A RIEMANN-ROCH THEOREM FOR DG ALGEBRAS

by

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Abstract. — Given a smooth proper dg-algebra $A$, a perfect dg $A$-module $M$ and an endomorphism $f$ of $M$, we define the Hochschild class of the pair $(M, f)$ with values in the Hochschild homology of the algebra $A$. Our main result is a Riemann-Roch type formula involving the convolution of two such Hochschild classes.

1. Introduction

An algebraic version of the Riemann-Roch formula was recently obtained by D. Shklyarov [28] in the framework of the so-called noncommutative derived algebraic geometry. More precisely, motivated by the well known result of A. Bondal and M. Van den Bergh about "dg-affinity" of classical varieties, D. Shklyarov has obtained a formula for the Euler characteristic of the Hom-complex between two perfect modules over a dg-algebra in terms of the Euler classes of the modules.

On the other hand, M. Kashiwara and P. Schapira [12] initiated an approach to the Riemann-Roch theorem in the framework of deformation quantization modules (DQ-modules) with the view towards applications to various index type theorems. Their approach is based on Hochschild homology which, in this setup, admits a description in terms of the dualizing complexes in the derived categories of coherent DQ-modules.

In this paper, we obtain a Riemann-Roch theorem in the dg setting, similarly as D. Shklyarov. However, our approach is really different of the latter one in that we avoid the categorical definition of the Hochschild homology, and use instead the Hochschild homology of the ring $A$ expressed in terms of dualizing objects. Our result is slightly more general than the one obtained in [28]. Instead of a kind of non-commutative Riemann-Roch theorem, we
rather prove a kind of non-commutative Lefschetz theorem. Indeed, it involves certain Hochschild classes of pairs \((M,f)\) where \(M\) is a perfect dg module over a smooth proper dg algebra and \(f\) is an endomorphism of \(M\) in the derived category of perfect \(A\)-modules. Moreover, our approach follows [12]. In particular, we have in our setting relative finiteness and duality results (Theorem 3.1.1 and Theorem 3.2.2) that may be compared with [12] Theorem 3.2.1 and [12] Theorem 3.3.3]. Notice that the idea to approach the classical Riemann-Roch theorem for smooth projective varieties via their Hochschild homology goes back at least to the work of N. Markarian [21]. This approach was developed further by A. Caldararu [6], [7] and A. Caldararu, S. Willerton [8] where, in particular, certain purely categorical aspects of the story were emphasized. The results of [6] suggested that a Riemann-Roch type formula might exist for triangulated categories of quite general nature, provided they possess Serre duality. In this categorical framework, the role of the Hochschild homology is played by the space of morphisms from the inverse of the Serre functor to the identity endofunctor. In a sense, our result can be viewed as a non-commutative generalization of A. Caldararu’s version of the topological Cardy condition [6]. Our original motivation was different though it also came from the theory of DQ-modules [12].

Here is our main result:

**Theorem.** — Let \(A\) be a proper, homologically smooth dg algebra, \(M \in D_{\text{per}}(A)\), \(f \in \text{Hom}_A(M, M)\) and \(N \in D_{\text{per}}(A^{\text{opp}})\), \(g \in \text{Hom}_{A^{\text{opp}}}(N, N)\). Then

\[
\text{hh}_k(N \otimes_A M, g \otimes_A f) = \text{hh}_{A^{\text{opp}}}(N, g) \cup \text{hh}_A(M, f),
\]

where \(\cup\) is a pairing between the corresponding Hochschild homology groups and where \(\text{hh}_A(M, f)\) is the Hochschild class of the pair \((M, f)\) with value in the Hochschild homology of \(A\).

The above pairing is obtained using Serre duality in the derived category of perfect complexes and, thus, it strongly resembles analogous pairings, studied in some of the references previously mentioned (cf. [6], [12], [29]). We prove that various methods of constructing a pairing on Hochschild homology lead to the same result. Notice that in [26], A. Ramadoss studied the links between different pairing on Hochschild homology.

To conclude, we would like to mention the recent paper by A. Polishchuk and A. Vaintrob [24] where a categorical version of the Riemann-Roch theorem was applied in the setting of the so-called Landau-Ginzburg models (the categories of matrix factorizations). We hope that our results, in combination with some results by D. Murfet [22], may provide an alternative way to derive
the Riemann-Roch formula for singularities.

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2. Review on differential graded modules

All along this paper $k$ is a field of characteristic zero. A $k$-algebra is a $k$-module $A$ equipped with an associative $k$-linear multiplication admitting a two sided unit $1_A$. We recall some basic definitions regarding dg-algebras and dg-modules. A much more detailed account of the subject can be found in \[17\].

2.1. Differential graded algebras and modules. — We recall some basic definitions. References for this subsection are made to \[4\], \[5\], \[17\]. By a module we understand a left module unless otherwise specified.

Definition 2.1.1. — A graded ring $A$ is a ring $A$ together with a family $(A^p)_{p \in \mathbb{Z}}$ of subgroups of the additive group of $A$, such that

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

and such that $A^p \cdot A^q \subseteq A^{p+q}$ for all $p, q \in \mathbb{Z}$.

Definition 2.1.2. — A graded $A$-module $M$ is an $A$-module $M$ together with a family $(M^p)_{p \in \mathbb{Z}}$ of subgroups of $M$, such that

$$M = \bigoplus_{p \in \mathbb{Z}} M^p$$

and such that $A^p \cdot M^q \subseteq M^{p+q}$ for all $p, q \in \mathbb{Z}$. The elements of the subgroups $(M^p)_{p \in \mathbb{Z}}$ are called homogeneous. An element of $M^p$ is of degree $p$. The degree of an homogeneous element $x \in M$ is denoted $|x|$.

Definition 2.1.3. — A morphism $f : M \rightarrow N$ of graded $A$-modules of degree $n$ is a morphism of $A$ modules such that $f(M^p) \subseteq N^{p+n}$ for every $p \in \mathbb{Z}$. 
Remark 2.1.1. — The field $k$ is considered as a graded ring concentrated in degree zero. A graded $k$-algebra $A$ is a graded ring and a graded module over $k$.

Definition 2.1.4. — Let $A$ be a graded $k$-algebra. The category of graded module $\text{Mod}^{gr}(A)$ is the category whose objects are the graded $A$-module. For two graded modules $M$, $N$, the space of morphisms $\text{Hom}_{\text{Mod}(A)}(M,N)$ is the graded $k$-module whose component of degree $n$ is the vector space of graded morphisms of degree $n$ between $M$ and $N$.

Definition 2.1.5. — A differential graded algebra (or dg algebra for short) is a graded algebra endowed with a degree one morphism of graded $k$-modules $d_A : A \to A$ called the differential of $A$. This morphism satisfies $d_A^2 = 0$ and if $a$ is an homogeneous element of $A$ and $b$ an other element of $A$ then

$$d_A(a \cdot b) = d_Aa \cdot b + (-1)^{|a|}a \cdot d_Ab$$

Definition 2.1.6. — Let $A$ be a dg algebra. A differential graded module $M$ over $A$ (or dg module for short) is a graded $A$-module endowed with a one morphism of graded $k$-modules $d_M : M \to M$ called the differential of $M$. This differential is such that $d_M^2 = 0$ and it satisfies the Leibniz relation for every $a \in A, m \in M$,

$$d_M(a \cdot m) = d_A(a) \cdot m + (-1)^{|a|}a \cdot d_M(m).$$

Definition 2.1.7. — If $A$ is a dg algebra we denote by $A^{op}$ the opposite dg algebra. It is equal to $A$ as a dg $k$-module but the multiplication on $A^{op}$ is given by $a \cdot b = (-1)^{|a||b|}ba$.

Definition 2.1.8. — Let $A$ be a dg $k$-algebra. The category $\text{Mod}(A)$ of dg modules is the category whose objects are the dg $A$-modules. For two dg modules $M$ and $N$, the space of morphisms $\text{Hom}_{\text{Mod}(A)}(M,N)$ is the dg $k$-module whose underlying graded $k$-module is $\text{Hom}_{\text{Mod}^{gr}(A)}(M,N)$ and whose differential is $d(f) = d_N \circ f - (-1)^{|f|}f \circ d_M$.

A morphism $f$ is called closed if $df = 0$. A degree zero closed morphism is a morphism of complexes.

Remark 2.1.2. — For two dg algebras $A$ and $B$, $A \otimes B^{op}$ bimodules will be considered as left $A \otimes B^{op}$ modules via the action

$$a \otimes b \cdot m = (-1)^{|b||m|}amb.$$  

If we want to emphasize the left (resp. right) structure of an $A \otimes B$ bimodule we will write $AM$ (resp. $MB$).
Definition 2.1.9. — The shifted (or suspended) dg module \( M[n] \) is the dg module defined by \( M[n]^p = M^{n+p} \) and endowed with the differential \((-1)^n d_M\).

