Classicalization by phase space measurements

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Abstract
This article provides an illustration of the measurement approach to the quantum–classical transition suitable for beginning graduate students. As an example, we apply this framework to a quantum system with a general quadratic Hamiltonian, and obtain the exact solution of the dynamics for an arbitrary measurement strength using phase space methods.

Keywords: quantum–classical transition, measurement master equation, phase space methods

1. Introduction

The study of the quantum–classical transition is an active field of research that has provided important insights into the foundations of quantum theory and plays a prominent role in the development of quantum technologies [1]. Some of its achievements have resulted from using the framework of generalised measurements [2] to model the dynamics of an open quantum system, i.e. a quantum system in interaction with a large number of quantum degrees of freedom, which are collectively called environment. This, for example, has led to a quantitative assessment of the macroscopic character of a superposition state based on the experimental observation of quantum effects [3].

The concepts and methods used in this field are usually not familiar to beginning graduate students. Fortunately, phase space methods and open-system dynamics are increasingly being discussed in the pedagogical literature [4–7]. In this article the two are brought together in the study of an analytically tractable model of the classicalization of a quantum system with a general quadratic Hamiltonian. To the best of the author’s knowledge, this model has not been discussed before at a level suitable for our intended readership.
The contributions of this work to physics education are twofold. First, it provides an accessible illustration of the measurement approach to the quantum–classical transition using phase space methods. This should be useful for instructors with an interest in introducing graduate students to current research topics in quantum mechanics. Second, it collects several results which are scattered throughout the literature, thus making them more accessible to both students and instructors.

The manuscript is organised as follows. In section 2, Dirac quantisation is briefly reviewed with a focus on physical dimensions and algebraic considerations. These two aspects will play an important role throughout the article. We remark that the use of algebraic methods in quantum mechanics has led to a deeper understanding of nature, while also offering an elegant and powerful framework to study a wide variety of systems [8, 9]. Section 3 contains a brief account of the phase space formulation of quantum mechanics, which is particularly useful for the study of the quantum–classical transition. In section 4 generalised measurements are defined and it is shown how to construct a dynamical equation for the state of a system subject to such a measurement. Section 5 contains an example of measurement-induced classicalization. In particular, we consider an imprecise, simultaneous measurement of two canonically conjugate observables of a quantum harmonic oscillator, which leads to the decay of all the coherences in a superposition state.

2. Dirac quantisation and the Heisenberg algebra

In analogy to classical Hamiltonian mechanics, in Dirac quantisation the observable quantities of an elementary quantum system are described by operator-valued functions of two self-adjoint operators \( Q = Q^\dagger, P = P^\dagger \) satisfying the canonical commutation relation

\[
[Q, P] = i\hbar /1,  \tag{2.1}
\]

which shows that their product has dimensions of action. The operators \( Q, P, \) and \( I \) (identity) are the canonical basis of the Heisenberg Lie algebra \( h(3) \) [10]. It is customary to choose units in which \( \hbar = 1 \). However, in this paper this will not be the case in order to make it easy to verify that equations have the correct dimensions.

The dimensionless basis of \( h(3) \) (strictly speaking, it is the basis of the complex extension or complexification of this algebra) is given by [11]

\[
a = \frac{1}{\sqrt{2}} \left( \frac{Q}{\lambda} + \frac{i}{\hbar} P \lambda \right), \quad a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{Q}{\lambda} - \frac{i}{\hbar} P \lambda \right), \quad I,  \tag{2.2}
\]

where \( \lambda \) has the same dimensions as \( Q, P \) has dimensions of \( \hbar /\lambda \) and \( a, a^\dagger \) satisfy the commutation relation

\[
[a, a^\dagger] = 1.  \tag{2.3}
\]

Correspondingly, the observable quantities of the system can be expressed as operator-valued functions of \( a \) and \( a^\dagger \). For example, the operator \( N = a^\dagger a \) together with the operators \( a, a^\dagger, I \) generates the Lie algebra \( h(4) \). From the commutators

\[
[a, N] = a, \quad [a^\dagger, N] = -a^\dagger  \tag{2.4}
\]

follows [10] that \( a \) and \( a^\dagger \) are ladder operators with respect to the eigenvectors \( |n\rangle = (a^\dagger)^n |0\rangle /\sqrt{n!} \) with \( n \geq 0 \), of \( N \):

\[
a |n\rangle = \sqrt{n} |n - 1\rangle, \quad a^\dagger |n\rangle = \sqrt{n + 1} |n + 1\rangle.  \tag{2.5}
\]
These eigenvectors form a complete and orthonormal \( \langle n|m \rangle = \delta_{nm} \) set that spans an infinite-dimensional Hilbert space \( \mathcal{H} \), in which one can represent the state of a system described by elements of \( \mathfrak{h}(4) \), e.g. the quantum harmonic oscillator.

