On weakly $\sigma$-permutable subgroups of finite groups*

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Abstract

Let $G$ be a finite group and $\sigma = \{\sigma_i | i \in I\}$ be a partition of the set of all primes $\mathbb{P}$. A set $\mathcal{H}$ of subgroups of $G$ with $1 \in \mathcal{H}$ is said to be a complete Hall $\sigma$-set of $G$ if every non-identity member of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$. A subgroup $H$ of $G$ is said to be $\sigma$-permutable if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. We say that a subgroup $H$ of $G$ is weakly $\sigma$-permutable in $G$ if there exists a $\sigma$-subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where $H_{\sigma G}$ is the subgroup of $H$ generated by all those subgroups of $H$ which are $\sigma$-permutable in $G$. By using this new notion, we establish some new criterias for a group $G$ to be a $\sigma$-soluble and supersoluble, and also we give the conditions under which a normal subgroup of $G$ is hypercyclically embedded.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a group. Moreover, $n$ is an integer, $\mathbb{P}$ is the set of all primes. The symbol $\pi(n)$ denotes the set of all primes dividing $n$ and $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$. $|G|_p$ denotes the order of the Sylow $p$-subgroup of $G$.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. $\Pi$ is always supposed to be a non-empty subset of $\sigma$ and $\Pi' = \sigma \setminus \Pi$. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

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1
Following [1,2], $G$ is said to be $\sigma$-primary if $G = 1$ or $|\sigma(G)| = 1$; $n$ is said to be a $\Pi$-number if $\pi(n) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$; a subgroup $H$ of $G$ is called a $\Pi$-subgroup of $G$ if $|H|$ is a $\Pi$-number; a subgroup $H$ of $G$ is called a Hall $\Pi$-subgroup of $G$ if $H$ is a $\Pi$-subgroup of $G$ and $|G : H|$ is a $\Pi$-number. A set $\mathcal{H}$ of subgroups of $G$ with $1 \in \mathcal{H}$ is said to be a complete Hall $\Pi$-set of $G$ if every non-identity member of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \Pi$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \Pi \cap \sigma(G)$. In particular, when $\Pi = \sigma$, we call $\mathcal{H}$ a complete Hall $\sigma$-set of $G$. $G$ is said to be $\Pi$-full if $G$ possesses a complete Hall $\Pi$-set; a $\Pi$-full group of Sylow type if every subgroup of $G$ is a $D_{\sigma_i}$-group for all $\sigma_i \in \Pi \cap \sigma(G)$. In particular, $G$ is said to be $\sigma$-full (or $\sigma$-group) if $G$ possesses a complete Hall $\sigma$-set; a $\sigma$-full group of Sylow type if every subgroup of $G$ is a $D_{\sigma_i}$-group for all $\sigma_i \in \sigma(G)$. A subgroup $H$ of $G$ is called [1] $\sigma$-subnormal in $G$ if there is a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_t = G$ such that either $H_{i-1}$ is normal in $H_i$ or $H_i/(H_{i-1})$ is $\sigma$-primary for all $i = 1, 2, \cdots, t$.

In the past 20 years, a large number of researches have involved finding and applying some generalized complemented subgroups. For example, a subgroup $H$ of $G$ is said to be $c$-normal [3] in $G$ if $G$ has a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_G$, where $H_G$ is the normal core of $H$. A subgroup $H$ of $G$ is said to be weakly $s$-permutable [4] in $G$ if $G$ has a subnormal subgroup $T$ such that $G = HT$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the largest $s$-permutable subgroup of $G$ contained in $H$ (note that a subgroup $H$ of $G$ is said to be $s$-permutable in $G$ if $HP = PH$ for any Sylow subgroup $P$ of $G$). A subgroup $H$ of $G$ is said to be $\sigma$-permutable [1] in $G$ if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $HA^x = A^xH$ for all $A \in \mathcal{H}$ and all $x \in G$. By using the above special supplemented subgroups and other generalized complemented subgroups, people have obtained a series of interesting results (see [1,3–11] and so on). Now, we consider the following new generalized supplemented subgroup:

**Definition 1.1.** A subgroup $H$ of $G$ is said to be weakly $\sigma$-permutable in $G$ if there exists a $\sigma$-subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{\sigma G}$, where $H_{\sigma G}$ is the subgroup of $H$ generated by all those subgroups of $H$ which are $\sigma$-permutable in $G$.

Following [12], $H_{\sigma G}$ is called $\sigma$-core of $H$.

It is clear that every $c$-normal subgroup, every $s$-permutable subgroup, every weakly $s$-permutable subgroup and every $\sigma$-permutable subgroup of $G$ are all weakly $\sigma$-permutable in $G$. However, the following example shows that the converse is not true.

