Sparsification of Matrices and Compressed Sensing

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June 12, 2015

Abstract

Finding practical matrix constructions and computationally efficient recovery algorithms for compressed sensing is an area of intense research interest. Many probabilistic matrix constructions have been proposed, and it is now well known that matrices with entries drawn from a suitable probability distribution are essentially optimal for compressed sensing.

Potential applications have motivated the search for constructions of sparse compressed sensing matrices (i.e. matrices containing few non-zero entries). Various constructions have been proposed, and simulations suggest that their performance is comparable to that of dense matrices. In this paper we study the effect of sparsification on compressed sensing for the first time. Extensive simulations suggest that sparsification leads to a marked improvement in compressed sensing performance for a large class of matrix constructions and for many different recovery algorithms.

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1 Introduction

Compressed sensing is a new paradigm in signal processing, developed in a series of groundbreaking publications by Donoho, Candès, Romberg, Tao and their collaborators over the past 10 years [Donoho, 2006, Candès et al., 2006a, Candès et al., 2006b]. Many real-world signals have the special property of being sparse – they can be stored much more concisely than a random signal. Instead of sampling the whole signal and then applying data compression algorithms, sampling and compression of sparse signals can be achieved simultaneously. This process requires dramatically fewer measurements than the number dictated by the Shannon-Nyquist Theorem, but requires complex measurements that are incoherent with respect to the signal. The compressed sensing paradigm has generated an explosion of interest over the past few years within both the mathematical and electrical engineering research communities.

A particularly significant application has been to MRI, for which compressed sensing can speed up scans by a factor of five, either allowing increased resolution from a given number of samples or allowing real-time imaging at clinically useful resolutions. A major breakthrough achieved with compressed sensing has been real-time imaging of the heart [Uecker et al., 2010, Lustig et al., 2008]. The US National Institute for Biomedical Imaging and Bioengineering published a news report in September 2014 describing compressed sensing as offering a ‘vast improvement’ in paediatric MRI imaging [Kern, 2014]. Emerging applications of compressed sensing in data mining and computer vision were described by Candès in a plenary lecture at the 2014 International Congress of Mathematicians [Candès, 2014].

The central problems in compressed sensing can be framed in terms of linear algebra. In this model, a signal is a vector $v$ in some high-dimensional vector space, $\mathbb{R}^N$. The sampling process can be described as multiplication by a specially chosen $n \times N$ matrix $\Phi$, called the sensing matrix. Typically we will have $n \ll N$, so that the problem of recovering $v$ from $\Phi v$ is massively under-determined.

A vector is $k$-sparse if it has at most $k$ non-zero entries. The set of $k$-sparse vectors in $\mathbb{R}^N$ plays the role of the set of compressible signals in a communication system. The problem now is to find necessary and sufficient conditions so that the inverse problem of finding $v$ given $\Phi$ and $\Phi v$ is efficiently solvable.

If $u$ and $v$ are distinct $k$-sparse vectors for which $\Phi u = \Phi v$, then one of them is not recoverable. Clearly, therefore, we require that the images of all $k$-sparse vectors under $\Phi$ are distinct, which is equivalent to requiring that the null-space of $\Phi$ not contain any $2k$-sparse vectors. There is no known polynomial time algorithm to certify this property. We refer to the problem of finding the sparsest solution $\hat{x}$ to the linear system $\Phi \hat{x} = \Phi x$ as the sparse recovery problem. Natarajan has shown that the problem is NP-hard when no additional assumptions are made about $\Phi$ [Natarajan, 1995].

Compressed sensing can be regarded as the study of methods for solving the sparse recovery problem and its generalisations (e.g., sparse approximations of non-sparse signals, solutions in the presence of noise, etc.) in a computationally efficient way. Most results in compressed sensing can be characterised either as certifications that the sparse recovery problem is solvable for a restricted class of matrices, or as the development of efficient computational methods for sparse recovery for some given class of matrices.

