An Ergodic Result

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Abstract

A rather general ergodic type scheme is presented on arbitrary sets \( X \), as they are generated by arbitrary mappings \( T : X \rightarrow X \). The structures considered on \( X \) are given by suitable subsets of the set of all of its finite partitions. Ergodicity is studied not with respect to subsets of \( X \), but with the inverse limits of families of finite partitions.

1. The Setup

Let \((X, \Sigma, T)\) be as follows: \( X \) is an arbitrary nonvoid set, \( \Sigma \subseteq \mathcal{P}(X) \) is a nonvoid set of subsets of \( X \), while \( T : X \rightarrow X \).

The issue considered, as usual in Ergodic Theory, is the behaviour of the sequence of iterates \( T^n(x), n \in \mathbb{N}_+ = \{1, 2, 3, \ldots\} \), for an arbitrary given \( x \in X \). Of a main interest in this regard is of course the case when \( X \) is infinite.

A simplest and natural way to follow is to consider a partition of \( X \), and see how the mentioned sequence of iterates may possibly move through the various sets of that partition. In this regard, a further simplest and natural case is when the partitions considered for \( X \) are finite, and thus at least one of their sets must contain infinitely many terms of any such sequence of iterates.

As it turns out, a number of properties can be obtained simply form
the finite versus infinite interplay as set up above, an interplay slightly extending the usual pigeon-hole principle. However, in order to obtain such properties, one may have to shift the usual focus which tends to be concerned with the relationship between the mapping $T$ and its iterates $T^n$, with $n \in \mathbb{N}_+$, and on the other hand, the various subsets $A \subseteq X$. Namely, this time one is dealing with the relationship between the mapping $T$ and its iterates $T^n$, with $n \in \mathbb{N}_+$, and on the other hand, whole families of finite partitions $\Delta$ of $X$.

Let us therefore consider

\[(1.1) \quad \mathcal{FP}(X, \Sigma)\]

the set of all finite partitions $\Delta$ of $X$ with nonvoid subsets in $\Sigma$, thus $\Delta \subseteq \Sigma$, $\Delta$ is finite, and $X = \bigcup_{A \in \Delta} A$, where for $A \in \Delta$ we have $A \neq \phi$, however in general, none of $A \in \Delta$ need to be finite.

Given $x \in X$ and $\Delta \in \mathcal{FP}(X, \Sigma)$, then obviously

\[(1.2) \quad \exists \ A \in \Delta : \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} \text{ is infinite} \]

since $\Delta$ is finite.

Let us therefore denote

\[(1.3) \quad \Delta(x) = \{ A \in \Delta \mid \{ n \in \mathbb{N}_+ \mid T^n(x) \in A \} \text{ is infinite} \} \]

and then (1.2) implies

\[(1.4) \quad \Delta(x) \neq \phi \]

**Problem 1**

Given $x \in X$, what happens with $\Delta(x)$, when $\Delta$ ranges over $\mathcal{FP}(X, \Sigma)$?

**Example 1**
Let $X = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$ and consider the following three cases of mappings $T : \mathbb{N} \rightarrow \mathbb{N}$, where here and in the sequel, we denote $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$:

1) $T$ is given by the usual shift $T(x) = x + 1$, $x \in \mathbb{N}$.

If $\Delta \in \mathcal{FP}(\mathbb{N}, \mathcal{P}(\mathbb{N}))$, then obviously there exists $A \in \Delta$ such that $A$ is infinite. Furthermore, for every $x \in \mathbb{N}$, we have

\[ \Delta(x) = \{ A \in \Delta \mid A \text{ is infinite} \} \neq \emptyset \]

2) If $T$ is the identity mapping then clearly

\[ \Delta(x) = \{ A \} \neq \emptyset, \quad \text{where } x \in A \in \Delta \]

