THE SPACE OF EMBEDDED MINIMAL SURFACES OF FIXED GENUS IN A 3-MANIFOLD IV; LOCALLY SIMPLY CONNECTED

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0. Introduction

This paper is the fourth in a series where we describe the space of all embedded minimal surfaces of fixed genus in a fixed (but arbitrary) closed 3-manifold. The key is to understand the structure of an embedded minimal disk in a ball in $\mathbb{R}^3$. This was undertaken in [CM3], [CM4] and the global version of it will be completed here; see [CM13] for discussion of the local case and [CM13], [CM14] where we have surveyed our results about embedded minimal disks.

Our main results are Theorems 0.1, 0.2 below.

**Theorem 0.1.** See fig. 1. Let $\Sigma_i \subset B_{R_i} = B_{R_i}(0) \subset \mathbb{R}^3$ be a sequence of embedded minimal disks with $\partial \Sigma_i \subset \partial B_{R_i}$ where $R_i \to \infty$. If $\sup_{B_{1} \cap \Sigma_i} |A|^2 \to \infty$, then there exists a subsequence, $\Sigma_j$, and a Lipschitz curve $S: \mathbb{R} \to \mathbb{R}^3$ such that after a rotation of $\mathbb{R}^3$:

1. $x_3(S(t)) = t$. (That is, $S$ is a graph over the $x_3$-axis.)
2. Each $\Sigma_j$ consists of exactly two multi-valued graphs away from $S$ (which spiral together).
3. For each $1 > \alpha > 0$, $\Sigma_j \setminus S$ converges in the $C^\alpha$-topology to the foliation, $\mathcal{F} = \{x_3 = t\}_t$, of $\mathbb{R}^3$.
4. For all $r > 0$, $t$, then $\sup_{B_r(S(t)) \cap \Sigma_j} |A|^2 \to \infty$.

In 2., 3. that $\Sigma_j \setminus S$ are multi-valued graphs and converges to $\mathcal{F}$ means that for each compact subset $K \subset \mathbb{R}^3 \setminus S$ and $j$ sufficiently large $K \cap \Sigma_j$ consists of multi-valued graphs over (part of) $\{x_3 = 0\}$ and $K \cap \Sigma_j \to K \cap \mathcal{F}$.

This theorem (as many of the results below) is modeled by the helicoid and its rescalings. The helicoid is the minimal surface $\Sigma^2$ in $\mathbb{R}^3$ given by $(s \cos t, s \sin t, -t)$ where $s, t \in \mathbb{R}$.

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Take a sequence $\Sigma_i = a_i \Sigma$ of rescaled helicoids where $a_i \to 0$. Since the helicoid has cubic volume growth, the density is unbounded. The curvature is blowing up along the vertical axis. The sequence converges (away from the vertical axis) to a foliation by flat parallel planes. The singular set $\mathcal{S}$ (the axis) then consists of removable singularities.

**Theorem 0.2.** See fig. 2. There exists $\epsilon > 0$, such that if $\Sigma^2 \subset B_{2r_0} \cap \{x_3 > 0\} \subset \mathbb{R}^3$ is an embedded minimal disk with $\partial \Sigma \subset \partial B_{2r_0}$, then for all components $\Sigma'$ of $B_{r_0} \cap \Sigma$ which intersect $B_{\epsilon r_0}$

$$\sup_{\Sigma'} |A_{\Sigma}|^2 \leq r_0^{-2}. \quad (0.3)$$

Using the minimal surface equation and that $\Sigma'$ has points close to a plane, it is not hard to see that, for $\epsilon > 0$ sufficiently small, (0.3) is equivalent to the statement that $\Sigma'$ is a graph over the plane $\{x_3 = 0\}$.

An embedded minimal surface $\Sigma$ which is as in Theorem 0.2 is said to satisfy the $(\epsilon, r_0)$-effective one-sided Reifenberg condition; cf. Appendix A. We will often refer to Theorem 0.2 as the one-sided curvature estimate.

Note that the assumption in Theorem 0.2 that $\Sigma$ is simply connected is crucial as can be seen from the example of a rescaled catenoid. The catenoid is the minimal surface in
\(\mathbb{R}^3\) given by \((\cosh s \cos t, \cosh s \sin t, s)\) where \(s, t \in \mathbb{R}\); see fig. 3. Under rescalings this converges (with multiplicity two) to the flat plane; see fig. 4.

An almost immediate consequence of Theorem 0.2 is:

**Corollary 0.4.** See fig. 5. There exist \(c > 1, \epsilon > 0\) so: Let \(\Sigma_1, \Sigma_2 \subset B_{cr_0} \subset \mathbb{R}^3\) be disjoint embedded minimal surfaces with \(\partial \Sigma_i \subset \partial B_{cr_0}\) and \(B_{cr_0} \cap \Sigma_i \neq \emptyset\). If \(\Sigma_1\) is a disk, then for all components \(\Sigma'_i\) of \(B_{r_0} \cap \Sigma_1\) which intersect \(B_{\epsilon r_0}\)

\[
\sup_{\Sigma'_i} |A|^2 \leq r_0^{-2}.
\]  

To explain how these theorems are proven using the results of \([CM3]–[CM5]\), and \([CM7]\) we will need some notation for multi-valued graphs. Let \(P\) be the universal cover of the punctured plane \(C \setminus \{0\}\) with global (polar) coordinates \((\rho, \theta)\) and set \(S_{r,s}^{\theta_1,\theta_2} = \{r \leq \rho \leq s, \theta_1 \leq \theta \leq \theta_2\}\). An \(N\)-valued graph \(\Sigma\) of a function \(u\) over the annulus \(D_s \setminus D_r\) is a (single-valued) graph (of \(u\)) over \(S_{r,s}^{-N\pi,N\pi} \Sigma^{\theta_1,\theta_2}\) will denote the subgraph of \(\Sigma\) over \(S_{r,s}^{\theta_1,\theta_2}\). The separation, see fig. 6, between consecutive sheets will be denoted by \(w\) so \(w(\rho, \theta) = u(\rho, \theta + 2\pi) - u(\rho, \theta)\). A multi-valued graph is embedded if and only if \(|w| > 0\). Note (see fig. 7) that one-half of the helicoid, i.e., each of the two components of \((s \cos t, s \sin t, -t) \setminus \{s = 0\}\), as an \(\infty\)-valued graph of a function given in polar coordinates by \(u(\rho, \theta) = -\theta, u(\rho, \theta) = -\theta + \pi\), respectively. In particular, \(w(\rho, \theta) = -2\pi\).

In this paper, as in \([CM7]\), we have normalized so embedded multi-valued graphs have negative separation. This can be achieved after possibly reflecting in a plane.

In \([CM4]\) we showed that an embedded minimal disk in a ball in \(\mathbb{R}^3\) with large curvature at a point contains an almost flat multi-valued graph nearby. Namely, we showed:

**Theorem 0.6.** (Theorem 0.2 of \([CM4]\)). See A. and B. in fig. 8. Given \(N \in \mathbb{Z}_+, \epsilon > 0\), there exist \(C_1, C_2 > 0\) so: Let \(0 \in \Sigma^2 \subset B_R \subset \mathbb{R}^3\) be an embedded minimal disk, \(\partial \Sigma \subset \partial B_R\). If \(\sup_{B_{r_0}} |A|^2 \geq 4 C_1^2 r_0^{-2}\) for some \(0 < r_0 < R\), then there exists (after a rotation) an \(N\)-valued graph \(\Sigma_{\theta} \subset \Sigma \cap \{x_3^2 \leq \epsilon^2 (x_1^2 + x_2^2)\}\) over \(D_{R/C_2} \setminus D_{2r_0}\) with gradient \(\leq \epsilon\).

An important consequence of Theorem 0.6 is (see theorem 5.8 of \([CM4]\)): Let \(\Sigma_i \subset B_{2R}\) be a sequence of embedded minimal disks with \(\partial \Sigma_i \subset \partial B_{2R}\). Clearly (after possibly going to a subsequence) either (1) or (2) occur:
Figure 6. The separation of a multi-valued graph. (Here the multi-valued graph is shown with negative separation.)

Figure 7. The helicoid is obtained by gluing together two $\infty$-valued graphs along a line. The two multi-valued graphs are given in polar coordinates by $u_1(\rho, \theta) = -\theta$ and $u_2(\rho, \theta) = -\theta + \pi$. In either case $w(\rho, \theta) = -2\pi$.

Figure 8. Proving Theorem 0.1. A. Finding a small $N$-valued graph in $\Sigma$. B. Extending it in $\Sigma$ to a large $N$-valued graph. C. Extend the number of sheets. (A. follows from [CM4] and B. follows from [CM3].)

(1) $\sup_{B_R \cap \Sigma_i} |A|^2 \leq C < \infty$ for some constant $C$.
(2) $\sup_{B_R \cap \Sigma_i} |A|^2 \to \infty$.

In (1) (by a standard argument) $B_s(y_i)$ is a graph for all $y_i \in B_R \cap \Sigma_i$, where $s$ depends only on $C$. In (2) (by theorem 5.8 of [CM4]) if $y_i \in B_R \cap \Sigma_i$ with $|A|^2(y_i) \to \infty$, then we can after passing to a subsequence assume that $y_i \to y$, each $\Sigma_i$ contains a 2-valued graph $\Sigma_{d,i}$ over $D_{R/C_2}(y) \setminus D_{\epsilon_i}(y)$ with $\epsilon_i \to 0$, and $\Sigma_{d,i}$ converges to a graph $y \in \Sigma_\infty$ over $D_{R/C_2}(y)$. In either case in the limit there is a smooth minimal graph through each point in the support.

These multi-valued graphs should be thought of as the basic building blocks for an embedded minimal disk. In fact, using a standard blow up argument, we showed in [CM4]...
(corollary 4.14 combined with proposition 4.15 there) that Theorem 0.6 was a consequence of the following that we will use to construct the actual building blocks starting off on the smallest possible scale:

**Theorem 0.7.** [CM4]. Given $N \in \mathbb{Z}_+$, $\epsilon > 0$, there exist $C_1, C_2, C_3 > 0$ so: Let $0 \in \Sigma^2 \subset B_R \subset \mathbb{R}^3$ be an embedded minimal disk, $\partial \Sigma \subset \partial B_R$. If $\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq 4C_2^2 r_0^{-2}$ and $|A|^2(0) = C_2^2 r_0^{-2}$ for some $0 < r_0 < R$, then there exists (after a rotation) an $N$-valued graph $\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \epsilon^2 (x_1^2 + x_2^2)\}$ over $D_{R/C_2} \setminus D_{r_0}$ with gradient $\leq \epsilon$ and separation $\geq C_3 r_0$ over $\partial D_{r_0}$.

It will be important for the application of Theorem 0.7 here that the initial separation of the sheets is proportional to the initial scale that the graph starts off on.

Theorems 0.1 and 0.2 deal with how the building blocks fit together. A consequence of Theorem 0.1 is that if an embedded minimal disk starts to spiral very tightly, then it can unwind only very slowly. That is, in a whole extrinsic tubular neighborhood it continues to spiral tightly and fills up almost the entire space.

Let us briefly outline the proof of the one-sided; i.e., Theorem 0.2. Suppose that $\Sigma$ is an embedded minimal disk in the half-space $\{x_3 > 0\}$. We prove the curvature estimate by contradiction; so suppose that $\Sigma$ has low points with large curvature. Starting at such a point, we decompose (see Corollary III.1.3) $\Sigma$ into disjoint multi-valued graphs using the existence of nearby points with large curvature (see Proposition I.0.1), a blow up argument, and [CM3], [CM4]. The key point is then to show (see Proposition III.2.2 and fig. 9) that we can in fact find such a decomposition where the “next” multi-valued graph starts off a definite amount below where the previous multi-valued graph started off. In fact, what we show is that this definite amount is a fixed fraction of the distance between where the two graphs started off. Iterating this eventually forces $\Sigma$ to have points where $x_3 < 0$. Which is the desired contradiction.

