On the Doi-Edwards and K-BKZ rheological models for polymer fluids: an existence result for shear flows.

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Abstract

This paper establishes the existence of smooth solutions for the Doi-Edwards rheological model of viscoelastic polymer fluids in shear flows. The problem turns out to be formally equivalent to a K-BKZ equation but with constitutive functions spanning beyond the usual mathematical framework. We prove, for small enough initial data, that the solution remains in the domain of hyperbolicity of the equation for all $t \geq 0$.

Keywords: Doi-Edwards polymer model; K-BKZ viscoelastic fluid; shear flows; convolution operator; evolutionary integro-differential equation.

1 Introduction.

Today’s modeling of non-Newtonian and viscoelastic industrial flows (and of the rheological behavior in general) relies heavily on molecular theories. The rheology of various linear/branched polymer liquids is very well described by the so-called tube-reptation theories initiated by Doi and Edwards (DE), see \textsuperscript{[7]}. At the heartcore of any kinetical model one finds a configurational probability diffusion equation (a parabolic PDE) the solution of which is needed to obtain the stress tensor, i.e. the corresponding constitutive equation (CE). For the full, non-linear DE model, in \textsuperscript{[5]} we proved the existence and uniqueness of solutions for the diffusion equation using the Schauder fixed point theorem and the Galerkin’s approximation method. Moreover, this work is related to that in \textsuperscript{[4]}.

Here we focus on an equally crucial issue, that of existence of solutions to shear flows. The corresponding constitutive equation is that of the simplified DE theory commonly called Independent Alignment Approximation (IAA). The governing equations for the shear flow are given below:

$$\frac{\partial v}{\partial t} = \frac{\partial \theta}{\partial x}$$

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\[
\theta = \int_0^1 \int_{S^2} u_1 u_2 F \, du \, ds
\]  
(1.2)

\[
\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial s^2} - \frac{\partial v}{\partial x} \frac{\partial}{\partial u} \cdot (G_0(u)F)
\]  
(1.3)

In the above, the notations are common to the mathematical and the related continuum mechanics, rheology, and polymer physics literature: \(v = v(x, t)\) is the scalar velocity field, \(\theta = \theta(x, t)\) is the stress, and \(F(t, u, s, x)\) the configurational probability function. The flow occurs in the \(x\) direction during time \(t\), \(s \in (0, 1)\) is the polymer chain’s primitive path curvilinear coordinate, and \(u = \begin{pmatrix} u_0, u_2, u_3 \end{pmatrix}\) the unitary vector pointing outwardly the unit sphere \(S^2\).

Similar to notations in [6], \(G_0(u) = M \cdot u - (M : uu)u\), with \(M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\)  
(1.4)

To the system of equations (1.1)-(1.3) we assign the following boundary and initial conditions:

\[
\begin{cases}
    v = 0, & \text{for } x \in \partial \Omega \\
    v = v_0, & \text{for } t = 0 \\
    F = \frac{1}{4\pi}, & \text{for } s = 0 \text{ or } s = 1 \\
    F = F_0, & \text{for } t = 0
\end{cases}
\]  
(1.5)

where \(\Omega \subset \mathbb{R}\) is the range for \(x\), while \(v_0(x)\) and \(F_0(u, s, x)\) are initial data.

From [6] one sees the equation (1.3) for \(F\) can be solved allowing the obtainment of \(\theta\) as a function of the velocity gradient \(\frac{\partial v}{\partial x}\). In particular, for \(F_0 = 1/(4\pi)\) one gets:

\[
\theta = -g_{DE} \left( \int_0^t \frac{\partial v}{\partial x} (x, \tau) d\tau \right) a_{DE}(t) + \int_0^t g_{DE} \left( \int_0^\tau \frac{\partial v}{\partial x} (x, r) dr \right) a_{DE}'(t - \tau) d\tau
\]  
(1.6)

with \(g_{DE} : \mathbb{R} \rightarrow \mathbb{R}\)

\[
g_{DE}(y) = -\int_{S^2} \frac{u_1 u_2}{\left[ (u_1 - u_2 y)^2 + u_2^2 + u_3^3 \right]^{3/2}} du, \forall y \in \mathbb{R}
\]  
(1.7)

and \(a_{DE} : \mathbb{R}_+ \rightarrow \mathbb{R}\)

\[
a_{DE}(t) = \sum_{p=1}^{+\infty} \frac{1}{(2p+1)^2} e^{-(2p+1)^2 t}
\]  
(1.8)

the relaxation function of the DE model.

From the above considerations one infers the shear flow problem under scrutiny is tantamount to solving for \(v\) the below integro-differential equation:

\[
\frac{\partial v}{\partial t} = -\frac{\partial}{\partial x} g_{DE} \left( \int_0^t \frac{\partial v}{\partial x} (x, \tau) d\tau \right) a_{DE}(t) + \frac{\partial}{\partial x} \int_0^t g_{DE} \left( \int_\tau^t \frac{\partial v}{\partial x} (x, r) dr \right) a_{DE}'(t - \tau) d\tau, \ t > 0
\]  
(1.9)
Now equation (1.9) - here obtained on molecular dynamics grounds - has been focused on within the area of viscoelastic fluids as it comes out when one studies shear flows for the K-BKZ fluids. There is no contingency here as in their 1978 original paper [6]. Doi and Edwards have shown the simplified IAA version of their nonlinear model actually enters the class of K-BKZ integral models, which are based on continuum mechanics concepts (for more on see [1], [2], [3]). Consequently, when undertaking the study of certain particular flows of DE fluids one may capitalize on previously obtained results for K-BKZ liquids.

In this paper we study equation (1.9) with more general functions $g$ and $a$ replacing $g_{DE}$ and $a_{DE}$, respectively. We prove a global in time solution existence result for small enough data. Uniqueness is the focus of an upcoming paper [11]. Equation (1.9) - as well as variants of it - was studied by various authors, see Renardy, Hrusa and Nohel [15], Engler [8], Brandon and Hrusa [7] and references cited therein.

The existence of local in time solutions [15] and of global solutions [6], [8] are known under more restrictive conditions compared to those stated in this paper. One of the assumptions in [6] and [8] is $g'(y) < -\gamma$, for any $y \in \mathbb{R}$, with $\gamma > 0$, which is not verified by the function $g = g_{DE}$. Here we make use of the less restrictive assumption $g'(y) < 0$, for any $y \in [-\theta, \theta]$, with $\theta > 0$, and show that the argument of $g'$ is confined to $[-\theta, \theta]$. The requirement $g' < 0$ is a necessary hyperbolicity condition for the solution local existence. For the work presented in this paper, this condition being valid only locally makes it necessary to control, w.r.t. time $t$, the argument $\int_0^t \frac{\partial v}{\partial x}(x, \tau) d\tau$ of $g'$. Observe that at a first sight, this argument may become large with increasing $t$.

Next, among the restrictive hypotheses invoked by the authors of [8], [15] for function $a$ is that $a'' \in L^1(0, +\infty)$, which $a = a_{DE}$ does not verify. Comparatively, here we shall place significantly less restrictions on $a$ and accordingly will construct a class of totally monotone functions, an element of which is $a = a_{DE}$.

The manuscript is organized as following:

In Section 2 we introduce the problem and enunciate the main result.

In Section 3 we are devoted to the proof of several necessary results such as a Gårding type inequality and an inversion formula for the operator $a \mapsto a * u$ which differs from the one given in [8].

In Section 4 we introduce an approximated problem and obtain useful estimates for its solution. In particular we obtain an estimate for the argument of $g'$ with the help of a maximal function. The proof of the main result is achieved in Section 5.

In the ending Section 6 we construct a class of totally monotone functions that is compatible with the hypothesis made about $a$.

### 2 Presentation of the problem and of the main results.

Let from now on $\Omega \subset \mathbb{R}$ be a bounded, open interval. Let the functions $f : \Omega \times [0, +\infty) \to \mathbb{R}$, $g : I \subset \mathbb{R} \to \mathbb{R}$, with $I \ni 0$ an open interval, $v_0 : \Omega \to \mathbb{R}$, $a : [0, +\infty) \to \mathbb{R}$.

The aim is to search for a solution $v : \Omega \times [0, +\infty) \to \mathbb{R}$ to the below given initial boundary value problem:

\[
v_t(x, t) = -a(t) \frac{\partial}{\partial x} \left( \int_0^t v_x(x, s) ds \right) + \frac{\partial}{\partial x} \int_0^t g \left( \int_s^t v_x(x, \tau) d\tau \right) a'(t - s) ds + f(x, t) \quad (2.1)\]

\[
v(x, t = 0) = v_0(x), \quad \forall x \in \Omega, \quad \text{and} \quad v(x, t) = 0, \quad \forall t < 0 \quad (2.2)\]

\[
v = 0, \quad \forall x \in \partial \Omega, \quad \forall t \geq 0 \quad (2.3)
\]

In the above, $v_x \equiv \frac{\partial v}{\partial x}$ and $a'$ stands for the derivative of $a$. Throughout this paper, any
function defined for \( t \geq 0 \) is understood as being set equal to 0 for \( t < 0 \), i.e. it has domain \( \mathbb{R} \). Moreover, for a function \( \varphi \in W^{k,1}(0, +\infty) \) we denote by \( \varphi^{(k)} \) the distributional derivative of \( \varphi \) on \( \mathbb{R}_+ \), derivative which is understood to be extended to \( \mathbb{R} \) by 0. Define
\[
\bar{v}_t(x, s) := \int_{t-s}^t v(x, \tau) d\tau, \quad 0 \leq s, t; \quad x \in \Omega
\]

Equation (2.1) now takes on a simpler form:
\[
v_t(x, t) = \int_0^{+\infty} a'(s) \frac{\partial}{\partial x} g (\bar{v}_x^t(x, s)) ds + f(x, t) \tag{2.4}
\]

Drawing inspiration from (2.1), (2.4) can be re-written as
\[
v_t(x, t) + g'(0) \int_0^t a(t-s)v_{xx}(x, s) ds = f(x, t) + \mathcal{G}(x, t) \tag{2.5}
\]

where
\[
\mathcal{G}(x, t) = \int_0^{+\infty} a'(s) \left[ g' (\bar{v}_x^t(x, s)) - g'(0) \right] \bar{v}_{xx}^t(x, s) ds
\]
\[
= \int_0^t v_{xx}(x, s) \int_s^{+\infty} a'(\tau) \left[ g' (\bar{v}_x^\tau(x, \tau)) - g'(0) \right] d\tau ds \tag{2.6}
\]

Convolution with respect to \( t \) is denoted as usually by \( * \); therefore (2.5) can be re-written in a more close form as
\[
v_t + g'(0)a * v_{xx} = f + \mathcal{G}
\]

We now proceed to presenting several constitutive assumptions. The function \( g \) is taken such that:

\((g_1)\). there exist \( \theta \in [0, 1] \) and \( K > 0 \), such that \( g \in C^3([-\theta, \theta], \mathbb{R}) \) and \( |g^{(3)}(y) - g^{(3)}(0)| \leq K|y|, \forall y \in [-\theta, \theta] \)

\((g_2)\). \( g(0) = g''(0) = 0 \)

\((g_3)\). \( g'(0) < 0 \)

The function \( f \) is such that

\((f_1)\). \( f, f_x, f_t \in C^0_b([0, +\infty); L^2(\Omega)) \cap L^2([0, +\infty); L^2(\Omega)) \),

\((f_2)\). \( f_t \in L^2([0, +\infty); L^2(\Omega)), \int_0^t f(x, s) ds \in C^0_b([0, +\infty); H^1(\Omega)) \),

where \( C^0_b([0, +\infty); X) \) is the set of all functions \( w : [0, +\infty) \to X \) which are bounded and continuous, and \( X \) is a Banach space.

Next, let \( v_0 \) be such that

\((v_0)_1\). \( v_0 \in H^2(\Omega) \).
We assume that \( f \) and \( v_0 \) are compatible with the already stated initial-boundary conditions:

\[
v_0(x) = f(x, t = 0) = 0, \quad \forall x \in \partial \Omega
\]  

(2.7) \hspace{1cm} \text{nbcel}

Let the measures associated to \( f \) and \( v_0 \) be defined as:

\[
F(f) := \sup_{t \geq 0} \int_{\Omega} \left[ f^2 + f_x^2 + f_t^2 + \left( \int_0^t f(x, s)\,ds \right)^2 + \left( \int_0^t f_x(x, s)\,ds \right)^2 \right] \,dx
\]  

(2.8) \hspace{1cm} \text{msf}

\[
+ \int_0^{+\infty} \int_{\Omega} \left( f^2 + f_x^2 + f_t^2 + f_{tt}^2 \right) (x, t)\,dx\,dt
\]  

(2.9) \hspace{1cm} \text{msf}

\[
V_0(v_0) = \| v_0 \|^2_{H^2(\Omega)} = \int_{\Omega} \left[ v_0^2 + (v_0')^2 + (v_0'')^2 \right] (x)\,dx
\]  

(2.10) \hspace{1cm} \text{msv}

For any function \( \varphi \in L^1((0, +\infty)) \) we denote by \( \mathcal{F}\varphi \) (or alternatively by \( \hat{\varphi} \)) and \( \mathcal{L}\varphi \) the corresponding Fourier and Laplace transforms, i.e.:

\[
\mathcal{F}\varphi(\omega) := \int_0^{+\infty} \varphi(t) e^{-i\omega t} \,dt, \quad \forall \omega \in \mathbb{R}
\]

\[
\mathcal{L}\varphi(z) := \int_0^{+\infty} \varphi(t) e^{-zt} \,dt, \quad \forall z \in \mathbb{C}, \text{Re} z \geq 0
\]

Let us now assume the function \( a \) is such that

(a1). \( a \in W^{1,1}(0, +\infty), \ a'(t) \leq 0 \text{ \ a.e. } t \geq 0, \)

There exists a sequence of functions \((a_n)_{n \in \mathbb{N}}, a_n \in C^2([0, +\infty) \cap W^{2,\infty}([0, +\infty)) \) s.t.

(a2). \( a_n'(t) \leq 0 \quad \forall t \geq 0, \) such that \((a_n)_{n \in \mathbb{N}} \) bounded in \( W^{1,1}(0, +\infty) \) and \( a_n \xrightarrow[\mathcal{F}(0, +\infty)]{n \to +\infty} a, \)

(a3). \( \sup_n \left[ \int_0^1 t |a_n''(t)| \,dt + \int_1^{+\infty} \sqrt{t} |a_n''(t)| \,dt + \int_1^{+\infty} t^2 |a_n'(t)| \,dt \right] < +\infty, \)

(a4). there exist constants \( M_1 > 0 \) and \( n_0 \in \mathbb{N} \) s.t. \( \text{Re}(\mathcal{F}a_n(\omega)) \geq \frac{M_1}{1 + \omega^2}, \forall n \in \mathbb{N}, n \geq n_0, \forall \omega \in \mathbb{R}; \) observe that this is a strong positivity condition, common for this type of problems (see [P1,2]).

(a5). there exist constants \( M_2 > 0 \) and \( p \in \mathbb{N}^* \) s.t. \( \frac{[\mathcal{F}(a_n')]^p}{\mathcal{F}a_n} \in \mathcal{F} (B_{L^1(\mathbb{R)}(0, M_2)}), \forall n \in \mathbb{N}, \) where \( B_{L^1(\mathbb{R)}(0, M_2)} \) denotes the ball in \( L^1(\mathbb{R}) \) centered at 0 and of radius \( M_2; \) this assumption will be used to obtain a representation for the solution \( u \) of \( a_n * u = b \) (see Theorem [P3.1]).

Remark 2.1. In Section 1, we shall construct a class of functions compliant with assumptions (a1) to (a5). This class contains the Doi-Edwards relaxation kernel \( a_{DE} : [0, +\infty) \to \mathbb{R}, \)

\[
a_{DE}(t) = \sum_{k \geq 1} \frac{1}{(2k+1)^2} e^{-(2k+1)^2 t}
\]  

(2.11) \hspace{1cm} \text{tila}

Also, since \( g_{DE} \in \mathcal{C}^\infty(\mathbb{R}) \) is an odd function and \( g_{DE}'(0) = -3 \int_{S_2} u_1^2 u_2^2 \,du < 0, \) then \( g_{DE} \)
also verifies (91)-(94) and this paper results equally apply to the function \( g_{DE}. \)
The main result of this paper is stated below:

**Theorem 2.1 (Main Result).** Assume that the hypotheses on the data given in \((g_1)-(g_4), (f_1)-(f_2), (v_0), (a_1)-(a_5)\) hold true. Then there exists a \(\delta > 0\) such that, if the additional smallness assumption \(F(f) + V_0(v_0) \leq \delta\) is verified, then there exists at least a solution

\[
v \in \left\{ \cap_{m=0}^{2} W^{m,\infty} ((0, +\infty); H^{2-m} (\Omega)) \right\} \cap \left\{ \cap_{m=0}^{2} W^{m,2} ((0, +\infty); H^{2-m} (\Omega)) \right\}
\]

with

\[
\int_0^t v(x,s)ds \in L^\infty ((0, +\infty); H^2 (\Omega))
\]

to the problem \((2.4), (2.2)-(2.3)\).

