mathbf{B})^{n/2}}.
\]

Now, substituting (8) and (16) in (15), the desired result is obtained, noting that \( \theta_i^2 = \text{ch}_i(\mathbf{Y}^* \mathbf{Y}) = \text{ch}_i(\mathbf{A}^+ \mathbf{V}' \mathbf{V} \Lambda^+) = \text{ch}_i(\mathbf{V}' \mathbf{V} (\mathbf{A}^+)^2) = \text{ch}_i(\mathbf{V}' \mathbf{V} \mathbf{B}^+) \) \( i = 1, \ldots, q \), where \( q \) is the rank of \( \mathbf{V} \mathbf{B}^+ \), \( q \leq \min(p, s) \). \( \square \)

### 3. Singular matrix variate generalized Birnbaum–Saunders distribution

As a first goal, the density of the matrix variate \( \mathbf{V} \) defined in (13) is obtained when \( \mathbf{Z} \) has a singular matrix variate elliptically contoured distribution. This distribution shall be termed **singular matrix variate generalized square root Birnbaum–Saunders distribution**. Then the main result is derived by finding the density of the **singular matrix variate generalized Birnbaum–Saunders distribution**. Finally, some special cases and basic properties of the singular matrix variate generalized Birnbaum–Saunders distribution can be derived.

Now, instead of making the change of variable (4), where \( Z \sim \mathcal{E}(0, 1; h) \), we can propose the transformation
\[
Y = \sqrt{\beta} \frac{V}{\sqrt{\beta} - \sqrt[3]{\mathbf{V}}}
\]
where \( Y \sim \mathcal{E}(0, \beta^2; h) \). Next, we extend this idea to the singular matrix variate case.

**Theorem 2.** Assume that \( Y \sim \mathcal{E}_{n \times m}^{m,s}(0_{n \times m}, I_n \otimes \mathbf{E}^2, h) \) and consider the following matrix transformation
where $\Xi^2 \in S^+_m(s)$ is the shape parameter matrix; $\Delta \in S^+_m(s)$, is the scale parameter matrix, such that $\Delta^2 = \beta$; and $V \in \mathcal{L}^n_{m,n}(p, p = \min(n,s)$. Then the density with respect to Hausdorff measure is

$$dF_V(V) = \frac{G(q, \theta^2)}{\prod_{j=1}^s \text{ch}_i(\Xi)^n \prod_{j=1}^q \text{ch}_j(\Theta)^{n/2}} 	imes h[\text{tr}\Xi^2 + (\Delta^+ V^T V \Delta^+ + \Delta (V^T V)^+ \Delta - 2 \Delta V^T V \Delta^+)](dV),$$

where $(dV)$ is defined in (6), $\Xi^2 = (\Xi^2)^+$ and

$$G(q, \theta^2) = \left\{ \begin{array}{l}
\prod_{i=1}^q (1 - \theta_i^{-2})^{n+m-2q} (1 + \theta_i^{-2})^{n+m-2q} \\
\prod_{i=1}^q \theta_i^{2(n+m-q)} (\theta_i^2 - 1) \prod_{i<j} \theta_i^2 \theta_j^2 - 1,
\end{array} \right. $$

$$\theta_i^2 = \text{ch}_i(V^T V \beta^+), i = 1, \ldots, q,$$ where $q$ denotes the rank of $V \Delta^+, q \leq \min(p,s)$.

**Proof.** By Definition 1, the density of $Y$ is

$$dF_Y(Y) = \frac{1}{\prod_{i=1}^s \text{ch}_i(\Xi)^n} h[\text{tr}\Xi^2 Y^T Y](dY).$$

Now, making the change of variable (17), with corresponding Jacobian established in Theorem 1, and observing that

$$\text{tr}\Xi^2 Y^T Y = \text{tr}\Xi^2 (V \Delta^+ - V^+ \Delta^+)^T (V \Delta^+ - V^+ \Delta^+)$$

$$= \text{tr}\Xi^2 (\Delta^+ V^T V \Delta^+ + \Delta (V^T V)^+ \Delta - 2 \Delta V^T V \Delta^+).$$

