Some new exact solutions for a generalized variable coefficients KdV equation

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Abstract
In this paper, the variable coefficients KdV equation with general power nonlinearities is proposed. Firstly, it is transformed into a generalized KdV equation with constant coefficients using a point transformation. Then, the traveling wave transformation is utilized to transform the obtained constant coefficients generalized KdV equation into a generalized ordinary differential equation. We give a classification for the obtained generalized ordinary differential equation using a suitable integrating factor. Some new solutions are obtained for the generalized KdV equation with constant coefficients. All the obtained solutions in this paper for the variable coefficients KdV equation with general power nonlinearities are new.

Keywords: Exact solutions, generalized KdV equation, traveling wave, solitons.

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where, \( k, n, m \) are constants and \( f(t), g(t), R(t) \) are some smooth functions of the variable \( t \). When \( f(t) = \text{const}, g(t) = \text{const}, R(t) = 1 \), Eq. (1.1) is studied in [20] and some compact solutions, periodic solutions and Jacobi elliptic functions solutions of Eq. (1.1) are obtained when \( k + m = n + 1 \).

In this paper, we first transform Eq. (1.1) into an equation with constant coefficients. Then we investigate the traveling wave solutions of the resultant equation to get exact traveling wave solutions of Eq. (1.1).

This paper has been organized as follows. In Section 2, we transform the variable coefficients KdV equation (1.1) into itself but with constant coefficients. In Section 3, we use the traveling wave transformation to obtain some new solutions of the variable coefficients KdV equation.

2. Transforming Eq. (1.1) into its constant coefficients equation

**Theorem 2.1.** The generalized variable coefficients KdV equation (1.1) can be transformed into the generalized constant coefficients KdV equation

\[
\bar{u}_\bar{t} + a\bar{u}^n\bar{u}_\bar{x} + b\bar{u}^{2n}\bar{u}_\bar{x} + (\bar{u}^k(\bar{u}^m)_{\bar{x}x})_{\bar{x}} = 0, \tag{2.1}
\]

under the transformation

\[
\bar{t} = \int R(t)dt, \bar{x} = x, \quad \bar{u} = u, \tag{2.2}
\]

with the constraints:

\[
f(t) = a \, R(t), \quad g(t) = b \, R(t),
\]

where \( a, b \) are arbitrary constants.

**Proof.** The transformation (2.2) leads to the change of variables

\[
\frac{\partial}{\partial t} = R(t)\frac{\partial}{\partial \bar{t}}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}. \tag{2.3}
\]

Applying the operators (2.3) to the function \( \bar{u}(\bar{x}, \bar{y}, \bar{t}) \), we obtain

\[
\frac{\partial \bar{u}}{\partial \bar{t}} = R(t)\frac{\partial \bar{u}}{\partial \bar{t}} , \quad \frac{\partial \bar{u}}{\partial \bar{x}} = \frac{\partial u}{\partial x} , \quad \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} = \frac{\partial^3 u}{\partial x^3}.
\]

Using (2.2), we can get

\[
u_t = R(t)\frac{\partial \bar{u}}{\partial \bar{t}}, \quad \nu_x = \bar{u}_\bar{x}, \quad \nu_{xx} = \bar{u}_{\bar{x}\bar{x}}. \tag{2.4}
\]

From Eq. (2.4) and Eq. (1.1), we can get

\[
\bar{u}_\bar{t} + \frac{f(t)}{R(t)}\bar{u}^n\bar{u}_\bar{x} + \frac{g(t)}{R(t)}\bar{u}^{2n}\bar{u}_\bar{x} + (\bar{u}^k(\bar{u}^m)_{\bar{x}x})_{\bar{x}} = 0. \tag{2.5}
\]

Equation (2.5) is an equation with constant coefficients, if and only if, \( f(t) = a \, R(t) \) and \( g(t) = b \, R(t) \). Hence, Eq. (2.5) becomes

\[
\bar{u}_\bar{t} + a\bar{u}^n\bar{u}_\bar{x} + b\bar{u}^{2n}\bar{u}_\bar{x} + (\bar{u}^k(\bar{u}^m)_{\bar{x}x})_{\bar{x}} = 0. \tag{2.6}
\]

3. Solutions of the variable coefficients Eq. (1.1)

In this section, we obtain solutions of the variable coefficients Eq. (1.1) by solving the constant coefficients Eq. (2.6).

