SIMPSON’S RULE REVISITED

SLAVKO SIMIĆ

ABSTRACT. In this article we give some refinements of Simpson’s Rule in cases when it is not applicable in its classical form i.e., when the target function is not four times differentiable on a given interval. Some sharp two-sided inequalities for an extended form of Simpson’s Rule are also proven.

1. Introduction

We begin with some notions from Classical Analysis which will be frequently needed in the sequel.

A function \( h : I \subset \mathbb{R} \to \mathbb{R} \) is said to be convex on a non-empty interval \( I \) if the inequality

\[
    h(px + qy) \leq ph(x) + qh(y)
\]

holds for all \( x, y \in I \) and all non-negative \( p, q ; p + q = 1 \).

If the inequality (1.1) reverses, then \( h \) is said to be concave on \( I \). [HLP]

The well-known convexity/concavity criteria says that if \( h \in C^2(I) \) and \( h''(x) \gtrless 0, x \in I \), then the function \( h \) is convex/concave on \( I \). [HLP]

Let \( h : I \subset \mathbb{R} \to \mathbb{R} \) be a convex function on an interval \( I \) and \( a, b \in I \) with \( a < b \). Then

\[
    h\left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b h(t)dt \leq \frac{h(a) + h(b)}{2}.
\]

This double inequality is known in the literature as the Hermite-Hadamard (HH) integral inequality for convex functions. See, for example, [NP] and references therein. There is a number of refinements and possible generalizations of HH inequality. Some recent trends can be found in [WQ] and [SE].

If \( h \) is a concave function then both inequalities in (1.2) hold in the reversed direction.

Closely connected to the HH inequality is the well-known Simpson’s Rule which is of great importance in numerical integration. It says that

**Lemma 1.3.** [U] For an integrable function \( g \), we have

\[
    \int_{x_1}^{x_3} g(t)dt = \frac{1}{3} h(g_1 + 4g_2 + g_3) - \frac{1}{90} h^5 g^{(4)}(\xi), \quad (x_1 < \xi < x_3),
\]

where \( g_i = g(x_i) \) and \( h =: x_2 - x_1 = x_3 - x_2 \).
Now, by taking $x_1 = a, x_2 = (a+b)/2, x_3 = b$, it follows that $h = (b-a)/2$ and therefore we obtain another form of Simpson’s Rule:

\[ (1.4) \quad \frac{g(a) + g(b)}{6} + \frac{2}{3}g\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b g(t)dt = \frac{g^{(4)}(\xi)}{2880}(b-a)^4, \quad a < \xi < b. \]

Note that the equation (1.4) explicitly supposes that $g \in C^{(4)}(I)$.

An interesting problem arises if $g \notin C^{(4)}(I)$, i.e., if $g$ is not a four times continuously differentiable function on $I$. How to approximate the expression

\[ T_g(a, b) := \frac{g(a) + g(b)}{6} + \frac{2}{3}g\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b g(t)dt \]

in this case?

In order to give an answer to this question, we shall consider the class of functions $g \in C^{(n)}(E)$ which are continuously differentiable up to $n$-th order on an interval $E := [a, b] \subset I$. Since $g^{(n)}(\cdot)$ is a continuous function on a closed interval, there exist numbers $m_n = m_n(a, b; g) := \min_{t \in E} g^{(n)}(t)$ and $M_n = M_n(a, b; g) := \max_{t \in E} g^{(n)}(t)$. These numbers will play an important role in further approximations.

Our task in this article is to demonstrate a method which improves Simpson’s Rule in some characteristic situations.

For example, let $g(\cdot)$ be a twice differentiable function on $E$. Some preliminary bounds for $T_g(a, b)$ in this case can be obtained by utilizing Hermite-Hadamard inequality (1.2) in a natural way.

