Abstract

For simple enough spatial topologies, at least four approaches to (2 + 1)-dimensional quantum gravity have been proposed: Wheeler-DeWitt quantization, canonical quantization in Arnowitt-Deser-Misner (ADM) variables on reduced phase space, Chern-Simons quantization, and quantization in terms of Ashtekar-Rovelli-Smolin loop variables. An important problem is to understand the relationships among these approaches. By explicitly constructing the transformation between the Chern-Simons and ADM Hilbert spaces, we show here that Chern-Simons quantization naturally gives rise to spinorial wave functions on superspace, whose time evolution is governed by a Dirac equation. Chern-Simons quantum gravity can therefore be interpreted as the Dirac square root of the Wheeler-DeWitt equation.
Over the past several years, it has become apparent that (2 + 1)-dimensional Einstein gravity can serve as a useful model for studying some of the conceptual problems of quantum gravity in four spacetime dimensions. While attempts to quantize four-dimensional gravity have thus far failed, several approaches have been worked out successfully, in various degrees of detail, in three dimensions. These include Wheeler-DeWitt quantization [1, 2, 3], quantization on reduced phase space using Arnowitt-Deser-Misner (ADM) variables and the York time slicing [4, 5], and quantization in terms of Chern-Simons variables and ISO(2,1) holonomies [6, 7] or the closely related Ashtekar-Rovelli-Smolin loop variables [8]. Three-dimensional gravity is not a terribly realistic model — among other shortcomings, it contains only finitely many degrees of freedom — but many of the basic issues of quantum gravity do not depend on the dimension of spacetime, and even a simple model may offer valuable insights.

The purpose of this paper is to relate the Chern-Simons quantization of (2+1)-dimensional gravity to the quantum theory arising from the Wheeler-DeWitt equation. This work is in part a continuation of reference [9]. In that paper, a start was made towards comparing the fundamental operators in the Chern-Simons theory to those of the ADM and Wheeler-DeWitt quantizations. Here, that comparison is extended to the Hilbert spaces and wave functions.

For spacetimes whose spatial topology is that of a torus $T^2$, this comparison can be made completely explicitly, and, as we shall see, the exact transformation between the two Hilbert spaces can be worked out. It is well known that the Wheeler-DeWitt equation is a Klein-Gordon-like equation, and that models based on the true physical degrees of freedom should be, in some sense, “square roots.” We shall see below that the ISO(2,1) quantum theory is a Dirac square root, with wave functions that are most naturally understood as spinors on superspace.

1. Moduli and Holonomies

We begin with a brief review of reference [9]. Let $M$ be a spacetime with the topology $[0, 1] \times \Sigma$, foliated by surfaces $\Sigma_\tau$ of constant mean extrinsic curvature $\tau$. The spatial metric $g_{ij}$ on a slice $\Sigma_\tau$ is conformal to one of constant intrinsic curvature (+1 for spherical topology, 0 for the torus, −1 for higher genus):
extrinsic curvature.) The modulus can be thought of as determining a parallelogram with vertices \( \{0, 1, m, m + 1\} \); a flat torus is obtained by identifying opposite sides. Not all values of \( m \) correspond to distinct geometries, however. The mapping class group \( \mathcal{D} \) — the group of diffeomorphisms of \( \Sigma \) not homotopic to the identity — acts on the upper half-plane \( U \) with generators

\[
S : m \to -\frac{1}{m}, \quad T : m \to m + 1.
\]  

(1.2)

The true moduli space is the quotient space \( M_{\mathbb{T}^2} = U/\mathcal{D} \), and the physical phase space, obtained by adjoining the conjugate momenta \( p \), is the cotangent bundle \( T^*M \).

The Einstein action can now be rewritten in terms of \( m \) and \( p \). As Moncrief and Hosoya and Nakao show, we obtain

\[
S = \int d\tau \left( p_\alpha \frac{\partial m^\alpha}{\partial \tau} - H \right),
\]  

with the Hamiltonian

\[
H = \tau^{-1} \left( h^{\alpha\beta} p_\alpha p_\beta \right)^{1/2},
\]  

(1.3)

(1.4)

where \( h_{\alpha\beta} \) is the constant negative curvature metric on \( U \), \( h_{11} = h_{22} = m_2^{-2} \). This action and Hamiltonian are the starting points for ADM quantization.

