AN $L_q(L_p)$-THEORY FOR DIFFUSION EQUATIONS WITH
SPACE-TIME NONLOCAL OPERATORS

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ABSTRACT. We present an $L_q(L_p)$-theory for the equation
\[ \partial_t^\alpha u = \phi(\Delta) u + f, \quad t > 0, \quad x \in \mathbb{R}^d; \quad u(0, \cdot) = u_0. \]
Here $p, q > 1$, $\alpha \in (0, 1)$, $\partial_t^\alpha$ is the Caputo fractional derivative of order $\alpha$, and $\phi$ is a Bernstein function satisfying the following: $\exists \delta_0 \in (0, 1]$ and $c > 0$ such that
\[ c \left( \frac{R}{r} \right)^{\delta_0} \leq \frac{\phi(R)}{\phi(r)}, \quad 0 < r < R < \infty. \]
We prove uniqueness and existence results in Sobolev spaces, and obtain maximal regularity results of the solution. In particular, we prove
\[ \| \partial_t^\alpha u \|_{L_q([0, T]; L_p)} + \| u \|_{L_q([0, T]; L_p)} + \| \phi(\Delta) u \|_{L_q([0, T]; L_p)} \leq N(\| f \|_{L_q([0, T]; L_p)} + \| u_0 \|_{B_{p,q}^{\alpha-2/\alpha q}}), \]
where $B_{p,q}^{\alpha-2/\alpha q}$ is a modified Besov space on $\mathbb{R}^d$ related to $\phi$.

1. Introduction

Many types of diffusion equations have been used to describe diverse phenomena in various fields including mathematics, engineering, biology, hydrology, finance, and chemistry. The classical heat equation $\partial_t u = \Delta u$ describes the heat propagation in homogeneous media. When $\alpha \in (0, 1)$, the equation $\partial_t^\alpha u = \Delta u$ describes the anomalous diffusion exhibiting subdiffusive behavior caused by particle sticking and trapping effects (e.g. [30, 31]). On the other hand, the spatial nonlocal operator $\phi(\Delta)$ describes long range jumps of particles, diffusions on fractal structures, and long time behavior of particles moving in space with quenched and disordered force field (e.g. [26] [34]).

The space-time fractional diffusion equation can be used to describe the combined phenomena, for instance, jump diffusions with a higher peak and heavier tails (see e.g. [10] [15]). The space-time fractional equation is also related to the scaling limit of continuous time random walk (see [11] [16] [28]).

In this article we study the space-time fractional equation
\[ \partial_t^\alpha u = \phi(\Delta) u + f, \quad t > 0; \quad u(0, \cdot) = u_0. \]

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For instance, if \( \phi \) is the Laplacian. It turns out that \( \phi \) is a Bernstein function satisfying \( \phi(0+) = 0 \), that is, \( \phi : (0, \infty) \to (0, \infty) \) such that
\[
(−1)^{n}\phi^{(n)}(λ) \leq 0, \quad ∀λ > 0, \quad n = 1, 2, \ldots.
\]

The operator \( \phi(∆) \) is defined by
\[
\phi(∆)u := −ϕ(−∆)u := −F_{d}^{-1}[ϕ(|ξ|^2)F_{d}(u)(ξ)], \quad u \in C_{c}^{∞}.
\]

For instance, if \( \phi(λ) = λ^{ν/2} \), \( ν ∈ (0, 2) \), then \( \phi(∆) = ∆^{ν} \) becomes the fractional Laplacian. It turns out that \( \phi(∆) \) is a type of integro-differential operator, and the class of \( \phi(∆) \) is characterized by the infinitesimal generators of subordinate Brownian motions. See Section 2 for details.

Probabilistic representation of solution to equation (1.1) has been introduced e.g. in [7, 8, 11, 28]. Actually the transition density of subordinate Brownian motion delayed by an inverse subordinator becomes the fundamental solution, and this transition density and its appropriate time-fractional derivative appear in the solution representation. See Sections 4 and 5.

The main purpose of this article is to present a Sobolev-regularity theory of equation (1.1). We prove the uniqueness and existence in Sobolev spaces and obtain the maximal regularity of higher order derivatives as well as negative or der derivatives of solutions. In particular, we prove for any \( p, q > 1 \),
\[
\|⟨\partial^α t⟩u⟩+|u|+⟨\phi(∆)u⟩\|_{L_q([0,T];L_p)} \leq N\|f\|_{L_q([0,T];L_p)} + N\|u_0\|_{B^{\frac{2-q}{2}}_{p,q}}, \quad (1.2)
\]
where \( B^{\frac{2-q}{2}}_{p,q} \) is a Besov space on \( \mathbb{R}^d \) related to \( \phi \). Moreover, we obtain the maximal regularity of higher order derivatives as well as negative order derivatives of solutions.

Our proof for (1.2) is mainly based on BMO-estimate if \( u_0 = 0 \), and Littlewood-Paley theory is used to treat the case \( u_0 \neq 0 \). Specifically speaking, we prove that if \( u_0 = 0 \) then
\[
|⟨\partial^α_t u⟩|^#(t,x)+|⟨\phi(∆)u⟩|^#(t,x)| \leq N\|f\|_{L_{∞}}, \quad t > 0, \quad x \in \mathbb{R}^d. \quad (1.3)
\]
Here \( ⟨\partial^α_t u⟩|^# \) denotes the sharp function of \( \partial^α_t u \). The BMO estimate and the Marcinkiewicz interpolation theorem lead to (1.2) for \( p = q \), and the case \( p \neq q \) is covered based on the vector-valued Calderón-Zygmund theorem. For the implement of these procedures, we rely on sharp upper bounds of arbitrary order space-time derivatives of the fundamental solution, which are obtained in Section 3. Due to the non-integrability of derivatives of the fundamental solution, our proofs of e.g. (1.3) are much more delicate than the proofs for PDEs with local operators. Condition (1.1) is a minimal assumption on \( \phi \) such that our derivative estimates of the fundamental solution hold for all \( t > 0 \). This is essential in this article because we are aiming to prove estimates for solutions which are independent of the time intervals where the solutions are defined.

Here are some related \( L_q(L_p) \)-theories for the diffusion equations with either time fractional derivative or spatial integro-differential operators. An \( L_q(L_p) \)-theory for the time fractional equation
\[
\partial^α_t u = \sum_{i,j=1}^{d}a^{ij}(t,x)u_{x_i x_j} + f
\]
was introduced in [6, 36] when \( a^{ij} = δ^{ij} \). The result of [6, 36] is based on semigroup theory, and similar approach is used in [39] to treat the equation with uniformly
continuous coefficients. Recently, the continuity condition of [39] is significantly relaxed in [12, 22]. For instance, if $p = q$ then [12] only requires that the coefficients are only measurable in $t$ and have small mean oscillation in $x$. The approach in [12] is based on the level set arguments. Regarding the equations with spatial integro-differential operators, an $L_p$-theory of the diffusion equation the type

$$u_t = \int_{\mathbb{R}^d} (u(x+y) - u(x) - \chi(y) y \cdot \nabla u(y)) J(x, dy) + f$$

was introduced in [34]. Here $\chi$ is a certain indicator function and the jump kernel $J(x, dy)$ is of the type $a(x, y)|y|^{-d-\alpha}$, where $a(x, y)$ is homogeneous of order zero and sufficiently smooth in $y$. Recently, the condition on $J(x, dy)$ has been generalized and weaken e.g. in [12, 22, 32, 33, 40].

This article is organized as follows. In Section 2, we introduce some basic facts on the fractional calculus, integro-differential operator $\phi(\Delta)$, and related function spaces. We also introduce our main result, Theorem 2.8 in Section 2. In Section 3 we obtain sharp upper bounds of space-time derivatives of the fundamental solution. In section 4 we study the zero initial data problem, and non-zero initial data problem is considered in Section 5. Finally we prove our main result in Section 6.

We finish the introduction with some notations. We use “:=” or “=:” to denote a definition. The symbol $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Also we use $\mathbb{Z}$ to denote the set of integers. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$. We set

$$B_r(x) := \{y \in \mathbb{R} : |x - y| < r\}, \quad \mathbb{R}^{d+1} := \{(t, x) \in \mathbb{R}^{d+1} : t > 0\}.$$ 

For $i = 1, \ldots, d$, multi-indices $\sigma = (\sigma_1, \ldots, \sigma_d)$, and functions $u(t, x)$ we set

$$\partial_{x^i} u = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\sigma u = D_1^{\sigma_1} \cdots D_d^{\sigma_d} u, \quad |\sigma| = \sigma_1 + \cdots + \sigma_d.$$ 

We also use the notation $D_x^m$ for arbitrary partial derivatives of order $m$ with respect to $x$. For an open set $\mathcal{O}$ in $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$, $C_\infty^\infty(\mathcal{O})$ denotes the set of infinitely differentiable functions with compact support in $\mathcal{O}$. By $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ we denote the class of Schwartz functions on $\mathbb{R}^d$. For $p > 1$, by $L_p$ we denote the set of complex-valued Lebesgue measurable functions $u$ on $\mathbb{R}^d$ satisfying

$$\|u\|_{L_p} := \left( \int_{\mathbb{R}^d} |u(x)|^p dx \right)^{1/p} < \infty.$$ 

Generally, for a given measure space $(X, \mathcal{M}, \mu)$, $L_p(X, \mathcal{M}, \mu; F)$ denotes the space of all $F$-valued $\mathcal{M}^\sigma$-measurable functions $u$ so that

$$\|u\|_{L_p(X, \mathcal{M}, \mu; F)} := \left( \int_X \|u(x)\|^p_\mathcal{F} \mu(dx) \right)^{1/p} < \infty,$$

where $\mathcal{M}^\sigma$ denotes the completion of $\mathcal{M}$ with respect to the measure $\mu$. If there is no confusion for the given measure and $\sigma$-algebra, we usually omit the measure and the $\sigma$-algebra. We denote $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the $d$-dimensional Fourier transform and the inverse Fourier transform respectively, i.e.

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) dx, \quad \mathcal{F}^{-1}(f)(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x) dx.$$
For any \(a, b > 0\), we write \(a \sim b\) if there is a constant \(c > 1\) independent of \(a, b\) such that \(c^{-1}a \leq b \leq ca\). Finally if we write \(N = N(\ldots)\), this means that the constant \(N\) depends only on what are in the parentheses. The constant \(N\) can differ from line to line.

2. MAIN RESULTS

First we introduce some definitions and facts related to the fractional calculus. For \(\alpha > 0\) and \(\varphi \in L_1((0, T))\), the Riemann-Liouville fractional integral of the order \(\alpha\) is defined as

\[
I_1^\alpha \varphi := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds, \quad 0 \leq t \leq T.
\]

We also define \(I_0^0 \varphi := \varphi\). By Jensen’s inequality, for \(p \in [1, \infty)\),

\[
\|I_1^\alpha \varphi\|_{L_p((0, T))} \leq N(T, \alpha) \|\varphi\|_{L_p((0, T))}.
\] (2.1)

Using Fubini’s theorem, one can easily check for any \(\alpha, \beta \geq 0\),

\[
I_1^\alpha I_1^\beta \varphi = I_1^{\alpha+\beta} \varphi, \quad (a.e.) \ t \leq T.
\] (2.2)

Let \(\alpha \in [n-1, n), n \in \mathbb{N}\). If \(\varphi(t)\) is \((n-1)\)-times differentiable and \((\frac{d}{dt})^{n-1} \varphi\) is absolutely continuous on \([0, T]\), then the Riemann-Liouville fractional derivative \(D_t^\alpha\) and the Caputo fractional derivative \(\partial_t^\alpha\) are defined as

\[
D_t^\alpha \varphi := \left(\frac{d}{dt}\right)^n (I_1^{n-\alpha} \varphi),
\] (2.3)

and

\[
\partial_t^\alpha \varphi = D_t^\alpha \left(\varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^{(k)}(0)\right).
\] (2.4)

Note that \(D_t^\alpha \varphi = \partial_t^\alpha \varphi\) if \(\varphi(0) = \varphi'(0) = \cdots = \varphi^{(n-1)}(0) = 0\). By (2.2) and (2.3), if \(\alpha, \beta \geq 0\),

\[
D_t^\alpha D_t^\beta \varphi = D_t^{\alpha+\beta} \varphi, \quad D_t^\alpha I_1^\beta \varphi = D_t^{\alpha-\beta} \varphi,
\]

where \(D_t^\alpha \varphi := I_t^{-\alpha} \varphi\) if \(\alpha < 0\). Also if \(\varphi(0) = \varphi^{(1)}(0) = \cdots = \varphi^{(n-1)}(0) = 0\) then

\[
I_1^\alpha \partial_t^\alpha u = I_1^\alpha D_t^\alpha u = u.
\] (2.5)

Next, we introduce our assumption on \(\varphi\) and some informations on the operator \(\varphi(\Delta)\). Recall that a function \(\varphi : \mathbb{R}^+ \to \mathbb{R}^+\) satisfying \(\varphi(0) = 0\) is a Bernstein function if there exist a constant \(b \geq 0\) and a Lévy measure \(\mu\) (i.e. \(\int_{(0, \infty)} (1 \wedge t) \mu(dt) < \infty\)) such that

\[
\phi(\lambda) = b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda t}) \mu(dt).
\] (2.6)

It is known that a function \(\varphi\) is a Bernstein function if and only if it is a Laplace exponent of a subordinator, that is, there exists a nonnegative real-valued Lévy process \(S_t\) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

\[
\mathbb{E} e^{-\lambda S_t} := \int_{\Omega} e^{-\lambda S_t(\omega)} \mathbb{P}(d\omega) = e^{-t\phi(\lambda)}.
\]

For \(f \in \mathcal{S}(\mathbb{R}^d)\), we define \(\varphi(\Delta)f := -\varphi(-\Delta)f\) as

\[
\phi(\Delta)f(x) = \mathcal{F}^{-1}(-\phi(|\xi|^2)\mathcal{F}(f)(\xi))(x).
\]
It turns out (see [18, Theorem 31.5]) that $\phi(\Delta)$ is an integro-differential operator defined by

$$\phi(\Delta)f(x) = b\Delta f + \int_{\mathbb{R}^d} (f(x+y) - f(x) - \nabla f(x) \cdot y 1_{|y| \leq 1}) J(y) dy,$$

where $J(x) = j(|x|)$ and $j : (0, \infty) \rightarrow (0, \infty)$ is given by

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(dt).$$

Furthermore, $\phi(\Delta)$ is the infinitesimal generator of the $d$-dimensional subordinate Brownian motion $X_t := W_t S_t$,

$$\phi(\Delta)f(x) = \lim_{t \to 0} \frac{E f(x + X_t) - f(x)}{t},$$

where $W_t$ is a $d$-dimensional Brownian motion independent of $S_t$. For instance, by taking $\phi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0, 2)$, we get the fractional Laplacian $\Delta^{\alpha/2} = -(-\Delta)^{\alpha/2}$, which is the infinitesimal generator of a rotationally symmetric $\alpha$-stable process in $\mathbb{R}^d$.

Using (2.6) one can check

$$\phi^{(n)}(\lambda) = b1_{n=1} - \int_{(0,\infty)} (-t)^n e^{-\lambda t} \mu(dt), \quad n \in \mathbb{N},$$

where $\phi^{(n)}$ is the $n$-th derivative of $\phi$. Therefore, $\phi'(\lambda) > 0$ and

$$(-1)^n \phi^{(n)}(\lambda) \leq 0, \quad \forall \lambda > 0, \ n \in \mathbb{N},$$

and, by the inequality $t^n e^{-t} \leq N(n)(1 - e^{-t})$, we also have for any $n \geq 1$

$$\lambda^n |\phi^{(n)}(\lambda)| \leq 1_{n=1} b\lambda + \int_0^\infty (\lambda t)^n e^{-\lambda t} \mu(dt) \leq 1_{n=1} b\lambda + N \int_0^\infty (1 - e^{-\lambda t}) \mu(dt) \leq N(n)\phi(\lambda). \quad (2.7)$$

Here is our assumption on $\phi$.

**Assumption 2.1.** There exist constants $\delta_0 \in (0, 1]$ and $c > 0$ such that

$$c \left( \frac{R}{r} \right)^{\delta_0} \leq \frac{\phi(R)}{\phi(r)}, \quad 0 < r < R < \infty. \quad (2.8)$$

Note that the case $\delta_0 = 1$ is included, and (2.8) is assumed to hold for all $0 < r < R < \infty$. In the literature, it is common to impose separate conditions on $\phi$ near zero and infinity. For instance, in [21, 24, 35], conditions (H1) and (H2) below are used for the study of the transition density of subordinate Brownian motion:

**H1:** $3 c_1, c_2 > 0$, and $0 < \delta_1 \leq \delta_2 < 1$ such that

$$c_1 \lambda^{\delta_1} \phi(t) \leq \phi(\lambda t) \leq c_2 \lambda^{\delta_2} \phi(t) \quad \lambda \geq 1, \quad t \geq 1.$$

**H2:** $\exists c_3 > 0$, and $0 < \delta_3 \leq 1$ such that

$$\phi(\lambda t) \leq c_3 \lambda^{\delta_3} \phi(t) \quad \lambda \leq 1, \quad t \leq 1.$$

Inequality (3.27) in [21] shows that (H1) and (H2) together give

$$\frac{c_1}{c_3} \left( \frac{R}{r} \right)^{\delta_1 + \delta_3} \leq \frac{\phi(R)}{\phi(r)}, \quad 0 < r < R < \infty. \quad (2.9)$$
Thus our assumption is weaker than (H1) combined with (H2).