![Figure 9. Two consecutive blow up points satisfying (III.2.1).](image)

To prove this key proposition (Proposition III.2.2) we use two decompositions and two kinds of blow up points. The first decomposition which is Corollary III.1.3 uses the more standard blow up points given by (III.1.1). These are pairs $(y, s)$ where $y \in \Sigma$ and $s > 0$ is such that $\sup_{B_{s}(y)} |A|^2 \leq 4|A|^2(y) = 4C_1^2 s^{-2}$. The point about such a pair $(y, s)$ is that by [CM3], [CM4] (and an argument in Part I which allows us replace extrinsic balls by intrinsic ones), then $\Sigma$ contains a multi-valued graph near $y$ starting off on the scale $s$. (This is assuming that $C_1$ is a sufficiently large constant given by [CM3], [CM4].) The second kind of blow up points are the ones satisfying (III.2.1). Basically (III.2.1) is (III.1.1) (except...
for a technical issue) where 8 is replaced by some really large constant \( C \). The point will then be that we can find blow up points satisfying (III.2.1) so that the distance between them is proportional to the sum of the scales. Moreover, between consecutive blow up points satisfying (III.2.1), we can find a bunch of blow up points satisfying (III.1.1); see fig. 10. The advantage is now that if we look between blow up points satisfying (III.2.1), then the height of the multi-valued graph given by such a pair grows like a small power of the distance whereas the separation between the sheets in a multi-valued graph given by (III.1.1) decays like a small power of the distance; see fig. 11. Now thanks to that the number of blow up points satisfying (III.1.1) (between two consecutive blow up points satisfying (III.2.1)) grows almost linearly then, even though the height of the graph coming from the blow up point satisfying (III.2.1) could move up (and thus work against us), then the sum of the separations of the graphs coming from the points satisfying (III.1.1) dominates the other term. Thus the next blow up point satisfying (III.2.1) (which lies below all the other graphs) is forced to be a definite amount lower than the previous blow up point satisfying (III.2.1).

Let \( x_1, x_2, x_3 \) be the standard coordinates on \( \mathbb{R}^3 \) and \( \Pi : \mathbb{R}^3 \to \mathbb{R}^2 \) orthogonal projection to \( \{ x_3 = 0 \} \). For \( y \in S \subset \Sigma \subset \mathbb{R}^3 \) and \( s > 0 \), the extrinsic and intrinsic balls are \( B_s(y) \), \( B_s(y) \) and \( \Sigma_{y,s} \) is the component of \( B_s(y) \cap \Sigma \) containing \( y \). \( D_s \) denotes the disk \( B_s(0) \cap \{ x_3 = 0 \} \). \( K_\Sigma \) the sectional curvature of a smooth compact surface \( \Sigma \) and when \( \Sigma \) is immersed \( A_\Sigma \) will be its second fundamental form. When \( \Sigma \) is oriented, \( n_\Sigma \) is the unit normal.

This paper completes the results announced in [CM11] and [CM12].

Using Theorems 1.1, 1.2, W. Meeks and H. Rosenberg proved that the plane and helicoid are the only complete properly embedded simply-connected minimal surfaces in \( \mathbb{R}^3 \), [MeR9].

\textbf{Part I. The proof of Theorem 0.1 assuming Theorem 0.2 and short curves}

In this part we will show how Theorem 0.1 follows from Theorem 0.2, the results about existence of multi-valued graphs from [CM3], [CM4] which were recalled in the introduction, corollary III.3.5 of [CM3], and the results about properness of embedded disks from [CM7] (once we see that the conditions in corollary 0.7 of [CM7] are satisfied). The remaining parts of this paper are devoted to showing Theorem 0.2 (Part III) and that corollary 0.7 of [CM7] applies (Part IV; see, in particular, Theorem 1.0.10 below).
We will use several times that given \( \alpha > 0 \), Proposition II.2.12 of \([\text{CM}3]\) gives \( N_\rho \) so if \( u \) satisfies the minimal surface equation on \( S_{e^{-N_\rho}, e^{N_\rho}} \) with \( |\nabla u| \leq 1 \), and \( w < 0 \), then \( \rho |\text{Hess}_u| + \rho |\nabla w|/|w| \leq \alpha \) on \( S_{1,R}^{1,2\pi} \). Theorem 3.36 of \([\text{CM}7]\) then yields \( |\nabla u - \nabla u(1,0)| \leq C\alpha \).

We can therefore assume (after rotating so \( \nabla u(1,0) = 0 \)) that
\[
|\nabla u| + \rho |\text{Hess}_u| + 4\rho |\nabla w|/|w| + \rho^2 |\text{Hess}_w|/|w| \leq \epsilon \leq 1/(2\pi).
\]

The bound on \( |\text{Hess}_w| \) follows from the other bounds and standard elliptic theory. In what follows, we will assume that \( w < 0 \). (This normalizes the graph of \( u \) to spiral downward; this can be achieved after possibly reflecting in a plane.)

If \( \Sigma \) is an embedded graph of \( u \) over \( S_{1/2,2R}^{-3\pi,N+3\pi} \), then \( E \) is the region over \( D_R \setminus D_1 \) between the top and bottom sheets of the concentric subgraph over \( S_{1,R}^{2\pi,N+2\pi} \) (recall that, possibly after reflection, we can assume \( w < 0 \)). Namely, when \( N \) is even, \( E \) is the set (see fig. 12) of all \((r \cos \theta, r \sin \theta, t)\) with \( 1 \leq r \leq R \), \(-2\pi \leq \theta < 0 \), and
\[
w(r, \theta + (N + 2\pi)) < t < u(r, \theta).
\]

To apply corollary 0.7 of \([\text{CM}7]\) we need the following result (which will be proven in Part \( \text{V} \)) on existence of “the other half” of an embedded minimal disk and short curves, \( \sigma_\theta \), connecting the two halves:

Theorem I.0.10. See fig. 13. There exist \( C, R_0, N_0, \epsilon > 0 \) so for \( N \geq N_0 \): Let \( \Sigma \subset B_{4R} \) be an embedded minimal disk, \( \partial \Sigma \subset \partial B_{4R}, R \geq R_0 \), and \( \Sigma_1 \subset \Sigma \) a graph of \( u_1 \) with \( |\nabla u_1| \leq \epsilon \) over \( S_{1/2,2R}^{3\pi,N+3\pi} \). Then \( E \cap \Sigma \setminus \Sigma_1 \) is a graph of \( u_2 \) over \( S_{1,R}^{0,N+2\pi} \) and \( u_1(1,2\pi) < u_2(1,0) < u_1(1,0) \). Moreover, for all \( 0 \leq \theta \leq N + 2\pi \), a curve \( \sigma_\theta \subset \{ x_1^2 + x_2^2 \leq 1 \} \cap \Sigma \) with length \( \leq C \) connects the image of \( u_1 \) over \( (1, \theta) \) with the image of \( u_2 \) over \( (1, \theta) \).

The main example of the “two halves” of an embedded minimal disk and short curves connecting them comes from the helicoid. Namely, let \( \Sigma \) be the helicoid, i.e., \( \Sigma = (\rho \cos \theta, \rho \sin \theta, -\theta) \) where \( \rho, \theta \in \mathbb{R} \), then \( \Sigma \setminus \{ \rho = 0 \} \) consists of two \( \infty \)-valued graphs \( \Sigma_1, \Sigma_2 \) and \( \sigma_\theta \) given by \( \Sigma \cap \{ x_3 = -\theta \} \) union \( \{ (-\cos \tau, -\sin \tau, -\tau) | \theta \leq \tau \leq \theta + \pi \} \) are short curves connecting the two halves. Theorem I.0.10 asserts that this is the general picture.

We will use the following result from \([\text{CM}3]\) to get nearby points with large curvature (here, as before, \( \Sigma_{y,a} \) is the component of \( B_4(y) \cap \Sigma \) containing \( y \):
Proposition I.0.11. (Corollary III.3.5 of [CM5]). See fig. 14. Given $C_1$, there exists $C_2$ so:

Let $0 \in \Sigma \subset B_{2C_2r_0}$ be an embedded minimal disk. Suppose $\Sigma_1, \Sigma_2 \subset \Sigma \cap \{x_3^2 \leq (x_1^2 + x_2^2)\}$ are graphs of $u_i$ satisfying (I.0.8) on $S_{r_0C_2r_0}^{-2\pi,2\pi}$, $u_1(r_0,2\pi) < u_2(r_0,0) < u_1(r_0,0)$, and $\nu \subset \partial \Sigma_0.2r_0$ a curve from $\Sigma_1$ to $\Sigma_2$. Let $\Sigma_0$ be the component of $\Sigma_0.,C_2r_0\setminus(\Sigma_1 \cup \Sigma_2 \cup \nu)$ which does not contain $\Sigma_0, r_0$. Suppose either $\partial \Sigma \subset \partial B_{2C_2r_0}$ or $\Sigma$ is stable and $\Sigma_0$ does not intersect $\partial \Sigma$.

Then

$$\sup_{x \in \Sigma_0 \setminus B_{4r_0}} |x|^2 |A|^2(x) \geq 4C_1^2.$$ (I.0.12)

Note that by the curvature estimate for stable surfaces, [Sc], [CM2], when $\Sigma$ is stable then the conclusion of Proposition I.0.11 is that no such surface exists for $C_1, C_2$ sufficiently large.

I.1. REGULARITY OF THE SINGULAR SET

If $\delta > 0$ and $z \in \mathbb{R}^3$, then we denote by $C_\delta(z)$ the (convex) cone with vertex $z$, cone angle $(\pi/2 - \arctan \delta)$, and axis parallel to the $x_3$-axis. That is, see fig. 15,

$$C_\delta(z) = \{x \in \mathbb{R}^3 \mid (x_3 - z_3)^2 \geq \delta^2 ((x_1 - z_1)^2 + (x_2 - z_2)^2)\}.$$ (I.1.1)

Figure 15. It follows from the one-sided curvature estimate that the singular set has the cone property and hence is a Lipschitz curve; see Lemma I.1.2.
Lemma I.1.2. See fig. 15. Let \( 0 \in S \subset \mathbb{R}^3 \) be a closed set such that for some \( \delta > 0 \) and each \( z \in S \), then \( S \subset C_\delta(z) \). If for all \( t \in x_3(S) \) and all \( \epsilon > 0 \), \( S \cap \{ t < x_3 < t + \epsilon \} \neq \emptyset \), \( S \cap \{ t - \epsilon < x_3 < t \} \neq \emptyset \), then \( S \cap \{ x_3 = t \} \) consists of exactly one point \( S_t \) for all \( t \in \mathbb{R} \), and \( t \to S_t \) is a Lipschitz parameterization of \( S \). In fact,

\[
|t_2 - t_1| \leq |S_{t_2} - S_{t_1}| \leq \sqrt{1 + \delta^{-2}}|t_2 - t_1|.
\]

\[(I.1.3)\]

**Proof.** First by the cone property it follows that \( S \cap \{ x_3 = t \} \) consists of at most one point for each \( t \in \mathbb{R} \). Assume that \( S \cap \{ x_3 = t_0 \} = \emptyset \) for some \( t_0 \). Since \( S \subset \mathbb{R}^3 \) is a nonempty closed set and \( x_3 : S \subset C_\delta(0) \to \mathbb{R} \) is proper, then \( x_3(S) \subset \mathbb{R} \) is also closed (and nonempty). Let \( t_s \in x_3(S) \) be the closest point in \( x_3(S) \) to \( t_0 \). The desired contradiction now easily follows since either \( S \cap \{ t_s < x_3 < t_0 \} \) or \( S \cap \{ t_0 < x_3 < t_s \} \) is nonempty by assumption.

It follows that \( t \to S_t \) is a well-defined curve (from \( \mathbb{R} \) to \( S \)). Moreover, since \( S_t \subset \{ x_3 = t_1 + (t_2 - t_1) \} \cap C_\delta(S_{t_1}) \subset B_{\sqrt{1 + \delta^{-2}}|t_2 - t_1|}(S_{t_1}) \), \[(I.1.3)\] follows.

We will refer loosely to a set \( S \) as in Lemma I.1.2 as having the cone property. Next we will see, by a very general compactness argument, that for any sequence of surfaces in \( \mathbb{R}^3 \), after possibly going to a subsequence, there is a well defined notion of points where the second fundamental form of the sequence blows up. The set of such points will below be the \( S \) in Lemma I.1.2 we observe in Corollary I.1.9 below that \( S \) has the cone property.

Lemma I.1.4. Let \( \Sigma_i \subset B_{R_i}, \partial \Sigma_i \subset \partial B_{R_i}, \) and \( R_i \to \infty \) be a sequence of (smooth) compact surfaces. After passing to a subsequence, \( \Sigma_j \), we may assume that for each \( x \in \mathbb{R}^3 \) either \( \sup_{B_r(x) \cap \Sigma_j} |A|^2 \to \infty \) for all \( r > 0 \) or \( \sup_j \sup_{B_r(x) \cap \Sigma_j} |A|^2 < \infty \) for some \( r > 0 \).