In the Wheeler-DeWitt approach, rather than solving the Hamiltonian constraint for \( \lambda \), we impose it as a constraint on the physical state space. We therefore need a new variable corresponding to \( e^{2\lambda} \), which can be regarded as being conjugate to a “time” variable \( T \). As Hosoya and Nakao [2] have shown (see also [3]), there is a gauge choice in which only the integral

\[
v = \int_\Sigma d^2 x \sqrt{\bar{g}} e^{2\lambda}
\]  

(1.5)

is relevant. For general topologies, this gauge involves a choice of time \( T \) that differs from the mean extrinsic curvature. For the torus, however, \( T = \tau \), and the Hamiltonian constraint takes the form

\[
\mathcal{H} = h^{\alpha\beta} p_\alpha p_\beta - v^2 \tau^2 = 0,
\]  

(1.6)

which must be imposed as an operator equation on the states. Note that \( \mathcal{H} \) is essentially the square of the Hamiltonian \( (1.4) \), a first indication of the square root structure of the quantum theory.

In the ISO(2,1) holonomy approach, we begin instead with the first order form of the Einstein action. As Witten has observed [3], the triad \( e^a_\mu \) and the spin connection \( \omega^a_{\mu b} \) together constitute an ISO(2,1) connection on \( M \). In terms of this connection, the Einstein action is the standard Chern-Simons action, and the field equations become the condition that the connection be flat.

Now, a flat connection is determined by its holonomies (the values of the Wilson loops), and the identity component of the group of diffeomorphisms of \( M \) acts on these holonomies by conjugation. A solution of the field equations is therefore characterized by a group of ISO(2,1) holonomies — i.e., a representation of \( \pi_1(M) \) in ISO(2,1) — modulo conjugation. Not all such holonomy groups can occur, however; it may be shown that the SO(2,1) projection of
any holonomy group arising from a solution of the Einstein equations must be Fuchsian. This condition picks out a connected component in the space of holonomy groups, and Mess \[11\] has shown that any group lying in this component corresponds to a solution of the field equations. (See also \[12\] for a description geared more towards physicists, and \[13\] for a closely related lattice approach).

The relationship between spacetimes and ISO(2,1) holonomies seems rather abstract, but in fact one can construct a spacetime fairly explicitly from a holonomy group. Any subgroup \(G\) of ISO(2,1) acts on Minkowski space \(V^{2,1}\) as a group of isometries. If the SO(2,1) projection of \(G\) is Fuchsian, this action is properly discontinuous in some region \(F\) of \(V^{2,1}\). The quotient \(F/G\) is then a flat spacetime with topology \([0,1] \times \Sigma\), and it (or some suitable maximal extension) is the desired spacetime.

The space of ISO(2,1) holonomies has the structure of a cotangent bundle, with a base space consisting of the SO(2,1) projections. To obtain the set of physically distinct points in the phase space, we must again divide out the mapping class group, which has an action on the holonomies induced by its action on \(\pi_1(\Sigma)\). It is not hard to show that in this representation, the Hamiltonian is identically zero, a result that follows essentially from the manifest diffeomorphism invariance of the holonomies \[14\].

For \(\Sigma = T^2\), \(\pi_1(M)\) is the Abelian group \(\mathbb{Z} \oplus \mathbb{Z}\), so a point in the physical phase space corresponds to a choice of two commuting Poincaré transformations. In the relevant topological component of the space of holonomy groups, the corresponding Lorentz transformations are parallel boosts, which by overall conjugation can be taken to be in the \(x\) direction. The holonomies are then

\[
\begin{align*}
\Lambda_1 & : (t, x, y) \rightarrow (t \cosh \lambda + x \sinh \lambda, x \cosh \lambda + t \sinh \lambda, y + a) \\
\Lambda_2 & : (t, x, y) \rightarrow (t \cosh \mu + x \sinh \mu, x \cosh \mu + t \sinh \mu, y + b) .
\end{align*}
\]

