Considerations on the ModMax electrodynamics in the presence of an electric and magnetic background

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(Dated: September 21, 2022)

The properties of the modified Maxwell electrodynamics (ModMax) are investigated in the presence of an external and uniform electromagnetic field. We expand the non-linear theory around an electromagnetic background up to second order in the propagating fields to obtain the permittivity and permeability tensors, the dispersion relations, the group velocity and the refractive index of the wave as functions of the uniform electric and magnetic fields. The case in which these background fields are perpendicular is analyzed. The birefringence phenomenon is discussed where the parameter of the non-linear theory has a fundamental role. We calculate the difference of the refractive indices as function of the ModMax parameter, and of the electric and magnetic fields. Finally, we compute the interaction energy for an axionic ModMax electrodynamics in the presence of an external magnetic background fields, within the framework of the gauge-invariant, but path-dependent variables formalism. Our results show that the interaction energy contains a linear term leading to the confinement of static probe charges.

I. INTRODUCTION

As is widely-known, non-linear electrodynamics has a long history beginning from the pioneering paper by Born and Infeld (BI) [1], who introduced their theory in order to overcome the intrinsic divergences in the Maxwell theory, at short distances. In passing we mention that, just like to Maxwell electrodynamics, the BI electrodynamics displays no birefringence in vacuum. We also recall here that, after the development of Quantum Electrodynamics (QED) [2–5], Heisenberg and Euler (HE) [6] obtained a new non-linear effective theory which includes quantum effects. Indeed, this new theory contains a striking prediction of the QED, that is, the light-by-light scattering arising from the interaction of photons with virtual electron-positron pairs. It should be further noted that one of the most interesting physical consequences of the HE result is vacuum birefringence. In other words, when the quantum vacuum is stressed by external electromagnetic fields behaves like a birefringent material medium. Let us also mention that this physical effect has been emphasized from different viewpoints [7–12]. Nevertheless, this optical phenomenon has not yet been confirmed [13–16].

It is pertinent to recall here that recently the ATLAS and CMS collaborations at the Large Hadron Collider (LHC) have reported on the high energy gamma-gamma pair emission from virtual gamma-gamma scattering in ultraperipheral Pb-Pb collisions [17, 18]. It is worth noting that in these results there is no modification of the optical properties of the vacuum [19]. Besides the coming of the laser facilities also provide us opportunities to probe quantum vacuum non-linearities [20, 21]. Among various experiments, a promising proposal is the DeLLight project [22], using the induced change in the refractive index due to non-linear electrodynamics.

In the spirit of the foregoing remarks, we have considered the physical effects presented by different models of (3 + 1)-D non-linear electrodynamics in the vacuum [23–25]. This has led us to fresh insights on quantum vacuum non-linearities in different contexts. For instance, in the generalized Born-Infeld and Logarithmic electrodynamics, our analysis reveals that the field energy of a point-like charge is finite, apart from displaying the vacuum birefringence phenomenon. Additionally, we have examined the lowest-order modifications of the static potential within the framework of the gauge-invariant but path-dependent variables formalism, which is an alternative to the Wilson loop approach [26].

Moreover, recently a novel nonlinear modification of Maxwell’s electrodynamics that preserves all its symmetries including the electric-magnetic duality and the conformal invariance has been proposed [27–29], which is called ModMax electrodynamics. Certainly, the interest in studying nonlinear electrodynamics is mainly due to its potentially significant contributions to light-by-light scattering and in the description of certain materials in condensed matter physics [30].

From the preceding considerations and given the inter-
est and importance related to photon-photon interaction physics, the purpose of this work is to further elaborate on the phenomenological consequences presented by the ModMax electrodynamics. More specifically, we will focus our attention on the birefringence, dispersion relations, as well as the computation of the static potential along the lines of the refs. [21,22,31,32]. We present the results of the ModMax ED in view of a classical field theory as the conservation law and the energy-momentum tensor. Using the result of the energy-momentum tensor, we obtain the spatial components, that are known as the ModMax stress tensor, and posteriorly, the general angular momentum tensor and the spin vector. The properties of a material medium ruled by the ModMax ED are studied in an external and uniform electromagnetic background.

The particular case in which the electric background is perpendicular to the magnetic one is of our interest because the CP-symmetry of the theory is preserved under this condition. Therefore, we study the wave propagation effects in the linearized ModMax ED, in the presence of perpendicular electric and magnetic background fields. We obtain the permittivity and permeability tensors, the dispersion relations, the refractive index and the group velocity of the free wave as function of the ModMax parameter, and of the external electric and magnetic fields. The phenomenon of birefringence in the electromagnetic background is studied in relation to the external magnetic field. Some particular cases of these results are discussed and the results of the recent literature in ModMax are recovered. In addition, all the results of the Maxwell ED are recovered when the ModMax parameter goes to zero.

In the final part of the paper, we introduce the ModMax ED coupled to an axion scalar field through an axion-photon (ModMax) coupling. In this case, we take the electric background null, and just consider the axionic ModMax ED in the presence of a magnetic background field. In this manner, we study the confinement properties of this new model by computing the interaction energy for a pair of static probe charges within the framework of gauge-invariant but path dependent variational energy for a pair of static probe charges within the ModMax properties of this new model by computing the interaction energy for a pair of static probe charges within the ModMax ED through the Lagrangian density:

\[ L_{MM} = \cosh \gamma \mathcal{F} + \sinh \gamma \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + J_\mu A^\mu, \]

where \( \mathcal{F} \) and \( \mathcal{G} \) denote the Lorentz and gauge invariants,

\[ \mathcal{F} = -\frac{1}{4} F_{\mu\nu}^2 = \frac{1}{2} (E^2 - B^2), \]

\[ \mathcal{G} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = E \cdot B, \]

with \( \gamma \) being a real parameter, that satisfies the condition \( \gamma \geq 0 \) to insure the causality and unitarity of the model, and \( J_\mu = (\rho, J) \) is a classical source of charge and current densities. The \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = (-E^t, -\epsilon^{ijk}B^k) \) is the skew-symmetric field strength tensor, and \( \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2 = (-B^t, \epsilon^{ijk}E^k) \) corresponds to the dual tensor. The usual Maxwell electrodynamics is recovered when \( \gamma \rightarrow 0 \).

