A Fixed Point Problem Under a Finite Number of Equality Constraints on $b$-Banach spaces

Monica-Felicia Bota$^a$, Erdal Karapınar$^{b,*}$

$^a$Department of Mathematics, Babeş-Bolyai University Kogălniceanu Street No.1, 400084, Cluj-Napoca, Romania.
$^b$Department of Medical Research, China Medical University Hospital, China Medical University, 40402, Taichung, Taiwan

Abstract. In this manuscript, we investigate a fixed point problem under a finite number of equality constraints involving a well-known Ćirić type mappings in the context of $b$-metric space. We obtain sufficient conditions for the existence of solutions of such problems. We also express some immediate consequences of our main results.

1. Introduction

A fixed point problem, $f(x) = x$, can be reformulated as in the form $F(x) = 0$. It is usually expected to find a unique solution of this equation. On the other hand, the equations, abstracted from the real world problems, need not to have a unique solution. In particular, it is very well known that in several cases, the nonlinear integral or differential equations (that corresponding to a fixed point problem) have periodic solution or more than one solutions. Thus, non-unique fixed point problem, is very natural and has a wide application potential.

In 1974, Ćirić [18] proved a fixed point theorem for certain operators that do not need to possess a unique fixed point.

Theorem 1.1. [Non-unique fixed point theorem of Ćirić [18]] Let $T$ be a self-map on a metric space $(X,d)$. Suppose also that

(i) $T$ is called orbitally continuous, that is,
\[
\lim_{i \to \infty} T^m x = z \quad \text{implies} \quad \lim_{i \to \infty} TT^m x = Tz \quad \text{for each} \quad x \in X,
\]

(ii) $(X,d)$ is called orbitally complete, that is, each Cauchy sequence of type $\{T^n x\}_{n \in \mathbb{N}}$ converges in $(X,d)$,

(iii) there is $k \in (0,1)$ such that
\[
\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\} - \min\{d(x,Ty),d(Tx,y)\} \leq kd(x,y),
\]

for all $x, y \in X$.

2010 Mathematics Subject Classification. Primary 47H10; 54H25
Keywords. fixed point problem $b$-Banach space, Ćirić-type operator, cone, level closed.
Received: 14 February 2019; Accepted: 23 April 2019
Communicated by Adrian Petrusel
*Corresponding author: Erdal Karapınar
Email addresses: bmonica@math.ubbcluj.ro (Monica-Felicia Bota), erdalkarapinar@yahoo.com (Erdal Karapinar)
Then, for each \( x_0 \in X \) the sequence \( \{T^n x_0\}_{n \in \mathbb{N}} \) converges to a fixed point of \( T \).

Here, the mapping \( T \) that satisfies (1) is called Ćirić-type-contraction. This remarkable result of Ćirić [18] initiated the trend of research on non-unique fixed points, and several authors have focused on this topic, see e.g. [2, 3, 16, 17, 19–24].

Very recently, Rakočević and Samet [25], considered a solution for a finite number of equality constraints involving in the setting of metric space as an application of this trend. More precisely, they consider a system

\[
\begin{align*}
T x &= x \\
q_i(x) &= 0_E, \; i = 1, 2, \ldots, r,
\end{align*}
\]

where \( T, q_i : E \to E, \; i = 1, 2, \ldots, r \) be a finite number of mappings defined on a Banach space \( (E, \| \cdot \|) \) with a cone \( P \), and \( 0_E \) is the zero vector of \( E \), and \( T \) is mapping satisfying a Ćirić-type-contraction.

The main aim of this paper is to reconsider the system (2) in the setting of \( b \)-metric space and \( b \)-normed space. It is clear that our results improve and extend the results of [25].

2. Preliminaries

In this section we recall and recollect the necessary basic definitions and fundamental results to construct the our problem in the framework of \( b \)-metric. Throughout the paper, we let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

We first express the definition of a \( b \)-metric space that was given by Bakhtin [4], Czerwik [12] and was announced earlier as quasi-metric by many authors, such as, Berinde [5–7, 26, 27].

**Definition 2.1.** (Bakhtin [4], Czerwik [12]) Let \( X \) be a set and let \( s \geq 1 \) be a given real number. A functional \( d : X \times X \to [0, \infty) \) is said to be a \( b \)-metric if the following conditions are satisfied:

1. \( d(x, y) = 0 \) if and only if \( x = y \),
2. \( d(x, y) = d(y, x) \),
3. \( d(x, z) \leq s(\alpha(x, y) + d(y, z)) \),

for all \( x, y, z \in X \). A pair \( (X, d) \) is called a \( b \)-metric space.

**Example 2.2.** ([5]) For \( 0 < p < 1 \), the set

\[
\ell^p(\mathbb{R}) := \{(x_n) \in \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\}
\]

together with the function

\[
d(x, y) := \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}, \text{ for each } x, y \in \ell^p(\mathbb{R})
\]

is a complete \( b \)-metric space with constant \( s = 2^{1/p} > 1 \).

