ABSTRACT. We discuss the construction of 8-manifolds with harmonic $\text{Sp}(2)\text{Sp}(1)$-structures. In particular, we find 10 new examples of nilmanifolds that admit a closed 4-form $\Omega$ whose stabiliser is $\text{Sp}(2)\text{Sp}(1)$. Our constructions entail the notion of $\text{SO}(4)$-structures on 7-manifolds. We present a thorough investigation of the intrinsic torsion of such structures; in addition to the construction of harmonic structures, this analysis leads to explicit Lie group examples with invariant intrinsic torsion.

1. Introduction

A quaternionic Kähler structure on a manifold $M$ of dimension $4n \geq 8$ is an $\text{Sp}(n)\text{Sp}(1)$-structure preserved by the Levi-Civita connection; equivalently, the 4-form $\Omega$ associated to the structure is required to be parallel. This condition makes $M$ a quaternionic Hermitian manifold, and forces the underlying metric to be Einstein [20].

Swann [24] was the first to realise that if the dimension of $M$ is at least 12, then $\Omega$ cannot be closed unless it is parallel. However, his arguments also showed that in dimension 8, the condition $d\Omega$ fails to fully determine the covariant derivative, thereby leaving open the possible existence of harmonic $\text{Sp}(2)\text{Sp}(1)$-structures, that is 8-manifolds admitting an $\text{Sp}(2)\text{Sp}(1)$-form which is closed but not parallel. This problem can be viewed as a quaternionic analogue of the search for strictly almost Kähler manifolds in almost Hermitian geometry.

This use of the adjective “harmonic”, consistent with [25, Definition 26], refers to harmonicity of $\Omega$: as $\Omega$ is self-dual, the condition $d\Omega = 0$ implies $d^*\Omega = 0$, which means $\Omega$ is a harmonic 4-form.

Quaternionic Kähler manifolds are very rigid objects. In fact, there are no non-symmetric compact quaternionic Kähler 8-manifolds of positive scalar curvature, and a conjecture by LeBrun and Salamon [17] asserts that the same result holds in dimensions $4n \geq 12$. This scarceness of examples motivates the study of almost quaternionic Hermitian geometries which are, in some sense, close to being quaternionic Kähler. In dimension 8, harmonic $\text{Sp}(2)\text{Sp}(1)$-structures constitute one such class.

It was Salamon [23] who first constructed an example of a harmonic $\text{Sp}(2)\text{Sp}(1)$-structure. The basic idea, which was developed further by Giovannini in his PhD thesis [11] (see also [3]), uses the identification of

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admitting a harmonic SO \((\alpha, \beta)\) similar obstructions in our setting so as to classify nilpotent Lie algebras \(\alpha\)-forms that the the form \(v\) tractable than expected. The main complication originates from the fact that, at the infinitesimal level, the SO \(\alpha\)-structure \(\beta\) stabilisers are different copies of the non-compact exceptional Lie group \(G_2\) with intersection isomorphic to SO\((4)\). Closedness of \(\Omega\) is enforced by requiring that both of the forms \(\alpha\) and \(\beta\) are closed. This condition, \(d\alpha = 0 = d\beta\), can be rephrased in terms of the intrinsic torsion \(\xi \in T^*N \otimes so(4)^{-1}\); it forces \(\xi\) to take its values in a 49-dimensional submodule. As \(*\alpha = \beta\), we have dubbed such SO\((4)\)-structures as harmonic.

Whilst our construction recovers one of the known examples of a harmonic Sp\((2)\) Sp\((1)\)-structure, it also enables us to find several new examples. More precisely, we show that if \(N_i, 1 \leq i \leq 11\), is a nilpotent Lie group associated with the algebra \(n_i\) in Table 1, then \(N_i\) admits a harmonic SO\((4)\)-structure, hence \(K_i = \mathbb{R} \times N_i\) admits a harmonic Sp\((2)\) Sp\((1)\)-form. By choosing a lattice \(\Gamma_i \subset K_i\), we obtain 11 examples of compact harmonic Sp\((2)\) Sp\((1)\)-manifolds. Except for the case corresponding to \((0, 0, 0, 0, 0, 0, 12, 13)\), these examples are new.

**Theorem 1.1.** Each of the nilmanifolds \(M_i = \Gamma_i \backslash K_i, 1 \leq i \leq 11\), admits a harmonic Sp\((2)\) Sp\((1)\)-structure.

As these nilmanifolds are not Einstein (cf. [19]), \(\Omega\) cannot be parallel. In fact, for any Sp\((2)\) Sp\((1)\)-structure on \(\mathbb{R} \times N\) induced by an SO\((4)\)-structure on \(N\), \(\Omega\) can be parallel only if the associated metric is hyper-Kähler. In this case, the underlying SO\((4)\)-structure is torsion-free, i.e., \(\xi = 0\). In the compact setting this implies that \(N\) is finitely covered (via a local isometry) by a flat torus or the product of a flat torus with a K3 surface (cf. Theorem 2.8).

Harmonic Sp\((2)\) Sp\((1)\)-structures appear to be fairly rigid objects. Indeed, case-by-case computations for the above nilmanifold examples show that, at the infinitesimal level, the SO\((8)\) orbit of the Sp\((2)\) Sp\((1)\)-structure contains no other harmonic structures than those obtained by Lie algebra automorphisms.

On a 7-dimensional nilpotent Lie algebra, there is a very useful obstruction to the existence of a calibrated G\(_2\)-structure [7 Lemma 3]. As the forms \(\alpha\) and \(\beta\) are very similar to the calibrated G\(_2\)-form, in that they have open GL\((7, \mathbb{R})\)-orbit and are calibrated, one might be tempted to look for similar obstructions in our setting so as to classify nilpotent Lie algebras admitting a harmonic SO\((4)\)-structures. This, however, turns out to be less tractable than expected. The main complication originates from the fact that the the form \(v \mapsto (v \lrcorner \alpha) \land (v \lrcorner \alpha) \land \alpha\), similarly for \(\beta\), has split signature. In addition to being closed and non-degenerate, the forms \(\alpha\) and \(\beta\) are required to satisfy certain compatibility conditions. This is reminiscent
of the case of half-flat SU(3)-structures, defined by a pair \((\psi, \sigma)\) of closed forms (cf. [13]). Nilmanifolds with a half-flat structure were classified in [3], using a cohomological obstruction based on the fact that a simple 2-form \(\gamma\) satisfying \(\gamma \wedge \psi = 0 = \gamma \wedge \sigma\) is necessarily zero. However, this way of expressing the compatibility conditions is not appropriate in the case of SO(4), because the equation \(\gamma \wedge \alpha = 0\) alone implies \(\gamma = 0\). In summary, a classification of harmonic SO(4)-structures on nilmanifolds remains an open problem.

As already mentioned, compact 7-manifolds with an SO(4)-structure whose holonomy reduces are very rare. It is therefore natural to consider generalisations of the torsion-free condition, e.g., the case of invariant intrinsic torsion. In our case, this means \(\xi\) lies in a trivial SO(4)-submodule of \(T^*N \otimes \mathfrak{so}(4)^{1}\); this condition is complementary to the harmonic condition in the sense that an SO(4)-structure which satisfies both criteria is torsion-free. Whilst being relatively unexplored, the invariant intrinsic torsion setting is already known to include interesting geometries such as nearly-Kähler manifolds and nearly parallel G\(_2\)-manifolds (see also [4]). As it is the case for these latter two types of geometries, we show (Proposition 4.1) that any SO(4)-structure with invariant intrinsic torsion, in fact, has constant torsion, i.e., \(\xi\) is independent of \(n \in N\).

In the final part of the paper, we explain the principle behind a potential classification of invariant intrinsic torsion SO(4)-structures on Lie groups, and use this approach to provide a number of examples. The basic idea is that for a Lie group \(H\), any left-invariant SO(4)-structure is obtained

\[
\begin{array}{cccc}
\mathbf{n}_1 & (0, 0, 0, 12, 23, -13, 26 - 34 - 16 + 25) & (3, 7, 10) \\
\mathbf{n}_2 & (0, 0, 12, 13, 12) & (5, 13, 21) \\
\mathbf{n}_3 & (0, 0, 0, 0, 12, 13 + 24, 14) & (4, 11, 16) \\
\mathbf{n}_4 & (0, 0, 0, 0, 12 + 34, 23, 24) & (4, 11, 17) \\
\mathbf{n}_5 & (0, 0, 0, 0, 12, 13, 34, 0) & (4, 11, 16) \\
\mathbf{n}_6 & (0, 0, 0, 0, 0, 12, 13, 13 + 14, 25 + 34) & (3, 6, 10) \\
\mathbf{n}_7 & (0, 0, 0, 0, 12, 13, 15 + 35, 25 + 34) & (3, 6, 10) \\
\mathbf{n}_8 & (0, 0, 0, 0, 0, 0, 12, 13, 14 + 23) & (5, 12, 18) \\
\mathbf{n}_9 & (0, 0, 0, 0, 0, 12, 13, 14 + 23) & (5, 12, 18) \\
\mathbf{n}_{10} & (0, 0, 0, 0, 0, 13 + 42, 14 + 23) & (5, 12, 18) \\
\mathbf{n}_{11} & (0, 0, 0, 0, 0, 0, 12, 13, 14 + 25) & (4, 9, 13) \\
\end{array}
\]

**Table 1.1.** The Lie algebras \(\mathfrak{n}_i = \mathbb{R} \oplus \mathfrak{n}_i\), with \(\mathfrak{n}_i\) listed above, all admit a harmonic Sp\((2)\)Sp\((1)\)-structure. The rightmost column gives the Betti numbers \(b_1(\mathfrak{n}_i), b_2(\mathfrak{n}_i), b_3(\mathfrak{n}_i)\). The notation should be understood as follows. The dual \(\mathfrak{n}_i^*\) of the Lie algebra \(\mathfrak{n}_i\) has a basis \(\mathfrak{n}_1, \ldots, \mathfrak{n}_7\) such that \(d\mathfrak{n}_1 = \cdots = d\mathfrak{n}_5, d\mathfrak{n}_6 = \mathfrak{n}_1 \wedge \mathfrak{n}_2\) and \(d\mathfrak{n}_7 = \mathfrak{n}_1 \wedge \mathfrak{n}_3\).
by specifying an adapted coframe $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4, w^1, w^2, w^3$ of the Lie algebra $\mathfrak{h}$. This structure has invariant intrinsic torsion if and only if the torsion, $\tau$, of the flat connection on $\mathbf{H}$ lies in $2\mathbb{R} \oplus \partial \mathbb{R}^7 \otimes \mathfrak{so}(4)$, where $\partial$ denotes the canonical alternating map (see (4.1)). In order to obtain concrete examples, we pick an adapted coframe and make a suitable ansatz for the form of $\tau$. Subsequently, we check that the equations specified by $\tau$ actually determine a Lie algebra structure.