Namely, for a given $g \in C^{(2)}(E)$ define an auxiliary function $h$ by $h(t) := g(t) - m_2 t^2/2$. Since $h''(t) = g''(t) - m_2 \geq 0$, we see that $h$ is a convex function on $E$. Therefore, applying Hermite-Hadamard inequality, we get

\[ g\left(\frac{a+b}{2}\right) - \frac{m_2}{2} \left(\frac{a+b}{2}\right)^2 \leq \frac{1}{b-a} \int_a^b g(t)dt - \frac{m_2}{2} \frac{b^3 - a^3}{3(b-a)} \leq \frac{g(a) + g(b)}{2} - \frac{m_2}{2} \frac{a^2 + b^2}{2}, \]

that is,

\[ (1.5) \quad g\left(\frac{a+b}{2}\right) + \frac{m_2}{24}(b-a)^2 \leq \frac{1}{b-a} \int_a^b g(t)dt \leq \frac{g(a) + g(b)}{2} - \frac{m_2}{12}(b-a)^2. \]

On the other hand, taking the auxiliary function $h$ to be $h(t) = M_2 t^2/2 - g(t)$, we see that it is also convex on $E$.

Applying Hermite-Hadamard inequality again, we get

\[ (1.6) \quad \frac{g(a) + g(b)}{2} - \frac{M_2}{12}(b-a)^2 \leq \frac{1}{b-a} \int_a^b g(t)dt \leq g\left(\frac{a+b}{2}\right) + \frac{M_2}{24}(b-a)^2. \]
Apart from the fact that those inequalities improves HH inequality in the cases if \( g \) is convex \((m_2 \geq 0)\) or concave \((M_2 \leq 0)\) function on \( E \), they also lead to an estimation of \( T_g(a, b) \), as follows.

Inequalities (1.5) and (1.6) give
\[
\frac{1}{b-a} \int_a^b g(t) \, dt \leq \frac{g(a) + g(b)}{2} - \frac{m_2}{12} (b - a)^2,
\]
\[
\frac{2}{b-a} \int_a^b g(t) \, dt \leq 2g\left(\frac{a + b}{2}\right) + \frac{M_2}{12} (b - a)^2.
\]

Hence,
\[
\frac{1}{b-a} \int_a^b g(t) \, dt - \left[ \frac{g(a) + g(b)}{6} + \frac{2}{3} g\left(\frac{a + b}{2}\right) \right] \leq \frac{M_2 - m_2}{36} (b - a)^2.
\]

Also, adjusting the left-hand sides of (1.5) and (1.6), we get
\[
2g\left(\frac{a + b}{2}\right) + \frac{m_2}{12} (b - a)^2 \leq \frac{2}{b-a} \int_a^b g(t) \, dt,
\]
\[
g\left(\frac{a + b}{2}\right) - \frac{M_2}{12} (b - a)^2 \leq \frac{1}{b-a} \int_a^b g(t) \, dt.
\]

Therefore,
\[
\frac{1}{b-a} \int_a^b g(t) \, dt - \left[ \frac{g(a) + g(b)}{6} + \frac{2}{3} g\left(\frac{a + b}{2}\right) \right] \geq -\frac{M_2 - m_2}{36} (b - a)^2,
\]
and we finally obtain that
\[
\tag{1.7} \left| T_g(a, b) \right| = \left| \frac{g(a) + g(b)}{6} + \frac{2}{3} g\left(\frac{a + b}{2}\right) - \frac{1}{b-a} \int_a^b g(t) \, dt \right| \leq \frac{1}{36} (M_2 - m_2) (b - a)^2,
\]
whenever \( g \in C^{(2)}(E) \).

Another and more efficient method to approximate \( T_g(a, b) \) is to use its integral representations in the cases when \( g \in C^{(1)}(E) \), \( g \in C^{(2)}(E) \) or \( g \in C^{(3)}(E) \). In this way we obtain the following estimations:

1. For \( g \in C^{(1)}(E) \), \( \left| T_g(a, b) \right| \leq \frac{5}{72} (b - a)(M_1 - m_1) \);
2. For \( g \in C^{(2)}(E) \), \( \left| T_g(a, b) \right| \leq \frac{1}{162} (b - a)^2(M_2 - m_2) \);
3. For \( g \in C^{(3)}(E) \), \( \left| T_g(a, b) \right| \leq \frac{1}{1152} (b - a)^3(M_3 - m_3) \).