**Proof.** For \( r > 0 \) and an integer \( n \), define a sequence of functions on \( \mathbb{R}^3 \) by

\[
A_{i,r,n}(x) = \min\{n, \sup_{B_r(x) \cap \Sigma_i} |A|^2\},
\]

where we set \( \sup_{B_r(x) \cap \Sigma_i} |A|^2 = 0 \) if \( B_r(x) \cap \Sigma_i = \emptyset \). Set

\[
D_{i,r,n} = \lim_{k \to \infty} 2^{-k} \sum_{m=0}^{2^k-1} A_{i,(1+m2^{-k})r,n},
\]

then \( D_{i,r,n} \) is continuous and \( A_{i,2r,n} \geq D_{i,r,n} \geq A_{i,r,n} \). Let \( \nu_{i,r,n} \) be the (bounded) functionals,

\[
\nu_{i,r,n}(\phi) = \int_{B_n} \phi D_{i,r,n} \text{ for } \phi \in L^2(\mathbb{R}^3).
\]

By standard compactness for fixed \( r, n \), after passing to a subsequence, \( \nu_{j,r,n} \to \nu_{r,n} \) weakly. In fact (since the unit ball in \( L^2(\mathbb{R}^3) \) has a countable basis), by an easy diagonal argument after passing to a subsequence we may assume that for all \( n, m \geq 1 \) fixed \( \nu_{j,2^{-m}n} \to \nu_{2^{-m}n} \) weakly. Note that if \( x \in \mathbb{R}^3 \) and for all \( m, n \) with \( n \geq |x| + 1 \), (identify \( B_{2^{-m}}(x) \) with its characteristic function)

\[
\nu_{2^{-m}n}(B_{2^{-m}}(x)) \geq n \text{ Vol}(B_{2^{-m}}),
\]

then for each fixed \( r > 0 \), \( \sup_{B_r(x) \cap \Sigma_j} |A|^2 \to \infty \). Conversely, if for some \( n \geq |x| + 1, m \), \((I.1.8)\) fails at \( x \), then \( \sup_j \sup_{B_r(x) \cap \Sigma_j} |A|^2 < \infty \) for \( r = 2^{-m-1} \). \(\square\)
To implement Lemma I.1.2 in the proof of Theorem I.1, we will need the following (direct) consequence of Theorem 0.2 with $\Sigma_d$ playing the role of the plane (and the maximum principle as in Appendix C):

**Corollary I.1.9.** See fig. 16. There exists $\delta_0 > 0$ so: Suppose $\Sigma \subset B_{2R}$, $\partial \Sigma \subset \partial B_{2R}$ is an embedded minimal disk containing a 2-valued graph $\Sigma_d \subset \{x_3^2 \leq \delta_0^2 (x_1^2 + x_2^2)\}$ over $D_R \setminus D_{r_0}$ with gradient $\leq \delta_0$. Then each component of $B_{R/2} \cap \Sigma \setminus (C_{\delta_0}(0) \cup B_{2r_0})$ is a multi-valued graph with gradient $\leq 1$.

Note that since $\Sigma$ is compact and embedded, the multi-valued graphs given by Corollary I.1.9 spiral through the cone. Namely, if a graph did close up, then the graph containing $\Sigma_d$ would be forced to accumulate into it, contradicting compactness.

Another result we need to apply Lemma I.1.2 is:

**Lemma I.1.10.** See fig. 17. There exists $c_0 > 0$ so: Let $\Sigma_i \subset B_{R_i}$, $\partial \Sigma_i \subset \partial B_{R_i}$ be a sequence of embedded minimal disks with $R_i \to \infty$. If $\Sigma_{d,i} \subset \Sigma_i$ is a sequence of 2-valued graphs over $D_{R_i/C} \setminus D_{\epsilon_i}$ with $\epsilon_i \to 0$ and $\Sigma_{d,i} \to \{x_3 = 0\} \setminus \{0\}$, then

$$\sup_{B_i \cap \Sigma \cap \{x_3 > c_0\}} |A|^2 \to \infty. \quad (I.1.11)$$

**Proof.** Suppose not, see fig. 18; so assume that for each $c_0 > 0$, there is a sequence of embedded minimal disks $\Sigma_i$ (and $C_1$ depending on both $c_0$ and the sequence) with

$$\sup_{B_i \cap \Sigma \cap \{x_3 > c_0\}} |A|^2 \leq C_1 < \infty \quad (I.1.12)$$

and 2-valued graphs $\Sigma_{d,i} \subset \Sigma_i$ over $D_{R_i/C} \setminus D_{\epsilon_i}$ with $\epsilon_i \to 0$, $\Sigma_{d,i} \to \{x_3 = 0\} \setminus \{0\}$. Increasing $\epsilon_i$ (yet still $\epsilon_i \to 0$) and replacing $R_i$ by $S_i \to \infty$, we can assume $\Sigma_{d,i} \subset \{x_3^2 \leq \epsilon_i^2 (x_1^2 + x_2^2)\}$ is a 2-valued graph over $D_{4e^{\kappa_2}S_i} \setminus D_{e^{-\kappa_1}e^{\kappa_2}}$ with gradient $\leq \epsilon_i$ ($N_g$ given before [I.0.8]).

By Corollary I.1.9, each component of $B_{2e^{\kappa_2}S_i} \cap \Sigma_i \setminus (C_{\delta_0}(0) \cup B_{e^{-\kappa_1}e^{\kappa_2}})$ is a graph. Hence, by the Harnack inequality, if $\alpha > 0$ is sufficiently small and $q_i \in B_{S_i} \cap \Sigma_i \setminus (C_{\alpha}(0) \cup B_{2\epsilon_i})$, then for $i$ large $\Sigma_i$ contains an $(N_g + 1)$-valued graph over $D_{e^{\kappa_2}q_i} \setminus D_{e^{-\kappa_1}q_i/2}$ with gradient $\leq \epsilon < 1/(4\pi)$ and so $q_i$ is in the image of $\{\theta \leq \pi\}$ for this graph. Consequently, each component of $B_{S_i} \cap \Sigma_i \setminus (C_{\alpha}(0) \cup B_{2\epsilon_i})$ is a multi-valued graph satisfying [I.0.8].
Fix $h, \ell$ with $0 < h < \alpha \ell$. We get points $z_i \in \{x_3 = h, x_i^2 + y_i^2 = \ell^2\} \cap \Sigma_i$ and multi-valued graphs $z_i \in \Sigma_{1,i} \subset \{x_3 > 0\} \cap \Sigma_i$ defined over $S_{\ell/2, \delta/2}^{-\pi, \pi}$, with $N_i \to \infty$, so $z_i$ is in the image of $S_{\ell, \delta}^{-\pi, \pi}$, and so $\Sigma_{1,i}$ spirals into $\{x_3 = 0\}$ (note that we have assumed that it spirals down; we can argue similarly in the other case). In particular, Theorem I.0.10 applies, giving the other multi-valued graphs $\Sigma_{2,i}$ so $\Sigma_{1,i}$ and $\Sigma_{2,i}$ spiral together and so $\Sigma_{2,i}$ is the only part of $\Sigma_i$ between the sheets of $\Sigma_{1,i}$. Moreover, Theorem I.0.11 also gives the short curves $\sigma_{\theta,i}$ connecting these. It now follows from corollary 0.7 of [CM7] that the separations of the graph $\Sigma_{1,i}$ at $z_i$ go to 0. Since this holds for all such $h$ and $\ell$, it follows that $\Sigma_i \setminus C_\alpha(0) \to \mathcal{F}$; where $\mathcal{F}$ is a foliation of $\mathbb{R}^3 \setminus C_\alpha(0)$ by minimal annuli (all graphs over part of $\{x_3 = 0\}$).

Theorem I.4 gives $0 < C_2 < \infty$ so, given $r_0 > 0$, if $y_i \in \Sigma_i \setminus B_{3r_0}$, $i$ is large, and

$$|y_i|^2 |A|^2(y_i) > C_2 \quad (I.1.13)$$

then there is a 2-valued graph $\Sigma_{d,i} \subset \Sigma_i \setminus B_{C_4|y_i|}$ starting in $B_{C_4|y_i|}(y_i) \subset \{x_3 > C_3 r_0\}$ (by Theorem I.7, $\Sigma_{d,i}$ starts in $B_{C_4|y_i|}(y_i)$ where $C_4 = C_4(C_2)$ and, by Corollary I.1.9, $y_i \in C_{\delta/2}(0)$). Let $C_2' = C_2'(C_2) > 1$ be given by Proposition I.0.11 and set $r_0 = 1/(4C_2')$.

Choose $h_i, \ell_i \to 0$ with $\epsilon_i < \ell_i < r_0/4$, $0 < h_i \leq \alpha \ell_i$ and let $z_i, \Sigma_{1,i}, \Sigma_{2,i}$ be as above. Since $\partial \Sigma_{i,z_i,2r_0}$ is a simple closed curve, it must pass between the sheets of $\Sigma_{1,i}$. Since $\Sigma_{2,i}$ is the only part of $\Sigma_i$ between the sheets of $\Sigma_{1,i}$, we can connect $\Sigma_{1,i}$ and $\Sigma_{2,i}$ by curves $\nu_i \subset \partial \Sigma_{i,z_i,2r_0}$ which are above $\Sigma_{1,i}$. We can now apply Proposition I.0.11 to get the points $y_i \in B_{1/2}(z_i) \cap \Sigma_i \setminus B_{2r_0}(z_i) \subset B_{1/2+4\epsilon_i} \setminus B_{3r_0}$ as in (I.1.13).

To get the desired contradiction, observe that if $c_0 < C_3 r_0$, then the 2-valued graphs $\Sigma_{d,i}$ given by (I.1.13) and Theorem I.7, have separation $\geq C_5 = C_5(C_1) > 0$ (since $B_{C_4|y_i|}(y_i) \subset \{x_3 > C_3 r_0\}$). Namely, this separation is on a fixed scale bounded away from zero even as $\Sigma_{d,i}$ extends out of $C_\alpha(0)$, contradicting $\Sigma_i \setminus C_\alpha(0) \to \mathcal{F}$ and the lemma follows. \qed
I.2. Proof of Theorem 0.1

Proof. (of Theorem 0.1) By Lemma I.1.4, after passing to a subsequence (also denoted by \( \Sigma_i \)) we can assume that for each \( x \in \mathbb{R}^3 \) either

\[
\sup_{B_r(x) \cap \Sigma_i} |A|^2 \to \infty \text{ for all } r > 0, \tag{I.2.1}
\]

or \( \sup_{B_r(x) \cap \Sigma_i} |A|^2 < \infty \) for some \( r > 0 \). Let \( S \subset \mathbb{R}^3 \) be the points where (I.2.1) holds. By assumption \( B_1 \cap S \neq \emptyset \). So after a possible translation we may assume that \( 0 \in S \) and it follows easily from the definition that \( S \) is closed. By theorem 5.8 of [CM3] (and Bernstein’s theorem; see for instance theorem 1.16 of [CMII]), there exists a subsequence \( \Sigma_j \) and 2-valued graphs \( \Sigma_{d,j} \subset \Sigma_j \) over \( D_{R_j/C} \setminus D_{\epsilon_j} \) with \( \epsilon_j \to 0 \) such that \( \Sigma_{d,j} \to \{ x_3 = 0 \} \setminus \{ 0 \} \) (after possibly rotating \( \mathbb{R}^3 \)). (This fixes the subsequence and the coordinate system of \( \mathbb{R}^3 \).

Again by theorem 5.8 of [CM3] (and Bernstein’s theorem) for each \( S_i \in S \) there are 2-valued graphs \( \Sigma_{d,j} \subset \Sigma_j \) over \( D_{R_j/C}(S_i) \setminus D_{\epsilon_j}(S_i) \) with \( \epsilon_j \to 0 \) such that \( \Sigma_{d,j} \to \{ x_3 = t \} \setminus \{ S_i \} \).

Hence, by Corollary I.1.5, \( S \subset C_\delta(S_i) \). By Lemma I.1.10 (and scaling), for all \( t \in x_3(S) \) and all \( \epsilon > 0 \), \( S \cap \{ t < x_3 < t + \epsilon \} \neq \emptyset \), \( S \cap \{ t - \epsilon < x_3 < t \} \neq \emptyset \). It follows from Lemma I.1.2 that \( t \to S_t \) is a Lipschitz curve and \( \Sigma_j \setminus S \to F \setminus S \) in the \( C^\alpha \)-topology for all \( \alpha < 1 \) (and with uniformly bounded curvatures on compact subsets of \( \mathbb{R}^3 \setminus S \); see also Appendix B).

