Fisher information, nonclassicality and quantum revivals

Elvira Romera

Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada,
Fuentenueva s/n, 18071 Granada, Spain

Departamento de Física Atómica, Molecular y Nuclear, Universidad de Granada,
Fuentenueva s/n, 18071 Granada, Spain

Francisco de los Santos

Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada,
Fuentenueva s/n, 18071 Granada, Spain

Departamento de Electromagnetismo y Física de la Materia, Universidad de Granada,
Fuentenueva s/n, 18071 Granada, Spain

Abstract

Wave packet revivals and fractional revivals are studied by means of a measure of nonclassicality based on the Fisher information. In particular, we show that the spreading and the regeneration of initially Gaussian wave packets in a quantum bouncer and an in the infinite square-well correspond, respectively, to high and low nonclassicality values. This result is in accordance with the physical expectations that at a quantum revival wave packets almost recover their initial shape and the classical motion revives temporarily afterward.

dlsantos@onsager.ugr.es +34 958 244 014

1. Introduction

Quantum systems may behave classically or quasiclassically under a variety of circumstances and, in this regard, the transition from quantum to classical mechanics still poses intriguing problems that attract considerable attention. Of particular interest are systems that display both classical and quantum periodic motions, with generally different, incommensurable periods, for in this case the interesting question arises as to how the classical
periodicity emerges from the quantum one in the appropriate limit. For instance, a particle of mass $m$ and energy $E$ in an infinite square-well potential of width $L$ initially oscillate with a classical period $T_{cl} = L\sqrt{2m/E}$. The classical oscillations gradually damp out as the wave packet representing the particle spreads more or less uniformly across the well. Quantum mechanically, the wave function regains exactly its initial form with a revival period $T_{rev} = 4mL^2/\pi\hbar$, after which the classical oscillations resume with period $T_{cl}$ again. At times that are rational fractions of $T_{rev}$, the wave packet temporarily splits into a number of scaled copies called fractional revivals [1, 2, 3]. In the same vein, an object of mass $m$ released from a height $z_0$ and subjected only to gravity, bounces up and down against an impenetrable flat surface with a classical period (in suitable units) $T_{cl} = 2\sqrt{z_0}$, while the wave function of the corresponding quantum bouncer almost returns to its initial shape after a revival time $T_{rev} = 4z_0^2/\pi$. After a revival has taken place, a new cycle of quasiclassical behavior and revivals commences again. The fact that in this case the revivals are only approximate does not make a difference. Revivals and fractional revivals received a great deal of attention over the last decades. Theoretical progress and experimental observations were made in atoms and molecules, and Bose-Einstein condensates [4, 5, 6, 7]. Recently, revivals have been theoretically investigated in low dimensional systems [8, 9, 10, 11, 12, 13] and have been related to quantum phase transitions [14].

Identifying the occurrence of wave packet revivals usually makes use of the autocorrelation function $A(t) = \langle \Psi(0)|\Psi(t) \rangle$, which is the overlap between the initial and the time-evolving wave packet. Within this approach, the occurrence of revivals and fractional revivals corresponds to, respectively, the return of $A(t)$ to its initial value of unity and the appearance of relative maxima in $A(t)$. Another method to study revival phenomena consists in monitoring the time evolution of the expectation values of some quantities [3, 15, 16], and an approach based on a finite difference eigenvalue method has been put forward that allows to predict the revival times directly [17]. Recently, information entropy approaches were proposed [18] based on the Shannon and Rényi entropies, complementary to the conventional ones. This technique was shown to be superior to analyses based on both the standard variance uncertainty product [19] and the autocorrelation function, inasmuch as it overcomes the difficulty that wave packets reform themselves at locations that do not coincide with their original ones. A complementary informational measure is the Fisher information [20] which has attracted substantial interest in physics, in particular in atomic and molecular physics (see for example...
In this paper we show that the analysis of the wave packet dynamics can be carried out using a new tool, namely, the nonclassicality $J_{nc}$, defined in terms of the Fisher information as we now discuss.

