Double criticality of the SK-model at $T = 0$.

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Numerical results up to 42nd order of replica symmetry breaking (RSB) are used to predict the singular structure of the SK spin glass at $T = 0$. We confirm predominant single parameter scaling and derive corrections for the $T = 0$ order function $q(a)$, related to a Langevin equation with pseudotime $1/a$. $a = 0$ and $a = \infty$ are shown to be two critical points for $\infty$-RSB, associated with two discrete spectra of Parisi block size ratios, attached to a continuous spectrum. Finite-RSB-size-scaling, associated exponents, and $T = 0$-energy are obtained with unprecedented accuracy.

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The low temperature limit usually simplifies considerably the properties of magnetically ordered phases. Recent in research has however shown that frustrated systems can have rich behaviour even at $T = 0$. Spin glasses are an extreme example from condensed matter, while others are a feature of computer and information science in problems such as hard satisfiability and error-correcting codes. In particular, even the potentially soluble infinite-range Ising spin glass model of Sherrington and Kirkpatrick has left open many puzzling questions. Parisi devised an ansatz for the order parameter of the SK-model, based on an infinite hierarchy of so-called replica symmetry breakings and related hierarchical to the distribution of overlaps of metastable solutions. The determining equations for this ansatz have recently been rigorously proven to be exact, but its explicit solution remains elusive. Also only recently has the $T = 0$ SK problem been recognized as a critical one-dimensional theory.

In view of the paradigmic role that the SK-model has played in the understanding and development of the statistical physics of complex systems, together with the potential that further comprehension of its subtleties has for extensions to other more-complicated systems in many fields of science, especially those involving zero- (or effectively zero-)temperature replica-symmetry-breaking transitions, it seems important to pursue the better understanding of $T = 0$ RSB in the SK model. This letter is concerned with such a study and the exposure of several known interesting new features to be identified.

Parisi’s order parameter is a function $q(x, T)$ on an interval $0 \leq x \leq 1$, the limit of a stepwise function $q_i(T)$ determined by extremization of a free energy. It provides the hierarchical distribution of pure state overlaps $P(q)$ through $P(q) = dx/dq$. Parisi’s original work considered numerically an approximation with a small finite number of steps, but most recent studies of the SK-model have been based on self-consistent solutions for his later non-trivial continuous order function, typically perturbatively in the deviation from the finite-temperature phase transition. Here the analysis is considered explicitly at $T = 0$ using very accurate studies of a very large sequence of RSB orders.

In the limit of zero temperature Parisi’s order function may be replaced by $q(a), 0 \leq a \leq \infty$, $q(a) = \lim_{T \to 0} q_{\text{parisi}}(aT)$ [11]. Already, in recent work to 5RSB we observed a predominant single parameter scaling for $q(a)$ and used this to predict a simple order function $q(a) = \sqrt{\pi a}/(2\xi) \text{erf}(\xi/a) = 1/(1/(2/3/2, -\xi^2/a^2) \text{a single correlation 'length' (in RSB-space) } \xi \approx 1.13 \text{ characterized the non-trivial scaling dependence on } a$ [18]. Now, by means of redesigned numerical procedures and analytical transformations we have been able to improve the previous 10-dimensional extremization (5-RSB) up to an 84-dimensional extremization of the SK-energy (42-RSB). To the best of our knowledge, this is by far the highest order calculation of RSB, allowing for the first time finite ‘size’ scaling on a rather large one-dimensional ‘lattice’ with up to 42 ‘sites’, where each site stands for one RSB-order [18]. This has enabled many hitherto unknown interesting new features to be identified.

Let us recall the SK Hamiltonian $H = \sum_{i<j} J_{ij} \delta_{ij}$ with Gaussian-distributed $J_{ij}$ independently distributed over all pairs of sites with mean zero and variance $N^{-1}$. The $T = 0$-limit of the free energy can be written as the extremization with respect to the Parisi order parameter plateaux $q_i$ and steps $a_i$ of

$$E_\kappa = \sum_{i=1}^\kappa a_i \left(q_i^2 - q_i^{2+1}\right) - \frac{\tilde{\cal{I}}_i^{(1)}}{a_\kappa} \ln \left(\prod_{i=1}^2 \tilde{\cal{C}}_i^{(r)}(\phi)\right)$$

where $\tilde{\cal{C}}_i^{(r)}$ acts like a RSB-transfer matrix as

$$\tilde{\cal{C}}_i^{(r)}(\phi) = \frac{1}{\sqrt{2\pi\delta q_i}} \int dh_i \exp\left(-\frac{1}{2\delta q_i}(h_{i+1} - h_i)^2\right) f(h_i)^r$$

with $\delta q_i = q_i - q_{i+1}, r_i = a_i/a_{i-1}, q_1 = 1, h_{i+2} = 0$.