**Remark 1.8.** A challenging task and an open problem is to improve the constants 5/72 and 1/162, if possible. We shall prove that 1/1152 is the best possible constant in part 3.
In the sequel we sharply refine Simpson’s Rule by assuming that \( f''(\cdot) \) is a convex function on \( E \). Then,

\[
0 \leq T_\phi(a, b) \leq \frac{(b - a)^2}{162} \left[ \frac{f''(a) + f''(b)}{2} - f''\left(\frac{a + b}{2}\right) \right],
\]

(Theorem 2.8 below).

Finally, applying the method described above, we shall give tight bounds for an improved form of Simpson’s Rule of the fourth order.

2. Results and Proofs

We begin with refinements of Simpson’s Rule in non-standard cases.

For this cause, the following integral representation of \( T_\phi(a, b) \) in the case \( \phi \in C^{(1)}(E) \) is of crucial value.

**Lemma 2.1.** The identity

\[
T_\phi(a, b) = \frac{b - a}{12} \int_0^1 (1 - 3t)[\phi'(u) - \phi'(v)]dt
\]

holds for any \( \phi \in C^{(1)}(E) \), where

\[
u := b^2 + a(1 - \t^2), v := b^2 + a(1 - \t^2).
\]

**Proof.** In the well-known formula

\[
\int_0^1 UdV = UV|_0^1 - \int_0^1 VdU,
\]

putting

\[
U = 1 - 3t, \quad dV = (\phi'(u) - \phi'(v))dt,
\]

we get

\[
dU = -3dt, \quad V = \frac{2}{b - a}(\phi(u) + \phi(v)).
\]

Also,

\[
\int_0^1 \phi(u)dt = \int_0^1 \phi\left(\frac{t}{2} + b(1 - \t^2)\right)dt = \frac{2}{b - a} \int_{\frac{a + b}{2}}^b \phi(t)dt;
\]

\[
\int_0^1 \phi(v)dt = \int_0^1 \phi\left(\frac{t}{2} + a(1 - \t^2)\right)dt = \frac{2}{b - a} \int_{\frac{a + b}{2}}^a \phi(t)dt.
\]

Therefore,

\[
\frac{b - a}{12} \int_0^1 (1 - 3t)[\phi'(u) - \phi'(v)]dt = \frac{b - a}{12} \left[ \frac{2}{b - a}(3t - 1)(\phi(u) + \phi(v))\right]_0^1 - \frac{6}{b - a} \int_0^1 (\phi(u) + \phi(v))dt
\]
\[ \frac{2}{3} \phi\left(\frac{a+b}{2}\right) + \frac{\phi(a) + \phi(b)}{6} - \frac{2}{b-a} \left( \int_{a}^{a+b/2} \phi(t) dt + \int_{a+b/2}^{b} \phi(t) dt \right) \]
\[ = \frac{\phi(a) + \phi(b)}{6} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} \phi(t) dt = T_\phi(a,b). \]

Our first contribution is the following

**Theorem 2.2.** For any \( \phi \in C^{(1)}(E) \), we have

\[ \left| \frac{\phi(a) + \phi(b)}{6} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} \phi(t) dt \right| \leq \frac{5}{12} (M_1 - m_1)(b-a), \]

where \( m_1 := \min_{t \in E} \phi'(t) \) and \( M_1 := \max_{t \in E} \phi'(t) \).

**Proof.** By the above Lemma, we obtain

\[ \left| T_\phi(a,b) \right| \leq \frac{b-a}{12} \int_{0}^{1} |1 - 3t||\phi'(u) - \phi'(v)| dt \]
\[ \leq \frac{b-a}{12} (M_1 - m_1) \int_{0}^{1} |1 - 3t| dt = \frac{5}{12} (M_1 - m_1)(b-a). \]

□

**Theorem 2.3.** For any \( \phi \in C^{(2)}(E) \), we have

\[ \left| \frac{\phi(a) + \phi(b)}{6} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_{a}^{b} \phi(t) dt \right| \leq \frac{1}{162} (M_2 - m_2)(b-a)^2, \]

where \( m_2 := \min_{t \in E} \phi''(t) \) and \( M_2 := \max_{t \in E} \phi''(t) \).

