A HIGHER CATEGORY OF COBORDISMS AND TOPOLOGICAL QUANTUM FIELD THEORY

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ABSTRACT. The goal of this work is to describe a categorical formalism for (Extended) Topological Quantum Field Theories (TQFTs) and present them as functors from a suitable category of cobordisms with corners to a linear category, generalizing 2d open-closed TQFTs to higher dimensions. The approach is based on the notion of an \(n\)-fold category by C. Ehresmann, weakened in the spirit of monoidal categories (associators, interchangers, Mac Lane’s pentagons and hexagons), in contrast with the simplicial (weak Kan and complete Segal) approach of Jacob Lurie. We show how different Topological Quantum Field Theories, such as gauge, Chern-Simons, Yang-Mills, WZW, Seiberg-Witten, Rozansky-Witten, and AKSZ theories, as well as sigma model, may be described as functors from the pseudo \(n\)-fold category of cobordisms to a pseudo \(n\)-fold category of sets.

INTRODUCTION

In recent years, there has been an increased interest to higher categories or \(n\)-categories [Lei02, CL04, Lur09a]. One of the motivating ideas for that development was Grothendieck’s idea of the fundamental \(n\)-groupoid of a topological space. Since any kind of associativity in the fundamental \(n\)-groupoid is expected to be satisfied only up to homotopy, similar to composition of based loops in a space, so-called “weak” \(n\)-categories have been of primary interest in the higher-category community. In this paper, we focus on a close relative of the fundamental \(n\)-groupoid, the category of cobordisms with corners. We introduce the notion of a pseudo \(n\)-fold category, which is a weak version of a classical strict notion by C. Ehresmann [Ehr63, EE78], and show that cobordisms with corners naturally possess the structure of a pseudo \(n\)-fold category, see Theorem 7.1. This weak version is similar to the familiar weakness in monoidal categories: associators and interchangers are part of the structure, and coherence axioms include pentagons and hexagons. One of the main results of the paper is a Regular Coherence theorem, Theorem 10.1 of the Appendix, indicates that our set of coherence axioms is complete, that is to say, no matter what coherence maps one chooses to go from one way of composing morphisms to another, the resulting coherence maps will be equal. This theorem is analogous to Mac Lane’s coherence theorems for monoidal and symmetric monoidal categories and Joyal-Street’s coherence theorem for braided monoidal categories and possibly a generalization thereof.

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Another goal of our paper is to present (extended) Topological Quantum Field Theories (TQFTs), including gauge, Chern-Simons, Yang-Mills, and Seiberg-Witten theories, and sigma model, as (contravariant, lax) monoidal functors from the monoidal pseudo $n$-fold category of cobordisms with corners to a certain monoidal $n$-fold category of spans of sets, which are set-theoretic counterparts of cobordisms with corners. The problem of presenting TQFTs as such functors was set up by Baez and Dolan [BD95], who formulated it as part of the extended TQFT hypothesis, which also anticipates that extended TQFTs are classified by their values on the one-point space. J. Lurie [Lur09b] described all the ingredients of such formalization and proved the extended TQFT hypothesis. S. Morrison and K. Walker [MW10] gave a very interesting definition of a weak $n$-category and based their (derived) TQFTs on $n$-cobordisms in which all inputs, boundaries, creases, and corners were essentially morphed into one boundary. On a more physical side, A. Kapustin [Kap10] indicated what one should expect from higher categories to guarantee that actual models of quantum field theory are indeed described as extended TQFT functors. Our main result here is a demonstration that numerous physical models, from gauge theory to sigma model, may indeed be described as extended TQFTs in our formalism, see Metatheorem 9.1.

We have chosen an approach to higher categories with concrete composition laws and coherences given by concrete associators and interchangers. Another approach to higher categories, invented by Boardman and Vogt [BV73] and developed in subsequent works of Joyal [Joy02] and Lurie [Lur09a], uses “fuzzy” compositions and coherences and has proven to be very successful. We could have followed that path as well and worked with the notion of an $n$-fold quasi-category, which we introduce in Section 8 but decided to develop an approach hinted on and attempted in the works of Baez and Dolan [BD95], Verity, Morton, and Grandis, to name a few. This approach has all the advantages of a hands-on method: since there exist canonical, familiar compositions in such examples as cospans of sets or cobordisms, it is tempting to have a theory based on them. We have found it amazing that this theory has not yet been developed to a fully coherent theory, and this is one of the goals of the present paper.

In this paper, we deliberately avoid the question of Baez-Dolan’s cobordism (or tangle) hypothesis [BD95], so as to reduce categorical considerations to a minimum and concentrate on the physical examples of Section 9. It would be interesting to introduce duals à la Baez and Dolan in our context and verify the cobordism hypothesis.

If the reader intends to skip the present paragraph, it may appear to him that we must run into all sorts of set-theoretic difficulties in the paper. Those may be resolved by using standard Grothendieck’s trick: assume the existence of two universes, one being an element of the other: $\mathcal{U} \in \mathcal{U}'$. Then by default, when we talk about sets, spaces, etc., we will assume those to be $\mathcal{U}$-small. Then the categories of such will automatically be $\mathcal{U}'$-small. These will be our default assumptions throughout the paper.

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1. Strict n-Fold Categories

In this section we give a brief account of (strict) n-fold categories of C. Ehresmann [Ehr63, EE78], including three equivalent definitions of an n-fold category. Details may be found in [FP10] and [Pfe07].

There is an algebraic gadget which describes the usual notion of a category. This gadget is called the theory of categories, denoted $\text{Th}(\text{Cat})$. It is an essentially algebraic theory, also known as a finitely complete category, which is just a category with finite limits, or equivalently, pullbacks and a terminal object. The statement that $\text{Th}(\text{Cat})$ describes the notion of a category means that a (U'-small, see the last paragraph of the Introduction) category may be defined as an algebra over $\text{Th}(\text{Cat})$, which means a left exact, i.e., finite-limit preserving, functor $\text{Th}(\text{Cat}) \to \text{Set}$, where $\text{Set}$ is the category of (U'-small) sets. The theory $\text{Th}(\text{Cat})$ of categories is generated (as a finitely complete category) by two objects $\text{Ob}$ and $\text{Mor}$ and four morphisms $s, t : \text{Mor} \to \text{Ob}$, $\text{id} : \text{Ob} \to \text{Mor}$, and $\circ : \text{Mor} \times_s \text{Mor} \to \text{Mor}$, with a number of relations imposed, including $\circ(\circ \times \text{id}) = \circ(\text{id} \times \circ)$, which encodes the associativity of morphism composition in a category, relations encoding what the source and the target of identity morphisms and compositions of morphisms should be, and those encoding the behavior of identity morphisms as units under composition, see details in [Pfe07].

**Definition 1.1.** An n-fold category is a left exact (i.e., limit preserving) functor $F : \text{Th}(\text{Cat})^n \to \text{Set}$.

In particular, a 0-fold category is a set and a 1-fold category is just a category. 2-Fold categories are also known as double categories.

**Remark.** Note that by definition, the (essentially algebraic) theory of n-fold categories is nothing but the nth Cartesian power of the theory $\text{Th}(\text{Cat})$ of categories.

The following, equivalent definition of an n-fold category is inductive. Set $\text{Cat}_0 := \text{Set}$ and assume we have defined the category $\text{Cat}_{n-1}$ of (n − 1)-fold categories and proven it is finitely complete.

**Definition 1.2.** An n-fold category is a left exact functor $F : \text{Th}(\text{Cat}) \to \text{Cat}_{n-1}$.

Equivalently, an n-fold category is an internal category in $\text{Cat}_{n-1}$.

Another equivalent definition of an n-fold category may be given by specifying sets of 0-, 1-, . . . , and n-morphisms and maps defining relationships between these sets.

**Definition 1.3.** An n-fold category consists of the following data satisfying the following axioms.

**Morphisms:**

- A set $X$ of 0-morphisms (also called the objects);
For each $i$, $1 \leq i \leq n$, a set $X_i$ of 1-morphisms in the direction labeled $i$;

For each $k$-combination of numbers between 1 and $n$, a set $X_{i_1 \ldots i_k}$ of $k$-morphisms in the directions $i_1, \ldots, i_k$, i.e., in the direction of the "$x_{i_1 \ldots i_k}$ plane;"

A set $X_{i_1 \ldots n}$ of $n$-morphisms (in all existing directions, labeled $1, \ldots, n$).

Let $kX$ denote the set of all $k$-morphisms.

Sources, targets, identities: For each $k$, $1 \leq k \leq n$, (multi)maps $s : kX \rightarrow (k-1)X$ for sources, $t : kX \rightarrow (k-1)X$ for targets, and $id : (k-1)X \rightarrow kX$ for identities. Here multimaps $s$, $t$, and $id$ denote collections of maps $s_{ik}, t_{ik} : X_{i_1 \ldots i_k} \rightarrow X_{i_1 \ldots i_{k-1}}$, and $id_{ik} : X_{i_1 \ldots i_{k-1}} \rightarrow X_{i_1 \ldots i_k}$ for each combination $\{i_1, \ldots, i_k\}$ of $k$ numbers between 1 and $n$.

Compositions: For each combination $\{i_1, \ldots, i_k\}$ of $k$ numbers between 1 and $n$, maps

$$\circ_{ik} : X_{i_1 \ldots i_k} \times X_{i_1 \ldots i_{k-1}} X_{i_1 \ldots i_k} \rightarrow X_{i_1 \ldots i_k}$$

of composition in direction $i_k$. Here $X_{i_1 \ldots i_k} \times X_{i_1 \ldots i_{k-1}} X_{i_1 \ldots i_k}$ is determined by a pullback diagram of sets, as follows:

$$X_{i_1 \ldots i_k} \times X_{i_1 \ldots i_{k-1}} X_{i_1 \ldots i_k} \xrightarrow{\pi_1} X_{i_1 \ldots i_k}$$

$$\pi_2 \downarrow \downarrow t_{ik} \quad s_{ik} \downarrow$$

$$X_{i_1 \ldots i_k} \rightarrow X_{i_1 \ldots i_{k-1}}$$

Axioms: Composition must be associative:

$$(x \circ_i y) \circ_i z = x \circ_i (y \circ_i z),$$

identities must behave as such under composition:

$$id_i s_i(x) \circ_i x = x = x \circ_i id_i t_i(x),$$

$$id_i(x \circ_j y) = id_i(x) \circ_j id_i(y) \quad \text{for } i \neq j,$$

and commute with each other and sources and targets:

$$id_i id_j = id_j id_i \quad \text{for } i \neq j,$$

$$s_i id_i(x) = t_i id_i(x) = x,$$

$$s_i id_j(x) = id_j s_i(x), \quad t_i id_j(x) = id_j t_i(x), \quad \text{for } i \neq j;$$

compositions in different directions $i \neq j$ must commute with each other (the interchange law):

$$(x \circ_i y) \circ_j (x' \circ_i y') = (x \circ_j x') \circ_i (y \circ_j y');$$

compositions must be compatible with the source and target maps in the following sense:

$$s_i(x \circ_i y) = s_i(x), \quad t_i(x \circ_i y) = t_i(y),$$

$$s_i(x \circ_j y) = s_i(x) \circ_j s_i(y), \quad t_i(x \circ_j y) = t_i(x) \circ_j t_i(y), \quad \text{for } i \neq j;$$

1Here a combination means an (unordered) subset.
Remark. There is still other, equivalent single-set definition, which describes the set of \( n \)-morphisms and treats lower-dimensional morphisms as degenerate \( n \)-morphisms, see [EE78].

Remark. Every strict \( n \)-category has the natural structure of an \( n \)-fold category, moreover, we have a functor \( n-\text{Cat} \to \text{Cat}_n \), where \( n-\text{Cat} \) is the category of strict \( n \)-categories, cf. [ML98] Section II.5. In terms of Definition 1.3 if we require that \( k \)-morphisms in every plane but the \( x_1 \ldots x_k \) plane be identities, we will get a definition of a strict \( n \)-category. This yields a morphism of theories \( \text{Th} (\text{Cat}_n) \to \text{Th} (n-\text{Cat}) \).

One advantage of Definition 1.3 is that it allows us to talk about \( \infty \)-fold (or \( \omega \)-fold) categories directly, without passing to the limit \( n \to \infty \): to define an \( \infty \)-fold category, we will just set \( n = \infty \) in Definition 1.3 while the index \( i \) for directions will still be finite, \( 1 \leq i < \infty \), and \( k \)-morphisms will also span in a finite number \( k \), \( 0 \leq k < \infty \), of directions \( \{i_1, \ldots, i_k\} \). We will not use \( \infty \)-fold categories in this paper, except in the following example.

