Tunable Coupler for Mediating Interactions between a Two-Level System and a Waveguide from a Decoupled State to the Ultra-Strong Coupling Regime

N. Janzen,1 X. Dai,1 S. Ren,1 J. Shi,1 and A. Lupascu1

1Institute for Quantum Computing, Department of Physics and Astronomy, and Waterloo Institute for Nanotechnology, University of Waterloo, Waterloo, ON, Canada N2L 3G1

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Two-level systems (TLS) coupled to waveguides are a fundamental paradigm for light-matter interactions and quantum networks. We introduce and experimentally demonstrate a method to tune the interaction between a TLS, implemented as a flux qubit, and a transmission line waveguide from a decoupled state to a coupling strength that is a significant fraction of the TLS transition frequency, near the ultra-strong coupling regime. The coupling, controlled via magnetic flux, is described by a normalized coupling strength $\alpha$ that is measured to range between $6.2 \times 10^{-5}$ and $2.19 \times 10^{-2}$, with larger attainable maximum values predicted by a circuit model of the device. This system enables future investigations in the dynamics of the spin-boson model, microwave photonics, and relativistic quantum information.

The investigation of light-matter interactions in the ultra-strong coupling (USC) regime [1, 2] enabled through specially designed electromagnetic environments is a growing area of research, with emerging applications that include the study of the spin-boson model [3], quantum computing [4, 5], and relativistic quantum information processing [6–11]. In superconducting devices, the USC regime was previously demonstrated for a two-level system (TLS) coupled to either single cavity modes [12], or to a continuum of modes [3].

Past implementations of strong coupling between a TLS and a continuum of modes in a waveguide were limited by the coupling range being either fixed or bound to a relatively narrow range. In this paper, we propose and demonstrate a device capable of mediating the light-matter coupling between a two-level system and a waveguide, achieving a large range of interaction strengths from a decoupled state to coupling that approaches the USC regime. The normalized coupling strength $\alpha$ is measured to range from $6.2 \times 10^{-5}$ to $2.19 \times 10^{-2}$, which enables exploring the dynamics of the interactions over a span of several qualitatively distinct regimes, including near the USC regime where the relaxation rate becomes comparable to the splitting frequency and the rotating wave approximation (RWA) breaks down [13, 14]. The design of the coupler is compatible with fast-switching of the interaction, enabling new experiments in spin-boson model physics [2] and relativistic quantum information [8, 9].

The system we investigate is formed of a TLS, implemented by a superconducting persistent current qubit (PCQ), coupled to a waveguide, realized using a superconducting coplanar waveguide transmission line (TL). The device is shown schematically in Fig. 1(a). It is formed of two superconducting loops with six Josephson junctions. The qubit loop, biased with magnetic flux $f_q$, implements the TLS. The coupling loop, biased with the flux $f_\beta$, effectively acts as a tunable inductor between the qubit loop and the transmission line carrying current $I_{TL}$. Qualitatively, the interaction between the qubit loop and the TL is analogous to previously explored devices used to tune interactions between two flux qubits [15, 16]. The tunable inductance between the TLS and the TL is given by $M_{eff} = M_{tq}M_{tl}/L_\beta$, where $M_{tq}$ is the mutual inductance between the coupling loop and TLS (TL) and $1/L_\beta$ is the susceptibility of the coupling loop. The susceptibility is $1/L_\beta = \frac{1}{\Phi_0} \partial f_q / \partial f_\beta$, with $I_q$ the coupler loop ground state current, and can be tuned by changing $f_\beta$. Further, $M_{tq}$ and $M_{tl}$ are predominantly supplied by the inductance of the shared junctions 4 and 5 respectively.

We note that while previous work [17] discussed switch-
able coupling circuits, our design is relatively simpler as it requires only one flux control to tune the coupling.

The TLS, which is identified based on the lowest two energies of the double loop device, has the effective Hamiltonian

\[ H = -\frac{\hbar \Delta}{2} \tau_x - \frac{\hbar c}{2} \tau_z, \]  

where \( \Delta \) is the minimum gap, \( \tau_x \) and \( \tau_z \) are the Pauli matrices, and \( \epsilon = 2I_p\Phi_0(f - f_{\text{sym}})/\hbar \) with \( I_p \) the persistent current in this loop. This Hamiltonian is expressed in the flux basis, formed of states with opposing currents in the qubit loop. The flux \( f_{\text{sym}} \) is the setting for which the potential energy defining the TLS has a symmetric double well shape. For a standard PCQ, formed of a single loop with three junctions, \( f_{\text{sym}} = 0.5 \). Our device has a symmetry point that is in general not at 0.5 biasing, due to the fact that the coupling loop effectively induces an additional flux component in the qubit loop. The qubit transition frequency close to the symmetry point is given by

\[ \omega_{10} = \sqrt{\Delta^2 + \epsilon^2}. \]  

The Hamiltonian of the interacting qubit and transmission line is \( H_{\beta\beta} = H_s + H_b + H_{\text{int}}, \) with \( H_s, H_b, \) and \( H_{\text{int}} \) the qubit, TL, and interaction Hamiltonians. We have

\[ H_s = -\frac{\hbar \omega_{10}}{2} \sigma_z, \]  

\[ H_b = \sum_k \hbar \omega_k a_k^\dagger a_k, \]  

\[ H_{\text{int}} = \sum_k (g_k^x \sigma_x + g_k^z \sigma_z)(a_k^\dagger + a_k). \]  

In these equations \( \omega_{10} \) is the transition frequency of the two-level system (TLS), \( \sigma_z \) and \( \sigma_x \) are the system’s Pauli operators in the energy eigenbasis, \( \omega_k, a_k^\dagger, a_k \) are frequency, creation, and annihilation operators of the \( k \)-th mode of the bosonic bath, and \( g_k^{x(z)} \) are the transverse (longitudinal) coupling of the TLS to mode \( k \). More details of the derivation of the Hamiltonian is provided in the Ref. [18].

