Coherent configurations and triply regular association schemes obtained from spherical designs

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March 30, 2009

Abstract

Delsarte-Goethals-Seidel showed that if $X$ is a spherical $t$-design with degree $s$ satisfying $t \geq 2s - 2$, $X$ carries the structure of an association scheme. Also Bannai-Bannai showed that the same conclusion holds if $X$ is an antipodal spherical $t$-design with degree $s$ satisfying $t = 2s - 3$. As a generalization of these results, we prove that a union of spherical designs with a certain property carries the structure of a coherent configuration. We derive triple regularity of tight spherical 4, 5, 7-designs, mutually unbiased bases, linked symmetric designs with certain parameters.

1 Introduction

Spherical codes and designs were studied by Delsarte-Goethals-Seidel [10]. There are two important parameters of finite set $X$ in the unit sphere $S^{d-1}$, that is, strength $t$ and degree $s$. In the paper [10], it is shown that $t \geq 2s - 2$ implies $X$ carries an $s$-class association scheme. Recently Bannai-Bannai [1] has shown that if $X$ is antipodal and $t = 2s - 3$, then $X$ carries an $s$-class association scheme.

Coherent configurations, that were introduced by D. G. Higman [11], are known as a generalization of association schemes. In Section 2, as an analogue of these results, we give a certain sufficient condition for a union of spherical designs to carry the structure of a coherent configuration. Our proof is based on the method of Delsarte-Goethals-Seidel [10, Theorem 7.4].

In Section 3, we consider triply regular association schemes which were introduced in connection with spin models by F. Jaeger [13] and have higher regularity than ordinary association schemes. Triple regularity is equivalent to the condition that the partition consisting of subconstituents relative to any point of the association scheme carries a coherent configuration whose parameters are independent of the point. In order to show that a symmetric association scheme is triply regular, we embed the scheme to the unit sphere $S^{d-1}$ by a primitive idempotent. This embedding has a partition of derived designs in $S^{d-2}$ for arbitrary point in the association scheme. Applying the main theorem of this paper to the union of derived designs, we obtain a sufficient condition for triple regularity of a symmetric association scheme.

In Sections 3-6, we consider tight spherical 4, 5, 7-designs, mutually unbiased bases (MUB), and linked symmetric designs with certain parameters. We note that tight spherical $t$-designs are classified except for $t = 4, 5, 7$. It is known that a tight spherical design, MUB, and a linked system of symmetric designs carry a symmetric association scheme [10, Theorem 7.4], [1, Theorem 1.1]. We will show that these symmetric association schemes are triply regular using our main theorem.
2 Coherent configurations obtained from spherical designs

Let $X$ be a finite set, we define $\text{diag}(X \times X) = \{(x, x) \mid x \in X\}$. Let $\{f_i\}_{i \in I}$ be a set of relations on $X$, we define $f_i^t = \{(y, x) \mid (x, y) \in f_i\}$. $(X, \{f_i\}_{i \in I})$ is a coherent configuration if the following properties are satisfied:

1. $\{f_i\}_{i \in I}$ is a partition of $X \times X$,
2. $f_i^t = f_i^*$ for some $i^* \in I$,
3. $f_i \cap \text{diag}(X \times X) \neq \emptyset$ implies $f_i \subset \text{diag}(X \times X)$,
4. for $i, j, k \in I$, the number $|\{z \in X \mid (x, z) \in f_i, (z, y) \in f_j\}|$ is independent of the choice of $(x, y) \in f_k$.

If moreover $f_0 = \text{diag}(X \times X)$ and $i^* = i$ for all $i \in I$, then we call $(X, \{f_i\}_{i \in I})$ a symmetric association scheme.

Let $X_1, \ldots, X_n$ be finite subsets of $S^{d-1}$. We denote by $\bigsqcup_{i=1}^n X_i$ the disjoint union of $X_1, \ldots, X_n$. We denote by $(x, y)$ the inner product of $x, y \in \mathbb{R}^d$. We define the nontrivial angle set $A(X_i, X_j)$ between $X_i$ and $X_j$ by

$$A(X_i, X_j) = \{(x, y) \mid x \in X_i, y \in X_j, x \neq \pm y\},$$

and the angle set $A'(X_i, X_j)$ between $X_i$ and $X_j$ by

$$A'(X_i, X_j) = \{(x, y) \mid x \in X_i, y \in X_j, x \neq \pm y\}.$$

If $i = j$, then $A(X_i, X_i)$ (resp. $A'(X_i, X_i)$) is abbreviated $A(X_i)$ (resp. $A'(X_i)$).

We define the intersection numbers on $X_j$ for $x, y \in S^{d-1}$ by

$$p^j_{\alpha, \beta}(x, y) = |\{z \in X_j \mid \langle x, z \rangle = \alpha, \langle y, z \rangle = \beta\}|.$$

For a positive integer $t$, a finite non-empty set $X$ in the unit sphere $S^{d-1}$ is called a spherical $t$-design in $S^{d-1}$ if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x)d\sigma(x)$$

for all polynomials $f(x) = f(x_1, \ldots, x_d)$ of degree not exceeding $t$. Here $|S^{d-1}|$ denotes the volume of the sphere $S^{d-1}$. When $X$ is a $t$-design and not a $(t+1)$-design, we call $t$ its strength.

We define the Gegenbauer polynomials $\{Q_k(x)\}_{k=0}^\infty$ on $S^{d-1}$ by

$$Q_0(x) = 1, \quad Q_1(x) = dx, \quad \frac{k+1}{d+2k}Q_{k+1}(x) = xQ_k(x) - \frac{d+k-3}{d+2k-4}Q_{k-1}(x).$$

Let $\text{Harm}(\mathbb{R}^d)$ be the vector space of the harmonic polynomials over $\mathbb{R}$ and $\text{Harm}_l(\mathbb{R}^d)$ be the subspace of $\text{Harm}(\mathbb{R}^d)$ consisting of homogeneous polynomials of total degree $l$. Let $\{\phi_{l,1}, \ldots, \phi_{l,h_l}\}$ be an orthonormal basis of $\text{Harm}_l(\mathbb{R}^d)$ with respect to the inner product

$$\langle \phi, \psi \rangle = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \phi(x)\psi(x)d\sigma(x).$$

Then the addition formula for the Gegenbauer polynomial holds [10] Theorem 3.3]:

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Lemma 2.1. \[ \sum_{i=1}^{h_l} \phi_{il}(x) \phi_{il}(y) = Q_l(\langle x, y \rangle) \] for any \( l \in \mathbb{N}, x, y \in S^{d-1} \).

We define the \( l \)-th characteristic matrix of a finite set \( X \subset S^{d-1} \) as the \( |X| \times h_l \) matrix
\[
H_l = (\phi_{il}(x))_{x \in X, 1 \leq i \leq h_l}.
\]

A criterion for \( t \)-designs using Gegenbauer polynomials and the characteristic matrices is known \cite{10}, Theorem 5.3, 5.5.

Lemma 2.2. Let \( X \) be a finite set in \( S^{d-1} \). The following conditions are equivalent:

1. \( X \) is a \( t \)-design,
2. \( \sum_{x,y \in X} Q_k(\langle x, y \rangle) = 0 \) for any \( k \in \{1, \ldots, t\} \),
3. \( H^T_k H_l = \delta_{k,l}|X|I \) for \( 0 \leq k + l \leq t \).

We define \( \{f_{\lambda,l}\}_{i=0}^\lambda \) as the coefficients of Gegenbauer expansion of \( x^\lambda \) for any nonnegative integers \( \lambda \), i.e., \( x^\lambda = \sum_{i=0}^\lambda f_{\lambda,l}Q_i(x) \), and let \( F_{\lambda,\mu}(x) = \sum_{i=0}^{\min\{\lambda,\mu\}} f_{\lambda,l} f_{\mu,i} Q_i(x) \), where \( \lambda, \mu \) are nonnegative integers.