**Example 1.2.** Let $G = (C_7 \rtimes C_3) \times A_5$, where $C_7 \rtimes C_3$ is a non-abelian group of order 21 and $A_5$ is the alternating group of degree 5. Let $B$ be a subgroup of $A_5$ of order 12 and $A$ be a Sylow 5-subgroup of $A_5$. Let $\sigma = \{\sigma_1, \sigma_2\}$, where $\sigma_1 = \{2, 3, 5\}$ and $\sigma_2 = \{2, 3, 5\}'$. Then $B$ is weakly $\sigma$-permutable in $G$. In fact, let $T = (C_7 \rtimes C_3) \times A$, then $C_7 \rtimes C_3 \leq T_G$ and
\[ G : C_7 \times C_3 \] is a σ1-number. Hence \( G/T_G \) is a σ1-group, and so \( T \) is σ-subnormal in \( G \). Since \( T \cap B = 1 \) and \( G = BT \), which means that \( B \) is weakly σ-permutable in \( G \). But \( B \) is neither weakly \( s \)-permutable in \( G \) nor \( c \)-normal in \( G \). In fact, if there exists a subnormal subgroup \( K \) of \( G \) such that \( G = BK \) and \( B \cap K \leq B_{sG} \). Then \( B_{sG} \) is subnormal in \( G \) by [4, Lemma 2.8], and so is subnormal in \( A_5 \) by [13, A, (14.1)]. It follows that \( B_{sG} = 1 \) for \( A_5 \) is a simple group. Hence \( |G : K| = |B| = 2^2 \cdot 3 \). But since \( 1 < A_5 < A_5C_7 < G \) is a chief series of \( G \) and also a composition series of \( G \), \( G \) has no subnormal subgroup \( K \) whose index is \( 2^2 \cdot 3 \) by Jordan-Hölder theorem. Therefore \( B \) is not weakly \( s \)-permutable in \( G \). Consequently, \( B \) is neither \( s \)-permutable nor \( c \)-normal in \( G \).

Now let \( H = BC_3 \). Then \( H \) is weakly σ-permutable in \( G \) but not σ-permutable in \( G \). Indeed, let \( T = C_7A_5 \). Then \( G = HT \), \( T \) is normal in \( G \) and \( H \cap T = B \). It is easy to see that \( H = \{A_5C_3, C_7\} \) is a complete Hall σ-set of \( G \). Since \( H_{sG} \) is σ-subnormal in \( G \) by Lemma 2.3 (4) below and [1, Theorem B], \( H_{sG} \leq O_{\sigma_1}(G) \) by Lemma 2.2(8) below. Clearly, \( O_{\sigma_1}(G) \leq C_G(O_{\sigma_1}(G)) = C_G(C_7) = C_7A_5 \). Hence \( H_{sG} \leq C_7A_5 \). But since \( B(A_5C_3)x = BA_5C_3^x = A_5C_3^x = C_3^xA_5 = (A_5C_3)^xB \) for all \( x \in G \), \( B \) is σ-permutable in \( G \) for \( C_7 \leq G \).

Hence \( B \leq H_{sG} \leq C_7A_5 \), which implies that \( B = H_{sG} \). So \( H \) is weakly σ-permutable in \( G \), but \( H \) is not σ-permutable in \( G \) for \( H_{sG} = B < H \).

Following [1], \( G \) is called: (i) σ-soluble if every chief factor of \( G \) is σ-primary; (ii) σ-nilpotent if \( H/K \cong (G/C_G(H/K)) \) is σ-primary for every chief factor \( H/K \) of \( G \).

The result in [1,3,4,14,15] are the motivation for the following:

**Question 1.3.** Let \( G \) be a σ-full group of Sylow type. What is the structure of \( G \) provided that some subgroups are weakly σ-permutable in \( G \)?

In this paper, we obtain the following results.

**Theorem 1.4.** Let \( G \) be a σ-full group of Sylow type and every Hall \( \sigma_i \)-subgroup of \( G \) is weakly σ-permutable in \( G \) for every \( \sigma_i \in \sigma(G) \). Then \( G \) is σ-soluble.

**Theorem 1.5.** Let \( G \) be a σ-full group of Sylow type and \( H = \{1,W_1,W_2, \ldots , W_t\} \) be a complete Hall σ-set of \( G \) such that \( W_i \) is a nilpotent \( \sigma_i \)-subgroup for all \( i = 1, \ldots , t \). Suppose that the maximal subgroups of any non-cyclic \( W_i \) is weakly σ-permutable in \( G \). Then \( G \) is supersoluble.

The following results now follow immediately from Theorems 1.4 and 1.5.

**Corollary 1.6.** If every sylow subgroup is weakly \( s \)-permutable in \( G \), then \( G \) is soluble.

**Corollary 1.7.** (See Huppert [16, Chap. VI, Theorem 10.3]) If every Sylow subgroup of \( G \) is cyclic, then \( G \) is supersoluble.
**Corollary 1.8.** (See Miao [17, Corollary 3.4]) If all maximal subgroups of every Sylow subgroup of $G$ are weakly $s$-permutable in $G$, then $G$ is supersoluble.

**Corollary 1.9.** (See Skiba [4, Theorem 1.4]) If all maximal subgroups of every non-cyclic Sylow subgroup of $G$ are weakly $s$-permutable in $G$, then $G$ is supersoluble.

**Corollary 1.10.** (See Srinivasan [15, Theorem 1]) If all maximal subgroups of every Sylow subgroup of $G$ are normal in $G$, then $G$ is supersoluble.

**Corollary 1.11.** (See Srinivasan [15, Theorem 2]) If all maximal subgroups of every Sylow subgroup of $G$ are $s$-permutable in $G$, then $G$ is supersoluble.

**Corollary 1.12.** (See Wang [3, Theorem 4.1]) If all maximal subgroups of every Sylow subgroup of $G$ are $c$-normal in $G$, then $G$ is supersoluble.

Recall that a normal subgroup $E$ of $G$ is called hypercyclically embedded in $G$ and is denoted by $E \leq Z_u(G)$ (see [18, p. 217]) if either $E = 1$ or $E \neq 1$ and every chief factor of $G$ below $E$ is cyclic, where the symbol $Z_u(G)$ is the $U$-hypercentre of $G$, that is, the product of all normal hypercyclically embedded subgroups of $G$. Hypercyclically embedded subgroups play an important role in the theory of groups (see [7, 8, 18, 19] ) and the conditions under which a normal subgroup is hypercyclically embedded in $G$ were found by many authors (see the books [7, 8, 18, 19] and the recent papers [10, 14, 20–23] ).

