ON THE MONOID OF MONOTONE INJECTIVE PARTIAL SELFMAPS OF $\mathbb{N}^2_\preceq$ WITH COFINITE DOMAINS AND IMAGES, II

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Abstract. Let $\mathbb{N}^2_\preceq$ be the set $\mathbb{N}^2$ with the partial order defined as the product of usual order $\preceq$ on the set of positive integers $\mathbb{N}$. We study the semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)$ of monotone injective partial selfmaps of $\mathbb{N}^2_\preceq$ having cofinite domain and image. We describe the natural partial order on the semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)$ and show that it coincides with the natural partial order which is induced from symmetric inverse monoid $\mathcal{F}_{\mathbb{N}\times\mathbb{N}}$ over the set $\mathbb{N}\times\mathbb{N}$ onto the semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)$. We proved that the semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)$ is isomorphic to the semidirect product $\mathcal{PO}_\infty^+(\mathbb{N}^2_\preceq) \times \mathbb{Z}_2$ of the monoid $\mathcal{PO}_\infty^+(\mathbb{N}^2_\preceq)$ of orientation-preserving monotone injective partial selfmaps of $\mathbb{N}^2_\preceq$ with cofinite domains and images by the cyclic group $\mathbb{Z}_2$ of the order two. Also we describe the congruence $\sigma$ on the semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)$ which is generated by the natural order $\preceq$ on the semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)$: $\alpha\sigma\beta$ if and only if $\alpha$ and $\beta$ are comparable in $(\mathcal{PO}_\infty(\mathbb{N}^2_\preceq), \preceq)$. We prove that the quotient semigroup $\mathcal{PO}_\infty^+(\mathbb{N}^2_\preceq)/\sigma$ is isomorphic to the free commutative monoid $\mathcal{AM}_\omega$ over an infinite countable set and show that the quotient semigroup $\mathcal{PO}_\infty(\mathbb{N}^2_\preceq)/\sigma$ is isomorphic to the semidirect product of the free commutative monoid $\mathcal{AM}_\omega$ by the group $\mathbb{Z}_2$.

We shall follow the terminology of [2] and [10].

In this paper we shall denote the first infinite cardinal by $\omega$ and the cardinality of the set $A$ by $|A|$. We shall identify every set $X$ with its cardinality $|X|$. By $\mathbb{Z}_2$ we shall denote the cyclic group of order two. Also, for infinite subsets $A$ and $B$ of an infinite set $X$ we shall write $A \subseteq^* B$ if and only if there exists a finite subset $A_0$ of $A$ such that $A \setminus A_0 \subseteq B$.

An algebraic semigroup $S$ is called inverse if for any element $x \in S$ there exists a unique $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. The element $x^{-1}$ is called the inverse of $x \in S$.

If $S$ is a semigroup, then we shall denote the subset of idempotents in $S$ by $E(S)$. If $S$ is an inverse semigroup, then $E(S)$ is closed under multiplication and we shall refer to $E(S)$ a band (or the band of $S$). If the band $E(S)$ is a non-empty subset of $S$, then the semigroup operation on $S$ determines the following partial order $\preceq$ on $E(S)$: $e \preceq f$ if and only if $ef = fe = e$. This order is called the natural partial order on $E(S)$. A semilattice is a commutative semigroup of idempotents.

If $\alpha: X \to Y$ is a partial map, then by $\text{dom} \alpha$ and $\text{ran} \alpha$ we denote the domain and the range of $\alpha$, respectively.

Let $\mathcal{F}_\lambda$ denote the set of all partial one-to-one transformations of an infinite set $X$ of cardinality $\lambda$ together with the following semigroup operation: $x(\alpha\beta) = (x\alpha)\beta$ if $x \in \text{dom}(\alpha\beta) = \{ y \in \text{dom} \alpha \mid y\alpha \in \text{dom} \beta \}$, for $\alpha, \beta \in \mathcal{F}_\lambda$. The semigroup $\mathcal{F}_\lambda$ is called the symmetric inverse semigroup over the set $X$ (see [2 Section 1.9]). The symmetric inverse semigroup was introduced by Vagner [18] and it plays a major role in the theory of semigroups. An element $\alpha \in \mathcal{F}_\lambda$ is called cofinite, if the sets $\lambda \setminus \text{dom} \alpha$ and $\lambda \setminus \text{ran} \alpha$ are finite.

Let $(X, \preceq)$ be a partially ordered set (a poset). For an arbitrary $x \in X$ we denote

$$\uparrow x = \{ y \in X : x \preceq y \}.$$ 

We shall say that a partial map $\alpha: X \to X$ is monotone if $x \preceq y$ implies $(x)\alpha \preceq (y)\alpha$ for $x, y \in \text{dom} \alpha$. 

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Let \( \mathbb{N} \) be the set of positive integers with the usual linear order \( \leq \). On the Cartesian product \( \mathbb{N} \times \mathbb{N} \) we define the product partial order, i.e.,

\[
(i, m) \leq (j, n) \quad \text{if and only if} \quad (i \leq j) \quad \text{and} \quad (m \leq n).
\]

Later the set \( \mathbb{N} \times \mathbb{N} \) with so defined partial order will be denoted by \( \mathbb{N}_\leq^2 \).

By \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) we denote the semigroup of injective partial monotone selfmaps of \( \mathbb{N}^2_\leq \) with cofinite domains and images. Obviously, \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) is a submonoid of the symmetric inverse semigroup \( \mathcal{I}_\omega \) and \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) is a countable semigroup.

Furthermore, we shall denote the identity of the semigroup \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) by \( I \) and the group of units of \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) by \( H(I) \).

For any positive integer \( n \) and an arbitrary \( \alpha \in \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) we denote:

\[
V^n = \{(n, j): j \in \mathbb{N}\}; \quad H^n = \{(j, n): j \in \mathbb{N}\};
\]

\[
V^n_{\text{dom} \alpha} = V^n \cap \text{dom} \alpha; \quad V^n_{\text{ran} \alpha} = V^n \cap \text{ran} \alpha;
\]

\[
H^n_{\text{dom} \alpha} = H^n \cap \text{dom} \alpha; \quad H^n_{\text{ran} \alpha} = H^n \cap \text{ran} \alpha,
\]

and

\[
(i_{\alpha[i,j]}, j_{\alpha[i,j]}) = (i, j)\alpha, \quad \text{for every} \quad (i, j) \in \text{dom} \alpha.
\]

It well known that each partial injective cofinite selfmap \( f \) of \( \lambda \) induces a homeomorphism \( f^*: \lambda^* \to \lambda^* \) of the remainder \( \lambda^* = \beta \lambda \setminus \lambda \) of the Stone–Čech compactification of the discrete space \( \lambda \). Moreover, under some set theoretic axioms (like \textsf{PFA} or \textsf{OCA}), each homeomorphism of \( \omega^* \) is induced by some partial injective cofinite selfmap of \( \omega \) (see [12–17]). So, the inverse semigroup \( \mathcal{I}_\lambda^{\text{cf}} \) of injective partial selfmaps of an infinite cardinal \( \lambda \) with cofinite domains and images admits a natural homomorphism \( h: \mathcal{I}_\lambda^{\text{cf}} \to \mathcal{H}(\lambda^*) \) to the homeomorphism group \( \mathcal{H}(\lambda^*) \) of \( \lambda^* \) and this homomorphism is surjective under certain set theoretic assumptions.

In the paper [9] algebraic properties of the semigroup \( \mathcal{I}_\lambda^{\text{cf}} \) are studied. It is showed that \( \mathcal{I}_\lambda^{\text{cf}} \) is a bisimple inverse semigroup and that for every non-empty chain \( L \in E(\mathcal{I}_\lambda^{\text{cf}}) \) there exists an inverse subsemigroup \( S \) of \( \mathcal{I}_\lambda^{\text{cf}} \) such that \( S \) is isomorphic to the bicyclic semigroup and \( L \subseteq E(S) \), the Green relations on \( \mathcal{I}_\lambda^{\text{cf}} \) are described and it is proved that every non-trivial congruence on \( \mathcal{I}_\lambda^{\text{cf}} \) is a group congruence. Also, the structure of the quotient semigroup \( \mathcal{I}_\lambda^{\text{cf}}/\sigma \) is described, where \( \sigma \) is the least group congruence on \( \mathcal{I}_\lambda^{\text{cf}} \).

The semigroups \( \mathcal{I}_\infty(\mathbb{N}) \) and \( \mathcal{I}_\infty(\mathbb{Z}) \) of injective isotone partial selfmaps with cofinite domains and images of positive integers and integers are studied in [7] and [3], respectively. It was proved that the semigroups \( \mathcal{I}_\infty(\mathbb{N}) \) and \( \mathcal{I}_\infty(\mathbb{Z}) \) have similar properties to the bicyclic semigroup: they are bisimple and every non-trivial homomorphic image \( \mathcal{I}_\infty(\mathbb{N}) \) and \( \mathcal{I}_\infty(\mathbb{Z}) \) is a group, and moreover the semigroup \( \mathcal{I}_\infty(\mathbb{N}) \) has \( \mathbb{Z}(+) \) as a maximal group image and \( \mathcal{I}_\infty(\mathbb{Z}) \) has \( \mathbb{Z}(+) \times \mathbb{Z}(+) \), respectively.

