Random matrix ensembles and the extensivity of the $S_q$ entropy

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We consider the joint density distribution of the elements of certain random matrix models which are example of globally correlated and asymptotically scale-invariant distributions. It is shown that in their cases, the nonadditive entropy $S_q$ is extensive only when the limit $q \to 1$ is taken. On the other hand, when restriction in the occupation of the phase space is imposed extensiveness is obtained for values of the entropic parameter different of one.

I. INTRODUCTION

In recent studies of random matrix theory (RMT) there has been an interest in a family of ensembles which at the same time generalizes the standard Wigner Gaussian ensemble and preserves its invariance under unitary transformations$^{[1]}$. This kind of ensembles is generated by superposing to the Gaussian fluctuations, an external source of randomness. Accordingly, this family has been named disordered ensembles$^{[2]}$. It first appeared as a result of applying to RMT the generalized maximum entropy principle$^{[3, 4]}$ associated to the nonadditive Tsallis $S_q$ entropy$^{[5]}$. Generalized in Ref. $^{[2, 6]}$, this more general family can be considered as an instance of the so-called superstatistics$^{[7]}$. Given a set of probabilities $p_i$, the Tsallis nonadditive entropy associated to them is defined as

$$S_q = \frac{1}{q-1} \left( 1 - \sum_i p_i^q \right),$$

which, in the $q \to 1$ limit, recovers the standard Boltzmann-Shannon (BS) entropy

$$S_1 = - \sum_i p_i \ln p_i.$$ 

The nonadditivity of (1) follows since, for a system composed of two independent sub-systems $A$ and $B$ such that the probabilities $p_i$ are product $p_i = p_i^A p_i^B$, $S_q \neq S_A + S_B$. Therefore, the definition (1) breaks the additivity property of the standard entropy. Nevertheless, it has been claimed$^{[8]}$ that, under certain conditions, $S_q$, though nonadditive, asymptotically becomes extensive with respect to the definition.

$$0 < \left| \lim_{f \to \infty} \frac{S_q}{f} \right| < \infty,$$

where $f$ is the degree of freedom, while $S_1$, under the same circumstance, may or may not be nonextensive. It also has been claimed$^{[8]}$ that this occurs for systems in which the phase space is occupied in a scale-invariant way. We show that, though the criterion of scale-invariance is satisfied by the joint density probability of the matrix elements of the
disordered ensembles, they only become extensive when the $q \to 1$ limit is taken. On the other hand, if restriction in the variation of the variables is introduced by forcing them to occupy only a fraction of the whole phase space then, situations occur which made the nonadditive entropy extensive for values of the entropic parameter different from one. Our analysis is performed, by considering joint density distribution of random variables continuous and discrete.

II. UNRESTRICTED OCCUPATION OF PHASE SPACE

Our starting point are two random matrices ensembles: the Gaussian Wigner ensembles for continuous case and for the discrete case, the ensemble of adjacency matrices of the Erdős-Renyi model of random graphs. Correlations are then introduced among the matrix elements by constructing mixture with some appropriate weight of these two type of matrices. In both cases, the scale invariant property is verified independent of the number of variables.

A. Disordered Gaussian ensembles

The joint density probability distribution of the matrix elements of the RMT Wigner Gaussian ensemble is\(^1\)

$$P_G(H) = \left(\frac{\beta}{2\pi}\right)^{f/2} \exp \left(-\frac{\beta}{2} \text{tr} H^2\right), \quad (4)$$

where $f = N + \beta N(N-1)/2$ is the number of independent matrix elements for the classes of real matrices ($\beta = 1$), complex matrices ($\beta = 2$) and quaternion matrices ($\beta = 4$). These three classes form, respectively, invariant Gaussian ensembles under orthogonal (GOE), unitary (GUE) and symplectic (GSE) transformations. The above distribution is normalized with respect to the measure $dH = \prod_i^N dH_{ii} \prod_{j>i}^\beta \prod_{k=1}^\beta \sqrt{2} dH_{ij}$. As matrix elements in (4) are statistically independent random variables, one should expect that their distribution should be additive with respect to the BS entropy and nonadditive and nonextensive with respect to $S_q$. In fact, with $q \neq 1$ we have