On the base of Theorem 1.5, we will prove the following result.

**Theorem 1.13.** Let $G$ be a $\sigma$-full group of Sylow type and $\mathcal{H} = \{1, W_1, W_2, \ldots, W_t\}$ be a complete Hall $\sigma$-set of $G$ such that $W_i$ is nilpotent for all $i = 1, \ldots, t$. Let $E$ be a normal subgroup of $G$. If every maximal subgroup of $W_i \cap E$ is weakly $\sigma$-permutable in $G$ for all $i = 1, 2, \cdots, t$, then $E \leq Z_u(G)$.

The following results directly follow from Theorem 1.13.

**Corollary 1.14.** Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let $E$ be a normal subgroup of $G$ with $G/E \in \mathfrak{F}$. Suppose that $G$ is a $\sigma$-full group of Sylow type and $\mathcal{H} = \{1, W_1, W_2, \cdots, W_t\}$ is a complete Hall $\sigma$-set of $G$ such that $W_i$ is nilpotent for all $i = 1, \cdots, t$. If every maximal subgroup of $W_i \cap E$ is weakly $\sigma$-permutable in $G$, for all $i = 1, 2, \cdots, t$, then $G \in \mathfrak{F}$.

**Corollary 1.15.** (See Asaad [24, Theorem 4.1]) Let $G$ be a group with a normal subgroup $E$ such that $G/E$ is supersoluble. If every maximal subgroup of every Sylow subgroups of $E$ is $s$-permutable in $G$, then $G$ is supersoluble.
Corollary 1.16. (See Asaad [25, Theorem 1.3]) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let $E$ be a normal subgroup of $G$ with $G/E \in \mathfrak{F}$. If the maximal subgroups of every Sylow subgroup of $E$ are $s$-permutable in $G$, then $G \in \mathfrak{F}$.

Corollary 1.17. (See Wei [11, Corollary 1]) Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and let $E$ be a normal subgroup of $G$ with $G/E \in \mathfrak{F}$. If the maximal subgroups of every Sylow subgroup of $E$ are $c$-normal in $G$, then $G \in \mathfrak{F}$.

All unexplained terminologies and notations are standard, as in [8] and [13].

2 Preliminaries

We use $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ to denote the classes of all $\sigma$-soluble groups and $\sigma$-nilpotent groups.

Lemma 2.1. (See [1, Lemma 2.1] The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the $\sigma$-soluble group by a $\sigma$-soluble group is a $\sigma$-soluble group as well.

Following [1] and [2], $O_\Pi(G)$ to denote the subgroup of $G$ generated by all its $\Pi'$-subgroups. Instead of $O_{\{\sigma_i\}}(G)$ we write $O_{\sigma_i}(G)$. $O_\Pi(G)$ to denote the subgroup of $G$ generated by all its normal $\Pi$-subgroups.

Lemma 2.2. (See [1, Lemma 2.6] and [2, Lemma 2.1]) Let $A, K$ and $N$ be subgroups of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$.

1. $A \cap K$ is $\sigma$-subnormal in $K$.
2. If $K$ is a $\sigma$-subnormal subgroup of $A$, then $K$ is $\sigma$-subnormal in $G$.
3. If $K$ is $\sigma$-subnormal in $G$, then $A \cap K$ and $\langle A, K \rangle$ are $\sigma$-subnormal in $G$.
4. $AN/N$ is $\sigma$-subnormal in $G/N$.
5. If $N \leq K$ and $K/N$ is $\sigma$-subnormal in $G/N$, then $K$ is $\sigma$-subnormal in $G$.
6. If $K \leq A$ and $A$ is $\sigma$-nilpotent, then $K$ is $\sigma$-subnormal in $G$.
7. If $\left|G:A\right|$ is a $\Pi$-number, then $O_\Pi(A) = O_\Pi(G)$.
8. If $G$ is $\Pi$-full and $A$ is a $\Pi$-group, then $A \leq O_\Pi(G)$.

Let $\mathcal{L}$ be some non-empty set of subgroups of $G$ and $E$ a subgroup of $G$. Then a subgroup $A$ of $G$ is called $\mathcal{L}$-permutable if $AH = HA$ for all $H \in \mathcal{L}$; $\mathcal{L}^E$-permutable if $AH^x = H^xA$ for all $H \in \mathcal{L}$ and all $x \in E$. In particular, a subgroup $H$ of $G$ is $\sigma$-permutable if $G$ possesses a complete Hall $\sigma$-set $\mathcal{H}$ such that $H$ is $\mathcal{H}^G$-permutable.

Lemma 2.3. (See [1, Lemma 2.8] and [2, Lemma 2.2]) Let $H, K$ and $N$ be subgroups of $G$. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a complete Hall $\sigma$-set of $G$ and $\mathcal{L} = \mathcal{H}^K$. Suppose that $H$ is $\mathcal{L}$-permutable and $N$ is normal in $G$. 

5
(1) If \( H \leq E \leq G \), then \( H \) is \( L^\ast \)-permutable, where \( L^\ast = \{H_1 \cap E, H_2 \cap E, \ldots, H_t \cap E\}^{K \cap E} \). In particular, if \( G \) is a \( \sigma \)-full group of Sylow type and \( H \) is \( \sigma \)-permutable in \( G \), then \( H \) is \( \sigma \)-permutable in \( E \).

(2) The subgroup \( HN/N \) is \( L^\ast\ast \)-permutable, where \( L^\ast\ast = \{H_1N/N, \ldots, H_tN/N\}^{K/N} \).