An eigenvector \( |\alpha\rangle \) of the operator \( a \) is called a coherent state. The corresponding eigenvalue, \( \alpha \), is a complex number, since this operator is not self-adjoint. In the basis \( |n\rangle \), a coherent state is expressed as \[ |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{2.6} \]

These states play a fundamental role in the study of the quantum–classical transition \[ [12] \]. This becomes apparent with the use of phase space methods, which will be discussed in the following section.

3. Quantum mechanics in phase space

Quantum mechanics can also be formulated in terms of unitary operators, i.e. operators that satisfy \( OO^\dagger = O^\dagger O = I \). In this framework, the operators associated with \( Q \) and \( P \) are \[ U(a) = e^{iQa}, \quad V(b) = e^{ibP}. \tag{3.1} \]

Since the argument of the exponential function must be dimensionless, \( a \) has the dimensions of \( P \) and \( b \) has the dimensions of \( Q \). The above operators are a special case of the Weyl operators (or displacement operators)

\[ D(a, b) = e^{iQ^a + bP}, \tag{3.2} \]

which satisfy the canonical commutation relation in Weyl (or integral) form \[ [13] \]

\[ D(a_1, b_1)D(a_2, b_2) = e^{i\pi[a_1b_2 - a_2b_1]}D(a_1 + a_2, b_1 + b_2). \tag{3.3} \]

In terms of the operators \( a \) and \( a^\dagger \), the displacement operator is given by \[ [10] \]

\[ D(\alpha) = e^{a^\dagger a^\alpha - a^\dagger a^\alpha}, \tag{3.4} \]

and the relation corresponding to (3.3) is

\[ D(\alpha)D(\beta) = e^{i\alpha\beta}D(\alpha + \beta). \tag{3.5} \]

The coherent states can also be defined as displaced ground states \( |\alpha\rangle := D(\alpha)|0\rangle \), as can be seen by comparing this expression with (2.6).

The operators acting on a representation space (such as the Hilbert space spanned by the vectors \( |n\rangle \)) belong to a Hilbert space called Liouville space, in which the scalar product is given by \( \langle A|B \rangle = \text{Tr}(A^\dagger B) \) \[ [13] \]. The set of displacement operators forms a delta-orthogonal basis of this space: \( \langle D(\alpha)|D(\beta) \rangle = \pi^{\delta(\alpha - \beta)} \) \[ [10] \]. In this basis, the density operator (or state operator) \( \rho \) of a quantum system is described by the Wigner characteristic function (also called ambiguity function or chord function) \[ [14, 15] \]

\[ \chi(\eta, \eta^*) = \text{Tr}[\rho \ D(\eta)], \tag{3.6} \]

which will figure prominently in section 5.

The following Fourier transform of \( \chi(\eta, \eta^*) \) yields the Wigner function \[ [16] \]:

\[ W(\xi, \xi^*) = \frac{1}{\pi^2} \int d^2\eta \ \chi(\eta, \eta^*) \ e^{i\xi^*\eta - \xi\eta^*}, \quad d^2\eta = d\text{Re}(\eta)d\text{Im}(\eta). \tag{3.7} \]
which can also be defined as $W(\xi, \xi^*) = \text{Tr}[\rho \Pi(\xi)]$, where

$$\Pi(\xi) = \frac{2}{\pi} D(\xi) \Pi_0 D^*(\xi)$$

is the Wigner operator (or displaced parity operator) [17], which is self-adjoint and therefore is an observable, unlike the displacement operator. This implies that the Wigner function is real, whereas the characteristic function is complex. The geometrical interpretation of these functions is discussed in [15]. For the experimental determination of the Wigner function we refer the reader to [18].

We remark that the definition (3.6) maps an operator-valued function of $a$ and $a^\dagger$, namely (3.4), to a complex-valued function of $\eta$ and $\eta^*$. This is known as a mapping from $q$-numbers to $c$-numbers and it is very useful in the study of the quantum–classical transition, as will be shown in section 5. For a thorough discussion of mappings of this kind we refer the reader to [19, 20].

For a coherent state $|\alpha\rangle$, the Wigner function is Gaussian:

$$W(\xi, \xi^*) = \frac{2}{\pi} \text{Tr}[D(\xi) \Pi_0 D^*(\xi) |\alpha\rangle \langle \alpha|] = \frac{2}{\pi} \text{Tr}[|\xi - \alpha\rangle \langle \xi - \alpha|] = \frac{2}{\pi} e^{-2|\xi - \alpha|^2}. \tag{3.9}$$

This expression is readily obtained using (3.5), the cyclic property of the trace, the definition of the parity operator, $\Pi_0 |\alpha\rangle = |\alpha\rangle$, the unitarity of the displacement operator, $D^*(\alpha) = D(-\alpha)$, and the inner product of coherent states:

$$\text{Tr}[|\alpha\rangle \langle \alpha|] = \langle \alpha | \langle \alpha | = \exp\left[-\frac{1}{2} |\xi - \alpha|^2 + \frac{1}{2} (\alpha^* \xi + \xi^* \alpha)\right]. \tag{3.10}$$

However, for a so-called cat state $|\psi\rangle$, the Wigner function takes negative values:

$$W(\xi, \xi^*) = \frac{2}{\pi} \mathcal{N} [e^{-2|\epsilon - \alpha|^2} + e^{-2|\epsilon + \alpha|^2} + 2 e^{-2|\epsilon|^2} \cos(4 \text{Im}(\alpha^* \xi))]. \tag{3.12}$$

Negativity of the Wigner function is a signature of a non-classical state [21]. Moreover, it can be proven that the only positive-definite Wigner functions describing pure states are Gaussian [22]. This is one reason why coherent states are considered the most classical quantum states.

