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Symplectic topology of Lagrangian submanifolds of $\mathbb{C}P^n$ with intermediate minimal Maslov numbers

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Abstract: We examine symplectic topological features of a certain family of monotone Lagrangian submanifolds in $\mathbb{C}P^n$. First we give cohomological constraints on a Lagrangian submanifold in $\mathbb{C}P^n$ whose first integral homology is $p$-torsion. In the case where $(n, p) = (5, 3), (8, 3)$, we prove that the cohomologies with coefficients in $\mathbb{Z}_2$ of such Lagrangian submanifolds are isomorphic to that of $SU(3)/(SO(3)\mathbb{Z}_3)$ and $SU(3)/\mathbb{Z}_3$, respectively. Then we calculate the Floer cohomology with coefficients in $\mathbb{Z}_2$ of a monotone Lagrangian submanifold $SU(p)/\mathbb{Z}_p$ in $\mathbb{C}P^{p-1}$.

Keywords: Lagrangian submanifold, Floer cohomology, homological rigidity, non-displaceability.

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1 Introduction and main results

Let $(M, \omega)$ be a symplectic manifold, i.e. $M$ is a smooth manifold with a closed nondegenerate two-form $\omega$. A submanifold $L$ of $M$ is called Lagrangian if $2 \cdot \dim_{\mathbb{R}} L = \dim_{\mathbb{R}} M$ and the restriction of $\omega$ on $L$ vanishes. Throughout this paper all symplectic manifolds are assumed to be tame, i.e. there exists an almost complex structure $J$ such that the bilinear form $\omega(\cdot, J \cdot)$ defines a Riemannian metric on $M$ which is geometrically bounded; see [4]. All Lagrangian submanifolds are assumed to be closed (i.e. compact and without boundary), connected and embedded.

In the complex projective space $\mathbb{C}P^n$, there are two familiar examples of Lagrangian submanifolds. One is the real form $\mathbb{R}P^n$ of it and the other is the Clifford torus defined by $\mathbb{T}_{\text{clif}}^n = \{ [z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid |z_0| = \cdots = |z_n| \}$.

For instance, the Arnold–Givental conjecture was first proved for the former example from the viewpoint of Floer theory (see [30]), and then researches of general cases were developed (see [32], [21], [23] and [26]). The latter example is a maximal orbit of the standard $n$-dimensional torus action on $\mathbb{C}P^n$. A calculation of the Floer cohomology of $\mathbb{T}_{\text{clif}}^n$ was carried out by Cho [16] and, at present, that of general Lagrangian torus orbits in toric Fano manifolds has been intensively studied. However, it seems that until quite recently there is no Floer theoretic study of concrete Lagrangian submanifolds in $\mathbb{C}P^n$ beyond $\mathbb{R}P^n$ and $\mathbb{T}_{\text{clif}}^n$. In this direction a recent paper [19] treats a special three-dimensional monotone Lagrangian submanifold in $\mathbb{C}P^3$; see also [15].

The present paper proposes symplectic topological research of a class of monotone Lagrangian submanifolds in $\mathbb{C}P^n$ which naturally includes the above two typical examples. In particular, we present new ideas to obtain results about homological rigidity (or uniqueness) and non-displaceability of such submanifolds.

We recall the definitions of monotoneness and the minimal Maslov number of a Lagrangian submanifold. For a Lagrangian submanifold $L$ in a symplectic manifold $(M, \omega)$, two homomorphisms

$I_{\mu, L} : \pi_2(M, L) \to \mathbb{Z}$ and $I_{\omega} : \pi_2(M, L) \to \mathbb{R}$

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are defined as follows. For a smooth map \( w : (D^2, \partial D^2) \to (M, L) \) in a class \( A \in \pi_2(M, L) \), \( I_{\mu, L}(A) \) is defined to be the Maslov number of the bundle pair \( (w^*TM, (w|\partial D^2)^*TL) \) and \( I_\omega(A) := \int_{D^2} w^* \omega \). Then \( L \) is said to be monotone if there exists a constant \( a > 0 \) such that \( I_\omega = a I_{\mu, L} \). The minimal Maslov number \( N_L \) of \( L \) is defined to be the positive generator of \( \text{Im}(I_{\mu, L}) \subset \mathbb{Z} \).

Here we focus on the case of the complex projective space \( (\mathbb{C}P^n, \omega_{\text{FS}}) \) with the standard Fubini-Study Kähler form \( \omega_{\text{FS}} \). First we provide certain homogeneous monotone Lagrangian submanifolds in \( \mathbb{C}P^n \).

**Proposition 1.** There exist the following four examples of monotone Lagrangian submanifolds \( L \) in \( \mathbb{C}P^n \):

1. \( L = \frac{\text{SU}(p)}{\mathbb{Z}_p} \) if \( n + 1 = p^2 \),
2. \( L = \frac{\text{SU}(p)}{\text{SO}(p)\mathbb{Z}_p} \) if \( n + 1 = \frac{p(p + 1)}{2} \),
3. \( L = \frac{\text{SU}(2p)}{\text{Sp}(p)\mathbb{Z}_{2p}} \) if \( n + 1 = p(2p - 1) \),
4. \( L = \frac{E_6}{F_4\mathbb{Z}_3} \) if \( n + 1 = 27 \),

where \( p \in \mathbb{N} \setminus \{1\} \). The minimal Maslov numbers of Examples (1) and (4) equal \( 2p \) and 18, respectively; that of Example (2) equals \( p + 1 \) if \( p \) is prime.

Note that the Euler characteristics of the above examples are zero. We observe the following inequality for the minimal Maslov number \( N_L \).

**Proposition 2.** Let \( L \) be a monotone Lagrangian submanifold of \( \mathbb{C}P^n \). Then we have

\[ 1 \leq N_L \leq n + 1, \quad (1.1) \]

and if \( N_L = n + 1 \) holds, then \( L \) is a \( \mathbb{Z}_2 \)-homological \( \mathbb{R}P^n \).

The equality condition where \( N_L = n + 1 \) rephrases a result of P. Biran (see Theorem 3 and Remark 5 below). By contrast \( N_{\text{FS}}^{\text{min}} = 2 \), which is the minimal value of \( N_L \) among orientable Lagrangian submanifolds of \( \mathbb{C}P^n \). Therefore each Lagrangian submanifold in Proposition 1 possesses an intermediate minimal Maslov number.

In the following subsections we explain several symplectic topological properties, e.g. the homological rigidity phenomenon, Floer cohomology, non-displaceability and uniruling of the Lagrangian submanifolds listed in Proposition 1. We believe that their model Lagrangian submanifolds also provide interesting examples for other aspects of symplectic topology as well as contribute to the classification of monotone Lagrangian submanifolds in \( \mathbb{C}P^n \).

### 1.1 Homological rigidity of Lagrangian submanifolds in \( \mathbb{C}P^n \)

In this subsection we treat homological rigidity or uniqueness of Lagrangian submanifolds. Roughly speaking, this phenomenon means that low-dimensional topological invariants of Lagrangian submanifolds determine their entire (co)homology. This phenomenon was first discovered by P. Seidel [35] for the case \( M = \mathbb{C}P^n \), which states that any Lagrangian submanifold \( L \subset \mathbb{C}P^n \) with \( H^1(L; \mathbb{Z}_{2m-2}) = \mathbb{Z}_2 \) must satisfy \( H^*(L; \mathbb{Z}_2) \equiv H^*(\mathbb{R}P^n; \mathbb{Z}_2) \) as graded vector spaces. Nowadays, many results concerning homological rigidity or more powerful ones are known especially for noncompact symplectic manifolds (see e.g. [13], [24], [17], [1] and references therein). However, for the compact case it seems that there are only few such results. Here we review the results of P. Biran and M. Damian; see also [9, Corollary 1.2.11].

**Theorem 3** (Biran [7], Theorem A). Let \( L \) be a Lagrangian submanifold of \( \mathbb{C}P^n \) such that \( H_1(L; \mathbb{Z}) \) is 2-torsion, i.e. \( 2H_1(L; \mathbb{Z}) = 0 \). Then

1. \( H^*(L; \mathbb{Z}_2) \equiv H^*(\mathbb{R}P^n; \mathbb{Z}_2) \) as graded vector spaces;
2. if \( n \) is even, then the isomorphism in (1) is an isomorphism as graded algebras.
Theorem 4 (Damian [17], Theorem 1.8c). Under the assumptions of Theorem 3, if \( n \) is odd, then the Lagrangian submanifold \( L \subset \mathbb{C}P^n \) satisfies \( \pi_1(L) \cong \mathbb{Z}_2 \) and the universal cover of \( L \) is homeomorphic to \( S^n \).

Remark 5. The assumption \( 2H_1(L; \mathbb{Z}) = 0 \) implies that \( L \subset \mathbb{C}P^n \) is monotone and \( N_L = n + 1 \). The proof [7, p. 313] of Theorem 3 actually shows that these two conditions yield the conclusion. Hence Theorem 3 can be viewed as a homological characterisation of monotone Lagrangian submanifolds \( L \subset \mathbb{C}P^n \) with the maximal \( N_L \) (see Proposition 2).

From these results, it is natural to ask whether a Lagrangian submanifold \( L \subset \mathbb{C}P^n \) such that \( H_1(L; \mathbb{Z}) \) is \( p \)-torsion will possess such a rigidity or not. Now we present a new class of homologically rigid Lagrangian submanifolds beyond the case of \( \mathbb{R}P^n \). The following is the main result.

Theorem 6. Let \( L \) be a Lagrangian submanifold of \( \mathbb{C}P^n \) such that \( H_1(L; \mathbb{Z}) \) is 3-torsion, i.e., \( 3H_1(L; \mathbb{Z}) = 0 \). Then \( L \) is monotone, orientable, \( 3 \mid n + 1 \) and \( n \geq 5 \). Moreover, the Euler characteristic \( \chi(L) \) of \( L \) is equal to zero. Furthermore, the following isomorphisms as graded algebras hold:

1. If \( n = 5 \), then
   
   \[ H^*(L; \mathbb{Z}_2) \cong H^*(\frac{\text{SU}(3)}{\text{SO}(3)\mathbb{Z}_3}; \mathbb{Z}_2) \].

