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Stress formulation of elastic wave motion

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Abstract: Propagation of elastic waves in anisotropic solids is solved through a pure stress formalism. The stress solutions lead to the three wave solutions expected from the displacement formulation plus three non-propagating stresses with zero wave speed which do not satisfy the strain compatibility conditions. This work provides a different perspective to modeling elastic waves and is expected to be better suited for certain types of problems. © 2021 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).

1. Introduction

The displacement form of the differential equation of motion governing elastic wave propagation traces back to Navier’s exposition in 1821 and later memoir.\textsuperscript{1} Hadamard’s treatise\textsuperscript{2} is the foundation of the modern theory of elastic waves in anisotropic solids, which formulated the Fresnel-Hadamard propagation condition, now more commonly known as the Christoffel equation. The vast majority of research activity in elastic waves in anisotropic solids has since been approached through the displacement formulation.\textsuperscript{3–7} Stress-based solutions to elastostatic problems are well-known, e.g., solutions that satisfy the Beltrami-Michell compatibility relations.\textsuperscript{8} A dual elastodynamic treatment using pure stress language is not common. Some work in this direction did begin to appear in the late 1950s.\textsuperscript{9} Recently, Ostoja-Starzewski offered an extensive review of the topic covering the time period since then and gave several new extensions, which elucidated several benefits of considering stress formulations.\textsuperscript{10} To our knowledge, a dual pure stress formulation analogous to the displacement-based Fresnel-Hadamard propagation condition or the Christoffel equation has not been developed. The aforementioned is developed here in addition to its solutions, which give the phase velocities and stress components of the stress wave. The phase velocities and stress components are functions of newly formed invariants containing elastic compliance constants and the propagation direction.

This work offers an alternative way to consider waves in elastic solids that could lead to new and unexpected results. As an example, the stress-based analysis indicates that in addition to the expected propagating stress wave solutions there are non-propagating stresses, i.e., with zero velocity, which correspond identically to all possible incompatible strains. The present formulation provides a simultaneous solution to the propagating (compatible) and non-propagating (incompatible) wave solutions for waves in anisotropic solids.

2. Displacement formalism

Traditionally, elastic wave motion is described in terms of the displacement of a traveling wave,\textsuperscript{2} $u(x, t) = \hat{u}g(n \cdot x - vt)$, where $\hat{u}$ is a constant vector and $g$ is twice differentiable with respect to space and time. In an infinite medium, Cauchy’s law of motion governs the wave displacement,

\[ \text{div} \ \sigma = \rho \ddot{u}. \] (1)

The linear elastic constitutive relation is Hooke’s law, $\sigma_{ij} = c_{ijkl}u_{kl}$, with $c_{ijkl} = c_{iklj} = c_{ijlk}$. Upon substitution of Hooke’s law and executing the derivatives in Eq. (1) leads to the Christoffel equation

\[ (Q - \lambda \delta) \ddot{u} = 0, \] (2)

where $Q = Q_{ik} = c_{iklj}\eta_l\eta_j$ is the acoustic tensor, $\lambda = \rho v^2$ are eigenvalues, and $\delta = \delta_{ik}$ is the Kronecker delta function. The eigenvalues $\lambda$ must satisfy

\[ \det(Q - \lambda \delta) = 0 \] (3)

or, equivalently, in terms of the characteristic polynomial

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\[
\lambda^3 - \lambda^2 I_1 + \lambda I_2 - I_3 = 0,
\]
where the principle invariants of the acoustic tensor are
\[
I_1 = \text{tr} \mathbf{Q}, \quad I_2 = \frac{1}{2} \left[ (\text{tr} \mathbf{Q})^2 - \text{tr} \mathbf{Q}^2 \right], \quad I_3 = \det \mathbf{Q}.
\]

Standard techniques for solving cubic equations can be used to solve Eq. (4) for the eigenvalues and, in turn, the phase velocities. The polarization \( \mathbf{u} \) is an eigenvector of \( \mathbf{Q} \).