Hall [36] has recently introduced a measure of nonclassicality, $J_{nc}$, in terms of the probability densities in position and momentum spaces, $\rho(x) = |\psi(x)|^2$ and $\gamma(p) = |\phi(p)|^2$, respectively. To be concrete, $J_{nc} \equiv (\hbar/2) \sqrt{I_\rho I_\gamma}$, where

$$I_\rho = 4 \int \left| \frac{d}{dx} \rho^{1/2}(x) \right|^2 dx, \quad I_\gamma = 4 \int \left| \frac{d}{dp} \gamma^{1/2}(p) \right|^2 dp. \tag{1}$$

Note that $I_\rho$ and $I_\gamma$ are the classical Fisher informations associated with the probability densities $\rho(x)$ and $\gamma(p)$ [20]. In the next section we shall show that the time evolution of a wave packet exhibiting revivals and fractional revivals is initially characterized by a classical behavior with a low $J_{nc}$ value, followed by a wave packet spreading with a higher value of $J_{nc}$. In the long time evolution, for times near $T_{rev}$, the wave packet (approximately) restores its initial form, exhibiting classical periodicity again accompanied by a decrease in $J_{nc}$. In the case of fractional revivals at $t = pT_{rev}/q$, several minipackets emerge whereupon a decrease in $J_{nc}$ is expected.

This paper is organized as follows. In Section II we shall consider the Fisher information as it applies to revival phenomena. In particular, we show the role of the Fisher information as a measure of nonclassicality in the dynamics of two model systems that exhibit fractional revivals: the so-called quantum ‘bouncer’, that is a quantum particle bouncing against a hard surface under the influence of gravity and the infinite square-well. Finally, some concluding remarks will be given in the last section.

2. Fisher information and nonclassicality

The Fisher information of single particle systems is defined as a functional of the density function in conjugate spaces by Eq. (1) and it has been shown to be a measure of nonclassicality [36]. Following Hall [36] and Mosna et al. [37], the Fisher information in position space can be expressed as $I_\rho = (4/\hbar)^2(\langle P^2 \rangle_\psi - \langle P^2 \rangle_\psi^p)$ where $P$ denotes the momentum operator and $P_\psi$ is a (state-dependent) classical momentum operator defined by

$$P_\psi \Psi(x) = \frac{\hbar}{2i} \left( \frac{\psi''(x)}{\psi(x)} - \frac{\psi''(x)}{\psi^*(x)} \right) = \hbar \left( \arg \psi(x) \right)'.$$
Hence, it is natural to separate the momentum operator in a classical \( P_{cl} \) and nonclassical \( P_{nc} \) contribution with \( P_{nc} \equiv P - P_{cl} \). The definition of the classical momentum observable is supported by the facts that \( \rho \) satisfies the classical continuity equation and that the expectation values of \( P \) and \( P_{cl} \) are equal for all wave functions \( \langle P \rangle_{\psi} = \langle P_{cl} \rangle_{\psi} \). The conjugate equality that relates the momentum Fisher information and the nonclassicality of the position operator can be obtained analogously, \( I_\gamma = (4/\hbar)^2 (\langle X^2 \rangle_{\phi} - \langle X^2_{cl} \rangle_{\phi}) \).

Finally, Hall introduced a measure of joint nonclassicality \( J_{nc} \) for a quantum state as

\[
J_{nc} = \frac{\hbar}{2} I_{\rho}^{1/2} I_{\gamma}^{1/2}.
\] (3)

It follows that \( J_{nc} = 1 \) for Gaussian distributions. For instance, the evolution of Gaussian wave packets in a harmonic oscillator follows a periodic motion in accordance with classical expectations [3], and \( J_{nc} = 1 \) for all times. For mixed states, \( J_{nc} \) can be arbitrarily small while for pure states Hall found \( J_{nc} \geq |1 + (i/\hbar)\langle [P_{cl}, X_{cl}] \rangle_{\psi}| \) [36].

