Online Adaptive Bin Packing with Overflow

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Abstract

Motivated by bursty bandwidth allocation [32] and by the allocation of virtual machines into servers in the cloud [28], we consider the online problem of packing items with random sizes into unit-capacity bins. Items arrive sequentially, but upon arrival an item’s actual size is unknown; only its probabilistic information is available to the decision maker. Without knowing this size, the decision maker must irrevocably pack the item into an available bin or place it in a new bin. Once packed in a bin, the decision maker observes the item’s actual size, and overflowing the bin is a possibility. An overflow incurs a large penalty cost and the corresponding bin is unusable for the rest of the process. In practical terms, this overflow models delayed services, failure of servers, and/or loss of end-user goodwill. The objective is to minimize the total expected cost given by the sum of the number of opened bins and the overflow penalty cost. We present an online algorithm with expected cost at most a constant factor times the cost incurred by the optimal packing policy when item sizes are drawn from an i.i.d. sequence of unknown length. We give a similar result when item size distributions are exponential with arbitrary rates.

1 Introduction

Bin Packing is one of the oldest problems in combinatorial optimization, and has been studied by multiple communities in a variety of forms. In the classical online formulation, \( n \) items with sizes in \([0, 1]\) arrive in an online fashion, and the objective is to pack the items into the fewest possible number of unit-capacity bins. The model has wide applicability in areas including cargo shipping [48], assigning virtual machines to servers [47], a variety of scheduling problems [11] [22] [46], and so on. In many of these applications, the items’ sizes may be uncertain, with this uncertainty often modeled via probability distributions. In much of the stochastic bin packing literature, an item’s size is observed before it must be packed, e.g. [28] [45]. Nevertheless, in many applications this assumption is unrealistic. For instance, in bandwidth allocation, connection requests are often bursty and deviate from their typical utilization. If the utilization of the request is higher than expected, it can jeopardize the stability of other connections sharing the same channel. Moreover, the only way to observe the actual traffic required by the connection is to first allocate the request and then observe the traffic pattern.

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Motivated by these considerations, we introduce an online adaptive bin packing problem that takes into account the following ingredients.

1. Arrivals are adversarial distributions and the length of the item sequence is unknown to the decision maker.

2. In contrast to existing work in the online and/or stochastic bin packing literature, when an item arrives, the decision maker only observes a probability distribution of its size.

3. The decision maker observes the item’s actual size only after irrevocably placing it in a bin; therefore, overflowing a bin is possible.

4. An overflowed bin incurs a penalty and renders the bin unusable from that point on. The objective is to minimize the expected cost given by the sum of the number of open bins and overflow penalty.

1.1 Motivating Applications

The online adaptive bin packing problem captures the uncertainty introduced by the online nature of the problem, and also the uncertainty introduced by learning the size of an item after it is packed in a bin. While the variant of the bin packing problem we consider is general and widely applicable, the following examples give some concrete applications:

**Bandwidth Allocation** An operator is in charge of assigning sequentially arriving independent connection requests. The operator can open new fixed-capacity connections (bins) of unit cost or try to use one of the available connections to pack the incoming request. Traffic on a connection may be bursty, requiring more than the available bandwidth. In this case, the connection suffers from the overflow of the channel, which could represent a monetary penalty or extra work involved in reassigning the request(s) to other connection(s). See also [32].

**Freight Shipping** A dispatcher in a fulfillment center is in charge of packing items into trucks for delivery. Truckloads must comply with a maximum weight limit, and our model applies when the dispatcher assigns items into trucks before their final weighing. An overweight truck incurs a penalty representing additional labor or possible fines. See also [33, 34, 38].

**Virtual Machines** A controller is in charge of assigning virtual machines (VM) to servers. The controller has statistical knowledge of the amount of resource a VM will utilize (CPU, RAM, I/O bandwidth, energy, etc.), learned via historical data. The actual resource usage is observed once the VM runs in a server. Excessive consumption of a resource by the VM could compromise the stability of the server and negatively affect other VM’s sharing the same infrastructure. See also [28].

**Operation Room Scheduling** In hospitals, an administrator is in charge of assigning incoming surgeries to different operation rooms. There may be a statistical estimation of a procedure’s duration, but the real time spent in the room is only learned once the operation has finished. Over-allocating a room could incur economic penalties and loss of patients’ good will. See also [20].
1.2 The Model

We consider the problem of sequentially packing items arriving in an online fashion into homogeneous bins of unit capacity. The input consists of a sequence of $n$ nonnegative independent random variables $X_1, \ldots, X_n$, released sequentially one at a time. Similar to the bin packing literature, we refer to items interchangeably either by their index $i$ or their corresponding random variable $X_i$. At iteration $i$, random variable $X_i$ arrives and we observe its distribution but not its outcome. We decide irrevocably to pack $X_i$ into an available bin with nonnegative remaining capacity (if any), or to place $X_i$ in a new bin and pay a unit cost. Once packed, we observe the outcome of the random variable $X_i = x_i$, and the chosen bin’s capacity is reduced by this amount. A bin overflows when the sizes of items packed in it sum to more than one; when this happens, we incur a cost $C \geq 1$ and the overflowed bin becomes unavailable for future iterations.

We measure the performance of an algorithm $\mathcal{P}$ based on the expected overall cost incurred and denote it $\text{cost}(\mathcal{P})$. Because of the online nature of the problem, we cannot expect to compute the optimal cost for an arbitrary sequence of distributions. Even if we knew all distributions in advance, computing the minimum-cost packing is still computationally challenging; the deterministic version reduces to the NP-hard offline bin packing problem. To quantify the quality of an online algorithm, we compare the expected cost incurred by the algorithm against the expected cost incurred by an optimal adaptive packing policy that knows all distributions in advance. This benchmark knows all size distributions in advance but not their outcomes, and must pack the items sequentially in the same order as the online algorithm. This measure of quality differs from the traditional online competitive ratio, cf. [2,9]. In the latter, we would compare the performance of an online algorithm against the performance of an extremely powerful optimal offline algorithm that knows all item sizes in advance. Consider $n$ i.i.d. random variables with distribution

$$
X_i = \begin{cases} 
1/n & \text{w.p. } 1 - 1/C \\
1 & \text{w.p. } 1/C.
\end{cases}
$$

We expect $n/C$ random variables to realize to 1. Therefore, the expected cost of an offline solution that observes the sizes is at most $n/C + 1$. In contrast, the cost incurred by any online algorithm (or even an offline algorithm that observes distributions but not sizes) is at least $n$. Therefore, when measured against the more powerful benchmark, no online algorithm can have a bounded competitive ratio, which motivates us to use a more refined benchmark that knows distributions but not outcomes before the items are packed.

1.3 Our Results and Contributions

We propose a heuristic algorithm, called Budgeted Greedy and denoted $\text{Alg}$ (Algorithm [H]). Budgeted Greedy uses a risk budget in each bin as a way to control the risk of overflowing the bins. If we consider packing item $i$ in bin $j$, this action’s risk is equal to the probability of overflowing the bin; Budgeted Greedy maintains a bin’s risk below its risk budget. At every step, similar to the bin’s capacity, when an item is packed in a bin, the bin’s budget is reduced by the probability of the current item overflowing the bin. If no currently opened bin has enough risk budget left, then a new bin is opened. Observe that the risk of packing item $i$ into any available bin depends on the realized sizes of items $1, \ldots, i-1$ and these items’ assignments.

1See Section 8 for a more detailed description of policies.
The risk as defined above can be calculated for any policy. While there are instances where the optimal policy incurs a large risk for certain bins it opens, our first structural result shows any policy can be converted to one with budgeted risk with at most a constant factor loss.

**Theorem 1.1.** Let $X_1, \ldots, X_n$ be an arbitrary sequence of independent nonnegative random variables (not necessarily identically distributed). For any policy $P$ that sequentially packs $X_1, \ldots, X_n$, there exists a risk-budgeted policy $P'$ packing the same items such that no bin surpasses the risk budget $1/C$, and with expected cost

$$\text{cost}(P') \leq 4 \text{cost}(P).$$

Theorem 1.1 is obtained by considering the decision tree for policy $P$ and updating the decision tree whenever the risk budget is violated by opening a new bin. The extra cost of the new opened bins is paid by a delicate charging argument.

While the cost of any policy involves two terms, the expected number of open bins and the penalty for overflowed bins, we show (Lemma 3.2) that for a budgeted policy, the cost of overflowed bins is at most the number of opened bins in expectation. This allows us to exclusively focus on the number of bins opened by the budgeted policy. A consequence of these structural results is the following.

**Theorem 1.2.** If the input sequence $X_1, \ldots, X_n$ is i.i.d., Budgeted Greedy minimizes the expected number of opened bins among all budgeted policies. As a consequence, $\text{cost}(\text{ALG}) \leq 8 \text{cost}(\text{OPT})$, where $\text{OPT}$ denotes the optimal policy that knows $n$ in advance.

This i.i.d. model can be interpreted in the following manner. Suppose there is a probability distribution over the nonnegative real numbers, say $D$. There are $n$ item sizes independently drawn from this distribution, $x_1, \ldots, x_n$. For each $i = 1, \ldots, n$, we are asked to pack the $i$-th item without observing its size. This is indeed a model for basic allocation systems where only a population distribution is known about the item’s size, which is a typical occurrence in practical applications if more granular information is not available.

As a second contribution, we show that for arbitrary exponential distributions, i.e. a sequence of random variables $X_1, \ldots, X_n$ with $\text{P}(X_i > x) = e^{-\lambda_i x}$, Budgeted Greedy incurs a cost that is at most a factor $O(\log C)$ times the benchmark cost. Moreover, if the exponential random variables are sufficiently small, this factor can be reduced to a constant.

**Theorem 1.3.** If each $X_i$ is exponentially distributed with rate $\lambda_i > 0$, Budgeted Greedy satisfies

$$\text{cost}(\text{ALG}) \leq O(\log C) \text{cost}(\text{OPT}).$$

Furthermore, if $\lambda_i \geq 2 \log C$ for all $i = 1, \ldots, n$, $\text{cost}(\text{ALG}) \leq O(1) \text{cost}(\text{OPT})$.

We show that Budgeted Greedy opens bin $i + 1$ if it either packs at least $O(1/\log C)$ in bin $i$ or the risk in bin $i$ is bounded by a small constant, which we obtain via an auxiliary non-convex maximization problem. With this, Budgeted Greedy’s cost is bounded by $O(\log C) \sum_i E[X_i] \leq O(\log C) \text{cost}(\text{OPT})$. When the exponential random variables are small enough, Budgeted Greedy opens bin $i + 1$ if a constant amount of mass in bin $i$ is packed, thereby reducing the $\log C$ factor to constant. We also give a $\Omega(\sqrt{\log C})$ lower bound for Budgeted Greedy’s competitive ratio in the case of exponentially distributed sizes.