Definition 2.1.10. — Let \( L \) and \( M \) be two \( A \)-modules and \( p : L \to M \) a degree zero closed morphism. The cone of \( p \) denoted \( \text{Cone}(p) \) is the \( A \)-module defined by

\[
\text{Cone}(p) := \left( L[1] \oplus M, \begin{pmatrix} d_L[1] & 0 \\ p & d_M \end{pmatrix} \right).
\]

There are morphisms

\[
r = (\text{id}_{L[1]} & 0) : \text{Cone}(p) \to L[1],
\]

\[
q = \begin{pmatrix} 0 \\ \text{id}_M \end{pmatrix} : M \to \text{Cone}(p).
\]

Definition 2.1.11. — The tensor product of a dg \( A^{op} \)-module \( M \) and dg \( A \)-module \( N \) is the dg \( k \)-module whose components are

\[
(M \otimes_A N)^n = \bigoplus_{p+q=n} M^p \otimes_A N^q, \quad n \in \mathbb{Z}
\]

The differential of \( M \otimes_A N \) is given by

\[
(d_M \otimes_A \text{id}_N + \text{id}_M \otimes_A d_N)(v \otimes w) = d_M(v) \otimes_A w + (-1)^{|w|} v \otimes_A d_N(w).
\]

The tensor product of two morphisms \( f : M \to M' \) and \( g : N \to N' \) is defined by

\[
(f \otimes_A g)(v \otimes_A w) = (-1)^{pq} f(v) \otimes g(w)
\]

if \( g \) is of degree \( p \) and \( v \) is of degree \( q \).

Notation. — We denote by \( A^e \) the dg algebra \( A \otimes A^{op} \) and by \( ^e A \) the dg algebra \( A^{op} \otimes A \).

Due to the differential structure of dg modules, we can take their cohomology. If \( M \) is a dg module we will denote by \( H(M) \) its total cohomology, that is the dg module \( \bigoplus_{n \in \mathbb{Z}} H^n(M)[-n] \) whose differential is zero.

Definition 2.1.12. — A differential graded or dg category \( \mathcal{A} \) is a \( k \)-linear category whose morphisms spaces are dg \( k \)-modules and whose composition

\[
\circ : \text{Hom}_\mathcal{A}(X,Y) \otimes \text{Hom}_\mathcal{A}(Y,Z) \to \text{Hom}_\mathcal{A}(X,Z)
\]

are morphisms of dg \( k \)-modules. Its differentials are such that if \( f \) is a morphism of degree \( p \) and \( g \) a morphism such that \( f \circ g \) exists then

\[
d(f \circ g) = df \circ g + (-1)^p f \circ dg.
\]

A morphism \( f \) in a dg category \( \mathcal{A} \) is called closed if \( df = 0 \).
Definition 2.1.13. — Let $A$ and $B$ two dg categories. A dg functor $F : A \to B$ is given by a map $F_0 : \text{Ob}(A) \to \text{Ob}(B)$ and for all $X, Y \in \text{Ob}(A)$, by a degree zero closed morphism of dg $k$-modules

$$F_1 : \text{Hom}_A(X, Y) \to \text{Hom}_B(F_0(X), F_0(Y)), \ X, Y \in \text{Ob}(A)$$

compatible with the compositions and identity.

Definition 2.1.14. — Let $A$ be a dg-category. Its homotopy category denoted by $H^0(A)$ is the category whose objects are the same as those of $A$ and whose morphisms are defined by

$$\text{Hom}_{H^0(A)}(X, Y) = H^0(\text{Hom}_A(X, Y)).$$

We come back to the study of dg modules over a dg algebra $A$.

Definition 2.1.15. — A distinguished triangle in $H^0(\text{Mod}(A))$ is a sequence

$$X \to Y \to Z \to X[1]$$

isomorphic in $H^0(\text{Mod}(A))$ to a sequence of the form

$$L \xrightarrow{p} M \xrightarrow{q} \text{Cone}(p) \xrightarrow{r} L[1]$$

where $L$, $M$ are dg modules, $p$ is a dg morphism and $q$, $r$ are given in definition 2.1.10.

2.2. Perfect modules. — References are made to [4, 5, 28].

Definition 2.2.1. — An $A$-module is finitely generated free if it is isomorphic to a finite direct sums of shifted copies of $A$.

Definition 2.2.2. — The category of finitely generated semi-free $A$-module is the smallest full subcategory of $\text{Mod}(A)$ containing the finitely generated free $A$-modules and closed under the operations of taking cones.

Definition 2.2.3. — A homotopy direct summand of an $A$-module $N$ is an $A$-module $L$ that satisfies the following property: there exist two closed degree zero morphisms $f : N \to L$, $g : L \to N$ such that $fg = \text{id}_L$ in $H^0(\text{Mod}(A))$.

Definition 2.2.4. — A perfect $A$-module is a homotopy direct summand of a finitely generated semi-free $A$-module. We denote by $\text{Perf}(A) \subset \text{Mod}(A)$ the category of such modules.

Remark 2.2.1. — The category $H^0(\text{Perf}(A))$ is a full triangulated subcategory of the triangulated category $H^0(\text{Mod}(A))$.

Proposition 2.2.1. — Let $A$ and $B$ be two dg algebras and $F : \text{Mod}(A) \to \text{Mod}(B)$ a dg functor. Assume that $F(A) \in \text{Perf}(B)$. Then for any $X \in \text{Perf}(A)$, $F(X) \in \text{Perf}(B)$. 

2.3. The derived category of perfect modules. — The main references for this subsection are [14] and [29].

Definition 2.3.1. — Let $A$ be a dg algebra. The derived category of $\text{Mod}(A)$, denoted $\mathcal{D}(A)$, is the localization of $\text{H}^0(\text{Mod}(A))$ with respect to the quasi-isomorphisms. We denote by $Q$ the localization functor: $Q : \text{H}^0(\text{Mod}(A)) \to \mathcal{D}(A)$.

Definition 2.3.2. — A dg module $P \in \text{H}^0(\text{Mod}(A))$ is homotopically projective if for every acyclic module $N \in \text{H}^0(\text{Mod}(A))$, $\text{Hom}_{\text{H}^0(\text{Mod}(A))}(P, N) = 0$.

Proposition 2.3.1. — Perfect modules are homotopically projective.

Proof. — The essential fact to notice is that $\text{Hom}_{\text{H}^0(\text{Mod}(A))}(A, N) \simeq \text{H}^0(\text{Hom}_{\text{Mod}(A)}(A, N)) \simeq \text{H}^0(N) \simeq 0$. □

The bifunctor $\text{Hom}_{\text{Mod}(A)}(\cdot, \cdot) : \text{Mod}(A)^{\text{op}} \times \text{Mod}(A) \to \text{Mod}(k)$ preserves homotopy classes of morphisms thus it induces a triangulated bifunctor $\text{Hom}_A(\cdot, \cdot) : \text{H}^0(\text{Mod}(A)^{\text{op}}) \times \text{H}^0(\text{Mod}(A)) \to \text{H}^0(\text{Mod}(k))$. For any $N \in \text{Mod}(A)$, the derived functor corresponding to $\text{Hom}_A(\cdot, N)$ is denoted $\text{RHom}(\cdot, N)$ and is defined using homotopically projective resolutions. Similarly the functor $\mathbb{L} \otimes_A \cdot : \text{Mod}(A)^{\text{op}} \times \text{Mod}(A) \to \text{Mod}(k)$ gives rise to a derived bifunctor $\mathbb{L} \otimes_A \cdot : \mathcal{D}(A)^{\text{op}} \times \mathcal{D}(A) \to \mathcal{D}(k)$. It is also defined using homotopically projective resolutions.

Proposition 2.3.2. — Let $k$ be a field. Let $A$ and $B$ be two dg $k$-algebras and assume that $N \in \text{H}^0(\text{Perf}(A \otimes B^{\text{op}}))$. The image of $N$ by the forgetful functor $\text{H}^0(\text{Perf}(A \otimes B^{\text{op}})) \to \text{H}^0(\text{Mod}(A))$ is homotopically projective.

Proof. — Let $M$ be an acyclic $A$-module.

(i) Assume that $N = A \otimes B^{\text{op}}$ then

$$
\text{Hom}_{\text{H}^0(A)}(N, M) = \text{Hom}_{\text{H}^0(A)}(A \otimes B^{\text{op}}, M) \\
\simeq \text{H}^0(\text{Hom}_k(B^{\text{op}}, \text{Hom}_A(A, M))) \\
\simeq \text{Hom}_k(B^{\text{op}}, M) \simeq 0.
$$

The last isomorphism holds because $k$ is a field.

(ii) Now assume that the result is true for finitely generated semi-free modules obtained by taking at most $n - 1$ cones of degree zero closed morphisms. Let $N$ be a finitely generated semi-free module obtained by taking at most $n$ cones. There exist finitely generated semi-free modules obtained by taking at most $n - 1$ cones $P$ and $Q$ and a degree zero closed morphism $f : P \to Q$ such that $N \simeq \text{Cone}(f)$. Thus we have a distinguished triangle

$$
P \longrightarrow Q \longrightarrow N \longrightarrow P[1].$$
By applying the functor $\text{Hom}_{H^0(A)}(\cdot,M)$ to the preceding triangle and using the induction hypothesis we see that $N$ is homotopically projective. If $N$ is perfect, it is homotopically equivalent to a direct summand of a finitely generated semi-free module which implies that $N$ is homotopically projective. \qed

We now introduce an operation on the set of strictly full subcategories of a triangulated category, see [2].

