INTEGRAL POINTS FOR DRINFELD MODULES

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Abstract. We prove that in the backward orbit of a non-preperiodic point under the action of a Drinfeld module of generic characteristic there exist at most finitely many points S-integral with respect to another nonpreperiodic point. This provides the answer (in positive characteristic) to a question raised by Sookdeo in [22]. We also prove that for each nontorsion point \( z \) there exist at most finitely many torsion points which are S-integral with respect to \( z \). This proves a question raised by Tucker and the author in [11], and it gives the analogue of Ih’s conjecture [3] for Drinfeld modules.

1. Introduction

Let \( k \) be a number field, let \( S \) be a finite set of places of \( k \) containing all its archimedean places, and let \( \mathfrak{o}_k,S \) be the subring of S-integers contained in \( k \). The following two conjectures were made by Ih (and refined by Silverman and Zhang) as an analogue of the classical diophantine problems of Mordell-Lang, Manin-Mumford, and Lang; for more details, see [3].

Conjecture 1.1. Let \( A \) be an abelian variety defined over \( k \), and let \( A_S/\text{Spec}(\mathfrak{o}_k,S) \) be a model of \( A \). Let \( D \) be an effective divisor on \( A \), defined over \( k \), at least one of whose irreducible components is not the translate of an abelian subvariety by a torsion point, and let \( D_S \) be its Zariski closure in \( A_S \). Then the set of all torsion points of \( A(k) \) whose closure in \( A_S \) is disjoint from \( D_S \), is not Zariski dense in \( A \).

The next conjecture is for algebraic dynamical systems, and it is modeled by Conjecture 1.1 for elliptic curves where torsion points are seen as preperiodic points for the multiplication-by-2-map. In general, for any rational map \( f \), we say that \( \alpha \) is a preperiodic point for \( f \) if its orbit under \( f \) is finite. As always in arithmetic dynamics, we denote by \( f_n \) the \( n \)-th iterate of \( f \). So, \( \alpha \) is preperiodic if and only if there exist nonnegative integers \( m \neq n \) such that \( f^m(\alpha) = f^n(\alpha) \) (for more details on the theory of arithmetic dynamics we refer the reader to Silverman’s book [21]).

Conjecture 1.2. Let \( f \) be a rational function of degree at least 2 defined over \( k \), and let \( \alpha \in \mathbb{P}^1(k) \) be non-preperiodic for \( f \). Then there are only finitely many preperiodic points which are S-integral with respect to \( \alpha \), i.e. whose Zariski closures in \( \mathbb{P}^1/\text{Spec}(\mathfrak{o}_k,S) \) do not meet the Zariski closure of \( \alpha \).

In [3], Baker, Ih and Rumely prove the first cases of the above conjectures. They prove Conjecture 1.1 for elliptic curves \( A \), which in particular provides a proof of Conjecture 1.2 for Lattès maps \( f \). Also in [3], the authors prove Conjecture 1.2 when \( f \) is a powering map (same proof works when \( f \) is a Chebyshev polynomial). The ingredients for proving their results are a strong form of equidistribution for

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torsion points of 1-dimensional algebraic groups (such as elliptic curves or $\mathbb{G}_m$), lower bounds for linear forms in (elliptic) logarithms, and a good understanding of the local heights associated to a dynamical system coming from an algebraic group endomorphism. At this moment it seems difficult to extend the above results beyond the case of 1-dimensional abelian varieties, or the case of rational maps associated to endomorphisms of 1-dimensional algebraic groups.

It is natural to consider the above conjectures in characteristic $p$, but one has to be careful in their reformulation. Indeed, if $A$ is defined over $\bar{\mathbb{F}}_p$ then all its torsion points are also defined over $\bar{\mathbb{F}}_p$ and thus one can find infinitely many torsion points which are $S$-integral with respect to a divisor $D$ of $A$. Similarly, if $f$ is a rational map defined over $\bar{\mathbb{F}}_p$, all its preperiodic points are also in $\bar{\mathbb{F}}_p$ and thus again one can find infinitely many points which are $S$-integral with respect to a non-preperiodic point $a$. On the other hand, the Drinfeld modules have always proven to be the right analogue in characteristic $p$ of abelian varieties. Therefore we propose to study in this paper analogues of the above conjectures for Drinfeld modules. One could consider an analogue of Conjecture 1.1 for $T$-modules acting on $\mathbb{G}_m$, but similar to the case over number fields, proving any result towards Conjecture 1.1 (or its analogues) for groups varieties of dimension larger than 1 would be difficult.

We start by defining Drinfeld modules (for more details, see Section 2). Let $p$ be a prime number, let $q$ be a power of $p$, and let $K$ be a finite extension of $\mathbb{F}_q(t)$. A Drinfeld module $\Phi$ (of generic characteristic) is a ring homomorphism from $\mathbb{F}_q[t]$ to $\text{End}_K(\mathbb{G}_a)$. We fix an algebraic closure $\bar{K}$ of $K$, and we let $K^{\text{sep}}$ be the separable closure of $K$ inside $\bar{K}$.

Since each Drinfeld module is isomorphic (over $K^{\text{sep}}$) to a Drinfeld module for which $\Phi_t := \Phi(t)$ is a monic polynomial, we assume from now on that $\Phi$ is indeed in normal form i.e., $\Phi_t$ is monic. Note that a Drinfeld module $\Psi$ is isomorphic to $\Phi$ (over $K^{\text{sep}}$) if there exists $\gamma \in K^{\text{sep}}$ such that $\Psi_t(x) = \gamma^{-1} \Phi_t(\gamma x)$. So, our results about $S$-integral points are not affected by replacing $\Phi$ with an isomorphic Drinfeld module since conjugating by $\gamma$ only affects the finitely many places where $\gamma$ is not an $S$-unit (for a precise definition for $S$-integrality we refer the reader to Subsection 2.3).

The points of $K^{\text{sep}}$ which have finite orbit under the action of $\Phi$ are called torsion; we denote by $\Phi_{\text{tor}}$ the set of all torsion points for $\Phi$.

A Drinfeld module may have complex multiplication (similar to the case of abelian varieties), i.e. there exist endomorphisms $g$ of $\mathbb{G}_a$ defined over $K^{\text{sep}}$ such that $g \circ \Phi_t = \Phi_t \circ g$. In our results we assume $\Phi$ does not have complex multiplication since we employ a strong equidistribution theorem from [4] for torsion points of Drinfeld modules which uses the assumption that $\Phi$ has no complex multiplication.

The places of $K$ split in two categories: infinite places and finite places depending on whether they lie over the place $v_{\infty}$ of $\mathbb{F}_q(t)$, for which $v_{\infty}(f/g) = \deg(g) - \deg(f)$, for all nonzero $f, g \in \mathbb{F}_q[t]$. We assume $\Phi$ has good reduction at all its finite places, i.e., for all finite places $v$ of $K$, the coefficients of $\Phi_t$ are $v$-integral (recall that we already assumed that $\Phi_t$ is monic). Also, for each place $v$ of $K$ we fix an extension of it to $K^{\text{sep}}$. Then we can prove the following result.

**Theorem 1.3.** Assume $\Phi$ is in normal form and it has good reduction at all finite places of $K$, and also assume that $\Phi$ has no complex multiplication. Let $\beta \in K$ be a nontorsion point for $\Phi$, and let $S$ be a finite set of places of $K$. Then there exist at most finitely many $\gamma \in \Phi_{\text{tor}}$ such that $\gamma$ is $S$-integral with respect to $\beta$. 
Theorem 1.4. Let \( \Phi \) be a Drinfeld module as in Theorem 1.3. Let \( \beta \in K \), and let \( \{ \gamma_n \} \subset \Phi_{\text{tor}} \) be an infinite sequence. Then
\[
\sum_{v \in \Omega_K} \lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K_{\text{sep}}/K)} \log |\beta - \gamma^n|_v = \hat{h}_\Phi(\beta).
\]

In the above result, and also later in the paper, a sum involving \( \delta^n \) over all \( \sigma \in \text{Gal}(K_{\text{sep}}/K) \) is simply a sum over all the Galois conjugates of \( \delta \).

Theorem 1.4 answers a conjecture of Tucker and the author (\cite[Conjecture 1.2]{tucker}). It is worth pointing out that there is no Bogomolov-type statement of Ih’s Conjecture for Drinfeld modules, i.e., it is possible to find infinitely many points \( x \) of small canonical height which are \( S \)-integral with respect to a given nontorsion point \( \beta \) (see Example 2.4).

Theorem 1.5 may be interpreted as follows: for a nontorsion point \( \beta \) there exist at most finitely many \( \gamma \) such that \( \Phi_Q(\gamma) = 0 \) for some nonzero polynomial \( Q \in \mathbb{F}_q[t] \), and \( \gamma \) is \( S \)-integral with respect to \( \beta \). In other words, there exist at most finitely many points in the backward orbit of 0 (under \( \Phi \)) which are \( S \)-integral with respect to \( \beta \). In general, for each \( \alpha \in K \) we let \( O_\Phi(\alpha) \) be the backward orbit of \( \alpha \) under \( \Phi \), i.e., the set of all \( \beta \in K_{\text{sep}} \) such that there exists some nonzero \( Q \in \mathbb{F}_q[t] \) such that \( \Phi_Q(\beta) = \alpha \). So, Theorem 1.5 yields that for each torsion point \( \alpha \) of \( \Phi \), there exist at most finitely many \( \gamma \in O_\Phi(\alpha) \) which are \( S \)-integral with respect to the nontorsion point \( \beta \). It is natural to ask whether Theorem 1.5 holds when the point \( \alpha \) is nontorsion.

Theorem 1.5. Let \( \Phi \) be a Drinfeld module without complex multiplication, and let \( \alpha, \beta \in K \) be nontorsion points. If \( \alpha \notin \Phi_{\text{tor}} \), then there exist at most finitely many \( \gamma \in O_\Phi(\alpha) \) such that \( \gamma \) is \( S \)-integral with respect to \( \beta \).

