Deep Learning-Based Least Square Forward-Backward Stochastic Differential Equation Solver for High-Dimensional Derivative Pricing

Jian Liang, Zhe Xu, Peter Li

7/23/2019

Abstract

We propose a new forward-backward stochastic differential equation solver for high-dimensional derivatives pricing problems by combining deep learning solver with least square regression technique widely used in the least square Monte Carlo method for the valuation of American options. Our numerical experiments demonstrate the efficiency and accuracy of our least square backward deep neural network solver and its capability to provide accurate prices for complex early exercise derivatives such as callable yield notes. Our method can serve as a generic numerical solver for pricing derivatives across various asset groups, in particular, as an efficient means for pricing high-dimensional derivatives with early exercises features.

Key Words: partial differential equation (PDE), forward-backward stochastic differential equation (FBSDE), deep neural network (DNN), least square regression (LSQ), derivative pricing, Bermudan option, callable yield note (CYN), high-dimensional derivative pricing

Contents

1 Introduction

2 Background Knowledge

2.1 Forward-backward stochastic differential equation

2.2 Bermudan options

2.3 Callable yield notes

1

2

4

\begin{thebibliography}{9}
\bibitem{1} Corporate Model Risk, Wells Fargo Bank, jian.liang@wellsfargo.com
\bibitem{2} Corporate Model Risk, Wells Fargo Bank, zhe.xu@wellsfargo.com
\bibitem{3} Corporate Model Risk, Wells Fargo Bank, peter.li@wellsfargo.com
\end{thebibliography}
1 Introduction

In finance, a derivative is a contract that derives its value from the performance of an underlying entity. This underlying entity can be an asset such as index, stock, commodity, or interest rate, etc. Derivatives can be used for a number of purposes, including insuring against price movements (hedging), increasing exposure to price movements for speculation or obtaining access to otherwise hard-to-trade assets or markets. The most common derivative types are futures contracts, forward contracts, swaps and options.

Derivative pricing has been widely studied in academia and industry. Except for simple derivatives such as futures, forwards, swaps, and European vanilla options, numerical methods have to be used for their valuations. Tree, PDE and Monte Carlo are the three major methods in pricing complex derivatives. However, both tree and the classical finite difference based PDE approach are infeasible for high-dimensional (such as >2) derivatives pricing due to the implementation complexity and the numerical burden. This is the well known “curse of dimensionality”. Therefore, Monte Carlo method is widely used in high-dimensional derivative pricing. Some additional numerical procedures have to be added in Monte Carlo method when pricing early exercisable products, e.g. American options, Bermudan options, callable structured notes, etc., as it is computational impractical to perform a sub-MC simulation at the early exercise time to compute the continuation
value. Barraquand et al [2] proposed the stratified state method which sorts the stock price paths according to a state variable (rather than the stock price) to determine payoff. Broadie et al [3] proposed a simulated tree method to price American options, which can derive the upper and lower bounds for American options. Longstaff et al [11] proposed the least square regression method to price American options, and in their approach a least square regression was introduced in the early exercise step. As far as we know, the least square Monte Carlo is the most widely used algorithm by practitioners for pricing high-dimensional derivatives with early exercise features.

More recently, researchers have utilized machine learning techniques in derivative pricing. Beck et al [3] utilized the deep neural network to solve high-dimensional stochastic differential equations (SDE) and Kolmogorov equations. Ali Al-Aradi et al [1] applied the deep Galerkin method to solve PDEs that arise in quantitative finance applications including option pricing. Weinan E et al [7, 9] proposed a new algorithm (we call it as forward DNN) utilizing deep neural network to solve non-linear parabolic PDE. They utilized the amazing generalized Feynman-Kac theorems to formulate the PDE into equivalent backward stochastic differential equations (BSDE), then utilized the deep neural network technique to solve the BSDE. Their innovative method can be directly used in European style high-dimensional derivative pricing. Raissi [13] proposed a different loss function comparing to [7], which also utilized the derivative terms in a step-wise, rolling-back fashion to solve BSDE. Note that Weinan E’s or Raissi’s method is more appropriate in European style derivative pricing, but not for derivatives with early exercise features. Fujii et al [8] demonstrated that the use of asymptotic expansion as prior knowledge in the forward DNN method could drastically reduce the loss function and accelerate the convergence speed. They also extended the forward DNN method for reflected BSDEs which could be used in American basket option pricing. Wang et al [14] proposed a backward DNN algorithm for pricing Bermudan swaptions under Libor market model. However, since there were no numerical studies in the work of Wang et al to compare the results from the backward DNN with those from the classical approaches such as least square Monte Carlo simulation, the validity and accuracy of their backward DNN for Bermudan swaptions is not clear.

In this paper we propose a deep learning based least square forward-backward stochastic differential equation solver for pricing high-dimensional derivatives, in particular, with early exercise features. The application of neutral networks combined with regression to tackle early exercise options such as American options pricing problems has been reported by Kohler et al [10]. In Kohler et al’s work neutral network was used as an optimization tool for non-parametric regression while in our work neutral network is used to solve the BSDE. Different from Wang et al’s work [14], our algorithm can be used for general drift functions and the least square regression is used to determine optimal condition for early exercises. Even though there have been many researches on using neural network to approximate the solutions of PDEs for the purpose of derivatives pricing, very little studies have been reported to assess its efficiency by comparing with classical numerical methods. Our work
also aims at closing this gap by comparing DNN based algorithms with classical Monte Carlo simulation and hence providing guidance on what situations DNN approach is more efficient.

The remainder of this paper is organized as follows. In section 2, we introduce some basic background knowledge for forward-backward stochastic differential equation (FBSDE), which is key knowledge to our least square backward DNN method. We also briefly explain the Bermudan option and callable yield note, which will be used as examples in our numerical testing. The forward DNN method ([7]) is described in section 3. In section 4 we outline the backward DNN method first and then introduce the least square backward DNN method. Numerical results for Bermudan options and callable yield notes are presented in section 5. We conclude our paper in section 6.