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2If there are items $X_i$ with $\text{P}(X_i > 1) > 1/C$, these are packed individually. Bins not containing these items have risk bounded by $1/C$. Details in Section 4.
1.4 Organization

The rest of the paper is organized as follows. We follow this introduction with a brief literature review. In Section 3 we present the Budgeted Greedy algorithm and introduce the necessary notation for the rest of the paper. In Section 4 we present the proof of Theorem 1.1 and use it to prove Theorem 1.2. In Section 5 we present the proof for Theorem 1.3 and the construction of the corresponding lower bound.

2 Related Work

In the classic one-dimensional bin packing problem, \( n \) items with sizes \( x_1, \ldots, x_n \) in \([0, 1]\) must be packed in the fewest unit-capacity bins without splitting any item into two or more bins. This is a well-studied \( \text{NP} \)-complete problem spanning more than sixty years of work [16, 22, 23, 29, 30, 31, 43]. For excellent surveys see [10, 14]. In the online version, the list of items \( L = (x_1, \ldots, x_n) \) is revealed online one item at a time. At round \( t \), we observe item \( x_t \) and we need to decide irrevocably and without knowledge of future arrivals whether to pack the item in an open bin with enough remaining space, or to open a new unit-capacity bin at unit cost. It is standard to measure an online algorithm’s performance via its (asymptotic) competitive ratio [2, 9, 14],

\[
\limsup_{|L| \to \infty} \frac{\text{cost}_{\text{alg}}(L)}{\text{cost}_{\text{OPT}}(L)},
\]

where \( \text{cost}_{\text{alg}}(L) \) is the cost incurred by the online algorithm with input \( L \), and \( \text{cost}_{\text{OPT}}(L) \) is the cost incurred by the optimal offline solution that knows \( L \) in advance. The best known competitive ratio is 1.57829 [3], and a universal lower bound is 1.5 [49].

In several real-world applications, exact item sizes are unknown to the decision maker at the time of insertion [18, 46]. This uncertainty is typically modeled via probability distributions on the items’ size. Several online and offline bin packing models introducing stochastic components have been studied [12, 15, 24, 27, 32, 35, 41, 45, 44]. These stochastic models have revealed connections with balls-into-bins problems [45], sums of squares [15], queuing theory [13], Poisson approximation [35], etc. For the online case, common to all these models is the assumption that the item size is observed before packing it. Nevertheless, observing the item size is unrealistic in many scenarios. For instance, in cloud computing, before running a job in a cluster, we may have some statistical knowledge of the amount of resource the job will utilize. However, the only way to observe the real utilization is to start the job. In this work, we propose a new model variant where items’ size distributions are revealed in an online fashion but each outcome is observed only after packing the item. We therefore relax the strict capacity constraint by allowing each bin to overflow at most once, at the expense of a penalty. Related to this kind of online input are the works [12, 15, 27, 41, 45, 44].

Our model also shares similarities with adaptive combinatorial optimization, particularly stochastic knapsack models introduced in [19]. Recent treatments began with [17]; a large body of work has now studied this model from several perspectives [4, 5, 6, 7, 8, 21, 26, 35, 36]. Most of these works assume complete knowledge of the input distributions, and online treatments are scarcer in the literature, see [11, 25, 37].
3 The Algorithm

3.1 Preliminaries

The problem’s input consists of \( n \) independent nonnegative random variables \( X_1, \ldots, X_n \). The (possible) bins to utilize are denoted by \( B_1, B_2, \ldots, B_n \). A state \( s \) for round \( i \in [n+1] \) is a sequence \((x_1, 1 \rightarrow j_1)(x_2, 2 \rightarrow j_2) \cdots (x_{i-1}, i - 1 \rightarrow j_{i-1})\), where \( x_k \) is an outcome of \( X_k \) for all \( k < i \). The pair \((x_k, k \rightarrow j)\) represents round \( k \), and refers to packing \( X_k \) in bin \( j \) and observing outcome \( X_k = x_k \). A state for round \( i \) represents the path followed by a decision maker packing items \( X_1, \ldots, X_n \) sequentially into bins and the outcomes for each of these decisions until round \( i - 1 \). States have a natural recursive structure: \( s = s'(x_{i-1}, i - 1 \rightarrow j_{i-1}) \), where \( s' \) is the state for round \( i - 1 \). The initial state \( s_0 \) is the empty state. Bin \( B_j \) is open by state \( s \) if some \((x_k, k \rightarrow j)\) appears in \( s \). The items packed into bin \( B_i \) by state \( s \) are \( B_i(s) = \{k : (x_k, k \rightarrow j) \text{ appears in } s \} \). The number of bins opened by state \( s \) is \(|\{j : (x_k, k \rightarrow j) \text{ appears in } s \}|\). The usage of bin \( B_j \) at the beginning of round \( i \) in state \( s \) is \( S_j^{-1}(s) = \sum_{k \leq i-1} x_k \), i.e., the sum of sizes of items packed in bin \( j \). A bin \( B_j \) is broken or overflowed in \( s \) if \( S_j^{-1}(s) > 1 \). In our model, we stop using bins that overflow. A state \( s \) is feasible if any overflowed bin by round \( k \) is never used again after \( k \), for any \( k < i \). The state space is the set of all feasible states, denoted \( S \). The set of all feasible states for rounds \( \leq n \) is denoted by \( S_n \).

A policy \( \mathcal{P} \) is a function \( \mathcal{P} : [n] \times S_n \rightarrow [n] \cup \{\emptyset\} \) such that \( \mathcal{P}(i, s) = \begin{cases} j & \text{if } s \text{ is for round } i \\ \emptyset & \text{if } s \text{ is for round } \neq i \end{cases} \) where \( \mathcal{P}(i, s) = j \) indicates that at feasible state \( s \in S_n \) for round \( i \) we pack item \( i \) into bin \( j \); we write this as \( i \rightarrow j \) when the policy and state are clear from the context. The policy is feasible if \( s' = s(x_i, i \rightarrow \mathcal{P}(i, s)) \) is a feasible state for any feasible \( s \in S_n \) for round \( i \) and outcome \( x_i \) of \( X_i \). From now on, we only consider feasible policies. A state \( s' \in S \) is reachable by the policy if \( s' = s_0 \) or \( s' = s(x_i, i \rightarrow \mathcal{P}(i, s)) \) with \( s \) reachable and \( x_i \) an outcome of \( X_i \). For a reachable state \( s \) for round \( i \in [n] \), we say that \( \mathcal{P} \) opens bin \( j \) if \( \mathcal{P}(i, s) = j \) and \( B_j \) is not open in \( s \). We say that the policy overflows bin \( B_j \) at state \( s \) if \( B_j \) overflows for \( s' = s(x_i, i \rightarrow \mathcal{P}(i, s)) \). We set the cost of a policy as

\[
\text{cost}(\mathcal{P}) = E[N_P] + C \cdot E[O_P],
\]

where \( N_P \) is the number of bins opened and \( O_P \) is the number of bins broken by reachable states for round \( n + 1 \). The randomness is over the items’ outcomes. Notice that non-reachable states in \( S \) are unimportant for \( \text{cost}(\mathcal{P}) \), hence we can always assume \( \mathcal{P}(i, s) = \emptyset \) for non-reachable \( s \in S_n \).

A policy specifies the actions to apply in any epoch of the sequential decision-making problem.

Any policy \( \mathcal{P} \) has a natural \((n+1)\)-level decision tree representation \( T_\mathcal{P} \), which we call the policy tree. The root, denoted \( r \), is at level 1 and represents item \( X_1 \) and state \( s_0 \). A node at level \( i \in [n] \) is labeled with \( \mathcal{P}(i, s) \) where \( s \) is the state of the system obtained by following the path from the root to the current node. There is a unique arc going out of the node for every possible outcome of \( X_i \) directed to a unique node in level \( i + 1 \). Nodes at level \( n + 1 \) are leaves denoting that the computation has ended. Nodes in levels \( i \in [n] \) are called internal nodes. To compute the cost(\( \mathcal{P} \)) using the policy-tree \( T_\mathcal{P} \), we add two labels to the tree:

- For an internal node \( u \), \( \ell_u = 1 \) if \( \mathcal{P} \) opens a new bin in node \( u \); 0 otherwise. For leaves we define \( \ell_u = 0 \).
- For arcs \( a = (u, v) \), we define \( c_a = C \) if the outcome of the random variable belonging to the level where \( u \) is located overflows the bin chosen by the policy at node \( u \); 0 otherwise.
We refer to this tree as cost-labeled tree $T_P$ with cost vectors $(\ell, c)$, or simply cost-labeled tree $T_P$ if the costs are clear from the context. The tree structure gives us a recursive way of computing the cost of the policy. Let $T_P(u)$ be the cost-labeled sub-tree of $T_P$ rooted at node $u$, then
\[
\text{cost}_{\ell,c}(T_P(u)) = \begin{cases} 
\ell_u + E_{X_i}[c(u,u_{X_i}) + \text{cost}_{\ell,c}(T_P(u_{X_i}))] & \text{if } u \text{ is at level } i = 1, \ldots, n \\
0 & \text{if } u \text{ is at level } n+1
\end{cases},
\]
where $u_{X_i}$ is the node at level $i+1$ connected to $u$. Thus, $\text{cost}(P) = \text{cost}_{\ell,c}(T_P(r))$. We define $\text{OPT} = \arg\min_P \text{cost}(P)$ as the optimal policy for sequentially packing items $X_1, \ldots, X_n$.

Note that we defined only deterministic policies, since the action $\mathcal{P}(i, s)$ is deterministic. If $\mathcal{P}(i, s)$ were random, then we would have a randomized policy. A standard result from Markov decision process theory ensures that any randomized policy has a deterministic counterpart incurring the same cost; hence, we only focus on deterministic policies. For more details see \cite{39, 40}.

The following proposition characterizes the expected number of bins overflowed by a policy. The proof appears in Appendix \[A\].

**Proposition 3.1.** Let $X_1, \ldots, X_n$ be nonnegative independent random variables, and let $\mathcal{P}$ be any policy that sequentially packs these items. The expected number of bins broken by the policy $\mathcal{P}$ is
\[
E[O_P] = \sum_{i=1}^{n} E_{X_{i-1}, \ldots, X_1} \left[ \sum_{i=1}^{n} P_{X_i}(X_i + S_1^{i-1} > 1) 1_{\mathcal{P}_{i-1} \rightarrow j} \right],
\]
where $S_1^{i-1}$ is the usage of bin $j$ at the beginning of iteration $i$ and $1_{\mathcal{P}_{i-1} \rightarrow j}$ is the indicator random variable of the event in which $\mathcal{P}$ packs item $X_i$ into bin $j$.