Theorem 1.5 is an analogue for Drinfeld modules of a result conjectured by Sookdeo in \cite{sookdeo}. Motivated by a question of Silverman for \( S \)-integral points in the forward orbit of a rational map defined over \( \mathbb{Q} \), Sookdeo \cite[Conjecture 1.2]{sookdeo} asks whether for a rational map \( f \) defined over \( \mathbb{Q} \), and for a given non-preperiodic point \( \alpha \) of \( \Phi \), there exist at most finitely many points \( \gamma \) in the forward orbit of \( \alpha \) (i.e., \( f^n(\gamma) = \alpha \) for some nonnegative integer \( n \)) such that \( \gamma \) is \( S \)-integral with a given non-preperiodic point \( \beta \). In \cite{sookdeo}, Sookdeo proves that his conjecture holds when \( f \) is either a powering map, or a Chebyshev polynomial. For an arbitrary rational map \( f \), the conjecture appears to be difficult. Furthermore, Sookdeo presents a general strategy of attacking his conjecture by reducing it to proving that the number of Galois orbits for \( f^n(\gamma) - \beta \) is bounded above independently of \( n \).

We observe that Theorems 1.5 and 1.3 combine to yield the following result.

Theorem 1.6. Let \( \Phi \) and \( K \) be as in Theorem 1.3, and let \( \alpha, \beta \in K \) such that \( \beta \) is not a torsion point for \( \Phi \). Then there exist at most finitely many \( \gamma \in O_\Phi(\alpha) \) such that \( \gamma \) is \( S \)-integral with respect to \( \beta \).

It is essential to ask that \( \beta \) is not a torsion point for \( \Phi \) in Theorem 1.0 since otherwise the result is false. Indeed, if \( \beta \) were equal to 0, say, then one can find infinitely many points \( \gamma \in O_\Phi(\alpha) \) which are \( S \)-integral with respect to 0, for any
set of places $S$ which contains the infinite places and also all the places where $\alpha$ is not a unit. Indeed, if $\Phi_Q(\gamma) = \alpha$ for some nonzero $Q \in \mathbb{F}_q[t]$ then for each $v \in S$, we have that if $|\gamma|_v > 1$ then $|\alpha|_v > 1$, while if $|\gamma|_v < 1$ then $|\alpha|_v < 1$ (because $\Phi$ has good reduction at $v$). Hence, each $\gamma \in \mathcal{O}_\Phi^r(\alpha)$ is $S$-integral with respect to 0.

Instead of considering $S$-integral points in the backward orbit of a point $\alpha$ under a rational map $f \in \mathbb{F}(z)$, one could consider $S$-integral points in its forward orbit. This question was settled by Silverman [20] who showed (as an application of Siegel’s classical theorem [19] on $S$-integral points on curves) that if there exist infinitely many $S$-integers in the forward orbit of $\alpha$, then $f \circ f$ is totally ramified at infinity, i.e., $f \circ f$ is a polynomial. In our setting for Drinfeld modules, studying $S$-integrality of points of the form $\Phi_Q(\alpha)$ (for arbitrary $Q \in \mathbb{F}_q[t]$) relative to a given element $\beta$ was done by Tucker and the author in [9] and [12] (where a Siegel-type theorem for Drinfeld modules was proven). The result from [12] will be used in the proof of Theorem 1.5. The results of our present paper complete the picture for Drinfeld modules by studying the $S$-integrality of points in the backward orbit of a point.

Here is the plan of our paper. In Section 2 we introduce the notation and also state a crucial result (of Häberli [14] and Pink [16]) on Galois orbits of points in the backward orbit of a given point $\alpha$ (see Theorem 2.2). This last result allows us to prove Theorem 1.5. In Section 3 we derive some results regarding torsion of Drinfeld modules which have everywhere good reduction away from the places at infinity. Finally, we conclude our proof of Theorems 1.3 and 1.4 in Section 4.

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2. Generalities

2.1. Drinfeld modules. Let $p$ be a prime number, and let $q$ be a power of $p$. We let $K$ be a finite extension of $\mathbb{F}_q(t)$, let $\overline{K}$ be a fixed algebraic closure of $K$ and $K^{\text{sep}}$ be the separable closure of $K$ inside $K^{\text{sep}}$.

Let $\Phi : \mathbb{F}_q[t] \rightarrow \text{End}_K(\mathcal{G}_a)$ be a Drinfeld module of generic characteristic, i.e., $\Phi$ is a ring homomorphism with the property that

$$\Phi(t)(x) := \Phi_t(x) = tx + \sum_{i=1}^r a_i x^i$$

where $r \geq 1$ and each $a_i \in K$. We call $r$ the rank of the Drinfeld module $\Phi$.

We note that usually a Drinfeld module is defined on a ring $A$ of functions defined on a projective irreducible curve $C$ defined over $\mathbb{F}_q$, which are regular away from a given point $\eta$ on $C$. In our definition, $C = \mathbb{P}^1$ and $\eta$ is the point at infinity. There is no loss of generality for using our definition since $\mathbb{F}_q[t]$ embeds into each such ring $A$ of functions.

At the expense of replacing $K$ with a finite extension, and replacing $\Phi$ with an isomorphic Drinfeld module $\Psi = \gamma^{-1} \circ \Phi \circ \gamma$, for a suitable $\gamma \in K^{\text{sep}}$, we may assume $\Phi_t$ is monic. We say in this case that $\Phi$ is in normal form.

2.2. Endomorphisms of $\Phi$. We say that $f \in K^{\text{sep}}[x]$ is an endomorphism of $\Phi$ if $\Phi_t \circ f = f \circ \Phi_t$, or equivalently if $\Phi_a \circ f = f \circ \Phi_a$ for all $a \in \mathbb{F}_q[t]$. The set of all endomorphisms of $\Phi$ is $\text{End}_{K^{\text{sep}}} (\Phi)$. Generically, $\Phi$ has no endomorphism other than $\Phi_a$ for $a \in \mathbb{F}_q[t]$; in this case $\text{End}_{K^{\text{sep}}} (\Phi) \approx \mathbb{F}_q[t]$.
2.3. **Torsion points.** For each nonzero $Q \in \mathbb{F}_q[t]$, the set of all $x \in K^\text{sep}$ such that $\Phi_Q(x) := \Phi(Q)(x) = 0$ is defined to be $\Phi[Q]$; each such point $x$ is called a torsion point for $\Phi$. The set of all torsion points for $\Phi$ is denoted by

$$\Phi_{\text{tor}} := \bigcup_{Q \in \mathbb{F}_q[t] \setminus \{0\}} \Phi[Q].$$

One can show that for each nonzero $Q \in \mathbb{F}_q[t]$, we have $\Phi[Q] \sim (\mathbb{F}_q[t]/(Q))^n$. Moreover, since each $\Phi_Q$ is a separable polynomial, we obtain that $\Phi_{\text{tor}} \subset K^\text{sep}$. Furthermore, since any polynomial $f \in \overline{K}[z]$ satisfying $f \circ \Phi_t = \Phi_t \circ f$ has the property that $f(\Phi_{\text{tor}}) \subset \Phi_{\text{tor}}$ we conclude that $f \in K^\text{sep}[z]$; i.e., all endomorphisms of $\Phi$ are indeed defined over $K^\text{sep}$. For more details on Drinfeld modules, see [13].

2.4. **Places of $K$.** Let $\Omega_K$ be the set of all places of $K$. The places from $\Omega_K$ lie above the places of $\mathbb{F}_q(t)$. We normalize each corresponding absolute value $| \cdot |_v$ so that we have a well-defined product formula for each nonzero $x \in K$:

$$\prod_{v \in M_K} |x|_v = 1.$$

Furthermore, we may assume $\log |t|_\infty = 1$, where $\infty := v_\infty$ is the place of $\mathbb{F}_q(t)$ corresponding to the degree of rational functions, i.e. (in exponential form), $v_\infty(f/g) = \deg(g) - \deg(f)$ for any nonzero $f, g \in \mathbb{F}_q[t]$. We let $S_\infty$ be the set of (infinite) places in $\Omega_K$ which lie above $v_\infty$. If $v \in \Omega_K \setminus S_\infty$, then we say that $v$ is a finite place for $\Phi$. For each $v \in \Omega_K$, we fix a completion $K_v$ of $K$ with respect to $v$, and also we fix an embedding of $K^\text{sep}$ into $\overline{K}_v$. Finally, we let $\mathbb{C}_v$ be the completion with respect to $| \cdot |_v$ of a fixed algebraic closure of $K_v$.

2.5. **$S$-integrality.** For a set of places $S \subset M_K$ and $\alpha \in K$, we say that $\beta \in K^\text{sep}$ is $S$-integral with respect to $\alpha$ if for every place $v \not\in S$, and for every morphism $\sigma \in \text{Gal}(K^\text{sep}/K)$ the following are true:

- if $|\alpha|_v \leq 1$, then $|\alpha - \beta^\sigma|_v \geq 1$.
- if $|\alpha|_v > 1$, then $|\beta^\sigma|_v \leq 1$.

For more details about the definition of $S$-integrality, we refer the reader to [3].

We note that if $\Psi$ is an arbitrary Drinfeld module and $\Phi$ is a normalized Drinfeld module isomorphic to $\Psi$ through the conjugation by $\gamma \in K^\text{sep}$ (i.e., $\Phi_t(x) = \gamma^{-1} \Psi_t(\gamma x)$), then all torsion points of $\Phi$ are of the form $\gamma^{-1} \cdot z$, where $z \in \Psi_{\text{tor}}$. So, if we let $K$ be a finite extension of $\mathbb{F}_q(t)$ containing $\gamma$, and let $S$ be a finite set of places of $K$ containing all places where $\gamma$ is not a unit, then $\alpha \in \Psi_{\text{tor}}$ is $S$-integral with respect to $\beta \in K$ if and only if $\gamma^{-1} \alpha \in \Phi_{\text{tor}}$ is $S$-integral with respect to $\gamma^{-1} \beta$. Therefore, our hypothesis from Theorems 1.3 and 1.5 (since a similar statement holds for the backward orbit of a nontorsion point) that $\Phi$ is in normal form is not essential. However, we prefer to work with Drinfeld modules in normal form since this makes easier to define the notion of good reduction for $\Phi$, and also simplifies some of the technical points in our proof.

