Mixing Coefficients Between Discrete and Real Random Variables: Computation and Properties

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Abstract

In this paper we study the problem of estimating the alpha-, beta- and phi-mixing coefficients between two random variables, that can either assume values in a finite set or the set of real numbers. In either case, explicit closed-form formulas for the beta-mixing coefficient are already known. Therefore for random variables assuming values in a finite set, our contributions are two-fold: (i) In the case of the alpha-mixing coefficient, we show that determining whether or not it exceeds a prespecified threshold is NP-complete, and provide efficiently computable upper and lower bounds. (ii) We derive an exact closed-form formula for the phi-mixing coefficient. Next, we prove analogs of the data-processing inequality from information theory for each of the three kinds of mixing coefficients. Then we move on to real-valued random variables, and show that by using percentile binning and allowing the number of bins to increase more slowly than the number of samples, we can generate empirical estimates that are consistent, i.e., converge to the true values as the number of samples approaches infinity.

I. INTRODUCTION

The notion of independence of random variables is central to probability theory. In [10, p. 8], Kolmogorov says:

“Indeed, as we have already seen, the theory of probability can be regarded from the mathematical point of view as a special application of the general theory of additive set functions.

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“Historically, the independence of experiments and random variables represents the very mathematical concept that has given the theory of probability its peculiar stamp.”

In effect, Kolmogorov is saying that, if the notion of independence is removed, then probability theory reduces to just measure theory.

Independence is a binary concept: Either two random variables are independent, or they are not. It is therefore worthwhile to replace the concept of independence with a more nuanced measure that quantifies the extent to which given random variables are dependent. In the case of stationary stochastic processes, there are various notions of ‘mixing’, corresponding to long term asymptotic independence. These notions can be readily adapted to define various mixing coefficients between two random variables. In this setting, the mixing rate of a stochastic process can be interpreted as the mixing coefficient between the semi-infinite ‘past’ and ‘future’ variables. Several such definitions are presented in [6, p. 3], out of which three are of interest to us, namely the $\alpha$-, $\beta$- and $\phi$-mixing coefficients. While the definitions themselves are well-known, there is very little work on actually computing (or at least estimating) these mixing coefficients in a given situation. The $\beta$-mixing coefficient is easy to compute but this is not the case for the $\alpha$- and the $\phi$-mixing coefficients.

Against this background, the present paper makes the following specific contributions: For random variables that assume values in a finite set:

1) In the case of the $\alpha$-mixing coefficient, it is shown that determining whether or not it exceeds a prespecified threshold is NP-complete, and efficiently computable upper and lower bounds are derived.

2) An efficiently computable exact formula is derived for the $\phi$-mixing coefficient.

3) We study the case of three random variables $X, Y, Z$, where $X, Z$ are conditionally independent given $Y$, or equivalently, $X \rightarrow Y \rightarrow Z$ is a short Markov chain. In this case a well-known inequality from information theory [4, p. 34], usually referred to as the ‘data processing inequality (DPI)’, states that

$$I(X, Z) \leq \min\{I(X, Y), I(Y, Z)\},$$

(1)

where $I(\cdot, \cdot)$ denotes the mutual information. We state and prove analogs of the DPI for each of the $\alpha$-, $\beta$- and $\phi$-mixing coefficients.
Next we turn to real-valued random variables.

1) Suppose $X, Y$ are real-valued random variables whose joint distribution has a density with respect to the Lebesgue measure, and that $\{(x_1, y_1), \ldots, (x_l, y_l)\}$ are independent samples of $(X, Y)$. If we compute the empirical joint distribution of $(X, Y)$ from these samples, then the Glivenko-Cantelli Lemma states that the empirical joint distribution converges with probability one to the true joint distribution; in other words, the empirical distribution gives a consistent estimate. However, it is shown here that if the empirical distribution is used to estimate the mixing coefficients, then with probability one both the estimated $\beta$-mixing coefficient and the estimated $\phi$-mixing coefficient approach one as $l \to \infty$, irrespective of what the true value might be. Thus a quantity derived from a consistent estimator need not itself be consistent.

2) On the other hand, if we bin the $l$ samples into $k_l$ bins and choose $k_l$ in such a way that $k_l \to \infty$ and $k_l/l \to 0$ as $l \to \infty$, and a few technical conditions are satisfied, then the empirically estimated $\alpha$-, $\beta$- and $\phi$-mixing coefficients converge to their true values as $l \to \infty$, with probability one.

The notion of a mixing process and various definitions of mixing coefficients originated in an attempt to establish the law of large numbers for stationary stochastic processes that are not i.i.d. The problem of determining (or at least bounding) the mixing coefficients of random variables and stochastic processes arises in various contexts, including system identification and statistical learning. Traditional theories of system identification are based on the assumption that the input sequence to the unknown system is i.i.d. However, it became clear over time that much of the theory continues to hold even if the input sequence is not i.i.d., but is mixing in an appropriate sense. See [28], [26] as just two examples of such an approach. Similarly, the standard formulation of PAC (probably approximately correct) learning in statistical learning theory is based on the assumption that the inputs are i.i.d. See [22], [20] for example. However, subsequently PAC learning theory has been extended to the case where the learning inputs are not i.i.d., but are mixing instead; see for example the book [23] and the references therein, as well as [15]. In adapting results in system identification or statistical learning theory from the i.i.d. case to the case of mixing processes, it becomes necessary to obtain at least upper bounds for the mixing coefficients, if not exact values. The results presented here have some relevance.
to this problem, as do other recent results such as [14]. We shall return to this topic in the concluding remarks.

Proving that various mixing coefficients satisfy analogs of the data processing inequality (DPI) is not just of academic interest. Recent work on reverse-engineering genome-wide interaction networks from gene expression data is based on first constructing a complete graph where each node corresponds to a gene, and then using the DPI to “prune” the graph. Among the first algorithms to use this approach is ARACNE [13], which is based on using mutual information as a measure of interaction between genes. However, because mutual information is a symmetric quantity, the resulting graphs are undirected, which is quite contrary to biological realism, because in reality the interactions between genes are not symmetric. This led the authors to explore whether the $\phi$-mixing coefficient, which is asymmetric, can be used as a measure of the interaction between two genes. Once it is established that the $\phi$-mixing coefficient satisfies an analog of the DPI (which is one of the principal results of this paper), it is possible to develop a method for constructing directed graphs that represent whole genome regulatory networks. However, this by itself is not sufficient. If there are $n$ genes in the study, this approach requires the computation of $n^2$ $\phi$-mixing coefficients. So for a typical genome-wide study involving 20,000 genes, it becomes necessary to compute 400 million $\phi$-mixing coefficients. Hence it is mandatory to have a method for the efficient computation of the $\phi$-mixing coefficient. Such a method is also provided in the present paper. Please see [24], [18], [25] for a discussion of how the methods presented here can be applied to reverse engineering gene regulatory networks.

II. Definitions of Mixing Coefficients

Definitions of the $\alpha$-, $\beta$- and $\phi$-mixing coefficients of a stationary stochastic process can be found, among other places, in [23, pp. 34-35]. The $\alpha$-mixing coefficient was introduced by Rosenblatt [17]. According to Doukhan [6, p. 5], Kolmogorov introduced the $\beta$-mixing coefficient, but it appeared in print for the first time in a paper published by some other authors. The $\phi$-mixing coefficient was introduced by Ibragimov [9].

Essentially, all notions of mixing processes try to quantify the idea that, in a stationary stochastic process of the form $\{X_t\}_{t=-\infty}^{\infty}$, the random variables $X_t$ and $X_\tau$ become more and more independent as $|t - \tau|$ approaches infinity, in other words, there is an asymptotic long-term near-independence. However, these very general notions can be simplified and readily adapted
to define mixing coefficients between a pair of random variables $X$ and $Y$. This is how they are defined in [6]. Note that, strictly speaking, mixing is a property not of the random variables $X$ and $Y$, but rather of the $\sigma$-algebras generated by $X$ and $Y$. Note also that, if $\{X_t\}_{t=-\infty}^{\infty}$ is a stationary stochastic process, then the $k$-th ($\alpha$, $\beta$ or $\phi$) mixing coefficient of the stochastic process is just the corresponding mixing coefficient as defined in [6] between the semi-infinite past $X_{-\infty}^0 := (X_t, t \leq 0)$ and the semi-infinite future $X_k^\infty := (X_t, t \geq k)$.

Though mixing coefficients can be defined for arbitrary random variables, in the interests of avoiding a lot of technicalities we restrict our attention in this paper to just two practically important cases: real-valued and finite-valued random variables. We first define mixing coefficients between real-valued random variables, and then between finite-valued random variables.

**Definition 1:** Suppose $X$ and $Y$ are real-valued random variables. Let $B$ denote the Borel $\sigma$-algebra of subsets of $\mathbb{R}$. Then we define

$$\alpha(X, Y) := \sup_{S,T \in B} |\Pr\{X \in S \& Y \in T\} - \Pr\{X \in S\} \cdot \Pr\{Y \in T\}|.$$  \hspace{1cm} (2)

$$\phi(X|Y) := \sup_{S,T \in B} |\Pr\{X \in S|Y \in T\} - \Pr\{X \in S\}|$$

$$= \sup_{S,T \in B} \left|\frac{\Pr\{X \in S \& Y \in T\}}{\Pr\{Y \in T\}} - \Pr\{X \in S\}\right|. \hspace{1cm} (3)$$

In applying the above definition, in case $\Pr\{Y \in T\} = 0$, we use the standard convention that

$$\Pr\{X \in S|Y \in T\} = \Pr\{X \in S\}.$$  

Note that the $\alpha$-mixing coefficient is symmetric: $\alpha(X, Y) = \alpha(Y, X)$. However, in general $\phi(X|Y) \neq \phi(Y|X)$.

The third coefficient, called the $\beta$-mixing coefficient, has a somewhat more elaborate definition, at least in the general case. Let $\theta$ denote the probability measure of the joint random variable $(X, Y)$, and let $\mu, \nu$ denote the marginal measures of $X$ and $Y$ respectively. Note that $\theta$ is a measure on $\mathbb{R}^2$ while $\mu, \nu$ are measures on $\mathbb{R}$. If $X$ and $Y$ were independent, then $\theta$ would equal $\mu \times \nu$, the product measure. With this in mind, we define

$$\beta(X, Y) = \rho(\theta, \mu \times \nu), \hspace{1cm} (4)$$
where $\rho$ denotes the total variation distance between two measures. That is, if $\theta, \eta$ are probability measures on a common measure space $(\Omega, \Sigma)$, then

$$\rho(\theta, \eta) := \sup_{S \in \Sigma} |\theta(S) - \eta(S)|.$$  

The $\beta$-mixing coefficient is also symmetric.

