EXAMPLES OF ITÔ CÀDLÀG ROUGH PATHS

CHONG LIU AND DAVID J. PRÖMEL

Abstract. Based on a dyadic approximation of Itô integrals, we show the existence of Itô càdlàg rough paths above general semimartingales, suitable Gaussian processes and non-negative typical price paths. Furthermore, Lyons-Victoir extension theorem for càdlàg paths is presented, stating that every càdlàg path of finite $p$-variation can be lifted to a rough path.

Key words: càdlàg rough paths, Gaussian processes, Lyons-Victoir extension theorem, semimartingales, typical price paths.

MSC 2010 Classification: Primary: 60H99, 60G17; Secondary: 91G99.

1. Introduction

Very recently, the notion of càdlàg rough paths was introduced by Friz and Shekhar [FS17] (see also [CF17, Che17]) extending the well-known theory of continuous rough paths initiated by Lyons [Lyo98]. These new developments significantly generalize an earlier work by Williams [Wil01]. While [Wil01] already provides a pathwise meaning to stochastic differential equations driven by certain Lévy processes, [FS17, CF17] develop a more complete picture about càdlàg rough paths, including rough path integration, differential equations driven by càdlàg rough paths and the continuity of the corresponding solution maps. We refer to [LCL07, FV10b, FH14] for detailed introductions to classical rough path theory.

A càdlàg rough path is analogously defined to a continuous rough path using finite $p$-variation as required regularity, see Definition 2.1 and 2.3, but (of course) dropping the assumption of continuity. Note that the notion of $p$-variation still works in the context of càdlàg paths without any modifications. Loosely speaking, for $p \in [2, 3)$ a càdlàg rough path is a pair $(X, X)$ given by a càdlàg path $X: [0, T] \to \mathbb{R}^d$ of finite $p$-variation and its “iterated integral”

$$X_{s,t} = \int_s^t (X_r - X_s) \otimes dX_r, \quad s, t \in [0, T],$$

which satisfies Chen’s relation and is of finite $p/2$-variation in the rough path sense. While the “iterated integral” can be easily defined for smooth paths $X$ as for example via Young integration [You36], it is a non-trivial question whether any paths of finite $p$-variation can be lifted (or enhanced) to a rough path. In the setting of continuous rough paths this question was answered affirmative by Lyons-Victoir extension theorem [LV07]. In Section 2 we prove the analogous result in the context of càdlàg rough paths stating that every càdlàg path of finite $p$-variation for arbitrary non-integer $p \geq 1$ can be lifted to a rough path.

The theory of càdlàg rough paths provides a novel perspective to many questions in stochastic analysis involving stochastic processes with jumps, which play a very important role in probability theory. For a long list of successful applications of continuous rough path theory we refer to the book [FH14]. However, for applications of rough path theory in probability
theory Lyons-Victoir extension theorem is not sufficient. Instead, it is of upmost importance to be able to lift stochastic processes to random rough paths via some type of stochastic integration.

In Section 3 we focus on stochastic processes with sample paths of finite \( p \)-variation for \( p \in (2,3) \), which is the most frequently used setting in probability theory, and construct the corresponding random rough paths using Itô(-type) integration. More precisely, we define for a stochastic process \( X \) the “iterated integral” \( X \) (cf. (1.1)) as limit of approximating left-point Riemann sums, which corresponds to classical Itô integration if \( X \) is a semimartingale. The main difficulty is to show that \( X \) is of finite \( p/2 \)-variation in the rough path sense. For this purpose we provide a deterministic criterion to verify the \( p/2 \)-variation of \( X \) based on a dyadic approximation of the path and its iterated integral, see Theorem 3.1. As an application of Theorem 3.1 we provide the existence of Itô càdlàg rough paths above general semimartingales (possibly perturbed by paths of finite \( q \)-variation), certain Gaussian processes and typical non-negative prices paths. Let us remark that related constructions of random càdlàg rough paths above stochastic processes are given in [FS17] and [CF17], on which we comment in more detail in the specific subsections.

Organization of the paper: In Section 2 the basic definitions and Lyons-Victoir extension theorem are presented. Section 3 provides the constructions of Itô càdlàg rough paths.

Acknowledgment: D.J.P. gratefully acknowledges financial support of the Swiss National Foundation under Grant No. 200021_163014 and was affiliated to ETH Zürich when this project was commenced.

2. CàDLÀG ROUGH PATH AND LYONS-VICTOIR EXTENSION THEOREM

In this section we briefly recall the definitions of càdlàg rough path theory as very recently introduced in [FS17, CF17] and present the Lyons-Victoir extension theorem in the càdlàg setting, see Proposition 2.4.

Let \( D([0,T]; E) \) be the space of càdlàg (right-continuous with left-limits) paths from \([0,T]\) into a metric space \((E,d)\). A partition \( \mathcal{P} \) of the interval \([0,T]\) is a set of essentially disjoint intervals covering \([0,T]\), i.e. \( \mathcal{P} = \{[t_i, t_{i+1}] : 0 = t_0 < t_1 < \cdots < t_n = T, n \in \mathbb{N} \} \). A path \( X \in D([0,T]; E) \) is of finite \( p \)-variation for \( p \in (0, \infty) \) if

\[
\|X\|_{p\text{-var}} := \left( \sup_{\mathcal{P}} \sum_{[s,t] \in \mathcal{P}} d(X_s, X_t)^p \right)^{\frac{1}{p}} < \infty,
\]

where the supremum is taken over all partitions \( \mathcal{P} \) of the interval \([0,T]\) and the sum denotes the summation over all intervals \([s,t] \in \mathcal{P} \). The space of all càdlàg paths of finite \( p \)-variation is denoted by \( D^{p\text{-var}}([0,T]; E) \). For a two-parameter function \( X : \Delta_T \to \mathbb{R}^{d \times d} \) we define

\[
\|X\|_{p/2\text{-var}} := \left( \sup_{\mathcal{P}} \sum_{[s,t] \in \mathcal{P}} |X_{s,t}|^2 \right)^{\frac{p}{2}}, \quad p \in (0, \infty),
\]

where \( \Delta_T := \{(s,t) \in [0,T] : s \leq t\} \) and \( d \in \mathbb{N} \). Furthermore, we use the shortcut \( X_{s,t} := X_t - X_s \) for \( X \in D([0,T]; \mathbb{R}^d) \).

For \( p \in [2,3) \) the fundamental definition of a càdlàg rough path was introduced in [FS17, Definition 12] and reads as follows.
Definition 2.1. For \( p \in [2, 3) \), a pair \( X = (X, \pi) \) is called càdlàg rough path over \( \mathbb{R}^d \) (in symbols \( X \in \mathcal{W}^p([0, T]; \mathbb{R}^d) \)) if \( X: [0, T] \to \mathbb{R}^d \) and \( \pi: \Delta_T \to \mathbb{R}^{d \times d} \) satisfy:

1. Chen’s relation holds: \( X_{s,t} - X_{s,u} - X_{u,t} = X_{s,u} \otimes X_{u,t} \) for \( 0 \leq s \leq u \leq t \leq T \).
2. The map \([0, T] \ni t \mapsto X_{0,t} + \pi_{0,t} \in \mathbb{R}^d \times \mathbb{R}^{d \times d} \) is càdlàg.
3. \( X = (X, \pi) \) is of finite \( p \)-variation in the rough path sense, i.e. \( \|X\|_{p\text{-var}} + \|\pi\|_{p/2\text{-var}} < \infty \).

