CONTINUUM LIMIT OF NONLINEAR DISCRETE SYSTEMS WITH LONG RANGE INTERACTION POTENTIALS

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Abstract

One-dimensional nonlinear lattices with harmonic long range interaction potentials (LRIP) of inverse power type are studied. For the nearest neighbour nonlinear interaction we shall consider the anharmonic potential of the Fermi-Pasta-Ulam problem and the $\phi^3 + \phi^4$ as well. The continuum limit is obtained following the method used by Ishimori [1], and several Boussinesq and KdV-type equations are found with supplementary Hilbert transform terms. This nonlocal terms are introduced by the LRIP. For the $\phi^3 + \phi^4$ nearest neighbour potential, the continuum approximation turns out to admit exact bilinearization in Hirota formalism. Exact rational nonsingular solutions are found. The connection with perturbed KdV equations is also discussed, and the dynamics of the KdV soliton is studied for one of these equations.

Introduction

Several years ago Ishimori [1] has discussed in the continuum approximation a one-dimensional nonlinear lattice with a long range interaction potential of Lennard-Jones type. This was done expanding the lattice displacements in Fourier series, performing the summations in the real space, and finally taking into account only the first terms in the wave vector expansion - the long wave limit. His conclusion was that the long range interaction manifests mostly in the linear dispersive terms, the effect on the nonlinear terms being less dramatic. Several integro-differential equations were obtained, some of them containing an Hilbert transform.

From another point of view, in the context of wave propagation in shallow water, Whitham [2] has extended the well known Korteweg-de Vries equation by including a nonlocal (integral) dispersive term. An example from plasma physics is given by the nonlinear ion acoustic waves where a Hilbert transform term is introduced into the KdV equation to describe the Landau damping [3].

Recently the interest in the study of nonlinear evolution equations with nonlocal dispersive terms was stimulated by the discovery that their solutions have very interesting picking and breaking phenomena [4]. On the other hand it is well known that several integro-differential equations of Benjamin-Ono (BO) type [5] are completely integrable and the main feature of their integrability character is the existence of rational solutions (algebraic solitons).

Following Ishimori [1] we shall consider a one-dimensional lattice with long range harmonic interaction between the atoms, the nonlinear interaction being restricted between nearest neighbours. The long range harmonic interaction potential is written

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\[ \frac{1}{2} \sum_{m,n} J_{mn} (u_m - u_n)^2, \]  
(1)

where

\[ J_{mn} = J_{|m-n|} = \frac{1}{2} \frac{J}{\zeta(p)} \frac{1}{|m-n|^p}, \]  
(2)

and in the summation the term \( m = n \) is excluded. Here \( u_n, u_m \) are the dimensionless lattice displacements at the lattice points \( m, n \), \( p \geq 2 \) is an integer describing the spatial decreasing of the long range interaction, and \( J \) measures its strength. In the definition of \( J_{mn} \) we introduced the Riemann sum

\[ \zeta(p) = \sum_{m=1}^{\infty} \frac{1}{m^p}, \]  
(3)

and consequently

\[ \sum_{m} J_{mn} = J. \]  
(4)

Also a long range potential with alternating sign will be considered. For this \( J_{mn} \) is given by

\[ J_{mn} = \frac{1}{2} \frac{J}{\zeta(p)} (-)^{|m-n+1|} \frac{1}{|m-n|^p}, \]  
(2a)

and in this case

\[ \sum_{m} J_{mn} = J(1 - \frac{1}{2p-1}). \]  
(4a)

The nonlinear interaction is assumed only between nearest neighbours

\[ \sum_{n} V(u_{n+1} - u_n) \]  
(5)

and the simplest form for \( V \) is

\[ V(u) = \frac{1}{2} u^2 - \frac{\alpha}{3} u^3, \]  
(6a)

\( \alpha \) being a parameter describing the strength of the anharmonic term. The expression (6) is the same as in the famous Fermi-Ulam-Pasta problem. We shall discuss, also the following form

\[ V(u) = \frac{1}{2} u^2 + \frac{\alpha}{3} u^3 + \frac{\beta}{4} u^4 \]  
(6b)

which is an anharmonic potential introduced by M. Wadati [11]. Obviously the continuum approximation for the equations of motion will involve Boussinesq type equations with long range interaction corrections.