It can be shown [21] that for a Hamiltonian with a general quadratic potential $V(q) = aq^2 + bq + c$, the evolution of the Wigner function is given by the classical Liouville equation for a probability distribution in phase space:

$$\frac{\partial}{\partial t} W(q, p, t) = \frac{\partial H}{\partial q} \frac{\partial W}{\partial p} - \frac{\partial H}{\partial p} \frac{\partial W}{\partial q}. \tag{3.13}$$

However, this does not represent classical behaviour unless the Wigner function is positive. For this reason, $W$ is called a quasi-probability phase space distribution. Moreover, as a consequence of Heisenberg’s uncertainty principle, the Wigner function cannot have a width smaller than the size of a Planck cell: $\Delta q \Delta p \geq \hbar$ [21]. However, it is worth noting that the sub-Planck structure of this function is important in the study of the quantum–classical transition [23].

One reason why quasi-probability distributions are useful is that they enable calculating quantum-mechanical expectation values similarly to averages in classical statistical mechanics. In particular, distributions belonging to the Cohen class [24], which includes the Wigner function, have the property that integrating them with respect to one canonical variable yields the probability distribution of the canonically conjugate variable.
In the literature, it is common to describe the transition to classical behaviour as ‘taking the limit $\hbar \to 0$’. However, this characterisation is misleading, since $\hbar$ is a constant of nature and it cannot be made arbitrarily small. What is meant by this statement is that one may form a dimensionless parameter involving $\hbar$ and other physical quantities, such that when this parameter is made arbitrarily small, a quantum equation reduces to a classical one (e.g. the Liouville equation above). This shows again the importance of dimensional considerations in quantum mechanics. We remark that this limiting procedure does not lead to the vanishing of negative regions in the Wigner function.

### 4. Quantum measurements and the measurement master equation

#### 4.1. Measurement in quantum mechanics

In the axiomatization of quantum mechanics carried out by von Neumann [25], given an observable with a discrete spectrum $O = \sum_n \lambda_n|\lambda_n\rangle\langle \lambda_n|$, the probability that a measurement of $O$ yields the result $\lambda_n$ is

$$\text{Prob}(\lambda_n) = \text{Tr}[\rho|\lambda_n\rangle\langle \lambda_n|],$$

(4.1)

and the state of the system after the measurement is $\rho = |\lambda_n\rangle\langle \lambda_n|$. If the measurement result is not known, the system is described by the mixed state

$$\rho = \sum_n \text{Prob}(\lambda_n)|\lambda_n\rangle\langle \lambda_n|.$$  

(4.2)

Instead of associating a projector with each measurement result $n$, in general one may associate a positive operator $\pi_n$ with it. These operators form a positive-operator-valued measure [2] and must be such that $\sum_n \pi_n = I$. Moreover, each operator $\pi_n$ may be decomposed in terms of pairs $(A_k, A_k^\dagger)$ of operators:

$$\pi_n = \sum_k A_k A_k^\dagger.$$  

(4.3)

In this framework, the probability that a measurement yields the result $n$ is $\text{Prob}(n) = \text{Tr}[\rho \pi_n]$ and the state of the system after the measurement is

$$\rho = (\text{Tr}[\rho \pi_n])^{-1} \sum_k A_{nk}^\dagger \rho A_{nk}.$$

(4.4)

If the measurement result is not known, the state of a system after performing a generalised measurement is given by

$$\rho = \sum_n \text{Prob}(n) \frac{\sum_k A_{nk}^\dagger \rho A_{nk}}{\text{Tr}[\rho \pi_n]} = \sum_{n,k} A_{nk}^\dagger \rho A_{nk}.$$  

(4.5)

We remark that this formalism is quite general. When the measurement result can take any real or complex value, the sums are replaced by corresponding integrals.