3. Stress formalism

The equation of motion given in Eq. (1) can be rewritten in terms of stresses only by using Hooke’s law in the inverse stress formulation, Eq. (7) is written in an equivalent matrix form
\[
\frac{1}{2} \mathbf{V} \, \text{div} \mathbf{\sigma} + \frac{1}{2} (\mathbf{V} \, \text{div} \mathbf{\sigma})^T = \rho \mathbf{\ddot{s}},
\]
where \( \mathbf{V} \) is the gradient. Consider a traveling stress wave of the form \( \mathbf{\sigma} = \tilde{\mathbf{\sigma}} f(\mathbf{n} \cdot \mathbf{x} - c t) \), where \( \tilde{\mathbf{\sigma}} \) is a constant second rank tensor and \( f \) is twice differentiable in space and time. Substituting the stress wave \( \mathbf{\sigma} \) into Eq. (6) and evaluating the derivatives leads to
\[
(\mathbf{N} - \lambda \mathbf{S}) : \mathbf{\ddot{s}} = 0,
\]
where
\[
N_{ijkl} = \frac{1}{4} (\delta_{ik} n_j n_l + \delta_{jk} n_i n_l + \delta_{il} n_j n_k + \delta_{jl} n_i n_k)
\]
has the same symmetry properties as the compliance and stiffness. To arrive at a characteristic equation of \( \lambda \) based on the stress formulation, Eq. (7) is written in an equivalent matrix form
\[
(\mathbf{\hat{N}} - \lambda \mathbf{\hat{S}}) : \mathbf{\ddot{s}} = 0,
\]
where notable hatted quantities are
\[
\tilde{\mathbf{\sigma}} = \mathbf{K} \mathbf{\sigma}, \quad \tilde{\mathbf{e}} = \mathbf{K} \mathbf{e}, \quad \hat{\mathbf{C}} = \mathbf{K} \mathbf{C} \mathbf{K}, \quad \hat{\mathbf{S}} = \mathbf{K} \mathbf{S} \mathbf{K}, \quad \hat{\mathbf{N}} = \mathbf{K} \mathbf{N} \mathbf{K},
\]
with \( \mathbf{K} \) being the \( 6 \times 6 \) matrix
\[
\mathbf{K} = \begin{pmatrix}
I_{3 \times 3} & 0 \\
0 & \sqrt{2}I_{3 \times 3}
\end{pmatrix}.
\]

The matrix entries denoted as \( I_{3 \times 3} \) are the elements of the \( 3 \times 3 \) identity matrix. The quantities \( \mathbf{\sigma}, \mathbf{e}, \mathbf{C}, \) and \( \mathbf{S} \) are written as the matrices
\[
\mathbf{\sigma} = \begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{pmatrix}, \quad \mathbf{e} = \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6
\end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix}
s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\
s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\
s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\
s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\
s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\
s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66}
\end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\
c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\
c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\
c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\
c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\
c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66}
\end{pmatrix},
\]
which are formed from the Voigt index reduction, which maps pairs of indices into a single index according to
\[
11 \rightarrow 1, \quad 22 \rightarrow 2, \quad 33 \rightarrow 3, \quad 23 \text{ or } 32 \rightarrow 4, \quad 13 \text{ or } 31 \rightarrow 5, \quad \text{and } 12 \text{ or } 21 \rightarrow 6.
\]
The propagation direction matrix \( \hat{\mathbf{N}} \) is
\[
\hat{\mathbf{N}} = \mathbf{K} \mathbf{N} \mathbf{K} = \begin{pmatrix}
n_1^2 & 0 & 0 & 0 & n_1 n_2 & n_1 n_3 \\
0 & n_2^2 & 0 & n_2 n_3 & 0 & n_1 n_2 \\
0 & 0 & n_3^2 & 0 & n_2 n_3 & 0 \\
0 & n_2 n_3 & n_2 n_3 & n_1^2 & 0 & 0 \\
n_3 n_1 & 0 & n_3 n_2 & 0 & n_1^2 + n_2^2 & n_1 n_3 \\
n_3 n_2 & 0 & n_3 n_1 & 0 & n_1 n_3 & n_2^2 + n_3^2
\end{pmatrix}.
\]
3.1 Wave speeds