The density of $V$ is obtained. \hfill \Box

The density of a singular matrix variate generalized Birnbaum–Saunders distribution is obtained in the following result. This fact shall be denoted as $T \sim \text{GBS}^p_m(q, s, n, \Xi, \beta, h)$, where $\Xi \in S^+_m(s)$, is the shape parameter matrix, $\Delta \in S^+_m(s)$, such that $\beta$ is the scale parameter matrix where $\Delta^2 = \beta$. Also, note that $q$ denotes the rank of $V \Delta^+, q \leq \min(p,s)$, and $p = \min(n,s)$.

**Theorem 3.** Suppose that $T \sim \text{GBS}^p_m(q, s, n, \Xi, \beta, h)$, $T \in S^+_m(p, \Xi \in S^+_m(s)$, and $\beta \in S^+_m(s)$; $\beta = (\Delta)^2$ and $p = \min(n,s)$. Then the density of $T$ with respect to the Hausdorff measure is

$$dF_T(T) = \frac{\pi^{np/2} G(q, \delta)}{2^n \Gamma_p[n/2] \prod_{i=1}^q \text{ch}_i(\Xi)^n \prod_{j=1}^n \text{ch}_j(\Theta)^{n/2}} |\Delta|^{(n-m-1)/2}$$

$$\times h[\text{tr}\Xi^2 (\Delta^+ V^T \Delta^+ + \Delta T^+ \Delta - 2 \Delta T^+ \Delta^+)](dT),$$
where if \( T = V'V = Q_1\Lambda Q_1', Q_1 \in \mathcal{V}_{p,m} \) and \( \Lambda = \text{diag}(\lambda_1, ..., \lambda_p), \lambda_1 > \cdots > \lambda_p > 0 \), then (Díaz-García and Gutiérrez 1997)

\[
(dT) = 2^{-p} \prod_{i=1}^{p} \lambda_i^{-m-p} \prod_{i<j} (\lambda_i - \lambda_j) (Q_1' dQ_1) \prod_{i=1}^{p} d\lambda_i,
\]

and

\[
G(q, \delta) = \begin{cases} 
\prod_{i=1}^{q} (1 - \delta_i^{-1})^{n+m-2q} (1 + \delta_i^{-1}) \prod_{i<j} (1 - \delta_i^{-1} \delta_j^{-1}) \prod_{i=1}^{q} \delta_i^{-(n+m-q)} (\delta_i - 1)^{n+m-2q} (1 + \delta_i) \prod_{i<j} (\delta_i \delta_j - 1). \end{cases}
\]

Here \( \delta_i = \text{ch}_j(\beta^+ T), i = 1, ..., q \) is the rank of \( \beta^+ T, q \leq \min(p, s) \), and \( \Gamma_p[\cdot] \) denotes de multivariate gamma function, see Muirhead (2005, Definition 2.1.10, 61),

\[
\Gamma_p[a] = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma[a - (i - 1)/2], \text{Re}(a) > (p - 1)/2
\]

where \( \text{Re}(\cdot) \) denotes de real part of the argument.

**Proof.** The density of \( V \) is

\[
dF_V(V) = \frac{G(q, \theta^2)}{\prod_{i=1}^{s} \text{ch}_i(\Xi)^n \prod_{j=1}^{s} \text{ch}_j(\beta)^{n/2}}
\times h[\text{tr} \Xi^{2} (\Lambda^+ V'V \Lambda^+ + \Lambda(V'V)^+ \Lambda - 2\Lambda V^+ V \Lambda^+)](dV)
\]

where

\[
G(q, \theta^2) = \begin{cases} 
\prod_{i=1}^{q} (1 - \theta_i^{-2})^{n+m-2q} (1 + \theta_i^{-2}) \prod_{i<j} (1 - \theta_i^{-2} \theta_j^{-2}) \prod_{i=1}^{q} \theta_i^{2(n+m-q)} (\theta_i^2 - 1)^{n+m-2q} (1 + \theta_i^2) \prod_{i<j} (\theta_i^2 \theta_j^2 - 1), \end{cases}
\]

\( \theta_i^2 = \text{ch}_i(V'V \beta^+), i = 1, ..., q \), and \( q \) is the rank of \( \text{VA}^+, q \leq \min(p, s) \).