The selfconsistently obtained $q$- and $a$-level distributions are shown in Fig.1. High RSB-orders $\kappa$ demonstrate several interesting features. We remark first upon the emergence of spectral bands and the formation of a dense region of $a$-levels close to $a \approx \xi$. The spectral character is further elaborated in Fig.2 by plotting the ratios $r_i^{(\kappa)} = a_i^{(\kappa)}/q_i^{(\kappa)}$ on the unit interval $0 \leq i/\kappa \leq 1$ for all RSB-orders $\kappa$. Already by eye one sees that the Parisi blocksize ratios approach characteristic limits as $\kappa \to \infty$, further justified by fitting to Padé approximants. These
FIG. 1: $q_i^{(s)}$-spectra (left) are plotted versus RSB-order $\kappa$; a-level spectra (right) are displayed by log($a_i^{(s)}$) versus $c \log(\kappa)$ with $c \approx 4/3$; asymptotic linear behavior of diverging log($a_i^{(s)}$) and a dense regime for $a_i = O(\kappa^0)$ are observed as $\kappa \to \infty$.

FIG. 2: Left figure shows results for Parisi block size ratios $r_{\kappa-1} r_{\kappa-1} r_{\kappa-1} \equiv a_{\kappa-1}^{(s)}/a_{\kappa-1}^{(s)}$ (black dots) normalized to the unit interval $0 \leq i/\kappa \leq 1$, connected by lines with period-10 alternating colors from 2-RSB (red, bottom) to 42-RSB (red, top); $a_0 = \infty, a_{\kappa+1} = 0$. Right figure shows extrapolations to three different spectra at $\infty$-RSB: a discrete spectrum on line $((0,0), (0,1))$, a continuous distribution of $r = 1$ along the line $((0,1), (1,1))$, and a second discrete spectrum from $(1,1)$ to $(1,0)$.

FIG. 3: Discrete spectra for Parisi block size ratios $r_i$ (ratios of largest $a$) and $r_1 \equiv r_{\kappa-i} (\text{ratios of smallest } a)$ as obtained for $i = 0, \ldots, 22$ in the $\kappa \to \infty$-limit (dots) compared with fit curves interpolating (discrete) analytical predictions demonstrate the existence of two discrete spectra on the lines $L_1 \equiv \{(0,0), (0,1)\}$ and $L_2 \equiv \{(1,0), (1,1)\}$, respectively connected at the singular accumulation points $(0,1)$ and $(1,1)$ to a continuum on $\{(0,1), (1,1)\}$. Explicitly, the continuation to $\kappa = \infty$ yields (i) a discrete spectrum of the form $\tilde{r}_i \equiv \lim_{\kappa \to \infty} r_i^{(s)} = 1 - 1/(i + 1)$ for integer-valued $i = 0,1,2,\ldots$ on $L_1$ and (ii) another discrete spectrum for $r_i \equiv \lim_{\kappa \to \infty} r_i^{(s)}$ on line $L_2$, well approximated by $r_i = (1 + \frac{2}{\kappa}/((i + \frac{1}{2})^\frac{3}{2} - \frac{3}{2}))^{-1/2}$, (see Fig 3) while (iii) continuations at fixed finite $i/\kappa$ (see Fig 2 right) demonstrate the continuous spectrum (as $\kappa \to \infty$). The two discrete spectra reflect the inequivalent non-analytic behaviors of the order function $q(a)$ at its limiting values at $a = \infty$ and $a = 0$. Fig 4 shows our recent conjecture for the analytic continuation of the $\kappa$-RSB stepped order function $q_i^{(s)}(a) = \sqrt{\pi} \xi_a \text{erf}(\sqrt{\kappa} + w_\kappa) / (\kappa w_\kappa)$ with $\lim_{\kappa \to \infty} w_\kappa = 0$ and $\xi_{\kappa} \to \xi \approx 1.13$ is confirmed as a good fit by the numerical solutions from $\kappa = 10$ to $\kappa = 42$. 