□

Part II. “The other half”

Theorem I.0.10 will follow by first showing that if an embedded minimal disk contains a multi-valued graph, then “between the sheets” of the graph the surface is another multi-valued graph - “the other half”. Second, we show an intrinsic version of Theorem 0.7 and, third, using this intrinsic version, we construct in Part IV the short curves connecting the two halves.

II.1. “THE OTHER HALF” OF AN EMBEDDED MINIMAL DISK

We show first that any point between the sheets of a multi-valued graph must connect to it within a fixed extrinsic ball:

Lemma II.1.1. There exist \( \epsilon_s > 0 \), \( C_s > 2 \) so: Let \( 0 \in \Sigma \subset B_R \) be an embedded minimal disk with \( \partial \Sigma \subset \partial B_R \), \( \Sigma_d \subset \{ x_3^2 \leq x_1^2 + x_2^2 \} \cap \Sigma \) a 2-valued graph over \( D_{3r_0} \setminus D_{r_0} \) with gradient \( \leq \epsilon_s \). If \( E_0 \) is the region over \( D_{2r_0} \setminus D_{r_0} \) between the sheets of \( \Sigma_d \), then \( E_0 \cap \Sigma \subset \Sigma_{0,C_s r_0} \).

Proof. Fix \( \epsilon_s > 0 \) small and \( C_s \) large enough to be chosen. If the lemma fails, then there are disjoint components \( \Sigma_a, \Sigma_b \) of \( B_{C_s r_0} \setminus \Sigma \) with \( \Sigma_d \subset \Sigma_a \) and \( y \in E_0 \cap \Sigma_b \). By the maximum principle, \( \Sigma_a, \Sigma_b \) are disks. Let \( \tilde{\eta}_y \) be the vertical segment (i.e., parallel to the \( x_3 \)-axis) through \( y \) connecting the sheets of \( \Sigma_d \). Fix a component \( \eta_y \) of \( \tilde{\eta}_y \setminus \Sigma \) connecting \( \Sigma_a \) to \( \Sigma \setminus \Sigma_b \). Let \( \Omega \) be the component of \( B_{C_s r_0} \setminus \Sigma \) containing \( \eta_y \) (so \( \partial \Sigma_b \) and \( \eta_y \) are linked in \( \Omega \)). [MeYa] gives a stable disk \( \Gamma \subset \Omega \) with \( \partial \Gamma = \partial \Sigma_b \). Using the linking, \( \Gamma \) intersects \( \eta_y \). [CM3] (cf. lemma I.0.9 of [CM3]) give \( C_s \) so any component \( \Gamma_y \) of \( B_{10r_0} \cap \Gamma \) intersecting \( \eta_y \) is a graph with bounded gradient over some plane; for \( \epsilon_s \) small, this plane must be almost horizontal. Hence, \( \Gamma_y \) is forced to “cut the axis” (i.e., intersect the curve in \( \Sigma_d \) over \( \partial D_{r_0} \) connecting the top and bottom sheets), giving the desired contradiction.

□
In the next proposition $\Sigma \subset B_{4R}$ with $\partial \Sigma \subset \partial B_{4R}$ is an embedded minimal surface and $\Sigma_1 \subset \{x_3^2 \leq x_1^2 + x_2^2\} \cap \Sigma$ an $(N + 2)$-valued graph of $u_1$ over $D_{2R} \setminus D_{r_1}$ with $|\nabla u_1| \leq \epsilon$ and $N \geq 6$. Let $E_1$ be the region over $D_R \setminus D_{2r_1}$ between the top and bottom sheets of the concentric $(N + 1)$-valued subgraph in $\Sigma_1$. To be precise, $E_1$ is the set of all $(r \cos \theta, r \sin \theta, t)$ with $2r_1 \leq r \leq R$, $(N - 1)\pi \leq \theta < (N + 1)\pi$, and

$$u_1(r, \theta) < t < u_1(r, \theta - 2N\pi).$$

(II.1.2)

**Proposition II.1.3.** There exist $C_0 > C_*$, $\epsilon_0 > 0$ so if $\Sigma$ is a disk as above, $R \geq C_0 r_1$, and $\epsilon_0 \geq \epsilon$, then $E_1 \cap \Sigma \setminus \Sigma_1$ is an (oppositely oriented) $N$-valued graph $\Sigma_2$.

**Proof.** Fix $z \in \Sigma_1$ over $\partial D_{r_1}$. Since $\partial \Sigma(z, 2r_1)$ is a simple closed curve, it must pass between the sheets of $\Sigma_1$ and hence through some other component $\Sigma_2$ of $E_1 \cap \Sigma$.

The version of the “estimate between the sheets” given in theorem III.2.4 of [CM3] gives $\epsilon_0 > 0$ so that $E_1 \cap \Sigma$ is locally graphical (i.e., if $z \in E_1 \cap \Sigma$, then $(\mathfrak{m}_2(z), (0, 0, 1)) \neq 0$). It follows that each component of $E_1 \cap \Sigma$ is an $N$-valued graph.

Fix a component $\Omega$ of $B_{4R} \setminus \Sigma$. We show next that $\Sigma_2$ is the only other component of $E_1 \cap \Sigma$ (i.e., $E_1 \cap \Sigma \subset \Sigma_1 \cup \Sigma_2$). If not, then there is a third component $\Sigma_3$ which is also an $N$-valued graph. An easy argument (using orientations) shows that there must then be a fourth component $\Sigma_4$ of $E_1 \cap \Sigma$. Using that each $\Sigma_i$ is a multi-valued graph, it follows easily that we can choose two of these four which cannot be connected in $\Omega \cap E_1$; call these $\Sigma_{i_1}, \Sigma_{i_2}$.

The rest of this argument uses these components to find a stable $\Gamma \subset \Omega$ which has points of large curvature by Proposition [I.0.11] contradicting the curvature estimates from stability. First, we construct $\partial \Gamma$. Let $\sigma_j \subset \Sigma_{i_j}$ be the images of $\{\theta = 0\}$ from $\{x_1^2 + x_2^2 = 4r_1^2\}$ to $\partial B_R$ and set $y_j = \{x_1^2 + x_2^2 = 4r_1^2\} \cap \partial \sigma_j$. By Lemma [I.1.1], $y_1, y_2$ can be connected by a curve $\sigma_0 \subset B_{C_1 r_1} \cap \Sigma$. By the maximum principle, each component of $B_{R} \cap \Sigma$ is a disk. Therefore, we can add a segment in $\partial B_R \cap \Sigma$ to $\sigma_0 \cup \sigma_1 \cup \sigma_2$ to get a closed curve $\sigma \subset \Sigma$. A result of MeYau then gives a stable embedded minimal disk $\Gamma \subset \Omega$ with $\partial \Gamma = \sigma$.

Now that we have $\Gamma$, we show that Proposition [I.0.11] applies. Namely, let (the disk) $\Gamma_{2C_1 r_1}(\sigma_0)$ be the component of $B_{2C_1 r_1} \cap \Gamma$ containing $\sigma_0$, so that $\partial \Gamma_{2C_1 r_1}(\sigma_0)$ contains a curve $\nu \subset \partial B_{2C_1 r_1}$ connecting $\sigma_1$ to $\sigma_2$. Since $\sigma_1, \sigma_2$ are in the middle sheets of $\Sigma_{i_1}, \Sigma_{i_2}$ (and $\Gamma$ is stable), $\Gamma$ contains two disjoint $(N/2 - 1)$-valued graphs $\Gamma_1, \Gamma_2$ in $E_1$ which spiral together and $\nu$ connects these (note that $E_1 \cap \Gamma$ may contain many components; at least two of these, say $\Gamma_1, \Gamma_2$, spiral together). Let $\Gamma_0$ be the component of $\Gamma_{R/2}(\sigma_0) \setminus (\nu \cup \Gamma_1 \cup \Gamma_2)$ which does not contain $\Gamma_{2C_1 r_1}(\sigma_0)$. It is easy to see that $\Gamma_0 \cap \partial \Gamma = \emptyset$; in fact, if $x \in \Gamma_0$, then $\text{dist}_t(x, \partial \Gamma) \geq |x|/2$. Therefore, for $R/r_1$ sufficiently large, Proposition [I.0.11] gives an interior point of large curvature, contradicting the curvature estimate for stable surfaces. We conclude that $E_1 \cap \Sigma \subset \Sigma_1 \cup \Sigma_2$. Finally, it follows easily that $\Sigma_2$ is oppositely oriented.

The proof of Proposition II.1.3 simplifies when $\Sigma$ is in a slab. In this case, [S], [CM2] and the gradient estimate (cf. lemma I.0.9 of [CM3]) force $\Gamma$ to spiral indefinitely if it leaves $E_1$.

### II.2. An intrinsic version of Theorem [I.0.7]

We will first show a “chord-arc” type result (relating extrinsic and intrinsic distances) assuming a curvature bound on an intrinsic ball.

**Lemma II.2.1.** (cf. lemma III.1.3 in [CM3]). Given $R_0$, there exists $R_1$ so: If $0 \in \Sigma \subset B_{R_1}$ is an embedded minimal surface, $\partial \Sigma \subset \partial B_{R_1}$, and $\sup_{B_{R_1}} |A|^2 \leq 4$, then $\Sigma_{0, R_0} \subset B_{R_1}$.
Proof. Let $\tilde{\Sigma}$ be the universal cover of $\Sigma$ and $\tilde{\Pi} : \tilde{\Sigma} \to \Sigma$ the covering map. With the definition of $\delta$-stable as in section 2 of [CM4], the argument of [CM2] (i.e., curvature estimates for 1/2-stable surfaces) gives $C > 0$ so if $B_{CR_0/2}(\tilde{z}) \subset \tilde{\Sigma}$ is 1/2-stable and $\tilde{\Pi}(\tilde{z}) = z$, then $\tilde{\Pi} : B_{5R_0}(\tilde{z}) \to B_{5R_0}(z)$ is one-to-one and $B_{5R_0}(z)$ is a graph with $B_{4R_0}(z) \cap \partial B_{5R_0}(z) = \emptyset$.

Corollary 2.13 in [CM4] gives $\epsilon = \epsilon(CR_0) > 0$ so if $|z_1 - z_2| < \epsilon$ and $|A|^2 \leq 4$ on (the disjoint balls) $B_{CR_0}(z_i)$, then each $B_{CR_0/2}(\tilde{z}_i) \subset \tilde{\Sigma}$ is 1/2-stable where $\tilde{\Pi}(\tilde{z}_i) = z_i$.

We claim that there exists $n$ so $\Sigma_{0,R_0} \subset B_{2(n+1)CR_0}$. Suppose not; we get a curve $\sigma \subset \Sigma_{0,R_0} \subset B_{R_0}$ from 0 to $\partial B_{2(n+1)CR_0}$. For $i = 1, \ldots, n$, fix points $z_i \in \partial B_{2iCR_0} \cap \sigma$. It follows that the intrinsic balls $B_{CR_0}(z_i)$ are disjoint, have centers in $B_{R_0} \subset \mathbb{R}^3$, and have $|A|^2 \leq 4$.

In particular, there exist $i_1, i_2$ with $0 < |z_{i_1} - z_{i_2}| < C' R_0 n^{-1/3} < \epsilon$, and, by corollary 2.13 in [CM4], each $B_{CR_0/2}(\tilde{z}_{i_j}) \subset \tilde{\Sigma}$ is 1/2-stable where $\tilde{\Pi}(\tilde{z}_{i_j}) = z_{i_j}$. By [CM2], each $B_{5R_0}(z_{i_j})$ is a graph with $B_{4R_0}(z_{i_j}) \cap \partial B_{5R_0}(z_{i_j}) = \emptyset$. In particular, $B_{R_0} \cap \partial B_{5R_0}(z_{i_j}) = \emptyset$. This contradicts that $\sigma \subset B_{R_0}$ connects $z_j$ to $\partial B_{CR_0}(z_{i_j})$.

An immediate consequence of Lemma [1.2.1] is that we can improve Theorem [0.7] (and hence also, by an intrinsic blow-up argument, Theorem [1.6]) by observing that the multi-valued graph can actually be chosen to be intrinsically nearby where the curvature is large (as opposed to extrinsically nearby):

**Theorem II.2.2.** Given $N \in \mathbb{Z}_+$, $\epsilon > 0$, there exist $C_1, C_2, C_3 > 0$ so: If $0 \in \Sigma^2 \subset B_R \subset \mathbb{R}^3$ is an embedded minimal disk, $\partial \Sigma \subset \partial B_R$, and $\sup_{B_{R_0}} |A|^2 \leq 4 |A|^2(0) = 4 C^2_2 r_{0}^{-2}$ for some $0 < r_0 < R$, then there exists (after a rotation) an $N$-valued graph $\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \epsilon^2 (x_1^2 + x_2^2)\}$ over $D_{R/C_2} \setminus D_{r_0}$ with gradient $\leq \epsilon$, separation $\geq C_3 r_0$ over $\partial D_{r_0}$, and $\dist(0, \Sigma_g) \leq 2 r_0$.

