Problems in application of LDPC codes to information reconciliation in quantum key distribution protocols

Ryutaroh Matsumoto
Dept. of Communications and Integrated Systems
Tokyo Institute of Technology, 152-8550 Japan
Email: ryutaroh@rmatsumoto.org
September 2009

Abstract
The information reconciliation in a quantum key distribution protocol can be studied separately from other steps in the protocol. The problem of information reconciliation can be reduced to that of distributed source coding. Its solution by LDPC codes is reviewed. We list some obstacles preventing the LDPC-based distributed source coding from becoming a more favorable alternative to the Cascade protocol for information reconciliation in quantum key distribution protocols. This exposition does not require knowledge of the quantum theory.

1 Introduction
The quantum key distribution (QKD) protocol invented in [1] is one of technologies nearest to practical realization among various quantum information processing technologies. The goal of a QKD protocol is to share a common random string, called key, between two legitimate users Alice and Bob secretly from the

*IEICE Technical Report (ISSN 0913-5685), IT2009-41, Sept. 2009. The author welcomes comments from readers.
eavesdropper Eve. Alice and Bob can use an authenticated public classical channel between them to achieve the goal, but Eve can see all the contents in the public channel. In addition to this classical channel, there is a quantum channel between Alice and Bob over which quantum objects are transmitted from Alice to Bob. Observe that this is a quantum extension of the model CW introduced by Ahlswede and Csiszár [18, Section 9.2] with the classical noisy channel replaced by the quantum noisy one.

As categorized in [26], a QKD protocol can usually be divided into four steps:

1. **Quantum transmission and reception:** Alice transmits randomly chosen quantum objects to Bob. Bob measures received objects by a randomly chosen measurement method. After this step, Alice and Bob have classical bits of the same length. The remaining steps in a QKD protocol are purely classical information processing, and all the processed data are classical.

2. **Channel parameter estimation:** Alice and Bob publicly announce parts of transmitted objects and measurement outcomes. From announced data, they estimate the channel parameters between them. Usually, part of parameters remains unknown. Remaining parts of Alice and Bob’s bits are used for generating secret key.

   The surprising feature of the quantum theory is that (quantum counterpart of) the joint probability distribution among Alice, Bob and Eve can be determined from the channel parameter only between Alice and Bob, which cannot be done within the classical secret key agreement.

3. **Information reconciliation:** Alice and Bob make their bits identical by conversation over the public channel.

4. **Privacy amplification:** Alice and Bob shorten their bits by multiplying a binary matrix to their identical bits. The resulting shortened bits are almost statistically independent of all the information possessed by Eve, which includes the conversation between Alice and Bob over the public channel.

Note that the third and fourth steps are essentially the same as the information theoretically secure key agreement introduced by Maurer, Ahlswede, and Csiszár [18, Chapter 9]. Thus, many parts of this exposition are also relevant to the information theoretically secure key agreement.

Traditional security proofs for QKD protocols, for example [28], combines the information reconciliation and the privacy amplification. Because of that,
we could not study the information reconciliation in QKD protocols separately
from the privacy amplification, for example, we could not investigate what kind
of the information reconciliation was suitable without considering the privacy
amplification. This situation was reversed by the several new security proofs
\cite{11,12,13,14,19,20,25,26,27,31}, which enabled us to study the informa-
tion reconciliation in QKD protocols without considering other steps in
QKD protocols.

The purpose of this exposition is to introduce the problem of information rec-
coniliation in QKD protocols in a form accessible to coding theorists without
background in the quantum theory except footnotes and to clarify what kind of
problems arises in LDPC codes used for information reconciliation. This exposi-
tion is organized as follows: Section \ref{sec:problem_statement} describes the problem statement and briefly
reviews the relevant research results. Section \ref{sec:slepian-wolf} reviews the Slepian-Wolf coding
\cite[Section 15.4]{6} and its relation to the information reconciliation. Section \ref{sec:solution LDPC}
reviews a solution by LDPC matrices and lists the problems whose solutions are
wanted (by this author). Section \ref{sec:conclusion} gives a conclusion.

\section{Problem statement} \label{sec:problem_statement}

We assume that physical objects with two-dimensional state spaces are transmit-
ted in the QKD protocols. This assumption is valid in one of several common
realization of QKD protocols. Another common realization of QKD protocols
uses infinite-dimensional objects \cite{9}. Information reconciliation in such a case
is discussed in \cite{2,16,17,24}. The information reconciliation in the infinite-
dimensional case seems more challenging than the two-dimensional case.