but on smaller $a$-scales a tiny correction is found, better resolved by the derivative $q'(a)$ which shows a maximum near $a \approx 0.344$. This correction can be incorporated in an improved analytical model fit function [22] by

$$q^{(\kappa)}(a) = \frac{a}{\sqrt{a^2+w_k}} F_1(\alpha, \gamma, -\xi_2/(a^2+w_k)).$$

The flow of the variational parameters as $\kappa \to \infty$ remains close to our previous proposal [5]: the limits slightly modified to $\alpha \equiv \alpha_\infty = 0.53 \pm 0.01$, $\gamma \equiv \gamma_\infty = 1.71 \pm 0.02$, $\xi \equiv \xi_\infty = 1.16 \pm 0.01$, and $w$ monotonically falling towards zero [22].

We observe that the large $a$-expansion $q^{\infty}(a) = 1 - \frac{a^2}{2} + \frac{a^4}{12} + O(a^{-6}) \approx 1 - 0.42a^2 + 0.16a^4 + O(a^{-6})$ of Eq.(2) agrees well with Pankov’s $q(1) = 1 - 0.41a^2$ obtained from the standard $\infty$-RSB differential equation and $T \to 0$ scaling [6]. The deviation of $\alpha$ and $\gamma$ from the rational values $\frac{2}{5}$ and $\frac{3}{5}$ of our original proposal [5] leads to a small correction to a linear rise of $q(a)$ at small $a$ and a pronounced maximum of $q^{(\kappa)}(a)$ at $a = 0.344$. This feature confirms low $T$ results of Crisanti and Rizzo [12], obtained perturbatively. It is also interesting to compare our results for $q(a)$ with the predictions of the Vannimenus-Toulouse-Parisi scaling ansatz $q_{\text{parisi}}(x,T) = f(x/T) \leq q_{\text{max}}(T)$ [7,8], combined with the Parisi-Toulouse hypothesis [7] and the knowledge of the behaviour on the de Almeida-Thouless line. The comparison is qualitatively quite good but not perfect, most noticeably deviating at low $a$, the VTP prediction being slightly lower and more curved.

Our analysis also gives strong hints for the existence of invariance-points in $a$-space, i.e. points where $q^{(\kappa)}(a)$ or one of its derivatives $\partial_a q^{(\kappa)}(a)$ does not vary under a change of RSB-order. They appear to be fixed points under the renormalization group which decimates the RSB-order [3]. For example $q^{(\kappa)}(a)$ does not change with $\kappa$ at $a = \bar{a} \approx 0.401$; for all calculated orders the changes $q^{(\kappa+1)}(a) - q^{(\kappa)}(a)$ are negative for $a < \bar{a}$ and positive for $a > \bar{a}$; they decay towards zero in $\infty$-RSB without change of sign, such that the contributions accumulate. As yet we have no physical explanation for these new critical “a-values”.

Our accurate results, $O(10^{-12})$, for such a large number of RSB orders also enables us to determine finite-RSB-size scaling exponents and the ground state energy to unprecedented accuracy. The energy of the SK-model is plotted versus RSB-order $\kappa$ in Fig.6. Within the orders studied both the energy and the susceptibility as a function of $\kappa$ have reached the asymptotic large-$\kappa$ scaling regime, fitting accurately the power law behaviors

$$1 - E_\infty/E_\infty \sim \kappa^{-z}, \quad \chi_1 \sim \kappa^{-z_\chi}$$

with $E_\infty = -0.7631667265(6)$, $z_\infty \approx 4$ and $z_\chi \approx \frac{5}{2}$ and $(\chi - 1)$ vanishing exponentially [23].

In an earlier 5-RSB study of the $T = 0$ distribution of local fields $P(h) = N^{-1} \sum_i (\delta(h-h_i) \phi_{ij}^q )$ there appeared to be a deviation from linearity in $|h|$ at very small $h$ [5]. In the light of the extension to 40-RSB, we are able to show that this is an artefact of finite RSB order and the power law $P(h) \sim h$ for small $h$ [13,14] is confirmed. In any specific finite-RSB order, small oscillations are observed around $P(h) \approx 0.3$ near $h = \chi_1$ but they shrink to zero as $\chi_1 \to 0$ in the $\infty$-RSB limit. Thus in the present context the pseudogap behavior $P(h) \sim |h|$ can be viewed as being generated by the decay of finite-$\kappa$ RSB gaps as $\kappa \to \infty$.

