CONGRUENCE AMALGAMATION OF LATTICES

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Abstract. J. Tůma proved an interesting “congruence amalgamation” result. We are generalizing and providing an alternate proof for it. We then provide applications of this result:
(i) A.P. Huhn proved that every distributive algebraic lattice $D$ with at most $\aleph_1$ compact elements can be represented as the congruence lattice of a lattice $L$. We show that $L$ can be constructed as a locally finite relatively complemented lattice with zero.
(ii) We find a large class of lattices, the $\omega$-congruence-finite lattices, that contains all locally finite countable lattices, in which every lattice has a relatively complemented congruence-preserving extension.

1. Introduction

The first congruence lattice characterization theorem is due to R.P. Dilworth (see G. Grätzer and E.T. Schmidt [7]):

Dilworth’ Theorem. Let $D$ be a finite distributive lattice. Then there exists a finite lattice $L$ such that the congruence lattice of $L$, $\text{Con} L$, is isomorphic to $D$.

The best extension of this result is due to A.P. Huhn [10]:

Huhn’s Theorem. Let $D$ be a distributive algebraic lattice. If $D$ has at most $\aleph_1$ compact elements, then there exists a lattice $L$ such that $\text{Con} L \cong D$.

An equivalent form of this result is the following: Let $S$ be a distributive join-semilattice with zero. If $|S| \leq \aleph_1$, then there exists a lattice $L$ such that the join-semilattice of compact congruences of $L$ is isomorphic to $S$.

By P. Pudlák [14], $S$ is a direct limit of its finite distributive $\{\lor, 0\}$-subsemilattices. So it is natural to attempt to prove Huhn’s result with a direct limit argument.

Assigning to a lattice $L$ its congruence lattice, $\text{Con} L$, determines a functor $\text{Con}$ from the category of lattices with lattice homomorphisms to the category of algebraic distributive lattices with morphisms the complete $\lor$-homomorphisms. Specifically, if $K$ and $L$ are lattices and $\varphi : K \to L$ is a lattice homomorphism, then the mapping $\text{Con} \varphi : \text{Con} K \to \text{Con} L$ is determined by setting

$$(\text{Con} \varphi) \Theta = \Theta_L (\langle \varphi x, \varphi y \rangle \mid x, y \in K, x \equiv y (\Theta)),$$

for each $\Theta \in \text{Con} K$.

J. Tůma [17] proved the following result:

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Túma’s Theorem. Let $L_0, L_1, L_2$ be finite atomistic lattices and let $\eta_1 : L_0 \to L_1$ and $\eta_2 : L_0 \to L_2$ be lattice embeddings preserving the zero such that $\text{Con} \eta_1$ and $\text{Con} \eta_2$ are injective. Let $D$ be a finite distributive lattice, and, for $i \in \{1, 2\}$, let $\psi_i : \text{Con} L_i \to D$ be $\{\lor, 0\}$-embeddings such that $\psi_1 \circ \text{Con} \eta_1 = \psi_2 \circ \text{Con} \eta_2$.

Then there is a finite atomistic lattice $L$, and there are lattice embeddings $\varphi_i : L_i \to L$, for $i \in \{1, 2\}$, that preserve the zero, satisfying $\varphi_1 \circ \eta_1 = \varphi_2 \circ \eta_2$,

and there is an isomorphism $\alpha : \text{Con} L \to D$ such that $\alpha \circ \text{Con} \varphi_i = \psi_i$, for $i \in \{1, 2\}$.

We extend Túma’s result by proving:

Theorem 1. Let $L_0, L_1, L_2$ be lattices and let $\eta_1 : L_0 \to L_1$ and $\eta_2 : L_0 \to L_2$ be lattice homomorphisms. Let $D$ be a finite distributive lattice, and, for $i \in \{1, 2\}$, let $\psi_i : \text{Con} L_i \to D$ be complete $\lor$-homomorphisms such that $\psi_1 \circ \text{Con} \eta_1 = \psi_2 \circ \text{Con} \eta_2$.

There is then a lattice $L$, there are lattice homomorphisms $\varphi_i : L_i \to L$, for $i \in \{1, 2\}$, with $\varphi_1 \circ \eta_1 = \varphi_2 \circ \eta_2$,

and there is an isomorphism $\alpha : \text{Con} L \to D$ such that $\alpha \circ \text{Con} \varphi_i = \psi_i$, for $i \in \{1, 2\}$.

If $L_0, L_1, L_2$ have zero and both $\eta_1, \eta_2$ preserve the zero, then $L$ can be chosen to have a zero and $\varphi_1, \varphi_2$ can be chosen to preserve the zero.

If $L_1$ and $L_2$ are finite, then $L$ can be chosen to be finite and atomistic.

This theorem is an extension of Túma’s theorem—we need only observe that if the $\psi_i$ are injective, then the $\varphi_i$ must be lattice embeddings. This fact follows from the elementary fact that a lattice homomorphism $\varphi : K \to L$ is an embedding iff $\text{Con} \varphi$ separates zero, that is, iff $(\text{Con} \varphi) \Theta = \omega_L$ implies that $\Theta = \omega_K$, for all $\Theta \in \text{Con} K$.

We shall apply Theorem 1 to prove the following strong form of Huhn’s Theorem:

Theorem 2. Let $D$ be a distributive algebraic lattice. If $D$ has at most $\aleph_1$ compact elements, then there exists a locally finite, relatively complemented lattice $L$ with zero such that $\text{Con} L \cong D$.

A lattice $L$ is congruence-finite, if $\text{Con} L$ is finite; it is $\omega$-congruence-finite, if $L$ can be written as a union,

$L = \bigcup \{ L_n \mid n < \omega \}$,

where $\{ L_n \mid n < \omega \}$ is an increasing sequence of congruence-finite sublattices of $L$.

