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Systematic construction of non-autonomous Hamiltonian
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Abstract

This is the second article in a suite of articles investigating relations between Stäckel-type sys-
tems and Painlevé-type systems. In this article we construct isomonodromic Lax representations for
Painlevé-type systems found in the previous paper [6] by Frobenius integrable deformations of Stäckel-
type systems. We first construct isomonodromic Lax representations for Painlevé-type systems in the
so called magnetic representation and then, using a multitime-dependent canonical transformation,
we also construct isomonodromic Lax representations for Painlevé-type systems in the non-magnetic
representation. Thus, we prove that the Frobenius integrable systems constructed in Part I are indeed
Painlevé-type. We also present isomonodromic Lax representations for all one-, two- and three-
dimensional Painlevé-type systems originating in our scheme. Based on these results we propose
complete hierarchies of Painlevé equations $P_I$ that follow from our construction.

Keywords: Painlevé equations; Stäckel systems; Frobenius integrability; non-autonomous Hamiltonian
equations, Lax representation

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1 Introduction

This is the second article in the suit of articles investigating a systematic way of constructing Painlevé-
type systems from an appropriate Stäckel-type systems. In the previous paper (Part I, [6]) we have
constructed multi-parameter families of Frobenius integrable non-autonomous Hamiltonian systems with
arbitrary number of degrees of freedom. Each of these families was written in two different representations
(two different coordinate systems), that we called an ordinary one and a magnetic one, respectively,
connected by a multi-time dependent canonical transformation [13].

In this paper (Part II) we construct the isomonodromic Lax equations for both representations of these
systems, thus proving that Frobenius integrable systems constructed in Part I are indeed Painlevé-type
systems. Based on these results we propose complete hierarchies of the celebrated Painlevé equations $P_I$,
Consider the St"ackel system (separable system) generated by the following hyperelliptic spectral curve on an \((x,y)\)-plane, where \(\sigma, \varphi\) and \(f\) are (arbitrary so far) smooth functions of one variable. By taking \(n\) copies of (2.1) at points \((x,y) = (\lambda_i, \mu_i), i = 1, \ldots, n\), we obtain a system of \(n\) linear equations (separation relations) for \(h_r\):

\[
\sum_{r=1}^{n} h_r \lambda_i^{n-r} = \frac{1}{2} f(\lambda_i) \mu_i^2 - \varphi(\lambda_i) \mu_i - \sigma(\lambda_i) \equiv \Phi(\lambda_i, \mu_i), \quad i = 1, \ldots, n. \tag{2.2}
\]

Solving this system (by inverting of the Vandermonde matrix \(\lambda_i^{n-r}\)) yields \(n\) functions (Hamiltonians)

\[
h_r = E_r + M^r_\sigma + V^\sigma_r, \quad r = 1, \ldots, n, \tag{2.3}
\]

depending on \(2n\) variables \((\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\). We will from now on assume that these variables parametrize a \(2n\)-dimensional smooth manifold \(\mathcal{M} = T^*Q\) in such a way that \(\lambda_i\) are coordinates on an \(n\)-dimensional configurational manifold \(Q\) and \(\mu_i\) are fiber coordinates (momenta) on \(T^*Q\). Explicitly, we obtain

\[
E_r = \frac{1}{2} \mu^T K_r G \mu, \quad r = 1, \ldots, n,
\]

where \(\mu = (\mu_1, \ldots, \mu_n)^T\) and where the \(n \times n\) \(\lambda\)-dependent matrix \(G\) can be interpreted as a contravariant metric tensor on \(Q\) (thus turning \(Q\) into a Riemannian manifold). The metric \(G\) is flat if \(f\) is a polynomial of order less then \(n + 1\) and of constant curvature if \(f\) is a polynomial of order \(n + 1\). The matrices \(K_r\) \((K_1 = \text{Id})\) can be shown to be (1,1)-Killing tensors for the metric \(G\) (for any given \(f\)) \([2, 3, 5]\). The functions \(E_r\) on \(\mathcal{M} = T^*Q\) are called geodesic St"ackel Hamiltonians.

Further, \(V^\sigma_r\) are functions on \(Q\) that we call separable potentials. In case that \(\sigma\) is a Laurent sum \(\sigma(x) = \sum_{\alpha} \varepsilon_\alpha x^\alpha, V^\sigma_r\) will be the corresponding Laurent sum \(V^\sigma_r(\lambda) = \sum_{\alpha} \varepsilon_\alpha V^{\sigma(\alpha)}_r\) of basic separable potentials \(V^{\sigma(\alpha)}_r\), that can be constructed by the formula \([4]\)

\[
V^{(\alpha)}_r = R^\alpha V^{(0)}, \quad V^{(\alpha)} = (V^{(\alpha)}_1, \ldots, V^{(\alpha)}_n)^T, \tag{2.4}
\]
where

$$R = \begin{pmatrix}
-\rho_1 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & 0 \\
-\rho_n & 0 & 0 & 0
\end{pmatrix} \quad (2.5)$$

with $V^{(0)} = (0, 0, \ldots, -1)^T$ and with $\rho_k(\lambda) = (-1)^k s_k(\lambda)$, $k = 1, \ldots, n$, where $s_k(\lambda)$ is the elementary symmetric polynomial in $\lambda_i$ of degree $k$ (so that $s_1 = \lambda_1 + \ldots + \lambda_n$ and so on). Further, $M^r_\lambda$ are some, in general complicated, functions on $\mathcal{M}$. In case that $\varphi$ is a Laurent sum $\varphi(x) = \sum \varepsilon_\gamma x^\gamma$, $M^r_\lambda$ will be the corresponding Laurent sum $M^r_\lambda(\lambda, \mu) = \sum \varepsilon_\gamma M^r_\lambda(\gamma)$ of basic separable magnetic potentials $M^r_\lambda(\gamma)$. They have the explicit form

$$M^r_\lambda(\gamma) = \sum_{i=1}^n \frac{\partial \rho_r \lambda_i \mu_i}{\partial \lambda_i} \Delta_i, \quad \Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j) \quad (2.6)$$

(see Part I [6] for more details) and are called magnetic since they depend linearly on momenta $\mu_i$.

Assume now that $(\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ are Darboux (canonical) coordinates for a time-independent Poisson tensor $\Pi$ on $\mathcal{M}$ (so that $\{\mu_i, \lambda_j\}_\Pi = \delta_{ij}$, $\{\lambda_i, \lambda_j\}_\Pi = \{\mu_i, \mu_j\}_\Pi = 0$, $i, j = 1, \ldots, n$). Then, the Hamiltonians $h_r$ generate $n$ separable autonomous evolution equations (Hamiltonian flows)

$$\frac{d\xi_r}{dt} = X_r(\xi) = \pi dh_r(\xi), \quad r = 1, \ldots, n \quad (2.7)$$

where $\xi \in \mathcal{M}$ and where $X_r$ are the related autonomous Hamiltonian vector fields $X_r = \pi dh_r$ (autonomous means in this context that $X_r$ do not depend explicitly on time variables $t_s$). From the very construction it follows that

$$\{h_r, h_s\} = 0 \quad \text{and thus} \quad [X_r, X_s] = 0, \quad r, s = 1, \ldots, n \quad (and hence the set of $n$ Hamiltonian systems (2.7) is a Liouville integrable system, we will refer to this set as Stäckel system) and moreover that the canonical coordinates $(\lambda, \mu)$ are separation coordinates for all the flows (2.7). Since (2.7) is autonomous, it is also Frobenius integrable, i.e. the set of $n$ equations in (2.7) has a common, unique solution $\xi(t_1, \ldots, t_n, \xi_0)$ through each point $\xi_0 \in \mathcal{M}$, depending in general on all the evolution parameters $t_r$.

In this section we derive isospectral Lax representations for each Hamiltonian flow in (2.7). In literature the reader can find Lax representation for flows related to various subcases of separation curves from the family (2.1) (see for example [9, 17, 19, 20] and references therein) but our general construction is new.

Let us consider the following family of non-equivalent (in the sense of lack of similarity transformation between them) Lax matrices $L \in \mathfrak{s}(2, \mathbb{R})$, parametrized by an arbitrary non-zero smooth function $g$:

$$L(x, \xi) = \begin{pmatrix}
u(x) - \frac{g(x)v(x)}{f(x)} & u(x) \\
u(x) + \frac{g(x)v(x)}{f(x)} & -v(x) \end{pmatrix}, \quad (2.8)$$

where

$$u(x) = \prod_{k=1}^{n} (x - \lambda_k) = \sum_{k=0}^{n} \rho_k x^{n-k}, \quad \rho_0 \equiv 1 \quad (2.9)$$

$v$ is a polynomial of order $n - 1$ such that $v(\lambda_i) = g(\lambda_i)\mu_i$, so that it takes the form

$$v(x) = \sum_{i=1}^{n} g(\lambda_i)\mu_i \prod_{k \neq i} \frac{x - \lambda_k}{\lambda_i - \lambda_k} = -\sum_{k=1}^{n} \left[ \sum_{i=1}^{n} \frac{\partial \rho_k}{\partial \lambda_i} \frac{g(\lambda_i)\mu_i}{\Delta_i} \right] x^{n-k}, \quad (2.10)$$

while $w$ is determined by the right hand side of the spectral curve (2.1) through

$$w(x) = -2\frac{g^2(x)}{f(x)} \left[ \frac{\Phi(x, v(x)/g(x))}{u(x)} \right]_+ \quad (2.11)$$
The operation \( [\cdot]_+ \) is defined as follows: for an arbitrary smooth function \( b \) and an arbitrary polynomial \( a \), \( \left[ \frac{b(x)}{a(x)} \right]_+ \) is a smooth function defined uniquely through

\[
\frac{b(x)}{a(x)} = \left[ \frac{b(x)}{a(x)} \right]_+ + \frac{r(x)}{a(x)}
\]

(2.12)

where \( r \) is a polynomial of degree \( \deg r < \deg a \). In case that \( b \) is a polynomial (Laurent polynomial) then \( \left[ \frac{b(x)}{a(x)} \right]_+ \) is a polynomial (Laurent polynomial) part of the division of the polynomial (Laurent polynomial) \( b \) by the polynomial \( a \) (see [1] for the details of this construction).

Note that in our notation \( u = u(x, \xi) \), \( v = v(x, \xi) \) and \( w = w(x, \xi) \) are functions depending not only on the spectral parameter \( x \) but also on the point \( \xi \) on \( \mathcal{M} \), but in the sequel we will omit this dependence on \( \xi \) in \( u \), \( v \) and \( w \) and we will write \( u(x) \), \( v(x) \) and \( w(x) \) in order to shorten the notation.

The Lax matrix \( L \) has the important property: it can be used to reconstruct the separation curve, as the next Lemma shows.

**Lemma 1** The element \( w \) in \( L \) can be written as

\[
w(x) = -\frac{v^2(x)}{u(x)} + 2\frac{g(x)\varphi(x)v(x)}{f(x)u(x)} + 2\frac{g^2(x)\sigma(x)}{f(x)u(x)} + 2\frac{g^2(x)}{f(x)u(x)} \sum_{k=1}^{n} h_k x^{-k} - \varphi(x)y - \sigma(x).
\]

(2.13)

**Proof.** The following identity with respect to \( x, \lambda, \mu \)

\[
\sum_{k=1}^{n} h_k x^{-k} = \Phi(x, v(x)/g(x)) \mod u(x) = \Phi(x, v(x)/g(x)) - u(x) \left[ \frac{\Phi(x, v(x)/g(x))}{u(x)} \right]_+.
\]

was proved in [1, Lemma 4.1]. Taking \( \Phi(x, y) \) in (2.1), that is \( \Phi(x, y) = \frac{1}{2} f(x) y^2 - \varphi(x)y - \sigma(x) \), and using (2.11), this identity can be rewritten in the form

\[
\sum_{k=1}^{n} h_k x^{-k} = \frac{1}{2} f(x) \frac{v^2(x)}{g^2(x)} - \varphi(x) \frac{v(x)}{g(x)} - \sigma(x) + \frac{f(x)}{2g^2(x)} u(x) w(x).
\]

Solving the above identity with respect to \( w \) yields (2.13). \( \blacksquare \)

**Theorem 1** For an arbitrary smooth function \( g(x) \), the separation curve (2.1) is reconstructed from the Lax matrix (2.8) in the sense that

\[
\det \left[ L(x, \xi) - g(x) \left( y - \frac{\varphi(x)}{f(x)} \right) \right] = 2 \frac{g^2(x)}{f(x)} \left( \Phi(x, y) - \sum_{r=1}^{n} h_r x^{-r} \right).
\]

(2.14)

**Proof.** We prove this theorem by direct calculation:

\[
\det \left[ L(x, \xi) - g(x) \left( y - \frac{\varphi(x)}{f(x)} \right) \right] = \det \left( v(x) - g(x)y \atop w(x) \right) \begin{pmatrix} u(x) & v(x) \\ -v(x) - g(x)y & \frac{u(x)}{f(x)} \end{pmatrix} = \frac{f(x)}{2g^2(x)} \frac{u(x)}{f(x)} - \frac{g^2(x)}{f(x)} y - \frac{2g^2(x)\varphi(x)}{f(x)}
\]

\[
= 2 g^2(x) y^2 - 2 g^2(x) \varphi(x) y - 2 g^2(x) \sigma(x) - 2 g^2(x) \sum_{k=1}^{n} h_k x^{-k}
\]

\[
= 2 g^2(x) \left( \frac{1}{2} f(x) y^2 - \varphi(x) y - \sigma(x) - \sum_{k=1}^{n} h_k x^{-k} \right).
\]

\( \blacksquare \)

We present now the main theorem of this section.
Theorem 2 Each Hamiltonian flow \( \frac{df}{dt} = X_r \) in the St"ackel system (2.7) has the isospectral Lax representation

\[
\frac{d}{dt} L(x, \xi) = [U_r(x, \xi), L(x, \xi)]
\]  

(2.15)

where

\[
U_r(x, \xi) = g_r(x) L(x, \xi) + \frac{1}{2u(x)} P_r(x),
\]  

(2.16)

and with \( L \) given by (2.8)-(2.11), \( g(x) \) in \( L \) being an arbitrary smooth function of one argument, and where \( g_r(x) \) is an arbitrary smooth function of \( x \) and of \( (\lambda, \mu) \).

Thus, each flow \( \frac{df}{dt} = X_r \) in the St"ackel system (2.7) has the isospectral Lax representation parametrized by two arbitrary functions: \( g \) (common for all flows) and \( g_r \) (that can be chosen uniquely for each flow).

**Proof.** To prove (2.15) note that

\[
[U_k(x), L(x)] = \frac{1}{2u(x, \xi)} \begin{pmatrix}
\{u(x), h_k\} & 0 & u(x) \\
-2\{v(x), h_k\} & \{u(x), H_k\} & -v(x) + \frac{g(x)\phi(x)}{f(x)} \\
u(x) & -2\{v(x), h_k\} & -\{u(x), h_k\}
\end{pmatrix}
\]  

\[
- \frac{1}{2u(x)} \begin{pmatrix}
v(x) - \frac{g(x)\phi(x)}{f(x)} & w(x) & u(x) \\
2v(x) - \frac{g(x)\phi(x)}{f(x)} & -w(x) & 0 \\
-2v(x) & 2 & -w(x)
\end{pmatrix}
\]  

\[
= \frac{1}{2u(x)} \begin{pmatrix}
\{v(x), h_k\} - 2u(x)\{v(x), h_k\} & \{u(x), h_k\} & -u(x)\{u(x), h_k\} \\
2(\{v(x), h_k\}) & -2u(x)\{v(x), h_k\} & \{v(x), h_k\}
\end{pmatrix}
\]  

\[
- \frac{1}{2u(x)} \begin{pmatrix}
\{v(x), h_k\} - 2\{v(x), h_k\} & \{u(x), h_k\} & -2\{v(x), h_k\}
\end{pmatrix}
\]

Since

\[
\{u(x)w(x), h_k\} = u(x)\{w(x), h_k\} + w(x)\{u(x), h_k\}
\]

and on the other hand

\[
\{u(x)w(x), h_k\} = -\{v^2(x), h_k\} + 2\frac{g(x)\phi(x)}{f(x)}\{v(x), h_k\} + 2\frac{g^2(x)}{f(x)} \sum_{i=1}^{n} \{h_i, h_k\} x^{n-1}
\]

\[
= -2v(x)\{v(x), h_k\} + \frac{2g(x)\phi(x)}{f(x)}\{v(x), h_k\}
\]

we get that

\[
\{w(x), h_k\} = -\frac{w(x)}{u(x)}\{u(x), h_k\} - 2\left(\frac{v(x)}{u(x)} - \frac{g(x)\phi(x)}{f(x)u(x)}\right)\{v(x), h_k\}.
\]

Hence

\[
[U_k(x), L(x)] = \begin{pmatrix}
\{v(x), h_k\} & u(x) \\
\{w(x), h_k\} & -\{v(x), h_k\}
\end{pmatrix} = \frac{d}{dt} L(x).
\]
In this article we will use two important specifications of the functions $g_r(x)$. These specifications are used when discussing Painlevé-type systems in next sections, as for Painlevé-type systems we have to specify $g$ and $g_r$ in a very concrete way.

**Lemma 2** If

$$g_r(x) = \frac{1}{2u(x)} \frac{f(x)}{g(x)} \left[ \frac{u(x)}{x^{n+1-r}} \right]_+$$  \hspace{1cm} (2.17)

then the auxiliary matrices $U_r(x)$ are of the form

$$U_r(x) = \begin{bmatrix} B_r(x) \\ u(x) \end{bmatrix}_+ \quad B_r(x) = \frac{1}{2} f(x) \left[ \frac{u(x)}{x^{n+1-r}} \right]_+ L(x)$$  \hspace{1cm} (2.18)

and if

$$g_r(x) = -\frac{1}{2u(x)} \frac{f(x)}{g(x)} \left[ \frac{u(x)}{x^{n+1-r}} \right]_-$$  \hspace{1cm} (2.19)

where

$$\left[ \frac{u(x)}{x^{n+1-r}} \right]_- = \frac{u(x)}{x^{n+1-r}} - \left[ \frac{u(x)}{x^{n+1-r}} \right]_+$$  \hspace{1cm} (2.20)

then the auxiliary matrices $U_r(x)$ are

$$U_r(x) = \begin{bmatrix} B_r(x) \\ u(x) \end{bmatrix}_+ \quad B_r(x) = -\frac{1}{2} f(x) \left[ \frac{u(x)}{x^{n+1-r}} \right]_- L(x).$$  \hspace{1cm} (2.21)

**Proof.** Indeed, from [1, Lemma 4.4] we get

$$\{u(x), h_r\} = \left( -\frac{f(x)}{g(x)} v(x) + \varphi(x) \right) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ \mod u(x),$$

$$\{v(x), h_r\} = \frac{f(x)}{2g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ \mod u(x).$$

Using these formulas we calculate that for $g_r(x)$ given by (2.17) we get

$$(U_r)_{21} = \frac{1}{2u(x)} \left( \frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ - 2\{v(x), h_r\} \right)$$

$$= \frac{1}{2u(x)} \left( \frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ - \frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ \mod u(x) \right)$$

$$= \left[ \frac{f(x)}{2g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ \right]_+ \mod u(x),$$

and the remaining components of $U_r(x)$ can be received in a similar fashion. On the other hand for $g_r(x)$ given by (2.19) we obtain

$$(U_r)_{21} = \frac{1}{2u(x)} \left( -\frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_- - 2\{v(x), h_r\} \right)$$

$$= \frac{1}{2u(x)} \left( -\frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_- + \frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_- \mod u(x) \right)$$

$$= \left[ -\frac{f(x)}{2g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_- \right]_+ \mod u(x).$$
where we used the fact that since \( \frac{f(x)}{g(x)} w(x) \frac{u(x)}{x^{n-r+1}} \mod u(x) = 0 \) we can write
\[
\frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ \mod u(x) = \frac{f(x)}{g(x)} w(x) \left( \left[ \frac{u(x)}{x^{n-r+1}} \right]_+ - \frac{u(x)}{x^{n-r+1}} \right) \mod u(x)
\]
\[
= - \frac{f(x)}{g(x)} w(x) \left[ \frac{u(x)}{x^{n-r+1}} \right]_- \mod u(x).
\]