**Example 2.3.** ([5]) For \( 0 < p < 1 \), the set

\[
L^p[a, b] := \{x : [a, b] \to \mathbb{R} : \int_a^b |x(t)|^p dt < \infty\},
\]

together with the function

\[
d(x, y) := \left( \int_a^b |x(t) - y(t)|^p dt \right)^{1/p}, \text{ for each } x, y \in L^p[a, b],
\]

is a complete \( b \)-metric space. Notice that in this case \( s = 2^{1/p} > 1 \).
Example 2.4. (9) Let $E$ be a Banach space and $P$ a normal cone in $E$ with $\text{int}(P) \neq \emptyset$. Denote by $\preceq$ the partially order relation generated by $P$. If $X$ is a nonempty set, then a mapping $d : X \times X \to E$ is called a cone metric on $X$ if the usual axioms of the metric take place with respect to $\preceq$. The cone $P$ is called normal if there is a number $K \geq 1$ such that, for all $x, y \in E$, the following implication holds:

$$0 \leq x \leq y \implies \|x\| \leq K\|y\|.$$ 

If the cone $P$ is normal with the coefficient of normality $K \geq 1$, then the functional

$$\hat{d} : X \times X \to \mathbb{R}_{+}, \hat{d}(x, y) := \|d(x, y)\|$$

is a $b$-metric on $X$ with constant $s := K$.

For more details and examples on $b$-metric spaces, see e.g. [7, 27].

Definition 2.5. (Bakhtin [4], Czerwik [12]) Let $X$ be a vector space over a field $\mathbb{F}$ (either $\mathbb{C}$ or $\mathbb{R}$) and let $s \geq 1$ be a given real number. A functional $\|x\| : X \to [0, \infty)$ is said to be a $b$-norm if the following conditions are satisfied:

(N$_1$) $\|x\| = 0$ if and only if $x = 0$,
(N$_2$) $\|cx\| = |c|\|x\|$, for all $c \in \mathbb{F}$ and for all $x \in X$
(N$_3$) $\|x + y\| \leq s(\|x\| + \|y\|)$, for all $x, y \in X$

for all $x, y, z \in X$. A pair $(X, d)$ is called a $b$-metric space.

In the context of a linear space $X$, the pair $(X, \| \cdot \|)$ is called a $b$-normed space with constant $s \geq 1$ if the third axiom of the norm has the following form

Notice that in Example 2.2, the functional

$$\|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$$

is a $b$-norm on $l^p(\mathbb{R})$ with constant $s = 2\frac{1}{p}$ and the pair $(l^p(\mathbb{R}), \| \cdot \|_p)$ is a $b$-Banach space. Something similar takes place for Example 2.3.

If the $b$-metric generated by the $b$-norm $\| \cdot \|$ (i.e., $d(x, y) := \|x - y\|$) is complete, then the space $(X, \| \cdot \|)$ is called a $b$-Banach space.

For $b$-metric spaces, the notions of convergent sequence, Cauchy sequence, completeness are similar to those given for usual metric spaces. Moreover, a $b$-metric generates (in a similar way to the case of usual metric spaces) a topology $\tau$ on $X$ (see [14]), but there are also major differences with the classical case of metric spaces. For example, the open ball $B(x_0; r) := \{x \in X : d(x_0, x) < r\}$ in a $b$-metric space $(X, d)$ is not necessarily an open set, while the closed ball $\tilde{B}(x_0; r) := \{x \in X : d(x_0, x) \leq r\}$ is not necessarily closed, in the usual sense. Moreover, a $b$-metric is not necessarily continuous, which induces several problems in different approaches.

The following Lemma is used in the proof of our main results.

Lemma 2.6. Let $(E, \| \cdot \|)$ be a $b$-Banach space and let $\{x_k\}_{k=0}^{\infty} \subset E$. Then:

$$\|x_n - x_0\| \leq s\|x_0 - x_1\| + \ldots + s^{n-1}\|x_{n-2} - x_{n-1}\| + s^{n-1}\|x_{n-1} - x_0\|.$$ 

Proof. Applying inequality (N$_3$) $n$-times we obtain the desired result.

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a comparison function if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by $\Phi$, the class of the comparison functions $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples, see e.g. [7, 27].
We denote by \( \Psi \) the family of nondecreasing functions \( \psi : [0, \infty) \to [0, \infty) \) such that \( \sum_{n=1}^{\infty} \psi^n(t) < \infty \) for each \( t > 0 \), where \( \psi^n \) is the \( n \)-th iterate of \( \psi \). It is clear that if \( \Phi \subset \Psi \) (see e.g. [15]) and hence, by Lemma 2.7 (3), for \( \psi \in \Psi \) we have \( \psi(t) < t \), for any \( t > 0 \).

We recall the following essential result.

**Lemma 2.7.** (Berinde [7], Rus [27]) If \( \varphi : [0, \infty) \to [0, \infty) \) is a comparison function, then:

1. each iterate \( \varphi^k \) of \( \varphi \), \( k \geq 1 \), is also a comparison function;
2. \( \varphi \) is continuous at 0;
3. \( \varphi(t) < t \), for any \( t > 0 \).

The notion of a \((b)\)-comparison function was given by Berinde [6] in order to expand some related fixed point results into the class of \( b \)-metric space.

**Definition 2.8.** (Berinde [6]) Let \( s \geq 1 \) be a real number. A mapping \( \varphi : [0, \infty) \to [0, \infty) \) is called a \((b)\)-comparison function if the following conditions are fulfilled

1. \( \varphi \) is monotone increasing;
2. there exist \( k_0 \in \mathbb{N} \), \( a \in (0,1) \) and a convergent series of nonnegative terms \( \sum_{k=1}^{\infty} v_k \) such that
\[
\varphi^{k+1}(t) \leq a \varphi^k(t) + v_k, \text{ for } k \geq k_0 \text{ and any } t \in [0, \infty).
\]

We denote by \( \Psi_b \) the class of \((b)\)-comparison function \( \varphi : [0, \infty) \to [0, \infty) \). In a special case of \( b = 1 \), \((b)\)-comparison function get a special name: \((c)\)-comparison function that has been used effectively in several fixed point results.

