Abstract—With the fast development of information technology, especially the popularization of Internet, multi-view learning becomes more and more popular in machine learning and data mining fields. As we all know that, multi-view semi-supervised learning, such as co-training, co-regularization has gained considerable attentions. Although recently, multi-view clustering (MVC) has developed rapidly, there are not a survey or review to summarize and analyze the current progress. Therefore, this paper sums up the common strategies of combining multiple views and based on that we proposed a novel taxonomy of the MVC approaches. We also discussed the relationships between MVC and multi-view representation, ensemble clustering, multi-task clustering, multi-view supervised and multi-view semi-supervised learning. Several representative real-world applications are elaborated. To promote the further development of MVC, we pointed out several open problems that are worth exploring in the future.

Index Terms—Multi-view learning, clustering, survey, nonnegative matrix factorization, k means, spectral clustering, subspace clustering, canonical correlation analysis, machine learning, data mining.

I. INTRODUCTION

Clustering [1] is a paradigm to classify the subjects into several groups based on their similarity information. As we know that clustering is a fundamental task in machine learning, pattern recognition and data mining fields and it has widespread applications. With the obtained groups by clustering methods, further analysis tasks can be conducted to achieve different ultimate goals. However, traditional clustering methods only use one feature set or one view information of the subjects while multiple feature sets or multiple view information of these subjects are available. The subjects of interest with multiple feature sets or multiple view information are the so-called multi-view data.

Multi-view data are very common in real-world applications due to the innate properties, or collecting from different sources. For instance, a web page can be described by the words appearing on the web page itself and the words underlying all links pointing to the web page from other pages in nature. In multimedia content understanding, the multimedia segments can be simultaneously described by their video signals from visual camera and audio signals from voice recorder devices. The existence of such multi-view data raised the interest of multi-view learning [2, 3, 4], which has been extensively studied in semi-supervised setting. However, for unsupervised learning, especially previous single view clustering methods cannot make full use of the information from multiple views, like running single view clustering algorithm on the concatenated features from multiple views that cannot distinguish the different significance of different views. To make full use of these multiple view information to boost clustering accuracy, multi-view clustering attracted more and more attentions in the past two decades, which makes it the time and necessary to summarize and sort out the current progress and open problems to guide its further advancement in the future.

Now, based on the above concepts, we give the definition of the multi-view clustering (MVC). MVC is a machine learning paradigm to classify the similar subjects into the same group and dissimilar subjects into different groups by combining the available multi-view feature information, which indicates that MVC searches for the consistent clusterings across different views. Consistent with the categorization of clustering algorithms in [1], we divide the existing MVC methods into two categories: generative (or model-based) approaches and discriminative (or similarity-based) approaches. Generative approaches try to learn a generative model from the subjects, with each model representing one cluster while discriminative approaches directly optimize an objective function that involves the pairwise similarities to minimize the average similarities within clusters and to maximize the average similarities between clusters. Due to a large number of discriminative approaches, based on how to combine the multi-view information, we further classified them into five groups: (1) common eigenvector matrix (mainly multi-view spectral clustering), (2) common coefficient matrix (mainly multi-view subspace clustering), (3) common indicator matrix (mainly multi-view nonnegative matrix factorization clustering), (4) direct combination (mainly multi-kernel clustering), (5) combination after projection (mainly canonical correlation analysis (CCA)). The first three groups have a commonality: sharing a similar structure to combine multiple views.

Research on MVC is motivated by the multi-view real applications. With the same motivation, multi-view representation, multi-view supervised and multi-view semi-supervised learning emerged and developed well. Therefore, the similarities and differences of them are worth exploring. The common similarity between them is that all of them are learned with the multi-view information. With regard to the differences, multi-view representation aims to learn a compact representation while MVC is to perform clustering, MVC is learned without any label information while multi-view supervised and multi-view semi-supervised learning have all or part of label information. Some of the view combining strategies in these related paradigms can be borrowed and adapted to MVC. In addition, the relationship between MVC and ensemble clustering and multi-task clustering are also elaborated due to their similar

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clustering style.

Recently, MVC has been applied to many applications such as computer vision, natural language processing, social multimedia, bioinformatics, health informatics and so on. At the same time, MVC papers appear largely in many top venues like the conferences ICML [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], NIPS [19], [20], CVPR [21], [22], [23], [24], ICCV [25], AAAI [26], [27], [28], [29], [30], IJCAI [31], [32], [33], [34], [35], [36], [37], SDM [38], [39], ICDM [40], [41], [42], [43], [44], and journals PAMI [45], TKDE [46], [47], [48], [49], TCYB [51], [52], TIP [53], TNNLS [54]. Although MVC has permeated into many fields and made great practical success, there are still many open problems to be further developed.

The remainder of this paper is organized as follows. In section II we review the existing generative models of MVC. Section II introduces several categories of discriminative models of MVC. In Section IV we analyze the relationship between MVC and some related topics. Section V presents the applications of MVC in different areas. In Section VI we list several open problems existing current MVC methods, which may help us to advance the further development of MVC. Finally, we make the conclusions.

II. GENERATIVE APPROACHES

Generative approaches aim to learn the generative models of each of which generates the data from one cluster. In most cases, generative clustering approaches are based on mixture models or expectation maximization (EM) [55]. Therefore, mixture models and EM algorithm are first of all introduced. Another popular single view clustering model named convex mixture models (CMMs) [56] is also introduced, which will be extended to multi-view case.

1) Mixture Models and CMMs: In generative approach, data are considered as sampling independently from a mixture model of multiple probability distributions. The mixture distribution can be written as

\[ p(x|\theta) = \sum_{k=1}^{K} \pi_k p(x|\theta_k), \]

where \( \pi_k \) is the prior probability of the \( k \)th component and satisfies \( \pi_k \geq 0 \), and \( \sum_{k=1}^{K} \pi_k = 1 \). \( \theta_k \) is the parameter of the \( k \)th probability density model and \( \theta = \{\pi_k, \theta_k, k = 1, 2, \ldots, K\} \) is the parameter set of the mixture model. For instance, \( \theta_k = \{\mu_k, \Sigma_k\} \) for Gaussian mixture model.

EM is a widely used algorithm in the parameter estimation of the mixture models. Suppose the observed data and unobserved data are denoted by \( X \) and \( Z \), respectively. \( \{X, Z\} \) and \( X \) are called complete data and incomplete data. In the E (expectation) step, the posterior distribution \( p(Z|X, \theta^{old}) \) of the unobserved data is evaluated with the current parameter values \( \theta^{old} \). According to maximum likelihood estimation, the E step calculates the expectation of the complete data log likelihood as

\[ Q(\theta, \theta^{old}) = \sum_{z} p(Z|X, \theta^{old}) \ln p(X, Z|\theta). \]

The M step updates the parameters by maximizing the function \( Q \)

\[ \theta = \arg \max_{\theta} Q(\theta, \theta^{old}). \]

Note that for clustering, \( X \) can be considered as the observed data while \( Z \) is the latent variable whose entry \( z_{nk} \) indicates the \( n \)th data point comes from the \( k \)th component. Note that the posterior distribution form used to be evaluated in E step and the expectation of the complete data log likelihood used to evaluate the parameters are different for different distribution assumptions.

CMMs [56] are simplified mixture models that result in soft assignments of data points to clusters after extracting the representative exemplars from the data set. By maximizing the log-likelihood, all instances compete to become the “center” (representative exemplar) of the clusters. The instances corresponding to the components that received the highest priors are selected exemplars and then the remaining instances are assigned to the “closest” exemplar. Note that the priors of the components are the only adjustable parameters of a CMM.

Given a data set \( X = x_1, x_2, \ldots, x_N \in \mathbb{R}^{d \times N} \), the CMM distribution is \( Q(x) = \sum_{j=1}^{N} q_j f_j(x), x \in \mathbb{R}^{d} \), where \( q_j \geq 0 \) denotes the prior probability of the \( j \)th component, satisfying the constraint \( \sum_{j=1}^{N} q_j = 1 \), and \( f_j(x) \) is an exponential family distribution, with its expectation parameters equal to the \( j \)th data point. Due to the bijection relationship between the exponential families and Bregman divergences [57], the exponential family \( f_j(x) = C_\phi(x)\exp(-\beta d_\phi(x, x_j)) \), with \( d_\phi \) denoting the Bregman divergence corresponding to the components’ distributions, \( C_\phi(x) \) being independent of \( x_j \), and \( \beta \) being a constant controlling the sharpness of the components.