(3) If \( G \) is a \( \sigma \)-full group of Sylow type and \( E/N \) is a \( \sigma \)-permutable subgroup of \( G/N \), then \( E \) is \( \sigma \)-permutable in \( G \).

(4) If \( K \) is \( L \)-permutable, then \( \langle H, K \rangle \) is \( L \)-permutable \([13, A, \text{Lemma 1.6(a)}] \). In particular, \( H_{\sigma G} \) is \( \sigma \)-permutable in \( G \). Moreover, if \( G \) is a \( \sigma \)-full group of Sylow type, then \( H_{\sigma G} \) is a \( \sigma \)-subnormal of \( G \) (see \([1, \text{Theorems B and C}]\)).

**Lemma 2.4.** (See \([1, \text{Lemma 3.1}]\)) Let \( H \) be a \( \sigma_1 \)-subgroup of a \( \sigma \)-full group \( G \). Then \( H \) is \( \sigma \)-permutable in \( G \) if and only if \( O^{\sigma_1}(G) \leq N_G(H) \).

**Lemma 2.5.** Let \( G \) be a \( \sigma \)-full group of Sylow type and \( H \leq K \leq G \).

(1) If \( H \) is weakly \( \sigma \)-permutable in \( G \), then \( H \) is weakly \( \sigma \)-permutable in \( K \).

(2) Suppose that \( N \) is a normal subgroup of \( G \) and \( N \leq H \). Then \( H/N \) is weakly \( \sigma \)-permutable in \( G/N \) if and only if \( H \) is weakly \( \sigma \)-permutable in \( G \).

(3) If \( N \) is a normal subgroup of \( G \), then for every weakly \( \sigma \)-permutable subgroup \( H \) of \( G \) with \( (|H|, |N|) = 1 \), \( HN/N \) is weakly \( \sigma \)-permutable in \( G/N \).

**Proof.** (1) Suppose that there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_{\sigma G} \). Since \( H \leq K \), \( K = H(K \cap T) \). By **Lemma 2.2** (1), \( K \cap T \) is \( \sigma \)-subnormal in \( K \). Moreover, \( H \cap (K \cap T) = H \cap T \leq H_{\sigma G} \leq H_{\sigma K} \) by **Lemma 2.3** (1)(4). Hence, \( H \) is weakly \( \sigma \)-permutable in \( K \).

(2) First assume that there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_{\sigma G} \). Then \( G/N = (H/N)(TN/N) \), \( TN/N \) is \( \sigma \)-subnormal in \( G/N \) by **Lemma 2.2** (4) and \( H/N \cap TN/N = (H \cap T)N/N \leq H_{\sigma G}N/N \leq (H/N)_{\sigma(G/N)} \) by **Lemma 2.3** (2). This shows that \( H/N \) is weakly \( \sigma \)-permutable in \( G/N \).

Conversely, assume that \( H/N \) is weakly \( \sigma \)-permutable in \( G/N \). Then \( G/N = (H/N)(T/N) \) and \( H/N \cap T/N \leq (H/N)_{\sigma(G/N)} \), where \( T/N \) is \( \sigma \)-subnormal in \( G/N \). So \( G = HT \) and \( T \) is \( \sigma \)-subnormal in \( G \) by **Lemma 2.2** (5). Let \( (H/N)_{\sigma(G/N)} = E/N \). Then \( E \) is \( \sigma \)-permutable in \( G \) by **Lemma 2.3** (3)(4). Hence \( H \cap T \leq E \leq H_{\sigma G} \). This shows that \( H \) is weakly \( \sigma \)-permutable in \( G \).

(3) Assume that there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_{\sigma G} \). Then \( G/N = (HN/N)(TN/N) \). Since \( (|H|, |N|) = 1 \), \( (HT \cap N : H \cap N) | HT \cap N : T \cap N | = (|HT \cap N)H : H \cap (HT \cap N)T : T | = 1 \). Hence \( N = N \cap HT = (N \cap H)(N \cap T) = N \cap T \) by \([13, A, \text{Lemma 1.6}]\). It follows that \( N \leq T \). Hence \( (HN/N) \cap (TN/N) = (H \cap T)N/N \leq H_{\sigma G}N/N \leq (H/N)_{\sigma(G/N)} \) by **Lemma 2.3** (2)(4).
Besides, by Lemma 2.2 (4), \( T/N \) is \( \sigma \)-subnormal in \( G/N \). Thus \( HN/N \) is weakly \( \sigma \)-permutable in \( G/N \).

\[ \square \]

**Lemma 2.6.** (See [26, Lemma 2.12]) Let \( P \) be a normal \( p \)-subgroup of a group \( G \). If \( P/\Phi(P) \leq Z_d(G/\Phi(P)) \), then \( P \leq Z_d(G) \).

## 3 Proof of Theorem 1.4

**Proof of Theorem 1.4.** Assume that this is false and let \( G \) be a counterexample of minimal order. Then \( |\sigma(G)| > 1 \).

1. \( G/N \) is \( \sigma \)-soluble, for every non-identity normal subgroup \( N \) of \( G \).

