TIGHTNESS FOR THE COVER TIME OF THE TWO DIMENSIONAL SPHERE

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Abstract. Let \( C^{*}_{\epsilon, S^2} \) denote the cover time of the two dimensional sphere by a Wiener sausage of radius \( \epsilon \). We prove that
\[
\sqrt{C^{*}_{\epsilon, S^2}} - \sqrt{\frac{2A_{S^2}}{\pi}} \left( \log \epsilon^{-1} - \frac{1}{4} \log \log \epsilon^{-1} \right)
\]
is tight, where \( A_{S^2} = 4\pi \) denotes the Riemannian area of \( S^2 \).

1. Introduction

Let \( M \) be a smooth, compact, connected, two-dimensional Riemannian manifold without boundary. For each \( x \in M \) let \( C_{x, \epsilon, M} \) be the amount of time needed for the Brownian motion to come within (Riemannian) distance \( \epsilon \) of \( x \). Then \( C^{*}_{\epsilon, M} = \sup_x C_{x, \epsilon, M} \) is the \( \epsilon \)-cover time of \( M \). It is shown in [16, Theorem 1.3] that
\[
\lim_{\epsilon \to 0} \frac{C^{*}_{\epsilon, M}}{(\log \epsilon)^2} = \frac{2A_{M}}{\pi} \quad \text{a.s.,}
\]
where \( A_{M} \) denotes the Riemannian area of \( M \). For the special case where \( M \) is the two dimensional torus \( T^2 \) with \( A_{T^2} = 1 \), [4] showed that the rescaled cover time \( C^{*}_{\epsilon, T^2}/(\frac{1}{\pi} \log \epsilon^{-1}) \) has a log-log correction term:
\[
\frac{C^{*}_{\epsilon, T^2}}{\frac{1}{\pi} \log \epsilon^{-1}} = 2 \log \epsilon^{-1} - \log \log \epsilon^{-1} + o(\log \log \epsilon^{-1}),
\]
see [4 (1.2)]. (An analogue of (1.2) for the cover time of the discrete torus by simple random walk was recently obtained in [1].) Note that, with \( c^*_M = \sqrt{\frac{2A_{M}}{\pi}} \) and
\[
m_{\epsilon, M} = c^*_M \left( \log \epsilon^{-1} - \frac{1}{4} \log \log \epsilon^{-1} \right),
\]
\[ \sqrt{C^*_{\epsilon, S^2}} - m_{\epsilon, S^2} = o(\log \log \epsilon^{-1}). \]

1.1. Tightness of cover time. In spite of recent progress concerning the study of the maximum of various correlated fields, see Section 1.3 for details, improving on (1.2) has remained elusive. Our goal in this paper is to improve on (1.1) and (1.2) by proving tightness, in the case of the standard two dimensional sphere, \( M = S^2 \), where \( A_{S^2} = 4\pi \) and \( c^*_{S^2} = 2\sqrt{2}. \) (We comment below on our choice of working with \( M = S^2 \).) Let \( P_x \) denote the probability measure for Brownian motion on the sphere starting at \( x \), and whenever probabilities do not depend on the starting point \( x \) of the Brownian motion, we write \( P \) instead of \( P_x \). We use \( \mathbb{B}_d(a, r) \) denote the ball in \( S^2 \) centered at \( a \) of radius \( r \) in the standard metric \( d \) for \( S^2 \), see Section 2 and (2.5). Our main result reads as follows.

**Theorem 1.1.** The sequence of random variables

\[ \sqrt{C^*_{\epsilon, S^2}} - m_{\epsilon, S^2} \]

is tight. More explicitly,

\[ \lim_{K \to \infty} \limsup_{\epsilon \to 0} \mathbb{P} \left( \left| \sqrt{C^*_{\epsilon, S^2}} - m_{\epsilon, S^2} \right| > K \right) = 0. \]

In addition, for any \( \mathbb{B}_d(a, r) \subseteq S^2 \), the same result holds if \( C^*_{\epsilon, S^2} \) is replaced by \( C^*_{\epsilon, S^2, \mathbb{B}_d(a, r)} \), the \( \epsilon \)-cover time of \( \mathbb{B}_d(a, r) \subseteq S^2 \) by Brownian motion on \( S^2 \).

Note that (1.5) is equivalent to the statement

\[ \lim_{K \to \infty} \limsup_{\epsilon \to 0} \mathbb{P} \left( \left| \frac{C^*_{\epsilon, S^2}}{2(c^*_{S^2})^2 \log \epsilon^{-1}} - (2 \log \epsilon^{-1} - \log \log \epsilon^{-1}) \right| > K \right) = 0. \]

With a slight abuse of language, we use the statement the cover time of \( M \) is tight to mean either (1.5) or (1.6), with \( S^2 \) replaced by \( M \).

As in [16] and subsequent work on finer results including [4] and [5], the key to Theorem 1.1 is obtaining good upper and lower bounds on the right tail of the distribution of the centered cover time (with possibly different centerings for the upper and lower bounds). Our main technical result is in this spirit, this time with precisely the correct centering for both bounds.

**Theorem 1.2.** On \( S^2 \), for some \( 0 < C, C', d < \infty \) and all \( z \geq 0 \),

\[ \limsup_{\epsilon \to 0} \mathbb{P} \left( \sqrt{C^*_{\epsilon, S^2}} - m_{\epsilon, S^2} \geq z \right) \leq C e^{-\sqrt{2z} + d\sqrt{z}}. \]

and

\[ \liminf_{\epsilon \to 0} \mathbb{P} \left( \sqrt{C^*_{\epsilon, S^2}} - m_{\epsilon, S^2} \geq z \right) \geq C' e^{-\sqrt{2z} - d\sqrt{z}}. \]
We believe, in analogy with [5], that the right side of (1.7) and (1.8) should be multiplied by a factor \((z \lor 1)\) and that the factor \(d \sqrt{z}\) should not be present in the exponent. Obtaining such precision is beyond our current method of proof, but would be important if one tried to establish a limit law for \(\sqrt{C^*_{\epsilon, S^2} - m_{\epsilon, S^2}}\). See subsection 1.4 below.

We believe that Theorem 1.2 should extend to other two-dimensional compact manifolds.\(^1\) While this is currently outside our reach\(^2\), the following may serve as an intermediate step. Let \(B_{\epsilon}(a, r)\) denote the ball in \(\mathbb{R}^2\) centered at \(a\) of radius \(r\). Let \(C^*_{\epsilon, P}\) denote the \(\epsilon\)-cover time of the unit disc by planar Brownian motion \(W_t\). For \(R > 0\), set

\[C^*_{\epsilon, P, R} = \int_0^{C^*_{\epsilon, P}} \mathbf{1}_{W_t \in B_{\epsilon}(0, R)} dt,\]

the amount of time the path needs to spend in \(B_{\epsilon}(0, R)\) to come within \(\epsilon\) of each point in the unit disc. Set \(c^*_{P, R} = \sqrt{2} R\) and, in analogy with (1.3), set

\[m_{\epsilon, P, R} = c^*_{P, R} \left( \log \epsilon^{-1} - \frac{1}{4} \log \log \epsilon^{-1} \right).\]

**Corollary 1.3.** Let \(R \geq 1\). Then the sequence of random variables

\[\sqrt{C^*_{\epsilon, P, R} - m_{\epsilon, P, R}}\]

is tight.

In the rest of the paper except for Section 7, we drop \(S^2\) from the notation and write \(C^*_{\epsilon}, c^*\) instead of \(C^*_{\epsilon, S^2}, c^*_{S^2}\).

**1.2. Methods and limitations.** Theorem 1.4 is the counterpart of our earlier result [5, Theorem 1.3] concerning the cover time of the binary tree of depth \(n\) by a random walk, and has the same form. In fact, the underlying method of proof is, at a high level, similar. The uninitiated reader may find it helpful to read [5, Section 5] prior to going over the details of the proofs in this paper. However, the tree possesses a certain decoupling property that is not present in \(S^2\), and for this reason the proof for \(S^2\) is much more intricate. Two crucial new ideas, which could be considered as the main innovations of this paper, are needed in order to handle the manifold case. First, continuity estimates, which provide control of correlations in short scales, are developed in Section 5. This is a point where the argument differs from all previous works, and it is precisely in order to derive such estimates that we work with \(S^2\) instead of a general manifold. Note that such issues are not present for the cover time of the binary tree, discussed in [5]. The second main new idea relates to decoupling at coarse scales, where

\(^1\)We expect that the case of the torus \(T^2\) could be handled by our methods, although we do not address this in the paper.

\(^2\)An earlier version of this article claimed such a result, based on a faulty reduction from general manifolds to planar Brownian motion.
we introduce and use an $L^1$-Wasserstein distance in order to couple certain dependent Brownian excursions to a collection of independent ones. This is explained below and in Section 4.5. In the rest of this subsection, we give an outline of our argument.

As in [16] and [4], rather than work directly with $C^*_\varepsilon$, we work with an object $t^*_L$ which we call the $h_L$-cover local excursion time. Here $h_L = 2 \arctan(r_0 e^{-l}/2)$ with $r_0$ small (the appearance of arctan is due to our use of isothermal coordinates, see Section 2 below), and $L$ is chosen so that $h_L \sim \varepsilon$, i.e. $L \sim \log(1/\varepsilon)$.

To define $t^*_L$, let $T^x_{l,n}$ be the number of excursions from $\partial B^d(x,h_l-1)$ to $\partial B^d(x,h_l)$ prior to completing the $n$'th excursion from $\partial B^d(x,h_0)$ to $\partial B^d(x,h_0)$. The processes $T^x_{l,n}, l \geq 0$, are in fact critical Galton-Watson processes with geometric offspring distribution, which explains why the estimates in [5] are relevant.

Let
\begin{equation}
(1.11)
\begin{aligned}
t^*_{x,L} = \inf \{ n \mid T^x_{n,L} \neq 0 \},
\end{aligned}
\end{equation}

the number of excursions from $\partial B^d(x,h_l)$ to $\partial B^d(x,h_0)$ needed for the Brownian motion to come within (Riemannian) distance $h_l$ of $x$, and let $t^*_L$ be the centers of an $h_l/1000$ minimal cover of $\mathbb{S}^2$. Then
\begin{equation}
(1.12)
\begin{aligned}
t^*_L := \sup_{x \in F_L} t^*_{x,L},
\end{aligned}
\end{equation}
can be thought of as a cover time, but with time measured locally, that is, in terms of the number of excursions from $\partial B^d(x,h_1)$ to $\partial B^d(x,h_0)$, for each $x$. Note that
\begin{equation}
(1.13)
\begin{aligned}
\left\{ t^*_L > t \right\} = \left\{ T^{y,t}_{L} = 0 \text{ for some } y \in F_L \right\}.
\end{aligned}
\end{equation}

\begin{equation}
(1.14)
\begin{aligned}
\rho_L = 2 - \frac{\log L}{2L},
\end{aligned}
\end{equation}

and
\begin{equation}
(1.15)
\begin{aligned}
t_z = t_{L,z} = (\rho_L L + z)^2/2 = L \left( 2L - \log L + 2z \right) + z^2/2 + O \left( z \log L \right).
\end{aligned}
\end{equation}

In terms of excursion counts, our main result is the following.

**Theorem 1.4.** Fix $r_0$ small. On $\mathbb{S}^2$, for some $C,C' < \infty$ and all $z \geq 0$,
\begin{equation}
(1.16)
\begin{aligned}
&\limsup_{L \to \infty} \mathbb{P} \left( T^{y,t_z}_{L} = 0 \text{ for some } y \in F_L \right) \leq C(1 + z)e^{-2z}.
\end{aligned}
\end{equation}

and
\begin{equation}
(1.17)
\begin{aligned}
&\liminf_{L \to \infty} \mathbb{P} \left( T^{y,t_z}_{L} = 0 \text{ for some } y \in F_L \right) \geq C'ze^{-2z}.
\end{aligned}
\end{equation}

Once Theorem 1.4 is established, in order to relate excursion counts to time, we use the fact that there are many excursions at the macroscopic level before the cover time, and hence a law of large numbers should allow one to transfer excursion counts to running time. The actual argument is
somewhat more complicated, mostly because one is dealing with excursion counts between circles of different centers, and so continuity considerations are important - one needs to show that, with high probability, the function $x \mapsto t_{x,L}^*$ is essentially continuous. Since the same issue also arises in the study of the upper bound (1.16), we discuss it in greater detail below, and only mention that this is one place where the assumption that $M = S^2$ is used crucially.

We now discuss the proof of the upper bound (1.16). The basic idea is simple, and reminiscent of similar computations done in the context of branching random walks, going back to [9], see [32, 25, 2] for a review; we follow [4] closely in the precise mapping of our cover time problem to the language of branching random walk. Using the fact that $T_{x,n}^y$ is a critical geometric Galton-Watson process, one may attempt a union bound of the form

$$
P \left( T_{y,t}^z L = 0 \text{ for some } y \in F_L \right) \leq \sum_{y \in F_L} P \left( T_{y,t}^z L = 0 \right).$$

By standard estimates for Galton–Watson processes, see (3.8) below, one sees that this computation is not sharp enough and would work only if one decreased the log $L$ correction term in the definition of $\rho_L$. (Indeed, this is what is done in [16], which only provides the correct leading order.) Instead, informed by the branching random walk analogy of [4], one observes that the process $l \rightarrow \sqrt{2T_{r}y,t} L$ should behave like a particle in branching random walk, and should therefore satisfy a barrier condition. Indeed, by barrier estimates for geometric Galton–Watson processes that were derived in our earlier work [5], see Lemma 8.1, the estimate (1.16) would follow by a union bound if instead of the event $\{ T_{y,t}^z L = 0 \}$, one would consider the event

$$\{ \sqrt{2T_{r}y,t} L \geq \alpha(l), l = 1, \ldots, L - 1, T_{y,t}^z L = 0 \}$$

where $\alpha(\cdot)$ is the barrier

$$\alpha(l) = \rho_L (L - l) - l L, \quad l = (l \wedge (L - l)), \quad \gamma = 0.4, \quad l \text{ integer}.$$  

Thus, it remains to handle the event that there exists $y \in F_L$ for which $\{ T_{y,t}^z L = 0 \}$ and $\{ \sqrt{2T_{r}y,t} L < \alpha(l) \}$ for some $l \leq L - 1$.

It is at this point that the strategy diverges from the case of branching random walks: one cannot take a union bound over all $y \in F_L$ concerning events at level $l$ - the decay probability for the event $T_{r}y,t^z L < \alpha(l)^2$ has exponential decay rate roughly $-2l$ but the exponential growth rate of $|F_L|$ is $2L$. Instead, we must effectively reduce the cardinality of points $y$ to consider for events involving scale $l$. This point was already present in previous work relating extreme problems to extremes of branching random walks, see e.g. [3], and appears also in the context of cover times in [4]. In the latter paper this problem was solved by allowing a margin of error and using a deterministic sandwiching of excursions between slightly smaller/larger
balls to relate $T_{l}^{y, t_{z}}$ and $T_{l}^{y', t_{z}}$ for $y, y'$ with $d(y, y') \ll e^{-2l}$. Here, we cannot afford such errors, and instead resort to a probabilistic estimate: because of the symmetry of the sphere, during one excursion between concentric circles $S_1, S_2$ centered at a point $x$ started uniformly on the inner circle $S_1$, the expected number of excursions between concentric smaller circles centered around a point $y$ well inside $S_1$ does not depend on $y$, see Lemma 5.4. This, together with a chaining argument (which can be traced back, in this context, to [3]), allows us to obtain concentration bounds, which we refer to as “continuity estimates”, that are strong enough to control the event that the barrier has been crossed by some “particle” at some intermediate level $l$.

A significant effort has to be invested in the proof of the lower bound (1.17). As in [16, 4, 5], we use a second moment method. Similarly to [4, 5], we apply it together with a (linear) barrier, which means that we need to compute probabilities of events of the form

$$\bigcap_{y \in \{y_1, y_2\}} \{\sqrt{2T_{l}^{y, t_{z}}} \geq \rho_L(L - l), l = 1, \ldots, L - 1, T_{L}^{y, t_{z}} = 0\}.$$  (1.19)

(Actually, we consider a more complicated notion of good event, see (4.22).)

In the case of the tree detailed in [5], there is complete decoupling between the excursions in different branches of the tree, given the number of excursions at the edges below the common root of the branches. This exact decoupling is not true on $M$. In [16] and [3], this obstacle was circumvented by disregarding several levels in the barrier (i.e., those layers corresponding roughly to $\log d(y_1, y_2)/2 \pm O(\log L)$), and applying an estimate on the Poisson kernel for Brownian motion. This results in a loss in the upper bound on the probability of the event in (1.19), which we cannot afford, especially when $d(y_1, y_2)$ is relatively large. Our way to circumvent this issue is to observe that when $l < L/2$, there are many excursions at level $l$ before the cover time, and hence the empirical measure of the angle between the starting and ending points of each excursion approaches the equilibrium measure (in Wasserstein distance). We then add the event that this Wasserstein distance is not large to our definition of good event. On the good event, we can use Poisson kernel estimates to obtain a good decoupling, sufficient for an application of the second moment method, see Lemma 4.13. Here too, working with $M = S^2$ somewhat simplifies the analysis, but not significantly: the proof of the lower bound carries over to general compact two-dimensional manifolds, although we do not carry this out here.

Having invested all this work, transferring the results from $S^2$ to the plane is then straightforward. The result concerning excursion counts extends immediately, by using stereographic projection. The control of cover time (measured only inside the disc $B_e(0, R)$) is then a concentration result. The details are provided in Section 7.
1.3. **Background.** There is convincing evidence that, at the leading order, the cover time of graphs and manifolds is closely related to extremes of the Gaussian free field on the same space. Perhaps most striking is the sequence \([20, 19, 31]\), where it is established that for discrete graphs with bounded degrees, the cover time normalized by the size of the edge set is asymptotic to a (universal) constant multiple of the (square of) the maximum of the Gaussian free field (GFF) on the graph. In dimension \(d = 2\), the GFF obtained by a discretization of \(M\) with finer and finer mesh is a logarithmically correlated centered Gaussian field. In recent years, a theory has emerged concerning the leading order of the maximum of such fields \([7]\), the tightness of the maximum \([12, 21]\), and even the fluctuations of the maximum, see \([10, 11, 21]\). In particular, one has a log-log correction term for the centering of the maximum, and the centered maximum has tight fluctuations as the mesh-size tends to 0. However, as pointed out already in \([22]\) (for the tree) and \([4]\) (for the torus), the log-log terms do not match what they are in the case of the GFF. As is clear from the latter papers and \([5]\), the mismatch in the log-log correction term is not due to a basic difference between the behavior of the cover time and the associated GFF. Rather, it is because Gaussian random walks (in the case of maximum of the GFF) are essentially replaced by Bessel processes. However, even after this replacement, much work needs to be done to obtain a properly decoupled tree structure, and it is precisely this extra step that the current paper addresses. (Compare with \([5]\), where cover time results for the tree are obtained in a relatively straight-forward way from the barrier estimates for critical geometric Galton Watson processes.)

1.4. **Open problems.** We expect that Theorem 1.1 extends to general smooth two-dimensional compact manifolds. In addition, based on the analogy with the extrema of Branching random walks and log-correlated Gaussian fields, one expects that Theorem 1.1 should be replaced by the statement

\[
\text{The sequence of random variables } \sqrt{C_{\varepsilon,M} - m_{\varepsilon,M}} \text{ converges in distribution to a randomly shifted Gumbel random variable.}
\]

(1.20)

As mentioned above, a key step in proving such convergence would be the improvement of the tail estimates in Theorem 1.2 for \(z\) large, which in turn would require a corresponding improvement of Theorem 1.1.

A first step in the direction of proving (1.20), by resolving the analogous question for random walk on the binary tree has recently been taken in \([15]\), by methods different from those employed in this paper. For a proof based on the methods here, see the forthcoming \([18]\).

1.5. **Structure of the paper.** Sections 2-6 are devoted to a proof of (1.16). The proof employs barrier estimates from \([5]\) which are adapted to our needs in Appendix 8 and a comparison of excursion counts to excursion times...
around different centers, see Theorem 3.1 (whose proof is given in Section 3.4) for a precise statement. The comparison heavily relies on the continuity estimates contained in Section 5, see Lemma 5.1. Theorem 3.1 is used again in Section 3.4 to show that (1.16) implies (1.7), which gives one side of (1.5). Section 4 is devoted to the proof of the lower bounds (1.8) and (1.17). After quickly showing, using again Theorem 3.1 that (1.17) implies (1.8), the rest of the section and most of the effort are devoted to the proof of (1.17); a key ingredient is Lemma 4.13 (the decoupling lemma), proved in subsection 4.5. Equipped with the lemma and the barrier estimates of Appendix 8, the argument employs a second moment method (of a counting statistic). Subsection 4.1 is devoted to a lower bound on the first moment of the statistic, while subsections 4.2-4.4 are devoted to an upper bound on the second moment, divided into cases according to the distance between the points involved. Finally, Section 7 is devoted to the proof of Corollary 1.3.

Acknowledgement We thank two anonymous referees for a detailed reading of the paper. We particularly thank one of the referees for raising doubts concerning our reduction of the general compact manifold case.
1.6. **Index of Notation.** The following are frequently used notation, and a pointer to the location where the definition appears.

- \( t^*_{x,L}, t^*_{L}, F_l \) (1.11), (1.12)
- \( \rho_L \) (1.14)
- \( t_z \) (1.15)
- \( l_x, \alpha(l) \) (1.18)
- \( B_2(x, r) \) (2.5)
- \( r_l, h_l \) (2.9)
- \( T^{y,n}_{x,l} = T^{y,r_l-1\rightarrow r_l}, T^{x,r_l-1\rightarrow r_l} \) (2.11)
- \( T^{x,n}_{l} = T^{x,r_l-1\rightarrow r_l} \) (2.10)
- \( \tau_x(m) \) (2.14)
- \( s(z), s_L(z) \) (2.16)
- \( I_u \) (2.18)
- \( \tilde{c}, q_0 \) (3.2)
- \( A,z,d \) (3.11)
- \( \hat{z} \) (3.15)
- \( B_{l,t} \) (3.20)
- \( b_l(L_1, z, \theta) \) (3.24)
- \( D_{0,t,l}(j) \) (3.30)
- \( Q \) (3.36)
- \( G_l \) (3.38)
- \( \tilde{I}_z \) (3.42)
- \( C_{0,t,l} \) (3.49)
- \( T^{0,t}_{y,r_{l1}}, T^{0,t}_{r_{l1}} \) (3.55)
- \( B_{l,t}^g \) (3.57)
- \( \gamma(l) \) (4.14)
- \( \mathcal{T}_{y,z}, \mathcal{I}_{y,z} \) (4.15), (4.22)
- \( \mathcal{W}_{y,k}(n) \) (4.19)
- \( N_{k,a} \) (4.20)
- \( \mathcal{H}_{k,a} \) (4.34)
- \( k^+, k^{++} \) (4.59)
- \( B_{y,k,L}, K_{k,p,a} \) (4.44), (4.45)
- \( \mathcal{B}_{y,k+3,k++}, \mathcal{B}_{y,k+++1,L} \) (4.60), (4.61)
- \( G_{k+}^{y} \) (4.63)
- \( \mathcal{B}_{y,k+3,k++} \) (4.69)
- \( \mathcal{V}_{y,k}(n), \mathcal{G}_{y,k}(N_{k,a}) \) (4.101), (4.102)
- \( \Phi_{k,a}, A_{N,k} \) (4.112), (4.116)
- \( \kappa_{a,b} \) (5.45)

2. **Isothermal coordinates and notation**

As explained in [16, Section 8], the existence of a smooth isothermal coordinate system in each small disc allows us to transform Brownian motion...
on $M$ to a time changed Brownian motion in the plane. Since hitting probabilities do not depend on a time change, it follows that in these coordinates, for $\rho_1 < \rho_2 < \rho_3$,

$$\mathbb{P}^{x \in \partial B_c(y, \rho_2)} (H_{\partial B_c(y, \rho_1)} < H_{\partial B_c(y, \rho_3)}) = \frac{\log (\rho_2 / \rho_3)}{\log (\rho_1 / \rho_3)},$$

where $H_A$ is the first hitting time for $A$ and $B_c(x,r)$ is the open Euclidean ball of radius $r$ centered at $x$.

For $S^2$, isothermal coordinates are nothing but stereographic projection. That is, we consider $S^2$ as the unit sphere in $\mathbb{R}^3$ centered at $(0,0,1)$, so that $S^2$ is tangent to $R^2 := \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. The stereographic projection of $p = (p_1, p_2, p_3) \in S^2$ is the point $\sigma(p)$ where the line between $p$ and the ‘North Pole’ $0 := (0,0,2)$ intersects $R^2$. Thus

$$\sigma(p_1, p_2, p_3) = \frac{(p_1, p_2)}{1 - p_3/2}.$$

We will refer to these as the isothermal coordinates centered at $(0,0,0)$. It can be shown that $\sigma$ maps circles into circles or lines. As shown in [29, p. 335], the stereographic projection is an isometry if we give $R^2$ the metric

$$\frac{1}{(1 + \frac{1}{4}(x^2 + y^2))^2} (dx \otimes dx + dy \otimes dy).$$

In this metric the distance from $(0,0)$ to $(\rho,0)$ is given by

$$d((0,0), (\rho,0)) = \int_0^\rho \frac{1}{(1 + \frac{1}{4}x^2)} dx = 2 \arctan(\rho/2).$$

Let

$$B_d(x,r) = \{y \in S^2 : d(x,y) < r\}$$

denote the open ball of radius $r$ in the spherical metric centered at $x$. Setting

$$h(r) = 2 \arctan(r/2),$$

we get from (2.1) that

$$\mathbb{P}^{x \in \partial B_d(0, h(\rho_2))} (H_{\partial B_d(0, h(\rho_1))} < H_{\partial B_d(0, h(\rho_3))}) = \frac{\log (\rho_2 / \rho_3)}{\log (\rho_1 / \rho_3)}.$$
2.1. Notation. Throughout this paper, unless otherwise stated, constants $c$, $c_i$ and $C_i$ may change from line to line but, unless noted otherwise, are universal and their value does not depend on additional parameters. Other constants (e.g., $\tilde{c}$, $q_0$, etc) will be fixed. Given $a, b > 0$ which may depend on other parameters, we write $a \simeq b$ if the ratio $a/b$ and $b/a$ are bounded above. We write $a \ll b$ if $a/b \to 0$ as function of an implicit parameter, which is always clear from the context.

Recall that $\mathbb{P}^x$ denotes the probability measure for Brownian motion on the sphere started at a point $x$. We let $X_t$ denote the canonical process under $\mathbb{P}^x$. When probabilities do not depend on $x$, we use $\mathbb{P}$ instead of $\mathbb{P}^x$.

We need to introduce notation for various hitting times, excursion counts and excursion times. For a set $A$ of positive capacity, we let $H_A$ denote the hitting time of $A$ by Brownian motion. We fix $r_0 \in (0, 10^{-6})$ small enough so that $r_l = r_0 e^{-l}$ and $h_l = h(r_l)$ satisfy $0.9 r_l \leq h_l \leq r_l$ for all $l = 0, 1, \ldots$, see (2.6).

In addition to the traversal counts $T^{x,n}_l$, the number of excursions from $\partial B_d(x, h_{l-1})$ to $\partial B_d(x, h_l)$ prior to completing the $n$th excursion from $\partial B_d(x, h_1)$ to $\partial B_d(x, h_0)$, we will often consider traversal counts between sometimes non-concentric annuli. The following notation will be particularly useful.

Definition 2.1. For any $0 < r < R < \tilde{r} < \tilde{R}$, let $T^{x,r_1 \to r}_y, r \to r_0 x, r_{l-1} \to r_l$ be the number of traversals $\partial B_d(u, h(R)) \to \partial B_d(u, h(r))$ during $n$ excursions $\partial B_d(x, h(\tilde{r})) \to \partial B_d(x, h(\tilde{R}))$.

Note that with this notation

(2.10) $T^{x,n}_l = T^{x,r_1 \to r_0}_{x, r_{l-1} \to r_l}$.

We will often abbreviate the notation in Definition 2.1, writing e.g.

(2.11) $T^{x,n}_x = T^{x,r_1 \to r_0}_{x, r_{l-1} \to r_l}$.

We will need to consider certain traversal processes that “start at lower scales”. For each $k \geq 1$ we define

(2.12) $T^{y,k,m}_l = T^{y,r_1 \to r_0}_{y, r_{k-1} \to r_l}$

to be the number of traversals from scale $l - 1$ to $l$ during the first $m$ excursions from scale $k$ to scale $k - 1$. Note the crucial “compatibility” property that

(2.13) $T^{y,k,m}_l = T^{y, l \to l}_l$ for $l \geq k$, on $\{ m = T^{y, t_l}_k \}$.

We also need to keep track of the time it takes to complete a prescribed number of excursions. We set

(2.14) $\tau_m(x) = \text{time needed for Brownian motion to complete } m \text{ excursions from } \partial B_d(x, h_1) \text{ to } \partial B_d(x, h_0)$.
Similarly, for \( 0 < a < b < \pi \) we set
\[
\tau_{x,a,b}(m) = \text{time needed for Brownian motion to complete } m \text{ excursions from } \partial B_d(x,a) \text{ to } \partial B_d(x,b).
\]

In addition to the notation \( t_z \), see (1.15), it is sometimes convenient to consider its rescaled linear approximation, defined as
\[
s(z) = s_L(z) = L(2L - \log L + z).
\]

We let \( P_n^{GW} \) denote the law of a critical Galton-Watson process with geometric offspring distribution with initial offspring \( n \). Using (2.7) and the strong Markov property, it is easy to see that
\[
\text{the } P\text{-law of } T^{x,n}_l, l \geq 0, \text{ is } P_n^{GW}.
\]

For any number \( u \) we write
\[
I_u = [u, u + 1).
\]

3. The upper bound

Let \( r_l \) be as in (2.9) and recall that \( F_l \) are the centers of a minimal \( h_l/1000 \) cover of \( M \). We can and will assume that \( F_l \subseteq F_{l+1} \). We record for future use that
\[
|F_l| \asymp r_l^{-2} = r_0^{-2} e^{2l}, l \geq 0.
\]

Recall also the excursion counts \( T_l^{y,z} \) and the notation \( t_z \) and \( s(z) \), see (2.10), (1.15) and (2.16). Let \( \tilde{c} = q_0/2 < 1/2 \) denote small constants, with \( q_0 \) chosen according to the deviation estimates in Lemma 5.6. For each \( l \) we then choose \( x_i \in F_L \) so that
\[
F_L = \bigcup_{i=1}^c \{ F_L \cap B_d(x_i, \tilde{c}h_l) \} \text{ for some } c = c(\tilde{c}) < \infty.
\]

Our goal in this section is to prove the upper bounds in Theorems 1.4 and 1.2, namely to prove (1.16) and (1.7). As it turns out, both parts rely on an accurate comparison of real time needed to complete roughly \( t_z \) traversals between concentric circles on the sphere. Recall that \( \tau_x(m) \) denotes the time needed to complete \( m \) excursions from \( \partial B_d(x, h_1) \) to \( \partial B_d(x, h_0) \), see (2.14).

**Theorem 3.1.** Fix \( r_0 > 0 \) small. Then there exist constants \( d, z_0, c \) (possibly depending on \( r_0 \)) such that for \( L \) sufficiently large and all \( z \) with \( z_0 \leq |z| \leq L^{1/2} \log^2 L \),
\[
\mathbb{P} \left( 4s_L \left( z - d\sqrt{|z|} \right) \leq \tau_x(s_L(z)) \leq 4s_L \left( z + d\sqrt{|z|} \right), \forall x \in F_L \right) \geq 1 - ce^{-4|z|}.
\]
The constant 4 plays no particular role - except that it is important that \( 4 > 2 \). The proof of Theorem 3.1 appears in Section 6 and is based on the continuity estimates provided in Section 5, see Lemma 5.11. Given Theorem 3.1 most of the work in this section is in the proof of the following proposition.

**Proposition 3.2.** There exists a constant \( c > 0 \) such that for all \( L \) sufficiently large, and all \( z \geq 0 \),

\[
\mathbb{P} \left( \inf_{y \in F_L} T_{L}^{y,t_z} = 0 \right) \leq c(1 + z)e^{-2z} e^{-\frac{z^2}{2L}}. \tag{3.5}
\]

To see what is involved in the proof, we begin with a simple estimate.

