Stability transitions for axisymmetric relative equilibria of Euclidean symmetric Hamiltonian systems

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Abstract
In the presence of noncompact symmetry, the stability of relative equilibria under momentum-preserving perturbations does not generally imply robust stability under momentum-changing perturbations. For axisymmetric relative equilibria of Hamiltonian systems with Euclidean symmetry, we investigate different mechanisms of stability: stability by energy–momentum confinement, KAM, and Nekhoroshev stability, and we explain the transitions between them. We apply our results to the Kirchhoff model for the motion of an axisymmetric underwater vehicle, and we numerically study dissipation induced instability of KAM stable relative equilibria for this system.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Relative equilibria of Hamiltonian systems with symmetry are special solutions which are equilibria of the symmetry reduced dynamics. Relative equilibria are called stable if they are stable equilibria for the symmetry reduced dynamics. In other words, in a Hamiltonian system with symmetry group \( \mathcal{G} \), a relative equilibrium is called stable (more precisely \( \mathcal{G} \)-stable \([13, 14, 16]\)) if every time-orbit close to the relative equilibrium stays close to the \( \mathcal{G} \)-orbit of the relative equilibrium. Since the \( \mathcal{G} \)-orbit of the relative equilibrium usually strictly contains its time-orbit, \( \mathcal{G} \)-stability is usually weaker than orbital stability.

The stability of relative equilibria is delicate for Hamiltonian systems with noncompact symmetry. We have established \([12]\) exactly why it is erroneous in general to conclude
stability of relative equilibria under momentum-changing perturbations from stability under momentum-preserving perturbations. In the presence of noncompact symmetry, there is a gap between these. In that gap, energy–momentum confinement fails, meaning that stability under momentum-changing perturbations cannot be established by energy–momentum Lyapunov functions.

Here we investigate the stability of axisymmetric relative equilibria of Hamiltonian systems which admit the Euclidean symmetry \( G = \text{SO}(2) \times \mathbb{R}^3 \times \text{SO}(2) \). Our work bears on some results of Leonard and Marsden [6] concerning a class of relative equilibria in the 12-dimensional Kirchhoff model for an axially symmetric underwater vehicle, in which the vehicle falls and spins. They derive a condition for energy–momentum confinement under arbitrary perturbations ([6], theorem 4.4), and they calculate that, at the boundary of this stability region, a Hamiltonian Hopf bifurcation occurs. We show that energy–momentum confinement under arbitrary perturbations actually occurs in a smaller region, but that, in the intervening gap, stability can be established by KAM methods. We also show that the coincidence of the gap boundary and the Hamiltonian Hopf bifurcation is to be generally expected when the phase space is 12-dimensional.

In the underwater vehicle system, we provide numerical evidence that the KAM stability in the gap is destroyed by small dissipation, whereas the stability by energy–momentum confinement is preserved. In this system, the transition from the energy–momentum region to the gap is spin independent, whereas the transition from the gap to spectral instability, at which the Hopf bifurcation occurs, does depend on spin. All the axisymmetric relative equilibria that are spin-stabilized are in the gap, and their KAM stability is destroyed by small momentum-preserving dissipation. The implication for gyroscopically stabilized devices is startling. In the presence of noncompact symmetry, robust stability may not be achievable by the use of spin: dissipation induced loss of stability of the relative equilibrium will occur even in the absence of dissipation of spin. When the symmetry is noncompact, a general understanding of the kinds of stability which generically occur, and the transitions between them, is necessary for the determination of robust stability criteria.

This paper is structured as follows: we begin in section 2 with a description of axisymmetric relative equilibria of Hamiltonian systems with Euclidean symmetry. We introduce coordinates related to the reduction of the system by its largest abelian subgroup \( K = \mathbb{R}^3 \times \text{SO}(2) \) of \( G \). We obtain a family of Hamiltonian systems parametrized by the corresponding momenta. This lays plain the essential difficulty, because it shows that perturbations to arbitrary momentum are \( \text{SO}(2) \) symmetry-breaking, for an additional \( \text{SO}(2) \) symmetry present in the system which does not commute with the abelian subgroup \( K \). The symmetry-breaking occurs because the additional \( \text{SO}(2) \) symmetry acts on the whole family of Hamiltonian systems, including the parametrizing momenta, and acts on one single Hamiltonian system of the parametrized family only for certain vertical momenta; only those Hamiltonian systems are \( \text{SO}(2) \)-symmetric. In section 3 we study the stability of axisymmetric relative equilibria. In general, we establish the presence of the gap and discuss different mechanisms of stability, and for Hamiltonian systems with 12-dimensional phase space, we establish the Hopf eigenvalue collision at the gap boundary. In section 4 we begin with a brief summary of the Kirchhoff model. We recover the stability criteria of [6] for the falling, spinning relative equilibria and prove stability of the relative equilibria within the gap by verifying the Moser twist condition for the corresponding equilibrium on the reduced space. In section 5 we numerically demonstrate that addition of small dissipation distinguishes energy–momentum from KAM stability, by observing that stability is maintained in the former and destroyed in the latter by the addition of momentum conserving dissipation.
2. Dynamics near axisymmetric relative equilibria

In this section we introduce axisymmetric relative equilibria of Hamiltonian systems, and introduce suitable coordinates near them for use in the stability analysis in section 3. We consider a general context that includes the underwater vehicle example in section 4. Let

\[ \dot{x} = f_H(x) \]  

be a Hamiltonian system defined by an energy \( H: \mathcal{M} \to \mathbb{R} \) on a connected symplectic manifold \( (\mathcal{M}, \omega) \), i.e.

\[ \omega(x)(f_H(x), w) = DH(x)w \quad \text{for all } x \in \mathcal{M}, w \in T_x \mathcal{M}. \]

Let \( e_1, e_2, e_3 \) denote the unit vectors along the \( x, y, z \) axis of \( \mathbb{R}^3 \), and for any \( v \in \mathbb{R}^3 \) denote

\[ v^\wedge = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \]

so that, for example,

\[ \exp(\mathfrak{e}_3 \phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Define the group

\[ \mathcal{G} = \text{SE}(2) \times \mathbb{R} \times \text{SO}(2) = \text{SO}(2) \ltimes \mathbb{R}^3 \rtimes \text{SO}(2), \]

where \( \ltimes \) stands for a semidirect product in that the multiplication for \( g = (\phi, a, \theta) \in \text{SO}(2) \ltimes \mathbb{R}^3 \rtimes \text{SO}(2) \) is

\[ g_1 g_2 = (\phi_1, a_1, \theta_1)(\phi_2, a_2, \theta_2) = (\phi_1 + \phi_2, a_1 + \exp(\mathfrak{e}_3 \phi_1) a_2, \theta_1 + \theta_2). \]

Assume \( \mathcal{G} \) acts symplectically on \( \mathcal{M} \), i.e. \( \omega \) is \( \mathcal{G} \)-invariant, and suppose that \( H \) is \( \mathcal{G} \)-invariant. This implies that the vector field \( f_H \) commutes with \( \mathcal{G} \), as does its flow, i.e. \( f_H \) and its flow are \( \mathcal{G} \)-equivariant. Also, assume that \( \mathcal{G} \) acts properly and faithfully.

In the following, \( \text{SO}(2)_R \) is the copy of \( \text{SO}(2) \) in \( \mathcal{G} \) which lies in the centre of \( \mathcal{G} \), so that \( \theta \in \text{SO}(2)_R \), and \( \text{SO}(2)_L \) is such that \( \phi \in \text{SO}(2)_L \). Here ‘\( R \)’ stands for right and ‘\( L \)’ for left. In the application to axisymmetric underwater vehicles, see section 4 below, the \( \text{SO}(2)_L \times \mathbb{R}^3 \) is related to spatial isotropy, it corresponds to a left multiplication, and it translates and rotates the body in space. The \( \text{SO}(2)_R \) action is related to a material symmetry, it corresponds to a right multiplication and it spins the body around a symmetry axis.

By Noether’s theorem there are \( \dim \mathcal{G} = 5 \) locally defined conserved quantities of (2.1). We assume these exist globally and organize them in the momentum map \( J: \mathcal{M} \to \mathfrak{g}^* \) such that \( \omega(x)(\xi x, w) = DJ_x(x)w \) for all \( x \in \mathcal{M}, w \in T_x \mathcal{M}, \xi \in \mathfrak{g} \). Here \( \mathfrak{g} = \mathfrak{so}(2)_L \times \mathbb{R}^3 \times \mathfrak{so}(2)_R = T_{id} \mathcal{G} \) is the Lie algebra of \( \mathcal{G} \). Denote the components of \( J \) by

\[ J(x) = (J^\phi, J^a, J^\theta)(x), \quad J^\phi(x), J^a(x), J^\theta(x) \in \mathfrak{so}(2)_L^*, \quad J^a(x) \in (\mathbb{R}^3)^*, \]

and the elements of \( \mathfrak{g}^* \) by

\[ \mu = (\mu^\phi, \mu^a, \mu^\theta) \in \mathfrak{g}^* = \mathfrak{so}(2)_L^* \times \mathbb{R}^3 \times \mathfrak{so}(2)_R^*. \]

In the underwater vehicle example of section 4, the component \( J^\phi \) is the linear momentum, \( J^a \) the angular momentum, and \( J^\theta \) the momentum of the spin of the body. As in that system, we will assume that the momentum mapping \( J \) transforms by the coadjoint action on \( \mathfrak{g}^* \), i.e. [8]

\[ J(gx) = (\text{Ad}_g^*)^{-1} J(x) \quad \text{for any } g \in \mathcal{G} \text{ and } x \in \mathcal{M}. \]
Here $\text{Ad}_g^*$ is defined by the condition $(\text{Ad}_g^*\mu)(\xi) = \mu(\text{Ad}_g\xi)$, $\mu \in g^*$, $\xi \in g$, $g \in G$ and $\text{Ad}_g\xi = g\xi g^{-1}$. A standard computation (see, for example, [8]) gives

\begin{align*}
J^0((\phi, a, \theta) x) &= J^0(x) - (\text{exp}(\mathcal{E}_3\phi)a)J^3(x) + (\text{exp}(\mathcal{E}_3\phi)a)J^2(x), \\
J^3((\phi, a, \theta) x) &= J^3(x) \cos \phi - J^2(x) \sin \phi, \\
J^2((\phi, a, \theta) x) &= J^2(x) \cos \phi + J^3(x) \sin \phi, \\
J^1((\phi, a, \theta) x) &= J^1(x), \\
J^0((\phi, a, \theta) x) &= J^0(x).
\end{align*}

(2.4)

2.1. Axisymmetric relative equilibria of Euclidean group actions

Let $x_e$ be a relative equilibrium of (2.1), i.e. let

$$
\xi_e = (\xi^\theta_e, \xi^a_e, \xi^\phi_e) \in \mathfrak{so}(2) \oplus \mathbb{R}^3 \oplus \mathfrak{so}(2) = g,
$$

and suppose $\text{exp}(\xi_e) x_e$ is a solution of the differential equations $\dot{x} = f_H(x)$. We assume $x_e$ is axisymmetric, meaning that its isotropy subgroup $G_{x_e} = \{g \in G : gx_e = x_e\}$ is the diagonal subgroup

$$
\text{SO}(2)_D = \{(\phi, 0, \phi) \in \text{SO}(2) \times \mathbb{R}^3 \times \text{SO}(2)\} \subseteq G.
$$

As is general for Hamiltonian systems with symmetry, if $\eta_e$ is in the isotropy algebra $\mathfrak{g}_{x_e}$, then

$$
(\xi^\theta_e + \eta_e) x_e = \xi_e x_e = f_H(x_e),
$$

because $\eta_e x_e = 0$, hence $\text{exp}(t(\xi_e + \eta_e)) x_e$ is also a solution of the Hamiltonian system. Thus $\xi_e$ is only determined up to addition of elements in

$$
g_{x_e} = \mathfrak{so}(2)_D = \{(\xi^\theta_e, 0, \xi^\theta_e) \in \mathfrak{so}(2) \times \mathbb{R}^3 \times \mathfrak{so}(2)\} \subseteq g,
$$

and so we can choose $\xi^\theta_e = 0$. In the underwater vehicle, $x_e$ is an equilibrium in a frame that comoves with the action of $\text{exp}(\xi_e)$, $\omega^{\text{rot}} = \xi^\phi_e - \xi^a_e = -\xi^a_e$ is the angular velocity, and $\xi^a_e$ is the translational velocity. The momentum value $\mu_e = J(x_e)$ of the relative equilibrium is of the form $\mu_e = (\mu^\theta_e, \mu^a_e, \mu^\phi_e)$ with $\mu^\phi_e \parallel \mathbf{e}_3$, because it is fixed under the action of $\text{SO}(2)_D$ that occurs in the transformation rule (2.4). The momentum map $J^{\text{SO}(2)_D}$ of the action of the symmetry group $\text{SO}(2)_D$ is

$$
J^{\text{SO}(2)_D}(x) = J^0(x) + J^0(x).
$$

It follows from $\xi^\theta_e = 0$ that $x_e$ is a relative equilibrium for the abelian symmetry group

$$
\mathcal{K} = \mathbb{R}^3 \times \text{SO}(2)_R = \{(0, a, \phi) \in \text{SO}(2) \times \mathbb{R}^3 \times \text{SO}(2)\} \subseteq G,
$$

(2.5)

and $x_e$ becomes an equilibrium after reduction by this group. The stability of this equilibrium implies stability of the relative equilibrium, see section 2.2 below. We will study the stability of such an axisymmetric relative equilibrium, which in the underwater vehicle example of section 4, corresponds to a vehicle spinning about its symmetry axis with angular velocity $\omega^{\text{rot}}$, and translating along its symmetry axis with translational velocity $\xi^a_e$.