The action principle yields the field equations

\[ \partial_\mu \left[ \cosh \gamma F^{\mu\nu} + \sinh \gamma \frac{F^{\mu\nu} \mathcal{F} + \tilde{F}^{\mu\nu} \mathcal{G}}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}} \right] = J^\nu, \]

and the dual tensor satisfies the Bianchi identity \( \partial_\mu F^{\mu\nu} = 0 \). The charge conservation obeys the continuity equation \( \partial_\mu J^\mu = 0 \), as in the Maxwell ED. The equations in terms of the electric and magnetic fields can be written as:

\[ \nabla \cdot D = \rho, \quad \nabla \times E + \partial_t B = 0, \]

\[ \nabla \cdot B = 0, \quad \nabla \times H - \partial_t D = J, \]

where \( D \) and \( H \) are defined by

\[ D = \cosh \gamma E + \sinh \gamma \frac{\mathcal{F} E + \mathcal{G} B}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}}, \]

\[ H = \cosh \gamma B + \sinh \gamma \frac{\mathcal{F} B - \mathcal{G} E}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}}. \]

Multiplying the Bianchi identity by \( F^{\mu\nu} \), and using the eq. (3), we arrive at the conservation law

\[ \partial_\mu \Theta^{\mu\nu} = J_\rho F^{\rho\nu}, \]

II. THE MODIFIED MAXWELL ELECTRODYNAMICS

We start off the classical description of the ModMax ED through the Lagrangian density:

\[ L_{MM} = \cosh \gamma \mathcal{F} + \sinh \gamma \sqrt{\mathcal{F}^2 + \mathcal{G}^2} + J_\mu A^\mu, \]

where \( \mathcal{F} \) and \( \mathcal{G} \) denote the Lorentz and gauge invariants,

\[ \mathcal{F} = -\frac{1}{4} F_{\mu\nu}^2 = \frac{1}{2} (E^2 - B^2), \]

\[ \mathcal{G} = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = E \cdot B, \]

with \( \gamma \) being a real parameter, that satisfies the condition \( \gamma \geq 0 \) to insure the causality and unitarity of the model, and \( J_\mu = (\rho, J) \) is a classical source of charge and current densities. The \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = (-E^t, -\epsilon^{ijk}B^k) \) is the skew-symmetric field strength tensor, and \( \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2 = (-B^t, \epsilon^{ijk}E^k) \) corresponds to the dual tensor. The usual Maxwell electrodynamics is recovered when \( \gamma \rightarrow 0 \).

The action principle yields the field equations

\[ \partial_\mu \left[ \cosh \gamma F^{\mu\nu} + \sinh \gamma \frac{F^{\mu\nu} \mathcal{F} + \tilde{F}^{\mu\nu} \mathcal{G}}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}} \right] = J^\nu, \]

and the dual tensor satisfies the Bianchi identity \( \partial_\mu F^{\mu\nu} = 0 \). The charge conservation obeys the continuity equation \( \partial_\mu J^\mu = 0 \), as in the Maxwell ED. The equations in terms of the electric and magnetic fields can be written as:

\[ \nabla \cdot D = \rho, \quad \nabla \times E + \partial_t B = 0, \]

\[ \nabla \cdot B = 0, \quad \nabla \times H - \partial_t D = J, \]

where \( D \) and \( H \) are defined by

\[ D = \cosh \gamma E + \sinh \gamma \frac{\mathcal{F} E + \mathcal{G} B}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}}, \]

\[ H = \cosh \gamma B + \sinh \gamma \frac{\mathcal{F} B - \mathcal{G} E}{\sqrt{\mathcal{F}^2 + \mathcal{G}^2}}. \]

Multiplying the Bianchi identity by \( F^{\mu\nu} \), and using the eq. (3), we arrive at the conservation law

\[ \partial_\mu \Theta^{\mu\nu} = J_\rho F^{\rho\nu}, \]
where the energy-momentum tensor of the ModMax ED is given by

\[
\Theta^\mu\nu = (F^\mu\rho F_\rho)^\nu - \eta^\mu\nu F
\]

\[\times \left[ \cosh \gamma + \frac{\sinh \gamma F}{\sqrt{F^2 + G^2}} \right]. \quad (7)
\]

This tensor have the properties of the index symmetry (\(\mu \leftrightarrow \nu\)) and gauge invariance. In the case of the ModMax in a free space (with no charge and current densities), the expression \(\Theta^{00}\) satisfies immediately the conservation law \(\partial_\mu \Theta^{\mu0} = 0\), where the \(\Theta^{00}\) and \(\Theta^{0i}\) components denote the conserved energy and momentum densities stored in the EM fields, namely,

\[
\Theta^{00} = \frac{1}{2} (E^2 + B^2) \left[ \cosh \gamma + \frac{\sinh \gamma (E^2 - B^2)}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right],
\]

\[
\Theta^{0i} = (E \times B)^i \left[ \cosh \gamma + \frac{\sinh \gamma (E^2 - B^2)}{\sqrt{(E^2 - B^2)^2 + 4(E \cdot B)^2}} \right]. \quad (8)
\]