**Example 1.4** (The fundamental \( \omega \)-fold groupoid of a topological space). We have learned this version of Grothendieck’s fundamental \( \omega \)-groupoid from U. Tillmann [Til08], see also R. Brown [Bro09]. For a topological space \( X \), let the set \( X_{i_1 \ldots i_k} \) of \( k \)-morphisms be the set of pairs \((I, f)\), where \( I = [a_1, b_1] \times [a_2, b_2] \times \ldots \), where \( a_1 \leq b_1, a_2 \leq b_2, \text{ etc.} \), so that \( a_i = b_i \) whenever \( i \notin \{i_1, \ldots, i_k\} \) and \( a_i = b_i = 0 \) for \( i >> 0 \), is a coordinate rectangular solid in \( \mathbb{R}^\infty \) of dimension at most \( k \) and \( f : I \to X \) is a continuous map. Composition of \( k \)-morphisms in the \( i \)th direction is defined, if and only if the rectangular solids fit together along their \( i \)th faces in \( \mathbb{R}^\infty \) to form a larger rectangular solid and the restrictions of the maps from the solids to \( X \) to the two faces are equal. In this case, the two maps glue to a map from their union to \( X \) by continuity. If we fix \( n \) and use \( \mathbb{R}^n \) instead of \( \mathbb{R}^\infty \) in the above, we will obtain a \( n \)-truncation of that construction, which will be an \( n \)-fold category.

For instance, for \( n = 1 \) we get the (Moore) path category of the topological space.

We will mention another example of a strict \( n \)-fold category, when we will discuss embedded cobordisms with corners below.

## 2. The Nerve of an \( n \)-Fold Category

The **nerve of a (strict) \( n \)-fold category** \( X_n \) is an \( n \)-fold simplicial set \( X : (\Delta^{op})^n \to \text{Set} \), where \( \Delta \) is the standard simplex category. The nerve may be constructed as the iterated nerve of a usual category, using the fact that an \( n \)-fold category is a category object in the category of \((n-1)\)-fold categories. Thus, in order to construct the simplicial nerve of an \( n \)-fold category, we first construct the nerve of it as a category object: for \( k > 0 \), a \( k \)-simplex in this simplicial object is a composable sequence of \( k \) morphisms in our category object, i.e., functors between \((n-1)\)-categories; for \( k = 0 \), this is just an object, i.e., an \((n-1)\)-category. Each \((n-1)\)-category gives rise to an \((n-1)\)-fold simplicial set by induction, and the above nerve, which is a simplicial object in the category of \((n-1)\)-fold categories, turns into a simplicial object in the category of \((n-1)\)-fold simplicial sets.
The following theorem gives a test for an \( n \)-fold simplicial set to be the nerve of an \( n \)-fold category via an inner-horn filling condition. Recall that for the standard simplex \( \Delta^k \), considered as a simplicial set, and any \( 0 \leq j \leq k \), the \( j \)th horn \( \Lambda^k_j \subset \Delta^k \) is obtained from \( \Delta^k \) by deleting the interior and the face opposite the \( j \)th vertex.

**Theorem 2.1.** An \( n \)-fold simplicial set \( X \) is the nerve of a strict \( n \)-fold category, if and only if it satisfies the unique inner horn-filling condition: for each sequence \( \sigma_1, \ldots, \sigma_n \), where each \( \sigma_i \) is either a simplex \( \Delta^{k_i} \), \( k_i \geq 0 \), or an inner horn \( \Lambda^k_{ij} \subset \Delta^{k_i} \), \( 0 < j < k_i \), any inner multihorn \( (\sigma_1, \ldots, \sigma_k) \to X \) may be completed to a multisimplex \( (\Delta^{k_1}, \ldots, \Delta^{k_n}) \to X \).

**Proof.** This theorem follows by iterating the \( n = 1 \) statement, known as Boardman-Vogt’s theorem [BV73]. \( \square \)

The geometric realization of the nerve of an \( n \)-fold category \( X_\bullet \) is called the classifying space of it, denoted \( BX_\bullet \).

**Example 2.2.** The classifying space of the fundamental \( n \)-fold groupoid of a topological space \( X \) is homotopy equivalent to the space. For \( n = 1 \), this follows from the observation that the path space of a topological space \( X \) is homotopy equivalent to \( X \). For higher \( n \)'s, the result follows by iteration.

## 3. Pseudo \( n \)-Fold Categories

For abstract cobordisms with corners, we will need to introduce “controlled weakness” into the notion of a strict \( n \)-fold category. We will call the corresponding notion a pseudo \( n \)-fold category, which will be a close relative of Marco Grandis’ notion of a symmetric weak \((n + 1)\)-cubical category, see [Gra07]. The principal difference is that we add a second hexagon axiom, which makes the whole notion coherent, see Theorems 3.2 and 10.1. Technical differences include our omitting the symmetric structure and using strict units.

**Definition 3.1.** A pseudo \( n \)-fold category is a certain weak model of an \( n \)-fold category in \( \text{Cat} \). This means that a pseudo \( n \)-fold category has the same structure of morphisms \( k\text{Mor} \) for \( k = 0, 1, \ldots, n \), sources and targets \( s, t : k\text{Mor} \to (k - 1)\text{Mor} \), identities \( \text{id} : (k - 1)\text{Mor} \to k\text{Mor} \), and compositions \( \circ : k\text{Mor} \times_{(k - 1)\text{Mor}} k\text{Mor} \to k\text{Mor} \), \( 1 \leq k \leq n \), as a (strict) \( n \)-fold category, see Definition 1.3 except that now \( k\text{Mor} \) is a category, rather than a set, and \( s, t, \text{id} \), and compositions are functors. Thus, a pseudo \( n \)-fold category has extra \((k + 1)\)-morphisms in a “transversal” direction, the morphisms of the categories \( k\text{Mor} \), \( k = 0, 1, \ldots, n \). Some of the axioms must be satisfied weakly, that is, up to natural isomorphism of functors. These natural isomorphisms are part of the structure, satisfying their own coherence axioms. We will go over this extra data and the axioms one by one in detail below. Letters \( x, y, z \), etc., will be placeholder for objects in categories \( k\text{Mor} \).

**Associators:** There must be natural isomorphisms

\[
\alpha_i : \circ_i (\circ_i \times \text{id}) \to \circ_i (\text{id} \times \circ_i),
\]

\[
(x \circ_i y) \circ_i z \Rightarrow x \circ_i (y \circ_i z),
\]

of functors

\[
k\text{Mor} \times_{(k - 1)\text{Mor}} k\text{Mor} \times_{(k - 1)\text{Mor}} k\text{Mor} \to k\text{Mor}
\]
for each $k = 1, \ldots, n$ and $i = 1, \ldots, n$. (To be more precise, $\alpha_i$ is a collection of isomorphisms depending on several other indices.)

**Interchangers:** For each $k = 1, \ldots, n$ and pair of distinct numbers $i$ and $j$ between 1 and $n$, there must be natural isomorphisms

$$
\beta_{ij} : \circ_j (\circ_i \times \circ_i) \to \circ_i (\circ_j \times \circ_j),
$$

$$
(x \circ_i y) \circ_j (x' \circ_i y') \mapsto (x \circ_j x') \circ_i (y \circ_j y'),
$$

of functors

$$
k\text{Mor}^4_{(k-1)\text{Mor}^4} \to k\text{Mor},
$$

where $k\text{Mor}^4_{(k-1)\text{Mor}^4}$ is the category pullback describing quadruples of $k$-morphisms composable horizontally (i.e., in the $i$th direction) and vertically (i.e., in the $j$th direction), as per the following square:

![Pentagon Diagram]

**Axioms:** First of all, we must have

$$
\beta_{ji} = \beta_{ij}^{-1}.
$$

Next, the *coherence axioms* below for the associators and interchangers must be satisfied. The diagrams involved are diagrams of natural transformations between morphism categories $k\text{Mor}$, and commutativity of diagrams means equality of the corresponding natural transformations.

The following *pentagon* diagram must commute for each $i$:

![Pentagon Diagram]

This axiom, in other words, requires that the two ways of moving parentheses from one extreme to the other in a sequence

$$
\bullet \xrightarrow{x} \bullet \xrightarrow{y} \bullet \xrightarrow{z} \bullet \xrightarrow{u} \bullet
$$

of composable $k$-morphisms, using associators, be equal.
The following *hexagon* diagram must commute for each pair of distinct $i$ and $j$:

$$
\begin{align*}
&((x \circ_i y) \circ_j (x' \circ_i y')) \circ_i (z \circ_j z') \\
&\quad \xrightarrow{\beta \circ_i \alpha} \\
&((x \circ_j x') \circ_i (y \circ_j y')) \circ_i (z \circ_j z')
\end{align*}
$$

$$
\begin{align*}
&((x \circ_i y) \circ_j (x' \circ_i y')) \circ_j ((x' \circ_i y') \circ_i z') \\
&\quad \xrightarrow{\beta \circ_j \alpha} \\
&(x \circ_j x') \circ_i ((y \circ_j y') \circ_i (z \circ_j z'))
\end{align*}
$$

This axiom requires that the two ways of moving parentheses from one extreme to
the other in a sequence of morphisms

![Hexagon Diagram](image)

using associators and interchangers, be equal.

There is a second *hexagon* axiom that must be satisfied. For each triple of distinct
$i$, $j$, and $k$, the following diagram must commute:

$$
\begin{align*}
&((x \circ_j x') \circ_i (y \circ_j y')) \circ_k ((x'' \circ_j x''') \circ_i (y'' \circ_j y'''')) \\
&\quad \xrightarrow{\beta_{jk} \circ_i \beta_{ij}} \\
&((x \circ_j x') \circ_k (x'' \circ_j x''') \circ_i (y'' \circ_j y''''))
\end{align*}
$$

$$
\begin{align*}
&((x \circ_i y) \circ_j (x' \circ_i y')) \circ_k ((x'' \circ_i y'') \circ_j (x''' \circ_i y''')) \\
&\quad \xrightarrow{\beta_{jk} \circ_i \beta_{ik}} \\
&((x \circ_k x'') \circ_i (y \circ_k y'') \circ_j (x''' \circ_i y''''))
\end{align*}
$$

This axiom requires that the two ways of moving parentheses from one extreme to
the other in a sequence of morphisms

![Hexagon Diagram](image)

using interchangers between different pairs of directions, be equal.
Like in a strict n-fold category, the following identities, where \( x, y, z \), etc., now stand for objects or morphisms in categories \( k\text{Mor} \) and \( i \neq j \), must hold:

1. \( \text{id}_i s_i(x) \circ_i x = x = x \circ_i \text{id}_i t_i(x) \),
2. \( \text{id}_i(x \circ_j y) = \text{id}_i(x) \circ_j \text{id}_i(y) \),
3. \( \text{id}_i \text{id}_j = \text{id}_j \text{id}_i \),
4. \( s_i \text{id}_i(x) = t_i \text{id}_i(x) = x \),
5. \( s_i \text{id}_j(x) = \text{id}_j s_i(x) \), \( t_i \text{id}_j(x) = \text{id}_j t_i(x) \),
6. \( s_i(x \circ_i y) = s_i(x) \), \( t_i(x \circ_i y) = t_i(y) \),
7. \( s_i(x \circ_j y) = s_i(x) \circ_j s_i(y) \), \( t_i(x \circ_j y) = t_i(x) \circ_j t_i(y) \),
8. \( s_is_j(x) = s_js_i(x) \), \( t_is_j(x) = sjt_i(x) \), \( t_it_j(x) = tjt_i(x) \).

There are extra axioms that require the identity morphisms be coherent with the associators and the interchangers in the following way:

\[ \alpha = \text{id} : (id_i s_i(x) \circ_i x) \circ_i y \rightarrow id_i s_i(x) \circ_i (x \circ_i y), \]

\[ \beta = \text{id} : (x \circ_i y) \circ_j (id_j t_j(x) \circ_i id_j t_j(y)) \rightarrow (x \circ_j id_j t_j(x)) \circ_i (y \circ_j id_j t_j(y)). \]

A similar argument to one in the case of monoidal categories shows that the first and the third equalities \( \alpha = \text{id} \) follow from the second one, but we will not worry here about the independence of our collection of axioms.

An important aspect in defining the notion of a weak higher category is whether there are “enough” coherence axioms. Such aspects are usually addressed by coherence theorems, which we will prove below. Suppose we have a composable combination of k-morphisms. Combinatorially, such a combination may be thought of as a k-dimensional cube in \( \mathbb{R}^n \) subdivided into rectangular blocks, placeholders for the morphisms to be composed. Composing these morphisms means specifying an order in which the blocks are being glued. The blocks are not supposed to be moved around in the process.

**Theorem 3.2 (Weak Coherence Theorem).** Any two functors \((k\text{Mor})^p \rightarrow k\text{Mor}\) in a pseudo n-fold category built from morphism composition are related by a natural isomorphism built from associators \( \alpha \) and interchangers \( \beta \). Here \( p \geq 1 \) is an integer and \((k\text{Mor})^p\) denotes the full subcategory in \( k\text{Mor} \) of morphisms composable in a diagram c given by a cube of dimension \( k \) made of \( p \) blocks.