2.6. **Good reduction for $\Phi$.** We say that the Drinfeld module $\Phi$ defined over $K$ has good reduction at a place $v$ of $K$, if each coefficient of $\Phi_t$ is integral at $v$, and moreover, the leading coefficient of $\Phi_t$ is a $v$-adic unit. It is immediate to see that if $\Phi$ has good reduction at $v$, then for each nonzero $Q \in \mathbb{F}_q[t]$ we have that $\Phi_Q$ has all its coefficients $v$-adic integers, while its leading coefficient is a $v$-adic unit. Clearly, a Drinfeld module cannot have good reduction at a place in $S_\infty$. Since we assumed
that \( \Phi \) is in normal form, the condition of having good reduction at \( v \) reduces to the fact that \( \Phi \) has all its coefficients \( v \)-adic integers.

2.7. The action of the Galois group. The Galois group \( \text{Gal}(K^{\text{sep}}/K) \) acts on each \( \Phi(Q) \). In particular, \( \text{Gal}(K^{\text{sep}}/K) \) acts on the associated Tate module \( T_{\Phi}(\Phi) \) for \( \Phi \) corresponding to irreducible, monic \( Q \in F_q[t] \); the Tate module is isomorphic to \( F_q(t)_{(Q)} \) (where \( F_q(t)_{(Q)} \) is the \( (Q) \)-adic completion of \( F_q[t] \)) at the prime ideal \( (Q) \). We let \( T := \prod Q T_Q \) (where the product is over all the monic irreducible polynomials \( Q \in F_q[t] \)) be the profinite completion of \( F_q[t] \). Therefore we obtain a continuous representation

\[
\rho : \text{Gal}(K^{\text{sep}}/K) \rightarrow \text{GL}_r(T).
\]

Pink and Rütsche [17] proved that the image of \( \rho \) is open, assuming that \( \Phi \) has no complex multiplication; their result is a Drinfeld module analogue of the classical Serre Openess Conjecture for (non-CM) abelian varieties. Initially, Breuer and Pink [3] proved that the image of \( \rho \) is open assuming \( K \) is a transcendental extension of \( F_q(t) \), which can be viewed as the function field version of the result from [17]. In particular, Pink-Rütsche result yields that for any torsion point \( \gamma \) of order \( Q \), where \( Q(t) \in F_q[t] \) is a polynomial of degree \( d \), there exists a positive constant \( c_\Phi \) such that,

\[
\frac{[K(\gamma) : K]}{\# \text{GL}_r(F_q[t]/(Q))} \geq c_\Phi.
\]

Hence, \([K(\gamma) : K] \gg q^{-d}\).

Now, let \( x \in K \) be a nontorsion point for \( \Phi \). For each monic irreducible \( Q \in F_q[t] \), for each \( \sigma \in \text{Gal}(K^{\text{sep}}/K) \), and for each sequence \( \{x_i\}_{i \in \mathbb{N}} \subset K^{\text{sep}} \) such that \( \Phi_Q(x_{i+1}) = x_i \) for each \( i \in \mathbb{N} \) while \( \Phi_Q(x_1) = x \), we consider the map

\[
\sigma \mapsto \{\sigma(x_i) - x_i\}_{i \in \mathbb{N}}.
\]

This yields another continuous representations

\[
\Psi_Q : \text{Gal}(K^{\text{sep}}/K) \rightarrow T_Q
\]

and more generally

\[
\Psi : \text{Gal}(K^{\text{sep}}/K) \rightarrow T.
\]

Furthermore, we have the following Galois action on the entire backward orbit of \( x \) under the action of \( \Phi \)

\[
\tilde{\Psi} : \text{Gal}(K^{\text{sep}}/K) \rightarrow T \times \text{GL}_r(T).
\]

Häberli [14] proved that the image of \( \tilde{\Psi} \) is open, again assuming that \( \Phi \) has no complex multiplication. The assumption regarding the endomorphism ring for \( \Phi \) is crucial; however Pink [16] proved an appropriately modified statement when \( \Phi \) has complex multiplication. In particular, Häberli’s result [14] yields that there exists a number \( d := d(x) \in \mathbb{N} \) (bounded above by the index of the image of \( \tilde{\Psi} \) in \( T \times \text{GL}_r(T) \)) such that for each nonzero \( Q \in F_q[t] \), there are at most \( d \) distinct Galois orbits containing all the preimages of \( x \) under \( \Phi_Q \). Hence, we have the following theorem.

**Theorem 2.2.** Let \( \Phi : F_q[t] \rightarrow \text{End}_K(G_a) \) be a Drinfeld module such that \( \text{End}_{K^{\text{sep}}}(\Phi) \rightarrow F_q[t] \). Then for each \( \alpha \in K \) which is not torsion, there exists a number \( d(\alpha) \) such that for each nonzero \( Q \in F_q[t] \) there exist at most \( d(\alpha) \) distinct Galois orbits of points \( y \in K^{\text{sep}} \) satisfying \( \Phi_Q(y) = x \).
Theorem 2.2 allows us to complete the proof of Theorem 1.5.

Proof of Theorem 1.5. Our proof follows the strategy from [22, Theorems 2.5 and 2.6]. For each nonzero \( Q \in \mathbb{F}_q[t] \) we let \( d_Q(\alpha) \) be the number of Galois orbits contained in \( \Phi_Q^{-1}(\alpha) \). Then \( d_Q | d_{Q_1} \) whenever \( Q_1 | Q_2 \), and \( d_{Q_1}d_{Q_2} = d_{Q_1}d_{Q_2} \) whenever \( \gcd(Q_1, Q_2) = 1 \). Indeed, since \( \gcd(Q_1, Q_2) = 1 \) there exist \( R_1, R_2 \in \mathbb{F}_q[t] \) such that \( R_1Q_1 + R_2Q_2 = 1 \). Then for each pair \( (\delta_1, \delta_2) \in \Phi_Q^{-1}(\alpha) \times \Phi_Q^{-1}(\alpha) \), we let \( \delta_{1,2} := \Phi_{R_1}(\delta_1) + \Phi_{R_2}(\delta_2) \in \Phi_Q^{-1}(\alpha) \). Furthermore, if \( (\delta_1', \delta_2') \) is another such pair such that \( \delta_i' \) is not Galois conjugate with \( \delta_i \) for some \( i \in \{1, 2\} \), then \( \delta_{1,2}' \) is not Galois conjugate with \( \delta_{1,2} \). Indeed, if there is some \( \sigma \in \text{Gal}(K^{sep}/K) \) such that \( \delta_{1,2}' = \Phi_{\delta_1, \delta_2}(\delta_{1,2}) \) and thus

\[
\Phi_{Q_1R_1}(\delta_{1,2}') + \Phi_{R_1}(\alpha) = \Phi_{Q_1R_2}(\delta_{1,2}') + \Phi_{R_1}(\alpha),
\]

which yields that \( \delta_{1,2}' - \delta_{1,2} \in \Phi_{Q_1R_2}(\alpha) \). On the other hand, \( \delta_{1,2}' - \delta_{1,2} \in \Phi_{Q_1}(\alpha) \) (since both are in \( \Phi_Q^{-1}(\alpha) \)). Because \( \gcd(Q_1, Q_2) = 1 \) we conclude that \( \delta_{1,2}' = \delta_{1,2} \); similarly we get that \( \delta_{1,2}' = \delta_{1,2} \) which yields that the pairs of points \( (\delta_1', \delta_2') \) and \( (\delta_1, \delta_2) \) are Galois conjugate, contrary to our assumption.

So indeed, \( d_QQ_2 = d_Qd_{Q_2} \) if \( \gcd(Q_1, Q_2) = 1 \). By Theorem 2.2 \( d_Q \) is bounded above independently of \( Q \), which yields that there exists a nonzero \( P := P(\alpha) \in \mathbb{F}_q[t] \) such that for all \( Q \in \mathbb{F}_q[t] \) we have \( d_Q(\alpha) = d_R(\alpha) \), where \( R = \gcd(P, Q) \). In particular, with the above notation, if \( \delta \in \Phi_R^{-1}(\alpha) \) then all points in \( \Phi_Q^{-1}(\alpha) \) are Galois conjugates.

At the expense of replacing \( S \) by a larger set we may assume it contains all places where either \( \beta \) or \( \alpha \) is not a unit (note that neither \( \alpha \) nor \( \beta \) are 0 since they are nontorsion). Thus for each \( v \notin S \) we have that each point in \( \Phi_Q^{-1}(\alpha) \) is also a \( v \)-adic unit. Therefore a point \( \gamma \in \mathcal{O}_Q^{-1}(\alpha) \) is \( S \)-integral with respect to \( \beta \) if and only if for all \( v \notin S \) we have \( |\gamma - \beta|_v = 1 \).

Now, assume \( \gamma \in \Phi_Q^{-1}(\alpha) \) is \( S \)-integral with respect to \( \beta \) for some \( Q \in \mathbb{F}_q[t] \). Let \( R = \gcd(P, Q) \) (where \( P := P(\alpha) \in \mathbb{F}_q[t] \) is defined as above for \( \alpha \)), and let \( \delta \in \Phi_R^{-1}(\alpha) \) such that \( \gamma \in \Phi_Q^{-1}(\alpha) \). Because \( \gamma \) is \( S \)-integral with respect to \( \beta \), then for each conjugate \( \gamma^\sigma \), and for each place \( v \notin S \), we have \( |\gamma^\sigma - \beta|_v = 1 \). Hence taking the product over all conjugates \( \gamma^\sigma \) satisfying \( \Phi_{Q/R}(\gamma^\sigma) = \delta \) we obtain

\[
|\Phi_{Q/R}(\beta - \gamma)|_v = |\Phi_{Q/R}(\beta - \gamma^\sigma)|_v = \prod_{\sigma} |\beta - \gamma^\sigma|_v = 1.
\]

In the above computation we used that \( \Phi_t \) is monic and therefore the leading coefficient of \( \Phi_{Q/R} \) is a \( v \)-adic unit since it is in \( \mathbb{F}_p \). Also, we used that

\[
|\Phi_{Q/R}(\beta - \gamma)|_v = \prod_{z \in \Phi_{Q/R}} |\beta - \gamma - z|_v = \prod_{\sigma} |\beta - \gamma^\sigma|_v,
\]

since all points in \( \Phi_{Q/R}^{-1}(\delta) \) are Galois conjugates, and therefore for each \( z \in \Phi_{Q/R} \) we have that \( \gamma + z = \gamma^\sigma \) for some \( \sigma \in \text{Gal}(K^{sep}/K) \).