Here are some examples related to our assumption on $\phi$:

1. Stable subordinators: $\phi(\lambda) = \lambda^\beta$, $0 < \beta \leq 1$.
2. Sum of stable subordinators: $\phi(\lambda) = \lambda^{\beta_1} + \lambda^{\beta_2}$, $0 < \beta_1, \beta_2 \leq 1$.
3. Stable with logarithmic correction: $\phi(\lambda) = \lambda^\beta (\log(1 + \lambda))^{\gamma}$, $\beta \in (0, 1), \gamma \in (-\beta, 1 - \beta)$.
4. Relativistic stable subordinators: $\phi(\lambda) = (\lambda + m^2)^{\beta}$, $\beta \in (0, 1), m > 0$.
5. Conjugate geometric stable subordinators: $\phi(\lambda) = (\log(1 + \lambda^{\beta/2}))^{\gamma}$, $\beta \in (0, 2)$.

One can check that Examples (1)-(4) satisfy both (H1) and (H2), and therefore Assumption 2.1 is also fulfilled due to (2.9). On the other hand, Example (5) satisfies Assumption 2.1 with $\delta_0 = 1 - \beta_2$, but condition (H1) fails to hold because

$$\lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda^{1-\varepsilon}} = \infty, \quad \forall \varepsilon > 0.$$ 

Next we introduce Sobolev and Besov spaces related to the operator $\phi(\Delta)$. For $\gamma \in \mathbb{R}$, denote

$$H^{\phi, \gamma}_p := (1 - \phi(\Delta))^{-\gamma/2} L_p.$$ 

That is, $u \in H^{\phi, \gamma}_p$ if

$$\|u\|_{H^{\phi, \gamma}_p} := \|(1 - \phi(\Delta))^{\gamma/2} u\|_{L_p} := \|\mathcal{F}^{-1}\{(1 + \phi(|\cdot|^2))^{\gamma/2} \mathcal{F}(u)(\cdot)\}\|_{L_p} < \infty.$$ 

Note that if $\phi(\lambda) = \lambda$, then $H^{\phi, \gamma}_p$ is the classical Bessel potential space $H^\gamma_p$.

The following lemma gives some basic properties of $H^{\phi, \gamma}_p$.

**Lemma 2.2.** (i) For any $\gamma \in \mathbb{R}$, $H^{\phi, \gamma}_p$ is a Banach space.

(ii) For any $\mu, \gamma \in \mathbb{R}$, the map $(1 - \phi(\Delta))^{\mu/2} : H^{\phi, \gamma}_p \to H^{\phi, \gamma-\mu}_p$ is an isometry.

(iii) If $\gamma_1 \leq \gamma_2$, then $H^{\phi, \gamma_2}_p \subset H^{\phi, \gamma_1}_p$, and there is a constant $N > 0$ so that

$$\|u\|_{H^{\phi, \gamma_1}_p} \leq N \|u\|_{H^{\phi, \gamma_2}_p}.$$ 

(iv) For any $\gamma \geq 0$,

$$\left(\|u\|_{L_p} + \|\phi(\Delta)^{\gamma/2} u\|_{L_p}\right) \sim \|u\|_{H^{\phi, \gamma}_p}.$$ 

(2.10)

**Proof.** See [21, Lemma 6.1]. We remark that [21, Lemma 6.1] is proved for arbitrary Bernstein functions $\phi$ with no drift, that is $b = 0$ (see (2.6)). The same proof works for us because it is proved based on (2.7). \(\square\)

Take a function $\Psi \in \mathcal{S}(\mathbb{R}^d)$ whose Fourier transform $\hat{\Psi}(\xi)$ is supported in a strip $\{1/2 \leq |\xi| \leq 2\}$, $\hat{\Psi} \geq 0$, and

$$\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1, \quad \xi \neq 0.$$ 

(2.11)

Define

$$\hat{\Psi}_j(\xi) = \hat{\Psi}(2^{-j}\xi), \quad j = \pm 1, \pm 2, \ldots,$$

$$\hat{\Psi}_0(\xi) = 1 - \sum_{j=1}^{\infty} \hat{\Psi}(\xi).$$ 

(2.12)

Also, for distributions (or functions) $f$ we define $f_j = \Psi_j * f$, where $*$ denotes the convolution.
\textbf{Definition 2.3.} (cf. \cite{32}) Let $1 < p, q < \infty$ and $s \in \mathbb{R}$. The Besov space $B_{p, q}^{\phi, s}(\mathbb{R}^d)$ is a closure of $\mathcal{S}$ under the norm
\[
\|u\|_{B_{p, q}^{\phi, s}} := \left\{ u_0\|_{L_p}^q + \sum_{j=1}^{\infty} \phi(2^{2j})^{s \cdot q/2} \|u_j\|_{L_p}^q \right\}^{1/q}.
\]

\textbf{Remark 2.4.} (i) If $\phi(\lambda) = \lambda$, then $B_{p, q}^{\phi, s}$ corresponds to the standard Besov space $B_{p, q}^s$. In general, since $\phi(\lambda) \leq N\lambda$ for $\lambda \geq 1$, for $u \in \mathcal{S}$ and $s \geq 0$ we have
\[
\|u\|_{B_{p, q}^{\phi, s}} \leq N\|u\|_{B_{p, q}^s}.
\]
This and \cite{11} Theorem 6.2.4 yield that if $\gamma > (s \vee 0)$ then
\[
H_\gamma^p \subset B_{p, q}^{\phi, s},
\] and the embedding is continuous.

(ii) Let $J^s u := (1 - \phi(\Delta))^{s/2} u$ and $\tilde{B}_{p, q}^{\phi, s}$ be the closure of $\mathcal{S}$ under the norm
\[
\|u\|_{\tilde{B}_{p, q}^{\phi, s}} = \left\{ \sum_{j=0}^{\infty} \| J^s \Psi_j \ast u \|_{L_p}^q \right\}^{1/p}.
\]

We now show that
\[
\tilde{B}_{p, q}^{\phi, s} = B_{p, q}^{\phi, s}, \quad 1 < p, q < \infty, \ s \in \mathbb{R},
\] (2.14)
First we prove that there is a constant $N$ such that
\[
\|J^s \Psi_j \ast u\|_{L_p} \leq N\phi(2^{2j})^{s/2}\|u_j\|_{L_p}, \quad \forall j \in \mathbb{N}_0.
\] (2.15)
If we denote $\zeta_j = \Psi_{j-1} + \Psi_j + \Psi_{j+1}$ for $j \in \mathbb{N}$ and $\zeta_0 = \Psi_0 + \Psi_1$, then
\[
(1 + \phi(|\xi|^2))^{s/2} \hat{\Psi}_j(\xi) = (1 + \phi(|\xi|^2))^{s/2} \hat{\Psi}_j(\xi) \hat{\zeta}_j(\xi).
\]
Fix $j$, and let $\xi$ be in the support of $\zeta_j$. Then for any $\nu \in \mathbb{R}$ and $m \in \mathbb{N}_0$, by (2.7)
\[
\frac{(1 + \phi(|\xi|^2))^{s/2}}{\phi(2^j)^\nu} \leq N,
\]
\[
\left| D_\xi^m \left( \frac{(1 + \phi(|\xi|^2))^{s/2}}{\phi(2^j)^m} \hat{\Psi}_j(\xi) \hat{\zeta}_j(\xi) \right) \right| \leq N|\xi|^{-m},
\]
where $N$ is independent of $j$. Hence combining the above inequalities, for any $m \in \mathbb{N}_0$ it follows that
\[
\left| D_\xi^m \left( \frac{(1 + \phi(|\xi|^2))^{s/2}}{\phi(2^j)^{s/2}} \hat{\Psi}_j(\xi) \hat{\zeta}_j(\xi) \right) \right| \leq N|\xi|^{-m},
\]
where $N$ is independent of $j$. Therefore, due to \cite{17} Theorem 5.2.7, (2.15) holds.
This implies that for any $s \in \mathbb{R}$
\[
B_{p, q}^{\phi, s} \subset \tilde{B}_{p, q}^{\phi, s}.
\]
Using (2.15) again, one can easily check
\[
\|u_j\|_{L_p} = \|J^s J^s u_j\|_{L_p} \leq N\phi(2^{2j})^{-s/2}\|J^s u_j\|_{L_p},
\]
and therefore (2.14) holds.

(iii) Since $(1 - \phi(\Delta))^{\nu/2}$ is an isometry from $\tilde{B}_{p, q}^{\phi, s}$ to $\tilde{B}_{p, q}^{\phi, s - \nu}$, we may consider this operator as an isometry from $B_{p, q}^{\phi, s}$ to $B_{p, q}^{\phi, s - \nu}$ due to (2.14).
For $p, q \in (1, \infty), \gamma \in \mathbb{R}$ and $T < \infty$, we denote
$$H_{q,p}^\phi(T) := L_q \left((0, T); H_{\phi}^{\gamma} \right), \quad L_{q,p}(T) := H_{q,p}^\phi(T).$$
We write $u \in C^\alpha_p([0, T] \times \mathbb{R}^d)$ if $D_x^m u, \partial_t^p D_x^m u \in C([0, T]; L_p)$ for any $m \in \mathbb{N}_0$. $C^\infty_p(\mathbb{R}^d)$ denotes the set of functions $u_0 = u_0(x)$ such that $D_x^m u_0 \in L_p$ for any $m \in \mathbb{N}_0$.

**Definition 2.5.** (i) For $\alpha \in (0, 1), 1 < p, q < \infty$, and $\gamma \in \mathbb{R}$, we write $u \in H_{q,p}^{\alpha, \phi, \gamma+2}(T)$ if there exists a sequence $u_n \in C^\infty_p([0, T] \times \mathbb{R}^d)$ satisfying
$$\|u - u_n\|_{H_{q,p}^{\alpha, \phi, \gamma+2}(T)} \to 0 \quad \text{and} \quad \|\partial_t^\alpha u_n - \partial_t^\alpha u_m\|_{H_{q,p}^{\alpha, \phi, \gamma}(T)} \to 0$$
as $n, m \to \infty$. We call this sequence $u_n$ a defining sequence of $u$, and we define
$$\partial_t^\alpha u = \lim_{n \to \infty} \partial_t^\alpha u_n \text{ in } H_{q,p}^{\alpha, \phi, \gamma}(T).$$
The norm in $H_{q,p}^{\alpha, \phi, \gamma+2}(T)$ is naturally given by
$$\|u\|_{H_{q,p}^{\alpha, \phi, \gamma+2}(T)} = \|u\|_{H_{q,p}^{\alpha, \phi, \gamma+2}(T)} + \|\partial_t^\alpha u\|_{H_{q,p}^{\alpha, \phi, \gamma}(T)}.$$ (ii) For $u \in H_{q,p}^{\alpha, \phi, \gamma+2}(T)$ and $u_0 \in B_{p,q}^{\phi, \gamma+2 - \frac{2}{\alpha}}$, we say $u(0, x) = u_0$ if there exists a defining sequence $u_n$ such that $u_n(0, \cdot) \in C^\infty_p(\mathbb{R}^d)$ and
$$u_n(0, \cdot) \to u_0 \text{ in } B_{p,q}^{\phi, \gamma+2 - \frac{2}{\alpha}}.$$ (iii) We write $u \in H_{q,p,0}^{\alpha, \phi, \gamma+2}(T)$, if there exists a defining sequence $u_n$ such that
$$u_n(0, x) = 0 \quad \forall x \in \mathbb{R}^d, \forall n \in \mathbb{N}.$$

**Remark 2.6.** (i) Obviously, $H_{q,p,0}^{\alpha, \phi, \gamma+2}(T)$ is a Banach space.
(ii) Applying (2.3) to $u_n(t, x) - u_n(0, x)$ and using (2.11), one can check that Definition 2.5 (ii) is independent of the choice of a defining sequence.
(iii) Actually, by Corollary 5.5 $u \in H_{q,p,0}^{\alpha, \phi, \gamma+2}(T)$ and $u(0, \cdot) = 0$ if and only if $u \in H_{q,p,0}^{\alpha, \phi, \gamma+2}(T)$.
(iv) Following [33, Remark 3], one can show that the embedding $H_{p}^{2n} \subset H_{p,2n}^{\alpha, \phi}$ is continuous for any $n \in \mathbb{N}$.

**Lemma 2.7.** Let $\alpha \in (0, 1), 1 < p, q < \infty, \gamma \in \mathbb{R}$, and $T < \infty$.
(i) The space $H_{q,p}^{\alpha, \phi, \gamma+2}(T)$ is a closed subspace of $H_{q,p}^{\alpha, \phi, \gamma+2}(T)$.
(ii) $C^\infty(\mathbb{R}^{d+1})$ is dense in $H_{q,p,0}^{\alpha, \phi, \gamma+2}(T)$.
(iii) For any $\gamma, \nu \in \mathbb{R}$, $(1 - \phi(\Delta))^{\nu/2} : H_{q,p}^{\alpha, \phi, \gamma+2}(T) \to H_{q,p}^{\alpha, \phi, \gamma-\nu+2}(T)$ is an isometry, and for any $u \in H_{q,p}^{\alpha, \phi, \gamma+2}(T)$
$$\partial_t^\alpha u = \partial_t^\alpha (1 - \phi(\Delta))^{\nu/2} u.$$
**Proof.** (i) The proof is straightforward and is left to the reader.
(ii) It suffices to show that for any given $u \in C^\infty_p([0, T] \times \mathbb{R}^d)$ with $u(0, \cdot) = 0$, there exists a sequence $u_n \in C^\infty_p(\mathbb{R}^{d+1})$ such that
$$\|u_n - u\|_{H_{q,p}^{\alpha, \phi, \gamma+2}(T)} \to \infty$$
as $n \to \infty$. By Remark 2.5 (iv) and considering a multiplication with smooth cut-off function of $x$, one may assume that $u$ has compact support, that is, with some
For $\varepsilon > 0$, we define

$$\eta_{1,\varepsilon}(t) = \varepsilon^{-1}\eta(t/\varepsilon),$$

$$u_{\varepsilon}(t, x) = \eta(t) \int_0^\infty u(s, x) \eta_{1,\varepsilon}(t-s) ds,$$

where $\eta \in C^\infty([0, \infty))$ such that $\eta(t) = 1$ for all $t \leq T + 1$ and vanishes for all large $t$. Then, due to $\eta_1 \in C^\infty((1, 2))$,

$$u_{\varepsilon}(t, x) = 0 \quad \forall t < \varepsilon, \quad \forall x \in \mathbb{R}^d,$$

and $u_{\varepsilon^1} \in C^\infty(\mathbb{R}^{d+1})$. Also using $u(0, x) = 0$ and Fubini’s theorem, one can prove

$$\partial_t^\alpha u_{\varepsilon}(t) = (\partial_t^\alpha u)(t), \quad t \leq T.$$

Therefore, for any $n \in \mathbb{N}$,

$$\|u_{\varepsilon} - u\|_{L_q([0, T]; H^m_p)} + \|\partial_t^\alpha u_{\varepsilon} - \partial_t^\alpha u\|_{L_q([0, T]; H^m_p)} \to 0$$

as $\varepsilon \downarrow 0$. This and Remark 2.6 (iv) yields (2.16) with $u_n := u^{1/n}$. Therefore, (ii) is proved.

(iii) We first prove the claims for functions $u \in C^\alpha_{\infty}([0, T] \times \mathbb{R}^d)$. Take $\eta \in C^\infty_c([0, \infty))$ such that $\eta = 1$ on $B_1$ and $\eta = 0$ outside of $B_2$. For $n \in \mathbb{N}$, define $\eta_n(x) = \eta(x/n)$ and

$$v_n := \eta_n(1 - \phi(\Delta))^{\nu/2} u \in C^\alpha_{\infty}([0, T] \times \mathbb{R}^d).$$

Then, for any $m \in \mathbb{N}$, $v_n \to (1 - \phi(\Delta))^{\nu/2} u$ in $L_q((0, T); H^m_p)$ as $n \to \infty$, and therefore by Remark 2.7 (iv), $v_n \to (1 - \phi(\Delta))^{\nu/2} u$ in $\mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$ as $n \to \infty$. Similarly,

$$\partial_t^\alpha v_n = \eta_n(1 - \phi(\Delta))^{\nu/2} \partial_t^\alpha u \to (1 - \phi(\Delta))^{\nu/2} \partial_t^\alpha u$$

in $\mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$ as $n \to \infty$. Therefore $u \in \mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$, and all the assertions of (iii) also follow.

Now, let $u \in \mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$. Take a defining sequence $u_n$ for $u$. Then, by the above result

$$(1 - \phi(\Delta))^{\nu/2} u_n \in \mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T), \quad \partial_t^\alpha (1 - \phi(\Delta))^{\nu/2} u_n \to (1 - \phi(\Delta))^{\nu/2} \partial_t^\alpha u_n.$$

From these and the fact $u_n \to u$ in $\mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$ as $n \to \infty$, it follows that $(1 - \phi(\Delta))^{\nu/2} u_n$ is a Cauchy sequence in $\mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$. Let $w$ denote the limit in this space. Then, since $(1 - \phi(\Delta))^{\nu/2} u_n \to (1 - \phi(\Delta))^{\nu/2} u$ in $\mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$, we conclude $w = (1 - \phi(\Delta))^{\nu/2} u \in \mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$. The claim for the isometry is obvious, and the other assertion of (iii) also follows. The lemma is proved.

