1. Introduction

The purpose of this paper is to study Bitsadze–Samarskii type nonlocal problem for the time-fractional diffusion–wave equation with the Heisenberg sub-Laplacian $\Delta_{\mathbb{H}}$ in the space variables.

In [1], Bitsadze and Samarskii established the solvability of the new class of nonlocal problems for the elliptic equations, which relate the values of the solution on parts of the boundary with its values inside the domain. Such problems are called the Bitsadze–Samarskii problems. For the motivation of studying the Bitsadze–Samarskii type nonlocal problems, we refer to [2–8] and references therein.

Certain types of physical problems can be modelled by heat and wave equations with Bitsadze–Samarskii type initial conditions. The time multi-point heat and wave problems can arise from studying the atomic reactors [9,10] and of some inverse heat conduction problems for determining the unknown physical parameters [11]. Well-posedness and numerical simulations of time multi-point heat and wave problems were studied in [9,10,12–15].
The version of such equations on the Heisenberg group serves as a basic model for the analysis of the sub-elliptic diffusion and wave propagation models, providing new insights and techniques for the whole problem.

Thus we consider the fractional integro-differential diffusion–wave equation

\[ D_{+0,t}^{\alpha} u(t,x) - I_{+0,t}^{\beta} \Delta_{\mathbb{H}^n} u(t,x) = f(t,x), \quad t > 0, \ x \in \mathbb{H}^n, \]

\[ \text{Equation (1.1)} \]

where \( f(t,x) \) is a sufficiently smooth function, \( I_{+0,t}^{\beta} \) is the Riemann–Liouville fractional integral of order \( \beta > 0 \) [16]

\[ I_{+0,t}^{\beta} u(t,x) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s,x) \, ds, \]

and \( \partial_{+0,t}^{\alpha} \) is the Riemann–Liouville fractional derivative of order \( 0 < \alpha \leq 2 \) ([16]) defined as

\[ D_{+0,t}^{\alpha} u(t,x) = \partial_t^{[\alpha]+1} + 1^{[\alpha]-\alpha} I_{+0,t}^{1+[\alpha]-\alpha} u(t,x), \]

where \([\alpha]\) is the integer part of \( \alpha \).

When \( 1 < \alpha < 2, \beta = 0 \), Equation (1.1) is the time-fractional wave equation and when \( 0 < \alpha < 1, \beta = 0 \), Equation (1.1) is the time-fractional diffusion equation. When \( \alpha = 2, \beta = 0 \), it represents the classical wave equation, while if \( \alpha = 1, \beta = 0 \), it represents the classical diffusion equation.

Many mathematical formulations of physical phenomena contain integro-differential equations, these equations arise in many fields such as fluid dynamics, biological models and chemical kinetics. If \( \alpha = 1 \), Equation (1.1) describes the heat conduction with memory [17,18], and many authors studied the analogue problems [19–25].

1.1. Heisenberg group

Let \( \mathbb{H}^n \) be the Heisenberg group, that is, the space \( \mathbb{R}^{2n+1} \) endowed with the group law

\[ \xi \circ \xi' = \left( x + x', y + y', s + s' + 2 \sum_{i=1}^n (x_i y'_i - x'_i y_i) \right), \]

where \( \xi = (z,s) = (x,y,s) = (x_1, \ldots, x_n, y_1, \ldots, y_n, s), \ z = (x,s), \ x \in \mathbb{R}^n, \ y \in \mathbb{R}^n, \ s \in \mathbb{R}, \ n > 1; \xi' = (x',y',s') \in \mathbb{R}^{2n+1} \). This group multiplication endows \( \mathbb{H}^n \) with a structure of a nilpotent Lie group. A family of dilations is defined as

\[ \delta_{\tau} (x,y,s) = (\tau x, \tau y, \tau^2 s), \quad \tau > 0. \]

The homogeneous dimension with respect to these dilations is \( Q = 2n + 2 \). The left invariant vector fields on the Heisenberg group are

\[ X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial s}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial s}, \quad i = 1, 2, \ldots, n. \]

The horizontal gradient is

\[ \nabla_{\mathbb{H}^n} = (X_1, \ldots, X_n, Y_1, \ldots, Y_n). \]
Hence, the sub-Laplacian $\Delta_{H^n}$ is denoted by
\[\Delta_{H^n} = \sum_{i=1}^{n} (X_i^2 + Y_i^2) = \nabla_{H^n} \cdot \nabla_{H^n}.\]