In the paper [6] we studied the semigroup \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \) of monotone injective partial selfmaps of the set of \( L_n \times_{\text{lex}} \mathbb{Z} \) having cofinite domain and image, where \( L_n \times_{\text{lex}} \mathbb{Z} \) is the lexicographic product of \( n \)-elements chain and the set of integers with the usual linear order. In this paper we described Green’s relations on \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \), showed that the semigroup \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \) is bisimple and established its projective congruences. Also, we proved that \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \) is finitely generated, every automorphism of \( \mathcal{IO}_\infty(\mathbb{Z}) \) is inner and showed that in the case \( n \geq 2 \) the semigroup \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \) has non-inner automorphisms. In [10] we also proved that for every positive integer \( n \) the quotient semigroup \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n)/\sigma \), where \( \sigma \) is a least group congruence on \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \), is isomorphic to the direct power \( (\mathbb{Z}(+))^{2n} \). The structure of the sublattice of congruences on \( \mathcal{IO}_\infty(\mathbb{Z}_{\text{lex}}^n) \) that are contained in the least group congruence is described in [11].

In the paper [13] we studied algebraic properties of the semigroup \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \). We described properties of elements of the semigroup \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) as monotone partial bijection of \( \mathbb{N}^2_\leq \) and showed that the group of units of \( \mathcal{PO}_\infty(\mathbb{N}^2_\leq) \) is isomorphic to the cyclic group of order two. Also in [5] the subsemigroup of
idempotents of \( \mathcal{PO}_\infty(N^2_\preceq) \) and the Green relations on \( \mathcal{PO}_\infty(N^2_\preceq) \) are described. In particular, here we proved that \( \mathcal{I} = \mathcal{J} \) in \( \mathcal{PO}_\infty(N^2_\preceq) \).

The present paper is a continuation of [5]. We describe the natural partial order \( \prec \) on the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \) and show that it coincides with the natural partial order which is induced from symmetric inverse monoid \( \mathcal{J}_{N \times N} \) over the set \( N \times N \) onto the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \). We proved that the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \) is isomorphic to the semidirect product \( \mathcal{PO}_\infty(N^2_\preceq) \times \mathbb{Z}_2 \) of the monoid \( \mathcal{PO}_\infty(N^2_\preceq) \) of orientation-preserving monotone injective partial selfmaps of \( \mathbb{N}^2_\preceq \) with cofinite domains and images by the cyclic group \( \mathbb{Z}_2 \) of the order two. Also we describe the congruence \( \sigma \) on the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \), which is generated by the natural order \( \prec \) on the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \): \( \alpha \sigma \beta \) if and only if \( \alpha \) and \( \beta \) are comparable in \( (\mathcal{PO}_\infty(N^2_\preceq), \prec) \). We prove that the quotient semigroup \( \mathcal{PO}_\infty^+(N^2_\preceq)/\sigma \) is isomorphic to the free commutative monoid \( \mathbb{AM}_\omega \) over an infinite countable set and show that quotient semigroup \( \mathcal{PO}_\infty(N^2_\preceq)/\sigma \) is isomorphic to the semidirect product of the free commutative monoid \( \mathbb{AM}_\omega \) by the group \( \mathbb{Z}_2 \).

The following proposition implies that the equations of the form \( a \cdot x = b \) and \( x \cdot c = d \) in the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \) have finitely many solutions. This property holds for the bicyclic monoid, many its generalizations and other semigroups (see corresponding results in [1, 3, 6, 7, 8, 9]).

**Proposition 1.** For every \( \alpha, \beta \in \mathcal{PO}_\infty(N^2_\preceq) \), both sets

\[
\{ \chi \in \mathcal{PO}_\infty(N^2_\preceq) \mid \alpha \cdot \chi = \beta \} \quad \text{and} \quad \{ \chi \in \mathcal{PO}_\infty(N^2_\preceq) \mid \chi \cdot \alpha = \beta \}
\]

are finite. Consequently, every right translation and every left translation by an element of the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \) is a finite-to-one map.

**Proof.** We consider the case of the equation \( \alpha \cdot \chi = \beta \). In the case of the equation \( \chi \cdot \alpha = \beta \) the proof is similar.

The definition of the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \) and the equality \( \alpha \cdot \chi = \beta \) imply that \( \text{dom} \beta \subseteq \text{dom} \alpha \) and \( \text{ran} \alpha \subseteq \text{ran} \chi \). Since any element of the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \) has a cofinite domain and a cofinite image in \( N \times N \), we conclude that if an element \( \chi_0 \) satisfies the equality \( \alpha \cdot \chi = \beta \) then for every other root \( \chi \) of the equation \( \alpha \cdot \chi = \beta \) there exist finitely many \( (i, j) \in (N \times N) \setminus \text{ran} \beta \) such that one of the following conditions holds:

1. \( (i, j)\chi \neq (i, j)\chi_0 \);
2. \( (i, j)\chi \) is determined and \( (i, j)\chi_0 \) is undetermined;
3. \( (i, j)\chi_0 \) is determined and \( (i, j)\chi \) is undetermined.

This implies that the equation \( \alpha \cdot \chi = \beta \) has finitely many solutions, which completes the proof of the proposition.

Later we shall describe the natural partial order “\( \succeq \)” on the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \). For \( \alpha, \beta \in \mathcal{PO}_\infty(N^2_\preceq) \) we put

\[
\alpha \succeq \beta \quad \text{if and only if} \quad \alpha = \beta \varepsilon \quad \text{for some} \quad \varepsilon \in E(\mathcal{PO}_\infty(N^2_\preceq)).
\]

We need the following proposition from [11].

**Proposition 2** ([11] p. 387, Corollary). For any semigroup \( S \) and its natural partial order \( \succeq \) the following conditions are equivalent:

1. \( a \preceq b \);
2. \( a = wb = bz, az = a \) for some \( w, z \in S^1 \);
3. \( a = xb = by, xa = ay = a \) for some \( x, y \in S^1 \).

**Proposition 3.** The relation \( \succeq \) is the natural partial order on the semigroup \( \mathcal{PO}_\infty(N^2_\preceq) \).
Proof. Suppose that $\alpha = \beta \varepsilon$ for some idempotent $\varepsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$. Then we have that
$$\alpha \varepsilon = (\beta \varepsilon) \varepsilon = \beta (\varepsilon \varepsilon) = \beta \varepsilon = \alpha.$$ Let $\iota: \text{dom}(\beta \varepsilon) \to \text{dom}(\beta \varepsilon)$ be the identity map of the set $\text{dom}(\beta \varepsilon)$. Then $\iota \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ and the definition of the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ implies that $\text{dom}(\beta \varepsilon) = \text{dom}(\iota \beta)$, because $\varepsilon$ is an idempotent of $\mathcal{P}O_\infty(\mathbb{N}_2^2)$. This implies that $(i, j) \iota \beta = (i, j) \beta \varepsilon$ for each $(i, j) \in \text{dom}(\iota \beta)$ and hence we get that $\alpha = \beta \varepsilon = \iota \beta$. Next we apply Proposition 2. \qed

Remark 4. Proposition 3 implies that the natural partial order on the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ coincides with the natural partial order which is induced from symmetric inverse monoid $\mathcal{I}_{\mathbb{N} \times \mathbb{N}}$ over the set $\mathbb{N} \times \mathbb{N}$ onto the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$.

We define a relation $\sigma$ on the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ in the following way:
$$\alpha \sigma \beta \quad \text{if and only if there exists} \quad \varepsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2)) \quad \text{such that} \quad \alpha \varepsilon = \beta \varepsilon,$$
for $\alpha, \beta \in \mathcal{P}O_\infty(\mathbb{N}_2^2)$.

Proposition 5. For $\alpha, \beta \in \mathcal{P}O_\infty(\mathbb{N}_2^2)$ the following conditions are equivalent:
(i) $\alpha \sigma \beta$;
(ii) there exist $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ such that $\alpha \zeta = \beta \upsilon$;
(iii) there exist $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ such that $\alpha \zeta = \upsilon \beta$;
(iv) there exists $\iota \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ such that $\iota \alpha = \iota \beta$;
(v) there exist $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ such that $\zeta \alpha = \upsilon \beta$.

Thus $\sigma$ is a congruence on the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$.

Proof. Implication $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (i)$ If we have that $\alpha \zeta = \beta \upsilon$ for some $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ then $\alpha \zeta (\upsilon \zeta) = \beta \upsilon (\upsilon \zeta)$. Since $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ is a subsemigroup of the symmetric inverse monoid $\mathcal{I}_{\mathbb{N} \times \mathbb{N}}$; the idempotents in the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ commute and hence $\alpha (\upsilon \zeta) = \beta (\upsilon \zeta)$. This implies that $\alpha \sigma \beta$.