2.2. Reduction by $\mathcal{K} = \mathbb{R}^3 \times \text{SO}(2)_R$

The following theorem provides the coordinates which we require for the subsequent stability analysis.

**Theorem 2.1.** In a $G$-invariant neighbourhood of an axisymmetric relative equilibrium $x_e$ of (2.1) there are coordinates $x = (a, \theta, v^a, v^\phi, w)$ such that

(a) $J^a = v^a$, $J^\phi = v^\phi$, and the differential equations (2.1) are

\begin{align*}
\dot{a} &= D_v H(v, w), \\
\dot{\theta} &= D_{v^a} H(v, w), \\
\dot{v}^a &= 0, \\
\dot{v}^\phi &= 0, \\
\dot{w} &= \text{JD}_a H(v, w),
\end{align*}

(2.6)

where $v = (v^a, v^\phi)$, $\text{J}$ is the standard symplectic structure matrix on the linear symplectic space $W = \mathbb{R}^{\dim M - 8}$ and $w \in W$. 


(b) The Hamiltonian \( H(v, w) \) is invariant under the action of \( \text{SO}(2)_D \), which takes the form
\[
\begin{align*}
\nu^\theta &\to \exp(\theta \hat{\mathbf{e}}_3)\nu^\theta, \\
v^\theta &\to v^\theta, \\
w &\to R_\phi(v, w),
\end{align*}
\]
i.e. the action on \( w \) generally depends on \( v \) and this equation defines \( R_\phi \).

(c) The coordinates of the axisymmetric relative equilibrium \( x_e \) are
\[
\begin{align*}
a = 0, \quad \theta = 0, \quad v = v_e = (J^\theta(x_e), J^\theta(x_e)), \quad w = 0,
\end{align*}
\]
and \( w = 0 \) is an equilibrium of the \( \dot{w} \)-equation of (2.6) at \( v = v_e \).

The \( a = 0, \theta = 0 \) plane is locally a slice at \( x_e \) through the action of \( K = \mathbb{R}^3 \times \text{SO}(2)_R \) and models the Poisson reduced space \( \mathcal{M}/K \). The symplectic leaves of this Poisson space are given by fixing \( v \). At fixed \( v \), the system
\[
\dot{w} = \mathbb{J}_w H(v, w) \tag{2.7}
\]
is the (Marsden–Weinstein) symplectic reduced system \([8]\) corresponding to the abelian subgroup \( K \) at momentum \( v \), obtained simply by ‘ignoring cyclic coordinates’. The additional 1-dimensional symmetry is expected because \( K \) is a codimension 1 subgroup of \( G \). Most of the proof follows from the general theory developed in \([15]\), but since the symmetry group \( K \) is abelian there is also the following elementary proof.

**Proof of theorem 2.1.** The four momenta \( J^1, J^2, J^3, J^\theta \) Poisson commute since \( K \) is abelian, and their derivatives are linearly independent at \( x_e \), since the isotropy group of \( x_e \) is bigger than \( \text{SO}(2)_D \). Define \( \nu^\theta = J^\theta, \nu^\theta = J^\theta \), and using the Darboux theorem \([18]\), choose four conjugate functions, i.e. four functions \( a_1, a_2, a_3, \theta \), such that
\[
\{a_i, v_j^\theta\} = \delta_{ij}, \quad \{\theta, v^\theta\} = 1, \quad \{a_i, v^\theta\} = 0.
\]
Find functions \( q_1, \ldots, q_8, p_1, \ldots, p_8, k = \frac{1}{2} (\dim \mathcal{M} - 8) \), that Poisson commute with the original 8 and satisfy
\[
\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0.
\]
Set \( w = (q, p) \), and translate so that \( x_e \) is at \( w = 0 \). The group \( K \) acts by addition on the conjugate coordinates \( a_1, a_2, a_3, \theta \) since this action is generated by the momenta \( \nu^\theta, \nu^\theta, \nu^\theta, \nu^\theta \). So \( H \) does not depend on the coordinates \( a_1, a_2, a_3, \theta \), and we have the \( v = (\nu^\theta, \nu^\theta) \) parametrized canonical system on the linear symplectic space \( W \) given by system (2.7) of (2.6). The other equations of (2.6) follow immediately.

The symmetry group \( \text{SO}(2)_D \) acts on the variables \( (w, v) \) independently of \( a, \theta \). Indeed, if \( f \) is a function which does not depend on \( a, \theta, \xi \), if \( x \) is one of \( a_1, a_2, a_3, \theta \) and the conjugate of \( x \) is \( \tilde{x} \), then
\[
\frac{\partial}{\partial x} \{J^\theta + J^\theta, f\} = \{\tilde{x}, [J^\theta + J^\theta, f]\},
\]
\[
= -\{J^\theta + J^\theta, \{f, \tilde{x}\}\} - \{f, [\tilde{x}, J^\theta + J^\theta]\}. \tag{2.8}
\]
The first term of (2.8) is zero because
\[
\{f, \tilde{x}\} = -\frac{\partial f}{\partial x} = 0.
\]
For the second term of (2.8), note that \( \tilde{x} \) is one of \( \nu^\theta, \nu^\theta, \nu^\theta, \nu^\theta \), which are momenta of \( K \). Since \( K \) is a normal subgroup of \( G \), its momenta are a Poisson ideal of the momenta \( J_\xi \), \( \xi \in g \). This can also be seen directly from (2.4) because the Poisson brackets of \( J^\theta \) with the other components of the momentum map \( J \) are found by differentiation in \( \phi \) at \( \phi = 0 \) of the right-hand sides of (2.4). The results are linear combinations of \( J^\theta \) and \( J_i^\theta, i = 1, 2, 3 \). In particular, \( \{\tilde{x}, J^\theta + J^\theta\} \) is a linear combination of \( \nu^\theta, \nu^\theta, \nu^\theta, \nu^\theta \), so the second term of (2.8) is
Remark 2.4. Equation (2.9) and the differential equation for \( \dot{J} \) of the symplectic form on \( \mathbb{R}^3 \times \mathfrak{so}(2) \) for the same reason as the first. Thus there is an action of \( \text{SO}(2)_D \) on the variables \( (v, w) \). Since \( J \) is equivariant and \( J = (\mathbf{J}^\rho, \nu) \), the resulting action on the \( \nu \) variables is independent of \( w \) and equal to the coadjoint action of \( (\theta, 0, \theta) \) on \( v \), which by (2.4) is a rotation of \( (v_1^\nu, v_2^\nu) \) by \( \theta \). The coordinates above can be restricted to an \( \text{SO}(2)_D \) invariant neighbourhood on the \( \mathcal{G} \)-invariant neighbourhood of \( \mathcal{G} x_e \).

\[ \square \]

**Definition 2.2.** A momentum \( v \in (\mathbb{R}^3 \times \mathfrak{so}(2))^\nu \) is vertical if it is fixed by the coadjoint action of \( \text{SO}(2)_D \), if i.e. \( v^\nu = v_3 \in \mathbb{R}^3 \).

For vertical \( v \), system (2.7) inherits an \( \text{SO}(2)_D \) symmetry from the full phase space \( \mathcal{M} \). It does not have this additional \( \text{SO}(2)_D \) symmetry for non vertical \( v \). Hence a perturbation of the \( \text{SO}(2)_D \)-symmetric Hamiltonian system (2.7) at a vertical momentum value \( v \) to a nonvertical momentum value is an \( \text{SO}(2)_D \)-symmetry breaking perturbation.

**Proposition 2.3.** For vertical momenta \( v \), the \( \text{SO}(2)_D \) action on \( W \) is symplectic with momentum map

\[ J_W = J^{\text{SO}(2)_D}(a, \theta, v, w) \big|_{v^\nu = 0, \theta = 0}. \]

In particular, at vertical momentum \( v \), system (2.7) is \( \text{SO}(2)_D \) symmetric and conserves \( J_W \).

**Proof.** The action of \( \text{SO}(2)_D \) in the coordinates \((a, \theta, v^a, v^\beta, w)\) of theorem 2.1 is symplectic since it is the action of a subgroup of \( \mathcal{G} \), which acts symplectically by assumption. It follows that \( \text{SO}(2)_D \) acts symplectically on \( W \), since the symplectic form on \( W \) at fixed \( v \) is the restriction of the symplectic form on \( \mathcal{M} \). Using the symmetry properties (2.4) of the momentum map,

\[ J_W(v, w) = J^{\text{SO}(2)_D}(0, 0, v^a, v^\beta, w), \]

\[ = (\mathbf{J}^\rho + \mathbf{J}^\beta)((v^\rho, 0, -\theta)(a, \theta, v^a, v^\beta, w)), \]

\[ = (\mathbf{J}^\rho + \mathbf{J}^\beta)(a, \theta, v^a, v^\beta, w) + \alpha_2 v_1^\nu - \alpha_1 v_2^\nu, \]

\[ = J^{\text{SO}(2)_D}(a, \theta, v^a, v^\beta, w) + \alpha_2 v_1^\nu - \alpha_1 v_2^\nu, \]

so \( J_W = J^{\text{SO}(2)_D} \) if \( v_1^\nu = v_2^\nu = 0 \). Thus for fixed vertical \( v \), \( J_W \) is equal to the momentum generating the \( \text{SO}(2)_D \) action on \( W \).

\[ \square \]

**Remark 2.4.** Equation (2.9) and the differential equation for \( \dot{a} \) in system (2.6) implies that the conservation of \( J_W \) is not typical for nonvertical momenta. Indeed, if \( J_W \) is conserved at some fixed nonvertical \( v = v_0 \), then adding any function \( \epsilon H(v, w) \) to \( H \) results in

\[ \frac{d}{dt}(\alpha_2 v_1^\nu - \alpha_1 v_2^\nu) = \epsilon \left( \frac{\partial H}{\partial v_1^\nu} v_1^\nu - \frac{\partial H}{\partial v_2^\nu} v_2^\nu \right). \]

We can choose an \( \text{SO}(2)_D \) invariant function \( \tilde{H} \) such that the right side of (2.10) is not everywhere zero for \( \epsilon \) arbitrarily small.

**Remark 2.5.** Locally, the symmetry reduced space takes the form \((v, w) \in \mathfrak{t}^\nu \oplus W \) where \( \mathfrak{t} \) is the Lie algebra of \( \mathcal{K} \). The space \( W \) is called the symplectic normal space at \( x_e \) with respect to the symmetry group \( \mathcal{K} \), see [15], since it is transversal to the group orbit \( \mathcal{K} x_e \) at \( x_e \) and is a symplectic space. Moreover, it is the largest symplectic subspace of the normal space \( \mathcal{N} \) to the group orbit \( \mathcal{G} x_e \). In the notation of [15], we have \( \mathcal{N}_0 = \mathfrak{t}^\nu, \mathcal{N}_1 = W \). An application of the results of [15] would mean a reduction by the group \( \mathcal{H} = \text{SO}(2)_D \ltimes \mathbb{R}^3 \) where

\[ \text{SO}(2)_{AD} = \{ (\phi, 0, -\phi) \in \text{SO}(2) \ltimes \mathbb{R}^3 \times \text{SO}(2) \} \]
is the antidiagonal embedding of $\text{SO}(2)$ in $\text{SO}(2)_L \times \text{SO}(2)_R$. But the resulting Poisson structure on $N_0 = h^* \simeq \mathfrak{so}(2)^* \oplus \mathbb{R}^n$ (where $h$ is the Lie algebra of $\mathcal{H}$) is nontrivial since the symmetry group $\mathcal{H}$ is nonabelian, so that $\mathcal{N} \simeq N_0 \oplus N_1$. In particular the symplectic leaves have nonconstant dimension at the axisymmetric relative equilibrium. For this reduction the conserved quantity $J_{\text{SO}(2)_D}$ is a Poisson momentum map for the $\text{SO}(2)_D$ action on $N_0 \oplus N_1$. However the nontrivial Poisson structure on $N_0$ would make the stability analysis more difficult. We would have to employ the general methods developed in [12] to study stability by energy–momentum confinement and we would have to desingularize the Poisson structure using blow up methods as in [14] to study the KAM stability of the axisymmetric relative equilibrium $x_e$.

**Remark 2.6.** In theorem 2.12 we could also construct an $\text{SO}(2)_D$ invariant Witt decomposition of the tangent space $T_{x_e}M$ and then use the equivariant Darboux theorem to obtain a model $\mathbb{K} \times \mathfrak{t}^* \times W$ of a $G$-invariant neighbourhood of $Gx_e$. In this construction the $\text{SO}(2)_D$ action on $W$ would be linear and $\nu$-independent, see [15]. Note, however, in this case $\mathfrak{t}^*$ is not the annihilator of $\mathfrak{g}_{x_e} = \text{SO}(2)_D$, so the situation here is not exactly the setting of [15].

### 3. Stability of axisymmetric relative equilibria

In this section we use the coordinates from theorem 2.1 to study the stability of the axisymmetric relative equilibrium $x_e$. We show that definiteness of the Hessian of $H(\nu_e, w)$ at the equilibrium $w = 0$ of (2.7) implies stability of $x_e$ under arbitrary perturbations. When this Hessian is not definite, for momentum-preserving perturbations, a constant times the momentum $J_W$ can be added to the energy to obtain a Lyapunov function of system (2.7). However, for nonvertical linear momentum, $J_W$ is not a conserved quantity of (2.7), so it cannot be used in a Lyapunov function to establish stability for perturbations to nonvertical linear momentum. The gap (sections 1 and 3.1) is defined to be those relative equilibria for which a constant times the momentum $J_W$ must be added to obtain a Lyapunov function. If $x_e$ is in the gap then the equilibrium $w = 0$ of (2.7) has imaginary spectrum and indefinite Hessian, and is amenable to classical KAM and Nekhoroshev results (section 3.1) to obtain stability.