3) Let us now assume that, for a given $x_\ast \in \mathbb{N}$, we have $T(x) = x_\ast$, with $x \in \mathbb{N}$. Then obviously

\[ \Delta(x) = \{ A \} \neq \emptyset, \quad \text{where } x_\ast \in A \in \Delta \]

**Remark 1**

The above general setup clearly contains as a particular case the following one which is of a wide interest in Ergodic Theory, namely, $(X, \Sigma, \nu)$, where $\Sigma$ is a $\sigma$-algebra on $X$, while $\nu$ is a probability on $(X, \Sigma)$. In that case, the mapping $T$ is supposed to satisfy the conditions

\[ T^{-1}(\Sigma) \subseteq \Sigma \]

and

\[ \nu(T^{-1}(A)) = \nu(A), \quad A \in \Sigma \]

We note that $\Sigma$ being a $\sigma$-algebra, we have in particular
\begin{align}
\forall \ A, A' \in \Sigma : \ A'' = A \cap A' \in \Sigma \\
\therefore \end{align}

(1.10)

\begin{align}
\psi, X \in \Sigma \\
\therefore
\end{align}

(1.11)

\begin{align}
\mathcal{FP}(X, \Sigma) \neq \phi
\end{align}

2. Towards a Solution

First we observe the following natural structure on \( \mathcal{FP}(X, \Sigma) \), given by the concept of refinement. Namely, if \( \Delta, \Delta' \in \mathcal{FP}(X, \Sigma) \), we define

(2.1) \[ \Delta \leq \Delta' \]

if and only if

(2.2) \[ \forall \ A' \in \Delta' : \ \exists \ A \in \Delta : \ A' \subseteq A \]

and in view of that, we can define the mapping

(2.3) \[ \psi_{\Delta', \Delta} : \Delta' \rightarrow \Delta \]

by

(2.4) \[ A' \subseteq A = \psi_{\Delta', \Delta}(A'), \ A' \in \Delta' \]

Then we obtain

Lemma 1

(2.5) \[ \psi_{\Delta', \Delta}(\Delta'(x)) \subseteq \Delta(x), \ x \in X \]

Proof

If \( A' \in \Delta'(x) \), then (1.3) gives
\[
\{ n \in \mathbb{N}_+ \mid T^n(x) \in A' \} \text{ is infinite}
\]

but in view of (2.4), we have

\[
A' \subseteq \psi_{\Delta', \Delta}(A')
\]

hence

\[
\{ n \in \mathbb{N}_+ \mid T^n(x) \in \psi_{\Delta', \Delta}(A') \} \text{ is infinite}
\]

thus (2.5).

\[
\square
\]

Let us pursue the consequences of the above result in (2.5). In this regard we note that in the usual particular case in Remark 1, the partial order (2.1) on $\mathcal{FP}(X, \Sigma)$ is in fact directed, and obviously has the following stronger property

\[
(2.6) \quad \forall \Delta, \Delta' \in \mathcal{FP}(X, \Sigma) : \exists \Delta \vee \Delta' \in \mathcal{FP}(X, \Sigma)
\]

since

\[
(2.7) \quad \Delta \vee \Delta' = \{ A \cap A' \mid A \in \Delta, A' \in \Delta', A \cap A' \neq \emptyset \}
\]

However, for a greater generality, let us consider in Problem 1 not only the whole of $\mathcal{FP}(X, \Sigma)$, but also arbitrary subsets of it. Let therefore $(\Lambda, \leq)$ be any partially ordered set, and consider a mapping

\[
(2.8) \quad \Lambda \ni \lambda \mapsto \Delta_\lambda \in \mathcal{FP}(X, \Sigma)
\]

such that

\[
(2.9) \quad \lambda \leq \lambda' \implies \Delta_\lambda \leq \Delta_{\lambda'}
\]

We call the family $(\Delta_\lambda)_{\lambda \in \Lambda}$ a refinement chain.

