Finite W symmetry in finite dimensional integrable systems

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Abstract

By generalizing the Drinfeld-Sokolov reduction a large class of W algebras can be constructed. We introduce ‘finite’ versions of these algebras by Poisson reducing Kirillov Poisson structures on simple Lie algebras. A closed and coordinate free formula for the reduced Poisson structure is given. These finitely generated algebras play the same role in the theory of W algebras as the simple Lie algebras in the theory of Kac-Moody algebras and will therefore presumably play an important role in the representation theory of W algebras. We give an example leading to a quadratic $sl_2$ algebra. The finite dimensional unitary representations of this algebra are discussed and it is shown that they have Fock realizations. It is also shown that finite dimensional generalized Toda theories are reductions of a system describing a free particle on a group manifold. These finite Toda systems have the non-linear finite W symmetry discussed above.

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Introduction

Nonlinear extensions of the Virasoro algebra, generally known as $W$ algebras, have turned up in various areas of mathematical physics (see [1] for a recent review). Unfortunately not too much is known about them, their interpretation, classification and representation theory is still far from complete. In this paper we discuss a very simple class of nonlinear algebras, which are basically 'finite' $W$ algebras and which may serve as instructive playground for the infinite case. However as we shall see they do have some interest of their own.

In [2] $W_n$ algebras, were shown to arise as Dirac bracket algebras on submanifolds of Kac-Moody algebras. This gave a clear understanding of how nonlinear $W$ algebras can arise as Poisson reductions of linear current algebras. However, $W_n$ algebras are certainly not the only $W$ algebras known in the literature, which leads one to ask whether others could be constructed in a similar way. The answer to this question turned out to be yes as it was shown in [3] that there are as many different Poisson reductions of an $sln$ current algebra leading to $W$ algebras as there are partitions of the number $n$. The reduction point of view has recently been investigated by a great number of people (for example [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). The classical covariant $W$ gravity theories for all the algebras which can be constructed as reductions of Kac-Moody algebras as well as their moduli spaces have been constructed in [15].

Having the picture of $W$ algebras as reductions of Kac-Moody algebras in mind one can ask oneself the question whether this is special for infinitely many dimensions, i.e. can one formulate a similar theory for finite dimensional Lie algebras? The answer to this question is affirmative as was shown in [16]. This lead to so called 'finite $W$ algebras'. It turns out that these are a very effective toy model for ordinary $W$ algebras but apart from that they are of some interest themselves. For example they are what one could consider to be the finite algebras underlying the $W$ algebras, just as the simple Lie algebras underly the Kac-Moody algebras. This means that they will play an important role in the representation theory of ordinary $W$ algebras since the subspace of singular vectors of a $W$ algebra will carry a representation of the underlying finite $W$ algebra.

Another area where finite $W$ algebras may play an important role is finite dimensional generalized Toda theory. These were originally introduced as dimensional reductions of self dual Yang-Mills theories in [17]. The general solution space of these models was however not constructed (only for some special examples). In this paper we will show that these generalized finite Toda systems are reductions of a system describing a free particle moving on a group manifold and that they have the finite $W$ symmetry discussed above. The general solution of the equations of motion of such a system is easily found and has the form of a transform by the symmetry group of a certain reference solution. This reference solution can be reduced to give a nontrivial solution of the generalized Toda theory and one would expect the orbits of finite $W$ transformations to provide the other solutions.
In this paper we shall give a brief account of the ideas described above deferring
details to our forthcoming paper [18].

1 Generalized Drinfeld-Sokolov reductions

In this section we briefly discuss the results of [3] in which the Drinfeld-Sokolov
reductions were generalized.

Let \( g \) be simple Lie algebra, \( \{ t_a \} \) a basis of \( g \) and \( f_{ab}^c \) the structure constants of
\( g \) in this basis. Consider the Poisson algebra

\[
\{ J^a(z), J^b(w) \} = f_{c}^{ab} J^c(z-w) - k g^{ab} \delta(z-w)
\]

where \( g^{ab} \) is the inverse matrix of \( g_{ab} = \text{Tr}(t_a t_b) \). This Poisson structure is actualy
nothing but the Kirillov Poisson structure on the affine Kac-Moody algebra over
\( g \).

