Essential Spectrum of a Fermionic Quantum Field Model

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Abstract. An interaction system of a fermionic quantum field is considered. The state space is defined by a tensor product space of a fermion Fock space and a Hilbert space. It is assumed that the total Hamiltonian is a self-adjoint operator on the state space and bounded from below. Then it is proven that a subset of real numbers is the essential spectrum of the total Hamiltonian. It is applied to the system of a Dirac field coupled to a Klein-Gordon field. Then the HVZ theorem for the system is obtained.

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1 Introduction and an Main Theorem

In this paper, an interaction system of a fermionic field is invested. Let $\mathcal{K}$ be a Hilbert space and $\mathcal{F}_f(\mathcal{K})$ the fermion Fock space over $\mathcal{K}$. Let $\mathcal{T}$ be a Hilbert pace. The state space of the interaction system is defined by

$$\mathcal{H} = \mathcal{F}_f(\mathcal{K}) \otimes \mathcal{T}. \quad (1)$$

Let $K$ be a self-adjoint operator on $\mathcal{K}$ with $\ker K = \{0\}$ and $d\Gamma_f(K)$ the second quantization of $K$. Let $T$ be a self-adjoint operator on $\mathcal{T}$. We assume that $K$ is non-negative and $T$ is bounded from below. The free Hamiltonian is given by

$$H_0 = d\Gamma_f(K) \otimes I + I \otimes T. \quad (2)$$

It is seen that $H_0$ is self-adjoint on $\mathcal{D}(H_0) = \mathcal{D}(d\Gamma_f(K) \otimes I) \cap \mathcal{D}(I \otimes T)$ and bounded from below. The total Hamiltonian is given by

$$H = H_0 + H_I, \quad (3)$$

where $H_I$ is a symmetric operator on $\mathcal{H}$.

We are interested in the essential spectrum of $H$. Locations of essential spectrum of quantum filed Hamiltonians are investigated by methods in scattering theory in [1, 5] and by a weak commutator method in [2]. In this paper, we apply the weak commutator method in [2], which is mentioned below. Let $X$ and $Y$ be densely defined linear operators on a Hilbert space $\mathcal{X}$. 

Assume that there exist a linear operator $Z$ and a dense subspace $M$ such that $M \subset D(Z) \cap D(X) \cap D(X^*) \cap D(Y) \cap D(Y^*)$ and for all $\Phi, \Psi \in M$,

$$(X^*\Phi, Y\Psi) - (Y^*\Phi, X\Psi) = (\Phi, Z\Psi).$$

Then the restriction of $Z$ to $M$ is called a weak commutator of $X$ and $Y$ on $M$, and denoted by $[X,Y]_M^0$.

We suppose conditions below:

(A.1) $H$ is self-adjoint on $D(H) = D(H_0) \cap D(H_1)$ and bounded from below.

(A.2) For all $h \in D(K)$, $[H_1, B(h) \otimes I]^{0}_{D(H)}$ and $[H_1, B^*(h) \otimes I]^{0}_{D(H)}$ exist, where $B(h)$ and $B^*(h)$ denote the annihilation operator and the creation operator, respectively. In addition, for any sequence $\{h_n\}_{n=1}^{\infty}$ of $D(K)$ such that $\text{w- lim}_{n \to \infty} h_n = 0$ and $\|h_n\| = 1$, $n \geq 1$, it follows that for all $\Psi \in D(H)$,

1. $\text{s- lim}_{n \to \infty} [H_1, B(h_n) \otimes I]^{0}_{D(H)} \Psi = 0$,
2. $\text{s- lim}_{n \to \infty} [H_1, B^*(h_n) \otimes I]^{0}_{D(H)} \Psi = 0$.

For a self-adjoint operator $X$, the spectrum of $X$ is denoted by $\sigma(X)$, and the essential spectrum by $\sigma_{\text{ess}}(X)$ From (A.1), it follows that $E_0(H) > -\infty$, where $E_0(H) = \inf \sigma(H)$ is the ground state energy of $H$.

**Theorem 1.1** Assume (A.1) and (A.2). Then 

$$\{E_0(H) + \lambda \mid \lambda \in \sigma_{\text{ess}}(K) \setminus \{0\}\} \subset \sigma_{\text{ess}}(H),$$

where $\overline{J}$ denotes the closure of $J \subset \mathbb{R}$.

In section 3 we consider an application of Theorem 1.1 to a system in the Yukawa theory. The Yukawa theory describes systems of fermionic fields coupled to bosonic fields (e.g., [4, 6, 8]). We consider the system of a Dirac field coupled a Klein-Gordon field. The total Hamiltonian is a defined on a boson-fermion Fock space. The existence of a positive spectral gap above the ground state energy is proven in [11]. By Theorem 1.1 and results in [2, 11], it is proven that a subset of $\mathbb{R}$ is equal to the essential spectrum of the total Hamiltonian. This result can be regarded as a quantum field version of the HVZ theorem for quantum mechanics systems [3].