We also apply Theorem 1 to prove the following:

Theorem 3. Every $\omega$-congruence-finite lattice $K$ has a $\omega$-congruence-finite, relatively complemented congruence-preserving extension $L$. Furthermore, if $K$ has a zero, then $L$ can be taken to have the same zero.
2. Preliminaries

2.1. Notation. Let $M_3$ be the five-element nondistributive modular lattice and let $2$ denote the two-element chain.

For any lattice $K$, we denote the set of join-irreducible elements of $K$ by $J(K)$.

2.2. Sectionally complemented lattices. We start with the following stronger form of Dilworth’ Theorem (G. Grätzer and E.T. Schmidt [8]).

**Theorem 4.** Let $D$ be a finite distributive lattice. Then there exists a finite sectionally complemented lattice $L$ such that $\text{Con} L$ is isomorphic to $D$ under an isomorphism $\alpha$. Moreover, $L$ contains a Boolean ideal generated by the atoms

$$\{d_p \mid p \in J(D)\},$$

and under $\alpha$, the congruence $\Theta_L(d_p,0)$ corresponds to $p$, for each $p \in J(D)$.

Let $K$ and $L$ be lattices, let $f : K \to L$ be a lattice homomorphism. We say that $f$ is relatively complemented, if for all $a, b, c \in K$ such that $a \leq b \leq c$, there exists a relative complement of $f(b)$ in the interval $[f(a), f(c)]$ of $L$.

If $f$ is the inclusion map from a sublattice $K$ to the lattice $L$, we say that $K$ is relatively complemented in $L$.

We need the following embedding results:

**Lemma 2.1.**

(i) For every lattice $L$, there is a bounded, simple, sectionally complemented extension $S(L)$ of $L$ with a dual atom $p$ such that $L$ is relatively complemented in $S(L)$.

(ii) If $L$ is finite, then there is a finite, simple, sectionally complemented extension $S(L)$ of $L$ with a dual atom $p$ such that $L$ is relatively complemented in $S(L)$.

For general lattices, by P.M. Whitman [21], every lattice can be embedded in a partition lattice and by O. Ore [12], a partition lattice is simple and sectionally complemented; it also obviously has a dual atom. The second statement of Lemma 2.1 follows from the following very deep result of P. Pudlák and J. Tůma [15]: *Every finite lattice can be embedded into a finite partition lattice.*

In G. Grätzer and E.T. Schmidt [8], it is pointed out that a version of this lemma can be proved almost trivially. The comment, and the simpler proof in [8], also applies to the present version.

2.3. Congruence-preserving extension. Let $L$ be a finite lattice. A finite lattice $K$ is a congruence-preserving extension of $L$, if $K$ is an extension and every congruence of $L$ has exactly one extension to $K$. Of course, then the congruence lattice of $L$ is isomorphic to the congruence lattice of $K$.

A major research tool was discovered by M. Tischendorf [16]:

**Tischendorf’s Theorem.** Every finite lattice has a congruence-preserving extension to a finite atomistic lattice.

A much stronger result was proved in G. Grätzer and E.T. Schmidt [8]:

**Theorem 5.**

(i) Every finite lattice has a congruence-preserving extension to a finite sectionally complemented lattice.
(ii) Every congruence-finite lattice has a congruence-preserving extension to a sectionally complemented lattice.

In the first statement, we cannot strengthen “sectionally complemented” to “relatively complemented”, because the congruence lattice of a finite relatively complemented lattice is always Boolean.

2.4. \( k \)-ladders. Let \( k \) be a positive integer. A \( k \)-ladder is a lattice \( L \) such that, for any \( a \in L \),

(i) \( \downarrow a \) is finite;
(ii) \( a \) covers at most \( k \) elements.

Note that every \( k \)-ladder has breadth at most \( k \) (see, for example, G. Grätzer [6] for the definition of breadth).

Every finite chain is a 1-ladder. The chain \( \omega \) of all non-negative integers is also a 1-ladder. Note that \( k \)-ladders are called \( k \)-frames in H. Dobbertin [4].

By using the Kuratowski Free Set Theorem, see [11], one can easily prove that every \( k \)-ladder has at most \( \aleph_{k-1} \) elements, see S.Z. Ditor [3]. See also H. Dobbertin [4] for the case \( k = 2 \) (his proof does not use the Kuratowski Free Set Theorem).

The converse is obviously true for \( k = 1 \); also for \( k = 2 \), by the following result of S.Z. Ditor [3] and by H. Dobbertin [4]:

Proposition 2.2. There exists a 2-ladder of cardinality \( \aleph_1 \).

Proof. For \( \xi < \omega_1 \) (the first uncountable ordinal), we construct inductively the lattices \( L_\xi \) with no largest element, as follows. Put \( L_0 = \omega \). If \( \lambda \) is countable limit ordinal, put \( L_\lambda = \bigcup \{ L_\xi \mid \xi < \lambda \} \). So assume that we have constructed \( L_\xi \), a countable 2-ladder with no largest element. Then \( L_\xi \) has a strictly increasing, countable, cofinal, sequence \( \{ a_n \mid n < \omega \} \). Let \( \{ b_n \mid n < \omega \} \) be a strictly increasing countable chain, with \( b_n \notin L_\xi \), for all \( n \). Define \( L_{\xi+1} \) by

\[ L_{\xi+1} = L_\xi \cup \{ b_n \mid n < \omega \}, \]

equipped with the least partial ordering containing the ordering of \( L_\xi \), the natural ordering of \( \{ b_n \mid n < \omega \} \), and all pairs \( a_n < b_n \), for \( n < \omega \). It is easy to verify that \( L = \bigcup \{ L_\xi \mid \xi < \omega_1 \} \) is a 2-ladder of cardinality \( \aleph_1 \). \( \square \)

3. Proving Theorem \( \text{[H]} \)

We prove Theorem \( \text{[H]} \) in several steps.