The wave speed parameter $\lambda$ must satisfy the characteristic equation

$$\det(\mathbf{N} - \lambda\mathbf{S}) = \lambda^6 \det \mathbf{S} - \lambda^5 a_1 + \lambda^4 a_2 - \lambda^3 a_3 + \lambda^2 a_4 - \lambda a_5 + a_6 = 0,$$  

where

$$a_1 = \frac{15}{6!} \left[ 12 (tr\mathbf{N})^2 (tr\mathbf{N}\mathbf{S})^2 - 24 tr\mathbf{N}^2 \mathbf{S}^2 - 48 tr\mathbf{N}^2 \mathbf{S}^2 - 48 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2 + (tr\mathbf{N})^4 (tr\mathbf{S})^2 + 3 (tr\mathbf{N})^2 (tr\mathbf{S})^2 + 48 tr\mathbf{N} tr\mathbf{N} \mathbf{S}^2 \mathbf{S}^2 + 24 tr\mathbf{N}^2 \mathbf{S}^2 + 12 tr\mathbf{N}^2 \mathbf{S}^2 + 12 tr\mathbf{N}^2 \mathbf{S}^2 + 6 tr\mathbf{N}^2 \mathbf{S}^2 + 48 tr\mathbf{N}^2 \mathbf{S}^2 + 6 tr\mathbf{N}^2 \mathbf{S}^2 - 12 tr\mathbf{N}^2 (tr\mathbf{N}\mathbf{S})^2 + 8 (tr\mathbf{N})^3 tr\mathbf{S}^2 + 24 (tr\mathbf{N})^2 tr\mathbf{N}^2 \mathbf{S}^2 - 12 (tr\mathbf{N})^2 tr\mathbf{N} \mathbf{S}^2 \mathbf{S} + (tr\mathbf{N})^4 tr\mathbf{S}^2 - 3 (tr\mathbf{N}^2)^2 tr\mathbf{S}^2 - 6 tr\mathbf{N}^2 (tr\mathbf{S})^2 + 24 (tr\mathbf{N}^2)^2 tr\mathbf{N} \mathbf{S}^2 - 6 (tr\mathbf{N}^2)^2 tr\mathbf{S}^2 - 24 tr\mathbf{N}^2 tr\mathbf{N} \mathbf{S}^2 - 48 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2 - 8 tr\mathbf{N} \mathbf{S}^2 \mathbf{S}^2 - 48 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2 - 24 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2\right];$$

$$a_2 = \frac{6}{120} \left[ 120 tr\mathbf{N} tr\mathbf{N} \mathbf{S}^2 - 120 tr\mathbf{N} \mathbf{S}^2 + 60 tr\mathbf{N} ^2 \mathbf{S}^2 + 30 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2 + 40 tr\mathbf{N}^3 \mathbf{S}^2 - 24 tr\mathbf{N}^3 \mathbf{S}^2 + 24 tr\mathbf{N}^3 \mathbf{S}^2 - 10 (tr\mathbf{N}^2)^4 tr\mathbf{N} \mathbf{S}^2 - 15 (tr\mathbf{N}^2)^4 tr\mathbf{N} \mathbf{S}^2 + 20 (tr\mathbf{N}^2)^3 tr\mathbf{N}^2 \mathbf{S} + (tr\mathbf{N}^2)^3 tr\mathbf{S} + 30 (tr\mathbf{N}^2)^3 tr\mathbf{N} \mathbf{S}^2 + 15 tr\mathbf{N} (tr\mathbf{N}^2)^2 tr\mathbf{S} + 20 (tr\mathbf{N}^2)^2 tr\mathbf{N}^2 \mathbf{S} + 20 (tr\mathbf{N}^2)^2 tr\mathbf{N} \mathbf{S}^2 + 30 tr\mathbf{N} \mathbf{N} \mathbf{S}^2 + 60 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2 - 40 tr\mathbf{N} \mathbf{S}^2 \mathbf{N} \mathbf{S}^2 - 30 tr\mathbf{N} \mathbf{N} \mathbf{S}^2 + 20 tr\mathbf{N} \mathbf{N} \mathbf{S}^2\right];$$

