Black brane in asymptotically Lifshitz spacetime and viscosity/entropy ratios in Horndeski gravity

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Abstract – We investigate black brane solutions in asymptotically Lifshitz spacetime in $(3 + 1)$-dimensional Horndeski gravity, which admit a critical exponent fixed at $z = 1/2$. The cosmological constant depends on $z$ as $\Lambda = -(1 + 2z)/L^2$. We compute the shear viscosity in the $(2 + 1)$-dimensional dual boundary field theory via holographic correspondence. We investigate the violation of the bound for a viscosity to entropy density ratio of $\eta/s \geq 1/(4\pi)$ at $z = 1/2$.

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Introduction. – The celebrated Einstein gravity has been supported by strong observational evidence in many astrophysical scenarios. Despite this, there are still fundamental problems such as dark matter, dark energy and inflationary phase of the early Universe to be well understood in this framework. One of the principal attempts to deal with such problems in Einstein gravity concerns coupling the theory to scalar fields. Such efforts have led to the development of the now well-known Galileon theories which are scalar-tensor theories [1]. Particularly, these studies have led to the rediscovery of Horndeski’s gravity that is the most general scalar-tensor theory that was originally discovered in 1974 [2–9]. It is characterized by a single scalar-tensor theory with second-order field equations and second-order energy-momentum tensor. The Lagrangian producing second-order equations of motion as discussed in [7,8,10–16] includes four arbitrary functions of the scalar field and its kinetic term. The term we shall focus our attention on includes a nonminimal coupling between the standard scalar kinetic term and the Einstein tensor. Besides the cosmological interest, this theory has also attracted attention in astrophysics. In a more recent investigation, it was shown to admit the construction of black holes that develop Hawking-Page phase transitions at a critical temperature [8,17]. Other examples of spherically symmetric solutions in Horndeski theory in the context of the Solar System and further astrophysical scenarios can also be found [18–20], for instance, in the study of perihelion shift and light bending [21] and in the issues involving properties of spinning gyroscope and the Gravity Probe B experiment [22].

The AdS/CFT correspondence (holographic description) [23–26] has become an important tool to explore strongly coupled field theories. Black branes are fundamental as gravity duals in this correspondence. They, for instance, can be dual to strongly coupled plasma and allow to determine hydrodynamic and thermodynamic properties. These objects have been proposed recently [16] in the context of the Horndeski gravity in a particular sector of the theory, well known as the K-essence sector, which is supported by axion scalar fields that depend on the horizon coordinates. Further scenarios involving descriptions of static black branes supported by axionic scalars/two 3-form fields were proposed in [27]. These solutions have planar (or toroidal) horizons.

In quantum critical systems develops the Lifshitz scaling $t \rightarrow \lambda^2 t$, $x_i \rightarrow \lambda x_i$, where $z$ stands for a critical exponent. This scaling is similar to the scaling invariance of the pure AdS spacetime ($z = 1$) in Poincaré coordinates. The holographic point of view suggests that such scaling realizes an isometry in the spacetime metric as long as the radial coordinate scales as $r \rightarrow \lambda^{-1} r$ [28–30]. Holographic description of models involving Lifshitz scaling has attracted strong interest in recent years, mainly due to applications in condensed matter systems [31]. Especially, planar black holes involving Lifshitz superconductors with
an axion field as proposed in [32] are of special interest in the context of the AdS/CFT correspondence due to the application in non-conventional superconductor systems.

More recently, several holographic issues in the context of the Horndeski gravity have been put forward in [33–37]. Among other quantities, an important relation well known as the shear viscosity to entropy density ratio [38–43] can be computed in the dual conformal field theory. This ratio, which is given by the bound \( \eta/s \geq \frac{1}{4\pi} \), can be violated by the addition of higher-order curvature terms [44,45]. However, as shown in [38,46] by using the Horndeski gravity this ratio can be simply violated by adjusting parameters, without the presence of any such curvature term in the bulk. In our analyses the violation of the “universal” Kovtun-Son-Starinet (KSS) [39,42] bound \( \eta/s = 1/(4\pi) \) is due to the critical exponent \( z = 1/2 \) and sufficiently large ratio of parameters \( [\alpha/\gamma\Lambda] \), which implies \( \eta/s < 1/(4\pi) \) and confirms the aforementioned violations.