**Proof.** By combining theorems 0.4 and 0.6 of [CM4], we get $C_0, C_2, C_3$ so if $\sup_{\Sigma_{0,R_0}} |A|^2 \leq 4 |A|^2(0) = 4 C^2_2 r_0^{-2}$, then we get (after a rotation) an $N$-valued graph $\Sigma_g \subset \Sigma \cap \{x_3^2 \leq \epsilon^2 (x_1^2 + x_2^2)\}$ over $D_{R/C_2} \setminus D_{r_0}$ with gradient $\leq \epsilon$, separation $\geq C_3 r_0$ over $\partial D_{r_0}$, and which intersects $\Sigma_{0,r_0}$. Namely, theorem 0.4 of [CM4] gives an initial $N$-valued graph contained in $\Sigma_{0,r_0}$ and then theorem 0.6 of [CM4] extends this out to $\partial D_{R/C_2}$. Let $C_1$ be the $R_1$ from Lemma [1.2.1] with $R_0 = C_0$. By rescaling, we can assume that $|A|^2(0) = 1$ and $\sup_{B_{C_1}} |A|^2 \leq 4$. By Lemma [1.2.1], $\Sigma_{0,C_0} \subset B_{C_1}$, hence $\sup_{\Sigma_{0,C_0}} |A|^2 \leq 4$. Theorems 0.4 and 0.6 of [CM4] now give the desired $\Sigma_g$.

A standard blowup argument gives points as in Theorem [II.2.2] (with $C_4 = 1$ and $s = r_0$):

**Lemma II.2.3.** (Lemma 5.1 of [CM4]). Given $C_1, C_4$, if $B_{C_1,C_4}(0) \subset \Sigma$ is an immersed surface and $|A|^2(0) \geq 4$, then there exists $B_{C_4}(z) \subset B_{C_1,C_4}(0)$ with

$$\sup_{B_{C_4}(z)} |A|^2 \leq 4 |A|^2(z) = 4 C^2_1 s^{-2}.$$  \hspace{1cm} (II.2.4)

**Proof.** This follows as in Lemma 5.1 of [CM4], except we define $F$ intrinsically on $B_{C_1,C_4}(0)$ by $F = d^2 |A|^2$ where $d(x) = C_1 C_4 - \dist(\Sigma, x)$ (so $F = 0$ on $\partial B_{C_1,C_4}(0)$, $F(0) \geq 4 (C_1 C_4)^2$). Let $F(z)$ be the maximum of $F$ and set $s = C_1 |A|^2(0)$. It follows that $\sup_{B_{d(z)/2}(z)} |A|^2 \leq 4 |A|^2(z)$ and (using $F(z) \geq (C_1 C_4)^2$) we get $2C_4 s \leq d(z)$, giving (II.2.4).
Part III. The stacking and the proof of Theorem 1.2

This part deals with how the multi-valued graphs given by [CM4] fit together. As mentioned in the introduction, a general embedded minimal disk with large curvature at some point should be thought of as obtained by stacking such graphs on top of each other.

III.1. Decomposing disks into multi-valued graphs

Fix $N > 6$ large, $1/10 > \epsilon > 0$ small. We will choose $\epsilon_g > 0$ small depending on $\epsilon$ and then let $N_g = N_g(\epsilon_g)$ be given by proposition II.2.12 of [CM3]. Below $\Sigma$ will be an embedded minimal disk. Theorem [1.2.2] gives $C_1, C_2, C_3$ (depending on $\epsilon_g, N,$ and $N_g$) so if $B_R(y) \cap \partial \Sigma = \emptyset$ and the pair $(y, s)$ satisfies

$$
\sup_{E_{a.s.}(y)} |A|^2 \leq 4 |A|^2(y) = 4 C^2 g^2 s^{-2}, \tag{III.1.1}
$$

then (after a rotation) we get an $(N + N_g + 4)$-valued graph $\hat{\Sigma}_1$ over $D_{2\epsilon g^{1/2}}(p) \setminus D_{e^{-N_g} s/2}(p)$ with gradient $\leq \epsilon_g$, separation $\geq C_3 s$ over $\partial D_*(p)$, and $\text{dist}_\Sigma(y, \hat{\Sigma}_1) \leq 2s$ (where $p = (y_1, y_2, 0)$). In particular, by proposition II.2.12 of [CM3] and the version of the “estimate between the sheets” given in theorem III.2.4 of [CM3], we can choose $\epsilon_g = \epsilon_g(\epsilon) > 0$ so that

1. the concentric $(N + 3)$-valued subgraph $\hat{\Sigma}_1$ over $D_{R/C_2}(p) \setminus D_s(p)$ satisfies (I.0.8) and
2. each component of $\Sigma$ between the sheets of $\hat{\Sigma}_1$ (as in (1.1.4)) is an $(N + 2)$-valued graph also satisfying (I.0.8). In the remainder of this section, $C_1, C_2, C_3$ will be fixed.

Let $\epsilon_0, C_0$ be from Proposition [1.1.3] and suppose $\epsilon < \epsilon_0$. If $s < R/(8C_2 C_0)$ for such a pair $(y, s)$, then Proposition [1.1.3] applies. Let $\hat{E}, E$ be the regions between the sheets of the concentric $(N + 2)$-valued and $(N + 1)$-valued, respectively, subgraphs of $\hat{\Sigma}_1$ (over $D_{R/C_2}(p) \setminus D_s(p)$).

By Proposition [1.1.3] and (2) above, $\hat{E} \cap \Sigma \setminus \hat{\Sigma}_1$ is an $(N + 1)$-valued graph $\hat{\Sigma}_2$; similarly, $E \cap \Sigma \setminus \hat{\Sigma}_1$ is an $N$-valued graph $\Sigma_2 \subset \hat{\Sigma}_2$. Let $\Sigma_1 \subset \hat{\Sigma}_1$ be the concentric $N$-valued subgraph. Since $\partial \Sigma_{y, 4s}$ is a simple closed curve, it must pass through $E \setminus \Sigma_1$. Therefore, since $\Sigma_2$ is the only other part of $\Sigma$ in $E$, we can connect $\Sigma_1$ and $\Sigma_2$ by curves $\nu_\pm \subset \partial B_{4s}(y) \cap \Sigma$ which are above and below $E$, respectively. This gives components $\Sigma_\pm$ of $\Sigma_{y, R/(2C_2)} \setminus (\Sigma_1 \cup \Sigma_2 \cup \nu_\pm)$ which do not contain $\Sigma_{y, s}$ and which are above and below $E$, respectively (these will be the $\Sigma_0$’s for Proposition [1.1.1]).

Given a pair satisfying (III.1.1), Proposition [0.11] and Lemma [2.3] easily give two nearby pairs (one above and one below):

**Lemma III.1.2.** There exists $C_4 > 1$ so: If $0 \in \Sigma \subset B_{3R}$ is an embedded minimal disk with $\partial \Sigma \subset \partial B_{3R}$; $(0, s)$ satisfies (III.1.1), and $s < \min\{R/(2C_4), R/(8C_2 C_0)\}$, then we get $(y_-, s_-)$ also satisfying (III.1.1) with $y_- \in \Sigma_-$ and $\Sigma_{y-, 4s_-} \subset \Sigma_{0, C_4} \setminus B_{4s}$. Moreover, the $N$-valued graphs corresponding to $(0, s), (y_-, s_-)$ are disjoint.

**Proof.** Proposition [0.11] gives $C_4 = C_4(C_1)$ and $z \in \Sigma_{0, C_4} \setminus B_{8s}$ with $|z|^2 |A|^2(z) \geq 4(8C_1)^2$. Since $E \cap \Sigma$ consists of the multi-valued graphs $\hat{\Sigma}_1, \hat{\Sigma}_2$, we have $|z|^2 |A|^2(z) \leq C$ on $\hat{E} \cap B_{C_4 s} \setminus B_{2s}$ for $C$ small ($C$ can be made arbitrarily small by choosing $\epsilon$ even smaller). Hence, $z \notin \hat{E}$ and so $B_{3|z|^2/2}(z) \cap E = \emptyset$. Applying Lemma [2.3] on $B_{|z|^2/2}(z)$, we get $(y_-, s_-)$ satisfying (III.1.1) with $B_{8s_-}(y_-) \subset B_{|z|^2/2}(z) \subset \Sigma_-$ and $\Sigma_{y-, 4s_-} \subset \Sigma_{0, C_4} \setminus B_{4s}$ and the corresponding $N$-valued graphs are disjoint. \[\square\]
Let $C_4$ be given by Lemma III.1.2. Iterating the construction of Lemma III.1.2, we can decompose an embedded minimal disk into basic building blocks ordered by heights (the points $p_i$ in Corollary III.1.3 are the projections to $\{x_3 = 0\}$ of the blowup points $y_i$):

**Corollary III.1.3.** There exist $C_5 > 1, \tilde{C}_3 > 0$ so: Let $\Sigma \subset B_{C_5R}$ be an embedded minimal disk, $\partial \Sigma \subset \partial B_{C_5R}$. If $(y_0, s_0)$ satisfies (III.1.1) with $B_{C_4}(y_0) \subset B_R$, then there exist $\{(y_i, s_i)\}$ (for $i > 0$) satisfying (III.1.1) with $y_i \in \Sigma$ and corresponding (disjoint) $N$-valued graphs $\Sigma_i \subset \Sigma$ of $u_i$ over $D_{2R}(0) \setminus D_{2s_i}(p_i)$ with gradient $\leq 2 \epsilon$, separation $\geq \tilde{C}_3 s_i$ over $\partial D_{2s_i}(p_i)$, if $i < j$ and both $u_i, u_j$ are defined at $p$, then $u_j(p) < u_i(p)$,

$$\Sigma_{y_{i+1},4s_{i+1}} \subset \Sigma_{y_i,c_{4s_i}} \setminus B_{4s_i}(y_i) \text{ and } \cup_i B_{C_4s_i}(y_i) \supset B_R \neq \emptyset. \tag{III.1.5}$$

**Proof.** Starting with $(y_0, s_0)$, we can apply Lemma III.1.2 repeatedly, until the second part of (III.1.3) holds, to find bottom $N$-valued graphs giving (III.1.4) and the first part of (III.1.3). Each $N$-valued graph is a graph over some plane with gradient $\leq \epsilon$. Since $\Sigma$ is embedded, we can take these to be graphs over a fixed plane with gradient $\leq 2 \epsilon$ (after possibly taking $C_5 > 3C_2 + 1$ larger). $\tilde{C}_3 > 0$ is now just a fixed fraction of $C_3$. \qed

In the next lemma and corollary, $\Sigma \subset B_{C_5R}$ is an embedded minimal disk, $\partial \Sigma \subset \partial B_{C_5R}$.

**Lemma III.1.6.** If $(y, s)$ satisfies (III.1.1), $B_s(y) \subset B_{R/2}$, then the corresponding 2-valued graph over $D_R(0) \setminus D_s(p)$ (after a rotation) has separation $\geq C_3(s/R)^{\epsilon} s/2$ at $\partial D_R(0)$.

**Proof.** By the discussion around (III.1.1), the separation $|w|$ is $\geq C_3 s$ at $\partial D_s(p)$ and $|\nabla \log |w|| \leq \epsilon/\rho_p$ on $D_{2R}(p) \setminus D_s(p)$. Since $D_s(p) \subset D_{R/2}(0)$, integrating gives

$$\min_{\partial D_R(0)} |w| \geq \min_{D_{2R}(p) \setminus D_{R/2}(p)} |w| \geq C_3 (s/(2R))^{\epsilon} s. \tag{III.1.7}$$

**Corollary III.1.8.** There exists $C_6 > 0$ so if $(0, s)$ satisfies (III.1.1) and $\sup_{B_{R}(0) \cap \Sigma_{+}} |A|^2 \leq 5C^2_2 s^{-2}$ for some $4C^2_2 s < \ell < R$, then there exists $(z, r)$ satisfying (III.1.1) with $\Sigma_{z,r} \subset \Sigma_{0,\ell/2}$, so the separation at $\partial D_{\ell}(0)$ between the 2-valued graphs $\Sigma_0, \Sigma_z$, corresponding to $(0, s)$, $(z, r)$, is $\geq C_6 (s/\ell)^{\epsilon} \ell$, and $\Sigma_z \subset \Sigma_{-}$.