From now on we fix \( g = \text{sl}_n \).

Let \( i : \text{sl}_2 \hookrightarrow \text{sl}_n \) be an embedding of \( \text{sl}_2 \) into \( \text{sl}_n \). Under the adjoint action of the
embedded \( \text{sl}_2 \) algebra the algebra \( \text{sl}_n \) branches into a direct sum of \( p \) irreducible \( \text{sl}_2 \)
multiplets. Let \( \{ t_{k,m} \}^{j_k}_{m=-j_k} \) be a basis of the \( k \)th multiplet where \( j_k \) is the highest
weight of this multiplet. The numbering is chosen such that \( t_{1,\pm 1} = t_\pm \) and \( t_{1,0} = t_3 \)
where \( \{ t_3, t_\pm \} \) are the Cartan, step up and step down elements of \( i(\text{sl}_2) \). An arbitrary
map \( J : S^1 \to \text{sl}_n \) can then be written as

\[
J(z) = \sum_{k=1}^{p} \sum_{m=-j_k}^{j_k} J^{k,m} t_{k,m} \tag{2}
\]

Impose now the constraints \( \phi^{1,1}(z) \equiv J^{1,1}(z) - 1 = 0 \) and \( \phi^{k,m}(z) = J^{k,m} = 0 \)
for \( m > 0, k \neq 1 \). The constraints \( \{ \phi^{k,m}(z) \}_{m \leq 1} \) are first class which means they
generate gauge invariance. This gauge invariance can be completely fixed by gauging
away the fields \( \{ J^{k,m}(z) \}_{m > -j_k} \). After constraining and gauge fixing the currents
look like

\[
J_{\text{fix}}(z) = \sum_{k=1}^{p} J^{k,-j_k} t_{k,-j_k} + t_+
\]

The Poisson bracket \( [\mathbb{H}] \) on the set of 'currents' of the form \( (2) \) induces a Poisson
bracket (which is in fact a Dirac bracket as first realized in \( [\mathbb{H}] \)) on the set of gauge
fixed currents \( (3) \). The algebra generated by the fields \( \{ J^{k,-j_k}(z) \} \) and equipped
with the Dirac bracket is then a \( W \) algebra with conformal weights \( \{ \Delta_k = j_k + 1 \} \).

The ordinary Drinfeld-Sokolov reductions which yield the Zamolodchikov \( W_n \)
algebras correspond to the case where one takes the principal embedding of \( \text{sl}_2 \) into
\( \text{sl}_n \). The algebra \( W_3^{(2)} \) introduced first by Polyakov and Bershadsky corresponds to
the only non-principal \( \text{sl}_2 \) embedding into \( \text{sl}_4 \). In general however there are as many
inequivalent \( \text{sl}_2 \) embeddings into \( \text{sl}_n \) as there are partitions of the number \( n \). This
gives a large number of possibilities.
2 Finite W algebras

In this section we introduce finite W algebras \([16]\) by reducing finite dimensional simple Lie algebras instead of KM algebras.

The starting point is again the Kirillov Poisson structure on a Lie algebra (actually it is on its dual but since simple Lie algebras carry a nondegenerate bilinear form we identify the Lie algebra with its dual). The coordinate free expression of this Poisson bracket is

\[ \{F, G\}(x) = (x, [\text{grad}_x F, \text{grad}_x G]) \]  

(4)

where \((.,.)\) is the Cartan-Killing form, \(F, G\) are smooth functions on \(g\) and \(\text{grad}_x F\) is defined by

\[ \frac{d}{d\epsilon} F(x + \epsilon x')|_{\epsilon=0} = (x', \text{grad}_x F) \quad \text{for all } x' \in g \]  

(5)

Using again the basis \(\{t_a\}\) an arbitrary element of \(g\) can be written as \(x = J^a(x) t_a\) where \(J^a\) is a smooth function on \(g\). In terms of these coordinate functions the Kirillov bracket reads

\[ \{J^a, J^b\} = f^c_{ab} J^c \]  

(6)

(compare to eq.(1)).

