HOW TO CALCULATE THE PROPORTION OF EVERYWHERE LOCALLY SOLUBLE DIAGONAL HYPERSURFACES

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Abstract. In this paper, we establish a strategy for the calculation of the proportion of everywhere locally soluble diagonal hypersurfaces of $\mathbb{P}^n$ of fixed degree. Our strategy is based on the product formula established by Bright, Browning and Loughran. Their formula reduces the problem into the calculation of the proportions of $\mathbb{Q}_v$-soluble diagonal hypersurfaces for all places $v$. As worked examples, we carry out our strategy in the cases of quadratic and cubic hypersurfaces. As a consequence, we prove that around 99.99% of diagonal cubic 4-folds have $\mathbb{Q}$-rational points under a hypothesis on the Brauer-Manin obstruction.

1. Introduction

In arithmetic geometry, for a given family of algebraic varieties, it is natural to ask how often its member has a rational point. Poonen and Voloch [19] gave a philosophy that, for a nice family, the proportion of members which have $\mathbb{Q}$-rational points can be represented by the product of the proportions of those which have $\mathbb{Q}_v$-rational points for every place $v$. Following their philosophy, the proportions have been calculated for some special families in e.g. [1, 2, 5, 6] under conditions on the Brauer-Manin obstruction. There are also other related works e.g. [3,4,16,18].

Among algebraic varieties, diagonal hypersurfaces of $\mathbb{P}^n$ have attracted special interest because of their remarkable arithmetic properties. For example, although cubic hypersurfaces of $\mathbb{P}^6$ may not have $\mathbb{Q}_v$-rational points in general, Lewis [14] proved that diagonal cubic hypersurfaces of $\mathbb{P}^6$ have $\mathbb{Q}_v$-rational points for every place $v$. Moreover, although cubic hypersurfaces of $\mathbb{P}^8$ may not have $\mathbb{Q}$-rational points in general, Hooley [11] proved that diagonal cubic hypersurfaces of $\mathbb{P}^8$ have $\mathbb{Q}$-rational points by establishing the Hasse principle for non-singular cubic hypersurfaces of $\mathbb{P}^8$ (see also [12]).

In this paper, we study the proportion of diagonal hypersurfaces of $\mathbb{P}^n$ which have $\mathbb{Q}$-rational points by following the above philosophy of Poonen and Voloch. Before our main results, let us explain some simple applications. Fix $n, k \in \mathbb{Z}_{\geq 2}$. For each $a = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$, let $X^k_a$ be a diagonal hypersurfaces of $\mathbb{P}^n$ defined by $\sum_{i=0}^n a_i x_i^k = 0$, and set $|a|$ as $\max\{|a_0|, |a_1|, \cdots, |a_n|\}$ with the Euclidean norm $|\cdot|$ on $\mathbb{R}$. Moreover, set

$$\rho(n, k) := \lim_{H \to \infty} \frac{\# \{ a \in \mathbb{Z}^{n+1} \mid |a| < H \text{ and } X^k_a(\mathbb{Q}) \neq \emptyset \}}{\# \{ a \in \mathbb{Z}^{n+1} \mid |a| < H \}}$$

if the limit exists.

Theorem 1.1. (1) For $k = 2$, we have the following table.

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For $k = 3$, we have the following table under the assumption that if $3 \leq n \leq 7$, then the Brauer-Manin obstruction is the only obstruction to the Hasse principle for diagonal cubic $(n-1)$-folds.

\[
\begin{array}{cccccc}
 n & 2 & 3 & 4 & 5 & \geq 6 \\
 \rho(n, 2) & 0 & 0.8268\ldots & 1 - 2^{-n} & \\
 \rho(n, 3) & 0 & 0.8964\ldots & 0.9965\ldots & 0.9999\ldots & 1
\end{array}
\]

(2) For $k = 3$, we have the following table under the assumption that if $3 \leq n \leq 7$, then the Brauer-Manin obstruction is the only obstruction to the Hasse principle for diagonal cubic $(n-1)$-folds.

For the proof of Theorem 1.1, see §3.3. Note that there is no known cubic $(n-1)$-fold $(n \geq 4)$ which violates the Hasse principle (cf. [10], p.49, [19], Conjecture 3.2 and Appendix A). For the detail on the Brauer-Manin obstruction, see e.g. [15].

In fact, some values in the above tables have been already known in the literature (e.g. [17],14,20), but the authors could not find any explicit references which contain the values $\rho(3, 2)$, $\rho(4, 3)$, $\rho(5, 3)$.

Thanks to [7, Theorem 1.4], under the condition in Theorem 1.1, the value $\rho(n, k)$ coincides with its local avatar $\rho_{loc}(n, k)$, which is defined by

\[
\rho_{loc}(n, k) := \lim_{H \to \infty} \frac{\{a \in \mathbb{Z}_{p}^{\otimes n+1} \mid |a| < H \text{ and } X_{a}(\mathbb{Q}_{p}) \neq \emptyset \text{ for every place } v\}}{\# \{a \in \mathbb{Z}_{p}^{\otimes n+1} \mid |a| < H\}}
\]

if the limit exists. For a sufficient condition so that $\rho(n, k) = \rho_{loc}(n, k)$ in the case $k \geq 4$ due to [8], see Remark 3.3. Moreover, set

\[
\rho_{v}(n, k) := \begin{cases} 
\mu_{p}\left(\{a \in \mathbb{Z}_{p}^{\otimes n+1} \mid X_{a}^{k}(\mathbb{Q}_{p}) \neq \emptyset\}\right) & \text{if } v = \text{prime } p, \\
2^{-n-1} \mu_{\infty}\left(\{a \in [-1,1]^{n+1} \mid X_{a}^{k}(\mathbb{R}) \neq \emptyset\}\right) & \text{if } v = \infty.
\end{cases}
\]

Here, $\mu_{p}$ is the Haar measure on $\mathbb{Z}_{p}$ normalized so that $\mu_{p}(\mathbb{Z}_{p}) = 1$ and $\mu_{\infty}$ is the Lebesgue measure on $\mathbb{R}$. We use the same letter $\mu_{p}$ (resp. $\mu_{\infty}$) also for the product measure on $\mathbb{Z}_{p}^{\otimes n+1}$ (resp. $\mathbb{R}^{\otimes n+1}$). The calculation of $\rho_{loc}(n, k)$ is reduced to that of $\rho_{v}(n, k)$ for every place $v$ by the following theorem of Bright, Browning and Loughran.