2.1. Quantum bouncer

Consider an object of mass \( m \) bouncing against a hard surface subjected only to the influence of the gravitational force directed downward along the \( z \) axis, that is, a particle in a potential \( V(z) = mgz \), if \( z > 0 \) and \( V(z) = +\infty \) otherwise. Gravitational quantum bouncers have been recently realized using neutrons [38] and atomic clouds [39]. Their revival behavior has been discussed in [16, 40] and an entropy-based approach was presented in [18, 19].

The time-dependent wave function for a localized quantum wave packet is expanded as a one-dimensional superposition of energy eigenstates as

\[
\psi(x, t) = \sum_n a_n u_n(x) e^{-iE_n t/\hbar}.
\] (4)

The eigenfunctions and eigenvalues are given by [40]

\[
E'_n = z_n; \quad u_n(z') = N_n \text{Ai}(z' - z_n); \quad n = 1, 2, 3, \ldots
\] (5)

where \( l_g = (\hbar/2gm^2)^{1/3} \) is a characteristic gravitational length, \( z' = z/l_g \), \( E' = E/mgl_g \), \( \text{Ai}(z) \) is the Airy function, \(-z_n\) denotes its zeros, and \( N_n = |\text{Ai}'(-z_n)| \) is the \( u_n(z') \) normalization factor. Accurate analytic approximations for \( z_n \) exist [40],

\[
z_n \simeq \frac{3\pi}{2} \left[ n - \frac{1}{4} \right]^{2/3}.
\] (6)
Consider now an initial Gaussian wave packet localized at a height $z_0$ above the surface, with a width $\sigma$ and a momentum $p_0$ (in the remainder of this paper the primes on the energy and position symbols will be dropped)

$$\psi(z, 0) = \frac{1}{\sqrt{\sigma \hbar \sqrt{\pi}}} e^{-(z-z_0)^2/2\sigma^2\hbar^2} e^{ip_0(z-z_0)/\hbar}. \quad (7)$$

If the lower bound of the integral is extended to $-\infty$, the associated coefficients of the time-dependent wave function for $p_0 = 0$ can be obtained analytically as [40]

$$C_n \simeq N_n \left( \frac{2}{\pi \sigma^2} \right)^{1/4} \int_{-\infty}^{\infty} Ai(z - z_n) e^{-(z-z_0)^2/\sigma^2} dz$$

$$= N_n \left( \frac{2}{\pi \sigma^2} \right)^{1/4} \sqrt{\pi} \sigma \exp \left[ \frac{\sigma^2}{4} \left( z_0 - z_n + \frac{\sigma^4}{24} \right) \right]$$

$$\times Ai \left( z_0 - z_n + \frac{\sigma^4}{16} \right). \quad (8)$$

The important time scales of a wave packet’s time evolution are in the coefficients of the Taylor series (see, for instance [1, 2, 3]) of the energy spectrum $E_n$ around the level the wave packet is centered around, let us say $\bar{n}$:

$$E_{\bar{n}} = E_n + 2\pi \hbar \left( \frac{(n - \bar{n})}{T_{cl}} + \frac{(n - \bar{n})^2}{T_{rev}} + \cdots \right). \quad (9)$$

The classical period and the revival time can be calculated to obtain $T_{cl} = 2\sqrt{z_0}$ and $T_{rev} = 4z_0^2/\pi$, respectively [40]. The temporal evolution of the wave packet in momentum-space was obtained numerically by the fast Fourier transform method.

