Some symbolic dynamics in real quadratic fields with applications to inhomogeneous minima

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Abstract

Let $K$ be a real quadratic field. We use a symbolic coding of the action of a fundamental unit on the torus $T_K = (K \otimes \mathbb{Q})/\mathcal{O}_K$ to study the family of subsets $X_t \subseteq T_K$ of norm distance $\geq t$ from the origin. As an application, we prove that inhomogeneous spectrum of $K$ contains a dense set of elements of $K$, and conclude that all isolated inhomogeneous minima lie in $K$.

1 Introduction

Let $D > 1$ be a square-free positive integer and let $K = \mathbb{Q}(\sqrt{D})$ be the associated real quadratic field with ring of integers $\mathcal{O}_K$. Let $N : K \to \mathbb{Q}$ denote the absolute norm $N(a) = |Nm_{K/{\mathbb{Q}}}(a)| = |a\pi|$, where $a \mapsto \pi$ is Galois conjugation, and recall that the ring $\mathcal{O}_K$ is called norm-Euclidean if for all $a \in K$ there exists $q \in \mathcal{O}_K$ such that $N(a - q) < 1$. The ring of integers $\mathcal{O}_K$ embeds as a lattice in the two-dimensional real vector space $V_K = K \otimes \mathbb{R}$, and we denote the quotient torus by $T_K = V_K/\mathcal{O}_K$. Galois conjugation extends linearly to $V_K$, and the absolute norm extends accordingly to an indefinite quadratic form on $V_K$ that we also denote by $N$. The norm is not $\mathcal{O}_K$-invariant, but the function defined by

$$M(P) = \inf_{Q \in \mathcal{O}_K} N(P - Q)$$

is, and descends to a function on the torus $T_K$ which we also denote by $M$. The function $M$ is upper-semicontinuous ([2], Theorem F).

The Euclidean minimum of $K$ is defined by $M_1(K) = \sup_{P \in K} M(P)$. In particular, $M_1(K) < 1$ implies that $\mathcal{O}_K$ is norm-Euclidean, while $M_1(K) > 1$ implies that it is not. The second Euclidean minimum is defined by

$$M_2(K) = \sup_{\substack{P \in K \\text{\colon} \ M(P) < M_1(K)}} M(P)$$

and $M_1(K)$ is said to be isolated if $M_2(K) < M_1(K)$. We may proceed in this fashion producing Euclidean minima $M_i(K)$ until we find a non-isolated one. Note that upper-semicontinuity ensures that each of these suprema is actually achieved by some collection of points on the torus. The points $P$ in the above suprema are constrained to $K$, but we may remove that restriction and define the inhomogeneous minimum of $K$. 

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by \( M_1(K) = \sup_{P \in \mathbb{T}_K} M(P) \) and proceed as above to define the inhomogeneous minima \( M_i(K) \). The \textit{inhomogeneous spectrum} of \( K \) is simply the image \( M(\mathbb{T}_K) \), and the \textit{Euclidean spectrum} of \( K \) is its subset \( M(K) \).

The inhomogeneous minima demonstrate a variety of behavior, in some cases producing an infinite sequence of isolated minima while in others we find that \( M_2(K) \) already fails to be isolated - see \cite{11} for an overview of results. Barnes and Swinnerton-Dyer proved in \cite{13} that \( M_1(K) = M_1(K) \) and conjectured that \( M_1(K) \) is always isolated and rational, and that \( M_2(K) \) is taken at a point with coordinates in \( K \). Numerous computations by other authors (e.g. \cite{6}, \cite{7}, \cite{10}, \cite{8}, \cite{9}, \cite{15}) suggest further that all inhomogeneous minima lie in \( K \).

Much is known about these minima and spectra in higher degree when the unit group furnishes more automorphisms. Cerri showed in \cite{5} that if \( K \) has unit rank at least two, then \( M_1(K) \) is taken at rational point, and hence rational. He showed further that if such a \( K \) is not CM, then \( M_1(K) \) is attained and isolated, and the Euclidean and inhomogeneous spectra coincide and consist of a sequence of rational numbers converging to 0. Building on Cerri’s work, Shapira and Wang proved in \cite{13} that if \( K \) has unit rank at least three then \( M_1(K) \) is isolated and attained.

Returning to real quadratic \( K \), the following is our main result.

\textbf{Theorem 1.} The set of \( M(\mathbb{T}_K) \cap K \) is dense in the inhomogeneous spectrum \( M(\mathbb{T}_K) \).