**Theorem 1.2** (a special case of [1] Theorem 1.3). The limit $\rho_{loc}(n, k)$ exists\(^a\) and is given by

\[
\rho_{loc}(n, k) = \prod_{v, \text{place}} \rho_{v}(n, k).
\]

Moreover, $\rho_{loc}(n, k)$ is positive whenever $n \geq 3$.

In this paper, we establish a strategy to calculate $\rho_{v}(n, k)$ for all $v$ for each fixed $n$ and $k$ (see Remark 2.4). Our strategy is regarded as a quantitative refinement of the argument in [7, §3]. As worked examples, we carry out our strategy in the cases $k = 2, 3$. In particular, Theorem 1.1 follows immediately from the following Theorems 1.3 and 1.4.

**Theorem 1.3** ($k = 2$). Suppose that $k = 2$.

1. If $n = 2$, then

\[
\rho_{p}(2, 2) = \begin{cases} 
\frac{7}{12} & \text{if } p = 2, \\
1 - \frac{3}{2} p^{-1} \left(\frac{1}{1 - p^{-1}} \right)^{2} & \text{otherwise}.
\end{cases}
\]

\(^{a}\) Note that $\rho_{loc}(2, k) = 0$ for every $k \geq 2$ as proven in [7, Theorem 1.1]. On the other hand, Proposition 2.3 implies that $\prod_{v, \text{place}} \rho_{v}(2, 2) = 0$ in the sense that $\sum_{v \in H} \log \rho_{v}(2, k) \to -\infty$ ($H \to \infty$) because $\sum_{p \leq H} \log \rho_{v}(2, k) \to -\infty$ ($H \to \infty$) (cf. [20], p75)).
(2) If \( n = 3 \), then
\[
\rho_p(3, 2) = \begin{cases} 
\frac{1231}{1296} & \text{if } p = 2, \\
1 - \frac{3}{2}b^{-2} \left( \frac{1 - p^{-1}}{1 - p^{-2}} \right)^4 & \text{otherwise.}
\end{cases}
\]

(3) (cf. [20 Corollary 2]) If \( n \geq 4 \), then \( \rho_p(n, 2) = 1 \).

Note that the value \( \rho_\infty(n, 2) \) equals \( 1 - 2^{-n} \) for every \( n \in \mathbb{Z}_{\geq 2} \).

**Theorem 1.4** (\( k = 3 \)). Suppose that \( k = 3 \).

1. If \( n = 2 \), then
\[
\rho_p(2, 3) = \begin{cases} 
\frac{13831}{19773} & \text{if } p = 3, \\
1 - 2p^{-1} \left( \frac{1 - p^{-1}}{1 - p^{-3}} \right) & \text{if } p \equiv 1 \text{ mod } 3, \\
1 - 6p^{-3} \left( \frac{1 - p^{-1}}{1 - p^{-3}} \right)^3 & \text{if } p \equiv 2 \text{ mod } 3.
\end{cases}
\]

2. (= [1 Theorem 2.2]) If \( n = 3 \), then
\[
\rho_p(3, 3) = \begin{cases} 
\frac{6391}{6591} & \text{if } p = 3, \\
1 - \frac{8}{3}b^{-2}(1 + p^{-1})^2 \left( \frac{1 - p^{-1}}{1 - p^{-3}} \right)^3 & \text{if } p \equiv 1 \text{ mod } 3, \\
1 & \text{if } p \equiv 2 \text{ mod } 3.
\end{cases}
\]

3. If \( n = 4 \), then
\[
\rho_p(4, 3) = \begin{cases} 
1 - \frac{40}{3}p^{-4} \left( \frac{1 - p^{-1}}{1 - p^{-3}} \right)^4 & \text{if } p \equiv 1 \text{ mod } 3, \\
1 & \text{otherwise.}
\end{cases}
\]

4. If \( n = 5 \), then
\[
\rho_p(5, 3) = \begin{cases} 
1 - \frac{80}{3}p^{-6} \left( \frac{1 - p^{-1}}{1 - p^{-3}} \right)^6 & \text{if } p \equiv 1 \text{ mod } 3, \\
1 & \text{otherwise.}
\end{cases}
\]

5. (cf. [14 Theorem 2]) If \( n \geq 6 \), then \( \rho_p(n, 3) = 1 \).

Note that the value \( \rho_\infty(n, 3) \) equals 1 for every \( n \in \mathbb{Z}_{\geq 2} \).

The plan of this paper is as follows. In §2 we give a general upper bound for the value \( \rho_p(n, k) \). It is sufficient for determination of the value \( \rho_p(n, k) \) for a generic prime. In §3 we calculate the values \( \rho_p(n, k) \) for pathological pairs \( (p, k) = (2, 2), (3, 3) \) and complete the proofs of Theorems 1.3 and 1.4 hence of Theorem 1.1. Our strategy in §3 works also for general \( n \) and \( k \) in principle. In §4 we discuss a consequence for the proportions of (uni)rationality.

