Dispersive detection of atomic ensembles in the presence of strong lensing

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Abstract

We experimentally and theoretically investigate in-medium propagation effects of off-resonant light in dense, spatially inhomogeneous ultracold atomic gases. Focussing on frequency modulation spectroscopy as the dispersive detection tool of atoms, we observe that the refractive gradient-index lenses presented by localised atomic ensembles can significantly modify the interpretation of the dispersive signal even for large probe detuning, owing to the collective response of the atoms. We identify criteria for distinguishing between thin and thick atomic lenses, leading to either diffraction-dominated and lensing dominated regimes for the outgoing probe beams. Our findings are consistent with experimental data and solutions of paraxial wave equation for light propagation. Our study provides important practical insights for dispersive, minimally intrusive optical detection and imaging schemes of ultracold atoms and will be valuable for choosing optimal parameter regimes in numerous applications.

1. Introduction

Strong atom–light interaction is the foundation of many emerging quantum technologies, such as quantum metrology beyond the standard quantum limit [1–5], continuous variable quantum communication via quantum non-demolition measurements [6, 7] and quantum memories [8]. A key parameter for strong atom-light coupling is the so-called co-operativity [9] which is a measure of the spatial mode overlap between the incident mode of light and the mode scattered by the atoms. High co-operativity is typically achieved by placing atoms in high-finesse resonant optical cavities. In recent years, strong atom–light interactions between single atoms and light in free-space and guided geometries have also been investigated using high-NA lenses [10, 11], optical nanofibers [12–14] and photonic crystal waveguides [15]. There has also been a substantial interest in designer atomic arrays to cooperatively enhance light-atom coupling [16–19].

Ensembles of randomly positioned atoms dispersively interacting with a paraxial light beam can potentially give rise to high free-space cooperativity. This is owing to the fact that scattered light fields from individual atomic dipoles interfere predominantly constructively in the forward direction, strongly enhancing the overlap of the scattered mode with the incident mode. As it is impossible to tell which atom scattered a particular photon detected in the forward direction, the light field is quantum-entangled with the collective atomic coherences. As a result its detection causes backaction on the atoms, leading to squeezing of quantum projection noise and potentially enhanced measurement precision [20].

Dispersive optical probing has proven an important minimally destructive diagnostic tool for ultracold atomic samples. This constitutes the basis of ‘real-time’ monitoring of the internal and external degrees of freedom of the trapped ultracold atoms [21–32], as well as for minimally destructive imaging of atoms [33–36] at or below the shot noise level [37, 38]. This underpins many interesting feedback loop or optimal...
Ultracold trapped atomic samples typically realised in the lab have inhomogeneous density profiles and finite spatial extents. This renders the one-dimensional Beer–Lambert type description of ensemble-light coupling insufficient and motivates a three-dimensional description. The effect of the geometry of the atomic ensemble and paraxial light field on the efficiency of cooperative scattering was for example theoretically discussed by Müller et al [46]. Along the same lines, Baragiola et al studied theoretically the effect of mode-matching between the atomic ensembles and the probe light field in the quantum regime in the context of spin-squeezing via continuous QND measurements. They interestingly concluded that spin-squeezing can even be enhanced in the presence of spatial inhomogeneity compared to homogeneous samples.

Large resonant optical density (ODres) is a key figure of merit for quantum-interfacing atomic ensembles with light. For spatially non-uniform samples of atoms, the distortion of an off-resonant light field during its propagation inside the sample can have a profound effect on the outgoing spatial mode of the light field. For example, in a cigar-shaped Gaussian atomic sample with high on-axis optical depth, a significant transverse gradient of optical phase can develop when an off-resonant plane wave probe propagates through the sample. According to the Poynting theorem, the local propagation direction of light is bent towards the direction of the phase gradient [69]. Thus the probe light field is diverted out of the incident mode by the atomic sample, which in this case behaves as a gradient-index lens. Such lensing by micron scale cold atomic samples has been studied recently in [48–50].

In this paper, we experimentally and theoretically study the effect of lensing in dispersive interfacing of atomic ensemble with paraxial light for a particular implementation of optical probing, based on frequency modulation spectroscopy (FMS) [28, 51, 63]. FMS offers the advantages of a zero baseline and excellent passive common-mode phase noise reduction between the reference and signal arms of an effective Mach–Zehnder interferometer. We observe that the effect of lensing in inhomogeneous ultracold samples can be substantial, even when the probe field is hundreds of linewidths away from the atomic resonance. While our study uses a specific implementation of dispersive probing, the results are relevant to a broad class of experiments probing atomic samples with off-resonant light. Such geometrical and in-medium propagation effects relevant in three-dimensional interfaces have not been studied before in the context of frequency modulation spectroscopy. Our studies are particularly important for atomic samples with large optical depths, small spatial size and strong density gradients, such as a Bose–Einstein condensate.

The paper is organised as follows. In section 2, we briefly introduce the basic light–atom interactions and how collective effects dictate the propagation of light fields in a medium consisting of dipoles. In section 3, we describe the macroscopic fields for homogeneous and inhomogeneous samples, derive an expression for the focal length atomic gradient index lens and the criterion where lensing plays a dominant role in dispersive probing. In section 4, we introduce frequency modulation spectroscopy, and derive expressions for FMS signal in the case of a three-dimensional interface, as in our experiment. We also discuss different regimes in dispersive probing arising from geometric factors, and carry out a brief comparison of FMS with phase-contrast imaging. In section 5, we discuss our experimental observations and interpret the data based on our analytical model as well as a paraxial wave equation model. In section 6, we summarise our results and provide an outlook on the implications of our findings and future studies.

2. Basic problem

When a ground state atom is placed in a propagating light field, which has its frequency in the neighbourhood of an atomic transition to an excited state, light will be scattered. For a sufficiently low intensity the scattered light will have the same frequency as the incoming light, it will have the same spectrum as the incoming light (i.e., it will not reflect the natural linewidth of the excited state), and the two fields will be mutually coherent [52, 53]. The scattered light will, however, generally be in a different spatial mode to the incoming field. In a typical experimental configuration, the incoming light would be in the form of a Gaussian laser beam and light would be scattered into a dipole radiation pattern. For example, a circularly polarised incoming field propagating along the z axis driving a circularly oscillating (rotating) electric dipole, the emitted field will be a spherical wave with its power distributed as $\propto (1 + \cos^2 \theta)$ [54]. Since the incoming and scattered fields are coherent they can interfere [55].

