Duality on \( q \)-Starlike Functions Associated with Fractional \( q \)-Integral Operators and Applications

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Abstract: In this paper, we make use of the Riemann–Liouville fractional \( q \)-integral operator to discuss the class \( S^\alpha_{\delta\beta}(a) \) of univalent functions for \( \delta > 0, a \in \mathbb{C} - \{0\} \), and \( 0 < |q| < 1 \). Then, we develop convolution results for the given class of univalent functions by utilizing a concept of the fractional \( q \)-difference operator. Moreover, we derive the normalized classes \( P_{\delta\beta}(\beta, \gamma) \) and \( P_{\beta\gamma}(\beta) \) \((0 < |q| < 1, \delta \geq 0, 0 \leq \beta \leq 1, \zeta > 0)\) of analytic functions on a unit disc and provide conditions for the parameters \( q, \delta, \zeta, \beta \), and \( \gamma \) so that \( P_{\delta\beta}(\beta, \gamma) \subset S^\alpha_{\delta\beta}(a) \) and \( P_{\beta\gamma}(\beta) \subset S^\alpha_{\delta\beta}(a) \) for \( a \in \mathbb{C} - \{0\} \). Finally, we also propose an application to symmetric \( q \)-analogues and Ruscheweyh’s duality theory.

Keywords: Riemann–Liouville; \( q \)-analogue; difference operator; \( q \)-starlike functions; duality principle; dual set; \( q \)-hypergeometric function

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1. Introduction

In recent decades, the theory of \( q \)-calculus has been applied to various areas of science and computational mathematics. The concept of \( q \)-calculus was used in quantum groups, \( q \)-deformed super algebras, \( q \)-transform analysis, \( q \)-integral calculus, optimal control, and many other fields, to mention but a few [1–4]. Soon after the concept of \( q \)-calculus was furnished, many basic \( q \)-hypergeometric functions, \( q \)-hypergeometric symmetric functions, and \( q \)-hypergeometric and hypergeometric symmetric function polynomials were discussed in geometric function theory [5]. Jackson [6] was the first to introduce and analyze the \( q \)-derivative and the \( q \)-integral operator. Later, various researchers applied the concept of the \( q \)-derivative to various sub-collections of univalent functions. Srivastava [7] used the \( q \)-derivative operator to describe some properties of a subclass of univalent functions. Agrawal et al. [8] extended a class of \( q \)-starlike functions to certain subclasses of \( q \)-starlike functions. Kanas et al. [9] used convolutions to define a \( q \)-analogue of the Ruscheweyh operator and studied some useful applications of their operator. Srivastava et al. [10] defined the \( q \)-Noor integral operator by following the concept of convolution. Purohit [11] introduced a subclass of univalent functions by using a certain operator of a fractional \( q \)-derivative. Aouf et al. [12] employed subordination results to discuss analytic functions associated with a new fractional \( q \)-analogue of certain operators. However, many extensions of different operators can be found in [13–29] and the references cited therein.
Here, we will make use of definitions and notations used in the literature [30,31]. For \(a, q \in \mathbb{C}\), the \(q\)-analogue of the Pochhammer symbol is defined by

\[
(a; q)_k = \begin{cases} 
\prod_{j=0}^{k-1} (1 - aq^j), & \text{if } k > 0, \\
1, & \text{if } k = 0, \\
\prod_{j=0}^{\infty} (1 - aq^j), & \text{if } k \to \infty,
\end{cases}
\]

and, hence, it is very natural to write \((a; q)_k = \frac{(a; q)_\infty}{(aq^k; q)_\infty}\) \((k \in \mathbb{N} \cup \{\infty\})\). The extension of the Pochhammer symbol to a real number \(\delta\) is given as

\[
(a; q)_\delta = \frac{(a; q)_\infty}{(aq^\delta; q)_\infty}, \quad (\delta \in \mathbb{R}).
\]

Therefore, for any real number \(\delta > 0\), the \(q\)-analogue of the gamma function is defined by

\[
\Gamma_q(a) = \frac{(q; q)_\infty}{(q^a; q)_\infty} (1 - q)^{1-\delta}.
\]

The \(q\)-analogue of the natural number \(n\) and the multiple \(q\)-shifted factorial for complex numbers \(a_1, \ldots, a_k\) are, respectively, defined by

\[
[n]_q = \frac{1 - q^n}{1 - q}, \quad 0 < |q| < 1, \text{ and } (a_1, \ldots, a_k)_n = \prod_{j=1}^k (a_j)_n.
\]