#### 4.2. Measurement master equation

The change in the state of a system subject to a generalised measurement can be modelled as a continuous-time stochastic process. That is, the state is considered to be a continuous time-dependent random variable that changes its value in discrete steps occurring at exponentially distributed times with expectation value $\gamma$. Such a stochastic process is called a Poisson process. We refer the interested reader to [26] for an in depth discussion of this so-called jump process.
Concretely, this is done as follows. We assume that in a short time interval $\Delta t$ the probability that a measurement occurs is $\gamma \Delta t$. The interval is assumed to be short enough, so that at most only one measurement takes place. If a measurement occurs, then the state at the time $t + \Delta t$ will be given by

$$\rho(t + \Delta t) = \rho(t) - \frac{i}{\hbar} [H, \rho(t)] \Delta t.$$  \hspace{1cm} (4.6)

Since the probability that this occurs is $1 - \gamma \Delta t$, the state of a system subject to this stochastic process is described at time $t + \Delta t$ by

$$\rho(t + \Delta t) = (1 - \gamma \Delta t) \rho(t) + \gamma \Delta t \sum_{n,k} A_{nk} \rho(t) A_{nk}^\dagger,$$  \hspace{1cm} (4.7)

which, to first order in $\Delta t$, can be written as

$$\frac{\rho(t + \Delta t) - \rho(t)}{\Delta t} = -\frac{i}{\hbar} [H, \rho(t)] + \gamma \sum_{n,k} A_{nk} \rho(t) A_{nk}^\dagger - \gamma \rho(t),$$  \hspace{1cm} (4.8)

and in the limit $\Delta t \to 0$ yields the measurement master equation [27]

$$\frac{\partial \rho_t}{\partial t} = -\frac{i}{\hbar} [H, \rho_t] + \gamma \left[ \sum_{n,k} A_{nk} \rho_t A_{nk}^\dagger - \rho_t \right].$$  \hspace{1cm} (4.9)

Though describing a general measurement, this equation is a special case of the Lindblad–Gorini–Kossakowski–Sudarshan master equation, which is widely used in the description of open quantum systems. For a thorough discussion of this topic, the reader is referred to [28].

### 5. Classicalization of systems with a quadratic Hamiltonian

A general quadratic Hamiltonian in the basis $Q, P$ is of the form:

$$H = c_1 Q^2 + c_2 P^2 + c_3 (QP + PQ) + c_4 Q + c_5 P, \quad c_i \in \mathbb{R}. \hspace{1cm} (5.1)$$

The corresponding expression in a dimensionless basis analogous to (2.2) is:

$$H = z_1 b^* b + z_2 b^2 + z_2^2 (b^*)^2 + z_3 b + z_3^2 b^*, \quad z_1, z_3 \in \mathbb{R}, \quad z_2 \in \mathbb{C}. \hspace{1cm} (5.2)$$

Using a canonical transformation (see appendix) one obtains from (5.2) the harmonic oscillator-like Hamiltonian ($z_0 = (z_1^2 - 4|z_2|^2)^{1/2}, \phi = \text{Arg}(z_2)$):

$$H = z_0 a^* a + c \mathbb{I},$$  \hspace{1cm} (5.3)

with

$$b = \left[ \frac{1}{2} \frac{z_1 + z_0}{z_0} \right]^{1/2} e^{i\phi/2} a + \left[ \frac{1}{2} \frac{z_1 - z_0}{z_0} \right]^{1/2} e^{i\phi/2} a^\dagger \quad \text{and} \quad \frac{2z_2 z_3^* - z_4 z_3}{z_0^2} \frac{2z_2^* z_3 - z_4^* z_3}{z_0^2}$$

and

$$c = \frac{1}{2} (z_0 - z_1) + \frac{1}{z_0} (z_2 (z_3^*)^2 + z_2^* z_3^2 - z_4 |z_3|^2).$$

We note that the constant $c$ in the Hamiltonian does not affect the dynamics and, therefore, in this basis the system behaves like a harmonic oscillator with ground-state energy zero. This is
a demonstration of the usefulness of algebraic methods in quantum mechanics. In the following we will set $z_0 = \hbar \omega$.

5.1. Simultaneous imprecise measurement of the quadratures of the harmonic oscillator

For the quantum harmonic oscillator (5.3) one can define the dimensionless self-adjoint quadrature operators:

$$X = \frac{1}{2}(a + a^\dagger), \quad Y = \frac{1}{2i}(a - a^\dagger), \quad [X, Y] = \frac{i}{2}. \quad (5.4)$$

For any state $\rho$ of the harmonic oscillator, the Heisenberg uncertainty product is given by $\Delta X \Delta Y \geq 1/4$, with $\Delta \mathcal{O} = (\langle O^2 \rangle - \langle O \rangle^2)^{1/2}$. States that satisfy the equality are called minimum-uncertainty states. For example, coherent states have this property: $\Delta X = \Delta Y = \frac{1}{2}$. This is another reason why coherent states are considered the most classical quantum states. The eigenvalues of $X$ and $Y$ are the real and imaginary parts of the amplitude $\alpha = \alpha_x + i\alpha_y$. A measurement of $\alpha$ can be described using the formalism of generalised measurements of section 4, as follows.