The log-likelihood needs to be maximized is given as

\[ L(X; \{q_j\}_{j=1}^{N}) = \frac{1}{N} \sum_{i=1}^{N} \log(\sum_{j=1}^{N} q_j f_j(x_i)) = \frac{1}{N} \sum_{i=1}^{N} \log(\sum_{j=1}^{N} q_j \exp(-\beta d_\phi(x_i, x_j))) + \text{const.} \]

With the empirical data set distribution definition \( P = \frac{1}{N} \), the log-likelihood maximization can be equivalently expressed in terms of Kullback-Leibler (KL) divergence among \( P \) and \( Q(x) \) as

\[ D(P||Q) = -\frac{1}{N} \sum_{i=1}^{N} \hat{P}(x_i) \log Q(x_i) - \mathbb{H}(\hat{P}) \]

where \( \mathbb{H}(\hat{P}) \) is the entropy of the empirical distribution \( \hat{P}(x) \) which does not depend on the parameter \( q_j \). Now, the problem is changed into minimizing \( D \), which is convex and can be solved with an iterative algorithm, whose updates for prior probabilities are given by

\[ q_j^{(t+1)} = q_j^{(t)} \frac{\hat{P}(x_i) f_j(x_i)}{\sum_{i=1}^{N} q_j^{(t)} f_j(x_i)}. \]

Grouping the data points into \( K \) disjoint clusters is done by requiring the instances with the \( K \) highest \( q_j \) values to serve
as exemplars and then assigning the remaining instances to the exemplar with the highest posterior probability. Note that the clustering performance is affected by the constant \( \beta \), in [56] a reference value \( \beta_0 \) is determined with the empirical rule \( \beta_0 = N^2 \log N / \sum_{i,j=1}^N d_\phi(x_i, x_j) \) to identify a reasonable range of \( \beta \).

2) Multi-View Clustering Based on Mixture Models or EM Algorithm: In [58], under the assumption that the two views are independent, multinomial distribution is adopted for document clustering problem. Take two-view case as an example, they execute M, E steps on each view and then interchange the posteriors in each iteration. The optimization process is terminated until some predefined stopping condition is satisfied. Two multi-view EM algorithm versions for finite mixture models are proposed in the paper [59]: the first version can be regarded as that it runs EM in each view and combines all the weighted probabilistic clustering labels generated in each view before each new EM iteration while the second version can be viewed as some probabilistic information fusion for components of two views.

Specifically, based on the CMMs for single-view clustering, the multi-view version proposed in [60] became much attractive because it can locate the global optimum and thus avoid the initialization and local optima problems of standard mixture models, which require multiple executions of the EM algorithms.

For multi-view CMMs, each \( x_v \) with \( m \) views is denoted by \( \{x_{v1}, x_{v2}, \cdots, x_{vm}\} \), \( x_v \in \mathbb{R}^{d_v} \), the mixture distribution for each view is given as \( Q_v(x_v) = \sum_{j=1}^N \hat{q}_j f_j(x_v) = C_\phi(x_v) \sum_{j=1}^N \hat{q}_j e^{-\beta t_{vj} d_v(x_v, x_v^j)} \). To pursue a common clustering across all views, all \( Q_v(x_v) \) share the same priors. In addition, an empirical data set distribution \( P_v(x_v) = 1/N, x_v \in \{x_{v1}, x_{v2}, \cdots, x_{vN}\} \), is associated with each view and the multi-view algorithm minimizes the sum of KL divergences between \( Q_v(x_v) \) and \( Q_v(x_v) \) across all views with the constraint \( \sum_{j=1}^N \hat{q}_j = 1 \)

\[
\min_{q_1, \cdots, q_N} \sum_{v=1}^m D(\hat{P}_v||Q_v) = \min_{q_1, \cdots, q_N} -\sum_{v=1}^m \sum_{i=1}^N \hat{P}_v(x_v^i) \log Q_v(x_v^i) - \sum_{v=1}^m H(\hat{P}_v). \tag{6}
\]

which is straightforward to see that the optimized objective is convex, hence the global minimum can be found. The prior undate rule is given as follows:

\[
q_j^{(t+1)} = \frac{q_j^{(t)}}{M} \sum_{v=1}^m \sum_{i=1}^N \frac{\hat{P}_v f_j(x_v^i)}{\sum_{j'=1}^N \hat{q}_{j'} f_{j'}(x_v^i)}. \tag{7}
\]

The prior \( q_j \) associated with the \( j \)th instance is a measure of how likely this instance is to be an exemplar, taking all views into account. The appropriate \( \beta_t \) values are identified in the range of an empirically defined \( \beta_0^t \) by \( \beta_0^t = N^2 \log N / \sum_{i,j=1}^N d_\phi(x_v^i, x_v^j) \). From Eq. (6), it can be found that all views contribute equally to the sum, without considering their different importance. To overcome this limitation, a weighted version of multi-view CMMs was proposed in [61].

### III. Discriminative Approaches

Compared with generative approaches, discriminative approaches directly optimize the objective to seek for the best clustering solution rather than first modelling the subjects then solving these models to determine clustering result. Directly focusing on the objective of clustering makes discriminative approaches gain more attentions and develop more comprehensively. Up to now, most of the existing MVC methods are discriminative approaches. Based on how to combine multiple views, we categorize MVC methods into five main groups and introduce the representative works in each group.

To facilitate the following discussion, we introduce the settings of MVC first. Assume that we are given the set data with \( m \) views. Let \( X = \{x_1^1, x_2^1, \cdots, x_N^1\}, x_v^i \in \mathbb{R}^{d_v}, d_v \) is the dimension of the \( v \)th view data. The aim of MVC is to cluster the \( N \) subjects into \( K \) groups. That is, finally we will get a membership matrix \( H \in \mathbb{R}^{N \times K} \) to indicate which subjects are in the same group while others in other groups, the sum of each row entries of \( H \) should be 1 to make sure each row is a probability. If only one entry of each row is 1 and all others are 0, it is the so-called hard clustering otherwise it is soft clustering.

#### A. Common Eigenvector matrix (Mainly Multi-View Spectral Clustering)

This group of MVC methods are based on a commonly used clustering technique spectral clustering. Since spectral clustering hinges crucially on the construction of the graph Laplacian and the resulting eigenvectors reflect the cluster structure of the data, this group of MVC methods guarantee to get a common clustering results by assuming that all the views share the common or similar eigenvector matrix. There are two representative methods: co-training spectral clustering [6] and co-regularized spectral clustering [19]. Before discussing them, we will introduce spectral clustering [62] first.

1) Spectral Clustering: Spectral clustering is a clustering technique to utilize the properties of the Laplacian of graph whose edges denote the similarities between the data points and solve a relaxation of the normalized min-cut problem on this graph [63]. Compared with other widely used method like k means clustering that only fits the spherical shaped clustering, spectral clustering can apply to arbitrary shaped clustering and demonstrate good performance.

Given \( G = (V, E) \) as a weighted undirected graph with vertex set \( V = v_1, \cdots, v_N \). The data adjacency matrix of the graph is defined to be \( W \) whose entry \( w_{ij} \) represents the similarity of two vertices \( v_i \) and \( v_j \). If \( w_{ij} = 0 \) it means that the vertices \( v_i \) and \( v_j \) are not connected. Apparently \( W \) is symmetric because \( G \) is an undirected graph. The degree matrix \( D \) is defined as the diagonal matrix with the degree \( d_1, \cdots, d_N \) on the diagonal, where \( d_i = \sum_{j=1}^N w_{ij} \). Generally, the graph Laplacian is \( D - W \) and the normalized graph Laplacian is \( L = D^{-1/2}(D - W)D^{-1/2} \). In many spectral clustering works like [62], [6], [19], \( L = D^{-1/2}WD^{-1/2} \) is also used to change a minimization problem [9] into a maximization problem [8] since \( L = I - L \) where \( I \) is
the identity matrix. Following the same way adopted in [62], [6], [19], we will name both \( L \) and \( \tilde{L} \) as normalized graph Laplacians afterwards. Now the single view spectral clustering approach can be formulated as follows:

\[
\begin{align*}
\max_{U \in \mathbb{R}^{N \times k}} & \; \text{tr}(U^T L U) \\
\text{s.t.} & \; U^T U = I,
\end{align*}
\]

which is also equivalent to the following problem:

\[
\begin{align*}
\min_{U \in \mathbb{R}^{N \times k}} & \; \text{tr}(U^T \tilde{L} U) \\
\text{s.t.} & \; U^T U = I,
\end{align*}
\]

where \( \text{tr} \) denotes the matrix trace. The rows of matrix \( U \) are the embeddings of the data points, which can be feed the \( k \) means to obtain the final clustering results. A version of the Rayleigh-Ritz theorem in [64] shows that the solution of the above optimization problem is given by choosing \( U \) as the matrix containing the largest or smallest \( K \) eigenvectors of \( L \) or \( \tilde{L} \) as columns. To understand the spectral clustering algorithm better, we briefly outline a commonly used procedure [62] to solve Eq. (8) as follows:

- Construct the adjacency matrix \( W \).
- Compute the normalized Laplacian matrix \( L = D^{-1/2}WD^{-1/2} \).
- Calculate the eigenvectors of \( L \) and stack the top \( K \) eigenvectors as the columns to construct a \( N \times K \) matrix \( U \).
- Normalize each row of \( U \) to obtain \( U_{\text{sym}} \).
- Run \( k \) means algorithm to cluster the row vectors of \( U_{\text{sym}} \).
- Assign subject \( i \) to cluster \( k \) if the \( i \)th row of \( U_{\text{sym}} \) is assigned to cluster \( k \) by the \( k \) means algorithm.