An important subclass of rough paths are the so-called weakly geometric rough paths: For \( N \geq 1 \) let \( G^N(\mathbb{R}^d) \subset \mathcal{T}^N(\mathbb{R}^d) := \sum_{k=0}^N (\mathbb{R}^d)^{\otimes k} \) be the step-\( N \) free nilpotent Lie group over \( \mathbb{R}^d \), embedded into the truncated tensor algebra \((\mathcal{T}^N(\mathbb{R}^d), +, \otimes)\) which is equipped with the Carnot-Carathéodory norm \( \|\cdot\| \) and the induced (left-invariant) metric \( d \). For more details we refer to [FY10b, Chapter 7]. A rough path \( X = (X, \pi) \in \mathcal{W}^p([0, T]; \mathbb{R}^d) \) for \( p \in [2, 3) \) is said to be a weakly geometric rough path if \( 1 + X_{0,t} + \pi_{0,t} \) takes values in \( G^2(\mathbb{R}^d) \).

Note, while the constructions of rough paths carried out in Section 3 lead in general to non-geometric rough paths, it is always possible to recover a weakly geometric one.

Remark 2.2. If \( N = 2 \) and \( p \in [2, 3) \), one can easily verify that if \( X = (X, \pi) \) is a càdlàg rough path, then there exists a càdlàg function \( F: [0, T] \to \mathbb{R}^{d \times d} \) of finite \( p/2 \)-variation such that \( 1 + X_{0,t} + \pi_{0,t} + F_t \) is a weakly geometric càdlàg rough path over \( \mathbb{R}^d \).

The notion of weakly geometric rough paths naturally extends to arbitrary low regularity \( p \in [1, \infty) \), see [CF17, Definition 2.2].

Definition 2.3. Let \( 1 \leq p < N + 1 \) and \( N \in \mathbb{N} \). Any \( X \in \mathcal{D}^{p\text{-var}}([0, T]; G^N(\mathbb{R}^d)) \) is called weakly geometric càdlàg rough path over \( \mathbb{R}^d \).

The next proposition is the Lyons-Victoir extension theorem (see in particular [LV07, Corollary 19]) in the context of càdlàg rough paths.

Proposition 2.4. Let \( p \in [1, \infty) \setminus \{2, 3, \ldots\} \) and \( N \in \mathbb{N} \) be such that \( p < N + 1 \). For every càdlàg path \( X: [0, T] \to \mathbb{R}^d \) of finite \( p \)-variation there exists a (in general non-unique) weakly geometric càdlàg rough path \( \tilde{X} \in \mathcal{D}^{p\text{-var}}([0, T]; G^N(\mathbb{R}^d)) \) such that \( \pi_1(\tilde{X}) = X \), where \( \pi_1: G^N(\mathbb{R}^d) \to \mathbb{R}^d \) is the canonical projection onto the first component.

Proof. Let \( X \) be a càdlàg \( \mathbb{R}^d \)-valued path of finite \( p \)-variation. By a slight modification of [CG98, Theorem 3.1], there exists a non-decreasing function \( \varphi: [0, T] \to [0, \varphi(T)] \) with \( \varphi(T) < \infty \) and a \( 1/p \)-Hölder continuous function \( g: [0, \varphi(T)] \to \mathbb{R}^d \) such that \( X = g \circ \varphi \). Since \( \varphi(t) \) is non-decreasing, the set \( N \) of discontinuity points of \( \varphi \) is at most countable. Let us define a function \( \phi \) such that \( \phi(t) = \varphi(t) \) for \( t \in ([0, T] \setminus N) \cup \{T\} \) and \( \phi(t) = \varphi(t+) := \lim_{s \uparrow t, s \in N} \varphi(s) \) if \( t \in N \). It is easy to verify that \( \phi \) is non-decreasing, càdlàg and \( \phi(T) = \varphi(T) \). Moreover, since \( X \) is right-continuous and \( g \) is continuous, we have \( g \circ \phi = X \).

By [LV07, Corollary 19] there exists a weakly geometric \( 1/p \)-Hölder continuous rough path \( \tilde{g} \) such that \( \pi_1(\tilde{g}) = g \). Now we define \( \tilde{X} := \tilde{g} \circ \phi \). Since \( \phi \) is càdlàg and \( \tilde{g} \) is continuous, \( \tilde{X} \) is also càdlàg. Furthermore, using [CG98, Theorem 3.1] again we conclude that \( \tilde{X} \) has finite \( p \)-variation and thus \( \tilde{X} \in \mathcal{D}^{p\text{-var}}([0, T]; G^p(\mathbb{R}^d)) \) with \( [p] := \max\{ n \in \mathbb{N} : n \leq p \} \). Finally, it is obvious that \( \pi_1(\tilde{X}) = \pi_1(\tilde{g}) \circ \phi = g \circ \phi = X \) and the extension of \( \tilde{X} \) to a weakly geometric càdlàg rough path \( \tilde{X} \in \mathcal{D}^{p\text{-var}}([0, T]; G^N(\mathbb{R}^d)) \) for every \( N \in \mathbb{N} \) with \( p < N + 1 \) is possible due to [FS17, Theorem 20].
denotes the left-continuous version of $X$, i.e. $X_-(t) := X_{t-} := \lim_{s \rightarrow t, s \leq t} X_s$ for $t \in (0, T]$ and $X_-(0) := X_{0-} := X_0$. We write $A_\theta \lesssim B_\theta$ meaning that $A_\theta \leq C B_\theta$ for some constant $C > 0$ independent of a generic parameter $\theta$ and $A_\theta \lesssim_\theta B_\theta$ meaning that $A_\theta \leq C(\theta) B_\theta$ for some constant $C(\theta) > 0$ depending on $\theta$. The indicator function of a set $A \subset \mathbb{R}$ or $A \subset D([0, T]; \mathbb{R}^d)$ is denote by $1_A$ and $x \wedge y := \min\{x, y\}$ for $x, y \in \mathbb{R}$.

3. Construction of Itô rough paths

In order to lift stochastic processes using Itô type integration, we first prove a deterministic criterion to check the $p/2$-variation of the corresponding lift. The construction of random rough paths above (stochastic) processes is presented in the following subsections.

