CIRCULAR MAXIMAL FUNCTIONS ON THE HEISENBERG GROUP

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Abstract. We prove the $L^p$ boundedness of the circular maximal function on the Heisenberg group $\mathbb{H}^1$ for $2 < p \leq \infty$. The proof is based on the square sum estimate associated with the $2 \times 2$ cone $|\langle \xi'_1, \xi'_2 \rangle| = |\langle \xi'_3, \xi'_4 \rangle|$ of the phase space arising from the vector fields $X_1, X_2, tX_3, \partial/\partial t$ on the Heisenberg group, rather than the $2 \times 1$ cone $|\langle \xi_1, \xi_2 \rangle| = |\xi_3|$ of the frequency space arising from $\partial/\partial x_1, \partial/\partial x_2, \partial/\partial t$ on the Euclidean space.

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1. Introduction

The Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n} \times \mathbb{R}$, whose group law is given by

$$(x, x_{2n+1}) \cdot (y, y_{2n+1}) = (x + y, x_{2n+1} + y_{2n+1} + \langle E(x), y \rangle)$$

where $E = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

As the group inverse of $(y, y_{2n+1})$ is given by $(-y, -y_{2n+1})$, the convolution $*_{\mathbb{H}}$ on $\mathbb{H}^n$ is defined by

$$f *_{\mathbb{H}} K(x, x_{2n+1}) = \int f(x - y, x_{2n+1} - y_{2n+1} - \langle E(x), y \rangle) K(y, y_{2n+1}) dy dy_{2n+1}.$$

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Let \( d\sigma \) be the surface-carried measure on the unit sphere \( S^{2n-1} \subseteq \mathbb{R}^n \), and set \([d\sigma]_t = d\sigma(\cdot/t)/t^{2n}\) with \( t > 0 \). Furthermore, let \( \delta_{2n+1} \) be the Dirac mass at \( 0 \in \mathbb{R} \) (in the last coordinate). Then, the measure \([d\sigma]_t \otimes \delta_{2n+1}\) is supported on the horizontal plane of \( \mathbb{H}^n \). The convolution of \( f \) with this measure is given by

\[
f *_H ([d\sigma]_t \otimes \delta_{2n+1})(x, x_{2n+1}) = \int_{y \in S^{2n-1}} f(x - ty, x_{2n+1} - (E(x), ty)) d\sigma(y),
\]

representing the average of \( f \) along the \( 2n - 1 \) dimensional sphere embedded in the hyperplane

\[
\pi_E(x, x_{2n+1}) := (x, x_{2n+1}) + \{(y, (E(x), y)) : y \in \mathbb{R}^{2n}\} \text{ contained in } \mathbb{R}^{2n+1}
\]

We define the spherical maximal operator \( \mathcal{M}_n \) on the horizontal plane of \( \mathbb{H}^n \) by

\[
\mathcal{M}_n f(x, x_{2n+1}) = \sup_{t \geq 0} f *_H ([d\sigma]_t \otimes \delta_{2n+1})(x, x_{2n+1}).
\]

In 1997, Nevo and Thangavelu \cite{10} initiated the study of the maximal average \( \mathcal{M}_n \) in \cite{13}, to prove the maximal and pointwise ergodic theorems associated with the spherical measure in the Heisenberg group \( \mathbb{H}^n \). They conjectured that the circular maximal average \( \mathcal{M}_1 \) is bounded in \( L^p(\mathbb{H}^1) \) if \( 2 < p \leq \infty \).

In 2004, Muller and Seeger \cite{9} viewed the average in \( \mathcal{M}_n \) as a Fourier integral operator with fold singularities, and proved the \( L^p \) boundedness of \( \mathcal{M}_n \) on \( \mathbb{H}^n \) for \( n \geq 2 \). In addition, Narayanan and Thangavelu \cite{11} obtained the same \( L^p \) range using spectral theorems associated with the sphere measure on the Heisenberg group. Their results \cite{9, 11} state that, for \( n \geq 2 \),

\[
\|\mathcal{M}_n\|_{L^p(\mathbb{H}^n)} \lesssim 1 \text{ if and only if } \frac{2n}{2n-1} < p \leq \infty.
\]

More recently in \cite{11}, the horizontal plane is replaced by the general hyperplane in the Heisenberg group \( \mathbb{H}^n \). The purpose of this paper is to prove the \( L^p(\mathbb{H}^1) \) boundedness of the circular maximal average as Nevo and Thangavelu conjectured \cite{13} with \( n = 1 \):

**Main Theorem 1.** For the circular measure \( d\sigma \) in \cite{12} and \cite{13} with \( n = 1 \), it holds that

\[
\|\mathcal{M}_1\|_{L^p(\mathbb{H}^1)} \lesssim 1 \text{ if and only if } 2 < p \leq \infty.
\]

In 1976, Stein \cite{15} utilized the decay rate of the Fourier transform of the spherical measure \( d\sigma \) to determine the \( L^p \) range \( p > n/(n - 1) \) for the maximal means over the sphere \( S^{n-1} \subseteq \mathbb{R}^n \). In 1986, Bourgain \cite{2} resolved the case for the circle \( S^1 \). In 1991, Muckenaupt, Seeger, and Sogge \cite{8} obtained a local smoothing theorem including the result of Bourgain via the analysis of the Fourier transform side-cone (which we denote as the \( C_{2 \times 1} \) cone) in \( \mathbb{R}^3 \), arising from the wave equation in \( \mathbb{R}^2 \). As an analogue of this frequency cone to be adapted in our phase space (consisting of \( x, x_3, t, \xi, \xi_3 \) of \( \mathbb{H}^1 \), we take a three-dimensional object called the “\( C_{2 \times 2} \) cone” in \( \mathbb{R}^4 \). This arises naturally from a variant of the wave equation in \( \mathbb{H}^1 \), whose solution is given by the circular average \( f *_H ([d\sigma]_t \otimes \delta_3)(x, x_3) \), as described in Section \cite{12}. We utilize this \( C_{2 \times 2} \) cone to obtain the \( L^4 \) estimates corresponding to \cite{8} and \cite{13}. 


**Organization.** In Sections 2 and 3, we reduce matters to the $L^4$ estimates of the Fourier integral operators with the phase function $(x, x_3) \cdot (\xi, \xi_3) + t|\xi + \xi_3 E(x)|$. In Section 4, we construct the "$C_{2 \times 2}$ cone" containing the singularities of this phase function, and present a strategy for the square sum estimates in $L^4$. In Section 5, we decompose the phase space according to small cubes attached to the $C_{2 \times 2}$ cone. We establish the $L^4$ estimates in Sections 6 through 9.

**Notation.** The skew-symmetric matrix is usually denoted by $J$. However, we denote $J$ by $-E$. We write the counterclockwise rotation of $x$ by $\pi/2$ as $E(x)$. Let $\rho > 0$. Then, given two vectors $u$ and $v$ in $\mathbb{R}^d$, we write $v = u + O(\rho)$ if there exists $C > 0$ independent of $u, v,$ and $\rho$ such that

$$|v - u| \leq C\rho. \tag{1.6}$$

Let $U$ and $V$ be two subsets of $\mathbb{R}^d$. Then, we write $U = V + O(\rho)$ if all $u \in U$ and $v \in V$ satisfy (1.6). Accordingly, we denote the sum $V + B(0, C\rho)$ of the two sets as

$$V + O(\rho) = \{ u + v : u \in U \text{ and } |v| \leq C\rho \} \tag{1.7}$$

without specifying the constant $C$. We employ the smooth cutoff functions

1. $\psi$ supported in $\{ u : |u| \leq 1 \} \subset \mathbb{R}^1$ or $\mathbb{R}^2$ or $\mathbb{R}^3$ with $\psi(u) \equiv 1$ in $|u| < 1/2$ and

2. $\chi$ supported in $\{ u : 1/2 \leq |u| \leq 2 \} \subset \mathbb{R}^1$ or $\mathbb{R}^2$ or $\mathbb{R}^3$,

allowing slight line-by-line modifications of $\chi$ and $\psi$. Because the Heisenberg group $\mathbb{H}^1$ is identical to $\mathbb{R}^3$ as a set, we often employ the norm of $L^p(\mathbb{R}^3)$ rather than $L^p(\mathbb{H}^1)$, so long as the convolution rule in (1.1) is maintained. We denote the Euclidean Fourier transform (or the inverse Fourier transform) of $f$ in $\mathcal{S}(\mathbb{R}^d)$ by $\hat{f}$ (or $f^\vee$). We do not employ the group Fourier transform on the Heisenberg group $\mathbb{H}^n$ in this study. Given two scalars $a, b$, we write $a \lesssim b$ if $a \leq Cb$ for some $C > 0$ independent of $a, b$. The notation $a \approx b$ denotes that $a \lesssim b$ and $b \lesssim a$.

## 2. Basic Decompositions

Let $M_{2 \times 2}$ be the set of $2 \times 2$ real matrices, and take a function $f$ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$. Rather than the skew-symmetric $E$ in (1.1), we work with a more general $A \in M_{2 \times 2}$ when defining the average

$$A_{S^1(A)}f(x, x_3, t) = \int_{y \in S^1} f(x - ty, x_3 - \langle A(x), ty \rangle) d\sigma(y), \tag{2.1}$$

where $(x, x_3) \in \mathbb{R}^2 \times \mathbb{R}$. Associated with $A \in M_{2 \times 2}$, we set

$$M^A f(x, x_3) = \sup_{t > 0} A_{S^1(A)}f(x, x_3, t) \quad \text{so that } M^E = M_1 \text{ in (1.5) if } E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$
2.1. Expression in Phase Space. Using the Euclidean Fourier transform of $f \in S(\mathbb{R}^3)$, we express the convolution in (2.4) as the pseudo-differential operator

\[ \mathcal{A}_{S^1}(A) f(x, x_3, t) = \int e^{2\pi i(x, x_3, t) \cdot \xi} \hat{\rho}(t(\xi + \xi_3 A(x))) \hat{f}(\xi, \xi_3) d\xi d\xi_3 \]

with the symbol $\hat{\rho}(t(\xi + \xi_3 A(x)))$, where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Here, $\hat{\rho}$ is the Euclidean Fourier transform of the measure $d\sigma$ on the unit circle $S^1$, given by

\[ \hat{\rho}(\xi) = 2\pi J_0(2\pi |\xi|) \text{ for } \xi \in \mathbb{R}^2, \]

where $J_0$ is the Bessel function. Using the asymptotic expansion of the Bessel function $J_0$, we can split

\[ 2\pi J_0(2\pi |\xi|) = m_0(\xi) + e^{2\pi i|\xi|} m_+(\xi) + e^{-2\pi i|\xi|} m_-(\xi), \]

where $m_0$ is a smooth function supported on $|\xi| \leq 2$, while $m_{\pm}$, supported on $|\xi| \geq 1$, satisfies

\[ m_{\pm}(\xi) = c|\xi|^{-1/2} + O(|\xi|^{-3/2}). \]

2.2. Basic Symbol Decomposition. The maximal average of (2.3) associated with the symbol $m_0(t(\xi + \xi_3 A(x)))$ in (2.4) is bounded in $L^p$ for any $p > 1$, as it is controlled by the Hardy–Littlewood maximal function associated with the moving plane $(x, x_3) - \{(y, A(x), y) : y \in \mathbb{R}^2\}$ with a non-vanishing rotational curvature $\det(A)$. Thus, we only have to work with $e^{\pm 2\pi i|\xi|} m_{\pm}(\xi)$ in (2.3), with $m_\pm$ in the form of (2.5). Because $|\xi|^{-3/2}$ gives a better bound on the support $|\xi| \geq 1$, it suffices to work only with the first term $m(\xi) = c|\xi|^{-1/2}$ of (2.5) supported $|\xi| \geq 1$. So, we set

\[ m(\xi) = |\xi|^{-1/2}(1 - \psi(\xi)) \text{ with } \psi \text{ defined in } (1) \]

and redefine $\mathcal{M}^A f(x, x_3) = \sup_{t > 0} \mathcal{A}_{S^1}(A) f(x, x_3, t)$ in (2.4), where

\[ \mathcal{A}_{S^1}(A) f(x, x_3, t) = \int e^{2\pi i(x, x_3, t) \cdot \xi} m(t(\xi + \xi_3 A(x))) \hat{f}(\xi, \xi_3) d\xi d\xi_3. \]

In (2.7), we only treat $|\xi + \xi_3 A(x)|$ among $|\xi + \xi_3 A(x)|$, by allowing $t < 0$. Given $A \in M_{2 \times 2}$, we write the wave propagation operator associated with a general symbol $\sigma \in C^\infty(\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3)$ as

\[ \mathcal{T}_\sigma f(x, x_3, t) = \int e^{2\pi i[(x, x_3, t) \cdot (\xi, \xi_3)]} \sigma(x, x_3, t, \xi, \xi_3) \hat{f}(\xi, \xi_3) d\xi d\xi_3 \text{ for } f \in S(\mathbb{R}^3). \]