For the groups $\mathbf{H}_i$, corresponding to the Lie algebras appearing in Table 1.2, we also describe the type of the associated product $\text{Sp}(2) \text{Sp}(1)$-structure on $S^1 \times \mathbf{H}_i$. On each of these Lie groups, the same adapted frame that determines the $\text{Sp}(2) \text{Sp}(1)$-structure also determines a reduction to $\text{Sp}(2)$.

In particular, $S^1 \times \mathbf{H}_i$ is an almost hyper-Hermitian manifold, meaning Riemannian with a compatible almost hypercomplex structure. Whilst many of the examples are, in fact, (integrable) hyper-Hermitian, only two of them are hyper-Kähler (hence, necessarily, flat).

In summary, our analysis of invariant intrinsic torsion $\text{SO}(4)$-structures leads to:

**Theorem 1.2.** Let $\mathbf{H}$ be a Lie group whose Lie algebra $\mathfrak{h}$ belongs to the list in Table 1.2. The left-invariant $\text{SO}(4)$-structure on $\mathbf{H}$ determined by the coframe $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3, \mathbf{e}^4, w^1, w^2, w^3 \in \mathfrak{h}^*$ has invariant intrinsic torsion.

If $\mathbf{H}$ corresponds to either $\mathfrak{h}_1$ or $\mathfrak{h}_5$, then the associated product structure on $S^1 \times \mathbf{H}$ is flat and hyper-Kähler. In the remaining cases, the non-zero components of the intrinsic torsion are $\mathbb{E}H$ and $\mathbb{K}H$, meaning these are quaternionic Hermitian structures which are not quaternionic Kähler.

2. Dimensional reduction

The key ingredient in our construction of harmonic $\text{Sp}(2) \text{Sp}(1)$-structures is a “dimensional reduction” in the sense that we break down the 8-dimensional geometry into $7 + 1$ dimensions and identify a suitable subgroup

$$\text{SO}(4) \cong \text{SO}(7) \cap \text{Sp}(2) \text{Sp}(1)$$

that characterises the 7-dimensional building blocks.

In terms of representation theory, it is useful to keep in mind the isomorphism

$$\text{SO}(4) \cong \text{Sp}(1) \otimes_{\mathbb{Z}/2\mathbb{Z}} \text{Sp}(1) = \text{Sp}(1) \text{Sp}(1).$$

Then it is clear that the real irreducible representations of $\text{SO}(4)$ take the form

$$S^{p,q} := [S^p H_+ \otimes S^q H_-],$$

where $p + q$ is an even number, $\dim S^{p,q} = (p + 1)(q + 1)$, and the square brackets, $[V]$, are used to indicate that we are considering the real vector space underlying $V$.

The 7-dimensional representation of $\text{SO}(4)$ can now be expressed as

$$T = \mathbb{R}^7 \cong \mathbb{R}^4 \otimes \Lambda_2 = S^{1,1} \oplus S^{2,0}.$$  

For later computations, it is worth recalling (see, e.g., [21, Theorem 1.3]) that the tensor product of $\text{SO}(4)$ representations can be worked out via
the formula

\[ S^{p,q} \otimes S^{r,s} \cong \bigoplus_{m,n=0}^{\min(p,r),\min(q,s)} S^{p+r-2m,q+s-2n}. \]

Let \( e_1, \ldots, e_7 \) denote the standard basis of \( \mathbb{R}^7 \) and \( e^1, \ldots, e^7 \) the dual basis of \( T^* \). Via the inclusion \( SO(4) \subset SO(7) \) we obtain splittings

\[ T^* \cong (\mathbb{R}^4)^* \times (\mathbb{R}^3)^* = \Lambda^{1,0} \oplus \Lambda^{0,1}, \quad \Lambda^k T^* \cong \bigoplus_{p+q=k} \Lambda^{p,q}. \]

In particular, we can naturally identify \((e^1, e^2, e^3, e^4)\) with \((\mathbb{R}^4)^* \subset T^*\), and so forth. Consider now the pair of forms \((\alpha, \beta) \in \Lambda^3 T^* \times \Lambda^4 T^*\) given by

\[
\begin{align*}
\alpha &= \omega_1 \wedge w^1 + \omega_2 \wedge w^2 + \omega_3 \wedge w^3 - 3w^{123}, \\
\beta &= -\omega_1 \wedge w^{23} - \omega_2 \wedge w^{31} - \omega_3 \wedge w^{12} - 3w^{1234},
\end{align*}
\]

where we have fixed a basis of the space \( \Lambda^2(\mathbb{R}^4)^* = \Lambda^2_+ \oplus \Lambda^2_- \) of the form

\[
\begin{align*}
\omega_1 &= e^{12} - e^{24}, \\
\omega_2 &= e^{13} - e^{42}, \\
\omega_3 &= e^{14} - e^{23},
\end{align*}
\]

\[
\begin{align*}
\omega_1 &= e^{12} - e^{24}, \\
\omega_2 &= e^{13} + e^{42}, \\
\omega_3 &= e^{14} + e^{23},
\end{align*}
\]

whilst \( w^s = e^{4+s}, s = 1, 2, 3 \), span the annihilator of \( \mathbb{R}^4 \subset \mathbb{R}^4 \times \mathbb{R}^3 \).

The forms \( \alpha \) and \( \beta \) are stable in the sense that the associated \( GL(7, \mathbb{R}) \)-orbits in \( \Lambda^k T^* \) are open. Straightforward computations show:

\[
\begin{array}{c|c|c}
\mathfrak{h}_1 & de^i = \sum_{j,s}(e_i \wedge \omega_s) \wedge w^j, dw^s = 0 & \\
\mathfrak{h}_2 & de^i = 0, dw^s = w_{s,3}w^{123} & \\
\mathfrak{h}_3 & de^i = 0, dw^s = \omega_s & \\
\mathfrak{h}_4 & de^i = \sum_{j,s}(e_i \wedge \omega_s) \wedge w^j, dw^s = -2w_{s,3}w^{123} & \\
\mathfrak{h}_5 & de^1 = 0 = de^2, de^3 = e^{14}, de^4 = -e^{13}, & dw^1 = a\omega_1, dw^2 = a\omega_2 + e^1 \wedge w^3, dw^3 = a\omega_3 - e^1 \wedge w^2 & \\
\mathfrak{h}_6 & de^1 = 0 = de^2, de^3 = e^{14}, de^4 = -e^{13}, & dw^1 = a\omega_1, dw^2 = a\omega_2 + e^1 \wedge w^3, dw^3 = a\omega_3 - e^1 \wedge w^2 & \\
\mathfrak{h}_7 & de^1 = 0, de^2 = e^{12}, de^3 = e^{13} + \kappa e^{14}, de^4 = e^{14} - \kappa e^{13}, & dw^1 = a\omega_3 - 1\alpha w_{s,3}w^{123} + \kappa (\omega^1 - e^2) \wedge w_{s,3}w^{23} & \\
\mathfrak{h}_8 & de^1 = 0, de^2 = e^{12} + \kappa e^{13}, de^3 = -\kappa e^{12} + e^{13}, de^4 = 2\omega_3, & dw^1 = a\omega_3 + (\kappa e^1 - e^4) \wedge w^2 + 2e^3 \wedge w^3 - 3\omega_{23}, & dw^2 = a\omega_2 + (-\kappa e^1 + e^4) \wedge w^1 - 2e^2 \wedge w^3 - 3\omega_{23}, & dw^3 = a\omega_3 - 2e^3 \wedge w^1 + 2e^2 \wedge w^2 - 3\omega_{12}.
\end{array}
\]
Lemma 2.1. The stabilisers of $\alpha$ and $\beta$ in $\mathfrak{gl}(7, \mathbb{R})$ are two different copies of $\mathfrak{g}_2^*$ that intersect in a copy of $\mathfrak{so}(4)$.

Proof. We have the well-known decomposition $\text{End} \ T \cong \mathbb{R} \oplus S^2_0 T \oplus \Lambda^2 T$, where
\[ S^2_0 T \cong \mathbb{R} \oplus S^{1,1} \oplus S^{2,2} \oplus S^{0,4} \oplus S^{1,3}, \quad \Lambda^2 T \cong \mathfrak{so}(4) \oplus S^{0,2} \oplus S^{1,1} \oplus S^{1,3}. \]

The stabilisers of $\alpha$ and $\beta$ are computed with respect to the action that identifies $\Lambda^2 T^*$ with $\mathfrak{so}(7)$ by letting $\Lambda^2 T^*$ act on $T$ as
\[ \Lambda^2 T^* \times T \ni (\omega, v) \mapsto v \lrcorner \omega, \quad (2.3) \]
meaning $e^{12}$ is identified with $e^1 \otimes e_2 - e^2 \otimes e_1$, and so forth.

Straightforward computations, using $(2.3)$, now show that the copy
\[ \mathfrak{so}(4) \cong \langle \omega_1, \omega_2, \omega_3 \rangle \oplus \langle \omega_1 - 2w^{23}, \omega_2 - 2w^{31}, \omega_3 - 2w^{12} \rangle, \quad (2.4) \]
n annihilates $\alpha$ and $\beta$.