Consider the transformation given in [1, 6]:

\[
\bar{u}(\bar{x}, \bar{t}) = y(z), \quad z = \alpha \bar{x} - \omega \bar{t}. \tag{3.1}
\]
Accordingly, Eq. (2.6) becomes,

\[-\omega + a \alpha y^n + b \alpha y^{2n} \frac{dy}{dz} + \alpha^3 \frac{d}{dz} \left(y^k \frac{d^2y^m}{dz^2}\right) = 0.\]  
(3.2)

Integrating Eq. (3.2), we get,

\[-\omega y + \frac{a \alpha}{n+1} y^{n+1} + \frac{b \alpha}{2n+1} y^{2n+1} + \alpha^3 y^k \frac{d^2y^m}{dz^2} = c_1,\]  
(3.3)

where \(c_1\) is a constant. Multiplying Eq. (3.3) by \(my^{-k-1}y'\), we obtain

\[-\omega \ m \ y^{m-k}y' + \frac{a \alpha}{n+1} \ m \ y^{m+n-k}y' + \frac{b \alpha}{2n+1} \ m \ y^{m+2n-k}y' + \alpha^3 \ m \ y^{m-1}y' \frac{d^2y^m}{dz^2} = c_1 \ m \ y^{m-k-1}y'.\]

Integrating above equation, we obtain,

\[y'(z)^2 = \frac{2c_1}{\alpha^3 m (m-k)} y^{-k-m+2} + \frac{2c_2}{\alpha^3 m^2} y^{-2m} + \frac{2}{\alpha^3 m} y^{-k-m+3} \times \left(\frac{\omega}{\omega - k + k + 1} - \frac{a \alpha}{(n+1) (-k + m + n + 1)y^n} - \frac{ab}{(2n+1) (-k + m + 2n + 1)y^{2n}}\right),\]  
(3.4)

where \(c_2\) is a constant. Equation (3.4) has the following solutions.

- For \(k = 1 - m, n = m\) and \(c_1 = 0\) and letting

\[y = H(z)\]  
(3.5)

Eq. (3.4) becomes

\[(H')^2 = \frac{2c_2}{\alpha^3} + \frac{\omega}{\alpha^3} H^2 - \frac{2a}{3(1+m)\alpha^3} H^3 - \frac{b}{2(1+2m)\alpha^3} H^4.\]  
(3.6)

Equation (3.6) has many solutions as mentioned in [2, 5, 16, 22]. One of the solutions is

\[H(z) = \frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^2}} \cn \left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha - 4\alpha\gamma^2}} z, \gamma\right),\]  
(3.7)

with

\[c_2 = \frac{-9\omega^3(1+m)^2}{32a^2\alpha^2(1-2\gamma^2)^2}, \ b = \frac{2a^2(1+2m)}{9\omega(1+m)^2},\]

where \(\cn\) is the Jacobi elliptic cosine function.

Substituting Eq. (3.7) into Eq. (3.5), we obtain

\[y = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^2}} \cn \left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha - 4\alpha\gamma^2}} z, \gamma\right)\right)^{\frac{1}{m}}.\]  
(3.8)

Substituting Eq. (3.8) into Eq. (3.1), we obtain

\[\tilde{\eta}(\bar{\tau}, \bar{\eta}) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega\gamma(1+m)}{a\alpha\sqrt{4\gamma^2}} \cn \left(\frac{\sqrt{\omega}}{\alpha\sqrt{2\alpha - 4\alpha\gamma^2}} z, \gamma\right)\right)^{\frac{1}{m}}.\]
Eventually, the solution of Eq. (1.1) can be expressed as

$$u(x,t) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega_1(1+m)}{a\alpha\sqrt{4\gamma^2-2}}\right) \left(\frac{\sqrt{\omega} (\alpha x - \omega \int R(t)dt)}{\alpha\sqrt{2\alpha - 4\gamma^2}}\right)^{\frac{1}{\gamma}}.$$  