Due to the inverse power form of the long range kernel, continuum limits of the long range interaction terms turn out to be in the form of a Hilbert transform and its derivatives [1]. Also the continuum limits of the local terms, restricted at the waves in only one direction, leads to the appearance of the KdV [7] equation for the local potential (6a), and KdV+mKdV [11] equation for the local potential (6b). In the first case, assuming \( J \) to be small a perturbatively approach is done in order to find what happened with the 1-soliton solution of KdV type. It is found that for \( p = 2 \) the first 5 conservation laws for KdV 1-soliton remain unchanged in the first order of perturbation.

In the second case the presence of the Hilbert transform which is a nonlocal dispersive term leads us to the possibility to consider rational solutions which appear in nonlinear integro-differential equations such as BO-hierarchy [5] and nonlocal AKNS-hierarchy [5]. These rational solutions behave like solitons and interact
elastically with no phase shift. For local (1+1) dimensional nonlinear evolution equations both continuous and discrete the rational solutions also exist but their character is rather singular \[12\], \[13\], \[14\], \[17\]. They have no free parameters and there are very few cases in which they are not singular \[12\], \[15\], \[16\] and, moreover, they interact elastically with no phase shift \[16\].

It turns out that for \( p = 4 \) and (6b) the continuum limit of the equation of motion admits exact bilinearization in Hirota formalism for certain values of the parameters \( \alpha \) and \( \beta \). The corresponding rational solutions are exactly the same as for mKdV with nonvanishing boundary conditions \[12\], \[15\]. In this way we are expecting that this system to be completely integrable although it does not belong to the BO-hierarchy or Nonlocal AKNS-hierarchy either.

**Continuum approximation for the equations of motion**

The equation of motion writes

\[
\frac{d^2 u_n}{dt^2} = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}) - 2\Sigma' J_{mn}(u_n - u_m).
\]  

(7)

In the linear approximation it becomes

\[
\frac{d^2 u_n}{dt^2} = (u_{n+1} - 2u_n + u_{n-1}) - 2J \left( u_n - \frac{1}{2\zeta(p)} \sum_{m=1}^{\infty} \frac{1}{m^p} (u_{n+m} + u_{n-m}) \right).
\]  

(8)

Looking for solutions of the form

\[ u_n \simeq e^{i(kn-\omega t)} \]

the following dispersion relation results

\[
\omega^2(k) = 4\sin^2\frac{k}{2} + 2J \frac{1}{\zeta(p)} F_p(k),
\]  

(9)

where

\[
F_p(k) = \sum_{m=1}^{\infty} \frac{1 - \cos mk}{m^p}.
\]  

(10)

This is an even function of \( k \) and can be calculated for each value of \( p \) \[6\]. In the following the results for several values of \( p \) will be presented.

**Case \( p=2 \)**

\[
F_2(k) = \frac{\pi}{2} |k| - \frac{1}{4} k^2, \quad \zeta(2) = \frac{\pi^2}{6}.
\]  

(11)

Then

\[
\omega^2(k) = 4 \sin^2\frac{k}{2} + 2J \left( \frac{3}{\pi} |k| - \frac{3}{2\pi^2} k^2 \right),
\]  

(12)

which in the long wave limit is approximated by

\[
\omega^2(k) \simeq \frac{6J}{\pi} |k| + (1 - \frac{3J}{\pi^2}) k^2 - \lambda k^4 + O(k^6),
\]  

(13)
where \( \lambda = \frac{1}{12} \) is a measure of the dispersion. This dispersion relation corresponds to a linear partial differential equation, namely

\[
\frac{\partial^2 u}{\partial t^2} = a H(u_x) + c^2 \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial^4 u}{\partial x^4},
\]

(14)

where the subscript indicates the derivative with respect to the corresponding variable, and the following notations were introduced

\[
a = \frac{6J}{\pi}, \quad c^2 = 1 - \frac{3J}{\pi^2}.
\]

(15)

Here \( H(f(x)) \) is the Hilbert transform of the function \( f(x) \)

\[
H(f(x)) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{f(x')}{x-x'} dx',
\]

(16)

and we have used the relation

\[
H(e^{ikx}) = i \text{sgn } k \ e^{ikx}.
\]

(16a)

The contribution of the anharmonic term \( \alpha u^3 \) in the equation of motion is

\[-\alpha(u_{n+1} - u_n)^2 + \alpha(u_n - u_{n-1})^2 = -\alpha(u_{n+1} - u_{n-1})(2u_n + u_{n-1})
\]

and in the continuum limit this goes to

\[-2\alpha u_x u_{xx} = -\alpha(u_x^2)_x.
\]

(17)

Then the complete nonlinear partial differential equation in the continuum limit is

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial}{\partial x} (u_x^2)_x - \lambda \frac{\partial^4 u}{\partial x^4} + a T(u_{xxx}) = 0.
\]

(18)

This is a Boussinesq type equation with a correction term in the form of a Hilbert transform because of the long range harmonic interaction.