This theorem is proven by introducing the intermediate collection of points \( K/\mathcal{O}_K \subseteq \widetilde{K}/\mathcal{O}_K \subseteq \mathbb{T}_K \), whose coordinates belong to \( K \), establishing that the minima of such points lie in \( K \), and finally proving that the associated spectrum \( M(\widetilde{K}) \) is dense in the inhomogeneous spectrum. In the isolated case, this establishes the following extension of the conjecture of Barnes and Swinnerton-Dyer above.

\textbf{Corollary 1.} If \( M_i(K) \) is isolated, then it is taken at point with coefficients in \( K \) and we have \( M_i(K) \in K \).

The method of proof also demonstrates that \( M_1(K) = M_1(K) \) is rational if it is isolated, but this was known already to Barnes and Swinnerton-Dyer (\cite{3}, Theorem M).

2 \hspace{1em} \textbf{The dynamical systems} \( X_t \)

By Dirichlet’s unit theorem, we have \( \mathcal{O}_K^\times = \pm \mathbb{Z} \) for some fundamental unit \( \varepsilon \) of infinite order. We will later fix an embedding of \( K \) into \( \mathbb{R} \) and assume that \( \varepsilon \) is chosen so that \( \varepsilon > 1 \). Multiplication by \( \varepsilon \) is absolute norm-preserving and extends by linearity to an endomorphism \( \phi \) of \( V_K \) that is also absolute norm-preserving. Since \( \phi \) preserves the lattice \( \mathcal{O}_K \), it descends to an endomorphism of the torus \( \mathbb{T}_K \) with the property that \( M(\phi(P)) = M(P) \) for all \( P \in \mathbb{T}_K \). The eigenvalues of \( \phi \) are the embeddings of \( \varepsilon \) into \( \mathbb{R} \) and hence not roots of unity, so \( \phi \) is an ergodic transformation of \( \mathbb{T}_K \). This dynamical system, and a symbolic coding of it obtained from a Markov partition of the torus, are our main resource. Note that the subset \( K/\mathcal{O}_K \), which coincides with the set of periodic points for \( \phi \), is traditionally referred to as the \textit{rational points} since they have rational \((x,y)\) coordinates (see Section 3).

For \( t > 0 \), the \( \phi \)-invariant set \( X_t = \{ P \in \mathbb{T}_K \mid M(P) \geq t \} \) is closed by upper semicontinuity. We can describe \( X_t \) alternatively by first noting that the open set

\[ \mathcal{U}(t) = \bigcup_{Q \in \mathcal{O}_K} \{ P \in V_K \mid N(P - Q) < t \} \]

is translation-invariant and descends to an open subset of \( \mathbb{T}_K \), and then observing that \( X_t \) is its complement. The sets \( X_t \) have Lebesgue measure zero for \( t > 0 \) since they are proper, closed, and \( \phi \)-invariant. It is natural to ask how the Hausdorff dimension \( \dim(X_t) \) varies with \( t \). That \( \dim(X_t) \to 2 \) as \( t \to 0 \) is a simple consequence of Theorem 2.3 of \cite{1}. We prove in Corollary 2 that \( \dim(X_t) \) is left-continuous everywhere. Right-continuity remains an open question.

We illustrate in the case \( K = \mathbb{Q}((\sqrt{5}) \). Davenport computed the Euclidean minima for this field in \cite{6} and \cite{7}, finding the infinite decreasing sequence of minima \( M_1 = 1/4 \), and for \( i \geq 1 \),

\[ M_{i+1} = \frac{F_{6i-2} + F_{6i-4}}{4(F_{6i-1} + F_{6i-3} - 2)} \]
where $F_k$ denotes the $k$th Fibonacci number. Each of these minima is obtained at a finite collection of elements of $K/O_K$, and we have $M_t \rightarrow t_\infty = (-1 + \sqrt{5})/8 \approx .1545$. A plot of $\dim(X_t)$ in this case is given below. The zero-dimensional region necessarily covers $t > t_\infty$, since the collection of points giving rise to the Euclidean minima is countable. We prove in [12] that $\dim(X_t) > 0$ for all $t < t_\infty$, while $\dim(X_{t_\infty}) = 0$. In particular, $\dim(X_t)$ is continuous at $t_\infty$.

The evident plateaus on this graph and its detail in Figure 1 have dynamical significance. The dimensions plotted here are actually upper bounds obtained by symbolically coding the torus dynamical system with a Markov partition and finding subshifts of finite type (SFTs) that contain the coding of $X_t$, as in Section 5. As we will make precise in Proposition 3, a plateau will occur wherever it is possible to make such a bound tight and $X_t$ can be described directly by an SFT. The longest such plateau in the positive-dimensional region occurs around $t = .15$ (see Figure 1 for a detail), and we determine its endpoints and give an explicit symbolic coding of $X_t$ on this plateau in [12].