In the present study, we find black brane solutions in asymptotically Lifshitz spacetime in Horndeski gravity. Lifshitz black holes with a time-dependent scalar field in this theory were studied in [47]. Here, we obtain static Lifshitz black holes for Horndeski parameters related to this theory and confirms the aforementioned violations.

We consider the following interesting sub-class with non-minimal kinetic coupling given by the action

\[
I[g_{\mu\nu}, \phi] = \int \sqrt{-g} d^4x \mathcal{L},
\]

\[
\mathcal{L} = \kappa(\alpha \mathcal{T}^{(1)}_{\mu\nu} + \gamma \mathcal{T}^{(2)}_{\mu\nu}) + \frac{1}{2\kappa}(\alpha g_{\mu\nu} - \gamma G_{\mu\nu}) \partial^\mu \phi \partial^\nu \phi.
\]

Note that we have a non-minimal scalar-tensor coupling where we can define a new field \( \phi' = \psi \). This field has a dimension of \( \text{(mass)}^2 \) and the parameters \( \alpha \) and \( \gamma \) control the strength of the kinetic couplings, \( \alpha \) is dimensionless and \( \gamma \) has a dimension of \( \text{(mass)}^{-2} \). Thus, the Einstein-Horndeski field equations can be formally written as in the usual way,

\[
G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2\kappa} T_{\mu\nu},
\]

where \( T_{\mu\nu} = \alpha T^{(1)}_{\mu\nu} + \gamma T^{(2)}_{\mu\nu} \) and \( \kappa = (16\pi G)^{-1} \) and the scalar field equation is given by

\[
\nabla_\mu[(\alpha g_{\mu\nu} - \gamma G_{\mu\nu}) \nabla_\nu \phi] = 0.
\]

The energy-momentum tensors \( T^{(1)}_{\mu\nu} \) and \( T^{(2)}_{\mu\nu} \) take the following form:

\[
T^{(1)}_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla^{\lambda} \phi,
\]

\[
T^{(2)}_{\mu\nu} = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R - 2 \nabla_\lambda \phi \nabla_\mu \phi R^{\lambda}_{\nu \lambda \rho} - \nabla^\lambda \phi \nabla_\lambda R^{\mu \nu}_{\rho \rho} + \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi (\nabla_\lambda \phi)^2 - \nabla_\mu \phi \nabla_\nu \phi \nabla_\lambda \phi + \nabla_\mu \phi \nabla_\nu \phi \nabla_\lambda \phi + \frac{1}{2} G_{\mu\nu} \nabla_\phi \phi^2.
\]

In our case for Einstein-Horndeski gravity, we consider the following Ansatz for a general four-dimensional Lifshitz black brane of the form

\[
ds^2 = L^2 \left( -r^{2\epsilon} f(r) dt^2 + r^2 d\Omega_2^2 + \frac{dr^2}{r^2 f(r)} \right).
\]