If we interpret $P_{X_i}(X_i + S_1^{i-1} > 1)$ as the risk that $X_i$ overflows bin $j$ if packed there, then, the result says that the number of overflowed bins is the expected aggregation of these risks. We define the risk of a bin $j$ as $\text{Risk}(B_j) = \sum_{i=1}^{n} P_{X_i}(X_i + S_1^{i-1} > 1) 1_{\mathcal{P}_{i-1} \rightarrow j}$. Then $E[O_P] = \sum_{j=1}^{n} E[\text{Risk}(B_j)]$.

A policy $\mathcal{P}$ is called risk-budgeted or simply budgeted with risk budget $r > 0$ if no bin incurs a risk larger than $r$. That is, for any $j \in [n]$, $\text{Risk}(B_j) \leq r$.

A deterministic online algorithm induces a policy, with non-reachable states simply mapped to $\emptyset$. Since online algorithms are not aware of the number of items $n$, we label the $j$-th bin opened by an online algorithm as $B_j$ in this case. The cost of an online algorithm is naturally defined as the cost of the corresponding induced policy.

### 3.2 The Budgeted Algorithm

In the **Budgeted Greedy** algorithm, we keep a risk budget for each bin that is initialized as $\gamma/C$, where $\gamma \geq 1$ is an algorithm parameter. We pack items in a bin as long as the usage of the bin is at most 1 and its risk budget has not run out. More formally, when opening a bin, say bin $j$ at round $i$, we initialize its risk of overflow at $r_j^{i-1} = 0$. At round $i$, when item $X_i$ arrives, we find a bin $j$ such that $r_j^{i-1} + p_i(S_j^{i-1}) \leq \gamma/C$, where $r_j^{i-1}$ is the accumulated risk of overflowing the bin until $i-1$, $S_j^{i-1}$ is the usage of the bin $j$ until the previous round and $p_i(S_j^{i-1}) = P_{X_i}(X_i + S_j^{i-1} > 1)$ is the risk that $X_i$ overflows bin $j$. If that bin $j$ exists, we pack the incoming
item into bin $j$, breaking ties arbitrarily, and we update the risk of overflow as $r^i_j = r^{i-1}_j + p_i(S^{i-1}_j)$ and $r^i_{j'} = r^{i-1}_{j'}$ for any $j' \neq j$. Such a bin may not exist, in which case we open a new bin $k$ with $r^i_k = p_i(0)$. Strictly speaking, Budgeted Greedy is not a budgeted policy with risk budget $\gamma/C$ unless all items satisfy $P(X_i > 1) \leq \gamma/C$; items with $P(X_i > 1) > \gamma/C$ are packed into individual bins. In Algorithm 1, we formally present the description of Budgeted Greedy.

Algorithm 1: ALG($\gamma, X_1, \ldots, X_n$)

1. Init: $I = \emptyset$.
2. for $i = 1, \ldots, n$ do
   3.   if $\exists j \in I$ such that $r^{i-1}_j + p_i(S^{i-1}_j) \leq \gamma/C$ then
      4.     $S^i_j = S^{i-1}_j + X_i$,
      5.     $r^i_j = r^{i-1}_j + p_i(S^{i-1}_j)$.
   else
      6.      Define $r^i_j = p_i(0)$ for $j$ such that $j = \inf\{j \geq 0 : j \notin I\}$.
      7.      $S^i_j = X_i$.
   for $j' \neq j$ do
      8.      $S^i_{j'} = S^{i-1}_{j'}$.
      9.      $r^i_{j'} = r^{i-1}_{j'}$.

Lemma 3.2. Let $\gamma \geq 1$ and assume that for all $i$, $P(X_i > 1) \leq \gamma/C$. For any bin $j$, Algorithm 1 guarantees

$$P(\text{ALG breaks bin } j) \leq \frac{\gamma}{C} P(\text{ALG opens bin } j).$$

Proof. Using Proposition 3.1

$$P(\text{ALG breaks bin } j) = E \left[ \left( \sum_{i=1}^{n} P(X_i + S^{i-1}_j > 1) \mathbf{1}_{\{\text{ALG}_{\{t+j\}}\}} \right) \mathbf{1}_{\{\text{ALG opens bin } j\}} \right]$$

$$= E \left[ \text{Risk}(B_j) \mathbf{1}_{\{\text{ALG opens bin } j\}} \right] \leq \frac{\gamma}{C} P(\text{ALG opens bin } j),$$

since once the bin has been opened, its risk never goes beyond $\gamma/C$. □

As a result, we have the following corollary, which implies that we only need to bound the expected number of bins opened by Budgeted Greedy in our analysis.

Corollary 3.3. Under the same assumptions of Lemma 3.2, $\text{cost}(\text{ALG}) \leq (1 + \gamma) E[N_{\text{ALG}}].$

4 A Policy-Tree Analysis for I.I.D. Random Variables

In this section we prove a slight generalization of Theorem 1.1 and use this result to prove Theorem 1.2. Our first result shows that any policy can be converted into a budgeted version, where a risk budget is never surpassed for any bin. This transformation can be carried out while only incurring a constant multiplicative loss. The proof relies on a charging scheme in the cost paid
by overflowing bins. Starting with the original policy tree, we increase the cost paid by overflowing bins by an amount $\delta > 0$. The overall cost of the tree increases multiplicatively by at most $(1 + \delta/C)$. We show that this additional $\delta$ allows us to pay for new bins whenever the risk of the bin goes beyond $\gamma/C$, for appropriate choice of $\delta$ and $\gamma$.

**Theorem 4.1.** Let $X_1, \ldots, X_n$ be an arbitrary sequence of independent nonnegative random variables (not necessarily identical). Fix $\delta > 0$ and let $\gamma \geq C/\delta$, where $C \geq 1$ is the penalty paid for overflowing bins. For any policy $P$ that sequentially packs items $X_1, \ldots, X_n$, there exists a policy $P'$ for the same items such that

- $P'$ packs items with $P(X_i > 1) > \gamma/C$ into individual bins, and bins not containing these items never exceed the risk budget $\gamma/C$.
- $P'$ satisfies $\text{cost}(P') \leq 2(1 + \delta/C)\text{cost}(P)$.

In particular, if all items satisfy $P(X_i > 1) \leq \gamma/C$, $P'$ is a risk-budgeted policy with risk budget $\gamma/C$.

**Proof.** The proof follows two phases. In the first and longest phase, we show that we can modify the policy $P$ in such a way that the risk of each bin exceeds $\gamma/C$ at most once. In the second phase, we show that with a multiplicative cost increase of at most 2, the item surpassing the risk budget in each bin can be packed into an individual bin.

In the rest of the proof we utilize the tree representation of the policy. We proceed as follows:

1. In the cost labeled tree $T_P$, increase the cost of overflowing the bins from $C$ to $C + \delta$. That is, $c_{(u,v)} = C + \delta$ if $c_{(u,v)} = C$ and 0 otherwise for any arc $(u,v)$ in $T_P$. Then,

$$\text{cost}_{u,v}(T_P(r)) \leq \left(1 + \frac{\delta}{C}\right)\text{cost}_{u,v}(T_P(r)) = \left(1 + \frac{\delta}{C}\right)\text{cost}(P).$$

2. Starting at the root of this new cost-labeled tree, find a node $u$ at level $i = 1, \ldots, n$ where the policy $P$ decides to open a new bin, say bin $j$. In each of the branches starting at node $u$ and directed to some leaf, find the sequence of nodes $u_1 = u, u_2, \ldots, u_k$ where the policy packs items into bin $j$ and node $u_k$ corresponds to the first node in the branch where the risk budget $\gamma/C$ is surpassed for bin $j$. Define $u_k$ as a leaf if in the branch the risk budget is not surpassed for bin $j$. Let $i_1 = i, i_2, \ldots, i_k$ be the items packed into bin $j$ in this branch; that is, node $u_{i_\ell}$ is at level $i_\ell$. Then, we have

$$\sum_{m=1}^{k-1} P(X_{i_m} + S_{i_{m-1}} > 1) \leq \frac{\gamma}{C}, \quad \text{and} \quad \sum_{m=1}^{k} P(X_{i_m} + S_{i_{m-1}} > 1) > \frac{\gamma}{C}$$

if node $u_k$ is not a leaf.

Consider the following modifications to the cost-labeled tree $T_P$: We start with the same tree as $P$ but in the subtree rooted at $u$, bin $j$ is utilized only in nodes $u_1, \ldots, u_k$ for the different branches. Any future utilization of bin $j$ after passing through node $u_k$ is moved to a new bin $j'$. Now, we update the cost labels as follows. For all the branches, we reduce the cost of $C + \delta$ appearing in the arcs going out from nodes $u_1, \ldots, u_k$ to $C$. We label the first node appearing after node $u_k$ where the bin $j'$ is opened with a 1. We reduce the labels of arcs
going out of nodes using bin $j'$ if they do not overflow the bin $j'$ anymore (bin $j'$ has smaller usage than bin $j$). Formally, for any branch and nodes $u_1, \ldots, u_k$ defined as before,

$$c'_{(a,b)} = \begin{cases} 
\frac{c}{c + \hat{c}(a,b)} & a = u_m \text{ for some branch starting at } u \\
\hat{c}_{a,b} & \text{otherwise}
\end{cases}$$

and for nodes,

$$\ell'_a = \begin{cases} 
1 & a \text{ is the first node packing into bin } j' \text{ in the subtree } \mathcal{T}_P(u_k) \\
\ell_a & \text{otherwise}
\end{cases}$$

We denote this new policy by $\mathcal{P}'$. Figure 1 displays the modification process.

We now argue that the changes applied to the cost-labeled tree $\mathcal{T}_P$ to transform it into $\mathcal{T}_{P'}$ do not increase the cost function $\text{cost}_\ell,\hat{c}(\mathcal{T}_P)$. Since we only modified labels in the subtree $\mathcal{T}_P(u)$ it is enough to study the cost change in this specific subtree for bins $j$ and bin $j'$.