**Definition 2.3.3.** — Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{A}$ and $\mathcal{B}$ be strictly full subcategories of $\mathcal{T}$. Let $\mathcal{A} \star \mathcal{B}$ be the strictly full subcategory of $\mathcal{T}$ whose object $X$ occurs in a triangle of the form

$$A \to X \to B \to A[[1]]$$

where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

**Remark 2.3.1.** — If $\mathcal{A}$ and $\mathcal{B}$ are stable by shifts (resp. direct sums) then $\mathcal{A} \star \mathcal{B}$ is stable by shifts (resp. direct sums).

Let $A$ be a dg algebra. We denote by $\text{Mod}_{ac}(A)$ the full subcategory of $\text{Mod}(A)$ whose objects are the acyclic dg modules.

**Definition 2.3.4.** — (i) We set $H^0_{\text{per}}(A) := H^0(\text{Perf}(A)) \star H^0(\text{Mod}_{ac}(A))$.

(ii) The derived category of perfect complexes $\mathcal{D}_{\text{per}}(A) \subset \mathcal{D}(A)$ is the image by the localization functor $Q$ of the category $H^0_{\text{per}}(A)$.

We recall a few standard results on the derived category of perfect modules and give proofs for the convenience of the reader.

**Proposition 2.3.3.** — The categories $H^0_{\text{per}}(A)$ and $\mathcal{D}_{\text{per}}(A)$ are triangulated.

**Proof.** — To prove that $H^0(\text{Perf}(A)) \star H^0(\text{Mod}_{ac}(A))$ is triangulated we only need to prove that it is stable by taking cones of morphisms. Let $f : M \to N$ be a morphism in $H^0_{\text{per}}(A)$. By definition there exists exact triangles in $H^0(\text{Mod}(A))$ such that

$$pM \xrightarrow{r_M} M \xrightarrow{aM} pM[1]$$

and

$$pN \xrightarrow{r_N} N \xrightarrow{aN} pN[1]$$

where $pM$ and $pM$ belong to $H^0(\text{Perf}(A))$ and $aM$ and $aN$ belong to $H^0(\text{Mod}_{ac}(A))$. The modules $pM$ and $pN$ are homotopically projective. Thus, we have an isomorphism

$$\text{Hom}_{H^0(\text{Mod}(A))}(pM,N) \xrightarrow{r_N^*} \text{Hom}_{H^0(\text{Mod}(A))}(pM,pN).$$
Consequently, using the axioms (TR4) of triangulated categories we can complete the diagram (2.1) and obtain

\[
\begin{array}{ccc}
pM & \xrightarrow{rM} & M \\
\downarrow \phi & & \downarrow f \\
pN & \xrightarrow{rN} & N
\end{array}
\quad \begin{array}{ccc}
aM & \to & pM[1] \\
\downarrow & & \downarrow \\
aN & \to & pN[1].
\end{array}
\]

Now, by (TR2) there is a distinguished triangle

\[
pM \xrightarrow{\phi} pN \to \text{Cone}(\phi) \to pM[1].
\]

It is clear that \(\text{Cone}(\phi)\) is a perfect module.

Using (TR4) again, we obtain the following diagram

\[
\begin{array}{ccc}
pM & \xrightarrow{\phi} & pN \\
\downarrow rM & & \downarrow rN \\
M & \xrightarrow{f} & N
\end{array}
\quad \begin{array}{ccc}
\text{Cone}(\phi) & \xrightarrow{\gamma} & \text{Cone}(f) \\
\downarrow & & \downarrow \\
pM[1] & \xrightarrow{} & M[1].
\end{array}
\]

It is not difficult to verify by applying the Five lemma to the previous diagram that \(\gamma\) is a quasi-isomorphism. Then, by applying (TR2) to \(\gamma\) we obtain a distinguished triangle

\[
\text{Cone}(\phi) \xrightarrow{\gamma} \text{Cone}(f) \to Z \to \text{Cone}(\phi)[1].
\]

Since \(\gamma\) is a quasi-isomorphism, then \(Z\) belongs to \(H^0(\text{Mod}_{ac}(A))\) which proves that \(\text{Cone}(f) \in H^0_{\text{per}}(A)\).

The fact that \(H^0_{\text{per}}(A)\) is triangulated implies immediately that \(D_{\text{per}}(A)\) is triangulated.

The following proposition is direct consequence of the construction of \(D_{\text{per}}(A)\), (see also [14] subsection 4.1).

**Proposition 2.3.4.** — The canonical functor \(H^0(\text{Perf}(A)) \to D_{\text{per}}(A)\) is an equivalence of category.

We recall the notion of thickness for a subcategory of a triangulated category as defined in [23] p.75 Definition 2.1.6. There is also a notion of thickness for subcategories of an abelian category, see [13] p. 184, Definition 8.3.21.

**Definition 2.3.5.** — A subcategory of a triangulated category \(\mathcal{T}\) is called thick if it is triangulated and contains all the direct summands of its objects. If \(\mathcal{E}\) is a subset of \(\mathcal{T}\) then the smallest thick category containing \(\mathcal{E}\) is denoted \(\langle \mathcal{E} \rangle\) and is called the thick envelope of \(\mathcal{E}\).
For more details see [2, section 2].

**Proposition 2.3.5.** — The category $D_{\text{per}}(A)$ is the smallest thick subcategory of $D(A)$ containing $A$.

**Proof.** — It is clear from the definition of $\langle A \rangle$ that $\langle A \rangle \subset D_{\text{per}}(A)$. Let $N \in D_{\text{per}}(A)$, there exists $pN \in H^0(\text{Perf}(A))$ such that $pN \sim N$ in $D_{\text{per}}(A)$. By definition of $pN$ there exists $pM$ such that $pN \oplus pM$ is isomorphic in $H^0(\text{Perf}(A))$ to a finitely generated semi-free $A$ module $S$. It is clear that $S \in \langle A \rangle$ and since $\langle A \rangle$ is thick, $N \in \langle A \rangle$.

3. Finiteness and duality for perfect modules

In this section, after recalling some basic facts concerning dg modules, we summarize some finiteness and duality results for perfect modules over a dg algebra satisfying suitable finiteness and regularity hypotheses.

3.1. Finiteness results for perfect modules. —

**Definition 3.1.1.** — A dg algebra $A$ is said to be proper if $\sum_n \dim H^n(A) < \infty$.

The next theorem, though the proof is much easier, can be thought as a dg analog to the theorem asserting the finiteness of proper direct images for coherent $\mathcal{O}_X$-modules.

**Theorem 3.1.1.** — Let $A$, $B$ and $C$ be dg algebras. Assume $B$ is a proper dg algebra. Then the functor $\cdot \otimes_B \cdot : \text{Mod}(A \otimes B^{\text{op}}) \otimes \text{Mod}(B \otimes C^{\text{op}}) \to \text{Mod}(A \otimes C^{\text{op}})$ induces a functor $\cdot \otimes_B \cdot : \text{Perf}(A \otimes B^{\text{op}}) \otimes \text{Perf}(B \otimes C^{\text{op}}) \to \text{Perf}(A \otimes C^{\text{op}})$.

**Proof.** — According to Proposition 2.2.1 we only need to check that $(A \otimes B^{\text{op}}) \otimes_B (B \otimes C^{\text{op}}) \simeq A \otimes B \otimes C^{\text{op}} \in \text{Perf}(A \otimes C^{\text{op}})$. In $\text{Mod}(k)$, $B$ is homotopically equivalent to $\text{H}(B)$ since $k$ is a field. Then in $\text{Mod}(A \otimes C^{\text{op}})$, $A \otimes B \otimes C^{\text{op}}$ is homotopically equivalent to $A \otimes \text{H}(B^{\text{op}}) \otimes C^{\text{op}}$ which is a finitely generated free $A \otimes C^{\text{op}}$-module since $B$ is proper.

We recall a regularity condition for dg algebra called homological smoothness, [18], [30].

**Definition 3.1.2.** — A dg-algebra $A$ is said to be homologically smooth if $A \in D_{\text{per}}(A^{\text{e}})$.

**Proposition 3.1.1.** — The tensor product of two homologically smooth dg-algebras is an homologically smooth dg-algebra.

**Proof.** — Obvious.
There is the following characterization of perfect modules over a proper homologically smooth dg algebra extracted from [29].

**Theorem 3.1.2.** — Let $A$ be a proper dg algebra. Let $N \in D(A)$.

(i) If $N \in D_{\text{per}}(A)$ then $\sum_n \dim H^n(N) < \infty$.

(ii) If $A$ is also homologically smooth and $\sum_n \dim H^n(N) < \infty$ then $N \in D_{\text{per}}(A)$.