The following three lemmas are used to prove Theorem 2.6 by using uniqueness of the solution of linear equations. Let \( A \) be a square matrix of size \( n \). For index sets \( I, J \subset \{1, \ldots, n\} \), we denote the submatrix that lies in the rows of \( A \) indexed by \( I \) and the columns indexed by \( J \) as \( A(I, J) \) and the complement of \( I \) as \( I' \). If \( I = \{i\} \) and \( J = \{j\} \), then \( A(I, J) \) is abbreviated \( A(i, j) \). A lemma which relates a minor of \( A^{-1} \) to that of \( A \) is the following:

Lemma 2.3. \cite{12} p.21] Let \( A \) be a nonsingular matrix, and let \( I, J \) be index sets of rows and columns of \( A \) with \( |I| = |J| \). Then
\[
\det A^{-1}(I', J') = (-1)^{\sum_{i \in I} i + \sum_{j \in J} j} \frac{\det A(I, J)}{\det A}.
\]

We define the \( k \)-th elementary symmetric polynomial \( e_k(x_1, \ldots, x_n) \) in \( n \) valuables \( x_1, \ldots, x_n \) by
\[
e_k(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } k = 0, \\ \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k} & \text{if } k \geq 1. \end{cases}
\]

We define the polynomial \( a_\lambda(x_1, \ldots, x_n) \) for a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) by
\[
a_\lambda(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \epsilon(\sigma) x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n)}^{\lambda_n},
\]
and the Schur function \( S_\lambda(x_1, \ldots, x_n) \) by
\[
S_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}(x_1, \ldots, x_n)}{a_\lambda(x_1, \ldots, x_n)},
\]
where \( \delta = (n-1, n-2, \ldots, 1, 0) \).

Lemma 2.4. Let \( A \) be a square matrix of order \( n \) with \( (i,j) \) entry \( a_i^{j-1} \), where \( \alpha_1, \cdots, \alpha_n \) are distinct. Then
\[
A^{-1}(i, j) = (-1)^{i+j} \prod_{1 \leq k < i} (\alpha_i - \alpha_k) \prod_{i < l \leq n} (\alpha_j - \alpha_l).
\]
Proof. Putting $\lambda = (1^{n-j}, 0^j)$, we have by [16, p.42],

$$A^{-1}(i,j) = (-1)^{i+j} \frac{\det A(\{j\}, \{i\})}{\det A} = (-1)^{i+j} \frac{\alpha_{\lambda+\delta}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)}{\det A} = \prod_{1\leq k<i} (\alpha_i - \alpha_k) \prod_{i<j\leq n} (\alpha_i - \alpha_j) \frac{\alpha_{\lambda+\delta}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)}{\det A} = \prod_{1\leq k<i} (\alpha_i - \alpha_k) \prod_{i<j\leq n} (\alpha_i - \alpha_j) S_\lambda(1^n, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n) = (-1)^{i+j} \prod_{1\leq k<i} (\alpha_i - \alpha_k) \prod_{i<j\leq n} (\alpha_i - \alpha_j) e_{n-j}(\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n)$$

Lemma 2.5. Let $A$ be a square matrix of order $n$ with $(i,j)$ entry $\alpha_{ij}$ and let $B$ be a square matrix of order $m$ with $(i,j)$ entry $\beta_{ij}$, where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ are distinct. Let $J, I$ be index sets of rows and columns, respectively, of $A \otimes B$ such that $J' = \{(n-1, m), (n, m-1), (n, m)\}$, $I' = \{(i_1, j_1), (i_2, j_2), (i_3, j_3)\}$. Then

$$\frac{\det (A \otimes B)(J, I)}{\det A \otimes B} = \pm \frac{\alpha_{i_1} \beta_{j_2} + \alpha_{i_2} \beta_{j_3} + \alpha_{i_3} \beta_{j_1} - \alpha_{i_1} \beta_{j_3} - \alpha_{i_2} \beta_{j_1} - \alpha_{i_3} \beta_{j_2}}{\prod_{1 \leq r < s \leq 3} (\alpha_i - \alpha_k) \prod_{i_1 < j_2 \leq n} (\alpha_i - \alpha_{i_2}) \prod_{1 \leq k < j_r \leq m} (\beta_i - \beta_{j_r})}.$$ 

Proof. We define $f(i, j) = \prod_{1 \leq k < i} (\alpha_i - \alpha_k) \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j) \prod_{1 \leq k < j_r \leq m} (\beta_i - \beta_{j_r})$. Using Lemmas 2.3 and 2.4,

$$\frac{\det (A \otimes B)(J, I)}{\det A \otimes B} = \pm \det (A \otimes B)^{-1}(I', J') = \pm \det (A^{-1} \otimes B^{-1})(I', J') = \pm \det \begin{pmatrix} (-1)^{i_1+n-j_1+1} \alpha_{i_1} \beta_{j_2} & (-1)^{i_1+n-j_1+1} \alpha_{i_1} \beta_{j_3} & (-1)^{i_1+n-j_1+1} \alpha_{i_1} \beta_{j_1} \\ (-1)^{i_2+n-j_2+1} \alpha_{i_2} \beta_{j_3} & (-1)^{i_2+n-j_2+1} \alpha_{i_2} \beta_{j_1} & (-1)^{i_2+n-j_2+1} \alpha_{i_2} \beta_{j_2} \\ (-1)^{i_3+n-j_3+1} \alpha_{i_3} \beta_{j_1} & (-1)^{i_3+n-j_3+1} \alpha_{i_3} \beta_{j_2} & (-1)^{i_3+n-j_3+1} \alpha_{i_3} \beta_{j_3} \end{pmatrix} \frac{1}{\prod_{1 \leq r < s \leq 3} f(i_r, j_r)} \det \begin{pmatrix} \alpha_{i_1} & \sum_{j \neq j_1} \beta_j & 1 \\ \sum_{i_1 \neq i_2} \alpha_i & \sum_{j \neq j_2} \beta_j & 1 \\ \sum_{i_1 \neq i_3} \alpha_i & \sum_{j \neq j_3} \beta_j & 1 \end{pmatrix} \frac{1}{\prod_{1 \leq r < s \leq 3} f(i_r, j_r)} \begin{pmatrix} \alpha_{i_1} \beta_{j_1} & 1 \\ \alpha_{i_2} \beta_{j_2} & 1 \\ \alpha_{i_3} \beta_{j_3} & 1 \end{pmatrix} \frac{1}{\prod_{1 \leq r < s \leq 3} f(i_r, j_r)} \prod_{1 \leq k < \ell \leq n} (\alpha_i - \alpha_k) \prod_{1 \leq k < \ell \leq m} (\beta_i - \beta_{\ell}).$$ 

$\square$
The following is the main theorem of this paper.

**Theorem 2.6.** Let $X_i \subset S^{d-1}$ be a spherical $t_i$-design for $i \in \{1, \ldots, n\}$. Assume that $X_i \cap X_j = \emptyset$ or $X_i = X_j$, and $X_i \cap (-X_j) = \emptyset$ or $X_i = -X_j$ for $i, j \in \{1, \ldots, n\}$. Let $s_{i,j} = |A(X_i, X_j)|$, $s_{i,j}^* = |A'(X_i, X_j)|$ and $A(X_i, X_j) = \{\alpha_{i,j}^1, \ldots, \alpha_{i,j}^{s_{i,j}}\}$, $\alpha_{i,j}^0 = 1$, when $-1 \in A'(X_i, X_j)$, we define $\alpha_{i,j}^{s_{i,j}} = -1$. We define $R_{i,j}^k = \{(x, y) \in X_i \times X_j \mid \langle x, y \rangle = \alpha_{i,j}^k\}$. If one of the following holds depending on the choice of $i, j, k \in \{1, \ldots, n\}$:

1. $s_{i,j} + s_{j,k} - 2 \leq t_j$,

2. $s_{i,j} + s_{j,k} - 3 = t_j$ and for any $\gamma \in A(X_i, X_k)$ there exist $\alpha \in A(X_i, X_j), \beta \in A(X_j, X_k)$ such that the number $p_{\alpha,\beta}(x, y)$ is independent of the choice of $x \in X_i, y \in X_k$ with $\gamma = \langle x, y \rangle$,

3. $s_{i,j} + s_{j,k} - 4 = t_j$ and for any $\gamma \in A(X_i, X_k)$ there exist $\alpha, \alpha' \in A(X_i, X_j), \beta, \beta' \in A(X_j, X_k)$ such that $\alpha \neq \alpha'$, $\beta \neq \beta'$ and the numbers $p_{\alpha,\beta}(x, y), p_{\alpha',\beta'}(x, y)$ and $p_{\alpha,\beta'}(x, y)$ are independent of the choice of $x \in X_i, y \in X_k$ with $\gamma = \langle x, y \rangle$,

then \(\bigcup_{i=1}^n X_i, \{R_{i,j}^k \mid 1 \leq i, j \leq n, 1 - \delta_{X_i,X_j} \leq k \leq s_{i,j}^*\}\) is a coherent configuration. The parameters of this coherent configuration are determined by $A(X_i, X_j), |X_i|, t_i, \delta_{X_i,X_j}, \delta_{X_i,-X_j}$, and when $s_{i,j} + s_{j,k} - 3 = t_j$ (resp. $s_{i,j} + s_{j,k} - 4 = t_j$), the numbers $p_{\alpha,\beta}(x, y)$ (resp. $p_{\alpha',\beta'}(x, y)$, $p_{\alpha,\beta'}(x, y)$) which are assumed be independent of $(x, y)$ with $\langle x, y \rangle = \gamma$.