Next we take on to introducing (and explaining) the proof stages for the aforementioned Theorem 2.1. In short, first we obtain a regularized problem \((P_n)\) obtained from \((2.5)\) with \(a\) being replaced by a sequence \(a_n\) satisfying hypotheses \((a_1)\) to \((a_4)\). Doing this allows to obtain a local in time existence and uniqueness result capitalizing on Renardy’s result in \((16)\) for any \(t\) close to 0 (a consequence of the assumption made on data \(v_0\) and \(f\)), then \(E(t)\) stays “small” for any \(t\). We do this by obtaining an inequality of the type

\[
E(t) \leq \frac{1}{2} E(t) + \text{“small enough” quantities depending uniquely on } V_0 \text{ and } F
\]  

Getting the second term in the rhs of \((2.12)\) requires previously calculated upper bounds of \(v\) and its up to second order derivatives in \(x\) and \(t\), and of \(u\) and its up to third order derivatives in \(x\) and \(t\). Equation \((2.5)\) is equivalently written as:

\[
v_t + g'(0)a \ast v_{xx} = f + G
\]

Next, we calculate three energy estimates (in a way similar in nature with that of Brandon and Hrusa) we derive \((2.13)\) \(i\)-times (with \(i \in \{0, 1, 2\}\)) w.r.t. time \(t\), then multiply the result by \(\frac{d^i v}{dt^i}\) and integrate on \(Q_t := \Omega \times (0,t)\). To calculate the second order derivative one uses a finite difference operator \(\triangle_h w(t) = w(t+h) - w(t)\), see \((1.1)\). We sum up the resulting three equations and get an equality in which the most important term originates from the convolution part in the lhs of \((2.13)\). This term reads

\[
g'(0) \left[ Q(v_x, t, a) + Q(v_{xxt}, t, a) + Q(v_{xtt}, t, a) \right]
\]

where \(Q(w, t, a) = \int_0^t \int_{\Omega} w(x,s) \ast w(x,s)dxds\) (see \((1.3)\)). We lower bound \((2.14)\) using the Plancherel-Parseval equality and assumption \((a_4)\) and get (with \(w = 0\) outside \((0, t)\))

\[
Q(w, t, a) \geq \int_{\mathbb{R}} \int_{\Omega} \frac{M_1}{\omega_1} |(\mathcal{F}w)(x, \omega)|^2 dxd\omega
\]
Notice the presence of $\frac{M_1}{1 + \omega^2}$ does not render the rhs of (2.10) sufficiently coercive, however we use it to obtain the necessary coercivity for $Q(w, t, a) + Q(w_t, t, a)$ instead of $Q(w, t, a)$. The procedure is given in sufficient detail in Lemma 3.4, which deals with a Gårding type inequality with a boundary term.

The terms denoted by $G$ in (2.11) can be controlled w.r.t. well chosen norms by carrying out an integration by parts w.r.t. time $t$ and switching the time derivatives onto $a$ and using the fact that $ta'' \in L^1(0, 1)$ (see assumption (a3)). Eventually one upper bounds w.r.t. $L^\infty L^2_x$ norms $v, v_t, v_x, v_{xt}, v_{tt}$, and w.r.t. $L^2 L^2_x$ norms $v, v_x, v_t, v_{xt}$. The results are gathered into $E_1$, see (1.6). We point out that the aforementioned energy estimates do not provide norm estimates for $v_{xx}$. To cope with this difficulty we use (2.12) which allows to express $v_{xx}$ as a function of $v_t$ and $G$ with the help of an inversion Theorem for the operator $w \mapsto a * w$ and using the previously obtained estimates. We cannot use the resolvent kernel technique like in Brandon and Hrusa because in this paper case $r' \notin L^1(\mathbb{R})$ (as $a'' \notin L^1(\mathbb{R}_+)$). Because of that we prove a point-wise inversion Theorem for the convolution of $a$ assuming pretty weak constraints on $a$: see Theorem 3.1.

3 Preliminaries.

We shall frequently employ the following inequalities:

$$|xy| \leq \mu x^2 + \frac{1}{4}y^2, \quad x, y \in \mathbb{R}, \mu > 0$$

(3.1) \text{iq1}

$$\|F_1 * F_2\|_{L^p(0,T)} \leq \|F_1\|_{L^1(0, +\infty)} \|F_2\|_{L^p(0,T)},$$

(3.2) \text{iq2}

The above is true for any $T > 0$, $F_1 \in L^1(0, +\infty)$, and $F_2 \in L^p(0, T)$, with $p \geq 1$. Functions $F_1$ and $F_2$ are extended to $\mathbb{R}$ by 0.

For any $T > 0$, $w \in \mathcal{C}^0([0, T]; L^2(\Omega))$, $b \in L^1(0, +\infty)$ and $t \in [0, T]$. We define

$$Q(w, t, b) := \int_0^t \int_{\Omega} w(x, s) \int_0^s b(s - \tau) w(x, \tau) d\tau dx ds$$

$$= \int_0^t \int_{\Omega} w(x, s) (b * w)(x, s) dx ds$$

(3.3) \text{qdf}

where $w$ is considered as extended by 0 on $(T, +\infty)$. For any $T > 0$ and $h \in (0, T)$, we define the finite difference operator $\Delta_h$

$$(\Delta_h w)(x, t) = w(x, t + h) - w(x, t)$$

(3.4) \text{fd1}

as a linear operator from $\mathcal{C}^0([0, T - h]; L^2(\Omega))$ onto $\mathcal{C}^0([0, T]; L^2(\Omega))$.

Moreover, if $X(J)$ denotes a space of functions defined on $J \subseteq \mathbb{R}$ and $I \subset J$, then $X_t(I)$ stands for the subspace of functions $X(J)$ the supports of which are included in $I$ (i.e. that vanish on $J - I$).

Recall that $b \in L^1(\mathbb{R}_+)$ is of positive type if, for any $t \geq 0$ and any $\varphi \in L^2(\mathbb{R}_+)$, it satisfies

$$\int_0^t \varphi(s) \int_0^s b(s - \tau) \varphi(\tau) d\tau ds \geq 0.$$ 

Next, $b$ is said to be of strong positive type if there exists $\epsilon > 0$ s.t. the function $b(t) - \epsilon e^{-t}$ is of positive type. Moreover, $Q_t := \Omega \times (0, t)$.

For future reference we prove the following Lemmas:
Lemma 3.1. Let the mappings $\varphi$ and $s \mapsto s\varphi(s)$ be elements of $L^1(\mathbb{R}_+ )$. Then the function $s \mapsto \int_0^{+\infty} \varphi(\tau)d\tau$ belongs to $L^1(\mathbb{R}_+ )$ and we have the estimate

$$\int_0^{+\infty} \left| \int_s^{+\infty} \varphi(\tau)d\tau \right| ds \leq \int_0^{+\infty} |s\varphi(s)| ds$$

Proof. The proof is a direct consequence of Fubini’s Theorem. \hfill \Box

Lemma 3.2. Let $\varphi \in L^1(\mathbb{R}_+)$. Then:

(i) for any $w_1, w_2 \in L^2(Q_t)$ we have

$$\left| \int_0^t \int_{\Omega} w_1(x, s)(w_2 * \varphi)(x, s)ds \right| \leq \| \varphi \|_{L^1(\mathbb{R}_+ )}\| w_1 \|_{L^2(Q_t)} \| w_2 \|_{L^2(Q_t)}$$

(ii) for any $w_3 \in L^2(\Omega)$, $w_4 \in L^\infty (0; L^2(\Omega))$ we have

$$\left| \int_{\Omega} w_3(x)(\varphi * w_4)(x, t)dx \right| \leq \| \varphi \|_{L^1(0, T)}\| w_3 \|_{L^2(\Omega)} \sup_{0 \leq \tau \leq t} \| w_4(\tau) \|_{L^2(\Omega)} , \text{ a.e. } t \in [0, T)$$

Proof. Part (i): observe that

$$\left| \int_0^t \int_{\Omega} w_1(x, s)(w_2 * \varphi)(x, s)ds \right| \leq \int_{\Omega} \| w_1(x, \cdot) \|_{L^2(0, t)} \| (w_2 * \varphi)(x, \cdot) \|_{L^2(0, t)} dx$$

$$\leq \| \varphi \|_{L^1(\mathbb{R}_+ )}\int_{\Omega} \| w_1(x, \cdot) \|_{L^2(0, t)} \| w_2(x, \cdot) \|_{L^2(0, t)} dx$$

which gives the result.

Part (ii): one has

$$\left| \int_{\Omega} w_3(x)(\varphi * w_4)(x, t)dx \right| \leq \| w_3 \|_{L^2(\Omega)} \int_0^t \| w_4(x, t - \tau) \|_{L^2(\Omega)} |\varphi(\tau)| d\tau$$

and the result follows. \hfill \Box

We continue by proving the following result:

Lemma 3.3. Assume $b \in W^{1, 1}((0, +\infty))$ verifies: there exists $M > 0$ s.t.

$$\text{Re}[\mathcal{F}b(\omega)] \geq \frac{M}{1 + \omega^2}, \forall \omega \in \mathbb{R}$$

Then:

(i) $b(0_+) \geq M$, 

(ii) $|\mathcal{L}b(z)| \geq \frac{M}{2(1 + |z|^2)}$, $\forall z \in \mathbb{C}$, $\text{Re}(z) \geq 0$,

(iii) $|\mathcal{F}b(\omega)| \geq \frac{\tilde{M}}{2(1 + |\omega|)}$, $\forall \omega \in \mathbb{R}$, where $\tilde{M}$ may depend on $b$. 

Proof. Part (a) is a direct consequence of

$$b(0_+) = \frac{1}{\pi} \lim_{k \to +\infty} \int_{-k}^{k} \mathcal{F}b(\omega) d\omega = \frac{1}{\pi} \lim_{k \to +\infty} \int_{-k}^{k} \text{Re} \mathcal{F}b(\omega) d\omega$$

and of (3.11).

Part (b): one has \( \text{Re} [(\mathcal{F} e^{-t} ) (\omega)] = \frac{1}{1 + \omega^2} \). This fact, together with Theorem 2.4 on page 494 of [4] imply that the function \( t \to b(t) - M e^{-t} \) is of positive type. From the same Theorem one also gets \( \text{Re} [\mathcal{L} (b - M e^{-t}) (z)] \geq 0 \), \( \forall z \in \mathbb{C} \) with \( \text{Re}(z) \geq 0 \). The later in turn implies \( \text{Re} [\mathcal{L}b(z)] \geq M \frac{1 + z_1}{(1 + z_1)^2 + z_2^2} \), \( \forall z = z_1 + iz_2 \) with \( z_1, z_2 \in \mathbb{R}, \ z_1 \geq 0 \). The statement in (b) now follows.

Part (c) is a consequence of (b) and the fact that \( b \in W^{1,1}(0, +\infty) \). Indeed, from \( |\mathcal{F}b(\omega)| \geq \frac{M}{2(1 + \omega^2)} \), \( \forall \omega \in \mathbb{R} \), it suffices to prove that there exist \( m_1, m_2 > 0 \) s.t. \( |\mathcal{F}b(\omega)| \geq \frac{m_1}{|\omega|}, \forall \omega \in \mathbb{R} \) with \( |\omega| \geq m_2 \). This follows from \( \mathcal{F}b(\omega) = \frac{1}{i\omega} [\mathcal{F}b'(\omega) + b(0_+)] \), the fact that \( \mathcal{F}b'(\omega) \to 0 \) and (b).

\[ \square \]

The following Lemma is a Gårding type inequality with boundary terms. It is proved in [5] using preliminary results due to Staffans [12] (see also [13] and [14]). Here we shorten the original proof of [5] and remove the extraneous assumptions \( b \in W^{3,1}(0, +\infty), b'' \geq 0 \).

\[ \textbf{Lemma 3.4.} \text{ Assume } b \in L^1_{\mathbb{R}^+} (\mathbb{R}) \text{ is such that } \text{Re} \left( \frac{b(\omega)}{\omega} \right) \geq \frac{M_1}{1 + \omega^2}, \text{ for any } \omega \in \mathbb{R}, \text{ where } M_1 > 0. \text{ Then, for any } T > 0, w \in C^1 ([0, T], L^2(\Omega)) \text{ and } t \in [0, T), \text{ we have} \]

$$\int_{\Omega} w^2(x,t) dx + \int_{0}^{t} \int_{\Omega} w^2(x,s) dx ds \leq C \left[ \frac{1}{M_1} Q(w,t,b) + \frac{1}{M_1} Q(w_t,t,b) + \int_{\Omega} w^2(x,0) dx \right]$$

(3.10) \[ \text{ah1} \]

with \( C > 0 \) independent of \( T, t, w \) and \( b \).

Moreover, if \( w \in C^0 ([0, T], L^2(\Omega)) \), then, for any \( t \in [0, T] \),

$$\int_{\Omega} w^2(x,t) dx + \int_{0}^{t} \int_{\Omega} w^2(x,s) dx ds \leq C \left[ \frac{1}{M_1} Q(w,t,b) + \frac{1}{M_1} \liminf_{h \to 0_+} \frac{1}{h^2} Q(\triangle_h w,t,b) + \int_{\Omega} w^2(x,0) dx \right]$$

(3.11) \[ \text{ah2} \]

\[ \text{Proof.} \text{ Assuming that inequality (3.11) holds true, we undertake to proving (3.11). Let } w \in C^0 ([0, T], L^2(\Omega)) \text{ and } t \in [0, T) \text{ be fixed.} \text{ For } 0 < h < (T-t)/2, \text{ define the function } w_h \in C^1 ([0, (t+T)/2], L^2(\Omega)) \text{ by} \]

$$w_h(s) := \frac{1}{h} \int_{s}^{s+h} w(\sigma) d\sigma, s \in [0, (t+T)/2)$$

(3.12) \[ \text{ah3} \]

Applying (3.11) to \( w_h \) and passing to the limit \( \liminf \) gives (3.11).
We now prove (3.11). Let \( w \in \mathcal{C}^1 ([0, T], L^2(\Omega)) \), \( t \in [0, t] \) be fixed, and let \( \hat{w} \in L^2_{[0,t]} (\mathbb{R}, L^2(\Omega)) \) be defined by \( \hat{w} = w \) a.e. in \([0, t]\) and \( \hat{w} = 0 \) outside. Denote by \( D\hat{w} \) the distributional derivative of \( \hat{w} \) and by \( w' \) its regular part, i.e.

\[
D\hat{w} = \hat{w}' + w(0)\delta_0 - w(t)\delta_t
\]  

(3.13)

Due to the Parseval identity we have

\[
Q(w, t, b) = \frac{1}{2\pi} \int_\mathbb{R} \int_\Omega \Re \left( \hat{b}(\tau) \right) \left| \hat{w}(x, \tau) \right|^2 \, dx \, d\tau
\]  

(3.14)

and a similar equation with \( w' \) instead of \( w \) as well. For \( \lambda > 0 \) (to be later determined) define \( I(w) \) by

\[
I(w) := Q(\hat{w}', t, b) + \lambda Q(\hat{w}, t, b) + \frac{3M_1}{2} \int_\Omega w^2(x, 0) \, dx
\]  

(3.15)

By (3.13) and (3.14) and the strong positivity of \( b \),

\[
I(w) \geq \frac{M_1}{2\pi} \int_\mathbb{R} \int_\Omega \left( |i\tau \hat{w}(\tau) - w(0) + w(t)e^{-i\tau t}|^2 + \lambda \left| \hat{w}(\tau) \right|^2 + 3|w(0)|^2 \right) \, dx \, \frac{d\tau}{1 + \tau^2}
\]  

(3.16)

Since for any \((a, b, c) \in \mathbb{C}^3\) we have \(|a + b + c|^2 \geq \frac{|a|^2 + |b|^2}{2} - 2|a||b| - 3|c|^2\), inequality (3.16) implies

\[
I(w) \geq \frac{M_1}{2\pi} \int_\mathbb{R} \int_\Omega \left( \frac{|\tau|^2 + 2\lambda}{2} \left| \hat{w}(\tau) \right|^2 + |w(t)||\tau|\beta \sqrt{|\tau|} - 2|w(t)||\tau| \left| \hat{w}(\tau) \right| \right) \, dx \, \frac{d\tau}{1 + \tau^2}
\]  

(3.17)

with

\[
\beta = \frac{1}{2} \left( \int_\mathbb{R} \frac{d\tau}{1 + \tau^2} \right) / \left( \int_\mathbb{R} \frac{\sqrt{|\tau|}}{1 + \tau^2} \, d\tau \right)
\]  

(3.18)

But:

\[
2|w(t)||\tau| \left| \hat{w}(\tau) \right| \leq \frac{\beta}{2} \sqrt{|\tau||w(t)|^2} + \frac{2}{\beta} |\tau|^{3/2} \left| \hat{w}(\tau) \right|^2
\]

\[
\leq \frac{\beta}{2} \sqrt{|\tau||w(t)|^2} + \left( \frac{\sqrt{|\tau|}}{4} + L \right) \left| \hat{w}(\tau) \right|^2
\]  

(3.19)

with \( L > 0 \) independent of \( t, w, b \). Choose \( \lambda = L + 1/4 \). By (3.17) and (3.19) we get

\[
I(w) \geq \frac{M_1}{2\pi} \int_\mathbb{R} \int_\Omega \left( \frac{|\tau|^2 + 1}{4} \left| \hat{w}(\tau) \right|^2 + \frac{\beta \sqrt{|\tau|}}{2} |w(t)|^2 \right) \, dx \, \frac{d\tau}{1 + \tau^2}
\]  

(3.20)

which is (3.10).
We now prove that, under suitable assumptions application \( w \mapsto b \ast w \) is invertible, and obtain an inversion formula. We use truncated Neumann series and a special assumption (see (b₃) below) in order to control the remainder term.