$$a_3 = \frac{1}{6!} \left[ (tr\mathbf{N})^6 - 15 (tr\mathbf{N})^4 tr\mathbf{N}^2 + 40 (tr\mathbf{N})^4 tr\mathbf{N}^2 + 45 (tr\mathbf{N})^4 tr\mathbf{N}^2 - 90 tr\mathbf{N}^4 tr\mathbf{N} + 120 tr\mathbf{N}^4 tr\mathbf{N}^2 - 120 tr\mathbf{N}^4 tr\mathbf{N}^2 + 144 tr\mathbf{N}^4 tr\mathbf{N}^2 + 40 (tr\mathbf{N}^2)^3 - 120 tr\mathbf{N}^2 + 20 tr\mathbf{N}^2\right],$$

where $I = I_{6x6}$. These three coefficients vanish due to the specific form of $\mathbf{N}$. In order to see this we first note the identity $\mathbf{N} (I - \mathbf{N}) (I - 2\mathbf{N}) = 0$.

This implies $tr\mathbf{N}'' = \frac{2}{3} tr\mathbf{N}'' - \frac{1}{3} tr\mathbf{N}'' - n$ for $n \geq 3$, which combined with $tr\mathbf{N} = 2$, $tr\mathbf{N}^2 = \frac{3}{2}$ yields $tr\mathbf{N}'' = 1 + 2^{1-n}$ for $n \geq 1$ and hence $a_6$ is zero. It also leads to the simplified expression

$$a_5 = \frac{180}{6!} tr(I - \mathbf{N})(I - 2\mathbf{N})^3 tr\mathbf{S};$$

which is zero on account of Eq. (18). Similarly, $a_4$ can be reduced to

$$a_4 = \frac{90}{6!} tr \left[ 4 \mathbf{N}^2 (\mathbf{N} \mathbf{S} - \mathbf{S} \mathbf{N}) - tr(\mathbf{N} \mathbf{S} - \mathbf{N} \mathbf{S}) - \mathbf{N} \mathbf{S} (\mathbf{N} \mathbf{S} - tr\mathbf{S}) \right].$$
Consider, for instance, \( n = (1, 0, 0) \) for which \( \hat{N} = \text{diag} \{ 1, 0, 0, 0, \frac{1}{2}, \frac{1}{2} \} \) and it is easy to verify that the expression for \( a_4 \) vanishes. This implies the general result for arbitrary \( n \) if we consider \( \hat{S} \) as the compliance in the rotated coordinate system. In summary, \( a_4 = 0, a_1 = 0 \) and \( a_0 = 0 \). Thus, a cubic equation for \( \lambda \) follows:

\[
\lambda^3 - \lambda^2 c_1 + \lambda c_2 - c_3 = 0, \quad \text{where} \quad c_4 = a_4/\det \hat{S}, \quad x = 1, 2, 3
\]

and using the properties of \( \hat{N} \) discussed above we find

\[
a_1 = \frac{6}{\ell_1} \left[ 48\text{tr}\hat{S}^3 - 120\text{tr}\hat{N}\hat{S}^5 + 30\text{tr}\hat{N}\hat{S}\hat{N}^4 - 40\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 + 60\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 - 120\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 - 60\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 - 40\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 \right]
\]

\[
a_2 = \frac{15}{\ell_1} \left[ 96\text{tr}\hat{N}^4 - 48\text{tr}\hat{N}^2 \hat{S}^3 - 15\text{tr}\hat{N}^2 \hat{S}^3 \hat{N} - 48\text{tr}\hat{N}^2 \hat{N}^2 \hat{N} + 12\text{tr}\hat{N}^2 \hat{N}^2 \hat{S} \right)
\]

\[
a_3 = \frac{20}{\ell_1} \left[ \text{tr}(3I - 63\hat{N} + 126N^2) \hat{S}^3 - 12\text{tr}(\hat{N}\hat{S})^3 + 72\text{tr}\hat{S}^2 \hat{N}\hat{N}^2(I - \hat{N}) + 9\text{tr}\hat{S}(3\hat{N} - 2N^2) S^2
\]

