AN ALGEBRAIC CHARACTERISATION FOR FINSLER METRICS OF
CONSTANT FLAG CURVATURE

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Abstract. In this paper we prove that a Finsler metrics has constant flag curvature if and only
if the curvature of the induced nonlinear connection satisfies an algebraic identity with respect
to some arbitrary second rank tensors. Such algebraic identity appears as an obstruction to the
formal integrability of some operators in Finsler geometry, [4, 7]. This algebraic characterisation
for Finsler metrics of constant flag curvature allows to provide yet another proof for the Finslerian
version of Beltrami’s Theorem, [2, 3].

1. Introduction

Riemannian metrics of constant curvature are well understood and classified. However, one
still can find new characterisations that shed new light on this topic. In this work we consider a
connected manifold, with dim $M > 2$. In [9], the following characterisation is proposed.

A manifold is a constant curvature manifold if and only if

$$b_{im} R^{m}_{i j k} + b_{km} R^{m}_{i j} + b_{jm} R^{m}_{i k i} = 0,$$

for all symmetric tensors $b$.

The geometric motivation of this characterisation is as follows, [8]. If $X$ and $Y$ are two eigenvec-
tors of a symmetric tensor $b$ that satisfies (1.1), then the curvature operator preserves the subspace
generated by the bivector $X \wedge Y$. Since at any point $p \in M$, the sectional curvature, in the direction
of the plane $P = \text{span} \{X, Y\} \subset T_{p} M$, can be viewed as

$$\kappa_{p}(X \wedge Y) = \frac{R_{p}(X, Y, X, Y)}{\|X\|_{p}^{2} \|Y\|_{p}^{2}} = \frac{X \wedge Y, R(X \wedge Y) >_{p}}{\|X \wedge Y\|_{p}^{2}},$$

it follows that for such eigenvectors, $\kappa$ does not depend on $X \wedge Y$. Having enough symmetric
tensors satisfying (1.1), we obtain that $\kappa$ is independent of any 2-plane and using Schur’s lemma
we have that the sectional curvature is constant.

This algebraic characterisation for Riemannian metrics of constant curvature can be used to
provide a very simple and direct proof of Beltrami’s Theorem. If two Riemannian metrics are
projectively equivalent, then their curvature tensors are related by

$$\bar{R}^{h}_{i j k} = R^{h}_{i j k} + \psi_{i j} \delta^{h}_{k} - \psi_{i k} \delta^{h}_{j},$$

for some symmetric tensor $\psi_{i j}$. Since $\psi_{i j} \delta^{h}_{k} - \psi_{i k} \delta^{h}_{j}$ automatically satisfies (1.1) it follows that $\bar{R}$
satisfies (1.1) if and only $R$ satisfies (1.1).

In this paper we provide a similar algebraic characterisation for Finsler metrics of constant flag
curvature. For a manifold $M$, we consider $(T M, \pi, M)$ its tangent bundle and denote by $T_{0} M =$
the tangent space with the zero section removed. We note that the second iterated tangent bundle has two vector bundle structures over \( TM \), \( TTM, \tau, TM \) and \( TTM, D\pi, TM \).

In Finsler geometry, most of the geometric structures live either on \( TM \) or \( T_0M \). The tangent structure (or vertical endomorphism) is the \( (1,1) \)-type tensor field \( J \) on \( TM \) defined as

\[
J(u) = (\tau(u) + tD\pi(u))'(0), \forall u \in TM.
\]

For a Finsler metric \( F \), we denote by \( S \in \mathfrak{X}(T_0M) \) its geodesic spray. We consider the geometric setting induced by \( S \), \( [8] [12] \), with \( h \) the horizontal projector on \( T_0M \) and

\[
R = \frac{1}{2}[h, h],
\]

the curvature (Frölicher-Nijenhuis) tensor of the horizontal distribution.

The vector valued, semi-basic 2-form \( R \) induces an algebraic derivation \( i_R \) of degree 1. We are interested on its action on the space of 2-forms, given by \( i_R : \Lambda^2(T_0M) \to \Lambda^3(T_0M) \),

\[
i_R \omega(X, Y, Z) = \omega(R(X, Y), Z) + \omega(R(Z, X), Y) + \omega(R(Y, Z), X), \forall X, Y, Z \in \mathfrak{X}(T_0M).
\]

The Hessian of the energy of a Finsler metric \( F \) gives a symmetric, second rank positive-definite tensor that can be used to define scalar products on any tensor space on \( T_0M \).