Define \( T = V'V \) with \( V \in L_{m,n}(p), V = H_1 \Lambda^{1/2} Q_1' \), where \( H_1 \in \mathcal{V}_{p,n} \) and \( Q_1 \in \mathcal{V}_{p,m} \) and \( \Lambda^{1/2} = \text{diag}(\lambda_1^{1/2}, ..., \lambda_p^{1/2}), \lambda_1^{1/2} > \cdots > \lambda_p^{1/2} > 0 \). Then \( T = V'V = Q_1 \Lambda Q_1' \). Note that in the singular value factorization under consideration, \( V = H_1 \Lambda^{1/2} Q_1' \), the matrices \( H_1 \) and \( Q_1 \) are defined in Mathai (1997, 115), see Theorem 2.12. Then by Díaz-García and González-Farías (2005a)

\[
(dV) = 2^{-p} |\Lambda|^{(n-m-1)/2} (dT)(H_1' dH_1).
\]
In particular, from Mathai (1997, 117),

\[
\int_{H_1 \in \mathcal{V}_{p,n}} (H_1'dH_1) = \frac{\pi^{p/2}}{\Gamma_p(n/2)}.
\]

(19)

In addition note that

\[
T^+T = (V'V)^+(V'V) = V^+V'V = V^+(VV')V = V^VV
\]

Hence, the joint density function of \(T\) and \(H_1\) is

\[
dF_{T,H_1}(T,H_1) = \frac{G(q,\delta)}{2^p \prod_{i=1}^s ch_i(\Xi)^{n_i} \prod_{j=1}^s ch_j(\beta)^{n_j/2}} \times h[\text{tr} \Xi^{n/2} (\Lambda^t T \Lambda^+ + \Lambda T^+ \Lambda - 2 \Lambda T^+ \Lambda^t)](dT)(H_1'dH_1),
\]

where

\[
G(q,\delta) = \begin{cases} 
\prod_{i=1}^q (1 - \delta_i^{-1})^{n + m - 2q} (1 + \delta_i^{-1}) \prod_{i<j}^q (1 - \delta_i^{-1} \delta_j^{-1}) \\
\prod_{i=1}^q \delta_i^{-(n+m-q)} (\delta_i - 1)^{n + m - 2q} (1 + \delta_i) \prod_{i<j}^q (\delta_i \delta_j - 1).
\end{cases}
\]

where \(\delta_i = ch_i(\beta^t T), i = 1, \ldots, q, q = \text{rank of } \beta^t T, q \leq \text{min}(p,s).\)

Integration with respect to \(H_1\), by using (19), provides the stated marginal density function for \(T\).

Three cases of particular interest are given next:

1. Nonsingular case. In this case \(q = p = s = m\).
2. The classic case, which is obtained assuming that \(Y\) as a singular matrix variate normal distribution in Theorem 1. Therefore, from Theorem 3 we obtain the singular matrix variate Birnbaum–Saunders distribution, which shall be denoted as \(T \sim BS_m^p(q, s, n, \Xi, \beta)\).
3. When \(\beta = \beta I_m, \beta > 0\), that is, when \(T \sim GBS_m^p(q, s, n, \Xi, \beta I_m, h)\).

The results are summarized in following Corollaries.