In the case of a 12-dimensional phase space $M$, system (2.7) has 2 degrees of freedom, and propositions 3.9–3.11 below show that formal stability (section 3.1) of the axisymmetric relative equilibrium $x_e$ fails exactly at an eigenvalue collision. Moreover, we show that this is exactly where a Hamiltonian Hopf bifurcation typically occurs and a loss of linear stability is expected. Thus, in the case of a 12-dimensional phase space, formal stability and linearized stability of axisymmetric relative equilibria are typically equivalent, explaining the observation of this by Leonard and Marsden in the Kirchhoff model of axisymmetric underwater vehicles [6].

#### 3.1. Nonlinear stability

The relative equilibrium $x_e$ is A-stable for some subset $A$ of $G$, if initial data starting sufficiently close to $x_e$ stay arbitrarily close to $Ax_e$ for all times. It is formally stable if $D^2 w H(\nu_e, 0)$ is definite for some $\lambda \in \mathbb{R}$. Recall from (2.5) that $\mathbb{K} = \mathbb{R}^3 \times \text{SO}(2)_R$.

**Proposition 3.1.** The axisymmetric relative equilibrium $x_e$ of (2.1) is K-stable if the Hessian $D^2 w H(\nu_e, 0)$ is definite.

**Proof.** If the Hessian $D^2 w H(\nu_e, 0)$ of the Hamiltonian system (2.7) is definite then, using the energy as a Lyapunov function, the equilibrium $w = 0$ of (2.7) is stable under perturbations
in $w$, i.e. within $v = v_e$. The equilibrium $w = 0$ persists to nearby momentum values $v$, and the corresponding equilibria of (2.7) at those momentum values are also stable. Since the momentum $v(t)$ is a conserved quantity of (2.6), we conclude that for initial data close to $v = v_e$, $w = 0$, the solution $\{v(t), w(t)\}$ of (2.6) stays close to $(v_e, 0)$ for all times. Since the only remaining variables are $a, \theta$, and these correspond to the action of $K$, the proof is complete. □

Remark 3.2. Better stability criteria cannot be obtained by relaxing to the weaker stability modulo the whole group $G$, instead of just the subgroup $K$, because the isotropy of $x_e$ implies that these two stabilities are equivalent.

Let $J^K = (J^a, J^\theta)$ denote the momentum map for the symplectic action of $K$ on $M$.

Proposition 3.3. The axisymmetric relative equilibrium $x_e$ of (2.1) is $K$-stable under perturbations that preserve the momentum $J^K$ of $K$ if it is formally stable.

Proof. A perturbation which preserves $J^K$ does not change $v$ and therefore is a perturbation of the stable equilibrium $w = 0$ of system (2.7). Since $J_w$ is a conserved quantity of (2.7) at $v = v_e$ by proposition 2.3, the equilibrium $w = 0$ of (2.7) is stable if there is some $\lambda \in \mathbb{R}$ such that $D^2 wH(0) + \lambda D^2 J_w(0)$ is definite. □

So we see that the presence of the $SO(2)D$ symmetry and its momentum $J_w(w)$ at vertical momenta $v \parallel e_3$ causes a gap between energy–momentum confinement under momentum-preserving and general perturbations of $x_e$.

Definition 3.4. Let $A_e$ be the set of axially symmetric relative equilibria. The EM-region is the subset $A^{EM}_e \subseteq A_e$ such that $D^2 wH(\nu_e, 0)$ is definite. The gap is the subset $A^{gap}_e \subseteq A_e$ that is formally stable but not in the EM-region.

Remark 3.5. If the symmetry group $G$ is compact then the gap is absent since formal stability implies $G$-stability, see e.g. [12, 13].

We will say that the axisymmetric relative equilibrium $x_e$ is nondegenerate if the Hessian $D^2 wH(\nu_e, 0)$ of the equilibrium $w = 0$ of (2.7) is invertible. Suppose that $x_e \in A^{gap}_e$ is nondegenerate. Then $w = 0$ is a stable equilibrium of (2.7), so the linearization $J D^2 wH(\nu_e, 0)$ of (2.7) has nonzero purely imaginary eigenvalues, i.e. it is elliptic. We call the relative equilibrium $x_e$ of (2.1) elliptic if the corresponding equilibrium $w = 0$ of (2.7) is elliptic.

Depending on the dimension of the phase space $M$, different scenarios are possible.

(a) If $\dim M = 10$ then (2.7) is a one degree of freedom Hamiltonian system. Equilibria of such Hamiltonian systems have definite Hessians if and only if they are elliptic. So, the Hessian $D^2 wH(\nu_e, 0)$ is definite, contradicting $x_e \in A^{gap}_e$. In particular, at this dimension there are no nondegenerate relative equilibria in the gap.

(b) If $\dim M = 12$ then, for each $\nu$, (2.7) is a two degree of freedom Hamiltonian system. The equilibrium $w = 0$ persists as elliptic equilibrium of (2.7) to nearby $\nu$. Elliptic equilibria of such Hamiltonian systems are expected to be KAM stable [2], and so the axisymmetric relative equilibrium $x_e$ is expected to be $K$-stable.

(c) If $\dim M \geq 14$ then the equilibrium $w = 0$ persists as in case (b), but for each $\nu$ system (2.7) has at least three degrees of freedom. The equilibrium $w = 0$ of (2.7) is therefore not expected to be stable because it will exhibit Arnold diffusion. However, Nekhoroshev stability [5, 11] is expected, which implies $K$ stability of the axisymmetric relative equilibrium $x_e$ over exponentially long times.
Remark 3.6. In case (b), in order to prove KAM stability of the axisymmetric relative equilibrium \( x_e \), one can verify the twist condition (see [9]) for the equilibrium \( w = 0 \) of the 4-dimensional SO(2) symmetric Hamiltonian system (2.7) at \( v = v_e \). If the twist condition holds at \( v = v_e \), then it holds for the equilibria persisting to nearby \( v \) and \( x_e \) is therefore \( \mathcal{K} \)-stable. The additional SO(2) symmetry of (2.7) at \( v = v_e \) can be used to express the reduced Hamiltonian of (2.7) in terms of SO(2)\(_D\)-invariant functions. This simplifies the verification of the twist condition, see section 4.4. But, as already emphasized, the conserved quantity \( J_w \) of (2.7) at \( v = v_e \) cannot be used because it is only conserved for vertical \( v \). System (2.7) for which the twist condition needs to be verified can be obtained in a concrete example by choosing any realization of the \( \mathbb{R}^3 \times \text{SO}(2)\_R \) Marsden–Weinstein reduced space.

Remark 3.7. If \( x_e \in \mathcal{A}^{\text{top}} \) then, as already noted above, \( x_e \) is a spectrally stable equilibrium with indefinite Hessian of the Hamiltonian system (2.7). As shown in [3, 4], adding arbitrarily small dissipation to such Hamiltonian systems results in spectral instability, because the negative eigendirection of the Hessian forces instability in the presence of decreasing energy. Consequently, adding a small SO(2)\(_D\) invariant dissipation for each vertical \( v \), and extending that to nonvertical \( v \), will result in dissipation induced instability at each persisting equilibrium of the Hamiltonian systems parametrized by \( v \). If the dissipation preserves also the SO(2)\(_D\) momentum \( J_w \), then it preserves all momenta resulting from the symmetries of the original system on \( \mathcal{M} \). In this case, and if \( D^2(H + \lambda J_w)(x_e) \) is positive definite for some \( \lambda \), then the persisting equilibria of (2.7) remain spectrally stable for nearby vertical \( v \) while, for nearby nonvertical \( v \), there is again spectral instability. In section 5 we verify this by numerical simulation of an \( \mathbb{R}^3 \times \text{SO}(2)\_R \) reduction of the underwater vehicle system.

3.2. Hamiltonian Hopf bifurcation

In this section we assume that the phase space \( \mathcal{M} \) is 12-dimensional, so that system (2.7) is 4-dimensional. The linearization of the action of SO(2)\(_D\) on \( W \) at the equilibrium \( w = 0 \) is a linear symplectic representation, about which we need a few elementary facts [10]. Consider a symplectic representation of SO(2) on a 2\( d \)-dimensional linear symplectic space \( \mathbb{W} \). These representations are classified by tuples of integers \( n_1 \leq n_2 \leq \cdots \leq n_d \). We say that the action is of type \( (n_1, n_2, \ldots, n_d) \) if there is a linear splitting of \( \mathbb{W} \) into the sum of two 2-dimensional invariant subspaces on which the action is isomorphic to \( z \mapsto e^{i\omega}z \) on the vector space \( \mathbb{R}^2 \cong \mathbb{C} \) with its standard symplectic structure. With respect to such a splitting, the associated momentum mapping is

\[
\frac{1}{2} n_1 |z_1|^2 + \cdots + \frac{1}{2} n_d |z_d|^2,
\]

and particularly, the quadratic momentum map for type \( (n_1, n_2, \ldots, n_d) \) actions has both positive and negative definite eigendirections if both positive and negative \( n_j \) occur in its type.

Suppose \( \mathbb{W} \) is a linear symplectic space and \( A: \mathbb{W} \to \mathbb{W} \) is infinitesimally symplectic. Let \( \pm i \omega \) be a simple complex conjugate pair of purely imaginary eigenvalues of \( A \). The corresponding (2-dimensional) real eigenspace is invariant for the symplectic flow \( \exp(At) \), which defines a symplectic representation of SO(2) of either type +1 or type −1. The Krein sign of \( \pm i \omega \) is the sign of its type. We adopt the convention that the frequency \( \omega \) has the same sign as the Krein sign of the eigenvalue \( \pm i \omega \), so that the corresponding quadratic Hamiltonian on the real eigenspace, with respect to a basis with the standard symplectic structure, is \( + \frac{1}{2} \omega (q^2 + p^2) \). Since the frequencies are signed, resonances between two frequencies are also signed e.g. there can be \( -1 : 1 \) and \( 1 : 1 \) resonances, and these are distinct cases.
Remark 3.8. Actions of SO(2) of type (−1, 1) occur frequently in dimension 4, and this case occurs in the underwater vehicle example, see (4.25). This case can be realized as \( q \mapsto R_\theta q \), \( p \mapsto R_\theta p \) where \( R_\theta \) is the action of SO(2) by counterclockwise rotations (see the proof of proposition 3.11). This is the standard SO(2) action on \( \mathbb{R}^4 \) obtained by lifting the standard action of SO(2) on \( \mathbb{R}^3 = \{q\} \) to the cotangent bundle \( T^*\mathbb{R}^3 = \{(q, p)\} \).

Proposition 3.9. Let \( x_\epsilon \) be an axisymmetric relative equilibrium of the SO(2) \( \times \mathbb{R}^3 \times \mathbb{R}^3 \) symmetric Hamiltonian system (2.1) on a 12-dimensional phase space \( \mathcal{M} \). Assume that \( x_\epsilon \) is semisimple and elliptic, and let the action of SO(2) be of type \( n_j \) on the real \( \pm i\omega_j \) eigenspace, where \( \omega_1, \omega_2 \) are the two normal frequencies of the equilibrium \( w = 0 \). Then \( x_\epsilon \) is formally stable if and only if there is not an \( n_1 \) : \( n_2 \)-resonance between \( \omega_1 \) and \( \omega_2 \), i.e. if and only if \( \omega_1 n_2 - \omega_2 n_1 \neq 0 \).

Proposition 3.9 implies that, in 12 dimensions, elliptic axisymmetric relative equilibria are typically formally stable because resonance is atypical in the linearization of the corresponding equilibrium in the reduction by \( \mathcal{K} \). The proof of this proposition follows by applying the following result to the linearized SO(2)\( _D \) action at \( x_\epsilon \), restricted to \( \mathbb{W} \).

Proposition 3.10. Let \( \mathbb{W} \) be a 4-dimensional linear symplectic space and suppose SO(2) acts linearly and symplectically on \( \mathbb{W} \) with momentum map \( J \). Suppose that \( H: \mathbb{W} \to \mathbb{R} \) is SO(2) invariant and that \( 0 \in \mathbb{W} \) is an elliptic semisimple equilibrium where the linearization \( \mathcal{J} \mathbb{D}^2H(0) \) has eigenvalues \( \pm \omega_j \), \( j = 1, 2 \). Suppose that \( \omega_1 < 0 \) and \( \omega_2 > 0 \), and let the action of SO(2) be of type \( n_j \) on the real \( \pm i\omega_j \) eigenspace. Then there is a \( \lambda \) such that \( \mathcal{J} \mathbb{D}^2H(0) + \lambda \mathcal{J} \mathbb{D}^2J(0) \) is definite (i.e. \( 0 \in \mathbb{W} \) is formally stable) if and only if

\[
(\omega_1 n_2 - \omega_2 n_1) \neq 0.
\]

(3.1)

Proof. Split \( \mathbb{W} = \mathbb{R}^2 \oplus \mathbb{R}^2 \) into the real eigenspaces of \( \mathcal{J} \mathbb{D}^2H(0) \) to the eigenvalues \( \pm \omega_j \), choosing a basis giving the standard symplectic structure matrix on each factor. Then for \( w = (w_1, w_2) \in \mathbb{R}^2 \oplus \mathbb{R}^2 \)

\[
\langle w, (\mathcal{J} \mathbb{D}^2H(0) + \lambda \mathcal{J} \mathbb{D}^2J(0))w \rangle = \frac{1}{2}(\omega_1 |w_1|^2 + \omega_2 |w_2|^2) + \frac{\lambda}{2}(n_1 |w_1|^2 + n_2 |w_2|^2),
\]

\[
= \frac{1}{2}((\omega_1 + \lambda n_1) |w_1|^2 + (\omega_2 + \lambda n_2) |w_2|^2).
\]

(3.2)

Since the action of SO(2) is nontrivial, one of \( n_1, n_2 \) is not zero, and

\[
\lambda \mapsto (\omega_1, \omega_2) + \lambda(n_1, n_2)
\]

is a line. To show that (3.2) is definite, it is sufficient that this line meets the first or third quadrant. But such a line is contained in the second and fourth quadrant only if it contains the origin, which is excluded by (3.1).