Next we deal with finite-valued random variables, and for this purpose we introduce some notation that is used throughout the remainder of the paper. The most important notational change is that, since probability distributions on finite sets can be represented by vectors, we use bold-face Greek letters to denote them, whereas we use normal Greek letters to denote measures on $\mathbb{R}$ or $\mathbb{R}^2$. For each integer $n$, let $S_n$ denote the $n$-dimensional simplex. Thus

$$S_n := \{ v \in \mathbb{R}^n : v_i \geq 0 \forall i, \sum_{i=1}^n v_i = 1 \}.$$  

If $A = \{a_1, \ldots, a_n\}$ and $\mu \in S_n$, then $\mu$ defines a measure $P_\mu$ on the set $A$ according to

$$P_\mu(S) = \sum_{i=1}^n \mu_i I_{S}(a_i),$$  

where $I_S(\cdot)$ denotes the indicator function of $S$.

Suppose $\mu, \nu \in S_n$ are probability distributions on a set $A$ of cardinality $n$. Then the total variation distance between $\mu$ and $\nu$ is defined as

$$\rho(\mu, \nu) := \max_{S \subseteq A} |P_\mu(S) - P_\nu(S)|.$$  

It is easy to give several equivalent closed-form formulas for the total variation distance.

$$\rho(\mu, \nu) = 0.5 \| \mu - \nu \|_1 = \sum_{i=1}^n (\mu_i - \nu_i)_+ = -\sum_{i=1}^n (\mu_i - \nu_i)_-, $$  

where as usual $(\cdot)_+$ and $(\cdot)_-$ denote the nonnegative and the nonpositive parts of a number:

$$(x)_+ = \max\{x, 0\}, (x)_- = \min\{x, 0\}.$$  

Now suppose $A, B$ denote sets of cardinality $n, m$ respectively, and that $\mu \in S_n, \nu \in S_m$. Then the distribution $\psi \in S_{nm}$ defined by $\psi_{ij} = \mu_i \nu_j$ is called the product distribution on $A \times B$, and is denoted by $\mu \times \nu$. In the other direction, if $\theta \in S_{nm}$ is a distribution on $A \times B$, then $\theta_A \in S_n, \theta_B \in S_m$ defined respectively by

$$(\theta_A)_i := \sum_{j=1}^m \theta_{ij}, (\theta_B)_j := \sum_{i=1}^n \theta_{ij}.$$  

are called the **marginal distributions** of $\theta$ on $A$ and $B$ respectively.

The earlier definitions of mixing coefficients become quite explicit in the case where $X,Y$ are random variables assuming values in the finite sets $A,B$ of cardinalities $n,m$ respectively. In this case it does not matter whether the ranges of $X,Y$ are finite subsets of $\mathbb{R}$ or some abstract finite sets. Definition 1 can now be restated in this context. Note that, since $A,B$ are finite sets, the associated $\sigma$-algebras are just the power sets, that is, the collection of all subsets.

**Definition 2:** With the above notation, we define

\[ \alpha(X,Y) := \max_{S \subseteq A, T \subseteq B} \left| P_{\theta}(S \times T) - P_{\mu}(S)P_{\nu}(T) \right|, \]  
\[ \beta(X,Y) := \rho(\theta, \mu \times \nu), \]  
\[ \phi(X|Y) := \max_{S \subseteq A, T \subseteq B} \left| \frac{P_{\theta}(S \times T)}{P_{\nu}(T)} - P_{\mu}(S) \right|. \]

Whether $X,Y$ are real-valued or finite-valued random variables, the mixing coefficients satisfy the following inequalities; see [6, p. 4]:

\[ \alpha(X,Y) \in [0, 0.25], \beta(X,Y) \in [0, 1], \phi(X,Y) \in [0, 1], \]  
\[ 0 \leq 2\alpha(X,Y) \leq \beta(X,Y) \leq \min\{\phi(X|Y), \phi(Y|X)\} \leq \max\{\phi(X|Y), \phi(Y|X)\} \leq 1. \]

Also, the following statements are equivalent:

1) $X$ and $Y$ are independent random variables.
2) $\alpha(X,Y) = 0$.
3) $\beta(X,Y) = 0$.
4) $\phi(X|Y) = 0$.
5) $\phi(Y|X) = 0$.

**III. Computation of Mixing Coefficients for Finite-Valued Random Variables**

From the definitions, it is clear that $\beta(X,Y)$ can be readily computed in closed form. As before, let us define $\psi = \mu \times \nu$ to be the product distribution of the two marginals, and define

\[ \gamma_{ij} := \theta_{ij} - \psi_{ij}, \Gamma := [\gamma_{ij}] \in [-1, 1]^{n \times m}. \]
Then it readily follows from (2) that
\[ \beta(X, Y) := \rho(\theta, \psi) = 0.5 \sum_{i=1}^{n} \sum_{j=1}^{m} |\gamma_{ij}| \]
\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} (\gamma_{ij})^+ \]
\[ = -\sum_{i=1}^{n} \sum_{j=1}^{m} (\gamma_{ij})^- . \]

In addition, there is a very useful upper bound on the \( \beta \)-mixing coefficient in terms of the so-called “Pinsker’s inequality”, though it may be appropriate to credit this inequality to Csiszár; see [5] or [4]. This inequality states that, for any two probability distributions \( \theta \) and \( \phi \) on a common set,
\[ \rho(\theta, \phi) \leq \sqrt{(1/2) D(\theta \parallel \phi)}, \]
where \( D(\cdot, \cdot) \) is the Kullback-Leibler divergence. Now apply this inequality with \( \phi = \mu \times \nu \).
This leads to
\[ \rho(\theta, \mu \times \nu) \leq \sqrt{(1/2) D(\theta \parallel \mu \times \nu)}, \]
However, \( \rho(\theta, \mu \times \nu) = \beta(X, Y) \) whereas \( D(\theta \parallel \mu \times \nu) = I(X, Y) \), the mutual information between \( X \) and \( Y \). Therefore
\[ \beta(X, Y) \leq \sqrt{(1/2) I(X, Y)}. \]

On the other hand, computing \( \alpha(X, Y) \) or \( \phi(X|Y) \) directly from Definition 2 would require \( 2^{n+m} \) computations, since \( S, T \) must be allowed to vary over all subsets of \( A, B \) respectively. Therefore the question arises as to whether this is an artefact of the definition, or an inherent barrier to efficient computation. In the present section, the following results are established:

- As stated in (9), the quantity \( \alpha(X, Y) \) always lies in the interval \([0, 0.25]\). It is shown that the problem of determining whether \( \alpha(X, Y) = 0.25 \) for a given pair of random variables \( X, Y \) is NP-complete. More generally, given any number \( \epsilon \in (0, 0.25] \), determining whether \( \alpha(X, Y) \geq \epsilon \) is NP-complete.
- Some efficiently computable upper and lower bounds are derived for \( \alpha(X, Y) \). These bounds become germane in view of the above complexity result.
- An exact and efficiently computable formula is derived for \( \phi(X, Y) \).
In proceeding further, the first step is to get rid of the absolute value signs in the definitions of the $\alpha$- and $\phi$-mixing coefficients.

**Theorem 1:** It is the case that

$$\alpha(X,Y) = \max_{S \subseteq A, T \subseteq B} \left[ P_\theta(S \times T) - P_\mu(S)P_\nu(T) \right],$$

(10)

$$\phi(X|Y) = \max_{S \subseteq A, T \subseteq B} \left[ \frac{P_\theta(S \times T)}{P_\nu(T)} - P_\mu(S) \right].$$

(11)

**Proof:** Define

$$\mathcal{R}_\alpha := \left\{ P_\theta(S \times T) - P_\mu(S)P_\nu(T), S \subseteq A, T \subseteq B \right\}.$$

Then $\mathcal{R}_\alpha$ is a finite subset of the real line consisting of at most $2^{n+m}$ elements. Now it is claimed that the set $\mathcal{R}_\alpha$ is symmetric; that is, $x \in \mathcal{R}_\alpha$ implies that $-x \in \mathcal{R}_\alpha$. If this claim can be established, then (10) follows readily. So suppose $x \in \mathcal{R}_\alpha$, and choose $S \subseteq A, T \subseteq B$ such that

$$P_\theta(S \times T) - P_\mu(S)P_\nu(T) = x.$$

Let $S^c$ denote the complement of $S$ in $A$. Then, using the facts that

$$P_\mu(S^c) = 1 - P_\mu(S),$$

$$P_\theta(S^c \times T) = P_\theta(A \times T) - P_\theta(S \times T)$$

$$= P_\nu(T) - P_\theta(S \times T),$$

it is easy to verify that

$$P_\theta(S^c \times T) - P_\mu(S^c)P_\nu(T) = -x.$$

So $\mathcal{R}_\alpha$ is symmetric and (10) follows. By analogous reasoning, the set

$$\mathcal{R}_\phi := \left\{ \frac{P_\theta(S \times T)}{P_\nu(T)} - P_\mu(S) : S \subseteq A, T \subseteq B \right\}$$

is also symmetric, which establishes (11).
A. NP-Completeness of Estimating the Alpha-Mixing Coefficient

We begin by revisiting the definition of $\alpha(X,Y)$, and determine the conditions under which it can attain its theoretical maximum value of 0.25.

**Theorem 2:** Suppose $X, Y$ are random variables assuming values in finite sets $\mathbb{A}, \mathbb{B}$ respectively, with marginal distributions $\mu, \nu$ respectively, and joint distribution $\theta$. Then $\alpha(X,Y) \leq 0.25$. Moreover, $\alpha(X,Y) = 0.25$ if and only if there exist subsets $S \subseteq \mathbb{A}, T \subseteq \mathbb{B}$ such that $P_\mu(S) = 0.5$, $P_\nu(T) = 0.5$, and $P_\theta(S \times T) = 0$.

