Eigenfunctions and Very Singular Similarity Solutions of Odd-Order Nonlinear Dispersion PDEs: Toward a “Nonlinear Airy Function” and Others

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Asymptotic properties of nonlinear dispersion equations

\[ u_t = (|u|^n u)_{xxx} \quad \text{and} \quad u_t = (|u|^n u)_{xxx} - |u|^{p-1} u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad (1) \]

with fixed exponents \( n > 0 \) and \( p > n + 1 \), and their \((2k+1)\)th-order analogies are studied. The global in time similarity solutions, which lead to “nonlinear eigenfunctions” of the rescaled ordinary differential equations (ODEs), are constructed. The basic mathematical tools include a “homotopy-deformation” approach, where the limit \( n \to 0^+ \) in the first equation in (1) turns out to be fruitful. At \( n = 0 \) the problem is reduced to the linear dispersion one:

\[ v_t = v_{xxx}, \]

whose oscillatory fundamental solution via Airy’s classic function has been known since the nineteenth century. The corresponding Hermitian linear non-self-adjoint spectral theory giving a complete countable family of eigenfunctions was developed earlier in [1]. Various other nonlinear operator and numerical methods for (1) are also applied. As a key alternative, the “super-nonlinear” limit \( n \to +\infty \), with the limit partial differential equation (PDE)

\[ (\text{sign } v)_t = v_{xxx}, \quad \text{in terms of the variable} \quad v = |u|^n u, \]

admitting three almost “algebraically explicit” nonlinear eigenfunctions, is performed.

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For the second equation in (1), very singular similarity solutions (VSSs) are constructed. In particular, a “nonlinear bifurcation” phenomenon at critical values \(\{p = p_i(n)\}_{i \geq 0}\) of the absorption exponents is discussed.

1. Introduction: nonlinear dispersion PDEs

1.1. NDEs: the models, first discussion, and some examples

In this paper, we study asymptotic properties of nonlinear dispersion equations (NDEs) of the following form:

\[
u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) - \lambda |u|^{p-1} u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad k = 1, 2, \ldots , (2)
\]

where \(\lambda = 0\) stands for a pure NDE, and \(\lambda = 1\) for an NDE with absorption. In both cases, \(n > 0\) and \(p > n + 1\) are fixed exponents. This continues the study begun in [1] for \(n = 0\), that is, for the linear and semilinear dispersion equation.

A continuous “homotopy” path \(n \to 0^+\), connecting this study with that in [1], turns out to be rather effective. In fact, regardless of their wide spread in applications and clear significance, such third-order nonlinear dispersion PDEs are quite poorly represented in general PDE theory, unlike their direct “neighboring” second- and fourth-order quasi-linear parabolic equations (the so-called porous medium equations, PMEs)

\[
u_t = (|u|^n u)_{xx} \quad \text{(the PME-2)} \quad \text{and} \quad \nu_t = -(|u|^n u)_{xxxx} \quad \text{(the PME-4)}. \quad (3)
\]

The main goal of this paper is to fill such a gap.

It is worth mentioning that all the above PDEs are degenerate at the singularity level \(\{u = 0\}\), so that, as in PME and conservation laws theory, all of them (as well as related ODEs for particular solutions) are understood in the weak sense via integration by parts. However, while both (3) have monotone operators in the \(L^2\)-metric (and powerful theory of monotone operators applies to guarantee existence and uniqueness of weak solutions), this is not the case for the NDE (2). This makes such equations to be interesting and principal for general PDE theory.

Let us clarify from the beginning a general role and the main purpose of this study in the theory of higher-order NDEs such as (2):

(i) We will study a countable family of sufficiently smooth continuous similarity solutions, of a fundamental and a VSS (very singular) type, which are defined for all \(t > 0\). It turns out that all such solutions are strongly oscillatory as \(x \to +\infty\), which is an inevitable property of such a nonlinear dispersion mechanism. In fact, for \(n = 0\), this oscillatory behavior corresponds to asymptotics of the Airy function, known since the middle of
the nineteenth century. Therefore, we confirm that (one-sided for $k = 1$) oscillatory solutions of (2) persist for arbitrary $n > 0$.

Concerning a physical meaning of such “nonlinear oscillations,” especially by comparing with a number of nonnegative compactons for the third- and higher-order NDEs constructed during the last 2–3 decades (see Section 1.2), we thus claim that such compactons may not describe some stable (generic) properties of their solutions.

(ii) In addition, (2), as an equation with nonlinear dispersion mechanism, contains and describes other key singularity phenomena such as a complicated formation of various shock and smoother rarefaction waves, which appear from discontinuous data, as well as a general nonuniqueness of “entropy solutions” after a single point gradient blow-up. We do not touch these difficult, even still often mathematically obscure, phenomena and refer to [2, 3, 4] for further details.

Overall, it may be said that the smooth similarity behavior and asymptotic patterns studied here occur in the NDE (2) for large times, when the gradient blow-up and/or shock wave influence have already been settled down via evolution, and hence become negligible.

Then, concerning local existence and uniqueness theory, including “smoothing results” for NDEs (i.e., for solutions without grad-blow-up, shocks, and other stronger singularities), see references and results in [2]. Because we are dealing with some special exact similarity solutions of (2), we do not use any advanced results of local and/or global regularity and any shock-entropy theory here, and always tackle continuous solutions.

Various applications of NDEs are characterized in detail in [3], so we will minimize extra references to previous physical and more formal investigations of such PDE models, and concentrate on their quite unusual mathematical aspects of our interest. It is worth mentioning here a couple of remarkable examples of NDEs from integrable PDE theory. The first one is the third-order Harry Dym equation

\[ u_t = u^3 u_{xxx}, \tag{4} \]

an integrable soliton equation; see [5, § 4.7] for survey and references therein. The second one is the fifth-order Kawamoto equation [6]

\[ u_t = u^5 u_{xxxx} + 5 u^4 u_x u_{xxx} + 10 u^5 u_{xx} u_{xxx}, \]

which has higher-degree algebraic terms. Shock waves for the pure fifth-order NDEs

\[ u_t = u^5 u_{xxxx} \quad \text{and} \quad u_t = (|u|^n u)_{xxxx} \]

are studied in [7]. Let us discuss further necessary aspects of NDEs and their standing in general PDE theory.
1.2. Compactons in NDEs: compactly supported traveling waves

Consider the higher odd-order NDE, with another divergent lower-order term:

\[ u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) + (|u|^n u)_x. \]  

Equation (5) is a generalization of the third-order Rosenau–Hyman (RH) equation

\[ u_t = (u^2)_{xxx} + (u^2)_x, \]  

which models the effect of nonlinear dispersion in the pattern formation of liquid drops (see [8]). For \( n = k = 1 \), (5) is the RH equation in classes of nonnegative solutions.

It is well known that the RH Equation (6) possesses explicit moving compactly supported soliton-type solutions, known as compactons. Compactons have the same structure as traveling wave solutions, given by

\[ u_c(x, t) = f(z), \quad z = x - \lambda t \implies -\lambda f'' = (-1)^{k+1} D_x^{2k+1}(|f|^n f) + (|f|^n f'). \]

After integrating once with zero constant (i.e., zero “flux”), we find that \( f(z) \) satisfies

\[ -\lambda f = (-1)^{k+1} D_x^{2k}(|f|^n f) + |f|^n f \]  

in the weak sense. For \( k = 1 \), the ODE (7) can be solved explicitly (see the expression (11) later), while for \( k \geq 2 \), this is a difficult variational problem, which admits various countable families of compactly supported oscillatory solutions \( f(z) \) of changing sign, [9].

While we note that compacton solutions may be found for NDEs, for nonconservative and nondivergent NDEs such as (2), TW solutions may be irrelevant for classes of bounded solutions of the Cauchy problem. Therefore, instead, we need to find other more complicated similarity solutions.

1.3. A relation to blow-up in reaction–diffusion theory

Surprisingly, the NDE (5) is related to parabolic even-order equations of a usual reaction–diffusion type:

\[ u_t = (-1)^{k+1} D_x^{2k}(|u|^n u) + |u|^n u \quad (u_t = (u^{n+1})_{xx} + u^{n+1} \text{ for } k = 1, \ u \geq 0). \]  

Indeed, Equation (8) admits blow-up self-similar solutions of the separate form

\[ u(x, t) = (T - t)^{-\frac{1}{n}} f(x), \]  

where \( T \) is the finite blow-up time, and the similarity profile \( f(x) \) solves the following ODE, which is obtained by substituting (9) to (5):

\[ (-1)^{k+1} D_x^{2k}(|f|^n f) + |f|^n f = \frac{1}{n} f. \]
For \( k = 1 \), Equation (10) possesses the explicit weak compactly supported solution

\[
f(x) = \begin{cases} 
\left[ \frac{2(n+1)}{n(n+2)} \cos^2 \left( \frac{nx}{2(n+1)} \right) \right]^{\frac{1}{n}} & \text{for } |x| < \frac{\pi(n+1)}{n}, \\
0 & \text{for } |x| \geq \frac{\pi(n+1)}{n}.
\end{cases}
\]  

(11)

Thus, (9), (11) yields the so-called standing wave blow-up solution (S-regime of blow-up) of (5), which have compact support for all times \( t \in (0, T) \). This exact Zmitrenko–Kurdyumov blow-up solution has been known since the middle of 1970s; see details in [10, Ch. 4]. As we have mentioned, for \( k \geq 2 \), (10) cannot be solved explicitly, but the ODE is shown to admit a countable set of compactly supported solutions obtained by variational Lusternik–Schnirel’man and fibering methods, [9].

Thus, comparing the ODEs (10) and (7), we see that these coincide provided the compacton velocity \( \lambda \) is given by

\[
\lambda = -\frac{1}{n}.
\]  

(12)

In other words, this means that some principles of compacton propagation for such NDEs (5) are directly related to blow-up formation mechanisms for reaction–diffusion equations (8). Moreover, in both equations, such asymptotic structures are expected to be structurally stable (for \( k = 1 \) in (8), this has been proved, [10, p. 260]). This is indeed surprising, because both classes of nonlinear PDEs seem to be responsible for entirely different physical phenomena (to say nothing of their different mathematical essence).

The lower-order case of Equation (8), for \( k = 1 \), which is just a standard reaction–diffusion PDE, is fairly well understood. However, the third-order NDE in (5), also for the minimal value \( k = 1 \), has not been studied extensively and some of its basic governing mathematical principles are still relatively unknown.

1.4. Two main model NDEs and layout of the paper

Thus, we will construct global similarity solutions of the following two NDEs. The first is the pure NDE:

\[
u_t = (-1)^{k+1} D_x^{2k+1} (|u|^n u) \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \quad n > 0,
\]  

(13)

which is studied in Sections 2–4. Our goal therein is to show that (13) admits an infinite countable family of self-similar solutions governed by “nonlinear eigenfunctions” of a rescaled operator. Moreover, in Section 3.6, we show that this countable family as \( n \to 0^+ \) can be described by eigenfunctions from
Hermitian spectral theory of the non-self adjoint operator \([1, \S \text{4}]\)

\[
B = (-1)^{k+1} D^2 y^{2k+1} + \frac{1}{2k+1} y D y + \frac{1}{2k+1} I. \tag{14}
\]

The above operator occurs after similar scaling of the linear dispersion equation (LDE)

\[
u_t = (-1)^{k+1} D^2 u_x^{k+1} \text{ in } \mathbb{R} \times \mathbb{R}_+; \quad k \geq 1. \tag{15}\]

In Section 4, we perform the opposite, more important “super-nonlinear” limit \(n \to +\infty\) in (13). Namely, we observe that the natural change of the independent variable leads to the following PDE:

\[
|u|^n u = v \implies (|v|^{-\frac{n}{n+1}} v)_t = (-1)^{k+1} D^2 x^{k+1} v. \tag{16}\]

The formal limit \(n \to +\infty\) leads to the following limit NDE:

\[
(\text{sign } v)_t = (-1)^{k+1} D^2 x^{k+1} v, \tag{17}\]

which admits analogous similarity patterns. Similar to the original NDE (2), (17) is to be understood in a natural weak sense. It turns out that, at least for \(k = 1\), where (17) takes a particularly simple form

\[
(\text{sign } v)_t = v_{xxx}, \tag{18}\]

the first three occurring ODEs for (18) reduce to an algebraic system, and we develop a geometric–algebraic approach to constructing first nonlinear eigenfunctions.

We postpone until the Appendix the study of VSSs of the NDE with absorption (2), for \(\lambda = 1\), where our main goal is to justify existence of the so-called \(p\)-bifurcation branches of VSS solutions, which appear from nonlinear Airy-type functions at some critical exponents \(p = p_l(n) > n + 1, l = 0, 1, 2, \ldots\).

2. Similarity solutions of the NDE (“nonlinear eigenfunctions”)

Here, we consider the pure NDE (13), which is connected to (5), but now we do not have the convection-like term \((|u|^n u)_x\), which is negligible in our asymptotics. This NDE may be compared with the even-order model, which represents the general higher-order porous medium equation (the PME–2m)

\[
u_t = (-1)^{m+1} \Delta^m (|u|^n u) \text{ in } \mathbb{R}^N \times \mathbb{R}_+, \quad m \geq 2. \tag{19}\]

The classic PME–2, for \(m = 1\), appears in a number of physical applications, such as fluid and gas flows, heat transfer, or (nonlinear) diffusion. Other applications have been proposed in mathematical biology, lubrication, boundary layer theory, and various other fields. Papers exploring delicate asymptotics
for the PME include [11], where extra references are available. For nonlinear eigenfunctions of (19), with $m = 2, N = 1$, see [12] and key references therein. Overall, the higher-order case $m \geq 2$ in (19) has been studied much less in the mathematical literature and represents a number of difficult open problems.

2.1. Self-similar solutions: toward a “nonlinear eigenvalue problem”

Our NDE (13) admits standard similarity solutions given by

$$u_{gl}(x, t) = t^{-\alpha} f(y),$$

where $y = x/t^\beta$, (20)

for some unknown real parameters $\alpha$ and $\beta$. After substitution (20) into the NDE, we obtain the ODE for the rescaled profile $f$,

$$-\alpha t^{-\alpha - 1} f - \beta t^{-\alpha - 1} f' y = (-1)^{k+1} t^{-\alpha(n+1)-\beta(2k+1)} D_y^{2k+1}(|f|^n f).$$

By equating powers of $t$, the parameter $\beta$ can be found in terms of $\alpha$ and is given by

$$\beta = \frac{1 - \alpha n}{2k + 1} > 0 \text{ for } \alpha < \frac{1}{n},$$

so that now $\alpha \in \mathbb{R}$ remains the only unknown nonlinear eigenvalue.

Thus, the ODE (21) takes the form

$$A(f, \alpha) \equiv (-1)^{k+1} D_y^{2k+1}(|f|^n f) + \frac{1 - \alpha n}{2k + 1} f' y + \alpha f = 0 \text{ in } \mathbb{R}. \quad (23)$$

To formulate a suitable nonlinear eigenvalue problem for the pairs $\{\alpha_l, f_l\}_{l \geq 0}$ for the ODE (23), one should specify the “boundary” conditions at $y = \pm \infty$, which is one of the main goals of this paper. It turns out that, loosely speaking, such conditions can be formulated as follows: for some special values of the real eigenvalues $\{\alpha_l = \alpha_l(n) > 0\}_{l \geq 0}$, the corresponding nonlinear eigenfunctions

$$\left\{ \begin{array}{l}
\text{(i) finite left-hand interface } (f_l(y) \equiv 0 \text{ for } y \ll -1), \\
\text{and (ii) “minimal oscillatory” behavior as } y \to +\infty.
\end{array} \right.$$  

Both such conditions will get proper explanations later on. Of course, at least for sufficiently large $l \geq 2k + 1$, the problem (23), (24) exhibits all typical features of self-similarity of the second kind (the first kind corresponds to standard dimensional analysis of PDEs), a notion introduced by Ya. B. Zel’dovich in 1956 [13]. Unlike previous and known examples, in general, the eigenvalues $\alpha_l(n), l \geq 2k + 1$ (and hence $\beta_l(n)$) cannot be found explicitly from any dimensional analysis. Thus, admitted (real) values of $\alpha = \alpha_l(n) > 0$ at this stage are still unknown and play a role of “nonlinear eigenvalues.”