**Lemma 3.3.** There exists a \( c' > 0 \) so that, for all \( y \in \mathbb{R}^2 \), \( x \not\in B_d(y, h) \) and \( z > 0 \),

\[
\mathbb{P}\left(T_{L}^{y,t_z} = 0 \right) \leq c' e^{-2L L e^{-\frac{z^2}{4L}}}. \tag{3.6}
\]

**Proof.** It follows from (2.7) with \( \rho_1 = r_0, \rho_2 = r_1 \) and \( \rho_3 = r_L \), and the strong Markov property, that

\[
\mathbb{P}\left(T_{L}^{y,t_z} = 0 \right) = \left(1 - \frac{1}{L} \right)^{t_z} \leq e^{-\frac{z}{4L}}. \tag{3.7}
\]

The estimate (3.6) then follows from (1.15) and the fact that the \( O(z \log L)/L \) term is bounded by \( 1 + z^2/4L \).

Using Lemma 3.3 and (3.1) with \( l = L \), a union bound would give

\[
\mathbb{P}\left(T_{L}^{y,t_z} = 0 \text{ for some } y \in F_L \right) \leq CLE^{-2z-z^2/4L}. \tag{3.8}
\]

The factor \( L \) on the right hand side destroys any chance of using (3.8) to obtain (3.5). However, if \( z > L^{1/2} \log L \) it is easily seen that (3.8) implies (3.5). It thus remains to prove (3.5) for \( 0 \leq z \leq L^{1/2} \log L \).

To improve on (3.8), we will use the fact that the events \( \{T_{L}^{y,t_z} = 0\}_{y = y_1, y_2} \) are correlated because if \( \log d(y_1, y_2) \ll r \) then the number \( T_{L-r}^{y_i,t_z} \) of traversals around \( y_i \) at level \( L-r \) will be almost the same for \( i = 1, 2 \). To deal effectively with this, we recall the barrier \( \alpha(l) \) and the notation \( l_L \), see (1.18):

\[
\alpha(l) = \rho_L(L - l) - \gamma \frac{l}{L}, \quad l_L = (l \wedge (L - l)), \quad \gamma = .4
\]

Since \( \alpha(L) = 0 \), Proposition 3.2 will follow from the next proposition.

**Proposition 3.4.** There exists \( c \) such that for all \( L \) sufficiently large, and all \( 0 \leq z \leq L^{1/2} \log^2 L \),

\[
\mathbb{P}\left( \exists x \in F_L \text{ and } 1 \leq l \leq L \text{ such that } \sqrt{2T_{l}^{x,t_z}} \leq \alpha(l) \right) \leq c(1+z)e^{-2z} e^{-\frac{z^2}{2L}}. \tag{3.9}
\]
The proof of Proposition 3.4 will be provided in Sections 3.1-3.3, and is split into two cases. For \( l \) which are not too large, i.e. \( l \leq L/2 \), we can deal with (3.9) one level at a time. This is the content of Section 3.1. For larger \( l \)'s, which are handled in Section 3.2, and in particular for \( l = L \), we need to proceed inductively and make use of the facts established for lower levels. This method can be traced back to Bramson’s work [9]. Some crucial auxiliary estimates are postponed to Section 3.3. Finally, Section 3.4 is devoted to the proof of (1.7).

3.1. \( l \) not too large. We begin with rephrasing the part of Proposition 3.4 pertaining to \( l \) not too large.

**Proposition 3.5.** There exists \( c < \infty \) so that, for all \( L \) sufficiently large and all \( 0 \leq z \leq L^{1/2} \log^2 L \),

\[
\Pr \left( \exists x \in F_L \text{ and } 1 \leq l \leq L/2 \text{ such that } \sqrt{2T_{x,t}^{x,t_z}} \leq \alpha(l) \right) \leq ce^{-2z}e^{-\frac{2}{20}z^2 L}.
\]

**Proof.** Note the statement always holds if \( z \leq z_0 \) by increasing \( c \) if necessary. Hence, it suffices to prove the claim for \( z \geq z_0 \), for some fixed \( z_0 \) to be determined.

Set

\[
A_{z,d} = \{ \tau_y(t_{z-d\sqrt{z}}) \leq \tau_x(t_{z}) \leq \tau_y(t_{z+d\sqrt{z}}), \forall x,y \in F_L \}.
\]

\( A_{z,d} \) is the good event in which the time to complete the “right” number of excursions is comparable for different balls.) Noting that \( t_z = s(2z) + O(z^2 + z \log L) \) for \( z \) in the stated range, it follows from Theorem 3.1 (by modifying \( d \) there if necessary) that we can find \( d, z_0, c \) such that for \( L \) sufficiently large and all \( z_0 \leq z \leq L^{1/2} \log^2 L \)

\[
\Pr (A_{z,d}) \geq 1 - ce^{-4z}.
\]

Therefore, we need only show that for \( z_0 \leq z \leq L^{1/2} \log^2 L \),

\[
\Pr \left( \exists x \in F_L \text{ and } 1 \leq l \leq L/2 \text{ such that } \sqrt{2T_{x,t}^{x,t_z}} \leq \alpha(l), A_{z,d} \right) \leq ce^{-2z}e^{-\frac{2}{20}z^2 L}.
\]

Since \( e^{-t_L} \) is summable, it thus suffices to show that for all \( 1 \leq l \leq L/2 \),

\[
P_1^{(l)} := \Pr \left( \exists x \in F_L \text{ such that } \sqrt{2T_{x,t}^{x,t_z}} \leq \alpha(l), A_{z,d} \right) \leq ce^{-2z-t_L}e^{-\frac{2}{20}z^2 L}.
\]

By a union bound, recall (3.2) and (3.3), and then using \( A_{z,d} \) and introducing the notation

\[
\hat{z} = z - d\sqrt{z}
\]
we have
\begin{equation}
(3.16) \mathbb{P}_1(l) \leq c e^{2l} \mathbb{P} \left( \exists x \in F_L \cap B_d(0, \hat{c}h_l) \text{ such that } \sqrt{2T_{x,t}^{x,t}} \leq \alpha(l), A_{x,t} \right)
\end{equation}
\begin{equation}
\leq c e^{2l} \mathbb{P} \left( \exists x \in F_L \cap B_d(0, \hat{c}h_l) \text{ such that } \sqrt{2T_{x,t}^{0,0}} \leq \alpha(l) \right) =: c e^{2l} \mathbb{P}_2(l),
\end{equation}
where 0 is a fixed point in $F_L$ (which could be taken as the south pole) and we recall that $T_{x,t}^{0,0}$ is the number of traversals from $\partial B_d(x, h_{l-1}) \rightarrow \partial B_d(x, h_l)$ during $n$ excursions from $\partial B_d(0, h_1) \rightarrow \partial B_d(0, h_0)$, see (2.11).

Hence to obtain (3.14) it suffices to show that
\begin{equation}
(3.17) \mathbb{P}_2(l) \leq c e^{-2l-\theta} e^{-\frac{\theta^2}{2l}},
\end{equation}
We write $\mathbb{P}_2(l) \leq \mathbb{P}_2^{(l)} + \mathbb{P}_2^{(l)}$ where
\begin{align}
(3.18) & \quad \mathbb{P}_2^{(l)} := \mathbb{P} \left( \sqrt{2T_{l-2}^{x,t}} \leq \alpha(l) + 1 \right) \\
(3.19) & \quad \mathbb{P}_2^{(l)} := \mathbb{P} \left( B_{x,t} \text{ and } \sqrt{2T_{x,t}^{0,0}} \geq \alpha(l) + 1 \right),
\end{align}
and
\begin{equation}
(3.20) B_{x,t} = \left\{ \exists x \in F_L \cap B_d(0, \hat{c}h_l) \text{ s.t. } \sqrt{2T_{x,t}^{0,0}} \leq \alpha(l) \right\},
\end{equation}
It is important to remember that $B_{x,t}$ involves traversal counts centered at points which can differ from 0.

Before proceeding we need to state some deviation inequalities of Gaussian type for the Galton-Watson process $T_l \geq 0$ under $P_n^{GW}$, see (2.17). The proof is very similar to [4, Lemma 4.6], and is therefore omitted.

**Lemma 3.6.** There exists a constant $c$ such that for all $n, l = 1, 2, 3, \ldots,$
\begin{equation}
(3.21) P_n^{GW} \left( \left| \sqrt{2T_l} - \sqrt{2T_0} \right| \geq \theta \right) \leq c e^{-\frac{\theta^2}{2l}}, \quad \theta \geq 0.
\end{equation}

Recall, see (2.17), that $\{T_l^{x,t}\}_{l \geq 0}$ under $\mathbb{P}$ is distributed like $\{T_l\}_{l \geq 0}$ under $P_n^{GW}$. Therefore, we obtain the following estimates from Lemma 3.6 for $z \geq 0$ and $\theta \in \mathbb{R}$:
\begin{equation}
(3.22) \quad \mathbb{P} \left( \sqrt{2T_l^{x,t}} \leq \alpha(l) + \theta \right) \leq c e^{-\left(\sqrt{2T_l^{x,t}} - \alpha(l) - \theta\right)^2 / 2l}, \text{ if } (\alpha(l) + \theta)^2 / 2 \leq t_z,
\end{equation}
and
\begin{equation}
(3.23) \quad \mathbb{P} \left( \sqrt{2T_l^{x,t}} \geq \alpha(l) + \theta \right) \leq c e^{-\left(\sqrt{2T_l^{x,t}} - \alpha(l) - \theta\right)^2 / 2l}, \text{ if } (\alpha(l) + \theta)^2 / 2 \geq t_z.
\end{equation}
Using the definitions of $\alpha(l)$ and $t_z$, see (1.13) and (1.18), we have that
\[
\frac{(\sqrt{2l_z} - \alpha(l) - \theta)^2}{2l} = \frac{(2l - (l/2L) \log L + z + l_L^\gamma - \theta)^2}{2l}
\]
\[
= 2l + 2(z + l_L^\gamma - \theta) - \frac{l \log L}{L} + \frac{(z + l_L^\gamma - \theta) - \frac{l}{2L} \log L)^2}{2l}
\]
\[
\geq 2l + 2(z + l_L^\gamma - \theta) - \frac{l \log L}{L} + \frac{(z + l_L^\gamma - \theta)^2}{4l} - o_L(1).
\]
Therefore, with
\[
b(l, L, z, \theta) := L^{1/l} e^{-2l(2(z + l_L^\gamma - \theta))^2/4l},
\]
we have
\[
P\left(\sqrt{2l_z^{x(t)}} \leq \alpha(l) + \theta\right) \leq cb(l, L, z, \theta), \text{ if } \alpha(l) + \theta)^2 / 2 \leq t_z,
\]
and
\[
P\left(\sqrt{2l_z^{x(t)}} \geq \alpha(l) + \theta\right) \leq cb(l, L, z, \theta), \text{ if } \alpha(l) + \theta)^2 / 2 \geq t_z.
\]

Applying (3.25) with $\theta = 1$ and $x = 0$, and using that $L^{1/l} e^{-l_L^\gamma/2} \leq 1$ for $1 \leq l \leq L/2$, we have that for $l$ in that range
\[
Q_{21}^{(l)} \leq ce^{-2l - 2z + 2d \sqrt{2} - 2\log L} - \frac{z^2}{4l}.
\]
With $\gamma = 0.4$, if $z^{1.4} \leq l$ then $z^{0.54} \leq l_L^\gamma = l_L^\gamma$, while if $z^{1.4} \geq l$ then $z^2 / l \geq z^0.6$. It follows that
\[
Q_{21}^{(l)} \leq o_2(1) e^{-2l - 2z - l_L^\gamma - z^2/8l}.
\]

Hence to conclude the proof of Proposition 3.5, it will suffice to show that
\[
Q_{22}^{(l)} \leq ce^{-2l - 2z - l_L^\gamma - \left(\frac{z^2}{2ml} + \lambda\right)}.
\]

To obtain (3.29), our strategy will be to replace the events $B_{t_z,l}$ by events involving excursions around 0. Toward this end, recall that $I_u = [u, u + 1)$, see (2.18), and define the "(l - 2)-endpoint event"
\[
D_{0,t,l}(j) = \left\{ \sqrt{2l_z^{0,t,l}} \in I_{\alpha(l) + j} \right\}.
\]
Then
\[
Q_{22}^{(l)} = \sum_{j=1}^{\infty} P\left(B_{t_z,l} \cap D_{0,t_z,l}(j) \right) = \sum_{j=1}^{\infty} P\left(B_{t_z,l} \mid D_{0,t_z,l}(j) \right) \cdot P\left(D_{0,t_z,l}(j) \right)
\]
\[
\leq \sum_{j=1}^{\infty} P\left(B_{t_z,l} \mid D_{0,t_z,l}(j) \right) \cdot ce^{-2l - 2z + 2d \sqrt{2} + 2j - 1.5l_L^\gamma - (z + l_L^\gamma - j)^2/4l},
\]
where the last inequality follows again from the deviations estimates (3.25) or (3.26) as appropriate. We now state the following lemma, whose proof is postponed to subsection 3.3.
Lemma 3.7. There exist positive constants \( \tilde{c} \) and \( j_0 \) so that, with \( B_{t_z,l} \) as in (3.20), one has that for all \( z \geq 0 \),

\[
\Pr(B_{t_z,l} \mid D_{0,t_z,l}(j)) \leq e^{-4j}, \text{ for all } j \geq j_0.
\]

Substitute (3.32) into (3.31) and consider separately the case where \( j \leq \tilde{z}/2 \) and \( j \geq \tilde{z}/2 \). In the first case we have the bound

\[
\leq \sum_{j=1}^{z/2} e^{-4j} c e^{-2l-2z+2d\sqrt{z}+2j-1.5l^2/16l}
\]

which can be bounded by (3.29) as before. In the second case we simply use

\[
\leq \sum_{j=z/2}^{\infty} e^{-4j} c e^{-2l-2z+2d\sqrt{z}+2j-1.5l^2/16l}.
\]

Since \( j \geq \tilde{z}/2 \) this gives (3.29) and completes the proof of Proposition 3.5.

3.2. \( l \) large and proof of Proposition 3.4. Recall that \( \tau_x(m) \) denotes the time needed to complete \( m \) excursions from \( \partial B_d(x,h_0) \) to \( \partial B_d(x,h_1) \), see (2.14). We begin by stating a (simpler) version of Theorem 3.1, whose proof is also given in Section 6.

Theorem 3.8. There exists \( c > 0 \) so that for \( L \) sufficiently large and all \( 0 \leq z \leq L^{1/2} \log^2 L \),

\[
\Pr(\tau_y(t_z - 10) \leq \tau_x(t_z) \leq \tau_y(t_z + 10), \forall x, y \in F_L, d(x, y) \leq h(r_{L/2})) \geq 1 - ce^{-L/2}.
\]

Set

\[
Q = \{\tau_y(t_z - 10) \leq \tau_x(t_z) \leq \tau_y(t_z + 10), \forall x, y \in F_L, d(x, y) \leq h(r_{L/2})\}.
\]

It follows from Theorem 3.8 that for \( L \) sufficiently large and all \( 0 \leq z \leq L^{1/2} \log^2 L \),

\[
\Pr(Q) \geq 1 - ce^{-L/2}.
\]

We fix a small constant \( \tilde{c} \), to be chosen later. Introduce the “barrier event”

\[
G_l = \left\{ \sqrt{2T_y^{t_z}} \geq \alpha(l') \text{ for all } l' = 1, \ldots, l \text{ and } \forall y \in F_L \cap B_d(0, \tilde{c}h_0) \right\}.
\]

Let \( L' = L/2 \). By Proposition 3.5 we have that for all \( L \) large and \( 0 \leq z \leq L^{1/2} \log^2 L \),

\[
\Pr(G_{l'}^c) \leq ce^{-2z} e^{-\frac{2z^2}{2m}}.
\]
We will prove in this section the following lemma. The proof of the lemma uses some continuity estimates from Section 5 below, and barrier estimates from [5] which are discussed in Appendix 8.

**Lemma 3.9.** There exists a constant \( c > 0 \) so that, for all \( l > L' \) and \( 1 \leq z \leq L'/2 \log^2 L \),

\[
\mathbb{P}(G^c_l \cap G_{l-2} \cap Q) \leq c(z + 1)e^{-t^*_{l-2}z}e^{-\frac{2}{20r}^\frac{r}{2}}.
\]  

(3.40)

Assuming Lemma 3.9, we can complete the proof of Proposition 3.4.

**Proof of Proposition 3.4.** From (3.40), one has

\[
\mathbb{P}(G^c_L \cap Q) \leq \sum_{l=L'+1}^L \mathbb{P}(G^c_l \cap G_{l-1} \cap Q) + \mathbb{P}(G^c_L') \\
\leq \sum_{l=L'+1}^L \mathbb{P}(G^c_l \cap G_{l-2} \cap Q) + \mathbb{P}(G^c_L') \\
\leq \sum_{l=L'+1}^L cze^{-t^*_{l-2}z}e^{-\frac{2}{20r}^\frac{r}{2}} + \mathbb{P}(G^c_L') \leq cze^{-2z}e^{-\frac{2}{20r}^\frac{r}{2}},
\]

where the last inequality used (3.39). Combined with (3.37), we conclude that \( \mathbb{P}(G^c_L) \leq cze^{-2z} \). A simple union bound (over \( \sim (1/\tilde{c}h_0)^2 \) balls) then completes the proof of Proposition 3.4.

We turn to proving Lemma 3.9.

**Proof of Lemma 3.9.** By a union bound as in (3.16), \( \mathbb{P}(G^c_l \cap G_{l-2} \cap Q) \) is bounded above by

\[
ce^{2l}\mathbb{P}\left(\exists x \in F_L \cap B_d(0, \tilde{c}h_l) \text{ s.t. } \sqrt{2T^x_{t^*}} \leq \alpha(l) \right) \cap G_{l-2} \cap Q). \]

(3.41)

On \( Q \) we have that \( \left\{ \exists x \in F_L \cap B_d(0, \tilde{c}h_l) \text{ s.t. } \sqrt{2T^x_{t^*}} \leq \alpha(l) \right\} \) implies the event, see (3.20),

\[
B_{t^*_x,l} = \left\{ \exists x \in F_L \cap B_d(0, \tilde{c}h_l) \text{ s.t. } \sqrt{2T^0_{x,l}} \leq \alpha(l) \right\},
\]

where \( \tilde{t}_x = t_x - 10 \) and, recall (2.11), \( T^0_{x,l} \) is the number of traversals from \( \partial B_d(x, h_{l-1}) \to \partial B_d(x, h_l) \) during \( n \) excursions from \( \partial B_d(0, h_1) \to \partial B_d(0, h_0) \). Hence to prove (3.40) it suffices to show that for all \( l > L' \)

\[
\mathbb{P}(B_{t^*_x,l} \cap G_{l-2}) \leq c(z + 1)e^{-2t^*_x l^* - 2z}e^{-\frac{2}{20r}^\frac{r}{2}}.
\]

(3.42)

Set

\[
A_l = T^0_{l^*} - T^0_{l} \text{ law } T^0_{l^*}.
\]

(3.43)
We bound
\[ P(B_{t_x,l} \cap \mathcal{G}_{l-2}) \leq P(B_{t_x,l}; A_{L/2} > 0) + P(B_{t_x,l} \cap \mathcal{G}_{l-2}; A_{L/2} = 0), \]
and estimate each term separately.

Using the independence of \( B_{t_x,l} \) and \( A_l \) we have that
\[ P(B_{t_x,l}; A_{L/2} > 0) = P(B_{t_x,l}) \cdot P(A_{L/2} > 0) \leq cP(B_{t_x,l} L^{-1}, \]
where we have used (3.25) and the fact that the survival probability up to\( P_{10} \) is bounded by \( cL^{-1} \).

We claim that
\[ P(B_{t_x,l}) \leq cLe^{-2l^{-2} - 2x}e^{-\frac{z^2}{4}} \] \( l \leq L \).
To see this, we bound \( P(B_{t_x,l}) \leq \mathfrak{P}_{31}^{(l)} + \mathfrak{P}_{32}^{(l)} \) where
\[ \mathfrak{P}_{31}^{(l)} := P\left( \sqrt{2T_{t_x,l}^0} \leq \alpha(l) + 1 \right) \]
\[ \mathfrak{P}_{32}^{(l)} := P\left( B_{t_x,l} \text{ and } \sqrt{2T_{t_x,l}^0} \geq \alpha(l) + 1 \right). \]

It follows from (3.25) and (3.26) that \( \mathfrak{P}_{31}^{(l)} \) is bounded by the right hand side of (3.45). To bound \( \mathfrak{P}_{32}^{(l)} \) we follow the proof of the bound (3.29) we obtained for \( \mathfrak{P}_{32}^{(l)} \). The only difference is that now in the analogue of (3.31) we obtain
an extra factor of \( L \).

Combining (3.45) with (3.44) we see that to establish (3.42) it suffices to show that for all \( l > L' \),
\[ P(B_{t_x,l} \cap \mathcal{G}_{l-2}; A_{L/2} = 0) \leq c(z + 1)e^{-2l^{-2} - 2x}e^{-\frac{z^2}{4}}. \]
However, \( A_{L/2} = 0 \) implies that \( T_{m}^{0,t_x} = T_{m}^{0,t_x} \) for all \( m \geq L/2 \). Since the \( x \)'s in \( B_{t_x,l} \) are all in \( B_d(0, \epsilon h_t) \), it follows that for such \( x \)'s, \( T_{x,l}^{0,t_x} = T_{x,l}^{0,t_x} \). Thus on \( \{ A_{L/2} = 0 \} \) we have \( B_{t_x,l} = B_{t_x,l} \).

Since \( \mathcal{G}_{l-2} \subset \mathcal{C}_{0,t_x,l} \), where
\[ \mathcal{C}_{0,t_x,l} := \left\{ \sqrt{2T_{l'}^{0,t_x}} \geq \alpha(l') \right\} \text{ for all } l' = 1, \ldots, l - 2 \}

it suffices to show that for \( l \geq L' \),
\[ P(B_{t_x,l} \cap \mathcal{C}_{0,t_x,l}) \leq c(z + 1)e^{-2l^{-2} - 2x}e^{-\frac{z^2}{4}}. \]

Let
\[ \mathcal{D}_{0,t_x,l}(j) = \left\{ \sqrt{2T_{l-2}^{0,t_x}} \in I_{\alpha(l)+j} \right\}. \]
Using (3.25)-(3.26), by (3.50) it suffices to show that

\[
\sum_{j=0}^{8L} \mathbb{P}(B_{t_{x},l} \cap C_{0,t_{x},l} \cap D_{0,t_{x},l}(j)) \leq c(z+1)e^{-2l_{L} - 2z - e^{-\frac{t_{x}^{2}}{16}}}.
\]

The following analogue of Lemma 3.7 will be proved in Section 3.3 below.

**Lemma 3.10.** There exist constants \( j_{0}, C, z_{0} \) such that, for all \( j \geq j_{0}, l \leq L \) and \( z \geq 0 \),

\[
\mathbb{P}(B_{t_{x},l} \mid C_{0,t_{x},l} \cap D_{0,t_{x},l}(j)) \leq Ce^{-4j}.
\]

We continue with the proof of Lemma 3.9. Note that \( C_{0,t_{x},l} \cap D_{0,t_{x},l}(j) \) is a barrier event in the sense discussed in [5]. Based on the latter paper, we develop in the appendix the barrier estimates in the form that we need here. In particular, it follows from (3.3) in the appendix that

\[
\mathbb{P}(C_{0,t_{x},l} \cap D_{0,t_{x},l}(j)) \leq ce^{-2l_{L} - 2z - 2l_{L}^{2} + 2j}
\]

\[
\times (1 + z + l_{L}) (1 + j) e^{-\frac{(z + l_{L} - j)^{2}}{4}}.
\]

We break the sum in (3.51) into a sum over two intervals, \([0, (z + l_{L})/2]\), and \([(z + l_{L})/2, 8L]\). In the first interval we use

\[
e^{-\frac{(z + l_{L} - j)^{2}}{4}} \leq e^{-\frac{(z + l_{L})^{2}}{16}} \leq e^{-\frac{z^{2}}{16}}.
\]

For the last interval we ignore the last factor in (3.53) and use \( e^{-j} \leq e^{-z/2} \).

Putting this all together, we can bound from above the left hand side of (3.51) by

\[
C(z+1)e^{-2l_{L} - 2z - l_{L}^{2}}e^{-\frac{z^{2}}{16}} \sum_{j=0}^{\infty} e^{-3j_{1}(j \geq j_{0}) + 2j(1 + |j|)},
\]

which proves (3.51) and completes the proof of Lemma 3.9.

### 3.3. Proof of the conditional barrier estimates

We prove in this section Lemmas 3.7 and 3.10 whose statements boil down to the estimates

\[
\mathbb{P}(B_{t_{x},l} \mid D_{0,t_{x},l}(j)) \leq e^{-4j}, \quad \text{and} \quad \mathbb{P}(B_{t_{x},l} \mid C_{0,t_{x},l} \cap D_{0,t_{x},l}(j)) \leq Ce^{-4j},
\]

see (3.32) and 3.32. We intend to give a proof that will cover both cases. It will be seen from the proof that the time, \( t_{x} \) or \( t_{l} \), does not play a role in the proof. Hence we shall write it as \( t \). In addition, we will see that the extra conditioning on \( C_{0,t_{x},l} \) present in (3.52) is not significant.

**Proof of Lemma 3.10**

Fix \( \beta \in (0, 1/2) \),

\[
\tilde{r}_{l-1}^{+} = r_{l-1}(1 + \beta), \quad \tilde{r}_{l} = r_{l}(1 - \beta),
\]

and consider the excursions count \( T_{0,l}^{0,t} := T_{y_{i},r_{l}}^{0,r_{l-1} \rightarrow r_{0}}, \) compare with (2.11), writing \( T_{0,l}^{0,t} = T_{y_{i},r_{l}}^{0,t}, \) compare with (2.10). Note that for \( y, y' \) with \( d(y, y') \leq
$\beta r_l/2$, we have (using (2.9)) that

$$B_d (y', h(\tilde{r}_l)) \subset B_d (y, h_l) \subset B_d (y, h_{l-1}) \subset B_d (y', h(\tilde{r}_{l-1}^+)),$$

and therefore, writing $t = t_z$ throughout,

\begin{equation}
(3.56) \quad T_{y, r_l}^{0, t} \leq T_{y', r_l}^{0, t} \quad \text{for all} \ y \text{ and } y' \text{ such that } d (y, y') \leq \beta r_l/2.
\end{equation}

Let

\begin{equation}
(3.57) \quad B_{t, l}^\beta = \left\{ \exists y \in F_k \cap B_d (0, \tilde{c}h_l) \text{ such that } \sqrt{2T_{y, r_l}^{0, t}} \leq \alpha (l) \right\}.
\end{equation}

From now on we fix

\begin{equation}
(3.58) \quad \beta = \frac{1}{\alpha (l) + j} \quad \text{and} \quad k = \log (2(\alpha (l) + j)) + l.
\end{equation}

We will show that with these values,

\begin{equation}
(3.59) \quad \mathbb{P}_3 = \mathbb{P}_3 (j) := \mathbb{P} \left( B_{t, l}^\beta \cap D_{0, t, l} (j) \right) \leq C e^{-4j}.
\end{equation}

Using (3.56) this will imply (3.52), since for each $y \in F_k \cap B_d (0, \tilde{c}h_l)$ there exists a representative $y' \in F_k \cap B_d (0, \tilde{c}h_l)$ such that

$$d (y, y') \leq r_k = \frac{1}{2(\alpha (l) + j)} r_l = \beta r_l/2.$$

We thus turn to proving (3.59). We bound

\begin{equation}
(3.60) \quad \mathbb{P}_3 \leq \mathbb{P} \left( \sqrt{2T_{r_l}^{0, t}} \leq \alpha (l) + \frac{j}{2} \right) \left[ B_{t, l}^\beta \cap D_{0, t, l} (j) \right] + \mathbb{P} \left( \sqrt{2T_{r_l}^{0, t}} > \alpha (l) + \frac{j}{2} \right) \left[ B_{t, l}^\beta \cap D_{0, t, l} (j) \right] =: \mathbb{P}_{31} + \mathbb{P}_{32}.
\end{equation}

(Do not confuse $\mathbb{P}_{31}$ and $\mathbb{P}_{32}$ with $\mathbb{P}_{31}^{(l)}$ and $\mathbb{P}_{32}^{(l)}$ from (3.46) and (3.47). Note also that both $\mathbb{P}_{31}$ and $\mathbb{P}_{32}$ depend on $j$, but we continue to suppress the dependence in the notation.)

We first bound $\mathbb{P}_{31}$. Note that given $T_{l, l}^{0, t}$ for all $l' = 1, \ldots, l - 2$, it follows from the Markov property that $T_{r_l}^{0, t}$ depends only on $T_{l-2}^{0, t}$, and if $m = T_{l-2}^{0, t}$ then $T_{r_l}^{0, t} = T_{0,r_l}^{0,t} r_{r_l-1}^{r_l} = T_{0,r_{l-2}}^{0,t} r_{r_{l-1}-1}^{r_{l-2}}$. Hence

\begin{equation}
(3.61) \quad \mathbb{P}_{31} = \mathbb{P} \left( \sqrt{2T_{r_l}^{0, t}} \leq \alpha (l) + \frac{j}{2} \right) D_{0, t, l} (j).
\end{equation}
Hence,

\[
\begin{align*}
\mathcal{P}_{31} &= \mathbb{P}\left(\sqrt{2T_{\tilde{r}_l}^{0,t}} \leq \alpha(l) + \frac{j}{2} \mid \sqrt{2T_{l-2}^{0,t}} \in I_{\alpha(l)+j}\right) \\
&= \mathbb{P}\left(\sqrt{2T_{\tilde{r}_l}^{0,r_{l-2},m}} \leq \alpha(l) + \frac{j}{2} \mid \sqrt{2T_{l-2}^{0,t}} \in I_{\alpha(l)+j}\right),
\end{align*}
\]

(3.62)

where we write \(T_{\tilde{r}_l}^{0,r_{l-2},m} = T_{0,\tilde{r}_{l-1} \to \tilde{r}_l}^{0,r_{l-2},m}\), compare with (2.12).

Set \(u = \alpha(l) + j + \zeta\), where \(0 \leq \zeta \leq 1\). It follows from [1] Lemma 4.6, after correcting a typo, that

\[
\mathbb{P}\left(\sqrt{2T_{\tilde{r}_l}^{0,r_{l-2},u^{2/2}}} \leq \alpha(l) + \frac{j}{2}\right) \leq e^{-\left(\sqrt{q(\alpha(l)+j+\zeta)} - \sqrt{e}(\alpha(l)+\frac{j}{2})\right)^2/2}
\]

where

\[
q := \frac{\log r_{l-3} - \log r_{l-2}}{\log r_{l-3} - \log (r_l (1 - \beta))} = \frac{1}{3 + O(\beta)}
\]

and

\[
p := \frac{\log (r_{l-1} (1 + \beta)) - \log (r_l (1 - \beta))}{\log r_{l-3} - \log (r_l (1 - \beta))} = \frac{1 + O(\beta)}{3 + O(\beta)}.
\]

Indeed, to apply Lemma 3.6 or [1] Lemma 4.6 it suffices to show that

\[\alpha(l) + \frac{j}{2} \leq (\alpha(l) + j) \sqrt{q/p} = (\alpha(l) + j)(1 - O(\beta)) = (\alpha(l) + j) - O(1),\]

since \(\beta (\alpha(l) + j) = 1\). Thus we can use (3.63) for all \(j \geq c_3\) for some \(c_3 < \infty\). For such \(j\) we therefore have

\[
\mathbb{P}\left(\sqrt{2T_{\tilde{r}_l}^{0,r_{l-2},u^{2/2}}} \leq \alpha(l) + \frac{j}{2}\right) \leq ce^{-\left(\frac{c}{2} + \zeta + O(\beta(\alpha(l)+\frac{j}{2}))\right)^2/2},
\]

and using again \(\beta (\alpha(l) + j) = 1\) we obtain

\[
\mathcal{P}_{31} \leq c'e^{-c_4 j^2}
\]

for all \(j \geq c_4\). By enlarging \(c'\) we then have (3.66) for all \(j\).