So, letting \( S(\delta) \) be the places of \( K(\delta) \) which lie above the places from \( S \), we conclude that \( \Phi_{Q/R}(\beta) \) is \( S(\delta) \)-integral with respect to \( \delta \) (where the ground field is now \( K(\delta) \)). On the other hand, since \( \beta \notin \Phi_{\text{tor}}, \) Theorem 2.5 yields that there exist at most finitely \( Q/R \in \mathbb{F}_q[t] \) such that \( \Phi_{Q/R}(\beta) \) is \( S(\delta) \)-integral with respect
to $\delta$. Finally, noting that there are only finitely many
$$\delta \in \bigcup_{R|P} \Phi_{R}^{-1}(\alpha).$$
we conclude our proof.  

2.8. Canonical height associated to $\Phi$. For a point $x$ in $K^{sep}$ its usual Weil height is defined as
$$h(x) = \sum_{v \in M_K} \frac{1}{[K(x) : K]} \sum_{\sigma \in \text{Gal}(K^{sep}/K)} \log^+ |x^\sigma|_v,$$
where by $\log^+ z$ we always denote $\log \max\{z, 1\}$ (for any real number $z$).

The global canonical height $\hat{h}_\Phi(x)$ associated to the Drinfeld module $\Phi$ was first introduced by Denis [7] (Denis defined the global canonical heights for general $T$-modules which are higher dimensional analogue of Drinfeld modules). For each $x \in K^{sep}$, the global canonical height is defined as
$$\hat{h}_\Phi(x) = \lim_{n \to \infty} \frac{h(\Phi_{\text{tor}}(x))}{q^n},$$
where $h$ is the usual (logarithmic) Weil height on $K^{sep}$. Denis [7] showed that $\hat{h}$ differs from the usual Weil height $h$ by a bounded amount, and also showed that $x \in \Phi_{\text{tor}}$ if and only if $\hat{h}_\Phi(x) = 0$.

Following Poonen [18] and Wang [24], for each $x \in C_v$, the local canonical height of $x$ is defined as follows
$$\hat{h}_{\Phi, v}(x) := \lim_{n \to \infty} \frac{\log^+ |\Phi_{\text{tor}}(x)|_v}{q^n}.$$ 
It is immediate that $\hat{h}_{\Phi, v}(\Phi_{\text{tor}}(x)) = q^r \hat{h}_{\Phi, v}(x)$ and thus $\hat{h}_{\Phi, v}(x) = 0$ whenever $x \in \Phi_{\text{tor}}$.

Now, if $f(x) = \sum_{i=0}^d a_i x^i$ is any polynomial defined over $K$, then $|f(x)|_v = |a_d x^d|_v > |x|_v$ when $|x|_v > M_v$, where
$$M_v = M_v(f) := \max \left\{ \left( \frac{1}{|a_d|} \right)^{\frac{1}{d-1}}, \max \left\{ \left\| \frac{a_i}{a_d} \right\|_v^{\frac{1}{d-1}} \right\} \right\}_{0 \leq i < d}.$$ 
Moreover, for a Drinfeld module $\Phi$, if $|x|_v > M_v(\Phi_i)$ then $\hat{h}_{\Phi, v}(x) = \log |x|_v + \frac{\log|a_d|_v}{d-1} > 0$. In the special case that $\Phi$ has good reduction at $v$, then $M_v(\Phi_i) = 1$ and so, if $|x|_v > 1$ then $\hat{h}_{\Phi, v}(x) = \log |x|_v$, while if $|x|_v \leq 1$ then $\hat{h}_{\Phi, v}(x) = 0$.

We define the $v$-adic filled Julia set $J_v$ be the set of all $x \in C_v$ such that $\hat{h}_{\Phi, v}(x) = 0$. So, we know that if $x \in J_v$, then $|x|_v \leq M_v$. In particular, if $v$ is a place of good reduction for $\Phi$, then $J_v$ is the unit disk.

As shown in [18] and [24], the global canonical height decomposes into a sum of the corresponding local canonical heights, as follows
$$\hat{h}_\Phi(x) = \sum_{v \in M_K} \frac{1}{[K(x) : K]} \sum_{\sigma \in \text{Gal}(K^{sep}/K)} \hat{h}_{\Phi, v}(x^\sigma).$$ 
We note that the theory of canonical height associated to a Drinfeld module is a special case of the canonical heights associated to morphisms on algebraic varieties
developed by Call and Silverman (see [6] for details). The definition for the canonical height functions given above seems to depend on the particular choice of the map \( \Phi \). On the other hand, one can define the canonical heights \( \hat{h}_\Phi \) as in [7] by letting

\[
\hat{h}_\Phi(x) = \lim_{\deg(R) \to \infty} \frac{h(\Phi_R(x))}{q^\deg(R)},
\]

and similar formula for canonical local heights \( \hat{h}_{\Phi,v} \) where \( R \) runs through all non-constant polynomials in \( \mathbb{F}_q[t] \). In [13] and [24] it is proven that both definitions yield the same height function.

Finally, we observe that Ingram [15] defined the local canonical height in a slightly different way, i.e., Ingram’s local canonical height \( h_v(x) \) for a point \( x \in \mathbb{C}_v \) is defined as follows:

\[
\lambda_v(x) := \hat{h}_{\Phi,v}(x) - \log |x|_v + c_v,
\]

where \( c_v := - \log |a_v|_v/(q' - 1) \), where \( a_v \) is the leading coefficient of \( \Phi \). Using the product formula, we conclude that if \( x \in K^{\text{sep}} \) then

\[
\hat{h}_\Phi(x) = \sum_{v \in \Omega_K} \frac{1}{[K(x) : K]} \sum_{\sigma \in \text{Gal}(K^{\text{sep}}/K)} \lambda_v(x^\sigma).
\]

Furthermore, for normalized Drinfeld modules, one has

\[
\lambda_v(x) = \hat{h}_{\Phi,v}(x) - \log |x|_v.
\]

The advantage in Ingram’s definition is that \( \lambda_v(x) := G_v(x, 0) \), where \( G_v(x, y) \) is the Green’s function for the dynamical system induced by the action of \( \Phi \) on the affine line.

2.9. Points of small canonical height. Next we give a construction showing that there is no Bogomolov-type statement of Ih’s Conjecture for Drinfeld modules, i.e., there exist infinitely many points of canonical height arbitrarily small which are \( S \)-integral with respect to a given nontorsion point \( \beta \).

Example 2.4. Indeed, let \( \Phi \) be the Carlitz module corresponding to \( \Phi_t(x) = tx+x^p \), where \( p \) is a prime number, and let \( \beta = 1 \). Then \( \beta \) is nontorsion for \( \Phi \) because \( \deg(\Phi_t(1)) = p^{n-1} \) for all \( n \geq 1 \). For each positive integer \( n \), consider \( x_n \in \mathbb{F}_p(t)^{\text{sep}} \) which is a root of the equation

\[
\Phi_t(n)(z) \cdot (z - 1) = 1.
\]

We let \( S = \{v_\infty\} \subset \Omega_{\mathbb{F}_p(t)} \). Then it is immediate to see that \( x_n \) must be \( v \)-integral for each \( v \notin S \) (otherwise \( \deg(\Phi_t(x_n))_v > 1 \) and also \( |x_n - 1|_v > 1 \), which is a contradiction). Furthermore, if \( |x_n - 1|_v < 1 \) for \( v \notin S \), then \( |\Phi_t(n)(x_n)|_v > 1 \), which is again a contradiction since it would imply that \( |x_n|_v > 1 \). Similarly, for each conjugate \( x_n^a \) we have \( |x_n^a - 1|_v = 1 \) for each \( v \notin S \). On the other hand, \( x_n \) has height tending to 0. Indeed, as shown by Denis [7] there exists a positive constant \( C \) such that \( |h(z) - \hat{h}_\Phi(z)| \leq C \) for all algebraic points \( z \). So,

\[
p^n \hat{h}_\Phi(x_n) = \hat{h}_\Phi(\Phi_t(n)(x_n)) = \hat{h}_\Phi \left( \frac{1}{x_n - 1} \right) \leq h \left( \frac{1}{x_n - 1} \right) + C = h(x_n) + C \leq \hat{h}(x_n) + 2C.
\]

Hence, \( \hat{h}_\Phi(x_n) \to 0 \) as \( n \to \infty \).
3. Preliminary Results on Torsion Points

In this section we assume that $\Phi$ is in normal form and that it has good reduction at all finite places of $K$.

**Lemma 3.1.** Let $s$ be a real number in $(0, 1)$. If $v \in \Omega_K \setminus S_\infty$, then there exist at most finitely many $x \in \Phi_{tor}$ such that $|x|_v < s$.

**Proof.** Let $P \in \mathbb{F}_q[t]$ be the unique irreducible monic polynomial such that $|P|_v < 1$, i.e., the place $v$ lies above the place corresponding to the polynomial $P$ in $\mathbb{F}_q(t)$.

We first observe that if $|x|_v < 1$, then for each $a \in \mathbb{F}_q(t)$ we have $|\Phi_a(x)|_v < s$ because each coefficient of $\Phi_a$ is integral at $v$.

Secondly, we claim that if $x \in \Phi_{tor}$ satisfies $|x|_v < 1$, then there exists $n \in \mathbb{N}$ such that $|\Phi_{P(t)}^{(n)}(x)|_v = 0$. Indeed, using our first observation it suffices to prove that if $|x|_v < 1$ and $Q \in \mathbb{F}_q[t]$ is relatively prime with $P(t)$, then $x = 0$. Because $|x|_v < 1$ and each coefficient of $\Phi_Q(t)$ is integral at $v$ while $|Q(t)| = 1$, we conclude that $|\Phi_Q(x)|_v = |Q(t)x|_v = |x|_v$. Hence, indeed $x = 0$ as claimed.