For \( b \in L^1(\mathbb{R}) \), let the \( k \)-times convolution defined as \( b^{*k} := b \ast b \ast \cdots \ast b \). For \( 1 \leq q \leq +\infty \) and \( t_0 \in (0, +\infty] \), the mapping \( \mathcal{R}_{t_0,q} \) is defined by:

\[
\begin{align*}
\mathcal{R}_{t_0,q} : \left\{ \begin{array}{c}
L^q_{[0,t_0)}(-\infty, t_0) \\
W^{1,q}_{[0,t_0)}(-\infty, t_0)
\end{array} \right\} & \rightarrow b \ast \mathcal{R}_{t_0,q} w
\end{align*}
\]

Here \( b \ast w(t) := \int_0^t b(t - s)w(s)\,ds \), for any \( t < t_0 \). We always write \( \mathcal{R} \) in place of \( \mathcal{R}_{+\infty,2} \).

Next, function \( b \) is assumed to comply with:

\( (b_1) \) \( b \in W^{1,1}(0, +\infty), \ b(0_+) \neq 0, \)

\( (b_2) \) there exists \( M > 0, \beta > 0 \) s.t.

\[
|\mathcal{L}b(z)| \geq \frac{M}{1 + |z|^{\beta}}, \ \forall z \in \mathbb{C}, \ \text{Re}(z) \geq 0 \quad (3.21)
\]

\( (b_3) \) there exists \( p \in \mathbb{N}^*, \ p \geq 2 \) s.t.

\[
\mathcal{F}^{-1} \left[ \left( \frac{(\mathcal{F}b')^p}{\mathcal{F}b} \right) \right] \in L^1(\mathbb{R}) \quad (3.22)
\]

Notice that (b₁) and (b₂) imply the following: there exists \( M > 0 \) s.t.

\[
|\mathcal{F}b(\omega)| \geq \frac{M}{1 + |\omega|}, \ \forall \omega \in \mathbb{R} \quad (3.23)
\]

(see the proof of part (iii) in Lemma 3.3).

Our goal is to prove the following inversion Theorem:

**Theorem 3.1 (Inversion Theorem).** Let the assumptions (b₁) - (b₃) hold true. Then:

(i) for any \( 1 \leq q \leq +\infty \) and \( t_0 \in (0, +\infty] \), the mapping \( \mathcal{R}_{t_0,q} \) is a Banach isomorphism;

(ii) functions \( B_1, \ B_2 \) that depend only on \( b \) and are being given by

\[
B_1 = \sum_{k=1}^{p-1} (-1)^k \frac{(b')^k}{b^{k+1}(0_+)}
\]

\[
B_2 = \frac{(-1)^p}{b^p(0_+)} \mathcal{F}^{-1} \left[ \left( \frac{(\mathcal{F}b')^p}{\mathcal{F}b} \right) \right]
\]

belong to \( L^1_{\mathbb{R}^+}(\mathbb{R}) \);

(iii) for any \( l \in W^{1,q}_{[0,t_0)}(-\infty, t_0) \), one has

\[
\mathcal{R}_{t_0,q}^{-1}(l) = \frac{l'}{b(0_+)} + B_1 \ast l' + B_2 \ast l
\]

For the proof we first need to introduce and prove two preliminary Lemmas.
Lemma 3.5. Assume that $b \in W^{1,1}(\mathbb{R}^+_t)$, $b(0+) \neq 0$. Let $1 \leq q \leq +\infty$, $t_0 \in (0, +\infty)$. Then $\mathcal{R}_{t_0,q}$ is a continuous injection.

Proof. We begin by showing $\mathcal{R}_{t_0,q}$ is well defined and continuous. Since $b \in W^{1,1}(\mathbb{R}^+_t)$, it is clear that for any $w \in L^q_{(0,t_0)}(-\infty, t_0)$, the function $b \ast w$ belongs to $W^{1,q}_{(0,t_0)}(-\infty, t_0)$. Moreover, $(b \ast w)' = [b(0+)w + b' \ast w]$. Hence

$$\|\mathcal{R}_{t_0,q}(w)\|_{W^{1,q}(0,t_0)} \leq \left[\|b(0+)\| + \|b\|_{W^{1,1}(\mathbb{R}^+_t)}\right] \|w\|_{L^q(0,t_0)} \tag{3.27}$$

which proves $\mathcal{R}_{t_0,q}$ is indeed continuous.

Next, assume $w \in L^q_{(0,t_0)}(-\infty, t_0)$ satisfies $\mathcal{R}_{t_0,q}(w) = 0$. Deriving the later leads to

$$w(s) + \int_0^s \frac{b'(s - \tau)}{b(0^+)} w(\tau)d\tau = 0, \text{ a.e. } s < t_0 \tag{3.28}$$

Multiply \(e^{-\theta s}, \theta > 0\), and set $w_1(s) = e^{-\theta s}w(s)$, $b_1(s) = \frac{b'(s)}{b(0^+)}e^{-\theta s}$. Equality \(\text{(3.28)}\) can now be re-written as

$$w_1(s) + \int_0^t b_1(s - \tau)w_1(\tau)d\tau = 0, \text{ a.e. } s < t_0 \tag{3.29}$$

It implies that

$$\|w_1\|_{L^q(0,t_0)} \leq \|b_1\|_{L^1(\mathbb{R}^+_t)} \|w_1\|_{L^q(0,t_0)} \tag{3.30}$$

Notice that $\|b_1\|_{L^1} = \int_0^{+\infty} e^{-\theta s} \frac{|b'(s)|}{b(0^+)} ds \xrightarrow{\theta \to +\infty} 0$. Pick up a $\theta > 0$ large enough s.t. $\|b_1\|_{L^1(\mathbb{R}^+_t)} < 1$. From \(\text{(3.30)}\) we get $\|w_1\|_{L^1(0,t_0)} = 0$. Finally $w = 0$ and $\mathcal{R}_{t_0,q}$ is an injection mapping.

\[\square\]

Lemma 3.6. The Theorem \(\text{I.1}\) holds true for $t_0 = +\infty$ and $q = 2$.

Proof. The proof consists of three steps.

Step 1.
First we prove $\mathcal{R}$ is a Banach isomorphism. Due to Lemma \(\text{3.5}\), one only needs to prove $\mathcal{R}$ is surjective. To begin with, one establishes that, for any $w \in L^2_{\mathbb{R}^+_t}(\mathbb{R})$, one has (with $M > 0$ the constant in \(\text{3.23}\))

$$\|w\|_{L^2(\mathbb{R})} \leq \frac{1}{\sqrt{\pi M}} \|\mathcal{R}(w)\|_{H^1(\mathbb{R})} \tag{3.31}$$

Actually using Parseval’s identity and \(\text{1.19}\) one gets

$$\sqrt{2\pi} \|w\|_{L^2(\mathbb{R})} = \|\mathcal{F}w\|_{L^2(\mathbb{R})} = \left\|\frac{\mathcal{F}\mathcal{R}(w)}{\mathcal{F}b}\right\|_{L^2(\mathbb{R})} \leq \frac{1}{M} \left\|\left(1 + |\omega|\right)\mathcal{F}\mathcal{R}(w)\right\|_{L^2(\mathbb{R})} \tag{3.32}$$

Since $(1 + |\omega|) \leq \sqrt{2(1 + \omega^2)}$, inequality \(\text{3.31}\) implies inequality \(\text{3.32}\). Next, inequalities \(\text{3.27}\) and \(\text{3.31}\) prove that $\mathcal{R}(L^2_{\mathbb{R}^+_t}(\mathbb{R}))$ is closed. Therefore, in order to prove that $\mathcal{R}$ is surjective it is sufficient to show that the dense subset of $(\mathcal{C}_c^\infty(0, +\infty))(\mathbb{R})$ of $H^1_{\mathbb{R}^+_t}(\mathbb{R})$ is included in $\mathcal{R}(L^2_{\mathbb{R}^+_t}(\mathbb{R}))$. 

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Let \( r \in (C_c^\infty)_{(0, +\infty)}(\mathbb{R}) \). We search for \( w \in L^2_{\mathbb{R}_+}(\mathbb{R}) \) s.t. \( b * w = r \). Since we are unable to identify the support of \( w \) by Fourier transform, we use Laplace transform instead. Consider the function

\[
    z \in \{ z \in \mathbb{C} / \Re(z) \geq 0 \} \mapsto \frac{Lr(z)}{Lb(z)} \in \mathbb{C}
\]

which is well defined based on \((b_2)\) and the fact that \( r \in (C_c^\infty)_{(0, +\infty)}(\mathbb{R}) \). This function is clearly continuous on \( \Re(z) \geq 0 \) and analytic on \( \Re(z) > 0 \). As for any \( z \in \mathbb{C} \) and \( \gamma \in \mathbb{N} \), \( Lr^{(\gamma)}(z) = z^{\gamma}Lr(z) \), and as \( r^{(\gamma)} \in L^1(\mathbb{R}) \), we deduce that there exists \( m_1 \geq 0 \) s.t.

\[
    |Lr(z)| \leq \frac{m_1}{1 + |z|^{p+2}}, \quad \forall z \in \mathbb{C}, \quad \Re(z) \geq 0
\]

Now it easily follows the existence of \( m_2 \geq 0 \) s.t.

\[
    \left| \frac{Lr(z)}{Lb(z)} \right| \leq \frac{m_2}{1 + |z|^2}, \quad \forall z \in \mathbb{C}, \quad \Re(z) \geq 0 \tag{3.33} \]

Next, with the help of Bromwich-Mellin formula, for any \( t \in \mathbb{R} \) and for fixed \( x > 0 \), define \( w \) as

\[
    w(t) := \frac{1}{2\pi i} \int_{\mathbb{R}} e^{t(x+iy)} \frac{Lr}{Lb}(x + iy)dy \tag{3.34}
\]

Owing to Cauchy’s formula and invoking \((b_2)\), \( w \) thus defined is independent of \( x > 0 \). Also, for fixed \( t < 0 \), letting \( x \to +\infty \) in \((3.34)\) leads to \( w(t) = 0 \). This is \( w(t) = 0 \) for any \( t < 0 \). Next, for any fixed \( t \in \mathbb{R} \), using Lebesgue’s Theorem we calculate the limit for \( x \to 0 \) of \((3.34)\) and obtain \( w = \mathcal{F}^{-1}\left( \frac{F_r}{F_b} \right) \). By Parseval’s identity and by \((3.34)\), \( w \) is clearly an element of \( L^2_{\mathbb{R}_+}(\mathbb{R}) \) and satisfies \( \mathcal{R}(w) = r \). Therefore \( \mathcal{R} \) is surjective.

**Step 2.**

The task now is proving the representation formula. Let \( w \in L^2_{\mathbb{R}_+}(\mathbb{R}) \) and set \( l = \mathcal{R}(w) \). Derivation of the later gives

\[
    w + \frac{b'}{b(0_+)} * w = \frac{l'}{b(0_+)} \tag{3.35}
\]

Convolute \((3.32)\) with the operator \( \sum_{k=0}^{p-1} (-1)^k \left( \frac{b'}{b(0_+)} \right)^k * \) (by convention \( \left( \frac{b'}{b(0_+)} \right)^0 = \delta_0 \)). We obtain:

\[
    w = \frac{l'}{b(0_+)} + \left( B_1 * l' \right) + \frac{(-1)^{p}}{b^p(0_+)} [(b')^p * w] \tag{3.36}
\]

Since \( l = b * w \), we get \( \mathcal{F}l = \mathcal{F}b \mathcal{F}w \). Hence

\[
    \mathcal{F} [(b')^p * w] = (\mathcal{F}b')^p \frac{\mathcal{F}l}{\mathcal{F}b} \tag{3.37}
\]

By hypothesis \((b_3)\), \( (\mathcal{F}b')^p / \mathcal{F}b \in L^\infty(\mathbb{R}) \), which proves that inequality \((3.37)\) holds in \( L^2(\mathbb{R}) \) since \( \mathcal{F}l \in L^2(\mathbb{R}) \). This fact allows to state that \( (b')^p * w = B_2 * l \) with \( B_2 \) given by \((3.37)\).

Now, for any \( w \in L^2_{\mathbb{R}_+}(\mathbb{R}) \) and \( l = \mathcal{R}(w) \), \((3.36)\) gives the representation formula

\[
    w = \frac{l'}{b(0_+)} + B_1 * l' + B_2 * l \tag{3.38}
\]
Step 3.
Let us now show that the support of $B_1$ and that of $B_2$ are included in $\mathbb{R}_+$. Since the support of $b'$ is in $\mathbb{R}_+$, $B_1$ also has its support in $\mathbb{R}_+$ due to formula (3.24). Let $\rho \in \mathcal{D}(\mathbb{R})$ and set $w = \mathcal{R}^{-1}(\rho)$ (see Step 1.). Equation (3.38) now ensures that, a.e. $t < 0$,

$$0 = w(t) = \frac{\rho'(t)}{b(0_+)} + (B_1 * \rho') (t) + (B_2 * \rho) (t) \quad (3.39)$$

Since $\rho'(s) = 0$ a.e. $s < 0$ and since $B_1$ has support in $\mathbb{R}_+$, we get

$$(B_2 * \rho) (t) = 0, \text{ a.e. } t < 0 \quad (3.40)$$

Take $\rho \geq 0$, $\rho \neq 0$, and set $\rho_n(t) = n \rho(nt)$, $n \in \mathbb{N}^*$, $t \in \mathbb{R}$. We know that:

$$B_2 * \rho_n \xrightarrow{n \to +\infty} \| \rho \|_{L^1(\mathbb{R})} B_2 \quad (3.41)$$

Taking $\rho = \rho_n$ in (3.40) and using (3.41) we obtain $B_2 = 0$ a.e. $t < 0$. Hence $B_2$ has support in $\mathbb{R}_+$. \qed

We are now in a position allowing to prove the previously stated Inversion Theorem 3.3.

Proof. Proof of the Inversion Theorem

Let $q \in [1, +\infty)$ and $t_0 \in \mathbb{R}_+ \cup \{+\infty\}$. Define the mapping $\mathcal{S}_{t_0,q}$ by:

$$\mathcal{S}_{t_0,q} = \begin{cases} \frac{W_{[0,t_0]}^1}{W_{[0,t_0]}^q}(-\infty, t_0) & \longrightarrow \frac{L_q^1}{L_q^q}(-\infty, t_0) \\ l & \mapsto \frac{l'}{b(0_+)} + B_1 * l' + B_2 * l \end{cases}$$

with $B_1, B_2 \in L^1_{\mathbb{R}_+}(\mathbb{R})$ given by (3.24)–(3.25). Clearly $\mathcal{S}_{t_0,q}$ is well defined and continuous. We begin by studying the case $t_0 = +\infty$.

Notice that $\mathcal{S}_{+\infty,q} \circ \mathcal{R}_{+\infty,q}$ restricted to $D = L_q^q(\mathbb{R}) \cap L^1_{\mathbb{R}_+}(\mathbb{R})$ is the identity (see Lemma 3.5). Since $D$ is dense in $L_q^q(\mathbb{R})$, and $\mathcal{S}_{+\infty,q}$ and $\mathcal{R}_{+\infty,q}$ are continuous, we find that $\mathcal{S}_{+\infty,q} \circ \mathcal{R}_{+\infty,q}$ is the identity on $L_q^q(\mathbb{R})$. Similarly, $\mathcal{R}_{+\infty,q} \circ \mathcal{S}_{+\infty,q}$ is the identity on $W_{\mathbb{R}_+}^1(\mathbb{R})$. This proves the Theorem for $t_0 = +\infty$.