$(ii) \Rightarrow (iii)$ Suppose that $\alpha \zeta = \beta \upsilon$ for some $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$. Let $\iota: \text{dom}(\beta \upsilon) \to \text{dom}(\beta \upsilon)$ be the identity map of the set $\text{dom}(\beta \upsilon)$. Then $\iota \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ and the definition of the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ implies that $\text{dom}(\beta \upsilon) = \text{dom}(\iota \beta)$, because $\upsilon$ is an idempotent of $\mathcal{P}O_\infty(\mathbb{N}_2^2)$. This implies that $(i, j) \iota \beta = (i, j) \beta \upsilon$ for each $(i, j) \in \text{dom}(\iota \beta)$ and hence we get that $\alpha \zeta = \beta \upsilon = \iota \beta$.

$(iii) \Rightarrow (ii)$ Suppose that $\alpha \zeta = \upsilon \beta$ for some $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$. Let $\iota: \text{ran}(\upsilon \beta) \to \text{ran}(\upsilon \beta)$ be the identity map of the set $\text{ran}(\upsilon \beta)$. Then $\iota \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ and the definition of the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ implies that $\text{ran}(\upsilon \beta) = \text{ran}(\iota \beta)$, because $\upsilon$ is an idempotent of $\mathcal{P}O_\infty(\mathbb{N}_2^2)$. Since all elements of the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ are partial bijections of $\mathbb{N} \times \mathbb{N}$ we get that $\text{dom}(\upsilon \beta) = \text{dom}(\iota \beta)$. This implies that $(i, j) \beta \iota = (i, j) \upsilon \beta$ for each $(i, j) \in \text{dom}(\beta \iota)$ and hence we get that $\alpha \zeta = \upsilon \beta = \beta \iota$.

The proofs of equivalences $(iii) \Leftrightarrow (iv)$ and $(iv) \Leftrightarrow (v)$ are similar.

It is obvious that $\sigma$ is a reflexive and symmetric relation on $\mathcal{P}O_\infty(\mathbb{N}_2^2)$. Suppose that $\alpha \sigma \beta$ and $\beta \sigma \gamma$ in $\mathcal{P}O_\infty(\mathbb{N}_2^2)$. Then there exist $\zeta, \upsilon \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ such that $\alpha \zeta = \beta \upsilon$ and $\beta \upsilon = \gamma \zeta$. This implies that $\alpha \zeta \upsilon = \beta \zeta \upsilon$ and $\beta \upsilon \zeta = \gamma \upsilon \zeta$, and since the idempotents in $\mathcal{P}O_\infty(\mathbb{N}_2^2)$ commute we get that $\alpha \zeta \upsilon = \beta \zeta \upsilon = \beta \upsilon \zeta = \gamma \upsilon \zeta$, and hence $\alpha \sigma \gamma$.

Suppose that $\alpha \sigma \beta$ for some $\alpha, \beta \in \mathcal{P}O_\infty(\mathbb{N}_2^2)$. Then by $(iv)$ there exists $\iota \in E(\mathcal{P}O_\infty(\mathbb{N}_2^2))$ such that $\iota \alpha = \iota \beta$. This implies that $\alpha \gamma = \iota \beta \gamma$ for each $\gamma \in \mathcal{P}O_\infty(\mathbb{N}_2^2)$ and hence by item $(iv)$ we get that $(\alpha \gamma) \sigma (\beta \gamma)$. The proof of the statement that $(\gamma \alpha) \sigma (\gamma \beta)$ for each $\gamma \in \mathcal{P}O_\infty(\mathbb{N}_2^2)$ is similar, and hence $\sigma$ is a congruence on the semigroup $\mathcal{P}O_\infty(\mathbb{N}_2^2)$.

Corollary 6. For $\alpha, \beta \in \mathcal{P}O_\infty(\mathbb{N}_2^2)$ the following condition are equivalent:
(i) $\alpha \sigma \beta$;
(ii) $\alpha \sigma \beta \sigma \alpha$;

(iii) $\alpha \sigma \beta$;
(iv) $\alpha \sigma \beta \sigma \alpha$;
(iii) \( \varpi \alpha \sigma \varpi \beta \).

Proof. (i) \( \Leftrightarrow (ii) \) If \( \alpha \sigma \beta \) in \( POC_\infty(N_2^n) \) then by Proposition 5 there exists \( i \in E(POC_\infty(N_2^n)) \) such that \( i \alpha = i \beta \). This implies that \( i \alpha \varpi = i \beta \varpi \) and hence \( (\alpha \varpi) \sigma (\beta \varpi) \). Conversely, if \( (\alpha \varpi) \sigma (\beta \varpi) \) then by Proposition 5 we have that \( \nu \alpha \varpi = \nu \beta \varpi \) for some \( \nu \in E(POC_\infty(N_2^n)) \), and hence \( \nu \alpha = \nu \alpha \varpi = \nu \beta \varpi = \nu \beta \), which implies that \( \sigma \alpha \beta \).

The proof of (i) \( \Leftrightarrow (ii) \) is similar. \( \square \)

Also the definition of the congruence \( \sigma \) on the semigroup \( POC_\infty(N_2^n) \) implies the following simple property of \( \sigma \)-equivalent elements of \( POC_\infty(N_2^n) \):

Corollary 7. Let \( \alpha, \beta \) be elements of the semigroup \( POC_\infty(N_2^n) \) such that \( \alpha \sigma \beta \). Then the following assertions hold:

(i) \( (H_{\text{dom} \alpha}) \alpha \subseteq H_1 \) if and only if \( (H_{\text{dom} \beta}) \beta \subseteq H_1 \);
(ii) \( (H_{\text{dom} \alpha}) \alpha \subseteq V^1 \) if and only if \( (H_{\text{dom} \beta}) \beta \subseteq V^1 \).

We define

\[
POC_\infty(N_2^n) = \{ \alpha \in POC_\infty(N_2^n) : (H_{\text{dom} \alpha}) \alpha \subseteq H_1 \}.
\]

Then Lemma 3 and Theorem 1 from [5] imply that \( POC_\infty(N_2^n) \) is a subsemigroup of \( POC_\infty(N_2^n) \).

The subsemigroup \( POC_\infty(N_2^n) \) is called the monoid of orientation-preserving monotone injective partial selfmaps of \( N_2 \) with cofinite domains and images. Moreover it is obvious that \( E(POC_\infty(N_2^n)) = E(POC_\infty(N_2^n)) \). Also, later by \( \preceq \) and \( \sigma \) we denote the corresponding induced relations of the relations \( \preceq \) and \( \sigma \) from the semigroup \( POC_\infty(N_2^n) \) onto its subsemigroup \( POC_\infty(N_2^n) \).

The proofs of the following propositions are similar to those of Propositions 3 and 5 respectively.

Proposition 9. The relation \( \preceq \) is the natural partial order on the semigroup \( POC_\infty(N_2^n) \).

Proposition 10. The relation \( \sigma \) is a congruence on the semigroup \( POC_\infty(N_2^n) \).

By \( \varpi \) we denote the bijective transformation of \( \mathbb{N} \times \mathbb{N} \) defined by the formula \( (i, j) \varpi = (j, i) \), for any \( (i, j) \in \mathbb{N} \times \mathbb{N} \). It is obvious that \( \varpi \) is an element of the semigroup \( POC_\infty(N_2^n) \) and \( \varpi \varpi = I \).

Remark 10. We observe that

(i) \( \alpha \in POC_\infty(N_2^n) \) if and only if \( \alpha \varpi, \varpi \alpha \in POC_\infty(N_2^n) \setminus POC_\infty(N_2^n) \);
(ii) \( \alpha \in POC_\infty(N_2^n) \) if and only if \( \alpha \varpi \varpi \in POC_\infty(N_2^n) \).

We define a map \( h : POC_\infty(N_2^n) \to POC_\infty(N_2^n) \) by the formula \( (\alpha)h = \varpi \alpha \varpi \), for \( \alpha \in POC_\infty(N_2^n) \).

Proposition 11. The map \( h : POC_\infty(N_2^n) \to POC_\infty(N_2^n) \) is an automorphism of the semigroup \( POC_\infty(N_2^n) \). Moreover its restriction \( h|_{POC_\infty(N_2^n)} : POC_\infty(N_2^n) \to POC_\infty(N_2^n) \) is an automorphism of the subsemigroup \( POC_\infty(N_2^n) \).

Proof. First we show that \( h : POC_\infty(N_2^n) \to POC_\infty(N_2^n) \) is a homomorphism. Fix arbitrary \( \alpha, \beta \in POC_\infty(N_2^n) \). Then we have that

\[
(\alpha \beta)h = \varpi(\alpha \beta)\varpi = \varpi(\alpha \varpi \beta)\varpi = (\varpi \alpha \varpi \beta)\varpi = (\alpha \varpi \beta)\varpi = (\alpha \varpi)h(\beta)h,
\]
and hence \( h : POC_\infty(N_2^n) \to POC_\infty(N_2^n) \) is a homomorphism.