Here \( d\Omega_2^2 \) is the metric for the unit \( S^2 \) sphere, plane or hyperboloid corresponding to \( \epsilon = 1, 0, -1 \), respectively [38]. Note that we can take \( d\Omega_2^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu \) to be the metric of constant curvature such that \( \nabla^2 = \tilde{g}_{\mu\nu} \). We can
also make $d\Omega^2$ as
\[
d\Omega^2_{2,\epsilon} = \frac{d\alpha^2}{1 - \epsilon u^2} + u^2 d\Omega^2_{1},
\] (11)
where $d\Omega^2_{1}$ is the metric of the unit 1-sphere. In [58] static spherically symmetric configurations of certain Galileons with shift invariance was first argued to admit a no-hair theorem. The no-hair theorem for Galileons requires that the square of the conserved current $J^\mu = (\partial g^{\mu\nu} - \gamma G^{\mu\nu})\nabla_\nu \phi$, defined in (8), should not diverge at horizon. Thus, to escape from the no-hair theorem [17,47,58,59], we need to impose that the radial component of the conserved current vanishes identically without restricting the radial dependence of the scalar field [47]
\[
\alpha g_{rr} - \gamma G_{rr} = 0. \tag{12}
\]
Recalling that $\phi'(r) \equiv \psi(r)$ we can easily note that this condition annihilates $\psi^2(r)$ regardless of its behavior at the horizon. The metric function $f(r)$ can be found by using eq. (12). Thus, one can show that the eq. (8) is satisfied by the following solution:
\[
f(r) = \frac{\alpha L^2}{\gamma (2 + 1)} - \left(\frac{r_0}{r}\right)^{2z+1}, \tag{13}
\]
\[
\psi^2(r) = -\frac{2L^2\epsilon(\alpha + \gamma \Lambda)}{\alpha \gamma^2 f(r)}, \tag{14}
\]
where the scalar field is real for a suitable relation between the parameters —see its rewritten form below. Finally, the Einstein-Horndeski field equation (7) is satisfied by these equations as long as $z = 1/2$ with $\epsilon = 0$ (the planar case). The former solution corresponds to black brane solution for asymptotically $AdS_4$ spacetime [28]. We shall focus on the latter case [32,38]. Note that when $\Lambda = -(1 + 2\epsilon)/L^2$ that is in accord with [28–30] we have the black hole solution in asymptotically Lifshitz spacetime
\[
f(r) = -\frac{\alpha}{\gamma \Lambda} - \left(\frac{r_0}{r}\right)^{2z+1}, \tag{15}
\]
\[
\psi^2(r) = \frac{2\epsilon(1 + 2\epsilon)(\alpha + \gamma \Lambda)}{\alpha \gamma \Lambda f(r)}, \tag{16}
\]
for $z = 1/2$. The parameters are defined in the range $-\infty < \alpha/\gamma \Lambda \leq -1$, with $\alpha, \gamma < 0$, or $-1 \leq \alpha/\gamma \Lambda < 0$, with $\alpha, \gamma > 0$.

We can still make the following transformation in the metric (10) with solution (15):
\[
f(r) \to -\frac{\alpha}{\gamma \Lambda} f(r), \quad r_0^{2z+1} \to -\frac{\alpha}{\gamma \Lambda} r_0^{2z+1},
\]
\[
L \to \left(\frac{\alpha}{\gamma \Lambda}\right)^{1/2} L, \quad t \to \frac{\gamma \Lambda \epsilon}{\alpha} t,
\]
\[
x \to \left(\frac{\gamma \Lambda \epsilon}{\alpha}\right)^{1/2} x, \quad y \to \left(\frac{\gamma \Lambda \epsilon}{\alpha}\right)^{1/2} y, \tag{17}
\]
in order to put the black hole solution in the standard form
\[
f(r) = 1 - \left(\frac{r_0}{r}\right)^{2z+1}, \tag{18}
\]
\[
\psi^2(r) = -\frac{2\epsilon(1 + 2z)(\alpha + \gamma \Lambda)}{\alpha \gamma^2 f(r)}. \tag{19}
\]
In addition, the fact that $\psi^2(r \to \infty) = 0$ into the action (6) ensures that this is a genuine vacuum solution. Note also that from eqs. (18), (19) the black hole geometry is regular everywhere (except at the central singularity), the scalar field derivative $\psi(r)$ diverges at horizon [8,38,59], but the scalar field does not explode at horizon since it approaches a constant near the horizon as $\phi^2(r) \sim ((\Lambda L^2(\alpha + \gamma \Lambda)/\alpha^2)^{1/2} f'(r_0)) (r - r_0) + const.$ These facts are in complete agreement with the no-hair theorem as previously discussed and evade the issues raised in [59]. The scalar field equation (19) is a real function outside the horizon since for $r > r_0$ we have $f(r > r_0) > 0$, and the scalar field is real, for example, in the interval $-1 < \alpha/\gamma \Lambda < 0$, with $\alpha, \gamma > 0$. We can see that at infinity the scalar field itself diverges as $\phi(r) \sim \ln r$, but not its derivatives $\psi$ that are the ones present in the action, which are finite at asymptotic infinity [59].