Then, $\mathcal{A}_{S^1}(A) f = \mathcal{T}_\sigma f$ in (2.7) if $\sigma(x, x_3, t, \xi, \xi_3) = m(t(\xi + \xi_3 A(x)))$ with $m$ in (2.6). For $j \in \mathbb{Z}_+$, let

\[ \tilde{\chi} \left( \frac{t(\xi + \xi_3 A(x))}{2^j} \right) = \left| \frac{t(\xi + \xi_3 A(x))}{2^j} \right|^{-1/2} \chi \left( \frac{t(\xi + \xi_3 A(x))}{2^j} \right) \]

and set for $k \in \mathbb{Z}$,

\[ m_{j,k}(x, x_3, t, \xi, \xi_3) = \chi \left( \frac{t}{2^{-j-k}} \right) \tilde{\chi} \left( \frac{t(\xi + \xi_3 A(x))}{2^j} \right). \]
We decompose \( m(t(\xi + \xi_3 A(x))) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-j/2} m_{j,k}(x, x_3, t, \xi, \xi_3) \) in (2.7). We set the symbol supported on the low and middle frequencies of \( \xi_3 \) as

\[
a_{j,k}(x, x_3, t, \xi, \xi_3) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-j/2} m_{j,k}(x, x_3, t, \xi, \xi_3)\]

(2.10)

For the extremely high frequency \( \xi_3 \), we set the symbol

\[
b_{j,k}(x, x_3, t, \xi, \xi_3) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{-j/2} m_{j,k}(x, x_3, t, \xi, \xi_3)\]

(2.11)

Then, we can split

\[m_{j,k}(x, x_3, t, \xi, \xi_3) = a_{j,k}(x, x_3, t, \xi, \xi_3) + b_{j,k}(x, x_3, t, \xi, \xi_3)\]

so that \( \mathcal{M}_{m_{j,k}} = \mathcal{M}_{a_{j,k}} + \mathcal{M}_{b_{j,k}} \) in the sense of (2.8). Thus, the maximal averages corresponding to \( a_{j,k} \) and \( b_{j,k} \) are defined in (2.10) and (2.11), respectively.

Then, for \( \mathcal{M}^A \) defined above (2.7), we have

\[
\mathcal{M}^A f(x, x_3) \leq \mathcal{M}^a f(x, x_3) + \mathcal{M}^b f(x, x_3).\]

(2.13)

First, we switch the supremum in (2.12) with the following vector-valued norms:

**Lemma 2.1.** For any \( 2 \leq p < \infty \), it holds that

\[
\| \mathcal{M}^a f \|_{L^p(\mathbb{R}^3)} \lesssim 2^{j(1/p - 1/2)} \left( \sum_{k \in \mathbb{Z}} |2^{k/p} \mathcal{M}_{a_{j,k}} f|^p \right)^{1/p} \|_{L^p(\mathbb{R}^3 \times \mathbb{R})}
\]

(2.14)

and

\[
\| \mathcal{M}^b f \|_{L^2(\mathbb{R}^3)} \lesssim \left( \sum_{k \in \mathbb{Z}} |2^{k/2} \mathcal{M}_{b_{j,k}} f|^2 \right)^{1/2} \|_{L^2(\mathbb{R}^3 \times \mathbb{R})}
\]

(2.15)

where \( a_{j,k} \) and \( b_{j,k} \) are defined in (2.10) and (2.11), respectively.

**Proof.** Consider \( \mathcal{T}_{a_{j,k}} f \) in (2.8) and (2.9), and apply the Sobolev inequality majorizing the \( L^\infty(dt) \) norm by the \( L^p_{1/p}(dt) \) norm of \( \mathcal{T}_{a_{j,k}} f(x, x_3, \cdot) \). Then, we lose the amount \( |\xi + \xi_3 A(x)|^{1/p} \approx 2^{j+k}/p \) owing to \((\partial/\partial t)^{1/p} \) of \( e^{2\pi t |\xi + \xi_3 A(x)|} \) on the support of \( \chi_0(2^{k/p} \mathcal{M}_{a_{j,k}} f \mathcal{M}_{a_{j,k}} f |)^{1/p} \) in (2.14) and (2.15).

Next, we replace \( \sup_k \mathcal{T}_{a_{j,k}} f \) by \( (\sum_k |\mathcal{T}_{a_{j,k}} f|^p)^{1/p} \), to obtain (2.14). We similarly obtain (2.15) for \( p = 2 \).
3. Statement of Main Estimates

3.1. Littlewood–Paley Decompositions.

**Definition 3.1.** Note that \( \psi(\xi, \xi_3) \) is supported on \(|(\xi, \xi_3)| \leq 1 \), and \( \psi(\xi, \xi_3) \equiv 1 \) on \(|(\xi, \xi_3)| \leq 1/2 \). Using the non-isotropic dilation

\[
(3.1) \quad \chi_k(\xi, \xi_3) = \psi \left( \frac{\xi}{2^{k+1}}, \frac{\xi_3}{2^{2(k+1)}} \right) - \psi \left( \frac{\xi}{2^k}, \frac{\xi_3}{2^{2k}} \right),
\]

we define the Littlewood–Paley projection \( P_k f \) as

\[
(3.2) \quad P_k f (y, y_3) = \int e^{2\pi i (\eta y + \eta_3 y_3)} \chi_k(\eta, \eta_3) f(\eta, \eta_3) d\eta d\eta_3.
\]

**Lemma 3.1.** For \( p = 4 \), we have that

\[
(3.3) \quad \left\| \left( \sum_k |P_k f|^p \right)^{1/p} \right\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.
\]

**Proof.** The case of \( p = 2 \) follows from the almost orthogonality estimate of \( \|P_{k_1}^* P_{k_2}\|_{op} = O(2^{-c|k_1 - k_2|}) \). Interpolating the results for this \( p = 2 \) and the trivial \( p = \infty \) yields \( (3.3) \) for \( p = 4 \). \( \square \)

By the composition of \( \sum_{\ell \in \mathbb{Z}} P_{j+k+\ell} = \text{Id} \) for fixed \( j, k \), we have the following in (2.1):

\[
(3.4) \quad \|M_j f\|_{L^p(\mathbb{R}^3)} \lesssim 2^{j(1/p - 1/2)} \left\| \left( \sum_k |2^{k/p} T_{a_{j,k}} P_{j+k+\ell} f|^p \right)^{1/p} \right\|_{L^p(\mathbb{R}^3 \times \mathbb{R})}.
\]

Let \( a = (a_k)_{k \in \mathbb{Z}} \) with \( a_k \in \mathbb{C} \), and define the \( \ell^p \) norm of \( a = (a_k) \) with a weight \( 2^{k/p} \) for each \( k \)

\[
(3.5) \quad \|a\|_{\ell^p} = \left( \sum_{k \in \mathbb{Z}} |2^{k/p} a_k|^p \right)^{1/p}.
\]

For each \( j \in \mathbb{Z}_+ \) and \( \ell \in \mathbb{Z} \), let us define a vector-valued function

\[
(3.6) \quad G^a_{j, \ell} f = (T_{a_{j,k}} P_{j+k+\ell} f)_{k \in \mathbb{Z}}.
\]

Then, \( \|G^a_{j, \ell} f\|_{L^p(|\ell| \mathbb{R}^3 \times \mathbb{R})} \) is the \( \ell^\text{th} \) piece of the summation in (3.4), given by

\[
(3.7) \quad \left\| G^a_{j, \ell} f \right\|_{L^p(|\ell| \mathbb{R}^3 \times \mathbb{R})} \left( \sum_k |2^{k/p} T_{a_{j,k}} P_{j+k+\ell} f|^p \right)^{1/p} \right\|_{L^p(\mathbb{R}^3 \times \mathbb{R})}.
\]

Let \( A = E \) and \( 2 < p \leq 4 \). Then, we shall prove that for some \( c, c(p) \geq 0 \) and sufficiently small \( \epsilon > 0 \), it holds that

\[
(3.8) \quad \|G^a_{j, \ell} f\|_{L^p(|\ell| \mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{-c|\ell|} 2^{j \left( \frac{1}{p} + \frac{1}{4} - \epsilon \right)} 2^{c(p)j} \|f\|_{L^p(\mathbb{R}^3)} \text{ for all } f \in S(\mathbb{R}^3).
\]

Then, by summing the estimates \( (3.8) \) over \( \ell \) in (3.4), we obtain that for \( A = E \),

\[
\|M_j^a\|_{L^p(\mathbb{R}^3) \rightarrow L^p(\mathbb{R}^3)} \lesssim 2^{j \left( \frac{1}{p} - \frac{1}{4} + \epsilon \right)} 2^{j \left( \frac{1}{p} + \frac{1}{4} - \epsilon \right)} 2^{c(p)j}.
\]
By now summing this over \( j \in \mathbb{Z}_+ \) in (2.12), for \( 2 < p \leq 4 \) we obtain that for \( A = E \),

\[
\|M^a\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \lesssim 1. \tag{3.9}
\]

Using the last part of Proposition 3.1 of [7], it holds for an invertible \( A \) that

\[
\|G^b_{j,\ell} f\|_{L^2(\mathbb{R}^3)} \lesssim 2^{-(j+|\ell|)} \|f\|_{L^2(\mathbb{R}^3)} \quad \text{for all } f \in L^2(\mathbb{R}^3)
\]

which implies that in (2.15) it holds that

\[
\|M^b_j\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim 2^{-cj} \tag{3.10}
\]

By the straightforward computation of the \( L^1 \) norm of the integral kernel in (2.12), we obtain

\[
\|M^b_j\|_{L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)} \lesssim 2^{j/2}.
\]

This with (3.10) gives

\[
\|M^b\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \lesssim 1 \quad \text{for all } 2 \leq p \leq 2 + c_0 \quad \text{for some } c_0 > 0.
\]

Using this with the bound of \( M^a \) in (3.9), for \( A = E \) in (2.13) we obtain

\[
\|M^A\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \lesssim 1 \quad \text{for } 2 < p \leq 2 + c_1 \quad \text{for some } c_1 > 0.
\]

Interpolating this and the case \( p = \infty \) yields the desired result of the main theorem \[1\]

\[
\|M^A\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \lesssim 1 \quad \text{for } 2 < p < \infty
\]

for \( A = E \). Hence, our aim is to prove (3.8).

3.2. Interpolation. To deal with (3.8), we require the following interpolation lemma.

**Lemma 3.2.** [Interpolation for vector-valued operators] Let \( \|a\|_{\ell^p} \) be defined as in (3.5). Suppose that a vector-valued map \( f \to U(f) := (U_k(f))_{k \in \mathbb{Z}} \) satisfies the following for \( f \in \mathcal{S}(\mathbb{R}^d) \):

\[
\|U(f)\|_{L^p_0(\ell^p_0)} \leq C_0 \|f\|_{L^p_0} \quad \text{and} \quad \|U(f)\|_{L^{p_1}(\ell^{p_1})} \leq C_1 \|f\|_{L^{p_1}}. \tag{3.11}
\]

Then, for \( 1/p = (1 - \theta)/p_0 + \theta/p_1 \) with \( \theta \in [0, 1] \) it holds that

\[
\|U(f)\|_{L^{\theta p}(\ell^p)} \leq C_0^{1-\theta} C_1^\theta \|f\|_{L^p}.
\]

Here, we write \( L^p(\mathbb{R}^d) \) and \( L^p(\ell^p(\mathbb{R}^d)) \) as \( L^p \) and \( L^p(\ell^p) \), for simplicity. The proof is given in the appendix.
3.3. Statement of Main Estimates.

Proposition 3.1 ($L^2$ estimates for square sum). Suppose that $\text{rank}(EA + (EA)^T) = 2$, which holds when $A = E$. Recall that $G_{j,\ell}^\alpha f = \{T_{a_{j,k}} P_{j+k+\ell} f\}_{k\in\mathbb{Z}}$ in (3.6). Then, for some $C > 0$ we have that for all $f \in L^2(\mathbb{R})$,

\[(3.12) \|G_{j,\ell}^\alpha f\|_{L^2(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{-\epsilon |\ell|} \|f\|_{L^2(\mathbb{R}^3)} \text{ for } |\ell| \geq Cj,
\]

\[(3.13) \|G_{j,\ell}^\alpha f\|_{L^2(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{\epsilon |\ell|} \|f\|_{L^2(\mathbb{R}^3)} \text{ for } |\ell| < Cj.
\]

Proof of (3.12) and (3.13). See Proposition 3.1 of [7].