**Notation.** For each prime \( p \), let \( \mathbb{Z}_p \) be the ring of \( p \)-adic integers and \( \mathbb{Q}_p \) be its field of fractions. We denote the (additive) \( p \)-adic valuation map by \( v_p : \mathbb{Q}_p^\times \to \mathbb{Z} \), and we use the same symbol also for its direct sum \( v_p : (\mathbb{Q}_p^\times)^{\oplus n+1} \to \mathbb{Z}^{\oplus n+1} \) defined by \( v_p(\mathbf{a}) := (v_p(a_0), \ldots, v_p(a_n)) \) for every \( \mathbf{a} = (a_0, \ldots, a_n) \in (\mathbb{Q}_p^\times)^{\oplus n+1} \).
2. $\rho_p(n,k)$ FOR GENERIC PRIMES

For every $k, r \in \mathbb{Z}_{\geq 0}$, set $[k] := \{0, 1, \ldots, k-1\} \subset \mathbb{Z}$, and let $[k]^{(r)}$ be the set of subsets of $[k]$ consisting of $r$ elements. For every $K = \{k_1, \ldots, k_d\} \subset [k]$, set $w(K) := k_1 + \cdots + k_d$.

In a similar manner to [1, §2.1.1], we define an equivalence relation $\simeq$ on $Q_p^{\oplus n+1}$ as follows. Set $\Gamma_p(n, k) := Q_p^X \times ((Q_p^k)^{\oplus n+1} \rtimes \mathcal{S}_{n+1})$. Here, the semi-direct product $(Q_p^k)^{\oplus n+1} \rtimes \mathcal{S}_{n+1}$ is defined by the natural left permutation action of $\mathcal{S}_{n+1}$. Define an action of $\Gamma_p(n, k)$ on $Q_p^{\oplus n+1}$ by

$$(\alpha; a_0^k, \ldots, a_n^k; \sigma): (a_0, \ldots, a_n) := (\alpha a_{\sigma(0)}(a_0)^k, \ldots, \alpha a_{\sigma(n)}(a_n)^k).$$

Define an equivalence relation $\simeq$ on $Q_p^{\oplus n+1}$ by $a \simeq b$ if there exists $\gamma \in \Gamma_p(n, k)$ such that $a = \gamma(b)$. Then, $X_a^k$ is isomorphic to $X_b^k$ over $Q_p$ if $a \simeq b$. Moreover, since the set $Q_p^{\oplus n+1} / \simeq$ is finite, we can calculate $\rho_p(n, k)$ for each fixed $n, k, p$ at least by tour de force analysis. In fact, by using the following Proposition 2.3, we can calculate $\rho_p(n, k)$ for many $n, k, p$ uniformly.

In what follows, the measure $\mu_p(\{a \in Z_p^{\oplus n+1} \mid \cdots\})$ is abbreviated by $\mu_p(\cdots)$. We set

$$\kappa_p(n, k) := \frac{\mu_p(v_p(a) = 0 \mod{k})}{\mu_p(v_p(a) = 0)} = \left( \sum_{e_0, \ldots, e_n \geq 0} p^{-ke_0-\cdots-ke_n} \right) = (1 - p^{-k})^{-n-1}.$$ 

Intuitively, this quantity gives the "expansion ratio by $(p^{kZ_{\geq 0}})^{\oplus n+1}$-action".

Let $\Delta_n$ be the set $\{a \in Z_p^{\oplus n+1} \mid \prod_{i=0}^n a_i = 0\}$. Then we have

$$\mu_p(X_a) \text{ is singular} = \mu_p(\Delta_n) = 0.$$ 

Therefore, it is sufficient to consider $a \in Z_p^{\oplus n+1} \setminus \Delta_n$, which are classified (not exclusively) into three types as follows.

**Definition 2.1.** Let $n, k \in \mathbb{Z}_{\geq 2}$, $p$ be a prime, and $a \in Z_p^{\oplus n+1} \setminus \Delta_n$.

1. If there exist some $u_0, \ldots, u_n \in Z_p^k$ and $k_3, \ldots, k_n \in [k]$ such that

$$a \simeq (u_0, u_1, u_2, p^{k_3}u_3, \ldots, p^{k_n}u_n),$$

then we say that $a$ is of type I.

2. If there exist some $u_1, \ldots, u_n \in Z_p^k$, $t \in Z_p^{\times k}$, and $k_2, \ldots, k_n \in [k]$ such that

$$a \simeq (u_1, -u_1t, p^{k_2}u_2, p^{k_3}u_3, \ldots, p^{k_n}u_n),$$

then we say that $a$ is of type II.

3. If there exist some $r \in \mathbb{Z}_{\geq 0}$, $u_1, \ldots, u_{n+1-r} \in Z_p^k$, $t_1, \ldots, t_r \in Z_p^{\times k}$, and distinct $k_1, \ldots, k_{n+1-r} \in [k]$ such that

$$a \simeq (p^{k_1}u_1, -p^{k_1}u_1t_1, \ldots, p^{k_r}u_{r+1}, -p^{k_r}u_{r+1}t_r, p^{k_{r+1}}u_{r+1}, \ldots, p^{k_{n+1-r}}u_{n+1-r}),$$

then we say that $a$ is of type III.

Moreover, by the Hasse-Weil bound for non-singular projective curves defined by $u_0x_0^k + u_1x_1^k + u_2x_2^k = 0$ $(u_0, u_1, u_2 \in Z_p^k)$, we obtain the following.

**Lemma 2.2.** Let $n, k \in \mathbb{Z}_{\geq 2}$, $p$ be a prime such that $\gcd(p,k) = 1$, and $a \in Z_p^{\oplus n+1} \setminus \Delta_n$. Then, the following statements hold.

1. Suppose that $p \geq (k-1)(k-2)$ or $\gcd(p-1,k) = 1$. If $a$ is of type I, then $X_a^k(Q_p) \neq \emptyset$.
2. If $a$ is of type II, then $X_a^k(Q_p) \neq \emptyset$.
3. If $a$ is of type III, then $X_a^k(Q_p) = \emptyset$. 