When the qubit is at the symmetry point, the interaction with the TL is dominantly transverse and the Hamiltonian becomes the standard spin-boson model. The environment coupling is captured by the bath spectral density function \( J(\omega) = \pi \alpha \omega \), where \( \alpha \) is the dimensionless coupling factor to the bath. For small values of \( \alpha \ll 1 \) at the symmetry point, the TLS radiative relaxation rate \( \Gamma_1 \) is given by

\[ \Gamma_1 = \pi \alpha \Delta. \]  

A scanning electron microscope (SEM) image of the device, shown in Fig. 1(b), depicts the qubit loop, the coupling loop, and the TL of the circuit. The six Josephson junctions in the two loops are each set to have a specific critical current controlled through its junction area. With a target critical current density of 3 \( \mu A/\mu m^2 \), the junction areas are designed to be \( A_1 = A_3 = 1.69 A_0, \) \( A_2 = 0.58 A_1, \) \( A_4 = A_5 = 3 A_0, \) and \( A_6 = 0.52 A_4, \) where \( A_0 = 0.0467 \mu m^2 \). The fluxes through the superconducting loops are controlled via two bias lines, and can be operated with both DC currents and pulses to fast-tune the coupling of the system. The bottom line is designed to bias the qubit loop and interacts through a simulated mutual inductance of \( M_c = 0.25 \) pH while the top bias line interacts with the coupler loop through a simulated mutual inductance \( M_\beta = 0.47 \) pH. The device includes a DC-SQUID, which is inductively coupled to the qubit loop and can be used for readout of the TLS. The switching pulses and readout system were not required for this experiment. The full device and the schematic of setup are displayed in Fig. 1(c). The bias line currents are controlled by external voltage sources and the transmission through TL is measured with a vector network analyzer (VNA).

In our device, which has multiple bias lines and superconducting loops in close proximity, there is unavoidable flux crosstalk during operation. Measuring crosstalk is needed in order to enable independent control of \( f_\beta \) and \( f_\epsilon \). The system is calibrated by measuring the periodicity of the transmission through the TL at a single frequency versus the currents along the two bias lines. Voltage applied to each line generates a current that induces flux through both loops, which tunes the TLS transition frequency \( \omega_{10} \) periodically. When the detuning, \( \delta = \omega_p - \omega_{10} \), between the probe frequency and the qubit gap approaches zero, the photons are absorbed producing a visible dip in the transmission \( |S_{21}| \). These dips form a closed contour when performing a sweep of \( f_\beta \) and \( f_\epsilon \). These contours occur near \( (f_\beta, f_\epsilon) = (0.5 + n, 0.5 + m) \) for integer values of \( n \) and \( m \).

The fluxes in the loops are given by

\[ \begin{pmatrix} f_\beta \\ f_\epsilon \end{pmatrix} = \begin{pmatrix} W_{\beta\beta} & W_{\beta\epsilon} \\ W_{\epsilon\beta} & W_{\epsilon\epsilon} \end{pmatrix}^{-1} \begin{pmatrix} I_\beta - I_{0,\beta} \\ I_\epsilon - I_{0,\epsilon} \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}, \]  

where \( W_{ij} \) are the elements of the crosstalk matrix, \( I_{c(i)} \) is the applied current in the bottom (top) bias line, and \( I_{0,\beta(i)} \) are the offset currents corresponding to the circuit biased at \( (f_\beta, f_\epsilon) = (0.5, 0.5) \). The crosstalk matrix is determined by scanning over a wide range of currents, and then using an image analysis routine to determine the periodicity of the data [19]. In addition, image inversion symmetries are used to measure the current offset. Fig. 2(a) shows the transmission through the TL for \( \omega_p = 7.5 \) GHz versus applied currents. We extract the translation vectors \( \mathbf{W}_1 = (W_{\beta\beta}, W_{\beta\epsilon}) \) and \( \mathbf{W}_2 = (W_{\beta\epsilon}, W_{\epsilon\epsilon}) \), which correspond to increasing the flux bias by one flux quantum in the qubit and coupling...
loop respectively. Fig. 2(b)-(c) shows additional scans, with a different offset, that illustrate how the contours change with frequency.

In contrast to the case of standard flux qubits, \( f_{c,sym} \) is not generally at \( f_c = 0.5 \) due to an effective bias resulting from the neighbouring persistent current in the coupling loop. We can determine \( f_{c,sym} \) for each \( f_\beta \) by measuring the \( f_c \) dependent qubit transition frequency from transmission spectroscopy and then fitting the Eq. (2) model as shown by Fig. 3(a)-(c). We note that the flux in the coupling loop affects the minimum transition energy \( \Delta \) and, to a lesser extent, the persistent current \( I_p \) of the TLS.

We develop a realistic model of the device that includes the full capacitance matrix between the metallic islands in the two loops and the surrounding ground plane [18]. This model is used to generate, through numerical diagonalization, the energy eigenstates of the system. These are fit to the spectroscopically determined transition frequencies collected over a range of \( f_\beta \) and \( f_c \) values, with the Josephson junction critical currents as fit parameters. We make the assumption that the capacitance matrix can be accurately determined based on electromagnetic simulations, whereas Josephson junction critical currents have significant uncertainties arising from the fabrication process. As shown in Fig. 3(d), the best fit is in excellent agreement with the data, which is a compelling result given the relatively high complexity of the circuit, containing six Josephson junctions. Further validation of the model is provided by comparing the predicted transition frequency with the spectroscopy data taken during the calibration measurements (see Fig. 2(b)-(c)).