In particular, our goal is to show by a combination of analytic, formal, and numerical tools that, for any $n > 0$, the eigenvalue problem (23), (24) has a countable set of pairs $\{\alpha_l, f_l(y)\}_{l \geq 0}$:

$$A(f_l, \alpha_l) = 0. \quad (25)$$
For the linear case with $n = 0$, the corresponding linear non-self-adjoint Hermitian spectral problem was developed in [1, § 4]; we present and use these results later.

2.2. Toward blow-up patterns

The present analysis admits a natural duality: because the NDE (13) is invariant under reflections,

\[
\begin{cases}
  x \mapsto -x, \\
  t \mapsto T - t,
\end{cases}
\]

the global patterns (20) lead to the corresponding reflected blow-up ones:

\[
u_{bl}(x, t) = (T - t)^{-\alpha} f(y), \quad y = -\frac{x}{(T - t)^{\alpha}}, \quad \text{where} \quad \beta = \frac{1 - \alpha n}{2k + 1} > 0.
\]

Therefore, the nonlinear eigenvalue problem (23), (24) is assumed to describe both countable families of global and blow-up patterns for the NDE (13). In what follows, for simplicity, we will mainly use a global treatment of such asymptotic patterns.

2.3. A homotopy path $n \rightarrow 0^+$ to Hermitian spectral theory: a route to countability of nonlinear eigenfunctions

For $n = 0$, $\alpha$ gives the eigenvalues $\{\lambda_l\}$ in linear Hermitian spectral theory, [1, § 4]. Indeed, for $n = 0$, (23) reads

\[
(-1)^{k+1} D_{y}^{2k+1} f + \frac{1}{2k + 1} f' y + \alpha f \equiv B f + \left(\alpha - \frac{1}{2k + 1}\right) f = 0.
\]

Therefore, this gives the eigenvalue equation for the linear operator $B$ in (14):

\[
B \psi = \lambda \psi, \quad \text{where} \quad \lambda = -\alpha + \frac{1}{2k + 1}.
\]

By Hermitian spectral theory [1], this defines a countable set of eigenvalues for $n = 0$:

\[
\alpha_l(0) = \frac{l + 1}{2k + 1}, \quad l = 0, 1, 2, \ldots
\]

The corresponding eigenfunctions are then the normalized derivatives of the rescaled kernel $F(y)$, of the dispersion operator $D_t + (-1)^{k} D_{x}^{2k+1}$:

\[
\psi_l(y) = \frac{(-1)^l}{\sqrt{l!}} D_{y}^{l} F(y), \quad l \geq 0.
\]

Moreover, the “adjoint” (not in the standard $L^2$-sense, but in a space with an indefinite metric; see [1, § 5.1]) operator
\[ B^* = (-1)^{k+1} D_y^{2k+1} - \frac{1}{2k+1} y D_y \]  
\[ \psi_l^*(y) = \frac{1}{\sqrt{l!}} \left[ y^l + (-1)^{k+1} \sum_{j=1}^{\lfloor \frac{l}{2k+1} \rfloor} \frac{1}{j!} D_y^{(2k+1)j} y^j \right]. \]  

Finally, the operator pair \( \{B, B^*\} \) has a bi-orthonormal set of eigenfunctions.

As a key conclusion, we expect that there exists a uniform and pointwise convergence, as \( n \to 0^+ \), of the nonlinear eigenfunctions to the linear ones, for \( B \) defined by (31). We postpone explanations of the main aspects of such a branching until Section 3.6, when we will have gained enough understanding of basic nonlinear eigenfunction theory involved. However, a rigorous proof of such a branching of eigenfunctions at \( n = 0 \) is a hard open problem. Nevertheless, this \( n \)-branching approach still remains a unique analytically convincing fact toward existence of an infinite countable discrete set of nonlinear eigenfunctions of the problem (23). In this case, branching theory as \( n \to 0^+ \) can be developed along the same lines as for the PME–4 (19), with \( m = 2 \) [12, § 6]. Such a construction then uses a specific Hermitian spectral theory created in [1]; see Section 3.6.

2.4. Conservation laws: explicit values of first nonlinear eigenvalues

As a first pleasant surprise, it turns out that some first nonlinear eigenvalues for any \( n > 0 \) can be calculated explicitly by conservation laws for the NDE.

- **First eigenvalue** \( \alpha_0(n), \ l = 0 \). Assuming that the solution \( u(x, t) \) is integrable in \( x \) over \( \mathbb{R} \) (this is a formal assumption), we have that (13) obeys the conservation of mass, that is,
  \[ \frac{d}{dt} \int_{\mathbb{R}} u(x, t) \, dx = 0. \]  

For similarity solutions (20), we have that
  \[ \int_{\mathbb{R}} u(x, t) \, dx = t^{\beta-\alpha} \int_{\mathbb{R}} f(y) \, dy, \]

where by \( f(y) \) we mean the first nonlinear Airy function \( f_0(y) \) of, say, unit mass, \( \int f_0 = 1 \). This satisfies (34), provided that, in addition to (22),
  \[ \beta - \alpha = 0 \quad \implies \quad \alpha_0(n) = \frac{1}{(2k + 1) + n}. \]
Hence, on substitution into (23), we get
\[-1]^{k+1} D^2_{y}(|f|^n f) + \frac{1}{(2k + 1) + n} f'y + \frac{1}{(2k + 1) + n} f = 0.

Integrating once with the zero constant (a zero-flux condition), we end up with the ODE
\[-1]^{k+1} D^2_{y}(|f|^n f) + \frac{1}{(2k + 1) + n} f'y = 0. \tag{36}

Note that, for \(n = 0\), we have exactly the linear ODE (see [1])
\[-1]^{k+1} F^{(2k)} + \frac{1}{2k+1} Fy = 0 \quad \text{for } y \in \mathbb{R}. \tag{37}

For convenience, we use in (36) the natural substitution
\[Y = |f|^n f \implies f = |Y|^{-\frac{1}{n}} Y, \tag{38}\]
to remove nonlinearities in the highest differential. Substitution yields
\[-1]^{k+1} D^2_{y} Y + \frac{1}{(2k + 1) + n} y|Y|^{-\frac{1}{n}} Y = 0. \tag{39}

- **Second eigenvalue** \(\alpha_1(n), l = 1\). Similarly, we have *conservation of the first moment*, with
\[
\frac{d}{dt} \int_{\mathbb{R}} x u(x, t) \, dx = 0, \quad \text{where } \int_{\mathbb{R}} x u(x, t) \, dx = t^{2\beta - \alpha} \int_{\mathbb{R}} yf(y) \, dy, \quad \text{so that}
\]
\[2\beta - \alpha = 0 \implies \alpha_1(n) = \frac{2}{(2k + 1) + 2n}. \tag{40}

Here \(f = f_1(y)\) is the second “dipole-like” nonlinear eigenfunction with the unit moment, \(\int yf_1(y) = 1\). Then \(f_1(y)\) solves the ODE
\[-1]^{k+1} D^2_{y}(|f|^n f) + \frac{1}{(2k + 1) + 2n} f'y + \frac{2}{(2k + 1) + 2n} f = 0. \tag{41}

However, we cannot simply integrate this equation, as we could before, to reduce the order of the ODE. Instead, we multiply (42) by \(y\), so that
\[-1]^{k+1} D^2_{y}(|f|^n f)y + \frac{1}{(2k + 1) + 2n} f'y^2 + \frac{2}{(2k + 1) + 2n} fy = 0,

and now it is possible to integrate by parts, to obtain
\[-1]^{k+1} D^2_{y}(|f|^n f)y + (-1)^k D^{k-1}(|f|^n f) + \frac{1}{(2k + 1) + 2n} fy^2 = 0. \tag{43}
• **Third Eigenvalue** $\alpha_2(n)$, $l = 2$. Consider the conservation of the second moment,

$$\frac{d}{dt} \int_{\mathbb{R}} x^2 u(x, t) \, dx = 0, \quad \text{where} \quad \int_{\mathbb{R}} x^2 u(x, t) \, dx = t^{3\beta - \alpha} \int_{\mathbb{R}} y^2 f(y) \, dy,$$

and $f = f_2(y)$ possesses the unit second moment $\int y^2 f_2 = 1$. Then, $3\beta - \alpha = 0$, so, by (22),

$$\alpha_2(n) = \frac{3}{2k + 1 + 3n} \quad \text{and} \quad \left( -1 \right)^{k+1} D_{y}^{2k+1}(|f|^n f) + \frac{1}{2k + 1 + 3n} f' y + \frac{3}{2k + 1 + 3n} f = 0. \quad (45)$$

Similarly, we multiply by $y^2$ and integrate to reduce the order, to obtain the ODE

$$(-1)^{k+1} D_{y}^{2k}(|f|^n f)y^2 + 2(-1)^k D_{y}^{2k-1}(|f|^n f)y$$

$$+ 2(-1)^{k+1} D_{y}^{2k-2}(|f|^n f) + \frac{1}{2k + 1 + 3n} f y^3 = 0. \quad (46)$$

These three conservation laws in particular are important, as we can explicitly find the first three (second-order) equations for the case $k = 1$ (corresponding to the first three nonlinear eigenvalues), but not for others. The case $k = 1$ is essential, because it is much easier to develop theory for the lower-order case, as well as being easier to solve numerically (see Section 2.5).

• **lth Eigenvalue** $\alpha_l(n)$. In general, for arbitrary $k \geq 1$, for all $l < 2k + 1$, where $l$ is the eigenvalue index as before, we have our moments conservation given by

$$\int_{\mathbb{R}} x^l u(x, t) \, dx = t^{(l+1)\beta - \alpha} \int_{\mathbb{R}} y^l f(y) \, dy,$$

where $f = f_l(y)$ has the unit $l$th moment, $\int y^l f_l = 1$. Therefore, our nonlinear eigenvalues are represented by

$$(l + 1)\beta - \alpha = 0 \quad \implies \quad \alpha_l(n) = \frac{l + 1}{2k + 1 + (l+1)n} \quad (48)$$

for any $0 \leq l < 2k + 1$.

The corresponding ODEs are

$$(-1)^{k+1} D_{y}^{2k+1}(|f|^n f) + \frac{1}{(2k + 1 + (l+1)n) f' y + \frac{l + 1}{2k + 1 + (l+1)n} f = 0. \quad (49)$$
The function \( Y = |f|^n f \) then solves the ODE

\[
(-1)^{k+1} Y^{(2k+1)} + \frac{1}{(2k + 1) + (l + 1)n} y(|Y|^{-\frac{n}{n+1}} Y)' + \frac{1}{(2k + 1) + (l + 1)n} |Y|^{-\frac{n}{n+1}} Y = 0.
\]

(50)

Thus, for such NDEs, one can always explicitly obtain \( 2k + 1 \) \( n \)-branches of nonlinear eigenvalues (48). Though of course the solvability of the corresponding eigenvalue equations remains a difficult open problem, especially for large \( l \). However, (48) shows one important feature of this nonlinear eigenvalue theory: the \( n \)-branches (48) are \textit{global} in \( n > 0 \), that is, exist for all \( n > 0 \). Existence of such a global continuation, with no turning points, is one of the most difficult questions in general nonlinear operator theory; see for example, [14, 15]. Recall that the eigenvalue problem (23) is by no means variational, neither contains any monotone nor coercive operators.

Next, we must admit that, for any larger \( l \geq 2k + 1 \), we cannot find \( \alpha_l(n) \) explicitly using conservation laws. Searching for \( n \)-branches of these nonlinear eigenfunctions, we should rely on the above local homotopy approach as \( n \to 0^+ \), and also use advanced numerical methods for global branches continuation phenomena.

\textit{Remark} 2: comparison with the linear case \( n = 0 \). Indeed, the conservation laws via (34), (40), (44), . . . , (47) express some \textit{normalization} (as well as \textit{orthogonality}) of the corresponding profiles \( f(y) \) to polynomials

\[
\psi^*_0(y) = 1, \quad \psi^*_1(y) = y, \quad \psi^*_2(y) = \frac{1}{\sqrt{2}} y^2, \ldots \psi^*_{2k}(y) = \frac{1}{\sqrt{2k!}} y^{2k}.
\]

(51)

For \( n = 0 \), this perfectly corresponds to the bi-orthogonality of the corresponding bases, \( \{\psi_k\} \) and \( \{\psi^*_k\} \), of a pair of non self-adjoint operators \( \{B, B^*\} \) given in (28) and (32); see [1, § 4] for extra details. Then the following holds:

\[
\langle \psi_k, \psi^*_l \rangle_* = \delta_{kl} \quad \Longrightarrow \quad \left\langle \psi_l, \frac{1}{\sqrt{l!}} y^l \right\rangle_* = 1,
\]

for all \( l \leq 2k \). In addition, there exists an infinite number of such normalization (and orthogonality) properties (in an indefinite metric), but for \( l \geq 2k + 1 \), the generalized Hermite polynomials get more sophisticated than those in (51).

Obviously, for \( n > 0 \), linear normalization (orthogonality) properties seem to make no sense. However, as we have seen, this is not the case: linear normalization-like relations still persist for simpler polynomials (51). In other

\footnote{1 Recall that we are dealing with \textit{nonlinear eigenvalues}, which give important scaling factors for the PDE, so that finding their possible exact values via conservation laws is already an essential success, regardless of the fact that corresponding \textit{nonlinear eigenfunctions} remain unclear at the moment.}

\footnote{2 The authors would like to thank the anonymous referee for a suggestion leading to this remark.}
words, we still observe $2k$ linear normalization relations, while others (an infinite number) cease to exist in any explicit form. Of course, these first ones do not correspond to any bases, but become intriguing nonlinear “remnants” of those linear bi-orthonormal bases that used to exist for $n = 0$.

2.5. Numerical construction of nonlinear eigenfunctions

We use a shooting method to find reliable profiles for the NDE (13). First, considering solutions satisfying conservation of mass, we have our first “nonlinear eigenvalue” (for $l = 0$), where $n = 0$ corresponds to the linear kernel, $\psi_0 = F(y)$ (where $F(y) = \text{Ai}(y)$ for $k = 1$), that is, $\lambda = 0$ in (29). Here the rescaled equation is given by (39).

Consider first the simpler case $k = 1$, when we have the second-order ODE for a “nonlinear Airy function”:

$$Y'' = -\frac{1}{n + 3} |Y|^{-\frac{2}{n+1}} Y, \quad y \in \mathbb{R}. \quad (53)$$

First, it can be shown (see Proposition 1 below) that there exists a finite left-hand interface, at which solutions are not oscillatory. In fact, the same is true for $n = 0$, that is, for the Airy function satisfying (60) later: as $y \to -\infty$ (naturally, the left-hand interface is at infinity, because there is no finite propagation in linear problems), $\text{Ai} = F(y)$ is positive and decaying exponentially fast.