Notice that both components in eq. (8) are not defined if the electric and magnetic fields, for example, satisfies the relation \(E^2 - B^2 = E \cdot B = 0\) in a particular case. To remove this singularity, it is convenient to define the Legendre transformation:

\[
\mathcal{H}(D, B) = E \cdot D - \mathcal{L}_{MM}(E, B),
\]

and using the ModMax Lagrangian (11), the hamiltonian density is

\[
\mathcal{H}_{MM} = \frac{1}{2} \cosh \gamma (D^2 + B^2)
\]

\[-\frac{1}{2} \sinh \gamma \sqrt{(D^2 - B^2)^2 + 4(D \cdot B)^2}, \quad (10)
\]

which is positive-definite if the ModMax parameter obeys the condition

\[
\tanh \gamma < \sqrt{\frac{(D^2 + B^2)^2}{(D^2 - B^2)^2 + 4(D \cdot B)^2}}. \quad (11)
\]

The quantity (10) is well defined for any \(D\) and \(B\). Furthermore, it is manifestly invariant under the duality transformations \(D' = iB' = e^{i\alpha} (D + iB)\), with \(\alpha\) being a real parameter. The Poynting vector from the \(\Theta^{0i}\)-component in terms of \(D\) and \(B\) fields is

\[
S_P = D \times B. \quad (12)
\]

The spatial component \(\nu = j\) in the conservation law \(\Theta^{0j}\) yields the expression

\[
\nabla \cdot \mathbf{T} - \partial_j S_P = f_L,
\]

where \(f_L = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}\) is the Lorentz force density, and the components of the Maxwell’s stress tensor \(\mathbf{T}\) are given by

\[
T^{ij} = \left[ \cosh \gamma + \frac{\sinh \gamma F}{\sqrt{F^2 + G^2}} \right] \times
\]

\[\times \left[ E^i E^j + B^i B^j - \delta^{ij} \frac{1}{2} (E^2 + B^2) \right]. \quad (14)
\]

The Lorentz force has the same definition of the Maxwell ED, namely, it is obtained by integrating the Lorentz force density in a region of the space.

Above, in eq. (7), we give the expression for the symmetric and gauge-invariant energy-momentum tensor of the ModMax ED. Now, to present the angular momentum tensor, we adopt the canonical approach and split it into an orbital (OAM) and spin (SAM) components as follows:

\[
M_{\mu \alpha \beta} = x_\alpha T^\mu_{\beta} - x_\beta T^\mu_{\alpha} + S^\mu_{\alpha \beta}, \quad (15)
\]

where \(T^\mu_{\beta}\) stands for the canonical energy-momentum tensor, whereas \(S^\mu_{\alpha \beta}\) expresses the spin piece:

\[
T^\mu_{\beta} = \left[ \cosh \gamma + \frac{\sinh \gamma F}{\sqrt{F^2 + G^2}} \right] \partial_\beta A_\lambda - \delta^\mu_\beta \mathcal{L}_{MM}.
\]

\[
S^\mu_{\alpha \beta} = \frac{\sinh \gamma G}{\sqrt{F^2 + G^2}} (A_\beta F^\mu_{\alpha} - F^\mu_{\alpha} A_\beta).
\]

The correspondent spin vector is read below

\[
\mathbf{S}_{spin} = \int d^3x \left\{ \left[ \cosh \gamma + \frac{\sinh \gamma F}{\sqrt{F^2 + G^2}} \right] (\mathbf{E} \times \mathbf{A}) + \frac{\sinh \gamma G}{\sqrt{F^2 + G^2}} (\mathbf{B} \times \mathbf{A}) \right\}, \quad (18)
\]

which can be recast as

\[
\mathbf{S}_{spin} = \int d^3x \mathbf{D} \times \mathbf{A}. \quad (19)
\]

Both the OAM and SAM components are not gauge-invariant and a physically unambiguous splitting into these two pieces is still controversial and object of debate. Actually, the formal separation into orbital and spin parts of an optical field first appeared in a paper by J. Humblet [33]. In 1932, C. G. Darwin pioneered the investigation of the angular momentum tensor of electromagnetic radiation, though he did not exploit the OAM-SAM splitting [34]. For an updated discussion, we address the interested readers to the papers [35, 36], where alternative decompositions into OAM and SAM components are presented.