Assume now that $t_0 > 0$ and $q \in [1, +\infty]$. We know from Lemma 3.12 that $\mathcal{R}_{t_0,q}$ is continuous and injective. We now prove that $\mathcal{R}_{t_0,q}$ is surjective and that $\mathcal{S}_{t_0,q}$ is its inverse. Let $l \in W_{[0,t_0]}^1(-\infty, t_0)$ and extend $l$ into $L \in W_{\mathbb{R}_+}^1(\mathbb{R})$ by reflexion

$$L(t) = \begin{cases} l(t) & \text{for } t < t_0 \\ l(2t_0 - t) & \text{for } t > t_0 \end{cases}$$

Let $W = (\mathcal{S}_{+\infty,q})(L)$ and define $w \in L_q^q(0, t_0)$ as the restriction of $W$ to $(-\infty, t_0)$. Then, $b * w = b * W = l$ on $(-\infty, t_0)$, and:

$$w = W = \frac{L'}{b(0_+)} + B_1 * L' + B_2 * L, \text{ on } (-\infty, t_0) \quad (3.42)$$

This is $w = \mathcal{S}_{t_0,q}(l)$. This proves the Theorem. \qed

Notice that from hypotheses $(a_4)$, $(a_5)$ and Lemma 3.3 the above Inversion Theorem can be used with $b = a_n$. 

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4 Approximated problems and estimates.

4.1 Approximated and local problems. Preliminary notions and estimates.

Remark that $a$ is not smooth enough to ensure a straightforward local in time existence result for a solution $v$ to our problem. As a consequence we study the following approximated problem which we denote by $P_n$.

Problem $P_n$: find $v_n : \Omega \times [0, +\infty) \rightarrow \mathbb{R}$ s.t.

\[
(P_n)_1 \quad (v_n)_t = \int_0^{+\infty} a_n'(s) \frac{\partial}{\partial x} g \left( (\nabla v_n) x \right) (x, s) ds + f(x, t) \]

\[
(P_n)_2 \quad v_n = 0 \text{ on } \partial \Omega, \quad v_n(t) = 0, \quad \forall t < 0 \]

\[
(P_n)_3 \quad v_n(x, 0) = v_0(x) \text{ for } x \in \Omega \]

Given the assumptions on $g$ we conclude there exist $\gamma > 0$ and $\theta \in [0, 1]$ s.t.

\[
g'(y) < -\gamma, \quad \forall y \in [-\theta, \theta] \tag{4.1} \]

Clearly we can take the same $\theta$ as in assumption $(g_1)$. Moreover, there exists $K > 0$ s.t.

\[
|g'(y) - g'(0)| \leq Ky^2, \quad \forall y \in [-\theta, \theta] \tag{4.2} \]

In the above one may consider the same $K$ as in $(g_1)$.

Let us denote, for almost every $x \in \Omega$,

\[
u(x, t) = \int_0^t v_n(x, s) ds. \]

The proof of the next Proposition is very similar to that of Theorem III.10 in [2] and is omitted.

\begin{proposition}
Assume that the hypotheses $(g_1)$-$(g_3)$, $(f_1)$-$(f_4)$, $(v_0)$, and $(a_1)$-$(a_5)$ on the data hold true. Then the initial value problem $(P_n)_1$, $(P_n)_2$, $(P_n)_3$ has a unique solution $v_n$ defined on a maximal time interval $[0, T_n)$, $T_n > 0$, and s.t. $v_n \in C^0 \left( [0, T_n); H^2(\Omega) \right)$, $(v_n)_t \in C^0 \left( [0, T_n); H^1(\Omega) \right)$, $(v_n)_t \in C^0 \left( [0, T_n); H^1(\Omega) \right)$, $(v_n)_{tt} \in C^0 \left( [0, T_n); H^1(\Omega) \right)$, and $u_n \in C^0 \left( [0, T_n); H^3(\Omega) \right)$. Moreover, if

\[
\sup_{t \in [0, T_n]} \left\{ \|v_n(\cdot, t)\|_{H^2(\Omega)}^2 + \|(v_n)_t(\cdot, t)\|_{H^1(\Omega)}^2 + \|(v_n)_u(\cdot, t)\|_{L^2(\Omega)}^2 + \|u_n(\cdot, t)\|_{H^3(\Omega)}^2 \right\} < \infty \tag{4.3}
\]

and

\[
\sup_{x \in \Omega} \{\|(u_n)_x(\cdot, t)\| \leq \frac{\theta}{2} \}
\]

with $\theta$ as in $(g_4)$, then $T_n = +\infty$.

Notice that our functional framework is different from that of [2]. As a consequence, here it is necessary to obtain new estimates on $\|u_n\|_{H^1(\Omega)}$.

In this Section we obtain the necessary estimates to proving $T_n = +\infty$. These estimates will be proved to be independent of $n$, fact which allows to pass to the limit as $n \rightarrow +\infty$. To simplify notations, we drop the subscript $n$ of $a_n$, $v_n$ and $T_n$. 

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Drawing inspiration from \[\text{bh1}\], we introduce the following expressions:

\[
\mathcal{E}(t) = \sup_{s \in [0,t]} \left[ \int_{\Omega} \left( v^2 + v_x^2 + v_t^2 + v_{xt}^2 + v_{tt}^2 + u^2 + u_x^2 + u_{xx}^2 \right) (x,s) \, dx \right]
\]

\[
+ \int_{0}^{t} \int_{\Omega} \left( v^2 + v_x^2 + v_t^2 + v_{xt}^2 + v_{tt}^2 \right) (x,s) \, dx \, ds \tag{4.4} \]

and

\[
\nu(t) = \sup_{x \in \Omega} \sup_{s \in [0,t]} \left[ \sqrt{v^2 + v_x^2 + v_t^2} (x,s) \right] + \sqrt{\int_{0}^{t} \sup_{x \in \Omega} (v_x(x,s))^2 \, ds} \tag{4.5} \]

For simplicity let us denote

\[
\mathcal{E}_1(t) = \sup_{s \in [0,t]} \left[ \int_{\Omega} \left( v^2 + v_x^2 + v_t^2 + v_{xt}^2 + v_{tt}^2 \right) (x,s) \, dx \right]
\]

\[
+ \int_{0}^{t} \int_{\Omega} \left( v^2 + v_x^2 + v_t^2 + v_{xt}^2 \right) (x,s) \, dx \, ds \tag{4.6} \]

In fact \(\mathcal{E}_1(t)\) collects the terms of \(\mathcal{E}(t)\) which will be estimated in a first step with the help of energy estimates.

Remark that, due to Sobolev inequalities, there exists a constant \(C_\Omega > 0\) s.t.

\[
\nu(t) \leq C_\Omega \sqrt{\mathcal{E}(t)}, \quad \forall t \in [0,T) \tag{4.7} \]

and

\[
\sup_{x \in \Omega} \left| u_x(x,t) \right| \leq C_\Omega \sqrt{\mathcal{E}(t)}, \quad \forall t \in [0,T) \tag{4.8} \]

Next, from \(\text{p6}\), we get

\[
\mathcal{G}_t(x,t) = v_{xx}(x,t) \int_{0}^{+\infty} a'(s) \left[ g'(\tau_x(x,s)) - g'(0) \right] \, ds
\]

\[
- \int_{0}^{t} v_{xx}(x,s) a'(t-s) \left[ g'(\tau_x(x,t-s)) - g'(0) \right] \, ds
\]

\[
+ \int_{0}^{t} v_{xx}(x,s) \int_{t-s}^{+\infty} a'(\tau) g''(\tau_x(x,\tau)) \left[ v_x(x,t) - v_x(x,t-\tau) \right] \, d\tau \, ds \tag{4.9} \]

All subsequent estimates will be obtained under the following smallness hypothesis on \(\mathcal{E}(t)\):

\[
\mathcal{E}(t) \leq \frac{\theta^2}{4C_\Omega^2}, \quad \forall t \in [0,T) \tag{4.10} \]

which implies

\[
\sup_{x \in \Omega} \sup_{0 \leq t \leq T} \left| u_x(x,t) \right| \leq \frac{\theta}{2} \tag{4.11} \]
Lemma 4.1. Let \( t \in [0, T) \), assume (4.10) is satisfied. Then:

\begin{enumerate}[(i)]
\item \[ |g^{(j)}(\vec{v}_x(x,s)) - g^{(j)}(0)| \leq K \min \{ \nu(t) r_0(s), \theta \} \text{ a.e. } x \in \Omega, s \in [0, t], j = 0, 1, 2, 3 \]
\item \[ |G(x,t)| \leq K \nu(t) |v_{xx}(x, \cdot) \ast \psi(t), \text{ a.e. } x \in \Omega \]
\item \[ |G_t(x,t)| \leq K \nu(t) |v_{xx}(x, \cdot) \ast \psi(t), \text{ a.e. } x \in \Omega, \]
\end{enumerate}

where

\[ \psi(t) = |a'(t)| r_0(t) + 2 \int_t^{+\infty} |a'(\tau)| r_0(\tau) d\tau \]

Remark 4.1. Lemma (4.12) and the assumptions made about function \( g \) grant the fact that \( \psi \) in (4.19) is s.t. \( \psi \in L^1(\mathbb{R}_+) \).

Proof. (i) On one hand, as a consequence of (4.1) and (4.11) we have

\[ |g^{(j)}(\vec{v}_x(x,s)) - g^{(j)}(0)| \leq K |\vec{v}_x(x,s)|, j = 0, 1, 2, 3 \]

On the other hand,

\[ |\overline{v}_x(x,s)| \leq \int_{t-s}^{t} |v_x(x, \lambda)| d\lambda \leq s \sup_{t-s \leq \lambda \leq t} |v_x(x, \lambda)| \leq s \nu(t) \]

and

\[ |\overline{v}_x(x,s)| \leq \sqrt{s} \left[ \int_{t-s}^{t} |v_x(x, \lambda)|^2 d\lambda \right]^{1/2} \leq \sqrt{s} \nu(t) \]

which gives the result.

(ii) From (4.12) and (4.13) above one gets:

\[ |G(x,t)| \leq K \nu(t) \int_0^t |v_{xx}(x,s)| \int_{t-s}^{+\infty} a'(\tau) \min\{\tau, \sqrt{\tau}\} d\tau ds \]

\[ \leq K \nu(t) \int_0^t |v_{xx}(x,s)| \psi(t-s) ds \]

from which the result follows.

(iii) We use (4.12), (4.11), the fact that \( g''(0) = 0 \) and \( 0 \leq \theta \leq 1 \) to obtain:

\[ |G_t(x,t)| \leq K |v_{xx}(x,t)| \nu(t) \int_0^{+\infty} |a'(s)| r_0(s) ds \]

\[ + K \nu(t) \int_0^t |v_{xx}(x,s)| |a'(t-s)| r_0(t-s) ds \]

\[ + 2K \theta \nu(t) \int_0^t |v_{xx}(x,s)| \int_{t-s}^{+\infty} |a'(\tau)| r_0(\tau) d\tau ds \]

which gives the result.
4.2 Energy estimates.

The next Lemmas give energy estimates for the terms in $\mathcal{E}_1(t)$ (see (4.9)), as in [1]. In what follows, the notation $C > 0$ stands for a generic constant that is independent of $n$.

Lemma 4.2. Assume the inequality (4.1) holds true. Then

$$\int_\Omega v^2(x, t)dx - 2g'(0)Q(v_x, t, a) \leq V_0 + 2\sqrt{F}\sqrt{\mathcal{E}(t)} + 2K\|\psi\|_{L_1(\mathbb{R}^n)}\nu(t)\mathcal{E}(t)$$  \hspace{1cm} (4.20)  

Proof. For a fixed $t \in (0, T_0)$, we multiply (4.2) by $v(x, t)$ and integrate on $\Omega$ and on $(0, t)$. We get

$$\frac{1}{2} \int_\Omega v^2(x, t)dx - \frac{1}{2} \int_\Omega v'_0dx - g'(0)Q(v_x, t, a)$$

$$= \int_0^t \int_\Omega f(x,s)v(x,s)dxds + \int_0^t \int_\Omega \mathcal{G}(x,s)v(x,s)dxds$$ \hspace{1cm} (4.21)  

Observe that $\int_0^t \int f vdxds \leq \|f\|_{L^2(Q_t)}\|v\|_{L^2(Q_t)} \leq \sqrt{F}\sqrt{\mathcal{E}}$.

Now, using Lemma 4.1, we get

$$\left| \int_0^t \int_\Omega \mathcal{G}(x,s)v(x,s)dxds \right| \leq K\nu(t) \int_0^t \int_\Omega |v(x,s)| (|v_x| * |\psi|)(x,s)dxds$$

Using part (i) of Lemma 4.2 with $w_1 = v$, $w_2 = v_{xx}$ and $\varphi = |\psi|$ one gets

$$\left| \int_0^t \int_\Omega \mathcal{G}(x,s)v(x,s)dxds \right| \leq K\nu(t)\|\psi\|_{L_1(\mathbb{R}^n)}\mathcal{E}(t),$$

thus ending the proof. \hfill \Box

Lemma 4.3. Let $\bar{\sigma}$ and $\psi$ be given by (4.4) and (4.5), respectively. Under the assumption that (4.1) is fulfilled, one has the following inequality:

$$\int_\Omega v_t^2(x, t)dx - 2g'(0)Q(v_{xt}, t, a) \leq F + 2\|a\|_{L_1(\mathbb{R}^n)}\sqrt{V_0}\sqrt{\mathcal{E}(t)}$$

$$+ 2\sqrt{F}\sqrt{\mathcal{E}(t)} + 2K\left(\|\psi\|_{L_1(\mathbb{R}^n)} + \bar{\sigma}\right)\nu(t)\mathcal{E}(t)$$  \hspace{1cm} (4.22)  

Proof. First, we derivate w.r.t. $t$ and obtain

$$v_{tt}(x, t) + g'(0)a(0)v_{xx}(x, t) + g'(0)\int_0^t a'(t-s)v_{xx}(x, s)ds = f_t + \mathcal{G}_t$$ \hspace{1cm} (4.23)  

Secondly, multiplying the above by $v_t$ and integrating on $\Omega$ and on $[0, t]$ leads to

$$\frac{1}{2} \int_\Omega v_t^2(x, t)dx - \frac{1}{2} \int_\Omega v_t^2(x, 0)dx - g'(0)a(0) \int_0^t \int_\Omega v_xv_{xt}dxds$$

$$- g'(0) \int_0^t \int_\Omega a'(s-\tau)v_x(\tau)d\tau v_{xt}(s)dxds = \int_0^t \int_\Omega f_tv_tdxds + \int_0^t \int_\Omega \mathcal{G}_tv_tdxds$$ \hspace{1cm} (4.24)  


Observe now that
\[ \int_0^s a'(s - \tau)v_x(\tau)d\tau = -a(0)v_x(s) + a(s)v_x(0) + \int_0^s a(s - \tau)v_{xt}(\tau)d\tau \tag{4.25} \]
One now gets:
\[ \frac{1}{2} \int_\Omega v_t^2(x,t)dx - g'(0)Q(v_{xt},t,a) = \frac{1}{2} \int_\Omega v_t^2(x,0)dx - g'(0) \int_0^t \int_\Omega a(s)v_0''(x)v_t(x,s)dxds \]
\[ + \int_0^t \int_\Omega (f_tv_t)(x,s)dxds + \int_0^t \int_\Omega (G_tv_t)(x,s)dxds \tag{4.26} \]
Notice that
\[ v_t(x,0) = f(x,0) \tag{4.27} \]
which gives
\[ \int_\Omega v_t^2(x,0)dx \leq F. \] We also have
\[ \left| \int_0^t \int_\Omega a(s)v_t''(x)v_t(x,s)dxds \right| \leq \|v_t''\|_{L^2(\Omega)} \|a\|_{L^1(\mathbb{R}^+)} \sup_{0 \leq s \leq t} \|v_t(s)\|_{L^2(\Omega)} \leq \|a\|_{L^1(\mathbb{R}^+)} \sqrt{V_0} \sqrt{\mathcal{E}(t)} \tag{4.28} \]
and
\[ \int_0^t \int_\Omega (f_tv_t)(x,s)dxds \leq \sqrt{F} \sqrt{\mathcal{E}(t)} \tag{4.29} \]
Finally, invoking part (iii) of Lemma 4.3 and part (i) of Lemma 4.4, we deduce that
\[ \int_0^t \int_\Omega (G_tv_t)(x,s)dxds \leq K \sqrt{\nu(t)\mathcal{E}(t)} + K \nu(t)\|\psi\|_{L^1(\mathbb{R}^+)}\mathcal{E}(t) \tag{4.30} \]
and with the obtainment of this last estimates the proof ends.

Next, in order to obtain energy estimates for \( \int_\Omega v_\Omega^2(x,t)dx \) we shall use the difference operator \( (\Delta_h w)(x,t) = w(x,t + h) - w(x,t), \) for \( h > 0 \) small enough.