The remaining components of \( U_r(x) \) can be received in a similar fashion. ■

3 Frobenius integrability of non-autonomous equations of Painlevé-type in magnetic representation

In this section we briefly sketch the construction of Frobenius integrable non-autonomous Hamiltonian systems through appropriate deformations of Stäckel systems. For details of this procedure, we refer the reader to Part I [6]. Consider a multitime-dependent spectral curve:
\[
\sum_{r=1}^{n} h_r x^{n-r} = \frac{1}{2} x^m y^2 - \sum_{\gamma=0}^{n+1} d_\gamma(t) x^\gamma - e(t) x^n = \Phi(x, y, t), \quad m \in \{0, \ldots, n+1\} \tag{3.1}
\]
with \( t = (t_1, \ldots, t_n), \ t_i \in \mathbb{R} \). It means that we specify \( f, \sigma \) and \( \varphi \) in (2.1) as follows:
\[
f(x) = x^m, \quad \sigma = \sigma(x, t) = e(t) x^n, \quad \varphi = \varphi(x, t) = \sum_{\gamma=0}^{n+1} d_\gamma(t) x^\gamma. \tag{3.2}
\]
where we allow for a direct dependence of \( \sigma \) and \( \varphi \) on all \( t \). The corresponding Hamiltonians \( h_r \) become then non-autonomous (directly multitime-dependent):
\[
h_r = E_r + \sum_{\gamma=0}^{n+1} d_\gamma(t_1, \ldots, t_n) M_{r}^{(\gamma)} + e(t_1, \ldots, t_n) V_{r}^{(n)}, \quad r = 1, \ldots, n, \tag{3.3}
\]
where \( M_{r}^{(\gamma)} \) are the magnetic potentials (2.6), \( V_{r}^{(n)} = -\rho_r \) is the first nontrivial ordinary potential in (2.4) and where \( d_\gamma \) and \( e \) are yet unspecified functions of in general all evolution parameters \( t_r \). Let us now perturb the Hamiltonians \( h_r \) in (3.3)
\[
h_1^B = h_1, \quad h_r^B = h_r + W_r, \quad r = 2, \ldots, n. \tag{3.4}
\]
by quasi-Stäckel terms \( W_r = W_r(\lambda, \mu) \), linear in momenta \( \mu_i \), and given by
\[
W_r = -\sum_{i=1}^{n} \left( \sum_{k=1}^{r-1} k \rho_{r-k-1} \frac{\lambda^{m+k-1}}{\Delta_i} \right) \mu_i = \sum_{i=1}^{n} J_{r}^i \mu_i, \quad r \in I_1^m \tag{3.5}
\]
and by
\[
W_r = -\sum_{i=1}^{n} \left( \sum_{k=1}^{n-r+1} k \rho_{r+k-1} \frac{\lambda^{m-k-1}}{\Delta_i} \right) \mu_i = \sum_{i=1}^{n} J_{r}^i \mu_i, \quad r \in I_2^m \tag{3.6}
\]
where for each \( m \in \{0, \ldots, n+1\} \) the index sets \( I_1^m \) and \( I_2^m \) are defined as follows:
\[
I_1^m = \{2, \ldots, n-m+1\}, \quad I_2^m = \{n-m+2, \ldots, n\}, \quad m = 0, \ldots, n+1 \tag{3.7}
\]
with the following degenerations for \( m = 0 \) and for \( m = n+1 \):
\[
I_1^0 = I_1^n = \{2, \ldots, n\}, \quad I_2^0 = I_2^n = \emptyset, \quad I_1^{n+1} = I_1^n = \emptyset, \quad I_2^{n+1} = I_2^n = \{2, \ldots, n\}.
\]
The vector fields \( J_r = J^i \frac{\partial}{\partial x_i} \) on \( Q \) are Killing vector fields for \( g = G^{-1} \). Thus, the terms \( W_r = \sum_{i=1}^{n} J^i \mu_i \) (these terms were first introduced in ([16])) constitute linear in momenta \( \mu_i \) constants of motion for the geodesic Stäckel Hamiltonian \( E_1 \).

The functions \( \mathcal{E}_r = E_r + W_r \) (called the geodesic quasi-Stäckel Hamiltonians [6],[16]) constitute a nilpotent Lie algebra \( g = \text{span}\{\mathcal{E}_r, \ r = 1, \ldots, n\} \) with the following commutation relations:

\[
\{\mathcal{E}_1, \mathcal{E}_r\} = 0, \quad r = 2, \ldots, n,
\]

and

\[
\{\mathcal{E}_r, \mathcal{E}_s\} = \begin{cases} 
0 & \text{for } r \in I_1^n \text{ and } s \in I_2^n, \\
(s-r)\mathcal{E}_{r+s-(n-m+2)} & \text{for } r, s \in I_1^n, \\
-(s-r)\mathcal{E}_{r+s-(n-m+2)} & \text{for } r, s \in I_2^n,
\end{cases} \tag{3.8}
\]

where we denote \( \mathcal{E}_r = 0 \) as soon as \( r \leq 0 \) or \( r > n \). The algebra \( g \) has an Abelian subalgebra

\[
a = \text{span} \{\mathcal{E}_1, \ldots, \mathcal{E}_{\kappa_1}, \mathcal{E}_{n-\kappa_2+1}, \ldots, \mathcal{E}_n\} \tag{3.9}
\]

where

\[
\kappa_1 = \left[ \frac{n + 3 - m}{2} \right], \quad \kappa_2 = \left[ \frac{m}{2} \right].
\]

The \( n \) Killing vector fields \( J_r \) were carefully chosen from the whole \( n(n+1)/2 \)-dimensional Lie algebra of all Killing vectors fields for \( g = G^{-1} \) precisely in order to guarantee that \( \mathcal{E}_r = E_r + \sum_{i=1}^{n} J^i \mu_i \) would become a nilpotent Lie-algebra.

Finally, we construct \( n \) new Hamiltonians \( H^B_r \) such that for \( r \in \{1\} \cup I_1^n \)

\[
H^B_r = h^B_r, \quad \text{for } r = 1, \ldots, \kappa_1,
\]

\[
H^B_r = \sum_{j=1}^{r} \zeta_{r,j}(t_1, \ldots, t_{r-1})h^B_j, \quad \zeta_{r,r} = 1, \quad \text{for } r = \kappa_1 + 1, \ldots, n - m + 1 \tag{3.10}
\]

and for \( r \in I_2^n \)

\[
H^B_r = \sum_{j=0}^{n-r} \zeta_{r,r+j}(t_{r+1}, \ldots, t_n)h^B_{r+j}, \quad \zeta_{r,r} = 1, \quad \text{for } r = n - m + 2, \ldots, n - \kappa_2,
\]

\[
H^B_r = h^B_r, \quad \text{for } r = n - \kappa_2 + 1, \ldots, n, \tag{3.11}
\]

where \( \zeta_{r,j} \) are some functions of some of the evolution parameters \( t_s \). Let us now demand that the Hamiltonians \( H^B_r \) constitute a Frobenius integrable system, i.e. that they satisfy the Frobenius integrability conditions for non-autonomous Hamiltonians

\[
\{H^B_r, H^B_s\} + \frac{\partial H^B_r}{\partial t_s} - \frac{\partial H^B_s}{\partial t_r} = 0, \quad r, s = 1, \ldots, n \tag{3.12}
\]

(we will thus call the Hamiltonians \( H^B_r \) the Frobenius integrable deformations of the quasi-Stäckel Hamiltonians \( h^B_r \)). The \( n + 2 \) functions \( d_\gamma \) and the function \( e \) that enter \( H^B_r \) through (3.3) can be determined directly from the Frobenius condition (3.12) or equivalently - as it was shown in Part I - from the following set of linear first order PDE’s:

1. For \( r \in \{1, \ldots, \kappa_1\} \subset \{1\} \cup I_1^n \)

\[
\frac{\partial d_\gamma}{\partial t_r} = 0, \quad \gamma \neq m, \ldots, m + r - 1, \tag{3.13}
\]

\[
\frac{\partial d_m}{\partial t_r} = (\gamma - m + 1)d_{n-m+2+\gamma-r}, \quad \gamma = m, \ldots, m + r - 1. \tag{3.14}
\]
2. For \( r \in \{\kappa_1 + 1, \ldots, n - m + 1\} \subset I^m_1 \)
\[
\frac{\partial d_r}{\partial t_r} = 0, \quad \gamma \neq m, \ldots, m + r - 1, \quad \text{for } \gamma = m, \ldots, m + r - 1.
\]
\[
\frac{\partial d_r}{\partial t_r} = (\gamma - m + 1) \sum_{j=\gamma-m+1}^r \zeta_{r,j}(t_1, \ldots, t_{r-1})d_{n-m+2+\gamma-j}, \quad \gamma = m, \ldots, m + r - 1.
\]

3. For \( r \in \{n - m + 2, \ldots, n - \kappa_2\} \subset I^m_2 \)
\[
\frac{\partial d_r}{\partial t_r} = 0, \quad \gamma \neq r - (n - m + 2), \ldots, m - 2,
\]
\[
\frac{\partial d_r}{\partial t_r} = -(\gamma - m + 1) \sum_{j=0}^{n-m+2+\gamma-r} \zeta_{r,r+j}(t_{r+1}, \ldots, t_n)d_{n-m+2+\gamma-r-j}, \quad \gamma = r - (n - m + 2), \ldots, m - 2.
\]

4. For \( r \in \{n - 1 - \kappa_2, \ldots, n\} \subset I^m_2 \)
\[
\frac{\partial d_r}{\partial t_r} = 0, \quad \gamma \neq r - (n - m + 2), \ldots, m - 2,
\]
\[
\frac{\partial d_r}{\partial t_r} = -(\gamma - m + 1)d_{n-m+2+\gamma-r}, \quad \gamma = r - (n - m + 2), \ldots, m - 2.
\]

5. For \( r = 1, \ldots, n \)
\[
\frac{\partial e}{\partial t_r} = 0, \quad m = 0, \ldots, n, \quad \frac{\partial e}{\partial t_r} = \epsilon \delta_{1,r}, \quad m = n + 1,
\]
where the functions \( \zeta_{r,j}(t_1, \ldots, t_{r-1}) \) and \( \zeta_{r,r+j}(t_{r+1}, \ldots, t_n) \) can be calculated from the first-order PDE’s resulting from the compatibility conditions for the above system of PDE’s
\[
\frac{\partial^2 d_r}{\partial t_r \partial t_s} = \frac{\partial^2 d_r}{\partial t_s \partial t_r}, \quad r, s = 1, \ldots, n
\]
provided that all integration constants in (3.22) are chosen to be zero. The details of the above construction the reader can find in Part I.

The main object of our interest in this paper is the Frobenius integrable, non-autonomous Hamiltonian system
\[
\frac{d^\xi}{dt_r} = Y_r^B(\xi, t) = \pi dH^B_r(\xi, t), \quad r = 1, \ldots, n
\]
where \( H^B_r \) are given by (3.10)-(3.11) with the functions \( d_r(t), \epsilon(t), \zeta_{r,j}(t_1, \ldots, t_{r-1}) \) and \( \zeta_{r,r+j}(t_{r+1}, \ldots, t_n) \) satisfying the set of PDE’s (3.14)-(3.21) and (3.22)). The main goal of this paper is to prove that the system (3.23) is of Painlevé-type and in order to do this we have to construct an isomonodromic Lax representation for each of the flows contained in the system (3.23). This is done in the following section.

## 4 Isomonodromic Lax representation for non-autonomous equations of Painlevé-type in the magnetic representation

In this section we present the main theorem of this paper, namely that each of the non-autonomous Hamiltonian flows in (3.23) has an isomonodromic Lax representation [8, 12, 13] and thus it is appropriate to call the system (3.23) a Painlevé-type system. The Lax matrices considered here will have the form of the explicitly time-dependent matrix \( L(x, \xi, t) \) obtained from (2.8)–(2.11) by choosing \( g(x) = f(x) \) and by assuming (3.2).

**Theorem 3** Each non-autonomous Hamiltonian flow \( \frac{d\xi}{dt_r} = Y_r^B(\xi, t) \) in (3.23) has the isomonodromic Lax representation
\[
\frac{d}{dt_r}L(x, \xi, t) = [U_r(x, \xi, t), L(x, \xi, t)] + 2x^m \frac{\partial}{\partial x}U_r(x, \xi, t)
\]
where now

\[
\frac{d}{dt_r} = \frac{\partial}{\partial t_r} + \{\cdot, H_r^B\}
\]  

(4.2)

is the evolutionary derivative along the Hamiltonian vector field \(Y_r^B\), the Lax matrix \(L(x, \xi, t)\) is given by

\[
L(x, \xi, t) = \begin{pmatrix}
v(x) - \sum_{\gamma=0}^{n+1} d_\gamma(t)x^\gamma & \frac{u(x)}{w(x, t)} \sum_{\gamma=0}^{n+1} d_\gamma(t)x^\gamma \\
v(x) + \sum_{\gamma=0}^{n+1} d_\gamma(t)x^\gamma & -v(x) + \frac{u(x)}{w(x, t)} \sum_{\gamma=0}^{n+1} d_\gamma(t)x^\gamma
\end{pmatrix}
\]  

(4.3)

with \(u(x)\) given by (2.9) as before, with

\[
v(x) = \sum_{i=1}^n \lambda_i^m \mu_i \prod_{k \neq i} \frac{x - \lambda_k}{\lambda_i - \lambda_k} = \sum_{k=1}^n \left[ \sum_{l=1}^n \frac{\partial \rho_k}{\partial \lambda_l} \frac{\lambda_l^{m_i} \mu_i}{\Delta_i} \right] x^{-k},
\]

while \(w\) is given by

\[
w(x, t) = -2x^m \left[ \frac{\Phi(x, v(x)x^{-m}, t)}{u(x)} \right] +
\]

with \(\Phi(x, y, t)\) given by (3.1) and by (3.2). Further, for \(r \in \{1\} \cup I_1^m\)

\[
\mathcal{U}_r(x, \xi, t) = U_r(x, \xi, t) \quad \text{for } r = 1, \ldots, \kappa_1,
\]

(4.4)

and for \(r \in I_2^m\)

\[
\mathcal{U}_r(x, \xi, t) = \sum_{j=1}^{n-r} \zeta_{r,j}(t_1, \ldots, t_{r-1}) U_{r+j}(x, \xi, t) \quad \text{for } r = \kappa_1 + 1, \ldots, n - m + 1
\]

(4.5)

with the matrix \(U_r(x, t)\) given by (2.18)–(2.21):

\[
U_r(x, \xi, t) = \begin{pmatrix}
\frac{B_r(x)}{u(x)} \\
\frac{B_r(x)}{u(x)}
\end{pmatrix}, \quad B_r(x) = \frac{1}{2} \left. \frac{u(x)}{x^{n+1-r}} \right|_+ L(x, \xi, t), \quad r \in \{1\} \cup I_1^m;
\]

(4.6)

\[
U_r(x, \xi, t) = \begin{pmatrix}
\frac{B_r(x)}{u(x)} \\
\frac{B_r(x)}{u(x)}
\end{pmatrix}, \quad B_r(x) = -\frac{1}{2} \left. \frac{u(x)}{x^{n+1-r}} \right|_- L(x, \xi, t), \quad r \in I_2^m.
\]

(4.7)

and with the functions \(\zeta_{r,j}\) and \(\zeta_{r,r+j}\) satisfying the set of PDE’s (3.22).

The proof of the theorem is technical and requires additional lemmas, so we shifted it to Appendix. The proof, as well as all the examples, are presented in Viète coordinates \((q, p)\) instead of the separation coordinates \((\lambda, \mu)\). Viète coordinates \((q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)\) are related with separation coordinates through the point transformation

\[
q_i = \rho_i(\lambda), \quad p_i = -\sum_{k=1}^n \lambda_k^{m - i} \mu_k \Delta_k, \quad i = 1, \ldots, n.
\]

(4.8)

In this specification \(u(x)\) and \(v(x)\) in (4.3) are expressed as follows:

\[
u(x) = \sum_{k=0}^n q_{n-k} x^k, \quad v(x) = \sum_{k=0}^{n-1} M_{n-k}^{(m)} x^k.
\]
where the magnetic potentials \( M_r^{(m)} \) attain the form

\[
M_r^{(m)} = -\sum_{j=1}^{n} \left[ \sum_{s=0}^{r-1} q_j V_j^{(r-s-1)} \right] p_j
\]

\[
= \begin{cases} 
  \sum_{j=0}^{r-1} q_j p_{n+1-m-r+j} & \text{for } r = 1, \ldots, n-m \\
  -\sum_{j=r}^{n} q_j p_{n+1-m-r+j} & \text{for } r = n-m+1, \ldots, n
\end{cases}
\]

\[ (4.9) \]

The first equality in (4.9) is shown in Part I while the second equality in (4.9) is proved in [16]. Note also that \( V_r^{(k)} \) are defined by the non-tensorial formula (2.4)-(2.5) that is valid in any coordinate system. These formulas yield easily the form of \( V_r^{(k)} \) in Viète coordinates \( q \). Finally, the quasi-Stackel terms \( W_r \) in Viète coordinates are given by

\[
W_r = \sum_{k=n-m+2}^{n-m} (n+1-m-k)q_{m+r-n-2+k} p_k, \quad r \in I_1^m
\]

\[
W_r = -\sum_{k=n-m+2}^{n-m} (n+1-m-k)q_{m+r-n-2+k} p_k, \quad r \in I_2^m.
\]

\[ (4.10) \]

For details, see Part I.

**Example 1** Let us explicitly show the isomonodromic Lax representation for the Painlevé-type system (3.23) given by \( n = 3, m = 1, b_0 = b_1 = b_2 = b_4 = b = 0 \) (see Example 4 in Part I). We have \( H_r^{(3)} = h_r^{(3)} \), \( U_r = U_r, \ r = 1, 2, H_3^{(3)} = h_3^{(3)} + t_2 h_1^{(3)} \), \( U_3 = U_3 + t_2 U_1 \) with \( h_r^{(3)} \) of the form

\[
h_r^{(3)} = E_r + b_3 M_r^{(3)} + 2 b_3 t_3 M_r^{(2)} + b_3 (t_3^2 + t_2) M_r^{(1)},
\]

where the geodesic quasi-Stackel Hamiltonians \( E_r = E_r + W_r \) are given explicitly by (5.6) while the magnetic potentials \( M_r^{(3)} \) are given by (4.9) (they are also given explicitly in (5.2)). The Lax matrix \( L \) takes the form

\[
L = \begin{pmatrix} 
-b_3 x^3 - (2 b_3 t_3 + p_2) x^2 - b_3 (t_3^2 + t_2) + p_1 + q_1 p_2 |x + q_3 p_3 \ x^3 + q_1 x^2 + q_2 x + q_3 \\
-2 b_3 p_2 x^2 - 2 b_3 (p_1 + 2 t_3 p_2) x - q_3 p_3^2 & -L_{11}
\end{pmatrix}
\]

while the auxiliary matrices are as follows

\[
U_1 = \begin{pmatrix} 
-\frac{1}{2} b_3 & \frac{1}{2} \\
0 & \frac{1}{2} b_3
\end{pmatrix}
\]

\[
U_2 = \begin{pmatrix} 
-\frac{1}{2} b_3 x - b_3 t_3 - \frac{1}{2} p_2 & \frac{1}{2} x + \frac{1}{2} q_1 \\
- b_3 p_2 & -(U_2)_{11}
\end{pmatrix}
\]

\[
U_3 = \begin{pmatrix} 
-\frac{1}{2} b_3 x^2 - (b_3 t_3 + \frac{1}{2} p_2) x \\
-\frac{1}{2} (b_3 (t_3^2 + t_2) + p_1 + q_1 p_2) & \frac{1}{2} x^2 + \frac{1}{2} q_1 x + \frac{1}{2} q_2 \\
- b_3 p_2 x - b_3 (p_1 + 2 t_3 p_2) - \frac{1}{2} p_1^2 & -(U_3)_{11}
\end{pmatrix}
\]

A direct calculation confirms that the matrices \( L, \overline{U}_1, \overline{U}_2, \overline{U}_3 \) do satisfy the isomonodromic Lax representation (4.1).
5 Isomonodromic Lax representations for one-, two- and three-dimensional Painlevé-type systems in the magnetic representation

In Section 7 and Section 9 of Part I we presented the complete list of all one-, two- and three-dimensional non-autonomous Frobenius integrable systems, originating from our deformation procedure, in the magnetic representation. Here we present again these one-, two- and three-dimensional systems, this time together with their isomonodromic Lax representations (4.1) in Viète coordinates. In each case $(n, m)$ we obtain a $(n + 3)$-parameter family of systems, parametrized by real constants $b_0, \ldots, b_{n+1}, \bar{b}$. We also remind the reader that $e(t) = \bar{b}$ for $m = 0, \ldots, n$ and $e(t) = b^e t^1$ for $m = n + 1$, due to (3.21).

5.1 One-dimensional systems

Let us first consider the case $n = 1$. In this case $H^B = h^B$, $\overline{U} = U$ for all $m = 0, 1, 2$ and we obtain for each $m$ a 4-parameter family of related Painlevé-type systems.