(3.9)

When $\gamma = 1$, solution (3.9) degenerates to

$$u(x,t) = \left(\frac{3\omega(1+m)}{2a\alpha} + \frac{3\omega_1(1+m)}{a\alpha\sqrt{2}}\right) \left(\frac{\sqrt{\omega} (\alpha x - \omega \int R(t)dt)}{\alpha\sqrt{-2\alpha}}\right)^{\frac{1}{\gamma}}.$$  

(3.10)

For $a < 0$ and $m > 0$, we obtain bright soliton solutions. For $a > 0$ and $m < 0$, we obtain dark soliton solutions.

Figure 1: Solution (3.10) when $\omega = 1, \alpha = -\frac{9}{2}, m = 2, a = -1$, and different forms of $R(t)$.

Figure 2: Solution (3.10) when $\omega = 1, \alpha = -\frac{3}{2}, a = 1, m = -2$, and different forms of $R(t)$.
Many solutions of Eq. (3.16) are mentioned in [2, 5, 16, 22]. One of them is

\[ G(z) = \frac{a x (\omega + 2 \gamma^2 (1 - 2 \gamma^2) \beta_3^2)}{2 (1 + 2 m) (\omega^2 - \alpha \beta_3 (1 - \gamma^2) \beta_3^2)} + \beta_2 \, \text{cn}^2 (\beta_3 z, \gamma), \]  

with \( \beta_2 = \frac{3 \alpha \gamma^2 \beta_3^2}{2 (1 + 2 m) (\omega^2 - \alpha^6 (1 - \gamma^2 + \gamma^4) \beta_3^4)}, \)
\[ c_2 = \frac{(1 + 2m)(-\omega^2 + \alpha^6(1 - \gamma^2 + \gamma^4)\beta_3^4)}{3\alpha \alpha}, \]
\[ b = \frac{\alpha^2(1 + 4m)\alpha(-\omega^3 + 3\alpha^6(1 - \gamma^2 + \gamma^4)\omega\beta_3^4 + \alpha^9(2 - 3\gamma^2 - 3\gamma^4 + 2\gamma^6)\beta_3^4)}{4(1 + 2m)^2(\omega^2 - \alpha^6(1 - \gamma^2 + \gamma^4)\beta_3^4)^2}, \]

where \( \beta_3 \) is an arbitrary constant. Substituting Eq. (3.17) into Eq. (3.15), we get

\[ y = \left( \frac{\alpha(\omega + \alpha^3(1 - 2\gamma^2)\beta_3^4)}{2(1 + 2m)(\omega^2 - \alpha^6(1 - \gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \operatorname{cn}^2(\beta_3 z, \gamma) \right)^{\frac{1}{6}}. \quad (3.18) \]

From Eq. (3.18) and Eq. (3.1), we can get

\[ \Pi(x, \bar{t}) = \left( \frac{\alpha(\omega + \alpha^3(1 - 2\gamma^2)\beta_3^4)}{2(1 + 2m)(\omega^2 - \alpha^6(1 - \gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \operatorname{cn}^2(\beta_3 (\alpha x - \omega \bar{t}), \gamma) \right)^{\frac{1}{6}}. \]

Using Eq. (2.2), we obtain

\[ u(x, t) = \left( \frac{\alpha(\omega + \alpha^3(1 - 2\gamma^2)\beta_3^4)}{2(1 + 2m)(\omega^2 - \alpha^6(1 - \gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \operatorname{cn}^2(\beta_3 (\alpha x - \omega \int R(t) \, dt), \gamma) \right)^{\frac{1}{6}}. \quad (3.19) \]