**Case p=3**

In this case (see the Appendix)

\[
F_3(k) = -\frac{k^2}{2} \ln |k| + \frac{3}{4} k^2 - \frac{1}{288} k^4 + O(k^6).
\]

(19)

The dispersion relation becomes

\[
\omega^2(k) = (1 + \frac{3J}{2\zeta(3)})k^2 - \frac{J}{\zeta(3)}k^2 \ln |k| - \frac{1}{12} \left(1 - \frac{J}{24\zeta(3)}\right) k^4 + O(k^6),
\]

(20)

and the corresponding Boussinesq type equation is

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + \alpha \frac{\partial}{\partial x} (u_x^2)_x - \lambda \frac{\partial^4 u}{\partial x^4} + a T(u_{xxx}) = 0,
\]

(21)

where

\[
c^2 = 1 + \frac{3J}{2\zeta(3)}
\]

\[
\lambda = \frac{1}{12} \left(1 - \frac{J}{24\zeta(3)}\right)
\]

\[
a = \frac{J}{\zeta(3)}
\]

(22)
and $T(f(x))$ is the following integral operator [1]

$$
T(f(x)) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \text{sgn} (x' - x) \left( \ln |x' - x| + \gamma \right) f(x') \, dx'
$$

(23)

$\gamma$ being the Euler constant. In deriving the above equation we have used

$$
T(e^{ikx}) = \frac{\ln |k|}{ik} e^{ikx}.
$$

(23a)

Case $p=4$

$$
F_4(k) = \frac{\pi^2}{12} k^2 - \frac{\pi}{12} |k^3| + \frac{1}{48} k^4, \quad \zeta(4) = \frac{\pi^4}{90},
$$

(24)

and the following equation is finally obtained.

$$
u_{tt} - c^2 u_{xx} + \alpha(u_x^2)_x - \lambda u_{xxxx} - aH(u_{xxx}) = 0.
$$

(25)

Here

$$
c^2 = 1 - \frac{15J}{\pi^2}, \quad a = \frac{15J}{\pi^3}, \quad \lambda = \frac{1}{12} (1 - \frac{45J}{\pi^4}).
$$

(25a)

For an alternating in sign long range interaction potential, when $J_{mn}$ is given by (2a), the same technique gives slightly different results. Instead of (10) $F_p(k)$ is now defined as

$$
F_p^{(a)}(k) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{1 - \cos km}{m^p}.
$$

(26)

As an example we shall consider the case $p = 2$ when [6]

$$
F_p^{(a)}(k) = \frac{\pi^2}{4} k^2.
$$

(27)

Consequently the dispersion relation in the long wave limit takes a very simple form

$$
\omega^2(k) = c^2 k^2 - \lambda k^4 + O(k^6)
$$

(28)

$$
c^2 = 1 + 3J
$$

and the corresponding nonlinear evolution equation in this continuum limit becomes

$$
u_{tt} - c^2 u_{xx} + \lambda u_{xxxx} + \alpha(u_x^2)_x = 0.
$$

(29)

No non-local corrections appear and the long range interaction has a small influence modifying only the wave velocity.

It is well known since the paper of Zabusky and Kruskal [7] that the KdV eq. is obtained as the uni-directional long time approximation of the Boussinesq equation. We have to assume that there is a competition between the dispersion, nonlinearity and we have to add now the new term in the form of the Hilbert transform, competition which manifests at long times. In order to describe it we shall assume

$$
\alpha \to \epsilon \alpha, \quad \lambda \to \epsilon \lambda, \quad a \to \epsilon a,
$$

(30)

where $\epsilon$ is a small parameter. The solution $u(x,t)$ is written as
\[ u(x, t) = f(\xi, T) + \epsilon u^{(1)}(x, t) \]  

where  
\[ \xi = x - ct \]  
\[ T = ct \]  

are the "slow variables". Then in the first order in \( \epsilon \) from (18) we get  
\[ u^{(1)}_{tt} - c^2 u^{(1)}_{xx} = 2c f_{\xi T} - \alpha (f_{\xi}^2)_{\xi} + \lambda f_{\xi \xi \xi \xi} + a H(f_{\xi}). \]  

