A statistical theory for inhomogeneous turbulent flow formulated in the mean-Lagrangian coordinates

Taketo Ariki & Fujihiro Hamba
Institute of Industrial Science, The University of Tokyo, Komaba, Meguro-ku, Tokyo, 153-8505, Japan

Abstract. An analytical way for investigating the inhomogeneous turbulent flow has been proposed by introducing Lagrangian-like view. The present method is developed based on TSDIA (Two-Scale Direct Interaction Approximation) which is one of the few theoretical works for the inhomogeneous turbulent flow formulated in the Eulerian frame work and has fundamental inconsistency with the general covariance under the coordinate transformation. The present work reached certain success in the following two aspects. First, the algebraic representation for the Reynolds stress derived by the present method has been proved to be generally covariant. Second the algebraic Reynolds stress is in much better agreement with the result of a DNS of the channel flow.

1. Introduction

The TSDIA (Two-Scale Direct Interaction Approximation) has been proposed to analyze various kinds of statistical quantities of inhomogeneous turbulent flow in an analytical way[1][2]. Though TSDIA has provided us analytical foundations and fairly amount of suggestions about turbulence modeling, some results have inconsistency in the law of coordinate transformations. More correctly saying, TSDIA is formulated in Euclidian-Eulerian coordinate system and cannot satisfy the covariance under the general coordinate transformation, which is one of the most important properties in continuum physics. In this paper the theory has been reformulated in the Lagrangian view and developed to satisfy the general covariance clearly.

2. Necessity for the general covariance in the turbulence theory

First we introduce the Eulerian-orthonormal coordinates \{z\} which are the most often used for the ordinary fluid dynamics. Next we introduce the coordinate system \{y\} defined by the following relation.

\[ y^i = y^i (z_1, z_2, z_3, t) \] (1)

This includes the coordinates with motion like rotation or translation relatively to \{z\}. Also \{y\} includes the coordinates moving much more freely with the coordinate frame getting curved in the time evolution shown in the Figure1. Thus the coordinates \{y\} introduced by (1) are exactly the general one. In this paper let us call any of coordinate systems like \{y\} introduced by (1) the general coordinate system.

In the continuum physics, any of the physically objective properties are represented as
tensors which are covariant under the general coordinate transformation. Tensors must satisfy the following rule of the transformation between the general coordinates \( \{ y \} \) and \( \{ y' \} \).

\[
C_{\mu}^{\nu} (y, t) = \frac{\partial y^{\nu}}{\partial y'^{\mu}} \cdots \frac{\partial y^{\nu}}{\partial y'^{\mu}} C_{\mu}^{\nu} (y', t)
\]  

(2)

However, the velocity field, which is the direct representative of the motion, does not satisfy the above requirement (2) under the general coordinate transformation since the motion is essentially relative as is known in the Newtonian mechanics. Under the general coordinate transformation, velocity transforms as follow.

\[
v^{\nu} (y, t) = \frac{\partial y^{\nu}}{\partial y'^{i}} v^{i} (y', t) + \frac{\partial y^{\nu}}{\partial t}
\]  

(3)

Thus the velocity itself is not generally covariant.

Then let us think about the statistical averaged flow and the velocity fluctuation. Taking the statistical average of (3), we obtain the following.

\[
V^{\nu} (\bar{y}, t) = \frac{\partial y^{\nu}}{\partial y'^{i}} V^{i} (y, t) + \frac{\partial y^{\nu}}{\partial t} \quad , \quad V^{i} (y, t) = \langle v^{i} (y, t) \rangle
\]  

(4)

By subtracting (4) from (3) we obtain the following.

\[
v^{\nu} (\bar{y}, t) = \frac{\partial y^{\nu}}{\partial y'^{i}} v^{i} (y, t)
\]  

(5)

Here the bracket \( \langle \rangle \) means the statistical average and the prime means the fluctuation from the averaged value. The relation (4) shows exactly the transformation rule of the mean velocity under the general coordinate transformation. It is remarkable thing that the mean velocity field transforms in the same manner as the original velocity. This indicates that the mean velocity itself behaves like a kind of continuums.