One can go now through the whole procedure again, i.e. choose an \(sl_2\) embedding, impose the constraints and gauge fix. Define the set of gauge fixed elements (which is a submanifold of \(g\)) by

\[ g_{\text{fix}} = \{t_+ + \sum_{k=1}^p y^{k,j_k} t_{k,-j_k} | y^{k,j_k} \in \mathbb{C}, \mathbb{R}\} \]  

(7)

Again the Kirillov Poisson bracket on \(g\) induces a Poisson bracket on \(g_{\text{fix}}\). In order to describe this bracket introduce the map

\[ L : g \rightarrow g \]  

(8)

which, on \(\text{Im}(ad_{t_+})\) is the inverse of the map \(ad_{t_+} : \text{Im}(ad_{t_-}) \rightarrow \text{Im}(ad_{t_+})\) and on the complement of \(\text{Im}(ad_{t_+})\) is the zero map. It is shown in [18] that for \(Q_1\) and \(Q_2\) smooth functions on \(g_{\text{fix}}\) and \(y \equiv t_+ + w \in g_{\text{fix}}\) we have

\[ \{Q_1, Q_2\}(y) = \left( y, [\text{grad}_y Q_1, \frac{1}{1 + L \circ ad_w} \text{grad}_y Q_2] \right) \]  

(9)

where \(\text{grad}_y Q \in \ker(ad_{t_+})\) is (uniquely) defined by

\[ \frac{d}{d\epsilon} Q(y + \epsilon y')|_{\epsilon=0} = (y', \text{grad}_y Q) \]  

(10)

for all \(y' \in \ker(ad_{t_-}).\) (This uniquely defines \(\text{grad}_y Q\) because \(\ker(ad_{t_-})\) and \(\ker(ad_{t_+})\) are nondegenerately paired by the Cartan-Killing form.)
Let us consider an example. The finite versions of $W_n$, corresponding to the principal $\mathfrak{sl}_2$ embeddings, give abelian Poisson algebras and are therefore not very interesting. The simplest nontrivial case is associated to the nonprincipal $\mathfrak{sl}_2$ embedding of $\mathfrak{sl}_2$ into $\mathfrak{sl}_3$. Under the adjoint action of this embedding $\mathfrak{sl}_3$ decomposes into a direct sum of a triplet, two doublets and a singlet (i.e. $k = 1, \ldots, 4$ and $j_1 = 1, j_2 = j_3 = \frac{1}{2}, j_4 = 0$). The reduced algebra will therefore have 4 generators $J^{1,-1}, J^{2,-1/2}, J^{3,-1/2}$ and $J^{4,0}$ (or equivalently $g_{fix}$ is 4 dimensional). The Poisson brackets (11) in terms of $c = -\frac{4}{3}(J^{1,-1} + 3(J^{4,0})^2), e = \sqrt{\frac{4}{3}} J^{2,1/2}, f = \sqrt{\frac{4}{3}} J^{3,-1/2}$ and $h = -4J^{4,0}$ read in this case

\[
\begin{align*}
\{h,e\} &= 2e \\
\{h,f\} &= -2f \\
\{e,f\} &= h^2 + c
\end{align*}
\]

and $c$ Poisson commutes with everything. This algebra is obviously a non-linear and centrally extended version of $\mathfrak{sl}_2$ and was first constructed in [19] as a solution of the Jacobi identities. We summarize the (real) representation theory of the commutator version of this algebra in the following theorem [16].

**Theorem 1** Let $p$ be a positive integer and $x$ a real number.

1. For every pair $(p, x)$ the algebra (11) has a unique highest weight representation $W(p; x)$ of dimension $p$ with highest weight $j(p; x) = p + x - 1$ and central value $c(p; x) = \frac{1}{3}(1 - p^2) - x^2$.

2. Let $k \in \{1, \ldots, p-1\}$ then $W(p; \frac{2}{3}k - \frac{1}{3}p)$ is reducible and its invariant subspace is isomorphic as a representation to $W(p - k; -\frac{1}{3}(k + p))$.