For $X \in D([0, T]; \mathbb{R}^d)$ or for (later) any càdlàg process $X$, we define the dyadic (stopping) times $(\tau^n_k)_{n, k \in \mathbb{N}}$ by

$$\tau^n_0 := 0 \quad \text{and} \quad \tau^n_{k+1} := \inf\{t \geq \tau^n_k : |X_t - X_{\tau^n_k}| \geq 2^{-n}\}.$$

Furthermore, for $t \in [0, T]$ and $n \in \mathbb{N}$ we introduce the dyadic approximation

$$X^n_t := \sum_{k=0}^{\infty} X_{\tau^n_k} 1_{(\tau^n_k, \tau^n_{k+1})}(t) \quad \text{and} \quad \int_0^t X^n_s \odot dX_s := \sum_{k=0}^{\infty} X^n_{\tau^n_k} \odot X^n_{\tau^n_k \land t, \tau^n_{k+1} \land t}.$$

Note that the integral $\int_0^t X^n_s \odot dX_s$ is well-defined and $\|X^n - X_\cdot\|_\infty \leq 2^{-n}$ for every $n \in \mathbb{N}$.

**Theorem 3.1.** Suppose that $X \in D^{p, \text{var}}([0, T]; \mathbb{R}^d)$ for every $p > 2$ and there exist a function $\int_0^t X_\cdot \odot dX \in D([0, T]; \mathbb{R}^{d \times d})$ and a dense subset $D_T$ containing $T$ in $[0, T]$ satisfying that for every $t \in D_T$ and for every $\varepsilon \in (0, 1)$, there exist an $N = N(t, \varepsilon) \in \mathbb{N}$ and a constant $c = c(p, \varepsilon)$ such that

$$\left| \int_0^t X^n_s \odot dX_s - \int_0^t X_\cdot \odot dX \right| \leq c2^{-n(1-\varepsilon)} \quad \text{for all } n \geq N.$$

Setting for $(s, t) \in \Delta_T$

$$X_{s,t} := \int_s^t X_{\cdot} \odot dX_{\cdot} - X_s \otimes X_{s,t} := \int_s^t X_{\cdot} \odot dX_{\cdot} - \int_0^s X_{\cdot} \odot dX_{\cdot} - X_s \otimes X_{s,t},$$

then $(X, X) \in \mathbb{W}^p([0, T]; \mathbb{R}^d)$ for every $p \in (2, 3)$.

To prove Theorem 3.1 we adapted some arguments used in the proof of [PP16, Theorem 4.12], in which the existence of rough paths above typical continuous price paths is shown, cf. Subsection 3.3 below. As a preliminary step, we need a version of Young’s maximal inequality (cf. [You36] or [LCL07, Theorem 1.16]) specific to the integral $\int X^n \odot dX$.

Recall that a function $c: \Delta_T \rightarrow [0, \infty)$ is called right-continuous super-additive if

$$c(s, u) + c(u, t) \leq c(s, t) \quad \text{for } \ 0 \leq s \leq u \leq t \leq T,$$

and $c(s, t)$ is right-continuous in $t$ for fixed $s$. Note that $X \in D^{p, \text{var}}([0, T]; \mathbb{R}^d)$ if and only if there exists a right-continuous super-additive function $c$ s.t. $|X_{s,t}|^p \leq c(s, t)$ for all $(s, t) \in \Delta_T$.

**Lemma 3.2.** Let $X \in D^{p, \text{var}}([0, T]; \mathbb{R}^d)$ for every $p > 2$. Then it holds

$$\left| \int_0^t X^n_s \odot dX_s - \int_0^t X^n_s \odot dX_s - X_s \otimes X_{s,t} \right| \leq \max\{2^{-n} c(s, t)^{1/q}, 2^{n(q-2)} c(s, t) + c(s, t)^{2/q}\},$$

for $q \in (2, 3)$ and every super-additive function $c: \Delta_T \rightarrow [0, \infty)$ (which may depend on $q$) such that $|X_{s,t}|^q \leq c(s, t)$ for all $(s, t) \in \Delta_T$. 


The proof follows the classical arguments used to derive Young’s maximal inequality.

Proof. Let \( X \in D^{p\text{-var}}([0,T];\mathbb{R}^d) \) and let \( X^n \) be its dyadic approximation as defined in (3.1).

1. If there exists no \( k \) such that \( \tau^n_k \in [s,t] \), then
\[
\left| \int_0^t X^n_r \otimes dX_r - \int_0^s X^n_s \otimes dX_r - X_s \otimes X_{s,t} \right| \leq 2^{-n} c(s,t)^{1/q}
\]
due to the estimate \( |X_{s,t}| \leq c(s,t)^{1/q} \).

2. If there exists a \( k \) such that \( \tau^n_k \in [s,t] \), we may assume that \( s = \tau^n_{k_0} \) for some \( k_0 \). Otherwise, we just add \( c(s,t)^2/q \) to the right-hand side. Let \( \tau^n_{k_0}, \ldots, \tau^n_{k_0+N-1} \) be those \( (\tau^n_k)_k \) which are in \( [s,t) \). W.l.o.g. we may further suppose that \( N \geq 2 \). Abusing notation, we write \( \tau^n_{k_0+N} = t \). The idea is now to successively delete points \( (\tau^n_{k_0+\ell})_\ell \) from \( \tau^n_{k_0}, \ldots, \tau^n_{k_0+N-1} \). Due to the super-additivity of \( c \), there exist \( \ell \in \{1,\ldots,N-1\} \) such that
\[
c(\tau^n_{k_0+\ell-1}, \tau^n_{k_0+\ell+1}) \leq \frac{2}{N-1} c(s,t)
\]
and thus
\[
|X^n_{\tau^n_{k_0+\ell-1}} \otimes X^n_{\tau^n_{k_0+\ell-1} \tau^n_{k_0+\ell}} + X^n_{\tau^n_{k_0+\ell}} \otimes X^n_{\tau^n_{k_0+\ell+1} \tau^n_{k_0+\ell+1}} - X^n_{\tau^n_{k_0+\ell-1}} \otimes X^n_{\tau^n_{k_0+\ell+1} \tau^n_{k_0+\ell+1}}| = |X^n_{\tau^n_{k_0+\ell-1} \tau^n_{k_0+\ell}} \otimes X^n_{\tau^n_{k_0+\ell+1} \tau^n_{k_0+\ell+1}}| \leq c(\tau^n_{k_0+\ell-1}, \tau^n_{k_0+\ell+1})^{2/q} \leq \left( \frac{2}{N-1} c(s,t) \right)^{2/q}.
\]
Successively deleting in this manner all the points except \( \tau^n_{k_0} = s \) and \( \tau^n_{k_0+N} = t \) from the partition generated by \( \tau^n_{k_0}, \ldots, \tau^n_{k_0+N} \) leads to the estimate
\[
\left| \int_0^t X^n_r \otimes dX_r - \int_0^s X^n_s \otimes dX_r - X_s \otimes X_{s,t} \right| \leq \sum_{k=2}^N \left( \frac{2}{k-1} c(s,t) \right)^{2/q} \lesssim N^{1-2/q} c(s,t)^{2/q} \lesssim \left( \# \{ k : \tau^n_k \in [s,t] \} \right)^{1-2/q} c(s,t)^{2/q} + c(s,t)^{2/q}
\]
since \( N \leq \# \{ k : \tau^n_k \in [s,t] \} \).