Let \(a_1, \ldots, a_r, b_1, \ldots, b_s\) be complex numbers; then, the \(q\)-hypergeometric series \(_r\phi_s\) is denoted as

\[
_r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} ; q, z \right) = \sum_{n=0}^\infty \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} z^n (-q)^{n(s-r+1)}. \tag{1}
\]

It is clear that the series representation of the function \(_r\phi_s\) converges absolutely for all \(z \in \mathbb{C}\) if \(r \leq s\) and converges only for \(|z| < 1\) if \(r = s + 1\). Now, let \(\mathcal{A}\) be the collection of all analytic functions in the open unit disc \(\mathcal{U} = \{z \in \mathbb{C}; |z| < 1\}\) expressed in the normalized form

\[
f(z) = z + \sum_{n=2}^\infty a_n z^n, \tag{2}
\]

and let \(\mathcal{A}_0\) be a collection comprising all functions \(g\) such that \(zg \in \mathcal{A}\) and \(g(0) = 1, z \in \mathbb{C}\). Then, the sub-collection of \(\mathcal{A}\) of functions that are univalent in \(\mathcal{U}\) is denoted by \(S\). However, in geometric function theory, a variety of sub-collections of univalent functions have been discussed. See the monographs published by [32,33] for details.

Let us consider the Riemann–Liouville fractional \(q\)-integral operator of a non-integer of order \(\delta\) defined by [34]

\[
I_q^\delta f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (x - [qt])^{\delta-1} f(t) dq t. \tag{3}
\]

Then, \(I_q^\delta f \to I_q f\) when \(\delta \to 1\), where \(I_q f\) is the \(q\)-Jackson integral defined by [6]

\[
I_q f(z) = \int_0^z f(t) dq t \quad z \in \mathcal{U}, z \neq 0, |q| < 1.
\]

With the concept of the Riemann–Liouville fractional \(q\)-integral of the non-integer order \(\delta\), we recall some rules associated with \(I_q^\delta\) by (3):

(i) \(I_q^\delta(cf) = cI_q^\delta f\), \(c \in \mathbb{C} - \{0\}, f \in \mathcal{A}\),
we recall the convolution (or the Hadamard product) introduced in [6] in the subsequent form:

\[ D_q^{δ}f(z) = \frac{1}{(1-q)^{z}} \sum_{n=0}^{∞} \frac{(q^{-δ};q)_n}{(q;q)_n} q^n f(q^n z). \]  

(ii) \( D_q^{δ}f \leq D_q^{δ}f \).

Agarwal [34] defined the \( q \)-analogue difference operator of a non-integer order \( δ \) as follows:

\[ D_q^{δ}f(z) = \frac{z^{n-δ}}{(1-q)^{z}} \sum_{n=0}^{∞} \frac{(q^{-δ};q)_n}{(q;q)_n} q^n f(q^n z). \]  

Note that \( D_q^{δ}f \rightarrow D_q f \) when \( δ \rightarrow 1 \). \( D_q f \) is the \( q \)-derivative of the function \( f \) introduced in [6] in the subsequent form:

\[ D_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad z \in U, z \neq 0, \ |q| < 1. \]  

Thus, for \( n \in \mathbb{N} \), through simple computations, we obtain

\[ D_q^{δ}z^n = \frac{z^{n-δ}}{(1-q)^{z}} \frac{(q^{1+n-δ};q)_n}{(1-q)^{z}} \text{ and } I_q^{δ}z^n = \frac{(q^{n+1+δ};q)_n}{(q^{n+1};q)_n} z^{n+δ}. \]  

Let \( 0 < |q| < 1, \ δ \geq 0, \ ζ > 0, \ 0 \leq β \leq 1, \) and \( 0 < γ \leq 1 \). By the definition of the \( q \)-analogue difference operator with the non-integer order \( δ \), the following rules of \( D_q^{δ} \) hold:

(i) \( D_q^{δ}(cf) = cD_q^{δ}f, \quad c \in \mathbb{C} - \{0\}, \ f \in A, \)

(ii) \( D_q^{δ}(f + g) = D_q^{δ}f + D_q^{δ}g, \quad f, g \in A. \)

We define \( P_{δ,β}(f,γ) \) as the class of all functions \( f \in A \) satisfying the following condition:

\[ \text{Re} \left\{ \frac{(1-q)^{δ} \left( D_q^{δ+1+γ}f(z) + \frac{q^γ z D_q^{δ+2+γ}f(z)}{1-q} \right) - β}{1-β} \right\} > 0, \quad |z| < 1. \]