2. If \( n = 8 \), then
   
   \[ H^*(L; \mathbb{Z}_2) \cong H^*(\frac{\text{SU}(3)}{\mathbb{Z}_3}; \mathbb{Z}_2) \].

3. If \( n = 26 \) and \( H^i(L; \mathbb{Z}_2) = 0 \) for \( i = 2, 3, 4 \), then
   
   \[ H^*(L; \mathbb{Z}_2) \cong H^*(\frac{E_6}{F_4\mathbb{Z}_3}; \mathbb{Z}_2) \].

Remark 7. In the following, we denote by \( \wedge \) and \( \wedge_2 \) exterior algebras over \( \mathbb{Z} \) and \( \mathbb{Z}_2 \), respectively, with generators in round brackets. It is known that

\[ H^*(\frac{\text{SU}(3)}{\text{SO}(3)\mathbb{Z}_3}; \mathbb{Z}_2) \cong \wedge_2(x_2, x_3) \], \quad H^*(\frac{\text{SU}(3)}{\mathbb{Z}_3}; \mathbb{Z}_2) \cong \wedge_2(x_3, x_5) \], \quad H^*(\frac{E_6}{F_4\mathbb{Z}_3}; \mathbb{Z}_2) \cong \wedge_2(x_9, x_{17})

as graded algebras (see Section 2 for details). Here the index \( i \) of \( x_i \) denotes the degree of \( x_i \).

### 1.2 Floer cohomology of a model Lagrangian submanifold

Next we turn to the problem whether model Lagrangians of \( \mathbb{C}P^n \) are displaceable or not. A diffeomorphism \( \phi \) of \( (M, \omega) \) is called Hamiltonian if it is the time-one map of the flow of a Hamiltonian vector field \( X_\phi \), which is defined by the equation \( \omega(X_\phi, \cdot) = -dH_\phi \), where \( H_\phi \) is a compactly supported time-dependent Hamiltonian function on \( M \). A Lagrangian submanifold \( L \subset M \) is said to be displaceable if there exists a Hamiltonian diffeomorphism \( \phi \in \text{Ham}(M, \omega) \) such that \( L \cap \phi(L) = \emptyset \). Otherwise, \( L \) is called non-displaceable. Non-vanishing of the Floer cohomology of \( L \) implies that \( L \) is non-displaceable in \( M \). By using general facts on the Floer cohomology, we shall show:

Theorem 8. Let \( L \) be the Lagrangian submanifold \( SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p^2-1} \) in Proposition 1. Then the Floer cohomology \( HF(L) \) with coefficients in \( \mathbb{Z}_2 \) is given as follows:

1. If \( p \) is a power of 2, then \( HF(L) \cong H^*(L; \mathbb{Z}_2) \oplus \Lambda \), where \( \Lambda = \mathbb{Z}_2[T, T^{-1}] \).

2. Otherwise, \( HF(L) = 0 \).

In particular, \( L \subset \mathbb{C}P^{p^2-1} \) is non-displaceable if \( p \) is a power of 2.

Note that \( SU(2)/\mathbb{Z}_2 \subset \mathbb{C}P^3 \) is nothing but a real form \( \mathbb{R}P^3 \). If \( p \) is not a power of 2, then the above theorem gives no information about the non-displaceability of \( L \). Nevertheless, using the Floer cohomology with coefficients in \( \mathbb{Z} \), we prove

Proposition 9. The monotone Lagrangian submanifolds \( SU(3)/\mathbb{Z}_3 \subset \mathbb{C}P^8 \) and \( SU(5)/\mathbb{Z}_5 \subset \mathbb{C}P^{24} \) are non-displaceable.
1.3 Uniruling of model Lagrangian submanifolds in \( \mathbb{C}P^n \)

For a specific model Lagrangian submanifold \( L \), we can prove the existence of a pseudo-holomorphic disc with its boundary on \( L \). We review here a convenient terminology for such results; see [9, Definition 1.1.2].

**Definition 10.** A Lagrangian submanifold \( L \subset (M, \omega) \) is said to be uniruled of order \( k \) if for any \( x \in L \), there exists a generic family of almost complex structures \( J \) with the property that for each \( f \in J \) there exists a nonconstant \( J \)-holomorphic disc \( u : (D^2, \partial D^2) \to (M, L) \) such that
\[
x \in u(\partial D^2) \quad \text{and} \quad \mu(u) \leq k,
\]
where \( \mu(u) = I_{\mu,L}(u) \) is the Maslov number of \( u \).

Combining a result of Biran and Cornea (see Theorem 25 (2) below) with Theorem 8 above, we obtain

**Corollary 11.** The monotone Lagrangian submanifold \( SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1} \) is uniruled of order \( 2p \), if \( p \) is not a power of \( 2 \).

1.4 Organisation of the paper

In Section 2, we summarise some elementary properties of model Lagrangian submanifolds, including a calculation of their minimal Maslov numbers. In Section 3, we briefly recall the Floer cohomology \( HF(L) \) for monotone Lagrangian submanifolds and the spectral sequence which converges to \( HF(L) \), including its multiplicative structure. In Section 4, we review the standard symplectic disc bundle, the Lagrangian circle bundle construction \( \Gamma_L \to L \) and related results. Here, the total space \( \Gamma_L \) is a Lagrangian submanifold in a Stein manifold. We also review a recent result of Biran and Khanevsky (see Proposition 14), which relates \( HF(L) \) with the classical \( \mathbb{Z}_2 \)-Euler class of the normal bundle \( N|_L \) defined by a subcritical polarisation \( (X, \omega, J; \mathbb{C}P^n) \). In our case, the polarisation is \( (\mathbb{C}P^n, \omega_{FS}, J; \mathbb{C}P^n) \) and we prove the vanishing of the \( \mathbb{Z}_2 \)-Euler class of \( N|_L \) under a suitable condition, which yields the vanishing of \( HF(L) \). After giving fundamental results about a Lagrangian submanifold in \( \mathbb{C}P^n \) whose first integral homology is prime-torsion, Section 5 is devoted to proving the main result (Theorem 6). To obtain topological constraints on a considered Lagrangian submanifold \( L \subset \mathbb{C}P^n \), we combine the fact that \( HF(L) = 0 \) with the above spectral sequence. This approach is more direct than the one used in [7], which combines \( HF(\Gamma_L) = 0 \) with the Gysin sequence of the circle bundle \( \Gamma_L \to L \). In Section 6, we calculate the Floer cohomology of a model Lagrangian submanifold \( SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1} \) by two approaches. One approach uses a general result for Floer cohomology by Biran, Cornea and Khanevsky (see Theorem 25 and Proposition 14). The other approach is based on the spectral sequence and the Floer cohomology with coefficients in \( \mathbb{Z} \). In Section 7, as an application we show the existence of a pseudo-holomorphic disc with its boundary on a model Lagrangian submanifold. In the last section, we discuss the geography of monotone Lagrangian submanifolds in \( \mathbb{C}P^n \) towards the classification.

2 Model Lagrangian submanifolds

We consider the Lagrangian submanifolds in the complex projective space introduced in the previous section:

\[
\begin{align*}
\text{SU}(p)/\mathbb{Z}_p, & \quad \text{SO}(p)/\mathbb{Z}_p, & \quad \text{SU}(2p)/\text{Sp}(p)/\mathbb{Z}_{2p}, & \quad E_6/F_4/Z_3.
\end{align*}
\]

These examples first appeared in [2] in the context of minimal submanifold theory. Note that R. Chiang [15] rediscovered the first example in the above list from the view point of the momentum map. Including \( R^n \) they are irreducible embedded minimal submanifolds in \( \mathbb{C}P^n \) (i.e. critical points of the volume functional of \( L \) with respect to the induced Riemannian metric from \( \mathbb{C}P^n \)). Moreover, their second fundamental forms are parallel, and hence they are not only homogeneous spaces but also symmetric spaces. Although the Clifford torus \( T_{clif}^n \)}
is not irreducible, it also has such a property. The local classification of symmetric Lagrangian submanifolds of $\mathbb{C}P^n$ was accomplished by Naitoh and Takeuchi; see [27, Theorem 4.5], [28]. The above examples seem to be good pieces in the set of all monotone Lagrangian submanifolds of $\mathbb{C}P^n$. So we call them model Lagrangian submanifolds in $\mathbb{C}P^n$.

Here we review the construction of the example $L = SU(p)/Z_p \subset C P^{p^2} - 1$, where $p \in \mathbb{N} \setminus \{1\}$. We denote by $M_p(C)$ the ring of all square matrices of size $p$ with entries in $C$. The unitary group $U(p)$, which is a compact real form of the general linear group $GL_p(C)$, is a Lagrangian submanifold of the complex Euclidean space $\mathbb{C}P^n$ with the standard symplectic structure $\omega_0$. It is also contained in a round hypersphere $S^{2p^2 - 1} \subset \mathbb{C}P^n$. Since the standard $S^1$-action on $\mathbb{C}P^n$ restricts to both $U(p)$ and $S^{2p^2 - 1}$, we can take the quotients of them. The induced symplectic form on $\mathbb{C}P^{p^2 - 1} \approx S^{2p^2 - 1}/S^1$ is the Fubini-Study symplectic form $\omega_{FS}$.

$$\begin{align*}
U(p) & \subset S^{2p^2 - 1} \subset (\mathbb{C}P^n, \omega_0) \\
\downarrow & \quad \quad \downarrow \\
U(p)/S^1 & \subset \mathbb{C}P^{p^2 - 1}
\end{align*}$$

The resulting submanifold $SU(p)/Z_p \equiv U(p)/S^1$ is Lagrangian in $\mathbb{C}P^{p^2 - 1}$ by the push-forward operation of Lagrangian submanifolds. Here $Z_p = \{e^{i\pi T_0} p \mid \theta = 2\pi/p, j = 0, 1, \ldots, p - 1\}$ is the center of $SU(p)$, where $I_p$ denotes the identity matrix of size $p$.

