QUANTUM HELLINGER DISTANCES REVISITED

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ABSTRACT. This short note aims to study quantum Hellinger distances introduced recently by Bhatia et al. [Lett. Math. Phys. (2019), in press, arXiv:1901.01378] with a particular emphasis on barycenters. We consider a quite large family of generalized quantum Hellinger divergences of the form

\[ \phi(A, B) = \text{Tr}((1-c)A + cB - A\sigma B), \]

where \( \sigma \) is an arbitrary Kubo-Ando mean, and \( c \in (0, 1) \) is the weight of \( \sigma \). We note that these divergences satisfy the data processing inequality (DPI), and hence are reasonable measures of dissimilarity from the quantum information theory viewpoint. We derive a characterization of the barycenter of finitely many positive definite matrices for these generalized quantum Hellinger divergences. We also note that in view of our results, the characterization of the barycenter as the weighted multivariate geometric mean, that appeared in the work of Bhatia et al. mentioned above, seems to be incorrect.

1. INTRODUCTION

1.1. Motivation, goals. Given a measure space \((X, \mathcal{A}, \mu)\) and probability measures \(\rho\) and \(\sigma\) that are absolutely continuous with respect to \(\mu\), the classical squared Hellinger distance or Hellinger divergence of \(\rho\) and \(\sigma\) is defined as

\[ d^2_H(\rho, \sigma) = \frac{1}{2} \int_X \left( \left( \frac{d\rho}{d\mu} \right)^{\frac{1}{2}} - \left( \frac{d\sigma}{d\mu} \right)^{\frac{1}{2}} \right)^2 d\mu, \]

where \(d\rho/d\mu\) and \(d\sigma/d\mu\) denote the Radon–Nikodym derivatives \([12]\). The Hellinger divergence is a special Csiszár-Morimoto \(f\)-divergence \([4, 18]\) generated by the convex function \(f(x) = (\sqrt{x} - 1)^2\), and it has several possible counterparts in quantum information theory. One of them is the squared Bures distance or Wasserstein metric, see, e.g., the most recent works of Bhatia et al. \([10]\) and Molnár \([17]\). Another important quantum analogue of the classical Hellinger divergence has been investigated in \([9]\), namely the quantity

\[ d^2_H(A, B) = \text{Tr} \left( \frac{1}{2} (A + B) - A\#B \right), \]

2010 Mathematics Subject Classification. Primary: 47A64. Secondary: 15A24, 81Q10.

Key words and phrases. quantum Hellinger distance, Kubo-Ando mean, weighted multivariate mean, barycenter, data processing inequality, convexity.

We are grateful to Milán Mosonyi and Miklós Pálfia for several discussions on the topic. J. Pitrik was supported by the Hungarian Academy of Sciences Lendület-Momentum grant for Quantum Information Theory, no. 96 141, and by the Hungarian National Research, Development and Innovation Office (NKFIH) via grants no. K119442, no. K124152, and no. KH129601. D. Virosztek was supported by the IST-FELLOW program of the Institute of Science and Technology Austria (project code IC1027FELL01) and partially supported by the Hungarian National Research, Development and Innovation Office (NKFIH) via grants no. K124152, and no. KH129601.

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where $A, B$ are density matrices representing quantum states, or even more generally, positive matrices, and $\#$ is the geometric mean introduced by Pusz and Woronowicz [20], which is a particularly important Kubo-Ando mean [2, 3, 14].

In this note, we consider a far-reaching generalization of the quantum Hellinger divergence [2]. We introduce the family of generalized quantum Hellinger divergences of the form

$$\phi(A, B) = \operatorname{Tr}((1 - c)A + cB - A\sigma B),$$

where $\sigma$ is an arbitrary Kubo-Ando mean, and $c \in (0, 1)$ is the weight of $\sigma$. We will note that these divergences satisfy the data processing inequality (DPI), and hence are reasonable measures of dissimilarity from the quantum information theory viewpoint. Then we derive the equation that characterizes the barycenter of finitely many positive definite matrices for these generalized quantum Hellinger divergences. We will also note that in view of our results, the characterization of the barycenter as the weighted multivariate geometric mean, that appeared in the work of Bhatia et al. [9, Thm. 9] seem to be incorrect.