For \( 0 < |q| < 1, \ δ > 0, \) and \( 0 \leq β \leq 1 \), the class \( P_{δ,β} \) consists of functions satisfying the following condition:

\[ \text{Re} \left\{ \frac{(1-q)^{δ} \left( D_q^{δ+1+β}f(z) + q^β z D_q^{δ+2+β}f(z) \right) - β}{1-β} \right\} > 0, \quad |z| < 1. \]

Now, for two functions

\[ f(z) = \sum_{n=0}^{∞} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{∞} b_n z^n, \]

we recall the convolution (or the Hadamard product) of \( f \) and \( g \), denoted by \( f \ast g \), which is given by

\[ (f \ast g)(z) = \sum_{n=0}^{∞} a_n b_n z^n, \quad z \in U. \]

For a set \( V \subseteq A_0 \), the dual set \( V^{∗} \) is defined by

\[ V^{∗} = \{ g \in A_0 : (f \ast g)(z) \neq 0, \ ∀ f \in V, \ z \in U \}. \]

However, the second dual of \( V \) is defined as \( V^{∗∗} = (V^{∗})^{∗} \). However, \( V \subseteq V^{∗∗} \). For basic reference to this theory, we may refer to the book by Ruscheweyh [35] (see also [36–38]).
In this paper, we define the class $S_{q,\delta}^\ast$ for $\delta > 0$, $0 < |q| < 1$, and establish the convolution condition of this class. Furthermore, we find conditions for $q, \delta, \xi, \beta,$ and $\gamma$ so that $\mathcal{P}_{q,\delta}^\ast(\beta, \gamma) \subset S_{q,\delta}^\ast(\alpha)$ and $\mathcal{P}_{q,\delta}^\ast(\beta) \subset S_{q,\delta}^\ast(\alpha)$.

2. Preliminary Lemmas

The following lemmas are very useful in our investigation.

**Lemma 1.** (Duality principle; see [35]) Let $\mathcal{V} \subseteq \mathcal{A}_0$ be compact; it has the following property:

$$f \in \mathcal{V} \implies \forall |x| \leq 1 : f_x \in \mathcal{V},$$

where $f_x(z) = f(xz)$. Then,

$$\phi(\mathcal{V}) = \phi(\mathcal{V}^\ast),$$

for all continuous linear functionals $\phi$ on $\mathcal{A}$, and

$$\text{co}(\mathcal{V}) \subseteq \text{co}(\mathcal{V}^\ast),$$

where $\text{co}$ stands for the closed convex hull of a set.

**Lemma 2** ([35]). Let $0 \leq \gamma < 1$ and $\beta \in \mathbb{R}, \beta \neq 1$. If

$$V_{\beta,\gamma} = \left\{ \gamma(1 - \beta) \frac{1 + xz}{1 - xz} + (1 - \gamma)(1 - \beta) \frac{1 + yz}{1 - yz} + \beta, \ |x| = |y| = 1, \ z \in \mathcal{U} \right\},$$

then

$$V_{\beta,\gamma}^\ast = \left\{ f \in \mathcal{A}_0 : \exists \zeta \in \mathbb{R}, \Re \left\{ g(z) - \frac{1 - 2\beta}{2(1 - \beta)} \right\} > 0, \ g(z) = f_x(z), \ |x| \leq 1 \right\},$$

and

$$V_{\beta,\gamma}^{\ast\ast} = \left\{ f \in \mathcal{A}_0; \Re \left\{ \frac{g(z) - \beta}{1 - \beta} \right\} > 0, \ g(z) = f_x(z), \ |x| \leq 1 \right\}.$$

We see that the set $V_{\beta,\gamma}$ in (7) does not satisfy the property (6), i.e., if $f \in V_{\beta,\gamma}$, then $f(xz) \in V_{\beta,\gamma}$ for all $|x| \leq 1$, as is required in the Duality Principle. However, the Duality Principle can be stated with a slightly weaker but more complicated condition that $V_{\beta,\gamma}$ can be seen to satisfy (see [35] for more details).

3. Main Results

**Definition 1.** Let $f \in \mathcal{A}, \delta > 0$, and $\alpha \in \mathbb{C} - \{0\}$. Then, a function $f$ is said to be in the class $S_{q,\delta}^\ast(\alpha)$ if it satisfies the following inequality:

$$\Re \left\{ 1 + \frac{1}{\alpha} \left( zD_{q}^{\delta+1}I_{q}^{\beta}f(z) - \frac{1}{1 - q^2} \right) \right\} > 0,$$

where the operators $D_{q}^{\delta}$ and $I_{q}^{\beta}$ are given by (4) and (3), respectively.