The relation (5) is the transformation rule for the velocity fluctuation under the general coordinate transformation. This clearly agrees with (2) and is the obvious proof that the velocity fluctuation is essentially the covariant quantity. Thus the general covariance is a fundamental property for the turbulence dynamics since the velocity fluctuation is the most basic quantity of the turbulence and we construct the various kinds of the statistical quantities from it as the dynamical variables of the turbulence. For example by taking the tensor products and averaging them we obtain

\[
\langle v^{\nu} (\bar{y}, t) v^{\nu} (\bar{y}, t) \rangle = \frac{\partial y^{\nu}}{\partial y'^{i}} \frac{\partial y^{\nu}}{\partial y'^{i}} \langle v^{i} (y, t) v^{i} (y, t) \rangle
\]  

(6)

\[
\langle v^{\nu} (\bar{y}, t) v^{\nu} (\bar{y}, t) v^{\nu} (\bar{y}, t) \rangle = \frac{\partial y^{\nu}}{\partial y'^{i}} \frac{\partial y^{\nu}}{\partial y'^{i}} \frac{\partial y^{\nu}}{\partial y'^{i}} \langle v^{i} (y, t) v^{i} (y, t) v^{i} (y, t) \rangle
\]  

(7)
and so on. Thus the various kinds of statistical quantities will be generally covariant. Thus the theoretical method for inhomogeneous turbulent flow should be formulated based on the general covariance. However, the theoretical results from TSDIA inevitably contain the non-covariant terms. For example TSDIA produces the following algebraic representation for the Reynolds stress [3].

\[ R^{ij} = \langle v^i v^j \rangle = \frac{2}{3} K g^{ij} - 0.12 \frac{K^3}{\varepsilon} S^{ij} + 0.043 \frac{K^3}{\varepsilon} \frac{D S^{ij}}{Dt} + 0.030 \frac{K^3}{\varepsilon} \left( S_i S^a - \frac{1}{3} S \cdot S g^{ij} \right) + 0.012 \frac{K^3}{\varepsilon} \left( S_i^j \Omega^a + S_i^j \Omega^a \right) \]  

(8)

where \( S, \Omega, K \) and \( \varepsilon \) are mean strain rate, mean vorticity, turbulence energy and its dissipation rate respectively.

\[ S_g = V_{i,j} + V_{j,i} \]  

(9)

\[ \Omega_g = V_{i,j} - V_{j,i} \]  

(10)

\[ K = \frac{1}{2} \langle v^{i,i} \rangle \]  

(11)

\[ \varepsilon = \nu \langle v^{i,i} \rangle_{i,j} \]  

(12)

The operator \( \frac{D}{Dt} \) is the Lagrangian derivative using the mean velocity \( V \) instead of the original velocity \( v \), and let us call this operator the mean-Lagrangian derivative.

\[ \frac{D}{Dt} = \frac{\partial}{\partial t} + V \cdot \nabla \]  

(13)

Although the Reynolds stress is proved to be generally covariant in (6), the result (8) contains non-covariant factors like the mean vorticity and the Lagrangian derivative for the strain. Thus we have to reformulate the theory to be consistent with the general covariance. For this purpose, let us investigate the dynamical equation for the velocity fluctuation, which will play the central role in the statistical analysis. By taking the fluctuating part of the Navier-Stokes equation, we obtain the equation for the velocity fluctuation given by

\[ \left( \frac{D}{Dt} - \nu \nabla^2 \right) v^{i'} + \left( v^{i'} v^{j'} \right)_{i,j} + p' = -V_{i,j} v^{j'} + R^{i,j} \]  

(14)

where \( p' \) is the pressure fluctuation. This form of equation is employed as the fundamental law for the velocity fluctuation in TSDIA. However, (14) is in the non-covariant form including the \( \frac{D}{Dt} \)-related term and velocity gradient which are non-covariant and will survive in the following works. Thus it is natural to begin with the generally covariant equation for making the result always generally covariant. For this reason, we change the equation (14) into the following form.

\[ \left( \frac{O}{Ot} - \nu \nabla^2 \right) v^{i'} + \left( v^{i'} v^{j'} \right)_{i,j} + p' = -\left( S_i^j + \Theta_j^i \right) v^{j'} + R^{i,j} \]  

(15)

\( \Theta \) is the mean-absolute vorticity which is the mean vorticity defined only in the inertial frame. The operator \( \frac{O}{Ot} \) is the Oldroyd's derivative using the mean velocity instead of the original velocity and let us call this the mean-Oldroyd's derivative. The mean-Oldroyd's derivative for arbitrary 2-rank tensor field is defined as follow.

\[ \frac{O}{Ot} C^i_{j} = \frac{D}{Dt} C^i_{j} - V_{i,k} C^k_{j} + V_{j,k} C^i_{k} \]  

(16)

All of the terms in (15) are covariant and thus (15) is exactly the covariant equation for the
velocity fluctuation. Although we cannot reformulate the equations for arbitrary quantities in general, it was well established that the equation for the velocity fluctuation has its covariant form since we had already known from (5) that the velocity fluctuation is the generally covariant factor.