In addition to this copy of $\mathfrak{so}(4)$, each of the forms $\alpha$ and $\beta$ is annihilated by a copy of the module $S^{1,3}$; neither of these (different) copies of $S^{1,3}$ is contained in $\mathfrak{so}(7)$. In particular, we conclude that each form has stabiliser $\mathfrak{g}_2^*$ rather than $\mathfrak{g}_2$. \qed

Remark 2.2. Starting from an $\text{SO}(4)$-structure $(\alpha, \beta)$, with associated metric $g = g^H + g^V$, we may use the splitting of $T$ into $\mathbb{R}^4 \times \mathbb{R}^3$ so as to obtain a family of metrics. Up to conformal transformation, this family is obtained by rescaling in the fibre directions $V$, meaning $w^\lambda \mapsto \lambda w^\lambda =: \tilde{w}^\lambda$, where $\lambda > 0$. The metric $\tilde{g}$ and defining forms, $(\tilde{\alpha}, \tilde{\beta})$, associated with this rescaling are:
\[ \tilde{g} = g^H + \lambda^2 g^V, \quad \tilde{\alpha} = \lambda(\omega_1 \wedge w^1 + \omega_2 \wedge w^2 + \omega_3 \wedge w^3 - 3\lambda^3 w^{123}), \]
\[ \tilde{\beta} = \lambda^2(-\omega_1 \wedge w^{23} - \omega_2 \wedge w^{31} - \omega_3 \wedge w^{12}) - 3\lambda e^{1234}. \]

From the pair $(\alpha, \beta)$ we construct a 4-form $\Omega$ on $\mathbb{R}^8 \cong \langle e_8 \rangle \oplus T$ as follows:
\[ \Omega = \alpha \wedge e^8 + \beta = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad (2.5) \]
where
\[ \sigma_1 = \omega_1 + e^{58} - e^{67}, \quad \sigma_2 = \omega_2 + e^{68} - e^{75}, \quad \sigma_3 = \omega_3 + e^{78} - e^{56}. \quad (2.6) \]
It is well known [20, Lemma 9.1] that $\Omega$ has stabiliser $\text{Sp}(2) \text{Sp}(1)$ in $\text{GL}(8, \mathbb{R})$.

Conversely, consider a manifold $M$ endowed with an $\text{Sp}(2) \text{Sp}(1)$-structure whose associated fundamental form is given by $2\Omega = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$. Let $i: N \to M$ be an oriented hypersurface, with induced metric $\tilde{g}$, and suppose the unit normal direction is given by $n = -e_8$. Then
\[ \tilde{\beta} = i^* \Omega = -(e^{12} - e^{34}) \wedge e^{67} - (e^{13} - e^{42}) \wedge e^{75} - (e^{14} - e^{23}) \wedge e^{56} - 3e^{1234}, \]
and its Hodge star is
\[ \alpha = i^*(n \lrcorner \Omega) = (e^{12} - e^{34}) \wedge e^5 + (e^{13} - e^{42}) \wedge e^6 + (e^{14} - e^{23}) \wedge e^7 - 3e^{567}. \]

Consequently, $N$ inherits an $\text{SO}(4)$-structure which is determined by fixing any two of $\alpha$, $\beta$, and $g$. 

As already mentioned, the 7-dimensional representation of SO(4) can be written as \( \mathbb{R}^4 \oplus \Lambda_2^2 \cong S^{1,1} \oplus S^{2,0} \), and in the proof of Lemma 2.1 we used this fact to write:

\[
\Lambda^2 T \cong 2S^{0,2} \oplus S^{2,0} \oplus S^{1,1} \oplus S^{1,3} \cong \Lambda^5 T.
\]

Another straightforward computation shows that

\[
\Lambda^3 T \cong 2\mathbb{R} \oplus S^{0,2} \oplus S^{0,4} \oplus 2S^{1,1} \oplus S^{1,3} \oplus S^{2,2} \cong \Lambda^4 T.
\]

(2.7)

In particular, the space of invariant forms \((\Lambda^* T^*)_{SO(4)}\) is 4-dimensional. Indeed, the decomposable forms \(v = e^{1234}\) and \(*v = w^{123}\) are clearly invariant.

The pair of forms \(\alpha + 4*v, \beta + 4*v \in (\Lambda^* T^*)_{SO(4)}\) are distinguished by having stabiliser the same copy of \(\mathfrak{g}_2 \subset \mathfrak{so}(7)\); this is an alternative way to phrase the fact that given any SO(4)-structure, the inclusion \(\text{SO}(4) \subset \mathfrak{g}_2\) determines a natural \(\mathfrak{g}_2\)-structure.

On a 7-manifold with an SO(4)-structure, defined as above, SO(7) acts on the pair \((\alpha, \beta)\) with stabiliser SO(4). Differentiation therefore gives a map

\[
\mathfrak{so}(7) \to \Lambda^3 T^* \oplus \Lambda^4 T^*, \ A \mapsto A \cdot \alpha + A \cdot \beta
\]

whose kernel is \(\mathfrak{so}(4)\). In particular, we have an inclusion of the orthogonal complement of \(\mathfrak{so}(4)\) in \(\mathfrak{so}(7)\):

\[
\mathfrak{so}(4)\perp \hookrightarrow \Lambda^3 T^* \oplus \Lambda^4 T^*, \ A \mapsto A \cdot \alpha + A \cdot \beta.
\]

The intrinsic torsion \(\xi\) of an SO(4)-structure takes values in the 105-dimensional space

\[T^* \otimes \mathfrak{so}(4)\perp \cong 2\mathbb{R} \oplus 3S^{0,2} \oplus 2S^{0,4} \oplus 3S^{1,1} \oplus 3S^{1,3} \oplus S^{1,5} \oplus S^{2,0} \oplus 2S^{2,2} \oplus S^{2,4},\]

and, due to the above inclusion, the map

\[
\xi \mapsto \nabla \alpha + \nabla \beta = \xi \cdot \alpha + \xi \cdot \beta
\]

(2.8)

is an isomorphism.

If we let \(a\) denote the alternation map \(T^* \otimes \Lambda^r T^* \to \Lambda^{r+1} T^*\), we have the well-known relation \(d = a \circ \nabla\). Taking into account the isomorphism (2.8), one expects that the exterior derivatives of \(\alpha\) and \(\beta\) could encode valuable information about \(\xi\). Indeed, a computation shows that \((d\alpha, d\beta)\) encodes the maximal possible amount of information in the following sense.

**Lemma 2.3.** The map

\[
T^* \otimes \mathfrak{so}(7) \to \Lambda^4 T^* \oplus \Lambda^5 T^*, \ e^i \otimes A \mapsto e^i \wedge A \cdot \alpha + e^i \wedge A \cdot \beta
\]

(2.9)

is surjective.

**Proof.** Set \(\tilde{T} := T \oplus (e_8)\), and write \(\Omega = \alpha \wedge e^8 + \beta\). The map appearing in the statement can be identified with the restriction to \(T^* \otimes \mathfrak{so}(7)\) of

\[
\tilde{T}^* \otimes \mathfrak{so}(8) \to \Lambda^5 \tilde{T}^*, \ e^i \otimes A \mapsto e^i \wedge A \cdot \Omega.
\]
By [24], this map is surjective with kernel $KS^3H \oplus \tilde{T}^* \otimes (sp(2) \oplus sp(1))$, and the component $KS^3H$ can be identified with the kernel of the skew-symmetrisation map $a: \tilde{T}^* \otimes (sp(2) \oplus sp(1)) \rightarrow \Lambda^3 \tilde{T}^*$. It therefore suffices to prove that the composition

$$T^* \otimes so(7) \xrightarrow{\pi} \tilde{T}^* \otimes (sp(2) \oplus sp(1)) \rightarrow \Lambda^3 \tilde{T}^*$$

is surjective.

If we write $so(7)$ as

$$so(7) = so(4) \oplus so(3) \oplus \mathbb{R}^4 \otimes \mathbb{R}^3, \quad so(3) = \langle \omega_1 + 2e^{67}, \omega_2 + 2e^{75}, \omega_3 + 2e^{56} \rangle,$$

we see that the projection $\pi$ is zero on $T^* \otimes so(4)$ and is the inclusion on $T^* \otimes so(3)$. On $T^* \otimes \mathbb{R}^4 \otimes \mathbb{R}^3$, it has the form

$$e^c \otimes e^{ab} \mapsto e^c \otimes \frac{1}{4}(3e^{ab} - e_{a \cup b} \sigma_a \wedge e_b \wedge \sigma_b).$$

Clearly, the restriction of $a \circ \pi$ to $\langle e^8 \rangle \otimes \mathbb{R}^4 \otimes \mathbb{R}^3$ is injective, with image $e^8 \wedge \Lambda^{1,1}$. It follows that

$$a \circ \pi: \langle (e^8) \oplus \Lambda^{0,1} \rangle \otimes \mathbb{R}^4 \otimes \mathbb{R}^3 \rightarrow \Lambda^{1,2} \oplus e^8 \wedge \Lambda^{1,1}$$

is surjective. Likewise,

$$a \circ \pi: \Lambda^{1,0} \otimes \mathbb{R}^4 \otimes \mathbb{R}^3 \rightarrow \Lambda^{2,1} \oplus e^8 \wedge \Lambda^{2,0}$$

is an isomorphism. We conclude that $a \circ \pi$ is surjective by considering its restrictions to $\Lambda^{1,0} \otimes so(3)$ and $\langle (e^8) \oplus \Lambda^{0,1} \rangle \otimes so(3)$.