Proof. Let $x \in X_i, y \in X_k$ be such that $\gamma = \langle x, y \rangle$. It is sufficient to show that the number $p_{\alpha,\beta}(x, y)$ depends only on $\gamma$ and does not depend on the choice of $x \in X_i, y \in X_k$ satisfying $\gamma = \langle x, y \rangle$.

For the ease of notation, let $\alpha_l = \alpha_{i,j}^l$ and $\beta_m = \alpha_{j,k}^m$.

We define a mapping $\phi_l : S^{d-1} \to \mathbb{R}^{H_l}$ by $\phi_l(x) = (\varphi_{l,1}(x), \ldots, \varphi_{l,H_l}(x))$. Let $H_l$ be the $l$-th characteristic matrix of $X_j$. For any non-negative integers $\lambda$ and $\mu$ satisfying $\lambda + \mu \leq t_j$, we calculate

$$\sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)H_l^t \left( \sum_{m=1}^{\mu} f_{\mu,m}H_m\phi_m(y)^t \right)$$

in two different ways.

First we use Lemma 2.2 and Lemma 2.4 in turn, to obtain the following equality:

\[
\left( \sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)H_l^t \right) \left( \sum_{m=1}^{\mu} f_{\mu,m}H_m\phi_m(y)^t \right) = \sum_{l=1}^{\min\{\lambda,\mu\}} f_{\lambda,l}f_{\mu,l}\phi_l(x)\phi_l(y)^t = \sum_{l=1}^{\min\{\lambda,\mu\}} f_{\lambda,l}f_{\mu,l}Q_l\langle x, y \rangle = |X_j|F_{\lambda,\mu}(\langle x, y \rangle).
\]

(2.1)

Next using Lemma 2.1 we obtain the following equality:

\[
\sum_{l=1}^{\lambda} f_{\lambda,l}\phi_l(x)H_l^t \left( \sum_{m=1}^{\mu} f_{\mu,m}H_m\phi_m(y)^t \right) = \sum_{z \in X_j} \sum_{l=1}^{\lambda} f_{\lambda,l}(\phi_l(x)\phi_l(z)^t) \left( \sum_{m=1}^{\mu} f_{\mu,m}(\phi_m(z)\phi_m(y)^t) \right)
\]

Next using Lemma 2.1, we obtain the following equality:
\[
\sum_{z \in X_j} \delta(2.3) \text{ is the following:} \\
A \therefore G_{i,j,k}(x, y) \text{ satisfies (2) i.e., for } 0 \leq \lambda \leq s_{i,j} - 1 \text{ and } 0 \leq \mu \leq s_{j,k} - 1, \text{ yields a system of } s_{i,j}s_{j,k} \text{ linear equations whose unknowns are} \\
\{ p^{i,j}_{\alpha_l,\beta_m}(x, y) | 1 \leq l \leq s_{i,j}, 1 \leq m \leq s_{j,k} \}. \\
\text{Its coefficient matrix } A \otimes B \text{ is nonsingular, where} \\
A = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_{s_{i,j}} \\ \vdots & \ddots & \vdots \\ \alpha_{s_{i,j}}^{-1} & \cdots & \alpha_{s_{i,j}}^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & \cdots & 1 \\ \beta_1 & \cdots & \beta_{s_{j,k}} \\ \vdots & \ddots & \vdots \\ \beta_{s_{j,k}}^{-1} & \cdots & \beta_{s_{j,k}}^{-1} \end{pmatrix}. \\
\text{Therefore } p^{i,j}_{\alpha_l,\beta_m}(x, y) \text{ for } 1 \leq l \leq s_{i,j}, 1 \leq m \leq s_{j,k} \text{ depends only on } \gamma \text{ and does not depend on the choice of } x, y \text{ satisfying } \gamma = \langle x, y \rangle, \text{ and is determined by } A(X_i, X_j, X_k), A(X_j, X_k), \gamma, |X_j|, t_j, \delta_{X_i,X_j}, \delta_{X_j,X_k}, \delta_{X_i,X_k}, \delta_{X_j} - X_k. \\
\text{In the case where } i, j, k \text{ satisfy (2) i.e., for } \langle x, y \rangle = \gamma \in A(X_i, X_k), \text{ there exist } \alpha_* \in A(X_i, X_j), \beta_{m^*} \in A(X_j, X_k) \text{ such that the number } p^{i,j}_{\alpha_*,\beta_{m^*}}(x, y) \text{ is uniquely determined. The linear equation (2.4) is the following:} \\
\sum_{\substack{l \leq s_{i,j} \leq s_{i,j} s_{j,k} \leq s_{j,k} \\l(m) \neq (l,m)^*}} \alpha_l^\lambda \beta_m^\mu p^{i,j}_{\alpha_*,\beta_{m^*}}(x, y) = |X_j| F_{\lambda,\mu}(\langle x, y \rangle) - G^{i,j,k}_{\lambda,\mu}(\langle x, y \rangle) - \alpha_l^\lambda \beta_m^\mu p^{i,j}_{\alpha_l,\beta_m}(x, y). \quad (2.4)
\]
For \(0 \leq \lambda \leq s_{i,j} - 1, 0 \leq \mu \leq s_{j,k} - 1\) and \((\lambda, \mu) \neq (s_{i,j} - 1, s_{j,k} - 1)\), (2.4) yields a system of \(s_{i,j}s_{j,k} - 1\) linear equations whose unknowns are

\[
\{p_{\alpha_1,\beta_m}^j(x, y) \mid 1 \leq l \leq s_{i,j}, 1 \leq m \leq s_{j,k}, (l, m) \neq (l^*, m^*)\}.
\]

The coefficient matrix \(C_1\) of these linear equations is the submatrix obtained by deleting the \((s_{i,j}, s_{j,k})\)-row and \((l^*, m^*)\)-column of \(A \otimes B\). Using Lemma 2.4, the determinant of \(C_1\) is, up to sign,

\[
\det C_1 = \pm ((s_{i,j}, s_{j,k}), (l^*, m^*)) - \text{cofactor of } A \otimes B
\]

\[
= \pm ((l^*, m^*), ((s_{i,j}, s_{j,k})) - \text{entry of } (A \otimes B)^{-1}) \det A \otimes B
\]

\[
= \pm ((l^*, s_{i,j}) - \text{entry of } A^{-1}) \times ((m^*, s_{j,k}) - \text{entry of } B^{-1}) \det A \otimes B
\]

\[
= \pm \frac{\prod_{1 \leq k < l^*} (\alpha_{l^*} - \alpha_k) \prod_{l^* < i \leq s_{i,j}} (\alpha_i - \alpha_{l^*}) \prod_{1 \leq k < m^*} (\beta_{m^*} - \beta_k) \prod_{m^* < k \leq s_{j,k}} (\beta_k - \beta_{m^*})}{\det A \otimes B}.
\]

Hence \(C_1\) is nonsingular.

Therefore \(p_{\alpha_1,\beta_m}^j(x, y)\) for \(1 \leq l \leq s_{i,j}, 1 \leq m \leq s_{j,k}; (l, m) \neq (l^*, m^*)\) depends only on \(\gamma\) and does not depend on the choice of \(x, y\) satisfying \(\gamma = \langle x, y \rangle\), and is determined by \(A(X_i, X_j), A(X_j, X_k), \gamma, |X_j|, t_j, \delta_{X_i,X_j}, \delta_{X_j,X_k}, \delta_{X_k,-X_i}, \delta_{X_j,-X_k}\), the numbers \(p_{\alpha_1,\beta_m}^j(x, y)\) which are assumed independent of \((x, y)\) with \(\langle x, y \rangle = \gamma\).