**Lemma 4.2.** $\text{cost}_\ell,\hat{c}(\mathcal{T}_P(u)) \geq \text{cost}_{\ell',c'}(\mathcal{T}_{P'}(u))$.

The proof of this lemma appears in Appendix A. With this result we have

$$\text{cost}_\ell,\hat{c}(\mathcal{T}_P(r)) - \text{cost}_{\ell',c'}(\mathcal{T}_{P'}(r)) = (\text{cost}_\ell,\hat{c}(\mathcal{T}_P(u)) - \text{cost}_{\ell',c'}(\mathcal{T}_{P'}(u))) \mathbb{P}(\text{Reach node } u) \geq 0$$

3. Now, starting from policy $\mathcal{P}'$ and cost-labeled tree $\mathcal{T}_{P'}$ with labels $\ell'$ in the nodes and $c'$ in the arcs, repeat step 2 until every bin exceeds the risk budget $\gamma/C$ at most once.
With the previous method, we construct a policy, which we still call $\mathcal{P}'$ for simplicity, that exceeds each bin’s risk budget at most once. For the second phase, we further modify $\mathcal{P}'$: If the policy tries to exceed some bin’s risk budget, we open a new bin for that item. Using the notation of the first phase, this means that whenever the policy reaches node $u_k$ in some branch starting at $u$, instead of packing the item in node $u_k$ into bin $j$, it opens a new bin $j''$ for it. We call this new policy $\mathcal{P}''$. This increases the number of opened bins by at most a factor of 2, $N_{\mathcal{P}''} \leq 2N_{\mathcal{P}'}$. Moreover, the risk of the new bins is no larger than the old risk, $\text{Risk}(B_{j''}) \leq \text{Risk}(B_j)$; therefore, $\mathbb{E}[O_{\mathcal{P}''}] \leq 2 \mathbb{E}[O_{\mathcal{P}'}]$. Hence,

$$\text{cost}(\mathcal{P}'') \leq 2 \text{cost}(\mathcal{P}') \leq 2 \left(1 + \frac{\delta}{C}\right) \text{cost}(\mathcal{P}).$$

When the input is an i.i.d. sequence of nonnegative random variables, Budgeted Greedy induces a policy tree that only keeps one bin opened at a time. This simple fact is crucial in the proof of our next result. The next lemma shows that among all budgeted policies, Budgeted Greedy opens the minimum expected number of bins whenever the input is an i.i.d. sequence of random variables. Intuitively, if we ignore the penalty paid by overflowing bins and all items are i.i.d., the optimal way to minimize the expected number of opened bins is by packing as many items as possible in each bin, as long as the risk budget is satisfied. This can of course be done sequentially, one bin at a time, which is what Budgeted Greedy does.

**Lemma 4.3.** Suppose $X_1, \ldots, X_n$ are nonnegative i.i.d. random variables. Then,

$$\mathbb{E}[N_{\text{ALG}}] = \min_{\mathcal{P} \text{ budgeted with } \text{risk budget } \gamma/C} \mathbb{E}[N_{\mathcal{P}}].$$

That is, among all risk-budgeted policies with budget $\gamma/C$, Budgeted Greedy (Algorithm 1) opens the minimum expected number of bins.

**Proof.** Consider any policy $\mathcal{P}$ for packing items such that the risk budget of each bin $\gamma/C$ is never surpassed. Consider its tree representation $T_\mathcal{P}$. We modify the policy tree so only one bin is utilized at a time. For this, we exhibit a sequence of operations ensuring that, whenever bin $j$ is opened, bins $1, \ldots, j-1$ are never utilized again. In the tree, this is equivalent to saying that any branch starting from the root directed to any leaf has labels $1 \to j_1, 2 \to j_2, \ldots, n \to j_n$ where $1 = j_1 \leq j_2 \leq \cdots \leq j_n$, where we recall that $i \to j$ means the policy packs item $i$ into bin $j$.

**Claim 1.** Let $j = 1, \ldots, n$ be any bin opened by the policy. Suppose that node $u$ in level $k$ is labeled $k \to j'$, where $j' \neq j$. Furthermore, suppose that at node $u$, bin $j$ is open, its usage does not exceed 1, and its risk budget can accommodate $k$. Then $u$ can be relabeled $k \to j$ without increasing the expected number of bins opened by the policy.

Before proving this claim, we show how to use it to conclude the result. Starting at the root $r$ of the policy tree $T_\mathcal{P}$, find the closest node $u$ to the root where we have the label $k \to j$, $j \neq 1$, but the usage of bin 1 is no more than 1 and its risk budget can accommodate $k$. Use the claim to relabel this node $k \to 1$ without increasing the expected number of open bins. Repeat this process until there are no nodes $u$ in this category. After this process has been finished, all branches starting at the root have the form $1 \to 1, 2 \to 1, \ldots, i \to 1, i + 1 \to 2, \ldots$ and from $i + 1$ onward, bin 1 is overflowed or does not have enough risk budget to receive any additional item. We repeat this process with bin 2,3,... After this process has been carried out, the resulting policy that is the one induced by Budgeted Greedy.
We prove the claim by backward induction on the level of node \( u \) in the policy tree. Fix an opened bin \( j = 1, \ldots, n \) and pick any node \( u \) at level \( n \) with label \( n \to j', j' \neq j \), and suppose bin \( j \) is open and satisfies the hypothesis (\( \sum_{i \in B_j} X_i \leq 1 \) and enough risk budget to receive item \( X_n \)). Labeling this node \( n \to j \) does not worsen the number of bins opened since the cost of bin \( j \) has already been paid at some previous node. Recall that we are only taking into account the cost paid by opening bins and not the cost of breaking bins.

Suppose the result holds for all levels \( k + 1, k + 2, \ldots, n \). Pick a node \( u \) at level \( k \) with label \( k \to j' \), \( j' \neq j \), and such that bin \( j \) is open and satisfies the hypothesis. If all its children are labeled \( k + 1 \to j \) then relabel all its children with \( k + 1 \to j' \) and relabel \( u \) with \( k \to j \). The cost remains the same after this operation since the distribution of \( X_k \) is the same as \( X_{k+1} \). Now, suppose that some child of \( u \), say \( v \), is labeled \( k \to m \) with \( m \neq j \). Since at node \( u \) bin \( j \) still has usage not exceeding 1 and risk budget left, at node \( v \) bin \( j \) still satisfies this condition. Therefore, by induction, we can relabel \( v \) with \( k + 1 \to j \) without increasing the cost. We can repeat this for any children of \( u \) until all of its children have been labeled \( k + 1 \to j \). We conclude by swapping the label of \( u \) with the label of its children as in the previous case. \( \square \)

**Theorem 4.4.** For \( \gamma = 1 \) and i.i.d. nonnegative random variables \( X_1, \ldots, X_n \), we have

\[
\text{cost}(\text{ALG}) \leq 8 \text{cost}(\text{OPT}).
\]

**Proof.** If \( P(X_1 > 1) > 1/C \), cost(\text{ALG}) = \( n(1 + C P(X_1 > 1)) \). On the other hand, \( \text{cost}(\text{OPT}) \geq nC P(X_1 > 1) \) since each item incurs at least this expected cost. Therefore, \( \text{cost}(\text{ALG}) \leq 2 \text{cost}(\text{OPT}) \).

Now assume \( P(X_1 > 1) \leq 1/C \). Let \( \gamma = C/\delta \). Using Theorem 4.3 and Lemma 4.3, we have

\[
\text{cost}(\text{ALG}) \leq (1 + \gamma) \mathbb{E}[N_{\text{ALG}}] \leq (1 + \gamma) \mathbb{E}[N_{\text{OPT}}] \leq 2(1 + \gamma) \left(1 + \frac{\delta}{C}\right) \text{cost}(\text{OPT})
\]

where \( \text{OPT}^* \) is the budgeted version of \( \text{OPT} \) with risk budget \( \gamma/C \) in each bin. Now,

\[
(1 + \gamma) \left(1 + \frac{\delta}{C}\right) = \left(1 + \frac{C}{\delta}\right) \left(1 + \frac{\delta}{C}\right) = 2 + \frac{\delta}{C} + \frac{C}{\delta}
\]

which is minimized at \( \delta = C \) with a value of 4 and with \( \gamma = 1 \). \( \square \)

## 5 Exponential Random Variables

In this section, we show that Budgeted Greedy incurs an expected cost at most \( O(\log C) \) times the optimal expected cost, when the item sizes are exponentially distributed. That is, for any \( X_i \) in the input sequence,

\[
P(X_i > x) = e^{-\lambda_i x}, \tag{1}
\]

for any \( x \geq 0 \), where \( \lambda_i > 0 \) is the rate. Recall that \( \mathbb{E}[X_i] = 1/\lambda_i \).

The proof is divided into two parts: First, as in the deterministic bin packing problem, we show that \( \mathbb{E}\left\{ \sum_{i=1}^n \min\{X_i, 1\} \right\} \) is a lower bound for \( \mathbb{E}[N_{\text{OPT}}] \). In the next step we show that the probability that Algorithm 1 opens bin \( k \geq 2 \) is related to the amount of mass packed into bin \( k - 1 \). To open bin \( k \geq 2 \), Algorithm 1 must have packed at least \( O(1/ \log C) \) mass in bin \( k - 1 \). Moreover,
when the exponential random variables are sufficiently small, \( \lambda_i \geq 2 \log C \), we can guarantee that the amount of mass packed into bin \( k - 1 \) is at least a constant, therefore improving the result to a constant factor of the optimal cost. We show that our analysis of Algorithm 1 for exponential random variables is almost tight by exhibiting an input sequence that forces Budgeted Greedy to incur a cost \( \Omega(\sqrt{\log C}) \) times the optimal cost.

In this section, we use the notation \( z(B) = \sum_{i \in B} z_i \) for a vector \( z = (z_1, \ldots, z_n) \). If \( X = (X_1, \ldots, X_n) \) is the vector of random variables and \( B = B_j \), then \( X(B) = S_j^n \) is the usage of bin \( B_j \).

The following proposition is in some sense the probabilistic successor of the well-known lower bound of sizes for the standard bin packing.

**Proposition 5.1.** For any sequence of nonnegative i.i.d. random variables \( X_1, \ldots, X_n \), for any bin \( B = B_j \) and any policy \( \mathcal{P} \), we have

\[
E\left[ \sum_{i \in B} E[X_i \wedge 1] \right] \leq 2 P(\mathcal{P} \text{ opens bin } B),
\]

where \( X_i \wedge 1 = \min\{X_i, 1\} \).

**Proof.** The proof follows from a result in [17], which we replicate here for completeness. Let \( \mu_i = E[X_i \wedge 1] \) be the normalized expected size of an item. Let \( B^t \) be the (random) items that the policy packs into bin \( B \) by time \( t \). We are interested in the expectation of \( \mu(B) = \sum_{i \in B^t} \mu_i = \mu(B^n) \). The random variables \( \mu(B^t) \) are nondecreasing in \( t \); by the monotone convergence theorem,

\[
E[\mu(B)] = \sup_{t \geq 0} E[\mu(B^t)].
\]

Now, the random variables \( Z^t = \sum_{i \in B^t} X_i \wedge 1 - \mu_i \) form a martingale. Indeed,

\[
E[Z^t | Z^{t-1}, t \to j] = Z^{t-1} + E[X_t \wedge 1] - \mu_t = Z^{t-1};
\]

then, for all \( t \), \( E[Z^t] = Z^0 = 0 \) and so \( E[\mu(B^t)] = E[\sum_{i \in B^t} X_i \wedge 1] \leq 2 P(\mathcal{P} \text{ opens bin } B) \); this last inequality holds because we break the bin at most once and we must have opened the bin. Therefore

\[
E[\mu(B)] = \sup_{t \geq 0} E[\mu(B^t)] \leq 2 P(\mathcal{P} \text{ opens bin } B). \quad \square
\]

### 5.1 A \( O(\log C) \) Multiplicative Factor for Arbitrary Exponential R.V.’s

Here we show that \( \text{cost}(\text{ALG}) \leq O(\log C) \text{cost}(\text{OPT}) \) whenever the input is an arbitrary sequence of exponential random variables. By using Proposition 3.1, we can assume \( P(X_i > 1) \leq 1/C \) for all \( i = 1, \ldots, n \) at the expense of an extra multiplicative loss of 2 in the cost incurred. This assumption translates into \( \lambda_i \geq \log C \) for all \( i \).