**Proof:** It is easy to see that the following relationship, which is the mirror image of (10), is true:

$$\alpha(X,Y) = -\min_{S \subseteq \mathbb{A}, T \subseteq \mathbb{B}} [P_\theta(S \times T) - P_\mu(S)P_\nu(T)].$$

(12)

Indeed, as shown in the proof of Theorem 1, if $S, T$ achieve the maximum in (10), then $S^c, T$ achieve the minimum in (12), and vice versa. Given sets $S \subseteq \mathbb{A}, T \subseteq \mathbb{B}$, define

$$a := P_\theta(S \times T), b := P_\theta(S \times T^c),$$
$$c := P_\theta(S^c \times T), d := P_\theta(S^c \times T^c).$$

Then it is evident that $a + b + c + d = 1$. Moreover,

$$P_\mu(S) = a + b, P_\nu(T) = a + c.$$

Therefore

$$P_\theta(S \times T) - P_\mu(S)P_\nu(T) = a - (a + b)(a + c)$$
$$= -a^2 + a(1 - b - c) - bc$$
$$=: f(a).$$

Let us think of the above quantity as a function of $a$ with $b, c$ fixed. This amounts to fixing the measures of the sets $S, T$ while adjusting the joint distribution to change $P_\theta(S \times T)$. Then $f(0) = -bc$, and $f'(0) = 1 - b - c \geq 0$. So $f(a)$ is nondecreasing at $a = 0$. The maximum permissible value of $a$ is $1 - b - c$ (amounting to setting $d = 0$), and $f(1 - b - c)$ again equals $-bc$. Simple high school algebra shows that $f(a)$ achieves a maximum at $a^* = (1 - b - c)/2$,
and then begins to decrease. Therefore it follows that \( f(a) \geq -bc \). Now \( b, c \) satisfy \( b + c \leq 1 \), whence it is immediate that

\[
\alpha(X, Y) \leq - \min_{b,c} -bc \text{ s.t. } b + c \leq 1 \\
= \max_{b,c} bc \text{ s.t. } b + c \leq 1 \\
= 0.25. \tag{13}
\]

Moreover, \( \alpha(X, Y) = 0.25 \) if only if the choice \( b = c = 0.5 \) (which in turn implies that \( a = d = 0 \)) is compatible with the given joint distribution. Recalling what these symbols represent shows that (i) \( \alpha(X, Y) \leq 0.25 \) always, and (ii) \( \alpha(X, Y) = 0.25 \) if and only if there exist subsets \( S \subseteq \mathbb{A}, T \subseteq \mathbb{B} \) such that \( P_\theta(S \times T) = 0, P_\mu(S) = P_\nu(T) = 0.5. \)

The next step is to map a problem that is known to be NP-complete into the problem of checking whether or not \( \alpha(X, Y) = 0.25 \). Our choice is the so-called “normalized partition” problem, which is a variant of the “partition problem”, which can be found in [7, p. 47], among other places. We begin by stating the partition problem.

**Problem Partition:**

**Instance:** A positive integer \( m \), and a set of positive integers \( a_1, \ldots, a_m \).

**Question:** Does there exist a subset \( I \subseteq \mathcal{M} := \{1, \ldots, m\} \) such that

\[
\sum_{i \in I} a_i = \sum_{j \in \mathcal{M} \setminus I} a_j ?
\]

This problem is known to be NP-Complete; see [7, p. 47]. For our purposes we modify the problem as follows:

**Problem Normalized Partition:**

**Instance:** A positive integer \( m \), and a set of positive rational numbers \( a_1, \ldots, a_m \) such that \( \sum_{i=1}^{m} a_i = 1 \).

**Question:** Does there exist a subset \( I \subseteq \mathcal{M} := \{1, \ldots, m\} \) such that

\[
\sum_{i \in I} a_i = \sum_{j \in \mathcal{M} \setminus I} a_j ?
\]

It is clear that this problem is equivalent to the partition problem, and is therefore NP-complete.

**Theorem 3:** The following problem is NP-complete:

**Problem:**

**Instance:** Positive integers \( n, m \) and a set of nonnegative rational numbers \( \theta_{ij}, i = 1, \ldots, n, j = \)

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1, \ldots, m \text{ such that } \sum_{i,j} \theta_{ij} = 1.

**Question:** Let \((X, Y)\) be random variables assuming values in \(\mathcal{N} := \{1, \ldots, n\}, \mathcal{M} := \{1, \ldots, m\}\) respectively with the joint distribution \(\Pr\{X = i \& Y = j\} = \theta_{ij}\). Is \(\alpha(X, Y) = 0.25\)?

**Proof:** By Theorem 2, we know that \(\alpha(X, Y) = 0.25\) if and only if there exist subsets \(S \subseteq \mathcal{N}, T \subseteq \mathcal{M}\), such that \(P_\mu(S) = 0.5, P_\nu(T) = 0.5\) and \(P_\theta(S \times T) = 0\), where \(P_\mu, P_\nu\) denote the marginals of \(P_\theta\). Hence, given a candidate solution in terms of sets \(S, T\), all one has to do is to verify the above three relationships, which can be done in polynomial time. So the problem is in \(\text{NP}\).

To show that it is \(\text{NP-complete}\), we map the normalized partition problem into it. Given positive rational numbers \(a_1, \ldots, a_m\) such that \(\sum_m a_i = 1\), define \(n = m\) and \(\theta_{ij} = a_i \delta_{ij}\), where \(\delta_{ij}\) is the Kronecker delta. Thus, under this joint distribution, \(n = m\), both \(X\) and \(Y\) have the vector \(a = [a_1, \ldots, a_m]\) as their marginal distributions, and \(\Pr\{X = Y\} = 1\). Given subsets \(S, T \subseteq \mathcal{M}\), it is easy to verify that \(P_\theta(S \times T) = P_a(S \cap T)\). (Note that \(P_\theta\) is a measure on \(\mathcal{M} \times \mathcal{M}\) while \(P_a\) is a measure on \(\mathcal{M}\).) Therefore \(\alpha(X, Y) = 0.25\) if and only if there exist subsets \(S, T \subseteq \mathcal{M}\) such that \(P_a(S) = 0.5, P_a(T) = 0.5,\) and \(P_a(S \cap T) = 0\). These conditions imply that \(S, T\) form a partition of \(\mathcal{M}\), and that \(I = S, \mathcal{M} \setminus I = T\) is a solution of the normalized partition problem. Hence this problem is \(\text{NP-complete}\). ■

**Corollary 1:** The following problem is \(\text{NP-complete}\):

**Problem:**

**Instance:** Positive integers \(n, m\), a set of nonnegative rational integers \(\theta_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\) such that \(\sum_{i,j} \theta_{ij} = 1\), and a rational number \(\epsilon \in (0, 0.25]\).

**Question:** Let \((X, Y)\) be random variables assuming values in \(\mathcal{N} := \{1, \ldots, n\}, \mathcal{M} := \{1, \ldots, m\}\) respectively with the joint distribution \(\Pr\{X = i \& Y = j\} = \theta_{ij}\). Is \(\alpha(X, Y) \geq \epsilon\)?

**Proof:** If we choose \(\epsilon = 0.25\), this problem reduces to that studied in Theorem 3, which is \(\text{NP-complete}\). Therefore the present problem is \(\text{NP-hard}\). On the other hand, given a candidate solution in terms of subsets \(S \subseteq \mathcal{N}, T \subseteq \mathcal{M}\), it is possible to verify in polynomial time that \(|P_\theta(S \times T) - P_\mu(S)P_\nu(T)| \geq \epsilon\). Therefore the problem is \(\text{NP-complete}\). ■

**B. Upper and Lower Bounds for the Alpha-Mixing Coefficient**

Since computing the \(\alpha\)-mixing coefficient is \(\text{NP-hard}\) (because merely testing whether it exceeds a prespecified threshold is \(\text{NP-complete}\)), it is worthwhile to have efficiently computable
upper and lower bounds for this mixing coefficient. The aim of this subsection is to present such bounds.

To contrast with later results on the $\phi$-mixing coefficient, we introduce a bit of notation. Suppose $\Gamma \in \mathbb{R}^{n \times m}$, and that $p, q \in [1, \infty]$. Then the induced norm $\| \Gamma \|_{p,q}$ is defined as

$$
\| \Gamma \|_{p,q} := \max_{\| v \|_p \leq 1} \| \Gamma v \|_q.
$$

Explicit closed-form formulas are available for $\| \Gamma \|_{p,q}$ when $(p, q) = (1, 1), (2, 2), (\infty, \infty), (2, \infty)$; see for example [21]. However, not much is known about other combinations.

**Theorem 4:** Suppose $X, Y$ are random variables over finite sets $\mathbb{A}, \mathbb{B}$ with joint distribution $\theta$ and marginals $\mu, \nu$ respectively. Define

$$
\Gamma = \Theta - \mu \nu^t \in \mathbb{R}^{n \times m},
$$

(14)

where $\Theta = [\theta_{ij}]$. Then

$$
\alpha(X, Y) = 0.25 \| \Gamma \|_{\infty,1}.
$$

(15)

**Proof:** Let $n$ denote $|\mathbb{A}|$, and define a map $h: \mathbb{A}^n \rightarrow \{0, 1\}^n$ as follows: For a subset $S \subseteq \mathbb{A}$

$$
h_i(S) = \begin{cases} 
1, & \text{if } a_i \in S, \\
0, & \text{if } a_i \notin S.
\end{cases}
$$

Note that by definition we have:

$$
h(S) + h(S^c) = e_n,
$$

where $e$ denotes a column vector whose components all equal one, and the subscript denotes its dimension. A similar map can be defined for $\mathbb{B}$ as well. With this notation, for any subsets $S \subseteq \mathbb{A}, T \subseteq \mathbb{B}$, we have that

$$
P_\mu(S) = \mu^t h(S) = [h(S)]^t \mu,
$$

$$
P_\nu(T) = \nu^t h(T) = [h(T)]^t \nu.
$$

Moreover, with the joint distribution $\theta$, we have that

$$
P_\theta(S \times T) = \mu^t \Theta \nu.
$$

Since the function $h$ is a bijection, it follows from (10) that

$$
\alpha(X, Y) = \max_{a \in \{0,1\}^n, b \in \{0,1\}^m} a^t \Theta b - a^t \mu \nu^t b
$$

$$
= \max_{a \in \{0,1\}^n, b \in \{0,1\}^m} a^t \Gamma b.
$$

(16)
Let \( b \in \mathbb{R}^m \) be any fixed vector; then \( a^t \Gamma b \) is maximized with respect to \( a \in \{0, 1\}^n \) by choosing \( a_i = 1 \) if \((\Gamma b)_i \geq 0\), and \( a_i = 0 \) if \((\Gamma b)_i < 0\). In other words,

\[
\max_{a \in \{0, 1\}^n} a^t \Gamma b = \sum_i ((\Gamma b)_i)_+.
\]

However, since the product distribution \( \mu \nu^t \) and the joint distribution \( \Theta \) have the same marginals, it follows that

\[
\Gamma e_m = 0_n, e_n^t \Gamma = 0_m.
\]

This implies that, for any vector \( a \in \{0, 1\}^n \) and any \( b \in \mathbb{R}^m \), we have

\[
a^t \Gamma b = -(e_n - a)^t \Gamma b,
\]

and also that

\[
\sum_i ((\Gamma b)_i)_+ = -\sum_i ((\Gamma b)_i)_- = 0.5 \| \Gamma b \|_1.
\]

Therefore

\[
\min_{a \in \{0, 1\}^n} a^t \Gamma b = -\max_{a \in \{0, 1\}^n} a^t \Gamma b,
\]

whence

\[
\max_{a \in \{0, 1\}^n} |a^t \Gamma b| = \max_{a \in \{0, 1\}^n} a^t \Gamma b
\]

\[
= \sum_i ((\Gamma b)_i)_+
\]

\[
= 0.5 \| \Gamma b \|_1.
\]

As a consequence, it now follows from (16) that

\[
\alpha(X, Y) = \max_{b \in \{0, 1\}^m} \sum_i ((\Gamma b)_i)_+
\]

\[
= 0.5 \max_{b \in \{0, 1\}^m} \| \Gamma b \|_1.
\]

The proof is completed by showing that the quantity on the right side of (17) equals \( \| \Gamma \|_{\infty, 1} \).