**Proof.** In this case, the integral representation of \( T_\phi(a,b) \) has the following form.

**Lemma 2.4.** The identity

\[ T_\phi(a,b) = \frac{(b-a)^2}{48} \int_{0}^{1} t(2 - 3t)[\phi''(u) + \phi''(v)] dt \]

holds for any \( \phi \in C^{(2)}(E) \), where \( u \) and \( v \) are defined as in Lemma 2.1.

**Proof.** Indeed, applying partial integration on the assertion from Lemma 2.1, we get

\[ T_\phi(a,b) = \frac{b-a}{12} \int_{0}^{1} (1 - 3t)[\phi'(u) - \phi'(v)] dt \]
\[ = \frac{b-a}{12} [(t - 3t^2/2)[\phi'(u) - \phi'(v)]_1 + \frac{b-a}{2} \int_{0}^{1} (t - 3t^2/2)[\phi''(u) + \phi''(v)] dt] \]
\[ = \frac{(b-a)^2}{48} \int_{0}^{1} t(2 - 3t)[\phi''(u) + \phi''(v)] dt. \]

□
Hence,

\[
\int_0^1 t(2-3t)[\phi''(u)+\phi''(v)]dt = \int_0^{2/3} t(2-3t)[\phi''(u)+\phi''(v)]dt - \int_{2/3}^1 t(3t-2)[\phi''(u)+\phi''(v)]dt
\]

\[
\leq 2M_2 \int_0^{2/3} t(2-3t)dt - 2m_2 \int_{2/3}^1 t(3t-2)dt = \frac{8}{27}(M_2 - m_2),
\]

since \( \int_0^{2/3} t(2-3t)dt = \int_{2/3}^1 t(3t-2)dt = \frac{4}{27} \).

Analogously,

\[
\int_0^1 t(2-3t)[\phi''(u)+\phi''(v)]dt = \int_0^{2/3} t(2-3t)[\phi''(u)+\phi''(v)]dt - \int_{2/3}^1 t(3t-2)[\phi''(u)+\phi''(v)]dt
\]

\[
\geq 2m_2 \int_0^{2/3} t(2-3t)dt - 2M_2 \int_{2/3}^1 t(3t-2)dt = -\frac{8}{27}(M_2 - m_2),
\]

and we get the desired result. \(\Box\)

A challenging task is to determine the best possible constant \(A\) such that the relation

\[
\bigg| T_\phi(a,b) \bigg| \leq A(M_2 - m_2)(b-a)^2,
\]

holds for any \(\phi \in C^{(2)}(E)\).

Note that the function \(\phi(\cdot)\), defined on \(E = [-x, x]\) by

\[
\phi(t) = \begin{cases} 
t^3/6, & t \geq 0; 
-t^3/6, & t \leq 0,
\end{cases}
\]

gives \(A \geq 1/288\). Hence \(A \in [1/288, 1/162]\).

Another important result concerns the functions which are only 3-times differentiable on \(E\).

**Theorem 2.5.** For any \(\phi \in C^{(3)}(E)\) we have

\[
\bigg| \frac{\phi(a) + \phi(b)}{6} + \frac{2}{3} \phi\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b \phi(t)dt \bigg| \leq \frac{1}{1152}(M_3 - m_3)(b-a)^3.
\]

The constant \(C = 1/1152\) is best possible.