### III. THE LINEARIZED MODMAX ELECTRODYNAMICS IN AN EM BACKGROUND

The ModMax ED can be linearized by expanding the \(A^\mu\)-potential around a uniform and constant EM field, as \(A_\mu = a_\mu + A_{0\mu}\), where \(a_\mu\) is interpreted as the photon field, and \(A_{0\mu}\) is the potential associated with the
The propagating and background fields, respectively. Therefore, by expanding the Lagrangian \( \mathcal{L}_\text{MM}^{(2)} \) around this background field, we obtain
\[
\mathcal{L}_\text{MM}^{(2)} = \frac{1}{4} c_1 f_{\mu\nu} f_{\mu\nu} - \frac{1}{4} c_2 \bar{f}_{\mu\nu} \bar{f}_{\mu\nu} - \frac{1}{2} G_{0\mu\nu} f_{\mu\nu} + \frac{1}{8} Q_{0\mu\nu\sigma\lambda} f_{\mu\nu} f_{\sigma\lambda},
\]
(20)
where \( G_{0\mu\nu} = c_1 F_{0\mu\nu} + c_2 \bar{F}_{0\mu\nu} \) and \( Q_{0\mu\nu\sigma\lambda} = d_1 F_{0\mu\nu} F_{0\sigma\lambda} + d_2 \bar{F}_{0\mu\nu} \bar{F}_{0\sigma\lambda} + d_3 F_{0\mu\nu} \bar{F}_{0\sigma\lambda} + d_3 \bar{F}_{0\mu\nu} F_{0\sigma\lambda} \) are tensors that depend on the components of the EM background fields. The \( \bar{f}_{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} f_{\alpha\beta}/2 = (-b_i, \epsilon^{ijk} e_{k}) \) and \( \bar{F}_{0\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{0\alpha\beta}/2 = (-B_0^i, \epsilon^{ijk} E_{k}) \) are the dual strength tensors of the propagating and background fields, respectively. The \( f_{\mu\nu} \) satisfies the Bianchi identity \( \partial_{\mu} f_{\nu\lambda} = 0 \) in that emerges the homogeneous field equations
\[
\nabla \cdot \mathbf{b} = 0 \quad \nabla \times \mathbf{e} = -\partial_t \mathbf{b}.
\]
(21)
The coefficients \( c_1, c_2, d_1, d_2 \) and \( d_3 \) of this expansion are defined by
\[
c_1 = \left. \frac{\partial \mathcal{L}_\text{MM}^{(2)}}{\partial F_{\mu\nu}^{0\mu\nu}} \right|_{\mathbf{E}_0, \mathbf{B}_0}, \quad c_2 = \left. \frac{\partial \mathcal{L}_\text{MM}^{(2)}}{\partial \bar{F}_{\mu\nu}^{0\mu\nu}} \right|_{\mathbf{E}_0, \mathbf{B}_0},
\]
\[
d_1 = \left. \frac{\partial^2 \mathcal{L}_\text{MM}^{(2)}}{\partial F_{\mu\nu}^{0\mu\nu} \partial F_{\delta\epsilon}^{0\delta\epsilon}} \right|_{\mathbf{E}_0, \mathbf{B}_0}, \quad d_2 = \left. \frac{\partial^2 \mathcal{L}_\text{MM}^{(2)}}{\partial \bar{F}_{\mu\nu}^{0\mu\nu} \partial \bar{F}_{\delta\epsilon}^{0\delta\epsilon}} \right|_{\mathbf{E}_0, \mathbf{B}_0},
\]
\[
d_3 = \left. \frac{\partial^2 \mathcal{L}_\text{MM}^{(2)}}{\partial \bar{G}_{\mu\nu}^{0\mu\nu} \partial \bar{G}_{\delta\epsilon}^{0\delta\epsilon}} \right|_{\mathbf{E}_0, \mathbf{B}_0},
\]
(22)
that also depend only on the EM background. Substituting the ModMax lagrangian \( \mathcal{L}_\text{MM}^{(2)} \), we obtain:
\[
c_1 = \cosh \gamma + \sinh \gamma \frac{\mathcal{F}_0}{\sqrt{F_0^2 + G_0^2}}, \quad c_2 = \sinh \gamma \frac{\mathcal{G}_0}{\sqrt{F_0^2 + G_0^2}},
\]
\[
d_1 = \frac{\sinh \gamma}{\sqrt{F_0^2 + G_0^2}} \frac{\mathcal{F}_0}{\sqrt{F_0^2 + G_0^2}}, \quad d_2 = \frac{\sinh \gamma}{\sqrt{F_0^2 + G_0^2}} \frac{\mathcal{G}_0}{\sqrt{F_0^2 + G_0^2}},
\]
\[
d_3 = \frac{\sinh \gamma}{\sqrt{F_0^2 + G_0^2}} \frac{\mathcal{F}_0 \mathcal{G}_0}{\sqrt{F_0^2 + G_0^2}},
\]
(23)
in which \( \mathcal{F}_0 = (\mathbf{E}_0^2 - \mathbf{B}_0^2)/2 \) and \( \mathcal{G}_0 = \mathbf{E}_0 \cdot \mathbf{B}_0 \) are the gauge and Lorentz invariants associated with the electric \( \mathbf{E}_0 \) and magnetic \( \mathbf{B}_0 \) fields.

The action principle applied to the lagrangian \( \mathcal{L}_\text{MM}^{(2)} \) yields the field equations:
\[
\partial_{\mu} \left[ c_1 f_{\mu\nu} + c_2 \bar{f}_{\mu\nu} - \frac{1}{2} Q_{0\mu\nu\sigma\lambda} f^{\sigma\lambda} \right] = 0.
\]
(24)

The usual Maxwell ED is recovered when the EM background fields are turn-off and the ModMax parameter goes to zero. The equations \( \partial_{\mu} \mathbf{D} = \epsilon_{\mu\nu\rho\sigma} \mathbf{E}_{\nu\rho} \) can be written into the same form of eqs. \( (16a) \) and \( (16b) \), with the auxiliary fields
\[
D_i = \varepsilon_{ij} e_j + \sigma_{ij} b_j, \quad H_i = -\sigma_{ij} e_j + (\mu_{ij})^{-1} b_j,
\]
(25a)
(25b)
where the permittivity symmetric tensor \( \varepsilon_{ij} \), the inverse of the permeability symmetric tensor \( (\mu_{ij})^{-1} \), and \( \sigma_{ij} \) are given in terms of the electric and magnetic background components, as follows
\[
\varepsilon_{ij} = c_1 \delta_{ij} + d_1 E_{0i} E_{0j} + d_2 B_{0i} B_{0j} + d_3 E_{0i} B_{0j} + d_3 B_{0i} E_{0i},
\]
(26a)

\[
\sigma_{ij} = c_2 \delta_{ij} - d_1 E_{0i} B_{0j} + d_2 B_{0i} E_{0j} - d_2 B_{0i} B_{0j} + d_3 E_{0i} E_{0j} - d_3 B_{0i} B_{0j},
\]
(26b)

\[
(\mu_{ij})^{-1} = c_1 \delta_{ij} - d_1 B_{0i} B_{0j} - d_2 E_{0i} E_{0j} - d_3 B_{0i} B_{0j} - d_3 E_{0i} E_{0j},
\]
(26c)

Notice that the case in which \( c_2 \neq 0 \) and \( d_2 \neq 0 \), the CP-symmetry is violated in the linearized theory. Thereby, we just consider the situation in which the electric \( E_0 \) and magnetic \( B_0 \) fields are perpendicular vectors between themselves.