Hence, 1. and 2., in combination with \( \# \{ k : \tau^n_k \in [s,t] \} \lesssim 2^{nq} c(s,t) \), imply the assertion. \( \square \)

With the auxiliary Lemma 3.2 at hand we come to the proof of Theorem 3.1.

Proof of Theorem 3.1. It is straightforward to check that \((X,X)\) satisfies condition (1) and (2) of Definition 2.1 and \( \|X\|_{p\text{-var}} < \infty \). Therefore, it remains to show the \( p/2 \)-variation (in the sense of (2.1)) of \( X \) for every \( p > 2 \).

Let \( c \) be a right-continuous super-additive function with \( |X_{s,t}|^{q} \leq c(s,t) \). Then for all \((s,t) \in \Delta_T \cap D_{\ell}^{t} \), using (3.2) and Lemma 3.2 for every \( \varepsilon > 0 \) and \( q \in (2,3) \) we get a constant \( c = c(p,q,\varepsilon) \) such that
\[
|X_{s,t}|^{q} \leq c \left( 2^{-n(1-\varepsilon)} + \left| \int_0^t X^n_r \otimes dX_r - \int_0^s X^n_s \otimes dX_r - X_s \otimes X_{s,t} \right| \right) \leq c \left( 2^{-n(1-\varepsilon)} + \max \{ 2^{-n} c(s,t)^{1/q}, 2^{-n(2-q)} c(s,t) + c(s,t)^{2/q} \} \right),
\]
for all \( n \geq N \), where \( N \in \mathbb{N} \) may depend on \( s, t \) and \( \varepsilon \).
In the case that $c(s,t) \leq 1$, we set $\alpha := p/2$ for $p \in (2,3)$ and choose $n \geq N$ such that $2^{-n} \leq c(s,t)^{1/(\alpha(1-\varepsilon))}$. Taking this $n$ in (3.3), we obtain

$$|X_{s,t}|^\alpha \leq c\left(c(s,t) + \max\left\{c(s,t)^{1/(\alpha-1)}c(s,t)^{\alpha/q}, c(s,t)^{(2-q)/(1-\varepsilon)+\alpha}+c(s,t)^{2\alpha/q}\right\}\right)$$

$$= c\left(c(s,t) + \max\left\{c(s,t)^{\frac{\alpha+\alpha(1-\varepsilon)}{q(1-\varepsilon)}}, c(s,t)^{\frac{2-q+\alpha(1-\varepsilon)}{1-\varepsilon}}+c(s,t)^{2\alpha/q}\right\}\right)$$

for some constant $c = c(\alpha, q, \varepsilon)$. Now we would like all the exponents in the maximum on the right-hand side to be larger or equal to 1. For the first term, this is satisfied as long as $\varepsilon < 1$. For the third term, we need $\alpha \geq q/2$. For the second term, we need $\alpha \geq (q-1-\varepsilon)/(1-\varepsilon)$. Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, it suffices if $\alpha > q - 1$. This means, choosing a $q_0 > 2$ close to 2 enough such that $p/2 = \alpha > \max\{q_0/2, q_0 - 1\}$, we obtain that $|X_{s,t}|^\alpha \leq c \cdot c(s,t)$ for some constant $c = c(p, q_0)$.

For the remaining case $c(s,t) > 1$, we simply estimate

$$|X_{s,t}|^\alpha \leq c\left(\int_0^\infty X_{r-} \otimes dX_r\right)^{\alpha/\alpha} + \|X\|^\alpha \leq c\left(\int_0^\infty X_{r-} \otimes dX_r\right)^{\alpha/\alpha} + \|X\|^\alpha c(s,t).$$

Therefore, $|X_{s,t}|^\alpha \leq c \cdot c(s,t)$ for some constant $c = c(p)$ and for every $(s,t) \in \Delta_T \cap D^2_T$. Moreover, for an arbitrary $(s,t) \in \Delta_T$, picking any sequences $(s_k)_{k \in \mathbb{N}}$ and $(t_k)_{k \in \mathbb{N}}$ in $D_T$ such that $s_k \downarrow s$ and $t_k \downarrow t$ as $k \to \infty$, we have

$$|X_{s,t}|^\alpha = \lim_{k \to \infty} |X_{s_k,t_k}|^\alpha \leq c(p) \limsup_{k \to \infty} c(s_k, t_k) \leq c(p) \lim_{k \to \infty} c(s, t_k) = c(p) c(s, t),$$

since $c(s,t)$ is right-continuous and super-additive. This ensures that $\|X\|_{p \text{-var}} < \infty$. \hfill $\square$

**Remark 3.3.** All arguments in the proofs of Theorem 3.1 and of Lemma 3.2 extend immediately from $\mathbb{R}^d$ to (infinite dimensional) Banach spaces. However, while the theory of continuous rough paths works for Banach spaces (cf. [Ly98], [LCL07]), the current results about càdlàg rough paths are developed in finite dimensional settings (cf. [FS17], [CF17]). For this reason we also focus only on $\mathbb{R}^d$-valued paths and stochastic processes.

### 3.1. Semimartingales.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t \in [0,T]}$ satisfying the usual conditions. For a $\mathbb{R}^d$-valued semimartingale $X$ we consider

$$X_{s,t} := \int_s^t (X_{r-} - X_s) \otimes dX_r = \int_0^t X_{r-} \otimes dX_r - \int_0^s X_{r-} \otimes dX_r - X_s \otimes X_{s,t}, \quad (s,t) \in \Delta_T,$$

where the integration $\int X_{r-} \otimes dX_r$ is defined as Itô integral. We refer to [Pro05] and [JS03] for more details on stochastic integration.

**Proposition 3.4.** Let $X$ be a $\mathbb{R}^d$-valued semimartingale. If $X$ is defined as in (3.4) via Itô integration, then $(X, X) \in \mathcal{W}^p([0,T]; \mathbb{R}^d)$ for every $p \in (2,3)$ $\mathbb{P}$-almost surely.