The flag curvature of a Finsler metric \( F \), in the direction of the flagpole \( y \) and the tangent plane \( P = \text{span}\{y, X\} \subset T_yM \), can be defined as

\[
\kappa_{(x, y)}(y \wedge X) = \frac{<y \wedge X, y \wedge R(y \wedge X)>_{(x, y)}}{||y \wedge X||^2_{(x, y)}}.
\]

A Finsler metric has scalar flag curvature if the flag curvature \( \kappa \) does not depend on the flag \( P \) and it has constant flag curvature if the function \( \kappa \) is constant. In view of formula (1.2), we can see that a Finsler metric has scalar (constant) flag curvature if and only if \( y \wedge R(y \wedge X) = \kappa y \wedge X \), for some function (constant) \( \kappa \) and any flag \( P = \text{span}\{y, X\} \).

In Finsler geometry there are various characterisations for metrics of constant flag curvature using some Weyl-type curvature tensors \( [1] [2] [11] \). In this paper we will prove the following algebraic characterisation for Finsler metrics of constant flag curvature.

**Theorem 1.1.** A Finsler metric is of constant flag curvature if and only if

\[
i_R \omega = 0, \quad \forall \omega \in \Lambda^2(T_0M), \text{ satisfying } i_{j\omega} = 0.
\]

As we will see in the proof of Theorem 1.1, the condition (1.3) can be written locally as

\[
b_{km}R^m_{jk} + b_{km}R^m_{ij} + b_{jm}R^m_{ki} = 0, \quad \forall b_{ij} \text{ symmetric.}
\]

The condition (1.3) appears as a first order obstruction to the formal integrability of the projective metrizability problem \( [4] \) Theorem 4.3, and as a second order obstruction to the formal integrability of the Euler-Lagrange operator \( [7] \) §5.2.

We will use the algebraic characterisation from Theorem 1.1 to provide a new proof for the Finslerian version of Beltrami’s theorem studied in \([2] [3]\).

2. **Finsler Metrics and their Curvature Tensors**

In this work, we consider \( M \) a smooth, \( n \)-dimensional, connected manifold, with \( n > 2 \). We denote by \( (TM, \pi, M) \) the tangent bundle, while \( T_0M = TM \setminus \{0\} \) denotes the tangent space with the zero section removed. We will use \((x^i)\) to denote local coordinates on \( M \) and \((x^i, y^i)\) for the induced local coordinates on \( TM \).
The canonical submersion $\pi$ induces a regular, $n$-dimensional, integrable distribution $VTM = \text{Ker}(D\pi)$, which is called the vertical distribution. There is a canonical vertical vector field $C = y^i \partial / \partial y^i$, called the Liouville vector field, or the dilation vector field. The vertical endomorphism has the following local expression $J = dx^i \otimes \partial / \partial y^i$.

In this work, we use the Frölicher-Nijenhuis theory of derivations as it is developed in [7, Ch. 2]. For a vector valued $k$-form $K$, we denote by $i_K$ the $i^*$-derivation of degree $k - 1$, and by $dK$ the $d$-derivation of degree $k$. For two vector valued $k$ and $l$-forms $K$ and $L$, we consider the Frölicher-Nijenhuis bracket $[K, L]$, which is a vector valued $(k + l)$-form.

**Definition 2.1.** A Finsler metric is a continuous, positive function $F : TM \to \mathbb{R}$ that satisfies:

i) $F$ is smooth on $T_0M$;

ii) $F$ is positively homogeneous in the fiber coordinates: $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$;

iii) the Hessian of the energy function:

\[
g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}(x, y)
\]

is non-degenerate.

The homogeneity condition ii) of a Finsler metric implies, using Euler’s Theorem, that

\[
F^2(x, y) = g_{ij}(x, y)y^i y^j, \quad g_{ij}y^j = \frac{1}{2} \frac{\partial F^2}{\partial y^i}.
\]

The regularity condition iii) from the definition of a Finsler metric assures that $ddJF^2$ is a symplectic structure on $T_0M$. Therefore, there is a unique vector field $S \in \mathfrak{X}(T_0M)$, satisfying

\[
i_S ddJF^2 = -dF^2.
\]

$S$ is called the geodesic spray of the Finsler metric, and it is given locally by

\[
S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.
\]

