A COMPUTATION OF THE RING STRUCTURE IN WRAPPED FLOER HOMOLOGY

HANWOOL BAE AND MYEONGGI KWON

Abstract. We give an explicit computation of the ring structure in wrapped Floer homology of real Lagrangians in $A_k$-type Milnor fibers. In the $A_k$-type plumbing description, those Lagrangians correspond to cotangent fibers or diagonal Lagrangians. The main ingredient of the computation is an idea of the Seidel representation. For a technical reason, we first carry out computations in v-shaped wrapped Floer homology, and this in turn gives the desired ring structure via the Viterbo transfer map.

1. Introduction

For an admissible Lagrangian $L$ in a Liouville domain, we consider a version of Lagrangian Floer homology, called wrapped Floer homology $HW_*(L)$, introduced in [3]. Wrapped Floer homology is equipped with a family of algebraic structures, called $A_\infty$-structures, which include a ring structure with unity. Even though there are some structural results, for example [1, 12], it is usually difficult to compute those structures explicitly as with other Floer theoretical invariants.

A Brieskorn Milnor fiber is a (completed) Liouville domain defined as a regular level set of a Brieskorn-type complex polynomial. They have played an important role in symplectic and contact topology, especially as computational examples in Floer theory. Various Floer theoretical invariants of Brieskorn Milnor fibers are studied in [13, 14, 16, 21, 28, 29, 30, 31].

In this paper, we give an explicit computation of the ring structure in wrapped Floer homology of real Lagrangians in $A_k$-type Brieskorn Milnor fibers. In Section 3.2, we define a class of real Lagrangians in $A_k$-type Milnor fibers which are of particular interest. This includes real Lagrangians considered in [20] where the graded group structure of wrapped Floer homology is computed. Note that $A_k$-type Milnor fibers can be seen as $k$-fold linear plumbings of cotangent bundles $T^* S^n$, see Section 3.3. The real Lagrangians we consider correspond to cotangent fibers away from plumbing regions or diagonal Lagrangians in a plumbing region, see Section 3.3.3.

The main idea of our computations comes from Uebele [28] who used the idea of the Seidel representation to study the ring structure in symplectic homology. For a closed symplectic manifold $(M, \omega)$, Seidel [26] introduced a representation of the fundamental group $\pi_1(\text{Ham}(M, \omega))$ of Hamiltonian diffeomorphisms on Hamiltonian Floer homology $HF_*(M, \omega)$. More precisely, for a loop of Hamiltonian diffeomorphisms on $M$, say $g: S^1 \to \text{Ham}(M, \omega)$, we have an associated group isomorphism $S_g: HF_*(M, \omega) \to HF_*(M, \omega)$ up to a degree shift. We call $S_g$ a Seidel operator. In particular the Seidel operator satisfies the so-called module property, namely $S_g(x \cdot y) = S_g(x) \cdot y$ for $x, y \in HF_*(M, \omega)$. A relative version of the Seidel representation in Lagrangian Floer homology for closed symplectic manifolds was studied in [17].

We carry out the idea of the Seidel representation in wrapped Floer theory. For a technical reason, we do not work directly with wrapped Floer homology, but instead we construct a Seidel operator in a v-shaped version of wrapped Floer homology, V-shaped wrapped Floer homology, denoted by $HW_*(L)$, is a variant of wrapped Floer homology which also admits a ring structure with unity, see Section 2 for definition. This can be regarded as a relative version of v-shaped symplectic homology introduced in [8]. We remark that v-shaped wrapped Floer homology is group isomorphic to wrapped Floer homology of trivial Lagrangian cobordisms [9] and to Lagrangian Rabinowitz Floer homology [10, 22].

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An important property of v-shaped wrapped Floer homology for our purpose is that under a certain index positivity condition on the contact type boundary, called product index positivity (Definition 2.14), its ring structure can be defined purely in the symplectization part of the completion of a Liouville domain, see Corollary 2.16. A related discussion can also be found in [9, Section 9.5]. This property allows us to define a Seidel operator in $\check{HW}^\ast_\ast(L)$ by taking a path of Hamiltonian diffeomorphisms on the symplectization part which extends the Reeb flow on the contact boundary. Such an extension of the Reeb flow to the whole Liouville filling is not possible in our examples, see Remark 4.18.

To formulate a Seidel operator in v-shaped wrapped Floer homology, let $(W, \lambda)$ be a Liouville domain with a Liouville form $\lambda$. Denote the contact type boundary by $(\Sigma, \xi = \ker \alpha)$ where $\alpha$ is a contact form given by $\alpha := \lambda|_{\Sigma}$. Let $L$ be an admissible Lagrangian, meaning that it is exact and intersects with $\Sigma$ in a Legendrian $\mathcal{L} := \partial L$. For simplicity of gradings in Floer homology, we put topological assumptions in (2.1).

We say that the Reeb flow $\phi^t_R$ on $(\Sigma, \alpha)$ is $\mathcal{L}$-periodic if there exists a positive real number $T_0$ such that $p \in \mathcal{L}$ if and only if $\phi^{T_0}_R(p) \in \mathcal{L}$. Then, extending the Reeb flow on $(\Sigma, \alpha)$, we can define a path of Hamiltonian diffeomorphisms $g : [0, 1] \to \text{Ham}(\mathbb{R}_+ \times \Sigma)$ on the symplectization $\mathbb{R}_+ \times \Sigma$ of the form

$$g_t(r, y) = (r, \phi^{f(t)}_R(y)).$$

Here $f$ is a smooth function on $[0, 1]$ such that $g_0(r, y), g_1(r, y) \in \mathbb{R}_+ \times \mathcal{L}$ if and only if $(r, y) \in \mathbb{R}_+ \times \mathcal{L}$. We use the path $g$ to define a Seidel operator on $\check{HW}^\ast_\ast(L)$ in the following theorem.

**Theorem 1.1.** Suppose that the triple $(\Sigma, \xi, \mathcal{L})$ is product index positive and the Reeb flow $\phi^t_R$ is $\mathcal{L}$-periodic. Then we have a graded group isomorphism

$$S_g : \check{HW}^\ast_\ast(L) \to \check{HW}^\ast_{\ast + I(g)}(L)$$

up to a degree shift $I(g)$ depending on $g$, and it satisfies a module property

$$S_g(x \cdot y) = S_{g_0}(x) \cdot S_{g}(y) = S_{g}(x) \cdot S_{g_1}(y)$$

for $x, y \in \check{HW}^\ast_\ast(L)$.

The degree shift $I(g)$ in the above theorem is given by a Maslov-type index for paths of Hamiltonian diffeomorphisms, see Section 4.1.3. Computing $I(g)$ is crucial for our purpose. In Section 4.2.1, we observe that if the triple $(\Sigma, \alpha, \mathcal{L})$ forms a real contact manifold and the Reeb flow $\phi^t_R$ is $\mathcal{L}$-periodic, then we can describe the shift $I(g)$ in terms of the Maslov index of principal Reeb chords, see Lemma 4.10. This will be enough to determine $I(g)$ explicitly in many cases.

**Outline of computations.** Here we outline our computation of the ring structure of (ordinary) wrapped Floer homology. We for example take the real Lagrangian $L_0$ in $A_k$-type Milnor fiber defined in Section 3.2. This can be seen as a cotangent fiber in $A_k$-type plumbing of $T^* S^n$, see Proposition 3.7. We can compute the group structure on $\check{HW}^\ast_\ast(L_0)$ using a Morse–Bott technique in Section 3.4.2 and the result is given as follows.

**Proposition 1.2.** For $n \geq 3$, we have

$$\check{HW}^\ast_\ast(L_0) = \begin{cases} \mathbb{Z}_2 & \ast = I \cdot N - n + 1, I \cdot N \text{ with } N \in \mathbb{Z}; \\ 0 & \text{otherwise}. \end{cases}$$

Here $I = (n - 2)(k + 1) + 2$.

An essential feature of the above graded group is that the generators of $\check{HW}^\ast_\ast(L_0)$ appear periodically due to the $\mathcal{L}$-periodicity of the Reeb flow, and the rank of $\check{HW}^\ast_\ast(L_0)$ in each degree is at most one. Because of this simplicity, the module property in Theorem 1.1 of the Seidel operator turns out to be enough to determine the full ring structure of $\check{HW}^\ast_\ast(L)$ algebraically.

**Proposition 1.3.** For $n \geq 3$ and $k \geq 2$, we have a graded ring isomorphism

$$\check{HW}^\ast_\ast(L_0) \cong \mathbb{Z}_2[x, y, y^{-1}] / (x^2),$$

where $|x| = I - n + 1$ and $|y| = I$ with $I = (n - 2)(k + 1) + 2$. 
Next, note that the ring structures on $HW_*(L)$ and $HW_*(\hat{L})$ are in general related by a ring homomorphism

$$HW_*(L) \to HW_*(\hat{L}),$$

which is called the Viterbo transfer map. This map is defined as a continuation map in a standard way in Floer theory, see Section [5.1]. We prove that, due to the simplicity of the graded group structure, the Viterbo morphism $k$ in terms of the homology of the path space of the base manifold in [1]. Therefore if $L = 0$ which is called the Viterbo morphism $k$ in terms of the homology of the path space of the base manifold in [1]. Therefore if $L = 0$ which is the case when $L_0$ is a cotangent fiber in $T^*S^n$, we already know that $HW_*(L_0) \cong \mathbb{Z}_2[x]$ as graded rings with $|x| = n - 1$.

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2. **V-shaped wrapped Floer homology**

2.1. **V-shaped admissible Hamiltonians.** Let $(W, \omega = d\lambda)$ be a Liouville domain with Liouville form $\lambda$. Its boundary $\Sigma := \partial W$ admits a contact form $\alpha := \lambda|_\Sigma$ and hence a contact structure $\xi = \ker \alpha$. Let $L$ be an exact Lagrangian in $W$ which is called admissible in the sense that $L$ intersects $\Sigma$ transversely in a Legendrian $L := L \cap \Sigma$, and the Liouville vector field $X_\lambda$ is tangent to $L$ near the boundary. For grading purpose, see [24, Remark 4.6], we assume throughout this paper the following topological conditions:

\[
\pi_1(W, L), \pi_1(W), \pi_1(\Sigma), \pi_1(\Sigma, L), c_1(TW), c_1(\xi), c_1(TW, TL), c_1(\xi, TL) \text{ vanish.}
\]

**Remark 2.1.** One can work with less topological assumptions, but the above makes constructions in Floer theory simpler and is fulfilled in our examples.

We denote the completions of $W$ and $L$ by

$$\hat{W} = W \cup_{\partial W} ([1, \infty) \times \Sigma), \quad \hat{L} = L \cup_{\partial L} ([1, \infty) \times L),$$

respectively. Denoting by $r$ the coordinate of $[1, \infty)$ in the completion, the Liouville form $\lambda$ is completed by $\hat{\lambda} = r\alpha$ in $[1, \infty) \times \Sigma$. We denote the spectrum of Reeb chords on the contact boundary by

$$\text{Spec}(\Sigma, \alpha, L) := \{T \in \mathbb{R} \mid T \text{ is a period of a Reeb chord on } (\Sigma, \alpha) \text{ whose end points are in } L\}.$$

**Definition 2.2.** A Hamiltonian $H : \hat{W} \to \mathbb{R}$ is called v-shaped admissible if

1. Hamiltonian 1-chords with end points in $L$ are non-degenerate;
2. $H \leq 0$ in a small neighborhood of the boundary $\partial W = \{r = 1\}$ and $H \geq 0$ elsewhere;
3. $H(r, p) = h(r)$ for a convex function $h$ on $\mathbb{R}_+$ such that $h(r) = ar + b$ for $r \gg 1$ where $a \notin \text{Spec}(\Sigma, \alpha, L), b \in \mathbb{R}$.

We denote the set of Hamiltonian 1-chords of $H$ in $\hat{W}$ relative to $\hat{L}$ by $P_L(H)$.

**Remark 2.3.** The first condition can always be achieved by perturbing the Lagrangian $\hat{L}$ or the perturbing Hamiltonian $H$, [24, Lemma 4.3]. The second condition is the main difference from the usual admissible Hamiltonians for ordinary wrapped Floer homology.

**Remark 2.4.** We use the convention that $\omega(X_H, \cdot) = -dH$.

We actually use a cofinal subclass of v-shaped admissible Hamiltonians of the form in Figure [1]. For a given action window, we can take the positive and negative slopes $\pm a$ of $h$ and its constant value $c$ near $r = 0$ so that the Hamiltonian 1-chords in the action window are of the following three types: Negatively parametrized Reeb chords relative to $L$, constant chords in $\Sigma$, and positively parametrized Reeb chords relative to $L$. For more details, see [10 Section 3.2.2].
2.2. Floer chain complex. Since we assume $\pi_1(W, L) = 0$, every Hamiltonian 1-chord relative to $L$, i.e. end points are in $L$, is contractible. Let $x : [0, 1] \to \hat{W}$ be a Hamiltonian 1-chord. By choosing a capping half disk of $x$, we have a symplectic trivialization $\tau_x : x^*\hat{T}\hat{W} \to [0, 1] \times \mathbb{R}^{2n}$ along $x$ such that $\tau_x(T_x(t)L) = t \times \Lambda_0$ where $\Lambda_0$ denotes the horizontal Lagrangian in $\mathbb{R}^{2n}$. We associate a path of symplectic matrices $\Phi_x : [0, 1] \to Sp(2n)$ to $x$ by

$$\Phi_x(t) := \tau_x(t) \circ d\phi_H(t) \circ \tau_x(0)^{-1}.$$ 

The Maslov index of $x$ is then defined by

$$\mu(x) := \mu_{RS}(\Phi_x, \Lambda_0),$$

where $\mu_{RS}$ denotes the Robbin-Salamon index for paths of Lagrangians, defined in [25]. Under the topological conditions (2.1), the Maslov index of $x$ does not depend on the choice of capping half disks. See [20, Section 2.1] for more details.

For a v-shaped admissible Hamiltonian $H$ and a time-dependent family of admissible (i.e. contact type at the cylindrical end) almost complex structures $J = \{J_t\}$ on $\hat{W}$, we define the filtered Floer chain complex $(CF_*^{<c}(L; H, J), \partial)$ with $c \not\in \text{Spec}(\Sigma, \alpha, L)$ as follows. Define a Hamiltonian action functional on the path space relative to $L$ by

$$A_H(x) := -g_L(x(1)) + g_L(x(0)) + \int_0^1 x^*\hat{\lambda} - \int_0^1 H(x(t))dt$$

where $\lambda|_L = dg_L$. Then the critical points of $A_H$ are exactly elements in $\mathcal{P}_L(H)$. The chain group $CF_*^{<c}(L; H, J)$ is a $\mathbb{Z}_2$-vector space generated by Hamiltonian 1-chords relative to $L$ with action less than $c$, in other words,

$$CF_*^{<c}(L; H, J) := \bigoplus_{x \in \mathcal{P}_L(H), A_H(x)<c} \mathbb{Z}_2(x).$$

We grade the chain complex by

$$|x| := \mu(x) - \frac{n}{2}.$$ 

The differential $\partial$ of the Floer chain complex counts the moduli space of Floer strips. We define

$$\widetilde{\mathcal{M}}(x_-, x_+; H, J) := \{u : \mathbb{R} \times [0, 1] \to \hat{W} \mid \partial_s u + J_t(\partial_t u - X_H(u)) = 0, \lim_{s \to \pm \infty} u(s, t) = x_\pm(t), u(s, 0), u(s, 1) \in \hat{L}\}.$$

For $x_- \neq x_+ \in \mathcal{P}_L(H)$, we have a free $\mathbb{R}$-action on $\widetilde{\mathcal{M}}(x_-, x_+; H, J)$ by translating the $s$-parameter. Denote the quotient space by

$$\mathcal{M}(x_-, x_+; H, J) := \widetilde{\mathcal{M}}(x_-, x_+; H, J)/\mathbb{R}.$$
For generic $J$, the moduli space $\mathcal{M}(x_-, x_+; H, J)$ is a smooth manifold of dimension $|x_+| - |x_-| - 1$. The differential $\partial : CF_k^c(L; H, J) \to CF_k^c(L; H, J)$ is defined by counting elements in $\mathcal{M}(x_-, x_+; H, J)$ modulo 2 as

$$\partial(x_+) = \sum_{x_+ \in \mathcal{P}_L(H)} \#_2 \mathcal{M}(x_-, x_+; H, J) \cdot x_-.$$

We define the chain complex of action window $(a, b)$, for $a < b \in \mathbb{R}$, by

$$CF_s^{(a,b)}(L; H, J) := CF_s^c(L; H, J)/CF_s^c(L; H, J)$$

with the induced differential by $\partial$. Then the filtered Floer homology $HF_s^{(a,b)}(L; H, J)$ is defined to be the homology of the chain complex $(CF_s^{(a,b)}(L; H, J), \partial)$.

2.3. **V-shaped wrapped Floer homology.** For two v-shaped admissible Hamiltonians $H_+ \leq H_-$ and generic almost complex structures $J_+$ and $J_-$ (not necessarily distinct), we have a continuation map

$$f_{H_+, H_-} : HF_{s}^{(a,b)}(L; H_+, J_+) \to HF_{s}^{(a,b)}(L; H_-, J_-).$$

Let us briefly review its construction. We take a one parameter family $\{(H_s, J_s)\}_{s \in \mathbb{R}}$ of pairs of Hamiltonians and almost complex structures in such a way that

- $(H_s, J_s) = \begin{cases} (H_-, J_-) & s \leq -R \\ (H_+, J_+) & s \geq R \end{cases}$

for some sufficiently large $R > 0$;

- $H_s$ is non-increasing with respect to $s$, i.e. $\frac{\partial H_s}{\partial s} \leq 0$.

We define the moduli space $\mathcal{M}(x_-, x_+; H_s, J_s)$ of parametrized Floer solutions, i.e.

$$\mathcal{M}(x_-, x_+; H_s, J_s) = \{ u : \mathbb{R} \times [0, 1] \to \hat{W} \mid \partial_s u + J_s(t)(\partial_t u - X_{H_s}(u)) = 0, \quad s \to \pm \infty \text{ } u(s, 0), u(s, 1) \in \mathfrak{L} \}$$

for $x_- \in \mathcal{P}_L(H_-)$ and $x_+ \in \mathcal{P}_L(H_+)$. For generic homotopies $\{(H_s, J_s)\}$, the moduli space is a smooth manifold of dimension $|x_+| - |x_-|$. The chain level continuation map $f_{H_+, H_-} : CF_{s}^{(a,b)}(L; H_+, J_+) \to CF_{s}^{(a,b)}(L; H_-, J_-)$ is then defined by

$$f_{H_+, H_-}(x_+) = \sum_{x_+ \in \mathcal{P}_L(H_-), \mu(x_+) = \mu(x_+)} \#_2 \mathcal{M}(x_-, x_+; H_s, J_s) x_-.$$

A standard compactness result in Floer theory, with the condition that $H_s$ is non-increasing with respect to $s$, shows that $f_{H_+, H_-}$ is a chain map. The induced map on homology is called a continuation map, which we denote still by $f_{H_+, H_-}$. This yields a direct system on Floer homology $HF_{s}^{(a,b)}(L; H, J)$ over the set of pairs of a v-shaped admissible Hamiltonian $H$ and an almost complex structure $J$.