For an arbitrary \( b \in \mathbb{R}^m \), define the associated vector \( z \in \mathbb{R}^m \) by \( z = 2b - e_m \), and observe that, as \( b \) varies over \( \{0, 1\}^m \), \( z \) varies over \( \{-1, 1\}^m \). Also,

\[
\Gamma b = 0.5(\Gamma z - \Gamma e_m) = 0.5 \Gamma z,
\]
because $\Gamma e_m = 0_m$. Therefore

$$\alpha(X, Y) = 0.5 \max_{b \in \{0,1\}^m} \|\Gamma b\|_1 = 0.25 \max_{z \in \{-1,1\}^m} \|\Gamma z\|_1.$$  

Now consider the optimization problem

$$\max \|\Gamma z\|_\infty \text{ s.t. } \|z\|_\infty \leq 1.$$  

Since the objective function is convex and the feasible region is convex and polyhedral, the optimum occurs at an extremum point. In other words,

$$\max_{\|z\|_\infty \leq 1} \|\Gamma z\|_1 = \max_{z \in \{-1,1\}^m} \|\Gamma z\|_1.$$  

However, by definition the left side equals $\|\Gamma\|_{\infty,1}$. This shows that

$$\alpha(X, Y) = 0.25 \|\Gamma\|_{\infty,1},$$  

which is the desired conclusion. ■

By combining Theorems 3 and 4, we can conclude that computing the induced norm $\|\cdot\|_{\infty,1}$ of an arbitrary matrix is NP-hard. However, this result is already shown in [16], which also gives an efficiently computable upper bound for this induced norm, with a guaranteed suboptimality. By adapting that result, we can derive efficiently computable upper and lower bounds for the $\alpha$-mixing coefficient.

**Theorem 5:** [16] Given a matrix $\Gamma \in \mathbb{R}^{n \times m}$, define $c(\Gamma)$ to be the value of the following optimization problem:

$$c(\Gamma) := 0.5 \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \|x\|_\infty + \|y\|_1$$  

subject to

$$\begin{bmatrix} \text{Diag}(x) & \Gamma^t \\ \Gamma & \text{Diag}(y) \end{bmatrix} \succeq 0,$$

where $M \succeq 0$ denotes that $M$ is positive semidefinite. Then

$$0.1086 \quad c(\Gamma) \leq \alpha(X, Y) \leq 0.25 \quad c(\Gamma), (19)$$  

**Proof:** It is shown in [16] that

$$\|\Gamma\|_{\infty,1} \leq c(\Gamma) \leq 2.3 \|\Gamma\|_{\infty,1}.$$  

Therefore

$$(1/2.3) \quad c(\Gamma) \leq \|\Gamma\|_{\infty,1} \leq c(\Gamma).$$
Combining this with (15) leads to the desired conclusion.

Note that the computation of \( c(\Gamma) \) requires the solution of a semidefinite optimization program. Efficient algorithms to solve semidefinite programs can be found in [19].

C. An Exact Formula for the Phi-Mixing Coefficient

We have seen in Section III-A that estimating the \( \alpha \)-mixing coefficient is NP-complete. Though the definition of the \( \phi \)-mixing coefficient resembles that of the \( \alpha \)-mixing coefficient in terms of (apparently) requiring an enumeration of all subsets, it turns out that there is an efficiently computable exact formula for the \( \phi \)-mixing coefficient.

**Theorem 6:** Suppose \( X, Y \) are random variables over finite sets \( \mathbb{A}, \mathbb{B} \) with joint distribution \( \theta \) and marginals \( \mu, \nu \) respectively. Then

\[
\phi(X, Y) = \max_j \frac{1}{\nu_j} \sum_i (\gamma_{ij})_+ = 0.5 \max_j \frac{1}{\nu_j} \sum_i |\gamma_{ij}| = 0.5 \|\Gamma[\text{Diag}(\nu)]^{-1}\|_{1,1} \tag{20}
\]

**Proof:** We already know from Theorem 1 that

\[
\phi(X|Y) := \max_{S \subseteq \mathbb{A}, T \subseteq \mathbb{B}} \Pr\{X \in S|Y \in T\} - \Pr\{X \in S\}.
\]

Now define

\[
g(S, T) := \Pr\{X \in S|Y \in T\} - \Pr\{X \in S\}
\]

Then

\[
\phi(X|Y) = \max_{T \subseteq \mathbb{B}} \max_{S \subseteq \mathbb{A}} g(s, T).
\]

Next, using obvious notation, let us rewrite \( g(S, T) \) as

\[
g(S, T) = P(S|T) - P(T) = \frac{P(S \times T) - P(S)P(T)}{P(T)}. \tag{21}
\]

Now, suppose \( T_1, T_2 \) are disjoint subsets of \( \mathbb{B} \). Then

\[
P(T_1 \cup T_2) = P(T_1) + P(T_2),
\]

\[
P(S \times (T_1 \cup T_2)) = P(S \times T_1) + P(S \times T_2)
\]
because the events $S \times T_1$ and $S \times T_2$ are also disjoint. Therefore

$$g(S, T_1 \cup T_2) = \frac{P(S \times (T_1 \cup T_2)) - P(T_1 \cup T_2)}{P(T_1 \cup T_2)}$$

$$= \frac{P(S \times T_1) - P(T_1)}{P(T_1) + P(T_2)} + \frac{P(S \times T_2) - P(T_2)}{P(T_1) + P(T_2)}$$

$$= \frac{P(S \times T_1)}{P(T_1)} \cdot \frac{P(T_1)}{P(T_1) + P(T_2)} + \frac{P(S \times T_2)}{P(T_2)} \cdot \frac{P(T_2)}{P(T_1) + P(T_2)}$$

$$= \lambda_1 g(S, T_1) + \lambda_2 g(S, T_2),$$

where

$$\lambda_1 = \frac{P(T_1)}{P(T_1) + P(T_2)}, \lambda_2 = \frac{P(T_2)}{P(T_1) + P(T_2)}.$$

Therefore $g(S, T_1 \cup T_2)$ is a convex combination of $g(S, T_1)$ and $g(S, T_2)$. There is nothing special about writing $T$ as a disjoint union of two subsets. In general, if $T = \{j_1, \ldots, j_k\}$, then the above reasoning can be repeated to show that

$$g(S, T) = \sum_{l=1}^{k} \frac{P(\{j_l\})}{P(T)} g(S, \{j_l\}),$$

that is, $g(S, T)$ is a convex combination of $g(S, \{j_l\})$. This shows that, if $T = \{j_1, \ldots, j_k\}$, then

$$g(S, T) \leq \max_{1 \leq l \leq k} g(S, \{j_l\}).$$

Hence, for a given subset $S \subseteq \mathcal{A}$, we have

$$\max_{T \subseteq B} g(S, T) = \max_{j \in \mathcal{B}} g(S, \{j\}).$$

The importance of the above equation lies in enabling us to replace a maximum over all subsets of $\mathcal{B}$ with a maximum over all elements of $\mathcal{B}$. This is how we break through the barrier of enumerating an exponential number of subsets. As a consequence we have

$$\phi(X|Y) = \max_{j \in \mathcal{B}} \max_{S \subseteq \mathcal{A}} g(S, \{j\}).$$
Now for a fixed subset $S \subseteq A$ and fixed element $j \in B$, we have

$$g(S, \{j\}) = \Pr\{X \in S|Y = j\} - \Pr\{X \in S\}$$

$$= \sum_{i \in S} \left[ \frac{\theta_{ij}}{\nu_j} - \mu_i \right]$$

$$= \frac{1}{\nu_j} \sum_{i \in S} [\theta_{ij} - \mu_i \nu_j]$$

$$= \frac{1}{\nu_j} \sum_{i \in S} \gamma_{ij}.$$ 

Hence, for a fixed $j \in B$, the summation is maximized with respect to $S$ by choosing $i \in S$ if $\gamma_{ij} \geq 0$ and $i \not\in S$ if $\gamma_{ij} < 0$. The resulting maximum value (for a fixed $j \in B$) is

$$\max_{S \subseteq A} g(S, \{j\}) = \frac{1}{\nu_j} \sum_i (\gamma_{ij})_+.$$ 

So finally

$$\phi(X|Y) = \max_{j \in B} \max_{S \subseteq A} g(S, \{j\})$$

$$= \max_j \frac{1}{\nu_j} \sum_i (\gamma_{ij})_+,$$ 

which is the first equation in (20). The second equation in (20) follows from the the fact that $e_n^t \Gamma = 0_m$, which implies in turn that, for each fixed $j \in B$, we have that

$$\sum_i (\gamma_{ij})_+ = -\sum_i (\gamma_{ij})_- = 0.5 \sum_i |\gamma_{ij}|.$$ 

Lastly, the fact that

$$\max_j \frac{1}{\nu_j} \sum_i |\gamma_{ij}| = \|\Gamma[\text{Diag}(\nu)]^{-1}\|_{1,1}$$

is standard and can be found in many places, e.g. [21].