Hence, for small solutions of $Y(y)$ as $y \to y_0^+$, with some interface point at $y = y_0 < 0$, we can approximate solutions of (53) as follows:\footnote{Because (53) is a second-order ODE, this representation of solutions with finite interfaces is not much different from classic ones for the second-order porous medium equation in (3) with a theory of compactly supported solutions developed by Oleinik et al. already in the 1950s; see [10, Ch. 1]. In general, we expect that finite propagation to the left for the NDEs (13) can be proved by a suitable energy method, see references and a survey in [16], but it can be rather technical and not straightforward.}

$$Y(y) = C_0 (y-y_0)^{\tilde{\alpha}} (1 + o(1)), \quad (54)$$

for some constant $C_0 > 0$ and exponent $\tilde{\alpha} > 0$. Here we have used the standard notation

$$(\cdot)_+ = \max\{0, \cdot\} \quad \Rightarrow \quad Y(y) \equiv 0 \text{ for } y < y_0.$$ 

Substituting (54) into the ODE (53), we obtain the leading equality

$$\tilde{\alpha}(\tilde{\alpha} - 1)C_0 (y-y_0)^{\tilde{\alpha}-2} = -\frac{1}{n + 3} C_0^{\frac{1}{n+1}} (y-y_0)^{\frac{2}{n+1}} y_0, \quad (55)$$

where we use $(y - y_0)$ to mean $(y - y_0)_+ (1 + o(1))$, as defined before. Hence, we must have from (55) that $\tilde{\alpha} = \frac{2(n+1)}{n}$, with $2 < \tilde{\alpha} \leq 4$ for $n \geq 1$. From this,
the constant $C_0$ is given by

$$C_0 = \left( \frac{n^2|y_0|}{2(n+1)(n+2)(n+3)} \right)^{\frac{n+1}{2}}$$

for any interface $y_0 < 0$. \hfill (56)

Therefore, our “nonlinear Airy function” has a positive interface expansion

$$Y(y) = \left( \frac{n^2|y_0|}{2(n+1)(n+2)(n+3)} \right)^{\frac{n+1}{2}} (y - y_0)^{\frac{2n+1}{2}}(1 + o(1)) \quad \text{as} \quad y \to y_0^+, \hfill (57)$$

with the derivative expansion obtained similarly (Proposition 1 will justify both),

$$Y'(y) = \frac{2(n+1)}{n} \left( \frac{n^2|y_0|}{2(n+1)(n+2)(n+3)} \right)^{\frac{n+1}{2}} (y - y_0)^{\frac{2n+2}{2}}(1 + o(1)). \hfill (58)$$

It follows from (57), that $Y(y)$ is sufficiently smooth at the interface. Indeed, with the trivial continuation (56) beyond, we see that, at least, for a small $\delta > 0$,

$$Y \in C^l(y_0 - \delta, y_0 + \delta), \quad \text{where} \quad l = \left\lceil \frac{2(n+1)}{n} \right\rceil - 1 \to +\infty \quad \text{as} \quad n \to 0^+. \hfill (59)$$

It is worth mentioning that because $\frac{2(n+1)}{n} > 2$, for all $n$, the expansion (57) actually gives a classical solution of the ODE (53), admitting near the interface all necessary derivatives (though the ODE is degenerate at $y = y_0$). For smaller $n > 0$, as (59) shows, solutions get arbitrarily smooth at the interface, somehow predicting $C^\infty$ and analytic smoothness of Airy’s function at $n = 0$.

We use the MatLab IVP solver ode15s to plot our profiles. Taking an arbitrary initial (interface) point $y = y_0 < 0$, our solution and the first derivative will be zero here. Because both initial values are zero at $y = y_0$, we will often find the solution $Y = 0$, while trying to solve numerically. To overcome this problem, we must look at some point $y_0 + \delta$ (for small $\delta \sim 10^{-3}$), close to this point, and after finding the Cauchy data there by (57) and (58), these are used as the initial conditions.

Obviously, due to the nature of the expansion (58), we must take a relatively large initial point (in our case we take $y_0 = -10$), for us to have initial conditions that are not negligible. Hence, the solution is a large rescaling of the “nonlinear fundamental solution” with the unit mass. Below are a few profiles that have been found, in which we have taken $\delta = 10^{-3}$. For $n \sim 0.5$, the derivative $Y'$ is very small and larger negative interface points must be used to find reliable profiles. This makes comparison, between different values of $n$, more difficult.

In Figures 1–4, we show the first nonlinear eigenfunction $Y_0(y)$, with $k = 1$, for various values of $n = 3, 2, 1$, and 0.7. Note that this is precisely the profile that, as $n \to 0^+$, must converge to the rescaled kernel $F(y) \equiv Ai(y)$,
Figure 1. Rescaled solution $Y(y)$ of the ODE (53) for $k = 1$, $l = 0$, with $n = 3$.

Figure 2. Rescaled solution $Y(y)$ of the ODE (53) for $k = 1$, $l = 0$, with $n = 2$. 
Figure 3. Rescaled solution $Y(y)$ of the ODE (53) for $k = 1, l = 0$, with $n = 1$.

Figure 4. Rescaled solution $Y(y)$ of the ODE (53) for $k = 1, l = 0$, with $n = 0.7$. 
representing Airy’s classic function satisfying (cf. (53) for \( n = 0 \))

\[
F'' + \frac{1}{3} Fy = 0, \quad \int F = 1, \quad \text{where } F \in L^2_\rho(\mathbb{R}),
\]

\[
\rho(y) = \begin{cases} 
  e^{ay^{3/2}} & \text{for } y < 0, \\
  e^{-ay^{3/2}} & \text{for } y > 0,
\end{cases}
\]  

(60)

where \( a > 0 \) is a small enough constant.

In general, for all values of \( l \), we have, for the lower-order case of \( k = 1 \), that the same expansion (57) and (58) hold near finite interfaces. Here \( l < 2k + 1 \) and hence, for \( k = 1 \), we can only take \( l = 0, 1, 2 \) for explicitly given nonlinear eigenvalues.

For \( l = 1 \), we have from (43), with \( Y(y) = |f|^n f \) and \( k = 1 \), that the “dipole” eigenfunction \( Y_1(y) \) solves the ODE

\[
Y'' = \frac{1}{y} Y - \frac{1}{2n + 3} |Y|^{-\frac{n}{n+1}} Y y.
\]  

(61)

For \( l = 2 \) and \( k = 1 \), we have from (46) that \( Y_2(y) \) solves

\[
Y'' = \frac{2}{y} Y' - \frac{2}{y^2} Y - \frac{1}{3n + 3} |Y|^{-\frac{n}{n+1}} Y y.
\]  

(62)

We use these to plot the profiles of our NDE with \( l = 1 \) and \( l = 2 \), which, for \( n = 0 \), coincide with the derivatives \( F'(y) \) and \( F''(y) \) respectively, of the linear kernel \( F(y) \).

Figures 5 and 6 show the “dipole-like” profiles for \( l = 1 \) as solutions of (61), while Figure 7 represents the even more oscillatory third eigenfunction \( f_2(y) \), for \( l = 2 \). Recall that, by a homotopy path as \( n \to 0^+ \), this function is expected to converge to the highly oscillatory eigenfunction of the linear operator \( B \) in (14)

\[
\psi_2(y) = \frac{1}{\sqrt{2}} F''(y).
\]

For \( k > 1 \), the solutions \( Y(y) \) are oscillatory (changing sign) as \( y \to y_0^+ \), so that expansions such as (54) must include oscillatory components as an extra multiplier:

\[
Y(y) = (y - y_0)^s(\phi(s) + o(1)), \quad \text{where } s = \ln(y - y_0),
\]  

(63)

and \( \phi(s) \) is a periodic solution of a \((2k + 1)\)th-order ODE. Examples of such oscillatory structures (63) are presented in [5, pp. 186–192] for \( k = 2 \) and \( k = 3 \).

We will use a similar asymptotic approach in Section 3.5 in the opposite limit \( y \to +\infty \).
Figure 5. Rescaled solution $Y(y)$ of the ODE (61) for $k = 1$, $l = 1$, with $n = 3$.

Figure 6. Rescaled solution $Y(y)$ of the ODE (61) for $k = 1$, $l = 1$, with $n = 4$. 
3. On some mathematical aspects of similarity profiles

3.1. Local existence and uniqueness for \( k = 1 \)

We need to show that the above-mentioned numerical construction and formal approaches can be justified, by carefully verifying some rigorous aspects of the asymptotic analysis. We first apply a fixed point approach for the equivalent nonlinear integral equation to prove the expansion (57) near the finite left-hand interface.

To this end, we look at our integrated second-order equation (53) \((k = 1)\), assuming that \( Y(y) > 0 \) is strictly monotone increasing (and hence nonoscillatory) sufficiently close to the interface at \( y = y_0^+ < 0 \):

\[
Y'' = -\frac{1}{n+3} |Y|^{-\frac{n}{n+1}} Y y \quad \text{or} \quad Y'' = -\frac{1}{n+3} Y^{\frac{1}{n+1}} y \quad \text{for} \quad Y > 0. \quad (64)
\]

Therefore, we can rewrite our derivatives of \( Y(y) \) in terms of the inverse function \( y(Y) \):

\[
Y' = \frac{dY}{dy} = \frac{1}{y'(Y)} \quad \text{and} \quad Y'' = \frac{d}{dy} \left( \frac{1}{y'} \right) = -\frac{y''}{(y')^3}.
\]
So, now we can reduce our differential equation (64) to its equivalent integral form,

\[-\frac{y''}{(y')^3} = -\frac{1}{n + 3} \frac{1}{Y^{n+1}} y \iff -\frac{1}{2(y')^2} = -\frac{1}{n + 3} \int_0^Y y(s)s^{\frac{1}{n+1}} ds\]

\[\iff y(Y) = y_0 + \int_0^Y \sqrt{\frac{n + 3}{2 \int_0^s y(r)s^{\frac{1}{n+1}} dr}} ds \equiv M(y).\] (65)

We next prove the following property of the integral operator \(M\) in (65):

**Proposition 1.** For small \(\delta > 0\), \(M(y)\) is a contraction in \(C[0, \delta]\), with the sup-norm, and therefore admits a unique fixed point \(y(Y) > 0\) on \((0, \delta)\), giving the unique positive solution of the ODE (64) on \((y_0, y_0 + \varepsilon)\), with some sufficiently small \(\varepsilon = \varepsilon(\delta) > 0\).

**Proof:** We need to show that, for \(M(y)\) to be a contraction, in the \(C\)-metric,

\[\|M(\xi_2) - M(\xi_1)\| < \mu \|\xi_2 - \xi_1\|,\] (66)

for some constant \(\mu = \mu(\delta) \in (0, 1)\). It is easy to see that \(M : Z_\delta \to Z_\delta\), for the space \(Z_\delta\) of continuous functions, \(Z_\delta = \{\xi(Y) \in C[0, \delta], \xi(0) = y_0\}\), with the sup-norm:

\[\|\xi\| := \sup_{Y \in (0, \delta)} |\xi(Y)|.\]

Now, take arbitrary \(\xi_1(Y), \xi_2(Y) \in Z_\delta\). Then from (65) we have that

\[\|M(\xi_2) - M(\xi_1)\| = \sqrt{\frac{n + 3}{2} \int_0^Y \left( \int \xi_2(s)s^{\frac{1}{n+1}} ds \right)^{\frac{1}{2}} ds - \left( \int \xi_1(s)s^{\frac{1}{n+1}} ds \right)^{\frac{1}{2}} ds} \] (67)

where we use the simplified notation for the integral \(\int\), without any limits of integration, to mean \(\int_0^Y\). This equality can now be written as

\[\|M(\xi_2) - M(\xi_1)\| = \sqrt{\frac{n + 3}{2} \int_0^Y \left( \int \xi_2(s)s^{\frac{1}{n+1}} ds \right)^{\frac{1}{2}} ds - \left( \int \xi_1(s)s^{\frac{1}{n+1}} ds \right)^{\frac{1}{2}} ds} \] (67)

\[\|M(\xi_2) - M(\xi_1)\| = \sqrt{\frac{n + 3}{2} \int_0^Y \left( \int \xi_2(s)s^{\frac{1}{n+1}} ds \right)^{\frac{1}{2}} ds - \left( \int \xi_1(s)s^{\frac{1}{n+1}} ds \right)^{\frac{1}{2}} ds} \] (67)
Denoting the exponent $\nu = \frac{1}{n+1}$, we see that

$$\|M(\zeta_2) - M(\zeta_1)\| \leq \sqrt{\frac{n+3}{2}} \int_0^Y \left\| \frac{\int [\zeta_2(s) - \zeta_1(s)] s^\nu \, ds}{\int \zeta_2(s) s^\nu \, ds \int \zeta_1(s) s^\nu \, ds \left[ \left( \int \zeta_2(s) s^\nu \, ds \right)^{-\frac{1}{2}} + \left( \int \zeta_1(s) s^\nu \, ds \right)^{-\frac{1}{2}} \right]} \right\| \, dr.$$ 

Because we deal with sufficiently small values of $Y$, so that always $\zeta_{1,2}(s) \approx y_0$, it is easy to estimate in the denominator to get that

$$\|M(\zeta_2) - M(\zeta_1)\| \leq \mu_0 \int_0^Y \left\| \frac{\|\zeta_2 - \zeta_1\| \int s^\nu \, ds}{\int \zeta_2(s) s^\nu \, ds \int \zeta_1(s) s^\nu \, ds \left[ \left( \int \zeta_2(s) s^\nu \, ds \right)^{-\frac{1}{2}} + \left( \int \zeta_1(s) s^\nu \, ds \right)^{-\frac{1}{2}} \right]} \right\| \, dr \leq \mu_0 \|\zeta_2 - \zeta_1\| \, Y^{\frac{n}{2(n+1)}}.$$

Here, $\mu_0$ is a constant dependent on $n$ and $y_0$. Because we take $\frac{1}{2} |y_0| \leq y \leq |y_0|$, then we have that $|Y| < 1$, and so, fixing $Y \in [0, y_0]$, with $\mu = \mu_0 |y_0| < 1$, we have that (66) holds true. Hence by Banach’s Fixed Point Theorem [17, p. 39], $M(y)$ has a unique fixed point in $Z_{\delta}$. ■

Similarly, we prove local existence and uniqueness of a positive solution for $l = 1, 2$.

For $k \geq 2$, the solutions are oscillatory (changing sign) close to interfaces, so that a contraction approach does not apply, and, as we have mentioned, we need to use other techniques of asymptotic analysis in both limits $y \to y_0^+$ and $y \to +\infty$ (the latter one includes $k = 1$). Such oscillatory structures near interfaces have been thoroughly studied for higher-order thin film equations; see [18, 19]. Examples of such oscillatory patterns for various NDEs can be found in [5, Ch. 4], so we do not address these questions here anymore, and concentrate on global existence and another more difficult limit.

3.2. Global existence and uniqueness for $k = 1$

We restrict our attention to the first nonlinear eigenfunction $Y_0$, and prove the following global existence of nonlinear Airy functions:

**Theorem 1.** For any $n > 0$ and a fixed interface point $y_0 < 0$, the problem (53), (57) admits a unique solution $Y_0(y)$, which is infinitely oscillatory as $y \to +\infty$. 
Proof: Once the local existence and uniqueness have been established in Proposition 1, its unique existence on the whole interval $(y_0, +\infty)$ follows from elementary checked local extension properties for the ODE (53), which is shown not to admit strong singularities ("blow-ups") at any finite point. Oscillatory character of such a solution will be shown later. ■

3.3. Local and global behavior of nonlinear eigenfunctions as $y \to +\infty$

At this moment, we do not know the behavior of nonlinear eigenfunctions $f_l(y)$ (or $Y_l(y)$) for $y \gg 1$. Because the first eigenfunction $Y_0(y)$ is expected to converge to our linear kernel $F(y)$ as $n \to 0^+$ (the Airy function (60) for $k = 1$), we also expect to have reasonably similar behavior as $y \to +\infty$. Recall that the rescaled kernel $F(y) = \psi_0(y)$ has the following oscillatory slow algebraic decay as $y \to +\infty$ [1, §2.2]: for some constant $\hat{c} \in \mathbb{R}$,

$$F(y) \sim y^{\frac{2k-1}{4}} \cos \left( d_k y^{\frac{2k+1}{4}} + \hat{c} \right) \quad \text{as} \quad y \to +\infty,$$

where $d_k = 2k \left( \frac{1}{2k+1} \right)^{\frac{2k+1}{4}}.$ (67)

Further eigenfunctions $\tilde{\psi}_l(y)$ given by the derivatives (31) have the corresponding asymptotics via differentiating (67), so these are much more oscillatory and unbounded for $y \gg 1$. On the other hand, the eigenvalue problem (29) admits polynomial solutions (cf. (33) for $B^*$)

$$\tilde{\psi}_l(y) \sim y^l + \cdots \quad \text{for} \quad \tilde{\lambda}_l = \frac{l+1}{2k+1}, \quad l = 0, 1, 2, \ldots. \quad (68)$$

Because $\tilde{\psi}_l \not\in L^2_{\rho}$, these are not proper eigenfunctions. However, this shows that the Equation (29) admits polynomially growing nonoscillatory solutions as $y \to +\infty$.

Due to the complicated nature of the nonlinear equations for $n > 0$, it is not that easy to predict possible behavior of solutions as $y \to +\infty$, where we expect them to be oscillatory as in the linear case $n = 0$. We first study existence and nonexistence of nonoscillatory solutions, which mimic the polynomials (68). Without loss of generality, we consider the ODE (39) for the first nonlinear eigenfunction $Y_0(y)$.