**Proof.** Set $(y_0, s_0) = (0, s)$ and let $(y_i, s_i), \Sigma_i, u_i, p_i$ be given by Corollary III.1.3. Let $i_0$ be the first $i$ with $B_{C_4s_{i_0}}(y_{i_0}) \setminus B_{\ell/2}(0) \neq \emptyset$. Set $(z, r) = (y_{i_0-1}, s_{i_0-1})$. It follows for $i < i_0$ that $B_{s_i}(y_i) \subset B_{\ell/2}(0)$ and $s_i \geq s/2$ since $\sup_{B_{R}(0) \cap \Sigma_{-}} |A|^2 \leq 5C^2_2 s^{-2}$. Hence, by Lemma III.1.6 (as in Corollary III.1.3), $\Sigma_i$ has separation $\geq \tilde{C}_3 (s/\ell)^{\epsilon} s_i/4$ at $\partial D_{\ell}(0)$ for $i < i_0$. By (III.1.3), $\ell/4 \leq \sum_{i \leq i_0} C_4 s_i \leq (1 + C_4) \sum_{i < i_0} C_4 s_i$. Since the $\Sigma_i$’s are ordered by height, the separation at $\partial D_{\ell}(0)$ between $\Sigma_0$ and $\Sigma_z = \Sigma_{i_0-1}$ is $\geq \sum_{i < i_0} \tilde{C}_3 (s/\ell)^{\epsilon} s_i/4 \geq C_6 (s/\ell)^{\epsilon} \ell$. \qed

**III.2. Stacking multi-valued graphs and Theorem 0.2**

If $(y, s)$ satisfies (III.1.1), then $\Sigma_y$ is the corresponding 2-valued graph and $\Sigma_{y,-}$ the portion of $\Sigma$ below $\Sigma_y$. Given $C > 8$, we will consider such pairs which in addition satisfy

$$\sup_{B_{C_7}(y) \cap \Sigma_{y,-}} |A|^2 \leq 4|A|^2(y) = 4C^2_7 s^{-2}. \tag{III.2.1}$$

Using Section III.1, we show next that a pair $(0, s)$ satisfying (III.2.1) has a nearby pair with a definite height below $\Sigma_0$. $\Sigma \subset B_{C_8R}$, $\partial \Sigma \subset \partial B_{C_8R}$ is an embedded minimal disk.
Proposition III.2.2. See fig. 9. There exist $C, C' > 10 C_1^2$ and $\delta > 0$ so if $(0, s)$ satisfies \((\text{III.2.1})\) with $s < R/\hat{C}$, $\Sigma_0 \subset \Sigma$ is over $D_R \setminus D_s$ (without a rotation), and $\nabla u((Rs)^{1/2}, 0) = 0$, then we get $(y, t)$ satisfying \((\text{III.2.1})\) with $y \in C_b(0) \cap \Sigma - \setminus B_{C_s/2}$.

Proof. We will choose $C$ large below and then set $\delta = \delta(C) > 0$, $\hat{C} = \hat{C}(C)$. Note first that (since $\nabla u((Rs)^{1/2}, 0) = 0$), corollary 1.14 of [CM7] gives $|\nabla u(\rho, \theta)| \leq C_a (\rho/s)^{-5/12}$ for $s \leq \rho \leq (Rs)^{1/2}$. Integrating this, we get for $s \leq \rho \leq (Rs)^{1/2}$

$$|u(\rho, \theta)| \leq s + C_a \int_0^\rho (\tau/s)^{-5/12} d\tau \leq (1 + 2C_a) (s/\rho)^{5/12} \rho. \quad \text{(III.2.3)}$$

Proposition (I.0.11) gives $C_b(C_1, C)$, $\Sigma_0 \subset B_{C_b \delta} \cap \Sigma_\setminus \setminus B_{1s}$ with $|A|^2(\Sigma_0) \geq 5 C^2 C_1^2 |\Sigma_0|^{-2}$. Set

$$A = \{x \in B_{C_b \delta} \cap \Sigma_\setminus |A|^2(x) \geq 5 C^2 C_1^2 |x|^{-2}\}, \quad \text{(III.2.4)}$$

(so $\Sigma_0 \in A$) and let $x_0 \in A$ satisfy $|x_0| = \inf_{x \in A} |x|$. So $|A|^2 \leq 5 C^2 s^{-2}$ on $B_{|x_0|} \cap \Sigma_\setminus$ by \((\text{III.2.7})\) and $C_b \leq |x_0| \leq C_b s$. An obvious extrinsic version of Lemma \((\text{II.2.3})\) (cf. Theorem \(A.9\)) gives $(y, t)$ satisfying \((\text{III.2.1})\) with $B_{C_b}(y) \subset B_{|x_0|/2}(y_0)$. We can assume $|p| \geq |y|/5$.

Since $|A|^2 \leq 5 C^2 s^{-2}$ on $B_{|x_0|/2} \cap \Sigma_\setminus$ and $(0, s)$ satisfies \((\text{III.2.1})\) hence \((\text{III.1.1})\), Corollary \((\text{II.1.8})\) (with $\ell = |p|$) gives $(z, r)$ also satisfying \((\text{III.1.1})\) with $\Sigma_z \subset \Sigma_\setminus$, $\Sigma_z \cap \Sigma_\setminus \cap \Sigma_\setminus \cap \Sigma_\setminus \cap \Sigma_\setminus$ and so the separation between $\Sigma_0$ and $\Sigma_z$ is at least $\bar{C}_z (s/|y|)^{\ell} |y|$ at $p$. However, since $\Sigma$ is embedded, then $\Sigma_y$ must be below both $\Sigma_0$ and $\Sigma_z$. Combining this with \((\text{III.2.3})\) gives

$$|x_3|/|y| \geq C (s/|y|)^{\ell} - (1 + 2 C_a) (s/|y|)^{5/12}. \quad \text{(III.2.5)}$$

Since $C \leq 2|y|/s \leq 3 C_b$, by choosing $C$ sufficiently large and then setting $\hat{C} = \hat{C}(C, C_5)$, $C_b = C_b(C)$ (where $\hat{C}$ is chosen so $C_b s \leq (Rs)^{1/2}$) the proposition follows from \((\text{III.2.3})\).

We show next Theorem \((\text{II.2})\). Namely, iterating Proposition \((\text{II.2.2})\), we show that if the curvature of an embedded minimal disk were large at a point, then it would be forced to grow out of the half-space it was assumed to lie in. First we need:

Lemma III.2.6. Given $C, \delta > 0$, there exists $\epsilon_1 > 0$ so: Let $\Sigma \subset B_{2R_0} \cap \{x_3 > 0\}$ be an embedded minimal disk, $\partial \Sigma \subset \partial B_{2R_0}$, and $\sup_{\Sigma \cap \{x_3 \leq \delta r_0\}} |A|^2 \leq C r_0^{-2}$, then $\sup_{\Sigma} |A|^2 \leq r_0^{-2}$ for all components $\Sigma'$ of $B_{2R_0} \cap \Sigma$ which intersect $B_{\epsilon_1 r_0}$.

Proof. If $y \in B_{R_0} \cap \Sigma \cap \{x_3 \leq \delta r_0/4\}$, then $\sup_{\Sigma \cap \{x_3 \leq \delta r_0/4\}} |\nabla x_3|^2 \leq C x_3^2 (\delta r_0)^{-2}$ (by the gradient estimate) and hence $\Sigma_{y, \delta r_0/2}$ is a graph for $x_3(y)/(\delta r_0)$ small; cf. Lemma \((\text{A.3})\). The lemma follows by applying this to a chain of balls as in the proof of lemma 2.10 in [CM8] or the theorem in [CM10].

Let $C_1, \ldots, C_6$ be as above and $\delta, C, \hat{C}$ be from Proposition \((\text{III.2.2})$.

Proof. (of Theorem \((\text{II.2})\)$ By Lemma \((\text{II.2.6})\)$ and scaling), it suffices to find $d > 0$ and $\hat{C} > 1$ so if $\Sigma \subset B_{4C_5 \hat{C} R} \cap \{x_3 > 0\}$ and $\partial \Sigma \subset \partial B_{4C_5 \hat{C} R}$, then

$$\sup_{B_{2dR} \cap \Sigma} |A|^2 \leq 4 C^2 C_1^2 (dR)^{-2}. \quad \text{(III.2.7)}$$

Suppose \((\text{III.2.7})\) fails; we will get a contradiction. An obvious extrinsic version of Lemma \((\text{II.2.3})\)$ gives $(y_0, s_0)$ satisfying \((\text{III.2.1})\)$ with $B_{C s_0}(y_0) \subset B_{2dR}$. Let $\Sigma_0$ be the corresponding $N$-valued graph of $u_0$ over $D_{\hat{C} R} \setminus D_{s_0}(p_0)$ and $\Sigma_\setminus$ the portion of $\Sigma$ below $\Sigma_0$. 

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To apply Proposition III.2.2 we will need that if \((y, s)\) satisfies (III.2.1) with \(y \in B_{2R} \cap \Sigma_-\) (where \(\Sigma_y\) is a graph of \(u\) over \(D_{\hat{C}R} \setminus D_s(p)\), then
\[
s \leq R/\hat{C} \quad \text{and} \quad |\nabla u((\hat{C}Rs)^{1/2}, 0)| < \delta/4.
\]
(III.2.8)

To see (III.2.8), observe first that by proposition II.2.12 of [CM3] (since \(u_0 > 0\), \(\sup_{D_{6R}} u_0 \leq 2dR(6/d)^{\epsilon} \leq 12 R d^{1-\epsilon}\). It follows that \(B_{6R} \cap \Sigma_- \subset \{0 < x_3 < 12 R d^{1-\epsilon}\}\); hence, \(s \leq C_a R d^{1-\epsilon}\) and by the gradient estimate \(\sup_{\partial D_{6d}} |\nabla u| \leq C_b d^{1-\epsilon}\). Lemma 1.8 of [CM7] and the mean value inequality (as in corollary 1.14 of [CM7]) gives \(|\text{Hess}_u| \leq C_c (\hat{C}R)^{-5/12} \rho^{-7/12}\) for \((\hat{C}Rs)^{1/2} \leq \rho \leq \hat{C}R\). Combining these gives at \(\rho = (\hat{C}Rs)^{1/2}, \theta = 0\)
\[
|\nabla u| \leq C_b d^{1-\epsilon} + C_c (\hat{C}R)^{-5/12} \int_0^{8R} t^{-7/12} dt = C_b d^{1-\epsilon} + C_c^2 \hat{C}^{-5/12}.
\]

(III.2.9)

In particular, for \(d > 0\) small and \(\hat{C}\) large, (III.2.9) gives (III.2.8).

Repeatedly applying Proposition III.2.2 (using (III.2.8)) gives \(((y_{i+1}, s_{i+1})\) satisfying (III.2.1) with \(y_{i+1} \in C_{\theta/2}(y_i) \cap \Sigma_- \setminus B_{C_s/2}(y_i)\). After choosing \(d > 0\) even smaller, it follows that the \(y_i\)’s must leave the half-space before they leave \(B_R\). \(\square\)

**Proof.** (of Corollary 4.3) Using \(\Sigma_1 \cup \Sigma_2\) as a barrier, [MeYa], and a linking argument (cf. Lemma 1.1.1) give a stable surface \(\Gamma \subset B_{cR_\theta} \setminus (\Sigma_1 \cup \Sigma_2)\) with \(\partial \Gamma \subset \partial B_{cR_\theta}\) and \(B_{cR_\theta} \cap \Gamma \neq \emptyset\). Estimates for stable surfaces give a graphical component of \(B_{2R_0} \cap \Gamma\) which intersects \(B_{cR_\theta}\). The corollary now follows from Theorem 4.2. \(\square\)

**Part IV. The short connecting curves and Theorem 1.0.10**

We first combine Lemmas 1.1.1 and 1.2.1 to see that any curve in \(\Sigma\) which intersects both above and below a multi-valued graph (with a curvature bound on an intrinsic ball) connects it to a fixed intrinsic ball:

**Corollary IV.0.10.** Given \(C_1\), there exists \(C_4 > 1\) so: Let \(\Sigma, \Sigma_d, E_0, r_0\) be as in Lemma 1.1.1. If \(\sup_{B_{C_4^2 r_0}} |A|^2 \leq 4 C_4^2 r_0^{-2}\) and \(\eta \subset B_{2r_0} \cap \Sigma\) connects points in \(\partial B_{2r_0}\) above and below \(E_0\), then \(\eta \subset B_{C_4 r_0}\).