**Proof.** For this case we need a new integral representation of \(T_\phi(a,b)\).

**Lemma 2.6.** If \(\phi \in C^{(3)}(E)\), then

\[
T_\phi(a,b) = \frac{(b-a)^3}{96} \int_0^1 t^2(1-t)[\phi'''(u) - \phi'''(v)]dt.
\]
Indeed, by partial integration we get

\[
\int_0^1 t(2 - 3t)[\phi''(u) + \phi''(v)]dt = t^2(1 - t)[\phi''(u) + \phi''(v)]|_0^1 - \int_0^1 t^2(1 - t)\frac{d}{dt}[\phi''(u) + \phi''(v)]dt,
\]

and, by Lemma 2.4, the proof follows.

Therefore,

\[
|T_\phi(a, b)| \leq \frac{(b - a)^3}{96} \int_0^1 t^2(1 - t)|\phi'''(u) - \phi'''(v)|dt 
\leq \frac{(b - a)^3}{96}(M_3 - m_3) \int_0^1 t^2(1 - t)dt = \frac{1}{1152}(M_3 - m_3)(b - a)^3.
\]

To prove that the constant \( C = 1/1152 \) is best possible, we consider the function \( d(\cdot) \) defined as:

\[
d(x) = \begin{cases} 
-x^3/6 - x/3, & x \leq -1; \\
x^4/24 + x^2/4 - x/6 + 1/24, & -1 \leq x \leq 1; \\
x^3/6, & x \geq 1.
\end{cases}
\]

It is easy to confirm that this function is 3-times continuously differentiable on the real line.

Applying the form of Simpson’s Rule for \( x \in [-a, a], a > 1 \), we obtain

\[
(2.7) \quad \left| \frac{d(-a) + d(a)}{6} + \frac{2}{3}d(0) - \frac{1}{2a} \int_{-a}^{a} d(x)dx \right| \leq 8Ca^3(M_3 - m_3).
\]

Since,

\[
d'''(x) = \begin{cases} 
-1, & x \leq -1; \\
x, & -1 \leq x \leq 1; \\
1, & x \geq 1,
\end{cases}
\]

we see that \( m_3 = -1, \quad M_3 = 1 \).

Therefore, by \(2.7\) we get

\[
C \geq \frac{|a^3/72 - a/36 + 1/36 - 1/120a|}{16a^3} 
= \frac{1}{1152}\left| 1 - \frac{2}{a^2} + \frac{2}{a^3} - \frac{3}{5a^4} \right|.
\]

Letting \( a \to \infty \), we obtain

\[
C \geq \frac{1}{1152},
\]

and the proof is done.
We shall give in the sequel precise estimation of an extended form of Simpson’s rule under a smoothness condition posed on the target function.

For example, supposing that $\phi''$ is convex on $E$, we obtain a clarification of the formula (1.4).

**Theorem 2.8.** For a $\phi \in C^4(E)$, let $\phi''(\cdot)$ be convex on $E$. Then

$$0 \leq \frac{\phi(a) + \phi(b)}{6} + \frac{2}{3} \phi\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b \phi(t)dt \leq \frac{(b - a)^2}{162} \left[\phi''(a) + \phi''(b) - \phi''\left(\frac{a + b}{2}\right)\right].$$

**Proof.** The left-hand side inequality follows from the convexity of $\phi''$ and (1.4). For the second inequality we need an interesting assertion from Convexity Theory [6].

**Lemma 2.9.** Let $h$ be a convex function on $E$ and for some $u, v \in E$, $u + v = a + b$.

Then,

$$2h\left(\frac{a + b}{2}\right) \leq h(u) + h(v) \leq h(a) + h(b).$$

**Proof.** For the left-hand side we have

$$h(u) + h(v) \geq 2h\left(\frac{u + v}{2}\right) = 2h\left(\frac{a + b}{2}\right).$$