Apart from the symmetric normalization operator \( U_{\text{sym}} \), there is also another normalization operator \( U_{\text{tr}} = D^{-1}W \). Refer to [65] for more details about spectral clustering.

2) Co-Training Multi-View Spectral Clustering: Co-training is a widely used idea in semi-supervised learning, where both labeled and unlabeled data are available. It assumes that the predictor functions in both views will give the same labels for the sample with high probability. There are two main assumptions to guarantee its success: (1) Sufficiency: each view is sufficient for classification on its own. (2) Conditional independence: the views are conditionally independent given the class labels. In the original co-training algorithm [66], two initial predictor functions \( f_1 \) and \( f_2 \) are trained on the labeled data, then co-training algorithm repeatedly runs the following steps: the most confident examples \( f_1 \) and \( f_2 \) are added to the labeled set for \( f_2 \) and vice versa, then retrain \( f_1 \) and \( f_2 \) on the enlarged labeled data. After a predefined number of iterations, \( f_1 \) and \( f_2 \) will agree with each other on labels.

For co-training multi-view spectral clustering, the motivation is similar: the clustering result in each view should be the same. In spectral clustering, the eigenvectors of the graph Laplacian encode the discriminative information of the clustering. Therefore, Co-training multi-view spectral clustering [6] uses the eigenvectors of the graph Laplacian in one view to cluster samples and then use the clustering result to modify the graph Laplacian in the other view.

Each column of the similarity matrix (also named adjacency matrix) \( W_{N \times N} \) can be considered as a \( N \)-dimensional vector that indicates the similarities of \( i \)th point with all the points in the graph. Since the largest \( K \) eigenvectors have the discriminative information for clustering, the similarity vectors can be projected along those directions to retain the discriminative information for clustering and ignore the within cluster details that might confuse the clustering. After that, the projected information is back-projected to the original \( N \)-dimensional space to get the modified graph. Due to the orthogonality of the projection matrix, the inverse projection is equivalent to the transpose operation.

To make the co-training spectral clustering algorithm clear, we borrowed Algorithm 1 from [6]. Note that a symmetrization operator on a matrix \( S \) is defined as \( \text{sym} (S) = (S + S^T)/2 \) in Algorithm 1.

**Algorithm 1** Co-training Multi-View Spectral Clustering

**Input:** Similarity matrices for two views: \( W^{(1)} \) and \( W^{(2)} \).

**Output:** Assignments to \( K \) clusters.

**Initialize:** \( L^{(v)} = D^{(v)(-1/2)}L^{(v)}D^{(v)(-1/2)} \) for \( v = 1, 2 \), \( U^{(v)} = \text{argmax}_{U \in \mathbb{R}^{N \times k}} \text{tr}(U^T L^{(v)} U) \) s.t. \( U^T U = I \) for \( v = 1, 2 \).

for \( i = 1 \) to \( T \) do
1. \( S^{(1)} = \text{sym} \left( (U^{(2)T})^{-1}U^{(1)T}W^{(1)} \right) \)
2. \( S^{(2)} = \text{sym} \left( (U^{(1)T})^{-1}U^{(2)T}W^{(2)} \right) \)
3. Use \( S^{(1)} \) and \( S^{(2)} \) as the new graph similarities and compute the graph Laplacians. Solve for the largest \( K \) eigenvectors to obtain \( U^{(1)} \) and \( U^{(2)} \).
end for
1. Normalize each row of \( U^{(1)} \) and \( U^{(2)} \).
5. Form matrix \( V = U^{(v)T} \), where \( v \) is the most informative view a priori. If there is no prior knowledge on the view informativeness, matrix \( V \) can also be set to be column-wise concatenation of the two \( U^{(v)} \)’s.
6. Assign example \( j \) to cluster \( K \) if the \( j \)th row of \( V \) is assigned to cluster \( K \) by \( k \) means algorithm.

3) Co-Regularized Multi-View Spectral Clustering: Co-regularization is a famous technique in semi-supervised multi-view learning. The core idea of co-regularization is minimizing the distinction between the predictor functions of two views acts as one part of the objective function. However, there are no predictor functions in unsupervised learning like clustering, so how to implement the co-regularization idea in clustering problem. Co-regularized multi-view spectral clustering [19] adopted the eigenvectors of graph Laplacian to play the similar role of predictor functions in semi-supervised learning scenario and proposed two co-regularized clustering approaches.

Let \( U^{(s)} \) and \( U^{(t)} \) be the eigenvector matrices corresponding to any pair of view graph Laplacians \( L^{(s)} \) and \( L^{(t)} \) \((1 \leq s, t \leq m, s \neq t)\). The first version uses a pair-wise co-regularization criteria that enforces \( U^{(s)} \) and \( U^{(t)} \) as close as
possible. The measure of clustering disagreement between the two views $s$ and $t$ is $D(U^{(s)}, U^{(t)}) = \|K^{(s)} - K^{(t)}\|_F^2$, where $K^{(s)} = U^{(s)}U^{(s)\top}$ using linear kernel is the similarity matrix of $U^{(s)}$. Since $\|K^{(s)}\|_F^2 = K$, where $K$ is the number of the clusters, the measure of disagreement becomes $D(U^{(s)}, U^{(t)}) = -tr(U^{(s)}U^{(s)\top}U^{(t)}U^{(t)\top})$. Integrating the measure of disagreement between any pair of views into the spectral clustering framework, the pair-wise co-regularized multi-view spectral clustering will be formed as the following optimization problem:

$$\max_{U^{(1)}, U^{(2)}, \ldots, U^{(m)} \in \mathbb{R}^{N \times K}} \sum_{s=1}^{m} (U^{(s)\top}L^{(s)}U^{(s)}) + \sum_{1 \leq s \leq m, s \neq t} \lambda tr(U^{(s)}U^{(s)\top}U^{(t)}U^{(t)\top})$$

s.t. $U^{(s)\top}U^{(s)} = I$, $\forall 1 \leq s \leq m$. \hspace{1cm} (10)

The hyperparameter $\lambda$ is used to trade-off the spectral clustering objectives and the spectral embedding disagreement terms.

The second version named centroid-based co-regularization enforces the eigenvector matrix from each view to be similar by regularizing them towards a common consensus eigenvector matrix. The corresponding optimization problem is formulated as

$$\max_{U^{(1)}, U^{(2)}, \ldots, U^{(m)} \in \mathbb{R}^{N \times K}} \sum_{s=1}^{m} (U^{(s)\top}L^{(s)}U^{(s)})$$

$$+ \lambda \sum_{s=1}^{m} \lambda tr(U^{(s)}U^{(s)\top}U^{(s)}U^{(s)\top})$$

s.t. $U^{(s)\top}U^{(s)} = I$, $\forall 1 \leq s \leq m$, $U^{(s)}U^{(s)} \leq I$. \hspace{1cm} (11)

Since relaxed kernel k means and spectral clustering are equivalent, by learning flexible weights automatically, Ye et al. [57] proposed a co-regularized kernel k means for multi-view clustering. With a multi-layer Grassmann manifold interpretation, Dong et al. [58] obtained the same formulation with the pair-wise co-regularized multi-view spectral clustering.

4) Others: Besides the above mentioned two representative multi-view spectral clustering methods, Wang et al. [58] enforces a common eigenvector matrix and formulates a multi-objective problem and then solve it using Pareto optimization.

B. Common Coefficient Matrix (Mainly Multi-View Subspace Clustering)

In many practical applications, even though the given data set is high dimensional, its intrinsic dimension is often much low. For example, the number of pixels in a given image can be large, yet only a few parameters are used to describe the appearance, geometry and dynamics of a scene. This motivates the development of finding the underlying low dimensional space. In practice, the data could be sampled from multiple subspaces. Subspace clustering [59] is the technique to find the underlying subspaces and then cluster the data points correctly.