We turn to bounding \(\mathcal{P}_{32}\). Assign to each \(y \in F_{l+m} \cap B_d(0, \hat{c}_l)\) a unique “parent” \(\check{y} \in F_{l+m-1} \cap B_d(0, \hat{c}_l)\) such that \(d(\check{y}, y) \leq r_{l+m}\). In particular, for \(m = 1\) we set \(\check{y} = 0\), and set \(\check{y} = y\) if \(y \in F_{l+1}\). Let \(q = q(\check{y}, y) = d(\check{y}, y) / r_l\) (not to be confused with (3.64)) and set

\[
\mathcal{A}_m = \left\{ \sup_{y \in F_{l+m} \cap B_d(0, \hat{c}_l)} \left| T_{y,\check{r}_l}^{0,t} - T_{\check{y},\check{r}_l}^{0,t}\right| \leq d_0 jm (\alpha(l) + j) \sqrt{q} \right\},
\]

where \(d_0\) will be chosen later, but small enough so that \(d_0 \sum_{m \geq 1} m e^{-m/2} \leq \frac{1}{8}\). In words, \(\mathcal{A}_m\) is the good event in which all neighboring (at scale \(l+m-1\))
excursion counts for balls whose centers are in a fixed ball at scale $l$ are not too distinct. We claim that

$$\bigcap_{m=1}^{k-l} A_m \cap \left\{ \sqrt{2 T_{y,\tilde{r}_l}^{0,t}} > \alpha(l) + \frac{j}{2} \right\}$$

$$\subseteq \left\{ \sqrt{2 T_{y,\tilde{r}_l}^{0,t}} > \alpha(l), \forall y \in F_k \cap B_d(0,\tilde{c}h(r_l)) \right\},$$

that is, under $A_m$, having excursion counts (centered at 0) larger than $\alpha(l) + j/2$ implies that all center counts for slightly off-center balls are larger than $\alpha(l)$.

Using that $q = d(y',y)/r_l \leq r_{l+m}/r_l = e^{-m}$ for $y \in F_{l+m}$ we see that on the event in the left hand side of (3.68), for any $y \in F_k \cap B_d(0,\tilde{c}h_l)$ one has

$$T_{y,\tilde{r}_l}^{0,t} \geq \left( \alpha(l) + \frac{j}{2} \right)^2 / 2 - j (\alpha(l) + j) d_0 \sum_{m \geq 1} me^{-m/2},$$

which, since $d_0 \sum_{m \geq 1} me^{-m/2} \leq \frac{1}{8}$, implies that

$$T_{y,\tilde{r}_l}^{0,t} \geq \left( \alpha(l) + \frac{j}{2} \right)^2 / 2 - \frac{1}{4} j (\alpha(l) + j) > \alpha^2(l)/2.$$

This establishes (3.68) and taking complements we see that

$$B_{t,l}^{\beta,k} \subseteq \bigcup_{m=1}^{k-l} A_m^c \cup \left\{ \sqrt{2 T_{y,\tilde{r}_l}^{0,t}} \leq \alpha(l) + \frac{j}{2} \right\}.$$

It follows that

$$B_{t,l}^{\beta,k} \cap \left\{ \sqrt{2 T_{y,\tilde{r}_l}^{0,t}} > \alpha(l) + \frac{j}{2} \right\} \subseteq \bigcup_{m=1}^{k-l} A_m^c.$$

For $y \in F_{l+m} \cap B_d(0,\tilde{c}h_l)$, write

$$A_{y,m}^{y,c} = \left\{ \left| T_{y,\tilde{r}_l}^{0,t} - T_{y,\tilde{r}_l}^{0,t} \right| \geq d_0 j m (\alpha(l) + j) \sqrt{q} \right\}.$$

We thus obtain that

$$\mathbb{P}_{j2} \leq \sum_{m=1}^{k-l} \mathbb{P} \left( \bigcup_{y \in F_{l+m} \cap B_d(0,\tilde{c}h_l)} A_{y,m}^{y,c} \left| C_{0,t,l} \cap D_{0,t,l}(j) \right. \right)$$

$$\leq \sum_{m=1}^{k-l} \left| F_{l+m} \cap B_d(0,\tilde{c}h_l) \right| \sup_{y \in F_{l+m} \cap B_d(0,\tilde{c}h_l)} \mathbb{P} \left( A_{y,m}^{y,c} \left| C_{0,t,l} \cap D_{0,t,l}(j) \right. \right)$$

$$\leq c \sum_{m=1}^{k-l} e^{2m} \sup_{y \in F_{l+m} \cap B_d(0,\tilde{c}h_l)} \mathbb{P} \left( A_{y,m}^{y,c} \left| C_{0,t,l} \cap D_{0,t,l}(j) \right. \right).$$
Note that if \( m = T_{l-2}^{0,t} \) then \( T_{y,\tilde{r}_l}^{0,r_{l-2}} = T_{y,\tilde{r}_l}^{0,r_{l-3}} = T_{y,\tilde{r}_l}^{0,r_{l-3}} \). We can thus write the last probability as

\[
(3.72) \quad P\left( \left| T_{y,\tilde{r}_l}^{0,r_{l-2}} - T_{y,\tilde{r}_l}^{0,r_{l-3}} \right| \geq d_0 jm (\alpha(l) + j) \sqrt{q} \left| C_{0,t,l} \cap D_{0,t,l}(j) \right| \right)
\]

\[
\leq \sup_{u \in I_{u(l)+j}} P\left( \left| T_{y,\tilde{r}_l}^{0,r_{l-2}} - T_{y,\tilde{r}_l}^{0,r_{l-3}} \right| \geq d_0 jm \sqrt{q}/2 \left| C_{0,t,l} \cap D_{0,t,l}(j) \right| \right)
\]

\[
(3.73) \quad \leq ce^{-4(j+m)}.
\]

Here \( \tilde{y} \) is the “parent” of \( y \) defined in the paragraph following (3.66).

Using (3.73) and substituting in (3.66) we see that for all \( j \geq j_0 \)

\[
(3.74) \quad \Psi_32 \leq c \sum_{m=1}^{k-l} e^{2m} e^{-4(j+m)} \leq Ce^{-4j},
\]

provided \( C \) is large enough. Combining (3.66) and (3.74) yields (3.59) and completes the proof of (3.52) and therefore of Lemma 3.10.

**Proof of Lemma 3.11.** The proof is almost identical to that of Lemma 3.10. Replace \( \Psi_3 \) by the same quantities with the extra conditioning on \( C_{0,t,l} \) omitted. For the analogue of \( \Psi_31 \), because of (3.61), we obtain exactly the same estimate, i.e. (3.66). For the analogue of \( \Psi_32 \), we follow the proof up to (3.72), and note that the application of Lemma 3.11 still works, since the conditioning on \( D_{0,t,l}(j) \) already specifies \( T_{l-2}^{0,t} \). This leads to (3.74) and completes the proof of (3.32) and hence of the lemma.

3.4. **From excursion counts to cover time.** This short section is devoted to the proof of (1.7).

**Proof of (1.7).** By the definitions of \( t_z \) and \( s_L(z) \), see (1.15) and (2.16), the estimate (1.16) that we have already proved is equivalent to the existence of a constant \( C \) so that

\[
(3.75) \quad \limsup_{L \to \infty} \mathbb{P}\left(T_L^{y,s_L(z)} = 0 \text{ for some } y \in F_L \right) \leq C(1+z) e^{-z}.
\]
With
\[ \tilde{C}_L = \sup_{x \in F_L} H_{\partial B_d(x, h_L)}, \]
it follows from Theorem 3.1 that
\[ \limsup_{L \to \infty} P \left( \tilde{C}_L \geq 4 s_L(z) \right) \leq C e^{-z + d \sqrt{z}}. \]

On the other hand, for some \( d_0 < \infty \)
\[ \tilde{C}_{[\log \epsilon - 1] - d_0} \leq C_* \leq \tilde{C}_{[\log \epsilon - 1] + d_0}, \]
so that (3.76) implies that
\[ \limsup_{\epsilon \to 0} P \left( C_* \geq 4 s_{[\log \epsilon - 1]}(z) \right) \leq C e^{-z + d \sqrt{z}}, \]
for a possibly larger \( C \) and \( d \). This is equivalent to (1.7).

4. LOWER BOUND

In this section we complete the proof of tightness of the cover time by proving the following.

**Proposition 4.1.** For any \( \delta > 0 \) there exists \( -\infty < z < 0 \) such that
\[ \liminf_{\epsilon \to 0} P \left( \sqrt{C_* - 2 \sqrt{2}\left( \log \epsilon^{-1} - \frac{1}{4} \log \log \epsilon^{-1} \right)} \geq z \right) \geq 1 - \delta. \]

In addition, for any \( B_d(a, r) \subseteq S^2 \), the same result holds if \( C_* \) is replaced by \( C_* \cap B_d(a, r) \), the \( \epsilon \)-cover time of \( B_d(a, r) \subseteq S^2 \) by Brownian motion on \( S^2 \).

Indeed, (4.1) together with (1.7) yield (1.3). Along the way, we will also obtain the estimates on the right tail of the cover time contained in (1.8).

As discussed in the introduction, the main technical step is the control on the right tail of the \( h_L \)-cover local excursion time \( t^*_L \), see (1.12), in the form of (1.17). We will in fact prove a more quantitative version of the latter, which is an analogue of [5, Lemma 5.3]. Recall the notation \( r_0 \) and \( r_l \), see (2.9).

**Proposition 4.2.** There exists a constant \( c \) so that for all \( 0 < r_0 < 1 \) and for all \( z \geq 1 \),
\[ \liminf_{L \to \infty} P \left( \inf_{y \in F_L} T_L^{y, t_L} = 0 \right) \geq \frac{(1 + z)e^{-2z}}{(1 + z)e^{-2z} + cr_0^2}. \]

In addition, for any \( B_d(a, r) \subseteq S^2 \), the same result holds if \( F_L \) is replaced by \( F_L \cap B_d(a, r) \).

The proof of Proposition 4.2 uses a modified second moment method and occupies most of this section. Before giving the proof, we show quickly how all the announced statements follow from Proposition 4.2 and Theorem 3.1.
Proof of Proposition 4.1 (assuming Proposition 4.2). Note that the inequality (4.1) is equivalent to the statement that for any \( \delta > 0 \) there exists \(-\infty < z < 0\) such that

\[
\liminf_{\epsilon \to 0} P \left( \frac{C^*_\epsilon}{4 \log \epsilon^{-1}} \geq z \right) \geq 1 - \delta.
\]

For \( z = 0 \) one has that \( t_{L,0} = s_L(0) \), see (1.15) and (2.16). Set \( r_0 \) small enough so that (4.2) with \( z = 1 \) guarantees

\[
\liminf_{L \to \infty} P \left( \inf_{y \in F_L} T^{y,s_L(0)}_L = 0 \right) \geq 1 - \delta/2,
\]

reducing \( r_0 \) further if necessary to ensure that (2.9) holds. Next define

\[
L = L(\epsilon) = \log \frac{r_0}{2\epsilon}.
\]

With this choice of \( L \) we have

\[
r_L = 2\epsilon \text{ and } 1.8 \epsilon \leq h_L \leq 2\epsilon,
\]
in addition to

\[
\sup_{y \in S^2} \inf_{x \in F_L} d(x,y) \leq \frac{h_L}{1000} \leq \frac{r_L}{1000} \leq \frac{\epsilon}{500}.
\]

This implies that for all \( y \in S^2 \) the ball \( B_d(y,\epsilon) \) is contained in \( B_d(x,h_L) \) for some \( x \in F_L \). Therefore on the event that \( \inf_{y \in F_L} T^{y,s_L(0)}_L = 0 \) the cover time \( C^*_\epsilon \) has not yet occurred at the time when the first \( x \in F_L \) registers \( s_L(0) \) excursions from \( \partial B_d(x,h_0) \) to \( \partial B_d(x,h_1) \), i.e. by time \( \inf_{x \in F_L} \tau_x(s_L(0)) \).

Thus from (4.4) it follows that

\[
\liminf_{L \to \infty} P \left( \inf_{x \in F_L} \tau_x(s_L(0)) \right) \geq 1 - \delta/2,
\]

where we recall that \( L \), and therefore also \( \tau_x(s_L(0)) \), depends on \( \epsilon \) through (4.5).

Recall, see (2.16), that \( s(\cdot) \) is monotone increasing, so that \( \tau_x(s_L(0)) \geq \tau_x(s_L(-z_0)) \) so that Theorem 3.1 implies that

\[
\liminf_{\epsilon \to 0} P \left( \tau_x(s_L(0)) \geq 4s_L(-z_0 - d\sqrt{\epsilon}) \right) \geq 1 - \delta/2.
\]

Combining (4.6) and (4.7) proves (4.3).

For any \( B_d(a,r) \subseteq S^2 \), the same proof shows that if (4.2) holds with \( F_L \) replaced by \( F_L \cap B_d(a,r) \) then (4.3) holds with \( C^*_{\epsilon,S^2} \) replaced by \( C^*_{\epsilon,S^2,B_d(a,r)} \).

Proof of (1.8) (assuming Proposition 4.2). We use Proposition 4.2 with \( r_0 \) sufficiently small to obtain that

\[
\liminf_{L \to \infty} P \left( \inf_{y \in F_L} T^{y,t_z}_L = 0 \right) \geq c(1 + z)e^{-2z} \text{ for all } z > 0,
\]
where we bounded above the denominator of the right-hand side of (4.2) by a constant. Set \( L = L(\epsilon) = \log (r_0/(2\epsilon)) \), as in (4.5), and argue as in the proof of Proposition 4.1 to obtain that

\[
\lim_{\epsilon \to 0} \inf \mathbb{P} \left( C^*_\epsilon \geq \inf_{x \in F_L} \tau_x(t_\epsilon) \right) \geq c e^{-2z} \text{ for all } z > 0.
\]

Now \( t_\epsilon = s_L(2z + o(1)) \) for \( z > 0 \) and large enough \( L \), and by Theorem 3.1

\[
\lim_{\epsilon \to 0} \inf \mathbb{P} \left( \inf_{x \in F_L} \tau_x(s_L(2z)) \geq 4s_L(2z - d\sqrt{z}) \right) \geq 1 - c'e^{-4z}.
\]

Also

\[
4s_L(2z - d\sqrt{z}) = 4 \log \frac{1}{2} \left( 2 \log \epsilon^{-1} - \log \log \epsilon^{-1} + 2z - d\sqrt{z} + o(1) \right).
\]

Together with (4.8) and (4.9), we obtain

\[
\lim_{\epsilon \to 0} \inf \mathbb{P} \left( C^*_\epsilon \geq 4 \log \frac{1}{2} \left( 2 \log \epsilon^{-1} - \log \log \epsilon^{-1} + 2z - d\sqrt{z} \right) \right) \geq c e^{-2z} - c'e^{-4z} \geq c e^{-2z} \text{ for all } z \geq z_0,
\]

which in turns implies that

\[
\lim_{\epsilon \to 0} \mathbb{P} \left( \frac{C^*_\epsilon}{4 \log \epsilon^{-1}} - (2 \log \epsilon^{-1} - \log \log \epsilon^{-1}) \geq z \right) \geq c'e^{-z} - c\epsilon \sqrt{z} \text{ for } z \geq z_0'.
\]

Due to the monotonicity in \( z \) of the left side of (4.11), this implies (1.8) for \( z \geq 0 \) (possibly reducing \( c' \) as needed).

We turn to the main object of this section, which is the proof of Proposition 4.2. We need to set up a second moment method, which means to attach to each \( y \in F_L \) an event \( I_{y,z} \) so that, with \( J_z = \sum_{y \in F_L} 1_{I_{y,z}} \), the following properties hold:

\[
I_{y,z} \subset \left\{ T_{L}^{y,z} = 0 \right\}.
\]

\[
\frac{\mathbb{E} J_z^2}{\mathbb{E} J_z^2} \geq \frac{(1 + z)e^{-2z}}{(1 + z)e^{-2z} + cr_0^2}.
\]

Indeed, (4.12) and (4.13) together imply Proposition 4.2 by an application of Cauchy-Schwarz.

The major difficulty in constructing such events as \( I_{y,z} \) is that the computation of \( \mathbb{E} J_z^2 \) involves probabilities of the form \( \mathbb{P}(I_{y,z} \cap I_{y',z}) \) for \( y \neq y' \), and one would like to have the events in the last probability decouple as much as possible. A standard method is to have \( I_{y,z} \) include a barrier event, similar to but different from the one used in the upper bound. Indeed, this was the approach of [4], and also the approach taken by us in [5]. However, due to the difficulties in decoupling excursions around non-concentric centers, this turns out not to be enough to obtain the degree of precision in (1.13). Our approach is to add to the barrier event information on the start and end points of excursions, in relatively large scales (i.e., small \( l \)).
We now begin with the construction, which culminates with (4.23) below. Recall the notation $\rho_L$ and $l_L$, see (1.14) and (1.18), and set
\begin{equation}
\gamma(l) = \gamma(l, L) = \rho_L(L - l) + l_L^{1/4}.
\end{equation}
We introduce the events $\mathcal{I}_{y,z}$, beginning with a barrier event. Set
\begin{equation}
\hat{\mathcal{I}}_{y,z} = \left\{ \gamma(l) \leq \sqrt{2} T_{y,t}^{y,t,z} \text{ for } l = 1, \ldots, L - 1 \text{ and } T_{L}^{y,t,z} = 0 \right\}.
\end{equation}

As discussed above, we need to augment $\hat{\mathcal{I}}_{y,z}$ by information on the angular increments of the excursions. Instead of keeping track of individual excursions, we track the empirical measure of the increments, by comparing it in Wasserstein distance to a reference measure. Recall that the Wasserstein $L^1$-distance between probability measures on $\mathbb{R}$ is given by
\begin{equation}
d_1^{Wa}(\mu, \nu) = \inf_{\xi \in \mathcal{P}_{L}^2(\mu, \nu)} \left\{ \int |x - y| \, d\xi(x, y) \right\},
\end{equation}
where $\mathcal{P}_{L}^2(\mu, \nu)$ denotes the set of probability measures on $\mathbb{R} \times \mathbb{R}$ with marginals $\mu, \nu$. If $\mu$ is a probability measure on $\mathbb{R}$ with finite support and if $\theta_i, 1 \leq i \leq n$ denote a sequence of i.i.d $\mu$-distributed random variables then it follows from [24, Theorem 2] that for some $c_0 = c_0(\mu)$
\begin{equation}
\text{Prob} \left\{ d_1^{Wa} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_i}, \mu \right) > \frac{c_0 x}{\sqrt{n}} \right\} \leq 2 e^{-x^2}.
\end{equation}

Let $W_t$ be Brownian motion in the plane. For each $k$ let $\nu_k$ be the probability measure on $[0, 2\pi]$ defined by
\begin{equation}
\nu_k(dx) = P^{(0,rk)} \left( \arg W_{H_{\partial B_d(0,rk)}} \in dx \right),
\end{equation}
where $\arg x$ for $x \in \mathbb{R}^2$ is the argument of $x$ measured from the positive $x$-axis and $P^w$ is the law of $W_t$ started from $w$.

Returning to $X_t$, our Brownian motion on the sphere, and using isothermal coordinates, see Section 2 let $0 \leq \theta_{k,i} \leq 2\pi$, $i = 1, 2, \ldots$ be the angular increments centered at $y$, mod $2\pi$, from $X_{H_{\partial B_d(y,h_k)}}$ to $X_{H_{\partial B_d(y,h_{k-1})}}$, the endpoints of the $i$'th excursion between $\partial B_d(y, h_k)$ and $\partial B_d(y, h_{k-1})$. By the Markov property the $\theta_{k,i}, i = 1, 2, \ldots$ are independent, and using Section 2 we see that each $\theta_{k,i}$ has distribution $\nu_k$. We set, for $n$ a positive integer,
\begin{equation}
W_{y,k}(n) = \left\{ d_1^{Wa} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_{k,i}}, \nu_k \right) \leq \frac{c_0 \log(k)}{2\sqrt{n}} \right\}.
\end{equation}

We are ready to define the good events $\mathcal{I}_{y,z}$. For $a \in \mathbb{Z}_+$ let
\begin{equation}
N_{k,a} = [(\rho_L(L - k) + a + 1)^2 / 2].
\end{equation}
We set
\begin{equation}
N_k = N_{k,a} \text{ if } \sqrt{2} T_{k}^{y,t,z} \in I_{\rho_L(L - k) + a}.
\end{equation}
With \( L, = 16 (\log L)^4 \) and \( d \) a constant to be determined below, let
\[
I_{y,z} = I_{y,z} \cap L^{2} \cap \mathcal{W}_{y,k} (N_{k}),
\]
and define the count
\[
J_{z} = \sum_{y \in F_{L}} 1_{I_{y,z}}.
\]
To obtain (4.13), we need a control on the first and second moments of \( J_{z} \), which is provided by the next two lemmas. Most of this section is devoted to their proof. We emphasize that in the statements of the lemmas, the implied constants are uniform in \( r_{0} \) smaller than a fixed small threshold.

**Lemma 4.3** (First moment estimate). *There is a large enough \( d^{\star} \), such that for all \( L \) sufficiently large, all \( 1 \leq z \leq (\log L)^{1/4} \), and all \( y \in F_{L} \),
\[
P(I_{y,z}) \approx (1 + z) e^{-2L L^{-2/4} z}.
\]

Let
\[
G_{0} = \{(y, y^{'}) : y, y^{'} \in F_{L} \text{ s.t. } d(y, y^{'}) > 2h_{0}\},
\]
\[
G_{k} = \{(y, y^{'}) : y, y^{'} \in F_{L} \text{ s.t. } 2h_{k} < d(y, y^{'}) \leq 2h_{k-1}\} \text{ for } 1 \leq k < L,
\]
\[
G_{L} = \{(y, y^{'}) : y, y^{'} \in F_{L} \text{ s.t. } 0 < d(y, y^{'}) \leq 2h_{L-1}\}.
\]

**Lemma 4.4** (Second moment estimate). *There are large enough \( d^{\star}, c^{'}, \) such that for all \( L \) sufficiently large, all \( 1 \leq z \leq (\log L)^{1/4} \) and all \( (y, y^{'}) \in G_{k}, 1 \leq k \leq L, \)
\[
P(I_{y,z} \cap I_{y^{'},z}) \leq c^{'} (1 + z) e^{-4L L^{-2/4} z} e^{-c k^{1/4}}.
\]

Before providing the proofs of Lemmas 4.3 and 4.4, we show how Proposition 4.2 follows from them.

**Proof of Proposition 4.2** (assuming Lemmas 4.3 and 4.4). It follows immediately from (3.1) and Lemma 4.3 that
\[
\mathbb{E}(J_{z}) \asymp r_{0}^{-2} (1 + z) e^{-2z}.
\]
Since \( J_{z} \) is a sum of indicators and hence at least 1 if not 0, an application of the Cauchy-Schwarz inequality to \( J_{z} \cdot 1_{J_{z} \geq 1} \) gives that
\[
(\mathbb{E}(J_{z}))^{2} \leq \mathbb{P} \left( \inf_{y \in F_{L}} T_{L}^{y, I_{y,z}} = 0 \right) \mathbb{E}(J_{z}^{2}).
\]
Hence (4.2) will follow from (4.27) and (4.28) once we show that
\[
\mathbb{E}(J_{z}^{2}) \leq (\mathbb{E}(J_{z}))^{2} + c \mathbb{E}(J_{z}).
\]
Since \( \mathbb{E}(J_{z}^{2}) = \mathbb{E}(J_{z}) + \sum_{y \neq y^{'} \in F_{L}} \mathbb{P}(I_{y,z} \cap I_{y^{'},z}) \), it suffices to show that
\[
\sum_{y \neq y^{'} \in F_{L}} \mathbb{P}(I_{y,z} \cap I_{y^{'},z}) \leq (\mathbb{E}(J_{z}))^{2} + c \mathbb{E}(J_{z}).
\]
Recall the sets \( G_k \), see (4.25). We have that \( \bigcup_{k=0}^{L} G_k = \{(y,y') \in F_L \times F_L : y \neq y'\} \) and therefore

\[
\sum_{y \neq y' \in F_L} \mathbb{P}(I_{y,z} \cap I_{y',z}) = \sum_{(y,y') \in G_0} \mathbb{P}(I_{y,z} \cap I_{y',z}) + \sum_{k=1}^{L} \sum_{(y,y') \in G_k} \mathbb{P}(I_{y,z} \cap I_{y',z}).
\]

By the Markov property, see [4, Lemma 5.3] for details, we have that

\[
\sum_{(y,y') \in G_k} \mathbb{P}(I_{y,z} \cap I_{y',z}) = \sum_{(y,y') \in G_0} \mathbb{P}(I_{y,z}) \mathbb{P}(I_{y',z}) \leq (\mathbb{E}(J_z))^2.
\]

To handle \((y,y') \in G_k, k = 1, \ldots, L\), we write

\[
\sum_{(y,y') \in G_k} \mathbb{P}(I_{y,z} \cap I_{y',z}) \leq cr_0^{-2} e^{4L-2k} \sup_{(y,y') \in G_k} \mathbb{P}(I_{y,z} \cap I_{y',z}),
\]

where we have used that for \(1 \leq k \leq L\),

\[
|G_k| \leq |F_L| \sup_{v \in F_L} |F_L \cap B_d(v, 2h_{k-1})| \leq c |F_L|^2 r_{k-1}^2 = c |F_L|^2 r_0^2 e^{-2k},
\]

and therefore \(|G_k| \leq cr_0^{-2} e^{4L-2k}\). Using this and (4.26), the right hand side of (4.33) is bounded above by \(cr_0^{-2} (1 + z) e^{-2z} e^{-ck L^{1/4}}\). Summing over \(k = 1, \ldots, L\), we obtain that

\[
\sum_{k=1}^{L} \sum_{(y,y') \in G_k} \mathbb{P}(I_{y,z} \cap I_{y',z}) \leq cr_0^{-2} (1 + z) e^{-2z} \sum_{k=1}^{L} e^{-ck L^{1/4}} \leq c \mathbb{E}(J_z),
\]

where the second inequality used (4.27). Combined with (4.31) and (4.32), we obtain (4.30) and thus complete the proof of (4.2).

For any \(B_d(a, r) \subseteq S^2\), the same proof shows that (4.2) holds with \(F_L\) replaced by \(F_L \cap B_d(a, r)\). \(\square\)

So, it remains to prove Lemmas 4.3 and 4.4. The proof of Lemma 4.3 is relatively straightforward, given the barrier estimates of [5], and is provided in subsection 4.1. The proof of Lemma 4.4 is much more intricate, and is divided to cases according to the distance between \(y\) and \(y'\), see subsections 4.2-4.4. The last of these sections uses a decoupling argument which is described in detail in subsection 4.5.

Before turning to these proofs, we introduce some notation and record a simple computation that will be useful in calculations. Recall (4.20)-(4.21), and for \(a \in \mathbb{Z}_+\) introduce the level-\(k\) event

\[
\mathcal{H}_{k,a} = \left\{ \sqrt{2T_k^{y,z}} \in I_{\rho_L(L-k) + a} \right\}.
\]

Note that on \(\mathcal{H}_{k,a}\) we have \(N_k = N_{k,a}\) where

\[
N_{k,a} = [(\rho_L(L-k) + a + 1)^2 / 2].
\]

The next lemma is the computation alluded to above.
Lemma 4.5. For all $L$ sufficiently large and $1 \leq k < L$,
\begin{equation}
\gamma^2(k)/2(L-k) \leq cL^{(L-k)/L}e^{-2(L-k)-2k^{1/4}}.
\end{equation}
In particular, if $k \geq \log^4 L$ then
\begin{equation}
\gamma^2(k)/2(L-k) \leq ce^{-2(L-k)-k^{1/4}}.
\end{equation}
Furthermore, for some $d^*$ sufficiently large, the same bounds hold for $e^{-\gamma^2(k)(1-\frac{2}{L+k})/2(L-k)}$ if $L - k \geq d^*$.

Proof. Recalling (1.14) and (4.14) we have
\begin{equation}
\frac{\gamma^2(k)}{2(L-k)} \geq 2(L-k) - \frac{(L-k)^2}{L} \log L + 2k^{1/4} + o_L(1) + k^{1/2}/L.
\end{equation}
Consequently
\begin{equation}
\frac{\gamma^2(k)}{2(L-k)} \geq 2(L-k) - \frac{(L-k)^2}{L} \log L + 2k^{1/4} + o_L(1) + k^{1/2}/L.
\end{equation}
This yields (4.36).

To see (4.37), note that if $\log^4 L \leq k \leq L - \log^4 L$ then $k^{1/4} \geq \log L$, while if $k \geq L - \log^4 L$ then $L(L-k)/L \leq c$.

For the last statement of the lemma, note that by (4.38),
\begin{equation}
\frac{\gamma^2(k)}{(L-k)^2} = \frac{4k^{1/4}}{L-k} + \frac{k^{1/2}}{(L-k)^2} + O(1).
\end{equation}
Since $L - k \geq d^*$, by taking $d^*$ sufficiently large, up to an error which is $O(1)$ this is dominated by the ‘extra term’ $k^{1/2}/2(L-k)$ in (4.39).

4.1. The first moment estimate. We prove in this subsection Lemma 4.3.

In the proof we will use barrier estimates from Section 8 and the decoupling lemma (Lemma 4.13) from section 4.5. Recall that our goal is to evaluate $P(I_{y,z})$ up to multiplicative constants.

Proof of Lemma 4.3. For the upper bound, we note that since $I_{y,z} \subseteq \widehat{I}_{y,z}$, the upper bound in (4.24) is immediate from the barrier estimate contained in Lemma 8.2 of Appendix I.

For the lower bound we have for all $0 \leq z \leq (\log L)^{1/4}$
\begin{equation}
P(I_{y,z}) \geq \mathbb{P}(\hat{I}_{y,z}) - \sum_{k=d^*}^{L_\infty} \mathbb{P}(\hat{I}_{y,z} \cap W_{y,k}^c (N_k))
\end{equation}
\begin{equation}
\geq c (1+z)e^{-2L}e^{-2z} - \sum_{k=d^*}^{L_\infty} \mathbb{P}(\hat{I}_{y,z} \cap W_{y,k}^c (N_k)),
\end{equation}
see (4.19) and (4.21) for notation, and where for \( P(\hat{T}_{y,z}) \) we have used the barrier estimate contained in Lemma 8.2 of Appendix I. We note that (4.41) \[
\Pr(\hat{T}_{y,z} \leq \wedge \hat{I}_{y,z}^{c} \wedge W_{C}^{y,k}(N_{k,a})) \leq \sum_{a \geq \lceil k^{1/4} \rceil} \Pr(\hat{T}_{y,z}^{k,a}) \]
where \( \hat{T}_{y,z}^{k,a} := \hat{T}_{y,z} \wedge H_{k,a} \wedge W_{C}^{y,k}(N_{k,a}) \), see (4.34) and (4.20) for notation and we have used (4.15) and (4.14) to restrict the sum to \( a \geq \lceil k^{1/4} \rceil \). We show below that for all \( d^{*} \leq k \leq L_{-} \), and all \( 0 \leq z \leq (\log L)^{1/4} \),

(4.42) \[
\sum_{a \geq \lceil k^{1/4} \rceil} \Pr(\hat{T}_{y,z}^{k,a}) \leq c(1 + z)e^{-2L}e^{-2z}e^{-c'(\log k)^{2}}.
\]

Furthermore, it is easily seen using Lemma 3.6 that the sum over \( a > \frac{3}{4}L \) is much smaller than the right hand side of (4.42) so it suffices to show that

(4.43) \[
\sum_{a = \lceil k^{1/4} \rceil} \Pr(\hat{T}_{y,z}^{k,a}) \leq c(1 + z)e^{-2L}e^{-2z}e^{-c'(\log k)^{2}} =: \mathcal{E}(z, k).
\]

Combining (4.43) with (4.40) and (4.41) yields the lower bound in (4.24) (when \( d^{*} \) is taken sufficiently large) and completes the proof of Lemma 4.3.