3. The mean-Lagrangian coordinate system

Until current days, the Eulerian description has been the major tool for the representation in the fluid dynamics for its simple handling. This is exactly because there is no need to derive the coordinates dependent on time since the simple continuums like Navier-Stokes fluid have no historical dependence in their material properties like the viscous constant, which are all determined by the present physical state. However, the decaying time of correlation of the turbulence is no more negligible relatively to the characteristic time-scale of the mean flow, and we inevitably have to consider about the historical dependence in turbulence phenomena. When we take the historical effects into account, it is quite useful to give certain identity for every element in continuum, which is called the Lagrangian coordinate system. Thus we introduce the Lagrangian coordinate system for the mean flow and let us call this the mean-Lagrangian coordinate system; the coordinate system \{x\} convected by the mean flow(Figure 2).

On the mean-Lagrangian coordinates, the equation for the fluctuation will take much simpler form as follow.

\[
\left(\frac{\partial}{\partial t} - \nu \nabla^2\right) v'' + \left(v'' \nabla v''\right)_\nu + p'' = -\left(S''_\nu + \Theta''\right) v'' + R''\nu. \tag{17}
\]

The time derivative is now only the partial time derivative, which makes the rest of the formulation much simpler. Using this newly-introduced coordinate system, we modify the TSDIA to derive covariant expressions for correlations in a theoretical way.

![Figure 2. Mean-Lagrangian coordinate system](image)

4. Theoretical results for the Reynolds stress

By applying the above method to the Reynolds stress, we obtain the following result.

\[
R''\nu = \frac{2}{3} K g''\nu - C_v K^2 \nu S''\nu + C_f K^3 \frac{\partial S''\nu}{\partial t} + S''_\nu S''\nu \]

\[
+ C_s \frac{K^3}{\nu} \left(S''_\nu S''\nu - \frac{1}{3} S \cdot S g''\nu\right) \]

\[
+ C_c \frac{K^3}{\nu^2} \left(S''_\nu \Theta'' + S''_\nu \Theta''\right) \tag{18}
\]

with the following set of coefficients.

\[
C_v = 0.082 , C_f = 0.033 , C_s = 0.0055 , C_c = 0.025 \tag{19}
\]
Although the theoretical result is calculated in the mean-Lagrangian coordinates, it never contradict with the general covariance. By a simple transformation, we can reach the result in any coordinate representations as follow.

\[
R^{\nu}(y,t) = \frac{\partial y^i}{\partial x^\nu} \frac{\partial y^j}{\partial x^\nu} R_{ij}(x,t)
\]

\[
= \frac{2}{3} K_g^g - C_i \frac{K}{\varepsilon} S_i^g + C_i \frac{K^2}{\varepsilon^2} \left( \frac{O S_i^g}{\partial t} + S_i^g S_i^g \right)
\]

\[
+ C_i \frac{K^3}{\varepsilon^3} \left( S_i^g S_i^g - \frac{1}{3} S \cdot S g^g \right)
\]

\[
+ C_i \frac{K^3}{\varepsilon} \left( S_i^g \Theta^g + S_i^g \Theta^g \right)
\]

The time derivative in the third term is the mean-Oldroyd's derivative which is exactly the covariant operation. We can easily recognize at least that the above representation is consistent with the general covariance, since all of terms in the right hand side are covariant respectively. Furthermore it is to be emphasized that the present theory gives the covariant results for any of the statistical quantities which are originally covariant, since we start from the covariant dynamical equation (17) and there are only covariant quantities in the whole process in the present work. This is the great advantage over the previous works like TSDIA, which uses always the mean velocity and its gradient as the essential effect of the mean flow so that the results are inevitably accompanied by the non-covariant terms, which contradict with the original property of the turbulence.

5. Comparison with the DNS data

To evaluate the validity of the present theory, we investigate the result (18) with using the DNS data[4] for the channel flow, which is one of the most basic and primitive cases(Figure3). In the present work, we compare the Reynolds stress directly calculated in the DNS with the algebraic representation (18) which the statistical quantities from the DNS are substituted into.