For a given action window $(a, b)$, we in particular consider a cofinal family of Hamiltonians such that $H = h(r)$ is of the form in Figure 1. We can choose the constants $\delta, \epsilon$ so that Hamiltonian 1-chords in the region (I) and (II) are not in the action window; see for example the proof in [8 Proposition 2.9]. Over this cofinal family, we define the **v-shaped wrapped Floer homology of $L$ of action $(a, b)$** to be the direct limit

$$HW_{s}^{(a,b)}(L) := \lim_{a \to b} HF_{s}^{(a,b)}(L; H, J).$$

We define the (full) **v-shaped wrapped Floer homology of $L$** by

$$HW_{s}(L) := \lim_{b \to a} \lim_{a \to \epsilon} HW_{s}^{(a,b)}(L).$$
Remark 2.5. As a group, v-shaped wrapped Floer homology defined here is equivalent to the Lagrangian Floer homology for a trivial cobordism defined in [9] (up to minor conventional differences). Moreover, in [10 Section 3.3.2], it is shown that the v-shaped wrapped Floer homology is isomorphic to Lagrangian Rabinowitz Floer homology. This is parallel to the fact that v-shaped symplectic homology is isomorphic to Rabinowitz Floer homology as groups.

2.4. Ring structure on $\tilde{HW}(L)$. In this section, we outline the construction of the ring structure on $V$-shaped wrapped Floer homology $\tilde{HW}_+(L)$ with unity. This is completely analogous to the case of the ordinary wrapped Floer homology and is defined by counting half pair of pants.

We first define a half pair-of-pants product

$$HF(L; H_1, J_1) \otimes HF(L; H_2, J_2) \to HF(L; H_0, J_0),$$

which is compatible with continuation maps so that we have a product structure on $\tilde{HW}_+(L)$. Here, $(H_j, J_j)$ with $j = 0, 1, 2$ are pairs of v-shaped admissible Hamiltonians and time-dependent almost complex structures.

Let $S$ be a two dimensional disk with three boundary points $z_0, z_1, z_2 \in \partial S$ removed and let $j$ be a complex structure on $S$. In order to introduce the notion of strip-like ends near the boundary punctures, let us consider the semi-infinite strips

$$Z^\pm = \mathbb{R}^\pm \times [0, 1].$$

This equipped with the standard complex structures, namely $j \frac{\partial}{\partial x} = \frac{\partial}{\partial x}$.

We consider a positive strip-like end near $z_i$ for $i = 1, 2$, that is, a holomorphic embedding

$$\epsilon_i : \mathbb{R}^+ \times [0, 1] \to S$$

such that

$$\epsilon_i^{-1}(\partial S) = \mathbb{R}^+ \times \{0, 1\} \text{ and } \lim_{s \to 1^\pm} \epsilon_i(s, \cdot) = z_i.$$

We also consider a negative strip-like end near $z_0$ in a similar way. We further require that the images of $\epsilon_i$ are pairwise disjoint.

We need to make a choice of the following data to define the moduli space of half pair of pants:

- A family of almost complex structures $J^S$ parametrized by $S$, which is of contact type at the cylindrical ends and satisfies $J^S_{\epsilon_i(s, t)} = J^i_s$ for each $i = 0, 1, 2$.
- A 1-form $\beta \in \Omega^1(S)$ such that $\epsilon_i^* \beta = dt$ and its restriction to the boundary $\partial S$ is zero.
- A family of Hamiltonians $H^S = \{H^S(z, \cdot) \in C^\infty(\hat{W})\}_{z \in S}$ parametrized by $S$ such that $H^S(z, \cdot)$ is a v-shaped admissible Hamiltonian on $\hat{W}$ for each $z \in S$ and that $H^S(\epsilon_i(s, t), \cdot) = H^i_t(\cdot)$ for each $i = 0, 1, 2$ and $d(H^S(\cdot, w) \beta) \leq 0$ for all $w \in \hat{W}$. The last property is necessary to apply maximum principle.

Applying the Stokes' theorem to $H^S(\cdot, w) \beta$ for some $w \in \hat{W}$, we observe that the sum of $H_1$ and $H_2$ is less than or equal to $H_0$. Let us assume the Hamiltonians $H_0, H_1, H_2$ satisfy this condition from now on.

**Definition 2.6.** Let $x_i \in P_L(H_i)$ be a Hamiltonian chord for $i = 0, 1, 2$. The moduli space of half pair of pants $M(x_0, x_1, x_2; \beta, H^S, J^S)$ consists of all maps $u : S \to \hat{W}$ of finite energy, which satisfies the following.

$$\begin{aligned}
&u(\partial S) \subset \hat{L}, \\
&\lim_{s \to -\infty} u(\epsilon_0(s, t)) = x_0(t), \\
&\lim_{s \to +\infty} u(\epsilon_i(s, t)) = x_i(t) \text{ for } i = 1, 2, \\
&(du - X_{H^S} \otimes \beta)^{0,1} := \frac{1}{2} ((du - X_{H^S} \otimes \beta) + J^S \circ (du - X_{H^S} \otimes \beta) \circ j) = 0,
\end{aligned}$$

where $X_{H^S}$ is the Hamiltonian vector field associated to $H^S$.

For a generic pair $(H^S, J^S)$, the moduli space $M(x_0, x_1, x_2; \beta, H^S, J^S)$ is a smooth manifold, and its dimension, in our index convention, is equal to

$$\dim M(x_0, x_1, x_2; \beta, H^S, J^S) = \mu(x_1) + \mu(x_2) - \mu(x_0) - \frac{n}{2}.$$
As a consequence, the moduli space is zero-dimensional if $|x_0| = |x_1| + |x_2|$. We define the half pair of pants product by

$$x_1 \cdot x_2 = \sum_{x_0 \in P_L(H_0)} \#_2 \mathcal{M}(x_0, x_1, x_2; \beta, H^S, J^S) \cdot x_0$$

for $x_i \in P_L(H_i), i = 1, 2$ and extend this $\mathbb{Z}_2$-linearly to Floer chain complexes.

The product structure descends to homology as it satisfies the Leibniz rule. Arguing as in [9, Section 10], we observe that the product structure actually defines a map

$$HF^{(a_1, b_1)}(L; H_1, J_1) \otimes HF^{(a_2, b_2)}(L; H_2, J_2) \to HF^{(a_0, b_0)}(L; H_0, J_0),$$

for any $a_i < b_i, i = 0, 1, 2$ such that $a_i < 0 < b_i$ for $i = 1, 2, a_0 = \max\{a_1 + b_2, a_2 + b_1\}$ and $b_0 = b_1 + b_2$.

Since the product structure is compatible with the continuation map, we first take the inverse limit as $a_1, a_2 \to -\infty$ and then take the direct limit as $b_1, b_2 \to \infty$ to get a product structure on the v-shaped wrapped Floer homology. Furthermore, the product structure is associative and carries a strict unit. We refer the readers to [2] and [24] for more details. In summary, the v-shaped wrapped Floer homology carries a unital associative ring structure.

**Remark 2.7.** The ring $HW_*(L)$, as well as $HW_*(L)$, is not in general commutative, see [24, Section 6].

2.5. **Independence of fillings.** In this section, we show that the ring structure in v-shaped wrapped Floer homology $HW_*(L)$ can be defined purely in the symplectization part $\mathbb{R}_+ \times \partial W$ of the completion $\hat{W}$, provided an index positivity in Definition 2.14.

2.5.1. **Index positivity.** To motivate the notion of index positivity of a contact manifold with a Legendrian, we first give the virtual dimension of moduli spaces which we shall consider later. As always in this paper, we assume the topological conditions (2.1) to have that all periodic Reeb orbits and chords are contractible and have well-defined indices. In addition, all of them are assumed to be non-degenerate.

Let $\chi^- = (x_1^-, \ldots, x_{k^-}^-)$ and $\chi^+ = (x_1^+, \ldots, x_{k^+}^+)$ be collections of Hamiltonian 1-chords in $\mathbb{R}_+ \times \mathcal{L}$ relative to $L = \mathbb{R}_+ \times \mathcal{L}$, and let $\Gamma = (\gamma_1, \ldots, \gamma_\ell)$ be a collection of contractible periodic Reeb orbits in $\Sigma$, and let $C = \{c_1, \ldots, c_m\}$ be a collection of contractible Reeb chords in $\Sigma$ relative to $\mathcal{L}$. Consider the punctured disk

$$S := \mathbb{D} \setminus \{z_1^-, \ldots, z_{k^-}^-, z_1^+, \ldots, z_{k^+}^+, w_1, \ldots, w_\ell, \overline{w}_1, \ldots, \overline{w}_m\}$$

where $z_j^-, z_j^+, w_\ell, \overline{w}_m \in \partial \mathbb{D}$ are boundary punctures ($z_j^+$ is positive, and $z_j^-$ and $w_\ell$ are negative) and $w_1, \ldots, w_\ell \in \mathbb{D}$ are interior negative punctures. Take a Floer data $(H, J, \beta)$ parametrized by $S$ defined analogously to the case for the pair of pants strips.

Define the moduli space $\mathcal{M}(\chi^-, \chi^+, \Gamma, C; \beta, H, J)$ of maps

$$u : S \to \mathbb{R}_+ \times \Sigma$$

such that $u$ satisfies the Floer equation, the Lagrangian boundary condition, and converges to the Hamiltonian 1-chord $x_j^\pm$ at $z_j^\pm$ in the sense of Hamiltonian Floer theory and converges to the periodic Reeb orbits $0 \times \gamma_j$ at $w_j$ in the sense of SFT and converges to the Reeb chord $0 \times c_i$ at $\overline{w}_i$ in the sense of (relative) SFT.

The following dimension formula, which is well-known, can be obtained by a standard way using the additive property of indices, together with virtual dimensions of moduli spaces given in [11, Section 2.2]. In the below the Maslov index $\mu(c)$ of a Reeb chord $c$ is the one defined in [20, Section 2.2.1]; see also the proof of Lemma 4.10.

**Theorem 2.8.** The virtual dimension of the moduli space $\mathcal{M}(\chi^-, \chi^+, \Gamma, C; \beta, H, J)$ equals

$$\sum_{j=1}^{k_\pm} \mu(x_j^\pm) - \sum_{j=1}^{k_-} \mu(x_j^-) + \frac{n}{2} (2 - k_+ - k_-) - \sum_{j=1}^{\ell} (\mu_c(\gamma_j) + n - 3) - \sum_{j=1}^{m} \left(\mu(c_j) + \frac{n - 3}{2}\right).$$

**Remark 2.9.** For a given Reeb chord $c$, Ekholm [11, Section 2.2] introduced a degree $|c|$ of $c$, which is the dimension of the moduli space of holomorphic disks that converge to the Reeb chord $c$ at one positive boundary puncture. The relation between our Maslov index $\mu(c)$ and the degree $|c|$ is given by

$$|c| = \mu(c) + \frac{n - 3}{2}.$$
Definition 2.10. The triple \((\Sigma, \xi, \mathcal{L})\) of contact manifold \((\Sigma, \xi)\) with a Legendrian \(\mathcal{L}\) is called index positive if for each \(T > 0\), there exists a contact form \(\alpha_T\) for \((\Sigma, \xi)\) such that
- every periodic Reeb orbit \(\gamma\) in \((\Sigma, \alpha_T)\) of period less than \(T\) is non-degenerate, and the Conley-Zehnder index \(\mu_{CZ}(\gamma)\) is greater than \(3 - n\);
- every Reeb chord \(c\) in \((\Sigma, \alpha_T, \mathcal{L})\) of period less than \(T\) is non-degenerate, and the Maslov index \(\mu(c)\) is greater than \(\frac{3-n}{2}\).

We note that if \((\Sigma, \xi, \mathcal{L})\) is index-positive, then the last two terms in \(2.3\) are negative for periodic Reeb orbits and Reeb chords in Definition 2.10.

2.5.2. Stretching the neck. The key technique to show that \(\hat{HW}_s(L)\) can be defined in the symplectization part is the stretch-the-neck operation in (relative) symplectic field theory. In this section, we briefly describe this operation following the idea in [9, Lemma 2.4] and its relative version given in [7, Section 2.6].

Let \(H : \hat{W} \to \mathbb{R}\) be a v-shaped admissible Hamiltonian. Take a small neighborhood \(I_{\delta_0} \times \Sigma\) of the contact hypersurface \(\{\delta_0\} \times \Sigma\) in \(\hat{W}\) where \(\delta_0 > 0\) is sufficiently small and \(I_{\delta_0}\) is a closed interval containing \(\delta_0\). Since \(H\) is v-shaped, we may assume that \(H\) is constant in the neighborhood \(I_{\delta_0} \times \Sigma\). Let \(J\) be a SFT-like compatible almost complex structure on \(\hat{W}\), namely
- \(J\) preserves the contact structure;
- \(J\) is invariant under the Liouville direction in the neighborhood \(I_{\delta_0} \times \Sigma\);
- \(J\) sends the Liouville vector field \(rd_t\) to the Reeb vector field \(R_\alpha\).

For a deformation parameter \(R > 0\), we pick a diffeomorphism \(\phi_R\) from the interval \([-R, R]\) to the interval \(I_{\delta_0}\) with \(\phi_R'(\pm R) = 1\). This induces a diffeomorphism \(\phi_R \times \text{id} : [-R, R] \times \Sigma \to I_{\delta_0} \times \Sigma\). Let \(J_0\) be a compatible almost complex structure on \([-R, R] \times \Sigma\) which is SFT-like in the above sense and such that \(J_0|_{\{0\} \times \Sigma}\) coincides with \(J|_{\{\delta_0\} \times \Sigma}\). We define an almost complex structure \(J_R\) on the completion \(\hat{W}\) by
\[
J_R = \begin{cases} (\phi_R \times \text{id})_* J_0 & \text{on } I_{\delta_0} \times \Sigma; \\ J & \text{elsewhere.} \end{cases}
\]

By the condition that \(\phi_R'(\pm R) = 1\) and that \(J_0|_{\{0\} \times \Sigma}\) coincides with \(J|_{\{\delta_0\} \times \Sigma}\), the almost complex structure \(J_R\) is well-defined. By stretching the neck operation we basically mean that we replace the given almost complex structure \(J\) on \(\hat{W}\) by \(J_R\), typically with an arbitrarily large \(R > 0\).

2.5.3. Group structure. Let \(x_-\) and \(x_+\) be Hamiltonian 1-chords of a v-shaped admissible Hamiltonian \(H\), with \(|x_+| - |x_-| = 1\), which are generators of the Floer chain complex \(CF^*_c(\alpha, \beta)(L; H, J)\). Denote the moduli space of Floer strips from \(x_+\) to \(x_-\) in the symplectization part \(\mathbb{R}_+ \times \Sigma\) by \(\mathcal{M}^{\mathbb{R}_+ \times \Sigma}(x_-, x_+; H, J)\). To stress the difference, we denote the usual moduli space by \(\mathcal{M}(x_-, x_+; H, J)\).

Proposition 2.11. If \((\Sigma, \xi, \mathcal{L})\) is index positive, then
\[
\mathcal{M}^{\mathbb{R}_+ \times \Sigma}(x_-, x_+; H, J) = \mathcal{M}^{\mathbb{R}_+ \times \Sigma}(x_-, x_+; H, J).
\]
In other words, all Floer strips in \(\hat{W}\) with asymptotics in the symplectization part \(\mathbb{R}_+ \times \Sigma\) are actually contained in \(\mathbb{R}_+ \times \Sigma\).

Proof. Due to invariance of (v-shaped) wrapped Floer theory under Liouville isotopies, we may assume that the contact form on the boundary is \(\alpha_T\) for some \(T > 0\) appeared in Definition 2.10. For a statement of invariance under Liouville isotopies we refer to [18]; note also that changing contact forms on the (cooriented) contact boundary results in a Liouville isotopy, see [16, Section 4.4].

Suppose, on the contrary, that there exists a Floer strip \(u : \mathbb{R} \times [0, 1] \to \hat{W}\) which is not entirely contained in \(\mathbb{R}_+ \times \Sigma\). In particular \(u\) intersects nontrivially with a neighborhood \(I_{\delta_0} \times \Sigma \subset \mathbb{R}_+ \times \Sigma\) considered in Section 2.5.2. Then the stretching the neck operation in Section 2.5.2 produces a sequence of Floer strips \(u_{R_k} : \mathbb{R} \times [0, 1] \to \hat{W}\), with \(R_k \to \infty\) as \(k \to \infty\), which is \(J_{R_k}\)-holomorphic (note that \(H\) is constant in \(I_{\delta_0} \times \Sigma\) in the neighborhood \(I_{\delta_0} \times \Sigma\). By the relative SFT compactness in [6, Section 11.3], \(u_{R_k}\) converges to a (partially-) holomorphic building, and in particular the top piece in the limit is an element in a moduli space of the form \(\mathcal{M}(x_-, x_+, \gamma_1, \ldots, \gamma_\ell, c_1, \ldots, c_m; \beta, H, J)\) in Section 2.5.1.
By Stokes’ theorem and positivity of energy of elements in \( \mathcal{M}(x_-, x_+, \gamma_1, \ldots, \gamma_\ell, c_1, \ldots, c_m; \beta, H, J) \), the action value of periodic Reeb orbits and Reeb chords which appear in the top piece are bounded from above by a constant \( T' > 0 \) which depends only on \( x_+ \) and \( H \). Since \( H \) is linear at infinity, there are only finitely many Hamiltonian 1-chords. Therefore we may assume that the upper bound \( T' \) in fact depends only on \( H \). Now, choosing the contact form \( \alpha_T \) at the very beginning so that \( T = T' \), it follows that all periodic Reeb orbits and Reeb chords appeared in the top piece are non-degenerate and satisfy the index condition in Definition 2.10.

By Theorem 2.8 the virtual dimension of the moduli space, after modding out the \( \mathbb{R} \)-action, is

\[
\mu(x_+) - \mu(x_-) - 1 - \sum_{j=1}^\ell (\mu_{CZ}(\gamma_j) + n - 3) - \sum_{j=1}^m \left( \mu(c_j) + \frac{n-3}{2} \right).
\]

Since we have assumed \( |x_+| = |x_-| = 1 \), the dimension equals to

\[
- \sum_{j=1}^\ell (\mu_{CZ}(\gamma_j) + n - 3) - \sum_{j=1}^m \left( \mu(c_j) + \frac{n-3}{2} \right).
\]

By the index positivity, it follows that the virtual dimension is necessarily negative. This gives a contradiction. \( \square \)

It is straightforward to see that the same argument as the proof of Proposition 2.11 also works for the case of parametrized Floer strips which we count to define continuation maps. We therefore conclude that \( \hat{v} \)-shaped wrapped Floer homology, as a graded group, can purely be defined in the symplectization part \( \mathbb{R}_+ \times \Sigma \):

**Corollary 2.12.** Under the index positivity, the group \( \hat{H}W_*(L) \) does not depend of the filling \( W \) and is defined in \( \mathbb{R}_+ \times \Sigma \in \hat{W} \).

**Remark 2.13.** By the topological assumptions in (2.1), the Maslov index of Hamiltonian 1-chords can be defined in terms of the Maslov index of the corresponding Reeb chords. Therefore the grading of \( \hat{v} \)-shaped wrapped Floer homology in fact does not appeal to the filling.

2.5.4. **Ring structure.** Under a stronger index positivity condition, the ring structure on the \( \hat{v} \)-shaped wrapped Floer homology is independent of the filling. An essentially the same definition is given in [9] Section 9.5).

**Definition 2.14.** The triple \( (\Sigma, \xi, \mathcal{L}) \) of contact manifold \( (\Sigma, \xi) \) with a Legendrian \( \mathcal{L} \) is called product index positive if for each \( T > 0 \), there exists a contact form \( \alpha_T \) for \( (\Sigma, \xi) \) such that

- every periodic Reeb orbit \( \gamma \) in \( (\Sigma, \alpha_T) \) of period less than \( T \) is non-degenerate, and the Conley-Zehnder index \( \mu_{CZ}(\gamma) \) is greater than \( 3 - n \);
- every Reeb chord \( c \) in \( (\Sigma, \alpha_T, \mathcal{L}) \) of period less than \( T \) is non-degenerate, and the Maslov index \( \mu(c) \) is greater than \( \frac{3}{2} \).