So, if $0 \neq x \in \Phi_{tor}$ satisfies $|x|_v < s < 1$ then there exists some $n \in \mathbb{N}$ such that $|\Phi_{P(t)}^{(n)}(x)|_v = 0$. Assume $n$ is the smallest such positive integer, and let $y = \Phi_{P(t)^{-1}}(x)$. Then $0 \neq y$ and $|\Phi_{P(t)}(y)| = 0$. Let $s_0 := |P(t)|_v^{1/(v-1)} < 1$; so, if $|y|_v < s_0$, then $|\Phi_{P(t)}(y)|_v = |P(t)y|_v \neq 0$ which yields a contradiction. Therefore, $|y|_v \geq s_0$. On the other hand, if $0 < |x|_v < 1$ then

\[
|\Phi_{P(t)}(z)|_v = \max\{|P(t)z|_v, |z|_v^v\} < |z|_v
\]

since each coefficient of $\Phi_{P(t)}$ is integral at $v$. Because $|P(t)|_v < 1$ and also $|z|_v < 1$, then $|P(t)z|_v < s_0$. Thus, if $n > n_0 := 1 + \log_q (\log_q (s_0))$ inequality (3.2) yields that $|\Phi_{P(t)^{-1}}(x)|_v = |y|_v < s_0$. This yields a contradiction with the fact that $y \neq 0$ but $\Phi_{P(t)}(y) = 0$. So, in conclusion, if $x \in \Phi_{tor}$ such that $|x|_v < s$ then $\Phi_{P(t)^{-n}}(x) = 0$, where $n_0$ is a positive integer depending only on $s$ and on $v$ (note that $s_0$ depends only on $v$). Thus there exist at most finitely many torsion points $x$ satisfying the inequality $|x|_v < s$.

Lemma 3.1 is a special case of [10, Theorem 2.10]; however, our result is more precise since we assume each coefficient of $\Phi_t$ is integral at $v$. In particular, the following result is an immediate corollary.

**Corollary 3.3.** Let $z \in K^{sep}$, let $v \in \Omega_K \setminus S_\infty$ and let $s$ be a real number such that $0 < s < 1$. Then there exist at most finitely many $x \in \Phi_{tor}$ such that $|z - x|_v < s$.

4. Ih’s Conjecture for Drinfeld Modules

Assume $\Phi$ is in normal form and also it has everywhere good reduction away from $S_\infty$. Also, using the notation from (2.3), for each place $v \notin S_\infty$, if $|x|_v > 1$ then $h_{\Phi, v}(x) = \log |x|_v > 0$ (since the leading coefficient of $\Phi_t$ is a $v$-adic unit). Let $\beta \in K$ be a nontorsion point. We’ll prove Theorem 1.3 as a consequence of Theorem 1.4.

**Proof of Theorem 1.3.** First, we enlarge $S$ so that it contains $S_\infty$; clearly enlarging $S$ can only increase the number of torsion points which are $S$-integral with respect to $\beta$. Then for all $v \notin S$ we know that for a torsion point $\gamma$ we have $|\gamma|_v \leq 1$ since
\( \Phi \) has good reduction at \( v \). Hence for each \( v \notin S \), if \( \gamma \in \Phi_{\text{tor}} \) is \( S \)-integral with respect to \( \beta \), then
\[
|\beta - \gamma_\sigma|_v = \max\{|\beta|_v, 1\},
\]
for each \( \sigma \in \text{Gal}(K^{\text{sep}}/K) \).

Assume there exist infinitely many torsion points \( \gamma_n \) which are \( S \)-integral with respect to \( \beta \). By Theorem 1.4, we know that
\[
\hat{h}_\Phi(\beta) = \lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K^{\text{sep}}/K)} \log |\beta - \gamma_\sigma|_v.
\]

On the other hand, since \( \gamma_n \) is \( S \)-integral with respect to \( \beta \), we know that
\[
\log |\beta - \gamma_\sigma|_v = \log^+|\beta|_v.
\]

Hence the above outer sum consists of only finitely many nonzero terms and therefore we may reverse the order of the summation with the limit, and conclude that
\[
\hat{h}_\Phi(\beta) = \lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K^{\text{sep}}/K)} \log |\beta - \gamma_\sigma|_v = 0,
\]
by the product formula applied to each \( \beta - \gamma_n \) (note that this element is nonzero since \( \beta \notin \Phi_{\text{tor}} \)). Thus we obtain that \( \hat{h}_\Phi(\beta) = 0 \), which contradicts the fact that \( \beta \notin \Phi_{\text{tor}} \). So, indeed there are at most finitely many torsion points which are \( S \)-integral with respect to \( \beta \).

□

We are left to proving Theorem 1.4. This will follow from the following result.

**Theorem 4.1.** Let \( \beta \in K \), let \( v \in M_K \), and let \( \{\gamma_n\} \subset K^{\text{sep}} \) be an infinite sequence of torsion points for the Drinfeld module \( \Phi \). Then
\[
\hat{h}_{\Phi,v}(\beta) = \lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K^{\text{sep}}/K)} |\beta - \gamma_\sigma|_v.
\]

We’ll prove Theorem 4.1 by analyzing two cases depending on whether \( v \in S_\infty \) or not.

**Proposition 4.2.** Theorem 4.1 holds if \( v \notin S_\infty \).

**Proof.** Firstly, since \( v \notin S_\infty \), then \( v \) is a place of good reduction for \( \Phi \) and then \( J_v \) consists of the unit disk. We denote by \( J_v \) the Julia set, which is the boundary of \( J_v \). Note that since \( J_v \) is closed, then \( J_v \subset J_v \); finally, because \( v \) is a place of good reduction and so, \( J_v \) is the unit disk, then \( J_v \) is its boundary and thus consists of all \( z \in \mathbb{C}_v \) such that \( |z|_v = 1 \).

There are two cases: either \( \beta \in J_v \), or not.

**Case 1.** Assume \( \beta \in J_v \).

Then \( \hat{h}_{\Phi,v}(\beta) = 0 \), since in particular \( \beta \in J_v \). If for each torsion point \( \gamma \) we have that \( |\beta - \gamma|_v = 1 \), then clearly
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K^{\text{sep}}/K)} |\beta - \gamma_\sigma|_v = 0 = \hat{h}_{\Phi,v}(\beta).
\]

Now, if there exists some torsion point \( \gamma \) such that \( |\beta - \gamma|_v < 1 \), let \( s \) be any real number satisfying
\[
|\beta - \gamma|_v < s < 1.
\]
By Lemma 3.3 we conclude that there exist finitely many torsion points \( \gamma' \) such that \( |\beta - \gamma'|_v < s \). In particular, for all \( n \) sufficiently large, and for all \( \sigma \in \Gal(K^\sep/K) \), we have \( |\beta - \gamma'_n|_v \geq s \) and thus
\[
\frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma} \log |\beta - \gamma'_n|_v \geq \log(s)
\]
and so, letting \( s \to 1 \) we obtain
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma: \Gal(K^\sep/K)} \log |\beta - \gamma'_n|_v \geq 0.
\]
On the other hand, since each torsion point \( \gamma' \) is in \( J_v \), we have \( |\gamma'|_v \leq 1 \). Because \( |\beta|_v = 1 \) (because \( \beta \in J_v \)) we conclude that \( |\beta - \gamma'|_v \leq 1 \) for all \( \gamma' \in \Phi_{tor} \). Hence
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma: \Gal(K^\sep/K)} \log |\beta - \gamma'_n|_v \leq 0
\]
and therefore, in conclusion
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma: \Gal(K^\sep/K)} \log |\beta - \gamma'_n|_v = 0 = \hat{h}_{\Phi, v}(\beta).
\]

**Case 2.** Assume \( \beta \notin J_v \).

In this case, the function \( z \mapsto \log |\beta - z|_v \) is continuous on \( J_v \) and the equidistribution result of Baker-Hsia [2] Proposition 4.5] yields that
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma: \Gal(K^\sep/K)} \log |\beta - \gamma'_n|_v = \int_{J_v} \log |\beta - z|_v d\mu_v(x),
\]
where \( \mu_v \) is the invariant measure on the Julia set. So, if \( |z|_v > 1 \), then the above limit equals \( \log |z|_v \), while if \( |z|_v < 1 \), then the above limit equals 0. In both cases, the limit is the local canonical height of \( z \) at \( v \).

**Remark 4.3.** The method of proof for Proposition 4.2 reveals the necessity for our hypothesis from Theorem 1.3 that \( \Phi \) has good reduction at all finite places. Indeed, this allows us to conclude that \( J_v \) is the unit disk in \( \mathbb{C}_v \) and moreover that for each \( z \in C_v \), we have \( \hat{h}_{\Phi, v}(z) = \log^+ |z|_v \). It is likely that Proposition 4.2 holds without the above hypothesis but one would need a different approach.

Assume now that \( v \in S_\infty \). In this case the setup is from our paper [9]. As shown by Theorem 4.6.9 of [13], there exists an \( \mathbb{F}_q[t]\)-lattice \( \Lambda_v \subset \mathbb{C}_v \) associated to the generic characteristic Drinfeld module \( \Phi \); \( \mathbb{C}_v \) is the completion of \( \overline{K}_v \) which is a complete, algebraically closed field. Let \( E_v \) be the exponential function defined in 4.2.3 of [13] which gives a continuous (in the \( v \)-adic topology) isomorphism
\[
E_v : \mathbb{C}_v/\Lambda \to \mathbb{C}_v.
\]
The torsion submodule of \( \Phi \) in \( \mathbb{C}_v \) is isomorphic naturally through \( E_v^{-1} \) to
\[
(\mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \Lambda_v) / \Lambda_v.
\]
Since each torsion point is repelling, we conclude that the corresponding Julia set \( J_v \) is the closure of the set of all torsion points. We also note that the completion of \( \mathbb{F}_q(t) \) with respect to the restriction of \( v \) on \( \mathbb{F}_q(t) \) is \( \mathbb{F}_q((\frac{1}{t})) \).