This paper is organized as follows. In Section 2, basic properties of Fock spaces are explained and the proof of Theorem 1.1 is given. In section 3, an application of Theorem 1.1 to the Yukawa model is considered.
2 Proof of Theorem 1.1

2.1 Fermion Fock Space and Boson Fock Space

First we introduce Fermion Fock space (e.g., [7, 12]). The fermion Fock space over a Hilbert space \( \mathcal{H} \) is defined by \( \mathcal{F}_f(\mathcal{H}) = \bigoplus_{n=0}^{\infty} (\otimes^n \mathcal{H}) \), where \( \otimes^n \mathcal{H} \) denotes the \( n \)-fold anti-symmetric tensor product of \( \mathcal{H} \) with \( \otimes^0 \mathcal{H} := \mathbb{C} \). The Fock vacuum is defined by \( \Omega_f = \{ 1, 0, 0, \cdots \} \in \mathcal{F}_f(\mathcal{H}) \). The annihilation operator is denoted by \( \mathcal{A}_f \), \( f \in \mathcal{H} \) and the creation operator by \( \mathcal{A}^*_f \), \( g \in \mathcal{H} \). For a dense subspace \( \mathcal{M} \subset \mathcal{H} \), the finite particle subspace \( \mathcal{F}^\text{fin}_f(\mathcal{M}) \) is the linear hull of vectors of the form \( \Psi = \mathcal{A}^*_f(f_1) \cdots \mathcal{A}^*_f(f_n) \Omega_f \), \( f_j \in \mathcal{M} \), \( j = 1, \cdots, n \in \mathbb{N} \). It is known that \( \mathcal{A}_f \) and \( \mathcal{A}^*_f \) is bounded with

\[
\| \mathcal{A}_f \| = \| f \|, \quad \| \mathcal{A}^*_f \| = \| g \|,
\]

respectively. They satisfy canonical anti-commutation relations:

\[
\{ \mathcal{A}_f, \mathcal{A}^*_g \} = (f, g)_{\mathcal{H}},
\]

\[
\{ \mathcal{A}_f, \mathcal{A}_g \} = \{ \mathcal{A}^*_f, \mathcal{A}^*_g \} = 0,
\]

where \( \{ X, Y \} = XY + YX \). Let \( \mathcal{H} \) be a self-adjoint operator on \( \mathcal{H} \). Suppose that \( \mathcal{H} \) is bounded from below. Then the second quantization \( d\Gamma_f(\mathcal{H}) \) is a self-adjoint on \( \mathcal{F}_f(\mathcal{H}) \), which acts for the vector \( \Psi = \mathcal{A}^*_f(f_1) \cdots \mathcal{A}^*_f(f_n) \Omega_f \) as \( d\Gamma_f(\mathcal{H}) \Psi = \sum_{j=1}^{n} \mathcal{A}^*_f(f_1) \cdots \mathcal{A}^*_f(Xf_j) \cdots \mathcal{A}^*_f(f_n) \Omega_f \). Let \( f \in \mathcal{D}(\mathcal{H}) \). Then it holds that on \( \mathcal{F}^\text{fin}_f(\mathcal{D}(\mathcal{H})) \),

\[
[d\Gamma_f(X f), \mathcal{A}_f] = -d\Gamma_f(X f),
\]

\[
[d\Gamma_f(X f), \mathcal{A}^*_f] = d\Gamma_f(X f).
\]

Next we introduce the boson Fock space. The boson Fock space over a Hilbert space \( \mathcal{Y} \) is defined by \( \mathcal{F}_b(\mathcal{Y}) = \bigoplus_{n=0}^{\infty} (\otimes^n \mathcal{Y}) \), where \( \otimes^n \mathcal{Y} \) denotes the \( n \)-fold symmetric tensor product of \( \mathcal{Y} \) with \( \otimes^0 \mathcal{Y} := \mathbb{C} \). The Fock vacuum is defined by \( \Omega_b = \{ 1, 0, 0, \cdots \} \in \mathcal{F}_b(\mathcal{Y}) \). The annihilation operator is denoted by \( \mathcal{B}_b \), \( f \in \mathcal{Y} \), and the creation operator by \( \mathcal{B}^*_b \), \( g \in \mathcal{Y} \). For a dense subspace \( \mathcal{N} \subset \mathcal{Y} \), the finite particle subspace \( \mathcal{F}^\text{fin}_b(\mathcal{N}) \) is the linear hull of vectors of the form \( \Psi = \mathcal{B}^*_b(f_1) \cdots \mathcal{B}^*_b(f_n) \Omega_b \), \( f_j \in \mathcal{N} \), \( j = 1, \cdots, n \in \mathbb{N} \). Creation operators and annihilation of bosonic field satisfy canonical commutation relations on the finite particle subspace \( \mathcal{F}^\text{fin}_b(\mathcal{Y}) \):

\[
[A(f, A^*_g)] = (f, g)_{\mathcal{Y}},
\]

\[
[A(f), A^*(g)] = [A^*(f), A^*(g)] = 0,
\]

where \( [X, Y] = XY - YX \). Let \( \mathcal{Y} \) be a self-adjoint operator on \( \mathcal{Y} \). We assume that \( \mathcal{Y} \) is non-negative. Then the second quantization \( d\Gamma_b(\mathcal{Y}) \) is self-adjoint on \( \mathcal{F}_b(\mathcal{Y}) \) which acts for the finite particle vector \( \Psi = \mathcal{B}^*_b(f_1) \cdots \mathcal{B}^*_b(f_n) \Omega_b \) as \( d\Gamma_b(\mathcal{Y}) \Psi = \sum_{j=1}^{n} \mathcal{A}^*_b(f_1) \cdots \mathcal{A}^*_b(Yf_j) \cdots \mathcal{A}^*_b(f_n) \Omega_b \). Let
\(f \in \mathcal{D}(Y)\). Then it follows that on \(\mathcal{F}_b^{\text{fin}}(\mathcal{D}(Y))\),
\[
[d\Gamma_b(Y), A(f)] = -d\Gamma_b(Y f),
\]
\(d\Gamma_b(Y), A^*(f) = d\Gamma_b(Y f).\)  \(\tag{11}\)
\(\tag{12}\)