**Proof.** We need to compare two functors. We will be talking about their values on objects, which are k-morphisms in the pseudo n-fold category, making sure the constructions are natural. This will imply the functors are related by a natural transformation.

Let us use induction on the number \( p \) of morphisms being composed. The statement is trivial for \( p = 1 \). For any \( p \geq 2 \), consider the last morphism compositions in each of the two functors. They must be compositions in either the same direction or two different ones.

If they are compositions in the same direction \( i \), then they are either the same, \( x \circ_i y \), in which case, the induction assumption applied to \( x \) and \( y \) separately
completes the argument, or different: $x \circ_i y'$ and $y'' \circ_i z$. Without loss of generality, by the induction hypothesis, we can assume that $S y' = y \circ z$ and $T y'' = x \circ y$ for some $y$ and natural transformations $S$ and $T$ made out of $\alpha$'s and $\beta$'s, in which case for the associator $\alpha_i : (x \circ y) \circ_i z \to x \circ_i (y \circ z)$, we have $\alpha_i(T y'' \circ_i z) = x \circ_i S y'$. Then the composition of $T \circ_i \text{id}, \alpha_i$, and $\text{id} \circ_j S^{-1}$ takes $y'' \circ_i z$ to $x \circ_i y'$.

If the last compositions in the two functors are in different directions, $x \circ_i y$ and $x' \circ_j y'$, we can assume by the induction hypothesis that $x = S(x_1 \circ_j x_2), y = T(y_1 \circ_j y_2)$, and $x' = S'(x_1 \circ_i y_1), y' = T'(x_2 \circ_i y_2)$ for some morphisms $x_1, x_2, y_1$, and $y_2$ and natural transformations $S, T, S'$, and $T'$, built out of associators and interchangers. Then the composition of these natural transformations and their inverses and an interchanger $\beta_{ij}$ will move $x' \circ_j y'$ to $x \circ_i y$, as we have $\beta_{ij}((S')^{-1} x' \circ_j (T')^{-1} y') = S^{-1} x \circ_i T^{-1} y$. \hfill \Box

A strong coherence theorem would state that a natural transformation relating two functors in Theorem 4.3 is unique. We prove an important particular case of such statement in the Appendix.

\section{Monoidal Versions}

4.1. Monoidal Pseudo $n$-Fold Categories.

\begin{definition}
A \textit{monoidal pseudo $n$-fold category} is a pseudo $(n+1)$-fold category with directions labeled $0, 1, \ldots, n$ and such that $\text{Mor}_{1i_1 \ldots i_k}$ is the category with one object $1_{i_1 \ldots i_k}$ and one morphism $1_{i_1 \ldots i_k}$ for any combination $\{i_1, \ldots, i_k\}$ which does not contain $0$. 
\end{definition}

\begin{remark}
In particular, the category $\emptyset \text{Mor}$ of “objects” must have only one object $1$. As concerns the category $1 \text{Mor} = \prod_{i=0}^{n} \text{Mor}_i$ of 1-morphisms, we have $\text{Mor}_i \cong (1, \text{id}_1)$ for each $i \geq 1$, while $\text{Mor}_0$ may be plentiful. We think of objects in $\text{Mor}_0$ as the “objects” of our monoidal pseudo $n$-fold category. Each nontrivial $(k + 1)$-morphism, which is an object in $\text{Mor}_{0i_1 \ldots i_k}$, has one and the same $1_{i_1 \ldots i_k}$ as the source and target in the 0th direction. Thus, any two such morphisms $x$ and $y$ may be composed in the 0th direction: we will denote this composition $x \otimes y$, and this will be the monoidal structure.
\end{remark}

\begin{definition}
A \textit{braided monoidal pseudo $n$-fold category} is a pseudo $(n+2)$-fold category with directions labeled $-1, 0, 1, \ldots, n$ and $\text{Mor}_{1i_1 \ldots i_k}$ being the category with one object $1 = 1_{i_1 \ldots i_k}$ and one morphism, as long as $\{-1, 0\} \not\subset \{i_1, \ldots, i_k\}$. We also require that the interchanger $\beta_{-1,0} : x \circ_0 y = (x \circ_{-1} 1) \circ_0 (1 \circ_{-1} y) \to (x \circ_0 1) \circ_{-1} (1 \circ_0 y) = x \circ_{-1} y$ be equal to the identity transformation, cf. the discussion of the Eckmann-Hilton argument below. Here $x$ and $y$ are any objects of $\text{Mor}_{1i_1 \ldots i_k}$ with $\{-1, 0\} \subset \{i_1, \ldots, i_k\}$. The monoidal structure will be given by $x \otimes y := x \circ_0 y$.
\end{definition}

\begin{remark}
In particular, the categories $0 \text{Mor}$ of “objects” and $\text{Mor}_i, \ i = -1, 0, \ldots, n$, of 1-morphisms will each have only one object. Each category $\text{Mor}_{ij}$ will also have only one object, except for $\text{Mor}_{-1,0}$.
\end{remark}

\begin{definition}
A \textit{symmetric monoidal pseudo $n$-fold category} is a pseudo $(n+3)$-fold category with directions labeled $-2, -1, 0, 1, \ldots, n$ and $\text{Mor}_{1i_1 \ldots i_k}$ being the category with one object $1_{i_1 \ldots i_k}$ and one morphism, as long as $\{-2, -1, 0\} \not\subset \{i_1, \ldots, i_k\}$. We also require that for $i, j \leq 0$ the interchangers $\beta_{ij} : (x \circ_i 1) \circ_j$...
The commutativity of the diagram implies that $\beta_{-2,-1}\beta_{-1,0} = \text{id}$, whence $\beta_{-2,-1} = \beta_{0,-1}$. Since the same argument works for any permutation $(i,j,k)$ of $(-2,-1,0)$,
we get $\beta_{ij} = \beta_{ik} = \beta_{jk} = \beta_{ji}$. It follows that all these braiding operators are equal and since $\beta_{ij} = \beta_{ji}^{-1}$, we also have $\beta^2 = \beta^2_{ij} = \text{id}$.

Remark. Classical monoidal categories may be described as particular examples of monoidal pseudo (1-fold) categories. For example, a braided monoidal category would be a braided monoidal pseudo category satisfying the following conditions: the associator $\alpha_1$ in the “spatial” direction 1 and the interchangers $\beta_{ii}$ for $i = -1, 0$ must be equal to identity. We anticipate that the classical coherence theorems of Mac Lane [ML63, ML98] for monoidal and symmetric monoidal categories and of Joyal-Street [JS93, ML98] for braided monoidal categories (in the unitary case) follow from our Regular Coherence Theorem 10.1.

5. Functors

A functor $F : C \to D$ between two pseudo $n$-fold categories $C$ and $D$ is a collection of functors $F : k\text{Mor}(C) \to k\text{Mor}(D)$ together with natural transformations

$$\phi_{x,y} : F(x \circ_i y) \to F(x) \circ_i F(y),$$

called the coherence maps, commuting with sources, targets, and identities, and satisfying the following coherence conditions, making sure the natural transformations are compatible with the identity axioms, associators, and interchangers:

$$\phi_{\text{id}_i, s_i, x, x} = \text{id}_{F(x)}, \quad \phi_{x, \text{id}_i, t_i, x} = \text{id}_{F(x)},$$

$$\phi_{F(x \circ_i y) \circ_i F(z), x, y, z} = \phi_{F(x), x, y, z} \circ F(\alpha_i),$$

$$\phi_{F(x), x, y, z} = \phi_{F(y), y, z} \circ F(\beta_{ij}),$$

A monoidal functor between monoidal pseudo $n$-fold categories is just a functor between the corresponding pseudo $(n+1)$-fold categories. Likewise, a braided (symmetric) monoidal functor between braided (symmetric, respectively) monoidal pseudo $n$-fold categories is a functor between the corresponding pseudo $(n+2)$-fold ($(n+3)$-fold, respectively) categories.

6. Higher Spans and Cospans

In this section, we will give basic examples of symmetric monoidal pseudo $n$-fold categories and prove the following theorem along the way.

**Theorem 6.1.** $k$-Spans of sets (more generally, topological spaces), $k$-cospans of sets (or topological spaces), $k$-cospans of (differential graded) algebras for $k \leq n \leq \infty$ form symmetric monoidal pseudo $n$-fold categories.
6.1. Higher Spans of Sets and Topological Spaces. We will limit the discussion here to higher spans of topological spaces only. This will cover the case of higher spans of sets, if you regard sets as spaces with discrete topology. For \( k \geq 0 \), a \( k \)-span (correspondence) of topological spaces is a commutative diagram of continuous maps between topological spaces, which is shaped as a barycentric subdivision of a \( k \)-dimensional cube:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

The edges in the diagram must be maps pointed away from the barycenter of each \( m \)-face, \( 1 \leq m \leq k \), toward the barycenters of the bounding \( (m - 1) \)-faces. All the squares of maps in the diagram must commute. To be included as \( k \)-morphisms of a pseudo \( n \)-fold category, \( k \)-spans also need to have directions \( 1 \leq i_1, \ldots, i_k \leq n \) assigned to the axes of the \( k \)-cube, so that the two \( (k - 1) \)-faces bounding the cube in direction \( i_p \) are given by the equations \( x_{i_p} = 0 \), the source face, and \( x_{i_p} = 1 \), the target face. Two \( k \)-spans may be composed in direction \( i_p \), whenever the target of one \( k \)-span coincides with the source of the other in this direction. To compose such \( k \)-spans, each sequence

\[
A_1 \leftarrow A_2 \to A_3 \leftarrow A_4 \to A_5
\]

of maps in direction \( i_p \) that will show up in the diagram for the composition needs to be converted into a sequence \( A_1 \leftarrow A' \to A_5 \) by taking the canonical pullback (a.k.a. fibered product)

\[
\begin{array}{c}
A_1 \\
A_2 \\
A_3 \\
A_4 \\
A_5
\end{array}
\]

where

\[
A' = A_2 \times_{A_3} A_4 := \{(a_2, a_4) \in A_2 \times A_4 \mid p_1(a_2) = p_2(a_4)\}.
\]

Note that each square of maps in the diagram of the resulting span will still be commutative.

We should be careful to use formal units \( \text{id}_i(x) \), rather than actual \( k \)-spans with all maps in the \( i \)th directions being identity maps. Otherwise, these units would be weak and we will have to burden our definition of a pseudo \( n \)-fold category by weak units. Thus, formal units \( \text{id}_i(x) \) for all \((k - 1)\)-morphisms \( x \) missing direction
\( i \) shall be added to the set of \( k \)-morphisms. This in fact means that a \( k \)-morphism in given directions will be represented by an \( m \)-span \( x \) for some \( m \leq k \) and, when \( m < k \), thought of as a formal identity morphism, based on the \( m \)-span \( x \), in the directions complementary to those present in \( x \). Formal units must be composed as strict units, i.e., in such a way that they satisfy the identities
\[
\text{id}_i \circ_i x = x = x \circ_i \text{id}_i.
\]

One can easily check that the formal units will satisfy the other axioms (2)-(5) involving units in the definition of a pseudo \( n \)-fold category.

Higher spans described above are objects of the categories of \( k \)-morphisms. Morphisms in these categories are (continuous) maps between higher spans, which are nothing but maps from the respective vertices of one higher span to another, commuting with all the maps between the vertices within the spans. The associators and interchangers, described in the next paragraph, will be morphisms between higher spans.

The associators and interchangers on compositions of higher spans are defined at the level of points by rearranging parentheses in the canonical pullbacks, as follows. Composition of three higher spans in one direction boils down to composing usual spans like
\[
A_1 \leftarrow A_2 \rightarrow A_3 \leftarrow A_4 \rightarrow A_5 \leftarrow A_6 \rightarrow A_7
\]
in that direction. The corresponding associator \( \alpha \) will act as follows:
\[
\alpha((a_2, a_4), a_6) := (a_2, (a_4, a_6)), \quad a_i \in A_i.
\]

If we compose four spans in two directions as per the following diagram:
\[
\begin{array}{ccccccc}
\bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\bullet & \leftarrow & A_1 & \rightarrow & \bullet & \leftarrow & A_2 & \rightarrow & \bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\bullet & \leftarrow & B_1 & \rightarrow & \bullet & \leftarrow & B_2 & \rightarrow & \bullet \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bullet & \leftarrow & \bullet & \rightarrow & \bullet & \leftarrow & \bullet & \rightarrow & \bullet
\end{array}
\]
the corresponding interchanger will again rearrange parentheses, as follows:
\[
\beta((a_1, a_2), (b_1, b_2)) := ((a_1, b_1), (a_2, b_2)), \quad a_i \in A_i, b_i \in B_i.
\]

The pentagon and both hexagon axioms will hold, because the arrows in each polygon are defined by the same rearrangement of parentheses for elements of the sets sitting at vertices of the spans as for the very spans participating in compositions, e.g., in the pentagon diagram, the map \( \alpha \circ_i \text{id} \) may be defined as follows:
\[
\alpha \circ_i \text{id} : ((x \circ_i y) \circ_i z) \circ_i u \rightarrow (x \circ_i (y \circ_i y)) \circ_i u,
\]
\[
((a_2, a_4), a_6, a_8) \mapsto ((a_2, (a_4, a_8)), a_8).
\]
Thus, the fact that going from the far left vertex of the polygon to the far right vertex along the upper and the lower paths results in one and the same vertex implies that these two paths are equal set-theoretically, and thereby the diagram commutes.