One of the most important early developments in compressed sensing was a series of results of Candès, Romberg, Tao and their collaborators. They established fundamental constraints for sparse recovery: one cannot hope to recover $k$-sparse signals of length $N$ in less than $\Theta(k \log N)$ measurements under any circumstances. (For $k = 1$, standard results from complexity theory show that $\Theta(\log N)$ measurements are required.) They then established that several classes of random matrices meet this bound asymptotically. Important examples are the random Gaussian ensemble, which has entries drawn from a standard normal distribution, and the...
random Fourier ensemble, a random selection of rows from the discrete Fourier transform matrix [Candès et al., 2006a, Candès et al., 2006b].

The main tool developed by Candès et al. is the concept of restricted isometry parameters (RIP), which measure how the sensing matrix $\Phi$ distorts the $\ell_2$-norm of sparse vectors. Specifically, $\Phi$ has the RIP$(t, \varepsilon)$ property if, for every $t$-sparse vector $v$, the following inequalities hold:

$$(1 - \varepsilon)|v|_2^2 \leq |\Phi v|_2^2 \leq (1 + \varepsilon)|v|_2^2.$$ 

Tools from Random Matrix Theory allow precise estimations of the RIP parameters of certain random matrices, such as those considered by Candès et al. [Candès et al., 2006a, Tao, 2012]. Unfortunately, explicitly evaluating or even estimating the RIP parameters for a given matrix is computationally difficult. The problem is essentially equivalent to the estimation of the largest eigenvalue of any $t \times t$-principal-submatrix of the $N \times N$ Gram matrix $\Phi^\top \Phi$.

As well as providing examples of asymptotically optimal compressed sensing matrices, Candès et al. provided an efficient recovery algorithm: they showed that, under modest additional assumptions on the RIP parameters of a matrix, $\ell_1$-minimisation (i.e. linear programming) can be used for signal recovery. Thus efficient signal recovery is possible when $\Phi$ has hundreds of rows and thousands of columns.

Storage of random matrices is expensive, and it is difficult to design efficient signal recovery algorithms capable of exploiting the structure of a random matrix. To be suitable for implementation in real-world systems, it is desirable that compressed sensing constructions produce matrices that are sparse (possess relatively few non-zero entries), structured, and deterministically constructed. Systems with these properties can be stored implicitly, and efficient recovery algorithms can be designed to take advantage of their known structure. Motivated by real-world applications, a number of papers have explored compressed sensing where the random Gaussian matrix of Candès et al. is replaced by a sparse matrix [Berinde et al., 2008, Baron et al., 2010], or by a matrix obtained from a deterministic construction [DeVore, 2007, Fickus et al., 2012]. But to date, constructions meeting all three criteria have either been asymptotic in nature (i.e., the results only produce matrices that are too large for practical implementations), or are known only to exist for a very restricted range of parameters.

Berinde, Indyk and their collaborators [Berinde et al., 2008] considered random binary matrices with constant column sum and related these to the incidence matrices of expander graphs. We reinterpret these matrices as sparsifications of the all-ones matrix below. Sarvotham, Baron and Baraniuk [Baron et al., 2010] have considered the use of LDPC matrices. While both groups obtained compressed sensing performance comparable to that of a Gaussian matrix, their constructions are essentially limited by the lack of known explicit constructions for expander graphs and LDPC codes respectively. Moghadam and Radha have previously considered a two step construction of sparse random matrices, involving construction of a random zero-one matrix followed by replacing each one with a sample from some probability distribution, [Moghadam and Radha, 2013, Moghadam and Radha, 2013].

In this paper we take a new approach. Rather than constructing a sparse matrix and examining its compressed sensing properties, we begin with a matrix which is known to possess good CS properties (with high probability) and explore the effect of sparsification on this matrix. That is, we set many of the entries in the original matrix to zero, and compare the performance of the sparse matrix with the original. This investigation was inspired by work of the second author on constructions of sparse compressed sensing matrices from pairwise balanced designs and complex Hadamard matrices [Bryant and Ó Catháin, 2015, Bryan et al., 2015].

In Section 2, we give a formal definition of sparsification, and describe algorithms used to generate random matrices and random vectors as well as the recovery algorithms. In Section 3 we describe the results of extensive simulations. These provide substantial computational evidence
which suggests that sparsification is a robust phenomenon, providing benefits in both recovery
time and proportion of successful recoveries for a wide range of random and structured matrices
occurring in the compressed sensing literature. In particular, Table 2 shows the benefits of sparsi-
fication for a range of matrix constructions, while Figure 2 illustrates the results of sparsification
for a range of recovery algorithms used in compressed sensing. Finally in Section 4, we conclude
with some observations and open questions motivated by our numerical experiments.