We turn to the proof of (4.43). We introduce the notation (4.44) \[
\mathcal{B}_{y,k,L} = \left\{ \gamma(l) \leq \sqrt{2T_{l}^{y,t_{z}}} \text{ for } l = k, \ldots, L - 1; T_{L_{-}}^{y,t_{z}} = 0 \right\},
\]
for the barrier condition from \( l \geq k \) to \( L \). Then, with

(4.45) \[
\mathcal{K}_{k,p,a} = H_{k-3,p} \cap H_{k,a} \cap W_{y,k}(N_{k,a}) \cap B_{y,k+1,L},
\]
see (4.34) and (4.35), we have

(4.46) \[
\Pr(\hat{T}_{y,z}^{k,a}) \leq \sum_{p = \lceil(k-3)^{1/4} \rceil}^{L^{3/4}} \Pr(\mathcal{K}_{k,p,a})
\]
plus a term which is much smaller than the right hand side of (4.43). Hence to prove (4.43) it suffices to show that for all \( d^{*} \leq k \leq L_{-} \), and all \( 0 \leq z \leq (\log L)^{1/4} \),

(4.47) \[
\sum_{a = \lceil k^{1/4} \rceil}^{L^{3/4}} \sum_{p = \lceil(k-3)^{1/4} \rceil}^{L^{3/4}} \Pr(\mathcal{K}_{k,p,a}) \leq c(1 + z)e^{-2L}e^{-2z}e^{-c'(\log k)^{2}}.
\]

Let

(4.48) \[
W_{y,k}^{C}(n) = \left\{ \delta_{\omega_{n}}^{1} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\theta_{k,i}}, t_{k} \right) \in \mathcal{C}_{2\sqrt{n}L_{-}} \right\},
\]
so that

(4.49) \[
W_{y,k}^{c}(N_{k,a}) \subseteq \bigcup_{m=\log k}^{\infty} W_{y,k}^{C}(N_{k,a}),
\]
and consequently, setting
\[(4.50) \quad \mathcal{L}_{k,m,p,a} = \mathcal{K}_{k,p,a} \cap \mathcal{W}_{y,k}^{<m}(N_{k,a}), \]
we have
\[(4.51) \quad \mathbb{P}(\mathcal{K}_{k,p,a}) \leq \sum_{m=\log k}^{\infty} \mathbb{P}(\mathcal{L}_{k,m,p,a}). \]

Thus to prove (4.47) it suffices to show that for all \(m \geq \log k\), all \(d^* \leq k \leq L_-, \text{and all } 0 \leq z \leq (\log L)^{1/4}, \)
\[(4.52) \quad \sum_{a=[k^{1/4}]}^{L^{3/4}} \sum_{p=\lceil(k-3)^{1/4} \rceil}^{L^{3/4}} \mathbb{P}(\mathcal{L}_{k,m,p,a}) \leq c(1 + z)e^{-2L}e^{-2z}e^{-c'm^2}. \]

Write
\[(4.53) \quad \mathcal{L}_{k,m,p,a} = \mathcal{H}_{k-3,p} \bigcap \mathcal{H}_{k,a} \bigcap \mathcal{W}_{y,k}^{<m}(N_{k,a}). \]

Since the \(\theta_{k,i}\) are i.i.d. \(\nu_k\)-distributed random variables, as explained in the paragraph before (4.19), it follows from (4.17) that
\[(4.54) \quad \mathbb{P}\left(\mathcal{W}_{y,k}^{<m}(N_{k,a}) \Bigg| T_{y,t}^{\rho_{L,L-k+3}}\right) \leq 2e^{-m^2/4}. \]

**Lemma 4.6.** There exist constants \(c, c' > 0\) so that
\[(4.55) \quad \mathbb{P}(\mathcal{L}'_{k,m,p,a}) \leq c e^{-2k-2(z-p)-(z-p)^2/4k} e^{-m^2/8} e^{-c'(a-p)^2}. \]

**Proof.** From Lemma 3.6 as in (3.25) and (3.26), we obtain
\[(4.56) \quad \mathbb{P}(\mathcal{H}_{k-3,p}) = \mathbb{P}\left(\sqrt{2T_{k-3}^{y,t}} \in I_{[\rho_L(L-k+3)]+p}\right) \leq c e^{-2k-2(z-p)-(z-p)^2/4k}, \]
and a slight variation of the same argument shows that for some \(c_1, c > 0\)
\[(4.57) \quad \mathbb{P}(\mathcal{H}_{k,a} | \mathcal{H}_{k-3,p}) = \mathbb{P}\left(\sqrt{2T_{3}^{y,t}} \in I_{[\rho_L(L-k)]+a}\right) \leq c e^{-c_1(a-p)^2}, \]
and hence
\[(4.58) \quad \mathbb{P}(\mathcal{H}_{k-3,p} \cap \mathcal{H}_{k,a}) \leq c e^{-2k-2(z-p)-(z-p)^2/4k} e^{-c_1(a-p)^2}. \]

Using (4.54) and (4.56) we obtain that
\[(4.59) \quad \mathbb{P}\left(\mathcal{H}_{k-3,p} \bigcap \mathcal{W}_{y,k}^{<m}(N_{k,a})\right) \leq c e^{-2k-2(z-p)-(z-p)^2/4k} e^{-m^2/4}. \]

Using the Cauchy-Schwarz inequality and (4.53), we have that \(\mathbb{P}(\mathcal{L}'_{k,m,p,a}) \leq \left(\mathbb{P}(\mathcal{H}_{k-3,p} \bigcap \mathcal{W}_{y,k}^{<m}(N_{k,a}))^{1/2} \mathbb{P}(\mathcal{H}_{k-3,p} \bigcap \mathcal{H}_{k,a})^{1/2}. \right. \]
The lemma follows by substituting (4.57) and (4.58). \(\square\)
Let
\[(4.59)\]
k^+ = k + \lfloor 10^{10} \log L \rfloor, \quad k^{++} = k + 2 \lfloor 10^{10} \log L \rfloor,
\[
\mathbb{B}_{y,k+3,k^+} = \left\{ \gamma(l) \leq \sqrt{2T_{y,t}^{y,t_z}} \text{ for } l = k + 3, \ldots, k^+, \right. \\
\left. \sqrt{2T_{k^+}^{y,t_z}} \in I_{\rho_L(L-k^+)+j'} \right\},
\]
and
\[
\hat{\mathcal{B}}_{y,k^{++}+1,L} = \left\{ \rho_L(L-l) \leq \sqrt{2T_{y,t}^{y,t_z}} \text{ for } l = k^{++} + 1, \ldots, L-1, \right. \\
\left. \sqrt{2T_{L}^{y,t_z}} = 0 \right\}.
\]
(Compare (4.61) to (4.44). The only difference is that a straight barrier is used.) We have that
\[
\mathbb{P}(L_{k,m,p,a}) = \mathbb{P} \left( L_{k,m,p,a} \cap \mathcal{B}_{y,k+1,L} \right) \leq \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k^{++})^{1/4}}^{L^{3/4}} \mathbb{P} \left( L_{k,m,p,a} \cap \mathbb{B}_{y,k+3,k^+} \cap \hat{\mathcal{B}}_{y,k^{++}+1,L} ; \sqrt{2T_{k^+}^{y,t_z}} \in I_{\rho_L(L-k^+)+j'} \right) + o(\mathcal{E}(z,k)),
\]
see (4.43) for the definition of \(\mathcal{E}(z,k)\), and the error is due to the restriction \(j', j'' \leq L^{3/4}\). Let
\[
\mathcal{G}_{k^+}^y = \sigma\text{-algebra generated by the excursions from } \partial B_d(y, h_{k+1}) \text{ to } \partial B_d(y, h_{k^+}).
\]
(Compare (4.63) to (3.38).) Note that
\[
(4.64) \quad A_{j'} := L_{k,m,p,a} \cap \mathbb{B}_{y,k+3,k^+} \in \mathcal{G}_{k^+}^y.
\]
Using (4.64) we have
\[
\mathbb{P} \left( A_{j'} ; \sqrt{2T_{k^+}^{y,t_z}} \in I_{\rho_L(L-k^{++})+j''} ; \hat{\mathcal{B}}_{y,k^{++}+1,L} \right) \leq \mathbb{P} \left( \sup_{s \in I_{\rho(L-k^{++})+j'} , \tau \in I_{\rho(L-k^{++})+j''} } \mathbb{P} \left( A_{j'} ; \sqrt{2T_{k^+}^{y,t_z}} \in I_{\rho(L-k^{++})+j''} ; \hat{\mathcal{B}}_{y,k^{++}+1,L} \right) \right). \]
(4.65)
By Lemma 9.1 with the \( k, k' \) there replaced by \( k^+, k^{++} \) and using that 
\((1 + 10(k^{++} - k^+)h_{k^{++}}/h_{k^+})^{4L^2} \) is bounded above uniformly, we have that for some universal \( c < \infty \)
\begin{equation}
\mathbb{P}\left( \mathcal{B}_{y,k^{++}+1,L} \mid T_{y,k^{++}+1} = v^2/2, G_{k^+}^y \right) \leq c \mathbb{P}\left( \mathcal{B}_{y,k^{++}+1,L} \mid T_{y,k^{++}+1} = v^2/2 \right).
\end{equation}

By the barrier estimate of Lemma 8.4 in Appendix I, we see that for \( j'' \) in the range of summation in (4.62), uniformly in \( s \in I_{\rho(L-k^+) + j'} \) and \( v \in I_{\rho(L-k^{++}) + j''} \),
\begin{equation}
\mathbb{P}\left( \mathcal{B}_{y,k^{++}+1,L} \mid T_{y,k^{++}+1} = v^2/2 \right) \leq c(1 + j'')e^{-2(L-k^+) - 2j''}.
\end{equation}
Thus, uniformly in \([k^{1/4}] \leq a \leq L^{3/4} \)
\begin{equation}
\mathbb{P}\left( \mathcal{L}_{k,m,p,a} \right) = \mathbb{P}\left( \mathcal{L}_{k,m,p,a} \cap \mathcal{B}_{y,k+3,L} \right) \leq ce^{-2(L-k^+)}
\end{equation}
\begin{equation}
\times \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k^{++})^{1/4}}^{L^{3/4}} j'' e^{-2j''} \mathbb{P}\left( \mathcal{L}_{k,m,p,a} \cap \mathbb{B}_{y,k+3,k^{++},k^{++}}^{j',j''} \right),
\end{equation}
where
\begin{equation}
\mathbb{B}_{y,k+3,k^{++},k^{++}}^{j',j''} = \left\{ \gamma(l) \leq \sqrt{2T_{y,l}^{y,l}} \text{ for } l = k + 3, \ldots, k^+, k^{++}, \right. \\
\left. \sqrt{2T_{k^{++}}^{y,l}} \in I_{\rho_L(L-k^+)+j'}, \sqrt{2T_{k^{++}}^{y,l}} \in I_{\rho_L(L-k^{++})+j''} \right\}.
\end{equation}

We first consider the case that \( m \geq (\log L)^4 \). Using (4.68), dropping the event \( \mathbb{B}_{y,k+3,k^{++},k^{++}}^{j',j''} \) and then using (4.55), we obtain
\begin{equation}
\mathbb{P}\left( \mathcal{L}_{k,m,p,a} \right) \leq ce^{-2(L-k^+)}L^{3/2} \mathbb{P}\left( \mathcal{L}_{k,m,p,a} \right) \leq ce^{-2L} L^{3/4} e^{2(k^{++} - k)} e^{-2(z-p) - (z-p)^2/4k} e^{-m^2/8}.
\end{equation}
Using that \( k \leq L_- = 16(\log L)^4 \) and our assumption that \( 0 \leq z \leq (\log L)^{1/4} \), it follows that (4.52) holds for all \( m \geq (\log L)^4 \).

We consider separately the case where \( \log k \leq m < \log \log L \) and the case where \( \log \log L \leq m < (\log L)^4 \).

To handle the last probability in (4.68) for \( \log k \leq m < \log \log L \), we use the following estimate, whose proof, given in sub-section 4.5, uses the decoupling lemma (Lemma 4.13), barrier estimates and the control on Wasserstein distance contained in (4.17).
Lemma 4.7. Let \( \log k \leq m < \log \log L \). For \( a, p, j', j'', k \) in the ranges specified above,

\[
\mathbb{P} \left( \mathcal{L'}_{k,m,p,a} \cap \mathbb{B}^{j',j''}_{y,k+3,k+,k++} \right) \leq G_{k,a,j',j''} \mathbb{P} \left( \mathcal{L'}_{k,m,p,a} \right) + e^{-\sqrt{T}},
\]

where

\[
G_{k,a,j',j''} = c(a + m^{5/2})e^{-2a}e^{-2(k++ - k)} \sum_{\{j'' | j'' \leq 2M_0 m\}} \bar{j} e^{-(j''-a)^2/(4(k++ - k))} \frac{e^{-(j''-j')^2/2(k+ - k)}}{(k+ - k)}.
\]

Combining Lemma 4.7 with (4.68) and (4.55) we see that

\[
\mathbb{P} \left( \mathcal{L}_{k,m,p,a} \right) \leq c e^{-2L - 2z} e^{m^2/8} e^{4M_0 m} \sum_{j'' \leq 2M_0 m} \frac{j' e^{-(j'-a)^2/(4(k++ - k))}}{(k+ - k)} \frac{j'' e^{-(j''-j')^2/2(k+ - k)}}{(k+ - k)}.
\]

which leads to

\[
\mathbb{P} \left( \mathcal{L}_{k,m,p,a} \right) \leq c e^{-2L - 2z} e^{m^2/16} e^{-(z-p)^2/4k} e^{-c'(p-a)^2} \sum_{j' \geq (k+)^{1/4}/2} \frac{j' e^{-(j'-a)^2/(4(k++ - k))}}{(k+ - k)} \sum_{j'' \geq (k+)^{1/4}/2} \frac{j'' e^{-(j''-j')^2/(2(k+ - k))}}{(k+ - k)}.
\]

We obtain (4.52) for \( \log k \leq m \leq \log \log L \) by an elementary, if tedious, calculation.

To handle the case when \( \log \log L \leq m \leq (\log L)^4 \), we use, instead of Lemma 4.7, the following a priori weaker estimate, whose proof, also given in sub-section 4.5, is a variation of the proof of Lemma 4.7.

Lemma 4.8. Let \( \log k \leq m \leq (\log L)^4 \). For \( a, p, j', j'', k \) in the ranges specified above,

\[
\mathbb{P} \left( \mathcal{L'}_{k,m,p,a} \cap \mathbb{B}^{j',j''}_{y,k+3,k+,k++} \right) \leq H_{k,a,j',j''} \mathbb{P} \left( \mathcal{L'}_{k,m,p,a} \right) + e^{-\sqrt{T}},
\]
where

\begin{equation}
H_{k,a,j,j'} = ce^{-2a} e^{-2(k^+ - k)} \sum_{\substack{j' \in [2M_0m] \setminus \{j'' \in [2M_0m] \}} \sum_{j'' \in [2M_0m]} j' e^{-(j' - a)^2/(4(k^+ - k))} (k^+ - k)^{1/2} e^{3j''} e^{-(j'' - j')^2/2(k^+ - k)} (k^+ - k)^{1/2}.
\end{equation}

Combining the last Lemma with (4.68) and (4.55) we obtain

\begin{equation}
P(L_{k,m,p,a}) \leq ce^{-2L - 2z} e^{-m^2/8} e^{4M_0m} \sum_{j' = (k^+)^{1/4}}^L \sum_{j'' = (k^+)^{1/4}}^L e^{-(z - p)^2/4k} e^{c'(p - a)^2} \times \sum_{j' \in [2M_0m]} \sum_{j'' \in [2M_0m]} j' e^{-(j' - a)^2/(4(k^+ - k))} (k^+ - k)^{1/2} \sum_{j'' \in [2M_0m]} j'' e^{-(j'' - j')^2/2(k^+ - k)} (k^+ - k)^{1/2} \leq ce^{-2L - 2z} e^{-m^2/16} e^{-(z - p)^2/4k} e^{c(p - a)^2} \times \sum_{j' \in [2M_0m]} j' e^{-(j' - a)^2/(4(k^+ - k))} (k^+ - k)^{1/2} \sum_{j'' \in [2M_0m]} j'' e^{-(j'' - j')^2/(2(k^+ - k))},
\end{equation}

where in the last inequality we used that for \( m \geq \log \log L \), we have \( e^{-m^2/16} \leq ce^{-m^2/20} (k^+ - k)^{-2} \). Note however that we can no longer assert that \( j' \geq (k^+)^{1/4} / 2 \). Another tedious summation now yields the inequality (4.52) for the case \( \log \log L \leq m \leq (\log L)^4 \). This completes the proof of Lemma 4.3.

\[ \square \]

4.2. Second moment estimate: branching in the bulk. We begin our proof of the second moment estimate contained in (4.26). The proof is divided to cases according to the value of \( k \). In this subsection we prove Lemma 4.4 for \( 2^4 \log^4 L \leq k < L - 2^4 \log^4 L \), that is, branching in the bulk. In this case, the curved boundary \( \gamma(\cdot) \) will play an important role and considerably simplify the proof compared to the case of very early branching treated below (where sophisticated decoupling, and the full barrier, needs to be used). Indeed, we will content ourselves with dropping the barrier almost entirely, and just bound the probability of the event

\begin{equation}
\{ T_L^{y,4z} = 0 \} \cap \{ T_L^{y',k,\gamma^2(k)/2} = 0 \},
\end{equation}
which contains the event $\mathcal{I}_{y,z} \cap \mathcal{I}_{y',z}$. In this, we follow the argument in [4]. Specifically, by the proof of [4, Corollary 6.7] it follows that

$$
\mathbb{P}\left( T_{y,t}^{y,z} L = 0, T_{y',k,\gamma^2(k)/2} L = 0 \right)
\leq \left( 1 - \frac{1}{L - k} \right) \frac{1}{2} \mathbb{P}\left( T_{y,t}^{y,z} = 0 \right)
\leq e^{-\gamma^2(k)(1 - \frac{2}{L - k})} \mathbb{P}\left( T_{y,t}^{y,z} = 0 \right).
$$

The contribution in (4.77) referring to $y$ is easily bounded by (3.6), giving

$$
\mathbb{P}\left( T_{y,t}^{y,z} = 0 \right) \leq c e^{-2L} e^{-2z - z^2/4L}.
$$

Using Lemma 4.4 and then the fact that $k_1^{1/4} \geq 2 \log L$ we see that

$$
\mathbb{P}(\mathcal{I}_{y,z} \cap \mathcal{I}_{y',z}) \leq c e^{-2L} e^{-2-2z - z^2/4L} e^{-2(L-k)-2k_1^{1/4}}
= c e^{-4L-2k_1^{1/4}} e^{-2z - z^2/4L}.
$$

**4.3. Second moment estimate: late branching.** In this subsection we prove Lemma 4.4 for $L - 16 \log^4 L \leq k < L - 1$. Consider first the case $L - 16 \log^4 L \leq k < L - d^*$. The proof is very similar to the bulk branching case, except that we no longer have $k_1^{1/4} \geq 2 \log L$ so we will need a barrier bound to control the factor $L$ on the right hand side (4.78). Thus we set

$$
\mathcal{B}_y^\dagger = \{ \rho_L (L - l) \leq \sqrt{2T_{y,t}^{y,z}} \text{ for } l = 1, \ldots, k - 3 \}.
$$

Once again, using the proof of [4, Corollary 6.7] and the calculations in (4.77) it follows that

$$
\mathbb{P}\left( \mathcal{B}_y^\dagger \cap \left\{ T_{y,t}^{y,z} = 0, T_{y',k,\gamma^2(k)/2} L = 0 \right\} \right)
\leq \left( 1 - \frac{1}{L - k} \right) \frac{1}{2} \mathbb{P}\left( \mathcal{B}_y^\dagger \cap \left\{ T_{y,t}^{y,z} = 0 \right\} \right)
\leq e^{-\gamma^2(k)(1 - \frac{2}{L - k})} \mathbb{P}\left( \mathcal{B}_y^\dagger \cap \left\{ T_{y,t}^{y,z} = 0 \right\} \right),
$$

where the last line follows from Lemma 4.4.

The contribution referring to $y$ is then bounded by the barrier estimate contained in Lemma 8.3 of Appendix I,

$$
\mathbb{P}\left( \mathcal{B}_y^\dagger \cap \left\{ T_{y,t}^{y,z} = 0 \right\} \right) \leq c(1 + z)(L - k)^{1/2} e^{-2L} e^{-2z - z^2/4L}.
$$
Hence
\[
\mathbb{P}(I_{y,z} \cap I_{y',z}) \leq c(1 + z)(L - k)^{1/2}e^{-2L}e^{-2z - z^2/4L}e^{-2(L - k) - 2k^{1/4}}
\]
(4.83)
\[
\leq c(1 + z)e^{-(4L - 2k) - k^{1/4}}e^{-2z - z^2/4L},
\]
where we have used part of the exponential in \(k^{1/4} = (L - k)^{1/4}\) to control the factor \((L - k)\).

For \(L - d^* \leq k < L - 1\) we simply bound the term \(\mathbb{P}(I_{y,z} \cap I_{y',z})\) by \(\mathbb{P}(I_{y,z})\) and obtain from (4.24) the following upper bound
(4.84)
\[
\mathbb{P}(I_{y,z} \cap I_{y',z}) \leq c(1 + z)e^{-2L}e^{-2z - z^2/4L} \leq cd^*(1 + z)e^{-(4L - 2k) - k^{1/4}}e^{-2z - z^2/4L}.
\]

4.4. Second moment estimate: early branching. In this subsection we prove Lemma 4.4 for \(1 \leq k \leq L_0 = 16 \log^4 L\). It is here that the distinction between \(I_{y,z}\) and \(\hat{I}_{y,z}\), see (4.15), plays an important role. This difference will be controlled by (4.17). We remark that for \(k\) not too small we can avoid this by applying the methods of [4, Proposition 6.16], but for \(k = o(\log \log L)^4\), that approach fails.

Consider first the case where \(d^* \leq k \leq L_0\). Recall \(\mathcal{B}_{y,k,L}\) from (4.44). Then
(4.85)
\[
\mathbb{P}(I_{y,z} \cap I_{y',z}) \leq \sum_{a \geq [k^{1/4}]} \mathbb{P}(W_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,L} \cap \mathcal{H}_{k,a} \cap I_{y',z}).
\]
Hence it suffices to show that if \(d(y, y') \geq 2h_k\) and \(k \geq d^*\) then
(4.86)
\[
\mathbb{P}(W_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,L} \cap \mathcal{H}_{k,a} \cap I_{y',z}) \leq ce^{-a}E(k),
\]
where we abbreviate
(4.87)
\[
E(k) = e^{-4L + 2k}e^{-2z}.
\]
By replacing \(I_{y',z}\) by the larger set \(\{T_L^{y',k^+,\gamma^2(k^+)^2/2} = 0\}\) it is easy to see that the probability in (4.85) for \(a > L^{3/4}\) is much smaller than the right hand side of (4.86), so we may assume that \(a \leq L^{3/4}\).

Set
(4.88)
\[
\hat{I}_{y',k,z} = \left\{ \gamma(l) \leq \sqrt{2T_L^{y',l,z}} \text{ for } l = 1, \ldots, k - 2, k + 1, \ldots L - 1; T_L^{y',l,z} = 0 \right\},
\]
that is, we drop the barrier in \(\hat{I}_{y',z}\) for \(l = k - 1, k\).
Recall the events $\mathcal{B}_{y,k+3,k^+}$ and $\mathcal{B}_{y,k++1,L}$, see (4.60) and (4.61), and the $\sigma$-algebra $G^y_{k^+}$, see (4.63). We have that

\[
P \left( W_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,L} \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',z} \right) \leq \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k++)^{1/4}}^{L^{3/4}} P \left( W_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,k^+} \cap \mathcal{B}_{y,k++1,L} \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',k,z}, \sqrt{2T_{k^+}^{y,t_s} \in I_{\rho L(L-k++)+j''}} + o(e^{-a}E(k)), \right)
\]

where, once again, the error term (see (4.86)) is coming from the restriction $j', j'' \leq L^{3/4}$.

Next, note that

\[(4.90) \quad A'_j := W_{y,k}(N_{k,a}) \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',k,z} \cap \mathcal{B}^y_{y,k+3,k^+} \in G^y_{k^+}, \]

This is the reason we introduced $\mathcal{T}_{y',k,z}$. Using (4.90) we have

\[
P \left( A'_j; \sqrt{2T_{k^+}^{y,t_s} \in I_{\rho L(L-k++)+j''}}; \mathcal{B}_{y,k++1,L} \right) \leq \sup_{s \in I_{\rho L(L-k^+)+j'}, \nu \in I_{\rho L(L-k++)+j''}} P \left( A'_j; \sqrt{2T_{k^+}^{y,t_s} \in I_{\rho L(L-k++)+j''}}; \mathcal{B}_{y,k++1,L} \mid T_{y,k^+}^{x,t_s} \mathcal{I}_{y,k++-1} = \nu^2 / 2, G^y_{k^+} \right), \]

Recall the estimates (4.66) and (4.67). We then have that uniformly in $[k^{1/4}] \leq a \leq L^{3/4}$,

\[
P \left( W_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,L} \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',z} \right) \leq ce^{-2(L-k^-)} \sum_{j'=(k^-)^{1/4}}^{L^{3/4}} \sum_{j''=(k++)^{1/4}}^{L^{3/4}} (1 + j''^2) e^{-2j''^2}
\times P \left( W_{y,k}(N_{k,a}) \cap \mathcal{B}^y_{y,k+3,k^+} \cap \mathcal{B}^y_{y,k++1,L} \cap \mathcal{H}_{k,a} \cap \mathcal{T}_{y',k,z} \right),
\]

where $\mathcal{B}^y_{j',j''}$ is as in (4.69).

To handle the last term in (4.92), we use the following estimate, whose proof, given in the following sub-section, uses the decoupling lemma (Lemma 4.13), barrier estimates and the control on Wasserstein distance contained in (4.17).

**Lemma 4.9.** For some $M_0 < \infty$ and $k,a,j'$ in the ranges specified above,

\[
P(W_{y,k}(N_{k,a}) \cap \mathcal{B}^y_{y,k+3,k^+} \cap \mathcal{B}^y_{y,k++1,L} \cap \mathcal{H}_{k,a} \cap \mathcal{T}_{y',k,z}) \leq F_{k,a,j',j''} \frac{1}{2} P(\mathcal{T}_{y',k,z}) + e^{-2L^2/L},
\]
where
\[
F_{k,a,j',j''} = c a e^{-2a} e^{-2(k^+ + k)} \sum_{\substack{(j',j'') \leq 2M_0 \log k \\ \{j''\} : |j''| \leq 2M_0 \log k}} \bar{j'} e^{-(j'-a)^2/(4(k^+ + k))} e^{j''} e^{-(j''-\bar{j}'')^2/2(k^+ + k)} \frac{(k^+ + k)}{(k - k)}.
\]

Assuming Lemma 4.9, we can now complete the proof of Lemma 4.4 for \( d^* \leq k \leq L_- = 16 \log^4 L \). In view of (4.89)-(4.91), (4.66), (4.67), and using the barrier estimate Lemma 8.2 for \( P(\mathcal{I}_{y',k,z}) \), we see that the contribution of \( F_{k,a,j',j''} \) to (4.86) is bounded by
\[
\sum_{\substack{(j',j'') \leq 2M_0 \log k \\ \{j''\} : |j''| \leq 2M_0 \log k}} \bar{j'} e^{-(j'-a)^2/(4(k^+ + k))} e^{j''} e^{-(j''-\bar{j}'')^2/2(k^+ + k)} \frac{(k^+ + k)}{(k - k)}.
\]

Using the fact that \( a \geq k^{1/4} \), and recalling that \( E(k) = e^{-4L^2 - 2k e^{-2z}} \), see (4.87), this is bounded by
\[
c e^{-3a/2} e^{-2a} e^{2M_0 \log k} \sum_{\substack{(j',j'') \leq 2M_0 \log k \\ \{j''\} : |j''| \leq 2M_0 \log k}} \bar{j'} e^{-(j'-a)^2/(4(k^+ + k))} e^{j''} e^{-(j''-\bar{j}'')^2/2(k^+ + k)} \frac{(k^+ + k)}{(k - k)}.
\]

The sum over \( j'' \) can be bounded by \( c(1 + j'/(k^+ + k)^{1/2}) \). Thus we can bound (4.96) by
\[
c e^{-3a/2} \sum_{j'} \left( j' e^{-(j'-a)^2/(4(k^+ + k))} \right) e^{-(j'-a)^2/(4(k^+ + k))} \frac{(k^+ + k)}{(k - k)^{3/2}}.
\]

Similarly, the contribution of the last term of (4.93) to (4.86) is bounded by
\[
c L^{3/2} e^{-2(L-k^{++})} e^{-2L - \sqrt{L}},
\]
which, after summation in \( k^{1/4} \leq a \leq L^{3/4} \), is easily seen to be bounded by \( c e^{-k^{1/4}} E(k) \). Together with the previous displays, this completes the proof of Lemma 4.4 for \( d^* \leq k \leq L_- \).

It thus remains to consider the case where \( 1 \leq k < d^* \). We show below that for some \( c, c' < \infty \),
\[
\text{for } 1 \leq k < d^*, \text{ we have } e^{-4L^2 + 2d^* e^{-2z} e^{-c(d_L^*)^{1/4}}},
\]
whenever \( 2h_{d^*} < d(y, y') \), that is, without the condition that \( d(y, y') \leq 2h_{d^*} - 1 \). Then, since for \( 1 \leq k < d^* \), \( 2h_k < d(y, y') \) implies \( 2h_{d^*} < d(y, y') \),
\[\text{Terms depending on } z \text{ are absorbed in the factor } \sqrt{L}.\]
and $d^*$ is fixed, by increasing the constant $c'$ in (4.99) we obtain (4.26) for $1 \leq k < d^*$.

To prove (4.99) solely under the condition that $2h_{d^*} < d(y, y')$, we return to the proof of (4.26) for $k = d^*$. The condition $2h_k < d(y, y') \leq 2h_{k-1}$ is needed to guarantee that $\hat{I}_{y',k,z} \in G_y^y$. If instead of just dropping the barrier in $\hat{I}_{y',k,z}$ for $l = k - 1$, $k$ we were to drop it for all $l \leq k$ and replace $\hat{I}_{y',k,z}$ by

$$
(4.100) \quad \hat{I}^{y',k,z} = \left\{ \gamma(l) \leq \sqrt{2T_{1}^{y,t_z}} \quad \text{for} \quad l = k, \ldots, L - 1; T_{L}^{y,t_z} = 0 \right\},
$$

then $\hat{I}^{y',k,z} \in G_y^y$. We did not use this for general $k$ since dropping the barrier for a unbounded number of $I$'s can effect our estimates. However, $d^*$ is fixed so we can replace $\hat{I}^{y',d^*,z}$ by $\hat{I}^{y',d^*,z}$ and following our proof of (4.26) for $k = d^*$ we will obtain (4.99) whenever $2h_{d^*} < d(y, y')$.

This completes the proof of Lemma 4.4. \qed

4.5. Decoupling. In this section we assume that $d^* \leq k \leq L^{1/2}$.

Let $\Psi = \{\psi_{k,i}, \ i = 1, 2, \ldots\}$ be a collection of independent $\nu_k$-distributed random variables, independent of the Brownian motion $X$. We set

$$
V_{y,k}(n) = \left\{ d^{1}_{wa} \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\psi_{k,i}, \nu_k} \right) \leq \frac{c_0 \log k}{\sqrt{n}} \right\},
$$

and define $W_{y,k}(n)$ similarly with the $\psi_{k,i}$ replaced by $\theta_{k,i}$. We set

$$
\mathbb{G}_{y,k}(N,k) = V_{y,k}(N_k) \cap W_{y,k}(N_k) \cap \mathcal{H}_{k,a}.
$$

We will often use the notation $P_{X,\Psi}$ instead of $\mathbb{P} \times \nu_k^{\mathbb{Z}^+}$, while emphasizing that we are dealing with two independent processes. It follows from (4.117) that for any event $\mathcal{E}$ which is measurable with respect to $X$,

$$
\mathbb{P}(W_{y,k}(N_k) \cap \mathcal{H}_{k,a} \cap \mathcal{E}) \leq 2P_{X,\Psi}(\mathbb{G}_{y,k}(N_k) \cap \mathcal{E}).
$$

Set $\tilde{\theta}_{k,N} = (\theta_{k,1}, \theta_{k,2}, \ldots, \theta_{k,N})$ where the $\theta_{k,i}$, as above, are the angular increments of the excursions at level $k$. Similarly we set $\tilde{\psi}_{k,N} = (\psi_{k,1}, \psi_{k,2}, \ldots, \psi_{k,N})$.

On $\mathbb{G}_{y,k}(N_k)$, by the triangle inequality we have that

$$
(4.104) \quad d^{1}_{wa} \left( \frac{1}{N_k,a} \sum_{i=1}^{N_k,a} \delta_{\psi_{k,i}, \nu_k} \frac{1}{N_k,a} \sum_{i=1}^{N_k,a} \delta_{\theta_{k,i}} \right) \leq \frac{c_0 \log k}{\sqrt{N_k,a}}.
$$

If $Q \in \mathbb{P}^2(\mu, \nu)$ with marginals $\mu = \frac{1}{N_k,a} \sum_{i=1}^{N_k,a} \delta_{\theta_{k,i}}$ and $\nu = \frac{1}{N_k,a} \sum_{i=1}^{N_k,a} \delta_{\psi_{k,i}}$ and distinct $\theta_{k,i}$ and $\psi_{k,i}$, then we can write

$$
Q = \frac{1}{N_k,a} \sum_{i,j=1}^{N_k,a} q_{i,j} \delta_{\theta_{k,i}} \times \delta_{\psi_{k,j}}.
$$
Thus, \( \{q_{i,j}\} \) is a doubly stochastic matrix. Hence, recall (4.106), (4.104) says that

\[
\inf_{A \in DS(N_{k,a})} \sum_{i,j=1}^{N_{k,a}} A_{i,j} |\theta_{k,i} - \psi_{k,j}| \leq c_0 \log k \sqrt{N_{k,a}},
\]

where the infimum is over the set of doubly stochastic \( N_{k,a} \times N_{k,a} \) matrices, denoted \( DS(N_{k,a}) \).