Let \(f \in \mathcal{D}(Y^{-1/2})\). Then it follows that for all \(\Psi \in \mathcal{D}(d\Gamma_b(Y)^{1/2})\),
\[
\|A(f)\Psi\| \leq \|Y^{-1/2}f\| \|d\Gamma_b(X)^{1/2}\Psi\|, \tag{13}\]
\[
\|A^*(f)\Psi\| \leq \|Y^{-1/2}f\| \|d\Gamma_b(Y)^{1/2}\| + \|f\| \|\Psi\|. \tag{14}\]

### 2.2 Proof of Theorem 1.1

#### Lemma 2.1

Let \(\{h_n\}_{n=1}^{\infty}\) be a sequence of \(\mathcal{K}\) such that \(w\)-lim \(h_n = 0\). Then for all \(\Psi \in \mathcal{F}_l(\mathcal{K})\),

1. \(s\)-lim \(n \to \infty (h_n)\Psi = 0\),
2. \(w\)-lim \(n \to \infty B^*(h_n)\Psi = 0\).

**Proof**

(1) Let \(\Psi = B^*(f_1) \cdots B^*(f_l)\Omega_f \in \mathcal{F}_l^{\text{fin}}(\mathcal{K})\) be a finite particle vector. From canonical anti-commutation relations \(\text{(5)}\) and \(\text{(6)}\), it is seen that
\[
B(h_n)\Psi = \sum_{j=1}^{l} (-1)^j (\begin{array}{c} h_n \end{array}) B^*(f_1) \cdots \widehat{B^*(f_j)} \cdots B^*(f_l)\Omega_f,
\]
where \(\widehat{X}\) stands for omitting the operator \(X\). Since \(w\)-lim \(h_n = 0\), it follows that \(\lim_{n \to \infty} \|B(h_n)\Psi\| = 0\). Then we see that for all finite vector \(\Psi \in \mathcal{F}_l^{\text{fin}}(\mathcal{K})\), \(s\)-lim \(n \to \infty B(h_n)\Psi = 0\). Note that \(\mathcal{F}_l^{\text{fin}}(\mathcal{K})\) is dense in \(\mathcal{F}_l(\mathcal{K})\) and \(\|B(h_n)\|\) is uniformly bounded with \(\|B(h_n)\| = \|h_n\| = 1\), for all \(n \in \mathbb{N}\). Then we see that \(s\)-lim \(n \to \infty B(h_n)\Psi = 0\) for all \(\Psi \in \mathcal{F}_l(\mathcal{K})\).

(2) Let \(\Psi \in \mathcal{F}_l(\mathcal{K})\). From canonical anti-commutation relations \(\text{(5)}\) and \(\text{(6)}\), we see that for \(\Phi = B^*(g_1) \cdots B^*(g_l)\Omega_f \in \mathcal{F}_l^{\text{fin}}(\mathcal{K})\),
\[
(\Phi, B^*(h_n)\Psi) = (B(h_n)\Phi, \Psi) = \sum_{j=1}^{l} (-1)^j (g_j, h_n) (B^*(g_1) \cdots \widehat{B^*(g_j)} \cdots B^*(g_l)\Omega_f, \Psi).
\]

From this equality and \(w\)-lim \(h_n = 0\), we have \(\lim_{n \to \infty} (\Phi, B^*(h_n)\Psi) = 0\). Note that \(\mathcal{F}_l^{\text{fin}}(\mathcal{K})\) is dense in \(\mathcal{F}_l(\mathcal{K})\) and \(\|B^*(h_n)\| = \|h_n\| = 1\) for all \(n \in \mathbb{N}\). Hence we see that for all \(\Phi \in \mathcal{F}_l(\mathcal{K})\), \(\lim_{n \to \infty} (\Phi, B^*(h_n)\Psi) = 0\). \(\blacksquare\)

#### Lemma 2.2

It follows that for all \(f \in \mathcal{D}(K)\),

1. \(\begin{cases} [H_0, B(f) \otimes I_{\mathcal{D}(H_0)}^0] = -B(Kf) \otimes I_{\mathcal{D}(H_0)}, \end{cases}\)
2. \(\begin{cases} [H_0, B^*(f) \otimes I_{\mathcal{D}(H_0)}^0] = B^*(Kf) \otimes I_{\mathcal{D}(H_0)}. \end{cases}\)
Hence we see that for all \( \lambda \in \sigma_{\text{ess}}(K) \setminus \{0\} \). From canonical anti-commutation relations (5) and (6),

\[
(\Phi, (B(f) \otimes I)\Phi, H_0\Phi) = -(\Phi, (B(Kf) \otimes I)\Phi).
\]

Note that \( \|h\| \leq 1 \) and \( \|\Xi\| = 1 \). From the commutation relation (7),

\[
(\Phi, (B(f) \otimes I)\Phi, H_0\Phi) - ((B^*(f) \otimes I)\Phi, H_0\Phi) = -(\Phi, (B(Kf) \otimes I)\Phi). \text{ Thus (i) holds. Similarly, we can prove (ii).}\]