1.2. Basic notions, notation. Operator monotone functions mapping the positive half-line $(0, \infty)$ into itself admit a transparent integral-representation by Löwner’s theory. In the seminal paper of Kubo and Ando [14], the following integral representation of was considered:

$$f(x) = \int_{[0, \infty]} \frac{x(1 + t)}{x + t} dm(t) \quad (x > 0),$$

where $m$ is some positive Radon measure on the extended half-line $[0, \infty]$. By a simple push-forward of $m$ by the transformation $T : [0, \infty] \to [0, 1]; t \mapsto \lambda := \frac{1}{t+1}$, we get the following integral-representation of positive operator monotone functions on $(0, \infty)$:

$$f_\mu(x) = \int_{[0, 1]} \frac{x}{(1 - \lambda)x + \lambda} d\mu(\lambda) \quad (x > 0),$$

where $\mu = T_\mu m$, that is, $\mu(A) = m(T^{-1}(A))$ for every Borel set $A \subseteq [0, 1]$. This representation is also well-known and appears — among others — in [6] and [21]. Note that if $m$ is absolutely continuous with respect to the Lebesgue measure and $dm(t) = \rho(t) dt$, then the density of $\mu = T_\mu m$ is given by $d\mu(\lambda) = \frac{1}{(1 - \lambda)^2} \rho\left(\frac{1}{1 - \lambda}\right) d\lambda$.

Throughout this note, $\mathcal{H}$ stands for a finite dimensional complex Hilbert space, $\mathcal{B}(\mathcal{H})$ denotes the set of all linear operators on $\mathcal{H}$, and $\mathcal{B}(\mathcal{H})^{sa}$ and $\mathcal{B}(\mathcal{H})^{++}$ stand for the set of all self-adjoint and positive definite operators, respectively. On $\mathcal{B}(\mathcal{H})^{sa}$ we consider the usual Löwner order induced by positivity. The Fréchet derivative of a map $\psi : \mathcal{B}(\mathcal{H})^{sa} \supseteq \mathcal{U} \to \mathcal{V}$ at the point $X \in \mathcal{U}$ is denoted by $D\psi(X)[\cdot]$. Here, $\mathcal{U}$ is an open subset of $\mathcal{B}(\mathcal{H})^{sa}$, usually the cone of positive definite operators, and the target space $\mathcal{V}$ is usually $\mathbb{R}$ or $\mathcal{B}(\mathcal{H})^{sa}$. Note that in the latter case $D\psi(X)[\cdot]$ is a linear map from $\mathcal{B}(\mathcal{H})^{sa}$ into itself. The symbol $I$ denotes the identity operator on $\mathcal{H}$.

For positive definite operators $A, B \in \mathcal{B}(\mathcal{H})^{++}$, the Kubo-Ando connection generated by the operator monotone function $f_\mu : (0, \infty) \to (0, \infty)$ is denoted by $A\sigma_{f_\mu}B$, and is defined by

$$A\sigma_{f_\mu}B = A^{\frac{1}{2}} f_\mu (A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$
A Kudo-Ando connection \( \sigma_{f_\mu} \) is a mean if and only if \( f(1) = \mu([0,1]) = 1 \). In the sequel, we will restrict our attention to means. We denote by \( \mathcal{P}([0,1]) \) the set of all Borel probability measures on \([0,1]\), and by \( c(\mu) := \int_{[0,1]} \lambda d\mu(\lambda) \) the center of mass of \( \mu \). There is a natural way to assign a weight parameter to a mean \( \sigma_{f_\mu} \), namely, \( W(\sigma_{f_\mu}) := f'(1) = c(\mu) \). More details about this weight parameter can be found in [21]. We only mention that for the weighted arithmetic, geometric, and harmonic means generated by \( c(\mu) \), we have \( W(\sigma_{a_\lambda}) = W(\sigma_{g_\lambda}) = W(\sigma_{h_1}) = \lambda \). That is, this weight parameter coincides with the usual one in the most important special cases.