**Corollary 1.** Let \(T \sim GBS_m(n, \Xi, \beta, h), T \in S_m, \Xi \in S_m\), and \(\beta \in S_m; \beta = (\Lambda)^2\). Then the density of \(T\) with respect to the Lebesgue measure on \(S_m\) is

\[
dF_T(T) = \frac{\pi^{nm/2} G(m,\delta)}{2^p \Gamma_m(n/2) |\Xi|^n |\beta|^{n/2}} |T|^{(n-m-1)/2} \times h[\text{tr} \Xi^{-2} (\Lambda^{-1} T \Lambda^{-1} + \Lambda T^{-1} \Lambda - 2 I_m)](dT),
\]

where
\[
G(m, \delta) = \begin{cases}
\prod_{i=1}^{m} (1 - \delta_i^{-1})^{n-m} (1 + \delta_i^{-1}) \prod_{i<j}^{m} (1 - \delta_i^{-1} \delta_j^{-1}) \\
\prod_{i=1}^{m} \delta_i^{-n}(\delta_i - 1)^{n-m} (1 + \delta_i) \prod_{i<j}^{m} (\delta_i \delta_j - 1),
\end{cases}
\]

and \( \delta_i = \text{ch}_i(\beta^- T), i = 1, \ldots, m. \)

**Proof.** The result follows by taking \( q = p = s = m \) in Theorem 3 and noting that 2AT^+T^+ = 2AT^-T^- = 2I_m. \( \square \)

**Corollary 2.** Suppose that \( T \sim \mathcal{BS}_m^p(q, s, n, \Xi, \beta), T \in S_m^+(p), \Xi \in S_m^+(s), \) and \( \beta \in S_m^+(s); \ \beta = (\Lambda)^2 \) and \( p = \min(n, s) \). Then the density of \( T \) with respect to the Hausdorff measure is

\[
dF_T(T) = \frac{\pi^{n(p-s)/2} G(q, \delta)}{2^{p+ns/2} \Gamma_p[n/2] \prod_{i=1}^s \text{ch}_i(\Xi)^n \prod_{j=1}^s \text{ch}_j(\beta)^{n/2}} |\Lambda|^{(n-m-1)/2} \times \text{etr} \left[ -\frac{1}{2} \Xi^{-2} (\Lambda^+ T \Lambda^+ + \Lambda T^+ \Lambda - 2 \Lambda T^+ T \Lambda^+) \right] (dT),
\]

where the element of volume \((dT)\) is defined in (18),

\[
G(q, \delta) = \begin{cases}
\prod_{i=1}^{q} (1 - \delta_i^{-1})^{n+m-2q} (1 + \delta_i^{-1}) \prod_{i<j}^{q} (1 - \delta_i^{-1} \delta_j^{-1}) \\
\prod_{i=1}^{q} \delta_i^{-(n+m-q)} (\delta_i - 1)^{n+m-2q} (1 + \delta_i) \prod_{i<j}^{q} (\delta_i \delta_j - 1),
\end{cases}
\]

and \( \delta_i = \text{ch}_i(\beta^+ T), i = 1, \ldots, q, \) \( q \) denotes the rank of \( \beta^+ T, q \leq \min(p, s), \) and \( \text{etr}(\cdot) \equiv \exp (\text{tr}(\cdot)) \).

**Proof.** Defining \( h(z) = (2\pi)^{-ns/2} \exp (-z/2) \), the proof follows straightforwardly from Theorem 3. \( \square \)

In following case, note that \( \beta = \Lambda^2 = \beta I_m \), hence \( \Lambda = \sqrt{\beta} I_m \).

**Corollary 3.** If \( T \sim \mathcal{GBS}_m^p(q, s, n, \Xi, \beta I_m, h), \) with \( p = \min(n, s), T \in S_m^+(p), \Xi \in S_m^+(s), \) and \( \beta > 0 \). Then the density of \( T \) with respect to the Hausdorff measure is

\[
dF_T(T) = \frac{\pi^{np/2} G(p, \delta)}{2^p \Gamma_p[n/2] \beta^{mn/2} \prod_{i=1}^s \text{ch}_i(\Xi)^n} |\Lambda|^{(n-m-1)/2} \times h \left[ \text{tr} \Xi^{-2} \left( \frac{1}{\beta} T + \beta T^+ - 2 T^+ T \right) \right] (dT),
\]

where the Hausdorff measure \((dT)\) is explicitly defined in (18) and
The result follows straightforwardly from Theorem 2.