**Proof.** — (i) is clear.

(ii) Assume that $N \in D(A)$ and $\sum_n \dim H^n(N) < \infty$. Let $pN$ be a homotopically projective resolution of $N$ as an $A$ module and let $pA$ be a perfect resolution of $A$ in $H^0_{\text{per}}(A^e)$. In $D(A)$, we have the isomorphisms

$$N \simeq A \otimes_A pN \simeq pA \otimes_A pN$$

The functor $\cdot \otimes_A pN$ is a triangulated functor from $H^0_{\text{per}}(A^e)$ to $H^0_{\text{per}}(A)$. Thus $\langle A^e \rangle \otimes_A pN \subset \langle A^e \rangle \otimes_A pN$. Moreover $A^e \otimes_A pN \simeq A \otimes_k pN$ and in $\text{Mod}(k)$, $pN$ is homotopically equivalent to $H(pN)$ thus in $D(A)$ there is an isomorphism between $A \otimes_k pN$ and $A \otimes_k H(pN)$. The dg $A$ module $A \otimes_k H(pN)$ is perfect and by the isomorphisms (3.1) it is clear that $N \in \langle A \otimes_k H(pN) \rangle \subset \langle A \rangle$. Thus $N$ is a perfect module. 

A direct consequence of the previous theorem is (see [29, Lemma 3.2])

**Lemma 3.1.1.** — If $A$ is a proper algebra then $H^0_{\text{per}}(A)$ is Ext-finite.

### 3.2. Serre duality for perfect modules.

In this subsection we recall some facts concerning Serre duality for perfect modules over a dg algebra and give various forms of the Serre functor in this context. References are made to [3], [10], [28].

If $M$ is a dg $k$-module, we define $M^* = \text{Hom}_k(M, k)$ where $k$ is considered as the dg $k$-module whose 0th-components is $k$ and other components are zero.

Let us recall the definition of a Serre functor, [31].

**Definition 3.2.1.** — Let $C$ be a $k$-linear Ext-finite triangulated category. A Serre functor $S : C \to C$ is an autoequivalence of $C$ such that there exist an isomorphism

$$\text{Hom}_C(Y, X)^* \simeq \text{Hom}_C(X, S(Y))$$

functorial with respect to $X$ and $Y$ where $*$ denote the dual with respect to $k$. If it exists, such a functor is unique up to isomorphism.

**Notation.** — We set $D_A = R\text{Hom}_A(\cdot, A) : (D(A))^{\text{op}} \to D(A^{\text{op}})$. 
Proposition 3.2.1. — The functor $\mathcal{D}_A$ preserves perfect modules and induces an equivalence $(\mathcal{D}_{\text{per}}(A))^{\text{op}} \to \mathcal{D}_{\text{per}}(A^{\text{op}})$. When restricted to perfect modules, $\mathcal{D}_A \circ \mathcal{D}_A \simeq \text{id}$.

Proof. — See [29, proposition A.1].

Proposition 3.2.2. — Suppose $N$ is a perfect $A$-module and $M$ is an arbitrary left $A \otimes B$-module, where $B$ is another algebra. Then there is a natural isomorphism of $B^{\text{op}}$-modules

$$(3.3) \quad (\text{RHom}_A(N, M))^L \simeq M^L \otimes_A N$$

We endow $A^*$ with a structure of left $A \otimes A^{\text{op}}$ module inherited from the structure of right $A^*$-module of $A$ given by $x \cdot a \otimes b = (-1)^{|b||x|}|x|bxa$.

Theorem 3.2.1. — In $\mathcal{D}_{\text{per}}(A)$, the functor $\text{RHom}_A(\cdot, A)^*$ is isomorphic to the functor $A^* \otimes_A \cdot$.

This result is a direct corollary of proposition 3.2.2 by choosing $M = A$ and $B = A^{\text{op}}$.

Lemma 3.2.1. — Let $B$ be a proper dg algebra, $M \in \mathcal{D}_{\text{per}}(A \otimes B^{\text{op}})$ and $N \in \mathcal{D}_{\text{per}}(B^{\text{op}} \otimes C)$ and $\mathcal{D}_{\text{per}}(A^{\text{op}} \otimes C)$:

$$(3.4) \quad \text{RHom}_{B \otimes C^{\text{op}}}(N, B \otimes C^{\text{op}}) \otimes_B B^* \simeq \text{RHom}_{C^{\text{op}}}(N, C^{\text{op}})$$

$$\text{RHom}_{A \otimes B^{\text{op}}}(M, A \otimes \text{RHom}_{C^{\text{op}}}(N, C^{\text{op}}))$$

$$(3.5) \quad \simeq \text{RHom}_{A \otimes C^{\text{op}}}(M \otimes_B N, A \otimes C^{\text{op}}).$$

Proof. — (i) Let us prove formula (3.4). Let $N \in \text{Perf}(B \otimes C^{\text{op}})$. There is a morphism of $B^{\text{op}} \otimes C$ modules

$\Psi_N : \text{Hom}_{B \otimes C^{\text{op}}}(N, B \otimes C^{\text{op}}) \otimes_B B^* \to \text{Hom}_{C^{\text{op}}}(N, C^{\text{op}})$

such that $\Psi_N(\phi \otimes \delta) = m((\delta \otimes \text{id}_{C^{\text{op}}}) \circ \phi)$ where $m : k \otimes C^{\text{op}} \to C^{\text{op}}$ and $m(\lambda \otimes c) = \lambda \cdot c$. Clearly, $\Psi$ is a natural transformation between the functor $\text{Hom}_{B \otimes C^{\text{op}}}(\cdot, B \otimes C^{\text{op}}) \otimes_B B^*$ and $\text{Hom}_{C^{\text{op}}}(\cdot, C^{\text{op}})$. 


Assume that $N = B \otimes C^{\text{op}}$. Then we have the following commutative diagram

\[
\begin{array}{cccccc}
\text{Hom}_{B \otimes C^{\text{op}}}(B \otimes C^{\text{op}}, B \otimes C^{\text{op}}) \otimes_B B^* & \xrightarrow{\Psi_{B \otimes C^{\text{op}}}} & \text{Hom}_{C^{\text{op}}}(B \otimes C^{\text{op}}, C^{\text{op}}) \\
B \otimes C^{\text{op}} \otimes_B B^* & & \text{Hom}_{k}(B, \text{Hom}_{C^{\text{op}}}(C^{\text{op}}, C^{\text{op}})) \\
C^{\text{op}} \otimes B^* & \sim & \text{Hom}_{k}(B, C^{\text{op}})
\end{array}
\]

which proves that $\Psi_{B \otimes C^{\text{op}}}$ is an isomorphism. The bottom map of the diagram is an isomorphism because $B$ is proper.

Now assume that the result is true for finitely generated semi-free module obtained by taking at most $n - 1$ cones. Let $N$ be a finitely generated semi-free module obtained by taking at most $n$ cones of degree zero closed morphisms. There exists a degree zero morphism $f : P \to Q$ such that $N \cong \text{Cone}(f)$ and where $P$ and $Q$ are finitely generated semi-free modules obtained by taking at most $n - 1$ cones. Thus we have a distinguished triangle in $H^0(\text{Perf}(B \otimes C^{\text{op}}))$

\[
P \xrightarrow{f} Q \to N \to P[1].
\]

If $X$ is a perfect $B \otimes C^{\text{op}}$-module, using proposition 2.3.2. we obtain that $\text{RHom}_{B \otimes C^{\text{op}}}(X, B \otimes C^{\text{op}}) \otimes_B B^* \cong \text{Hom}_{B \otimes C^{\text{op}}}(X, B \otimes C^{\text{op}}) \otimes_B B^*$. For short we set $F(X) = \text{RHom}_{B \otimes C^{\text{op}}}(X, B \otimes C^{\text{op}}) \otimes_B B^*$ and $G(X) = \text{RHom}_{C^{\text{op}}}(X, C^{\text{op}})$. Applying these functors to the above triangles and using the fact that $\Psi$ is a natural transformation, we obtain the following commutative diagram where the rows are distinguished triangles. By the induction hypothesis, $\Psi_P$ and $\Psi_Q$ are isomorphisms thus $\Psi_N$ is also an isomorphism.

\[
\begin{array}{cccccc}
F(P) & \xrightarrow{\Psi_P} & F(Q) & \xrightarrow{\Psi_Q} & F(N) & \xrightarrow{\Psi_N} & F(P)[1] \\
G(P) & \xrightarrow{\Psi_P} & G(Q) & \xrightarrow{\Psi_Q} & G(N) & \xrightarrow{\Psi_N} & G(P)[1].
\end{array}
\]

If $N$ is a perfect $B \otimes C^{\text{op}}$-module it is a direct summand of a finitely generated semi-free module in $H^0(\text{Perf}(B \otimes C^{\text{op}}))$ which implies the desired result.