In a semiclassical treatment, the steady-state dipole moment of a two-level atom perturbed by a weak, near resonant light field, which is $E(t) = E_0 \cos \omega t$ at the position of the atom, is found to be [56]
\[ D(t) = \frac{\mu^2/\hbar}{\sqrt{\Delta^2 + \Gamma^2/4}} E_0 \cos(\omega t - \eta), \] (1)

where \( \eta = \arctan(\Gamma/2\Delta) \) is the phase lag of the dipole with respect to the driving electric field and \( \mu \) is the magnitude of the electric (transition) dipole matrix element corresponding to the transition which is the related to the natural linewidth \( \Gamma \) of the transition by \( \Gamma = k^2 \mu^2 / (3\pi\epsilon_0 h) \). Here \( k = 2\pi/\lambda \), where \( \lambda \) is the wavelength of the transition. In particular, we note that on resonance \( \Delta = 0 \) the dipole lags the driving field by \( \pi/2 \). For the oscillating induced dipole, the radiated electric field has a magnitude at the point \( \mathbf{r} = (r, \theta, \phi) \) given by

\[ E(\mathbf{r}) = \frac{D(t)}{8\pi\epsilon_0} \left(1 + \cos^2 \theta\right) \frac{k^2}{r}. \] (2)

In the case of an ensemble of atomic dipoles inside the volume \( V' \) illuminated by an incident field \( E_{\text{inc}}(\mathbf{r}, t) \) and scattering coherently, the situation is complicated by the fact that each dipole sees not only the incident field but also the dipolar scattered field owing to all other dipoles present. The total field is then given by \([57, 58]\)

\[ E(\mathbf{r}, t) = E_{\text{inc}}(\mathbf{r}, t) + \int_{V'} \nabla \times \left( \nabla \times \frac{\rho(\mathbf{r}')\alpha\mathbf{E}(\mathbf{r}', t - R/c)}{R} \right) \, dV', \] (3)

where \( \rho \) is the local number density of the scatterers, \( \alpha \) is the polarizability and \( R = |\mathbf{r} - \mathbf{r}'| \) is the distance between a scatterer and the observation point. The complex polarizability of a two-level atom can be easily derived from equation (1), by writing \( D(t) \) as the real part of a complex number, to be

\[ \alpha = -\frac{\mu^2}{\hbar} \frac{1}{\Delta + i\Gamma/2}. \] (4)

Note that the total field \( E \) appears on both sides of equation (3), so the integral equation must be solved self-consistently to obtain the total field.

For a dilute ensemble of dipoles, where the total scattered field from all dipoles at the location of a dipole is small compared to the incident field, one can make the simplifying assumption that all dipoles are illuminated by the incident light field and scatter independently, giving rise to a resultant electric field that then interferes with the incident light field at the observation point. This approximation is frequently made for dispersive probing of ultracold atoms \([46, 47]\). This reduces the problem of light scattering from an atomic ensemble to that of diffraction of the light field by the medium. Notably, if the geometry of the atomic ensemble is fixed then the scattered light field in the forward direction is cooperatively enhanced by a factor of \( N \), where \( N \) is the number of dipoles in the ensemble. This linear dependence of the dispersive signal on the number of scatterers forms the basis of quantum non-demolition measurement of collective atomic variables using dispersive probing.

There are, however, common practical situations where the dipoles are not only driven by \( E_{\text{inc}} \). A trivial example is a system of dipoles with a large physical extent \( l \) along incident light field axis such that \( l > 1/n\sigma \), where \( n \) is the atomic density and \( \sigma \) is the light scattering cross-section. In this case the incident field will be significantly extinguished within the medium due to the large optical depth of the medium. In typical dispersive probing experiments with ultracold atoms one uses atomic samples with an off-resonant optical depth \( \ll 1 \), but even in this case, if the sample is inhomogeneous in the transverse direction, it can significantly alter the transverse phase profile of the electric field \( \text{within} \) the medium. Such gradient-index lensing effect can significantly affect the interpretation of the dispersive signal, as we show here.

As we will show below, two regimes for lensing exist. In the first regime, the atomic cloud acts as a thin lens where the focal length is much larger than the axial dimension of the sample, and the linearity between the dispersive signal and the atom numbers hold true. In the other regime where the focal length is comparable or smaller than the axial dimension, the linearity breaks down. Hence the relevant parameter is the lens Fresnel number \( F = a^2/f\lambda \), where \( a \) is the transverse dimension of the sample and \( f \) is the focal length. In regime corresponding to \( F \ll 1 \), a thin atomic lens is realized.

3. Propagation of light in atomic gases

3.1. Homogeneous ensembles

To set the scene, we first consider a homogeneous dilute ensemble of atomic scatterers. Since these are discrete point scatterers, there will be randomness present at some level, so we assume that only the average density of atoms is constant and the positions of the scatterers are otherwise random. The fluctuations of
atomic positions lead to incoherent (diffuse) scattering in non-forward directions [47]. In a truly continuous medium, as is well-known [59], such diffuse scattering is absent and the description of wave propagation is strictly defined by refraction, reflection and transmission. To model the propagation of light in a homogeneous medium with discrete random scatterers with a mean inter-particle distance $D$, we consider a plane wave light field polarized in the $x$-direction, given by

$$E_{\text{inc}}(z, t) = \hat{x} E_{\text{inc}}(kz - \omega t)$$

be incident in the medium and focus on a thin slice at $z = 0$ with width $dz \gg D \gg 1/k$ (figure 1). The total field driving the dipoles inside this slab is the sum of the external incident field and the dipolar field coming from all dipoles inside the medium—both the forward-propagating scattered field owing to the slab $z < 0$ and backward-propagating field from slabs at $z > 0$. According to the Ewald–Oseen extinction theorem of classical optics [59, 60], the total scattered field at any point inside the medium can be decomposed into two parts: one that exactly cancels the external field and the other that propagates in the forward direction with an amplitude $2E_{\text{inc}}/(n + 1)$ and a phase $e^{i(kz - \omega t)}$, where $n$ is the refractive index given by

$$n = \sqrt{1 + \frac{1}{\varepsilon_0 \rho \alpha}}. \quad (5)$$

This is a remarkable result in that even though most of the space between the scatterers is empty, the medium acts as having a continuous refractive index determined by the average density and this holds no matter how dilute the system is [59]. Making the assumption that $|n - 1| \ll 1$, we can write the field right after the first slab as $E(z) = E(0)e^{i(k_0 z)}$ which is the effective field driving the dipoles in the following slab. The field after passing through $m$ such slabs so that $l = m\delta z$ the field is $E(z = l) = E(0)e^{ik_0zl} = E(0)e^{ikz}$. The assumption in equation (6) also allows us to expand equation (5) as $n \simeq 1 + \frac{\rho \alpha}{2\varepsilon_0}$, such that,

$$E(z = l) = E(0)e^{i(k_0 + \frac{\rho \alpha}{2\varepsilon_0})l} = E(0)e^{i(k_0 + \frac{\rho \alpha}{2\varepsilon_0})l}. \quad (7)$$

With the further assumption that

$$\frac{k\rho|\alpha|l}{2\varepsilon_0} \ll 1, \quad (8)$$

equation (7) reduces to

$$E(z) \simeq E(0)e^{i(k_0 + \frac{k\rho \alpha}{2\varepsilon_0})z}. \quad (9)$$

In this regime, the scattered light field in the forward direction (the second term in equation (9)) is a linear function of the total number of atoms (for a fixed sample volume), as is frequently required in QND experiments [4, 5, 9, 42, 45]. For instance, assuming an ensemble with a density $\rho = 2 \times 10^{19} \text{ m}^{-3}$, $\Delta = 50\Gamma$ and $\lambda = 780 \text{ nm}$, one must have $l \ll 30 \mu\text{m}$ to fulfill condition 8. In the following we consider the regimes that go beyond this assumption.