The associators and interchangers on compositions of morphisms involving units should be set to equal identity, so that the axioms (9) are satisfied.

The argument above ignores the monoidal structure and so far shows that $k$-spans of topological spaces form a pseudo $n$-fold category. Essentially, the monoidal structure is given by Cartesian product, but to add it within our framework, we will need to shift $n$ by 3 and think of $k$-spans of topological spaces in directions $i_1, \ldots, i_k$ between 1 and $n$ as $(k+3)$-morphisms (as well as degenerate $(l+3)$-morphisms for $l > k$) of the symmetric monoidal pseudo $n$-fold category in directions $-2, 1, 0, i_1, \ldots, i_k$. The degenerate morphisms play the role of formal units in positive directions, but we also need to add formal units for the monoidal structure: a single morphism $1$ in every multi-direction missing 0, $-1, -2$. Compositions of morphisms in positive directions are given by composition of higher spans, as above. Compositions of higher spans in negative directions $-2, -1,$ and 0 are all given by Cartesian products of higher spans and denoted $\otimes$: one takes Cartesian products of the respective entries in the composed spans and products of maps for the edges of the product span. When composing with formal units $1$ in negative directions, we should be careful to define composition formally as $x \otimes 1 := x$ and $1 \otimes x := x$, rather than use the actual one-point space instead of $1$. This avoids the issue of weak units for the monoidal structure. For all $i, j \leq 0$, we define the interchangers $\beta_{ij} : x \otimes y = (x \circ_i 1) \circ_j (1 \circ_j y) \rightarrow (x \circ_j 1) \circ_i (1 \circ_i y) = x \otimes y$ as the identity transformations and the interchangers $\beta_{ij} : x \otimes y = (1 \circ_i x) \circ_j (y \circ_i 1) \rightarrow (1 \circ_j y) \circ_i (x \circ_j 1) = y \otimes x$ as the canonical set-theoretic interchange law for Cartesian products. (When either $x$ or $y$ is a unit itself, we will define the interchanger to be the identity.) The polygonal coherence axioms will be satisfied for the same reason they were satisfied for compositions of higher spans above: now we are using Cartesian product, as opposed to more general fibered product.

6.2. Higher Cospans of Sets and Topological Spaces. Here we will “dualize” the previous example and consider $k$-cospans (a.k.a. “cocorrespondences”) of topological spaces (or sets), which will be commutative $k$-dimensional cubical diagrams of spaces (respectively, sets) at the barycenters of all faces and continuous (respectively, arbitrary) maps directed from to barycenters of faces from the barycenters of the bounding faces. The monoidal structure will be given by disjoint union, which may be defined as follows in our set-theoretic universe:

$$x \coprod y := x \times \{x\} \cup y \times \{y\},$$

where $\{a\}$ is the one-point set whose only element is $a$. The monoidal units will again be formal, rather than the empty set. Compositions of morphisms will be
defined by using canonical set-theoretic pushouts

\[ A' = A_2 \cup_{A_1} A_4 := (A_2 \coprod A_4) / \{ i_1(a_3) \sim i_2(a_3) \text{ for all } a_3 \in A_3 \}. \]

The rest of the argument goes exactly the same way as for higher spans above.

6.3. Higher Cospans of Algebras. This example may be thought of as another dualization of the example of higher spans of sets. A \( k \)-cospans of differential graded (dg) algebras (over a field) is a commutative \( k \)-cubical diagram of dg associative algebras placed at the barycenters of all faces and dg-algebra morphisms directed toward the barycenter of each face from the barycenters of the bounding faces. The monoidal structure is given by tensor product \( x \otimes y \) of the dg algebras sitting at the respective vertices of the two \( k \)-cospans. The monoidal units will again be formal, rather than the ground field literally. Compositions of morphisms will be defined by using canonical tensor products over a dg associative algebra: \( A_2 \otimes_{A_3} A_4 \). The rest of the argument repeats the arguments for higher spans and cospans above.

6.4. Higher Spans of Coalgebras. This example is a linearization of the example of higher spans of sets or topological spaces and also linear dual to the example of higher cospans of algebras. More precisely, if you have a higher span of topological spaces, applying the functors of chains or homology, you get higher spans of dg coalgebras. A \( k \)-span of dg coalgebras over a field is a commutative \( k \)-cubical diagram of dg coassociative coalgebras sitting at the barycenters of all faces and dg-coalgebra morphisms directed from the barycenter of each face to the barycenters of the bounding faces. The monoidal structure is given by tensor product of the dg coalgebras. The monoidal units are formal, and composition of spans is defined using the tensor product of dg coalgebras relative to a morphism to another coalgebra.

7. Cobordisms with Corners

In this section we are going to prove the following theorem.

**Theorem 7.1.** Abstract \( k \)-cobordisms for \( k \leq n \leq \infty \) form a symmetric monoidal pseudo \( n \)-fold category.

**Remark.** There is a notion of cobordisms with corners, embedded nicely in coordinate rectangular solids in \( \mathbb{R}^\infty \), which leads to a strict \( n \)-fold category, see Tillmann [Til08]. We will not use it here, as we are ultimately interested in extended TQFTs, which will be functors from a suitable \( n \)-fold category of cobordisms with corners to one of the pseudo \( n \)-fold categories of higher (co)spans. We are not aware of any strict model of the latter, therefore there will be little benefit in using a strict model of the former.
An $n$-dimensional manifold with corners is roughly a space smoothly modeled on $\mathbb{R}^n_+$, where $\mathbb{R}_+ = [0, \infty)$ is the closed positive real line.

**Definition 7.2.** A (smooth) $n$-**manifold with corners** is a second-countable, Hausdorff topological space $M$ with a sheaf $\mathcal{F}$ of $\mathbb{R}$-algebras, which is a subsheaf of the sheaf of $\mathbb{R}$-algebras of continuous real-valued functions on $M$, such that $(M, \mathcal{F})$ is locally isomorphic to $(\mathbb{R}^n_+, C^\infty)$. For each $k = 0, 1, \ldots, n$, an open $k$-face of $M$ is a connected component of the locally closed submanifold of $M$ consisting of points corresponding to an open $k$-dimensional face of $\mathbb{R}^n_+$. A (closed) $k$-face of $M$ is the closure of an open $k$-face.

**Definition 7.3.** An $n$-**cobordism $M$ with corners**, see Figure 1, is a compact $n$-manifold with corners with the following extra data. The cobordism $M$ must be locally modeled on $(I^n, C^\infty)$, so that the information about which open face of the cube each point of $M$ corresponds to is part of the data. Thus, each open (and thereby closed) $k$-face of $M$ is labeled by a combination \( \{i_1, i_2, \ldots, i_{n-k}\} \) of $n-k$ out of $n$ numbers $1, 2, \ldots, n$ and a sequence $(\varepsilon_{i_1}, \ldots, \varepsilon_{i_{n-k}})$ of zeros and ones, which define the $k$-face of the cube $I^n$ by the equations

$$
  x_{i_1} = \varepsilon_{i_1}, x_{i_2} = \varepsilon_{i_2}, \ldots, x_{i_{n-k}} = \varepsilon_{i_{n-k}}.
$$

These labelings must be compatible between faces in the sense that each $k$-face labeled by the equations (10) must locally be the intersection of $k$ $(n-1)$-faces labeled by the equations $x_{i_1} = \varepsilon_{i_1}, x_{i_2} = \varepsilon_{i_2}, \ldots, x_{i_{n-k}} = \varepsilon_{i_{n-k}}$. Each $k$-face $F$ of $M$ must be provided with a **collar**, which is the germ of a diffeomorphism of an open neighborhood of $F \times \{ (\varepsilon_{i_1}, \ldots, \varepsilon_{i_{n-k}}) \}$ in $F \times I^{n-k}$ with an open neighborhood of $F$ in $M$. Coordinates used in this copy of $I^{n-k}$ are $(x_{i_1}, x_{i_2}, \ldots, x_{i_{n-k}})$. Here a germ is an equivalence class of such diffeomorphisms; two such diffeomorphisms of open neighborhoods of $F$ in $M$ are equivalent, if they are equal over the intersection of the neighborhoods. The collars at different faces must be compatible with each other, that is, for any $(k-1)$-subface $F'$ of each $k$-face $F$ given by the equation $x_{i_{n-k+1}} = \varepsilon_{i_{n-k+1}}$, the restriction of the collar $F \times I^{n-k} \to M$ to $F' \times I^{n-k}$ must coincide with the restriction of the collar $F' \times I^{n-k+1} \to M$ to the subset $F' \times I^{n-k}$ given by the equation $x_{i_{n-k+1}} = \varepsilon_{i_{n-k+1}}$.

For each $j = 1, \ldots, n$, the $(j,+)$-**boundary** $\partial^+_j M$ of $M$ is the union of all the $(n-1)$-faces given by the equation $x_j = 1$. Similarly, the $(j,-)$-**boundary** $\partial^-_j M$ is the union of the $(n-1)$-faces given by the equation $x_j = 0$. The $(j,+)$-boundary inherits the structure of an $(n-1)$-cobordism with corners, with the model $(n-1)$-cube $I^{n-1} \subset I^n$ given by the equation $x_j = 1$. The same for the $(j,-)$-boundary and the equation $x_j = 0$.

**Remark.** An $n$-cobordism with corners is automatically a manifold with faces under the terminology of [Jan68, Lau00] and moreover a manifold with $n$ distinguished
faces $\partial^+_i M \cup \partial^-_i M, \ldots, \partial^+_n M \cup \partial^-_n M$, see [Gra08], also known as an \langle n \rangle-manifold, see [Jan68] [Lau00].

Abstract $n$-cobordisms $M$ and $N$ with corners may be composed to an $n$-cobordism $M \circ_j N$ with corners along the $j$th boundary for any $j = 1, \ldots, n$, if the $(j, +)$-boundary $\partial^+_j M$ coincides with the $(j, -)$-boundary $\partial^-_j N$ of $N$ (literally, as sets with the structure of a smooth $(n - 1)$-cobordism with corners). As a topological space, $M \circ_j N$ is the canonical topological pushout

$$
\begin{array}{ccc}
F_j & \xrightarrow{f_c} & N \\
\downarrow f_- & & \downarrow \\
M & \longrightarrow & M \circ_j N,
\end{array}
$$

where $F_j = \partial^+_j M = \partial^-_j N$ is the seam, the common boundary of $M$ and $N$ along which the composition occurs. The pushout is defined as a quotient of the disjoint union, as follows:

$$
M \circ_j N := M \coprod N/\sim,
$$

where $x \sim y$, if there exists $z \in F_j$ such that $f_-(z) = x$ and $f_+(z) = y$. The smooth structure along the seam $F_j$ is defined by requiring that the map of a neighborhood of $F_j \times \{0\}$ in $F_j \times \mathbb{R}$ to $M \circ_j N$ given by taking the union of the collar of $F_j$ in $M$ (after a coordinate change $x_j \mapsto x_j - 1$) and the collar of $F_j$ in $N$ be smooth. The faces of the composition cobordism $M \circ_j N$ are just the unions of faces of $M$ and $N$ except the faces belonging to the seam, which are regarded as canceling out. The faces of $M \circ_j N$ are labeled the same way as their components on $M$ and $N$. These labels at the lower-dimensional faces on the seam $F_j$ also match, because it was assumed to be equal to the outgoing $j$th boundary of $M$ and the incoming $j$th boundary of $N$ as a cobordism with corners, including labelings of all the faces. The collars on the faces are obtained from the collars of the corresponding faces of $M$ and $N$. The $(i, +)$-boundary of $M \circ_j N$ is the union of the $(i, +)$-boundaries of $M$ and $N$ for $i \neq j$ and just the $(i, +)$-boundary of $N$ for $j = i$. Analogously, the $(i, -)$-boundary of $M \circ_j N$ is the union of the $(i, -)$-boundaries of $M$ and $N$ for $i \neq j$ and just the $(i, -)$-boundary of $M$ for $j = i$.