### 3 Coordinates and $K$-points

Let us now take $K$ to be a subset of $\mathbb{R}$ by fixing an embedding, and take $\varepsilon$ to be a fundamental unit with $\varepsilon > 1$. Recall that $\{1, \alpha_K\}$ is a $\mathbb{Z}$-basis of $O_K$, where

$$\alpha_K = \begin{cases} \sqrt{D} & D \equiv 2, 3 \pmod{4} \\ \frac{1+\sqrt{D}}{2} & D \equiv 1 \pmod{4} \end{cases}$$

Coordinates with respect to this basis will be denoted $(x, y)$. The choice of embedding gives an isomorphism

$$V_K = K \otimes_{\mathbb{Q}} \mathbb{R} \sim \rightarrow \mathbb{R} \times \mathbb{R}$$

$$a \otimes 1 \mapsto (x, y)$$

of $\mathbb{R}$-algebras, and thus another coordinate system. Multiplication by $\varepsilon$ has the effect of multiplying by $\varepsilon = \pm \varepsilon^{-1}$ in the first coordinate and $\varepsilon$ in the second coordinate. Accordingly, these are known as the stable and unstable coordinates and denoted $(s, u)$. Note that the absolute norm is simply $N(s, u) = |su|$ in these coordinates, and that the coordinate transformations between $(x, y)$ and $(s, u)$ coordinates are $K$-linear.

A point $P \in \mathbb{T}_K$ is called determinate if it has a representative $Q \in V_K$ with $N(Q) = M(P)$. It is shown in [4] (Theorem 2.6) that the set of determinate points is a meagre $F_\sigma$ set of measure zero and Hausdorff dimension 2. For a general point $P \in \mathbb{T}_K$, the following two lemmas help relate the value $M(P)$ to the more concrete values $N(Q)$ for $Q \in V_K$. 

![Fig. 1: Detail near $t = .15$](image-url)
Lemma 1 ([3], Lemma 4.2). Suppose that \( P \in \mathbb{T}_K \) satisfies \( M(P) < t \). There exists a point \( Q = (s, u) \in V_K \) representing an element of the orbit of \( P \) satisfying
\[
|s|, |u| < \sqrt{\varepsilon t}
\]
such that \( N(Q) = |su| < t \).

Lemma 2. Let \( P \in \mathbb{T}_K \). There exists \( Q \in V_K \) representing an element of the orbit closure of \( P \) satisfying
\[
N(Q) = M(Q) = M(P)
\]

Proof. Let \( R_{\text{big}} \) denote the rectangle in \( V_K \) given by \( |s|, |u| < \sqrt{\varepsilon(M_1(K) + 1)} \). By Lemma 1 there is for each \( n \in \mathbb{N} \) a point \( Q_n \in R_{\text{big}} \) representing an element of orbit of \( P \) with
\[
N(Q_n) < M(P) + \frac{1}{n}
\]
Since \( R_{\text{big}} \) is bounded, there exists a subsequence \( Q_{k_n} \) converging to some point \( Q \). Observe that
\[
M(Q) \leq N(Q) = \lim_{k \to \infty} N(Q_{k_n}) \leq M(P)
\]
where the last inequality follows from [1]. The definition of \( Q \) ensures that it represents an element of the orbit closure of \( P \). But this implies that \( M(Q) \geq M(P) \) by upper-semicontinuity, since the value \( M(P) \) is common to the entire orbit of \( P \), and the result follows. \( \square \)

Let us call a point in \( V_K \) with rational \((x, y)\) coordinates a \( \mathbb{Q}\)-point. Similarly a \( K\)-point is one whose \((x, y)\) coordinates lie in \( K \), or equivalently whose \((s, u)\) coordinates lie in \( K \). The set of \( \mathbb{Q}\)-points coincides with \( K/\mathbb{Q}_K \), which is also the set of periodic points for \( \phi \). In particular, if \( P \) is a \( \mathbb{Q}\)-point then the previous lemma immediately implies that \( P \) is determinate and \( M(P) \in \mathbb{Q} \).

Proposition 1. Let \( P \in \mathbb{T}_K \) be a \( K\)-point.

1. There exists \( N \in \mathbb{N} \) such that
\[
\phi^k(NP) \to 0 \quad \text{as} \quad |k| \to \infty
\]
2. \( M(P) \in K \)

Proof.

1. Since \( P \) has \((s, u)\) coordinates in \( K \), there exists \( N \in \mathbb{N} \) such that \( NP \) has \((s, u)\) coordinates in \( \mathbb{Q}_K \). In \((s, u)\) coordinates, the lattice \( \mathbb{Q}_K \subseteq V_K \) is given by the set of pairs \((\pi, a)\) for \( a \in \mathbb{Q}_K \). It follows by subtracting such elements that \( NP \) has a representative whose stable coordinate vanishes, as well as a representative whose unstable coordinate vanishes. Now \( \phi^k(NP) \to 0 \) as \( |k| \to \infty \) follows immediately.