**Proof.** First note that every semimartingale possesses càdlàg sample paths of finite $p$-variation for any $p > 2$ (see e.g. [Pro05], Chapter II.1) and [Lep76] and $\int X_{r-} \otimes dX_r$ has càdlàg sample paths (see e.g. [JS03], Theorem I.4.31]). Therefore, in order to deduce Proposition 3.3 from Theorem 3.1 it is sufficient to verify that the condition (3.2) holds $\mathbb{P}$-almost surely for $\int X_{r-} \otimes dX_r$ and its dyadic approximation $\int X^n \otimes dX_r$ defined via (3.1).
EXAMPLES OF İTO CÁDLÁG ROUGH PATHS

1. Let us assume that \( X = M \) is a square integrable martingale and denoted by \( M^n \) its approximation defined as in \([3.1]\). By Burkholder-Davis-Gundy inequality we observe

\[
(3.5) \quad C(M,n) := \mathbb{E} \left[ \left\| \int_0^\infty M^n \otimes dM - \int_0^\infty M_\cdot \otimes dM \right\|_\infty^2 \right] \lesssim 2^{-2n}, \quad n \in \mathbb{N},
\]

where the constant depends on the quadratic variation of \( M \). Combining Chebyshev’s inequality with \((3.5)\) we get

\[
\mathbb{P} \left( \left\| \int_0^\infty M^n \otimes dM - \int_0^\infty M_\cdot \otimes dM \right\|_\infty \geq 2^{-n(1-\varepsilon)} \right) \leq 2^{2n(1-\varepsilon)} C(M,n) \lesssim 2^{-2n\varepsilon}.
\]

Since the right-hand side is summable in \( n \), the Borel-Cantelli lemma gives

\[
\left\| \int_0^\infty M^n \otimes dM - \int_0^\infty M_\cdot \otimes dM \right\|_\infty \lesssim_{\omega,\varepsilon} 2^{-n(1-\varepsilon)} \quad \mathbb{P}\text{-a.s.}
\]

2. Let \( X = M \) be a locally square integrable martingale. Let \( (\sigma_k)_{k \in \mathbb{N}} \) be a localizing sequence of stopping times for \( M \) such that \( \sigma_k \leq \sigma_{k+1} \), \( \lim_{k \to \infty} \mathbb{P}(\sigma_k = T) = 1 \), and for every \( k \), the stopped process \( M^{\sigma_k} \) is a square integrable martingale. Thanks to 1. applied to every \( M^{\sigma_k} \), for every \( k \) there exists a \( \Omega_k \subset \Omega \) with \( \mathbb{P}(\Omega_k) = 1 \) such that for all \( \omega \in \Omega_k \), it holds that

\[
\left\| \int_0^{\sigma_k} M^n \otimes dM - \int_0^{\sigma_k} M_\cdot \otimes dM \right\|_\infty \lesssim_{\omega,k} 2^{-n(1-\varepsilon)}
\]

for any \( n \). It follows immediately that \( (3.2) \) holds for any \( \omega \in \bigcup_{k \in \mathbb{N}} (\{ \sigma_k = T \} \cap \Omega_k) \) and it holds that \( \mathbb{P}(\bigcup_{k \in \mathbb{N}} (\{ \sigma_k = T \} \cap \Omega_k)) = 1 \).

3. By [Pro65] Theorem III.29, every semimartingale \( X \) can be decomposed as \( X = X_0 + M + A \), where \( X_0 \in \mathbb{R}^d \), \( M \) is a locally square integrable martingale and \( A \) has finite variation. By 2. we obtain that \( \left\| \int_0^T X^n \otimes dM - \int_0^T X_\cdot \otimes dM \right\|_\infty \lesssim_{\omega,k} 2^{-n(1-\varepsilon)} \mathbb{P}\text{-a.s.} \); on the other hand, since \( \| X^n - X_\cdot \|_\infty \leq 2^{-n} \), we also have \( \left\| \int_0^T X^n \otimes dM - \int_0^T X_\cdot \otimes dM \right\|_\infty \lesssim_{\omega,k} 2^{-n} \). \( \square \)

Remark 3.5. \(^1\) Very recently, Chevyrev and Friz proved that every semimartingale can be lifted via the “Marcus lift” to a weakly geometric càdlàg rough path based on a new enhanced Burkholder-Davis-Gundy inequality, see [CF17] Section 4). Their result allows for deducing the existence of Itô rough paths due to [FS17] Proposition 16. However, let us emphasize that our approach directly provides the existence of an Itô rough path only relying on classical Itô integration and fairly elementary analysis (cf. Theorem 3.1). Moreover, it is independent of the results from [CF17] [FS17].

Two natural generalizations of semimartingales are semimartingales perturbed by paths of finite \( q \)-variation for \( q \in [1,2) \) and Dirichlet processes. While these stochastic processes are beyond the scope of classical Itô integration, one can still construct corresponding random rough paths as limit of approximating Riemann sums.

For \( Y \in D^{p,\text{var}}([0,T];\mathbb{R}^d) \) with \( q \in [1,2) \), the Young integral

\[
\int_0^\infty Y_{r^-} \otimes dY_r := \lim_{n \to \infty} \sum_{\{a_i\} \in \mathcal{P}^n} Y_{a^-} \otimes Y_{a^+ - a^-}.
\]

exists along suitable sequences of partition \( (\mathcal{P}^n)_{n \in \mathbb{N}} \) and belongs to \( D^{p,\text{var}}([0,T];\mathbb{R}^{d \times d}) \), see for instance [You36] or [FS17] Proposition 14]. In this case the Young integral can also

\(^1\)After completion of the present work, it was pointed out in [FZ17] that also the Itô lift of semimartingales can be constructed using an enhanced version of Burkholder-Davis-Gundy inequality.
obtained via the dyadic approximation \((Y^n)\) as defined in \([3.1]\). Indeed, using the Young-Loeve inequality (see e.g. [FS17] Theorem 2)) and a standard interpolation argument, one gets

\[
\left\| \int_0^t Y_r^\sigma \otimes dY_r - \int_0^t Y_r^- \otimes dY_r \right\|_\infty \lesssim \|Y^n - Y^-\|_{q'\text{-var}} \|Y\|_{q'\text{-var}}.
\]

\[
\lesssim \|Y^n - Y^-\|_{q/q'} \|Y - Y^n\|_\infty \lesssim \|Y\|_{q'\text{-var}} \lesssim \|Y\|_{q'/q'-1} 2^{-n(1-q/q')}
\]

for \(1 \leq q < q' < 2\) and thus \(\lim_{n \to \infty} \left\| \int_0^t Y_r^\sigma \otimes dY_r - \int_0^t Y_r^- \otimes dY_r \right\|_\infty = 0\).

As a consequence of Proposition 3.4 and the previous discussion, it follows that semimartingale perturbed by paths of finite \(q\)-variation admits a rough path lift, cf. [FH14, Exercise 2.12]. Here \(C^{q,2\text{-var}}(0,T;\mathbb{R}^d)\) denotes the closure of smooth paths on \([0,T]\) w.r.t. \(|\cdot|_{2\text{-var}}\).

### 3.2. Gaussian processes

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with filtration \((\mathcal{F}_t)_{t \in [0,T]}\) satisfying the usual conditions and let \(X = (X^1, \ldots, X^d) : \Omega \times [0,T] \to \mathbb{R}^d\) be a \(d\)-dimensional Gaussian process. A natural candidate for the corresponding \(X = (X^{i,j})_{i,j=1,\ldots,d}\) is

\[
(3.6) \quad X^{i,j}_{s,t} := \int_0^t X^i_{r-} dX^j_r - \int_0^s X^i_{r-} dX^j_r - X^i_s X^j_s, \quad (s, t) \in \Delta_T,
\]

where \(i \neq j\) and where the integral is given as an \(L^2\)-limit of left-point Riemann-Stieltjes approximations. For more details on Gaussian processes in the context of rough path theory we refer to [FV10b] Chapter 15.