Proposition 2.3. Let $n, k \in \mathbb{Z}_{\geq 2}$, and $p$ be a prime. Suppose that $\gcd(p, \kappa) = 1$. Then, we have

$$
\rho_p(n, k) \leq 1 - (n + 1)! \left(1 - \frac{1}{p-1}\right)^n \sum_{r \geq 0} \left(1 - \frac{1}{2 \cdot \gcd(p-1, \kappa)}\right)^r \sum_{\substack{K \subset \mathbb{Q}_{\kappa}^{(r)} \\
L \subset \mathbb{Q}_{\kappa}^{(n+1-2r)} \cap K \cap L = \emptyset}} p^{-2 \cdot w(K)-w(L)}
$$

Moreover, if $p \geq (k-1)(k-2)$ or $\gcd(p-1, \kappa) = 1$, then the equality holds. \[\square\]

Here, the sum with respect to $r$ is finite which runs over $\max\{n - k + 1, 0\} \leq r \leq \min\{\frac{n+1}{2}, k\}$, where $[x]$ denotes the maximal integer not exceeding $x$.

Proof. The whole statement is a direct consequence of Lemma 2.2. Indeed, since the $\mathcal{O}_{n+1}$-orbit of $(p^{k_1}, p^{k_2}, \ldots, p^{k_r}, p^{k_{r+1}}, p^{k_{r+2}}, \ldots, p^{k_{n+1-r}})$ with distinct $k_i \in [k]$ consists of $(n + 1)!/2^r$ vectors, we obtain

$$
\rho_p(n, k) \leq 1 - \mu_p (a \text{ is of type III})
$$

$$
= 1 - \sum_{r \geq 0} \sum_{\{k_1, \ldots, k_r\} \subset [k] \{k_{r+1}, \ldots, k_{n+1-r}\} \subset [k] \setminus \{k_1, \ldots, k_r\}} \mu_p (a \text{ satisfies formula } \mathbb{Q} \text{ with some } u_i, t_i)
$$

$$
= 1 - \sum_{r \geq 0} \sum_{\substack{K = \{k_1, \ldots, k_r\} \subset [k]^{(r)} \\
L = \{k_{r+1}, \ldots, k_{n+1-r}\} \subset [k]^{(n+1-2r)} \cap K \cap L = \emptyset}} \bigg(\frac{n + 1)!}{2^r} p^{-2k_1-\cdots-2k_r-k_{r+1}-\cdots-k_{n+1-r}}
$$

$$
\times \kappa_p(n, k) \cdot \mu_p (a = (u_1, -u_1t_1, \ldots, u_r, -u_rt_r, u_{r+1}, \ldots, u_{n+1-r}) \text{ with some } u_i, t_i)
$$

$$
= 1 - (n + 1)! \sum_{r \geq 0} \frac{1}{2^r} \sum_{\substack{K \subset \mathbb{Q}_{\kappa}^{(r)} \\
L \subset \mathbb{Q}_{\kappa}^{(n+1-2r)} \cap K \cap L = \emptyset}} p^{-2 \cdot w(K)-w(L)}
$$

$$
\times (1 - p^{-k})^{-(n+1)} \left(1 - \frac{1}{\gcd(p-1, \kappa)}\right)^r (1 - p^{-1})^{n+1}
$$

as desired. \[\square\]

Remark 2.4. Thanks to Proposition 2.3 in order to determine $\rho_p(n, k)$ for all primes $p$, it is sufficient to consider the following two kinds of pathological primes:

(1) $\gcd(p, \kappa) \neq 1$.
(2) $p < (k-1)(k-2)$ and $\gcd(p-1, \kappa) \neq 1$.

The problems of these cases are as follows:

- In the case (1), every $\mathbb{F}_p$-rational point on a variety $X^k_a \mod p$ is singular.
- In the case (2), a scheme $X^k_a \mod p$ may not have a $\mathbb{F}_p$-rational point.

Anyway, since the number of pathological primes $p$ for each fixed $n$ and $k$ is finite, we can determine $\rho_p(n, k)$ for all primes $p$ by tour de force and eventually obtain $\rho_{\text{loc}}(n, k)$.

3. $\rho_p(n, k)$ for pathological primes

In this section, we carry out tour de force analysis in order to calculate $\rho_p(n, k)$ for pathological primes $p$. Although we consider only the cases $k = 2$ and $k = 3$, the same method also works for any $k$ in principle.

\[\text{In fact, if one of the following conditions holds, then } X^k_a(\mathbb{Q}_p) \neq \emptyset \text{ for all } a \in \mathbb{Z}^{\geq k} \setminus \Delta_k:\]

(1) $p \geq (k-1)(k-2)$ and $n \geq 2k$.
(2) $\gcd(p-1, k) = 1$ and $n \geq k$.

In particular, if (1) or (2) holds, then we obtain $\rho_p(n, k) = 1$. 

\[\text{as desired.} \]
If \( k = 2 \) (resp. 3), then there is no prime number of second kind in Remark 2.4. Therefore, it is sufficient to calculate \( \rho_2(n, 2) \) (resp. \( \rho_3(n, 3) \)) as we will do in what follows.

3.1. The case of \( k = 2 \) and \( p = 2 \).

**Proposition 3.1** \( (p = 2 \text{ and } n = 2) \). Let \( u_0, u_1, u_2 \in \mathbb{Z}_2^\times \).

1. \( X^2_{(u_0,a_1,u_2)} \) has a \( \mathbb{Q}_2 \)-rational point if and only if
   \[
   (u_0, u_1, u_2) \simeq (1, 1, 3), (1, 1, 7), (1, 3, 7).
   \]

2. \( X^2_{(u_0,u_1,2u_2)} \) has a \( \mathbb{Q}_2 \)-rational point if and only if
   \[
   (u_0, u_1, 2u_2) \simeq (1, 1, 6), (1, 1, 14), (1, 5, 2), (1, 7, 2), (1, 7, 6).
   \]

In particular, \( X^2_a \) has a \( \mathbb{Q}_2 \)-rational point if and only if
\[
a \simeq (1, 1, 3), (1, 1, 7), (1, 3, 7), (1, 1, 6), (1, 1, 14), (1, 5, 2), (1, 7, 2), (1, 7, 6).
\]

**Proof.**

1. We may assume that \( u_0 = 1 \) and \( u_1, u_2 \in \{1, 3, 5, 7\} \). Moreover, if \( u_1 = 7 \) or \( u_2 = 7 \), then \( X^2_{(1,u_1,u_2)} \) has a rational point. Therefore, it is sufficient to consider the following six cases.
   