As a consequence of the above observation, $da$ and $d\beta$ can only both be zero if $\tilde{\xi}$ takes values in the modules

$$S^{0,4} \oplus S^{1,3} \oplus S^{1,5} \oplus S^{2,2} \oplus S^{2,4}.$$

In summary:

**Proposition 2.4.** The conditions $da = 0 = d\beta$ characterise $SO(4)$-structures whose intrinsic torsion takes values in the 49-dimensional submodule

$$\mathcal{W} = \mathcal{W}^{0,4} \oplus \mathcal{W}^{1,3} \oplus \mathcal{W}^{1,5} \oplus \mathcal{W}^{2,2} \oplus \mathcal{W}^{2,4}$$

with each component being isomorphic to the $SO(4)$-module $S^{p,q}$ with the same indices.

We shall say an $SO(4)$-structure is harmonic if the forms $\alpha$ and $\beta$ are closed. This terminology reflects the fact that, as $\alpha = \ast \beta$, the conditions $da = 0 = d\beta$ imply that $\alpha$ and $\beta$ are harmonic forms.

There are obvious obstructions to the existence of a harmonic $SO(4)$-structure on a 7-manifold:

**Proposition 2.5.** If $N$ is a 7-manifold with an $SO(4)$-structure, then $N$ is spin.

If, in addition, $N$ is compact and the structure is harmonic then $H^3(N)$ and $H^4(N)$ are non-trivial.

**Proof.** The first assertion follows since $SO(4)$ is contained in $G_2$. The statement concerning harmonic structures follows by observing that $\alpha \wedge \beta$ is a volume form; in particular, neither $\alpha$ nor $\beta$ can be exact. \qed
Proposition 2.6. For a harmonic $\text{SO}(4)$-structure, one can identify $d\nu$ with the component $W^{1,3}$ of the intrinsic torsion, and $d^*\nu$ with the components $W^{2,2} \oplus W^{0,4}$.

Proof. At each point, $d\nu$ is determined by the intrinsic torsion via an $\text{SO}(4)$-equivariant map. By Schur’s lemma, this implies that $d\nu$ takes values in the component isomorphic to $S^{1,3}$. Computing the rank of $T^* \otimes \text{so}(7) \to \Lambda^4 T^* \oplus \Lambda^5 T^*$, $e^i \otimes A \mapsto e^i \wedge A \cdot \alpha + e^i \wedge A \cdot \beta + e^i \wedge A \cdot \nu$ we conclude that the kernel of (2.9) does not contain this $S^{1,3}$.

A similar argument applies for $d^*\nu$. □

Following [2], we shall say a subgroup $G$ of $\text{GL}(n, \mathbb{R})$ is admissible if $G$ is the largest subgroup that fixes the space $(\Lambda^n \mathbb{R}^n)^G$ of invariant forms on $\mathbb{R}^n$. $G$ is strongly admissible if, on $G$-structures, the closedness of the invariant forms is equivalent to the vanishing of the intrinsic torsion. For instance, $\text{Sp}(2)\text{Sp}(1) \subset \text{SO}(8)$ is admissible, but not strongly admissible. Similarly, the above observations imply:

Corollary 2.7. $\text{SO}(4) \subset \text{SO}(7)$ is admissible, but not strongly admissible. □

As a final general remark on $\text{SO}(4)$-structures, note that for any given harmonic structure, the product $\text{Sp}(2)\text{Sp}(1)$-structure can be Einstein only when it is Ricci-flat, due to flatness of the $S^1$ factor. Thus, the holonomy can reduce to $\text{Sp}(2)\text{Sp}(1)$ only when the 8-manifold is hyper-Kähler, and so the 7-manifold has holonomy contained in $\text{SO}(4)$. On the other hand, if the holonomy reduces to $\text{SO}(4)$, the product 8-manifold is hyper-Kähler, and so the Ricci tensor is zero.

In the compact case, we can use standard arguments from Riemannian geometry to completely characterise the 7-manifolds whose holonomy reduces to $\text{SO}(4)$:

Theorem 2.8. Let $N$ be a compact 7-manifold with a torsion-free $\text{SO}(4)$-structure. Then, up to a finite cover, $N$ is isometric to $T^3 \times K$, where $T^3$ is a flat torus, and $K$ is either a flat torus or a K3 surface.

Proof. By a splitting theorem due to Cheeger and Gromoll (see [1, Corollary 6.67]), the universal cover of $N$ has the form $\tilde{N} \times \mathbb{T}^k$, where $\tilde{N}$ is a compact, simply-connected Ricci-flat manifold, and $\mathbb{T}^k$ is a flat torus. If $N$ is flat then $k = 7$, so $N$ is covered by a torus. Otherwise, since $\tilde{N} \times \mathbb{T}^k \to N$ is a local isometry, $\tilde{N} \times \mathbb{T}^k$ has non-trivial holonomy contained in $\text{SO}(4)$. As the holonomy representation is reducible, the holonomy of $\tilde{N} \times \mathbb{T}^k$, hence of $\tilde{N}$, is in fact contained in $\text{SU}(2)$. In addition, $\tilde{N}$ is irreducible, otherwise de Rham’s theorem would give $\tilde{N} \cong \tilde{N}' \times \mathbb{R}$. Also note that $\tilde{N}$ cannot be locally symmetric, by Ricci-flatness. According to Berger’s holonomy classification, this implies that $\tilde{N}$ is a 4-dimensional manifold with holonomy $\text{SU}(2)$, accordingly a K3 surface (cf. [16, Theorem 7.3.13]). □

3. Explicit harmonic $\text{SO}(4)$-structures

We have already noticed that any $\text{SO}(4)$-structure determines a $G_2$-structure with the same underlying metric. Hence, it might be tempting
to look for harmonic $SO(4)$-structures on the total space of the bundle of anti-self-dual forms over a self-dual 4-manifold [20]. By duplicating and modifying the construction of $G_2$-metrics in [20, Theorem 11.10], we can find an $SO(4)$-structure on a domain of the total space of the bundle $\Lambda^2 T^*X \xrightarrow{\pi} X$ over a self-dual 4-manifold $X$. In contrast with the $G_2$ case, however, the closedness of the defining forms lead to two incompatible equations when $X$ has non-zero scalar curvature. More precisely, we can always achieve either $da = 0$ or $d\beta = 0$, separately, but not $da = 0 = d\beta$. In addition, the examples always have $dv = 0$; one finds that $d*v$ vanishes if and only if $X$ has zero scalar curvature. The fact that we can always achieve $d\beta = 0 = dv$ is clearly equivalent to the following result by Salamon (cf. [22]):

**Proposition 3.1.** If $X$ is a self-dual 4-manifold, an open subset of $\Lambda^2 T^*X$ admits a cocalibrated $G_2$-structure. \hfill $\square$

3.1. **Proof of Theorem 1.1.** Another possible setting for producing examples is that of Lie algebras. In fact, we have found two ways of manufacturing harmonic $SO(4)$-structures on semidirect products of Lie algebras. In the following, we explain these constructions and use them to exhibit a number of harmonic structures on nilpotent and solvable algebras; the nilpotent ones are those appearing in Table 1.1.

The construction methods are based on relations between harmonic $SO(4)$-structures and certain types of half-flat structures on hypersurfaces thereof. Recall that on $\mathbb{R}^3$ we can define an $SU(2,1)$- or $SL(3, \mathbb{R})$-structure by the stabiliser of the forms

$$\omega = e^{12} + e^{34} + e^{56}, \quad \psi^+ = \text{Re}(e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 - ie^6)$$

or

$$\omega = e^{12} + e^{34} + e^{56}, \quad \rho = e^{135} + e^{246},$$

respectively. In each case another 3-form is induced by

$$\psi^- = \text{Im}(e^1 + ie^4) \wedge (e^2 + ie^5) \wedge (e^3 - ie^6)$$

and $\rho = e^{135} - e^{246}$.

The adjective “half-flat” refers to the condition that defining forms $\psi^+$ (resp. $\rho$) and $\omega^2$ are closed.

Notice that if we take an $SU(2,1)$-structure and fix an extra 2-form, say $\gamma = e^{56}$, the structure group is reduced to $S(U(2) \times U(1))$. Also observe that the structure group $SL(3, \mathbb{R})$ fixes an additional 3-form, namely

$$\gamma = (e^1 - e^2) \wedge (e^3 - e^4) \wedge (e^5 - e^6).$$

Our first construction of harmonic structures is as follows.

**Proposition 3.2.** Let $(\alpha, \beta)$ be a harmonic $SO(4)$-structure on a Lie algebra $\mathfrak{g}$. Let $\eta$ be a unit 1-form in $\Lambda^0 \mathfrak{g}$ such that $\mathfrak{n} = \ker \eta$ is a subalgebra (i.e. $d\eta \wedge \eta = 0$). Then $\mathfrak{n}$ has an induced half-flat $S(U(2) \times U(1))$-structure such that

$$\alpha|_{\mathfrak{n}} = \sqrt{3}\psi^+, \quad \beta|_{\mathfrak{n}} = \frac{3}{2}\omega^2, \quad \gamma = \frac{1}{8}(\eta^4 \cdot \alpha - \omega).$$

Assume, in addition, that $\mathfrak{n}$ is an ideal in $\mathfrak{g}$ (i.e $\eta$ is closed). Then the derivation $b \in \text{Der}(\mathfrak{n})$ associated to the adjoint action of $\sqrt{3}\eta^4$ on $\mathfrak{n}$ satisfies

$$b \cdot \psi^+ - d\omega - 8d\gamma = 0 = \frac{1}{2}b \cdot \omega^2 + d\psi^-.$$

(3.1)
Conversely, given a 6-dimensional Lie algebra $\mathfrak{n}$ with a derivation $b$ and a half-flat $S(U(2) \times U(1))$-structure satisfying (3.1). Then the semidirect product $\mathfrak{g} = \mathfrak{n} \ltimes \mathbb{R}$ has a harmonic $SO(4)$-structure.