After the channel parameter estimation, Alice has an \(n\)-bit binary string \(X^n = (X_1, \ldots, X_n)\), Bob has \(Y^n = (Y_1, \ldots, Y_n)\), and they know an estimate of the joint
probability distribution \(P_{XY}\) assuming that \((X_i, Y_i)\) are i.i.d. for all \(i = 1, \ldots, n\). The goal of the information reconciliation is for Bob to produce a string \(\hat{X}^n\) by
(possibly two-way) conversation with Alice over the public channel. The entire
content of their conversation depends on \(X^n\) and \(Y^n\), and \(c(X^n, Y^n)\) denotes the
entire conversation. The desirable properties of the information reconciliation are

- Make \(\Pr[X^n = \hat{X}^n]\) sufficiently close to one.

- Make the mutual information \(I(X^n; c(X^n, Y^n))\) as small as possible.

The reason behind the second property is that we must subtract \(I(X^n; c(X^n, Y^n))\) bits from the length of the final secret key \cite{12,27}, because \(I(X^n; c(X^n, Y^n))\) is
the amount of information leaked to Eve during the conversation over the public channel. Note that decreasing $I(X^n; c(X^n, Y^n))$ is totally different from decreasing the number of bits in the conversation $c(X^n, Y^n)$. For example, the famous information reconciliation protocol Cascase \[3, 29\] exchanges many bits between Alice and Bob, while keeping $I(X^n; c(X^n, Y^n))$ relatively small.

We restrict ourselves to the one-way conversation, that is, only Alice sends information to Bob and Bob sends nothing to Alice. In the one-way conversation, $c(X^n, Y^n)$ is a function of $X^n$, denoted by $c(X^n)$. We have $I(X^n; c(X^n)) \leq H(c(X^n)) \leq$ the number of bits in $c(X^n)$. We can find a good information reconciliation method by saving the number of bits in $c(X^n)$ while enabling Bob to decode $X^n$ from $c(X^n)$ and $Y^n$. This is a kind of data compression problem, called the Slepian-Wolf problem. So we shall review it in the next section.

### 3 Slepian-Wolf coding

A simplified version of the general Slepian-Wolf problem [6, Section 15.4] is given in Figure 1. The main information $X^n$ is statistically correlated with the side information $Y^n$. The encoder (data compressor) can only use $X^n$ for generating the codeword (compressed data) $f_{SW}(X^n)$ of some fixed length $m$. On the other hand, the decoder (decompresser) can use both $f_{SW}(X^n)$ and $Y^n$.

If $Y^n$ is unavailable by the decoder, the compression rate $m/n$ must be $> H(X)$, the entropy of $X^n$, in order for the decoding error probability $Pr[X \neq \hat{X}^n]$ to be negligible. The availability of $Y^n$ improves the optimal compression rate to $H(X|Y)$ from $H(X)$. The encoder and the decoder are assumed to know (a good estimate

---

1Although the Cascade \[3, 29\] does not asymptotically yield more key, it is also known that use of two-way conversation increases the amount of key \[8, 33\], which are quantum counterparts of the two-way conversation over the public channel proposed in \[21\], but we do not discuss the two-way conversation here, because the information reconciliation with two-way conversation seems rarely used.
of the joint probability distribution $P_{XY}$, and they are usually optimized for a particular $P_{XY}$. This special form of the Slepian-Wolf coding is called asymmetric Slepian-Wolf coding [10], because the roles of $X$ and $Y$ are asymmetric at the decoder.

We return to the information reconciliation. Recall that Alice has $X^n$ and Bob has $Y^n$. If Alice sends the codeword $f_{SW}(X^n)$, then Bob can recover $X^n$ with high probability by the Slepian-Wolf decoder and $Y^n$. The amount of information leaked to Eve is estimated as $I(X^n; f_{SW}(X^n)) \leq H(f_{SW}(X^n)) \leq$ the number of bits in $f_{SW}(X^n)$. Thus, if the compression rate is better\(^2\), then the upper bound on the leaked information is smaller.

### 4 Use of LDPC codes and open issues

The application of LDPC codes to the Slepian-Wolf coding with full side information can be done as follows [5, 15]. Let $M$ be an $m \times n$ sparse matrix, and $X^n$ be the source information. The codeword $f_{SW}(X^n)$ is $MX^n$. Decoding of $X^n$ given $MX^n$ and $Y^n$ can be done by the sum-product (belief propagation) algorithm over the Tanner graph of $M$. The difference to the channel decoding by the sum-product algorithm over the binary symmetric channels is as follows:

- $Y^n$ can be regarded the received word with the transmitted word\(^3\) $X^n$ over the channel $P_{Y|X}$ with exception that the syndrome of $X^n$ is not the zero vector but $MX^n$.