3.1. Theorem \( \text{[H]} \) for \( D = 2 \). In this section, let \( D = 2 \).

We first state and prove the following special case:

Lemma 3.1. Let \( L_0', L_1', L_2' \) be lattices and let \( \eta_1' : L_0' \to L_1' \) and \( \eta_2' : L_0' \to L_2' \) be lattice embeddings. Let \( D \) be the two-element chain, and, for \( i \in \{1, 2\} \), let \( \psi_i' : \operatorname{Con} L_i \to D \) satisfy

\[ \psi_i' \Theta = 0_D \iff \Theta = \omega_{L_i'}. \]

There is then a lattice \( L \) with 1 and with a dual atom, there are lattice embeddings \( \varphi_i' : L_i' \to L \), for \( i \in \{1, 2\} \), with \( \varphi_i' \circ \eta_1' = \varphi_i' \circ \eta_2' \), and there is an isomorphism \( \alpha : \operatorname{Con} L \to D \) such that \( \alpha \circ \operatorname{Con} \varphi_i' = \psi_i' \) for \( i \in \{1, 2\} \).
If \(L'_0', L'_1', L'_2\) have zero and both \(\eta'_1, \eta'_2\) preserve the zero, then \(L\) can be chosen to have a zero and \(\varphi'_1, \varphi'_2\) can be chosen to preserve the zero.

If \(L'_1\) and \(L'_2\) are finite, then \(L\) can be chosen to be finite.

**Proof.** There is a lattice \(K\) amalgamating \(L'_1, L'_2\) over \(L'_0\). If \(L'_0, L'_1, L'_2\) have zero and \(\eta'_1, \eta'_2\) preserve the zero, then we can choose \(K\) so that \(L'_1\) and \(L'_2\) are zero-preserving sublattices of \(K\). Observe, also, that if \(L'_1\) and \(L'_2\) are finite, then \(K\) can be chosen finite.

As we pointed out in Lemma 2.1, we can embed \(K\) into a simple lattice \(L\) that has a 1 and a dual atom, where this embedding preserves the zero, if \(K\) has a zero, and where \(L\) is finite, if \(K\) is.

For each \(i \in \{1, 2\}\), let \(\varphi'_i: L'_i \to L\) be the composition of the embedding of \(L'_i\) into \(K\) with the embedding of \(K\) into \(L\). Then

\[\varphi'_i \circ \eta'_i = \varphi'_2 \circ \eta'_2.\]

Since \(L\) is simple, we have an isomorphism \(\alpha: \text{Con} L \to D\) such that

\[\alpha \Theta = 0_D \quad \text{iff} \quad \Theta = \omega_L.\]

For each \(i \in \{1, 2\}\) and each \(\Theta \in \text{Con} L'_i\),

\[(\text{Con} \varphi'_i) \Theta = \omega_L \quad \text{iff} \quad \Theta = \omega_{L'_i},\]

since \(\varphi'_i\) is an embedding.

Thus,

\[\alpha \circ \text{Con} \varphi'_i = \psi'_i,\]

concluding the proof of the lemma. \(\square\)

We proceed to prove Theorem 4 for \(D = 2\). For each \(i \in \{1, 2\}\), set

\[\Theta_i = \bigvee(\Theta \in \text{Con} L_i \mid \psi_i \Theta = 0_D),\]

and set

\[\Theta_0 = \{ \langle x, y \rangle \in L_0 \mid \eta_1 x \equiv \eta_1 y (\Theta_1) \} = \{ \langle x, y \rangle \in L_0 \mid \eta_2 x \equiv \eta_2 y (\Theta_2) \}.\]

For each \(i \in \{0, 1, 2\}\), set \(L'_i = L_i/\Theta_i\) and let \(\pi_i: L_i \to L'_i\) be the canonical surjection. Note that \(\Theta_i = \ker \pi_i\). We then have lattice embeddings \(\eta'_1: L'_0 \to L'_1\) and \(\eta'_2: L'_0 \to L'_2\) such that

\[\pi_i \circ \eta_i = \eta'_i \circ \pi_0, \quad \text{for } i \in \{1, 2\}.\]

Furthermore, we have mappings \(\psi'_i: \text{Con} L'_1 \to D\) and \(\psi'_2: \text{Con} L'_2 \to D\) with

\[\psi'_i \Theta = 0_D \quad \text{iff} \quad \Theta = \omega_{L'_i}, \quad \text{for } i \in \{1, 2\}\]

such that

\[\psi'_i \circ \text{Con} \pi_i = \psi_i, \quad \text{for } i \in \{1, 2\}.\]

We apply Lemma 3.1 to get the lattice \(L\), the embeddings \(\varphi'_i: L'_i \to L\), and the isomorphism \(\alpha: \text{Con} L \to D\) with

\[\alpha \circ \text{Con} \varphi'_i = \psi'_i, \quad \text{for } i \in \{1, 2\}.\]

For each \(i \in \{1, 2\}\), set

\[\varphi_i = \varphi'_i \circ \pi_i: L_i \to L.\]
Then
\[ \varphi_1 \circ \eta_1 = \varphi'_1 \circ \pi_1 \circ \eta_1 = \varphi'_1 \circ \eta'_1 \circ \pi_0 \]
\[ = \varphi'_2 \circ \eta'_2 \circ \pi_0 = \varphi'_2 \circ \pi_2 \circ \eta_2 = \varphi_2 \circ \eta_2, \]
and
\[ \alpha \circ \Con \varphi_i = \alpha \circ (\Con \varphi'_i) \circ (\Con \pi_i) = \psi'_i \circ \Con \pi_i = \psi_i, \]
for \( i \in \{1, 2\} \), concluding the proof of Theorem 1 for \( D = 2 \).