The phase space is parameterized two pairs of canonically conjugate variables \((a, \mu)\) and \((b, -\lambda)\), and it is not hard to show that the action of the mapping class group corresponding to \(1.2\) is

\[
\begin{align*}
S & : (a, \lambda) \rightarrow (b, \mu), \quad (b, \mu) \rightarrow (-a, -\lambda) \\
T & : (a, \lambda) \rightarrow (a, \lambda), \quad (b, \mu) \rightarrow (b + a, \mu + \lambda) .
\end{align*}
\]

The action of the group generated by \(\Lambda_1\) and \(\Lambda_2\) on Minkowski space is worked out in detail in \[9\]. It is shown that the quotient \(F/\langle \Lambda_1, \Lambda_2 \rangle\) is indeed a spacetime with the topology \([0,1] \times T^2\), and that the corresponding moduli and conjugate momenta of the slice \(\Sigma_\tau\) of mean extrinsic curvature \(\tau\) are

\[
\begin{align*}
m & = \left( a + \frac{i\lambda}{\tau} \right)^{-1} \left( b + \frac{i\mu}{\tau} \right) , \\
p & = -i\tau \left( a - \frac{i\lambda}{\tau} \right)^2 .
\end{align*}
\]

(1.7)
In terms of the holonomies, the Hamiltonian (1.4) is

\[ H = \frac{a\mu - \lambda b}{\tau}, \]

and it is not hard to check that the moduli and momenta (1.9) and (1.10) obey Hamilton’s equations of motion.

The relations (1.9)–(1.11) establish the classical equivalence of the ADM and Chern-Simons formulations of (2 + 1)-dimensional gravity on \( M = [0, 1] \times T^2 \). We now turn to the quantum theories.

2. Operators and Hilbert Spaces

From now on, let us fix \( M \) to have the topology \([0, 1] \times T^2\). We begin with the simplest means of quantization, the Wheeler-DeWitt method. In this approach, we impose the Hamiltonian constraint (1.6) on wave functions \( \psi(m, \bar{m}, \tau) \), with the operator substitutions

\[ p_\alpha = -i \frac{\partial}{\partial m^\alpha}, \quad v = -i \frac{\partial}{\partial \tau}. \]  

With the standard operator ordering, the resulting Wheeler-DeWitt equation is

\[ \mathcal{H} \psi(m, \bar{m}, \tau) = \left( \tau \frac{\partial}{\partial \tau} \right)^2 + \Delta_0 \psi(m, \bar{m}, \tau) = 0, \]

where

\[ \Delta_0 = -m_2^2 \left( \partial_1^2 + \partial_2^2 \right). \]

is the scalar Laplacian on the upper half-plane with respect to the Poincaré metric. We should further require that our wave functions \( \psi \) be invariant under the modular transformations (1.2) — that is, that they be automorphic forms of weight zero — and that they be square-integrable over a fundamental region for the modular group. The Laplacian restricted to this domain is nonnegative, symmetric, and has a unique self-adjoint extension. Its spectrum has been studied by mathematicians, and a fair amount is known about its eigenfunctions, which are known as the weight zero Maass forms [15, 16].

The connection between Wheeler-DeWitt quantization and reduced phase space ADM quantization is straightforward. If we start with the action (1.3), wave functions \( \psi(m, \bar{m}, \tau) \) will again be square-integrable automorphic forms of weight zero, with a time evolution now governed by the Hamiltonian (1.4), i.e.,

\[ \hat{H} = \tau^{-1} \Delta_0^{1/2}. \]

The (positive) square root is defined by the spectral decomposition of \( \Delta_0 \). The Schrödinger equation in this approach is simply

\[ i \frac{\partial \psi}{\partial \tau}(m, \bar{m}, \tau) = \hat{H} \psi(m, \bar{m}, \tau), \]
and any solution of (2.5) will clearly also obey the Wheeler-DeWitt equation. The ADM theory can thus be viewed as the ordinary square root of the Wheeler-DeWitt equation.