Conversely, suppose that \( \omega_1 n_2 - \omega_2 n_1 = 0 \). Then there is a \( \kappa \) such that \( (n_1, n_2) = \kappa (\omega_1, \omega_2) \) and (3.2) becomes

\[
\langle w, (\mathcal{J} \mathbb{D}^2H(0) + \kappa \mathcal{J} \mathbb{D}^2J(0))w \rangle = \frac{1}{2}(\kappa + \lambda) (\omega_1 |w_1|^2 + \omega_2 |w_2|^2),
\]

and this is not definite for any \( \lambda \) because \( \omega_1 < 0 < \omega_2 \) \( \Box \).

We now consider the case when condition (3.1) is violated.

Proposition 3.11. Assume the setting of proposition 3.10 again, but let \( H(v, w) \) depend on a parameter \( v \in \mathbb{R} \) and assume, as before, that the SO(2) action is of type \( (n_1, n_2) \). Assume that (i) the equilibrium is elliptic for \( v < 0 \), \( v \approx 0 \);
(ii) condition (3.1) holds for $\nu < 0$, $\nu \approx 0$, but is violated at $\nu = 0$.

Then

(a) if $n_1 \neq -n_2$ then the equilibrium is linearly stable for $\nu > 0$, $\nu \approx 0$; and
(b) if $n_1 = -n_2$ and if the transversality condition (3.3) holds then a Hamiltonian Hopf bifurcation occurs and the equilibrium 0 becomes spectrally unstable for $\nu > 0$.

**Proof.** For part (a), if $n_1 \neq \pm n_2$ then the condition $\omega_1 n_2 - \omega_2 n_1 = 0$ at $w = 0$, $\nu = 0$ implies that the eigenvalues $i\omega_j$ do not collide at $\nu = 0$. Therefore the symplectic eigenvalue theorem [1] implies that the eigenvalues are imaginary in a neighbourhood of $\nu = 0$. If $\nu < 0$, then $n_1 \neq n_2$, and so $\omega_1 = \omega_2$. This contradicts the assumption $\omega_1 < 0 < \omega_2$. For part (b), assuming $n_1 = n_2 = -n_1$, there is an equivariant Lagrangian splitting $W = \mathbb{R}^2 \oplus \mathbb{R}^2$ such that the SO(2) action is $z \mapsto e^{in\theta} z$ on each factor $\mathbb{R}^2 \simeq \mathbb{C}$. The standard symplectic structure matrix blocks with respect to this splitting, is the identity on the (1,2)-block, and minus the identity on the (2,1)-block. Indeed, with respect to the symplectic splitting of proposition 3.10, the basis

$$
eq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}, \quad \neq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix},$$

$$\neq \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}, \quad \neq \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix},$$

accomplishes this, as is easily verified. With respect to this basis, write the linearization at $w = 0$ as

$$L = \mathbb{J}D^2 H(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a, b, c, d$ are SO(2)-equivariant $2 \times 2$ matrices, which can therefore be identified with complex numbers. Since $L$ is infinitesimally symplectic, $d = -a^2$ and $b, c$ are symmetric, i.e. $d = -\bar{a}$ and $b, c \in \mathbb{R}$, and the eigenvalues of $L$ are

$$\lambda = i\text{Im}(a) \pm \sqrt{-\text{Im}(a)^2 - (ad - bc)} = i\text{Im}(a) \pm \sqrt{\text{Re}(a)^2 + bc}.$$ 

So $L$ is spectrally stable if the discriminant $D = \text{Re}(a)^2 + bc$ is negative, unstable if $D$ is positive, and an eigenvalue collision occurs if $D = 0$. Since $D = 0$ when $\nu = 0$ and $D < 0$ for $\nu < 0$, $\nu \approx 0$, $D$ changes sign under the transversality condition

$$\frac{dD}{d\nu} \bigg|_{\nu=0} \neq 0,$$

whereupon the equilibrium becomes spectrally unstable for $\nu > 0$, $\nu \approx 0$, and undergoes a Hamiltonian Hopf bifurcation at $\nu = 0$, see e.g. [17].

**4. Application to the Kirchhoff model**

In this section we use the theory from section 3 to analyse the stability of vertically falling, spinning relative equilibria of a neutrally buoyant submerged axisymmetric body. In the Kirchhoff approximation (see e.g. [6, 7]), this is a Lagrangian system with configuration space $\text{SE}(3)$, such that the position and orientation of the body in configuration $(A, b) \in \text{SE}(3)$ are obtained relative to a reference body, by the Euclidean transformation $x \mapsto Ax + b$. 


The reference body is chosen with a vertical axis of symmetry, so that in the configuration \((A, b)\), the submerged body has axis of symmetry in the direction \(Ae_3\). The Lagrangian is

\[
L(A, b, \Omega, v) = \frac{1}{2} \Omega^T I \Omega - ml e_3 \cdot \Omega \times v + \frac{1}{2} v^T M v + mgl(e_3 \cdot Ae_3).
\] (4.1)

Here \((\Omega, v) \in TSE(3)\) are the body-referenced angular and translational velocities obtained by left translation, i.e. \((\tilde{\Omega}, v) = (A, b)^{-1}(\dot{A}, \dot{b})\), or

\[
\tilde{\Omega} = A^{-1} \dot{A}, \quad v = A^{-1} \dot{b}.
\] (4.2)

The added inertia matrix \(I\) and added mass matrix \(M\) are the 3 \(\times\) 3 diagonal matrices

\[
I = \begin{pmatrix}
    I_1 & 0 & 0 \\
    0 & I_1 & 0 \\
    0 & 0 & I_3
\end{pmatrix}, \quad M = \begin{pmatrix}
    M_1 & 0 & 0 \\
    0 & M_1 & 0 \\
    0 & 0 & M_3
\end{pmatrix},
\]

and are determined from the shape and mass distribution of the body [7]. The reference body is such that the centre of buoyancy is at the origin and the centre of mass is at distance \(l\) below the centre of buoyancy, so the vehicle is bottom heavy when \(l > 0\) and top heavy when \(l < 0\).

We will assume, as in the physical situation, that \(I, M, m\) and \(l\) satisfy

\[
I_1 > 0, \quad I_3 > 0, \quad M_1 > 0, \quad M_3 > 0, \quad I_1 M_1 - m^2 l^2 > 0.
\] (4.3)

The Lagrangian is positive definite quadratic in \(\Omega, v\) under assumptions (4.3).

The system admits the symmetries of

(a) spatial isotropy: the left action of the subgroup \(SE(2) \times \mathbb{R}\), corresponding to translations in any direction and rotations about the vertical and

(b) material symmetry: the action of the subgroup \(SO(2)\) by multiplication of the inverse on the right, corresponding to rotating the body about its axis.

Thus the system has the symmetry \(\mathcal{G}\) considered in section 3. The point \(x_e\) defined by

\[
x_e : A = \text{id}, \quad b = 0, \quad \Omega = \frac{S_e}{I_3} e_3, \quad v = \frac{P_e}{M_3} e_3,
\]
is an axisymmetric relative equilibrium, corresponding to the motion where the vehicle spins at angular velocity \(S_e/I_3\) and angular momentum \(S_e\) about its (vertical) symmetry axis, and translates along that axis with vertical velocity \(P_e/M_3\) and vertical momentum \(P_e\). We will apply the general stability theory developed in section 3 to this family of relative equilibria.

4.1. Hamiltonian formulation of the Kirchhoff model

We briefly review the derivation of the Hamiltonian and the equation of motion for the Kirchhoff model. See [7] for more details. The conjugate variables \(\Pi, P\) to \(A, b\) are

\[
\Pi = \frac{\partial L}{\partial \dot{\Omega}}\bigg|_{(A,b)=(\text{id},0)} = I \dot{\Omega} + ml e_3 \times v, \quad P = \frac{\partial L}{\partial \dot{v}}\bigg|_{(A,b)=(\text{id},0)} = M \dot{v} - ml e_3 \times \Omega,
\]

and these equations can be inverted to give

\[
\Omega_1 = \frac{M_1 \Pi_1 + ml P_2}{I_1 M_1 - m^2 l^2}, \quad \Omega_2 = \frac{M_1 \Pi_2 - ml P_1}{I_1 M_1 - m^2 l^2}, \quad \Omega_3 = \frac{\Pi_3}{I_3},
\]

\[
v_1 = \frac{I_1 P_1 - ml \Pi_2}{I_1 M_1 - m^2 l^2}, \quad v_2 = \frac{I_1 P_2 + ml \Pi_1}{I_1 M_1 - m^2 l^2}, \quad v_3 = \frac{P_3}{M_3}.
\]
The Hamiltonian is
\[
H(A, b, \Pi, P) = (\Omega, \Pi) + (v, P) - L,
\]
\[
= \frac{1}{2}(\Omega, \Pi) + \frac{1}{2}(v, P) - mgl(e_3 \cdot A e_3),
\]
\[
= \frac{1}{2}M_1 \left(\Pi_1^2 + \Pi_2^2\right) + \frac{1}{2I_3} \Pi_3^2 + \frac{I_1}{2(I_1M_1 - m^2l^2)} (P_1^2 + P_2^2)
\]
\[
+ \frac{1}{2M_3} P_3^2 + ml \frac{\Pi_1 P_2 - \Pi_2 P_1}{I_1M_1 - m^2l^2} - mgl\Gamma_3,
\]
where \(\Gamma = AT e_3\). The equations of motion are
\[
\frac{d\Pi}{dt} = \Pi \times \Omega + P \times v - mgl \Gamma \times e_3,
\]
\[
\frac{dP}{dt} = \Pi \times \Omega,
\]
\[
\frac{d\Gamma}{dt} = \Gamma \times \Omega.
\]
This system is Poisson with bracket
\[
\{ f, k \} = \nabla f^i \mathcal{J}(\Pi, P, \Gamma) \nabla k, \quad \text{where} \quad \mathcal{J}(\Pi, P, \Gamma) = \begin{pmatrix} \Pi^\wedge & P^\wedge & \Gamma^\wedge \\ P^\wedge & 0 & 0 \\ \Gamma^\wedge & 0 & 0 \end{pmatrix}.
\]

The momentum map \(J = (J^a, J^b, J^c)\), cf (2.3), is given by
\[
J^a(A, b, \Pi, P) = (A\Pi + b \times AP) \cdot e_1,
\]
\[
J^b(A, b, \Pi, P) = AP,
\]
\[
J^c(A, b, \Pi, P) = -\Pi \cdot e_1.
\]

4.2. Reduction by \(\mathbb{R}^3 \times SO(2)_R\)

As shown in section 3, the stability of the relative equilibrium \(x_c\) can be established by a study of the stability of the equilibrium of (2.7) at the parameter value \(v = (\mu^\alpha_\gamma, \mu^\alpha_\gamma) = (P_e e_3, S_e)\). System (2.7) is the Marsden–Weinstein reduction (see [8]) of the full Hamiltonian system by the group \(\mathbb{R}^3 \times SO(2)_R\), at the momentum level \((\mu^\alpha_\gamma, \mu^\alpha_\gamma)\), i.e. (2.7) is obtained from the full Hamiltonian system (2.1) by fixing the values of the conserved quantities \(J^a\) and \(J^c\) to their respective values \(\mu^\alpha_\gamma\) and \(\mu^\alpha_\gamma\), and eliminating variables along the group \(\mathbb{R}^3 \times SO(2)_R\). For the Kirchhoff model, equations (4.5) cannot be directly used because they are the result of a reduction by the group \(SO(2)_e \times \mathbb{R}^3\). In this section we compute the reduced spaces near \(x_c\), in a way that relates to the variables \(\Pi, P, \Gamma\) of (4.5), by comoving with the action of \(SO(2)_R\).

Given \(q_1, q_2\), define \(q = q_1 e_1 + q_2 e_2\) and
\[
A_{q_1, q_2} = \text{id} + \tilde{x} + \tilde{x} \tilde{x}^2 \bigg|_{x=-e_3 \times q}
\]
where \(\tilde{x}\) is defined in (2.2). The matrix \(A_{q_1, q_2}\) in (4.8) is orthogonal because \(f\) satisfies the equation \(\|q\|^2 f^2 - 2f + 1 = 0\) and \(f\) is the solution of this quadratic that is smooth at \(q = 0\).

The choice \(x = -e_3 \times q\) is such that \(A_{q_1, q_2} e_3 = q_1 e_1 + q_2 e_2 + \sqrt{1 - q_1^2 - q_2^2} e_3\). The map \((q_1, q_2, \theta) \mapsto A_{q_1, q_2} \exp(-\theta e_3)\) coordinatizes \(SO(3)\) near the identity and provides an angle \(\theta\), which may be used to define the coordinates \((a, \theta, q_1, q_2)\) by
\[
(A, b) = (a, \theta) \cdot (A_{q_1, q_2}, mlM^{-1} q) = (A_{q_1, q_2} \exp(-\theta e_3), mlM^{-1} q + a).
\]

The choice of \(b\) is such that the two sections \(a = 0, \theta = 0\) and \(q_1 = q_2 = 0\) are orthogonal (at their intersection point) in the kinetic energy metric. This is an example of a general procedure advocated in [16]. The group \(\mathbb{R}^3 \times SO(2)_R\) acts by addition in \(a, \theta,\) and
\[
\Gamma = \exp(\theta e_3)(q_1 e_1 + q_2 e_2) + \Gamma_3 e_3,
\]
so \((q_1, q_2)\) comoves with \((\Gamma_1, \Gamma_2)\).
Let the conjugate momenta to $a, \theta, q_1, q_2$ be $\nu^a, \nu^\theta, p_1, p_2$. The coordinates $a$ and $\theta$ are cyclic and so $\nu^a, \nu^\theta$ are conserved. Thus the $\mathbb{R}^3 \times \text{SO}(2)_R$ reduction results in a Hamiltonian in terms of canonical coordinates $q_i, p_i$, which is parameterized by $\nu^a, \nu^\theta$. This is computed in the following proposition.