The (Kaplan) distance function on $H^n$ is given by
\[\text{dist}(\xi, \xi') = \left\{ \left( (x - x')^2 + (y - y')^2 \right) \right\}^{1/4}, \quad \xi, \xi' \in H^n.\]
If $\xi' = 0$, then the distance function is
\[\text{dist}(\xi) = \left\{ \left( |x|^2 + |y|^2 \right)^2 + s^2 \right\}^{1/4} = \left\{ |z|^4 + s^4 \right\}^{1/4}, \quad |z| = \sqrt{|x|^2 + |y|^2}.\]

1.2. Group Fourier transform

We begin with a reminder of the definition of the group Fourier transform on the Heisenberg group (see many sources, but e.g. [26,27] for its use in similar contexts). For $f \in \mathcal{S}(H^n)$ the group Fourier transform is defined as
\[\hat{f}(\lambda) := \int_{H^n} f(x) \pi_\lambda(x)^* \, dx,\]
with the Schrödinger representations
\[\pi_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\]
for all $\lambda \in \mathbb{R}_+: = \mathbb{R} \setminus \{0\}$. The inverse group Fourier transform formula can be written as
\[f(x) = \int_{\lambda \in \mathbb{R}_+} \text{Tr} \left[ \hat{f}(\lambda) \pi_\lambda(x) \right] |\lambda|^n \, d\lambda,\]
where Tr is the trace operator. The Plancherel formula becomes
\[\|f\|^2_{L^2(H^n)} = \int_{\lambda \in \mathbb{R}_+} \|\hat{f}(\lambda)\|_{HS[L^2(\mathbb{R}^n)]}^2 |\lambda|^n \, d\lambda,\]
where $\| \cdot \|^2_{HS[L^2(\mathbb{R}^n)\]}$ is the Hilbert–Schmidt norm on $L^2(\mathbb{R}^n)$. For more details on Plancherel formula and the Hilbert Schmidt norm, we refer to [28, Chapter 6, Proposition 6.2.7].

2. Main results

In this paper, we will work in the functional space $C([0, T]; L^2(H^n))$ with the norm
\[\|u\|_{C([0, T]; L^2(H^n))} := \sup_{t \in [0, T]} \|u(t, \cdot)\|_{L^2(H^n)}\]
for all $u \in C([0, T]; L^2(H^n))$. 
Problem 2.1: Assume that \( f \in C([0, T]; L^2(\mathbb{H}^n)) \). In a domain \( \Omega = \{(t, x) : (0, T) \times \mathbb{H}^n\} \) consider Equation (1.1) with Bitsadze–Samariskii type time-nonlocal conditions

\[
I_{1+\alpha}^{-\alpha} u(0, x) + \sum_{i=1}^{m} \mu_i I_{1+\alpha}^{-\alpha} u(T_i, x) = 0, \quad [\alpha] D_{1+\alpha}^{-1} u(0, x) = 0, \quad x \in \mathbb{H}^n, \tag{2.1}
\]

where \( m \in \mathbb{N}, \mu_i \in \mathbb{R}, 0 < T_1 \leq T_2 \leq \cdots \leq T_m = T \).

We seek a solution \( I_{1+\alpha}^{-\alpha} u \in C([0, T]; L^2(\mathbb{H}^n)) \) of the problems (1.1) and (2.1) such that

\[
D_{1+\alpha}^{-1} u \in C([0, T]; L^2(\mathbb{H}^n)) \quad \text{and} \quad \Delta u \in C([0, T]; L^2(\mathbb{H}^n)).
\]

The condition (2.2) can be interpreted as a multi-point non-resonance condition. Note that a similar problem for the time-fractional multi-term diffusion–wave equation was investigated by the authors in [29].

Theorem 2.2: Let \( f \in C([0, T]; L^2(\mathbb{H}^n)), D_{1+\alpha}^{-1} f \in C([0, T]; L^2(\mathbb{H}^n)), \) and assume that the conditions

\[
\left| 1 + \sum_{i=1}^{m} \mu_i T_i^{\alpha-\alpha} - \alpha - 1 \right|^{-1} E_{\alpha+\beta, \alpha-1} \left( -|\lambda| v_i T_i^{\alpha+\beta} \right) \geq M > 0 \tag{2.2}
\]

hold for all \( l \in \mathbb{N}^n \) (where \( M \) is a constant), where

\[
v_i = \sum_{j=1}^{n} (2l_j + 1), \quad l = (l_1, \ldots, l_n) \in \mathbb{N}^n.
\]