In the following, we address the issue of singularity through the curvature invariant given by the Ricci scalar
\[
R = \frac{r^2 f''(r) + 3zf'(f) + 2z^2 f(r)}{L^2} + \frac{5rf'(r) + 4zf(r) + 6f(r)}{L^2}, \tag{20}
\]
\[
= \frac{1}{L^2} \left[ 2z^2 + 4z + 6 - 3(z - 1) \left(\frac{r_0}{r}\right)^{2z+1} \right]. \tag{21}
\]
We can see that for $r = 0$ and $z = 1/2$ we find a curvature singularity since the Ricci scalar diverges, which implies that we have found a solution that is in fact a black brane solution. However, for $z = 1$, we have $R = -12/L^2$, that is, $R = 4\Lambda$ which characterizes the $AdS_4$ spacetime [8].

**Viscosity/entropy density ratio.** – In this section, we present the computation of the shear viscosity in the boundary field theory through holographic correspondence [38,39,41]. We do this in the Horndeski gravity context [38,60], where a black brane solution is found in the presence of asymptotically Lifshitz spacetime. In the gravity side, this planar black brane plays the role of the gravitational dual of a certain fluid. To compute the shear viscosity through holographic correspondence, we need to linearize the field equations [39–41]. Thus, the effective hydrodynamics in the boundary field theory is constructed in terms of conserved currents and energy-momentum tensor by considering small fluctuations around the black brane background $g_{xy} \to g_{xy} + h_{xy}$, where $h_{xy} = h_{xy}(t, x, r)$ [39,41,60–62] is a small perturbation —related issues in braneworlds at Einstein and Horndeski gravity can be seen, e.g., in [49,50]. For the
metric (10) in the planar case, i.e., \(\epsilon = 0\), we find the fluctuations of Ricci tensor in the form
\[
\delta^{(1)} R_{xy} = -\frac{r^2 f(r)}{2L^2} h_{xy} + \frac{h_{xy}}{2r^2 z f(r) L^2} - \left(\frac{r f(r) + 3}{2r^2 z f(r)}\right) h_{xy}'.
\] (22)

Here we have disregarded the dependence of \(h_{xy}\) on \(\vec{x}\). Recalling that \(T_{xy} = \delta^{(1)} T_{xy}/(2\kappa)\) and using the Einstein-Horndeski equation in the Ricci form then \(\delta^{(1)} R_{xy} = \delta^{(1)} T_{xy}/(2\kappa)\), such that we find a Klein-Gordon-like equation with a position-dependent mass as follows:
\[
\frac{1}{\sqrt{-g}}(\sqrt{-g} g^\alpha \beta \partial_\alpha h_{xy}) = -m^2(r) h_{xy},
\] (23)

where
\[
m^2(r) = \frac{2(1 + 2z)}{3L^2} \left(4 + 7z + z^2\right) \left(-\alpha(1 + 2z)/\gamma + 3 + 6z\right) - \frac{2 + 2z - 2z^2}{3L^2(-\alpha(1 + 2z)/\gamma + 3 + 6z)(2z + 1)} \bigg(\frac{r_0}{r}\bigg)^{2z+1}.
\] (24)

Now considering the following Ansatz:
\[
h_{xy}(x, r) = \int \frac{d^3k}{(2\pi)^3} e^{i k x} \chi(r, k),
\] (25)

with \(x = (t, \vec{x})\), and \(k = (\omega, \vec{q})\), the equation of motion for the fluctuations assumes the following form:
\[
\frac{1}{\sqrt{-g}}(\sqrt{-g} g^{\alpha \beta} \partial_\alpha \chi(r, k)) = -m^2(r) \chi(r, k).
\] (26)

In addition to the mass term, in general, this equation also contains the contribution \(k^2 = q^2 - \omega^2\), but we have considered the limit \(\omega \to 0\) and spatial momentum \(q = 0\). After these considerations, we shall simply consider \(\chi(r, k) \equiv \chi(r)\).