We have computed the temporal evolution of the autocorrelation function and the nonclassicality $J_{nc}$ for the initial conditions $z_0 = 100$, $\sigma = 1$ and $p_0 = 0$. Figure 1 displays the early time evolution of both quantities and the location of the main fractional revivals. The top panel shows how the autocorrelation function initially follows the first classical periods of motion. The nonclassicality, $J_{nc}$, describes precisely this same behavior, with peaks at the wave packet’s collapses and minima at the multiples of the classical period. In the long time limit, the wave packet eventually spreads out and collapses, only to reform at multiples of the revival time. In between, fractional revivals take place. All this is reflected in the maxima and relative maxima of $|A(t)|^2$.
as shown in the top panel of Fig. 2. The alternative description in terms of $J_{nc}$ is shown in the bottom panel of Fig. 2. The slightly nonclassical behavior or, equivalently, the quasiclassical behavior that takes place at full and fractional revivals is described by, respectively, the minima and the relative minima of $J_{nc}$. Finally, notice that the occurrence of fractional revivals at, for instance, $t = 4/5T_{rev}$ or $t = 5/6T_{rev}$ is clearly better indicated by $J_{nc}$ than by $|A(t)|^2$.

2.2. Infinite square-well system

The one-dimensional infinite square-well potential confines a particle of mass $m$ to a box of width $L$ and is described by $V(x) = 0$ for $0 < x < L$ and $V(x) = +\infty$ otherwise.

The normalized eigenstates and the corresponding eigenvalues are given by,

$$u_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right), \quad E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2}. \quad (10)$$

The classical period and the revival time can be computed as $T_{cl} = 2mL^2/\hbar \pi n$ and $T_{rev} = 4mL^2/\hbar \pi$, respectively. Note that due to the fact that in this particular case the quantized energy levels are exactly quadratic
Figure 2: Autocorrelation function (top panel) and nonclassicality (bottom panel) for a quantum bouncer with $z_0 = 100$, $a = 1$, and $P_0 = 0$. The revival time is $T_R \approx 12 732.4$ and the main fractional revivals are denoted by vertical dotted lines.

In $n$, there is no super-revival (or higher-order effects in the evolution), nor does the revival period depend on the mean energy level $\bar{n}$. The occurrence of revivals and fractional revivals can now be illustrated by simply taking $t = kT_{\text{rev}}$ in the expansion (4), which immediately entails $\psi(x, t = kT_{\text{rev}}) = \psi(x, t = 0)$, with $k$ an integer [2]. Therefore, the time evolution of the infinite square-well is periodic with period $T_{\text{rev}}$, and this period is an exact revival time too. It is also easy to see by direct substitution in (4) that $\psi(L - x, T_{\text{rev}}/2) = -\psi(-x, 0)$, so at time $t = T_{\text{rev}}/2$ a copy of the initial state reforms itself, reflected around the center of the well [2].

We shall consider initial Gaussian wave packets centered at a position $x_0$, with a momentum $p_0$ and a variance $\sigma$,

$$\psi(x, 0) = \frac{1}{\sqrt{\sigma \sqrt{\pi}}} e^{-(x-x_0)^2/2\sigma^2} e^{ip_0(x-x_0)/\hbar}. \quad (11)$$

Assuming that the integration region can be extended to the whole real axis due to the exponential functional form of (11) [3], the expansion coefficients can be approximated with high accuracy by the analytic expression

$$a_n \approx 2 \frac{b\sqrt{\pi}}{L} e^{-b^2 [p_0^2 + (n\pi/L)^2]} \sin \left[ \frac{n\pi}{L} \left( x_0 + \frac{ib^2 p_0}{\hbar} \right) \right], \quad (12)$$
where \( b = \sigma \hbar \), \( p_n = n\pi \hbar /L \) and \( p_0 = n_0\pi \hbar /L \).

To calculate the corresponding time dependent, momentum wave function we use the Fourier transform of the equation (11), and the momentum-space normalized eigenstates

\[
\phi_n(p) = \sqrt{\frac{\hbar}{\pi L p^2 - p_n^2}} \left[ (-1)^n e^{ipL/\hbar} - 1 \right].
\]  

(13)

The initial wave packet in momentum space is then given by the Fourier transform of the equation (11), which leads again to a Gaussian expression,

\[
\Phi(p, 0) = \sqrt{\frac{\sigma}{\sqrt{\pi}}} e^{-\sigma^2(p-p_0)^2/2} e^{-ipx_0/\hbar}.
\]  

(14)

Without loss of generality, we shall henceforth take \( 2m = \hbar = L = 1 \), and \( \sigma = 1/\sqrt{200} \) for the initial wave packet.