Fix an arbitrary \( \alpha \in POC_\infty(N_2^n) \). Then the definition of \( h \) implies that

\[
(\varpi \alpha \varpi)h = \varpi \alpha \varpi \varpi = \varpi \alpha = \alpha,
\]
and hence the map \( h : POC_\infty(N_2^n) \to POC_\infty(N_2^n) \) is surjective. Suppose that \( (\alpha)h = (\beta)h \) for some \( \alpha, \beta \in POC_\infty(N_2^n) \). Then

\[
\alpha = \varpi \varpi = \varpi \varpi \alpha \varpi = ((\alpha)h)h = ((\beta)h)h = \varpi \varpi \beta \varpi = \varpi \beta = \beta,
\]
and hence the map $\mathfrak{h}: \mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \to \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is injective. Thus the map $\mathfrak{h}$ is an automorphism of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$.

Now, Remark $\text{[4]}$ implies that the restriction $\mathfrak{h}|_{\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)}: \mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \to \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is an automorphism of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$, too. \hfill $\Box$

For the automorphism $\mathfrak{h}: \mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \to \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ we have that $\mathfrak{h}^2 = \text{id}_{\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)}$ is the identity automorphism of $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$. This implies that the element $\mathfrak{h}$ generates the group which is isomorphic to the cyclic group of order two $\mathbb{Z}_2$. By Proposition $4$ from $\text{[5]}$ the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is isomorphic to $\mathbb{Z}_2$. We define a map $\mathfrak{Q}$ from $H(\mathbb{I})$ into the group $\text{Aut}(\mathcal{P}_\infty^+(\mathbb{N}_\leq^2))$ of automorphisms of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ in the following way $\text{(II)} \mathfrak{Q} = \text{id}_{\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)}$ and $(\infty) \mathfrak{Q} = \mathfrak{h}$. It is obvious that so defined map $\mathfrak{Q}: H(\mathbb{I}) \to \text{Aut}(\mathcal{P}_\infty^+(\mathbb{N}_\leq^2))$ is an injective homomorphism.

Let $S$ and $T$ be semigroups and let $\mathfrak{F}$ be a homomorphism from $T$ into the semigroup of endomorphisms $\text{End}(S)$ of $S$, $\mathfrak{F}: t \mapsto \mathfrak{F}_t$. Then the Cartesian product $S \times T$ with the following semigroup operation

$$(s_1, t_1) \cdot (s_2, t_2) = (s_1 \cdot (s_2) \mathfrak{F}_t_1, t_1 \cdot t_2), \quad s_1, s_2 \in S, \quad t_1, t_2 \in T,$$

is called a semidirect product of the semigroup $S$ by $T$ and is denoted by $S \ltimes T$. We remark that if $1_T$ is the unit of the semigroup $T$ then $(1_T) \mathfrak{F} = \mathfrak{F}_1$ is the identity homomorphism of $S$ and in the case when $T$ is a group then $(t) \mathfrak{F} = \mathfrak{F}_t$ is an automorphism of $S$ for any $t \in T$.

**Theorem 12.** The semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is isomorphic to the semidirect product $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \ltimes \mathfrak{Q} H(\mathbb{I})$ of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ by the group $H(\mathbb{I})$.

**Proof.** We define a map $\mathfrak{I}: \mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \ltimes \mathfrak{Q} H(\mathbb{I}) \to \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ by the formula $(\alpha, g) \mathfrak{I} = \alpha g$. Then for all $\alpha_1, \alpha_2 \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ and $g_1, g_2 \in H(\mathbb{I})$ we have that

$$[((\alpha_1, g_1) \cdot (\alpha_2, g_2)) \mathfrak{I} = (\alpha_1 \cdot (\alpha_2) g_1) \mathfrak{Q}, g_1 \cdot g_2) \mathfrak{I} = (\alpha_1 \cdot \alpha_2 g_1, g_1 \cdot g_2) = \alpha_1 \cdot \alpha_2 g_1, g_1 \cdot g_2 = \alpha_1 \cdot \alpha_2 g_1, g_1 \cdot g_2 =$$

$$= (\alpha_1, g_1) \mathfrak{I} \cdot (\alpha_2, g_2) \mathfrak{I},$$

because $g^2 = \mathbb{I}$ for any $g \in H(\mathbb{I})$, and hence the map $\mathfrak{I}: \mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \ltimes \mathfrak{Q} H(\mathbb{I}) \to \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is a homomorphism.

By Lemma $3$ from $\text{[6]}$ for every $\alpha \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ there exist $\alpha^+ \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ and $g_\alpha \in H(\mathbb{I})$ such that $\alpha = \alpha^+ g_\alpha$. Indeed,

$(a)$ in the case when $(H^1_{\text{dom}})\alpha \subseteq H^1$ we put $\alpha^+ = \alpha$ and $g_\alpha = \mathbb{I}$;

$(b)$ in the case when $(H^1_{\text{dom}})\alpha \subseteq \mathbb{V}^1$ we put $\alpha^+ = \alpha \omega$ and $g_\alpha = \omega$.

Let $\alpha^+, \beta^+ \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ and $g_\alpha, g_\beta \in H(\mathbb{I})$ be such that $\alpha^+ g_\alpha = (\alpha^+, g_\alpha) \mathfrak{I} = (\beta^+, g_\beta) \mathfrak{I} = \beta^+ g_\beta$. Since $(H^1_{\text{dom}})\alpha \subseteq H^1$ and $(H^1_{\text{dom}})\beta \subseteq H^1$, Lemma $3$ from $\text{[5]}$ implies that $g_\alpha = g_\beta$. By Proposition $4$ from $\text{[5]}$ the group of units $H(\mathbb{I})$ of the semigroup $\mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is isomorphic to $\mathbb{Z}_2$ and hence $\alpha^+ = \alpha^+ g_\alpha g_\beta = \beta^+ g_\beta^2 = \beta^+$. Therefore, we get that so defined map $\mathfrak{I}: \mathcal{P}_\infty^+(\mathbb{N}_\leq^2) \ltimes \mathfrak{Q} H(\mathbb{I}) \to \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ is an isomorphism. \hfill $\Box$

By Theorem $2(ii_1)$ from $\text{[5]}$ for every $\alpha \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ there exists a smallest positive integer $n_\alpha$ such that $(i, j) \alpha = (i, j)$ for each $(i, j) \in \text{dom} \alpha \cap \uparrow(n_\alpha, n_\alpha)$.

**Lemma 13.** For every $\alpha \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ there exists $\alpha_\mathfrak{I} \in \mathcal{P}_\infty^+(\mathbb{N}_\leq^2)$ such that the following assertions hold:

$(i) \quad \alpha \sigma \alpha_\mathfrak{I};$

$(ii) \quad (i + 1)\alpha_\mathfrak{I}[s, j] - (i + 1) = i \alpha_\mathfrak{I}[s, j] - i$ for arbitrary $(i, j) \in \text{dom} \alpha_\mathfrak{I}$ with $j < n_\alpha$, $(i, j) \alpha_\mathfrak{I} = (i \alpha_\mathfrak{I}[s, j], j \alpha_\mathfrak{I}[i, s])$ and $(i + 1, j) \alpha_\mathfrak{I} = ((i + 1) \alpha_\mathfrak{I}[s, j], j \alpha_\mathfrak{I}[i + 1, s])$, i.e., $\alpha_\mathfrak{I}$ acts as a partial shift on the set $H^1$.\hfill $\Box$
Moreover, there exist smallest positive integers \( \hat{h}_\alpha, \hat{v}_\alpha \leq n_\alpha \) such that \((i, j)\alpha_\tau = (i, j)\) for arbitrary \((i, j) \in \text{dom } \alpha_\tau\) with \(i \geq \hat{h}_\alpha \) and \((k, l)\alpha_\tau = (k, l)\) for arbitrary \((k, l) \in \text{dom } \alpha_\tau\) with \(l \geq \hat{v}_\alpha \).

**Proof.** Fix an arbitrary element \(\alpha\) of the semigroup \(\mathcal{PH}_\infty^+(\mathbb{N}_\alpha^2)\). Then by Theorem 1(1) from [3] we get that \(\mathcal{H}_{\text{dom } \alpha} \subseteq ^* \mathcal{H}^n\) and \(\mathcal{V}_{\text{dom } \alpha} \subseteq ^* \mathcal{V}^n\) for any positive integer \(n\). Also, the definition of the semigroup \(\mathcal{PH}_\infty^+(\mathbb{N}_\alpha^2)\) and Theorem 2(ii) of [3] imply that there exists a smallest positive integer \(n_\alpha\) such that \((i, j)\alpha = (i, j)\) for each \((i, j) \in \text{dom } \alpha \cap \uparrow(n_\alpha, n_\alpha)\), and hence for arbitrary positive integers \(i, j < n_\alpha\) there exist smallest positive integers \(h^i_\alpha\) and \(v^j_\alpha\) such that the following conditions hold:

\[
\mathcal{H}_{\text{ran } \alpha}^i \cap \{(p, i) : p \geq h^i_\alpha\} = \{(p, i) : p \geq h^i_\alpha\};
\]

\[
\mathcal{V}_{\text{ran } \alpha}^j \cap \{(j, q) : q \geq v^j_\alpha\} = \{(j, q) : q \geq v^j_\alpha\};
\]

and

\[
(k, i), (j, l) \in \text{dom } \alpha, \quad (k, i)\alpha \in \mathcal{H}^i, \quad (j, l)\alpha \in \mathcal{V}^j,
\]

for all positive integers \(k \geq h^i_\alpha\) and \(l \geq v^j_\alpha\).