## IV. Wave Propagation for Perpendicular External Fields

If we consider the case of \( E_0 \) perpendicular to \( B_0 \), the second Lorentz and gauge invariant \( \mathcal{G}_0 = 0 \), and the coefficients from eq. \( (22) \) are given by
\[
c_1 = \cosh \gamma + \sinh \gamma \frac{\mathcal{F}_0}{\sqrt{F_0^2 + G_0^2}}, \quad c_2 = \sinh \gamma \frac{\mathcal{G}_0}{\sqrt{F_0^2 + G_0^2}},
\]
\[
c_2 = c_1 = d_3 = 0,
\]
(27)
in which \( \text{sgn}(\mathcal{F}_0) \) is the signal function of \( \mathcal{F}_0 \), where \( \text{sgn}(\mathcal{F}_0) = 1 \) if \( |E_0| > |B_0| \), and \( \text{sgn}(\mathcal{F}_0) = -1 \) if \( |B_0| > |E_0| \). Under these conditions, the permittivity tensor \( \varepsilon_{ij} \) and the permeability tensor related to eq. \( (20b) \) are read below:
\[
\varepsilon_{ij} = c_1 \delta_{ij} + d_2 B_{0i} B_{0j}, \quad \mu_{ij} = \frac{1}{c_1} \left[ \delta_{ij} + d_E E_{0i} E_{0j} \right],
\]
(28a)
(28b)
where the coefficient \( d_E \) is
\[
d_E = \frac{d_2}{c_1} = \frac{\tanh(\gamma)}{|\mathcal{F}_0| + \tanh(\gamma) |\mathcal{F}_0|},
\]
(29)
in which
\[
d_E = \frac{1 - e^{-2\gamma}}{E_0^2 - B_0^2} \quad \text{if} \quad |E_0| > |B_0|,
\]
(30a)

\[
d_E = \frac{e^{2\gamma} - 1}{B_0^2 - E_0^2} \quad \text{if} \quad |B_0| > |E_0|,
\]
(30b)
The eigenvalues of the matrices \(\varepsilon_{ij}\) and \(\mu_{ij}\) are given by:

\[
\begin{align*}
\lambda_{1x} &= \lambda_{2x} = c_1, \\
\lambda_{3x} &= c_1 + d_2 B_0^2, \\
\lambda_{1\mu} &= \lambda_{2\mu} = \frac{1}{c_1}, \\
\lambda_{3\mu} &= \frac{1}{c_1 - d_2 E_0^2},
\end{align*}
\tag{31a}
\]

\[
\begin{align*}
\lambda_{1i} &= \lambda_{2i} = 1, \\
\lambda_{3i} &= 1 - d_2 E_0^2, \\
\lambda_{1j} &= \lambda_{2j} = 1, \\
\lambda_{3j} &= 1 - d_2 E_0^2.
\end{align*}
\tag{31b}
\]

in which the electric permittivity and magnetic permeability are positive-definite if the eigenvalues satisfies the conditions \(c_1 > 0, 1 + d_2 B_0^2 > 0,\) and \(1 - d_2 E_0^2 > 0.\)

Substituting the plane wave solutions, \(e(x, t) = e_0 e^{i(k \cdot x - \omega t)}\) and \(b(x, t) = b_0 e^{i(k \cdot x - \omega t)}\) in eq. (24), the wave equation for the components of the electric amplitude \(e_{0i}\) is

\[M_{ij} e_{0j} = 0.\tag{32}\]

The matrix elements \(M_{ij}\) have the form

\[M_{ij} = (\omega^2 - k^2) \delta_{ij} + k_i k_j + u_i v_j,\tag{33}\]

where the vectors are defined by \(u_i = \omega B_{0i} - (k \times E_0)_i\) and \(v_i = d_2 E_{0i}.\) The determinant of the \(M\)-matrix is

\[
\det M = (\omega^2 - k^2) \left(\omega^2 - k^2 + u \cdot v - (u \cdot k)(v \cdot k)\right),
\]

where \(R = d_E (k \times E_0)^2 - k^2 - d_E (B_0 \cdot k)^2.\) The roots of eq. (33) are \(\omega = 0,\) and the non-trivial solutions:

\[
\begin{align*}
\omega^{(-)}(k) &= \frac{d_E B_0 \cdot (k \times E_0) - \sqrt{k^2 - d_E (k \times E_0)^2 - d_E^2 (E_0^2 - B_0^2) (k \cdot B_0)^2 + d_E (k \cdot B_0)^2 + d_E B_0^2 k^2}}{1 + d_E B_0^2}, \tag{36a} \\
\omega^{(+)}(k) &= \frac{d_E B_0 \cdot (k \times E_0) + \sqrt{k^2 - d_E (k \times E_0)^2 - d_E^2 (E_0^2 - B_0^2) (k \cdot B_0)^2 + d_E (k \cdot B_0)^2 + d_E B_0^2 k^2}}{1 + d_E B_0^2}. \tag{36b}
\end{align*}
\]

The limit of usual Maxwell ED (\(\gamma \to 0\)) in eqs. (36a) and (36b) yields the roots \(\lim_{\gamma \to 0} \omega^{(\pm)}(k) = \pm |k|,\) where the photon dispersion relation is recovered. Thereby, we choose the \(\omega^{(+)}\) frequency for the analysis of the refractive index of the medium. Notice that the refractive index \(n^{(+)} = |k|/\omega^{(+)}(k)\) correspondent to \(\omega^{(+)}\) does not depend on the wavelength \(\lambda = 2\pi/|k|.)\ It is important to remark that the frequencies in eqs. (36a) and (36b) has an asymmetry in which \(|\omega^{(-)}(k)| \neq |\omega^{(+)}(k)|.\) The regime of a strong magnetic field, \(i.e., |B_0| \gg |E_0|,\) the previous frequencies are reduced to

\[\omega^{(\pm)}(k) \approx \pm |k| e^{-\gamma} \sqrt{1 + 2 e^{\gamma} \sinh(\gamma) \cos^2 \theta}, \tag{37}\]

where \(\cos \theta = k \cdot B_0.\)