Then the restriction of \( E_v \) on \( (\mathbb{F}_q((\frac{1}{t})) \otimes_{\mathbb{F}_q[t]} \Lambda_v) / \Lambda_v \) gives an isomorphism between \( (\mathbb{F}_q((\frac{1}{t})) \otimes_{\mathbb{F}_q[t]} \Lambda_v) / \Lambda_v \) and \( J_v \).
Let $r$ be the rank of $\Lambda$, which is the same as the rank of the Drinfeld module $\Phi$. Then $(\mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \Lambda_v) / \Lambda_v \to (\mathbb{F}_q(t) / \mathbb{F}_q[t])^r$. Let $\omega_1, \ldots, \omega_r$ be a fixed $\mathbb{F}_q[t]$-basis of $\Lambda_v$. Furthermore, we may assume the $\omega_i$’s is a basis of successive minima as defined by Taguchi [23], i.e., for each $P_1, \ldots, P_r \in \mathbb{F}_q[t]$ we have

$$|P_1(t)\omega_1 + \cdots + P_r(t)\omega_r|_v = \max_{i=1}^r |P_i(t)\omega_i|_v. $$

Using Proposition 4.6.3 of [13], $(\mathbb{F}_q ((\frac{1}{t})) \otimes_{\mathbb{F}_q[t]} \Lambda_v) / \Lambda_v$ is isomorphic to $(\mathbb{F}_q((\frac{1}{t}))/\mathbb{F}_q[t])^r$. Then we have the isomorphism

$$E : \left(\mathbb{F}_q \left(\left(\frac{1}{t}\right)\right) / \mathbb{F}_q[t] \right)^r \to J_v$$

given by

$$E(\gamma_1, \ldots, \gamma_r) := E_v(\gamma_1\omega_1 + \cdots + \gamma_r\omega_r),$$

for each $\gamma_1, \ldots, \gamma_r \in \mathbb{F}_q \left(\left(\frac{1}{t}\right)\right) / \mathbb{F}_q[t]$.

We construct the following group isomorphism

$$\tau : \mathbb{F}_q \left(\left(\frac{1}{t}\right)\right) / \mathbb{F}_q[t] \to \frac{1}{t} \cdot \mathbb{F}_q \left(\left(\frac{1}{t}\right)\right),$$

given by

$$\tau \left(\sum_{i \geq -n} \alpha_i \left(\frac{1}{t}\right)^i\right) = \sum_{i \geq 1} \alpha_i \left(\frac{1}{t}\right)^i,$$

for every natural number $n$ and for every $\sum_{i \geq -n} \alpha_i \left(\frac{1}{t}\right)^i \in \mathbb{F}_q \left(\left(\frac{1}{t}\right)\right)$ (obviously, $\tau$ vanishes on $\mathbb{F}_q[t]$). The group $\frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]]$ is a topological group with respect to the restriction of $v$ on $\frac{1}{t} \cdot \mathbb{F}_q[[\frac{1}{t}]]$. Hence, the isomorphism $\tau^{-1}$ induces a topological group structure on $\mathbb{F}_q \left(\left(\frac{1}{t}\right)\right) / \mathbb{F}_q[t]$. Therefore, $\tau$ becomes a continuous isomorphism of topological groups. We endow $(\mathbb{F}_q \left(\left(\frac{1}{t}\right)\right) / \mathbb{F}_q[t])^r$ with the corresponding product topology. The isomorphism $\tau$ extends diagonally to another continuous isomorphism, which we also call

$$\tau : \mathbb{F}_q \left(\left(\frac{1}{t}\right)\right) / \mathbb{F}_q[t] \to \left(\frac{1}{t} \mathbb{F}_q \left(\left[\frac{1}{t}\right]\right)\right)^r =: G.$$ 

Moreover, using that $E_v$ is a continuous morphism, we conclude

$$E \tau^{-1} : G \to J_v$$

is a continuous isomorphism.

Since $\omega_1, \ldots, \omega_r$ is a basis of $\Lambda$ formed by successive minima, for each $a_1, \ldots, a_r \in \mathbb{F}_q[[1/t]]$ we have

$$\left| \sum_{i=1}^r a_i(1/t)\omega_i \right|_v = \max_{i=1}^r |a_i(1/t)\omega_i|_v.$$

Indeed, without loss of generality, we assume $a_i \neq 0$ for $i = 1, \ldots, s$ and $a_i = 0$ for $i > s$ (for some $s \leq r$). Let $N \in \mathbb{N}$ such that $|1/t^N|_v = \min_{i=1}^s |a_i(1/t)|_v$. Then there exist nonzero $P_1, \ldots, P_s \in \mathbb{F}_q[t]$ of degree at most equal to $N$ such that for each $i = 1, \ldots, s$ we have

$$a_i(1/t) = P_i(t)/t^N + b_i(1/t),$$

where each $b_i(1/t) \in 1/t^{N+1} \cdot \mathbb{F}_q[[1/t]]$. Then

$$\left| \sum_{i=1}^s b_i(1/t)\omega_i \right|_v \leq |1/t^{N+1}|_v \cdot \max_{i=1}^s |\omega_i|_v < |1/t^N|_v \cdot \max_{i=1}^s |\omega_i|_v.$$
On the other hand,
\[
\left| \sum_{i=1}^{s} \frac{P_i(t)}{|t|^N} \omega_i \right| = \max_{i=1}^{s} \frac{|P_i(t)|\omega_i}{|t|^N} \geq \max_{i=1}^{s} \frac{|\omega_i|}{|t|^N},
\]
since \(|P_i(t)|_v \geq 1\) because each \(P_i\) is nonzero. Therefore,
\[
\left| \sum_{i=1}^{r} a_i(1/t)\omega_i \right| = \max_{i=1}^{r} \frac{|P_i(t)|\omega_i}{|t|^N} = \max_{i=1}^{r} \frac{|a_i(1/t)\omega_i|}{|t|^N}.
\]

**Notation.** Let \(\nu_v\) be the Haar measure on \(G\), normalized so that its total mass is 1. Let \(\mu_v := (\mathbb{E}\sigma^{-1})_\nu \nu_v\) be the induced measure on \(J_v\) (i.e. \(\mu_v(V) := \nu_v(\sigma\mathbb{E}^{-1}(V))\) for every measurable \(V \subset J_v\)).

Because \(\nu_v\) is a probability measure, then \(\mu_v\) is also a probability measure. Because \(\nu_v\) is a Haar measure on \(G\) and \(\mathbb{E}\sigma^{-1}\) is a group isomorphism, then \(\mu_v\) is a Haar measure on \(J_v\).

**Definition 4.6.** Given \(x \in K^{\text{sep}}\), we define a probability measure \(\delta_x\) on \(\mathbb{C}_v\) by
\[
\delta_x = \frac{1}{[K(x) : K]} \sum_{\sigma \in \text{Gal}(K^{\text{sep}}/K)} \delta_{x^\sigma},
\]
where \(\delta_y\) is the Dirac measure on \(\mathbb{C}_v\) supported on \(\{y\}\).

Before we can state the equidistribution result from [9, Theorem 2.7] (see our Theorem 4.8), we need to define the concept of weak convergence for a sequence of probability measures on a metric space.

**Definition 4.7.** A sequence \(\{\lambda_k\}\) of probability measures on a metric space \(S\) weakly converges to \(\lambda\) if for any bounded continuous function \(f : S \to \mathbb{R}\), \(\int S f \lambda_k \to \int S f \lambda\) as \(k \to \infty\). In this case we use the notation \(\lambda_k \xrightarrow{w} \lambda\).

**Theorem 4.8.** Let \(\Phi : A \to K\{\tau\}\) be a Drinfeld module of generic characteristic such that \(\text{End}_{K^{\text{sep}}}\Phi) \simeq \mathbb{F}_q[\tau]\). Let \(\{x_k\}\) be a sequence of distinct torsion points in \(\Phi\). Then \(\delta_{x_k} \xrightarrow{w} \mu_v\).

**Remark 4.9.** Theorem 4.8 is stated in [9, Theorem 2.7] when \(K\) is a transcendental extension of \(\mathbb{F}_q(t)\) since the author needed to use the fact that the image of the Galois group is open in the ad`elic Tate module for a Drinfeld module, and at that moment the result was only known under the assumption that \(K\) is transcendental over \(\mathbb{F}_q(t)\) (see [5]). However, since then R"utische and Pink [17] removed the assumption on \(K\), and thus Theorem 4.8 holds in the above generality (see also [9, Remarks 3.2]).

Furthermore the proof from [9] yields that the points in \(\Phi[Q]\) are equidistributed in the Julia set \(J_v\) with respect to \(d\mu_v\) as \(\deg(Q) \to \infty\). This follows as the main result of [9] using the fact that \(J_v\) is isomorphic (as a topological group) to \(G\) and the points in \(\Phi[Q]\) correspond in \(G\) to all points of the form
\[
\left( \frac{P_1}{Q}, \ldots, \frac{P_r}{Q} \right)
\]
where the monic polynomials \(P_i\) have degrees less than \(\deg(Q)\). More precisely, for a generic open subset of \((\mathbb{F}_q[[t]])^r\), which is of the form (see [9] page 847,
for some polynomials \( a_i \in \mathbb{F}_q \left[ \left[ \frac{1}{t} \right] \right] \) of degree at most \( n_i \), we have to show (see [9, page 847, equation (7)]) that the number of tuples \( \left( \frac{P_1}{Q_1}, \ldots, \frac{P_r}{Q_r} \right) \in U \) where \( \deg(P_i) < \deg(Q) = d \) is asymptotic to

\[
q^{d - \sum_{i=1}^{r} n_i} \quad \text{as} \quad d \to \infty.
\]

As argued in [9], it suffices to prove this claim when \( r = 1 \), in which case the above statement reduces to show that (as \( d \to \infty \)) there are \( q^d \) distinct polynomials \( P_1 \) of degree less than \( d \) satisfying

\[
\frac{P_1}{Q_1} - a_1 \left( \frac{1}{t} \right) \in \frac{1}{t^{n_1+1}} \cdot \mathbb{F}_q \left[ \left[ \frac{1}{t} \right] \right].
\]

This last statement follows at once since this last condition induces \( n_1 \) conditions on the \( d \) coefficients of \( P_1 \).

Furthermore, a strong equidistribution result for torsion points is obtained in the proof of the main result from [9] (a similar result was proven in a more general context by Favre and Rivera-Letelier [8]).