In our present scenario, we analyze viscosity/entropy density ratio for the critical exponent \(z = 1/2\). Non-integer critical exponents have appeared in previous studies, e.g., [47,63–67] — and references therein. Especially in [63,64] the authors argue that although the Lifshitz critical exponents in the action of a quantum field theory developing Lifshitz symmetry are assumed to be an integer, there is no such limitation indeed since the obtained dispersion relation in the Hamiltonian density associated with the quantized theory shows the exact analytic continuation to non-integer values of \(z\). Furthermore, in [65] the authors also consider non-integer \(z > 1\) in the problem of entanglement propagation in Lifshitz-type scalar theories.

The general explicit solution for the differential equation (26) can be given by [60]
\[
\chi(r) = C_1 F_1 \left(\frac{a_-, a_+ , b_-; \left(\frac{r_0}{r}\right)^{2z+1}}{r/r_0} \right)^{3z} + C_2 (r/r_0)^{3z} F_1 \left(\frac{d_-, d_+ , b_+; \left(\frac{r_0}{r}\right)^{2z+1}}{r/r_0} \right)^{3z+1},
\] (27)

\[
\beta_\mp = \pm \sqrt{9\gamma L - 3\alpha z + 2\sqrt{9\gamma L - 3\alpha}} + \sqrt{2 - \alpha z - 3\alpha z + 2\sqrt{9\gamma L - 3\alpha}}.
\] (28)

On the horizon \(r = r_0\), we can see that \(\chi\) in the eq. (27) diverges for general values of \(C_1\) and \(C_2\), because the argument \(r_0/r\) of the hypergeometric functions becomes equal to unity. In order to remove this divergence, we first rewrite eq. (27) in terms of gamma functions, such that on the horizon reads
\[
\chi(r_0) = \frac{C_1 \Gamma(b_-) \Gamma(a_- + a_+ - b_-)}{\Gamma(b_- - a_-) \Gamma(b_- - a_+)} + \frac{C_2 \Gamma(b_+) \Gamma(d_- + d_+ - b_+)}{\Gamma(b_+ - d_+) \Gamma(b_+ - d_-)}. \] (32)

We can now remove the divergence by choosing
\[
\frac{C_2}{C_1} = \frac{\Gamma(b_-) \Gamma(b_+ - d_-) \Gamma(b_+ - d_+)}{\Gamma(b_- - a_-) \Gamma(b_- - a_+)}.
\] (33)

This allows us to find the following regularized solution:
\[
\chi(r_0) = \frac{2\pi C_1 \Gamma(b_-)}{\Gamma(b_- - a_-) \Gamma(b_- - a_+)},
\] (34)

that will be crucial to our following considerations. The integration constant \(C_1\) can be set to unity without loss of generality. To obtain the final result in the form (34) we have made use of the parameters \(a_\pm\), \(b_\pm\) and \(d_\pm\) definitions given above and gamma function properties.