Let \( x_0, x_1, x_2 \) be Hamiltonian 1-chords in \( \mathbb{R}_+ \times \Sigma \) relative to \( L = \mathbb{R}_+ \times \mathcal{L} \). Consider the moduli space \( \mathcal{M}(x_0, x_1, x_2; \beta, H^S, J^S) \) of half pair of pants.

**Proposition 2.15.** If \( (\Sigma, \xi, \mathcal{L}) \) is product index positive, then

\[
\hat{\mathcal{M}}(x_0, x_1, x_2; \beta, H^S, J^S) = \mathcal{M}^{\mathbb{R}_+ \times \Sigma}(x_0, x_1, x_2; \beta, H^S, J^S).
\]

In other words, all half pair of pants in \( \hat{W} \) with asymptotics in the symplectization part \( \mathbb{R}_+ \times \Sigma \) lie in the symplectization.

**Proof.** As in the proof of Proposition 2.11 suppose on the contrary that there exists a half pair of pants \( u \) in \( \mathcal{M}(x_0, x_1, x_2; \beta, H^S, J^S) \) that enters into the filling for some Hamiltonian chords \( x_0, x_1, x_2 \) with \( \mu(x_0) + \mu(x_2) - \mu(x_0) - \frac{3}{2} = 0 \). Using the neck stretching operation again, we get a sequence of half pair of pants, and by the relative SFT compactness, the sequence converges to a holomorphic building. In contrast to the Floer strip case, we may have disconnected top piece of the holomorphic building. We argue by separating cases.
We first consider the case when the top piece of the holomorphic building is connected. Then it must be an element in the moduli space of the form \( \mathcal{M}(x_0, x_1, x_2, \gamma_1, \ldots, \gamma_l, c_1, \ldots, c_m; \beta, H, J) \). The expected dimension of the space is

\[
\mu(x_1) + \mu(x_2) - \mu(x_0) - \frac{n}{2} - \sum_{j=1}^{l} (\mu_{CZ}(\gamma_j) + n - 3) - \sum_{j=1}^{m} \left( \mu(c_j) + \frac{n-3}{2} \right).
\]

Due to the assumption \( \mu(x_1) + \mu(x_2) - \mu(x_0) - \frac{n}{2} = 0 \), the dimension is equal to

\[
- \sum_{j=1}^{l} (\mu_{CZ}(\gamma_j) + n - 3) - \sum_{j=1}^{m} \left( \mu(c_j) + \frac{n-3}{2} \right),
\]

which is negative by product index positivity. This gives a contradiction.

The remaining case is actually when there are exactly two connected components in the top piece. Indeed, every component of the top piece must contain at least one of \( x_1 \) and \( x_2 \) due to the maximum principle.

Let us assume that the top piece has exactly two components and let \( u_1 \) and \( u_2 \) be the maps corresponding to those two components. Without loss of generality, we may assume that \( u_1 \) belongs to \( \mathcal{M}(x_0, x_1, \Gamma_1, \Gamma_2; \beta, H, J) \) and \( u_2 \) belongs to \( \mathcal{M}(\emptyset, x_2, \Gamma_2, C_2; \beta, H, J) \) for some sets \( \Gamma_j \) of periodic Reeb orbits and \( C_j \) of Reeb chords, \( j = 1, 2 \).

Consider the case when \( \Gamma = \emptyset = \Gamma_2 \) and \( |C_1| = 1 = |C_2| \). All the other cases can be treated in a similar way. Write \( C_1 = \{c_1\} \) and \( C_2 = \{c_2\} \). Considering that the moduli space \( \mathcal{M}(x_0, x_1, \Gamma_1, \Gamma_2; \beta, H, J) \) has nonnegative dimension, we get

\[
\mu(x_1) - \mu(x_0) - \mu(c_1) - \frac{n-3}{2} \geq 0.
\]

Similarly regarding \( \mathcal{M}(\emptyset, x_2, \Gamma_2, C_2; \beta, H, J) \), we get

\[
\mu(x_2) + \frac{n}{2} - \mu(c_2) - \frac{n-3}{2} \geq 0.
\]

Plugging the equality \( \mu(x_1) - \mu(x_0) = \frac{n}{2} - \mu(x_2) \) into the first inequality, we get two inequalities

\[
-\mu(x_2) - \mu(c_1) + \frac{3}{2} \geq 0
\]

\[
\mu(x_2) - \mu(c_2) + \frac{3}{2} \geq 0.
\]

By the product index positivity assumption, the first inequality leads to \( \mu(x_2) < 0 \) while the second one leads to \( \mu(x_2) > 0 \), which is a contradiction.

**Corollary 2.16.** Under the product index positivity, the ring \( \hat{HW}_*(L) \) does not depend on the filling \( W \) and is defined purely in \( \mathbb{R}_+ \times \Sigma \in \hat{W} \).

### 3. Real Lagrangians in \( A_k \)-type Milnor fibers

#### 3.1. Lefschetz thimbles.

Let \( V_k \) be the (completed) \( A_k \)-type Milnor fiber of dimension \( 2n \) defined by

\[
V_k = \{ z \in \mathbb{C}^{n+1} \mid z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 1 \}.
\]

The Milnor fiber \( V_k \) admits an explicit Lefschetz fibration structure given by

\[
\pi : V_k \to \mathbb{C}, \quad z \mapsto z_0.
\]

Note that \( \pi \) has exactly \( k + 1 \) critical values; the \( (k+1) \)-th roots of unity, say \( \xi_0 = 1, \xi_1 = e^{\frac{2\pi i}{k+1}}, \ldots, \xi_k = e^{\frac{2\pi i k}{k+1}} \). We associate a Lefschetz thimble to each critical value \( \xi_j \) in the following way: Consider a half straight line \( \Gamma_j : [1, \infty) \to \mathbb{C} \) starting from \( \xi_j \) i.e. \( \Gamma_j(t) = \xi_j t \). For \( t = 1 \), the fiber \( \pi^{-1}(\Gamma_j(t)) \) is a singular fiber given by

\[
\pi^{-1}(\Gamma_j(1)) = \{ z \in \mathbb{C}^n \mid z_1^2 + \cdots + z_n^2 = 0 \}.
\]

This can be identified with \( T^*S^{n-1} \) with the zero section collapsed to a point. For \( t \neq 1 \), the value \( \Gamma(t) \) is a regular value of \( \pi \), and the fiber \( \pi^{-1}(\Gamma_j(t)) \) is identified with \( T^*S^{n-1} \).
For later use, we describe the identification \( \pi^{-1}(\Gamma_j(t)) = T^*S^{n-1} \) in more detail. The fiber is given by
\[
\pi^{-1}(\Gamma(t)) = \{ z \in \mathbb{C}^n \mid z_1^2 + \cdots + z_n^2 = 1 - \Gamma(t)^{k+1} \}.
\]
More generally, for \( c \in \mathbb{C} \) nonzero, consider the level set of the quadratic polynomial
\[
Q_c := \{ z \in \mathbb{C}^{n+1} \mid z_1^2 + \cdots + z_n^2 = c \}.
\]
The value \( \sqrt{c} \) is not uniquely determined, but if we fix the angle range \([0, 2\pi)\), then we have only two of them. From now on, by \( \sqrt{c} \) we always mean the one with the smaller angle. For example, we have \( \sqrt{-1} = i \), while \(-i\) also satisfies \((-i)^2 = -1\). We now define a map \( \Phi_c : Q_c \to T^*S^{n-1} \) as follows:

We first rotate the standard coordinate system by the angle of \( \sqrt{c} \) on each coordinate, and then we send the rotated coordinate, say \( u + iv \), to \(|u|^{-1}u, |u|v\). The resulting map \( \Phi_c : Q_c \to T^*S^{n-1} \) is then an exact symplectomorphism.

Under the identification \( \pi^{-1}(\Gamma_j(t)) = T^*S^{n-1} \) via \( \Phi_{\Gamma_j(t)}^{k+1} \), we define a subset \( L_j \subset V_k \) by
\[
L_j := \bigsqcup_{t \in [1, \infty)} \{ \text{the zero section of } \pi^{-1}(\Gamma_j(t)) \},
\]
for each \( 0 \leq j \leq k \). Then it is well known that \( L_j \) is a Lagrangian in \( V_k \), which is called a Lefschetz thimble associated to the critical value \( \xi_j \); see [19, 27].

### 3.2 Real Lagrangians
We describe the Lefschetz thimbles \( L_i \)'s as fixed point sets of anti-symplectic involutions on \( V_k \). Consider the straight line \( \ell_j \) in \( \mathbb{C} \) which connects the origin and \( \xi_j \), for each \( 0 \leq j \leq k \).

We denote the reflection on \( \mathbb{C} \) whose axis is \( \ell_j \) by \( R_j : \mathbb{C} \to \mathbb{C} \). We define an involution on \( \mathbb{C}^{n+1} \) by
\[
\rho_j : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \quad z \mapsto (R_j(z_0), -\overline{z}_0, \ldots, -\overline{z}_n).
\]

Then \( \rho_j \) is an exact anti-symplectic involution on \( \mathbb{C}^{n+1} \) with respect to the standard Liouville form. Observe that \( \rho_j \) restricts to an anti-symplectic involution on the Milnor fiber \( V_k \) with respect to the induced Liouville 1-form by \( \mathbb{C}^{n+1} \). We denote the restricted involution by the same notation \( \rho_j : V_k \to V_k \).

Since \( \rho_j \) is an anti-symplectic involution, the fixed point set \( \text{Fix}(\rho_j) \) is a Lagrangian in \( V_k \), which is called a real Lagrangian. We note that if \( k \) is even, then \( \text{Fix}(\rho_j) \) is connected, and if \( k \) is odd, \( \text{Fix}(\rho_j) \) consists of exactly two connected components.

Now we consider the Lefschetz fibration \( \pi : V_k \to \mathbb{R} \) in (3.1). Observe that the image \( \pi(L_j) \) lies on the line \( \ell_j \subset \mathbb{C} \). More precisely, each connected component of the real Lagrangians is mapped to the half line \( \xi_j \cdot [1, \infty) \subset \mathbb{C} \) for some \( j \). Therefore, we can label the connected components of real Lagrangians using the corresponding \( \xi_j \), say \( \tilde{L}_0, \ldots, \tilde{L}_k \).

**Example 3.1.** Consider the case when \( k = 1 \). The anti-symplectic involutions are given by
\[
\rho(z) := \rho_0(z) = \rho_1(z) = (\overline{z}_0, -\overline{z}_0, \ldots, -\overline{z}_n).
\]

Its fixed point set is then
\[
\text{Fix}(\rho) = \{ z \in \mathbb{C}^{n+1} \mid x_0^2 - y_1^2 - \cdots - y_n^2 = 1 \}
\]
where \( z_j = x_j + iy_j \). This has two connected components, namely \( \{ x_0 > 0 \} \) and \( \{ x_0 < 0 \} \). The images of them under the Lefschetz fibration \( \pi : V_1 \to \mathbb{C} \) correspond to \( \{ x \geq 1 \} \) and \( \{ x \leq -1 \} \) respectively, and \( \xi_0 = 1 \) and \( \xi_1 = -1 \) are the critical values of \( \pi \). We label the connected components as
\[
\tilde{L}_0 = \{ x_0 > 0 \}, \quad \tilde{L}_1 = \{ x_0 < 0 \}.
\]

**Proposition 3.2.** The connected component \( \tilde{L}_j \) coincides with the Lefschetz thimble \( L_j \).

**Proof.** This is just a direct computation. We write down the case \( j = 0 \). Remind that the involution which defines \( \tilde{L}_0 \) is given by
\[
\rho_0(z) = (\overline{z}_0, -\overline{z}_1, \ldots, -\overline{z}_n).
\]

Its fixed point set has possibly two connected components, but in any case the connected component \( \tilde{L}_0 \) is given by
\[
\tilde{L}_0 = \{ z \in \mathbb{C}^{n+1} \mid x_0^{k+1} - y_1^2 - \cdots - y_n^2 = 1, \ x_0 \geq 1 \}
\]
\[
= \{ z \in \mathbb{C}^{n+1} \mid y_1^2 + \cdots + y_n^2 = x_0^{k+1} - 1, \ x_0 \geq 1 \}.
\]
On the other hand, the Lefschetz thimble $L_0$ is the disjoint union of the zero sections of $\pi^{-1}(\Gamma_0(t))$, $1 \leq t < \infty$, under the identification $\pi^{-1}(\Gamma_0(t)) = T^nS^{n-1}$ (except for $t = 1$). This identification is precisely given by the map $\Phi_{1-\Gamma_0(t)+1}$. Since $\Gamma_0(t) = t$ and hence $1 - \Gamma_0(t)^{k+1} = 1 - t^{k+1}$ is a positive real number in this case, the map $\Phi_{1-\Gamma_0(t)+1}$ is given by the formula

$$z \in \pi^{-1}(\Gamma_0(t)) \mapsto (-|y|^{-1}y, |y|x) \in T^nS^{n-1}.$$ 

In particular the zero section of $T^nS^{n-1}$ corresponds to the set

$$\pi^{-1}(\Gamma_0(t)) \cap \{x_1 = \cdots = x_n = 0\} = \{z \in \mathbb{C}^{n+1} \mid y_1^2 + \cdots + y_n^2 = \Gamma_0(t)^{k+1} - 1\}.$$ 

It follows that

$$L_0 = \bigcup_{t \in [1, \infty)} \{\text{zero section of } \pi^{-1}(\Gamma_0(t))\}$$

$$= \bigcup_{x_0 \in [1, \infty)} \{z \in \mathbb{C}^{n+1} \mid y_1^2 + \cdots + y_n^2 = x_0^{k+1} - 1\}$$

$$= \{z \in \mathbb{C}^{n+1} \mid y_1^2 + \cdots + y_n^2 = x_0^{k+1} - 1, \ x_0 \geq 1\} = \tilde{L}_0.$$ 

This completes the proof. \qed

In short, the Lefschetz thimbles $L_j$’s are (connected components of) real Lagrangians. From now on, we will use the notation $L_j$, $0 \leq j \leq k$, for the real Lagrangians.

3.3. **Real Lagrangians in plumbings.** The Milnor fiber $V_k$ is Stein deformation equivalent to the linear plumbing $\#^kT^nS^n$ of cotangent bundles over the sphere, called $A_k$-type plumbing. In this section, we give a description of this fact, and we observe that the real Lagrangian $L_j$, $0 \leq j \leq k$, can be seen as a cotangent fiber or a diagonal Lagrangian in the $A_k$-type plumbing $\#^kT^nS^n$.

3.3.1. $A_k$-type Plumbings. To fix notations, we briefly describe the plumbing of two cotangent bundles. The definition of the plumbing associated to the $A_k$-type Dynkin diagram is then a straightforward generalization.

Let $Q_0$ and $Q_1$ be closed manifolds of dimension $n$. Following [4, Section 2], we define a local model of the plumbing region as follows.

\begin{equation}
(3.2) \quad R := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid |x||y| \leq 1\}.
\end{equation}

Consider the ball $B^n \subset \mathbb{R}^n$ of radius $1/2$. Its disk cotangent bundle $DT^nB^n$ is identified with $B^n \times B^n$ via $\sum y_i dx_i = (x_1, \ldots, x_n, y_1, \ldots, y_n)$, where $x = (x_1, \ldots, x_n)$ is the base coordinate and $y = (y_1, \ldots, y_n)$ is the fiber coordinate. Furthermore, the disk cotangent bundle $DT^nB^n$ has the symplectic form $\sum dx_i \wedge dy_i$. We glue two copies of $DT^nB^n$ via the symplectomorphism $(x, y) \mapsto (y, -x)$. This is exactly how the local model $R$ looks like near the origin.

For each $j = 0, 1$, we take a metric $g_j$ on $Q_j$ locally flat near a fixed point $q_j$ and consider an isometric embedding

$$(B^n_j, 0) \hookrightarrow (Q_j, q_j)$$

of the ball of radius $1/2$ centered at the origin. This induces a symplectic embedding

$$DT^nB^n_j \hookrightarrow DT^nQ_j$$

for $j = 0, 1$. We glue the disk cotangent bundles $DT^nQ_0$ and $DT^nQ_1$ using the map $(x, y) \mapsto (y, -x)$ on $DT^nB^n$ described above. Smoothing along the corner of the resulting space, we get a Liouville domain $DT^nQ_0\#T^nQ_1$. Its completion is called the plumbing of $T^nQ_0$ and $T^nQ_1$ and is denoted by $T^nQ_0\#T^nQ_1$.

In the plumbing $T^nQ_0\#T^nQ_1$, there are two types of typical Lagrangians: cotangent fibers (away from the plumbing region) and diagonal Lagrangians. By a cotangent fiber in the plumbing, we mean the completion of a cotangent fiber in $DT^nQ_j$ (for some $j = 0, 1$) away from the plumbing region. In the plumbing region, on the other hand, we have another type of Lagrangians which can be written in the local model as

\begin{equation}
(3.3) \quad D := \{(x, y) \in R \mid \text{either } x_j = y_j \text{ or } x_j = -y_j \text{ for each } 1 \leq j \leq n\}.
\end{equation}

We call Lagrangians obtained by completing $D$ diagonal Lagrangians.
3.3.2. The Milnor fiber $V_k$ as a plumbing. To describe the Milnor fiber $V_k$ as a plumbing, we use the following neighborhood theorem. We provide a proof of the theorem for later use in Section 3.3.3 although it is fairly well-known.

**Proposition 3.3.** Let $S_j$, $1 \leq j \leq k$, be closed exact Lagrangians in a symplectic manifold $(W, \omega)$. Suppose $S_j$'s are in the $A_k$-configuration, i.e. $S_j$ and $S_{j+1}$ with $1 \leq j \leq k-1$ intersect transversely at one point and there exist no other intersections between these Lagrangians. Then there exists a symplectic embedding $\Phi : \nu(\bigcup_j S_j) \rightarrow W$ from a neighborhood $\nu(\bigcup_j S_j)$ in the $A_k$-type plumbing $\#_j DT^* S_j$ into $W$, which maps $S_j$ in the plumbing $\#_j DT^* S_j$ identically to $S_j$ in $W$.

For a proof, we use the following lemma from [23] Proposition 3.4.1.

**Lemma 3.4 (Poźniak).** Let $(M, \omega)$ be a symplectic manifold of dimension $2n$ and let $L_0, L_1$ be two Lagrangian submanifolds which intersect transversely at $q \in L_0 \cap L_1$.

There exist a neighborhood $V$ of $q$ in $\mathbb{R}^n \times \mathbb{R}^n$, a neighborhood $U$ of $q$ in $M$ and a symplectomorphism $\psi : (U, \omega) \rightarrow (V, \omega_{std})$ such that

$$\psi(L_0 \cap U) = K_0 \cap V \text{ and } \psi(L_1 \cap U) = K_1 \cap V,$$

where $K_0 = \mathbb{R}^n \times \{0\}$ and $K_1 = \{0\} \times \mathbb{R}^n$.

**Proof of Proposition 3.3.** For the proof of the first statement, we provide an argument for the case when $k = 2$; the general case follows from the same argument.