3.2. Transversely inhomogeneous ensembles

Trapped atomic ensembles, in practice, often presents a spatially localised sample typically with substantial density gradients both in the radial and axial directions with respect to a probe light beam. Let us focus on
the radial dimension. An atomic cloud with a finite radial extent acts as a diffractive aperture to the incoming beam. Moreover, any refractive index gradient causes the incoming beam to acquire a spatially varying phase shift, causing an effective lensing similar to a gradient-index (GRIN) lens (figure 2). Assuming a Gaussian radial density distribution of atoms \( \rho(r) = \rho_0 \exp(-r^2/\sigma^2_r) \) and substituting into equation (7), we obtain the field after passing through a slab of length \( \delta z \) through the sample:

\[
E(z = \delta z) = E(0)e^{ik(1 + \frac{\alpha}{\rho_0} \delta z)} = E(0)e^{ik\frac{\alpha}{\rho_0} \delta z} e^{-i\frac{k\rho_0^2}{2\sigma^2_r} \delta z}.\]

Taking only into account the radial extent \( r \ll \sigma_r \), such that \( \rho(r) \simeq \rho_0(1 - r^2/\sigma^2_r) \),

\[
E(z = \delta z) = \left\{ E(0)e^{ik\frac{\alpha}{\rho_0} \delta z} e^{-i\frac{k\rho_0^2}{2\sigma^2_r} \delta z} \right\} e^{-i\frac{k\alpha}{\rho_0} \delta z r^2}.\]

This equation clearly shows that the effect of the atomic cloud is to convert a plane wavefront to a paraxial-spherical one, just as a thin lens would do. Comparing equation (10) to the phase transformation of a thin lens with a focal length \( f \) [61]

\[
E \rightarrow E e^{-\frac{ikr^2}{2f}},\]

we obtain an expression for the focal length of the atomic gradient-index lens:

\[
f_a = \frac{\epsilon_o \sigma^2_r}{\alpha \rho_0 \delta z}.\]

Equation (12) reduces to the widely used formula for the focal length of a ball lens [33, 48] in the case \( \sigma_r = \delta z \). We note that for a trapped Bose–Einstein condensate with a Thomas-Fermi (inverted parabolic) radial distribution, the expression is exact. As the light propagates through more and more slabs, the effect of diffraction and lensing has to be taken into account concurrently for each individual slab. For long samples and high atomic densities, the sample will cease to be a thin lens, as light bends significantly within the medium. The expression for the focal length given by equation (12) stays valid for a thick lens as long as the paraxial approximation is valid, and the phase factor in the term inside the curly bracket of equation (10) is less than \( \pi \) [61]:

\[
\frac{k\alpha}{\rho_0} = \frac{\epsilon_o \sigma^2_r}{\alpha \rho_0 \delta z},\]

which is equivalent to requiring \( F \ll 1 \), where the lens Fresnel number \( F \) is defined by

\[
F = \frac{\sigma^2_r}{\lambda f_a}.\]

The lens Fresnel number thus provides a qualitative measure for distinguishing between the thin and thick lens regimes. The left-hand side of equation (13) is the ‘peak phase shift’ in the one-dimensional picture, as evident from equation (9), and thus a peak phase-shift of \( \pi \) delineates between the thin and thick lens regimes.

### 4. Heterodyne detection of dispersive response: phase modulation spectroscopy

Phase modulation spectroscopy (PMS) was developed in the 1980s as an ultra-sensitive probe for small absorption and phase shifts of light [62, 63]. For this, a single-frequency laser source is phase modulated
electro-optically at microwave frequencies and the resultant light beam has frequency sidebands imprinted on it. For weak modulation, only the first order sidebands are prominent, and phase modulation is equivalent to frequency modulation \[64\] (the reason why the technique is also sometimes known as frequency modulation spectroscopy). The microwave frequency is chosen such that one of the sidebands (the probing sideband) is close to an optical resonance of the atoms while the carrier and all other sidebands are far-detuned from the resonance. In the absence of the atoms, the beat notes arising from the interference of the carrier with the positive and negative frequency sidebands, respectively, cancel each other, resulting in a null signal at the heterodyne frequency. When the atoms are present, the probing sideband incurs both extinction and phase shifts, which results in a non-zero signal at the modulation frequency. More recently, the technique has been employed to perform minimally destructive dispersive probing of trapped ultracold samples to probe fast dynamical processes in-situ \[28–31, 42, 51, 65\]. Phase modulation spectroscopy offers several important benefits, passively phase-stable interference arms, excellent common-mode phase noise and low-frequency noise (such as flicker noise) reduction and its relatively simple implementation.

### 4.1. 1D interface: plane wave, homogeneous samples

Let us assume a plane wave \( E_0 = E_0 e^{i(\mathbf{k} z - \omega t)} \) that is electro-optically phase-modulated by a microwave field with frequency \( \Omega \). For a sufficiently small modulation depth \( m \), the resultant field to a good approximation is given by

\[
E_{\text{in}} = \tilde{E}_c e^{-i\mathbf{k} z} + \tilde{E}_b e^{-i(\omega+\Omega)t} + \tilde{E}_r e^{-i(\omega-\Omega)t}. \tag{15}
\]

Here the subscripts ‘c’, ‘b’ and ‘r’ stand for carrier frequency and blue (higher frequency) and red (lower frequency) sidebands and \( |E_b| = |E_c| = \frac{\omega}{\Omega} |E_0| \). The minus sign in front of the red sideband term reflects an additional phase of \( \pi \) that arises from phase-modulation. When this beam passes through a homogeneous ensemble, the probing sideband (which we assume to the red one) acquires a (complex) phase shift \( \delta = \delta_1 + i \delta_t \), such that \( \tilde{E}_r = E_0 e^{i\delta_r} e^{\delta i} \) (see equation (7)), while the carrier and the blue sidebands leave the sample unchanged. The outgoing wave

\[
E_{\text{out}} = \tilde{E}_c e^{i\mathbf{k} z} + \tilde{E}_b e^{-i(\omega+\Omega)t} + \tilde{E}_r e^{-i(\omega-\Omega)t} e^{\delta i} \tag{16}
\]

is detected on a fast photodiode which measures the resultant intensity

\[
I = \frac{1}{2} c \epsilon_o E_{\text{out}}^* E_{\text{out}}. \tag{17}
\]

Filtering out the dc and the \( 2\Omega \) components from the photodetector signal, one obtains the PMS signal

\[
S(\Omega) = \frac{1}{4} m \kappa \epsilon c e_o \left\{ |E_c|^2 (1 - e^{-i\delta}) e^{-i\Omega t} + \text{c.c.} \right\}. \tag{18}
\]

Here \( \kappa \) is the photodetector responsivity and \( \mathcal{A} \) is the detector area. For small phase shifts, \( |\delta| \ll 1 \), the amplitude of the sinusoidal PMS signal is proportional to the phase shift:

\[
S(\Omega) = \frac{1}{2} \mathcal{A} \kappa \epsilon \epsilon_o |E_c|^2 \cos(\Omega t)(i \delta_t - \delta_t). \tag{19}
\]

The quadrature and the in-phase parts of the signal, therefore, measure the absorption and the (real) phase shift respectively—yielding information about the optical response of the sample.