The next result shows that the probability that Algorithm 1 opens a bin, besides the first bin, is related to the amount of mass packed in the previous bin.

**Proposition 5.2.** Suppose \( \gamma = 2 \). Then, for any \( k \geq 2 \), Budgeted Greedy guarantees

\[
P(\text{ALG opens bin } B_k) \leq 5 \log C E \left[ \sum_{i \in B_{k-1}} X_i \wedge 1 \right] + \left( \frac{1}{3} + 20 \sqrt{\frac{\log C}{C}} \right) P(\text{ALG opens bin } B_{k-1}).
\]
Proof. Since the item sizes are continuous random variables, we have $P(\text{ALG opens bin } B_k) = P(X(B_k) > 0)$. Now, we have

$$P(\text{ALG opens bin } B_k) \leq P \left( X(B_{k-1}) > \frac{1}{5 \log C} \right) + P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, X(B_k) > 0 \right).$$

We bound each term separately. To bound the first term we use Markov’s inequality:

$$P \left( X(B_{k-1}) > \frac{1}{5 \log C} \right) = P \left( X(B_{k-1}) \wedge 1 > \frac{1}{5 \log C} \right) \leq 5 \log C E \left[ X(B_{k-1}) \wedge 1 \right].$$

For the second term, we proceed as follows. Let $E$ be the event all items packed in $B_{k-1}$ have rate $\lambda_i \geq 2 \log C$. Then

$$P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, X(B_k) > 0 \right) \leq P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, X(B_k) > 0, E \right)$$

$$+ P \left( X(B_{k-1}) \leq \frac{1}{5 \log C} | \overline{E} \right) P(X(B_{k-1}) > 0),$$

since $\overline{E}$, the event that some item in $B_{k-1}$ has rate $\leq 2 \log C$, is contained in the event Algorithm[1] opens bin $B_{k-1}$.

Claim 2. We have $P \left( X(B_{k-1}) \leq \frac{1}{5 \log C} | \overline{E} \right) \leq 1 - e^{-2/5} \leq 1/3$

Proof. If $X_i, \ldots, X_{i_m}$ are all the large items with rates $\lambda_{i_p} \leq 2 \log C$, then, the events

$$M_p = \{ X_{i_p} \text{ is the first large item packed into } B_{k-1} \}$$

satisfy $\overline{E} = \bigcup_{p=1}^{m} M_p$. Then,

$$P \left( X(B_{k-1}) \leq \frac{1}{5 \log C} | \overline{E} \right) = \sum_{p=1}^{m} P \left( X(B_{k-1}) \leq \frac{1}{5 \log C} | \overline{E}, M_p \right) P(M_p | \overline{E})$$

$$\leq \sum_{p=1}^{m} P \left( X_{i_p} \leq \frac{1}{5 \log C} \right) P(M_p | \overline{E}),$$

since the outcome is observed after packing the item in the bin. The rest of the proof follows from simple calculations using [1].

Claim 3. $P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, X(B_k) > 0, E \right) \leq 20 \sqrt{\log C \log \frac{C}{\log C}} P(X(B_{k-1}) > 0)$

Proof. In this case, bin $B_k$ has been opened even though $B_{k-1}$ still has available space. That means that the element that opens bin $B_k$ surpasses the budget of $B_{k-1}$. From here we obtain,

$$\frac{2}{C} \leq \text{Risk}(B_{k-1}) + P(X_i > 1 - X(B_{k-1})) \leq \text{Risk}(B_{k-1}) + \frac{e^{1/5}}{C},$$
where $X_t$ is the first item packed into $B_k$, we used the fact that $\lambda_t \geq \log C$ and formula (1). Let $F_\beta$ be the event $\{\sum_{i \in B_{k-1}} E[X_i] > \beta\}$; by Markov’s inequality,

$$
P(F_\beta) \leq \frac{1}{\beta} \mathbb{E}\left[ \sum_{i \in B_{k-1}} E[X_i] \right]
$$

$$
\leq \frac{C}{C - 1} \frac{1}{\beta} \mathbb{E}\left[ \sum_{i \in B_{k-1}} E[X_i \wedge 1] \right] \quad (E[X_i \wedge 1] = (1 - e^{-\lambda_i}) E[X_i])
$$

$$
\leq 2 \frac{C}{C - 1} \frac{1}{\beta} \mathbb{P}(X(B_{k-1}) > 0).
$$

Thus,

$$
P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, X(B_k) > 0, E \right) \leq P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, \text{Risk}(B_{k-1}) > 2 - e^{1/5}, E \right)
$$

$$
\leq \frac{2C}{\beta(C - 1)} P(X(B_{k-1}) > 0)
$$

$$
+ P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, \text{Risk}(B_{k-1}) > 2 - e^{1/5}, F_\beta, E \right)
$$

$$
\leq \frac{2C}{\beta(C - 1)} P(X(B_{k-1}) > 0)
$$

$$
+ \frac{C}{2 - e^{1/5}} \mathbb{E} \left[ \text{Risk}(B_{k-1}) \mid X(B_{k-1}) \leq \frac{1}{5 \log C}, E, \overline{F_\beta} \right].
$$

**Claim 4.** $\mathbb{E} \left[ \text{Risk}(B_{k-1}) \mid X(B_{k-1}) \leq \frac{1}{5 \log C}, E, \overline{F_\beta} \right] \leq 10 \beta \log C C \mathbb{P}(X(B_{k-1}) > 0)$.

**Proof.** Given $X(B_{k-1}) \leq \frac{1}{5 \log C}$, the event $E$, the event $\overline{F_\beta}$ and the event $X(B_{k-1}) > 0$ the value

$$
\text{Risk}(B_{k-1}) = \sum_{i=1}^{n} P(X_i > S_{k-1}^{-1} > 1) \mathbb{1}_{\{i \rightarrow k-1\}} \leq \sum_{i=1}^{n} e^{-\lambda_i (1 - 1/5 \log C)} \mathbb{1}_{\{i \rightarrow k-1\}}
$$

can be upper bounded by the non-convex problem:

$$
\max \left\{ \sum_{i=1}^{n} e^{-x_i (1 - 1/5 \log C)} : \sum_{i=1}^{n} 1/x_i \leq \beta, x_i \geq 2 \log C \right\} \leq 10 \beta \log C C^2,
$$

from which the result follows.

Therefore,

$$
P \left( X(B_{k-1}) \leq \frac{1}{5 \log C}, X(B_k) > 0, E \right) \leq \left( \frac{2C}{\beta(C - 1)} + 10 \beta \log C C \right) \mathbb{P}(X(B_{k-1}) > 0).
$$

the RHS is minimized at $\beta = \sqrt{\frac{C}{5 \log C}}$.

Putting Claims 2 and 3 together we obtain

$$
P(X(B_k) > 0) \leq 5 \log C \mathbb{E} \left[ \sum_{i \in B_{k-1}} X_i \wedge 1 \right] + \left( \frac{1}{3} + 20 \sqrt{\frac{\log C}{C}} \right) \mathbb{P}(X(B_{k-1}) > 0). \quad \Box$$
Proposition 5.3. Let \( X_1, \ldots, X_n \) be arbitrary exponential random variables with \( \lambda_i \geq \log C \). Algorithm with \( \gamma = 2 \) guarantees

\[
\frac{1}{12} \left( \frac{2}{3} - 20 \sqrt{\frac{\log C}{C}} \right) \text{cost}(\text{ALG}) \leq (5 \log C) \text{cost}(\text{OPT}),
\]

where OPT is the optimal policy that knows \( n \) and the rates of all the sizes \( X_1, \ldots, X_n \) in advance.

Proof. For any policy \( \mathcal{P} \), we have the bound:

\[
\mathbb{E}[N_{\mathcal{P}}] = \sum_{k=1}^{n} \mathbb{P}(\mathcal{P} \text{ opens bin } k)
\]

\[
\geq 1 + \sum_{k=2}^{n} \frac{1}{2} \mathbb{E} \left[ \sum_{i \in B_{k}^{\mathcal{P}}} \mathbb{E}[X_i \wedge 1] \right]
\]

\[
\geq \frac{1}{4} \left( 1 + \sum_{i=1}^{n} \mathbb{E} \left[ \sum_{i \in B_{i}^{\mathcal{P}}} \mathbb{E}[X_i \wedge 1] \right] \right)
\]

\[
= \frac{1}{4} \left( 1 + \mathbb{E} \left[ \sum_{i=1}^{n} X_i \wedge 1 \right] \right),
\]

where we used Proposition 5.1 and where \( B_{k}^{\mathcal{P}} \) denotes the items packed into bin \( k \) by policy \( \mathcal{P} \). We also used the fact that \( 1 = \mathbb{P}(\mathcal{P} \text{ opens bin } 1) = \mathbb{P}(X(B_{1}^{\mathcal{P}}) > 0) \). On the other hand, we have

\[
5 \log C \mathbb{E} \left[ \sum_{i=1}^{n} \mathbb{E}[X_i \wedge 1] \right] = 5 \log C \sum_{k=1}^{n} \mathbb{E} \left[ \sum_{i \in B_{k}^{\text{ALG}}} \mathbb{E}[X_i \wedge 1] \right]
\]

\[
\geq \sum_{k=2}^{n} \mathbb{P}(X(B_{k}^{\text{ALG}}) > 0) - \left( \frac{1}{3} + 20 \sqrt{\frac{\log C}{C}} \right) \mathbb{P}(X(B_{k-1}^{\text{ALG}}) > 0)
\]

\[
\geq \left( \frac{2}{3} - 20 \sqrt{\frac{\log C}{C}} \right) \sum_{k=2}^{n} \mathbb{P}(X(B_{k}^{\text{ALG}}) > 0)
\]

\[
- \left( \frac{1}{3} + 20 \sqrt{\frac{\log C}{C}} \right) \mathbb{P}(X(B_{1}^{\text{ALG}}) > 0).
\]

From here

\[
5 \log C \text{cost}(\mathcal{P}) \geq \frac{5 \log C}{4} \left( 1 + \mathbb{E} \left[ \sum_{i=1}^{n} X_i \wedge 1 \right] \right)
\]

\[
\geq \frac{1}{4} \left( \frac{2}{3} - 20 \sqrt{\frac{\log C}{C}} \right) \sum_{k=1}^{n} \mathbb{P}(X(B_{k}^{\text{ALG}}) > 0)
\]

\[
= \frac{1}{4} \left( \frac{2}{3} - 20 \sqrt{\frac{\log C}{C}} \right) \mathbb{E}[N_{\text{ALG}}],
\]

where we used \( 1 = \mathbb{P}(\text{ALG opens bin } 1) \). The conclusion follows from here using \( \text{cost}(\text{ALG}) \leq 3 \mathbb{E}[N_{\text{ALG}}] \). \qed
5.2 A $\mathcal{O}(1)$ Multiplicative Factor for Small Arbitrary Exponential R.V.’s

We next show that $\text{cost(ALG)} \leq \mathcal{O}(1) \leq \text{cost(OPT)}$ whenever the item sizes are independent exponential random variables with rates satisfying $\lambda_i \geq 2 \log C$. In this case, $\mathbb{E}[\sum_i X_i \wedge 1]$ is a better approximation for $\mathbb{E}[N_{\text{ALG}}]$ than in the general case. The following results shows that we can improve Proposition 5.2 by a logarithmic factor.