In the case where \(i, j, k\) satisfy (3) i.e., for \(\langle x, y \rangle = \gamma \in A(X_i, X_k)\) there exist \(\alpha_{l_1}, \alpha_{l_2} \in A(X_i, X_j), \beta_{m_1}, \beta_{m_2} \in A(X_j, X_k)\) such that the numbers \(p_{\alpha_1,\beta_m}^j(x, y), p_{\alpha_1,\beta_m}^j(x, y), p_{\alpha_1,\beta_m}^j(x, y)\) are uniquely determined. The linear equation (2.3) is the following:

\[
\sum_{1 \leq l \leq s_{i,j}} \alpha_l^\lambda \beta_m^\mu p_{\alpha_1,\beta_m}^j(x, y) = |X_j| F_{\lambda,\mu}(\langle x, y \rangle) - G_{\lambda,\mu}^{i,j,k}(\langle x, y \rangle) - \alpha_l^\lambda \beta_m^\mu p_{\alpha_1,\beta_m}^j(x, y)
\]

\[
- \alpha_l^\lambda \beta_m^\mu p_{\alpha_1,\beta_m}^j(x, y) - \alpha_l^\lambda \beta_m^\mu p_{\alpha_1,\beta_m}^j(x, y).
\]

(2.5)

For \(0 \leq \lambda \leq s_{i,j} - 1, 0 \leq \mu \leq s_{j,k} - 1\) and \((\lambda, \mu) \neq (s_{i,j} - 2, s_{j,k} - 1), (s_{i,j} - 1, s_{j,k} - 2), (s_{i,j} - 1, s_{j,k} - 1)\), (2.5) yields a system of \(s_{i,j}s_{j,k} - 3\) linear equations whose unknowns are

\[
\{p_{\alpha_1,\beta_m}^j(x, y) \mid 1 \leq l \leq s_{i,j}, 1 \leq m \leq s_{j,k}, (l, m) \neq (l_1, m_1), (l_1, m_2), (l_2, m_1)\}.
\]

The coefficient matrix \(C_2\) of these linear equations is the submatrix obtained by deleting the \((s_{i,j} - 1, s_{j,k} - 1), (s_{i,j}, s_{j,k} - 1), (s_{i,j}, s_{j,k})\)-rows and \((l_1, m_1), (l_1, m_2), (l_2, m_1)\)-columns of \(A \otimes B\). Let \(J, I\) be index sets of rows and columns, respectively, of \(A \otimes B\) such that

\[J' = \{(s_{i,j} - 1, s_{j,k}), (s_{i,j}, s_{j,k} - 1), (s_{i,j}, s_{j,k})\}\]

and

\[I' = \{(l_1, m_1), (l_1, m_2), (l_2, m_1)\}.
\]

Setting \((i_1, j_1), (i_2, j_2), (i_3, j_3)\) to be \((l_1, m_1), (l_1, m_2), (l_2, m_1)\) respectively, we have

\[
\alpha_{i_1} \beta_{j_2} + \alpha_{i_2} \beta_{j_3} + \alpha_{i_3} \beta_{j_1} - \alpha_{i_1} \beta_{j_3} - \alpha_{i_2} \beta_{j_1} - \alpha_{i_3} \beta_{j_2} = (\alpha_{i_1} - \alpha_{i_2})(\beta_{m_1} - \beta_{m_2}).
\]

Hence \(C_2\) is nonsingular by Lemma 2.5. Therefore \(p_{\alpha_1,\beta_m}^j(x, y)\) for \(1 \leq l \leq s_{i,j}, 1 \leq m \leq s_{j,k}, (l, m) \neq (l_1, m_1), (l_1, m_2), (l_2, m_1)\) depends only on \(\gamma\) and does not depend on the choice of \(x, y\) satisfying \(\gamma = \langle x, y \rangle\), and is determined by \(A(X_i, X_j), A(X_j, X_k), \gamma, |X_j|, t_j, \delta_{X_i,X_j}, \delta_{X_j,X_k}, \delta_{X_k,-X_i}, \delta_{X_j,-X_k}\), the numbers \(p_{\alpha_1,\beta_m}^j(x, y), p_{\alpha_1,\beta_m}^j(x, y), p_{\alpha_1,\beta_m}^j(x, y)\) which are assumed independent of \((x, y)\) with \(\langle x, y \rangle = \gamma\). \(\blacksquare\)
Several results known for the case \( n = 1 \) are derived from Theorem 2.6. We consider the case where \( n = 1 \) and \( X = X_1 \) is a \( t \)-design of degree \( s \). Then \( t = t_1 \) and

\[
s_{1,1} = \begin{cases} 
  s - 1 & \text{if } X \text{ is antipodal,} \\
  s & \text{if } X \text{ otherwise.}
\end{cases}
\]

Suppose \( t \geq 2s - 2 \). If \( X \) is antipodal, then \( t_1 \geq 2s_{1,1} \), and if \( X \) is not antipodal, then \( t_1 \geq 2s_{1,1} - 2 \). Thus \( X \) satisfies the assumption (1) of Theorem 2.6 and hence \( X \) carries a symmetric association scheme. So Theorem 2.6 contains the first half of [10, Theorem 7.4] as a special case.

Suppose \( t = 2s - 3 \) and \( p_{\gamma,\gamma}(x, y) \) is uniquely determined for any fixed \( \gamma = (x, y) \in A'(X) \). If \( X \) is antipodal, then \( t_1 = 2s_{1,1} - 1 \), and if \( X \) is not antipodal, then \( t_1 = 2s_{1,1} - 3 \). Thus \( X \) also satisfies the assumption (1) or (2) of Theorem 2.6 and hence \( X \) carries a symmetric association scheme. So Theorem 2.6 contains the second half of [10, Theorem 7.4] as a special case.

Suppose that \( t = 2s - 3 \). If \( X \) is antipodal, then \( t_1 = 2s_{1,1} - 1 \). Thus \( X \) satisfies the assumption (1) of Theorem 2.6 and hence \( X \) carries a symmetric association scheme. So Theorem 2.6 contains [1, Theorem 1.1] as a special case.

Next, we consider triple regularity of a symmetric association scheme. This concept was introduced in connection with spin models [13].

**Definition 2.7.** Let \((X, \{R_i\}_{i=0}^d)\) be a symmetric association scheme. Then the association scheme \( X \) is said to be triply regular if, for all \( i, j, k, l, m, n \in \{0, 1, \ldots, d\} \), and for all \( x, y, z \in X \) such that \((x, y) \in R_i, (y, z) \in R_j, (z, x) \in R_k\), the number \( p_{i,j,k} := |\{w \in X \mid (w, x) \in R_i, (w, y) \in R_j, (w, z) \in R_k\}| \) depends only on \( i, j, k, l, m, n \) and not on \( x, y, z \).

Let \((X, \{R_i\}_{i=0}^d)\) be an association scheme. We define the \( i \)-th subconstituent with respect to \( z \in X \) by \( R_i(z) := \{y \in X \mid (y, z) \in R_i\} \). We denote by \( R_{i,j}(z) \) the restriction of \( R_k \) to \( R_i(z) \times R_j(z) \). The following lemma gives an equivalent definition of a triply regular association scheme. We omit its easy proof.

**Lemma 2.8.** A symmetric association scheme \((X, \{R_i\}_{i=0}^d)\) is triply regular if and only if for all \( z \in X \), \((\bigcup_{i=0}^d R_i(z), \{R_{i,j}^z(z) \mid 1 \leq i, j \leq d, 0 \leq k \leq d, p_{i,j,k}^z = 0\})\) is a coherent configuration whose parameters are independent of \( z \).

Let \( X \) be a spherical \( t \)-design in \( S^{d-1} \) with degree \( s \), and \( A'(X) = \{\alpha_1, \ldots, \alpha_s\} \). For \( z \in X \) and \( i \in \{1, \ldots, s\} \), \( X_i(z) \) will denote the orthogonal projection of \( \{y \in X \mid \langle y, z \rangle = \alpha_i\} \) to \( z^\perp = \{y \in \mathbb{R}^d \mid \langle y, z \rangle = 0\} \), rescaled to lie in \( S^{d-2} \) in \( z^\perp \). \( X_i(z) \) is called the derived design. In fact \( X_i(z) \) is a \((t + 1 - s^*)\)-design by [10, Theorem 8.2], where \( s^* = |A'(X) \setminus \{\alpha_1\}| \). We define \( \alpha_{i,j}^k = \frac{\alpha_i - \alpha_j}{\sqrt{(1 - \alpha_i^2)(1 - \alpha_j^2)}} \). If \( \langle x, z \rangle = \alpha_i, \langle y, z \rangle = \alpha_j \) and \( \langle x, y \rangle = \alpha_k \), then the inner product of the orthogonal projection of \( x, y \) to \( z^\perp \) rescaled to lie in \( S^{d-2} \), is \( \alpha_{i,j}^k \).