2 Background Knowledge

In this section, we first introduce some basics of forward-backward stochastic differential equation (FBSDE) and then describe the Bermudan option and callable yield note (CYN), as we will use these instruments to perform our numerical tests in section 5.

2.1 Forward-backward stochastic differential equation

Many pricing and optimization problems in financial mathematics can be reformulated in terms of backward stochastic differential equations (BSDEs). These equations are non-anticipating terminal value problems for stochastic differential equations (SDE) of the form

\[ -dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t \]
\[ Y_T = \xi \]  \hspace{1cm} (2.1)

where \( W_t \) is a standard \( d \)-dimensional Brownian motion defined on a complete probability space, the square-integrable terminal condition \( \xi \) (measurable with respect to filtration generated up to time \( T \) by the Brownian motion) and the so-called drift term \( f \) is given.

When BSDEs are used in financial mathematics, \( Y_t \) corresponds to the derivative value and \( Z_t \) is related to the hedging portfolio. In many portfolio optimization problems, \( Y_t \) corresponds to the value process while an optimal control can often be derived from \( Z_t \). Finally, BSDEs can also be applied in order to obtain Feynman-Kac type representation formulas for nonlinear parabolic PDEs. Here \( Y_t \) and \( Z_t \) correspond to the solution and the gradient of the PDE, respectively.
In this paper, we will focus on a forward-backward stochastic differential equation (FBSDE) of the form:

\[ dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t \]
\[ X_0 = x \]
\[ -dY_t = f(t, X_t, Y_t, Z_t) \, dt - Z_t \, dW_t \]
\[ Y_T = g(X_T) \] (2.2)

The name forward-backward comes from that \( X \) moves forward as its initial value is given, \( Y \) moves backward as its terminal value is given. Suppose \( X_t \) is the stock value and it follows

\[ dX_t = (r - q) X_t dt + \sigma X_t dW_t \] (2.3)

for simplicity, we assume \( r \) is the constant discount rate, \( q \) is the constant dividend and \( \sigma \) is the constant volatility. We only use subscript when it is necessary. Proceeding in the same fashion as in the derivation of the Black-Scholes PDE, we construct a portfolio \( \Pi = Y - \Delta X \), \( \Delta \) will be selected so that the value of the portfolio is deterministic.

\[ d\Pi = dY - \Delta dX - q \Delta X dt = \left( \frac{\partial Y}{\partial t} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 Y}{\partial X^2} \right) dt + \frac{\partial Y}{\partial X} dX - \Delta dX - q \Delta X dt \]

The term \( q \Delta X dt \) arises since the stock pays dividends which decreases the value of the portfolio by the amount of the dividend. If we select \( \Delta = \frac{\partial Y}{\partial X} \), we then have

\[ d\Pi = \left( \frac{\partial Y}{\partial t} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 Y}{\partial X^2} \right) dt - q \Delta X dt \]

Since the value of the portfolio is risk free, we must have

\[ d\Pi = r \Pi dt = r (Y - \Delta X) \, dt \]

This leads to the following Black Scholes PDE

\[ \frac{\partial Y}{\partial t} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 Y}{\partial X^2} + (r - q) X \frac{\partial Y}{\partial X} - rY = 0 \] (2.4)

From Itô's Lemma, we have

\[ dY = \left( \frac{\partial Y}{\partial t} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 Y}{\partial X^2} \right) dt + \frac{\partial Y}{\partial X} dX \]
\[ = \left( rY - (r - q) X \frac{\partial Y}{\partial X} \right) dt + \frac{\partial Y}{\partial X} \left( (r - q) X dt + \sigma X dW \right) \]
\[ = rY dt + \sigma X \frac{\partial Y}{\partial X} dW \]
\[ -dY = -rY dt - \sigma X \frac{\partial Y}{\partial X} dW \quad (2.5) \]

that is \( f = -rY \), and \( Z = \sigma X \frac{\partial Y}{\partial X} \) in Eq (2.2).

The above statement can be easily extended to a high-dimensional derivative pricing \((Y = Y (X_1, X_2, \ldots, X_d))\), and we have (neglecting subscript \( t \))

\[
\begin{align*}
  dX^i &= \mu^i (t, X^i) \, dt + \sigma^i (t, X^i) \, dW^i \\
  X^i_0 &= x^i \\
  -dY &= -rY dt - \sum \sigma^i X^i \frac{\partial Y}{\partial X^i} dW^i \\
  Y_T &= g (X^1_T, X^2_T, \ldots, X^d_T) \\
  \text{cov} (dW^i, dW^j) &= \rho^{ij} dt \quad |\rho^{ij}| < 1
\end{align*}
\]

2.2 Bermudan options

A Bermudan option is a type of exotic option that can only be exercised on predetermined dates. The Bermudan option is exercisable on the date of expiration, and on certain specified dates that occur between the purchase date and the date of expiration. Bermudan option is a hybrid of American options (exercisable on any dates before and including expiration) and European options (exercisable only at expiration). The payoff function of Bermudan call at expiration if not exercised early is given by

\[
V (T) = \max \left( \sum_{i=1}^{d} \omega_i X^i (T) - K, 0 \right) \quad (2.7)
\]

where \( K \) is the strike of the option and weight \( \omega_i \) are given constants. When an exercise event happens, the option expires and the holder will receive its intrinsic value. Given the exercise times as \( t < t_1 < t_2 < \ldots < t_n \leq T \), the value of a Bermudan option at time \( t \) can be written as:

\[
V(t) = D(t, T) \sup_{\tau \in T(t)} E^Q [ V(\tau) | \mathcal{F}_t ] \quad (2.8)
\]

where \( D(t, T) \) is the discount factor, \( T(t) \) is the set of exercise times, and the expectation is taken under risk neutral measure.
2.3 Callable yield notes