Putting $\delta = 0$ into Definition 1 leads to the following definition.

**Definition 2.** The function $f \in \mathcal{A}$ is said to be in the class of $q$-starlike functions of order $\alpha$, $S_{q}^\ast(\alpha)$, if it satisfies the following inequality:

$$\Re \left\{ 1 + \frac{1}{\alpha} \left( zD_{q}\frac{f(z)}{f(z)} - 1 \right) \right\} > 0, \ \alpha \in \mathbb{C} - \{0\},$$
where $D_q f(z)$ is given by (5).

**Theorem 1.** Let $f \in \mathcal{A}$, $\delta > 0$, $\alpha \in \mathbb{C} - \{0\}$, and $|z| < R < 1$. Then, $f \in S^*_q(\alpha)$ if and only if

$$\frac{f(z)}{z} \neq \frac{1 + qz\left(\frac{x+1}{2\alpha - qz} - 1\right)}{(1-z)(1-qz)}$$

where $|x| = 1$ and $x \neq -1$.

**Proof.** Since $\frac{zD_q^{\delta+1} f(z)}{f(z)} - \frac{1}{(1-q)^2} = 0$ at $z = 0$, we have

$$1 + \frac{1}{\alpha}\left(\frac{zD_q^{\delta+1} f(z)}{f(z)} - \frac{1}{(1-q)^2}\right) \neq \frac{x - 1}{x + 1}, \quad |x| = 1, \; x \neq -1.$$

By following simple computations, we can rewrite this as

$$(x + 1)(1-q)^{\delta}zD_q^{\delta+1} f(z) - \left(2\alpha(1-q)^{\delta} - x - 1\right)f(z) \neq 0. \quad (8)$$

Since the function $f$ satisfies (2), we obtain

$$zD_q^{\delta+1} f(z) = \frac{1}{(1-q)^2}\left(z + \sum_{n=2}^{\infty} [n]_q a_n z^n \right) = \frac{1}{(1-q)^2}\left(f(z) \ast \frac{z}{(1-z)(1-qz)}\right).$$

Now, as Equation (8) is equivalent to

$$\left(f(z) \ast \frac{(x+1)z}{(1-z)(1-qz)}\right) + \left(f(z) \ast \frac{z(2\alpha(1-q)^{\delta} - x - 1)}{1-z}\right) \neq 0,$$

it simplifies to

$$f(z) \ast \frac{(x+1)z + z(1-qz)(2\alpha(1-q)^{\delta} - x - 1)}{(1-z)(1-qz)} \neq 0.$$

Hence, the required result has been proven. \(\square\)

Putting $\delta = 0$ into Theorem 1, we get the following corollary.

**Corollary 1.** Let $\alpha \in \mathbb{C} - \{0\}$, $|x| = 1$, and $x \neq -1$. Then, the function $f$ is a $q$-starlike function of order $\alpha$ if and only if

$$\frac{f(z)}{z} \neq \frac{1 + qz\left(\frac{x+1}{\alpha \alpha} - 1\right)}{(1-z)(1-qz)} \neq 0, \ |z| < R \leq 1.$$

**Theorem 2.** Let $\delta > 0$, $0 < q < 1$, $\alpha \in \mathbb{C} - \{0\}$, $\zeta > 0$, $0 \leq \beta < 1$, $0 < \gamma < 1$, and $|x| = 1$ with $x \neq -1$. Then, $\mathcal{P}^q_\delta(\beta, \gamma) \subseteq S^*_q(\alpha)$ if and only if

$$\text{Re}\left\{F(x,z)\right\} > -\frac{(1-q)^{\delta}}{\zeta \gamma (1-\beta)},$$

where

$$F(x,z) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(x+1)[n+1]_q + 2\alpha(1-q)^{\delta} - (x+1)}{[n+1]_q (\zeta \gamma + (1-\gamma)(n+1]_q)} z^n, \ |z| < R \leq 1, \ |z| < R \leq 1. \quad (11)$$
**Proof.** Let the function $f$ be in the class $\mathcal{P}_{\alpha,\beta}^{\gamma,\alpha}(\beta, \gamma), |z| < R \leq 1$. If we denote

$$ g(z) = (1-q)\delta \left( D_{\delta}^{\alpha+1} \phi f(z) + \frac{1-\gamma}{\zeta \gamma} z D_{\delta}^{\alpha+2} \phi f(z) \right), $$