**Proof.** Let \(\Sigma_{2r_0}(\eta)\) be the component of \(B_{2r_0} \cap \Sigma\) containing \(\eta\). By the maximum principle, \(\Sigma_{2r_0}(\eta)\) is a disk and so \(\partial \Sigma_{2r_0}(\eta)\) must pass through \(E_0\) (to connect the points on opposite sides of \(E_0\)). Hence, by Lemma 1.1.1, \(\Sigma_{2r_0}(\eta) \subset \Sigma_{0, C_4 r_0}\). Finally, by Lemma 1.2.1, \(\Sigma_{0, C_4 r_0} \subset B_{C_4 r_0}\), giving the corollary. \(\square\)

**Proof.** (of Theorem 1.0.10). Fix \(\epsilon > 0\) with \(\epsilon < \min\{\epsilon_0, \epsilon_s\}\) (\(\epsilon_0\) given by Proposition 1.1.3 and \(\epsilon_s\) from Lemma 1.1.1). Choose \(N_0, R_0\) large so that Proposition 1.1.3 gives “the other half” \(\Sigma_2\). If \(\Sigma_1\) comes from an intrinsic blow up point, then it follows from Lemma 1.1.1 that there are short curves connecting \(\Sigma_1\) and \(\Sigma_2\). While it is a priori not clear that every multi-valued graph arises this way, Theorem 1.2.2 implies that every multi-valued graph is intrinsically near one of these. We use this below to produce the short curves \(\sigma_\theta\) in general.

Suppose that no \(\sigma_\theta\) with length \(\leq C\) exists for some \(\theta\); we get \(y_i \in \{x_1^2 + x_2^2 = 1\} \cap \Sigma_i\) for \(i = 1, 2\) with \(\text{dist}_\Sigma(y_1, y_2) > C\) and so \(y_1, y_2\) are in consecutive sheets of \(\Sigma\) (i.e., \(y_1\) and \(y_2\) can be connected by a segment parallel to the \(x_3\)-axis which does not otherwise intersect \(\Sigma\)). See fig. 19. We will get a contradiction from this for \(C\) large.
Similarly, Lemma II.2.3 gives pairs $(\eta_1, \eta_2)$ from $\Sigma_2$ and $\hat{\eta}$.

It follows easily from Corollary IV.0.10 that $\hat{\eta}$ intersects only one side of each of $\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3$. Combining Corollary IV.0.10, Length$(\eta_i) \leq C'$, and dist$_{\Sigma}(\eta_1, \eta_2) > C - 2C'$, it is easy to see that, for $C$ large, $\eta_1$ intersects only one side of $\hat{\Sigma}_1 \cup B_{2s_1}(z_1)$; similarly, $\eta_2$ intersects only one side of $\hat{\Sigma}_2 \cup B_{2s_2}(z_2)$.

We will next find a third pair $(z_3, s_3)$ satisfying (II.2.4) which is between $\hat{\Sigma}_1 \cup B_{2s_1}(z_1)$ and $\hat{\Sigma}_2 \cup B_{2s_2}(z_2)$ but which is intrinsically far from $\eta_1, \eta_2$; Corollary IV.0.10 will then give a contradiction. Since dist$_{\Sigma}(\eta_1, \eta_2) > C - 2C'$, $\eta_1$ intersects only one side of $\hat{\Sigma}_1 \cup B_{2s_1}(z_1)$ and $\eta_2$ intersects only one side of $\hat{\Sigma}_2 \cup B_{2s_2}(z_2)$. (This last condition means that there is a curve $\eta_{y_{3}}$ from $B_{2s_1}(z_1)$ to $B_{2s_2}(z_2)$ so $y_{3}$ is $\eta_{y_{3}}$ and $\eta_{y_{3}}$ intersects only one side of each of $\hat{\Sigma}_1 \cup B_{2s_1}(z_1), \hat{\Sigma}_2 \cup B_{2s_2}(z_2)$.) As above, Lemma II.2.3 gives a pair $(z_3, s_3)$ satisfying (II.2.4) with $B_{C_4s_3}(z_3) \subset B_{C'}(y_{3})$ and then Theorem II.2.2 gives corresponding $\hat{\Sigma}_3, \hat{\Sigma}_3$. Since $C'$ does not depend on $C$, we can assume that

$$\text{dist}_{\Sigma}(z_3, \{\eta_1, \eta_2\}) > C/4.$$  \hspace{1cm} (IV.0.11)

It follows easily from Corollary IV.0.10 that $\hat{\Sigma}_3$ is between $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$ (since $\hat{\Sigma}_3$ is close to $y_3$ and $y_3$ is far from $\hat{\Sigma}_1, \hat{\Sigma}_2$). Moreover, it is easy to see that at least one of $\eta_1, \eta_2$ must
intersect both sides of \( \hat{E}_3 \cup B_{2\lambda_3}(z_3) \) and, therefore, Corollary [IV.0.10] gives
\[
\text{dist}_\Sigma(\hat{\Sigma}_3, \{\eta_1, \eta_2\}) \leq C''
\] (IV.0.12)
\((C'' \text{ independent of } C)\). For \( C \) large, [IV.0.11] contradicts [IV.0.12], giving the theorem. \( \square \)

APPENDIX A. ONE-SIDED REIFENBERG CONDITION AND CURVATURE ESTIMATES

We will show here curvature estimates for minimal hyper-surfaces, \( \Sigma^{n-1} \subset M^n \), which on all sufficiently small scales lie on one side of, but come close to, a hyper-surface with small curvature. Such a minimal hyper-surface is said to satisfy the one-sided Reifenberg condition. Note that no assumption on the topology is made. Inspired by the classical Reifenberg condition (cf. [ChC] and references therein) we make the definition:

**Definition A.1.** A subset, \( \Gamma \), of \( M^n \) satisfies the \((\delta, r_0)\)-one-sided Reifenberg condition at \( x \in \Gamma \) if for every \( 0 < \sigma \leq r_0 \) and every \( y \in B_{r_0-\sigma}(x) \cap \Gamma \), there is a connected hyper-surface, \( L_{x,\sigma}^{n-1} \), with \( \partial L_{y,\sigma} \subset \partial B_{r_0}(y) \),
\[
B_{\delta \sigma}(y) \cap L_{y,\sigma} \neq \emptyset, \quad \sup_{B_{\sigma}(y) \cap L} |A_L|^2 \leq \delta^2 \sigma^{-2},
\] (A.2)
and the component of \( B_{\sigma}(y) \cap \hat{\Gamma} \) through \( y \) lies on one side of \( L_{y,\sigma} \).

**Lemma A.3.** There exist \( r_1(i_0, k, n) > 0, \ 0 < \epsilon_0 < 1, C = C(n) \) so for \( \epsilon \leq \epsilon_0, \ r_0 \leq r_1 \):
If \( z \in \Sigma^{n-1} \subset B_{r_0} = B_{r_0}(z) \subset M^n \) is an embedded minimal hyper-surface, \( \partial \Sigma \subset \partial B_{r_0} \), and there is a connected hyper-surface, \( L_{y,0}^{n-1} \), with \( \partial L \subset \partial B_{r_0}, B_{r_0} \cap L \neq \emptyset \)
\[
\sup_{B_{r_0} \cap L} |A_L|^2 \leq \epsilon^2 r_0^{-2},
\] (A.4)
\[
\sup_{B_{r_0} \cap \Sigma} |A|^2 \leq \epsilon_0^2 r_0^{-2},
\] (A.5)
and \( B_{r_0} \cap \Sigma \) lies on one side of \( L \), then \( |A(z)|^2 \leq \epsilon \epsilon_0^2 r_0^{-2} \).

**Proof.** We will prove this for \( B_{r_0} = B_{r_0}(0) \subset \mathbb{R}^n \) \((z = 0, \ r_1 = \infty)\); the general case is similar (cf. [CM1]). Choose \( \epsilon_0 > 0 \) so: If \( B_{2\lambda}(y) \subset \Sigma, s \sup_{B_{2\lambda}(y)} |A| \leq 4 \epsilon_0 \), and \( t \leq 9s/5 \), then \( z \in \Sigma \) is a graph over \( T_y \Sigma \) with gradient \( \leq t/s \) and
\[
\inf_{y' \in B_{2\lambda}(y)} |y' - y|/\text{dist}_\Sigma(y, y') > 9/10.
\] (A.6)
Using \( B_{r_0} \cap L \neq \emptyset \), let \( L_{y_1} \) be a component of \( B_{r_0} \cap L \) containing some \( y_L \in B_{r_0} \cap L \). By (A.4) and (A.3), \( L_{y_1} \subset B_{3\lambda}(y_L) \). Hence, by (A.4), we can rotate so \( L_{y_1} \) is a graph over \( \{x_n = 0\} \) with \(|\nabla x_n| \leq \epsilon \) and \(|x_n(L)| \leq 4 \epsilon r_0 \). Since \( L \cap \Sigma = \emptyset \), \( x_n + 4 \epsilon r_0 > 0 \) is harmonic on \( B_{2\lambda} \subset \Sigma \). By (A.3), the Harnack inequality (and \( 0 \in \Sigma \)) yields \( C = C(n) \) so
\[
0 < \sup_{B_{\lambda}} (x_n + 4 \epsilon r_0) \leq C \inf_{B_{\lambda}} (x_n + 4 \epsilon r_0) \leq 4 C \epsilon r_0.
\] (A.7)
Since \( B_{r_0} \) is a graph with bounded gradient, elliptic estimates give
\[
\int_{B_{\lambda}} |A|^2 \leq C' r_0^{-4} \int_{B_{\lambda}} |x_n|^2,
\] (A.8)
where $C' = C'(n)$. Combining (A.1) and (A.8), the lemma follows from the mean value inequality since $\Delta |A|^2 \geq -2 |A|^4$ (see (CMI)). \hfill \Box

**Theorem A.9.** (Curvature estimate). There exist $\epsilon_1(i_0, k, n), r_1(i_0, k, n) > 0$ so: If $r_0 \leq r_1$, $\Sigma^{n-1} \subset B_{r_0} = B_{r_0}(x) \subset M^n$ is an embedded minimal hyper-surface, $\partial \Sigma \subset \partial B_{r_0}$, and $\Sigma$ satisfies the $(\epsilon_1, r_0)$-one-sided Reifenberg condition at $x$, then $\sup_{B_{r_0-\sigma}\cap \Sigma} |A|^2 \leq \sigma^{-2}$ for $0 < \sigma \leq r_0$.

**Proof.** Take $r_1 > 0$ as in Lemma A.3, and set $F = (r_0-r)^2 |A|^2$. Since $F \geq 0$, $F|\partial B_{r_0}\cap \Sigma = 0$, and $\Sigma$ is compact, $F$ achieves its supremum at $y \in \partial B_{r_0-\sigma} \cap \Sigma$ with $0 < \sigma \leq r_0$. If $F \leq 1$, the theorem follows trivially. Hence, we may suppose $F(y) = \sup_{B_{r_0}\cap \Sigma} F > 1$. With $\epsilon_0 \leq 1$ as in Lemma A.3, define $s > 0$ by $s^2 |A(y)|^2 = \epsilon_0^2/4$. Since $F(y) = \sigma^2 |A(y)|^2 > 1$ and $\epsilon_0 \leq 1$, we have $2s < \sigma$. Since $F(y) > 1$,

$$\sup_{B_s(y)\cap \Sigma} \left( \frac{\sigma}{2} \right)^2 |A|^2 \leq \sup_{B_{s/2}(y)\cap \Sigma} \left( \frac{\sigma}{2} \right)^2 |A|^2 \leq \sup_{B_{s/2}(y)\cap \Sigma} F = \sigma^2 |A(y)|^2. \tag{A.10}$$

Multiplying (A.10) by $4s^2/\sigma^2$ gives $\sup_{B_s(y)\cap \Sigma} s^2 |A|^2 \leq \epsilon_0^2$. Hence, the $(\epsilon_1, r_0)$-one-sided Reifenberg assumption, Lemma A.3, contradicting the choice of $s$ if $C \epsilon_1^2 < \epsilon_0^2/4$. Therefore, $F \leq 1$ for this $\epsilon_1$, and the theorem follows. \hfill \Box

Letting $r_0 \to \infty$ in Theorem A.9 gives the Bernstein type result:

**Corollary A.11.** There exists $\epsilon(n) > 0$ so any connected properly embedded minimal hyper-surface satisfying the $(\epsilon, \infty)$-one-sided Reifenberg condition is a hyper-plane.

