A REMARK ON GLOBAL REGULARITY OF 2D GENERALIZED MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. In this paper we study the global regularity of the following 2D (two-dimensional) generalized magnetohydrodynamic equations

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p + b \cdot \nabla b - \nu(-\triangle)^\alpha u, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u - \kappa(-\triangle)^\beta b,
\end{align*}
\]

and get global regular solutions when \(0 \leq \alpha < 1/2, \ \beta \geq 1, \ 3\alpha + 2\beta > 3,\) which improves the results in [5]. In particular, we obtain the global regularity of the 2D generalized MHD when \(\alpha = 0\) and \(\beta > \frac{3}{2}\).

Keywords: Generalized Magnetohydrodynamic equations, Global regularity

1. Introduction

Consider the Cauchy problem of the 2D (two-dimensional) generalized magnetohydrodynamic equations:

\[
\begin{align*}
    u_t + u \cdot \nabla u &= -\nabla p + b \cdot \nabla b - \nu(-\triangle)^\alpha u, \\
    b_t + u \cdot \nabla b &= b \cdot \nabla u - \kappa(-\triangle)^\beta b, \\
    \nabla \cdot u &= \nabla \cdot b = 0, \\
    u(x,0) &= u_0(x), \quad b(x,0) = b_0(x)
\end{align*}
\]

for \(x \in \mathbb{R}^2\) and \(t > 0\), where \(u = u(x,t)\) is the velocity field, \(b = b(x,t)\) is the magnetic field, \(p = p(x,t)\) is the pressure, and \(u_0(x), b_0(x)\) with div\(u_0(x) = \nabla b_0(x) = 0\) are the initial velocity and magnetic field, respectively. Here \(\nu, \kappa, \alpha, \beta \geq 0\) are nonnegative constants and \((-\triangle)^{1/2}\) is defined through the Fourier transform by

\[((-\triangle)^{1/2}f)(\xi) = |\xi| \hat{f}(\xi),\]
where $\wedge$ denotes the Fourier transform. Later, we will use the inverse Fourier transform $\vee$. As usual, we write $(-\Delta)^{1/2}$ as $\Lambda$. To simplify the presentation, we will assume $\nu = \kappa = 1$ when $\alpha > 0$ and $\nu = 0$, $\kappa = 1$ when $\alpha = 0$.

There have been extensive studies on the global regularity of solutions to (1) (see e.g. [6], [7], [5] and references therein). It follows from [7] that the problem (1) has a unique global regular solution if

$$\alpha \geq 1, \ \beta > 0, \ \alpha + \beta \geq 2.$$  

Recently, Tran, Yu and Zhai [5] proved that if

$$\alpha \geq 1/2, \ \beta \geq 1 \ \text{or} \ \ 0 \leq \alpha < 1/2, \ \ 2\alpha + \beta > 2 \ \text{or} \ \alpha \geq 2, \ \beta = 0,$$

then the solution is global regular.

In this paper, we will prove the global regularity to (1) when $0 \leq \alpha < 1/2, \beta \geq 1, 3\alpha + 2\beta > 3$. Denote the vorticity by $\omega = -\partial_2 u_1 + \partial_1 u_2$ and the current by $j = -\partial_2 b_1 + \partial_1 b_2$. We will prove that $\omega, j \in L^2(0, T; L^\infty)$ and hence obtain the global regularity of the solution due to BKM type criterion (see [1]). To this end, based on the estimates of $\omega, j$ in $L^\infty(0, T; L^2)$ in [5], we will prove a new estimate on $\Lambda^r j$ for some $r > 0$ in this paper (see Lemma 2). Our result improves ones in [5] and in particular, if $\alpha = 0, \beta > \frac{3}{2}$, the problem (1) has a global regular solution.

Our main result is stated as

**Theorem 1.** Suppose that $(u_0, b_0) \in H^k$ with $k > \max\{2, \alpha + \beta\}$. If

$$0 \leq \alpha < 1/2, \beta \geq 1, 3\alpha + 2\beta > 3,$$

then the Cauchy problem has a unique global regular solution.

**Remark 1.** When $\alpha + \beta > 2, k > 2$, the global regularity has been proved in [5].

**Remark 2.** For the 2D MHD equations, it remains open to prove the global regularity when $\alpha = 0, \beta = 1$ or $\beta = 0, \alpha = 1$.