Using G. Birkhoff’s theorem that any doubly stochastic matrix is a convex combination of permutation matrices, \([6, 28]\), (4.105) implies that there is a permutation \( \pi = \pi_{\theta_{k,N_{k,a}}} \psi_{k,N_{k,a}} \) of \([1, N_{k,a}]\) such that, on \( G_{y,k}(N_{k,a}) \),

\[
\sum_{i=1}^{N_{k,a}} |\theta_{k,i} - \psi_{k,\pi(i)}| \leq c_0 \log k \sqrt{N_{k,a}}.
\]

Let \( C(y, k+2) \) denote the space of finite sequences of continuous paths from \( \partial B_d(y, h_{k+2}) \) to \( \partial B_d(y, h_{k+1}) \). We define an equivalence relation \( \mathcal{R} \) on \( C(y, k+2) \) by saying that two sequences in \( C(y, k+2) \) are equivalent if they differ by some rotation around \( y \). Let \( \tilde{C}(y, k+2) = C(y, k+2)/\mathcal{R} \), and let \( \Delta \) denote an auxiliary point, the ‘cemetery state’. To each \( \alpha \in [0, 2\pi) \), we associate a random variable \( \tilde{Y}^\alpha \) with values in \( \tilde{C}(y, k+2) \cup \{\Delta\} \), as follows. Consider the Brownian excursion started at some point in \( \partial B_d(y, h_k) \) until exiting \( \partial B_d(y, h_k-1) \), conditioned so that \( \alpha \) is the angular increment between its initial and terminal point. If this excursion reaches \( \partial B_d(y, h_{k+2}) \), we let \( \tilde{Y}^\alpha \) be the element of \( \tilde{C}(y, k+2) \) generated by our excursion. (There may be several excursions from \( \partial B_d(y, h_{k+2}) \) to \( \partial B_d(y, h_{k+1}) \) until exiting \( \partial B_d(y, h_{k-1}) \), which is why \( C(y, k+2) \) involves sequences of excursions). If our excursion exits \( B_d(y, h_k-1) \) before reaching \( \partial B_d(y, h_{k+2}) \), we set \( \tilde{Y}^\alpha = \Delta \). In this manner, for each \( \psi_{k,j}, j = 1, 2, \ldots \) we define \( \tilde{Y}_{\psi_{k,j}} \in \tilde{C}(y, k+2) \cup \{\Delta\} \). Recalling the definition of \( \theta_{k,i} \), it follows from the Markov property that the excursions \( Y^{\theta_{k,i}} \in \tilde{C}(y, k+2) \cup \{\Delta\}, i = 1, 2, \ldots \), generated by the Brownian motion \( X \), have the same law as \( \tilde{Y}^{\theta_{k,i}} \). Note that on the event \( \mathcal{H}_{k,a} \) we have that \( T^{\theta_{k,i}} \leq N_{k,a} \).

By a direct computation using the Poisson kernel in (2.8), we see that there exists a universal constant \( c_1 \) so that if \( x, x' \in \partial B_d(0, h_{k-1}) \) then

\[
\max_{u, z \in \partial B_d(0, h_k)} \frac{P_{B_d(0, h_{k-1})}(z, x)}{P_{B_d(0, h_{k-1})}(u, x)} - \frac{P_{B_d(0, h_{k-1})}(z, x')}{P_{B_d(0, h_{k-1})}(u, x')} 
\leq c_1 d_a(x, x'),
\]

\[
\max_{u, z \in \partial B_d(0, h_k)} \frac{\sin^2(d(u, x)/2)}{\sin^2(d(z, x)/2)} - \frac{\sin^2(d(u, x')/2)}{\sin^2(d(z, x')/2)} 
\leq c_1 d_a(x, x'),
\]
where \( d_a(x, x') \) denotes the difference of arguments of \( x \) and \( x' \), which can be taken by definition (since \( h_{k-1} \) is small and the ratio \( h_{k-1}/h_k \) is fixed) as \( d(x, x')/h_{k-1} \), and we used the fact that \( \frac{\sin^2(d(u, x)/2)}{\sin^2(d(z, x)/2)} \) is Lipschitz in \( x \), uniformly in \( u, z \in \partial B_d(0, h_k) \).

With \( c_1 \) as in (4.107), let \( p_i = \min(\theta_{k,i} - \psi_{k,\pi(i)}, 1) \). Note that by (4.106), with \( c_2 = c_0 c_1 \), we have that, on \( G_{y,k}(N_{k,a}) \),

\[
\sum_{i=1}^{N_{k,a}} p_i \leq c_2 \log k \sqrt{N_{k,a}}.
\]

Let \( B_i, i = 1, \ldots, N_{k,a} \) be independent Bernoulli random variables with mean \( p_i \), independent of the Brownian motions. We can use \( B_i \) to create a coupling between \( Y^{\theta_{k,i}} \) and \( Y^{\psi_{k,\pi(i)}} \), as follows.

**Lemma 4.10.** For each \( 1 \leq i \leq N_{k,a} \) there exist random variables \( \bar{Y}_i, Z_i, \tilde{Z}_i \) in \( \mathbb{C}(y, k + 2) \cup \{\Delta\} \) so that

\[
Y^{\theta_{k,i}} = (1 - B_i) \bar{Y}_i + B_i Z_i,
\]

\[
Y^{\psi_{k,\pi(i)}} = (1 - B_i) \bar{Y}_i + B_i \tilde{Z}_i.
\]

The random variables \( B_i \) and \( \bar{Y}_i \) can be taken independent of each other.

We refer to the joint law of \( B_i, \bar{Y}_i, Z_i, \tilde{Z}_i \) in the lemma as \( \mathbb{P}^{\theta_{k,i}, \psi_{k,\pi(i)}} \).

**Proof.** Without loss of generality we take \( y = 0 \). (4.109) will follow from [26] Theorem 5.2.\footnote{The formulation in [26] Theorem 5.2] allows for the independence of \( Z_i \) and \( \tilde{Z}_i \) of each other, with the change that the parameter of the Bernoulli \( B_i \) equals the variation distance. By splitting \( B_i \) in that formulation, we arrive at the current statement.} once we show that the variational distance between the distributions \( Y^{\theta_{k,i}} \) and \( Y^{\psi_{k,\pi(i)}} \) is less than \( c_1 |\theta_{k,i} - \psi_{k,\pi(i)}| \). The latter follows from an application Poisson kernel estimates, similarly to the proof of Lemma 9.1. We provide the details below.

Let \( \tau = \inf\{ t : X_t \in \partial B_d(0, h_{k-1}) \} \), \( \tau_0 = 0 \) and for \( i = 0, 1, \ldots \) define

\[
\tau_{2i+1} = \inf\{ t \geq \tau_{2i} : X_t \in \partial B_d(0, h_{k-1}) \cup \partial B_d(0, h_{k+2}) \},
\]

\[
\tau_{2i+2} = \inf\{ t \geq \tau_{2i+1} : X_t \in \partial B_d(0, h_{k+1}) \}.
\]

If \( \tau_{2i+1} < \tau \), let \( \mathcal{E}_i \) denote the excursion from time \( \tau_{2i+1} \) until \( \tau_{2i+2} \), while if \( \tau_{2i+1} \geq \tau \), let \( \mathcal{E}_i = \emptyset \). Then let \( \mathcal{I}_j \) denote the \( \sigma \)-algebra of events measurable with respect to \( \mathcal{E}_i \), \( 0 \leq i \leq j \), which are rotationally invariant around 0. We set \( \mathcal{I} = \bigcup_{j=0}^\infty \mathcal{I}_j \).

If \( A \subseteq \mathcal{I}_j - \mathcal{I}_{j-1} \), using the Markov property we have, for any \( u \in \partial B_d(0, h_k) \),

\[
\mathbb{P}^u(A \mid X_{\tau} = x) = \mathbb{E}^u(A \mid \tau_{2j+2} > \tau; \tau_{2j+2} < \tau; p_{B_d(0, h_{k-1})}(x_{\tau_{2j+2}}, x), p_{B_d(0, h_{k-1})}(u, x))
\]
see (2.8). Using (4.107), we obtain that for \( A \subseteq I_j - I_{j-1} \),

\[
|\mathbb{P}^u (A \mid X_\tau = x) - \mathbb{P}^u (A \mid X_\tau = x')| \leq c_1 d_a(x, x') P^u(A),
\]

uniformly in \( A \) and \( j \). Hence if \( B = \bigcup_{j=0}^{\infty} A_j \) with \( A_j \subseteq I_j - I_{j-1} \), then

\[
|\mathbb{P}^u (B \mid X_\tau = x) - \mathbb{P}^u (B \mid X_\tau = x')| \leq \sum_{j=0}^{\infty} |\mathbb{P}^u (A_j \mid X_\tau = x) - \mathbb{P}^u (A_j \mid X_\tau = x')| \leq c_1 d_a(x, x') \sum_{j=0}^{\infty} \mathbb{P}^u(A_j) = c_1 d_a(x, x') \mathbb{P}^u(B),
\]

uniformly in \( B \).

Finally we need to consider the event that \( \{ \tau_1 = \bar{\tau} \} \). Since

\[
\mathbb{P}^u (\tau_1 = \bar{\tau} \mid X_\tau = x) = 1 - \mathbb{P}^u (\tau_1 \neq \bar{\tau} \mid X_\tau = x),
\]

and since

\[
\{ \tau_1 \neq \bar{\tau} \} = \bigcup_{j=0}^{\infty} \{ \tau_{2j+2} < \bar{\tau} \} \in \mathcal{I},
\]

the desired result follows from the previous paragraph.

We define

\[
(4.110) \quad \tilde{P} = \tilde{P}^{\theta_{k,N_k,a}, \bar{\psi}_{k,N_k,a}} = \bigotimes_{i=1}^{N_k,a} \tilde{P}^{\theta_{k,i,a}, \psi_{k,a}(i)},
\]

where \( \pi = \pi^{\theta_{k,N_k,a}, \bar{\psi}_{k,N_k,a}} \), see (4.106) and \( \tilde{P}^{\theta_{k,i,a}, \psi_{k,a}(i)} \) is as in Lemma 4.10.

If \( \mathcal{U} \) is an event measurable on \( Y_{\theta_{k,i}} \), \( i = 1, \ldots, T_k^{y_{k,i}} \), then for any event \( \mathcal{V} \) which depends only on excursions outside of \( B_d(y, h_k) \) we have

\[
P_{X,\psi} (\mathcal{G}_{y,k}(N_{k,a}) \cap \mathcal{U} \cap \mathcal{V}) = E_{X,\psi} \left( \tilde{P}^{\theta_{k,N_k,a}, \bar{\psi}_{k,N_k,a}} (\mathcal{U}), \mathcal{G}_{y,k}(N_{k,a}) \cap \mathcal{V} \right).
\]

We use \( \tilde{P}^{\psi} \) to denote the law of \( \sum_{i=1}^{N_{k,a}} \delta_{\gamma \psi_{k,i}} \) under \( \tilde{P} \). We note that

\[
(4.111) \quad \tilde{P}^{\psi} \text{ does not depend on } \hat{\theta}_{k,N_k,a}.
\]

\( Y^{\theta_{k,i}} \) and \( \tilde{Y}^{\psi_{k,a}(i)} \) will coincide with probability at least \( 1 - p_i \) under \( \tilde{P} \). We will call this a success. Let \( \Gamma_{k,a} = \Gamma_{k,a}(\hat{\theta}_{k,N_k,a}, \bar{\psi}_{k,N_k,a}) \) be the number of successes among the \( N_{k,a} \) excursions. We have the following.

**Lemma 4.11.** There exists \( d^* \) sufficiently large so that, for \( d^* \leq k \leq L^{1/2} \), and all \( L \) large,

\[
\tilde{P} (\Gamma_{k,a} < N_{k,a} - 2c_2 L \log k) \leq e^{-10L}.
\]
Proof. The number of failures is at most \( \sum_{i=1}^{N_{k,a}} B_i \), and using \( \lambda > 0 \) for the first inequality, \( 1 + px \leq e^{\lambda x} \) for the third and (4.108) for last inequality, we obtain

\[
\tilde{P} \left( \sum_{i=1}^{N_{k,a}} B_i \geq 2c_2 \log k \sqrt{N_{k,a}} \right) \leq e^{-2\lambda c_2 \log k \sqrt{N_{k,a}}} \prod_{i=1}^{N_{k,a}} \tilde{E}(e^{\lambda B_i})
\]

\[
= \prod_{i=1}^{N_{k,a}} \left( 1 + p_i (e^\lambda - 1) \right) e^{-2\lambda c_2 \log k \sqrt{N_{k,a}}}
\]

\[
\leq e^{\sum_{i=1}^{N_{k,a}} p_i (e^\lambda - 1)} e^{-2\lambda c_2 \log k \sqrt{N_{k,a}}} \leq e^{c_2 \log k \sqrt{N_{k,a}}} (e^\lambda - 1 - 2\lambda).
\]

Hence taking \( \lambda > 0 \) small, recalling that \( N_{k,a} \geq L^2 \) see (4.20), we have that for some \( c_3 > 0 \)

\[
\tilde{P} \left( \sum_{n=1}^{N_{k,a}} B_n \geq 2c_2 \log k \sqrt{N_{k,a}} \right) \leq e^{-c_3 L \log k} \leq e^{-10L}
\]

by choosing \( d^* \) sufficiently large so that \( \log k \) is sufficiently large. \( \square \)

Thus, if we set

(4.112) \( \Phi_{k,a} = \Phi_{k,a}(\theta_{k,N_{k,a}}, \tilde{\psi}_{k,N_{k,a}}) = \{ \Gamma_{k,a} \geq N_{k,a} - 2c_2 L \log k \} \),

we have

(4.113) \( P_{X,\psi}(G_{y,k}(N_{k,a}) \cap U \cap V) \leq E_{X,\psi}(\tilde{P}(U \cap \Phi_{k,a}), G_{y,k}(N_{k,a}) \cap V) + e^{-10L} \).

Recall (4.34) and assume that \( a \leq L^{3/4} \). Let \( N = T_k^{y,1} \). On the event \( \mathcal{H}_{k,a} \) we have \( N_{k,a} - 1 \leq N \leq N_{k,a} \) and \( N_{k,a} - N_{k,a} - 1 \leq 4L \). Let

(4.114) \( A = A_\Theta = \{ i \mid \theta_{k,i} \in \hat{\theta}_{k,N} \text{ that did not successively couple } \} \)

and with \( A_N^c = \{ 1, \ldots, N \} \cap A^c \) set

(4.115) \( B = B_\Psi = \{ i \mid \psi_{k,i} \in \tilde{\psi}_{k,N_{k,a}} \text{ that did not successively couple with } A_N^c \} \).

Then under \( \Phi_{k,a} \), recalling that \( N = T_k^{y,1} \),

\[
\tilde{\tau}_{k,N} = \{ \theta_{k,i}, i \in A \} \cup \{ \psi_{k,j}, j \in B \}
\]

satisfies that \( |\tilde{\tau}_{k,N}| \leq 4c_2 L \log k \). Recall the notation (2.12) and use \( T_l^{y,k,\hat{\tau}_{k,N}} \) to denote the number of excursions at level \( l \) that occur during the \( |\tilde{\tau}_{k,N}| \) excursions \( \{ Y_{\theta_{k,i}}, i \in A \} \cup \{ \hat{Y}_{\psi_{k,j}}, j \in B \} \). With \( M_0 \) to be defined below, see Lemma 4.14, let

(4.116) \( A_{N,k} = \bigcup_{l=k+3}^{k++} \{ T_l^{y,k,\hat{\tau}_{k,N}} \geq M_0 \rho L (L - l) \log k \} \), with \( k++ = k + 200 \log L \).
In words, $A_{N,k}$ is the event that for some $l \in [k + 3, k++]$, the uncoupled excursions generated an excessive number of excursions at level $l$. We have

$$U \subseteq \left( U \cap A_{N,k}^c \right) \cup A_{N,k}. \quad (4.117)$$

We can now state our main decoupling lemma. It will be under the following assumption.

**Assumption 4.12.** $U$ is measurable with respect to $Y^{\theta_k,i}$, $i = 1, \ldots, T_k^{g,t_x}$, and there exists an event $\tilde{U}$ measurable on $\sum_{i=1}^{N_{k,a}} \delta_{Y^{\psi_k,i}}$ so that

$$U \cap A_{N,k} \cap \Phi_{k,a} \subseteq \tilde{U} \left( \sum_{i=1}^{N_{k,a}} \delta_{Y^{\psi_k,i}} \right). \quad (4.118)$$

In words, the assumption allows for events that, whenever there are not too many uncoupled excursions at level $k$ and the latter do not generate too many excursions at a level in $[k, k++]$, can be dominated by an event using only the (empirical measure of the) coupled excursions. In a typical application, $U$ will be a barrier event for the original excursions, and $\tilde{U}$ will be a barrier event for the coupled excursions, with a slightly modified barrier.

**Lemma 4.13** (decoupling lemma). Let $U$ satisfy Assumption 4.12, and let $V$ be an event which depends only on excursions outside of $B_d(y, h_k)$. Then for all $d^* \leq k \leq L^{1/2}$ and $a \leq L^{3/4}$

$$\Prob(W_{y,k}(N_{k,a}) \cap \mathcal{H}_{k,a} \cap U \cap V) \leq c \Prob \left( \tilde{U} \left( \sum_{i=1}^{N_{k,a}} \delta_{Y^{\psi_k,i}} \right) \right) \times \Prob(W_{y,k}(N_{k,a}) \cap \mathcal{H}_{k,a} \cap V) + ce^{-5L \log k / (\log L) \Prob(W_{y,k}(N_{k,a}) \cap \mathcal{H}_{k,a} \cap V) + e^{-10L}.} \quad (4.119)$$

**Proof.** By (4.103)

$$\Prob(W_{y,k}(N_{k,a}) \cap \mathcal{H}_{k,a} \cap U \cap V) \leq 2 \Prob(X,\psi(G_{y,k}(N_{k,a}) \cap U \cap V), \quad (4.113)$$

and by (4.113)

$$P_{X,\psi}(G_{y,k}(N_{k,a}) \cap U \cap V) \leq E_{X,\psi}(\tilde{P}(U \cap \Phi_{k,a}), G_{y,k}(N_{k,a}) \cap V) + e^{-10L}. \quad (4.120)$$

Using (4.117) we can bound

$$E_{X,\psi}(\tilde{P}(U \cap \Phi_{k,a}), G_{y,k}(N_{k,a}) \cap V) \leq E_{X,\psi}(\tilde{P}(U \cap A_{N,k}^c \cap \Phi_{k,a}) G_{y,k}(N_{k,a}) \cap V) + E_{X,\psi}(\tilde{P}(A_{N,k} \cap \Phi_{k,a}) G_{y,k}(N_{k,a}) \cap V). \quad (4.121)$$
Using our assumption (4.118) we have

\[
E_{X,y} \left( \tilde{P} \left( \mathcal{U} \cap A_{N,k}^{\circ,\Psi} \cap \Phi_{k,a} \right) , \mathcal{G}_{y,k,(N_k,a) \cap V} \right) \\
\leq \tilde{P}^\psi \left( \tilde{U} \left( \sum_{i=1}^{N_k,a} \delta_{\tau^{y,k,i}} \right) \right) \times \mathbb{P} \left( \mathcal{W}_{y,k,(N_k,a) \cap \mathcal{H}_{k,a} \cap V} \right) \\
= \mathbb{P} \left( \tilde{U} \left( \sum_{i=1}^{N_k,a} \delta_{\tau^{y,k,i}} \right) \right) \times \mathbb{P} \left( \mathcal{W}_{y,k,(N_k,a) \cap \mathcal{H}_{k,a} \cap V} \right),
\]

where the decoupling in the inequality came from Assumption 4.12 and (4.111), and the final equality comes from the fact that \( \sum_{i=1}^{N_k,a} \delta_{\tau^{y,k,i}} \) and \( \sum_{i=1}^{N_k,a} \delta_{\tau^{y,k,i}} \) have the same distribution.

We continue by bounding the probabilities of the last term on the right side of (4.121). Toward this end we need the following lemma.

**Lemma 4.14.** There exists \( M_0 < \infty \) independent of \( k \) and \( L \) such that for all \( l = k + 3, \ldots, k^{+} = k + 10^{10} \log L \), under \( \Phi_{k,a} \)

\[
\tilde{P} \left( \left\{ T_{l}^{y,k,\tau_{k,N}} \geq M_0 \rho_{L}(L-l) \log k \right\} \mid \tau_{k,N} \right) \leq e^{-10L \log k / \log L}.
\]

**Proof of Lemma 4.14.** We note that conditional on \( \tau_{k,N} \), the \( |\tau_{k,N}| \) excursions that are used to construct \( T_{l}^{y,k,\tau_{k,N}} \) are independent. We enumerate these excursions by \( i = 1, \ldots, |\tau_{k,N}| \). Write \( T_{l}^{y,k,(i)} \) for the number of excursions at level \( l \) that occur during the \( i \)th excursion from \( \tau_{k,N} \). By Lemma 9.4 with \( n = 1 \) (which is nothing but a standard use of Poisson kernel estimates as in [17]), there exists a constant \( c \) (independent of \( i, l, k \)) so that

\[
\tilde{P}(T_{l}^{y,k,(i)} = j) \leq \frac{c}{(l-k)^2} \left( 1 - \frac{1}{l-k} \right)^{j-1}, \quad j \geq 1.
\]

Let \( T_{l-k}^{(j)}, j \geq 1 \) denote independent copies of \( T_{l-k} \) under \( P_{1}^{GW} \). Equation (4.124) implies that \( T_{l}^{y,k,(i)} \) is stochastically dominated by a (finite) sum of independent copies of \( T_{l-k} \) plus possibly a constant, in the sense that there exist integers \( n_1, n_2 \) so that if \( l-k > n_1 \) then for all \( s \geq 0 \) integer,

\[
\tilde{P}(T_{l}^{y,k,(i)} \geq s) \leq P_{1}^{GW} \otimes n_2 \left( \sum_{j=1}^{n_2} T_{l-k}^{(j)} \geq s \right).
\]

(We use inclusion-exclusion and the fact that \( l-k \geq n_1 \) so that the probabilities in (4.125) are small even for \( s = 1 \), see (4.124).) Similarly, if \( l-k \leq n_1 \) then

\[
\tilde{P}(T_{l}^{y,k,(i)} \geq s) \leq P_{1}^{GW} \otimes n_2 \left( \sum_{j=1}^{n_2} T_{l-k}^{(j)} \geq s - n_2 \right).
\]
(Here, we use the additive integer $n_2$ in order to make sure that even for small $s \geq 1$ when the left side in (4.126) is large, the inequality remains true.) All in all, (4.126) holds for all $l - k$. Therefore, applying (4.126), the Markov property, and Lemma 3.6 (or more directly, \[4, \text{Lemma 4.6}\]), we obtain that

\[
\tilde{P}(T_{y,k}^{g,k},\bar{\tau}_{k,N} \geq M_0 \rho_L(L - l) \log k \mid \bar{\tau}_{k,N}) \\
\leq P^{G_N \otimes \tau_{2[\tau_{k,N}]}(n_2, [\tau_{k,N}])} \left( \sum_{j=1}^{n_2[\tau_{k,N}]} T_{l-k}^{(j)} \geq M_0 \rho_L(L - l) \log k - n_2[\bar{\tau}_{k,N}] \right) \\
\leq e^{-\left(\sqrt{M_0 \rho_L(L - l) \log k - n_2[\bar{\tau}_{k,N}]} - \sqrt{n_2[\bar{\tau}_{k,N}]} \right)^2/(l - k)}.
\]

Since $\rho_L(L - l) \sim 2L, \mid \bar{\tau}_{k,N} \mid \leq 4c_2L \log k$ under $\Phi_{k,a}$, and $l - k \leq 10^{10} \log L$, the lemma follows by choosing $M_0$ large enough.

We continue with the proof of Lemma 4.13. Summing in (4.123) over $l = k + 3, \ldots, k^{++}$ while considering $N$ with $\sqrt{N} \in I_{\rho_L(L - k)+a}$, and using the fact that $L/(\log L) \gg L^{3/4} \gg k^{++}$, we obtain that

\[
\tilde{P}(A_{N,k} \cap \Phi_{k,a}) \leq e^{-5L \log k/(\log L)}.
\]

Using (4.127) we see that

\[
E_{X,\Psi} \left( \tilde{P}(A_{N,k} \cap \Phi_{k,a}), G_{y,k}(N_{k,a}) \cap \mathcal{V} \right) \\
\leq ce^{-5L \log k/(\log L)} \mathbb{P}(W_{y,k}(N_{k,a}) \cap \mathcal{H}_{k,a} \cap \mathcal{V} \).
\]

Combining (4.119)-(4.122) and (4.128) completes the proof of Lemma 4.13.

As in the notation above, let $T_{y,k}^{g,k,\psi_{l,k,N_{k,a}}}$ denote the number of excursions at level $l$ that occur during the $N_{k,a}$ excursions $Y_{\psi_{k,j}}, j = 1, 2, \ldots, N_{k,a}$. In our applications, the functional on the right hand side of (4.118) will be a functional of the $T_{y,k}^{g,k,\psi_{l,k,N_{k,a}}}$, for $l = k + 3, \ldots$, which are functionals of $\sum_{i=1}^{N_{k,a}} \delta_{Y_{\psi_{k,i}}}$. 

\textbf{Proof of Lemma 4.9.} We begin the proof with a use of the decoupling lemma, Lemma 4.13. Recall (4.69) and (4.88). Let $\mathcal{U} = \mathbb{B}_{y,k+3,k^{++}}^{\psi_{l,k,N_{k,a}}}$ and $\mathcal{V} = \ldots$
\[ \mathcal{I}_{y',k,z}. \] We first claim, see (4.116) and (4.112) for notation, that
\[ \mathcal{U} \cap \mathcal{A}_{N,k}^{c} \cap \Phi_{k,a} \]
\[ \subseteq \bigcup \{ s' \in \mathbf{Z}_+ : |s' - (\rho_{L}(L-k^{+})+j'')| \leq 2M_{0} \log k \} \{ s'' \in \mathbf{Z}_+ : |s'' - (\rho_{L}(L-k^{+})+j'')| \leq 2M_{0} \log k \} \]
\[ \left\{ \rho_{L}(L-l) \leq \sqrt{2T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}, k^{++}, \right. \]
\[ \sqrt{2T_{k^{+}}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}, k^{++}, \}
\[ \bigcup \left\{ \sqrt{2T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}, k^{++}, \right\} \}
\]

To see (4.129), first note that \( |T_{l}^{y,t_{z}} - T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}| \leq T_{l}^{y,k,\tilde{\psi}_{k,N}} \) by (4.109) and the definition of \( \tilde{\tau}_{k,N} \). Hence by (4.116), on \( \mathbb{P}_{y,k+3,k^{+},k^{++}} \cap \mathcal{A}_{N,k}^{c} \), for all \( k + 3 \leq l \leq k^{+}, k^{++} \) we have
\[ |T_{l}^{y,t_{z}} - T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}| \leq M_{0} \rho_{L}(L-l) \log k \leq .01 l^{1/4} \rho_{L}(L-l) \]

for \( d^{*} \) sufficiently large. Since on \( \mathbb{P}_{y,k+3,k^{+},k^{++}} \) we have that \( \gamma(l) \leq \sqrt{2T_{l}^{y,t_{z}}} \) for \( l = k + 3, \ldots, k^{+}, k^{++} \), it follows that
\[ \sqrt{2T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}} \geq \sqrt{2T_{l}^{y,t_{z}}} - .02 l^{1/4} \rho_{L}(L-l) \geq \rho_{L}(L-l) + .9l^{1/4}, \]
which completes the proof of (4.129).

Note next that the \( T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}} \) in (4.129) are measurable on \( \sum_{i=1}^{N_{k,a}} \delta_{\mathbf{y}_{i}} \mathbf{v}_{k,i} \). Therefore, Assumption (4.12) is satisfied and we can apply Lemma (4.13) together with the fact that for \( l = k + 3, \ldots, k^{+} \) we have \( T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}} = T_{l}^{y,k,N,k,a} \)

to deduce that
\[ \mathbb{P}(W_{y,k}(N_{k,a}) \cap \bigcap \mathbb{P}_{y,k+3,k^{+},k^{++}} \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',k,z}) \]
\[ \leq c \sum \{ s' \in \mathbf{Z}_+ : |s' - (\rho_{L}(L-k^{+})+j'')| \leq 2M_{0} \log k \} \{ s'' \in \mathbf{Z}_+ : |s'' - (\rho_{L}(L-k^{+})+j'')| \leq 2M_{0} \log k \} \]
\[ \mathbb{P}\left( \rho_{L}(L-l) \leq \sqrt{2T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}; \right. \]
\[ \left. \sqrt{2T_{k^{+}}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}, k^{++}, \}
\[ \bigcup \left\{ \sqrt{2T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}, k^{++}, \right\} \}
\]

Using Lemma (8.3) we obtain
\[ \mathbb{P}\left( \rho_{L}(L-l) \leq \sqrt{2T_{l}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}; \right. \]
\[ \left. \sqrt{2T_{k^{+}}^{y,k,\tilde{\psi}_{k,N,k,a}}}, \text{ for } l = k + 3, \ldots, k^{+}, k^{++}, \}
\[ \leq \frac{ca e^{-2a}}{(k^{+} - k)^{3/2}} \gamma^{2} e^{-2(k^{+} - k) + 2j' - (j'' - a)^2/(4(k^{+} - k))}. \]
It follows from \[5\] Proposition 1.4 that
\[
(4.134) \quad \sup_{v \in I_{\rho_L(L-k++)+\tilde j'}} \mathbb{P} \left( \sqrt{2T_{k++}^{y,k,N_k,a}} \in I_{\rho_L(L-k++)+\tilde j''} \bigg| \sqrt{2T_{k+}^{y,k,N_k,a}} = v \right) \leq c \frac{1}{(k++ - k+)^{1/2}} e^{-2(k++ - k+)} e^{-2(\tilde j' - \tilde j'')/2(k++ - k+)}. \]

Noting that \(k++ - k+ = k+ - k\) we can bound the sum in (4.132) by
\[
(4.135) \quad c a e^{-2a} e^{-2(k++ - k+)} \sum_{\{j',|j' - j'| \leq 2M_0 \log k\}} \frac{\tilde j' e^{-2(\tilde j' - \tilde j'')/(4(k++ - k+))}}{(k+ - k)^2(k++ - k+)}.
\]
This is the term \(F_{k,a,j'}\) in (4.93). The last term in (4.132) gives the second term in the right side of (4.93). \(\square\)

**Proof of Lemma 4.7** The proof is very similar to that of Lemma 4.9. We again take \(U = B_j^{y,k+3,k++}\) but now take \(V = L_{k,m,p,a}^I\) and replace \(\log k\) by \(m\) in (4.101) and all corresponding expressions. The important consequence for us is that now in (4.130) we will only have \(M_0 \rho_L(L - l)m \leq .01 l^{1/4} \rho_L(L - l)\) for \(l \geq cm^4\), so that we have to drop the barrier from \(k+3\) to \(cm^4 << k+\). In consequence, in (4.133) we need to replace the factor \(a\) by \(a + m^2\), compare the proofs of Lemmas 8.5 and 8.3 \(\square\)

**Proof of Lemma 4.8** The proof is very similar to that of Lemmas 4.9 and 8.7 except that now we drop the barrier in (4.132) and instead of (4.133) we use the analog of (4.134):
\[
(4.136) \quad \mathbb{P} \left( \sqrt{2T_{k+}^{y,k,N_k,a}} \in I_{\rho_L(L-k++)+\tilde j'} \right) \leq c \frac{1}{(k+ - k)^{1/2}} e^{-2(k+ - k)} e^{-2(a - \tilde j')/2(k+ - k)}.
\]
\(\square\)

### 5. Continuity estimates

This section is devoted to the statement and proof of a general continuity result for excursion counts, Lemma 5.1. We begin by introducing notation.