- While the generation of messages from a check node assumes the parity of the bits is always zero in the channel decoding, the parity of a check node in the Slepian-Wolf decoding is determined from $MX^n$.

- The initial log-likelihood ratio at a variable node $X_i$ is determined from $P_{X|Y}$ and $Y_i$ in the Slepian-Wolf decoding.

Under the maximum likelihood decoding, the sparse matrix is shown to asymptotically achieve the optimum compression rate [23]. The use of sparse matrices for

\(^2\)Strictly speaking, the use of the Slepian-Wolf coding and the simple minimization of the number of bits in $f_{SW}(X^n)$ neglect the optimization of the auxiliary random variables $U$ and $V$ in [26, which are the quantum counterparts of $U$ and $Q$ in [18] Theorem 9.2].

\(^3\)It is also possible to regard that the concatenation of $X^n$ and $MX^n$ is the transmitted word. See [10] for more detail.
information reconciliation as Slepian-Wolf encoders seems to be first considered by Muramatsu [22].

As a consumer of LDPC matrices for the information reconciliation, there are at least the following problems.

1. For a given distribution \( P_{XY} \), an optimized matrix \( M \) is not available (on the Internet). A consumer has to find an optimized matrix by himself using the density evolution or its alternative.

2. It is convenient to have a single matrix \( M \) and puncture (or shorten) \( M \) for various different rates \( H(X|Y) \).

3. For a fixed compression rate \( R \), there are infinitely many distributions \( P_{XY} \) such that \( H(X|Y) = R \), when we do not assume that \( Y^n \) is the output of a binary symmetric channel. It is convenient to have a single \( nR \times n \) matrix \( M \) such that the encoder by \( M \) yields small decoding error probability with all the distributions \( P_{XY} \) with \( H(X|Y) \approx R \).

Problem 1 can be solved by a slightly modified version of the density evolution. Under the assumption that \( Y \) is the output of a binary symmetric channel, good sparse matrices were found by Elkouss et al. [7]. The codes in [7] outperform the Cascade [3, 29], which seems the most popular method for the information reconciliation in QKD protocols when this exposition is written. Thus, the use of LDPC matrices looks promising for QKD protocols.

Problems 2 and 3 are large disadvantages compared to the Cascade [3, 29], because the Cascade is in a sense universal and we do not have to adjust it to different \( P_{XY} \). In order for the LDPC method to become more favorable as an alternative to the Cascade in the QKD application, these problems may have to be solved.

Problem 2 was considered by Varodayan et al. [30], in which an accumulator is serially connected to an LDPC encoder. However, the performance is still a bit distant from the theoretical optimum, and there seems to be a room for improvement. Several other solutions have been proposed and can be found in [10].

Although Coleman [4] provided a Shannon theoretic solution to Problem 3 with the expander code and the minimum entropy decoder by the linear programming, an efficient solution has not been provided as far as the author knows.

\footnote{Be careful that some security proofs cannot take advantage of the nonzero difference between conditional probabilities \( P_{Y|X}(1|0) \) and \( P_{Y|X}(0|1) \). References [11, 12, 25, 26, 27] are known to be capable of utilizing this difference in order to improve the compression rate in the Slepian-Wolf coding, as pointed out in [32, Remark 1].}
5 Conclusion

The standard error-correction scheme, such as LDPC codes and turbo codes, seems less popular than the Cascade protocol [3, 29] for the information reconciliation in quantum key distribution protocols. The author guessed the difficulty in selecting optimized codes as the reason for its unpopularity, and gave three specific difficulties.

Acknowledgment

The author would like to thank Dr. Manabu Hagiwara for helpful comments on an earlier manuscript, Mr. Anthony Leverrier for drawing his attention to [16, 17], and Mr. Tetsunao Matsuta for providing Fig. 1.