To the geodesic spray we associate the horizontal projector, [6],

\[
h = \frac{1}{2} (\text{Id} - [S, J]) = \left( \frac{\partial}{\partial x^i} - N^i_j(x, y) \frac{\partial}{\partial y^i} \right) \otimes dx^i, \quad N^i_j = \frac{\partial G^i}{\partial y^j}.
\]

The equation (2.2) that uniquely gives the geodesic spray $S$ of a Finsler metric $F$ is equivalent to the following equation:

\[
dh F^2 = 0.
\]

The image of the horizontal projector $h$, $HTM$, is a regular $n$-dimensional distribution that is supplementary to the vertical distribution $VTM$. The obstruction to the integrability of the horizontal distribution is given by the curvature tensor:

\[
R = \frac{1}{2} [h, h] = R^i_{jk}(x, y) \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k.
\]

In this work we will use the following properties of the curvature tensor $R$:

- $R_1$) Vector-valued, semi-basic 2-form: $R(X, Y) = -R(Y, X) = R(hX, hY)$, $\forall X, Y \in \mathfrak{X}(T_0M)$;
- $R_2$) Satisfies first Bianchi identity: $[J, R] = 0$;
- $R_3$) Satisfies second Bianchi identity: $[h, R] = 0$. 


In Finsler geometry, there are some other useful curvature tensors. One is the Jacobi endomorphism, \( \Phi = v \circ [S, h] \), connected to the curvature tensor \( R \) by the following formulae:
\[
\Phi = i_S R, \quad 3R = [J, \Phi].
\]

Similar to the notion of sectional curvature from Riemannian geometry, in Finsler geometry we have the concept of flag curvature. For \((x, y) \in T_0 M\), consider the 2-dimensional plane \( P \subset T_x M\), \( P = \text{span}\{y, X\} \). The flag curvature of the flag \( \{P, y\} \) can be defined as, \([10]\).

\[
(2.5) \quad \kappa_{(x, y)}(y \wedge X) = \frac{X^i g_{ij} R^l_{jk} y^j X^k}{g_{ij} y^j y^k g_{ij} X^i X^j - (g_{ij} y^j X^j)^2} = \frac{< y \wedge X, y \wedge R(y \wedge X) >_{(x, y)}}{||y \wedge X||^2_{(x, y)}}.
\]

We say that a Finsler metric \( F \) has scalar flag curvature (SFC) if the flag curvature \( \kappa = \kappa(x, y) \) does not depend on the flag \( P \), and it has constant flag curvature (CFC) if the flag curvature \( \kappa \) is a constant. In recent years many geometers obtained new characterisations for Finsler metrics of constant flag curvature, \([1\ 2\ 11]\), using some Weyl-type curvature tensors.

### 3. Proof of Theorem 1.1

In this section, we will provide the proof of Theorem 1.1 using the following lemma that characterises Finsler metrics of constant flag curvature.

**Lemma 3.1.** A Finsler metric has constant flag curvature if and only if there exists a semi-basic 1-form \( \xi \) such that
\[
(3.1) \quad R = \xi \wedge J.
\]

**Proof.** Assume that the Finsler metric \( F \) has constant flag curvature \( \kappa \). Then, we can rewrite formula \((2.5)\) as follows:
\[
g_{ij} R^l_{jk} y^j X^i X^k = \kappa \left( F^2 g_{ik} - g_{il} y^l g_{jk} y^j \right) X^i X^k, \ \forall X^i, X^k.
\]

Above formula implies
\[
(3.2) \quad g_{ij} R^l_{jk} y^j = \kappa \left( F^2 g_{ik} - g_{il} y^l g_{jk} y^j \right),
\]

since both sides are symmetric second rank tensors. We denote by \( R^l_{ik} := R^l_{jk} y^j \), the components of the Jacobi endomorphism \( \Phi = i_S R \). In terms of the Jacobi endomorphism, formula \((3.2)\) can be expressed as follows
\[
(3.3) \quad R^l_{ik} = \kappa \left( F^2 \delta^l_k - g_{jk} y^j y^i \right).
\]

Using the fact that one can recover the curvature tensor from the Jacobi endomorphism, \( R = [J, \Phi]/3 \), we get from formula \((3.3)\), the following expression for the curvature tensor:
\[
R^l_{jk} = \frac{1}{3} \left( \frac{\partial R^l_{ik}}{\partial y^j} - \frac{\partial R^l_{ij}}{\partial y^k} \right) = \kappa \left( g_{kj} y^s \delta^l_k - g_{jk} y^s \delta^l_1 \right).
\]

If we consider the semi-basic 1-form \( \xi = \kappa d_j F^2 / 2 = \kappa F d_j F = \kappa g_{1s} y^s dx^4 \), then the curvature tensor \( R \) is given by formula \((3.1)\).