2. By the previous part, the orbit closure of the \( K\)-point \( P \) consists of the orbit of \( P \) together with a finite collection of \( N\)-torsion points on the torus. By Lemma 2 there exists \( Q \in V_K \) representing an element of this orbit closure with \( N(Q) = M(P) \). Should \( Q \) represent an \( N\)-torsion point, then \( M(P) \in \mathbb{Q} \) since torsion points are \( \mathbb{Q}\)-points. On the other hand, if \( Q = (s, u) \) represents an element of the orbit of \( P \) then \( Q \) is also a \( K\)-point, so we have \( M(P) = N(Q) = |su| \in K \).

\( \square \)

4 Markov partitions

For each \( K \), the dynamical system \((T_K, \phi)\) admits a Markov partition consisting of two open rectangles. Such a partition \( \{R_0, R_1\} \) for \( K = \mathbb{Q}(\sqrt{5}) \) is pictured in Figure 2 in \((x, y)\) coordinates. Figure 3 furnishes a uniform description in \((s, u)\) coordinates of a two-rectangle Markov partition for any \( K \). This description is simply the one provided by Adler in [11] translated into \((s, u)\) coordinates. See also [14], where the construction may originate.
These two-rectangle partitions are typically not generators essentially because the intersections $R \cap \phi(S)$ are generally disconnected. In the case of $Q(\sqrt{5})$ however, the original partition $P_0 = \{R_0, R_1\}$ is a generator. Moreover, $R_0 \cap \phi(R_0) = \emptyset$, while the remaining intersections consist of a single nonempty rectangle each. Let $\Sigma$ denote the subset of $\{0, 1\}^\mathbb{Z}$ that avoids the string 00 and let $\sigma : \Sigma \to \Sigma$ be the shift operator $\sigma(s)_i = s_{i+1}$. The Markov generator property furnishes a map

$$\pi : \Sigma \to \mathbb{T}_K$$

intertwining $\phi$ and the shift operator on $\Sigma$ that sends each string of coordinates to the unique point in $\mathbb{T}_K$ whose orbit has these coordinates:

$$\pi(s) = \bigcap_{n \in \mathbb{N}} \bigcap_{i=-n}^{n} \phi^{-1}(s_i) = \bigcap_{i \in \mathbb{Z}} \phi^{-1}(\pi_i)$$

**Remark 1.** This construction ensures that $\phi^k\pi(s) \in \overline{s(k)}$ for all $k$. It follows that if the coordinate word of $A \in P_n$ occurs in $s \in \Sigma$, then $\phi^k\pi(s) \in \overline{A}$ for a suitable $k \in \mathbb{Z}$.

The map $\pi$ is continuous, surjective, bounded-to-one, and essentially one-to-one. Moreover, if $X \subseteq \mathbb{T}_K$ is a closed, invariant subset then $\pi$ restricts to a map

$$\pi^{-1}(X) \to X$$

with the same properties, from which it follows that the entropy of $\phi|_X$ coincides with the entropy of the shift restricted to $\pi^{-1}(X)$. In the case of $X = \mathbb{K}_t$, this entropy can be approximated by approximating the set $\pi^{-1}(\mathbb{K}_t)$ by subshifts obtained by refining the partition $P_0$ and omitting some rectangles. The refinements are defined by taking $P_n$ to consist of all nonempty intersections of the form

$$\phi^n(A_{-n}) \cap \cdots \cap \phi(A_{-1}) \cap A_0 \cap \phi^{-1}(A_1) \cap \cdots \cap \phi^{-n}(A_n), \quad A_i \in P_0,$$

and we say that this particular rectangle has coordinate word $A_{-n} \cdots A_0 \cdots A_n$. When a representative rectangle in the plane $V_K$ is needed for a member of $P_n$, we take the one contained in the original footprint $R_0 \cup R_1$.

The refinement $P_n$ is also a Markov generator, and we have a refined coding $\pi_n : \Sigma_n \to \mathbb{T}_K$ by the set of admissible strings in the alphabet $P_n$. Note that $\Sigma_n$ is simply a “block form” of $\Sigma$ and there is a canonical bijection $\Sigma \cong \Sigma_n$ compatible with the shift operator and the two codings of $\mathbb{T}_K$. While for general $K$, the partition $\{R_0, R_1\}$ is not a generator, in all cases the connected components of $A_0 \cap \phi^{-1}(A_1)$ for $A_i \in \{R_0, R_1\}$ do comprise a Markov generator (see the proof of Theorem 8.4 of [1]). Thus for any $K$ other than $Q(\sqrt{5})$ we may let $P_0$ denote this generator and then proceed as in the previous paragraph to produce refinements $P_n$. In all cases, the diameter of $P_n$ tends to zero as $n \to \infty$.