For $m = 0$ we get

$$
h^B = \frac{1}{2} p^2 + (b_2 q^2 - b_1 q + b_2 t + b_0) p + \bar{b} q,
$$

$$
L = \begin{pmatrix}
-b_2 x^2 - b_1 x - b_2 t - b_0 - p & x + q \\
-2(b_2 p x - b_2 q p + b_1 p - \bar{b}) & -L_{11}
\end{pmatrix},
U = \begin{pmatrix}
-\frac{1}{2} (b_2 x - b_2 q + b_1) & \frac{1}{2} \\
-b_2 p & -U_{11}
\end{pmatrix}.
$$

For $m = 1$ we obtain

$$
h^B = -\frac{1}{2} q p^2 + \left[ b_2 q^2 - (b_2 t + b_1) q + b_0 \right] p + \bar{b} q,
$$

$$
L = \begin{pmatrix}
-b_2 x^2 - (b_2 t + b_1) x + q p - b_0 & x + q \\
2(b_2 q p + \bar{b}) x - q^2 p^2 + 2 b_0 p & -L_{11}
\end{pmatrix},
U = \begin{pmatrix}
-\frac{1}{2} (b_2 x - b_2 q + b_2 t + b_1) & \frac{1}{2} \\
b_2 q p + \bar{b} & -U_{11}
\end{pmatrix},
$$

while for $m = 2$ we obtain

$$
h^B = \frac{1}{2} q^2 p^2 + (b_2 e^t q^2 - b_1 q + b_0) p + \bar{b} e^t q,
$$

$$
L = \begin{pmatrix}
-b_2 e^t x^2 - b_1 x - q^2 p - b_0 & x + q \\
2 b e^t x^2 + (q^2 p^2 - 2 b_1 q p + 2 b_0 p) x - q^3 p^2 - 2 b_0 q p & -L_{11}
\end{pmatrix},
U = \begin{pmatrix}
-\frac{1}{2} (b_2 e^t x - b_2 e^t q + b_1) & \frac{1}{2} \\
\bar{b} e^t x + \frac{1}{2} q^2 p^2 - b_1 q p + b_0 p - \bar{b} e^t q & -U_{11}
\end{pmatrix}.
$$

In particular they contain the isomonodromic Lax representation for Painlevé-II, Painlevé-IV and Painlevé-III respectively, in the magnetic representation (see Part I).

5.2 Two-dimensional systems

For $n = 2$ again $H^B = h^B$ and $\overline{U} = U$, for all $m = 0, ..., 3$ and we obtain for each $m$ a 5-parameter family of related Painlevé-type systems. We use here the notation $p_0 = -q_1 p_1 - q_2 p_2$ (which follows formally from (4.8) with $i = 0$).
For $m = 0$ we get

$$h_1^B = p_1 p_2 + \frac{1}{2} q_1 p_2^2 + [b_3(q_1^2 - q_2 - 2t_2) - b_2 q_1 + b_1]p_1 + [b_3(q_1 q_2 + t_1) - b_2(q_2 - t_2) + b_0]p_2 + b q_1,$$

$$h_2^B = \frac{1}{2} q_1^2 + q_1 p_2 + \frac{1}{2} (q_2^2 - q_2) p_2^2 + [b_3(q_1 q_2 + t_1) - b_2(q_2 - t_2) + b_0]p_1 + [b_3(q_2^2 + t_1 q_1 - 2t_2 q_2) + b_2 t_2 q_1 - b_1 q_2 + b_0 q_1 + 1]p_2 + b q_2,$$

with

$$L = \begin{pmatrix}
    L_{11} & x^2 + q_1 x + q_2 \\
    L_{21} & -L_{11}
\end{pmatrix},$$

(5.1)

where

$$L_{11} = -b_3 x^3 - b_2 x^2 - (p_2 + 2b_3 t_2 + b_1)x - (p_1 + q_1 p_2 + b_3 t_1 + b_2 t_2 + b_0),$$

$$L_{21} = -2b_3 p_2 x^2 - 2(b_3 p_1 + b_2 p_2)x - p_2^2 - 2[b_3(p_0 + t_2 p_2) + b_2 p_1 + b_1 p_2 + b],$$

$$U_1 = \begin{pmatrix}
    -\frac{1}{2} b_3 x + \frac{1}{2} (b_3 q_1 - b_2) & \frac{1}{2} \\
    -b_3 p_2 & -(U_1)_{11}
\end{pmatrix},$$

$$U_2 = \begin{pmatrix}
    -\frac{1}{2} b_3 x^2 - \frac{1}{2} b_2 x + b_3 (\frac{1}{2} q_2 - t_2) - \frac{1}{2} b_1 - \frac{1}{2} p_2 & \frac{1}{2} x + \frac{1}{2} q_1 \\
    -b_3 p_2 x - b_3 p_1 - b_2 p_2 & -(U_2)_{11}
\end{pmatrix}.$$
For $m = 2$ we get
\[ h_1^B = \frac{1}{2} q_1 p_1^2 - q_2 p_2 p_1 + [b_3(q_1^2 - q_2 - t_1 q_1) - b_2 q_1 + b_1] p_1 + [b_3(q_1 - t_1) q_2 - b_2 q_2 + b_0 e^{t_2}] p_2 + \overline{b} q_1, \]
\[ h_2^B = \frac{1}{2} q_2 p_2^2 + \frac{1}{2} q_2 p_2^2 + [b_3(q_1 q_2 - t_1 q_2) - b_2 q_2 + b_0 e^{t_2}] p_1 + [b_3 q_2^2 - b_1 q_2 + b_0 e^{t_2} q_1 + q_2] p_2 + \overline{b} q_2, \]
while $L$ is given by (5.1), where now
\[ L_{11} = -b_3 x^3 - (b_3 t_1 + b_2) x^2 - (p_0 + b_1) x + q_2 p_1 - b_0 e^{t_2}, \]
\[ L_{21} = -2(b_3 p_0 - \overline{b}) x^2 + 2(b_1 p_1 + b_0 e^{t_2} p_2 - \frac{1}{2} q_1 p_1^2 - q_2 p_1 p_2) x + 2 b_0 e^{t_2} p_1 - q_2 p_1^2 \]
and moreover
\[ U_1 = \begin{pmatrix} -\frac{1}{2} b_3 x + \frac{1}{2} [b_3(q_1 - t_1) - b_2] & \frac{1}{2} \\ -b_3 p_0 + \overline{b} & -(U_{1})_{11} \end{pmatrix}, \]
\[ U_2 = \begin{pmatrix} \frac{1}{2} b_3 q_2 + \frac{1}{2} (b_0 e^{t_2} - q_2 p_1) x^{-1} & -\frac{1}{2} q_2 x^{-1} \\ (\frac{1}{2} q_2 p_2^2 - b_0 e^{t_2} p_1) x^{-1} & -(U_{2})_{11} \end{pmatrix}. \]

For $m = 3$ we get
\[ h_1^B = \frac{1}{3} p_0^2 - \frac{1}{2} q_2 p_1^2 + [b_3 e^{t_1} (q_1^2 - q_2) - b_2 q_1 + b_1 + b_0 t_2] p_1 + (b_3 e^{t_1} q_1 q_2 - b_2 q_2 + b_0) p_2 + \overline{b} e^{t_1} q_1, \]
\[ h_2^B = \frac{1}{3} q_1 q_2 p_2^2 + q_2 p_1^2 + [b_3 e^{t_1} q_1 q_2 - b_2 q_2 + b_0 + q_2] p_1 + [b_3 e^{t_1} q_2^2 - b_1 q_2 + b_0 (q_1 - t_2 q_2)] p_2 + \overline{b} e^{t_1} q_2, \]
while $L$ again is given by (5.1), where
\[ L_{11} = -b_3 e^{t_1} x^3 - b_2 x^2 - (b_1 + b_0 t_2 - q_1 p_0 - q_2 p_1) x + q_2 p_0 - b_0, \]
\[ L_{21} = 2 b_0 e^{t_1} x^3 + [2 b_2 p_0 + 2 b_1 p_0 + 2 b_0 (t_2 p_1 + p_2) + p_0^2 - q_2 p_0^2] x^2 + 2 [b_1 p_0 + b_0 (t_2 p_0 + p_1) - \frac{1}{2} q_1 p_0^2 - q_2 p_1 p_0] x + 2 b_0 p_0 - q_2 p_0 \]
and moreover
\[ U_1 = \begin{pmatrix} -\frac{1}{2} b_3 e^{t_1} x + \frac{1}{2} (b_3 e^{t_1} q_1 - b_2) & \frac{1}{2} \\ \overline{b} e^{t_1} x + b_2 p_0 + b_1 p_1 + b_0 (t_2 p_1 + p_2) + \frac{1}{2} p_0^2 - \frac{1}{2} q_2 p_2^2 - \overline{b} e^{t_1} q_1 & -(U_{1})_{11} \end{pmatrix}, \]
\[ U_2 = \begin{pmatrix} \frac{1}{2} b_3 e^{t_1} q_2 - \frac{1}{2} (q_2 p_0 - b_0) x^{-1} & -\frac{1}{2} q_2 x^{-1} \\ -\overline{b} e^{t_1} q_2 + (-b_0 p_0 + \frac{1}{2} q_2 p_0^2) x^{-1} & -(U_{2})_{11} \end{pmatrix}. \]

### 5.3 Three-dimensional systems

For each $m = 0, \ldots, 4$ we obtain a 6-parameter family of Painlevé-type systems satisfying (3.12). We use the notation $p_0 = -q_1 p_1 - q_2 p_2 - q_3 p_3$ (again, see (4.8) with $i = 0$). For all families the magnetic potentials $M_{(i)}^{(j)}$ in (4.9) are given explicitly by
\[
\begin{align*}
M_1^{(0)} &= p_3, & M_2^{(0)} &= p_2 + q_1 p_3, & M_3^{(0)} &= p_1 + q_1 p_2 + q_2 p_3, \\
M_1^{(1)} &= p_2, & M_2^{(1)} &= p_1 + q_1 p_2, & M_3^{(1)} &= -q_3 p_3, \\
M_1^{(2)} &= p_1, & M_2^{(2)} &= -q_2 p_2 - q_3 p_3, & M_3^{(2)} &= -q_3 p_2, \\
M_1^{(3)} &= p_0, & M_2^{(3)} &= -q_2 p_1 - q_3 p_2, & M_3^{(3)} &= -q_3 p_1, \\
M_1^{(4)} &= -q_1 p_0 - q_2 p_1 - q_3 p_2, & M_2^{(4)} &= -q_2 p_0 - q_3 p_1, & M_3^{(4)} &= -q_3 p_0. 
\end{align*}
\]

\[14\]
while $W_r$ are given by (4.10).

For $m = 0$ we have $a = g$ so $H_r^B = h_r^B$ and $U_r = U_r$ for all $r$ and our procedure yields

$$h_r^B = \mathcal{E}_r + b_4 M_r^{(4)} + b_3 M_r^{(3)} + (b_2 + 3 b_4 t_3) M_r^{(2)} + (b_1 + 2 b_3 t_3 + 2 b_4 t_2) M_r^{(1)}$$
$$+ [b_0 + b_2 t_3 + b_3 t_2 + b_4 (\frac{1}{2} t_3^2 + t_1)] M_r^{(0)} + b_0 q_r,$$

where

$$\mathcal{E}_1 = q_1 p_2 p_3 + p_1 p_3 + \frac{1}{2} b_4 p_3^2 + \frac{1}{2} q_3 p_2^2,$$
$$\mathcal{E}_2 = p_1 p_2 + q_1 p_1 p_3 + q_2 p_3^2 + \frac{1}{2} (q_1 q_2 - q_3) p_3^2 + p_3,$$
$$\mathcal{E}_3 = \frac{1}{2} b_4 p_1^2 + q_1 p_1 p_2 + q_2 p_1 p_3 + \frac{1}{2} q_3 p_2^2 + \frac{1}{2} (q_2^2 - q_3 q_1) p_3^3$$
$$+ (q_1 q_2 - q_3) p_2 p_3 + 2 p_2 + q_1 p_3,$$

(so that $W_1 = 0$, $W_2 = p_3$, $W_3 = 2 p_2 + q_1 p_3$). The isomonodromic Lax representation is given by

$$L = \begin{pmatrix}
L_{11} & x^3 + q_1 x^2 + q_2 x + q_3 \\
L_{21} & -L_{11}
\end{pmatrix},$$

where

$$L_{11} = -b_4 x^4 - b_3 x^3 - (3 b_4 t_3 + b_2 + p_3) x^2 - (2 b_4 t_2 + 2 b_3 t_3 + b_1 + p_2 + q_1 p_3) x$$
$$- [b_4 (\frac{1}{2} t_2^2 + t_1) + b_3 t_2 + b_2 t_3 + b_0 + q_1 p_2 + q_2 p_3 + p_1],$$
$$L_{21} = -2 b_4 p_3 x^3 - 2 (b_4 p_2 + b_3 p_3) x^2 - 2 [b_4 (p_1 + 3 t_3 p_3) + b_3 p_2 + b_2 p_3 + \frac{1}{2} p_3^2] x$$
$$- 2 [b_4 (p_0 + 3 t_3 p_3 + 2 t_2 p_3) + b_3 (p_1 + 2 t_3 p_3) + b_2 p_2 + b_1 p_3 + \frac{1}{2} q_1 p_3^2 + p_2 p_3 + 2 - b],$$

and

$$U_1 = \begin{pmatrix}
-\frac{1}{2} b_4 x + \frac{1}{2} (b_4 q_1 - b_3) & \frac{1}{2} \\
-b_4 p_3 & -(U_1)_{11}
\end{pmatrix},$$
$$U_2 = \begin{pmatrix}
-\frac{1}{2} b_4 x^2 - \frac{1}{2} b_3 x - \frac{1}{2} [b_4 (3 t_3 - q_2) + b_2 + p_3] & \frac{1}{2} x + \frac{1}{2} q_1 \\
-b_4 p_3 x - b_4 p_2 - b_3 p_3 & -(U_2)_{11}
\end{pmatrix},$$
$$U_3 = \begin{pmatrix}
-\frac{1}{2} b_4 x^3 - \frac{1}{2} b_3 x^2 - \frac{1}{2} (3 b_4 t_3 + b_2 + p_3) x^2 - b_4 (2 t_2 - q_3) + 2 b_3 t_3 + b_1 + p_2 + q_1 p_3 \\
-\frac{1}{2} b_4 (2 t_3 - q_3) + b_3 t_3 + b_1 + p_2 + q_1 p_3 & \frac{1}{2} x^2 + \frac{1}{2} q_1 x + \frac{1}{2} q_2 \\
-b_4 p_3 x^2 - (b_4 p_2 + b_3 p_3) x - [b_4 (p_1 + 3 t_3 p_3) + b_3 p_2 + b_2 p_3 + \frac{1}{2} p_3^2] & -(U_3)_{11}
\end{pmatrix}.$$
The Lax matrix $L$ is given by (5.4) with

$\begin{align*}
L_{11} &= -b_4 x^4 - (3b_4 t_3 + b_3) x^3 - [b_4(3t_3^2 + 2t_2) + 2b_3 t_3 + b_2 + p_2] x^2 \\
&\quad - [b_4(t_3^4 + 3t_2 t_3 + t_1) + b_3(t_3^2 + t_2) + b_2 t_3 + b_1 + p_1 + q_1 p_2] x + q_3 p_3 - b_0, \\
L_{21} &= -2b_4 p_2 x^3 - 2[b_4(p_1 + 3t_3 p_2) + b_3 p_2] x^2 \\
&\quad - 2[b_4(p_0 + 3t_3 p_1 + 2t_2 p_2 + 3t_3^2 p_2) + b_3(p_1 + 2t_3 p_2) + b_2 p_2 + \frac{1}{2} p_2^2 - \bar{b}] x - q_3 p_3^2 + 2b_0 p_3
\end{align*}$

while

$U_1 = \begin{pmatrix}
-\frac{1}{2} b_4 x + \frac{1}{2} [b_4(q_1 - 3t_3) - b_3] \\
-\frac{1}{2} b_4 p_2 \\
-(U_1)_{11}
\end{pmatrix},$

$U_2 = \begin{pmatrix}
-\frac{1}{2} b_4 x^2 - \frac{1}{2} [b_4(t_3^2 + b_3)] x - \frac{1}{2} [b_4(3t_3^2 + 2t_2) + 2b_3 t_3 + b_2 + p_2] x \\
-\frac{1}{2} [b_4(t_3^4 + 3t_2 t_3 + t_1 - q_3) + b_3(t_3^2 + t_2) + b_2 t_3 + b_1 + q_1 p_2] x \\
-\frac{1}{2} b_4 p_2 x - b_4(p_1 + 3t_3 p_2) - b_3 p_2 \\
-(U_2)_{11}
\end{pmatrix},$

$U_3 = \begin{pmatrix}
-\frac{1}{2} b_4 x^3 - \frac{1}{2} (3b_4 t_3 + b_3) x^2 - \frac{1}{2} [b_4(3t_3^2 + 2t_2) + 2b_3 t_3 + b_2 + p_2] x \\
-\frac{1}{2} [b_4(t_3^4 + 3t_2 t_3 + t_1 - q_3) + b_3(t_3^2 + t_2) + b_2 t_3 + b_1 + q_1 p_2] x \\
-\frac{1}{2} b_4 p_2 x^2 - [b_4(p_1 + 3t_3 p_2) + b_3 p_2] x - b_4([3b_3 p_1 + 2t_3 p_2 + 3t_3^2 p_2] + p_0) \\
-b_3(p_1 + 3t_3 p_2) - b_2 p_2 - \frac{1}{2} p_2^2 + \bar{b} \\
-(U_3)_{11}
\end{pmatrix}.$

For $m = 2$ we have $a = g$ so $H_r^B = h_r^B$ and $\overline{U}_r = U_r$ for all $r$ and so

$h_r^B = E_r + b_4 M_r^{(4)} + (b_3 + 2b_4 t_2) M_r^{(3)} + [b_2 + b_3 t_2 + b_4(t_2^2 + t_1)]M_r^{(2)} + b_1 M_r^{(1)} + b_0 e^{t_3} M_r^{(0)},$

where

$$\begin{align*}
E_1 &= \frac{1}{2} p_1^2 - \frac{1}{2} q_2 p_2^2 - q_3 p_2 p_3, \\
E_2 &= -q_2 p_1 p_2 - q_3 p_1 p_3 - q_4 q_2 p_3 - \frac{1}{2} (q_1 q_2 + q_3) p_2^2 + p_1, \\
E_3 &= -q_3 p_1 p_2 - \frac{1}{2} q_1 q_2 p_2^2 + \frac{1}{2} q_2 p_3^2 + q_3 p_3.
\end{align*}$$

(5.7)

The Lax matrix $L$ is given by (5.4) with

$L_{11} = -b_4 x^4 - (2b_4 t_2 + b_3) x^3 - [b_4(t_2^2 + t_1) + b_3 t_2 + b_2 + p_1] x^2 + (q_2 p_2 + q_3 p_2 - b_1) x + q_3 p_2 - b_0 e^{t_3},$

$L_{21} = -2b_4 p_1 x^3 - 2[b_4(p_0 + 2t_2 p_1) + b_3 p_2 - \bar{b}] x^2 + (2b_1 p_2 + b_0 e^{t_3} p_1 - \frac{1}{2} q_2 p_2^2 - q_3 p_2 p_3) x - q_3 p_2^2 + 2b_0 e^{t_3} p_2$

while

$U_1 = \begin{pmatrix}
-\frac{1}{2} b_4 x + \frac{1}{2} [b_4(q_1 - 2t_2) - b_3] \\
-\frac{1}{2} b_4 p_1 \\
-(U_1)_{11}
\end{pmatrix},$

$U_2 = \begin{pmatrix}
-\frac{1}{2} b_4 x^2 - \frac{1}{2} (2b_4 t_2 + b_3) x - \frac{1}{2} [b_4(t_2^2 + t_1 - q_2) + b_3 t_2 + b_2 + p_1] x \\
-\frac{1}{2} [b_4(t_2^4 + 3t_2 t_3 + t_1 - q_3) + b_3(t_2^2 + t_2) + b_2 t_3 + b_1 + q_1 p_2] x \\
-\frac{1}{2} b_4 p_1 x - b_4(p_0 + 2t_2 p_1) - b_3 p_1 + \bar{b} \\
-(U_2)_{11}
\end{pmatrix},$

$U_3 = \begin{pmatrix}
\frac{1}{2} b_4 q_3 + \frac{1}{2} (b_0 e^{t_3} - q_3 p_2) x^{-1} \\
-\frac{1}{2} q_3 x^{-1} \\
-(U_3)_{11}
\end{pmatrix}.$