### 4.2. 3D interface: focussed Gaussian beam and trapped inhomogeneous ensembles

For atomic ensembles with small spatial extent, it is necessary to focus the probe light beam to achieve high overlap of the light beam and the atomic distribution (figure 4). The incident light beam is typically derived from a TEM\(_{00}\) mode of a laser, the spatial part of the electric field of which is given by

\[
\tilde{E}(r) = \left[ \epsilon E_0 - \frac{w_0}{w(z)} \exp\left( -\frac{r^2}{w^2(z)} \right) \right] \exp\left( -i[kz - G(z) + \frac{k^2 r^2}{2R(z)}] \right) \tag{20}
\]

where \( \epsilon \) is the polarization of the light, \( w_0 \) is the beam waist, \( w(z) = w_0 \sqrt{1 + (z/z_R)^2} \) is the radius of curvature \( R(z) = z[1 + (z_R/z)^2] \) and Gouy phase \( G(z) = \arctan(z/z_R) \), where \( z_R = \pi w_0^2 / \lambda \) is the Rayleigh range. As in section 4.1, the incident beam is then given by \( E_{\text{in}}(r) = \tilde{E}(r) e^{i\mathbf{k} z} + \tilde{E}_b(r) e^{-i(\omega+\Omega)t} \]

\[-\tilde{E}_r(r) e^{-i(\omega-\Omega)t}, \]

where \( \tilde{E}_{b,c}(r) \) all have the same spatial (Gaussian) mode as defined in equation \(20\).
Assuming again that the carrier and the blue sideband interact negligibly with the atoms, the outgoing beam is given by

$$E_{\text{out}}(r) = \tilde{E}_c(r)e^{-i\omega t} + \tilde{E}_b(r)e^{-i(\omega + \Omega)t} - \{\tilde{E}_c(r) + \tilde{E}_b(r)\}e^{-i(\omega - \Omega)t}. \quad (21)$$

Here

$$\tilde{E}_c(r) = \tilde{E}_{s,\text{tot}}(r) - \tilde{E}_c(r), \quad (22)$$

is the spatial part of the total scattered field, and \(\tilde{E}_{s,\text{tot}}(r)\) is the spatial part of the total field for red sideband in the presence of the atoms. The intensity at a detection point \(r\) in the far field is then given by

$$I_\Omega(r, t) = \frac{1}{2}\epsilon_0 [\tilde{E}_c^*(r)\tilde{E}_c(r)\exp\{i\Omega t\} + \text{c.c.}] \quad (23)$$

and the PMS heterodyne signal is given by

$$S(\Omega) = \frac{1}{2}\kappa\epsilon_0 \left[\exp\{i\Omega t\} \int_\mathcal{A} r^3 \tilde{E}_c^*(r)\tilde{E}_c^*(r) + \text{c.c.}\right],$$

where \(\mathcal{A}\) is the effective aperture of the system, and \(\kappa\) is the responsivity of the photodetector. The signal measures the overlap of the scattered mode with the incident reference mode (carrier), or equivalently by equation (22), any deviation of the probe (red) sideband from the incident mode. It is thus sensitive to extinction, phase shift or any propagation effects such as lensing, of the red sideband due to the presence of the atoms.

### 4.3. FMS signal detection at the baseband

The PMS spectroscopy signal is typically in the microwave domain \((\Omega > 1 \text{ GHz})\). It is therefore convenient to down-convert it to the baseband by mixing the signal with a reference microwave signal (local oscillator (LO)) that is phase-coherent to the signal employed for phase-modulation: \(S_{\text{ref}} = V_0 \cos(\Omega t + \theta) = (V_0/2)(\exp(i\Omega t + \theta) + \text{c.c.})\). The baseband signal is then given by

$$B_t = \frac{1}{4}\kappa\epsilon_0 V_0 \cos \theta \Re\{\int_\mathcal{A} r^3 \tilde{E}_c^*(r)\tilde{E}_c^*(r)\}. \quad (25)$$

The signal in equation (25) can be maximised by adjusting the LO phase such that \(\theta = n\pi\) with \(n = 0, 1, 2, \ldots\). In our implementation, we use IQ demodulation where we generate a second signal by mixing the heterodyne signal with \(S_{\text{ref}} = V_0 \sin(\Omega t + \theta) = (V_0/2i)(\exp(i\Omega t + \theta) - \text{c.c.})\), resulting in the baseband signal

$$B_Q = \frac{1}{4}\kappa\epsilon_0 V_0 \sin \theta \Im\{\int_\mathcal{A} r^3 \tilde{E}_c^*(r)\tilde{E}_c^*(r)\}. \quad (26)$$

From equations (25) and (26), we eliminate \(\theta\):

$$B = \sqrt{B_t^2 + B_Q^2} = \frac{1}{4}\kappa\epsilon_0 V_0 \left|\int_\mathcal{A} r^3 \tilde{E}_c^*(r)\tilde{E}_c^*(r)\right|. \quad (27)$$

### 4.4. Comparison with phase-contrast imaging

It is instructive to compare the PMS signal with dispersive imaging methods. Without much loss of generality, we focus on phase-contrast imaging (PCI) \([34–38]\) as a representative system. In PCI, a single-frequency off-resonant probe \(E_0\) with transverse dimension much greater than the atomic sample is employed, thus the last two phase terms in equation (20) can be considered constants. Moreover, the outgoing light is detected in a spatially resolved way, so spatial integration of equation (24) is absent. The theoretical description is often \([34, 36]\) given using a 1D interface, similar to section 4.1, where the probe light acquires a transversely dependent excess (complex) phase \(\delta\) after passing through the atoms. The outgoing field is given by \(E_r \exp(-i\delta) = E_0 + E_r[\exp(-i\delta) - 1]\) and the measurement of the outgoing intensity bears no information about the dispersive response of the atoms, i.e., real(\(\delta\)). This information can, however, be retrieved by selectively shifting the phase of \(E_0\) by a known amount \(\delta_{\text{ref}}\) subsequent to the atoms by, e.g., placing a phase-mask at the focal place of a lens, such that the total field becomes \(E_r \exp(-i\delta_{\text{ref}}) + E_r[\exp(-i\delta) - 1]\). This then allows the transverse phase distribution \(\delta\) and thereby the atomic column density to be mapped.

We note that in the regime where the focal length of the atomic lens is comparable to the longitudinal dimension of the atomic cloud, the above 1D picture is insufficient. For example in the extreme case where
Figure 3. Geometrical effects in atom-light interfacing. (a) For the transverse size of the sample much larger than the probe beam waist ($\sigma_r \gg w_0$), the overlap of the scattered mode and incident mode is perfect, but many atoms do not participate in the process. (b) For $\sigma_r \ll w_0$, the scattered mode creates a diffraction cone that partly falls out of the incident mode. (c) A pencil-shaped sample results in perfect mode overlap. (d) As shown in this paper, there exist regimes where atomic lensing in pencil shaped samples causes light to bend out the diffraction cone.