3.2. Theorem 1 for \( D \) Boolean. In this section, let \( D \) be a finite Boolean lattice.

We prove Theorem 1 with the following addition:

**Addition for \( D \) Boolean.** \( L \) contains a Boolean dual ideal isomorphic to \( D \) with a set
\[ \{ d_p \mid p \in J(D) \} \]
its set of dual atoms. For each \( p \in J(D) \),
\[ \alpha_\Theta_L(d_p, 1) = p. \]

**Proof.** The set \( J(D) \) is the set of atoms of \( D \). For each \( p \in J(D) \), we have a zero-preserving lattice surjection \( \beta_p: D \to 2 \) such that \( \beta_p(x) = 1 \) iff \( p \leq x \). Then
\[ \beta = \prod (\beta_p \mid p \in J(D)): D \to \prod (2 \mid p \in J(D)) \]
is an isomorphism.

For each \( p \in J(D) \), set \( \psi_{pi} = \beta_p \circ \psi_i \), for \( i \in \{1, 2\} \) and apply the case \( D = 2 \) to the configuration \( \eta_i: L_0 \to L_i, \psi_{pi}: \Con L_i \to 2 \) to obtain a simple lattice \( L_p \) with a 1 and a dual atom \( d'_p \), lattice homomorphisms \( \varphi_{pi}: L_i \to L_p \) with \( \varphi_{pi} \circ \eta_i = \varphi_{pi} \circ \eta_2 \), and an isomorphism \( \alpha_p: \Con L_p \to 2 \) with
\[ \alpha_p \circ \Con \varphi_{pi} = \psi_{pi} = \beta_p \circ \psi_i. \]

We then set
\[ L = \prod (L_p \mid p \in J(D)) \]
and set
\[ \varphi_i = \prod (\varphi_{pi} \mid p \in J(D)): L_i \to L. \]
Then
\[ \varphi_1 \circ \eta_1 = \varphi_2 \circ \eta_2. \]

Now,
\[ \Con \varphi_i = \prod (\Con \varphi_{pi} \mid p \in J(D)). \]
Thus,
\[ \prod (\alpha_p \mid p \in J(D)) \circ \Con \varphi_i = \prod (\alpha_p \circ \Con \varphi_{pi} \mid p \in J(D)) \]
\[ = \prod (\beta_p \circ \psi_i \mid p \in J(D)) \]
\[ = \prod (\beta_p \mid p \in J(D)) \circ \psi_i \]
\[ = \beta \circ \psi_i. \]

Setting \( \alpha = \beta^{-1} \circ \prod (\alpha_p \mid p \in J(D)) \), we thus get an isomorphism \( \alpha: \Con L \to D \) with
\[ \alpha \circ \Con \varphi_i = \psi_i. \]
For each \( q \in J(D) \), we define \( d_q \in L = \prod (L_p \mid p \in J(D)) \) by setting
\[
(d_q)_p = \begin{cases} 
  d'_q, & \text{if } p = q; \\
  1_{L_p}, & \text{otherwise}.
\end{cases}
\]

Then each \( d_q \) is a dual atom of \( L \), and the dual ideal of \( L \) generated by \( \{ d_p \mid p \in J(D) \} \) is
\[
\prod (\{ d'_p, 1_{L_p} \mid p \in J(D) \})
\]
a Boolean lattice with \( \{ d_p \mid p \in J(D) \} \) its set of dual atoms.

Now, \( \text{Con} L = \prod (\text{Con} L_p \mid p \in J(D)) \) and each \( L_p \) is simple. Thus, for \( p, q \in J(D) \), the \( p \)-th component of \( \Theta_L(d_q, 1_L) \) satisfies
\[
(\Theta_L(d_q, 1_L))_p = \begin{cases} 
  \Theta_{L_q}(d'_q, 1_{L_q}) = \iota_{L_p}, & \text{if } p = q; \\
  \Theta_{L_p}(1_{L_p}, 1_{L_p}) = \omega_{L_p}, & \text{otherwise}.
\end{cases}
\]

Then, for each \( p \in J(D) \),
\[
\beta_p q = \alpha_p (\Theta_L(d_q, 1))_p,
\]
that is,
\[
\beta q = \prod (\alpha_p \mid p \in J(D)) \Theta_L(d_q, 1),
\]
that is,
\[
q = \alpha \Theta_L(d_q, 1).
\]

Since finite direct products preserve the zero and finiteness, the proof is completed. \( \square \)

### 3.3. The General Proof

We let \( B \) be the Boolean lattice generated by \( D \), and let \( \eta: D \to B \) be the canonical embedding. For each \( x \in B \), let \( gx \) denote the smallest element of \( D \) containing \( x \). Then we get a \( \{ \lor, 0 \} \)-homomorphism \( \varrho: B \to D \) such that
\[
\varrho \circ \eta = \text{id}_D.
\]

Note that \( \varrho \mid_{J(B)} \) is just the usual dual of \( \eta \) in the duality between distributive lattices and posets. In our case of \( B \) being the Boolean lattice generated by \( D \), we get an isotone bijection
\[
\varrho \mid_{J(B)}: J(B) \to J(D).
\]