Thus, a $k$-cobordism with corners for $k \leq n$ (whose directions are labeled by the choice of a coordinate $k$-cube $I^k \subset I^n$) will be regarded as a $k$-morphism of the pseudo $n$-fold category of cobordisms. More precisely, a $k$-cobordism with corners will be an object of the category of $k$-morphisms, with morphisms given by smooth maps, respecting the corner structure, i.e., mapping faces to faces, preserving labelings and collars. We will also need to add formal units with respect to composition, which will result in treating $m$-cobordisms with corners for $m < k$ as identity $k$-morphisms in specified complementary directions, as in the example of cospans of topological spaces above.

The monoidal structure will be given by disjoint union of cobordisms with corners.

**Proof of Theorem 7.1**. Abstract cobordisms with corners are nothing but higher cospans of sets with certain structure. Given that higher cospans of sets form a symmetric monoidal pseudo $n$-fold category, we just need to make sure that the structure glues under composition, i.e., the smooth structure is naturally defined
on the composition of cobordisms with corners, as explained in the construction of cobordism composition above. This fact is guaranteed by the collars. Smooth maps between cobordisms with corners glue under composition as well.

8. Comparison to Quasi-Categories

Cobordisms with corners can also be handled by quasi-categories \([\text{Joy02}]\) (also known as \(\infty\)-categories \([\text{Lur09a}]\) or weak Kan complexes \([\text{BV73}]\)). In our approach, the objects were compact 0-dimensional manifolds, i.e., finite sets, the 1-morphisms were represented by compact one-dimensional manifolds with boundary, i.e., disjoint unions of intervals and circles, the 2-morphisms were given by surfaces with corners, . . . , the \(n\)-morphisms were given by \(n\)-dimensional cobordisms with corners. Remember that \(k\)-morphisms in a pseudo \(n\)-fold category actually make up a usual category; what we have just described are the objects in the categories of \(k\)-morphisms, the morphisms are given by smooth maps, respecting the cobordism structure (the corners, labels, and collars). A quasi-category version of the category of \(n\)-cobordisms with corners will have \((n−1)\)-dimensional manifolds with corners as objects, \(n\)-dimensional cobordisms with corners as 1-morphisms, pairs of composable \(n\)-cobordisms with corners along with a choice for their composition (a pushout) as 2-morphisms, and certain towers of pushouts (all being \(n\)-dimensional manifolds with corners) of \(k\) composable cobordisms as \(k\)-morphisms for higher values of \(k\). While this inclusion of \(n\)-cobordisms with corners into the framework of quasi-categories might be interesting on its own, it largely ignores cobordisms of dimension other than \(n−1\) and \(n\) for a fixed number \(n\). To include other dimensions along these lines, one needs a mixture of the \(n\)-fold (or \(\infty\)-fold) category approach with that of quasi-categories, i.e., some kind of an \(n\)-fold quasi-category, which is what we are going to introduce in this section.

We would like to outline the following simplicial approach to weak \(n\)-fold categories, in the spirit of Boardman-Vogt’s approach to weak \(n\)-categories, see \([\text{BV73, Joy02, Lur09a}]\). We would call them \(n\)-fold quasi-categories, where \(0 \leq n \leq \infty\). An \(n\)-fold quasi-category is a \(n\)-fold simplicial set \(\mathbf{S} : (\Delta^{\text{op}})^n \to \text{Set}\) satisfying the weak Kan or inner horn-filling condition: for each sequence \(\sigma_1, \ldots, \sigma_n\), where each \(\sigma_i\) is either a simplex \(\Delta^{k_i}\), \(k_i \geq 0\), or an inner horn \(\Lambda^{k_i}_{j<i} \subset \Delta^{k_i}\), \(0 \leq j < k_i\), any inner multihorn \((\sigma_1, \ldots, \sigma_n) \to \mathbf{S}\) may be completed to a multisimplex \((\Delta^{k_1}, \ldots, \Delta^{k_n}) \to \mathbf{S}\). (Compare this to the unique inner horn-filling condition in characterizing the nerve of a strict \(n\)-fold category in Theorem 2.1.) Equivalently, one can iteratively define an \(n\)-fold quasi-category as a weak Kan (or quasi-category) object in the category of \((n−1)\)-fold quasi-categories, starting with 0-fold quasi-categories, which are just sets by definition. Simply put, an \(n\)-fold quasi-category is what it is, that is to say, an \(n\)-fold iteration of the notion of a quasi-category. The notion of an \(n\)-fold quasi-category might be regarded as a model of an \((\infty, n)\)-category, alternative to Barwick and Lurie’s \([\text{Lur09b}]\) version of it as an \(n\)-fold (complete) Segal space.

An \(n\)-fold quasi-category \(\text{Cob}_n\) of \(n\)-cobordisms with corners may be described by defining a multisimplex \((\Delta^{k_1}, \ldots, \Delta^{k_n}) \to \text{Cob}_n\) as a multisimplicial diagram of composable \(n\)-dimensional cobordisms with corners. Start with a pasting diagram in the form of an \(n\)-dimensional rectangular solid consisting of strings of \(k_1\) composable \(n\)-cobordisms in direction 1, \(k_2\) composable \(n\)-cobordisms in direction 2, . . . , \(k_n\) composable \(n\)-cobordisms in direction \(n\). Then add choices of all possible pushouts, so that for each triangular prism \([p_1q_1] \times \cdots \times [p_{i−1}q_{i−1}] \times [p_iq_ir]\) ×
\[ [p_{i+1}q_{i+1}] \times \cdots \times [p_nq_n], \quad p_i < q_i < r_i, \]

in the multisimplex, the cobordism along the face \([p_1q_1] \times \cdots \times [p_iq_i] \times \cdots \times [p_nq_n] \) is a pushout of the cobordisms along the faces \([p_1q_1] \times \cdots \times [p_iq_i] \times \cdots \times [p_nq_n] \) and \([p_1q_1] \times \cdots \times [q_ir_i] \times \cdots \times [p_nq_n] \) (not necessarily the canonical way we used in Section 7). Inner multihorns will be similar towers of pushouts, possibly missing the pushouts along an inner face of the \(k_i\)-simplex in the \(i\)th direction for some values of \(i\), \(1 \leq i \leq n\). Since pushouts for \(n\)-cobordisms with corners exist (referred to as compositions of cobordisms with corners in Section 7), all multihorns in triangular prisms can always be filled. As concerns more general multihorns, we can use canonical pushouts to add missing edges and coherence maps \(\alpha\) and \(\beta\) to realize missing faces as pushouts. The existence of such compositions is guaranteed by our Weak Coherence theorem, Theorem 3.2.

Remark. In the above we have assumed implicitly that all \(k_1, \ldots, k_n \geq 1\). If one of the \(k_i\)'s is equal to 0, we are talking about an \((n-1)\)-cobordism with corners in all the directions except \(i\). If more of the \(k_i\)'s are 0, it will be a cobordism of lower dimension, missing these directions.

The above arguments prove the following result.

**Theorem 8.1.** \(n\)-Cobordisms with corners form an \(n\)-fold quasi-category.

Of course, one would also be interested in taking into account the monoidal structure coming from disjoint union. This might be addressed via \((n+1)\)-fold (or higher) quasi-categories in a way similar to what we did with monoidal pseudo \(n\)-fold categories above, but we would rather not discuss this issue, as we will not be using quasi-categories and Segal spaces in this paper. In the case of Segal spaces, Lurie discusses the notion of symmetric monoidal \((\infty,1)\)-categories in [Lur11]. Another feature that we would like to have in the context of \(n\)-fold quasi-categories is including smooth maps and in particular diffeomorphisms in the picture. It is not obvious how to do that, whereas smooth maps are organic part of the treatment of cobordisms within the framework of pseudo \(n\)-fold categories, developed in this paper.

9. **Topological Quantum Field Theories**

**Metatheorem 9.1.** An (extended) \(n\)-dimensional Topological Quantum Field Theory (TQFT) is a symmetric monoidal functor from the symmetric monoidal pseudo \(n\)-fold category of cobordisms with corners to the opposite of the symmetric monoidal pseudo \(n\)-fold category of higher spans of sets (topological spaces, manifolds, orbifolds, or stacks) or to the symmetric monoidal pseudo \(n\)-fold category of higher cospans of dg associative algebras.

A functor to the opposite of a category may be called contravariant, as usual. Here we will be talking about TQFTs as contravariant functors, having in mind that by taking appropriate function algebras or de Rham or (singular) cochain algebras in concrete situations, the contravariant functor with values in higher spans of sets (or more interesting geometric objects) will induce a (covariant) functor with values in higher cospans of (dg) algebras.

The rest of the section is a case study, meant to be a “metaproof” of the metatheorem. We will discuss gauge theory in greater detail and then briefly go over Wess-Zumino-Witten (WZW) theory and a nonlinear sigma model, including perhaps the most general AKSZ model, all of which will repeat the same features. In principle,
all the three basic theories may be regarded as variations on the theme of sigma models: a sigma model is based on maps $M \to X$ from cobordisms $M$ with corners to a fixed target space $X$, whereas the WZW model is based on maps $M \to G$ to a Lie group and gauge theory on maps $M \to BG$ to the classifying space $BG$ of a Lie group. The fact that they all give rise to TQFT functors in the sense of Metatheorem 9.1 is, roughly speaking, based on the fact that maps $M \to X$ restrict naturally to faces of $M$, as well as to the components of $M$, if it happens to be a composition of cobordisms or disjoint union. Our formalism also allows us to consider mixed theories, for example, gauge theory combined with a sigma model, based on maps $M \to X \times BG$, or twisted ones, such as a theory based on pairs of maps $M \to X$.

We would also like to make a few remarks about quantum versus classical "path space" $\Phi(M)$. In our description, we will construct the "quantum path space" $\Phi(M)$ first and then describe examples of action functionals defined on the quantum path space. The Euler-Lagrange equations for these functionals define "classical trajectories," which form a "classical path space" $\Phi_c(M) \subset \Phi(M)$. We name the TQFTs obtained from $\Phi_c(M)$ after the action functionals, e.g., the Yang-Mills functional gives rise to Yang-Mills theory. In principle, to obtain TQFTs in a more classical, operator formalism, one wishes to perform Feynman integration over the whole path space $\Phi(M)$, using the action functional as a density function. This integral localizes around the classical path space $\Phi_c(M)$ via Feynman expansion. Usually, a combination of fiberwise integration (e.g., Faddeev-Popov or BV quantization procedures) and further localization is performed to reduce the path integral to a finite-dimensional one and finally obtain a single vector or operator between state spaces. However, this procedure depends on the specifics of the setup: sometimes getting such a vector is problematic because of divergences or anomalies, sometimes one gets finite-dimensional spaces of such vectors instead of uniquely defined vectors (cf. conformal blocks and modular functors). The goal of our approach is to bring to surface a common geometric background featured by all TQFTs irrespective of their particulars.

9.1. **Gauge Theory.** Fix a compact Lie group $G$ and, for any $k$-cobordism $M$ with corners, consider the set

$$\Phi(M) = \{(P, \nabla)/\sim\},$$

where $P$ is a (smooth) principal $G$-bundle over $M$, $\nabla$ is a connection on $P$, and $\sim$ denotes gauge equivalence, given by $G$-equivariant isomorphisms $P \xrightarrow{\sim} P'$ lifting the identity isomorphism from the base $M$ and respecting the connections on $P$ and $P'$.