1) Subspace clustering: Subspace clustering uses the self-expression property [70] of the data set to represent itself as:

$$X = XZ + E$$

where $Z = \{z_1, z_2, \cdots, z_N\} \in \mathbb{R}^{N \times N}$ is the subspace coefficient matrix (representation matrix), and each $z_i$ is the representation of the original data point $x_i$ based on the subspace, $E \in \mathbb{R}^{N \times N}$ is the noise matrix.

The subspace clustering can be formulated as the following optimization problem:

$$\min_{Z} \|X - XZ\|_F^2$$

s.t. $Z(i, i) = 0$, $Z^\top 1 = 1$. \hspace{1cm} (13)

The constraint $Z(i, i) = 0$ is to avoid the case that a data point is represented by itself while $Z^\top 1 = 1$ denotes that the data point lies in a union of affine subspaces. The nonzero elements of $z_i$ correspond to data points from the same subspace.

After getting the subspace representation $Z$, the similarity matrix $W = \|Z_i + |Z|^{-1}\|_2$ can be obtained to further construct the graph Laplacian and then run spectral clustering on that graph Laplacian to get the final clustering results.

2) Multi-View Subspace Clustering: With multi-view information, each subspace representation $Z_v$ can be obtained from each view. To get a consistent clustering result from multiple views, Yin et al. [71] shares the common coefficient matrix by enforcing the coefficient matrices from each pair of views as similar as possible. The optimization problem is formulated as

$$\min_{Z^{(s)}, s=1,2,\cdots,m} \sum_{s=1}^{m} \|X^{(s)} - X^{(s)}Z^{(s)}\|_F^2$$

$$+ \alpha \sum_{s=1}^{m} \|Z^{(s)}\|_1 + \beta \sum_{1 \leq s \leq t \leq m} \|Z^{(s)} - Z^{(t)}\|_1$$

s.t. $\text{diag}(Z^{(s)}) = 0$, $\forall s \in \{1, 2, \cdots, m\}$. \hspace{1cm} (14)

where $\|Z^{(s)} - Z^{(t)}\|_1$ is the $l_1$-norm based pairwise co-regularization constraint that can alleviate the noise problem. $\|Z\|_1$ is used to enforce sparse solution. $\text{diag}(Z)$ denotes the diagonal elements of matrix $Z$, and the zero constraint is used to avoid trivial solution (each data point represents by itself).

Wang et al. [72] enforced the similar idea to combine multi-view information. Apart from that, it adopted a multi-graph regularization with each graph Laplacian regularization characterizing the view-dependent non-linear local data similarity. At the same time, it assumes that the view-dependent representation is low rank and sparse and considers the sparse noise in the data. Wang et al. [53] proposed an angular based similarity to measure the correlation consensus in multiple views and obtained a robust subspace clustering for multi-view data. Different from the above approaches, These three works [35, 40, 73] adopted general nonnegative matrix factorization formulation but shared a common representation matrix for the samples with both views and kept each view representation matrix specific. Zhao et al. [26] adopted a novel semi-negative matrix factorization to perform multi-view clustering, in the last layer a common coefficient matrix is enforced to exploit the multi-view information.

C. Common Indicator Matrix (Mainly Multi-View Nonnegative Matrix Factorization Clustering)

1) Nonnegative Matrix Factorization: For nonnegative data matrix $X \in \mathbb{R}^{d \times N}$, Nonnegative Matrix Factorization (NMF) [74] aims to seek two nonnegative matrix factors
$U \in \mathbb{R}^{d \times K}$ and $V \in \mathbb{R}^{N \times K}$ whose product is a good approximation to $X$:

$$X \approx UV^T,$$  \hspace{1cm} (15)

where $K$ denotes the desired reduced dimension (for clustering, it is the number of clusters). Here, $U$ is the basis matrix while $V$ is the indicator matrix.

Due to the nonnegative constraints, NMF can learn a part-based representation. This is very intuitive and meaningful in many applications like the face recognition in [74]. The observations in many applications like information retrieval [74] and pattern recognition [75] can be explained as an additive linear combinations of nonnegative basic vectors. It had been applied successfully to clustering [74], [76] and achieved the state-of-the-art performance.

In addition, k means clustering can be formulated using NMF by introducing an indicator matrix $H$. The NMF formulation of k-means clustering is

$$\begin{align*}
\min_{H,G} & \|X^T - HG^T\|_F^2 \\
\text{s.t.} & \quad H_{i,k} \in \{0,1\}, \sum_{k=1}^K H_{i,k} = 1, \forall i = 1, 2, \ldots, N
\end{align*}$$

where $G \in \mathbb{R}^{d \times K}$ is the cluster centroid matrix.

2) Multi-View Nonnegative Matrix Factorization Clustering: By enforcing the indicator matrices from different views to the same, Akata et al. [77] extended the NMF [74] to multi-view settings.

To combine multi-view information in the NMF framework, Akata et al. [77] enforces a shared indicator matrix among different views to perform multi-view clustering in NMF framework. However, the indicator matrix $V^{(v)}$ might not be comparable at the same scale. In order to keep the clustering solutions across different views meaningful and comparable, Liu et al. [78] enforces a constraint to push each view-dependent indicator matrix towards a common indicator matrix and another normalization constraint inspired by the connection between NMF and probability latent semantic analysis. The final optimization problem is formulated as:

$$\begin{align*}
\min_{U^{(v)},V^{(v)_{v=1,2,\ldots,m}}} & \sum_{v=1}^m \|X^{(v)} - U^{(v)}V^{(v)}\|_F^2 \\
& + \sum_{v=1}^m \lambda_v \|V^{(v)} - V^*\|_F^2 \\
\text{s.t.} & \quad 1 \leq k \leq K, \|U^{(v)_{v=1,2,\ldots,m}}_{i,k}\|_1 = 1, U^{(v)}, V^{(v)}, V^* \geq 0.
\end{align*}$$

The constraint $\|U^{(v)}_{i,k}\|_1 = 1$ is used to guarantee $V^{(v)}$ within the same range for different $v$ such that the comparison between the view-dependent indicator matrix $V^{(v)}$ and the consensus indicator matrix $V^*$ is reasonable.

After obtaining the consensus matrix $V^*$, the cluster label of data point $i$ can be computed as $\text{argmax}_k V^*_{i,k}$.

As we aforementioned, for subspace learning, there are two steps: calculate the subspace representation and run spectral clustering on the graph Laplacian computed from the obtained subspace representation. To get a consistent clustering from multiple views, Gao et al. [79] merged the two steps in subspace clustering and enforced a common indicator matrix across different views. The formulation is as follows:

$$\begin{align*}
\min_{Z^{(v)_{v=1,2,\ldots,m}},W^{(v)_{v=1,2,\ldots,m}}} & \sum_{v=1}^m \|X^{(v)} - X^{(v)}Z^{(v)} - E^{(v)}\|_F^2 \\
& + \lambda_1 \text{tr}(H^T(D^{(v)} - W^{(v)})H) + \lambda_2 \sum_{v=1}^m \|E^{(v)}\|_1 \\
\text{s.t.} & \quad Z^{(v)_{v=1,2,\ldots,m}} \geq 0, Z^{(v)}(i,i) = 1, H^T H = I
\end{align*}$$

where $Z^{(v)}$ is the subspace representation matrix of the $v$th view, $W^{(v)} = [Z^{(v)}]_+ Z^{(v)T}$. $D^{(v)}$ is a diagonal matrix with diagonal elements defined as $d_{v,i} = \sum_j w_{v,i,j}$, $H$ is the common indicator matrix which can result in a consistent clustering result across different views. Although this multi-view subspace clustering is based on subspace clustering, it does not enforce the common coefficient matrix but shares a common indicator matrix for different views. That is why we categorize it into this group. Similar things happen to spectral clustering later.

Since spectral clustering works on graph and the eigen decomposition is time consuming, it is inappropriate for large scale application. k means algorithm does not subject to this limitation, thus it becomes a good choice for large scale application. To deal with large scale multi-view data, Cai et al. [81] proposed a multi-view k means clustering method by adopting a common indicator matrix for different views. The optimization problem is formulated as follows:

$$\begin{align*}
\min_{F^{(v)_{v=1,2,\ldots,m}},H} & \sum_{v=1}^m (\gamma_v) \|X^{(v)} - HG^T\|_{2,1} \\
\text{s.t.} & \quad H_{i,k} \in \{0,1\}, \sum_{k=1}^K H_{i,k} = 1, \sum_{v=1}^m \gamma_v = 1.
\end{align*}$$

where $\gamma_v$ is the weight factor for the $v$th view and $\gamma$ is the parameter to control the weights distribution. By learning the weights for different views, the important views will get large weight during multi-view clustering.