Next we look at the topology of model Lagrangian submanifolds. The cohomology rings of $SU(p)$, $SU(p)/SO(p)$ and $E_6/F_4$, i.e. universal covering spaces of the model spaces, are known, see e.g. [12] and [3, Proposition 2.5]:

$$H^*(SU(p); \mathbb{Z}) \equiv \wedge(x_3, x_5, \ldots, x_{2p-1}), \quad H^*(SU(p)/SO(p); \mathbb{Z}_2) \equiv \wedge_2(x_2, x_3, \ldots, x_p), \quad H^*(E_6/F_4; \mathbb{Z}) \equiv \wedge(x_9, x_{17}).$$

Let us also treat the case of model Lagrangian submanifolds. For instance, consider $SU(p)/Z_p$, where $p \geq 3$ is an odd number. Since it has no 2-torsion in its homology, we have

$$H^*(SU(p)/Z_p; \mathbb{Z}_2) \equiv H^*(SU(p); \mathbb{Z}_2) \equiv \wedge_2(x_3, x_5, \ldots, x_{2p-1}). \quad (2.2)$$

We also obtain

$$H^*(SU(3)/SO(3); \mathbb{Z}_3) \equiv \wedge_2(x_2, x_3), \quad H^*(E_6/F_4; \mathbb{Z}_2) \equiv \wedge(x_9, x_{17}).$$

Now we give a proof of Proposition 1.

**Proof of Proposition 1.** For each $L$, the monotonicity of $L$ and the calculation of $N_L$, where $p$ is odd prime, are consequences of more general facts (see Lemma 18 and Proposition 19 in Section 5). Hence it suffices to calculate $N_L$ of the Case (1) for any positive integer $p$.

Before calculating $N_L$ we give a general scheme to compute it. We define two subgroups $\Gamma_\omega$, $\Gamma_{\omega, L}$ of $\mathbb{R}$ by

$$\Gamma_\omega = \{[\omega](A) \mid A \in H_2(M; \mathbb{Z}), \quad \Gamma_{\omega, L} = \{[\omega](B) \mid B \in H_2(M, L; \mathbb{Z})\}.$$ 

Recall that a symplectic manifold $(M, \omega)$ is said to be prequantisable if $\Gamma_\omega$ is either trivial or discrete. A Lagrangian submanifold $L \subset M$ is called cyclic if $\Gamma_{\omega, L} \subset \mathbb{R}$ is discrete. In particular, a closed minimal Lagrangian submanifold $L$ in a Kähler-Einstein manifold $M$ with positive Ricci curvature is cyclic; see [31, Theorem II]. Note that $\Gamma_\omega$ is a subgroup of $\Gamma_{\omega, L}$. When $L$ is cyclic, we can define a positive integer $n_L := \#(\Gamma_{\omega, L}/\Gamma_\omega)$ as in [31, p. 473]. It is well-known that if $(M, \omega)$ is prequantisable, then there is a principal $\mathbb{R}/\Gamma_\omega$-bundle $\pi : Q \to M$ with a connection form $\theta$ such that its curvature form $d\theta$ satisfies

$$d\theta = \pi^* \omega.$$ 

Given any Lagrangian submanifold $L \subset M$ the connection $\theta$ on $Q|_L$ is flat. Thus we can define the holonomy group of $Q|_L$ with a base point $x \in L$ as follows:

$$G_L(x) := \text{im}(\pi_1(L, x) \to \mathbb{R}/\Gamma_\omega).$$
In the present setting, the principal $S^1$-bundle is none other than the Hopf fibration $\pi : S^{2n+1} \to \mathbb{C}P^n$. Let $L \subset (\mathbb{C}P^n, \omega_{FS})$ be one of the Lagrangian submanifolds in Proposition 1. Note that $L$ is monotone and cyclic, and the integer $n_L$ is the order of the cyclic group $G_L(x) \subset S^1$. Here we use the following formula from [34, Proposition 3.6]:

$$2(n + 1) = n_L N_L.$$  \hspace{1cm} (2.3)

From (2.3) we obtain the inequality

$$N_L \geq 2(n + 1)/p \hspace{1cm} (2.4)$$

for Examples (1), (2) and (4) even if $p$ is not necessarily prime, because the order $n_L$ of the group $G_L(x)$ is at most $p$. For the Example (3) we have $N_L \geq (n + 1)/p$.

Now we consider the case where $L = SU(p)/\mathbb{Z}_p$. To show that $N_L = 2p$, it is sufficient to construct a holomorphic disc $w : (D^2, \partial D^2) \to (\mathbb{C}P^{p-1}, L)$ satisfying that $\mu (w) = 2p$. To achieve this, we consider the maximal torus $A$ of $SU(p)$ defined by

$$A = \left\{ \begin{array}{c} e^{\sqrt{-1} \theta_1} \\ \vdots \\ e^{\sqrt{-1} \theta_p} \end{array} \right\} \begin{array}{c} \theta_1 + \theta_2 + \cdots + \theta_p = 0 \pmod{2\pi} \\ 0 \leq \theta_i < 2\pi \quad (i = 1, \ldots, p) \end{array}.$$

Then its projection $\pi(A)$ by the Hopf fibration $\pi : S^{2p-1} \to \mathbb{C}P^{p-1}$ is

$$\left\{ \left[ e^{\sqrt{-1} \theta_0} : 0 : \cdots : 0 : e^{\sqrt{-1} \theta_p} : 0 : \cdots : 0 \right] \right\},$$

which is a maximal torus on $L$. Note that $\pi(A) \equiv A/\mathbb{Z}_p$. Let $w_0 : D^2 \to \mathbb{C}P^{p-1}$ be the holomorphic disc defined by

$$[1/z : 0 : \cdots : 0 : 1 : 0 : \cdots : 0 : 1 : 0 : \cdots : 0 : z] = [1 : 0 : \cdots : 0 : z : 0 : \cdots : 0 : z : 0 : \cdots : 0 : z^2].$$

Its boundary $w_0(\partial D)$ is described as

$$\left[ e^{-\sqrt{-1} \theta_0} : 0 : \cdots : 0 : 1 : 0 : \cdots : 0 : 1 : 0 : \cdots : 0 : e^{-\sqrt{-1} \theta_p} \right].$$

(0 $\leq \theta_p < 2\pi$) and it is a circle in $\pi(A) \subset L$. From these expressions we see that $\mu(w_0) = 2(p - 2) + 4 = 2p$, which completes the proof.

\[\square\]

### 3 Floer cohomology and the quantum cup product on it

In this section we briefly review the Lagrangian Floer theory as developed in [29], [33], [7] and [14].

#### 3.1 Lagrangian Floer cohomology

Let $(M, \omega)$ be a tame symplectic manifold and let $L$ be a closed monotone Lagrangian submanifold of $M$. Assume that $N_L \geq 2$. Under these conditions, it is well known that the self Floer cohomology $HF(L)$ of $L$ is well defined; see [20] and [29, Theorems 4.4, 4.5]. In the present paper, we take $\mathbb{Z}_2$ as the ground ring of the coefficients of the Floer cohomology, except for the proof of Proposition 27.

Let $U(L)$ be a Weinstein neighbourhood of $L$ in $M$. Let $L_\varepsilon \subset U(L)$ be a Hamiltonian perturbation of $L$ constructed by a $C^2$-small Morse function $f : L \to \mathbb{R}$. Assume that $f$ has exactly one relative minimum $x_0$. Denote by $C_f^*$ the Morse complex of $f$. We use $x_0$ as a base intersection point for the Floer complex $(CF(L) := CF(L, L_\varepsilon), d_f)$, where $d_f$ is the Floer’s boundary operator and $J$ is a generically chosen time-dependent family $J = \{J_t\}_{0 \leq t \leq 1}$ of almost complex structures on $M$ compatible with $\omega$. It has been shown that

$$CF(i \mod N_L)(L) = \bigoplus_{j \equiv i \mod N_L} C_f^j$$
(see [33, Section 8]) and that \( d_f \) is decomposed into \( d_f = \sum_{j \in \mathbb{Z}} \partial_j \), where \( \partial_j : C_f^* \to C_f^{*+1-NL} \). An index computation shows that

\[
d_f = \partial_0 + \partial_1 + \cdots + \partial_v,
\]

where \( v := \lfloor (\dim L + 1)/N_L \rfloor \). Note that for a suitable choice of \( J \) and a Riemannian metric on \( L \), the operator \( \partial_0 : C_f^* \to C_f^{*+1} \) can be identified with the Morse differential, and hence \( H^*(C_f, \partial_0) \cong H^*(L; \mathbb{Z}_2) \); see [33, Section 4].

### 3.2 Oh-Biran’s spectral sequence

We now recall a spectral sequence which enables us to calculate the Floer cohomology \( HF(L) \) using the operators \( \partial_1, \ldots, \partial_v \). For the details see [7, Section 5].

Let \( A = \mathbb{Z}_2[T, T^{-1}] \) be the algebra of Laurent polynomials over \( \mathbb{Z}_2 \). We define the degree of \( T \) to be \( N_L \). Then \( A \) is decomposed as \( A = \bigoplus_{i \in \mathbb{Z}} A^i \), where \( A^i \) is the subspace of homogeneous elements of degree \( i \). We refine the Morse complex \( C_f \) as \( C = C_f \otimes A \), i.e.

\[
\tilde{C}^l = \bigoplus_{k \in \mathbb{Z}} C_f^{l-kN_L} \otimes A^{kN_L}
\]

for every \( l \in \mathbb{Z} \) and the operator \( d_f \) as \( \tilde{d} : \tilde{C}^* \to \tilde{C}^{*+1} \):

\[
\tilde{d} = \partial_0 \otimes 1 + \partial_1 \otimes T + \cdots + \partial_v \otimes T^v,
\]

where \( T^i : A^* \to A^{*+iN_L} \) is the multiplication by \( T^i \). Then a simple calculation shows that \( \tilde{d} \circ \tilde{d} = 0 \) and

\[
H^*(\tilde{C}, \tilde{d}) \cong HF(\text{mod } N_L)(L)
\]

for every \( l \in \mathbb{Z} \). Let \( \{E_r^{p, q}, d_r\} \) be the spectral sequence defined by the following decreasing filtration on \( \tilde{C} \):

\[
F^p \tilde{C} = \left\{ \sum x_i \otimes T^{n_i} \mid x_i \in C_f^r, n_i \geq p \right\}.
\]

Since \( C_f^j = 0 \) for \( j < 0 \) and for \( \dim L < j \), the filtration \( F^p \tilde{C} \) is bounded. Note that this filtration was used by Biran in [7, p. 309], which is different from the one originally used by Oh [33, p. 338], and see Theorem D and Chapter 6 in [22] for a more general situation.