### 2. A Priori estimates

In this section, we will give a priori estimates for $\omega$ and $j$. For convenience, we use the same notation as in [5]. Let $\omega$ and $j$ denote the vorticity and the current respectively, where $\omega = \nabla ^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2$ and $j = \nabla ^\perp \cdot b = -\partial_2 b_1 + \partial_1 b_2$. Applying $\nabla ^\perp$, to the equations (1), we obtain the following equations for $\omega$ and $j$:

$$\omega_t + u \cdot \nabla \omega = b \cdot \nabla j - \Lambda^{2\alpha} \omega,$$

$$j_t + u \cdot \nabla j = b \cdot \nabla \omega + T(\nabla u, \nabla b) - \Lambda^{2\beta} j,$$

where

$$T(\nabla u, \nabla b) = 2\partial_1 b_1 (\partial_1 u_2 + \partial_2 u_1) + 2\partial_2 u_2 (\partial_1 b_2 + \partial_2 b_1).$$

The $L^\infty(0, T; L^2(\mathbb{R}^2))$ estimates for $\omega, j$ are obtained in [5], which is

**Lemma 1.** (5) Suppose that $\alpha \geq 0, \beta \geq 1$. Let $u_0, b_0 \in H^1$. For any $T > 0$, we have

$$\|\omega\|_{L^2}^2 (t) + \|j\|_{L^2}^2 (t) + \int_0^t \left(\|\Lambda^\alpha \omega\|_{L^2}^2 + \|\Lambda^\beta j\|_{L^2}^2\right) \, d\tau \leq C(T).$$

The following is a key lemma of this paper which will be used later.
Lemma 2. Suppose that $0 \leq \alpha < \frac{1}{2}$, $\beta \geq 1$, $r = \alpha + \beta - 1 > 0$ and $k \geq \alpha + \beta$. Let $u_0, b_0 \in H^k$. Then for any $T > 0$, we have
\begin{equation}
\|\Lambda^r j\|_{L^2}^2 (t) + \int_0^t \|\Lambda^{\beta+r} j\|_{L^2}^2 \, dt \leq C (u_0, b_0, T). \tag{5}
\end{equation}

Proof. Applying $\Lambda^r$ on both sides of the equation (3), we obtain
\begin{equation}
(\Lambda^r j)_t + \Lambda^r (u \cdot \nabla j) = \Lambda^r (b \cdot \nabla \omega) + \Lambda^r (T (\nabla u, \nabla b)) - \Lambda^{2\beta+r} j. \tag{6}
\end{equation}

Multiplying (6) by $\Lambda^r j$ and integrating with respect to $x$ in $\mathbb{R}^2$, we obtain
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|\Lambda^r j\|_{L^2}^2 + \|\Lambda^{\beta+r} j\|_{L^2}^2 = -\int_{\mathbb{R}^2} \Lambda^r (u \cdot \nabla j) \Lambda^r j \, dx + \int_{\mathbb{R}^2} \Lambda^r (b \cdot \nabla \omega) \Lambda^r j \, dx + \int_{\mathbb{R}^2} \Lambda^r T (\nabla u, \nabla b) \Lambda^r j \, dx \equiv I_1 + I_2 + I_3. \tag{7}
\end{equation}