**Proof of Theorem 1.1**

Let \( \lambda \in \sigma_{\text{ess}}(K) \setminus \{0\} \). Then there exists a Weyl sequence \( \{h_n\}_{n=1}^{\infty} \) for \( K \) and \( \lambda \), i.e., \( h_n \in \mathcal{D}(K) \) and \( \|h_n\| = 1 \) for all \( n \in \mathbb{N} \), \( \text{w-} \lim_{n \to \infty} h_n = 0 \), \( \text{s-} \lim_{n \to \infty} (K - \lambda)h_n = 0 \). By this sequence, we construct a Weyl sequence for \( H \) and \( E_0(H) + \lambda \). Let us set

\[
\Psi_{n,\varepsilon} = ((B(h_n) + B^*(h_n)) \otimes I)\Xi_{\varepsilon},
\]

where \( \Xi_{\varepsilon} \in \text{Ran}(E_{H}([0,\varepsilon))) \), \( \|\Xi_{\varepsilon}\| = 1 \), \( 0 < \varepsilon \leq 1 \). Here \( E_{H} \) denotes the spectral projection of \( H \). From canonical anti-commutation relations (5) and (6),

\[
\|\Psi_{n,\varepsilon}\|^2 = (\Xi_{\varepsilon}, (B(h_n)^2 + \{B(h_n), B^*(h_n)\} + B^*(h_n)^2) \otimes I)\Xi_{\varepsilon}) = \|h_n\|^2 \|\Xi_{\varepsilon}\|^2.
\]

Since \( \|h_n\| = 1 \) and \( \|\Xi_{\varepsilon}\| = 1 \), we see that \( \|\Psi_{n,\varepsilon}\| = 1 \) for all \( n \geq 1 \) and \( 0 < \varepsilon \leq 1 \). From Lemma 2.2 and the assumption (A,2), it holds that for all \( \Phi, \Theta \in \mathcal{D}(H) \) and for all \( h \in \mathcal{D}(K) \),

\[
(H\Phi, ((B(h) + B^*(h)) \otimes I)\Theta) - ((B(h) + B^*(h)) \otimes I)\Phi, H\Theta)
= \left(\Phi, \left((B^*(Kh) - B(Kh)) \otimes I + [H_1, B^*(h) \otimes I]_{\mathcal{D}(H)}^0 + [H_1, B(h) \otimes I]_{\mathcal{D}(H)}^0\right)\Theta\right).
\]

Hence we see that for all \( \Phi \in \mathcal{D}(H) \),

\[
(H\Phi, \Psi_{n,\varepsilon}) = (\Phi, ((B(h_n) + B^*(h_n)) \otimes I)H\Xi_{\varepsilon}) + (\Phi, ((B^*(Kh_n) - B(Kh_n)) \otimes I)\Xi_{\varepsilon})
+ (\Phi, [H_1, (B^*(h_n) \otimes I)]_{\mathcal{D}(H)}^0\Xi_{\varepsilon}) + (\Phi, [H_1, (B(h_n) \otimes I)]_{\mathcal{D}(H)}^0\Xi_{\varepsilon})).
\]

Then \( \Psi_{n,\varepsilon} \in \mathcal{D}(H) \), and hence, \( \Psi \in \mathcal{D}(H) \), since \( H \) is self-adjoint. Then we have

\[
H\Psi_{n,\varepsilon} = ((B(h_n) + B^*(h_n)) \otimes I)H\Xi_{\varepsilon} + (B^*(Kh_n) - B(Kh_n)) \otimes I)\Xi_{\varepsilon}
+ [H_1, B^*(h_n) \otimes I]_{\mathcal{D}(H)}^0\Xi_{\varepsilon} + [H_1, B(h_n) \otimes I]_{\mathcal{D}(H)}^0\Xi_{\varepsilon}.
\]

From (18),

\[
\| (H - (\lambda + E_0(H)))\Psi_{n,\varepsilon}\|
\leq \|(B(h_n) + B^*(h_n)) \otimes I)(H - E_0(H))\Xi_{\varepsilon}\| + \|(B^*(Kh_n - \lambda h_n) \otimes I)\Xi_{\varepsilon}\|
+ \|B(Kh_n + \lambda h_n)\Xi_{\varepsilon}\| + \|[H, B(h_n) \otimes I]_{\mathcal{D}(H)}^0\Xi_{\varepsilon}\| + \|[H, B^*(h_n) \otimes I]_{\mathcal{D}(H)}^0\Xi_{\varepsilon}\|. \tag{19}
\]
We see that
\[ \|(B(h_n) + B^*(h_n)) \otimes I)(H - E_0(H))\| \leq 2\|h_n\| \|(H - E_0(H))\| \leq 2\varepsilon, \]  
and
\[ \|(B^*(Kh_n - \lambda h_n) \otimes I)\| \leq \|(K - \lambda)h_n\|. \]  
Since \( B(K + \lambda)h_n = B(K - \lambda)h_n + 2\lambda B(h_n) \), we also see that
\[ \|(B((K + \lambda)h_n) \otimes I)\| \leq \|(K - \lambda)h_n\| + 2|\lambda| \|(B(h_n) \otimes I)\|. \]  
By applying (20), (21) and (22) to (19),
\[ \|(H - (\lambda + E_0(H)))\| \leq 2\varepsilon + 2\|\|(K - \lambda)h_n\| + 2|\lambda| \|(B(h_n) \otimes I)\|. \]  
Since \( s- \lim \frac{\lambda}{n} \), we see that \( w- \lim \Psi_{n,\varepsilon} = 0 \) by Lemma 2.1. Thus
\[ \lim \sup_{\varepsilon \to 0} \| (H - (\lambda + E_0(H)))\| = 0. \]  
Then, we can take a subsequence \( \{\Psi_{n_j,\varepsilon_j}\}_{j=1}^{\infty} \) satisfying \( \lim \| (H - (\lambda + E_0(H)))\| = 0 \). In addition, from the definition of \( \Psi_{n,j} \) and Lemma 2.1, we see that \( w- \lim \Psi_{n_j,\varepsilon_j} = 0 \).