then we have $g \in V_{\beta,\gamma}^{**}$. If $f$ satisfies 2, then we obtain

$$ g(z) = 1 + \sum_{n=2}^{\infty} [n]_{q} a_{n} z^{n-1} + \sum_{n=2}^{\infty} \frac{1-\gamma}{\zeta \gamma} [n]_{q} [n-1]_{q} a_{n} z^{n-1} $$

$$ = 1 + \sum_{n=2}^{\infty} [n]_{q} a_{n} \left( 1 + \frac{1-\gamma}{\zeta \gamma} [n-1]_{q} \right) z^{n-1} $$

$$ = 1 + \sum_{n=2}^{\infty} [n]_{q} a_{n} \left( \frac{\zeta \gamma + (1-\gamma) [n-1]_{q}}{\zeta \gamma} \right) z^{n-1}. $$

Therefore,

$$ \frac{f(z)}{z} = 1 + \sum_{n=2}^{\infty} a_{n} z^{n-1} = g(z) \ast \left( 1 + \sum_{n=2}^{\infty} [n]_{q} (\zeta \gamma + (1-\gamma) [n-1]_{q}) z^{n-1} \right). $$

We now obtain a one-to-one correspondence between $\mathcal{P}_{\alpha,\beta}^{\gamma,\alpha}(\beta, \gamma)$ and $V_{\beta,\gamma}^{**}$. Thus, by Theorem 1, $\mathcal{P}_{\alpha,\beta}^{\gamma,\alpha}(\beta, \gamma) \subseteq S_{\alpha,\beta}^{\gamma,\alpha}(\alpha)$ if and only if

$$ g(z) \ast \left( 1 + \sum_{n=2}^{\infty} [n]_{q} (\zeta \gamma + (1-\gamma) [n-1]_{q}) z^{n-1} \right) \ast \frac{1+qz\left( \frac{x+1}{2a(1-q)^{\delta}} - 1 \right)}{(1-z)(1-qz)} \neq 0. \quad (12) $$

For $z \in \mathcal{U}$, consider the continuous linear functional $\lambda_{z} : \mathcal{A}_0 \rightarrow \mathbb{C}$ such that

$$ \lambda_{z}(h) = h(z) \ast \left( 1 + \sum_{n=2}^{\infty} [n]_{q} (\zeta \gamma + (1-\gamma) [n-1]_{q}) z^{n-1} \right) \ast \frac{1+qz\left( \frac{x+1}{2a(1-q)^{\delta}} - 1 \right)}{(1-z)(1-qz)} \neq 0. $$

By the Duality Principle, we have $\lambda_{z}(V) = \lambda_{z}(V_{\beta,\gamma}^{**})$. Therefore, (12) holds if and only if

$$ \left( 1 + 2(1-\beta) \sum_{k=1}^{\infty} z^{k} \right) \ast \left( 1 + \sum_{n=1}^{\infty} \frac{\zeta \gamma}{[n+1]_{q} (\zeta \gamma + (1-\gamma) [n]_{q})} z^{n} \right) \ast \left( 1 + \sum_{n=1}^{\infty} ([n+1]_{q} + \left( \frac{x+1}{2a(1-q)^{\delta}} - q \right) [n]_{q}) z^{n} \right) \neq 0. $$

Using the properties of convolution, we obtain

$$ 1 + \frac{2(1-\beta)\zeta \gamma}{2a(1-q)^{\delta}} \sum_{n=1}^{\infty} \frac{2a(1-q)^{\delta} [n+1]_{q} + (q(x+1) - 2a(1-q)^{\delta}) [n]_{q}}{[n+1]_{q} (\zeta \gamma + (1-\gamma) [n]_{q})} z^{n} \neq 0. $$

Since $[n+1]_{q} = 1 + q[n]_{q}$, we get

$$ 1 + \frac{2(1-\beta)\zeta \gamma}{2a(1-q)^{\delta}} \sum_{n=1}^{\infty} \frac{(x+1)[n+1]_{q} + 2a(1-q)^{\delta} - (x+1)}{[n+1]_{q} (\zeta \gamma + (1-\gamma) [n]_{q})} z^{n} \neq 0. $$