In this condition, the frequency depends on the direction of the magnetic field with the \(k\)-wave vector, and consequently, the refractive index also depends on the \(\theta\)-angle. In addition, the frequency (37) does not depend on the magnetic field magnitude when \(|B_0| \to \infty.\) We also point out that the frequencies (36a) and (36b) are degenerate at \(\omega^{(+)} = \omega^{(-)} = d_E B_0 \cdot (k \times E_0)\) if the ModMax parameter and the EM background field satisfy the condition

\[
1 + d_E (k \cdot B_0)^2 + d_E B_0^2 = d_E (k \times E_0)^2 + d_E (k \cdot B_0)^2 + d_E B_0^2 k^2.
\]

If the EM background field from the condition (33) satisfies the inequality

\[
1 + d_E (k \cdot B_0)^2 + d_E B_0^2 > d_E (k \times E_0)^2 + d_E (k \cdot B_0)^2 + d_E B_0^2 k^2,
\]

both the dispersion relations (36a) and (36b) are real. Otherwise, if the expression (39) has the inequality signal \(\langle\),\ the dispersion relations have imaginary parts. The refractive index associated with the \(\omega^{(+)}\)-frequency is defined by

\[
n = \frac{1 + d_E B_0^2}{d_E B_0 \cdot (k \times E_0) + \sqrt{1 - d_E (k \times E_0)^2 - d_E^2 (E_0^2 - B_0^2) (k \cdot B_0)^2 + d_E (k \cdot B_0)^2 + d_E B_0^2}}.
\tag{40}\]

where the condition (39) can be imposed to obtain a real refractive index. The inequality (39) with the opposite
signal \(<\) implies into a dichroism effect in the refractive index \(40\). In the regime of a strong magnetic field \(\|B_0\| \gg \|E_0\|\), the previous refractive index leads to

\[
n(\theta) \simeq \frac{e^\gamma}{\sqrt{1 + \left( e^{2\gamma} - 1 \right) \cos^2 \theta}},
\]

(41)

that depends only on the \(\theta\)-angle emerged in \(37\).

The correspondent group velocity associated with the polynomial equation \(36\) is

\[
v_g = \frac{\omega}{2\omega^2 - k^2 + d_E \omega B_0 - k \times E_0} \cdot \left( 1 - d_E B_0^2 \right) \kappa + d_E \left( E_0 \cdot \kappa \right) E_0 + d_E \left( B_0 \cdot \kappa \right) B_0 + d_E \omega \left( E_0 \times B_0 \right) \right] \]

\[
v_g = \frac{\omega}{2\omega^2 - k^2 + d_E \omega \left( B_0 \cdot \kappa \right) B_0 - k \times E_0} \cdot d_E \omega \left( B_0 - k \times E_0 \right) - d_E \left( B_0 \cdot \kappa \right) k^2 \]

(42)

For a strong magnetic field, the group velocity \(46\) decays with the ModMax parameter and propagates only on the \(k\)-direction : \(v_g \simeq e^{-\gamma} k (B_0 \gg E_0)\), that confirms the result \(43\), when \(\theta = \pi/2\).

V. BIREFRINGENCE IN PRESENCE OF ELECTRIC AND MAGNETIC BACKGROUNDS

The phenomenon of birefringence analysis in the ModMax ED starts in an electromagnetic background field in which we assume the variation of the refractive index in relation to the magnetic background field. We consider the external magnetic field on the \(z\)-direction and the electric field on the \(y\)-direction, i.e. \(B_0 = B_0 \hat{z}\) and \(E_0 = E_0 \hat{y}\), respectively. Initially, the first situation is with the plane wave solution for the propagate electric field into the form \(e(x, t) = e_{03} \hat{z} e^{i(kx - \omega t)}\), whose propagation direction is \(k = k \hat{x}\) and the wave amplitude is parallel to the magnetic background. Under these conditions, the wave equation \(32\) yields the parallel refractive index :

\[
n_\parallel = \sqrt{\mu_{22}(E_0, B_0) \varepsilon_{33}(E_0, B_0)} = \sqrt{\frac{1 + d_E B_0^2}{1 - d_E E_0^2}}.
\]

(47)

In the second situation, the plane wave solution has the amplitude perpendicular to the magnetic background, \(e(x, t) = e_{02} \hat{y} e^{i(kx - \omega t)}\). In this case, the perpendicular refractive index is

\[
n_\perp = \sqrt{\mu_{33}(E_0, B_0) \varepsilon_{22}(E_0, B_0)} = 1.
\]

(48)

The birefringence is defined by the difference of the refractive indices :

\[
\Delta n = n_\parallel - n_\perp = \left| \sqrt{\frac{1 + d_E B_0^2}{1 - d_E E_0^2}} - 1 \right|.
\]

(49)

In the limit \(\gamma \to 0\), the birefringence disappears in eq. \(49\), in which \(\Delta n = 0\). Using the definitions of \(d_E\) in
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VI. INTERACTION ENERGY FOR AN AXIONIC MODMAX ELECTRODYNAMICS UNDER A UNIFORM MAGNETIC FIELD

We shall now discuss the interaction energy between static point-like sources for an axionic ModMax electrodynamics under a uniform magnetic field, along the lines of refs. 