Next, we focus on the transmission through the TL while setting the flux biases at symmetry points, corresponding to transverse coupling. As shown in previous work [3, 20], the transmission linewidth and the minimum transmission on resonance are related to the radiative loss from the qubit to the TL, as well as other loss channels coupled to the qubit. We extend previous work by including both the effects of finite temperature and large drive amplitude [18]. With these extensions, the transmission is given by

\[
t = \frac{1 - r_0 + \delta^2 T_2^2 + 2 T_1 T_3 \Gamma_1 N_{in} + i r_0 \delta T_2}{1 + \delta^2 T_2^2 + 2 T_1 T_3 \Gamma_1 N_{in}},
\]

where \( \delta = \omega_p - \Delta \) is the detuning of the probe frequency from the qubit gap, \( T_{1(2)} \) is the qubit energy relaxation (dephasing) time due to all possible noise channels, \( \Gamma_1 \) is the relaxation rate to the TL in the zero temperature limit, and \( N_{in} \) is the average number of incoming photons per second in the driving tone. The minimum transmission occurs when the probe frequency is on resonance and the incoming photon number \( N_{in} \) is small, and is given by \( 1 - r_0 \) where

\[
r_0 = \frac{1}{2} \frac{1 - b}{1 + b} T_2 \Gamma_1.
\]

Here \( b = \exp(-\hbar \Delta / k_B T) \), where \( T \) is the qubit effective temperature. It can be shown that \( T_2 \to 2/\Gamma_1 \) when the qubit temperature approaches zero and decoherence due to noises other than the TL becomes negligible and
Thus $r_0$ goes to 1 in this limit. Under the condition of strong coupling between the TLS and the TL and at a very low temperature, it is expected that we should reach this limit where the transmission through the TL goes to zero.

The transmission data at the symmetry point for a range of $f_\beta$ values and powers $P$, is fit to Eq. (8) to extract $\Gamma_1$, as well as other relevant parameters. For each $f_\beta$, the transmission at different power is fitted simultaneously, with the rate of input photon given by $N_{in} = 10^{(P-A)/10-3}/\hbar\omega_p$, where $A$ is the attenuation. The parameters $T$ and $A$ for different $f_\beta$ are treated as independent parameters, and are found to be consistent with the estimates of the noise temperature from the bias line and the attenuation along the signal delivery chain [18]. An example of the fit is shown in Fig. 4 for $(f_\beta, f_r) = (0.41, 0.433)$.

The coupling between the TL and the TLS is characterized and compared to the circuit model simulation. In Fig. 5(a), plots for both the qubit splitting $\Delta$ and $f_{\epsilon, \text{sym}}$ versus $f_\beta$ are shown, which are predicted well by the circuit model. The zero temperature relaxation rate $\Gamma_1$, a measure of the coupling strength, is shown in Fig. 5(b). Finally, the normalized coupling strength $\alpha$ is plotted in Fig. 5(c) following Eq. (6). The minimum measured value of $\Gamma_1 = 8.7$ MHz is at $f_\beta = 0.36$ while the maximum measured value is $\Gamma_1 = 1.85$ GHz at $f_\beta = 0.44$. These produce a range of values for $\alpha$ where $\alpha_{\text{min}} = 6.2 \times 10^{-5}$ and $\alpha_{\text{max}} = 2.19 \times 10^{-2}$. Although there is some deviation from the expected coupling, we still expect the maximum coupling can reach the USC regime at $\alpha \sim 0.1$ [1, 3] by projecting the measured data to $f_\beta = 0.5$. The circuit model also predicts that the minimum coupling is likely well below the smallest experimental value. At the decoupling point ($f_\beta \approx 0.365$), the interaction strength is very small causing any dip in transmission to have a narrow width and become dominated by the residual dephasing, thus no smaller coupling is measured. This result demonstrates that we can effectively turn the coupling on or off by changing the flux bias through the coupling loop.

Regarding the relaxation and decoherence due to non-radiative sources, we make estimates based on the typical flux noise in superconducting flux qubits as well as the thermal noise due to the bias lines. We find that these estimates agree within typical noise parameters and are not expected to affect the performance of the device [18]. We observed a discrepancy between the measured and predicted coupling strength at large coupling strength. We considered reflections in the transmission lines outside the device as a potential cause [18]; we find that very large reflection values, not realistic for our setup, would be necessary to explain this discrepancy.

In summary, we implemented a system that tunably couples a TLS to a waveguide from a decoupled state near the USC regime. This coupler enables several avenues for future research using fast-switching of the TLS-waveguide interaction. Fast switching can be used to directly measure the light-matter time dynamics in the USC regime by implementing experiments proposed in Ref. [21]. Further, the device has application in photon packet production where the TLS can be excited and then rapidly coupled to the TL to emit shaped single photons [22, 23]. Finally, the device can be used for relativistic quantum information experiments where two of the couplers allow spatially separated qubits to independently interact with a field over very short time scales to demonstrate entanglement harvesting from a vacuum [10, 11].

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FIG. 5. The extracted (dots) versus simulation (continuous lines) of key parameters for the coupled system. For each $f_\beta$ value, $f_\alpha$ is set to the corresponding symmetry point. (a) The qubit gap frequency. The inset shows values for $f_{\alpha,\text{sym}}$. (b) The relaxation rate $\Gamma_1$ measured by fitting spectroscopy data using Eq. (8). (c) The coupling $\alpha$ between the TLS and the TL. The measured coupling ranges from $\alpha_{\min} = 6.2 \times 10^{-5}$ and $\alpha_{\max} = 2.19 \times 10^{-2}$.