Let us now focus on the computation of the shear viscosity of the fluid in \((2 + 1)\)-dimensional dual boundary
field theory by proceeding as follows. We can see that we can propose an effective action for eq. (26) written as

\[ S = -\frac{1}{16\pi G} \int \frac{d^3k}{(2\pi)^3} \left( \frac{N(r)\partial r\partial r - M(r)}{2} \right) \]

\[ -\frac{1}{16\pi G} \int d^3k \partial_r \partial_r G_{xy,xy}^{R}(\omega,0) \bigg|_{\chi(r) = \chi(r,0)} \bigg|_{r_0}, \]

where \( N(r) = \sqrt{-g_g^{rr}} \) and \( M(r) = \sqrt{-g_m^{rr}} \). This action evaluated on-shell reduces to the surface term

\[ S = \left. \frac{1}{16\pi G} \int \frac{d^3k}{(2\pi)^3} \left( \frac{1}{2} N(r)\partial_r \chi(r,0) \right) \right|_{r=r_0}. \]

From the holographic correspondence [23–26,51,52,56,57, 68] the Green’s retarded function can be computed via two-point function from the generator of connected correlation functions on the boundary which is given in terms of the classical action (36).

Thus, as is well known, we have that the retarded Green’s function reads

\[ G_{xy,xy}^{R}(\omega,0) = -\frac{2}{16\pi G} \left( \sqrt{-g_g^{rr}} \right)_{r_0} \chi(r,0) \partial_r \chi(r,0) \bigg|_{r=r_0}, \]

where we have admitted spatial momentum \( q = 0 \). Since the imaginary part of the Green’s function does not depend on the radial coordinate, we have conveniently chosen to compute it at horizon \( r = r_0 \).

Now, from regularity at the horizon, the derivative of \( \chi \) at the horizon is given in terms of \( \chi(r_0) \) at leading order in \( \omega \) [69]. As such, we can write eq. (37) as follows:

\[ G_{xy,xy}^{R}(\omega,0) = -\frac{2\omega L^2 r_0^2}{16\pi G} \chi^2(r_0). \]

The shear viscosity [42,43,51,56,57] is then given by

\[ \eta = -\lim_{\omega \to 0} \frac{1}{2\omega} \text{Im} G_{xy,xy}^{R} \]

\[ = \frac{L^2 r_0^2}{16\pi G} \chi^2(r_0), \]

where \( s = r_0^3 L^2/4G \) is the entropy density [32]. Thus, we can now write the shear viscosity/entropy density ratio in the form

\[ \frac{\eta}{s} = \frac{1}{4\pi} \left[ \frac{2\pi \Gamma(b_-)}{\Gamma(b_- - a_-) \Gamma(b_- - a_+)} \right]^2. \]

One should notice that from eq. (40), that is

\[ \frac{\eta}{s} = \frac{1}{4\pi} \chi^2(r_0), \]

our analyses show that for the fixed critical exponent \( z = 1/2 \) the “universal” bound \( \eta/s \geq 1/(4\pi) \) is not violated, for example, as \(-\alpha/\gamma \Lambda = 5, 10\), but it is violated if \( |\alpha/\gamma \Lambda| \) is sufficiently large, for example, as \(-\alpha/\gamma \Lambda = 15, 40\) —see table 1. Moreover, for the case \( \alpha = -\gamma \Lambda \) this bound saturates, i.e., \( \eta/s = 1/(4\pi) \). This is because in this regime eq. (19) gives \( \psi(r) = 0 \) which in turns substituting into eq. (26) yields \( m(r) = 0 \). Thus, by solving eq. (26), we find \( \chi(r) = \text{const.} \), everywhere. Then, on the horizon one can assume the solution normalized to unit \( \chi(r_0) = 1 \), which from (42) clearly saturates the bound.

**Conclusions.** In this paper, we show that black brane with asymptotically Lifshitz spacetime and viscosity/entropy ratios in Horndeski gravity

| \( -\alpha/(\gamma \Lambda) \) | \( \eta/s \) |
|---|---|
| 5 | 13.14/4\pi > 1/4\pi |
| 10 | 2.44/4\pi > 1/4\pi |
| 15 | 0.57/4\pi < 1/4\pi |
| 40 | 0.053/4\pi < 1/4\pi |

Table 1: The table shows the shear viscosity to entropy density ratio \( \eta/s \) for some values of \(-\alpha/\gamma \Lambda\) with critical exponent \( z = 1/2 \).

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