We assume now that for the curvature tensor \( R \) there is a semi-basic 1-form \( \xi \) that satisfies formula \((3.1)\). Using formula \((2.4)\), as well as the form of the curvature tensor, we obtain
\[
0 = d^2 \xi = d_R F^2 = d_\xi J F^2 = \xi \wedge d_j F^2.
\]
From the last formula we obtain that the semi-basic 1-forms \( \xi \) and \( d_JF^2 \) are proportional and hence there exists \( \kappa \in C^\infty(T_0M) \) such that \( \xi = \kappa d_JF^2/2 \). Hence, the curvature tensor is given by

\[
R = \frac{\kappa}{2} d_JF^2 \wedge J.
\]

The curvature tensor satisfies the first Bianchi identity \([J, R] = 0\), which implies \( d_J\xi \wedge J = 0 \). We take the trace of this vector-valued 3-form to obtain \((n-2)d_J\xi = 0\), that gives \( d_J\xi = 0 \), since we made the assumption that \( n > 2 \). From the expression of the curvature 1-form, we obtain that \( d_J\kappa = 0 \).

The curvature tensor also satisfies the second Bianchi identity \([h, R] = 0\), which yields \( d_h\xi \wedge J = 0 \). Again we take the trace to obtain \((n-2)d_h\xi = 0\), which in view of our assumption regarding the dimension, gives \( d_h\xi = 0 \). Using the expression of the curvature 1-form, we obtain that \( d_h\kappa = 0 \).

The two conditions \( d_J\kappa = 0 \) and \( d_h\kappa = 0 \) assures that \( \kappa \) is a constant, \([12, \text{Theorem 9.4.11}]\). For this constant, the curvature tensor \( R \) satisfies formula \((2.5)\) and therefore the Finsler metric has constant sectional curvature \( \kappa \).

One can view Lemma \([3.1]\) as a reformulation, in dimension \( n \geq 3 \), of the Finslerian version of Schur Lemma \([3, \text{Theorem 3.2}]\), the expression \((3.1)\) of the curvature tensor contains two information: the geodesic spray is isotropic and the curvature 1-form \( \xi \) satisfies \( d_J\xi = 0 \).

### 3.1. Proof of Theorem \([1.1]\)

We provide now the proof of Theorem \([1.1]\). The simplest implication is the sufficiency. We assume that the Finsler metric has constant flag curvature. Then, using Lemma \([3.1]\) the curvature tensor is given by \( R = \xi \wedge J \), for some semi-basic 1-form \( \xi \). For any \( \omega \in \Lambda^2(T_0M) \), satisfying \( i_J\omega = 0 \), we have

\[
i_R \omega = i_{\xi \wedge J} \omega = \xi \wedge i_J \omega = 0.
\]

We assume now that for a Finsler metric \( F \), its curvature tensor \( R \) satisfies the identity \((1.3)\), \( i_R \omega = 0 \), \( \forall \omega \in \Lambda^2(T_0M) \) with \( i_J \omega = 0 \).

We first characterise the 2-forms \( \omega \) satisfying the condition \( i_J \omega = 0 \). For arbitrary vector fields \( X, Y \in \mathfrak{X}(T_0M) \), it means

\[
\omega(JX, Y) + \omega(X, JY) = 0.
\]

If \( Y = JZ \) is a vertical vector field, it follows that \( \omega(JX, JZ) = 0 \). Therefore \( \omega \) vanishes on any pair of vertical vector fields (the vertical distribution is a Lagrangian distribution for \( \omega \)). Consider \( \{dx^i, \delta y^i := dy^i + N^i_\ell dx^\ell \} \) a local basis, adapted to the horizontal and vertical distributions. With respect to this basis, a 2-form \( \omega \) satisfying \( i_J \omega = 0 \) can be expressed as

\[
\omega = a_{ij} dx^i \wedge dx^j + b_{ij} dx^i \wedge \delta y^j.
\]

Since \( i_J \omega = (b_{ij} - b_{ji}) dx^i \wedge dx^j \), the condition \( i_J \omega = 0 \) implies \( b_{ij} = b_{ji} \). We have now

\[
i_R \omega = \left( b_{im} R_{jk}^m + b_{km} R_{ij}^m + b_{jm} R_{ki}^m \right) dx^i \wedge dx^j \wedge dx^k.
\]