Then there exists a unique solution of Problem 2.1, and it can be written as

\[
u(t, x) = \int_{\mathbb{R}_+} \text{Tr}[\mathcal{K}(t, \lambda) \pi_\lambda(x)] d\lambda, \tag{2.3}
\]

where

\[
\mathcal{K}(t, \lambda)_{l,k} = \hat{\mathcal{K}}(t, \lambda)_{l,k} - \sum_{i=1}^{m} \mu_i \hat{\mathcal{K}}(T_i, \lambda)_{l,k} T_i^{\alpha-\alpha} - \alpha - 1 \left( -|\lambda| v_i T_i^{\alpha+\beta} \right) \frac{1}{1 + \sum_{i=1}^{m} \mu_i T_i^{\alpha-\alpha} - \alpha - 1 \left( -|\lambda| v_i T_i^{\alpha+\beta} \right)},
\]

for all \( \lambda \in \mathbb{R}_+ \) and \( k \in \mathbb{N} \). Here

\[
\hat{\mathcal{K}}(t, \lambda)_{l,k} = \int_{0}^{t} s^{\alpha-1} E_{\alpha+\beta, \alpha} \left( -|\lambda| v_i s^{\alpha+\beta} \right) \hat{f}(t-s, \lambda)_{l,k} ds,
\]

and

\[
E_{\alpha, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}
\]

is the Mittag–Leffler function [16].
2.1. Proof of Theorem 2.2

2.1.1. Proof of the existence result
Let us take the group Fourier transform from Section 1.2 with respect to $x \in \mathbb{H}^n$, that is,
\begin{equation}
D_{+0,t}^{\alpha} \widehat{u}(t, \lambda) + \sigma_{\Delta_{\mathbb{H}^n}}(\lambda) I_{+0,t}^{\beta} \widehat{u}(t, \lambda) = \widehat{f}(t, \lambda),
\end{equation}
where $\sigma_{\Delta_{\mathbb{H}^n}}(\lambda)$ is the symbol of the Heisenberg sub-Laplacian. It has the following form:
\begin{equation}
\sigma_{\Delta_{\mathbb{H}^n}}(\lambda) = |\lambda| H_\tau \equiv |\lambda| \left(-\Delta_\tau + |\tau|^2\right),
\end{equation}
where $H_\tau$ is a harmonic oscillator operator for the variable $\tau \in \mathbb{R}^n$. For more information about the operator $H_\tau$, we refer to [28,30].

It is known that the operator $H_\tau$ is essentially self-adjoint in $L^2(\mathbb{R}^n)$ with a discrete spectrum $\nu_i$, $\lambda = (l_1, \ldots, l_n) \in \mathbb{N}^n$.

Corresponding to $\nu_i$, the harmonic oscillator operator has the complete system of orthonormal eigenfunctions $\{e_l\}_{l \in \mathbb{N}}$ on $L^2(\mathbb{R}^n)$. They take the form
\begin{equation}
e_l(\tau) = \prod_{j=1}^n P_j(\tau_j)e^{-|\tau|^2/2},
\end{equation}
where $P_m(\cdot)$ is the Hermite polynomial of order $m$. That is,
\begin{equation}
P_m(t) = c_m e^{\frac{|t|^2}{2}} \left(t - \frac{d}{dt}\right)^m e^{-\frac{|t|^2}{2}}, \quad t > 0, \quad c_m = 2^{-m/2}(m!)^{-1/2}\pi^{-1/4}.
\end{equation}
For more details, see [30].