As the right hand side depends only on \( \xi \) and not on \( \eta = x + ct \) it has to vanish, in order to prevent the linear rising in \( \eta \) of \( u^{(1)}(x, t) \). Defining  
\[ q = \frac{1}{3} f_{\xi} \]  
\[ \tau = \frac{T}{2c}, \]  
we finally obtain  
\[ q_{\tau} - 6qq_{\xi} + \lambda q_{\xi \xi \xi} + a H(q) = 0 \]  

which in a KdV equation with an additional term (long range correction) in the form of a Hilbert transform.

If the same procedure is applied to the equation (25) we get  
\[ q_{\tau} - 6qq_{\xi} + \lambda q_{\xi \xi \xi} - a H(q_{\xi \xi}) = 0. \]  

Here the correction term has the same form as the Hilbert transform appearing in the Benjamin-Ono equation [5], which is actually obtained if the dispersive term is dropped out.

A quite surprising result emerges in the case of long range interaction with alternating signs. As shown before in this case the continuum limit is the usual Boussinesq equation (29) and from it the KdV eq. is immediately derived. It seems that the alternating signs in the long range interaction potential (we can call it an "antiferromagnetic" interaction) has a very small effect on the nature of elementary excitations in the long wave limit. This conjecture has to be verified for other values of \( p \) in (2a).

Perturbatively approach

Now for small values of the parameter "\( a \)" one can ask ourselves in what way the soliton of the KdV eq. is perturbed. We shall discuss briefly this problem for eq. (35) (we shall put \( \lambda = 1 \)) following one of the methods used to the analytical description of soliton dynamics in nearly integrable systems [8]. The approach is based on calculating the changes (evolution in time) of the first conservation laws. It is easily seen from eq. (35) that the first two, the mass and impulse, remain unchanged  
\[ \int_{-\infty}^{+\infty} q(\xi, \tau) d\xi = ct. \]  
\[ \int_{-\infty}^{+\infty} q^2(\xi, \tau) d\xi = ct. \]  

(37)
In order to find the time evolution of the KdV soliton energy we have to multiply eq. (35) by $3q^2 + q_r \partial \frac{\partial}{\partial \xi}$ and integrate over $\xi$. One obtains

$$\frac{dE_k}{d\tau} + a \int_{-\infty}^{+\infty} d\xi (3q^2 + q_r \partial \frac{\partial}{\partial \xi}) H(q(\xi)) = 0$$

(38)

where for the one soliton solution

$$q_k(\xi \tau) = -2k^2 \text{sech}^2 (k(\xi - 4k^2 \tau))$$

$$E_k = \int_{-\infty}^{+\infty} \left( \frac{1}{2} q^2 + q^4 \right) d\xi = -\frac{32}{5} k^5.$$  

(39)

The second term in (38) separates in two integrals, namely:

$$I_1 = \frac{3}{\pi P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi d\xi' q^2(\xi) q(\xi') \frac{q(\xi')}{\xi' - \xi}$$

(40)

and

$$I_2 = \frac{1}{\pi P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi d\xi' \frac{q(\xi)}{\xi' - \xi} \frac{\partial q(\xi)}{\partial \xi} \left( \frac{q(\xi')}{\xi' - \xi} \right)$$

(41)

After a partial integration on $\xi$ in $I_2$ we get

$$I_2 = -\frac{1}{\pi P} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi d\xi' \frac{q(\xi)}{\xi' - \xi} \frac{q(\xi')}{\xi - \xi}.$$  

(41a)

But for the KdV soliton

$$q_{\xi\xi} = 4k^2 q + 3q^2.$$  

(42)

Then using the fact that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi d\xi' \frac{q(\xi)}{\xi' - \xi} \frac{q(\xi')}{\xi - \xi} = 0,$$

(43)

relation which was already used for proving the momentum conservation (37), we obtain

$$I_1 + I_2 = 0$$

(44)

and consequently also the soliton energy is conserved in a first order perturbation calculation.

We can perform the same procedure for the following conserved quantity:

$$T_4 = 5q^4 + 10qq^2 + q^2$$

Multiply eq (35) by:

$$20q^3 + 10q^2 \partial \frac{\partial}{\partial \xi} + 2qq \frac{\partial^2}{\partial \xi^2}$$

and integrating over $\xi$ using the same procedure it is found that $T_4$ remains conserved. Lenghty but straightforward calculations lead to the same result for the next conserved quantity:

$$T_5 = 21q^5 + 105q^2q^2 + 21qq^2 + \frac{3}{2} q^2$$

(45)
We can proceed to any conserved quantity but the calculations are very much involved.