The correspondence $M \mapsto \Phi(M)$ may be completed to a functor, a TQFT, to be more precise. First of all, we need to introduce more structure on $\Phi(M)$. Recall that connections on $P$ form an affine space (i.e., a principal homogeneous space) over the vector space $C^\infty(M, \Omega^1(P))$ of $P = (G \times \text{Ad} G)$-valued 1-forms, $\mathfrak{g}$ being the Lie algebra of $G$ and $\text{Ad} G$ the adjoint action of $G$ on $\mathfrak{g}$. Using an inner product on $\mathfrak{g}$, we can introduce topology on the vector space and the affine space over it and thereby the set

$$\Phi_P(M) = \{ \text{connections } \nabla \text{ on } P \}/G^M$$
of connections on a fixed principal $G$-bundle $P$, up to the action of the gauge group $G^\infty_M = C^\infty(M, G) = H^0(M, G)$, where $\mathcal{G} := C^\infty(M, G)$ is the sheaf of (germs of) smooth functions $M \to G$, and $C^\infty(M, G)$ is the space of global smooth functions $M \to G$. The gauge group $G^\infty_M$ might also be understood as non-abelian cohomology $H^0(M, \mathcal{G})$ in degree zero. Finally, redefine $\Phi(M)$ as an affine bundle over the space of gauge classes of principal $G$-bundles $P$ with fiber $\Phi_P(M)$, i.e.,

$$\Phi(M) = \prod_{P \in H^1(M, \mathcal{G})} \Phi_P(M) \quad \text{and} \quad \Phi(M) = \prod_{P \in [M, BG]} \Phi_P(M),$$

where $P$ in the first disjoint union runs over the set $H^1(M, \mathcal{G})$, non-abelian cohomology in degree one, which is, by definition, the set of gauge classes of principal $G$-bundles over $M$, and $[M, BG]$ in the second disjoint union is the set of homotopy classes of maps from $M$ to the classifying space $BG$ of the Lie group $G$. Moreover, instead of treating $\Phi(M)$ as a topological space, one has to adopt the “stacky” point of view and regard $\Phi(M)$ as a space with a group action, rather than an orbit space. In this case, it will be an affine bundle over $H^1(M, \mathcal{G})$ with fibers being affine spaces over the infinite dimensional vector space $C^\infty(M, \Omega^1(\mathfrak{g}_M))$, endowed with, say, Fréchet topology, with an action of a (generally, infinite dimensional) Lie group, $G^\infty_M$. This suggests that one can view $\Phi(M)$ as a step toward the Borel quotient

$$H^1(M, \mathcal{G})/H^0(M, \mathcal{G}).$$

To extend the correspondence $M \mapsto \Phi(M)$ to a TQFT, we would like to associate an $n$-span of stacks to a given $n$-cobordism $M$ with corners. The core (the object at the center) of the $n$-span will be the stack $\Phi(M)$ of gauge classes of connections on principal $G$-bundles. The object at the barycenter of each face of the $n$-span will be the stack of gauge classes of connections on principal $G$-bundles over the union of faces of the $n$-cobordism labeled by the face of the $n$-cube $I^n$ corresponding to the face of the $n$-span. The maps from the barycenter of the face toward the barycenters of the bounding faces will be the restriction map: a connection on a principal $G$-bundle over a cobordism with corners induces a principal $G$-bundle with a connection on all the source and target manifolds of the cobordism. This restriction map is manifestly equivariant with respect to the notion of gauge equivalence. We will abuse the notation and let $\Phi(M)$ denote the resulting $n$-span, rather than just its core. The correspondence $M \mapsto \Phi(M)$ is in fact a functor, as it takes $k$-cobordisms with corners to $k$-spans of sets, functorially with respect to smooth maps of cobordisms and maps of spans, whereas compositions $M \circ_j N$ of cobordisms induce natural coherence maps

$$\Phi(M \circ_j N) \to \Phi(M) \circ_j \Phi(N),$$

where on the left-hand side, we have the set of (classes of) connections over the composition $M \circ_j N$, whereas on the right-hand side, we have the set of pairs of (classes of) connections on $M$ and $N$, coinciding (up to gauge equivalence) on the common, $j$th boundary of $M$ and $N$. The flip of the direction of the map as compared to Section 5 is due to the fact that $\Phi$ is contravariant. It is easy to see that the coherence conditions of Section 5 are satisfied.

We could have imposed boundary conditions on principal $G$-bundles and connections along the collars, so that they glue nicely when the cobordisms are composed. The natural choice of boundary conditions would be compatibility with collars at all faces of the cobordism $M$: the pair $(P, \nabla)$ along a collar must be a trivial extension.
of a principal $G$-bundle with connection from the face to its collar. Gauge transformations would have to be constant along the collars. Gluing along composable cobordisms would result in coherence maps

$$\Phi(M) \circ_j \Phi(N) \to \Phi(M \circ_j N),$$

which would mean that the TQFT functor is a cofunctor, rather than a functor, by definition. However, we anticipate that since we are talking about gauge classes of connections, irrespective of the boundary conditions, the TQFT functor is actually a strong functor, that is to say, all coherence maps are invertible.

We also need to see that the gauge theory functor $M \mapsto \Phi(M)$ is symmetric monoidal. Remember that now we have to think about $k$-cobordisms with corners and $k$-spans of sets as $(k + 3)$-morphisms spanning $k$ positive, spatial, directions and the three nonpositive ones: $-2, -1, \text{ and } 0$. Compositions in the nonpositive directions are all given by disjoint union for cobordisms and Cartesian product for spans. Thus, for $j \leq 0$ the coherence maps

$$\Phi(M \circ_j N) \to \Phi(M) \circ_j \Phi(N),$$

will represent (the gauge class of) a principle $G$-bundle with a connection on $M \bigsqcup N$ as a pair $((P_M, \nabla_M), (P_N, \nabla_N))$ of the same on $M$ and $N$ separately. Again, the coherence conditions of Section 5 involving nonpositive directions is easy to check.

There exist different reincarnations of gauge theory, whose description within our formalism is given below.

9.1.1. The Dijkgraaf-Witten Toy Model. This is a version of gauge theory in the case when the group $G$ is finite. We will indicate the main ingredients of the construction, leaving out those details which are similar to those in the case of the general treatment of gauge theory above.

The functor $M \mapsto \Phi(M)$ assigns to a cobordism $M$ with corners the set

$$\Phi(M) = \text{Hom}(\Pi_1(M), \mathfrak{G})/\sim$$

of morphisms from the fundamental groupoid $\Pi_1(M)$ of $M$ to the groupoid $\mathfrak{G}$ of principal homogeneous spaces for group $G$, up to natural isomorphism. The objects of the fundamental groupoid $\Pi_1(M)$ are points of $M$, and the morphisms are homotopy classes of paths in $M$. The objects of $\mathfrak{G}$ are principal homogeneous spaces, or torsors, i.e., sets with free, transitive (left) action of $G$. The morphisms of $\mathfrak{G}$ are $G$-equivariant maps. Again, we should think about $\Phi(M)$ as a stack, rather than a set: it is the groupoid of functors $\Pi_1(M) \to \mathfrak{G}$ with natural isomorphisms as morphisms. One can represent this stack as a global quotient by the "conjugation" action of the group $G$ via automorphisms of the groupoid $\mathfrak{G}$: for an element $g \in G$ and a $G$-torsor $X$, a new torsor $gX$ is defined as the set $\{gx \mid x \in X\}$ along with $G$-action, as follows:

$$h(gx) := (hg)x = g(g^{-1}hg)x \quad \text{for } h \in G \text{ and } x \in X.$$  

The isotropy subgroup of a point will then consist of those elements of $G$ which commute with the monodromy group.

As in the general case of gauge theories above, the $G$-space $\Phi(M)$ should be extended to a higher span of $G$-spaces, with the span structure induced by restriction of morphisms $\Pi_1(M) \to \mathfrak{G}$ to faces of $M$.

In this case, it is easy to see that the coherence maps

$$\Phi(M \circ_j N) \to \Phi(M) \circ_j \Phi(N),$$
will be $G$-equivariant isomorphisms: a pair of morphisms $\Pi_1(M) \to \mathfrak{g}$, $\Pi_1(N) \to \mathfrak{g}$, equal on the common boundary of $M$ and $N$, extend uniquely to a morphism $\Pi_1(M \circ_j N) \to \mathfrak{g}$ by Seifert-van Kampen argument.

In a similar way, the coherence maps for $\circ_j$ compositions for $j \leq 0$, i.e., the natural maps

$$\Phi(M \coprod N) \to \Phi(M) \times \Phi(N)$$

are also isomorphisms of spans.

**Remark.** A key ingredient of Dijkgraaf-Witten’s toy model is the choice of a class $\alpha$ in the cohomology $H^3(BG, \mathbb{U}(1))$ of the classifying space $BG$ under the assumption that $\dim M = 3$. This class may be used to obtain a “partition function,” also known as the Dijkgraaf-Witten invariant

$$Z(M) := \sum_{\gamma \in \Phi(M)} \gamma^*(\alpha)([M]) \in \mathbb{U}(1),$$

where we use the additive notation for $\mathbb{U}(1) = \mathbb{R}/\mathbb{Z}$ and understand summation in the orbifold sense, i.e.,

$$\sum_{\gamma \in \Phi(M)} \gamma^*(\alpha)([M]) := \frac{1}{|G|} \sum_{\gamma \in \text{Hom}(\pi_1(M), G)} \gamma^*(\alpha)([M]),$$

representing $\Phi(M)$ as a global quotient $\text{Hom}(\pi_1(M), G)/G$ by the conjugation action of $G$. More generally, if $\partial_2 M = \partial_3 M = \emptyset$, then one can define a TQFT in a more traditional sense. This includes metrized complex lines $L(\partial^-_1 M)$ and $L(\partial^+_1 M)$ corresponding to the incoming and outgoing boundary, respectively, and a ”tunneling amplitude,” a linear map $\Phi_M : L(\partial^-_1 M) \to L(\partial^+_1 M)$, see [DW90, FQ93, FHLT10]. The above summation formula is a simple case of path integration, which is needed to obtain a number, such as $Z(M)$, or a map, such as $\Phi_M$, from the span $\Phi(M)$.

9.1.2. Yang-Mills Theory. In Yang-Mills theory, one starts with the Yang-Mills action (functional):

$$\int_M \text{Tr}(F \wedge *F),$$

defined on the space $\{(P, \nabla) \mid \sim \}$ of equivalence classes of principal $G$-bundles $P$ with connection $\nabla$ for a compact Lie group $G$. To define this functional, we need to consider cobordisms $M$ with corners and a Riemannian metric which makes all faces orthogonal to each other at intersections, so that the Hodge dual $*F$ of the curvature form $F$ of the connection $\nabla$ is defined, whereas $\text{Tr}$ denotes the Killing form on the Lie algebra $\mathfrak{g}$. The Euler-Lagrange equations

$$d\nabla F = d\nabla *F = 0,$$

for the Yang-Mills functional are manifestly gauge invariant and called the Yang-Mills equations. Then the functor $\Phi_{YM}(M)$ defining Yang-Mills theory as a TQFT consists of all solutions to the Euler-Lagrange equations, i.e.,

$$\Phi_{YM}(M) = \{(P, \nabla) \mid d\nabla *F = 0\}/\sim.$$
The first Yang-Mills equation $d_\nabla F = 0$ is the Bianchi identity, which is satisfied automatically, and thereby redundant. As above, it is easy to check that we get a TQFT in the sense of this paper.

9.1.3. Yang-Mills Theory in Lower Dimensions. Donaldson theory is a version of Yang-Mills theory in the case of $\dim M = 4$. The advantage of four dimensions is that in this case both $F$ and $\ast F$ are 2-forms and moreover the Yang-Mills functional minimizes on self-dual and anti-self-dual connections, also known as instantons — those whose curvature $F$ satisfies

$$\ast F = \pm F.$$ 

Thereby this equation implies the Yang-Mills equations. In particular, Donaldson considered anti-self-dual connections. Thus, one can define Donaldson theory as a 4d TQFT (i.e., one defined on the pseudo four-fold category of cobordisms of dimension $\leq 4$) by specifying

$$\Phi_D(M) = \left\{ (P, \nabla) \mid \ast F = -F \right\} / \sim$$

for $\dim M = 4$ and using the standard Yang-Mills equation $d_\nabla \ast F = 0$ for $\dim M \leq 3$. When $\dim M = 3$, Yang-Mills theory is known to turn into Chern-Simons theory with the corresponding TQFT functor (on the pseudo three-fold category of cobordisms of dimension $\leq 3$) defined as follows:

$$\Phi_{CS}(M) = \left\{ (P, \nabla) \mid F = 0 \right\} / \sim.$$ 

Seiberg-Witten theory is a version of Yang-Mills theory in four dimensions, alternative to Donaldson theory. Seiberg-Witten theory is described by certain gauge fields, called monopoles, on 4-manifolds with complex spin structure. Restriction of monopoles to the boundary also generates a 4d TQFT in the sense of the present work.

9.2. Wess-Zumino-Witten Theory. In Wess-Zumino-Witten (WZW) theory one starts with a compact Lie group $G$ and assigns to an $n$-cobordism $M$ the mapping space

$$\Phi(M) := \{ \text{smooth maps } M \to G \} / \sim,$$

where two maps $f$ and $g$ are equivalent, if they fit into a commutative diagram

$$\begin{array}{ccc}
M & \xrightarrow{f} & G \\
\phi \searrow & & \nearrow \\
& M & \\
\end{array}$$

where $\phi$ is a diffeomorphism respecting the structure of a cobordism with corners. Out of a given $n$-cobordism, we are actually getting an $n$-span $\Phi(M)$ of stacks, where the maps making up the span come from restriction of maps to $G$ from faces of $M$ to faces of lower dimension. The coherence maps

$$\begin{aligned}
\Phi(M \circ_j N) & \to \Phi(M) \circ_j \Phi(N), \\
\Phi(M \coprod N) & \to \Phi(M) \times \Phi(N)
\end{aligned}$$

are given by restriction from the composition (or union) of two cobordisms to the individual cobordisms.
WZW theory for a complex semi-simple Lie group $G$ and 2-cobordisms $M$ provided with complex structure is defined by an action whose Euler-Lagrange equations require that solution fields $f : M \to G$ be holomorphic. Thus, under these assumptions, we can consider only classical trajectories in our path space and redefine the TQFT functor on the pseudo 2-fold category of 2-cobordisms with complex structure as follows:

$$\Phi_{cl}(M) := \{ \text{holomorphic maps } M \to G \}/\sim,$$

where two maps are equivalent, if they are related by a complex isomorphism of $M$ preserving the cobordism with corners structure.