**Proposition 4.1.** The Hamiltonian of the $\mathbb{R}^3 \times \text{SO}(2)_R$ reduction of the Kirchhoff model (4.4) in the canonical coordinates $q_i, p_i$ is

$$H(q_1, q_2, \dot{p}_1, \dot{p}_2) = \frac{M_1}{2(I_1 M_1 - m l^2)} \left( \dot{p}_1^2 + \dot{p}_2^2 - (q_2 \dot{p}_1 - q_1 \dot{p}_2)^2 \right) + \frac{1}{2} \left( \frac{1}{M_1} - \frac{I_1}{M_1 M_1 - m l^2} \right) (\nu^a \cdot \Gamma^-)^2 + \frac{ml}{M_1 M_1 - m l^2} \left( (q_1 \dot{p}_2 - q_2 \dot{p}_1)(\nu^a \cdot \Gamma^-) + v_i \dot{p}_1 - v_i \dot{p}_2 \right) - mg l \Gamma_3,$$

where

$$\dot{p}_1 = p_2 - f q_1 \nu^\theta - \frac{ml}{M_1} v_i, \quad \dot{p}_2 = -p_1 - f q_2 \nu^\theta + \frac{ml}{M_1} v_i,$$

$$\Gamma^- = A'_{q_1, q_2} e_3 = q_1 e_1 + q_2 e_2 - \Gamma_3 e_3, \quad \Gamma_3 = \sqrt{1 - q_i^2 - q_i^2}. \quad \text{(4.10)}$$

**Proof.** The $\mathbb{R}^3 \times \text{SO}(2)_R$ reduced Hamiltonian can be computed by computing $p_i$ in terms of $\Pi$ and $q_1, q_2$ and eliminating $P$ and $\Pi_3$ using (see (4.7)) $P = A' \nu^a$ and $\Pi_3 = -\nu^\theta$. Since the Hamiltonian is invariant, it does not depend on $a$ or $\theta$, so we set $a = 0$ and $\theta = 0$. By definition of canonical coordinates,

$$p_i = \left( A' \frac{\partial A}{\partial q_i} \right)^\vee \cdot \Pi + \left( A' \frac{\partial b}{\partial q_i} \right) \cdot P,$$

where the superscript $\vee$ recovers a vector from an antisymmetric matrix by inverting the $\wedge$ operation defined by (2.2). Then

$$A'_{q_1, q_2} \frac{\partial A}{\partial q_1} = \left( -e_2 + \frac{f^2 q_1}{f - 1} e_3 \times q + f q_2 e_3 \right)^\wedge,$$

$$A'_{q_1, q_2} \frac{\partial A}{\partial q_2} = \left( e_1 - \frac{f^2 q_2}{f - 1} e_3 \times q - f q_1 e_3 \right)^\wedge,$$

from which

$$p_1 = -\frac{f^2 q_1 q_2}{f - 1} \Pi_1 + \left( -1 + \frac{f^2 q_1^2}{f - 1} \right) \Pi_2 - f q_2 \nu^\theta + \frac{ml}{M_1} v_i,$$

$$p_2 = \left( 1 - \frac{f^2 q_2^2}{f - 1} \right) \Pi_1 + \frac{f^2 q_1 q_2}{f - 1} \Pi_2 + f q_1 \nu^\theta + \frac{ml}{M_1} v_i.$$

These linear equations may be solved for $\Pi_1, \Pi_2$, with the result

$$\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} = \begin{pmatrix} 1 - f q_2^2 & f q_1 q_2 \\ f q_1 q_2 & 1 - f q_1^2 \end{pmatrix} \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix}, \quad \text{(4.12)}$$

where $\tilde{\rho}_i$ are defined by (4.11). The matrix in (4.12) is the same as the upper $2 \times 2$ submatrix of $-e_3 A'_{q_1, q_2} e_3$, which is otherwise sparse, so setting $\tilde{\rho} = \tilde{p}_1 e_1 + \tilde{p}_2 e_2$,

$$\Pi = -e_3 \times \left( A'_{q_1, q_2} (e_3 \times \tilde{\rho}) \right) - \nu^\theta e_3, \quad \text{(4.13)}$$
and

\[ \Pi_1^2 + \Pi_2^2 = -e_3 \times (A_{q_1}^t \cdot (e_3 \times \tilde{p})) = \tilde{p}_1^2 + \tilde{p}_2^2 - (q_2 \tilde{p}_1 - q_1 \tilde{p}_2)^2, \]

\[ \Pi_1 P_2 - \Pi_2 P_1 = e_1 \cdot (\Pi \times A_{q_1}^t) = (q_1 \tilde{p}_2 - q_2 \tilde{p}_1)(\nu^\theta \cdot \Gamma') + \nu^\theta \tilde{p}_1 - \nu^\theta \tilde{p}_2. \]

\[ P_3^2 = (\nu^\theta \cdot A_{q_1}^t \cdot e_3)^2. \]

The Hamiltonian (4.4) is obtained from (4.10) after substituting these and deleting inessential constants.

As already noted, the case of \( \nu^\theta \) vertical is important for the stability and KAM analysis, and we specialize to this now.

**Proposition 4.2.** At vertical momentum \( \nu^\theta_1 = \nu^\theta_2 = 0 \), the reduced Hamiltonian system (4.4) is

\[ H(q_1, q_2, p_1, p_2) = \frac{F_p}{2}((p_1 + F_l \Gamma_3 q_1 + \nu^\theta f q_2)^2 - (q_1 p_1 + q_2 p_2)^2 + (p_2 + F_l \Gamma_3 q_2 - \nu^\theta f q_1)^2 + \frac{E_2}{2} ||q||^2 + mg l \left( f - \frac{1}{2} \right) ||q||^2 + \frac{1}{2} F_p F_l^2 ||q||^4, \]

where

\[ F_p = \frac{M_1}{I_1 M_1 - m^2 I_2^2}, \quad F_\theta = mgl - (\nu^\theta)^2 \left( \frac{1}{M_3} - \frac{1}{M_1} \right), \quad F_l = \frac{m l v^\theta}{M_1}. \]

**Proof.** Using (4.11), the Hamiltonian (4.10) can be written in the canonical coordinates \( q_1, q_2, p_1, p_2 \) as follows:

\[ H(q_1, q_2, p_1, p_2) = \frac{M_1}{2(I_1 M_1 - m^2 I_2^2)} \left( \tilde{p}_1^2 + \tilde{p}_2^2 - (q_2 \tilde{p}_1 - q_1 \tilde{p}_2)^2 \right) + \frac{1}{2} \left( \frac{1}{M_3} - \frac{I_1}{(I_1 M_1 - m^2 I_2^2)} \right) (\nu^\theta_3 \Gamma_3)^2 \]

\[ - \frac{m l}{I_1 M_1 - m^2 I_2^2} (q_1 \tilde{p}_2 - q_2 \tilde{p}_1)(\nu^\theta_3 \Gamma_3) - mgl \Gamma_3, \]

\[ = \frac{M_1}{2(I_1 M_1 - m^2 I_2^2)} \left( \tilde{p}_1 + \frac{m l v^\theta_3}{M_1} q_2 \Gamma_3 \right)^2 + \left( \tilde{p}_2 - \frac{m l v^\theta_3}{M_1} q_1 \Gamma_3 \right)^2 - (q_2 \tilde{p}_1 - q_1 \tilde{p}_2)^2 \]

\[ - \frac{m l}{M_1} \left( 1 - ||q||^2 \right) \left( \frac{1}{M_3} - \frac{I_1}{(I_1 M_1 - m^2 I_2^2)} \right) ||q||^2 - mgl \Gamma_3, \]

\[ = \frac{M_1}{2(I_1 M_1 - m^2 I_2^2)} \left( p_1 + \frac{m l v^\theta_3}{M_1} q_1 \Gamma_3 + \nu^\theta q_2 f \right)^2 - (q_1 p_1 + q_2 p_2)^2 + \left( p_2 + \frac{m l v^\theta_3}{M_1} q_2 \Gamma_3 - \nu^\theta q_1 f \right)^2 \]

\[ + \left( 2mgl f - (\nu^\theta)^2 \left( \frac{1}{M_3} - \frac{1}{M_1} - \frac{m^2 I_2^2 ||q||^2}{M_1 (I_1 M_1 - m^2 I_2^2)} \right) \right) ||q||^2, \]

where a constant has been deleted at the final equality. Substitution of (4.15) then gives the Hamiltonian (4.14).
Remark 4.3. In accord with proposition 2.3, the Hamiltonian (4.14) admits an additional $SO(2)$ symmetry because $\nu^v$ is vertical, which we can take as the diagonal action of $SO(2)$ on $(q_1, q_2), (p_1, p_2)$, with the standard conserved momentum $q_1 p_2 - q_2 p_1$. From (4.12)

$$\Pi \cdot \Gamma = q_1 \dot{p}_1 + q_2 \dot{p}_2,$$

$$= q_1 (p_2 - f q_1 \nu^v) + q_2 (- p_1 - f q_2 \nu^v) + \Pi_3 \Gamma_3,$$

$$= q_1 p_2 - q_2 p_1 - f \nu^v (q_1^2 + q_2^2) - \nu^0(1 - \|q\|^2 f),$$

$$= (q_1 p_2 - q_2 p_1) - \nu^0.$$

Consequently, the additional conserved momentum (conserved for vertical $\nu$ only) is equivalent to the subcasimir $\Pi \cdot \Gamma$ of [6].

4.3. Energy–momentum confinement, spectral stability

Theorem 4.4. The axisymmetric relative equilibrium $x_e$, with linear momentum $P_e$ and spin momentum $S_e$, of the Kirchhoff model (4.5) is $G$-stable by energy–momentum confinement if

$$mgl > \left( \frac{1}{M_3} - \frac{1}{M_1} \right) P_e^2,$$

and spectrally stable if

$$mgl > \left( \frac{1}{M_3} - \frac{1}{M_1} \right) P_e^2 - \frac{M_1}{4(I_1 M_1 - m^2 l^2)} S_e^2.$$

Proof. By proposition 3.1, it is sufficient to consider the equilibrium $q = 0, p = 0$ of the $\mathbb{R}^3 \times SO(2)_K$ reduced Kirchhoff Hamiltonian at the parameter values $\nu^v = P_e$ and $\nu^0 = S_e$. The relevant Hessian may be obtained from (4.14) by deleting terms of higher than order two, resulting in

$$H = \frac{F_p}{2} |p + F q|^2 + \frac{F_q}{2} |q|^2,$$

where, for convenience, we use the notations

$$q = q_1 + i q_2, \quad p = p_1 + i p_2, \quad F = F_1 - \frac{i}{2} S_e.$$

By assumption $F_p > 0$, so the Hamiltonian (4.18) is positive definite when $F_q > 0$, a condition which, in view of (4.15), is equivalent to (4.16).

To show (4.17), note that the linearized equations at $q = 0, p = 0$ are obtained from (4.18) by Hamilton’s equations:

$$\frac{dq}{dt} = \frac{dq_1}{dt} + i \frac{dq_2}{dt} = \frac{\partial H}{\partial p_1} + i \frac{\partial H}{\partial p_2} = F_p F q + F_p p,$$

$$\frac{dp}{dt} = \frac{dp_1}{dt} + i \frac{dp_2}{dt} = -\frac{\partial H}{\partial q_1} - i \frac{\partial H}{\partial q_2} = -(F_p |F|^2 + F_q) q - F_p \bar{F} p.$$

Due to $SO(2)$-equivariance, computing the eigenvalues of the linearization reduces to solving the equation

$$0 = \det \begin{pmatrix} F_p F - \lambda & F_q \\ -(F_p |F|^2 + F_q) & -F_p F - \lambda \end{pmatrix} = \lambda^2 + i F_p S_e \lambda + F_p F_q.$$

This has roots

$$\lambda = \frac{1}{2} \left( -i F_p S_e \pm \sqrt{-F_p^2 S_e^2 - 4 F_p F_q} \right).$$

(4.19)
so spectral stability holds if and only if
\[ F_p^2 S_e^2 + 4F_p F_q = 4F_p \left( mgl - \left( \frac{1}{M_3} - \frac{1}{M_1} \right) P_e^2 + \frac{M_1}{4(I_1 M_1 - m^2 I_2^2)} S_e^2 \right) > 0, \tag{4.20} \]
which is equivalent to (4.17).

\[ \square \]

**Remark 4.5.** Energy–momentum confinement under momentum-preserving perturbations, and spectral stability, fail coincidentally as predicted in general by proposition 3.11, and as observed in this example by [6]. We can check this by determining the conditions such that \( H + F_p \lambda (q_1p_2 - q_2p_1) \) is positive definite for some \( \lambda \in \mathbb{R} \), where \( H \) is the Hessian in (4.18).