**Lemma 4.4.** Under the assumption that (4.10) is fulfilled, one has:
\[ \int_\Omega v_\Omega^2(x,t)dx - 2g'(0) \lim_{h \to 0^+} \frac{1}{h^2} Q(\Delta_h v_{xt},t,a) \leq C \left\{ F + \sqrt{F} \sqrt{\mathcal{E}(t)} \right. \]
\[ + \left. [\nu(t) + \nu^3(t)] \mathcal{E}(t) + \sqrt{V_0} \mathcal{E}(t) \right\} \tag{4.31} \]
For the Proof, see the Appendix Section.
Since \( \nu(t) \) and \( \mathcal{E}(t) \) are non-increasing functions in \( t \), we obtain as a consequence of Lemma 4.1 and Sobolev embeddings, that:

**Lemma 4.5.** Under the assumption stated in (4.10) one has
\[ \mathcal{E}_1(t) \leq C \left\{ V_0 + F + \left( \sqrt{V_0} + \sqrt{F} \right) \sqrt{\mathcal{E}(t)} + [\nu(t) + \nu^3(t)] \mathcal{E}(t) + \sqrt{V_0} \mathcal{E}(t) \right\} \tag{4.32} \]
4.3 Non-energy estimates.

In the following we obtain estimates for the other constitutive terms of \( \mathcal{E}(t) \).

Now, from \((4.3)\) and using for a.e. \( x \in \Omega \) the result of Theorem \((4.1)\) with \( b = a \),
\[ l(t) = \frac{1}{g'(0)} \left[ f(x, t) + \mathcal{G}(x, t) - v_t(x, t) \right], \]
and \( w(t) = v_{xx}(x, t) \), we deduce the equality

\[ v_{xx} = \frac{1}{g'(0)} \left[ \frac{1}{a(0)} \left( f_t + \mathcal{G}_t - v_t \right) + A_1 \ast (f_t + \mathcal{G}_t - v_t) + A_2 \ast (f + \mathcal{G} - v_t) \right] \quad (4.33) \]

where \( A_1, A_2 \in L^1_{[0, +\infty)}(\mathbb{R}) \) are two functions that depend on \( a_n \), with bounded \( L^1 \) norms which are independent of \( n \), due to \((a_2)\) and \((a_5)\).

We have the following estimate:

**Lemma 4.6.** Under the assumption stated in \((4.1)\) one has

\[
\int_\Omega v_{xx}(x, t)dx + \int_0^t \int_\Omega v_{xx}^2(x, s)dxds + \int_0^t \int_\Omega v_t^2(x, s)dxds \leq C [F + \mathcal{E}_1(t) + v(t)\mathcal{E}(t)] \quad (4.34)
\]

**Proof.** **Step 1.**

We multiply \((4.33)\) by \( v_{xx} \) and integrate on \( \Omega \). It is clear that, for any \( \eta > 0 \), we have

\[
\left| \int_\Omega (f_t - v_t) v_{xx} dx \right| \leq \eta \int_\Omega v_{xx}^2 dx + \frac{1}{2\eta} \int_\Omega (f_t^2 + v_t^2) dx \quad (4.35)
\]

From part (iii) in Lemma \((4.1)\) we obtain

\[
\left| \int_\Omega \mathcal{G}_t v_{xx} dx \right| \leq K\nu(t) \int_\Omega |v_{xx}(x, t)| (|v_{xx}| \ast |\psi|)(x, t) dx + \pi K\nu(t) \int_\Omega |v_{xx}(x, t)|^2 dx \quad (4.36)
\]

Further, with the help of part (ii) in Lemma \((4.1)\) we obtain

\[
\left| \int_\Omega \mathcal{G}_t v_{xx} dx \right| \leq K\nu(t) \| v_{xx}(\cdot, t) \|_{L^2(\Omega)} \| \psi \|_{L^1(\mathbb{R}^+)} \sup_{0 \leq \tau \leq t} \| v_{xx}(\cdot, \tau) \|_{L^2(\Omega)}
\]

\[ + \pi K\nu(t) \| v_{xx}(\cdot, t) \|_{L^2(\Omega)}^2 \leq K\nu(t) \left[ \| \psi \|_{L^1(\mathbb{R}^+)} + \pi \right] \mathcal{E}(t) \quad (4.37)
\]

For any \( \eta > 0 \) one has

\[
\left| \int_\Omega A_1 \ast (f_t - v_t) v_{xx} dx \right| \leq \| A_1 \|_{L^1(\mathbb{R}^+)} \| v_{xx}(\cdot, t) \|_{L^2(\Omega)} \sup_{0 \leq \tau \leq t} \left[ \| f_t(\cdot, \tau) \|_{L^2(\Omega)} + \| v_t(\cdot, \tau) \|_{L^2(\Omega)} + \right]
\]

\[ \leq \eta \| v_{xx}(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{2\eta} \| A_1 \|_{L^2(\mathbb{R}^+)} \sup_{0 \leq \tau \leq t} \left[ \| f_t(\cdot, \tau) \|_{L^2(\Omega)}^2 + \| v_t(\cdot, \tau) \|_{L^2(\Omega)}^2 \right] \quad (4.38)
\]

and also

\[
\left| \int_\Omega A_2 \ast (f_t - v_t) v_{xx} dx \right|
\]
\[ \eta \|v_{xx}(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2\eta} \|A_2\|_{L^1(\mathbb{R}^+)}^2 \sup_{0 \leq \tau \leq t} \left[ \|f(\cdot, \tau)\|_{L^2(\Omega)} + \|v_t(\cdot, \tau)\|_{L^2(\Omega)}^2 \right] \geq 0. \tag{4.39} \]

We now have:

\[
\left| \int_\Omega (A_1 \ast \mathcal{G}_t)(x, t)v_{xx}(x, t)dx \right| \leq \pi K \nu(t) \int_\Omega (|A_1| \ast |v_{xx}|)(x, t) |v_{xx}(x, t)| dx
\]
\[
+ K \nu(t) \int_\Omega (|A_1| \ast |\psi| \ast |v_{xx}(x, t)|)(x, t) |v_{xx}(x, t)| dx \tag{4.40} \]

Then:

\[
\left| \int_\Omega (A_1 \ast \mathcal{G}_t)(x, t)v_{xx}(x, t)dx \right| \leq K \nu(t) \left[ \pi \|A_1\|_{L^1(\mathbb{R}^+)} + \|A_1\|_{L^1(\mathbb{R}^+)} \right] \|v_{xx}(\cdot, t)\|_{L^2(\Omega)} \sup_{0 \leq \tau \leq t} \|v_{xx}(\cdot, \tau)\|_{L^2(\Omega)}. \tag{4.41} \]

This gives

\[
\left| \int_\Omega (A_1 \ast \mathcal{G}_t)(x, t)v_{xx}(x, t)dx \right| \leq C \nu(t) t \mathcal{E}(t) \tag{4.42} \]

Likewise,

\[
\left| \int_\Omega (A_2 \ast \mathcal{G}_t)(x, t)v_{xx}(x, t)dx \right| \leq C \nu(t) t \mathcal{E}(t) \tag{4.43} \]

Now, from the above estimates, one gets for any \( \eta > 0 \) small enough leads to

\[
\sup_{0 \leq s \leq t} \int_\Omega v_{xx}^2(x, s)dx \leq C [F + \mathcal{E}_1(t) + \nu(t) \mathcal{E}(t)] \tag{4.44} \]

Step 2.

We multiply (1.10) by \( v_{xx} \) and integrate on \((0, t)\) and on \(\Omega\). Proceeding as in Step 1., using part (i) in Lemma 4.3, one gets for any \( \eta > 0 \) that

\[
\int_{Q_t} \left[ f_t + \mathcal{G}_t + A_1 \ast f_t + A_2 \ast (f - v_t) + A_1 \ast \mathcal{G}_t + A_2 \ast \mathcal{G} \right] v_{xx} dx ds
\]
\[
\leq \eta \int_{Q_t} v_{xx}^2 dx ds + \frac{C}{\eta} \left[ F + \mathcal{E}_1(t) \right] + C \nu(t) \mathcal{E}(t) \tag{4.45} \]

We are left to focus on terms that contain \( v_{tt} \). Invoking density arguments,

\[
\int_{Q_t} (v_{tt} v_{xx})(x, s) dx ds = \int_\Omega (v_{xx} v_t)(x, t) dx - \int_\Omega v_0''(x)v_t(x, 0)dx + \int_{Q_t} v_{xx}^2 dx ds \tag{4.46} \]

which gives, using (4.27),

\[
\left| \int_{Q_t} (v_{tt} v_{xx})(x, s) dx ds \right| \leq \|v_{xx}(\cdot, t)\|_{L^2(\Omega)} \|v_t(\cdot, t)\|_{L^2(\Omega)}
\]
\[ + \|v_0''\|_{L^2(\Omega)} \|f(\cdot, 0)\|_{L^2(\Omega)} + \int_{Q_t} v_{xx}^2(x, s)dxds \quad (4.47) \]

Finally we have:

\[ \int_{Q_t} (A_1 \ast v_t)(x, s)v_{xx}(x, s)dxds = \int_{Q_t} (A_1 \ast v_t)_t v_{xx}(x, s)dxds \]
\[ - \int_{Q_t} A_1(s)v_t(x, 0)v_{xx}(x, s)dxds \quad (4.48) \]

Again, calling in the density arguments leads to

\[ \int_{Q_t} (A_1 \ast v_t)_t (x, s)v_{xx}(x, s)dxds = \int_{Q_t} (A_1 \ast v_t)(x, t)v_{xx}(x, t)dx \]
\[ + \int_{Q_t} (A_1 \ast v_{xt}) v_{xt}dxds \quad (4.49) \]

From equalities (4.48) and (4.49) one easily gets:

\[ \left| \int_{Q_t} (A_1 \ast v_t) v_{xx}(x, s)dxds \right| \leq \|A_1\|_{L^1(\mathbb{R}^+)} \]
\[ \left[ \int_{Q_t} v_{xx}^2dxds + \|v_{xx}(\cdot, t)\|_{L^2(\Omega)} \sup_{0 \leq \tau \leq \theta} \|v_t(\cdot, \tau)\|_{L^2(\Omega)} + \|f(\cdot, 0)\|_{L^2(\Omega)} \sup_{0 \leq \tau \leq \theta} \|v_{xx}(\cdot, \tau)\|_{L^2(\Omega)} \right] \quad (4.50) \]

Now, adding inequalities (4.45), (4.44) and (4.50) and upon using (4.47) it allows us to get

\[ \int_{Q_t} v_{xx}^2(x, t)dx \leq C [F + \mathcal{E}_1(t) + \nu(t)\mathcal{E}(t)] \quad (4.51) \]

**Step 3.**

We now multiply (4.48) by \( v_{tt} \) and integrate on \( Q_t \). We have the listed below results:

\[ \left| \int_{Q_t} v_{xx}v_{tt}dxds \right| \leq \eta \int_{Q_t} v_{tt}^2dxds + \frac{1}{4\eta} \int_{Q_t} v_{xx}^2dxds \quad (4.52) \]

\[ \int_{Q_t} (a' \ast v_{xx}) v_{tt}dxds \leq \|a'\|_{L^1(\mathbb{R}^+)} \|v_{xx}\|_{L^2(Q_t)} \|v_t\|_{L^2(Q_t)} \leq \eta \|v_t\|_{L^2(Q_t)} + \frac{1}{4\eta} \|a'\|_{L^1(\mathbb{R}^+)}^2 \|v_{xx}\|_{L^2(Q_t)}^2 \quad (4.53) \]

\[ \int_{Q_t} f_t v_{tt}dxds \leq \|v_t\|_{L^2(Q_t)}^2 + \frac{1}{4\eta} \|f_t\|_{L^2(Q_t)}^2 \quad (4.54) \]

\[ \int_{Q_t} \mathcal{G}_t v_{tt}dxds \leq \mathbf{\pi} \nu(t) \int_{Q_t} |v_{xx}| |v_t| dxds + \nu(t) \int_{Q_t} (|v_{xx}| \ast |\psi|) |v_{tt}| dxds \leq \nu(t) (\mathbf{\pi} + \|\psi\|_{L^1(\mathbb{R}^+)}) \mathcal{E}(t) \quad (4.55) \]

We then obtain, taking \( \eta \) small enough and using (4.51), that

\[ \int_{Q_t} v_{xx}^2(x, t)dxds \leq C [F + \mathcal{E}_1(t) + \nu(t)\mathcal{E}(t)] \quad (4.56) \]

Now from estimates (4.41), (4.51) and (4.56) we obtain the result of Lemma 4.10.
Now we take on to obtaining estimates for \( u \) defined as \( u(x,t) = \int_0^t v(x,s)ds \). The idea is to integrate \((4.33)\) w.r.t. \( t \); one gets:

\[
u_{xx} = \frac{1}{g'(0)} \left\{ \frac{f + \mathcal{G}}{a(0)} - v_t + \int_0^t [A_1 * (f_t + \mathcal{G}_t - v_t)](x,s)ds + \int_0^t [A_2 * (f + \mathcal{G} - v_t)](x,s)ds \right\}
\]

(4.57)

We shall use in the following the below Lemma:

**Lemma 4.7.** Suppose that \( A \in L^1(0,T), \varphi \in W^{1,1}(0,T) \). Then, for any \( t \in (0,T) \), we have

\[
\int_0^t (A * \varphi')(s)ds = A * [\varphi - \varphi(0)]H
\]

(4.58)

*Proof.* The proof is a direct consequence of Fubini’s Theorem.

Recall from \((1.27)\) that \((f + \mathcal{G} - v_t)(x,0) = 0\). Then \((4.57)\) can be re-written in the form

\[
u_{xx} = \frac{1}{g'(0)} \left\{ \frac{f + \mathcal{G}}{a(0)} - v_t + A_1 * (f + \mathcal{G} - v_t) + A_2 * \left[ \int_0^t f(x,s)ds + \int_0^t \mathcal{G}(x,s)ds - v + v_0 \right] \right\}
\]

(4.59)

We deduce from the above equation that

\[
u_{xxx} = \frac{1}{g'(0)} \left\{ \frac{f_x + \mathcal{G}_x - v_{xt}}{a(0)} + A_1 * (f_x + \mathcal{G}_x - v_{xt}) + A_2 * \left[ \int_0^t f_x(x,s)ds + \int_0^t \mathcal{G}_x(x,s)ds - v_x + v_0' \right] \right\}
\]

(4.60)

We can now prove the following:

**Lemma 4.8.** Assume the assumption formulated in \((4.10)\) holds true. Then

\[
\sup_{0 \leq s \leq t} \|u_{xx}(\cdot, s)\|_{L^2(\Omega)}^2 \leq C \left\{ V_0 + F + \nu^2(t)\mathcal{E}(t) + \mathcal{E}^3(t) + \mathcal{E}_1(t) \right\}
\]

(4.61)

and

\[
\sup_{0 \leq s \leq t} \|u_{xxx}(\cdot, s)\|_{L^2(\Omega)}^2 \leq C \left\{ V_0 + F + \nu^2(t)\mathcal{E}(t) + \nu^2(t)\mathcal{E}^2(t) + \mathcal{E}^3(t) + \mathcal{E}_1(t) \right\}
\]

(4.62)

where \( C > 0 \) is a constant which is independent of \( n \).
Proof. The proof is performed in two steps.

**Step 1.**

Here we obtain the necessary estimates for $G(t)$, $\int_0^t G(s)ds$, $G_x(t)$ and for $\int_0^t G_x(s)ds$. Using (4.6) and part (i) of Lemma 4.1, we have

$$|G(t)| \leq K \nu(t) \int_0^{+\infty} |a'(s)| r_0(s) |u_{xx}(x, t) - u_{xx}(x, t - s)| ds$$

(4.63)

and this gives

$$\|G(\cdot, t)\|_{L^2(\Omega)} \leq 2K \nu(t) \int_0^{+\infty} |a'(s)| r_0(s) ds \left( \sup_{0 \leq s \leq t} \|u_{xx}(\cdot, s)\|_{L^2(\Omega)} \right) \leq C \nu(t) \sqrt{\mathcal{E}(t)}$$

(4.64)

On the other hand, using (4.6) and (4.7), we have that

$$\left| \int_0^t G(x, s)ds \right| \leq K \int_0^{+\infty} |a'(\tau)| \int_0^t |\nabla^2_x(x, \tau)|^2 |u_{xx}(x, s) - u_{xx}(x, s - \tau)| ds d\tau$$

(4.65)

which implies, taking the $L^2(\Omega)$-norm, that

$$\left\| \int_0^t G(\cdot, s)ds \right\|_{L^2(\Omega)} \leq 2K \left( \sup_{0 \leq \tau \leq t} \|u_{xx}(\cdot, \tau)\|_{L^2(\Omega)} \right) \int_0^{+\infty} |a'(\tau)| \int_0^t |\nabla^2_x(\cdot, \tau)|^2 ds d\tau$$

(4.66)

Now we have by Sobolev inclusions:

$$\|\nabla^2_x(\cdot, \tau)\|_{L^\infty(\Omega)} \leq C \int_{s-\tau}^s \|\nabla^2(\cdot, \lambda)\|_{H^2(\Omega)} d\lambda \leq 2C \tau \mathcal{M} \left( \|\nabla\|_{H^2(\Omega)} \right)(s)$$

(4.67)

where $\nabla(x, s)$ is the function defined on $\Omega \times \mathbb{R}$ by

$$\nabla(x, s) = \begin{cases} v(x, s) & \text{for } s \in [0, t) \\ 0 & \text{for } s \in \mathbb{R} \setminus [0, t) \end{cases}$$

(4.68)

and

$$\mathcal{M} \left( \|\nabla\|_{H^2(\Omega)} \right)(s) = \sup_{\rho > 0} \frac{1}{2\rho} \int_{s-\rho}^{s+\rho} \|\nabla(\cdot, \tau)\|_{H^2(\Omega)} d\tau$$

(4.69)

is the maximal function of $s \mapsto \|\nabla(\cdot, s)\|_{H^2(\Omega)}$ (see (4.7)).