1.3. **Convex order.** The convex order is a well-known relation between probability measures; for \( \mu, \nu \in \mathcal{P}([0,1]) \), we say that \( \mu \preceq \nu \) if for all convex functions \( u : [0,1] \rightarrow \mathbb{R} \) we have \( \int_{[0,1]} u d\mu \leq \int_{[0,1]} u d\nu \). It is clear that for all \( \mu \in \mathcal{P}([0,1]) \) with \( c(\mu) = \lambda \) we have \( A \preceq \mu \preceq (1-\lambda)\delta_0 + \lambda\delta_1 \), where \( \delta_x \) denotes the Dirac mass concentrated on \( x \). For any fixed \( x > 0 \), the map \( \lambda \mapsto \frac{x}{1-x+x} \) is convex. Therefore, if \( \mu \preceq \nu \), then \( f_\mu(x) \leq f_\nu(x) \) for all \( x > 0 \), and hence \( A \sigma_{f_\mu} B \preceq A \sigma_{f_\nu} B \) for all \( A, B \in \mathcal{B}(\mathcal{H})^+ \). Consequently, if \( \nu = (1 - c(\mu))\delta_0 + c(\mu)\delta_1 \), then \( A \sigma_{f_\mu} B \preceq A \sigma_{f_\nu} B \) is always positive, in particular, \( \text{Tr} \left( A \sigma_{f_\mu} B - A \sigma_{f_\nu} B \right) \geq 0 \). This quantity is exactly the one we are interested in.

2. **Basic properties of quantum Hellinger distances**

We are interested in divergences of the form

\[
\phi_\mu(A, B) := \text{Tr} \left( (1 - c(\mu)) A + c(\mu) B - A \sigma_{f_\mu} B \right) \quad (A, B \in \mathcal{B}(\mathcal{H})^+) \tag{7}
\]

where \( \mu \in \mathcal{P}([0,1]) \). To avoid trivialities, we assume in the sequel that the support of \( \mu \) is strictly larger than \([0,1]\), and therefore, \( f_\mu \) is non-affine—in fact, it is strictly concave.

If \( \mu \) is the arcsine distribution, that is, \( d\mu(\lambda) = \frac{1}{\pi \sqrt{\lambda(1-\lambda)}} d\lambda \), then

\[
\phi_\mu(A, B) = \text{Tr} \left( \frac{1}{2} (A + B) - A \# B \right),
\]

where \# is the Pusz-Woronowicz geometric mean [20]. The square root of this quantity (up to an irrelevant multiplicative constant) was considered in [9] as a possible quantum (or matrix) version of the classical Hellinger distance. Therefore, we will call the quantities of the form \( \phi_\mu \) generalized quantum Hellinger divergences.

We easily get that

\[
\phi_\mu(A, B) = \text{Tr} \left\{ A \cdot g_\mu \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) \right\}, \tag{8}
\]

where \( g_\mu : (0, \infty) \rightarrow [0, \infty) \) is defined by

\[
g_\mu(x) = (1 - c(\mu)) + c(\mu) x - f_\mu(x). \tag{9}
\]

Note that \( g_\mu \) is operator convex as \( f_\mu \) is operator concave. Now we check that \( \phi_\mu \) defined in eq. (7) is a divergence in the sense of [11] Sec. 1.2 & 1.3).

**Proposition 1.** For any \( \mu \in \mathcal{P}[0,1] \), the map

\[
\phi_\mu : \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ \rightarrow [0, \infty); (A, B) \mapsto \phi_\mu(A, B) \tag{10}
\]

satisfies the followings.

(i) \( \phi_\mu(A, B) \geq 0 \) and \( \phi_\mu(A, B) = 0 \) if and only if \( A = B \).
(ii) The first derivative of \( \Phi_\mu \) in the second variable vanishes at the diagonal, that is, 
\[
\mathbf{D} \left( \phi_\mu (A, \cdot) \right) (A) = 0 \in \mathrm{Lin} (\mathcal{B}(\mathcal{H})^{sa}, \mathbb{R}) \text{ for all } A \in \mathcal{B}(\mathcal{H})^{+}. 
\]