We put

\[
\hat{h}_\alpha = \max \{h^i_\alpha : i = 1, \ldots, n_\alpha - 1\} \quad \text{and} \quad \hat{v}_\alpha = \max \{v^j_\alpha : j = 1, \ldots, n_\alpha - 1\}.
\]

The above arguments imply that

\[
\mathcal{H}_{\text{ran } \alpha}^i \cap \{(p, i) : p \geq \hat{h}_\alpha\} = \{(p, i) : p \geq \hat{h}_\alpha\};
\]

\[
\mathcal{V}_{\text{ran } \alpha}^j \cap \{(j, q) : q \geq \hat{v}_\alpha\} = \{(j, q) : q \geq \hat{v}_\alpha\};
\]

and

\[
(k, i), (j, l) \in \text{dom } \alpha, \quad (k, i)\alpha \in \mathcal{H}^i, \quad (j, l)\alpha \in \mathcal{V}^j,
\]

for all positive integers \(k \geq \hat{h}_\alpha\) and \(l \geq \hat{v}_\alpha\).

Next we put

\[
D_\alpha = (\mathbb{N} \times \mathbb{N}) \setminus \{(i, j) : i \leq \hat{h}_\alpha \text{ and } j \leq n_\alpha\} \cup \{(i, j) : i \leq n_\alpha \text{ and } j \leq \hat{v}_\alpha\}.
\]

We define \(\alpha_\tau = \alpha|_{D_\alpha}\), i.e.,

\[
\text{dom } \alpha_\tau = D_\alpha, \quad \text{ran } \alpha_\tau = (D_\alpha)\alpha \quad \text{and} \quad (i, j)\alpha_\tau = (i, j)\alpha \quad \text{for all } (i, j) \in \text{dom } \alpha_\tau.
\]

Since \(\varepsilon_\alpha\alpha_\tau = \varepsilon_\alpha\alpha\) for the identity partial map \(\varepsilon_\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}\) with \(\text{dom } \varepsilon_\alpha = \text{ran } \varepsilon_\alpha = D_\alpha\), Proposition 5 implies that \(\alpha\alpha_\tau\).

Then condition [1] and the definition of the positive integer \(\hat{h}_\alpha\) imply that

\[
(\hat{h}_\alpha + 2)\alpha_{[i, 1]} = (\hat{h}_\alpha + 1)\alpha_{[i, 1]} + 1,
\]

and by similar arguments and induction we have that \((i + 1)\alpha_{[i, 1]} = (i, 1)\alpha_{[i, 1]} + 1\) for arbitrary \(i \geq \hat{h}_\alpha + 1\). Next, if we apply condition [1] and induction for arbitrary \(j < n_\alpha\) then we get that

\[
(i + 1)\alpha_{[j, 1]} = (i)\alpha_{[j, 1]} + 1\]

for arbitrary \(i \geq \hat{h}_\alpha + 1\). This implies assertion (ii).

The proof of item (iii) is similar to (ii).

The last statement of the lemma follows from the above arguments and Theorem 2(1) from [3]. □

For every positive integer \(n\) we define partial maps \(\gamma_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}\) and \(\nu_n : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N}\) in the following way:

\[
\text{dom } \gamma_n = \mathbb{N} \times \mathbb{N} \setminus \{(1, i) : i = 1, \ldots, n\},
\]

\[
\text{dom } \nu_n = \mathbb{N} \times \mathbb{N} \setminus \{(i, 1) : i = 1, \ldots, n\},
\]

\[
\text{ran } \gamma_n = \text{ran } \nu_n = \mathbb{N} \times \mathbb{N}.
\]
and
\[ (i, j)γ_n = \begin{cases} (i - 1, j), & \text{if } j \leq n; \\ (i, j), & \text{if } j > n \end{cases} \quad \text{for } (i, j) \in \text{dom } γ_n, \]
\[ (i, j)ν_n = \begin{cases} (i, j - 1), & \text{if } i \leq n; \\ (i, j), & \text{if } i > n \end{cases} \quad \text{for } (i, j) \in \text{dom } ν_n. \]

Simple verifications show that \( γ_n, v_n \in \mathcal{PO}_+^+(\mathbb{N}_2^2) \) for every positive integer \( n \), and moreover the subsemigroups \( \langle γ_k | k \in \mathbb{N} \rangle \) and \( \langle v_k | k \in \mathbb{N} \rangle \) of the semigroup \( \mathcal{PO}_+^+(\mathbb{N}_2^2) \), generated by the sets \( \{γ_k : k \in \mathbb{N}\} \) and \( \{v_k : k \in \mathbb{N}\} \), respectively, are isomorphic to the free Abelian semigroup over an infinite countable set.

**Lemma 14.** For every \( α \in \mathcal{PO}_+^+(\mathbb{N}_2^2) \) there exist finitely many elements \( γ_{k_1}, \ldots, γ_{k_1} \) and \( v_{l_1}, \ldots, v_{l_j} \) of the semigroup \( \mathcal{PO}_+^+(\mathbb{N}_2^2) \), with \( k_1 < \ldots < k_i, l_1 < \ldots < l_j \), such that
\[ ασ(γ_{k_1}^{p_1} \cdots γ_{k_i}^{q_i} v_{l_1}^{q_1} \cdots v_{l_j}^{q_j}), \]
for some positive integers \( p_1, \ldots, p_i, q_1, \ldots, q_j \). Moreover if
\[ ασ(γ_{k_1}^{p_i} \cdots γ_{k_i}^{q_i} v_{l_1}^{q_1} \cdots v_{l_j}^{q_j}) \quad \text{and} \quad βσ(γ_{k_1}^{b_i} \cdots γ_{k_i}^{b_i} v_{l_1}^{d_1} \cdots v_{l_j}^{d_j}) \]
for some \( α, β \in \mathcal{PO}_+^+(\mathbb{N}_2^2) \) then \((α, β) \not∈ σ\) if and only if
\[ νγ_{k_i}^{p_i} \cdots γ_{k_i}^{q_i} v_{l_1}^{q_1} \cdots v_{l_j}^{q_j} \neq νγ_{k_i}^{b_i} \cdots γ_{k_i}^{b_i} v_{l_1}^{d_1} \cdots v_{l_j}^{d_j} \]
for any idempotent \( ν \in \mathcal{PO}_+^+(\mathbb{N}_2^2) \).

**Proof.** Fix an arbitrary element \( α \) of the semigroup \( \mathcal{PO}_+^+(\mathbb{N}_2^2) \). Let \( α_ε \) be the element of \( \mathcal{PO}_+^+(\mathbb{N}_2^2) \) defined in the proof of Lemma 13. By Theorem 3 from [5] and the second statement of Lemma 13 there exist smallest positive integers \( \tilde{h}_α, \tilde{v}_α \leq n_α \) such that \((i, j)α_ε = (i, j)\) for arbitrary \((i, j)\) \( \in \text{dom } α_ε \) with \( i \geq \tilde{h}_α \) and \((k, l)α_ε = (k, l)\) for arbitrary \((k, l)\) \( \in \text{dom } α_ε \) with \( l \geq \tilde{v}_α \).

By Lemma 13 and Theorem 1(1) of [5] we have that
\[ (j, \tilde{h}_α - 1)α_ε = (j, \tilde{h}_α - 1) - (j, \tilde{h}_α - 1) \quad \text{and} \quad (j + 1)α_ε[s, \tilde{h}_α - 1] - jα_ε[s, \tilde{h}_α - 1] = 1, \]
for arbitrary \((j, \tilde{h}_α - 1), (j + 1, \tilde{h}_α - 1) \) \( \in \text{dom } α_ε \). Then we put \( p_{\tilde{h}_α - 1} = j - jα_ε[s, \tilde{h}_α - 1] \). Next, for \( s = 2, \ldots, \tilde{h}_α - 2 \) we define integers \( p_{\tilde{h}_α - s} , \ldots, p_1 \) by induction,
\[ p_{\tilde{h}_α - s} = j - jα_ε[s, \tilde{h}_α - s] - (p_{\tilde{h}_α - 1} + \cdots + p_{\tilde{h}_α - s + 1}), \]
where \((j, \tilde{h}_α - s)α_ε = (j, \tilde{h}_α - s) - (j, \tilde{h}_α - s)\) for arbitrary \((j, \tilde{h}_α - s) \) \( \in \text{dom } α_ε \).