Our main estimate in this study is the following $L^4$ inequality for all $f \in L^4(\mathbb{R}^3)$:

\[(3.14) \|G_{j,\ell}^\alpha f\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{\left(\frac{1}{2} + \frac{1}{4} + \epsilon\right) j} \|f\|_{L^4(\mathbb{R}^3)} \text{ for sufficiently small } \epsilon > 0.
\]

Proof of (3.14) under the assumption of (3.14). Assume that (3.14) holds, and let $2 < p < 4$ and $1/p = (1 - \theta)/2 + \theta/4$. The interpolation of (3.12) and (3.14) via Lemma 3.2 gives

\[(3.15) \|G_{j,\ell}^\alpha f\|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{\left(\frac{1}{2} + \frac{1}{4} + \epsilon\right) j} 2^{-c|\ell| (1 - \theta)} \|f\|_{L^p(\mathbb{R}^3)} \text{ for } |\ell| \geq Cj.
\]

The similar interpolation of (3.14) and (3.14) yields

\[(3.16) \|G_{j,\ell}^\alpha f\|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{\left(\frac{1}{2} + \frac{1}{4} + \epsilon\right) j} 2^{c|\ell| (1 - \theta)} \|f\|_{L^p(\mathbb{R}^3)} \text{ for } |\ell| < Cj.
\]

From $\theta = 2 - \frac{4}{p}$, we have that \(\left(\frac{1}{2} + \frac{1}{4} + \epsilon\right) j = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{2p}\right) j + \left(2 - \frac{4}{p}\right) \epsilon j\) in the exponents of (3.15) and (3.16). Combined with the conditions $|\ell| < Cj$ and $\epsilon \ll 1$, this gives the desired estimate (3.14).

Therefore, our goal is to prove (3.14). To this end, it suffices to show that for each $k$ it holds that

\[(3.17) \|2^{k/4} T_{a_{j,k}} f\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{\left(\frac{1}{2} + \frac{1}{4}\right) j} 2^{|\ell|} \|f\|_{L^4(\mathbb{R}^3)},
\]

because (3.17) and (3.3) imply that

\[
\left\| \left( \sum_k \|2^{k/4} T_{a_{j,k}} P_{j+k+\ell} f\|^4 \right)^{1/4} \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{\left(\frac{1}{2} + \frac{1}{4}\right) j} 2^{|\ell|} \left\| \left( \sum_k |P_{j+k} f|^4 \right)^{1/4} \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})}.
\]

On the other hand, we have the following useful localization property.

Lemma 3.3 (Localization). Let $B(0,r) = \{(x, x_3) \in \mathbb{R}^3 : |(x, x_3)| < r\}$, and fix $j \geq 0$. As in (2.10), let

\[ T_{a_{j,k}} f(x, x_3, t) = \int 2\pi i (\xi x + \xi x_3 + \xi + \xi A(x)) a_{j,k}(x, x_3, t, \xi, \xi_3) \hat{f}(\xi, \xi_3) d\xi d\xi_3. \]
where \( a_{j,k} \) is defined in (2.8). Let \( I = [-2, -1] \cup [1, 2] \). Then
\[
\|2^{k/p}T_{a_{j,k}}f\|_{L^p(\mathbb{R}^3 \times \mathbb{R})} \leq C_p \|f\|_{L^p(\mathbb{R}^3)} \quad \text{for all } f \in L^p
\]
if and only if
\[
\|T_{a_{j,0}}f\|_{L^p(B(0,1) \times I)} \leq C_p \|f\|_{L^p(B(0,100))} \quad \text{for all } f \in L^p.
\]

**Proof.** See Proposition 4.1 of Section 4 in [7]. \(\square\)

The Lemma 3.3 tells us that (3.17) for \( k \in \mathbb{Z} \) is equivalent to the case with \( k = 0 \). Thus we state our goal in the following theorem.

**Theorem 3.1.** Suppose that \( A = E \). To define \( T_{a_{j,0}} \) in (2.8), let
\[
a_{j,0}(x, x_3, t, \xi, \xi) = \psi(x, x_3) \chi(t) \left( \frac{t(\xi + \xi_1 A(x))}{2^j} \right) \psi \left( \frac{||\xi_1||}{2^{j+\epsilon_0}} \right).
\]
Then, for a sufficiently small \( \epsilon > 0 \) we have that for all \( f \in L^4(\mathbb{R}^3), \)
\[
(3.18) \quad \|T_{a_{j,0}}f\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{(1+\epsilon/2)2^j} \|f\|_{L^4(B(0,1) \times \mathbb{R})}.
\]

**Remark 3.1.** Here \( 1/8 \) arises from the square sum estimate of \( T_{a_{j,0}} \) in Theorem 6.1. The other exponent \( 1/16 \) appears as the square root of the \( L^2 \) norm of the Nikodym maximal operator on the Heisenberg group \( H^1 \) in Proposition 8.1.

### 4. Idea of Proof

We introduce the key idea for adapting the method of [8] to the Heisenberg group \( H^1 \) in Section 4.4. We start with an elementary geometric property of quadruples of vectors, which are required for the square sum estimates in \( L^4(H^3) \) as well as in \( L^4(\mathbb{R}^2) \).

#### 4.1. Conjugate Pairings in an Ellipse.

**Definition 4.1.** [Ellipse and Elliptical Rings] Collect pairs of non-parallel vectors of \( \mathbb{R}^2 \) in acute angles:
\[
V = \left\{ (\eta, \eta') \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\}) : 0 < \frac{\eta'}{||\eta'||} - \frac{\eta}{||\eta||} < \pi/2 \right\}.
\]
Given \( (\xi, \xi') \in V \), let \( E(\xi, \xi') \) be the ellipse in \( \mathbb{R}^2 \) with the two foci \( 0 \) and \( \xi + \xi' \) containing both \( \xi \) and \( \xi' \). By giving a width \( \rho \) to the ellipse \( E(\xi, \xi') \), we define an elliptical ring \( E(\xi, \xi') + O(\rho) \) as in (1.7).

**Definition 4.2** (Elliptic Conjugates in \( \mu S^1 \times \nu S^1 \)). Let \( \mu S^1 = \{ \xi \in \mathbb{R}^2 : |\xi| = \mu \} \) and \( \nu S^1 = \{ \xi' \in \mathbb{R}^2 : |\xi'| = \nu \} \) be two circles in \( \mathbb{R}^2 \). We say that two pairs \( (\xi, \xi') \in V \) and \( (\eta, \eta') \in V \) are elliptic conjugates in \((\mu S^1) \times (\nu S^1)\) if
\[
(4.1) \quad (\xi, \xi'), (\eta, \eta') \in \mu S^1 \times \nu S^1 \quad \text{and} \quad \eta + \eta' = \xi + \xi'.
\]
Denote the relation (4.1) by \((\xi, \xi') \sim_{\mu, \nu} (\eta, \eta')\). Then, given \((\xi, \xi') \in \mu S^1 \times \nu S^1\) we call the pair \((\eta, \eta')\) of (4.1) such that \((\eta, \eta') \neq (\xi, \xi')\) the **elliptic conjugate** of \((\xi, \xi')\).

**Lemma 4.1** (The elliptic conjugate of \((\xi, \xi')\) is unique). Given a pair \((\xi, \xi') \in V\), there exists a unique elliptic conjugate \((\eta, \eta')\) of \((\xi, \xi')\). Moreover, consider \(N : \mu S^1 \times \nu S^1 \cap V \to \mu S^1 \times \nu S^1 \cap V\) whose value \(N(\xi, \xi') = (\eta, \eta')\) of \((\xi, \xi')\) as in (4.1). Then, \(N\) is a one-to-one and onto map, whose value at \((\xi, \xi')\) is exactly

\[
N(\xi, \xi') \in \bigg((|\xi|S^1 \cap \mathbb{E}(\xi, \xi') \setminus \{\xi\}) \times (|\xi'|S^1 \cap \mathbb{E}(\xi, \xi') \setminus \{\xi'\})\bigg) \cap V
\]

where \(\mathbb{E}(\xi, \xi')\) is an ellipse containing \(\xi, \xi'\) whose foci are 0 and \(\xi + \xi'\), as in Definition 4.1. We shall write this elliptic conjugate of \((\xi, \xi')\) as \(N(\xi, \xi') = (\eta(\xi, \xi'), \eta'(\xi, \xi'))\).

**Proof.** Let \((\xi, \xi') \in \mu S^1 \times \nu S^1\). We first show the existence of the elliptic conjugate of \((\xi, \xi')\) by the construction of (4.2). We observe that the intersections of the ellipse \(\mathbb{E}(\xi, \xi')\) in Definition 4.1 with each circle \(\mu S^1\) and \(\nu S^1\) are doubletons, which are expressed as \(\mathbb{E}(\xi, \xi') \cap \mu S^1 = \{\xi, \eta\}\) and \(\mathbb{E}(\xi, \xi') \cap \nu S^1 = \{\xi', \eta'\}\) respectively. Namely,

\[
\eta \in [\mu S^1 \cap \mathbb{E}(\xi, \xi') \setminus \{\xi\}] \text{ and } \eta' \in [\nu S^1 \cap \mathbb{E}(\xi, \xi') \setminus \{\xi'\}]
\]

We can observe that \(\frac{1}{2}(\xi + \xi')\) and \(\frac{1}{2}(\eta + \eta')\) coincide, which means that \((\eta, \eta')\) satisfies

\[
(\xi, \xi') \sim_{\mu, \nu} (\eta, \eta') \text{ in (4.1) with } (\eta, \eta') \neq (\xi, \xi')
\]

which shows the existence of an elliptic conjugate of \((\xi, \xi')\). Next, we demonstrate the uniqueness of the elliptic conjugate of \((\xi, \xi')\) by showing that it coincides with \(N(\xi, \xi')\) in (4.2). If \((\eta, \eta')\) is the pair in (4.4), this implies that \(|\xi| + |\xi'| = |\eta| + |\eta'|\) and \(\xi + \xi' = \eta + \eta'\). Then, from the definition of the ellipse this implies that both of \(\eta, \eta'\) must be on the ellipse \(\mathbb{E}(\xi, \xi')\) passing through \(\xi\) and \(\xi'\). Thus, \((\eta, \eta')\) is uniquely determined as that in (4.4). This proves the uniqueness of the elliptic conjugate of \((\xi, \xi')\). The above proof yields that the elliptic conjugate of \((\xi, \xi')\) is the exact value of \(N(\xi, \xi')\) defined in (4.2). Now, we switch the roles of \((\xi, \xi')\) and \((\eta, \eta')\) in (4.3). Then, \(N(\eta, \eta) = (\xi, \xi')\), and we observe that \(N\) is a reflection satisfying

\[
N \circ N(\xi, \xi') = (\xi, \xi').
\]

This implies that \(N\) is a bijective map on \((\mu S^1 \times \nu S^1) \cap V\).

**Remark 4.1.** We start with \((\xi, \xi') \in V\) in the above proof, and choose \(\mu = |\xi|, \nu = |\xi'| > 0\) such that \((\xi, \xi') \in \mu S^1 \times \nu S^1\). This enables us to extend the domain \(N\) to the union of \((\mu S^1 \times \nu S^1) \cap V\) over all possible \((\mu, \nu) \in \mathbb{R}^2_+\). The extended function \(N\) is a reflection and bijection on \(V\) satisfying (4.2). \(\square\)
Definition 4.3. Fix \( j \in \mathbb{Z}_+ \). Then, for \( m \in \mathbb{Z}_+ \) with \( 2^{j/2} \leq m \leq 2^{j/2+1} \), denote \( J(m) = m2^{j/2} \). Define a circle and an annulus in the sense of \( \mathbb{1.7} \):

\[
S_m := \{ \xi \in \mathbb{R}^2 : |\xi| = J(m) \} \text{ and } S_m + O(2^{j/2})
\]

We equivalently write \( S_m \times S_n + O(2^{j/2}) \) instead of \((S_m + O(2^{j/2})) \times (S_n + O(2^{j/2})) \). Suppose that

\[
(4.6) \quad \eta + \eta' = \xi + \xi' + O(2^{j/2}) \text{ and } (\eta, \eta') \in S_m \times S_n + O(2^{j/2}).
\]

We denote this relation by \( (\xi, \xi') \sim_{(m,n)} (\eta, \eta') \) within \( O(2^{j/2}) \). If \( (\xi, \xi') \sim_{(m,n)} (\eta, \eta') \) within \( O(2^{j/2}) \) and \( (\eta, \eta') \neq (\xi, \xi') + O(2^{j/2}) \), then we say that \( (\eta, \eta') \) is an elliptic conjugate of \( (\xi, \xi') \) in \( S_m \times S_n + O(2^{j/2}) \).

Lemma 4.2 (Elliptic Conjugate of \( (\xi, \xi') \) in \( S_m \times S_n + O(2^{j/2}) \)). Fix \( (\xi, \xi') \in V \), and let \( N \) be defined in \( \mathbb{2.2} \). If \( (\eta, \eta') \sim_{(m,n)} (\eta, \eta') \) within \( O(2^{j/2}) \) as \( \mathbb{4.6} \), then

\[
(4.7) \quad (\eta, \eta') = (\xi, \xi') + O(2^{j/2}) \text{ or } (\eta, \eta') = N(\xi, \xi') + O(2^{j/2})
\]

which is written as \( (\eta, \eta') \in \{(\xi, \xi'), N(\xi, \xi')\} + O(2^{j/2}) \).