4.5. Comparison with purely diffractive models

We here briefly summarise the purely diffractive model of three-dimensional light–atom interfaces, as employed in [46, 47] and compare that with the strongly lensing regime we are interested in. In QND experiments, both the atomic ensemble and the probe light beams are localised for enhanced collective effects. In figure 3, we show a number of possible geometric configurations for this. The probe light beam which has typically a Gaussian profile defines the reference mode for interferometric detection and in the far field has an angular width defined by

$$\theta_G = \frac{\lambda}{\pi w_0}. \quad (28)$$

The interferometric signal is maximised when the scattered light from the atom has an angular width $\lesssim \theta_G$. This is readily achieved in the setting of figure 3(a), where the transverse size of the atomic ensemble $\sigma_r$ is much larger than the probe beam. This, however, is not ideally suited for QND experiments, as most of the atoms will not experience the quantum backaction from light. In the opposite extreme where $w_0 \gg \sigma_r$ (figure 3(b)), the atomic sample acts as an effective aperture that diffracts light into a 'diffraction cone', the cone defined by the first minimum in the Airy pattern resulting from the diffraction [61]. This has an angular width

$$\theta_D = \frac{\lambda}{2\sigma_r}, \quad (29)$$

that exceeds the $\theta_G$. It was argued in [46, 47] that in an elongated pencil-shaped atomic ensemble with $\sigma_r \simeq w_0$ (figure 3(c)), the two angles are well-matched and results in maximal interferometer signal. As we showed in section 3.2, however, in this configuration there exists regimes where strong atomic lensing results in light to focus or de-focus at distances comparable to the length of the atomic sample (figure 3(d)), diverting light to an angular distribution characterised by the deflection angle of the lens

$$\theta_L = \frac{\sigma_r}{f_n}. \quad (30)$$

In this case, if $\theta_L > \theta_D$ light–atom interface is sub-optimal despite having the optimal geometry in the diffractive regime. From equations (12), (29) and (30), we see that the condition $\theta_L > \theta_D$ is equivalent to the lens Fresnel number $F \gtrsim 0.5$. We also note that $F$ is linearly proportional to the length of the ensemble.
compared to the wavelength of light (equations (12) and (14)) meaning that at a given peak density, elongated ensembles are particularly susceptible to this.

5. Experiments

5.1. Experimental setup
We perform experiments with ultracold samples of $^{87}\text{Rb}$ atoms in the $|F=2, m_F=2\rangle$ spin-state confined by an Ioffe–Pritchard (IP) magnetic trap with trapping frequencies $\omega_x = \omega_y = 2\pi \times 166$ Hz and $\omega_z = 2\pi \times 17$ Hz, loaded from a magneto-optical trap (MOT) using a movable quadruple trap. The details of our setup have been reported elsewhere [30, 31]. For ultracold temperatures considered here, the atoms reside close to the trap minimum where the magnetic field direction is predominantly in the $z$-direction, which we choose as the quantisation axis. We control the number of atoms in the samples by adjusting the MOT loading while their temperatures are controlled by adjusting the final point of the radio-frequency ramp used for evaporative cooling. The setup for phase-modulation spectroscopy is shown schematically in figure 4. In short, a master laser is frequency-locked to the $F=2 \rightarrow F'=3$ atomic transition of $^{87}\text{Rb}$ using saturation absorption spectroscopy. The laser light used as the carrier for dispersive probing is derived from a narrow-linewidth commercial diode laser and its frequency is stabilised using an offset beat-note lock using light from the master laser as reference. The probe laser can be locked to up to 7 GHz away from the resonance on either side. We set the carrier frequency to be on the blue (higher frequency) side of the $F=2 \rightarrow F'=3$ resonance. The light passes through a fiber-coupled electro-optic modulator which is driven with a microwave frequency $\Omega = 2\pi \times 3.7$ GHz, and at a low modulation depth such that only the first-order frequency sidebands are predominant and contain about 5% of the total power each. The red sideband is placed close to the $F=2 \rightarrow F'=3$ resonance with a detuning $\Delta_r$ that is adjusted by changing the offset-lock frequency. The light is focussed to a $1/e^2$ waist of $w_0 = 36$ $\mu m$ onto the atomic sample and is collected with a fast photodetector following an optical system with an effective numerical aperture of 0.025 and with a collection efficiency of 75% as measured without any atoms present. The photodetector signal is amplified before being demodulated using an IQ-mixer and sampled on a digitizer. The probe beam triplet is $\sigma^+$ polarized and has a total power of 20 $\mu W$ before the atoms which we illuminate for a duration of 5 $\mu s$. For $\Delta_r = 2\pi \times 300$ MHz, the smallest detuning used in this work, this corresponds to about 0.25 photon spontaneously scattered per atom per pulse. Due to the cycling nature of the $\sigma$ transition, optical pumping to other ground spin-states is negligible. The samples are subsequently imaged using resonant absorption imaging in time-of-flight to deduce atom number.

Spontaneous scattering of probe photons and the resulting fluctuations of the radiation force results in recoil heating which is the predominant source of heating in our experiments. Every spontaneous photon scattering event adds $2E_{\text{rec}}$ amount of heat on average per atom where $E_{\text{rec}} = \frac{\hbar \omega^2}{2m}$ is the recoil energy, $m$ being the mass of the atom [52]. Even for the smallest frequency detuning $\Delta_r = 50\Gamma$ used here, this amounts only to heating of the atomic cloud by about 90 nK during the probe pulse from the red sideband. As spontaneous photon scattering rate scales as $\sim 1/\Delta^2$, the corresponding heating from the carrier and other sidebands is much smaller than this. Inelastic heating due to Raman processes leading to the change of the atomic ground state is absent in our case. Due to the large depth of our magnetic trap, we observe no discernible atom losses from the probe pulse.
The in-phase and quadrature components of the signal recorded with the digitizer are processed using equation (27) and integrated over the length of the pulse to obtain the dispersive signal. For low atom numbers, this signal is a direct measure of the phase shift of the red sideband due to the atomic medium in the limit where the phase shift of the carrier and the blue sideband are negligible, as discussed in sections 4.1 and 4.2. The general treatment of the case where the carrier and the blue sideband acquire phase shifts due to atoms as well as the red sideband is given in appendix B. In this case our experimental setup measures the sum of the phase shifts acquired by the red and the blue sidebands relative to the carrier phase shift. This means the measured dispersive signal is smaller than what it would be in the ideal case where \( \Omega \gg \Delta_r \) approximately by a factor \( h = 1/(1 + 1/(\pi \eta^2) - 1/(2\Gamma^2)) \), where \( \eta = \Omega/\Delta_r \). For the largest \( \Delta_r \) (smallest \( \eta \)) employed here, this leads to \( \sim 40\% \) reduction of the signal compared to the idealised case \( \eta \to \infty \), whereas for the case of the smallest \( \Delta_r \), it is around 12\%. These factors are incorporated in our data analysis.