Given a positive operator with unit trace $\Psi$, one can define the operators [29]:

$$\pi_\alpha = \frac{1}{\pi} D(\alpha) \Psi \mathcal{D}^\dagger(\alpha), \quad \int d^2 \alpha \pi_\alpha = 1. \quad (5.5)$$

The probability density of obtaining the result $\alpha$ is $\text{Prob}(\alpha) = \text{Tr}[\pi_\alpha \rho]$, and its moments are given by

$$M_{ij} = \int d^2 \alpha \; \text{Prob}(\alpha) \alpha_i^j \alpha_i^j, \quad d^2 \alpha = d\alpha_x \; d\alpha_y. \quad (5.6)$$

The expectation values of $\alpha_x$ and $\alpha_y$ are given by

$$\langle \alpha_x \rangle = M_{10} = \langle X \rangle_\rho - \langle X \rangle_\psi, \quad \langle \alpha_y \rangle = M_{01} = \langle Y \rangle_\rho - \langle Y \rangle_\psi, \quad (5.7)$$

so in order for the measurement to be unbiased, the expectation values with respect to $\Psi$ must be zero. The second-order moments give the mean square uncertainty for the simultaneous measurement of $\alpha_x$ and $\alpha_y$:

$$\langle (\Delta \alpha_x)^2 \rangle = M_{20} - (M_{10})^2 = \langle X^2 \rangle_\rho + \langle X^2 \rangle_\psi, \quad \langle (\Delta \alpha_y)^2 \rangle = M_{02} - (M_{01})^2 = \langle Y^2 \rangle_\rho + \langle Y^2 \rangle_\psi, \quad (5.8)$$

which shows that the measurement apparatus, as described by $\Psi$, contributes (classical) noise to the uncertainty (quantum noise) of the measured state $\rho$. We note that this is an imprecise, simultaneous measurement of two non-commuting observables. In the following, we will show that this interaction leads to the classicalization of a superposition state of the harmonic oscillator.

5.2. Measurement-induced classicalization

From the theory of generalised measurements in section 4, the operators (5.5) can be written as:

$$\pi_\alpha = A_\alpha^\dagger A_\alpha, \quad A_\alpha = \frac{1}{\sqrt{\pi}} D(\alpha) \mathcal{D}^\dagger(\alpha). \quad (5.9)$$
In turn, \( A_\alpha \) can be written in the basis of displacement operators:

\[
A_\alpha = \int d^2 \beta \, D(\beta) \text{Tr}[D^\dagger(\beta) A_\alpha] = \frac{1}{\sqrt{\pi}} \int d^2 \beta \, D(\beta) e^{\beta^\dagger \alpha - \alpha^\dagger \beta} \chi_{1/2}(\beta),
\]

(5.10)

where we used the cyclic property of the trace, the definition (3.6) and the identity

\[
D(\zeta)D(\zeta)D(\zeta) = D(\zeta) \text{exp}(\xi^\dagger \zeta - \zeta^\dagger \xi).
\]

(5.11)

In this representation,

\[
\int d^2 \alpha \, A_\alpha \rho \, A_\alpha^\dagger = \pi \int d^2 \lambda \, D(\lambda) \rho \, D^\dagger(\lambda) [\chi_{\psi(\zeta)}(\lambda)]^2,
\]

(5.12)

where we used the identity

\[
\delta^{(2)}(\beta - \lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} d^2 \alpha \, \text{exp}[\alpha \beta^\dagger - \alpha^\dagger \beta] \text{exp}[\alpha^\dagger \lambda - \alpha \lambda^\dagger].
\]

(5.13)

Defining the rotation-invariant, positive-definite function \( g(|\lambda|) = N [\chi_{\psi(\zeta)}(\lambda)]^2 \), where \( N \) is a normalisation constant, we finally arrive at the master equation corresponding to the measurement described above:

\[
\frac{\partial \rho_t}{\partial t} = -\frac{i}{\hbar}[H, \rho_t] + \gamma \int d^2 \lambda \, g(|\lambda|) D(\lambda) \rho_t D^\dagger(\lambda) - \gamma \rho_t,
\]

(5.14)

where \( H \) is defined in (5.3). The integral term can be interpreted as describing phase space ‘kicks’ occurring with rate \( \gamma \) and with a \( g \)-distributed strength \( \lambda \).

Recalling the definition of an operator in the interaction picture

\[
\hat{O}_t = e^{iHt/\hbar} \hat{O} e^{-iHt/\hbar}, \quad \hat{O}_t = e^{-iHt/\hbar} \hat{O} e^{iHt/\hbar},
\]

(5.15)

and using the identity \([30]\)

\[
e^{x^\dagger a} f(\alpha, a^\dagger) e^{-x^\dagger a^\dagger} = f(\alpha e^{-x}, a^\dagger e^x),
\]

(5.16)

the master equation in the interaction picture is given by

\[
\frac{\partial \hat{\rho}_t}{\partial t} = \gamma \int d^2 \lambda \, g(|\lambda|) \hat{D}(\lambda) \hat{\rho}_t \hat{D}^\dagger(\lambda) - \gamma \hat{\rho}_t,
\]