(ii) Let us prove formula (3.5). We first notice that there is a morphism of $A \otimes C^{\text{op}}$-modules functorial in $M$ and $N$

\[
\Theta : \text{Hom}_{A \otimes C^{\text{op}}}(M, A \otimes \text{Hom}_{C^{\text{op}}}(N, C^{\text{op}})) \to \text{Hom}_{A \otimes C^{\text{op}}}(M \otimes_B N, A \otimes C^{\text{op}})
\]
defined by $\psi \mapsto (\Psi : m \otimes n \mapsto \psi(m)(n))$.
If $M = A \otimes B^{\text{op}}$ and $N = B \otimes C^{\text{op}}$, then they induce an isomorphism. By applying an argument similar to the previous one (but in this case we have to do two inductions, one on the number of cones in $N$ and another one on the number of cone in $M$) we are able to establish the isomorphism \[3.3\].

The next relative duality theorem can be compared to \([12, \text{Thm 3.3.3}]\) in the framework of DQ-modules though the proof is completely different.

**Theorem 3.2.2.** — Assume that $B$ is proper. Let $M \in \mathcal{D}_{\text{per}}(A \otimes B^{\text{op}})$ and $N \in \mathcal{D}_{\text{per}}(B \otimes C^{\text{op}})$. There is a natural isomorphism in $\mathcal{D}_{\text{per}}(A^{\text{op}} \otimes C)$

\[
\mathcal{D}_{A \otimes B^{\text{op}}}(M) \otimes_{B^{\text{op}}}(B^{\text{op}})^* \otimes_{B^{\text{op}}} \mathcal{D}_{B \otimes C^{\text{op}}}(N) \simeq \mathcal{D}_{A \otimes C^{\text{op}}}(M \otimes_B N).
\]

**Proof.** — We obtain

\[
\mathcal{D}(M) \otimes_{B^{\text{op}}}(B^{\text{op}})^* \otimes_{B^{\text{op}}} \mathcal{D}(N)
\]

\[
\simeq \mathcal{RHom}(M, A \otimes B^{\text{op}}) \otimes_{B^{\text{op}}}(B^{\text{op}})^* \otimes_{B^{\text{op}}} \mathcal{RHom}(N, B \otimes C^{\text{op}})
\]

\[
\simeq \mathcal{RHom}_{A \otimes B^{\text{op}}}(M, A \otimes B^{\text{op}}) \otimes_{B^{\text{op}}} \mathcal{RHom}_{C^{\text{op}}}(N, C^{\text{op}})
\]

\[
\simeq \mathcal{RHom}_{A \otimes C^{\text{op}}}(M \otimes_B N, A \otimes C^{\text{op}}).
\]

One has (see for instance \([10]\))

**Theorem 3.2.3.** — Let $A$ be a proper homologically smooth dg algebra. The functor $S_A(N) := A^* \otimes_A N, \mathcal{D}_{\text{per}}(A) \to \mathcal{D}_{\text{per}}(A)$ is a Serre functor. Moreover, $S_A(N) \simeq \mathcal{RHom}(N, A)^*.$

**Proof.** — According to Lemma \[3.1.1\] $\mathcal{D}_{\text{per}}(A)$ is an Ext-finite category. Moreover using Theorem \[3.1.2\] and Proposition \[3.2.1\] one sees that $\mathcal{RHom}(-, A)^*$ is an equivalence on $\mathcal{D}_{\text{per}}(A)$ and so is the functor $A^* \otimes_A -$. By applying Theorem \[3.2.2\] with $A = C = k, B = A$ and $M = \mathcal{RHom}_A(N, A)$ one obtains

\[
\mathcal{RHom}_A(N, A^* \otimes_A M) \simeq \mathcal{RHom}_A(M, N)^*.
\]

The Serre functor can also be expressed in term of dualizing objects. They are defined by \([18], [1], [10]\). Related results can also be found in \([11]\). One sets:
\( \omega^{-1} := \text{RHom}_{A^e}(A^{\text{op}}, A^e) \) and \( \omega := \text{RHom}_A(\omega^{-1}, A) \).

Proposition 3.2.1 ensures that \( \omega \) and \( \omega^{-1} \) are perfect \( A^e \)-modules (see for instance [28]).

In view of the preceding results, the structure of left \( A^e \)-module of \( \omega^{-1} \) is defined as follows. The algebra \( A \) has a structure of left \( A^e \)-module, consequently \( A^{\text{op}} \) as a structure of right \( A^e \)-module. Thus according to Proposition 3.2.1, the object \( \text{RHom}_{A^e}(A^{\text{op}}, A^e) \) has a structure of a left \( A^e \)-module. Similarly, \( \omega \) is an \( A^e \)-module. Moreover \( A \) is a smooth dg algebra. As a consequence \( \omega^{-1} \) and \( \omega \) are perfect dg \( A^e \)-modules.

It follows from the definition of \( \omega \) and \( \omega^{-1} \) that:

**Proposition 3.2.3.** — The functor \( \omega^{-1} \Rightarrow \text{L}( \omega \otimes_A - ) \) is left adjoint to the functor \( \omega \Rightarrow \text{L}( \omega^{-1} \otimes_A - ) \).

**Proof.** — Let \( M \) and \( N \in \text{D}_{\text{per}}(A) \) then one has

\[
\text{Hom}_A(M, \omega \otimes_A N) \simeq \text{Hom}_A(M, \text{RHom}_A(\omega^{-1}, N)) \\
\simeq \text{Hom}_A(\omega^{-1} \otimes_A M, N).
\]

One also has, [10]

**Theorem 3.2.4.** — The two functors \( \omega^{-1} \Rightarrow \text{L}( \omega \otimes_A - ) \) and \( S_A \) from \( \text{D}_{\text{per}}(A) \) to \( \text{D}_{\text{per}}(A) \) are quasi-inverse.

**Proof.** — The functor \( S_A \) is an autoequivalence. Thus it is a right adjoint of its inverse. We prove that \( \omega^{-1} \Rightarrow \text{L}( \omega \otimes_A - ) \) is a left adjoint to \( S_A \). On the one hand we have for every \( N, M \in \text{D}_{\text{per}}(A) \) the isomorphism

\[
\text{Hom}_{\text{D}_{\text{per}}(A)}(N, S_A(M)) \simeq (\text{Hom}_{\text{D}_{\text{per}}(A)}(M, N))^*.
\]

On the other hand we have the following isomorphisms

\[
\text{RHom}_A(\omega^{-1} \otimes_A N, M) \simeq (M^* \otimes_A \omega^{-1} \otimes_A N)^* \\
\simeq ((M^* \otimes_k N) \otimes_{A^e} \omega^{-1})^* \\
\simeq \text{RHom}_{A^e}(\text{RHom}_{A^e}(\omega^{-1}, A^e), M^* \otimes_k N)^* \\
\simeq \text{RHom}_{A^e}(A, M^* \otimes_k N)^*.
\]
Using the isomorphism
\[
\text{Hom}_{D_{\text{per}}(A^e)}(A, N \otimes_k M^*) \simeq \text{Hom}_{D_{\text{per}}(A)}(M, N),
\]
we obtain the desired result.

The two preceding theorems lead to the next corollary.

**Corollary 3.2.1.** — Let \( A \) be a proper homologically smooth dg algebra. The two objects \( A^* \) and \( \omega_A \) of \( D_{\text{per}}(A^e) \) are isomorphic.

**Remark 3.2.1.** — Since \( \omega_A \) and \( A^* \) are isomorphic as \( A^e \)-modules, we will use \( \omega_A \) to denote both \( A^* \) and \( \text{RHom}_A(\omega_A^{-1}, A) \) considered as the dualizing complexes of the category \( D_{\text{per}}(A) \).

**Theorem 3.2.5.** — Let \( A \) be a proper homologically smooth dg algebra. We have the isomorphisms of \( A^e \)-modules
\[
A^* \overset{L}{\otimes}_A \omega^{-1}_A \simeq A, \quad \omega^{-1}_A \overset{L}{\otimes}_A A^* \simeq A.
\]

The previous results allow us to build an "integration" morphism.

**Proposition 3.2.4.** — There exists a natural "integration" morphism in \( D_{\text{per}}(k) \)
\[
\int : \omega_{A^e} \overset{L}{\otimes}_{A^e} A \to k
\]

**Proof.** — There is a natural morphisms \( k \to \text{RHom}_{A^e}(A, A) \). Applying \( (\cdot)^* \) and formula (3.3), we obtain a morphism \( A^* \otimes_{A^e} A \to k \). Now, we notice that \( A^* \) and \( (A^\text{op})^* \) are isomorphic right \( A^e \)-modules. Indeed, on one hand, \( A \) is a left \( A^e \)-module via the action described in Remark 2.1.2, thus \( A^* \) is a right \( A^e \)-module via the action
\[
(\phi(a \otimes b))(x) = (-1)^{|b||x|} \phi(axb).
\]
where \( \phi \in A^*, x \in A \) and \( a \otimes b \in A^e \). On the other hand, denoting by \( \cdot \) the multiplication in \( A^\text{op} \), it is endowed with a structure of left \( A^e \)-module. The action is given by
\[
(a \otimes b)x = (-1)^{|a||b|+|x|} (b \cdot x \cdot a).
\]
Then \( (A^\text{op})^* \) is a right \( A^e \)-module. In the category of dg \( k \)-module, \( A^* \) and \( (A^\text{op})^* \) are the same objects. The identity morphism between \( A^* \) and \( (A^\text{op})^* \) as dg \( k \)-module lifts to an isomorphism of right \( A^e \)-module.
4. Hochschild homology and Hochschild classes

4.1. Hochschild homology. — In this subsection we recall the definition of Hochschild homology, (see [15, 19]) and prove that it can be expressed in terms of dualizing objects, (see [6, 8, 12, 18]).