When \( \gamma = 1 \), the solution (3.19) degenerates to

\[ u(x, t) = \left( \frac{a \omega + \alpha^3(1 - 2\gamma^2)\beta_3^4}{2(2m + 1)(\alpha^3\beta_3^4 + \omega)} + \frac{3\alpha^4\beta_3^4}{2(1 + 2m)(\omega^2 - \alpha^6\beta_3^4)} \right) \left( \frac{\operatorname{sech}^2(\beta_3 (\alpha x - \omega \int R(t) \, dt))}{\beta_3^4} \right)^{\frac{1}{6}}. \quad (3.20) \]

![Figure 3](image-url)  
(a) \( R(t) = 1 \)

(b) \( R(t) = te^{t^2} \)

Figure 3: Solution (3.20) when \( \beta_3 = \alpha = a = 1, m = \omega = 2, \) and different forms of \( R(t) \).

From Figures 3 and 4, it is clear that we can retrieve bright solitons, bright rogue waves, dark solitons, and dark rogue waves from solution (3.20) according to the values of \( \omega, \beta_3, \alpha, a, m, \) and \( R(t) \). Also, Eq. (3.16) has the following solution

\[ G(z) = \frac{a \omega + \alpha^3(1 + \gamma^2)\beta_3^4}{2(1 + 2m)(\omega^2 - \alpha^6(1 - \gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \operatorname{sn}^2(\beta_3 z, \gamma), \quad (3.21) \]
with

$$
\begin{align*}
\beta_2 &= \frac{3a\alpha^4\gamma^2\beta_3^2}{2(1+2m)(-\omega^2 + \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} , \\
c_2 &= \frac{3a\alpha}{(1+2m)(-\omega^2 + \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} , \\
b &= \frac{\alpha^2(1+4m)\alpha(-\omega^3 + 3\alpha^6(1-\gamma^2 + \gamma^4)\omega\beta_3^4 + \alpha^6(2-3\gamma^2 - 3\gamma^4 + 2\gamma^6)\beta_3^6)}{4(1+2m)^2(\omega^2 - \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} ,
\end{align*}
$$

where $\beta_3$ is an arbitrary constant. Substituting Eq. (3.21) into Eq. (3.15), we get

$$
y = \left( \frac{a\alpha(\omega + \alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2 - \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \text{ sn}^2 (\beta_3 \gamma, \gamma) \right) \left( \frac{1}{2m} \right) . \tag{3.22}
$$

From Eq. (3.22) and Eq. (3.1), we can get

$$
\mathcal{U}(x, t) = \left( \frac{a\alpha(\omega + \alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2 - \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \text{ sn}^2 (\beta_3 (\alpha x - \omega \int R(t) dt), \gamma) \right) \left( \frac{1}{2m} \right).
$$

Using Eq. (2.2), we obtain

$$
u(x, t) = \left( \frac{a\alpha(\omega + \alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2 - \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \text{ tanh}^2 \left( \beta_3 \left( \alpha x - \omega \int R(t) dt \right) \right) \right) \left( \frac{1}{\gamma} \right) . \tag{3.23}
$$

When $\gamma = 1$, the solution (3.23) degenerates to

$$
u(x, t) = \left( \frac{a\alpha(\omega + \alpha^3(1+\gamma^2)\beta_3^2)}{2(1+2m)(\omega^2 - \alpha^6(1-\gamma^2 + \gamma^4)\beta_3^4)} + \beta_2 \text{ tanh}^2 \left( \beta_3 \left( \alpha x - \omega \int R(t) dt \right) \right) \right) \left( \frac{1}{\gamma} \right) . \tag{3.24}
$$
For $k = 1 - m$, $c_1 = c_2 = 0$, Eq. (3.4) becomes

$$\left(\frac{y'}{y}\right)^2 = \frac{-b}{a^2 m (2n + 1) (m + n)} \left( \frac{a (2n + 1) (m + n)}{b (n + 1) (2m + n)} + y^m \right)^2,$$

(3.25)

with the constraint

$$\omega = -\frac{a^2 m (m + n) (1 + 2n) \alpha}{b(1 + n)(2m + n)^2},$$

(3.25) has the solution

$$y = \left( \frac{a (2n + 1) (m + n)}{2b (n + 1) (2m + n)} \right) \tan \left( \frac{an \sqrt{2n + 1} \sqrt{m + n}}{2n (n + 1) \sqrt{-b m (2m + n)}} \right) - \frac{a (2n + 1) (m + n)}{2b (n + 1) (2m + n)} \right) \frac{1}{2}.$$