**Proposition 5.4.** Let $\gamma = 1$. For $k \geq 1$,

$$\Pr(\text{ALG opens bin } k + 1) \leq 4 \mathbb{E} \left[ \sum_{i \in B_k} X_i \wedge 1 \right] + 8 \frac{\sqrt{\log C}}{C^{1/4}} \Pr(\text{ALG opens bin } k).$$

**Proof.** We have

$$\Pr(\text{ALG opens bin } B_{k+1}) = \Pr(X(B_{k+1}) > 0) \leq \Pr(X(B_k) > 1/4) + \Pr(X(B_{k+1}) > 0, X(B_k) \leq 1/4) \leq 4 \mathbb{E} [X(B_k) \wedge 1] + \Pr(X(B_{k+1}) > 0, X(B_k) \leq 1/4).$$

We only focus on bounding the second term in the rest of the proof. Algorithm 1 opens bin $B_{k+1}$ ($X(B_{k+1}) > 0$) if there are no available bins ($\forall i \leq k, X(B_i) \geq 1$) or there is an item that does not fit because of the budget. The first case cannot happen when the event $X(B_k) \leq 1/4$ happens so we are only left with the budget case. In particular, for bin $k$, we open bin $B_{k+1}$ because for some item $X_i$ we have

$$\frac{1}{C} < \text{Risk}(B_k) + \Pr(X_t + X(B_k) > 1) \leq \text{Risk}(B_k) + \Pr(X_t > 3/4) \leq \text{Risk}(B_k) + \frac{1}{C^{3/2}}$$

where we used the information from the event $X(B_{k+1}) \leq 1/4$. Therefore,

$$\Pr(X(B_{k+1}) > 0, X(B_k) \leq 1/4) \leq \Pr(\text{Risk}(B_k) > 1/C - 1/C^{3/2}, 0 < X(B_k) < 1/4) \leq \frac{C^{3/2}}{C^{1/2} - 1} \mathbb{E}[\text{Risk}(B_k) \mid X(B_k) < 1/4, X(B_k) > 0] \Pr(X(B_k) > 0).$$

Now, as in the previous proof, let $F_\beta = \{\sum_{i \in B_k} X_i > \beta\}$; by Markov’s inequality and Proposition 5.1

$$\Pr(F_\beta) \leq \frac{2C^2}{\beta(C^2 - 1)} \Pr(X(B_k) > 0).$$

**Claim 5.** $\mathbb{E}[\text{Risk}(B_k) 1_{\{X(B_k) < 1/4\}} \mid X(B_k) < 1/4, X(B_k) > 0, \nabla_\beta] \leq \frac{2\beta \log C}{C^{3/2}}$.

**Proof.** Given $X(B_k) < 1/4, X(B_k) > 0, \nabla_\beta$, the risk

$$\text{Risk}(B_k) = \sum_{i=1}^n \Pr(X_i + S_k^{i-1} > 1) 1_{\{i \rightarrow k\}} \leq \sum_{i=1}^n e^{-\lambda_i \cdot \frac{3}{2}} 1_{\{i \rightarrow k\}}$$

is bounded by the non-convex problem:

$$\max \left\{ \sum_i e^{-x_i \beta} : \sum_i \frac{1}{x_i} \leq \beta, x_i \geq 2 \log C \right\} \leq \frac{2\beta \log C}{C^{3/2}}.$$
With this claim,

\[ E[\text{Risk}(B_k) \mid X(B_k) < 1/4, X(B_k) > 0] \leq \frac{1}{C} P(F_\beta) + \frac{2\beta \log C}{C^{3/2}} P(F_\beta) \]

\[ \leq \frac{2C}{\beta(C^2 - 1)} + \frac{2\beta \log C}{C^{3/2}}, \]

since \( \text{Risk}(B_k) \leq \frac{1}{C} \) always. Thus,

\[ P(X(B_{k+1}) > 0, X(B_k) \leq 1/4) \leq \frac{C^{3/2}}{C^{1/2} - 1} \left( \frac{2C}{\beta(C^2 - 1)} + \frac{2\beta \log C}{C^{3/2}} \right) P(X(B_k) > 0). \]

Now, optimizing over \( \beta \) with \( \beta = \frac{C^{3/4}}{\sqrt{C^2 - 1} \log C} \) we obtain

\[ P(X(B_{k+1}) > 0, X(B_k) \leq 1/4) \leq 8 \frac{\sqrt{\log C}}{C^{1/4}} P(X(B_k) > 0). \]

**Proposition 5.5.** For \( \gamma = 1 \), Algorithm [1] guarantees \( \frac{1}{6} \left( 1 - 8 \frac{\sqrt{\log C}}{C^{1/4}} \right) \text{cost(ALG)} \leq \text{cost(OPT)} \), where OPT is the optimal policy that knows \( n \) and all item sizes rates in advance.

**Proof.** Adding the result in Proposition 5.4 for \( k = 1, \ldots, n - 1 \), we obtain

\[ \sum_{k=1}^{n-1} P(\text{ALG opens bin } k + 1) \leq 4 \sum_{k=1}^{n-1} E \left[ \sum_{i \in B_k} X_i \wedge 1 \right] + 8 \frac{\sqrt{\log C}}{C^{1/4}} \sum_{k=1}^{n-1} P(\text{ALG opens bin } k) \]

\[ = \sum_{k=1}^{n-1} E \left[ \sum_{i \in B_k} E[X_i \wedge 1] \right] + 8 \frac{\sqrt{\log C}}{C^{1/4}} \sum_{k=1}^{n-1} P(\text{ALG opens bin } k) \]

\[ = \sum_{k=1}^{n} E \left[ \sum_{i \in B_k} E[X_i \wedge 1] \right] + 8 \frac{\sqrt{\log C}}{C^{1/4}} \sum_{k=1}^{n-1} P(\text{ALG opens bin } k) \]

\[ = E \left[ \sum_{i} X_i \wedge 1 \right] + 8 \frac{\sqrt{\log C}}{C^{1/4}} \sum_{k=1}^{n-1} P(\text{ALG opens bin } k). \]

Then, for any policy \( P \),

\[ \text{cost}(P) \geq \frac{1}{2} E \left[ \sum_{i} X_i \wedge 1 \right] + P(P \text{ opens bin 1}) \]

\[ \geq \frac{1}{2} \left( \sum_{k=1}^{n-1} P(\text{ALG opens bin } k) - 8 \frac{\sqrt{\log C}}{C^{1/4}} \sum_{k=1}^{n-1} P(\text{ALG opens bin } k) \right) + 1 \]

\[ \geq \frac{1}{2} \left( 1 - 8 \frac{\sqrt{\log C}}{C^{1/4}} \right) \sum_{k=1}^{n} P(\text{ALG opens bin } k) \]

\[ \geq \frac{1}{2} \left( 1 - 8 \frac{\sqrt{\log C}}{C^{1/4}} \right) E[N_{\text{ALG}}] \]

\[ \geq \frac{1}{6} \left( 1 - 8 \frac{\sqrt{\log C}}{C^{1/4}} \right) \text{cost(ALG)}. \]

We used the fact that for any policy \( P(\text{ALG opens bin 1}) = 1 = P(P \text{ opens bin 1}). \) \qed
5.3 \( \Omega(\sqrt{\log C}) \) Lower Bound for Algorithm 1 with Exponential R.V.’s

In this subsection, we present a hard input of exponential random variables for Budgeted Greedy. The sequence contains two kind of independent exponential random variables, those with rates \( \mu = \beta \log C \), \( \beta \geq 2 \) and those with rates \( \lambda = (1 + \varepsilon) \log C \), with \( \varepsilon \in (0, 1) \). This sequence has \( n_1 \) items with rate \( \lambda \) and \( n_2 = kn_1 \) items with rate \( \mu \), presented to Algorithm 1 as,

\[ X_{\mu_1,1} \cdots X_{\mu_1,k} X_{\lambda_1,1} \cdots X_{\lambda_1,k} X_{\mu_2,1} \cdots X_{\mu_n,1} \cdots X_{\mu_n,n_1} X_{\lambda_n,1} \cdots X_{\lambda_n,n_1}, \]

where \( X_{\mu_i,j} \sim \exp(\mu) \) and \( X_{\lambda_i,j} \sim \exp(\lambda) \) for all \( i, j \). With the choices of \( \beta = 6 \frac{n_1 \log C}{\varepsilon} \) and \( k = 3\varepsilon \mu = 18n_1 (\log C)^2 \), we show that Budgeted Greedy incurs an expected cost of at least \( \frac{1}{2} n_1 \). For the same choices of \( \beta \) and \( k \) and optimizing over the choice of \( \varepsilon \), we show that \( \text{cost(OPT)} \leq O \left( \frac{1}{\sqrt{\log C}} \right) n_1 \). This choice of \( \varepsilon \) is independent of \( n_1 \), which allows us to scale the result for any input size.

We prove each bound separately; the main results are stated here.

**Proposition 5.6.** Let \( \varepsilon > 0 \) and set \( \beta = 6 \frac{n_1 \log C}{\varepsilon} \) and \( k = 3\varepsilon \mu \). Then, running Budgeted Greedy with \( \gamma = 1 \) on the input described above yields

\[ \text{cost(ALG)} \geq \frac{1}{2} n_1. \]

**Proposition 5.7.** Using the same parameters as in the previous Proposition, for any \( \varepsilon > 0 \) such that \( \varepsilon \log C \geq 4 \), we have

\[ \text{cost(OPT)} \leq 48n_1 \left( \frac{k}{\beta \log C} + \frac{1}{\varepsilon \log C} \right) = 48n_1 \left( 3\varepsilon + \frac{1}{\varepsilon \log C} \right). \]

The result now follows by taking \( \varepsilon = \sqrt{\frac{1}{3 \log C}} \). The proofs are in Appendix A.
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A Appendix

A.1 Missing Proofs

Here we present the missing proofs form the main article.

