Some consequences of the definitions of trapezoid

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Abstract. There are two definitions of trapezoid. One says that a trapezoid is a quadrilateral with exactly one pair of opposite parallel sides, and the other says that a trapezoid is a quadrilateral with at least one pair of opposite parallel sides. The difference is that the first uses the word “exactly” and the second uses the word “at least”. These two definitions have some different consequences. We discuss some consequences of the definitions in some problems and solutions.

1. Introduction

One day when I taught a course for a first semester undergraduate students at our mathematics department, I asked the students a question “What is the definition of trapezoid?” Most of them said that a trapezoid is a quadrilateral with exactly one pair of opposite parallel sides. This definition would be different from the definition of trapezoid (using “at least one” instead of “exactly one”) that would be used in some courses in the department.

Consider the following two quadrilaterals $ABCD$ and $EFGH$ as in Figure 1.1. It is obvious that the quadrilateral $ABCD$ is a trapezoid, but the quadrilateral $EFGH$ is not a trapezoid when we use “exactly one” in the definition and it is trapezoid when we use “at least one” in the definition.

![Figure 1.1. Quadrilaterals](image)

The existences of two different definitions of trapezoid is like what be mentioned in [1, p.xiii]. It said that there are two definitions of trapezoid can be found in mathematics textbooks. The definitions are:

1. A trapezoid is a quadrilateral with exactly one pair of opposite parallel sides.
2. A trapezoid is a quadrilateral with at least one pair of opposite parallel sides.

They say that the first definition is more common in high school books, and the second is less common in high school books but is very common in college texts. Further, they call the first and the second definition as an exclusive definition and inclusive definition, respectively.

Josefsson [2] mentions the reason for the two possible definitions, and mentions their significance. He says that when students first encounter shape like a trapezoid or rhombus, they can get confused if a
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rhombus also can be called rhombus. Hence the exclusive definition is introduced. But this definition has some disadvantage when the students study further mathematics. For example, there is a trapezoid rule for calculating integral, but the trapezoids do not always exactly one pair of opposite parallel sides. Further, if a property holds for a trapezoid, then the property also holds for all quadrilaterals with two pairs parallel sides. Hence the inclusive definition is applied.

The two different definitions of trapezoid will also have some consequences. One of them is when we have a problem concerning trapezoid, we could have different answers of the problem, depend on the definition that we use. In this paper we discuss some consequences of these two definitions in some problems and solutions.

2. Discussion

The existence of two definitions of trapezoid will have some consequences. If we want to have a quadrilateral concept map, we will have different concept maps. We cannot ask a question concerning definition of trapezoid without mention the definition of trapezoid. For example, in the college entrance test, we cannot ask whether a parallelogram is a trapezoid or not without mention the definition of trapezoid. In numerical analysis (see e.g. [3]), we have trapezoidal rule to approximate a definite integral. The integral is approximated by the sum of trapezoidal areas. In this case we use inclusive definition of trapezoid since sometimes some of the trapezoid has two pairs of opposite parallel sides.

Now let we use the inclusive definition. Consider the following Problem 2.1 and its solution.

**Problem 2.1.** Given an isosceles trapezoid with its area is 64cm$^2$. Find the minimum perimeter of the trapezoid.

**Solution:** Let $ABCD$ be a trapezoid having area 64cm$^2$, as in **Figure 2.1**, $E$ and $F$ be the midpoints of $AD$ and $BC$, respectively.

![Figure 2.1. Trapezoid ABCD](image)

Let $h$ cm be the length of the height of the trapezoid. Since the area of trapezoid $ABCD$ is $\frac{AB+CD}{2}h = 64$ cm$^2$, then we have

$$AB + CD = \frac{128}{h}.$$  

Since $\Delta AEG \cong \Delta DH$, then the perimeter will be minimum only if the sides $AD$ and $BC$ are perpendicular to the side $AB$, when $AD = BC = h$. Let the perimeter be $P(h)$ cm. We have

$$P(h) = AB + BC + CD + DA$$

$$= \frac{128}{h} + 2h$$

$$= \frac{128+2h^2}{h}.$$  

The perimeter will be minimum if

$$\frac{dP(h)}{dh} = 0,$$

$$\frac{2h^2-128}{h^2} = 0,$$
The minimum perimeter is $P(8) = 32$ cm.