(3.26)

From Eq. (3.26) and Eq. (3.1), we can get

$$\bar{u}(x, t) = \left( \frac{a (2n + 1) (m + n)}{2b (n + 1) (2m + n)} \right) \tan \left( \frac{an \sqrt{2n + 1} \sqrt{m + n}}{2n (n + 1) \sqrt{-b m (2m + n)}} \right) \left( \alpha x - \omega t \right) - \frac{a (2n + 1) (m + n)}{2b (n + 1) (2m + n)} \right) \frac{1}{2}.$$

(3.27)

Using Eq. (2.2), we obtain

$$u(x, t) = \left( \frac{a (2n + 1) (m + n)}{2b (n + 1) (2m + n)} \right) \tan \left( \frac{an \sqrt{2n + 1} \sqrt{m + n}}{2n (n + 1) \sqrt{-b m (2m + n)}} \right) \left( \int \frac{R(t) dt}{R(t)} \right) - \frac{a (2n + 1) (m + n)}{2b (n + 1) (2m + n)} \right) \frac{1}{2}.$$

(3.27)

Figure 5: Solution (3.27), when $n = 2$, $m = 1$, $b = -1$, $a = 1$ and different forms of $R(t)$.

Figure 5 represents the kink wave solution (3.27).

For $m + k = 2n + 1$, $c_1 = c_2 = 0$, letting

$$y = H(z) \frac{1}{2}$$

(3.28)

and substituting Eq. (3.28) in Eq. (3.4), we get

$$H'(z)^2 = \frac{-bn^2}{a^2 m^2 (2n + 1)} H(z)^2 - \frac{2an^2}{a^2 m (n + 1) (2m - n)} H(z)^6 + \frac{n^2 \omega}{a^3 m (m - n)} H(z)^4.$$

(3.29)
Equation (3.29) can be solved easily to get
\[ H(z) = \frac{c^2 (1 + n) (2m^2 - mn) \alpha^2}{an^2} \frac{1}{1 + B e^{c\alpha z}}, \] (3.30)
when
\[ \omega = -\frac{a^2 n^2 (-m + n)}{c^2 m (1 + n)^2 (-2m + n)^2 \alpha}, \quad \beta = -\frac{c^2 m^2 (1 + 2n) \alpha^2}{n^2}, \]
where B, c are constants. Substituting Eq. (3.30) into Eq. (3.28), we obtain
\[ y = \left( \frac{c^2 (1 + n) (2m^2 - mn) \alpha^2}{an^2} \frac{1}{1 + B e^{c\alpha z}} \right)^{\frac{1}{n}}. \] (3.31)
From Eqs. (3.31) and (3.1), we can get
\[ \pi(x, \tilde{t}) = \left( \frac{c^2 (1 + n) (2m^2 - mn) \alpha^2}{an^2} \frac{1}{1 + B e^{c\alpha x - \omega \tilde{t}}(\alpha x - \omega \tilde{t})} \right)^{\frac{1}{n}}. \]
Using Eq. (2.2), we obtain
\[ u(x, t) = \left( \frac{c^2 (1 + n) (2m^2 - mn) \alpha^2}{an^2} \frac{1}{1 + B e^{c\alpha x - \omega t}} \int R(t) \, dt \right)^{\frac{1}{\alpha}}. \] (3.32)

![Figure 6](image1.png)
(a) R(t) = cos t

![Figure 6](image2.png)
(b) R(t) = tanh t

Figure 6: Solution (3.32) when \( \alpha = B = c = 1, a = -1, m = 2, n = -2 \) and different forms of \( R(t) \).

Figure 6 represents the kink wave solution (3.32).