Consequently, Equation (2.4) can be rewritten as
\begin{equation}
D_{+0,t}^{\alpha} \widehat{u}(t, \lambda)_{l,k} + |\lambda| \nu_l I_{+0,t}^{\beta} \widehat{u}(t, \lambda)_{l,k} = \widehat{f}(t, \lambda)_{l,k},
\end{equation}
for all $\lambda \in \mathbb{R}^n$, and any $l, k \in \mathbb{N}$. Here
\begin{equation}
\widehat{u}(t, \lambda)_{l,k} = (\widehat{u}(t, \lambda), e_l, e_k)_{L^2(\mathbb{R}^n)}
\end{equation}
and
\begin{equation}
\widehat{f}(t, \lambda)_{l,k} = (\widehat{f}(t, \lambda), e_l, e_k)_{L^2(\mathbb{R}^n)}.
\end{equation}
According to [31], the solution for Equation (2.6) satisfying initial conditions
\begin{equation}
I_{+0,t}^{2-\alpha} \widehat{u}(0, \lambda)_{l,k} = C, \quad [\alpha] D_{+0,t}^{\alpha-1} \widehat{u}(0, \lambda)_{l,k} = 0,
\end{equation}
can be represented in the form
\begin{equation}
\widehat{u}(t, \lambda)_{l,k} = \int_0^t s^{\alpha-1} E_{\alpha+\beta,\alpha} \left(-|\lambda| \nu_l s^{\alpha+\beta}\right) \widehat{f}(t - s, \lambda)_{l,k} ds + CE_{\alpha+\beta,\alpha-1} \left(-|\lambda| \nu_l T_t^{\alpha+\beta}\right).
\end{equation}
Then, it is not difficult to show that the solutions of Equation (2.6) satisfying the following conditions:
\begin{equation}
I_{+0,t}^{2-\alpha} \widehat{u}(0, \lambda)_{l,k} + \sum_{i=1}^m \mu_i I_{+0,t}^{2-\alpha} \widehat{u}(T_i, \lambda)_{l,k} = 0, \quad [\alpha] D_{+0,t}^{\alpha-1} \widehat{u}(0, \lambda)_{l,k} = 0,
\end{equation}
can be represented in the form

\[
\tilde{u}(t, \lambda)_{l,k} = \tilde{F}(t, \lambda)_{l,k} - \frac{\sum_{i=1}^{m} \mu_i \tilde{F}(T_i, \lambda)_{l,k} t^{\alpha-|\alpha|-1} E_{\alpha+\beta,\alpha-1} \left(-|\lambda| v_1 t^{\alpha+\beta}\right)}{1 + \sum_{i=1}^{m} \mu_i T_i^{\alpha-|\alpha|-1} E_{\alpha+\beta,\alpha-1} \left(-|\lambda| v_1 T_i^{\alpha+\beta}\right)},
\]

(2.9)

where

\[
\tilde{F}(t, \lambda)_{l,k} = \int_{0}^{t} s^{\alpha-1} E_{\alpha+\beta,\alpha} \left(-|\lambda| v_1 s^{\alpha+\beta}\right) \tilde{f}(t-s, \lambda)_{l,k} ds.
\]

Indeed, the formula (2.9) can be checked by the direct calculation from (2.7) under the conditions (2.8).

Now, applying the inverse group Fourier transform, we obtain the solution of Problem 2.1 in the form (2.3).

We note that the above expression is well defined in view of the non-resonance conditions (2.2). Finally, based on (2.9), we rewrite our formal solution as (2.3).

### 2.1.2. Convergence of the formal solution

Here, we prove convergence of the obtained integrals corresponding to functions \(t^{1+|\alpha|-\alpha} u(x, t), D_{+0,t}^{\alpha} u(x, t)\) and \(\Delta_{\|u\|} u(x, t)\). To prove the convergence, we use the estimate for the Mittag–Leffler function

\[
|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}.
\]

(2.10)

Let us first prove the convergence of (2.3). From the estimate (2.10), we have the following inequalities:

\[
|\tilde{F}(t, \lambda)_{l,k}| \leq C_1 \frac{||f(t, \lambda)_{l,k}||}{1 + |\lambda| v_1},
\]

\[
|E_{\alpha+\beta,\alpha-1} \left(-|\lambda| v_1 t^{\alpha+\beta}\right)| \leq \frac{C_2}{1 + |\lambda| v_1 t^{\alpha+\beta}}
\]

for some constants \(C_1, C_2 > 0\). Hence, from these estimates it follows that

\[
|t^{1+|\alpha|-\alpha} \tilde{u}(t, \lambda)_{l,k}|^2 \leq \left|t^{1+|\alpha|-\alpha} \tilde{F}(t, \lambda)_{l,k}\right|^2
\]

\[
+ \frac{1}{M} \sum_{i=1}^{n} |\mu_i|^2 |\tilde{F}(T_i, \lambda)_{l,k}|^2 |E_{\alpha+\beta,\alpha-1} \left(-|\lambda| v_1 T_i^{\alpha+\beta}\right)|^2
\]