\[
-9\text{tr}\hat{N}\hat{S}^3 + 36\text{tr}\hat{N}\hat{N}(\hat{N} - I) S^2 + 18\text{tr}\hat{N}\hat{S}^3 \hat{S}^2 \hat{N}^2 - 32\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 \hat{S}^2 + 8\text{tr}\hat{N}\hat{S}^3 \hat{S}^2 \hat{N} \hat{S}^2(\hat{S} - 24\text{tr}\hat{N} \hat{S}^2 (\hat{S} - 12\text{tr}\hat{N}\hat{S}^3 \hat{N}^2 - 15\text{tr}\hat{S}^3 - 24\text{tr}\hat{N} \hat{S}^2 \hat{S} \hat{N}^2 + \frac{5}{2} \text{tr}\hat{S}^3)
\]

while

\[
\det \hat{S} = \frac{1}{\ell_1} \left[ (\text{tr}S)^6 - 15(\text{tr}S)^4 \text{tr}S^2 + 40(\text{tr}S)^2 \text{tr}S^2 - 90(\text{tr}S)^4 \text{tr}S^2 + 45(\text{tr}S)^2 \text{tr}S^2 - 120\text{tr}S^3 \text{tr}S^3
\]

\[
+144\text{tr}S^5 - 15(\text{tr}S)^3 \text{tr}S^2 - 90\text{tr}S^3 \text{tr}S^2 + 40(\text{tr}S)^2 \text{tr}S^2 - 120\text{tr}S^6 \right].
\]

Equation (21) together with the constants \( c_x \) completely define the characteristic equation for a completely stress-based formalism of elastic wave propagation and is the primary result of this letter.

3.2 Consistency with displacement formalism wave speeds

The stress-based approach is consistent with the displacement formalism if it gives the same wave speed and stress polarization corresponding to each displacement solution. We first show that the speeds match and discuss polarizations in Sec. 4.

Comparison of Eq. (21) and Eq. (4) indicates that \( c_1, c_2, \) and \( c_3 \) must be equal to the principle invariants \( I_1, I_2, \) and \( I_3 \) seen in Eq. (5), respectively. To show this, consider the identity

\[
\epsilon_{ijklmn} \tilde{S}_{ij} \tilde{S}_{jq} \tilde{S}_{kr} \tilde{S}_{lu} \tilde{S}_{mv} \tilde{S}_{nw} = \epsilon_{pqrsnt} \det \tilde{S}.
\]

Multiplying both sides of Eq. (24) by \( \tilde{S}^{-1} \) and \( \tilde{N} \) gives

\[
\epsilon_{ijklmn} \tilde{S}_{ij} \tilde{S}_{jq} \tilde{S}_{kr} \tilde{S}_{lu} \tilde{S}_{mv} \tilde{S}_{nw} = \epsilon_{pqrsnt} \tilde{N}_{vw} \det \tilde{S}.
\]

Next, multiply both sides of Eq. (25) by \( \epsilon_{pqrsnt} \) to give

\[
\epsilon_{pqrsnt} \epsilon_{ijklmn} \tilde{S}_{ij} \tilde{S}_{jq} \tilde{S}_{kr} \tilde{S}_{lu} \tilde{S}_{mv} \tilde{S}_{nw} = \epsilon_{pqrsnt} \epsilon_{ijklmn} \tilde{S}_{ij} \tilde{S}_{jq} \tilde{S}_{kr} \tilde{S}_{lu} \tilde{S}_{mv} \tilde{S}_{nw}.
\]

It can be shown that

\[
\epsilon_{pqrsnt} \epsilon_{ijklmn} \tilde{S}_{ij} \tilde{S}_{jq} \tilde{S}_{kr} \tilde{S}_{lu} \tilde{S}_{mv} \tilde{S}_{nw} = 5! \delta_{nw}.
\]
\[
\hat{S}_{m}^{-1} \hat{N}_{\mu \nu} \text{det} \hat{S} = \frac{1}{2 \alpha} \epsilon_{pqrs} \epsilon_{ijklmn} \hat{S}_{pq} \hat{S}_{ij} \hat{S}_{kl} \hat{S}_{mr} \hat{N}_{nu}.
\]

With the form of \( a_i \) in Eq. (15) together with Eq. (28), it is observed that

\[
c_i = \hat{S}_{ij} \hat{N}_{ji} = \hat{C}_{ij} \hat{N}_{ji} = \text{tr} \hat{Q} = I_1.
\]