Wang et al. [7] integrates multi-view information via a common indicator matrix and simultaneously take the varied importance of different features to different data clusters into consideration. Its formulation is

$$\begin{align*}
\min_{H,F} & \|XTW + 1_Nb^T - H\|_F \\
& + \gamma_1 \|W\|_{1,2} + \gamma_2 \|W\|_{2,1}
\end{align*}$$

where $X = \{x_1, x_2, \ldots, x_N\} \in \mathbb{R}^{d \times N}$, each $x_i$ is the input including the features from all the $m$ views and each view has $d_j$ dimension such that $d = \sum_{j=1}^m d_j$. $W = [w_1^1, \ldots, w_1^K, \ldots, w_m^1, \ldots, w_m^K] \in \mathbb{R}^{d \times K}$ is the weights of each feature for $K$ clusters. $b \in \mathbb{R}^{K \times 1}$ is the intercept vector, $1_N$ is $N \times 1$ constant vector of all 1’s, $H = [h_1, \ldots, h_N] \in \mathbb{R}^{N \times K}$ is the cluster indicator matrix. $\|W\|_{1,2} = \sum_{k=1}^K \sum_{j=1}^m \|w_k^j\|^2$ is the group $l_2$ regularization to learn the group-wise feature importance of one view on each cluster while $\|W\|_{2,1} = \sum_{j=1}^m \|w_j\|_2$ is the $l_2,1$ norm to learn the individual weight across different clusters.

Before centroid-based co-regularization, a similar work [80] used the same idea to perform multi-view spectral clustering. The main difference is that [80] used $\text{tr}(U^* - U^{(s)})^T(U^* - U^{(s)})$ as the disagreement measure between each view.
eigenvector matrix and the common eigenvector matrix while co-regularized multi-view spectral clustering \[19\] adopted \(tr(U^s(s)U^s(s)^T U^s(s)U^s(s)^T)\). The optimization problem \[80\] is formulated as

\[
\begin{aligned}
\max_{U(s), s=1,2,\ldots,m, U^*} & \sum_{s=1}^{m} tr(U^s(s)^T L(s)U^s(s)) \\
+ \lambda \sum_{s=1}^{m} tr((U^* - U^s(s))^T(U^* - U^s(s))) & s.t. \quad U^s(s) \geq 0, \quad U^s(s)^T U^s(s) = I.
\end{aligned}
\]

where \(U^* \geq 0\) makes \(U^*\) become the final cluster indicator matrix. Different from general spectral clustering that get eigenvector matrix first and then run clustering (such as k means that is sensitive to initialization condition) to assign clusters, Cai et al. \[80\] directly solves the final cluster indicator matrix, thus it will be more robust to the initial condition.

In \[31\], a matrix factorization approach was adopted to reconcile the groups arising from the individual views. Specifically, a matrix that contains the partitioning of every individual view is created and then decomposed into two matrices, the one showing the contribution of those partitionings to the final multi-view clusters, called meta-clusters, and the other the assignment of instances to the meta-clusters. Tang et al. \[40\] considered multi-view clustering as clustering with multiple graphs, each of which is approximated by matrix factorization with a graph-specific factor and a factor common to all graphs. Qian et al. \[82\] used the relaxed common indicator matrix (make each view indicator matrix as close as possible) to combine multi-view information and employed the Laplacian regularization to maintain the latent geometric structure of the views simultaneously. Apart from using common indicator matrix, \[83, 84, 85\] introduced a weight matrix to indicate whether the corresponding matrix whose entry is missing such that it can tackle the missing value problem, the multi-view self-paced clustering \[84\] takes the complexities of the samples and views into consideration to alleviate the local minima problem. Tao et al. \[82\] enforces a common indicator matrix and seeks for the consensus clustering among all views in an ensemble clustering way. Apart from multi-view nonnegative matrix factorization clustering, there exist some other methods to utilize the common indicator matrix to combine multiple views for clustering like \[21\] who additionally borrowed the linear discriminant analysis idea and weighted each view automatically. For graph-based clustering methods, each similarity matrix for each view is obtained first, Nie et al. \[83\] assumes a common indicator matrix and then solves the problem by minimizing the differences between the common indicator matrix and each similarity matrix.

D. Direct Combination (Mainly Multi-Kernel Based Multi-View Clustering)

Apart from sharing a common structure from different views, the direct view combination is a good way to perform multi-view clustering. A natural approach is to define a kernel for each view information and then convexly combine these kernels. \[8, 86, 87\].

1) Kernel Functions and Kernel Combination Forms:

Kernel is a trick to learn nonlinear problem just by linear learning algorithm, since kernel function \(K : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}\) can directly give the inner products in feature space without explicitly defining the nonlinear transformation \(\phi\). There are some common kernel functions as follows:

- Linear kernel: \(K(x_i, x_j) = (x_i \cdot x_j)\),
- Polynomial kernel: \(K(x_i, x_j) = (x_i \cdot x_j + 1)^d\),
- Gaussian kernel (Radial basis kernel): \(K(x_i, x_j) = \exp(-\frac{\|x_i-x_j\|^2}{2\sigma^2})\),
- Sigmoid kernel: \(K(x_i, x_j) = \tanh(\eta x_i \cdot x_j + \nu)\).

In machine learning, common kernel functions can be viewed as similarity functions \[88\], such that we can use kernel trick to deal with spectral clustering and kernel k means problems. So far, there exist some works on multi-kernel learning for clustering \[89, 90, 91\], however, they are all for single-view clustering. If each kernel is derived from each view and then combined elaborately to deal with the clustering problem, it will become the multi-kernel learning for multi-view clustering. Obviously, multi-kernel learning \[92\], \[93, 94, 95\] can be considered as the most important part of multi-view clustering. There are three main categories of multi-kernel combination \[95\]:

- Linear combination: It includes two basic subcategories: unweighted sum \(K(x_i, x_j) = \sum_{i=1}^{m} k_i(x_i, x_j)\) and weighted sum \(K(x_i, x_j) = \sum_{i=1}^{m} w_i k_i(x_i, x_j)\) where \(w_i \in \mathbb{R}^+\) denotes the kernel weight for the \(i\)th view and \(\sum_{i=1}^{m} w_i = 1\), \(q\) is the hyperparameter to control the distribution of the weights,
- Nonlinear combination: It uses some nonlinear functions of kernels, namely, multiplication, power, and exponentiation,
- Data-dependent combination: It assigns specific kernel weights for each data instance, which can identify the local distributions in the data and learn proper kernel combination rules for each region.

2) Kernel K Means and Spectral Clustering: Kernel k means \[27\] and spectral clustering \[98\] are two kernel-based clustering methods for optimizing the intra-cluster variance. Let \(\phi(\cdot) : \mathcal{X} \rightarrow \mathcal{H}\) be a feature mapping which maps \(x\) onto a reproducing kernel Hilbert space \(\mathcal{H}\). The kernel k means problem is formulated as the following optimization problem,

\[
\begin{aligned}
\min_{\mathbf{H}} & \sum_{i=1}^{N} \sum_{k=1}^{K} H_{ik} \| \phi(x_i) - \mu_k \|^2 \quad s.t. \quad \sum_{k=1}^{K} H_{ik} = 1, \quad \mu_k = \frac{1}{n_k} \sum_{i=1}^{n_k} H_{ik} \phi(x_i) \quad \text{the number and centroid of the kth cluster.} \;
\end{aligned}
\]

With a kernel matrix \(K\) whose entry is \(K_{ij} = \phi(x_i)^T \phi(x_j)\), \(L = \text{diag}([n_1^{-1}, n_2^{-1}, \cdots, n_K^{-1}])\) and \(1_i \in \mathbb{R}^l\) that is a column vector with all elements 1, Eq. \[22\] can be equivalently rewritten as the following matrix-vector form,

\[
\begin{aligned}
\min_{\mathbf{H}} & \quad tr(K) - tr(L^+ H^TKHL^+) \quad s.t. \quad H_1 k = 1_N.
\end{aligned}
\]

For the above kernel k means matrix-factor form, the matrix \(H\) is discrete, which makes the optimization problem difficult to solve. By relaxing the matrix \(H\) to take arbitrary real
values, the above problem can be approximated. Specifically, by defining $U = HL^T$ and letting $U$ take real values, further considering $\text{Tr}(K)$ is constant, Eq. (23) will be relaxed to

$$\max_U \text{Tr}(U^T K U) \quad \text{s.t.} \quad U^T U = 1_K.$$  

(24)

The fact $H^T H = L^{-1}$ leads to the orthogonality constraint on $U$ which further tells us that the optimal $U$ can be obtained by the top $K$ eigenvectors of the kernel matrix $K$. Therefore, Eq. (24) can be considered as the generalized optimization matrix form of spectral clustering. Note that Eq. (24) is equivalent to Eq. (8) if the kernel matrix $K$ takes the normalized Gram matrix form.