**Corollary 2.9.** Let \( X \subset S^{d-1} \) be a finite set and \( A'(X) = \{\alpha_1, \ldots, \alpha_s\} \). Assume that \((X, \{R_k\}_{k=0}^s)\) is a symmetric association scheme, where \( R_k = \{(x, y) \in X \times X \mid \langle x, y \rangle = \alpha_k\} \) \((0 \leq k \leq s)\) and \( \alpha_0 = 1 \). Then

1. \( A(X_i(z), X_j(z)) = \{\alpha_{i,j}^k \mid 0 \leq k \leq s, p_{i,j}^k \neq 0, \alpha_{i,j}^k \neq \pm 1\} \).
2. \( X_i(z) = X_j(z) \) or \( X_i(z) \cap X_j(z) = \emptyset \), and \( X_i(z) = -X_j(z) \) or \( X_i(z) \cap -X_j(z) = \emptyset \) for any \( z \in X \) and any \( i, j \in \{1, \ldots, s\} \). And \( \delta_{X_i(z), X_j(z)} \), \( \delta_{X_i(z), -X_j(z)} \) are independent of \( z \in X \).
3. \( X_i(z) \) has the same strength for all \( z \in X \).
Moreover if the assumption (1), (2) or (3) of Theorem 2.6 is satisfied for \( \{X_i(z)\}_{i=1}^t \), and when \((i,j,k)\) satisfies (2) (resp. (3)) the numbers \( p_{\alpha,\beta}(x,y) \) (resp. \( p_{\alpha,\beta'}(x,y), p_{\alpha',\beta}(x,y), p_{\alpha',\beta'}(x,y) \)) which are assumed to be independent of \((x,y)\) with \( \gamma = (x,y) \) are independent of the choice of \( z \), then \((X,\{R_k\}_{k=0})\) is a triply regular association scheme.

Proof. Let \( z \in X \). (1) is immediate from the definition of \( \alpha_{i,j}^k \).

We define \( R_{i,j}^k(z) = \{ (x,y) \in X_i(z) \times X_j(z) \mid \langle x,y \rangle = \alpha_{i,j}^k \} \). Then

\[
\{(x,y) \mid x \in X_i(z), y \in X_j(z)\} \ni \pm 1 \\
\exists k \quad \alpha_{i,j}^k = \pm 1 \text{ and } p_{i,j}^k \neq 0 \\
\exists k \quad \alpha_{i,j}^k = \pm 1, \text{ and} \\
\forall x \in X_i(z) \exists y \in X_j(z) \text{ s.t. } (x,y) \in R_{i,j}^k(z) \text{ and} \\
\forall y \in X_j(z) \exists x \in X_i(z) \text{ s.t. } (x,y) \in R_{i,j}^k(z) \\
\iff X_i(z) = \pm X_j(z).
\]

Since

\[
\{(x,y) \mid x \in X_i(z), y \in X_j(z)\} = \{\alpha_{i,j}^k \mid 0 \leq k \leq s, p_{i,j}^k \neq 0\}
\]

is independent of \( z \in X \), (2) holds.

By Lemma 2.2, \( X_i(z) \) is a spherical \( t \)-design if and only if \( \sum_{x,y \in X_i(z)} Q_k(\langle x,y \rangle) = 0 \) for \( k = 1, \ldots, t \). Since the number of \( y \in X_i(z) \) satisfying \( \langle x,y \rangle = \frac{\alpha_i - \alpha_j^2}{1 - \alpha_i^2} \) is \( p_{i,j}^k \) for any \( x \in X_i(z) \), the latter condition is equivalent to \( \sum_{0 \leq j \leq s} Q_k(\frac{\alpha_i - \alpha_j^2}{1 - \alpha_i^2}) p_{i,j}^k = 0 \) for \( k = 1, \ldots, t \), which is independent of \( z \). Hence \( X_i(z) \) has the same strength for all \( z \in X \). Therefore (3) holds.

Moreover if the assumption (1), (2) or (3) of Theorem 2.6 is satisfied for \( \{X_i(z)\}_{i=1}^t \), then \((\bigcup_{i=1}^t X_i(z), \{R_{i,j}^k(z) \mid 0 \leq i,j,k \leq s, p_{i,j}^k \neq 0\})\) is a coherent configuration. Clearly, \( |X_i(z)| \) is independent of \( z \in X \). Also, \( A(X_i(z), X_j(z)) \) is independent of \( z \in X \) by (1), \( t_i \) is independent of \( z \in X \) by (3), and \( \delta_{X_i(z), X_j(z)}, \delta_{X_i(z), -X_j(z)} \) are independent of \( z \in X \) by (2). It follows from Theorem 2.6 that the parameters of the coherent configuration are independent of \( z \in X \). Therefore, \((X, \{R_k\}_{k=0})\) is a triply regular association scheme by Lemma 2.8. \( \square \)

3. Tight designs

Let \( X \) be a \( t \)-design in \( S^{d-1} \). It is known [10] Theorems 5.11, 5.12] that there is a lower bound for the size of a spherical \( t \)-design in \( S^{d-1} \). Namely, if \( X \) is a spherical \( t \)-design, then

\[
|X| \geq \binom{d + t/2 - 1}{t/2} + \binom{n + t/2 - 2}{t/2}
\]

if \( t \) is even, and

\[
|X| \geq 2 \binom{d + (t - 3)/2}{(t - 1)/2}
\]

if \( t \) is odd. If \( X \) is a \( t \)-design for which one of the lower bounds is attained, then \( X \) is called a tight \( t \)-design. It was proved in [2, 3, 10] that if \( X \) is a tight \( t \)-design with degree \( s \) in \( S^{d-1} \), then the following statements hold.

1. If \( t \) is even, then \( t = 2s \),
2. If \( t \) is odd, then \( t = 2s - 1 \) and \( X \) is antipodal,
(3) if $d = 2$, then $X$ is the regular $(t + 1)$-gon,

(4) if $d \geq 3$, then $t \leq 5$ or $t = 7, 11$.

If $X$ is a tight 11-design in $S^{d-1}$ where $d \geq 3$, then $d = 24$ and $X$ is the set of minimum vectors of the Leech lattice [5]. We consider tight 4-, 5-, 7-designs in $S^{d-1}$ where $d \geq 3$.

Let $X \subset S^{d-1}$ be a tight $2s$-design, and let $A'(X) = \{\alpha_i \mid 1 \leq i \leq s\}$. For any $z \in X$, $X_i(z)$ is a $t_i := t + 1 - s^* = (s + 1)$-design in $S^{d-2}$. Then the degrees $s_{i,j} = |A(X_i(z), X_j(z))|$ satisfy $s_{i,j} \leq s$, and the following holds:

$$2s - 2 \leq s + 1 \iff s \leq 3 \iff t = 2, 4, 6.$$ In particular, if $t = 4$, then $s_{i,j} + s_{j,k} - 2 \leq t_j$ holds, i.e., the assumption (1) of Theorem 2.6 holds for all $i, j, k$. By Corollary 2.9, we obtain the following result.

**Corollary 3.1.** Every tight 4-design carries a triply regular association scheme.

The same argument shows that a spherical 3-design with degree 2 i.e., a strongly regular graph with $a^*_1 = 0$ carries a triply regular association scheme. This is already known (see [9]).

Let $X \subset S^{d-1}$ be a tight $(2s - 1)$-design, and let $A'(X) = \{\alpha_i \mid 1 \leq i \leq s\}$ where $a_s = -1$. For any $z \in X$ and $i \neq s$, $X_i(z)$ is a $t_i := t + 1 - s^* = (s + 1)$-design in $S^{d-2}$.

Then the degrees $s_{i,j} = |A(X_i(z), X_j(z))|$ satisfy $s_{i,j} \leq s - 1$, and the following holds:

$$2s - 4 \leq s + 1 \iff s \leq 5 \iff t = 1, 3, 5, 7, 9.$$ In particular, if $t = 5, 7$, then $s_{i,j} + s_{j,k} - 2 \leq t_j$ holds, i.e., the assumption (1) of Theorem 2.6 holds for all $i, j, k$. By Corollary 2.9, we obtain the following result.

**Corollary 3.2.** Every tight 5- or 7-design carries a triply regular association scheme.

The same argument shows that an antipodal spherical 3-design with degree 3 carries a triply regular association scheme i.e., subconstituents of a Taylor graph are strongly regular graphs. This is already known (see [6, Theorem 1.5.3]).

### 4 Derived designs of $Q$-polynomial association schemes

The reader is referred to [4] for the basic information on $Q$-polynomial association schemes. The following lemma is used to prove Lemma 4.2.

**Lemma 4.1.** Let $X = (X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme of class $d$. Let $B_i = (p_{i,j}^k)$ be its $i$-th intersection matrix, and $Q = (q_j(i))$ be the second eigenmatrix of $X$. Then

$$(Q'B_i)(h, i) = \frac{k_i q_h(i)^2}{m_h} \quad (0 \leq h, i \leq d).$$

**Proof.** See [4, p.73 (4.2) and Theorem 3.5(i)].

The following lemma gives a property of derived designs of the embedding of a $Q$-polynomial association scheme into the first eigenspace.
Lemma 4.2. Let \((X, \{R_i\}_{i=0}^s)\) be a \(Q\)-polynomial association scheme, and we identify \(X\) as the image of the embedding into the first eigenspace by \(E_1 = \frac{1}{|X|} \sum_{j=0}^s \theta_j^* A_j\). Then, for \(i \in \{1, \ldots, s\}\) with \(\theta_i^* \neq -\theta_0^*\), the derived design \(X_i(z)\) is a 2-design in \(S^{\theta_0^*-2}\) for any \(z \in X\) if and only if \(a_i^*(\theta_i^* + 1) = 0\).