Obviously, in view of the above, $\mathcal{FP}(X, \Sigma)$ itself is such a refinement chain, namely, with $\Lambda = \mathcal{FP}(X, \Sigma)$, the partial order in (2.1), and
with the identity mapping in (2.8).

The main point to note is the following. Given \( x \in X \), then (1.4) implies

\[
\Delta_{\lambda}(x) \neq \phi, \quad \lambda \in \Lambda
\]

Hence in view of (2.5), (2.9), we have for \( \lambda \leq \lambda' \)

\[
\phi \neq \psi_{\Delta_{\lambda'}, \Delta_{\lambda}(\Delta_{\lambda'}(x))} \subseteq \Delta_{\lambda}(x)
\]

Now, based on (2.10), let us use the notation

\[
\Delta_{\lambda}(x) = \{ A_{\lambda,1}(x), \ldots, A_{\lambda,m_{\lambda}}(x) \}
\]

where \( m_{\lambda} \geq 1 \), and \( \phi \neq A_{\lambda,j}(x) \in \Sigma \), with \( 1 \leq j \leq m_{\lambda} \).

**Problem 2**

A more precise reformulation of Problem 1 is as follows. We can investigate whether for a given \( x \in X \), one or the other of the following two properties may hold, namely

\[
\exists \Lambda \ni \lambda \mapsto A_{\lambda} \in \Delta_{\lambda}(x) : \bigcap_{\lambda \in \Lambda} A_{\lambda} \neq \phi
\]

or what appears to be a milder property

\[
\lim_{\lambda \in \Lambda} \Delta_{\lambda}(x) \neq \phi
\]

The *inverse limit*, [1, p. 191], in (2.14) is of the family

\[
(\Delta_{\lambda}(x) \mid \lambda \in \Lambda)
\]

with the mappings, see (2.8), (2.10)

\[
\psi_{\lambda', \lambda, x} : \Delta_{\lambda'}(x) \longrightarrow \Delta_{\lambda}(x)
\]
for $\lambda, \lambda' \in \Lambda$, $\lambda \leq \lambda'$ where

$$
(2.17) \quad \psi_{\lambda', \lambda} : \Delta_{\lambda'} \longrightarrow \Delta_{\lambda}
$$

is given by

$$
(2.18) \quad \psi_{\lambda', \lambda} = \psi_{\Delta_{\lambda'}, \Delta_{\lambda}}
$$

while

$$
(2.19) \quad \psi_{\lambda', \lambda, x} = \psi_{\lambda', \lambda} \mid_{\Delta_{\lambda'}(x)}
$$

In order to establish (2.14), we recall the definition of the inverse limit, namely

$$
(2.20) \quad \lim_{\lambda \in \Lambda} \Delta_{\lambda}(x) = \left\{ (A_{\lambda} \mid \lambda \in \Lambda) \in \prod_{\lambda \in \Lambda} \Delta_{\lambda}(x) \mid \forall \lambda, \lambda' \in \Lambda, \lambda \leq \lambda' : \psi_{\lambda', \lambda, x}(A_{\lambda'}) = A_{\lambda} \right\}
$$

We note that in the definition of the inverse limit, the partial order $\leq$ on $\Lambda$ can be arbitrary, and in fact, it can be a mere pre-order.

Further we note, [1, Exercise 4, no. 4, p. 252], that an inverse limit such as for instance in (2.20), can be void even when all sets $\Delta_{\lambda}(x)$ are nonvoid and all mappings $\psi_{\lambda', \lambda, x}$ are surjective.

However, as seen in Theorem 1 in the sequel, this is not the case in (2.14).

Meanwhile, for the sake of further clarification, we consider (2.20) in the following particular case.