The relationship between these quantizations and the Chern-Simons formulation is more subtle, since the fundamental canonical variables are different. Our strategy will be to construct operators representing the moduli $m(\tau)$ and their conjugate momenta from the holonomies, and to reexpress the wave functions in terms of eigenfunctions of these operators. Note that from this point of view, $\tau$ is simply a parameter labeling families of operators; its physical significance as mean extrinsic curvature comes from the correspondence principle.

The canonical variables in the Chern-Simons formulation on the manifold $[0, 1] \times T^2$ are $\{a, b, \lambda, \mu\}$. The cotangent bundle structure on the space of flat ISO(2,1) connections induces a corresponding bundle structure on this parameter space, with $\mu$ and $\lambda$ serving as coordinates on the base space (see, for example, [17]). We quantize the system by making the substitutions

$$a = \frac{i}{2} \frac{\partial}{\partial \mu}, \quad b = -\frac{i}{2} \frac{\partial}{\partial \lambda}. \quad (2.6)$$

Wave functions will now be functions $\chi(\mu, \lambda)$, invariant under the modular transformations (1.8) and square-integrable over a suitable fundamental region. The existence of such invariant functions is not obvious, but they will be explicitly constructed below.

Our first task is to construct operators to represent the moduli $m(\tau)$. The classical correspondence (1.9) allows us to do so, up to questions of operator ordering. The appropriate ordering is largely fixed by mapping class group invariance: we must require that the transformations (1.8) of the holonomies induce the transformations (1.2) of the moduli. The ordering in (1.9) has been carefully chosen so that this is the case. I do not know whether this choice (and, trivially, its transpose) are unique in this respect, but if other such orderings exist, it is easy to see that they must be quite complicated, and in particular non-rational.

We therefore define moduli and momentum operators $\hat{m}$ and $\hat{p}$ by replacing the holonomy variables in (1.9)–(1.10) with the corresponding operators. It is easy to check that $\hat{m}$ and $\hat{p}$ obey the correct canonical commutation relations. Note, however, that the momentum operators now transform in a complicated way under the mapping class group:

$$S : \hat{p} \rightarrow \hat{m}^2 \hat{p} - i \hat{m}, \quad \hat{p}^\dagger \rightarrow \hat{m}^2 \hat{p}^\dagger - 3i \hat{m}. \quad (2.7)$$

It is therefore inconsistent to represent these momenta as derivatives with respect to the moduli.

Indeed, $\hat{p}$ and $\hat{p}^\dagger$ do not even transform as adjoints. This, at least, can be cured by forming the combinations $\tilde{p} = \hat{p} + \hat{m}^{-1}$ and $\tilde{p}^\dagger = \hat{p}^\dagger - \hat{m}^{-1}$, which still obey the correct canonical commutation relations, and which transform as

$$S : \tilde{p} \rightarrow \tilde{m}^2 \tilde{p} + i \tilde{m}, \quad \tilde{p}^\dagger \rightarrow \tilde{m}^2 \tilde{p}^\dagger - i \tilde{m}. \quad (2.8)$$

The requirement that $\tilde{p}$ and $\tilde{p}^\dagger$ be adjoints then determines the inner product

$$< \tilde{\chi}_1 | \tilde{\chi}_2 > = \int \frac{d^2 m}{m_2} \tilde{\chi}_1(m, \tilde{m})\tilde{\chi}_2(m, \tilde{m}), \quad (2.9)$$
which is the expected Petersson inner product induced from the Poincaré metric on the upper half-plane.