Callable yield note (CYN), also called worst of issuer callable, is a yield enhancement product, whose performance is capped by a coupon that is guaranteed by an issuer. As the name implies, the issuer, at its discretion, can call the product, usually on predefined observation dates. The underlying entities are generally composed of several stocks or stock indices, thus making it a product based on a worst-of function. The call notice dates for a CYN are often identical to the coupon record dates. We denote the coupon record dates as $t_i, i = 1, 2, \ldots, N$, with $t_N = T$ being equal to the expiry date $T$. The coupon payments are subject to a barrier condition and the knock-in barrier is observed at expiry. The coupon payment per unit of notional are

$$c(t_i) = r_i \Theta (p(t_i) - B_i) \quad \text{for } i = 1, 2, \ldots, N - 1$$
$$c(t_N) = r_N \Theta (p(T) - B_N) - \Theta (B - p(T)) \max (K - p(T), 0)$$

(2.9)

where $r_i$ is the contingent coupon with coupon barrier $B_i$ on $i$th coupon day, $B$ is the knock-in barrier at expiry, $K$ is the knock-in put strike, and $p(t)$ is the relevant performance since trade inception. $p(t)$ is defined as:

$$p(t) = \min_{j \in \{1, 2, \ldots, d\}} \left[ \frac{X_j(t)}{X_j(0)} \right]$$

(2.10)

and $\Theta(x)$ is the Heaviside function

$$\Theta(x) = \begin{cases} 
1 & \text{for } x \geq 0 \\
0 & \text{otherwise}
\end{cases}$$

(2.11)

Furthermore, upon redemption (at the scheduled expiry or early issuer call) the principal notional is returned to the holder. That is

$$\text{payoff}(T) = \text{notional} + c(t_N)$$
$$\text{callvalue}(t_i) = \text{notional} \quad \text{for } i = 1, 2, \ldots, N - 1$$

(2.12)

Given the call times as $t < t_1 < t_2 < \ldots < t_n \leq T$, the value of callable yield note at time $t$ is

$$V(t) = D(t, T) \inf_{\tau \in \mathcal{T}(t)} E^Q[V(\tau)|\mathcal{F}_t]$$

(2.13)

where $D(t, T)$ is the discount factor, $\mathcal{T}(t)$ is the set of call times, and the expectation is taken under risk neutral measure.
3 Forward DNN Method

Forward solvers using deep neural network (DNN) have been developed mainly by Weinan E et al [7, 9]. FBSDE (Eq (2.2)) can be numerically solved in the following way:

- Simulate sample paths for the FBSDE using a standard Monte Carlo method.
- Approximate $Z$ using a deep neural network (DNN), then plug into the FBSDE to propagate along time.

We briefly describe the forward DNN method in this section. More details can be found in [7, 9, 14]. For simplicity, we use 1D (single underlier) case as an example. High-dimensional case is similar. Specifically, consider a time discretization

$$\pi = \{t_0, \ldots, t_N\}$$

of the interval $[0, T]$, i.e. $0 = t_0 < t_1 < \cdots < t_N = T$, where we assume valuation date=0 and expiration date= T. Denoting $h_i = t_{i+1} - t_i$ and $dW_i = W_{t_{i+1}} - W_{t_i}$.

1. $M$ Monte Carlo (MC) paths of the underlying stock $X_i$ (short for $X_{t_i}$, similarly for other notations) are sampled by an Euler scheme through

$$X_{i+1} = X_i + \mu(t_i, X_i) h_i + \sigma(t_i, X_i) dW_i$$  \hspace{1cm} (3.1)

This step is the same as Monte Carlo pricer. Other discretization schemes can be used, for instance, log-Euler discretization and Milstein discretization [12].

2. At time $t_0 = 0$, $Y_0$ and $Z_0$ are randomly picked.

3. For $t_i \in \pi$, we have

$$Y_i - Y_{i+1} = f(t_i, X_i, Y_i, Z_i) h_i - Z_i dW_i$$  \hspace{1cm} (3.2)

or

$$Y_{i+1} = Y_i - f(t_i, X_i, Y_i, Z_i) h_i + Z_i dW_i$$  \hspace{1cm} (3.3)

At each time step $t_i$, given $Y_i$, a deep neural network (DNN) approximation is used for $Z_i$ as $Z_i(\theta_i)$ for some hyper-parameter $\theta_i$ using sampled data $X_i$. Then the FBSDE is propagating forward in time direction from $t_i$ to $t_{i+1}$ as

$$Y_{i+1} = Y_i - f(t_i, X_i, Y_i, Z_i(\theta_i)) h_i + Z_i(\theta_i) dW_i$$  \hspace{1cm} (3.4)

Along each Monte Carlo path, as propagating forward from time 0 to $T$, one can estimate $Y_N^{(j)}$ as $Y_N^{(j)}(Y_0, Z_0, \theta^{(j)})$ where $\theta^{(j)} = \{\theta_0^{(j)}, \ldots, \theta_{N-1}^{(j)}\}$ are all hyper-parameters for neural network at each time steps for the $j$th MC path.
4. A natural loss function will be
\[ L_{\text{Forward}} = \text{Mean all paths} \left( Y_{N}^{(j)} \left( Y_0, Z_0, \theta^{(j)} \right) - g \left( X_{N}^{(j)} \right) \right)^2 \] (3.5)

5. The Adam optimization (in TensorFlow library) is used to minimize the loss function \( L_{\text{Forward}} \) and estimate \( Y_0 \) as
\[
\tilde{Y}_0 = \arg \min_{Y_0} \text{Mean all paths} \left( Y_{N}^{(j)} \left( Y_0, Z_0, \theta^{(j)} \right) - g \left( X_{N}^{(j)} \right) \right)^2
\] (3.6)

The estimated \( \tilde{Y}_0 \) is the desired derivative value at \( t = 0 \). More details about the use of Adam optimization to solve the above minimization problem can be found in [7, 9, 14].