The following lemma has an important role in the proof of our main result.

**Lemma 2.9.** (Berinde [5]) If \( \varphi : [0, \infty) \to [0, \infty) \) is a \((b)\)-comparison function, then we have the following

1. the series \( \sum_{k=0}^{\infty} s^{k+1}(t) \) converges for any \( t \in \mathbb{R}_+ \);
2. the function \( s_b : [0, \infty) \to [0, \infty) \) defined by \( s_b(t) = \sum_{k=0}^{\infty} s^k(t) \), \( t \in [0, \infty) \), is increasing and continuous at 0.

In what follows we recollect the notion of cone: A nonempty closed convex set \( P \subset E \) is said to be a cone if it satisfies the following conditions:

\( \text{(P1)} \) \( \lambda \geq 0, x \in P \Rightarrow \lambda x \in P \).
\( \text{(P2)} \) \( -x, x \in P \Rightarrow x = 0_E \).

We note that any \((b)\)-comparison function is a comparison function.

Throughout this paper we consider the \( b \)-Banach space \( (E, \| \cdot \|) \) which is supposed to be partially ordered by a cone \( P \). We define the partial order \( \preceq_p \) in \( E \) induced by the cone \( P \) by
\[
(x, y) \in E \times E, \ x \preceq_p y \Leftrightarrow y - x \in P.
\]

Throughout this paper we consider the \( b \)-Banach space \( (E, \| \cdot \|) \) which is supposed to be partially ordered by a cone \( P \). We define the partial order \( \preceq_p \) in \( E \) induced by the cone \( P \) by
\[
(x, y) \in E \times E, \ x \preceq_p y \Leftrightarrow y - x \in P.
\]
Definition 2.10. Let $\varphi : E \to E$ be a given mapping. We say that $\varphi$ is $0_{E}$-level closed from the left, if and only if the set

$$
\mathcal{L}(\varphi_{\geq}) := \{ x \in E : \varphi(x) \geq p_{0_{E}} \}
$$

is nonempty and closed. Moreover, we say that $\varphi$ is $0_{E}$-level closed from the right, if and only if the set

$$
\mathcal{L}(\varphi_{\leq}) := \{ x \in E : \varphi(x) \leq p_{0_{E}} \}
$$

is nonempty and closed.

Definition 2.11. Let $T : E \to E$ be a given mapping. For a given $x \in E$, we denote by $O(x)$ the orbit of $x$, that is,

$$
O(x) = \{ T^{n}x : n = 0, 1, 2, \ldots \}.
$$

We say that $T$ is orbitally continuous on $F \subset E$, if and only if $T$ is continuous on $O(x)$, for every $x \in F$.

Definition 2.12. Let $T, \varphi_{i} : E \to E$ $(i = 1, 2, \ldots, r)$ be a finite number of mappings. We say that $T$ is a Ćirić-type operator with respect to $\{ \varphi_{i} \}_{i=1}^{r}$, if and only if there exists a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi \in \Psi_{b}$, such that

$$
\min\{||Tx - Ty||, ||x - Tx||, ||y - Ty||\} - \min\{||x - Ty||, ||y - Tx||\} \leq \psi(||x - y||),
$$

for every $(x, y) \in E \times E$ such that:

$$
\varphi_{i}(x) \leq p_{0_{E}} \text{ and } \varphi_{i}(y) \geq p_{0_{E}}, \ i = 1, 2, \ldots, r.
$$

Remark 2.13. Let $T, \varphi_{i} : E \to E$ $(i = 1, 2, \ldots, r)$ be a finite number of mappings. If $T$ is a Ćirić-type contraction with respect to $\{ \varphi_{i} \}_{i=1}^{r}$, then by symmetry, we have

$$
\min\{||Tx - Ty||, ||x - Tx||, ||y - Ty||\} - \min\{||x - Ty||, ||y - Tx||\} \leq \psi(||x - y||),
$$

where $\psi \in \Psi_{b}$, for every $(x, y) \in E \times E$ such that:

$$
\varphi_{i}(x) \geq p_{0_{E}} \text{ and } \varphi_{i}(y) \leq p_{0_{E}}, \ i = 1, 2, \ldots, r.
$$

3. Main results

In the first part of this paragraph we consider the analog of the system (2) for the case $r = 1$ and $\varphi_{1} = \varphi$ in the context of $b$-Banach space:

$$
\begin{cases}
    T x = x \\
    \varphi(x) = 0_{E},
\end{cases}
$$

(3)

where $T, \varphi$ be self-mappings on a $b$-Banach space $(E, || \cdot ||)$ with a cone $P$ and $0_{E}$ is the zero vector of $E$, and $T$ forms a Ćirić-type-contraction.