For the right-hand side let $u = pa + qb; p, q \geq 0, p + q = 1$. Then $v = qa + pb$ and, applying (1.1), we get

$$h(u) + h(v) = h(pa + qb) + h(qa + pb) \leq (ph(a) + qh(b)) + (qh(a) + ph(b)) = h(a) + h(b),$$

as desired. $\square$

Now, Lemma 2.4 gives

$$T_\phi(a, b) = \frac{(b - a)^2}{48} \int_0^1 t(2 - 3t)[\phi''(u) + \phi''(v)]dt$$

$$= \frac{(b - a)^2}{48} \left(\int_0^{2/3} t(2 - 3t)[\phi''(u) + \phi''(v)]dt - \int_{2/3}^1 t(3t - 2)[\phi''(u) + \phi''(v)]dt\right).$$

Since $u, v \in [a, b]$ and $u + v = a + b$, applying Lemma 2.9 to both integrals separately, we obtain

$$T_\phi(a, b) \leq \frac{(b - a)^2}{48} \left(\phi''(a) + \phi''(b)\right) \int_0^{2/3} t(2 - 3t)dt - 2\phi''\left(\frac{a + b}{2}\right) \int_{2/3}^1 t(3t - 2)dt$$

$$= \frac{(b - a)^2}{324} [\phi''(a) + \phi''(b) - 2\phi''\left(\frac{a + b}{2}\right)],$$

since $\int_0^{2/3} t(2 - 3t)dt = \int_{2/3}^1 t(3t - 2)dt = \frac{4}{27}$. $\square$
Applying the method which was demonstrated in Introduction, we obtain an improved form of Simpson’s Rule.

**Theorem 2.10.** For any \( \psi \in C^4(E) \), we have

\[
\left| T_\psi(a,b) - \frac{(b-a)^2}{360} \left[ \frac{\psi''(a) + \psi''(b)}{2} - \psi''\left( \frac{a+b}{2} \right) \right] \right| \leq \frac{11}{57600} (M_4 - m_4)(b-a)^4,
\]

with \( m_4 = \min_{t \in E} \psi'(t) \), \( M_4 = \max_{t \in E} \psi'(t) \).

**Proof.** Take that \( \phi(t) = \psi(t) - m_4^4/24, t \in E \). Since \( \phi'(t) = \psi'(t) - m_4 \geq 0 \), we conclude that \( \phi''(\cdot) \) is a convex function on \( E \).

Therefore, applying Theorem 2.8 along with the identity

\[
a^4 + b^4 + 2 \left( \frac{a+b}{2} \right)^4 - \frac{b^5 - a^5}{5(b-a)} = \frac{(b-a)^4}{120},
\]

which follows from (1.4), we get

\[
(2.11) \quad m_4 \frac{(b-a)^4}{2880} \leq T_\psi(a,b) \leq \frac{(b-a)^2}{162} \left[ \frac{\psi''(a) + \psi''(b)}{2} - \psi''\left( \frac{a+b}{2} \right) \right] - \frac{11}{9} m_4 \frac{(b-a)^4}{2880}.
\]

Consequently, taking \( \phi(t) = M_4 t^4/24 - \psi(t), t \in E \) we have that \( \phi''(\cdot) \) is a convex function on \( E \). Hence, using Theorem 2.8 again, we obtain

\[
(2.12) \quad \frac{(b-a)^2}{162} \left[ \frac{\psi''(a) + \psi''(b)}{2} - \psi''\left( \frac{a+b}{2} \right) \right] - \frac{11}{9} M_4 \frac{(b-a)^4}{2880} \leq T_\psi(a,b) \leq M_4 \frac{(b-a)^4}{2880}.
\]

Now, (2.11) and (2.12) give

\[
\frac{9}{20} T_\psi(a,b) \leq \frac{(b-a)^2}{360} \left[ \frac{\psi''(a) + \psi''(b)}{2} - \psi''\left( \frac{a+b}{2} \right) \right] - \frac{11}{20} m_4 \frac{(b-a)^4}{2880},
\]

and

\[
\frac{11}{20} T_\psi(a,b) \leq \frac{11}{20} M_4 \frac{(b-a)^4}{2880}.
\]

Adding these inequalities, we obtain

\[
T_\psi(a,b) - \frac{(b-a)^2}{360} \left[ \frac{\psi''(a) + \psi''(b)}{2} - \psi''\left( \frac{a+b}{2} \right) \right] \leq \frac{11}{57600} (M_4 - m_4)(b-a)^4.
\]