4 Least Square Backward DNN Method

Since the forward DNN method above cannot be applied to price options with early exercise features, for instance, Bermudan options, Wang et al [14] proposed a backward DNN method to price Bermudan swaptions under LIBOR market model. At any future time of an exercise date, according to the dynamic programming principle for optimality, the continuation value of the option must be known as well. Given a numerical scheme for pricing, forward estimation of the continuation value could be arduous. Wang et al [14] mainly focus on a pricing model for the backward process with a vanishing drift term (\( f = 0 \) in Eq (2.2)). We extend their method to general drift functions and apply the least square regression to determine the optimal exercise decision.

4.1 Backward DNN method

We would like to propagate backward in time direction and apply the call/put and coupon events to the derivative value. From Eq (3.4), we have
\[
Y_i = Y_{i+1} + f \left( t_i, X_i, Y_i, Z_i (\theta_i) \right) h_i - Z_i (\theta_i) dW_i
\] (4.1)

As we propagate backward in time direction from \( t_{i+1} \) to \( t_i \), \( Y_{i+1} \) is known while \( Y_i \) is to be determined. We use 1st order Taylor expansion to do the approximation.
\[
Y_i \approx Y_{i+1} + \left( f \left( t_i, X_i, Y_{i+1}, Z_i (\theta_i) \right) - \frac{\partial f}{\partial Y} \left( t_i, X_i, Y_{i+1}, Z_i (\theta_i) \right) (Y_{i+1} - Y_i) \right) h_i - Z_i (\theta_i) dW_i
\] (4.2)

which leads to
\[
Y_i \approx Y_{i+1} + \frac{1}{1 - \frac{\partial f}{\partial Y} \left( t_i, X_i, Y_{i+1}, Z_i (\theta_i) \right) h_i} \left( f \left( t_i, X_i, Y_{i+1}, Z_i (\theta_i) \right) h_i - Z_i (\theta_i) dW_i \right)
\] (4.3)
One can use higher order Taylor expansion to achieve more precise approximation. For our particular equations (Eq (2.5)), 1st order Taylor expansion approximation is indeed the exact solution. And we have

\[ Y_i = \frac{Y_{i+1} - Z_i (\theta_i) dW_i}{1 + rh_i} \]  

(4.4)

Starting from \( t_N = T \), we can propagate backward in time direction to \( t_0 = 0 \), and obtain the estimated initial value \( Y_0^{(j)} (\theta^{(j)}) \) for each sampled path, where \( \theta^{(j)} = \{ \theta_0^{(j)}, \ldots, \theta_{N-1}^{(j)} \} \) are all hyper-parameters for neural network at each time steps for the \( j \)th MC path. The ideal case will be all the estimated initial value \( Y_0^{(j)} (\theta^{(j)}) \) concentrate to one point. Therefore, the loss function is defined as

\[ L_{\text{Backward}} = \text{Mean all paths} \left( Y_0^{(j)} (\theta^{(j)}) - \text{Mean all paths} \left( Y_0^{(j)} (\theta^{(j)}) \right) \right)^2 \]  

(4.5)

This implies that we are trying to minimize the variance of the estimated initial values. The Adam optimizations is used to minimize the loss function \( L_{\text{Backward}} \) and estimate \( Y_0 \) as

\[ \widetilde{Y}_0 = \text{Mean all paths} \left( Y_0^{(j)} (\tilde{\theta}^{(j)}) \right) \]  

(4.6)

where

\[ \tilde{\theta}^{(j)} = \arg \min_\theta L_{\text{Backward}} \]  

(4.7)

And the estimated \( \widetilde{Y}_0 \) is our desired derivative value at \( t = 0 \).

### 4.2 Least square regression

We use a Bermudan call option to explain how the conditional expectation of the payoff estimated from least square regression is used to determine optimal strategy at an early exercise time. The readers are referred to the classical paper by Longstaff et al [11] for more details. Without loss of generality, we assume the exercise time \( t_k \in \pi = \{ t_0, \ldots, t_N \} \). The main idea is to employ a regression equation, e.g.,

\[ Y_k = a + bX_k + cX_k^2 + v \]  

(4.8)

where \( v \) is the white noise and \( v \sim N(0, \eta^2) \). The expected derivative value is estimated as

\[ \mathbb{E}Y_k = a + bX_k + cX_k^2 \]  

(4.9)

At an exercise time, the above least square regression is performed over all the in-the-money paths that have positive call values. Note that other basis functions could be used in the least square regression, e.g. weighted Lauerre polynomials, which are used in the paper by Longstaff et al [11]. We use the monic polynomials as basis functions in our numerical tests.
(section 5) for illustration purpose. Our testing results indicate that the monic polynomial can produce satisfactory results for the products in our study based on comparison with results from a finite difference PDE solver.

The optimal strategy at an exercise time can be determined by comparing the call value (i.e. the immediate exercise value) with the expectation of the derivative value from continuation:

$$Y_k = \begin{cases} Y_k & \text{if } EY_k \geq \text{callvalue}(t_k, X_k) \\ \text{callvalue}(t_k, X_k) & \text{if } EY_k < \text{callvalue}(t_k, X_k) \end{cases} \quad (4.10)$$

We summarize our least square backward DNN method as follow

1. $M$ Monte Carlo (MC) paths of the underlying stock $X_i$ (short for $X_{t_i}$, similarly for other notations) are sampled by an Euler scheme through Eq (3.1). This step is the same as forward DNN method.