$I_1$ is estimated as follows:
\begin{align*}
|I_1| &= \left| \int_{\mathbb{R}^2} \Lambda^r (u \cdot \nabla j) \Lambda^r j \, dx \right| \\
&= \left| \int_{\mathbb{R}^2} \Lambda^{-1} \nabla \cdot \Lambda^\alpha (uj) \Lambda^{\beta+r} j \, dx \right| \\
&\leq \left\| \Lambda^{-1} \nabla \cdot \Lambda^\alpha (uj) \right\|_{L^2} \left\| \Lambda^{\beta+r} j \right\|_{L^2} \\
&\leq C \left\| \Lambda^\alpha (uj) \right\|_{L^2} \left\| \Lambda^{\beta+r} j \right\|_{L^2} \\
&= C \left\| \int_{\mathbb{R}^2} |\xi|^\alpha \hat{u}(\xi - \eta) \hat{j}(\eta) \, d\eta \right\|_{L^2} \left\| \Lambda^{\beta+r} j \right\|_{L^2} \\
&\leq C \left( \left\| \int_{\mathbb{R}^2} |\xi - \eta|^\alpha \hat{u}(\xi - \eta) \right| \hat{j}(\eta) \, d\eta \right)_{L^2} + \left\| \int_{\mathbb{R}^2} |\xi - \eta| \hat{u}(\xi - \eta) \left| \hat{j}(\eta) \right| \, d\eta \right\|_{L^2} \left\| \Lambda^{\beta+r} j \right\|_{L^2} \\
&\leq C \left( \left\| \Lambda^\alpha (\hat{u}^\gamma) \right\|_{L^4} \left\| \hat{j}^\gamma \right\|_{L^4} + \left\| \Lambda^\alpha \left( \hat{j}^\gamma \right) \right\|_{L^4} \left\| \Lambda^{\beta+r} j \right\|_{L^2} \right) \\
&\leq C \left( \left\| \Lambda^\alpha (\hat{u}^\gamma) \right\|_{L^4} \left\| \hat{j}^\gamma \right\|_{L^4} \right) \left\| \Lambda^{\beta+r} j \right\|_{L^2} \\
&\leq C \left( \left\| u \right\|_{L^2}^{\frac{1}{2} - \alpha} \left\| \nabla u \right\|_{L^2}^{\frac{1}{2} + \alpha} \left\| j \right\|_{L^2}^{\frac{1}{2}} + \left\| u \right\|_{L^2}^{\frac{1}{2}} \left\| \nabla u \right\|_{L^2}^{\frac{1}{2}} \left\| j \right\|_{L^2}^{\frac{1}{2}} \right) \left\| \Lambda^{\beta+r} j \right\|_{L^2} \\
&\leq C \left( \left\| \Lambda^{\beta+r} j \right\|_{L^2}^2 + C(\varepsilon) \left\| u \right\|_{L^2}^{1 - 2\alpha} \left\| \omega \right\|_{L^2}^{1 + 2\alpha} \left\| j \right\|_{L^2}^{2 - \frac{1}{2}} \left\| \Lambda^{\beta} j \right\|_{L^2}^\frac{1}{2} + C(\varepsilon) \left\| u \right\|_{L^2} \left\| \omega \right\|_{L^2} \left\| j \right\|_{L^2} \left\| \Lambda^{\beta} j \right\|_{L^2} \right) \left\| \Lambda^{\beta+r} j \right\|_{L^2} \right). 
\end{align*}
Similarly, we can deal with where we have used the following Gagliardo-Nirenberg inequalities

\[
\| \Lambda^\alpha (|\hat{u}|^\nu) \|_{L^4} \leq C \| |\hat{u}|^\nu \|_{L^2}^{1-\alpha} \| \nabla (|\hat{u}|^\nu) \|_{L^2}^{\frac{1+\alpha}{4}} \leq C \| u \|_{L^2}^{1-\alpha} \| \nabla u \|_{L^2}^{\frac{1+\alpha}{4}},
\]

\[
\| |\hat{u}|^\nu \|_{L^4} \leq C \| |\hat{u}|^\nu \|_{L^2} \| \nabla (|\hat{u}|^\nu) \|_{L^2}^{\frac{1}{2}} \leq C \| u \|_{L^2}^{\frac{1}{2}} \| \nabla u \|_{L^2}^{\frac{1}{2}};
\]

\[
\| j \|_{L^4} \leq C \| j \|_{L^2}^{1-\frac{\nu - \rho}{2}} \| \Lambda^\beta (|j|^\nu) \|_{L^2}^{\frac{\rho}{2}} \leq C \| j \|_{L^2}^{1-\frac{\nu - \rho}{2}} \| \Lambda^\beta j \|_{L^2}^{\frac{\rho}{2}},
\]

\[
\| \Lambda^\alpha (|j|^\nu) \|_{L^4} \leq C \| j \|_{L^2}^{\frac{2\nu - 1}{2(\nu + \rho) - 1}} \| \Lambda^\beta (|j|^\nu) \|_{L^2}^{\frac{\rho}{2}} \leq C \| j \|_{L^2}^{\frac{2\nu - 1}{2(\nu + \rho) - 1}} \| \Lambda^\beta j \|_{L^2}^{\frac{\rho}{2}}.
\]

Similarly, we can deal with \( I_2 \) as follows.