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Supplemental Information for “Tunable Coupler for Mediating Interactions between a Two-Level System and a Waveguide from a Decoupled State to the Ultra-Strong Coupling Regime”

N. Janzen,1 X. Dai,1 S. Ren,1 J. Shi,1 and A. Lupascu1
1Institute for Quantum Computing, Department of Physics and Astronomy, and Waterloo Institute for Nanotechnology, University of Waterloo, Waterloo, ON, Canada N2L 3G1
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I. SAMPLE PREPARATION AND MEASUREMENT SETUP

The device is fabricated using a processes that contains 3 aluminium layers consisting of waveguides, Josephson Junctions, and bridges/bandages respectively. The waveguide layer is patterned using negative photo resist on a silicon wafer, followed by evaporating 100 nm of aluminum. The junction layer, which includes the qubit and the coupling loops, is made using standard double angle shadow evaporation with two aluminium depositions of thickness 40 nm and 70 nm respectively. The final layer serves a dual purpose of creating air bridges to act as ground plane interconnects as well as a layer for generating bandages to ensure galvanic contact between the circuit layer and the junction layer. For this step, argon milling is first used to galvanically connect the deposited aluminium to the previous layers. Then, 450 nm of aluminium is deposited on PMMA scaffolds created with gray scale e-beam lithography [1].

The device is mounted in a copper sample box placed at the mixing chamber plate of a dilution refrigerator with a base temperature of 30 mK. The sample is placed inside a magnetic shield formed of three layers of paramagnetic material. Microwave components near the sample are chosen to be non-magnetic to minimize potential noise and offset magnetic flux at the device. The signals sent to and received from the device along the transmission line (TL) are each filtered through a 4-8 GHz band pass filter to reduce quasi particle noise from outside this bandwidth. The TL is connected to an Agilent E5071C vector network analyser for transmission measurements.

The on chip bias lines are connected to both DC and microwave signals, which are combined using a bias-tee placed at the mixing chamber of the dilution refrigerator. The microwave lines are designed for pulses used to fast-switch the coupler in and out of the USC regime for future experiments, but are not required for validation of the coupling range. The DC lines are connected to Yokogawa 7651 external power supplies. The currents applied through the bias lines are larger than initially anticipated, due to misestimates of the mutual inductances and the malfunctioning of a global flux control. This led to significant heating of the DR during operation, with temperature reaching \( \sim 50 \) mK during the transmission measurements. This heating effect can be prevented in future work by replacing the bias wires with superconducting wires.

II. SPIN-BOSON MODEL FOR SUPERCONDUCTING CIRCUITS

In this section, we discuss details of the derivation of the coupling between a two-level system and a transmission line (TL), and its correspondence with the spin-boson model [2]. A description of this form has been used in previous literature [3–5].

The spin-boson model Hamiltonian is given by

\[
H_{SB} = H_s + H_b + H_{int},
\]

where \( H_s \) is the Hamiltonian for an atom implemented as a TLS, \( H_b \) is the Hamiltonian for a bath of bosonic modes, and \( H_{int} \) is the Hamiltonian describing their interaction. We have

\[
H_s = -\frac{\hbar \omega_{10}}{2} \sigma_z, \tag{2}
\]

\[
H_b = \sum_k \hbar \omega_k a_k^\dagger a_k \quad \text{and} \quad \tag{3}
\]

\[
H_{int} = \sum_k (g_k^x \sigma_x + g_k^z \sigma_z) (a_k^\dagger + a_k), \tag{4}
\]

where \( \omega_{10} \) is the transition frequency of the TLS, \( \sigma_z \) and \( \sigma_x \) are the system’s Pauli operators in the energy eigenbasis, \( g_k^x, g_k^z \) are the transverse and longitudinal coupling strength of the \( k \)th mode of the bosonic bath, and \( \omega_k, a_k^\dagger, a_k \) are frequency, creation, and annihilation operators of the \( k \)th mode of the bosonic bath.

In the Born-Markov limit, the interaction leads to qubit relaxation with the rate

\[
\Gamma_1 = \frac{2\pi}{\hbar^2} \sum_k g_k^2 \delta(\omega - \omega_k) = J(\omega), \tag{5}
\]

where \( \Gamma_1 \) is the relaxation rate when the bath temperature is zero, which we also associate with the bath spectral density \( J(\omega) \). For an ohmic bath, a case that applies to our open transmission line, \( J(\omega) \) can be written in terms of the dimensionless coupling constant \( \alpha_{SB} \)

\[
J(\omega) = \pi \alpha_{SB} \omega. \tag{6}
\]
The spin-boson model is realized with a superconducting flux qubit tunably coupled to an open transmission line. The Hamiltonian is given by

\[ H = \int_{-L/2}^{L/2} \left\{ \frac{q(x)^2}{2c_0} + \frac{\partial_x \Phi(x)^2}{2l_0} \right\} dx + H_c + H_{\text{int}}. \] (7)

Here \( q(x) \) and \( \Phi(x) \) are the charge and phase at position \( x \) along the line, \( c_0 \) and \( l_0 \) are the characteristic capacitance and inductance of the line, and \( L \) is the length of the TL. The component \( H_c \) is the coupler circuit Hamiltonian consisting of the capacitive and Josephson energies in the coupling and qubit loops, as described in Sec. III. The interaction Hamiltonian is given by

\[ H_{\text{int}} = \frac{1}{l_0} \gamma_5 \phi_0 \partial_x \Phi(x = 0), \] (8)

which describes the coupling between the harmonic modes and the tunable coupler circuit via the Josephson phase of the shared junction, \( \gamma_5 \), assuming the coupler circuit is at \( x = 0 \).