(iii) The second derivative of \( \Phi_\mu \) in the second variable is positive at the diagonal, that is, 
\[
\mathbf{D}^2 \left( \phi_\mu (A, \cdot) \right) (A)[Y, Y] \geq 0 \text{ for all } Y \in \mathcal{B}(\mathcal{H})^{sa}. 
\]

**Proof.** We check the above mentioned properties step-by-step.

(i) It is clear that \( g_\mu \geq 0 \), hence by recalling that \( \phi_\mu (A, B) \) can be expressed as the right hand side of (8), we get that \( \phi_\mu (A, B) \geq 0 \). Furthermore, \( f_\mu \) is strictly concave, hence \( g_\mu (x) = 0 \) if and only if \( x = 1 \). Therefore, as \( A \) is strictly positive, we deduce that \( \phi_\mu (A, B) = 0 \) if and only if \( A = B \).

(ii) \( g_\mu \) is analytic, and hence can be written in the form
\[
g_\mu (1 + t) = \sum_{k=0}^{\infty} \frac{g_\mu^{(k)}(1)}{k!} t^k 
\]
in some neighbourhood of 1. By (9), \( g_\mu (1) = 0 \) and \( g_\mu '(1) = 0 \), hence
\[
g_\mu (1 + t) = O\left( t^2 \right). 
\]

For any \( Y \in \mathcal{B}(\mathcal{H})^{sa} \) we have
\[
\phi_\mu (A, A + Y) - \phi_\mu (A, A) = \mathrm{Tr} \left\{ A \cdot g_\mu \left( A^{-\frac{1}{2}} (A + Y) A^{-\frac{1}{2}} \right) \right\} 
\]
\[
= \mathrm{Tr} \left\{ A \cdot g_\mu \left( I + A^{-\frac{1}{2}} Y A^{-\frac{1}{2}} \right) \right\} = \mathrm{Tr} \left\{ A \cdot O \left( \left( A^{-\frac{1}{2}} Y A^{-\frac{1}{2}} \right)^2 \right) \right\} = O\left( Y^2 \right), 
\]
which shows that \( \mathbf{D} \left( \phi_\mu (A, \cdot) \right) (A) = 0 \).

(iii) For any \( Y \in \mathcal{B}(\mathcal{H})^{sa} \),
\[
\mathbf{D}^2 \left( \phi_\mu (A, \cdot) \right) (A)[Y, Y] = \frac{d^2}{dt^2} \left( \phi_\mu (A, A + tY) \right)_{t=0} 
\]
\[
= \frac{d^2}{dt^2} \left( \mathrm{Tr} \left\{ A \cdot g_\mu \left( A^{-\frac{1}{2}} (A + tY) A^{-\frac{1}{2}} \right) \right\} \right)_{t=0} = \mathrm{Tr} \left\{ A \cdot \frac{d^2}{dt^2} \left( g_\mu \left( A^{-\frac{1}{2}} (A + tY) A^{-\frac{1}{2}} \right) \right) \right\} = 0. 
\]
because the operator convexity of \( g_\mu \) implies that
\[
\frac{d^2}{dt^2} \left( g_\mu \left( I + t A^{-\frac{1}{2}} Y A^{-\frac{1}{2}} \right) \right)_{t=0} \geq 0. 
\]

\( \square \)

In fact, generalized quantum Hellinger divergences are in an intimate relation with operator valued Bregman divergences. Note that \( h_\mu := -f_\mu \) is an operator convex function, and that
\[
g_\mu (x) = \left( 1 - c \left( \mu \right) \right) + c \left( \mu \right) x + h_\mu (x) = h_\mu (x) - h_\mu (1) - h_\mu ' (1) (x - 1). 
\]
The operator valued Bregman divergence generated by the operator convex function \( h_\mu \) reads as follows:
\[
H^{(op)}_{h_\mu} (X, Y) = h_\mu (X) - h_\mu (Y) - \mathbf{D}h_\mu (Y) [X - Y]. 
\]
In particular,
\[
H^{(op)}_{h_\mu} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, I \right) = h_\mu \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) - h_\mu (I) - \mathbf{D}h_\mu (I) \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} - I \right). 
\]
As \( D_{h_\mu}(I) \) coincides with the multiplication by the constant \(-c(\mu)\), and \( h'_\mu(I) = -c(\mu) I \), we get that

\[
H^{(op)}_{h_\mu} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, I \right) = g_\mu(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}).
\]

Therefore, we obtain the following claim.