2 Sparsification

We begin with a formal definition of sparsification.

**Definition 1.** If $\Phi$ and $\Phi'$ are matrices such that $\Phi'_{ij} = \Phi_{ij}$ for every non-zero entry of $\Phi'$, then
we say that $\Phi'$ is a sparsification of $\Phi$. The density of $\Phi$, $\delta(\Phi)$, is the proportion of non-zero entries
that it contains, and the relative density of $\Phi'$ is the ratio $\delta(\Phi')/\delta(\Phi)$. We write $Sp(\Phi, s)$ for the
set of all sparsifications of $\Phi$ of relative density $s$.

In general, we have that $Sp(\Phi, 1) = \Phi$, and that $Sp(\Phi, 0)$ is the zero matrix. We also have a
transitive property: if $\Phi' \in Sp(\Phi, s_1)$ and $\Phi'' \in Sp(\Phi', s_2)$ then $\Phi'' \in Sp(\Phi, s_1s_2)$. Two independent
sparsifications will not in general be comparable – there is a partial ordering on the set of
sparsifications of a matrix, but not a total order.

We illustrate our notation. Consider a Bernoulli random variable which takes value 1 with
probability $p$ and value 0 with probability $1-p$; let $\Phi$ be an $n \times N$ matrix with entries drawn
from this distribution, in short a Bernoulli ensemble with expected value $p$. Then the expected
density of $\Phi$ is $p$. Writing $J$ for the all-ones matrix, we have $\Phi \in Sp(J, p)$. If $\Phi'$ is a random
sparsification (i.e. all non-zero entries of the matrix have an equal probability to be set to zero) of
$\Phi$ with relative density $p'$, then $\Phi'$ is easily seen to be a Bernoulli ensemble with expected value
$p p'$. So we have both $\Phi' \in Sp(\Phi, p')$ and $\Phi' \in Sp(J, p p')$.

Bernoulli ensembles have previously been considered in the compressed sensing literature, see [Rauhut et al., 2008] for example, though note that the matrices here take values in $\{0, 1\}$, not $\{-1, 1\}$. Such $\{-1, 1\}$-matrices are an affine transformation of ours: $M' = 2M - J$; as a result, compressed sensing performance of either matrix is essentially the same.

In this paper, we will mostly be interested in pseudo-random sparsifications of an $n \times N$ compressed sensing matrix $\Phi$. Specifically, for $s = \frac{k}{N}$, we obtain a matrix $\Phi' \in Sp(\Phi, s)$ by generating
a pseudo-random $\{0, 1\}$-matrix $S$ with $sn$ randomly located ones per column, and returning the
entry-wise product $\Phi' = \Phi \ast S$. We will generally re-normalise $\Phi'$ so that every column has unit
$\ell_2$-norm.

Given a matrix $\Phi$, we test its compressed sensing performance by running simulations. Since
many different methodologies occur in the literature, we specify ours here.

Our $k$-sparse vectors always contain exactly $k$ non-zero entries, in positions chosen uniformly
at random from the $\binom{N}{k}$ possible supports of this size. The entries, unless otherwise specified, are
drawn from a uniform distribution on the open interval $(0, 1)$. The vector is then scaled to have
unit $\ell_2$-norm. Simulations where the non-zero entries were drawn from the absolute value of a
Gaussian distribution produced similar results. Note that many authors use $(0, 1)$- or $(0, \pm 1)$-vectors for their simulations. Appropriate combinations of matrices and algorithms often exhibit
dramatic improvements of performance on this restricted set of signals.