Here is the main result of this article.

**Theorem 2.8.** Let $\gamma \in \mathbb{R}, \alpha \in (0, 1)$, $p, q \in (1, \infty)$. Suppose Assumption 2.7 holds. Then for any $u_0 \in B^{\phi, \gamma, \nu+2-2/\alpha q}_{p, q}$ and $f \in \mathbb{H}_{\alpha, p}^{\phi, \gamma}(T)$, the equation

$$\partial_t^\alpha u = \phi(\Delta) u + f, \quad t > 0; \quad u(0, \cdot) = u_0$$

admits a unique solution $u$ in the class $\mathbb{H}_{\alpha, p}^{\phi, \gamma, \nu+2}(T)$, and we have
\[ \|u\|_{H^{\alpha,\varphi,\gamma+2}_{p,q}((T))} \leq N \left( \|f\|_{H^{\alpha,\varphi,\gamma}_{p,q}((T))} + \|u_0\|_{H^{\alpha+2}_{p,q}((T))} \right), \] (2.18)

where \( N = N(\alpha, d, \phi, p, q, \gamma, T) \). Moreover, if \( u_0 = 0 \) then
\[ \|\phi(\Delta)u\|_{H^{\alpha,\varphi,\gamma}_{p,q}((T))} \leq N_0 \|f\|_{H^{\alpha,\varphi,\gamma}_{p,q}((T))}, \] (2.19)

where \( N_0 = N_0(\alpha, d, \phi, p, q, \gamma) \) is independent of \( T \).

3. Estimates of the fundamental solution

In this section we obtain sharp bounds of arbitrary order derivatives of the fundamental solution to equation (2.17).

We first study the derivatives of the transition density of \( d \)-dimensional subordinate Brownian motion.

Lemma 3.1. Let Assumption 2.1 hold. Then there exists a constant \( N = N(c, \delta_0) \) such that
\[ \int_{\lambda^{-1}}^{\infty} r^{-1} \phi(r^{-2}) dr \leq N \phi(\lambda^2), \quad \forall \lambda > 0. \] (3.1)

Proof. Note first that Assumption 2.1 combined with the concavity of \( \phi \) gives
\[ c \left( \frac{R}{r} \right)^{\delta_0} \leq \frac{\phi(R)}{\phi(r)} \leq \frac{R}{r}, \quad 0 < r < R < \infty. \] (3.2)

By the change of variables and (3.2),
\[ \int_{\lambda^{-1}}^{\infty} r^{-1} \phi(r^{-2}) dr = \int_{1}^{\infty} r^{-1} \phi(\lambda^2 r^{-2}) dr = \int_{1}^{\infty} r^{-1} \phi(\lambda^2) \frac{\phi(\lambda^2)}{\phi(\lambda^2)} dr \]
\[ \leq N \int_{1}^{\infty} r^{-1-2\delta_0} dr \phi(\lambda^2) = N \phi(\lambda^2). \]

The lemma is proved. \( \square \)

Recall that \( S = (S_t)_{t \geq 0} \) is a subordinator with Laplace exponent \( \phi \) and \( W = (W_t)_{t \geq 0} \) is a Brownian motion in \( \mathbb{R}^d \), independent of \( S \). We call \( X_t := W_S \) the subordinate Brownian motion. It is known (see e.g. [3, 18]) that \( X_t \) is rotationally invariant Lévy process in \( \mathbb{R}^d \) with characteristic exponent \( \phi(|\xi|^2) \). That is,
\[ \mathbb{E} \left[ e^{i\xi \cdot X_t} \right] = e^{-t\phi(|\xi|^2)}, \quad \forall \xi \in \mathbb{R}^d, \quad t > 0. \] (3.3)

Using (3.3) and the equality
\[ e^{-|z|^2} = (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi \cdot z} e^{-|\xi|^2/4} d\xi, \]
we have for \( \lambda > 0 \)
\[ \mathbb{E}[e^{-\lambda |X_t|^2}] = (4\pi)^{-d/2} \int_{\mathbb{R}^d} \mathbb{E}[e^{i\sqrt{\lambda} \xi \cdot X_t}] e^{-|\xi|^2/4} d\xi \]
\[ = (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t\phi(|\xi|^2)} e^{-|\xi|^2/4} d\xi. \]
Therefore, for $t, \lambda > 0$,
\[
0 \leq E[e^{-\lambda|X_t|^2} - e^{-2\lambda|X_t|^2}]
\]
\[
= (4\pi)^{-d/2} \int_{\mathbb{R}^d} (e^{-t\phi(\lambda)|\xi|^2} - e^{-t\phi(2\lambda)|\xi|^2}) e^{-|\xi|^2/4} d\xi
\]
\[
= N(d) \int_0^\infty (e^{-t\phi(\lambda r^2)} - e^{-t\phi(2\lambda r^2)}) e^{-r^2/4} r^{d-1} dr
\]
\[
= N(d) g_t(\lambda).
\]
(3.4)

The following lemma is a version of [21, Lemma 2.1].

**Lemma 3.2.** Let Assumption 2.1 hold. Then, there exists a constant $N = N(c, \delta_0, d)$ such that for every $t, v > 0$
\[
g_t(v^{-1}) \leq N t \phi(v^{-1}).
\]

**Proof.** Note that by (3.2) for any $r > 0$
\[
\frac{1}{t\phi(v^{-1})} \leq \frac{\phi(2v^{-1}r^2) + \phi(v^{-1}r^2)}{\phi(v^{-1})} \frac{1}{t|\phi(2v^{-1}r^2) - \phi(v^{-1}r^2)|}
\]
\[
\leq N(v^2 \vee r^{2\delta_0}) - \frac{1}{t|\phi(2v^{-1}r^2) - \phi(v^{-1}r^2)|},
\]
Thus using the inequality $|e^{-a} - e^{-b}| \leq |a - b|$, $a, b > 0$, we have
\[
\frac{g_t(v^{-1})}{t\phi(v^{-1/2})}
\]
\[
\leq N \int_0^\infty \frac{|e^{-t\phi(v^{-1}r^2)} - e^{-t\phi(2v^{-1}r^2)}|}{t|\phi(2v^{-1}r^2) - \phi(v^{-1}r^2)|} e^{-r^2/4} r^{d-1} (r^2 \vee r^{2\delta_0}) dr
\]
\[
\leq N \int_0^\infty e^{-r^2/4} r^{d-1} (r^2 \vee r^{2\delta_0}) dr < \infty.
\]

Therefore the lemma is proved. □

Let $p(t, x) = p_d(t, x)$ be the transition density of $X_t = W_{S_t}$, the $d$-dimensional subordinate Brownian motion corresponding to $\phi$. Then it is known ([3, Section 5.3.1]) that for any $t > 0, x \in \mathbb{R}^d$,
\[
p(t, x) = p_d(t, x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-t\phi(|\xi|^2)} d\xi
\]
\[
= \int_{(0, \infty)} (4\pi s)^{-d/2} \exp \left( -\frac{|x|^2}{4s} \right) \eta_t(ds),
\]
(3.6)

where $\eta_t(ds)$ is the distribution of $S_t$. Thus $X_t$ is rotationally invariant. We put $p(t, r) := p(t, x)$ if $r = |x|$ for notational convenience. It follows from (3.6) that $r \to p(t, r)$ is a decreasing function.

**Lemma 3.3.** Let Assumption 2.1 hold. Then there exists a constant $N = N(c, \delta_0, d)$ such that
\[
p_d(t, r) \leq N tr^{-d} \phi(r^{-2}), \quad (t, r) \in (0, \infty) \times (0, \infty).
\]
Proof. Fix $t \in (0, \infty)$. For $r \geq 0$ define $f_t(r) = r^{d/2}p(t, r^{1/2})$. Since $r \to p(t, r)$ is decreasing, for $r \geq 0$,
\[
\mathbb{P}(\sqrt{r/2} < |X_t| < \sqrt{r}) = \int_{\sqrt{r/2} < |y| < \sqrt{r}} p(t, |y|) \, dy \geq |B_1(0)| (1 - 2^{-d/2})r^{d/2}p(t, r^{1/2}) = |B_1(0)| (1 - 2^{-d/2})f_t(r).
\]
Denoting $\mathcal{L}f_t(\lambda)$ the Laplace transform of $f_t$, we have
\[
\mathcal{L}f_t(\lambda) = N \int_0^{\infty} \mathbb{P}(\sqrt{r/2} < |X_t| < \sqrt{r})e^{-\lambda r} \, dr = N \lambda \int_{|X_t|^2} e^{-\lambda |X_t|^2} \, dr = N \lambda^{-1} g_t(\lambda), \quad \lambda > 0
\] (3.7)
from (3.3). Furthermore, for any $v > 0$
\[
\mathcal{L}f_t(v^{-1}) = \int_0^{\infty} e^{-av^{-1}} f_t(a) \, da = v \int_0^{\infty} e^{-s} f_t(sv) \, ds \geq v \int_{1/2}^{1} e^{-s} f_t(sv) \, ds = v \int_{1/2}^{1} e^{-s} s^{d/2} v^{d/2} \left( t, s^{1/2} v^{1/2} \right) \, ds \geq v^2 e^{-d/2} v^{d/2} t \int_{1/2}^{1} e^{-s} \, ds = 2^{-d/2} v f_t(v) \left( \int_{1/2}^{1} e^{-s} \, ds \right).
\]
Thus
\[
f_t(v) \leq 2^{d/2} v^{-1} \frac{\mathcal{L}f_t(v^{-1})}{e^{-1/2} - e^{-1}}.
\] (3.8)
Finally, combining (3.7) and (3.8) with Lemma 3.2 we conclude
\[
p(t, r) = r^{-d} f_t(r^{2}) \leq N r^{-d-2} \mathcal{L}f_t(r^{-2}) \leq N r^{-d} g_t(r^{-2}) \leq N tr^{-d}\phi(r^{-2}).
\]
The lemma is proved. \(\square\)

The difference of the following result from those in the literature is that it only concerns the upper bound rather than two-side estimate, but it holds under relatively weaker assumption and gives estimate which holds for all $t > 0$. See Remark 3.3 for some related results.

**Lemma 3.4.** Let Assumption 2.1 hold. Then, there exists a constant $N = N(d, c, \delta_0)$ such that for $(t, x) \in (0, \infty) \times \mathbb{R}^d$,\n\[
p_d(t, x) \leq N \left( \phi^{-1}(t^{-1}) \right)^{d/2} \wedge t \frac{\phi(|x|^{-2})}{|x|^d}.
\] (3.9)

Proof. By Lemma 3.3 we only need to prove
\[
p(t, x) \leq N \left( \phi^{-1}(t^{-1}) \right)^{d/2}.
\]
We modify the proof of [21 Corollary 3.5]. Note that
\[
t\phi(|\xi|^2) = \frac{\phi(|\xi|^2)}{\phi(\phi^{-1}(t^{-1}))}.
\]
If $|\xi|^2 > \phi^{-1}(t^{-1})$, then by (3.2),
\[
\frac{\phi(|\xi|^2)}{\phi(\phi^{-1}(t^{-1}))} \geq c \left( \frac{|\xi|^2}{\phi^{-1}(t^{-1})} \right)^{\delta_0}.
\] (3.10)
Using (3.9) and (3.10),
\[
p(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i \xi \cdot x} e^{-\phi(|\xi|^2)} d\xi
\]
\[
\leq \left| \frac{1}{(2\pi)^d} \int_{|\xi|^2 > \phi^{-1}(t^{-1})} e^{i \xi \cdot x} e^{-\phi(|\xi|^2)} d\xi \right|
+ \left| \frac{1}{(2\pi)^d} \int_{|\xi|^2 \leq \phi^{-1}(t^{-1})} e^{i \xi \cdot x} e^{-\phi(|\xi|^2)} d\xi \right|
\leq \frac{1}{(2\pi)^d} \int_{|\xi|^2 > \phi^{-1}(t^{-1})} e^{-\frac{c_1^2}{\phi^{-1}(c_1 - 1)}} d\xi + \frac{1}{(2\pi)^d} \int_{|\xi|^2 \leq \phi^{-1}(t^{-1})} d\xi
\leq N \left( \phi^{-1}(t^{-1}) \right)^{d/2}.
\]

The lemma is proved. \(\square\)

**Remark 3.5.**
(i) Inequality (3.9) was introduced in [21, Corollary 3.5] on finite time interval \([0, T]\) with a constant \(N\) depending also on \(T\). Condition (H1) is used in [21, Corollary 3.5].
(ii) Assume that there exist constants \(0 < \delta_1 \leq \delta_2 < 1\) and \(c_1, c_2 > 0\) such that
\[
c_1 \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\phi(R)}{\phi(r)} \leq c_2 \left( \frac{R}{r} \right)^{\delta_2}, \quad 0 < r < R < \infty.
\]  
(3.11)

Then, by [24, Section 3], we have the sharp two-sided estimate
\[
p(t, x) \sim \left( \phi^{-1}(t^{-1}) \right)^{d/2} \wedge t \left( \frac{\phi(|x|^2)}{|x|^d} \right).
\]  
(3.12)

Note that (3.11) is stronger than Assumption (21-1). It turns out that Assumption (21) is insufficient for equivalence relation (3.12). See [3] Theorem 4.1).

The following result is a consequence of (3.9) and Lemma 3.3

**Lemma 3.6.** Let Assumption (27) hold. Then there exists a constant \(N = N(d, c, \delta_0, m)\) so that for any \((t, x) \in (0, \infty) \times \mathbb{R}^d\),
\[
|D_x^n p_d(t, x)| \leq N \sum_{m-2n \geq 0, n \in \mathbb{N}_0} |x|^{m-2n} \left( \phi^{-1}(t^{-1}) \right)^{d/2 + m-n} \wedge t \left( \frac{\phi(|x|^2)}{|x|^{d+2(m-n)}} \right).
\]

**Proof.** For \(n \in \mathbb{N}\), let \(0_{2n}\) denote the origin in \(\mathbb{R}^{2n}\). Thus, for \(x \in \mathbb{R}^d\) we have \((x, 0_{2n}) \in \mathbb{R}^{d+2n}\). By the dominated convergence theorem,
\[
\partial_x^k p_d(t, x) = \int_{(0, \infty)} (4\pi s)^{-d/2} \partial_x^k \exp \left( -\frac{|x|^2}{4s} \right) \eta_t(ds)
= -\frac{x^i}{2} \int_{(0, \infty)} s^{-1} (4\pi s)^{-d/2} \exp \left( -\frac{|x|^2}{4s} \right) \eta_t(ds)
= -2\pi x^i \partial_{d+2}(t, (x, 0_{2n})).
\]
The last equality is due to (3.6).
Similarly,  
\[ \partial_{x^i} p_d(t, x) = 4\pi^2(x^i)^2 p_{d+4}(t, (x, 0_4)) - 2\pi p_{d+2}(t, (x, 0_2)), \]
and, for \( i \neq j \),  
\[ \partial_{x^i x^j} p_d(t, x) = 4\pi^2 x^i x^j p_{d+4}(t, (x, 0_4)). \]
Repeating the product rule of differentiation (or using the induction argument), one can check that  
\[ D^m p_d(t, x) \]
is a sum of functions of the type  
\[ x^{\sigma} p_{d+2(m-n)}(t, (x, 0_{2(m-n)})}, \quad |\sigma| = m - 2n \geq 0. \]
Thus we get the claim of the lemma by applying (3.9) for dimensions \( d + 2, \ldots, d + 2m \). □

Next we study the fundamental solution \( q(t, x) \) to the equation  
\[ \partial_t \alpha u(t, x) = \phi(\Delta) u(t, x), \quad t > 0; \quad u(0, x) = u_0. \quad (3.13) \]
That is, \( q(t, x) \) is the function such that under appropriate smoothness condition on \( u_0 \), the function  
\[ u(t, x) := (q(t, \cdot) * u_0(\cdot))(x) \]
satisfies equation (3.13).

Recall that \( X_t \) is the \( d \)-dimensional subordinate Brownian motion with transition density \( p_d(t, x) \). Let \( Q_t \) be an increasing Lévy process independent of \( X_t \) having the Laplace transform  
\[ E \exp(-\lambda Q_t) = \exp(-t \lambda^\alpha). \]
Let  
\[ R_t := \inf\{s > 0 : Q_s > t\} \]
be the inverse process of the subordinator \( Q_t \), and let \( \phi(t, r) \) denote the probability density function of \( R_t \). Then, as is shown in Lemma 5.1 (cf. [7, Theorem 1.1]), the function  
\[ q(t, x) := \int_0^\infty p(r, x) d_r \mathbb{P}(R_t \leq r) = \int_0^\infty p(r, x) \phi(t, r) dr \quad (3.14) \]
becomes the fundamental solution to equation (3.13). Actually the definition of \( q(t, x) \) implies that \( q(t, x) \) is the transition density of \( Y_t := X_{R_t} \), which is called subordinate Brownian motion delayed by an inverse subordinator.

