ON THE CLASSICAL LIMIT
OF THE BALANCED STATE SUM

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I. Introduction

In [1], a new state sum model (the “balanced” model) was suggested as a
quantum theory of gravity. In [2], a proposal was made to construct a geomet-
rical interpretation of it, and to demonstrate that it gives general relativity in
the classical limit.

The purpose of this note is to make several advances in the interpretation
of the balanced model. First, we outline a shortcoming of the definition of
the model in [1] pointed out to us by John Barrett and John Baez [3], and
explain how to correct it. Second, we show that the classical limit of our state
sum reproduces the Einstein-Hilbert lagrangian whenever the term in the state
sum to which it is applied has a geometrical interpretation. Next we outline
a program to demonstrate that the classical limit of the state sum is in fact
dominated by terms with geometrical meaning. This uses in an essential way
the alteration we have made to the model in order to fix the shortcoming discussed in
the first section. Finally, we make a brief discussion of the Minkowski signature
version of the model.

This note is not intended to be self contained, it can only be read after
familiarizing oneself with [1] and [2].

II. Fixing a Hole

As pointed out in [3], the argument leading to the definition of the balanced
model overlooked a geometric fact. Specifically, if we have four bivectors
\( b_i \); \( i=1,2,3,4 \), which satisfy the constraints proposed in [1], namely

\[
|b_{i+}| = |b_{i-}|
\]

\[
|(b_i + b_j)+| = |(b_i + b_j)_-|
\]

\[
b_1 + b_2 + b_3 + b_4 = 0,
\]

it does not follow that they are the four bivectors corresponding to the faces of
a tetrahedron in \( \mathbb{R}^4 \) as stated in [1]. Another configuration is also possible, in
which the planes associated to the 4 bivectors all intersect in a common line.

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In the classical theory it is easy enough to eliminate this other set of possibilities. An elegant way to do so is to form the trivector whose components are the square roots of the four determinants defined in [2]. In the desired case, when the four bivectors are associated to the faces of a tetrahedron, we obtain the volume trivector of the tetrahedron by this formula. In the other case, which we wish to avoid, the components of the trivector are all zero. Thus, in order to eliminate the unwanted cases, we project out the intersection of the null spaces of the operators given by the four determinants of [2]. This also eliminates tetrahedra of volume zero, which we also do not want in our theory.

This prescription has a natural extension to the quantum theory, since the definitions of the four determinants are unplagued by ordering problems.

Thus, to our list of constraints above we add, on each tetrahedron, the condition \(|F| > 0\), where \(F\) is the trivector of the (quantum) tetrahedron.

Of course, it remains to be seen by a careful analysis exactly what quantum states this leaves us.

III. The Classical Lagrangian

A critical step in the argument of [1] is motivated only by heuristic arguments, and by the extreme naturality of the proposal. Namely, after replacing the bivectors on the faces of a 4-simplex with pairs of spins and replacing the classical constraints with operators in the category of representations of \(\text{so}(4)\), we close the maps so defined into a spin net, called a balanced 15j symbol, which we evaluate to obtain a number. A state sum is now formed as a sum of products of these numbers, which we propose as a quantum version of general relativity.

Our immediate purpose here is to show that this proposal does in fact correspond to a discrete approximation to the Einstein-Hilbert action in a suitable classical limit. Our argument closely parallels the 3d analysis of [4].

More precisely, let us imagine that we have a term in our state sum, i.e. an assignment of representations to the faces and tetrahedra of a triangulated 4-manifold whose absolute values closely approximate the numbers calculated from the areas and dihedral angles of a classical state of the regge calculus for the triangulation, or otherwise put, of a choice of flat metrics on each 4-simplex which match at all the tetrahedra where they intersect. Let us call such a term a “geometric configuration”. We want to show that under a small modification of the data, the evaluation of the balanced 15j symbol varies in a way which closely approximates the variation of the Einstein-Hilbert action.

Our point of departure in demonstrating this is the well-known asymptotic formula for the evaluation of a 6j symbol for ordinary representations of \(\text{su}(2)\):

\[
6j \sim \frac{1}{\sqrt{12\pi V}} \cos(\sum J_i \theta_i + \pi/4),
\]
where we have used the values of the casimir on the six irreducible representations as lengths for a euclidean tetrahedron, and the $\theta_i$ and $V$ are the dihedral angles and volume. This approximation is in a picture where the couplings of the spins are around faces rather than at vertices, as is more usual; thus it is a topologically dual picture. This is important in what follows. All of this is corresponding to the classically permitted region for a 6j symbol.