We conclude this section by observing that the $\alpha$-mixing coefficient is proportional to the $(\infty, 1)$-induced norm of the matrix $\Gamma$, whereas the $\phi$-mixing coefficient is proportional to the $(1, 1)$-induced norm of the matrix $\Gamma[\text{Diag}(\nu)]^{-1}$. Therefore the reason for the NP-hardness of computing the $\alpha$-mixing coefficient and the efficient computability of the $\phi$-mixing coefficient lies in the nature of the induced norms that need to be computed.
IV. DATA PROCESSING-TYPE INEQUALITIES FOR MIXING COEFFICIENTS

In this section we study the case where two finite-valued random variables are conditionally independent given a third finite-valued random variable, and prove inequalities of the data processing-type for the associated mixing coefficients. The nomenclature ‘data processing-type’ is motivated by the well-known data processing inequality in information theory.

Definition 3: Suppose $X, Y, Z$ are random variables assuming values in finite sets $A, B, C$ respectively. Then $X, Z$ are said to be conditionally independent given $Y$ if, $\forall i \in A, j \in B, k \in C$, it is true that

$$\Pr\{X = i & Z = k | Y = j\} = \Pr\{X = i | Y = j\} \times \Pr\{Z = k | Y = j\}. \quad \text{(22)}$$

If $X, Z$ are conditionally independent given $Y$, we denote this by $(X \perp Z) | Y$. Some authors also write this as ‘$X \rightarrow Y \rightarrow Z$ is a short Markov chain’, ignoring the fact that the three random variables can belong to quite distinct sets. In this case, it makes no difference whether we write $X \rightarrow Y \rightarrow Z$ or $Z \rightarrow Y \rightarrow X$, because it is obvious from (22) that conditional independence is a symmetric relationship. Thus

$$\text{(}X \perp Z) | Y \Leftrightarrow \text{(}Z \perp X) | Y.$$  

Also, from the definition, it follows readily that if $(X \perp Z) | Y$, then $\forall S \subseteq A, j \in B, U \subseteq C$ we have that

$$\Pr\{X \in S & Z \in U | Y = j\} = \Pr\{X \in S | Y = j\} \times \Pr\{Z \in U | Y = j\}. \quad \text{(23)}$$

However, in general, it is not true that, $\forall S \subseteq A, T \subseteq B, U \subseteq C$,

$$\Pr\{X \in S & Z \in U | Y \in T\} = \Pr\{X \in S | Y \in T\} \times \Pr\{Z \in U | Y \in T\}.$$  

In fact, by setting $T = B$, it would follow from the above relationship that $X$ and $Z$ are independent, which is a stronger requirement than conditional independence.

Given two random variables $X, Y$ with joint distribution $\theta$ and marginal distributions $\mu, \nu$ of $X, Y$ respectively, the quantity

$$H(\mu) := -\sum_{i=1}^{n} \mu_i \log \mu_i$$
is called the **entropy** of \( \mu \), with analogous definitions for \( H(\nu) \) and \( H(\theta) \); and the quantity

\[
I(X, Y) = H(\mu) + H(\nu) - H(\theta)
\]

is called the **mutual information** between \( X \) and \( Y \). It is clear that \( I(X, Y) = I(Y, X) \). The following well-known inequality, referred to as the **data-processing inequality**, is the motivation for the contents of this section; see [4, p. 34]. Suppose \( (X \perp Z)|Y \). Then

\[
I(X, Z) \leq \min\{I(X, Y), I(Y, Z)\}.
\]  

**Theorem 7:** Suppose \( (X \perp Z)|Y \). Then

\[
\alpha(X, Z) \leq \min\{\alpha(X, Y), \alpha(Y, Z)\}.
\]  

**Theorem 8:** Suppose \( (X \perp Z)|Y \). Then

\[
\beta(X, Z) \leq \min\{\beta(X, Y), \beta(Y, Z)\}.
\]  

**Theorem 9:** Suppose \( (X \perp Z)|Y \). Then

\[
\phi(X|Z) \leq \min\{\phi(X|Y), \phi(Y|Z)\},
\]

\[
\phi(Z|X) \leq \min\{\phi(Z|Y), \phi(Y|X)\}.
\]

**Proof of Theorem 7:** Let \( S \subseteq A, U \subseteq C \) be arbitrary, and define

\[
r_\alpha(S, U) := \Pr\{X \in S & Z \in U\} - \Pr\{X \in S\} \Pr\{Z \in U\}.
\]

Then

\[
\alpha(X, Y) = \max_{S \subseteq A, U \subseteq C} r_\alpha(S, U).
\]

Recall from (17) that

\[
\alpha(X, Y) = \max_{b \in \{0,1\}^n} \sum_i ((\Gamma b)_i)_+.
\]

Using the definition of the matrix \( \Gamma \) and the one-to-one relationship between vectors in \( \{0,1\}^n \) and subsets of \( B \), we can rewrite the above equation equivalently as

\[
\alpha(X, Y) = \max_{T \subseteq B} \sum_{i=1}^n \left[ \Pr\{X = i & Y \in T\} - \Pr\{X = i\} \Pr\{Y \in T\}\right]_+.
\]  

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Now we manipulate the quantity $r_\alpha(S,U)$ for arbitrary subsets $S \subseteq A, U \subseteq C$ to prove the desired conclusion.\footnote{Due to the width limitations of the two-column format, the long equations that follow have been split across two and sometimes three lines.}

$$r_\alpha(S,U) = \sum_{j=1}^{m} \left[ \Pr\{X \in S \& Y = j \& Z \in U\} - \Pr\{X \in S\} \Pr\{Y = j\} \Pr\{Z \in U\} \right] = \sum_{j=1}^{m} \left[ \Pr\{X \in S\} \Pr\{Y = j\} \Pr\{Z \in U\} \right] \	imes \Pr\{Y = j\} - \Pr\{X \in S\} \Pr\{Y = j\} \Pr\{Z \in U\} \right] + \frac{m}{\max_{U \subseteq C} \sum_{j=1}^{m} \left[ \Pr\{Z \in U \& Y = j\} \right] - \Pr\{Y = j\} \Pr\{Z \in U\} \right] + \frac{m}{\max_{U \subseteq C} \sum_{j=1}^{m} \left[ \Pr\{Z \in U \& Y = j\} \right] - \Pr\{Y = j\} \Pr\{Z \in U\} \right] = \alpha(Y,Z).$$

Since $S$ and $U$ are arbitrary, this implies that $\alpha(X,Z) \leq \alpha(Y,Z)$ whenever $X \rightarrow Y \rightarrow Z$ is a short Markov chain. Since $X \rightarrow Y \rightarrow Z$ is the same as $Z \rightarrow Y \rightarrow X$, it also follows that $\alpha(Z,X) \leq \alpha(Y,X)$. Finally, since $\alpha$ is symmetric, the desired conclusion (25) follows. \hfill \blacksquare

**Proof of Theorem 8:** Suppose that $A, B, C$ have cardinalities $n, m, l$ respectively. (The symbols $n, m$ have been introduced earlier and now $l$ is introduced.) Let $\delta$ denote the joint distribution of $(X,Y,Z)$, $\zeta$ the joint distribution of $(X,Z)$, $\eta$ the joint distribution of $(Y,Z)$, and as before,
the joint distribution of \((X, Y)\). Let \(\xi\) the marginal distribution of \(Z\), and as before, let \(\mu, \nu\) denote the marginal distributions of \(X\) and \(Y\). Finally, define

\[
c_{jk} = \frac{n_{jk}}{\nu_j} = \Pr\{Z = k|Y = j\}.
\]

As can be easily verified, the fact that \((X \perp Z)|Y\) (or (22)) is equivalent to

\[
d_{ijk} = \frac{\theta_{ij}n_{jk}}{\nu_j} = \theta_{ij}c_{jk}, \ \forall i, j, k.
\]

Also note the following identities:

\[
\sum_{i=1}^{n} \theta_{ij} = \nu_j, \ \sum_{j=1}^{m} \theta_{ij} = \mu_i, \ \sum_{j=1}^{m} d_{ijk} = \zeta_{ik}, \ \forall i, j, k.
\]

Now it follows from the various definitions that

\[
\beta(X, Z) = \sum_{i=1}^{n} \sum_{k=1}^{l} (\zeta_{ik} - \mu_i \xi_k)_+
\]

\[
= \sum_{i=1}^{n} \sum_{k=1}^{l} \left( \sum_{j=1}^{m} (d_{ijk} - \theta_{ij} \xi_k) \right)_+
\]

\[
\leq \sum_{i=1}^{n} \sum_{k=1}^{l} \sum_{j=1}^{m} (d_{ijk} - \theta_{ij} \xi_k)_+
\]

\[
= \sum_{i=1}^{n} \sum_{k=1}^{l} \sum_{j=1}^{m} (\theta_{ij}c_{jk} - \theta_{ij} \xi_k)_+
\]

\[
= \sum_{k=1}^{l} \sum_{j=1}^{m} \left[ \sum_{i=1}^{n} \theta_{ij} \right] (c_{jk} - \xi_k)_+
\]

\[
= \sum_{k=1}^{l} \sum_{j=1}^{m} (\nu_j c_{jk} - \nu_j \xi_k)_+
\]

\[
= \sum_{k=1}^{l} \sum_{j=1}^{m} (n_{jk} - \nu_j \xi_k)_+
\]

\[
= \beta(Y, Z).
\]

Now the symmetry of \(\beta(\cdot, \cdot)\) serves to show that \(\beta(X, Z) \leq \beta(X, Y)\). Putting both inequalities together leads to the desired conclusion.