We recover signals using $\ell_1$-minimisation. In this paper we will use the MATLAB LP-solver and
the implementations of OMP (Orthogonal Matching Pursuit) and CoSaMP (Compressive Sampling
Matching Pursuit) developed by Needell and Tropp [Needell and Tropp, 2009]. Specifically,
given a matrix $\Phi$ and signal vector $x$, we compute $y = \Phi x$, and solve the $\ell_1$-minimisation problem
$\Phi \hat{x} = y$ for $\hat{x}$. The objective function is the $\ell_1$-norm of $\hat{x}$ and it is assumed that all variables are
non-negative. We consider the recovery successful if $|x - \hat{x}| \leq c$ for some constant $c$. We take $c = 10^{-6}$ in all the simulations presented in this paper.

We conclude this section with an example illustrating the potential benefits of sparsification. In Figure 1, we explore the effect of sparsification on a $200 \times 2000$ matrix $\Phi$ with entries uniformly distributed on $(0, 1)$. The results for this case were compared with matrices drawn from $\text{Sp}(\Phi, 0.1)$ and $\text{Sp}(\Phi, 0.05)$. For each sparsity between 1 and 60, we generated 500 standard random vectors, and recorded the number of successful recoveries using the matlab LP-solver. To avoid bias we generated a new random matrix for each trial.

![Figure 1: Signal recovery comparison of Sp($\Phi$, 1), Sp($\Phi$, 0.1) and Sp($\Phi$, 0.05)](image)

We observe that for signal vector sparsities between 45 and 55, matrices in $\text{Sp}(\Phi, 0.05)$ achieve substantially better recovery than those from $\text{Sp}(\Phi, 1)$. The code used to generate this simulation as well as others in this paper is available in full, along with data from multiple simulations at a webpage dedicated to this project: http://fintanhegarty.com/compressed_sensing.html.

3 Results

Our simulations produce large volumes of data. To highlight the interesting features of these data-sets, we propose the following measure for acceptable signal recovery in practice.

**Definition 2.** For a matrix $\Phi$ and for $0 \leq t \leq 1$, we define the $t$-recovery threshold, denoted $R_t$, to be the largest value of $k$ for which $\Phi$ recovers $k$-sparse signal vectors with probability exceeding $t$.

We construct an estimate $\hat{R}_t$ for $R_t$ by running simulations. As the number of simulations that we run increases, $\hat{R}_t$ converges to $R_t$. In practice this convergence is rapid. The definition of $R_t$ generalises naturally to a space of matrices (say $n \times N$ Gaussian ensembles): it is simply the expected value of $R_t$ for a matrix chosen uniformly at random from the space. To estimate $R_t$ with reasonably high confidence, we proceed as follows: beginning with signals of sparsity $k = 1$, we attempt 50 recoveries. We increment the value $k$ by 1 and repeat until we reach the first sparsity $k_0$ where less than $50t$ signals are recovered. Beginning at $k_0 - 3$, we attempt 200 recoveries at each signal sparsity. When we reach a signal sparsity $k_1$ where less than $200t$ signals are recovered, we attempt 1000 signal recoveries at each signal sparsity starting at $k_1 - 3$. When we reach a signal sparsity $k_2$ where less than $1,000t$ signals are recovered, we set $\hat{R}_t = k_2 - 1$.

We typically find that $k_1 = k_2$, which gives us confidence that $\hat{R}_t = R_t$. Unless otherwise specified, we use the assumptions outlined in Section 2.
3.1 Recovery algorithms with sparsification

As suggested already in Figure 1, taking $\Phi' \in \text{Sp}(\Phi, s)$ for some value of $s \sim 0.05$ seems to offer considerable improvements when using linear programming for signal recovery. Similar results also hold for OMP and CoSaMP, though note that in each case we supply these algorithms with the sparsity of the signal vector. (While there is an option to withhold this data, the recovery performance of CoSaMP seems to suffer substantially without it – and we wish to be able to perform comparisons with linear programming.) In Figure 2 we graph $R_{0.98}$ of $\text{Sp}(\Phi, s)$ as a function of $s$, where $\Phi$ is a $200 \times 2000$ matrix with entries drawn from the absolute values of samples from a standard normal distribution.