We choose the values for the $\gamma$-ModMax parameter: $\gamma = 0.1$ (black line), $\gamma = 0.5$ (blue line) and $\gamma = 1.0$ (red line) in both cases of $E_0 > B_0$ and $B_0 > E_0$. The birefringence manifests itself in the region below the curves in the fig. 1. Under an intense magnetic background, $B_0 \gg E_0$ (or when $E_0 \to 0$) and $\gamma \ll 1$, the first solution in eq. (50a) yields the same result $\Delta n_1 \simeq \gamma$. The curves go to infinity when $x \to e^\gamma$ (left panel) and $x \to e^{-\gamma}$ (right panel).

where we define the D’Alembertian operator $\Box \equiv \partial_\mu \partial^\mu$.

Since we are interested in estimating the lowest-order interaction energy, we will linearize the above effective theory following the procedure that led to the equation (20). Thus, in the case of a pure magnetic background, we make $E = 0$, in which the effective Lagrangian density simplifies to

$$
L = -\frac{1}{4} e^{-\gamma} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} e^{-\gamma} F_{\mu\nu} f^{\mu\nu} + \frac{1}{2} \sinh (\gamma) B_0^2 \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} + \frac{g_{a\gamma\gamma}^2}{4} \tilde{F}_{\alpha\beta} \tilde{F}_{\alpha\beta} \frac{1}{(\Box + m_a^2)} \tilde{F}_{\mu\nu} f^{\mu\nu}
$$

(54)

In passing, we note that the effective model described by the Lagrangian (54) is a theory with non-local time derivative. However, as we have already explained in refs. 

By this hand, the canonical quantization of this effective theory from the Hamiltonian analysis point of view is straightforward. The canonical momenta reads

$$
\Pi^\mu = -e^{-\gamma} f^{0\mu} - e^{-\gamma} F_B^{0\mu} + \frac{\sinh (\gamma)}{B_0^2} \tilde{F}_B^\nu \tilde{F}_{\nu\mu} + \frac{g_{a\gamma\gamma}^2}{2} \tilde{F}_B^\nu f^{\nu\mu}
$$

(55)

which produces the usual primary constraint

$$
\Pi^0 = 0,
$$

(56)

and

$$
\Pi_i = -\left\{ e^{-\gamma} \delta_{ij} + 2 B_i B_j \left[ \frac{\sinh (\gamma)}{B_0^2} + \frac{g_{a\gamma\gamma}^2}{(\Box + m_a^2)} \right] \right\} e_j,
$$

(57)

Let us also mention here that the electric field due to the fluctuation takes the form

$$
e_i = \frac{1}{u \det D} \left( \frac{1}{\Omega^2} B_i B_j \right) \Pi_j,
$$

(58)

where $u = e^{-\gamma}$ and $\det D = 1 + \frac{B_i^2}{\Omega^2}$, whereas

$$
\frac{1}{\Omega^2} = 2 e^{-\gamma} \left[ \frac{\sinh (\gamma)}{B_0^2} + \frac{g_{a\gamma\gamma}^2}{(\Box + m_a^2)} \right].
$$

(59)

Recalling again that $\mathbf{B}$ represents the external (background) magnetic field around which the $a^\mu$-field fluctuates.

The canonical Hamiltonian can be worked as usual and is given by

$$
H_C = \int d^3x \left[ \Pi_i \partial_0 a_i + \frac{\Pi^2}{2u} + \frac{1}{2} u b^2 + u \mathbf{B} \cdot \mathbf{b} \right] - \int d^3x \frac{1}{2u \Omega^2 \det D} (\mathbf{B} \cdot \mathbf{b})^2.
$$

(60)
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Time conserving the primary constraint, $\Pi_0$, immediately yields the secondary constraint, $\Gamma_1 \equiv \partial_i \Pi' = 0$, which is the Gauss constraint, and together displays the first-class structure of the theory. The extended Hamiltonian that generates translations in time is now found to be

$$H = H_C + \int d^3x \left[ c_0 (x) \Pi_0 (x) + c_1 (x) \Gamma_1 (x) \right],$$

where $c_0 (x)$ and $c_1 (x)$ are the Lagrange multipliers. As before, neither $a^0$ nor $\Pi^0$ are of interest in describing the system and may be discarded of the theory. Thus we are left with the following expression for the Hamiltonian

$$H = \int d^3x \left[ c(x) \partial_i \Pi^i + \frac{\Pi^2}{2u} - \frac{1}{2u \Omega^2 \det D} (\mathbf{B} \cdot \mathbf{b})^2 \right],$$

we have defined $c(x) = c_1 (x) - a_0 (x)$.

To fix gauge symmetry we adopt the gauge discussed previously $^{26}$, that is,

$$\Gamma_2 (x) \equiv \int dz'' a_\nu (z) \equiv \int d\lambda x^i \partial_i (\lambda x) = 0.$$  \hspace{1cm} (63)

Here $\lambda (0 \leq \lambda \leq 1)$ is the parameter describing the space-like straight path $x^i = \zeta^i + \lambda (x - \zeta)^i$, and $\zeta^i$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\zeta = 0$. With such a choice, the fundamental Dirac bracket is given by

$$\{ a_i (x), \Pi^j (y) \} = \delta_i^j \delta (3) (x - y)$$

$$- \partial_x^i \int_0^1 d\lambda x^j \delta (3) (\lambda x - y).$$  \hspace{1cm} (64)

Next, we recall that $^{31,32}$ the physical states $\Phi$ are gauge-invariant. In that case we consider the stringy gauge-invariant state

$$\Phi \equiv | \Psi (y) \Psi (y') \rangle = | \Psi (y) \exp \left( i q \int_y^{y'} dz'' a_i (z) \right) \Psi (y') | 0 \rangle, \hspace{1cm} (65)$$

where the line integral is along a space-like path on a fixed time slice, $q$ is the fermion charge and $| 0 \rangle$ is the physical vacuum state.