**Definition 4.1.1.** — The Hochschild homology of a dg algebra is defined by

\[ \mathcal{H}(A) := A^{\text{op}} \otimes_{A^e} A. \]

The Hochschild homology groups are defined by \( \mathcal{H}_n(A) = H^{-n}(A^{\text{op}} \otimes_{A^e} A) \).

**Proposition 4.1.1.** — If \( A \) is a proper and homologically smooth dg algebra then there is a natural isomorphism

\[ \mathcal{H}(A) \simeq \text{RHom}_{A^e}(\omega_A^{-1}, A). \]

**Proof.** — We have

\[
\begin{align*}
A^{\text{op}} \otimes_{A^e} A &\simeq (\mathbb{D}_{A^e} \circ \mathbb{D}_{A^e}(A^{\text{op}})) \otimes_{A^e} A \\
&\simeq \mathbb{D}_{A^e}(\omega_A^{-1}) \otimes_{A^e} A \\
&\simeq \text{RHom}_{A^e}(\omega_A^{-1}, A).
\end{align*}
\]

4.2. The Hochschild class. — In this subsection, following [12], we construct the Hochschild class of an endomorphism of a perfect module and describe the Hochschild class of this endomorphism when the Hochschild homology is expressed in term of dualizing objects.

To build the Hochschild class we need the following two natural morphisms \( \eta \) and \( \varepsilon \) in \( D_{\text{per}}(A^e) \).

**Lemma 4.2.1.** — Let \( M \) be a perfect \( A \)-module. There is a natural isomorphism

\[ \text{RHom}_{A^e}(M, M) \simeq \text{RHom}_{A^e}(\omega_A^{-1}, M \otimes A). \]

The morphism \( \eta \) in \( D_{\text{per}}(A^e) \) is induced by the identity of \( M \) and \( \varepsilon \) in \( D_{\text{per}}(A^e) \) is obtained from \( \eta \) by duality.

\[ \eta : \omega_A^{-1} \to M \otimes_k \mathbb{D}_A M, \]

\[ \varepsilon : M \otimes_k \mathbb{D}_A M \to A. \]
The map $\eta$ is called the coevaluation map and $\varepsilon$ the evaluation map.

Before proving this lemma recall the following canonical isomorphism.

**Proposition 4.2.1.** — Let $A$ and $B$ be a dg algebras. Let $M \in \mathcal{D}_{\text{per}}(A)$ and $S \in \mathcal{D}_{\text{per}}(B)$ and $N \in \mathcal{D}(A)$ and $M \in \mathcal{D}(B)$ then

\[
\text{RHom}_{A \otimes B}(M \otimes S, N \otimes T) \simeq \text{RHom}_{A}(M, N) \otimes \text{RHom}_{B}(S, T).
\]

**Proof of Lemma 4.2.1.** —

\[
\text{RHom}_{A}(M, M) \simeq \mathcal{D}_{A} M \otimes_{A} M
\]

\[
\simeq A^{\text{op}} \otimes_{A^{\text{e}}}(M \otimes \mathcal{D}_{A} M)
\]

\[
\simeq \text{RHom}_{A^{\text{e}}}(\omega^{-1}_{A}, M \otimes \mathcal{D}_{A} M).
\]

Thus we get an isomorphism

\[
(4.5) \quad \text{RHom}_{A}(M, M) \xrightarrow{\sim} \text{RHom}_{A^{\text{e}}}(\omega^{-1}_{A}, M \otimes \mathcal{D}_{A} M)
\]

The image of the identity in $\text{Hom}_{A}(M, M)$ by $(4.2)$ gives us the morphism $(4.3)$. Then applying $\mathcal{D}_{A^{\text{e}}}$ to $(4.3)$ we obtain a map

\[
\mathcal{D}_{A^{\text{e}}}(M \otimes \mathcal{D}_{A} M) \to A
\]

Then using the isomorphism of Proposition 4.2.1

\[
\mathcal{D}_{A^{\text{e}}}(M \otimes \mathcal{D}_{A} M) \simeq \mathcal{D}_{A}(M) \otimes \mathcal{D}_{A^{\text{op}}}(\mathcal{D}_{A} M)
\]

\[
\simeq \mathcal{D}_{A}(M) \otimes M.
\]

Finally, we get morphism $(4.4)$.

Let us define the Hochschild class. We have the following chain of morphisms

\[
\text{RHom}_{A}(M, M) \simeq \mathcal{D}_{A} M \otimes_{A} M
\]

\[
\simeq A^{\text{op}} \otimes_{A^{\text{e}}}(M \otimes_{k} \mathcal{D}_{A} M)
\]

\[
\simeq A^{\text{op}} \otimes_{A^{\text{e}}} A.
\]

We get a map

\[
(4.6) \quad \text{Hom}_{\mathcal{D}_{\text{per}}(A)}(M, M) \to \text{HH}_{0}(A).
\]

**Definition 4.2.1.** — The image of an element $f \in \text{Hom}_{\mathcal{D}_{\text{per}}(A)}(M, M)$ by the map $(4.7)$ is called the Hochschild class of $f$ and is denoted $\text{hh}_{A}(M, f)$. The Hochschild class of the identity is denoted $\text{hh}_{A}(M)$. 
Remark 4.2.1. — If $A = k$, then
\[ \text{hh}_k(M, f) = \sum_i (-1)^i \text{Tr}(H^i(f : M \to M)), \]
see for instance [12].

Lemma 4.2.2. — The isomorphism (4.1) sends $\text{hh}_A(M, f)$ to the image of $f$ under the composition
\[ \text{RHom}_A(M, M) \xrightarrow{\sim} \text{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}_A M) \xrightarrow{\epsilon^\circ} \text{RHom}_{A^e}(\omega_A^{-1}, A) \]
where the first morphism is defined in (4.2) and the second morphism is induced by the evaluation map.

Proof. — This follows from the commutative diagram:
\[
\begin{array}{ccc}
\text{RHom}_A(M, M) & \xrightarrow{\sim} & A^\text{op} \otimes_{A^e} (M \otimes_k \mathbb{D}_A M) \\
\text{id} & & \downarrow \\
\text{RHom}_A(M, M) & \xrightarrow{\sim} & \text{RHom}_{A^e}(\omega_A^{-1}, M \otimes \mathbb{D}_A M) \xrightarrow{\epsilon} \text{RHom}_{A^e}(\omega_A^{-1}, A).
\end{array}
\]

The trace of $f \in \text{Hom}_A(M, M)$ is the chain of composition
\[ k \xrightarrow{\text{id}} \text{RHom}_A(M, A) \xrightarrow{\text{id} \otimes f} \text{RHom}_A(M, A) \otimes_A M \xrightarrow{\epsilon} A \otimes_{A^e} A \]

\[ \cdots \xrightarrow{\text{id} \otimes f} \text{RHom}_A(M, A) \otimes_A M \xrightarrow{\epsilon} A \otimes_{A^e} A \]

Remark 4.2.2. — Our definition of the Hochschild class is equivalent to the definition of the trace of a 2-cell in [25]. This equivalence allows us to use string diagrams to prove some properties of the Hochschild class, see [25] and [6], [8].

Proposition 4.2.2. — Let $M, N \in \mathcal{D}_{\text{per}}(A)$, $g \in \text{Hom}_A(M, N)$ and $h \in \text{Hom}_A(N, M)$ then
\[ \text{hh}_A(N, g \circ h) = \text{hh}_A(M, h \circ g). \]

The proof of Proposition 4.2.2 uses string diagrams and will not be given here.
5. A pairing on Hochschild homology

In this section, we build a pairing on Hochschild homology. It acts as the Hochschild class of the diagonal, (see [12], [6], [8], [28]). Using this result, we prove our Riemann-Roch formula. We follow the approach of [12].

5.1. Hochschild homology and bimodules. — We show that a perfect $A \otimes B^{op}$-module induces a morphism from $\mathcal{H}(B)$ to $\mathcal{H}(A)$. We need the following technical lemma which generalizes Lemma 4.2.1.