5.2. Experimental observations and qualitative interpretation

We measure the dispersive signal as a function of atom number for atomic samples at three different nominal temperatures \( T = 1 \) (\( \pm 0.2 \)), 2.5 (\( \pm 0.5 \)) and 12 (\( \pm 2 \)) \( \mu \)K. For each temperature, we conduct experiments at four different detunings of the probe sideband from the resonance \( (\Delta_r = \pm 50, \pm 100, \pm 170, \pm 235\Gamma) \); the carrier and the blue sidebands are detuned from the resonance by \( \Delta_r = \Delta_\sigma + \Omega \) and \( \Delta_\sigma = \Delta_\sigma + 2\Omega \), respectively. Figure 5 presents our results as a 4 \( \times \) 3 matrix of subplots, where \( T \) is constant along each column and \( \Delta_r \) is constant along each row. Evidently, in each of the twelve subplots of figure 5, the signal initially grows linearly with atom number. Beyond a certain point in the atom number which depends on both the temperature and the probe detuning, however, the signal deviates from linearity, and eventually reaches saturation. In all cases, the peak optical depth remains low (for \( N = 5 \times 10^6 \), \( T = 1 \) \( \mu \)K and \( \Delta_r = 2\pi \times 100\Gamma \), peak OD \( \sim 0.03 \)) accounting for a small fraction of incident light to be diffusely scattered at large angles. In addition, large probe detuning means that even at the highest peak atomic densities presented here (\( \rho \sim 5 \times 10^{19} \text{ m}^{-3} \)) the effect of light-induced dipole–dipole interaction is completely negligible [17]. Hence the observed plateauing of the signal is neither a result of extinction of the probe side band due to the sample being optically thick nor can it be attributed to a change in the many-body optical response. It is clear that observed signal reaches saturation even at very large detuning (\( \Delta_r = +235\Gamma \)) and relatively high temperature (\( T = 2.5 \) \( \mu \)K).

It is instructive to analyse the observations in figure 5 qualitatively using the theory presented in sections 3.2 and 4.5. At 12 \( \mu \)K, the transverse size of the atomic sample (\( \sigma_t = 44 \mu \)m) is well-matched to the probe beam waist size, whereas the longitudinal size (\( \sigma_l = 450 \mu \)m) is much larger than that, thus realising the scenario in figure 3(c). This means that the diffraction cone stays well within the reference spatial mode of the probe beam. For a fixed mode overlap between the reference beam and the scattered light beam (equation (24)), one then expects that as the number of scatterers grows, the observed signal would grow linearly. This is indeed the case for \( \Delta_r = 235\Gamma \) and 170\( \Gamma \). This linear behaviour with near-perfect mode-matching between the scattered mode and the reference mode is qualitatively similar to the linear, homogeneous case discussed in section 3.1 where a homogeneous ensemble is illuminated with a plane wave light field. However, at smaller detunings \( \Delta_r = 100\Gamma \) and 50\( \Gamma \), a deviation from this expected linearity is manifest.

At lower temperatures, the transverse size of the atomic cloud is smaller than the probe waist (\( \sigma_t = 20 \mu \)m and 13 \( \mu \)m, for \( T = 2.5 \) \( \mu \)K and 1 \( \mu \)K, respectively). In this case the diffraction angle \( \theta_D \) (equation (29)) exceeds the angle subtended by the reference Gaussian beam in the far-field, realising a scenario similar to figure 3(b). As a result the scattered light field partly resides outside the reference mode, and thus the overlap the field in equation (24) is modified. Nevertheless in this regime, as the temperature and therefore the spatial extent stays the same, the signal should grow linearly with atom number, since in this diffraction-dominated regime, the atoms act as an aperture that cooperatively diffracts light and as long as the diffraction angle stays constant, the signal is expected be linear with atom number initially and we observe this behaviour for high detuning and low atom numbers. For higher atom numbers and lower detunings, the signal should be oscillatory. However, the observed signal rather, beyond a certain critical atom number, shows strong plateauing.

We ascribe the saturating behaviour of the signal to atomic GRIN lensing. For low atom numbers and high detuning, the focal length of the atomic lens is long, and as a result, the deflection angle of the lens \( \theta_L \) (equation (30)) is smaller than both \( \theta_D \) and \( \theta_D \). This means lensing is a much less prominent effect than diffraction. As the atom number grows and/or the frequency detuning is reduced, the focal length of the lens gradually decreases and at some point \( \theta_L \) will exceed \( \theta_D \) and consequently the mode-overlap in equation (24) will become dependent on atom number even for a fixed spatial extent of the atomic sample.
Figure 5. Dispersive signal as a function of atom number. Each column represents a fixed temperature and each row represents a fixed probe detuning. Hence, sample temperature decreases from left to right and the red sideband detuning of the probe decreases from top to bottom. The bottom horizontal axis on each plot shows the atom number whereas the top horizontal axis shows the Fresnel number \( F \) based on equation (12). The black circles indicate the measured signals. The solid lines show the results of the simulations based on a paraxial wave equation. The vertical dashed lines indicate \( F = 0.5 \) — distinguishing the lensing-dominated regime to its right from the diffraction-dominated regime to its left.

This is the lensing dominated regime, and as shown in figure 5, can be reached even at moderate atom numbers and large detunings at low temperatures. In this regime, the probe beam is nearly-entirely diverted out of the reference mode by the atomic ensemble. Adding atoms to the sample then adds very little to the observed signal.

We expect from the discussion in section 3.2 that the diffraction dominated and lensing dominated regimes result depending on whether that atomic GRIN lens is a thin lens or a thick lens and is conveniently characterised by a single parameter, the lens Fresnel number. In figure 5, we show the Fresnel number for the atomic GRIN lenses formed under the experimental conditions of the data (top x-axis), calculated using equations (12) and (14). Evidently, the approach of the dispersive signal to saturation is linked to the Fresnel number approaching 1. The saturation behaviour manifest beyond \( F \sim 0.5 \), shown as dashed vertical lines in figure 5. This, as mentioned in section 3.2, is equivalent to a ‘peak phase shift’ of \( \pi \) of light within the radius of the probe beam due to the atomic medium. The \( F = 0.5 \) line thus qualitatively separates the diffraction-dominated regime \( (F \lesssim 0.5) \) from the lensing-dominated regime \( (F > 0.5) \).

In figure 6, we compare the 2.5 \( \mu \text{K} \) dispersive signals for the four detunings explored in our experiments by rescaling the signals proportionally to detuning. For low atom numbers all curves coalesce on each other and follow a common straight line. As the atom number grows, the curve corresponding to the smallest detuning \( (\Delta r = 50\Gamma) \) diverts away, followed consecutively by curves corresponding to the higher detuning.
Figure 6. Rescaled dispersive signals as a function of atom number at temperature $T = 2.5 \mu K$. Here the data from the second column of figure 5 are rescaled by factors of 235/50, 170/50, 2 and 1 for $\Delta r = 235, 170, 100$ and $50 \Gamma$, shown as hexagrams, diamonds, squares and circles respectively. The solid lines show the results of the simulations based on a paraxial wave equation, which are scaled by the same factors. Vertical dashed lines are guide to eyes showing where a pair of curves diverge from each other.

doing the same at exceedingly higher atom numbers. This also shows that atomic lensing causes the rescaled dispersive signal to plateau beyond a critical atom number that is a function on atomic detuning. This critical atom number can be surprisingly low even for relatively large detuning (for $\Delta r = 170 \Gamma$ this takes place at $N = 3 \times 10^6$).