   (a) If \( (u_1, u_2) = (1, 1) \), then we can check that \( X^2_{(1,1,1)} \) has no \( \mathbb{Q}_2 \)-rational point by the standard infinite descent argument with a help of modulo 8 calculation.
   
   (b) If \( (u_1, u_2) = (1, 3) \), then \( X^2_{(1,1,3)} \) has a \( \mathbb{Q}_2 \)-rational point \([\sqrt{-7} : 2 : 1]\).
   
   (c) If \( (u_1, u_2) = (1, 5) \), then we can check that \( X^2_{(1,1,5)} \) has no \( \mathbb{Q}_2 \)-rational point by the standard infinite descent argument with a help of modulo 8 calculation.
   
   (d) If \( (u_1, u_2) = (3, 3) \), then \( X^2_{(1,3,3)} \) is isomorphic to \( X^2_{(1,1,3)} \) over \( \mathbb{Q}_2 \) and has a \( \mathbb{Q}_2 \)-rational point.
   
   (e) If \( (u_1, u_2) = (3, 5) \), then \( X^2_{(1,3,5)} \) is isomorphic to \( X^2_{(1,7,3)} \) over \( \mathbb{Q}_2 \) and has a \( \mathbb{Q}_2 \)-rational point.
   
   (f) If \( (u_1, u_2) = (5, 5) \), then \( X^2_{(1,5,5)} \) is isomorphic to \( X^2_{(1,1,5)} \) over \( \mathbb{Q}_2 \) and has no \( \mathbb{Q}_2 \)-rational point.

2. We may assume that \( u_2 = 1 \) and \( u_0, u_1 \in \{1, 3, 5, 7\} \). Note that if \( x_0x_1 \equiv 0 \) (mod 2), then \( x_0 \equiv x_1 \equiv 0 \) (mod 2) and \( x_2 \equiv 0 \) (mod 2). Therefore, it is sufficient to consider rational points such that \( x_0, x_1 \in \mathbb{Z}_2^\times \).

   (a) If \( u_1 \equiv u_0 \) (mod 8), i.e., \( u_1 = u_0 \), then we have \( 2u_0 + 2x_2^3 \equiv 0 \) (mod 8), which has a \( \mathbb{Z}_2 \)-solution only if \( u_0 \equiv 3 \) (mod 4), i.e., \( (u_0, u_1) = (3, 3), (7, 7) \). We can check that \( X^2_{(3,3,2)} \) (resp. \( X^2_{(7,7,2)} \)) has a \( \mathbb{Q}_2 \)-rational point \([x_0 : x_1 : x_2] = [1 : 3 : \sqrt{-15}] \) (resp. \([1 : 1 : \sqrt{-7}] \)).

   (b) If \( u_1 \equiv 3u_0 \) (mod 8), then we have \( 4u_0 + 2x_2^3 \equiv 0 \) (mod 8), which is impossible.

   (c) If \( u_1 \equiv 5u_0 \) (mod 8), then we have \( 6u_0 + 2x_2^3 \equiv 0 \) (mod 8), which has a solution only if \( u_0 \equiv 1 \) (mod 4), i.e., \( (u_0, u_1) = (1, 5), (5, 1) \). We can check that \( X^2_{(1,5,2)} \) has a \( \mathbb{Q}_2 \)-rational point \([x_0 : x_1 : x_2] = [1 : 3 : \sqrt{-23}] \), and \([5, 1, 2] \simeq (1, 5, 2) \).

   (d) If \( u_1 \equiv 7u_0 \) (mod 8), then \( X^2_{(3,0,1,u_2)} \) has a \( \mathbb{Q}_2 \)-rational point \([x_0 : x_1 : x_2] = [1 : \sqrt{-7} : 0] \). Note that \( (1, 7, 14) \simeq (1, 7, 2) \) and \( (1, 7, 10) \simeq (1, 7, 6) \).

This completes the proof. \( \square \)

**Proposition 3.2** \( (p = 2 \text{ and } n = 3) \). Let \( a = (a_0,a_1,a_2,a_3) \in \mathbb{Z}_p^{3\times} \setminus \Delta_3 \). Then, \( X^2_a \) has no \( \mathbb{Q}_2 \)-rational point if and only if
\[
a \simeq (1, 1, 1, 1), (1, 1, 5, 5), (1, 1, 2, 2), (1, 1, 10, 10), (1, 3, 2, 6), (1, 3, 10, 14), (1, 5, 6, 14).
\]

**Proof.** First of all, note that we may assume that \( a_0 = 1 \), \( v_2(a_1) = 0 \leq v_2(a_2), v_2(a_3) \leq 1 \), and \( a_ip^{-v_p(a_i)} \in \{1, 3, 5, 7\} \).
Proposition 3.4

Remark 3.3. In fact, the \( \Gamma(3, 2) \)-orbits of the 7 vectors in the statement of Proposition 3.2 do not intersect each other. We can check it by noting that

(a) Suppose that \( a = (1, 1, 1) \). Then, \( X_a^2 \) has no \( \mathbb{Q}_2 \)-rational point.

(b) Suppose that \( a = (1, 1, 3) \). Then, \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point \( [x_0 : x_1 : x_2 : x_3] = [\sqrt{-7} : 2 : 0 : 1] \).