$$S_q(f) = \frac{1}{q-1} \left[1 - \int dh_1 dh_2 \ldots dh_f \left(\frac{\beta}{2\pi}\right)^{qf/2} \exp \left(-\frac{q\beta}{2} \sum_{i=1}^f h_i^2\right)\right], \quad (5)$$

where, with $i = 1, 2, \ldots, N$, $h_i = H_{ii}$, for the diagonal elements and, with $i = N + 1, N = 2, \ldots, f$, $h_i = \sqrt{2} H_{ij}$ for the off-diagonal forming a set $h$ of $f$ independent random variables. Performing the integrals

$$S_q(f) = \frac{1}{q-1} \left[1 - \left(\frac{q\beta}{2\pi}\right)^{(q-1)f/2} q^{-f/2}\right], \quad (6)$$

such that it is immediately seen that the ratio $S_q/f$ diverges or vanishes in the limit $f \to \infty$. On the other hand, if the limit $q \to 1$ is first taken we get
\[ S_1(f) = \frac{1}{2} \left( 1 - \frac{\beta}{2\pi} \right) = S_1(1) f \] (7)

which, as expected, shows additiveness.

Consider now matrices constructed by the relation

\[ H(\xi) = \frac{H_G}{\sqrt{\xi}/\bar{\xi}} \] (8)

where \( \xi \) is a positive random variable with a normalized density probability distribution \( w(\xi) \) with average \( \bar{\xi} \). From (1) and (8), the joint density probability distribution

\[ P(H) = \int d\xi w(\xi) \left( \frac{\beta \xi}{2\pi \xi} \right)^{f/2} \exp \left( -\frac{\beta \xi \text{tr} H^2}{2\xi} \right) \] (9)

follows which, expressed in terms of the reduced set of matrix elements, becomes

\[ P_f(h_1, h_2, ..., h_f) = \int d\xi w(\xi) \left( \frac{\beta \xi}{2\pi \xi} \right)^{f/2} \exp \left( -\frac{\beta \xi \sum_{i=1}^{f} h_i^2}{2\xi} \right). \] (10)

Clearly, \( h \) is now a set of \( f \) correlated variables with distribution which has the scale-invariance property since removing by integration any one of the variables, say \( h_k \), produces the same joint distribution \( P_{f-1} \), that is

\[ \int dh_k P_f(h_1, h_2, ..., h_f) = P_{f-1}(h_1, h_2, ..., h_{f-1}). \] (11)

The Tsallis entropy of \( P_f(h) \) is by definition

\[ S_q(f) = \frac{1}{q - 1} \left[ 1 - \int dh_1 dh_2 ... dh_f P_f^q(h_1, h_2, ..., h_f) \right], \] (12)

which through the substitution \( h'_i = \sqrt{\frac{\beta}{2\pi \xi}} h_i \) becomes

\[ S_q(f) = \frac{1}{q - 1} \left[ 1 - \left( \frac{\beta}{2\pi \xi} \right)^{(q-1)f/2} \int dh'_1 dh'_2 ... dh'_f P_f^q(h'_1, h'_2, ..., h'_f) \right]. \] (13)

If the integral in (13) has a finite limit when \( f \to \infty \), the entropy will diverge or vanish for \( q \neq 1 \). In this case, extensivity can only happen if the limit \( q \to 1 \), is first taken or, if the limit \( f \to \infty \) is taken keeping fixed the quantity

\[ \lambda = \frac{1}{q - 1} - \frac{f}{2}. \] (14)

In this case, concomitantly we have the limits \( q \to 1 \), \((q - 1)f \to 2\) and

\[ \lim_{f \to \infty} \frac{S_q}{f} = \frac{1}{2} \left( 1 - \frac{\beta I}{2\pi \xi} \right). \] (15)

are implied, where \( I \) denotes the limiting value of the integral.