The answer will be different if we use the exclusive definition. If we use the exclusive definition, in the solution of Problem 1, then we cannot have both sides $AD$ and $BC$ are perpendicular to the side $AB$, we have no minimum perimeter of the trapezoid.

Let we use the exclusive definition and slightly modify the problem by giving a restriction that the length of the sides are integers. We show that we can have the minimum of the perimeter.

**Problem 2.2.** Given an isosceles trapezoid with its area is $64\text{cm}^2$ and the length of the sides are integers. Find the minimum perimeter of the trapezoid.

**Solution:** Let $ABCD$ be a trapezoid having area $64\text{cm}^2$, as in Figure 2.2, $E$ and $F$ be the midpoint of sides $AD$ and $BC$, respectively, and $AG = HD = x$.

Let $h$ cm be the length of the height of the trapezoid. Then $AD = \sqrt{h^2 + 4x^2}$, with $x = \frac{AB-CD}{4} > 0$ (if $x = 0$, then $ABCD$ is not a trapezoid). Since the area of trapezoid $ABCD$ is $\frac{AB + CD}{2} h \text{cm}^2$, then we have

$$AB + CD = \frac{128}{h}.$$

Let the perimeter be $P(h, x)$ cm. We have

$$P(h, x) = AB + BC + CD + AD = \frac{128}{h} + 2\sqrt{h^2 + 4x^2}$$

If we take $h = 8$ and $x = 3$, then we have $AD = 10$ and $P(h, x) = 36$. Let the minimum $P(h, x)$ be $p$. Then $p \leq 36$.

We will show that $p = 36$. Since the length of the sides are integers and $AB + CD = \frac{128}{h}$, then $h = 1, 2, 4, 8, 16, 32, \text{or } 128$. Further, since $x = \frac{AB-CD}{4}$, then $x$ is a multiplication of $\frac{1}{4}$. It is easy to see that if $h = 1, 2, 4, 16, 32, \text{or } 128$, then $P(h, x) > 36$. Thus $h = 8$, and so $AD = \sqrt{h^2 + 4x^2} = \sqrt{64 + 4x^2} > 8$. If $AD = 9$, then $4x^2 = 17$, and $x$ is not a multiplication of $\frac{1}{4}$. Thus $AD = 10$, and the minimum of $P(h, x)$ is $p = 36$.

Let we still use the exclusive definition and modify Problem 2.2 by giving the area of the trapezoid be $4\text{cm}^2$ instead of $64\text{cm}^2$. The problem is as follows.

**Problem 2.3.** Find the minimum perimeter of an isosceles trapezoid with its area is $4\text{cm}^2$ and the lengths of the sides are integers.

It is easy to show that in Problem 2.3 we cannot find the minimum perimeter since there is no isosceles trapezoid with its area is $4\text{cm}^2$ and the length of the sides are integers. If we use the inclusive definition, then the answer of Problem 2.3 is $8\text{cm}$. We can also have the similar problems to Problems 2.1, 2.2, and 2.3 by asking the maximum area when the perimeter of a trapezoid is given.
3. Conclusion

Base on the introduction and discussion, we can have the following summary:

1. Since we have more than one definitions of trapezoid, when we state a problem concerning trapezoid, we have to give information which definition we use. The “same” problem can have different solutions when we use different definitions.

2. By using the exclusive definition of trapezoid, an isosceles trapezoid with given integer area and integer lengths of sides may or may not exist. Further, by using this definition, the minimum perimeter (or maximum area, respectively) of an isosceles trapezoid with given area (or perimeter, respectively) does not exist; it may exist when its value is integer. Thus, we can make an interesting problem by using exclusive definition of trapezoid.

3. In high school, it is very common to use exclusive definition of trapezoid, and in college, it is very common to use inclusive definition of trapezoid. In college, when we talk about trapezoidal rule we use the inclusive definition.

We conclude this paper by stating the following exercise (we use the exclusive definition) and suggestions.

Problem 3.1. Find the necessary and sufficient conditions for a positive integer $a$ such that there exists an isosceles trapezoid with integer lengths of sides and the area is $a$. If $a$ exists, find the minimum perimeter related to the area $a$.

We can also have the similar problems to Problem 3.1 by asking the existence of an isosceles trapezoid with integer lengths of sides and its perimeter is given.

Since in college we use inclusive definition of trapezoid, while in high school it is very common to use exclusive definition of trapezoid, it is very important to mention to the college students that the second definition is used in the courses, rather than the first.

References

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