At lower temperatures and at smaller detunings, the dispersive signals in figure 5 exhibit small residual oscillations for high atom numbers. This can be explained by noting that in equation (10) the term $e^{\delta m_o \rho_r / \alpha r}$ inside the curly bracket is oscillatory. This term arises from the first term in the expansion of the Gaussian atom distribution: $\rho(r) \simeq \rho_0(1 - r^2 / \sigma_r^2)$ and owe itself to the fact that at the centre of the radial distribution, the density gradient is zero. Atoms near $r = 0$ therefore give rise to little lensing, behave like those in a uniform distribution and the small oscillatory behaviour reflects the multiple $\pi$ phase shifts of light caused by this small subset of atoms.

5.3. Quantitative analysis: paraxial propagation model

To model the data in figure 5 quantitatively, we solved the paraxial wave equation for each of the propagating (scalar) electric fields [66] corresponding to the carrier, and the two sidebands:

$$\frac{\partial \tilde{E}}{\partial z} = \frac{i}{2k} \nabla^2 \tilde{E} + \frac{1}{2}i k \chi \tilde{E},$$

(31)

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Laplacian and $\chi = \rho \alpha / \epsilon_o$ is the susceptibility of the medium, where we have assumed that $\rho \alpha \ll 1$, so the Lorentz–Lorenz correction term [61] can be neglected. For a radially symmetric system that is not necessarily Cartesian separable, as in our case, the equation is most conveniently solved using cylindrical co-ordinates and a split-step Hankel method (see appendix A).

In order to directly compare the results of the theoretical model with our observations, a number of experimental constants of the part of the apparatus following the detection of light must be taken into account. This includes the quantum efficiency of the detector, gain of the electronic amplifiers and the conversion coefficient of the demodulation stage. In our experiments, we operated these experimental components well within their linear response regimes, and as a result, their contribution to the observed signal is a constant calibration factor. To determine this factor, we divided the results of our simulation for the largest temperature and detuning by the corresponding observed signals (the top-left subfigure of figure 5, i.e., the regime where the signal is linear in atom number). Once fixed, the same calibration factor is then used to scale the theoretical results for all other subplots.
Figure 7. Transition from the diffraction-dominated regime to the lensing-dominated regime. (a) shows the intensity distribution of the light propagating along the z-direction through an atomic cloud (dashed ellipse at origin) of temperature $T = 2.5 \, \mu \text{K}$ for (top to bottom): $(N = 1 \times 10^5, \Delta r = 50 \Gamma)$, $(N = 5 \times 10^5, \Delta r = 50 \Gamma)$, $(N = 1.5 \times 10^6, \Delta r = 50 \Gamma)$, $(N = 2.5 \times 10^6, \Delta r = 50 \Gamma)$, $(N = 3.5 \times 10^6, \Delta r = 50 \Gamma)$, and $(N = 3.5 \times 10^6, \Delta r = -50 \Gamma)$. (b) and (c) show the correspondent intensity and phase distributions of light in the detection plane in our experiment. The dashed lines show the case with no atoms present.

For each experimental data point, we calculate the dispersive signal from the fields obtained from the solutions of equation (31) with experimental parameters (number of atoms, temperature) as inputs. In figures 5 and 6, the results of the modeling are shown as lines that connect between each calculated point (small kinks in the model curves are an artefact from performing calculations at the actual measured temperatures which have small fluctuations about the nominal sample temperature). The simulations are generally in excellent agreement with our experimental observations and capture several important features and in particular underpin the validity of the qualitative criterion established in sections 3 and 4 on the impact of atomic lensing on the signal. We stress that due to the calibration procedure described above, our model has no free parameters and all the subplots and their vertical axes are directly comparable to each other. We also note that for the regime $F \gtrsim 1$, the agreement between the experiments and the theoretical model becomes slightly weaker. We attribute this to light travelling at large angles with respect to the optical axis which leads to uncertainties in the collected fraction of light that we did not account for in our theory, originating from, e.g., the non-paraxial parameters of the collecting lenses.

Figure 7 shows the propagation of an incident Gaussian light beam through atomic samples of varying atom numbers calculated using the paraxial wave equation. For large atom numbers, light deviates significantly inside the medium (figure 7(a)) showing that the atomic medium acts as a negative or positive lens depending on the sign of the detuning. This leads to a significant redistribution of the intensity and phase of the light field at the detection plane and a reduction of its overlap with the incident mode. The light distribution of the detected field in our experiment is at the focal plane of a lens and therefore shows the Fourier transform of the far-field distribution. The enhanced intensity peaks at the centre of the
detection plane in figure 7(b) correspond to the light travelling outside the diffraction cone in the far-field (prior to the collection lens) due to the lensing action of the atomic ensemble.

5.4. Implications for dispersive imaging techniques

We finally discuss the implications of our findings in relation to dispersive imaging techniques considering phase-contrast imaging as a prototypical system, as introduced in section 4.4. In such imaging schemes the reference mode (the incoming probe beam) occupies a large area in the detection plane compared to the scattered light mode (see for example, figure 1 of [36]). The dispersive image results from the interference of the two in the detection plane. Such imaging schemes often use the axis with the lower optical depth as the imaging axis [34–38]. As a result the deflection angle of light owing to atomic lensing is often well within the reference mode in the detection plane, and interference can take place. The case of a strong atomic lens is equivalent to scattered light travelling through an extra lens on its way to the camera, resulting in the acquisition camera to be out-of-focus and the ensuing image significantly blurred. Image degradation occurs where the light bent by the atomic lens does not make it to the detection region due, for example, to the finite aperture of the collection optics. This inevitably leads to information about atomic column density to be lost.

An important consideration for phase-contrast imaging of dense, small atomic samples therefore is the numerical aperture of the collection optics compared to the atomic lens deflection angle. In [36], for example, the maximum lens deflection angle is 0.015 radian for the largest atom number in the smaller atomic dimension whereas the numerical aperture of their system is 0.1. As a result, multiple modulo of π phase zones of light passing through the atoms could be observed, along with a slight image deformation. In contrast, in our setup (i) we probe along the longest dimension of the atomic cloud, and the largest atomic lens deflection angle can be substantial compared to the effective numerical aperture of the system and (ii) the reference mode (the carrier light field) is highly localised both in the object and the detection planes.

6. Summary and outlook

We have theoretically and experimentally studied the effect of atomic lensing on the interferometric, dispersive detection of ultracold atomic ensembles in an effective zero baseline heterodyne Mach–Zender arrangement. We derived an expression for the focal length of the atomic lens and have shown that the lens Fresnel number provides a useful and intuitive measure of the distinction between diffractive and strong-lensing regimes that lead to quantitatively different dependence of the dispersive signal on the atom number. We stress that while our experimental observations result from a setup based on frequency modulation spectroscopy, both the experimental and theoretical results are relevant to any interferometric detection of coherently scattered light from an atomic ensemble. We also derived an expression for the FMS signal for a three-dimensional interface of the sample and the probe and showed that it is equivalent to zero-baseline Mach–Zender interferometers that measure the spatial-mode overlap of the coherently scattered light field with a reference field and are realized in many different ways [2, 43].