Proof. The result follows straightforwardly from Theorem 2.

Now, two basic properties of the singular matrix variate generalized Birnbaum–Saunders distribution are summarized in the follow result.

Theorem 4. Assume that \( T \sim \mathcal{GBS}_m^\rho(q, s, n, \Xi, \beta, h) \), then

i. Consider the singular matrix transformation \( S = T^+, T, S \in \mathcal{S}_m^\rho \). Then the density of \( S \) with respect to Hausdorff is given by

\[
dF_S(S) = \frac{\pi^{n/2} G(q, \psi)}{2^p \Gamma_p[n/2] \prod_{i=1}^s \chi_i(\Xi)^n \prod_{j=1}^s \chi_j(\beta)^{n/2}} \times h[\text{tr}\Xi^{2}(\Delta^+ S^+ \Delta^+ + \Delta S \Delta - 2\Delta SS^+ \Delta^+)](dS),
\]

where the measure \( (dS) \) is given by (18), \( S = Q_1 \Omega Q_1^T \), \( Q_1 \in \mathcal{V}_{p,m} \), \( \Omega = \text{diag}(\omega_1, \ldots, \omega_p) \), \( \omega_1 > \cdots > \omega_p > 0 \),

\[
G(q, \psi) = \begin{cases}
\prod_{i=1}^q (1 - \psi_i^{-1})^{n+m-2q} (1 + \psi_i^{-1}) \prod_{i<j} (1 - \psi_i^{-1} \psi_j^{-1}) \\
\prod_{i=1}^q \psi_i^{(n+m-q)} (\psi_i - 1)^{n+m-2q} (1 + \psi_i) \prod_{i<j} (\psi_i \psi_j - 1),
\end{cases}
\]

and \( \psi_i = \chi_i(\beta^+ S^+) \), \( i = 1, \ldots, q \), \( q \) denotes the rank of \( \beta^+ S^+ \), \( q \leq \min(p, s) \).

ii. Let \( Y, T \in \mathcal{S}_m^\rho(n) \) and define \( Y = C^T TC \), \( C \in \mathcal{L}_{m,m} \). Then the density of \( Y \) with respect to Hausdorff measure is

\[
dF_Y(T) = \frac{\pi^{n/2} G(q, \eta)}{2^n \Gamma_n[n/2] \prod_{i=1}^s \chi_i(\Xi)^n \prod_{j=1}^s \chi_j(\beta)^{n/2} |C|^n} \times h[\text{tr}\Xi^{2}(\Delta C^+ Y \Delta C^+ + \Delta CY^+ (\Delta C)^' - 2\Delta CY^+ Y (\Delta C)^+)](dY),
\]

where the element of volume \( (dY) \) is defined in (18), \( Y = P_1 \Lambda P_1' \), \( P_1 \in \mathcal{V}_{n,m} \), \( \Lambda = \text{diag}(l_1, \ldots, l_n) \), \( l_1 > \cdots > l_n > 0 \).
Proof.

i. Let $S = T^+$, then by Díaz-García and Gutiérrez (2006a) we have that:

$$(dT) = |\Omega|^{-(2m+p-1)}(dS),$$

where $S = Q_1 \Omega Q_1^t, Q_1 \in \mathcal{V}_{p,m}, \Omega = \text{diag}(\omega_1, \ldots, \omega_p), \omega_1 > \cdots > \omega_p > 0$. Given that $T = Q_1 \Lambda Q_1^t$, we obtain that $\Lambda = \Omega^{-1}$; and then $|\Lambda|^{(n-m-1)/2} = |\Omega|^{-(n-m-1)/2}$. In addition, by Theorem 3, we have that $\delta_i = \text{ch}_i(\beta^T Y)$, then under the change of variables $S = T^+$, we define $\psi_i = \text{ch}_i(\beta^T S^+), i = 1, \ldots, q$. Substituting these results into the density of $T$ in Theorem 3, the desired result is obtained.