**Example 2**

In the case 1) of Example 1, let us consider $(\Lambda, \leq) = \mathbb{N}$, and take the following sequence of finite partitions of $\mathbb{N}$
\[ (2.21) \quad \mathbb{N} \ni \lambda \mapsto \Delta_{\lambda} \in \mathcal{FP}(\mathbb{N}, \mathcal{P}(\mathbb{N})) \]

where

\[
\begin{align*}
\Delta_0 &= \{ \mathbb{N} \} \\
\Delta_1 &= \{ \{0\}, \{1, 2, 3, \ldots\} \} \\
\Delta_2 &= \{ \{0\}, \{1\}, \{2, 3, 4, \ldots\} \} \\
\Delta_3 &= \{ \{0\}, \{1\}, \{2\}, \{3, 4, 5, \ldots\} \} \\
&\cdots\cdots\cdots\cdots
\end{align*}
\]

thus clearly \( \Delta_0 \leq \Delta_1 \leq \Delta_2 \leq \Delta_3 \leq \ldots \)

Now if we take \( x = 0 \in \mathbb{N} = X \), then, see (2.12)

\[ (2.23) \quad \Delta_{\lambda}(x) = \{ A_{\lambda, x} = \{\lambda, \lambda + 1, \lambda + 2, \ldots\} \}, \quad \lambda \in \Lambda \]

Therefore (2.13) fails to hold, since obviously

\[ (2.24) \quad \bigcap_{\lambda \in \Lambda} A_{\lambda, x} = \phi \]

On the other hand, regarding (2.14), in view of (2.20), (2.23), as well as (2.16) - (2.19), we obtain

\[ (2.25) \quad \lim_{\lambda \in \Lambda} \Delta_{\lambda}(x) = \{ (A_{\lambda, x} \mid \lambda \in \Lambda) \} \neq \phi \]

In the case 2) of Example 1, for \( x \in \mathbb{N} \), we have, see (1.6)

\[ (2.26) \quad \Delta_{\lambda}(x) = \{ A_{\lambda, x} \} \]

where

\[ (2.27) \quad A_{\lambda, x} = \begin{cases} \{x\} & \text{if } x < \lambda \\ \{\lambda, \lambda + 1, \lambda + 2, \ldots\} & \text{if } x \geq \lambda \end{cases} \]

thus (2.13) will this time hold, since (2.26), (2.27) obviously yield for \( x \in \mathbb{N} \)

\[ (2.28) \quad \bigcap_{\lambda \in \Lambda} A_{\lambda, x} = \{x\} \neq \phi \]
As for (2.14), the relations (2.26), (2.27) applied to (2.20) give

\( \lim_{\lambda \in \Lambda} \Delta(\lambda) = \{ A_{\lambda,x} \mid \lambda \in \Lambda \} \neq \phi \)

which in view of (2.26) - (2.28) means essentially that

\( \lim_{\lambda \in \Lambda} \Delta(\lambda) = \{ x \} \neq \phi \)

Lastly, in the case 3) of Example 1, we have, see (1.7)

\( \Delta(\lambda) = \{ A_{\lambda,x} \}, \quad x \in \mathbb{N} \)

where

\( A_{\lambda,x} = \begin{cases} \{ x_\ast \} & \text{if } x_\ast < \lambda \\ \{ \lambda, \lambda + 1, \lambda + 2, \ldots \} & \text{if } x_\ast \geq \lambda \end{cases} \)

hence (2.13) holds again, since

\( \bigcap_{\lambda \in \Lambda} A_{\lambda,x} = \{ x_\ast \} \neq \phi \)

while (2.14) takes the form

\( \lim_{\lambda \in \Lambda} \Delta(\lambda) = \{ A_{\lambda,x} \mid \lambda \in \Lambda \} \neq \phi \)

which in view of (2.32) means essentially that

\( \lim_{\lambda \in \Lambda} \Delta(\lambda) = \{ x_\ast \} \neq \phi \)

**Remark 2**

The three instances in Example 2 above, with their respective versions (2.25), (2.29), (2.30), (2.34) and (2.35) of problem (2.14) as formulated in Problem 2, can give a motivation for the use of the inverse limits in Ergodic Theory. Indeed, in each of these three cases, the corresponding inverse limits reflect in a nontrivial manner obvious ergodic properties of the specific mappings \( T \) involved.