3) Multi-Kernel Based Multi-View Clustering: Assume there are $m$ kernel matrices available, each of which corresponds to one view. To make full use of all views, the weighted combination $K = \sum_{v=1}^{m} w_v^p K^{(v)}$, $w_v \geq 0$, $\sum_{v=1}^{m} w_v = 1$, $p \geq 1$ will be used in kernel k means and spectral clustering (8) to obtain the corresponding multi-view kernel $K$ means and multi-view spectral clustering in paper [41]. With the same nonlinear combination by specifically setting $p = 1$.

Guo et al. [99] extended the spectral clustering to multi-view clustering by further employing the kernel alignment. Due to the potential redundancy of the selected kernels, Liu et al. [28] introduced a matrix-induced regularization to reduce the redundancy and enhance the diversity of the selected kernels to attain the final goal of boosting the clustering performance. By replacing the original Euclidean norm metric in fuzzy c-means with a kernel-induced metric in the data space and adopting the weighted kernel combination, Zhang et al. [100] successfully extended the fuzzy c-means to multi-view clustering that is robust to noise and outliers. In the case with incomplete multi-view data set existing, by optimizing the alignment of shared instances of the data sets, Shao et al. [43] collectively completes the kernel matrices of incomplete data sets. To overcome the cluster initialization problem associated with kernel k means, Tzortzis et al. [54] proposed a global kernel k means algorithm, a deterministic and incremental approach that adds one cluster each stage, through a global search procedure consisting of several executions of kernel k means from suitable initiations.

4) Others: Besides multi-kernel based multi-view clustering, there are some other methods that use the direct combination to perform multi-view clustering like [21], [33]. In [46], two-level weights: view weights and variable weights are assigned to the clustering algorithm for multi-view data to identify the importance of the corresponding views and variables. To extend fuzzy clustering method to multi-view clustering, each view is weighted and the multi-view versions of fuzzy c-means and fuzzy k means are obtained [42] and [51], respectively.

E. Combination After Projection (Mainly CCA-Based Multi-View Clustering)

For homogeneous multi-view data, it is reasonable to directly combine them together. However, in real-world applications, the multiple representations may come from different feature vector spaces. For instance, in bioinformatics, gene information can be one view while the clinical symptoms can be the other view in patient clustering [13]. Obviously, these information cannot be combined directly. Moreover, high dimension and noise are difficult to handle. To solve the above problems, the last yet important combination way is introduced: combination after projection. The most commonly used technique is Canonical Correlation Analysis (CCA) and the kernel version of CCA (KCCA).

1) CCA and KCCA: To better understand this style of multi-view combination, CCA and KCCA are briefly introduced (refer to [101] for more detail). Given two data sets $S_x = \{x_1, x_2, \cdots, x_N\} \in \mathbb{R}^{d_x \times N}$ and $S_y = \{y_1, y_2, \cdots, y_N\} \in \mathbb{R}^{d_y \times N}$ each entry $x$ or $y$ with zero mean, CCA aims to find a projection $w_x \in \mathbb{R}^{d_x}$ for $x$ and another projection $w_y \in \mathbb{R}^{d_y}$ for $y$ such that the correlation between the projection of $S_x$ and $S_y$ on $w_x$ and $w_y$ are maximized,

$$\rho = \max_{w_x, w_y} \frac{w_x^T C_{xy} w_y}{\sqrt{(w_x^T C_{xx} w_x)(w_y^T C_{yy} w_y)}}$$  

(25)

where $\rho$ is the correlation and $C_{xy} = E[xy^T]$ denotes the covariance matrix of $x$ and $y$ with zero mean. Observing that $\rho$ is not affected by scaling $w_x$ or $w_y$ either together or independently, CCA can be reformulated as

$$\max_{w_x, w_y} \quad w_x^T C_{xy} w_y$$  

(26)

$$\text{s.t.} \quad w_x^T C_{xx} w_x = 1,$$

$$w_y^T C_{yy} w_y = 1.$$  

With the method of Lagrange multiplier, the two lagrange multipliers $\lambda_x$ and $\lambda_y$ are equal to each other, that is $\lambda_x = \lambda_y = \lambda$. If $C_{xy}$ is invertible, $w_y$ can be obtained as $w_y = \frac{1}{\lambda} C_{yy}^{-1} C_{xy} w_x$ and $C_{xy} = (C_{yy})^{-1} C_{xy} C_{xx}^{-1} w_x$. For different eigen values (from large to small), different eigen vectors are obtained, which can be considered as a successive process.

The above canonical correlation problem can be transformed into a distance minimization problem. For ease of derivation, the successive formulation of the canonical correlation is replaced by the simultaneous formulation of the canonical correlation. Assume the number of projections is $p$, the matrices $W_x$ and $W_y$ denote $(w_{x1}, w_{x2}, \ldots, w_{xp})$ and $(w_{y1}, w_{y2}, \ldots, w_{yp})$, respectively. The simultaneous formulation is the optimization problem with $p$ iteration steps:

$$\max_{(w_{x1}, w_{x2}, \ldots, w_{xp}), (w_{y1}, w_{y2}, \ldots, w_{yp})} \quad \frac{\sum_{i=1}^{p} w_{xi}^T C_{xy} w_{yi}}{w_{xi}^T C_{xx} w_{xi} + \sum_{j=1}^{p} w_{yi}^T C_{yx} W_{xy} w_{yi}}$$  

(27)

$$\text{s.t.} \quad \begin{cases} w_{xi}^T C_{xx} w_{xi} = 1 & \text{if } i=j, \\ w_{xi}^T C_{xy} w_{yi} = 0 & \text{otherwise}, \\ w_{yi}^T C_{yx} w_{yi} = 1 & \text{if } i=j, \\ w_{yi}^T C_{yy} w_{yi} = 0 & \text{otherwise}, \\ i, j = 1, 2, \ldots, p. \end{cases}$$
The matrix formulation to the optimization problem \(^{(27)}\) is

\[
\begin{aligned}
\max_{W_x, W_y} & \quad \text{Tr}(W_x^T C_{xy} W_y) \\
\text{s.t.} & \quad W_x^T C_{xx} W_x = I, \\
& \quad W_y^T C_{yy} W_y = I, \\
& \quad w_{xi}^T C_{xy} w_{xj} = 0, \\
& \quad w_{yi}^T C_{yx} w_{yj} = 0, \\
& \quad i, j = 1, ..., p, j \neq i.
\end{aligned}
\]

(28)

where \(I\) is an identity matrix with size \(p \times p\). Maximizing the objective function of equation \(^{(28)}\) can be transformed into the equivalent form:

\[
\min_{W_x, W_y} \left\| W_x^T S_x - W_y^T S_y \right\|_F,
\]

(29)

which is used widely in many works \(^{[35], [73], [102], [103]}\).

KCCA uses the “kernel trick” to maximize the correlation between two non-linear projected variables. Corresponding to Eq. \(^{(25)}\), the optimization problem for KCCA is formulated as follows:

\[
\begin{aligned}
\max_{w_x, w_y} & \quad w_x^T K_x K_y w_y \\
\text{s.t.} & \quad w_x^T K_x w_x = 1, \\
& \quad w_y^T K_y w_y = 1.
\end{aligned}
\]

(30)

In contrast with linear CCA that works by solving an eigen-decomposition of the covariance matrix, the eigenvalue problem for KCCA is given by

\[
\begin{pmatrix}
0 & K_y \\
K_x & 0
\end{pmatrix}
\begin{pmatrix}
w_x \\
w_y
\end{pmatrix} = \lambda
\begin{pmatrix}
K_x \\
0
\end{pmatrix}
\begin{pmatrix}
w_x \\
w_y
\end{pmatrix}
\]

(31)

2) CCA Based Multi-View Clustering: Since clustering data in high dimension is a tough problem, Chaudhuri et al. \(^{[10]}\) firstly gets the projections via CCA for multi-view data and then clusters in the projected low dimensional subspace. Under the assumption that multiple views are uncorrelated given the cluster label, it shows a weaker separation conditions required to guarantee the algorithm successful. Blaschko et al. \(^{[24]}\) projected the data onto the top directions obtained by kernel CCA across the views and applied k means to clustering the projections.