Theorem 3.1. Let $(E, || \cdot ||)$ be a $b$-Banach space and let $T, \varphi : E \to E$ be two given mappings. Suppose the following conditions are satisfied:

(i) $T$ is orbitally continuous on $\mathcal{L}(\varphi_{\geq})$,

(ii) $T$ is a Ćirić-type contraction with respect to $\varphi$, i.e.

$$
\min\{||Tx - Ty||, ||x - Tx||, ||y - Ty||\} - \min\{||x - Ty||, ||y - Tx||\} \leq \psi(||x - y||),
$$

where $\psi \in \Psi_{b}$, for every $(x, y) \in E \times E$ such that $\varphi(x) \leq p_{0_{E}} \text{ and } \varphi(y) \geq p_{0_{E}}$.\]
(iii) \( \varphi \) is \( E \)-level closed from left,

(iv) there exists \( x_0 \in E \) such that \( \varphi(x_0) \leq_p 0_E \),

(v) for every \( x \in E \) we have,

\[
\varphi(x) \leq_p 0_E \Rightarrow \varphi(Tx) \geq_p 0_E,
\]

and

\[
\varphi(x) \geq_p 0_E \Rightarrow \varphi(Tx) \leq_p 0_E.
\]

Then \( \{T^n(x_0)\} \) converges to a solution of (3).

**Proof.** On account of (iv), there exists \( x_0 \in E \) such that \( \varphi(x_0) \leq_p 0_E \). By employing (v), we find

\[
\varphi(x_0) \leq_p 0_E \Rightarrow \varphi(Tx_0) \geq_p 0_E.
\]

We construct an iterative sequence \( \{x_n\} \) in \( E \) by

\[
x_{n+1} = Tx_n, \quad n \in \mathbb{N}_0.
\]

Taking (v) into account, we observe \( \varphi(x_1) = \varphi(Tx_0) \geq_p 0_E \Rightarrow \varphi(Tx_1) \leq_p 0_E \), and further, we have \( \varphi(x_2) = \varphi(Tx_1) \leq_p 0_E \Rightarrow \varphi(Tx_2) \geq_p 0_E \).

Iteratively, we derive that

\[
\varphi(x_{2n}) \leq_p 0_E, \quad \text{and} \quad \varphi(x_{2n+1}) \geq_p 0_E, \quad \text{for all } n \in \mathbb{N}_0.
\]

Regarding Remark 2.13, together with (ii) and (4), we find that

\[
\min\{\|Tx_{n-1} - Tx_n\|, \|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_n\|\} - \min\{\|x_{n-1} - Tx_{n-1}\|, \|x_n - Tx_{n-1}\|\}
\]

\[
\leq \psi(\|x_{n-1} - x_n\|),
\]

for all \( n \in \mathbb{N} \), that is

\[
\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|, \|x_n - x_{n+1}\|\} - \min\{\|x_{n-1} - x_{n+1}\|, \|x_n - x_n\|\}
\]

\[
\leq \psi(\|x_{n-1} - x_n\|),
\]

for all \( n \in \mathbb{N} \). Accordingly, we have

\[
\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq \psi(\|x_{n-1} - x_n\|),
\]

for all \( n \in \mathbb{N} \).

We examine the inequality above in three cases.

**Case 1.** If \( x_{2N} = x_{2N+1} \), for some \( N \in \mathbb{N}_0 \), then we have

\[
x_{2N} = Tx_{2N}.
\]

Moreover, from (4), we find

\[
\varphi(x_{2N}) \leq_p 0_E \quad \text{and} \quad \varphi(x_{2N}) = \varphi(Tx_{2N}) = \varphi(x_{2N+1}) \geq 0_E.
\]

Consequently, we have \( \varphi(x_{2N}) \in P \) and \( -\varphi(x_{2N}) \in P \). Regarding that \( P \) is a cone, we conclude that \( \varphi(x_{2N}) = 0_E \). Thus, \( x_{2N} \in E \) forms a solution for (3).

**Case 2.** If \( x_{2N+1} = x_{2N+2} \), for some \( N \in \mathbb{N}_0 \), then, by verbatim with the Case 1 we conclude that \( x_{2N+1} \) is a solution to (3).

**Case 3.** Suppose that \( x_n \neq x_{n+1} \), for every \( n \in \mathbb{N} \). Now, to examine the inequality below

\[
\min\{\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|\} \leq \psi(\|x_{n-1} - x_n\|)
\]
we assume that \( \min(\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|) = \|x_{n-1} - x_n\| \). It yields that
\[
\|x_{n-1} - x_n\| \leq \psi(\|x_{n-1} - x_n\|),
\]
which is a contradiction (see Lemma 2.7).

Consequently, we observe that \( \min(\|x_n - x_{n+1}\|, \|x_{n-1} - x_n\|) = \|x_n - x_{n+1}\| \), and hence
\[
\|x_n - x_{n+1}\| \leq \psi(\|x_{n-1} - x_n\|).
\]

for every \( n \in \mathbb{N} \).

Recursively, we derive that
\[
\|x_n - x_{n+1}\| \leq \psi^n(\|x_0 - x_1\|), \quad \text{for all } n \in \mathbb{N}.
\] (5)

From (5) and using the triangular inequality, for all \( p \geq 1 \), we have:
\[
\|x_n - x_{n+p}\| \leq s\|x_n - x_{n+1}\| + s^2\|x_{n+1} - x_{n+2}\| + ... + s^{p-2}\|x_{n+p-3} - x_{n+p-2}\|
\]
\[
+ s^{p-1}\|x_{n+p-2} - x_{n+p-1}\| + s^{p-1}\|x_{n+p-1} - x_{n+p}\|
\]
\[
\quad \leq s\psi^n(\|x_0 - x_1\|) + s^2\psi^{n+1}(\|x_0 - x_1\|) + ... + s^{p-2}\psi^{n+p-3}(\|x_0 - x_1\|)
\]
\[
+ s^{p-1}\psi^{n+p-2}(\|x_0 - x_1\|) + s^{p-1}\psi^{n+p-1}(\|x_0 - x_1\|)
\]
\[
= \frac{1}{s^{p-1}}[s^n\psi^n(\|x_0 - x_1\|) + s^{n+1}\psi^{n+1}(\|x_0 - x_1\|) + ... + s^{n+p-2}\psi^{n+p-2}(\|x_0 - x_1\|)
\]
\[
\quad + s^{n+p-1}\psi^{n+p-1}(\|x_0 - x_1\|)]
\]