Let \( E_{\alpha, \beta} \) be the two-parameter Mittag-Leffler function defined as  
\[ E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}, \alpha > 0, \beta \in \mathbb{C}, \]
and \( E_{\alpha} := E_{\alpha, 1} \). For later use, we note the recurrence relation which follows immediately from the definition:  
\[ E_{\alpha, \beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha, \beta+\alpha}(z), \quad (3.15) \]
where \( \frac{1}{\Gamma(\beta)} := 0 \) if \( \beta = 0, -1, -2, \ldots. \)
For \( \beta \in \mathbb{R} \), denote  
\[ \varphi_{\alpha, \beta}(t, r) := D_t^{\beta-\alpha} \varphi(t, r) := (D_t^{\beta-\alpha} \varphi(\cdot, r))(t), \]
and for \( (t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\} \) define  
\[ q_{\alpha, \beta}(t, x) := \int_0^\infty p(r, x) \varphi_{\alpha, \beta}(t, r) dr. \]
Lemma 3.7. Let $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$.

(i) $\varphi_{\alpha,\beta}(t,r)$ has the series representation

$$\varphi_{\alpha,\beta}(t,r) = t^{-\beta} \sum_{k=0}^{\infty} \frac{(-rt^{-\alpha})^k}{\Gamma(1-\beta-\alpha k)} = t^{-\beta} \Phi_1 \left( a, b \right),$$

where $\Phi_1 \left( a, b \right)$ is the Wright function.

(ii) There exist constants $c, N > 0$ depending only on $\alpha, \beta$ such that

$$|\varphi_{\alpha,\beta}(t,r)| \leq N t^{-\beta} e^{-c rt^{-\alpha}^{1/(1-\alpha)}}$$

for $rt^{-\alpha} \geq 1$, and

$$|\varphi_{\alpha,\beta}(t,r)| \leq \begin{cases} N rt^{-\alpha-\beta} & \beta \in \mathbb{N} \\ N t^{-\beta} & \beta \notin \mathbb{N} \end{cases}$$

for $rt^{-\alpha} \leq 1$.

(iii) For any $(t,x) \in (0,\infty) \times \mathbb{R}^d \setminus \{0\}$,

$$D_t^{\beta-\alpha} q(t,x) = q_{\alpha,\beta}(t,x).$$

(iv) For any $t > 0, \xi \in \mathbb{R}^d$,

$$q_{\alpha,\beta}(t,\xi) = t^{\alpha-\beta} E_{\alpha,1-\alpha+\beta} (-t^{\alpha} \phi(|\xi|^2)).$$

Proof. (i) By [2] Proposition 1(a) (or [4] Theorem 4.3, [29] Remark 3.1)]

$$e^{-sR_t} = \int_{(0,\infty)} e^{-st} \varphi(t,r) dr = \sum_{k=0}^{\infty} \frac{(-s t^{-\alpha})^k}{\Gamma(\alpha k + 1)} = E_{\alpha}(-s t^{\alpha}).$$

Formula (3.16) in [25] and the equality below (3.17) in [25] lead to the following Wright function representation

$$\varphi(t,r) = t^{-\alpha} \Phi_1 \left( a, b \right) - rt^{-\alpha}.$$  

Therefore, (3.16) follows from [19] Theorem 3.1 if $\beta \leq \alpha$ and from [19] Theorem 3.5 if $\beta \geq \alpha$.

(ii) If $rt^{-\alpha} \geq 1$, then (3.17) is a consequence of the asymptotic behavior of the Wright function (see e.g. [38] Theorem 1) or [27] Theorem 25).

Also, if $rt^{-\alpha} \leq 1$, then the series representation of $\varphi_{\alpha,\beta}(t,r)$ easily yields (3.18).

(iii) By Lemma 3.4

$$|p(r,x)| \leq N(d,c,\delta_0, x)r.$$

This, (3.17) and (3.18) easily yield

$$\int_0^\infty |p(r,x)\varphi_{\alpha,\beta}(t,r)| dr \leq N \int_0^\infty r |\varphi_{\alpha,\beta}(t,r)| dr < \infty.$$  

Now we prove (3.19). First assume $\beta < \alpha$. Then, since $\varphi(t,\cdot) \geq 0$, (3.22) implies

$$\int_0^\infty \int_0^t (t-s)^{\alpha-\beta-1} \varphi(s,r) |p(r,x)| ds dr < \infty,$$

and therefore Fubini's theorem yields the desired result.
Next we assume $\beta > \alpha$. Take $n \in \mathbb{N}$ such that $n - 1 \leq \beta - \alpha < n$. By the above result, we have

$$I^{n-(\beta-\alpha)}_t q(t, x) = D^{(\beta-n)-\alpha}_t q(t, x) = q_{\alpha,\beta-n}(t, x).$$

Hence, by the definition of fractional derivative, it remains to prove that

$$\left( \frac{d}{dt} \right)^n q_{\alpha,\beta-n}(t, x) = q_{\alpha,\beta}(t, x).$$

For this, due to the dominated convergence theorem, we only need to show for any $0 < \varepsilon < T$ and $k = 0, 1, 2, \ldots, n$,

$$\int_0^\infty |p(r, x)| \times \left( \sup_{t \in [\varepsilon, T]} |\varphi_{\alpha,\beta-k}(t, r)| \right) dr < \infty. \quad (3.23)$$

For $t \in [\varepsilon, T]$, by (3.17) and (3.18), there exist $c, N > 0$ such that

$$|\varphi_{\alpha,\beta-k}(t, r)| \leq Ne^{-c r^{1/(1-\alpha)}} \quad \text{if} \quad r \geq \varepsilon^\alpha,$$

and

$$|\varphi_{\alpha,\beta-k}(t, r)| \leq N \quad \text{if} \quad r \leq T^\alpha,$$

where constants $c, N$ depend on $\alpha, \beta, k, \varepsilon$ and $T$. Combining this with (3.21) we have (3.23). The claim of (iii) is proved.

(iv) By (34) of [15], we have

$$\int_0^\infty e^{-sr} \varphi_{\alpha,\beta}(t, r) dr = t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}(-st^\alpha). \quad (3.24)$$

Therefore, by (3.24) and Fubini’s theorem

$$\hat{q}_{\alpha,\beta}(t, \xi) = \int_0^\infty \varphi_{\alpha,\beta}(t, r) \left[ \int_{\mathbb{R}^d} e^{-ix \cdot \xi} p(r, x) dx \right] dr$$

$$= \int_0^\infty \varphi_{\alpha,\beta}(t, r) e^{-r \phi(|\xi|^2)} dr$$

$$= t^{\alpha-\beta} E_{\alpha,1-\beta+\alpha}(-t^\alpha \phi(|\xi|^2)).$$

The lemma is proved. \hfill \Box

**Lemma 3.8.** Let $\alpha \in (0, 1)$, $\beta \in \mathbb{R}$, and $m \in \mathbb{N}_0$. Then there exists a constant $N = N(\alpha, \beta, d, c, \delta_0, m)$ such that

(i) for any $t > 0$, $x \in \mathbb{R}^d \setminus \{0\}$

$$|D^m_x q_{\alpha,\beta}(t, x)| \leq \frac{Nt^{2\alpha-\beta} \phi(|x|^2)}{|x|^{d+m}}, \quad (3.25)$$

(ii) furthermore, if $t^\alpha \phi(|x|^2) \geq 1$ then

$$\left| D^m_x q_{\alpha,\beta}(t, x) \right| \leq N \int_{(\phi(|x|^2))^{-1}} \left( \phi^{-1}(r^{-1}) \right)^{(d+m)/2} r^{d-\alpha-\beta} dr, \quad \beta \in \mathbb{N},$$

and

$$\left| D^m_x q_{\alpha,\beta}(t, x) \right| \leq N \int_{(\phi(|x|^2))^{-1}} \left( \phi^{-1}(r^{-1}) \right)^{(d+m)/2} t^{-\beta} dr, \quad \beta \notin \mathbb{N}.$$
Proof. Take \( x \in \mathbb{R}^d \setminus \{0\} \). Then for any \( r > 0 \) and \( y \neq 0 \) sufficiently close to \( x \), we have
\[
|D^\sigma p(r, y)| \leq N(\phi, x, d, m, r), \quad |\sigma| \leq m
\]
due to Lemma 3.6. Using (3.22) and the dominated convergence theorem, we get
\[
D_x^m q_{\alpha, \beta}(t, x) = \int_0^\infty D_x^m p(r, x) \varphi_{\alpha, \beta}(t, r) dr. \tag{3.26}
\]

For \( n \leq m/2 \), set
\[
P_{m, n}(r, x) := |x|^{m-2n} \left((\phi^{-1}(r^{-1}))^{d/2+m-n} \& r \frac{\phi(|x|^{-2})}{|x|^{d+2(m-n)}}\right). \tag{3.27}
\]

By Lemma 3.3 and (3.26), to estimate \( |D_x^m q_{\alpha, \beta}(t, x)| \), it is enough to control
\[
\int_0^\infty |P_{m, n}(r, x)\varphi_{\alpha, \beta}(t, r)| dr = I + II,
\]
where
\[
I := \int_0^{t^\alpha} |P_{m, n}(r, x)\varphi_{\alpha, \beta}(t, r)| dr, \quad II := \int_{t^\alpha}^\infty |P_{m, n}(r, x)\varphi_{\alpha, \beta}(t, r)| dr. \tag{3.28}
\]

(i) We prove (3.29) following the argument in the proof of [3, Theorem 1.4]. Note that \( |\varphi_{\alpha, \beta}(t, r)| \leq Nt^{-\beta} \) for \( r \leq t^\alpha \) due to (3.18). Thus,
\[
I \leq N \int_0^{t^\alpha} r^{-\beta} \frac{\phi(|x|^{-2})}{|x|^{d+m}} dr \leq N \int_0^{t^\alpha} \phi(|x|^{-2}) \leq \frac{N}{|x|^{d+m}}.
\]

For II, by (3.17) and the change of variables,
\[
II \leq N \int_{t^\alpha}^\infty |P_{m, n}(r, x)\varphi_{\alpha, \beta}(t, r)| dr = N \int_{t^\alpha}^\infty r^{-\beta} \frac{\phi(|x|^{-2})}{|x|^{d+m}} e^{-c(r-t^{-1})^{1/(1-\alpha)}} dr
\]
\[
= N \int_{t^\alpha}^\infty r^{-\beta} \frac{\phi(|x|^{-2})}{|x|^{d+m}} r e^{-c r^{-1/(1-\alpha)}} dr \leq N \int_{t^\alpha}^\infty r^{-\beta} \frac{\phi(|x|^{-2})}{|x|^{d+m}} dr.
\]

Hence, (3.25) is proved.

(ii) We follow the argument in the proof of [3, Theorem 4.8].

As in the proof of (i), we estimate \( I \) and \( II \) which are defined in (3.28). First, assume \( \beta \in \mathbb{N} \). From (3.18) we have
\[
I \leq N \int_0^{t^\alpha} |P_{m, n}(r, x)| r^{-\alpha-\beta} dr =: I_1 + I_2,
\]
where
\[
I_1 := \int_0^{(\phi(|x|^{-2}))^{-1}} |P_{m, n}(r, x)| r^{-\alpha-\beta} dr, \quad I_2 := \int_0^{t^\alpha} |P_{m, n}(r, x)| r^{-\alpha-\beta} dr.
\]

By (3.27),
\[
I_1 \leq N \int_0^{(\phi(|x|^{-2}))^{-1}} r^2 t^{-\alpha-\beta} \frac{\phi(|x|^{-2})}{|x|^{d+m}} dr \leq N t^{-\alpha-\beta} \frac{(\phi(|x|^{-2}))^{-2}}{|x|^{d+m}}. \tag{3.29}
\]
Note that if 
\[(\phi(|x|^{-2}))^{-1} \leq r \leq 2 \left(\phi(|x|^{-2})\right)^{-1},\]
then 
\[\phi^{-1}(r^{-1}) \leq |x|^{-2} \leq \phi^{-1}(2r^{-1}).\]
Applying \((3.2)\) with \(R = \phi^{-1}(2r^{-1})\) and \(r = \phi^{-1}(r^{-1})\) we have 
\[\phi^{-1}(r^{-1}) \leq |x|^{-2} \leq N(c, \delta_0)\phi^{-1}(r^{-1}).\] (3.30)
Therefore, by \((3.29)\) and \((3.30)\)
\[I_1 \leq Nt^{-\alpha-\beta} \left(\frac{\phi(|x|^{-2})}{|x|^{d+m}}\right)^{-2} \leq N \int_{\phi(|x|^{-2})^{-1}}^{2\phi(|x|^{-2})^{-1}} \left(\phi^{-1}(r^{-1})\right)^{(d+m)/2}rt^{-\alpha-\beta} dr \]
\[\leq N \int_{\phi(|x|^{-2})^{-1}}^{2^{\alpha}} \left(\phi^{-1}(r^{-1})\right)^{(d+m)/2}rt^{-\alpha-\beta} dr.\] (3.31)
To estimate \(I_2\), we note that since \(\phi\) and \(\phi^{-1}\) are increasing, \(r\phi(|x|^{-2}) \geq 1\) implies 
\[|x|^2 \leq \left(\phi^{-1}(r^{-1})\right)^{-1}.\] (3.32)
Applying \((3.32)\) and \((3.27)\) again,
\[I_2 \leq N|x|^{m-2n} \int_{\phi(|x|^{-2})^{-1}}^{2^{\alpha}} \left(\phi^{-1}(r^{-1})\right)^{(d/2+m-n)/2}t^{-\alpha-\beta} dr \]
\[\leq N \int_{\phi(|x|^{-2})^{-1}}^{2^{\alpha}} \left(\phi^{-1}(r^{-1})\right)^{(d+m)/2}rt^{-\alpha-\beta} dr.\] (3.33)
Now we estimate \(II\). By \((3.2)\) with \(R = \phi^{-1}(t^{-\alpha})\) and \(r = \phi^{-1}(r^{-1})\) we find that 
\[\phi^{-1}(r^{-1}) \leq t^{\alpha}r^{-1}\phi^{-1}(t^{-\alpha}) \quad \text{if} \quad t^{\alpha} \leq r.\]
Therefore, by the change of variables \(r \to t^{\alpha}r,\)
\[II \leq N|x|^{m-2n} \int_{t^{\alpha}}^{\infty} \left(\phi^{-1}(r^{-1})\right)^{(d/2+m-n)/2}t^{-\beta}e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} dr \]
\[\leq N|x|^{m-2n} \int_{t^{\alpha}}^{\infty} \left(t^{\alpha}r^{-1}\phi^{-1}(t^{-\alpha})\right)^{(d/2+m-n)/2}t^{-\beta}e^{-c(rt^{-\alpha})^{1/(1-\alpha)}} dr \]
\[= N|x|^{m-2n} t^{\alpha-\beta} \left(\phi^{-1}(t^{-\alpha})\right)^{(d/2+m-n)} \int_{1}^{\infty} r^{-d/2+m-n}e^{c^{1/(1-\alpha)}} dr \]
\[\leq N|x|^{m-2n} t^{\alpha-\beta} \left(\phi^{-1}(t^{-\alpha})\right)^{(d/2+m-n)} \]
\[\leq Nt^{\alpha-\beta} \left(\phi^{-1}(t^{-\alpha})\right)^{(d+m)/2}.\] (3.34)
For the last inequality above, we used \(t^{\alpha}\phi(|x|^{-2}) \geq 1.\)
Note that if \(t^{\alpha} \leq r \leq 2t^{\alpha},\) then 
\[\phi^{-1}(t^{-\alpha}) \leq \phi^{-1}(2r^{-1}) \leq N(c, \delta_0)\phi^{-1}(r^{-1}),\] (3.35)
where the last inequality is from \((3.2)\) with \(R = \phi^{-1}(2r^{-1})\) and \(r = \phi^{-1}(r^{-1}).\)
Therefore, we have 
\[t^{\alpha-\beta} \left(\phi^{-1}(t^{-\alpha})\right)^{(d+m)/2} \leq N \int_{t^{\alpha}}^{2^{\alpha}} \left(\phi^{-1}(r^{-1})\right)^{(d+m)/2}rt^{-\alpha-\beta} dr \]
\[\leq N \int_{\phi(|x|^{-2})^{-1}}^{2^{\alpha}} \left(\phi^{-1}(r^{-1})\right)^{(d+m)/2}rt^{-\alpha-\beta} dr,\]
and this gives the desired result in (ii) for $\beta \in \mathbb{N}$.