Suppose $S_0$ and $S_1$ intersect at $q \in W$ transversely. Applying Lemma 3.4 to the Lagrangians $S_0$ and $S_1$ we have an open neighborhood $U$ of $q$ in $W$ as in the lemma. Shrinking $U$ and scaling $\omega$ if necessary, we may identify $U_{0} := U \cap S_0$ with $B^n \times \{0\} \subset \mathbb{R}^n \times \{0\}$ and $U_{1} := U \cap S_1$ with $\{0\} \times B^n \subset \{0\} \times \mathbb{R}^n$ where $B^n$ is the open ball of radius $1/2$. Accordingly, we have identifications $DT^* U_0 = B^n \times B^n = DT^* U_1 \subset \mathbb{R}^n \times \mathbb{R}^n$, which is exactly same as what happened in the plumbing region.

We take a metric $g$ on $W$ such that $g = \sum dx_i \otimes dx_i + dy_i \otimes dy_i$ on $U$. The metric $g$ induces a bundle isomorphism $\psi_j : T^* S_j \rightarrow T^* S_j$, $v^* \mapsto u$ where $u \in T_p S_j^{\#}$ is chosen so that $\omega(\cdot, u) = v^* \in T_p S_j$. We define a smooth embedding $\Psi_j : DT^* S_j \rightarrow W$ by

$$\Psi_j(q_j, v_j^*) = \exp_{q_j}(\psi(v_j^*)) \text{ for } (q_j, v_j^*) \in DT^* S_j.$$

We may shrink the radius of each of the disk cotangent bundles to ensure that both maps $\Psi_0$ and $\Psi_1$ are embeddings.

Under the identification $DT^* U_0 = B^n \times B^n$, the restriction of $\Psi_0$ to $DT^* U_0 \subset DT^* S_0$ is identified with the inclusion map $B^n \times B^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$ in the neighborhood obtained in Lemma 3.4. Similarly, the restriction of $\Psi_1$ to $DT^* U_1$ can be identified with the inclusion map $B^n \times B^n \hookrightarrow \mathbb{R}^n \times \mathbb{R}^n$. This shows that $\Psi_0$ and $\Psi_1$ glue together and yield a map $\Psi : DT^* S_0 \# DT^* S_1 \rightarrow W$.

Furthermore we have $\Psi^* \omega = \omega_{can}$ on each $S_j \subset DT^* S_j$. Since $[\Psi^* \omega] = [\omega_{can}] \in H^2(DT^* S_0 \# DT^* S_1)$, we apply the Moser’s trick to find a symplectic embedding $\Phi : \nu(\bigcup_j S_j) \rightarrow W$ of a neighborhood $\nu(S_0 \cup S_1) \subset DT^* S_0 \# DT^* S_1$ of $S_0 \cup S_1$ into $W$. Since $\Psi^* \omega|_{S_j} = \omega_{can}|_{S_j}$ and $\Psi$ sends $S_j \subset DT^* S_0 \cup DT^* S_1$ identically to $S_j \subset W$ for each $j = 0, 1$, so does the map $\Phi$. \hfill \Box

**Remark 3.5.** In the lemma above, if $(W, \omega = d\lambda)$ is a Liouville domain, then the neighborhood $\nu(\bigcup_j S_j)$ is equipped with two different Liouville structures: one from that of $(W, \omega = d\lambda)$ and the other one from that of the plumbing $\bigcup_j DT^* S_j$. If $H^1(S_j, \mathbb{R}) = 0$ for all $1 \leq j \leq k$, then two Liouville structures are isomorphic.

We can now identify the Milnor fiber $V_k$ with the plumbing of $T^* S^n$'s associated to the $A_k$-type Dynkin diagram. Consider the Lefschetz fibration

$$\pi : V_k \rightarrow C, \quad z \mapsto z_0.$$

As in Section 3.1, denote the critical values by $\xi_0 = 1, \xi_1, \ldots, \xi_k = e^{\frac{2\pi i}{k+1}}$; $k + 1$-th roots of unity. We define a subset $S_j \subset V_k$ for $0 \leq j \leq k$ as follows. Consider the line segment in the base $\mathbb{C}$ which connects $\xi_j$ and $\xi_{j+1}$ (here we conventionally put $\xi_{k+1} = \xi_0$). Observe that at each point on the line segment except for the end points, the fiber is a copy of $T^* S^n$, and at end points the fiber is a copy of $T^* S^n$ with the zero section collapsed to a point. Define $S_j$ to be the union of the zero section on each fibers over the segment. In
terms of Picard-Lefschetz theory, \( S_j \) is called a matching cycle in the total space of the Lefschetz fibration, and it is an exact Lagrangian sphere.

Now we pick \( k \) Lagrangian spheres in \( V_k \) out of \( S_j \)'s, say \( S_0, \ldots, S_{k-1} \). Note that they are in the \( A_k \) configuration by construction. Consider the \( A_k \)-type plumbing of cotangent bundles over \( S_j \)'s. We can embed a neighborhood of the union of \( S_j \)'s in the plumbing into a neighborhood of the union of \( S_j \)'s in \( V_k \) by Proposition 3.3. Since the completion of the neighborhood of \( S_j \)'s in \( V_k \) contains all critical points of \( \pi \), we conclude that \( V_k \) is exact symplectomorphic to the \( A_k \)-type plumbing \( \#^k T^* S_j = \#^k T^* S^0 \).

Remark 3.6. The identification of \( V_k \) with the \( A_k \)-type plumbing \( \#^k T^* S^0 \) depends on the choice of matching cycles of the Lefschetz fibration \( \pi : V_k \rightarrow C \). Our discussion below is with respect to the above choice of matching cycles \( S_0, \ldots, S_{k-1} \).

3.3.3. Cotangent fibers and diagonal Lagrangians. In the Milnor fiber \( V_k \), we have defined real Lagrangians, \( L_j \) with \( 0 \leq j \leq k \), which can be seen as Lefschetz thimbles. In this section, we want to observe the following.

Proposition 3.7. Under the identification \( V_k = \#^k T^* S_j = \#^k T^* S^0 \), the two real Lagrangians \( L_0 \) and \( L_k \) are cotangent fibers of \( T^* S_0 \) and \( T^* S_{k-1} \), respectively, away from the plumbing regions, and the other real Lagrangians are diagonal Lagrangians in the plumbing regions.

Proof. This follows from the intersection configurations between \( L_j \)'s and \( S_j \)'s. For a simper presentation, let us assume that \( k = 2 \) (the other cases are then straightforward). We have two Lagrangian spheres \( S_0 \) and \( S_1 \) in the \( A_2 \)-configuration and we have three Lagrangians \( L_0, L_1, \) and \( L_2 \) located as in Figure 2.

![Figure 2. Real Lagrangians with matching cycles](image)

In particular \( L_0 \) intersects with \( S_0 \) transversely at the critical point \( \xi_0 = 1 \). In view of the proof of Proposition 3.3, we can readily modify the embedding \( \Phi \) in the statement of Proposition 3.3 to have the property that \( \Phi \) sends a local neighborhood of \( \xi_0 \) in \( L_0 \) to a fiber in the plumbing \( T^* S_0 \# T^* S_1 \) which comes from a fiber of \( T^* S_0 \)-factor. The same argument applies to \( L_2 \) since \( L_2 \) intersects transversely with \( S_1 \) at one point.

Now let us consider the other Lagrangian \( L_1 \). This Lagrangian intersects with \( S_0 \) and \( S_1 \) transversely and simultaneously at one point, namely \( \xi_1 \). In view of Proposition 3.3, we can take a local chart \( U \) of \( \xi_1 \) in \( V_2 \), identified symplectically with (a neighborhood of) the model plumbing \( R \) defined in (3.2) such that \( S_0 \) corresponds to the \( x \)-coordinates (zero section) and \( S_1 \) to the \( y \)-coordinates (cotangent fiber).

In this chart, since \( L_1 \) is transverse to \( S_1 \), it can be written as the graph of an exact 1-form, say \( df \), where \( f \) is a smooth function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) of the \( x \)-coordinates. Since \( L_1 \) intersects \( S_0 \) at the origin \( (x, y) = (0, 0) \) (in the chart), it follows that \( df = 0 \) at the origin. In other words, the origin in \( R \) is the unique critical point of \( f \). Furthermore, since the intersection is transverse, it follows that the origin is a non-degenerate critical point of \( f \).

Note, from the equation (3.3), that the Lagrangian \( L_1 \) is diagonal in \( R \) if and only if the 1-form \( df \) is of the following form

\[
df = \sum_{i=1}^{n} \lambda_i x_idx_i
\]

where \( \lambda_i = \pm 1 \). That is, the coefficient functions of \( df \) are all linear with slope 1 or \(-1 \). This is precisely the case when we took the \( x \)-coordinates for \( S_0 \) so that \( f \) is written near the origin as a complex quadratic
3.4. Computing group structures of (v-shaped) wrapped Floer homology. A benefit of regarding the Lefschetz thimbles as real Lagrangians is that this enables us to compute their wrapped Floer homology explicitly. In this section, we outline the computation of the group structure of (v-shaped) wrapped Floer homology of $L_j$’s.

3.4.1. Wrapped Floer homology of $L_j$. The group structure of the wrapped Floer homology of $L_0$ is computed in [20, Section 7.4.2] using a Morse–Bott spectral sequence. Actually, we can compute $HW_\ast(L_j)$ for each $j$ using the same computational technique. One then finds that the groups $HW_\ast(L_j)$ are isomorphic to each other for all $0 \leq j \leq k$. This is because the Morse–Bott setup, i.e. Morse–Bott submanifolds and their Maslov indices are the same, so they have the same $E^1$-page of the spectral sequence in [20, Theorem 2.1]. We therefore have the following computation.

**Proposition 3.8.** Let $k \geq 1$ and $n \geq 3$. For every $0 \leq j \leq k$, the wrapped Floer homology group of $L_j$ is given by

$$HW_\ast(L_j) = \begin{cases} \mathbb{Z}_2 & \ast = 0, \{(n-2)(k+1) + 2\}N - n + 1, \{(n-2)(k+1) + 2\}N \text{ with } N \in \mathbb{N}; \\
0 & \text{otherwise.} \end{cases}$$

**Remark 3.9.** For $n \geq 3$, the topological conditions [2,1] are all satisfied with $(V_k, L_j)$.

3.4.2. V-shaped wrapped Floer homology of $L_j$. We can also compute v-shaped wrapped Floer homology groups of $L_j$’s. We use an analogous Morse–Bott spectral sequence converging to $HW(L_j)$. To state a spectral sequence we fix notations as follows. Let $(W, d\lambda)$ be a Liouville domain with an admissible Lagrangian $L$. Denote its Legendrian boundary in $\partial W$ by $L$. For technical simplicity (to have well-defined Maslov indices of Reeb chords), we assume in addition to (2.1) the following topological conditions: $\pi_1(L) = 0$ so that the Maslov class vanishes.

Assume that every Reeb chords in $(\partial W, \alpha, L)$ is of Morse–Bott type, see [20, Definition 2.6] for a definition. Denote the v-spectrum of $(\partial W, \alpha)$ by

$$\tilde{\text{Spec}}(\partial W, \alpha, L) := \{T \in \mathbb{R} \mid \phi^T_R(x) \in L \text{ for some } x \in L\},$$

where $\phi^T_R(x)$ denotes the time-$t$ Reeb flow of the contact form $\alpha$. For each $T \in \tilde{\text{Spec}}(\partial W, \alpha, L)$, we define the corresponding Morse–Bott submanifold by

$$L_T := \{x \in L \mid \phi^T_R(x) \in L\}.$$

Finally, we arrange the v-spectrum as

$$\text{Spec}(\partial W, \alpha, L) = \{\ldots, T_{-2} < T_{-1} < T_0 = 0 < T_1 < T_2, \ldots\}.$$

**Remark 3.10.** If one has arranged the (non v-shaped) spectrum as

$$\text{Spec}(\Sigma, \alpha, L) = \{T_0 = 0 < T_1 < T_2 < \cdots\},$$

then the v-shaped spectrum is arranged just by putting $T_{-p} = -T_p$ for $p \in \mathbb{N}$.

**Theorem 3.11.** There exists a spectral sequence $\{\{E^p, d^p\}\}_{p \in \mathbb{Z}}$ converging to v-shaped wrapped Floer homology $HW_\ast(L; W)$ such that

$$E^1_{pq} = \begin{cases} H_{p+q-\text{shift}(L_T_p)+\frac{1}{2} \dim L(L_T_p; \mathbb{Z}_2)} & p \neq 0, \\
H_{q+\dim L(L; \mathbb{Z}_2)} & p = 0, \end{cases}$$

where $\text{shift}(L_T_p) = \begin{cases} \mu(L_T_p) - \frac{1}{2}(\dim L_T_p - 1) & p > 0 \\
\mu(L_T_p) - \frac{1}{2}(\dim L_T_p + 1) & p < 0 \end{cases}$.

**Remark 3.12.**

1. Our grading convention is such that $HW_\ast(-\epsilon, \epsilon)(L) \cong H_{s+n-1}(L)$ for sufficiently small $\epsilon > 0$.
2. Here $\mu(L_T_p)$ denotes the Maslov index (also referred as Robbin–Salamon index) of a Morse–Bott component $L_T_p$, see [20, Section 2.2.2] for a definition. By the inverse property of the index, we have $\mu(L_{-T_p}) = -\mu(L_T_p)$ for each non-zero $p \in \mathbb{Z}$. 

polynomial, which is always possible due to the Morse lemma. It follows that $L_1$ can be written as a diagonal Lagrangian in a plumbing region. This completes the proof.  

□
(3) The construction of the Morse–Bott spectral sequence is essentially the same as the non-v-shaped wrapped Floer homology case. Namely we use the action filtration, on v-shaped wrapped chain complexes, which is adapted to the Morse–Bott setup.

Applying the spectral sequence to \( L_j \) in \( V_k \), we can compute the group structure of its v-shaped wrapped Floer homology. First of all, the computation of the \( E^1 \)-page can be done exactly as in wrapped Floer homology case [20 Section 7.4.2]. The only computational differences would be that we have homology of the Legendrian \( L_j \) in the column \( p = 0 \) instead of the relative homology of \( (L_j, L_j) \), and we have homology of Morse–Bott submanifold \( L_{T_p} \) for negative \( p \in \mathbb{Z} \). The latter in particular is the same as the homology of \( L_{T_p} \) with positive \( p \) but with a different degree shift. However, as noted in Remark 3.12 it just involves the same Robin-Salamon index of the corresponding Robin-Salamon index for positive \( p \) with the opposite sign.

3.4.3. Computing the group \( \tilde{HW}_*(L_j) \). We present the computation of \( \tilde{HW}_*(L_0) \) in more detail. Firstly, to fit our situation into a Morse–Bott setup, we deform the (non-completed) Milnor fiber \( V_k \cap B^{2n+2} \) into the following Liouville domain

\[
W = W_k := \{ z \in \mathbb{C}^{n+1} \mid z_0^{k+1} + z_1^2 + \cdots + z_n^2 = \zeta(|z|) \} \cap B^{2n+2}
\]

where \( B^{2n+2} \subset \mathbb{C}^4 \) is the unit ball and \( \zeta : \mathbb{R} \to \mathbb{R} \) is a monotone smooth function such that \( \zeta(r) = 1 \) for \( 0 \leq r \leq 1/4 \) and \( \zeta(r) = 0 \) for \( 3/4 \leq r \leq 1 \). The boundary \( \partial W =: \Sigma \) is then an \( A_k \)-type Brieskorn manifold, and it carries a Morse–Bott type contact form \( \alpha \) given by

\[
\alpha := \frac{i}{2} \left\{ (k+1)(z_0dz_0 - z_0dz_0) + 2 \sum_{j=1}^n (z_jdz_j - z_jdz_j) \right\}
\]

restricted to \( \Sigma \). The corresponding Reeb flow is given as

\[
\phi^t_R(z) = (e^{it/k+1}z_0, e^{it/2}z_1, \ldots, e^{it/2}z_n).
\]

The Legendrian \( L = L_0 \subset \partial W \) is written as

\[
L = \{ z \in \mathbb{C}^{n+1} \mid z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 0, \ |z|^2 = 1, \ y_0 = x_1 = \cdots = x_n = 0 \},
\]

where \( z_j = x_j + iy_j \). The v-spectrum is given by

\[
\tilde{\text{Spec}}(\Sigma, \alpha, L) = \{ N \cdot 2(k+1)\pi \mid N \in \mathbb{Z} \}.
\]

It is straightforward to see that, for each \( T \in \tilde{\text{Spec}}(\Sigma, \alpha, L) \), the corresponding Morse–Bott component is identical to \( L \), i.e. \( L_T = L \) as sets, which is topologically equivalent to \( S^{n-1} \). Furthermore, by [20 Lemma 7.2], we obtain the indices \( \mu(L_T) \) as follows.

\[
\mu(L_{N \cdot 2(k+1)\pi}) = -\mu(L_{-N \cdot 2(k+1)\pi}) = \{ 2 + (n-2)(k+1) \} N.
\]

Using these data, we can compute the \( E^1 \)-page of the spectral sequence. It turns out that if \( n \geq 3 \), the \( E^1 \)-page terminates at the \( E^1 \)-page for degree reasons. For example if \( k = 2 \) or \( 3 \) and \( n = 3 \) the \( E^1 \)-page is given as Figure 3. Therefore we get \( \tilde{HW}_*(L_0) \) directly from the \( E^1 \)-page, namely

\[
\tilde{HW}_*(L_0) = \begin{cases} \mathbb{Z}_2 & \ast = \cdots - 7, -5, -2, 0, 3, 5, \cdots; \\ 0 & \text{otherwise.} \end{cases}
\]

In the same way, we can check that for each \( n \geq 3 \) and \( k \geq 1 \) the spectral sequence terminates at the \( E^1 \)-page as in the above example. Furthermore, for a fixed \( k \) and \( n \), the \( E^1 \)-page for \( \tilde{HW}_*(L_j) \) does not depend on \( 0 \leq j \leq k \). We therefore obtain the following computation.

**Proposition 3.13.** For \( n \geq 3 \) and \( 0 \leq j \leq k \),

\[
\tilde{HW}_*(L_j) = \begin{cases} \mathbb{Z}_2 & \ast = \{ (n-2)(k+1) + 2 \} N - n + 1, \{ (n-2)(k+1) + 2 \} N \text{ with } N \in \mathbb{Z}; \\ 0 & \text{otherwise.} \end{cases}
\]
3.5. Index positivity of the real Legendrians in $A_k$-type Brieskorn manifolds. Note that the boundary of an $A_k$-type Milnor fiber $V_k$ is, up to exact symplectic isotopy, an $A_k$-type Brieskorn manifold, see Section 3.4.3. In this section, for $n \geq 3$ and $k \geq 1$, we show that $A_k$-type Brieskorn manifolds with the real Legendrians are product index positive in the sense of Definition 2.14 (which also implies the index positivity in Definition 2.10). For the sake of simplicity we only deal with the case of the first real Legendrian $L := L_0 = \partial L_0$ and the case $k$ is even; the other cases follow by the same observations with minor changes.

3.5.1. Morse–Bott setup in $A_k$-type Brieskorn manifolds. Recall that the $A_k$-type Brieskorn manifold is given by

$$\Sigma = \{ z \in \mathbb{C}^{n+1} \mid z_0^{k+1} + z_1^2 + \cdots + z_n^2 = 0, \ |z| = 1 \}.$$  

The real Legendrian $L = L_0$ is written by

$$L = \{ z \in \mathbb{C}^{n+1} \mid z \in \Sigma, \ y_0 = x_1 = \cdots = x_n = 0 \}$$

where $z = x + iy$. As we have seen in Section 3.4.3, we have a contact form $\alpha$ on $\Sigma$ whose Reeb flow is periodic and hence is of Morse–Bott type.