In this regard, the relevance of the inverse limit is particularly clear.
in the first instance in Example 2, namely, when \( T : \mathbb{N} \rightarrow \mathbb{N} \) is the usual shift, and when problem (2.13), as formulated in Problem 2, has a solution in (2.24) which does not give much information about \( T \), since the same relation may be obtained for many other mappings of \( \mathbb{N} \) into itself.

On the other hand, the inverse limit in (2.25) does give an information which is clearly more revealing about the specific feature of \( T \).

Of course, in analyzing the ergodic features of mappings \( T \) of \( \mathbb{N} \) into itself, one can use a variety of other refinement chains, than the particular one in (2.21), (2.22). We consider next such an example of a different refinement chain in the case 1) of Example 1.

**Example 3**

Let \( X = \mathbb{N}, \Sigma = \mathcal{P}(\mathbb{N}) \) and consider the mapping \( T : \mathbb{N} \rightarrow \mathbb{N} \) given by the usual shift \( T(x) = x + 1, \ x \in \mathbb{N} \).

Let \( \mathcal{U} \) be a *free ultrafilter* on \( X = \mathbb{N} \) which, we recall, means a filter with the following two properties

\[
\forall \ A \subseteq \mathbb{N} : \text{ either } A \in \mathcal{U}, \text{ or } \mathbb{N} \setminus A \in \mathcal{U}
\]

\[
\bigcap_{U \in \mathcal{U}} U = \phi
\]

These two conditions imply that

\[
\forall \ U \in \mathcal{U} : \ U \text{ is infinite}
\]

Furthermore, we also have that

\[
\exists \ U \in \mathcal{U} : \ \mathbb{N} \setminus U \text{ is infinite}
\]

For \( U \in \mathcal{U} \), let us consider the set of finite partitions \( \Delta \) of \( \mathbb{N} \) which contain \( U \), that is, given by

\[
\mathcal{F}(\mathbb{N}, \mathcal{P}(\mathbb{N}))(U) = \{ \Delta \in \mathcal{F}(\mathbb{N}, \mathcal{P}(\mathbb{N})) \mid U \in \Delta \}
\]
Further, let us consider
\[ FP_u(N, \mathcal{P}(N)) = \bigcup_{U \in \mathcal{U}} FP_U(N, \mathcal{P}(N)) = \]
\[ = \{ \Delta \in FP(N, \mathcal{P}(N)) \mid \exists U \in \mathcal{U} : U \in \Delta \} \]

We shall take now \((\Lambda, \leq) = FP_u(N, \mathcal{P}(N))\) endowed with the partial order \(\leq\) in (2.1) which corresponds to the usual refinement of partitions. Finally, the mapping (2.8), (2.9) will simply be the identity mapping
\[ (2.42) \quad \Lambda \ni \lambda = \Delta \mapsto \Delta \in FP_u(N, \mathcal{P}(N)) \]

Given \(U \in \mathcal{U}, \Delta \in FP_u(N, \mathcal{P}(N)), \) with \(U \in \Delta,\) as well as \(x \in N,\) it follows easily that
\[ (2.43) \quad U \in \Delta(x) \]

and in fact, we have the stronger property, similar with (1.5), namely
\[ (2.44) \quad \Delta(x) = \{ A \in \Delta \mid A \text{ is infinite} \} \]

Now it is easy to see that, in view of (2.20), we obtain
\[ (2.45) \quad (A_\lambda \mid \lambda \in \Lambda) \in \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) \]

where for \(\lambda = \Delta \in FP_U(N, \mathcal{P}(N)),\) we have
\[ (2.46) \quad \Delta_\lambda = \Delta, \quad A_\lambda = U \]

hence
\[ (2.47) \quad \varprojlim_{\lambda \in \Lambda} \Delta_\lambda(x) \neq \phi \]