The field in the TL can be quantized and the interaction is given by [4]

\[ H_{\text{int}} = \frac{1}{l_0} \phi_0 \gamma_5 \sum_k \frac{\hbar}{2c_0 \omega_k L} \left( i a_k e^{i(kx - \omega_k t)} - i a_k^\dagger e^{-i(kx - \omega_k t)} \right), \] (9)

where angular wavenumber \( k \) is related to the frequency via \( k = \omega_k / v \), with \( v = \sqrt{1/(l_0 c_0)} \) the speed of light in the TL. Comparing Eq. (9) with the spin-boson interaction Hamiltonian the transverse coupling constant is obtained as

\[ g_k^x = \frac{1}{l_0} \phi_0 |\gamma_{5, 10}| \frac{1}{\sqrt{2 \pi}} \sqrt{\frac{\hbar \omega_k}{2c_0 v^2}}, \] (10)

where \( \gamma_{5, 10} = \langle 1 |\gamma_5|0 \rangle \) is the off-diagonal matrix element between the first excited state and the ground state of the \( \gamma_5 \) phase operator. Finally, to obtain the spectral density or equivalently the relaxation rate at the symmetry point, one takes the limit \( L \rightarrow \infty \), replaces the summation in Eq. (5) with an integral (see Supplementary information of Ref. [4]), and uses the minimum qubit transition frequency \( \Delta \), which gives

\[ \Gamma_1 = \frac{\phi_0^2}{\hbar Z_0} |\gamma_{5, 10}|^2 \Delta, \] (11)

where \( Z_0 = \sqrt{l_0 / c_0} \) is the characteristic impedance of the line.

Besides the transverse coupling, the coupling loop also mediates a longitudinal coupling between the qubit and the TL, even when the qubit is at the symmetry point. The ratio of the coupling strengths satisfies

\[ \frac{|g_k^z|}{g_k^x} = \frac{2|\gamma_{5, 10}|}{|\gamma_{511} - \gamma_{500}|}. \] (12)

and is plotted in Fig. 1. It can be seen that the transverse coupling strength \( g_k^x \) dominates except near \( f_\beta \approx 0.365 \), where \( g_k^z \rightarrow 0 \). We therefore expect the longitudinal coupling to have small influence on the tunable coupler device when operated in the strong to ultra-strong coupling limit. However, we note that this longitudinal coupling leads to pure dephasing, which we discuss in more detail in Sec. V.

### III. TUNABLE COUPLER CIRCUIT MODEL

In this section, we discuss the Hamiltonian of the coupler-qubit circuit. The phases across the junction are constrained by the fluxoid quantization condition, given by

\[ \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + 2\pi f_x = 0 \quad \text{and} \quad \gamma_4 + \gamma_5 + \gamma_6 - 2\pi f_\beta = 0, \] (13)

where \( \gamma_i \) is the phase across junction \( i \), and \( f_{x(\beta)} \) is the flux quanta through the qubit (coupling) loop. The circuit Hamiltonian is given by

\[ H = \frac{1}{2\phi_0} \mathbf{p}^T C^{-1} \mathbf{p} + U, \] (14)

where \( C \) is the capacitance matrix of the system, \( \mathbf{p} \) is the momentum vector that can be described in terms of the charge number \( p_i = \hbar n_i \) for each island of the superconducting circuit and \( U \) is the potential energy stored in the Josephson junctions. The potential energy
is given by

\[
U = -\phi_0 [I_{c1} \cos \gamma_2 + I_{c3} \cos \gamma_2 + I_{c5} \cos(\gamma_1 + \gamma_2 + \gamma_3 + 2\pi f_\phi) + I_{c4} \cos \gamma_4 + I_{e5} \cos \gamma_3 + I_{e6} \cos(\gamma_4 + \gamma_5 - 2\pi f_\phi)].
\]

(15)

The Hamiltonian in Eq. (14) can be represented in the charge basis. A suitably truncated representation, with a large enough number of charge states, is used to numerically calculate relevant properties, including transition frequencies, matrix elements, and symmetry points. Using this numerical model, we fit the transition frequencies \( \omega_{10} \) versus the applied fluxes \( f_\beta \) and \( f_\phi \). The Josephson junction critical currents \( I_{cj} \) are taken as fit parameters whereas the capacitance matrix is taken as a constant because the the junction critical currents have a much larger variance as a result of the fabrication process. We fit the junction critical currents to be \( I_{c1} = 0.236 \) mA, \( I_{c2} = 0.131 \) mA, \( I_{c3} = 0.236 \) mA, \( I_{c4} = 0.411 \) mA, \( I_{c5} = 0.584 \) mA, and \( I_{c6} = 0.1846 \) mA. Most of the junction values are in reasonable agreement to their design target with an error of < 12%, except for \( I_{c5} \), which was designed to be the same size as \( I_{c3} \) and has an error of approximately 40%. This discrepancy had an effect on the symmetry points \( f_\phi, f_\beta \) sym, but a negligible effect on other metrics.

IV. INFLUENCE OF FINITE TEMPERATURE AND RESIDUAL DECOHERENCE ON THE SCATTERING PARAMETER

Our expression for the transmission can be obtained by extending the derivation in Ref. [6] to include explicitly the temperature of the qubit, as well as thermalization and dephasing due to noise channels other than the TL.