**Lemma 5.1.1.** — Let $K \in D_{\text{per}}(A \otimes B^{op})$. Let $C = A \otimes B^{op}$. Then, there are natural morphisms in $D_{\text{per}}(A)$ which coincide with the coevaluation and evaluation morphisms when $B = k$

$$\omega_{A}^{-1} \rightarrow K \otimes_{B} D_{C}K$$

(5.1)

$$K \otimes_{B} \omega_{B} \otimes_{B} D_{C}K \rightarrow A.$$

**Proof.** — By [4,3] we have a morphism in $D_{\text{per}}(A \otimes B^{op} \otimes B \otimes A^{op})$

$$\omega_{A}^{-1} \rightarrow K \otimes_{C} D_{C}K.$$

Applying the functor $- \otimes_{B^{e}} B$, we obtain

$$\omega_{C}^{-1} \otimes_{B^{e}} B \rightarrow K \otimes_{C} D_{C}K \otimes_{B^{e}} B$$

and by proposition (4.2.1)

$$\omega_{C}^{-1} \simeq \omega_{A}^{-1} \otimes \omega_{B^{op}}^{-1} \simeq \omega_{A}^{-1} \otimes D_{B^{e}}(B).$$

Then there is a sequence of isomorphisms

$$\omega_{A}^{-1} \otimes \text{RHom}_{B^{e}}(B,B) \simeq \omega_{A}^{-1} \otimes (D_{B^{e}} B \otimes_{B^{e}} B) \simeq \omega_{C}^{-1} \otimes_{B^{e}} B$$

and there is a natural arrow $\omega_{A}^{-1} \text{id} \otimes \omega_{A}^{-1} \otimes \text{RHom}_{B^{e}}(B,B)$. Composing these maps we obtain the morphism

$$\omega_{A}^{-1} \rightarrow \omega_{A}^{-1} \otimes (D_{B^{e}} B \otimes_{B^{e}} B) \rightarrow \omega_{C}^{-1} \otimes_{B^{e}} B \rightarrow K \otimes_{C} D_{C}K \otimes_{B^{e}} B.$$  

For the map (5.1), we have a morphism in $D_{\text{per}}(A \otimes B^{op} \otimes B \otimes A)$ given by the map [4,4]

$$K \otimes_{C} D_{C}K \rightarrow C.$$

Then applying the functor $- \otimes_{B^{e}} \omega_{B}$, we obtain

$$(K \otimes_{C} D_{C}K) \otimes_{B^{e}} \omega_{B} \rightarrow C \otimes_{B^{e}} \omega_{B} \simeq (A \otimes B^{op}) \otimes_{B^{e}} \omega_{B}.$$
Composing with the natural "integration" morphism of Proposition 3.2.4, we get

\[(K \otimes D_C K) \otimes_B \omega_B \to A.\]

We show that an object in \(D_{\text{perf}}(A \otimes B^{\text{op}})\) induces a morphism between the Hochschild homology of \(A\) and that of \(B\).

Let \(K \in D_{\text{perf}}(A \otimes B^{\text{op}})\). We set \(C = A \otimes B^{\text{op}}\). We define the morphism

\[\Phi_K : \mathcal{H}(B) \to \mathcal{H}(A).\]

as the composition of the sequence of maps

\[
\begin{align*}
\text{RHom}_{B^e}(\omega_B^{-1}, B) &\to \text{RHom}_{A^e}(K \otimes_B \omega_B^{-1} \otimes_B \omega_B \otimes_B D_C K, K \otimes_B B \otimes_B \omega_B \otimes_B D_C K) \\
&\simeq \text{RHom}_{A^e}(K \otimes_B D_C K, K \otimes_B \omega_B \otimes_B D_C K) \\
&\to \text{RHom}(\omega_A^{-1}, A).
\end{align*}
\]

The first morphism is obtained by replacing \(\omega_B^{-1}\) by a homotopically projective resolution in \(D_{\text{perf}}(B^e)\). We denote it by \(P\omega^{-1}\). We get the following isomorphism

\[
\text{RHom}_{B^e}(P\omega^{-1}, B) \cong \text{Hom}_{B^e}(P\omega^{-1}, B).
\]

Let \(P_K\) be a homotopically projective resolution of \(K\) in \(D_{\text{perf}}(A \otimes B^{\text{op}})\). In view of Lemma 3.1.1 it is also a homotopically projective resolution in \(D_{\text{perf}}(A)\) and \(D_{\text{perf}}(B^{\text{op}})\). Similarly we define \(P\omega\) in \(D_{\text{perf}}(B \otimes B^{\text{op}})\). Then we have the following isomorphism

\[
\text{RHom}_{A^e}(P_K \otimes_B P\omega^{-1} \otimes_B P\omega \otimes_B P\omega \otimes_B P\omega \otimes_B D_K, P_K \otimes_B B \otimes_B P\omega \otimes_B P\omega \otimes_B D_K).
\]

Indeed by Lemma 3.1.1 \(P_K \otimes_B P\omega^{-1} \otimes_B P\omega \otimes_B P\omega \otimes_B D_K\) is an homotopically projective resolution of \(K \otimes_B \omega_B^{-1} \otimes_B \omega_B \otimes_B D_C K\) in \(D_{\text{perf}}(A^e)\). The first arrow is then obtain by applying the functor \(M \mapsto P_K \otimes_B M \otimes_B P\omega \otimes_B P\omega \otimes_B P\omega \otimes_B D_K\).

5.2. A pairing on Hochschild homology. — In this subsection, we build a pairing on Hochschild homology. It acts as the Hochschild class of the diagonal, (see [12, 6, 8, 28]). We also relate \(\Phi_K\) to this pairing.
Proposition 5.2.1. — Let $A, B, C$ be three proper homologically smooth dg algebras. There is a natural map
$$\mathcal{H}(A \otimes B^{\text{op}}) \otimes \mathcal{H}(B \otimes C^{\text{op}}) \to \mathcal{H}(A \otimes C^{\text{op}})$$
inducing an operation
$$\cup_B : \mathcal{H}_0(A \otimes B^{\text{op}}) \otimes \mathcal{H}_0(B \otimes C^{\text{op}}) \to \mathcal{H}_0(A \otimes C^{\text{op}})$$
such that for every $\lambda \in \mathcal{H}_0(B \otimes C^{\text{op}})$, $H^0(\Phi_K) (\lambda) = hh(A \otimes B^{\text{op}}) \cup_B \lambda$.

Before proving Proposition 5.2.1, let us state a technical lemma.

Lemma 5.2.1. — Let $M \in D_{\text{per}}(A)$. There is an isomorphism in $D_{\text{per}}(k)$
$$\omega_A \otimes_k M \cong M \otimes_{A^{\text{op}}} \omega_A.$$ 

Proof. — Obvious. \hfill $\Box$

The next proof explains the construction of $\cup_B : \mathcal{H}_0(A \otimes B^{\text{op}}) \otimes \mathcal{H}_0(B \otimes C^{\text{op}}) \to \mathcal{H}_0(A \otimes C^{\text{op}})$. We also prove the equality $H^0(\Phi_K)(\lambda) = hh(A \otimes B^{\text{op}}) \cup_B \lambda$.

Proof of Proposition 5.2.1 — (i) We identify $(A \otimes B^{\text{op}})^{\text{op}}$ and $A^{\text{op}} \otimes B$.

We have
$$\mathcal{H}(A \otimes B^{\text{op}}) \cong \text{RHom}_{A^{\otimes B^{\text{op}}}}(\omega^{-1}_{A^{\otimes B^{\text{op}}}}, A \otimes B^{\text{op}})$$
$$\cong \text{RHom}_{A^{\otimes B^{\text{op}}}}(\omega^{-1}_{A^{\otimes B^{\text{op}}}} \otimes_{B^{\text{op}}} \omega_{B^{\text{op}}}, A \otimes B^{\text{op}} \otimes_{B^{\text{op}}} \omega_{B^{\text{op}}})$$
$$\cong \text{RHom}_{A^{\otimes B^{\text{op}}}}(\omega^{-1}_{A^{\otimes B^{\text{op}}}} \otimes B^{\text{op}}, A \otimes \omega_{B^{\text{op}}}.$$ 

Let $S_{AB} = \omega^{-1}_{A} \otimes B^{\text{op}}$ and $T_{AB} = A \otimes \omega_{B^{\text{op}}}$. Similarly, we define $S_{BC}$ and $T_{BC}$. Then we get
$$\mathcal{H}(A \otimes B^{\text{op}}) \otimes \mathcal{H}(B \otimes C^{\text{op}})$$
$$\cong \text{RHom}_{A^{\otimes B^{\text{op}}}}(S_{AB}, T_{AB}) \otimes \text{RHom}_{B^{\otimes C^{\text{op}}}}(S_{BC}, T_{BC})$$
$$\to \text{RHom}_{A^{\otimes C^{op}}}(S_{AB} \otimes B^{\text{op}}, S_{BC}, T_{AB} \otimes B^{\text{op}}, T_{BC}).$$

Using the morphism $k \to \text{RHom}_{B^{\text{op}}}(B^{\text{op}}, B^{\text{op}})$, we get
$$k \to B^{\text{op}} \otimes_{B^{\text{op}}} \omega_B^{-1}.$$ 