3. The representation $W(p; x)$ is unitary iff $x > \frac{1}{3}p - \frac{2}{3}$

It is well known that it is possible to realize the finite dimensional irreducible representations of any simple Lie algebra on a Fock space. Consider for example the realization on $\mathbb{C}[z]$ of the algebra $\mathfrak{sl}_2$

\[
\begin{align*}
\sigma_\Lambda(t_+) &= \frac{d}{dz} \\
\sigma_\Lambda(t_-) &= (\Lambda, \alpha)z - z^2 \frac{d}{dz} \\
\sigma_\Lambda(t_0) &= (\Lambda, \alpha) - 2z \frac{d}{dz}
\end{align*}
\]  

(12)

where $\Lambda$ is a weight and $\alpha$ is the positive root of $\mathfrak{sl}_2$. For $\Lambda$ a principal dominant weight the representation $\sigma_\Lambda$ is reducible and in fact the subspace

\[
V = \{ P(z) \in \mathbb{C}[z] \mid (\frac{d}{dz})^{(\Lambda, \alpha)+1} P(z) = 0 \}
\]  

(13)
is isomorphic to the $(\Lambda, \alpha)+1$ dimensional irreducible representation of $sl_2$ (what we have presented here is the first term of a Fock resolution of the $(\Lambda, \alpha)+1$ dimensional irrep of $sl_2$).

Similar realizations exist for the representations $W(p; x)$ of the algebra $[11]$. Define the representation $\hat{\sigma}_\Lambda$ by

\[
\hat{\sigma}_\Lambda(h) = (\Lambda, \alpha_1 - \alpha_2) + \frac{1}{3} - 2z \frac{d}{dz}
\]

\[
\hat{\sigma}_\Lambda(e) = -3(\Lambda, \alpha_2) \frac{d}{dz} - 2z \frac{d^2}{dz^2}
\]

\[
\hat{\sigma}_\Lambda(f) = (\Lambda, \alpha_1)z - \frac{2}{3} z^2 \frac{d}{dz}
\]

\[
\hat{\sigma}_\Lambda(c) = -(\Lambda, \alpha_1)^2 - (\Lambda, \alpha_2)^2 - (\Lambda, \alpha_1)(\lambda, \alpha_2) + \frac{2}{3}(\Lambda, \alpha_2 - \alpha_1) - \frac{1}{9}
\]

where $\alpha_1$ and $\alpha_2$ are the simple roots of $sl_3$ and $\Lambda$ is a weight of $sl_3$. It is easy to check that these operators satisfy the algebra $[11]$. For $(\Lambda, \alpha_1) = \frac{2}{3}(p - 1)$ and $(\Lambda, \alpha_2) = \frac{2}{3} - \frac{1}{3}p - x$ the Fock representation $\hat{\sigma}_\Lambda$ has a $p$ dimensional invariant subspace

\[
V = \{ P(z) \in C[z] \mid (\frac{d}{dz})^p P(z) = 0 \}
\]

isomorphic to $W(p; x)$. This provides a Fock realization of the representation $W(p; x)$.

In general (that is for arbitrary embeddings) it is possible to find Fock realizations of finite $W$ algebras by a generalized Miura transformation $[18]$. Here we shall not go into this however.

### 3 Finite W symmetries in generalized Toda theories

Consider the system of a particle moving on a group manifold $G (= SL(n))$. The action of such a particle can be taken to be

\[
S[g] = \frac{1}{2} \int dt \ Tr \left( g^{-1} \frac{dg}{dt} g^{-1} \frac{dg}{dt} \right)
\]

where $g : R \rightarrow G$ is the world line of the particle. The equations of motion of this action are

\[
\frac{d}{dt} \left( g^{-1} \frac{dg}{dt} \right) = 0,
\]

or equivalently,

\[
\frac{d}{dt} \left( \frac{dg}{dt} g^{-1} \right) = 0.
\]
In local coordinates $\{x^i\}$ the action looks like

$$S = \frac{1}{2} \int dt \; g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$$

(18)

where $g_{ij} = R^a_i R^b_j K_{ab}$ and $g^{-1} \frac{dg}{dt} = R^a_i \frac{dx^i}{dt} t_a$ (remember that $\{t_a\}$ is a basis of $g$ and $K_{ab} = Tr(t_a t_b)$). From this we conclude that the action (28) describes a free particle moving in a curved background.