For $m = 3$, we have $H_r^B = h_r^B, \overline{U}_r = U_r,$ $r = 1, 3, H_2^B = h_2^B + t_3 h_3^B, \overline{U}_2 = U_2 + t_3 U_3$ and our procedure yields

$h_r^B = E_r + b_4 M_r^{(4)} + (b_3 + t_2 q_1) M_r^{(3)} + b_2 M_r^{(2)} + (b_0 t_3 e^{2t_2} + b_1 e^{t_2}) M_r^{(1)} + b_0 e^{2t_2} M_r^{(0)},$
where

\[
\mathcal{E}_1 = -\frac{1}{2} q_1 p_1^2 - q_2 p_1 p_2 - q_3 p_1 p_3 - \frac{1}{2} q_3 p_2^2, \\
\mathcal{E}_2 = -\frac{1}{2} q_2 p_1^2 - q_3 p_1 p_2 + \frac{1}{2} (q_2^2 - q_1 q_3) p_2^2 + q_2 q_3 p_2 p_3 + \frac{1}{2} q_3^2 p_3^2 + q_3 p_2, \\
\mathcal{E}_3 = -\frac{1}{2} q_3 p_1^2 + \frac{1}{2} q_2 q_3 p_2^2 + q_3^2 p_2 p_3 + q_2 p_2 + 2 q_3 p_3.
\]

The Lax matrix \(L\) is given by (5.4) with

\[
L_{11} = -b_4 x^4 - (b_1 t_1 + b_3) x^3 - (p_0 + b_2) x^2 - (b_1 e^{t_2} + b_0 t_3 e^{2t_2} - q_2 p_2 x - b_0 e^{2t_2} + q_3 p_1,
\]

\[
L_{21} = 2(-b_4 p_0 + b) x^3 + 2[b_2 p_1 + b_1 e^{t_2} p_2 + b_0 e^{2t_2} (t_3 p_2 + p_3) + p_1 p_0 + \frac{1}{2} q_3 p_1^2 - \frac{1}{2} q_3 p_2^2] x^2
\]

\[+2[b_1 e^{t_2} p_1 + b_0 e^{2t_2} (t_3 p_1 + p_2) - \frac{1}{2} q_2 p_1^2 - q_3 p_1 p_2] x - q_3 p_1^2 + 2 b_0 e^{2t_2} p_1
\]

while

\[
U_1 = \left(-\frac{1}{2} b_4 x + \frac{1}{2} [b_4 (q_1 - t_1) - b_3] \quad \frac{1}{2} \\
-b_4 p_0 + b \quad -(U_1)_{11}
\right),
\]

\[
U_2 = \left(\begin{array}{ccc}
\frac{1}{2} b_4 q_3 + \frac{1}{2} (b_1 e^{t_2} + b_0 t_3 e^{2t_2} - q_2 p_1 - q_3 p_2) x^{-1} & -\frac{1}{2} q_2 x^{-1} & \frac{1}{2} q_3 x^{-2} \\
-(b_1 e^{t_2} p_2 + b_0 e^{2t_2} (t_3 p_1 + p_2) - \frac{1}{2} q_2 p_1^2 - q_3 p_1 p_2) x^{-1} & -(U_2)_{11} & -\frac{1}{2} q_3 p_1^2 x^{-2}
\end{array}\right),
\]

\[
U_3 = \left(\begin{array}{ccc}
\frac{1}{2} b_4 q_1 + \frac{1}{2} (b_0 e^{2t_2} - q_3 p_1) x^{-1} & -\frac{1}{2} q_3 x^{-1} & -\frac{1}{2} (2 b_0 e^{2t_2} - q_3 p_1) p_1 x^{-1} \\
-\frac{1}{2} (b_0 e^{2t_2} - q_3 p_1) p_1 x^{-1} & -(U_3)_{11} & -\frac{1}{2} q_3 p_1^2 x^{-2}
\end{array}\right).
\]

Finally, for \(m = 4\) we have again \(a = q\) so \(H_r^B = H_r^B\) and \(\overline{U}_r = U_r\) for all \(r\) and our procedure yields

\[
h_r^B = \mathcal{E}_r + b_4 e^{t_2} M_1^{(4)} + b_3 M_1^{(3)} + [b_2 + b_1 t_2 + b_0(t_2^2 + t_3)] M_1^{(2)} + (b_1 + 2 b_0 t_2) M_1^{(1)} + b_0 M_1^{(0)},
\]

where

\[
\mathcal{E}_1 = \frac{1}{2} p_0^2 - \frac{1}{2} q_2 p_2^2 - q_3 p_3 p_2, \\
\mathcal{E}_2 = -q_3 p_3 p_1 - q_3 p_1 p_2 + \frac{1}{2} (q_1 q_2 + q_3) p_2^2 - q_1 q_3 p_1 p_2 + q_2 p_1 + 2 q_3 p_2, \\
\mathcal{E}_3 = -q_3 p_3 p_1 - \frac{1}{2} q_1 q_3 p_2^2 + \frac{1}{2} q_3^2 p_2^2 + q_3 p_3.
\]

The Lax matrix \(L\) is given by (5.4) with

\[
L_{11} = -b_4 e^{t_1} x^4 - b_3 x^3 - [b_2 + b_1 t_2 + b_0(t_2^2 + t_3) - q_1 p_0 - q_2 p_1 - q_3 p_2] x^2
\]

\[-(b_1 + 2 b_0 t_2 - q_2 p_0 - q_3 p_1) x + q_3 p_0 - b_0,
\]

\[
L_{21} = 2 b_4 e^{t_1} x^4 + 2[b_3 p_0 + b_2 p_1 + b_1 (t_2 p_1 + p_2) + b_0((t_2^2 + t_3)p_1 + 2 t_2 p_2 + p_3) + \frac{1}{2} p_0^2 - \frac{1}{2} q_2 p_1^2 - q_3 p_1 p_2] x^3
\]

\[+2[b_2 p_0 + b_1 (t_2 p_0 + p_1) + b_0((t_2^2 + t_3)p_0 + 2 t_2 p_1 + p_2) - \frac{1}{2} q_1 p_0^2 - (q_2 p_1 + q_3 p_2)p_0 - \frac{1}{2} q_3 p_1^2] x^2
\]

\[+2[b_1 p_0 + b_0(2 t_2 p_0 + p_1) - \frac{1}{2} q_2 p_1^2 - q_3 p_1 p_0] x + 2 b_0 p_0 - q_3 p_0^2
\]

while

\[
U_1 = \left(-\frac{1}{2} b_4 e^{t_1} x + \frac{1}{2} (b_4 e^{t_1} q_1 - b_3) \quad \frac{1}{2} \\
\frac{1}{2} b_4 e^{t_1} x + b_3 p_0 + b_2 p_1 + b_1 (t_2 p_1 + p_2) + b_0[(t_2^2 + t_3)p_1 + 2 t_2 p_2 + p_3] + \frac{1}{2} p_0^2 - \frac{1}{2} q_2 p_1^2 - q_3 p_1 p_2 - \frac{1}{2} b_4 e^{t_1} q_1 \quad -(U_1)_{11}
\right).
\]
we find 6 Painlevé-type systems with ordinary potentials where $n \geq 3$ for the case with magnetic potentials. First, taking $c_\alpha$ provided that the functions generated by $h_r = h_r'(\lambda, \mu') = \frac{1}{2} \mu'^T K_r \mu' + \sum_{\alpha = -m}^{2n-m+2} c_\alpha(t) V_r^{(\alpha)}$, $r = 1, \ldots, n$.\(\text{In the next step we perturb the Hamiltonians } h_r', \text{ defined through } (6.2), \text{ to the quasi-Stäckel Hamiltonians } h_r^A = h_r' + W_r', \text{ where } W_r = W_r(\lambda, \mu') \text{ with } W_r \text{ given by } (3.5)-(3.6). \text{ Next, we deform the Hamiltonians } h_r^A \text{ to the Frobenius integrable Hamiltonians } H_r^A \text{ through the deformations } (3.10)-(3.11). \text{ In Part I [6] we also proved that the Hamiltonians } H_r^A \text{ with ordinary potentials and } H_r^B \text{ with magnetic potentials are related by the multitime-dependent canonical transformation (rational symplectic transformation [13])}

\begin{align*}
\lambda_i' &= \frac{\partial F(\lambda, \mu', t)}{\partial \mu_i'} = \lambda_i, \\
\mu_i &= \frac{\partial F(\lambda, \mu', t)}{\partial \lambda_i} = \mu_i' + \sum_{\gamma = 0}^{n+1} d_\gamma(t) \lambda_i^{\gamma-m}, \quad i = 1, \ldots, n, \quad (6.3)
\end{align*}

\text{generated by}

\begin{align*}
F(\lambda, \mu', t) &= \sum_{i=1}^{n} \left( \lambda_i \mu_i' + \sum_{\gamma = 0, \gamma \neq m-1}^{n+1} \frac{1}{\gamma - m + 1} d_\gamma(t) \lambda_i^{\gamma-m} + d_{m-1}(t) \ln \lambda_i \right), \quad (6.4)
\end{align*}

\text{provided that the functions } c_\alpha \text{ and } d_\gamma, e \text{ are related by the polynomial (w.r.t } x) \text{ identity}

\begin{align*}
\sum_{\alpha = -m}^{2n-m+2} c_\alpha(t) x^\alpha &= \frac{1}{2} x^m \left( \sum_{\gamma = 0}^{n+1} d_\gamma(t) x^{\gamma-m} \right)^2 + (e(t) - d_{n+1}(t)) x^n, \quad (6.5)
\end{align*}

\text{It means, that in the above notation and up to terms independent of the coordinates on } \mathcal{M}:

\begin{align*}
H_r^A(\lambda, \mu', t) &= H_r^B(\lambda, \mu', t) + \frac{\partial F(\lambda, \mu', t)}{\partial t_r}, \quad r = 1, \ldots, n. \quad (6.6)
\end{align*}

\text{For details of this construction, see Part I [6].}

\text{As a result of the above considerations, we obtain the following corollary}
Corollary 4 Each non-autonomous Hamiltonian flow on $M$

$$\frac{d\xi}{dt_r} = Y^A_r(\xi, t) = \pi dH^A_r(\xi, t)$$  \hfill (6.7)

(cf. (3.23)) has the isomonodromic Lax representation

$$\frac{d}{dt_r} L(x, \xi, t) = [U_r(x, \xi, t), L(x, \xi, t)] + 2x^m \frac{\partial}{\partial x} U_r(x, \xi, t)$$

with the evolutionary derivative given by

$$\frac{d}{dt_r} = \frac{\partial}{\partial t_r} + \{\cdot, H^A_r\}$$  \hfill (6.8)

(cf. (4.2)), with the Lax matrix $L(x)$ given by

$$L(x, \xi, t) = \begin{pmatrix} v(x) & u(x) \\ w(x, t) & -v(x) \end{pmatrix},$$  \hfill (6.9)

$u(x)$ given by (2.9), $v(x)$ given by (2.10) with $g(\lambda_i) = f(\lambda_i) = \lambda_i^m$, $w(x, t) = -2x^m \frac{\Psi(x, v(x)x^{-m}, t)}{u(x)} + .$$  \hfill (6.10)

with $\mu$ replaced by $\mu'$ and with $\Psi(x, y, t)$ given in (6.1). Finally, the matrices $U_r$ are given by (4.4)-(4.7) with the same functions $\zeta_{r,j}$ and $\zeta_{r,r+j}$ as for the corresponding (i.e. with the same $n$ and $m$) magnetic flow.

The system (6.7) can be considered as the non-magnetic representation of the corresponding (i.e. with the same $n$ and $m$) system (3.23). In the sequel we will omit $'$ at $\mu$ when writing our systems in the non-magnetic representation (6.7).

In order to find an explicit form of the part of the Lax element $L_{21} = w(x, t)$ in (6.9) that is generated by the ordinary potentials in (6.10) (i.e. by the term $\sum \alpha c_\alpha x^\alpha$ in $\Psi$; note that the operation $\llbracket \cdot \rrbracket_+$ defined in (3.14) is linear) we need the following lemma.

Lemma 3 (i) For $s \in \mathbb{N}$

$$\begin{bmatrix} x^{n+s} \\ u(x) \end{bmatrix}_+ = -\sum_{r=0}^s V_1^{(n+r-1)} \lambda^{s-r} = -\sum_{r=0}^s V_1^{(n+s-r-1)} x^r.$$  \hfill (6.11)

(ii) For $s \in \mathbb{N}$

$$\begin{bmatrix} x^{-s} \\ u(x) \end{bmatrix}_+ = \sum_{r=1}^s V_1^{(-r)} \lambda^{s+r-1} = \sum_{r=1}^s V_1^{(-s+r-1)} x^{-r}.$$  \hfill (6.12)

Proof. The basic ordinary potentials satisfy the following recursion relations:

$$V_k^{(r+1)} = V_k^{(r)} - \rho_k V_1^{(r)}, \quad r \in \mathbb{N},$$  \hfill (6.13)

$$V_k^{(-r-1)} = V_k^{(-r)} - \frac{\rho_{r-1}}{\rho_n} V_n^{(-r)}, \quad r \in \mathbb{N},$$

that follow directly from (2.4)-(2.5). The proof of this Lemma is by induction with respect to $s$.

(i) Due to (2.4), (2.5) we have $V_1^{(n-1)} = -1$ and $V_k^{(n)} = \rho_k$. So, according to (2.9)

$$\frac{x^n}{u(x)} = 1 - \sum_{k=1}^n \rho_k x^{n-k} \frac{u(x)}{u(x)} = -V_1^{(n-1)} - \sum_{k=1}^n V_k^{(n)} x^{n-k} \frac{u(x)}{u(x)}.$$  \hfill (6.14)
Assume now that for a fixed \( s \in \mathbb{N} \)

\[
\frac{x^{n+s}}{u(x)} = - \sum_{r=0}^{s} V_1^{(n+r-1)} x^{s-r} - \sum_{k=1}^{n} \frac{V_k^{(n+s)} x^{n-k}}{u(x)}.
\]  
(6.15)

Then

\[
x^{n+s} \frac{1}{u(x)} = -x \sum_{r=0}^{s} V_1^{(n+r-1)} x^{s-r} - x \sum_{k=1}^{n} \frac{V_k^{(n+s)} x^{n-k}}{u(x)}
\]

\[
= - \sum_{r=0}^{s} V_1^{(n+r-1)} x^{s-r+1} - \sum_{k=1}^{n} \left( V_k^{(n+s)} - \rho_k V_1^{(n+s)} \right) x^{n-k}
\]

(6.13)

\[
= - \sum_{r=0}^{s+1} V_1^{(n+r-1)} x^{s-r+1} - \sum_{k=1}^{n} \frac{V_k^{(n+s+1)} x^{n-k}}{u(x)}.
\]

Thus, by induction, (6.15) is true for any \( s \in \mathbb{N} \) and using the definition of \( [\cdot]_{+} \) in (2.12) we obtain (6.11).

(ii) By (2.4), (2.5) we have, \( V_1^{(-1)} = \frac{1}{\rho_n} \) and \( V_k^{(-1)} = \frac{\rho_{k-1}}{\rho_n} \), . So, due to (2.9)

\[
x^{-1} \frac{1}{u(x)} = x^{-1} - \sum_{k=1}^{n} \frac{\rho_{k-1}}{\rho_n} x^{n-k} = V_1^{(-1)} x^{-1} - \sum_{k=1}^{n} \frac{V_k^{(-1)} x^{n-k}}{u(x)}.
\]

Assume now that for a fixed \( s \in \mathbb{N} \)

\[
\frac{x^{-s}}{u(x)} = \sum_{r=1}^{s} V_1^{(-r)} x^{-s+r-1} - \sum_{k=1}^{n} \frac{V_k^{(-s)} x^{n-k}}{u(x)}.
\]  
(6.16)

Then

\[
x^{-1} \frac{x^{-s}}{u(x)} = x^{-1} \sum_{r=1}^{s} V_1^{(-r)} x^{-s+r-1} - x^{-1} \sum_{k=1}^{n} \frac{V_k^{(-s)} x^{n-k}}{u(x)}
\]

\[
= \sum_{r=1}^{s} V_1^{(-r)} x^{-(s+1)+r-1} - \sum_{k=1}^{n} \frac{V_k^{(-s)} x^{n-k-1}}{u(x)} - \sum_{k=1}^{n} \frac{V_k^{(-s)} x^{n-k}}{u(x)}
\]

(6.14)

\[
= \sum_{r=1}^{s+1} V_1^{(-r)} x^{-(s+1)+r-1} - \sum_{k=1}^{n} \frac{V_k^{(-s-1)} x^{n-k}}{u(x)}.
\]

Thus, (6.16) is valid for any \( s \in \mathbb{N} \) and it implies, by the definition of \( [\cdot]_{+} \) in (2.12), that (6.12) is true.

\begin{itemize}
\item
\end{itemize}

**Example 2** The system from Example 1 (i.e. given by \( n = 3, m = 1, b_0 = b_1 = b_2 = b_4 = 5 = 0 \)) has in the non-magnetic representation (see Example 2 in Part I) the form \( H^A = h^A_r \), \( \overline{U} = U_r \), \( r = 1, 2 \), \( H_3^A = h_3^A + t_2 h_1^A \), \( \overline{U}_3 = U_3 + t_2 U_1 \) with \( h^A_r \) given by (up to terms independent on coordinates on \( \mathcal{M} \))

\[
h^A_r = \mathcal{E}_r + 2 a_5 (3 h_3^2 + t_2) V_r^{(3)} + 4 a_5 t_3 V_r^{(4)} + a_5 V_r^{(5)}
\]
with \(a_5 = \frac{1}{2} b_3^2\) (this follows from the map (6.5)) where the geodesic quasi-Stäckel Hamiltonians \(E_r\) are the same and are given by (5.6) while the ordinary potentials \(V^{(a)}_t\) are given by (2.4) and (2.5). Explicitly, in Viète coordinates:

\[
V^{(5)} = \begin{pmatrix}
q_1^2 - 2q_1q_2 + q_3 \\
q_1^2q_2 - q_1q_3 - q_2^2 \\
q_1^2q_3 - q_2q_3
\end{pmatrix},
\quad V^{(3)} = \begin{pmatrix}
q_2 - q_1^2 \\
q_3 - q_1q_2 \\
-q_1q_3
\end{pmatrix},
\quad V^{(3)} = \begin{pmatrix}
q_1 \\
q_2 \\
q_3
\end{pmatrix}
\]

Further, the Lax matrix \(L\) takes the form

\[
L = \begin{pmatrix}
-p_2 x^2 - (q_1p_2 + p_1)x + q_3p_3 & x^3 + q_1 x^2 + q_2 x + q_3 \\
-2a_5 x^2 - 2a_5(q_1 - 4t_3)x^2 & -L_{11}
\end{pmatrix}
\]

(\(L_{11}\) here and in the sequel we will omit \('\) at \(p\) when writing down the isomonodromic Lax representation in the non-magnetic case) while

\[
U_1 = \begin{pmatrix}
0 \\
a_5 \\
\frac{1}{2}
\end{pmatrix},
\quad U_2 = \begin{pmatrix}
-\frac{1}{2}p_2 \\
ap_5(x - q_1 + 4t_3) \\
\frac{1}{2}p_2
\end{pmatrix}
\]

\[
U_3 = \begin{pmatrix}
-\frac{1}{2}p_2 x - \frac{1}{2}q_1 p_2 - \frac{1}{2} p_1 \\
a_5 x^2 - a_5(q_1 - 4t_3)x + a_5 (q_1^2 - 4t_3q_1 - q_2 + 6t_2^2 + 2t_2) - \frac{1}{2} p_2^2 \\
\frac{1}{2}x^2 + \frac{1}{2}q_1 x + \frac{1}{2}q_2
\end{pmatrix}
\]

A direct calculation confirms that the matrices \(L, \overline{U}_1, \overline{U}_2, \overline{U}_3\) do satisfy the isomonodromic Lax equation (4.1) with the time derivative given by (6.8). The explicit form of the multitime-dependent transformation (6.3) between both systems has been presented in Example 5 in Part I.

### 7 Isomonodromic Lax representations for one-, two- and three-dimensional Painlevé-type systems in the ordinary representation

In Sections 5 and 9 of Part I we presented a complete list of all one-, two- and three-dimensional non-autonomous Frobenius integrable systems originating from our deformation procedure in the non-magnetic representation (i.e. with ordinary potentials). Here we present these one-, two- and three-dimensional systems together with their isomonodromic Lax representations (4.1) in Viète coordinates. All Hamiltonians are given up to terms independent of the coordinates on \(M\). In particular we propose below four complete (in the sense explained in Introduction) Painlevé hierarchies of \(P_t - P_{1t}\).