Now, the maximal inequality (see Theorem 1, page 5 in (4.8) in this case leads to

$$\int_\mathbb{R} \mathcal{M} \left( \|\nabla(\cdot, s)\|_{H^2(\Omega)}^2 \right)(s) ds \leq 2\sqrt{10} \int_\mathbb{R} \|\nabla(\cdot, s)\|_{H^2(\Omega)}^2 (x, s) ds$$

$$= 2\sqrt{10} \int_0^t \|v(\cdot, s)\|_{H^2(\Omega)}^2 (x, s) ds$$

(4.70)

Then, from (4.7) and (4.10) by Sobolev inclusions we have that:

$$\int_0^t \|\nabla^2_x(\cdot, \tau)\|_{L^\infty(\Omega)}^2 d\tau \leq C \tau^2 \int_0^t \|v(\cdot, s)\|_{H^2(\Omega)}^2 ds$$

(4.71)
Next, with the help of (4.10) we deduce

\[ \left\| \int_0^t G(\cdot, s) ds \right\|_{L^2(\Omega)} \leq CK \sup_{0 \leq s \leq t} \left\| u_{xx}(\cdot, \tau) \right\|_{L^2(\Omega)} \int_0^t \left\| v(\cdot, s) \right\|^2_{H^2(\Omega)} ds \int_0^{+\infty} |a'(\tau)| \tau^2 d\tau \]  

(4.72)  

that is

\[ \left\| \int_0^t G(\cdot, s) ds \right\|_{L^2(\Omega)} \leq CE^{3/2}(t) \]  

(4.73)  

Next, let \( G_x(x, t) = I_1 + I_2 \), where

\[ I_1 = \int_0^{+\infty} a'(s)g'' \left( \overline{v}_x(s) \right) \left| \overline{v}_{xx}(s) \right|^2 ds \]  

(4.74)  

\[ I_2 = \int_0^{+\infty} a'(s) \left[ g' \left( \overline{v}_x(s) \right) - g'(0) \right] \overline{v}_{xxx}(s) ds \]  

(4.75)  

and also \( \int_0^t G_x(x, s) ds = I_3 + I_4 \), where

\[ I_3 = \int_0^t \int_0^{+\infty} a'(\tau)g'' \left( \overline{v}_x(\tau) \right) \left| \overline{v}_{xx}(\tau) \right|^2 d\tau ds \]  

(4.76)  

\[ I_4 = \int_0^t \int_0^{+\infty} a'(\tau) \left[ g' \left( \overline{v}_x(\tau) \right) - g'(0) \right] \overline{v}_{xxx}(\tau) d\tau ds \]  

(4.77)  

Since \( \overline{v}'(s) = u(t) - u(t-s) \), using again part (i) in Lemma 4.1, we obtain

\[ \left\| I_1 \right\|_{L^2(\Omega)} \leq 2K\nu(t) \int_0^{+\infty} \left| a'(s) \right| r_0(s) \left[ \left\| u_{xx}^2(\cdot, t) \right\|_{L^2(\Omega)} + \left\| u_{xx}^2(\cdot, t-s) \right\|_{L^2(\Omega)} \right] ds \]  

\[ \leq 4K\nu(t) \sup_{0 \leq s \leq t} \left\| u_{xx}(\cdot, s) \right\|_{L^4(\Omega)}^2 \int_0^{+\infty} \left| a'(s) \right| r_0(s) ds \]  

(4.78)  

This gives further down by Sobolev inclusion:

\[ \left\| I_1 \right\|_{L^2(\Omega)} \leq 4K \left( \int_0^{+\infty} \left| a'(s) \right| r_0(s) ds \right) \nu(t)E(t) \]  

(4.79)  

Next, as in (4.10), one easily obtains that

\[ \left\| I_2 \right\|_{L^2(\Omega)} \leq 2K \left( \int_0^{+\infty} \left| a'(s) \right| r_0(s) ds \right) \nu(t)\sqrt{E(t)} \]  

(4.80)  

Moreover,

\[ \left\| I_3 \right\|_{L^2(\Omega)} \leq \]  

\[ K \int_0^t \int_0^{+\infty} \left| a'(\tau) \right| \left\| \overline{v}_x(\cdot, \tau) \right\|_{L^\infty(\Omega)} \left\| u_{xx}(\cdot, s) - u_{xx}(\cdot, s-\tau) \right\|_{L^\infty(\Omega)} \left\| \overline{v}_{xx}(\cdot, \tau) \right\|_{L^2(\Omega)} d\tau ds \]  

(4.81)  

As in the proof of (4.10) we have the following estimates:
\[
\|\bar{v}_x^s(\tau)\|_{L^\infty(\Omega)} \leq 2\tau \mathcal{M}\left(\|\bar{v}_x\|_{L^\infty(\Omega)}\right)(s)
\]
\[
\|\bar{v}_{xx}^s(\tau)\|_{L^2(\Omega)} \leq 2\tau \mathcal{M}\left(\|\bar{v}_{xx}\|_{L^2(\Omega)}\right)(s)
\]

which give
\[
\|I_3\|_{L^2(\Omega)} \leq 8K \sup_{0 \leq s \leq t} \|u_{xx}(\cdot, s)\|_{L^\infty(\Omega)} \int_0^{+\infty} |d'(\tau)| \tau^2 d\tau
\]
\[
\frac{\sqrt{\int_0^t \mathcal{M}\left(\|\bar{v}_x\|_{L^\infty(\Omega)}\right)^2(s) ds}}{\int_0^t \mathcal{M}\left(\|\bar{v}_{xx}\|_{L^2(\Omega)}\right)^2(s) ds}
\]

Using again the maximal inequality from \[^{4.87}\] and the Sobolev embeddings leads to
\[
\|I_3\|_{L^2(\Omega)} \leq C \sup_{0 \leq s \leq t} \|u(\cdot, s)\|_{H^\nu(\Omega)} \int_0^t \|v(\cdot, s)\|^2_{H^2(\Omega)} ds
\]

that is
\[
\|I_3\|_{L^2(\Omega)} \leq CE^{3/2}(t)
\]

Finally, for \(I_4\) we proceed as for obtaining \[^{4.85}\] and get
\[
\|I_4\|_{L^2(\Omega)} \leq CE^{3/2}(t)
\]

The above estimates lead to the below ones:
\[
\|G_x(\cdot, t)\|_{L^2(\Omega)} \leq C\nu(t) \left(\mathcal{E}(t) + \sqrt{\mathcal{E}(t)}\right)
\]
\[
\left\|\int_0^t G_x(\cdot, s) ds\right\|_{L^2(\Omega)} \leq CE^{3/2}(t)
\]

Step 2

From \[^{4.88}\] we obtain:
\[
\|u_{xx}(\cdot, t)\|_{L^2(\Omega)} \leq \frac{1}{|g'(0)|} \left\{ \frac{1}{a(0)} \left[ \|f(\cdot, t)\|_{L^2(\Omega)} + \|G(\cdot, t)\|_{L^2(\Omega)} + \|v_t(\cdot, t)\|_{L^2(\Omega)} \right] \right.
\]
\[
+ \left. \|A_1\|_{L^1(\mathbb{R}^+)} \sup_{0 \leq s \leq t} \left[ \|f(\cdot, s)\|_{L^2(\Omega)} + \|G(\cdot, s)\|_{L^2(\Omega)} + \|v_t(\cdot, s)\|_{L^2(\Omega)} \right] \right.
\]
\[
+ \left. \|A_2\|_{L^1(\mathbb{R}^+)} \sup_{0 \leq s \leq t} \left[ \left\| \int_0^s f(\cdot, \tau) d\tau \right\|_{L^2(\Omega)} + \left\| \int_0^s G(\cdot, \tau) d\tau \right\|_{L^2(\Omega)} + \|v(\cdot, s)\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)} \right] \right\}
\]

Using now \[^{4.86}\] and \[^{4.87}\] and the fact that \(\nu(t)\) and \(\mathcal{E}(t)\) are increasing functions we obtain \[^{4.81}\]. Next, \[^{4.82}\] is obtained in a similar manner: one produces an equality like that of \[^{4.85}\] satisfied by \(\|u_{xx}(\cdot, t)\|_{L^2(\Omega)}\) with \(f_x, G_x, v_{xx}, v_x, v'_0\) in place of \(f, G, v_t, v, v_0\). Using \[^{4.86}\] and \[^{4.87}\] we get \[^{4.62}\].
4.4 Smallness estimates.

The next Proposition proves the uniform boundedness of $\mathcal{E}(t)$.

**Proposition 4.2.** There exist two numbers $\overline{\mathcal{E}} > 0$ and $\delta > 0$ independent of $n$ such that, whenever $v_0$ and $f$ verify $F(f) + V_0(v_0) \leq \delta$, one has

$$\mathcal{E}(t) \leq \frac{\overline{\mathcal{E}}}{2}, \forall t \in [0, T)$$

(4.89)

**Proof.** Remark first that, capitalizing on \((4.23)\) and \((4.24)\), one has $v_t(x, 0) = f(x, 0)$, $v_{xt}(x, 0) = f_x(x, 0)$, $v_{tt}(x, 0) = -g''(0)a(0)v_0''(x) + f_t(x, 0)$. From the definition of $\mathcal{E}(t)$ we deduce

$$\mathcal{E}(0) \leq \left[1 + 2a^2(0) |g''(0)|^2 \right] \|v_0\|^2_{H^2(\Omega)} + \int_\Omega \left[f^2(x, 0) + f_x^2(x, 0) + 2f_t^2(x, 0)\right] dx$$

(4.90)

Therefore

$$\mathcal{E}(0) \leq 2 \left[1 + a^2(0) |g''(0)|^2 \right] (F + V_0)$$

(4.91)

We now use the fact that the seminorm $w \in H^2(\Omega) \mapsto \|w_{xx}\|_{L^2(\Omega)}$ is a norm on $H^2(\Omega) \cap H_0^1(\Omega)$, equivalent to the usual norm in $H^2(\Omega)$. We shall as well make use of the inequality

$$\left(\sqrt{V_0} + \sqrt{F}\right) \sqrt{\mathcal{E}(t)} \leq \eta \mathcal{E}(t) + \frac{1}{\eta}(V_0 + F), \text{ with } \eta > 0 \text{ small enough.}$$

From Lemmas \(4.5\) and \(4.6\), we deduce

$$\mathcal{E}(t) \leq C \left\{ V_0 + F + [\nu(t) + \nu^3(t)] \mathcal{E}(t) + \sqrt{V_0} \mathcal{E}(t) + \mathcal{E}^3(t) + \nu^2(t)\mathcal{E}^2(t) \right\}$$

(4.92)

provided \((4.92)\) holds true. Recall also the inequality \((4.1)\):

$$\nu(t) \leq c_3 \sqrt{\mathcal{E}(t)}, \forall t \in [0, T)$$

(4.93)

Then, we deduce from \((4.92)\) that

$$\mathcal{E}(t) \leq c_1 \left[ V_0 + F + \mathcal{E}^3(t) \right]$$

(4.94)

with $c_1 > 0$ a constant independent of $n$.

Now observe that we can choose $\overline{\mathcal{E}} > 0$ and $\delta > 0$ such that

$$\begin{align*}
&\begin{cases}
&c_1 \overline{\mathcal{E}}^2 \leq \frac{1}{2} \\
&\overline{\mathcal{E}} < \frac{\theta^2}{4C_\Omega} \\
&c_1 \delta \leq \frac{\overline{\mathcal{E}}}{4} \\
&2 \left[1 + a^2(0) |g''(0)|^2 \right] \delta \leq \frac{\overline{\mathcal{E}}}{2}
\end{cases} \quad (4.95)
\end{align*}$$

Let us now prove that, for any $t \in [0, T)$, \((4.90)\) holds true. Indeed, if the contrary were true, then invoking the continuity w.r.t. time there exists $t_2 \in (0, T)$ s.t. $\mathcal{E}(t) \leq \overline{\mathcal{E}}$, for any $t \in (0, t_2)$, but inequality \((4.89)\) is false on an interval $(t_1, t_2)$ with $0 < t_1 < t_2$. From the second inequality in \((4.95)\) we deduce that \((4.94)\) is satisfied on $[0, t_2]$. Using once more \((4.95)\) one gets
\[
\mathcal{E}(t) \leq \frac{\mathcal{E}(t)}{2} + \frac{\mathcal{E}}{4}
\] which triggers \( \mathcal{E}(t) \leq \frac{\mathcal{E}}{2} \) on \([0,t_2]\), hence a contradiction. This later fact ends the proof.

\[\square\]

5 Proof of the main result.

Remark that from Proposition 5.2 we actually deduce that for \( v_n \) - solution of \( (P_n)_1, (P_n)_2, (P_n)_3 \) - we have the following upper bounds:

\[
\begin{align*}
\sup_{t \in [0,T_n]} \left[ \|u_n(\cdot,t)\|_{H^3(\Omega)}^2 + \|u_n(\cdot,t)\|_{H^2(\Omega)}^2 + \|(u_n)u(\cdot,t)\|_{L^2(\Omega)}^2 + \|u_n)u(\cdot,t)\|_{L^2(\Omega)}^2 \right] \\
+ \int_0^{T_n} \left\{ \|v_n(\cdot,t)\|_{H^2(\Omega)}^2 + \|v_n(\cdot,t)\|_{H^1(\Omega)}^2 + \|v_nu(\cdot,t)\|_{L^2(\Omega)}^2 \right\} dt \leq \frac{\mathcal{E}}{2} \tag{5.1} \quad \text{(ub1)}
\end{align*}
\]

and

\[
\sup_{0 < s < t < T_n} \left| \int_{t-s}^t (v_n)_x (x,\tau)d\tau \right| \leq \theta \tag{5.2} \quad \text{(ub2)}
\]

We then deduce from Proposition 5.2 that \( T_n = +\infty \), so (5.1) and (5.2) are valid upon replacing \( T_n \) by \( +\infty \). It follows that there exist two limits

\[
u \in \bigcap_{m=0}^3 W^{m,\infty} \left( (0, +\infty); H^{3-m}(\Omega) \right)
\]

and

\[
v \in \left\{ \bigcap_{m=0}^2 W^{m,\infty} \left( (0, +\infty); H^{2-m}(\Omega) \right) \right\} \cap \left\{ \bigcap_{m=0}^2 W^{m,2} \left( (0, +\infty); H^{2-m}(\Omega) \right) \right\}
\]

with \( u(x,t) = \int_0^t v(x,s) ds \) s.t. (up to a subsequence of \( n \)) we have

\[
\frac{d^m u_n}{dt^m} \rightharpoonup \frac{d^m u}{dt^m} \quad \text{weakly * in} \quad L^\infty \left( (0, +\infty); H^{3-m}(\Omega) \right), \quad m = 0,1,2,3
\]

and

\[
\frac{d^m u_n}{dt^m} \rightharpoonup \frac{d^m v}{dt^m} \quad \text{weakly in} \quad L^2 \left( (0, +\infty); H^{2-m}(\Omega) \right), \quad m = 0,1,2.
\]

By the trace theorem we have \( v = 0 \) for \( x \in \partial\Omega, t \geq 0 \), and \( v(x,0) = v_0(x) \), for \( x \in \Omega \). Now remark that the equation \( (P_n)_1 \) can be written in the form

\[
(v_n)_t(x,t) = -\partial_x \int_0^t a_n(t-s)g' \left[ (u_n)_x(x,t) - (u_n)_x(x,s) \right] (v_n)_x(x,s) ds + f(x,t) \tag{5.3} \quad \text{(ub3)}
\]

We now pass to the limit in (5.3) above, for any fixed \( t \geq 0 \). By the trace theorem it is clear that \( (v_n)_t(\cdot,t) \xrightarrow{L^2(\Omega)} v_t(\cdot,t) \) weakly. Next, we take on to proving that

\[
\int_0^t a_n(t-s)g' \left[ (u_n)_x(x,t) - (u_n)_x(x,s) \right] (v_n)_x(x,s) ds
\]
weakly converges in $L^2(\Omega)$ towards

$$\int_0^t a(t-s)g'(u_x(x,t) - u_x(x,s))v_x(x,s)ds$$

Let $\phi \in L^2(\Omega)$ be fixed; we have to prove that

$$E_n \xrightarrow{n \to +\infty} E$$

where

$$E_n = \int_{Q_t} \phi(x)a_n(t-s)g'(u_n(x,t) - u_n(x,s))(v_n)_x(x,s)dxds$$

$$E = \int_{Q_t} \phi(x)a(t-s)g'(u_x(x,t) - u_x(x,s))v_x(x,s)dxds$$

By Sobolev compact inclusion we have that

$$(u_n)_x \xrightarrow{C(Q_t)} u_x$$

strongly and

$$(u_n)_x(\cdot,t) \xrightarrow{C(\Omega)} u_x(\cdot,t)$$

also strongly. From (5.2), with $T_n = +\infty$ we deduce

$$\sup_{0<s<t} \left| \int_{t-s}^t v_x(x,\tau)d\tau \right| \leq \theta$$

Making use of (5.7) leads to the strong convergence

$$g'((u_n)_x(x,t) - (u_n)_x(x,s)) \xrightarrow{C(Q_t)} g'(u_x(x,t) - u_x(x,s)).$$

Since $(v_n)_x \xrightarrow{L^2(Q_t)} v_x$ strongly and $a_n \xrightarrow{L^2(0,\infty)} a$ strongly (consequence of assumption (a2)), one easily gets (5.8) which ends the proof of Theorem 2.1.