**Proposition 3.7.** Let \((X_t)_{t \in [0,T]}\) be a \(d\)-dimensional separable centered Gaussian process with independent components and c\(\acute{a}d\)l\(\acute{a}\)g sample paths. If for every \(q > 1\)

\[
(3.7) \quad \sup_{p,p'} \sum_{[s,t] \in \mathcal{P} \setminus [u,v] \in \mathcal{P}'} |\mathbb{E}[X_{s,t} \otimes X_{u,v}]|^q < \infty,
\]

then \((X, \mathcal{X}) \in \mathcal{W}^p([0,T];\mathbb{R}^d)\) for every \(p \in (2,3)\) \(\mathbb{P}\)-almost surely, where \(\mathcal{X}\) is defined as in (3.6) and \(X^{i,j}\) exists in the sense of an \(L^2\)-limit of Riemann-Stieltjes approximations for \(i \neq j\).

**Proof.** Proceeding as in [FS17] Section 10.3], the sample paths of \(X\) have finite \(p\)-variation for any \(p > 2\) due to (3.7) and there exists a centered Gaussian process \(\tilde{X}\) with continuous sample paths such that \(\tilde{X} \circ F = X\), where \(F(t) := \sup_p \sum_{[u,v] \in \mathcal{P}} |X^k_u - X^k_v|^2\) for every \(k = 1, \ldots, d\).

By [FV10a] Theorem 35 (iv)] the integral \(\int_0^t \tilde{X}^i_r d\tilde{X}^j_r\) exists as the \(L^2\)-limit of Riemann-Stieltjes approximation and has continuous sample paths. Furthermore, using Young-Towghi’s
maximal inequality (see [FV11, Theorem 3]) it can be verified that
\[
(3.8) \quad \int_0^t \tilde{X}_i^j \, d\tilde{X}_j^i \circ O(t) = \lim_{|P| \to 0} \sum_{[u,v] \in \mathcal{P}} X_u^i(X_v^{j, -} - X_u^{j, \wedge}) = \lim_{|P| \to 0} \sum_{[u,v] \in \mathcal{P}} X_u^{j, -}(X_v^{j, \wedge} - X_u^{j, \wedge}),
\]
for \( i, j = 1, \ldots, d \) with \( i \neq j \), where the limits are taken in \( L^2 \) and in Refinement Riemann-Stieltjes sense (cf. [FS17, Definition 1]). We denote by \( \int_0^t X_{-j}^i \, dX_j^i \) the integral from (3.8), which has càdlàg sample paths.

It remains check condition (3.2) for \( \int_0^t X_{-j}^i \, dX_j^i \), which then implies the proposition by Theorem 3.1. With an abuse of notation, we write now \( X \) for \( X^i \) and \( \tilde{X} \) for \( X^j \). Let \( X^n \) be given as in (3.1) such that \( \|X^n - X\|_{\infty} \leq 2^{-n} \). We define for \( Y^n := X^n - X \) and for \( s, t \in [0, T] \) we set \( R^n(s, t) := E[Y_{0,s}^n Y_{0,u}^n] \) and \( \tilde{R}(s) := E[X_{0,s} X_{0,u}] \). Thanks to (3.7), \( \tilde{R} \) has finite \( q \)-variation for any \( q > 1 \). We claim that \( R^n \) has finite \( p \)-variation for any \( p > 2 \).

Indeed, for every rectangle \( [s, t] \times [u, v] \subset [0, T]^2 \), we have \( \|E[Y^n_{s,t} Y^n_{u,v}]\|^p \leq \|Y^n_{s,t}\|^p \|Y^n_{u,v}\|^p \). Using [BOW16, Proposition 1.7] and the definition of \( X^n \) we obtain that \( E[\|Y^n\|^p_{p\text{-var}}] \lesssim E[\|X\|^p_{p\text{-var}}] < \infty \). By Jensen’s inequality we deduce that
\[
E[\|Y^n\|^p_{p\text{-var}}] \geq \sup_{p \in \mathcal{P}} \sum_{[s,t] \in \mathcal{P}} E[|Y^p_{s,t}|^p] \geq \sup_{p \in \mathcal{P}} \sum_{[s,t] \in \mathcal{P}} (E[|Y^p_{s,t}|^2])^{p/2} = \sup_{p \in \mathcal{P}} \sum_{[s,t] \in \mathcal{P}} \|Y^p_{s,t}\|^p_{L^2},
\]
which means that \( Y^n \) has finite \( p \)-variation w.r.t. the \( L^2 \)-distance. Let \( \|\|Y^n\|\|_{p\text{-var}} \) denote the \( p \)-variation norm of \( Y^n \) in the \( L^2 \)-distance, then \( c_n(s, t) := \|\|Y^n\|\|_{p\text{-var};[s,t]} \) is super-additive and \( c_n(0, T) \lesssim E[\|X\|^p_{p\text{-var}}] \) for all \( n \). Hence, for any partitions \( \mathcal{P}, \mathcal{P}' \) of \( [0, T] \) and for
\[
R^n(s, t, u, v) := R^n(s, t) - R^n(s, u) - R^n(u, t) + R^n(u, v), \quad u, v, s, t \in [0, T],
\]
it holds that
\[
\sum_{[s,t] \in \mathcal{P}, [u,v] \in \mathcal{P}'} R^n(s, t, u, v) \leq \sum_{[s,t] \in \mathcal{P}, [u,v] \in \mathcal{P}'} c_n(s, t) c_n(u, v) \leq c_n(0, T)^2 \lesssim E[\|X\|^p_{p\text{-var}}].
\]
Now, for any \( p > 2 \), we can choose any \( q > 1 \) close to \( 1 \) enough such that \( 1/p + 1/q > 1 \). Since \( Y^n \) and \( X \) are independent, applying Young-Towghi’s maximal inequality to the discrete integrals \( E[(\sum_{i \in \mathcal{P}} Y^n_{t_i} X_{t_{i+1}})^2] \) and then sending \( |\mathcal{P}| \) to zero, by Fatou’s lemma we obtain that
\[
E \left[ \left( \int_0^t Y^n_{0,r} \, d\tilde{X}_r \right)^2 \right] \lesssim V_p(R^n) V_q(\tilde{R}), \quad t \in [0, T],
\]
where \( V_p \) denotes \( p \)-variation on \( [0, T]^2 \) in the sense of [FV11, Definition 1], given by
\[
V_p(R) := \sup_{\mathcal{P}, \mathcal{P}'} \left( \sum_{[s,t] \in \mathcal{P}, [u,v] \in \mathcal{P}'} R^n(s, t, u, v) \right)^{1/p},
\]
for a function \( R: [0, T]^2 \to \mathbb{R} \). By an interpolation argument we have for \( p' > p \),
\[
V_{p'}(R^n) \leq V_p(R^n)^{p/p'} \left( \sup_{s \neq t, u \neq v} \left| R^n(s, t, u, v) \right| \right)^{1-p/p'}.
\]
Hence, noting that $|R^n(s,t)| = |\mathbb{E}[Y^n_{s,t}Y^n_{s,n}]| \lesssim 2^{-2n}$ due to $\|Y^n\|_{\infty} \leq 2^{-n}$, the above inequality applied for $p$ and $q$ with $1/p' + 1/q > 1$ gives

$$\mathbb{E} \left[ \left( \int_0^t Y^n_{0,r}d\tilde{X}_r \right)^2 \right] \lesssim V_p(R^n)^{p/p'} V_q(\tilde{\rho}) 2^{-2n(1-p/p')}.$$ 