This is quite surprising because for an equation not very different from (35), describing the Landau damping of an ion acoustic wave [3], namely

\[ n_\tau - 6nn_\xi + n_{\xi\xi\xi} + aH(n_\xi) = 0 \quad (45) \]

only the first conservation law (particle conservation) is not destroyed, the other ones decaying in time.

In the same time we have to stress that our result is valid only in the first order of a perturbation theory and the resulting conclusions have to be understood only in this sense. More elaborated techniques have to be used to go beyond this approximation [8], [9].

That the situation is much more complicated was proven ten years ago by Birnir [10]. Discussing an equation of the same form as (35), with the coefficient \( a \) in front of the Hilbert transform a periodic function, satisfying mild conditions, he was able to prove the existence of chaotic rational solutions. They look like their integrable brothers, but are moving in a chaotic fashion and also can disappear and reappear [10]. This result shows how complex the problem of the perturbed KdV equation (35) is.

**Exact bilinearization**

In this case the local potential is given by:

\[ V(u) = \frac{1}{2}u^2 + \frac{\alpha}{3}u^3 + \frac{\beta}{4}u^4 \quad (46) \]

where we assume \( \beta \geq 0 \) For \( p = 4 \) the continuum approximation will have one more nonlinear term due to the \( u^4 \) anharmonic term in the potential, so:

\[ u_{tt} - c^2u_{xx} - \alpha(u_x^2)_x - \beta(u_x^3)_x - \lambda u_{xxxx} - aHu_{xxxx} = 0 \quad (47) \]

Following the same procedure, we consider:

\[ u(x, t) = f(\xi, T) + eu(1)(x, t) \quad (48) \]

and one gets:

\[ u_{tt}^{(1)} - c^2u_{xx}^{(1)} = 2cf_{\xi T} + \alpha(f_{\xi}^2)_{\xi} + \beta(f_{\xi}^3)_{\xi} + \lambda f_{\xi\xi\xi\xi} + aHf_{\xi\xi\xi} = 0 \quad (48) \]

For \( \tau = T/2c \) the equation becomes, after one integration:

\[ f_{\tau} + \alpha f_{\xi}^2 + \beta f_{\xi}^3 + \lambda f_{\xi\xi\xi\xi} + aHf_{\xi\xi\xi} = 0 \quad (49a) \]

which is a potential version of KdV+mKdV equation with nonlocal dispersion. For the sake of simplicity we shall consider \( J \) such that \( \lambda = a = 1 \). One can see for \( \tau = T/2c \) and \( q(\xi, \tau) = \frac{1}{\delta}f_{\xi} \), (48) becomes:

\[ q_{\tau} + 6\alpha q_{\xi} + 27\beta q_{\xi}^2 + q_{\xi\xi\xi} + Hq_{\xi\xi\xi\xi} = 0 \quad (49b) \]

In order to bilinearize the equation (49a) we shall use the following substitution:

\[ f(\xi, \tau) = i\log\frac{g_+(\xi, \tau)}{g_-(\xi, \tau)} \quad (50) \]
where
\[ g_-(\xi, \tau) = \prod_{n=1}^{N} (\xi - z_n(\tau)) \quad (51a) \]
\[ g_+(\xi, \tau) = \prod_{n=1}^{N} (\xi - z_n^*(\tau)) \quad (51b) \]
and \( \text{Im} z_n(\tau) \geq 0 \)

With this ansatz
\[ H_{f_\xi} = -\frac{\partial^2}{\partial \xi^2} \log (g_+g_-) \]
and introducing in (49a) the following expression appears:
\[ g_3^3 g_2^2 [(iD_\tau - D_\xi^2 + iD_\xi^3)g_+ \cdot g_-] - (\alpha - 1)g_2^2 g_1^2 (D_\xi g_+ \cdot g_-)^2 - 3ig_1^2 g_2^2 (D_\xi g_+ \cdot g_-) (D_\xi^2 g_+ \circ g_-) + i(2 - \beta)g_+g_- (D_\xi g_+ \cdot g_-)^3 = 0 \quad (52) \]
where \( D_\xi^n f \cdot g = (\partial_x - \partial_y)^n f(x)g(y) \mid_{x=y} \) are the Hirota bilinear operators.