The consistency relations for \( c_2 \) and \( c_3 \) follow a similar procedure. The results can be cast into a general formula for \( c_3 \),

\[
c_s = \frac{1}{2!} \begin{vmatrix} 
\delta_{h,1} & \ldots & \delta_{h,d} \\
\vdots & \ddots & \vdots \\
\delta_{h,1} & \ldots & \delta_{h,d} 
\end{vmatrix} \hat{S}_{h,h} \hat{N}_{h,h} \hat{S}_{h,f} \ldots \hat{S}_{h,h} \hat{N}_{h,h} = \frac{1}{2!} \begin{vmatrix} 
\delta_{h,1} & \ldots & \delta_{h,d} \\
\vdots & \ddots & \vdots \\
\delta_{h,1} & \ldots & \delta_{h,d} 
\end{vmatrix} \hat{C}_{h,h} \hat{N}_{h,h} \hat{C}_{h,h} \ldots \hat{C}_{h,h} \hat{N}_{h,h}.
\]

Then, the following consistency relations are easily obtained:

\[
c_1 = \hat{S}^{-1} : \hat{N} = \hat{C} : \hat{N} = \text{tr} \hat{Q}
\]

\[
= I_1,
\]

\[
c_2 = \frac{1}{2} \left( \left( \text{tr} \hat{S}^{-1} \hat{N} \right)^2 - \text{tr} (\hat{S}^{-1} \hat{N})^2 \right)
\]

\[
= \frac{1}{2} \left( \left( \text{tr} \hat{C} \hat{N} \right)^2 - \text{tr} (\hat{C} \hat{N})^2 \right)
\]

\[
= \frac{1}{2} \left( \left( \text{tr} \hat{Q} \right)^2 - \text{tr} \hat{Q}^2 \right)
\]

\[
= I_2,
\]

\[
c_3 = \frac{1}{6} \left( \left( \text{tr} \hat{S}^{-1} \hat{N} \right)^3 - 3 \text{tr} \hat{S}^{-1} \hat{N} \text{tr} (\hat{S}^{-1} \hat{N})^2 + 2 \text{tr} (\hat{S}^{-1} \hat{N})^3 \right)
\]

\[
= \frac{1}{6} \left( \left( \text{tr} \hat{C} \hat{N} \right)^3 - 3 \text{tr} \hat{C} \hat{N} \text{tr} (\hat{C} \hat{N})^2 + 2 \text{tr} (\hat{C} \hat{N})^3 \right)
\]

\[
= \frac{1}{6} \left( \left( \text{tr} \hat{Q} \right)^3 - 3 \text{tr} \hat{Q} \text{tr} \hat{Q}^2 + 2 \text{tr} \hat{Q}^3 \right)
\]

\[
= \text{det} \hat{Q}
\]

\[
= I_3.
\]

4. Stress polarizations

4.1 Non-propagating stresses

Equation (9) is a generalized eigenvalue problem that can be transformed into standard form

\[
(\tilde{\mathbf{C}} - \lambda \mathbf{I}) \mathbf{\tilde{\Sigma}} = \mathbf{0},
\]

where

\[
\tilde{\mathbf{C}} = \tilde{\mathbf{S}}^{-1/2} \mathbf{\tilde{N}} \tilde{\mathbf{S}}^{-1/2}, \quad \mathbf{\tilde{\Sigma}} = \tilde{\mathbf{S}}^{1/2} \mathbf{\tilde{\sigma}}
\]

and \( \tilde{\mathbf{C}} = \tilde{\mathbf{S}}^{-1} \) is the stiffness. The \( 6 \times 6 \) symmetric matrix \( \tilde{\mathbf{C}} \) has three positive eigenvalues, the roots of the characteristic cubic equation, and three zero eigenvalues. The eigenvectors corresponding to the positive eigenvalues give the three propagating wave solutions. The stress states corresponding to the zero eigenvalues represent solutions that do not propagate because their speed is zero.