We apply the special case where \( D \) is Boolean to the system \( L_0, L_1, L_2, B \) with the complete \( \lor \)-homomorphisms \( \eta \circ \psi_i: \text{Con} L_i \to B, i \in \{1, 2\} \), and obtain a lattice \( K_0 \) and lattice homomorphisms \( \varphi'_i: L_i \to K_0, i \in \{1, 2\} \), satisfying
\[
\varphi'_1 \circ \eta_1 = \varphi'_2 \circ \eta_2,
\]
and an isomorphism
\[
\alpha_0: \text{Con} K_0 \to B
\]
such that
\[
\alpha_0 \circ \text{Con} \varphi'_i = \eta \circ \psi_i, \quad \text{for } i \in \{1, 2\}.
\]

Furthermore, \( K_0 \) contains a finite Boolean dual ideal \( H \) with \( |J(B)| \) dual atoms \( d'_p, p \in J(B) \), such that
\[
\alpha_0 \Theta_{K_0}(d'_p, 1) = p,
\]
for each \( p \in J(B) \).
By Theorem 4, there is a finite lattice $K_1$ and there is an isomorphism
\[ \alpha_1 : \text{Con } K_1 \rightarrow D \]
such that $K_1$ contains a Boolean ideal $I$ with $|J(D)|$ dual atoms $d_p$, $p \in J(D)$, and
\[ \alpha_1 \Theta_{K_1}(d_p, 1_I) = p, \]
for each $p \in J(D)$.

In view of the bijection (3.2), there is an isomorphism of the dual ideal $H$ of $K_0$ with the ideal $I$ of $K_1$, whereby $d'_p \in H$ corresponds to $d_p$, for each $p \in J(B)$. We let $L$ be the lattice obtained by gluing $K_1$ to the top of $K_0$ by identifying $H$ with $I$ under this isomorphism, so that $K_1$ is a subset of $L$. We then have an embedding $\varepsilon_0 : K_0 \rightarrow L$, where
\[ \varepsilon_0 : d'_p \mapsto d_p, \]
for $p \in J(B)$, and an embedding $\varepsilon_1 : K_1 \rightarrow L$, where
\[ \varepsilon_1 : d_p \mapsto d_p, \]
for $p \in J(D)$. Then $\text{Con } \varepsilon_1 : \text{Con } K_1 \rightarrow \text{Con } L$, whereby
\[ \text{Con } \varepsilon_1 : \Theta_{K_1}(d'_p, 1_{I_0}) \mapsto \Theta_L(d_p, 1_I), \]
is an isomorphism, and the $\{ \lor, 0 \}$-homomorphism $\text{Con } \varepsilon_0 : \text{Con } K_0 \rightarrow \text{Con } L$ satisfies
\[ \text{Con } \varepsilon_0 : \Theta_{K_0}(d'_p, 1_{K_0}) \mapsto \Theta_L(d_p, 1_I). \]

For each $i \in \{ 1, 2 \}$, we set
\[ \varphi_i = \varepsilon_0 \circ \varphi'_i : L_i \rightarrow L. \]
Then
\[ \varphi_1 \circ \eta_1 = \varepsilon_0 \circ \varphi'_1 \circ \eta_1 = \varepsilon_0 \circ \varphi'_2 \circ \eta_2 = \varphi_2 \circ \eta_2. \]

We have an isomorphism $\alpha : \text{Con } L \rightarrow D$ defined by
\[ \alpha = \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1}. \]

We proceed to show that $\alpha \circ \text{Con } \varphi_i = \psi_i$, for $i \in \{ 1, 2 \}$. By the definition of $\alpha$ and $\varphi_i$,
\[ \alpha \circ \text{Con } \varphi_i = \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) \circ (\text{Con } \varphi'_i). \]
By (3.4) and (3.5),
\[ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) : \Theta_{K_0}(d'_p, 1_{K_0}) \mapsto \Theta_{K_1}(d_p, 1_I), \]
for each $p \in J(B)$. Thus,
\[ \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) \circ \alpha_0^{-1} : p \mapsto gp, \]
for each $p \in J(B)$. Therefore,
\[ \alpha_1 \circ (\text{Con } \varepsilon_1)^{-1} \circ (\text{Con } \varepsilon_0) \circ \alpha_0^{-1} = \varrho, \]
since both sides are $\{ \lor, 0 \}$-homomorphisms. Thus, for $i \in \{ 1, 2 \}$,
\[ \alpha \circ \text{Con } \varphi_i = \varrho \circ \alpha_0 \circ (\text{Con } \varphi'_i), \]
by (3.6) and (3.7),
\[ = \varrho \circ \eta \circ \psi_i, \]
by (3.4),
\[ = \psi_i, \]
by (3.1).

This concludes the proof for arbitrary lattices $L_i$ and homomorphisms $\eta_i$. 


We note that if \( K_0 \) has a zero, then \( \varepsilon_0 : K_0 \to L \) preserves the zero. Thus, by the special case \( D \) is Boolean, if the \( L_i \) each have a zero and if the \( \eta_i \) preserve the zero, then the \( \varphi'_i \) and, consequently, the \( \varphi_i \) preserve the zero.

We note, also, that if \( L_1 \) and \( L_2 \) are finite, then so is \( K_0 \) and thus so is \( L \). Then, using Tischendorf’s Theorem, we can replace \( L \) by a finite atomistic lattice.

This concludes the proof of the Theorem 4.

4. Proving Theorem 2

4.1. Congruence-preserving extensions. We shall now establish two results, the first a strengthening of both parts of Theorem 5, and the second a strengthening of Theorem 5(ii) in a different direction:

**Lemma 4.1.** Let \( K \) be a congruence-finite lattice. Then \( K \) has a congruence-preserving relatively complemented embedding into a sectionally complemented lattice \( K' \). If \( K \) has a zero, then one can assume that \( K' \) has the same zero. If \( K \) is finite, then \( K' \) can be chosen to be finite.