The action (28) has a left and right $G$ symmetry $g(t) \rightarrow a. g(t). b^{-1}$. From the equations of motion we immediately find that the conserved quantities are

$$J = \frac{dg}{dt} g^{-1} \equiv J^a t_a \quad \text{and} \quad J = g^{-1} \frac{dg}{dt} \equiv J^a t_a$$

(19)

The conserved quantities $\{J^a\}$ form a Poisson algebra

$$\{J^a, J^b\} = f_{c}^{ab} J^c$$

(20)

(and similar equations for $\bar{J}$) which, as we have seen, is nothing but the Kirillov Poisson bracket written out in coordinates. Let’s now consider what happens to the theory (13) when we reduce it.

Define $g^{(+)} = \text{span}\{t_{k,m}\}_{m>0}$, $g^{(0)} = \text{span}\{t_{k,0}\}$, $g^{(-)} = \text{span}\{t_{k,m}\}_{m<0}$. Obviously $g = g^{(-)} \oplus g^{(0)} \oplus g^{(+)}$. Let $G^-$, $G^0$, $G^+$ be the corresponding subgroups of $G$ and let $\pi_{\pm}$ be the projections of $g$ onto $g^{(\pm)}$. The constraints we impose are (as before)

$$\pi_{-}(J) = t_- \quad \text{and} \quad \pi_{+}(\bar{J}) = t_+$$

(21)

Inserting the generalized (local) Gauss decomposition $g = g_- g_0 g_+$ where $g \in G$, $g_{\pm} \in G^{\pm}$ and $g_0 \in G^0$ into eqns.(21), we find

$$g_0 t_+ g_0^{-1} = g_+^{-1} \frac{dg_+}{dt}$$

(22)

$$g_0^{-1} t_- g_0 = \frac{dg_-}{dt} g_-^{-1}$$

(23)

This means that the constrained currents look like

$$J = g_-^{-1} (g_0^{-1} \frac{dg_0}{dt} + t_+) g_- + g_-^{-1} \frac{dg_-}{dt}$$

(24)

$$J = g_+ (\frac{dg_0}{dt} g_0^{-1} + t_-) g_+^{-1} + \frac{dg_+}{dt} g_+^{-1}$$

(25)

Note that now the equations of motion (16) can be written as

$$[\frac{d}{dt}, \frac{d}{dt} + J] = 0$$

(26)

Conjugating this equation by $g_-$, using eqns. (22), (23), (24) and writing out the commutator we find

$$\frac{d}{dt} \left( g_0^{-1} \frac{dg_0}{dt} \right) = [g_0^{-1} t_- g_0, t_+]$$

(27)
This evolution equation describes the gauge invariant part of the constrained theory. The action corresponding to this equation is

\[
S[g_0] = \frac{1}{2} \int dt \, \text{Tr} \left( g_0^{-1} \frac{dg_0}{dt} g_0^{-1} \frac{dg_0}{dt} \right) + \int dt \, \text{Tr} \left( g_0^{-1} t_+ g_0 t_+ \right)
\]  

(28)

which describes a particle moving on the subgroup \( G_0 \) of \( G \) with some selfinteraction. It can be shown that the theory (28) has the nonlinear 'finite' \( W \) symmetry corresponding to the \( sl_2 \) subalgebra \( \{ t_3, t_\pm \} \) of \( g \).

Strictly speaking the above arguments only work when the \( sl_2 \) subalgebra which one considers provides an 'integral grading' of the Lie algebra \( g \) because then there are only first class constraints. However it can be shown that it is always possible to find a set of first class constraints that give the same constrained and gauge fixed manifold \( g_{\text{fix}} \). This is done by imposing only 'half' (there are always an even number of second class constraints) of the constraints that turned out to be second class such that they become first class. The other half can then be imposed as gauge fixing conditions (see for a treatment of this in the present context [3]).