Proposition 2. For any $n > 0$, the ODE (39) for even $k = 2, 4, \ldots$ admits a bundle of algebraically growing solutions as $y \to +\infty$

$$Y(y) \sim \pm Ay^m, \quad \text{where} \quad m = 2k + \frac{n+2}{n+1} \quad \text{and} \quad A = A(n, k) \neq 0, \quad \text{(69)}$$

and does not admit such nonoscillatory solutions for odd $k = 1, 3, \ldots$. 
Proof: As a straightforward computation, we substitute into (39) the asymptotic expression from (69) to get, by balancing the leading terms:

\[ |A|^{-\frac{n}{m+1}} = (-1)^k m(m-1) \cdots (m-2k+1)(2k+1) + n, \]

from which we get the explicit expression for \( A \) and the result. A standard asymptotic argument via fixed point theorems completes the proof of existence of such asymptotics.

Thus, for odd \( k \), all the admitted behavior as \( y \to +\infty \) are not of algebraic growth, and, more plausibly, are oscillatory (as well as for all even \( k \)). To show this, we separately consider the particularly interesting case \( k = 1 \).

**Proposition 3.** All the solutions of the ODE (39) for \( k = 1 \) are oscillatory as \( y \to +\infty \), i.e., have infinitely many sign changes in any neighborhood of \( +\infty \).

Proof: Assume first that, for \( k = 1 \),

\[ Y(y) \to +\infty \quad \text{as} \quad y \to +\infty. \]

Then (39) yields

\[ Y'' = -\frac{1}{n+3} y Y^{\frac{1}{n+1}} \ll -\frac{y}{n+3} \quad \text{for} \quad y \gg 1 \]

\[ \implies Y(y) \ll -\frac{y^3}{6(n+1)} \to -\infty, \]

whence the contradiction. Further cases are studied similarly. The oscillatory character of all the solutions follows form of the ODE in (71), because sign \( Y'' = -\text{sign} \ Y \) (cf. \( Y'' = -Y \) for harmonic oscillations).

3.4. A priori bounds for \( Y(y) \) for \( k = 1 \): nonlinear oscillatory tail

We continue to study oscillatory properties of nonlinear eigenfunctions \( Y_l(y) \), as \( y \to +\infty \), for the basic case \( k = 1 \). Let us fix two successive local extremum points \( 1 \ll y_1 < y_2 \) of \( Y(y) \), where \( Y'(y_1) = Y'(y_2) = 0 \). Let us characterize the size of oscillations of \( Y(y) \) at those points.

We initially look at the equation for the first nonlinear eigenfunction, with the eigenvalue \( \alpha_0(n) \), for \( k = 1 \). Multiplying (53) by \( Y' \) and integrating over \((y_1, y_2)\) yields

\[ \int_{y_1}^{y_2} Y'' Y' = -\frac{1}{n+3} \int_{y_1}^{y_2} |Y|^{-\frac{n}{m+1}} Y Y'. \]

Simplifying this, we see that

\[ \frac{1}{2} [(Y')^2]_{y_1}^{y_2} = -\frac{(n+1)}{(n+2)(n+3)} \int_{y_1}^{y_2} (|Y|^{\frac{n}{m+1}})' y. \]
After integrating by parts, we obtain
\[
\frac{1}{2} [(Y')^2]_{y_1} = - \frac{(n+1)}{(n+2)(n+3)} [Y^{n+2}_{n+1}]_{y_1} + \frac{(n+1)}{(n+2)(n+3)} \int_{y_1}^{y_2} |Y^{n+2}_{n+1}|_{y_1}.
\]
However, because we are looking at extremum points, where \( Y'(y_1) = Y'(y_2) = 0 \),
\[
[(Y')^2]_{y_1} = 0 \quad \text{and} \quad |Y(y_2)|^{n+2}_{n+1} y_2 - |Y(y_1)|^{n+2}_{n+1} y_1 = \int_{y_1}^{y_2} |Y^{n+2}_{n+1}| dy.
\]
Because \( \int_{y_1}^{y_2} |Y|^{n+2}_{n+1} dy > 0 \), we have
\[
|Y(y_2)|^{n+2}_{n+1} y_2 > |Y(y_1)|^{n+2}_{n+1} y_1, \quad \text{or, on rearranging,} \quad \left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{n+2}_{n+1} > \left( \frac{y_1}{y_2} \right).
\]
(72)
This gives a lower estimate on the character of oscillations.

Let us now look at our second nonlinear eigenfunction with \( \alpha_1(n) \), where our equation is given by (43), \( k = 1 \). Multiplying by \( Y' \) and integrating yields
\[
Y' Y'' y - (Y')^2 + \frac{1}{3+2n} |Y|^{-\frac{n}{n+1}} Y Y' y^2 = 0
\]
\[
\Longleftrightarrow \quad \frac{1}{2} [(Y')^2] y - (Y')^2 + \frac{n+1}{(n+2)(3+2n)} (|Y|^{n+2}_{n+1}) y^2 = 0.
\]
Integrating between \( y_1 \) and \( y_2 \) again, we have that
\[
\frac{1}{2} [(Y')^2]_{y_1} - \frac{1}{2} \int_{y_1}^{y_2} (Y')^2 + \frac{n+1}{(n+2)(3+2n)} [|Y|^{n+2}_{n+1}]_{y_1} y^2
- \frac{2(n+1)}{(n+2)(3+2n)} \int_{y_1}^{y_2} |Y|^{n+2}_{n+1} y = 0
\]
\[
\Longleftrightarrow -\frac{3}{2} \int_{y_1}^{y_2} (Y')^2 - \frac{2(n+1)}{(n+2)(3+2n)} \int_{y_1}^{y_2} |Y|^{n+2}_{n+1} y
+ \frac{n+1}{(n+2)(3+2n)} (|Y(y_2)|^{n+2}_{n+1} y_2^2 - |Y(y_1)|^{n+2}_{n+1} y_1^2) = 0.
\]
One can see that
\[
\frac{3}{2} \int_{y_1}^{y_2} (Y')^2 + \frac{2(n+1)}{(n+2)(3+2n)} \int_{y_1}^{y_2} |Y|^{n+2}_{n+1} y > 0,
\]
because \( y > 0 \). Hence, we have a similar estimate:
\[
|Y(y_2)|^{n+2}_{n+1} y_2^2 > |Y(y_1)|^{n+2}_{n+1} y_1^2, \quad \text{or} \quad \left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{n+2}_{n+1} > \left( \frac{y_1}{y_2} \right)^2.
\]
(73)
Finally, for the third eigenfunction with $\alpha_2(n)$, using (46), $k = 1$ and multiplying by $Y'$ yields

\[
Y''Y y^2 - 2(Y')^2 y + 2YY' + \frac{1}{3 + 3n} Y^{-\frac{n}{n+1}} Y' y^3 = 0
\]

\[
\Rightarrow \quad \frac{1}{2} [(Y')^2] y^2 - 2(Y')^2 + (Y^2)' + \frac{n + 1}{(n + 2)(3 + 3n)} (|Y|^{\frac{n+2}{n+1}})' y^3 = 0.
\]

Integrating over $(y_1, y_2)$ yields

\[
\frac{1}{2} [(Y')^2]_{y_1}^{y_2} y^2 - 2 \int_{y_1}^{y_2} (Y')^2 y - 2 \int_{y_1}^{y_2} [Y^2]_{y_1}^{y_2} + \frac{n + 1}{(n + 2)(3 + 3n)} \int_{y_1}^{y_2} |Y|^{\frac{n+2}{n+1}} y^3 = 0.
\]

It then follows that

\[
\left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{\frac{n+2}{n+1}} > \left( \frac{y_1}{y_2} \right)^3, \text{ and, for any } l, \left( \frac{|Y(y_2)|}{|Y(y_1)|} \right)^{\frac{n+2}{n+1}} > \left( \frac{y_1}{y_2} \right)^{l+1} \quad \text{(74)}
\]

Thus, the three estimates (72), (74), and (75) characterize the behavior of the nonlinear oscillatory tail for $k = 1$, which can be compared with numerical evidence in Section 2.5.

3.5. More on oscillatory structure and periodicity: toward a “nonlinear focus”

As mentioned before, at least for small $n > 0$, we expect that nonlinear eigenfunctions exhibit, as $y \to +\infty$, a behavior which is structurally similar to the linear kernel and its derivatives. We therefore expect to have a special oscillatory behavior for $y \gg 1$, as we have already seen before. Hence, we intend to describe this oscillatory structure and, as a first natural attempt, we will try to find if these oscillations are given by periodic functions. Let us introduce the oscillatory component $\phi(s)$ such that, as $y \to +\infty$,

\[
Y(y) = y^{\gamma} \phi(s), \quad \text{where } s = \ln y,
\]

for some power $\gamma \in \mathbb{R}$. Here the term $y^{\gamma}$ gives the rate of any growth/decay of the oscillations and may be compared to the controlling factor $y^{-\frac{2k-1}{n}}$ found in the linear asymptotics (67) for $n = 0$.

Let us begin with the simpler case $k = 1$, where substituting into (64), we obtain

\[
(y^{\gamma} \phi)' + \frac{1}{n + 3} y^{1 + \frac{\gamma}{n+1}} |\phi|^{-\frac{\gamma}{n+1}} \phi = 0.
\]

Expanding this expression and equating powers of $y$, we find that

\[
\gamma - 2 = 1 + \frac{\gamma}{n+1} \quad \Rightarrow \quad \gamma = \frac{3(n + 1)}{n}. \quad \text{(77)}
\]
This gives us a second-order ODE for the oscillatory component:

\[ P_2(\phi) \equiv \phi'' + (2\gamma - 1)\phi' + \gamma(\gamma - 1)\phi = -\frac{1}{n + 3} |\phi|^{-\frac{n}{n+1}} \text{ in } \mathbb{R}. \] (78)

A most typical orbit describing oscillations should be a periodic orbit of (79). However, because \(\gamma > 0\) in (78), the behavior (76) describes unbounded oscillations as \(y \to +\infty\), which are not acceptable for a source-type solution; cf. the linear decaying one (67) for \(n = 0\). Hence, we conclude that oscillatory behavior as \(y \to +\infty\) is not given by periodic oscillatory components \(\phi(\ln y)\) as in (76).

Similarly, we arrive at a contradiction, applying (76) to the general Equation (50):

\[ (-1)^k D^{2k+1}_y (y^\gamma \phi) = \frac{1}{(2k + 1) + (l + 1)n} y(y^\gamma |\phi|^{-\frac{n}{n+1}} \phi)' + \frac{l + 1}{(2k + 1) + (l + 1)n} y^\gamma |\phi|^{-\frac{n}{n+1}} \phi. \]

Balancing the polynomial terms yields

\[ \gamma - (2k + 1) = \frac{\gamma}{n + 1} \implies \gamma = \frac{(2k + 1)(n + 1)}{n} > 0. \] (79)

This leaves us with an ODE of the order \(2k + 1\), for \(\phi(s)\):

\[ (-1)^k P_{2k+1}(\phi) = \frac{1}{(2k + 1) + (l + 1)n} (|\phi|^{-\frac{n}{n+1}} \phi)' + \frac{1}{n} |\phi|^{-\frac{n}{n+1}} \phi. \] (80)

Here, \(P_{2k+1}(\phi)\) is a polynomial operator on \(\phi\), induced by the term \(D^{2k+1}_y (y^\gamma \phi)\). For the case \(k = 1\), the polynomial operator is given by

\[ P_3(\phi) = \phi''' + \frac{3(2n + 3)}{n} \phi'' + \frac{9n^2 + 7n + 27}{n} \phi' + \frac{3(n + 1)(2n + 3)(n + 3)}{n^3} \phi. \]

In general, \(P_{2k+1}(\phi)\) is defined by the recursion

\[ P_{2k+1}(\phi) = \frac{d^2}{ds^2} P_{2k-1}(\phi) + (2\gamma - 4k + 1) \frac{d}{ds} P_{2k-1}(\phi) + (\gamma - 2k)(\gamma - 2k + 1) P_{2k-1}(\phi). \]

As usual, periodic solutions of (81) are at most simple oscillatory components. However, as before, because \(\gamma > 0\) in (80), oscillatory structures of the form (76), for any \(k \geq 2\), are not applicable, at least for the first nonlinear eigenfunction \(Y_0(y)\), which is assumed to be integrable as \(y \to +\infty\).

For higher-order nonlinear eigenfunctions \(Y_l(y)\), with \(l \geq 1\), proving existence of periodic solutions of the ODE (81) is the first step in understanding the oscillatory behavior. This is a difficult mathematical problem, which nevertheless can be solved for orders of \(k\) that are not too large. We refer to
[18, 19, 20] for key references and recent results on existence–uniqueness of periodic solutions of even-order ODEs such as (81), which occur in parabolic thin film theory. We also refer to [12, §4] for existence results of periodic orbits for oscillatory solutions of the PME–4 (19), $m = 2$.

Overall, we rule out the “periodic” structures (76) as $y \to +\infty$, for the first nonlinear eigenfunction $Y_0(y)$, which is expected to have an oscillatory decay and be integrable (not in the absolute sense, that is, it is not measurable there). Therefore, in this case, the oscillatory behavior may be more complicated and corresponds to “nonlinear focus,” which we are going to describe for large $n \gg 1$, and also to try obtain using numerical methods. For $l \geq 1$, such a behavior (76) with almost periodic oscillatory components $\phi(\ln y)$ is still plausible (but remains rather unlikely).

3.6. Branching of nonlinear eigenfunctions at $n = 0$

As we have promised, we now apply another classic idea to trace out the behavior of all the nonlinear eigenfunctions, for small $n > 0$. Namely, we are going to show that there exists branching at $n = 0^+$ of solutions with respect to the parameter $n$. In other words, we show that, as $n \to 0$, there exists certain convergence to solutions (driven by the eigenfunctions of the linear operator $B$ in (14) of the LDE (15).

To this end, let us look at the general ODE given by (23). We first expand $|f|^n$ to formally get

$$|f|^n = 1 + n \ln |f| + O(n^2). \quad (81)$$

This is pointwise and uniformly true in any bounded positivity/negativity subset $\{|f| \geq \delta_0 > 0\}$. However, at this moment we are not going to discuss a rigorous functional meaning of this expansion for changing sign functions $f(y)$ defined in the whole $\mathbb{R}$. Note that (82) can then be understood in a weak sense, which may be sufficient for passing to the limit in the equivalent integral equations; see [18, §7.6] for asymptotic details.

Thus, using the formal expansion (82), (23) reduces to

$$(-1)^{k+1} D_y^{2k+1}[\ln |f| f] + \frac{1 - \alpha n}{2k + 1} f' + \alpha f + O(n^2) = 0.$$  

Expanding coefficients for small $n > 0$ yields

$$(B - \lambda_l I) f + (-1)^{k+1} D_y^{2k+1}(n \ln |f| f) + \left(\alpha - \frac{l + 1}{2k + 1}\right) f - \frac{\alpha n}{2k + 1} f' = O(n^2), \quad (82)$$

where $B$ is the linear operator (14) and $\lambda_l = -\frac{l}{2k+1}$ is its $(l + 1)$th eigenvalue.
For \( l < 2k + 1 \), we can find our eigenvalues \( \alpha_l(n) \) explicitly as in (48), so that

\[
\alpha_l(n) = \frac{l + 1}{(2k + 1) + n(l + 1)} = \frac{l + 1}{2k + 1} \left[ 1 + \frac{n(l + 1)}{2k + 1} \right]^{-1}
\]

\[
= \frac{l + 1}{2k + 1} \left[ 1 - \frac{n(l + 1)}{2k + 1} \right] + O(n^2).
\]

Then (83) reduces to

\[
(B - \lambda_l I)f + (-1)^{k+1} D_{y}^{2k+1}(n \ln |f| f) - \frac{n(l + 1)^2}{(2k + 1)^2} f - \frac{n(l + 1)}{(2k + 1)^2} f' y
\]

\[
+ O(n^2) = 0.
\]

Hence using the Lyapunov–Schmidt method [15] by setting

\[
f = \psi_l + n\phi_l + O(n^2),
\]

we obtain, within the order \( O(n) \), the following inhomogeneous equation:

\[
(B - \lambda_l I)\phi_l = (-1)^k D_{y}^{2k+1}(\ln |\psi_l| \psi_l) + \frac{(l + 1)^2}{(2k + 1)^2} \psi_l + \frac{(l + 1)}{(2k + 1)^2} \psi'_l y = h. \tag{84}
\]

Using Hermitian spectral theory for \( B \) and completeness-closure of the eigenfunctions subset \( \Phi_1 = \{\psi_l\}_{l \geq 0} \) [1, §4], for the unique solvability of (85) for \( \phi_l \), it now remains to demand that the right-hand side \( h \) is orthogonal to \( \psi^*_l \), that is,

\[
\langle h, \psi^*_l \rangle_a = 0. \tag{85}
\]

Here we have to use the corresponding indefinite metric \( \langle \cdot, \cdot \rangle_a \), in which the pair \([B, B^*]\) comprises the operator \( B \) and its adjoint (this metric can be reduced to the standard dual \( L^2 \)-one, [1, §5]). Here, (86) is known as a scalar bifurcation equation in the classic Lyapunov–Schmidt method [15].