The following explicit construction of $\pi$ will be useful below. Here, $P$ can be any Markov generator on $\mathbb{T}_K$ arising from a collection of rectangles in the plane $V_K$ with sides parallel to the stable and unstable axes. In particular we suppose we have a chosen representative in the plane for each member of $P$, or equivalently
a choice of stable and unstable interval of which this member is the product. Let \( s \in \Sigma \), the set of all admissible bi-infinite strings in the alphabet \( \mathcal{P} \). First we show how to compute the unstable coordinate of \( \pi(s) \). The intersections

\[
\begin{align*}
& r_0 = s_0 \\
& r_1 = s_0 \cap \phi^{-1}(s_1) \\
& r_2 = s_0 \cap \phi^{-1}(s_1) \cap \phi^{-2}(s_2) \\
& \vdots
\end{align*}
\]

on the torus can be viewed in the plane as a sequence of rectangles within \( s_0 \) whose stable interval is constant (and equal to that of \( s_0 \)) and whose unstable interval is shrinking. Up to similarity, the footprint of the unstable interval of \( r_{i+1} \) inside that of \( r_i \) depends only on the rectangles \( s_i \) and \( s_{i+1} \) and is independent of \( i \). This is because \( \phi \) simply scales by the positive number \( \varepsilon \) in the unstable direction, preserving similarity.

Given a rectangle in the plane with sides parallel to the stable and unstable axes, let us denote its stable and unstable intervals by \([\alpha_s(A), \beta_s(A)]\) and \([\alpha_u(A), \beta_u(A)]\), and let \( \ell_s(A) = \beta_s(A) - \alpha_s(A) \) denote the corresponding lengths. For each pair \( A, B \in \mathcal{P} \) with \( AB \) admissible, we define

\[
\rho_u(A,B) = \frac{\alpha_u(A \cap \phi^{-1}(B)) - \alpha_u(A)}{\ell_u(A)}
\]

Pictured in the \((s,u)\) plane, this is the height of the bottom of the subrectangle \( A \cap \phi^{-1}(B) \) inside \( A \), expressed as a fraction of the total height of \( A \), and is a measure of the footprint of this subrectangle in \( A \) alluded to above. The left endpoint of the unstable interval of \( r_i \) is then equal to

\[
\alpha_u(s_0) + \rho_u(s_0, s_1)\ell_u(s_0) + \rho_u(s_1, s_2)\ell_u(s_1) + \cdots + \rho_u(s_{i-1}, s_i)\ell_u(s_{i-1}) + \frac{\ell_u(s_i)}{\varepsilon},
\]

so the unstable coordinate of \( \pi(s) \) is given by the series

\[
\alpha_u(s_0) + \rho_u(s_0, s_1)\ell_u(s_0) + \rho_u(s_1, s_2)\ell_u(s_1) + \rho_u(s_2, s_3)\ell_u(s_2) + \cdots \tag{3}
\]

The stable coordinate works the same way if \( \varepsilon > 0 \). Some additional care must be taken if \( \varepsilon < 0 \), since then \( \phi \) is orientation-reversing in the stable direction and the footprints alternate with their mirror images up to similarity instead of being independent of \( i \). In that case we define coefficients

\[
\rho_s^+(A,B) = \frac{\alpha_s(A \cap \phi(B)) - \alpha_s(A)}{\ell_s(A)}
\]

and

\[
\rho_s^-(A,B) = \frac{\beta_s(A) - \beta_s(A \cap \phi(B))}{\ell_s(A)},
\]

and the stable coordinate alternates between these:

\[
\alpha_s(s_0) + \rho_s^+(s_0, s_{-1})\ell_s(s_0) + \rho_s^+(s_{-1}, s_{-2})\ell_s(s_{-1}) + \rho_s^+(s_{-2}, s_{-3})\ell_s(s_{-2}) + \cdots \tag{4}
\]

Let us now return to the partitions \( \mathcal{P}_n \) derived from the two-rectangle partition above. If \( s \in \Sigma \) is periodic, then the image \( \pi(s) \in \mathbb{T}_K \) has periodic orbit, and hence is a \( \mathbb{Q} \)-point. The following lemma furnishes a similar description of some \( K \)-pts.