* For \( c_1 = c_2 = 0 \), Eq. (3.4) becomes
\[ (y')^2 = \frac{-2b}{\alpha^2 m (2n + 1) (-k + m + 2n + 1)} y^{-k-m+3} \left( \frac{\alpha(2n + 1) (-k + m + 2n + 1)}{2b (n + 1) (-k + m + n + 1)} + y^n \right)^2, \] (3.33)
with the constraint
\[ \omega = \frac{a^2(1-k+m)(-1+k-m-2n)(1+2n)\alpha}{4b(1+n)^2(1-k+m+n)^2}. \]
Equation (3.33) has the implicit solution:

\[
c_3 + z = \frac{2\sqrt{2}\sqrt{-bm(1+n)(1-k+m+n)\alpha}}{a(-1+k+m)\sqrt{(1+2n)(1-k+m+2n)}}y^{2(k+m-1)}
\times 2F_1\left(1, \frac{k+m-1}{2n}; \frac{k+m-1}{2n} + 1; \frac{2b(n+1)(-k+m+n+1)y^n}{a(k-m-2n-1)(2n+1)}\right). \tag{3.34}
\]

From Eqs. (3.34) and (3.1), we can get

\[
c_3 + \alpha x - \omega t = \frac{2\sqrt{2}\sqrt{-bm(1+n)(1-k+m+n)\alpha}}{a(-1+k+m)\sqrt{(1+2n)(1-k+m+2n)}}u^{2(k+m-1)}
\times 2F_1\left(1, \frac{k+m-1}{2n}; \frac{k+m-1}{2n} + 1; \frac{2b(n+1)(-k+m+n+1)u^n}{a(k-m-2n-1)(2n+1)}\right).
\]

Using Eq. (2.2), we can retrieve

\[
c_3 + \alpha x - \omega t = \frac{a^2(1-k+m)(-1+k-m-2n)(1+2n)\alpha}{4b(1+n)^2(1-k+m+n)^2}\int R(t) \, dt
= \frac{2\sqrt{2}\sqrt{-bm(1+n)(1-k+m+n)\alpha}}{a(-1+k+m)\sqrt{(1+2n)(1-k+m+2n)}}u^{2(k+m-1)}
\times 2F_1\left(1, \frac{k+m-1}{2n}; \frac{k+m-1}{2n} + 1; \frac{2b(n+1)(-k+m+n+1)u^n}{a(k-m-2n-1)(2n+1)}\right). \tag{3.35}
\]

In case of \(m + k = n + 1\), the implicit solution (3.35) becomes

\[
u(x,t) = \left(\frac{a(2n+1)(-2m-n)}{4bm(n+1)}\tanh^2\left(\frac{\sqrt{an}}{2\sqrt{2am}\sqrt{n+1}}\right)\int R(t) \, dt + c_3 + \alpha x \right)^{\frac{1}{n}}. \tag{3.36}
\]

4. Conclusions

In this paper, using the transformation (2.2), the variable coefficients generalized KdV equation (1.1) is transformed into the constant coefficients KdV equation (2.1). Then we used the traveling wave transformation (3.1) to investigate the exact solutions of the KdV equation (2.1). We have obtained some new solutions for Eq. (2.1) that are not reported in [20]. The new obtained solutions are given by Eq. (3.10) and Eq. (3.14) when \(k = 1 - m\), \(n = m\), Eq. (3.20) and Eq. (3.24) when \(k = 1 - m\), \(n = 2m\), Eq. (3.27), when \(k = 1 - m\), \(c_1 = c_2 = 0\), Eq. (3.32) \(k + m = 2n + 1\), \(c_1 = c_2 = 0\) and Eq. (3.35) for arbitrary \(n\), \(m\) and \(k\). Using the function \(R(t)\) we can get many types of solutions for Eq. (1.1). Some of these types are bright solitons (Figure 1 (a) and Figure 4 (a)), dark solitons (Figure 2 (a) and Figure 3 (a)), bright rogue wave solution (Figure 1 (b) and Figure 4 (b)), dark rogue wave solution (Figure 2 (b) and Figure 3 (b)), and kink solution (Figures 5 and 6). To our knowledge, the obtained solutions in this paper for Eq. (1.1) are new.

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