(c) Suppose that \( a = (1, 1, 5) \). Then, \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point \( [x_0 : x_1 : x_2 : x_3] = [\sqrt{-7} : 1 : 1 : 1] \).

(d) Suppose that \( a = (1, 1, 5, 5) \). Then, \( X_a^2 \) has no \( \mathbb{Q}_2 \)-rational point.

(2) Suppose that \( v_2(a) = (0, 0, 0, 1) \). Then, by the proof of Proposition 3.1 (2), it is sufficient to consider the cases \( (a_0, a_1, a_2) = (1, 1, 1), (1, 1, 5) \). Moreover, by Proposition 3.1 (1), it is sufficient to consider the cases \( (a_0, a_1, a_2, a_3) = (1, 1, 1, 2), (1, 1, 1, 10) \).

In each case, \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point \( [x_0 : x_1 : x_2 : x_3] = [\sqrt{-7} : 1 : 2 : 1], [\sqrt{-15} : 1 : 2 : 1] \) respectively.

(3) Suppose that \( v_2(a) = (0, 0, 1, 1) \). Then, by Proposition 3.1 (1), it is sufficient to consider the cases \( a_1 \in \{1, 3, 5\} \).

(a) Suppose that \( a_1 = 1 \). Then, by Proposition 3.1 (1), it is sufficient to consider the cases \( a_2, a_3 \in \{2, 10\} \). If \( (a_2, a_3) = (2, 10) \) (resp. \( (10, 2) \)), then \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point \( [x_0 : x_1 : x_2 : x_3] = [2 : 0 : \sqrt{-7} : 1] \) (resp. \( [2 : 0 : 1 : \sqrt{-7}] \)).

(b) Suppose that \( a_1 = 3 \). Then, by Proposition 3.1 (1), it is sufficient to consider the cases \( (a_2, a_3) = (2, 6), (10, 14) \). In both cases, \( X_a^2 \) has no \( \mathbb{Q}_2 \)-rational point.

(c) Suppose that \( a_1 = 5 \). Then, by Proposition 3.1 (1), it is sufficient to consider the cases \( a_2, a_3 \in \{6, 14\} \). If \( a_2 = a_3 = 6 \) (resp. \( a_2 = a_3 = 14 \)), then \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point \( [x_0 : x_1 : x_2 : x_3] = [6 : 0 : \sqrt{-7} : 1] \) (resp. \( [14 : 0 : \sqrt{-15} : 1] \)). If \( a_2 \neq a_3 \), then \( X_a^2 \) has no \( \mathbb{Q}_2 \)-rational point.

This completes the proof.

Remark 3.3. In fact, the \( \Gamma(3, 2) \)-orbits of the 7 vectors in the statement of Proposition 3.2 do not intersect each other. We can check it by noting that

- each orbit has a representative whose components lie in \( \{1, 3, 5, 7, 2, 6, 10, 14\} \),
- in terms of these representatives, the \( \Gamma(3, 2) \)-action is reduced to the action of a finite group \( \mathbb{Z}_2^2 / \mathbb{Z}_2^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_2 \times S_3 \).

Proposition 3.4 \((p = 2 \text{ and } n \geq 4)\). Let \( n \in \mathbb{Z}_{\geq 4} \) and \( a = (a_0, \ldots, a_n) \in \mathbb{Z}_2^{n+1} \setminus \Delta_n \).

Then, \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point.

Proof. By the case (b) in the proof for Proposition 3.2, it is sufficient to consider the case \( v_2(a) = 0 \).

Moreover, by the case (a) in the proof for \( n = 3 \), it is sufficient to consider the cases \( n = 4 \) and \( a = (1, 1, 1, 1, 1), (1, 1, 3, 5, 5) \). In each case, \( X_a^2 \) has a \( \mathbb{Q}_2 \)-rational point \( [x_0 : x_1 : x_2 : x_3 : x_4] = [\sqrt{-7} : 2 : 1 : 1 : 1], [0 : 0 : \sqrt{-15} : 3 : 0] \).

Therefore, for \( p \neq 2 \), the statement is immediate from Proposition 2.3. For \( p = 2 \), the statement is a direct consequence of Propositions 3.1, 3.2, and 3.4 as follows.

(1) For \( n = 2 \), we have

\[
\rho(2, 2) = \mu_2(a) \simeq (1, 1, 3, 1, 1, 7, 1, 1, 3, 7, 1, 1, 6, 1, 1, 14, 1, 5, 2, 1, 7, 2, 1, 7, 6) = \mu_2(v_2(a) = (0, 0, 0, 1, 1, 1)) \\
+ \mu_2(v_2(a) = (0, 0, 0, 1, 1, 1)) = \frac{2^6}{3^3} \cdot \frac{3}{4} \cdot \left( \frac{1}{2^3} + \frac{1}{2^6} \right) + \frac{2^6}{3^3} \cdot \frac{3}{2} \cdot \left( \frac{1}{2^4} + \frac{1}{2^5} \right) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}.
\]
Here, in order to obtain the second equality, we use, for example,
\[
\{a \in \mathbb{Z}_p^{\oplus 3} \mid a \simeq (1, 5, 2)\} = \prod_{k_0, k_1, k_2 \geq 0} (2^{2k_0}, 2^{2k_1}, 2^{2k_2}) \mathcal{S}_3 A
\]
and
\[
\mu_2(A) = \frac{3! \cdot \#(\mathbb{Z}_2/\mathbb{Z}_2^2)}{\#(\mathbb{Z}_2/\mathbb{Z}_2^2)^3} \mu_2(v_2(a) = (0, 0, 1), (1, 1, 0)),
\]
where
\[
A := \{a \in \mathbb{Z}_p^{\oplus 3} \mid a = (ut_0, 5ut_1, 2ut_2), (2ut_0, 10ut_1, ut_2) \text{ with } t_i \in \mathbb{Z}_2^\times, u \in \mathbb{Z}_2^\times\}.
\]
(2) Similarly, for \(n = 3\), we have
\[
\rho_2 = 1 - \mu_2(a \simeq (1,1,1),(1,1,5),(1,1,2),(1,1,10),(1,3,2),(1,10,10),(1,3,10),(1,6,14),(1,5,6,14))
= 1 - \kappa_2(3,2) \cdot \frac{4 + \left(\frac{4}{3}\right) \cdot 2}{4^4} \cdot \mu_2(v_2(a) = (0,0,0),(1,1,1))
- \kappa_2(3,2) \cdot \frac{4 + \left(\frac{4}{3}\right) \cdot 4 + \left(\frac{4}{3}\right) = 1 + 7 + 48 = 1296.
\]
(3) For \(n = 4\), the statement is obvious from Proposition 3.4.