Next, we assume $\beta \notin \mathbb{N}$. We repeat the above argument used to estimate $I$ and $II$. For $I$, by (3.18), we need to estimate

\[ \tilde{I}_1 := \int_0^{(\phi(|x|^2))^{-1}} |P_{m,n}(r,x)|t^{-\beta}dr, \quad \tilde{I}_2 := \int_{(\phi(|x|^2))^{-1}}^{t^\alpha} |P_{m,n}(r,x)|t^{-\beta}dr, \]

instead of $I_1$ and $I_2$ respectively. As in (3.29) and (3.31), one can prove

\[ \tilde{I}_1 \leq Nt^{-\beta}(\phi(|x|^2))^{-1} \leq N \int_{(\phi(|x|^2))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2}t^{-\beta}dr. \]

One can handle $\tilde{I}_2$ as in (3.33) and prove

\[ \tilde{I}_2 \leq N \int_{(\phi(|x|^2))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{d/2}t^{-\beta}dr. \]

Finally we consider $II$. Note that (3.34) holds even if $\beta \notin \mathbb{N}$. Hence, by (3.35),

\[ t^\alpha t^{-\beta}(\phi^{-1}(t^{-\alpha}))^{(d+m)/2} \leq N \int_{t^\alpha}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2}t^{-\beta}dr \]

\[ \leq N \int_{(\phi(|x|^2))^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{(d+m)/2}t^{-\beta}dr. \]

Thus (ii) is also proved. \qed

**Corollary 3.9.** Let $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$.

(i) There exists a constant $N = N(\alpha, \beta, d, c, \delta_0)$ such that

\[ \int_{\mathbb{R}^d} |q_{\alpha,\beta}(t,x)|dx \leq N t^{\alpha-\beta}, \quad t > 0. \]

(ii) For any $0 < \varepsilon < T < \infty$,

\[ \int_{\mathbb{R}^{d+1}} \sup_{[\varepsilon,T]} |q_{\alpha,\beta}(t,x)|dx < \infty. \]
Proof. (i) Due to the similarity, we only consider the case $\beta \in \mathbb{N}$. By Lemma 3.1, Lemma 3.8 and Fubini's theorem,

$$
\int_{\mathbb{R}^d} |q_{\alpha, \beta}(t, x)| dx = \int_{|x| \geq \phi^{-1}(t^{-\alpha}))-1/2} |q_{\alpha, \beta}(t, x)| dx \\
+ \int_{|x| < \phi^{-1}(t^{-\alpha}))-1/2} |q_{\alpha, \beta}(t, x)| dx \\
\leq N \int_{|x| \geq \phi^{-1}(t^{-\alpha}))-1/2} \frac{\phi(|x|^{-2})}{|x|^d} dx \\
+ N \int_{|x| < \phi^{-1}(t^{-\alpha}))-1/2} \int_{\phi(|x|^{-2})^{-1}}^{2t^\alpha} (\phi^{-1}(r^{-1}))^{d/2} r t^{-\alpha-\beta} dr dx \\
\leq N \int_{r \geq \phi^{-1}(t^{-\alpha}))-1/2} \frac{t^{2\alpha-\beta} \phi(r^{-2})}{r} dr \\
+ N \int_{\phi(|x|^{-2})^{-1} \leq r} \frac{t^{2\alpha-\beta} \phi(r^{-2})}{r} dr dx \\
\leq N t^{\alpha-\beta} + N t^{2\alpha-\beta} \leq N t^{\alpha-\beta}.
$$

(ii) Again we only prove the case $\beta \in \mathbb{N}$. Let $0 < \varepsilon < T < \infty$. Since $t^{2\alpha-\beta} \leq N(\varepsilon, T, \alpha, \beta)$ for $t \in [\varepsilon, T]$, by Lemma 3.8,

$$
|q_{\alpha, \beta}(t, x)| \leq N(\alpha, \beta, d, c, \delta_0, \varepsilon, T) \frac{\phi(|x|^{-2})}{|x|^d}, \quad t \in [\varepsilon, T].
$$

Also, if $\varepsilon^\alpha \phi(|x|^{-2}) \geq 1$, and $t \in [\varepsilon, T]$, then using Lemma 3.8 again, we get

$$
|q_{\alpha, \beta}(t, x)| \leq N(\alpha, \beta, d, c, \delta_0, \varepsilon, T) \int_{\phi(|x|^{-2})^{-1}}^{2T^\alpha} (\phi^{-1}(r^{-1}))^{d/2} r dr.
$$

As in the proof of (i),

$$
\int_{\mathbb{R}^d} \sup_{[\varepsilon, T]} |q_{\alpha, \beta}(t, x)| dx = \int_{|x| \geq \phi^{-1}(t^{-\alpha}))-1/2} \sup_{[\varepsilon, T]} |q_{\alpha, \beta}(t, x)| dx \\
+ \int_{|x| < \phi^{-1}(t^{-\alpha}))-1/2} \sup_{[\varepsilon, T]} |q_{\alpha, \beta}(t, x)| dx \\
\leq N \int_{|x| \geq \phi^{-1}(t^{-\alpha}))-1/2} \frac{\phi(|x|^{-2})}{|x|^d} dx \\
+ N \int_{|x| < \phi^{-1}(t^{-\alpha}))-1/2} \int_{\phi(|x|^{-2})^{-1}}^{2T^\alpha} (\phi^{-1}(r^{-1}))^{d/2} r dr dx \\
\leq N + N \int_{0}^{2T^\alpha} \int_{\phi(|x|^{-2})^{-1} \leq r} (\phi^{-1}(r^{-1}))^{d/2} r dx dr \\
\leq N + N \int_{0}^{2T^\alpha} r dr < \infty.
$$

The corollary is proved. \qed
4. KEY ESTIMATES: BMO AND $L_q(L_p)$-ESTIMATES

In this section we prove some a priori estimates for solutions to the equation with zero initial condition

$$\partial_t^\alpha u = \phi(\Delta)u + f, \quad t > 0; \quad u(0, \cdot) = 0. \quad \text{(4.1)}$$

We first present the representation formula.

**Lemma 4.1.** (i) Let $u \in C_c^\infty(\mathbb{R}_+^{d+1})$ and denote $f := \partial_t^\alpha u - \phi(\Delta)u$. Then

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} q_{\alpha, 1}(t - s, x - y)f(s, y)dyds. \quad \text{(4.2)}$$

(ii) Let $f \in C_c^\infty(\mathbb{R}_+^{d+1})$ and define $u$ as in (4.2). Then $u$ satisfies equation (4.1) for each $(t, x)$.

**Proof.** See [22, Lemma 3.5], which treats the case $\phi(\lambda) = \lambda$. The proof for the general case is same. The only difference is one needs to use formula (3.20) and Corollary 3.9 in place of their corresponding results when $\phi(\lambda) = \lambda$. \hfill $\square$

Recall that due to Corollary 3.9 (i), $q_{\alpha, 1}$ is integrable on $[0, T] \times \mathbb{R}^d$ for any $T > 0$. Also note that $\phi(\Delta)f$ is bounded for any $f \in C_c^\infty(\mathbb{R}_+^{d+1})$. Therefore the operators $G_0$ and $G$ are well defined on $C_c^\infty(\mathbb{R}_+^{d+1})$:

$$G_0f := \int_{-\infty}^t \int_{\mathbb{R}^d} q_{\alpha, 1}(t - s, y)f(s, y)dyds,$$

$$Gf := \phi(\Delta)G_0f = \int_{-\infty}^t \int_{\mathbb{R}^d} q_{\alpha, 1}(t - s, y)\phi(\Delta)f(s, y)dyds.$$  

For each fixed $s$ and $t$ such that $s < t$, define

$$T_{t,s}f(x) := \int_{\mathbb{R}^d} q_{\alpha, 1}(t - s, x - y)\phi(\Delta)f(s, y)dy,$$

and

$$G_{t,s}f(x) := \int_{\mathbb{R}^d} q_{\alpha, 1+\alpha}(t - s, x - y)f(s, y)dy.$$  

Note that, by Corollary 3.9, $T_{t,s}f$ and $G_{t,s}f$ are square integrable. Moreover, from (3.15) and (3.20) we have

$$F_d\{T_{t,s}f\}(\xi) = -\phi(|\xi|^2)\hat{q}_{\alpha, 1}(t - s, \xi)\hat{f}(s, \xi) = \hat{q}_{\alpha, 1+\alpha}(t - s, \xi)\hat{f}(s, \xi) = F_d\{G_{t,s}f\}(\xi).$$

This implies that $T_{t,s}f = G_{t,s}f$ for $s < t$. Therefore, we have

$$Gf(t, x) = \phi(\Delta)G_0f = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{t-\varepsilon} \left( \int_{\mathbb{R}^d} q_{\alpha, 1+\alpha}(t - s, x - y)f(s, x - y)dy \right) ds.$$  

The following result concerns the $L_2$-boundedness of the operator $G$.

**Lemma 4.2.** There exists a constant $N$ depending only on $\alpha$ and $d$ such that for any $f \in C_c^\infty(\mathbb{R}_+^{d+1})$

$$\|Gf\|_{L_2(\mathbb{R}_+^{d+1})} \leq N\|f\|_{L_2(\mathbb{R}_+^{d+1})}. \quad \text{(4.3)}$$

Consequently, operator $G$ is continuously extended to $L_2(\mathbb{R}_+^{d+1})$. 


Proof. We follow the proof of [22, Lemma 3.6] which treats the case \( \phi(\lambda) = \lambda \).

Denote \( q_M := q_{\alpha,1} 1_{0 < t < M} \). By Corollary 3.9 \( q_M \) is integrable in \( \mathbb{R}^{d+1} \). Denote
\[
G_M f = q_M * \phi(\Delta) f.
\]

Then by Parseval’s identity
\[
\|G_M f\|_{L_2}^2 = N(d) \int_{\mathbb{R}^{d+1}} |\mathcal{F}_{d+1}(q_M * \phi(\Delta) f)|^2 \, d\tau d\xi,
\]
where \( \mathcal{F}_{d+1} \) is the \((d+1)\)-dimensional Fourier transform. By the properties of the Fourier transform,
\[
\mathcal{F}_{d+1}(q_M * \phi(\Delta) f)(\tau, \xi) = -N(d) \phi(|\xi|^2) \mathcal{F}_{d+1}(q_M)(\tau, \xi) \mathcal{F}_{d+1}(f)(\tau, \xi) = -N(d) J_M(\tau, \xi) \mathcal{F}_{d+1}(f)(\tau, \xi),
\]
where
\[
J_M(\tau, \xi) := - \phi(|\xi|^2) \mathcal{F}_{d+1}(q_M)(\tau, \xi)
\]
\[
= \int_0^M - \phi(|\xi|^2) e^{-i\tau t} \mathcal{F}(q_M)(t, \xi) \, dt
\]
\[
= \int_0^M - \phi(|\xi|^2) e^{-i\tau t} D_t^{1-\alpha} E_{\alpha}(-t^\alpha \phi(|\xi|^2)) \, dt.
\]
The last equality is due to (3.20). By [14, Proposition 3.25],
\[
- \phi(|\xi|^2) D_t^{1-\alpha} E_{\alpha}(-t^\alpha \phi(|\xi|^2)) = \frac{d}{dt} E_{\alpha}(-t^\alpha \phi(|\xi|^2)).
\]

Therefore,
\[
J_M(\tau, \xi) = \int_0^M e^{-i\tau t} \frac{d}{dt} E_{\alpha}(-t^\alpha \phi(|\xi|^2)) \, dt
\]
\[
= \int_0^M i\tau e^{-i\tau t} E_{\alpha}(-t^\alpha \phi(|\xi|^2)) \, dt + e^{-i\tau M} E_{\alpha}(-M^\alpha \phi(|\xi|^2)) - 1
\]
\[
= \int_{|\tau| M} e^{-i\tau M} E_{\alpha}(-M^\alpha \phi(|\xi|^2)) - 1
\]
\[
= IM(\tau, \sqrt{\phi(|\xi|^2)}),
\]
where \( IM(\tau, \lambda) \) is taken from formula (3.17) in [22], and by the inequality above (3.16) in [22],
\[
\sup_{M, \lambda > 0, \tau \in \mathbb{R}} |IM(\tau, \lambda)| < \infty.
\]

This, (4.2), (4.3) and Fatou’s lemma lead to (4.4). The lemma is proved. \( \square \)

We define an increasing function \( \kappa : (0, \infty) \to (0, \infty) \) by
\[
\kappa(b) = (\phi(b^{-2}))^{-1/\alpha}, \quad b > 0.
\]
For \((t, x) \in \mathbb{R}^{d+1} \) and \( b > 0 \), denote
\[
Q_b(t, x) = (t - \kappa(b), t + \kappa(b)) \times B_b(x),
\]
and
\[
Q_b = Q_b(0, 0), \quad B_b = B_b(0).
\]
For measurable subsets $Q \subset \mathbb{R}^{d+1}$ with finite measure and locally integrable functions $h$, define

$$h_Q = \int_Q h(s,y)dyds = \frac{1}{|Q|} \int_Q h(s,y)dyds,$$

where $|Q|$ is the Lebesgue measure of $Q$.

In the following lemmas, Lemmas 4.3 - 4.7, we estimate the mean oscillation of $\mathcal{G}f$ on $Q_b$. For this, we consider the cases

- (Lemma 4.4) $f$ has support in $(-3\kappa(b), \infty) \times \mathbb{R}^d$;
- (Lemma 4.5) $f$ has support in $(-\infty, -2\kappa(b)) \times \mathbb{R}^d$;

The second case above is further divided into the cases

- (Lemma 4.6) $f$ has support in $(-\infty, -2\kappa(b)) \times B_{2b}$;
- (Lemma 4.7) $f$ has support in $(-\infty, -2\kappa(b)) \times B_{3b}$.

**Lemma 4.3.** Let $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-3\kappa(b), 3\kappa(b)) \times B_{3b}$ for some $b > 0$. Then,

$$\int_{Q_b} |\mathcal{G}f(t,x)|dxdt \leq N\|f\|_{L_\infty(\mathbb{R}^{d+1})},$$

where $N$ depends only on $\alpha$ and $d$.

**Proof.** By the assumption and (3.2)

$$\|f\|_{L_2(\mathbb{R}^{d+1})} \leq N(\alpha, d)|Q_b|^{1/2}\|f\|_{L_\infty(\mathbb{R}^{d+1})}. $$

Thus, by Hölder’s inequality and Lemma 4.2

$$\int_{Q_b} |\mathcal{G}f(t,x)|dxdt \leq \left( \int_{Q_b} |\mathcal{G}f(t,x)|^2dxdt \right)^{1/2} |Q_b|^{1/2} \leq \|\mathcal{G}f\|_{L_2(\mathbb{R}^{d+1})}|Q_b|^{1/2} \leq N\|f\|_{L_2(\mathbb{R}^{d+1})}|Q_b|^{1/2} \leq N\|f\|_{L_\infty(\mathbb{R}^{d+1})}|Q_b|.$$ 

The lemma is proved. \(\square\)

Here is a generalization of Lemma 4.3.

**Lemma 4.4.** Let $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-3\kappa(b), \infty) \times \mathbb{R}^d$ for some $b > 0$. Then,

$$\int_{Q_b} |\mathcal{G}f(t,x)|dxdt \leq N\|f\|_{L_\infty(\mathbb{R}^{d+1})},$$

where $N$ depends only on $\alpha, d, c$ and $\delta_0$.

**Proof.** Take $\zeta_0 = \zeta_0(t) \in C_c^\infty(\mathbb{R})$ such that $0 \leq \zeta_0 \leq 1$, $\zeta_0(t) = 1$ for $t \leq 2\kappa(b)$, and $\zeta_0(t) = 0$ for $t \geq 5/2\kappa(b)$. Note that $\mathcal{G}f = \mathcal{G}(f\zeta_0)$ on $Q_b$ and $|f\zeta_0| \leq |f|$. This implies that to prove the lemma it is enough to assume $f(t, x) = 0$ if $|t| \geq 3\kappa(b)$.