3. A General Inverse Limit Ergodic Result

As seen in the theorem next, the result in (2.25) is in fact a particular case of a rather general one.
Theorem 1

Let \((X, \Sigma, T)\) be as at the beginning of section 1. Further, let \((\Lambda, \leq)\) be a directed partial order, together with a mapping, see (2.8)

\[
\Lambda \ni \lambda \mapsto \Delta_\lambda \in \mathcal{FP}(X, \Sigma)
\]

which satisfies (2.9), as well as the following condition:

\begin{equation}
\exists \Lambda_0 \subseteq \Lambda : \Lambda_0 \text{ is countable and cofinal in } \Lambda
\end{equation}

Then for every \(x \in X\), we have

\begin{equation}
\lim_{\lambda \in \Lambda} \Delta_\lambda(x) \neq \phi
\end{equation}

Proof.

It follows from Proposition 5 in [1, p. 198], whose conditions are satisfied, as shown next.

Indeed, given \(x \in X\), in view of (2.10), we have

\begin{equation}
\Delta_\lambda(x) \neq \phi, \quad \lambda \in \Lambda
\end{equation}

Further, (2.16) gives for \(\lambda, \lambda' \in \Lambda\), \(\lambda \leq \lambda'\) the mapping

\begin{equation}
\psi_{\lambda', \lambda, x} : \Delta_\lambda' \longrightarrow \Delta_\lambda(x)
\end{equation}

and obviously, see (2.17) - (2.19), (2.3), (2.4)

\begin{equation}
\psi_{\lambda, \lambda, x} = id_{\Delta_\lambda(x)}
\end{equation}

while for \(\lambda, \lambda', \lambda'' \in \Lambda\), \(\lambda \leq \lambda' \leq \lambda''\) we have

\begin{equation}
\psi_{\lambda', \lambda, x} \circ \psi_{\lambda'', \lambda', x} = \psi_{\lambda'', \lambda, x}
\end{equation}

Lastly, the mappings (3.3) are surjective. Indeed, let \(A \in \Delta_\lambda(x)\), then we have to find \(A' \in \Delta_{\lambda'}(x)\), such that
ψ_{\lambda',\lambda,x}(A') = A

But (1.3) yields

\begin{equation}
\{ \ n \in \mathbb{N}_+ \mid T^n(x) \in A \ \}\text{ is infinite}
\end{equation}

Therefore (2.1) - (2.4) will give \( A' \in \Delta_{\lambda'}(x) \), such that \( A' \subseteq A \), which means precisely (3.7).

Remark 3

1) An important fact in Theorem 1 above is that there are no conditions whatsoever required on the mappings \( T : X \rightarrow X \).

2) In general, when \( \Sigma \) is uncountable - a case which is often of interest in applications - the set \( \mathcal{F}P(X, \Sigma) \) of all finite partitions of \( X \) with subsets in \( \Sigma \), see (1.1), will also be uncountable. Furthermore, when considered with the natural partial order (2.1), (2.2), the set \( \mathcal{F}P(X, \Sigma) \) does not have a countable cofinal subset. Therefore, in such a case one cannot take in Theorem 1

\begin{equation}
(\Lambda, \leq) = (\mathcal{F}P(X, \Sigma), \leq)
\end{equation}

as the directed partial order, and instead, one has to limit oneself to smaller directed partial orders \( (\Lambda, \leq) \), namely, to those which satisfy condition (3.1).

3) The set \( \Sigma \) can be uncountable even when \( X \) is countable, since one can take, for instance, \( \Sigma = \mathcal{P}(X) \), that is, the set of all subsets of \( X \).

References

[1] Bourbaki N : Elements of Mathematics, Theory of Sets. Springer, 2004