We then use the adjoint polynomial eigenfunctions \( \psi^*_l \) given by (33). Then, for \( l < 2k + 1 \), we have that the generalized Hermite polynomials are simple [1, §5],

\[
\psi^*_l(y) = \frac{1}{\sqrt{l!}} y^l \quad (0 \leq l < 2k + 1).
\]

Hence, for all \( l < 2k + 1 \), (86) is indeed valid:

\[
\langle h, \psi^*_l \rangle_a = \frac{1}{\sqrt{l!}} \int \left[ (-1)^k D_{y}^{2k+1}(\ln |\psi_l| \psi_l)y^l + (-1)^l \frac{(l + 1)^2}{(2k + 1)^2} \psi_l y^l + (-1)^l \frac{(l + 1)}{(2k + 1)^2} \psi'_l y^{l+1} \right] dy
\]
\[
+ (-1)^l \left( \frac{l+1}{(2k+1)^2} \psi'_y y' + (-1)^l \frac{l+1}{(2k+1)^2} \psi''_{l+1} \right) dy
\]

\[
= \frac{1}{\sqrt{l!}} \int \left[ (-1)^k D_y^{2k+1} (\ln |\psi|)|\psi| y' + (-1)^l \frac{l+1}{(2k+1)^2} (\psi'_y y') \right] dy = 0.
\]

Recall that, for \( l \geq 2k+1 \), we do not know nonlinear eigenvalues \( \alpha_l(n) \) explicitly. In this case, we expand \( \alpha_l(n) \) as follows:

\[
\alpha_l(n) = \alpha_0 + \alpha_1 n + O(n^2),
\]

where \( \alpha_0 = \alpha_l(0) \) in (30) comes from linear Hermitian theory, and \( \alpha_1 \) is a new unknown. As before, we use (83) and now we substitute (87), as well as (84), to obtain

\[
n(B - \lambda_l I) \phi_l = (-1)^k D_y^{2k+1} (n \ln |\psi|)|\psi| + \left( \alpha_0 + n \alpha_1 - \frac{l+1}{2k+1} \right) \psi_l
\]

\[
+ n \left( \alpha_0 - \frac{l+1}{2k+1} \right) \phi_l - \frac{n \alpha_0}{2k+1} \psi'_y + O(n^2) = 0.
\]

Equating as usual the terms of the order \( O(n) \), we can find the value of \( \alpha_0 \), with

\[
\alpha_0 - \frac{l+1}{2k+1} = 0, \quad \text{and} \quad n(B - \lambda_l I) \phi_l
\]

\[
= (-1)^k D_y^{2k+1} (n \ln |\psi|)|\psi| + n \alpha_1 \psi_l + n \left( \alpha_0 - \frac{l+1}{2k+1} \right) \phi_l - \frac{n \alpha_0}{2k+1} \psi'_y.
\]

Hence, substituting into (88) and passing to the limit \( n \to 0^+ \) yield

\[
(B - \lambda_l I) \phi_l = (-1)^k D_y^{2k+1} (\ln |\psi|)|\psi| + \alpha_1 \psi_l - \frac{l+1}{(2k+1)^2} \psi'_y \equiv h.
\]

Then, the orthogonality condition (86) becomes an algebraic equation for \( \alpha_1 \) in (87). Namely, taking the inner product with \( \psi_l^* \) and noting that \( \langle h, \psi_l^* \rangle = 0 \),

\[
\langle \psi_l, \psi_l^* \rangle = 1 \quad \text{yield}
\]

\[
\alpha_1 = - \left( (-1)^k D_y^{2k+1} (\ln |\psi|)|\psi| - \frac{l+1}{(2k+1)^2} \psi'_y, \psi_l^* \right).
\]

Thus, this uniquely defines the second coefficient \( \alpha_1 \) in the expansion (87) and then, as usual, (89) gives a unique function \( \psi_l \) in the eigenfunction expansion (84), etc.

In the analytic or even finite regularity cases, solvability conditions and existence of expansions such as (87) usually rigorously justify the actual presence of branching. Our case is more delicate in view of the “weakness” of the expansion (82). However, for the variable \( Y = |f|^n f \), the expansion (82) is easier to justify, especially now the equation becomes semilinear and can
be reduced to an integral equation with compact Hammerstein–Uryson-type operators. Therefore, in the present case, a full justification of the \( n \)-branching method, though being rather technical, is achievable and does not represent a principally nonsolvable problem of nonlinear integral operator theory.

4. Mesa-like problem: “super-nonlinear” limit \( n \to \infty \) for \( k = 1 \) and \( l = 0 \)

While we dealt before with a “homotopy path” construction of nonlinear eigenfunctions in the limit of small \( n \to 0^+ \), we now consider the opposite “super-nonlinear” limit \( n \to +\infty \), which also helps to understand properties of the nonlinear eigenvalue problem.

This is the key result of the paper: passing to the more difficult limit \( n \to +\infty \) allows us, unlike the simpler and standard semilinear one \( n \to 0^+ \), to describe the actual **strongly nonlinear** behavior of solutions of the NDEs. Moreover, we then get a unique opportunity to describe, in an explicit **algebraic–geometric** manner, the actual oscillatory tails of nonlinear Airy-like functions for large \( n > 0 \). Recall that, in Sections 3.3–3.5, we failed this before by trying to revive a nonlinear spiral-focus-like behavior by more standard mathematical techniques.

Note that, this super-nonlinear limit for the PME-2 (3) used to be a special important open problem of nonlinear PME theory, which was resolved, leading to a **mesa problem**, at the end of the 1980s only in the works by well-known mathematicians in nonlinear PDE/semigroup theory such as Friedman, Caffarelly, Bénilan, Boccardo, Herrero, King, Elliot, Ockendon, and others; see [21, Ch. 2] for a history, references (e.g., [37, 67, 104, 122], etc. therein) and results.

Thus, we present here a rigorous analysis of the “mesa-like problem” corresponding to the third-order NDE (2) in the ODE representation.

4.1. Reducing to an algebraic problem

Now, we are going to perform the opposite **highly nonlinear limit** \( n \to +\infty \), for nonlinear eigenfunctions of the NDE. Rather surprisingly, it turns out that this “limit nonlinear case” admits a more profound analysis, and for \( l = 0, 1, 2 \) we are able to tackle the nonlinear eigenvalue problems by a simpler geometric–algebraic, allowing us to obtain a number of analytical and explicit expressions. Most importantly, for the first time, we will still also catch rather obscure oscillatory behavior of the nonlinear Airy function and next ones.

Consider the ODE (39), with \( k = 1 \) and where \( l = 0 \), that is, (53). Because we are dealing with \( n \gg 1 \), it is necessary to scale out any coefficients containing large \( n \)’s. To do this, we let

\[
Y(y) = C \tilde{Y}(y), \quad \text{where} \quad C = C(n) > 0 \quad \text{is a constant.} \tag{89}
\]
Substituting (90) into (53) yields:

\[ C \tilde{Y}'' = -\frac{1}{n + 3} |C|^{\frac{n}{n+1}} C |\tilde{Y}|^{-\frac{n}{n+1}} \tilde{Y} y \quad \implies \quad C = (n + 3)^{-\frac{n+1}{n}}. \]  

(90)

so that, on scaling out such a \( C(n) \), we obtain the rescaled ODE

\[ \tilde{Y}'' = -|\tilde{Y}|^{-\frac{n}{n+1}} \tilde{Y} y. \]  

(91)

Here we can pass to the limit \( n \to +\infty \), because the only \( n \)-dependent exponent satisfies \(-\frac{n}{n+1} \to -1\). At \( n = +\infty \), the ODE becomes simpler and contains a bounded discontinuous nonlinearity:

\[ B_\infty(\tilde{Y}) \equiv \tilde{Y}'' + \text{sign} \tilde{Y} y = 0 \quad \text{in} \quad \mathbb{R}. \]  

(92)

Of course, passing to the limit to arrive at (93) might be a delicate mathematical problem. A potentially dangerous situation occurs in those subsets, where \( \tilde{Y} \) vanishes. However, if both \( \tilde{Y}(y) \) and \( \tilde{Y}'(y) \) have a.a. zeros transversal in the natural sense, then passage to the limit is straightforward. A sufficient “transversality” of zeros of the limit function \( \tilde{Y}(y) \) can be checked a posteriori, after completing our algebraic construction.

Solving the “almost linear” ODE (93) yields expressions dependent on the sign of \( \tilde{Y} \):

\[
\begin{align*}
\tilde{Y} > 0 : & \quad \tilde{Y}_+(y) = -\frac{1}{6} y^3 + c_1 y + c_2, \\
\tilde{Y} < 0 : & \quad \tilde{Y}_-(y) = \frac{1}{6} y^3 + d_1 y + d_2.
\end{align*}
\]  

(93)

Here \( c_1, \ c_2, \ d_1, \ d_2 \) are all constants, but not necessarily positive ones. Knowing the conditions of continuity for the function \( \tilde{Y}(y) \) and \( \tilde{Y}'(y) \), we must have that all one-sided limits coincide, that is,

\[ \tilde{Y}_+ = \tilde{Y}_- \quad \text{and} \quad \tilde{Y}'_+ = \tilde{Y}'_- \quad \text{at any zero, where} \quad \tilde{Y} = 0. \]  

(94)

Let the points \( \{ y = y_i \}_{i \geq 0} \) be successive zeros, i.e., \( \tilde{Y}(y_i) = 0 \), for \( i = 0, 1, 2, \ldots \). Hence, for \( \tilde{Y}_+(y_i) \) given in (94), for a fixed isolated zero with an \( i \geq 1 \) (as usual, \( y_0 < 0 \) corresponds to the left-hand interface),

\[ \tilde{Y}_+(y_i) = -\frac{1}{6} y_i^3 + c_{1i} y_i + c_{2i} = 0. \]

We now rearrange this to find one of the unknown parameters, in terms of \( y_i \) and the parameter \( c_{2i} \), such that \( c_{1i} = \frac{1}{6} y_i^2 - \frac{c_{2i}}{y_i} \). We also have that \( \tilde{Y}'_+(y_i) = \tilde{Y}'_-(y_i) \), hence

\[ -\frac{1}{2} y_i^2 + c_{1i} = \frac{1}{2} y_i^2 + d_{1i}. \]
From this, we find a second parameter in terms of $y_i$ and $c_{2i}$, where $d_{1i} = -\frac{5}{6} y_i^2 - \frac{c_{2i}}{y_i}$. Now, we see that from $\tilde{Y} = \tilde{Y}$,

$$c_{1i} y_i + d_{1i} y_i + c_{2i} + d_{2i} = 0.$$  

So, substituting in known values, we have our third parameter $d_{2i}$ given by $d_{2i} = \frac{2}{3} y_i^3 + c_{2i}$.

We now see that, after substituting values for $c_{1i}, d_{1i}$, and $d_{2i}$, (94) can be written as

$$\begin{cases} 
\tilde{Y} > 0 : & \tilde{Y}_+ (y) = -\frac{1}{6} y^3 + \left( \frac{1}{6} y_i^2 - \frac{c_{2i}}{y_i} \right) y + c_{2i}, \\
\tilde{Y} < 0 : & \tilde{Y}_- (y) = \frac{1}{6} y^3 - \left( \frac{5}{6} y_i^2 + \frac{c_{2i}}{y_i} \right) y + \frac{2}{3} y_i^3 + c_{2i}.
\end{cases}$$

From the above, it is noted that $y_i \neq 0$, for any $i$, unless $c_{2i} = 0$.

4.2. Existence, uniqueness, and zero properties of $\tilde{Y}_0 (y)$

We now resolve the algebraic system to get a complete description of some important properties of the solution $\tilde{Y}_0 (y)$ obtained by such a geometric approach. First, recall the scaling invariance of the ODE (93):

$$\tilde{Y}(y) \text{ is a solution} \implies \pm a \tilde{Y} \left( \frac{y}{a} \right) \text{ is a solution for any } a > 0.$$  

Therefore, choosing the interface at $y_0 = -1$, we put two conditions there

$$\tilde{Y}(-1) = \tilde{Y}'(-1) = 0,$$  

and prove the following:

**Theorem 2.** The problem (97), (92) admits a unique nontrivial solution $\tilde{Y}_0 (y)$, which has transversal zeros $\{y_i\}_{i \geq 1}$ such that

$$y_0 = -1, \quad y_1 = 2|y_0| = 2, \quad y_2 = 3\sqrt{2} - 1 = 3.2426 \ldots,$$

and

$$y_{i+1} = \frac{\sqrt{17y_i^2 - 4y_i y_{i-1} - 4y_{i-1}^2} - y_i}{2} \text{ for any } i \geq 2.$$  

**Proof:** As promised, we construct such a solution using pure algebraic manipulations. Let us begin with the first interval of positivity ($y_0 = -1, y_1$), where according to (98), the solution reads

$$\tilde{Y}_+ (y) = -\frac{1}{6} (y + 1)^2(y - y_1).$$
Because (93) implies no quadratic term $\sim y^2$ in the cubic polynomial, this uniquely gives $y_1 = -2y_0 = 2$, and hence

$$\frac{1}{6} (y + 1)^2(y - 2) > 0 \quad \text{on} \quad (-1, 2). \quad (100)$$

On the next interval $(y_1 = 2, y_2)$, the negative solution takes the form:

$$\tilde{Y}_-(y) = \frac{1}{6} (y - 2)(y - y_2)(y + c_1) < 0. \quad (101)$$

Similarly to the above, we then conclude that

$$c_1 = 2 + y_2. \quad (102)$$

Then matching of the first derivatives (94) at $y = y_1 = 2$ implies

$$\frac{1}{6} 3^2 = \frac{1}{6} (2 - y_2)(4 + y_2) \implies y_2^2 + 2y_2 - 17 = 0 \implies y_2 = 3\sqrt{2} - 1, \quad (103)$$

and this procedure can be continued.