Hence, we have

$$ \frac{1}{a} \sum_{n=1}^{\infty} \frac{(x+1)[n+1]_{q} + 2a(1-q)^{\delta} - (x+1)}{[n+1]_{q} (\zeta \gamma + (1-\gamma) [n]_{q})} z^{n} \neq \frac{(1-q)^{\delta}}{\zeta \gamma (1-\beta)}. \quad (13) $$
where \( z \in \mathcal{U}, |x| = 1 \). The equality on the right side of Equation (13) takes its value on the line \( \text{Rew} \neq \frac{(1-q)^\delta}{\zeta \gamma (1-\beta)} \), and so (13) is equivalent to (10). □

**Remark 1.** Under the hypothesis of Theorem 3.5, the inequality (10) can be written in the form

\[
\frac{(1-q)^\delta}{\zeta \gamma (1-\beta)} + \text{Re} \left\{ \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\} \\
+ (1-q)^\delta \text{Re} \left\{ \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\} - \text{Re} \left\{ \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\} \\
\geq \text{Re} \left\{ \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} - \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\}.
\]

Therefore, for more clarification, we can see that this satisfies the inequality when

\[
\frac{(1-q)^\delta}{\zeta \gamma (1-\beta)} + \text{Re} \left\{ \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\} \\
+ (1-q)^\delta \text{Re} \left\{ \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\} - \text{Re} \left\{ \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right\} \\
\geq \left| \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} - \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q} \right|.
\]

Assume that the function \( \psi \) is given by

\[
\psi(z) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{z^n}{\zeta \gamma + (1-\gamma) |n| q}.
\]

Then, inequality (14) can be written in the form

\[
\frac{(1-q)^\delta}{\zeta \gamma (1-\beta)} + \text{Re} \{D_q z \psi(z) + (2\alpha (1-q)^\delta - 1) \psi(z) \} \geq |D_q z \psi(z) + \psi(z)|.
\]

Hence, \( \psi(z) \in S^*_q,\delta(\alpha) \) if and only if (15) is satisfied.

Putting \( \delta = 0 \) into Theorem 2 leads to the following corollary.

**Corollary 2.** Let \( 0 < q < 1, \alpha \in \mathbb{C} - \{0\}, \zeta > 0, 0 \leq \beta < 1, 0 < \gamma < 1, \) and \( |x| = 1 \) with \( x \neq -1 \). Then, \( \mathcal{P}_{q,\delta}(\beta, \gamma) \subseteq S^*_q(\alpha) \) if and only if

\[
\text{Re} \left\{ F_1(x, z) \right\} > \frac{1}{\zeta \gamma (1-\beta)},
\]

where

\[
F_1(x, z) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(x + 1)[n + 1]_q + 2\alpha - (x + 1)}{[n + 1]_q (\zeta \gamma + (1-\gamma) |n| q)} z^n, |z| < R \leq 1.
\]

Similarly, from Theorem 2, we get the following theorem.

**Theorem 3.** Let \( \delta > 0, 0 < q < 1, \alpha \in \mathbb{C} - \{0\}, 0 \leq \beta < 1, \) and \( |x| = 1 \) with \( x \neq -1 \). Then, \( \mathcal{P}_{q,\delta}(\beta) \subseteq S^*_q,\delta(\alpha) \) if and only if

\[
\text{Re} \left\{ F_1(x, z) \right\} > -\frac{(1-q)^\delta}{(1-\beta)},
\]

(16)
where
\[
F_1(x, z) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(x + 1)[n + 1]_q + 2\alpha(1 - q)^\delta - (x + 1)}{[n + 1]_q^2} z^n, \quad |z| < R \leq 1.
\] (17)

Putting \(\delta = 0\) into Theorem 3 leads to the following corollary.

**Corollary 3.** Let \(0 < q < 1, \alpha \in \mathbb{C} \setminus \{0\}, 0 \leq \beta < 1,\) and \(|x| = 1\) with \(x \neq -1\). Then, \(P_{q,0}(\beta, \alpha) \subseteq S_q^{\alpha}(\alpha)\) if and only if
\[
\text{Re}\{F_2(x, z)\} > -\frac{1}{1 - \beta},
\]
where
\[
F_2(x, z) = \frac{1}{\alpha} \sum_{n=1}^{\infty} \frac{(x + 1)[n + 1]_q + 2\alpha - (x + 1)}{[n + 1]_q^2} z^n, \quad |z| < R \leq 1.
\]

**Remark 2.** The function \(F_1(x, z)\) can be represented in terms of a \(q\)-hypergeometric function as follows:
\[
F_1(x, z) = \frac{x + 1}{\alpha} 2\Phi_1\left(\frac{q}{q^2}; q^2; q, z\right) + \frac{2\alpha(1 - q)^\delta - (x + 1)}{\alpha} 2\Phi_1\left(\frac{q}{q^2}; q^2; q, z\right).
\]