9.3. Sigma Model. In a (nonlinear) sigma model, one fixes a target space $X$, usually a smooth compact manifold, and defines a TQFT, using smooth maps to $X$:

$$\Phi(M) := \{ \text{smooth maps } M \to X \}/\sim,$$

with the same equivalence relation as in the WZW model above: two maps $f$ and $g$ are equivalent, if they fit into a commutative diagram:

$$\begin{align*}
M & \xrightarrow{f} X, \\
\phi & \downarrow \\
M & \xrightarrow{g}
\end{align*}$$

for a diffeomorphism $\phi$. As above, restriction to faces defines a span $\Phi(M)$ of stacks and restriction to components defines coherence maps (11)–(12).

If we consider 2d cobordisms with complex structure and a complex target space $X$, we will be getting what is known as (open-closed) Gromov-Witten theory, defined by a “classical path space”

$$\Phi_{GW}(M) := \{ \text{holomorphic maps } M \to X \}/\sim,$$

the equivalence relation given by complex isomorphisms preserving the structure of cobordism with corners.

Another useful theory, called the Rozansky-Witten model [RW97] comes in 3 dimensions ($\dim M = 3$). In the Rozansky-Witten model, one considers oriented 3d cobordisms $M$ with corners, a complex symplectic manifold $X$ as target space, maps $f : M \to X$, and sections $\eta : M \to f^*T^{0,1}X$ and $\rho : M \to f^*T^{1,0}X \otimes T^*M$, see more details in [Kap10]. To accommodate this theory, one can modify the TQFT functor to include the whole quantum path space $\Phi(M) := \{(f, \eta, \rho)\}/\sim$ or restrict the attention to classical paths: $\Phi_{RW}(M) := \{(f, \eta)\}/\sim$, i.e., those satisfying the Euler-Lagrange equations of the Rozansky-Witten action, see [Kap10] [RW97] or a more general (as per [QZ09]) AKSZ formalism below.

The AKSZ model [ASZK97] [Roy07] uses maps from $(T[1]M, d_{DR})$ to $(X, Q)$, where $(T[1]M, d_{DR})$ is the dg manifold whose sheaf of functions is the de Rham algebra $\Omega^*_M := S^* T^*_M[1]$, where $T^*_M[1]$ is the cotangent sheaf of $M$ shifted by degree 1 (also known as the desuspended cotangent sheaf). The dg structure is given by the de Rham differential $d_{DR}$. Also in the above, $(X, Q)$ is a dg symplectic manifold [Roy07], a dg manifold $(X, Q)$ with a graded skew form $\omega : T_X \otimes T_X \to \mathbb{R}[n-1]$ of degree $n - 1$, where $n = \dim M$, and a Hamiltonian differential $Q$, i.e., one given by a function $\Theta$ on $X$ of degree $n$ via $Q = \{\Theta, -\}$, satisfying the ”classical master equation" $Q^2 = 0$ or, equivalently, $\{\Theta, \Theta\} = 0$, where the bracket is the Poisson
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bracket associated to $\omega$. The classical path space $\Phi_{\text{AKSZ}}(M)$ will then be defined as the space of dg maps

$$
\Phi_{\text{AKSZ}}(M) := \text{Map}((T[1]M, d_{\text{DR}}), (X, Q))/\sim
$$

up to the same equivalence relation as in the usual sigma model above. Since the degree of the symplectic form $\omega$ depends on the dimension $n$ of the source $M$ and in a TQFT the dimension of $M$ varies, it is not obvious how to generate a TQFT out of the AKSZ model. We can only speculate that such a TQFT may be given by fixing the target space to be a graded manifold $X$ with a collection of symplectic forms $\{\omega_i : T_X \otimes T_X \to \mathbb{R}[i-1] | i = 0, 1, \ldots, n\}$ and differentials $Q_0, Q_1, \ldots Q_n$, such that $Q_i = \{\Theta_i, -\}$, where $\{\cdot, \cdot\}_i$ is the Poisson bracket associated with $\omega_i$ and $\Theta_i$ is a function on $X$ of degree $i$, satisfying the classical master equation $\{\Theta_i, \Theta_i\} = 0$. Then for a $n$-cobordism $M$ with corners, the TQFT functor in the AKSZ model might be defined as the space of dg maps $\phi : (T[1]M, d_{\text{DR}}) \to (X, Q_n)$ such that restriction of $\phi$ to a face $F_i$ of $M$ of dimension $i$ induces a dg map $(T[1]F_i, d_{\text{DR}}) \to (X, Q_i)$. This way, the structure of a span would be tautological and we would get a TQFT functor, indeed.

9.4. Variations. One could consider various modifications of TQFT theory, whose physical sense may be quite different. For example, one could take oriented cobordisms with corners. In some cases, such as Yang-Mills theory, we needed to bring in Riemannian metrics on cobordisms. Riemannian cobordisms are also needed for Stolz and Teichner’s construction [ST04] of Segal’s elliptic object. For conformal field theories, one could talk about complex cobordisms [Mil60, Rav86]. It would also make sense to consider symplectic/contact cobordisms, such as those in [Gin89, Gin92]. One can also include D-branes into the picture by labeling different connected components of the boundary (or even faces) of cobordisms in selected directions by elements of a fixed set of “D-branes.” For example, in the case of 2-cobordisms with corners, one can interpret the boundary $\partial_1 M$ in direction 1 as closed and open strings, depending on whether the component of the boundary is closed (circle) or not (interval). The boundary $\partial_2 M$ in direction 2 may be regarded as “free” boundary and each component of $\partial_2 M$ may be labeled by a D-brane. Further extension of the notion might be in the spirit of Topological Conformal Field Theories (TCFTs), [Seg04, Get94, Vor94, Cos07], when the source pseudo $n$-fold category of cobordisms with corners is replaced with a pseudo $n$-fold category in which a $k$-morphism is a (e.g., singular) chain of $k$-cobordisms with corners and the target category also has a dg structure, such as the pseudo $n$-fold category of spans of dg coalgebras, or cospans of dg algebras (or derived versions thereof). On top of that, 2d TCFTs cobordisms must also be provided with extra structure, such as that of a complex manifold. However, we prefer not to overload this paper with further generalizations in this direction.

10. Appendix: the Regular Coherence Theorem

Theorem 10.1 (Regular Coherence Theorem). Let $c$ be a regular gluing diagram for $k$-morphisms in pseudo $n$-fold category, that is to say, a diagram made by cutting a $k$-dimensional cube up by hyperplanes into $p = p_1 \ldots p_k$ rectangular blocks of equal size. Let $(k\text{Mor})^p_c$ denote the full subcategory of $k\text{Mor}^p$ of morphisms composable in the diagram $c$. Then any two functors $(k\text{Mor})^p_c \to k\text{Mor}$ built from morphism
composition are related by a unique natural isomorphism built from associators $\alpha$ and interchangers $\beta$.

**Proof.** The proof will be based on the following enhancement of Newman’s Diamond lemma in graph theory. This enhancement seems to be new and interesting on its own. Let us start with the classical Diamond lemma.

First of all, recall some terminology related to directed graphs. A **directed graph** is a set of vertices and a set of edges endowed with a sense of direction, along with an incidence relation between the endpoints of the edges and the vertices, so that each endpoint of an edge is incident to a vertex. We will consider paths between vertices in a graph. A **path** from $x$ to $y$ is given by a finite sequence of consecutive undirected edges from $x$ to $y$. If all the edges along a path are directed from $x$ to $y$, the path is called **directed**. A **descending path** starting at a vertex $x$ is an at most countable sequence of consecutive directed edges starting at $x$. A path is **trivial** when it comprises no edges. A **terminal vertex** is one with no edges originating from it.

**Lemma 10.2 (Diamond Lemma).** Suppose that we are given a directed graph with a descending chain condition — namely, that any directed path from a vertex has finite length, — and a diamond condition — namely, any two directed edges starting at a vertex may be extended to directed paths that end up at the same point. Then every connected graph component has a unique terminal vertex.

For the enhanced version of the lemma, we will need to consider graphs with an equivalence relation on the set $\text{Path}(x, y)$ of (undirected) paths from $x$ to $y$ for each pair of vertices $x$ and $y$. We will assume that this relation $\sim$ satisfies the following conditions:

- **trivial cycles:** For any path $f$, we have $f^{-1}f \sim \text{id}$, where $\text{id}$ is the trivial path at the starting vertex of $f$;
- **invariance:** Two paths from $x$ to $y$ are equivalent, if and only if their extensions by an edge adjacent to $x$ (or $y$) are equivalent.

**Lemma 10.3 (Enhanced Diamond Lemma).** Suppose that we have a directed graph with an equivalence relation on paths, as above. Suppose also that the graph satisfies the descending chain condition and the following enhanced diamond condition: any two directed edges originating at a vertex may be extended to equivalent directed paths ending at one and the same vertex. Then every connected component of the graph has a unique terminal vertex, and moreover any two paths between any two vertices are equivalent.

**Proof of Lemma.** We will prove the Enhanced Diamond lemma only, as the proof of Newman’s classical Diamond lemma is standard and moreover follows from our proof of the enhanced version. The proof will follow the same sequence of steps as the proof of the classical Diamond lemma, see [Hue80], based on well-founded (also known as Noetherian) induction.

**Step 1:** Local diamond condition implies global. First of all, let us show that under our assumptions, the following “global” version of the diamond condition takes place: any two finite directed paths starting at a vertex may be extended to equivalent directed paths that end up at one and the same vertex.

The proof follows from the well-founded induction principle, which states that if we have a property $P$ of vertices in a directed graph with a descending chain
condition, so that for each vertex $t$, property $P(t)$ holds whenever it holds for all ends of nontrivial finite directed paths emanating from $t$, then $P$ holds for all vertices of the graph.

In our case, the property $P(t)$ is the above global diamond condition for directed paths originating from a given vertex $t$. The necessary condition, that is, if $P(t)$ is true for a vertex $t$, then it is true for all vertices underneath, follows from the invariance property. To check the sufficient condition, we assume that we have two directed paths $t \rightarrow u$ and $t \rightarrow v$, where the asterisk distinguishes a path from an edge, and that the property $P$ holds for all vertices strictly under $t$; then we have to find a vertex $w$ with directed paths $u \rightarrow w$ and $v \rightarrow w$ such that the two composite directed paths from $t$ to $w$ are equivalent. The cases $t = u$ and $t = v$ are trivially resolved by setting $w = v$ and $w = u$, respectively.

Otherwise, we may assume the paths from $t$ to $u$ and $v$ to be nontrivial and thereby write them as $t \rightarrow u_1 \rightarrow u$ and $t \rightarrow v_1 \rightarrow v$. By the (local) diamond condition, there exists $w_1$ with directed paths $u_1 \rightarrow w_1$ and $v_1 \rightarrow w_1$, so that the two composite paths from $t$ to $w_1$ are equivalent. Using the induction hypothesis on $u_1$, there exists $w_2$ with directed paths $u \rightarrow w_2$ and $w_1 \rightarrow w_2$, so that the two composite paths from $u_1$ to $w_2$ are equivalent.

Now we have a composite path $v_1 \rightarrow w_1 \rightarrow w_2$ and the path $v_1 \rightarrow v$, starting from $v_1$, to which we apply the induction hypothesis again and get $w$ with paths $w_2 \rightarrow w$ and $v \rightarrow w$, with the two composite paths from $v_1$ to $w$ being equivalent. In the process, we have obtained three diamonds, each with equivalent paths from the top to the bottom:

The big, outer diamond is the one we have been looking for, and the paths down along its edges are equivalent, because the smaller diamonds are bounded by equivalent paths.

**Step 2: The uniqueness of a normal form.** This means that for any vertex $x$, there exists a unique terminal vertex $y$ with a directed path from $x$ to $y$, unique up to equivalence. This $y$ is called the normal form of $x$. The existence is obvious from the descending chain condition. The uniqueness of a normal form and a directed path to it, up to equivalence, follows trivially from the enhanced global diamond property.

**Step 3: If two vertices are connected by a path, then their normal forms are the same and the path between the two vertices is unique up to equivalence.** Note that the two
vertices may be the same, in which case the uniqueness of a path up to equivalence is still a nontrivial statement. It implies that any closed path is equivalent to the trivial path.

The exact statement we are going to prove will be more constructive: if two vertices \( x \) and \( y \) are connected by a path, then they have one and the same normal form \( z \) and the path is equivalent to a path \( x \rightarrow z \leftarrow y \).