Proof. The angle set of \(X_i(z)\) consists of

\[
\frac{\theta_j^* - \theta_i^*}{1 - (\theta_j^* \theta_i^*)^2} = \frac{\theta_j^* - \theta_i^*}{\theta_j^2 - \theta_i^2} \quad (0 \leq k \leq s, \ p_{i,i}^k \neq 0).
\]

Thus, Lemma 2.2 implies that \(X_i(z)\) is a 2-design in \(S^{\theta_0^*-2}\) if and only if

\[
\sum_{j=0}^s Q_k(\frac{\theta_j^* - \theta_i^*}{\theta_j^2 - \theta_i^2}) p_{i,j}^k = 0 \quad (k = 1, 2),
\]

where \(Q_k(x)\) is the Gegenbauer polynomial of degree \(k\) in \(S^{\theta_0^*-2}\).

Since \(Q_1(x) = (\theta_0^* - 1)x, \sum_{j=0}^s p_{i,j}^j = k_i\) and

\[
\sum_{j=0}^s \theta_j^* p_{i,j}^j = (Q^j B_i)(1, i) = \frac{k_i q_1(i)^2}{m_1} \quad \text{(by Lemma 4.1)}
\]

we have

\[
\sum_{j=0}^s Q_1(\frac{\theta_j^* - \theta_i^*}{\theta_j^2 - \theta_i^2}) p_{i,j}^1 = \frac{\theta_0^* - 1}{\theta_0^2 - \theta_i^2} \left( \theta_0^* \sum_{j=0}^s \theta_j^* p_{i,j}^j - \theta_i^2 \sum_{j=0}^s p_{i,j}^j \right)
\]

\[
= 0.
\]

Since \(Q_2(x) = (\theta_0^* - 1)x^2 - 1, \sum_{j=0}^s p_{i,j}^j = k_i, \text{(4.1)}\) and

\[
\sum_{j=0}^s \theta_j^* p_{i,j}^j = \sum_{j=0}^s (c_2 q_2(i) + a_i^* q_1(i) + b_i^* q_0(i)) p_{i,j}^j
\]

\[
= c_2^* (Q^j B_i)(2, i) + a_i^* k_i \theta_i^2 + \theta_0^* k_i \quad \text{(by (4.1))}
\]

\[
= c_2^* (Q^j B_i)(2, i) + k_i (\theta_i^2 + \theta_0^* - \theta_i^* - \theta_0^*)
\]

\[
= k_i \left( \frac{((\theta_i^* - a_i^*)^2 + \theta_i^* - \theta_0^*)^2}{(\theta_i^* - a_i^*)^2 + \theta_i^*} \right),
\]

we have

\[
\sum_{j=0}^s Q_2(\frac{\theta_j^* - \theta_i^*}{\theta_j^2 - \theta_i^2}) p_{i,j}^j = \frac{\theta_0^* - 1}{(\theta_0^* - \theta_i^*)^2} \left( \theta_0^* \sum_{j=0}^s \theta_j^* p_{i,j}^j - 2 \theta_0^* \theta_i^2 \sum_{j=0}^s \theta_j^* p_{i,j}^j + \theta_i^4 \sum_{j=0}^s p_{i,j}^j \right) - k_i
\]

\[
= 0.
\]
Lemma 5.2. from MUB.

An equivalence relation $S$ define $x$ real mutually unbiased bases (MUB) if any two vectors in $S$ then the assumption (1) of Theorem 2.6 holds. We remark that Definition 5.1. therefore $X$ is a 2-design in $S^{m-2}$ if and only if $a_1^2(\theta^* + 1) = 0$.

5 Real mutually unbiased bases

Definition 5.1. Let $M = \{M_i\}_{i=1}^f$ be a collection of orthonormal bases of $\mathbb{R}^d$. $M$ is called real mutually unbiased bases (MUB) if any two vectors $x$ and $y$ from different bases satisfy $\langle x, y \rangle = \pm 1/\sqrt{d}$.

It is known that the number $f$ of real mutually unbiased bases in $\mathbb{R}^d$ can be at most $d/2 + 1$. We call $M$ a maximal MUB if this upper bound is attained. Constructions of maximal MUB are known only for $d = 2^m + 1$, $m$ odd [7]. Throughout this section, we assume $M = \{M_i\}_{i=1}^f$ is an MUB, put $X^{(i)} = M_i \cup (-M_i)$ and $X = M \cup (-M)$. The angle set of $X$ is $A'(X) = \{1/\sqrt{d}, 0, -1/\sqrt{d}, -1\}$.

We set $a_0 = 1$, $a_1 = 1/\sqrt{d}$, $a_2 = 0$, $a_3 = -1/\sqrt{d}$, $a_4 = -1$, and we define $R_k = \{(x, y) \in X \times X \mid \langle x, y \rangle = a_k\}$.

Since $X^{(i)}$ is a spherical 3-design in $S^{d-1}$ for any $i \in \{1, \ldots, f\}$, $X$ is also a spherical 3-design in $S^{d-1}$. It is shown in [14] that $(X, \{R_k\}_{k=0}^4)$ is a $Q$-polynomial association scheme with $a_1^2 = 0$. $X$ is imprimitive and the set $\{X^{(i)}, \ldots, X^{(j)}\}$ is a system of imprimitivity with respect to the equivalence relation $R_0 \cup R_2 \cup R_4$.

By Lemma 4.2 for any $z \in X$ the derived design $X_i = X_i(z)$ is a $t_i = 2$-design in $S^{d-2}$. We define $s_{i,j} = |A(X_i, X_j)|$. Then the matrix $(s_{i,j})_{1 \leq i, j \leq 3}$ is

$$
\begin{pmatrix}
3 & 2 & 3 \\
2 & 1 & 2 \\
3 & 2 & 3
\end{pmatrix}.
$$

If $s_{i,j} + s_{j,k} - 2 \leq 2$, that is, when

$$(i, j, k) \in \{(1, 2, 1), (1, 2, 2), (1, 2, 3), (2, 1, 2), (2, 2, 1), (2, 2, 2), (2, 3, 2), (3, 2, 1), (3, 2, 2), (3, 3, 3)\},$$

then the assumption (1) of Theorem 2.6 holds. We remark that $X_2$ is in fact a 3-design because $X_2$ is a cross polytope in $\mathbb{R}^{d-1}$, but this fact does not improve the proof.

The following Lemma is used to determine intersection numbers of derived designs obtained from MUB.

Lemma 5.2. We define $X_i(x, \alpha) = \{w \in X_i \mid \langle x, w \rangle = \alpha\}$, and $X_i(x, \alpha; y, \beta) = X_i(x, \alpha) \cap X_i(y, \beta)$. Then the following equalities hold:

1. $X_i(x, -\alpha) = X_i(-x, \alpha)$,
2. $-X_i(x, \alpha) = X_{4-i}(x, -\alpha)$,
3. $|X_i(x, \alpha; y, \beta)| = |X_i(-x, -\alpha; y, \beta)| = |X_i(x, \alpha; -y, -\beta)| = |X_{4-i}(x, -\alpha; y, -\beta)|$. 

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Proof. (1) and (2) are immediate from the definition.

By (1), \( X_i(x, \alpha; y, \beta) = X_i(-x, -\alpha; y, \beta) = X_i(x, \alpha; -y, -\beta) \) holds. By (2), \(-X_i(x, \alpha; y, \beta) = X_{-i}(x, -\alpha; y, -\beta)\) holds. This proves (3). \(\square\)

If \( s_{i,j} + s_{j,k} - 3 = 2 \), that is, when

\[
(i, j, k) \in \{(1, 1, 2), (1, 3, 2), (2, 1, 1), (2, 1, 3), (2, 3, 1), (2, 3, 3), (3, 1, 2), (3, 3, 2)\}, \tag{5.1}
\]

Lemma 5.2 implies that the intersection numbers on \( X_j(z) \) for \( x \in X_i(z), y \in X_k(z) \) are determined by the intersection numbers on \( X_i(z) \) for \( x' \in X_1(z), y' \in X_2(z) \). And the intersection numbers \( p_{\alpha_{1,1}, \alpha_{1,2}}^1(x, y), p_{\alpha_{1,1}, \alpha_{1,2}}^3(x, y) \) for \( x, y \in X_1(z) \) are uniquely determined by \( \gamma = \langle x, y \rangle \) as follows:

\[
p_{\alpha_{1,1}, \alpha_{1,2}}^1(x, y) = \begin{cases} \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{1,2}^1, \\ \frac{d}{2} & \text{if } \langle x, y \rangle = \alpha_{3,2}^1, \end{cases} \quad p_{\alpha_{1,1}, \alpha_{1,2}}^3(x, y) = \begin{cases} \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{1,2}^1, \\ \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{3,2}^1. \end{cases}
\]

These numbers are independent of \( z \in X \). Hence the assumption (2) of Theorem 2.6 holds for \( (i, j, k) \) in (5.1).