We can now represent $\hat{p}$ and $\hat{p}^\dagger$ as derivatives, or equivalently set

$$\hat{p} = -2i \frac{\partial}{\partial m} - \frac{1}{m_2}, \quad \hat{p}^\dagger = -2i \frac{\partial}{\partial \bar{m}} + \frac{1}{m_2},$$

(2.10)

provided that these operators act on wave functions that transform nontrivially under the mapping class group,

$$S: \tilde{\chi} \to \left(\frac{m}{\bar{m}}\right)^{1/2} \tilde{\chi}.$$  (2.11)

The momenta can then be viewed as covariant derivatives; the phase in (2.11) accounts for the inhomogeneous terms in the transformation (2.7) and (2.8). Strictly speaking, to be consistent with the relations among $S$ and $T$ viewed as elements of $\text{SL}(2,\mathbb{Z})$ — i.e., $S^2 = (S T)^3 = -I$ — we must include additional constant phases, and require that

$$S: \tilde{\chi} \to e^{-\frac{i\pi}{4} \left(\frac{m}{\bar{m}}\right)^{1/2}} \tilde{\chi}$$

$$T: \tilde{\chi} \to e^{\frac{i\pi}{4} \tilde{\chi}}.$$  (2.12)

This is precisely the requirement that the wave functions $\tilde{\chi}$ transform as automorphic forms of weight $1/2$, that is, spinors on moduli space $\mathbb{H}$.  

To check this behavior, we can examine the Hamiltonian (1.11). It is easy to show that

$$\hat{H} = \tau^{-1} \left( m_2 (\hat{p}^\dagger \hat{p})^{1/2} - \frac{1}{2} (\hat{p}^\dagger \hat{p})^{-1/2} \hat{p} \right),$$

(2.13)

or, inserting (2.10),

$$\hat{H}^2 = \tau^{-2} \left( \Delta_0 + i m_2 \partial_1 - \frac{1}{4} \right) = \tau^{-2} \Delta_{\frac{1}{2}},$$

(2.14)

where $\Delta_{\frac{1}{2}}$ is the Maass Laplacian $[16]$ for automorphic forms of weight $1/2$. The holonomy formulation of $(2+1)$-dimensional quantum gravity thus naturally gives rise to wave functions that behave as spinors.

### 3. Changing Representations

The results of the previous section can be confirmed by another approach. Inserting the operator representations (2.6) directly into the expression (1.9) for the moduli, we can look for the simultaneous eigenfunctions of $\hat{m}$ and $\hat{m}^\dagger$. The eigenfunction with eigenvalues $m$ and $\bar{m}$ can be shown to be

$$K(m, \bar{m}; \lambda, \mu, \tau) = \frac{\mu - m \lambda}{\pi m_2^{1/2} \tau} \exp \left\{ -\frac{i}{m_2 \tau} |\mu - m \lambda|^2 \right\}.$$  (3.1)
The $\mu$ and $\lambda$ dependence of $K$ is determined by the eigenvalue equations, while the prefactor is fixed by the normalization

$$
\int d\mu d\lambda K(m', \bar{m}'; \mu, \lambda, \tau) K(m, \bar{m}; \mu, \lambda, \tau) = m_2^2 \delta^2(m - m').
$$

(3.2)

(The factor of $m_2^2$ on the right hand side of this expression converts the ordinary delta function into a delta function with respect to the Petersson inner product.)

An arbitrary wave function can now be expanded in terms of $K$,

$$
\chi(\mu, \lambda) = \int \frac{d^2m}{m_2^2} K(m, \bar{m}; \lambda, \mu, \tau) \tilde{\chi}(m, \bar{m}, \tau).
$$

(3.3)

Note, however, that $K$ is not invariant under the simultaneous mapping class group transformations (1.2) and (1.8), but rather transforms, up to constant phases, as an automorphic form of weight $-\frac{1}{2}$. For (3.3) to be well-defined, we must therefore once again require that $\tilde{\chi}$ be an automorphic form of weight $\frac{1}{2}$. As promised, we then have an explicit construction for wave functions $\chi(\mu, \lambda)$ invariant (up to constant phases) under modular transformations of the holonomies.