The dependence of this expression on the cosine of the dihedral angle is what allowed Regge and Ponzano to construct 3d quantum gravity from a state sum of spin nets in [4]. The key point is that for the state sum built of 6j symbols the variation as a single spin (quantum edge length) is varied vanishes if the sum of the dihedral angles around the edge is $2\pi$.

What we would really like to show for the balanced state sum is that, as we vary as single edge length, the variation of the phase of a term corresponding to a geometric configuration is a discretized approximation to the Ricci curvature of the 4d discrete metric of the configuration. Unfortunately, that is quite difficult to show. The reason is that there is no quantity in the balanced state sum which directly represents an edge length. As we shall discuss below, there is a natural edge length operator, but we have not yet investigated it.

Thus, we are going to show only that for a certain family of deformations of a geometric configuration, the change in the 15j symbol is the same as the corresponding change in the Einstein-Hilbert lagrangian. Since the family of deformations is sufficient to generate all allowable changes in the term, this suggests strongly that the effective lagrangian of the theory is a discrete version of E-H. The demonstration that we in fact have a quantum theory whose classical solutions correspond to discretized general relativity (i.e. the recovery of Einstein’s equations) will still require considerable work. We give a program for demonstrating that below.

Accordingly, then, let us assume that we have a 4-simplex with balanced spin labels on its faces and tetrahedra, which correspond to the geometric data on a euclidean 4-simplex. Now let us consider a certain type of perturbation. The specific type of perturbation we consider is to increase or decrease all three of the spins on the faces incident to a single edge by 1. If we choose to divide the tetrahedra incident to the line in such a way that the three faces are on the same side of each division, then the move we are making corresponds to adding a spin which circulates around a simple three edged loop. (We can always choose to devide each tetrahedron any way we like, because the sum of projections on the balanced summands in any two subdivisions are equal.) It is known from the theory of the binor calculus that every spin net can be decomposed as a superposition of such circulating spins. Hence our family of deformations, combined with judicious redivisions of the tetravalent vertices on tetrahedra, can generate all deformations of a 15j symbol. Let us call such a perturbation a “basic” perturbation.

Now let us use the decomposition theory of $so(4)$ spin networks to rewrite the balanced 15j symbol as a product of a 6j and a 12j symbol, where the 6j unites
the spins on the 3 faces surrounding an edge with the 3 bivectors representing their sums. (See Figure 1.)

This has the effect of isolating all the spins to be changed in our simple perturbation onto a single 6j symbol. Thus, the asymptotic formula for the evaluation of a 6j symbol (or more precisely its square, since we are considering representations of $so(4) \cong su(2) \oplus su(2)$) allows us to approximate the effect of our deformation.

Let us now imagine a euclidean tetrahedron whose edge lengths are dual to the spins on the 6j symbol we have isolated; so that the edges corresponding to the three faces meet at a vertex instead of forming a triangle. This dualization allows us to use the above formula to study the behavior of the associated 6j symbol under perturbation. Let us call this the “ancillary tetrahedron.” The effect of our deformation now can be approximated very simply as the sum of the three areas to be deformed each times the derivative of the cosine of the corresponding dihedral angle for the ancillary tetrahedron.

Now the crucial step in the analysis is the geometrical interpretation of the dihedral angles of the ancillary tetrahedron. On the one hand, the three planar angles connecting the edges corresponding to the three faces are approximately the angles between the three classical bivectors corresponding to the three faces. On the other, they form a spherical triangle with the three dihedral angles of the ancillary tetrahedron.

Thus, the three dihedral angles of the ancillary tetrahedron are determined from the three angles between the three classical bivectors by the laws of spherical trigonometry.

On the other hand, if we think of the geometry of the classical 4-simplex, the angles between the three bivectors on the faces around an edge have the same relationship with the three hyperdihedral angles on those faces: they are the edge angles and spherical angles of a spherical triangle. To see this, we pick any point interior to the edge, and intersect the 4-simplex with the hyperplane orthogonal to the edge at the point. The intersection with the three faces and tetrahedra around the edge form three planes, whose intersection with a small sphere centered at the point is a spherical triangle. (See Figure 2)

Thus, since the dihedral angles of the 4-simplex are closely approximated by the face angles of the ancillary tetrahedron, it follows that the dihedral angles of the ancillary triangle are good approximations to the hyperdihedral angles of the 4-simplex. Thus the variation formula for the 6j symbol of the ancillary tetrahedron shows us that the effect of one of our basic perturbations is the sum of the areas of the faces times the derivatives of the cosines of the hyperdihedral angles on them.