Note that the periods of simple periodic Reeb orbits in $(\Sigma, \alpha)$ are $T_0 := 4\pi$ and $T_1 := 4(k + 1)\pi$. (We have assumed $k$ is even.) Denote the corresponding Morse–Bott submanifolds by

$$\Sigma_{T_j} := \{ z \in \Sigma \mid \phi_{R_\alpha}^{T_j}(z) = z \}, \quad j = 0, 1.$$  

We have $\Sigma_{N:T_0} = \{ z \in \Sigma \mid z_0 = 0 \}$ for $N \not\equiv (k + 1)N$ and $\Sigma_{N:T_1} = \Sigma$ for $N \in \mathbb{N}$. Recall that periods of Reeb chords in $(\Sigma, \alpha, L)$ are given by $2(k + 1)\pi \cdot N$ for $N \in \mathbb{N}$ and the corresponding Morse–Bott submanifold of Reeb chords is given identically by $L_{2(k+1)\pi \cdot N} = L$. We in particular note that $L \cap \Sigma_{T_0} = \emptyset$.

The Reeb flow $\phi_{R_\alpha}$ naturally defines an $S^1$-action on $\Sigma_{T_j}$ for each $j = 0, 1$ and an $\mathbb{Z}_2$-action on $L$. The corresponding quotient spaces by $Q_{T_0} := \Sigma_{T_0}/S^1$ and $Q := L/\mathbb{Z}_2$ respectively. Note that $Q_{T_0}$ and $Q$ are smooth whereas $Q_{T_1}$ is an orbifold, and $Q_{T_0}$ and $Q$ can be seen as subsets in $Q_{T_1} = \Sigma/S^1$.

3.5.2. A Morse–Bott perturbation. Following [5, Section 2.2] (see also [21, Appendix C]), we shall perturb $\alpha$ using a Morse function on $Q_{T_1} = \Sigma/S^1$. First, take a positive Morse function $f_{T_0}$ on $Q_{T_0}$. We extend $f_{T_0}$ to a positive Morse function $f_{T_1}$ on $Q_{T_1}$ such that the Hessian of $f_{T_1}$ restricted to $Q_{T_0}$ is positive definite. In particular, we may assume that the restriction $f_{T_1}|Q$ is also a Morse function on $Q$ such that $\text{Crit}(f_{T_1}|Q) = \text{Crit}(f_{T_1}) \cap Q$; this can be done without any further difficulties since the orbifold singularities $Q_{T_0}$ in $Q_{T_1}$ does not intersect with $Q$. Note also that the Morse function $f_{T_1}$ on $Q_{T_1}$ is nothing but a special case of the construction in [5, Section 2.2]. Let $\tilde{f}_{T_1} : \Sigma_{T_1} = \Sigma \to \mathbb{R}$ be the $S^1$-lift of $f_{T_1}$.

For a small positive number $\lambda$, we perturb $\alpha$ by

$$\alpha_\lambda := (1 + \lambda \tilde{f}_{T_1})\alpha.$$

\begin{figure}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\hline
4 & 1 & 5 \\
\hline
2 & 0 & 3 \\
\hline
0 & 2 & 1 \\
\hline
-2 & 4 & 9 \\
\hline
-4 & -6 & -7 \\
\hline
\end{tabular}
\begin{tabular}{|c|c|c|}
\hline
5 & 1 & 3 \\
\hline
1 & 1 & 3 \\
\hline
-1 & -3 & -5 \\
\hline
-3 & -7 & -9 \\
\hline
\end{tabular}

(A) $k = 2$ and $n = 3$ \hspace{1cm} (B) $k = 3$ and $n = 3$

\caption{$E^1$-pages in examples}
\end{figure}
As shown in [5, Lemma 2.3], for a given $T > 0$, we can find a sufficiently small perturbation parameter $\lambda$ such that every Reeb orbit $\gamma$ in $(\Sigma, \alpha_\lambda)$ of period less than $T$ is non-degenerate and corresponds to a critical point of $f_{T_1}$. More precisely, each periodic Reeb orbit, say $\gamma_q$ in $(\Sigma, \alpha)$ over a critical point $q$ of $f_{T_1}$ in $Q_T$, is a non-degenerate periodic Reeb orbit in $(\Sigma, \alpha_\lambda)$, and for sufficiently small $\lambda > 0$ there are no more periodic Reeb orbits in $(\Sigma, \alpha_\lambda)$ of period less than $T$.

An important benefit of this particular perturbation is that Conley-Zehnder index of the periodic Reeb orbit $\gamma_q$ can be computed explicitly as in [5] Lemma 2.4 in terms of Robbin-Salamon index of $\gamma_q$ and Morse index of $q$ with respect to $f_{T_1}$.

In the case when a critical point $q$ lie in $Q$ and hence is a critical point of $f_{T_1}|_Q$, we have two corresponding Reeb chords, say $c_q^\pm$, in $(\Sigma, \alpha, L)$ given by the “half” of $\gamma$ starting in the $\mathbb{Z}_2$-orbit $\gamma \cap L$. A directly analogous argument to the periodic Reeb orbit case shows that for sufficiently small $\lambda > 0$, those Reeb chords (associated to each critical point of $f_{T_1}|_Q$) are Reeb chords in $(\Sigma, \alpha_\lambda, L)$ which are non-degenerate, and there are no more Reeb chords of period less than $T$. Also, the computation of Maslov index of $c_q^\pm$ is completely analogous to the case of $\gamma_q$, it can be written explicitly in terms of Robbin-Salamon index of $c_q^\pm$ and Morse index of $q$ with respect to $f_{T_1}|_Q$.

We can formulate the result of the perturbation as follows.

**Proposition 3.14.** Let $T > 0$ be given. For sufficiently small $\lambda > 0$, periodic Reeb orbits in $(\Sigma, \alpha_\lambda)$ and Reeb chords in $(\Sigma, \alpha_\lambda, L)$ of period less than $T$ are non-degenerate and correspond to critical point of $f_{T_1}$ and $f_{T_1}|_Q$, respectively. If $\gamma_q$ is a periodic Reeb orbit in $(\Sigma, \alpha_\lambda)$ corresponding to $q \in \text{Crit}(f_{T_1}) \cap Q_T$, then we have

$$\mu_{CZ}(\gamma_q) = \mu_{RS}(\gamma_q) - \frac{1}{2} \dim \Sigma_{T_j} + \text{ind}_{f_{T_1}}(q).$$

If $c_q$ be a Reeb chord in $(\Sigma, \alpha_\lambda, L)$ corresponding to $q \in \text{Crit}(f_{T_1}) \cap Q = \text{Crit}(f_{T_1}|_Q)$, then we have

$$\mu(c_q) = \mu_{RS}(c_q) - \frac{1}{2} (n - 1) + \text{ind}_{f_{T_1}|_Q}(q).$$

**Proof.** The statements about periodic Reeb orbits are just a reformulation of results in [5] Lemma 2.3, Lemma 2.4, and the Reeb chords case is also completely analogous.

**Remak 3.15.** In Proposition 3.14 iterations of $\gamma_q$ correspond to the same critical point $q \in \text{Crit}(f_{T_1})$, but the index $\mu_{CZ}(\gamma_q)$, and hence the index $\mu_{RS}(\gamma_q)$, changes under iteration. The same also applies to $c_q$.

**Corollary 3.16.** For $n \geq 3$ and $k \geq 1$, the triple $(\Sigma, \xi, L)$ is product index positive.

**Proof.** This follows from the fact that the Robbin-Salamon indices of $\gamma_q$ and $c_q$ in Proposition 3.14 are large enough. The computation of $\mu_{RS}(\gamma_q)$ is in [21] Section 5.3, and index positivity with respect to periodic Reeb orbits has been discussed in several precedent works, for example [21] Section 5.4, [29] Lemma 1. Here we check the Reeb chords case.

Recall that a Reeb chord $c_q$ over $q \in \text{Crit}(f_{T_1}|_Q)$ in the statement of Proposition 3.14 is a half of the periodic Reeb orbit $\gamma_q$, and considering their Robbin-Salamon index $\mu_{RS}(c_q)$ we regard it as a Reeb chord in $(\Sigma, \alpha, L)$. In this case the index computation is already done in [20], namely

$$\mu_{RS}(c_q) = ((n - 2)(k + 1) + 2)N.$$  

where $c_q$ is understood as the $N$-th iteration of a principal Reeb chord in $(\Sigma, \alpha, L)$ (see Remark 3.15). Since $0 \leq \text{ind}_{f_{T_1}|_Q}(q) \leq \dim Q = n - 1$, we have from Proposition 3.14 that

$$\mu(c_q) \geq (n - 2)(k + 1)N + 2 - \frac{1}{2} (n - 1) > \frac{3}{2}$$

provided that $n \geq 3$ and $k \geq 1$.

4. **Seidel operator on V-shaped wrapped Floer homology**

4.1. **Seidel operator.** In this section, we describe an open string analogue of the Seidel representation for Liouville domains and admissible Lagrangians. This associates to a path of Hamiltonian diffeomorphisms a graded group isomorphism on Floer homology, as in Theorem 4.17. An important feature of the Seidel representation for our purpose is the module property in Theorem 4.20.
An open string version of the Seidel representation was introduced in [17] for closed symplectic manifolds and Lagrangians, and a version for symplectic homology was discussed in [28]. Our case is basically analogous to their works.

4.1.1. Setup. Let \((W, d\lambda)\) be a Liouville domain and \(L\) an exact admissible Lagrangian in \(W\) with the topological assumptions (2.1). Let \(g : [0, 1] \to \text{Ham}(\hat{W})\) be a path of Hamiltonian diffeomorphisms on \(\hat{W}\). We require that

\[
(4.1) \quad p \in \hat{L} \text{ if and only if } g_0(p) \in \hat{L} \text{ and } g_1(p) \in \hat{L}.
\]

Denote the space of paths in \(\hat{W}\) relative to \(\hat{L}\), i.e. end points are in \(\hat{L}\), by

\[
\mathcal{P}_L(\hat{W}) := \{ x : [0, 1] \to \hat{W} \mid x(0), x(1) \in \hat{L} \}.
\]

Then \(g\) acts on \(\mathcal{P}_L(\hat{W})\) by

\[
(g \cdot x)(t) := g_t(x(t)).
\]

4.1.2. Correspondences. Let \(H\) be a \(v\)-shaped admissible Hamiltonian on the completion \(\hat{W}\). Let \(K^\varphi : [0, 1] \times \hat{W} \to \mathbb{R}\) be a Hamiltonian which generates the path \(g_t\), that is,

\[
dK^\varphi(t, g_t(x))(\cdot) = \omega\left(\cdot, \left.\frac{\partial g_t(x)}{\partial t}\right|_{t=0}\right).
\]

We define a Hamiltonian \(g^\ast H : [0, 1] \times \hat{W} \to \mathbb{R}\), which we call the push forward of \(H\) by \(g\), as follows.

\[
(g^\ast H)(t, p) := H(t, g_t^{-1}(p)) + K^\varphi(t, p).
\]

Then the Hamiltonian flow of \(g^\ast H\) is given by the composition \(g_t \circ \phi_t^H\). In particular, due to the condition (4.1), for a Hamiltonian 1-chord \(x\) of \(H\) relative to \(L\), the path \(g \cdot x\) is a Hamiltonian 1-chord of \(g^\ast H\) relative to \(L\).

Let \(J\) be an admissible almost complex structure on \(\hat{W}\). Define the push forward \(g^\ast J\) of \(J\) by \(g\) to be

\[
g^\ast J := dg_t \circ J \circ d\hat{g}_t^{-1}.
\]

Let \(u : \mathbb{R} \times [0, 1] \to \hat{W}\) be a Floer strip with respect to the pair \((H, J)\) from \(x_- \in \mathcal{P}_L(H)\) to \(x_+ \in \mathcal{P}_L(H)\). Then it is straightforward to see that the map \(g \cdot u\) is a Floer strip from \(g \cdot x_-\) to \(g \cdot x_+\) with respect to the pair \((g^\ast H, g^\ast J)\). The following can be shown exactly the same argument as in [17].

**Lemma 4.1.**

1. The map \(x \mapsto g \cdot x\) gives a one-to-one correspondence

\[
\mathcal{P}_L(H) \leftrightarrow \mathcal{P}_L(g^\ast H).
\]

2. The map \(u \mapsto g \cdot u\) gives a one-to-one correspondence

\[
\mathcal{M}(x_-, x_+ ; H, J) \leftrightarrow \mathcal{M}(g \cdot x_-, g \cdot x_+ ; g^\ast H, g^\ast J).
\]

3. The pair \((H, J)\) is regular if and only if the pair \((g^\ast H, g^\ast J)\) is regular.

The above lemma allows us to define a Floer homology of the pair \((g^\ast H, g^\ast J)\). The chain complex \(CF(L; g^\ast H, g^\ast J)\) is generated by elements in \(\mathcal{P}_L(g^\ast H)\) and the differential \(\partial\) counts elements in the moduli space \(\mathcal{M}(g \cdot x_-, g \cdot x_+ ; g^\ast H, g^\ast J)\). We get a well-defined homology \(HF(L; g^\ast H, g^\ast J)\) of the chain complex \((CF(L; g^\ast H, g^\ast J), \partial)\), which is isomorphic to the Floer homology \(HF(L; H, J)\) as a group under the map \(x \mapsto g \cdot x\). We denote the isomorphism by

\[
S_g : HF(L; H, J) \to HF(L; g^\ast H, g^\ast J),
\]

which we call a Seidel operator.
4.1.3. *Degree shift by \( S_g \).* The isomorphism \( S_g \) comes with a degree shift given by a Maslov type index \( I(g) \) for the path \( g \). The index \( I(g) \) of \( g \) is defined as follows. (See also [17].) Note that for each \( v \)-shaped admissible Hamiltonian \( H \), the set \( \mathcal{P}_L(H) \) is non-empty since there exist constant chords which generates \( HW^{(-r,c)}(L) \cong H(\partial L) \neq 0 \) for sufficiently small \( \epsilon > 0 \). Pick \( x \in \mathcal{P}_L(H) \). Under our topological assumptions \([2,1]\), the paths \( x \) and \( g \cdot x \) are contractible. As in Section \([2,2]\) we take a capping half disk for \( x \), and this induces a trivialization of \( \hat{T}W \) along \( x \), say
\[
\tau_x(t) : T_x(t)\hat{T}W \to \mathbb{R}^{2n}.
\]
We also take a capping half disk for \( g \cdot x \). Then we obtain a trivialization \( T\hat{W} \) along \( g \cdot x \), which we denote by
\[
\tau_{g\cdot x}(t) : T_{g\cdot x}(t)\hat{T}W \to \mathbb{R}^{2n}.
\]
Define a path of symplectic matrices \( \ell_x : [0, 1] \to Sp(2n) \) by
\[
\ell_x(t) := \tau_{g\cdot x}(t) \circ dg_0(g_0(x(t))) \circ \tau_{g_0\cdot x}(t)^{-1}.
\]
We define the *index* \( I_x(g) \) of \( g \) with respect to \( x \) by
\[
I_x(g) := \mu_{RS}(\ell_x\Lambda_0, \Lambda_0)
\]
where \( \Lambda_0 \subset \mathbb{R}^{2n} \) is the horizontal Lagrangian, and \( \mu_{RS} \) denotes the Robbin-Salamon index in \([25]\). The indices are independent of choices of capping half disks due to the topological conditions \([2,1]\).

**Lemma 4.2.** The index \( I_x(g) \) does not depend on the choice of \( x \).

*Proof.* Let \( x_0 \) and \( x_1 \) be contractible paths in \( \mathcal{P}_L(W) \). Then we have a homotopy \( x_s \) with \( s \in [0, 1] \) in \( \mathcal{P}_L(W) \) from \( x_0 \) to \( x_1 \). This yields a homotopy between \( g \cdot x_0 \) and \( g \cdot x_1 \) and hence a homotopy \( \ell_{x_s} \) of paths of symplectic matrices. Note that \( \ell_{x_s} \) is not a loop, but the corresponding path of Lagrangians \( \ell_{x_s}\Lambda_0 \) in \( \mathbb{R}^{2n} \) is a loop. This is because of the assumption \([4,1]\) on \( g \); indeed we have that \( Y \in T_{x_s(j)}L \) if and only if \( dg_j(x_s(j))(Y) \in T_{g \cdot x_s(j)}L \) for \( j = 0, 1 \). Therefore \( \ell_{x_s}\Lambda \) with \( s \in [0, 1] \) provides a homotopy of loops of Lagrangians in \( \mathbb{R}^{2n} \) from \( \ell_{x_0}\Lambda_0 \) to \( \ell_{x_1}\Lambda_0 \). It follows from the homotopy invariance of Maslov index for loops of Lagrangians that
\[
I_{x_0}(g) = \mu(\ell_{x_0}\Lambda_0, \Lambda_0) = \mu(\ell_{x_1}\Lambda_0, \Lambda_0) = I_{x_1}(g).
\]
This completes the proof. \( \square \)

We abbreviate the notation by \( I(g) := I_x(g) \). For the next proposition we introduce the following additional condition on \( g \).

\[
(4.2) \quad \text{The starting point } g_0 \text{ commutes with the Hamiltonian flow } \phi_H^t, \text{ i.e. } g_0^{-1} \circ \phi_H^t \circ g_0 = \phi_H^t.
\]
This condition is obviously true if \( g_0 = \text{id} \).

**Proposition 4.3.** Assume the condition \((4.2)\). For a contractible Hamiltonian 1-chord \( x \) relative to \( L \), the Maslov index of \( g \cdot x \) is given by
\[
\mu(g \cdot x) = \mu(x) + I(g).
\]

*Proof.* Let \( \Phi_x : [0, 1] \to Sp(2n) \) be a path of symplectic matrices defined by
\[
\Phi_x(t) := \tau_x(t) \circ d\phi_H^t(x(0)) \circ \tau_x(0)^{-1}.
\]
Then the Maslov index of \( x \) is by definition given by
\[
\mu(x) = \mu_{RS}(\Phi_x\Lambda_0, \Lambda_0).
\]
Similarly, the Maslov index of \( g \cdot x \) is defined by
\[
\mu(g \cdot x) = \mu_{RS}(\Phi_{g \cdot x}\Lambda_0, \Lambda_0)
\]
where the path \( \Phi_{g \cdot x} \) is given by
\[
\Phi_{g \cdot x}(t) := \tau_{g \cdot x}(t) \circ dg_0(\phi_H^t(g \cdot x(0))) \tau_{g \cdot x}(0)^{-1}.
\]
We compute, using the assumption that $\phi_H^t \circ g_0 = g_0 \circ \phi_H^t$,
\[
\Phi_{g,x}(t) = \tau_{g,x}(t) \circ d(g_t \circ \phi_H^t)(g \cdot x(0)) \circ \tau_{g,x}(0)^{-1}
\]
\[
= \tau_{g,x}(t) \circ dg_t(\phi_H^t(g \cdot x(0))) \circ d\phi_H^t(g \cdot x(0)) \circ \tau_{g,x}(0)^{-1}
\]
\[
= \tau_{g,x}(t) \circ dg_t(g_0(x(t))) \circ d\phi_H^t(g \cdot x(0)) \circ \tau_{g,x}(0)^{-1}
\]
\[
= \tau_{g,x}(t) \circ dg_t(g_0(x(t))) \circ \tau_{g_0 \cdot x}(t)^{-1} \circ \tau_{g_0 \cdot x}(t) \circ d\phi_H^t(g \cdot x(0)) \circ \tau_{g,x}(0)^{-1}
\]
\[
= \ell_x(t) \circ \Phi_{g_0 \cdot x}(t).
\]

We claim that $\Phi_{g_0 \cdot x}(t) = \Phi_x(t)$. For this, we observe that since $g_0$ is a Hamiltonian diffeomorphism with the boundary condition (4.1), it sends a capping half disk of $\tau$ along $x$ to a capping half disk of $g_0 \cdot x$. Therefore we may assume that the trivialization $\tau_{g_0 \cdot x}$ along $g_0 \cdot x$ is the same as the pushforward of the trivialization $\tau_x$ along $x$ by $d g_0$. In other words, we may assume that $\tau_{g_0 \cdot x}(t) = \tau_x(t) \circ d g_0(x(t))^{-1}$. Together with the assumption $\phi_H^t \circ g_0 = g_0 \circ \phi_H^t$, we now compute that
\[
\Phi_{g_0 \cdot x}(t) = \tau_{g_0 \cdot x}(t) \circ d\phi_H^t(g \cdot x(0)) \circ \tau_{g,x}(0)^{-1}
\]
\[
= \tau_x(t) \circ d g_0(x(t))^{-1} \circ d\phi_H^t(g_0 \cdot x(0)) \circ \tau_x(0)^{-1}
\]
\[
= \tau_x(t) \circ d(g_0^{-1} \circ \phi_H^t \circ g_0)(x(0)) \circ \tau_x(0)^{-1}
\]
\[
= \tau_x(t) \circ d(\phi_H^t(x(0))) \circ \tau_x(0)^{-1}
\]
\[
= \ell_x(t) \circ \Phi_x(t).
\]

Therefore we have
\[
\Phi_{g,x}(t) = \ell_x(t) \circ \Phi_x(t).
\]

Using the loop property of the Robbin-Salamon index, e.g. [15, Proposition 3.1], we conclude
\[
\mu(g \cdot x) = \mu(x) + I(g).
\]

The upshot is the following.