If \( s_{i,j} + s_{j,k} - 4 = 2 \), that is, when

\[
(i, j, k) \in \{(1, 1, 1), (1, 3, 3), (1, 3, 1), (1, 3, 3), (3, 1, 1), (3, 1, 3), (3, 3, 1), (3, 3, 3)\}, \tag{5.2}
\]

Lemma 5.2 implies that the intersection numbers on \( X_j(z) \) for \( x \in X_i(z), y \in X_k(z) \) are determined by the intersection numbers on \( X_1(z) \) for \( x' \in X_1(z), y' \in X_1(z) \). And the intersection numbers \( \{p_{\alpha, \beta}(x, y) | \alpha = \alpha_{2,1} \text{ or } \beta = \alpha_{2,1}\} \) are given in Table 1. These numbers are independent of \( z \in X \). Hence the assumption (3) of Theorem 2.6 holds for \( (i, j, k) \) in (5.2). By Corollary 2.9, we obtain the following result.

**Corollary 5.3.** Every MUB carries a triply regular association scheme.

| \((\alpha, \beta)\) | \(p_{\alpha, \beta}(x, y)\) |
|-------------------|------------------|
| \((\alpha_{2,1}^1, \alpha_{2,1}^2)\) | \begin{cases} 0 & \text{if } \langle x, y \rangle = \alpha_{1,1}^1, \\ d - 2 & \text{if } \langle x, y \rangle = \alpha_{2,1}^1, \\ 0 & \text{if } \langle x, y \rangle = \alpha_{3,1}^1. \end{cases} |
| \((\alpha_{2,1}^1, \alpha_{1,1}^2), \alpha_{1,1}^1, \alpha_{2,1}^2)\) | \begin{cases} \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{1,1}^1, \\ \frac{d}{2} & \text{if } \langle x, y \rangle = \alpha_{2,1}^2, \\ \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{3,1}^2. \end{cases} |
| \((\alpha_{2,1}^1, \alpha_{2,1}^3), \alpha_{1,1}^2, \alpha_{2,1}^2)\) | \begin{cases} \frac{d}{2} & \text{if } \langle x, y \rangle = \alpha_{1,1}^1, \\ \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{2,1}^2, \\ \frac{d}{2} - 1 & \text{if } \langle x, y \rangle = \alpha_{3,1}^3. \end{cases} |

6 Linked systems of symmetric designs

**Definition 6.1.** Let \((\Omega_i, \Omega_j, I_{i,j})\) be an incidence structure satisfying \(\Omega_i \cap \Omega_j = \emptyset, I_{i,j} = I_i \cup I_j\) for any distinct integers \(i, j \in \{1, \ldots, f\}\). We put \(\Omega = \bigcup_{i=1}^f \Omega_i, I = \bigcup_{i \neq j} I_{i,j}\). \((\Omega, I)\) is called a linked system of symmetric \((v, k, \lambda)\) designs if the following conditions hold:
(1) for any distinct integers $i, j \in \{1, \ldots, f\}$, $(\Omega_i, \Omega_j, I_{i,j})$ is a symmetric $(v, k, \lambda)$ design,

(2) for any distinct integers $i, j, l \in \{1, \ldots, f\}$, and for any $x \in \Omega_i, y \in \Omega_j$, the number of $z \in \Omega_l$ incident with both $x$ and $y$ depends only on whether $x$ and $y$ are incident or not, and does not depend on $i, j, l$.

We define the integers $\sigma, \tau$ by

$$|\{z \in \Omega_l | (x, z) \in I_{i,l}, (y, z) \in I_{j,l}\}| = \begin{cases} \sigma & \text{if } (x, y) \in I_{i,j}, \\ \tau & \text{if } (x, y) \notin I_{i,j}, \end{cases}$$

where $i, j, l \in \{1, \ldots, f\}$ are distinct and $x \in \Omega_i, y \in \Omega_j$.

By [8, Theorem 1], we may assume that

$$\sigma = \frac{1}{v}(k^2 - \sqrt{n}(v - k)), \quad \tau = \frac{k}{v}(k + \sqrt{n}),$$

where $n = k - \lambda$. It is easy to see that $(\Omega, \{R_i\}_{i=0}^3)$ is a 3-class association scheme, where

- $R_0 = \{(x, x) | x \in \Omega\}$,
- $R_1 = \{(x, y) | x \in \Omega_i, y \in \Omega_j, (x, y) \in I_{i,j} \text{ for some } i \neq j\}$,
- $R_2 = \{(x, y) | x, y \in \Omega_i, x \neq y \text{ for some } i\}$,
- $R_3 = \{(x, y) | x \in \Omega_i, y \in \Omega_j, (x, y) \notin I_{i,j} \text{ for some } i \neq j\}$.

We note that the second eigenmatrix $Q$ is given in [17] as follows:

$$Q = \begin{pmatrix} 1 & \frac{v-1}{k} & (f-1)(v-1) & f-1 \\ 1 - \sqrt{(v-1)(v-k)} & \frac{(v-1)(v-k)}{k} & -1 \\ 1 & -1 & -f+1 & f-1 \\ 1 & \frac{(v-1)k}{v-k} & -\sqrt{(v-1)(v-k)} & -1 \end{pmatrix},$$

and hence the Krein matrix $B^*_1 = (q_{i,j}^k)_{0 \leq j \leq 3}$ is given as follows:

$$B^*_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ v-1 & \frac{k(v-k)(v-2)+(f-1)(2k-v)\sqrt{k(v-k)(v-1)}}{f(k(v-k))} & \frac{k(v-k)(v-2)+(v-2k)\sqrt{k(v-k)(v-1)}}{f(k(v-k))} & 0 \\ 0 & \frac{(f-1)(k(v-k)(v-2)+(v-2k)\sqrt{k(v-k)(v-1)}}{f(k(v-k))} & \frac{f(k(v-k))}{f(k(v-k))} & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore $(\Omega, \{R_i\}_{i=0}^3)$ is a $Q$-polynomial association scheme. $(\Omega, \{R_i\}_{i=0}^3)$ is imprimitive and the set $(\Omega_1, \ldots, \Omega_f)$ is a system of imprimitivity with respect to the equivalence relation $R_0 \cup R_2$.

In the rest of this section, we assume that $a^*_i = 0$ i.e., $f = 1 + \frac{(v-2)\sqrt{k(v-k)}}{(v-2k)\sqrt{k(v-k)}}$. Examples of linked symmetric designs satisfying this assumption are known for $(v, k, \lambda) = (2^{2m}, 2^{2m-1}, 2^{2m-1})$ with $f = 2^{2m-1}$ for any $m > 1$ [8].

Let $X$ be the embedding of $\Omega$ into the first eigenspace. The angle set of $X$ is

$$A'(X) = \left\{ \frac{\theta^*_k}{\theta^*_0} | 1 \leq k \leq 3 \right\},$$
and we set $\alpha_k = \theta_k^* / \theta_0^*$. We consider the derived design $X_i(z)$ for $z \in X$. By $a_i^* = 0$, Lemma 4.2 implies $X_i(z)$ is a 2-design in $S^{n-3}$. We define $s_{i,j} = |A'(X_i(z), X_j(z))|$. Then the matrix $(s_{i,j})_{1 \leq i,j \leq 3}$ is

$$
\begin{pmatrix}
3 & 2 & 3 \\
2 & 1 & 2 \\
3 & 2 & 3 \\
\end{pmatrix}.
$$

Since $\{\Omega_1, \ldots, \Omega_f\}$ is a system of imprimitivity, we obtain Table 2. Table 3

If $s_{i,j} + s_{j,l} - 2 \leq 2$, that is, when

$$(i, j, l) \in \{(1, 1, 2), (1, 2, 1), (1, 2, 3), (2, 1, 2), (2, 2, 1), (2, 2, 2), (2, 3, 2), (3, 2, 1), (3, 2, 3)\},$$

then the assumption (1) of Theorem 2.6 holds.