In particular, for a given $p > 2$ and $\varepsilon > 0$, we choose $p' = p/\varepsilon$, and a corresponding parameter $q$ close to 1 enough such that $1/p' + 1/q > 1$, we get

$$\mathbb{E} \left[ \left( \int_0^t Y^n_{0,r}d\tilde{X}_r \right)^2 \right] \lesssim \varepsilon 2^{-2n(1-\varepsilon)}.$$ 

Then by Chebyshev’s inequality, for each $n$ and each $t \in [0, T]$ we have (note that $Y^n_0 = 0$)

$$\mathbb{P} \left( \left| \int_0^t Y^n_r d\tilde{X}_r \right| \geq 2^{-n(1-2\varepsilon)} \right) \leq 2^{2n(1-2\varepsilon)} \mathbb{E} \left[ \left( \int_0^t Y^n_{0,r}d\tilde{X}_r \right)^2 \right] \lesssim 2^{-2n\varepsilon}.$$ 

Since the right-hand side of the above inequality is summable over $n \in \mathbb{N}$, by the Borel-Cantelli lemma, we conclude that for every $t \in [0, T]$ there exists a $\Omega_t \subset \Omega$ with $\mathbb{P}[\Omega_t] = 1$ such that for every $\omega \in \Omega_t$, when $n$ large enough ($n$ may depend on $\omega$ and $t$),

$$\left| \int_0^t (X^n - X_{r-}) d\tilde{X}_r \right| \leq 2^{-n(1-\varepsilon)}$$

holds. Let $D_T$ be any countable dense subset in $[0, T]$ containing $T$ and $\tilde{\Omega} := \bigcap_{t \in D_T} \Omega_t$. Therefore, condition $\mathbf{E2}$ is satisfied for every $\omega \in \tilde{\Omega}$, which finishes the proof. $\square$

**Remark 3.8.** The Gaussian rough path as constructed in Proposition 3.7 is in fact a weakly geometric càdlàg rough path, which coincides with the one given in \cite{FS17, Theorem 60}. However, while the proof of \cite[Theorem 60]{FS17} is entirely based on time-change arguments and on corresponding well-known results for continuous Gaussian rough paths, the above proof gives a direct verification of the required rough path regularity via Theorem 3.1.

### 3.3. Typical price paths.

In recent years, initiated by Vovk, a model-free, hedging-based approach to mathematical finance emerged that uses arbitrage considerations to investigate which sample path properties are satisfied by “typical price paths”, see for instance \cite{Vovk08, TKT09, PP16}. In particular, Vovk’s framework allows for setting up a model-free Itô integration, see \cite{PP16, LPP18, Vovk16}. Based on this integration, we show in the present subsection that “typical price paths” can be lifted to càdlàg rough paths.

Let $\Omega_+ := D([0, T]; \mathbb{R}^d_+)$ be the space of all non-negative càdlàg functions $\omega: [0, T] \rightarrow \mathbb{R}^d_+$. The space $\Omega_+$ can be interpreted as all possible price trajectories on a financial market. For each $t \in [0, T]$, $\mathcal{F}_t^\sigma$ is defined to be the smallest $\sigma$-algebra on $\Omega_+$ that makes all functions $\omega \mapsto \omega(s), s \in [0, t]$, measurable and $\mathcal{F}_t$ is defined to be the universal completion of $\mathcal{F}_t^\sigma$. Stopping times $\tau: \Omega_+ \rightarrow [0, T] \cup \{\infty\}$ w.r.t. the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ and the corresponding $\sigma$-algebras $\mathcal{F}_\tau$ are defined as usual. The coordinate process on $\Omega_+$ is denoted by $S = (S^1, \ldots, S^d)$, i.e. $S_t(\omega) := \omega(t)$ and $S_t^i(\omega) := \omega^i(t)$ for $\omega = (\omega^1, \ldots, \omega^d) \in \Omega_+, t \in [0, T]$ and $i = 1, \ldots, d$.

A process $H: \Omega_+ \times [0, T] \rightarrow \mathbb{R}^d$ is a simple (trading) strategy if there exist a sequence of stopping times $0 = \sigma_0 < \sigma_1 < \sigma_2 < \ldots$ such that for every $\omega \in \Omega_+$ there exist an $N(\omega) \in \mathbb{N}$ such that $\sigma_n(\omega) = \sigma_{n+1}(\omega)$ for all $n \geq N(\omega)$, and a sequence of $\mathcal{F}_{\sigma_n}$-measurable bounded
functions $h_n: \Omega_+ \to \mathbb{R}^d$, such that $H_t(\omega) = \sum_{n=0}^{\infty} h_n(\omega)1_{(\sigma_n(\omega), \sigma_{n+1}(\omega))}(t)$ for $t \in [0,T]$. Therefore, for a simple strategy $H$ the corresponding integral process

$$(H \cdot S)_t(\omega) := \sum_{n=0}^{\infty} h_n(\omega)S_{\sigma_n \land t, \sigma_{n+1} \land t}(\omega)$$

is well-defined for all $(t, \omega) \in [0,T] \times \Omega_+$. For $\lambda > 0$ we write $\mathcal{H}_\lambda$ for the set of all simple strategies $H$ such that $(H \cdot S)_t(\omega) \geq -\lambda$ for all $(t, \omega) \in [0,T] \times \Omega_+$.

**Definition 3.9.** Vovk’s outer measure $\overline{P}$ of a set $A \subset \Omega_+$ is defined as the minimal super-hedging price for $1_A$, that is

$$\overline{P}(A) := \inf \left\{ \lambda > 0 : \exists (H^n)_{n \in \mathbb{N}} \subset \mathcal{H}_\lambda \text{ s.t. } \forall \omega \in \Omega_+ : \liminf_{n \to \infty} (\lambda + (H^n \cdot S)_T(\omega)) \geq 1_A(\omega) \right\}.$$  
A set $A \subset \Omega_+$ is called a null set if it has outer measure zero. A property $(P)$ holds for typical price paths if the set $A$ where $(P)$ is violated is a null set.

Note that $\overline{P}$ is indeed an outer measure, which dominates all local martingale measures on the space $\Omega_+$, see [LPP18, Lemma 2.3 and Proposition 2.5]. For more details about Vovk’s outer measure we refer for example to [LPP18, Section 2].