For the case of paired views with class labels, CCA can still be applied ignoring the class labels, however, the performance can be ineffective. To take good advantage of the class label information, Rasiswasaia et al. \(^{[11]}\) proposed two solutions with CCA: mean-CCA and cluster-CCA. Consider two data sets each of which is divided into \(K\) different but corresponding classes or clusters. Given \(S_x = \{x_1, x_2, \cdots, x_K\}\) and \(S_y = \{y_1, y_2, \cdots, y_K\}\), where \(x_k = \{x_k^1, x_k^2, \cdots, x_k^{|x_k|}\}\) and \(y_k = \{y_k^1, y_k^2, \cdots, y_k^{|y_k|}\}\) are the data points in the \(k\)th cluster for the first and second views, respectively. The first point is to establish correspondences between the mean cluster vectors of the two view data sets. With the cluster means \(m_x^k = \frac{1}{|x_k|} \sum_{i=1}^{|x_k|} x_k^i\) and \(m_y^k = \frac{1}{|y_k|} \sum_{i=1}^{|y_k|} y_k^i\), mean-CCA is formulated as

\[
\rho = \max_{w_x, w_y} \frac{w_x^T V_{xy} w_y}{\sqrt{(w_x^T V_{xx} w_x)(w_y^T V_{yy} w_y))}}
\]

where \(V_{xy} = \frac{1}{K} \sum_{k=1}^{K} m_x^k m_y^k\), \(V_{xx} = \frac{1}{K} \sum_{k=1}^{K} m_x^k m_x^k\) and \(V_{yy} = \frac{1}{K} \sum_{k=1}^{K} m_y^k m_y^k\). The second solution is to establish a one-to-one correspondence between all pairs of data points in given cluster across the two view data sets and then standard CCA is used to learn the projections.

For multi-view data with at least one complete view (features for this view is available for all data points), Anusua et al. \(^{[104]}\) borrowed the idea from Laplacian regularization to complete the incomplete kernel matrix and then applied KCCA to perform multi-view clustering. In multi-view clustering, multiple pattern matrices \(A(v) \in \mathbb{R}^{N \times K}, v = 1, 2, \cdots, K\) each of which corresponds to one view are obtained as intermediate step and then a consensus pattern matrix that is close to each pattern matrix as much as possible should be learned. Due to the unsupervised property, they are not directly comparable. Based on the idea from CCA equivalent form Eq. \(^{(29)}\), Long et al. \(^{[39]}\) projected one view pattern matrix first before comparing with other view pattern matrix. With the same idea but tackling incomplete view problem (there are no complete views), take two-view case as example, these two works \(^{[35], [73]}\) organized the data into the data with both views available and the data with one view available only and then projected each view data matrix to be close to the final indicator matrix. Multi-view information is connected by the common indicator matrix corresponding to the projected data matrix with both views available. Wang et al. \(^{[105]}\) provided a multi-view clustering with extreme learning machine that mapped the normalized feature space onto a higher dimensional feature space.

F. Discussion

In Fig. III-F we give the taxonomy of the multi-view clustering methods. This is also the organization of the multi-view clustering survey. For III-A, III-B, III-C in fact, a common property is that they share a similar structure to combine multiple views. Apart from them, there are also some methods to share other similar structure to perform multi-view clustering. By introducing a common vector to enforce the same clustering result, Sun et al. \(^{[13], [106], [107]}\) extended the bi-clustering \(^{[108]}\) method to multi-view settings. Wang et al. \(^{[47]}\) chose the Jaccard similarity to measure the cross-view clustering consistency and simultaneously considered the within-view clustering quality to cluster multi-view data.

Besides these categorized methods, there are some other multi-view clustering methods. Different from exploiting the consensus information of multi-view data, Cao et al. \(^{[109]}\) utilizes a Hilbert Schmidt Independence Criterion as a diversity term to explore the complementarity of multi-view information. It reduces the redundancy of multi-view information to improve the clustering performance. Based on “minimizing disagreement” idea, De Sa \(^{[12]}\) proposed a two-view spectral clustering that creates a bipartite graph of the views. Zhou et al. \(^{[9]}\) defined a mixture of Markov chains on similarity graph of each view and generalized spectral clustering to multiple views.

In \(^{(29)}\), a transition probability matrix is constructed from each single view, and all these transition probability matrices are used to recover a shared low-rank transition
probability matrix as a crucial input to the standard Markov chain method for clustering. By fusing the similarity data from different views, Lange et al. [20] formulates a nonnegative matrix factorization problem and adopts an entropy-based mechanism to control the weights of multi-view data. Liu et al. [48] chose tensor to represent multi-view data and then perform clustering via tensor methods.

IV. RELATIONSHIPS TO RELATED TOPICS

As we mentioned previously, MVC is to perform clustering with multi-view feature information. It is a basic task in machine learning and thus can be used for different further analysis tasks. In machine learning and data mining fields, there are several closely related learning topics like multi-view representation learning, ensemble clustering, multi-task clustering, multi-view supervised and semi-supervised learning. In the following, we will elaborate the relationships between MVC and them.

Multi-view representation [110] is the task to learn a more comprehensive or meaningful representation from multi-view data. According to [111], representation learning (also named feature engineering) is a way to take advantage of human ingenuity and prior knowledge to extract some useful but far-removed feature representation from the ultimate objective, thus representation is also unsupervised, which is the same with clustering in the sense that does not use label information. Multi-view representation can be considered as a more basic task than multi-view clustering, since multi-view representation can be further used for more broad use like classification or clustering and so on. However, the clustering based on multi-view representation may be not ideal due to the blindness of multi-view representation. In [110], multi-view representation methods are categorized into mainly two classes: the shallow methods and the deep methods. The shallow methods are mainly based on CCA, which may correspond to our subsection [111]. For the deep methods, there exist a large number of works [112], [15], [113], [114], [14], [115], [116] on multi-view representation. However, for multi-view deep clustering, there are just a few like [117], [17]. As we mentioned above, Sequentially learning multi-view representation and clustering is the nature way to perform multi-view clustering, but the ultimate performance is usually not well because of the blindness of multi-view representation. Therefore, how to integrate clustering and multi-view representation learning into a simultaneous process is an intriguing direction up to date, especially for deep multi-view representation.

Ensemble clustering [118] (also named consensus clustering or aggregation of clustering) is to reconcile clustering information about the same data set coming from different sources or from different runs of the same clustering results to find a single consensus clustering that is a better fit in some sense than any one. If ensemble clustering works on the clustering information about the multi-view data, it will become multi-view clustering. Therefore, all of the ensemble clustering techniques like [119], [120], [121], [122], [49] can be applied to multi-view clustering. [32], [123] are two multi-view ensemble clustering methods.

Multi-task clustering aims to improve the performance of unsupervised learning task clustering, such as [124], [125], [126], [127], [128]. If each task corresponds to one view of the same data set, multiple clustering results will be obtained, then ensemble clustering methods can be adopted to fuse these clustering results. Therefore, multi-task clustering combined with ensemble clustering can implement multi-view clustering when each task becomes one view of a same data set. Moreover, multi-task clustering and multi-view clustering can be conducted simultaneously to improve the clustering performance [37], [50].

Different from multi-view clustering, multi-view supervised learning [3] can just use the labeled data to learn classifier while multi-view semi-supervised learning [2], [3] can learn the classifier with both the labeled and unlabeled data. The commonality between them lies in the way to combine multiple views. Many widely used view combining techniques like co-training [129], co-regularization [130], mar-
gin consistency [131], [132] can be borrowed to multi-view clustering after a substitute information similar with label information in semi-supervised learning is found.

V. APPLICATIONS

MVC has been successfully applied to various applications including computer vision, natural language processing, social multimedia, bioinformatics and health informatics and so on.

A. Computer Vision

Multi-view clustering has been widely used in image categorization [71], [72], [79], [109], [120], [133] and motion segmentation tasks [25]. There are several feature types like CENTRIST [134], ColorMoment [135], HOG [136], LBP [137] and SIFT [138] that can be extracted from the images, see the Fig. 2 [79]. Yin et al. [71] proposed a pairwise sparse subspace representation for multi-view image clustering, which harnesses the prior information and maximizes the correlation between the representations of different views. Wang et al. [72] enforced view agreement in an iterative way to perform multi-view spectral clustering for image. Gao et al. [79] assumed a common low dimensional subspace representation for different views to reach the goal of multi-view clustering for computer vision applications. Cao et al. [109] adopted Hilbert Schmidt Independence Criterion as a diversity term to exploit the complementary information of different views and performed well on both image and video face clustering tasks. Jin et al. [30] utilized CCA to perform multi-view image clustering for large-scale annotated image collections.

Ozay et al. [120] used consensus clustering to fuse image segmentations. Méndez et al. [133] adopted the ensemble way to perform multi-view clustering for MRI image segmentation. Nonnegative matrix factorization was adopted in [77] to perform multi-view clustering for motion segmentation. Djelouah et al. [25] addressed the motion segmentation problem by propagating segmentation coherence information in both space and time.