If $s_{i,j} + s_{j,l} - 3 = 2$, that is, when

$$(i, j, l) \in \{(1, 1, 2), (1, 3, 2), (2, 1, 3), (2, 1, 1), (2, 1, 3), (2, 3, 1), (3, 1, 1), (3, 1, 3)\},$$

Table 2 implies that the numbers $p_{\alpha, j, \alpha_j^l}^i(x, y)$ or $p_{\alpha, j, \alpha_j^l}^i(x, y)$ are independent of $z \in X$ and $(x, y) \in X_i(z) \times X_j(z)$ with $\gamma = \langle x, y \rangle$. Hence the assumption (2) of Theorem 2.6 holds for $(i, j, l)$ in (6.1).

If $s_{i,j} + s_{j,l} - 4 = 2$, that is, when

$$(i, j, l) \in \{(1, 1, 2), (1, 1, 3), (1, 3, 3), (1, 1, 3), (1, 3, 1), (3, 1, 1), (3, 3, 1)\},$$

Table 3 implies that the numbers $p_{\alpha, j, \alpha_j^l}^i(x, y)$, $p_{\alpha, j, \alpha_j^l}^i(x, y)$ and $p_{\alpha, j, \alpha_j^l}^i(x, y)$ are independent of $z \in X$ and $(x, y) \in X_i(z) \times X_j(z)$ with $\gamma = \langle x, y \rangle$. Hence the assumption (3) of Theorem 2.6 holds for $(i, j, l)$ in (6.2). By Corollary 2.3, we obtain the following result.

**Corollary 6.2.** Every linked system of symmetric design satisfying $f = 1 + \frac{(v-2)\sqrt{k(v-k)}}{(v-2)\sqrt{k-1}}$ carries a triply regular association scheme.

**Table 2:** the values of $p_{\alpha, \beta}^j(x, y)$, where $x \in X_i(z)$, $y \in X_j(z)$

| $i, j, l$ | $(\alpha, \beta)$ | $p_{\alpha, \beta}^j(x, y)$ | $i, j, l$ | $(\alpha, \beta)$ | $p_{\alpha, \beta}^j(x, y)$ |
|-----------|------------------|-----------------|-----------|------------------|-----------------|
| $(1, 1, 2)$ | $(\alpha_1^2, \alpha_2^1)$ | $\lambda - 1$ $\langle x, y \rangle = \alpha_2^1$ $\langle x, y \rangle = \alpha_1^2$ | $(2, 1, 1)$ | $(\alpha_2^1, \alpha_1^1)$ | $\lambda - 1$ $\langle x, y \rangle = \alpha_2^1$ $\langle x, y \rangle = \alpha_1^2$ |
| $(1, 3, 2)$ | $(\alpha_1^2, \alpha_3^1)$ | $\lambda$ $\langle x, y \rangle = \alpha_3^1$ $\lambda$ $\langle x, y \rangle = \alpha_3^1$ | $(2, 3, 1)$ | $(\alpha_2^1, \alpha_3^1)$ | $\lambda$ $\langle x, y \rangle = \alpha_2^1$ $\lambda$ $\langle x, y \rangle = \alpha_3^1$ |
| $(3, 1, 2)$ | $(\alpha_1^2, \alpha_3^1)$ | $\lambda$ $\langle x, y \rangle = \alpha_3^1$ $\lambda$ $\langle x, y \rangle = \alpha_3^1$ | $(2, 1, 3)$ | $(\alpha_2^1, \alpha_1^1)$ | $\lambda$ $\langle x, y \rangle = \alpha_2^1$ $\lambda$ $\langle x, y \rangle = \alpha_1^1$ |
| $(3, 3, 2)$ | $(\alpha_3^2, \alpha_3^1)$ | $\lambda$ $\langle x, y \rangle = \alpha_3^1$ $\lambda$ $\langle x, y \rangle = \alpha_3^1$ | $(2, 3, 3)$ | $(\alpha_2^1, \alpha_3^1)$ | $\lambda$ $\langle x, y \rangle = \alpha_2^1$ $\lambda$ $\langle x, y \rangle = \alpha_3^1$ |

**Acknowledgements**

The author would like to thank Professor Akihiro Munemasa for helpful discussions. This work was supported by Grant-in-Aid for JSPS Fellows.
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Table 3: the values of $p_{\alpha,\beta}^i(x, y)$, where $x \in X_i(z)$, $y \in X_1(z)$

| $(i, j, l)$ | $(\alpha, \beta)$ | $p_{\alpha,\beta}^i(x, y)$ | $(i, j, l)$ | $(\alpha, \beta)$ | $p_{\alpha,\beta}^i(x, y)$ |
|------------|-------------------|----------------|------------|-------------------|----------------|
| (1, 1, 1)  | $(\alpha_{1,1}^2, \alpha_{1,1}^2)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,1}^1 \\ k-2 & \langle x, y \rangle = \alpha_{1,1}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,1}^3 \end{cases}$ | (1, 3, 3)  | $(\alpha_{1,3}^2, \alpha_{3,3}^2)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,3}^1 \\ v-k-1 & \langle x, y \rangle = \alpha_{1,3}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,3}^3 \end{cases}$ |
|            | $(\alpha_{1,1}^2, \alpha_{1,1}^1)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,1}^1 \\ \sigma & \langle x, y \rangle = \alpha_{1,1}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,1}^3 \end{cases}$ |            | $(\alpha_{1,3}^2, \alpha_{3,3}^3)$ | $\begin{cases} k-\tau & \langle x, y \rangle = \alpha_{1,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{1,3}^2 \\ k-\tau & \langle x, y \rangle = \alpha_{1,3}^3 \end{cases}$ |
|            | $(\alpha_{1,1}^1, \alpha_{1,1}^2)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,1}^1 \\ \sigma & \langle x, y \rangle = \alpha_{1,1}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,1}^3 \end{cases}$ |            | $(\alpha_{1,3}^2, \alpha_{3,3}^3)$ | $\begin{cases} k-\sigma & \langle x, y \rangle = \alpha_{1,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{1,3}^2 \\ k-\sigma & \langle x, y \rangle = \alpha_{1,3}^3 \end{cases}$ |
| (1, 1, 3)  | $(\alpha_{1,3}^2, \alpha_{1,3}^2)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,3}^1 \\ k-1 & \langle x, y \rangle = \alpha_{1,3}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,3}^3 \end{cases}$ | (1, 3, 3)  | $(\alpha_{3,3}^2, \alpha_{3,3}^1)$ | $\begin{cases} \tau & \langle x, y \rangle = \alpha_{3,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^2 \\ \tau & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |
|            | $(\alpha_{1,1}^2, \alpha_{1,3}^1)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,1}^1 \\ \sigma & \langle x, y \rangle = \alpha_{1,1}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,1}^3 \end{cases}$ |            | $(\alpha_{3,3}^2, \alpha_{3,3}^3)$ | $\begin{cases} \tau & \langle x, y \rangle = \alpha_{3,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^2 \\ \tau & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |
| (1, 3, 1)  | $(\alpha_{1,3}^2, \alpha_{3,3}^1)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{1,3}^1 \\ v-k & \langle x, y \rangle = \alpha_{1,3}^2 \\ 0 & \langle x, y \rangle = \alpha_{1,3}^3 \end{cases}$ | (3, 3, 1)  | $(\alpha_{3,3}^1, \alpha_{3,3}^2)$ | $\begin{cases} k-\tau & \langle x, y \rangle = \alpha_{3,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^2 \\ k-\tau & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |
|            | $(\alpha_{1,3}^1, \alpha_{3,3}^1)$ | $\begin{cases} k-\sigma & \langle x, y \rangle = \alpha_{1,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{1,3}^2 \\ k-\sigma & \langle x, y \rangle = \alpha_{1,3}^3 \end{cases}$ |            | $(\alpha_{3,3}^1, \alpha_{3,3}^2)$ | $\begin{cases} k-\tau & \langle x, y \rangle = \alpha_{3,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^2 \\ k-\tau & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |
| (3, 1, 1)  | $(\alpha_{3,3}^2, \alpha_{1,1}^1)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{3,3}^1 \\ k-1 & \langle x, y \rangle = \alpha_{3,3}^2 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ | (3, 3, 3)  | $(\alpha_{3,3}^2, \alpha_{3,3}^1)$ | $\begin{cases} k-\tau & \langle x, y \rangle = \alpha_{3,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^2 \\ k-\tau & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |
|            | $(\alpha_{3,3}^1, \alpha_{1,1}^1)$ | $\begin{cases} 0 & \langle x, y \rangle = \alpha_{3,3}^1 \\ \sigma & \langle x, y \rangle = \alpha_{3,3}^2 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |            | $(\alpha_{3,3}^2, \alpha_{3,3}^1)$ | $\begin{cases} k-\tau & \langle x, y \rangle = \alpha_{3,3}^1 \\ 0 & \langle x, y \rangle = \alpha_{3,3}^2 \\ k-\tau & \langle x, y \rangle = \alpha_{3,3}^3 \end{cases}$ |