6 A class of totally monotone functions compliant with hypotheses (a1) to (a5).

The goal here is to introduce a large class of functions $a$ compliant with assumptions (a1)-(a5). The following Lemma deals with sufficient conditions so that (a5) holds.

**Lemma 6.1.** Assume that $b \in W^{1,1}(0, +\infty)$ satisfies the following conditions

(i) $tb' \in L^1(0, +\infty)$

(ii) there exists $M_3 > 0$ and $\alpha_1 > 0$ s.t. $|Fb(\omega)| \geq \frac{M_3}{1 + |\omega|^{\alpha_1}}, \forall \omega \in \mathbb{R}$

(iii) there exists $M_4 > 0$ and $\alpha_2 > 0$ s.t. $|Fb'(\omega)| \leq \frac{M_4}{1 + |\omega|^{\alpha_2}}, \forall \omega \in \mathbb{R}$

(iv) there exists $\alpha_3 \in \mathbb{R}$ s.t. the function $\mathbb{R} \ni t \mapsto tb(t) \in \mathbb{R}$ is an element of $H^{\alpha_3}(\mathbb{R})$
Then there exists $M_5 > 0$ depending only on $M_3, M_4, \alpha_1, \alpha_2$ and $\alpha_3$, and $p \in \mathbb{N}^*$ depending only on $\alpha_1$ and $\alpha_2$ and $\alpha_3$, s.t.

$$
\frac{(\mathcal{F}b')^p}{\mathcal{F}b} \in \mathcal{F}\left(B_{L^1(\mathbb{R})}(0, M_6)\right)
$$

where

$$
M_6 = M_5 \left[ 1 + \|tb'\|_{L^1(\mathbb{R})} + \|tb\|_{H^\alpha(\mathbb{R})} \right]
$$

Proof. Since $H^1(\mathbb{R}) \subset \mathcal{F}L^1(\mathbb{R})$ and $\|\mathcal{F}^{-1}w\|_{L^1(\mathbb{R})} \leq C\|w\|_{H^1(\mathbb{R})}$, $\forall w \in H^1(\mathbb{R})$ (see [12]), it suffices to consider the $H^1$ norm of $E = \frac{[\mathcal{F}b']^p}{\mathcal{F}b}$. From hypotheses (ii) and (iii) it is clear that, for $p$ large enough depending on $\alpha_1$ and $\alpha_2$, we have

$$
\|E\|_{L^2(\mathbb{R})} \leq M_5
$$

where $M_5$ depends on $M_3, M_4$ and $\alpha_2$. We also have $E' = E_1 - E_2$, with

$$
E_1 := p\frac{[\mathcal{F}b']^{p-1}[\mathcal{F}b']}{\mathcal{F}b}
$$

$$
E_2 := \frac{[\mathcal{F}b']^{p-1}[\mathcal{F}b']}{(\mathcal{F}b)^2}
$$

Since $|([\mathcal{F}b'])' = |\mathcal{F}(tb')| \in L^{\infty}(\mathbb{R}^+)$, from the above mentioned assumptions we get there exists $p$ large enough depending on $\alpha_1$ and $\alpha_2$ s.t.

$$
\|E_1\|_{L^2(\mathbb{R})} \leq M_5\|tb'\|_{L^1(\mathbb{R})}
$$

From assumption (iv) and the fact that $|([\mathcal{F}b'])' = |\mathcal{F}(tb)|$ we have that the function $\omega \mapsto (1 + \omega^2)^{\alpha_3/2}(\mathcal{F}b)'(\omega) \in L^2(\mathbb{R})$, and, $\left\|(1 + \omega^2)^{\alpha_3/2}(\mathcal{F}b)'(\omega)\right\|_{L^2(\mathbb{R})} = \|tb\|_{H^\alpha(\mathbb{R})}$.

Then there exists $p$ large enough depending on $\alpha_1, \alpha_2$ and $\alpha_3$ s.t.

$$
\|E_2\|_{L^2(\mathbb{R})} \leq M_5\|tb\|_{H^\alpha(\mathbb{R})}
$$

with $M_5$ as before. From (6.1), (6.4) and (6.5) the claimed result follows.

Let $\mu$ be a positive, finite and non-zero Borel measure on $\mathbb{R}^+$, satisfying

$(\mu_1)$: the function $\mathbb{R}^+ \ni \rho \mapsto \frac{1}{\rho^2}$ is an element of $L^1_{\mu}(0, +\infty)$

$(\mu_2)$: there exists $\gamma \in (0, 1)$ s.t. the function $\mathbb{R}^+ \ni \rho \mapsto \rho^\gamma$ is an element of $L^1_{\mu}(0, +\infty)$

Remark that, as a consequence of these hypotheses, the function $\mathbb{R}^+ \ni \rho \mapsto \rho^\beta$ is an element of $L^1_{\mu}(0, +\infty)$ for any $\beta \in [-2, \gamma]$.

We now consider the following totally monotone function (see [12])

$$
\hat{a} : [0, +\infty) \to \mathbb{R}, \hat{a}(t) = \int_{\mathbb{R}^+} e^{-\rho t} d\mu(\rho), \forall t \geq 0
$$

This Section main result is contained in the below theorem:
Theorem 6.1. Assume the hypotheses \((\mu_1)\) and \((\mu_2)\) hold true. Then the function \(\tilde{a}_n\) given by \((\mu_3)\) satisfies the hypotheses \((a_1)-(a_3)\) of Section \(\S\) with

\[
\tilde{a}_n(t) = \int_{[0,n]} e^{-\rho t} d\mu(\rho), \forall t \geq 0, \forall n \in \mathbb{N}^*
\]

Proof. Since the measure \(\mu\) is finite, it is clear that \(\tilde{a}_n \in C^\infty(\mathbb{R}_+)\), and for any \(t \in \mathbb{R}_+\) and \(k \in \mathbb{N}\), \((\tilde{a}_n)^{(k)}(t) = \int_{[0,n]} (-1)^k \rho^k e^{-\rho t} d\mu(\rho)\). This gives \(\tilde{a}_n \in W^{p,\infty}(0, +\infty)\), for any \(p \in \mathbb{N}\) and also \(\tilde{a}_n' < 0\).

Let \(k \in \mathbb{N}\) and \(q \in \mathbb{R}_+\). Then

\[
\int_0^{+\infty} t^q (\tilde{a}_n)^{(k)}(t) dt = (-1)^k \int_0^{+\infty} t^q \int_{[0,n]} \rho^k e^{-\rho t} d\mu(\rho) dt = (-1)^k \int_{[0,n]} \rho^k \left( \int_0^{+\infty} t^q e^{-\rho t} dt \right) d\mu(\rho)
\]

Taking \(\tau = \rho t\) in the integral w.r.t. \(t\) leads to

\[
\int_0^{+\infty} t^q \left| (\tilde{a}_n)^{(k)}(t) \right| dt = \int_0^{+\infty} \tau^q e^{-\tau} d\tau \int_{[0,n]} \rho^{k-q-1} d\mu(\rho) \tag{6.9} \]

Invoking hypotheses \((\mu_1)\) and \((\mu_2)\) gives

\[
\int_{[0, +\infty)} \rho^{k-q-1} d\mu(\rho) < \infty \tag{6.10} \]

provided that

\[
0 \leq q + 1 - k \leq 2 \tag{6.11}
\]

For \(q = 0\) and \(k = 0\) or \(k = 1\) one sees that \((\mu_1)\) is verified, therefore \((a_1)\) and \((a_2)\) are valid.

For \(q = 2\) and \(k = 1\) \((\mu_1)\) is also verified, then \(\int_0^{+\infty} t^2 |\tilde{a}_n'(t)| dt\) is bounded. The same for \(q = 1\) and \(k = 2\), with this time \(\int_0^{+\infty} t |\tilde{a}_n''(t)| dt\) bounded. The later grants \((a_3)\) is valid.

Next, by Fubini’s theorem we obtain, for \(\omega \in \mathbb{R}\),

\[
\mathcal{F}\tilde{a}_n(\omega) = \int_0^{+\infty} \int_{[0,n]} e^{-\rho t} d\mu(\rho) e^{-i\omega t} dt = \int_{[0,n]} \frac{d\mu(\rho)}{\rho + i\omega}
\]

from which one gets

\[
\text{Re} [\mathcal{F}\tilde{a}_n(\omega)] = \int_{[0,n]} \frac{\rho}{\rho^2 + \omega^2} d\mu(\rho)
\]

Now, assumption \((\mu_1)\) gives \(\mu\left(\{0\}\right) = 0\), so, there exists \(\underline{\mu}\) and \(\overline{\mu}\) s.t. \(0 < \underline{\mu} < \overline{\mu}\) and \(\mu\left([\underline{\mu}, \overline{\mu}]\right) > 0\). Take \(n > \overline{\mu}\) to get

\[
\text{Re} [\mathcal{F}\tilde{a}_n(\omega)] \geq \frac{\underline{\mu}}{\overline{\mu}^2 + \omega^2} \mu\left([\underline{\mu}, \overline{\mu}]\right), \forall \omega \in \mathbb{R}
\]

which proves \((a_4)\).

Now we prove that the hypotheses of Lemma \((\mu_4)\) are verified for \(b = \tilde{a}_n\), with constants independent of \(n\).
The last inequality also proves that (ii) of Lemma 6.1 is verified with $M_4$ independent of $n$ and $\alpha_1 = 2$. Taking $q = k = 1$ (which satisfy (6.11)) we deduce that part (i) of Lemma 6.1 is also verified, and that $\|\hat{t}_\alpha n\|_{L^1(0, +\infty)}$ is bounded.

Next, on one hand, we easily calculate

$$\mathcal{F}\tilde{a}_n'(\omega) = -\int_{[0,n)} \frac{\rho}{\rho + i\omega} d\mu(\rho)$$

which gives

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leq \int_{[0,n)} \frac{\rho}{\sqrt{\rho^2 + \omega^2}} d\mu(\rho)$$

We deduce that

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leq \int_{[0,n)} d\mu(\rho)$$

On the other hand now, we use the fact that

$$\rho^{2(1-\gamma)}|\omega|^{2\gamma} \leq \gamma|\omega|^2 + (1-\gamma)\rho^2 \leq |\omega|^2 + \rho^2$$

to get from (6.12), for $\omega \neq 0$, we get

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leq \int_{[0,n)} \frac{\rho}{|\omega|^\gamma} d\mu(\rho) = \frac{1}{|\omega|^\gamma} \int_{[0,n)} \rho^\gamma d\mu(\rho)$$

Invoke (\mu_2) to get, for $\omega \neq 0$,

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leq \frac{1}{|\omega|^\gamma} \int_{\mathbb{R}_+} \rho^\gamma d\mu(\rho)$$

Then, (6.13) and (6.14) give

$$|\mathcal{F}\tilde{a}_n'(\omega)| \leq \frac{2}{1 + |\omega|^\gamma} \int_{\mathbb{R}_+} (1 + \rho^\gamma) d\mu(\rho)$$

Then the assumption formulated in (iii) of Lemma 6.1 is verified with $\alpha_2 = \gamma$ and a constant $M_4$ independent of $n$.

Finally, the inequality (6.11) is verified with $q = 1$ and $k = 0$. From (6.3) and assumption (\mu_2) we get

$$\|\hat{t}_\alpha n\|_{L^1(\mathbb{R}_+)} \leq \int_0^{+\infty} \tau e^{-\tau} d\tau \int_{\mathbb{R}_+} \rho^{-2} d\mu(\rho) < \infty$$

The above entails $\hat{t}_\alpha n$ is bounded in $H^{-1}(\mathbb{R})$; consequently hypothesis (iv) of Lemma 6.1 is verified with $\beta = -1$. We then deduce that the conclusion of Lemma 6.1 is verified with a constant $M_6 > 0$ independent of $n$. Then hypothesis (a_5) is verified.

Remark 6.1. The relaxation function of the Doi-Edwards theory, $a_{DE}(t) = \sum_{k=1}^{+\infty} \frac{1}{(2k + 1)^2} e^{-(2k+1)^2 t}$, $t \geq 0$, is actually a particular case of (6.8) with the measure $\mu_{DE} = \sum_{k=1}^{+\infty} \frac{1}{(2k + 1)^2} \delta_{(2k+1)^2}$, where $\delta_{(2k+1)^2}$ is Dirac’s measure at $(2k+1)^2$.

It is easy to see that the assumptions (\mu_1), (\mu_2) are verified for this measure, and this paper results can be applied for the $a_{DE}$ function.
Lemma 7.1. Invoking the above defined notations,

(i) one has: \( \xi \in C^1(D_T; H^1(\Omega)) \), \( \frac{\partial^2 \xi}{\partial t^2} \in C^0(D_T; L^2(\Omega)) \);

(ii) assuming (7.1) holds true, one has the following estimates a.e. \( x \in \Omega, s \in [0, +\infty) \)

\[
|\xi(s, t, x)| \leq K\nu(t) |a'(s)| r_0(s) \tag{7.1}
\]

\[
\left| \frac{\partial \xi}{\partial t}(s, t, x) \right| \leq 2K\theta \nu(t) |a'(s)| \tag{7.2}
\]

\[
\left| \frac{\partial \xi}{\partial s}(s, t, x) \right| \leq K\nu(t) \left[ |a''(s)| r_0(s) + \theta |a'(s)| \right] \tag{7.3}
\]

\[
\left| \frac{\partial^2 \xi}{\partial t^2}(s, t, x) \right| \leq 4\nu^2(t) \left[ K\theta + |g^{(3)}(0)| \right] |a'(s)| + K\nu(t) |a'(s)| r_0(s) |v_{xt}(x, t)| + |v_{xt}(x, t - s)| \tag{7.4}
\]

The above derivatives may be considered in the classical sense, as they are defined for \( s \neq t \).

Proof. Observe that

\[
\frac{\partial \xi}{\partial t} = a'(s) g''(\overline{v}_x(s)) \left[ v_{xt}(t) - v_x(t - s) \right]
\]

\[
\frac{\partial \xi}{\partial s} = a''(s) \left[ g'(\overline{v}_x(s)) - g'(0) \right] + a'(s) g''(\overline{v}_x(s)) v_x(t - s)
\]

\[
\frac{\partial^2 \xi}{\partial t^2} = a'(s) g^{(3)}(\overline{v}_x(s)) \left[ v_{xt}(t) - v_x(t - s) \right]^2 + a'(s) g''(\overline{v}_x(s)) \left[ v_{xt}(t) - v_{xt}(t - s) \right]
\]

Repeated use of part (i) of Lemma 7.1 triggers the result. \( \square \)

For sake of clarity and - last but not least - reader’s convenience, we restate Lemma’s content and then achieve its proof.