Elastic waves are, in practice, the result of some excitation. A common situation is that the flat surface of a solid is subject to prescribed motion. For instance, if displacements are excited uniformly across the surface, the displacements will propagate in the direction normal to the surface. Any displacement excitation will propagate since the three displacement polarization vectors of Eq. (2) span the space of three-dimensional vectors. The stress eigenvectors are elements of a six-dimensional space, which can be split into complementary three-dimensional subspaces of propagating and non-propagating stresses.

We examine these subspaces by considering the case of isotropy with two positive elastic constants, the bulk \( K \) and the shear \( \mu \) moduli, for which
The matrix \( \tilde{C}^{1/2} \) follows from the properties \( A^2 = A, B^2 = B, AB = BA = 0 \) as
\[
\tilde{C}^{1/2} = \begin{pmatrix}
\sqrt{3}K + \sqrt{2}\mu & 0 \\
0 & -\sqrt{2}\mu
\end{pmatrix}.
\]

Taking \( n = (1, 0, 0) \) we find the matrix \( \tilde{C}_n \) of Eq. (35) is
\[
\tilde{C}_n = \begin{pmatrix}
X & 0 \\
0 & Y
\end{pmatrix},
\]
where
\[
X = \begin{pmatrix}
2\mu & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{\sqrt{2}\mu}{3} \left( \sqrt{3}K - \sqrt{2}\mu \right) \begin{pmatrix}
2 & 1 & 1 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix} + \frac{\left( \sqrt{3}K - \sqrt{2}\mu \right)^2}{3} A, \quad Y = \begin{pmatrix}
0 & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \mu
\end{pmatrix}.
\]

The eigenvectors \( \tilde{\Sigma} \) of \( \tilde{C}_n \) can then be determined, from which the stress vectors follow using \( \tilde{\sigma} = \tilde{C}^{1/2} \tilde{\Sigma} \). We find that the unnormalized propagating stresses are
\[
\tilde{\sigma}^{(l)} = \begin{pmatrix}
3K + 4\mu \\
3K - 2\mu \\
0 \\
0 \\
0
\end{pmatrix}, \quad \tilde{\sigma}^{(T)}_1 = \begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}, \quad \tilde{\sigma}^{(T)}_2 = \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0
\end{pmatrix}, \quad \tilde{\sigma}^{(T)}_3 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix},
\]

where the form of \( \tilde{\sigma}^{(l)}_1 \) and \( \tilde{\sigma}^{(l)}_2 \) are determined from the orthogonality condition discussed next.

Note that the stresses of Eqs. (40) and (41) are not orthogonal in the sense of 6-vectors. However, the associated \( \tilde{\Sigma} = S^{1/2} \tilde{\sigma} \) are orthogonal of the form
\[
\tilde{\Sigma} = \begin{pmatrix}
\sqrt{3}K + 2\sqrt{2}\mu & 0 & \left( -\sqrt{3}K + \sqrt{2}\mu \right) \\
\sqrt{3}K - \sqrt{2}\mu & 0 & 0 \\
\sqrt{3}K - \sqrt{2}\mu & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad \tilde{\Sigma}^{(l)}_1 = \begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}, \quad \tilde{\Sigma}^{(l)}_2 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
1 \\
0
\end{pmatrix}, \quad \tilde{\Sigma}^{(l)}_3 = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]
\(\sigma_b\) are taken from the six stresses in Eqs. (40) and (41). Thus \(\sigma_a : \epsilon_a > 0\) while \(\sigma_a : \epsilon_b = 0\) for \(a \neq b\). For instance, \(\sigma^{(L)} : \epsilon_j^{(T)} = 0, \sigma^{(L)} : \epsilon_j^{(0)} = 0, \sigma^{(0)} : \epsilon_2^{(0)} = 0\), etc.