\[
\leq C \frac{||f(t, \lambda)_{l,k}||^2}{(1 + |\lambda| v_1)^2} + C \frac{1}{M} \sum_{i=1}^{n} |\mu_i|^2 \frac{||f(T_i, \lambda)_{l,k}||^2}{(1 + |\lambda| v_1)^2} \frac{1}{(1 + |\lambda| v_1 t^{\alpha+\beta})^2}.
\]
Thus, since for any Hilbert–Schmidt operator $A$ one has
\[
\|A\|_{\text{HS}}^2 = \sum_{l,k} |(A\phi_l, \phi_k)|^2
\]
for any orthonormal basis $\{\phi_1, \phi_2, \ldots\}$, then we can consider the infinite sum over $l, k$ of the inequalities provided by (2.9). This gives
\[
\|t^{1+\alpha}\hat{a}u(t, \lambda)\|_{\text{HS}}^2 \leq C\|(1 + \sigma_{\Delta n}(\lambda))^{-1}\hat{f}(t, \lambda)\|_{\text{HS}}^2,
\]
(2.11)
since $\sup_{t \in [0, T]} 1/(1 + |\lambda|t^{\alpha+\beta}) = 1$. Thus integrating both sides of (2.11) against the Plancherel measure on $\mathbb{R}_+$ and using the Plancherel identity [28], we obtain
\[
\|t^{1+\alpha}\hat{a}u(t, x)\|_{C([0, T]; L^2(\mathbb{H}^n))} \leq C\|(I + \Delta_{\mathbb{H}^n})^{-1}f\|_{C([0, T]; L^2(\mathbb{H}^n))}
\]
and
\[
\|\Delta_{\mathbb{H}^n}u\|_{C([0, T]; L^2(\mathbb{H}^n))} \leq C\|f\|_{C([0, T]; L^2(\mathbb{H}^n))}.
\]
Since $f \in C([0, T]; L^2(\mathbb{H}^n))$, we get
\[
\|t^{1+\alpha}\hat{a}u(t, x)\|_{C([0, T]; L^2(\mathbb{H}^n))} < \infty
\]
and
\[
\|\Delta_{\mathbb{H}^n}u(t, x)\|_{C([0, T]; L^2(\mathbb{H}^n))} < \infty.
\]
The convergence of the integral corresponding to $D^{\alpha}_{+0,t}u(x, t)$ can be shown in a similar way.

To show the uniqueness of the solution, let us assume that there are two different functions $u$ and $v$ satisfying Problem 2.1. Now we introduce a new function $w$ as the difference of the solutions $u$ and $v$, that is, $w = u - v$.

Indeed, $w$ satisfies the homogeneous equation
\[
D^{\alpha}_{+0,t}w(t, x) - t^\beta_{+0,t}\Delta_{\mathbb{H}}w(t, x) = 0, \quad t > 0, \; x \in \mathbb{H}^n,
\]
(2.12)
with boundary conditions
\[
t^{2-\alpha}_{+0,t}w(0, x) + \sum_{i=1}^m \mu_i t^{-\alpha}_{+0,t}w(T_i, x) = 0, \quad [\alpha]D^{\alpha-1}_{+0,t}w(0, x) = 0, \quad x \in \mathbb{H}^n,
\]
(2.13)
where $\mu_i \in \mathbb{R}$, $0 < T_1 \leq T_2 \leq \cdots \leq T_m = T$.

Again, repeating the same arguments, for the solution $w$ of the problems (2.12) and (2.13) we obtain the estimate
\[
\|t^{1+[\alpha]-\alpha}w\|_{C([0, T]; L^2(\mathbb{H}^n))} \leq 0.
\]
Thus $0 = w = u - v$. The proof is complete.

**Disclosure statement**

No potential conflict of interest was reported by the authors.
Funding
The authors were supported in parts by the FWO Odysseus Project. The first author was supported in parts by the EPSRC grant EP/R003025/1 and by the Leverhulme Trust Grant RPG-2017-151. The second author was supported in parts by the Ministry of Education and Science of the Republic of Kazakhstan (MESRK) Grant AP05130994. The third author was supported by MESRK Grant AP05131756. No new data was collected or generated during the course of research.