To illustrate the above discussion, take for the weight function \( w(\xi) \) the gamma distribution.
\[ w(\xi) = \exp(-\xi)\xi^{\xi-1}/\Gamma(\xi) \]  

(16)

with variance \( \sigma_\xi = \sqrt{\xi} \) of previous studies of disorder ensembles. This choice entails power-law behavior for statistic measures of the ensemble. Substituting (16) in the above expressions, the integrals are readily performed such that (10) gives

\[ P_f(h; \bar{\xi}) = \left( \frac{\beta}{2\pi \xi} \right)^{\frac{f}{2}} \frac{\Gamma(\bar{\xi} + f/2)}{\Gamma(\bar{\xi})} \left( 1 + \frac{\beta}{2\xi} \sum_{i=1}^{f} h_i^2 \right)^{-\xi-f/2} \]  

(17)

for the ensemble distribution and

\[ p(h; \bar{\xi}) = \left( \frac{\beta}{2\pi \xi} \right)^{\frac{1}{2}} \frac{\Gamma(\bar{\xi} + 1/2)}{\Gamma(\bar{\xi})} \left( 1 + \frac{\beta}{2\xi} h^2 \right)^{-\xi-1/2} \]  

(18)

for the density distribution of a given matrix element. Since for large \(|h|\), \( p(\beta; h; \bar{\xi}) \sim 1/|h|^{2\bar{\xi}+1} \), the distribution (18) does not have moments of order superior to \( 2\bar{\xi} \). This fact makes the value \( \bar{\xi} = 1 \) critical since below it (18) does not have second moment[3].

Substituting (16) in (13), the integrals are easily performed and the analytic expression

\[ S_q(f) = \frac{1}{q-1} \left[ 1 - \left( \frac{\Gamma(\bar{\xi} + f/2)}{\Gamma(\bar{\xi})} \right)^q \left( \frac{\beta}{2\pi \xi} \right)^{(q-1)f/2} \frac{\Gamma(q\bar{\xi} + (q-1)f/2)}{\Gamma(q(\bar{\xi} + f/2))} \right] \]  

(19)

for the entropy is obtained. In order to \( S_q \) be extensive, the second term inside the parenthesis has to become proportional to \( f \) for very large \( f \). Replacing the gamma functions by its Stirling approximation, it is simple to show that this proportionality does not happen if \( q \neq 1 \). Instead, if first the limit \( q \to 1 \) is taken with the degree of freedom \( f \) fixed, the BS entropy is

\[ S_1(f) = -\ln \left[ \frac{\Gamma(\bar{\xi} + f/2)}{\Gamma(\bar{\xi})} \right] - f \ln(\frac{\beta}{2\pi \xi}) - (\bar{\xi} + f/2) \left[ \Psi(\bar{\xi}) - \Psi(\bar{\xi} + f/2) \right], \]  

(20)

where \( \Psi(x) \) denotes the logarithmic derivative of the gamma function. Taking now the limit \( f \to \infty \), we obtain

\[ S_1(f) = \frac{f}{2} \left[ 1 - \ln(\frac{\beta}{2\pi \xi}) - \Psi(\bar{\xi}) \right] = fS_1(1), \]  

(21)

showing the additiveness of the Boltzmann-Shannon entropy.

Let us now take the limit \( f \to \infty \) imposing the condition (14) with \( \lambda = \bar{\xi} \) then the limit expression

\[ \lim_{f \to \infty} \frac{S_q}{f} = \frac{1}{2} \left( 1 - \frac{\beta}{2\pi e} \right) \]  

(22)

for the entropy is obtained (\( e = 2.71828... \)) proving its extensivity. In Fig. 1, the exact and the asymptotic expressions of the \( S_q \) entropy are plotted as function of \( f \).
B. Disordered random graphs

A graph is an array of points (nodes) connected by edges. It is completely defined by its adjacency matrix, $A$, whose elements, $A_{ij}$, have value $1(0)$ if the pair $(ij)$ of nodes is connected (disconnected). The diagonal elements are taken equal to zero, i.e. $A_{ii} = 0$. Adjacency matrices are real symmetric random matrices for graphs in which the connections between pairs of nodes are randomly set. The classical graph of this kind was proposed by Erdős and Rényi (ER) in which pair of nodes are independently connected with a fixed probability $p$\cite{11}. The properties of this classical model are strongly dependent on the value of the probability $p$ compared with the total number of nodes $N$ and it is usual to assume the scaling $p \sim N^{-z}$ ($z > 0$). Two important statistical properties in the studies of graphs are the eigenvalue density of the adjacency matrix and the degree probability that gives the probability distribution of the number of connections of a given node.