Thus we get
$$\omega^{-1}_{A} \otimes C^{\text{op}} \to (\omega^{-1}_{A} \otimes B^{\text{op}}) \otimes_{B^{\text{op}}} (\omega_B^{-1} \otimes C^{\text{op}}).$$
We know by Proposition 3.2.4 that there is a morphism

$$\omega_{B^{op}} \otimes_{B^{op}} B \to k.$$ we deduce a morphism

$$(A \otimes \omega_{B^{op}}) \otimes_{B^{op}} (B \otimes \omega_{C^{op}}) \to A \otimes \omega_{C^{op}}.$$ Therefore we obtain the following commutative diagram

$$\xymatrix{ \text{RHom}_{A \otimes C}(S_{AB} \otimes_{B^{op}} S_{BC}, T_{AB} \otimes_{B^{op}} T_{BC}) & \text{RHom}_{A \otimes C}(S_{AC}, T_{AC}) \\ \text{RHom}_{A \otimes C}(S_{AC}, T_{AB} \otimes_{B^{op}} T_{BC}) & \text{RHom}_{A \otimes C}(S_{AC}, T_{AC}) }$$

We get

$$\mathcal{H}(A \otimes B^{op}) \otimes \mathcal{H}(B \otimes C^{op}) \to \mathcal{H}(A \otimes C^{op}).$$ Finally, taking the cohomology we obtain,

$$\text{HH}_0(A \otimes B^{op}) \otimes \text{HH}_0(B \otimes C^{op}) \to \text{HH}_0(A \otimes C^{op}).$$

(ii) We follow the proof of [12]. We set $C = A \otimes B^{op}$. Let $\alpha = \text{hh}_{A \otimes B^{op}}(\mathcal{K})$ and let $\lambda \in \text{HH}_0(B)$. By Proposition 4.2.2 $\alpha$ can be viewed as a morphism of the form

$$\omega_{A \otimes B^{op}}^{-1} \to \mathcal{K} \otimes \mathbb{D}_C(\mathcal{K}) \to A \otimes B^{op}.$$ We consider $\lambda$ as a morphism $\omega_{B}^{-1} \to B$. Then following the construction of $\Phi_{\mathcal{K}}$, we observe that $\Phi_{\mathcal{K}}(\lambda)$ is obtained as the composition

$$\xymatrix{ \mathcal{K} \otimes_{B} \mathbb{D}_C \mathcal{K} & \mathcal{K} \otimes_{B} \omega_{B} \otimes_{B} \mathbb{D}_C \mathcal{K} \ar[l]^{\lambda} \ar[r]^{\omega_{A}^{-1}} & A }$$ We have the following commutative diagram in $D_{\text{per}}(k)$
From this natural operation we are able to deduce a pairing on Hochschild homology. Indeed using Proposition \ref{prop:par1} we obtain a paring
(5.2) \[ \cup : \text{HH}_0(A^{\text{op}}) \otimes \text{HH}_0(A) \to \text{HH}_0(k) \simeq k. \]

An other natural construction to obtain a pairing is the following one. We consider \( A \) as a perfect \( k - \text{e}A \) bimodule. This provides a map \( \Phi_A : \mathcal{H}H(\text{e}A) \to \mathcal{H}H(k) \).

**Proposition 5.2.2.** — Let \( A \) and \( B \) be proper homologically smooth dg algebras. There is a natural isomorphism

\[ R\text{Hom}_{A^{\text{op}}}(\omega_A^{-1}, A) \otimes R\text{Hom}_{B^{\text{op}}}(\omega_B^{-1}, B) \sim R\text{Hom}_{A^{\text{op}} \otimes B^{\text{op}}}(\omega_A^{-1} \otimes \omega_B^{-1}, A \otimes B) \]

We compose \( \Phi_A \) with \( \mathfrak{s}_{A^{\text{op}}, A} \) and get

\[ \text{HH}(A^{\text{op}}) \otimes \text{HH}(A) \to k. \]

Taking the zero degree homology, we obtain

(5.4) \[ \langle \cdot, \cdot \rangle : \text{HH}_0(A^{\text{op}}) \otimes \text{HH}_0(A) \to k. \]

In other words \( \langle \cdot, \cdot \rangle = H^0(\Phi_A) \circ H^0(\mathfrak{s}_{A^{\text{op}}, A}) \).

Yet another possible construction of a pairing is the following one. Proposition 5.2.1 gives us a map

\[ \cup_{\text{e}A} : \text{HH}_0(A^{\text{e}}) \otimes \text{HH}_0(\text{e}A) \to \text{HH}_0(k) \simeq k. \]

Then there is a morphism

\[ \text{HH}_0(A^{\text{op}}) \otimes \text{HH}_0(A) \to k \]
\[ \lambda \otimes \mu \mapsto hh_{A^{\text{op}}}(A) \cup_{\text{e}A} (\lambda \otimes \mu) \]

Using Proposition 5.2.1, we get that

\[ H^0(\Phi_A)(\lambda \otimes \mu) = hh_{A^{\text{op}}}(A) \cup_{\text{e}A} (\lambda \otimes \mu). \]

By Lemma 5.2.2, we have

\[ hh_{A^{\text{op}}}(A) \cup_{\text{e}A} (\lambda \otimes \mu) = (hh_{A \otimes A^{\text{op}}}(A) \cup_{A^{\text{op}}} \lambda) \cup_A \mu \]
\[ = (hh_{A^{\text{op}} \otimes A}(A^{\text{op}}) \cup_{A^{\text{op}}} \lambda) \cup_A \mu \]
\[ = \lambda \cup \mu. \]

**Remark 5.2.1.** — The isomorphism \( s : A \otimes A^{\text{op}} \to A^{\text{op}} \otimes A \) defined by \( x \otimes y \mapsto y \otimes x \) induces an isomorphism \( \text{HH}_0(A \otimes A^{\text{op}}) \sim \text{HH}_0(A^{\text{op}} \otimes A) \) that sends the Hochschild class of \( A \) to the Hochschild class of \( A^{\text{op}} \).
This proves that these three ways to define a pairing lead to the same pairing. It also shows that the pairing is equivalent to the action of the Hochschild class of the diagonal.

5.3. Riemann-Roch formula for dg algebras. — In this section we prove the Riemann-Roch formula announced in the introduction.

**Proposition 5.3.1.** — Let $M \in \mathcal{D}_{\text{per}}(eA)$ and let $f \in \text{Hom}_A(M, M)$. Then

$$\text{hh}_k(A \otimes_{eA} M, \text{id}_A \otimes_{eA} f) = \text{hh}_{A^e}(A) \cup \text{hh}_{eA}(M, f)$$

**Proof.** — Let $\lambda = \text{hh}_{eA}(M, f) \in \text{HH}_0(eA) \simeq \text{RHom}(\omega_{eA}^{-1}, eA)$. As previously we set $B = eA$. We denote by $\tilde{f}$ the image of $f$ in $\text{RHom}_{B^e}(\omega_B^{-1}, M \otimes D_B M)$ by the isomorphism (4.2). We obtain the commutative diagram below.

By Lemma 4.2.2, the composition of the arrows on the bottom is $\text{hh}_k(A \otimes_{eA} M, \text{id}_A \otimes_{eA} f)$ and the composition of the arrow on the top is $\text{H}^0(\Phi_A(\text{hh}_{eA}(M, f)))$. It results from the commutativity of the diagram that

$$\text{hh}_k(A \otimes_{eA} M, \text{id}_A \otimes_{eA} f) = \text{H}^0(\Phi_A(\text{hh}_{eA}(M, f))).$$

Then using Proposition 5.2.1 we get

$$\text{hh}_k(A \otimes_{eA} M, \text{id}_A \otimes_{eA} f) = \text{hh}_{A^e}(A) \cup \text{hh}_{eA}(M, f).$$

We state and prove our main result which can be viewed as a noncommutative generalization of A. Calderaru’s version of the topological Cardy condition [6].

**Theorem 5.3.1.** — Let $M \in \mathcal{D}_{\text{per}}(A)$, $f \in \text{Hom}_A(M, M)$ and $N \in \mathcal{D}_{\text{per}}(A^{op})$, $g \in \text{Hom}_{A^{op}}(N, N)$. Then

$$\text{hh}_k(N \otimes_A M, g \otimes_A f) = \text{hh}_{A^{op}}(N, g) \cup \text{hh}_A(M, f).$$
where \( \cup \) is the pairing defined by formula (5.2).

**Proof.** — Let \( u \) be the canonical isomorphism from \( A \otimes_A^L (N \otimes M) \) to \( N \otimes_A M \).

By definition of the pairing we have

\[
\langle \text{hh}^A \otimes_A^L (N \otimes M, g \otimes f) \rangle = H^0(\Phi_A)(\text{hh}^A \otimes_A^L (N \otimes M, g \otimes f))
\]

The last equality is a consequence of Proposition 4.2.2.

**Remark 5.3.1.** — By adapting the proof of Proposition 5.3.1, we also obtain the following result to be compared to [12, Theorem 4.3.4]

**Theorem 5.3.2.** — Let \( A, B, C \) be proper homologically smooth dg algebras. Let \( K_1 \in D_{\op}(A \otimes B) \) and \( K_2 \in D_{\op}(B \otimes C) \). Then

\[
\text{hh}^A \otimes_{B \otimes C}^L (K_1 \otimes_B K_2) = \text{hh}^A \otimes_{B \otimes C}^L (K_1) \cup_B \text{hh}^B \otimes_{C \otimes D}^L (K_2).
\]

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