Proof. Choose an adapted frame $e^1, \ldots, e^7$ such that $\eta = e^7$. Then

$$a|_n = \omega_1 \wedge e^5 + \omega_2 \wedge e^6, \beta|_n = -\omega_3 \wedge e^{56} - 3e^{1234},$$

and consequently

$$\psi^+ = \text{Re}(e^1 + ie^4) \wedge (e^2 - ie^3) \wedge (\sqrt{\frac{1}{3}}e^5 + \sqrt{\frac{1}{3}}e^6),$$

$$\omega = e^{14} - e^{23} - \frac{1}{3}e^{56}, \gamma = -\frac{1}{3}e^{56}.$$

These forms determine an $S(U(2) \times U(1))$-structure on $\mathfrak{n}$ which has an adapted coframe $(E^1, \ldots, E^6) = (e^1, e^4, e^2, -e^3, \sqrt{\frac{1}{3}}e^5, -\sqrt{\frac{1}{3}}e^6)$, and is half-flat because $\psi^+$ and $\omega^2$ are restrictions of closed forms.

If $\mathfrak{n}$ is an ideal in $\mathfrak{g}$, we can express the exterior derivative $d$ on the exterior algebra over the semidirect product $\mathfrak{g} = \mathfrak{n} \ltimes \mathbb{R}$ as

$$d\chi = \eta \wedge \text{ad}^*(\eta^2) \cdot \chi + d_n\chi = \tilde{\eta} \wedge b \cdot \chi + d_n\chi,$$

where $\tilde{\eta} = \frac{1}{\sqrt{3}}\eta$.

In terms of the adapted coframe, we have

$$\psi^- = \text{Im}(e^1 + ie^4) \wedge (e^2 - ie^3) \wedge (\sqrt{\frac{1}{3}}e^5 + \sqrt{\frac{1}{3}}e^6) = \frac{1}{\sqrt{3}}(e^{126} - e^{135} + e^{425} + e^{436}),$$

so that $\alpha$ and $\beta$ can be expressed as

$$\alpha = \sqrt{3}(\psi^+ + \tilde{\eta} \wedge (\omega + 8\gamma)), \beta = 3(\frac{1}{2}\omega^2 - \psi^- \wedge \tilde{\eta})$$

Thus

$$\eta^2 \cdot d\alpha = b \cdot \psi^+ - d_n\omega - 8d_n\gamma, \eta^4 \cdot d\beta = \sqrt{3}(\frac{1}{2}b \cdot \omega^2 - d_n\psi^-),$$

from which (3.1) follows.

For the last part of the statement, let $\eta$ be a 1-form on $\mathfrak{g} = \mathfrak{n} \ltimes \mathbb{R}$ that annihilates $\mathfrak{n}$ and satisfies $d\chi = \frac{1}{\sqrt{3}}\eta \wedge b \cdot \chi + d_n\chi$.

If $E^1, \ldots, E^6$ is a coframe on $\mathfrak{n}$, adapted to the given $S(U(2) \times U(1))$-structure, set

$$(e^1, \ldots, e^7) = (E^1, E^3, -E^4, E^2, \sqrt{3}E^5, -\sqrt{3}E^6, \eta).$$

The considerations above imply that the $SO(4)$-structure on $\mathfrak{g}$ associated to this coframe is harmonic. \hfill $\square$

Remark 3.3. The 2-form $\gamma$ can be expressed in terms of $*v$. Inspection shows that $\gamma = \eta^4 \cdot v$. In particular, it follows that if $d*v = 0$, then $\gamma$ is closed. In that case, (3.1) reads

$$b \cdot \psi^+ = d\omega, b \cdot \omega^2 = -2d\psi^-.$$

These equations are algebraic analogues of the flow equations one encounters in the study of half-flat $SU(3)$-structures in the context of $G_2$-holonomy metrics (cf. [13]).

The method described above enables us to construct harmonic $SO(4)$-structures on two nilpotent Lie algebras, corresponding to the first two algebras of Table [1.1].
3.1.1. The algebra $n_1$. The nilpotent Lie algebra
\[(0, 0, f^{27} + f^{17}, f^{12}, f^{14} + f^{23} + f^{57}, 0)\]
admits a harmonic $\text{SO}(4)$-structure whose adapted coframe is given by
\[(e^1, \ldots, e^7) = (f^1 + f^2, f^6, -f^3, f^3, \sqrt{3}f^4, -\sqrt{3}f^1, \sqrt{3}f^7).\]

In terms of the classifications of \[12\] \[74\], this Lie algebra is the distinguished member of the 1-parameter family of Lie algebras, denoted by 147E1, which carries a calibrated $G_2$-structure. Specifically, it can be written as
\[n_1 = (0, 0, 0, 12, 23, -13, 26 - 34 - 16 + 25).\]

3.1.2. The algebra $n_2$. The second example of a harmonic structure is found on the Lie algebra \((0, 0, 0, f^{27} + f^{12}, f^{23} + f^{13}, 0)\). The adapted coframe is
\[(e^1, \ldots, e^7) = (f^2 - f^6, f^1, -f^4, f^5, \sqrt{3}f^3, -\sqrt{3}f^6, \sqrt{3}f^7).\]

The underlying Lie algebra can also be expressed as $n_2 = (0, 0, 0, 0, 12, 13)$ and is clearly a decomposable Lie algebra. In fact, it can be decomposed as
\[\langle f^1, f^2, f^3, f^5, f^6, f^7 \rangle \oplus \langle f^4 \rangle;\]
the 6-dimensional component inherits an $\text{SO}(3)$-structure, in the sense of \[11\]. Its defining forms are
\[-e^{45} - e^{16} + e^{27}, e^{125} - e^{426} + e^{147} - 3e^{567}, -e^{467} - e^{175} - e^{256} + 3e^{124}.\]
The associated $\text{Sp}(2) \text{Sp}(1)$-structure is already known due to \[23\] \[11\].

In this case, there is a 4-dimensional subalgebra $\text{so}(3) \oplus \text{u}(1)$ of $\text{so}(8)$ that preserves the Lie algebra structure; its intersection with $\text{sp}(2) \oplus \text{sp}(1)$ is the 1-dimensional component. By construction, the infinitesimal action of $\text{so}(3)$ on the closed 4-form gives a space of closed forms; the infinitesimal orbit under the action of $\text{so}(8)$ contains no other closed form.

Similar techniques as in the proof of the Proposition \[3.2\] lead to:

**Proposition 3.4.** Let $(\alpha, \beta)$ be a harmonic $\text{SO}(4)$-structure on a Lie algebra $\g$. Let $\eta \in \Lambda^{1,0}$ be a 1-form on $\g$ such that $n = \ker \eta$ is a subalgebra. Then $n$ has an induced half-flat $\text{SL}(3, \mathbb{R})$-structure such that
\[a|_n = 4\sqrt{2} \rho, \beta|_n = -\frac{1}{8} \omega^2.\]
Assume, in addition, that $n$ is an ideal. Then the derivation $b \in \text{Der}(n)$ associated to $\text{ad}(\sqrt{2}\eta^3)$ satisfies
\[b \cdot \rho + d\omega = 0, -\frac{1}{3} b \cdot \omega^2 + d\check{\rho} + 2d\gamma = 0.\tag{3.2}\]

Conversely, given a 6-dimensional Lie algebra $n$ with a derivation $b$ and a half-flat $\text{SL}(3, \mathbb{R})$-structure satisfying \(3.2\), then the semidirect product $\g = n \rtimes \mathbb{R}$ has a harmonic $\text{SO}(4)$-structure.

**Remark 3.5.** Expressing the 3-form $\gamma$ in terms of $\nu$, $\gamma = 2\sqrt{2}\eta^{i\ldots} \nu$, we see that if $d\nu = 0$ then $\gamma$ is closed. Consequently, the system \(3.2\) reduces to the equations
\[b \cdot \nu = -d\omega, \frac{1}{4} b \cdot \omega^2 = d\check{\nu}.\]
The method of Proposition 3.4 results in harmonic $SO(4)$-structures on 9 different nilpotent Lie algebras; these are the last 9 algebras of Table 1. In addition, we find harmonic structures on different (non-nilpotent) solvable Lie algebras, as explained in the next section.

3.1.3. The algebras $n_3, \ldots, n_7$. A harmonic structure is found on the family of Lie algebras given by
\begin{equation*}
(0, 0, (cf^1 - bf^2)f^2, (3bf^2 - 2cf^1)f^3 + f^{12}, (af^1 + 3f^2 - 3bf^3 - bf^4)f^2 + f^{13},
-(3f^1 + af^2 + cf^3 + 2cf^3)f^2 + f^{23}, 0)
\end{equation*}
where $a, b, c \in \mathbb{R}$. The first Betti number is three and, in addition, the space \( V \) has dimension two. When \( b \neq 0 \) and \( c \neq 0 \), there are three cases, depending on the value of \( |a| \).

First, let us assume both \( b \) and \( c \) are zero. Then the space of exact 2-forms is a 3-dimensional subspace \( V \subset \Lambda^2(f^1, f^2, f^3, f^7) \). The form \( df^4 \) is decomposable, and the subset \( \{ \gamma \in V \mid \gamma^2 = 0 \} \) is the union of straight lines parallel to \( df^4 \). Taking its image in the quotient space \( V/(df^4) \), we obtain the trivial space when \( |a| < 3 \), a line when \( |a| = 3 \), and two lines when \( |a| > 3 \). Now, using the classification of nilpotent Lie algebras, we conclude that our Lie algebra is isomorphic, respectively, to
\begin{align*}
n_3 &= (0, 0, 0, 12 - 34, 13 + 24, 14), \\
n_4 &= (0, 0, 0, 12 + 34, 23, 24), \\
n_5 &= (0, 0, 0, 12, 13, 34).
\end{align*}

Secondly, let us consider the case when exactly one of \( b \) or \( c \) is zero. Then the Lie algebra is isomorphic to \( n_6 = (0, 0, 0, 12, 13, 23 + 14, 25 + 34) \). This can be seen, e.g., by considering the coframe
\begin{equation*}
(f^2, b f^2, -\frac{a}{\sqrt{2}} f^3, -\frac{2a}{\sqrt{3}} f^4 + 3 f^3 - \frac{a}{\sqrt{3}} f^1, -f^6 + \frac{a}{\sqrt{2}} f^5 + \frac{a}{\sqrt{3}} f^3 f^3)
\end{equation*}
when \( b \neq 0 \) and
\begin{equation*}
(f^1, -c f^2, -\frac{a}{\sqrt{2}} f^2, -\frac{2a}{\sqrt{3}} f^3, -\frac{a}{\sqrt{2}} f^4 + 2 f^3 + \frac{a}{\sqrt{3}} f^2, -f^5 + \frac{a}{\sqrt{2}} f^3 + \frac{a}{\sqrt{3}} f^3 f^3)
\end{equation*}
when \( c \neq 0 \).