For not too small values of $p$, that is not too rarefied ER graphs, the eigenvalue density obeys the Wigner semi-circle law\cite{1, 10}

$$\rho_{ER}(E, \alpha) = \begin{cases} \frac{1}{2\pi \sigma^2} \sqrt{4N\sigma^2 - E^2}, & \text{if } |E| < \sqrt{4N\sigma^2} \\ 0, & \text{if } |E| > \sqrt{4N\sigma^2} \end{cases},$$

(23)

where

$$\sigma^2 = p(1-p)$$

(24)

is the variance of the matrix elements. Deviations from this law appear\cite{12, 13} if $p \sim 1/N$ ($z \sim 1$). The degree distribution, that is the number $k_i$ of connections of a given node $i$ is obtained from the adjacency matrix as

$$k_i = \sum_{j=1}^{N} A_{ij}$$

(25)

and the probability $P_k$ of having $k_i$ equal to $k$ is

$$P_k = \frac{(N-1)!}{k!(N-1-k)!} p^k (1-p)^{N-1-k}.$$  

(26)

For large $N$, it can be assumed that the nodes are statistically independent in such a way that (26) also gives the probability of finding the average number of nodes with $k$ connections in the graph\cite{10}.

In Ref. 2, it has been shown that the joint distribution of the matrix elements of the ER adjacency matrix can be written in terms of its trace as

$$P_{ER}(A, \alpha) = [1 + \exp(-\alpha)]^{-f} \exp\left(-\frac{\alpha}{2} \text{tr}A^2\right)$$

(27)

where $f = \frac{N(N-1)}{2}$ is the number of independent matrix elements for a graph with $N$ nodes. Eq. (27) shows that the ER model can be considered as a special case of the RMT Gaussian ensembles in which matrix elements are Bernoulli random variables. Focussing only on the set $a = a_1, a_2, \ldots, a_f$ of $f$ independent elements, (27) becomes
\[ P_{ER}(a) = [1 + \exp(-\alpha)]^{-f} \exp \left( -\alpha \sum_{i=1}^{f} a_i \right), \quad (28) \]

from which the probability of one given element, say \( a_k \), is

\[ p(a_k) = \frac{\exp(-\alpha a_k)}{1 + \exp(-\alpha)} = \begin{cases} \frac{\exp(-\alpha)}{1+\exp(-\alpha)}, & \text{if } a_k = 1 \\ \frac{1}{1+\exp(-\alpha)}, & \text{if } a_k = 0, \end{cases} \quad (29) \]

(29) implies in the relation

\[ \alpha = \ln \left( \frac{1}{p} - 1 \right). \quad (30) \]

between the parameter \( \alpha \) and the probability \( p \). Since \( p \) is defined in the interval \([0, 1]\), the correspondent domain of variation of \( \alpha \) is \((\infty, -\infty)\).

As \( a \) is a set of binary discrete variables, the joint probability \( P_{ER}(a, \alpha) \) has a finite set of \( f+1 \) possibilities which can be arranged in the Leibnitz triangle

\[
\begin{array}{cccccc}
(f=0) & & & & & 1 \\
(f=1) & & & p & & 1-p \\
(f=2) & & p^2 & & p(1-p) & (1-p)^2 \\
(f=3) & p^3 & & p^2(1-p) & & p(1-p)^2 & (1-p)^3, \\
\end{array}
\]

which has the property of adding two adjacent entries of a row, the above entry on the immediate row is obtained. This property is equivalent to say that the probability distribution, Eq. (28), is scale-invariant.\(^8\) As the set \( a \) is also statistically independent, the BS entropy of its distribution should be extensive. In fact, the nonadditive entropy of (28) is easily calculated to be given by