One computes
\[ H + \lambda(q_1p_2 - q_2p_1) = \frac{1}{2} (F_p |p + Fq|^2 + 2\lambda(q_1p_2 - q_2p_1)) + \frac{F_q}{2} |q|^2, \]
\[ = \frac{1}{2} F_p (p + Fq) + i\lambda q + 2\Im(F_p (p + Fq)(\lambda q)), \]
where we have used
\[ |F_p (p + Fq) + i\lambda q|^2 = F_p^2 |p + Fq|^2 + \lambda^2 |q|^2 + 2\Im(F_p (p + Fq)(\lambda q)), \]
\[ = F_p^2 |p + Fq|^2 + \lambda^2 |q|^2 + 2\lambda F_p (q_1p_2 - q_2p_1) + |q|^2 \Im F. \]
In (4.21), the coefficient of \( |q|^2 \) is positive for some \( \lambda \) if and only if \( F_p F_q + \lambda F_p S_e - \lambda^2 \) has a real root, i.e. if and only if
\[ F_p^2 S_e^2 - 4(F_p F_q)(-1) = F_p^2 S_e^2 + 4F_p F_q > 0, \]
which is exactly the condition for spectral stability as in (4.20).

**4.4. KAM stability**

In this section we verify that (at vertical \( \nu^v \)) the equilibrium \( q = 0, p = 0 \) of the \( \mathbb{R}^3 \times \text{SO}(2) \) reduced Hamiltonian (4.10) satisfies the twist condition of the Arnold stability theorem (see [9]) in the gap between (4.16) and (4.17):

**Theorem 4.6.** The axisymmetric relative equilibrium \( x_e \) of the Kirchhoff model (4.5) (with linear momentum \( P_e \) and spin momentum \( S_e \)) is stable if
\[ p_e^2 \left( \frac{1}{M_3} - \frac{1}{M_1} \right) > mgl > p_e^2 \left( \frac{1}{M_3} - \frac{1}{M_1} \right) - \frac{M_1}{4(I_1 M_1 - m^2 I_2^2)} S_e^2. \]

For the proof we use Arnold’s theorem, see e.g. [9, chapter IX.E]. This requires us to bring the Hamiltonian into the normal form
\[ H = H_2 + H_4 \cdots + H_{2N} + H^1, \tag{4.22} \]
where
(a) \( H_{2k}, 1 \leq k \leq 2N \) are homogeneous degree \( k \) in the polynomials \( I_4 = \frac{1}{2} (q_1^2 + p_1^2) \) and \( I_2 = \frac{1}{2} (q_2^2 + p_2^2); \)
(b) \( H^1 \) is at least order \( 2N + 1 \) in \( q_1, q_2, p_1, p_2 \)
(c) \( H_2 = \omega_1 I_4 - \omega_2 I_2, \) where \( \omega_1, \omega_2 \neq 0. \)

**Theorem (Arnold stability theorem).** The origin is stable for the Hamiltonian system (4.22), provided for some \( k, 2 \leq k \leq N, \) the twist condition \( D_{2k} = H_{2k}(\omega_2, \omega_1) \neq 0 \) is satisfied.
Proof of theorem 4.6. Necessarily $\nu^0 = S_e \neq 0$, or else there is no gap. By choice of time scale, we can assume that $S_e = 1$. Also, $F_q < 0$ in view of (4.15) and (4.17). The proof requires four steps.

1. Consolidate parameters. The conditions $D_{2k} \neq 0$ of the Arnold stability theorem are algebraically complicated in the parameters. The number of these conditions that is expected to be required is one more than the number of free parameters in the Hamiltonian, so as to have the reasonable expectation that there will be no value of the free parameters for which all conditions vanish. To facilitate the symbolic computation of the required normal form, it is necessary to reduce as much as possible the number of parameters, and to cast the remaining free parameters so that they appear as simple powers.

First, the parameter $F_l$ of the Hamiltonian (4.14) may be eliminated by the substitution

$$p_1 = u_1 - \frac{F_l q_1}{\Gamma_3}, \quad p_2 = u_2 - \frac{F_l q_2}{\Gamma_3}.$$

This is a symplectic transformation because it is a shift by the exact form

$$-F_l \, d\Gamma_3 = \frac{F_l q_1}{\Gamma_3} \, dq_1 + \frac{F_l q_2}{\Gamma_3} \, dq_2.$$

By a simple direct computation

$$(p_1 + \frac{F_l}{\Gamma_3} q_1 + f q_1)^2 - (q_1 p_1 + q_2 p_2)^2 + (p_2 + F_l \Gamma_3 q_2 - f q_1)^2$$

$$= (u_1 + f q_2)^2 - (q_1 u_1 + q_2 u_2)^2 + (u_2 - f q_1)^2 - F_l^2 |q|^4,$$

so (4.10) is transformed to

$$H(q_1, q_2, u_1, u_2) = \frac{F_p}{2} (u_1 + f q_2)^2 - (q_1 u_1 + q_2 u_2)^2 + (u_2 - f q_1)^2$$

$$+ \frac{F_q}{2} \|q\|^2 + mgl \left( f - \frac{1}{2} \right) \|q\|^2,$$

where, evidently, $F_l$ is absent.

The frequencies of the linearization of $q = 0, p = 0$ are related to $F_p, F_q$ by (4.19):

$$f_1 = \frac{1}{2} (F_p + \sqrt{F_p^2 + 4F_p F_q}), \quad f_2 = \frac{1}{2} (F_p - \sqrt{F_p^2 + 4F_p F_q}),$$

and these can be inverted to give

$$F_p = f_1 + f_2, \quad F_q = -\frac{f_1 f_2}{f_1 + f_2}. \quad (4.23)$$

Note that $f_1, f_2 > 0$ and $f_1 - f_2 = \sqrt{F_p^2 + 4F_p F_q} > 0$ so that $f_1 > f_2 > 0$. Replacing $mgl$ with $-\mu$, and deleting an inessential constant, it is sufficient to consider the Hamiltonian

$$H(q_1, q_2, u_1, u_2) = \frac{f_1 + f_2}{2} ((u_1 + f q_2)^2 - (q_1 u_1 + q_2 u_2)^2 + (u_2 - f q_1)^2)$$

$$- \frac{f_1 f_2}{2(f_1 + f_2)} \|q\|^2 + \mu \left( \sqrt{1 - \|q\|^2} + \frac{1}{2} \|q\|^2 \right) \quad (4.24)$$

under the assumptions that $f_1 > f_2 > 0$.

The number of free parameters is 2, because an inessential multiplier of the Hamiltonian can reduce the triple $f_1, f_2, \mu$ by one. Thus, three twist conditions are expected to be required.
2. Linear normal form. The linearization at the origin with respect to the coordinates \((q_1, q_2, u_1, u_2)\) is

\[
\begin{pmatrix}
0 & \frac{1}{2}(f_1 + f_2) & f_1 + f_2 & 0 \\
-\frac{1}{2}(f_1 + f_2) & 0 & 0 & f_1 + f_2 \\
-\frac{(f_1 - f_2)^2}{4(f_1 + f_2)} & 0 & 0 & \frac{1}{2}(f_1 + f_2) \\
0 & -\frac{(f_1 - f_2)^2}{4(f_1 + f_2)} & -\frac{1}{2}(f_1 + f_2) & 0
\end{pmatrix},
\]

and four linearly independent eigenvectors are

\[
e_1 = \begin{pmatrix} 2(f_1 + f_2) & 0 & 0 & f_2 - f_1 \end{pmatrix},
\]

\[
e_2 = \begin{pmatrix} 0 & 2(f_1 + f_2) & f_1 - f_2 & 0 \end{pmatrix},
\]

\[
e_3 = \begin{pmatrix} 0 & 2(f_1 + f_2) & f_2 - f_1 & 0 \end{pmatrix},
\]

\[
e_4 = \begin{pmatrix} 2(f_1 + f_2) & 0 & 0 & f_1 - f_2 \end{pmatrix},
\]

with respect to which the symplectic form is canonical with multiplier \(4(f_1 - f_2)(f_1 + f_2)\), which can be ignored. The coordinates \((Q_1, P_1, Q_2, P_2)\) defined by

\[
(q_1, q_2, u_1, u_2) = Q_1 e_1 + P_1 e_2 + Q_2 e_3 + P_2 e_4,
\]

normalizes the quadratic part of \(H\) to

\[
\omega_1^2 (Q_1^2 + P_1^2) - \frac{\omega_2}{2} (Q_2^2 + P_2^2).
\]

where

\[
\omega_1 = 4(f_1 - f_2)(f_1 + f_2)f_1, \quad \omega_2 = 4(f_1 - f_2)(f_1 + f_2)f_2.
\]

The SO(2) action for vertical \(\nu\) explained in remark 4.3 is anticlockwise rotations on \((Q_1, P_1)\) and clockwise rotations on \((Q_2, P_2)\). The momentum of this action can be taken as

\[
J = -\frac{1}{2}(Q_1^2 + P_1^2) + \frac{1}{2}(Q_2^2 + P_2^2) = -I_1 + I_2.
\]

3. Nonlinear normal form. Since the Hamiltonian \(H\) is invariant under an SO(2) symmetry, we choose to seek normalizations within the corresponding class of invariant functions. Convenient invariants are

\[
w_1 = I_1 = \frac{1}{2}(Q_1^2 + P_1^2), \quad w_2 = I_2 = \frac{1}{2}(Q_2^2 + P_2^2),
\]

\[
w_3 = \frac{1}{\sqrt{2}}(Q_1 Q_2 - P_1 P_2), \quad w_4 = \frac{1}{\sqrt{2}}(Q_1 P_2 + Q_2 P_1),
\]

and these satisfy the relation

\[
2w_1 w_2 - w_3^2 - w_4^2 = 0.
\]

The Poisson bracket of the \(w_i\) is closed and the matrix \([w_i, w_j]\) is

\[
\{w_i, w_j\} = \begin{pmatrix}
0 & 0 & w_4 & -w_3 \\
0 & 0 & w_4 & -w_3 \\
-w_4 & -w_4 & 0 & -w_1 - w_2 \\
w_3 & w_3 & w_1 + w_2 & 0
\end{pmatrix}.
\]
Since $H$ is invariant it can be expanded in a Taylor series in $w_1, w_2, w_3, w_4$ as

$$H = H_2 + H_4 + \cdots + H_{2N} + H^1,$$

where

$$H_2 = \omega_1 \mathcal{L}_1 - \omega_2 \mathcal{L}_2 = \omega_1 w_1 - \omega_2 w_2,$$

and the $H_{2i}$ are degree $i$ homogeneous polynomials in $w_1, w_2, w_3, w_4$. The normalization process is formally the same in the space of invariants as it is in the space of functions of $q_1, p_1, q_2, p_2$. See [9, chapter VII.C]. The normal form is achieved at degree $k \geq 2$ successively in $k$. After normalizing the terms of degree smaller than or equal to $k$, we start with the Hamiltonian $H_{2k}$ from the previous step. The Hamiltonian $H_{2k}$ is in normal form up to order $k$, i.e. it is the sum of homogeneous degree $i \leq k$ polynomials in $w_1, w_2$ and homogeneous degree $i > k$ polynomials in $w_1, w_2, w_3, w_4$. We now look for a homogeneous polynomial $G$ of degree $k + 1$ in $w_1, w_2, w_3, w_4$ such that the time 1 flow of $G$ transforms $H$ into a function

$$\tilde{H} = H + \{G, H\} + \frac{1}{2!}\{G, \{G, H\}\} + \cdots,$$

which is in normal form at degree $k + 1$. The degree $k + 1$ term of $\tilde{H}$ is $H_{2(k+1)} + \{G, H_2\}$ because

$$\text{deg}[G, H_{2i}] = \text{deg} DG + \text{deg} DH_{2i} + 1 = k + (i - 1) + 1 = k + i,$$

so $\text{deg}[G, H_{2i}] = k + 1$ only for $i = 1$. The coefficients of $G$ are adjusted so that

$$\tilde{H}_{2k} = H_{2k} + \{G, H_2\} \quad (4.28)$$

is a function only of $w_1$ and $w_2$, after all powers of $w_4$ greater than 1 are eliminated using (4.26). The Poisson brackets here are computed using (4.27). Then $H$ is replaced by $\tilde{H}$ and the computation proceeds to step $k + 1$.

At step $k$, there always exists a $G$ such that $\tilde{H}_{2k}$ is normalized. Indeed, $\{\cdot, H_2\}$ maps the finite dimensional vector space $\mathbb{P}_{k+1}$ of homogeneous degree $k + 1$ polynomials to itself. The space $\mathbb{P}_{k+1}$ can be regarded as an inner product space by

$$\langle f, g \rangle = \int_{B_{12} \times B_{34}} fg,$$

where $B_{12}, B_{34}$ are the unit disks in the $w_1, w_2$ and $w_3, w_4$ variables. The linear map $\{\cdot, H_2\}$ is antisymmetric with respect to this inner product because

$$\langle f, \{g, H_2\} \rangle + \langle \{f, H_2\}, g \rangle = \int_{B_{12} \times B_{34}} \{fg, H_2\},$$

$$= \int_{B_{12} \times B_{34}} (\omega_1 - \omega_2) \left( -w_4 \frac{\partial (fg)}{\partial w_3} + w_3 \frac{\partial (fg)}{\partial w_4} \right),$$

$$= -(\omega_1 - \omega_2) \int_{B_{12}} \oint_{\partial B_{34}} fg (w_3 dw_3 + w_4 dw_4),$$

$$= 0,$$

because the vector field in the line integral is always orthogonal to its path. Consequently, the image of $\{\cdot, H_2\}$ is orthogonal to $\ker \{\cdot, H_2\}$ and $G$ in (4.28) can be chosen such that $\tilde{H}_{2k} \in \ker \{\cdot, H_2\}$. Moreover, $f \in \mathbb{P}_{k+1}$ is in $\ker \{\cdot, H_2\}$ if and only if

$$(\omega_1 - \omega_2) \left( -w_4 \frac{\partial f}{\partial w_3} + w_3 \frac{\partial f}{\partial w_4} \right) = 0.$$
from which $f = \bar{f}(w_1, w_2, w_3^2 + w_4^2)$, since $\omega_1 \neq \omega_2$. But in view of the relation (4.26), this implies that $f$ is a function of $w_1, w_2$, as required. We note that only the nonresonance condition $\omega_1 \neq \omega_2$ is required and the normalization could in theory be achieved at any order.