If \( \alpha = 1, \beta = 2 \) and decouples (52) one gets:
\[ (iD_\tau - D_\xi^2 + iD_\xi^3)g_+ \cdot g_- = 0 \]
\[ D_\xi^2 g_+ \cdot g_- = 0 \]
which relaxes to
\[ (D_\tau + D_\xi^3)g_+ \cdot g_- = 0 \]
\[ D_\xi^2 g_+ \cdot g_- = 0 \]
and it is nothing but bilinear form for mKdV equation. There is no solution of the form (51a), (51b).

If \( \beta = 2 \) and \( \alpha \neq 1 \) one obtains:
\[ (iD_\tau - D_\xi^2 + iD_\xi^3)g_+ \cdot g_- = 0 \]
\[ \left( D_\xi^2 - \frac{i}{3}(\alpha - 1)D_\xi \right) g_+ \cdot g_- = 0 \quad (53) \]

The equation (53) is equivalent with
\[ \left( iD_\tau - \frac{i}{3}(\alpha - 1)D_\xi + iD_\xi^3 \right) g_+ \cdot g_- = 0 \]
\[ \left( D_\xi^2 - \frac{i}{3}(\alpha - 1)D_\xi \right) g_+ \cdot g_- = 0 \quad (53) \]
which represents the bilinear form for the following mKdV equation with non-vanishing boundary condition:
\[ u_\tau + 3(\alpha - 1)u^2 u_\xi + u_{\xi\xi\xi} = 0 \quad (54) \]
and \( u \to -1/3 \) when \( \xi \to \pm \infty \). This equation is completely integrable and it admits \( N \)-soliton solution. By the long-wave limit procedure of Ablowitz and Satsuma it admits also nonsingular real rational solutions in the form (51a) and (51b) and accordingly they are solutions of (49). Due to the fact that (54) admits rational nonsingular solutions for every odd \( N \), we can conclude that the set of rational
solutions for (49) is the same with the set of rational solutions for (54). The 1-rational and 3-rational solutions for (49a) are given by:

\[ g_{\pm}(\xi, \tau) = \theta(\xi, \tau) \mp \frac{3i}{\alpha - 1} + d \]

and

\[ g_{\pm}(\xi, \tau) = \theta(\xi, \tau)^3 + 12\tau - \frac{27}{(\alpha - 1)^2}\theta(\xi, \tau) \mp \frac{9i}{\alpha - 1}\left[\theta(\xi, \tau)^2 + \frac{9}{(\alpha - 1)^2}\right] + d \]

where \( \theta(\xi, \tau) = (\xi + (\alpha - 1)\tau/3 + c) \) and \( c, d \) are arbitrary constants. Thus, \( q = i\partial_t \log(g_+/g_-) \) is a nonsingular rational solution. We can see that only for \( N = 1 \) the solution is a solitary wave. The velocity is fixed and the 3-rational solution is not a superposition of solitary waves.

**Appendix**

In calculating \( F_k \) we start from

\[ F_3(k) = \int_0^\infty \left( \sum_{m=1}^{\infty} \frac{\sin(mk')}{m^2} \right) dk' \]

and the integral

\[ \sum_{m=1}^{\infty} \frac{\sin(mk)}{m^2} = -\int_0^k \ln(2 \sin \frac{t}{2}) dt. \]

We have

\[ F_3(k) = -\int_0^k dk' \int_0^{k'} \ln(2 \sin \frac{t}{2}) dt \]

\[ = -\int_0^k (k - t) \ln(2 \sin \frac{t}{2}) dt \]

\[ = -\frac{k^2}{2} \ln(2 \sin \frac{k}{2}) + \frac{3}{2} \int_0^k (kt - \frac{t^2}{2}) \cot \frac{t}{2} dt. \]

But

\[ \ln(2 \sin \frac{k}{2}) \simeq \ln k + \ln(1 - \frac{k^2}{24} + ...) \simeq \ln k - \frac{k^2}{24} + O(k^4) \]

and

\[ \cot \frac{t}{2} \simeq \frac{2}{t} \left(1 - \frac{t^2}{12} + O(t^4)\right). \]

Then, by straightforward integration, we get \( k > 0 \)

\[ F_3(k) = -\frac{k^2}{2} \ln k + \frac{3}{4}k^2 + \frac{1}{288}k^4 + O(k^5). \]
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