**Claim 2.** The generalized Hellinger divergence \( \phi_\mu \) defined in (7) can be expressed by an operator valued Bregman divergence as follows:

\[
\phi_\mu(A, B) = \text{Tr} \left\{ A \cdot H^{(op)}_{h_\mu} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}}, I \right) \right\} \quad (A, B \in \mathcal{B}(\mathcal{H})^{sa}).
\]

For a detailed study of Bregman divergences on matrices we refer to [19].

### 3. Data Processing Inequality, Convexity

It is well-known that Kubo-Ando means are jointly concave on \( \mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H})^{++} \), that is,

\[
((1 - t)A_1 + t A_2) \sigma_{f_\mu} ((1 - t)B_1 + t B_2) \geq (1 - t)A_1 \sigma_{f_\mu} B_1 + t A_2 \sigma_{f_\mu} B_2
\]

holds for all Kubo-Ando mean \( \sigma_{f_\mu} \) and for all \( A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})^{++}, t \in [0, 1] \). Consequently, the quantum Hellinger divergences are jointly convex.

**Proposition 3.** The generalized quantum Hellinger divergence \( \phi_\mu \) defined in (7) is jointly convex on \( \mathcal{B}(\mathcal{H})^{++} \times \mathcal{B}(\mathcal{H})^{++} \).

**Proof.** By the joint concavity of the Kubo-Ando means, we have

\[
\phi_\mu((1 - t)A_1 + t A_2, (1 - t)B_1 + t B_2)
\]

\[
= \text{Tr} \left\{ (1 - c(\mu)) ((1 - t)A_1 + t A_2) + c(\mu) ((1 - t)B_1 + t B_2) - ((1 - t)A_1 + t A_2) \sigma_{f_\mu} ((1 - t)B_1 + t B_2) \right\}
\]

\[
\leq (1 - t) \text{Tr} \left\{ (1 - c(\mu)) A_1 + c(\mu) B_1 + t \text{Tr} \left((1 - c(\mu)) A_2 + c(\mu) B_2\right) - \right\}
\]

\[
- (1 - t) \text{Tr} \left((1 - t)A_1 \sigma_{f_\mu} B_1\right) - t \text{Tr} \left((1 - t)A_2 \sigma_{f_\mu} B_2\right)
\]

for all Kubo-Ando mean \( \sigma_{f_\mu} \) and for all \( A_1, A_2, B_1, B_2 \in \mathcal{B}(\mathcal{H})^{++}, t \in [0, 1] \). \( \Box \)

It is also clear that \( \phi_\mu(\cdot, \cdot) \) is positive homogeneous, that is,

\[
\phi_\mu(rA, rB) = r \phi_\mu(A, B) \quad (r > 0, A, B \in \mathcal{B}(\mathcal{H})^{++}).
\]

It is known that for homogeneous divergences, the joint convexity and the monotonicity under quantum channels (or data processing inequality) is equivalent, see, e.g., [15] remarks after Def. 2.3. So we obtained the following claim.

**Claim 4.** Let \( T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \) be a quantum channel, that is, a completely positive and trace preserving (CPTP) map. Let \( \mu \in \mathcal{P}[0, 1] \) be arbitrary. Then

\[
\phi_\mu(T(A), T(B)) \leq \phi_\mu(A, B)
\]

holds for every \( A, B \in \mathcal{B}(\mathcal{H})^{++} \).
4. Barycenters

The notion of barycenter (or least squares mean) plays a central role in averaging procedures related to various topics in mathematics and mathematical physics. Given a metric space \((X, \rho)\) and an \(m\)-tuple \(a_1, \ldots, a_m\) in \(X\) with positive weights \(w_1, \ldots, w_m\) such that \(\sum_j w_j = 1\), the barycenter (or Fréchet mean or Karcher mean or Cartan mean) is defined to be

\[
\arg\min_{x \in X} \sum_{j=1}^m w_j \rho^2(a_j, x).
\]

In our setting, \(X = \mathcal{B}(\mathcal{H})^++\), and the generalised quantum Hellinger divergence \(\phi_\mu\) plays the role of the squared distance \(\rho^2\), although it is not the square of any true metric in general.