(5.17)

with the interaction-picture displacement operator

\[
\hat{D}(\lambda) = \text{exp}(\lambda^\dagger a^\dagger - \lambda^* e^{-iHt/\hbar} a).
\]

(5.18)

Now, using the definition of the characteristic function (3.6) and the identity (5.11), from (5.17) one obtains the equation

\[
\frac{\partial}{\partial t} \tilde{\chi}_\rho(\eta, \eta^*, t) = -\gamma \left(1 - \int d^2 \lambda \, g(|\lambda|) e^{\eta^\dagger \lambda - \eta \lambda^\dagger} \right) \tilde{\chi}_\rho(\eta, \eta^*, t).
\]

(5.19)

Defining the characteristic function, \( \phi_{\hat{g}} \), of a complex variable \( Z \) with probability density \( g_Z \) as \([29]\)

\[
\phi_{\hat{g}}(\zeta) = \int d^2 z \, g(z) e^{\zeta z - \zeta^* z},
\]

(5.20)

the equation of motion (5.19) can be written as

\[
\frac{\partial}{\partial t} \tilde{\chi}_\rho(\eta, \eta^*, t) = -\gamma [1 - \phi_{\hat{g}}(\eta)] \tilde{\chi}_\rho(\eta, \eta^*, t).
\]

(5.21)
It follows that the quantum characteristic function corresponding to the solution of (5.17) is
\[
\tilde{\chi}_0(\eta, \eta^*, t) = \chi_0(\eta, \eta^*) \exp(-\gamma t[1 - \phi_\eta(|\eta|)]). \tag{5.22}
\]

We recognise the exponential term as the characteristic function of a compound Poisson process with rate $\gamma$ and jump-size distribution $g$ [26]. Since $\phi_\eta$ is the characteristic function of a probability distribution, it satisfies $\phi_\eta(0) = 1$ and in the limit $|\eta| \to \infty$ it vanishes for all $t$; \( \phi_\eta(|\eta|) \to 0 \). Therefore, at any time the exponential term is a rotation-invariant function that asymptotically decays in phase space towards the plane \( f(\eta) = e^{-\gamma t} \). This plane in turn decays with time towards \( f(\eta) = 0 \).

Let us assume that at time \( t = 0 \) the system is prepared in the cat state (3.11). In order to obtain its characteristic function, we first calculate
\[
\exp(-\frac{1}{2}|\eta|^2) \exp(-\gamma t). \tag{5.23}
\]

This expression is readily obtained using (3.4), (3.6), (3.10), the cyclic property of the trace and \( e^{az} = e^{az} \). Noting that \( |\eta - (\alpha - \beta)|^2 = |\eta|^2 - (\alpha^* - \beta^*)^2 \) and \( |\alpha|^2 - (\alpha^* - \beta^*)^2 \),
\[
\chi_{|\eta;\alpha}(\eta, \eta^*) = \exp\left(-\frac{1}{2}|\eta|^2 - (\alpha - \beta)^2 \right) \tag{5.24}
\]
\[\exp\left(-\frac{1}{2}(\eta^* \beta - \eta^* \beta^* + \eta^* \eta^* - \eta^* \eta^* + \beta \alpha^* - \alpha \beta^*) \right). \tag{5.25}
\]

The characteristic function of the cat state is then
\[\chi_0(\eta, \eta^*) = \mathcal{N}[e^{-\frac{1}{2}|\eta|^2 + 2\cos(2 \text{ Im}(\eta^* \alpha^*))} + e^{-\frac{1}{2}|\eta|^2 - 2\cos(2 \text{ Im}(\eta^* \alpha^*))}]. \tag{5.26}
\]

This expression is a sum of two Gaussian functions corresponding to the coherences of the state, located at \( 2\alpha_0 \) and \( -2\alpha_0 \), and a modulated Gaussian at the origin of phase space. Therefore, the decoherence of this state can be associated with the decay of the outer Gaussians.

Provided the side peaks are sufficiently separated, the only terms that contribute to the squared modulus of (5.26) are the products of Gaussians with a common centre:
\[\chi_0(\eta, \eta^*)^2 = \mathcal{N}[e^{-\frac{1}{2}|\eta|^2} + e^{-\frac{1}{2}|\eta|^2} + 4e^{-\frac{1}{2}|\eta|^2} \cos^2(2 \text{ Im}(\eta^* \alpha^*))]. \tag{5.27}
\]

The square root of this expression is plotted in the top panel of figure 1, with \( \alpha_0 = 3i \). A snapshot of its time evolution as given by (5.21) is shown in the bottom panel for \( \gamma t = 1 \). For illustration purposes, the distribution $g$ is assumed to be Gaussian. The noticeable suppression of the side peaks shows that the measurement drives a superposition state of the harmonic oscillator towards a mixture of Gaussian states, which corresponds to a classical phase space function.