Throughout the section we let \(0 < a < b < 1\) be fixed constants and let \(0 < r < R < \tilde r < \tilde R\) with
\[
(5.1) \quad h(r)/h(R), h(\tilde r)/h(\tilde R), h(R)/h(\tilde r) \in [a,b].
\]

Let 0 denote a fixed point on the sphere, for instance the “south pole”, which is identified with 0 ∈ \(R^2\) when using isothermal coordinates, see Section 2. Let \(\mu_{0,h(\tilde r)}\) denote the uniform probability measure on \(\partial B_d(0,h(\tilde r))\).
Let $Y_1, Y_2, \ldots$ denote a collection of excursions $\partial B_d(0, h(\hat{R})) \to \partial B_d(0, h(\hat{r}))$, and let $I_1, \ldots, I_n$ be a collection of rotationally invariant subsets of excursions $\partial B_d(0, h(\hat{R})) \to \partial B_d(0, h(\hat{r}))$ for which the densities

\begin{equation}
\inf_{i=1, \ldots, n} \frac{\mathbb{P} \left( X_{H\partial B_d(0,h(\hat{r}))} \in du \mid X_{-H\partial B_d(0,h(\hat{r}))} \in I_i \right)}{\mu_{0,h(\hat{r})}(du)} \geq a_0 > 0,
\end{equation}

for any (hence all, by rotational invariance) $w \in \partial B_d(0, h(\hat{R}))$. Note that $a_0$ is determined by $a, b$ and hence fixed throughout this section. With $a, b$ fixed, the results in the remainder of this section are uniform in choices of $r, R, \hat{r}, \hat{R}$ satisfying (5.1) and (5.2). Recall the notation $T_{x,R \to r}$, see Definition 2.1 for the number of traversals $\partial B_d(x, h(R)) \to \partial B_d(x, h(r))$ during $n$ excursions $\partial B_d(0, h(\hat{r})) \to \partial B_d(0, h(\hat{R}))$. The following lemma is the main result of this section.

**Lemma 5.1.** Fix $0 < a < b < 1$. Let $r < R < \hat{r} < \hat{R}$ so that (5.2) and (5.1) hold. Let $x, y$ be such that $B_d(x, h(R)), B_d(y, h(R)) \subset B_d(0, h(\hat{R}))$. Then for any $C_0 < \infty$ there exist small $c_0, q_0 > 0$, depending on $a, b, C_0$ only, such that if $q = d(x,y)/R \leq q_0$, and $\theta \leq c_0 \sqrt{(n-1)}$, then

\begin{equation}
\mathbb{P} \left( \left| T_{x,R \to r}^{0,\tilde{r},\tilde{R}} - T_{y,R \to r}^{0,\tilde{r},\tilde{R}} \right| \geq \theta \sqrt{q(n-1)} \right) \leq \exp \left( -C_0 \theta^2 \sqrt{q} \right).
\end{equation}

**Proof.** The proof of Lemma 5.1 will involve several steps. We begin by restating the lemma in terms of certain traversals counts. Let $\tilde{D}_{0,i}$ and $\tilde{R}_{0,i}$ the successive arrivals to $\partial B_d(0, h(\hat{R}))$ and $\partial B_d(0, h(\hat{r}))$. That is, $\tilde{R}_{0,1} = H_{\partial B_d(0,h(\hat{r}))}$, and for $i \geq 1$,

\begin{equation}
\tilde{D}_{0,i} = H_{\partial B_d(0,h(\hat{r}))} \circ \theta_{\tilde{R}_{0,i}} + \tilde{R}_{0,i}
\end{equation}

and

\begin{equation}
\tilde{R}_{0,i+1} = H_{\partial B_d(0,h(\hat{r}))} \circ \theta_{\tilde{D}_{0,i}} + \tilde{D}_{0,i},
\end{equation}

where $\theta_t$ denotes time shift by $t$. Thus

$\tilde{R}_{0,1} < \tilde{D}_{0,1} < \tilde{R}_{0,2} < \tilde{D}_{0,2} < \ldots$

Let $A_{x,i}$ and $B_{y,i}$ denote the number of traversals from $\partial B_d(x, h(R))$ to $\partial B_d(x, h(r))$, respectively $\partial B_d(y, h(R))$ to $\partial B_d(y, h(r))$, during the $i$'th excursion from $\partial B_d(0, h(\hat{r}))$ to $\partial B_d(0, h(\hat{R}))$ (i.e. between time $R_{0,i}$ and $D_{0,i}$).

We have that

\begin{equation}
T_{x,R \to r}^{0,\tilde{r},\tilde{R}} = \sum_{i=1}^{n} A_{x,i}, \quad T_{y,R \to r}^{0,\tilde{r},\tilde{R}} = \sum_{i=1}^{n} B_{y,i}.
\end{equation}
Hence Lemma 5.1 is equivalent to the statement that there exist small $c_0, q_0 > 0$, depending on $a, b, C_0$ only, such that if $q = d(x, y)/R \leq q_0$, and $\theta \leq c_0 \sqrt{(n - 1)}$,

$$\mathbb{P}\left(\sum_{i=1}^{n} (A_{x,i} - B_{y,i}) \geq \theta \sqrt{(n - 1)q} \mid Y_m \in \mathcal{I}_m, m = 1, \ldots, n\right) \leq \exp\left(-C_0 \theta^2 \sqrt{q}\right).$$

Before proving (5.7), we develop some necessary material. We assume that $q_1 = q_1(a, b)$ is sufficiently small so that with $q = d(x, y)/R \leq q_1$, (5.8)

$$B_d(x, h(r)) \subset B_d(y, h(R)), \quad B_d(y, h(r)) \subset B_d(x, h(R)).$$

**Lemma 5.2.** There exists $q_1 = q_1(a, b) < 1/2$ and $c_2 < \infty$ so that, for all $q < q_1$ and all $k \geq 1$,

$$\sup_{u \in \partial B_d(0, h(\tau))} \mathbb{P}^u (A_{x,1} - B_{y,1} \geq k) \leq c_2 q e^{-2k}.$$

**Proof.** Let $R_{x,1}, R_{x,2}, \ldots$ be the successive hitting times of $\partial B_d(x, h(r))$ after departure times $D_{x,1}, D_{x,2}, \ldots$ from $\partial B_d(x, h(R))$. That is, set $D_{x,1} = H_{\partial B_d(x, h(r))}$, and for $i \geq 1$,

$$R_{x,i} = H_{\partial B_d(x, h(r))} \circ \theta_{D_{x,i}} + D_{x,i}$$

and

$$D_{x,i+1} = H_{\partial B_d(x, h(R))} \circ \theta_{R_{x,i}} + R_{x,i}.$$ 

Thus

$$D_{x,1} < R_{x,1} < D_{x,2} < R_{x,2} < \ldots.$$ 

Let $B_{y,1}(j)$ denote the number of traversals from $\partial B_d(y, h(R))$ to $\partial B_d(y, h(r))$ up till $H_{\partial B_d(0, h(R))}$ that take place between $D_{x,j}$ and $D_{x,j+1}$. Then

$$B_{y,1} \geq \sum_{j \geq 1} B_{y,1}(j),$$

so

$$A_{x,1} - B_{y,1} \leq A_{x,1} - \sum_{j \geq 1} B_{y,1}(j) = \sum_{j : D_{x,j+1} < H_{\partial B_d(0, h(R))}} (1 - B_{y,1}(j)).$$

(5.11)

Since

$$\{H_{\partial B_d(y, h(R))} \circ \theta_{D_{x,j}} < R_{x,j}\} \cap \{H_{\partial B_d(y, h(r))} \circ \theta_{R_{x,j}} < D_{x,j+1}\} \subseteq \{B_{y,1}(j) \neq 0\},$$

we have by taking complements that

$$\{B_{y,1}(j) = 0\} \subseteq \{R_{x,j} < H_{\partial B_d(y, h(R))} \circ \theta_{D_{x,j}}\} \cup \{D_{x,j+1} < H_{\partial B_d(y, h(r))} \circ \theta_{R_{x,j}}\}.$$
Hence,\[
\sup_{u \in \partial B_d(x,h(R))} \mathbb{P}^u (B_{y,1} (j) = 0) \leq \sup_{u \in \partial B_d(x,h(R))} \mathbb{P}^u (H_{\partial B_d(x,h(r))} < H_{\partial B_d(y,h(R))})
\]
\begin{equation}
(5.12)
\end{equation}
\[
+ \sup_{v \in \partial B_d(x,h(r))} \mathbb{P}^v (H_{\partial B_d(x,h(r))} < H_{\partial B_d(y,h(r))}) .
\]

Further, with \( d = d(x,y) \), we have that \( B_d(y,h(R)) \subseteq B_d(x,h(R) + d) \) and therefore, \( H_{\partial B_d(y,h(R))} < H_{\partial B_d(x,h(R)+d)} \) for any path starting at \( u \in \partial B_d(x,h(R)) \cap B_d(y,h(R)) \). On the other hand, for \( u \in \partial B_d(x,h(R)) \cap B_d(y,h(R))^c \) we have that \( H_{\partial B_d(y,h(R))} < H_{\partial B_d(x,h(r))} \) by (5.8). All in all, we obtain
\[
\sup_{u \in \partial B_d(x,h(R))} \mathbb{P}^u (H_{\partial B_d(x,h(r))} < H_{\partial B_d(y,h(R))})
\]
\begin{equation}
(5.13)
\end{equation}
\[
\leq \sup_{u \in \partial B_d(x,h(R))} \mathbb{P}^u (H_{\partial B_d(x,h(r))} < H_{\partial B_d(x,h(R)+d)}).
\]

Similarly, since \( B_d(x,h(r) - d) \subseteq B_d(y,h(r)) \), we have
\[
\sup_{v \in \partial B_d(x,h(r))} \mathbb{P}^v (H_{\partial B_d(x,h(r))} < H_{\partial B_d(y,h(r))})
\]
\begin{equation}
(5.14)
\end{equation}
\[
\leq \sup_{u \in \partial B_d(x,h(R))} \mathbb{P}^u (H_{\partial B_d(x,h(r))} < H_{\partial B_d(x,h(r) - d)}).
\]

By (2.7) and (5.11), we have that the right sides of (5.13) and (5.14) are bounded by \( c_1 q \), for some \( c_1 = c_1(a,b) \) (which is fixed in what follows). Combining this with (5.12), it follows that
\[
\sup_{u \in \partial B_d(x,h(R))} \mathbb{P}^u (B_{y,1} (j) = 0) \leq c_1 q .
\]

Therefore, if we set
\[
1 - p_1 := \sup_{r < R < r_0} \sup_{v \in \partial B_d(x,h(R))} \mathbb{P}^u (R_{x,1} < H_{\partial B_d(0,h(R))}) < 1 ,
\]
then the sum (5.11) can be stochastically dominated by \( \chi := \sum_{j=1}^{G-1} I_j \), where \( G, I_i \) are independent, \( G \) is geometric with success probability \( p_1 = p_1(a,b) > 0 \), and \( I_i \) are Bernoulli with success probability \( c_1 q \). Using that \( \mathbb{P}(G - 1 = k) = (1 - p_1)^k p_1 \leq e^{-k p_1} \) we have that
\[
E(e^{\lambda \chi}) = \frac{p_1}{1 - (1 - p_1)(1 - c_1 q + c_1 q e^{\lambda})} \leq 4
\]
if one assumes that \( \lambda > 0 \) and \( c_1 q e^\lambda = p_1/4 \) (which requires that \( q_1 \leq p_1/(4c_1) \)). Using Chebycheff's inequality one then obtains that
\[
P(\sum_{j=1}^{G-1} I_j \geq k) \leq 4 e^{-\lambda k} = 4 \left( \frac{4c_1 q}{p_1} \right)^k \leq c_2 q e^{-2k},
\]
with \( c_2 = 16c_1 e^2/p_1 \) if one assumes that \( q < q_1(a,b) \) with \( 4c_1 q_1/p_1 < e^{-2} \). \( \square \)
Corollary 5.3. For some $c_4 < \infty$, with quantifiers as in Lemma 5.2, if $J$ is a geometric random variable with success parameter $p_2 > 0$, independent of $\{A_{x,i} - B_{y,i}\}$, then for $\lambda \leq p_2/2$,

\begin{equation}
\sup_{u \in \partial B_d(0,h(\tilde{r}))} \mathbb{E}^u \left( \exp \left( \lambda \sum_{i=1}^{J-1} |(A_{x,i} - B_{y,i})| \right) \right) \leq e^{c_4 q \lambda / p_2}.
\end{equation}

Note the linear in $\lambda$ behavior of the right side in (5.16). This is essentially due to mean of $|A_{x,i} - B_{y,i}|$. We will later need to improve on this and obtain a quadratic in $\lambda$ behavior when considering the same variables without absolute values, see Lemma 5.30.

Proof. We assume throughout that $q < 1/2$. We begin by a moment computation. Let $m \geq 1$. Then, by Lemma 5.2

\begin{equation}
\mathbb{E}^u(\langle A_{x,1} - B_{y,1} \rangle^m) \leq \sum_{k \geq 1} k^m (\mathbb{P}^u(A_{x,1} - B_{y,1} \geq k) + \mathbb{P}^u(B_{y,1} - A_{x,1} \geq k)) \leq 2c_2 \sum_{k \geq 1} k^m q e^{-2k} \leq 2c_2 q m! \sum_{k \geq 1} e^{-k} = c_3 q m!.
\end{equation}

For $m = 0$ we trivially bound the left hand side of (5.17) by 1.

To prove (5.16) we adapt the proof of [4, (8.27)]. For any $m \geq 1$,

\begin{equation}
\mathbb{E}^u\left( \left( \sum_{i=1}^{J-1} |(A_{x,i} - B_{y,i})| \right)^m \right) = \sum_{m_1, m_2, \ldots: \sum m_j = m} \prod_{j=1}^{\infty} \mathbb{P}^u_{m_j} \left( 1_{\{J-1 \geq \sup \{j: m_j \neq 0\} \}} \prod_{j=1}^{\infty} |(A_{x,j} - B_{y,j})|^{m_j} \right),
\end{equation}

so that using the Markov property and (5.17) we get that the last expression is bounded above by

\begin{equation}
m! \sum_{m_1, m_2, \ldots: \sum m_j = m} (1 - p_2)^{\sup \{j: m_j \neq 0\} - 1} \prod_{j=1}^{\infty} (c_3 q)^{1_{\{m_j > 0\}}}
\leq c_3 q m! \sum_{m_1, m_2, \ldots: \sum m_j = m} (1 - p_2)^{\sup \{j: m_j \neq 0\} - 1} = c_3 q \frac{m!}{p_2^{m_2}}.
\end{equation}

As pointed out by the referee, there is a typo in the latter; in the bottom of page 538 and top of page 539, all sums of the form $\sum_{i_1, i_2, \ldots: \sum i_j = k}$ are missing the multiplicative factor $k! / \prod_j i_j!$. With these extra factors, the derivation in [4] gives the result claimed there.
where the last equality follows because

\[ m! \sum_{m_1, m_2, \ldots : \sum m_j = m} (1 - p_2)^{\sup\{ j : m_j \neq 0 \}} \]

\[ = \sum_{m_1, m_2, \ldots : \sum m_j = m} m! \prod_{j=1}^{\infty} (1 - p_2)^{\sum m_j} \]

is the \( m \)-th moment of a geometric sum of standard exponentials, which is itself exponential with mean \( p_2^{-1} \), and for the inequality we assumed that \( q < 1/c_3 \). It follows from (5.19) that

\[ (5.20) \]

\[ \mathbb{E} u \left( \exp \left( \lambda \sum_{i=1}^{J-1} |(A_{x,i} - B_{y,i})| \right) \right) \leq 1 + c_3q \sum_{m \geq 1} \left( \frac{\lambda}{p_2} \right)^m \leq 1 + c_4q \left( \frac{\lambda}{p_2} \right), \]

for \( \lambda \leq p_2/2 \), which proves (5.16).

We can now return to the proof of (5.7). Consider the excursions \( \tilde{X}^i = X_{(\tilde{R}_{0,i} + \cdot) \wedge \tilde{R}_{0,i}} \) of the Brownian motion \( X \) from \( \partial B_d(0, h(\tilde{r})) \) to \( \partial B_d(0, h(\tilde{R})) \). Note that \( Y_i = X_{(\tilde{R}_{0,i} + \cdot) \wedge \tilde{R}_{0,i}} \). The variables \( A_{x,i}, B_{y,i} \) are measurable with respect to \( X_{(\tilde{R}_{0,i} + \cdot) \wedge \tilde{R}_{0,i}} \). The excursions \( X_{(\tilde{R}_{0,i} + \cdot) \wedge \tilde{R}_{0,i}} \) are dependent through the starting and ending points of successive excursions. As in [3, Section 8] we will construct renewal times that give some independence.

Let \( \mu_{x,r} \) denote the uniform measure on \( \partial B_d(x, r) \). For Brownian motion on \( S^2 \) it is clear from symmetry that for any \( x \) and \( r < R \),

\[ (5.21) \]

\[ \mathbb{P}^{\mu_{x,r}} \left( X_{\partial B_d(x, r)} \right) = \mu_{x,r} \left( dw \right), \]

\[ \mathbb{P}^{\mu_{x,R}} \left( X_{\partial B_d(x, R)} \right) = \mu_{x,R} \left( dw \right), \]

In addition, it follows from the rotation invariance of \( \mathcal{I}_i \) that

\[ (5.22) \]

\[ \mathbb{P}^{\mu_{0,h(\tilde{r})}} \left( X_{\partial B_d(0, h(\tilde{r}))} \right) = \mu_{0,h(\tilde{r})} \left( dw \right) \]

This reflects the symmetry of the sphere, and is the main reason why we work with \( M = S^2 \).

We postpone the proof of the following lemma to later in this section.

**Lemma 5.4.** On \( S^2 \) we have

\[ (5.23) \]

\[ \mathbb{E}^{\mu_{0,h(\tilde{r})}} \left( (A_{x,1} - B_{y,1}) \right) = 0. \]

Let

\[ p_3 = \inf_{\mu_{0,h(\tilde{r})}} \mathbb{P}^{\mu} \left( X_{\partial B_d(0, h(\tilde{r}))} \right) \]

By our assumption (5.22) we have that

\[ p_3 \geq a_0 > 0. \]
For \( u \in \partial B_d \left( 0, h(\tilde{R}) \right) \) define the measure,

\[
\nu^i_u (dw) = \frac{\mathbb{P}^u \left( X_{H_{\partial B_d(0,h(\tilde{r}))}} \in dw \mid X_{\wedge H_{\partial B_d(0,h(\tilde{r}))}} \in I_i \right) - p_3 \mu_{0,h(\tilde{r})} (dw)}{1 - p_3}.
\]

(We assume that \( p_3 < 1 \), otherwise we simply take \( \nu^i_u = \mu_{0,h(\tilde{r})} \).) By the definition of \( p_3 \) we have that \( \nu^i_u \geq 0 \) and by construction \( \nu^i_u \) is a probability measure on \( \partial B_d \left( 0, h(\tilde{r}) \right) \). Furthermore, by (5.22), when \( \nu^i_u \) is averaged over \( u \) distributed as \( \mu_{0,h(\tilde{r})} \) we recover \( \mu_{0,h(\tilde{r})} \):

\[
(5.25) \quad \nu^i_{\mu_{0,h(\tilde{r})}} (dw) = \frac{\mathbb{P}^{\mu_{0,h(\tilde{r})}} \left( X_{H_{\partial B_d(0,h(\tilde{r}))}} \in dw \mid X_{\wedge H_{\partial B_d(0,h(\tilde{r}))}} \in I_i \right) - p_3 \mu_{0,h(\tilde{r})} (dw)}{1 - p_3} = \mu_{0,h(\tilde{r})} (dw).
\]

Now construct a sequence \( X^1, X^2, \ldots \) of excursions from \( \partial B_d \left( 0, h(\tilde{r}) \right) \) to \( \partial B_d \left( 0, h(\tilde{R}) \right) \) as follows: Let \( X^1 = X_{\wedge H_{\partial B_d(0,h(\tilde{r}))}} \) under \( \mathbb{P} \), and let \( I_2, I_3, \ldots \), be i.i.d. Bernoulli random variables with success probability \( p_3 \), independent of the Brownian motion \( X \). Then,

If \( I_2 = 1 \), let \( X^2 = X_{\wedge H_{\partial B_d(0,h(\tilde{r}))}} \) under \( \mathbb{P}^{\mu_{0,h(\tilde{r})}} \).

(5.26)

If \( I_2 = 0 \), let \( X^2 = X_{\wedge H_{\partial B_d(0,h(\tilde{r}))}} \) under \( \mathbb{P}^{\nu^1_{\mu_{0,h(\tilde{r})}}} \).

Here we have used the abbreviation \( X^1_\infty = X_{H_{\partial B_d(0,h(\tilde{R}))}} \), which comes from the definition of \( X^1 \). We iterate this construction to get \( X^3, X^4, \ldots \). It follows as in [1] Lemma 8.5 that

\[
(5.27) \quad (X^i)_{i \geq 1} \overset{\text{law}}{=} \left( X_{(\tilde{R}_0,\ldots,\tilde{R}_i) \wedge \tilde{D}_{0,\infty}} \right)_{i \geq 1} \text{ under } \mathbb{P} \left( \cdot \mid Y_i \in I_i, i = 1, \ldots, n \right).
\]

Hence, to bound

\[
\mathbb{P} \left( \sum_{i=1}^n (A_{x,i} - B_{y,i}) \geq \theta \sqrt{(n-1)q} \mid Y_i \in I_i, i = 1, \ldots, n \right)
\]

we may instead bound

\[
\mathbb{P} \left( \sum_{i=1}^n (A_{x,1} - B_{y,1}) (X^i) \geq \theta \sqrt{(n-1)q} \right).
\]

Consider the renewal times

\( J_0 = 1 \) and \( J_{i+1} = \inf \{ j > J_i : I_j = 1 \} \).
We have that
\[(5.28) \quad \left( X^{J_i}, X^{J_i+1}, \ldots, X^{J_i+1-1} \right), i \geq 0, \]
are an independent sequence of vectors of excursions, whose lengths are distributed as a geometric random variable on \( \{1, 2, \ldots\} \) with parameter \( p_3 \). Furthermore, the sequences \((5.28)\) are identically distributed for \( i \geq 1 \).

Note that
\[X^{J_1} = X_{\partial B_d(0, h(\tilde{R}))} \quad \text{under} \quad \mathbb{P}_{\mu_0,h(\tilde{r})}, \]
so that \(X^{\infty} \) has distribution \( \mu_{0,h(\tilde{R})} \). Hence by \((5.25)\) we have
\[P_{\nu_1} X^{J_1} = P_{\mu_0,h(\tilde{r})}. \]
Thus, whether \( I^{J_1+1}_1 = 0 \) or \( I^{J_1+1}_1 = 1 \), we have that
\[X^{J_1+1} = X_{\partial B_d(0, h(\tilde{R}))} \quad \text{under} \quad \mathbb{P}_{\mu_0,h(\tilde{r})}, \]
and this will continue for all \( X^{J_i}, i \geq 1 \). In particular, it follows from \((5.23)\) that
\[(5.29) \quad E \left( \sum_{i=J_1}^{J_2-1} (A_{x,1} - B_{y,1}) (X^i) \right) = 0. \]

This leads to the following improvement on \((5.16)\).

**Lemma 5.5.** If \( \lambda \leq p_3/2 \) then
\[(5.30) \quad E \left( \exp \left( \lambda \sum_{i=J_1}^{J_2-1} (A_{x,1} - B_{y,1}) (X^i) \right) \right) \leq e^{c_4 q (\lambda/p_3)^2}. \]

**Proof.** Using \((5.29)\) for the first moment and bounding the other moments by their absolute values we see that
\[E \left( \exp \left( \lambda \sum_{i=J_1}^{J_2-1} (A_{x,1} - B_{y,1}) (X^i) \right) \right) \leq 1 + \sum_{m \geq 2} \frac{\lambda^m}{m!} E \left( \left( \sum_{i=J_1}^{J_2-1} |(A_{x,1} - B_{y,1}) (X^i)| \right)^m \right). \]
As in \((5.20)\) this is bounded by
\[1 + c_3 q \sum_{m \geq 2} (\lambda/p_3)^m \leq 1 + c_4 q (\lambda/p_3)^2, \]
for \( \lambda \leq p_3/2 \) and \((5.30)\) follows. \( \square \)

Let \( U_n = \sup \{ i > 1 : J_i \leq n \} \), the number of renewals up till time \( n \). We have the upper bound
\[(5.31) \quad \sum_{i=1}^{n} (A_{x,1} - B_{y,1}) (X^i) \leq \sum_{i=1}^{J_1-1} |(A_{x,1} - B_{y,1}) (X^i)| \]
\[+ \sum_{i=J_1}^{J_{U_n}-1} (A_{x,1} - B_{y,1}) (X^i) + \sum_{i=J_{U_n}}^{J_{U_n+1}-1} |(A_{x,1} - B_{y,1}) (X^i)|. \]
Set \( n' = n - 1 \). Since \( U_n = \text{Bin}(n', p_3) \), there exists a constant \( \tilde{c} \) (independent of all other parameters) so that

\[
\begin{align*}
P(U_n \geq n'p_3 (1 + \delta)) & \leq e^{-\delta^2 n'}, \\
P(U_n \leq n'p_3 (1 - \delta)) & \leq e^{-\delta^2 n'}.
\end{align*}
\]

Let

\[
\delta = \sqrt{\frac{1}{cn'}\theta} \leq \frac{c_0}{\sqrt{c}} \leq 1/2,
\]

by the assumptions on \( \theta \), see the statement of Lemma 5.1, after taking \( c_0 \) sufficiently small. With \( u_1 = n'p_3 (1 - \delta) \) and \( u_2 = n'p_3 (1 + \delta) - 1 \) let

\[
\Phi = \sum_{i=1}^{J_1-1} |(A_{x,1} - B_{y,1})(X^i)| + \sum_{i=J_1}^{J_2-1} (A_{x,1} - B_{y,1})(X^i) + \sum_{i=J_2}^{u_2-1} |(A_{x,1} - B_{y,1})(X^i)|.
\]

By (5.32) we have

\[
P \left( \sum_{i=1}^{n} (A_{x,1} - B_{y,1})(X^i) \geq \theta \sqrt{n'q} \right) \leq P \left( \Phi \geq \theta \sqrt{n'q} \right) + 2 \exp(-\theta^2).
\]

Using the independence properties of the sequences (5.28) we have

\[
E(e^{\lambda \Phi}) = E \left( \exp \left( \lambda \sum_{i=1}^{J_1-1} |(A_{x,1} - B_{y,1})(X^i)| \right) \right) \times \prod_{j=1}^{J_1-1} E \left( \exp \left( \lambda \sum_{i=J_j}^{J_{j+1}-1} (A_{x,1} - B_{y,1})(X^i) \right) \right) \times \prod_{j=J_1}^{u_2-1} E \left( \exp \left( \lambda \sum_{i=J_j}^{J_{j+1}-1} |(A_{x,1} - B_{y,1})(X^i)| \right) \right).
\]

It follows that for all \( \lambda > 0 \),

\[
E(e^{\lambda \Phi}) \leq \left( E \left( \exp \left( \lambda \sum_{i=J_1}^{J_2-1} (A_{x,1} - B_{y,1})(X^i) \right) \right) \right)^{u_1-1} \times \left( \sup_{u \in \partial B_d(0, h(\tilde{c}))} E_u \left( \exp \left( \lambda \sum_{i=1}^{J_1} |A_{x,1} - B_{y,1}|(X^i) \right) \right) \right)^{u_2-u_1+1} =: A_1 \times A_2.
\]

Using (5.16), the definition of \( u_1, u_2 \) and then (5.33) we have

\[
A_2 \leq \exp(c_4q\lambda(u_2 - u_1 + 1)/p_3) = \exp(2c_4q\lambda n'\delta) = \exp \left( 2c_4q\lambda \sqrt{n'}/\tilde{c}\theta \right),
\]
for \( \lambda \leq p_3/2 \). For such \( \lambda \), using (5.30),
\[
A_1 \leq \exp \left( c_4 q (\lambda/p_3)^2 u_1 \right) \leq \exp \left( c_4 q \lambda^2 n'/p_3 \right).
\]
Thus we get that for \( \lambda \leq p_3/2 \),
\[
(5.39) \quad \mathbb{P} \left( \Phi \geq \sqrt{n'q} \right) \leq \exp \left( c_4 q \lambda^2 n'/p_3 + 2c_4 q \lambda \sqrt{n'/\tilde{c}} \theta - \lambda \theta \sqrt{n'q} \right).
\]
If \( \theta \leq (p_3/4C_0) \sqrt{n'} \) we set
\[
\lambda = 2C_0 \frac{\theta}{\sqrt{n'}},
\]
and conclude that
\[
(5.40) \quad \mathbb{P} \left( \Phi \geq \sqrt{n'q} \right) \leq \exp \left( 2C_0 \theta^2 \sqrt{q} \left( 2c_4 C_0 \sqrt{q}/p_3 + 2c_4 \sqrt{q}/\tilde{c} - 1 \right) \right),
\]
which together with (5.35) gives (5.7) for \( q \) sufficiently small.

The next lemma is a variant of Lemma 5.1, allowing one to consider larger values of \( \theta \) (in particular, exhibiting the transition from Gaussian moderate deviations to exponential large deviations). It is useful in the proof of Lemma 5.1.

**Lemma 5.6.** Fix \( 0 < a < b < 1 \) and \( 0 < a_0 < 1 \). Let \( r < R < \tilde{r} < \tilde{R} \) so that (5.2) and (5.11) hold. Let \( x, y \) be such that \( B_d(x, h(R)) \) and \( B_d(y, h(\tilde{R})) \) hold. Then for any \( C_0 < \infty \) there exist small \( c_0, q_0 > 0 \) and \( c_1 > 0 \), depending on \( a, b, C_0 \) only, such that if \( q = d(x, y)/R \leq q_0 \), \( c_1 \leq \theta \leq c_0(n-1) \), and \( \theta \leq (n-1)q^2 \), then
\[
\mathbb{P} \left( \left| T^{0,\tilde{r} \to \tilde{R}}_{x,R \to r} - T^{0,\tilde{r} \to \tilde{R}}_{y,R \to r} \right| \geq \theta \sqrt{q(n-1)} \mid Y_m \in \mathcal{I}_m, m = 1, \ldots, n \right) \leq \exp (-C_0 \theta).
\]

**Proof.** Let \( n' = n - 1 \). We return to (5.33) but with \( \tilde{c} \) as in (5.32) we now take
\[
(5.42) \quad \delta = \sqrt{\frac{C_0 \theta}{cn'}} \leq \sqrt{\frac{c_0 C_0}{\tilde{c}}} \leq 1/2,
\]
by our assumptions on \( \theta \), and after taking \( c_0 \) sufficiently small. Then instead of (5.35) we obtain
\[
(5.43) \quad \mathbb{P} \left( \sum_{i=1}^n (A_{x,i} - B_{y,i}) (X_i) \geq \theta \sqrt{n'q} \right) \leq \mathbb{P} \left( \Phi \geq \theta \sqrt{n'q} \right) + 2 \exp (-C_0 \theta).
\]
With this choice of \( \delta \), instead of (5.38) we obtain
\[
A_2 \leq \exp \left( 2c_4 q \lambda n' \delta \right) = \exp \left( 2c_4 q \lambda \sqrt{n'q}/\tilde{c} \right),
\]
for \( \lambda \leq p_3/2 \). Thus, instead of (5.39) we see that for \( \lambda \leq p_3/2 \),
\[
\mathbb{P} \left( \Phi \geq \theta \sqrt{n'q} \right) \leq \exp \left( c_4 q \lambda^2 n'/p_3 + 2c_4 q \lambda \sqrt{C_0 n'/\tilde{c}} - \lambda \theta \sqrt{n'q} \right).
\]
If \( \theta \leq (n'q)^2 \) we set \( \lambda = \sqrt[4]{\theta}\sqrt{2} \) and see that

\[
\mathbb{P} \left( \Phi \geq \sqrt{n'q} \right) \\
\leq \exp \left( c_4 \theta^{1/2} p_3 / 4 + c_4 \sqrt{\theta^{3/4} C_0 / \theta^{5/4} p_3 / 2} \right) \leq \exp (-C_0 \theta),
\]

for \( q_0 \) sufficiently small and all \( \theta \geq c_1 \) sufficiently large.

Before proceeding to the proof of Lemma 5.4 we state a preliminary Lemma. Let

\[
\kappa_{a,b} = \mathbb{E}^u \left( H_{\partial B_d(x,b)} \right) + \mathbb{E}^v \left( H_{\partial B_d(x,a)} \right), \quad u \in \partial B_d(x,a), v \in \partial B_d(x,b).
\]

By symmetry, \( \kappa_{a,b} \) does not depend on \( x, u \) or \( v \).

**Lemma 5.7.** On \( S^2 \), for all \( 0 < a < b < \pi \) we have that

\[
\kappa_{a,b} = 4 \log \left( \frac{\tan(b/2)}{\tan(a/2)} \right).
\]

In addition, for any \( x \in S^2 \) and \( 0 < b < \pi \),

\[
\sup_{y \in S^2} \mathbb{E}^y \left( H_{\partial B_d(x,b)} \right) < \infty,
\]

and \( H_{\partial B_d(x,b)} \) has an exponential tail.