Proof. It suffices to work with \( N(\xi, \xi') \neq (\xi, \xi') + O(2^{j/2}) \). Given such \( (\xi, \xi) \), let \( (\eta, \eta') \) be a pair satisfying \( \mathbb{4.6} \). If \( (\eta, \eta') = (\xi, \xi') + O(2^{j/2}) \), then this satisfies \( \mathbb{4.7} \). Thus, it suffices to show that \( (\eta, \eta') = N(\xi, \xi') + O(2^{j/2}) \) if \( (\eta, \eta') \neq (\xi, \xi') + O(2^{j/2}) \). We set a width \( 2^{j/2} \) for the two circles \( S_m, S_n \) and the ellipse \( E(\xi, \xi') \) in Lemma \( \mathbb{4.1} \). Then, \( (\eta, \eta') \) satisfying \( \mathbb{4.6} \) is contained in

\[
(4.8) \quad \left( [E(\xi, \xi') \cap S_m] \times [E(\xi, \xi') \cap S_n] + O(2^{j/2}) \right) \cap \left( [\xi, \xi'] + O(2^{j/2}) \right)
\]

which also contains the pair \( N(\xi, \xi') \) defined in Lemma \( \mathbb{4.1} \). On the other hand, from the comparability of the two radii \( n2^{j/2} \) and \( m2^{j/2} \) of \( S_n \) and \( S_m \), where \( 2^{j/2} \leq m, n \leq 2^{j/2+1} \), we observe that each of the two circles \( S_n \) and \( S_m \) intersects with the ellipse \( E(\xi, \xi') \) transversally. Hence, there exists \( C > 0 \) such that

\[
|E(\xi, \xi') - N(\xi, \xi')| \leq C 2^{j/2} \text{ for all } (\xi, \xi') \text{ contained in } \mathbb{4.8}.
\]

This implies that \( (\eta, \eta') = N(\xi, \xi') + O(2^{j/2}) \). \( \square \)

Remark 4.2. In \( \mathbb{4.6} \) and \( \mathbb{4.7} \), we can replace the width \( 2^{j/2} \) by a slightly larger \( 2^{j/2+\epsilon} \) for some small \( \epsilon > 0 \) whenever both \( S_n \) and \( S_m \) still intersect the ellipse \( E(\xi, \xi') \) transversally.

4.2. Solutions and Darboux Equation. Let \( f \in \mathcal{S}(\mathbb{R}^2) \) and \( (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \). Then, the circular average \( A_f(x, t) = \int_{y \in S} f(x - ty) d\sigma(y) \) in the Euclidean space is the solution of the initial value problem

\[
(4.9) \quad \left( \frac{\partial}{\partial x_1} \right)^2 + \left( \frac{\partial}{\partial x_2} \right)^2 u = \left( \frac{\partial}{\partial t} + \frac{1}{t} \frac{\partial}{\partial \theta} \right) u \text{ with } u(x, 0) = f(x) \text{ and } u_t(x, 0) = 0.
\]
This equation is a variant of the wave equation called the Darboux equation \[ \text{[4.9]} \]. Now, we can derive the general version of the Darboux equation for the Heisenberg group \( \mathbb{H}^1 \). To state the equation, we take the basis \( \{X_1, X_2, X_3\} \) of the Heisenberg algebra \( \mathfrak{h}^1 \),

\[
(4.10) \quad X_1 = \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}, \quad \text{and} \quad X_3 = \frac{\partial}{\partial x_3},
\]

satisfying the canonical relations of the commutators \([X_1, X_2] = 2X_3\) with the other Lie brackets vanishing. We can obtain the analogous Darboux equation of \( \text{[4.9]} \) in the Heisenberg group \( \mathbb{H}^1 \) as follows:

\[
(4.11) \quad (X_1^2 + X_2^2 - (tX_3)^2) u = \left( \frac{\partial}{\partial t} \right)^2 + \frac{1}{t} \frac{\partial}{\partial t} \right) u \quad \text{for} \quad u(x, x_3, 0) = f(x, x_3) \quad \text{and} \quad u_t(x, x_3, 0) = 0
\]

where \( f \in S(\mathbb{R}^3) \). We can then obtain the following lemma.

**Lemma 4.3.** \([\text{Darboux Equation}]\) The circular average \( u \) given by \( u(x, x_3, t) = A_{S^1(A)}(f)(x, x_3, t) \) in \([2,3]\) with \( A = E \) is the solution of the initial value problem of \( (4.11) \).

The proof of Lemma 4.3 is given in the appendix. The vector fields in each equation \([4.9]\) and \([4.11]\) enable us to determine the singularities of the circular averages (their solutions) in \( \mathbb{R}^2 \) and \( \mathbb{H}^1 \), respectively.

- The three vector fields \( \partial_{x_1}, \partial_{x_2}, \partial_t \) in the equation \([4.9]\) form the critical \( 2 \times 1 \) cone of \( \mathbb{R}^3 \) of the frequency space associated with its solution \( Af(x, t) \). See Section 4.3.
- The four vector fields \( X_1, X_2, tX_3, \partial_t \) in the equation \([4.11]\) form the critical \( 2 \times 2 \) cone in the phase space associated with its solution \( A_{S^1(A)}f(x, x_3, t) \). See Section 4.4.

### 4.3. Brief Review of the Euclidean Proof

Let us examine the core idea of Muckenhoupt, Seeger, and Sogge in \([8]\). For \( n \in \mathbb{Z} := [2^{j/2}, 2^{j/2+2}] \cap \mathbb{Z} \) and \( \theta \in U_0 := [2^{j/2}, 2^{j/2+2}] \cap \mathbb{Z} \), let

\[
J(n) = n2^{j/2} \in [2^j, 2^{j+1}] \quad \text{and} \quad e(\theta) = (\cos 2^{-j/2} \theta, \sin 2^{-j/2} \theta) \in S^1.
\]

Furthermore, let \( \psi_{n, \theta} \) be supported on the intersection of a thin radial annulus and thin angular sector:

\[
S_n + O(2^{j/2}) = \left\{ \xi \in \mathbb{R}^2 : |\xi| = J(n) + O(2^{j/2}) \right\}, \quad C_\theta(2^{-j/2}) = \left\{ \xi \in \mathbb{R}^2 : \frac{\xi}{|\xi|} = e(\theta) + O(2^{-j/2}) \right\}.
\]

For \( (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \), we decompose wave propagator

\[
(4.12) \quad T_{n, \theta}f(x, t) = \chi(t)\psi(x) \int e^{2\pi i \xi \cdot (x + i t \theta)} \psi_{n, \theta}(\xi) \widehat{f}(\xi) d\xi \quad \text{and} \quad T_n f(x, t) = \sum_{\theta \in U_0} T_{n, \theta} f(x, t).
\]

Using \( \partial_{x_1}, \partial_{x_2}, \) and \( \partial_t \) from the equation \([4.11]\), we define the vector field \( D = (\partial_t, \partial_{x_1}, \partial_{x_2}) \) mapping a smooth real-valued function \( \Phi \) on \( \mathbb{R}^3 \) to a vector-valued function \( (\partial_t \Phi, \partial_{x_1} \Phi, \partial_{x_2} \Phi) \). We apply this
vector field $D$ to the phase function $\xi \cdot x + t|\xi|$ in \(1.12\), to form the set of all normal vectors of the level set \(\{(t, x) : \xi \cdot x + t|\xi| = c\}\),

\[\{(|\xi|, \xi) = D(\xi \cdot x + t|\xi|) \in \mathbb{R} \times \mathbb{R}^2\}\].

We refer to this characteristic cone in the frequency space \(\mathbb{R}^3\) as the \(C_{2 \times 1}\) cone. Then, this \(C_{2 \times 1}\) cone is split into small patches \((S_n + O(2^{j/2})) \cap C_0(2^{-j/2})\). The main estimate in the Euclidean space

\[\text{Step 1. The radial square sum estimate for } p = 4,\]

\[\left\| \sum_{n \in U_1} T_nf \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R})} \lesssim 2^{j/8} \left\| \sum_{n \in U_1} |T_nf|^2 \right\|_{L^p(\mathbb{R}^2 \times \mathbb{R})}^{1/2},\]

is mainly a result of the \(L^2\) orthogonality arising from the difference of the two radii \(|\xi| = J(m) + O(2^{j/2})\) and \(|\eta| = J(n) + O(2^{j/2})\) in the frequency sides of the integrals for \(p = 2\) of \(1.15\), given by

\[\int T_mf(x,t)\overline{T_nf(x,t)}dxdt \lesssim 2^{-Nj}\|f\|^2_{L^2(\mathbb{R}^2)}\text{ for }|m-n| > 3\text{ and }N \gg 1.\]

The interpolation of the bound \(O(1)\) for \(p = 2\) from \(1.16\) and the bound \(O(2^{j/2})\) for \(p = \infty\) from the Schwartz inequality yields the bound \(2^{j/8}\) in \(1.15\).

\textbf{Step 2. The angular square sum estimate}

\[\left\| \left( \sum_{n \in U_1} \sum_{\theta \in U_0} T_n,\theta f \right)^2 \right\|_{L^4(\mathbb{R}^2 \times \mathbb{R})}^{1/2} \lesssim \left\| \sum_{(n,\theta) \in U_1 \times U_0} |T_n,\theta f|^2 \right\|_{L^4(\mathbb{R}^2 \times \mathbb{R})}^{1/2}.\]

is obtained from the orthogonality of the two functions with indices \(((m, \theta_1), (n, \theta_1'))\) and \(((m, \theta_2), (n, \theta_2'))\) in the following expression of the left-hand side of \(1.17\):

\[\sum_{(m,\theta_1),(n,\theta_1'),(m,\theta_2),(n,\theta_2')} \left( \left\langle T_{m,\theta_1}f(\cdot) T_{n,\theta_1'}(\cdot), T_{m,\theta_2}f(\cdot) T_{n,\theta_2'}(\cdot) \right\rangle_{x,t} \right).\]

The above inner product, defined by \(\langle F, G \rangle_{x,t} = \int F(x,t)\overline{G(x,t)}dxdt\), contains the key part

\[\psi_{m,\theta_1}(\xi)\psi_{n,\theta_1'}(\xi') \psi_{m,\theta_2}(\eta)\psi_{n,\theta_2'}(\eta') \int e^{2\pi i \left((\xi + \xi') - (\eta + \eta')\right)x + \left((|\xi| + |\xi'|) - (|\eta| + |\eta'|)\right)t} \chi(t)\psi(x)dxdt,\]

where \((\xi, \xi'), (\eta, \eta') \in S_m \times S_n + O(2^{j/2})\). The gradient \(D_x = (\partial_{x_1}, \partial_{x_2})\) of the phase function in \(1.19\) is given by \((\xi + \xi') - (\eta + \eta')\). This yields that \(1.19\) \(\lesssim 2^{-Nj}\) away from all possible conjugate pairs

\[\langle \xi, \xi' \rangle \sim (m, n) \langle \eta, \eta' \rangle \text{ within } O(2^{j/2})\text{ in }1.19,\]
that is,

\begin{align}
\eta + \eta' &= \xi + \xi' + O(2^{j/2}) \quad \text{(we call this the ellipse condition)} \\
(\xi, \xi') \quad \text{and} \quad (\eta, \eta') &\in S_m \times S_n + O(2^{j/2}) \quad \text{(we call this the circle condition)}.
\end{align}

From (4.21), we observe that \((\eta, \eta')\) of (4.21) and (4.22) is determined by \((\xi, \xi')\) as

\begin{equation}
(\eta, \eta') \in \{ (\xi, \xi'), \mathcal{N}(\xi, \xi') \} + O(2^{j/2}) \quad \text{where} \quad \mathcal{N}(\xi, \xi') \quad \text{is the conjugate pair of} \quad (\xi, \xi') \quad \text{in} \quad (4.2).
\end{equation}

Therefore, we only take the summation (4.18) over \((m, \theta_1), (n, \theta_2) \sim (m, n) \quad ((m, \theta_2), (n, \theta_2'))\), representing (4.20) in terms of indices, which implies (4.17).