We have computed the measure of nonclassicality \( J_{nc} \) and the autocorrelation function for an initial wave packet with \( x_0 = 0.5 \) and \( p_0 = 400\pi \). The results are shown in Fig. 3. At early times, the Gaussian wave packet evolves quasiclassically but in a few periods the quantum and classical wave packet trajectories start moving apart, the classical component of the wave function being defined as \( \psi_{cl}(x, t) = \sum_n a_n u_n(x) e^{-i2\pi nt/T_{cl}} \). For long time
scales, a large amplitude modulation with a set of relative minima is superimposed on the quasiperiodic oscillations (Fig. 3). In this long-time regime, the wave packet initially spreads and delocalizes while undergoing a sequence of fractional revivals with the creation of correlated sets of sub-packets located along the classical orbit, each of them similar to the initial one so that the nonclassicality reaches a relative minimum, the most important fractional revivals are denoted by vertical dashed-lines. Notice that at one half of the revival period there is an absolute minimum which corresponds to a single rejected copy of the initial wave function.

3. Summary

To summarize, we have shown how the measure of nonclassicality $J_{nc}$ accounts for the regeneration of initially well localized wave packets during their time evolution. In particular, $J_{nc}$ shows the spreading (high nonclassicality values) and the resuming of the classical periodic motion (low nonclassicality values) of wave packets in two example cases, namely, the quantum bouncer and the infinite square-well. This approach overcome the generic difficulty that the autocorrelation function misses to detect some fractional revivals because the mini-packets emerge at arbitrary positions that do not coincide with the original one. This appears to be a common advantage of all the entropy-based approaches.

This work was supported by the Spanish Projects No. MICINN FIS2009-08451, No. FQM-02725 (Junta de Andalucía), and No. MICINN FIS2011-24149.

References

[1] I.Sh . Averbukh, J.F. Perelman, Phys. Lett. A 139 (1989) 449; Acta Phys. Pol. A 78 (1990) 33.

[2] D.L. Aronstein, C.R. Stroud, Jr., Phys. Rev. A 55 (1997) 4526 .

[3] R.W. Robinett, Phys. Rep. 392 (2004) 1.

[4] J.H. Eberly, N.B. Narozhny, J.J. Sánchez-Mondragón, Phys. Rev. Lett. 44 (1980) 1323.
[5] G. Rempe, H. Walther, N. Klein, Phys. Rev. Lett. 58 (1987) 353; J.A. Yeazell, M. Mallalieu, and C.R. Stroud, Jr., Phys. Rev. Lett. 64 (1990) 2007; T. Baumert, V. Engel, C. Röttgermann, W.T. Strunz, G. Gerber, Chem. Phys. Lett. 191 (1992) 639; M.J.J. Vrakking, D. M. Villeneuve, A. Stolow, Phys. Rev. A 54 (1996) R37-R40; A. Rudenko, Th. Ergler, B. Feuerstein, K. Zrost, C.D. Schrter, R. Moshammer, J. Ullrich, Chem. Phys. 329 (2006) 193.

[6] I.Sh. Averbukh, M.J.J. Vrakking, D.M. Villeneuve, A. Stolow, Phys. Rev. Lett. 77 (1996) 3518.

[7] M. Mehring, K. Müller, I.Sh. Averbukh, W. Merkel, W.P. Schleich, Phy. Rev. Lett. 98 (2007) 120502; M. Gilowski, T. Wendrich, T. Müller, Ch. Jentsch, W. Ertmer, E.M. Rasel, W.P. Schleich, ibid. 100 (2008) 030201; D. Bigourd, B. Chatel, W.P. Schleich, B. Girard, ibid. 100 (2008) 030202.