Thus \( \{\Psi_{n,j,\varepsilon_j}\}_{j=1}^{\infty} \) is a Weyl sequence for \( E_0(H) + \lambda \). Then Weyl’s criterion (\[9\]) says that \( E_0(H) + \lambda \in \sigma_{\text{ess}}(H) \). Since \( \sigma_{\text{ess}}(H) \) is closed in \( \mathbb{R} \), the proof is obtained.

3 Application

We consider an application of Theorem 1.1 to the system of a Dirac field interacting with a Klein-Gordon field. The state space of Dirac field and Klein-Gordon field are given by \( \mathcal{H}_D = \mathcal{F}_1(L^2(\mathbb{R}_p^3, \mathbb{C}^4)) \) and \( \mathcal{H}_{KG} = \mathcal{F}_b(L^2(\mathbb{R}_k^3)) \), respectively. The state space is defined by
\[ \mathcal{H}_Y = \mathcal{H}_D \otimes \mathcal{H}_{KG}. \]

The free Hamiltonians of the Dirac field and the Klein-Gordon field are given by \( H_D = d\Gamma_t(\omega_M) \) with \( \omega_M(p) = \sqrt{p^2 + M^2}, \ M > 0 \), and \( H_{KG} = d\Gamma_b(\omega_m) \) with \( \omega_m(k) = \sqrt{k^2 + m^2}, \ m > 0 \), respectively. The total Hamiltonian is defined by
\[ H_\kappa = H_D \otimes I + I \otimes H_{KG} + \kappa H_1, \ \ \kappa \in \mathbb{R}, \]
where \( H_1 \) is the symmetric operator on \( \mathcal{H}_Y \) such that for all \( \Phi \in \mathcal{H}_Y \), and for all \( \Psi \in \mathcal{D}(H_0) \),
\[ (\Phi, H_1 \Psi) = \int_{\mathbb{R}^3} \chi_1(x) \left( \Phi, \overline{\varphi(x)} \varphi(x) \otimes \phi(x) \Psi \right) dx. \]

Here \( \varphi(x) \) and \( \phi(x) \) are field operators of the Dirac field and Klein-Gordon field, respectively, and \( \overline{\varphi(x)} = \varphi^*(x)\beta \). Dirac matrices \( \alpha_j, \ j = 1, 2, 3, \) and \( \beta \) are the \( 4 \times 4 \) hermitian matrices.
satisfying anti-commutation relations \( \{ \alpha^l, \alpha^l \} = 2 \delta_{j,l} \), \( \{ \alpha_j, \beta \} = 0 \) and \( \beta^2 = I \). The definitions of \( \psi(x) \) and \( \phi(x) \) are as follows. First we consider the Dirac field’s operators. Let \( B(\xi), \xi = (\xi_1, \cdots, \xi_4) \in \mathcal{H}_D \), and \( B^*(\eta), \eta = (\eta_1, \cdots, \eta_4) \in \mathcal{H}_D \), be the annihilation operator and the creation operator on \( \mathcal{H}_D \), respectively. The field operator of the Klein-Gordon field is defined by

\[
\psi(x) = \sum_{s=\pm 1/2} (b_s(f_{s,x}) + d_s^*(g_{s,x})), \quad l = 1, \ldots, 4,
\]

where \( f_{s,x}(p) = f_s(p)e^{-ip \cdot x} \) with \( f_s(p) = \frac{\chi_D(p)u_s(p)}{\sqrt{(2\pi)^3 \omega(p)}} \) and \( g_{s,x}(p) = g_s(p)e^{-ip \cdot x} \) with \( g_s(p) = \frac{\chi_D(p)v_s(-p)}{\sqrt{(2\pi)^3 \omega(p)}} \). Here \( \chi_D \) denotes an ultraviolet cutoff. Next we define the Klein-Gordon filed’s operators. Let \( a(f), f \in \mathcal{H}_{KG} \), and \( a^*(g), g \in \mathcal{H}_{KG} \), be the annihilation operator and the creation operator, respectively. Then it is seen that on \( \mathcal{F}_b^{\text{fin}}(\mathcal{H}_{KG}) \),

\[
[a(f), a^*(g)] = 0 \quad [a(f), a(g)] = [a^*(f), a^*(g)] = 0.
\]

The field operator of the Klein-Gordon field is defined by

\[
\phi(x) = \frac{1}{\sqrt{2}} \left( a(h_x) + a^*(h_x) \right),
\]

where \( h_x(k) = h(k)e^{ik \cdot x} \) with \( h(k) = \frac{\chi_{KG}(k)}{\sqrt{(2\pi)^3 \omega(k)}} \), and \( \chi_{KG} \) is an ultraviolet cutoff.

We suppose the following conditions.