**Step 3.** The Littlewood–Paley inequality associated with the intervals of the same size, combined with an appropriate vector-valued inequality (handled by the Nikodym-type maximal function whose direction is restricted on the cone), gives

\[
\left\| \left( \sum_{(n, \theta) \in \mathcal{U}_1 \times \mathcal{U}_0} |T_{n, \theta} f|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^2 \times \mathbb{R}^2)} \lesssim \| f \|_{L^4(\mathbb{R}^2)}.
\]

4.4. **Heisenberg Group Analogue.** As the analogue of the cone \(C_{2 \times 1}\) in the frequency space for the Euclidean case (4.18), we find the \(2 \times 2\) cone defined by \(C_{2 \times 2} := \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^2 \times \mathbb{R}^2 : |(u_3, u_4)| = |(u_1, u_2)|\} \) in the phase space \(\mathbb{R}^4\). Consider the following operator derived from Theorem 3.1

\begin{equation}
T_{\sigma} f(x, x_3, t) = \int e^{2\pi i [x \cdot \xi + x_3 \xi_3 + t(\xi + \xi_3 A(x))] } \sigma(x, x_3, t, \xi, \xi_3) f(\xi, \xi_3) d\xi d\xi_3
\end{equation}

with a suitably decomposed symbol \(\sigma(x, x_3, t, \xi, \xi_3)\). Rather than \(D = (\partial_1, \partial_{x_1}, \partial_{x_2})\) in (4.13), we employ \(X_1, X_2, t X_3, \partial / \partial t\) in the Darboux equation (4.11) of \(\mathbb{H}^1\) to define the differential vector field as

\begin{equation}
D_A = (\partial_1, tX_1, X_1, X_2) = (\partial_1, t \partial_{x_3}, \partial_{x_1} + A(x) \cdot e_1 \partial_{x_3}, \partial_{x_2} + A(x) \cdot e_2 \partial_{x_3}),
\end{equation}

where \(A(x) \cdot e_1 = x_2\) and \(A(x) \cdot e_2 = -x_1\) for the case of the Heisenberg group with \(A = E\). Then, the corresponding critical region of (4.13) in our phase space of (4.24) is the set of \(\mathbb{R}^2 \times \mathbb{R}^2\)-vectors:

\begin{equation}
D_A \left( x \cdot \xi + x_3 \xi_3 + t |\xi + \xi_3 A(x)| \right) = \left[ \left( |\xi + \xi_3 A(x)|, t \xi_3 \right), \left( |\xi + \xi_3 A(x)|, t \xi_3 A \left( \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} \right) \right) \right] \in \mathbb{R}^2 \times \mathbb{R}^2.
\end{equation}

To simplify the notation in (4.26), let

- \(\tilde{\xi} = \xi + \xi_3 A(x) \in \mathbb{R}^2\), where we note that \(\tilde{\xi}\) depends on \(x\) as well as \((\xi, \xi_3)\).
- \(U(\tilde{\xi}, t\xi_3) = \tilde{\xi} + t\xi_3 A^T(\tilde{\xi} / |\tilde{\xi}|) \in \mathbb{R}^2\) in the second vector in (4.26).
Let $A = E$ be the skew-symmetric matrix. Then, we utilize $\tilde{\xi} \perp A^T (\tilde{\xi}/|\tilde{\xi}|)$ in $U(\tilde{\xi}, t\xi_3)$ to observe in (4.26) that

$$U(\tilde{\xi}, t\xi_3) = \left| (|\tilde{\xi}|, t\xi_3) \right| \text{ in } D_A[x \cdot \xi + x_3\xi_3 + t|\xi + \xi_3 A(x)|] = \left( (|\tilde{\xi}|, t\xi_3), U(\tilde{\xi}, t\xi_3) \right).$$

These vectors form the cone $C_{2\times2} \subset \mathbb{R}^2 \times \mathbb{R}^2$. We use the three parameters $(k, \ell, \theta)$ to decompose the vectors in (4.26) and (4.27),

(D-1) $\tilde{\xi}/|\tilde{\xi}| = e([\theta]) + O(2^{-j/2})$ where $e([\theta]) = (\cos(2^{-j/2}\theta), \sin(2^{-j/2}\theta))$.
(D-2) $|\tilde{\xi}|, t\xi_3) = J(k, \ell) + O(2^{j/2})$ where $J(k, \ell) = (J(k), J(\ell))$, with $J(k) = k2^{j/2}$ and $J(\ell) = \ell2^{j/2}$.
(D-3) $U(\tilde{\xi}, t\xi_3) = \left( e([\theta]) A^T e([\theta]) \right) J(k, \ell) + O(2^{j/2})$, which follows from (D-1) and (D-2).

**Remark 4.3.** In (D-3), note that $\left( \cdot \right)$ is a $2 \times 2$ matrix, and $J(k, \ell)$ is a column vector of $(J(k), J(\ell))$.

The condition (4.27), together with (D-2) and (D-3), is expressed as

$$|J(k, \ell)| = \left( e([\theta]) A^T e([\theta]) \right) J(k, \ell)$$

because

$$\left( e([\theta]) A^T e([\theta]) \right)$$

is a unitary matrix for all $\theta$.

This only occurs if $A$ is the skew-symmetric matrix $cE$, which is the case of the Heisenberg group.

**Step 1.** We prove the radial square sum estimate of (4.16) over $n$ for the decomposition of $|J(k, \ell)| = J(n) + O(2^{j/2})$ in (D-2). This estimate results from the $L^2$ orthogonality arising from the difference of the two radii $||(\tilde{\xi}|, t\xi_3)| = J(m) + O(2^{j/2})$ and $||(\tilde{\eta}|, t\eta_3)| = J(n) + O(2^{j/2})$ for the corresponding (4.16). For this step, we also lose $2j/8$.

**Step 2.** We establish the analogue of the angular square sum estimate (4.17). Here, the parameter $\theta$ is replaced by $(k, \ell, \theta)$ in (D-1) and (D-2). The cone $C_{2\times2}$ in (4.27) and (4.28) has the following crucial effect. Once the first vector $(|\tilde{\xi}|, t\xi_3)$ of the pair in (4.27) is restricted in the circle $S_m$,

$$\text{the full pair } \left( (|\tilde{\xi}|, t\xi_3), U(\tilde{\xi}, t\xi_3) \right)$$

forms a torus $S_m \times S_m$.

Let $\tilde{\eta} = \eta + \eta_3E(x)$ and $\tilde{\xi}^\prime, \tilde{\eta}^\prime$. Then, the corresponding oscillatory integral of (4.19) is

$$\int e^{2\pi i [(|\tilde{\xi}^\prime| - (\eta + \eta^\prime)) x + (|\xi_3 + \xi_3^\prime| - (\eta_3 + \eta_3^\prime)) x_3 + ((|\tilde{\xi}^\prime| + |\xi_3^\prime| - (|\tilde{\eta}^\prime| + |\eta_3^\prime|)) t)} a(x, x_3, t, \xi, \xi_3, \xi_3^\prime, \eta, \eta_3, \eta^\prime, \eta_3^\prime) dx dx_3 dt$$

with the amplitudes $a = a_{k, \ell, \theta}$ cutoff functions for (D-1) to (D-3) above. To integrate by parts, the vector field $D_A$ in (4.25) and (4.27) is applied to the phase function of the above integral to give

$$\left[ \left( (|\tilde{\xi}|, t\xi_3), U(\tilde{\xi}, t\xi_3) \right) + \left( (|\tilde{\xi}^\prime|, t\xi_3^\prime), U(\tilde{\xi}^\prime, t\xi_3^\prime) \right) \right] - \left[ \left( (|\tilde{\eta}|, t\eta_3), U(\tilde{\eta}, t\eta_3) \right) + \left( (|\tilde{\eta}^\prime|, t\eta_3^\prime), U(\tilde{\eta}^\prime, t\eta_3^\prime) \right) \right].$$
This gives the bound $2^{-jN}$ on the integral away from the elliptic conjugate conditions corresponding to (4.21) and (4.22) as:

\[
\begin{align*}
\text{elliptic conditions} & \quad \left\{ \left( |\tilde{\xi}|, t\xi_3 \right), \left( |\tilde{\xi}'|, t\xi_3' \right), = \left( |\tilde{\eta}|, t\eta_3 \right), + O(2^{j/2}2^{2cj}) \right. \\
U(\tilde{\xi}, t\xi_3) + U(\tilde{\xi}', t\xi_3') & = U(\tilde{\eta}, t\eta_3) + U(\tilde{\eta}', t\eta_3') + O(2^{j/2}2^{2cj})
\end{align*}
\]

and

\[
\begin{align*}
\text{circle conditions} & \quad \left\{ \left( |\tilde{\xi}|, t\xi_3 \right), \left( |\tilde{\xi}'|, t\xi_3' \right), \in S^2 + O(2^{j/2}) \right. \\
\left( |\tilde{\eta}|, t\eta_3 \right), U(\tilde{\eta}, t\eta_3) & \quad \left. \in S^2 + O(2^{j/2}) \right.
\end{align*}
\]

As in (4.23) with the application of (4.7) twice now, once the vectors involving $\tilde{\xi}, \xi_3, \tilde{\xi}', \xi_3'$ are fixed, we determine the vectors involving $\tilde{\eta}, \eta_3, \tilde{\eta}', \eta_3'$ as

\[
\begin{align*}
(4.29) & \quad \left\{ \left( |\tilde{\xi}|, \eta_3 \right), \left( |\tilde{\eta}|, \eta_3' \right), \in \left\{ \left( |\tilde{\xi}|, \xi_3, \left( |\tilde{\xi}'|, \xi_3' \right), N^\left( |\tilde{\xi}|, \xi_3, \left( |\tilde{\xi}'|, \xi_3' \right), + O(2^{j/2}2^{2cj}) \right) \right. \\
U(\tilde{\eta}, t\eta_3) & \quad \left. U(\tilde{\eta}', t\eta_3') \right. \in \left\{ \left( U(\tilde{\xi}, t\xi_3), U(\tilde{\xi}', t\xi_3') \right), N^\left( U(\tilde{\xi}, t\xi_3), U(\tilde{\xi}', t\xi_3') \right), + O(2^{j/2}2^{2cj}) \right.
\end{align*}
\]

This enables us to obtain the corresponding square sum estimate of (4.17) over the indices $(k, \ell, \theta)$ in (D-1) to (D-3) without losing $2^{cj}$ for meaningful $c > 0$.

**Step 3.** We obtain the corresponding vector-valued inequality and the Littlewood–Paley inequality for the same-sized interval on $\mathbb{H}^1$ to complete the proof. For this purpose, we require that

\[
(4.30) \quad (k, \ell, \theta) \rightarrow \left[ \left( e(\theta) \right) \left| Ae(\theta) \right| \right] J(k, \ell), J(\ell) \text{ is a one to one correspondence.}
\]

This holds for $A = E$, which is the Heisenberg group case. It suffices to fix $\ell$ in (4.30). Indeed, for each $k$,

\[
(4.31) \quad \left\{ \left( e(\theta) \right) \left| Ae(\theta) \right| \right\} J(k, \ell) = J(k)e(\theta) + J(\ell)Ae(\theta) + O(2^{j/2} : \theta \in [0, 2\pi])
\]

are annuli of the radii $\sqrt{J(k)^2 + J(\ell)^2}$ with width $O(2^{j/2})$. These are disjoint for those $k$’s. Thus, for fixed $\ell$ we observe that $(k, \theta) \rightarrow \left[ e(\theta) \right] Ae(\theta) J(k, \ell)$ is one-to-one. Therefore, (4.30) holds whenever $A = E$. However, for this step, we lose $2^{j/16}$ due to the $L^2$-norm of the corresponding Nikodym maximal function over tubes $2^{-j/2} \times 2^{-j/2} \times 2^{-j/2} \times 1$.

5. Torus According to Four Vector Fields $\frac{\partial}{\partial x}, tX_3, X_1, X_2$

Recall our main target $\| \mathcal{T}_{a,j}f \|_{L^1(B(0,100))} < 2^{(n/2)}2^{2k}j \| f \|_{L^1(B(0,100))}$ of Theorem 3.1 dealing with

\[
T_{a,j}f(x, 3, t) = e^{2\pi i \left( \xi x + \xi_3 x + |\xi + \xi_3 A(x)\right)} \mathcal{A}_{j,0}(x, 3, t, \xi_3) \tilde{f}(\xi, \xi_3) d\xi d\xi_3,
\]
where supp($f$) $\subset B(0, 100)$ such that $f = f\chi_{B(0,100)}$, and the symbol $a_{j,0}$ is given by

\[ a_{j,0}(x, x_3, t, \xi, \xi_3) = \chi(t) \psi(x, x_3) \chi\left(\frac{t[\xi + \xi_3 A(x)]}{2^{j/2}}\right) \psi\left(\frac{\xi_3}{2^{(j+\epsilon)}}\right). \tag{5.2} \]

In the support of $a_{j,0}$,

- $([\xi + \xi_3 A(x)], \xi_3)$ is contained in $[2^j, 2^{j+1}] \times [-2^{(j+\epsilon)}, 2^{(j+\epsilon)}]$;
- $(x, x_3, t)$ is contained in $[-1, 1]^3 \times I$ with $I = [-2, -1] \cup [1, 2]$.

We decompose the symbol $a_{j,0}$ into the equal-sized pieces, restricting some vectors in (1.26).