\[
|I_2| = \left| \int_{\mathbb{R}^2} \Lambda^\gamma (b \cdot \Lambda^\omega) \Lambda^\gamma j \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}^2} \Lambda^{-1} \nabla \cdot \Lambda^\alpha (b \Lambda^\omega) \Lambda^{\beta+r} j \, dx \right|
\]

\[
\leq \| \Lambda^{-1} \nabla \cdot \Lambda^\alpha (b \Lambda^\omega) \|_{L^2} \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
\leq C \| \Lambda^\alpha (b \Lambda^\omega) \|_{L^2} \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
= C \left( \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \xi^\alpha \hat{b}(\xi - \eta) \hat{\omega}(\eta) \, d\eta \right| \right) \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
\leq C \left( \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \xi^\alpha \hat{b}(\xi - \eta) \hat{\omega}(\eta) \, d\eta \right| + \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} \hat{b}(\xi - \eta) |\eta|^\alpha |\omega(\eta)| \, d\eta \right| \right) \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
= C \left( \left\| \Lambda^\alpha \left( |\hat{b}|^\nu \right) \hat{\omega} \right\|_{L^2} + \left\| \Lambda^\alpha \left( |\hat{\omega}|^\nu \right) |\hat{b}| \right\|_{L^2} \right) \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
\leq C \left( \left\| \Lambda^\alpha \left( |\hat{b}|^\nu \right) \hat{\omega} \right\|_{L^\infty} \| \omega \|_{L^2} + \left\| \hat{b} \|_{L^\infty} \| \Lambda^\alpha \omega \|_{L^2} \right\| \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
\leq C \left( \| b \|_{L^2}^{\beta + \alpha} \left\| \Lambda^{\beta+1} b \right\|_{L^2} \| \omega \|_{L^2} + \| b \|_{L^2} \| \Lambda^{1+r} b \|_{L^2} \| \Lambda^\alpha \omega \|_{L^2} \right\| \| \Lambda^{\beta+r} j \|_{L^2}
\]

\[
\leq C e \left( \| b \|_{L^2}^{\beta + \alpha} \left\| \Lambda^{\beta+1} b \right\|_{L^2} \| \omega \|_{L^2} + \| b \|_{L^2} \| \Lambda^{1+r} b \|_{L^2} \| \Lambda^\alpha \omega \|_{L^2} \right),
\]

where we have used the following Gagliardo-Nirenberg inequalities

\[
\| \Lambda^\alpha (|\hat{b}|^\nu) \|_{L^\infty} \leq C \| |\hat{b}|^\nu \|_{L^2} \| \Lambda^{\beta+1} (|\hat{b}|^\nu) \|_{L^2} \leq C \| b \|_{L^2} \| \Lambda^{\beta+1} b \|_{L^2},
\]

\[
\| |\hat{b}|^\nu \|_{L^\infty} \leq C \| |\hat{b}|^\nu \|_{L^2} \| \Lambda^{1+r} (|\hat{b}|^\nu) \|_{L^2} \leq C \| b \|_{L^2} \| \Lambda^{1+r} b \|_{L^2}.
Now, we give estimate of $I_3$.

$$
|I_3| = \left| \int_{\mathbb{R}^2} \Delta^r T(\nabla u, \nabla b) \Delta^r j \, dx \right|
$$

$$
= \left| \int_{\mathbb{R}^3} T(\nabla u, \nabla b) \Delta^{2r} j \, dx \right|
$$

$$
\leq C \|\nabla u\|_{L^2} \|\nabla b\|_{L^\infty} \|\Delta^{2r} j\|_{L^2}
$$

$$
\leq C \|\nabla u\|_{L^2} \|\nabla b\|_{L^{\frac{3+2\alpha}{1+2\alpha}}} \|\Delta^{\beta+r+1} b\|_{L^2} \| j \|_{L^\frac{2}{2\alpha}} \|\Delta^{\beta+r} j\|_{L^2}
$$

$$
\leq C \|\nabla u\|_{L^2} \| j \|_{L^{\frac{2(3+2\alpha)}{2\alpha}}} \|\Delta^{\beta+r} j\|_{L^2}
$$

where we have used the following Gagliardo-Nirenberg inequalities

$$
\|\nabla b\|_{L^\infty} \leq C \|\nabla b\|_{L^{\frac{3+2\alpha}{1+2\alpha}}} \|\Delta^{\beta+r+1} b\|_{L^2},
$$

$$
\|\Delta^{2r} j\|_{L^2} \leq C \| j \|_{L^2} \|\Delta^{\beta+r} j\|_{L^2}.
$$

Substituting estimates of $I_1 - I_3$ into (7), we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\Delta^r j\|_{L^2}^2 + \|\Delta^{\beta+r} j\|_{L^2}^2 \leq C \|\Delta^{\beta+r} j\|_{L^2}^2 + C(\epsilon) \| u \|_{L^2} \|\omega\|_{L^\frac{1+2\alpha}{2\alpha}} \| j \|_{L^2} \|\Delta^{\beta+r} j\|_{L^2}
$$