Denoting \( S_n = \sum_{k=0}^{n} s^k\psi^k(\|x_0 - x_1\|) \), \( n \geq 1 \) we obtain:
\[
\|x_n, x_{n+p}\| \leq \frac{1}{s^{p-1}}[S_{n+p-1} - S_{n-1}], \quad n \geq 1, \quad p \geq 1.
\] (6)

Using Lemma 2.9, we conclude that the series \( \sum_{k=0}^{n} s^k\psi^k(\|x_0, x_1\|) \) is convergent. Thus there exists \( S = \lim_{n \to \infty} S_n \in [0, \infty) \). Regarding \( s \geq 1 \) and by (6), we obtain that \( \{x_n\}_{n \geq 0} \) is a Cauchy sequence in the \( b \)-Banach space \((E, \|\cdot\|)\). Therefore, there exists some \( x^* \in E \) such that
\[
\lim_{n \to \infty} \|x_n - x^*\| = 0
\] (7)

Since \( T \) is orbitally continuous on \( \mathcal{L}(\varphi_{\leq s}) \), then \( T \) is continuous on \( O(x_0) \). Therefore, by (7), we deduce that
\[
\lim_{n \to \infty} \|x_{n+1} - Tx^*\| = 0
\] (8)

By the uniqueness of the limit, we obtain
\[
Tx^* = x^*
\] (9)

On the other hand, from (4), we have
\[
x_{2n+1} \in \mathcal{L}(\varphi_{\leq s}), \quad n = 0, 1, ...
\]
Since \( \varphi \) is 0\(_E\)-level closed from the left, we deduce from (7) that \( x^* \in \mathcal{L}(\varphi_{\leq p}) \), that is,
\[
\varphi(x^*) \geq_{p} 0_E.
\]
However, by (v) and (9), we derive that
\[
\varphi(x^*) = \varphi(Tx^*) \leq_{p} 0_E.
\]
As a result, we have \( \varphi(x^*) \in P \) and \( -\varphi(x^*) \in P \). By recalling that \( P \) is a cone, we get
\[
\varphi(x^*) = 0_E.
\]
Finally, (9) and (10) imply that \( x^* \in E \) is a solution to (11). \( \square \)

The next result holds if we replace the condition (iii), “\( \varphi \) is 0\(_E\) level closed from left” in Theorem 3.1 with the condition (iii)', that is, “\( \varphi \) is 0\(_E\) level closed from right”.

**Theorem 3.2.** Let \( (E, \| \cdot \|) \) be a b-Banach space and let \( T, \varphi : E \to E \) be two given mappings. Suppose the following conditions are satisfied:

(i) \( T \) is orbitally continuous on \( \mathcal{L}(\varphi_{\leq p}) \),

(ii) \( T \) is a \( \text{Cirić-type contraction} \) w.r.t \( \varphi \), i.e.
\[
\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),
\]
where \( \psi \in \Psi_{b} \), for every \((x, y) \in E \times E \) such that \( \varphi(x) \leq_{p} 0_E \) and \( \varphi(y) \geq_{p} 0_E \).

(iii)' \( \varphi \) is 0\(_E\) level closed from right

(iv) There exists \( x_0 \in E \) such that \( \varphi(x_0) \leq_{p} 0_E \)

(v) For every \( x \in E \) we have:
\[
\varphi(x) \leq_{p} 0_E \Rightarrow \varphi(Tx) \geq_{p} 0_E
\]
and
\[
\varphi(x) \geq_{p} 0_E \Rightarrow \varphi(Tx) \leq_{p} 0_E
\]

Then \( \{T^n(x_0)\} \) converges to a solution of (3).

**Proof.** To obtain the desired result just replace \( \varphi \) by \( -\varphi \) in Theorem 3.1. \( \square \)

**Remark 3.3.** We can observe that Theorems 3.1 and 3.2 are still valid if we replace condition (i) by the following condition: (i') \( T \) is orbitally continuous on \( \mathcal{L}(\varphi_{\geq p}) \).

In the second part of this paragraph we study the solvability of system (2) in the setting of b-Banach spaces. More precisely, we investigate a system
\[
\begin{align*}
&Tx = x \\
&\varphi_i(x) = 0_E, \ i = 1, 2, \ldots, r,
\end{align*}
\]
where \( T, \varphi_i : E \to E, \ i = 1, 2, \ldots, r \) be a finite number of mappings defined on a b-Banach space \( (E, \| \cdot \|) \) with a cone \( P \), and 0\(_E\) is the zero vector of \( E \), and \( T \) is mapping satisfying a \( \text{Cirić-type-contraction} \). For \( r = 1 \), this system coincide with (3).

The following is the first result of this part.

**Theorem 3.4.** Let \( (E, \| \cdot \|) \) be a b-Banach space and let \( T, \varphi_i : E \to E (i = 1, 2, \ldots, r) \) be a finite number of mappings. Suppose the following conditions are satisfied:
Proof. Due to (iv) there exists $\varphi_i(x_0) \leq p_{E}, i = 1, 2, ..., r$

Then $\{T^n(x_0)\}$ converges to a solution of (11).