That is, we consider the optimization problem

\[
\arg\min_{X \in \mathcal{B}(\mathcal{H})^++} \sum_{j=1}^m w_j \phi_\mu(A_j, X),
\]

where the positive definite operators \(A_1, \ldots, A_m\) and the weights \(w_1, \ldots, w_m\) are fixed. By the strict concavity of \(f_\mu\), the function

\[
X \mapsto \phi_\mu(A, X) = \text{Tr}\left(\left(1 - c(\mu)\right)A + c(\mu)X - A^{\frac{1}{2}} \mu \left(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}\right)
\]

is strictly convex on \(\mathcal{B}(\mathcal{H})^++\), see, e.g., [11] 2.10. Thm.]. Therefore, there is a unique solution \(X_0\) of (16), and it is necessarily a critical point of the function \(X \mapsto \sum_{j=1}^m w_j \phi_\mu(A_j, X)\).

That is, it satisfies

\[
\mathbf{D}\left(\sum_{j=1}^m w_j \phi_\mu(A_j, \cdot)\right)(X_0)[Y] = 0 \quad (Y \in \mathcal{B}(\mathcal{H})^{sa}).
\]

Easy computations give that

\[
\mathbf{D}\left(\sum_{j=1}^m w_j \phi_\mu(A_j, \cdot)\right)(X)[Y] = c(\mu) \text{Tr} Y - \sum_{j=1}^m w_j \text{Tr} \mathbf{D}_{\mu, A_j}(X)[Y],
\]

where for a positive definite operator \(A\), the map \(F_{\mu, A} : \mathcal{B}(\mathcal{H})^+ \to \mathcal{B}(\mathcal{H})^+\) is defined by

\[
F_{\mu, A}(X) := A^{\frac{1}{2}} \mu \left(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.
\]

By differentiating (5), we have

\[
\mathbf{D}f_\mu(X)[Y] = \int_{[0,1]} \lambda ((1 - \lambda)X + \lambda I)^{-1}Y ((1 - \lambda)X + \lambda I)^{-1} d\mu(\lambda)
\]

for \(X \in \mathcal{B}(\mathcal{H})^+, Y \in \mathcal{B}(\mathcal{H})^{sa}\). Consequently,

\[
\mathbf{D}F_{\mu, A_j}(X)[Y] = \int_{[0,1]} \lambda A^{\frac{1}{2}} \left((1 - \lambda)A_j^{-\frac{1}{2}}XA_j^{-\frac{1}{2}} + \lambda I\right)^{-1} A^{\frac{1}{2}} Y A^{\frac{1}{2}} \left((1 - \lambda)A_j^{-\frac{1}{2}}XA_j^{-\frac{1}{2}} + \lambda I\right)^{-1} A^{\frac{1}{2}} d\mu(\lambda)
\]

\[
= \int_{[0,1]} \lambda \left((1 - \lambda)X + \lambda I\right)^{-1} Y \left((1 - \lambda)X^{-1} + \lambda I\right)^{-1} d\mu(\lambda).
\]
By the linearity and the cyclic property of the trace, we get from (18) and (21) that (17) is equivalent to
\[
\text{(22) } \text{Tr} \left[ Y \left( c(\mu) I - \sum_{j=1}^{m} w_j \int_{[0,1]} \lambda \left| (1 - \lambda) A_j^{-1} X + \lambda I \right|^{-2} d\mu(\lambda) \right) \right] = 0 \quad (Y \in \mathcal{B}(\mathcal{H})^{sa}),
\]
where \(|\cdot|\) stands for the absolute value of an operator, that is, \(|Z| = (Z^* Z)^{1/2}\). This latter equation amounts to
\[
\text{(23) } c(\mu) I = \sum_{j=1}^{m} w_j \int_{[0,1]} \lambda \left| (1 - \lambda) A_j^{-1} X + \lambda I \right|^{-2} d\mu(\lambda).
\]
So we obtained the following characterization of the barycenter.