**Theorem 4.4.** The Seidel operator, under the assumption [4.2], gives a graded group isomorphism
\[
S_g : HF_*(L; H, J) \rightarrow HF_{*+1}(g; L; g, H, g, J).
\]

### 4.2. Seidel operator and $\mathcal{L}$-periodic Reeb flows

Aiming for Theorem 4.17, we now consider a Seidel operator on $v$-shaped wrapped Floer homology of a specific path of Hamiltonian diffeomorphisms coming from the Reeb flow on the boundary. Let $(\bar{W}, \lambda)$ be a Liouville domain and $L$ an admissible Lagrangian as before. We further assume that the contact boundary $(\Sigma, \xi, \mathcal{L})$ with the Legendrian $\mathcal{L} = \partial L$ is product index positive, see Definition 2.14. In this case, as we have discussed in Section 2.3, $v$-shaped wrapped Floer homology can be defined in the symplectization $\mathbb{R}_+ \times \Sigma$ with the Lagrangian $\mathbb{R}_+ \times \mathcal{L}$. We can therefore put $\bar{W} = \mathbb{R}_+ \times \Sigma$ and $\bar{L} = \mathbb{R}_+ \times \mathcal{L}$ in Section 4.1.

Denote the Reeb flow of the contact form $\alpha$ on $\Sigma$ by $\phi^t H$. Then we have a specific path of Hamiltonian diffeomorphisms $g_t$ on $\mathbb{R}_+ \times \Sigma$ given by
\[
g_t(r, y) := (r, \phi^t H(y)).
\]

**Remark 4.5.** This will be extended to a bit more general case in Section 4.3.4, but to simplify discussions we first work with $g_t$ here.

**Definition 4.6.** For a Legendrian $\mathcal{L}$ in a contact manifold $(\Sigma, \alpha)$, the Reeb flow $\phi^t H$ is called $\mathcal{L}$-periodic if there exists $T > 0$ such that $p \in \mathcal{L}$ if and only if $\phi^T H(p) \in \mathcal{L}$.

Assume that our Reeb flow is $\mathcal{L}$-periodic. Then the corresponding path $g_t$ satisfies the condition (4.1), and it is generated by a (time-independent) Hamiltonian $K^\theta : \mathbb{R}_+ \times \Sigma \rightarrow \mathbb{R}$ given by
\[
K^\theta(r, y) = r + c
\]
for some constant $c$. Let $H : \mathbb{R}_+ \times \Sigma \rightarrow \mathbb{R}$ be a $v$-shaped admissible Hamiltonian. The push forward $g_* H : [0, 1] \times \mathbb{R}_+ \times \Sigma \rightarrow \mathbb{R}$ is as before given by
\[
g_* H(t, r, y) = H(t, g_t^{-1}(r, y)) + K^\theta(t, r, y) = h(r) + r + c
\]
4.2.1. The index $I(g)$ for real contact manifolds. Suppose further that the contact type boundary $(\Sigma, \alpha)$ admits a real structure, i.e. an anti-contact involution $\rho: \Sigma \to \Sigma$, and the Legendrian $L$ is given by (a connected component of) the fixed point set of $\rho$. Throughout this section, the Reeb flow $\phi_R^T$ is assumed to be periodic and $L$-periodic. In this case, we can compute the index $I(g)$ in terms of the Maslov index of a principal Reeb chord relative to $L$. To define the notion of “principal Reeb chords”, we observe the following. Let $\gamma$ be a periodic Reeb orbit of period $2T$ which starts at $\gamma(0) \in L$.

Lemma 4.7. We have $\gamma(T) \in L$.

Proof. We observe that $\rho(\gamma(T)) = \rho(\phi_R^T(\gamma(0))) = \phi_R^{-T} \circ \rho(\gamma(0))$. Since $\gamma(0) \in L = Fix(\rho)$ we have $\rho(\gamma(T)) = \phi_R^{-T}(\gamma(0))$. Since $\gamma$ has the period equal to $2T$, we see that $\gamma(2T) = \gamma(0)$ or equivalently $\phi_R^T(\gamma(0)) = \phi_R^{-T}(\gamma(0))$. It follows that $\rho(\gamma(T)) = \phi_R^T(\gamma(0)) = \phi_R^{-T}(\gamma(0)) = \gamma(T)$ and hence $\gamma(T) \in Fix(\rho) = L$. \square

Definition 4.8.

1. For a periodic Reeb orbit $\gamma$ with period $2T$ and $\gamma(0) \in L$, we define its *half Reeb chord* $x: [0, T] \to \Sigma$ by $x(t) := \gamma(t)$ (with the domain is the half of the domain of $\gamma$).

2. Let $2 \cdot T_P$ be the least common period of the periodic Reeb flow. A periodic Reeb orbit $\gamma$ is called *principal* if $\gamma$ has the period $2 \cdot T_P$. Its half Reeb chord $c$ (of period $T_P$) is called a *principal Reeb chord*. We call $T_P$ the *principal period* of Reeb chords.

Remark 4.9. In [20], the following relation of the indices is proved. For a principal Reeb orbit $\gamma$ and its half Reeb chord $c$, we have

$$2\mu(c) = \mu(\gamma).$$

In addition, since $L$ itself forms a Morse–Bott connected component consisting of Reeb chords of period $T_P$, the index $\mu(x)$ does not depend on the choice of $c \in L$ (where $L$ is identified to be a space of chords).

Now the index $I(g)$ is simply given by the following.

Lemma 4.10. The index $I(g)$ is equal to the Maslov index of a principal Reeb chord.

Proof. We recall the definition of the Maslov index $\mu(c)$ of a Reeb chord $c$. Take a trivialization of the contact structure $\xi$ along $c: [0, T] \to \Sigma$,

$$\eta_c: e^s \xi \to [0, T] \times \mathbb{R}^{2(n-1)},$$

such that $\eta_c(\xi TL) = \Gamma_0$ for the horizontal Lagrangian $\Gamma_0 \subset \mathbb{R}^{2(n-1)}$. Then the Maslov index $\mu(c)$ is defined by

$$\mu(c) := \mu_{RS}(\Phi_c, \Gamma_0, \Gamma_0)$$

where

$$\Phi_c(t) = \eta_c(t) \circ d\phi_R^t(c(0))|_{\xi} \circ \eta_c(0)^{-1}.$$
The trivialization \( \tau_x \) sends \( TL \) to the extended Lagrangian \( \Lambda_0 := \mathbb{R} \times \{0\} \times \Gamma_0 \subset \mathbb{R}^2 \times \mathbb{R}^{2(n-1)} = \mathbb{R}^{2n} \).

The index \( I(g) \) of \( g \) is by definition given as follows (note that \( g_0 = \text{id} \) in this case).

\[
I(g) = \mu_{RS}(\ell \Lambda_0, \Lambda_0)
\]

where

\[
\ell : [0, T] \to \text{Sp}(2n), \quad \ell(t) = \tau_{g,x}(t) \circ dg_t(x(t)) \circ \tau_x(t)^{-1},
\]

and \( T \) is the period of \( c \). Since \( g_t(r, p) = (r, \phi^t_H) \) by definition, the linearization \( dg_t \) acts on \( \langle \partial_r, R \rangle \) trivially. Since \( \tau_x \) is extended from \( \eta_c \) trivially on \( \langle \partial_r, R \rangle \), we can split the path \( \ell \) in such a way that

\[
(4.4) \quad \ell(t) = \begin{bmatrix} \tilde{\ell}_c(t) & O \\ O & \text{id} \end{bmatrix}
\]

where \( \tilde{\ell}_x \) is given by

\[
\tilde{\ell}_c(t) = \eta_{\phi_{R \cdot c}}(t) \circ d\phi^t_{R \cdot c}(c(t))|_{\xi} \circ \eta_c(t)^{-1}, \quad t \in [0, T]
\]

with \( (\phi_R \cdot c)(t) = \phi^t_R(c(t)) \). Applying the same argument as in the proof of Proposition 4.3, it is straightforward that \( \mu_{RS}(\tilde{\ell}_c \Gamma_0, \Gamma_0) + \mu_{RS}(\Phi_c \Gamma_0, \Gamma_0) = \mu_{RS}(\Phi_{\phi_{R \cdot c}} \Gamma_0, \Gamma_0) \).

(In this case, the commuting condition (4.2) obviously holds.)

Now we put \( c \) to be a principal Reeb chord. Then the path \( \phi_R \cdot c \) is actually a Reeb chord of the (time-scaled) Reeb flow \( \phi^t_R \) of the same period as \( c \). The Maslov index \( \mu_{RS}(\Phi_{\phi_{R \cdot c}}, \Gamma_0, \Gamma_0) \) of \( \phi_R \cdot c \) is therefore the same as the index of the second iteration \( c^2 \). Since the second iteration \( c^2 \) is exactly the same as the principal periodic Reeb orbit which has \( c \) as the half Reeb chord, we know from [20, Proposition 3.1] that \( \mu(c^2) = 2\mu(c) \); see Remark 4.9. It follows that

\[
\mu_{RS}(\tilde{\ell}_c \Gamma_0, \Gamma_0) + \mu(c) = 2\mu(c)
\]

and hence \( \mu_{RS}(\tilde{\ell}_c \Gamma_0, \Gamma_0) = \mu(c) \). By the direct sum property of the Robbin-Salamon index applied to the splitting (4.4), we conclude that

\[
I(g) = \mu_{RS}(\ell \Lambda_0, \Lambda_0) = \mu_{RS}(\tilde{\ell}_c \Gamma_0, \Gamma_0) = \mu(c).
\]

\[
\square
\]

4.3. V-shaped wrapped Floer homology using \( g_* H \). Let \( H : \mathbb{R}_+ \times \Sigma \to \mathbb{R} \) be a v-shaped admissible Hamiltonian and let \( J \) be an admissible almost complex structure. In this section, we show that the v-shaped wrapped Floer homology can be defined using a cofinal family of Floer data of the form \((g_* H, g_* J)\) even though \( g_* H \) is not precisely v-shaped.

4.3.1. Shape of \( g_* H \). Recall that a v-shaped admissible Hamiltonian \( H \) be given, up to smoothing, as follows (see Figure 1).

\[
(4.5) \quad H(r) = \begin{cases} 
ar r + b_1 & r \geq 1 
0 & r = 1 
-ar + b_2 & \delta \leq r \leq 1 
c & r \leq \delta \end{cases}
\]

for some constants \( a, c > 0, b_1, b_2 \) and sufficiently small \( \delta > 0 \). Note that the generating Hamiltonian \( K^g : \mathbb{R}_+ \times \Sigma \to \mathbb{R} \) of \( g \), defined as (4.3), is given by

\[
(4.3) \quad K^g(r, y) = r + c
\]

for some constant \( c \). Taking \( c = -1 \), the push-forward \( g_* H = K^g + H \) is given, up to smoothing, by the following formula, see Figure 4.3.

\[
(4.6) \quad (g_* H)(r) = \begin{cases} 
(a + 1)r + b_1 - 1 & r \geq 1 
0 & r = 1 
(-a + 1)r + b_2 - 1 & \delta \leq r \leq 1 
r + c - 1 & r \leq \delta
\end{cases}
\]
Recall that for a given action window \((a, b)\), we can take \(\epsilon\) sufficiently small and \(\delta\) sufficiently large so that the filtered chain group \(\text{CF}^{(a, b)}_*(L; H, J)\) is generated only by the Hamiltonian chords of \(H\) in the region \((1 - \epsilon, 1 + \epsilon) \times \Sigma\) for sufficiently small \(\epsilon > 0\). In what follows, for the sake of simplicity, we abbreviate “in the region \((1 - \epsilon, 1 + \epsilon) \times \Sigma\) for sufficiently small \(\epsilon > 0\)” by “near \(r = 1\)”.

**Lemma 4.11.** For the above choice of \(\delta\) and \(\epsilon\), the chain group \(\text{CF}^{(a+1, b+1)}_*(L; g_*H, g_*J)\) is generated by the Hamiltonian chords of \(g_*H\) in the region near \(r = 1\).

**Proof.** Note first that since \(\tilde{L} = \mathbb{R}_+ \times \mathcal{L}\) and \(\tilde{\lambda} = r \tilde{\lambda}\), the restriction \(\lambda|_{\tilde{L}}\) vanishes. Therefore we can put the primitive \(f : L \to \mathbb{R}\) to be constant so that the \(f\)-related terms in the action functional \(A_H\) do not appear. Then we compute the action of \(g \cdot x\) as follows.

\[
A_{g,H}(g \cdot x) = \int_0^1 (g \cdot x)^*\tilde{\lambda} - \int_0^1 (g_*H)((g \cdot x)(t))dt \\
= \int_0^1 x^*\tilde{\lambda} + 1 - \int_0^1 H(x(t))dt - \int_0^1 K^g_\Sigma(x(t))dt \\
= A_H(x) + 1 - \int_0^1 K^g_\Sigma(x(t))dt,
\]

Note that near \(r = 1\), we have \(K_g = r - 1 \approx 0\). We conclude that

\[
A_{g,H}(g \cdot x) \approx A_H(\gamma) + 1.
\]

By the choice \(\delta\) and \(\epsilon\) with respect to the action window \((a, b)\), the generators \(\text{CF}_*^*(L; g_*H, g_*J)\) (which should come from the region near \(r = 1\)) are in the action window \((a + 1, b + 1)\).

### 4.3.2. Being constant at the negative end.

The shape of \(g_*H\) in (4.6) is almost v-shaped, but it is not constant at the negative end (i.e. the region in \(\mathbb{R}_+ \times \Sigma\) with \(r \leq \delta\)). Analogously to [28, Section 3.4], we modify \(g_*H\) to make it constant near \(r = 0\) without changing the chain complex \(\text{CF}_*(L; g_*H, g_*J)\). Note that being constant at the negative end is necessary to perform the stretching-the-neck operation, and this is to show that the the v-shaped wrapped Floer homology is defined purely in the symplectization \(\mathbb{R}_+ \times \Sigma\).

Let \((g_*H)_R : \mathbb{R}_+ \times \Sigma \to \mathbb{R}\) be the Hamiltonian which is the same as \(g_*H\) in \((\mathbb{R}, \infty) \times \Sigma\) and is constant in \((0, R) \times \Sigma\) for some \(0 < R < \delta\) (up to smoothing), see Figure 4b. Let \((a + 1, b + 1)\) be the action window as in Lemma 4.11. Then since \((g_*H)_R\) is the same as \(g_*H\) near \(r = 1\), the Hamiltonian chords of \((g_*H)_R\) in this region are identically the same as those of \(g_*H\) in the same region. Therefore the chain groups \(\text{CF}^{(a+1, b+1)}_*(L; g_*H, g_*J)\) and \(\text{CF}^{(a+1, b+1)}_*(L; (g_*H)_R, g_*J)\) have the same generators. Let \(x_-\) and \(x_+\) be two generators in \(\text{CF}^{(a+1, b+1)}_*(L; g_*H, g_*J)\) with index difference by 1. The proof of the following lemma is completely analogous to that in [28, Lemma 3.18].

**Lemma 4.12.** Let \(x_-, x_+ \in \mathcal{P}_L(H)\) be two Hamiltonian chords such that \(|x_+| - |x_-| = 1\). If \(R > 0\) is sufficiently small, then there is a bijection

\[
\mathcal{M}(x_-, x_+; g_*H, g_*J) \leftrightarrow \mathcal{M}(x_-, x_+; (g_*H)_R, g_*J).
\]
This implies that $CF^*_{(a+1,b+1)}(L; g, H, g, J)$ is the same as $CF^*_{(a+1,b+1)}(L; (g_+, H)_R, g_+, J)$ as a chain complex for sufficiently small $R > 0$. Since we have an isomorphism

$$S_g : HF^*_{(a,b)}(L; H, J) \rightarrow HF^*_{(a+1,b+1)}(L; (g_+, H)_R, g_+, J),$$

we obtain an induced isomorphism (with the same notation)

$$S_g : HF^*_{(a,b)}(L; H, J) \rightarrow HF^*_{(a+1,b+1)}(L; (g_+, H)_R, g_+, J).$$

4.3.3. Taking direct limit. We shall take a direct limit of $HF^*_{(a+1,b+1)}(L; (g, H)_R, g, J)$ over a cofinal family of $(g, H)_R$’s. Note that if two admissible Hamiltonians $H_-$ and $H_+$ satisfy $H_- \leq H_+$, then $(g_+, H_+) \leq (g, H_-)_R$. So we can describe a family of $(g, H)_R$’s in terms of $H$’s. We take a family of $H$’s by increasing the slope $a$ but keeping $\delta$ sufficiently small so that the Hamiltonian chords within the action window $(a, b)$ are all near $r = 1$. Then the induced family $(g, H)_R$’s has a property that Hamiltonian chords within the action window $(a + 1, b + 1)$ are all near $r = 1$. In particular, for two Hamiltonians $(g_+, H_+)_R \leq (g, H_-)_R$ in this family, we have a continuation map

$$f_{H_-, H_+} : HF^*_{(a+1,b+1)}(L; (g_+, H_+)_{R_1}, g_+, J_+) \rightarrow HF^*_{(a+1,b+1)}(L; (g, H_-)_R, g_+, J_-).$$

**Lemma 4.13.** Let $(H_-, J_-)$ and $(H_+, J_+)$ be pairs of v-shaped admissible Hamiltonians and admissible almost complex structures. Then the following diagram commutes.

$$
\begin{array}{ccc}
HF^*_{(a,b)}(L; H_-, J_+) & \xrightarrow{S_g} & HF^*_{(a+1,b+1)}(L; (g_+, H)_R, g_+, J) \\
\downarrow{f_{H_-, H_+}} & & \downarrow{f_{H_-, H_+}} \\
HF^*_{(a,b)}(L; H_-, J_-) & \xrightarrow{S_g} & HF^*_{(a+1,b+1)}(L; (g, H_-)_R, g_+, J_-)
\end{array}
$$

**Proof.** Let $x_- \in P_L(H_-)$ and $x_+ \in P_L(H_+)$ be Hamiltonian chords with $|x_+| - |x_-| = 0$. Choose a regular pair $\{(H_s, J_s)\}_{s \in \mathbb{R}}$ of monotone decreasing family of v-shaped admissible Hamiltonians and admissible almost complex structures such that

$$(H_s, J_s) = \begin{cases}
(H_-, J_-) & s < -M \\
(H_+, J_+) & s > M
\end{cases}$$

for some large $M > 0$. Since the set $\{s \in \mathbb{R} \mid H_s \text{ is strictly decreasing}\}$ is strictly decreasing is compact, we can choose a sufficiently small $\delta > 0$ such that $H_s$ is constant on $(0, \delta) \times \Sigma$ for all $s \in \mathbb{R}$.