5.1. Decompositions of Four Indices. We decompose these four mixed variables of $a_{j,0}$ in (5.2) as follows:

- $t$ and $\frac{\xi + \xi_3 A(x)}{[\xi + \xi_3 A(x)]}$ in the scale of $2^{-j/2}$
- $|\xi + \xi_3 A(x)|$ and $t\xi_3$ in the scale of $2^{j/2}$

**Definition 5.1.** Recall that $e(2^{-j/2}) = (\cos(2^{-j/2}), \sin(2^{-j/2}))$ for $\theta \in \mathbb{U}_0$. Let

\[ \mathbb{U}_0 = [0, 2^{j/2+2}] \cap \mathbb{Z}, \mathbb{U}_1 = [2^{j/2}, 2^{j/2+2}] \cap \mathbb{Z} \text{ and } \mathbb{U}_0^* = [0, 2^{j/2+2j}] \cap \mathbb{Z}. \tag{5.3} \]

By using $\psi$ supported on $|u| \leq 1$ in $\mathbb{R}$ or $\mathbb{R}^2$, we decompose $t$ and $\frac{\xi + \xi_3 A(x)}{[\xi + \xi_3 A(x)]}$ in the scale of $2^{-j/2}$ as

\[ \sum_{q \in \mathbb{U}_1} \psi\left(\frac{t - q 2^{-j/2}}{2^{-j/2}}\right) \equiv 1, ~ \sum_{\theta \in \mathbb{U}_0} \psi\left(\frac{\xi + \xi_3 A(x)}{[\xi + \xi_3 A(x)]} - e(2^{-j/2})\right) \equiv 1 \tag{5.4} \]

and decompose $|\xi + \xi_3 A(x)|$ and $t\xi_3$ in the scale of $2^{j/2}$ as

\[ \sum_{k \in \mathbb{U}_1} \psi\left(\frac{|\xi + \xi_3 A(x)| - k 2^{j/2}}{2^{j/2}}\right) \equiv 1, ~ \sum_{\ell \in \mathbb{U}_0^*} \psi\left(\frac{q 2^{-j/2} \xi_3 - \ell 2^{j/2}}{2^{j/2}}\right) \equiv 1. \tag{5.5} \]

Here we work with $t \in [1, 2]$ and $t\xi_3 \in [0, 2^{(1+\epsilon)}]$. The other cases can be treated similarly. To simplify the notation in cases with no confusion, we set

\[ J(k) := k 2^{j/2}, \ J(\ell) := \ell 2^{j/2} \text{ with } J(k, \ell) = (J(k), J(\ell)) \text{ and } [q] := q 2^{-j/2}, \ [\theta] := \theta 2^{-j/2} \tag{5.6} \]

where $k, q$ start from $2^{j/2}$ and $\ell, \theta$ from 0. These are used for decomposing each of the four quantities

$|\xi + \xi_3 A(x)| \approx 2^j, \ |t\xi_3| \approx 2^{j+\epsilon}, \ t \approx 1$ and $\frac{\xi + \xi_3 A(x)}{[\xi + \xi_3 A(x)]} \in S^1$ of $a_{j,0}$ in (5.2) respectively into approximately $2^{j/2}$ pieces. For each $(q, k, \ell, \theta) \in \mathbb{U}_1 \times \mathbb{U}_1 \times \mathbb{U}_0 \times \mathbb{U}_0^*$, we use (5.4)–(5.6) to define

\[ a_{k,\ell,\theta}^j(x, x_3, t, \xi, \xi_3) \]

\[ = \psi(x, x_3) \psi\left(\frac{t - [q]}{2^{-j/2}}\right) \psi\left(\frac{|\xi + \xi_3 A(x)| - J(k)}{2^{j/2}}\right) \psi\left(\frac{[q] \xi_3 - J(\ell)}{2^{j/2}}\right) \psi\left(\frac{\xi + \xi_3 A(x)}{[\xi + \xi_3 A(x)]} - e([\theta])\right). \tag{5.7} \]
The support of $a_{k,t,\theta}^q$ above is restricted within
\begin{equation}
(x + \xi_3 A(x), \xi_3) \text{ in the cube } C_{k,t,\theta}^q \text{ of diameter } 2^{j/2}, \text{ and } t \text{ in the interval of length } 2^{-j/2},
\end{equation}
where $C_{k,t,\theta}^q$ is shaped like a cube located around $2^j$ far from the origin, in the following sense:
\[ C_{k,t,\theta}^q := \left\{ (\xi + \xi_3 A(x), [q]\xi_3) : \xi + \xi_3 A(x) = J(k)e(\theta) + O(2^{j/2}) \text{ and } [q]\xi_3 = J(\ell) + O(2^{j/2}) \right\}. \]
This gives our main decomposition $a_{j,0}$ of (5.2):
\[ a_{j,0}(x, x_3, t, \xi, \xi_3) = \sum_{q \in U_1} \sum_{k \in U_1} \sum_{\ell \in U_0} \sum_{\theta \in U_0} a_{k,t,\theta}^q(x, x_3, t, \xi, \xi_3). \]

With each symbol $a_{k,t,\theta}^q$, we associate an operator
\begin{equation}
T_{k,t,\theta}^q f(x, x_3, t) = \int e^{2\pi i \Phi(x, x_3, t, \xi, \xi_3)} a_{k,t,\theta}^q(x, x_3, t, \xi, \xi_3) \hat{f}(\xi, \xi_3) d\xi d\xi_3
\end{equation}
where the phase function is
\begin{equation}
\Phi(x, x_3, t, \xi, \xi_3) = x \cdot \xi + x_3 \xi_3 + t(\xi + \xi_3 A(x)).
\end{equation}
From (5.10) with the space localization (3.18) of (5.10), we observe that $\hat{f}$ in (5.9) is replaced by
\begin{equation}
\hat{f}_j(\xi, \xi_3) = [f \chi_{B(0,100)}](\xi, \xi_3) \chi\left(\frac{|\xi|}{2^{j+\epsilon_j+10}}\right) \psi\left(\frac{\xi_3}{2^{j+\epsilon_j+10}}\right).
\end{equation}

Now, we can decompose our operator $T_{a_{j,0}}$ in (5.3) as
\begin{equation}
T_{a_{j,0}} = \sum_{q \in U_1} \sum_{k \in U_1} \sum_{\ell \in U_0} \sum_{\theta \in U_0} T_{k,t,\theta}^q \text{ and } T_{k,t,\theta} = \sum_{q \in U_1} T_{k,t,\theta}^q.
\end{equation}

**Lemma 5.1.** For each fixed $(k, \ell, \theta)$, the integral operator $T_{k,t,\theta}^q$ is expressed as the integral operator
\[ T_{k,t,\theta}^q f(x, x_3, t) = \int H_{k,t,\theta}^q(x, x_3, t, y, y_3) f(y, y_3) dy dy_3 \]
satisfying that
\begin{equation}
|H_{k,t,\theta}^q(x, x_3, t, y, y_3)| \lesssim 2^{3j/2} |(x - x_3 + te(\theta), x_3, y - y_3 + tA(x) \cdot e(\theta))| 2^{j/2} + 1)^N.
\end{equation}

**Proof.** In view of (5.10), the integral kernel of $T_{k,t,\theta}^q$ is given by
\begin{equation}
H_{k,t,\theta}^q(x, x_3, t, y, y_3) = \int e^{2\pi i ((x + x_3 \xi)(x_3 + \xi) + t(\xi + \xi_3 A(x) - \xi_3 A(x)))} a_{k,t,\theta}^q(x, x_3, t, \xi, \xi_3) d\xi d\xi_3
\end{equation}
with $a_{k,t,\theta}^q(x, x_3, t, \xi, \xi_3)$ given in (5.7). We use the gradient $\nabla_{\xi,\xi_3}$ of the phase function in (5.14)
\[ (x - y + te(\theta), x_3, y_3 + tA(x) \cdot e(\theta)) + O(2^{-j/2}) \]
and $\nabla_{\xi,\xi_3} a_{k,t,\theta}^q = O(2^{-j/2})$ to apply integration by parts. Repeat this $N$ times to obtain (5.13).
5.2. Vector Field $D_A$ and the $2 \times 2$ Cone. Note that $\langle E(x), e_1 \rangle = -x_2$ and $\langle E(x), e_2 \rangle = x_1$, and recall that in (4.10),

$$X_1 = \left( \frac{\partial}{\partial x_1} + \langle A(x), e_1 \rangle \frac{\partial}{\partial x_3} \right), \quad X_2 = \left( \frac{\partial}{\partial x_2} + \langle A(x), e_2 \rangle \frac{\partial}{\partial x_3} \right), \quad X_3 = \frac{\partial}{\partial x_3} \text{ where } A = E.$$

As in (4.26), we apply the vector field $D_A$ defined below to a function $F = F(x, x_3, t)$:

$$D_A := (\partial_t, tX_3, X_1, X_2) = \left( \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_3}, \left( \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) + A(x) \frac{\partial}{\partial x_3} \right) \right) \text{ at each } (x, x_3, t).$$

Recall that $U: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is given by

$$U(\xi, t\xi_3) = \xi + t\xi_3 A^T(\xi/|\xi|) \text{ in } (5.27) \text{ where } \xi = \xi + \xi_3 A(x),$$

and

$$U(\xi + \xi_3 A(x), t\xi_3) := \xi + \xi_3 A(x) + t\xi_3 A^T \left( \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} \right) \text{ in } (5.16).$$

As in (4.26), by computing the partial derivatives of the phase function $\Phi$ in (5.10), we obtain

$$D_A \Phi(x, x_3, t, \xi, \xi_3) = \left[ \frac{\partial}{\partial x_1} \left| \xi + \xi_3 A(x) \right|, t\xi_3, U(\xi + \xi_3 A(x), t\xi_3) \right] \in \mathbb{R}^2 \times \mathbb{R}^2.$$

If $A = E$, then $A^T = -E$. So, the two terms $\xi + \xi_3 E(x)$ and $E \left( \frac{\xi + \xi_3 E(x)}{|\xi + \xi_3 E(x)|} \right)$ in (5.16) are perpendicular. Thus,

$$|U(\xi + \xi_3 E(x), t\xi_3)| = |(|\xi + \xi_3 E(x)|, t\xi_3)| \text{ as } (5.17).$$

Hence the vectors $D_E \Phi(x, x_3, t, \xi, \xi_3)$ in (5.17) are contained in

$$C_{2 \times 2} \text{ cone := } \{ (u_1, u_2, u_3, u_4) \in \mathbb{R}^2 \times \mathbb{R}^2 : |(u_1, u_2)| = |(u_3, u_4)| \}.$$

Lemma 5.2. Let $(x, x_3, t, \xi, \xi_3)$ be in the support of the integral in (5.9). Furthermore, let $J(k, \ell) \in \mathbb{R}^2$ have entries $J(k) = k2^{j/2} \in (2^j, 2^{j+2})$ and $J(\ell) = \ell 2^{j/2} \in (0, 2^{j+2})$, as in (4.6). To each pair $(A, \theta)$ of a $2 \times 2$ matrix $A$ and $\theta \in U_0$, we assign a $2 \times 2$ matrix $A_\theta$ defined by

$$A_\theta := \left( c(|\theta|) \quad A^T c(|\theta|) \right) \text{ with } |\theta| = \theta 2^{-j/2}.$$

Then, in the support of (5.7) the derivative $D_A \Phi(x, x_3, t, \xi, \xi_3)$ in (5.14) satisfies the size condition

$$D_A \Phi(x, x_3, t, \xi, \xi_3) = \left[ \left( \frac{\partial}{\partial x_1} \left| \xi + \xi_3 A(x) \right|, t\xi_3, A_\theta \left( \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} \right) \right] + O(2^{j/2}) \right.$$

$$= \left( J(k, \ell), A_\theta J(k, \ell) \right) + O(2^{j/2}) \text{ in } \mathbb{R}^2 \times \mathbb{R}^2.$$

If $A = E$ (which is the case for the Heisenberg group), then for all $\theta$ it holds that

$$A_\theta \text{ in } (5.19) \text{ is the unitary matrix satisfying } |A_\theta J(k, \ell)| = |J(k, \ell)|,$$
which implies $D_E\Phi(x, x_3, t, \xi_3) \in C_{2 \times 2}$ as in (4.28) and (5.13). Finally, regard $\theta$ as a continuous variable. If $\det(AE + (AE)^T) > 0$ (which holds when $A = E$), then there exists $c > 0$ such that

$$
\left| \frac{\partial}{\partial \theta} (A_0 x) \right| = 2^{-j/2} \left( e^{+}([\theta]) \left| A^T e^{+}([\theta]) \right| - c 2^{-j/2} |x| \right) \text{ for all } x \in \mathbb{R}^2,
$$

(5.22)

Proof. By the support condition of (5.14) and (5.6),

$$
|\xi + \xi_3 A(x)| = J(k) + O(2^{j/2}), \quad t\xi_3 = J(\ell) + O(2^{j/2}) \quad \text{and} \quad \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} = e([\theta]) + O(2^{-j/2})
$$

with $(k, \ell, \theta) \in U_1 \times U_0 \times U_0$. Therefore, by (5.19), (5.23), and (5.20), we have that

$$
U(\xi + \xi_3 A(x), t\xi_3) = |\xi + \xi_3 A(x)| e([\theta]) + t\xi_3 A^T e([\theta]) + O(2^{j/2})
$$

$$
= A_0 \left( |\xi + \xi_3 A(x)|, t\xi_3 \right) + O(2^{j/2}).
$$

Together with (5.17) and (5.23), this again shows (5.20). If $A = E$, then $E e([\theta]) = e^{+}([\theta])$ yields $A_0$ in (5.19) is unitary in (5.21). Finally, the equality of (5.22) follows from

$$
\frac{\partial}{\partial \theta} (e([\theta])) = 2^{-j/2} (-\sin(2^{j/2} \theta), \cos(2^{j/2} \theta)) = 2^{-j/2} e^{+}([\theta]).
$$

In addition, the inequality of (5.22) follows from

$$
\det \left( e^{+}([\theta]) \left| A^T e^{+}([\theta]) \right| \right) = \langle A E e([\theta]), e([\theta]) \rangle = \left\langle \frac{(AE + (AE)^T)}{2} e([\theta]), e([\theta]) \right\rangle \gtrless 1,
$$

because $\det(AE + (AE)^T) > 0$ in our hypothesis. \hfill $\Box$

6. Square Sum over Radial Decompositions

The first part of this section is the $L^2$ orthogonality estimates over $(k, \ell, \theta) \in U_1 \times U_0 \times U_0$.