3.2. The case of \(k = 3\) and \(p = 3\).

**Proposition 3.5** (\(p = 3\) and \(n = 2\)). Let \(u_0, u_1, u_2 \in \mathbb{Z}_3^\times\).

1. \(X^3_{(u_0, u_1, u_2)}\) has no \(\mathbb{Q}_3\)-rational point.
2. \(X^3_{(u_0, u_1, u_2)}\) has a \(\mathbb{Q}_3\)-rational point if and only if \(u_0 \equiv \pm u_1 \pmod{9}\).
3. \(X^3_{(u_0, u_1, u_2)}\) has a \(\mathbb{Q}_3\)-rational point.
4. \(X^3_{(u_0, u_1, u_2)}\) has a \(\mathbb{Q}_3\)-rational point if and only if \(\{u_0, u_1, u_2\} \neq \{\pm 1, \pm 2, \pm 4\} \pmod{9}\).

**Proof.** First of all, note that since \(\mathbb{Z}_3^\times = \pm 1 + 9\mathbb{Z}_3\), we may assume that \(u_0, u_1, u_2 \in \{1,2,4\}\) by replacing \(x_i\) to \(w_i x_i\) with some \(w_i \in \mathbb{Z}_3^\times\) if necessary.

1. (1) These are immediate by an infinite descent with a help of modulo 9 reduction.
2. (2) If \(u_0 = u_1\), then our curve has a \(\mathbb{Q}_3\)-rational point such that \(x_0 = -x_1\) and \(x_2 = 0\). Therefore, it is sufficient to prove that for every \((u_0, u_1) = (1,2),(1,4),(2,4)\)
\[
u_0 x_0^3 + u_1 x_1^3 + 3 \cdot 1^3 = 0
\]
has a \(\mathbb{Z}_3\)-solution, for instance, \((x_0, x_1) = (1,1,1),(1,1,1),((-7/2)^{1/3},1)\).
3. (4) By the above argument, it is sufficient to prove that if \((u_0, u_1, u_2) = (1,2,4)\),
then \(X^3_{(1,2,4)}\) has no \(\mathbb{Q}_3\)-rational point (by contradiction as follows). Suppose that \(X^3_{(1,2,4)}\) has a \(\mathbb{Q}_3\)-rational point \([x_0 : x_1 : x_2]\) with \(x_i \in \mathbb{Z}_3\). Then, the congruence
\[
x_0^3 + 2x_1^3 + 4x_2^3 \equiv 0 \pmod{9}
\]
implies \(x_0 \equiv x_1 \equiv x_2 \equiv 0 \pmod{3}\), which is a contradiction by the infinite descent. This completes the proof. □

We define an equivalence relation \(\sim\) on the group \(\text{Im } v_3 = \mathbb{Z}^{\oplus n+1}\) as the induced equivalence relation by \(\simeq\) on \((\mathbb{Q}_3^\times)^{\oplus n+1}\) (cf. [1] §2.2.1).

**Proposition 3.6** (\(p = 3\) and \(n \geq 3\)). Let \(n \in \mathbb{Z}_{\geq 3}\) and \(a = (a_0, \ldots, a_n) \in \mathbb{Z}_3^{\oplus n+1} \setminus \Delta_n\).
1. Suppose that $n = 3$. Then, $X^k_a$ has a $\mathbb{Q}_3$-rational point if and only if one of the following conditions hold:
   - $v_2(a) \sim (0, 0, 0)$, $(0, 0, 0)$, $(0, 0, 1, 1, 1, 1)$, or $(0, 0, 1, 2)$.
   - $v_3(a) \sim (0, 0, 0)$, and if one normalizes $v_3(a)$ so that $v_3(a) = (0, 0, 0)$, then $\{\pm a_0, \pm a_1, \pm a_2\} \neq \{\pm 1, \pm 2, \pm 4\}$ (mod 9).
2. Suppose that $n \geq 4$. Then, $X^k_a$ has a $\mathbb{Q}_3$-rational point.

Proof. (1) For the detail of the case $n = 3$, see [1, §2.1.2].
(2) For $n \geq 4$, it is sufficient to consider the case $n = 4$ and $v_3(a) = (0, 0, 0, 2, 2)$. In this case, $X^3_{(a_0, a_3, a_4)} \subset X^3_a$ has a $\mathbb{Q}_3$-rational point by Proposition 3.5 (3).

This completes the proof.

Proof of Theorem 1.4. We can prove it in a similar manner as the proof of Theorem 1.3.
1. Suppose that $n = 2$. By Proposition 2.3, it is sufficient to consider the case $p = 3$.
   By Proposition 3.5, we have
   
   \[
   \rho_3(2, 3) = 1 - \mu_3(a \simeq (1, 2, 4)) \]
   
   \[= 1 - \kappa_3(2, 3) \cdot \frac{3! \cdot 1}{3^3} \cdot \mu_3(v_3(a) = (0, 0, 0), (1, 1, 1), (2, 2, 2)) \]
   
   \[= 13831 \quad 19773. \]

2. For the detail of the case $n = 3$, see [1, §2.1].
3. For $n \geq 4$, the statement is an immediate consequence of Propositions 2.3 and 3.6.