**Outline of proof.** We follow the original proof in G. Grätzer and E.T. Schmidt [8], with just one small addition. If \( K \) is a congruence-finite lattice, the congruence-preserving sectionally complemented extension of \( K \) is constructed as follows. Since \( \text{Con} K \) is a finite distributive lattice, we can associate with it the finite sectionally complemented lattice \( L_0 \) of Theorem 4 such that \( \text{Con} L_0 \cong \text{Con} K \). On the other hand, denote by \( M(\text{Con} K) \) the set of all meet-irreducible congruences of \( K \). The rectangular extension of \( K \) is defined by

\[
\mathbb{R}(K) = \prod (K/\Theta \mid \Theta \in M(\text{Con} K)).
\]

Let \( K_\Theta \) be a simple sectionally complemented extension of \( K/\Theta \) such that, in addition, \( K/\Theta \) is relatively complemented in \( K_\Theta \) (we use Lemma 2.1). If \( K \) is finite we choose \( K_\Theta \) finite. Put

\[
\hat{\mathbb{R}}(K) = \prod (K_\Theta \mid \Theta \in M(\text{Con} K)).
\]

Note that the diagonal map from \( K \) into \( \hat{\mathbb{R}}(K) \), that sends every \( x \in K \) to \(( [x]\Theta \mid \Theta \in M(\text{Con} K)) \), has the congruence extension property, but it is not necessarily congruence-preserving (the congruence lattice of \( \hat{\mathbb{R}}(K) \) is Boolean). However, the sectionally complemented extension \( K' \) constructed in [8] is obtained by considering the lattice of finitely generated ideals of the chopped lattice \( L_0 \cup \hat{\mathbb{R}}(K) \), with the two isomorphic Boolean sublattices of \( L_0 \) and \( \hat{\mathbb{R}}(K) \) identified. Since \( K \) is already relatively complemented in \( \hat{\mathbb{R}}(K) \), it is a fortiori relatively complemented in \( K' \).

If \( K \) has a zero, then the above construction preserves this zero. Furthermore, if \( K \) is finite, then \( K' \) is finite.

**Theorem 6.** Let \( K \) be a congruence-finite lattice. Then \( K \) has a congruence-preserving embedding into a relatively complemented lattice \( L \). Furthermore, if \( K \) has a zero, then one can assume that \( L \) has the same zero.

**Proof.** We use Lemma 4.1 to construct a sequence \(( K^{(n)} \mid n < \omega \) of lattices. Set \( K^{(0)} = K \), and, proceeding inductively, for each \( n \) set \( K^{(n+1)} = (K^{(n)})' \), the lattice \((K^{(n)})'\) being the extension of \( K^{(n)} \) guaranteed by Lemma 4.1. To conclude the proof, it suffices to take \( L = \bigcup (K^{(n)} \mid n < \omega \) ).

4.2. Proving Theorem 28. Let $S$ be the $\{\lor, 0\}$-semilattice of all compact elements of $D$. By definition, $S$ is distributive. By P. Pudlák’s Lemma (see 14), every finite subset of $S$ is contained in a finite distributive $\{\lor, 0\}$-subsemilattice of $S$. We use this to construct a direct system of finite distributive subsemilattices of $S$ as follows. First, by Proposition 2.2, there exists a 2-ladder of cardinality $\aleph_1$, say, $(I, \leq)$. Let $\pi : I \to S$ be a surjective map such that $\pi(0_I) = 0_S$. We define a family $(S_i \mid i \in I)$ of finite distributive $\{\lor, 0\}$-subsemilattices of $S$, as follows. We put $S_0 = \{0\}$, and, for all $i \in I$, we let $S_i$ be a finite distributive $\{\lor, 0\}$-subsemilattice of $S$ containing the subset

$$\bigcup \{ S_j \mid j < i \} \cup \{ \pi(i) \}.$$

Since $\pi(0_I) = 0_S$, we can take $S_0 = \{0\}$. Then $S$ is the directed union of all $S_i$, for $i \in I$. We denote by $\phi_i$ the inclusion map from $S_i$ into $S_j$, for all $i \leq j$ in $I$.

Let $\varrho : I \to \omega$ be any strictly increasing map from $I$ to $\omega$ (for example, the height function on $I$). We put $I_n = \{ i \in I \mid \varrho(i) \leq n \}$, for all $n < \omega$. By induction on $n$, we construct a family of finite lattices $L_i$, maps $\varepsilon_i : \text{Con} L_i \to S_i$, for $i \in I_n$, and $\{0\}$-lattice homomorphisms $f_{ij} : L_i \to L_j$, for $i \leq j$ in $I_n$, satisfying the following properties:

(a) $f_{ii}^i = \text{id}_{L_i}$, for all $i \in I_n$.
(b) $f_{ij}^k = f_{ij}^k \circ f_{ij}^j$, for all $i, j, k \in I_n$.
(c) $\varepsilon_i$ is an isomorphism from $\text{Con} L_i$ onto $S_i$, for all $i \in I_n$.
(d) $\varepsilon_j \circ \text{Con} f_{ij}^j = \varepsilon_j \circ \varepsilon_i$, for all $i \leq j$ in $I_n$.
(e) $f_{ij}[L_i]$ is relatively complemented in $L_j$, for all $i < j$ in $I_n$.