It is interesting to consider the case of very frequent (\( \gamma \to \infty \)) and very small phase-space kicks (\( \sigma \to 0 \)), where \( \sigma \) is the second moment of the distribution $g$, such that \( \gamma \sigma = \kappa \).

Let $r = |\eta|$. A Taylor-series approximation of $\phi_\eta$ to second order yields
\[\phi_\eta(r) = \phi_\eta(0) + r \phi'_\eta(0) + \frac{1}{2} r^2 \phi''_\eta(0). \tag{5.28}
\]

Since this function has a maximum at \( r = 0 \) then \( \phi'_\eta(0) = 0 \) and \( \phi''_\eta(0) < 0 \), and, by definition, \( \sigma = -\phi''_\eta(0) \). It follows that
Therefore, in this approximation one obtains a master equation describing diffusion in phase space \(31\). This can easily be understood by considering that the Fourier transform of the solution to this equation is a convolution of the Wigner function \(3.12\) with a Gaussian function, resulting in the broadening of the side peaks and the washing out of the oscillations in the centre. In fact, one can describe this dynamics in terms of the Brownian motion of the state vector of the system in Hilbert space \(32\). This and other master equations describing Gaussian dynamics are thoroughly discussed in \(11\). It is interesting to note that under this kind of evolution the Wigner function becomes positive everywhere in a finite time \(33\), in contrast to the infinite time required in the general case \(5.21\).

As the example discussed in this section illustrates, the framework of open quantum systems explains why it is so hard to observe a quantum system in a superposition state: the quantum coherence in this state is quickly destroyed by the interaction with another system. Moreover, this framework affords a dynamical description of the transition from quantum to classical behaviour. However, it does not solve the so-called measurement problem, since it does not provide an explanation for the emergence of a probability distribution of measurement outcomes, each one associated with a state of the system after the measurement, from the unitary dynamics of a superposition state evolving with the Schrödinger equation.

\[
\frac{\partial}{\partial t} \tilde{\chi}_\rho(\eta, \eta^*, t) = -\frac{1}{2} \gamma |\eta|^2 \tilde{\chi}_\rho(\eta, \eta^*, t). \tag{5.29}
\]

Figure 1. Top: \(|\tilde{\chi}_\rho(\eta, \eta^*, 0)|\) for the cat state \(3.11\) with \(a_0 = 3i\). Bottom: Snapshot of \(|\tilde{\chi}_\rho(\eta, \eta^*, t)|\) at \(\gamma t = 1\) for the cat state subject to an imperfect, simultaneous measurement of the quadratures assuming \(g\) to be a Gaussian distribution. The strong damping of the side peaks indicates that the measurement is classicalizing the state.
6. Conclusions

The model presented in section 5 illustrates the measurement approach to the classicalization of a quantum system by means of a master equation that can be solved exactly for an arbitrary measurement strength using phase space methods. Throughout the article, we aimed at keeping the presentation general and, at the same time, accessible. This should enable newcomers to the field to apply the measurement approach to classicalization to more complex systems.

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Appendix. Diagonalization of a quadratic Hamiltonian

Following [34], we consider the canonical transformation

\[ b = \mu a + \nu a' + \delta. \]

From \([a, a'] = [b, b'] = 1\), follows that \(|\mu|^2 - |\nu|^2 = 1\). Substituting in (5.2) and collecting terms, in order to get a diagonal operator the following equations must be satisfied:

\[ z_1 \nu^* \delta + z_1 \delta^* \mu + 2z_2 \nu^* \delta^* + 2z_2 \delta^* \mu^* + z_3 \mu^* = 0, \]

\[ z_1 \mu^* \delta + z_1 \delta^* \nu + 2z_2 \nu^* \delta + 2z_2 \delta^* \mu^* + z_3 \nu^* + z_3 \mu^* = 0, \]

\[ z_1 \nu^* \mu + z_2 \mu^* + z_3 (\nu^* )^2 = 0. \]

Subtracting (A.2) multiplied by \(\nu\) from (A.3) multiplied by \(\mu\), yields \(\delta = -(2z_2 \delta^* + z_3) / z_1\). Substituting in (A.2) we obtain \(\delta^* = (2z_2^* z_3 - z_1 z_3^*) / z_0^2\), with \(z_0 = (z_1^2 - 4|z_2|^2)^{1/2}\). From this expression we arrive at the condition \(z_1 > 2|z_2|\).