Following Eq. (47, 50) of Ref.[6], we write the input and output field from the left of the of the qubit as

\[
V_{\text{in}} = \Omega_p \sin(\omega_p t),
\]

(16)

\[
V_{\text{out}} = \frac{1}{2} \sqrt{Z_0 \hbar \omega_{10} \Gamma_1} \langle \sigma_x \rangle,
\]

(17)

where \( \omega_p \) is the drive frequency and \( \langle \sigma_x \rangle \) is the expectation value of the Pauli operator. The input field acts as a drive on the qubit, inducing Rabi oscillations with frequency \( \omega_r \). The Rabi frequency can be related to the input voltage amplitude \( \Omega_p \) via the coupling strength between the TL and the TLS in terms of \( \Gamma_1 \), given by

\[
\omega_r = \frac{\Omega_p \phi_0}{\hbar Z_0} |\gamma_{5,10}| \]

(18)

\[
= \Omega_p \sqrt{\frac{\Gamma_1}{Z_0 \hbar \omega_{10}}}.
\]

(19)

Based on Ref. [6], the finite temperature and residual decoherence only affects the output field via a change of the equilibrium state of the qubit only. The equilibrium state of the qubit can be found by solving a generic master equation including relaxation, dephasing and a coherent drive on the qubit. The master equation is given by

\[
\dot{\rho} = -\frac{i}{\hbar}[H_{\text{in}} + H_{\text{drive}}, \rho] + \Gamma_1 D(\sigma_+) \rho + \Gamma_0 D(\sigma_-) \rho + \frac{1}{2} \Gamma_\phi D(\sigma_z) \rho,
\]

(20)

where \( \Gamma_1, \Gamma_0 \), and \( \Gamma_\phi \) are the noise induced excitation, relaxation, and pure dephasing rates respectively due to all sources of noise including coupling to the open TL, and \( D(\sigma_+), D(\sigma_-), \) and \( D(\sigma_z) \) are the corresponding Lindblad-form super-operator. We also introduce \( \Gamma_0/\Gamma_1 = b = \exp(-\hbar \omega_{10}/k_B T) \), where \( T \) is the qubit effective temperature, defined in terms of the detailed balance of the total relaxation and excitation rates. The system Hamiltonian is given by

\[
H_{\text{in}} = -\frac{\hbar \omega_{10}}{2} \sigma_z,
\]

(22)

and the drive Hamiltonian is

\[
H_{\text{drive}} = \hbar \omega_r \sigma_y \sin(\omega_p t).
\]

(23)

Employing the rotating wave approximation we can find the steady-state solution of the master equation. In particular, the \( \langle \sigma_x \rangle \) expectation value is given by

\[
\langle \sigma_x \rangle = \frac{b - 1}{1 + \delta^2 T_2^2 + T_1 T_2 \omega_p^2} \sin(\omega_p t - \delta T_2 \cos \omega_p t),
\]

(24)

where \( \delta \) is the detuning \( \omega_p - \omega_{10} \) and we have defined the total relaxation and dephasing time, given respectively by

\[
T_1 = \frac{1}{\Gamma_0 + \Gamma_1} \quad \text{and} \quad T_2 = \frac{1}{\Gamma_\phi + 1/(2T_1)},
\]

(25)

(26)

Finally the reflection coefficient is given by

\[
r = -r_0 \frac{1 - i s T_2}{1 + \delta^2 T_2^2 + 2T_1 T_2 \Gamma_1 N_{\text{in}}},
\]

(27)

where we have defined the average number of incoming photons

\[
N_{\text{in}} = \frac{\Omega_p^2}{2Z_0 \hbar \omega_{10}},
\]

(28)

and

\[
r_0 = \frac{1}{2} \frac{1 - b}{b + 1} T_2 \Gamma_1.
\]

(29)

Substituting Eq. (27) into \( t = 1 + r \) allows us to obtain the transmission coefficient \( t \), which is used to fit the measured \( S_{21} \) after removing the background transmission. We note that in the zero temperature limit, the expression for the reflection Eq. (27) reduces to the form derived in Ref. [6] (see Eq. (53) of the reference).
V. PURE DEPHASING AND RESIDUAL RELAXATION

In this section we discuss the qubit relaxation and dephasing rates extracted from the transmission fit, as well as noise channels that could account for the rates.

We first discuss the pure dephasing rates. As mentioned in Sec. II, the TL couples to the qubit longitudinally even when the qubit is at the symmetry point. Low frequency noise in the TL hence causes dephasing, with the dephasing rate given by

\[ \Gamma_\phi^{\text{TL}} = \frac{\phi_0^2}{\hbar^2} |\gamma_{5,11} - \gamma_{5,00}|^2 S_f(0) \]  

(30)

where \( S_f(0) \) is the current noise in the TL at DC. The current noise is given by the Johnson-Nyquist noise of the impedance \( Z_0 \) from the TL input, noting the output has a bandpass filter which is shorted to ground at DC. Therefore, the dephasing due to the TL is given by

\[ \Gamma_\phi^{\text{TL}} = \frac{2k_B T \phi_0^2}{\hbar^2} |\gamma_{5,11} - \gamma_{5,00}|^2, \]  

(31)

and it is plotted versus \( f_3 \) in Figure. 2(a). The estimated pure dephasing due to the TL agrees well with the dephasing rate found by the transmission fits. We note that pure dephasing due to the TL can be significantly reduced by adding a DC block, without changing relevant aspects of the TLS-waveguide interactions significantly.

Another important source of dephasing for flux tunable devices is the intrinsic 1/f flux noise. We assume the flux noise is of the form \( A_{i,\beta}/(\omega/2\pi) \). The noise amplitudes are \( \sqrt{A_\alpha} = 1.2 \mu \Phi_0/\sqrt{\text{Hz}} \), \( \sqrt{A_\beta} = 1.1 \mu \Phi_0/\sqrt{\text{Hz}} \), estimated based on scaling the geometry of the loops compared to a previous device fabricated using similar process [7]. The estimated dephasing due to 1/f flux noise is plotted in Figure. 2(a), and found to be much smaller than the dephasing extracted from the transmission fits.