**Lemma 3.** Suppose that \( s \) is eventually periodic in both directions. Then \( \pi(s) \) is a \( K \)-point.

**Proof.** First observe that all members of our partitions \( \mathcal{P}_n \) have coordinates in the field \( K \). If \( s \) is eventually periodic in both directions, then the series (3) and (4) (and its analog in case \( \varepsilon > 0 \)) decompose into finitely many geometric series with all terms and coefficients expressible in terms of these coordinates, and the result follows. \( \square \)
Lemma 4. If \( t' < t \) and \( X_t \subset X_{t'} \), then there exists a finite word occurring in \( \pi^{-1}(X_{t'}) \) that does not occur in \( \pi^{-1}(X_t) \).

Proof. Suppose to the contrary that every word appearing in \( \pi^{-1}(X_{t'}) \) also occurs in \( \pi^{-1}(X_t) \). We claim this forces \( \pi^{-1}(X_{t'}) \) to be contained in the closure of \( \pi^{-1}(X_t) \), which is a contradiction since the latter is closed and assumed distinct from the former. Let \( s \in \pi^{-1}(X_{t'}) \), and for \( k \in \mathbb{N} \) let \( u_k \) be the word \( s(-k) \cdots s(0) \cdots s(k) \).

By hypothesis, this word occurs in \( \pi^{-1}(X_t) \), and by applying \( \phi \) we may assume that it occurs centrally in some element \( x_k \in \pi^{-1}(X_t) \). In particular, \( x_k \) and \( s \) agree on the index interval \([-k, k]\), and it follows that \( x_k \to s \) as \( k \to \infty \), so \( s \) lies in the closure of \( \pi^{-1}(X_t) \). \( \square \)

5 Upper bounds via trapping rectangles

Given a collection of rectangles \( \mathcal{C} \subset \bigcup_n \mathcal{P}_n \), we denote by \( \Sigma(\mathcal{C}) \) the subshift of \( \Sigma \) that avoids the coordinate words of elements of \( \mathcal{C} \). If \( \mathcal{C} \) is finite, then there is a largest \( n \) for which \( \mathcal{P}_n \) contains an element of \( \mathcal{C} \). Now every element of \( \mathcal{C} \) breaks up into rectangles in \( \mathcal{P}_n \), and we let \( \mathcal{C}' \subset \mathcal{P}_n \) denote the collection of rectangles occurring in this fashion. Under the identification \( \Sigma \approx \Sigma_n \), the subshift \( \Sigma(\mathcal{C}) \) can alternately be described as the collection of \( s \in \Sigma_n \) for which \( s(k) \notin \mathcal{C}' \) for all \( k \in \mathbb{Z} \).

Let \( I \subset \mathcal{O}_K \) be a finite set of lattice points and let

\[
\mathcal{U}(t, I) = \bigcup_{Q \in I} \{ P \in \mathcal{V}_K \mid N(P - Q) < t \}
\]

and let

\[
\mathcal{T}_n(t, I) = \{ A \in \mathcal{P}_n \mid \overline{A} \subset \mathcal{U}(t, I) \}
\]

be the collection of rectangles in \( \mathcal{P}_n \) whose closures are trapped within the norm-distance \( t \) “neighborhood” of some lattice point in \( I \). The following lemma says that \( \Sigma(\mathcal{T}_n(t, I)) \) is an upper bound not only for \( X_t \) but for \( X_{t-\eta} \) for some \( \eta > 0 \).

Lemma 5. There exists \( \eta > 0 \) such that \( \pi^{-1}(X_{t-\eta}) \subset \Sigma(\mathcal{T}_n(t, I)) \).

Proof. The elements of \( \mathcal{T}_n(t, I) \) have closures contained in the \( \mathcal{U}(t, I) \) and thus in \( \mathcal{U}(t-\eta, I) \) for some \( \eta > 0 \) since \( I \) is finite. If \( s \in \Sigma \) contains the coordinates of \( A \in \mathcal{T}_n(t, I) \), then \( \phi^k \pi(s) \) lies in \( \overline{A} \) for some \( k \), by Remark 7. But then \( M(\pi(s)) = M(\phi^k \pi(s)) < t - \eta \). Thus \( s \notin \pi^{-1}(X_{t-\eta}) \). \( \square \)

The entropy of \( \phi \) on \( X_t \) is thus bounded above by the shift entropy of \( \Sigma(\mathcal{T}_n(t, I)) \), which is computable by Perron-Frobenius theory. These upper bounds depend on the set \( I \subset \mathcal{O}_K \) and improve as \( I \) grows. The following proposition and its corollary ensure that it is possible to choose \( I \) so that the bounds are tight in the limit as \( n \to \infty \).