The representations (2.10) of the momenta can now be checked explicitly. For instance,

$$
\hat{p} \chi(\mu, \lambda) = -i\tau \left( a - \frac{i\lambda}{2} \right)^2 \chi(\mu, \lambda) = i\tau \left( \frac{1}{\mu} \frac{\partial}{\partial \mu} - \frac{\lambda}{\tau} \right)^2 \chi(\mu, \lambda)
$$

$$
= \int \frac{d^2m}{m_2^2} \left( \frac{3}{2m_2} - \frac{i}{m_2^2} (\mu - m\lambda)^2 \right) K(m, \bar{m}; \lambda, \mu, \tau) \tilde{\chi}(m, \bar{m}, \tau)
$$

$$
= \int \frac{d^2m}{m_2^2} \left\{ \left( \frac{2i}{\bar{m}} + \frac{1}{m_2} \right) K(m, \bar{m}; \lambda, \mu, \tau) \right\} \tilde{\chi}(m, \bar{m}, \tau)
$$

$$
= \int \frac{d^2m}{m_2^2} K(m, \bar{m}; \lambda, \mu, \tau) \left( -2i \frac{\partial}{\partial \bar{m}} - \frac{1}{m_2} \right) \tilde{\chi}(m, \bar{m}, \tau),
$$

(3.4)

in agreement with (2.10). Moreover, the time dependence of $\tilde{\chi}(m, \bar{m}, \tau)$ can be determined by the requirement that $\chi(\mu, \lambda)$ be $\tau$-independent. For the Hamiltonian given by (1.11), it is easy to check that

$$
\hat{H} K(m, \bar{m}; \lambda, \mu, \tau) = -i \frac{\partial}{\partial \tau} K(m, \bar{m}; \lambda, \mu, \tau),
$$

(3.5)

so

$$
0 = \frac{\partial \chi}{\partial \tau}(\mu, \lambda) = \int \frac{d^2m}{m_2^2} \left\{ i\hat{H} K(m, \bar{m}; \lambda, \mu, \tau) + K(m, \bar{m}; \lambda, \mu, \tau) \frac{\partial}{\partial \tau} \right\} \tilde{\chi}(m, \bar{m}, \tau)
$$

$$
= \int \frac{d^2m}{m_2^2} iK(m, \bar{m}; \lambda, \mu, \tau) \left\{ \hat{H}[\hat{m}, \hat{p}] - i \frac{\partial}{\partial \tau} \right\} \tilde{\chi}(m, \bar{m}, \tau).
$$

(3.6)

As expected, the time evolution of $\tilde{\chi}(m, \bar{m}, \tau)$ is thus generated by $\hat{H}$, now viewed as a functional of the moduli and momenta $\hat{m}$ and $\hat{p}$.
4. A Dirac Equation for Wave Functions

The appearance of spinorial wave functions suggests that it should be possible to express time evolution in the Chern-Simons model in terms of a Dirac equation. To see that this is so, first note that the Laplacian $\Delta^\frac{1}{2}$ can be written in terms of the Maass operators \[ K_n = (m - \bar{m}) \frac{\partial}{\partial m} + n, \quad L_n = (\bar{m} - m) \frac{\partial}{\partial \bar{m}} - n = K_n^\dagger \] as

$$ \Delta^\frac{1}{2} = -K_n^\dagger L_n^\dagger. $$

Here, $K_n$ maps forms of weight $n$ to forms of weight $n+1$, while $L_n$ maps forms of weight $n$ to forms of weight $n-1$. $K$ and $L$ have natural interpretations as chiral Dirac operators; indeed, if we had demanded invariance under some Fuchsian group $G$ rather than the modular group, they would be the standard Dirac operators on the Riemann surface $U/G$.

Now consider a wave function $\tilde{\chi}(m, \bar{m}, \tau)$ that is an (instantaneous) eigenfunction of the Hamiltonian (2.13) with nonvanishing eigenvalue $E$. Since $\hat{H}$ is $\tau$-dependent, $E$ will be as well. Note, however, that $[\hat{H}(\tau), \hat{H}(\tau')] = 0$, so the Hamiltonian can be simultaneously diagonalized for all values of $\tau$. (This also means that no $\tau$-ordering operation is needed in the definition of the evolution operator \[19\].) Let us define a new automorphic form $\tilde{\pi}$ of weight $-\frac{1}{2}$ by

$$ L_n^\frac{1}{2} \tilde{\chi}(m, \bar{m}, \tau) = -\tau E \tilde{\pi}(m, \bar{m}, \tau). \tag{4.3} $$