6.1. $L^2$ orthogonality.

Proposition 6.1. Let $T_{k, \ell, \theta} = \sum_{\theta' \in U_1} T'_{k, \ell, \theta}$. If $|(k_1, \ell_1, \theta_1) - (k_2, \ell_2, \theta_2)| \gtrsim 2^{c_1}$, then for a sufficiently large $M > 0$ it holds that

$$
\langle T_{k_1, \ell_1, \theta_1, f}, T_{k_2, \ell_2, \theta_2, f} \rangle \lesssim 2^{-j M} \|f\|_{L^2(\mathbb{R}^3)}^2.
$$

(6.1)

To prove Proposition 6.1 we require the non-overlapping condition of $(J(k, \ell), A_0 J(k, \ell))_{k, \ell, \theta}$.

Lemma 6.1. Recall that $J(k) = k 2^{j/2}, J(\ell) = \ell 2^{j/2}$ and $J(k, \ell)$ in (5.10), with $(k, \ell, \theta) \in U_1 \times U_0 \times U_0$. Suppose that $\det(AE + (AE)^T) > 0$ (which holds when $A = E$), and let $A_0$ in (5.14). If $|(k, \ell, \theta) - (k', \ell', \theta')| \gtrsim 2^{c_1}$ in (5.20), then there exists $c > 0$ such that

$$
\left| \left( J(k, \ell), A_0 J(k, \ell) \right) - \left( J(k', \ell'), A_0 J(k', \ell') \right) \right| \gtrsim e^{2^{c_1} 2^{j/2}}.
$$

(6.2)
Proof. Let \(|(k, \ell) - (k', \ell')| \geq c_1 2^{2j}\) for some \(c_1 > 0\). Then, (6.2) holds, because \(|J(k, \ell) - J(k', \ell')| = 2^{j/2}|(k, \ell) - (k', \ell')| \geq c_1 2^{2j}2^{j/2}\). Thus, we consider the case that \(|(k, \ell) - (k', \ell')| < c_1 2^{2j}\) (where \(0 < c_1 < 1/10\) will be fixed later). For this case, it holds that \(|\theta - \theta'| \geq 2^{2j-1}\), as \(|(k, \ell, \theta) - (k', \ell', \theta')| \geq 2^{2j}\) in our hypothesis. Hence, it suffices to show that

\[
(6.3) \quad |(k, \ell) - (k', \ell')| < c_1 2^{2j} \quad \text{and} \quad |\theta - \theta'| \geq 2^{2j-1} \implies |A_\theta J(k, \ell) - A_{\theta'} J(k', \ell')| \geq c 2^{j/2+2j}. \]

By using \(|(k, \ell) - (k', \ell')| < c_1 2^{2j}\) for the middle part of (6.6), we obtain

\[
(6.4) \quad |A_{\theta'}(J(k, \ell) - J(k', \ell'))| \leq C|J(k, \ell) - J(k', \ell')| = C2^{j/2}|(k, \ell) - (k', \ell')| \leq C c_1 2^{j/2+2j}. \]

On the other hand, by applying the size condition of (5.22), we obtain

\[
(6.5) \quad |(A_\theta - A_{\theta'}) J(k, \ell)| \geq c_2 |\theta - \theta'| 2^{-j/2} |J(k, \ell)| \geq (c_2/2) 2^{j/2+2j}. \]

The last inequality above follows from \(2^{2j-1} \leq |\theta - \theta'|\) in (6.3) and \(|J(k, \ell)| \geq J(k) \geq 2^j\), owing to the support condition (5.6). Therefore, there exists \(c > 0\) such that

\[
(6.6) \quad |A_\theta J(k, \ell) - A_{\theta'} J(k', \ell')| \geq |(A_\theta - A_{\theta'}) J(k, \ell)| - |A_{\theta'} (J(k, \ell) - J(k', \ell'))| \geq c 2^{j/2+2j} \]

where (6.4) and (6.5) with \(c_1\) such that \(c_2/2 \gg C c_1\) yield (6.6).

\[\square\]

Proof of Proposition 6.7. Let us compute

\[
\langle T_{k_2, \ell_2, \theta_2, f}^{q} \rangle = \int \sum_{q \in U_1} H^q(\xi, \eta) \hat{f}(\xi, \eta) d\xi d\eta. \tag{6.7} \]

where \(H^q(\xi, \eta)\) is given by

\[
(6.8) \quad H^q(\xi, \eta) = \int e^{2\pi i \Psi(x, t, \xi, \eta)} a_{k_2, \ell_2, \theta_2}^q(x, t, \xi, \eta) dx dt. \]

Here \(a_{k_2, \ell_2, \theta_2}^q\) is given in (5.7). As \(\Phi(x, t, \xi) = x \cdot \xi + x_3 \xi_3 + t|\xi + \xi_3 A(x)|\), the phase function above is given by

\[
(6.9) \quad \Psi(x, t, \xi) = \Phi(x, t, \xi) - \Phi(x, t, \eta) = x \cdot (\xi - \eta) + x_3 (\xi_3 - \eta_3) + t(|\xi + \xi_3 A(x)| - |\eta + \eta_3 A(x)|). \]

Suppose that \(|(k_1, \ell_1, \theta_1) - (k_2, \ell_2, \theta_2)| \geq 2^{2j}\). Then, we claim that for sufficiently large \(M > 0\) it holds that

\[
(6.10) \quad \sum_{q \in U_1} |H^q(\xi, \eta)| \lesssim 2^{-Mj}. \]
Assume that (6.10) holds, and insert (6.10) into (6.7) with the support \( \{ |\xi|, |\xi_3|, |\eta|, |\eta_3| \lesssim 2^{j+\epsilon} \} \) in (6.11):
\[
\langle T_{k_1,\ell_1,\theta_1}^a f, T_{k_2,\ell_2,\theta_2}^a f \rangle \lesssim 2^{-M j} \int_{\{ |\xi|, |\xi_3|, |\eta|, |\eta_3| \lesssim 2^{j+\epsilon} \}} \left( |\tilde{f}(\xi, \xi_3)|^2 + |\tilde{f}(\eta, \eta_3)|^2 \right) \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \\
\lesssim 2^{-M' j} \left( \int |\tilde{f}(\xi, \xi_3)|^2 \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} + \int |\tilde{f}(\eta, \eta_3)|^2 \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \partial \tilde{f} \right)
\]
which yields (6.1). We shall prove (6.10) using integration by parts for \( H_q(\xi, \xi_3, \eta_3) \) in (6.3). Recall \( \hat{\xi} = \xi + \xi_3 A(x) \) and \( \hat{\eta} = \xi + \xi_3 A(x) \). In view of (5.20), we apply the differential vector field \( D_A = (\partial_t, t \partial x_3, x_2 \partial x_3, \partial_{x_2}, x_1 \partial x_3) \) in (5.15) to the phase function \( \Psi \) in (6.8) and (6.9):
\[
D_A \Psi(x, x_3, t, \xi, \xi_3, \eta, \eta_3) = \left( (|\hat{\xi}|, t|\xi_3|), U(\hat{\xi}, t\xi_3) \right) - \left( (|\hat{\eta}|, t\eta_3), U(\hat{\eta}, t\eta_3) \right) \\
= \left( J(k_1, \ell_1), A_{\theta_1}(J(k_1, \ell_1)) \right) - \left( J(k_2, \ell_2), A_{\theta_2}(J(k_2, \ell_2)) \right) + O(2^{j/2}).
\]
To apply integration by parts, we require the sizes of the three types of derivatives below:

- The lower bound of the first derivative of the phase function:
  \[
  |D_A(\Psi(x, x_3, t, \xi, \xi_3, \eta, \eta_3))| \gtrsim 2^{j/2+2\epsilon j}
  \]
  which follows from \( |(k_1, \ell_1, \theta_1) - (k_2, \ell_2, \theta_2)| \gtrsim 2^{2\epsilon j} \) for the lower bound (6.2) in (6.11).

- The upper bound of the derivative of the amplitude in (6.3) and (6.8):
  \[
  D_A (a) (x, x_3, t, \eta, \eta_3) = O(2^{j/2+\epsilon j} |\tilde{a}(x, x_3, t, \eta, \eta_3)|)
  \]
  and \( D_A (\tilde{a}) = O(2^{j/2+\epsilon j} |\tilde{a}|) \)
  where \( a := a_{k_1, \ell_1, \theta_1}(x, x_3, t, \xi, \xi_3) \equiv a_{k_2, \ell_2, \theta_2}(x, x_3, t, \eta, \eta_3) \), with the modifications of \( \tilde{a} \) and \( \tilde{\tilde{a}} \).
  This follows from the repeated new factors \( |\xi_3|/2^{j/2} \) generated from the application of the \( x \)-derivative:
  \[
  \left| \nabla_x \left( \left( \frac{(\xi + \xi_3 A(x))}{|\xi + \xi_3 A(x)|} - e(\theta) \right)^{2-j/2} \right) \right| \lesssim |\xi_3|/2^{j/2} = O(2^{j/2+\epsilon j}) \]
  since \( |\xi_3| \lesssim 2^{j+\epsilon} \) in (5.2).

- The upper bound on the derivatives of the phase functions:
  \[
  |D_A \cdot D_A \Psi| \lesssim C2^{(1+\epsilon)j+\epsilon j} \] implying \( \frac{|D_A \cdot D_A \Psi|}{|D_A \Psi|^2} = O(2^{-2\epsilon j}) \).

Proof of (6.14). Using (5.20), we compute \( D_A \cdot D_A \Phi \) given by
\[
\left( \partial_t, t \partial x_3, \partial_{x_1} + x_2 \partial x_3, \partial_{x_2} - x_1 \partial x_3 \right) \cdot \left( (|\xi + \xi_3 A(x)|, t\xi_3), \xi + \xi_3 A(x) + t\xi_3 A^T \left( \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} \right) \right).
\]
Here, \( |\xi_3| = O(2^{j+\epsilon j}) \) in (5.2) yields \( \nabla_{x_1, x_2} \left( \xi_3 A^T \left( \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} \right) \right) \) \( = O(2^{(1+\epsilon)j/2}) \) implying (6.14). \( \square \)
We apply integration by parts with \((D_Ae^{2\pi i\Psi}) \cdot \left(\frac{D_A\Psi}{|D_A\Psi|^2}\right) = 2\pi ie^{2\pi i\Psi}\) involving the vector field \(D_A\): 

\[
H^q(\xi, \xi_3, \eta, \eta_3) = \int e^{2\pi i\Psi a} = \int \left(\frac{D_Ae^{2\pi i\Psi}}{2\pi i}\right) \frac{D_A\Psi}{|D_A\Psi|^2} a = \frac{-1}{2\pi i} \int e^{2\pi i\Psi} D_A \cdot \left(\frac{D_A\Psi}{|D_A\Psi|^2} a\right)
\]

where \(a = a_{k_1,\ell_1,\theta_1}(x, x_3, t, \xi, \xi_3)\) in \((6.18)\). We define the last term as the differential operator 

\[
D_a = \frac{-1}{2\pi i} D_A \cdot \left(\frac{D_A\Psi}{|D_A\Psi|^2} a\right)
\]

so that 

\[
\int e^{2\pi i\Psi} a \, dxdx dt = \int e^{2\pi i\Psi} [D_a] dxdx dt.
\]

Then, we apply \((6.12)\) and \((6.13)\) to obtain \(2^{-(1/2+2\epsilon)}j\) with the loss of \(2^{(1/2+\epsilon)}j\), and \((6.14)\) to obtain \(2^{-2\epsilon}j\). Thus, we have that 

\[
(6.15) \quad D(a) = O(2^{-\epsilon}).
\]

Using this, we are able to repeat the above integration by parts procedure \(N\) times, to obtain 

\[
|H(\xi, \xi_3, \eta, \eta_3)| \leq \sum_{q \in U_1} |H^q(\xi, \xi_3, \eta, \eta_3)| \leq \sum_{q \in U_1} \int |D^N a| \, dxdx dt 
\]

\[
\lesssim \text{card}(U_1) \, 2^{-jN\epsilon} = (2^{j/2})2^{-jN\epsilon} \leq 2^{-jM}.
\]

where \((N\epsilon)\) is a sufficiently large number. This proves \((6.10)\). Therefore, we have proved Proposition \(6.1\). \(\square\)

6.2. Radial Rearrangement. The radial decomposition is made for the absolute value

\[
\left|\left|\left|\xi, t\xi_3\right|\right|\right| = |J(k, \ell)| \text{ for } \tilde{\xi} = \xi + \xi_3 A(x) \text{ in } (5.20) \text{ and } (5.21).
\]

First, we rearrange the \(T_{k,\ell,\theta}^q\) terms in \(\sum a_{k,\ell,\theta} \, T_{k,\ell,\theta}^q\) according to those satisfying the radial sizes

\[
(6.16) \quad \left|\left|\left|\xi, t\xi_3\right|\right|\right| = |J(n)| + O(2^{j/2}) \text{ that means } \left(\tilde{\xi}, t\xi_3\right) \in S_n + O(2^{j/2}).
\]

In the support of the amplitude \(a_{k,\ell,\theta}(x, x_3, t, \xi, \xi_3)\) for the kernel of \(T_{k,\ell,\theta}^q\) in \((5.7)\), we observe the following:

- \(|\tilde{\xi}| = J(k) + O(2^{j/2}) \in [2^j, 2^{j+1}]\) for \(k \in U_1 = [2^{j/2}, 2^{j/2+1}] \cap \mathbb{Z}\);
- \(t\xi_3 = J(\ell) + O(2^{j/2}) \in [0, 2^{(1+\epsilon)}] \) for \(\ell \in U_0 = [0, 2^{j/2+\epsilon}] \cap \mathbb{Z}\).