For $n = 0$, we just take $L_0 = \{0\}$ (because $S_0 = \{0\}$). Let us assume that we have performed the construction at level $n$; we show how to extend it to the level $n + 1$. So, let $i \in I_{n+1} - I_n$. Since $I$ is a 2-ladder, $i$ has (at most) two immediate predecessors in $I$, say, $i_1$ and $i_2$. Note that $i_1$ and $i_2$ need not be distinct. For $k \in \{1, 2\}$, the map

$$\psi_k = \phi_{i_k}^i \circ \varepsilon_{i_k}$$

is a $\{\lor, 0\}$-embedding from $\text{Con} L_{i_k}$ into $S_i$, and the equality

$$\psi_1 \circ \text{Con} f_{i_1}^{i_1 \land i_2} = \psi_2 \circ \text{Con} f_{i_2}^{i_1 \land i_2}$$

holds. By Theorem 1 there is a finite lattice $L_i$, there are $\{0\}$-lattice homomorphisms $g_k : L_{i_k} \to L_i$, for $k \in \{1, 2\}$, and and there is an isomorphism $\varepsilon_i : \text{Con} L_i \to S_i$ such that

$$(4.1) \quad g_1 \circ f_{i_1}^{i_1 \land i_2} = g_2 \circ f_{i_2}^{i_1 \land i_2},$$

$$(4.2) \quad \varepsilon_i \circ \text{Con} g_k = \psi_k, \quad \text{for } k \in \{1, 2\},$$

hold. Furthermore, if $i_1 = i_2$, then replacing $g_2$ by $g_1$ does not change the validity of (4.1) and (4.2). Thus we may define $f_{i_k}^{i_1} = g_k$, for $k \in \{1, 2\}$, and (4.1), (4.2) take the following form:

$$(4.3) \quad f_{i_1}^{i_1 \land i_2} = f_{i_2}^{i_1 \land i_2},$$

$$(4.4) \quad \varepsilon_i \circ \text{Con} f_{i_k}^{i_k} = \psi_k, \quad \text{for } k \in \{1, 2\}.$$
So we have defined $f^j_i$, for all $i$ and $j$ in $I_{n+1}$ such that $j$ is an immediate predecessor of $i$ in $I_{n+1}$. We extend this definition to arbitrary $i$, $j$ in $I_{n+1}$ such that $j \leq i$. If $j = i$, then we put $f^j_i = \text{id}_{L_i}$. Now assume that $j < i$ in $I_{n+1}$, with $i \notin I_n$. There exists an index $k \in \{1, 2\}$ such that $j \leq i_k$. The only possible choice for $f^j_i$ is to define it as

\begin{equation}
(4.5) \quad f^j_i = f^{i_k}_i \circ f^j_{i_k},
\end{equation}

except that this should be independent of $k$. This means that if $j \leq i_1 \land i_2$, then the equality

\begin{equation}
(4.6) \quad f^{i_1}_i \circ f^{i_2}_i = f^{i_2}_i \circ f^{i_1}_i
\end{equation}

should hold. We compute:

\begin{align*}
f^{i_1}_i \circ f^{i_2}_i & = f^{i_1}_i \circ f^{i_1,\land i_2}_i \circ f^{i_2,\land i_1}_i \\
& = f^{i_2}_i \circ f^{i_1,\land i_2}_i \circ f^{i_1,\land i_2}_i \quad \text{(by (4.3))} \\
& = f^{i_2}_i \circ f^{i_2}_i,
\end{align*}

which establishes (4.6).

At this point, the $\{0\}$-lattice embeddings $f^j_i : L_j \to L_i$ are defined for all $j \leq i$ in $I_{n+1}$. The verification of conditions (a)–(c) above is then straightforward. Let us verify (d). Let $i \leq j$ in $I_{n+1}$, we prove that

\begin{equation}
(4.7) \quad \varphi^j_i \circ \varepsilon_i = \varepsilon_j \circ \text{Con} f^j_i.
\end{equation}

The only nontrivial case happens if $j \in I_{n+1} - I_n$ and $i < j$. It suffices then to verify (1.4) for the pairs $(i, j_*)$ and $(j_*, j)$, where $j_*$ is any immediate predecessor of $j$ such that $i \leq j_*$. For the pair $(i, j_*)$, this follows from the induction hypothesis, while for the pair $(j_*, j)$, this follows from (4.4).

Hence the construction of the $L_i$, $\varepsilon_i$, $f^j_i$ is carried out for the whole poset $I$. Let $L$ be the direct limit of all the $L_i$, $i \in I$, with the transition maps $f^j_i$, for $i \leq j$ in $I$. Then $\text{Con}_c L$ is the direct limit of the $\text{Con}_c L_i$, with the transition maps $\text{Con}_c f^j_i$, in the category of distributive $\{\lor, 0\}$-semilattices and $\{\lor, 0\}$-homomorphisms. Thus, by (c) and (d), $\text{Con}_c L$ is isomorphic to the direct limit of the $S_i$ with the transition maps $\varphi^j_i$, for $i \leq j$ in $I$. Hence, $\text{Con}_c L \cong S$, from which it follows that $\text{Con} L \cong D$. The fact that $L$ is relatively complemented follows from condition (e) above.