**Theorem 4.10.** Given \( \gamma \in \Phi_{tor} \), and also given an open subset \( \mathbb{E} \tau^{-1}(U) \) of \( J_v \), where \( U \subset G \) is defined as above:

\[
U := \left( a_1 \left( \frac{1}{t} \right), \ldots, a_r \left( \frac{1}{t} \right) \right) + \left( \frac{1}{t^{n_1+1}} \cdot \mathbb{F}_q \left[ \left[ \frac{1}{t} \right] \right], \ldots, \frac{1}{t^{n_r+1}} \cdot \mathbb{F}_q \left[ \left[ \frac{1}{t} \right] \right] \right),
\]

we let

\[
N(\gamma, U) := \{ \sigma \in \text{Gal}(K_{sep}/K) : \gamma^\sigma \in \mathbb{E} \tau^{-1}(U) \}.
\]

Let \( \delta \) be a real number in the interval \((0, 1)\). Then for all \( \gamma \in \Phi_{tor} \) and for all open subsets \( U \) as above,

\[
\frac{N(\gamma, U)}{[K(\gamma) : K]} = \mu_v(U) + O_\delta \left( (K(\gamma) : K)^{-\delta} \right).
\]

**Proof.** This is the Drinfeld module analogue of the strong equidistribution result from [3 Proposition 2.4], and it is essentially proven in [9] when deriving formula (7), page 847. One simply needs to be more careful when estimating the error term in [9, (39), page 854] since this time we don’t fix the open set \( U \). The differences are as follows. In [9, (33), page 852], one estimates the number of all polynomials \( q_1^i \) (for \( i = 1, \ldots, r \)) to be

\[
q^i \sum_{i=1}^r \deg(b) - \deg(b') - \deg(d) - n_i + O(1),
\]

where \( O(1) \) is independent of all previously defined quantities. Then the error term in [9, (39), page 854] is bounded above by the number of divisors of the polynomial \( b \in \mathbb{F}_q[t] \) (which is the order of \( \gamma \)). Finally, noting that the number of divisors of \( b \) is bounded above by \( \deg(b) \), and that [9, (44), page 855] is bounded below (see also [9 Remarks 3.2, page 856]) by the number of polynomials relatively prime with \( b \) and of degree less than \( \deg(b) \), and this number is larger than \( q^{(1 - \epsilon) \deg(b)} \), for any positive real number \( \epsilon \), we obtain the conclusion of Proposition 4.10. \( \Box \)
The following result is a simple consequence of Theorem 4.10 since $E r^{-1}$ induces a local isometry between $G$ and $U$, and therefore there exists a positive real number $r_v$ such that for all subsets $U$ of $J_v$ of diameter at most $r_v$, (i.e., for each $x, y \in V$ we have $\log|x - y|_v \leq r_v$), $E r^{-1}$ is a distance-preserving isomorphism between $U$ and $E r^{-1}(U)$.

**Corollary 4.11.** Let $\delta \in (0, 1)$ and let $\beta \in J_v$. Then for all open subsets $U \subset J_v$ containing $\beta$ and of diameter at most $r_v$, and for all $\gamma \in \Phi_{tor}$ we have
\[
\#\{\sigma \in \text{Gal}(K^{sep}/K) : \gamma^\sigma \in U\} = \mu_\nu(U) + O_\delta \left(\frac{[K(\gamma) : K]}{r_v}\right).
\]

The following result follows from the powerful lower bound for linear forms in logarithms for Drinfeld modules established by Bosser [4].

**Fact 4.12.** Assume $\infty \in \Omega_K$ is an infinite place. Let $\beta \in K$ be a nontorsion point and let $\gamma \in \Phi[Q]$ where $Q \in F_q[t]$ is a monic polynomial of degree $d$. Then there exist (negative) constants $C_0$ and $C_3$ (depending only on $\Phi$ and $\beta$) such that
\[
\log|\gamma - \beta|_\infty \geq C_0 + C_3d \log d.
\]

**Proof.** In [12] Fact 3.1, Tucker and the author showed that Bosser’s result yields the existence of some (negative) constants $C_2$ and $C_3$ such that for all polynomials $P \in F_q[t]$ we have
\[
(4.13) \quad \log|\Phi_P(\beta)|_\infty \geq C_2 + C_3 \deg(P) \log \deg(P).
\]

On the other hand, if $|y|_\infty$ is sufficiently small but positive, then
\[
\log|\Phi(y)|_\infty = \log|ty|_\infty = \log|y|_\infty + |t|_\infty.
\]

Note that $|t|_\infty > 0$ since $\infty$ is an infinite place. So assuming that $d$ is sufficiently large, say $d \geq d_0 \geq 3$, if
\[
\log|\beta - \gamma|_\infty < C_2 + (C_3 - 1)d \log d
\]
then
\[
\log|\Phi_Q(\beta - \gamma)|_\infty = \log|\Phi_Q(\beta)|_\infty = \log|Q\beta|_\infty = d\log|t|_\infty + \log|\beta|_\infty < C_2 + C_3d \log d
\]
contradicting thus (4.13). Therefore for all $d \geq d_0$ we have that
\[
\log|\beta - \gamma|_\infty \geq C_2 + (C_3 - 1)d \log d.
\]

Since $\beta \notin \Phi_{tor}$ we conclude that there exists $C_4$ such that for all torsion points $\gamma \in \Phi[Q]$ for some monic polynomial $Q$ of degree less than $d_0$ we have
\[
\log|\beta - \gamma|_\infty \geq C_4.
\]

In conclusion, Fact 4.12 holds with $C_0 := \min\{C_2, C_4\}$ and $C_1 := C_3 - 1$. \qed

**Proposition 4.14.** Theorem 4.1 holds if $v \in S_\infty$.

**Proof.** Again we’ll split our analysis into two cases depending on whether $\beta$ is in the Julia set $J_v$ or not.

**Case 1.** Assume $\beta \notin J_v$.

As previously discussed, if $\beta \notin J_v$, then $f(z) = \log|z - \beta|_v$ is a continuous function on $J_v$ and therefore using the result of [9] Theorem 2.7] (see our Theorem 4.8 above, or alternatively use [2] Proposition 4.5)]
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K^{sep}/K)} \log|\beta - \gamma_n^\sigma|_v = \int_{J_v} \log|\beta - z|d\mu_\nu(z).
\]
Because $\deg(Q_n) \to \infty$ (since the torsion points $\gamma_n$ are distinct), we know that 
\{\Phi[Q_n]\}_n$ is equidistributed in $J_v$ and thus
\[
\int_{J_v} \log |\beta - z|d\mu_v(z) = \lim_{\deg(Q)\to\infty} \frac{1}{q^{\deg(Q)}} \sum_{\Phi_Q(z) = 0} \log |\beta - z|_v = \lim_{\deg(Q)\to\infty} \frac{\log |\Phi_Q(\beta)|_v}{q^{\deg(Q)}}.
\]
By [11, Corollary 3.13] we conclude that
\[
\lim_{\deg(Q)\to\infty} \frac{\log |\Phi_Q(z)|_v}{q^{\deg(Q)}} = \hat{h}_{\Phi,v}(z)
\]
which yields that indeed
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K_{sep}/K)} \log |\beta - \gamma_n^{\sigma}|_v = \hat{h}_{\Phi,v}(\beta).
\]

**Case 2.** Assume $\beta \in J_v$.
First we note that in this case $\hat{h}_{\Phi,v}(\beta) = 0$. We need to show that
\[
\lim_{n \to \infty} \frac{1}{[K(\gamma_n) : K]} \sum_{\sigma \in \text{Gal}(K_{sep}/K)} \log |\beta - \gamma_n^{\sigma}|_v = 0 = \hat{h}_{\Phi,v}(\beta).
\]
For each $n \in \mathbb{N}$, let $Q_n \in \mathbb{F}_q[t]$ be the monic polynomial of minimal degree $d_n$ such that $\Phi_Q(\gamma_n) = 0$. Then we know that $e_n = [K(\gamma_n) : K] >> q^{d_n}$ since 
$q \notin \text{Gal}(\mathbb{F}_q[t]/[Q_n])$ is bounded above by [17].
We claim that $\int_{J_v} \log |z|_v d\mu_v = 0$. Indeed, for each point $z$ we have
\[
\log |z|_v = \hat{h}_{\Phi,v}(z) - \lambda_v(z) + c_v(\Phi),
\]
where $\lambda_v$ is the local height as defined by Ingram [15] and
\[
c_v(\Phi) := \frac{-\log |a_v|_v}{q^r - 1},
\]
where $a_v$ is the leading coefficient of $\Phi_v(x)$ (for more details see [15]). Since we assumed $a_v = 1$, then for each point $z \in J_v$ we have
\[
\log |z|_v = -\lambda_v(z)
\]
because $\hat{h}_{\Phi,v}(\beta) = 0$. However $\lambda_v(z) = g_{\mu_v}(z, 0)$ where $g_{\mu_v}(x, y)$ is the Arakelov-Green function as defined in [11, Section 10.2]. Therefore, by [11, Proposition 10.12] we have
\[
\int_{J_v} \lambda_v(z)d\mu_v = 0
\]
since the invariant measure $\mu_v$ is supported on the Julia set $J_v$. Thus indeed
\[
\int_{J_v} \log |z|_v d\mu_v = 0.
\]

Next we employ the strategy of proof from [3]. Let $\epsilon := |t^m|_v^{-1} < 1$ be sufficiently small such that the analytic map $E_v : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by
\[
E_v(u) := u \cdot \prod_{\omega \in \Lambda \setminus \{0\}} \left(1 - \frac{u}{\omega}\right)
\]
is an isomorphism restricted on the closed ball $\overline{B}(0, \epsilon)$.
We consider
\[
J_{v,\beta,\epsilon} := \{z \in J_v : |z - \beta|_v \leq \epsilon\}.
\]
Also, we define $h_{\beta, \epsilon} : J_v \to \mathbb{R}$ as follows

$$h_{\beta, \epsilon}(z) := \min \left\{ 0, \log \left( \frac{|z - \beta|}{\epsilon} \right) \right\}.$$  

Then $h_{\beta, \epsilon}$ is supported on $J_{v, \beta, \epsilon}$ and it has a logarithmic singularity at $\beta$. So, there exists a continuous function $g_{\beta, \epsilon} : J_v \to \mathbb{R}$ such that $|z - \beta| = g_{\beta, \epsilon}(z) + h_{\beta, \epsilon}(z)$.