A.1.1 Missing Proofs From Section 3

**Proposition 3.1** Let $X_1, \ldots, X_n$ be nonnegative independent random variables and any (deterministic) policy $P$ for packing these items sequentially. Then, the expected number of broken bins by the policy $P$ is given by

$$E[B_P] = \sum_{j=1}^{n} E \left[ \sum_{i=1}^{n} P_{X_i}(X_i + S^{i-1}_j > 1)1^P_{\{i \rightarrow j\}} \right]$$

where $S^{i-1}_j$ is the level of bin $j$ at the beginning of iteration $i$ and $1^P_{\{i \rightarrow j\}}$ is the 0/1 indicator random variable of the event: Policy $P$ packs item $X_i$ into bin $j$.

*Proof.* We just need to show that $P(P$ breaks bin $j) = E \left[ \sum_{i=1}^{n} P_{X_i}(X_i + S^{i-1}_j > 1)1^P_{\{i \rightarrow j\}} \right]$.

We have,

$$E \left[ \sum_{i=1}^{n} P_{X_i}(X_i + S^{i-1}_j > 1)1^P_{\{i \rightarrow j\}} \right] = \sum_{i=1}^{n} E \left[ P_{X_i}(X_i + S^{i-1}_j > 1)1^P_{\{i \rightarrow j\}} \right]$$

Observe that $S^{i-1}_j$ only depends on the outcomes of $X_1, \ldots, X_{i-1}$, therefore

$$E \left[ P_{X_i}(X_i + S^{i-1}_j > 1)1^P_{\{i \rightarrow j\}} \right] = E \left[ P_{X_i}(X_i + S_j > 1) \mathbb{1}_{\{i \rightarrow j\}} \right] = P(X_i \text{ breaks bin } j \mid i \rightarrow j) P(i \rightarrow j) = P(X_i \text{ breaks bin } j, i \rightarrow j) = P(X_i \text{ breaks bin } j),$$

thus,

$$E \left[ \sum_{i=1}^{n} P_{X_i}(X_i + S^{i-1}_j > 1)1^P_{\{i \rightarrow j\}} \right] = \sum_{i=1}^{n} P(X_i \text{ breaks bin } j) = P(P$ breaks bin $j)$$

since the last sum uses the fact that the bin is overflowed at most once and therefore the events $\{X_i \text{ breaks bin } j\}; i$ are disjoint. \qed
Lemma 4.2. \( \text{cost}_{\ell,\hat{c}}(T_P(u)) \geq \text{cost}_{\ell',\hat{c}'}(T_{P'}(u)) \).

Proof. We define

\[
\text{cost}_{\ell,\hat{c}}(T_P(u))_j = 1 + \mathbb{E} \left[ (C + \delta) \sum_{i=1}^{n} 1^P_{i-j} 1 \{X_i + S_i^{j-1} > 1\} \mid \text{Reach node } u \right]
\]

which is the original cost paid in \( T_P \) when packing items into bin \( j \) after reaching node \( u \) in the tree. We also define

\[
\text{cost}_{\ell',\hat{c}'}(T_{P'}(u))_j = 1 + \mathbb{E} \left[ C \sum_{i=1}^{n} 1^{P'}_{i-j} 1 \{X_i + S_i^{j-1} > 1\} + (C + \delta) \sum_{i=1}^{n} 1^{P'}_{i-j} 1 \{X_i + S_i^{j-1} > 1\} + 1 \{\text{Open bin } j'\} \mid \text{Reach node } u \right]
\]

which is the new cost paid by \( T_{P'} \) when packing items into bin \( j \) after reaching node \( u \) and the new cost incurred by packing items into bin \( j' \).

Therefore, the variation of the cost \( \text{cost}_{\ell,\hat{c}}(T_P(u)) - \text{cost}_{\ell',\hat{c}'}(T_{P'}(u)) \) is given by

\[
\text{cost}_{\ell,\hat{c}}(T_P(u)) - \text{cost}_{\ell',\hat{c}'}(T_{P'}(u)) = (\text{cost}_{\ell,\hat{c}}(T_P(u))_j - \text{cost}_{\ell',\hat{c}'}(T_{P'}(u))_j).
\]

Now, we always have

\[
1^P_{i-j} = 1^{P'}_{i-j} + 1^{P'}_{i-j'},
\]

for all \( i = 1, \ldots, n \). Indeed, if we are in a branch not containing \( u \), then \( P \) and \( P' \) behave the same and there is no bin \( j' \). If we are in a branch containing \( u \), and if we pack \( i \) into \( j \), we either pack \( i \) into \( j \) before surpassing the risk budget in which case \( P \) and \( P' \) behave the same or we do it after surpassing the risk budget in which case \( i \) goes to \( j' \). With this fact we have,

\[
\text{cost}_{\ell,\hat{c}}(T_P(u)) - \text{cost}_{\ell',\hat{c}'}(T_{P'}(u)) = (\text{cost}_{\ell,\hat{c}}(T_P(u))_j - \text{cost}_{\ell',\hat{c}'}(T_{P'}(u))_j)
\]

\[
\geq \mathbb{E} \left[ \delta \sum_{i=1}^{n} 1^{P'}_{i-j} 1 \{X_i + S_i^{j-1} > 1\} - 1 \{\text{Open bin } j'\} \mid \text{Reach node } u \right],
\]

in the last inequality we used the fact that the cost of breaking the bin \( j' \) is smaller than the cost of breaking \( j \) at that point of the computation. This is true since the usage of bin \( j' \) is at most the usage of \( j \) at the same point of computation. Now,

\[
\mathbb{E} \left[ 1^{P'}_{i-j} 1 \{X_i + S_i^{j-1} > 1\} \mid \text{Reach node } u \right] = \mathbb{E}_{X_i, X_{i-1}} \left[ \mathbb{E}_{X_i} \left[ 1^{P'}_{i-j} 1 \{X_i + S_i^{j-1} > 1\} \mid \text{Reach node } u \right] \right] 
\]

\[
= \mathbb{E}_{X_i, X_{i-1}} \left[ 1^{P'}_{i-j} \mathbb{E}_{X_i} \left[ 1 \{X_i + S_i^{j-1} > 1\} \mid \text{Reach node } u \right] \right] 
\]

\[
= \mathbb{E}_{X_i, X_{i-1}} \left[ 1^{P'}_{i-j} \mathbb{P}(X_i + S_i^{j-1} > 1) \mid \text{Reach node } u \right].
\]
Thus,

\[
\text{cost}_{\ell, \epsilon}(T_P(u)) - \text{cost}_{\ell, \epsilon'}(T_{P'}(u)) = \mathbb{E} \left[ \sum_{i=1}^{n} \mathbf{1}_{|i-j|} P(X_i + S_{j-1} > 1) | \text{Reach node } u \right] \\
\geq \left( \frac{\gamma}{C} - 1 \right) P(\text{Open bin } j' | \text{Reach node } u) \\
\geq 0. 
\] (Using $\gamma \geq \frac{C}{\epsilon}$.)

\[\square\]

A.1.3 Missing Proofs From Section 5

**Proposition 5.6.** Let $\epsilon > 0$ and set $\beta = 6^{n_1 \log C}$ and $k = 3\epsilon \mu$. Then, running Budgeted Greedy with $\gamma = 1$ in the input described in Subsection 5.3 we have

\[
\text{cost(ALG)} \geq \frac{1}{2} n_1. 
\]

**Proof.** First, we argue that $\text{cost(ALG)} \geq \frac{1}{2} n_1$. We show that (w.h.p.) Algorithm 1 packs each item $X_{j, i}^\mu$ individually. This is achieved by showing that in between two $X_{j, i}^\lambda$ and $X_{j+1, i}^\mu$, there is enough mass introduced by the elements $X_{j, i}^\mu$. Therefore, not allowing the items $X_{j, i}^\lambda$ to be packed together.

Now, let $k = 3\epsilon \mu = 3\beta \log C$. Then,

\[
\mathbb{E} \left[ \sum_{i=1}^{k} X_{j, i}^\mu \right] = \frac{k}{\beta \log C} = 3\epsilon,
\]

thus

\[
\mathbb{P} \left( \sum_{i=1}^{k} X_{j, i}^\mu \leq 2\epsilon \right) \leq \mathbb{P} \left( \left| \sum_{i=1}^{k} X_{j, i}^\mu - 3\epsilon \right| \geq \epsilon \right) \leq \frac{1}{\epsilon^2 (\beta \log C)^2} \leq \frac{3}{\epsilon \beta \log C}. 
\]

Chebyshev inequality

Pick $\beta = 3^{2n_1 \log C}$ and so,

\[
\mathbb{P} \left( \exists j = 1, \ldots, n_1 : \sum_{i=1}^{k} X_{j, i}^\mu \leq 2\epsilon \right) \leq n_1 \cdot \frac{3}{\epsilon \beta \log C} = \frac{1}{2}.
\]

That is, with probability at least $\frac{1}{2}$, all blocks $X_{j, 1}^\mu, \ldots, X_{j, k}^\mu$ add at least $2\epsilon$ mass. Consider the event

\[
E = \left\{ \forall j = 1, \ldots, n_1 : \sum_{i=1}^{k} X_{j, i}^\mu > 2\epsilon \right\}.
\]

then, we just proved that $\mathbb{P}(E) \geq \frac{1}{2}$.
Claim 6. Given event $E$, Algorithm $[\text{I}]$ with $\gamma = 1$ never packs $X_i^\lambda$ and $X_{i+1}^\lambda$ together for any $i$.

Proof. Suppose that Algorithm $[\text{I}]$ packs $X_i^\lambda$ and $X_{i+1}^\lambda$ together for some $i$. This means that the algorithm had enough budget and space to allocate $X_i^\lambda$ and $X_{i+1}^\lambda$. Since $P(X_{i+1}^\lambda > 1 - x) = e^{-\mu(1-x)} > e^{-\mu(1-x)} = P(X_{ij}^\mu > 1 - x)$ for any $x > 0$, then Budgeted Greedy must have packed all the $X_{ij}^\mu$ in between $X_i^\lambda$ and $X_{i+1}^\lambda$. However, under event $E$, these items increase the usage of the bin by at least $2\epsilon$. Then, the budget utilized by $X_{i+1}^\lambda$ is at least

$$P(X_{i+1}^\lambda > 1 - 2\epsilon) = e^{-\lambda(1-2\epsilon)} > e^{-(1-\epsilon^2)\log C} > \frac{1}{C}$$

which is a contradiction to the risk budget of Budgeted Greedy. \hfill \Box

Claim 7. Using the same choices of $\beta$ and $k$ as before, we have $\text{cost(Alg)} \geq \frac{1}{2} n_1$.