Similarly, by Lemma 13 and Theorem 1(1) of [5] we have that
\[ (\tilde{v}_α - 1, i)α_ε = (\tilde{v}_α - 1, i) - (\tilde{v}_α - 1, i) \quad \text{and} \quad (i + 1)α_ε[\tilde{v}_α - 1, s] - iα_ε[\tilde{v}_α - 1, s] = 1, \]
for arbitrary \((\tilde{v}_α - 1, i), (\tilde{v}_α - 1, i + 1) \) \( \in \text{dom } α_ε \). Then we put \( q_{\tilde{v}_α - 1} = i - iα_ε[\tilde{v}_α - 1, s] \). Next, for \( t = 2, \ldots, \tilde{v}_α - 2 \) we define integers \( q_{\tilde{v}_α - t} , \ldots, q_1 \) by induction,
\[ q_{\tilde{v}_α - t} = i - iα_ε[\tilde{v}_α - t, s] - (q_{\tilde{v}_α - 1} + \cdots + q_{\tilde{v}_α - t + 1}), \]
where \((\tilde{v}_α - t, i)α_ε = (\tilde{v}_α - t, i) - (\tilde{v}_α - t, i)\) for arbitrary \((\tilde{v}_α - t, i) \) \( \in \text{dom } α_ε \).

For any \( α \in \mathcal{PO}_+^+(\mathbb{N}_2^2) \) put \( ε_α : \mathbb{N} × \mathbb{N} \) be the identity partial map with \( \text{dom } ε_α = \text{ran } ε_α = D_α \), where the set \( D_α \) is defined by formula [3]. Simple verification shows that \( ε_α α = ε_α(γ_{k_1}^{p_1} \cdots γ_{h_α - 1}^{q_1} v_{l_1}^{q_1} \cdots v_{l_j}^{q_j}) \) and hence
\[ ασ(γ_{k_1}^{p_1} \cdots γ_{h_α - 1}^{q_1} v_{l_1}^{q_1} \cdots v_{l_j}^{q_j}), \]
which implies that relation (4) holds.
Since \( \gamma_m^0 = v_m^0 = I \) for any positive integer \( m \), without loss of generality we may assume that \( p_1, \ldots, p_i, q_1, \ldots, q_j \) are positive integers in formula (11).

Also, the last statement of the lemma follows from the definition of the congruence \( \sigma \) on the semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) \).

\[ \square \]

Lemma 15. Let be \( \alpha \sigma \left( v_1^{p_1} \ldots v_i^{q_i} \ldots v_j^{q_j} \right) \) for \( \alpha \in \mathcal{P}^+\mathcal{O}_\infty(N^2) \) and positive integers \( p_1, \ldots, p_i, q_1, \ldots, q_j, k_1 < \ldots < k_i, l_1 < \ldots < l_j \). Then there exists an idempotent \( \hat{\varepsilon}_\alpha \in \mathcal{P}^+\mathcal{O}_\infty(N^2) \) such that

\[ \hat{\varepsilon}_\alpha \alpha = \varepsilon_\alpha \hat{n}_1 \ldots \gamma_{k_i} v_{l_1}^{q_1} \ldots v_{l_j}^{q_j} = \hat{v}_1^{q_1} \ldots v_{l_j}^{q_j} \gamma_{k_i} \ldots \gamma_{k_i} \]

Proof. Put

\[ \bar{m}_\alpha = n_\alpha + h_\alpha + \bar{v}_\alpha + p_1 + \ldots + p_1 + q_1 + \ldots + q_j, \]

where \( h_\alpha \) and \( \bar{v}_\alpha \) are the positive integers defined in the proof of Lemma 13. We define the identity partial map \( \hat{\varepsilon}_\alpha : N \times N \to N \times N \) with dom \( \hat{\varepsilon}_\alpha = \text{ran} \hat{\varepsilon}_\alpha = M_\alpha \), where

\[ M_\alpha = (N \times N) \setminus \{(i, j) : i \leq \bar{m}_\alpha \text{ and } j \leq \bar{m}_\alpha \}. \]

Then \( \hat{\varepsilon}_\alpha \leq \varepsilon_\alpha \) where \( \varepsilon_\alpha \) is the idempotent of the semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) \) defined in the proof of Lemma 13. This implies that

\[ \hat{\varepsilon}_\alpha \alpha = \varepsilon_\alpha \hat{\varepsilon}_\alpha \alpha = \varepsilon_\alpha \alpha \hat{n}_1 \ldots \gamma_{k_i} v_{l_1}^{q_1} \ldots v_{l_j}^{q_j} = \hat{\varepsilon}_\alpha \hat{n}_1 \ldots \gamma_{k_i} v_{l_1}^{q_1} \ldots v_{l_j}^{q_j} \]

and the equality

\[ \hat{\varepsilon}_\alpha \gamma_{k_1} \ldots \gamma_{k_i} v_{l_1}^{q_1} \ldots v_{l_j}^{q_j} = \varepsilon_\alpha v_{l_1}^{q_1} \ldots v_{l_j}^{q_j} \gamma_{k_i} \ldots \gamma_{k_i} \]

follows from the definition of the idempotent \( \hat{\varepsilon}_\alpha \in \mathcal{P}^+\mathcal{O}_\infty(N^2) \).

The following theorem describes the quotient semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) / \sigma \) over an infinite countable set.

Theorem 16. The quotient semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) / \sigma \) is isomorphic to the free commutative monoid \( \mathcal{A}M_\omega \) over an infinite countable set.

Proof. Let \( X = \{ a_i : i \in N \} \cup \{ b_j : j \in N \} \) be a countable infinite set.

We define the map \( \delta_\sigma : \mathcal{P}^+\mathcal{O}_\infty(N^2) \to \mathcal{A}M_X \) in the following way:

(a) if \( \alpha \sigma \left( \gamma_{k_1} \ldots \gamma_{k_i} v_{l_1}^{q_1} \ldots v_{l_j}^{q_j} \right) \) for some positive integers \( p_1, \ldots, p_i, q_1, \ldots, q_j, k_1 < \ldots < k_i, l_1 < \ldots < l_j \), then

\[ (a) \delta_\sigma = (\gamma_{k_1} \ldots \gamma_{k_i} v_{l_1}^{q_1} \ldots v_{l_j}^{q_j}) \delta_\sigma = \alpha^{p_1} \ldots \alpha^{p_i} b_{l_1}^{q_1} \ldots b_{l_j}^{q_j}, \]

(b) \( (I) \delta_\sigma = e \), where \( e \) is the unit of the free commutative monoid \( \mathcal{A}M_X \).

Then Lemmas 13 and 15 imply that \( (a) \delta_\sigma = (\beta) \delta_\sigma \) if and only if \( \alpha \sigma \beta \) in \( \mathcal{P}^+\mathcal{O}_\infty(N^2) \) and hence the quotient semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) / \sigma \) is isomorphic to the free commutative monoid \( \mathcal{A}M_X \).

The following corollary of Theorem 16 shows that the semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) \) has infinitely many congruences similar as the free commutative monoid \( \mathcal{A}M_\omega \) over an infinite countable set.

Corollary 17. Every countable (infinite or finite) commutative monoid is a homomorphic image of the semigroup \( \mathcal{P}^+\mathcal{O}_\infty(N^2) \).

It is obvious that every non-unit element \( u \) of the free commutative monoid \( \mathcal{A}M_\omega \) over the infinite countable set \( \{ a_i : i \in \omega \} \cup \{ b_j : j \in \omega \} \) can be represented in the form \( u = a_{i_1}^{p_1} \ldots a_{i_k}^{p_k} b_{j_1}^{q_1} \ldots b_{j_l}^{q_l} \), where \( i_1, \ldots, i_k, j_1, \ldots, j_l \) are positive integers. We define a map \( \mathcal{A}M_\omega \to \mathcal{A}M_\omega \) by the formula

\[ (a_{i_1}^{p_1} \ldots a_{i_k}^{p_k} b_{j_1}^{q_1} \ldots b_{j_l}^{q_l}) f = a_{i_1}^{p_1} \ldots a_{i_k}^{p_k} b_{j_1}^{q_1} \ldots b_{j_l}^{q_l}, \]

for \( u = a_{i_1}^{p_1} \ldots a_{i_k}^{p_k} b_{j_1}^{q_1} \ldots b_{j_l}^{q_l} \in \mathcal{A}M_\omega \) and \( (e) f = e \), for unit element \( e \) of \( \mathcal{A}M_\omega \).

Proposition 18. The map \( \mathcal{A}M_\omega \to \mathcal{A}M_\omega \) is an automorphism of the free commutative monoid \( \mathcal{A}M_\omega \).
Proof. First we show that $\mathcal{AM}_\omega \to \mathcal{AM}_\omega$ is a homomorphism. Fix arbitrary elements $u, v \in \mathcal{AM}_\omega$. Without loss of generality we may assume that
\[ u = a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p} \quad \text{and} \quad v = a_1^{s_1} \ldots a_p^{s_p} b_1^{t_1} \ldots b_p^{t_p} \]
for some non-negative integers $p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p$, where $a^i = b^j = e$ for $i = 0$.