4.2 Non-propagating stresses are incompatible

An elastic strain \(\varepsilon = S\dot{\sigma}\) is compatible if it is derived from a displacement \(u\) as \(\varepsilon_{ij} = \frac{1}{2}(u_{ij} + u_{ji})\). The necessary and sufficient conditions for compatibility\(^6\) are \(\varepsilon_{ikp} \varepsilon_{jq} \varepsilon_{ilq} = 0\). Assuming uni-dimensional spatial dependence \(\varepsilon = \dot{\varepsilon} f(\mathbf{n} \cdot \mathbf{x}, t)\) the compatibility conditions become \(M_{ijkl} \ddot{\varepsilon}_{kl} = 0\), where the fourth rank tensor

\[
M_{ijkl} = \frac{1}{2} (\varepsilon_{ikp} \varepsilon_{jq} - \varepsilon_{ipq} \varepsilon_{jk}) \rho_n \eta_n
\]

possesses the symmetries of fourth rank elastic tensors. The non-propagating stress solutions with zero velocity are therefore compatible if and only if they satisfy the simultaneous conditions

\[
\begin{align*}
N T \dot{\sigma} &= 0, \quad (44a) \\
M T \ddot{\sigma} &= 0, \quad (44b)
\end{align*}
\]

where \(\ddot{\sigma} = S \dot{\sigma}\). We now show that the only solution of Eq. (44) is the trivial one \(\ddot{\sigma} = 0, \dot{\sigma} = 0\).

The simultaneous equations (44) can be written

\[
\begin{align*}
N T \dot{\Sigma} &= 0, \quad (45a) \\
M T \ddot{\Sigma} &= 0, \quad (45b)
\end{align*}
\]

where

\[
\begin{align*}
N &= C^{1/2} NC^{1/2}, \quad M = S^{1/2} MS^{1/2}, \quad \dot{\Sigma} = S^{1/2} \ddot{\sigma} = C^{1/2} \ddot{\sigma}.
\end{align*}
\]

It may be easily checked that \(M\), like \(N\) is trimodular\(^8\) in the sense that three of its six Kelvin moduli\(^16\) are zero. Also, \(MN = NM = 0\) implying \(MN = NM = 0\) and that the three-dimensional null spaces of \(M\) and \(N\) are distinct. A non-zero solution of one of the two conditions (45a) cannot be a solution of the other. The only solution of both is the trivial one \(\ddot{\Sigma} = 0\), and hence the non-propagating stress solutions are not compatible.

The same arguments prove that the propagating stress solutions are compatible. Thus, the stress satisfies \(\ddot{\Sigma} = \lambda^{-1} N T \dot{\Sigma}\) and the compatibility condition (45b) is therefore satisfied since \(M T \ddot{\Sigma} = \lambda^{-1} M T N T \dot{\Sigma} = 0\).

5. Discussion and conclusion

Wave solutions can be superimposed by virtue of the linearity of the equations of motion. A longitudinal wave traveling in the \(x-\) direction in an isotropic solid is any solution of the form \(\sigma = \sigma^{(L)} f^{(L)}(x - c_1 t)\) for a sufficiently smooth but arbitrary function \(f^{(L)}\). Similarly, a transverse wave is any solution of the form \(\sigma = \sigma^{(T)} f^{(T)}(x - c_2 t) + \sigma^{(T)} f^{(T)}(x - c_3 t)\).

These observations generalize to anisotropy. Thus, for a given direction of propagation in an anisotropic solid the propagating zero velocity stress solutions partition the six-dimensional stress into two three-dimensional parts identified as propagating and non-propagating. The six states of stress may be defined in terms of an orthonormal set of stresses \(\{\sigma_a\}\) such that \(\sigma_a : \epsilon_b = \delta_{ab}\) for \(a, b \in \{1, \ldots, 6\}\). Hence, the strain energy of any linear combination of these propagating and non-propagating solutions is simply the sum of the strain energies for each of the six elements. Non-propagating stress satisfies, from Eq. (7), \(N : \sigma^{(0)} = 0\), which is equivalent to the zero traction condition \(\sigma^{(0)} n = 0\). Propagating solutions satisfy the compatibility condition \(M : \varepsilon = 0\). It is interesting how the dual conditions yield the dual sets of stress and strain: null vectors of \(M\) and \(N\) are, respectively, propagating stresses and non-propagating strains. Further considerations of stress waves in anisotropic solids will be the subject of future work.

Considering both the stress and displacement formulation provides a complete treatment of stress waves in infinite elastic solids. Future analysis of the stress-based solutions are expected to provide new and exciting insights. In some cases, solving stress wave problems using the stress formulation might be more natural, for example, in handling certain boundary value problems or when a prestress is prescribed in the solid. Further applications of the stress formulation are under development by the authors.

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