5.1 Physical condition

The statistical properties depend only on the distance from the wall and are homogeneous in the direction parallel to the walls. The flow is characterized by the Reynolds number defined as

\[
Re_{\tau} = \frac{U_{\tau} H}{\nu} = 590
\]

where \( H \) is the half-width of the channel and \( U_{\tau} \) is called the friction velocity which is defined as follow.

\[
U_{\tau} = \sqrt{-\mu \frac{dU}{dy}}
\]

In the following discussions, all of the quantities are normalized by \( U_{\tau} \) and \( H \). We apply the coordinate system shown in the Figure3: \( x \) for the streamwise, \( y \) for the wall-normal and \( z \) for the spanwise respectively.
5.2 Results

In the following context, we use the shortened words for simplicity: PM for the present model (18), TSDIA-M for the model (8) from TSDIA, SKEM for the standard \( K \cdot \varepsilon \) model, DNS-d for the DNS data respectively, where the standard \( K \cdot \varepsilon \) model for the Reynolds stress is given by

\[
R_{ij} = \frac{2}{3} K_{ij} - 0.09 \frac{K^2}{\varepsilon} S_{ij}
\]  

(23)

5.2.1 Normal components

The normal components of the Reynolds stress are shown in Figures 4-6. According to Figures 4 and 5, both PM and TSDIA-M show the same tendency as the DNS-d in Figure 6 in the area apart from the wall. PM successfully reproduces the order of the strength for the components \( R_x > R_z > R_y \), which agrees with the DNS-d, while TSDIA-M results in the different order \( R_y > R_z > R_x \). Both PM and TSDIA-M come to be isotropic while the DNS-d has still anisotropy in the center of the channel. This is because the anisotropic terms of both PM and TSDIA-M contain only \( S \) and \( \Theta \), which are zero in the center. This deficiency cannot be removed even if we derive the higher order non-linear algebraic model of \( S \) and \( \Theta \). Thus we gain the important suggestion from this case that we have to incorporate the anisotropy transported by the inhomogeneity of \( S \), \( \Theta \) or scalar fields.

5.2.2 Shear components

The shear components of the respective Reynolds stresses are plotted in Figure 7. Except for the near-wall region, PM is in much better agreement with DNS-d than TSDIA-M and SKEM. To see in more detail, we plotted the turbulence viscosity (Figure 8). For the DNS-d, we introduced the effective turbulence viscosity.

\[
\nu_{t, \text{DNS}} = -\frac{K_{ij}^{\nu_{\text{DNS}}}}{dV / dy}
\]  

(24)

PM is in the much better agreement with DNS-d than TSDIA-M which is almost three-seconds as large as DNS-d.

5.2.3 Near-wall region

For both normal and shear components, PM cannot be applicable in the near-wall region as well as TSDIA-M. This is because the present theory is highly based on the analysis of the homogeneous isotropic turbulence. In the wall region, statistical properties have measurable slope and anisotropy, which is easily recognized from Figures 4-6. Thus the method based on the homogeneity and isotropy is fundamentally difficult to apply to the wall region. If we employ the perturbative method like the present work, we have to consider the effect of anisotropy and inhomogeneity of infinite order to overcome this difficulty by using the scheme like renormalization.
6. Conclusions

A new statistical theory for inhomogeneous turbulence has been developed to be consistent with the general covariance under the coordinate transformation. The present theory has been clarified to have some possibilities to produce much better scheme for inhomogeneous turbulent flow in terms of both theoretical and numerical agreements.

We have to pay much attention that the above comparison does not necessarily mean that the present work is already much superior to any other method as the application in the practical cases. We have shown the possibility that the present work will provide us the key to the analysis of the turbulence not only on the theoretical but also on the practical stages. To confirm the advantage of the present work, we have to apply this method to any other cases.

References

[1] Yoshizawa, A.: Statistical analysis of the deviation of the Reynolds stress from its eddy-viscosity representation. *Phys. Fluids* **27**:1377-1387, 1984

[2] Yoshizawa, A.: *Hydrodynamic and Magnetohydrodynamic Turbulent Flow: Modeling and Statistical Theory*, Kluwer Academic Publishers, 1999

[3] Okamoto, M.: Theoretical investigation of an eddy-viscosity-type representation of the Reynolds stress. *J. Phys. Soc. Jpn.* **63**:2102-2122

[4] Moser, R. D., Kim, J., Mansour, N. N.: Direct numerical simulation of turbulent channel flow up to Re$_\tau$ = 590. *Phys. Fluids* **11**:943-945
Figure 4. The Normal stress of DNS-d

Figure 5. The Normal stress of PM

Figure 6. The Normal stress of TSDIA-M
**Figure 7.** The comparison of the shear stresses

**Figure 8.** The comparison of the turbulence viscosity