**Proof.** From the definition of \(F_1(x, z)\) introduced in (11), we infer that
\[
F_1(x, z) = \frac{x + 1}{\alpha} \sum_{n=0}^{\infty} \frac{(q^2)_n}{(q^2)_n} z^n + \frac{2\alpha(1 - q)^\delta - (x + 1)}{\alpha} \sum_{n=0}^{\infty} \frac{(q^2)_n}{(q^2)_n} z^n - 2(1 - q).
\] (18)

Since \([n + 1]_q = \frac{(q)_n}{(q^2)_n}\), we have
\[
F_1(x, z) = \frac{x + 1}{\alpha} \sum_{n=0}^{\infty} \frac{(q^2)_n}{(q^2)_n} z^n + \frac{2\alpha(1 - q)^\delta - (x + 1)}{\alpha} \sum_{n=0}^{\infty} \frac{(q^2)_n}{(q^2)_n} z^n - 2(1 - q).
\]

Hence, by using the definition of \(\Phi_q\) from (1), the proof of the corollary is complete. \(\square\)

Putting \(\delta = 0\) into Remark 2 leads to the following corollary.

**Corollary 4.** The function \(F_2(x, z)\) can be expressed in terms of the \(q\)-hypergeometric function as follows:
\[
F_2(x, z) = \frac{x + 1}{\alpha} 2\Phi_1\left(\frac{q}{q^2}; q^2; q, z\right) + \frac{2\alpha - (x + 1)}{\alpha} 2\Phi_1\left(\frac{q}{q^2}; q^2; q, z\right).
\]

We now consider the Riemann–Liouville fractional \(q\)-integral and obtain the following corollary.

**Remark 3.** The function \(F_1(x, z)\) can be expressed in terms of the Riemann–Liouville fractional \(q\)-integral as follows:
\[
F_1(x, z) = \frac{x + 1}{\alpha} \int_0^1 \frac{1}{1 - tz} d_q t + \frac{2\alpha(1 - q)^\delta - (x + 1)}{\alpha} \int_0^1 \int_0^1 \frac{1}{1 - vtz} d_q v d_q t - 2(1 - q)
\]
Theorem 4. Let
\[ F_1(x, z) = \frac{x + 1}{a} \sum_{n=0}^{\infty} t^n dz^n + \frac{2 \alpha (1 - q)^\delta - (x + 1)}{a} \int_0^1 v^n dq_0 \int_0^1 t^n dt_z^n. \]

\[ = \frac{x + 1}{a} \int_0^1 \left( \sum_{n=0}^{\infty} t^n z^n \right) dq_0 t + \frac{2 \alpha (1 - q)^\delta - (x + 1)}{a} \int_0^1 \int_0^1 \left( \sum_{n=0}^{\infty} v^n t^n z^n \right) dq_0 dq_0 t. \]

This completes the proof of the corollary. \(\square\)

Proof. Since Equation (18) is satisfied, we have
\[ F_2(x, z) = \frac{x + 1}{a} \int_0^1 \frac{1}{1 - tz} dq_0 t + \frac{2 \alpha - (x + 1)}{a} \int_0^1 \int_0^1 \frac{1}{1 - vtz} dq_0 dq_0 t - 2(1 - q). \]

Corollary 5. The function \( F_2(x, z) \) can be expressed in terms of the Riemann–Liouville fractional \( q \)-integral in the following form:
\[ F_2(x, z) = \frac{x + 1}{a} \int_0^1 \frac{1}{1 - tz} dq_0 t + \frac{2 \alpha - (x + 1)}{a} \int_0^1 \int_0^1 \frac{1}{1 - vtz} dq_0 dq_0 t - 2(1 - q). \]

Theorem 4. Let \( 0 < |q| < 1, \delta > 0, 0 \leq \beta \leq 1, \) and \( f \in P_{\beta, \gamma}(\beta). \) We define
\[ K_q = \int_0^1 \frac{dq_0 t}{1 - t}. \]

If
\[ \beta \geq \frac{1 - 2K_q}{2(1 - K_q)}, \]

then \( f \in P_{\beta, \gamma}(0), \) and, hence, it is univalent.