This leads us to the expectation value $\langle H \rangle_\Phi$

$$\langle H \rangle_\Phi = \langle H \rangle_0 + \langle H \rangle_0^{(1)}, \hspace{1cm} (66)$$

where $\langle H \rangle_0 = \langle 0 \| H \| 0 \rangle$, whereas the $\langle H \rangle_0^{(1)}$ term is given by

$$\langle H \rangle_0^{(1)} = - \frac{e^{-\gamma}}{2} \Phi \bigg| \int d^3x \Pi^i \frac{(\nabla^2 - m^2)}{(\nabla^2 - M^2)} \Pi_i \Phi \bigg|, \hspace{1cm} (67)$$

where $M^2 = m^2_0 + g_d \gamma \gamma \mathbf{B}^2 e^{-\gamma}$.

Following our earlier procedure $^{31,32}$, when $g_d \gamma \gamma \rightarrow 0$, the static potential profile for two opposite charges located at $y$ and $y'$ then reads

$$V (L) = - \frac{q^2}{4\pi} e^{-\gamma} \frac{1}{L}, \hspace{1cm} (68)$$

where $L \equiv | y - y' |$ is the distance that separates the two charges. It is also, up to the $e^{-\gamma}$ factor, just the Coulomb potential. This result agrees with that of ref. $^{29}$, and finds here an independent derivation. While in the case $g_d \gamma \gamma \neq 0$, the interaction energy takes the form

$$V (L) = - \frac{q^2}{4\pi} e^{-\gamma} e^{-\frac{ML}{L}} + \frac{q^2 m^2_0}{8\pi} e^{-\gamma} \left[ \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) \right] L, \hspace{1cm} (69)$$

where $\Lambda$ is a cutoff. The next step is to give a physical meaning to the cutoff $\Lambda$. Proceeding in the same way as we did before $^{31,32}$, we recall that our effective model for the electromagnetic field is an effective theory that
arises from the integration over the $a$-field, whose excitation is massive. In this case, the Compton wavelength of this excitation ($\ell = m_a^{-1}$) defines a correlation distance. In view of this situation, we see that physics at distances of the order or lower than $m_a$ must necessarily take into account a microscopic description of the $a$-fields. By this we mean that, if we work with energies of the order or higher than $m_a$, our effective description with the integrated effects of $a$ is no longer sensible. As a consequence of this, we can identify $\Lambda$ with $m_a$. This then implies that the static potential profile takes the form

$$V(L) = -\frac{q^2}{4\pi} e^{-\gamma} \frac{e^{-ML}}{L} + \frac{q^2 m_a^2}{8\pi} e^{-\gamma} \left[ \ln \left( 1 + \frac{m_a^2}{M^2} \right) \right] L. \tag{70}$$

Again, up to the $e^{-\gamma}$ factor, we mention that similar forms of interaction potentials have been reported before from different viewpoints. For example, in the context of the Standard Model with an anomalous triple gauge boson couplings [31], in connection with anomalous photon and $Z$-boson self couplings from the Born-Infeld weak hypercharge action [32], also in a theory of antisymmetric tensor fields that results from the condensation of topological defects [37], and in a Higgs-like model [38].

VII. CONCLUSIONS

In this contribution, we study the modified Maxwell electrodynamics (ModMax ED) and their propagation properties in a uniform electromagnetic (EM) background field. The ModMax lagrangian is expanded up to second order in the propagating fields around an uniform EM field. We obtain the permittivity and permeability tensors, the properties of the wave propagation in the presence of a general EM background. We discuss the results of the dispersion relations, refractive index and the group velocity as functions of the wave propagation direction, of the external electric and magnetic. In the regime of a strong magnetic field, the dispersion relations for plane wave solutions do not depend on the electric and magnetic field magnitude, and it depends only on the ModMax parameter and on the direction in which the magnetic field does with the wave propagation direction. The particular case in which the wave vector, the electric, and the magnetic backgrounds are perpendicular among themselves, the refractive index and the group velocity of the medium decays with the ModMax parameter in a regime of strong magnetic field. The phenomenon of birefringence gives the difference of the refractive index when the wave amplitude is parallel and perpendicular to the external magnetic field. We obtain the region of birefringence in the plots of the fig. (1). In both regimes of strong magnetic field, or strong electric field, the birefringence is approximately given by the ModMax parameter, that confirm the result obtained previously in the ref. [28]. All the results known of the Maxwell ED are recovered when the ModMax parameter goes to zero.

Finally, using the gauge-invariant but path-dependent variables formalism, we have computed the static potential profile for an axionic ModMax electrodynamics under a uniform magnetic field. Once again, we have exploited a correct identification of field degrees of freedom with observable quantities. Interestingly, the static potential profile contains a linear potential leading to the confinement of static charges. As already expressed similar forms of interaction potentials have been reported before from different viewpoints [31] [32] [37] [38].

Having in mind the possible relevance of the ModMax model to describe non-linear electromagnetic effects, we call into question its application to study a number of physical properties of Dirac materials. In the work of Ref. [32], the authors show that the latter may display electromagnetic non-linearities at magnetic fields as low as 1T. We point out that reassessing the inspection of magnetic enhancement of the dielectric constant of insulators and, on the other hand, possible electric modulation of magnetization could be a good path to further investigate the potentialities of ModMax.

A direct contact we might establish between ModMax and electroweak physics could be through the issue of the photon and $Z$-boson self-couplings by associating the weak hypercharge symmetry to a ModMax description. In the work of Ref. [32], we adopt a Born-Infeld description for the weak hypercharge and consider the $Z$-decay channel into three photons to constrain the Born-Infeld parameter. By going along the same lines with ModMax, we could get a bound on the gamma-parameter by considering the $Z$-three photon anomalous vertex and the decay of the $Z$-boson into three photons.

ACKNOWLEDGMENTS

L.P.R. Ospedal is grateful to the CNPq for his postdoctoral fellowship. P. Gaete was partially supported by ANID PIA / APOYO AFB180002 (Chile).

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