**Proof of Theorem 9:** Suppose \((X \perp Z)|Y\). Since the \(\phi\)-mixing coefficient is not symmetric, it is necessary to prove two distinct inequalities, namely: (i) \(\phi(X|Z) \leq \phi(X|Y)\), and (ii) \(\phi(X|Z) \leq \phi(Y|Z)\).
**Proof that** $\phi(X|Z) \leq \phi(X|Y)$: For $S \subseteq \mathbb{A}$, define

$$r_\phi(S) := \max_{T \subseteq \mathbb{B}} \Pr\{X \in S|Y \in T\},$$

and observe that

$$\phi(X|Y) = \max_{S \subseteq \mathbb{A}} [r_\phi(S) - P_\mu(S)].$$

Suppose $S \subseteq \mathbb{A}, U \subseteq \mathbb{C}$ are arbitrary. Then

$$\Pr\{X \in S \& Z \in U\} = \sum_{j=1}^{m} \Pr\{X \in S \& Y = j \& Z \in U\}$$

$$= \sum_{j=1}^{m} \Pr\{X \in S|Y = j\} \times \Pr\{Z \in U|Y = j\} \Pr\{Y = j\}$$

$$= \sum_{j=1}^{m} \Pr\{X \in S|Y = j\} \times \Pr\{Z \in U \& Y = j\}$$

$$\leq r_\phi(S) \sum_{j=1}^{m} \Pr\{Z \in U \& Y = j\}$$

$$= r_\phi(S) \Pr\{Z \in U\}.$$

Dividing both sides by $\Pr\{Z \in U\}$ leads to

$$\Pr\{X \in S|Z \in U\} \leq r_\phi(S),$$

$$\Pr\{X \in S|Z \in U\} - P_\mu(S) \leq r_\phi(S) - P_\mu(S)$$

Taking the maximum of both sides with respect to $S \subseteq \mathbb{A}, U \subseteq \mathbb{C}$ shows that

$$\phi(X|Z) \leq \phi(X|Y).$$

**Proof that** $\phi(X|Z) \leq \phi(Y|Z)$: We begin by rewriting the expression for $\phi(X|Y)$. In order to make the equations fit, for $S \subseteq \mathbb{A}, T \subseteq \mathbb{B}$ we will use $P(S|T)$ as a shorthand for $\Pr\{X \in S|Y \in T\}$, and so on. With this convention, for $S \subseteq \mathbb{A}, T \subseteq \mathbb{B}$, we have that

$$P(S|T) - P(S) = \sum_{i \in S}[P(i|T) - P(i)]$$

$$\leq \sum_{i \in S}[P(i|T) - P(i)]_+$$

$$\leq \sum_{i \in \mathbb{A}}[P(i|T) - P(i)]_+.$$
Therefore
\[
\phi(X|Y) = \max_{S,T} [P(S|T) - P(S)] \\
\leq \max_T \sum_{i \in A} [P(i|T) - P(i)]_+.
\] (30)

Actually this can be shown to be an equality, and not an inequality, but we will not expend space on that.

Let us define
\[
c(S,U) := \Pr\{X \in S|Z \in U\} - P_{\mu}(S),
\]
and reason as follows:
\[
c(S,U) = \Pr\{X \in S|Z \in U\} - \Pr\{X \in S\} \\
= \sum_{j=1}^{m} [\Pr\{X \in S \& Y = j|Z \in U\} \\
- \Pr\{X \in S \& Y = j\}] \\
= \sum_{j=1}^{m} [\Pr\{X \in S|Y = j \& Z \in U\} \\
\times \Pr\{Y = j|Z \in U\} \\
- \Pr\{X \in S|Y = j\} \Pr\{Y = j\}] \\
\leq \sum_{j=1}^{m} [\Pr\{X \in S|Y = j\} \Pr\{Y = j|Z \in U\} \\
- \Pr\{Y = j}\} + \\
\leq \sum_{j=1}^{m} [\Pr\{Y = j|Z \in U\} - \Pr\{Y = j\}] + \\
\leq \max_{U \subseteq C} \sum_{j=1}^{m} [\Pr\{Y = j|Z \in U\} - \Pr\{Y = j\}] + \\
\leq \phi(Y|Z),
\]
where the last step follows from (30). Since the right side is independent of both $S$ and $U$, the desired conclusion follows. \[\blacksquare\]
V. INCONSISTENCY OF AN ESTIMATOR FOR MIXING COEFFICIENTS

Suppose $X, Y$ are real-valued random variables with some unknown joint distribution, and suppose we are given an infinite sequence of independent samples $\{(x_i, y_i), i = 1, 2, \ldots\}$. The question studied in this section and the next is whether it is possible to construct empirical estimates of the various mixing coefficients that converge to the true values as the number of samples approaches infinity.

Let

$$
\hat{\Phi}_{X,Y}(a, b) = \Pr\{X \leq a \& Y \leq b\}
$$

denote the true but unknown joint distribution function of $X$ and $Y$, and let $\Phi_X(\cdot), \Phi_Y(\cdot)$ denote the true but unknown marginal distribution functions of $X, Y$ respectively. Using the samples $\{(x_i, y_i), i = 1, 2, \ldots\}$, we can construct three ‘stair-case functions’ that are empirical estimates of $\Phi_X, \Phi_Y$ and $\hat{\Phi}_{X,Y}$ based on the first $l$ samples, as follows:

$$
\hat{\Phi}_X(a; l) := \frac{1}{l} \sum_{i=1}^{l} I\{x_i \leq a\},
$$

$$
\hat{\Phi}_Y(b; l) := \frac{1}{l} \sum_{i=1}^{l} I\{y_i \leq b\},
$$

$$
\hat{\Phi}_{X,Y}(a, b; l) := \frac{1}{l} \sum_{i=1}^{l} I\{x_i \leq a \& y_i \leq b\},
$$

where as usual $I$ denotes the indicator function. Thus $\hat{\Phi}_X(a; l)$ counts the fraction of the first $l$ samples that are less than or equal to $a$, and so on. With this construction, the well-known Glivenko-Cantelli lemma (see [8], [3] or [11, p. 20]) states that the empirical estimates converge uniformly and almost surely to their true functions as the number of samples $l \to \infty$. Thus $\hat{\Phi}_{X,Y}$ is a consistent estimator of the true joint distribution. Thus one might be tempted to think that an empirical estimate of any (or all) of the three mixing coefficients based on $\hat{\Phi}_{X,Y}$ will also converge to the true value as $l \to \infty$. The objective of this brief section is to show that this is not so. Hence estimates of mixing coefficients derived from a consistent estimator of the joint distribution need not themselves be consistent.

**Theorem 10:** Suppose $\hat{\Phi}_{X,Y}$ is defined by (33), and that $x_i \neq x_j$ and $y_i \neq y_j$ whenever $i \neq j$. Let $\hat{\beta}_i$ denote the $\beta$-mixing coefficient associated with the joint distribution $\hat{\Phi}_{X,Y}(\cdot, \cdot; l)$. Then $\hat{\beta}_i = (l - 1)/l$. 

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**Proof:** Fix the integer $l$ in what follows. Note that the empirical distribution $\hat{\Phi}_{X,Y}(\cdot,\cdot;l)$ depends only on the totality of the $l$ samples, and not the order in which they are generated. Without loss of generality, we can replace the samples $(x_1,\ldots,x_n)$ by their ‘order statistics’, that is, the same samples arranged in increasing order, and do the same for the $y_i$. Thus the assumption is that $x_1 < x_2 < \ldots < x_l$ and similarly $y_1 < y_2 < \ldots < y_l$. With this convention, the empirical samples will be of the form $\{(x_1,y_{\pi(1)}),\ldots,(x_l,y_{\pi(l)})\}$ for some permutation $\pi$ of $\{1,\ldots,l\}$. Therefore the probability measure associated with the empirical distribution $\hat{\Phi}$ is purely atomic, with jumps of magnitude $1/l$ at the points $\{(x_1,y_{\pi(1)}),\ldots,(x_l,y_{\pi(l)})\}$. So we can simplify matters by replacing the real line on the $X$-axis by the finite set $\{x_1,\ldots,x_l\}$, and the real line on the $Y$-axis by the finite set $\{y_1,\ldots,y_l\}$. With this redefinition, the joint distribution $\theta$ assigns a weight of $1/l$ to each of the points $(x_i,y_{\pi(i)})$ and a weight of zero to all other points $(x_i,y_j)$ whenever $j \neq \pi(i)$, while the marginal measures $\mu, \nu$ of $X$ and $Y$ will be uniform on the respective finite sets. Thus the product measure $\mu \times \nu$ assigns a weight of $1/l^2$ to each of the $l^2$ grid points $(x_i,y_j)$. From this, it is easy to see that

$$\hat{\beta}_l = \rho(\theta,\mu \times \nu) = (l-1)/l.$$ 

This is the desired conclusion. ■

**Corollary 2:** Suppose the true but unknown distribution $\Phi_{X,Y}$ has a density with respect to the Lebesgue measure. Then $\hat{\beta}_l \to 1, \hat{\phi}_l \to 1$ almost surely as $l \to \infty$.

**Proof:** If the true distribution has a density, then it is nonatomic, which means that with probability one, samples will be pairwise distinct. It now follows from Theorem 10 that

$$\hat{\phi}_l \geq \hat{\beta}_l = \frac{l-1}{l} \to 1 \text{ as } l \to \infty.$$ 

This is the desired conclusion. ■

VI. **Consistent Estimators for Mixing Coefficients**

The objective of the present section is to show that a simple modification of the ‘naive’ algorithm proposed in Section V does indeed lead to consistent estimates, provided appropriate technical conditions are satisfied.

The basic idea behind the estimators is quite simple. Suppose that one is given samples $\{(x_i,y_i), i \geq 1\}$ generated independently and at random from an unknown joint probability
measure $\theta \in \mathcal{M}(\mathbb{R}^2)$. Given $l$ samples, choose an integer $k_l$ of bins. Divide the real line into $k_l$ intervals such that each bin contains $\lfloor l/k_l \rfloor$ or $\lfloor l/k_l \rfloor + 1$ samples for both $X$ and $Y$. In other words, carry out percentile binning of both random variables. One way to do this (but the proof is not dependent on how precisely this is done) is as follows: Define $m_l = \lfloor l/k_l \rfloor$, $r = l - k_l m_l$, and place $m_l + 1$ samples in the first $r$ bins and $m_l$ samples in the next $m_l - r$ bins. This gives a way of discretizing the real line for both $X$ and $Y$ such that the discretized random variables have nearly uniform marginals. With this binning, compute the corresponding joint distribution, and the associated empirical estimates of the mixing coefficients. The various theorems below show that, subject to some regularity conditions, the empirical estimates produced by this scheme do indeed converge to their right values with probability one as $l \to \infty$, provided that $m_l \to \infty$, or equivalently, $k_l/l \to 0$, as $l \to \infty$. In other words, in order for this theorem to apply, the number of bins must increase more slowly than the number of samples, so that the number of samples per bin must approach infinity. In contrast, in Theorem 10, we have effectively chosen $k_l = l$ so that each bin contains precisely one sample, which explains why that approximation scheme does not work.

To state the various theorems, we introduce a little bit of notation, and refer the reader to [2] for all concepts from measure theory that are not explicitly defined here. Let $\mathcal{M}(\mathbb{R}), \mathcal{M}(\mathbb{R}^2)$ denote the set of all measures on $\mathbb{R}$ or $\mathbb{R}^2$ equipped with the Borel $\sigma$-algebra. Recall that if $\theta, \eta \in \mathcal{M}(\mathbb{R})$ or $\mathcal{M}(\mathbb{R}^2)$, then $\theta$ is said to be absolutely continuous with respect to $\eta$, denoted by $\theta \ll \eta$, if for every measurable set $E$, $\eta(E) = 0 \Rightarrow \theta(E) = 0$.