2. As we have the sampled $X^{(j)}_N$ ($j = 1, 2, \cdots, M$) available, we could calculate the payoff at expiry for the $j$th sampled path

   $$Y^{(j)}_N = g \left(X^{(j)}_N\right) \quad (4.11)$$

3. At each time step $t_i$, given $Y_{i+1}$, a deep neural network (DNN) approximation is used for $Z_i$ as $Z_i(\theta_i)$ for some hyper-parameter $\theta_i$ using sampled data $X_i$. Then the FBSDE is propagating backward using Eq (4.4) in time direction from $t_{i+1}$ to $t_i$. Along each Monte Carlo path, as propagating backward from time $T$ to 0, one could estimate $Y^{(j)}_0$ as $Y^{(j)}_0(X_0, Z_0, \theta^{(j)})$ where $\theta^{(j)} = \{\theta^{(j)}_0, \cdots, \theta^{(j)}_{N-1}\}$ are all hyper-parameters for neural network at each time steps for the $j$th MC path.

4. During propagating from time $T$ to 0, at call time $t_k$, the least square regression is performed and the derivative value at each path is reset using Eq (4.10).

5. Set $L_{\text{Backward}}$ as the loss function.

6. The Adam optimization is used to minimize the loss function $L_{\text{Backward}}$ and estimate $Y_0$ as Eq (4.6). The estimated $\tilde{Y}_0$ is our desired derivative value at $t = 0$. More details about usage of Adam optimization to solve the above minimization problem could be found in [14].

Notice that the above least square backward DNN method could be easily extended to high-dimensional derivative pricing ($Y = Y \left(X^1, X^2, \cdots, X^d\right)$).
5 Numerical Results

In this section we use Bermudan option and CYN as examples to test our least square backward DNN method, and compare with PDE and Monte Carlo results. Our finite difference PDE solver is only implemented for 1D and 2D cases. For MC solver, we use $M = 1,000,000$ sampling paths to estimate the means. Note that we have the relative differences between 1M and 500K less than 0.5%.

The market data setting used in all our testing examples is given in Table 5.1. All of our tested examples are with $T = 1$ and time step $N = 100$. Therefore, we have time step size $h_i = 0.01$.

| interest rate $r$ | $r = 0.01$ | time step $N$ | $N = 100$ |
|------------------|------------|---------------|------------|
| time step $N$    | stock 1    | stock 2       | stock 3    | stock 4    | stock 5    | stock 6    | stock 7    | stock 8    | stock 9    | stock 10   |
| spot $X_0$       | 100        | 150           | 200        | 175        | 125        | 100        | 150        | 200        | 175        | 125        |
| dividend rate $q$| 0.03       | 0.02          | 0.05       | 0.00       | 0.04       | 0.03       | 0.02       | 0.05       | 0.00       | 0.04       |
| volatility $\sigma$ | 0.2       | 0.3           | 0.25       | 0.24       | 0.15       | 0.2        | 0.3        | 0.25       | 0.24       | 0.15       |
| correlation $\rho$ | 0.3 for all $\rho_{ij}$ |               |             |             |             |             |             |             |             |             |

The deep neural network setting in our tests is as follows: each of the sub-neural network approximating $Z_i(\theta_i)$ consists of 4 layers (1 input layer [d-dimensional], 2 hidden layers [d+10-dimensional], and 1 output layer [d-dimensional], where $d$ is the number of underlying entities). In the test, we run 5,000 optimization iterations of training and validate the trained DNN every 100 iterations. This produces 50 results. We use the mean of the 10 results with the least loss function value as our derivative value. The MC sampling size is $M = 5,000$.

5.1 Forward vs backward DNN method

We first use a 1Y ATM single underlying stock European call option to compare the performance of the forward DNN method and the backward DNN method. We use stock 1 in Table 5.1 as the only underlying stock. The expiration $T = 1$ and strike $K = 100$. The results are given in Table 5.2 and Figure 5.1. Both the forward and the backward DNN methods could provide close results to the Black-Scholes price. Both methods converge fast. Small oscillations in prices can be observed in the forward DNN method. In contrary, the Backward DNN method is more stable and converges slightly faster than the forward DNN method.
Table 5.2: Comparison between forward and backward DNN on European call option

|                | Black Scholes | Forward DNN | Backward DNN |
|----------------|---------------|-------------|--------------|
| NPV            | 6.8669        | 6.8688      | 6.8575       |
| rel diff to BS | 0.03%         | -0.14%      |              |

Figure 5.1: Comparison between forward and backward DNN on European call option

5.2 Tests on Bermudan option

In this section we use 1Y ATM Bermudan options to test the performance of our least square backward DNN method. We test Bermudan call options on a single underlying stock, 2 underlying stocks (stock 1, 2), 3 underlying stocks (stock 1, 2, 3) and 5 underlying stocks (stock 1, 2, 3, 4, 5). The strike is chosen as $\sum_d \omega_i X_i^0$ with equal weight $\omega_i = 1/d$ so that the option is ATM. The Bermuda option can be exercised quarterly, or at 0.25, 0.5, 0.75, 1.0. We compare the prices from PDE, Monte Carlo and the least square backward DNN method. The results are presented in Table 5.3 and from Figure 5.2 to 5.5. It can be seen that the backward DNN method converges very fast and the convergence rate is not sensitive to the dimensions of the problem. It can also be seen that the backward DNN method can produce very accurate prices for Bermudan call options.
Table 5.3: Comparison among PDE, Monte Carlo and least square backward DNN method on Bermudan options

| 1Y ATM Bermudan Call | PDE  | MC   | LSQ BDNN | rel diff to PDE | rel diff to MC |
|----------------------|------|------|----------|-----------------|---------------|
| single stock (stock 1) | 7.0012 | 6.9911 | 6.9863 | -0.21% | -0.07%          |
| 2 stocks (stock 1, 2) | 9.9538 | 9.9520 | 9.9488 | -0.05% | -0.03%          |
| 3 stocks (stock 1, 2, 3) | 9.7031 | 9.6804 |         | -0.23% |                 |
| 5 stocks (stock 1, 2, 3, 4, 5) | 8.2856 | 8.2795 |         |         | -0.07%          |

Figure 5.2: Bermudan Call, single underlying stock

Figure 5.3: Bermudan Call, 2 underlying stocks
5.3 Tests on callable yield note

In this section we use 1Y CYNs to test the performance of our least square backward DNN method for complex payoffs. We test CYNs on a single underlying stock, 2 underlying stocks (stock 1, 2), 3 underlying stocks (stock 1, 2, 3) and 5 underlying stocks (stock 1, 2, 3, 4, 5). Some key contract parameters of the tested CYNs are given in Table 5.4. We compare the prices from PDE, Monte Carlo and the least square backward DNN method. The results are in Table 5.5 and from Figure 5.6 to 5.9. It can be seen that all the tested samples converge fast and the differences in prices between the backward DNN approach and PDE or MC method is very small. This set of tests indicates the validity and accuracy of the least square backward DNN method.