![Graph showing signal recovery as a function of matrix density for LP, OMP and CoSaMP](image)

For each algorithm, $R_{0.98}$ appears to obtain a maximum for matrices of density between 0.12 and 0.04. It is perhaps interesting to note that the percentage improvement obtained by CoSaMP is far greater than that for either of the other algorithms. Linear programming is an order of magnitude slower than either of the other algorithms for these parameters.

| $k$ | Time for 100 recovery attempts | % of vectors successfully recovered |
|-----|---------------------------------|-----------------------------------|
|     | CoSaMP $\delta = 0.05$ | LP $\delta = 0.05$ | CoSaMP $\delta = 0.05$ | LP $\delta = 0.05$ |
|     | $\delta = 1$ | $\delta = 1$ | $\delta = 1$ | $\delta = 1$ |
| 1   | 1.49 | 11.25 | 92.95 | 100 | 100 | 100 | 100 |
| 10  | 1.59 | 15.4  | 125.76 | 100 | 100 | 100 | 100 |
| 20  | 1.75 | 20.67 | 157.6 | 100 | 100 | 100 | 100 |
| 30  | 2.62 | 20.66 | 141.94 | 100 | 96  | 100 | 100 |
| 40  | 8.7  | 23.11 | 184.96 | 100 | 56  | 99  | 98  |
| 50  | 16.15| 25.47 | 333.08 | 96  | 10  | 85  | 38  |
| 60  | 20.1 | 33.15 | 336.92 | 22  | 0   | 6   | 0   |
| 70  | 23.48| 30.11 | 301.35 | 0   | 0   | 0   | 0   |

Table 1: Sparsification timing and recovery benefits

Table 1 shows the time taken to recover 100 vectors with a $200 \times 2000$ matrix with entries drawn from the absolute values of samples from a normal distribution, over a range of vector sparsities. We observe an improvement in running time of an order of magnitude for linear programming when using sparsified matrices. The running time for CoSaMP is slightly higher
for sparsified matrices, but this is due to runtimes including the time taken to sparsify matrices. This seems adequately compensated by the improvement in recovery obtained.

### 3.2 Matrix constructions under sparsification

In this section we demonstrate that the improvement in compressed sensing performance arising from sparsification is a robust phenomenon which occurs for many different constructions proposed for compressed sensing matrices.

We have already encountered the Gaussian, Uniform and Bernoulli ensembles. We will also consider some structured random matrices, which still have entries drawn from a probability distribution, but the matrix entries are no longer independent. The partial circulant ensemble [Rauhut et al., 2012] consists of rows sampled randomly from a circulant matrix, the first row of which contains entries drawn uniformly at random from some suitable probability distribution. Table 2 compares $R_{0.98}$ for $\text{Sp}(\Phi, 1)$ and $\text{Sp}(\Phi, 0.05)$ for $200 \times 2000$ matrices from each of the classes listed. Note that in the case of the Bernoulli ensemble, we actually compare $\text{Sp}(J_{200,2000}, 0.5)$ with $\text{Sp}(J_{200,2000}, 0.05)$, where $J_{200,2000}$ is an all-ones matrix. The entries of the partial circulant matrix were drawn from a normal distribution.

We denote by $\hat{k}$ the signal sparsity $k$ for which the greatest difference in recovery between $\Phi$ and $\Phi' \in \text{Sp}(\Phi, 0.05)$ occurs.

| Construction   | $R_{0.98}$ $\delta = 1$ | $R_{0.98}$ $\delta = 0.05$ | Maximal performance difference $\hat{k}$ $\delta = 1$ | $\delta = 0.05$ |
|---------------|-------------------------|-----------------------------|---------------------------------------------|----------------|
| Normal        | 39                      | 46                          | 51                                          | 25  81        |
| Interval      | 39                      | 45                          | 51                                          | 24  73        |
| Bernoulli*    | 39                      | 42                          | 49                                          | 38  67        |
| Partial Circulant | 39                      | 46                          | 52                                          | 22  76        |

Table 2: Sparsification benefit for different matrix constructions

### 3.3 Varying the matrix parameters

Finally, we investigate the effect of sparsification on matrices of varying parameters. In particular, we explore the effect of sparsification on a family of matrices with entries drawn from the absolute value of the Gaussian distribution. First we explore the effect of sparsification as the ratio of columns to rows in the sensing matrix increases. For this graph, we use signal vectors whose entries were drawn from the absolute value of the normal distribution. We observe a modest improvement in performance which appears to persist.