Proof. By the previous result, under event $E$, no $X_i^\lambda$ and $X_{i+1}^\lambda$ are packed together. Therefore, at least $n_1$ open bins are needed. Since $E$ occurs w.p. $\geq \frac{1}{2}$ we conclude the desired result. \hfill \Box

Proposition 5.7. Using the same parameters and the same input as in the previous result, for any $\epsilon > 0$ such that $\epsilon \log C \geq 4$, we have

$$\text{cost(Opt)} \leq 48n_1 \left( \frac{k}{\beta \log C} + \frac{1}{\epsilon \log C} \right) = 48n_1 \left( 3\epsilon + \frac{1}{\epsilon \log C} \right).$$

Proof. In order to show an upper bound for $\text{cost(Opt)}$ it is enough to exhibit a policy with cost bounded by the desired value. We consider the following budgeted policy $\mathcal{P}$ with risk budget $\frac{2}{3}$: Pack items with rate $\lambda$ separately of items with rate $\mu$. We are going to show that $\mathcal{P}$ opens at most

$$\frac{n_2}{\beta \log C} + \frac{n_1}{\epsilon \log C} = n_1 \left( \frac{k}{\beta \log C} + \frac{1}{\epsilon \log C} \right)$$

bins in expectation (up to a constant). Since $\mathcal{P}$ is budgeted with budget $\frac{2}{3}$ we have $\text{cost(Opt)} \leq \text{cost(P)} \leq 3\mathbb{E}[N_{\mathcal{P}}]$ from which the result follows. In what follows we prove the bound over the number of bins.

Let us analyze the policy $\mathcal{P}$. Policy $\mathcal{P}$ opens two kind of bins; the first kind of bins only contain items following exponentials distribution of rate $\lambda$; the second kind of bins only contain items following exponential distribution of rate $\mu$. We have $N_{\mathcal{P}} = N_{\mathcal{P}}^1 + N_{\mathcal{P}}^2$ where $N_{\mathcal{P}}^1$ is the number of bins of type 1 and $N_{\mathcal{P}}^2$ is the number of bins of type 2. An equivalent way to see this process is that policy $\mathcal{P}$ runs two copies of Algorithm $[\text{I}]$ one for the rate $\lambda$ and one for the rate $\mu$. Then, $N_{\mathcal{P}}^1$ equals $N_{\text{ALG}}$ over the sequence $X_1^\lambda, \ldots, X_{n_1}^\lambda$ and $N_{\mathcal{P}}^2$ equals $N_{\text{ALG}}$ over the sequence $X_{1,1}^\mu, \ldots, X_{n_1,k}^\mu$.

The following lemma is a general result that allows us to bound the number of bins used in a nonnegative i.i.d. sequence of items under Algorithm $[\text{I}]$. For the sake of clarity, the proof has been moved to the end of this subsection.

Lemma A.1. Suppose $X_1, \ldots, X_n$ are i.i.d. sequence of items, then $\mathbb{E}[N_{\text{ALG}}] \leq \frac{2n-1}{\mathbb{E}[B_1]}$, where $|B_1|$ is the number of items packed in the first bin.
Claim 8. Let $X_1, \ldots, X_n$ be $n$ independent exponential random variables with rate $\lambda = (1 + \varepsilon) \log C$, with $\varepsilon \log C \geq 4$, then
\[ E[|B_1|] \geq \frac{1}{8} \varepsilon \log C, \]
where $|B_1|$ is the number of items $X_1, \ldots, X_n$ packed in the first bin by Algorithm 1 with risk budget $= 2/C$.

Proof. Let $\ell = \frac{4}{\varepsilon} \log C \geq 1$. Then,
\[ P(|B_1| \leq \ell) = P\left(|B_1| \leq \ell, X(B_1) > \frac{\varepsilon}{2(1+\varepsilon)}\right) + P\left(|B_1| \leq \ell, X(B_1) \leq \frac{\varepsilon}{2(1+\varepsilon)}\right). \]

We bound each term separately. First, we have
\[ P\left(\sum_{i=1}^{\ell} X_i > \frac{\varepsilon}{2(1+\varepsilon)}\right) \leq \frac{2(1+\varepsilon)}{\varepsilon} E\left[\sum_{i=1}^{\ell} X_i\right] = \frac{2(1+\varepsilon)}{\varepsilon} \ell \frac{1}{(1+\varepsilon) \log C} = \frac{1}{2}. \]

For the other term we have that $|B_1| \leq \ell$, given $X(B_1) \leq \frac{\varepsilon}{2(1+\varepsilon)}$, only if $B_1$ runs out of budget. That is,
\[ \frac{2}{C} \leq \sum_{i=1}^{\ell+1} P(X_i > 1 - \alpha_i) \]
\[ \leq (\ell + 1) P\left(X_i > 1 - \frac{\varepsilon}{2(1+\varepsilon)}\right) \]
\[ \leq \left(\frac{\varepsilon}{4} \log C + 1\right) \frac{1}{C^{1+\varepsilon/2}} \]
\[ < \frac{2}{C} \]
using the assumption $\varepsilon \log C \geq 2$. From here we obtain that $P(|B_1| \leq \ell, X(B_1) \leq \frac{\varepsilon}{2(1+\varepsilon)}) = 0$. Therefore,
\[ P(|B_1| \leq \ell) \leq \frac{1}{2}. \]

Then,
\[ E[|B_1|] \geq \frac{1}{2} \ell = \frac{1}{8} \varepsilon \log C. \]

Claim 9. Let $X_1, \ldots, X_m$ be $m$ independent exponential random variables with rate $\mu = \beta \log C$, $\beta \geq 4$, then
\[ E[|B_1|] \geq \frac{1}{8} \beta \log C \]
where $|B_1|$ is the number of items $X_1, \ldots, X_m$ packed in the first bin by Algorithm 1 with risk budget $= 2/C$. 

\[ \square \]
Proof. Let $\ell = \frac{\beta}{4} \log C$. Then,
\[
P(|B_1| \leq \ell) = P(|B_1| \leq \ell, X(B_1) > 1/2) + P(|B_1| \leq \ell, X(B_1) \leq 1/2) \
\leq P \left( \sum_{i=1}^{\ell} X_i > \frac{1}{2} \right) + P \left( |B_1| \leq \ell, X(B_1) \leq 1/2 \right)
\]

Now, given the event $X(B_1) \leq \frac{1}{2}$, the only way that $|B_1| \leq \ell$ is by running out of budget. We have then
\[
\frac{2}{C} \leq \sum_{i=1}^{\ell+1} P(X_i > 1 - \alpha_i) \\
\leq (\ell + 1) P(X_i > 1/2) \\
\leq \left( \frac{\beta}{4} \log C + 1 \right) \frac{1}{C^{\beta/2}} \\
< \frac{1}{C}
\]

which cannot happen. Therefore, $P(|B_1| \leq \ell, X(B_1) \leq 1/2) = 0$ and then
\[
P(|B_1| \leq \ell) \leq 2 E \left[ \sum_{i=1}^{\ell} X_i \right] = 2 \ell \frac{1}{\beta \log C} = \frac{1}{2}.
\]

Therefore,
\[
E[|B_1|] \geq \frac{1}{2} \ell = \frac{\beta}{8} \log C.
\]

Remark 1. Both proofs follow the same scheme: Show that the probability of packing few items is upper bounded by some deviation from the capacity used. In the latter proposition, when $\beta \geq 4$, we know that most of the items fit in the bin since the budget consumed by each item is comparatively small. The crucial step is to use the budget to say that using a small fraction of the capacity of the bin is unlikely with few items.

Claim 10. The cost of $\mathcal{P}$ is $\text{cost}(\mathcal{P}) \leq 3 E[N_P] \leq 48n_1 \left( \frac{k}{\beta \log C} + \frac{1}{\epsilon \log C} \right)$

Proof. Putting all the results together we obtain
\[
E[N_P] = E[N_P^1] + E[N_P^2] \\
\leq \frac{2n_1}{E[|B_1^1|]} + \frac{2n_2}{E[|B_1^2|]} \tag{Proposition A.1} \\
\leq 16 \frac{n_1}{\epsilon \log C} + 16 \frac{n_2}{\beta \log C} \\
= 16n_1 \left( \frac{1}{\epsilon \log C} + \frac{k}{\beta \log C} \right).
\]
Here we present the proof of Lemma A.1.

**Proof of Lemma A.1.** We use the following fictitious experiment. Consider \( n \) independent copies of the random variables \( X_1, \ldots, X_n \) and run Algorithm 1 until its first bin is closed or the sequence fits entirely on the first bin. We denote by \( \tilde{B}_i \) the items packed in the first bin in the \( i \)-th trial of this experiment. The process \( |\tilde{B}_1|, \ldots, |\tilde{B}_n| \) is i.i.d.

We have the following identities:

\[
|B_1| = |\tilde{B}_1|
\]
\[
|B_2| = \min\{n - |B_1|, \tilde{B}_2\}
\]
\[
\vdots
\]
\[
|B_n| = \min\{n - |B_1| - \cdots - |B_{n-1}|, \tilde{B}_n\}.
\]

Observe that \( N_{ALG} = \min\{k : \sum_{i=1}^k |B_i| = n\} \) is a stopping time for \( |B_1|, \ldots, |B_n| \) so also is a stopping time for \( |\tilde{B}_1|, \ldots, |\tilde{B}_n| \). By Wald’s equation (see Theorem A.2 below) we have

\[
E[N_{ALG}] E[|\tilde{B}_1|] = E \left[ \sum_{i=1}^{N_{ALG}} |\tilde{B}_i| \right].
\]

Additionally, we have \( E[|\tilde{B}_1|] = E[|B_1|] \) by construction. Now, until time \( N_{ALG} - 1 \) we must have \( |B_1| = |\tilde{B}_1|, \ldots, |B_{N_{ALG}-1}| = |\tilde{B}_{N_{ALG}-1}| \), all of these values at least 1. Then,

\[
\sum_{i=1}^{N_{ALG}-1} |\tilde{B}_i| = \sum_{i=1}^{N_{ALG}-1} |B_i| \leq n - 1.
\]

Therefore,

\[
E[N_{ALG}] E[|B_1|] = E \left[ \sum_{i=1}^{N_{ALG}-1} |B_i| + |\tilde{B}_{N_{ALG}}| \right] \leq 2n - 1,
\]

which concludes the proof.

**Remark 2.** The result of Lemma A.1 is almost tight as the following deterministic example shows. Consider \( X_i = \frac{1}{n-i} \), then \( N_{ALG} = 2 \). Then, the upper bound is \( \frac{2n-1}{|B_1|} = \frac{2n-1}{n-1} = 2 + \frac{1}{n-1} = N_{ALG} + \frac{1}{n-1} \).

### A.2 Wald’s Equation

**Theorem A.2** (Wald’s equation). If \( X_1, X_2, \ldots \) are i.i.d. random variables with finite mean and \( N \) is a stopping time with \( E[N] < \infty \), then

\[
E \left[ \sum_{n=1}^{N} X_n \right] = E[N] E[X_1].
\]

Proof can be found in [42].