B. Natural Language Processing

In natural language processing, the documents can be obtained in multiple languages. It is natural to use multi-view clustering to conduct document categorization [6], [19], [78], [79], [139], [140] with each language as one view. Borrowed from the co-training and co-regularization ideas, Kumar et al. [6], [19] proposed co-training multi-view clustering and co-regularization multi-view clustering, respectively. The performance comparison on multilingual data demonstrates the superiority of these two methods. Liu et al. [78] extended nonnegative matrix factorization to multi-view settings for clustering multilingual documents. Kim et al. [139] obtained the clustering results on each view and then constructed a consistent cluster by voting. Jiang et al. [140] proposed a collaborative PLSA method that combines individual PLSA models in different views and imports a regularizer to force both the clustering results in different views agree across different views. Hussain [141] utilized an ensemble way to perform multi-view clustering for documents.

C. Social Multimedia

Currently, with the fast development of the social multimedia, there are huge information available. How to make full use of these large quantity of social multimedia information is a challenging problem, especially match them to the “real-world concepts” like the “social event detection”. Fig. 3 shows two such events like a concert, a NBA game, etc. The pictures showed there is just one view, other textural features such as tags and titles form the other view. Such social event detection problem is a typical multi-view clustering problem. Petkos et al. [142] adopted a multi-view spectral clustering method to detect the social event and additionally utilized the known supervisory signals. Samangooei et al. [143] did the feature selection first before constructing the similarity matrix and applied a density based clustering to the fused similarity matrix. Petkos et al. [144] proposed a graph-based multi-view clustering to cluster the data from social multimedia.
Multi-view clustering also applied to multimedia collection grouping [22] and news story clustering [145].

Fig. 3: Some pictures from two social events: concerts (top row) and NBA game (bottom row).

D. Bioinformatics and Health Informatics

In order to seek for the genetic variant for substance dependence, Sun et al. [13], [106], [107] designed three multi-view co-clustering methods to subtype the patients and then look for the genetic variant. To deal with missing values, Chao et al. [146] proposed a generalized multi-view co-clustering to analyze the heroin treatment effect. The three views of the heroin dependence patients can be seen from Fig. 4. Yu et al. [45], [147] designed the multi-kernel combination to fuse different view information and applied to disease data set to show its superior performance. In [148], a multi-view clustering based on Grassmann manifold was proposed to deal with disease gene identification.

Fig. 4: Three views from health informatics: vital sign (left), urine drug screen (middle) and craving measure (right)).

VI. OPEN PROBLEMS

A. Large Scale Problem (size and dimension)

In modern times, a large quantity of data are generated every day. For instance, several million posts are shared per minute in Facebook, which include multiple forms (views): videos, images and texts. At the same time, a large amount of news appear in different languages, which can also be considered as multi-view data with each language as one view. However, most of the existing multi-view clustering methods can only deal with small data, which raises a challenging problem: how to extend them to the large scale application. For example, it is impracticable for the existing multi-view clustering methods based on spectral clustering to work on large scale data due to the expensive computation of graph construction and eigen-decomposition. Some previous works like [52], [149], [150], [151] are proposed to accelerate the spectral clustering to adapt it to large scale scenarios. Therefore, it is intriguing to extend them into multi-view settings in the future.

Another important aspect is large in the dimension, like in the bioinformatics, each person has millions of genetic variants as genetic features. Compared with the dimension of the features, the number of the samples are not high. Combined with the phenotype features as phenotype view, this is a multi-view clustering problem. How to deal with such kind of clustering problems is tough due to the over-fitting problem. Although feature selection [152], [153] or feature dimension reduction like PCA is commonly used to alleviate this problem in single-view settings, there are not convincing methods up to now, especially deep learning cannot cope with it due to the properties: small size and high feature dimension. It may recall new theory to appear to handle this problem.

B. Incomplete Views or Missing Value

Multi-view clustering has been successfully applied to many applications as shown in Section V. However, there is an underlying problem hidden behind: what if one or more views are incomplete. This is very common in real applications. For example, in multi-lingual documents, many documents may just have one or two language versions; in social multimedia, some sample may lost visual or audio information due to the sensor failure; in health informatics, some patients do not like to take the urine drug screen or special vital sign item to cause missing views or missing values. Some missing items are random while the others are non-random. Simply imputing with zero or mean value [154] is the natural way to deal with this problem, and multiple imputation [155] is another way. However, without considering the differences of random and non-random effects, the final performance is not ideal [146].

Up to now, there are already some multi-view works [23], [35], [36], [43], [73], [83], [85], [104] developed in this direction. These three works [43], [83], [85] introduced a weight matrix to indicate whether the corresponding values are missing to deal with this problem. For two-view case, [35] reorganized the multi-view data to include three parts: data with both two views, data with only view 1 and data with only view 2 to reach the goal to handle missing values. By assuming there is at least a complete view, Trivedi et al. [104] completed the kernel matrix with missing values from the kernel matrix with all values borrowing the graph Laplacian idea. It is noted that all these methods just can deal with the settings that the subjects with incomplete views, they cannot deal with any missing values. That is, if some subject misses even one feature in a view, these aforementioned methods cannot use the other feature values in this view. Obviously, these methods have significant limitations that cannot make full use of the available multi-view incomplete information. Therefore, it is intriguing to develop more general multi-view incomplete algorithms to overcome these problems. In addition, just as we mention above, all these methods did not take the difference between random and non-random effects into consideration. Therefore, it would be interesting to design some new strategies with this taking into account. Moreover, it is noteworthy to explore how these algorithms perform with different missing degree.
C. Local Minima

For multi-view clustering based on partition methods like k means, the initial clusters are very important to the final results. So how to carefully select the initial clusters is still a bottleneck to multi-view clustering, even the basic clustering problem.

Most NMF methods are non-convex problems, and thus are prone to local minima problem, especially when missing value and outliers exist. Self-Paced learning [27] is a possible solution, and Xu et al. [54] applied it to multi-view clustering to alleviate the local minima problem.

The generative method convex clustering [56] is an interesting approach to avoid the local minima problem. In [60], the corresponding multi-view version is proposed. Therefore, this kind of generative method may be another good solution.

D. Deep Learning

Recently, Deep learning has demonstrated outstanding performance in many applications such as speech recognition, image segmentation, object detection and so on. However, they mainly aimed to supervised or semi-supervised learning, especially perform well on learning representation for classification; there are few deep learning works on clustering, let alone multi-view clustering.

These works [18], [156], [157] borrowed the supervised deep learning idea to perform supervised clustering, in fact, exactly they can be considered as semi-supervised learning. So far, there are only several truly deep clustering works [117], [17]. Tian et al. [117] proposed a deep clustering algorithm that is based on spectral clustering, but replaces eigenvalue decomposition with deep auto-encoder. Xie et al. [17] proposed a clustering approach using deep neural network which can learn representation and perform clustering simultaneously. Obviously, how to extend the existing single-view deep clustering to multi-view settings and design some special deep clustering or multi-view deep clustering methods targeting at certain real-world applications are promising future directions.

E. Mixed Data Types

Multi-view data may not necessarily just contain numerical or categorical features. They can also have other types like nominal, ordinal, binary etc. These different types can appear simultaneously in the same view, let alone in different views. How to take good advantage of each type to perform multi-view clustering is worthy to carefully deal with. Converting all of them to categorical type is the simple way to adopt. However, much information will be lost during such processing. For example, the difference of the continuous values categorized into the same category is ignored. Statistical perspectives may be a route to such a challenge.

F. Multiple Solutions

Most of the existing multi-view clustering, even single-view clustering algorithms only output one clustering solution. However, in real-world applications, data can often be grouped in many different ways and all these solutions are reasonable and interesting from different perspectives. For example, it is both reasonable to group the fruits apple, banana and grape according to the fruit type or color. Until now, to the best of our knowledge, there are only two works on this direction [44], [16]. Cui et al. [44] proposed to partition the multi-view data by projecting it to a space that is orthogonal to current solution so that multiple non-redundant solutions are obtained. In Niu et al. Niu et al. [16], Hilbert-Schmidt Independence Criterion was adopted to measure the dependence across different views and then one clustering solution was found in each view. Therefore, multi-view clustering algorithms that give multiple solutions should attract more attentions in the future.

VII. Conclusion

In this survey paper, we have reviewed two mainly MVC methods: discriminative methods and generative methods. For discriminative methods, based on the way to integrate multiple views, we split into five main categories, the first three of which have a commonality: sharing certain similarity structure, the fourth one is the direct combination (mainly including multi-kernel based methods) while the fifth one is combination after projection (mainly including CCA based methods). As for generative methods, we can find that they developed far less sufficiently than discriminative ones. To better understand multi-view clustering, we elaborate the relationships between MVC and several closely related learning methods. We also introduced several real-world applications of MVC and pointed out some interesting and challenging directions.

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