\[ S_q(f) = \frac{1}{q-1} \left[ 1 - \frac{(1 + \exp(-q\alpha))^f}{(1 + \exp(-\alpha))^{qf}} \right], \quad (31) \]

which divide by \( f \), diverges or vanishes in the limit \( f \to \infty \) if \( q < 1 \) or \( q > 1 \), respectively. On the other hand, taking first the limit \( q \to 1 \), we find

\[ S_1(f) = f \left[ 1 - \frac{\alpha}{(1 + \exp(\alpha))} - \ln (1 + \exp(-\alpha)) \right] = fS_1(1), \quad (32) \]

such that the additiviness of BS entropy follows.

Proceeding as in the Gaussian ensemble, correlations are introduced in the ER model by defining a disordered model constructed by superposing with a given weight ER graphs with different probabilities. Explicitly, we consider a random graph whose adjacency matrix has joint distribution of elements given by

\[ P(A; \alpha) = \int d\xi w(\xi) \frac{\exp \left( -\frac{\alpha \xi}{\xi} \text{tr} A^2 \right)}{[1 + \exp(-\alpha \xi/\xi)]}, \quad (33) \]

Eq. (33) extends to discrete variables the formalism defined by Eq. (9). Again the width of the distribution of \( w(\xi) \) is a controlling parameter but here the parameter \( \alpha \) defining the mean ER also plays an important role.
As in the case of the Gaussian ensembles, statistics measures of the averaged graph, i.e. the generalized model, are obtained as averages over the statistics measures of the ER model. For instance, the eigenvalue density is given by

\[
\rho(E; \alpha) = \int_0^{\xi_m} d\xi w(\xi) \frac{1}{2\pi p(\xi)[1 - p(\xi)]} \sqrt{4Np(\xi)[1 - p(\xi)] - E^2}
\]

(34)

where

\[
p(\xi) = \frac{\exp(-\alpha\xi)}{[1 + \exp(-\alpha\xi)]}
\]

(35)

and

\[
\xi_m = \frac{2}{\alpha} \cosh^{-1}(\sqrt{\frac{N}{E}});
\]

(36)

while the degree distribution is given by

\[
P_k(\alpha) = \frac{(N - 1)!}{k!(N - 1 - k)!} \int_0^{\infty} d\xi w(\xi) \frac{\exp\left(-\frac{k\alpha k}{\xi}\right)}{[1 + \exp\left(-\frac{\alpha k}{\xi}\right)]^{N-1}}
\]

(37)

In terms of the set \(a\) of independent matrix elements, (33) becomes

\[
P_f(a; \alpha) = \int d\xi w(\xi) \left[1 + \exp\left(-\frac{\alpha k}{\xi}\right)\right]^{1-f} \exp\left(-\frac{\alpha k}{\xi} \sum_{i=1}^{f} a_i\right)
\]

(38)

Due to the linearity property of integration, the finite number of probabilities defined by (38) can also be arranged in a Leibnitz triangle in such a way that the relations between elements of neighboring rows are preserved. Therefore, this generalization of the ER graph introduces correlation among the variables but their joint probability distribution still satisfies the scale-invariant criterion, namely

\[
\sum_{a_k=0}^{1} P_f(a; \alpha) = P_{f-1}(a; \alpha).
\]

(39)

The \(S_q\) entropy of (38) is

\[
S_q(f) = \frac{1}{q - 1} \left[1 - \sum_{a_1a_2...a_f} P_f^q(a_1, a_2, ..., a_f; \alpha)\right],
\]

(40)

or, explicitly,

\[
S_q(f) = \frac{1}{q - 1} \left[1 - \sum_{k=0}^{f} \frac{f!}{k!(f - k)!} \left(\int d\xi w(\xi) \left[1 + \exp\left(-\frac{\alpha k}{\xi}\right)\right]^{1-f} \exp\left(-\frac{\alpha k}{\xi}\right)\right)^q\right]
\]

(41)

To proceed, we choose the weight distribution \(w(\xi)\) to be given by Eq. (16). In Ref. [2] it is shown that with this choice the generalized model has features of a scale-free graph.
In particular, the eigenvalue density exhibits a crossover from the Wigner semi-circle law to a distribution highly peaked with heavy exponential tails.