Consider an arbitrary interval $(y_{i-1}, y_i)$ of positivity (or negativity), $i \geq 2$, with

$$\tilde{Y}_+(y) = (y - y_{i-1})(y - y_i)\left(-\frac{1}{6} y + c_{i-1}\right), \quad \text{where} \quad c_{i-1} = -\frac{1}{6} (y_i + y_{i-1}). \quad (104)$$

Similarly, on the next negativity (or respective positivity) interval $(y_i, y_{i+1})$,

$$\tilde{Y}_-(y) = (y - y_i)(y - y_{i+1})\left(\frac{1}{6} y + c_i\right), \quad \text{where} \quad c_i = \frac{1}{6} (y_i + y_{i+1}). \quad (105)$$

Therefore, the matching at $y = y_i$ yields

$$(y_i - y_{i-1})\left(-\frac{1}{6} y_i - \frac{1}{6} (y_i + y_{i-1})\right) = (y_i - y_{i+1})\left(\frac{1}{6} y_i + \frac{1}{6} (y_i + y_{i+1})\right)$$

$$\implies y_{i+1}^2 + y_i y_{i+1} - 4y_i^2 + y_i y_{i-1} + y_{i-1}^2 = 0. \quad (106)$$

which yields the final result in (99). Thus, the unique solution can be extended indefinitely for arbitrary $y \gg 1$ and is infinitely oscillatory as $y \to +\infty$. ■

**Remark:** The quadratic equation in (107) reduces to a two-dimensional linear discrete equation:

$$\alpha_{i,j} \equiv y_i y_j \implies \alpha_{i+1,i+1} + \alpha_{i,i+1} - 4\alpha_{i,i} + \alpha_{i,i-1} + \alpha_{i-1,i-1} = 0 \quad \text{for} \quad i \geq 2. \quad (107)$$
It is not difficult to find some particular solutions:

$$\alpha_{i,j} = \mu^i v^j \Rightarrow \mu(\mu + 1)v^2 - 4\mu v + \mu + 1 \Rightarrow \nu_{\pm}(\mu) = \frac{2\mu \pm (\mu - 1)\sqrt{-\mu}}{\mu(\mu + 1)}.$$  \hspace{1cm} (108)

Therefore, denoting by $\mathcal{M}$ a proper subset of parameters $\mu$ such that $\{\mu^i\}_{\mu \in \mathcal{M}}$ is complete/closed, the general solution of (109) is represented by a converging infinite series

$$\alpha_{i,j} = \sum_{\mu \in \mathcal{M}} C_{\mu} \mu^i v^j,$$ \hspace{1cm} (109)

where $\{C_{\mu}\}$ are constants and $\nu = \nu(\mu)$ take values according to (109). Here, (110) is a discrete analogy of eigenfunction expansions of solutions, of a linear PDE with two independent variables $(x, t)$. A proper posing of “boundary conditions” for (108), to specify the corresponding Sturm–Liouville problem and next initial conditions, to get the corresponding eigenfunction expansion (110) is a difficult and uncertain problem.\(^4\) It is also unlikely to provide us with any useful finite explicit formulae.

However, analytic relationships such as (99) can promise extra asymptotic properties of the first rescaled eigenfunction $\tilde{Y}_0(y)$, especially the decay rate of the minimal behavior indicated in (24). In the present case $l = 0$, unlike the simpler one $l = 2$ in Section 6, some computations are not expected to be easy, because such algebraic relations are quadratic and hence not always explicitly solvable.

4.3. Numerics for $n \gg 1$: $k = 1$ and $l = 0$

As usual, very sharp proper numerics can help to detect further properties of $\tilde{Y}_0(y)$, and hence avoid trying to get too complicated and exhaustive results concerning the corresponding algebraic system. Moreover, which is even more important, we can also check the character of convergence of solutions as $n \to +\infty$, which, after scaling (90), turns out to be rather fast. To deal with the ODE (93), it is possible to just use a simple shooting method using the ODE solver ode45, to find suitable profiles and nonlinear eigenfunctions.

We use a similar shooting method as that applied for the general case of $n > 0$, set out in Section 2.5. We recall that, close to the interface at some

\(^4\) It seems, a suitable behavior at infinity, as $i, j \to +\infty$, of $\alpha_{i,j}$ might include something like the “minimality” condition in (24), which is hard to take into account.
The first nonlinear eigenfunction $Y_0(y), n=\infty$: oscillations, decay, envelopes

**Figure 8.** The first nonlinear eigenfunction $\tilde{Y}_0(y)$ as a solution of the problem (92), (110); $k = 1, l = 0$.

For $y = y_0 < 0$, we look for small solutions of $\tilde{Y}(y)$ such that, as $y \to y_0^+$,

$$\tilde{Y}(y) = C_0(y - y_0)^{1/2}(1 + o(1)), \quad \text{where} \quad C_0 = \frac{1}{2}|y_0| > 0,$$

and

$$\tilde{Y}'(y) = |y_0|(y - y_0)(1 + o(1)) \quad (k = 1). \quad (110)$$

The proof of this expansion is similar and even simpler than that of Proposition 1.

In Figure 8, we show the general view of the first eigenfunction $\tilde{Y}_0(y)$ on the large interval $[y_0 = -1, 100]$. The envelope of the decaying oscillations are governed by the algebraic curve

$$L_0(y) \approx \pm 0.7 y^{-\frac{1}{3}} \quad \text{as} \quad y \to +\infty. \quad (111)$$

It seems that, this decay can be seen from the above algebraic system. We have checked that, for $n = 100$, the corresponding nonlinear eigenfunction (after scaling) is practically indistinguishable.

In Figure 9, we show $\tilde{Y}_0(y)$ on a smaller interval $y \in [-1, 7]$, indicating all the first zeros, which will coincide with those given by explicit algebraic expressions in (98).

All the computations have been performed with the enhanced accuracy and tolerances $\sim 10^{-10}$, so these are quite reliable. Let us present the first 15 zeros.
The first nonlinear eigenfunction $Y_0(y)$, $n=\infty$: first zeros

Figure 9. The first nonlinear eigenfunction $\tilde{Y}_0(y)$ as a solution of the problem (92), (110); $k=1$, $l=0$.

of $\tilde{Y}_0(y)$:

\begin{align*}
  y_0 &= -1, \quad y_1 = 2|y_0| = 2, \quad y_3 = 3.2426\ldots, \quad y_4 = 5.0777\ldots, \\
  y_5 &= 5.8426\ldots, \quad y_6 = 6.5459\ldots, \quad y_7 = 7.2021\ldots, \\
  y_8 &= 7.8207\ldots, \quad y_9 = 8.4083\ldots, \quad y_{10} = 8.9697\ldots, \\
  y_{11} &= 9.5086\ldots, \quad y_{12} = 10.0279\ldots, \quad y_{13} = 10.5299\ldots, \\
  y_{14} &= 10.0164\ldots, \\
\end{align*}

4.4. Branching at $n = +\infty$

Introducing the small parameter in (92) for $n \gg 1$,

\[ \epsilon = 1 - \frac{n}{n+1} \to 0^+ \quad \text{as} \quad n \to +\infty, \]

and performing a standard (formal, as usual, at least in the pointwise sense) linearization, yields the following problem:

\[ |\tilde{Y}|^{-\frac{n}{n+1}} \equiv |\tilde{Y}|^{\epsilon^{-1}} = \frac{1 + \epsilon \ln |\tilde{Y}| + O(\epsilon^2)}{|\tilde{Y}|} \]

\[ \implies B_{\infty}(\tilde{Y}) = -\text{sign} \tilde{Y} y \ln |\tilde{Y}| + O(\epsilon), \]  

(112)
where $B_\infty$ is the unperturbed operator in (92). Next, studying the branching at $\varepsilon = 0$ from the first nonlinear eigenfunction $\tilde{Y}_0$, we obtain the corresponding linear problem:

$$\tilde{Y} = \tilde{Y}_0 + \varepsilon \phi + O(\varepsilon^2) \implies B'_\infty(\tilde{Y}_0)\phi = h = -(\text{sign} \tilde{Y}_0)y \ln |\tilde{Y}_0|,$$

(113)

where the symmetric linearized operator $B'_\infty(\tilde{Y}_0)$ is given by

$$B'_\infty(\tilde{Y}_0) = D_y^2 + (\text{sign} \tilde{Y}_0) y \ I, \quad (\text{sign} \tilde{Y}_0)' = \delta(y - y_0) + 2 \sum_{(i \geq 1)} (-1)^i \delta(y - y_i).$$

(114)

Here $\{y_i\}$ are zeros of $\tilde{Y}_0(y)$ as explained in Theorem 2.

At this stage, one then needs proper spectral theory for a self-adjoint extension of the operator $B'_\infty(\tilde{Y}_0)$. Then the branching condition reads as the orthogonality

$$h \perp \ker B'_\infty(\tilde{Y}_0).$$

(115)

However, developing such a proper spectral theory faces some hard algebraic difficulties. Indeed, looking for eigenfunctions $\varphi_k$,

$$\varphi_k'' - \lambda_k \varphi_k = C_k = \text{const.} \equiv -(y_0 \varphi_k(y_0) + 2 \sum_{(i \geq 1)} (-1)^i y_i \varphi_k(y_i)), \quad \varphi_k(y_0) = 0,$$

(116)

for $\lambda_k < 0$, we obtain the solution and an algebraic equation for such eigenvalues:

$$\varphi_k(y) = \sin \left(\sqrt{|\lambda_k|}(y - y_0)\right) \implies \lambda_k : \sum_{(i \geq 1)} (-1)^i y_i \sin \left(\sqrt{|\lambda_k|}(y_i - y_0)\right) = 0.$$

(117)

According to (118), we require that each eigenfunction $\varphi_k(y)$ should be purely oscillatory as $y \to +\infty$ about zero, meaning a “minimal” (“zero-average”) behavior, or zero value in a weak sense. For $\lambda_0 = 0$, we take $\varphi_0(y) = y - y_0$ (again up to a normalization multiplier), leading to another algebraic problem, which is expected to be nonproper because the eigenfunctions are not oscillatory and do not satisfy the “zero condition” at infinity.

Proving that the algebraic equation in (118) has a discrete family of solutions $\{\lambda_k < 0\}$ is a difficult open problem. Nevertheless, at least this analysis shows a principal possibility to detect branching of nonlinear eigenfunctions at $n = +\infty$, where a simpler algebraic treatment is available, and more practical asymptotic and other properties of the nonlinear eigenfunctions than in Theorem 1, for any $n > 0$, are known.
4.5. The higher-order case \( k \geq 2 \) for \( l = 0 \)

For the general case of the higher-order ODEs (39), again for \( l = 0 \), the equations as \( n \to \infty \) are much the same. Here the scaling constant \( C(n) \), for \( \tilde{Y} = C(n) \tilde{Y} \) is given by

\[
C(n) = [(2k + 1) + n]^{-\frac{n+1}{n}}.
\]

After scaling, this yields the ODE

\[
B_{\infty}^{(2k)}(\tilde{Y}) \equiv (-1)^{k+1} D_{\tilde{y}}^{2k} \tilde{Y} + \text{sign} \tilde{Y} y = 0,
\]

which deserves further study by deriving the corresponding algebraic structures. Note that, close to the interface at \( y = y_0^+ \), for \( k \geq 2 \) (the case \( k = 1 \) is not oscillatory, as we have seen in Section 3), a stabilization to a periodic oscillatory component \( \phi(s) \), with the typical structure near the interface at \( y = y_0^+ \),

\[
\tilde{Y}(y) = (y - y_0)^{2k}(\phi(s) + o(1)), \quad \text{where} \quad s = \ln(y - y_0) \to -\infty,
\]

of solutions of (118) is most plausible. See [18, 19], where such oscillatory sign changing solutions such as (119), with a periodic component \( \phi(s) \) satisfying a nonlinear higher-order ODE, have been found for the fourth- and sixth-order thin film equations.

However, explicitly solving the ODE (118) for any \( k \geq 2 \) in the positivity and negativity domains leads to much more complicated algebraic systems, which do not admit such a clear explicit resolving as for \( k = 1 \). Moreover, even numerical shooting methods lead to difficult and unclear results, so we do not present such an analysis here.

5. The second eigenfunction \( \tilde{Y}_1(y) \) for \( n = +\infty \): algebraic approach

Using the same method, the equations governing the behavior as \( n \to \infty \), relating to the second nonlinear eigenfunction \( \tilde{Y}_1(y) \), can easily be found. Namely and analogously, for \( l = 1 \) in the ODE (43), with \( k = 1 \), the limit equation at \( n = +\infty \) is given by

\[
\tilde{Y}''y - \tilde{Y}' + \text{sign} \tilde{Y} y^2 = 0.
\]

This ODE (120) can be written in a singular Sturm–Liouville form

\[
\left( \frac{\tilde{Y}'}{y} \right)' = -\text{sign} \tilde{Y},
\]

where the weight \( \rho(y) = \frac{1}{y} \not\in L_{\text{loc}}^1 \). Therefore, \( y = 0 \) is a singular inner point, where an extra condition must be posed. This is done by checking the functional weighted \( L^2 \)-space, corresponding to the operator in (121), and its available
asymptotics as \( y \to 0 \):
\[
\int_0^\infty \rho(y)(\tilde{Y}'(y))^2 \, dy < \infty \quad \implies \quad \tilde{Y}'(0) = 0.
\] (122)

Next, integrating (121) twice yields, in the positivity and negativity domains, the following:
\[
\tilde{Y}_\pm(y) = \mp \frac{y^3}{3} + ay^2 + b, \quad a, b, \in \mathbb{R}.
\] (123)

Note that, unlike (93) for \( l = 0 \), here the linear term \( \sim y \) is absent in the cubic polynomial. The analysis of the algebraic matching system corresponding to (123) is similar in many places, but there exists a couple of differences, which we will concentrate upon now.

**Theorem 3.** The problem (121), (97) admits a unique nontrivial solution \( \tilde{Y}_1(y) \) with transversal zeros at \( \{y_i\}_{i \geq 1} \), where
\[
y_0 = -1, \quad y_1 = \frac{1}{2}, \quad y_2 = \frac{5 + 3\sqrt{5}}{4} = 2.927050 \ldots,
\] (124)
and triples of zeros \( \{y_{i-1}, y_i, y_{i+1}\}_{i \geq 2} \) satisfy a cubic homogeneous algebraic equation,
\[
y_{i-1}y_{i+1}^2 + y_i^2y_{i+1} + y_i^2y_{i-1} + y_i^2y_{i+1} = y_{i-1}y_iy_{i-1} + y_i^2y_{i+1} + y_{i-1}y_i^2 + y_i^3
\] for \( i \geq 2 \).
(125)

**Remark:** Equation (125) reduces to a three-dimensional linear discrete equation for \( \alpha_{i,j,k} = y_iy_jy_k \), but, similar to (106), this does not essentially help to get any finite explicit expressions for zeros \( \{y_i\} \).

**Proof:** The first step is elementary: on \((-1, y_1)\), in view of (123) (cf. also (122)),
\[
\tilde{Y}_+(y) = (y + 1)^2(y - y_1) \left( -\frac{1}{3} \right), \quad \text{where} \quad y_1 = \frac{1}{2}.
\] (126)
Next, on \((y_1, y_2)\) by (123),
\[
\tilde{Y}_-(y) = (y - y_1)(y - y_2) \left( \frac{1}{5}y + c_1 \right), \quad \text{where} \quad c_1 = \frac{1}{5} \frac{y_1y_2}{y_1 + y_2},
\] (127)
so that the matching (94) at \( y = y_1 \) yields
\[
(y_1 + 1)^2 \left( -\frac{1}{3} \right) = (y_1 - y_2) \left( \frac{1}{3}y_1 + c_1 \right) \quad \implies \quad 4y_2^2 - 10y_2 - 5 = 0,
\] (128)
The second nonlinear eigenfunction $Y_1(y)$, $n=\infty$: oscillations, slow growth, envelopes

![Figure 10](image)

**Figure 10.** The second nonlinear eigenfunction $\tilde{Y}_1(y)$ as a solution of the problem (121), (110); $k = 1, \ l = 1$.

giving the desired value of the root $y_2$ in (124).

Similarly, in the general case, on $(y_{i−1}, y_i)$,

$$\tilde{Y}_-(y) = (y - y_{i-1})(y - y_i) \left(\frac{1}{3} y + c_{i-1}\right), \quad \text{where} \quad c_{i-1} = \frac{1}{3} \frac{y_{i-1}y_i}{y_{i-1} + y_i},$$

(129)

and on $(y_i, y_{i+1})$,

$$\tilde{Y}_+(y) = (y - y_i)(y - y_{i+1}) \left(-\frac{1}{3} y + c_i\right), \quad \text{where} \quad c_i = -\frac{1}{3} \frac{y_iy_{i+1}}{y_i + y_{i+1}}.$$

(130)

Then the standard matching, via (95), of such $\tilde{Y}_\pm(y)$ yields the homogeneous cubic algebraic equation (126) for the zero triple $\{y_{i-1}, y_i, y_{i+1}\}$. ■

Figure 10 shows the general structure of the second (dipole) eigenfunction $\tilde{Y}_1(y)$ on the large interval $[y_0 = -1, 200]$. The envelope of the decaying oscillations are governed by the algebraic curve

$$L_1(y) \approx \pm 0.89 y^{\frac{1}{3}} \quad \text{as} \quad y \to +\infty,$$

(131)

which can be associated with the algebraic manipulations given earlier.
The second nonlinear eigenfunction $\tilde{Y}_1(y)$ as a solution of the problem (122), (110); $k = 1, l = 1$.