References

[1] C. H. Bennett and G. Brassard. Quantum cryptography: Public key distribution and coin tossing. In Proc. IEEE Intl. Conf. on Computers, Systems, and Signal Processing, pages 175–179, 1984.
[2] M. Bloch, A. Thangaraj, S. W. McLaughlin, and J.-M. Merolla. LDPC-based Gaussian key reconciliation. In Proc. 2006 IEEE Information Theory Workshop, pages 116–120, Punta del Este, Uruguay, Mar. 2006. doi:10.1109/ITW.2006.1633793.
[3] G. Brassard and L. Salvail. Secret-key reconciliation by public discussion. In T. Helleseth, editor, Advances in Cryptology–EUROCRYPT’93, volume 765 of Lecture Notes in Computer Science, pages 410–423. Springer-Verlag, 1994. doi:10.1007/3-540-48285-7_35.
[4] T. P. Coleman. Low-Complexity Approaches to Distributed Data Dissemination. PhD thesis, Massachusetts Institute of Technology, Feb. 2006.
[5] T. P. Coleman, A. H. Lee, M. Médard, and M. Effros. Low-complexity approaches to Slepian-Wolf near-lossless distributed data compression. IEEE Trans. Inform. Theory, 52(8):3546–3561, Aug. 2006. doi:10.1109/TIT.2006.878215
[6] T. M. Cover and J. A. Thomas. Elements of Information Theory. Wiley Interscience, 2nd edition, 2006.
[7] D. Elkouss, A. Leverrier, R. Alléaume, and J. J. Boutros. Efficient reconciliation protocol for discrete-variable quantum key distribution. In Proc. 2009 IEEE
[8] D. Gottesman and H.-K. Lo. Proof of security of quantum key distribution with two-way classical communications. *IEEE Trans. Inform. Theory*, 49(2):457–475, Feb. 2003. [arXiv:quant-ph/0105121](https://arxiv.org/abs/quant-ph/0105121) doi:10.1109/TIT.2002.807289.

[9] F. Grosshans, G. Van Assche, J. Wenger, R. Brouni, N. J. Cerf, and P. Grangier. Quantum key distribution using gaussian-modulated coherent states. *Nature*, 421(6920):238–241, Jan. 2003. [arXiv:quant-ph/0312016](https://arxiv.org/abs/quant-ph/0312016) doi:10.1038/nature01289.

[10] C. Guillemot and A. Roumy. Toward constructive Slepian-Wolf coding schemes. In P. L. Dragotti and M. Gastpar, editors, *Distributed Source Coding*, chapter 6, pages 131–156. Elsevier, Feb. 2009. doi:10.1016/B978-0-12-374485-2.00011-1.

[11] M. Hayashi. Practical evaluation of security for quantum key distribution. *Phys. Rev. A*, 74(2):022307, Aug. 2006. [arXiv:quant-ph/0602113](https://arxiv.org/abs/quant-ph/0602113) doi:10.1103/PhysRevA.74.022307.

[12] M. Koashi. Upper bounds of eavesdropper’s performances in finite-length code with the decoy method. *Phys. Rev. A*, 76(1):012329, July 2007. [arXiv:quant-ph/0702250](https://arxiv.org/abs/quant-ph/0702250) doi:10.1103/PhysRevA.76.012329.

[13] M. Koashi. Simple security proof of quantum key distribution based on complementarity. *New J. Phys.*, 11:045018, Apr. 2009. doi:10.1088/1367-2630/11/4/045018.

[14] M. Koashi and J. Preskill. Secure quantum key distribution with an uncharacterized source. *Phys. Rev. Lett.*, 90(5):057902, Feb. 2003. [arXiv:quant-ph/0208155](https://arxiv.org/abs/quant-ph/0208155) doi:10.1103/PhysRevLett.90.057902.

[15] A. D. Leveris, Z. Xiong, and C. N. Georghiades. Compression of binary sources with side information at the decoder using LDPC codes. *IEEE Communications Letters*, 6(10):440–442, Oct. 2002. doi:10.1109/LCOMM.2002.804244.

[16] A. Leverrier, R. Aléaume, J. Boutros, G. Zémor, and P. Grangier. Multidimensional reconciliation for a continuous-variable quantum key distribution. *Phys. Rev. A*, 77(4):042325, Apr. 2008. [arXiv:0812.4246](https://arxiv.org/abs/0812.4246) doi:10.1103/PhysRevA.77.042325.

[17] A. Leverrier and P. Grangier. Unconditional security proof of long-distance continuous-variable quantum key distribution with discrete modulation. *Phys. Rev. Lett.*, 102(18):180504, May 2009. [arXiv:0812.4246](https://arxiv.org/abs/0812.4246) doi:10.1103/PhysRevLett.102.180504.