In conclusion, we proved the theorem.

3.3. Proof of Theorem 1.11

Proof of Theorem 1.11. Under the assumption with [1, Theorem 1.3] and [5, Theorem 1.4], we have \(\rho(n, k) = \rho_{\text{loc}}(n, k) = \prod_{v \text{place}} \rho_v(n, k)\). Here, note that the Hasse principle holds for the case $k = 2$ (resp. $k = 3$ and $n \geq 9$) due to [20, p.48, Theorem 8] (resp. [11, Theorem]). We can estimate the last infinite product by using the Riemann zeta function \(\zeta(s) = \prod_{p \text{prime}}(1 - p^{-s})^{-1} \) (s $\in \mathbb{R}_{>1}$). For example, if $n = k = 3$, we obtain the following inequalities

\[\zeta(2)^{-4} \prod_{p < 10^6} (1 - p^{-2})^{-4} \prod_{p < 10^6} \rho_p(3, 3) < \prod_{p \text{prime}} \rho_p(3, 3) < \prod_{p < 10^6} \rho_p(3, 3),\]

which give the desired approximation.

Remark 3.7. In [1], Bright, Browning and Loughran obtained the formulas of \(\rho_p(3, 3)\) (\(\sigma_p\) in the notation of [1]) correctly. These formulas give the approximation \(\rho(3, 3) = \rho_{\text{loc}}(3, 3) \approx 0.8964\ldots\). However, they stated that \(\rho_{\text{loc}}(3, 3)\) (\(\sigma\) in the notation of [1]) equals 0.8605\ldots, which is incorrect.

Remark 3.8. For $k \geq 4$ and $n \geq 3k + 2$, [8, Theorem 1.3] implies that

\[\rho_{\text{loc}}(n, k) - \rho(n, k) \leq \lim_{H \to \infty} cH^{n+1-\theta} (2H + 1)^{n+1} = 0\]

for some $c, \theta \in \mathbb{R}_{>0}$, hence we obtain \(\rho(n, k) = \rho_{\text{loc}}(n, k)\) in this case too.
4. Concluding remarks

Recall that an algebraic variety defined over $\mathbb{Q}$ is said to be $\mathbb{Q}$-rational if it is birationally equivalent to $\mathbb{P}^n$ over $\mathbb{Q}$ for some $n$. Set

$$\delta(n, k) := \lim_{H \to \infty} \frac{\{a \in \mathbb{Z}^{n+1} \mid |a| < H \text{ and } X^k_a \text{ is } \mathbb{Q}-\text{rational}\}}{\{a \in \mathbb{Z}^{n+1} \mid |a| < H\}}.$$ 

**Proposition 4.1.**

$$\delta(n, 2) = \rho(n, 2) = \begin{cases} 0 & \text{if } n = 2, \\ 0.8268\ldots & \text{if } n = 3, \\ 1 & \text{if } n \geq 4. \end{cases}$$

Proposition 4.1 follows immediately if we apply the following proposition for nonsingular quadratic hypersurfaces.

**Proposition 4.2.** Let $Q \subset \mathbb{P}^n$ be a quadratic hypersurface defined over $\mathbb{Q}$ and $n \in \mathbb{Z}_{\geq 1}$. Then, $Q$ is $\mathbb{Q}$-rational if and only if it has a non-singular $\mathbb{Q}$-rational point.

**Proof.** The only if part is trivial. We prove the if part. Indeed, the given non-singular $\mathbb{Q}$-rational point has an open neighborhood isomorphic to an affine quadratic hypersurface $Q' \subset \mathbb{A}^n$ passing through the origin $O = (0, \ldots, 0)$. Then, we can take a Zariski dense subset $U$ of $\mathbb{A}^n$ so that for every $\mathbb{Q}$-rational point $A$ on $U$ the line $OA$ intersects with $Q' \setminus O$ exactly once. This induces a birational (i.e., generically 1 : 1 and dominant) map of $\mathbb{P}^{n-1}$ to $Q'$, hence $Q$ itself is $\mathbb{Q}$-rational. \qed

In a similar manner, we can prove a $v$-adic version of Proposition 4.1. More precisely, if we set $\delta_v(n, 2) := \mu_v(X^2_a \text{ is } \mathbb{Q}_v\text{-rational})$, then we obtain $\delta_v(n, 2) = \rho_v(n, 2)$ for every place $v$. Therefore, by combining it with Proposition 4.1 we obtain the product formula

$$\delta(n, 2) = \prod_{v : \text{place}} \delta_v(n, 2).$$

Moreover, the whole argument works also for the family of all quadratic hypersurfaces of $\mathbb{P}^n$ for every fixed $n$ (cf. [2]). It is a natural question whether or not similar product formulas hold for other families of geometrically rational algebraic varieties. However, as far as the authors know, there is no reference answering this question even for all or diagonal cubic surfaces.

On the other hand, if we replace “$\mathbb{Q}$-rational” to “$\mathbb{Q}$-unirational”, then similar product formulas hold for the families of all or diagonal cubic hypersurfaces of $\mathbb{P}^n$ (n $\geq 3$) (cf. [13, Theorem 1.2], see also [9, Remark 2.3.1]). The latter argument works also for the quartic del Pezzo surfaces defined by

$$\begin{cases} x_0x_1 = x_2x_3 \\ 4 \sum_{i=0}^4 a_ix_i = 0 \end{cases}$$

with $a_i \in \mathbb{Q}$ such that $\prod_{i=0}^3 a_i(a_0a_1 - a_2a_3) \neq 0$ (cf. [16] and [15, Theorem 29.4 and Theorem 30.1]).

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