The normal form cannot be hand-computed, but can be computed using a symbolic manipulator. The result up to and including $H_8$ (the notation $1 \leftrightarrow 2$ means the preceding fragment with $w_1$ and $w_2$ exchanged) is

\[ H_4 = 8(f_1 + f_2)^3(A_{20}w_1^2 + \frac{1}{2}A_{11}w_1w_2 + 1 \leftrightarrow 2), \]

\[ H_6 = -\frac{32(f_1 + f_2)^5}{(f_1 - f_2)^2}(A_{30}w_1^4 + A_{31}w_1^2w_2 + 1 \leftrightarrow 2), \]

\[ H_8 = -\frac{64(f_1 + f_2)^7}{(f_1 - f_2)^4}(A_{40}w_1^4 + A_{41}w_1^2w_2 + \frac{1}{2}A_{42}w_1^2w_2^2 + 1 \leftrightarrow 2), \]

\[ A_{20} = (-f_1 - f_2)\mu + (f_1 + f_2)f_2, \]

\[ A_{21} = (-4f_1 - 4f_2)\mu + 8f_1f_2, \]

\[ A_{30} = 2(f_1 + f_2)^3\mu^2 - 2(f_1 + f_2)(3f_1 + f_2)\mu f_2 + 4(f_1 + f_2)f_1f_2^2, \]

\[ A_{31} = 15(f_1 + f_2)^4\mu^2 - (f_1 + f_2)(5f_1^2 + 44f_1f_2 + 11f_2^2)\mu + (5f_1^2 + 38f_1f_2 + 17f_2^2)f_1f_2, \]

\[ A_{32} = 15(f_1 + f_2)^4\mu^2 - (f_1 + f_2)(11f_1^2 + 44f_1f_2 + 5f_2^2)\mu + (17f_1^2 + 38f_1f_2 + 5f_2^2)f_1f_2, \]

\[ A_{40} = -16(f_1 + f_2)^3\mu^2 + 2(f_1^2 + 36f_1f_2 + 11f_2^2)(f_1 + f_2)^2\mu^2 - 2(f_1 + f_2)(2f_1^3 + 55f_1^2f_2 \]

\[ + 36f_1f_2^2 + 3f_2^3)\mu f_2 + 2(f_1 + f_2)(f_1^3 + 26f_1f_2 + 5f_2^2)f_1f_2^2, \]

\[ A_{41} = -182(f_1 + f_2)^3\mu^3 + 26(3f_1^2 + 31f_1f_2 + 8f_2^2)(f_1 + f_2)^2\mu^2 \]

\[ - 2(13f_1 + f_2)(f_1 + f_2)(8f_1^2 + 49f_1f_2 + 21f_2^2)\mu f_2 \]

\[ + 2(65f_1^2 + 367f_1f_2 + 267f_1f_2 + 29f_2^2)f_1f_2^2, \]

\[ A_{42} = -354(f_1 + f_2)^3\mu^3 + 3(95f_1^2 + 518f_1f_2 + 95f_2^2)(f_1 + f_2)^2\mu^2 - 3(f_1 + f_2) \]

\[ \times (9f_1^4 + 274f_1^3f_2 + 850f_1^2f_2^2 + 274f_1f_2^3 + 9f_2^4)\mu + 3(9f_1^4 + 204f_1^3f_2 \]

\[ + 518f_1^2f_2^2 + 204f_1f_2^3 + 9f_2^4)f_1f_2. \]

4. Twist condition. The twist condition in the Arnold stability theorem is $D_3 = H_2(\omega_2, \omega_1) \neq 0$ for some $k \geq 2$. From the normal form, the first three $D_3$ are

\[ D_4 = 128(f_1 - f_2)^2(f_1 + f_2)^3(-(f_1 + f_2)(f_1^2 + 4f_1f_2 + f_2^2)\mu + (f_1^2 + 10f_1f_2 + f_2^2)f_1f_2^2), \]

\[ D_6 = -2048(f_1 + f_2)^3((2f_1^2 + 13f_1f_2 + 2f_2^2)(f_1 + f_2)^2\mu^2 - (f_1 + f_2)(11f_1^2 \]

\[ + 46f_1f_2 + 11f_2^2)\mu f_1f_2 + (9f_1^2 + 50f_1f_2 + 9f_2^2)f_1f_2^2)(f_1 - f_2), \]

\[ D_8 = 16384(f_1 + f_2)^3(-2(8f_1^4 + 91f_1^3f_2 + 177f_1^2f_2^2 + 91f_1f_2^3 + 8f_2^4)(f_1 + f_2)^3 \mu^3 \]

\[ + (2f_1^6 + 150f_1^5f_2 + 1113f_1^4f_2^2 + 1970f_1^3f_2^3 + 1113f_1^2f_2^4 \]

\[ + 150f_1f_2^5 + 2f_2^6)(f_1 + f_2)^2\mu^2 - (f_1 + f_2)(4f_1^6 + 345f_1^5f_2 + 2226f_1^4f_2^2 \]

\[ + 3850f_1^3f_2^3 + 2226f_1^2f_2^4 + 345f_1f_2^5 + 4f_2^6)\mu f_1f_2 + (2f_1^6 + 211f_1^5f_2 \]

\[ + 1466f_1^4f_2^3 + 2642f_1^3f_2^4 + 1466f_1^2f_2^5 + 211f_1f_2^6 + 2f_2^7)f_1f_2^2). \]
We assume $D_4, D_6, D_8$ vanish and argue by contradiction. The expressions are homogeneous in $f_1, f_2$, so one can assume $f_1 = 1$. Since $D_4$ is linear in $\mu$, if the coefficient of $\mu$ vanishes, then the assumption $D_4 = 0$ implies the constant term in $\mu$ also vanishes, i.e.
\[-(f_1 + 1)(1 + 4f_1 + f_1^2) = 0, \quad \text{and} \quad f_1(f_1^2 + 10f_1 + 1) = 0.\]
But these two polynomials in $f_1$ cannot simultaneously vanish because their gcd is 1, as can be computed with the Euclidean algorithm. So the coefficient of $\mu$ in $D_2$ does not vanish. Solving $D_4$ for $\mu$ and substituting into $D_6$ and $D_8$ results in
\[
0 = 10240(f_1 + 1)^6 f_1^3 (f_1 + 5)(5f_1 + 1)(f_1 - 1)^3, \quad 0 = -245760(f_1 + 1)^{11} f_1^4 (21f_1^4 + 172f_1^4 + 334f_1^2 + 172f_1 + 21)(f_1 - 1)^4.
\]
But these cannot simultaneously vanish because their gcd is $10240(f_1 + 1)^6 f_1^3 (f_1 - 1)^3$, which is nonzero in view of $f_1 > f_2 = 1$.

**Remark 4.7.** The computer generated normal form used in theorem 4.6 requires an independent check, and one possible check follows. The reduction by $\text{SO}(2)$ of the $\text{SO}(2)$-invariant Hamiltonian (4.24) gives a one parameter family of 2-dimensional Hamiltonian systems parametrized by the $\text{SO}(2)$ angular momentum. All trajectories of these reduced Hamiltonian systems near $q = 0, u = 0$ are periodic. The periods of these orbits depend only on the energy and the $\text{SO}(2)$ momentum, and may be numerically computed. On the other hand, the successive normal form Hamiltonians
\[
H_{4}^{nf} = H_2 + H_4, \quad H_{6}^{nf} = H_2 + H_4 + H_6, \quad H_{8}^{nf} = H_2 + H_4 + H_6 + H_8,
\]
are functions of $w_1$ and $w_2$. Using the Poisson bracket (4.27), the differential equations for $w_1$ are
\[
\frac{dw_1}{dt} = \frac{dw_2}{dt} = 0, \quad \frac{dw_3}{dt} = -\left(\frac{\partial H_{21}^{nf}}{\partial w_1} + \frac{\partial H_{21}^{nf}}{\partial w_2}\right) w_4, \quad \frac{dw_4}{dt} = \left(\frac{\partial H_{21}^{nf}}{\partial w_1} + \frac{\partial H_{21}^{nf}}{\partial w_2}\right) w_5.
\]
These differential equations are the $\text{SO}(2)$ reduction of the Hamiltonian system (4.24), and the orbits have periods
\[
T_{21}^{nf} = \left|\frac{\partial H_{21}^{nf}}{\partial w_1} + \frac{\partial H_{21}^{nf}}{\partial w_2}\right|.
\]
which also depend only on the energy and the $\text{SO}(2)$ momentum. For energy and momentum obtained from initial conditions $\epsilon q^0, \epsilon p^0$, the periods of $H_{21}^{nf}$, computed directly from (4.30), can be compared with numerically computed periods of the Hamiltonian (4.24). If the normal form is correct, then the difference of these must fall as $\epsilon^2$, because the period from $H_{21}^{nf}$ is accurate to order $i - 1$ in $\epsilon$, and therefore its difference falls as order $i$, which is order $2i$ in $\epsilon$. If the normal form is not correct, then it is not likely that the anharmonic periods it predicts will agree at the proper order with the periods of its nonnormalized precursor.

We have implemented this check for the data
\[
f_1 = \frac{\sqrt{5}}{2}, \quad f_2 = 1, \quad \mu = \frac{1}{2}, \quad q^0 = (0.5, 1.0), \quad p^0 = (-0.75, 0), \quad \epsilon = 0.0001, 0.0002, 0.0004, 0.0008, 0.0016.
\]
The choice of $f_1$ is convenient because then $4(f_1 - f_2)(f_1 + f_2) = 1$ and the symplectic multiplier of theorem 4.6 does not have to be accounted for. In table 1 are the numerically
Table 1. Comparison, at the same energy–momentum values, of numerically computed periods, and periods computed from the normal forms $H_{nf}^2$. The correct normal form is indicated because the values of $r$ in the columns are nearly the corresponding values of $2i$.

| $10^4\epsilon$ | $10^0\epsilon$ | $2i = 4$ | $2i = 6$ | $2i = 8$ |
|----------------|---------------|---------|---------|---------|
|                | $T(\epsilon)/T_0$ | $T_{nf}^2(\epsilon)/T_0$ | $r_{k,\epsilon}$ | $T_{nf}^2(\epsilon)/T_0$ | $r_{k,\epsilon}$ | $T_{nf}^2(\epsilon)/T_0$ | $r_{k,\epsilon}$ |
| 1              | 0.999 541 055 368 8 | 0.999 541 055 70 | 4.0 | 0.999 541 055 368 4 | 8.2 |
| 2              | 0.998 171 547 772 2 | 0.998 171 569 06 | 6.0 | 0.998 171 547 656 3 | 8.0 |
| 4              | 0.992 800 538 653 8 | 0.992 801 852 44 | 5.8 | 0.992 800 510 140 6 | 7.8 |
| 8              | 0.972 866 988 796 4 | 0.972 940 314 78 | 5.4 | 0.972 860 722 212 6 | 7.2 |
| 16             | 0.910 847 344 469 3 | 0.913 852 682 17 | 5.8 | 0.909 879 230 387 2 |

Computed periods $T(\epsilon)$, and the periods $T_{nf}^2(\epsilon)$ computed using the normal forms, as ratios of the period

$$T_0 = \frac{2\pi}{f_1 - f_2} = \frac{2\pi}{\sqrt{5} - 1},$$

of the linearization. Also tabulated are the values called $r_{2i,\epsilon}$, which are the base two logarithms of the successive (as $\epsilon$ is successively doubled) ratios of the differences between the periods $T_{nf}^2(\epsilon)$ and $T(\epsilon)$, i.e.$$
r_{2i,\epsilon} = \log_2 \left| \frac{T_{nf}^2(2\epsilon) - T(2\epsilon)}{T_{nf}^2(\epsilon) - T(\epsilon)} \right|.$$

In the column corresponding to $H_{nf}^2$, there is agreement between $r_{2i,\epsilon}$, which is the observed order that the difference falls as, and $2i$, which is the predicted order. We conclude that there is numerical evidence that the normal forms $H_{nf}^2$ are correct, because they predict the correct anharmonic period with an error of order $2i$.

5. Numerics

In this section we simulate small dissipation directly in the $\mathbb{R}^3 \times SO(2)$ reduction of the Kirchhoff model, as computed in proposition 4.1. We verify that, after addition of small dissipation,

(a) stability is maintained for the axisymmetric relative equilibria $x_\epsilon$ which are in the EM-region determined by (4.16) and

(b) stability is destroyed for the axisymmetric relative equilibria $x_\epsilon$ which are in the gap determined by (4.16) and (4.17).

This confirms that the stability of the system in the gap is qualitatively less robust than its stability in the EM-region, and therefore confirms that energy–momentum confinement fails in the gap, with physically significant consequences. The approach of simulating the $\mathbb{R}^3 \times SO(2)$ reduced system, rather than the full Kirchhoff model (4.1), is consistent with our development in sections 2 and 3, and is convenient for the simulation of energy dissipation which preserves the $\mathbb{R}^3 \times SO(2)$ momentum and symmetry (see remark 3.7).