**Theorem 12** ([7], Theorem 5.2.A and [22], Theorem D). *The spectral sequence \( \{E_r^{p, q}, d_r\} \) has the following properties:

1. \( E_0^{p, q} = C_f^{p+q-pN_L} \otimes A^{pN_L}, \quad d_0 = [\partial_0] \otimes 1 \).
2. \( E_1^{p, q} = H^{p+q-pN_L}(L; \mathbb{Z}_2) \otimes A^{pN_L}, \quad d_1 = [\partial_1] \otimes T \), where

\[
[\partial_1] : H^{p+q-pN_L}(L; \mathbb{Z}_2) \to H^{p+1+q-pN_L}(L; \mathbb{Z}_2)
\]

is induced from \( \partial_1 \).
3. For every \( r \geq 1 \), \( E_r^{p, q} \) has the form \( E_r^{p, q} = V_r^{p, q} \otimes A^{pN_L} \) with \( d_r = \delta_r \otimes T^r \), where each \( V_r^{p, q} \) is a vector space over \( \mathbb{Z}_2 \) and \( \delta_r \) is a homomorphism \( \delta_r : V_r^{p, q} \to V_r^{p+r, q-r+1} \) defined for every \( p, q \) and satisfies \( \delta_r \circ \delta_r = 0 \). Moreover,

\[
V_r^{p, q} = \frac{\ker(\delta_r : V_r^{p, q} \to V_r^{p+r, q-r+1})}{\im(\delta_r : V_r^{p-r, q-r+1} \to V_r^{p, q})}.
\]
4. \( E_r^{p, q} \) collapses at page \( v + 1 \), where \( v = \lfloor (\dim L + 1)/N_L \rfloor \), namely \( d_r = 0 \) for every \( r \geq v + 1 \), and the sequence converges to \( HF(L) \), i.e., for every \( l \in \mathbb{Z} \) we have

\[
\bigoplus_{p+q=l} E_{\infty}^{p, q} \cong HF(\text{mod } N_L)(L).
\]
5. For all \( p \in \mathbb{Z} \) we have \( \oplus_{q \in \mathbb{Z}} E_{\infty}^{p, q} = HF(L) \).
3.3 A multiplicative structure

The spectral sequence \( \{ E^{p,q}_r, d_r \} \) possesses a multiplicative structure. Note that this product structure is used only in the proof of the latter part of Theorem 6.

We consider generic Morse functions \( f, g, h : L \to \mathbb{R} \) and denote by \((CF_f, d^f), (CF_g, d^g)\) and \((CF_h, d^h)\) the corresponding Floer complexes. In [14] Buhovsky defined a quantum product only in the proof of the latter part of Theorem 6. The spectral sequence

\[ 3.3 \]

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\[ \star \]

for every \( a \in CF_f \), \( b \in CF_g \). This product is compatible with the filtrations on \( CF_f \), \( CF_g \) and \( CF_h \) such that \( \star : F^p CF_f \otimes F^p CF_g \to F^{p+q} CF_h \). It induces the product on \( \{ E^{p,q}_r, d_r \} \) satisfying that \( \star : E^{p,q}_r \otimes E^{p',q'}_r (g) \to E^{p+p'+q+q'}_r (h) \) at each page. Then the differential \( d_r \) satisfies the Leibniz rule with respect to this product, and the product at the page \( r+1 \) comes from the one at page \( r \). Moreover, these products induce products \( V^{p,q}_r (f) \otimes V^{p',q'}_r (g) \to V^{p+p'+q+q'}_r (h) \) and the differential \( \delta_r : V^{p,q}_r \to V^{p+r,q-r+1}_r \) satisfies the Leibniz rule. Then the following theorem is proved.

**Theorem 13** (Buhovsky [14], Theorem 5). The product on \( V_1 \) induced from \( \star \) coincides with the classical cup product on \( H^* (L; \mathbb{Z}_2) \).

4 A vanishing result of the Floer cohomology

In this section, we give a result about the vanishing of the Floer cohomology for a certain class of monotone Lagrangian submanifolds in \( \mathbb{C}P^n \) (Proposition 17), which is one of the key facts to prove Theorem 6. The proof works for the following more general setting.

Let \( (M, \omega_M) \) be a closed symplectic manifold. Throughout this section, we assume that it can be embedded in a polarised Kähler manifold \( \mathcal{P} = (X, \omega, J; M) \) as a complex hypersurface satisfying that \( \omega_M = \omega|_M \) and that the complement \( X \setminus M \) is a subcritical Stein manifold. A Stein manifold is an open complex manifold \((Y, I)\) equipped with a smooth exhaustion, namely proper and bounded from below, plurisubharmonic function \( \varphi : Y \to \mathbb{R} \) which means that the two-form \( \Omega_\varphi \) on \( Y \) defined by \( \Omega_\varphi = -dd^c \varphi \) is an \( I \)-positive symplectic form, where \( d^c \varphi := d \varphi \ast I \). The associated Riemannian metric \( g_\varphi (\cdot, \cdot) := \Omega_\varphi (\cdot, I \cdot) \) on \( Y \) is Kähler. It is known that if a plurisubharmonic function \( \varphi : Y \to \mathbb{R} \) is Morse, then

\[ \text{ind}_c (\varphi) \leq \text{dim}_C (Y) \]

holds for every critical point \( z \in \text{Crit}(\varphi) \) of \( \varphi \), where \( \text{ind}_c (\varphi) \) denotes the Morse index of \( z \in \text{Crit}(\varphi) \); see e.g. [18]. A Stein manifold \((Y, I)\) is said to be subcritical if it admits an exhausting plurisubharmonic Morse function \( \varphi : Y \to \mathbb{R} \) satisfying \( \text{ind}_c (\varphi) < \text{dim}_C (Y) \) for any \( z \in \text{Crit}(\varphi) \).

Let \( k \) be the degree of a hyperplane section \( M \subset X \), i.e. \( PD [M] = k [\omega] \in H^2 (X; \mathbb{Z}) \). The simplest example of polarised Kähler manifolds with a subcritical complement \( X \setminus M \) is \( \mathcal{P} = (\mathbb{C}P^{n+1}, \omega_{FS}, J; \mathbb{C}P^n) \), which is the main target of the present paper. In this case, \( X \setminus M \) is an open ball \( B_{2n+2} \) and \( k = 1 \). For further examples see e.g. [8, Section 2.2].

Next we recall the standard symplectic disc bundle based on [7, Section 3.3.1] and [11, Section 2.2]. Let \( \pi : N \to M \) be a complex line bundle over \( M \) with \( c_1 (N) = k [\omega_M] \), where \( c_1 \) stands for the first Chern class of a complex vector bundle. Choose an Hermitian metric \( || \cdot || \) and an Hermitian connection \( \nabla \) on \( N \) with curvature \( (\sqrt{-1}/2\pi)k\omega_M \). Denote by \( H^\nabla \) the horizontal distribution of \( \nabla \) and by \( a^\nabla \) the associated transgression one-form on \( N \setminus 0_M \) defined by

\[ a^\nabla|_{H^\nabla} = 0, \quad a^\nabla_u (u) = 0, \quad a^\nabla_u (\sqrt{-1}u) = \frac{1}{2\pi} \]

for \( u \in N \). Then we have \( da^\nabla = -\pi^* (k\omega_M) \). Moreover, define a symplectic form \( \omega_{can} \) on \( N \setminus 0_M \) as

\[ \omega_{can} = -d (e^{-r^2} a^\nabla) = e^{-r^2} \pi^* (k\omega_M) + 2re^{-r^2} dr \wedge a^\nabla, \]
where \( r \) denotes the distance to the zero section \( 0_M \). Note that \( \omega_{\text{can}} \) extends smoothly to \( 0_M \) and all fibres of \( N \) are symplectic and have area one with respect to \( \omega_{\text{can}} \). The symplectic structure \( \omega_{\text{can}} \) is independent of the above auxiliary data up to symplectomorphism. The closed disc bundle

\[ E_r := \{ u \in N \mid \| u \| \leq r \} \]

equipped with \( \omega_{\text{can}} \) is called a standard symplectic disc bundle over \((M, kw_M)\). There exists a symplectic embedding \( F : (N, (1/k)\omega_{\text{can}}) \to (X, \omega) \) satisfying the following:

- The zero section \( 0_M \) is isomorphic to \( M \).
- \( \Delta := X \setminus F(\mathbb{N}) \) is an isotropic CW-complex with respect to \( \omega \) (see [6, p. 413] for the definition).
- \( (X \setminus F(\text{Int} E_r), \omega) \) is a Stein domain for any \( r > 0 \).

See [6, Theorem 1.A]. Based on that, we identify \( \mathbb{N} \) with \( X \setminus \Delta \) via \( F \) and denote by \( \pi \) the projection \( \pi : F^{-1} : X \setminus \Delta \to M \). Under this identification, \( \mathbb{N} \) can be viewed as the normal line bundle over \( M \) in \( X \).

We also review the Lagrangian circle bundle construction; see [7, Section 4.1] and [11, Section 2.4]. We denote by \( \pi_{\mathbb{N}} : P_{\mathbb{N}} \to M \) the circle bundle associated to \( \pi : \mathbb{N} \to M \), which is defined by elements \( u \in \mathbb{N} \) with \( \| u \| = r_0 \). Then it is easy to check that for any Lagrangian submanifold \( L \subset M \), \( \Gamma_L := \pi_{\mathbb{N}}^{-1}(L) \) is a Lagrangian submanifold of a Stein manifold \((X \setminus M, f|_{X \setminus M}, \omega|_{X \setminus M})\). Furthermore, assume that \( \dim \mathbb{C}M \geq 2 \) or \( X \setminus M \) is subcritical. If \( L \subset M \) is monotone, then \( \Gamma_L \subset X \setminus M \) is also monotone with \( N_{\Gamma_L} = N_L \); see [7, Proposition 4.1.A]. This construction yields the proof of Proposition 2 in Section 1:

**Proof of Proposition 2.** Let \( L \) be a monotone Lagrangian submanifold of \( \mathbb{C}P^n \). Take a polarisation \( \mathcal{P} = (\mathbb{C}P^{n+1}, \omega_{\text{FS}}, J; \mathbb{C}^{n+1}) \) and consider the Lagrangian circle bundle \( \Gamma_L \to L \). Then \( \Gamma_L \) is a monotone Lagrangian submanifold with \( N_{\Gamma_L} = N_L \) in a subcritical Stein manifold \( X \setminus M = \mathbb{C}P^{n+1} \setminus \mathbb{C}P^n \equiv B^{2n+2}(1) \). On the other hand, we have \( 1 \leq N_{\Gamma_L} \leq n + 1 \) from [33, Theorem 5.3], which yields (1.1).