Finally, we are left with the case when \( b \) and \( c \) are both non-zero. Then the first Betti number is three and, in addition, the space \( [g, [g, g]] \) is 2-dimensional. Going through the classification of [12] (see also [7]), this leaves us with 22 possibilities. We can easily rule out 14 of these, namely by observing that the space \( \langle df^i \land df^j \mid 1 \leq i, j \leq 7 \rangle \) has dimension two. For each of the remaining 8 possibilities, we then consider the unique (up to non-zero multiple) element \( \tilde{f} \) of \( g \) satisfying
\begin{equation*}
\dim (B^2 \land \tilde{f} + \Lambda^3([g, g]^o)) = 2.
\end{equation*}
In our case, \( \tilde{f} = -cf^1 + bf^2 - bc f^2 \) and \( \tilde{f} \land d([g, g], g]^o) = \langle bc f^{127} \rangle \). Since, by assumption, we have \( bc \neq 0 \), this space is 1-dimensional. These observations leave us with only two possibilities which are
\begin{align*}
(0, 0, 0, 12, 13, 35 + 24 + 15 + 14, 34 + 25), \\
n_7 &= (0, 0, 0, 12, 13, 35 + 15, 34 + 25).
\end{align*}
Considering finally the dimension of \( B^3(g) \), we can rule out the first of these.
3.1.4. The algebras $n_8, \ldots, n_{10}$. The family of nilpotent Lie algebras

$$(0,0,0,0,af^{27} + f^{'12} + 2f^{37}, -af^{37} + e^{13} - 2f^{27}, 0),$$
a $\in \mathbb{R}$, admits a harmonic SO(4)-structure with adapted coframe given by

$$\frac{1}{\sqrt{2}}(f^3 - f^5, f^6 - f^2, f^1, 2f^7, -\frac{1}{\sqrt{3}}(f^1 + 2f^4), \frac{1}{\sqrt{3}}(f^2 + f^6), \frac{1}{\sqrt{3}}(f^3 + f^5)).$$

This is a decomposable Lie algebra which is isomorphic to

$$n_8 = (0,0,0,0,12,34), \quad n_9 = (0,0,0,0,12,14+23),$$

$$n_{10} = (0,0,0,0,13+42,14+23),$$

according to whether $|a| > 2, a = \pm 2$, or $|a| < 2$. We might as well assume that $a \geq 0$, since the symmetry $f^1 \mapsto -f^1, f^2 \mapsto -f^2, f^6 \mapsto -f^6, f^4 \mapsto -f^4$ corresponds to an element of $\text{Sp}(2)\text{Sp}(1)$ whose only effect is to change the sign of $a$.

As in the section 3.1.2, we choose a decomposition of the Lie algebra so as to obtain an SO(3)-structure on the 6-dimensional component. The defining forms, however, will not be closed in this case. This explains why the algebra is absent from [11].

A straightforward computation shows that, in terms of the adapted frame, the Ricci tensor of the associated 8-dimensional metric is given by

$$\begin{pmatrix}
\frac{4}{3} + \frac{e^2}{6} & \frac{4\sqrt{3} + \sqrt{3}a^2}{6} & 0 & 0 & 0 & - \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0 \\
\frac{4a}{3} & \frac{4}{3} + \frac{e^2}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - \frac{8}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & - \frac{8}{3} & \frac{2a^2}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2a^2}{3} & 0 & 0 & 0 & 0 \\
- \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0 & 0 & 0 & - \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0 \\
- \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0 & 0 & 0 & - \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0 \\
- \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0 & 0 & 0 & - \frac{4\sqrt{3}}{3} & - \frac{\sqrt{3}a^2}{6} & 0
\end{pmatrix}.$$
3.1.5. The algebra $n_{11}$. The proof of Theorem 111 is completed by considering the family of nilpotent Lie algebras
\[ (-af^{27}, 0, af^{27}, 0, -bf^{47} - af^{17} - af^{37} - f^{27}, bf^{27} + f^{47} + f^{12}, 0), \]
carrying an harmonic $SO(4)$-structure with adapted coframe
\[
\frac{1}{\sqrt{2}}(f^4 - f^6, f^5 - f^2, f^1, 2f^7, -\frac{1}{\sqrt{3}}(f^4 + 2f^3), \frac{1}{\sqrt{3}}(f^2 + f^5), \frac{1}{\sqrt{2}}(f^4 + f^6)).
\]
For the parameter values $a = 0 = b$, the Lie algebra is decomposable and isomorphic to $(0, 0, 0, 0, 12, 14 + 23)$. When $a = 0 = b$, we obtain $(0, 0, 0, 0, 12, 34)$. Finally, if $a \neq 0$, we see that the underlying Lie algebra is isomorphic to
\[ n_{11} = (0, 0, 0, 0, 12, 13, 14 + 25). \]
Again, in the reducible case, the structures cannot be decomposed so as to fit into the construction of 111.

If we set $a = 0$, the Lie algebra is
\[ (0, 0, 0, 0, -bf^{47} - f^{27}, bf^{27} + f^{47} + f^{12}, 0). \]
We can assume that $b \geq 0$, since the symmetry $f^1 \mapsto -f^1, f^2 \mapsto -f^2, f^3 \mapsto -f^3, f^5 \mapsto -f^5$ is an element of $SO(4)$ with the only effect that it changes the sign of $b$.

Note that the resulting $SO(4)$-structures are not the same as those that appear in section 3.1.4. However, the resulting harmonic $Sp(2)$ $Sp(1)$-structures are related through an automorphism of the 8-dimensional Lie algebra which only acts on the 2-dimensional center.

More precisely, denote by $e_1, \ldots, e_7$ the adapted frame we have fixed on the present Lie algebra, and by $\tilde{e}_1, \ldots, \tilde{e}_7$ the adapted frame considered in section 3.1.4. If we now set $a = 2\sqrt{b^2 + 1}$, we can construct a linear isomorphism between the two Lie algebras that maps eigenvectors of the Ricci tensor to eigenvectors of the same eigenvalues, as follows:
\[
\tilde{e}_3 \mapsto \frac{2}{a}(be_3 + e_4), \quad \tilde{e}_4 \mapsto \frac{2}{a}(-e_3 + be_4), \quad \tilde{e}_5 \mapsto e_5,
\]
\[
\tilde{e}_1 \mp \sqrt{3}\tilde{e}_6 + \sqrt{3}\tilde{e}_7 \mapsto \frac{1}{\sqrt{a}}\left(\sqrt{a+2e_1} \pm \sqrt{a+2e_2} \pm \sqrt{3}\sqrt{a+2e_6} + \sqrt{3}\sqrt{a+2e_7}\right),
\]
\[
\mp \sqrt{3}\tilde{e}_1 + \sqrt{3}\tilde{e}_2 + \tilde{e}_6 \mp \tilde{e}_7 \mapsto \frac{1}{\sqrt{a}}\left(-\sqrt{3}\sqrt{a+2e_1} \pm \sqrt{3}\sqrt{a+2e_2} \pm \sqrt{a+2e_6} + \sqrt{a+2e_7}\right).
\]
Up to a scale factor, this mapping is a Lie algebra isomorphism, and it maps the 4-form $\Omega$ corresponding to section 3.1.4 to
\[
e^{3467} - e^{3458} - 3e^{5678} - e^{1267} - 3e^{1234} + e^{1258} + e^{147}(ke^8 - 2he^5) + e^{23}(ke^5 + 2he^8)\eta^7 + e^{136}(ke^8 - 2he^5) - e^{14}(ke^5 + 2he^8)e^6 + e^{13}(ke^5 + 2he^8)e^7 - e^{237}(ke^8 - 2he^5),
\]
where $k = \frac{a^2 - 8}{a^2}$ and $h = 2\frac{\sqrt{a^2 - 4}}{a^2}$. As $k^2 + 4h^2 = 1$, it now suffices to set
\[
\tilde{e}_5 = ke_5 - 2he_8, \quad \tilde{e}_8 = 2he_5 + ke_8,
\]
so as to obtain an isomorphism, up to scale, of the two 8-dimensional Lie algebras that preserves the $Sp(2)$ $Sp(1)$-structures. However, the $SO(4)$-structures are not equivalent because this transformation mangles the possibility to separate $e_5$ from $e_8$. 

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This completes the proof of Theorem 1.1.

3.2. Examples on solvable Lie algebras. Using the same procedures as above enables us to find harmonic SO(4)-structures on the following families solvable Lie algebras.