Proof. Let \( \zeta > 0 \) and \( \gamma > 0; \) we define
\[ \phi_q(z) = 1 + \sum_{n=1}^{\infty} |n + 1|_q z^n, \]
and
\[ \psi_q(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{|n + 1|_q} z^n = 1 + \sum_{n=1}^{\infty} \int_0^1 t^n dq_0 t z^n \]
\[ = \int_0^1 \left( 1 + \sum_{n=1}^{\infty} t^n z^n \right) dq_0 t = \int_0^1 \frac{1}{1 - tz} dq_0. \]

In view of these representations, we can write
\[ D^\delta_q + 1 q^\beta f(z) + q D^\delta q^\beta q f(z) = D^\delta q^\beta + 1 q^\beta f(z) \ast \phi_q(z) \]
and
\[ \left( D^\delta q^\beta + 1 q^\beta f(z) + q D^\delta q^\beta q f(z) \right) \ast \psi_q(z) = D^\delta q^\beta + 1 q^\beta f(z). \]

Let \( f \in P_{\beta, \gamma}(\beta). \) Then, by using Lemma 2, we may restrict our attention to the function \( f \in P_{\beta}(\beta, \gamma) \) for which
\[ (1 - q)^\delta \left( D^\delta q^\beta + 1 q^\beta f(z) + q D^\delta q^\beta q f(z) \right) = \gamma(1 - \beta)^{1 + \frac{x}{1 - x}} + (1 - \beta)(1 - \gamma)^{1 + \frac{y}{1 - y}}z + \beta. \]
Thus, we obtain

\[(1 - q)^{\beta}D^{\delta+1}_q f(z) = \left( \gamma(1 - \beta) \frac{1 + xz}{1 - xz} + (1 - \gamma)(1 - \gamma) \frac{1 + yz}{1 - yz} + \beta \right) * \psi_q(z). \]  \tag{20}

Hence, Equation (20) is equivalent to

\[(1 - q)^{\beta}D^{\delta+1}_q f(z) = \left( \gamma \frac{1 + xz}{1 - xz} + (1 - \gamma) \frac{1 + yz}{1 - yz} \right) * \left((1 - \beta)\psi_q(z) + \beta\right).
= \left( \gamma \frac{1 + xz}{1 - xz} + (1 - \gamma) \frac{1 + yz}{1 - yz} \right) \left( \int_0^1 \left((1 - \beta) \frac{1}{1 - tz} + \beta \right) dt \right)
= \left( \gamma \frac{1 + xz}{1 - xz} + (1 - \gamma) \frac{1 + yz}{1 - yz} \right) G_q(z) \tag{21}

where

\[G_q(z) = \int_0^1 \left((1 - \beta) \frac{1}{1 - tz} + \beta \right) dt.

Therefore,

\[\text{Re}(G_q(z)) = \int_0^1 \left((1 - \beta) \frac{1}{1 - t} + \beta \right) dt = (1 - \beta)k_q + \beta,
\]

where \(K_q\) is defined by (19). Note that if \(\beta \geq (1 - 2K_q)/2(1 - K_q)\), then \(\text{Re}G(z) \geq 1/2\). Functions with real parts greater than 1/2 are known to preserve the closed convex hull under convolution [10, p. 23]. Therefore, from (21), we have

\[(1 - q)^{\beta}D^{\delta+1}_q f(z) = \gamma(2 \frac{1}{1 - xz} - 1) * G_q(z) + (1 - \gamma)(2 \frac{2}{1 - yz} - 1) * G_q(z)
= 2\gamma G_q(xz) - \gamma + 2(1 - \gamma)G_q(yz) - (1 - \gamma)
= 2\gamma G_q(xz) + 2(1 - \gamma)G_q(yz) - 1.

In addition, since \(\text{Re}\{D^{\delta+1}_q f(z)\} > 0\), we have \(f \in \mathcal{P}_{q,\delta\beta}(0)\). This completes the proof of the theorem. \(\Box\)

4. Conclusions

In this article, a new class of univalent functions was introduced by using Riemann–Liouville fractional \(q\)-integrals and \(q\)-difference operators of non-integer orders. Then, some convolution results for such a class of univalent functions were obtained. In addition, two classes of normalized analytic functions in the unit disc were derived, and some conditions on \(q, \delta, \zeta, \beta,\) and \(\gamma\) were given so that the new classes satisfied \(\mathcal{P}_{q,\delta\beta}(\bar{\beta}, \gamma) \subset S^*_q(\alpha)\) and \(\mathcal{P}_{q,\delta\beta}(\bar{\beta}) \subset S^*_q(\alpha)\).

The result obtained during this research can be further used for writing fractional differential and integral operators in order to extend the results of analytic functions.

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