   Let \( N \) be a non-identity normal subgroup of \( G \) and \( H/N \) is any Hall \( \sigma_i \)-subgroup of \( G/N \), where \( \sigma_i \cap \pi(G/N) \neq \emptyset \). Then \( H/N = H_iN/N \) for some Hall \( \sigma_i \)-subgroup \( H_i \) of \( G \). By the hypothesis, there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = H_iT \) and \( H_i \cap T \leq (H_i)_{\sigma G} \). Then \( G/N = (H_iN/N)(TN/N) = (H/N)(TN/N) \). Since \( |H_iN \cap T : H_i \cap T| = |(H_iN \cap T)H_i : H_i| \) is a \( \alpha_i \)-number, \( |(H_iN \cap T : H_i \cap T)|, |H_iN \cap T : N \cap T| \rangle = 1 \). Hence \( H_iN \cap T = (H_i \cap T)(N \cap T) \) by [13, A, Lemma 1.6]. Consequently, \( (H_iN/N) \cap (TN/N) \leq (H_i \cap T)N/N \leq (H_iN/N)_{\sigma G} \) by Lemma 2.3 (2)(4). By Lemma 2.2 (4), \( TN/N \) is \( \sigma \)-subnormal in \( G/N \). Therefore \( H/N \) is weakly \( \sigma \)-permutable in \( G/N \). This shows that \( G/N \) satisfies the hypothesis. The minimal choice of \( G \) implies that \( G/N \) is \( \sigma \)-soluble.

2. \( G \) is not a simple group.

   Suppose that \( G \) is a non-abelian simple group. Then 1 is the only proper \( \sigma \)-subnormal subgroup of \( G \). Let \( H_i \) be a non-identity Hall \( \sigma_i \)-subgroup of \( G \), where \( \sigma_i \in \sigma(G) \). By the hypothesis and \( |\sigma(G)| > 1 \), we have \( G = H_iG \) and \( H_i = H_i \cap G \leq (H_i)_{\sigma G} \). By Lemma 2.3 (4), \( (H_i)_{\sigma G} \) is \( \sigma \)-subnormal in \( G \), so \( H_i = (H_i)_{\sigma G} = 1 \), a contradiction. So we have (2).

3. Let \( R \) be a minimal normal subgroup of \( G \), then \( R \) is \( \sigma \)-soluble.

   Let \( H \) be any Hall \( \sigma_i \)-subgroup of \( R \), where \( \sigma_i \cap \pi(R) \neq \emptyset \). Then there exists a Hall \( \sigma_i \)-subgroup \( H_i \) of \( G \) such that \( H = H_i \cap R \). By the hypothesis, there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = H_iT \) and \( H_i \cap T \leq (H_i)_{\sigma G} \). Since \( |H_iT \cap R : H_i \cap R| = |(H_iT \cap R)H_i : H_i| \) is a \( \alpha_i \)-number, \( |H_iT \cap R : H_i \cap R|, |H_iT \cap R : T \cap R| \rangle = 1 \). Hence, \( R = R \cap H_iT = (H_i \cap R)(R \cap T) \leq H(R \cap T) \) by [13, A, Lemma 1.6(c)]. Since \( R \cap T \) is \( \sigma \)-subnormal in \( R \) by Lemma 2.2 (1) and \( H \cap R \cap T = (R \cap H_i)(R \cap T) \leq (H_i)_{\sigma G} \cap R \leq (H)_{\sigma R} \) by Lemma 2.3 (1)(4), \( R \) satisfies the hypothesis. The minimal choice of \( G \) implies that \( R \) is \( \sigma \)-soluble.

4. Final contradiction.

   In view of (1), (2) and (3), we have that \( G \) is \( \sigma \)-soluble by Lemma 2.1. The final contra-
diction completes the proof of the theorem.

4 Proof of Theorem 1.5

First we prove the following proposition, which is a main step of the proof of Theorem 1.5.

Proposition 4.1. Let $G$ be a $\sigma$-full group of Sylow type and $H = \{1, W_1, W_2, \ldots, W_t\}$ be a complete Hall $\sigma$-set of $G$ such that $W_i$ is a nilpotent $\sigma_i$-subgroup for all $i = 1, \ldots, t$, and let the smallest prime $p$ of $\pi(G)$ belongs to $\sigma_1$. If every maximal subgroup of $W_1$ is weakly $\sigma$-permutable in $G$, then $G$ is soluble.

Proof. First note that if $G$ is $\sigma$-soluble, then every chief factor $H/K$ of $G$ is $\sigma$-primary, that is, $H/K$ is a $\sigma_i$-group for some $i$. But since $W_i$ is nilpotent, $H/K$ is a elementary abelian group. It follows that $G$ is soluble. Hence we only need to prove that $G$ is $\sigma$-soluble. Suppose that the assertion is false, and let $G$ be a counterexample of minimal order. Then clearly $t > 1$, and $p = 2 \in \pi(W_1)$ by the well-known Feit-Thompson’s theorem. Without loss of generality, we can assume that $W_i$ is a $\sigma_i$-group for all $i = 1, 2, \ldots, t$.

(1) $O_{\sigma_i}(G) = 1$.

Assume that $N = O_{\sigma_1}(G) \neq 1$. Note that if $W_1 = N$, then $G/N$ is a $\sigma_1'$-group, so $G/N$ is soluble by the well-known Feit-Thompson’s theorem and so $G$ is $\sigma$-soluble. We may, therefore, assume that $W_1 \neq N$. Then $W_1/N$ is a non-identity Hall $\sigma_1$-subgroup of $G/N$. Let $M/N$ be a maximal subgroup of $W_1/N$. Then $M$ is a maximal subgroup of $W_1$. By the hypothesis and Lemma 2.5 (2), $M/N$ is weakly $\sigma_i$-permutable in $G/N$. The minimal choice of $G$ implies that $G/N$ is $\sigma$-soluble. Consequently, $G$ is $\sigma$-soluble. This contradiction shows that (1) holds.

(2) $O_{\sigma_1}(G) = 1$.