Example 3.6. The family of unimodular solvable algebras
\((c f^{27} - 4f^{17}, 4f^{27}, -4f^{37}, 4f^{47} - af^{37}, bf^{27} + af^{17} + f^{12} + f^{37}, f^{34} - be^{37} - ce^{47} - f^{27}, 0)\),
a, b, c ∈ \(\mathbb{R}\), admits a harmonic structure whose adapted coframe is given by
\[\frac{1}{\sqrt{2}}(f^5 - f^3, f^2 - f^6, f^1 - f^4, 2f^7, -\frac{1}{\sqrt{3}}(f^1 + f^4), \frac{1}{\sqrt{3}}(f^2 + f^6), \frac{1}{\sqrt{3}}(f^3 + f^5)).\]

Example 3.7. A harmonic structure can be put on the family of unimodular solvable algebras given by
\((2f^{17}, -4f^{27}, -af^{27} - f^{17} -bf^{47}, 6f^{47}, f^{12} + bf^{17} - 2f^{37}, af^{17} + f^{27} + f^{15}, 0)\),
for a, b ∈ \(\mathbb{R}\). The associated coframe is
\[\frac{1}{\sqrt{2}}(f^6 - f^2, f^5 - f^4, f^1 - f^3, 2f^7, -\frac{1}{\sqrt{3}}(f^1 + f^3), \frac{1}{\sqrt{3}}(f^4 + f^5), \frac{1}{\sqrt{3}}(f^2 + f^6)).\]

4. Invariant intrinsic torsion

The intrinsic torsion of an SO(4)-structure is dubbed invariant if it takes values in the trivial submodule \(2\mathbb{R} \subset T^* \otimes \mathfrak{so}(4)^{\perp}\). We may think of the latter space as the cokernel of the alternating map
\[\partial: T^* \otimes \mathfrak{so}(4) \hookrightarrow T^* \otimes T^* \otimes T \rightarrow \Lambda^2 T^* \otimes T.\] (4.1)

Then the trivial module \(2\mathbb{R}\) is the image of \(\langle \tau_1, \tau_2 \rangle \subset T^* \otimes \mathfrak{so}(7)\), where
\[\tau_1 = \sum_{i=1}^{7} e^i \otimes e_i \cdot \alpha, \quad \tau_2 = w^1 \otimes w^{23} + w^2 \otimes w^{31} + w^3 \otimes w^{12},\]
and we have identified vectors with covectors using the metric.

Using the alternation map \(a\) (cf. section 2), we compute
\[a(\tau_1 \cdot \alpha) = -2\beta + 6\nu, \quad a(\tau_2 \cdot \alpha) = 2\beta + 6\nu, \quad a(\tau_1 \cdot *v) = 2\beta + 6\nu, \quad a(\tau_2 \cdot *v) = 0.\]

Thus, if the intrinsic torsion, has the form \(\lambda \tau_1 + \mu \tau_2\), then we obtain
\[da = 2(-\lambda + \mu)\beta + 6(\lambda + \mu)\nu, \quad d(*v) = 2\lambda(\beta + 3\nu).\]

In particular, the “\(\lambda\) component” of the intrinsic torsion is completely determined by \(d(*v)\).

Since there are no invariants of degree 5, the forms \(\nu\) and \(\beta\) are closed. Using this fact and by imposing \(d^2 = 0\), we obtain the equations
\[2(-d\lambda + d\mu) \wedge \beta + 6(d\lambda + d\mu) \wedge \nu = 0 = 2d\lambda \wedge (\beta + 3\nu),\]
which, by injectivity of the map \(T^* \ni \delta \mapsto \delta \wedge (\beta + 3\nu) \in \Lambda^5 T^*\), implies that \(\lambda\) and \(\mu\) are necessarily constant.

Proposition 4.1. Any SO(4)-structure with invariant intrinsic torsion has constant torsion. \(\square\)
4.1. **The proof of Theorem 1.2.** The simplest examples of SO(4)-structures with invariant intrinsic torsion are given by Lie groups $H$. Let $e^1, \ldots, e^4$, $w^1, w^2, w^3$ be a basis of $h^*$ adapted to an SO(4)-structure. The torsion, $\tau$, of the flat connection can be expressed as $d\omega^\mathfrak{h} \otimes w^\mathfrak{h} + de^i \otimes e^j$, and its projection to $T^* \otimes so(4)^\perp$ is invariant if and only if $\tau$ lies in $2R \oplus \partial(T^* \otimes so(4))$, where $so(4)$ is given by (2.4), and the 2R component is

$$\theta = a\omega_3 \otimes w^5 + ce_{j,\perp} \omega_3 \wedge w^5 \otimes e^j.$$  

Throughout the section, we shall use the Einstein summation convention.

4.1.1. **The algebra $\mathfrak{h}_1$.** Consider the element $\sum_{s,t} w^s \otimes \omega_t$ of the module $S^{2,2} \subset T^* \otimes so(4)$. Its image under the map $\partial$ is given by:

$$\begin{align*}
(e^2 + e^3 + e^4) \wedge w \otimes e^1 - (e^1 - e^3 + e^4) \wedge w \otimes e^2 \\
(e^3 + e^2 - e^4) \wedge w \otimes e^1 - (e^1 - e^2 + e^3) \wedge w \otimes e^4,
\end{align*}$$

where $w = w^1 + w^2 + w^3$. It follows that the solvable Lie algebra $\mathfrak{h}_1$, given by

$$\begin{align*}
de^1 &= (e^2 + e^3 + e^4) \wedge w, 
&= (e^1 - e^3 + e^4) \wedge w, 
\end{align*}$$

$$\begin{align*}
de^3 &= -(e^1 + e^2 - e^4) \wedge w, 
&= -(e^1 - e^2 + e^3) \wedge w 
&= dw^1 = dw^2 = dw^3,
\end{align*}$$

has an SO(4)-structure with vanishing intrinsic torsion.

The three 2-forms of (2.6) are globally defined and hence determine an almost hypercomplex structure on $S^1 \times H_1$ which is compatible with the product metric. In other words, we have an Sp(2)-structure on $S^1 \times H_1$. As $d\sigma = 0$, we see that this structure is, in fact, hyper-Kähler (cf. [14, Lemma 6.8]).

To obtain examples with a non-flat metric, consider next the element

$$\chi = w^1 \otimes (e^{12} - e^{34} - 2w^{23}) + w^2 \otimes (e^{13} - e^{42} - 2w^{31}) + w^3 \otimes (e^{14} - e^{23} - 2w^{12})$$

that spans the trivial component $R \subset S^{0,2} \otimes S^{0,2} \subset T^* \otimes so(4)$. Straightforward computations show that

$$\tau = \theta + b\partial(\chi) = (a\omega_3 - 4bw_{s,\perp} w^{123}) \otimes w^5 + (b + c)(e_{j,\perp} \omega_3 \wedge w^5) \otimes e^j.$$  

Now, let us write $p = a$, $q = -4b$, and $r = b + c$ and, in addition, impose $d^2 = 0$ so as to get:

**Lemma 4.2.** If $p, q, r \in R$ satisfy the equations

$$pq = 0 = pr = r(q + 2r),$$  

then the relations

$$de^j = re_{j,\perp} \omega_3 \wedge w^5, 
\text{d}w^5 = pw_3 + qw_{s,\perp} w^{123}$$

are the structural equations of a Lie algebra with an SO(4)-structure whose intrinsic torsion is invariant.
Note that, in terms of \(a, b, c\), we find that \(\lambda = -a/2\) and \(\mu = a - 2c\). In
particular, the intrinsic torsion is independent of the component \(\partial(\chi)\) of \(\tau\),
as expected.

Up to a scale factor, (4.3) gives rise to exactly three non-trivial examples.

4.1.2. The algebra \(h_2\). If we take \((p, q, r) = (0, 1, 0)\) then \(de^i = 0\), \(dw^s =
\omega_{s,i}\omega^{123}\) which characterises the product \(H_2 = \text{Sp}(1) \times T^4\). Using the
relations
\[
\lambda = -\frac{p}{2}, \quad \mu = p - \frac{q}{2} - 2r,
\]
we see that \((\lambda, \mu) = (0, -1/2)\) and hence \(da = -\beta - 3\nu\), \(d(\ast v) = 0\).

The product of \(H_2\) with an \(S^1\), \(M = S^1 \times \text{Sp}(1) \times T^4\), can also be viewed
as the product of two quaternionic manifolds of dimension 4; one of the
factors is hyper-Kähler and the other is locally conformal hyper-Kähler. It
is not difficult to verify that the 2-forms \(\sigma_i\) define a hypercomplex structure
on \(M\). In fact, the associated hyper-Hermitian structure is “hyper-Kähler
with torsion” (briefly hkt), which, by [18, Proposition 6.2], means it satis-
ifies the condition \(I_1d\sigma_1 = I_2d\sigma_2 = I_3d\sigma_3\).

In the notation of [24], the two factors of \(M\) have intrinsic torsion in \(EH\),
and the product has intrinsic torsion in \(EH \oplus KH\). This is consistent with
the fact that the ideal generated by \(\sigma_1\), \(\sigma_2\), and \(\sigma_3\) is not a differential ideal.

A more systematic way of determining the intrinsic torsion is as follows.
The Levi-Civita connection is determined by \(\omega^1 \otimes \omega^{23} - \omega^2 \otimes \omega^{13} + \omega^3 \otimes
\omega^{12}\), and the intrinsic torsion is minus its projection to \(T^*_M \otimes (\mathfrak{sp}(2) +
\mathfrak{sp}(1))^{\perp}\). Composing with the skew-symmetrisation map \(a: T^*_M \otimes \mathfrak{so}(8) \to
\Lambda^3T^*_mM\), we obtain
\[
\frac{3}{2}(w^1 \wedge \sigma_1 + w^2 \wedge \sigma_2 + w^3 \wedge \sigma_3) + \gamma, \quad \gamma \in KH.
\]
The second term is not zero, and the first term lies in a submodule iso-
morphic to \(EH\), namely the image of the equivariant map
\[
EH \to \Lambda^3T^*_M, \quad v \mapsto (v, \ast \sigma_5) \wedge \sigma_5.
\]

Now observe that the kernel of \(a\) intersects \(T^*_M \otimes (\mathfrak{sp}(2) \oplus \mathfrak{sp}(1))^{\perp}\)
in \(KS^3H\), so this determines all of the intrinsic torsion except possibly for
\(KS^3H\). Since we have already observed that the structure is quaternionic,
this component is forced to be zero. Summing up, the only non-trivial
components in the intrinsic torsion are \(KH\) and \(EH\).

4.1.3. The algebra \(h_3\). Consider the case when \((p, q, r) = (1, 0, 0)\). Then
we have the structural equations \(de^i = 0\), \(dw^s = \omega_{s,i}\), giving the quater-
nionic Heisenberg group \(H_7\). In this case, the intrinsic torsion is \((\lambda, \mu) =
(1/2, 1)\) which implies \(da = 3\beta + 3\nu\), \(d(\ast v) = -\beta - 3\nu\).