With this choice, the integral

\[ I_{f_k} = \frac{1}{\Gamma(\bar{\xi})} \int_0^\infty d\xi \frac{\exp \left[ -(1 + \frac{k\alpha}{\bar{\xi}})\xi \right]}{\left[ 1 + \exp\left( -\frac{\alpha\xi}{\bar{\xi}} \right) \right]^f} \]

in (41), asymptotically can be calculated in the following way. We put the integrand in an exponential form \( \exp(-F_{f_k}) \) with

\[ F_{f_k}(\xi) = (1 + \frac{k\alpha}{\bar{\xi}})\xi - (\bar{\xi} - 1) \log \xi + f \log \left[ 1 + \exp\left( -\frac{\alpha\xi}{\bar{\xi}} \right) \right] \]

For \( \bar{\xi} > 1 \), \( F_{f_k}(\xi) \) has a parabolic shape such that a new variable \( t \) can be introduced through the mapping

\[ F_{f_k}(\xi) = F_{f_k}(\xi_s) + \frac{t^2}{2}, \]

where \( \xi_s \) is the minimum value obtained by finding the root of the equation \( F'_{f_k}(\xi) = 0 \).

In terms of \( t \) the integral becomes

\[ I_{f_k} = \frac{w(\xi_s) \exp(-k\alpha\xi_s/\bar{\xi})}{[1 + \exp(-\alpha\xi_s/\bar{\xi})]^f} \int_{-\infty}^\infty dt \frac{d\xi}{d\xi} \exp(-\frac{t^2}{2}). \]

Replacing \( d\xi/dt \) by its value at \( t = 0 \), the asymptotic approximation

\[ I_{f_k} = \frac{w(\xi_s) \exp(-k\alpha\xi_s/\bar{\xi})}{[1 + \exp(-\alpha\xi_s/\bar{\xi})]^f} \sqrt{\frac{2}{\pi F''_{f_k}(\xi_s)}} \]

is obtained which replaced in (41) leads to the asymptotic approximation

\[ S_q(f) = \frac{1}{q-1} \left[ 1 - \sum_{k=0}^f \frac{f!}{k!(f-k)!} \left( \frac{w(\xi_s) \exp(-\alpha\xi_s k/\bar{\xi})}{[1 + \exp(-\alpha\xi_s/\bar{\xi})]^f} \sqrt{\frac{2}{\pi G''_{f_k}(\xi_s)}} \right)^q \right] \]

for the entropy. In Fig. 2, (47) is plotted as a function of \( f \) imposing the condition (14) with \( \bar{\xi} = 6 \). The linear increase shows the extensivity of \( S_q \) under these condition.

### III. RESTRICTED OCCUPATION OF PHASE SPACE

The previous section shows that correlations among the random variables are not sufficient to make the \( S_q \) entropy of their joint distribution extensive for \( q \neq 1 \). The calculations have been performed allowing the variables to freely occupy the phase space in an unrestricted way. In this section, we again start considering the Wigner and the Erdős-Rényi models but imposing the condition that the random variations of their constituents are restricted to a region of their phase space. By doing this, we find situations in which \( S_q \), in both cases, becomes extensive for values of the entropic parameter different from one. The scale-invariant property is asymptotically obeyed but in a less strong form.
A. Continuous case

Consider the joint distribution of the matrix elements of the Wigner ensemble, Eq. (4), but with the additional constraint that the trace is restricted to be less than a maximum value $R^2$. In terms of the reduced set $h$ of elements this implies in the condition

$$\sum h_i^2 < R^2,$$

such that the elements are forced to be inside a hypersphere of radius $R$. As a consequence, the joint density distribution has to be renormalized as

$$P_f(h_1, h_2, ..., h_f) = Z_f^{-1} \exp \left( -\frac{\beta}{2} \sum_{i=1}^{f} h_i^2 \right),$$

where

$$Z_f(\beta, R) = \frac{2\pi}{\beta \Gamma(f/2)} \gamma\left( \frac{f}{2}, \frac{\beta R^2}{2} \right),$$

with $\gamma(a, x)$ being the incomplete gamma function. The entropy of this distribution is

$$S_q = \frac{1}{q-1} \left[ 1 - Z_f(q\beta, R) \frac{Z_f(\beta, R)}{Z_q^f(\beta, R)} \right].$$