We close by giving a condition which implies the one-sided Reifenberg condition. Its proof (left to the reader) relies on a simple barrier argument (as in the proof of Corollary 0.4).

**Lemma A.12.** There exist $\epsilon_0(i_0, k), r_1(i_0, k) > 0, c(i_0, k) \geq 1$ so: Let $\Sigma^2 \subset B_{r_0} = B_{r_0}(x) \subset M^3$ be an embedded minimal disk, $\partial \Sigma \subset \partial B_{r_0}$, and $r_0 \leq r_1$. If for some $\epsilon < \epsilon_0$, all $\sigma < r_0$ and all $y \in B_{r_0-\sigma} \cap \Sigma$ there is a minimal surface $\Sigma y, \sigma \subset B_{\sigma}(y) \setminus \Sigma$ with $\partial \Sigma y, \sigma \subset \partial B_{\sigma}(y)$ and $\Sigma y, \sigma \cap B_{\epsilon, \sigma}(y) \neq \emptyset$, then $\Sigma$ satisfies the $(\epsilon, \infty, r_0)$-one-sided Reifenberg condition at $x$.

**Appendix B. Laminations**

A codimension one lamination on a 3-manifold $M^3$ is a collection $\mathcal{L}$ of smooth disjoint surfaces (called leaves) such that $\cup_{\Lambda \in \mathcal{L}} \Lambda$ is closed. Moreover, for each $x \in M$ there exists an open neighborhood $U$ of $x$ and a coordinate chart, $(U, \Phi)$, with $\Phi(U) \subset \mathbb{R}^3$ so in these coordinates the leaves in $\mathcal{L}$ pass through $\Phi(U)$ in slices of the form $(\mathbb{R} \times \{t\}) \cap \Phi(U)$. A foliation is a lamination for which the union of the leaves is all of $M$ and a minimal lamination is a lamination whose leaves are minimal. Finally, a sequence of laminations is said to converge if the corresponding coordinate maps converge. Note that any (compact) embedded surface (connected or not) is a lamination.

**Proposition B.1.** Let $M^3$ be a fixed 3-manifold. If $\mathcal{L}_i \subset B_{2R}(x) \subset M$ is a sequence of minimal laminations with uniformly bounded curvatures (where each leaf has boundary contained in $\partial B_{2R}(x)$), then a subsequence, $\mathcal{L}_j$, converges in the $C^\alpha$ topology for any $\alpha < 1$ to a (Lipschitz) lamination $\mathcal{L}$ in $B_R(x)$ with minimal leaves.
Proof. For convenience we will assume that each lamination $\mathcal{L}_i$ has only finitely many leaves where the number of leaves may depend on $i$. This is all that is needed in the application of this proposition anyway. Fix $x_0 \in B_R(x)$. The proposition will follow once we construct uniform coordinate charts $\Phi_i$ on a ball $B_{r_0} = B_{r_0}(x_0)$, where $4r_0 \leq R$ is to be chosen.

By assumption, there exists $C$ so that $\sup_{B_{4r_0} \cap \Lambda} |A|^2 \leq C r_0^{-2}$ for each $i$ and every $\Lambda \in \mathcal{L}_i$. Replacing $r_0 > 0$ with a smaller radius, we may assume that $C > 0$ and $r_0 \sqrt{k}$ are as small as we wish and $r_0 < \frac{d}{2}$ ($i_0$ being the injectivity radius and $k$ a bound for the curvature of $M$ in $B_{4r_0}$). In fact, if $(x_1, x_2, x_3)$ are exponential normal coordinates centered at $x_0$ on $B_{r_0}$, then $\cup_{\Lambda \in \mathcal{L}_i} B_{r_0} \cap \Lambda$ gives a sequence of disconnected small curvature surfaces in these coordinates. By standard estimates for normal coordinates, the curvature is also small with respect to the Euclidean metric. Going to a further subsequence (possibly with $r_0$ even smaller), for each $i$ every sheet of $\cup_{\Lambda \in \mathcal{L}_i} B_{2r_0}(0) \cap \Lambda$ is a graph with small gradient over a subset of the $\mathbb{R}^2 \times \{0\}$ plane containing a ball of radius $r_0$ centered at the origin.

We claim that, in this ball, the sequence of laminations converges in the $C^\alpha$ topology to a lamination for any $\alpha < 1$. The coordinate chart $\Phi$ required by the definition of a lamination will be given by the Arzela-Ascoli theorem as a limit of a sequence of bi-Lipschitz maps $\Phi_i : (x_j) \to \mathbb{R}^3$ with bi-Lipschitz constants close to one and defined on a slightly smaller concentric ball $B_{sr_0}$ for some $s > 0$ to be determined. Furthermore, we will show that for each $i$ fixed $\Phi_i^{-1}(B_{sr_0} \cap \cup_{\Lambda \in \mathcal{L}_i} \Lambda)$ is the union of planes which are each parallel to $\mathbb{R}^2 \times \{0\}$ plane. The domain of $f_{i,k}$ contains the ball of radius $r_0$ around the origin in the $\mathbb{R}^2 \times \{0\}$ plane. With slight abuse of notation, we will also denote balls in $\mathbb{R}^3$ with radius $t$ and center 0 by $B_t$. Set $w_{i,k} = f_{i,k+1} - f_{i,k}$. In the next equation, $\Delta, \nabla,$ and $\text{div}$ will be with respect to the Euclidean metric on $\mathbb{R}^2 \times \{0\}$. By a standard computation (cf. [Si], (7) on p. 333 or chapter 1 of [CMII]),

$$\Delta w_{i,k} = \text{div} (a \nabla w_{i,k}) + b \nabla w_{i,k} + c w_{i,k}. \quad (B.2)$$

Here $a$ is a matrix valued function, $b$ is a vector valued function and $c$ is a function. Although $a$, $b$, and $c$ depend on $i$, their scale invariant norms are small when $C$ and $\sqrt{k}r_0$ are. By (B.2), the Schauder estimates and Harnack inequality (e.g., 6.2 and 8.20 of [G1]) applied to the positive function $w_{i,k}$ give

$$sr_0 \sup_{B_{sr_0}} |\nabla w_{i,k}| \leq C \sup_{B_{sr_0}} w_{i,k} \leq \exp(\epsilon_0 s^\beta) \inf_{B_{sr_0}} w_{i,k}. \quad (B.3)$$

Where $\epsilon_0$ and $\beta > 0$ depend on the scale invariant norms of $a, b,$ and $c$. Set $M_{i,k} = f_{i,k}(0)$. In the region $\{(y_1, y_2, y_3) \in B_{r_0} \times [M_{i,k}, M_{i,k+1}]\},$ define $\phi_i$ by

$$\phi_i(y_1, y_2, y_3) = f_{i,k}(y_1, y_2) + \frac{y_3 - M_{i,k}}{M_{i,k+1} - M_{i,k}} w_{i,k}(y_1, y_2). \quad (B.4)$$

Hence

$$\nabla \phi_i = \nabla f_{i,k} + \frac{y_3 - M_{i,k}}{M_{i,k+1} - M_{i,k}} \nabla w_{i,k} + \frac{w_{i,k}}{M_{i,k+1} - M_{i,k}} \frac{\partial}{\partial y_3}. \quad (B.5)$$
It follows easily from (B.3) and (B.3) that for each \( i \) the map \( \Phi_i \) restricted to \( B_{sr_0}(0) \subset \mathbb{R}^3 \) is bi-Lipschitz with bi-Lipschitz constant close to one if \( s \) is sufficiently small. By the Arzela-Ascoli theorem, a subsequence of \( \Phi_i \) converges in the \( C^\alpha \) topology for any \( \alpha < 1 \) to a Lipschitz coordinate chart \( \Phi \) with the properties that are required. The leaves in \( B_{r_0} \) are \( C^{1,\alpha} \) limits of minimal graphs with bounded gradient, and hence minimal by elliptic regularity. \( \square \)

Trivial examples show that the Lipschitz regularity above is optimal.

**Appendix C. A standard consequence of the maximum principle**

Using the maximum principle and the convexity of small extrinsic balls we can bound the topology of the intersection of a minimal surface with a ball:

**Lemma C.1.** Let \( \Sigma^2 \subset M^n \) be an immersed minimal surface, \( \partial \Sigma \subset \partial B_{r_0}(x) \), \( K_{M^n} \leq k \), and injectivity radius of \( M \geq i_0 \). If \( r_0 < \min\{\frac{\pi}{4}, \frac{\pi}{4\sqrt{k}}\} \), \( B_t(y) \subset B_{r_0}(x) \), and \( \gamma \subset B_t(y) \cap \Sigma \) is a closed one-cycle homologous to zero in \( B_{r_0}(x) \cap \Sigma \), then \( \gamma \) is homologous to zero in \( B_t(y) \cap \Sigma \).

**Proof.** Apply the maximum principle to \( f = \text{dist}^2_M(y, \cdot) \) on the 2-current that \( \gamma \) bounds. \( \square \)

By Lemma C.1 if \( y \in B_t(x) \cap \Sigma \) is connected, then \( \chi(B_s(y) \cap \Sigma) \geq \chi(B_t(x) \cap \Sigma) \) for \( s + \text{dist}_M(x, y) \leq t < \min\{\frac{\pi}{4}, \frac{\pi}{4\sqrt{k}}\} \). (The Euler characteristic is monotone.)

**Appendix D. A generalization of Proposition I.1.3**

In [CM6], the next proposition is needed when we deal with the analog of the genus one helicoid (cf. [HoKrWe]) where \( \Sigma \) (as above (I.1.2)) is not a disk. The *genus* of a surface \( \Sigma \) \( (\text{gen}(\Sigma)) \) is the genus of the closed surface given by adding a disk to each boundary circle.

**Proposition D.1.** There exist \( C_0, \epsilon_0 \) so if \( 0 \in \Sigma, \partial \Sigma \) is connected, \( \text{gen}(\Sigma) = \text{gen}(\Sigma_{0,r_1}) \), \( R \geq C_0r_1 \), and \( \epsilon_0 \geq \epsilon \), then \( E_1 \cap \Sigma \setminus \Sigma_1 \) is an (oppositely oriented) \( N \)-valued graph \( \Sigma_2 \).

**Proof.** Note that, by the maximum principle and elementary topology (as in part I of [CM3]), \( \Sigma \setminus \Sigma_{0,t} \) is an annulus for \( r_1 \leq t < 4R \). The proof now follows that of Proposition I.1.3.

First, (a slight extension of) the “estimate between the sheets” given in theorem III.2.4 of [CM3] gives \( \epsilon_0 \) so that \( E_1 \cap \Sigma \) is locally graphical (this extension uses that \( \Sigma \setminus \Sigma_{0,r_1} \) is an annulus instead of that \( \Sigma \) is a disk; the proof of this extension is outlined in appendix A of [CM8]). As before, we get the second (oppositely oriented) multi-valued graph \( \Sigma_2 \subset \Sigma \).

Second, we argue by contradiction to show that there are no other components of \( E_1 \cap \Sigma \). Fix \( \sigma_1, \sigma_2 \) as before. The proof of Lemma I.1.3 applies virtually without change (since at least one of \( \Sigma_{0,1}, \Sigma_{0,2} \) must be a disk), so \( \sigma_1 \) and \( \sigma_2 \) connect in \( \Sigma_{0,C,r_1} \). Hence, \( \sigma_0 \subset \partial \Sigma_{0,C,r_1} \) connects \( \sigma_1 \) and \( \sigma_2 \). Replace \( \sigma_i \) with \( \sigma_i \setminus B_{C,r_1} \), so that \( \sigma_0 \cup \sigma_1 \cup \sigma_2 \subset \Sigma \setminus \Sigma_{0,r_1} \) is a simple curve and \( \partial(\sigma_0 \cup \sigma_1 \cup \sigma_2) \subset \partial \Sigma_{0,R} \). Let \( \Sigma \) be the component of \( \Sigma_{0,R} \setminus (\sigma_0 \cup \sigma_1 \cup \sigma_2) \) which does not intersect \( \Sigma_{0,r_1} \). It follows that \( \Sigma \) has genus zero and connected boundary; i.e., it is a disk. Solve as above for the stable disk \( \Gamma \) with \( \partial \Gamma = \partial \Sigma \) so \( \Gamma \) contains two disjoint \((N/2 - 1)\)-valued graphs in \( E_1 \) which spiral together. For \( R/r_1 \) large, Proposition I.0.1 gives the point of large curvature, contradicting the curvature estimate for stable surfaces. \( \square \)
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