**Proof.** By the last formula in [13, Section 3], for \( u \in \partial B_d(x,b) \),

\[
\mathbb{E}^u \left( H_{\partial B_d(x,a)} \right) = 2 \log \left( \frac{1 - \cos(b)}{1 - \cos(a)} \right).
\]

This formula requires \( a < b \). If \( x^* \) denotes the antipode of \( x \in S^2 \) then by symmetry, for \( u \in \partial B_d(x,a) \) and \( v \in \partial B_d(x^*,\pi-a) \)

\[
\mathbb{E}^u \left( H_{\partial B_d(x,b)} \right) = \mathbb{E}^v \left( H_{\partial B_d(x^*,\pi-b)} \right).
\]

Hence using (5.45), for \( u \in \partial B_d(x,a) \),

\[
\mathbb{E}^u \left( H_{\partial B_d(x,b)} \right) = 2 \log \left( \frac{1 - \cos(\pi-a)}{1 - \cos(\pi-b)} \right) = 2 \log \left( \frac{1 + \cos(a)}{1 + \cos(b)} \right).
\]

Then using the half-angle formula for tangents, \( \frac{1-\cos(a)}{1+\cos(a)} = \tan^2(u/2) \) we obtain, using (5.45),

\[
\kappa_{a,b} = 2 \log \left( \frac{1-\cos(b)}{1+\cos(b)} \right) = 2 \log \left( \frac{\tan^2(b/2)}{\tan^2(a/2)} \right),
\]

which gives (5.46).

(5.47) follows from (5.48) and (5.49). By the Kac moment formula [23] this implies that \( H_{\partial B_d(x,b)} \) has an exponential tail.
Proof of Lemma 5.4. Recall, see (5.4), that $\tilde{D}_{0,n}$ is the time until the $n$'th excursion from $\partial B_d(0, h(\tilde{r}))$ to $\partial B_d(0, h(\tilde{R}))$. Let $\tilde{D}_{0,0} = 0$ and note that we can write

$$\tilde{D}_{0,n} = \sum_{i=1}^{n} T_i,$$

where

$$T_i = \left( H_{\partial B_d(0, h(\tilde{R}))} \circ \theta_{H_{\partial B_d(0, h(\tilde{r}))}} + H_{\partial B_d(0, h(\tilde{R}))} \right) \circ \theta_{\tilde{D}_{0,i-1}}.$$

Using the symmetry of the sphere and the Markov property we see that $T_2, T_3, \ldots$ are iid with $E(T_i) = \kappa_{h(\tilde{r}), h(\tilde{R})}$, $i = 2, 3, \ldots$ by (5.45). Hence by the Strong Law of Large numbers

$$\frac{\tilde{D}_{0,n}}{n} \to \kappa_{h(\tilde{r}), h(\tilde{R})} \text{ a.s.} \tag{5.53}$$

Similarly, if $D_{x,m}$ denotes the time until the $m$'th excursion from $\partial B_d(x, h(r))$ to $\partial B_d(x, h(R))$, then for any $x \in S^2$

$$\frac{D_{x,m}}{m} \to \kappa_{h(r), h(R)} \text{ a.s.} \tag{5.54}$$

Recalling Definition 2.1 let

$$V_{x,n} = \sum_{i=1}^{n} A_{x,i} = T_{x,R \to r}^{0,\tilde{r},n} \tilde{R}$$

be the number of traversals from $\partial B_d(x, h(R))$ to $\partial B_d(x, h(r))$ before $\tilde{D}_{0,n}$. Then,

$$D_{x,V_{x,n}} \leq \tilde{D}_{0,n} \leq D_{x,V_{x,n}+1}. \tag{5.55}$$

Hence using (5.53) and (5.54) we see that

$$\kappa_{h(\tilde{r}), h(\tilde{R})} = \lim_{n \to \infty} \frac{\tilde{D}_{0,n}}{n} = \lim_{n \to \infty} \frac{D_{x,V_{x,n}}}{V_{x,n}} \frac{V_{x,n}}{n} = \kappa_{h(r), h(R)} \lim_{n \to \infty} \frac{V_{x,n}}{n}. \tag{5.56}$$

It follows that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A_{x,i} = \kappa_{h(\tilde{r}), h(\tilde{R})}/\kappa_{h(r), h(R)}. \tag{5.57}$$

Since this holds for any $x$ with $\partial B_d(x, h(R)) \subset \partial B_d(0, h(\tilde{r}))$ it follows that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A_{x,i} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} B_{y,i}. \tag{5.58}$$

The increments $A_{x,i} - E_{X_{R0,i}}(A_{x,1}), i = 1, 2, \ldots$, are orthogonal by the strong Markov property, and have bounded second moment by Lemma 5.4.
Therefore Rajchman’s strong law of large numbers, \cite[Theorem 5.1.1]{14}, implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( A_{x,i} - E^{\bar{R}_{0,i}} (A_{x,1}) \right) = 0,$$

with a similar result for $A_{x,i}$ replaced by $B_{y,i}$. It then follows from (5.58) that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E^{\bar{R}_{0,i}} (A_{x,1}) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E^{\bar{R}_{0,i}} (B_{y,1}),$$

and by the Strong Law of Large Numbers for a general state space Markov Chain applied to the chain $X_{\bar{R}_{0,i}}$ in $\partial B_{d} (0, h(\tilde{r}))$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E^{\bar{R}_{0,i}} (A_{x,1}) = E^{\mu_{0, (\tilde{r})}} (A_{x,1}),$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E^{\bar{R}_{0,i}} (B_{y,1}) = E^{\mu_{0, (\tilde{r})}} (B_{y,1}).$$

This proves (5.23). \hfill \Box

We turn now to the proof of Lemma 3.11.

**Proof of Lemma 3.11.** This is a direct application of Lemma 5.6, taking $\tilde{R} = r_{i-3}$, $\tilde{r} = r_{i-2}$, $\tilde{R} = \tilde{r}_{l-1}$, $\tilde{r} = \tilde{r}_{l-1}$, $r = r_{l}$, $x = y$, $y = \tilde{y}$, $n = u^{2}/2$ with $u \in I_{\alpha(l) + j}$, and $\theta = d_{0}jm/2$. To apply the lemma, we must verify several points.

First, by taking $\tilde{y}_{0}$ sufficiently large we will have $\theta > c_{1}$. Next we need to verify that $\theta \leq c_{0}(n-1)$. By halving $c_{0}$ it suffices to show that $\theta \leq c_{0}n$ which is $d_{0}jm/2 \leq c_{0}(\alpha(l) + j)^{2}/2$. For this it is enough to show that $d_{0}jm/2 \leq c_{0}(\alpha(l) + j)/2$, which follows from the fact that $m \leq k - l = \log(2(\alpha(l) + j))$, see (3.58), and taking $\tilde{y}_{0}$ sufficiently large.

Secondly, we need to show that $\theta \leq ((n-1)q)^{2}$. Since we have already seen that $\theta \leq c_{0}(n-1)$, it suffices to show that $(n-1)q^{2} \geq c_{2}^{2}$ for some $c_{2} > 0$, or equivalently that $\sqrt{2n} \, q \geq c_{2} > 0$. That is, $(\alpha(l) + j) d(\tilde{y}, y)/r_{l} \geq c_{2} > 0$. Assume that $d(\tilde{y}, y) \geq c_{3}r_{k}$ for a small $c_{3} > 0$, so that, see (3.58),

$$(\alpha(l) + j) d(\tilde{y}, y)/r_{l} \geq c_{3} (\alpha(l) + j) e^{-(k-l)} = c_{3}/2.$$  

Recall that $\tilde{y}$ is the “parent” of $y$ defined in the paragraph following (3.66). With our construction of $F_{1}$, either $d(\tilde{y}, y) \geq c_{3}r_{k}$ for a small universal $c_{3} > 0$, or $y = \tilde{y}$, in which case the corresponding term in the sum in (3.71) is zero. Also, because $\tilde{c} = q_{0}/2$, see (3.2), we will have $d(\tilde{y}, y)/r_{l} \leq q_{0}$. \hfill \Box
6. Excursion time and real time

We prove in this section Theorem 3.1, which is used to control the relation between excursion counts and real time, and compare various excursions with different centering. It was crucially used in the proof of the upper bound.

Recall that for $0 < a < b < \pi$, $\tau_{x,a,b}(n)$ is the time needed to complete $n$ excursions in $S^2$ from $\partial B_d(x,a)$ to $\partial B_d(x,b)$, see (2.15). Recall $\kappa_{a,b}$ from (5.45)-(5.46).

Lemma 6.1. For any $0 < a < b < \pi$ there exists a $c = c(a,b)$ such that for $\delta \in (0,1)$ and $x \in S^2$,

(6.1) \[ P(\tau_{x,a,b}(n) \leq (1 - \delta)\kappa_{a,b}n) \leq e^{-c\delta^2 n} \]

and

(6.2) \[ P(\tau_{x,a,b}(n) \geq (1 + \delta)\kappa_{a,b}n) \leq e^{-c\delta^2 n}. \]

Proof. By symmetry we can take $x = 0$, and then as in the first paragraph of the proof of Lemma 5.4, $\tau_{x,a,b}(n) = \sum_{i=1}^{n} T_i$ where the $T_2, T_3, \ldots$ are iid with $E(T_i) = \kappa_{a,b}$, $i = 2, 3, \ldots$ and all $T_i$ have exponential tails. $\square$

Throughout this section we assume that $z_0 \leq z \leq L^{1/2} \log^2 L$. Recall the notation $s(z)$, see (2.16). It follows from Lemma 6.1 with $n = s(z)$ and $\delta = d\sqrt{z}/(2L)$ that for some $d < \infty$,

(6.3) \[ P(s(z - d\sqrt{z}) \kappa_{a,b} \leq \tau_{x,a,b}(s(z)) \leq s(z + d\sqrt{z}) \kappa_{a,b}) \geq 1 - e^{-10z}, \]

uniformly in $x \in S^2$.

It follows from (6.3), that uniformly in $x \in S^2$,

(6.4) \[ P(4s(z - d\sqrt{z}) \log(b/a) \leq \tau_{x,h(a),h(b)}(s(z)) \leq 4s(z + d\sqrt{z}) \log(b/a)) \geq 1 - e^{-10z}. \]

This is like Theorem 3.1, except it applies to one $x$ and not to all $x \in F_L$ simultaneously. We can not derive Theorem 3.1 from (6.4) via a union bound over all $x \in F_L$, since there are far too many elements in $F_L$. To reduce the number of $x$ that need to be considered we will use a chaining argument, contained in Lemma 6.2 below. Before stating it, we set notation. Throughout the argument, we fix $r_0$ small, e.g. $r_0 < 10^{-6}$. Then we define $\tilde{r}_1 < \tilde{r}_0 < \tilde{r}_{-1} < \tilde{r}_{-2}$, by

$\tilde{r}_0 = r_0 + \frac{1}{L}$ and $\tilde{r}_1 = r_1 - \frac{1}{L},$

$\tilde{r}_{-1} = 2\tilde{r}_0$ and $\tilde{r}_{-2} = 2\tilde{r}_0 \times \left(\frac{\tilde{r}_0}{\tilde{r}_1}\right)$. It follows from (6.4) that for some $d < \infty$

(6.5) \[ P(4s(z - d\sqrt{z}) \leq \tau_{0,h(\tilde{r}_{-1}),h(\tilde{r}_{-2})}(s(z)) \leq 4s(z + d\sqrt{z})) \geq 1 - e^{-10z}, \]
and
\[ \mathbb{P}(4s(z - d\sqrt{z}) \leq \tau_{0,h(\hat{r}_1),h(\tilde{r}_0)}(s(z)) \leq 4s(z + d\sqrt{z})) \geq 1 - e^{-10z}. \]

Here we used the fact that
\[ \log \frac{\tilde{r}_2}{\tilde{r}_1} = \log \frac{\tilde{r}_0}{\tilde{r}_1} = \log \frac{r_0 + \frac{1}{2}}{r_1 - \frac{1}{2}} = \log \left[ \frac{r_0}{r_1} \left( 1 + O \left( \frac{1}{\tilde{r}_1} \right) \right) \right] = 1 + O \left( \frac{1}{\tilde{r}_1} \right), \]
so that
\[ s(z \pm d\sqrt{z}) \log \frac{\tilde{r}_0}{\tilde{r}_1} = L \left( 2L - \log L + z \pm d\sqrt{z} + O(1) \right). \]

Recall Definition 2.1 and abbreviate \( T_{x,\tilde{r}_1}^{x,\tilde{r}_1,n} = T_{y,\tilde{r}_0}^{x,\tilde{r}_1,n+1} \), the number of traversals from \( \partial B_d(y, h(\tilde{r}_0)) \) to \( \partial B_d(y, h(\tilde{r}_1)) \) during \( n \) excursions from \( \partial B_d(x, h(\tilde{r}_1)) \to \partial B_d(x, h(\tilde{r}_2)) \). Once again, possibly enlarging \( d \), it follows from (6.5) and (6.6) that
\[ P \left( s(z - d\sqrt{z}) \leq T_{0,\tilde{r}_1}^{0,\tilde{r}_1,s(z)} \leq s(z + d\sqrt{z}) \right) \geq 1 - ce^{-5z}. \]

Let \( a_0 = \pi^2/3 \). The following lemma uses a chaining argument in its proof.

**Lemma 6.2.** There exist constants \( \bar{c}, z_0 < \infty \) such that for \( L \) sufficiently large and all \( 0 \leq z \leq L^{1/2} \log^2 L \),
\[ \mathbb{P} \left[ \exists y \in F_{\frac{1}{2} \log L} \cap B_d(0, \bar{c} h(\tilde{r}_0)) \right. \]
\[ \left. \text{s.t. } \left| T_{0,\tilde{r}_1}^{0,\tilde{r}_1,s(z)} - T_{y,\tilde{r}_1}^{0,\tilde{r}_1,s(z)} \right| \geq a_0 \sqrt{z} L \right] \leq ce^{-4z}. \]

**Proof of Lemma 6.2.** We use Lemma 5.1 from Section 5. Taking \( R = \tilde{r}_0 \), \( r = \tilde{r}_1 \), \( \tilde{R} = \tilde{r}_2 \), \( \tilde{r} = \tilde{r}_1 \) and \( n = s(z) \), the lemma shows that for any \( C_0 > 0 \) there exist small \( c_0, q_0 > 0 \) such that if \( q = d(x,y)/\tilde{r}_0 \leq q_0 \), and \( \theta \leq c_0 \sqrt{(n - 1)} \),
\[ \mathbb{P} \left[ \left| T_{x,\tilde{r}_1}^{0,\tilde{r}_1,s(z)} - T_{y,\tilde{r}_1}^{0,\tilde{r}_1,s(z)} \right| \geq \theta \sqrt{(n - 1)q} \right] \leq \exp \left( -C_0 \theta^2 \sqrt{q} \right), \]
where \( n = s(z) \sim 2L^2 \). Recalling \( \bar{c} \) from (5.2), we see that for any \( x, y \in B_d(0, \bar{c} h(\tilde{r}_0)) \), we have \( q = d(x,y)/\tilde{r}_0 \leq q_0 \) and \( B_d(x, h(\tilde{r}_0)), B_d(y, h(\tilde{r}_0)) \subseteq B_d(0, h(\tilde{r}_1)) \). Furthermore, since in our present application of Lemma 5.1 there are no \( I_i \)’s, condition (5.2) is easy to verify: just use the Poisson kernel (2.8) and the fact that the outside of a circle centered at the south pole is the inside of a circle centered at the north pole.

As is standard in continuity estimates, we use a chaining, that is, we construct a tree of points that are embedded in \( S^2 \), and “cover” \( B_d(0, \bar{c} h(\tilde{r}_0)) \) in the sense that such that the \( k \)th level of the tree has size \( 256^k \) and the largest distance from any point in \( B_d(0, \bar{c} h(\tilde{r}_0)) \) to a point in the \( k \)th level is at most \( 16^{-k} \). We further require that the last (i.e. \( k = 3 \log L/(2 \log 16) \)) level of the tree contains \( F_{\frac{1}{2} \log L} \cap B_d(0, \bar{c} h(\tilde{r}_0)) \). An explicit construction of the tree is obtained by choosing for the \( k \)th level an arbitrary net \( N_k \) of \( 256^k \)
Proof of Theorem 3.1. For any $x \in \mathcal{N}_k$ to be the element of $\mathcal{N}_{k-1}$ closest to it, with ties broken e.g. by lexicographic order, and $N_0 = 0$. Since the spacing of $F_{1/2 \log L}$ is $1/L^{3/2}$, we can easily choose the last level to satisfy the constraint.

Using this tree, we can connect a point $y \in F_{1/2 \log L}$ to 0 by a unique geodesic in the tree. This allows us to bound the left hand side of (6.10) by

$$\sum_{k=0}^{2 \log L \log 16} 256^k \sup_{x,y \in B_d(0, \tilde{c} \tilde{h}(	ilde{r}_0))}, \frac{d(x,y)}{zL} \geq 0. \quad \text{Since the spacing of} \quad zL$$

Using this tree, we can connect a point $\theta$ that (6.12) is at most $\log 16$. This gives the claim (6.10).

Thus (6.13) will hold as long as $z \neq 0. \quad \text{Thus we need}$

(6.13) $2\sqrt{zL} \leq c_0(k+1)^2(n-1)\sqrt{q}$.

But (6.14) $c_0(n-1)\sqrt{q} \geq c_0L^2\sqrt{d(x,y)/\tilde{r}_0} \geq c_0L^24^{-k}$

and

(6.15) $(k+1)^24^{-k} \geq c \log^2(L)4^{-1/2 \log L \log 16} = c \log^2(L)L^{-3/4}$.

Thus (6.13) will hold as long as $z \leq cL^{1/2} \log^4(L)$.

Clearly we can find $c_1$ such that $c_14^k(k+1)^{-4} \geq 2(k+1)$ for all $k \geq 0$. Then, using the fact that $n \leq 2L^2$, we can choose $C_0$ sufficiently large so that (6.12) is at most

$$\sum_{k=0}^{\infty} 256^k e^{-C_0 \left( \frac{2\sqrt{zL}(k+1)^{-2}}{\sqrt{n-1}q} \right)^2} \leq \sum_{k=0}^{\infty} 256^k e^{-c_14z(k+1)^{-4} \frac{\log L}{2x4^{-k}}}$$

(6.16) $= \sum_{k=0}^{\infty} 256^k e^{-4z(k+1)} \leq ce^{-4z},$

as long as $z_0 > \log 16$. This gives the claim (6.10). \hfill \Box

We have completed all preparatory steps and can turn to our main goal.

Proof of Theorem 3.1. For any $x \in F_L \cap B_d(0, \tilde{c} \tilde{h}(	ilde{r}_0))$, there exists a $y \in F_{1/2 \log L}$ such that

(6.17) $B_d(y, h(\tilde{r}_1)) \subset B_d(x, h(r_1)) \subset B_d(x, h(r_0)) \subset B_d(y, h(\tilde{r}_0)),$
since the “spacing” of $F_{\frac{2}{3} \log L}$ is $\frac{1}{L^{3/2}}$ and
\[
\frac{1}{L^{3/2}} + h(\bar{r}_1) \leq h(r_1) \quad \text{and} \quad h(r_0) + \frac{1}{L^{3/2}} \leq h(\bar{r}_0).
\]

It follows from (6.17), (6.10) and (6.9) that
\[
\mathbb{P} \left[ T_{x,\bar{r}_1}^{0,-1, s(z)} \geq s \left( z - d\sqrt{z} - a_0\sqrt{z}L \right) \right] \geq 1 - ce^{-4z}.
\]

Since the event $T_{x,\bar{r}_1}^{0,-1, s(z)} \geq s \left( z - d\sqrt{z} - a_0\sqrt{z}L \right)$ implies that
\[
\tau_x(s(z - (d + a_0)\sqrt{z})) \leq \tau_{0,h(\bar{r}_1),h(\bar{r}_2)}(s(z)),
\]
it then follows from (6.15) that for $d' = 2d + a_0$
\[
P \left( \tau_x(s(z)) \leq 4s \left( z + d'\sqrt{z} \right) \right) \geq 1 - ce^{-4z}.
\]
Since we can cover $S^2$ by a finite number of discs of radius $\tilde{c} \, h(\bar{r}_0)$ this implies that
\[
P \left( \tau_x(s(z)) \leq 4s \left( z + d'\sqrt{z} \right) \right) \geq 1 - ce^{-4z},
\]
To get the other direction of (3.4) we proceed as above but taking now $\bar{r}_0 = r_0 - \frac{1}{L}$ and $\bar{r}_1 = r_1 + \frac{1}{L}$.

We had to work hard, using the full force of the continuity estimates, to obtain Theorem 3.4. If we restrict to $x,y$ very close, as in Theorem 3.8, the situation is much simpler.

**Proof of Theorem 3.8.** Let $x, y$ be such that $d(x,y) \leq \Delta := h(r_{L/2}) \sim e^{-L/2}$. By (5.15) the probability that an excursion from $\partial B_d(x, h_0)$ to $\partial B_d(x, h_1)$ and back does not contain an excursion from $\partial B_d(y, h_0)$ to $\partial B_d(y, h_1)$ is $O(\Delta)$. Hence during $2L^2$ excursions from $\partial B_d(x, h_1)$ to $\partial B_d(x, h_0)$ the number of ‘missed opportunities’ for excursions from $\partial B_d(y, h_1)$ to $\partial B_d(y, h_0)$ is bounded by $\text{Bin}(2L^2, c\Delta)$. Since this has mean $\lambda = cL^2 \Delta \to 0$ as $L \to \infty$ we can use the Poisson approximation to bound
\[
P(\tau_x(t_z) < \tau_y(t_z - 10)) \leq c^{10} \lambda L^{20} e^{-5L}.
\]
Since there are $ce^{4L}$ pairs $x,y \in F_L$, (3.35) follows.

7. **Proof of Corollary 1.3 from $S^2$ to $R^2$**

The starting point for the proof is the following lemma concerning Brownian motion on $S^2$. We use $s$ to denote the south pole of $S^2$ and $*$ to denote the north pole. Set $\Lambda = \rho L$, see (1.14), with $L = \log(1/\epsilon)$. The next lemma follows by combining Theorem 1.11 for $B_d(s, h_2) \subseteq S^2$ with (6.14).

**Lemma 7.1.** Fix $t > 0$. Let $N_L$ be the number of excursions between $\partial B_d(s, h_1)$ and $\partial B_d(s, h_0)$ needed before $B_d(s,t)$ is $h_L$-covered. Then $\sqrt{2N_L - \lambda_L}$ is tight.
We next use stereographic projection $\sigma : S^2 \setminus \{\ast\} \to \mathbb{R}^2$. With $X_t$ Brownian motion on the sphere, let $W_t = \sigma(X_t)$. Because $\sigma$ provides a system of isothermal coordinates in $\mathbb{R}^2$, we have that $W_t$ is a time-changed version of standard Brownian motion in $\mathbb{R}^2$.

Let $N^P_L$ be the number of excursions between $\partial B_e(0, r_1)$ and $\partial B_e(0, r_0)$ needed before $B_e(0, 1)$ is $r_L$-covered. Since balls of radius $h_L$ in $B_d(s, h(1))$ are of Euclidean radius in $[r_L/c, cr_L]$ for some fixed $c$, we have from the lemma, simple inclusion and the fact that $|\lambda_{L+n} - \lambda_L| \leq cn$ for $L \gg n$, that $\sqrt{2N^P_L - \lambda_L}$ is tight.

Now note that $\sum_{k=1}^{N^P_L-1} S_k \leq C_{*}^{\varepsilon, P, R} \leq \sum_{k=1}^{N^P_L} S_k$ where $S_k$ are the times spent by planar Brownian motion inside $B_e(0, R)$ during an excursion starting at $\partial B_e(0, r_1)$, visiting $\partial B_e(0, r_0)$, and then returning to $\partial B_e(0, r_1)$. The $S_k$ are, by rotational invariance, i.i.d., of mean $a_R$, and with exponential tails. We immediately conclude that

$$\sqrt{C_{*}^{\varepsilon, P, R} - \sqrt{a_R \lambda_L}} \leq \frac{\lambda_L}{\sqrt{2}}$$

is tight.

It only remains to compute $a_R$. By Doob’s stopping theorem, the time to hit radius $r_0$ when starting at $r_1$ is $(r_0^2 - r_1^2)/2$. On the other hand, with $\tau_1$ denoting the hitting time of $\partial B_e(0, r_1)$,

$$u(x) = E_x\left(\int_{0}^{\tau_1} 1_{|W_t| \leq R} dt\right)$$

satisfies the boundary value problem

$$\frac{1}{2}(u''(r) + \frac{u'(r)}{r}) = -1_{r \leq R}, \quad u(r_1) = 0, u'(R) = 0,$$

with solution

$$u(r) = \frac{r_1^2}{2} + R^2 \log \left(\frac{r}{r_1}\right) - \frac{r^2}{2}, \quad r \leq R.$$ 

In particular, summing $u(r_0)$ with $(r_0^2 - r_1^2)/2$ gives $a_R = R^2$, which completes the proof.

8. Appendix I: Barrier estimates

Recall that $\alpha(l) = \rho_L(L - l) - l_L^2, \gamma < 1/2$, see (1.14) and (1.18), where

$$(8.1) \quad \rho_L = 2 - \frac{\log L}{2L},$$

and recall that

$$(8.2) \quad t_z = \frac{(\rho_L L + z)^2}{2}.$$ 

Recall the notation $I_y = [y, y + 1)$. (Note that in [5], $I_y$ is denoted $H_y$.)
Lemma 8.1. For any $k < L$ and all $j \geq 0$ and $0 \leq z \leq 100k$ with $t_z \in \mathbb{N}$,
\begin{equation}
K_1 := \mathbb{P} \left( \alpha(l) \leq \sqrt{2T_l^{0,t_z}}, \ l = 1, \ldots, k-1; \ \sqrt{2T_k^{0,t_z}} \in I_{\alpha(k+j)} \right)
\end{equation}
\begin{equation}
\leq ce^{-2k-2z-2k_L^\gamma + 2j} \times (1 + z + k_L^\gamma) (1 + j) e^{\frac{(z+k_L^\gamma-j)^2}{4k}}.
\end{equation}

Proof of Lemma 8.1. Recall that we denote by $T_l, l = 0, 1, 2, \ldots$ the critical Galton–Watson process with geometric distribution, and let $P_n^{GW}$ denote its law when $T_0 = n$. With $v = \rho_L(L-k) - k_L^\gamma$ we rewrite $K_1$ as
\begin{equation}
K_1 = P_n^{GW} \left( \rho_L(L-l) - l_L^\gamma \leq \sqrt{2T_l} \text{ for } l = 1, \ldots, k-1; \ \sqrt{2T_k} \in I_{v+j} \right).
\end{equation}

We show below that for any $0 < \gamma < 1$, and all $1 \leq l \leq k-1$
\begin{equation}
l_L^\gamma \leq k_L^\gamma + l_k^\gamma,
\end{equation}
from which it follows that
\begin{equation}
\rho_L(L-l) - l_L^\gamma \geq \rho_L(L-k) - k_L^\gamma + \rho_L(k-l) - l_k^\gamma.
\end{equation}
Hence
\begin{equation}
K_1 \leq P_n^{GW} \left( v + \rho_L(k-l) - l_L^\gamma \leq \sqrt{2T_k}, l = 1, \ldots, k-1; \ \sqrt{2T_k} \in I_{v+j} \right),
\end{equation}
and by (8.2), $\sqrt{2T_k} = \rho_L(L+z) = \rho_L(k+v+z+k_L^\gamma)$.

Thus using [5] Theorem 1.1, with $a = \rho_L k + v, x = \sqrt{2T_z}$ and $b = v, y = v+j$,
\begin{equation}
K_1 \leq K_2 = ce^{-2k-2z-2k_L^\gamma + 2j} \times (1 + z + k_L^\gamma) (1 + j) e^{-\frac{(z+k_L^\gamma-j)^2}{4k}}.
\end{equation}

We write out
\begin{equation}
(\rho_L k + z + k_L^\gamma - j)^2 / 2k
\end{equation}
\begin{equation}
= 2k + (z + k_L^\gamma - j) - k \log(L) / 2L)^2 / 2k
\end{equation}
\begin{equation}
= 2k + 2(z + k_L^\gamma - j) - k \log(L) / L + ((z + k_L^\gamma - j) - k \log(L) / 2L)^2 / 2k.
\end{equation}

Using the concavity of the logarithm, we have $k \log(L) / L \leq \log k$, and using $(r - s)^2 \geq r^2 / 2 - s^2$ we see that
\begin{equation}
((z + k_L^\gamma - j) - k \log(L) / 2L)^2 / 2k \geq (z + k_L^\gamma - j)^2 / 4k - \alpha(L)
\end{equation}

It follows that
\begin{equation}
K_2 \leq c \times (1 + z + k_L^\gamma) (1 + j) e^{-2k-2z-2k_L^\gamma + 2j} e^{-\frac{(z+k_L^\gamma-j)^2}{4k}}.
\end{equation}

We now prove (8.5). This certainly holds if $l \leq k/2$, since then $l_L^\gamma = l_k^\gamma = l^\gamma$. If $L/2 \leq l \leq k$, then (8.5) says that
\begin{equation}
(L-l)^\gamma \leq (L-k)^\gamma + (k-l)^\gamma,
\end{equation}
which follows from concavity.
\( (8.3) \) holds if \( k \leq L/2 \), since then \( l_1^L = l^\gamma \leq k^\gamma = k_1^L \). Finally, if \( k/2 \leq l \leq L/2 < k \), then \( (8.5) \) says that

\[
(8.9) \quad l^\gamma \leq (L - k)^\gamma + (k - l)^\gamma.
\]

Since for \( l \leq L/2 \) we have \( l^\gamma \leq (L - l)^\gamma \), \( (8.9) \) follows from \( (8.8) \).

Recall that

\[
(8.10) \quad \gamma (l) = \gamma (l, L) = \rho_L(L - l) + l_1^L \frac{1}{4},
\]

see \( (4.14) \).

**Lemma 8.2.** For all \( L \) sufficiently large, and all \( 0 \leq z \leq 10L \) with \( t_z \in \mathbb{N} \),

\[
\mathbb{P} \left( \gamma (l) \leq \sqrt{2T_i^{0,t_z}} \text{ for } l = 1, \ldots, L - 1; T_L^{0,t_z} = 0 \right) \leq c(1 + z)(L - k)^{1/2}e^{-2L - 2z - z^2/4L}.
\]

(8.11) follows immediately from [5, Theorem 1.1], with \( a = \rho_L, x = \rho_L + z, \ b = y = 0 \), since \( \{T_L = 0\} = \{T_L \in [0, 1]\} \).

For the last statement, we simply note that following the proof of [5, Lemma 2.3] we can show that the analogue of [5, Theorem 1.1] holds where we skip some fixed finite interval.

**Proof of Lemma 8.3.** We rewrite this in terms of the critical Galton-Watson process with geometric offspring distribution, \( T_i, l = 0, 1, 2, \ldots \), as

\[
P_{t_z}^{GW} \left( \rho_L(L - l) + l_1^L \frac{1}{4} \leq \sqrt{2T_i} \text{ for } l = 1, \ldots, L - 1; T_L = 0 \right).
\]

(8.13) follows immediately from [5, Theorem 1.1], with \( a = \rho_L, x = \rho_L + z, \ b = y = 0 \), since \( \{T_L = 0\} = \{T_L \in [0, 1]\} \).