Assume that $R = O_{\sigma_1}(G) \neq 1$. Then $W_1 R/R$ is a Hall $\sigma_1$-subgroup of $G/R$. Let $M/R$ be a maximal subgroup of $W_1 R/R$. Then $M = (M \cap W_1) R$. Since $W_1$ is nilpotent and $|W_1 R/R : M/R| = |W_1 R/R : (M \cap W_1) R/R| = |W_1 : M \cap W_1|$, $M \cap W_1$ is a maximal subgroup of $W_1$. By the hypothesis and Lemma 2.5 (3), $M/R = (M \cap W_1) R/R$ is weakly $\sigma_i$-permutable in $G/R$. This shows that $G/R$ satisfies the hypothesis. The choice of $G$ implies that $G/R$ is $\sigma$-soluble. By the well-known Feit-Thompson’s theorem, we know that $R$ is soluble. It follows that $G$ is $\sigma$-soluble, a contradiction.

(3) If $R \neq 1$ is a minimal normal subgroup of $G$, then $R$ is not $\sigma$-soluble and $G = RW_1$.

If $R$ is $\sigma$-soluble, then $R$ is a $\sigma_i$-group for some $\sigma_i \in \sigma(G)$. So $R \leq O_{\sigma_1}(G)$ or $R \leq O_{\sigma_1'}(G)$, a contradiction. Therefore, $R$ is not $\sigma$-soluble. Assume that $RW_1 < G$. Then by the hypothesis and Lemma 2.5 (1), $RW_1$ satisfies the hypothesis. Hence $RW_1$ is $\sigma$-soluble by the choice of $G$. It follows from Lemma 2.1 that $R$ is $\sigma$-soluble. This contradiction shows that $G = RW_1$.

(4) $G$ has a unique minimal normal subgroup $R$. 

8
By (3), $G = RW_1$ for every non-identity minimal normal subgroup $R$ of $G$. Then clearly, $G/R$ is $\sigma$-soluble. Hence by Lemma 2.1 $G$ has a unique minimal normal subgroup, which is denoted by $R$.

(5) $W_1$ is a 2-group.

Let $q \in \pi(W_1) \setminus \{2\}$. As $W_1$ is nilpotent, there exists two maximal subgroups $M_1$ and $M_2$ of $W_1$ such that $|W_1 : M_1| = q$ and $|W_1 : M_2| = 2$. By the hypothesis, there exists $\sigma$-subnormal subgroups $T_i$ of $G$, such that $G = M_iT_i$ and $M_i \cap T_i \leq (M_i)_{\sigma G}$, $i = 1, 2$. By Lemma 2.3 (4), $(M_i)_{\sigma G}$ is $\sigma$-subnormal in $G$. Then by Lemma 2.2 (8), $(M_i)_{\sigma G} \leq O_{\sigma_i}(G) = 1$, $i = 1, 2$. Hence $M_i \cap T_i = 1, i = 1, 2$. Consequently, $|G : T_i| = |M_i : M_i \cap T_i| = |M_i|$, $i = 1, 2$, which implies that $|G : T_i|$ is a $\sigma_i$-number for $i = 1, 2$. Hence $O^{\sigma_i}(T_i) = O^{\sigma_i}(G)$ for $i = 1, 2$ by Lemma 2.2 (7). Since $t > 1$, $O^{\sigma_i}(G) > 1$. It follows that $1 \neq O^{\sigma_i}(G) \leq (T_i)_G$ for $i = 1, 2$. Then by (4), $R \leq (T_1)_G \cap (T_2)_G \leq T_1 \cap T_2$. It is clear that $W_1 \cap R$ is a Hall $\sigma_i$-subgroup of $R$, and $W_1 \cap R \neq 1$ by (2). So $1 \neq W_1 \cap R \leq T_1 \cap T_2 \cap W_1$. Since $G = M_1T_1 = W_1T_1 = M_2T_2 = W_1T_2$, where $M_1 \cap T_1 = 1$ and $M_2 \cap T_2 = 1$, we have that $|W_1 \cap T_1| = |W_1 : M_i| = q$ and $|W_1 \cap T_2| = |W_1 : M_2| = 2$. Therefore $(W_1 \cap T_1) \cap (W_1 \cap T_2) = 1$, which implies that $1 \neq W_1 \cap R \leq T_1 \cap T_2 \cap W_1 = (T_1 \cap W_1) \cap (T_2 \cap W_1) = 1$. This contradiction shows that $W_1$ is a 2-group.

(6) Final contradiction.

Let $P_1$ be a maximal subgroup of $W_1$. Then $|W_1 : P_1| = 2$. By the hypothesis, there exists a $\sigma$-subnormal subgroup $K$ of $G$ such that $G = P_1K$ and $P_1 \cap K \leq (P_1)_{\sigma G}$. By (1) and Lemma 2.2 (8), $(P_1)_{\sigma G} = 1$, Hence $|K| = 2$, and so $K$ is 2-nilpotent by [16, IV, Theorem 2.8]. Let $K_2'$ be the normal Hall $2'$-subgroup of $K$. Then $1 \neq K_2'$ is $\sigma$-subnormal in $G$, and so $K_2' \leq O_{\sigma_i}(G) = 1$ by Lemma 2.2(8). The final contradiction completes the proof. □

Proof of Theorem 1.5. Assume that the assertion is false, and let $G$ be a counterexample of minimal order.

(1) $G$ is soluble.

Let $q$ is the smallest prime dividing $|G|$. Without loss of generality, we may assume that $q \in \pi(W_1)$. If $W_1$ is cyclic, then the Sylow $q$-subgroup of $G$ is cyclic. Hence $G$ is $q$-nilpotent by [16, IV, Theorem 2.8] and so $G$ is soluble. If $W_1$ is non-cyclic, then by Proposition 4.1, $G$ is soluble. Hence we always have that $G$ is soluble.