The next Figure 11 shows the first zeros of $\tilde{Y}_1(y)$ on the interval $[-1, 5]$.

In general, for all values of $k \geq 1$, we have the limit equation at $n = +\infty$ for $l = 1$ in the form:

$$(-1)^{k+1} D^{2k} \tilde{Y} + (-1)^k D^{2k-1} \tilde{Y} + \text{sign} \tilde{Y} = 0.$$ 

This can be integrated once, however, one cannot expect any reasonably easy algebraic manipulations leading to some explicit representation of asymptotic properties of $\tilde{Y}_1(y)$, for any $k \geq 2$.

6. The third eigenfunction $\tilde{Y}_2(y)$ for $n = +\infty$: algebraic approach

Similarly, for the third eigenfunction, where $l = 2$, we have, in the lower-order case $k = 1$, the limit ODE

$$\tilde{Y}'' y^2 - 2 \tilde{Y}' y + 2 \tilde{Y} + \text{sign} \tilde{Y} y^3 = 0. \quad (132)$$

The corresponding Sturm–Liouville form

$$\left( \frac{\tilde{Y}'}{y^2} \right)' = -\frac{2}{y^4} \tilde{Y} - \frac{1}{y} \text{sign} \tilde{Y}, \quad (133)$$
reveals even more singularity at \( y = 0 \) as it used to be in (121) for \( \tilde{Y}_1(y) \). The necessary extra condition at \( y = 0 \) is presented later at (141).

Writing (132) down for \( \tilde{Y}_\pm(y) \) in the form

\[
Y'' = \mp y + 2 \left( \frac{\tilde{Y}}{y} \right)'
\]  

(134)

and integrating twice yields

\[
Y_\pm(y) = \mp \frac{y^3}{2} + ay^2 + by, \quad a, b \in \mathbb{R}.
\]  

(135)

Surprisingly, the matching at zeros \( y = y_i \) by using the cubic polynomials (135) having \( y = 0 \) as a fixed zero always leads to a simpler mathematics.

**Theorem 4.** The problem (132), (97) admits a unique nontrivial solution \( \tilde{Y}_2(y) \) with transversal zeros at \( \{y_i\}_{i \geq 1} \), where

\[
y_0 = -1, \quad y_1 = 0, \quad y_2 = 1 + \sqrt{2} = 2.4142 \ldots,
\]

\[
y_3 = 1 + 3\sqrt{2} = 5.2426 \ldots,
\]

(136)

and further zeros are given by the second-order linear discrete equation: for any \( i \geq 3 \),

\[
y_{i+1} - 2y_i + y_{i-1} = 0 \implies y_i = C_1 + C_2i, \quad \text{where} \quad C_1 = 1 - 3\sqrt{2},
\]

\[
C_2 = 2\sqrt{2}.
\]

(137)

In particular, the distribution of zeros is uniform:

\[
y_{i+1} - y_i = 2\sqrt{2} = 2.8284 \ldots \quad \text{for all} \quad i \geq 2.
\]

(138)

**Proof:** On \((-1, y_1)\), in view of (135),

\[
\tilde{Y}_+(y) = y(y + 1)^2 \left( -\frac{1}{2} \right) \implies y_1 = 0.
\]

(139)

Next, on \((y_1 = 0, y_2)\) by (135),

\[
\tilde{Y}_-(y) = y(y - y_2) \left( \frac{1}{2} y + c_1 \right), \quad \text{where} \quad c_1 = \frac{1}{2y_2}.
\]

(140)

Unlike the previous cases, it is not possible to find \( y_2 \) and \( c_1 \) by using the standard matching conditions (94). The point is that the differential operator in (133) is strongly singular at \( y = 0 \), where the weight \( \rho(y) = \frac{1}{y^2} \not\in L^p(-1, 1) \) for any \( p \geq 1 \).

It then follows from (136), due to the singular setting (133), that the two usual conditions at \( y = 0 \), such as the values of \( \tilde{Y}(0) = 0 \) and of a given “flux” \( \tilde{Y}'(0) \),
are not sufficient to determine a unique local solution for \( y > 0 \) and \( y < 0 \). To get a unique solution, the value of \( Y''(0) \) should be prescribed. A further analysis shows that a proper stronger continuity condition of matching at \( y = 0 \) is necessary, and this includes the equality of the second-order derivatives:

\[
Y''(0^-) = Y''(0^+). \tag{141}
\]

Overall, this yields the following quadratic equation for \( y_2^2 \):

\[
Y''(0) = -2 = Y''(0) = 2c_1 - y_2, \quad c_1 = \frac{1}{2y_2} \quad \implies \quad y_2^2 - 2y_2 - 1 = 0, \tag{142}
\]

and this uniquely defines \( y_2 \) shown in (136).

Next, we use (140) with the obtained values of \( y_2 \) and \( c_1 \) to match (now, in a standard way) with the solution representation on \((y_2^i, y_3^i)\),

\[
\tilde{Y}_+(y) = y(y - y_2)(y - y_3)^{-\frac{1}{2}}, \tag{143}
\]

to get at \( y = y_2 \)

\[
y_2 \left( \frac{1}{2} y_2 + c_1 \right) = y_2(y_2 - y_3)\left(-\frac{1}{2}\right) \quad \implies \quad y_3 = 2y_2 + 2c_1 = 1 + 3\sqrt{2}. \tag{144}
\]

Finally, in the general case, on \((y_{i-1}, y_i)\) for \( i \geq 3 \),

\[
\tilde{Y}_+(y) = y(y - y_{i-1})(y - y_i)^{-\frac{1}{2}}, \tag{145}
\]

and on \((y_{i-1}, y_{i+1})\),

\[
\tilde{Y}_-(y) = y(y - y_{i-1})(y - y_{i+1})^{-\frac{1}{2}}. \tag{146}
\]

By the standard matching at \( y = y_i \) via (95) of such \( Y_\pm(y) \) yields

\[
y_i(y_i - y_{i-1})\left(-\frac{1}{2}\right) = y_i(y_i - y_{i+1})\frac{1}{2}, \tag{147}
\]

and the linear difference relation (137) follows, with some constants \( C_1 \) and \( C_2 \). These are uniquely obtained from the linear algebraic system,

\[
\begin{cases}
y_2 = 1 + \sqrt{2} = C_1 + 2C_2, \\
y_3 = 1 + 3\sqrt{2} = C_1 + 3C_2
\end{cases} \quad \implies \quad \begin{cases}
C_1 = 1 - 3\sqrt{2}, \\
C_2 = 2\sqrt{2},
\end{cases} \tag{148}
\]

completing the proof.
Figure 12. The third nonlinear eigenfunction $\tilde{Y}_2(y)$ as a solution of the problem (134), (110); $k = 1$, $l = 2$.

Figure 12 shows the general structure of the third eigenfunction $\tilde{Y}_2(y)$ on the large interval $[y_0 = -1, 200]$. Directly connected with (137), the envelope of the decaying oscillations is linear

$$L_2(y) \approx \pm y \quad \text{as} \quad y \to +\infty. \quad (149)$$

The first four zeros of $\tilde{Y}_2(y)$, on the interval $[-1, 7]$, are shown in Figure 13. For arbitrary $k \geq 2$, the limit ODE for $\tilde{Y}_2(y)$ is

$$(-1)^{k+1} D_y^{2k} \tilde{Y} y^2 + 2(-1)^k D_y^{2k-1} \tilde{Y} y + 2(-1)^{k+1} D_y^{2k-2} \tilde{Y} + \frac{\tilde{Y}}{|	ilde{Y}|} y^3 = 0,$$

which does not admit a simple geometric–algebraic method of solution.

Finally, we recall that profiles such as $\tilde{Y}_0(y)$ in Figure 8, $\tilde{Y}_1(y)$ in Figure 10, and $\tilde{Y}_2(y)$ in Figure 12 are not just some functions defining some self-similar solutions in the super-nonlinear limit ($n = +\infty$) Equation (18), for $k = 1$, but these are the most stable asymptotic pattern of such a nonlinear PDE. Indeed, $\tilde{Y}_0(y)$ is the most stable and is expected to attract, as $t \to +\infty$, a.a. solutions, excluding only those with data satisfying some extra orthogonality conditions. For instance, those having zero mass (and also the zero moment for $\tilde{Y}_2(y)$ to play a role), so that an extra time-scaling is necessary to get convergence...
The third nonlinear eigenfunction $Y_2(y)$, $n=\infty$: first zeros

Figure 13. The third nonlinear eigenfunction $\tilde{Y}_2(y)$ as a solution of the problem (134), (110); $k = 1$, $l = 2$.

to the second nonlinear eigenfunction $\tilde{Y}_1(y)$, etc. Of course, these stability questions are far beyond the scope of this paper, are very difficult, and remain open for most higher-order NDEs.

**Appendix: Nonlinear Dispersion Equation with Absorption**

*A full quasilinear NDE: homotopy deformation to linear PDEs*

The final natural progression, from the nonlinear model (13), is to go to the odd-order NDE with absorption (2), $\lambda = 1$. This links up both the nonlinear model and the semilinear one [1],

$$u_t = (-1)^{k+1}D_x^{2k+1}u - |u|^{p-1}u \quad \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \quad (n = 0). \quad (A.1)$$

However, the NDE (2) with $\lambda = 1$ is indeed a more difficult quasi-linear equation than (A.1), with phenomena of nonlinear bifurcation and branching that are not that well understood. Therefore, we should pay less effort here toward rigorously establishing some of the key analytical aspects concerning basic similarity solutions of (2).
The NDE (2) \((\lambda = 1)\) for any \(n > 0\), \(p > n + 1\), after the similarity scaling,

\[
u_{gl}(x, t) = t^{-\frac{1}{p-1}} f(y), \quad y = \frac{x}{t^\beta}, \quad \text{where} \quad \beta = \frac{p - (n + 1)}{(p - 1)(2k + 1)}, \tag{A.2}
\]

reduces to the ODE

\[
(-1)^{k+1} D_x^{2k+1}(|f|^n f) + \frac{p - (n + 1)}{(p - 1)(2k + 1)} f' y + \frac{1}{p - 1} f - |f|^{p-1} f = 0
\]

in \(\mathbb{R}\). \tag{A.3}

As usual, a proper setting for (A.3) assumes a finite left-hand interface at some \(y = y_0 < 0\), corresponding to a finite propagation due to the quasi-linear degeneracy of this NDE, and an admissible oscillatory behavior as \(y \to +\infty\), which was under scrutiny earlier.

Recall that, using the reflections (26), simultaneously, we construct blow-up solutions

\[
u_{bl}(x, t) = (T - t)^{-\frac{1}{p-1}} f(y), \quad y = -\frac{x}{(T - t)^\beta}, \quad \text{where} \quad \beta = \frac{p - (n + 1)}{(p - 1)(2k + 1)} \tag{A.4}
\]

(because \(\beta > 0\) for \(p > n + 1\), formally, this is a single point blow-up) of the NDE with source,

\[
u_t = (-1)^{k+1} D_x^{2k+1}(|u|^n u) + |u|^{p-1} u, \tag{A.5}
\]

so that profiles \(f(y)\) describe both global and blow-up asymptotics of such NDEs.

In the case \(n = 0\) in (A.3), we have a simpler semilinear ODE corresponding to the model (A.1). It is important also to note that, in addition to the homotopy path \(n \to 0^+\) in (2), it is useful to apply an extra limit \(p \to 1^+\). For \(n = 0\) and \(p = 1\) in (2), we have a linear equation

\[
u_t = (-1)^{k+1} D_x^{2k+1} u - u.
\]

Using a substitution reduces it to the standard LDE:

\[
u(x, t) = e^{-t} v(x, t) \implies v_t = (-1)^{k+1} D_x^{2k+1} v. \tag{A.6}
\]

The extra scaling yields a semigroup with the infinitesimal generator \(B\) in (14):

\[
u(x, t) = t^{-\frac{1}{2k+1}} w(y, t), \quad y = x/t^{\frac{1}{2k+1}}, \quad \tau = \ln t \implies w_\tau = Bw. \tag{A.7}
\]

Therefore, linear Hermitian theory developed in [1, § 4] (see also [22, § 9]) can be used, and this leads to an efficient approach based on comparison of linear eigenfunction structures of (A.6), and the nonlinear ones of (2). Actually, this implies existence of a certain “homotopy” of those PDEs as \(n \to 0^+\) (already noted in Section 3.6) and as \(p \to 1^+\). Overall, this establishes a countable
nature of the nonlinear eigenfunction family for (2), which is difficult to prove rigorously. We refer to [23], where such a homotopy is discussed for a related fourth-order PDE.

Thus, the continuous double limit, \( n \to 0^+ \) and \( p \to 1^+ \), by reducing to the LDE (A.6) with a countable set of rescaled eigenfunctions (31), in general, justifies that the ODE (A.3) admits a countable set of so-called \( p \)-bifurcation branches of solutions, which blow-up as \( p \to (n + 1)^+ \). Further study of such nonlinear phenomena is of importance. For \( n = 0 \), this branching phenomenon has been studied in [1].

It is worth mentioning that, in general, such a full branching approach assumes existence of a countable number of two-dimensional bifurcation surfaces defined in the two-dimensional parameter space of \((n, p)\), and the critical point \((0, 1)\) is expected to be a singular “cusp” induced by such a collection of bifurcation surfaces. This is a very difficult bifurcation-branching phenomenon, which in two-dimensional was not fully understood and remains an open problem. Therefore, later, we study another formal, but one-dimensional, “nonlinear bifurcation” phenomenon to explain existence of a countable number of \( p \)-bifurcation branches in the problem (A.3).

**Local “nonlinear” bifurcations at critical exponents \( p_l(n) \)**

We consider the VSS solutions (A.2) of (2), and develop a formal nonlinear version of such a one-dimensional \( p \)-bifurcation (branching) analysis for any fixed \( n > 0 \). As usual, according to classic branching theory [14, 15], a justification (if any) is performed for the equivalent semilinear integral equation with compact operators in suitable metrics. For simplicity, we present computations in the differential setting, which does not change anything essentially. Note that, for nonlinear odd-order operators, some issues of compactness can be rather tricky.

Let us compare the time-factor structure of the VSS (A.2) and that for the pure NDE in (20). It follows that the critical bifurcation exponents \( \{ p_l = p_l(n) \} \) are then determined from the equality of the exponents:

\[
- \frac{1}{p_l - 1} = -\alpha_l \quad \Rightarrow \quad p_l(n) = 1 + \frac{1}{\alpha_l(n)} \quad l = 0, 1, 2, \ldots, \quad (A.8)
\]

where \( \alpha_l(n) \) are the critical exponents as in (48) obtained explicitly and further fully nonlinear ones that cannot be determined dimensionally (via conservation laws).

In particular, for the semilinear case \( n = 0 \), the eigenvalues \( \alpha_l(0) \) are given by (30), and this leads to the critical bifurcation exponents

\[
p_l(0) = 1 + \frac{2k + 1}{l + 1}, \quad l = 0, 1, 2, \ldots, \quad (A.9)
\]
for the semilinear equation (A.1) studied in [1, §7.4]. In the present nonlinear case with a fixed $n > 0$, such a standard linearized approach is not suitable.

However, the first steps of this “nonlinear” bifurcation theory are straightforward. We next use an expansion relative to the small parameter $\varepsilon = p_l - p$, so that, as $\varepsilon \to 0$,

$$\alpha = \frac{1}{p - 1} = \frac{1}{p_l - 1 - \varepsilon} = \alpha_l + \alpha_l^2 \varepsilon + O(\varepsilon^2),$$

$$\beta = \frac{p - (n + 1)}{(p - 1)(2k + 1)} = \frac{1 - \alpha_l n}{2k + 1}$$

$$- \frac{n\alpha_l^2}{2k + 1} \varepsilon + O(\varepsilon^2), \quad |f|^{p_l - 1} f = |f|^{p_l - 1 - \varepsilon}$$

$$f = |f|^{p_l - 1} f (1 - \varepsilon \ln |f| + O(\varepsilon^2)).$$

Note that, unlike the case (81), the last expansion has a clearer functional validity, because at $f = 0$, there occurs standard issues of convergence, which makes sense suitable for passing to the limit in the integral operators.