Lemma 7.2. Under the assumption that (7.1) is fulfilled, one has:

\[
\int_\Omega v_n^2(x, t) dx - 2g'(0) \lim_{h \to 0} \frac{1}{h^2} Q(\Delta_h v_{xt}, t, a) \leq C \left\{ F + \sqrt{F} \sqrt{E(t)} + [\nu(t) + \nu^3(t)] E(t) + \sqrt{V_0} E(t) \right\} \tag{7.5}
\]
Proof. Derivate \( \frac{\partial}{\partial t} \) w.r.t. \( t \) and apply \( \Delta_h \) on the resulting equation. One gets:

\[
\Delta_h v_t = \int_0^{+\infty} a'(s) \Delta_h \left(g \left( \overline{w}_x(s) \right) \right) xt \, ds + \Delta_h f_t
\]  

Multiply the above by \( \Delta_h v_t \), integrate on \( \Omega \times [0, t] \) to obtain

\[
\frac{1}{2} \int_\Omega [\Delta_h v_t(x, t)]^2 \, dx - \frac{1}{2} \int_\Omega [\Delta_h v_t(x, 0)]^2 \, dx
\]

leads to

\[
- \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h g \left( \overline{w}_x(x, \tau) \right) \Delta_h v_{xt}(x, s) \, d\tau \, dx \, ds
\]

where:

\[
I_1 = - \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_{xt}(x, s) \Delta_h g' \left( \overline{w}_x(x, \tau) \right) \left[ v_x(s + h) - v_x(s + h - \tau) \right] \, d\tau \, dx \, ds
\]

\[
I_2 = - \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_{xt}(x, s) \left[ g' \left( \overline{w}_x(x, \tau) \right) - g'(0) \right] \Delta_h v_x(x, s) \, d\tau \, dx \, ds
\]

\[
I_3 = g'(0) \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_{xt}(x, s) \left[ \Delta_h v_x(s - \tau) - \Delta_h v_x(s) \right] \, d\tau \, dx \, ds
\]

\[
I_4 = \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_{xt}(x, s) \left[ g' \left( \overline{w}_x(x, \tau) \right) - g'(0) \right] \Delta_h v_x(s - \tau) \, d\tau \, dx \, ds
\]

Integrating by parts w.r.t. \( s \) leads to \( I_1 = I_{11} + I_{12} \), where:

\[
I_{11} = - \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_x(x, t) \Delta_h g' \left( \overline{w}_x(x, \tau) \right) \left[ v_x(t + h) - v_x(t + h - \tau) \right] \, d\tau \, dx
\]

\[
+ \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_x(x, s) \Delta_h \left[ g'' \left( \overline{w}_x(x, \tau) \right) \left( v_x(s, x) - v_x(s, x - \tau) \right) \right]
\]

\[
[v_x(x, s + h) - v_x(x, s + h - \tau)] \, d\tau \, dx \, ds
\]

\[
+ \int_0^t \int_\Omega \int_0^{+\infty} a'(\tau) \Delta_h v_x(x, s) \Delta_h g' \left( \overline{w}_x(x, \tau) \right) \left[ v_{xt}(s + h) - v_{xt}(s + h - \tau) \right] \, d\tau \, dx \, ds
\]
and
\[
I_{12} = \int_{\Omega} \int_{0}^{+\infty} a'(\tau) \Delta_h v_x(0) \Delta_h g' \left( \mathfrak{p}_x(x, \tau) \right) [v_x(x, h) - v_x(x, h - \tau)] \, d\tau dx \\
- \int_{0}^{t} \int_{\Omega} a'(s + h) \Delta_h g' \left( \int_{0}^{s} v_x(x, \lambda) d\lambda \right) v_0'(x) \Delta_h v_x(x, s) dx ds
\]
(7.14)  

Observe that
\[
\int_{\Omega} \int_{0}^{+\infty} a'(\tau) \Delta_h v_x(0) \Delta_h g' \left( \mathfrak{p}_x(x, \tau) \right) [v_x(x, h) - v_x(x, h - \tau)] \, d\tau dx \\
= \int_{\Omega} [v_x(h) - v_x(0)] \int_{0}^{h} a'(\tau) \left[ g' \left( \int_{0}^{h} v_x(\lambda) d\lambda \right) - g'(0) \right] [v_x(h) - v_x(h - \tau)] \, d\tau \\
- \int_{\Omega} a(h) [v_x(h) - v_x(0)] \left[ g' \left( \int_{0}^{h} v_x(\lambda) d\lambda \right) - g'(0) \right] v_x(h) dx
\]
(7.15)  

By integrating the first term by parts w.r.t. \( \tau \) one gets
\[
I_{12} = \int_{\Omega} [v_x(h) - v_x(0)]^2 a(h) \left[ g' \left( \int_{0}^{h} v_x(\lambda) d\lambda \right) - g'(0) \right] dx \\
- \int_{\Omega} [v_x(h) - v_x(0)] \int_{0}^{h} a(\tau) g'' \left( \int_{0}^{h} v_x(\lambda) d\lambda \right) v_x(h - \tau) [v_x(h) - v_x(h - \tau)] \, d\tau dx \\
- \int_{\Omega} [v_x(h) - v_x(0)] \int_{0}^{h} a(\tau) \left[ g' \left( \int_{0}^{h} v_x(\lambda) d\lambda \right) - g'(0) \right] v_x(h - \tau) d\tau dx \\
- \int_{\Omega} a(h) [v_x(h) - v_x(0)] \left[ g' \left( \int_{0}^{h} v_x(\lambda) d\lambda \right) - g'(0) \right] v_x(h) dx \\
- \int_{0}^{t} \int_{\Omega} a'(s + h) \Delta_h g' \left( \int_{0}^{s} v_x(\lambda) d\lambda \right) v_0'(x) \Delta_h v_x(s) dx ds
\]
(7.16)  

Next, dividing the above by \( h^2 \), passing to the limit for \( h \to 0_+ \) and using the fact that \( v \) and its derivatives up to order 2 belong to \( \mathcal{C}^2 ([0, T]; L^2(\Omega)) \) leads to
\[
\frac{1}{h^2} I_{12} \xrightarrow{h \to 0_+} J_1 + J_{01}
\]
(7.17)  

where
\[
J_1 = - \int_{\Omega} \int_{0}^{+\infty} \partial_2 \xi(\tau, t, x) v_{xt}(x, t) [v_x(x, t) - v_x(x, t - \tau)] \, d\tau dx \\
+ \int_{0}^{t} \int_{\Omega} \int_{0}^{+\infty} \partial_2 \xi(\tau, s, x) v_{xt}(x, s) [v_x(x, s) - v_x(x, s - \tau)] \, d\tau dx ds \\
+ \int_{0}^{t} \int_{\Omega} \int_{0}^{+\infty} \partial_2 \xi(\tau, s, x) v_{xt}(x, s) [v_{xt}(x, s) - v_{xt}(x, s - \tau)] \, d\tau dx ds
\]
(7.18)  

and
\[
J_{01} = - \int_{0}^{t} \int_{\Omega} a'(s) v_{xt}(x, s) g'' \left( \mathfrak{p}_x(x, s) \right) v_x(x, s) v_0'(x) dx ds
\]
(7.19)
The term $I_2$ can be re-written as

$$ I_2 = -\frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \xi(\tau, t, x) \frac{\partial}{\partial s} |\Delta_h v_x|^2 (x, s) d\tau dx ds $$

$$ = -\frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \xi(\tau, t, x) |\Delta_h v_x(x, t)|^2 d\tau dx $$

$$ + \frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \partial_2 \xi(\tau, t, x) |\Delta_h v_x(x, s)|^2 d\tau dx ds $$

(7.20)  

Dividing by $h^2$ and passing to the limit for $h \to 0_+$ one obtains

$$ \frac{1}{h^2} I_2 \xrightarrow[h \to 0_+]{} J_2 $$

(7.21)  

where

$$ J_2 = -\frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \xi(\tau, t, x) |v_{xt}(x, t)|^2 d\tau dx $$

$$ + \frac{1}{2} \int_0^t \int_{\Omega} \int_0^{+\infty} \partial_2 \xi(\tau, t, x) |v_{xt}(x, s)|^2 d\tau dx ds $$

(7.22)  

Next, $I_3 = I_{31} + I_{32} + I_{33}$, where

$$ I_{31} = g'(0) \int_0^t \int_{\Omega} \int_0^{s} a'(\tau) \Delta_h v_{xt}(x, s) \Delta_h v_x(x, s - \tau) d\tau dx ds $$

(7.23)  

$$ I_{32} = g'(0) \int_0^t \int_{\Omega} \int_s^{s+h} a'(\tau) \Delta_h v_{xt}(x, s) v_x(x, s + h - \tau) d\tau dx ds $$

(7.24)  

$$ I_{33} = g'(0) a(0) \int_0^t \int_{\Omega} \Delta_h v_{xt}(x, s) \Delta_h v_x(x, s) dx ds $$

(7.25)  

Upon integration by parts w.r.t. $\tau$ leads to

$$ I_{31} = g'(0) \int_0^t \int_{\Omega} a(s) \Delta_h v_{xt}(x, s) \Delta_h v_x(x, 0) dx ds $$

$$ - g'(0) a(0) \int_0^t \int_{\Omega} \Delta_h v_{xt}(x, s) \Delta_h v_x(x, s) dx ds $$

$$ + g'(0) Q(\Delta_h v_{xt}, a, t) $$

(7.26)  

The above implies, upon simplification and integration by parts w.r.t. $s$, that

$$ I_3 = g'(0) Q(\Delta_h v_{xt}, a, t) + g'(0) \int_{\Omega} a(t) \Delta_h v_x(t) \Delta_h v_x(0) dx $$

$$ - g'(0) a(0) \int_{\Omega} (\Delta_h v_x(0))^2 dx - g'(0) \int_0^t \int_{\Omega} a'(s) \Delta_h v_x(s) \Delta_h v_x(0) dx ds $$

$$ - g'(0) \int_0^t \int_{\Omega} \int_s^{s+h} a'(\tau) \Delta_h v_t(x, s) v_{xx}(x, s + h - \tau) d\tau dx ds $$

(7.27)
Divide the above by $h^2$ and taking the lower limit for $h \to 0_+$ gives

$$\liminf_{h \to 0_+} \frac{1}{h^2} I_3 = g'(0) \liminf_{h \to 0_+} \frac{1}{h^2} Q(\Delta_h v_{xt}, a, t) + J_3$$  \hspace{1cm} (7.28) \hspace{1cm} \text{(lies164)}$$

where

$$J_3 = g'(0) \left\{ a(t) \int_\Omega v_{xt}(x, t)v_{xt}(x, 0)dx - a(0) \int_\Omega v_{xt}^2(x, 0)dx - \int_0^t \int_\Omega a'(s)v_{xt}(x, s)v_{xt}(x, 0)dxdt - \int_0^t \int_\Omega a'(s)v_{xt}(x, s)v_{xt}^2(x, 0)dxdt \right\}$$  \hspace{1cm} (7.29) \hspace{1cm} \text{(lies165)}$$

Next we end up with the same result as in (7.28) with $\left( \liminf_{h \to 0_+} \right)$ being replaced by $\left( \limsup_{h \to 0_+} \right)$.

Now we can write $I_4$ in the form:

$$I_4 = \int_0^t \int_\Omega \left[ \int_0^s \xi(\tau, s)\Delta_h v_x(s - \tau)d\tau + \int_s^{s+h} \xi(\tau, s)v_x(s + h - \tau)d\tau \right] \Delta_h v_{xt}(x, s)dxd\tau$$  \hspace{1cm} (7.30) \hspace{1cm} \text{(lies166)}$$

An integration by parts w.r.t. $s$ gives

$$I_4 = I_{41} + I_{42} + I_{43} + I_{44}$$  \hspace{1cm} (7.31) \hspace{1cm} \text{(lies167)}$$

where

$$I_{41} = -\int_0^t \int_\Omega \int_0^s \partial_2 \xi(\tau, s)\Delta_h v_x(x, s - \tau) + \xi(\tau, s)\Delta_h v_{xt}(x, s - \tau) \right] d\tau \Delta_h v_x(x, s)dxd\tau$$  \hspace{1cm} (7.32) \hspace{1cm} \text{(lies168)}$$

$$I_{42} = -\int_0^t \int_\Omega \int_s^{s+h} \partial_2 \xi(\tau, s)v_x(x, s + h - \tau) + \xi(\tau, s)v_{xt}(x, s + h - \tau) \right] d\tau \Delta_h v_x(x, s)dxd\tau$$  \hspace{1cm} (7.33) \hspace{1cm} \text{(lies169)}$$

$$I_{43} = -\int_0^t \int_\Omega \left[ \xi(s + h, s) - \xi(s, s) \right] v_0(x)\Delta_h v_x(x, s)dxd\tau$$  \hspace{1cm} (7.34) \hspace{1cm} \text{(lies6n0)}$$

and

$$I_{44} = \left[ \int_\Omega \int_0^{s+h} \xi(\tau, s)\Delta_h v_x(x, s - \tau)\Delta_h v_x(x, s)d\tau dxd\tau \right]_{s=0}^{s=t}$$  \hspace{1cm} (7.35) \hspace{1cm} \text{(lies6n1)}$$

We now deal with the second term in $I_{41}$; we have:

$$-\int_0^s \xi(\tau, s)\Delta_h v_{xt}(x, s - \tau)d\tau = \xi(s, s) [v_x(h) - v_x(0)] - \int_0^s \partial_1 \xi(\tau, s)\Delta_h v_x(x, s - \tau)d\tau$$  \hspace{1cm} (7.36) \hspace{1cm} \text{(lies6n2)}$$

fact that allows to get

$$I_{41} = -\int_0^t \int_\Omega \int_0^s \left[ \partial_1 \xi(\tau, s) + \partial_2 \xi(\tau, s) \right] \Delta_h v_x(x, s - \tau)\Delta_h v_x(x, s)d\tau dxd\tau$$
\[ + \int_0^t \int_{\Omega} \xi(s, s) [v_x(h) - v_x(0)] \Delta_h v_x(x, s) \, dx \, ds \]  

(7.37)

Now we obtain

\[ \frac{1}{h^2} J_4 \to J_4 + J_{04} \]  

(7.38)

where

\[ J_4 = - \int_0^t \int_{\Omega} \left[ \partial_1 \xi(\tau, s) + \partial_2 \xi(\tau, s) \right] v_{xt}(x, s - \tau) v_{xt}(x, s) \, d\tau \, dx \, ds \]

(7.39)

and

\[ J_{04} = - \int_0^t \int_{\Omega} \left[ \partial_1 \xi(s, s) + \partial_2 \xi(s, s) \right] v_0'(x) v_{xt}(x, s) \, dx \, ds \]  

(7.40)

Now, from \( J_1 - J_4 \) being given by \( (7.18), (7.22), (7.25) \) and \( (7.38) \), respectively. One now needs to appropriately bound the terms \( J_1 - J_4, J_{01} \) and \( J_{04} \). It may be easily seen, using Lemma \( (7.1) \) that all terms \( J_1, J_2 \) and \( J_4 \) can be bounded by one of the following type of expressions:

\[ c v^k(t) \int_{\Omega} |w_1(x, t)| |w_2(x, t)| \, dx \]  

(7.42)

or

\[ c v^k(t) \int_0^t \int_{\Omega} |w_1(x, s)| |w_2(x, s)| \, dx \, ds \]  

(7.43)

or

\[ c v^k(t) \int_0^t \int_{\Omega} \varphi(\tau) |w_1(x, t - \tau)| |w_2(x, t)| \, d\tau \, dx \]  

(7.44)

or

\[ c v^k(t) \int_0^t \int_0^s \varphi(\tau) |w_1(x, s - \tau)| |w_2(x, s)| \, d\tau \, dx \]  

(7.45)

where \( \varphi \geq 0 \) is a given function in \( L^1(\mathbb{R}_+) \) depending on \( a, c > 0 \) is a constant, \( w_1, w_2 \) stand for either \( v \) or one of its derivatives up to second order, and \( k \in \{1, 2, 3\} \). This is a consequence of assumption \( (a_3) \).
Terms like (7.42) and (7.43) can easily be bounded by $c\nu(t)\mathcal{E}(t)$. Using Lemma 3.2, terms like (7.44) and (7.45) can also be easily bounded by $c\nu(t)\mathcal{E}(t)$. We then obtain that there exists a constant $c > 0$ s.t.

$$J_1 + J_2 + J_4 \leq c \left[ \nu(t) + \nu^3(t) \right] \mathcal{E}(t)$$

(7.46)

The estimates for $J_3$, $J_{01}$ and $J_{04}$ are simpler to obtain since they contain initial data. Using (4.27) we get $v_{xt}(x,0) = f(x,0)$. It easily follows that

$$|J_3| \leq |g'(0)| \left( |a(t)| + \|a'\|_{L^1} \right) \left[ \left( \sqrt{F} + \sqrt{V} \right) \sqrt{\mathcal{E}(t)} + a(0)F \right]$$

(7.47)

and

$$|J_{01}| \leq K\theta\|v_{0}\|_{H^2(\Omega)} \mathcal{E}(t)$$

(7.48)

and

$$|J_{04}| \leq 3K \left( \theta\|a'\|_{L^1} + \|a''r_0\|_{L^1} \right) \|v_0\|_{H^2(\Omega)} \nu(t) \sqrt{\mathcal{E}(t)}$$

(7.49)

From (7.41), (7.40) and (7.47), the result stated in the Lemma now follows.

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