The product \(M = S^1 \times H_7\) is \(\text{hkt}\). It is, however, a so-called \(\text{qkt}\)
manifold (see [18, Definition 7.1]) and interestingly appears in [9].

The same arguments as used for \(h_2\) show that the non-vanishing com-
ponents of the intrinsic torsion are \(KH\) and \(EH\).
4.1.4. The algebra $h_4$. If $(p, q, r) = (0, -2, 1)$, we obtain the Fino-Tomassini example $H_4 = Sp(1) \ltimes \mathbb{R}^4$ (cf. [10]), meaning the Lie group whose Lie algebra is given by $de^i = e_{ij} \omega^j \wedge \omega^i$, $d\omega^i = -2\omega_{i\mathbb{J}} w^{123}$. Note that we may view $H_4$ as $Sp(1) \ltimes \mathbb{H}$ by defining multiplication by $i, j, k$ via the adjoint action of $w_1, w_2,$ and $w_3$. The manifold $H$, notably, has a compact quotient.

Again, we can compute the exterior derivatives of $\alpha$ and $\ast v$ by using $(\lambda, \mu) = (0, -1)$. We find $\partial x = -2\beta - 6v$ and $d(\ast v) = 0$.

The same arguments we have used before apply to the product $Sp(2)Sp(1)$-structure and show that it has non-zero intrinsic torsion in both components of $EH \oplus KH$. In particular, we have provided another example of a QKT structure which is not HKT.

Remark 4.3. The equations used to construct invariant intrinsic torsion structures on $h_2, h_3, h_4$ also apply to give homogeneous examples that are not Lie groups. For instance, the 7-dimensional sphere, $S^7$, viewed as the homogeneous space $Sp(2)Sp(1)/Sp(1)Sp(1)$ (cf. [6, Example 5]), admits a natural $SO(4)$-structure with invariant intrinsic torsion. This corresponds to $(p, q, r) = (1, 0, 1)$.

The remaining classes of examples, appearing in Theorem 1.2, are closely tied to self-dual Einstein Lie groups. In order to arrive at these classes, one can impose suitable conditions on the torsion of the flat connection in order to ensure $\mathfrak{h}$ has structural equations given by

\[
\begin{align*}
\partial e^i &= -\phi(w_i) \wedge e_{ij} \omega^j - \psi(w_j) \wedge e_{ij} \omega^i, \\
\partial \omega_j &= p\omega_j + q\omega_{j\mathbb{J}} w^{123} - \phi(w_1) \wedge \omega_j - (\partial w_1 \wedge (w_1 \wedge w_3)).
\end{align*}
\]

In the above, $\phi, \psi : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ are linear maps representing the component of the torsion in $\partial((\mathbb{R}^3)^* \otimes so(4))$, and as before we identify vectors and covectors via $e_i \mapsto e^i$, $w_j \mapsto w^j$. These equations define a Lie algebra if and only if $d^2 = 0$; then the standard coframe defines an $SO(4)$-structure with invariant intrinsic torsion.

In this case, we can view $\langle e^1, \ldots, e^4 \rangle$ as the dual of a Lie algebra $\mathfrak{t}$, and the projection $\mathfrak{h} \rightarrow \mathfrak{t}$ is a Lie algebra homomorphism. The exterior derivative is required to satisfy

\[
0 = d^2 w^1 = pq(\omega_2 \wedge w^3 - \omega_3 \wedge w^2) + 4\phi(w_1) \wedge (\phi(w_2) \wedge w^2 + \phi(w_3) \wedge w^3) - \phi(w_2) \wedge (-2\phi(w_2) \wedge \omega_1 \wedge w^3 + 2\phi(w_3) \wedge \omega_2 \wedge w^2) - \psi(w_3) \wedge (-2\phi(w_2) \wedge \omega_1 \wedge w^3 + 2\phi(w_3) \wedge \omega_2 \wedge w^2),
\]

which, by considering the cyclic permutations, leads to the relation:

\[
\phi(w_1) \wedge \phi(w_2) \wedge \omega_1 + \psi(w_1) \wedge \phi(w_3) \wedge \omega_1 - \phi \wedge \phi, w_{i\mathbb{J}} w^{123} - \frac{1}{2} pq \omega_j. \quad (4.4)
\]

Thus, $\mathfrak{h}$ is a Lie algebra if and only if $\mathfrak{t}$ is a Lie algebra and (4.4) holds.

Decreeing $e^1, \ldots, e^4$ to be an oriented orthonormal basis of $\mathfrak{t}$, we can interpret this equation in terms of the Levi-Civita connection, $-\phi(w_i) \otimes$
\[ \omega_s - \psi(w_s) \otimes \omega_s, \] on \( \mathfrak{k} \). Indeed, its associated curvature is given by
\[ R = -d\phi(w_s) \otimes \omega_s - d\psi(w_s) \otimes \omega_s + \frac{1}{2} \phi(w_s) \wedge \phi(w_1) \otimes [\omega_s, \omega_1] \]
\[ + \frac{1}{2} \psi(w_s) \wedge \psi(w_1) \otimes [\omega_s, \omega_1] \]
which means that (4.4) is equivalent to
\[ R = -\frac{1}{2} pq \omega_s \otimes \omega_s - d\psi(w_s) \otimes \omega_s + (\psi \wedge \psi, w_{1,3}w^{123}) \otimes \omega_s. \]
In particular, \( \mathfrak{k} \) is Einstein and self-dual with scalar curvature 3pq (see [21]).

From the classification of flat and anti-self-dual metrics on 4-dimensional Lie groups in [19] and [8], we deduce that there are essentially three non-trivial possibilities.

4.1.5. The families \( h^s_2, h^s_3 \). Assume first that \( \mathfrak{k} \) is flat, but not Abelian (otherwise we would recover the examples \( h_2 \) and \( h_3 \)). Then, by [19], we can assume \( \mathfrak{k} = (0, 0, 14, -13) \) so that
\[ \phi(w_1) = -\psi(w_1) = \frac{1}{2} e_1, \]
\[ \phi(w_2) = 0 = \psi(w_2) = \phi(w_3) = \psi(w_3). \]
By construction \( pq = 0 \); for \( (p, q) = (a, 0) \) we obtain \( h^s_2 \), and \( h^s_3 \) for \( (p, q) = (0, a) \). In either case, the almost hypercomplex structure defined by the forms \( \sigma_i \) on \( S^1 \times H \) is not integrable; the non-vanishing components of the intrinsic torsion are \( EH \) and \( KH \), apart from the case \( a = 0 \) when \( h^s_0 = h^s_0 \) is flat and hyper-Kähler.

4.1.6. The family \( h^{s, \kappa}_\mathfrak{k} \). If \( \mathfrak{h} \) is only conformally flat, but not flat, the Einstein condition implies that it has constant sectional curvature. The 4-sphere has no Lie group structure, so we are dealing with hyperbolic space, which has a one-parameter family of Lie group structures (cf. [15]) corresponding to \( \mathfrak{k} = (0, 12, 13 + \kappa 14, 14 - \kappa 13) \). We then compute
\[ \phi(w_1) = \frac{1}{2} e_1 - \frac{1}{2} e_2, \]
\[ \phi(w_2) = -\frac{1}{2} e_3 = \psi(w_2), \phi(w_3) = -\frac{1}{2} e_4 = \psi(w_3), \]
and \( pq = -1 \). The corresponding 8-manifold \( S^1 \times H \) is only hypercomplex when \( \kappa = 0 \); regardless of \( \kappa \), the non-zero intrinsic torsion is in \( EH \oplus KH \).

4.1.7. The family \( h^{s, \kappa}_\mathfrak{k} \). If \( \mathfrak{k} \) is not conformally flat, based on the study of [8], we deduce that \( \mathfrak{k} \) is a solvable algebra belonging to the family:
\[ \mathfrak{k} = (0, f^{12} + \kappa f^{13}, -\kappa f^{12} + f^{13}, 2f^{14} + f^{23}). \]
A multiple of \( (f^1, \frac{1}{2} f^2, \frac{1}{2} f^3, \frac{1}{2} f^4) \) forms an orthonormal basis. In other words, up to scale, we have an orthonormal basis \( e^i \) such that
\[ \mathfrak{k} = (0, e^{12} + \kappa e^{13}, -\kappa e^{12} + e^{13}, 2e^{14} + 2e^{23}). \]

Considering the connection form of \( \mathfrak{k} \), we deduce that
\[ \phi(w_1) = -e_2, \phi(w_2) = -e_3, \phi(w_3) = -\frac{1}{2} e_4 + \frac{1}{2} e_1, \]
\[ \psi(w_1) = 0 = \psi(w_2), \psi(w_3) = -\frac{1}{2} e_4 - \frac{1}{2} e_1. \]
So in this case, the relation (4.4) is satisfied with \( pq = -2 \), and the curvature form of \( \mathfrak{h} \) is given by
\[ \omega_s \otimes \omega_s + 3 \omega_3 \otimes \omega_3. \]
In summary, we have obtained a 2-parameter family of 7-dimensional Lie algebras with a constant intrinsic torsion SO(4)-structure; actually, there are three parameters, namely $p, q, \kappa$, but these are subject to the constraint $pq = -2$.

The Sp(2)-structure, induced on the 8-manifold $S^1 \times H$ via the 2-forms $\sigma_i$, does not correspond to an integrable hypercomplex structure nor satisfies the differential ideal condition. In terms of the intrinsic torsion, we find that the only non-zero components are $EH$ and $KH$.

We emphasise that the only Einstein examples, obtained above, are the flat ones, i.e., the structures on $h_1$ and $h_0^5 = h_0^6$. Similarly, one can check directly that among the Sp(2) Sp(1)-structures constructed in this section, the only ones that satisfy the differential ideal condition are those corresponding to these two, consistently with the intrinsic torsion calculations.

This concludes the proof of Theorem 1.2.

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