In order to find \(\mu\) and \(\nu\) from (A.4) we use the polar representations

\[ z_2 = |z_2| e^{i \phi}, \quad \mu = U e^{i \phi}, \quad \nu = V e^{i \phi}, \]

and choose \(\phi = \phi_0 = \frac{1}{2} \phi\) in order to obtain an equation with real variables. We now use the parametrizations \(U = \cosh(\Theta)/2\) and \(V = \sinh(\Theta)/2\), and recall the identities

\[ \cosh^2(x) + \sinh^2(x) = \cosh(2x), \quad 2 \sinh(x) \cosh(x) = \sinh(2x). \]

In terms of \(\Theta\), (A.4) has the form \((z_1 \sinh(\Theta)) / 2 + |z_2| \cosh(\Theta) = 0\). Using \(2 \tanh(x) = \log((1 + x)/(1 - x))\), we obtain \(\Theta = \log((z_1 - 2|z_2|)/(z_1 + 2|z_2|))\). Now we can calculate \(\sinh(\Theta) = -2|z_2| / z_0^2\) and \(\cosh(\Theta) = z_1 / z_0^2\). Recalling the identities

\[ \cosh \frac{x}{2} = \sqrt{(\cosh x + 1)/2}, \quad \sinh \frac{x}{2} = \sqrt{(\cosh x - 1)/2}, \]

we finally arrive at

\[ U = \left[ \frac{1}{2} \frac{z_1 + z_0}{z_0} \right]^{1/2}, \quad V = \left[ \frac{1}{2} \frac{z_1 - z_0}{z_0} \right]^{1/2}. \]

Substituting all of the above in (5.2), one obtains (5.3).
References

[1] Schlosshauer M 2007 Decoherence: and the Quantum-to-Classical Transition (Heidelberg: Springer-Verlag)
[2] Busch P, Lahti P, Pellonpää J and Ylinen K 2016 Quantum Measurement (Berlin: Springer)
[3] Nimmrichter S 2014 Macroscopic Matter Wave Interferometry (Berlin: Springer)
[4] Case W B 2008 Am. J. Phys. 76 937–46
[5] Lovett B W and Nazir A 2009 Eur. J. Phys. 30 S89
[6] Pearle P 2012 Eur. J. Phys. 33 805
[7] Xu Y-J, Li C, Ma Y-H and Li R-S 2018 Eur. J. Phys. 39 015303
[8] Woit P 2017 Quantum Theory, Groups and Representations: An Introduction (Berlin: Springer)
[9] Thyssen P and Ceulemans A 2017 Shattered Symmetry: Group Theory from the Eightfold Way to the Periodic Table (Oxford: Oxford University Press)
[10] Klimov A and Chumakov S 2009 A Group-Theoretical Approach to Quantum Optics (New York: Wiley)
[11] Serafini A 2017 Quantum Continuous Variables: A Primer of Theoretical Methods (Boca Raton, FL: CRC Press)
[12] Zurek W H, Habib S and Paz J P 1993 Phys. Rev. Lett. 70 1187–90
[13] Tarasov V 2008 Quantum Mechanics of Non-Hamiltonian and Dissipative Systems (Amsterdam: Elsevier)
[14] de Gosson M A 2017 Emergence of the Quantum from the Classical: Mathematical Aspects of Quantum Processes (Singapore: World Scientific)
[15] Ozorio de Almeida A M, Vallejos R O and Saraceno M 2005 J. Phys. A: Math. Gen. 38 1473
[16] Barnett S and Radmore P 2002 Methods in Theoretical Quantum Optics (Oxford: Clarendon)
[17] Bishop R F and Vourdas A 1994 Phys. Rev. A 50 4488–501
[18] Haroche S and Raimond J 2006 Exploring the Quantum: Atoms, Cavities, and Photons (Oxford: Oxford University Press)
[19] Klauder J and Sudarshan E 2006 Fundamentals of Quantum Optics (New York: Dover)
[20] Agarwal G S and Wolf E 1970 Phys. Rev. D 2 2161–86
[21] Kim Y and Noz M 1991 Phase Space Picture of Quantum Mechanics: Group Theoretical Approach (Singapore: World Scientific)
[22] Soto F and Claverie P 1983 J. Math. Phys. 24 97–100
[23] Zurek W H 2001 Nature 412 712–7
[24] Cohen L 1966 J. Math. Phys. 7 781–6
[25] Von Neumann J 1955 Mathematical Foundations of Quantum Mechanics (Princeton, NJ: Princeton University Press)
[26] Hanson F 2007 Applied Stochastic Processes and Control for Jump Diffusions: Modeling, Analysis, and Computation (Philadelphia, PA: Society for Industrial and Applied Mathematics)
[27] Cresser J D, Barnett S M, Jeffers J and Pegg D T 2006 Opt. Commun. 264 352–61
[28] Breuer H and Petruccione F 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[29] Walker N 1987 J. Mod. Opt. 34 15–60
[30] Louisell W 1990 Quantum Statistical Properties of Radiation (New York: Wiley)
[31] Agarwal G S 1971 Phys. Rev. A 4 739–47
[32] Jacobs K and Steck D A 2006 Contemp. Phys. 47 279–303
[33] Brodier O and Ozorio de Almeida A M 2004 Phys. Rev. E 69 016204
[34] Zelevinsky V 2011 Quantum Physics: Volume 1: from Basics to Symmetries and Perturbations (New York: Wiley)