Choose a function $\zeta = \zeta(x) \in C_c^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ in $B_{7b/3}$, $\zeta = 0$ outside of $B_{8b/3}$ and $0 \leq \zeta \leq 1$. Set $f_1 = \zeta f$ and $f_2 = (1-\zeta)f$. Then $\mathcal{G}f = \mathcal{G}f_1 + \mathcal{G}f_2$. Since
\( \mathcal{G}f_1 \) can be estimated by Lemma 3.3 to prove the lemma, we may further assume that \( f(t, y) = 0 \) if \( y \in B_{2b} \). Therefore, for any \( x \in B_b \),

\[
\int_{\mathbb{R}^d} |q_{\alpha,1+\alpha}(t-s,x-y)f(s,y)| \, dy = \int_{|y-x| \geq 2b} |q_{\alpha,1+\alpha}(t-s,y)f(s,x-y)| \, dy \\
\leq \int_{|y| \geq b} |q_{\alpha,1+\alpha}(t-s,y)f(s,x-y)| \, dy.
\]

By (3.1) and (3.25),

\[
\int_{|y| \geq b} |q_{\alpha,1+\alpha}(t-s,y)f(s,x-y)| \, dy \\
\leq \|f\|_{L^\infty(\mathbb{R}^{d+1})} 1_{|s| \leq 3\kappa(b)} \int_{|y| \geq b} |q_{\alpha,1+\alpha}(t-s,y)| \, dy \\
\leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})} 1_{|s| \leq 3\kappa(b)} \int_b^\infty (t-s)^{\alpha-1} \frac{\phi(\rho^2)}{\rho^2} \rho^{d-1} \, d\rho \\
\leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})} 1_{|s| \leq 3\kappa(b)} (t-s)^{\alpha-1} \phi(b^{-2}).
\]

Note that if \( |t| \leq \kappa(b) \) and \( |s| \leq 3\kappa(b) \) then \( |t-s| \leq 4\kappa(b) \). It follows that for any \((t,x) \in Q_b\),

\[
|\mathcal{G}f(t,x)| \leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})} \phi(b^{-2}) \int_{|t-s| \leq 4\kappa(b)} |t-s|^{\alpha-1} \, ds \\
\leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

This implies the desired estimate. \( \square \)

**Lemma 4.5.** Let \( f \in C^\infty_c(\mathbb{R}^{d+1}) \) have a support in \((-\infty,-2\kappa(b)) \times \mathbb{R}^d\) for some \( b > 0 \). Then, for any \((t_1,x),(t_2,x) \in Q_b\),

\[
|\mathcal{G}f(t_1,x) - \mathcal{G}f(t_2,x)| \leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})},
\]

where \( N \) depends only on \( \alpha,d,c \) and \( \delta_0 \). In particular, we have

\[
\int_{Q_b} \left| \int_{Q_b} |\mathcal{G}f(t_1,x) - \mathcal{G}f(t_2,x)| \, dx \right| dt_1 dt_2 \leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

**Proof.** Without loss of generality, we assume \( t_1 > t_2 \). Then, since \( f(s,x) = 0 \) for \( s \geq -2\kappa(b) \) and \( t_1, t_2 \geq -\kappa(b) \), it follows that

\[
|\mathcal{G}f(t_1,x) - \mathcal{G}f(t_2,x)| \\
= \left| \int_{-\infty}^{t_1} \int_{\mathbb{R}^d} q_{\alpha,1+\alpha}(t_1-s,y)f(s,y)dyds \\
- \int_{-\infty}^{t_2} \int_{\mathbb{R}^d} q_{\alpha,1+\alpha}(t_2-s,y)f(s,y)dyds \right| \\
= \left| \int_{-\infty}^{-2\kappa(b)} \int_{\mathbb{R}^d} (q_{\alpha,1+\alpha}(t_1-s,x-y) - q_{\alpha,1+\alpha}(t_2-s,x-y)) f(s,y)dyds \right|.
\]
By the fundamental theorem of calculus and (3.19), we have
\[
\left| \int_{-\infty}^{-2\kappa(b)} \int_{\mathbb{R}^d} (q_{\alpha,1+\alpha}(t_1 - s, x - y) - q_{\alpha,1+\alpha}(t_2 - s, x - y)) f(s, y) dy ds \right|
= \left| \int_{-\infty}^{-2\kappa(b)} \int_{\mathbb{R}^d} \int_{t_2}^{t_1} q_{\alpha,2+\alpha}(t - s, x - y) f(s, y) dt dy ds \right|.
\]

By Corollary 3.9 (i),
\[
\int_{\mathbb{R}^d} \int_{t_2}^{t_1} |q_{\alpha,2+\alpha}(t - s, x - y) f(s, y)| dt dy \leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})} \int_{t_2}^{t_1} (t - s)^{-2} dt.
\]

Therefore, if \(-\kappa(b) \leq t_2 < t_1 \leq \kappa(b)\),
\[
\left| \int_{-\infty}^{-2\kappa(b)} \int_{\mathbb{R}^d} \int_{t_2}^{t_1} q_{\alpha,2+\alpha}(t - s, x - y) f(s, y) dt dy ds \right|
\leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})} \left( \int_{t_2}^{t_1} \int_{-\infty}^{-2\kappa(b)} (t - s)^{-2} ds dt \right)
\leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})} \left( \int_{t_2}^{t_1} \frac{1}{\kappa(b)} dt \right) \leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

This certainly proves the lemma. \(\square\)

**Lemma 4.6.** Let \(f \in C^\infty_c(\mathbb{R}^{d+1})\) have a support in \((-\infty, -2\kappa(b)) \times B^c_2\) for some \(b > 0\). Then, for any \((t, x_1), (t, x_2) \in Q_b\),
\[
|\mathcal{G}f(t, x_1) - \mathcal{G}f(t, x_2)| \leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})},
\]

where \(N\) depends only on \(\alpha, d, c\) and \(\delta_0\). In particular, we have
\[
\int_{Q_b} \int_{Q_b} |\mathcal{G}f(t, x_1) - \mathcal{G}f(t, x_2)| dx_1 dt dx_2 dt \leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

**Proof.** Recall \(f(s, y) = 0\) if \(s \geq -2\kappa(b)\) or \(|y| \leq 2b\). Thus, if \(t > -\kappa(b)\),
\[
|\mathcal{G}f(t, x_1) - \mathcal{G}f(t, x_2)|
= |\int_{-\infty}^{-2\kappa(b)} \int_{|y| \geq 2b} (q_{\alpha,1+\alpha}(t - s, x_1 - y) - q_{\alpha,1+\alpha}(t - s, x_2 - y)) f(s, y) dy ds|.
\]

By the fundamental theorem of calculus, for any \(x_1, x_2 \in B_2\) and \(t > -\kappa(b)\),
\[
|\int_{-\infty}^{-2\kappa(b)} \int_{|y| \geq 2b} (q_{\alpha,1+\alpha}(t - s, x_1 - y) - q_{\alpha,1+\alpha}(t - s, x_2 - y)) f(s, y) dy ds|
\leq |\int_{-\infty}^{-2\kappa(b)} \int_{|y| \geq 2b} \int_{0}^{1} \nabla q_{\alpha,1+\alpha}(t - s, \theta(x_1, x_2, u) - y) \cdot (x_2 - x_1) f(s, y) du dy ds|
\leq Nb \|f\|_{L^\infty(\mathbb{R}^{d+1})} \int_{-\infty}^{-2\kappa(b)} \int_{|y| \geq 2b} \left| \nabla q_{\alpha,1+\alpha}(t - s, y) \right| dy ds
\leq Nb \|f\|_{L^\infty(\mathbb{R}^{d+1})} \int_{\kappa(b)}^{\infty} \int_{|y| \geq 2b} \left| \nabla q_{\alpha,1+\alpha}(s, y) \right| dy ds,
\]
(4.6)
where \( \theta(x_1, x_2, u) = (1 - u)x_1 + ux_2 \). By Lemma 3.8

\[
\int_{|y|\geq b} \int_{\kappa(b)}^{\infty} \nabla q_{s,1+\alpha}(s,y) dy ds \\
\leq N \int_{\kappa(b)}^{\infty} \int_{\phi^{-1}(s^{-\alpha})}^{\infty} s^{\alpha-1} \frac{\phi(p^{-2})}{p^2} dp ds \\
+ N \int_{\kappa(b)}^{\infty} \int_{\phi(b)^{-1}}^{2s^\alpha} \int_{\phi(p^{-2})}^{\phi^{-1}(r^{-1})} (\phi^{-1}(p^{-1}))^{(d+1)/2} s^{-\alpha-1} p^{d-1} dp dr ds.
\]

We now estimate the last two integrals above. First, by (3.1),

\[
\int_{\phi^{-1}(s^{-\alpha})}^{\infty} s^{\alpha-1} \frac{\phi(p^{-2})}{p^2} dp \\
\leq (\phi^{-1}(s^{-\alpha}))^{1/2} \int_{\phi^{-1}(s^{-\alpha})}^{\infty} s^{\alpha-1} \frac{\phi(p^{-2})}{p} dp \\
\leq (\phi^{-1}(s^{-\alpha}))^{1/2} s^{-1}.
\]

Therefore, by the change of the variables \( s^\alpha \to s \),

\[
\int_{\kappa(b)}^{\infty} \int_{\phi^{-1}(s^{-\alpha})}^{\infty} s^{\alpha-1} \frac{\phi(p^{-2})}{p^2} dp ds \\
\leq N \int_{\kappa(b)}^{\infty} (\phi^{-1}(s^{-\alpha}))^{1/2} s^{-1} ds = N \int_{\kappa(b)}^{\infty} (\phi^{-1}(s^{-1}))^{1/2} s^{-1} ds.
\]

Second, by Fubini’s theorem,

\[
\int_{\kappa(b)}^{\infty} \int_{\phi^{-1}(s^{-\alpha})}^{\infty} s^{\alpha-1} \frac{\phi(p^{-2})}{p^2} dp ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{\phi^{-1}(s^{-\alpha})}^{\phi^{-1}(r^{-1})} (\phi^{-1}(p^{-1}))^{(d+1)/2} s^{-\alpha-1} p^{d-1} dp dr ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{\phi^{-1}(r^{-1})}^{\phi^{-1}(r^{-1})} (\phi^{-1}(r^{-1}))^{1/2} s^{-\alpha-1} dr ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{\phi^{-1}(r^{-1})}^{\phi^{-1}(r^{-1})} (\phi^{-1}(r^{-1}))^{1/2} s^{-\alpha-1} dr ds \\
\leq N \int_{\kappa(b)}^{\infty} (\phi^{-1}(r^{-1}))^{1/2} r^{-1} dr.
\]

Note that if \( s \geq (\kappa(b))^{\alpha} \), then by (3.2) with \( R = \phi^{-1}((\kappa(b))^{-\alpha}) \) and \( r = \phi^{-1}(s^{-1}) \), we have

\[ \phi^{-1}(s^{-1}) \leq s^{-1} (\kappa(b))^{\alpha} \phi^{-1}((\kappa(b))^{-\alpha}). \]

Therefore,

\[
\int_{\kappa(b)}^{\infty} (\phi^{-1}(s^{-1}))^{1/2} s^{-1} ds \leq N (\phi^{-1}((\kappa(b))^{-\alpha}))^{1/2} (\kappa(b))^{\alpha/2} \int_{\kappa(b)}^{\infty} s^{-3/2} ds \\
= N (\phi^{-1}((\kappa(b))^{-\alpha}))^{1/2} = Nb^{-1}.
\]

Combining this with (4.7) and (4.8), and going back to (4.1), we get

\[ |Gf(t, x_1) - Gf(t, x_2)| \leq Nb\|f\|_{L^\infty(R^{d+1})} b^{-1} = N\|f\|_{L^\infty(R^{d+1})}. \]
Therefore, the lemma is proved. □

Lemma 4.7. Let \( f \in C_c^\infty(\mathbb{R}^{d+1}) \) have a support in \((-\infty, -2\kappa(b)) \times B_{3b}\) for some \( b > 0 \). Then for any \((t, x) \in Q_b\)

\[
|\mathcal{G} f(t, x)| \leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})},
\]

where \( N \) depends only on \( \alpha, d, c \) and \( \delta_0 \). In particular,

\[
\int_{Q_b} |\mathcal{G} f(t, x)| \, dx \, dt \leq N \|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

Proof. By assumption, \((t, x) \in Q_b\),

\[
|\mathcal{G} f(t, x)| \leq \int_{-\infty}^{-2\kappa(b)} \int_{B_{3b}} |q_{\alpha,1+\alpha}(t-s, x-y) f(s, y)| \, dy \, ds
\]

\[
\leq N \|f\|_{L^\infty} \int_{-\infty}^{-2\kappa(b)} \int_{B_{3b}} |q_{\alpha,1+\alpha}(t-s, x-y)| \, dy \, ds
\]

\[
\leq N \|f\|_{L^\infty} \int_{\kappa(b)}^{\infty} \int_{B_{4b}} |q_{\alpha,1+\alpha}(s, y)| \, dy \, ds
\]

\[
\leq N \|f\|_{L^\infty} (I + II),
\]

where

\[
I = \int_{\kappa(b)}^{\kappa(4b)} \int_{B_{4b}} |q_{\alpha,1+\alpha}(s, y)| \, dy \, ds,
\]

\[
II = \int_{\kappa(4b)}^{\infty} \int_{B_{4b}} |q_{\alpha,1+\alpha}(s, y)| \, dy \, ds.
\]

By Corollary 3.9 (i) and 3.2

\[
I \leq N \int_{\kappa(b)}^{\kappa(4b)} s^{-1} \, ds \leq N \log \left( \frac{\kappa(4b)}{\kappa(b)} \right)
\]

\[
= N \log \left( \frac{\phi(b^{-2})}{\phi(b^{-2}/16)} \right)
\]

\[
\leq N \log (16).
\]

By Lemma 3.8

\[
II \leq N \int_{\kappa(4b)}^{\infty} \int_{B_{4b}} \int_{\phi(|y|-2)}^{2s^\alpha} (\phi^{-1}(r^{-1}))^{d/2} s^{-\alpha-1} \, dr \, dy \, ds.
\]
By Fubini’s theorem, if $s > \kappa(4b)$,
\[
\int_{B_{4b}} \int_{[\phi(|y|^2)]^{-1}}^{2s^\alpha} \left( \phi^{-1}(r^{-1}) \right)^{d/2} s^{-\alpha - 1} dr dy
\]
\[
= \int_{B_{4b}} \int_{[\phi(|y|^2)]^{-1}}^{2s^\alpha} \left( \phi^{-1}(r^{-1}) \right)^{d/2} s^{-\alpha - 1} dr dy
\]
\[
+ \int_{B_{4b}} \int_{[\phi(b^{-2}/16)]^{-1}}^{2s^\alpha} \left( \phi^{-1}(r^{-1}) \right)^{d/2} s^{-\alpha - 1} dr dy
\]
\[
\leq \int_{0}^{2s^\alpha} \int_{[\phi(|y|^2)]^{-1}}^{[\phi(b^{-2}/16)]^{-1}} \left( \phi^{-1}(r^{-1}) \right)^{d/2} s^{-\alpha - 1} dy dr
\]
\[
+ \int_{[\phi(b^{-2}/16)]^{-1}}^{2s^\alpha} \int_{B_{4b}} \left( \phi^{-1}(r^{-1}) \right)^{d/2} s^{-\alpha - 1} dy dr
\]
\[
\leq N \left( \phi(b^{-2}/16) \right)^{-1} s^{-\alpha - 1} + Nb^d s^{-\alpha - 1} \int_{[\phi(b^{-2}/16)]^{-1}}^{2s^\alpha} \left( \phi^{-1}(r^{-1}) \right)^{d/2} dr.
\]
Obviously,
\[
\int_{\kappa(4b)}^{\infty} \left( \phi(b^{-2}/16) \right)^{-1} s^{-\alpha - 1} ds \leq \frac{1}{\alpha}
\]
Also, by Fubini’s theorem and (3.2) with $R = b^{-2}/16$ and $r = \phi^{-1}(r^{-1})$
\[
\int_{\kappa(4b)}^{\infty} b^d s^{-\alpha - 1} \int_{[\phi(b^{-2}/16)]^{-1}}^{2s^\alpha} \left( \phi^{-1}(r^{-1}) \right)^{d/2} dr ds
\]
\[
\leq \int_{[\phi(b^{-2}/16)]^{-1}}^{\infty} \int_{[r/2]^{1/\alpha}}^{\infty} b^d s^{-\alpha - 1} \left( \phi^{-1}(r^{-1}) \right)^{d/2} ds dr
\]
\[
\leq N \int_{[\phi(b^{-2}/16)]^{-1}}^{\infty} b^d r^{-1} \left( \phi^{-1}(r^{-1}) \right)^{d/2} dr
\]
\[
\leq N \int_{[\phi(b^{-2}/16)]^{-1}}^{\infty} \left( \phi(b^{-2}/16) \right)^{-d/2} r^{-d/2 - 1} dr \leq N.
\]
Therefore, $I, II$ are bounded by a constant independent of $b$, and the lemma is proved. 

**Corollary 4.8.** Let $f \in C^\infty_c(\mathbb{R}^{d+1})$ and $b > 0$. Then,
\[
\int_{Q_b} \int_{Q_b} |\mathcal{G} f(t, x) - \mathcal{G} f(s, y)| dt dx ds dy \leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})},
\]
where $N$ depends only on $\alpha, d, c$ and $\delta_0$.

**Proof.** Step 1. Suppose that $f$ has a support in $(-\infty, -2\kappa(b)) \times \mathbb{R}^d$. 

Take $\zeta \in C^\infty_c(\mathbb{R}^d)$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B_{7b/3}$ and $\zeta = 0$ outside of $B_{8b/3}$. Then,
\[
|\mathcal{G}f(t, x) - \mathcal{G}f(s, y)| \leq |\mathcal{G}f(t, x) - \mathcal{G}f(t, y)| + |\mathcal{G}f(t, y) - \mathcal{G}f(s, y)|
\leq |\mathcal{G}(\zeta f)(t, x) - \mathcal{G}(\zeta f)(t, y)|
+ |\mathcal{G}(1 - \zeta)f(t, x) - \mathcal{G}(1 - \zeta)f(t, y)| + |\mathcal{G}f(t, y) - \mathcal{G}f(s, y)|
\leq |\mathcal{G}(\zeta f)(t, x) + |\mathcal{G}(\zeta f)(t, y)|
+ |\mathcal{G}(1 - \zeta)f(t, x) - \mathcal{G}(1 - \zeta)f(t, y)| + |\mathcal{G}f(t, y) - \mathcal{G}f(s, y)|.
\]

Applying Lemma 4.5, Lemma 4.6 and Lemma 4.7 to estimate $|\mathcal{G}(\zeta f)(t, x)|$ and $|\mathcal{G}(\zeta f)(t, y)|$, we get the desired estimate.