[18] Y. Liang, H. V. Poor, and S. Shamai (Shitz). *Information Theoretic Security*. NOW Publishers, Hanover, MA, USA, 2009. doi:10.1561/0100000036.
[19] H.-K. Lo. Method for decoupling error correction from privacy amplification. *New J. Phys.*, 5:36, Apr. 2003. [arXiv:quant-ph/0201030](https://arxiv.org/abs/quant-ph/0201030) doi:10.1088/1367-2630/5/1/336.

[20] Z. Luo and I. Devetak. Efficiently implementable codes for quantum key expansion. *Phys. Rev. A*, 75(1):010303, Jan. 2007. [arXiv:quant-ph/0608029](https://arxiv.org/abs/quant-ph/0608029). doi:10.1103/PhysRevA.75.010303

[21] U. Maurer. Secret key agreement by public discussion from common information. *IEEE Trans. Inform. Theory*, 39(3):733–742, May 1993. [doi:10.1109/18.256484](https://doi.org/10.1109/18.256484).

[22] J. Muramatsu. Secret key agreement from correlated source outputs using low density parity check matrices. *IEICE Trans. Fundamentals*, E89-A(7):2036–2046, July 2006. doi:10.1093/ietfec/e89-a.7.2036.

[23] J. Muramatsu, T. Uyematsu, and T. Wadayama. Low-density parity-check matrices for coding of correlated sources. *IEEE Trans. Inform. Theory*, 51(10):3645–3654, Oct. 2005. doi:10.1109/TIT.2005.855604.

[24] K.-C. Nguyen, G. Van Assche, and N. J. Cerf. Side-information coding with turbo codes and its application to quantum key distribution. In *Proc. 2004 International Symposium on Information Theory and its Applications*, pages 1274–1279, Parma, Italy, Oct. 2004. [arXiv:cs.IT/0406001](https://arxiv.org/abs/cs.IT/0406001).

[25] R. Renner. Security of quantum key distribution. *International Journal on Quantum Information*, 6(1):733–742, May 2008. (originally published as Ph.D thesis, ETH Zürich, Switzerland, 2005). [arXiv:quant-ph/0512258](https://arxiv.org/abs/quant-ph/0512258) doi:10.1142/S0219749908003256.

[26] R. Renner, N. Gisin, and B. Kraus. Information-theoretic security proof for quantum-key-distribution protocols. *Phys. Rev. A*, 72(1):012332, July 2005. [arXiv:quant-ph/0502064](https://arxiv.org/abs/quant-ph/0502064) doi:10.1103/PhysRevA.72.012332.

[27] V. Scarani and R. Renner. Security bounds for quantum cryptography with finite resources. In Y. Kawano and M. Mosca, editors, *Theory of Quantum Computation, Communication, and Cryptography*, volume 5106 of *Lecture Notes in Computer Science*, pages 83–95. Springer-Verlag, Nov. 2008. [arXiv:0806.0120](https://arxiv.org/abs/0806.0120) doi:10.1007/978-3-540-89304-2_8.

[28] P. W. Shor and J. Preskill. Simple proof of security of the BB84 quantum key distribution protocol. *Phys. Rev. Lett.*, 85(2):441–444, July 2000. [arXiv:quant-ph/0003004](https://arxiv.org/abs/quant-ph/0003004) doi:10.1103/PhysRevLett.85.441.

[29] T. Sugimoto and K. Yamazaki. A study on secret key reconciliation protocol “Cascade”. *IEICE Trans. Fundamentals*, E83-A(10):1987–1991, Oct. 2000.
[30] D. Varodayan, A. Aaron, and B. Girod. Rate-adaptive codes for distributed source coding. *Signal Processing*, 86(11):3123–3130, Nov. 2006. doi:10.1016/j.sigpro.2006.03.012.

[31] S. Watanabe, R. Matsumoto, and T. Uyematsu. Noise tolerance of the BB84 protocol with random privacy amplification. *International Journal on Quantum Information*, 4(6):935–946, Dec. 2006. arXiv:quant-ph/0412070, doi:10.1142/S0219749906002316.

[32] S. Watanabe, R. Matsumoto, and T. Uyematsu. Tomography increases key rates of quantum-key-distribution protocols. *Phys. Rev. A*, 78(4):042316, Oct. 2008. arXiv:0802.2419, doi:10.1103/PhysRevA.78.042316.

[33] S. Watanabe, R. Matsumoto, T. Uyematsu, and Y. Kawano. Key rate of quantum key distribution with hashed two-way classical communication. *Phys. Rev. A*, 76(3):032312, Sept. 2007. arXiv:0705.2904, doi:10.1103/PhysRevA.76.032312.