(Y.1) (Ultraviolet cutoff)

\[
\int_{\mathbb{R}^3} \left| \chi_D(p)u_s(p) \right|^2 dp < \infty, \quad \int_{\mathbb{R}^3} \left| \chi_D(p)v_s(-p) \right|^2 dp < \infty, \quad \int_{\mathbb{R}^3} \left| \chi_{KG}(k) \right|^2 dk < \infty.
\]

(Y.2) (Spatial cutoff) \( \int_{\mathbb{R}^3} |\chi_1(x)| dx < \infty \).
From the boundedness (4), (13), (14) and \( \psi(x) \psi(x) = \sum_{l,l'} \beta_{l,l'} \psi^*_l(x) \psi_{l'}(x) \), we see by (Y.1) that

\[
\sup_{x \in \mathbb{R}^3} \| \psi(x) \psi(x) \| \leq \sum_{l,l'=1}^4 \sum_{s,s' = \pm 1/2} |\beta_{l,l'}| \left( \| f^l_s \| + \| g^l_s \| \right) \left( \| f^l_{s'} \| + \| g^l_{s'} \| \right) , \tag{28}
\]

\[
\sup_{x \in \mathbb{R}^3} \| \phi(x) \psi \| \leq \sqrt{2} \frac{h}{\omega_m} \| H^{1/2} \psi \| + \frac{1}{\sqrt{2}} \| h \| \| \psi \|. \tag{29}
\]

Then from (28), (29), (Y.2) and \( \| H^{1/2} \Phi \| \leq \varepsilon \| H \psi \| + \frac{1}{2\varepsilon} \| \Phi \| \), it holds that for all \( \psi \in \mathcal{D}(H_0) \),

\[
\| H_1 \psi \| \leq \varepsilon \sqrt{2} \| \chi \| \left\| \frac{h}{\omega_m} \| H_0 \psi \| + \frac{1}{\sqrt{2}} \| h \| \right\| \chi \| \left( \frac{1}{\sqrt{2} \varepsilon} \| h \| + \frac{1}{\sqrt{2}} \right) \| \psi \|
+ \sum_{l,l'=1}^4 \sum_{s,s' = \pm 1/2} |\beta_{l,l'}| \left( \| f^l_s \| + \| g^l_s \| \right) \left( \| f^l_{s'} \| + \| g^l_{s'} \| \right) \| \psi \|. \tag{30}
\]

Thus \( H_1 \) is relatively bounded with respect to \( H_0 \) with the infinitely small bound, and then the Kato-Rellich theorem shows that \( H_\kappa \) is self-adjoint and essentially self-adjoint on any core of \( H_0 \). In particular \( H_\kappa \) is essentially self-adjoint on \( \mathcal{F}^\text{fin}(\mathcal{D}(\omega_D)) \otimes_{\text{alg}} \mathcal{F}_{\text{fin}}^\text{b}(\mathcal{D}(\omega_{KG})) \), where \( \otimes_{\text{alg}} \) stands for the algebraic tensor product.

Let \( \nu = \min\{m,M\} \). From Theorem[1.1] the next assertion follows.

**Theorem 3.1**

Assume (Y.1) and (Y.2). Then \( [E_0(H_\kappa) + \nu, \infty) \subset \sigma_{\text{ess}}(H_\kappa) \) for all \( \kappa \in \mathbb{R} \).

Before proving Theorem[3.1] we explain a result of the essential spectrum of the Yukawa model and state a corollary. In [11] the following theorem has been proven:

**Theorem A (11 ; Theorem 2.1)**

Assume (Y.1), (Y.2) and \( \int_{\mathbb{R}^3} |x| |\chi_l(x)| dx < \infty \). Then it follows that for all \( \kappa \in \mathbb{R} \),

\( \sigma_{\text{ess}}(H_\kappa) \subset [E_0(H_\kappa) + \nu, \infty) \).

From Theorem A and Theorem[3.1] the next corollary follows.

**Corollary 3.2 (HVZ theorem for the Yukawa model)**

Assume (Y.1), (Y.2) and \( \int_{\mathbb{R}^3} |x| |\chi_l(x)| dx < \infty \). Then \( [E_0(H_\kappa) + \nu, \infty) = \sigma_{\text{ess}}(H_\kappa) \) for all \( \kappa \in \mathbb{R} \).

To prove Theorem[3.1] we prepare for some lemmas.
Lemma 3.3 Let $A$ and $B$ be self-adjoint operators on Hilbert spaces $X$ and $Y$, respectively. Assume that $A$ and $B$ are non-negative. Let $X(x), x \in \mathbb{R}^d$, be a linear operator on $\mathcal{F}_f(X)$, and $Y(y), y \in \mathbb{R}^d$, a linear operator on $\mathcal{F}_b(Y)$, which satisfy

$$\sup_{x \in \mathbb{R}^d} \|X(x)\Psi\| \leq \text{const.}(\|d_G(A)^{1/2}\Psi\| + \|\Psi\|), \quad \Psi \in D(d_G(A)^{1/2}), \quad (30)$$

$$\sup_{x \in \mathbb{R}^d} \|Y(x)\Psi\| \leq \text{const.}(\|d_G(b)^{1/2}\Psi\| + \|\Psi\|), \quad \Psi \in D(d_G(b)^{1/2}), \quad (31)$$

respectively. Then there exists a linear operator $Z$ on $\mathcal{F}_f(X) \otimes \mathcal{F}_b(Y)$ such that $D(d_G(A) \otimes I) \cap D(I \otimes d_G(b)) \subset D(Z)$ and for all $\Phi \in \mathcal{F}_f(X) \otimes \mathcal{F}_b(Y)$ and for all $\Psi \in D(d_G(A) \otimes I) \cap D(I \otimes d_G(b))$,

$$(\Phi, Z\Psi) = \int_{\mathbb{R}^d} g(x)(\Phi, (X(x) \otimes Y(x))\Psi)dx,$$

where $g$ is the Borel function satisfying $\|g\|_{L^1} < \infty$.