The following is a direct consequence of the Floer-Gysin exact sequence for the circle bundle \( \pi_{\mathbb{N}}|_{\Gamma_L} : \Gamma_L \to M \); see [11, Theorem 1.1].

**Proposition 14** ([11], Corollary 1.3 and 1.4). Let \( M \) be a smooth hyperplane section in a Kähler manifold \((X, \omega, J)\) such that \( X \setminus M \) is subcritical. Let \( L \subset M \) be a monotone Lagrangian submanifold with \( N_L \geq 2 \) satisfying \( HF(L) \neq 0 \). Then

1. \( HF^*(L) \) is 2-periodic, i.e. \( HF^i(L) \equiv HF^{i+2}(L) \) for any \( i \in \mathbb{Z} \).
2. Moreover if \( N_L \geq 3 \), then the classical \( \mathbb{Z}_2 \)-Euler class \( e(N|_L) \in H^2(L; \mathbb{Z}_2) \) of the restriction \( \pi|_L : N|_L \to L \) is non-trivial.

It follows from Proposition 14 that if we can find a monotone Lagrangian submanifold \( L \) satisfying that \( N_L \geq 3 \) and \( e(N|_L) = 0 \) in such a hyperplane section \( M \), then \( HF(L) = 0 \) holds. A symplectic manifold \((M, \omega_{\text{FS}}(M))\) satisfying the assumption of Proposition 14, for instance, appears as a smooth hyperplane section of a complex projective manifold \( X \subset \mathbb{C}P^N \) with a positive defect.

Let \( X \subset \mathbb{C}P^N \) be a smooth complex projective manifold. The dual variety \( X^* \subset (\mathbb{C}P^N)^* \) of \( X \) is the space of all hyperplanes \( H \subset \mathbb{C}P^N \) which are not transverse to \( X \). The defect of \( X \) is defined by

\[ \text{def}(X) := \text{codim}_{\mathbb{C}P^N} X^*-1. \]

Note that for almost all manifolds \( X, X^* \) is a hypersurface and the defect of \( X \) is zero. If \( \text{def}(X) > 0 \), then the manifold \( X \) is said to have a positive defect. Interestingly, this feature is closely related to the subcriticality of the complement \( X \setminus M \) as follows.

**Theorem 15** ([10], Theorem 6.1). Let \( X \subset (\mathbb{C}P^N, \omega_{\text{FS}}) \) be a projective algebraic manifold with \( \text{def}(X) > 0 \), and let \( M \subset X \) be a smooth hyperplane section of \( X \), i.e. \( M = X \cap H \) for a hyperplane \( H \subset \mathbb{C}P^n \). Then the complement \( X \setminus M \) is a subcritical Stein manifold.

For a list of algebraic manifolds with a positive defect satisfying that \( b_2(X) = 1 \), see e.g. [10, p. 4418]. Here and in what follows, for a smooth manifold \( M \) we denote by \( H^*(M) \) the free part of \( H^*(M; \mathbb{Z}) \), and by \( b_k(M) \) the \( k \)-th Betti number of \( M \). For a complex manifold \( X \) we denote by \( c_k(X) := c_k(TX) \) the first Chern class of \( X \).
Lemma 16. Under the assumptions of Theorem 15, suppose also that \( b_2(X) = 1 \) and \( \dim \mathbb{C}(X) \geq 3 \). If \( L \) is a Lagrangian submanifold of \( M \) such that \( H_1(L; \mathbb{Z}) \) has no even torsion, then \( e(\mathcal{N}|_{L}) = 0 \).

Proof. Let \( X \subset \mathbb{C}P^N \) be a projective algebraic manifold with \( \text{def}(X) > 0 \). By the Lefschetz Hyperplane Theorem, \( b_2(X) = 1 \) yields \( H^2(M) \cong \mathbb{Z} \). It is generated by the class defined by the restriction to \( M \) of the class \( [\omega_{FS}] \in H^2(\mathbb{C}P^n; \mathbb{Z}) \).

Now we take a Lagrangian submanifold \( L \subset M \) and represent \( H_1(L; \mathbb{Z}) \) as \( \mathbb{Z}^{b_1(L)} \oplus T_1 \), where \( T_1 \) denotes its torsion part. The following argument is a modification of the one in the proof of [10, Theorem 8.5].

It follows from the universal coefficient theorem that \( H^2(L; \mathbb{Z}) \cong \mathbb{Z}^{b_1(L)} \oplus T_1 \). We denote \( c := c_1(N) \in H^2(M; \mathbb{Z}) \) and its image in \( H^2(M; \mathbb{R}) \) by \( c_\mathbb{R} \). Consider their restrictions \( c|_{L} \) and \( c_\mathbb{R}|_{L} \) to \( L \). Since \( c_\mathbb{R} = [\omega_{FS}|_{M}] \) and \( L \subset M \) is Lagrangian, we have \( c_\mathbb{R}|_{L} = 0 \in H^2(L; \mathbb{R}) \). Hence \( c|_{L} \) is in the torsion part of \( H^2(L; \mathbb{Z}) \), namely \( T_1 \). Note that \( T_1 \) consists of odd torsion only. Since \( e(\mathcal{N}|_{L}) \) is the modulo 2 reduction of \( c|_{L} \), we obtain \( e(\mathcal{N}|_{L}) = 0 \in H^2(L; \mathbb{Z}_2) \).

From Proposition 14 (2) and Lemma 16 we obtain

Proposition 17. Let \( M \) be a complex hyperplane section in a projective algebraic manifold \( X \subset \mathbb{C}P^N \) satisfying \( \text{def}(X) > 0 \), \( b_2(X) = 1 \) and \( \dim \mathbb{C}(X) \geq 3 \). Let \( L \subset M \) be a monotone Lagrangian submanifold with \( N_L \geq 3 \). If \( H_1(L; \mathbb{Z}) \) has no even torsion, then \( HF(L) = 0 \).

In particular, we can apply this proposition to monotone Lagrangian submanifolds of \( (M = \mathbb{C}P^n, \omega_{FS}) \).

5 Homological constraints on a Lagrangian submanifold whose \( H_1 \) is torsion

We obtain basic topological properties of a Lagrangian submanifold in \( \mathbb{C}P^n \) whose first integral homology is torsion. In subsection 5.1 we prove the homological rigidity of a Lagrangian submanifold \( L \) in \( \mathbb{C}P^n \) whose \( H_1(L; \mathbb{Z}) \) is 3-torsion, as stated in the introduction. In subsection 5.2, using a similar idea, we give certain homological constraints on a Lagrangian submanifold \( L \) in \( \mathbb{C}P^n \) whose \( H_1(L; \mathbb{Z}) \) is 5-torsion. We start with a preliminary lemma.

Lemma 18. Let \( L \) be a Lagrangian submanifold in \( \mathbb{C}P^n \). If \( H_1(L; \mathbb{Z}) \) has no free part, then \( L \) is monotone.

Proof. We denote by \( H^2_L(\mathbb{C}P^n, L) \) the image of the Hurewicz homomorphism \( \pi_2(\mathbb{C}P^n, L) \to H_2(\mathbb{C}P^n, L; \mathbb{Z}) \). Consider the following exact sequence

\[
\cdots \to H_2(L; \mathbb{Z}) \overset{i_*}{\to} H_2(\mathbb{C}P^n; \mathbb{Z}) \overset{j_*}{\to} H_2(\mathbb{C}P^n, L; \mathbb{Z}) \overset{\partial_*}{\to} H_1(L; \mathbb{Z}) \to \cdots.
\]

Since \( H_1(L; \mathbb{Z}) \) is a torsion group, we may assume that it is isomorphic to \( \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_t} \), where \( m_i \in \mathbb{N} \setminus \{1\} \) and \( m_i|m_{i+1} \). Then for any \( a \in H^2_L(\mathbb{C}P^n, L) \) there exists \( S \in H_2(\mathbb{C}P^n; \mathbb{Z}) \) such that \( j_*(S) = m_i a \in H_2(\mathbb{C}P^n, L; \mathbb{Z}) \). Since \( \mu(j_*(S)) = 2c_1(S) \), see [29, Remark 2.3 (ii)], we have \( \mu(a) = 2c_1(S)/m_i \), where \( c_1(S) = \langle c_1(\mathbb{C}P^n), S \rangle \in \mathbb{Z} \) is the first Chern number of \( S \).

Moreover, there exists a constant \( v > 0 \) such that \( c_1 = v I_\omega \), because \( (\mathbb{C}P^n, \omega_{FS}) \) is monotone as a symplectic manifold. Hence \( c_1(S) = v \int_S \omega = v m_i I_\omega(a) \). Therefore we obtain \( \mu(a) = 2v I_\omega(a) \).

It is well-known that \( \mathbb{R}P^n \subset \mathbb{C}P^n \) is monotone and that \( N_{\mathbb{R}P^n} = n + 1 \). The following is a generalisation of this fact.

Proposition 19. Let \( L \) be a Lagrangian submanifold in \( \mathbb{C}P^n \) and let \( p \) be a prime. If \( H_1(L; \mathbb{Z}) \) is \( p \)-torsion, then \( L \) is monotone and its minimal Maslov number \( N_L \) equals \( 2(n+1)/p \).

Proof. By Lemma 18, \( L \) is monotone. For any \( a \in H^2_L(\mathbb{C}P^n, L) \), it follows from the above proof of Lemma 18 that \( \mu(a) = 2c_1(S)/p \), because \( m_i = p \). Let \( S_0 \) be a generator of \( H_2(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z} \). We have \( S = kS_0 \) for some
Let \( L \subset \mathbb{C}P^n \) be a Lagrangian submanifold which satisfies that \( 3H_1(L; \mathbb{Z}) = 0 \). By Proposition 19, \( L \) is monotone and \( N_L = 2(n+1)/3 \). Hence \( 3 \mid n+1 \) and \( N_L \) is even. Moreover, the universal coefficient theorem yields that \( H^1(L; \mathbb{Z}) = 0 \). It means that \( L \) is orientable. In the case where \( n = 2 \), \( N_L = 2 \) holds. An orientable Lagrangian surface \( L \subset \mathbb{C}P^2 \) with \( H^1(L; \mathbb{Z}) = 0 \) must be a Lagrangian sphere. But such an \( L \) cannot exist. Indeed, if \( L \) is simply connected, then we obtain \( n_L = \#G_L(x) = 1 \), and hence by (2.3) we have \( N_L = 6 \), which is a contradiction. Thus we have proved the first part of Theorem 6.