The functions \(c_n(t)\) are expressed by functions \(d_i(t)\) and \(e(t)\) through (6.5) which - through comparison of coefficients at equal powers of \(x\) - induces the map

\[
(b_0, \ldots, b_{n+1}, \tilde{b}) \rightarrow (a_{-m}, \ldots, a_{2n-m+2})
\]

(7.1) between the parameters \((b_0, \ldots, b_{n+1}, \tilde{b})\) of the magnetic representation and the dynamical parameters \((a_{-m}, \ldots, a_{-1}, a_m, \ldots, a_{2n-m+2})\) and the non-dynamical parameters \((a_0, \ldots, a_{n-1})\) of the non-magnetic (ordinary) representation.

#### 7.1 One-dimensional systems

Let us first consider the case \(n = 1\). In this case \(H^A = h^A\) and \(\overline{U} = U\) for each \(m = 0, \ldots, 2\) and we obtain three 4-parameter families of the related Painlevé-type systems.

For \(m = 0\) we get

\[
h^A = \frac{1}{2} p^2 - a_4q^4 + a_3q^3 - (2a_4t + a_2)q^2 + (a_3t + a_1)q,
\]
while the dynamical part of the map (7.1) is

\[ a_4 = \frac{1}{2}b_2^2, \; a_3 = b_1b_2, \; a_2 = \frac{1}{2}b_1^2 + b_0b_2, \; a_1 = b_0b_1 + b - b_2. \]

For \( m = 1 \)

\[ h^A = -\frac{1}{2}qp^2 + a_3q^3 - (2a_3t + a_2)q^2 + (a_3t^2 + a_2t + a_1)q + a_{-1}q^{-1}, \]

\[ L = \begin{pmatrix} -p & x + q \\ 2a_4x^3 - 2(a_4q - a_3)x^2 + 2[a_4(q^2 + 2t) - a_3q + a_2]x \\ -2a_4(q^2 + 2tq) + 2a_4(q^2 + t) - 2a_2q + 2a_1 \end{pmatrix}, \]

\[ U = \begin{pmatrix} 0 \\ a_4x^2 - (2a_4q - a_3)x + a_4(3q^2 + 2t) - 2a_3q + a_2 \end{pmatrix}, \]

with the dynamical part of the map (7.1)

\[ a_3 = \frac{1}{2}b_2^2, \; a_2 = b_1b_2, \; a_1 = \frac{1}{2}b_1^2 + b_0b_2 + b - b_2, \; a_{-1} = \frac{1}{2}b_0. \]

For \( m = 2 \)

\[ h^A = \frac{1}{2}q^2p^2 - a_2e^{2t}q^2 + a_1e^tq + a_{-1}q^{-1} - a_{-2}q^{-2}, \]

\[ L = \begin{pmatrix} -q^2p & x + q \\ 2a_2e^{2t}x^3 - 2(a_2e^{2t}q - a_1e^t)x^2 \\ +2(a_{-1}q^{-1} - a_{-2}q^{-2} + \frac{1}{2}q^2p^2)x - q^2p^2 + 2a_{-2}q^{-1} \end{pmatrix}, \]

\[ U = \begin{pmatrix} 0 \\ a_2e^{2t}x^2 - (2a_2e^{2t}q - a_1e^t)x \\ +\frac{1}{2}q^2p^2 + 2a_2e^{2t}q^2 - a_1e^tq + a_{-1}q^{-1} - a_{-2}q^{-2} \end{pmatrix}, \]

with the dynamical part of the map (7.1)

\[ a_2 = \frac{1}{2}b_2^2, \; a_1 = b_1b_2 + b - b_2, \; a_{-1} = b_0b_1, \; a_{-1} = \frac{1}{2}b_0. \]

In particular the above formulas contain the isomonodromic Lax representation for Painlevé-I and Painlevé-II (for \( m = 0 \)), Painlevé-IV (for \( m = 1 \)) and Painlevé-III (for \( m = 2 \)) in the non-magnetic representation.
7.2 Two-dimensional systems

In the case of \( n = 2, g = a \) for each \( m = 0, \ldots, 3 \) and thus \( H^A_r = h^d_r \) and \( U_r = U_r \) for each \( m = 0, \ldots, 3 \). For each \( m \) we obtain a 5-parameter family of the related Painlevé-type systems in ordinary representation (as usual we denote \( p_0 = -q_1 p_1 - q_2 p_2 \)).

For \( m = 0 \) we get

\[
h^4_1 = p_1 p_2 + \frac{1}{4} q_1^2 p_2^2 + a_6(q_1^5 - 4q_1 q_2 + 3q_1 q_2^2) - a_5(q_1^4 - 3q_1^2 q_2 + q_2^3) + (4a_5 t_2 + a_4)(q_1^3 - 2q_1 q_2) \\
+ (2a_6 t_1 + 3a_5 t_2 + a_3)(q_2 - q_1^2) + (4a_6 t_2^2 + 2a_4 t_2 + a_5 t_1 + a_2) q_1,
\]

\[
h^4_2 = \frac{1}{2} p_1^2 + q_1 p_1 p_2 + \frac{1}{2} (q_1^2 - q_2) p_2^2 + p_2 + a_6(q_1 q_2 - 3q_1^2 q_2 + q_2^2) + a_5(2q_1 q_2 - q_1^2 q_2) + (4a_6 t_2 + a_4)(q_1^2 q_2 - q_2^2) \\
- (2a_6 t_1 + 3a_5 t_2 + a_3) q_2 q_1 + (4a_6 t_2 + a_4 t_1 + 2a_4 t_2 + a_2) q_2
\]

\[
L = \begin{pmatrix}
-p_2 x - q_1 p_2 - p_1 & x^2 + q_1 x + q_2 \\
L_{21} & p_2 x + q_1 p_2 + p_1
\end{pmatrix},
\]

where

\[
L_{21} = 2a_6 x^4 - 2(a_6 q_1 - a_5) x^3 + 2[a_6(q_1^2 - q_2 - 4t_2) - a_5 q_1 + a_4] x^2 \\
- 2[a_6(q_1^3 - 2q_1 q_2 + 4t_2 q_1 - 2t_1) - a_5(q_1^2 - q_2 + 3t_2) + a_4 q_1 - a_3] x \\
+ 2a_6(q_1^4 - 3q_1^2 q_2 + 4t_2 q_1^2 - 2t_1 q_1 + q_2^3 - 4t_2 q_2^2 + 4t_2 q_1^2) \\
+ 2a_4(q_2^2 - q_2 + 2t_2) - 2a_5 q_1 + 2a_2 - p_2^2,
\]

\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
a_6 x^2 - (a_6 q_1 - a_5) x + a_6(3q_1^2 - 2q_2 + 4t_2) - 2a_5 q_1 + a_4 & 0
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
0 & \frac{1}{2} x + \frac{1}{2} p_2 \\
a_6 x^3 - (a_6 q_1 - a_5) x^2 + [a_6(q_1^2 - 2q_2 + 4t_2) - a_5 q_1 + a_4] x \\
- a_6(q_1^3 - 4q_1 q_2 + 4t_2 q_1 - 2t_1) + a_5(q_1^2 - 2q_2 + 3t_2) - a_4 q_1 + a_3 & 1/2 p_2
\end{pmatrix}
\]

while the dynamical part of the map (7.1) becomes

\[
a_6 = \frac{1}{2} b_2^2, \quad a_5 = b_2 b_3, \quad a_4 = b_1 b_3 + \frac{1}{2} b_2^2, \quad a_3 = b_1 b_2 + b_0 b_3, \quad a_2 = b_0 b_2 + \frac{1}{2} b_1^2 - b_3 + b.
\]

For \( m = 1 \) we get

\[
h^4_1 = \frac{1}{2} p_1^2 - \frac{1}{2} q_2 p_2^2 - a_5(q_1^4 - 3q_1^2 q_2 + q_2^2) + (4a_5 t_2 + a_4)(q_1^3 - 2q_1 q_2) \\
+ [2a_5(3t_2^2 + t_1) + 3a_4 t_2 + a_3](q_2 - q_1^2) + [4a_5(t_3^2 + t_1 t_2) + a_4(3t_2^2 + t_1) + 2a_3 t_2 + a_2] q_1 + a_{-1} q_2^{-1},
\]

\[
h^4_2 = -q_2 p_1 p_2 - \frac{1}{2} q_1 q_2 p_2^2 + p_1 - a_5(q_1^2 q_2 - 2q_1 q_2^2) + (4a_5 t_2 + a_4)(q_1^2 q_2 - q_2^2) \\
- (2a_5(3t_2^2 + t_1) + 3a_4 t_2 + a_3) q_2 q_1 + (4a_5(t_3^2 + t_1 t_2) + a_4(3t_2^2 + t_1) + 2a_3 t_2 + a_2) q_2 + a_{-1} q_1 q_2^{-1}
\]

\[
L = \begin{pmatrix}
-p_1 x + q_2 p_2 & x^2 + q_1 x + q_2 \\
L_{21} & p_1 x - q_2 p_2
\end{pmatrix},
\]

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with
\[
L_{21} = 2a_5 x^4 + 2[a_5(4t_2 - q_1) + a_4]x^3 + 2[a_5(q_1^2 - q_2 - 4t_2q_1 + 6t_2^2 + 2t_1) + a_4(3t_2 - q_1) + a_3]x^2
- 2[a_5(q_1^3 - 2q_1q_2 - 4t_2q_1^2 + (6t_2^2 + 2t_1)q_1 + 4t_2q_2 - 4(t_2^3 + t_1t_2))
- a_4(q_1^2 - q_2 - 3t_2q_1 + 3t_2^2 + t_1) + a_3(q_1 - 2t_2) - a_2]x + 2a_1q_2^{-1} - q_2p_2^2.
\]

\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
a_5x^2 + [2a_5(2t_2 - q_1) + a_4]x + a_5(3q_1^2 - 2q_2 - 8t_2q_1 + 6t_2^2 + 2t_1) - a_4(2q_1 - 3t_2) + a_3 & 0
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
0 & \frac{1}{2} \\
-\frac{1}{2}p_1 & \frac{1}{2}x + \frac{1}{2}q_1
\end{pmatrix},
\]

and with the dynamical part of the map (7.1) given by
\[
a_5 = \frac{1}{2}b_3^2, \quad a_4 = b_2b_3, \quad a_3 = \frac{1}{2}b_2^2 + b_1b_3, \quad a_2 = b_0b_3 + b_1b_2 + \bar{b} - b_3, \quad a_{-1} = \frac{1}{2}b_0.
\]

For \( m = 2 \) we get
\[
h_1^a = -\frac{1}{2}q_1 p_1^2 - q_2 p_1 p_2 + a_4(q_1^3 - 2q_2 - 2t_1q_1)(q_2 - q_1^2) + (a_4t_2^2 + a_3t_1 + a_2)q_1 + a_{-1}e^{4t_2 q_2^{-1}} - a_{-2}e^{2t_2 q_1 q_2^{-2}}
\]
\[
h_2^a = -\frac{1}{2}q_2 p_1^2 + \frac{1}{2}q_2 p_2^2 + q_2 p_2 + a_4(q_1^3 q_2 - q_2^2) - (2a_4 t_1 + a_3) q_1 q_2 + (a_4 t_2^2 + a_3 t_1 + a_2) q_2 + a_{-1}e^{4t_2 q_1 q_2^{-1}} - a_{-2}e^{2t_2(q_1^2 - q_2)q_2^{-2}}
\]

\[
L = \begin{pmatrix}
-p_0 x + q_2 p_1 & x^2 + q_1 x + q_2 \\
L_{21} & p_0 x - q_2 p_1
\end{pmatrix}
\]

with
\[
L_{21} = 2a_4 x^4 + 2[a_4(2t_1 - q_1) + a_3]x^3 + 2[a_4(q_1^2 - q_2 - 2t_1q_1 + t_1^2) + a_3(t_1 - q_1) + a_2]x^2
+ 2(a_{-1}e^{4t_2 q_2^{-1}} - a_{-2}e^{2t_2 q_1 q_2^{-2}} - \frac{1}{2}q_1 p_1^2 - q_2 p_1 p_2)x + 2a_{-1}e^{2t_2 q_1 q_2^{-2}} - q_2 p_1^2,
\]

\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
a_4x^2 + [2a_4(t_1 - q_1) + a_3]x + a_4(3q_1^2 - 2q_2 - 4t_1q_1 + t_1^2) + a_3(t_1 - 2q_1) + a_2 & 0
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
0 & \frac{1}{2} \\
-\frac{1}{2}q_2 p_1 x^{-1} & -\frac{1}{2}q_2 x^{-1}
\end{pmatrix},
\]

and with the dynamical part of the map (7.1) in the form
\[
a_4 = \frac{1}{2}b_3^2, \quad a_3 = b_2b_3, \quad a_2 = b_1b_3 + \frac{1}{2}b_2^2 + \bar{b} - b_3, \quad a_{-1} = b_0b_3, \quad a_{-2} = \frac{1}{2}b_0^2.
\]
For $m = 3$ we get
\[
\begin{align*}
\hat{h}_1^A &= \frac{1}{2} p_0 + \frac{1}{2} q_2 p_2^2 + a_3 e^{2t_1} (q_2 - q_1^2) + a_2 e^{t_1} q_1 + (a_{-1} + a_{-2} t_2 + a_{-3} t_3) q_2^{-1} \\
&\hspace{1cm} - (a_{-2} + 2a_{-3} t_2) q_1 q_2^{-3} + a_{-3} (q_1^2 - q_2) q_2^{-3} \\
\hat{h}_2^A &= \frac{1}{2} q_1 p_2^2 + q_2^2 p_1 p_2 + q_2 p_1 - a_3 e^{2t_1} q_1 q_2 + a_2 e^{t_1} q_2 + (a_{-1} + a_{-2} t_2 + a_{-3} t_3) q_1 q_2^{-1} \\
&\hspace{1cm} - (a_{-2} + 2a_{-3} t_2) (q_1^2 - q_2) q_2^{-2} + a_{-3} q_1 (q_1^2 - 2q_2) q_2^{-3},
\end{align*}
\]
\[
L = \begin{pmatrix}
(q_1 p_0 + q_2 p_1) x + q_2 p_0 & x^2 + q_1 x + q_2 \\
L_{21} & -(q_1 p_0 + q_2 p_1) x - q_2 p_0
\end{pmatrix},
\]
\[
L_{21} = 2a_3 e^{2t_1} x^4 - 2(a_3 e^{2t_1} q_1 - a_2 e^{t_1}) x^3 \\
+ 2[a_{-1} q_2^{-2} + a_{-2} (t_2 q_2 - q_1) q_2^{-2} + a_{-3} ((t_2 q_2 - q_1)^2 - q_2) q_2^{-3} + \frac{1}{2} p_0^2 - \frac{1}{2} q_2 p_1^2] x^2 \\
+ 2[a_{-2} q_2^{-1} + a_{-3} (2t_2 q_2 - q_1) q_2^{-2} - \frac{1}{2} q_1 p_0^2 - q_2 p_1 x + 2a_{-3} q_2^{-1} - q_2^2],
\]
\[
U_1 = \begin{pmatrix}
0 \\
-\frac{1}{2} q_2 p_0 x^{-1} & \frac{1}{2} q_2 x^{-1}
\end{pmatrix},
\]
\[
U_2 = \begin{pmatrix}
a_3 e^{2t_1} x^2 - (2a_3 e^{2t_1} q_1 - a_2 e^{t_1}) x + a_3 e^{2t_1} (2q_1^2 q_2 - q_2) - a_2 e^{t_1} q_1 \\
+ a_{-1} q_2^{-1} - a_{-2} (q_1 - t_2 q_2) q_2^{-2} + a_{-3} ((t_2 q_2 - q_1)^2 - q_2) q_2^{-3} + \frac{1}{2} p_0^2 - \frac{1}{2} q_2 p_1^2 & 0
\end{pmatrix},
\]
and with the dynamical part of the map \((7.1)\)
\[
a_3 = \frac{1}{2} b_3^2, \quad a_2 = b_2 b_3 + b - b_3, \quad a_{-1} = b_0 b_2 + \frac{1}{2} b_1^2, \quad a_{-2} = b_0 b_1, \quad a_{-3} = \frac{1}{2} b_0^2.
\]

### 7.3 Three-dimensional systems

For each \(m = 0, \ldots, 4\) we obtain a 5-parameter family of Painlevé-type systems. We use here the already introduced abbreviation \(p_0 = -q_1 p_1 - q_2 p_2 - q_3 p_3\).

For \(m = 0\) we have \(a = \emptyset\) so \(\hat{h}_r^A = \hat{h}_r^B\) and \(\overline{U}_r = U_r\) for all \(r\) and our procedure yields
\[
\hat{h}_r^A = \mathcal{E}_r + a_8 V_6^{(8)} + a_7 V_7^{(7)} + (6 a_6 t_3 + a_6) V_r^{(6)} + (5 a_7 t_3 + 4 a_8 t_2 + a_5) V_r^{(5)} \\
+ [a_4 + 2 a_8 (6 t_3^2 + t_1) + 3 a_7 t_2 + 4 a_6 t_3] V_r^{(4)} + [a_3 + 12 a_8 t_3 t_2 + a_7 (t_1 + \frac{15}{2} t_2^2) + 2 a_6 t_2 + 3 a_5 t_3] V_r^{(3)}
\]
where the geodesic quasi-Stickel Hamiltonians \(\mathcal{E}_r\) are given by \((5.3)\) and basic ordinary potentials \(V_r^{(\alpha)}\) are given by \((2.4)\) and \((2.5)\). Then, the Lax matrix \(L\) is
\[
L = \begin{pmatrix}
-p_3 x^2 - (p_2 + q_1 p_3) x - (q_1 p_2 + q_2 p_3 + p_1) x^3 + q_1 x^2 + q_2 x + q_3 \\
L_{21} & -L_{11}
\end{pmatrix},
\]
while

\[
U_1 = \begin{pmatrix}
0 & 1 \\
\left(a_7 t_3 - 2q_1, a_6 \right) & 0 \\
\end{pmatrix},
\]

\[
U_2 = \begin{pmatrix}
-a_7 t_3 x^2 + [a_6(6t_3 - q_1) + a_7] x + a_8(6t_3 - 3V_4^{(4)} + q_2) - 2a_7 q_1 + a_6 & \frac{1}{2} \\
\frac{1}{2} x + \frac{1}{2} q_1 & 0 \\
\end{pmatrix},
\]

\[
U_3 = \begin{pmatrix}
\frac{1}{2} p_3 x - \frac{1}{2} (p_2 + q_1 p_3) & \frac{1}{2} x^2 + \frac{1}{2} q_1 x + \frac{1}{2} q_2 \\
(U_3)_{21} & \frac{1}{2} p_3 x + \frac{1}{2} (p_2 + q_1 p_3) \\
\end{pmatrix},
\]

\[
(U_3)_{21} = a_8 x^4 + (-a_8 q_1 + a_7) x^3 + [a_8(6t_3 - V_1^{(4)}) - a_7 q_1 + a_6] x^2 + [a_8(4t_2 - 6t_3 q_1 - V_1^{(5)}) - q_3] + a_7(5t_3 - V_1^{(4)}) - a_6 q_1 + a_5 + \frac{1}{2} p_3^2,
\]

The dynamical part of the map (7.1) becomes

\[
\begin{align*}
a_8 &= \frac{1}{2} b_2^2, & a_7 &= b_3 b_4, & a_6 &= \frac{1}{2} b_2^2 + b_2 b_4, & a_5 &= b_1 b_4 + b_3 b_3, \\
a_4 &= \frac{1}{2} b_2^2 + b_2 b_4 + b_1 b_4, & a_3 &= b_3 b_3 + b_1 b_2 - b_4 + b_5.
\end{align*}
\]