ii. Define $Y = C'TC$, where $C$ is nonsingular (according to the hypothesis, $C \in \mathcal{L}_{m,m}(m)$); then by Díaz-García and Gutiérrez (1997)

$$(dT) = |A|^{-(m+1-n)/2}|C|^{-(m+1-n)/2}(dY),$$

where $T = Q_1 \Lambda Q_1^t, Q_1 \in \mathcal{V}_{n,m}, \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n), \lambda_1 > \cdots > \lambda_n > 0$ and $Y = P_1 \Lambda P_1^t, P_1 \in \mathcal{V}_{n,m}, L = \text{diag}(l_1, \ldots, l_n), l_1 > \cdots > l_n > 0$. Moreover, note that $T = C^{-1} Y C^{-1}$, thus $T^+ = C Y^+ C'$. Finally, defining $\eta_i = \text{ch}_i((C \beta C')^+ Y), i = 1, \ldots, q, q \leq \min(n, s)$ and substituting these results into the density of $T$, defined in Theorem 3, the density of $Y$ is achieved.

Finally, the reader can apply this work by using particular kernels of the most common elliptical distributions, such as Kotz, Pearson VII, and II type distributions. The distributions just follow by changing $h(\cdot)$ with the corresponding model in a similar way of the proof given for Corollary 2.

4. Conclusions

A version of the singular matrix variate generalized Birnbaum–Saunders distribution with respect to Hausdorff measure has been proposed in this article. Several basic properties and particular cases of this distribution were also studied. The work has contributed to the unification of the matrix variate Birnbaum–Saunders in a general setting that includes families of singular and nonsingular elliptical contoured distributions. During the analysis, the associated Jacobian theory has been established and solving the problems of conceptualization of the literature about different pairs of densities and measures. Finally, the work is easily extended to real normed division algebras, which is valid for complex, quaternions and some particular octonions cases.
Acknowledgements

The authors wish to thank the Editor and the anonymous reviewers for their constructive comments on the preliminary version of this article.