Replacing the incomplete gamma function by its asymptotic expression

$$\gamma(a, x) = \frac{\exp(-x)x^a}{a}$$

the entropy assumes the correspondent asymptotic expression

$$S_q = \frac{1}{q-1} \left[ 1 - (\pi R^2)^{(1-q)f/2} \Gamma^{q-1}(1 + f/2) \right].$$

If in this expression, the radius $R$ is made dependent on the degree of freedom $f$ as

$$\pi R^2 = \left( \frac{f}{2} + 1 \right) \exp(-1)$$

and the gamma function is replaced by its Stirling approximation

$$\Gamma(z) = \sqrt{2\pi z^{z-\frac{1}{2}}} \exp(-z),$$

the $S_q$ becomes

$$S_q = \frac{1}{q-1} \left( 1 - 2\pi \left( \frac{f}{2} + 1 \right)^{\frac{q-1}{2}} \exp(1 - q) \right).$$

Therefore for the value $q = 3$ of the entropic parameter, $S_3$ increases linearly with $f$ as

$$S_3 = \frac{\pi}{2e^2} f.$$
This is illustrated in the figure where \( S_q \) is plotted as function of \( f \) for the values 2, 3 and 4 of the entropic parameter.

To investigate how the restriction in the occupation of the phase space affects the scale-invariant property of the joint density probability distribution, let us integrate it over one variable of the set of \( f \) variables. As the variables appear equivalently in the distribution, we take by convenience the last one, i.e. \( h_f \), integrating it over the available domain, we obtain the distribution

\[
P_f'(h_1, ..., h_{f-1}) = Z_f^{-1} \exp \left(-\frac{\beta}{2} \sum_{i=1}^{f-1} h_i^2\right) 2 \int_0^{R^2 - \sum_{i=1}^{f-1} h_i^2} dh_f \exp\left(-\frac{\beta}{2} h_f^2\right).
\]

Using the asymptotics of the functions we get

\[
P_f'(h_1, ..., h_{f-1}) = \left(\frac{\beta}{2\pi}\right)^{f/2} \exp \left[-\frac{\beta}{2} \left( R^2 + \sum_{i=1}^{f-1} h_i^2 \right) \right] \frac{\text{erf} \left( \beta \sqrt{R^2 - \sum_{i=1}^{f-1} h_i^2} \right)}{(\pi R^2)^{-f/2} \Gamma(1 + f/2)}.
\]

At the bulk of the space when \( f \to \infty \) the error function tends to one. Taking the ratio of this distribution with the distribution of for \( f-1 \) after replacing the gamma functions by its Stirling approximation we obtain

\[
\lim_{f \to \infty} \frac{P_f'(h_1, h_2, ..., h_{f-1})}{P_{f-1}(h_1, h_2, ..., h_{f-1})} = \sqrt{\frac{\beta}{2\pi}}.
\]

We remark that from Eq. (54), \( R \sim \sqrt{f} \), such that the restricted domain occupied by the distribution expands concomitantly with the expansion of the phase space. Therefore, asymptotically, in the limit \( f \to \infty \), the whole phase space is occupied.