If we now consider a geometric configuration for the state sum on a whole triangulated 4-manifold, the result of the forgoing analysis is that the basic deformations have the same effect as the change in the Einstein-Hilbert lagrangian.

Let us try to see why this is so. The Einstein-Hilbert lagrangian for a 4 dimensional manifold is the sum of the sectional curvatures of the manifold, with
the measure given by the metric. When one considers the setting of a discretized metric, the curvature is distributional, and concentrated on the 2-simplices of the triangulation. Thus the Regge version of the discretized E-H action is the sum of the areas of the 2-simplices times the curvature of a section transverse to them, which is just the deficit angle, i.e., the sum of the hyperdihedral angles surrounding the face minus $2\pi$. The terms $2\pi A$ can be neglected in a discretized setting, since the areas are integral. We can then rewrite the discretized EH lagrangian for a triangulated 4-manifold as a sum of expressions for each 4-simplex, where the expression for each 4-simplex is just the sum of the area of each 2-simplex bounding the 4-simplex, times the corresponding hyperdihedral angle.

Now by a well known theorem in classical geometry [4], the variation in this expression as we vary one area is just the corresponding hyperdihedral angle. Thus the variation in the contribution to our lagrangian from one 4-simplex when we make our basic perturbation is just the sum of the three hyperdihedral angles incident on the edge we have chosen.

To relate this to the result using the 6j symbol on the ancillary tetrahedron, it is necessary to recall that the 15j symbol is not a contribution to the lagrangian, but to a path integral, hence to the complex exponential of the lagrangian. Thus we are interested not in the variation of the 6j symbol itself, but of its phase, $\Sigma J_i \theta_i$.

The analogous classical theorem in 3d now gives us that its variation is also $\theta_i$ when we vary a single $J$, or the sum of the three angles for the basic deformation.

This argument is a strong indication, at this relatively early stage, that the balanced state sum is really a quantum version of general relativity. From the standpoint of proof (even as physicists use the term), the argument is not very powerful. What is needed is an argument that geometric configurations are plentiful, and in fact produce all the classical solutions.

Now we sketch a program for repairing these shortcomings.

IV. Towards a quantum geometry

Since we have included in our revision of the balanced state sum the constraint that the trivector operators on the tetrahedra are nonzero, we can form the components of four of the trivectors into a $4 \times 4$ matrix and compute its determinant. This gives us an operator for the hypervolume of the 4-simplex. The next thing we would like to investigate would be the inverse of this matrix; this would generate operators corresponding to edge lengths. At this point operator ordering problems will have to be faced in the quantum theory.

The task of showing that we have many geometrical configurations in the classical limit boils down to a study of the operator algebra of the operators corresponding to all the classical geometrical quantities on a 4-simplex: lengths, areas, dihedral angles, volumes and hyperdihedral angles and hypervolume. If
the commutators between these operators can be neglected, then their simultaneous eigenvectors will provide geometrical configurations. The observation in [5] that the commutators of the operators determining dihedral angles vanish because of cancellations in balanced tetrahedra make us optimistic. Much work will be needed in this direction.

V. The Minkowski signature case

The reduction of the deformations of 15j symbols to 6j symbols above suggests that the extension of the euclidean model above to a lorentzian one will parallel the situation in 3d. There, the classically forbidden regime of 6j symbols models the geometry of lorentzian tetrahedra much the same way that classically allowed 6j symbols model euclidean ones. This allowed Barrett and Foxon to construct 3d lorentzian general relativity as a categorical state sum [6].

It seems highly likely that a similar procedure will work in 4d as well. The state sum will be over the same labelling set, but a wick rotation of the evaluations will give the dominant weight to lorentzian rather than euclidean configurations.

The first step in the analysis will be relating the lorentzian geometry of the 4-simplex with that of the ancillary tetrahedron.

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4-simplex ABCDE

4-simplex cut by hyperplane A’B’C’
perpendicular to DE

ancillary tetrahedron as the intersection

Figure 2
15j symbol as spin-network geometrically associated to a 4-simplex

Figure 1