One may further relate our results to a pseudo-dynamical field theory. The order function $q(a)$ in Eq.(2) can be recognised for $w = 0$ as the solution of the confluent hypergeometric differential equation $(a^2/\xi^2) \partial_a q(a) + ((3-2\gamma)/\xi^2 - 2) a \partial_a q(a) + 4 a q(a) = 0$, or given by

$$\frac{\partial}{\partial(\xi/a)} \phi(a) = \frac{\delta H}{\delta \phi(a)}, \quad \phi(a) = \frac{1}{\sqrt{2\alpha}} \frac{\partial}{\partial (\xi/a)} \log(q(a))$$

$$H = 8\alpha \sinh(\phi(a)) - ((\gamma - 1/2)a/\xi + \xi/\alpha) \phi^2(a)$$

provided $\sinh(\phi)$ is truncated at third order. In fact, however, the numerical solution of the sinh($\phi$)-model [9] provides as good a fit to our data as Eq.(2) and thus mimics similarly well the SK-model at $T = 0$. The relationship to dynamics follows from an inverse relationship between the Parisi ‘length’ $x$ (or $a$ as here) and the timescale to explore the corresponding clusters of pure states in dynamics, with the characteristic timescales increasing sharply as $x$ is reduced [14]. Taking $1/a$ as a pseudo-time Eq.(4) is identified as a (Langevin) pseudo-dynamical equation. The parameters $\alpha$ and $\gamma$, which model the order function $q(a)$ analytically in between the singular points $a = 0$, $a = \infty$, appear as two new coupling constants in the Hamiltonian. We also recall here the known relationship between replica-symmetry breaking and slow dynamics,

![Graph](image-url)
much in analogy with the Halperin-Hohenberg theory\cite{9} of slow-dynamical (and spatially-critical) behavior.

In summary, we have obtained several new results for the paradigmic Sherrington-Kirkpatrick spin-glass at $T = 0$: (i) we have found that the Parisi order function can be represented by a 1D field theory with two inequivalent critical limits $a = 0$ and $a = \infty$, the order parameter $q(a)$ showing critical behavior in both limits of long ($a = 0$) and short ($a = \infty$) pseudotimes. The key issue for criticality is the presence of two discrete spectra for the Parisi block size ratios (at $a = 0$ and at $a = \infty$), attached to a continuous spectrum for finite $a$. (ii) All orders of RSB up to 42nd order have been solved to great numerical accuracy to enter deep into the large-RSB scaling regime, enabling us to obtain new power laws, the SK-energy up to $O(10^{-10})$, and corrections to single parameter scaling in a hypergeometric order function model.

It would be interesting to extend to systems, both range-free and finite-range, and to include real temporal dynamics and quantum effects. As examples of potential non-trivial range-free extensions one might consider K-SAT or LDPC code problems under dynamical algorithmic optimization using either classical or quantum dynamical annealing \cite{24}. The existence of true RSB in finite-range problems at equilibrium is still controversial but our study could potentially usefully extend current Ginzburg-Landau-Wilson functional integral theory\cite{10}.

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\bibitem{17} Or for many hierarchically structured metastable phases
\bibitem{18} \begin{equation*}
F_1(a, b, c) \text{ is the confluent hypergeometric function}
\end{equation*}
\bibitem{19} The new procedure reduces the naïvely ‘exponentially hard’ problem to one where the CPU time grows polynomially with RSB-order. Details will be given elsewhere.
\bibitem{20} This choice of a modified relativistic Dirac Coulomb spectrum was motivated by several Coulomb analogies
\bibitem{21} A minimal parameter-number choice is made which fits best all available numerical details at $T = 0$; in addition this choice confirmed the expected existence of an effective field theory with physical meaning.
\bibitem{22} $w = w_\infty = 0$ implies $q'(a) \rightarrow 0$ in an exponentially small $a$-regime which cannot be resolved by the numerical data so that a small but finite $w_\infty$ cannot be ruled out
\bibitem{23} The susceptibility $\chi_1$ derives from the variation of the SK energy (1) with respect to $q_1$ (largest-$a$ regime)
\bibitem{24} In particular, although several effectively mean-field (range-free) problems, with higher-order ($p > 2$) interactions have IRSB thermodynamics near their onsets from their finite-temperature ergodic regimes, they normally exhibit full RSB at $T = 0$, while finite connectivity also increases the size of the order parameter space.
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