Proposition 2. There exists a finite set \( I_K \) such that if \( I_K \subset I \) and \( t' < t \), then for \( n \) sufficiently large we have

\[
\pi^{-1}(X_t) \subset \Sigma(\mathcal{T}_n(t, I)) \subset \pi^{-1}(X_{t'})
\]

In particular, for such \( I \) we have

\[
\pi^{-1}(X_t) = \bigcap_{n \geq 0} \Sigma(\mathcal{T}_n(t, I))
\]

Proof. The second assertion here follows immediately from the first. Let \( R_{\text{big}} \) denote the rectangle in \( \mathcal{V}_K \) given by \( |s|, |u| < \sqrt{\varepsilon(M_1(K) + 1)} \) and let \( I_K \) be the set of all \( q \in \mathcal{O}_K \) such that \( R - q \) meets \( R_{\text{big}} \) for some \( R \in \mathcal{P}_0 \). The set \( I_K \) is finite and necessarily contains any \( q \) for which there exists some \( A \in \mathcal{P}_n \) such that \( A - q \) meets \( R_{\text{big}} \). Since the diameter of \( \mathcal{P}_n \) tends to zero, there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies that every translate of \( A \) that meets the region defined by \( N < t' \) in \( R_{\text{big}} \) must have closure entirely contained within the region \( N < t \).

The first containment in \( \square \) is clear from the preceding lemma, and we prove the second by contrapositive. Suppose that \( s \in \Sigma \) is not in \( \pi^{-1}(X_{t'}) \). Then with \( P = \pi(s) \) we have \( M(P) < t' \), so

\[
M(P) < t' = \min(t', M_1(K) + 1)
\]
Thus we may take \( Q = (s, u) \) as in Lemma 4 representing an element of the orbit of \( P \) with \( N(Q) < t'' \) and
\[
|s|, |u| < \sqrt{\varepsilon t''}
\]
In particular, \( Q \in R_{big} \). For each \( n \), the point \( Q \) lies in the \( O_K \)-translates of the \( K \) closures of one or more members of the partition \( \mathcal{P}_n \). Let \( A \in \mathcal{P}_n \) and \( q \in O_K \) such that \( Q \in A - q \). Thus \( A - q \) meets the region defined by \( N < t' \) in \( R_{big} \), which requires that \( A - q \) meet this region since \( A \) is open, and hence \( q \in I_K \subseteq I \). Now if \( n \geq N \), it follows that \( A \in T_n(t, I) \).

By Remark 1 the 0th symbolic coordinate of any element of \( \pi_n^{-1}(Q) \) must be a member of \( T_n(t, I) \), which implies that each element of \( \pi_n^{-1}(P) \) has some symbolic coordinate in \( T_n(t, I) \). This is to say that each element of \( \pi^{-1}(P) \), including \( s \), contains the coordinates of some element of \( T_n(t, I) \), and thus \( s \notin \Sigma(T_n(t, I)) \). \( \square \)

**Corollary 2.** If \( I_K \subseteq I \), then
\[
h(\phi|X_t) = \lim_{n \to \infty} h(\sigma|\Sigma(T_n(t, I)))
\]

**Proof.** Let \( \mu_n \) be a measure of maximal entropy for \( \Sigma(T_n(t, I)) \). Extended to \( \Sigma \), this sequence of measures has some weak-* limit point \( \mu \) in the convex, compact space of invariant probability measures on \( \Sigma \). The measure \( \mu \) is supported on the intersection \( \pi^{-1}(X_t) \), and by upper semi-continuity of entropy in subshifts we have
\[
h(\sigma|\pi^{-1}(X_t)) \geq h(\mu|\sigma) \geq \limsup h(\mu_n|\sigma) = \limsup h(\sigma|\Sigma(T_n(t, I))) \geq \liminf h(\sigma|\Sigma(T_n(t, I))) \geq h(\sigma|\pi^{-1}(X_t))
\]
This implies that \( \mu \) is a measure of maximal entropy for \( \pi^{-1}(X_t) \), as well as the claim. \( \square \)

**Corollary 3.** The function \( t \mapsto \dim(X_t) \) is left-continuous at each point.

**Proof.** The dimension of a closed, invariant subset \( X \subseteq T_K \) is related to the entropy of \( \phi \) on \( X \) via
\[
\dim(X) = \frac{2h(\phi|X)}{\log(\varepsilon)},
\]
so it suffices to prove that \( t \mapsto h(\phi|X_t) \) is left continuous. Since this function is decreasing, left-discontinuity at \( t \) would imply there exists \( B > 0 \) such that
\[
h(\phi|X_{t-\eta}) - h(\phi|X_t) \geq B \quad \text{for all } \eta > 0
\]
By the previous corollary we know there exists \( n \in \mathbb{N} \) with
\[
h(\sigma|\Sigma(T_n(t, I_K))) - h(\phi|X_t) < B
\]
Now Lemma 5 ensures that \( \Sigma(T_n(t, I_K)) \) contains \( \pi^{-1}(X_{t-\eta}) \) for some \( \eta > 0 \), which implies
\[
h(\sigma|\Sigma(T_n(t, I_K))) \geq h(\sigma|\pi^{-1}(X_{t-\eta})) = h(\phi|X_{t-\eta})
\]
contradicting the inequalities above. \( \square \)