For each $n \in \mathbb{N}$ we define $\mu_n := \delta_{\gamma_n}$ be the probability measure on $J_v$ supported on the Galois orbit of $\gamma_n$ (see Definition 4.6). Then for each continuous function $f : J_v \to \mathbb{R}$ we have

$$\int_{J_v} f d\mu_n := \frac{1}{[K(\gamma_n) : K]} \cdot \sum_{\sigma \in \text{Gal}(K^{sep}/K)} f(\gamma_n^\sigma).$$

**Lemma 4.16.**

$$\int_{J_v} h_{\beta, \epsilon} d\mu_v = -\frac{\epsilon r}{q^r - 1}.$$  

**Proof.** Since we know that $h_{\beta, \epsilon}$ is supported on $J_{v, \beta, \epsilon}$ and moreover we know that $J_v$ is closed under translations (and $\mu_v$ is translation-invariant) it suffices to prove

$$\int_{J_{v, 0, \epsilon}} h_{0, \epsilon} d\mu_v = -\frac{\epsilon r}{q^r - 1},$$  

i.e., we may assume $\beta = 0$. So, we have to prove that

$$\int_{J_{v, 0, \epsilon}} \log \left( \frac{|z|}{\epsilon} \right) d\mu_v = -\frac{\epsilon r}{q^r - 1}.$$  

By our assumption, $E_v$ induces an isometric analytic automorphism of $\overline{D}(0, \epsilon)$; so using the change of variables $z = E_v(u)$ we compute

$$\int_{J_{v, 0, \epsilon}} \log \left( \frac{|z|}{\epsilon} \right) d\mu_v = \int_{E_v^{-1}(J_{v, 0, \epsilon})} \log \left( \frac{|E_v(u)|}{\epsilon} \right) d\nu,$$

where $\nu$ is the measure on $E_v^{-1}(J_{v, 0, \epsilon})$ which is isomorphic to

$$(\mathbb{F}_q((1/t)) \otimes_{\mathbb{F}_q[t]} A) \cap B(0, \epsilon) = \oplus_{i=1}^r B \left( 0, \epsilon |\omega_i|^{r-1} \right) \cdot \omega_i,$$

Furthermore, we recall that for $x_1, \ldots, x_r \in \mathbb{F}_q[[1/t]]$ we have that

$$\log |x_1 \omega_1 + \cdots + x_r \omega_r|_v = \max_{i=1}^r |x_i \omega_i|.$$  

Making another change of variables $x_i = \frac{u_i}{\omega_i t^m}$ and noting that $\epsilon = |1/t^m|_v$ and that the measure $\nu$ has mass equal to 1, we are left to show that

$$I := \int_{(\mathbb{F}_q[[1/t]])^r} \log \max_{i=1}^r |x_i|_v d\nu(x_1) \cdots d\nu(x_r) = -\frac{1}{q^r - 1}.$$  

Indeed, we let

$$S_1 := (\mathbb{F}_q[[1/t]])^r \setminus (1/t \cdot \mathbb{F}_q[[1/t]])^r.$$
We note that $\max_{i=1}^r \log |x_i|_v$ restricted on $S_1$ equals 0, and so we obtain
\[ I = \int_{(1/t \mathbb{F}_v[[1/t]])^r} \max_{i=1}^r \log |x_i|_v \, \nu(x_1) \cdots \nu(x_r) \]
\[ = \frac{1}{q^r} \cdot \int_{\mathbb{F}_v[[1/t]]^r} -1 + \max_{i=1}^r \log |y_i|_v \, \nu(y_1) \cdots \nu(y_r) \text{ (by the change of variables } x_i = y_i/t) \]
\[ = -\frac{1}{q^r} + \frac{r}{q^r}. \]
So, $I = -\frac{1}{q^r} - r$. \hfill \Box

So, using (4.15) coupled with Lemma 4.10 we obtain that
\[ \int_{J_v} g_{\beta, \epsilon}(z) \, d\mu_v = -\int_{J_v} h_{\beta, \epsilon}(z) \, d\mu_v = \frac{e^r}{q^r - 1}. \]
Using the fact that the measures $\mu_n$ converge (weakly) to $\mu_v$ (according to [9]; see also Theorem 4.8) we conclude that for $n$ sufficiently large we have
\begin{equation}
(4.17) \quad \left| \int_{J_v} g_{\beta, \epsilon}(z) \, d\mu_n \right| < 2e^r.
\end{equation}

It remains to bound $\left| \int_{J_v} h_{\beta, \epsilon}(z) \, d\mu_n \right|$.

Since $h_{\beta, \epsilon}$ is supported on $J_{v, \beta, \epsilon}$ it suffices to analyze the conjugates of $\gamma_n$ which land in $J_{v, \beta, \epsilon}$. For this we let $D_n$ be the smallest integer larger than $[K(\gamma_n) : K]^{1/2r}$ and we split $J_{v, \beta, \epsilon}$ into $D_n$ subsets as follows. For each interval $[c, d] \subset [0, 1]$ let $J_v([c, d])$ be the subset of $J_{v, \beta, \epsilon}$ containing all $z \in J_v$ such that $c \leq |z - \beta|_v < d$. Then
\[ J_{v, \beta, \epsilon} = \bigcup_{i=1}^{D_n} J_v \left( \left\lfloor \frac{i - 1}{D_n}, \frac{i}{D_n} \right\rfloor \right). \]
We note that $\mu_v(J_v([c, d])) = (de)^r - (cd)^r$ for each $0 \leq c < d \leq 1$ since $J_{v, \beta, \epsilon}$ is isomorphic to $(1/t \mathbb{F}_v[[1/t]])^r$ and $|1/t^m|_v = \epsilon$. So, for large $n$, Corollary 4.11 (applied with $\delta := 2/3$, say) yields each such subset contains at most
\[ 2\mu_v(J_v([c, d])) \cdot [K(\gamma_n) : K] \leq 2e^r (d^r - c^r) D_n^{2r}. \]
conjugates of $\gamma_n$. The reason for which $\delta = 2/3$ works is that in this case
\[ \mu_v(J_v([c, d])) \cdot [K(\gamma_n) : K] \gg e^r \cdot D_n^r \gg D_n^{2r(1-\delta)} \gg [K(\gamma_n) : K]^{1-\delta}, \]
and thus Corollary 4.11 yields that $2\mu_v(J_v([c, d])) \cdot [K(\gamma_n) : K]$ is the main term for computing the number of conjugates of $\gamma_n$ landing in $J_v([c, d])$ (note that $\epsilon$ is fixed for this computation).

We analyze the first interval: $J_v([0, 1/D_n])$. We recall that $d_n$ is the degree of the minimal monic polynomial $Q_n$ such that $\Phi_{Q_n(t)}(\gamma_n) = 0$. Without loss of generality we assume $\gamma_n \in J_v([0, 1/D_n])$ is the conjugate of $\gamma_n$ closest to $\beta$. By Bosser’s theorem (see [4] and also our Fact 4.12), we have
\[ \log |\gamma_n - \beta|_v \geq C_0 + C_1 d_n \log d_n. \]
On the other hand, there are at most
\[ 2\mu_v(J_v([0, 1/D_n]))[K(\gamma_n) : K] = \frac{2e^r [K(\gamma_n) : K]}{D_n^r} \leq 2e^r [K(\gamma_n) : K]^{1/2} \]
conjugates of $\gamma_n$ in $J_v([0,1/D_n])$. We denote by $I_0$ the set of all $\sigma \in \text{Gal}(K^{sep}/K)$ such that $\gamma_n^\sigma \in J_v([0,1/D_n])$. Using also that $\epsilon < 1$, we conclude that

$$0 \geq \int_{J_v([0,1/D_n])} h_{\beta,\epsilon}(z) d\mu_n = \sum_{\sigma \in I_0} \frac{\log |\beta - \gamma_n^\sigma|}{|K(\gamma_n):K|} \geq 4\epsilon^r \cdot (C_0 + C_1 d_n \log d_n) \geq -\epsilon^r,$$

for large $n$ since $D_n = [K(\gamma_n):K]^{1/2} \gg q^{d_n/2}$ (by [17]).

Finally, consider the remaining subsets of $J_{v,\beta,\epsilon}$. For each $i = 2, \ldots, D_n$ there are at most $2\mu_v(J_v([(i-1)/D_n,i/D_n)]) \cdot [K(\gamma_n):K]$ conjugates of $\gamma_n$ in $J_v([(i-1)/D_n,i/D_n])$ and for each such conjugate $\gamma_n^\sigma$ we have

$$h_{\beta,\epsilon}(\gamma_n^\sigma) = \log |\beta - \gamma_n^\sigma| \geq \log((i-1)/D_n).$$

Using that $J_{v,\beta,\epsilon}$ is isomorphic to $(1/t^m \cdot \mathbb{F}_q[[1/t]])^r$ we conclude

$$0 \geq \int_{J_v \setminus J_v([0,1/D_n])} h_{\beta,\epsilon}(z) d\mu_n \geq \sum_{i=1}^{D_n-1} \log \left( \frac{i}{D_n} \right) \cdot 2\mu_v(J_v([(i-1)/D_n,i/D_n])) \geq 2 \int_{J_{v,\beta,\epsilon}} \log \left( \frac{|z|_v}{\epsilon} \right) = -\frac{2\epsilon^r}{q^r - 1},$$

by Lemma 4.16. Hence, using also inequality (4.18) we obtain that

$$\left| \int_{J_v} h_{\beta,\epsilon}(z) d\mu_n \right| < 3\epsilon^r.$$

Combining inequalities (4.17) with (4.19) we conclude that

$$\left| \int_{J_v} \log |z - \beta|_v d\mu_n \right| < 5\epsilon^r,$$

for all $n$ sufficiently large, and so,

$$\lim_{n \to \infty} \frac{1}{[K(\gamma_n):K]} \cdot \sum_{\sigma \in \text{Gal}(K^{sep}/K)} \log |\gamma_n^\sigma - \beta| = 0 = \hat{h}_{\Phi,v}(\beta),$$

if $\beta \in J_v$. This concludes our proof. \hfill \square

Propositions 4.12 and 4.14 finish the proof of Theorem 4.1.

Remark 4.20. In the proof of Proposition 4.14 we use in an essential way the hypothesis that $\Phi$ has no complex multiplication. However, we expect that Theorem 4.13 holds with this hypothesis on $\Phi$, but the proof would be harder.

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