**Step 2.** General case. Choose a function $\xi = \xi(t) \in C^\infty_c(\mathbb{R})$ such that $0 \leq \xi \leq 1$, $\xi = 1$ on $(-\infty, -8\kappa(b)/3)$ and $\xi(t) = 0$ for $t \geq -7\kappa(b)/3$. Let $f_1 = \xi f$ and $f_2 = (1 - \xi)f$. Then $\mathcal{G}f = \mathcal{G}f_1 + \mathcal{G}f_2$ in $\mathbb{R}^{d+1}$. Note that $f_1$ satisfies the assumption in Step 1, and $f_2$ satisfies the condition in Lemma 4.4. Therefore,
\[
\int_{Q_b} |\mathcal{G}f(t, x) - (\mathcal{G}f)_{Q_b}|dtdx
\leq \int_{Q_b} |\mathcal{G}f_1(t, x) - (\mathcal{G}f_1)_{Q_b}|dtdx + \int_{Q_b} |\mathcal{G}f_2(t, x) - (\mathcal{G}f_2)_{Q_b}|dtdx
\leq N\|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

The corollary is proved. □

For locally integrable functions $h$ on $\mathbb{R}^{d+1}$, we define the BMO semi-norm of $h$ on $\mathbb{R}^{d+1}$ as
\[
\|h\|_{BMO(\mathbb{R}^{d+1})} = \sup_{Q \in \mathcal{Q}} \int_{Q} |h(t, x) - h_Q|dtdx
\]
where $h_Q = \frac{1}{|Q|} \int_{Q} h(t, x)dtdx$ and
\[
\mathcal{Q} := \{Q_b(t_0, x_0) : b > 0, (t_0, x_0) \in \mathbb{R}^{d+1}\}.
\]

For measurable functions $h(t, x)$ on $\mathbb{R}^{d+1}$, we define the sharp function
\[
h^\#(t, x) = \sup_{(t,x) \in Q_{b}(r,z)} \int_{Q_{b}(r,z)} |h(s, y) - h_{Q_{b}(r,z)}|dsdy.
\]

**Theorem 4.9** (Fefferman-Stein Theorem). For any $1 < p < \infty$ and $h \in L^p(\mathbb{R}^{d+1})$,
\[
N^{-1} \|h^\#\|_{L^p(\mathbb{R}^{d+1})} \leq \|h\|_{L^p(\mathbb{R}^{d+1})} \leq N\|h^\#\|_{L^p(\mathbb{R}^{d+1})},
\]
where $N > 1$ depends on $\alpha, d, p, \kappa$ and $\delta_0$.

**Proof.** See [37, Theorem I.3.1, Theorem IV.2.2]. We only remark that due to (12), the balls $Q_b(s,y)$ satisfy the conditions (i)-(iv) in [37, Section 1.1]:

(i) $Q_c(t, x) \cap Q_b(s,y) \neq \emptyset$ implies $Q_c(s,y) \subset Q_{N_1c}(t, x)$;

(ii) $|Q_{N_1c}(t, x)| \leq N_2|Q_c(t, x)|$;

(iii) $\bigcap_{c>0} Q_c(t, x) = \{(t, x)\}$ and $\bigcup_{c} Q_c(t, x) = \mathbb{R}^{d+1}$;

(iv) for each open set $U$ and $c > 0$, the function $(t, x) \mapsto |Q_c(t, x) \cap U|$ is continuous.
Here is the main result of this section.

**Theorem 4.10.** (i) For any \( f \in L_2(\mathbb{R}^{d+1}) \cap L_\infty(\mathbb{R}^{d+1}) \),
\[
\|Gf\|_{\text{BMO}(\mathbb{R}^{d+1})} \leq N(\alpha, d, c, \delta_0) \|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\] (4.9)
(ii) For any \( p, q \in (1, \infty) \) and \( f \in C_c^\infty(\mathbb{R}^{d+1}) \),
\[
\|Gf\|_{L_q(R;L_p(\mathbb{R}^{d}))} \leq N(\alpha, d, c, \delta_0, p) \|f\|_{L_q(R;L_p(\mathbb{R}^{d}))}.
\] (4.10)

**Proof.** We follow the proof of [22, Theorem 4.4] which treats the case \( \phi(\lambda) = \lambda \).
(i) Note that for any \( (t_0, x_0) \in \mathbb{R}^{d+1} \),
\[
Gf(t + t_0, x + x_0) = \int_{-\infty}^{t+\alpha} \int_\mathbb{R}^d q_{\alpha,1+\alpha}(t + t_0 - s, x + x_0 - y)f(s, y)dyds
\]
\[
= \int_{-\infty}^t \int_\mathbb{R}^d q_{\alpha,1+\alpha}(t - s, x - y)f(s + t_0, x_0 + y)dxdyds
\]
\[
= G(f(\cdot + t_0, \cdot + x_0))(t, x).
\]
Therefore, with \( \tilde{f}(t, x) := f(t + t_0, x + x_0) \),
\[
\int_{Q_b(t_0, x_0)} |Gf(t, x) - (Gf)_{Q_b(t_0, x_0)}|dtdx
\]
\[
= \int_{Q_b} |Gf(t, x) - (Gf)_{Q_b}|dtdx.
\]
Due to this and the translation invariant property of \( L_\infty \)-norm, for the proof of (i) it suffices to prove that there exists \( N = N(\alpha, d, c, \delta_0) \) such that
\[
\int_{Q_b} |Gf(t, x) - (Gf)_{Q_b}|dtdx \leq N\|f\|_{L_\infty(\mathbb{R}^{d+1})}, \quad b > 0.
\] (4.11)
Recall that we already have (4.11) due to Corollary 4.8 if \( f \in C_c^\infty(\mathbb{R}^{d+1}) \).
Now we consider the general case, that is \( f \in L_2(\mathbb{R}^{d+1}) \cap L_\infty(\mathbb{R}^{d+1}) \). We choose a sequence of functions \( f_n \in C_c^\infty(\mathbb{R}^{d+1}) \) such that \( Gf_n \to Gf \) (a.e.), and \( \|f_n\|_{L_\infty(\mathbb{R}^{d+1})} \leq \|f\|_{L_\infty(\mathbb{R}^{d+1})} \). Then by Fatou’s lemma,
\[
\int_{Q_b} |Gf(t, x) - (Gf)_{Q_b}|dtdx
\]
\[
\leq \int_{Q_b} \int_{Q_b} |Gf(t, x) - Gf(s, y)|dtdxdy
\]
\[
\leq \liminf_{n \to \infty} \int_{Q_b} \int_{Q_b} |Gf_n(t, x) - Gf_n(s, y)|dtdxdy
\]
\[
\leq N \liminf_{n \to \infty} \|f_n\|_{L_\infty(\mathbb{R}^{d+1})} \leq N\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\] (4.11)

**Step 1.** We prove (4.10) for the case \( p = q \). First assume that \( p \geq 2 \). Then by Lemma 4.2 and Theorem 4.9 for any \( f \in L_2(\mathbb{R}^{d+1}) \cap L_\infty(\mathbb{R}^{d+1}) \), it holds that
\[
\|Gf\|_{L_2(\mathbb{R}^{d+1})} \leq N\|f\|_{L_2(\mathbb{R}^{d+1})}.
\]
By (4.9),
\[
\|Gf\|_{L_\infty(\mathbb{R}^{d+1})} \leq N\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]
Note that the map \( f \to (\mathcal{G} f)^\# \) is sublinear since \( \mathcal{G} \) is linear. Hence by a version of the Marcinkiewicz interpolation theorem, for any \( p \in [2, \infty) \) there exists a constant \( N \) such that
\[
\| (\mathcal{G} f)^\# \|_{L_p(\mathbb{R}^{d+1})} \leq N \| f \|_{L_p(\mathbb{R}^{d+1})}
\]
for all \( f \in L_2(\mathbb{R}^{d+1}) \cap L_\infty(\mathbb{R}^{d+1}) \). Finally by Theorem 4.9 we get
\[
\| \mathcal{G} f \|_{L_p(\mathbb{R}^{d+1})} \leq N \| f \|_{L_p(\mathbb{R}^{d+1})},
\]
Therefore (4.10) is proved for \( p \in [2, \infty) \).

Now let \( p \in (1, 2) \). Take \( f, g \in C_c^\infty(\mathbb{R}^{d+1}) \) and \( p' = \frac{p}{p-1} \in (2, \infty) \). By Parseval’s identity, Fubini’s theorem, and the change of variables,
\[
\int_{\mathbb{R}^{d+1}} g(t, x) \mathcal{G} f(t, x) dx dt = \int_{\mathbb{R}^{d+1}} g(t, x) \phi(\Delta) \mathcal{G}_0 f(t, x) dx dt
\]
\[
= N(d) \int_{\mathbb{R}^{d+1}} \mathcal{F}_d(g)(t, \xi) \phi(|\xi|^2) \mathcal{F}_d(\mathcal{G}_0 f)(t, \xi) d\xi dt
\]
\[
= \int_{\mathbb{R}^{d+1}} \mathcal{F}_d(g)(t, \xi) t \int_{\mathbb{R}^{d+1}} 1_{t-s>0} q_{0,1}(t-s, x-y) f(s, y) dy ds dx dt
\]
\[
= \int_{\mathbb{R}^{d+1}} \mathcal{F}_d(g)(t, \xi) \int_{\mathbb{R}^{d+1}} \mathcal{F}_d(\mathcal{G}_0 f)(t, \xi) 1_{t-s>0} q_{0,1}(t-s, y-x) f(s, y) dy ds dx dt
\]
\[
= \int_{\mathbb{R}^{d+1}} \mathcal{F}_d(g)(t, \xi) \int_{\mathbb{R}^{d+1}} \mathcal{F}_d(\mathcal{G}_0 f)(t, \xi) 1_{t-s>0} q_{0,1}(t-s, y-x) f(s, y) dy ds dx dt
\]
where \( \tilde{g}(t, x) = g(-t, -x) \). By Hölder’s inequality,
\[
\left| \int_{\mathbb{R}^{d+1}} g(t, x) \mathcal{G} f(t, x) dx dt \right|
\]
\[
\leq \| f \|_{L_p(\mathbb{R}^{d+1})} \| \mathcal{G} \|_{L_p(\mathbb{R}^{d+1})} \| g \|_{L_{p'}(\mathbb{R}^{d+1})}.
\]
Since \( g \in C_c^\infty(\mathbb{R}^{d+1}) \) is arbitrary, we have \( \mathcal{G} f \in L_p(\mathbb{R}^{d+1}) \) and (4.11) is also proved for \( p \in (1, 2) \).

**Step 2.** Now we prove (4.10) for general \( p, q \in (1, \infty) \). Define \( q_{\alpha, \beta}(t) := 0 \) for \( t \leq 0 \). For each \( (t, s) \in \mathbb{R}^2 \), we define the operator \( \mathcal{K}(t, s) \) as follows:
\[
\mathcal{K}(t, s) f(x) := \int_{\mathbb{R}^d} q_{\alpha,1+\alpha}(t-s, x-y) f(y) dy, \quad f \in C_c^\infty(\mathbb{R}^d).
\]
Let \( p \in (1, \infty) \). Then,
\[
\| \mathcal{K}(t, s) f \|_{L_p(\mathbb{R}^d)} = \left\| \int_{\mathbb{R}^d} q_{\alpha,1+\alpha}(t-s, x-y) f(y) dy \right\|_{L_p(\mathbb{R}^d)}
\]
\[
\leq \| f \|_{L_p(\mathbb{R}^d)} \int_{\mathbb{R}^d} |q_{\alpha,1+\alpha}(t-s, y)| dy \leq N(t-s)^{-1} \| f \|_{L_p(\mathbb{R}^d)}.
\]
Hence the operator \( \mathcal{K}(t, s) \) is uniquely extendible to \( L_p(\mathbb{R}^d) \) for \( t \neq s \). Denote \( Q := [t_0, t_0 + \delta) \), \( Q^* := [t_0 - \delta, t_0 + 2\delta) \), \( \delta > 0 \).

Note that for \( t \notin Q^* \) and \( s_1, s_2 \in Q \), we have
\[
|s_1 - s_2| \leq \delta, \quad |t - (t_0 + \delta)| \geq \delta.
\]
Thus, for any \( f \in L_p(\mathbb{R}^d) \) such that \( \|f\|_{L_p(\mathbb{R}^d)} = 1 \),
\[
\|K(t, s_1)f - K(t, s_2)f\|_{L_p(\mathbb{R}^d)} \\
= \left\| \int_{\mathbb{R}^d} (g_{\alpha, t_1}(x, y) - g_{\alpha, t_2}(x, y)) f(y) dy \right\|_{L_p(\mathbb{R}^d)} \\
\leq \|f\|_{L_p(\mathbb{R}^d)} \int_{\mathbb{R}^d} |g_{\alpha, t_1}(x, y) - g_{\alpha, t_2}(x, y)| dy dx \\
\leq N \int_{\mathbb{R}^d} \int_0^1 |g_{\alpha, t_1}(x, y) - g_{\alpha, t_2}(x, y)| dy dx \\
\leq \frac{N|s_1 - s_2|}{(t - (t_0 + \delta))^2}
\]
due to Corollary 3.9 (i). Here, recall that \( K(t, s) = 0 \) if \( t \leq s \). Hence,
\[
\|K(t, s_1) - K(t, s_2)\|_{\Lambda} \leq \frac{N|s_1 - s_2|}{(t - (t_0 + \delta))^2},
\]
where \( \| \cdot \|_{\Lambda} \) denotes the operator norm of \( \Lambda \) on \( L_p(\mathbb{R}^d) \). Therefore,
\[
\int_{\mathbb{R}\setminus Q} \|K(t, s_1) - K(t, s_2)\|_{\Lambda} dt \leq N \int_{\mathbb{R}\setminus Q} \frac{|s_1 - s_2|}{(t - (t_0 + \delta))^2} dt \\
\leq N|s_1 - s_2| \int_{t-(t_0+\delta)} \frac{1}{(t - (t_0 + \delta))^2} dt \leq N\delta \int_0^\infty t^{-2} dt \leq N.
\]
Furthermore, by following the proof of [26 Theorem 1.1], one can easily check that for almost every \( t \) outside of the support of \( f \in C_0^\infty(\mathbb{R}; L_p(\mathbb{R}^d)) \),
\[
Gf(t, x) = \int_{-\infty}^\infty K(t, s)f(s, x) ds
\]
where \( G \) denotes the extension to \( L_p(\mathbb{R}^{d+1}) \) which is verified in Step 1. Hence by the Banach space-valued version of the Calderón-Zygmund theorem (e.g. [26 Theorem 4.1]), our assertion is proved for \( 1 < q < p \).

For \( 1 < p < q < \infty \), define \( p' = \frac{p}{p-1} \) and \( q' = \frac{q}{q-1} \). By (4.12) and Hölder’s inequality,
\[
\left| \int_{\mathbb{R}^{d+1}} g(t, x)Gf(t, x) dx dt \right| = \left| \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} G\tilde{g}(s, y)f(-s, -y) dy \right) ds \right| \\
\leq \int_{\mathbb{R}} \|f(-s, \cdot)\|_{L_p(\mathbb{R}^d)} \|G\tilde{g}(s, \cdot)\|_{L_{p'}(\mathbb{R}^d)} ds \\
\leq N \|f\|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))} \|g\|_{L_{q'}(\mathbb{R}; L_{p'}(\mathbb{R}^d))}
\]
for any \( g \in C_0^\infty(\mathbb{R}^{d+1}) \), where the last inequality holds due to \( 1 < q' < p' \). Since \( g \) is arbitrary, we have
\[
\|Gf\|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))} \leq N \|f\|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))}.
\]
The theorem is proved. \( \square \)
5. Homogeneous equation

In this section we study the homogeneous equation with non-zero initial condition

\[ \partial_t^\alpha u = \phi(\Delta)u, \quad u(0, x) = u_0(x). \]

(5.1)

We first show that \( q \) is a fundamental solution to equation (5.1).

**Lemma 5.1.** Let \( u_0 \in C_p^\infty(\mathbb{R}^d) \), and define \( u \) as

\[ u(t, x) := \int_{\mathbb{R}^d} q(t, x - y)u_0(y)dy. \]

(i) As \( t \downarrow 0 \), \( u(t, \cdot) \) converges to \( u_0(\cdot) \) uniformly on \( \mathbb{R}^d \) and also in \( H^n_p \) for any \( n \in \mathbb{N}_0 \).

(ii) \( u \in C_p^\infty([0, T] \times \mathbb{R}^d) \) and \( u \) satisfies \( \partial_t^\alpha u = \phi(\Delta)u \) for \( t > 0 \).

**Proof.** (i) By (3.24), for any \( \delta > 0 \),

\[ \int_{\mathbb{R}^d} q(t, y)dy = \hat{q}(t, 0) = E_\alpha(0) = 1. \]

For any \( \delta > 0 \),

\[
\begin{align*}
&\left| \int_{\mathbb{R}^d} q(t, x - y)u_0(y)dy - u_0(x) \right| \\
&= \left| \int_{\mathbb{R}^d} q(t, y)(u_0(x - y) - u_0(x))dy \right| \\
&\leq \int_{|y| \leq \delta} |q(t, y)(u_0(x - y) - u_0(x))| dy + \int_{|y| > \delta} |q(t, y)(u_0(x - y) - u_0(x))| dy \\
&=: I_1(\delta, x) + I_2(\delta, x).
\end{align*}
\]

Since \( u_0 \in C_p^\infty(\mathbb{R}^d) \), for any \( \varepsilon > 0 \), one can take a small \( \delta \) so that \( I_1(\delta, x) < \varepsilon \). Moreover due to (3.1) and Lemma 3.8 we have

\[ \int_{|y| \geq \delta} q(t, y)dy \leq N t_\alpha \phi(\delta^{-2}), \]

and therefore

\[ I_2(\delta, x) \leq N t_\alpha \phi(\delta^{-2}) \|u_0\|_{L_\infty} \rightarrow 0 \quad \text{as} \quad t \downarrow 0. \]

This shows \( u(t, x) \rightarrow u_0(x) \) as \( t \rightarrow 0 \) uniformly in \( x \). Also,

\[
\begin{align*}
\|u(t) - u_0\|^p_{L_p} &\leq 2^p \|I_1(\delta)\|^p_{L_p} + 2^p \|I_2(\delta)\|^p_{L_p} \\
&\leq N \sup_{|z| \leq \delta} \|u_0(\cdot - z) - u_0(\cdot)\|^p_{L_p} + N \|u_0\|^p_{L_p} \int_{|y| > \delta} |q(t, y)|dy.
\end{align*}
\]

This and (5.2) certainly proves the \( L_p \) convergence. Considering \( D^\alpha_x \), we have \( u(t, \cdot) \rightarrow u_0 \) in \( H^n_p \) for any \( n \).