# 6 Applications to the inhomogeneous spectrum
The plot of \( \dim(X_t) \) contains a number of plateaus as illustrated in the case \( K = \mathbb{Q}(\sqrt{5}) \) above. Sometimes these are actually set-theoretic plateaus, and the following proposition demonstrates that \( \pi^{-1}(X_t) \) is particularly simple in such cases.

**Proposition 3.** Suppose that \( X_t = X_{t-\eta} \) for some \( \eta > 0 \). Then \( \pi^{-1}(X_t) \) is a subshift of finite type.

**Proof.** By Proposition 2 we may choose \( n \in \mathbb{N} \) so that
\[
\pi^{-1}(X_t) \subseteq \Sigma(T_n(t, I)) \subseteq \pi^{-1}(X_{t-\eta}) = \pi^{-1}(X_t)
\]
Thus \( \pi^{-1}(X_t) = \Sigma(T_n(t, I)) \), which is expressible directly as an SFT via a 0-1 matrix when viewed in block form in \( \Sigma_m \) for some \( m \) (namely, any \( m \geq n - 1 \)). \( \square \)
Finally, we prove the main density result.

**Proof of Theorem 7.** First suppose that \( t \in M(T_K) \) is an isolated point. By the previous proposition, \( \pi^{-1}(X_t) \) is a subshift of finite type, which is to say that it can be described by a 0-1 transition matrix when viewed in block form \( \Sigma_m \) for some \( m \). Since \( t \) is isolated, we know by Lemma 4 that \( \pi^{-1}(X_t) \) contains a finite word \( w \) that does not occur in \( \pi^{-1}(X_{>t}) \). Let \( s = uvw \in \Sigma \) with \( M(\pi(s)) = t \). Viewed in \( \Sigma_m \), there is by the Pigeonhole Principle a repeated block in both \( u \) and \( v \). We can then truncate \( u \) and \( v \) and loop the segment between these books indefinitely to produce an element \( s' \in \pi^{-1}(X_t) \) that contains \( w \) and is eventually periodic in both directions. Then \( \pi(s') \) is a \( K \)-point by Lemma 3 and \( M(\pi(s')) = t \) since \( s' \) contains \( w \).

Now suppose that \( t \in M(T_K) \) is not isolated, so there is a strictly monotone sequence \( (t_k) \) in \( M(T_K) \) with \( t_k \to t \). Fixing \( k \in \mathbb{N} \), we will show that there is a \( K \)-point \( P \) with such that \( M(P) \) lies between \( t \) and \( t_k \), which will finish the density claim. First suppose that \( (t_k) \) increases to \( t \). Since \( t_{k+1} \in M(T_K) \), Lemma 4 ensures that there exists \( s \in \pi^{-1}(X_{t_{k+1}}) \) containing a word \( w \) that does not occur in \( \pi^{-1}(X_t) \). Now take \( n \) large enough so that
\[
\pi^{-1}(X_{t_{k+1}}) \subseteq \Sigma(J_n(t_{k+1}, I)) \subseteq \pi^{-1}(X_t)
\]
as in Proposition 2. Since \( s \) belongs to the SFT \( \Sigma(J_n(t_{k+1}, I)) \), we can modify it by looping its ends as in the previous paragraph to obtain another element \( s' \) of this SFT that also contains \( w \). But then we have \( t_k \leq M(\pi(s')) < t \), so \( P = \pi(s') \) is the desired \( K \)-point.

Now suppose that \( (t_k) \) is decreasing. Since \( t_{k+1} \in M(T_K) \), Lemma 4 ensures there is word \( w \) occurring in \( \pi^{-1}(X_{t_{k+1}}) \) that does not occur in \( \pi^{-1}(X_t) \). Now take \( n \) large enough so that
\[
\pi^{-1}(X_{t_{k+1}}) \subseteq \Sigma(J_n(t_{k+1}, I)) \subseteq \pi^{-1}(X_t)
\]
and proceed as before to produce \( s' \in \Sigma(J_n(t_{k+1}, I)) \) that contains \( w \) and is eventually periodic in both directions. We have \( t \leq M(\pi(s')) < t_k \), and again \( P = \pi(s') \) is the desired \( K \)-point. \( \Box \)

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