![Figure 3: Recovery capability of matrices with 100 rows and varying number of columns under sparsification](image-url)
Now we fix the ratio of columns to rows of $\Phi$ to be 10, and vary the number of rows. We know from the results of Candès et al., that $R_{0.98} = O(n / \log(n))$ in all cases. Nevertheless, the clear difference in slopes for recovery at different sparsities offers compelling evidence that the benefits of sparsification persist for large matrices.

![Figure 4: Recovery with CoSaMP for matrices with fixed row to column ratio under sparsification](image)

### 4 Conclusion

The main open problems in the theory of compressed sensing relate to the development of efficient matrix constructions and effective recovery algorithms for sparse recovery. Deterministic constructions are essentially limited by the Welch bound: using known methods it is not possible to guarantee recovery of vectors of sparsity exceeding $O(\sqrt{n})$, where $n$ is the number of rows in the recovery matrix (see e.g. [Bryant and Ó Catháin, 2015]). Probabilistic constructions are much better: the Candès-Tao theory of restricted isometry parameters allows the provable recovery of vectors of sparsity $n / \log(n)$ with $n$ measurements. Such guarantees hold with overwhelming probability for Gaussian ensembles and many other classes of random matrices. But the random nature of these matrices can make the design of efficient recovery algorithms difficult. In this paper we have demonstrated that sparsification offers potential improvements for computational compressed sensing. In particular, Figures 1 and Figure 4 show that sparsification results in the recovery of vectors of slightly higher sparsity. Table 1 shows a substantial improvement in runtimes for linear programming arising from sparsification. These are robust phenomena, which persist under a variety of recovery algorithms and matrix constructions. At the problem sizes that we explored, matrices with densities between 0.05 and 0.1 seemed to provide optimal performance.

We conclude with a small number of observations and conjectures which we believe to be suitable for further investigation. Since a Bernoulli ensemble in our terminology can be regarded as a sparsification of the all-ones matrix, it is clear that sparsification can improve compressed sensing performance. The necessary decay in compressed sensing performance as the density approaches zero shows that the effect of sparsification cannot be monotone. Extensive simulations suggest that when recovery is achieved with a general purpose linear programming solver, matrices with approximately 10% non-zero entries have substantially better compressed sensing properties than dense matrices. A catastrophic decay of compressed sensing performance occurs in many of the examples we investigated between densities of 0.05 and 0.01. We conjecture that this collapse is due to the number of non-zero entries in each column of our matrices becom-
ing very small (< 10) and that as \( n, N \to \infty \), the density of a matrix will tend to zero before impairment of performance occurs.

**Question 1:** As the number of rows in \( \Phi \) increases, the optimal matrix density appears to decrease. Is there a function \( \Gamma(n, N) \) of the matrix parameters which describes the optimal level of sparsification? We propose that the asymptotics of \( \Gamma \) are independent of the matrix construction and of the recovery algorithm. We conjecture that when \( N < n^\alpha \) that the optimal sparsity of a compressed sensing matrix will be approximately \( 1 - \alpha n^{-\frac{1}{2}} \).

**Question 2:** Previous work suggests that sparse matrices have compressed sensing properties similar to that of unsparsified matrices [Moghadam and Radha, 2013, Moghadam and Radha,]. Rigorous results can be achieved using RIP-based methods. But such approaches seem to be too coarse to detect an improvement resulting from sparsification. What is the asymptotic behaviour of the improvement in compressed sensing performance resulting from sparsification? Is it possible to derive rigorous theoretical arguments for the benefit of sparsification?

**Question 3:** We have considered pseudo-random sparsifications in this paper. In general, this should not be necessary. Are there deterministic constructions for \((0, 1)\)-matrices with the property that their entry-wise product with a compressed sensing matrix improves compressed sensing performance? A natural class of candidates would be the incidence matrices of \( t-(v, k, \lambda) \) designs (see [Beth et al., 1999] for example). Some related work is contained in [Bryant et al., 2015, Bryant and Ó Catháin, 2015].

**Acknowledgements**

The first and third authors have been supported by the Engineering and Physical Sciences Research Council grant EP/K00946X/1. The second author acknowledges the support of the Australian Research Council via grant DP120103067.

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