References
[1] Bitsadze AV, Samarskii AA. On some simplest generalizations of linear elliptic problems. Dokl Akad Nauk SSSR. 1969;185:739–740.
[2] Avalishvili G, Avalishvili M, Miara B. Nonclassical problems with nonlocal initial conditions for second-order evolution equations. Asymptotic Anal. 2012;76:171–192.
[3] Il’in VA, Moiseev EI. Nonlocal boundary-value problem of the 1st kind for a Sturm–Liouville operator in its differential and finite-difference aspects. Differ Equ. 1987;23(7):803–811.
[4] Il’in VA, Moiseev EI. Nonlocal boundary-value problem of the 2nd kind for a Sturm–Liouville operator. Differ Equ. 1987;23(8):979–987.
[5] Il’in VA, Moiseev EI. 2-D Nonlocal boundary value problem for Poisson’s operator in differential and difference variants. Mate Model. 1990;2(8):139–156.
[6] Kharibegashvili S, MIDodashvili B. One nonlocal problem in time for a semilinear multidimensional wave equation. Lith Math J. 2017;57(3):331–350.
[7] Martin-Vaquero J, Sajavicius S. The two-level finite difference schemes for the heat equation with nonlocal initial condition. Appl Math Comput. 2019;342:166–177.
[8] Pao CV. Reaction diffusion equations with nonlocal boundary and nonlocal initial conditions. J Math Anal Appl. 1995;195:702–718.
[9] Byselewski L. Theorem about the existence and uniqueness of solution of a semilinear evolution nonlocal Cauchy problem. J Math Anal Appl. 1991;165:494–505.
[10] Byselewski L. Uniqueness of solutions of parabolic semilinear nonlocal boundary problems. J Math Anal Appl. 1992;165:472–478.
[11] Chadam J, Yin HM. Determination of an unknown function in a parabolic equation with an overspecified condition. Math Meth Appl Sci. 1990;13:421–430.
[12] Bysewski L. Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation $u_{tt} = F(x, t, u, u_x)$. J Appl Math Stoch Anal. 1990;3:163–168.
[13] Chabrowski J. On non-local problems for parabolic equations. Nagoya Math J. 1984;93:109–131.
[14] Dehghan M. Numerical schemes for one-dimensional parabolic equations with nonstandard initial condition. Appl Math Comput. 2004;147:321–331.
[15] Dehghan M. Implicit collocation technique for heat equation with non-classic initial condition. Int J Nonlin Sci Numer Simul. 2006;7:447–450.
[16] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. North-Holland: Elsevier; 2006.
[17] Gurtin ME, Pipkin AC. A general theory of heat conduction with finite wave speeds. Arch Rat Mech Anal. 1968;31:113–126.
[18] Miller RK. An integrodifferential equation for rigid heat conductors with memory. J Math Anal Appl. 1978;66:313–332.
[19] Tanabe H. Linear Volterra integral equations of parabolic type. Hokkaido Math J. 1983;12:265–275.
[20] Tanabe H. Remarks on Linear Volterra integral equations of parabolic type. Osaka J Math. 1985;22:519–531.
[21] Schneider WR, Wyss W. Fractional diffusion and wave equations. J Math Phys. 1989;30:134–144.
[22] Fujita Y. Integrodifferential equation which interpolates the heat equation and the wave equation. Osaka J Math. 1990;27:309–321.
[23] Fujita Y. Integrodifferential equation which interpolates the heat equation and the wave equation II. Osaka J Math. 1990;27:797–804.

[24] Borikhanov M, Kirane M, Torebek BT. Maximum principle and its application for the nonlinear time-fractional diffusion equations with Cauchy–Dirichlet conditions. Appl Math Lett. 2018;81:14–20.

[25] Kirane M, Torebek BT. Extremum principle for the Hadamard derivatives and its application to nonlinear fractional partial differential equations. Fract Calc Appl Anal. 2019;22(2):358–378.

[26] Ruzhansky M, Tokmagambetov N. Wave equation for operators with discrete spectrum and irregular propagation speed. Arch Ration Mech Anal. 2017;226:1161–1207.

[27] Ruzhansky M, Tokmagambetov N. Nonlinear damped wave equations for the sub-Laplacian on the Heisenberg group and for Rockland operators on graded Lie groups. J Differ Equ. 2018;265(10):5212–5236.

[28] Fischer V, Ruzhansky M. Quantization on nilpotent Lie groups. Vol. 314 of Progress in Mathematics. Basel, Switzerland: Birkhäuser/Springer; 2016 [Open access book].

[29] Ruzhansky M, Tokmagambetov N, Torebek BT. On a non-local problem for a multi-term fractional diffusion–wave equation. arXiv:1812.01336.

[30] Nicola F, Rodino L. Global pseudo-differential calculus on Euclidean spaces. Vol. 4 of Pseudo-Differential Operators. Theory and Applications. Basel: Birkhäuser Verlag; 2010.

[31] Al Saqabi BN, Tuan VK. Solution of a fractional differintegral equation. Integral Transform Spec Funct. 1996;4:321–326.