Figure 5.4: Bermudan Call, 3 underlying stocks

Figure 5.5: Bermudan Call, 5 underlying stocks

Figure 5.6: Bermudan Call, 3 underlying stocks

Figure 5.7: Bermudan Call, 5 underlying stocks
Table 5.4: CYN contract parameters

| Parameter                  | Value          |
|----------------------------|----------------|
| contingent coupon          | $r_i = 5\%$   |
| coupon barrier             | $B_i = 70\%$  |
| knock-in barrier           | $B = 50\%$    |
| knock-in put strike        | $K = 100\%$   |
| call/coupon schedule      | quarterly or 0.25, 0.5, 0.75, 1.0 |

Table 5.5: Comparison among PDE, Monte Carlo and least square backward DNN method on CYNs

| 1Y CYN                  | PDE   | MC     | LSQ BDNN | rel diff to PDE | rel diff to MC |
|-------------------------|-------|--------|----------|-----------------|----------------|
| single stock (stock 1)  | 1.0474| 1.0474 | 1.0474   | 0.00\%          | 0.00\%         |
| 2 stocks (stock 1, 2)   | 1.0456| 1.0458 | 1.0465   | 0.09\%          | 0.07\%         |
| 3 stocks (stock 1, 2, 3)| 1.0453| 1.0452 |          | -0.02\%         |                |
| 5 stocks (stock 1, 2, 3, 4, 5)| 1.0449| 1.0448 |          | 0.00\%          |                |

Figure 5.6: CYN, single underlying stock
Figure 5.7: CYN, 2 underlying stocks

Figure 5.8: CYN, 3 underlying stocks

Figure 5.9: CYN, 5 underlying stocks
5.4 Efficiency test

In this section, we use 1Y European and Bermudan options to compare the computation efficiency between the DNN method and classical Monte Carlo method. We select a single underlying stock and 10 underlying stocks (stock 1, 2, ... , 10) in our tests. The results from a single process Monte Carlo (called “sequential Monte Carlo” below) and a parallel Monte Carlo are both presented. Note that in order to have a “fair” comparison between the DNN approach and the classical Monte Carlo, a parallel Monte Carlo is needed since the DNN approach (mainly TensorFlow library) is parallelized. The parallel Monte Carlo method is based on sub-sampling approach and implemented as follows:

(1) Divide the $M$ sampling paths to $n$ groups of sub-sampling paths with equal size,
(2) Run the Monte Carlo method on these $n$ groups in parallel,
(3) Aggregate these $n$ prices.

We test on a desktop (Intel Xeon Silver 4108) which has 8 cores/16 threads and 24GB RAM. To maximize the computational power, we set $n = 16$ in the parallel Monte Carlo. From the previous tests, it is obvious that 5,000 iterations of training in our least square backward DNN method is more than enough to reach convergence and 500 iterations already has sufficient accuracy. Therefore we use 200, 300, and 500 iterations for DNN approach in this section.

5.4.1 Efficiency test on European options

We test on a single underlying and 10 underlying European call options, using sequential Monte Carlo, parallel Monte Carlo and backward DNN method. Maturity is chosen as 1Y and strike is chosen as $\sum_i \omega_i X_0^i$ with equal weight $\omega_i = 1/d$ to make the option ATM. Some key market parameters are given in Table 5.1. Results are presented in Table 5.6 and 5.7. The results indicate that for simple product (including high-dimensional case), sequential Monte Carlo and parallel Monte Carlo can produce identical prices at the same number of sampling paths and the parallel Monte Carlo is most efficient among the three tested methods. Based on the time consumed (500K) for a 10 dimensional 1Y European option in the parallel MC method, it can be estimated that a 100 dimensional 10Y European option takes around 647s, which is similar to the computational time taken for a 10 dimensional 1Y European option in the DNN approach at iteration of 200. Apparently, in general, DNN method is not efficient in pricing European style options. This is primarily caused by the high cost in the DNN initialization and optimization.
Table 5.6: Comparison among sequential Monte Carlo, parallel Monte Carlo and backward DNN method on 1D European option

|               | Sequential Monte Carlo | Parallel Monte Carlo | Backward DNN |
|---------------|------------------------|----------------------|--------------|
|               | # path | Mean | StdError | Time | StdError/Mean | # path | Mean | StdError | Time | StdError/Mean | # path | Mean | StdError | Time | StdError/Mean | MaxIter | Price | Time |
| 5K            |        | 6.5450 | 0.1642 | 0.08 | 2.51% | 500K    | 6.8755 | 0.0171 | 0.68 | 0.25% | 200 | 6.8674 | 208.63 |
| 10K           |        | 6.7329 | 0.1180 | 0.14 | 1.74% | 1M      | 6.8521 | 0.0123 | 2.39 | 0.18% | 300 | 6.8570 | 262.78 |
| 20K           |        | 6.8666 | 0.0847 | 0.29 | 1.23% | 2M      | 6.8585 | 0.0086 | 2.89 | 0.12% | 500 | 6.8577 | 326.68 |
| 50K           |        | 6.9225 | 0.0540 | 0.69 | 0.78% | 5M      | 6.8645 | 0.0054 | 6.90 | 0.08% |     |        |        |
| 100K          |        | 6.8534 | 0.0381 | 1.39 | 0.56% |         |        |        |      |        |     |        |        |
| 200K          |        | 6.8762 | 0.0270 | 2.71 | 0.39% |         |        |        |      |        |     |        |        |
| 500K          |        | 6.8555 | 0.0173 | 6.80 | 0.25% |         |        |        |      |        |     |        |        |
| 1M            |        | 6.8521 | 0.0121 | 14.14 | 0.18% |         |        |        |      |        |     |        |        |