For \( m = 1 \), we have \( H_r^A = h_r^A, \overline{U}_r = U_r, r = 1, 2, H_3^A = h_3^A + t_2 h_4^A, \overline{U}_3 = U_3 + t_2 U_1 \) and our procedure yields

\[
\begin{align*}
h_r^A &= \mathcal{E}_r + a_7 V_1^{(4)} + \left[a_6 + 6a_7 t_3 \right] V_2^{(6)} + \left[a_5 + 5a_6 t_3 + a_7(4t_2 + 15t_3^2) \right] V_3^{(5)} + \left[a_4 + 4a_5 t_3 + a_6(3t_2 + 10t_3^2) \right] \\
&+ 2a_7(15t_3^2 + 8t_2 t_3 + 2t_1) q_1 + a_7(4t_2 + 15t_3^2) - \left[a_3 + 3a_4 t_3 + 2a_5(3t_2 + 3t_3^2) \right] + a_6 \left(t_1 + 10t_2 t_3 + 10t_3^2 \right) \\
&+ a_7(4t_3^2 + 6t_1 t_3 + 30t_2 t_3 + 15t_3^2) \right] V_4^{(1)} + a_7 V_2^{(1)} - 1,
\end{align*}
\]

where the geodesic quasi-Stückel Hamiltonians \( \mathcal{E}_r \) are given by (5.6) and basic ordinary potentials \( V_r^{(a)} \) are given by (2.4) and (2.5). In this case \( L \) is given by

\[
L = \begin{pmatrix}
-p_2 x^2 - (p_1 + q_1 p_2) x + q_3 p_3 & x^3 + q_1 x^2 + q_2 x + q_3 \\
L_{21} & p_2 x^2 + (p_1 + q_1 p_2) x - q_3 p_3 \\
\end{pmatrix}
\]

with

\[
L_{21} = 2a_7 x^5 + 2[a_7(6t_3 - q_1) + a_6] x^4 + 2[a_7(15t_3^2 + 4t_2 - 6t_3 q_1 - V_4^{(4)}) + a_6(5t_3 - q_1) + a_5] x^3 \\
+ 2[a_7(20t_3^2 + 18t_2 t_3 + 2t_1 - (15t_3^2 + 4t_2) q_1 - 6t_3 V_4^{(4)} - V_1^{(5)}) + a_6(10t_3^2 + 3t_2 - 5t_3 q_1 - V_1^{(4)}) \\
+ a_5(5t_3 - q_1) + a_4] x^2 + 2[a_7(15t_3^2 + 30t_2 t_3 + 6t_1 t_3 + 4t_2^2 - (20t_3^2 + 18t_2 t_3 + 2t_1) q_1 \\
- (15t_3^2 + 4t_2) V_4^{(4)} - 6t_3 V_4^{(5)} - V_1^{(6)}) + a_6(10t_3^2 + 10t_2 t_3 + t_1 - (10t_3^2 + 3t_2) q_1 - 5t_3 V_4^{(4)} - V_1^{(5)}) \\
+ a_5(6t_3 + 2t_2 - 4t_3 q_1 - V_4^{(1)}) + a_4(3t_3 - q_1) + a_3 - \frac{1}{2} p_3^2(2) x + 2a_1 q_3^{-1} - q_3 p_3^2,
\]

while

\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
\left(a_7(6t_3 - 2q_1) + a_6 \right) x + a_7(15t_3^2 + 4t_2 - 12t_3 q_1 + q_2 - 3V_4^{(4)}) + a_6(5t_3 - 2q_1) + a_5 & 0 \\
\end{pmatrix},
\]
\[
U_2 = \begin{pmatrix}
-\frac{1}{2}p_2 & \frac{1}{2}\lambda + \frac{1}{2}q_1 \\
\frac{1}{2}p_2 & \frac{1}{2}p_2
\end{pmatrix},
\]

\[
(U_2)_{21} = a_7x^3 + [a_7(6t_3 - q_1) + a_6]x^2 + [a_7(15t_3^2 + 4t_2 - 6t_3q_1 - q_2 - V_1^{(4)}) + a_6(5t_3 - q_1) + a_5]x
\]
\[+ a_7[20t_3^3 + 18t_2t_3 + 2t_1 - (15t_3^2 + 4t_2)q_1 - 6t_3(V_1^{(4)} + q_2) + 3q_3 + 2V_2^{(4)} - V_1^{(5)}] + a_6(10t_3^3 + 3t_2 - 5t_3q_1 - q_2 - V_1^{(4)}) + a_5(4t_3 - q_1) + a_4,
\]

\[
U_3 = \begin{pmatrix}
-\frac{1}{2}p_2x - \frac{1}{2}(p_1 + q_1p_2) & \frac{1}{2}\lambda^2 + \frac{1}{2}q_1\lambda + \frac{1}{2}q_2 \\
\frac{1}{2}p_2x + \frac{1}{2}(p_1 + q_1p_2) & \frac{1}{2}p_2
\end{pmatrix},
\]

\[
(U_3)_{21} = a_7x^4 + [a_7(6t_3 - q_1) + a_6]x^3 + [a_7(15t_3^2 + 4t_2 - 6t_3q_1 - V_1^{(4)}) + a_6(5t_3 - q_1) + a_5]x^2
\]
\[+ [a_7(20t_3^3 + 18t_2t_3 + 2t_1 - (15t_3^2 + 4t_2)q_1 - 6t_3V_1^{(4)} - q_3 - V_1^{(5)})] + a_6(10t_3^3 + 3t_2 - 5t_3q_1 - V_1^{(4)}) + a_5(4t_3 - q_1) + a_4\]
\[+ 4t_2 - (20t_3^3 + 18t_2t_3 + 2t_1)q_1 - (15t_3^2 + 4t_2)V_1^{(4)} - 6t_3(V_1^{(5)} + q_3) - 2V_2^{(4)} - V_1^{(6)}] + a_6(10t_3^3 + 10t_2t_3 + t_1 - (10t_3^2 + 3t_2)q_1 - 5t_3V_1^{(4)} - q_3 - V_1^{(5)})
\]
\[+ a_5(6t_3^3 + 2t_2 - 4t_3q_1 - V_1^{(4)}) + a_4(3t_3 - q_1) + a_3 - \frac{1}{2}p_2^2.
\]

The dynamical part of the map (7.1) is
\[a_7 = \frac{1}{2}b_1^2, \quad a_6 = b_3b_4, \quad a_5 = \frac{1}{2}b_2^2 + b_2b_3, \quad a_4 = b_1b_4 + b_2b_3,
\]
\[a_3 = \frac{1}{2}b_2^2 + b_0b_4 + b_1b_3 + b_4, \quad a_{-1} = \frac{1}{2}b_0^2.
\]

For \(m = 2\) we have \(a = g\) so \(H_r^a = H_r^g\) and \(U_r = U_r\) for all \(r\) and so
\[h_r^A = \mathcal{E}_r + a_6V_r^{(6)} + (a_5 + 4a_6t_2)V_r^{(5)} + [a_4 + 3a_5t_2 + 2a_6(t_1 + 3t_2^2)]V_r^{(4)}
\]
\[+ [a_3 + 2a_4t_2 + a_5(t_1 + 3t_2^2) + 4a_6(t_1t_2 + t_2^3)]V_r^{(3)} + a_{-1}e^{t_3}V_r^{(-1)} + a_{-2}e^{2t_3}V_r^{(-2)},
\]

where the geodesic quasi-Stückel Hamiltonians \(\mathcal{E}_r\) are given by (5.7) and basic ordinary potentials \(V_r^{(\alpha)}\) are given by (2.4) and (2.5). In this case \(L\) is given by
\[
L = \begin{pmatrix}
-p_1x^2 + (q_2p_2 + q_3p_3)x + q_3p_2 & x^3 + q_1x^2 + q_2x + q_3 \\
0 & -L_{21}
\end{pmatrix},
\]

where
\[
L_{21} = 2a_6x^5 + [a_6(4t_2 - q_1) + a_5]x^4 + 2[a_6(6t_2^2 + 2t_1 - 4t_2q_1 - V_1^{(4)}) + a_5(3t_2 - q_1) + a_4]x^3
\]
\[+ [a_6(4t_2^3 + 4t_1t_2 - (6t_2^2 + 2t_1)q_1 - 4t_2V_1^{(4)} - V_1^{(5)}) + a_5(3t_2^2 + t_1 - 3t_2q_1 - V_1^{(4)})] + a_4(2t_2 - q_1) + a_3\]
\[+ 2a_5(2t_2 - q_1 + a_3)]x^2 + 2[a_{-1}e^{t_3}V_1^{(-1)} + a_{-2}e^{2t_3}V_1^{(-2)} - \frac{1}{2}q_2p_2^2 - q_3p_2p_3|x
\]
\[+ 2a_{-2}e^{2t_3}V_1^{(-1)} - q_3p_2^2,
\]

while
\[
U_1 = \begin{pmatrix}
0 & \frac{1}{2} \\
a_6x^2 + [a_6(4t_2 - 2q_1) + a_5]x + a_6(6t_2^2 + 2t_1 - 8t_2q_1 + q_2 - 3V_1^{(4)}) + a_5(3t_2 - 2q_1) + a_4 & 0
\end{pmatrix},
\]

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\[ U_2 = \begin{pmatrix} -\frac{1}{2} p_1 & \frac{1}{2} \lambda + \frac{1}{2} q_1 \\ (U_2)_{21} & \frac{1}{2} p_1 \end{pmatrix}, \]

\[
(U_2)_{21} = a_6 x^3 + [a_6 (4t_2 - q_1) + a_5] x^2 + [a_6 (6t_2^2 + 2t_1 - 4t_2 q_1 - q_2 - V_1^{(4)}) + a_5 (3t_2 - q_1) + a_4] x
\]
\[ + a_6 [4t_3^2 + 4t_1 t_2 - (6t_2^2 + 2t_1) q_1 - 4t_2 (q_2 + V_1^{(4)}) + q_3 - 2V_2^{(4)} - V_1^{(5)}] \]
\[ + a_5 (3t_2^2 + t_1 - 3t_2 q_1 - q_2 - V_1^{(4)}) + a_4 (2t_1 - q_1) + a_3, \]

\[
U_3 = \begin{pmatrix} -\frac{1}{2} q_3 p_2 x^{-1} & -\frac{1}{2} q_3 x^{-1} \\ -a_6 g_3 x + 2a_6 g_3 (q_1 - 2t_2) - a_5 g_3 - [a_{-2} e^{2t_2} V_1^{(-1)} - \frac{1}{2} q_3 g_3^2] x^{-1} & -\frac{1}{2} g_3^2 x^{-1} \end{pmatrix}. \]

The dynamical part of the map (7.1) is

\[
a_6 = \frac{1}{2} b_4^2, \quad a_5 = b_3 b_4, \quad a_4 = \frac{1}{2} b_3^2 + b_2 b_4, \quad a_3 = b_1 b_4 + b_2 b_3 - b_4 + b, \quad a_{-1} = b_0 b_1, \quad a_{-2} = \frac{1}{2} b_0^2. \]

For \( m = 3 \), we have \( H_r^A = h_r^A, \overline{U}_r = U_r, r = 1, 3, \) \( H_r^A = h_r^A + t_3 h_r^A, \overline{U}_r = U_r + t_3 U_3 \) and our procedure yields

\[
h_r^A = \mathcal{E}_r + a_5 V_r^{(5)} + (a_4 + 2a_5 t_1) V_r^{(4)} + (a_3 + a_4 t_1 + a_5 t_2^2) V_r^{(3)}
\]
\[ + [a_{-1} e^{2t_2} + a_{-2} e^{2t_2} + t_3 e^{3t_2}] V_r^{(-1)} + (a_{-2} e^{3t_2} + 2a_{-3} t_3 e^{4t_2}) V_r^{(-2)} + a_{-3} e^{4t_2} V_r^{(-3)}, \]

where the geodesic quasi-Stückel Hamiltonians \( \mathcal{E}_r \) are given by (5.8) and basic ordinary potentials \( V_r^{(a)} \) are given by (2.4) and (2.5). In this case \( L \) is given by

\[
L = \begin{pmatrix} -p_0 x^2 + (q_2 p_1 + q_3 p_2) x + q_3 p_1 & x^3 + q_1 x^2 + q_2 x + q_3 \\ L_{21} & -L_{11} \end{pmatrix}, \]

with

\[
L_{21} = 2a_5 x^5 + 2[a_5 (5t_1 - q_1) + a_4)] x^4 + 2[a_5 (t_1^2 - 2t_1 q_1 - V_1^{(4)}) + a_4 (t_1 - q_1) + a_3] x^3
\]
\[ + 2[a_{-1} e^{2t_2} V_1^{(-1)} + a_{-2} e^{2t_2} (t_3 e^{3t_2}) V_1^{(-1)}] + e^{3t_2} V_1^{(-2)} + a_{-3} e^{4t_2} (t_3 V_1^{(-1)} + 2t_3 V_1^{(-2)}) + V_1^{(-3)} \]
\[ + p_0 p_1 + \frac{1}{2} q_1 p_2^2 - \frac{1}{2} q_3 p_3^2] x^2 + 2[a_{-2} e^{3t_2} V_1^{(-1)} + a_{-3} e^{4t_2} (t_3 V_1^{(-1)} + V_1^{(-2)}) - \frac{1}{2} q_2 p_1^2 - q_3 p_4 p_2] x
\]
\[ + 2a_{-3} e^{4t_2} V_1^{(-1)} - q_3 p_1^2, \]

while

\[
U_1 = \begin{pmatrix} 0 \\ a_5 x^2 + [2a_5 (t_1 - q_1) + a_4] x + a_5 (t_1^2 - 4t_1 q_1 + q_2 - 3V_1^{(4)}) + a_4 (t_1 - 2q_1) + a_3 \end{pmatrix},
\]

\[
U_2 = \begin{pmatrix} -\frac{1}{2} (q_2 p_1 + q_3 p_2) x^{-1} - \frac{1}{2} q_3 p_1 x^{-2} & -\frac{1}{2} q_2 x^{-1} - \frac{1}{2} q_3 x^{-2} \\ (U_2)_{21} & \frac{1}{2} (q_2 p_1 + q_3 p_2) x^{-1} + \frac{1}{2} q_3 p_1 x^{-2} \end{pmatrix},
\]

\[
(U_2)_{21} = -a_5 q_2 x - a_5 (2t_1 q_2 - 2q_1 q_2 + q_3) - a_4 q_2 - [a_{-2} e^{3t_2} V_1^{(-1)} + a_{-3} e^{4t_2} (2t_3 V_1^{(-1)} + V_1^{(-2)}) - \frac{1}{2} q_2 p_1^2 - q_3 p_4 p_2] x^{-1} - (a_{-3} e^{4t_2} V_1^{(-1)} - \frac{1}{2} q_3 p_1^2) x^{-2}. \]
The dynamical part of the map (7.1) is

\[
U_3 = \begin{pmatrix}
-\frac{1}{2}q_3p_1x^{-1} & -\frac{1}{2}q_3x^{-1} \\
-\frac{1}{2}a_3q_3x - [2a_3(t_1 - q_1) + a_4]q_3 - (a_{-3}e^{4t_2}V_1^{(-1)} - \frac{1}{2}q_3p_1^2)x^{-1} & \frac{1}{2}q_3p_1x^{-1}
\end{pmatrix}.
\]

The dynamical part of the map (7.1) is

\[
a_5 = \frac{1}{2}b_4^2, \quad a_4 = b_3b_4, \quad a_3 = \frac{1}{4}b_3^2 + b_3b_4 + \bar{b} - b_4, \quad a_{-1} = \frac{1}{2}b_4^2 - b_0b_1 + b_0b_2, \\
a_{-2} = b_0b_1, \quad a_{-3} = \frac{1}{2}b_0^2.
\]

Finally, for \(m=4\) we have \(a = \varrho\) so \(H_r^A = h_r^A\) and \(\overline{U}_r = U_r\) for all \(r\) and our procedure yields

\[
h_r^A = E_r + a_4e^{2t_1}V_r^{(4)} + (a_3e^{t_1} + a_4e^{t_1})V_r^{(3)} + \left(a_{-1} + 2a_{-2}t_2 + a_{-3}t_3 + 3t_2^2 + 4a_{-4}(t_2t_3 + t_2^2)\right)V_r^{(-1)} + \left[a_{-2} + 3a_{-3}t_2 + 2a_{-4}(t_3 + 3t_2^2)\right]V_r^{(-2)} + \left(a_{-3} + 4a_{-4}t_2\right)V_r^{(-3)} + a_{-4}V_r^{(-4)},
\]

where geodesic quasi-Stäckel Hamiltonians \(E_r\) are given by (5.9) and basic ordinary potentials \(V_r^{(\alpha)}\) are given by (2.4) and (2.5). The matrix \(L\) is given by

\[
L = \begin{pmatrix}
(q_1p_0 + q_2p_1 + q_3p_2)x^2 + (q_2p_0 + q_3p_1)x + q_3p_0 & x^3 + q_1x^2 + q_2x + q_3 \\
L_{21} & -L_{11}
\end{pmatrix},
\]

with

\[
L_{21} = 2a_4e^{2t_1}x^5 + 2[a_4(e^{t_1} - 2e^{t_1}q_1) + a_3e^{t_1}]x^4 + 2[a_{-1}V_1^{(-1)} + a_{-2}(2t_2V_1^{(-1)} + V_1^{(-2)})
\]

\[
+ a_{-3}(3t_2^2 + t_3)V_1^{(-1)} + 3t_2V_1^{(-2)} + V_1^{(-3)}) + a_{-4}(4t_2^3 + 2t_3)V_1^{(-1)} + (6t_2^2 + 2t_3)V_1^{(-2)} + 4t_2V_1^{(-3)}
\]

\[
+ V_1^{(-4)}) + \frac{1}{2}p_0^2 - \frac{1}{2}q_3p_0^2q_1x^3 + 2[a_{-2}V_1^{(-1)} + a_{-3}(3t_2V_1^{(-1)} + V_1^{(-2)}) + a_{-4}(6t_2^2 + 2t_3)V_1^{(-1)}
\]

\[
+ 4t_2V_1^{(-2)} + V_1^{(-3)}) - \frac{1}{2}q_3p_0^2q_2 - q_2p_0p_1 - q_3p_0p_2 - \frac{1}{2}q_3p_0^2[x^2 + 2[a_{-1}V_1^{(-1)} + a_{-4}(4t_2V_1^{(-1)} + V_1^{(-2)})
\]

\[
- \frac{1}{2}q_3p_0^2]q_1x + 2a_{-1}V_1^{(-1)} - q_3p_0^2
\]

while

\[
U_1 = \begin{pmatrix} 0 & \frac{1}{2} \\ (U_1)_{21} & 0 \end{pmatrix},
\]

\[
(U_1)_{21} = a_4e^{2t_1}x^2 + [a_4(e^{t_1} - 2e^{t_1}q_1) + a_3e^{t_1}]x - a_4e^{t_1}(2e^{t_1}V_1^{(4)} + e^{t_1}q_2 - q_1) - a_3e^{t_1}q_1
\]

\[
+ a_{-1}V_1^{(-1)} + a_{-2}(2t_2V_1^{(-1)} + V_1^{(-2)}) + a_{-3}(3t_2^2 + t_3)V_1^{(-1)} + 3t_2V_1^{(-2)} + V_1^{(-3)})
\]

\[
+ a_{-4}(4t_2^3 + 2t_3)V_1^{(-1)} + (6t_2^2 + 2t_3)V_1^{(-2)} + 4t_2V_1^{(-3)} + V_1^{(-4)}) + \frac{1}{2}p_0^2 - \frac{1}{2}q_3p_0^2 - q_3p_0p_2;
\]

\[
U_2 = \begin{pmatrix}
-\frac{1}{2}(q_2p_0 + q_3p_1)x^{-1} + \frac{1}{2}q_3p_0x^{-2} & \frac{1}{2}q_2x^{-1} - \frac{1}{2}q_3x^{-2} \\
(U_2)_{21} & \frac{1}{2}(q_2p_0 + q_3p_1)x^{-1} - \frac{1}{2}q_3p_0x^{-2}
\end{pmatrix},
\]

\[
(U_2)_{21} = -a_4e^{2t_1}q_2x + a_4e^{t_1}(2e^{t_1}q_1 + q_2 - e^{t_1}q_3) - q_2 - a_3e^{t_1}q_2 + [-a_{-3}V_1^{(-1)} - a_{-4}(4t_2V_1^{(-1)} + V_1^{(-2)})
\]

\[
+ \frac{1}{2}q_3p_0^2 + q_3p_0p_1)x^{-1} + (-a_4V_1^{(-1)} + \frac{1}{2}q_3p_0^2)x^{-2},
\]

\[
U_3 = \begin{pmatrix}
-\frac{1}{2}q_3p_0x^{-1} & -\frac{1}{2}q_3x^{-1} \\
-a_4e^{2t_1}q_3 + a_4e^{t_1}(2e^{t_1}q_1 + q_2 - e^{t_1}q_3 + (-a_{-4}V_1^{(-1)} + \frac{1}{2}q_3p_0^2)x^{-1} & \frac{1}{2}q_3p_0x^{-1}
\end{pmatrix}.
\]

The dynamical part of the map (7.1) reads

\[
a_4 = \frac{1}{2}b_4^2, \quad a_3 = -\frac{1}{4}b_4^2 + b_3b_4 + \bar{b} - b_4, \quad a_{-1} = b_0b_3 + b_1b_2, \quad a_{-2} = \frac{1}{2}b_4^2 + b_0b_2, \\
a_{-3} = b_0b_1, \quad a_{-4} = \frac{1}{2}b_0^2.
\]
7.4 Painlevé $P_l - P_{IV}$ hierarchies.