This statement may be proven using usual induction on the length of the path. If the path is trivial, the statement follows from Step 2 and the trivial cycles property. Suppose the statement is true for all paths of length at most \( n \geq 0 \). If we have a path from \( x \) to \( y \) consisting of \( n + 1 \) edges, consider the vertex \( x' \) one edge away from \( x \) in the direction of \( y \) along the path. By induction, \( x' \) and \( y \), connected by a path of length \( n \), have the same normal form \( z \) and the path from \( x' \) to \( y \) is equivalent to \( x' \rightarrow z \leftarrow y \). Whichever way the edge between the vertices \( x \) and \( x' \) is directed, a normal form of one of them will automatically be a normal form of the other. For instance, if the edge is directed from \( x' \) to \( x \), a directed path from \( x \) to a normal form extended by the edge from \( x' \) to \( x \) will serve as a directed path from \( x' \) to a normal form thereof. Because of the uniqueness of a normal form, it will be the vertex \( z \). Because of the uniqueness of a path to a normal form, the path \( x' \rightarrow z \leftarrow y \) will be equivalent to the path \( x' \rightarrow x \rightarrow z \leftarrow y \). Note that the path \( x \rightarrow x' \) is equivalent to \( x \rightarrow z \leftarrow x \rightarrow x' \). Thus, we see that the path from \( x \) to \( y \) is equivalent to the path \( x \rightarrow z \leftarrow x' \rightarrow z \leftarrow y \), in which the two middle arrows may be “canceled,” so that it becomes equivalent to the path \( x \rightarrow z \leftarrow y \). The case when the edge between \( x \) and \( x' \) is directed from \( x \) to \( x' \) is treated similarly. \( \square \)

Now we are ready to prove the Regular Coherence theorem. The existence statement is a particular case of the Weak Coherence theorem, Theorem 3.2, and all we need is to prove uniqueness.

We would like to apply the Enhanced Diamond Lemma, in which the graph is formed by functors \((\mathcal{C} \mathbf{Mor})^p \rightarrow \mathbf{C} \mathbf{Mor}\) built from morphism composition as vertices and applicable natural transformations \(\alpha_i\) and \(\beta_{ji}\) with \(i < j\) (combined with identities in all the remaining variables) as directed edges. We say that two paths in this graph are equivalent, if the resulting compositions of natural transformations are equal. This is an equivalence relation satisfying the trivial cycles and invariance conditions, see the discussion before Lemma 10.3.

Let us check that this graph satisfies the descending chain condition. Note that each associator \(\alpha\) moves a pair of parentheses to the right. Thus, in a descending path of \(\alpha\)s and \(\beta\)s, there should be a finite number of \(\alpha\)s. Note that each interchanger \(\beta_{ji}\) with \(i < j\) leaves the parentheses intact, but moves morphism composition \(\circ_j\) with greater index \(j\) from the inside of a pair of parentheses to the outside. Even though some of the \(\alpha\)s participating in a path may move compositions \(\circ_j\) with greater indices \(j\) deeper inside parentheses, there are only finitely many \(\alpha\)s and thereby there should be only finitely many \(\beta\)s.

Thus, to show the uniqueness in the Regular Coherence theorem, we just need to verify the enhanced diamond condition. This is far from being obvious and is the core argument in the proof of the Regular Coherence theorem.

The enhanced diamond condition starts with two directed edges emanating from a vertex. This translates into two natural transformations \(\gamma\) and \(\delta\), each of
the type \( \alpha_i \) or \( \beta_{ij} \) with \( i < j \), applied to a composition of some number \( p \) of \( k \)-morphisms, or strictly speaking, a functor \((k\text{Mor})^p \to k\text{Mor})\) built out of morphism compositions. We need to show that these transformations may be appended by transformations of same kind, so that the resulting sequence of transformations will end at one and the same composition functor \((k\text{Mor})^p \to k\text{Mor})\) and the corresponding paths will be equivalent.

To check the enhanced diamond condition, we will use induction on the number \( p \) of terms in our functors \((k\text{Mor})^p \to k\text{Mor})\), starting with \( p = 2 \), when the gluing diagram \( c \) consists of two blocks attached in a particular direction \( i \), \( 1 \leq i \leq n \), and the only composition functor applied to two morphisms fitting into this diagram: \( \circ_i \). In this case, the graph consists of just a single vertex, and the enhanced diamond condition is vacuously true. Suppose we have proven the statement for any number less than \( p \) of terms in our functors. We need to check if the statement is true for \( p \) terms.

Note that the induction assumption implies the existence of a normal form for any regular composition of less than \( p \) morphisms. It is easy to describe what this normal form has to be. Without loss of generality, let us assume that morphisms in the composition are composed in directions \( 1 \), \( 2 \), \( \ldots \), \( n \), for some \( d \), \( 1 \leq d \leq k \). Then in a normal form, firstly, all compositions in direction \( 1 \) are made, with all the parentheses moved to the right, as in \( a_1 \circ_1 (a_2 \circ_1 (\cdots a_{q-1} \circ_1 a_q) \cdots) \), then all compositions in direction \( 2 \) are made, again with all the parentheses moved to the right, etc., the last compositions are made in direction \( d \), with all parentheses on the right. This is a normal form, because in our directed graph it represents a terminal vertex: no forward moves, i.e., moves along directed edges of the graph, \( \alpha_i \) or \( \beta_{ij} \) with \( i < j \), can be applied to this composition.

Next, observe that our functor \((k\text{Mor})^p \to k\text{Mor})\) is a composition two such functors, say \( a \circ_i b \), each with a strictly less than \( p \) number of terms. One has to consider a number of cases, depending on what the natural transformations \( \gamma \) and \( \delta \) are. There are the following possibilities for \( \gamma \):

1. \( \gamma \) acts inside \( a \);
2. \( \gamma \) acts inside \( b \);
3. \( a = a_1 \circ_1 a_2 \), and \( \gamma = \alpha_i : (a_1 \circ_1 a_2) \circ_i b \to a_1 \circ_i (a_2 \circ_i b) \) is an associator;
4. \( a = a_1 \circ_1 a_2, b = b_1 \circ_1 b_2 \), and \( \gamma = \beta_{ji} : (a_1 \circ_j a_2) \circ_i (b_1 \circ_j b_2) \to (a_1 \circ_i b_1) \circ_j (a_2 \circ_i b_2) \) with \( i < j \) is an interchanger.

The same possibilities are there for \( \delta \). If they act both within \( a \) or both within \( b \), we are done by the induction assumption. If \( \gamma \) acts within one term and \( \delta \) within the other, then they commute because of the functoriality of \( \circ_i \), and we get a commutative square. If both \( \gamma \) and \( \delta \) fall under Cases 3 and 4, then the only possibility for this to happen is when \( \gamma = \delta \), and the enhanced diamond condition holds trivially. Thus, what remains to be done is to check the enhanced diamond condition when \( \gamma \) is an associator or interchanger and \( \delta \) acts within either \( a \) or \( b \), or vice versa.

Suppose \( \gamma = \alpha_i \) is an associator, as in Case 3. If \( \delta \) works entirely within \( a_1, a_2, \) or \( b \), then \( \gamma \) and \( \delta \) commute by the naturality of \( \alpha_i \). The only remaining cases here are (1) \( a_1 = a_{11} \circ_1 a_{12} \) and \( \delta \) is the associator \(((a_{11} \circ_1 a_{12}) \circ_i a_{2}) \circ_i b \to (a_{11} \circ_i (a_{12} \circ_i a_{2})) \circ_i b \), and (2) when \( a_1 = a_{11} \circ_j a_{12} \) and \( a_2 = a_{21} \circ_j a_{22} \) for some \( j > i \) and \( \delta \) is the interchanger \( \beta_{ji} : ((a_{11} \circ_j a_{12}) \circ_i (a_{21} \circ_j a_{22})) \to (a_{11} \circ_i a_{21}) \circ_j (a_{12} \circ_i a_{22}) \) appended with \( \circ_i b \). In the first case, the two associators \( \gamma \) and \( \delta \) are the first
two edges in the coherence pentagon, and we can continue them as paths around the pentagon to the opposite vertex. The resulting compositions of the natural transformations will be equal, because of the pentagon axiom. The above argument repeats a standard argument in the proof of Mac Lane’s coherence theorem for monoidal categories, see [Arm07] or [ML98].

In the second case, we have \( \gamma = \alpha_i : (a_1 \circ_i a_2) \circ_i b \rightarrow a_1 \circ_i (a_2 \circ_i b) \) and \( \delta = \beta_j \circ_i \text{id} : ((a_{11} \circ_j a_{12}) \circ_i (a_{21} \circ_j a_{22})) \circ_i b \rightarrow ((a_{11} \circ_i a_{21}) \circ_j (a_{12} \circ_i a_{22})) \circ_i b, \) where \( a_1 = a_{11} \circ_j a_{12} \) and \( a_2 = a_{21} \circ_j a_{22}, \) and need to check the enhanced diamond condition for these two edges. If \( b \) were factored as \( b_1 \circ_j b_2, \) with \( b_1 \) and \( b_2 \) fitting in direction \( i \) with \( a_{21} \) and \( a_{22}, \) respectively, we would then proceed as in the previous case and just use the first hexagon axiom. If the composition diagram \( c \) were not regular, \( b \) would not factor like that in general, even after applying a few associators and interchangers. Given that our composition diagram is regular, we could use coherence transformations, existing by the induction assumption, to rearrange how \( b \) is composed and indeed factor it as \( b_1 \circ_j b_2 \) to fit with \( a_{21} \) and \( a_{22}, \) as above, but the problem is that it would not help to check the diamond condition, as this rearrangement would not necessarily be achieved by forward moves. To deal with this problem, we will use the induction assumption to put each factor \( a_{11}, a_{12}, a_{21}, a_{22}, b \) in a normal form by forward moves. In particular, since we assumed that compositions in our composition diagram \( c \) are performed in directions \( 1, 2, \ldots, d, \) a normal form of the above five terms will always be a composition in direction \( d: - \circ_d -, \) and we will also have \( i < j \leq d. \) We will have to consider two cases: \( j = d \) and \( j < d. \) If \( j = d, \) we will only need to put \( a_{11}, a_{21}, \) and \( b \) in a normal form:

\[
\begin{align*}
  a_{11} &= a_{111} \circ_j a_{112}, \\
  a_{21} &= a_{211} \circ_j a_{212}, \\
  b &= b_1 \circ_j b_2.
\end{align*}
\]

Because of the regularity of our initial diagram \( c, \) morphisms in it will fit each other as follows:

Then we will apply to \( (((a_{111} \circ_j a_{112}) \circ_j a_{12}) \circ_i ((a_{211} \circ_j a_{212}) \circ_j a_{22})) \circ_i (b_1 \circ_j b_2), \) as per this diagram, several successive forward moves with the idea of splitting off the top row \( a_{111} \circ_i (a_{211} \circ_i b_1) \) and using induction on the remaining smaller diagram. This process may be described by the following scheme, see Figure 2: we will extend the edges \( \gamma \) and \( \delta, \) which start at vertex 1, to directed paths ending at vertices 3 and 4, respectively. We will construct directed paths from vertex 1 to these vertices via a new, common vertex 2 in such a way that these new paths are equivalent to the respective previous ones via a sequence of coherence diagrams. The paths from 2 to 3 and 2 to 4 will actually be directed edges of the type \( \text{id} \circ_d \gamma' \)
and $\text{id} \circ d \delta'$ (remember that $d = j$), and those may be extended to two equivalent paths to a common vertex 5 by the induction assumption, applied to $\gamma'$ and $\delta'$.

In our case, when $\gamma = \alpha_i$ and $\delta = \beta_{ji} \circ \text{id}$, this scheme is realized in the diagram in Figure 3.

This completes considering the case when $\gamma = \alpha_i$ and $\delta = \beta_{ji}$ with $j = d$.

In principle, the same case with $j < d$ and the remaining Case (4), when $\gamma = \beta_{ji}$ and $\delta$ acts entirely within $a$ or $b$, or vice versa, are done similarly, following the general idea of Figure 2 except that there are extra dimensions involved, which is something new as compared to classical coherence theorems for monoidal categories. We will present the case which we found most complicated and leave the other cases as an exercise for the reader.

In this case, we will use the diagram in Figure 4 in which $\delta = \beta_{ji} : (a_1 \circ_j a_2) \circ_i (b_1 \circ_j b_2) \to (a_1 \circ_i b_1) \circ_j (a_2 \circ_i b_2)$ with $i < j$ and $\gamma$ acts within $a_1 \circ_j a_2$ as $\gamma = \beta_{kj} \circ_i \text{id} : ((a_{11} \circ_k a_{12}) \circ_j (a_{21} \circ_k a_{22})) \circ_i (b_1 \circ_j b_2) \to ((a_{11} \circ_j a_{21}) \circ_k (a_{12} \circ_j a_{22})) \circ_i (b_1 \circ_j b_2)$ with $j < k < d$. Again, the general scheme may be described by Figure 5.

□

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Figure 3. Verification of the Diamond Condition: $\alpha$ and $\beta$
Figure 4. Verification of the Diamond Condition: Two $\beta$s
Figure 5. General Scheme of Diagram in Figure 4.
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