Since the Lagrangian submanifold \( L \) in question satisfies the assumption of Proposition 17, \( HF(L) = 0 \) holds. We can deduce from it the following constraints on the cohomology of \( L \).

**Proposition 20.** Let \( L \) be a Lagrangian submanifold in \( \mathbb{C}P^n \) with \( 3H_1(L; \mathbb{Z}) = 0 \). Then \( 3 \mid n+1 \), \( N_L = 2(n+1)/3 \), and \( n \geq 5 \) hold and the cohomology of \( L \) with coefficients in \( \mathbb{Z} \) satisfies the following:

1. \( H^0(L; \mathbb{Z}) \cong H^\leq 1 (L; \mathbb{Z}) \equiv H^\leq 2 (L; \mathbb{Z}) \cong H^0(L; \mathbb{Z}) \cong \mathbb{Z}, \)
2. \( H^q(L; \mathbb{Z}) = 0 \) for \( q = \frac{n+5}{2}, \frac{n+7}{2}, \ldots, \frac{2n-7}{2} \) and \( n \geq 8, \)
3. \( H^q(L; \mathbb{Z}) \cong H^{q+\lfloor \frac{n-1}{2} \rfloor} (L; \mathbb{Z}) \) for \( q = 1, 2, \ldots, \frac{n-4}{2}. \)

In particular, the Euler characteristic \( \chi(L) \) of \( L \) is equal to zero.

**Proof.** The former part has already been proved. For the latter part, we use the spectral sequence \( \{ E^{p,q}_r, d_r \} \). Recall that

\[
E^{0,q}_2 = \ker([\partial_1] : H^q(L; \mathbb{Z}) \to H^{q+1-N_L(L; \mathbb{Z})}) / \text{im}([\partial_1] : H^{q-1+N_L(L; \mathbb{Z})} \to H^q(L; \mathbb{Z})).
\]

Since \( \partial_q E^{0,q}_2 = HF(L) = 0 \) holds by Theorem 12 (5), we obtain the following exact sequence

\[
H^{q+\frac{n-1}{2}} (L; \mathbb{Z}) \xrightarrow{[\partial_1]} H^q(L; \mathbb{Z}) \xrightarrow{[\partial_1]} H^{q+\frac{n-1}{2}} (L; \mathbb{Z}) \tag{5.5}
\]

for any \( q \in \mathbb{Z} \). From it we have the following isomorphisms

\[
[\partial_1] : H^{\frac{n-1}{2}} (L; \mathbb{Z}) \to H^0(L; \mathbb{Z}) \equiv \mathbb{Z}, \quad [\partial_1] : H^n(L; \mathbb{Z}) \equiv \mathbb{Z} \to H^{\frac{n-1}{2}} (L; \mathbb{Z}),
\]

which yield item (1). Similarly, items (2) and (3) are easily obtained from (5.5). Finally, by item (2) we obtain

\[
\chi(L) = \sum_{q=0}^n (-1)^q \dim \mathbb{Z}_q H^q(L; \mathbb{Z})
\]

\[
= \sum_{q=0}^{n+1} (-1)^q \dim \mathbb{Z}_q H^q(L; \mathbb{Z}) + \sum_{q=0}^{n-1} (-1)^q \dim \mathbb{Z}_q H^{q+\frac{n-1}{2}} (L; \mathbb{Z}).
\]

Since \( (2n-1)/3 \) is odd, we have \( \chi(L) = 0 \) from items (1) and (3).
Now we are in a position to prove the latter part of Theorem 6. The proofs of Theorem 6 (1) and (2) are essentially the same, so we present here that of the Case (2) only.

**Proof of Theorem 6 (2).** By Proposition 20 we obtain

\[
H^0(L;\mathbb{Z}_2) \cong H^1(L;\mathbb{Z}_2) \cong H^5(L;\mathbb{Z}_2) \cong H^8(L;\mathbb{Z}_2) \cong \mathbb{Z}_2,
\]

\[
H^6(L;\mathbb{Z}_2) \cong 0, \quad H^1(L;\mathbb{Z}_2) \cong H^6(L;\mathbb{Z}_2), \quad H^5(L;\mathbb{Z}_2) \cong H^7(L;\mathbb{Z}_2).
\]

Since \(H_1(L;\mathbb{Z})\) is 3-torsion, the universal coefficient theorem yields \(H_1(L;\mathbb{Z}_2) \cong 0\) and \(H^1(L;\mathbb{Z}_2) \cong 0\). Hence we have \(H^6(L;\mathbb{Z}_2) \cong 0\) and \(H^7(L;\mathbb{Z}_2) \cong H^1(L;\mathbb{Z}_2) \cong 0\). Therefore

\[
H^q(L;\mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 & \text{for } q = 0, 3, 5, 8 \\
0 & \text{otherwise}.
\end{cases}
\]

It is actually isomorphic to \(H^*(SU(3)/\mathbb{Z}_3;\mathbb{Z}_2)\) as graded vector spaces.

Next we consider the product structure on \(H^*(L;\mathbb{Z}_2)\). Denote by \(1, x_3, x_5\) and \(x_8\) the generator of \(H^q(L;\mathbb{Z}_2)\) for \(q = 0, 3, 5\) and 8, respectively. We first claim that \(x_3 \ast x_5 = x_8\). Observe that

\[
[\partial_1](x_8) = x_3, \quad [\partial_1](x_5) = 1, \quad [\partial_1](x_3) = 0 \in H^*(L;\mathbb{Z}_2),
\]

where we used the fact that \([\partial_1] : H^6(L;\mathbb{Z}_2) \to H^5(L;\mathbb{Z}_2)\) and \([\partial_1] : H^5(L;\mathbb{Z}_2) \to H^6(L;\mathbb{Z}_2)\) are isomorphisms (see Proposition 20). By the Leibniz rule, we have

\[
[\partial_1](x_3 \ast x_5) = [\partial_1](x_3) \cdot x_5 + x_3 \cdot [\partial_1](x_5) = x_3 \cdot 1 = x_3.
\]

Hence \(x_3 \ast x_5 = x_8\). Moreover, since \(H^6(L;\mathbb{Z}_2) = 0\) and \(H^1(L;\mathbb{Z}_2) = 0\), we have \(x_3 \ast x_3 = 0\) and \(x_5 \ast x_5 = 0\). This completes the proof of (2) in Theorem 6, because on \(V_1 = H^1(L;\mathbb{Z}_2)\) the induced product from \(\ast\) coincides with the cup product on \(H^*(L;\mathbb{Z}_2)\).

**Proof of Theorem 6 (3).** Using the assumption that \(H^i(L;\mathbb{Z}_2) = 0\) for \(i = 2, 3, 4\), as in the proof of (2) above we obtain

\[
H^0(L;\mathbb{Z}_2) \cong H^2(L;\mathbb{Z}_2) \cong H^17(L;\mathbb{Z}_2) \cong H^{26}(L;\mathbb{Z}_2) \cong \mathbb{Z}_2,
\]

and for the other values of \(q\) we have \(H^q(L;\mathbb{Z}_2) \cong 0\) except for \(q = 5, 6, 7, 22, 23\) and 24.

Since \(\dim_{\mathbb{Z}_2} H^{26-q}(L;\mathbb{Z}_2) = \dim_{\mathbb{Z}_2} H^q(L;\mathbb{Z}_2) = 0\) for \(q = 2, 3, 4\), we obtain

\[
H^q(L;\mathbb{Z}_2) \cong H^{q+17}(L;\mathbb{Z}_2) \cong 0
\]

for \(q = 5, 6\) and 7. Therefore

\[
H^q(L;\mathbb{Z}_2) = \begin{cases} 
\mathbb{Z}_2 & \text{for } q = 0, 9, 17, 26 \\
0 & \text{otherwise}.
\end{cases}
\]

The proof of the isomorphism as algebras is completely same as that of (2), so we omit it.

### 5.2 The case where \(H_1(L;\mathbb{Z})\) is 5-torsion

In this case the spectral sequence \(\{E_r^{pq}, d_r\}\) collapses at the third page (not at \(r = 2\)); however from this we can deduce several topological constraints on such a Lagrangian submanifold \(L \subset \mathbb{C}P^n\).

**Proposition 21.** Let \(n > 5\) and let \(L\) be a Lagrangian submanifold in \(\mathbb{C}P^n\) with \(5H_1(L;\mathbb{Z}) = 0\). Then \(L\) is monotone, \(N_L = 2(n + 1)/5\) and \(5 | n + 1\). Moreover, \(HF(L) = 0\) holds and \(H^*(L;\mathbb{Z}_2)\) satisfies that

1. \(\mathbb{Z}_2 \to H^{4n+1}(L;\mathbb{Z}_2) \to H^{4n+3}(L;\mathbb{Z}_2) \to 0\) is exact,
2. \(H^q(L;\mathbb{Z}_2) \cong H^{q+\frac{2n-1}{2}}(L;\mathbb{Z}_2)\) for \(q = \frac{2n-1}{2}, \ldots, \frac{2n+3}{2}\), if \(n \geq 19\),
3. \(0 \to H^{4n-6}(L;\mathbb{Z}_2) \to H^{4n-3}(L;\mathbb{Z}_2) \to \mathbb{Z}_2\) is exact.
Proof. The first part is a direct consequence of Proposition 19. We have $HF(L) = 0$ from Proposition 17. Since $n = \frac{(\dim L + 1)/N_L}{2}$, the spectral sequence $\{L^r, d_l\}$ collapses at the third page. It yields an exact sequence

$$V_2^{-2,q+1} \to V_2^{0,q} \to V_2^{2,q-1}$$

for any $q \in \mathbb{Z}$. Recall that

$$V_2^{-2,q+1} = \ker([\partial_1] : H^{q+1-2N_1}(L; \mathbb{Z}) \to H^{q+1-2N_1}(L; \mathbb{Z})),$$

$$V_2^{2,q-1} = \ker([\partial_1] : H^{q+1-2N_1}(L; \mathbb{Z}) \to H^{q-1-2N_1}(L; \mathbb{Z})).$$

When $n_{16} \leq q \leq \frac{4n_{16}}{5}$, both $H^{q+1-2N_1}(L; \mathbb{Z})$ and $H^{q-1-2N_1}(L; \mathbb{Z})$ vanish. Hence we have $V_2^{-2,q+1} = V_2^{2,q-1} = 0$. It follows that $L^{0,q} = V_2^{0,q} = 0$ for such $q$. Consequently, we obtain the following exact sequence

$$H^{q+1-2N_1}(L; \mathbb{Z}) \to H^q(L; \mathbb{Z}) \to H^{q+1-2N_1}(L; \mathbb{Z})$$

(5.6)

for $q = \frac{n_{16} + n_{11}}{5}, \ldots, \frac{4n_{16}}{5}$.