Proof. Due to (iv) there exists $x_0 \in E$ such that $\varphi_i(x_0) \leq p_{E}, i = 1, 2, ..., r$. Applying (v), we get that

$$\varphi_i(x_0) \leq p_{E} \Rightarrow \varphi_i(Tx_0) \geq p_{E}, i = 1, 2, ..., r$$

Consider the iterative sequence $\{x_n\}$ in $E$ by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}_0.$$ 

By employing (v) again, we find

$$\varphi_i(x_1) \geq p_{E} \Rightarrow \varphi_i(Tx_1) \leq p_{E}, i = 1, 2, ..., r,$$

and hence

$$\varphi_i(x_2) \leq p_{E} \Rightarrow \varphi_i(Tx_2) \geq p_{E}, i = 1, 2, ..., r.$$ 

Recursively, we find

$$\varphi_i(x_{2n}) \leq p_{E}, \text{ and } \varphi_i(x_{2n+1}) \geq p_{E}, \quad n \in \mathbb{N}_0, i = 1, 2, ..., r.$$ 

(12)

Regarding the symmetry property, discussed in Remark 2.13 together with (ii) and (12), we get

$$\min\{||x_{n+1} - x_n||, |x_{n} - x_{n+1}|, |x_n - T_{n-1}x_{n+1}|\} \leq \psi(|x_{n+1} - x_n|), n \in \mathbb{N},$$

that is,

$$\min\{||x_{n+1} - x_n||, |x_{n+1} - x_n|, |x_{n} - x_{n+1}|\} \leq \psi(|x_{n+1} - x_n|), \quad n \in \mathbb{N}.$$ 

Consequently, we find that

$$\min\{||x_{n+1} - x_n||, |x_{n+1} - x_n|, |x_{n} - x_{n+1}|\} \leq \psi(|x_{n+1} - x_n|), n \in \mathbb{N},$$

which is equivalent to

$$\min\{||x_{n+1} - x_n||, |x_{n+1} - x_n|\} \leq \psi(|x_{n+1} - x_n|), n \in \mathbb{N}.$$
For analyzing the inequality above, we shall consider three cases.

**Case 1.** If $x_{2N} = x_{2N+1}$, for some $N \in \mathbb{N}_0$, then $x_{2N} = T x_{2N}$. Hence, from (12), we find

$$
\phi_i(x_{2N}) \leq p \epsilon, \quad \phi_i(x_{2N}) = \phi_i(T x_{2N}) = \phi_i(x_{2N+1}) \geq 0 \epsilon, \quad i = 1, 2, ..., r.
$$

Therefore, $\phi_i(x_{2N}) \in P$ and $-\phi_i(x_{2N}) \in P$, for every $i = 1, 2, ..., r$. Regarding that $P$ is a cone, we conclude that $\phi_i(x_{2N}) = 0 \epsilon, \quad i = 1, 2, ..., r$. Hence, $x_{2N} \in E$ is a solution to (11).

**Case 2.** If $x_{2N+1} = x_{2N+2}$, for some $N \in \mathbb{N}_0$, by verbatim, we deduce that $x_{2N+1}$ is a solution to (11).

**Case 3.** Suppose that $x_n \neq x_{n+1}$, for every $n = 1, 2, ...$ with

$$
\min\{||x_n - x_{n+1}||, ||x_{n-1} - x_n||\} \leq \psi(||x_{n-1} - x_n||).
$$

On account of function $\psi$ we have

$$
||x_n - x_{n+1}|| \leq \psi(||x_{n-1} - x_n||).
$$

Iteratively, we find that

$$
||x_n - x_{n+p}|| \leq s||x_n - x_{n+1}|| + s^2||x_{n+1} - x_{n+2}|| + ... + s^{p-1}||x_{n+p-3} - x_{n+p-2}|| + s^p||x_{n+p-2} - x_{n+p-1}|| + s^{p-1}||x_{n+p-1} - x_{n+p}||
$$

$$
\leq s\psi^p(||x_0 - x_1||) + s^2\psi^{p-1}(||x_0 - x_1||) + ... + s^{p-2}\psi^{p-3}(||x_0 - x_1||) + s^{p-1}\psi^{p-2}(||x_0 - x_1||)
$$

$$
+ s^p\psi^{p-1}(||x_0 - x_1||) + s^{p-1}\psi^{p-2}(||x_0 - x_1||) + ... + s^{p+2}\psi^{p+3}(||x_0 - x_1||) + s^{p+1}\psi^{p+2}(||x_0 - x_1||)
$$

$$
= \frac{1}{s^p} [s^p\psi^p(||x_0 - x_1||) + s^{p+1}\psi^{p+1}(||x_0 - x_1||) + ... + s^{p+2}\psi^{p+2}(||x_0 - x_1||)]
$$

By letting $S_n = \sum_{k=0}^{n} s^k \psi^k(||x_0 - x_1||)$, $n \geq 1$ we obtain:

$$
||x_n, x_{n+p}|| \leq \frac{1}{s^p} ||S_{n+p-1} - S_{n-1}||, \quad n \geq 1, \ p \geq 1.
$$