Then we have that
\[
(uf)(v) = (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p})f = \]
\[= (a_1^{i_1+s_1} \ldots a_p^{i_p+s_p} b_1^{j_1+t_1} \ldots b_p^{j_p+t_p})f = \]
\[= a_1^{i_1+t_1} \ldots a_p^{i_p+t_p} b_1^{j_1+s_1} \ldots b_p^{j_p+s_p} = \]
\[= (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p})b(a_1^{s_1} \ldots a_p^{s_p} b_1^{t_1} \ldots b_p^{t_p})f = \]
\[= (u)f(v)f.\]

It is obvious that $f: \mathcal{AM}_\omega \to \mathcal{AM}_\omega$ is a bijective map and hence $f: \mathcal{AM}_\omega \to \mathcal{AM}_\omega$ is an automorphism. \qed

The relationships between elements of the subsemigroup $\langle \gamma_k \mid k \in \mathbb{N} \rangle$ and of the subsemigroup $\langle \nu_k \mid k \in \mathbb{N} \rangle$ in $\mathcal{PO}_\infty^+(N^2_2)$ is described by the following proposition.

We observe that the cyclic group $\mathbb{Z}_2$ acts on the free commutative monoid $\mathcal{AM}_\omega$ over the infinite countable set $\{a_i : i \in \omega\} \cup \{b_j : j \in \omega\}$ in the following way
\[\mathcal{AM}_\omega \times \mathbb{Z}_2 \to \mathcal{AM}_\omega : (u, g) \mapsto v = \begin{cases} u, & \text{if } g = 0; \\ (u)f, & \text{if } g = 1, \end{cases}\]
where the map $f: \mathcal{AM}_\omega \to \mathcal{AM}_\omega$ is defined by formula\[.\] By Proposition 18 the map $f$ is an automorphism of the free commutative monoid $\mathcal{AM}_\omega$.

Proposition 19. Let $p_1, \ldots, p_i, k_1, \ldots, k_i$ be some positive integers such that $k_1 < \ldots < k_i$. Then the following assertions hold:

(i) $\omega_k^{p_1} \ldots \omega_{k_i}^{p_i} = v_{k_1}^{p_1} \ldots v_{k_i}^{p_i}$;
(ii) $\gamma_k^{p_1} \ldots \gamma_{k_i}^{p_i} = \nu v_{k_1}^{p_1} \ldots v_{k_i}^{p_i}$;
(iii) $\omega_k^{p_1} \ldots \omega_{k_i}^{p_i} = v_{k_1}^{p_1} \ldots v_{k_i}^{p_i}$;
(iv) $\omega u_{k_1}^{p_1} \ldots u_{k_i}^{p_i} = \gamma_{k_1}^{p_1} \ldots \gamma_{k_i}^{p_i}$.

Proof. Assertion (i) follows from the definitions of the elements of the semigroups $\langle \gamma_k \mid k \in \mathbb{N} \rangle$ and $\langle \nu_k \mid k \in \mathbb{N} \rangle$. Other assertions follow from (i) and the equality $\omega \omega = \nu$.

Later we assume that $\mathbb{Z}_2 = \{0, 1\}$.

The following theorem describes the quotient semigroup $\mathcal{PO}_\infty^+(N^2_2)/\sigma$.

Theorem 20. The semigroup $\mathcal{PO}_\infty^+(N^2_2)/\sigma$ is isomorphic to the semidirect product $\mathcal{AM}_\omega \times_{\Delta} \mathbb{Z}_2$ of the free commutative monoid $\mathcal{AM}_\omega$ over an infinite countable set by the cyclic group $\mathbb{Z}_2$.

Proof. We define a map $\mathcal{J}: \mathcal{PO}_\infty^+(N^2_2)/\sigma \to \mathcal{AM}_\omega \times_{\Delta} \mathbb{Z}_2 : x \mapsto (u, g)$ in the following way. Let $\mathcal{P}_\sigma : \mathcal{PO}_\infty^+(N^2_2) \to \mathcal{PO}_\infty^+(N^2_2)/\sigma$ be the natural homomorphism generated by the congruence $\sigma$ on the semigroup $\mathcal{PO}_\infty^+(N^2_2)$. Then for every $x \in \mathcal{PO}_\infty^+(N^2_2)/\sigma$ for any $\alpha \in \mathcal{PO}_\infty^+(N^2_2)$ such that $(\alpha x)\mathcal{P}_\sigma = x$ only one of the following conditions holds:

1. $(H^1_{\text{dom } \alpha}) \alpha x \subseteq H^1$;
2. $(H^1_{\text{dom } \alpha}) \alpha x \subseteq V^1$. 
We put

\[(x)\mathcal{J} = \begin{cases} 
((\alpha x)\mathcal{H}_{\sigma x}, 0), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq H^1; \\
((\alpha x)\mathcal{H}_{\sigma x}, 1), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq V^1.
\end{cases}\]

for all \(\alpha x \in \mathcal{PC}_\infty(N^2_{\leq})\) with \((\alpha x)\mathcal{P}_\sigma = x\). Then the definition of the congruence \(\sigma\) on the semigroup \(\mathcal{PC}_\infty(N^2_{\leq})\) and Corollary \(\mathcal{C}\) imply that the map \(\mathcal{J} : \mathcal{PC}_\infty(N^2_{\leq})/\sigma \to \mathcal{P}C_{\infty} \times \mathcal{Z}_2\) is well defined.

We observe that formula (6) implies that \((x_1)\mathcal{J} = (e, 0)\) for \(x_1 = (\Pi)\mathcal{P}_\sigma\) and \((x_\omega)\mathcal{J} = (e, 1)\) for \(x_\omega = (\omega)\mathcal{P}_\sigma\). Hence we have that

\[
(x)\mathcal{J} \cdot (x_1)\mathcal{J} = \begin{cases} 
((\alpha x)\mathcal{H}_{\sigma x}, 0) \cdot (e, 0), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq H^1; \\
((\alpha x)\mathcal{H}_{\sigma x}, 1) \cdot (e, 0), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq V^1.
\end{cases}
\]

and

\[
(x_1)\mathcal{J} \cdot (x)\mathcal{J} = \begin{cases} 
(e, 0) \cdot ((\alpha x)\mathcal{H}_{\sigma x}, 0), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq H^1; \\
(e, 0) \cdot ((\alpha x)\mathcal{H}_{\omega x}, 1), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq V^1.
\end{cases}\]

Also, since \(\sigma\) is congruence on \(\mathcal{PC}_\infty(N^2_{\leq})\), we get

\[
(x)\mathcal{J} \cdot (x_\omega)\mathcal{J} = \begin{cases} 
((\alpha x)\mathcal{H}_{\sigma x}, 0) \cdot (e, 1), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq H^1; \\
((\alpha x)\mathcal{H}_{\sigma x}, 1) \cdot (e, 1), & \text{if } (H_{\text{dom } \alpha x})^1 \alpha x \subseteq V^1.
\end{cases}
\]

and
(i) in the case when \((H^1_{\text{dom} \alpha x})\alpha x \subseteq H^1\) for \(\alpha x = \gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p}\), for some non-negative integers \(p, i_1, \ldots, i_p, j_1, \ldots, j_p\), where \(\gamma^0 = v^0 = 1\), we get that \((H^1_{\text{dom} \alpha x})\alpha x \subseteq V^1\),

\[
(x_{\varnothing}) \cdot (x) \H = (e, 1) \cdot ((\alpha x) H_{\sigma}, 0) =
\]

\[
= (e, 1) \cdot ((\gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p}) H_{\sigma}, 0) =
\]

\[
= (e, 1) \cdot (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}, 0) =
\]

\[
= (e \cdot (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}) \H, 1) =
\]

\[
= (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}, 1)
\]

and by Proposition 19

\[
(x_{\varnothing} \cdot x) \H = ((\omega \alpha x \omega) H_{\sigma}, 1) =
\]

\[
= ((\omega \gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p}) H_{\sigma}, 1) =
\]

\[
= ((v_1^{j_1} \ldots v_p^{j_p} \omega \omega \gamma_1^{i_1} \ldots \gamma_p^{i_p}) H_{\sigma}, 1) =
\]

\[
= (b_1^{j_1} \ldots b_p^{j_p} a_1^{i_1} \ldots a_p^{i_p}, 1)
\]

(ii) in the case when \((H^1_{\text{dom} \alpha x})\alpha x \subseteq V^1\) we get for \(\alpha x = \gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \omega\), for some non-negative integers \(p, i_1, \ldots, i_p, j_1, \ldots, j_p\), where \(\gamma^0 = v^0 = 1\), we get that \((H^1_{\text{dom} \alpha x \omega})\alpha x \omega \subseteq H^1\),

\[
(x_{\varnothing}) \cdot (x) = (e, 1) \cdot ((\alpha x \omega) H_{\sigma}, 1) =
\]

\[
= (e, 1) \cdot ((\gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \omega) H_{\sigma}, 1) =
\]

\[
= (e, 1) \cdot (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}, 1) =
\]

\[
= (e \cdot (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}) \H, 1) =
\]

\[
= (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}, 0)
\]

and by Proposition 19

\[
(x_{\varnothing} \cdot x) \H = ((\omega \alpha x \omega) H_{\sigma}, 0) =
\]

\[
= ((\omega \gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \omega) H_{\sigma}, 0) =
\]

\[
= ((v_1^{j_1} \ldots v_p^{j_p} \omega \omega \gamma_1^{i_1} \ldots \gamma_p^{i_p}) H_{\sigma}, 0) =
\]

\[
= (b_1^{j_1} \ldots b_p^{j_p} a_1^{i_1} \ldots a_p^{i_p}, 0)
\]

\[
= (a_1^{i_1} \ldots a_p^{i_p} b_1^{j_1} \ldots b_p^{j_p}, 0)
\]

which implies that \((x_{\varnothing} \cdot x) \H = (x_{\varnothing}) \cdot (x) \H\).