Next, let $\theta$ denote the joint probability measure of $(X,Y)$, and let $\mu, \nu$ denote the marginal measures. Thus, for every measurable subset $S \subseteq \mathbb{R}$, the measure $\mu(S)$ is defined as $\theta(S \times \mathbb{R})$ and similarly for all $T \subseteq \mathbb{R}$, the measure $\nu(T)$ is defined as $\theta(\mathbb{R} \times T)$. Now the key assumption made here is that the joint measure $\theta$ is absolutely continuous with respect to the product measure $\mu \times \nu$. In the case of finite-valued random variables, this assumption is automatically satisfied. Suppose that for some pair of indices $i, j$, it is the case that $\mu_i \cdot \nu_j = 0$. Then either $\mu_i = 0$ or $\nu_j = 0$. If $\mu_i = 0$, then it follows from the identity $\sum_{j'} \theta_{ij'} = \mu_i$ that $\theta_{ij'} = 0$ for all $j'$, and in particular $\theta_{ij} = 0$. Similarly if $\nu_j = 0$, then it follows from the identity $\sum_{i'} \theta_{i'j} = \nu_j$ that $\theta_{i'j} = 0$ for all $i'$, and in particular $\theta_{ij} = 0$. In either case it follows that $\theta_{ij} = 0$, so that $\theta \ll \mu \times \nu$.

$^2$Hereafter we drop this adjective; it is assumed that all sets that are encountered are measurable.
However, in the case of real random variables, this need not be so. For example, replace $\mathbb{R} \times \mathbb{R}$ by the unit square, and let $\theta$ be the diagonal measure. Then both marginals $\mu, \nu$ are the uniform measures on the unit interval, and the product $\mu \times \nu$ is the uniform measure on the unit square – and $\theta$ is singular with respect to the uniform measure.

Next we introduce symbols for the various densities. Since $\theta \ll \mu \times \nu$, it follows that $\theta$ has a Radon-Nikodym derivative with respect to $\mu \times \nu$, which is denoted by $f(\cdot, \cdot)$. So for any sets $S, T \subseteq \mathbb{R}$, it follows that

$$\theta(S \times T) = \int_S \int_T f(x, y) d\nu(y) d\mu(x) = \int_T \int_S f(x, y) d\mu(x) d\nu(y).$$

For any $T \subseteq \mathbb{R}$ with $\nu(T) > 0$, the conditional probability $\Pr\{X \in S | Y \in T\}$ is given by

$$\Pr\{X \in S | Y \in T\} = \frac{\Pr\{X \in S & Y \in T\}}{\Pr\{Y \in T\}} = \frac{\theta(S \times T)}{\nu(T)} = \int_S \left[ \int_T \frac{f(x, y)}{\nu(T)} d\nu(y) \right] d\mu(x).$$

**Theorem 11:** Suppose $\theta \ll \mu \times \nu$, and that $k_l \to \infty$, $k_l/l \to 0$ as $l \to \infty$. Then the empirically estimated $\beta$-mixing coefficient $\hat{\beta}_l$ converges almost surely to the true value $\beta$ as $l \to \infty$.

**Theorem 12:** Suppose $\theta \ll \mu \times \nu$, and in addition that the density $f(\cdot, \cdot)$ belongs to $L_{\infty}(\mathbb{R}^2)$. Suppose that $k_l \to \infty$, $k_l/l \to 0$ as $l \to \infty$. Then the empirically estimated $\alpha$-mixing coefficient $\hat{\alpha}_l$ converges almost surely to the true value $\alpha$ as $l \to \infty$, and the empirically estimated $\phi$-mixing coefficient $\phi_l$ converges almost surely to the true value $\phi$ as $l \to \infty$.

Note that the absolute continuity assumption $\theta \ll \mu \times \nu$ guarantees that the density $f(\cdot, \cdot) \in L_1(\mathbb{R}^2, \mu \times \nu)$. So no additional technical assumptions are needed to ensure that the sequence of empirical estimates $\hat{\beta}_l$ converges to its true value. However, in order to establish that the sequences of empirical estimates $\hat{\alpha}_l$ and $\phi_l$ converge to their true values, we have added an assumption that the density $f$ is bounded almost everywhere. This condition is intended to ensure that conditional densities do not ‘blow up’. In the case of finite-valued variables, we have already seen that the condition $\theta \ll \mu \times \nu$ holds automatically, which means that the ‘density’ $f_{ij} \theta_{ij}/(\mu_i \nu_j)$ is always well-defined. Since there are only finitely many values of $i$ and $j$, this ratio is also bounded. However, in the case of real-valued random variables, this condition needs to be imposed explicitly.
The proofs of these two theorems are based on arguments in [12], [27]. In the proof of Theorem 11, we can use those arguments as they are, whereas in the proof of Theorem 12, we need to adapt them. To facilitate the discussion, we first reprise the relevant results from [12], [27].

**Definition 4:** Let \((\Omega, \mathcal{F})\) be a measurable space, and let \(Q\) be a probability measure on \((\Omega, \mathcal{F})\). Suppose \(\{I_1, \ldots, I_L\}\) is a finite partition of \(\Omega\), and that \(\{I_1^{(m)}, \ldots, I_L^{(m)}\}\) is a sequence of partitions of \(\Omega\). Then \(\{I_1^{(m)}, \ldots, I_L^{(m)}\}\) is said to **converge to** \(\{I_1, \ldots, I_L\}\) with respect to \(Q\) if, for every probability measure \(P\) on \((\Omega, \mathcal{F})\) such that \(P \ll Q\), it is the case that

\[
P(I_i^{(m)}) \to P(I_i) \quad \text{as} \quad m \to \infty.
\]

See [27, Definition 1].

**Theorem 13:** Suppose \(Q\) is a probability measure on \((\mathbb{R}, \mathcal{B})\) that is absolutely continuous with respect to the Lebesgue measure, \(L\) is a fixed integer, and that \(\{I_1, \ldots, I_L\}\) is an equiprobable partitioning of \(\mathbb{R}\). In other words, choose numbers

\[-\infty = a_0 < a_1 < \ldots < a_{L-1} < a_L = +\infty\]

such that the semi-open intervals \(I_i = (a_{i-1}, a_i]\) satisfy

\[
Q(I_i) = 1/L, \quad i = 1, \ldots, L.
\]

Suppose \(\{y_1, \ldots, y_m\}\) are i.i.d. samples generated in accordance with \(Q\), and that \(m = l_m T\) with \(l_m \in \mathbb{N}\), an integer. Let \(\{I_1^{(m)}, \ldots, I_L^{(m)}\}\) denote the empirical equiprobable partitioning associated with the samples \(\{y_1, \ldots, y_m\}\). Then \(\{I_1^{(m)}, \ldots, I_L^{(m)}\}\) converges to \(\{I_1, \ldots, I_L\}\) with respect to \(Q\) as \(m \to \infty\).

**Proof:** See [27, Lemma 1].

**Theorem 14:** Let \((\Omega, \mathcal{F})\) be a measurable space, and let \(Q\) be a probability measure on \((\Omega, \mathcal{F})\). Suppose \(\{I_1^{(m)}, \ldots, I_L^{(m)}\}\) is a sequence of partitions of \(\Omega\) that converges with respect to \(Q\) to another partition \(\{I_1, \ldots, I_L\}\) as \(m \to \infty\). Suppose \(\{x_1, \ldots, x_n\}\) are i.i.d. samples generated in accordance with a probability measure \(P \ll Q\), and let \(P_n\) the empirical measure generated by these samples. Then

\[
\lim_{m \to \infty} \lim_{n \to \infty} P_n(I_i^{(m)}) = P(I_i), \quad \text{a.s.} \quad \forall i.
\]

**Proof:** See [27, Lemma 2].
Before proceeding to the proofs of the two theorems, we express the three mixing coefficients in terms of the densities. As stated in (4), we have that

$$\beta(X, Y) = 0.5 \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y) - 1| d\mu(x) d\nu(y).$$

(34)

Here we take advantage of the fact that the ‘density’ of \(\mu\) with respect to itself is one, and similarly for \(\nu\). Next, as in Theorem 1, we can drop the absolute value signs in the definitions of \(\alpha(X, Y)\) and of \(\phi(X|Y)\). Therefore the various mixing coefficients can be expressed as follows:

$$\alpha(X, Y) = \sup_T \sup_S \int_S \int_T [f(x, y) - 1] d\nu(y) d\mu(x),$$

(35)

$$\phi(X, Y) = \sup_T \sup_S \left[ \int_S \int_T \frac{f(x, y)}{\nu(T)} d\nu(y) - 1 \right] d\mu(x).$$

(36)

Now, for each fixed set \(T\), let us define signed measures \(\kappa_T\) and \(\delta_T\) as follows:

$$\kappa_T(x) = \int_T [f(x, y) - 1] d\nu(y),$$

$$\delta_T(x) = \int_T \frac{f(x, y)}{\nu(T)} d\nu(y) - 1,$$

and associated support sets

$$A_+(T) = \{x \in \mathbb{R} : \kappa_T(x) \geq 0\},$$

$$B_+(T) = \{x \in \mathbb{R} : \delta_T(x) \geq 0\}.$$ 

Then it is easy to see that, for each fixed set \(T\), the supremum in (35) is achieved by the choice \(S = A_+(T)\) while the supremum in (36) is achieved by the choice \(S = B_+(T)\). Therefore

$$\alpha(X, Y) = \sup_T \int_{A_+(T)} \kappa_T(x) d\mu(x)$$

$$= \sup_T \int_{A_+(T)} [\kappa_T(x)]_+ d\mu(x),$$

(37)

$$\phi(X|Y) = \sup_T \int_{B_+(T)} \delta_T(x) d\mu(x)$$

$$= \sup_T \int_{B_+(T)} [\delta_T(x)]_+ d\mu(x).$$

(38)

These formulas are the continuous analogs of (29) and (30) respectively.