We next discuss qubit relaxation, which we distinguish between radiative relaxation to the TL and non-radiative relaxation to all other noise sources. The qubit relaxation and excitation rates are separated into

\[ \Gamma_{10(01)} = \Gamma_{10(01)}^{\text{nr}} + \Gamma_{10(01)}^{\text{nr}}. \]  

(32)

It needs to be noted that while the transmission line is likely thermalized at the DR base temperature, the bias lines have limited attenuations along the signal delivery chain and are likely to have much higher noise temperature. Hence the qubit effective temperature, extracted from the transmission fits, is a result of balancing the radiative and non-radiative thermalization rates and temperatures of the respective noise environment. We assume that the TL is in thermal equilibrium with the DR, at temperature \( T_{\text{TTL}} = 50 \text{ mK} \). This gives

\[ \Gamma_0^{\text{nr}} = \frac{\Gamma_{10}^{\text{nr}}}{\exp(h\omega_{01}/k_B T_{\text{TTL}}) - 1}. \]  

(33)

Assuming the noise channels other than the TL is also in thermal equilibrium at some temperature \( T_{\text{nr}} \), we have

\[ \frac{\Gamma_{10}^{\text{nr}}}{\Gamma_0^{\text{nr}}} = \exp \left( -\frac{h\omega_{01}}{k_B T_{\text{nr}}} \right) \]  

(35)

The extracted residual relaxation \( \Gamma_{10}^{\text{nr}} \) is shown in Fig. 2(b). We note that the residual noise temperatures found by the transmission fits is on the order of hundreds of mK, consistent with the estimated noise temperature based on the attenuation applied along the bias lines.

We next consider the possible causes of the residual relaxation. One important relaxation channel is the flux bias lines which is necessary for fast switching operations. The relaxation rate is proportional to the square of the mutual coupling between bias lines and the flux bias matrix elements \( \langle 0|\partial H/\partial f_i|1 \rangle, \langle 0|\partial H/\partial f_3|1 \rangle \). Using the mutual inductance values found from the calibration measurement, the calculated relaxation rate due to bias line thermal noise is shown in Fig. 2(b). It can be seen that this noise channel introduces relaxation rate of about 100 kHz, much smaller than the relaxation found from the transmission fits.

Another possible source of relaxation is intrinsic flux noise, which in general follows the 1/f spectrum up to a few GHz. Using the same noise amplitudes used for the dephasing estimates, we found the relaxation due to intrinsic flux noise below 10 kHz, which is negligible.

Finally we consider relaxation due to quasiparticle tunneling across the junctions. Following Ref. [8], the relaxation rate is given by

\[ \Gamma_{10}^{\text{qp}} = \sum_{i=1}^{6} \left| \langle 0|\sin(\gamma_i/2)|1 \rangle \right|^2 \frac{8x_{\text{qp}}E_{I_i}}{\hbar\pi} \left( \frac{2\Delta_{\text{Al}}}{h\omega} \right)^\frac{1}{2}, \]  

(36)

where \( x_{\text{qp}} \) is the quasiparticle density normalized by the density of superconducting electrons, \( E_{I_i} = I_c\phi_0 \) is the Josephson energy of junction \( i \) and \( \Delta_{\text{Al}} \) is the superconducting gap of Al. Assuming a modest value of the normalized quasiparticle density \( x_{\text{qp}} = 5 \times 10^{-7} [9, 10] \) on all qubit islands, and using the tunneling matrix elements \( \langle 0|\sin(\gamma_i/2)|1 \rangle \) on all six junctions computed from the circuit model, we found that the quasiparticle tunneling induced relaxation is about an order of magnitude smaller than the residual relaxation found from fitting the qubit transmission data (see Figure. 2(b)). However, it is possible that our device has a much larger quasiparticle density, due to the relatively high noise temperature on the bias lines or other sources. We expect future iterations of the experiments would have improved signal delivery setup, which could reduce the noise temperature and hence improve the qubit relaxation times.

VI. TRANSMISSION LINE MODES WITH FINITE REFLECTION

As discussed in Section. II, the spin-boson model can be obtained from the system of flux qubit coupled to...
The qubit is placed at position \( z = 0 \) on the TL. The coupling strength is obtained based on the assumption of an infinite TL. This assumption is valid when there is no reflection in the TL, or the signal becomes significantly attenuated before reflection occurs. In our experiment setup, the filters on the output line can cause significant reflection. In this section, we discuss how finite reflection affects the TL modes and estimate the reflection required to explain the discrepancy of \( \Gamma \) presented in the main text.

We model the filters and the entire microwave signal chain behind it as a generic component labeled as \( F \), with scattering parameter \( S^F \), placed at \( z = 0 \) on the TL. The qubit is placed at position \( z = z_q \). In Fig. 3 the schematic of this model is shown, with TL length \( d \). To simplify the analysis we will also assume that the component \( F \) is reciprocal and lossless.

The scattering parameter of the full system is related to \( S^F \) by a shift of reference plane, given by

\[
S(\omega) = \begin{pmatrix} e^{-2i\omega d/v} S^F_{11} & e^{-i\omega d/v} S^F_{12} \\ e^{-i\omega d/v} S^F_{21} & S^F_{22} \end{pmatrix},
\]

where \( \omega \) the angular frequency and \( v \) the speed of light.

In line with the periodic boundary condition, the allowed modes need to satisfy \( V_1^+ = V_2^- \), \( V_1^- = V_2^+ \). This leads to

\[
\begin{pmatrix} V_1^- \\ V_2^- \end{pmatrix} = S(\omega) \begin{pmatrix} V_1^+ \\ V_2^+ \end{pmatrix},
\]

and the allowed frequencies need to satisfy

\[
\det \left[ S(\omega) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = 0.
\]

Solving the above equation gives the allowed frequencies

\[
\omega_{s,n} = \frac{\theta v}{d} + (-1)^s \frac{v}{d} 2\pi n
\]

where \( n \) is an integer, \( s \in 0, 1 \) indicates the parity of the mode, and \( \theta \) is a factor which goes to zero when there is no reflection at \( F \). The allowed modes are in general superpositions of left and right travelling waves with their amplitudes satisfying

\[
S_{11}(\omega_s) V_{s,1}^- + (S_{12}(\omega_s) - 1) V_{s,1}^+ = 0,
\]

\[
\frac{V_{s,1}^-}{V_{s,1}^+} = \frac{S_{11}(\omega_s)}{1 - S_{12}(\omega_s)}.
\]

To simplify notation, the subscript \( n \) is dropped unless specified otherwise, since the values of the scattering matrix elements are independent of \( n \). The density of states in the finite reflection case is unchanged from the zero reflection case, given by \( 2\pi v/d \) for each parity.