(2) The hypothesis holds on $G/R$ for every non-identity minimal normal subgroup $R$ of $G$. Consequently $G/R$ is supersoluble.

It is clear that $\mathcal{H} = \{1, W_1R/R, W_2R/R, \cdots, W_iR/R\}$ is a complete Hall $\sigma$-set of $G/R$ and $W_iR/R \simeq W_i/W_i \cap R$ is nilpotent. By (1), $R$ is an elementary abelian $p$-group for some prime $p$. Without loss of generality, we can assume that $R \leq W_1$. If $W_1/R$ is non-cyclic, then $W_1$ is non-cyclic. For every maximal subgroup $M/R$ of $W_1/R$, we have that $M$ is a
maximal subgroup of \( W_1 \). Then by the hypothesis and Lemma 2.5 (2), \( M/R \) is weakly \( \sigma \)-permutable in \( G/R \). Now assume that \( W_i R/R \) is non-cyclic for \( i \neq 1 \) and \( M/R \) be a maximal subgroup of \( W_i R/R \). Then \( M = (M \cap W_i) R \). As \( W_i \) is nilpotent, \( |W_i R/R : M/R| = |W_i R/R : (M \cap W_i) R/R| = |W_i : M \cap W_i| \) is a prime. Hence \( M \cap W_i \) is a maximal subgroup of \( W_i \). By the hypothesis and Lemma 2.5 (3), \( M/R = (M \cap W_i) R/R \) is weakly \( \sigma \)-permutable in \( G/R \). This shows that the hypothesis holds for \( G/R \). Hence \( G/R \) is supersoluble by the choice of \( G \).

3) \( R \) is the unique minimal normal subgroup of \( G \), \( \Phi(G) = 1 \), \( C_G(R) = R = F(G) = O_p(G) \) and \( |R| > p \) for some prime \( p \) (It follows from (2)).

4) For some \( i \in \{1, 2, \cdots, t\} \), \( W_i \) is a \( p \)-group. Without loss of generality, we may assume that \( W_1 \) is a \( p \)-group.

Since \( R \) is a \( p \)-group, \( R \leq W_i \) for some \( i \in \{1, 2, \cdots, t\} \). Moreover, since \( C_G(R) = R \) and \( W_i \) is a nilpotent group, we have that \( W_i \) is a \( p \)-group.

5) Final contradiction.

Since \( \Phi(G) = 1 \), \( R \not\in \Phi(W_1) \) [16, Chapter III, Lemma 3.3]. Hence there exists a maximal subgroup \( V \) of \( W_1 \) such that \( W_1 = RV \). Let \( E = R \cap V \). Then \( |R : E| = |RV : V| = |W_i : V| = p \). Hence \( E \) is a maximal subgroup of \( R \) and \( 1 \neq E \leq W_1 \). Since \( |R| > p \) and \( R \leq W_1 \), \( W_1 \) is non-cyclic. Hence by the hypothesis, there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = VT \) and \( V \cap T \leq V_{\sigma G} \). Since \( |G : T| \) is \( p \)-number, \( O^p(T) = O^{\sigma_1}(T) = O^\sigma(T) \) by Lemma 2.2 (7). So \( |G : T_G| \) is \( p \)-number. It follows that \( T_G \neq 1 \) and \( R \leq T_G \leq T \) by (2). Since \( V_{\sigma G} \) is \( \sigma \)-subnormal in \( G \) by Lemma 2.3 (4), we have that \( V_{\sigma G} \leq O_{\sigma_1}(G) = O_p(G) = R \) by Lemma 2.2 (8). Hence \( E = R \cap V \leq T \cap V \leq V_{\sigma G} \leq R \). But since \( E \) is a maximal subgroup of \( R \), it follows that \( V_{\sigma G} = R \) or \( V_{\sigma G} = E \). In the former case, we have that \( R \leq V \), a contradiction. In the latter case, \( E = V_{\sigma G} \) is \( \sigma \)-permutable in \( G \) by Lemma 2.3 (4) and \( E \) is a \( \sigma_1 \)-group. It follows from Lemma 2.4 that \( O^\sigma_1(G) \leq N_G(E) \). Hence \( E \leq G \), which contradicts the minimality of \( R \). The final contradiction completes the proof of the theorem.

5 Proof of Theorem 1.13

Assume that the assertion is false and let \( (G, E) \) be a counterexample with \( |G| + |E| \) minimal. Without loss of generality, we can assume that \( W_i \) is a \( \sigma_i \)-group for all \( i = 1, 2, \cdots, t \). We now proceed with the proof via the following steps.

1) \( E \) is supersoluble.

In fact, \( \{1, W_1 \cap E, W_2 \cap E, \cdots, W_t \cap E\} \) is a complete Hall \( \sigma \)-set of \( E \) and \( W_i \cap E \) is nilpotent. Consequently, \( E \) is a \( \sigma \)-full group of Sylow type. By Lemma 2.5 (1) and Theorem 1.5. Hence \( E \) is supersoluble.
(2) If \( R \) is a minimal normal subgroup of \( G \) contained in \( E \), then \( R \) is a \( p \)-group for some prime \( p \) and the hypothesis holds for \((G/R, E/R)\). Therefore \( E/R \leq Z_u(G/R) \).