Analogously, adjusting the left-hand sides of (2.11) and (2.12), we get

\[
\frac{9}{20} T_\psi(a,b) \geq \frac{(b-a)^2}{360} \left[ \frac{\psi''(a) + \psi''(b)}{2} - \psi''\left( \frac{a+b}{2} \right) \right] - \frac{11}{20} M_4 \frac{(b-a)^4}{2880},
\]

and

\[
\frac{11}{20} T_\psi(a,b) \geq \frac{11}{20} m_4 \frac{(b-a)^4}{2880}.
\]
Therefore,
\[ T_\psi(a, b) = \frac{(b - a)^2}{360} \left[ \psi''(a) + \psi''(b) - \frac{\psi''(a + b)}{2} \right] \geq \frac{11}{57600} (M_4 - m_4) (b - a)^4, \]
and the proof is done.

\[ \square \]

3. Applications

As an illustration of results given in this article, we shall prove the next assertions.

**Theorem 3.1.** For \( x > y > 0 \), we have

\[ \frac{2}{3} - \frac{\coth x - \coth y}{x - y} - \frac{16}{243} (x^2 + xy + y^2) \leq \frac{1}{x - y} \int_y^x \frac{\coth t}{t} dt \leq \frac{2}{3} - \frac{\coth x - \coth y}{x - y}. \]

**Proof.** In Theorem 2.8 take \( \phi(u) = \phi''(u) = \cosh u, \ u \in [-2t, 2t] \).

We obtain,

\[ \frac{\cosh 2t}{3} + \frac{2}{3} - \frac{8}{81} t^2 (\cosh 2t - 1) \leq \frac{\sinh 2t}{2t} \leq \frac{\cosh 2t}{3} + \frac{2}{3}. \]

Since \( \sinh 2t = 2 \sinh t \cosh t \) and \( \cosh 2t = 1 + 2 \sinh^2 t \), dividing both sides of (3.2) by \( \sinh^2 t \), we get

\[ \coth^2 t - \frac{1}{3} - \frac{16}{81} t^2 \leq \frac{\coth t}{t} \leq \coth^2 t - \frac{1}{3}. \]

Integrating (3.3) over \( t \in [y, x], \ y > 0 \), the desired result follows. \( \square \)

In this case, Theorem 2.10 gives

**Theorem 3.4.** For \( x > y > 0 \), we have

\[ \left| \frac{2}{3} - \frac{\coth x - \coth y}{x - y} - \frac{4}{45} (x^2 + xy + y^2) - \frac{1}{x - y} \int_y^x \frac{\coth t}{t} dt \right| \leq \frac{22}{1125} \frac{x^5 - y^5}{x - y}. \]

Proof is left to the reader.

**Remark 3.5.** In the same way it is possible to approximate integrals of the form

\[ \int_a^b \frac{g(t)}{t \tanh t}, \]

where \( g(\cdot) \) is a non-negative function on \( E \).
4. Conclusion

The results of this paper are of purely theoretical nature. Namely, we considered the cases when the classical Simpson's Rule is not applicable, although they are rare in practice. An open problem of determining best possible constants in Theorems 2.2 and 2.3 and our solution in Theorem 2.5 are of the same kind. Comparison of the classical form $T_\phi(a,b)$ given in (1.4) and new form $T'_\phi(a,b)$ from Theorem 2.10 clearly shows that the later is much more precise.

For instance,

$$T_{x^4}(a,b) = (b-a)^4/120, \ T'_{x^4}(a,b) = 0; \ T_{x^5}(a,b) = (a+b)(b-a)^4/48, \ T'_{x^5}(a,b) = 0.$$

Further analysis and the composite form of the new Simpson's Rule is left to the interested reader.

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Mathematical Institute SANU, Kneza Mihaila 36, Belgrade, Serbia

*Email address: ssimic@turing.mi.sanu.ac.rs*