Since we will be taking the \( d \to \infty \) limit, and we are ultimately interested in the current fluctuation at the position of the qubit, it is useful to relate the ratio of left and right travelling voltage amplitudes at \( z = -d \) and at just at the left of the component \( F \).

\[
\frac{V_{s,F}^-}{V_{s,F}^+} = S_{11}^F + \frac{S_{12}^F S_{21}^F}{1 - S_{12}^F S_{21}^F} = r_s.
\]
\[ V_s(z, t) = V_{s,F}^+ \{ \exp [i\omega_s(t - z/v)] \]  
\[ + r_s \exp [i\omega_s(t + z/v)] \} + \text{c.c.,} \quad (45) \]
\[ I_s(z, t) = i \frac{V_{s,F}^+}{Z_0} \{ \exp [i\omega_s(t - z/v)] \]  
\[ - r_s \exp [i\omega_s(t + z/v)] \} + \text{c.c.,} \quad (46) \]

where c.c. stands for complex conjugate, added to ensure the voltage and current on the line are real. To quantize the field on the TL, we first consider the total energy stored in the line, given by
\[ H_s = \int_a^0 \left[ \frac{c_0}{2} V_s(z, t)^2 + \frac{1}{2Z_0} I_s(z, t)^2 \right] dz, \quad (47) \]
\[ = 2c_0d[V_{s,F}^+]^2(1 + |r_s|^2), \quad (48) \]

where in the first equality assumes that component \( F \) does not store energy. This is valid anticipating that we will take the \( d \to \infty \) limit, where the energy at any particular point on the line approaches zero. In the second equality, we discard terms oscillatory in \( z \) as they too approach zero in the \( d \to \infty \) limit. The energy expression Eq. (47) can be related to the Hamiltonian of a harmonic oscillator by introducing the dynamical variables
\[ q_s = N_s(V_{s,F}^+ + V_{s,F}^-), \quad (49) \]
\[ p_s = \dot{q}_s = -i\omega_s N_s(V_{s,F}^+ - V_{s,F}^-), \quad (50) \]

where \( V_{s,F}^± = V_{s,F}^±(t) \exp (i\omega_s t) \) and \( N_s \) is some normalization constant. By setting
\[ N_s = \sqrt{\frac{c_0d(1 + |r_s|^2)}{\omega_s^2}}, \quad (51) \]

the energy expression in Eq. (47) becomes
\[ H_s = \frac{1}{2} p_s^2 + \frac{1}{2} \omega_s^2 q_s^2. \quad (52) \]

We note that the equations of motion of \( q_s, p_s \) satisfy the Hamiltonian equation with Eq. (52). Therefore, following the canonical quantization of a harmonic oscillator, the current and voltages on the line can be given in terms of the bosonic ladder operators. Specifically, the current at the position of the qubit is
\[ I_s(z = q, t) = \sqrt{\frac{\hbar \omega}{2d(1 + |r_s|^2)}} \{ i\alpha \} \exp (-i\omega_s z q/v) \]
\[ - r_s \exp (i\omega_s z q/v) + \text{h.c.} \}. \quad (53) \]

The coupling strength between the qubit and a mode on the open TL is proportional to the product of the quantum fluctuation of the current at the position of the qubit and the junction phase matrix element, given by
\[ g_s = \phi_0 |\gamma_{11,10}| \sqrt{\frac{\hbar \omega_s}{2d(1 + |r_s|^2)}} \]
\[ \times |\exp (-i\omega_s z q/v) - r_s \exp (i\omega_s z q/v)|. \quad (54) \]

Finally, the relaxation rate can be obtained by inserting Eq. (54) into Eq. (5), and take the \( d \to \infty \) limit, which gives
\[ \Gamma(\omega) = \frac{2\pi}{\hbar^2} \sum_{n,s} \delta^2(\omega - \omega_{n,s}) \]
\[ = \sum_s \frac{\phi_0^2 |\gamma_{11,10}|^2}{2dZ_0(1 + |r_s|^2)} |\exp (-i\omega z q/v) \]
\[ - r_s \exp (i\omega z q/v)|^2. \quad (56) \]

To make the estimate concrete, we consider the component \( F \) to be reciprocal and characterized by a voltage standing wave ratio (VSWR), which allow us to obtain the ratio between left and right travelling wave amplitudes. Taking reasonable estimates of the qubit position, \( z_q = -20 \text{cm} \), we calculate the radiative relaxation rate \( \Gamma_1 \) as a function of \( f_{\beta} \), for VSWR = 1, 2, 4. As shown in Fig. 4, increasing reflection leads to oscillations in \( \Gamma_1 \) with increasing amplitudes around the predicted values in the no reflection limit (VSWR = 1). Taking the lower values in the oscillation, VSWR \( \gtrsim 4 \) is required to explain the discrepancy between measured values compared to the theoretical values in the no reflection limit. Room temperature characterization of the measurement setup indicate that the VSWR due to components on the output of the transmission line is roughly 1.5, smaller than the value required to explain the discrepancy.

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FIG. 4. The measured (blue dots with error bar) and predicted radiative relaxation rate $\Gamma_1$ using Eq. 56 assuming VSWR=1 (orange line), 2 (black crosses) and 4 (red stars).

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