By (1), \( R \) is a \( p \)-group for some \( p \). Without loss of generality, we can assume that \( R \leq W_1 \cap E \). It is clear that \( \mathcal{T} = \{1, W_1/R, W_2/R, \ldots, W_t/R\} \) is a complete Hall \( \sigma \)-set of \( G/R \) and \( W_i/R/R \cong W_i/W_i \cap R \) is nilpotent. Let \( M/R \) be a maximal subgroup of \((W_1 \cap E)/R\). Then by the hypothesis and Lemma 2.5 (2), \( M/R \) is weakly \( \sigma \)-permutable in \( G/R \). Now let \( V/R \) be a maximal subgroup of \( W_i/R/R \cap E/R = (W_i \cap E)/R \), \( i = 2, \ldots, t \). Then \( V = (V \cap W_i)/R \). Since \( W_i/R/R \cap E/R \) is nilpotent, \(|W_i \cap E : V \cap W_i| = |W_i/R \cap E : (V \cap W_i)/R| = |W_i/R/R \cap E/R : V/R| \) is a prime, so \( V \cap W_i \) is a maximal subgroup of \( W_i \cap E \). Then by the hypothesis and Lemma 2.5 (3), \( V/R = (V \cap W_i)/R \) is weakly \( \sigma \)-permutable in \( G/R \), \( i = 2, \ldots, t \). This shows that \((G/R, E/R)\) satisfies the hypothesis. Thus \( E/R \leq Z_u(G/R) \) by the choice of \((G,E)\).

(3) \( R \) is the unique minimal normal subgroup of \( G \) contained in \( E \), \(|R| > p \) and \( O_p'(E) = 1 \).

Let \( L \) be a minimal normal subgroup of \( G \) contained in \( E \) such that \( R \neq L \). Then \( E/R \leq Z_u(G/R) \) and \( E/L \leq Z_u(G/L) \) by (2), and clearly, \(|R| > p \). It follows that \( LR/L \leq Z_u(G/L) \), so \(|R| = p \) by the \( G \)-isomorphism \( RL/L \simeq R \), a contradiction. Hence \( R \) is the unique minimal normal subgroup of \( G \) contained in \( E \). Consequently, \( O_p'(E) = 1 \). Hence (3) holds.

Without loss of generality, we may assume \( p \in \pi(W_1) \).

(4) \( E \) is a \( p \)-group and so \( E \cap W_1 = E \) and \( E \cap W_i = 1 \) for \( i = 2, 3, \ldots, t \).

Let \( q \) be the largest prime dividing \(|E|\) and let \( Q \) be a Sylow \( q \)-subgroup of \( E \). Since \( E \) is supersoluble by (1) (see [16, Chapter VI, Theorem 9.1]), \( Q \) is characteristic in \( E \). Then \( Q \) is normal in \( G \). Hence by (3) we have that \( q = p \) and \( F(E) = Q = O_p(E) = P \) is a Sylow \( p \)-subgroup of \( E \). Thus \( C_E(P) \leq P \) (see [27, Theorem 1.8.18]). But since \( P \leq W_1 \cap E \) and \( W_1 \cap E \) is nilpotent, we have that \( P = W_1 \cap E \). Since \( P \cap W_1 = P = W_1 \cap E \) and \( P \cap W_i = 1 \) for all \( i = 2, \ldots, t \), the hypothesis holds for \((G,P)\). If \( P < E \), then \( R \leq P \leq Z_u(G) \) by the choice of \((G,E)\). It follows that \(|R| = p \), a contradiction. Hence \( E = P \) is a \( p \)-group, and so \( E \leq W_1 \).

(5) \( \Phi(E) = 1 \), so \( E \) is elementary abelian \( p \)-group.

Assume that \( \Phi(E) \neq 1 \). Then clearly, \((G/\Phi(E), E/\Phi(E))\) satisfies the hypothesis. Hence \( E/\Phi(E) \leq Z_u(G/\Phi(E)) \). It follows from (4) and Lemma 2.6 that \( E \leq Z_u(G) \), a contradiction. Thus we have (5).

(6) Final contradiction.

Let \( R_1 \) be a maximal subgroup of \( R \) such that \( R_1 \leq W_1 \). Then \(|R_1| > 1 \) by (3). Claim (5) implies that \( R \) has a complement \( S \) in \( E \). Let \( V = R_1S \). Then \( R \cap V = R_1 \) and \( V \) is a maximal subgroup of \( E \). Hence by (4) and the hypothesis, there exists a \( \sigma \)-subnormal subgroup \( T \) of \( G \) such that \( G = VT \) and \( V \cap T \leq V_{\sigma G} \). Then \( G = VT = ET \) and \( E = V(E \cap T) \). By (5), it is easy to see that \( 1 \neq E \cap T \leq G \). Hence \( R \leq E \cap T \) by (3), and so \( R_1 = R \cap V \leq E \cap T \cap V = \)}
\[ V \cap T \leq V_\sigma G. \] Consequently, \( R_1 \leq V_\sigma G \cap R \leq R. \) It follows that \( R = V_\sigma G \cap R \) or \( R_1 = V_\sigma G \cap R. \) In the former case, \( R \leq V, \) which contradicts the fact that \( R_1 = R \cap V. \) Thus \( R_1 = V_\sigma G \cap R. \)

By Lemma 2.3(4), we have that \( V_\sigma G \) is \( \sigma \)-permutable in \( G, \) so \( O^{\sigma_1}(G) \leq N_G(V_\sigma G) \) by Lemma 2.4. Hence \( O^{\sigma_1}(G) \leq N_G(V_\sigma G \cap R) = N_G(R_1). \) Moreover, since \( R_1 \unlhd W_1, \) we obtain that \( R_1 \leq G. \) This implies that \( R_1 = 1. \) The final contradiction completes the proof.

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