In particular, combining (5.6) for $q = \frac{n_{16}}{5}$ and $q = \frac{3n_{16}}{5}$ we obtain item (1). Similarly, (5.6) for $q = \frac{2n_{16}}{5}$ and $q = \frac{4n_{16}}{5} - 6$ yield item (3) and finally, (5.6) for $q = \frac{n_{16}}{5} + 1, \ldots, \frac{2n_{16}}{5} - 1$ and $q = \frac{3n_{16}}{5} + 1, \ldots, \frac{4n_{16}}{5} - 1$ imply item (2).

Example 22. Consider the case where $L = SU(5)/\mathbb{Z}_5 \subset \mathbb{C}P^{24}$. From Proposition 21 we have the following constraints on its Floer cohomology $HF^*(L; \mathbb{Z})$:

$$H^{16}(L; \mathbb{Z}) \cong H^7(L; \mathbb{Z}_2),$$

$$H^{17}(L; \mathbb{Z}) \cong H^9(L; \mathbb{Z}_2),$$

On the other hand, we know $HF^*(L; \mathbb{Z}_2)$ from (2.2): it is isomorphic to $\wedge_2(x_1, x_3, x_7, x_9)$. In particular, we have

$$H^6(L; \mathbb{Z}_2) \cong 0, \quad H^7(L; \mathbb{Z}_2) = H^8(L; \mathbb{Z}_2) = H^9(L; \mathbb{Z}_2) \cong \mathbb{Z}_2,$$

$$H^{15}(L; \mathbb{Z}_2) = H^{16}(L; \mathbb{Z}_2) = H^{17}(L; \mathbb{Z}_2) \cong \mathbb{Z}_2, \quad H^{18}(L; \mathbb{Z}_2) \cong 0,$$

which agrees with the above constraints.

6 Floer cohomology of a model Lagrangian submanifold

In this section, we calculate the Floer cohomology of a model Lagrangian submanifold with coefficients in $\mathbb{Z}_2$, in particular for the example $SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1}$. First we point out that the vanishing result (Proposition 17) yields

Corollary 23. Let $L$ be one of the following Lagrangian submanifolds:

(1) $L = SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1}$,

(2) $L = SU(p)/SO(p)\mathbb{Z}_p \subset \mathbb{C}P^{p(p+1)/2-1}$,

(3) $L = E_6/(F_4\mathbb{Z}_3) \subset \mathbb{C}P^{26}$,

where $p \geq 3$ is an odd number. Then the Floer cohomology $HF(L)$ with $\mathbb{Z}_2$-coefficients vanishes.

Proof. (1) The Lagrangian submanifold $L = SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1}$ is monotone with $N_L = 2p$ by Proposition 1. Since $H_1(L; \mathbb{Z}) \cong \mathbb{Z}_p$, it has no even torsion by the assumption, we obtain $HF(L) = 0$ by Proposition 17.

(2) The monotone Lagrangian submanifold $L = SU(p)/SO(p)\mathbb{Z}_p \subset \mathbb{C}P^{p(p+1)/2-1}$ satisfies $N_L \geq p + 1$ by (2.4). Since $p$ is odd, $H_1(L; \mathbb{Z}) \cong \mathbb{Z}_p$ has no even torsion. Hence $HF(L) = 0$.

(3) The proof is exactly same as that for (1).
We recall the following useful terminologies introduced by Biran and Cornea in [9, Definition 1.2.1].

**Definition 24.** A Lagrangian submanifold \( L \subset (M, \omega) \) is said to be narrow if \( HF(L) = 0 \), and wide if there exists an isomorphism \( HF(L) \cong H^*(L; \mathbb{Z}_2) \otimes A \), where \( A = \mathbb{Z}_2[T, T^{-1}] \).

All known monotone Lagrangian submanifolds are either narrow or wide. By Corollary 23, \( SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1} \) is narrow for every odd number \( p \geq 3 \). Now we study the case where \( p \) is even; here the following theorem is essential.

**Theorem 25** (Biran-Cornea [9, Theorem 1.2.2]). Let \( L^n \subset (M^{2n}, \omega) \) be a monotone Lagrangian submanifold. Assume that its singular cohomology \( H^*(L; \mathbb{Z}_2) \) is generated as a ring with the cup product by \( H^1(L; \mathbb{Z}_2) \).

1. If \( N_L > 1 \), then \( L \) is either wide or narrow. Moreover, if \( N_L > l + 1 \), then \( L \) is wide.
2. If \( L \) is narrow, then \( L \) is uniruled of order \( k \) with \( k = \max(l + 1, n + 1 - N_L) \) if \( N_L < l + 1 \), and with \( k = l + 1 \) if \( N_L = l + 1 \).

We apply this to the monotone Lagrangian submanifold \( L := SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1} \), where \( p \) is even. The cohomology ring of \( L \) with \( \mathbb{Z}_2 \)-coefficients was calculated by Baum and Browder; see [5, Corollary 4.2]. For \( p = 2n' + 2 \mid n' \), we have

\[
H^*(L; \mathbb{Z}_2) \cong \wedge^2(x_1, x_3, \ldots, x_{2^2-1}, \ldots, x_{2p-1}) \otimes \mathbb{Z}_2[y]/(y^2),
\]

(6.7)

where \( ^\sim \) denotes omission, \( \deg y = 2 \), and with the additional relation that \( y = x_1 \) if \( r = 1 \). In addition, by (6.7) the Poincaré polynomial \( P_L(t) = \sum_k \dim \mathbb{Z}_2 H^k(L; \mathbb{Z}_2) t^k \) of \( L \) is given by

\[
P_L(t) = (1 + t)(1 + t^2) \cdots (1 + t^{2^{p-1}}) \cdots (1 + t^{2p-1})(1 + t^2 + t^4 + \cdots + t^{2p-1}).
\]

If \( p = 2r \) for a natural number \( r \), then by (6.7) the singular cohomology of \( L \) is generated as a ring by \( H^0(L; \mathbb{Z}_2) \), where \( l = 2p - 3 \). Hence we have \( N_L = 2p > l + 1 \), which implies that \( L \) is wide by Theorem 25 (1).

To complete the proof of Theorem 8, it suffices to prove the following

**Claim.** If \( p \) is even and not a power of 2, then \( HF(L) = 0 \).

**Proof.** In this case, (6.7) implies that \( H^*(L; \mathbb{Z}_2) \) is generated as a ring by \( H^* (L; \mathbb{Z}_2) \), where \( l = 2p - 1 \). Therefore \( N_L = 2p > l = 2p - 1 \), and hence \( L \) is either wide or narrow. Suppose by contradiction that \( L \) is wide, i.e. \( HF(L) = H^*(L; \mathbb{Z}_2) \otimes A \). Since \( N_L = 2p \), the essential part of \( HF^*(L) \) is a power of \( HF^q(L) \). We write it down as follows:

\[
HF^0(L) \cong \bigoplus_{i=0}^{p-1} H^{2p+i}(L; \mathbb{Z}_2) T^{-i}, \quad HF^1(L) \cong \bigoplus_{i=0}^{p-1} H^{2p+i+1}(L; \mathbb{Z}_2) T^{-i}, \quad \ldots, \quad HF^{2p-1}(L) \cong \bigoplus_{i=0}^{p-1} H^{2p+i(2p-1)}(L; \mathbb{Z}_2) T^{-i}.
\]

Here we observe the coefficient of \( T^{-1} \) in \( HF^{2p-1}(L) \). Since the equation

\[
\dim \mathbb{Z}_2 H^{2p+i+2}(L; \mathbb{Z}_2) = \dim \mathbb{Z}_2 H^{p+2i}(L; \mathbb{Z}_2)
\]

holds and since \( H^{p+2i}(L; \mathbb{Z}_2) \) is the coefficient of \( T^{-(p+i-1)} \) in \( HF^0(L) \), we have

\[
\dim \mathbb{Z}_2 HF^0(L) = \dim \mathbb{Z}_2 HF^{2p-1}(L).
\]

Moreover, from the fact that \( HF^q(L) \) is 2-periodic (see Proposition 14) it follows that

\[
\dim \mathbb{Z}_2 HF^0(L) = \dim \mathbb{Z}_2 HF^q(L)
\]

(6.8)

for \( 0 \leq q \leq 2p - 1 \). On the other hand, we have

\[
\sum_{q=0}^{2p-1} \dim \mathbb{Z}_2 HF^q(L) = \sum_{i=0}^{p-1} \dim \mathbb{Z}_2 H^i(L; \mathbb{Z}_2) = P_L(1) = 2^{p-1+r}.
\]
By equation (6.8) we have $2p \dim_{\mathbb{Z}} HF^0(L) = 2^{p-1+r}$, which is a contradiction as $\dim_{\mathbb{Z}} HF^0(L)$ is an integer. Hence we conclude that $L$ is not wide.

This completes the proof of Theorem 8. In particular, we obtain

**Corollary 26.** Let $p$ be a power of 2. Then $L = SU(p)/\mathbb{Z}_p \subset \mathbb{C}P^{p-1}$ is non-displaceable. Moreover, for any $\phi \in \text{Ham}(\mathbb{C}P^{p-1}, \omega_{FS})$ such that $L$ and $\phi L$ intersect transversally, we have

$$\#(L \cap \phi L) \geq p \cdot 2^{p-1}.$$

On the other hand, in the case where $p$ is odd, the vanishing of the Floer cohomology of $L$ with $\mathbb{Z}_2$-coefficients provides no information about its non-displaceability. However, at least in the case where $p = 3$ or 5, we can show the following.