5.1. Splitting method

The Hamiltonian (4.10) may be split as

$$H = H_0 + H_1 + H_2 + H_3,$$
where
\[
H_0 = H(q_1, q_2, 0, 0),
\]
\[
H_1 = \frac{F_p}{2} (p_1^2 + p_2^2 - 2q_2 p_2 q_1 p_1 - q_1^2 p_1^2 - q_2^2 p_2^2),
\]
\[
H_2 = F_p F_l(q_1 p_1 + q_2 p_2) \sqrt{1 - q_1^2 - q_2^2},
\]
\[
H_3 = -\frac{F_p \nu^\theta (p_2 q_1 - p_1 q_2)}{1 + \sqrt{1 - q_1^2 - q_2^2}}.
\]
Only $H_0$ depends on $\nu$, and all but $H_0$ are $SO(2)$ invariant under the diagonal action of $SO(2)$ on $(q_1, q_2)$, $(p_1, p_2)$, whereas $H_0$ has this invariance only for vertical $\nu$ (see remark 4.3).

There are exact formulas for the flows of each $H_i$, as follows:

(a) $H_0$ is a function of $q$ only and hence its flow is $p \mapsto p - i \nabla H_0$.

(b) The flow of $H_1$ corresponds to the motion of a particle of mass $1/F_p$ freely moving on a sphere of radius 1. Indeed, for the top half of the sphere with coordinates
\[
x = q_1 e_1 + q_2 e_2 + \sqrt{1 - q_1^2 - q_2^2} \, e_3,
\]
the kinetic energy metric of the particle motion, and its inverse, are, respectively,
\[
\frac{1}{F_p \Gamma^2} \begin{pmatrix} 1 - q_2^2 & q_1 q_2 \\ q_1 q_2 & 1 - q_1^2 \end{pmatrix}, \quad \frac{F_p}{\Gamma^2} \begin{pmatrix} 1 - q_1^2 & -q_1 q_2 \\ -q_1 q_2 & 1 - q_2^2 \end{pmatrix},
\]
the second of which is twice the matrix corresponding to the quadratic form $H_1$.

(c) Since $H_2$ is linear in $p$, and $p_1$, $p_2$ are separated, the differential equations for $q_1$, $q_2$ close.

As is easily verified, $q_1/q_2$ is conserved for these equations and so $q_1$, $q_2$ evolve along radial lines. The radial equation can be separated and integrated, and the evolution of $p_1$, $p_2$ follows that from conservation of energy and momentum.

(d) As is easily verified, $q_1^2 + q_2^2$ is conserved for the flow of $H_3$. Using this, the differential equations corresponding to $H_3$ become linear and may be integrated.

For a dissipation vector field, one could use a positive multiple of the negative gradient in $p_1$, $p_2$ of the reduced Hamiltonian (4.10), i.e. the vector field
\[
\left(- (1 - q_1^2) p_1 + q_1 q_2 p_2 - \frac{m \nu^\theta \Gamma_3}{M_1} q_1 - \frac{\nu^\theta}{1 + \Gamma_3} q_2\right) \frac{\partial}{\partial p_1} \\
+ \left(q_1 q_2 p_1 - (1 - q_2^2) p_2 - \frac{m \nu^\theta \Gamma_3}{M_1} q_2 + \frac{\nu^\theta}{1 + \Gamma_3} q_1\right) \frac{\partial}{\partial p_2}, \tag{5.1}
\]
This does not preserve the momentum $q_1 p_2 - p_1 q_2$ which occurs for vertical $\nu$ (see remark 4.3). But the direction of (5.1) relative to the ker $DH$ hyperplane (for $q_1$, $q_2$ constant) is unaltered by projection: any projection of (5.1) onto any subspace will still dissipate energy. So we project onto the line tangent to the level set of the function $q_1 p_2 - p_1 q_2$, i.e. onto the vector $(q_1, q_2)$. Since the inner product of (5.1) and $(q_1, q_2)$ is
\[
(-1 + q_1^2 + q_2^2) q_1 p_1 + (-1 + q_1^2 + q_2^2) q_2 p_2 - \frac{m \nu^\theta \Gamma_3}{M_1} (q_1^2 + q_2^2)
\]
\[
= - \Gamma_3^2 (q_1 p_1 + q_2 p_2) - \frac{m \nu^\theta \Gamma_3}{M_1} (q_1^2 + q_2^2),
\]
we use for the dissipative perturbation vector field $\epsilon R$, where $\epsilon$ is small and positive, and
\[
R = (- (q_1 p_1 + q_2 p_2) \Gamma_3 - F_l(q_1^2 + q_2^2) q_1 \frac{\partial}{\partial p_1} + (- (q_1 p_1 + q_2 p_2) \Gamma_3 - F_l(q_1^2 + q_2^2) q_2 \frac{\partial}{\partial p_2},
\tag{5.2}
\]
The differential equations corresponding to this vector field are linear and easily solved exactly.

The full system, dissipation included, is given by the vector field
\[ \epsilon R + \sum_{i,s} \left( \frac{\partial H_s}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H_s}{\partial q_i} \frac{\partial}{\partial p_i} \right). \] (5.3)

Letting the flows of the Hamiltonians \( H_s \), \( 0 \leq s \leq 3 \), be \( F^H_s \), and the flow of the dissipation \( \epsilon R \) be \( G \), the concatenation
\[ G_{\Delta t/2} F^H_{\Delta t/2} F^H_{\Delta t/2} F^H_{\Delta t/2} F^H_{\Delta t/2} F^H_{\Delta t/2} F^H_{\Delta t/2} F^H_{\Delta t/2} G_{\Delta t/2} \]
is a second order one step method, which we use to simulate the system. At vertical \( \nu \), this method preserves the conserved quantity \( q_1 p_2 - q_2 p_1 \) because every one of its steps does that separately. The method is symplectic if \( \epsilon = 0 \).

### 5.2. Simulations

The simulations that we report in this section are done at parameter values
\[ I_1 = 4, \quad M_1 = 1, \quad M_3 = \frac{1}{2}, \quad m = 1, \quad l = 1, \quad g = 1, \quad S_e = 6, \quad \nu a_3 = P_e, \quad \nu a_1 = S_e, \] (5.4)
i.e. units are such that \( g = 1 \). The initial conditions are chosen so that
\[ q_2 = 0, \quad p_1 = 0, \quad p_2 = 0, \quad \nu a_2 = 0. \]

The perturbation from the equilibrium is achieved by choosing small nonzero \( q_1 \) and \( \nu a_1 \).

From (4.4), energy–momentum confinement occurs for \( P_e^2 < C_1 \) and spectral stability occurs for \( P_e^2 < C_2 \), where
\[ C_1 = \frac{M_1 M_3}{M_1 - M_3} mgl, \quad C_2 = \frac{M_1 M_3}{M_1 - M_3} \left( mgl + \frac{M_1}{4(I_1 M_1 - m^2 l^2)} S_e^2 \right). \]

For the parameter values (5.4), \( C_1 = 1 \) and \( C_2 = 4 \), so
\[ 2 < P_e: \quad \text{spectral instability;} \]
\[ 1 < P_e < 2: \quad \text{gap; spectral stability;} \]
\[ 0 < P_e < 1: \quad \text{EM-region; stability.} \]

#### 5.2.1. Stability

To the left, figure 1 shows a regular time sampling of the radius \( r = \sqrt{q_1^2 + q_2^2} \) versus time. The top and bottom graphs correspond, respectively, to \( P_e = 1.5 \) (which is inside the gap) and \( P_e = .5 \) (which is inside the EM-region). The graphs have been obtained from
\[ \epsilon = .05, \quad q_1 = 0.05/r, \quad \nu a_i = 0.01/k, \quad k = 1, 1.5, 2, 3, 4, \]
\[ \epsilon = .1, \quad q_1 = 0.05/r, \quad \nu a_i = 0.01/k, \quad k = \sqrt{2}, \quad (\sqrt{2})^2, \quad (\sqrt{2})^3, \quad (\sqrt{2})^6, \quad (\sqrt{2})^{10}, \]
and for \( 3 \times 10^5 \) and \( 1.2 \times 10^6 \) periods, respectively. Here, 'period' means the smallest period of the normal modes of the linearization at the equilibrium \( q = p = 0 \), i.e. \( 2\pi/\omega \) where \( \omega \) is the largest of the imaginary parts of the eigenvalues computed from (4.19). The top and bottom of figure 1 correspond to the first and second lines of initial data above, respectively. The time step for the simulations was \( \Delta t \approx .04453 \). The simulations were stopped if \( r \) exceeded 0.5, corresponding to a vehicle configuration which is skewed 30° to the vertical.
Figure 1. Top left: instability in the gap. Bottom left: stability in the EM-region. Dissipation is added in both the top and bottom, and the energy decay is shown in the graphs on the right.

On the top left, instability is indicated in the gap because large deviations from the equilibrium are observed for a sufficiently long time. At the bottom left, stability is indicated in the EM-region because of the decreasing deviation from the equilibrium for decreasing initial conditions, irrespective of the elapsed simulation time. At the right, energy dissipation is demonstrated by plotting the energy minus the initial energy ($\Delta H$) against time.

5.2.2. Transitions. On the bottom left, figure 2 shows the maximum value of $r$ over $1.2 \times 10^6$ periods, for

$$\epsilon = 0, \quad q_1 = 0.0125, \quad \nu^d = 0.0025.$$  

$\Delta t$ was adjusted to 40 time steps per period. On the right is the same except for $\epsilon = 0.05$. For the no-dissipation runs on the left, stability in the gap between $C_1$ and $C_2$ is indicated by the small maximum values for $r$. On the right, large deviations from the equilibrium occur over the entire gap between $C_1$ and $C_2$.

The peak at $C_1$ in the left graph indicates loss of stability at that transition, even in the purely Hamiltonian context. This is due to the presence of a zero eigenvalue in the linearization (see (4.19) when $F_q = 0$). At $C_1$, KAM stability as discussed in sections 3.1 and 4.4 is not present because it requires perturbation from an elliptic equilibrium but the equilibrium is not elliptic, it has a 0 eigenvalue.

The equilibria of the $\dot{w}$-equation (5.3) at $q = p = 0$ and vertical momentum $\nu$ persist to nearby nonvertical momenta. As further evidence of dissipation induced instability, we have computed the corresponding spectrum of the linearization of these equilibria, which were found numerically by Newton’s method with start at $q = p = 0$. As shown in the top of
Stability transitions for axisymmetric relative equilibria

Figure 2. Destruction of KAM stability of an axisymmetric relative equilibria in the gap between $C_1$ and $C_2$ as a result of adding small dissipation. Left: no dissipation, right: added dissipation. Top: plots the real parts of the spectrum of nearby persisting relative equilibria. The scale on the lower parts of the graphs differs from the scale of the upper parts so that the sign change at $C_1$ is visible. The computation was done with the program MAPLE at 18 digits of accuracy. Bottom: large $r$ indicates instability.

figure 2, at zero dissipation, we observe zero real parts in the spectrum throughout both the EM-region and the gap. For nonzero dissipation, the spectrum splits and has negative real parts in the EM-region and positive real parts in the gap. The real parts in the gap are small as compared with the real parts after the Hopf bifurcation, necessitating the two vertical scales in the figure.

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References

[1] Abraham R and Marsden J E 1978 Foundations of Mechanics 2nd edn (Reading, MA: Addision-Wesley)
[2] Arnold V I 1978 Mathematical Methods of Classical Mechanics (Berlin: Springer)
[3] Bloch A, Krishnaprassad P S, Marsden J E and Ratiu T S 1994 Dissipation induced instabilities Ann. Inst. H. Poincaré. Anal. Nonlinéaire 11 37–90
[4] Derks G and Ratiu T 2002 Unstable manifolds of relative equilibria in Hamiltonian systems with dissipation Nonlinearity 15 531–49
[5] Fassò F, Guzzo M and Benettin G 1998 Nekhoroshev-stability of elliptic equilibria of Hamiltonian systems Commun. Math. Phys. 197 347–60
[6] Leonard N E and Marsden J E 1997 Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry Physica D 105 130–62
[7] Leonard N E 1997 Stability of a bottom-heavy underwater vehicle Autom.—J. IFAC 33 331–46
[8] Marsden J E and Ratiu T S 1994 Introduction to Mechanics and Symmetry (Texts in Applied Mathematics) vol 17 (Berlin: Springer)
[9] Meyer K R and Hall G R 1991 Introduction to Hamiltonian Dynamical Systems and the $N$-Body Problem (Berlin: Springer)
[10] Montaldi J A, Roberts R M and Stewart I N 1988 Periodic solutions near equilibria of symmetric Hamiltonian systems Phil. Trans. R. Soc. Lond. A 325 237–93
[11] Niederman L 1998 Nonlinear stability around an elliptic equilibrium point in a Hamiltonian system Nonlinearity 11 1465–79
[12] Patrick G W, Roberts R M and Wulff C 2004 Stability of Poisson equilibria and Hamiltonian relative equilibria by energy methods Arch. Ration. Mech. Anal. 174 36–52
[13] Patrick G W 1992 Relative equilibria in Hamiltonian systems: the dynamic interpretation of nonlinear stability on the reduced phase space J. Geom. Phys. 9 111–9
[14] Patrick G W 2003 Stability by KAM confinement of certain wild, nongeneric relative equilibria of underwater vehicles with coincident centers of mass and bouyancy SIAM J. Appl. Dyn. Syst. 2 36–52
[15] Roberts M, Wulff C and Lamb J 2002 Hamiltonian systems near relative equilibria J. Diff. Eqns 179 562–604
[16] Roberts R M, Schmah T and Stoica C 2006 Relative equilibria in systems with configuration space isotropy J. Geom. Phys. 56 762–79
[17] van der Meer J-C 1990 Hamiltonian Hopf bifurcation with symmetry Nonlinearity 3 1041–56
[18] Weinstein A 1977 Lectures on Symplectic Manifolds (Regional Conference Series in Mathematics vol 29) (Providence, RI: American Mathematical Society)