Substituting these expansions into (A.3) and collecting $O(1)$ and $O(\varepsilon)$-terms yields

$$A(f, \alpha_l) - |f|^{p_l - 1} f + \varepsilon \mathcal{L} f + \varepsilon |f|^{p_l - 1} f \ln |f| + O(\varepsilon^2) = 0,$$

$$\mathcal{L} = - \frac{n\alpha_l^2}{2k + 1} yD_y + \alpha_l^2 I,$$

(A.10)

where $A(f, \alpha_l)$ is the nonlinear operator in (23). Recall that, at each nonlinear eigenvalue $\alpha = \alpha_l$, there exists the corresponding nonlinear eigenfunction $f_l$ such that (25) holds. At least, we are going to use this conclusion, which was not completely proved. The fact is that the operator $A(f, \alpha)$, with $\alpha = \alpha_l$ in (A.10) of the rescaled pure NDE, correctly describes the essence of a “nonlinear bifurcation phenomenon” to be revealed.

To this end, we use the additional invariant scaling of the operator $A(f, \alpha)$ by introducing the new unknown function $F(\cdot)$ as follows:

$$f(y) = bF(y/b^{\frac{1}{2k+1}}),$$

(A.11)

where $b = b(\varepsilon) > 0$ is also an unknown parameter to be determined from a scalar branching equation and satisfying $b(\varepsilon) \to 0$ as $\varepsilon \to 0$. Substituting (A.11) into (A.10), and under natural (but not proved) regularity assumptions on such expansions, omitting all higher-order terms (including the one with the logarithmic multiplier $\ln |b(\varepsilon)|$), yields

$$A(F, \alpha_l) - b^{p_l - 1} |F|^{p_l - 1} F + \varepsilon \mathcal{L} F = 0.$$

(A.12)
Finally, we perform linearization about the nonlinear eigenfunction $f_i(y)$ by setting $F = f_i + Y$. This gives the following linear nonhomogeneous problem:

$$A'(f_i, \alpha_i)Y = b^{p_i-1}|f_i|^{p_i-1}f_i - \varepsilon L f_i. \quad (A.13)$$

Here, the derivative is given by

$$A'(f_i, \alpha_i)Y = (n + 1)(-1)^{k+1} D_{y}^{k+1}(|f_i|^n Y) + \frac{1 - \alpha_i n}{2k + 1} Y' y + \alpha_i Y. \quad (A.14)$$

As usual in bifurcation theory, the rest of the analysis crucially depends on assumed good spectral properties of the linearized operator $A'(f_i, \alpha_i)$, which are not easy at all, and in fact are much more complicated than those for the pair $\{B, B^*\}$ in [1]. Many aspects of such a theory remain quite obscure. However, we proceed to explain the key final results on possible bifurcations, and follow the same lines. A proper functional setting of this operator is more understandable in the present one-dimensional case, where, using the behavior of $f_i(y) \to 0$ as $y \to y_0^+$ and $y \to +\infty$, it is, at least formally, possible to check whether the resolvent is compact in a suitable weighted $L^2$-space. In general, this is a difficult open problem, especially because we are not aware of precise asymptotic properties of all the eigenfunctions $f_i$.

We assume that such a proper functional setting is available for $A'(f_i, \alpha_i)$. Therefore, we deal with operators having solutions with “minimal” singularities at the boundary point of the support at $y = y_0 < 0$, where the operator is degenerate and singular. The same is assumed at $y = +\infty$, where the necessary admitted bundle of solutions should be identified to pose singular boundary conditions; see Naimark’s monographs on ordinary differential operators [24, 25] as a guide.

Namely (cf. [1, § 4]), we assume that the linear odd-order operator $A'(f_i, \alpha_i)$ has a discrete spectrum, and a complete and closed set of eigenfunctions denoted again by $\{\psi_\gamma\}$. We also assume that the kernel is finite-dimensional and we are able to determine the spectrum, eigenfunctions $\{\psi_\gamma^*\}$, and the kernel of the “adjoint” operator $(A'(f_i, \alpha_i))^*$ defined in a natural way using the topology of the dual space $L^2$ (or, equivalently and possibly, of a space with an indefinite metric) and having the same point spectrum. The latter is true for compact operators in suitable spaces in a more standard setting, [26, Ch. 4]. We also require that the bi-orthonormal eigenfunction subset $\{\psi_\gamma\}$ of the operator $A'(f_i, \alpha_i)$ is complete and closed in a suitable weighted $L^2$-space (for $n = 0$, such results are available [1]). Note that, often, this “spectral collection” is too exhaustive in nonlinear operator theory; see Deimling [17, p. 412] for general bifurcation results.

Thus, by a typical Fredholm-like alternative, the unique solvability of (A.13) requires the orthogonality of the inhomogeneous term therein to the $\ker A'(f_i, \alpha_i)$. For simplicity, let it be one-dimensional with the eigenfunction
\( \phi_l \), so the right-hand side in (A.13) satisfies
\[
 b^{p_l - 1} |f_i|^{p_l - 1} f_i - \varepsilon \mathcal{L} f_i \perp \ker A'(f_i, \alpha_l) = \text{Span} \{ \phi_l \}. \quad (A.15)
\]
Then, multiplying (A.15) by \( \phi_l^* \) in \( L^2 \) (or within the equivalent indefinite metric) yields the orthogonality condition (Lyapunov–Schmidt’s algebraic branching equation [15, § 27]):
\[
 b^{p_l - 1} \langle |f_i|^{p_l - 1} f_i, \phi_l^* \rangle = \varepsilon \langle \mathcal{L} f_i, \phi_l^* \rangle. \quad (A.16)
\]
Similar to (80), one needs to check whether the constants are nonzero:
\[
\langle |f_i|^{p_l - 1} f_i, \phi_l^* \rangle \neq 0 \quad \text{and} \quad \langle \mathcal{L} f_i, \phi_l^* \rangle \neq 0, \quad (A.17)
\]
which is not an easy problem and can lead to some restrictions for such a behavior, though is crucial for any hope to see a bifurcation point.

Under the conditions (A.17), the parameter \( b(\varepsilon) \) in (A.11), for \( p \approx p_l \), is given by
\[
 b(\varepsilon) \sim \left[ \gamma_l (p_l - p) \right]^{\alpha_l(n)} \left( \frac{1}{p_l - 1} = \alpha_l \right), \quad \varepsilon = p_l - p, \quad (A.18)
\]
The direction of developing in \( p \) of each \( p_l \)-branch, and whether the bifurcation is sub- or supercritical, depend on the sign on the coefficient \( \gamma_l \). This can be checked numerically only, but in general we expect that \( \gamma_l > 0 \), so that these nonlinear bifurcations are subcritical and the \( p_l \)-branches exist for \( p < p_l \).

Overall, the above formal analysis detects a number of key assumptions, which are necessary for such a nonlinear bifurcation to occur at the critical exponents \( p_l \) given by (A.8). Recall again that, for \( n = 0 \), a more rigorous justification of the corresponding linearized bifurcation analysis is done in [1], where a countable number of \( p \)-branches were shown to originate at bifurcation points (A.9).

Overall, we claim that, for any \( n \geq 0 \), the ODE (A.3), with proper setting as \( y \to \pm \infty \),
\[
\text{admits not more than countable number of } p \text{-branches of solutions,} \quad (A.19)
\]
which are originated at the critical exponents (A.8) (no rigorous proof is available still).

Numerical experiments for \( k = 1 \)

We briefly attempt to find numerical solutions of the Equation (A.3). As usual, we look at the lower-order case, \( k = 1 \). Once again, to remove the nonlinearity
in the highest (third)-order operator, the substitution \( Y = |f|^n f \) is used. This yields the semilinear third-order equation

\[
f = |Y|^{-\frac{n}{n+1}} Y : Y''' + \frac{p - (n + 1)}{3(p - 1)(n + 1)} |Y|^{-\frac{n}{n+1}} Y'\ln |Y| - \frac{1}{p - 1} |Y|^{-\frac{n}{n+1}} Y - |Y|^{\frac{p - (n + 1)}{n+1}} Y = 0.
\] (A.20)

Due to the complexity of the equation, which remains to be of the third order, there is not much hope of solving this problem using a shooting method, because, in fact, we do not know in detail the “nonlinear bundle” as \( y \to +\infty \). As in the semilinear case \( n = 0 \), there is a difficulty in finding proper boundary points for \( y > 0 \), such that correct oscillation structures are detected.

Hence, we return to the boundary value problem (BVP) setting, trying to “optimize” and “minimize” the oscillatory bundle for \( y \gg 1 \). However, even using this approach, which was rather effectively implemented in the simpler semilinear case \( n = 0 \) in [1, § 6], it is difficult to produce reliable numerics, due to the highly nonlinear/oscillatory nature of the problem, even for smaller values of \( p \) and \( n \). In addition, we must be careful when plotting, as we must avoid approaching any nonlinear bifurcation points in \( p \) given in (A.8), which are not actually known explicitly.

As a first example, in Figure A1, we present an “almost converging” VSS profile for \( p = 5 \) and small \( n = 0.1 \). This example is of particular importance:
its tail for $y \sim 35$ is clearly not symmetric about $\{Y = 0\}$ and is positive. According to [1, § 6], such a solution is not a VSS profile in $\mathbb{R}$, and actually corresponds to some (rather obscure) BVP setting on a bounded interval, which is of no interest here. In general, one should avoid such solutions in the future, even if these have been obtained with a perfect convergence up to the tolerances $\sim 10^{-3}$. However, “almost converging” here means that, while the oscillatory tail is structurally not perfect, the rest of the profile does exhibit proper convergence.

The next, Figure A2, with the same $p = 5$ and a larger $n = 0.6$, shows a better converging VSS profile, with a good tail. We see that, for such larger $n$, the tail for $y \gg 1$ gets smaller and stays symmetric.

For larger $n \geq 1$, the numerics get less reliable, though we have found a number of similar “almost converging” results. For instance, in Figure A3, this is done for $p = 10$ and $n = 1$. The tail is now larger for $y \sim 12$ and remains symmetric.

Further increasing $n$ requires also increasing $p$. In the next Figures, A4 and A5, we show VSS profiles for $p = 12, n = 2$ and $p = 16, n = 3$, respectively. In the former one, Figure A4, we present two different profiles (solid and dotted lines), showing that nonconverging tails do not affect the convergence in the dominant positive part of the profiles.

Figure A2. A VSS similarity solution $Y(y)$ of the ODE (A.20) for $p = 5$ and $n = 0.6$. 

$VSS: p=5, n=0.6; \text{"almost" converging tail (100000 points, max. res.=0.037)}$
Figure A3. A VSS similarity solution $Y(y)$ of the ODE (A.20) for $p = 10$ and $n = 1$.

Figure A4. A VSS similarity solution $Y(y)$ of the ODE (A.20) for $p = 12$ and $n = 2$. 
Finally, we must admit that, because the correct “oscillatory bundle behavior” as \( y \to +\infty \) is still poorly understood (except large \( n \gg 1 \); see Section 4), we cannot control it by choosing proper “minimal nonlinear components” (a good symmetry is not enough; see [1, §6.3]). Therefore, we must confess that, even having good enough numerical convergence, it cannot be guaranteed that the above numerical examples get into the countable family indicated in (A.19). In other words, without a full use of correct “minimal oscillatory bundle” as \( y \to +\infty \) (meaning the nonposing of any condition at the singular end-point \( y = +\infty \)), the family of VSS profiles becomes continuous. Then, for any fixed value of \( p > n + 1 \), “solutions” \( f(y) \) represent a continuous line, instead of a predicted at most countable subset of points, lying on the above \( p \)-branches.

On the other hand, it is plausible that those figures correctly describe a general geometry of VSS profiles. In particular, we have always observed a strong “stability” of the first maximal positive hump, which turned out to be rather independent of the resulting tail for \( y \gg 1 \). In other words, we do note that, while the oscillatory part is difficult to obtain (and properly justify mathematically), the nonoscillatory structure is rather stable and does not change much, when changing the length in which the problem is evaluated on. In particular, the numerical profile in the last Figure A5 was obtained with the worst maximal residual = 10.23 (actually meaning no convergence at all). However, we guarantee that this nonconvergence takes place in the tail only,
while the dominant positive part remains stable (hence “almost convergent”),
so we do not hesitate to present such a figure here. However, in almost all
other figures, the convergence is much better and is no worse than 2–5%.
It may be also noted that this confirms that such oscillatory (at least, symmetric)
tails, even when these are large in amplitude, are essentially zero in a natural
“weak” sense associated with Riemann’s Lemma in Fourier Integral Theory.
Naturally, this means that, for any test function \( g \in C_0(\mathbb{R}) \), the following holds:

\[
\int_{\mathbb{R}} f(y)g(L + y) \, dy \to 0 \quad \text{as} \quad L \to +\infty.
\]

**A nonlinear limit \( n \to +\infty: an example**

Finally, let us note that, as in Section 4, one way of finding proper oscillatory
patterns \( f(y) \), would be to look at the behavior as \( n \to +\infty \) and reduce the
ODE to a simpler one. We present an example of such a limit along the
following straight line on the \( \{n, p\} \)-plane:

\[
p = n + 4 \to +\infty \quad \text{as} \quad n \to +\infty.
\]

Then the ODE (A.20) reads

\[
Y''' + \frac{1}{n + 3} (|Y|^{-\frac{n}{n+3}} Y y)' - |Y|^{-\frac{3}{n+1}} Y = 0,
\]

so that, on scaling, one gets

\[
Y = (n + 3)^{-\frac{n+1}{n}} \tilde{Y} \quad \implies \quad \tilde{Y}''' + (|\tilde{Y}|^{-\frac{n}{n+3}} \tilde{Y} y)' - (n + 3)^{-\frac{1}{2}} |\tilde{Y}|^{-\frac{1}{n+1}} \tilde{Y} = 0
\]

(A.23)

We pass to the limit \( n \to +\infty \), in the ODE of (A.23), in the class of uniformly
bounded solutions. Using that \( (n + 3)^{-\frac{1}{2}} \to 1 \), this then yields the two terms
as in (92), with an extra linear one:

\[
\tilde{Y}''' + (|\tilde{Y}|^{-\frac{n}{n+3}} \tilde{Y} y)' - \tilde{Y} = 0.
\]

(A.24)

However, unlike (92), this is a third-order ODE, which means an algebraic
treatment of the first nonlinear eigenfunction \( \tilde{Y}_0(y) \), as in Theorem 2, becomes
rather elusive. Anyway, this shows a principal possibility to study the “nonlinear”
limits \( n \to +\infty \).

On the other hand, also scaling the independent variable \( y \),

\[
Y = C \tilde{Y}, \quad y = a \tilde{y}, \quad \text{and} \quad C = (n + 3)^{-\frac{n+1}{n}} a^{\frac{3(n+1)}{n}},
\]

(A.25)
yields, instead of (A.23),

\[
\tilde{Y}''' + (|\tilde{Y}|^{-\frac{n}{n+3}} \tilde{Y} \tilde{y})' - (n + 3)^{-\frac{1}{2}} a^3 |\tilde{Y}|^{\frac{1}{n+1}} \tilde{Y} = 0.
\]

(A.26)
Therefore, passing to the limit \( n \to +\infty \) in (A.26), after integrating once, we arrive precisely at Equation (93), provided that \( a = a(n) \to 0^+ \) as \( n \to +\infty \). Therefore, we obtain the same nonlinear eigenfunctions for \( l = 0, 1, 2 \) via the above algebraic–geometric approach, but, according to (A.25), on expanding subsets in the independent \( \tilde{y} \)-variable (for \( y \) on bounded intervals).

Recall that the above limit as \( n \to +\infty \), with a possible study of branching as in Section 4.4, occurs along the straight line (A.21) (or in its “small neighborhood”) in the two-dimensional parametric \( \{n, p\} \)-plane. Along other lines, the limits can be different and lead to other patterns \( \tilde{Y}_1(y), \tilde{Y}_2(y) \), etc.

Finally, overall, as we have seen, the ODE (A.20) represents a serious theoretical challenge with respect to both analytical study (\( n \)-branching, \( p \)-bifurcation diagrams, and \( p \)-branches; the latter are known for \( n = 0 \) [1, §7], etc.), as well as even a numerical one.

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