ORCID

José A. Díaz-García http://orcid.org/0000-0001-5406-8789

References

Balakrishnan, N., and D. Kundu. 2018. Birnbaum-Saunders distribution: A review of model, analysis and applications (with discussion). Appl. Stoch. Models Bus. Ind. (To appear).
Billingsley, P. 1986. Probability and measure, 2nd ed. New York: John Wiley & Sons.
Birnbaum, Z. W., and S. C. Saunders. 1969. A new family of life distributions. Journal of Applied Probability 6:637–52.
Bodnar, T., and Y. Okhrin. 2008. Properties of the singular, inverse and generalized inverse partitioned Wishart distributions. Journal of Multivariate Analysis 99 (10):2389–405. doi:10.1016/j.jmva.2008.02.024.
Caro-Lopera, F. J., V. Leiva, and N. Balakrishnan. 2012. Connection between the Hadamard and matrix products with an application to matrix-variate Birnbaum-Saunders distributions. Journal of Multivariate Analysis 104 (1):126–39. doi:10.1016/j.jmva.2011.07.004.
Caro-Lopera, F. J., and J. A. Díaz-García. 2016. Diagonalization matrix and its application in distribution theory. Statistics 50 (4):870–80. doi:10.1080/02331888.2015.1104312.
Desmond, A. 1985. Stochastic models of failure in random environments. Canadian Journal of Statistics 13 (3):171–83. doi:10.2307/3315148.
Díaz-García, J. A. 2007. A note about measures and Jacobians of singular random matrices. Journal of Multivariate Analysis 98 (5):960–69. doi:10.1016/j.jmva.2005.09.013.
Díaz-García, J. A., and J. R. Caro-Lopera. 2021. Matrix variate Birnbaum-Saunders distribution under elliptical models. Journal of Statistical Planning and Inference 210:100–13. doi:10.1016/j.jspi.2020.04.012.
Díaz-García, J. A., and F. J. Caro-Lopera. 2018. Discussion of “Birnbaum-Saunders distribution: A review of models, analysis, and applications” by N. Balakrishnan and D. Kundu. Applied Stochastic Models in Business and Industry 35 (1):104–109.
Díaz-García, J. A., and J. R. Domínguez Molina. 2006. Some generalisations of Birnbaum-Saunders and Sinh-normal distributions. International Mathematical Forum 1 (35):1709–27.
Díaz-García, J. A., and J. R. Domínguez Molina. 2007. A new family of life distributions for dependent data: Estimation. Computational Statistics & Data Analysis 51 (12):5927–39. doi:10.1016/j.csda.2006.11.010.
Díaz-García, J. A., and G. González-Farías. 2005a. Singular random matrix decompositions: Jacobians. Journal of Multivariate Analysis 93 (2):296–312.
Díaz-García, J. A., and G. González-Farías. 2005b. Singular random matrix decompositions: Distributions. Journal of Multivariate Analysis 94 (1):109–22. doi:10.1016/j.jmva.2004.08.003.
Díaz-García, J. A., and J. R. Gutiérrez. 1997. Proof of the conjectures of H. Uhlig on the singular multivariate beta and Jacobian of a certain matrix transformation. The Annals of Statistics 25 (5):2018–23. doi:10.1214/aos/1069362383.
Díaz-García, J. A., R. G. Jáimez, and K. V. Mardia. 1997. Wishart and pseudo-Wishart distributions and some applications to shape theory. Journal of Multivariate Analysis 63 (1):73–87. doi:10.1006/jmva.1997.1689.
Díaz-García, J. A., and J. R. Gutiérrez. 2005. Functions of singular random matrices and its applications. TEST 14 (2):475–87. doi:10.1007/BF02595414.
Díaz-García, J. A., and J. R. Gutiérrez. 2006a. Distribution of the generalised inverse of a random matrix and its applications. *Journal of Statistical Planning and Inference* 136 (1):183–92. doi:10.1016/j.jspi.2004.06.032.

Díaz-García, J. A., and J. R. Gutiérrez. 2006b. Wishart and Pseudo-Wishart distributions under elliptical laws and related distributions in the shape theory context. *Journal of Statistical Planning and Inference* 136 (12):4176–93. doi:10.1016/j.jspi.2005.08.045.

Díaz-García, J. A., and V. Leiva-Sánchez. 2005. A new family of life distributions based on elliptically contoured distributions. *Journal of Statistical Planning and Inference* 128 (2):445–57. doi:10.1016/j.jspi.2003.11.007.

Díaz-García, J. A., and V. Leiva-Sánchez. 2006. Erratum to “A new family of life distributions based on the elliptically contoured distributions.” [J. Statist. Plann. Inference 128(2) (2005) 445-457]. *Journal of Statistical Planning and Inference* 137 (4):1512–1513. doi:10.1016/j.jspi.2003.11.007.

Khatri, C. G. 1968. Some results for the singular normal multivariate regression models. *Sankhyā A* 30:267–80.

Mathai, A. M. 1997. Jacobian of matrix transformations and functions of matrix argument, World Scientific, Singapore.

Muirhead, R. J. 2005. *Aspects of multivariate statistical theory*. New York: John Wiley & Sons.

Rao, C. R. 2005. *Linear statistical inference and its applications*. 2nd ed. New York: John Wiley & Sons.

Uhlig, H. 1994. On singular Wishart and singular multivariate beta distributions. *The Annals of Statistics* 22 (1):395–405. doi:10.1214/aos/1176325375.

Sánchez, L., V. Leiva, F. Caro-Lopera, and F. J. Cysneiros. 2015. On matrix-variate Birnbaum-Saunders distributions and their estimation and application. *Brazilian Journal of Probability and Statistics* 29 (4):790–812.

Zhang, Z. 2007. Pseudo-inverse multivariate/matrix-variate distributions. *Journal of Multivariate Analysis* 98 (8):1684–92. doi:10.1016/j.jmva.2006.04.002.