Therefore we have showed that \((x_1) \H\) is the identity element of \(\mathcal{P}^\infty((\mathbb{N}^2_\omega) / \sigma\) and \((x_{\varnothing}) \cdot (x_{\varnothing} \H) =

\[
(x_1) \H
\]

Next we shall show that so defined map \(\H\) is a homomorphism from \(\mathcal{P}^\infty((\mathbb{N}^2_\omega) / \sigma\) into the semigroup \(\mathcal{AM}_\omega \times_{\sigma} \mathbb{Z}_2\). Fix arbitrary elements \(x\) and \(y\) of \(\mathcal{P}^\infty((\mathbb{N}^2_\omega) / \sigma\). We consider the following four possible cases:
(i) \((H_{\text{dom}\alpha})_x \subseteq H^1\) and \((H_{\text{dom}\alpha})_y \subseteq H^1\) for any \(\alpha, \alpha_y \in \mathcal{P}_\infty(N^2_\leq)\) such that \((\alpha)\Psi_x = x\) and \((\alpha_y)\Psi_y = y\);

(ii) \((H_{\text{dom}\alpha})_x \subseteq V^1\) and \((H_{\text{dom}\alpha})_y \subseteq V^1\) for any \(\alpha, \alpha_y \in \mathcal{P}_\infty(N^2_\leq)\) such that \((\alpha)\Psi_x = x\) and \((\alpha_y)\Psi_y = y\);

(iii) \((H_{\text{dom}\alpha})_x \subseteq H^1\) and \((H_{\text{dom}\alpha})_y \subseteq V^1\) for any \(\alpha, \alpha_y \in \mathcal{P}_\infty(N^2_\leq)\) such that \((\alpha)\Psi_x = x\) and \((\alpha_y)\Psi_y = y\);

(iv) \((H_{\text{dom}\alpha})_x \subseteq V^1\) and \((H_{\text{dom}\alpha})_y \subseteq V^1\) for any \(\alpha, \alpha_y \in \mathcal{P}_\infty(N^2_\leq)\) such that \((\alpha)\Psi_x = x\) and \((\alpha_y)\Psi_y = y\).

Assume that (i) holds. Then we have that \(\alpha, \alpha_y, \alpha_x \alpha_y \in \mathcal{P}_\infty^+(N^2_\leq)\). Since \(\sigma\) is a congruence on the semigroup \(\mathcal{P}_\infty(N^2_\leq)\), we may choose an element \(\alpha_{xy} = \alpha_x \alpha_y \in \mathcal{P}_\infty^+(N^2_\leq)\). Then \((\alpha_{xy})\Psi_x = xy\). Also, since \(\Psi_x: \mathcal{P}_\infty^+(N^2_\leq) \to \mathcal{P}_\infty(N^2_\leq)/\sigma\) is the natural homomorphism generated by the congruence \(\sigma\) on the semigroup \(\mathcal{P}_\infty(N^2_\leq)\) we get that

\[
(xy)\mathcal{J} = ((\alpha_{xy})\Psi_x)\mathcal{J} = ((\alpha_{xy})\mathcal{J}, 0) = ((\alpha_{xy})\mathcal{J}, 0) = ((\alpha_{xy})\Psi_x, 0, 0) = ((\alpha_x)\Psi_x, 0) \cdot ((\alpha_y)\Psi_y, 0) = (x)\mathcal{J} \cdot (y)\mathcal{J}.
\]

If (ii) holds then by Propositions 1 and 3 from \([5]\), \(\alpha, \alpha_{xy} \alpha_y \in \mathcal{P}_\infty^+(N^2_\leq)\) and by Lemma \([14]\) without loss of generality we may assume that

\[
\alpha_x = \gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \quad \text{and} \quad \alpha_y = \gamma_1^{s_1} \ldots \gamma_p^{s_p} v_1^{t_1} \ldots v_p^{t_p},
\]

for some non-negative integers \(p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p\), where \(\gamma^0 = v^0 = 1\). This and the fact that \(\sigma\) is a congruence on the semigroup \(\mathcal{P}_\infty(N^2_\leq)\), Proposition \([13]\) imply that

\[
(xy)\mathcal{J} = ((\alpha_{xy})\mathcal{J}, 1) = ((\gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \alpha_y) \mathcal{J}, 1) = ((\gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \class{\mathcal{J}}{\alpha_y}, 1) = (\alpha_{xy} \mathcal{J}, 1) = (x)\mathcal{J} \cdot (y)\mathcal{J}.
\]

If (iii) holds then by Propositions 1 and 3 from \([5]\), \(\alpha, \alpha_{xy} \alpha_y \in \mathcal{P}_\infty^+(N^2_\leq)\) and by Lemma \([14]\) without loss of generality we may assume that

\[
\alpha_x = \gamma_1^{i_1} \ldots \gamma_p^{i_p} v_1^{j_1} \ldots v_p^{j_p} \quad \text{and} \quad \alpha_y = \gamma_1^{s_1} \ldots \gamma_p^{s_p} v_1^{t_1} \ldots v_p^{t_p},
\]

for some non-negative integers \(p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p\), where \(\gamma^0 = v^0 = 1\). Since \(\sigma\) is a congruence on the semigroup \(\mathcal{P}_\infty(N^2_\leq)\), this and Proposition \([19]\) imply that
Assume that $(iv)$ holds. Then by Propositions 1 and 3 from [5] we have that $\alpha_x, \alpha_y, \alpha_x \alpha_y, \alpha_x \alpha_y \in \mathcal{P}(N_2)$ and by Lemma [1] without loss of generality we may assume that

$$\alpha_x = \gamma_1^i \ldots \gamma_p^i v_1^j \ldots v_p^j \circ \quad \text{and} \quad \alpha_y = \gamma_1^s \ldots \gamma_p^s v_1^t \ldots v_p^t \circ$$

for some non-negative integers $p, i_1, \ldots, i_p, j_1, \ldots, j_p, s_1, \ldots, s_p, t_1, \ldots, t_p$, where $\gamma^0 = v^0 = \mathbb{I}$. Since $\sigma$ is a congruence on the semigroup $\mathcal{P}(N_2)$, this and Proposition [19] imply that

$$(x y) \mathcal{J} = ((\alpha_x \alpha_y) \mathcal{J}, 0) = \gamma_1^i \ldots \gamma_p^i v_1^j \ldots v_p^j \circ$$

Thus the map $\mathcal{J} : \mathcal{P}(N_2) / \sigma \rightarrow \mathcal{M}_\omega \times_\mathbb{Z} 2$ is a homomorphism. Also, since $(x) \mathcal{J} = (e, \bar{0})$, $(x \circ) \mathcal{J} = (e, \bar{1})$ and for any $\alpha_x = \gamma_1^i \ldots \gamma_p^i v_1^j \ldots v_p^j$, where $p, i_1, \ldots, i_p, j_1, \ldots, j_p$ are some positive integers, our above arguments imply that

$$(x) \mathcal{J} = (a_1^i \ldots a_p^i b_1^i, 0) \quad \text{and} \quad (y) \mathcal{J} = (a_1^i \ldots a_p^i b_1^i, \bar{1})$$

where $x = (\alpha_x) \mathcal{P}_\sigma$ and $y = (\alpha_x \circ) \mathcal{P}_\sigma$. This implies that the homomorphism $\mathcal{J}$ is surjective.

Now suppose that $(x) \mathcal{J} = (y) \mathcal{J} = (u, g)$ for some $x, y \in \mathcal{P}(N_2) / \sigma$. Then there exist $\alpha_x, \alpha_y \in \mathcal{P}(N_2)$ such that $(\alpha_x) \mathcal{P}_\sigma = x$ and $(\alpha_y) \mathcal{P}_\sigma = y$ in the case when $g = \bar{0}$, and $(\alpha_x \circ) \mathcal{P}_\sigma = x$ and $(\alpha_y \circ) \mathcal{P}_\sigma = y$ in the case when $g = \bar{1}$. If $g = \bar{0}$ then $x, y \in \mathcal{P}(N_2)$ and the condition $\alpha_x \circ \alpha_y \circ$ in $\mathcal{P}(N_2)$ implies the equality $x = y$. Similarly, if $g = \bar{1}$ then $x, y \in \mathcal{P}(N_2) \setminus \mathcal{P}(N_2)$ and the condition $\alpha_x \circ \alpha_y \circ$ in $\mathcal{P}(N_2)$ implies the equality $x = y$. Hence $\mathcal{J} : \mathcal{P}(N_2) / \sigma \rightarrow \mathcal{M}_\omega \times_\mathbb{Z} 2$ is an isomorphism.
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