**B. Discrete case**

Consider now the joint distribution of matrix elements of the adjacent matrix of the ER model. Following the same scheme we impose the condition that the the trace can not be greater than a value \( d \), that is the variations of the set \( a \) of random binary variables are constrained to obey the condition

\[
\sum_{i=1}^{f} a_i = k \leq d.
\]

The probability that the sum be equal to \( k \) has to be redefined as

\[
P_f(\alpha, k) = \begin{cases} 
\frac{\exp(-\alpha k)}{1 + \exp(-\alpha)^f}, & \text{if } f \leq d \\
\frac{\exp(-\alpha k)}{W_f[1 + \exp(-\alpha)]^f}, & \text{if } f > d \text{ and } k \leq d \\
0, & \text{if } f > d \text{ and } k > d
\end{cases}
\]

In (62), the normalization constant is given by
$$W_f(\alpha, d) = \sum_{k=0}^{d} \frac{f!}{k!(f-k)!} \frac{\exp(-k\alpha)}{[1 + \exp(-\alpha)]^f}. \tag{63}$$

The entropy of this distribution is

$$S_q = \frac{1}{q-1} \left(1 - \frac{1}{W_f^q} \sum_{k=1}^{d} \frac{f!}{k!(f-k)!} \frac{\exp(-qk\alpha)}{[1 + \exp(-\alpha)]^q f}\right). \tag{64}$$

For $f >> d$, the sums in (63) and (64) can be approximated by their last terms as

$$W_f(\alpha, d) \sim \frac{f^d \exp(-d\alpha)}{d! \left[1 + \exp(-\alpha)\right]^f} \tag{65}$$

and

$$\sum_{k=1}^{d} \frac{f!}{k!(f-k)!} \frac{\exp(-qk\alpha)}{[1 + \exp(-\alpha)]^q f} \sim \frac{f^d \exp(-qd\alpha)}{d! \left[1 + \exp(-\alpha)\right]^q f}, \tag{66}$$

which substituted in (64) lead to the approximated expression

$$S_q \sim \frac{1}{q-1} \left(1 - \frac{f^{(1-q)d}}{(d!)^{1-q}}\right) \tag{67}$$

for the entropy. Immediately we conclude that making

$$q = 1 - \frac{1}{d} \tag{68}$$

$S_q$ becomes extensive in the limit $f \to \infty$.

To discuss the scale-invariant property of the distribution (62), we first note that apart from the normalization $W_f(\alpha, d)$ the expressions are the original probabilities of the Erdős-Renyi graph which satisfy the scale-invariant property. Therefore, it is enough to consider that extra term, that is the ratio between its value when $f$ changes to $f + 1$. Using its approximation by the last term we obtain

$$\frac{W_{f+1}(\alpha, d)}{W_f(\alpha, d)} \sim \frac{1}{1 + \exp(-\alpha)} \left(\frac{f + 1}{f}\right)^d, \tag{69}$$

which using the relation (30) becomes

$$\frac{W_{f+1}(\alpha, d)}{W_f(\alpha, d)} \sim (1 + p) \left(\frac{f + 1}{f}\right)^d. \tag{70}$$

Assuming that the probability scales as $p \sim N^{-z}$, we conclude that the distribution satisfies asymptotically the scale-invariant property.
IV. CONCLUSION

We discuss the extensivity of the nonadditive entropy, $S_q$, using the joint density distribution of certain random matrix ensembles. This extensivity has previously been investigated, using artificial distributions constructed with that objective. Here we are studying it resorting to probability distributions of statistical models with applications in many area. Our main result is to set the role the restriction in the variations of the random variables plays, in order to have extensivity for values different from one of the entropic parameter. We also verified that when this happens asymptotically the scale-invariant property condition is satisfied. Of course, restriction in the occupation of phase space concomitantly entails correlations among the variables.

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**Figure Captions**

Fig 1. Eqs. (19) and (22) for the exact and the asymptotic expressions of the $S_q$ entropy of the disordered ($\beta = 1$) Gaussian ensemble are plotted, as a function of the number of variables $f$, fixing $\frac{1}{q-1} + \frac{f}{2} = \tilde{\xi}$ with $\tilde{\xi} = 4$.

Fig 2. The expression of the $S_q$ entropy of the disordered adjacency matrices is plotted, as a function of number of variables $f$, fixing $\frac{1}{q-1} + \frac{f}{2} = \xi$ with $\xi = 6$.

Fig 3. The $S_q$ entropy of the restricted ($\beta = 1$) Gaussian ensemble is plotted as a function of the number of variables $f$ for $q = 2, 3$ and 4, The dashed line is the approximating asymptotic expression Eq. (57).