**Proof of Lemma 8.3.** We rewrite this in terms of the critical Galton–Watson process with geometric offspring distribution, \( T_i, l = 0, 1, 2, \ldots \), as

\[
P_{t_z}^{GW} \left( \rho_L(L - l) \leq \sqrt{2T_i} \text{ for } l = 1, \ldots, k; T_L^{0,t_z} = 0 \right).
\]
We condition on $T_k$ as follows: let $v = \rho_L (L - k)$. Then
\[
J_1 := P_{tz}^{GW} \left( \rho_L (L - l) \leq \sqrt{2T_l}, l = 1, \ldots, k; T_L = 0 \right)
\]
\[
\leq \sum_{j=0}^{\infty} P_{tz}^{GW} \left( \rho_L (L - l) \leq \sqrt{2T_l}, l = 1, \ldots, k; \sqrt{2T_k} \in I_{v+j} \right)
\]
\[
\times \sup_{u \in I_j} P_{t+u}^{GW} (T_{L-k} = 0)
\]
\[
= \sum_{j=0}^{\infty} P_{tz}^{GW} \left( \rho_L (L - l) \leq \sqrt{2T_l}, l = 1, \ldots, k; \sqrt{2T_k} \in I_{v+j} \right)
\]
\[
\times \sup_{u \in I_j} \left( 1 - \frac{1}{L-k} \right)^{\frac{(v+u)^2}{2}}.
\]
Since, by (3.21),
\[
P_{tz}^{GW} \left( \sqrt{2T_k} \geq 100L \right) \leq c e^{-(\rho_L L + z - 100L)^2/2L} \leq e^{-100L},
\]
we obtain
\[
J_1 \leq \sum_{j=0}^{100L} P_{tz}^{GW} \left( \rho_L (L - l) \leq \sqrt{2T_l}, l = 1, \ldots, k; \sqrt{2T_k} \in I_{v+j} \right)
\]
\[
\times e^{-\frac{(v+j)^2}{2(L-k)}} + e^{-100L} =: J_{1,1} + e^{-100L}.
\]
By [5, Theorem 1.1], with $a = \rho_L L$, $x = \rho_L L + z$ recall (8.2), and $b = v, y = v + j$,
\[
J_{1,1} \leq \sum_{j=0}^{100L} c \frac{(1+z)(1+j)}{k} \sqrt{\frac{\rho_L L}{k(v+j)}} e^{-\frac{(\rho_L L + z - v - j)^2}{2k}} e^{-\frac{(v+j)^2}{2(L-k)}}
\]
\[
\leq c \sum_{j=0}^{100L} \frac{(1+z)(1+j)}{k} \sqrt{\frac{L}{k(v+j)}} e^{-\frac{(\rho_L k + z - j)^2}{2k}} e^{-\frac{(v+j)^2}{2(L-k)}}
\]
\[
\leq c(1+z)e^{-\frac{L^2}{k}}e^{-2z} \sum_{j=0}^{100L} \frac{(1+j)}{k} \sqrt{\frac{L}{k(v+j)}} e^{-\frac{(1-z)^2}{2k}} e^{-\frac{j^2}{2(L-k)}}
\]
\[
\leq c(1+z)e^{-2L-2z-z^2/2k} \sum_{j=0}^{100L} \frac{(1+j)}{\sqrt{L-k}} e^{-\frac{j^2}{2(L-k)}} e^{jz/k}.
\]
But
\[
e^{-z^2/4k} e^{-\frac{L^2}{4(L-k)}} e^{jz/k} \leq 1
\]
which follows by considering separately $\frac{j}{k} \geq 4(L-k)z/k$ and $\frac{j}{k} < 4(L-k)z/k$, since in this case $j < z/4$ because our condition on $k$ implies that $(L-k)/k$
is tiny. It then follows that
\[ J_{1,1} \leq c(1 + z)e^{-2L-2z-z^2/4L}\sum_{j=0}^{100L}(1 + j)\frac{1}{\sqrt{L-k}}e^{-\frac{j^2}{(L-k)}} \leq c(1 + z)e^{-2L-2z-z^2/4L}\sqrt{L-k}. \]

**Lemma 8.4.** If \( k \log L/L = o_L(1) \) and \( m = \rho_L(L - k) + j \), then for all \( L \) sufficiently large, with \( m^2/2 \in \mathbb{N} \),
\[ \mathbb{P}\left( \rho_L(L - l) \leq \sqrt{2T_l^{0,k,m^2/2}} \text{ for } l = k + 1, \ldots, L - 1; T_L^{0,k,m^2/2} = 0 \right) \leq c(1 + j)e^{-2(L-k)-2j-\frac{j^2}{4(L-k)}}. \]

(8.15)
and if \( j \leq \eta L \) and we skip the barrier from \( k+1 \) to \( k+s \), with \( s \leq \eta' \log L \), for some \( \eta, \eta' < \infty \), then
\[ \mathbb{P}\left( \rho_L(L - l) \leq \sqrt{2T_l^{0,k,m^2/2}} \text{ for } l = k + s, \ldots, L - 1; T_L^{0,k,m^2/2} = 0 \right) \leq c (1 + j + \sqrt{s})e^{-2(L-k)-2j-\frac{j^2}{4(L-k)}}. \]

(8.16)

**Proof.** We first turn to the probability in (8.15). By the Markov property, the probability in question can be rewritten in terms of the critical Galton-Watson process \( T_l, l = 0, 1, 2, \ldots \) as
\[ M_1 := P_{m^2/2}^{GW} \left( \rho_L((L-k)-l) \leq \sqrt{2T_l}, l = 1, \ldots, L - k - 1; T_{L-k} = 0 \right). \]
By [5] Theorem 1.1, with \( a = \rho_L(L - k), x = m \) and \( b = y = 0 \),
\[ M_1 \leq c \frac{(1 + j)}{L-k}e^{-\frac{(\rho_L(L-k))^2}{2L-k}}. \]

(8.17)
We have
\[ e^{\frac{(\rho_L(L-k))^2}{2L-k}} = e^{-\frac{(\rho_L(L-k))^2}{2L-k}}e^{-j\rho_L-L}\leq ce^{-\frac{2(L-k)}{2L-k}L}e^{-2(L-k)-2j-\frac{j^2}{4(L-k)}}. \]

Using the concavity of the logarithm, we get that
\[ M_1 \leq c (1 + j)e^{-2(L-k)-2j-\frac{j^2}{4(L-k)}}, \]
which gives (8.15).

For (8.16) with \( k' = k + s, \nu = \rho_L(L - k') \) we bound
\[ \mathbb{P}\left( \rho_L(L - l) \leq \sqrt{2T_l^{0,k,m^2/2}} \text{ for } l = k + s, \ldots, L - 1; T_L^{0,k,m^2/2} = 0 \right) \leq \sum_{j'=0}^{\infty} P_{m^2/2}^{GW} \left( \sqrt{2T_{s} \in I_{\nu+j'}} \right) M'_1 \]

(8.18)
where

\[ M'_I = \sup_{u \in I_{v+j'}} P^{GW}_{m^2/2} \left( \rho_L \left( (L - k') - l \right) \leq \sqrt{2T_l}, l = 1, \ldots, L - k' - 1; T_{L-k'} = 0 \right). \]

By [5 Proposition 1.4, Remark 2.2]

\[ P^{GW}_{m^2/2} \left( \sqrt{2T_s} \in I_{v+j'} \right) \leq c \sqrt{\frac{m}{(v + j')s}} e^{-\frac{(m-(v+j'))^2}{2s}} \]

\[ \leq c \sqrt{\frac{1}{s}} e^{-\frac{(\rho L s + (j-j'))^2}{2s}} \leq c \sqrt{\frac{1}{s}} e^{-2s-2(j-j')-(j-j')^2/3s} \]

and as in the first part of this proof

\[ M'_1 \leq c (1 + j') e^{-2(L-k')-2j'-\frac{j'^2}{3(L-k')}}. \]

Thus

\[ \sum_{j'=0}^{\infty} P^{GW}_{m^2/2} \left( \sqrt{2T_s} \in I_{v+j'} \right) M'_1 \]

\[ \leq ce^{-2(L-k)-2j} \sqrt{\frac{1}{s}} \sum_{j'=0}^{\infty} (1 + j') e^{-\frac{j'^2}{3(L-k')}} e^{-(j-j')^2/3s}. \]

Considering separately the cases where \( j' \leq 1.1j \) and \( j' > 1.1j \) we see that for \( L \) sufficiently large,

\[ e^{-\frac{j'^2}{3(L-k')}} e^{-(j-j')^2/3s} \leq ce^{-\frac{j'^2}{4(L-k')}} e^{-(j-j')^2/4s}, \]

and (8.16) follows.

**Lemma 8.5.** If \( v = \rho_L(L - k) + u \) then for \( L \) large with \( v^2/2 \in \mathbb{N} \), and \( 1 \leq k, \overline{k}, u, j \leq L^{3/4}, \)

\[ \mathbb{P} \left( \rho_L(L - l) \leq \sqrt{2T_l^{0,k,v^2/2}}, l = k + 1, \ldots, k + \overline{k}; \sqrt{2T_{k+\overline{k}}^{0,k,v^2/2}} \in I_{\rho_L(L-k-\overline{k})+j} \right) \]

\[ \leq C \frac{(1 + u) (1 + j)}{k^{3/2}} e^{-2\overline{k}-2(u-j) - \frac{(u-j)^2}{4k}}. \]

In addition, if we skip the barrier from \( k + 1 \) to \( k + 3 \),

\[ \mathbb{P} \left( \rho_L(L - l) \leq \sqrt{2T_l^{0,k,v^2/2}}, l = k + 3, \ldots, k + \overline{k}; \sqrt{2T_{k+\overline{k}}^{0,k,v^2/2}} \in I_{\rho_L(L-k-\overline{k})+j} \right) \]

\[ \leq C \frac{(1 + u) (1 + j)}{k^{3/2}} e^{-2\overline{k}-2(u-j) - \frac{(u-j)^2}{4k}}. \]
Proof. By the Markov property, the probability in question can be rewritten in terms of the critical Galton-Watson process \( T_l, l = 0, 1, 2, \ldots \) as

\[
V_1 := P_{\nu/2} \left( \rho_L \left( L - k - l \right) \leq \sqrt{2T_l}, l = 1, \ldots, \tilde{k}; \right. \\
\left. \sqrt{2T_{\tilde{k}}} \in I_{\rho_L (L - k - \tilde{k}) + j} \right).
\]

This is a linear barrier of length \( \tilde{k} \). At the start of the barrier it is at distance \( u \) from the starting point, and at the end it is at distance \( j \) from the end point. Therefore by [3, Theorem 1.1] we have that the probability is at most

\[
c \frac{(1+u)(1+j)}{\tilde{k}^{3/2}} \sqrt{\frac{\rho_L (L-k) + u}{\rho_L (L-k-\tilde{k}) + j}} e^{-\frac{(\rho_L (L-k) + u - (\rho_L (L-k-\tilde{k}) + j))^2}{2k}}.
\]

where we have bounded the square root using the fact that \( \tilde{k}, \tilde{k}, u, j < L^{3/4} \) so that the ratio inside the square root is \( \asymp 2L/2L = 1 \). Using \( \tilde{k}, \tilde{k}, u, j < L^{3/4} \) again we see that

\[
e^{-\frac{(\rho_L k + u - j)^2}{2k}} \leq Ce^{-\frac{(2k - \log L)^2}{2k}} e^{-2(u-j)} e^{-\frac{(u-j)^2}{2k}} \leq Ce^{-2k} e^{-2(u-j)} e^{-\frac{(u-j)^2}{2k}}.
\]

This gives (8.23).

(8.24) follows as in the proof of the previous Lemma.

9. Appendix II: Conditional excursion probabilities

The following is a modification of [17] adapted to our situation. Fix \( k \geq 1 \), \( k' \geq k + 10 \) and let \( \mathcal{G}_k' \) to be the \( \sigma \)-algebra generated by the excursions from \( \partial B_d(y, h_{k-1}) \) to \( \partial B_d(y, h_k) \) (and if we start outside \( \partial B_d(y, h_{k-1}) \) we include the initial excursion to \( \partial B_d(y, h_{k-1}) \)). Note that \( \mathcal{G}_k' \) includes the information on the end points of the excursions from \( \partial B_d(y, h_k) \) to \( \partial B_d(y, h_{k-1}) \).

Recall that \( T_{y, l-1 ightarrow l}^{y,k',h_{k-1}} \) is the number of excursions from \( \partial B_d(y, h_{l-1}) \rightarrow \partial B_d(y, h_l) \) during the first \( n \) excursions from \( \partial B_d(y, h_k) \rightarrow \partial B_d(y, h_{k-1}) \), see (2.11).

Lemma 9.1. For any \( L - 2k \geq k' > k + 10 \geq 11 \) and \( n > 1 \), let \( A_k' \) denote an event, measurable on the excursions of the Brownian motion inside \( B_d(y, h_{k'}) \) during the first \( n \) excursions from \( \partial B_d(y, h_k) \rightarrow \partial B_d(y, h_{k-1}) \).
Then,

$$\mathbb{P}(A_{k'} \mid T_{y,k'}^{y,k-1} = m_{k'}, G_k)$$

(9.1)

$$= \left(1 + O\left((k' - k + 1) \frac{h_{k'-1}}{h_k}ight)\right)^{m_{k'}} \mathbb{P}(A_{k'} \mid T_{y,k'}^{y,k-1} = m_{k'}).$$

In particular, for all $m_l; l = k', \ldots, L$, and all $y \in S^2$,

$$\mathbb{P}(T_{y,l}^{y,k-1} = m_l; l = k' + 1, \ldots, L \mid T_{y,k'}^{y,k-1} = m_{k'}, G_k)$$

(9.2)

$$= \left(1 + O\left((k' - k + 1) \frac{h_{k'-1}}{h_k}ight)\right)^{m_{k'}} \mathbb{P}(T_{y,l}^{y,k-1} = m_l; l = k' + 1, \ldots, L \mid T_{y,k'}^{y,k-1} = m_{k'}).$$

In Lemma 9.1 we have $O\left((k' - k + 1) \frac{h_{k'-1}}{h_k}\right) \leq 50 \left((k' - k + 1) \frac{h_{k'-1}}{h_k}\right)$.

The key to the proof of Lemma 9.1 is the following Lemma.

**Lemma 9.2.** For $k \geq 0, k' \geq k + 10$ and a Brownian path $X$, starting at $z \in \partial B_d(y, h_{k'-1}),$ let $\bar{\tau} = \inf\{t > 0 : X_t \in \partial B_d(y, h_k)\}$ and let $F$ denote an event measurable with respect to the path of the Brownian motion inside $B_d(y, h_{k'}),$ prior to $\bar{\tau}.$ Then, uniformly in $z \in \partial B_d(y, h_{k'-1}),$ $v \in \partial B_d(y, h_k)$ and $y$,

$$\left| \frac{\mathbb{E}^z(F \mid X_\tau = v)}{\mathbb{E}^z(F)} - 1 \right| \leq 4.8(k' - k) \frac{h_{k'-1}}{h_k}. \tag{9.3}$$

If $Z_{k'} = T_{y,k'}^{y,k-1} - 1$ denotes the number of excursions of the path from $\partial B_d(y, h_{k'-1}) \to \partial B_d(y, h_k),$ prior to $\bar{\tau} = \inf\{t > 0 : X_t \in \partial B_d(y, h_k)\},$ then, uniformly in $z \in \partial B_d(y, h_{k'-1}),$ $v \in \partial B_d(y, h_k),$ $j$ and $y$,

$$\left| \frac{\mathbb{E}^z(F \mid Z_{k'} = j \mid X_\tau = v)}{\mathbb{E}^z(F \mid Z_{k'} = j)} - 1 \right| \leq 4.8(k' - k) \frac{h_{k'-1}}{h_k}, \tag{9.4}$$

and

$$\left| \frac{\mathbb{E}^z(F \mid Z_{k'} = j, X_\tau = v)}{\mathbb{E}^z(F \mid Z_{k'} = j)} - 1 \right| \leq 9.8(k' - k) \frac{h_{k'-1}}{h_k}. \tag{9.5}$$

Further, uniformly in $x \in \partial B_d(y, h_{k+1}),$ $v \in \partial B_d(y, h_k),$ $j \geq 1,$ and $y$,

$$\left| \frac{\mathbb{E}^x(F \mid Z_{k'} = j, X_\tau = v)}{\mathbb{E}^x(F \mid Z_{k'} = j)} - 1 \right| \leq 28.2(k' - k) \frac{h_{k'-1}}{h_k}. \tag{9.6}$$

where $\lambda_{k'-1}$ denotes uniform measure on $\partial B_d(y, h_{k'-1}).$

In words, conditioning by the endpoint of the excursion at level $k$ has only minor influence on the probability of events involving pieces of excursions inside the ball of radius $h_{k'}.$
Proof of Lemma 9.2} We first prove (9.4). (9.3) follows from it by multiplying both sides by \( E^z(F; Z_{k'} = j) \) and summing over \( j \).

Toward the proof of (9.4), it is enough to prove that
\[
\left| E^z(F; Z_{k'} = j) - E^z(F; Z_{k'} = j) \right| \leq 4.8(k' - k) \frac{h_{k' - 1}}{h_k} E^z(F; Z_{k'} = j).
\]

Without loss of generality we can take \( y = 0 \). Fixing \( k \geq 0 \) and \( z \in \partial B_d(0, h_{k' - 1}) \), consider a positive continuous function \( g \) on \( \partial B_d(0, h_k) \). Let \( \bar{\tau} = \inf\{t \geq t_0 \in \partial B_d(0, h_k) : \tau_0 = 0 \) and for \( i = 0, 1, \ldots \) define
\[
\bar{\tau}_{2i + 1} = \inf\{t \geq \tau_{2i} : \partial B_d(0, h_k) \cup \partial B_d(0, h_k) \}
\]
\[
\bar{\tau}_{2i + 2} = \inf\{t \geq \tau_{2i + 1} : \partial B_d(0, h_k) \}
\]

Then, by the strong Markov property at \( \tau_{2i} \),
\[
E^z(g(X_{\tau_{2j}}); F, Z_{k'} = j) = E^z\left[ E^{X_{\tau_{2j}}}(g(X_{\tau_{2j}}); Z_{k'} = 0); F, Z_{k'} = j, \bar{\tau} \geq \tau_{2j} \right]
\]
and, substituting \( g = 1 \),
\[
P^z(F, Z_{k'} = j) = E^z\left[ P^{X_{\tau_{2j}}}(Z_{k'} = 0); F, Z_{k'} = j, \bar{\tau} \geq \tau_{2j} \right]
\]

Consequently,
\[
P^z(F, Z_{k'} = j) \inf_{x \in \partial B_d(0, h_{k' - 1})} \frac{E^z(g(X_{\tau_{2j}}); Z_{k'} = 0)}{P^z(Z_{k'} = 0)} \leq E^z(g(X_{\tau_{2j}}); F, Z_{k'} = j)
\]
\[
\leq P^z(F, Z_{k'} = j) \sup_{x \in \partial B_d(0, h_{k' - 1})} \frac{E^z(g(X_{\tau_{2j}}); Z_{k'} = 0)}{P^z(Z_{k'} = 0)},
\]
and, using again the strong Markov property at time \( \tau_{2j} \),
\[
E^z(g(X_{\tau_2}); Z_{k'} = 0) = E^z\left( g(X_{\tau_2}) \right) - E^z\left( E^{X_{\tau_2}}(g(X_{\tau_2}); Z_{k'} \geq 1) \right)
\]
\[
\leq E^z\left( g(X_{\tau_2}) \right) - E^x(Z_{k'} \geq 1) \inf_{y \in \partial B_d(0, h_{k' - 1})} E^y(g(X_{\tau_2}))
\]
with the reversed inequality if the inf is replaced by sup. Writing \( p = P^x(Z_{k'} \geq 1) = 1 - 1/(k' - k) \) whenever \( x \in \partial B_d(0, h_{k' - 1}) \), c.f. (2.7), it thus follows that
\[
E^z(g(X_{\tau_2}); F, Z_{k'} = j) \leq P^z(F, Z_{k'} = j) \sup_{x \in \partial B_d(0, h_{k' - 1})} \frac{E^z(g(X_{\tau_2}))}{P^x(Z_{k'} = 0)} (1 - p)
\]

with the reversed inequality if the inf and sup are interchanged.

Let \( p_{B_d(0, h_k)}(z, x) \) denote the Poisson kernel for \( B_d(0, h_k) \subseteq S^2 \), see (2.8). Then,
\[
E^x g(X_{\tau_2}) = \int_{\partial B_d(0, h_k)} p_{B_d(0, h_k)}(z', u) g(u) du.
\]
Therefore, we get the Harnack inequality

$$\sup_{x \in \partial B_d(0, h_{k'} - 1)} \mathbb{E}^z(g(X_{\tau})) \leq \frac{\max_{x \in \partial B_d(0, h_{k'} - 1), u \in \partial B_d(0, h_k)} P_B(0, h_k)(x, u)}{\min_{y \in \partial B_d(0, h_{k'} - 1), u \in \partial B_d(0, h_k)} P_B(0, h_k)(y, u)} \max_{y \in \partial B_d(0, h_{k'} - 1), u \in \partial B_d(0, h_k)} \sin^2(d(u, y)/2)}{\min_{x \in \partial B_d(0, h_{k'} - 1), u \in \partial B_d(0, h_k)} \sin^2(d(u, x)/2)} \leq \frac{\sin^2((h_k + h_{k'} - 1)/2)}{\sin^2((h_k - h_{k'} - 1)/2)} =: B_{k, k'}^2. \tag{9.9}$$

Writing $\alpha = (h_k - h_{k'} - 1)/2, \beta = h_{k'} - 1$ we have

$$B_{k, k'} = \frac{\sin(\alpha + \beta)}{\sin(\alpha)} = \cos(\beta) + \frac{\sin(\beta) \cos(\alpha)}{\sin(\alpha)}. \tag{9.10}$$

Using the bounds $\sin(\alpha) \geq .9\alpha, \sin(\beta) \leq \beta, \cos(\alpha), \cos(\beta) \leq 1$, which are valid for $0 < \alpha, \beta < 0.01$, we obtain that $B_{k, k'}$ is bounded above by

$$1 + \beta/9\alpha = 1 + 2 h_{k'} - 1/9(h_k - h_{k'} - 1). \tag{9.11}$$

Since $k' \geq k + 10$ we have that $99 h_k \leq h_k - h_{k'} - 1$, so that (9.11) is bounded by $1 + 2.9 h_{k'} - 1/9 h_k$, and therefore $B_{k, k'}^2$ is bounded above by $1 + 4.8 h_{k'} - 1/9 h_k$ while its reciprocal is bounded below by $1 - 4.8 h_{k'} - 1/9 h_k$. Substituting this bound into (9.8) (and its reversed version with inequality and sup/inf interchanged), and using the value of $p$, yields (9.7).

For (9.5) we have by (9.7), first for $F$ and then replacing $F$ by $1$

$$\mathbb{E}(F; Z_{k'} = j \mid X_{\tau} = v) \leq \left(1 + 4.8(k' - k)\frac{h_{k'} - 1}{h_k}\right) \mathbb{E}(F; Z_{k'} = j).$$

and

$$\mathbb{E}(Z_{k'} = j \mid X_{\tau} = v) \geq \left(1 - 4.8(k' - k)\frac{h_{k'} - 1}{h_k}\right) \mathbb{E}(Z_{k'} = j).$$

This gives the upper bound

$$\mathbb{E}(F \mid Z_{k'} = j, X_{\tau} = v) = \frac{\mathbb{E}(F; Z_{k'} = j \mid w_{\tau} = v)}{\mathbb{E}(Z_{k'} = j \mid w_{\tau} = v)} \mathbb{E}(F; Z_{k'} = j).$$

(9.12)

The lower bound is obtained similarly.

We finally turn to the proof of (9.6). Let $\tau = \inf\{t > 0 : X_t \in \partial B_d(0, h_{k'} - 1)\}$. We will need to show that the law of $X_{\tau}$, started at $x \in \partial B_d(0, h_{k} - 1)$ and conditioned on the event $D = \{\tau < \tau\}$, is close to uniform. Toward this end, we begin with an unconditional statement. Let $0^*$ denote the antipode of $0 \in S^2$. Note that $\tau = \inf\{t > 0 : X_t \in \partial B_d(0, h_{k'} - 1) = \partial B_d(0^*, \pi - h_{k'} - 1)\}$. Then for any positive function $h$ on $\partial B_d(0, h_{k'} - 1) = \partial B_d(0^*, \pi - h_{k'} - 1)$.
\[ \partial B_d(0^*, \pi - h_{k' - 1}), \text{ and } x \in \partial B_d(0, h_{k+1}) = \partial B_d(0^*, \pi - h_{k+1}), \text{ writing } \bar{h} = h(X \bar{\tau}), \]

\[ \mathbb{E}^\pi \bar{h} = \int_{\partial B_d(0, h_{k' - 1})} p_{B_d(0^*, \pi - h_{k' - 1})}(x, u) h(u) d\lambda_{k' - 1}(u), \]

where \( \lambda_{k' - 1} \) is uniform measure on \( \partial B_d(0, h_{k' - 1}) \). We have, see (2.8),

\[ p_{B_d(0^*, \pi - h_{k' - 1})}(x, u) = \frac{\sin^2(\pi/2 - h_{k' - 1}/2) - \sin^2(\pi/2 - h_{k+1}/2)}{\sin^2(d(u, x)/2)} \]

\[ = \frac{\sin^2(h_{k+1}/2) - \sin^2(h_{k' - 1}/2)}{\sin^2(d(u, x)/2)} . \]

For the upper bound, using \( \sin^2(a) - \sin^2(b) = (a + b) \sin(a - b) \) again we note that

\[ p_{B_d(0^*, \pi - h_{k' - 1})}(x, u) \leq \sin((h_{k+1} + h_{k' - 1})/2) \]

\[ \leq \frac{1}{B_{k+1, k' - 1}} . \]

It follows that

\[ p_{B_d(0^*, \pi - h_{k' - 1})}(x, u) \geq 1 - 2.4 \left( \frac{h_{k' - 1}}{h_{k+1}} \right) . \]

Combining this with (9.13), we conclude that with \( h^\lambda = \int h(u) d\lambda_{k' - 1}(u), \)

\[ |\mathbb{E}^\pi(\bar{h}) - h^\lambda| \leq 2.4 \left( \frac{h_{k' - 1}}{h_{k+1}} \right) h^\lambda =: \delta_{k', k+1} h^\lambda . \]

We next turn to proving the analogous conditional statement. Recall the event \( D = \{ \bar{\tau} < \tilde{\tau} \} \). By the Markov property \( \mathbb{E}^\pi(\bar{h}, \{ \bar{\tau} < \tilde{\tau} \}) = \mathbb{E}^{\mathbb{E}^\pi}(\bar{h}, \{ \bar{\tau} < \tilde{\tau} \}) \), and hence by (9.14) with \( k + 1 \) replaced by \( k \)

\[ |\mathbb{E}^\pi(\bar{h}|D^c) - h^\lambda| \leq \delta_{k' - 1, k} h^\lambda . \]

Note by (2.7) that \( \mathbb{P}^\pi(D) = 1/(k' - k - 1) =: \delta'_{k', k} \) and is independent of \( x \).

We have

\[ \mathbb{E}^\pi(\bar{h}) = \mathbb{E}^\pi(\bar{h}|D) \mathbb{P}^\pi(D) + \mathbb{E}^\pi(\bar{h}|D^c) \mathbb{P}^\pi(D^c) \]

and therefore, using the last display and (9.19) and (9.20) in the inequality,

\[ |\mathbb{E}^\pi(\bar{h}|D) - h^\lambda| \leq \frac{1}{\mathbb{P}^\pi(D)} \left( \mathbb{E}^\pi(\bar{h}) - \mathbb{E}^\pi(\bar{h}|D^c) \right) + \mathbb{E}^\pi(\bar{h}|D^c) - h^\lambda \]

\[ \leq \left( \frac{\delta'_{k' - 1, k+1} + \delta_{k' - 1, k}}{\delta'_{k', k}} \right) h^\lambda \leq 9.6(k' - k) \left( \frac{h_{k' - 1}}{h_{k}} \right) h^\lambda . \]
This gives the desired conditional estimate.

To obtain (9.6) we first consider $H = F_1(z_{k' = j})g(X_\tau)$. For all $j \geq 1$ we have that $H = H \circ \theta_\tau 1_{\tau < \tau}$. Hence by the strong Markov property, $E^x(F_1(z_{k' = j})g(X_\tau)) = E^x(E^{X_\tau}(F_1(z_{k' = j})g(X_\tau)), D)$. Let $h(y) = E^y(F_1(z_{k' = j})g(X_\tau))$. (9.21) then shows that

\[(9.22) \quad |E^x(F_1(z_{k' = j})g(X_\tau)) - E^x(F_1(z_{k' = j})g(X_\tau))\mathbb{P}(D)| \leq 9.6(k' - k) \left( \frac{h_{k' - 1}}{h_k} \right) E^x(F_1(z_{k' = j})g(X_\tau))\mathbb{P}(D).\]

Using this also with $F = 1$, proceeding as in the proof of (9.5), and then letting $g \to \delta_v$ gives

\[(9.23) \quad |E^z(F_{i, v}) Z_{k'} = j, X_\tau = v) - E^x(F_{i, v}) Z_{k'} = j, X_\tau = v)| \leq 19.3(k' - k) \left( \frac{h_{k' - 1}}{h_k} \right) E^x(F_{i, v}) Z_{k'} = j, X_\tau = v),\]

and (9.6) then follows from (9.5).

**Proof of Lemma 9.1.** This follows from Lemma 9.2 in the same manner as [17, Lemma 7.3] was derived from [17, Lemma 7.4]: using the strong Markov property, each excursion from $\partial B_d(y, h_{k'})$ to $\partial B_d(y, h_{k-1})$ will contribute a multiplicative factor $(1 + 50(k' - k + 1)h_{k' - 1}/h_{k-1})$ to the probability.

(We emphasize that no rotation invariance of the event $A_{k'}$ is used in the argument, since none was imposed on $F$ in Lemma 9.2.)

In more detail, for the $i$'th excursion from $\partial B_d(y, h_k)$ to $\partial B_d(y, h_{k-1})$ let $x_i, v_i$ denote the the starting and endpoint of the excursion and let $Z_{k'}^i$ denote the number of excursions from $\partial B_d(y, h_{k'-1})$ to $\partial B_d(y, h_{k'})$. It suffices to show that

\[(9.24) \quad \mathbb{P}(A_{k'} \mid x_i, v_i, Z_{k'}^i = j; i \in [1, n]) = \left( 1 + O((k' - k + 1)\frac{h_{k' - 1}}{h_{k-1}}) \right)^{\sum_{i=1}^n j_i} \mathbb{P}(A_{k'} \mid Z_{k'}^i = j; i \in [1, n]) ,\]

with $|O((k' - k + 1)\frac{h_{k' - 1}}{h_{k-1}})| \leq 50(k' - k + 1)\frac{h_{k' - 1}}{h_{k-1}}$.

Let $F = \prod_{i=1}^n F_i$ where $F_i$ depends on the excursions inside $B_d(y, h_{k'})$ during the $i$'th excursion from $\partial B_d(y, h_k)$ to $\partial B_d(y, h_{k-1})$. We set $F_i = 1$ if $Z_{k'}^i = 0$. Then to prove (9.24) it suffices to show that

\[(9.25) \quad \mathbb{E}(F \mid x_i, v_i, Z_{k'}^i = j; i \in [1, n]) = \left( 1 + O((k' - k + 1)\frac{h_{k' - 1}}{h_{k-1}}) \right)^{\sum_{i=1}^n j_i} \mathbb{E}(F \mid Z_{k'}^i = j; i \in [1, n]) .\]
By the Markov property and (9.6) with \( k \) replaced by \( k - 1 \) we have

\[
\mathbb{E}
\left(
\prod_{i=1}^{n} F_i \mid x_i, v_i, Z_i^{j_i} = j_i; i \in [1, n]\right)
\]

\[
= \mathbb{E}
\left(
\prod_{i=1}^{n-1} F_i \mid x_i, v_i, Z_i^{j_i} = j_i; i \in [1, n-1]ight) \mathbb{E}^{Z_n}(F_n \mid v_n, Z_n^{j_n} = j_n)
\]

\[
\leq \left(1 + 28.2 \left(\frac{h_{k'} - 1}{h_{k-1}}\right)\right)^{1\{j_n > 0\}} \mathbb{E}
\left(
\prod_{i=1}^{n-1} F_i \mid x_i, v_i, Z_i^{j_i} = j_i; i \in [1, n-1]\right) \mathbb{E}^{\lambda_{k'-1}}(F_n \mid Z_n^{j_n} = j_n),
\]

and by induction

\[
\mathbb{E}
\left(
\prod_{i=1}^{n} F_i \mid x_i, v_i, Z_i^{j_i} = j_i; i \in [1, n]\right)
\]

\[
\leq \left(1 + 28.2 \left(\frac{h_{k'} - 1}{h_{k-1}}\right)\right)^{\sum_{i=1}^{n} j_i} \prod_{i=1}^{n} \mathbb{E}^{\lambda_{k'-1}}(F_i \mid Z_i^{j_i} = j_i).
\]

Similarly, by the analogue of (9.23), but without conditioning on the endpoint, which follows from (9.22) with \( g \equiv 1 \), we have

\[
\mathbb{E}
\left(
\prod_{i=1}^{n} F_i \mid Z_i^{j_i} = j_i; i \in [1, n]\right)
\]

\[
\geq \left(1 - 19.3 \left(\frac{h_{k'} - 1}{h_{k-1}}\right)\right)^{\sum_{i=1}^{n} j_i} \prod_{i=1}^{n} \mathbb{E}^{\lambda_{k'-1}}(F_i \mid Z_i^{j_i} = j_i).
\]

Together this gives the upper bound in (9.25) and the lower bound is proven similarly.

\[\square\]

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