By Lemma 3.4, the map $M(x_-, x_+; H_s, J_s) \rightarrow M(g \cdot x_-, g \cdot x_+; g, H_s, g, J_s)$, $u \rightarrow g \cdot u$ is a one-to-one correspondence. In particular $M(g \cdot x_-, g \cdot x_+; g, H_s, g, J_s)$ is compact and due to the condition $|x_+| - |x_-| = 0$, there are only finitely many elements, say $u_1, \ldots, u_m$ in $M(g \cdot x_-, g \cdot x_+; g, H_s, g, J_s)$.

Now proving commutativity of the diagram in the statement reduces to show that the moduli spaces $M(g \cdot x_-, g \cdot x_+; g, H_s, g, J_s)$ and $M(g \cdot x_-, g \cdot x_+; (g, H)_R, g, J_s)$ are the same.

Since every curve in $M(x_-, x_+; H_s, J_s)$ is contained in the symplectization, so are the curves in $M(g \cdot x_-, g \cdot x_+; g, H_s, g, J_s)$. Therefore there is a sufficiently small $R > 0$ such that the image of $u_j$ lies in $(R, \infty) \times \Sigma$ for all $j = 1, \ldots, m$. We use this number $R$ for the truncated function $(g, H)_R$. Then all elements $u_j$’s actually belong to $M(x_-, x_+; (g, H)_R, g, J)$ since $(g, H)_R$ and $g, H$ coincide in the region $(R, \infty) \times \Sigma$.

For the converse inclusion, following the idea in [28] Proof of Proposition 3.18, suppose that the latter moduli space has a further element, say $u$. This implies that the image of $u$ escapes from $(R, \infty) \times \Sigma$. Applying $g^{-1}$ to $u$, we get an element

$$g^{-1} \cdot u \in M(x_-, x_+; g^*(g, H)_R, g^*(g, J_s)),$$

which also escapes from $(R, \infty) \times \Sigma$ since the diffeomorphism $g_t$ preserves the radial coordinate for all $t$. Since $g^*(g, H)_R = H$ is constant on $(R, \delta) \times \Sigma$, we can apply the neck-stretching argument to $g^{-1}u$ at the contact hypersurface $\{R\} \times \Sigma$. This leads to a contradiction due to the index positivity assumption. This completes the proof. \qed
It is apparent that the direct limit of $HF_*^{(a+1,b+1)}(L; (g_*H)_R, g_*J)$ as $a \to \infty$ is the filtered v-shaped wrapped Floer homology, i.e.
\[
\lim_{a \to \infty} HF_*^{(a+1,b+1)}(L; (g_*H)_R, g_*J) = HF_*^{(a+1,b+1)}(L).
\]
Indeed, within a fixed action window, the truncated Hamiltonians give the same chain complex as genuine v-shaped Hamiltonians. Now by the commutative diagram in Lemma 4.13
the operator $S_g$ induces an isomorphism to the direct limits
\[
S_g : HW_*^{(a,b)}(L) \to HW_*^{(a+1,b+1)}(L).
\]
Taking direct limits along $a \to -\infty$ and $b \to \infty$ we finally conclude the following.

**Corollary 4.14.** There exists a graded group isomorphism
\[
S_g : HW_*(L) \to HW_{+ I(g)}(L).
\]

**4.3.4. More general case.** In the above, we have defined $HW_*(L)$ using Hamiltonians of the form $g_*H$ where $g$ is given by the Reeb flow as (4.3). The same idea basically applies to a bit more general case of $g$ given by
\[
g_t(r, y) = (r, \phi_R^{f(t)}(y))
\]
for some smooth function $f : [0, 1] \to \mathbb{R}$ satisfying $f(0), f(1) \in T_P \mathbb{Z}$, where $T_P$ is the principal period of Reeb chords. Note that $g$ in (4.7) satisfies conditions (4.1) and (4.2) for such a function $f$.

A technical difference is that the push forward $g_*H$ may not have a time-independent slope at ends. For this reason, we homotope the given path $g$ to another path $\tilde{g}$ such that the push forward $\tilde{g}_*H$ has time-independent slope. This in turn provides a cofinal family for the direct limit of $HF_*(L; \tilde{g}, H, \tilde{g}, J)$'s. The following is a straightforward adaptation of [28, Lemma 3.17].

**Lemma 4.15.** Let $g_1$ and $g_2$ be homotopic paths of Hamiltonian diffeomorphisms relative to their ends. For two Hamiltonians $H_1$, $H_2$ as in (4.5) with the slope of $H_2$ steeper than that of $H_1$ and $J_1$, $J_2$ regular almost complex structures, there exists a continuation map
\[
HF(L; g_1, H_1, g_1, J_1) \to HF(L; g_2, H_2, g_2, J_2).
\]

Suppose we are given a path of Hamiltonian diffeomorphisms $g_t(r, y) = (r, \phi_R^{f(t)}(y))$ for some smooth function $f : [0, 1] \to \mathbb{R}$ satisfying $f(0), f(1) \in T_P \mathbb{Z}$. Then the Hamiltonian $K^g$ associated to $g$ is given by
\[
K^g(t, r, y) = f'(t)r.
\]
We observe that the path $g$ is homotopic (relative to the ends) to a path $\tilde{g}$ defined by
\[
\tilde{g}_t(r, y) = (r, \phi_R^{f(1) - f(0)t}(y)).
\]
A generating Hamiltonian $K^{\tilde{g}}$ of $\tilde{g}$ is given by
\[
K^{\tilde{g}}(t, r, y) = (f(1) - f(0))r.
\]
Therefore the push forward $\tilde{g}_*H$ is time independent for any time independent Hamiltonian $H$.

Let $a < b$ be given. By Lemma 4.15, we have a map
\[
HF(L; g_*H_a, g_*J_a) \to HF(L; \tilde{g}, H_b, \tilde{g}, J_b),
\]
where $H_a$ is a v-shaped Hamiltonians described in (4.5) and $J_a$, $J_b$ are regular almost complex structures. Taking the direct limit as $a, b \to \infty$, we get a map
\[
\lim_{a \to \infty} HF(L; g_*H_a, g_*J_a) \to \lim_{b \to \infty} HF(L; \tilde{g}, H_b, \tilde{g}, J_b).
\]
Changing the role of $g$ and $\tilde{g}$ in the above, we get the inverse map of (4.8), namely
\[
\lim_{b \to \infty} HF(L; \tilde{g}, H_b, \tilde{g}, J_b) \to \lim_{a \to \infty} HF(L; g_*H_a, g_*J_a),
\]
by which we conclude the following.
Proposition 4.16. There is an isomorphism
\[ \lim_{a} HF(L; g_{a}H_{a}, g_{a}J_{a}) \cong \lim_{b} HF(L; \tilde{g}_{b}H_{b}, \tilde{g}_{b}J_{b}). \]

Furthermore, since \( \tilde{g}_{b}H_{b} \) is time independent for all \( b \), the argument in the previous section shows that the latter one \( \lim_{b} HF(L; \tilde{g}_{b}H_{b}, \tilde{g}_{b}J_{b}) \) is isomorphic to the v-shaped wrapped Floer homology \( HW_{\ast}(L) \).

We finally conclude the following.

Theorem 4.17. Let \( g \) be a path of Hamiltonian diffeomorphisms on \( \mathbb{R}_{+} \times \Sigma \) of the form \((4.7)\). Then there is a graded group isomorphism
\[ S_{g} : HW_{\ast}(L) \to HW_{\ast+t(g)}(L). \]

Remark 4.18. If the path of Hamiltonian diffeomorphism of the form \( \text{id} \times \phi_{B}^{f(t)} \) on \( \mathbb{R}_{+} \times \Sigma \) extends to a path of Hamiltonian diffeomorphisms on the completion \( \hat{W} \), then the wrapped Floer homology \( HW_{\ast}(L) \) must be of finite dimensional. This can be easily shown using an argument analogous to \([28\text{ Lemma 3.4}]\). In our examples, we already know that \( HW_{\ast}(L) \) is infinite dimensional as in Proposition 3.8. As suggested in \([28\text{, Proposition 3.25}]\), we therefore work with v-shaped wrapped Floer homology instead of working with ordinary wrapped Floer homology directly.

4.4. Module property. Let \( g \) and \( h \) be paths of Hamiltonian diffeomorphisms in \((4.7)\), after normalizing so that \( 0 \leq t \leq 1 \). We define their product \( gh \) by
\[ (gh)_{t} := g_{t} \circ h_{t}, \quad 0 \leq t \leq 1. \]

Note that the product \( gh \) still satisfies the conditions (4.1) and (4.2). In particular \( gg^{-1} = \text{id} \), so \( S_{gg^{-1}} = \text{id} \). From the definition of the operator \( S_{g} \), it is apparent that \( S_{gh} = S_{g} \circ S_{h} \). In particular, we have \( S_{g^{-1}} = (S_{g})^{-1} \). The following is basically the same as \([17\text{ Section 3.2}]\) adapted to the open case as in \([28\text{ Proposition 3.25}]\).

Lemma 4.19. Let \( g \) and \( h \) have the same end points. Suppose that \( g \) and \( h \) are homotopic relative to the end points in \( \text{Ham}(\hat{W}) \). Then \( S_{g} = S_{h} \).

We want to prove the following module property of the operator \( S_{g} \) which is crucial for computing the ring structure. In the following, we assume in addition to (4.2) that the end point \( g_{1} \) also commutes with v-shaped admissible Hamiltonians.

Theorem 4.20. The operator \( S_{g} : HW_{\ast}(L) \to HW_{\ast+t(g)}(L) \) satisfies
\[ S_{g}(x \cdot y) = S_{g_{1}}(x) \cdot S_{g}(y) = S_{g}(x) \cdot S_{g_{1}}(y) \]
where \( x, y \in HW_{\ast}(L) \) and \( S_{g_{1}} \) denotes the operator of the constant path \( g_{j} \) for \( j = 0, 1 \).

An immediate corollary is the following.

Corollary 4.21. If \( g \) is a constant path (not necessarily the identity), then \( S_{g} \) is a ring isomorphism on \( HW_{\ast}(L) \) without degree shifting.

Proof of Theorem 4.20. We present a proof of the first part, i.e. \( S_{g}(x \cdot y) = S_{g_{1}}(x) \cdot S_{g}(y) \). The second part follows from the same argument. We argue analogously to \([17\text{ Proposition 3.8}]\), \([26\text{ Proposition 6.3}]\), and \([28\text{ Proposition 3.29}]\). The idea of the proof is that we choose a specific disk as the domain of half pair of pants taking into account the homotopy property in Lemma 4.19.

Lemma 4.19 allows us to assume that \( g_{(0,1/4)} \equiv g_{0} \) since we can homotope \( g \) to be constant for \( 0 \leq t \leq 1/4 \) without changing the operator \( S_{g} \). Now we take a specific disk with three points removed on the boundary, namely
\[ S := (\mathbb{R} \times [0, 1]) \setminus \{(0, 0)\}. \]

For a holomorphic chart near \((0, 0)\), we take
\[ (s, t) \mapsto (1/4e^{-\pi s} \cos(\pi (1 - t)), 1/4e^{-\pi s} \sin(\pi (1 - t)) \]
where \((s, t) \in \mathbb{R}_{+} \times [0, 1] \). Note that the two boundary punctures \((0, 0)\) and \( s = \infty \) are positive and the other puncture \( s = -\infty \) is negative.
The coefficient \((x \cdot y, z) \in \mathbb{Z}_2\) counts the elements in the moduli space \(\mathcal{M}(z, x, y; \beta, H^S, J^S)\), i.e. half pair of pants from the domain \(S\) with positive ends converging to \(x\) and \(y\) and the negative end converging to \(z\). We arrange that \(x\) is the asymptotic at \((0,0)\), \(y\) at \(s = \infty\), and \(z\) at \(s = -\infty\). (Here \((s,t)\) is the coordinate of \(\mathbb{R} \times [0,1]\).)

Let \(u\) be an element in \(\mathcal{M}(z, x, y; \beta, H^S, J^S)\). The mapping \(u \mapsto g_\ast u\) gives one-to-one correspondence

\[\mathcal{M}(z, x, y; \beta, H^S, J^S) \leftrightarrow \mathcal{M}(g_\ast z, g_\ast x, g_\ast y; \beta, g_\ast H^S, g_\ast J^S).\]

In particular, since \(g_{[0,1/4]} \equiv g_0\), we have \(g_\ast x = g_0_\ast x\). So we have a correspondence

\[\mathcal{M}(z, x, y; \beta, H^S, J^S) \leftrightarrow \mathcal{M}(g_\ast z, g_0 \ast x, g_\ast y; \beta, g_\ast H^S, g_\ast J^S).\]

This implies that

\[\langle x \cdot y, z \rangle = \langle g_0 \ast x \cdot g_\ast y, g_\ast z \rangle,\]

and hence on the homology level that

\[\langle x \cdot y, z \rangle = (S_{g_0}(x) \cdot S_g(y), S_g(z)).\]

Since \(S_g\) is an isomorphism of \(\mathbb{Z}_2\)-modules, we have \(\langle S_{g_0}(x) \cdot S_g(y), S_g(z) \rangle = \langle x \cdot y, z \rangle\). From this we conclude

\[\langle S_{g_0}(x) \cdot S_g(y), S_g(z) \rangle = \langle S_g(x \cdot y), S_g(z) \rangle,\]

which implies \(S_{g_0}(x) \cdot S_g(y) = S_g(x \cdot y)\).

\[\square\]

4.5. Computing ring structure of v-shaped wrapped Floer homology. We now turn to the \(A_k\)-type Milnor fibers \(V_k\) and real Lagrangians \(L_j\), discussed in Section 3. We actually work with \(W_k\) in Section 3.4.3 which does not matter due to invariance of (v-shaped) wrapped Floer homology under Liouville isotopies. In Section 3.4 we have computed the graded group structure of v-shaped wrapped Floer homology of the real Lagrangian \(L_j\), namely Proposition 3.13. In Section 3.5 we have shown that the contact boundary of \(A_k\)-type Milnor fiber is product index positive if \(n \geq 3\). Furthermore the Reeb flow \(\xi_j^\ast\) on the boundary is \(L_j\)-periodic for each \(j\). We therefore have a well-defined Seidel operator in v-shaped wrapped Floer homology \(HW_\ast(L_j)\) as in Theorem 4.17

\[S_g : HW_\ast(L_j) \rightarrow HW_{\ast+1}(L_j)\]

for each \(0 \leq j \leq k\).

Since the index \(I(g)\) is given by the Maslov index of principal Reeb chords as in Lemma 4.10 we have

\[\mu(x_P) = I(g) = 2 + (n - 2)(k + 1)\]

Using these data we can compute the ring structure of \(HW_\ast(L_j)\) as follows.

**Theorem 4.22.** For all \(0 \leq j \leq k\), \(n \geq 3\), and \(k \geq 2\), we have a graded ring isomorphism

\[HW_\ast(L_j) \cong \mathbb{Z}_2[x, y, y^{-1}]/(x^2),\]

where \(|x| = I(g) - n + 1\) and \(|y| = I(g)\).

Before proving the above theorem, we observe the following. A crucial feature of the graded group structure of \(HW_\ast(L_j)\) in Proposition 3.13 is that on each non-trivial degree we have only one generator. Recalling

\[I(g) = 2 + (n - 2)(k + 1)\]

we have one generator for each degree

\[\ldots, -I(g) - n + 1, -I(g), -n + 1, 0, I(g) - n + 1, I(g), 2I(g) - n + 1, 2I(g), 3I(g) - n + 1, 3I(g), \ldots\]

Starting from the generators of degree \(I(g) - n + 1\) and \(I(g)\), the other generators are obtained by shifting the degree by \(\pm I(g)\). We label all generators in the form either \(C_{I(g)m}\) or \(C_{I(g)m-n+1}\) for \(m \in \mathbb{Z}\), where \(C_0\) denotes a generator of the group \(HW_k(L_j)\) of degree \(k\). In particular \(C_0\) is the unit element. In fact, since the \(S_g\) operator is a group isomorphism with degree shifting by \(I(g)\), we may arrange the generators in such a way that

\[C_{I(g)m} = S_g^m(C_0) \quad \text{and} \quad C_{I(g)m-n+1} = S_g^m(C_{I(g)-n+1})\]

for each \(m \in \mathbb{Z}\).
Lemma 4.23. For the constant path \( g_1 \), the corresponding operator \( S_{g_1} : \hat{HW}_*(L_j) \to \hat{HW}_*(L_j) \) is the identity.

Proof. This directly follows from the fact that \( S_{g_1} \) is a ring isomorphism without grading shift by Corollary 4.21 and the fact that for all \( k \in \mathbb{Z} \), the \( k \)-th group \( \hat{HW}_k(L_j) \) is a one-dimensional vector space over \( \mathbb{Z}_2 \) if it is not vanishing. In this situation, we must have that \( S_{g_1}(C_k) = C_k \) for all \( k \in \mathbb{Z} \), i.e. \( S_{g_1} = \text{id} \). \( \square \)

We now prove Theorem 4.22.

Proof of Theorem 4.22. For degree reasons, if \( k \geq 2 \), we already know that \( C_{I(g)m-n+1} \cdot C_k = C_k \cdot C_{I(g)m-n+1} = 0 \) for all \( k \neq I(g) \mathbb{Z} \) and \( m \in \mathbb{Z} \). It only remains to determine the product of the form
\[
C_{I(g)m} \cdot C_k \quad \text{and} \quad C_k \cdot C_{I(g)m}
\]
for \( m \in \mathbb{Z} \) and \( k = I(g) \ell \) or \( k = I(g) \ell - n + 1 \) for some \( \ell \in \mathbb{Z} \). (Note that the ring \( \hat{HW}_*(L_j) \) may not be commutative in general.)

We claim that \( C_{I(g)m} \cdot C_k = C_k \cdot C_{I(g)m} = C_{I(g)m+k} \). It suffices to show that \( C_{I(g)} \cdot C_k = C_k \cdot C_{I(g)} = C_{I(g)+k} \) since the claim follows by applying this several times. We compute, using Theorem 4.20
\[
C_k \cdot C_{I(g)} = C_k \cdot S_g(C_0) = S_g(C_k \cdot C_0) = S_g(C_k) = C_{I(g)+k}.
\]

We next compute, using Theorem 4.20 and Lemma 4.23
\[
C_{I(g)} \cdot C_k = S_g(C_0) \cdot C_k = S_g(C_0 \cdot C_k) = S_g(C_k) = C_{I(g)+k}.
\]

This completes the proof of the claim.