Using Lemma 2.9, we conclude that the series $\sum_{k=0}^{n} s^k \psi^k(||x_0, x_1||)$ is convergent. Thus there exists $S = \lim_{n \to \infty} S_n \in [0, \infty)$. Since $s \geq 1$ and taking (6) into account, we obtain that $\{x_n\}_{n \geq 0}$ is a Cauchy sequence in the $b$-Banach space $(E, \cdot || \cdot ||)$. Thus, there exists some $x' \in E$ such that

$$
\lim_{n \to \infty} ||x_n - x'|| = 0
$$

(15)

Since $T$ is orbitally continuous on $\cap_{i=1}^{r} L(\varphi_i, \mathbb{S}_p)$, then $T$ is continuous on $O(x_0)$. Therefore, by (15), we deduce that

$$
\lim_{n \to \infty} ||x_{n+1} - Tx'|| = 0
$$

(16)

By the uniqueness of the limit, we obtain

$$
Tx' = x'
$$

(17)
On the other hand, from (12), we have
\[ x_{2n+1} \subset \cap_{i=1}^{r} L(\phi_i, p), \quad n = 0, 1, \ldots \]
Since \( \phi \) is \( 0_E \)-level closed from the left, we deduce from (15) that \( x^{*} \in \cap_{i=1}^{r} L(\phi_i, p) \), that is, \( \phi_i(x^{*}) \geq p \) \( 0_E \) \( i = 1, 2, \ldots, r \).

But by (v) and (17), we obtain
\[ \phi_i(x^{*}) = \phi_i(Tx^{*}) \leq p \] \( 0_E \), \( i = 1, 2, \ldots, r \).
Hence, we have \( \phi_i(x^{*}) \in P \) and \( -\phi_i(x^{*}) \in P \), \( i = 1, 2, \ldots, r \).

So we have that \( x^{*} \in E \) is a solution to (11).

The next result holds if we replace the condition (iii) in the above theorem.

**Theorem 3.5.** Let \((E, \|\cdot\|) \) be a b-Banach space and let \( T, \phi_i : E \rightarrow E \) \( i = 1, 2, \ldots, r \) be a finite number of mappings. Suppose the following following conditions are satisfied:

(i) \( T \) is orbitally continuous on \( \cap_{i=1}^{r} L(\phi_i, p) \),

(ii) \( T \) is a Ćirić-type contraction w.r.t \( \{\phi_i\}_{i=1}^{r} \), i.e.

\[
\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} = \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(\|x - y\|),
\]
where \( \psi \in \Psi_b \), for every \((x, y) \in E \times E \) such that \( \phi_i(x) \leq p \) \( 0_E \) and \( \phi_i(y) \geq p \) \( 0_E \), \( i = 1, 2, \ldots, r \).

(iii) \( \phi_i, \ i = 1, 2, \ldots, r \) is \( 0_E \) level closed from right,

(iv) There exists \( x_0 \in E \) such that \( \phi_i(x_0) \leq p \) \( 0_E \), \( i = 1, 2, \ldots, r \),

(v) For every \( x \in E \) we have:

\[ \phi_i(x) \leq p \] \( 0_E \), \( i = 1, 2, \ldots, r \) \( \Rightarrow \) \( \phi_i(Tx) \geq p \) \( 0_E \), \( i = 1, 2, \ldots, r \)

and

\[ \phi_i(x) \geq p \] \( 0_E \), \( i = 1, 2, \ldots, r \) \( \Rightarrow \) \( \phi_i(Tx) \leq p \) \( 0_E \), \( i = 1, 2, \ldots, r \)

Then \( \{T^n(x_0)\} \) converges to a solution of (11).

**Remark 3.6.** We can observe that Theorems 3.4 and 3.5 are still valid if we replace condition (i) by the following condition:

(i') \( T \) is orbitally continuous on \( \cap_{i=1}^{r} L(\phi_i, p) \)

4. Consequences

We present first a fixed point result that can be deduced from Theorem 3.1.

**Corollary 4.1.** Let \((E, \|\cdot\|) \) be a b-Banach space and let \( T : E \rightarrow E \) be a given mappings. Suppose the following conditions are satisfied:

(i) \( T \) is orbitally continuous on \( E \);
(ii) \( T \) is a \( Ciri\-type \) contraction, i.e. there exists a function \( \psi \in \Psi_b \), such that for every \((x, y) \in E \times E\),
\[
\min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(||x - y||)
\]
Then \( \{T^n(x)\} \) converges to a fixed point of \( T \), for every \( x \in E \).

An interesting consequence can be obtained applying the main results from the previous paragraph. We consider the following common fixed point problem, where \( T, F : E \to E \) are two given mappings.

\[
\begin{cases}
    x = Tx \\
    x = Fx
\end{cases}
\]
This system is equivalent to
\[
\begin{cases}
    x = Tx \\
    \varphi(x) = 0_E,
\end{cases}
\]
where \( \varphi : E \to E \) is defined by \( \varphi(x) = Fx - x, \ x \in E \)

We define the sets:
\[
H_1 := \{x \in E : Fx \leq x, \ x \in E\}
\]
and
\[
H_2 := \{x \in E : Fx \leq x\}.
\]
and we can state the following result.