Choosing $\epsilon = \frac{1}{2C}$, we get

$$
\frac{d}{dt} \|\Delta^r j\|_{L^2}^2 + \|\Delta^{\beta+r} j\|_{L^2}^2 \leq C(\epsilon) \| u \|_{L^2} \|\omega\|_{L^\frac{1+2\alpha}{2\alpha}} \| j \|_{L^2} \|\Delta^{\beta+r} j\|_{L^2}
$$

$$
\leq \left( C(\epsilon)\| u \|_{L^2} \|\omega\|_{L^\frac{1+2\alpha}{2\alpha}} \| j \|_{L^2} \right) \|\Delta^{\beta+r} j\|_{L^2}
$$

$$
+ C(\epsilon) \| j \|_{L^2} \|\omega\|_{L^\frac{2(3+2\alpha)}{2\alpha}}.
$$

By assumptions of the lemma, we have $0 \leq \alpha < \frac{1}{2}$, $\beta \geq 1$, $r = \alpha + \beta - 1 > 0$, and hence

$$
\frac{1}{\beta} \leq 1, \frac{1+2\alpha}{\beta} \leq 2, \frac{2(1+\alpha)}{1+r} \leq 2, \frac{2}{1+r} \leq 2.
$$

Thus, due to Lemma 1, we have

$$
\|\Delta^r j\|_{L^2}, \|\Delta^{\beta+r} j\|_{L^2}, \|\Delta^{\beta+r+1} b\|_{L^2}, \|\Delta^{\beta+r} j\|_{L^2} \in L^1(0, T).
$$

Using the Gronwall’s inequality in (8), we obtain

$$
\|\Delta^r j\|_{L^2}^2(t) + \int_0^t \|\Delta^{\beta+r} j\|_{L^2}^2 \, d\tau \leq C(u_0, b_0, T).
$$
The proof of the lemma is complete. □

3. Proof of Theorem 1

In this section, we prove Theorem 1. We will prove that $\omega, j \in L^1(0, T; L^\infty(\mathbb{R}^2))$ and Theorem 1 is then followed from the BKM-type criterion. Two cases will be considered respectively: $0 < \alpha < 1/2, \beta \geq 1, 3\alpha + 2\beta > 3$ and $\alpha = 0, \beta > \frac{3}{2}$.

**Case I:** $0 < \alpha < 1/2, \beta \geq 1, 3\alpha + 2\beta > 3$

We will first give $L^\infty(0, T; H^1)$ estimates for $\omega, j$. Differentiating with respect to $x_i (i = 1, 2)$ on both sides of (2) and (3) respectively, we get

$$
(\partial_t \omega)_t + u \cdot \nabla (\partial_t \omega) = -(\partial_t u) \cdot \nabla \omega + (\partial_t b) \cdot \nabla j + b \cdot \nabla (\partial_t j) - \Lambda^{2\alpha} (\partial_t \omega),
$$

where we have used $\nabla \cdot u = \nabla \cdot b = 0$, and we denote

$$
\begin{align*}
A_1 &= \int_{\mathbb{R}^2} |\nabla u| |\nabla \omega|^2 \, dx, \\
A_2 &= \int_{\mathbb{R}^2} |\nabla b| |\nabla j| |\nabla \omega| \, dx, \\
A_3 &= \int_{\mathbb{R}^2} |\nabla u| |\nabla j|^2 \, dx, \\
A_4 &= \int_{\mathbb{R}^2} |\nabla b| |\nabla \omega| |\nabla j| \, dx, \\
A_5 &= \int_{\mathbb{R}^2} \left[ |\nabla^2 u| |\nabla b| + |\nabla u| |\nabla^2 b| \right] |\nabla j| \, dx.
\end{align*}
$$