Hence, the identity \((1.3)\) is satisfied if and only if

\[
(3.4) \quad b_{im} R_{jk}^m + b_{km} R_{ij}^m + b_{jm} R_{ki}^m = 0,
\]

for any symmetric tensor \( b_{ij} \). The identity \((3.4)\) can be written in the following equivalent form

\[
b_{sl} (\delta^s_i R_{jk}^l + \delta^s_j R_{ij}^l + \delta^s_k R_{ki}^l) = 0, \quad \forall b_{sl}.
\]

The above identity implies that the \((2, 3)\) type tensor \( \delta^s_i R_{jk}^l + \delta^s_j R_{ij}^l + \delta^s_k R_{ki}^l \) is skew-symmetric in the 2 contravariant indices. This means

\[
(3.5) \quad \delta^s_i R_{jk}^l + \delta^s_j R_{ij}^l + \delta^s_k R_{ki}^l = 0.
\]
For the $(1,2)$-type curvature tensor $R$, we consider its trace, the semi-basic 1-form $(n-1)\xi_k := R^i_{sk} = -R^i_{ks}$. In (3.8), if we take the trace $i=s$, we obtain:

$$nR^i_{jk} + R^i_{kj} + R^i_{jk} - (n-1)\delta^i_k\xi_j + (n-1)\delta^i_j\xi_k = 0.$$ 

Last formula can be written as

$$R^i_{jk} = \xi^i \delta^j_k - \xi^k \delta^i_j,$$

which means that the curvature tensor $R$ is given by formula (3.1) and in view of Lemma 3.1 we obtain that the Finsler metric has constant flag curvature.

The above proof of the Theorem [1,4] can be used to provide a similar result for an arbitrary semi-basic, vector valued 2-form $K$ on $T_0M$, not necessary the curvature tensor $R$.

**Proposition 3.2.** A semi-basic, vector valued 2-form $K$ on $T_0M$ satisfies the identity

$$i_K\omega = 0, \quad \forall \omega \in \Lambda^2(T_0M), \quad i_J\omega = 0$$

if and only if there exists a semi-basic 1-form $\xi \in \Lambda^1(T_0M)$ such that

$$K = \xi \wedge J.$$ 

The semi-basic 1-form $\xi = \xi_idv^i$ from formula (3.7), if it exists, is unique, being given by

$$(n-1)\xi_i = K^j_{ji} = -K^i_{jji}.$$ 

**3.2. A new proof of the Finslerian version of Beltrami’s Theorem.** Two Finsler metrics $F$ and $\tilde{F}$ are projectively related if their geodesics coincide as unparameterised oriented curves, their geodesic sprays $S$ and $\tilde{S}$ being related by $\tilde{S} = S - 2PC$. The function $F \in C^\infty(T_0M)$ is positively 1-homogeneous in the fiber coordinates and it is called the projective factor.

We will use now the algebraic characterisation for Finsler metrics of constant flag curvature given by Theorem [1,4] to provide another proof of the Finslerian version of Beltrami’s Theorem.

**Theorem 3.3.** Consider $F$ and $\tilde{F}$ two projectively related Finsler metrics and assume that one of them is of constant flag curvature. Then, the other metric is of constant curvature as well if and only if the projective factor $P$ satisfies the Hamel equation $d_\eta d_J P = 0$.

**Proof.** For two projectively related Finsler metrics $F$ and $\tilde{F}$, their curvature tensors $R$ and $\tilde{R}$ are related by [1,4] (4.8)

$$\tilde{R} = R + \eta \wedge J - d_J\eta \otimes C, \quad \eta = Pd_J P - d_\eta P.$$ 

We assume that the Finsler metric $F$ has constant flag curvature and hence we obtain that the curvature tensor $\tilde{R}$ is given by $\tilde{R} = \xi \wedge J$. Then formula (3.8) can be written as follows

$$\tilde{R} = (\xi + \eta) \wedge J - d_J\eta \otimes C.$$ 

Using this form of the curvature tensor $\tilde{R}$ and Theorem [1,4] we obtain that $\tilde{F}$ has constant curvature if and only if $d_J\eta = 0$. Since $d_J\eta = d_\eta d_J P$, we have that $\tilde{F}$ has constant flag curvature if and only if the projective factor $P$ satisfies Hamel’s equation $d_\eta d_J P = 0$. \hfill $\Box$

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