**Proposition 3.10.** Typical price paths belonging to $\Omega_+$ can be enhanced to càdlàg rough paths $(S, A) \in \mathcal{W}(\mathbb{R}^d)$ for every $p > 2$ where

$$A_{s,t} := \int_0^s S_r \otimes dS_r - \int_0^t S_r \otimes dS_r - S_s \otimes S_{s,t}, \text{ (}s,t) \in \Delta_T,$$

and $\int S_r \otimes dS$ denotes the model-free Itô integral from [LPP18, Theorem 4.2].

**Proof.** It follows from [Vov11, Theorem 1] that typical price paths belonging to $\Omega_+$ are of finite $p$-variation for every $p > 2$. Hence, it remains to check condition (3.2) of Theorem 3.1 to prove the assertion.

Let $S^n$ be the dyadic approximation of $S$ as defined in (3.11) for $n \in \mathbb{N}$ and let us recall that [LPP18, Corollary 4.9] extends to the estimate

$$\overline{P}\left(\left\{ \left. \left\| \int_0^t (S^n - S_-) \otimes dS \right\|_\infty \geq a_n \right\} \cap \left\{ \|S_I\| \leq b \right\} \cap \left\{ \|S\| \leq b \right\} \right\} \leq 6(\sqrt{b} + 2 + 2b) \frac{c_n}{a_n},$$

where $c_n := \|S^n - S\|_\infty \lesssim 2^{-n}$, $\|S_I\| := \left(\sum_{i,j=1}^d [S^i, S^j]_T^2\right)^{1/2}$ and $[S^i, S^j]$ denotes the quadratic co-variation as defined in [LPP18, Corollary 3.11]. Due to the countable subadditivity of $\overline{P}$, it is enough to consider a fixed $b > 0$. Setting $a_n := 2^{-(1-\varepsilon)n}$ for $\varepsilon \in (0,1)$ and applying the Borel-Cantelli lemma for $\overline{P}$ (see [LPP18, Lemma A.1]), we get $\overline{P}(B_b) = 0$ with

$$B_b := \bigcap_{m \in \mathbb{N}} \bigcup_{n \geq m} A_{b,n}$$

and

$$A_{b,n} := \left\{ \left. \left\| \int_0^t (S^n - S_-) \otimes dS \right\|_\infty \geq a_n \right\} \cap \left\{ \|S_I\| \leq b \right\} \cap \left\{ \|S\| \leq b \right\}.$$

In particular, for typical price paths (belonging to $\Omega_+$) we have

$$\left\| \int_0^t (S^n - S_-) \otimes dS \right\|_\infty \lesssim \omega 2^{-(1-\varepsilon)n}$$

for all $n \in \mathbb{N}$ and thus typical price paths satisfy condition (3.2). \qed
Let us briefly comment on various aspects of Proposition 3.10.

Remark 3.11.

(1) Proposition 3.10 implies the (robust) existence of Itô càdlàg rough paths in the sense that the set of all non-negative càdlàg paths which do not possess an Itô rough path has measure zero with respect to all local martingale measures on \( \Omega_+ \). This justifies to take the existence of Itô rough paths above price paths as an underlying assumption in model-free financial mathematics.

(2) The non-existence of Itô càdlàg rough paths above non-negative price paths leads to an pathwise arbitrage of the first kind, cf. [LPP18, Proposition 2.6].

(3) In the case of continuous (price) paths the assertion of Proposition 3.10 was obtained in [PP16, Theorem 4.12].

(4) Proposition 3.10 can be generalized in a straightforward manner from \( \Omega_+ \) to the more general sample spaces considered in [LPP18].

References

[BOW16] Andreas Basse-O'Connor and Michel Weber, On the \( \Phi \)-variation of stochastic processes with exponential moments, Trans. London Math. Soc. 3 (2016), no. 1, 1–27.

[CF17] Ilya Chevyrev and Peter K. Friz, Canonical RDEs and general semimartingales as rough paths, Preprint arXiv:1704.08053 (2017).

[CG98] V. V. Chistyakov and O. E. Galkin, On maps of bounded \( p \)-variation with \( p > 1 \), Positivity 2 (1998), no. 1, 19–45.

[Che17] Ilya Chevyrev, Random walks and Lévy processes as rough paths, Probability Theory and Related Fields (2017).

[CMS03] F. Coquet, J. Mémin, and L. Slomiński, On Non-continuous Dirichlet Processes, Journal of Theoretical Probability 16 (2003), no. 1.

[FH14] Peter K. Friz and Martin Hairer, A course on rough paths, Universitext, Springer, Cham, 2014, With an introduction to regularity structures.

[FS17] Peter K. Friz and Atul Shekhar, General rough integration, Lévy rough paths and a Lévy–Khintchine-type formula, Ann. Probab. 45 (2017), no. 4, 2707–2765.

[LCL07] Terry J. Lyons, Michael Caruana, and Thierry Lévy, Differential equations driven by rough paths, Lecture Notes in Mathematics, vol. 1908, Springer, Berlin, 2007.

[Lép76] D. Lépingle, La variation d’ordre \( p \) des semi-martingales, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 36 (1976), no. 4, 295–316.

[LPP18] Rafał M. Łochowski, Nicolas Perkowski, and David J. Prümel, A superhedging approach to stochastic integration, Stochastic Processes and their Applications (2018).

[LV07] Terry Lyons and Nicolas Victoir, An extension theorem to rough paths, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007), no. 5, 835–847.

[Lyo98] Terry J. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoam. 14 (1998), no. 2, 215–310.

[PP16] Nicolas Perkowski and David J. Prümel, Pathwise stochastic integrals for model free finance, Bernoulli 22 (2016), no. 4, 2486–2520.

[Pro05] Philip E. Protter, Stochastic integration and differential equations, Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2005, Second edition. Version 2.1, Corrected third printing.

[Str88] C. Stricker, Variation conditionnelle des processus stochastiques, Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), no. 2, 295–305.
[TKT09] Kei Takeuchi, Masayuki Kumon, and Akimichi Takemura, A new formulation of asset trading games in continuous time with essential forcing of variation exponent, Bernoulli 15 (2009), no. 4, 1243–1258.

[Vov08] Vladimir Vovk, Continuous-time trading and the emergence of volatility, Electron. Commun. Probab. 13 (2008), 319–324.

[Vov11] , Rough paths in idealized financial markets, Lith. Math. J. 51 (2011), no. 2, 274–285.

[Vov16] V. Vovk, Purely pathwise probability-free Itô integral, Mat. Stud. 46 (2016), no. 1, 96–110.

[Wil01] David R. E. Williams, Path-wise solutions of stochastic differential equations driven by Lévy processes, Rev. Mat. Iberoamericana 17 (2001), no. 2, 295–329.

[You36] Laurence C. Young, An inequality of the Hölder type, connected with Stieltjes integration, Acta Math. 67 (1936), no. 1, 251–282.

Chong Liu, Eidgenössische Technische Hochschule Zürich, Switzerland
E-mail address: chong.liu@math.ethz.ch

David J. Prömel, University of Oxford, United Kingdom
E-mail address: proemel@maths.ox.ac.uk