In the case where the \( sl_2 \) embedding is the 'principal' embedding of \( sl_2 \) into \( g = sl_n \) [20], equation (27) reduces to the ordinary finite Toda equations

\[
\frac{d^2 q_i}{dt^2} + \exp \left( \sum_{j=1}^{n-1} K_{ij} q_j \right) = 0
\]

(29)

where \( i = 1, \ldots, \text{rank}(g) = n-1 \), \( K_{ij} \) is the Cartan matrix of \( sl_n \) and \( g_0 = \exp(q_i H_i) \).

The generalized finite Toda theories (27) were already derived in [17] as dimensional reductions of the selfdual Yang-Mills equations. For some examples the solutions were constructed, however the general solution space is to our knowledge still not known. The finite \( W \) algebras may provide a new tool in this research since they are expected to transform solutions of (27) to (different) solutions of (27), just like the general solution of (13) can be found by letting the symmetry group \( G \times G \) act on the simplest solution \( g = e^{tX} \).

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References

[1] P. Bouwknegt, K. J. Schoutens, ‘\( W \) symmetry in CFT’, CERN-TH. 6583/92

[2] J. Balog, L. Feher, P. Forgac, L. O’Raifeartaigh and A. Wipf, Ann. Phys. 203 (1990) 76
[3] F. A. Bais, T. Tjin, P. van Driel, Nucl. Phys. B357 (1991) 632

[4] L. Feher, L. O’Raifeartaigh, P. Ruelle and I. Tsutsui, ‘Generalized Toda theories and W algebras associated with integral gradings’, ETH-TH/91-16, DIAS-STP-91-17; Phys. Lett. B 283 (1992) 243

[5] J. Balog, L. Feher, P. Forgac, L. O’Raifeartaigh and A. Wipf, Phys. Lett. B227 (1989) 214; Phys. lett. B244 (1990) 435

[6] L. Feher, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, ’On the general structure of Hamiltonian reductions of the WZW theory’, DIAS-STP-91-29, UdeM-LPN-TH-71/91

[7] L. O’Raifeartaigh, P. Ruelle and I. Tsutsui,’Quantum equivalence of constrained WZW and Toda models’, DIAS-STP-91-01

[8] L. O’Raifeartaigh, A. Wipf, Phys. Lett. B 251 (1990) 361

[9] L. O’Raifeartaigh, P. Ruelle, I. Tsutsui, A. Wipf, ’W algebras for generalized Toda theories’, Dublin preprint.

[10] M. Bershadski, H. Ooguri, Comm. Math. Phys. 126 (1989) 49

[11] M. Bershadsky, Comm. Math. Phys. 139 (1991) 71

[12] L. Frappat, E. Ragoucy, P. Sorba, ’W algebras and superalgebras from constrained WZW models: a group theoretical classification’, Enslapp-AL-391/92

[13] F. Delduc, E. Ragoucy, P. Sorba; Enslapp-L-352/91; Enslapp-al-362/92; Enslapp-L-352/91

[14] B. L. Feigin, E. Frenkel, Phys. Lett. B246(1990) 75

E. Frenkel, ’W algebras and Langlands-Drinfeld correspondence’, Harvard Preprint 1991

E. Frenkel, V. Kac, M. Wakimoto, ’Characters and fusion rules for W algebras via quantized Drinfeld-Sokolov reductions’, Harvard preprint

[15] J. de Boer, J. Goeree, ’Covariant W gravity and its moduli space’, THU-92/14

F. A. Bais, T. Tjin, P. van Driel, J. de Boer, J. Goeree, ’W algebras, W gravities and their moduli spaces’, THU-92/26, ITFA-92/24

[16] T. Tjin ,’Finite W algebras’, To be published in Phys. Lett B

[17] F. A. Bais, W. P. G. van Veldhoven, Physica 139A(1986)326

[18] J. de Boer, J. Goeree, T. Tjin , to be published.

[19] M. Rocek, Phys. Lett. B vol. 255, no.4, p.554

[20] E. B. Dynkin, Amer. Math. Soc. Transl. 6[2](1967)111