[8] E. Romera, F. de los Santos, Phys. Rev. B 80 (2009) 165416.

[9] V. Krueckl, T. Kramer, New J. Phys. 11 (2009) 093010.

[10] J.J. Torres, E. Romera, Phys. Rev. B 82 (2010) 155419.

[11] J. Schliemann, New J. Phys. 10 (2008) 043024.

[12] V.Ya. Demikhovskii, G.M. Maksimova, A.A. Perov, A.V. Telezhnikov, Phys. Rev. A 85 (2012) 022105.

[13] T. García, S. Rodríguez-Bolívar, N.A. Cordero, E. Romera, J. Phys.: Condens. Matter 25 (2013) 235301.

[14] F. de los Santos, E. Romera, Phys. Rev. A 87 (2013) 013424.

[15] C. Sudheesh, S. Lakshmibala, V. Balakrishnan, Phys. Lett. A 329 (2014) 14.

[16] M.A. Donchesk, R.W. Robinett, Am. J. Phys. 69 (2011) 1084.

[17] D.L. Aronstein, C.R. Stroud Jr., Laser Phys. 15 (2005) 1496.

[18] E. Romera, F. de los Santos, Phys. Rev. Lett. 99 (2007) 263601 (2007); Phys. Rev. A 78 (2008) 013837.
[19] F. de los Santos, C. Guglieri, E. Romera, Physica E 42 (2010) 303.

[20] R.A. Fisher, Proc. Cambridge Philos. Soc. 22 (1925) 700, [reprinted in
Collected Papers of R. A. Fisher, edited by J. H. Bennett, University of
Adelaide Press, South Australia, 1972, pp. 1540].

[21] S.B. Sears, R.G. Parr, U. Dinur, Isr. J. Chem. 19 (1980) 165.

[22] Á. Nagy, Chem. Phys. Lett. 449 (2007) 212.

[23] P. Sanchez-Moreno, A.R. Plastino, J.S. Dehesa, J. Phys. A 44 (2011)
065301.

[24] R. Nalewajski, Chem. Phys. Lett. 372 (2003) 28.

[25] Á. Nagy, J. Chem. Phys. 119 (2003) 9407.

[26] B. R. Frieden, Physics form Fisher Information. A Unification (Cam-
bridge University Press, Cambridge, 1998).

[27] B. R. Frieden, Am. J. Phys. 57 (1989) 1004.

[28] M. Reginatto, Phys. Rev. A 58 (1998) 1775.

[29] R. Nalewajski, Chem. Phys. Lett. 372 (2003) 28.

[30] E. Romera, P. Sánchez-Moreno, J.S. Dehesa, Chem. Phys. Lett. 414
(2005) 468.

[31] Á. Nagy, K.D. Sen, Phys. Lett. A 360 (2006) 291.

[32] E. Romera, J.S. Dehesa, J. Chem. Phys. 120 (2004) 8906.

[33] E. Romera, Mol. Phys. 100 (2002) 3325.

[34] S.B. Liu, J. Chem. Phys. 126 (2007) 191107.

[35] J. B. Szabó, K.D. Sen, Á. Nagy, Phys. Lett. A 372 (2008) 2428.

[36] M.J.W. Hall, Phys. Rev. A 62 (2000) 012107.

[37] R.A. Mosna, I.P. Hamilton, L. Delle Site, J. Phys. A 38 (2005) 3869.
[38] V. V. Nesvizhevsky et al., Nature (London) 415 (2002) 297; V. V. Nesvizhevsky, A. K. Petukhov, K. V. Protasov, A. Y. Voronin, Phys. Rev. A 78 (2008) 033616.

[39] K. Bongs, S. Burger, G. Birkl, K. Sengstock, W. Ertmer, K. Rzążewski, A. Sanpera, M. Lewenstein, Phys. Rev. Lett. 83 (1999) 3577.

[40] J. Gea-Banacloche, Am. J. Phys. 67 (1999) 776; O. Vallée, Am. J. Phys. 68 (2000) 672.