(Proof) Let $\ell$ be a linear functional on $(\mathcal{F}_f(X) \otimes \mathcal{F}_b(Y)) \times (\mathcal{F}_f(D(A)) \otimes \mathcal{F}_b(D(B)))$ defined by

$$\ell(\Phi, \Psi) = \int_{\mathbb{R}^d} g(x)(\Phi, (X(x) \otimes Y(x))\Psi)dx. \quad (32)$$

It is seen that $|\ell(\Phi, \Psi)| \leq c\|g\|_{L^1} \|\Phi\| (\|d_G(A) \otimes I + I \otimes d_G(b)\Psi\| + \|\Psi\|)$ with some constant $c > 0$. From the Riesz representation theorem, the assertion holds. \[\blacksquare\]

From canonical anti-commutation relations (25) and (26), it is proven that commutation relations below follow in (10; Lemma 3.1):

$$[\psi^*_f(x) \psi_f(x), b_s(\xi)] = -\langle \xi, f^f_s(x) \rangle \psi_f(x), \quad (33)$$

$$[\psi^*_f(x) \psi_f(x), d_s(\xi)] = \langle \xi, g^f_s(x) \rangle \psi_f(x). \quad (34)$$

From (33), (34), and $[X, Y]^* = -[X^*, Y^*]$, it is seen that

$$[\psi_r(x) \psi^*_r(x), b_s(\eta)] = (f^r_s(x), \eta) \psi^*_r(x), \quad (35)$$

$$[\psi_r(x) \psi^*_r(x), d_s(\eta)] = -(g^r_s(x), \eta) \psi^*_r(x). \quad (36)$$

Here note that

$$\sup_{x \in \mathbb{R}^d} \|\psi_f(x)\| \leq \sum_{s = \pm 1/2} (\|f^f_s\| + \|g^f_s\|). \quad (37)$$

From commutation relations (33) - (36) and bounds (29) and (37), Lemma 3.3 yields that the next lemma follows.
Lemma 3.4
(I) There exist \([H_1, b_s(\xi) \otimes I]_D^0 \) and \([H_1, d_s(\xi) \otimes I]_D^0 \) such that for all \(\Phi \in H_Y\) and for all \(\Psi \in D(H_0)\),

\[
(\Phi, [H_1, b_s(\xi) \otimes I]_D^0(\Psi)) = -\sum_{l,l'} \beta_{l,l'} \int_{\mathbb{R}^3} \chi_l(x)(\xi, f_{s,x}^l(\Phi, (\psi_l(x) \otimes \phi(x))\Psi) dx,
\]

and

\[
(\Phi, [H_1, d_s(\xi) \otimes I]_D^0(\Psi)) = \sum_{l,l'} \beta_{l,l'} \int_{\mathbb{R}^3} \chi_l(x)(\xi, g_{s,x}^l(\Phi, (\psi_l(x) \otimes \phi(x))\Psi) dx.
\]

(II) There exist \([H_1, b_s^e(\eta) \otimes I]_D^0 \) and \([H_1, d_s^e(\eta) \otimes I]_D^0 \) such that for all \(\Phi \in H_Y\) and for all \(\Psi \in D(H_0)\),

\[
(\Phi, [H_1, b_s^e(\eta) \otimes I]_D^0(\Psi)) = \sum_{r,r'} \beta_{r,r'} \int_{\mathbb{R}^3} \chi_r(x)(f_{s,x}^r, \eta)(\Phi, (\psi_r(x) \otimes \phi(x))\Psi) dx,
\]

and

\[
(\Phi, [H_1, d_s^e(\eta) \otimes I]_D^0(\Psi)) = -\sum_{r,r'} \beta_{r,r'} \int_{\mathbb{R}^3} \chi_r(x)(g_{s,x}^r, \eta)(\Phi(\psi_r(x) \otimes \phi(x))\Psi) dx.
\]

Similarly, the next proposition follows from (27) and (28) to Lemma 3.3.

Lemma 3.5 There exits \([H_1, I \otimes a^s(\xi)]_D^0 \) satisfying for all \(\Phi \in H_Y\) and for all \(\Psi \in D(H_0)\),

\[
(\Phi, [H_1, I \otimes a^s(\xi)]_D^0(\Psi)) = \int_{\mathbb{R}^3} \chi_l(x)(f_x, \xi)(\Phi, (\psi_l(x) \otimes \phi(x))\Psi) dx.
\]

(Proof of Theorem 3.1)

Let us apply to Theorem [1.1] to the Yukawa model. Since \(H_X\) is self-adjoint on \(D(H_0)\) and bounded from below, (A.1) is satisfied. Let us check (A.2). Let \(\{h_n\}_{n=1}^{\infty}\) be a sequence such that \(h_n \in D(\omega_M), \|h_n\| = 1\) for all \(n \in \mathbb{N}\), and \(w^{lim}_{n \rightarrow \infty} h_n = 0\). From (38), it is seen that for all \(\Psi \in D(H)\),

\[
\| [H_1, b_s(h_n) \otimes I]_D^0(\Psi) \| \leq \sum_{l,l'} |\beta_{l,l'}| \int_{\mathbb{R}^3} \chi_l(x)(h_n, f_{s,x}^l(\Phi, (\psi_l(x) \otimes \phi(x))\Psi) dx.
\]