Table 5.7: Comparison among sequential Monte Carlo, parallel Monte Carlo and backward DNN method on 10D European option

|               | Sequential Monte Carlo | Parallel Monte Carlo | Backward DNN |
|---------------|------------------------|----------------------|--------------|
|               | # path | Mean | StdError | Time | StdError/Mean | # path | Mean | StdError | Time | StdError/Mean | # path | Mean | StdError | Time | StdError/Mean | MaxIter | Price | Time |
| 5K            |        | 7.3532 | 0.1789 | 0.73 | 2.43% | 500K    | 7.2302 | 0.0178 | 6.47 | 0.25% | 200 | 7.2538 | 600.76 |
| 10K           |        | 7.4738 | 0.1258 | 1.32 | 1.73% | 1M      | 7.2371 | 0.0126 | 12.92 | 0.17% | 300 | 7.2461 | 158.84 |
| 20K           |        | 7.2304 | 0.0888 | 2.65 | 1.26% | 2M      | 7.2474 | 0.0089 | 25.49 | 0.12% | 500 | 7.2750 | 1054.74 |
| 50K           |        | 7.1857 | 0.0562 | 6.67 | 0.78% | 5M      | 7.2389 | 0.0056 | 64.00 | 0.08% |     |        |        |
| 100K          |        | 7.2233 | 0.0398 | 12.93 | 0.55% |         |        |        |      |        |     |        |        |
| 200K          |        | 7.2114 | 0.0252 | 26.38 | 0.39% |         |        |        |      |        |     |        |        |
| 500K          |        | 7.2302 | 0.0178 | 64.51 | 0.25% |         |        |        |      |        |     |        |        |
| 1M            |        | 7.2371 | 0.0120 | 128.10 | 0.17% |         |        |        |      |        |     |        |        |

5.4.2 Efficiency test on Bermudan options

It is well known that the parallelization of least square Monte Carlo, widely used in high dimensional American/Bermudan option pricing in practice, is a challenging task due to that the regression consumes most of the computation time for American options and Bermuda options with many early exercise times. However, since regression at each exercise date requires the cross-sectional information from all paths, regression step is not straightforward to parallelize. Realizing this characteristics, Choudhury et al [6] parallelized the singular value decomposition in the regression step. Together with path generation parallelized, an efficient ratio (speed up factor/number of processors) of 56% is achieved on a IBM Blue Gene. Chen et al [5] proposed to apply space decomposition (the same as the sub-sampling approach used in this paper) to both the path generation phase and the regression/valuation phases. In Chen et al’s work each sub-sample is an independent least square MC run. The authors found that the speedup efficiency can be around 100% for 8 processes and around 80% for 64 processes. Though there is significant improvement in parallelization efficiency, pricing bias is observed and the magnitude of the bias increases with the increase in the number of processes.
We test on a single underlying and 10 underlying Bermudan call options, using sequential Monte Carlo, parallel Monte Carlo and backward DNN method. The contract has a maturity of 1Y and is exercisable monthly (a total of 12 early exercise dates). The strike is chosen as $\sum_{d} \omega_i X_i^0$ with equal weight $\omega_i = 1/d$ to make the option ATM. Some key market parameters are given in Table 5.1. Results are presented in Table 5.8 and 5.9.

When there is only one underlying asset (i.e. at low dimensions), all three approaches produce very close option values and the parallel Monte Carlo is most efficient among the three tested methods. When there are 10 underlyings (i.e. at high-dimensions), the observations are interesting. First, different from efficiency test with European options, the computation time for Monte Carlo increases non-linearly with respect to the number of paths as the least square regression for high-dimensional case with a large number of sampling paths needs much more computation time. Second, the sub-sampling based parallel Monte Carlo, though much more efficient than sequential Monte Carlo, can produce pricing bias as large as 2-5%. This is due to that the least square regression only uses sub-sampling paths in the parallel Monte Carlo. The number of these sub-sampling paths is only 1/16 of the paths in the corresponding sequential Monte Carlo and this leads to sub-optimal exercise strategy. Third, it can be seen that the least square backward DNN can produce much closer option values to sequential Monte Carlo with much higher efficiency. For instance, the time taken for the 200-iteration run is only around 1/3 of the time taken for parallel Monte Carlo with 500K paths. Our testing indicates that the least square backward DNN method is an efficient algorithm for pricing high-dimensional American/Bermudan options.

Table 5.8: Comparison among sequential Monte Carlo, parallel Monte Carlo and least square backward DNN method on 1D Bermudan option