Multiplying $\partial_t \omega$ and $\partial_t j$ on both sides of (9) and (10) respectively, integrating with respect to $x$ in $\mathbb{R}^2$ and summing up $i=1,2$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \|\nabla \omega\|^2_{L^2} + \|\nabla j\|^2_{L^2} \right) = -\sum_{i=1}^{2} \int_{\mathbb{R}^2} [(\partial_t u) \cdot \nabla \omega] \partial_t \omega \, dx + \sum_{i=1}^{2} \int_{\mathbb{R}^2} [\partial_t b] \cdot \nabla j \partial_t \omega \, dx \\
- \sum_{i=1}^{2} \int_{\mathbb{R}^2} [(\partial_t u) \cdot \nabla j] \partial_t j \, dx + \sum_{i=1}^{2} \int_{\mathbb{R}^2} [\partial_t b] \cdot \nabla \omega \partial_t j \, dx \\
+ \sum_{i=1}^{2} \int_{\mathbb{R}^2} [\partial_t (\nabla u, \nabla b)] \partial_t j \, dx \\
- \|\Lambda^\alpha \nabla \omega\|^2_{L^2} - \|\Lambda^\beta \nabla j\|^2_{L^2} \\
\leq C (A_1 + A_2 + A_3 + A_4 + A_5) - \|\Lambda^\alpha \nabla \omega\|^2_{L^2} - \|\Lambda^\beta \nabla j\|^2_{L^2},
$$

where we have used $\nabla \cdot u = \nabla \cdot b = 0$, and we denote

$$
\begin{align*}
A_1 &= \int_{\mathbb{R}^2} |\nabla u| |\nabla \omega|^2 \, dx, \\
A_2 &= \int_{\mathbb{R}^2} |\nabla b| |\nabla j| |\nabla \omega| \, dx, \\
A_3 &= \int_{\mathbb{R}^2} |\nabla u| |\nabla j|^2 \, dx, \\
A_4 &= \int_{\mathbb{R}^2} |\nabla b| |\nabla \omega| |\nabla j| \, dx, \\
A_5 &= \int_{\mathbb{R}^2} \left[ |\nabla^2 u| |\nabla b| + |\nabla u| |\nabla^2 b| \right] |\nabla j| \, dx.
\end{align*}
$$
A_2 - A_5 can be estimated in a straight way (see also [5]), which are
\[ A_2 = A_4 \leq C(\varepsilon) \|j\|_{L^2}^2 + \|\nabla j\|_{L^2} \|\nabla \omega\|_{L^2} + C\varepsilon \|\Lambda \nabla j\|_{L^2} + C\varepsilon \|\Lambda^\beta \nabla j\|_{L^2}, \]
\[ A_3 \leq C(\varepsilon) \|\omega\|_{L^2}^2 \|\nabla j\|_{L^2}^2 + C\varepsilon \|\Lambda \nabla j\|_{L^2}, \]
\[ A_5 \leq C(\varepsilon) \|j\|_{L^2}^2 + C(\varepsilon) \|\omega\|_{L^2}^2 \|\nabla j\|_{L^2}^2 + C\varepsilon \|\Lambda \nabla j\|_{L^2}, \]
where we have used the following Gagliardo-Nirenberg inequality:
\[ \|\nabla \omega\|_{L^2} \leq C \|\nabla \omega\|_{L^2}^{1-\frac{2}{3\alpha}} \|\Lambda^\alpha \nabla \omega\|_{L^2}^{\frac{2}{3\alpha}}. \]
Now we deal with \( A_1 \).
\[ A_1 = \int_{\mathbb{R}^2} |\omega| \|\nabla \omega\|^2 \, dx \leq \|\omega\|_{L^p} \|\nabla \omega\|_{L^q} \]
\[ \leq C \|\omega\|_{L^p} \|\nabla \omega\|_{L^2}^{2-\frac{2}{3\alpha}} \|\Lambda^\alpha \nabla \omega\|_{L^2}^{\frac{2}{3\alpha}} \]
\[ \leq C\varepsilon \|\Lambda^\alpha \nabla \omega\|_{L^2}^2 + C(\varepsilon) \|\omega\|_{L^p} \|\nabla \omega\|_{L^2}^2, \]
where we have used the following Gagliardo-Nirenberg inequality:
\[ \|\nabla \omega\|_{L^2} \leq C \|\nabla \omega\|_{L^2}^{1-\frac{2}{3\alpha}} \|\Lambda^\alpha \nabla \omega\|_{L^2}^{\frac{2}{3\alpha}}. \]
Here
\[ \frac{1}{\alpha} < p < \infty, \quad \frac{1}{p} + \frac{2}{q} = 1, \]
and \( p \) is to be determined later.