It is then easy to check that

$$ \left( \begin{array}{cc} 0 & K_n^\frac{1}{2} \\ L_n^\frac{1}{2} & 0 \end{array} \right) \left( \begin{array}{c} \tilde{\chi} \\ \tilde{\pi} \end{array} \right) = \tau E \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{c} \tilde{\chi} \\ \tilde{\pi} \end{array} \right), \tag{4.4} $$

which may be recognized as the time-independent Dirac equation. Note that the extra spin degree of freedom causes no difficulty in interpretation: the wave function $\tilde{\chi}$ is the unique weight $1/2$ (chiral) projection of a two-component spinor.

Like any Dirac equation, (4.4) has both positive and negative energy solutions. Because of the simple $\tau$-dependence of the Hamiltonian, however, the negative energy solutions have a particularly easy interpretation. $H$ is odd in $\tau$, so given an eigenfunction $\tilde{\chi}(\tau)$ with eigenvalue $E(\tau)$, the time-reversed wave function $\tilde{\chi}(-\tau)$ will have an eigenvalue $-E(\tau)$. Classically, $\tau$ is the mean extrinsic curvature, that is, the fractional rate of change of the volume of space with respect to proper time. Hence reversing the sign of $\tau$, or $E$, amounts to replacing an expanding universe with a contracting one.

If $\tilde{\chi}$ is a zero-mode of $\hat{H}$, the definition (4.3) breaks down. In that case, however,

$$ 0 = \int \frac{d^2m}{m_2^2} \tilde{\chi} K_n^\frac{1}{2} L_n^\frac{1}{2} \tilde{\chi} = \int \frac{d^2m}{m_2^2} |L_n^\frac{1}{2} \tilde{\chi}|^2, \tag{4.5} $$

so the Dirac equation (4.4) holds with $\tilde{\pi} = 0$. In fact, the zero-mode can be computed explicitly: it is

$$ \tilde{\chi}_0(m, \bar{m}) = m_2^{1/2} \eta^2(m), \tag{4.6} $$
where $\eta(m)$ is the Dedekind eta function,

$$\eta(m) = e^{\pi im/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inm}).$$

(4.7)

This is an interesting wave function, representing a time-independent toroidal universe with vanishing spatial volume but with a nonzero expectation value for the modulus $m_2$.

5. Conclusion

We have seen that for the simplest nontrivial topology, $M = [0, 1] \times T^2$, Chern-Simons quantization of $(2 + 1)$-dimensional gravity can be naturally understood as the Dirac square root of Wheeler-DeWitt quantization. This contrasts sharply with reduced phase space ADM quantization, which is essentially an ordinary positive square root. This difference may help explain the comparative simplicity of the Chern-Simons formulation.

This phenomenon depends on a choice of operator ordering, of course. Our ordering was dictated by simplicity and modular invariance, but it is possible that a more complicated choice — in which the moduli are not rational functions of the holonomy operators — might reproduce ADM quantization. There seems to be no particular justification for such a choice, however, except to match the ADM theory.

An important question is whether these results are peculiar to genus one. For higher genus spaces, even the classical relationship between moduli and ISO(2,1) holonomies becomes much more complicated, and the connection between the corresponding quantum theories is quite difficult to study. One obvious conjecture can be made: Chern-Simons wave functions for spaces of genus greater than one could be higher dimensional automorphic forms on the corresponding Siegel upper half-spaces. It is possible that this conjecture can be checked for the genus two case, for which the components of the period matrix provide a good set of coordinates for moduli space.

Of course, these simple $(2 + 1)$-dimensional results will not generalize in any very straightforward manner to realistic $(3 + 1)$-dimensional gravity. The “moduli space” for a four-manifold, Wheeler’s superspace, is infinite dimensional, and the corresponding mapping class group is likely to have complicated representations. The $(2 + 1)$-dimensional model does, however, provide a dramatic illustration of the importance of the choice of canonical variables in the formulation of quantum gravity, and it once again emphasizes the key role of the mapping class group in quantization.

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