The above list gives us a possibility of constructing the complete Painlevé $P_l - P_{IV}$ hierarchies (up to some rescaling, see Part I, Section 9), in the following way. Fixing $m = 0$, choosing $a_{2n+1} = -1$ and remaining $a_i = 0$ and letting $n$ vary we obtain the $P_l$ hierarchy with the first three members given by Hamiltonians

$$n = 1 : \quad H = \frac{1}{2} p^2 - q^3 - tq$$

$$n = 2 : \quad H_1 = p_1 p_2 + \frac{1}{2} q_1 p_2^2 + \frac{1}{4} (q_1^2 - 4 q_1 q_2 + 3 q_1 q_2^2) + t_2 (q_1^3 - 2 q_1 q_2) + \left( \frac{1}{2} t_1 - \alpha \right) (q_2 - q_1^2) + t_2^2 q_1$$

$$H_2 = \frac{1}{2} p_3^2 + q_1 p_1 p_2 + \frac{1}{2} (q_1^2 - q_2) p_2^2 + p_2 + \frac{1}{4} (q_1^2 q_2 - 3 q_1^2 q_2 + q_2^3) + t_2 (q_1^3 q_2 - q_2^2) - \left( \frac{1}{2} t_1 - \alpha \right) q_2 q_1 + t_2^2 q_2$$

$$n = 3 : \quad H_r = E_r - V_r^{(7)} - 5 t_3 V_r^{(5)} - 3 t_2 V_r^{(4)} - \left( t_1 + \frac{15}{2} t_3^2 \right) V_r^{(3)}, \quad r = 1, 2, 3$$

where $E_r$ are given by (5.3).

Further, fixing $m = 0$ but choosing $a_{2n+2} = \frac{1}{2}$, $a_{2n-1} = -\alpha$ and remaining $a_i = 0$ and letting $n$ vary we obtain the $P_{II}$ hierarchy with the first three members given by

$$n = 1 : \quad H = \frac{1}{2} p^2 - \frac{1}{4} q^4 - \frac{1}{4} t q^2 - \alpha q$$

$$n = 2 : \quad H_1 = p_1 p_2 + \frac{1}{2} q_1 p_2^2 + \frac{1}{4} (q_1^2 - 4 q_1 q_2 + 3 q_1 q_2^2) + t_2 (q_1^3 - 2 q_1 q_2) + \left( \frac{1}{2} t_1 - \alpha \right) (q_2 - q_1^2) + t_2^2 q_1$$

$$H_2 = \frac{1}{2} p_3^2 + q_1 p_1 p_2 + \frac{1}{2} (q_1^2 - q_2) p_2^2 + p_2 + \frac{1}{4} (q_1^2 q_2 - 3 q_1^2 q_2 + q_2^3) + t_2 (q_1^3 q_2 - q_2^2) - \left( \frac{1}{2} t_1 - \alpha \right) q_2 q_1 + t_2^2 q_2$$

$$n = 3 : \quad H_r = E_r + \frac{1}{4} V_r^{(8)} + \frac{3}{2} t_3 V_r^{(6)} + (t_2 - \alpha) V_r^{(5)} + (3 t_3^2 + \frac{1}{2} t_1) V_r^{(4)} + (3 t_2 t_3 - 3 \alpha t_3) V_r^{(3)}, \quad r = 1, 2, 3$$

with the same $E_r$ given by (5.3).

For $m = 1$ we fix $a_{2n+1} = 1$, we let $a_{-1} = \alpha$ and $a_{2n-1} = \beta$ be free, the remaining $a_i = 0$ and we let $n$ vary. This way we obtain the $P_{IV}$ hierarchy with the first three members given by

$$n = 1 : \quad H = -\frac{1}{2} q p^2 + q^3 - 2 t q^2 + (t^2 + \beta) q + \alpha q^{-1}$$

$$n = 2 : \quad H_1 = \frac{1}{2} p_2^2 - \frac{1}{2} q_2 p_2^2 - (q_1^3 - 3 q_1^2 q_2 + q_2^2) + 4 t_2 (q_1^3 - 2 q_1 q_2) + (2 (3 t_2^2 + t_1) + \beta) (q_2 - q_1^2)$$

$$+ 4 (t_3^2 + t_1 t_2) + 2 \beta t_2) q_1 + \alpha q_2^{-1}$$

$$H_2 = -q_2 p_1 p_2 - \frac{1}{2} q_1 q_2 p_2^2 + p_1 - (q_1^3 - 2 q_1 q_2^2) + 4 t_2 (q_1^3 q_2 - q_2^2)$$

$$- 2 (3 t_2^2 + t_1) + \beta) q_1 q_2 + (4 (t_3^2 + t_1 t_2) + 2 \beta t_2) q_2 + \alpha q_1 q_2^{-1}$$

$$n = 3 : \quad h_r = E_r + t_3 V_r^{(7)} + 6 t_3 V_r^{(6)} + (\beta + 4 t_2 + 15 t_3^2) V_r^{(5)} + [4 \beta t_3 + 2 (t_1 + 9 t_2 t_3 + 10 t_3^2)] V_r^{(4)}$$

$$+ [2 \beta (t_2 + 3 t_3^2) + (4 t_2^2 + 6 t_1 t_3 + 30 t_2 t_3^2 + 15 t_3^4)] V_r^{(3)} + \alpha V_r^{(-1)} + \epsilon v_r^{(-1)}, \quad r = 1, 2, 3$$

and $H_r = h_r$ , $r = 1, 2$, $H_3 = h_3 + t_2 h_1$ and where $E_r$ are given by (5.6).

Finally, for $m = 2$ we let $a_{2n} = \alpha$, $a_{2n-1} = \beta$, $a_{-1} = \gamma$ and $a_{-2} = \delta$ be free, the remaining $a_i = 0$ and
we let \( n \) vary. This leads to the \( P_{III} \) hierarchy with its first three members given by:

\[
\begin{align*}
n = 1 & : H = \frac{1}{2} q^2 p^2 - \alpha e^{2t} q^2 + \beta e^t q + \gamma q^{-1} - \delta q^{-2} \\
n = 2 & : H_1 = -\frac{1}{2} q_1 p_1^2 - q_2 p_1 p_2 + \alpha (q_1^2 - 2 q_1 q_2) + (2 \alpha t_1 + \beta)(q_2 - q_1^2) + (\alpha t_1^2 + \beta t_1) q_1 \\
& + \gamma e^{t_2} q_2^{-1} - \delta e^{2t_2} q_2 q_2^{-2} \\
H_2 &= -\frac{1}{2} q_2 p_1^2 + \frac{1}{2} q_2^2 p_2^2 + q_2 p_2 + \alpha (q_2^2 q_2 - q_2^3) - (2 \alpha t_1 + \beta) q_1 q_2 + (\alpha t_1^2 + \beta t_1) q_2 \\
& + \gamma e^{t_2} q_1 q_2^{-1} - \delta e^{2t_2} (q_1^2 - q_2) q_2^{-2} \\
n = 3 & : H_r = E_r + \alpha V_r^{(6)} + (\beta + 4 \alpha t_2) V_r^{(5)} + [3 \beta t_2 + 2 \alpha (t_1 + 3 t_2^2)] V_r^{(4)} \\
& + [\beta (t_1 + 3 t_2^2) + 4 \alpha (t_1 t_2 + t_2^3)] V_r^{(3)} + \gamma e^{t_3} V_r^{(-1)} + \delta e^{2t_3} V_r^{(-2)}, \quad r = 1, 2, 3
\end{align*}
\]

and where \( E_r \) are given by (5.7). The basic separable potentials \( V_r^{(\alpha)} \) in the formulas above are given by (2.4)-(2.5).

According with Remark 3 in Part I we can rescale the time \( t_{n+2-m} \) in \( P_{III} \) above through \( t_{n+2-m}' = \exp(t_{n+2-m}) \) which turns the \( P_{III} \)-systems to

\[
\begin{align*}
n = 1 & : H = \frac{1}{t} \left( \frac{1}{2} t^2 p^2 - \alpha t^2 q^2 + \beta t q + \gamma q^{-1} - \delta q^{-2} \right) \\
n = 2 & : H_1 = -\frac{1}{2} q_1 p_1^2 - q_2 p_1 p_2 + \alpha (q_1^2 - 2 q_1 q_2) + (2 \alpha t_1 + \beta)(q_2 - q_1^2) + (\alpha t_1^2 + \beta t_1) q_1 \\
& + \gamma t_2 q_2^{-1} - \delta t_2^2 q_2 q_2^{-2} \\
H_2 &= \frac{1}{t} \left( -\frac{1}{2} q_2 p_1^2 + \frac{1}{2} q_2^2 p_2^2 + q_2 p_2 + \alpha (q_2^2 q_2 - q_2^3) - (2 \alpha t_1 + \beta) q_1 q_2 + (\alpha t_1^2 + \beta t_1) q_2 \\
& + \gamma t_2 q_1 q_2^{-1} - \delta t_2^2 (q_1^2 - q_2) q_2^{-2} \right) \\
n = 3 & : H_r = E_r + \alpha V_r^{(6)} + (\beta + 4 \alpha t_2) V_r^{(5)} + [3 \beta t_2 + 2 \alpha (t_1 + 3 t_2^2)] V_r^{(4)} \\
& + [\beta (t_1 + 3 t_2^2) + 4 \alpha (t_1 t_2 + t_2^3)] V_r^{(3)} + \gamma e^{t_3} V_r^{(-1)} + \delta e^{2t_3} V_r^{(-2)}, \quad r = 1, 2, 3
\end{align*}
\]

Note that while the Frobenius condition (3.12) does not change after the transformation \( t_{n+2-m}' = \exp(t_{n+2-m}) \), in the Lax formulation we have to replace \( \frac{d}{dt_{n+2-m}} \) in (6.8) with \( \frac{d}{dt_{n+2-m}'} \). For higher \( m \) the related Painlevé hierarchies start from \( n \) higher than one. Besides, the hierarchies \( P_{II} - P_{IV} \) can also be written in the magnetic representation using the multi-time canonical transformation (6.3)-(6.6).

This can’t be done for \( P_1 \) which has no magnetic representation, due to the fact that the map (7.1) is not bijective.

### 8 Conclusions

In this article we constructed the isomonodromic Lax representations for all Frobenius integrable systems constructed in Part I, thus proving that they are of Painlevé-type. We also proposed, based on our construction, complete (in the sense explained in Introduction) Painlevé \( P_1 - P_{IV} \) hierarchies. An interesting question, which will be a subject of separate research, is to what extent these hierarchies are related to
Painlevé hierarchies that can be found in literature. We may expect that the hierarchies existing in literature are sub-hierarchies within our scheme, which is thus more general, as our hierarchies contain - for each fixed number \( n \) of degrees of freedom - \( n \) different systems satisfying the Frobenius integrability condition.

**Appendix**

In order to prove Theorem 3 we need a couple of lemmas. In what follows all formulas are calculated in Viète coordinates (4.8). Below, to shorten the notations, we write simply \( L_{ij} \) although in reality \( L_{11}, L_{21} \) and \( L_{22} \) depend in general on \( x, t, q \) and \( p \) while \( L_{12} \) depends on \( x \) and \( q \).

**Lemma 4** Denote \( q_0 = 1 \) and \( p_0 = - \sum_{i=1}^{n} q_i p_i \) (this follows formally from (4.8) for \( i = 0 \)). The entries of the Lax matrix \( L(x) \) in (4.3) are given explicitly as follows:

\[
L_{11}(x) = v(x) - \varphi(x) = - \sum_{k=0}^{n-1} M_{n-k}^{(m)} x^k - \sum_{\gamma=0}^{n+1} d_\gamma (t_1, \ldots, t_n) x^\gamma, \quad (A.1)
\]

where \( L_{12}(x) \) is given by

\[
L_{12}(x) = u(x) = \sum_{k=0}^{n} q_{n-k} x^k \quad (A.2)
\]

and finally

\[
L_{21}(x) = -x^m \left[ \frac{v^2(x) x^{-m}}{u(x)} \right] + 2x^m \left[ \frac{v(x) \varphi(x) x^{-m}}{u(x)} \right] + 2x^m \left[ \frac{e(t)x^n}{u(x)} \right] +

= -w_1(x) + 2w_2(x) + 2w_3(x), \quad (A.3)
\]

where for \( m = 0, \ldots, n + 1 \)

\[
w_1(x) = \sum_{s=0}^{m-1} \left( \sum_{k=n-s}^{n} q_{k} \sum_{j=n-m+1}^{k+s-m+1} p_j p_{n+k+s-2m+2-j} \right) x^s \quad (A.4)
\]

\[
+ \sum_{s=m}^{n-2} \left( \sum_{k=s-m+2}^{n-s-2} q_{k} \sum_{j=k+s-m}^{n-m} p_j p_{n+k+s-2m+2-j} \right) x^s,
\]

\[
w_2(x) = \sum_{s=0}^{m-1} \left( \sum_{k=0}^{s} d_k p_{n+1-m-s-k} \right) x^s - \sum_{s=m}^{n+1} \left( \sum_{k=s+1}^{n+1} d_k p_{n+1-m-s-k} \right) x^s, \quad (A.5)
\]

\[
w_3(x) = e(t)x^n, \quad m = 0, \ldots, n + 1. \quad (A.6)
\]

Note that the above formulas mean that \( w_3 \) depends on times \( t_r \) only, \( w_2 \) depends both explicitly on times \( t_r \) and on coordinates \( (q, p) \) on manifold \( \mathcal{M} \), while \( w_1 \) depends only on coordinates \( (q, p) \) on manifold (of course, they all also depend on the spectral parameter \( x \)).

**Proof.** Formula (A.6) is obvious. We will thus only prove (A.5) as the proof of (A.4) is similar. From (2.9), (2.10) and (3.2) we have that

\[
v(x) = \sum_{i=1}^{n} \frac{u(x)}{x - \lambda_i} \frac{\lambda_i^m \mu_i}{\Delta_i}, \quad \varphi(x) = \sum_{k=0}^{n+1} d_k x^k.
\]

Using these formulas we can write \( w_2(x) \) in the following form

\[
w_2(x) = x^m \left[ \frac{v(x) \varphi(x) x^{-m}}{u(x)} \right] + x^m \sum_{k=0}^{n+1} \sum_{i=1}^{n} d_k \left[ \frac{x^{k-m}}{x - \lambda_i} \right] \frac{\lambda_i^m \mu_i}{\Delta_i}.
\]
Further

\[
x^m \left[ \frac{x^{k-m}}{x - \lambda_i} \right] = x^m \frac{x^{k-m} - \lambda_i^{k-m}}{x - \lambda_i} = \frac{x^k}{x - \lambda_i} - x^m \lambda_i^{k-m}
\]

\[
= \sum_{s=0}^{k-1} \lambda_i^{k-s-1} x^s + \frac{\lambda_i^k}{x - \lambda_i} - \left( \sum_{s=0}^{m-1} \lambda_i^{m-s-1} x^s + \frac{\lambda_i^m}{x - \lambda_i} \right) \lambda_i^{k-m}
\]

\[
= \sum_{s=0}^{k-1} \lambda_i^{k-s-1} x^s - \sum_{s=0}^{m-1} \lambda_i^{k-s-1} x^s = \begin{cases} 
- \sum_{s=k}^{m-1} \lambda_i^{k-s-1} x^s & \text{for } k < m \\
\sum_{s=m}^{k-1} \lambda_i^{k-s-1} x^s & \text{for } k > m \\
0 & \text{for } k = m
\end{cases}
\]

From this result we obtain

\[
w_2(x) = - \sum_{k=0}^{m-1} \left( \sum_{i=1}^{n} \sum_{s=k}^{m-1} d_k \lambda_i^{m-k-1} x^s \frac{\lambda_i^m \mu_i}{\Delta_i} \right) + \sum_{k=m+1}^{n} \left( \sum_{i=1}^{n} \sum_{s=m}^{k-1} d_k \lambda_i^{m-k-1} x^s \frac{\lambda_i^m \mu_i}{\Delta_i} \right)
\]

\[
= - \sum_{k=0}^{m-1} m-1 \sum_{s=k}^{m-1} d_k \sum_{i=1}^{n} \frac{\lambda_i^{m-k-1} \mu_i}{\Delta_i} x^s + \sum_{k=m+1}^{n} \sum_{s=m}^{k-1} d_k \sum_{i=1}^{n} \frac{\lambda_i^{m-k-1} \mu_i}{\Delta_i} x^s.
\]

After passing to Viète coordinates (4.8) we receive

\[
w_2(x) = \sum_{k=0}^{m-1} \sum_{s=k}^{m-1} d_k p_{n-m-k+s+1} x^s - \sum_{k=m+1}^{n} \sum_{s=m}^{k-1} d_k p_{n-m-k+s+1} x^s
\]

and interchanging the order of summation in the above equation we finally obtain (A.5). ■

We will now calculate the entries of \( U_r \) in (4.6) and (4.7). For \( r \in I_1^n \), directly from (4.6) we obtain

\[
(U_r)_{ij}(x; t) = \frac{1}{2} \left[ \frac{u(x)}{x^{n-r+1}} L_{ij}(x; t) + L_{ij}(x; t) \right] = \frac{1}{2} \left[ x^{n-r+1} \frac{u(x)}{x^{n-r+1} u(x)} \right] + \frac{1}{2} \left[ L_{ij}(x; t) \right] + \frac{1}{2} \left[ L_{ij}(x; t) \right] + \frac{1}{2} \left[ L_{ij}(x; t) \right] + \frac{1}{2} \left[ L_{ij}(x; t) \right], \quad (A.7)
\]

where

\[
r(x) = u(x) \mod x^{n-r+1} = \rho_r x^{n-r} + \cdots + \rho_n
\]

is a polynomial of degree \( n - r \). Thus, if \( L_{ij}(x; t) \) is a polynomial of degree less than \( n + 1 \) then

\[
(U_r)_{ij}(x; t) = \frac{1}{2} \left[ L_{ij}(x; t) \right] + . \quad (A.9)
\]
For $r \in I_2^m$, directly from (4.7) we obtain

$$(U_r)_{ij}(x; t) = \frac{1}{2} \left[ \frac{u(x)}{x^{n-r+1}} - L_{ij}(x; t) \right] = -\frac{1}{2} \left[ \frac{u(x)}{x^{n-r+1}} L_{ij}(x; t) - \frac{u(x)}{x^{n-r+1}} + L_{ij}(x; t) \right]_+$$

$$= -\frac{1}{2} \left[ \frac{u(x)x^{-n+r-1}L_{ij}(x; t)}{u(x)} \right] + \frac{1}{2} \left[ \frac{u(x)}{x^{n-r+1}} L_{ij}(x; t) \right]_+$$

$$= -\frac{1}{2} \left[ L_{ij}(x; t) \right]+ \frac{1}{2} \left[ r(x)L_{ij}(x; t) \right] +$$

$$= -\frac{1}{2} \left[ L_{ij}(x; t) \right]+ \frac{1}{2} \left[ r(x)L_{ij}(x; t) \right] +$$

$$= \frac{L_{ij}(x; t)}{x^{n-r+1}} - \frac{1}{2} \left[ r(x)L_{ij}(x; t) \right]_+$$

and again, if $L_{ij}(x; t)$ is a polynomial of degree less than $n + 1$ then

$$(U_r)_{ij}(x; t) = -\frac{1}{2} \left[ L_{ij}(x; t) \right]_-$$. (A.10)

The above results make it possible to calculate the entries of $U_r$ in Viète coordinates.