To estimate \( \|\omega\|_{L^p} \), we multiply on the both sides of (2) by \( \|\omega\|^{p-2} \omega \) and integrate with respect to \( x \) in \( \mathbb{R}^2 \) to obtain
\[ \frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p + \int_{\mathbb{R}^2} (\Lambda^\alpha \omega) \|\omega\|^{p-2} \omega \, dx \leq \int_{\mathbb{R}^2} |b| |\nabla j| \|\omega\|^{p-1} \, dx, \]
where we have used \( \nabla \cdot u = 0 \). Thanks to the following inequality (see [1][2][3][5])
\[ \int_{\mathbb{R}^2} (\Lambda^\alpha \omega) \|\omega\|^{p-2} \omega \, dx \geq 0, \]
we get
\[ \frac{d}{dt} \|\omega\|_{L^p} \leq \|b \cdot \nabla j\|_{L^p} \leq \|b\|_{L^\infty} \|\nabla j\|_{L^p}. \]
\[ (12) \]
It follows from Lemma 2 that
\[ j \in L^2(0, T; H^{\beta+r}). \]
Consequently, we have
\[ b \in L^2(0, T; L^\infty) \quad \text{and} \quad \nabla j \in L^2(0, T; L^p) \quad \text{for some} \quad p > \frac{1}{\alpha}. \]
In fact, if \( \beta + r \geq 2 \), i.e. \( \alpha + 2\beta \geq 3 \), we can choose \( \frac{1}{\alpha} < p < \infty \), such that
\[ \nabla j \in L^2(0, T; L^p); \]
if \( \beta + r < 2 \), i.e. \( \alpha + 2\beta < 3 \), we have
\[ \frac{2}{\alpha - (\beta + r)} = \frac{2}{3 - (2\beta + \alpha)} > \frac{1}{4}, \quad \text{so we choose} \quad \frac{1}{\alpha} < p < \frac{2}{\alpha - (\beta + r)}, \]
such that
\[ \nabla j \in L^2(0, T; L^p). \]
where we have used the assumption $3\alpha + 2\beta > 3$ in the theorem. Integrating on $t$ in $(0, t)$ on both sides of (12) yields

\[
\|\omega\|_{L^p} \leq \|\omega_0\|_{L^p} + \int_0^t \|b\|_{L^\infty} \|\nabla j\|_{L^p} \, dt \leq \|\omega_0\|_{L^p} + \|b\|_{L^2(0, T; L^\infty)} \|\nabla j\|_{L^2(0, T; L^p)} \leq C(\omega_0, T),
\]

which implies that $\|\omega\|_{L^p} \in L^\infty(0, T)$.

Putting the estimates of (13) and lemma 1, taking $\epsilon$ so that $C\epsilon = \frac{1}{2}$, and utilizing the Gronwall’s inequality, we get

\[
\|\nabla \omega\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha \nabla \omega\|_{L^2}^2 + \|\Lambda^\beta \nabla j\|_{L^2}^2 \, dt \leq C(T)
\]

which implies that

\[
\nabla \omega, \nabla j \in L^\infty(0, T; L^2), \nabla \omega \in L^2(0, T; H^\alpha), \nabla j \in L^2(0, T; H^\beta).
\]

Thus, by the Sobolev imbedding, we have $\omega, j \in L^2(0, T; L^\infty)$. Applying the BKM type criterion for global regularity (see [1]), we get the proof of Theorem 1.

**Case II:** $\alpha = 0, \beta > \frac{3}{2}$

In this case, since $\alpha = 0, \beta > \frac{3}{2}$, we have $r = \alpha + \beta - 1 > \frac{1}{2}$ and $\beta + r > 2$. It follows from Lemma 2 that $\nabla j \in L^2(0, T; L^\infty)$. Since

\[
\omega_t + u \cdot \nabla \omega = b \cdot \nabla j,
\]

we can prove that $\omega \in L^\infty(0, T; L^\infty)$ by using the particle trajectory method. By taking advantage of the BKM type criterion for global regularity (see [1]), we finish the proof of Theorem 1.

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