**Proposition 27.** The Floer cohomology of the monotone Lagrangian submanifold $SU(3)/\mathbb{Z}_3 \subset \mathbb{C}P^5$ with coefficients in $\mathbb{Z}$ is nonvanishing. In particular, it is non-displaceable in $\mathbb{C}P^5$. The same result holds for $SU(5)/\mathbb{Z}_5 \subset \mathbb{C}P^8$.

**Proof.** Denote by $L$ the monotone Lagrangian submanifold $SU(3)/\mathbb{Z}_3 \subset \mathbb{C}P^5$. Then $N_2 = 6$ and $L$ is orientable. The cohomology ring of $SU(p)/\mathbb{Z}_p$ ($r \in \mathbb{N}$) with $\mathbb{Z}_p$-coefficients, where $p$ is an odd prime, was calculated by A. Borel: by [12, p. 309] we have

$$H^*\left(\frac{SU(3)}{\mathbb{Z}_3}; \mathbb{Z}\right) \cong \wedge(x_1, x_3) \otimes \mathbb{Z}[y]/(y^3),$$

where $\deg x_1 = 1$, $\deg x_3 = 3$ and $\deg y = 2$. This implies that $x_1 \wedge x_3$ and $y^2$ generate $H^5(L; \mathbb{Z}_3)$, and hence

$$H^5(L; \mathbb{Z}_3) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3.$$  \hspace{1cm} (6.9)

On the other hand, since $L$ is spin, the Floer cohomology is well-defined with $\mathbb{Z}$-coefficients, see [22], and the spectral sequence in Theorem 12 works with $\mathbb{Z}$-coefficients, see [17]. Suppose by contradiction that $HF(L; \mathbb{Z}) = 0$. Since $v = [(\dim L + 1)/N_2] = 1$, the spectral sequence $\{E^{p,q}_r, d_r\}$ collapses at the second page, hence $E^{p,q}_2 = \cdots = E^{p,q}_{23}$. Recall that

$$E^{0,q}_2 = \frac{\ker([\partial_1] : H^q(L; \mathbb{Z}) \rightarrow H^{q+1-N_2}(L; \mathbb{Z}))}{\text{im}([\partial_1] : H^{q-N_2}(L; \mathbb{Z}) \rightarrow H^q(L; \mathbb{Z})).}$$

Since for every $p \in \mathbb{Z}$ we have $\bigoplus_{q \in \mathbb{Z}} E^{p,q}_2 \cong HF(L; \mathbb{Z}) = 0$, we obtain the following exact sequences for $q \in \mathbb{Z}$:

$$H^{q+5}(L; \mathbb{Z}) \xrightarrow{[\partial_1]} H^q(L; \mathbb{Z}) \xrightarrow{[\partial_1]} H^{q-5}(L; \mathbb{Z}),$$

which yield that $H^6(L; \mathbb{Z}) \cong 0$ and $H^7(L; \mathbb{Z}) \cong H^0(L; \mathbb{Z}) \cong \mathbb{Z}$. Here we consider the cohomology of $L$ with coefficients in $\mathbb{Z}$, By the universal coefficient theorem we have $H^4(L; \mathbb{Z}_3) \cong 0$, which is a contradiction to (6.9). Therefore $HF(L; \mathbb{Z}) \neq 0$, and $L$ is non-displaceable in $\mathbb{C}P^8$.

Next we treat the case where $L := SU(5)/\mathbb{Z}_5 \subset \mathbb{C}P^{24}$. Although the method is similar, the calculation is slightly complicated because in this case we must use the collapsin of $\{E^{p,q}_r, d_r\}$ at the third page. As above, from [12, p. 309] we obtain

$$H^*\left(\frac{SU(5)}{\mathbb{Z}_5}; \mathbb{Z}_5\right) \cong \wedge(x_1, x_3, x_5, x_7) \otimes \mathbb{Z}_5[y]/(y^5),$$

which yields

$$H^7(L; \mathbb{Z}_5) \cong (\mathbb{Z}_5)^4, \quad H^{16}(L; \mathbb{Z}_5) \cong (\mathbb{Z}_5)^5. \hspace{1cm} (6.10)$$

Suppose by contradiction that $HF(L; \mathbb{Z}) = 0$. Since $v = 2$, $\{E^{p,q}_r, d_r\}$ collapses at the third page. Hence we have

$$E^{0,q}_3 = V^{0,q}_3 = 0 \hspace{1cm} (6.11)$$
for any \( q \in \mathbb{Z} \). Here we can use the calculation in Section 5.2 replacing the coefficients \( \mathbb{Z}_2 \) by \( \mathbb{Z} \). Then (6.11) yields
\[
H^{16}(L; \mathbb{Z}) \equiv H^7(L; \mathbb{Z}), \quad H^{17}(L; \mathbb{Z}) \equiv H^8(L; \mathbb{Z}),
\]
see Proposition 21 (2). Combining these isomorphisms with the universal coefficient theorem, we obtain
\[
H^{16}(L; \mathbb{Z}_2) \equiv H^7(L; \mathbb{Z}_2).
\]
However, this is a contradiction to (6.10).

This completes the proofs of all the results in subsection 1.2. From these results the following natural question arises:

**Question.** Calculate the Floer cohomology \( HF(L) \) of the model Lagrangian submanifolds \( L \) in Proposition 1 other than \( SU(p)/\mathbb{Z}_p \). Are all Lagrangian submanifolds of \( CP^n \) in Proposition 1 non-displaceable?

### 7 Uniruling of model Lagrangian submanifolds

In this section, as a further application we show the existence of a pseudo-holomorphic disc with its boundary on a model Lagrangian submanifold \( L \). The result (Corollary 11) is an immediate consequence of arguments in the previous section.

**Proof of Corollary 11.** Let \( L \) be the monotone Lagrangian submanifold \( SU(p)/\mathbb{Z}_p \subset CP^{n-1} \), where \( p \) is not a power of 2. Then we know that \( N_L = 2p \) and \( L \) is narrow. The singular cohomology \( H^*(L; \mathbb{Z}_2) \) is generated as a ring by \( H^{2p-1}(L; \mathbb{Z}_2) \) from (2.2) and (6.7). Therefore Theorem 25 (2) yields that \( L \) is uniruled of order \( 2p \).

**Remark 28.** The same argument as above is applicable also to the following two cases:
- \( L \subset CP^6 \) which satisfies \( 3H_1(L; \mathbb{Z}) = 0 \),
- \( L \subset CP^{26} \) which satisfies \( 3H_1(L; \mathbb{Z}) = 0 \) and \( H^i(L; \mathbb{Z}_2) \equiv 0 (i = 2, 3, 4) \).
Both Lagrangian submanifolds \( L \) are uniruled of order \( N_L \).

### 8 Concluding remarks

Finally, we note that results about homological rigidity as in Theorem 3 and Theorem 6 are useful in understanding the geography of monotone Lagrangian submanifolds of \( CP^n \), at least for small \( n \).

As we pointed out in Remark 5, a monotone Lagrangian submanifold \( L \subset CP^n \) with \( N_L = n+1 \) is a \( \mathbb{Z}_2 \)-homological \( RP^n \). It seems that a Lagrangian submanifold \( L \) with relatively large \( N_L \) tends to be \( \mathbb{Z}_2 \)-homologically rigid. Consider a Lagrangian submanifold \( L \subset CP^8 \). The Condition \( 3H_1(L; \mathbb{Z}) = 0 \) implies that \( L \subset CP^8 \) is monotone and \( N_L = 6 \). Unfortunately, the argument in Theorem 6 (2) does not imply that a monotone Lagrangian submanifold \( L \subset CP^8 \) with \( N_L = 6 \) satisfies \( H^*(L; \mathbb{Z}_2) \equiv H^*(SU(3)/\mathbb{Z}_3; \mathbb{Z}_2) \), because the proof essentially uses the assumption \( 3H_1(L; \mathbb{Z}) = 0 \) to deduce that \( H^1(L; \mathbb{Z}_2) = 0 \). To overcome this difficulty Damian’s lifted Floer homology theory, see [17], may be useful.

By contrast let us consider the case where \( N_L = 2 \). For instance, \( CP^3 \) admits monotone Lagrangian submanifolds with distinct \( \mathbb{Z}_2 \)-homological types, \( \omega_{\text{cliff}} \) and \( SO(3)/D_3 \), where \( D_3 \) is the dihedral group; see [15]. The minimal Maslov number of the latter example is as follows.

**Lemma 29.** The Lagrangian submanifold \( L := SO(3)/D_3 \subset CP^3 \) is monotone and \( N_L = 2 \).

This result was obtained by Evans and Lekili [19, Lemma 4.4.1] including more general examples. However, for self-containment, we give here a proof following the lines discussed in the present paper.
Proof. The integral homology group of $L$ is
\[ H_0(L; \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(L; \mathbb{Z}) \cong \mathbb{Z}_4, \quad H_2(L; \mathbb{Z}) \cong 0, \quad H_3(L; \mathbb{Z}) \cong \mathbb{Z}; \]
see [15, Section 3]. Since $L$ is orientable, $N_L$ is even. Moreover, $L$ is monotone by Lemma 18. Hence we have $n_1N_L = 8$ by (2.3). Combining this with $N_L \leq 4$, we obtain $N_L = 2$ or 4. If $N_L = 4$, then we have $n_1(L) \equiv 2$ by Theorem 4 and Remark 5. This is impossible, so $N_L = 2$ holds.

Damian proved that a monotone oriented Lagrangian submanifold $L$ of $\mathbb{C}P^n$ which is aspherical satisfies $N_L = 2$; see [17, Theorem 1.6]. However, since $SO(3)/D_1$ is spherical, the above lemma shows that the converse of Damian’s result does not hold.

We finally note that Biran and Cornea [9, Section 6.4] gave a method to construct Lagrangian submanifolds in $\mathbb{C}P^n$ with $N_L = 2$. Compared to the case where $N_L = 2$, it seems to be difficult to construct an example of Lagrangian submanifolds in $\mathbb{C}P^n$ with large $N_L$ except for homogeneous examples like the model one.

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