| # path | Sequential Monte Carlo | Parallel Monte Carlo | LSQ Backward DNN |
|--------|------------------------|----------------------|------------------|
|        | # path | Mean | StdError | Time | StdError/Mean | # path | Mean | StdError | Time | StdError/Mean | MaxIter | Price | Time |
| 5K     | 7.0098 | 0.1351 | 0.25 | 1.91% | 100K | 7.0324 | 0.0143 | 2.58 | 0.20% | 200 | 7.0308 | 237.81 |
| 10K    | 7.1826 | 0.1012 | 0.51 | 1.41% | 500K | 7.0072 | 0.0194 | 5.20 | 0.14% | 500 | 7.0241 | 257.49 |
| 20K    | 7.0098 | 0.0720 | 0.97 | 1.03% | 1M | 7.0091 | 0.0071 | 10.56 | 0.10% | 300 | 7.0253 | 377.38 |
| 50K    | 7.0341 | 0.0450 | 2.36 | 0.64% | 2M | 7.0177 | 0.0045 | 25.14 | 0.06% | 500 | 7.0253 | 377.38 |
| 100K   | 7.0577 | 0.0318 | 4.71 | 0.45% | 5M | 7.0177 | 0.0045 | 25.14 | 0.06% | 500 | 7.0253 | 377.38 |
| 200K   | 7.0114 | 0.0225 | 9.36 | 0.32% | 1M | 7.0177 | 0.0045 | 25.14 | 0.06% | 500 | 7.0253 | 377.38 |
| 500K   | 6.9873 | 0.0142 | 24.98 | 0.20% | 10M | 7.0000 | 0.0014 | 74.51 | 0.10% | 500 | 7.0253 | 377.38 |
| 1M     | 7.0153 | 0.0101 | 47.39 | 0.14% | 50M | 7.0000 | 0.0014 | 74.51 | 0.10% | 500 | 7.0253 | 377.38 |
Table 5.9: Comparison among sequential Monte Carlo, parallel Monte Carlo and least square backward DNN method on 10D Bermudan option

| # path | Mean  | StdError | Time  | StdError/Mean |
|--------|-------|----------|-------|---------------|
| 5K     | 6.5215| 0.1367   | 30.00 | 2.10%         |
| 10K    | 8.5074| 0.0139   | 91.93 | 1.48%         |
| 20K    | 7.1418| 0.0727   | 199.50| 1.02%         |
| 50K    | 7.4101| 0.0473   | 674.50| 0.64%         |
| 100K   | 7.4443| 0.0332   | 1391.75| 0.45%         |
| 200K   | 7.4163| 0.0235   | 3428.15| 0.45%         |
| 500K   | 7.4387| 0.0149   | 12751.68| 0.20%        |

6 Conclusion

In this work, we have developed a deep learning-based least square forward-backward stochastic differential equation solver, which can be used in high-dimensional derivatives pricing. Our deep learning implementation follows a similar approach to the ones explored by Weinan E et al [7, 9] and Wang et al [14]. However, the forward DNN method is more suitable for European style derivative pricing and the backward DNN method from Wang et al may not adequately account for the early exercise features. In our approach we embed the least square regression technique similar to that in the least square Monte Carlo method ([11]) to the backward DNN algorithm. Numerical testing results on Bermudan options and callable yield notes indicate that our least square backward DNN method can produce accurate results based on comparisons to PDE and Monte Carlo methods, with the relative difference as small as 0.3% or lower. This method can be used in most of the derivative pricing, including Barrier option, American option, convertible bonds, etc. In conclusion, our least square backward DNN algorithm can serve as a generic numerical solver for pricing derivatives, and it is most efficient for high-dimensional derivatives with early exercises features, as demonstrated by our computational efficiency tests.

References

[1] Ali Al-Aradi, Adolfo Correia, Danilo Naiff, Gabriel Jardim, and Yuri Saporito. Solving nonlinear and high-dimensional partial differential equations via deep learning. abs/1811.08782, 2018.

[2] JÃ©rÃ©me Barraquand and Didier Martineau. Numerical valuation of high dimensional multivariate american securities. The Journal of Financial and Quantitative Analysis, 30(3):383–405, 1995.
[3] Christian Beck, Sebastian Becker, Philipp Grohs, Nor Jaafari, and Arnulf Jentzen. Solving stochastic differential equations and kolmogorov equations by means of deep learning. *CoRR*, abs/1806.00421, 2018.

[4] Mark Broadie and Paul Glasserman. A stochastic mesh method for pricing high-dimensional american options. *Journal of Computational Finance*, 7(4):35–72, 2004.

[5] Ching-Wen Chen, Kuan-Lin Huang, and Yuh-Dauh Lyuu. Accelerating the least-square monte carlo method with parallel computing. *The Journal of Supercomputing*, 71(9):3593–3608, Sep 2015.

[6] A. R. Choudhury, A. King, S. Kumar, and Y. Sabharwal. Optimizations in financial engineering: The least-squares monte carlo method of longstaff and schwartz. In 2008 *IEEE International Symposium on Parallel and Distributed Processing*, pages 1–11, April 2008.

[7] Weinan E, Jiequn Han, and Arnulf Jentzen. Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations. *Communications in Mathematics and Statistics*, 5(4):349–380, Dec 2017.

[8] Masaaki Fujii, Akihiko Takahashi, and Masayuki Takahashi. Asymptotic expansion as prior knowledge in deep learning method for high dimensional bsdes. *Asia-Pacific Financial Markets*, Mar 2019.

[9] Jiequn Han, Arnulf Jentzen, and Weinan E. Solving high-dimensional partial differential equations using deep learning. *Proceedings of the National Academy of Sciences*, 115(34):8505–8510, 2018.

[10] Michael Kohler, Adam Krzyżak, and Nebojsa Todorovic. Pricing of high-dimensional american options by neural networks. *Mathematical Finance*, 20(3):383–410, 2010.

[11] Francis A. Longstaff and Eduardo S. Schwartz. Valuing American Options by Simulation: A Simple Least-Squares Approach. *The Review of Financial Studies*, 14(1):113–147, 06 2015.

[12] G. N. Mil’shtejn. Approximate Integration of Stochastic Differential Equations. *Theory of Probability & Its Applications*, 19(3):557–562, 1975.

[13] Maziar Raissi. Forward-backward stochastic neural networks: Deep learning of high-dimensional partial differential equations. abs/1804.07010, 2018.

[14] Haojie Wang, Han Chen, Agus Sudjianto, Richard Liu, and Qi Shen. Deep Learning-Based BSDE Solver for Libor Market Model with Applications to Bermudan Swaption Pricing and Hedging. 2018.