**Lemma 5** The entries of $U_r$ in (4.6) and (4.7) are as follows.

$$(U_r)_{12}(x) = \frac{1}{2} \left[ \frac{u(x)}{x^{n+1-r}} \right] = \frac{1}{2} \left[ \frac{u(x)}{x^{n+1-r}} \right]_+ - \frac{1}{2} \left[ q_r - \frac{1}{2} \sum_{k=0}^{r-1} q_{r-k-1} x^k \right] \quad r \in \{1\} \cup I_1^m, \quad (A.12)$$

$$(U_r)_{12}(x) = -\frac{1}{2} \left[ \frac{u(x)}{x^{n+1-r}} \right]_+ - \frac{1}{2} \left[ \phi(x) \right]_{x^{n-r+1}} + \frac{1}{2} \left[ \frac{u(x)\phi(x)}{u(x)} \right]_+ = \frac{1}{2} a_r(x) - \frac{1}{2} b_r(x) + \frac{1}{2} c_r, \quad (A.13)$$

For $r \in \{1\} \cup I_1^m$,

$$(U_r)_{11}(x) = \frac{1}{2} \left[ \frac{v(x)}{x^{n+1-r}} \right]_+ - \frac{1}{2} \left[ \phi(x) \right]_{x^{n-r+1}} + \frac{1}{2} \left[ \frac{r(x)\phi(x)}{x^{n-r+1}u(x)} \right]_+ = -\frac{1}{2} a_r(x) + \frac{1}{2} b_r(x) + \frac{1}{2} c_r, \quad (A.14)$$

where

$$a_r(x) = -\sum_{k=0}^{r-2} M_{r-k-1} x^k, \quad b_r(x) = \sum_{k=0}^{r} d_{n-r-k+1} x^k, \quad c_r = q_r d_{n+1} \quad (A.15)$$

while for $r \in I_2^m$,

$$(U_r)_{11}(x) = -\frac{1}{2} \left[ \frac{v(x)}{x^{n+1-r}} \right]_+ - \frac{1}{2} \left[ \phi(x) \right]_{x^{n-r+1}} + \frac{1}{2} \left[ \frac{r(x)\phi(x)}{x^{n-r+1}u(x)} \right]_+ = -\frac{1}{2} a_r(x) + \frac{1}{2} b_r(x) + \frac{1}{2} c_r, \quad (A.16)$$

where

$$a_r(x) = \sum_{k=1}^{n+1-r} M_{r-k-1} x^{-k}, \quad b_r(x) = \sum_{k=1}^{n+1-r} d_{n-r-k+1} x^{-k}, \quad c_r = q_r d_{n+1} \quad (A.17)$$

Further, denoting (in accordance with (A.3), (A.7) and (A.10))

$$(U_r)_{21} = -(U_r)_{21}(w_1(x)) + (U_r)_{21}(w_2(x)) + (U_r)_{21}(w_3(x)), \quad (A.18)$$

(where $(U_r)_{21}(w_1(x))$ denotes the part of $(U_r)_{21}$ generated by the first term in (A.3) and so on) we receive for $r \in \{1\} \cup I_1^m$,

$$(U_r)_{21}(w_1(x)) = \frac{1}{2} \left[ \frac{w_1(x)}{x^{n-r+1}} \right]_+$$

$$= \frac{1}{2} \sum_{s=n-r+1}^{n-2} \left[ \sum_{k=0}^{s} q_k \sum_{j=k+s-m+2}^{n-m} p_{j+k+s-2m+2-j} \right] x^{s-n+r-1}, \quad (A.19)$$
and for \( r \in I_2^m \)

\[
(U_r)_{21}(w_1(x)) = -\frac{1}{2} \left[ \frac{w_1(x)}{x^{n+r+1}} \right]_-
\]

\[
= -\frac{1}{2} \sum_{s=0}^{m-1} \left( \sum_{k=n-s}^{n} q_k \sum_{j=n-m+1}^{k+s-m+1} p_j p_{n+k+s-2m+2-j} \right) x^{s-n+r+1}
\]

\[
= -\frac{1}{2} \sum_{s=m}^{n-r} \left( \sum_{k=0}^{n-s-2} q_k \sum_{j=k+s-m+2}^{n-m} p_j p_{n+k+s-2m+2-j} \right) x^{s-n+r+1}. \tag{A.20}
\]

For \( r \in \{1\} \cup I_1^m \)

\[
(U_r)_{21}(2w_2(x)) = \left[ \frac{w_2(x)}{x^{n+1-r}} \right]_+
\]

\[
= -\sum_{s=0}^{r-1} \left( \sum_{k=1}^{r-s} d_{n+1-r+s+k} p_{n+1-m-k} \right) x^s, \tag{A.21}
\]

and for \( r \in I_2^m \)

\[
(U_r)_{21}(2w_2(x)) = -\left[ \frac{w_2(x)}{x^{n-r+1}} \right]_-
\]

\[
= -\sum_{s=1}^{n+1-r} \left( \sum_{k=1}^{n+1-r-s} d_{n+1-r-s-k} p_{n+1-m+k} \right) x^{-s}. \tag{A.22}
\]

Finally

\[
(U_r)_{21}(2w_3(x)) = e(t) \delta_{r,n-m+1} \text{ for } m = 0, \ldots, n \tag{A.23}
\]

\[
(U_r)_{21}(2w_3(x)) = -e(t) q_r + e(t) x \delta_{1,r} \text{ for } m = n + 1.
\]

The formulas (A.12) and (A.13) follow directly from (A.9) and (A.11), respectively. The formulas (A.14)-(A.17) are easy consequences of (A.1). Finally, the formulas (A.19)-(A.23) follow from (A.4)-(A.6).

**Lemma 6** Poisson brackets \( \{ L_{ij}, W_r \} \) in Viète coordinates become

\[
\{ L_{11}, W_r \} = \{ v(x), W_r \} = \left\{ \begin{array}{ll}
- \sum_{k=1}^{r-2} kM_{r-k-1}^{(m)} x^{m+k-1} & \text{for } r \in I_1^m \\
\sum_{k=1}^{n+1-r} kM_{r+k-1}^{(m)} x^{m-k-1} & \text{for } r \in I_2^m
\end{array} \right. \tag{A.24}
\]

\[
\{ L_{12}, W_r \} = \{ u(x), W_r \} = \left\{ \begin{array}{ll}
r \sum_{k=1}^{r} kq_{r-k-1} x^{m+k-1} & \text{for } r \in I_1^m \\
\sum_{k=1}^{n+1-r} kq_{r+k-1} x^{m-k-1} & \text{for } r \in I_2^m
\end{array} \right. \tag{A.25}
\]

and, due to (A.3),

\[
\{ L_{21}, W_r \} = -\{ w_1(x), W_r \} + 2 \{ w_2(x), W_r \} + 2 \{ w_3(x), W_r \}
\]
with

\[
\{w_1(x), W_r\} = \begin{cases}
\sum_{s=n-r+2}^{n-2} \sum_{j=0}^{n-s} \sum_{k=0}^{n-m} (s-n+r-1)q_{k}p_{j}p_{n+k+s-2m+2-j}x^{m+s-n-r-2} & \text{for } r \in I^m_1 \\
-\sum_{s=0}^{m-1} \sum_{k=0}^{k+s+1} (s-n+r-1)q_{k}p_{n+k+s-2m+2-j}x^{m+s-n-r-2} & \text{for } r \in I^m_2 \\
-\sum_{s=m}^{n-r} \sum_{j=0}^{n-s-2} \sum_{k=0}^{n-m} (s-n+r-1)q_{k}p_{n+k+s-2m+2-j}x^{m+s-n-r-2} & \text{for } r \in I^m_2
\end{cases}
\]  

(A.26)

\[
\{w_2(x), W_r\} = \begin{cases}
x^m \sum_{s=1}^{r-1} \left( \sum_{k=1}^{r-s} k d_{n+1-r+s+k} p_{n+1-m-k} \right) x^{s-1} & \text{for } r \in I^m_1 \\
-x^m \sum_{s=0}^{n+1-r} \left( \sum_{k=0}^{n+1-r-s} k d_{n+1-r-s-k} p_{n+1-m+k} \right) x^{s-1} & \text{for } r \in I^m_2
\end{cases}
\]  

(A.27)

while \{w_3(x), W_r\} = 0.

All the equations in Lemma 6 follow from Lemma 4 and from the explicit form of perturbations terms \(W_r\) in Viète coordinates (4.10). For example, in order to prove (A.25) we note that for \(r \in I^m_1\) and due to (4.10)

\[
\{u(x), W_r\} = \sum_{s=n-m-r+2}^{n-m} x^{n-s} \{q_s, W_r\} = \sum_{s=n-m-r+2}^{n-m} (n+1-m-s)x^{n-s}q_{m+r-n-2+s} = \\
= \sum_{k=1}^{r-1} kq_{r-k-1}x^{m+k-1},
\]

where the last equality is obtained through the substitution \(k = n+1-m-s\). The case \(r \in I^m_2\) is treated similarly.

**Proof.** (of Theorem 3) The equation (4.1) reads componentwise as

\[
\frac{\partial L_{ij}}{\partial x} + \{L_{ij}, H^{B}_{r}\} = [U_r, L]_{ij} + 2x^m \frac{\partial}{\partial x} (U_r)_{ij}, \quad r = 1, \ldots, n,
\]  

(A.28)

with \(H^{B}_{r}\) given by (3.10) and (3.11) (we remind also that \(h^{B}_{r} = h_{r} + W_{r}\) for all \(r\)). Thus, for \(r = 1, \ldots, \kappa_1\) and for \(r = n - \kappa_2 + 1, \ldots, n\) the equation (A.28) reads

\[
\frac{\partial L_{ij}}{\partial x} + \{L_{ij}, h_{r}\} + \{L_{ij}, W_{r}\} = [U_r, L]_{ij} + 2x^m \frac{\partial}{\partial x} (U_r)_{ij}, \quad r = 1, \ldots, n,
\]

and since due to (2.15) \(\{L_{ij}, h_{r}\} = [U_r, L]_{ij}\) (this follows from the fact that \(h_{r}\) are Stäckel Hamiltonians when the remaining times \(t_{1}, \ldots, t_{r-1}, t_{r+1}, \ldots, t_{n}\) are considered as parameters), we obtain that for \(r = 1, \ldots, \kappa_1\) and for \(r = n - \kappa_2 + 1, \ldots, n\) the equation (A.28) becomes

\[
\frac{\partial L_{ij}}{\partial x} + \{L_{ij}, W_{r}\} = 2x^m \frac{\partial}{\partial x} (U_r)_{ij}, \quad r = 1, \ldots, n.
\]  

(A.29)

Similarly, and again due to the fact that \(h_{r}\) are Stäckel Hamiltonians, the equation (A.28) for \(r = \kappa_1 + 1, \ldots, n - m + 1\) reads

\[
\frac{\partial L_{ij}}{\partial x} + \sum_{s=1}^{r} \zeta_{r,s} \{L_{ij}, W_{s}\} = 2x^m \sum_{s=1}^{r} \zeta_{r,s} \frac{\partial}{\partial x} (U_{s})_{ij}, \quad r = 1, \ldots, n,
\]  

(A.30)

while for \(r = n - m + 2, \ldots, n - \kappa_2\) it becomes

\[
\frac{\partial L_{ij}}{\partial x} + \sum_{s=0}^{r-1} \zeta_{r,s} \{L_{ij}, W_{s}\} = 2x^m \sum_{s=0}^{r-1} \zeta_{r,s} \frac{\partial}{\partial x} (U_{r+s})_{ij}, \quad r = 1, \ldots, n.
\]  

(A.31)
Thus, our task is to show that (A.29)-(A.31) do follow from the assumptions of our theorem. We will prove it componentwise. For the component (1, 2) we have \( \frac{\partial L_{12}}{\partial t_r} = 0 \), so by (A.25) and (A.12)-(A.13) the equations (A.29)-(A.31) are identically satisfied. Note that in case when \( L_{ij} \) does not depend on \( t_r \) all the equations (A.29)-(A.31) coincide. Next, since \( \frac{\partial v(x)}{\partial t_r} = 0 \) while \( \varphi(x) \) does not depend on coordinates on manifold \( M \), the (1, 1)-component in equations (A.29)-(A.31) splits into two parts. For the \( v(x) \)-part it reduces to the equation

\[
\{v(x), W_r\} = 2x^m \frac{\partial}{\partial x_r} (U_r)_{11}, \quad r = 1, \ldots, n
\]

and by comparing (A.24) with (A.14)-(A.17) we see that it is identically satisfied. Further, for \( r \in \{1, \ldots, \kappa_1\} \), the \( \varphi(x) \)-part becomes

\[
\frac{\partial \varphi(x)}{\partial t_r} = x^n \frac{\partial b_r(x)}{\partial x},
\]

which reads explicitly as the polynomial in \( x \) equation

\[
\sum_{\gamma=0}^{n+1} \frac{\partial d_x}{\partial t_r} x^\gamma = \sum_{j=1}^r j d_{n-r+j+1} x^{j+m-1} = \sum_{\gamma=m}^{m+r-1} (\gamma - m + 1) d_{n-m-r+\gamma+2} x^\gamma
\]

and comparing the coefficients at \( x^\gamma \) we obtain:

\[
\frac{\partial d_x}{\partial t_r} = 0, \quad \gamma \neq m, \ldots, m + r - 1,
\]

\[
\frac{\partial d_x}{\partial t_r} = (\gamma - m + 1) d_{n-m-\gamma-r}, \quad \gamma = m, \ldots, m + r - 1,
\]

i.e. exactly the PDE’s (3.13)-(3.14) that are satisfied by the assumptions of the theorem. Similarly, for \( r \in \{n+1 - \kappa_2, \ldots, n\} \), the \( \varphi(x) \)-part becomes

\[
\frac{\partial \varphi(x)}{\partial t_r} = -x^m \frac{\partial b_r(x)}{\partial x}
\]

which is equivalent to

\[
\sum_{\gamma=0}^{n+1} \frac{\partial d_x}{\partial t_r} x^\gamma = \sum_{j=1}^{n+1-r} j d_{n-r+j+1} x^{m-j-1} = \sum_{\gamma=(n-m+2)} d_{n-m-r+\gamma+2} x^\gamma
\]

which in turn is equivalent to

\[
\frac{\partial d_x}{\partial t_r} = 0, \quad \gamma \neq r - (n - m + 2), \ldots, m - 2,
\]

\[
\frac{\partial d_x}{\partial t_r} = -(\gamma - m + 1) d_{n-m+2+\gamma-r}, \quad \gamma = r - (n - m + 2), \ldots, m - 2,
\]

which recover exactly the PDE’s (3.19)-(3.20) and are thus satisfied by the assumptions of the theorem. Further, for \( r \in \{\kappa_1, \ldots, n - m + 1\} \) the \( \varphi(x) \)-part becomes

\[
\frac{\partial \varphi(x)}{\partial t_r} = x^m \sum_{j=1}^r \zeta_{r,j} \frac{\partial b_j(x)}{\partial x},
\]

which is

\[
\sum_{\gamma=0}^{n+1} \frac{\partial d_x}{\partial t_r} x^\gamma = \sum_{j=1}^r \zeta_{r,j} \sum_{k=1}^j k d_{n-k+1} x^{k+m-1}
\]

\[
= \sum_{j=1}^r \zeta_{r,j} \sum_{\gamma=m}^{m+j-1} (\gamma - m + 1) d_{n-m-j+\gamma+2} x^\gamma
\]

\[
= \sum_{\gamma=m}^{m+r-1} (\gamma - m + 1) \left( \sum_{j=\gamma-m+1}^r \zeta_{r,j} (t_1, \ldots, t_{r-1}) d_{n-m+2+\gamma-j} \right) x^\gamma
\]

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which is equivalent to
\[
\frac{\partial d_\gamma}{\partial t_r} = 0, \quad \gamma \neq m, \ldots, m + r - 1,
\]
\[
\frac{\partial d_\gamma}{\partial t_r} = (\gamma - m + 1) \sum_{j=m+1}^r \zeta_{r,j}(t_1, \ldots, t_{r-1}) d_{n-m+2+\gamma-j} \gamma = m, \ldots, m + r - 1.
\]

The above equations are exactly the PDE’s (3.15)-(3.16) and thus are satisfied by the assumptions of the theorem. For \( r \in \{n-m+2, \ldots, n-\kappa_2\} \) the \( \varphi(x) \)-part becomes
\[
\frac{\partial \varphi(x)}{\partial t_r} = -x^m \sum_{j=0}^{n-r} \zeta_{r,r} \frac{\partial b_{r+j}(x)}{\partial x},
\]
that is
\[
\sum_{\gamma=0}^{n+1} \frac{\partial d_\gamma}{\partial t_r} x^\gamma = \sum_{j=0}^{n-r} \zeta_{r,r+j} \sum_{k=1}^{n+1-r-j} k d_{n+1-r-k-j} x^{m-k-1}
\]
\[
= \sum_{j=0}^{n-r} \zeta_{r,r+j} \sum_{\gamma=r-(n-m+2-j)}^{m-2} (m-\gamma-1) d_{n-m-r-j+\gamma+2} x^\gamma
\]
\[
= \sum_{\gamma=r-(n-m+2)}^{m-2} (m-\gamma-1) \left( \sum_{j=0}^{n-m+2-r+\gamma} \zeta_{r,r+j} d_{n-m+2-r-j+\gamma} \right) x^\gamma
\]
which is equivalent to
\[
\frac{\partial d_\gamma}{\partial t_r} = 0, \quad \gamma \neq r - (n-m+2), \ldots, m-2,
\]
\[
\frac{\partial d_\gamma}{\partial t_r} = -(\gamma - m + 1) \sum_{j=0}^{n-m+2+\gamma-r} \zeta_{r,r+j}(t_1, \ldots, t_n) d_{n-m+2+\gamma-r-j} \gamma = r - (n-m+2), \ldots, m-2.
\]

The above equations are exactly the PDE’s (3.17)-(3.18) and thus are satisfied by the assumptions of the theorem. Thus, the (1,1)-component of the equations (A.29)-(A.31) is satisfied. Let us finally turn to the (2,1)-component of (A.29)-(A.31). Since by (A.6) \( w_3 \) does not depend on coordinates on \( \mathcal{M} \), the \( w_3 \)-part of the (2,1)-component of equations (A.29)-(A.31) read
\[
\frac{\partial w_3(x)}{\partial t_r} = x^m \frac{\partial}{\partial x}(U_r)_{21}(2w_3(x)) \quad (A.33)
\]
which is identically satisfied due to (A.23) and (3.21). Next we prove the \( w_2 \)-part of the (2,1)-component of equations (A.29)-(A.31). Since \( w_2 \) depends both on times \( t_r \) and on the coordinates on \( \mathcal{M} \), we have to consider four separate cases. Consider first the case \( r = 1, \ldots, \kappa_1 \). Then the \( w_2 \)-part of the (2,1)-component of (A.29) is
\[
\frac{\partial w_2(x)}{\partial t_r} + \{ w_2(x), W_r \} = x^m \frac{\partial}{\partial x}(U_r)_{21}(2w_2(x)) \quad (A.34)
\]
Differentiating (A.5) w.r.t \( t_r \) with the help of PDE’s (3.13)-(3.14) yields that for \( m = 0, \ldots, n+1 \)
\[
\frac{\partial w_2(x)}{\partial t_r} = - \sum_{s=m}^{m+r-2} \sum_{k=s+1}^{m+r-1} (k-m+1) d_{n+2-m-r+k} \frac{p_{n+1-m-s-k}}{p_{n+1-m-k}} x^s
\]
which after the reparametrization of indices \( s \to s + m - 1 \) and then \( k \to k + s + m - 1 \) reads
\[
\frac{\partial w_2(x)}{\partial t_r} = -x^m \sum_{s=1}^{r-1} \left( \sum_{k=1}^{r-s} (k+s) d_{n+1-r+s+k} \frac{p_{n+1-m-k}}{p_{n+1-m-s-k}} \right) x^{s-1}, \quad (A.35)
\]
Combining this result with (A.21) and (A.27) yields (A.34). Consider now the case \( r = \kappa_1 + 1, \ldots, n - m + 1 \). Then the \( w_2 \)-part of the \((2,1)\)-component of (A.30) is

\[
\frac{\partial w_2(x)}{\partial t_r} + \sum_{s=1}^{r} \zeta_{r,s} \{ w_2(x), W_s \} = x^m \sum_{s=1}^{r} \zeta_{r,s} \frac{\partial}{\partial x} (U_s)_{21}(2w_2(x)) \tag{A.36}
\]

Differentiating (A.5) w.r.t. \( t_r \) with the help of PDE’s (3.15)-(3.16) yields that for \( m = 0, \ldots, n \)

\[
\frac{\partial w_2(x)}{\partial t_r} = - \sum_{s=m}^{m+r-2} \sum_{k=s+1}^{r} \sum_{j=k-m+1}^{r} (k - m + 1) \zeta_{r,j} d_{n+2-m-j+k} p_{n+1-m+s-k} x^s
\]

Substituting this result, (A.21) and (A.27) to (A.36) yields

\[
\sum_{s=m}^{m+r-2} \sum_{k=s+1}^{r} \sum_{j=k-m+1}^{r} \sum_{l=0}^{s-l} \sum_{k=1}^{d_{n+1-s+l+k}} p_{n+1-m-k} x^{s-l} \ell \to
\]

that can be shown, through suitable changes of summation indices and careful changes of the summation order, to be identically satisfied. The proof of the two remaining cases when \( r \in \Pi^m_2 \) is similar. Let us now turn into the \( w_1 \)-part of the \((2,1)\)-component of (A.29)-(A.31). Since \( w_1 \) does not depend on times \( t_r \), this part reads

\[
\{ w_1(x), W_r \} = 2x^m \frac{\partial}{\partial x} (U_r)_{21}(w_1(x)). \tag{A.37}
\]

Comparing (A.19) and (A.26) we get that (A.37) is satisfied. Similarly we prove the case \( r \in \Pi^m_2 \). This concludes the proof.

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