We define a map \( \varphi : R \to \hat{HW}_*(L_j) \) by assigning
\[
\varphi(x) = C_{I(g)-n+1}, \quad \varphi(y) = C_{I(g)}
\]
where \( R \) is a ring given by \( R = \mathbb{Z}_2[x,y,y^{-1}] \). By the above product computations, we see that \( \varphi \) is a ring homomorphism. It also follows from the computation that \( \text{ker } \varphi \) is generated by \( x^2 \). Indeed, we have seen that \( C_{I(g)-n+1}^2 = 0 \) and \( C_{I(g)} \cdot C_{-I(g)} = C_0 \). We therefore conclude that the ring \( \hat{HW}_*(L_j) \) is isomorphic to the quotient ring \( \mathbb{Z}_2[x,y,y^{-1}]/(x^2) \) with grading preserved. \( \square \)

Remark 4.24. The condition \( k \geq 2 \) is used at the beginning of the proof of Theorem 4.22. If \( k = 1 \) our technique determines the ring structure of \( \hat{HW}_*(L_j) \) only partially.

5. Computing ring structure on wrapped Floer homology

5.1. Viterbo transfer map. Let \( L \) be an admissible Lagrangian in a Liouville domain \((W, \lambda)\). Following the constructions in [8], [9], we recall a definition of the Viterbo transfer map from \( HW_*(L) \) to \( HW_*(L) \).

Consider an admissible Hamiltonian \( K_a \) on \( \tilde{W} \) which is of the form, up to smoothing,
\[
K_a(r) = \begin{cases} 
ar + b & r \geq 1 \\
0 & r \leq 1
\end{cases}
\]
for some constants \( a > 0 \) and \( b \). We may assume that \( K_a \) has positive slope only in \( \{ r \geq 1 \} \). Let \( H_a \) be a v-shaped admissible Hamiltonian of the form [4.5] such that \( K_a \leq H_a \) for \( r \leq 1 \) and \( K_a = H_a \) for \( r \geq 1 \). Then we can take a non-increasing family of Hamiltonians \( \{ G_s \}_{s \in \mathbb{R}} \) such that
\[
G_s = \begin{cases} 
H_a & s \leq -R \\
K_a & s \geq R
\end{cases}
\]
for some sufficiently large \( R > 0 \). Additionally, we also choose a family of almost complex structure \( \{ J_s \}_{s \in \mathbb{R}} \) such that
\[
J_s = \begin{cases} 
J_H & s \leq -R \\
J_K & s \geq R
\end{cases}
\]
where \( (H_a, J_H) \) and \( (K_a, J_K) \) are regular Floer data. These data define a continuation map as Section 2.3 namely
\[
f_{K_a, H_a}^{(a,b)} : HF^*(a,b)(L; K_a, J_K) \to HF^*(a,b)(L; H_a, J_H)
\]
for each action window \((a, b)\).

We define the Viterbo transfer map with action window \((a, b)\) by the direct limit of continuation maps

\[
f^{(a,b)} : \HW^* (a,b) (L) \to \HW^* (a,b) (L), \quad f^{(a,b)} = \lim_{a \to -\infty} f^{(a,b)}_{K_a, H_a}.
\]

Since the transfer map \(f^{(a,b)}\) is essentially a continuation map, which is compatible with enlarging the action window, we can further take direct/inverse limits as \(a \to -\infty\) and \(b \to \infty\). We consequently have a map

\[
f : \HW(L) \to \HW(L), \quad f = \lim_{b \to +\infty} \lim_{a \to -\infty} f^{(a,b)},
\]
call the Viterbo transfer map.

It is shown in [9, Theorem 10.2] that the Viterbo transfer map in symplectic homology respects the ring structure given by pair of pants products. The same argument directly applies to the half pair of pants product, which yields the following.

**Proposition 5.1.** The Viterbo transfer map \(f : \HW(L) \to \HW(L)\) is a ring homomorphism.

5.2. Computing ring structure on \(\HW(L_j)\). Let \(L_j\) be the real Lagrangian in \(A_k\)-type Milnor fiber \(V_k\), defined in Section 3. We shall show that the Viterbo transfer map \(\HW(L_j) \to \HW(L_j)\) is in fact an injective ring homomorphism. For this, we get the ring structure of \(\HW(L_j)\) out of the ring structure of \(\HW(L_j)\) which is computed in Theorem 4.22.

More precisely, the positive degree parts of the wrapped Floer homology \(\HW(L_j)\) and the \(v\)-shaped wrapped Floer homology \(\HW(L_j)\) can be identified in a sense that both \(\HW(L_j)\) and \(\HW(L_j)\) are isomorphic when \(v \geq 0\). We will show that the Viterbo transfer map \(f : \HW(L_j) \to \HW(L_j)\) maps each generator to a generator of the same degree. This will be enough to determine the full ring structure on \(\HW(L_j)\).

5.2.1. Specifying the Viterbo transfer map. We now consider the Viterbo transfer map in our examples; the real Lagrangians \(L_j\) in \(A_k\)-type Milnor fibers. We can specify the Viterbo transfer morphism by realizing the generators of \(\HW(L_j)\) and \(\HW(L_j)\) in view of Morse–Bott technique.

As in Section 3.4.3 we work with a deformed domain \(\hat{W}_k\) whose boundary is the \(A_k\)-type Brieskorn manifold \(\Sigma_k\). The Reeb flow on the boundary is given as (3.5), and Reeb chords in \((\Sigma_k, \alpha, L_j)\) are of Morse–Bott type. The \(v\)-spectrum is given by (3.6), and the (ordinary) spectrum of Reeb chords is obtained by taking positive periods in the \(v\)-spectrum, i.e.

\[
\Spec(\Sigma_k, \alpha, L_j) = \{ N \cdot 2(k + 1)\pi \mid N \in \mathbb{N} \}.
\]

We also recall that for \(T \in \Spec(\Sigma_k, \alpha, L_j) \subset \Spec(\Sigma_k, \alpha, L_j)\), the corresponding Morse–Bott submanifold \(\mathcal{L}_T\) of Reeb chords forms a \(S^{n-1}\)-family.

Let \(0 < a \notin \Spec(\Sigma_k, \alpha, L)\). We take a \(v\)-shaped admissible Hamiltonian \(K_a\) and an admissible Hamiltonian \(H_a\) on \(\hat{W}_k\) defined as in Section 5.1. Their Hamiltonian 1-chords form \(\mathcal{L}_T\) of Reeb chords in \(L\) since the Reeb chords on the boundary are of Morse–Bott type. Note that \(K_a\) and \(H_a\) are chosen to be identical to each other in the region (roughly the region \(r \geq 1\)) where the Morse–Bott families of Hamiltonian 1-chords corresponding to non-constant Reeb chords appear. Therefore the Morse–Bott families of Hamiltonian 1-chords of \(K_a\) in this region can be canonically identified with the Morse–Bott families of Hamiltonian 1-chords of \(H_a\). Furthermore they both correspond to Morse–Bott families \(\mathcal{L}_T\) of Reeb chords for \(T \in \Spec(\Sigma_k, \alpha, L_j)\) with \(0 < T < a\).

By taking Morse functions on each Morse–Bott submanifold of Hamiltonian 1-chords we can perturb \(K_a\) and \(H_a\) in a standard way (e.g. [23]) to make them non-degenerate. Denote the resulting perturbed Hamiltonians by \(\hat{K}_a\) and \(\hat{H}_a\) respectively. Hamiltonian 1-chords of them correspond to critical points of Morse functions on Morse–Bott submanifolds. In particular, in the region where \(K_a = H_a\), we may take the same perturbing Morse functions for both. Then \(\hat{K}_a\) and \(\hat{H}_a\) are still the same in this region and hence Hamiltonian 1-chords in there canonically correspond to each other.

Since \(\mathcal{L}_T\) forms a \(S^{n-1}\)-family, we can take perturbing Morse functions with exactly two critical points; the minimum and the maximum. As a result, for each \(T \in \Spec(\Sigma_k, \alpha, L)\) with \(T < a\), we have two non-degenerate Hamiltonian 1-chords, say \(x_T\) (minimum) and \(y_T\) (maximum), of \(\hat{K}_a\) and two Hamiltonian
1-chords, say $x_T'$ (minimum) and $y_T'$ (maximum), of $\widetilde{H}_a$. Even though we denote them with different notations, they are actually the same chords since $\widetilde{K}_a = \widetilde{H}_a$ where the chords appear. Moreover the gradings of $x_T'$ and $y_T'$ are same as those of $x_T$ and $y_T$, respectively, for each $T$.

**Proposition 5.2.** The continuation map $f_{\widetilde{K}_a, \widetilde{H}_a}$ at the chain level satisfies

$$f_{\widetilde{K}_a, \widetilde{H}_a}(x_T) = x_T' \quad \text{and} \quad f_{\widetilde{K}_a, \widetilde{H}_a}(y_T) = y_T'$$

for $T \in \text{Spec}(\Sigma_k, \alpha, \mathcal{L}_j)$ with $T < a$.

**Proof.** We prove the assertion for $x_T$ and $x_T'$, and the argument for $y_T$ and $y_T'$ is verbatim.

Let $u \in \mathcal{M}(x_T, x_T'; G_a, J_a)$ be an element of the parametrized moduli space in the definition of $f_{\widetilde{K}_a, \widetilde{H}_a}$. We claim that the curve $u: \mathbb{R} \times [0, 1] \to V_k$ stays at $x_T = x_T'$, i.e. $u(s, t) = x_T(t) = x_T'(t)$ for all $s \in \mathbb{R}$. This implies that there is a unique element in $\mathcal{M}(x_T, x_T'; G_a, J_a)$ and hence the assertion will follow.

To prove the claim, we consider the energy of $u$ as follows:

$$E(u) = \frac{1}{2} \int_{\mathbb{R} \times [0, 1]} |du - X_{G_a} \otimes dt|^2 ds \wedge dt$$

$$= \int_{\mathbb{R} \times [0, 1]} u^* \omega - (u^* dG_a) \wedge dt$$

$$= \int_{\mathbb{R} \times [0, 1]} d(u^* \lambda - u^* G_a dt) + \frac{\partial G_a}{\partial s} ds \wedge dt$$

$$\leq \int_{\partial (\mathbb{R} \times [0, 1])} u^* \lambda - u^* G_a dt$$

$$= A_{\widetilde{K}_a}(x_T) - A_{\widetilde{H}_a}(x_T') = 0.$$

Here, the inequality in the fourth line follows from the fact that $\frac{\partial G_a}{\partial s} \leq 0$. The last equality follows from the fact that both Hamiltonians $\widetilde{K}_a$ and $\widetilde{H}_a$ are equal on the region where the chords appear.

The computation shows that the energy $E(u)$ must be zero, which implies that $du - X_{G_a} dt$ is constantly zero and hence that $u(s, t) = x_T(t) = x_T'(t)$ for all $s \in \mathbb{R}$. This completes the proof. \qed

In view of the computation of the $E^1$-page of the spectral sequence in Section 3.4.3, namely differentials vanish for degree reasons, the 1-chords $x_T$ and $y_T$ are precisely the generators of the group $HF_k(L_j; \widetilde{H}_a)$ of degree $k > 0$, and likewise the 1-chords $x_T$ and $y_T$ are the generators of the group $HF_k(L_j; \widetilde{K}_a)$ of degree $k > 0$. Therefore Proposition 5.2 says that the Viterbo transfer map

$$f_{\widetilde{K}_a, \widetilde{H}_a}: HF_*(L_j; \widetilde{K}_a) \to HF_*(L_j; \widetilde{H}_a)$$

at homology level is the obvious inclusion.

To understand what happens when we take the direct limit $\lim_{n \to \infty} HF_*(L_j; \widetilde{K}_a)$ as $a \to \infty$, we choose a specific cofinal family of Hamiltonians as follows. First take an increasing sequence $\{a_n \mid n \in \mathbb{N}\}$ of slopes at cylindrical ends such that $\lim_{n \to \infty} a_n = \infty$ and the distance between $a_n$ and the spectrum $\text{Spec}(\Sigma_k, \alpha, \mathcal{L}_j)$ is greater than a number, say $\epsilon_0$, for every $n \in \mathbb{N}$. We now inductively define a sequence of Hamiltonians $\{\widetilde{K}_{a_n}\}_{n \in \mathbb{N}}$. First we choose an admissible Hamiltonian $\widetilde{K}_{a_1}$ such that $\widetilde{K}_{a_1}(r, y) = k_1(r)$ for some function $k_1$ in the cylindrical coordinate $r$ satisfying

- $k_1'(r) \geq 0$
- $k_1'(r) = \begin{cases} 0 & r \leq 1 \\ a_1 & r \geq 1 + \frac{\epsilon_0}{2} \end{cases}$

Suppose we have chosen the Hamiltonians $\widetilde{K}_{a_n}$ for $k < n$. We choose the $n$-th Hamiltonian $\widetilde{K}_{a_n}$ in such a way that $K_{a_n}(r, y) = k_n(r)$ for some function $k_n$ in the cylindrical coordinate $r$ satisfying

- $k_n'(r) \geq 0$
- $k_n(r) = k_{n-1}(r)$ for $r \leq 1 + \epsilon_0 \cdot (\sum_{i=1}^{n-1} \frac{1}{i})$
- $k_n'(r) = a_n$ for $r \geq 1 + \epsilon_0 \cdot (\sum_{i=1}^{n} \frac{1}{i})$. 


Then the sequence \( \{ K_{a_n} \} \) \( n \in \mathbb{N} \) form a cofinal family of admissible Hamiltonians. A point of the construction is that \( K_{a_n} = K_{a_n} \) for all \( m \leq n \) in the region \( \{ 1 \leq r \leq 1 + \epsilon_0 \cdot \left( \sum_{i=1}^{m} \frac{1}{2^i} \right) \} \). So we can perturb each Hamiltonian the sequence \( \{ K_{a_n} \} \) \( n \in \mathbb{N} \) to get a cofinal family of nondegenerate Hamiltonians \( \{ K_{a_n} \} \) \( n \in \mathbb{N} \) in such a way that \( \tilde{K}_{a_n} = \tilde{K}_{a_n} \) for all \( m \leq n \) in the region \( \{ 1 \leq r \leq 1 + \epsilon_0 \cdot \left( \sum_{i=1}^{m} \frac{1}{2^i} \right) \} \). In this case, in view of the proof of Proposition 5.2, the continuation map \( f_{m,n} : HF_*(L_j; \tilde{K}_{a_n}) \rightarrow HF_*(L_j; \tilde{K}_{a_n}) \) is the obvious inclusion. It follows that the homology classes of \( x_T \) and \( y_T \) in \( HF_*(L_j; \tilde{K}_{a_n}) \) survive in the process of direct limit as \( n \rightarrow \infty \) and hence define nonzero homology classes in \( \mathcal{H}F_*(L_j) \).

Analogous arguments for \( v \)-shaped Hamiltonians show that Hamiltonian chords \( x'_T \) and \( y'_T \) define nonzero homology classes in \( \mathcal{H}F_*(L_j) \). As a result we have

Lemma 5.3. Let \( N \) be a positive integer. Denote \( T_0 := 2(k + 1)\pi \).

1. The homology class of \( x_{NT_0} \) and \( y_{NT_0} \) generate \( \mathcal{H}F_\pi((n-2)(k+1)+2)N-n+1(L_j) \) and \( \mathcal{H}F_\pi((n-2)(k+1)+2)N-n+1(L_j) \) respectively.

2. The homology class of \( x_{NT_0} \) and \( y_{NT_0} \) generate \( \mathcal{H}F_\pi((n-2)(k+1)+2)N-n+1(L_j) \) and \( \mathcal{H}F_\pi((n-2)(k+1)+2)N-n+1(L_j) \) respectively.

Let us denote by \( B_d \) the homology class of a generator of \( \mathcal{H}F_\pi(L_j) \) for \( d \geq 0 \). Lemma 5.3 implies that \( x_{NT_0} \) is a representative of \( B_{((n-2)(k+1)+2)N-n+1} \) and \( y_{NT_0} \) is a representative of \( B_{((n-2)(k+1)+2)N-n+1} \) for \( N > 0 \). The zero degree part \( \mathcal{H}F_\pi(L_j) \) is generated by the unit element \( B_0 \). Likewise \( x_{NT_0} \) is a representative of \( C((n-2)(k+1)+2)N-n+1 \) and \( y_{NT_0} \) is a representative of \( B_{((n-2)(k+1)+2)N-n+1} \) for \( N \in \mathbb{Z} \), where \( C_j \) is defined in (4.11).

Combining Lemma 5.3 with Proposition 5.2 we have the following.

Theorem 5.4. The Viterbo transfer map \( f : \mathcal{H}F_\pi(L_j) \rightarrow \mathcal{H}F_\pi(L_j) \) is given by

\[
B_d \mapsto C_d
\]

for each \( d = (n-2)(k+1)+2 \) \( N \) with \( N \geq 0 \) and \( d = (n-2)(k+1)+2 \) \( N-n+1 \) with \( N \geq 1 \). In particular, the Viterbo transfer map is an injective ring homomorphism.

Proof. For the last statement, recall that the wrapped Floer homology \( \mathcal{H}F_\pi(L_j) \) is generated by \( B_0 \) and \( B_d \)’s for \( d = (n-2)(k+1)+2 \) \( N \) and \( d = (n-2)(k+1)+2 \) \( N-n+1 \), \( N > 0 \) as a \( \mathbb{Z}_2 \)-module, see Proposition 3.8.

5.2.2. The ring structure on wrapped Floer homology \( \mathcal{H}F_\pi(L_j) \). We now compute the ring structure on the wrapped Floer homology \( \mathcal{H}F_\pi(L_j) \). The following computation is analogous to Theorem 4.22 (The index \( I(g) \) below is given in (4.10)).

Theorem 5.5. For \( 0 \leq j \leq k \), \( n \geq 3 \), and \( k \geq 2 \), we have a graded ring isomorphism

\[
\mathcal{H}F_n(L_j) \cong \mathbb{Z}_2[x, y]/(x^2)
\]

where \( |x| = I(g) - n + 1 \) and \( |y| = I(g) \).

Proof. From Theorem 4.22 and Theorem 5.4 we observe

\[
B_{MI(g)} \cdot B_{NI(g)} = B_{(M+N)I(g)}
\]

for all \( M, N \geq 0 \). Indeed, we compute

\[
f(B_{MI(g)} \cdot B_{NI(g)}) = f(B_{MI(g)}) \cdot f(B_{NI(g)}) = C_{MI(g)} \cdot C_{NI(g)} = C_{(M+N)I(g)} = f(B_{(M+N)I(g)}).
\]

The first equality follows from the fact that \( f \) is a ring homomorphism. Next the second equality follows from Theorem 5.4 and the third equality follows from Theorem 4.22. Finally the assertion follows from the fact that the transfer map \( f \) is injective.

Similar arguments show that

\[
B_{MI(g)} \cdot B_{NI(g)-n+1} = B_{(M+N)I(g)-n+1} = B_{NI(g)-n+1} \cdot B_{MI(g)}
\]

for all \( M \geq 0 \) and \( N \geq 1 \). Then, the ring homomorphism \( \mathbb{Z}_2[x, y]/(x^2) \rightarrow \mathcal{H}F_\pi(L_j) \) defined by

\[
x \mapsto B_{I(g)-n+1}, y \mapsto B_{I(g)}
\]

gives an isomorphism.
Remark 5.6. If \( k = 1 \), then \( V_k \) can be identified with \( T^*S^n \), and \( L_j \) with \( j = 0, 1 \) is a cotangent fiber in \( T^*S^n \). Therefore by the result in [1], we know that \( HW_*(L_j) \cong \mathbb{Z}_2[x] \) as graded rings with \( |x| = n - 1 \).

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A COMPUTATION OF THE RING STRUCTURE IN WRAPPED FLOER HOMOLOGY

The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

E-mail address: hwbae@math.cuhk.edu.hk

Mathematisches Institut, Justus-Liebig-Universität Gießen, Arndtstrasse 2, 35398 Gießen, Germany

E-mail address: kwon.Myeonggi@math.uni-giessen.de