By (37) and (29), we have \(\sup_{x \in \mathbb{R}^3} \| (\psi_l'(x) \otimes \phi(x))\Psi \| < \infty\). We also see that \(\int_{\mathbb{R}^3} \chi_l(x)|dx < \infty\) by (A.1) and \(\| h_n, f_{s,x}^l \| \leq \| f_s \| \). Then from \(w^{lim}_{n \rightarrow \infty} h_n = 0\) and the Lebesgue dominated convergence theorem, (42) yields that \(\lim_{n \rightarrow \infty} [H_1, b_s(h_n) \otimes I]_D^0(\Psi) = 0, s = \pm \frac{1}{2}\). Similarly we can prove that \(\lim_{n \rightarrow \infty} [H_1, d_s(h_n) \otimes I]_D^0(\Psi) = 0, s = \pm \frac{1}{2}\). Then the condition (1) in (A.2) is satisfied. In addition, we can also prove that for \(s = \pm \frac{1}{2}\), \(\lim_{n \rightarrow \infty} [H_1, b_s'(h_n) \otimes I]_D^0(\Psi) = 0\) and \(\lim_{n \rightarrow \infty} [H_1, d_s'(h_n) \otimes I]_D^0(\Psi) = 0\), and then the condition (2) in (A.2) is satisfied. Hence from
Theorem 1.1 it follows that \( E_0(H_\kappa) + M, \infty \subset \sigma_{\text{ess}}(H_\kappa) \). Next we show \( E_0(H_\kappa) + m, \infty \subset \sigma_{\text{ess}}(H_\kappa) \), and then Theorem 3.1 is proven. Here note that \( \omega_m^r \) is bounded for all \( r > 0 \), since \( \omega_m > 0 \). Let \( \{f_n\}_{n=1}^\infty \) be a sequence of \( \mathcal{H}_m \) such that \( f_n \in \mathcal{D}(\omega_m), \|f_n\| = 1 \) for all \( n \in \mathbb{N} \), and \( \text{w- lim } h_n = 0 \). From Lemma 3.5, it is seen that for all \( \Psi \in \mathcal{D}(H) \),

\[
\| [H_1, I \otimes a^*(f_n)]^0_{\mathcal{D}(H)} \Psi \| \leq \int_{\mathbb{R}^3} |\chi_1(x)||\langle f_n, f_n \rangle ||(\overline{\psi(x)} \psi(x) \otimes I)\Psi| dx. \tag{43}
\]

We see that \( \chi_1 \in L^1, |\langle h_x, f_n \rangle| \leq \|f\|, \text{w- lim } f_n = 0 \) and \( \sup_{x \in \mathbb{R}^3} \|\overline{\psi(x)} \psi(x)\| < \infty \). Then we have \( \lim_{n \to \infty} \| [H_1, I \otimes a^*(f_n)]^0_{\mathcal{D}(H)} \Psi \| = 0 \) by (43) and the Lebesgue dominated convergence theorem. Then (S.2) of Theorem I in Appendix is satisfied, and hence \( E_0(H_\kappa) + m, \infty \subset \sigma_{\text{ess}}(H_\kappa) \). \( \blacksquare \)

**Appendix ([2]; Theorem 1.2)**

Let

\[ \mathcal{H} = \mathcal{R} \otimes \mathcal{F}_b(S) \]

where \( \mathcal{R} \) is a Hilbert space and \( \mathcal{F}_b(S) \) the boson Fock space over a Hilbert space \( S \). Let \( R \) be a self-adjoint operator on \( \mathcal{R} \) and \( S \) a self-adjoint operator on \( S \) with \( \ker S = \{0\} \). Additionally, we assume that \( R \) is bounded from below and \( S \) is non-negative. Let

\[ H = R \otimes I + I \otimes d\Gamma_b(S) + H_1, \]

where \( H_1 \) is a symmetric operator on \( \mathcal{H} \). Let \( H_0 = R \otimes I + I \otimes d\Gamma_b(S) \). We suppose following conditions:

- **(S.1)** \( H \) is self-adjoint on \( \mathcal{D}(H) = \mathcal{D}(H_0) \cap \mathcal{D}(H_1) \) and bounded from below.
- **(S.2)** For all \( h \in \mathcal{D}(S) \cap \mathcal{D}(S^{-1/2}) \), the weak commutator \( [H_1, I \otimes A^*(h)]^0_{\mathcal{D}(H)} \) exists. Moreover, for all sequences \( \{h_n\}_{n=1}^\infty \) of \( \mathcal{D}(S) \cap \mathcal{D}(S^{-1/2}) \) such that \( \|f_n\| = 1, n \geq 1 \), and \( \text{w- lim } h_n = 0 \), it follows that for all \( \Psi \in \mathcal{D}(H) \),

\[
\text{s- lim } [H_1, I \otimes A^*(h_n)]^0_{\mathcal{D}(H)} \Psi = 0.
\]

Then the next theorem follows.

**Theorem I ([2]; Theorem 1.2)**

Assume (S.1) and (S.2). Then

\[
\{E_0(H) + \lambda \mid \lambda \in \sigma_{\text{ess}}(S) \backslash \{0\} \subset \sigma_{\text{ess}}(H),
\]

where \( \mathcal{J} \) denotes the closure of \( J \subset \mathbb{R} \).
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