5. Proving Theorem 5

By definition, $K$ can be written as a union,

\[ K = \bigcup (K_n \mid n < \omega), \]

where $(K_n \mid n < \omega)$ is an increasing sequence of congruence-finite sublattices of $K$. Furthermore, if $K$ has a zero, then we can assume that 0 belongs to $K_n$, for all $n < \omega$. Denote by $e_n$ the inclusion map from $K_n$ into $K_{n+1}$. For $n < \omega$, let us assume that we have constructed a relatively complemented lattice $L_n$ and a congruence-preserving embedding $u_n : K_n \hookrightarrow L_n$ such that $u_n$ preserves the zero if $K_n$ has a zero. We apply Theorem 4 to the lattice homomorphisms

\[ u_n : K_n \hookrightarrow L_n, \quad e_n : K_n \hookrightarrow K_{n+1}, \]
the semilattice \( D = \text{Con} K_{n+1} \), and the \( \{\lor, 0\} \)-semilattice homomorphisms

\[
\varphi = \text{Con} u_n : \text{Con} K_n \to \text{Con} L_n,
\psi = (\text{Con} e_n) \circ (\text{Con} u_n)^{-1} : \text{Con} L_n \to \text{Con} K_{n+1}.
\]

We obtain a lattice \( L_{n+1} \), lattice homomorphisms \( f_n : L_n \to L_{n+1} \), \( u_{n+1} : K_{n+1} \to L_{n+1} \),

and an isomorphism \( \alpha_n : \text{Con} L_{n+1} \to \text{Con} K_{n+1} \) such that the following equalities hold:

\[
(5.1) \quad u_{n+1} \circ e_n = f_n \circ u_n,
(5.2) \quad \alpha_n \circ \text{Con} f_n = (\text{Con} e_n) \circ (\text{Con} u_n)^{-1},
(5.3) \quad \alpha_n \circ \text{Con} u_{n+1} = \text{id}_{\text{Con} K_{n+1}}.
\]

By Theorem 6, one can further assume that \( L_{n+1} \) is relatively complemented. By (5.3), the map \( \text{Con} u_{n+1} \) is an isomorphism and so \( u_{n+1} \) is congruence-preserving.

By (5.2), the map \( \text{Con} f_n \) separates zero (because \( \text{Con} e_n \) does), that is, \( f_n \) is a lattice embedding.

Let \( L \) be the direct limit of all the \( L_n \), with the transition maps

\[
f_m \circ \cdots \circ f_{n-1} : L_m \to L_n,
\]

for \( m < n \) in \( \omega \). Denote by \( g_n : L_n \to L \) the corresponding limiting maps.

By (5.1) and the fact that all the \( u_n \) are congruence-preserving embeddings, the sequence \( (u_n \mid n < \omega) \) defines a congruence-preserving embedding \( u : K \to L \) by

\[
u(x) = g_n \circ u_n(x), \quad \text{if } x \in K_n, \text{ for } n < \omega.
\]

Since all the \( L_n \) are relatively complemented, \( L \) is relatively complemented. If \( K \) has a zero, then all the maps \( u_n \) and \( f_n \) preserve the zero, thus \( L \) has the same zero as \( K \).

If \( K \) is locally finite, then we can assume that all the \( K_n \) are finite, and we can then take all the \( L_n \) finite. In particular, \( L \) is also locally finite. This concludes the proof of Theorem 3.

6. Discussion

6.1. Theorem 1. In Theorems [1, 3] the bound zero is preserved. We do not know whether the theorems of this paper have analogues for bounded lattices:

Problem 1. In the statement of Theorem 1 let us assume that \( L_0, L_1, \) and \( L_2 \) are bounded lattices and that \( \eta_1, \eta_2 \) are \( \{0, 1\} \)-preserving. Can the lattice \( L \) of the conclusion be taken bounded, with both \( \varphi_1 \) and \( \varphi_2 \) \( \{0, 1\} \)-preserving? In addition, if \( L_0, L_1, \) and \( L_2 \) are finite, can \( L \) be taken finite?

6.2. Theorem 2. From the results of the third author in [18, 19], the \( \aleph_1 \) bound in the statement of Theorem 2 is optimal, because there are algebraic distributive lattices with \( \aleph_2 \) compact elements that cannot be represented as congruence lattices of relatively complemented lattices.

There are stronger forms of Theorem 2. For example, a result of K.R. Goodearl and F. Wehrung [5] states that every distributive \( \{\lor, 0\} \)-semilattice is the direct limit of a family of finite Boolean \( \{\lor, 0\} \)-semilattices and \( \{\lor, 0\} \)-homomorphisms. Since every finite lattice embeds into a finite geometric lattice, one can prove that the lattice \( L \) of Theorem 2 can be assumed to be a direct limit of finite geometric lattices.
lattices. Similarly, using P. Pudlák and J. Tůma [15], we can prove that \( L \) can be assumed to be a direct limit of lattices, each of which is a finite product of finite partition lattices.

In neither of these cases is \( L \) modular. However, using the results of [20], one can show that the lattice \( L \) of Theorem 2 can be taken to be sectionally complemented and modular: in addition, in this case, \( L \) can be assumed to be bounded, if the largest element of \( D \) is compact. The local finiteness of \( L \) is lost.

**Problem 2.** If the lattice \( L \) has at most \( \aleph_1 \) compact congruences, does \( L \) have a relatively complemented congruence-preserving extension.

A variant of this problem, was raised by the first and the last author at the August 1998 Szeged meeting:

**Problem 3.** Let \( L \) be an infinite lattice with \( |L| \leq \aleph_1 \). Does \( L \) have a congruence-preserving extension to a (sectionally complemented) relatively complemented lattice?

6.3. **Theorem 3.** The countability assumption of the statement of Theorem 3 is essential: by M. Ploščica, J. Tůma, and F. Wehrung [13], the free lattice with \( \aleph_2 \) generators in the variety generated by \( M_3 \) (or any finite, nondistributive lattice) does not have a congruence-preserving embedding into a relatively complemented lattice.

Not every countable lattice is \( \omega \)-congruence-finite: take any finitely generated, non congruence-finite lattice, for example, the free lattice on \( n \) generators, where \( n \geq 3 \).

**Problem 4.** Is it true that every bounded, \( \omega \)-congruence-finite lattice \( L \) has a congruence-preserving extension into a relatively complemented lattice that preserves the bounds?

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