(ii) By (3.20) (recall (2.4)),

\[ \partial_t^\alpha u(t, x) = \int_{\mathbb{R}^d} q_{\alpha, 2\alpha}(t, y)u_0(x - y)dy - \frac{1}{\Gamma(1 - \alpha)} t^{-\alpha} u_0(x). \]
By (3.15) and (3.20), we have
\[
\mathcal{F}(\partial^2_t u(t, \cdot))(\xi) = \hat{q}_{\alpha,2\alpha}(t, \xi)\hat{u}_0(\xi) - \frac{1}{\Gamma(1-\alpha)}t^{-\alpha}\hat{u}_0(\xi) \\
= \phi(|\xi|^2) \left( R^{-1}E_{\alpha,1-\alpha}(-R) - R^{-1}\frac{1}{\Gamma(1-\alpha)} \right)\hat{u}_0(\xi) \\
= -\phi(|\xi|^2) (E_{\alpha,1}(-R))\hat{u}_0(\xi) \\
= -\phi(|\xi|^2)\hat{q}_{\alpha,\alpha}(t, \xi)\hat{u}_0(\xi) \\
= \mathcal{F}(\phi(\Delta)u(t, \cdot))(\xi),
\]
where \( R = R(t, \xi) = t^\alpha \phi(|\xi|^2) \). This together with (i) implies \( u \) satisfies (5.1).

Now we show that for any polynomial \( Q(z) \) of degree \( m \) and constant \( \sigma > 0 \), we have
\[
Q(z)e^{-\sigma z} \leq N(\sigma, m)z^{-1}.
\]
This together with (5.6), and (3.2) easily yields
\[ |F\hat{q}_j(t, \xi)| \leq N\phi(2^{2j})^{-\frac{1}{q}} t^{-1} 1_{1/2 \leq |\xi| \leq 2}. \]

Similarly, using (5.5) and following above computations, for any multi-index \( \gamma \) we get
\[ |D^\gamma F\hat{q}_j(t, \xi)| \leq N(\alpha, \gamma, d, \phi)\phi(2^{2j})^{-\frac{1}{q}} t^{-1} 1_{1/2 \leq |\xi| \leq 2}. \]

Therefore, we finally have
\[
\|\hat{q}_j(t, \cdot)\|_{L_1} = \int_{\mathbb{R}^d} (1 + |x|^{2d})^{-1} (1 + |x|^{2d})|\hat{q}_j(t, x)| dx \\
\leq N \int_{\mathbb{R}^d} (1 + |x|^{2d})^{-1} \sup_{\xi} \|(1 + \Delta^d_\xi)F(\hat{q}_j)(t, \xi)| dx \\
\leq N\phi(2^{2j})^{-\frac{1}{q}} t^{-1}.
\]

The lemma is proved. \( \square \)

**Theorem 5.3.** Let \( p > 1, 0 < \alpha < 1 \) and \( f \in C_c^\infty(\mathbb{R}^d) \). Then we have
\[
\int_0^T \|q \ast f(t, \cdot)\|_{L_q}^q dt \leq N\|f\|_{B_{p,q}^\gamma}^q,
\]
where the constant \( N \) depends only on \( \alpha, d, \phi, p, q, T \).

**Proof.** Note that
\[
\hat{\Psi}_j = \hat{\Psi}_j(\hat{\Psi}_{j-1} + \hat{\Psi}_{j+1}), \quad j \in \mathbb{N},
\]
\[ \hat{\Psi}_0 = \hat{\Psi}_0(\hat{\Psi}_0 + \hat{\Psi}_1). \]

Using this and the relation \( F(f_1 \ast f_2) = F(f_1)F(f_2), \)
\[
\int_0^T \|q \ast f(t, \cdot)\|_{L_q}^q dt \leq N \int_0^T (\|q_0(t, \cdot)\|_{L_1} + \|q_1(t, \cdot)\|_{L_1})^q f_0\|_{L_q}^q dt \\
+ N \int_0^T (\sum_{j=1}^{j+1} \|q_i(t, \cdot)\|_{L_1} f_j\|_{L_p})^q dt.
\]

By (5.4)
\[
\int_0^T (\|q_0(t, \cdot)\|_{L_1} + \|q_1(t, \cdot)\|_{L_1})^q f_0\|_{L_q}^q dt \leq N(T)f_0\|_{L_q}^q,
\]
and
\[
\int_0^T (\sum_{j=1}^{j+1} \|q_i(t, \cdot)\|_{L_1} f_j\|_{L_p})^q dt \leq N \int_0^T (\sum_{j=1}^\infty (\phi(2^{2j})^{-\frac{1}{q}} t^{-1} 1)) f_j\|_{L_p})^q dt.
\]

Observe that
\[
\int_0^T (\sum_{j=1}^\infty (\phi(2^{2j})^{-\frac{1}{q}} t^{-1} 1)) f_j\|_{L_p})^q dt \\
\leq 2^q \int_0^T (\sum_{j=1}^\infty 1_j(t, j) f_j\|_{L_p})^q dt + 2^q \int_0^T (\sum_{j=1}^\infty 1_j(t, j) \phi(2^{2j})^{-\frac{1}{q}} t^{-1} 1 f_j\|_{L_p})^q dt,
\]
where \( J = \{(t, j)|\phi(2^{2^j})^{-\frac{1}{a}}t^{-1} \geq 1\} \). By Hölder’s inequality,
\[
\int_0^T \left( \sum_{j=1}^{\infty} 1_j \|f_j\|_{L_p} \right)^q dt = \int_0^T \left( \sum_{j \in J(t)} \phi(2^{2^j}) \right)^{\frac{q}{p^*}} \phi(2^{2^j}) \|f_j\|_{L_p}^q dt \leq \int_0^T \left( \sum_{j \in J(t)} \phi(2^{2^j}) \right)^{\frac{q}{p^*}} \left( \sum_{j \in J(t)} \phi(2^{2^j}) \|f_j\|_{L_p}^q \right) dt,
\]
where \( a \in (-\frac{1}{q}, 0) \), \( q' = \frac{q}{q-1} \), and \( J(t) = \{ j = 1, 2, \ldots |(t, j) \in J\} \).

Fix \( t > 0 \) and let \( j_0(t) \) be the largest integer such that \( \phi(2^{2^{j_0}})^{-\frac{1}{a}}t^{-1} \geq 1 \). Then the above summation on \( J(t) \) is the summation over \( \{ j \leq j_0 \} \). Moreover, by (5.2) we have
\[
2^{-2} \leq \frac{\phi(2^{2^{j-2}})}{\phi(2^{2^j})} \leq c^{-1}2^{-2q_0}.
\] (5.9)
This yields that
\[
\sum_{j \in J(t)} \phi(2^{2^j})^{-\frac{a_0}{a}} \leq N(q, c, \delta_0)t^{a_0q'}.
\]
Hence, we have
\[
\int_0^T \left( \sum_{j=1}^{\infty} 1_j \|f_j\|_{L_p} \right)^q dt \leq N \sum_{j=1}^{\infty} \int_0^\infty \phi(2^{2^j})^{-\frac{1}{a}}t \phi(2^{2^j})^{\frac{q}{p^*}} \|f_j\|_{L_p}^q dt \leq N \sum_{j=1}^{\infty} \phi(2^{2^j})^{-\frac{1}{a}} \|f_j\|_{L_p}^q.
\] (5.10)
By Hölder’s inequality again, for \( b \in (-1, -\frac{1}{q'}) \) and \( q' = \frac{q}{q-1} \),
\[
\int_0^T \left( \sum_{j=1}^{\infty} 1_j \phi(2^{2^j})^{-\frac{1}{a}}t^{b-1} \|f_j\|_{L_p} \right)^q dt = \int_0^T \left( \sum_{j \notin J(t)} \phi(2^{2^j})^{-\frac{1}{a}}t \phi(2^{2^j})^{\frac{q}{p^*}} \|f_j\|_{L_p} \right)^q dt \leq t^{-q} \left( \sum_{j \notin J(t)} \phi(2^{2^j})^{-\frac{1}{a}}t \phi(2^{2^j})^{\frac{q}{p^*}} \|f_j\|_{L_p} \right)^q dt.
\]
The summation over \( J(t)^c \) is the summation over \( \{ j > j_0 \} \). Hence by (5.9) we have
\[
\sum_{j \notin J(t)} \phi(2^{2^j})^{-\frac{(b+1)q'}{a}} \leq N(q)t^{(b+1)q'}.
\]
Therefore, we have
\[
\int_0^T \left( \sum_{j=1}^{\infty} 1_j \phi(2^{2^j})^{-\frac{1}{a}}t^{-1} \|f_j\|_{L_p} \right)^q dt \leq N \sum_{j=1}^{\infty} \int_{\phi(2^{2^j})^{-\frac{1}{a}}}^\infty t^{-q}t^{(b+1)q} \phi(2^{2^j})^{\frac{q}{p^*}} \|f_j\|_{L_p}^q dt = N \sum_{j=1}^{\infty} \phi(2^{2^j})^{-\frac{1}{a}} \|f_j\|_{L_p}^q.
\] (5.11)
Combining (5.8), (5.10) and (5.11), we have (5.7). The theorem is proved. □
Lemma 5.4. Let $0 < \alpha < 1$, $1 < q, p < \infty$, $\gamma \in \mathbb{R}$ and $T < \infty$. Then, for any $u_0 \in B_{p,q}^{\phi,\gamma+2/\alpha q}$ equation (5.1) has a solution $u \in \mathcal{H}_{q,p}^{\alpha,\phi,\gamma+2}(T)$ satisfying
\[
\|u\|_{\mathcal{H}_{q,p}^{\alpha,\phi,\gamma+2}(T)} \leq N\|u_0\|_{B_{p,q}^{\phi,\gamma+2/\alpha q}},
\] (5.12)
where the constant $N$ depends only on $\alpha, d, p, q, \phi, \gamma$, and $T$.

Proof. By Remark 2.24 and Lemma 2.7 (iii), it is enough to prove the case $\gamma = -2$.

If $u_0 \in C_p^\infty(\mathbb{R}^d)$, then we define
\[
u(t, x) = \int_{\mathbb{R}^d} q(t, x-y)u_0(y)dy.
\] (5.13)

Then $u \in C_p^{\alpha,\infty}([0,T] \times \mathbb{R}^d)$ and all the claims of the lemma hold with $u$ due to Lemma 5.1 (ii) and Lemma 5.3.

In general, for $u_0 \in B_{p,q}^{\phi,\gamma+2/\alpha q}$ we take a sequence $u_n^0 \in \mathcal{S}$ such that $u_n^0 \to u_0$ in $B_{p,q}^{\phi,\gamma+2/\alpha q}$, and we define $u_n \in C_p^{\alpha,\infty}([0,T] \times \mathbb{R}^d) \cap H_{q,p,0}^{\alpha,\phi,\gamma}(T)$ corresponding to $u_n^0$ using (5.13). Then inequality (5.12) applied to $u_n - u_m$ shows that $u_n$ is a Cauchy sequence in $H_{q,p,0}^{\alpha,\phi,\gamma}(T)$. Finally one gets the claims of the lemma by considering the limit. The lemma is proved. \hfill \Box

Corollary 5.5. $u \in H_{q,p}^{\alpha,\phi,\gamma+2}(T)$ and $u(0, \cdot) = 0$ if and only if $u \in H_{q,p,0}^{\alpha,\phi,\gamma+2}(T)$.

Proof. We only prove “only if” part. The “if” part is obvious by definition. Suppose $u \in H_{q,p}^{\alpha,\phi,\gamma+2}(T)$ and $u(0, \cdot) = 0$. Then there exists a defining sequence $u_n \in C_p^{\alpha,\infty}([0,T] \times \mathbb{R}^d)$ of $u$ such that $u_n(0, \cdot) \in C_p^\infty$ and $u_n(0, \cdot) \to 0$ in $B_{p,q}^{\phi,\gamma+2/\alpha q}$.

By Lemmas 5.1 and 5.4 we can choose $v_n \in C_p^{\alpha,\infty}([0,T] \times \mathbb{R}^d)$ such that $v_n(0, \cdot) = u_n(0, \cdot)$ and
\[
\|v_n\|_{\mathcal{H}_{q,p}^{\alpha,\phi,\gamma+2}(T)} \leq N\|u_n(0, \cdot)\|_{B_{p,q}^{\phi,\gamma+2/\alpha q}} \to 0
\]
as $n \to \infty$. This implies that $u_n - v_n$ is also a defining sequence of $u$, and therefore we have $u \in H_{q,p,0}^{\alpha,\phi,\gamma}(T)$ because $(u_n - v_n)(0, \cdot) = 0$.

\hfill \Box

6. Proof of Theorem 2.8

Due to Remark 2.23 and Lemma 2.7 (iii), we only need to prove case $\gamma = 0$.

Step 1 (Uniqueness). Let $u \in H_{q,p}^{\alpha,\phi,2}(T)$ be a solution to equation (2.17) with $f = 0$ and $u_0 = 0$. Then by Corollary 5.5 $u \in H_{q,p,0}^{\alpha,\phi,2}(T)$. Hence, by Lemma 2.7 (ii), there exists $u_n \in C_p^c(\mathbb{R}^{d+1})$ such that $u_n \to u$ in $H_{q,p,0}^{\alpha,\phi,2}(T)$. Due to Lemma 4.1 it also holds that
\[
u_n(t, x) = \int_0^t \int_{\mathbb{R}^d} q_{0,1}(t-s, x-y)f_n(s, y)dyds,
\] (6.1)
where $f_n := \partial^\alpha_t u_n - \phi(\Delta)u_n$. Note
\[
\|f_n\|_{L_{q,p}(T)} = \|\partial^\alpha_t (u_n - u) - \phi(\Delta)(u_n - u)\|_{L_{q,p}(T)} \leq \|\partial^\alpha_t u_n - \partial^\alpha_t u\|_{L_{q,p}(T)} + \|\phi(\Delta)u_n - \phi(\Delta)u\|_{L_{q,p}(T)} \to 0
\]
as $n \to \infty$. Thus by Minkowski’s inequality we have
\[
\|u_n\|_{L_{q,p}(T)} \leq N(T)\|f_n\|_{L_{q,p}(T)}.
\]
Letting $n \to \infty$, we get $u = 0$ since $H_{q,p,0}^{\alpha,\phi,2}(T) \subset L_{q,p}(T)$. 

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Step 2 (Existence and estimate with zero initial condition). First assume \( f \in C_c^\infty \left( \mathbb{R}^{d+1}_+ \right) \), and let
\[
  u(t, x) := \int_0^t \int_{\mathbb{R}^d} q_{\alpha,1}(t-s, x-y)f(s, y)dyds.
\]

Using Remark 2.6 and the integrability of \( q_{\alpha,1} \) one can easily check \( D^m_x u, \phi(\Delta)D^m_x u \in C([0, T]; L_p) \), and therefore \( u \in C^{m, \infty}_p([0, T] \times \mathbb{R}^d) \). Also, by Lemma 4.1, \( u \) satisfies equation (2.17) with \( u(0, \cdot) = 0 \).

Now we show (2.18). By Minkowski’s inequality and Corollary 3.9 (i), one has
\[
  \|u\|_{L_{q,p}(T)} \leq N(T)\|f\|_{L_{q,p}(T)}.
\]

Also the estimate
\[
  \|\phi(\Delta)u\|_{L_{q,p}(T)} \leq N\|f\|_{L_{q,p}(T)}
\]
follows from Theorem 4.10. These two inequalities with (2.17) lead to (2.18) and (2.19).

For general \( f \), we take a sequence of functions \( f_n \in C_c^\infty \left( \mathbb{R}^{d+1}_+ \right) \) such that \( f_n \to f \) in \( L_{q,p}(T) \). Let \( u_n \) denote the solution to equation (6.2) with \( f_n \) in place of \( f \). Then (2.18) applied to \( u_n - u_m \) shows that \( u_n \) is Cauchy in \( \mathbb{H}^{\alpha, \phi, 2}_{q,p,0}(T) \). By taking \( u = \lim_{n \to \infty} u_n \) in \( \mathbb{H}^{\alpha, \phi, 2}_{q,p,0}(T) \), we find that \( u \) satisfies the equation \( \partial_t^\alpha u = \phi(\Delta)u + f \), and (2.18) and (2.19) also hold for \( u \).

Step 3 (Existence and estimate with nonzero initial condition). Let \( v \in \mathbb{H}^{\alpha, \phi, \gamma+2}_{q,p}(T) \) denote the solution to the homogeneous equation taken from Lemma 5.4, and let \( u \in \mathbb{H}^{\alpha, \phi, \gamma+2}_{q,p,0}(T) \) be taken from Step 2. Then \( \bar{u} := v + u \in \mathbb{H}^{\alpha, \phi, \gamma+2}_{q,p}(T) \) satisfies (2.17), and (2.18) also holds. The theorem is proved.

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