**Corollary 4.2.** Let \((E, \| \cdot \|)\) be a \( b\)-Banach space and let \( T, F : E \to E \) be two given mappings. Suppose the following conditions are satisfied:

(i) \( T \) is orbitally continuous on \( H_1 \);

(ii) There exists a function \( \psi : [0, \infty) \to [0, \infty), \ \psi \in \Psi_b \), such that for every \((x, y) \in E \times E\),
\[
Fx \leq_0 0_E, \ Fy \geq_0 0_E \Rightarrow \min\{\|Tx - Ty\|, \|x - Tx\|, \|y - Ty\|\} - \min\{\|x - Ty\|, \|y - Tx\|\} \leq \psi(||x - y||),
\]

(iii) \( H_2 \) is a closed subset of \( E \)

(iv) There exists \( x_0 \in E \) such that \( Fx_0 \leq x_0 \)

(v) For every \( x \in E \) we have:
\[
Sx \leq_0 x \Rightarrow F(Tx) \geq_0 Tx
\]
and
\[
Sx \geq_0 x \Rightarrow F(Tx) \leq_0 Tx
\]
Then the Picard sequence \( \{T^n(x_0)\} \) converges to a solution of (19).

**Proof.** Applying Theorem 3.1 with the function \( \varphi \) defined by \( \varphi(x) = Fx - x, \ x \in E \) the conclusion follows. \( \square \)

5. Conclusions

Notice that we extend the results in [25] in two-folds. First, we expanded the abstract spaces from Banach to \( b\)-Banach. Secondly, by using \( b\)-comparison function, we consider more general form of \( Ciri\) operators. Thus, even, if we let \( s = 1 \), we get more general results when we compare the corresponding results in [25].
Competing interests

The authors declare that they have no competing interests.

Authors contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

References

[1] H. Aydi, M-F. Bota, E. Karapınar and S. Moradi, A common fixed point for weak φ-contractions on b-metric spaces, Fixed Point Theory 13(2012) No 2 337–346.
[2] B. Alqahtani, A. Fulga; E. Karapinar, Non-Unique Fixed Point Results in Extended b-Metric Space, Mathematics 2018 6(5) 68.
[3] H. H. Alsulami, E. Karapınar, V. Rakocevic, Ćirić Type Nonunique Fixed Point Theorems on b-Metric Spaces, Filomat 31:11 (2017) 3147–3156.
[4] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30(1989) 26–37.
[5] V. Berinde, Generalized contractions in quasimetric spaces, Seminar on Fixed Point Theory Preprint no. 3(1993) 3–9.
[6] V. Berinde, Sequences of operators and fixed points in quasimetric spaces, Stud. Univ. “Babeş-Bolyai” Math. 16(4)(1996) 23–27.
[7] V. Berinde, Contractii generalizate și aplicații, Editura Club Press 22, Baia Mare, 1997.
[8] M. F. Bota-Borcea, A. Petrușel, Ulam-Hyers stability for operatorial equations, An. Univ. AL. I. Cuza 57(2011) 65–74.
[9] M. Bota, A. Molnar, Cs. Varga, On Ekeland’s variational principle in b-metric spaces, Fixed Point Theory 12(2011) 21–28.
[10] N. Bourbaki, Topologie Générale, Hermann, Paris, 1974.
[11] S. Czerwik, Contraction mappings in b-metric spaces, Acta Mathematica et Informatica Universitatis Ostraviensis 1(1993), 5–11.
[12] S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Mat. Univ. Modena 46(1998) 263–276.
[13] J. Heinonen, Lectures on Analysis on Metric Spaces, Springer Berlin, 2001.
[14] M.A. Khamsi, N. Hussain, KKM mappings in metric type spaces, Nonlinear Anal. 73(2010) 3123–3129.
[15] N. Hussain, Z. Kadelburg, S. Radenović, and F.Al-Solamy, Comparison Functions and Fixed Point Results in Partial Metric Spaces, Abstract and Applied Analysis, vol. 2012, Article ID 605781, 15 2012.
[16] J. Achari, On Ćirić’s nonunique fixed points, Mathematicki vesnik 13 (1976) 255–257.
[17] J. Achari, Results on non-unique fixed points, Publications de L’lnstitut Mathematique 26 (1978) 5–9.
[18] Lj. Ćirić’, On some maps with a nonunique fixed point, Publications de L’institut Mathematique 17 (1974) 52–58.
[19] Lj. Ćirić’, N. Jotić, A further extension of maps with non-unique fixed points, Mathematicki vesnik 50 (1998) 1–4.
[20] E. Karapinar, Ćirić type nonunique fixed point theorems on partial metric spaces, Journal of Nonlinear Science and Applications 5 (2012) 74–83.
[21] E. Karapinar, S. Romaguera, Nonunique fixed point theorems in partial metric spaces, Filomat 27 (2013) 1305–1314.
[22] E. Karapinar, Some Nonunique Fixed Point Theorems of Ćirić type on Cone Metric Spaces” Abstr. Appl. Anal., vol. 2010, Article ID 125904, 14 pages (2010).
[23] E. Karapinar, Ćirić Type Nonunique Fixed Points Results: A Review, Applied and Computational Mathematics an International Journal 1 (2019) 3–21.
[24] B.G. Pachpatte, On Ćirić type maps with a nonunique fixed point, Indian Journal of Pure and Applied Mathematics 10 (1979) 1039–1043.
[25] V. Rakocevic, B. Samet, A Fixed Point Problem Under a Finite Number of Equality Constraints Involving a Ćirić Operator, Filomat, 31(11)(2017), 3192–3202.
[26] I. A. Rus, The theory of a metrical fixed point theorem: theoretical and applicative relevances, Fixed Point Theory 9(2008), No. 2 541–559.
[27] I. A. Rus, Generalized contractions and applications, Cluj University Press, Cluj-Napoca, 2001.