**Proof of Theorem 11:** For a fixed integer \(L \geq 2\), choose real numbers

$$-\infty = a_0 < a_1 < \ldots < a_{L-1} < a_L = +\infty,$$
\[ -\infty = b_0 < b_1 < \ldots < b_{L-1} < b_L = +\infty \]
such that the semi-open intervals \( I_i = (a_{i-1}, a_i], J_i = (b_{i-1}, b_i] \) satisfy
\[
\mu(I_i) = 1/L, \nu(J_i) = 1/L, i = 1, \ldots, L.
\]

Now define the equiprobable partition of \( \mathbb{R}^2 \) consisting of the \( L \times L \) grid \( \{ I_i \times J_j, i, j = 1, \ldots, L \} \). Next, based on the \( l \)-length empirical sample \( \{(x_1, y_1), \ldots, (x_l, y_l)\} \), construct empirical marginal distributions \( \hat{\mu} \) for \( X \) and \( \hat{\nu} \) for \( Y \). Based on these empirical marginals, divide both the \( X \)-axis \( \mathbb{R} \) and \( Y \)-axis \( \mathbb{R} \) into \( L \) bins each having nearly equal fractions of the \( l \) samples in each bin. This gives an empirical \( L \times L \) partitioning of \( \mathbb{R}^2 \), which is denoted by \( \{ I_i^{(L)} \times J_j^{(L)}, i, j = 1, \ldots, L \} \).

Using this grid, compute the associated empirical joint distribution \( \hat{\theta}_l \) on \( \mathbb{R}^2 \). Then the proof of [27, Lemma 1] can be adapted to show that the empirical partition \( \{ I_i^{(L)} \times J_j^{(L)}, i, j = 1, \ldots, L \} \) converges to the true partition \( \{ I_i \times J_j, i, j = 1, \ldots, L \} \) as \( l \to \infty \), with respect to the product measure \( \mu \times \nu \). The only detail that differs from [27] is the computation of the so-called ‘growth function’. Given a set \( A \subseteq \mathbb{R}^2 \) of cardinality \( m \), the number of different ways in which this set can be partitioned by a rectangular grid of dimension \( L \times L \) is called the growth function, denoted by \( \Delta_m \). It is shown in [27, Eq. (15)] that when the partition consists of \( L \) intervals and the set being partitioned is \( \mathbb{R} \), then \( \Delta_m \) is given by the combinatorial parameter
\[
\Delta_m = \binom{m + L}{L} = \frac{(m + L)!}{m!L!}.
\]

It is also shown in [27, Eq. (21)] that
\[
\frac{1}{m} \log \left( \binom{m + L}{L} \right) \leq 2mh(1/L),
\]
where \( h(\cdot) \) is defined by
\[
h(x) = -x \log x - (1 - x) \log(1 - x), \ \forall x \in (0, 1).
\]

When \( \mathbb{R} \) is replaced by \( \mathbb{R}^2 \) and a set of \( L \) intervals is replaced by a grid of \( L^2 \) rectangles, it is easy to see that the growth function is \textit{no larger than}
\[
\Delta_m \leq \left( \binom{m + L}{L} \right)^2.
\]
Therefore
\[
\frac{\log \Delta_m}{m} \leq 4mh(1/L).
\]
In any case, since \( L \), the number of grid elements, approaches \( \infty \) as \( l \to \infty \), it follows that
the growth condition proposed in [12] is satisfied. Therefore the empirical partition converges to the true partition as \( l \to \infty \).

Next, let \( \{I_i \times J_j, i, j = 1, \ldots, L\} \) denote, as before, the true equiprobable \( L \times L \) gridding of \( \mathbb{R}^2 \). Suppose that, after \( l \) samples \((x_r, y_r), r = 1, \ldots, l\) have been drawn, the data is put into \( k_l \) bins. Then the expression (34) defining the true \( \beta \)-mixing coefficient can be rewritten as
\[
\beta(X, Y) = 0.5 \sum_{i=1}^{k_l} \sum_{j=1}^{k_l} \int_{I_i} \int_{J_j} |f(x, y) - 1| d\mu(x) d\nu(y).
\]
Now suppose \( l \) is an exact multiple of \( k_l \). Then the empirical estimate based on the \( k_l \times k_l \) empirical grid can be written as
\[
\hat{\beta}_l = 0.5 \sum_{i=1}^{k_l} \sum_{j=1}^{k_l} |C_{ij} - 1| k_l^{-2},
\]
where \( C_{ij} \) denotes the number of samples \((x_r, y_r)\) in the \( ij \)-th cell of the empirical (not true) equiprobable grid. If \( l \) is not an exact multiple of \( k_l \), then some bins will have \( \lfloor l/k_l \rfloor \) elements while other bins will have \( \lfloor l/k_l \rfloor + 1 \) elements. As a result, the term \( k_l^{-2} \) gets replaced by \((s_i t_j)/l^2\) where \( s_i \) is the number of samples in \( I_i^{(l)} \) and \( t_j \) is the number of samples in \( J_j^{(l)} \).

Now, just as in [27, Eq. (36) et seq.], the error \( |\hat{\beta}_l - \beta(X, Y)| \) can be bounded by the sum of two errors, the first of which is caused by the fact that the empirical equiprobable grid is not the same as the true equiprobable grid (the term \( e_1 \) of [27]), and the second is the error caused by approximating an integral by a finite sum over the true equiprobable grid (the term \( e_2 \) of [27]). Out of these, the first error term goes to zero as \( l \to \infty \) because, if \( k_l/l \to 0 \) so that each bin contains increasingly many samples, the empirical equiprobable grid converges to the true equiprobable grid. The second error terms goes to zero because the integrand in (34) belongs to \( L_1(\mathbb{R}^2, \mu \times \nu) \), as shown in [27, Eq. (37)].

**Proof of Theorem 12:** The main source of difficulty here is that, whereas the expression for \( \beta(X, Y) \) involves just a single integral, the expressions for \( \alpha(X, Y) \) and for \( \phi(X, Y) \) involve the supremum over all sets \( T \subseteq \mathbb{R} \). Thus, in order to show that the empirical estimates converge to the true values, we must show not only that empirical estimates of integrals of the form...
\[ \int_{\mathbb{R}} [\kappa_T]_+ d\mu(x) \text{ and } \int_{\mathbb{R}} [\delta_T]_+ d\mu(x) \] converge to their correct values for each fixed set \( T \), but also that the convergence is in some sense uniform with respect to \( T \). This is where we use the boundedness of the density \( f(\cdot, \cdot) \). The details are fairly routine modifications of arguments in [27]. Specifically, (switching notation to that of [27]), suppose that in their Equation (27), we have not just one measure \( \mu \), but rather a family of measures \( \mu_T \), indexed by \( T \), and suppose there exists a finite constant \( c \) such that for every set \( S \) we have \( \mu_T(S) \leq cQ(S) \). Then it follows from Equation (27) et seq. of [27] that

\[ \mu_T((a_i \land a_i^m, a_i \lor a_i^m)) \leq cQ((a_i \land a_i^m, a_i \lor a_i^m)), \forall T. \]

Therefore

\[ \lim_{m \to \infty} \sup_T \mu_T((a_i \land a_i^m, a_i \lor a_i^m)) = 0. \]

With this modification, the rest of the proof in [27] can be mimicked to show the following: In the interests of brevity, define

\[ r_T = \int_{\mathbb{R}} [\kappa_T]_+ d\mu(x) \]

and let \( \hat{r}_{T,l} \) denote its empirical approximation. Then, using the above modification of the argument in [27], it follows that

\[ \lim_{l \to \infty} \sup_T |r_T - \hat{r}_T| = 0. \]

As a consequence,

\[ \lim_{l \to \infty} \sup_T \hat{r}_T = \sup_T r_T = \alpha(X, Y). \]

The proof for the \( \phi \)-mixing coefficient is entirely similar. ■

VII. CONCLUDING REMARKS

In this paper we have studied the problems of computing and estimating the mixing coefficients between two random variables in two important cases, namely: finite-valued and real-valued random variables. Three different mixing coefficients were studied, namely \( \alpha \)-mixing, \( \beta \)-mixing and \( \phi \)-mixing coefficients. In the case of finite-valued random variables, it has been shown that determining whether the \( \alpha \)-mixing coefficient exceeds a prespecified threshold is an NP-complete problem. Efficiently computable upper and lower bounds for the \( \alpha \)-mixing coefficients have been derived. In contrast, an explicit and efficiently computable formula has been derived for the \( \phi \)-mixing coefficient. Analogs of the data-processing inequality from information theory have been
established for each of the three kinds of mixing coefficients. In the case of real-valued random variables, by using percentile binning and allowing the number of bins to increase more slowly than the number of samples, we can generate empirical estimates that converge to the true values for all the three kinds of mixing coefficients.

Several interesting questions are thrown up by the contents of this paper. As mentioned in the introduction, mixing coefficients were originally introduced as a way of extending the law of large numbers to stochastic processes that are not i.i.d. The problem studied in Section VI is to estimate the mixing coefficient between two real-valued random variables \( X \) and \( Y \), based on i.i.d. samples of the pair. A counterpoint to this problem is that studied in [14], where the objective is to estimate the \( \beta \)-mixing rate of a stationary stochastic process \( \{X_t\} \) from a single sample path. For a fixed integer \( k \), the rate \( \beta(k) \) can be interpreted as the \( \beta \)-mixing coefficient between the semi-infinite past \( X_{-\infty}^0 \) and the semi-infinite future \( X_k^\infty \). However, the techniques presented here do not work for that problem, whereas [14] presents a comprehensive solution in terms of “blocking” and “histogramming”, that is, estimating the joint distribution of \( dl \) consecutive random variables, when \( l \) samples are available in all. It is interesting to note that the convergence results in [14] also depend on letting \( dl \) grow more slowly than \( l \). Specifically, as shown in [14, Theorem 2.3], the estimator converges to the true value provided \( dl = O(\exp[W(\log l)]) \), where \( W \) is the Lambert \( W \) function. More details can be found in [14]. It would be worthwhile to explore whether similar estimators can be constructed for the \( \alpha \)-mixing rate of a stochastic process.

Another direction is to explore whether analogs of the data processing inequality, namely (25) through (28), hold for real-valued random variables. The proof techniques in Section IV make heavy use of the finiteness of the underlying sets where the various random variables assume their values. On the other hand, there are analogous formulas for real-valued random variables, namely (37) and (38). It might therefore be possible to extend the proofs in Section IV making use of these formulas. However, the technicalities may prove to be formidable.

In the consistency theorems of Section VI, the requirement that the bins should consist of empirically equiprobable (or percentile) samples is really not necessary. A close examination of the proof techniques used in [27] shows that, so long as the minimum number of samples in each bin approaches infinity as \( l \to \infty \), the results would still hold. We leave it to the reader to state and prove such results. The later parts of the paper [27] contain some proposals on how to
speed up the convergence of the empirical estimates of the Kullback-Leibler divergence between two unknown measures. It would be worthwhile to explore whether similar speed-ups can be found for the algorithms proposed here for estimating mixing coefficients from empirical data.

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