Our work augments the theoretical work on geometrical effects on dispersive detection of spin-ensembles [46, 47] to include the effect of the atomic gradient-index lensing and experimentally demonstrate the effect. The experimental data are well-described by a wave-equation model, but more importantly, are well-supported by analytical criteria based on the atomic GRIN lens focal length and Fresnel number. This provides a useful guide to the design choice for the atomic and probe beam geometries for dispersive detection. It is typically assumed, for instance, that for a pencil-shaped atomic sample and a probe beam of similar transverse size as the sample leads to an optimal detection of the coherently scattered field in the far-field [47, 67]. Our analysis and experimental data show, however, that for a given density if the length of the sample exceeds a certain point, the scattered field can encroach beyond the diffraction cone. Under such conditions, the detection signal will saturate beyond a critical atom number dependent on the Fresnel number of the atomic lens. The linearity of the signal for high atom numbers could be re-established by setting the probe detuning further from resonance, but this reduces the signal-to-noise ratio for probing lower atom numbers, and will be a limiting factor especially in cases where the atom numbers are dynamically changing [24, 28, 30].

The complementary effect of atomic lensing is the mechanical backaction on atoms resulting from the conservation of momentum. This causes a redistribution of atomic momenta when an incident light beam is re-directed collectively by an atomic ensemble. Such optomechanical forces can lead to interesting phenomena such as electrostriction, self-structuring and effective light-induced two-body
interactions between atoms [68–70]. In the strong lensing regime, we have observed a novel geometry-dependent optomechanical self-propelling effect, which will be the subject of an upcoming publication [71]. In future studies we will explore the possible use of atomic lensing for high numerical aperture imaging and to the degree co-operative effects owing to long-range interactions can influence atomic lensing.

Finally quantum statistics of the atoms can have a substantial impact on the optical response of atomic ensembles as theoretically discussed in [72, 73] and experimentally observed in [74] for bosonic atoms. We have recently conducted measurements of the effect of quantum statistics on the optical response in fermions. The interpretation of our observations is significantly compounded by atomic lensing effects, and are currently under investigation.

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Appendix A. Paraxial wave equation in radial co-ordinates and the Hankel transform

We consider the field in the cylindrical co-ordinates

\[ E(x, y, z) \leftrightarrow u(r, \theta, z). \]  

(32)

For a field whose radial profile is of the form \( u(r) \exp \{ \text{i} m \theta \} \), the two-dimensional Fourier transform is given by

\[
\mathcal{F}\{ u(r)e^{\text{i} m \theta} \} (k_r, \phi) = \mathcal{H}_m \{ u(r) \} (k_r) e^{\text{i} m \phi},
\]

(33)

where \( r, \theta \leftrightarrow k_r, \phi \) are the variable pairs in the real and the transform space respectively and

\[
\mathcal{H}_m \{ u(r) \} (k_r) = \int_0^\infty u(r) J_m (rk_r) \, dr
\]

(34)

is the Hankel transform of the order \( m \), where \( J_m \) is the \( m \)th order Bessel function [75].

In cylindrical co-ordinates, the transverse Laplacian operator in equation (31) reads

\[ \nabla^2_T = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \]

(35)

Substituting \( E(x, y) \rightarrow u(r)e^{\text{i} m \theta} \) in equation (31) and noting that for our situation \( m = 0 \) (zero-fold radial symmetry), we obtain the paraxial wave equation in radial co-ordinates

\[
\frac{\partial u}{\partial z} = -\frac{i}{2k} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) u(r) + \frac{1}{2} k \chi (r) u(r).
\]

(36)

A split-step spectral method [76] can now be readily applied where the field is propagated along \( z \)-axis with steps of \( \delta z \) the diffraction term (first term on the right hand side of equation (31)) is applied for half the propagation length:

\[
u_{j+1/2} = \mathcal{H}^{-1} \left[ e^{-\frac{1}{2}k\delta z} \mathcal{H}(u_j) \right].
\]

(37)

Here \( \mathcal{H}^{-1} \) is the inverse Hankel transform. The phase shift term is then applied

\[ v_{j+1/2} = e^{\text{i} \delta k \chi(r) \delta z} u_{j+1/2}, \]

(38)

which is followed by applying the diffraction term again for another half-step, yielding the field at \( z = z + \delta z \):

\[ u_{j+1} = \mathcal{H}^{-1} \left[ e^{-\frac{1}{2}k\delta z} \mathcal{H}(v_{j+1/2}) \right]. \]

(39)

The method is computationally efficient and third-order accurate. The Hankel transform and its inverse are efficiently evaluated using a quasi-fast Hankel algorithm [77, 78]. The logarithmic radial grid used in this algorithm has the advantage that it samples densely in the region near \( r = 0 \), where our atomic density is the highest and sparsely far-away from the paraxial region. This, combined with the one-dimensional...
nature of the problem, allows us to evaluate the field over a large transverse region and propagate it directly to the far-field through external lens elements and apertures.

Appendix B. Phase shifts of the carrier and the non-probing sideband

In general, the carrier and the non-probing sideband also acquire non-zero phase shifts and extinctions upon passing through the atoms. The outgoing beam in this case has the form, in contrast to equation (16),

$$ E_{\text{out}} = e^{-i(\omega t + \delta_c)} \left[ E_c + E_b e^{-i(\Omega t - \phi_b)} - E_c e^{i(\Omega t + \phi_c)} \right], \tag{40} $$

where $\phi_c(b) = \delta_c(b) - \delta_c$ are the relative phase shifts of the red (blue) sideband with respect to the carrier. The intensity $I = \frac{1}{4} c c^*_0 |E_{\text{out}}|^2$, upon filtering out dc $2\Omega$ components is given by

$$ I = c c^*_0 \{ |E_c||E_b| \cos(\Omega t + \phi_c) - |E_c||E_b| \cos(\Omega t - \phi_b) \}. \tag{41} $$

The quadrature components are obtained by mixing this with a local oscillator as in section 4.3, resulting in baseband signals $B_I = \frac{K}{2} \{ \cos(\phi_c + \theta) - \cos(\phi_c - \theta) \}$ and $B_Q = \frac{K}{2} \{ \sin(\phi_c + \theta) - \sin(\phi_c - \theta) \}$, where $K = A \kappa c c^*_0 V_0 |E_c||E_b|$, assuming $|E_c| = |E_b|$. The dispersive signal is then given by

$$ S = K \sqrt{2-2 \cos(\phi_c + \phi_b) + \sin(\phi_c + \phi_b)} = K \frac{\sin(\phi_c + \phi_b)}{2} \sim \frac{K}{4} (\phi_c + \phi_b), \tag{42} $$

in the limit of small phase shifts.

For three-dimensional interfaces, the field magnitudes $|E_{b,c}|$ and relative phase shifts $|\phi_{b,c}|$ in equation (14) depend on spatial co-ordinates and are derived from the solutions of the paraxial wave equations. In the general case where $|E_c| \neq |E_b|$ at the detector plane for all $r$, equation (42) above becomes

$$ S = Q \sqrt{\int_A d^2 r E_c(r) \left[ E_b^2(r) + E_c^2(r) - 2 E_b(r) E_c(r) \cos(\phi_c(r) + \phi_b(r)) \right]}^{1/2}, \tag{43} $$

where $Q$ is a constant.

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