**Theorem 5.** Let \(\mu \in \mathcal{P}[0,1]\) and let \(\phi_\mu\) be the generalized quantum Hellinger divergence generated by \(\mu\), that is,
\[
\phi_\mu(A,B) = \text{Tr} \left[ (1 - c(\mu)) A + c(\mu) B - A\sigma_f B \right] \quad (A,B \in \mathcal{B}(\mathcal{H})^{++}).
\]
Then the barycenter (or Cartan mean or Fréchet mean or Karcher mean) of \(A_1, \ldots, A_m\) with weights \(w_1, \ldots, w_m\) with respect to \(\phi_\mu\), i.e.,
\[
\arg\min_{X \in \mathcal{B}(\mathcal{H})^{++}} \sum_{j=1}^{m} w_j \phi_\mu(A_j, X)
\]
coincides with the unique positive definite solution of the matrix equation
\[
\text{(24) } c(\mu) I = \sum_{j=1}^{m} w_j \int_{[0,1]} \lambda \left| (1 - \lambda) A_j^{-1} X + \lambda I \right|^{-2} d\mu(\lambda).
\]

5. Remarks

In view of our results, Theorem 9 in [9] seems to be incorrect. By the particular choice of the arcsine distribution, \(d\mu(\lambda) = \frac{1}{\pi \sqrt{4(1-\lambda)}} d\lambda\), we do not recover the equation
\[
\text{(25) } X = \sum_{j=1}^{m} w_j A_j \# X
\]
which defines the \((w_1, \ldots, w_m)\)-weighted power mean of order \(\frac{1}{2}\) of \(A_1, \ldots, A_m\), (see [16, Def. 3.2]) and which is claimed to characterize the barycenter for \(\phi_\mu\) in [9, Thm. 9].

To demonstrate the difference between the barycenter and the weighted power mean defined by Lim and Pálfi [16], we take the following example. Set \(m = 2, w_1 = w_2 = \frac{1}{2}\), and
\[
A_1 := \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 := 4 \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + 1 \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{bmatrix}.
\]
Then numerical optimization performed by Wolfram Mathematica [22] shows that
\[
\hat{X}_0 := \arg\min_{X \in \mathcal{B}(\mathcal{H})^{++}} \frac{1}{2} \sum_{j=1}^{2} \frac{1}{2} \phi_\mu(A_j, X)
\]
\[
= \arg\min_{X \in \mathcal{B}(\mathcal{H})^{++}} \frac{1}{2} \text{Tr} \left\{ \frac{1}{2} (A_1 + A_2) + X - (A_1 \# X + A_2 \# X) \right\} = \begin{bmatrix} 2.99035 & 0.634419 \\ 0.634419 & 1.72151 \end{bmatrix}.
\]
Note that both \(A_1\) and \(A_2\) have real entries. Therefore, \(A_j \# X = A_j \# \overline{X}\), and hence \(\phi_\mu(A_j, X) = \phi_\mu(A_j, \overline{X})\) holds for every \(X \in \mathcal{B}(\mathcal{H})^{++}\) and \(j \in \{1, 2\}\), where \(\overline{X}\) denotes
the entrywise complex conjugate of \( X \). Consequently, the strict convexity of the functions \( X \mapsto \phi_\mu(A_j,X), j \in \{1,2\} \) implies that \( \arg\min_{X \in B(H)^{++}} \sum_{j=1}^2 \frac{1}{2} \phi_\mu(A_j,X) \) has real entries. So it is enough to minimize numerically over the cone of positive definite \( 2 \times 2 \) matrices with real entries \([22]\).

However, the barycenter obtained numerically in \([26]\) does not coincide with the weighted power mean of order 1/2 as

\[
\frac{1}{2} \left( A_1\# \hat{X}_0 + A_2\# \hat{X}_0 \right) = \begin{bmatrix} 3.02915 & 0.673215 \\ 0.673215 & 1.68272 \end{bmatrix} \neq \hat{X}_0.
\]

In our view, the proof of \([9, \text{Thm } 9.]\) contains a gap, namely — using their notation from now on — the fact that \( I \) is a critical point for \( g \) does not imply that \( X_0 \) is a critical point for \( f \), although formula (54) in \([9]\) is correct.

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