Thus, the range of \(|\tilde{\xi}|, t\xi_3|\) is contained in \([2^j, 2^{j+2j+1}]\) as \(|\tilde{\xi}| \geq 2^j\). Consider the following partition of \([2^j, 2^{j+2j+1}]\):

\[
\left\{ J(n) = n2^{j/2} \right\}_{n=2^{j/2}}^{2^{j+2j+1}} \text{ of the interval } [2^j, 2^{j+2j+1}] \text{ where } N_0 = 2^{j+2j+1}.
\]

For each \((k, \ell) \in U_1 \times U_0\), we can find \(n \in U_1^* = [2^{j/2}, 2^{j+2j+1}] \cap \mathbb{Z}\) such that

\[
(6.17) \quad |J(k, \ell)| \in |J(n)|, J(n) + 2^{j/2}| \text{ where } J(n) = n2^{j/2} \in [2^j, 2^{j+j+1}] \text{ and } J(k, \ell) = (J(k), J(\ell)),
\]
because $J(k) \geq 2^j$. This enables us define the map

$$n : \mathbb{U}_1 \times \mathbb{U}_0^* \rightarrow \mathbb{U}_1$$

mapping $(k, \ell) \mapsto n(k, \ell) \in \mathbb{U}_1$ satisfying $n = n(k, \ell)$ in \textcolor{red}{(6.17)}. This $n$ indicates the circle $S_n$ around which the support of \textcolor{red}{(6.16)} associated with $T_{k,\ell,\theta}^q$ is restricted.

**Definition 6.1** (Radial Decomposition). In view of \textcolor{red}{(6.16)} and \textcolor{red}{(6.17)}, we rearrange $T_{a,j_0}$ in \textcolor{red}{(5.12)} as

\begin{equation}
T_{a,j_0} = \sum_{n \in \mathbb{U}_1} T_n \quad \text{with} \quad T_n = \sum_{(q,k,\ell,\theta) \in \mathbb{U}_1 \times \mathbb{U}_1 \times \mathbb{U}_0 \times \mathbb{U}_0} T_{k,\ell,\theta}^q \text{ for } |J(k, \ell)| = J(n) + O(2^{j/2}).
\end{equation}

This decomposition of the first two components $\left(\hat{\xi}, t\xi_3\right)$ over the circles $S_n + O(2^{j/2})$ is actually the decomposition of $\left(\hat{\xi}, t\xi_3\right)$ over the tori $S_n \times S_n$ of width $O(2^{j/2})$. Indeed, from \textcolor{red}{(6.18)} and \textcolor{red}{(6.17)} with \textcolor{red}{(5.20)} and \textcolor{red}{(5.21)}, we observe that in the support of the amplitude for $T_n = \sum_{(q,k,\ell,\theta) ; n(k,\ell) = n} T_{k,\ell,\theta}^q$ in \textcolor{red}{(5.24)} the following holds:

\begin{equation}
D_A \Phi(x, x_3, t, \xi, \xi_3) = \left(\left|\hat{\xi}\right|, t\xi_3\right) + O(2^{j/2+\epsilon_j})
\end{equation}

$$= \left(J(k, \ell), A_\theta J(k, \ell)\right) + O(2^{j/2}) \in S_n \times S_n + O(2^{j/2})$$

for $A = E$, which forms a torus $S_n \times S_n$ with an error $O(2^{j/2})$.

The following lemma asserts that $(k, \ell) \mapsto n = n(k, \ell)$ is essentially a one-to-one correspondence.

**Lemma 6.2.** If $|(k_2, \ell_2) - (k_1, \ell_1)| \leq 2^j$, then $|n(k_2, \ell_2) - n(k_1, \ell_1)| \leq 2^{10}2^{j/2}$.

**Proof.** We write $k_2 = k_1 + O(2^{j})$ with $k_1, k_2 \in [2^{j/2}, 2^{j/2+1}]$ and $\ell_2 = \ell_1 + O(2^{j})$. From this and the definition of $n(k, \ell)$ in \textcolor{red}{(6.17)}, it holds that

$$n(k_2, \ell_2)2^{j/2} = \sqrt{|k_2|2^{j/2}|^2 + |\ell_2|2^{j/2}|^2 + O(2^{j})}$$

$$= \sqrt{|k_1|2^{j/2}|^2 + |\ell_1|2^{j/2}|^2 + O(2^{j/2+\epsilon_j}) + O(2^{j})}$$

$$= \sqrt{|k_1|2^{j/2}|^2 + |\ell_1|2^{j/2}|^2 + O(2^{j/2+\epsilon_j})}$$

$$= n(k_1, \ell_1)2^{j/2} + O(2^{j/2+\epsilon_j}).$$

Thus, $|n(k_2, \ell_2) - n(k_1, \ell_1)| = O(2^{j/2})$.

\[\Box\]

6.3. **$L^4$ Square Sum over Tori with Loss of $2^{j/8}$**. Now, we can reduce $\sum_n T_n$ to the square sum of $(\sum_n |T_n|^2)^{1/2}$ over the circles $S_n$ (hence, the tori $S_n \times S_n$).
Theorem 6.1 (Reduction to the Square Sum over Circles $S_n$). Let $A = E$. Suppose that $T_n$ is defined in (6.18). Then,

$$(6.20) \left\| \sum_{n \in U_1^1} T_n f \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{j(\frac{1}{2}+\epsilon)} \left\| \left( \sum_{n \in U_1^1} |T_n f|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} + 2^{-Mj} \|f\|_{L^4}.$$ 

Proof of (6.20). Note that $U_1^1 := [2^{j/2}, 2^{j+1/2}+1] \cap \mathbb{Z}$ with $N_0 = 2^{j+2j/2+1}$. Applying the Schwartz inequality to the summation, we have

$$(6.21) \left\| \sum_{n \in U_1^1} T_n f \right\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R})} \lesssim N_0^{1/2} \left\| \left( \sum_{n \in U_1^1} |T_n f|^2 \right)^{1/2} \right\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R})}.$$ 

On the other hand, by Lemmas 6.1 and 6.2 we obtain for $|n_1 - n_2| \geq 2^{j\epsilon}$ that

$$(6.22) \int T_{n_1} f(x,x_3,t) \overline{T_{n_2} f(x,x_3,t)} dx dx_3 dt \lesssim 2^{-Mj} \|f\|_{L^2}$$

for sufficiently large $M$.

This implies that

$$(6.23) \left\| \sum_{n \in U_1^1} T_n f \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{j\epsilon} \left\| \left( \sum_{n \in U_1^1} |T_n f|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^3 \times \mathbb{R})} + 2^{-Mj} \|f\|_{L^2}.$$ 

The interpolation of (6.23) and (6.21) implies (6.20). \hfill \Box

7. Square Sum over Cubes $C_{k,\ell,\theta}$

Recall the cubes $C_{k,\ell,\theta}$ in (5.8), where $a_{k,\ell,\theta,q}(x,x_3,t,\xi)$ is in (5.7) with $J(k) = k2^{j/2}, J(\ell) = \ell2^{j/2}$. We also recall that in (5.9) and (5.10) we have that

$$(7.1) T_{k,\ell,\theta,q} f(x,x_3,t) = \int e^{2\pi i \Phi(x,x_3,t,\xi,\xi_3)} a_{k,\ell,\theta,q}(x,x_3,t,\xi,\xi_3) \hat{f}(\xi,\xi_3) d\xi d\xi_3$$

where $\Phi(x,x_3,t,\xi,\xi_3) = x \cdot \xi + x_3 \xi_3 + t|\xi + \xi_3 A(x)|$ and $A = E$. In this section, we replace the square sum over the tori $S_n \times S_n$ by that over the cubes $C_{k,\ell,\theta}$ in (5.5):

Theorem 7.1. Let $A = E$. As in (6.18), let $T_n f(x,x_3,t) = \sum_{n(k,\ell,\theta)} T_{k,\ell,\theta,q} f(x,x_3,t).$ Then,

$$(7.2) \left\| \left( \sum_{n \in U_1^1} |T_n f|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})} \lesssim 2^{20\epsilon j} \left\| \left( \sum_{q \in U_1} \sum_{(k,\ell,\theta) \in U_0} |T_{k,\ell,\theta,q} f|^2 \right)^{1/2} \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})}.$$
7.1. Quadruples. To simplify the notations, we set

\[ [x] = (x, x_3) \text{ and } [\xi] = (\xi, \xi_3), [\xi'] = (\xi', \xi_3'), [\eta] = (\eta, \eta_3), [\eta'] = (\eta', \eta_3'). \]

By writing out the integral \( \int dx dx_3 dt = \int d[x] dt \), we express the left-hand side of (7.2) as

\[
\left\| \left( \sum_{n \in \mathbb{N}} |T_n f|^2 \right) \right\|_{L^4(\mathbb{R}^3 \times \mathbb{R})}^{1/2} = \int \left( \sum_{m} |T_m f([x], t)|^2 \right) \left( \sum_{n} |T_n f([x], t)|^2 \right) d[x] dt
\]

\[
= \sum_{(m,n) \in \mathcal{U}_m \times \mathcal{U}_n} Q_{mn}.
\]

Here, \( Q_{mn} \) are the quadruple integrals

\[ Q_{mn} := \int T_m f([x], t) \overline{T_m f([x], t)} T_n f([x], t) \overline{T_n f([x], t)} dx dt. \]

Recall the circle \( S_m = \{ \xi : |\xi| = J(m) \} \) of radius \( J(m) = m2^{1/2} \), and observe that

\[ (n(k, \ell), n(k', \ell')) = (m, n) + O(1) \text{ if and only if } (J(k, \ell), J(k', \ell')) \in S_m \times S_n + O(2^{1/2}) \]

from (6.14) and (6.15). To each \( S_m \times S_n \), we assign a set of pairs of triple indices making \( T_m, T_n \)

\[ \mathcal{U}_{mn}^6 = \left\{ ([k, \ell, \theta], [k', \ell', \theta']) \in (\mathcal{U}_1 \times \mathcal{U}_2^2) : (J(k, \ell), J(k', \ell')) \in S_m \times S_n + O(2^{1/2}) \right\} \]

where \( n(k, \ell) = m \) and \( n(k', \ell') = n \). Hence, the left-hand side of (7.2) is the sum over \( (m, n) \) given by

\[ Q_{mn} = \sum_{q} \sum_{\mathcal{U}_{mn}^6 \times \mathcal{U}_{mn}^6} \int T_{k_1, \ell_1, \theta_1}^q f([x], t) \overline{T_{k_1, \ell_1, \theta_1}^q f([x], t)} T_{k_2, \ell_2, \theta_2}^q f([x], t) \overline{T_{k_2, \ell_2, \theta_2}^q f([x], t)} dx dt \]

where the sum is over \( q \), and \( (k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1') \in \mathcal{U}_{mn}^6 \) and \( (k_2, \ell_2, \theta_2), (k_2', \ell_2', \theta_2') \in \mathcal{U}_{mn}^6 \). We denote \( d[\xi] = d\xi d\xi_3 \) and \( d[\eta] = d\eta d\eta_3 \). Then, the integral in (7.4) is

\[ \int H_{(k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1')} ([\xi], [\eta], [\eta']) f([\xi]) f([\xi']) \overline{f([\eta])} \overline{f([\eta'])} d[\xi] d[\xi'] d[\eta] d[\eta']. \]

Using (7.1), the symbol \( H_{(k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1')} ([\xi], [\xi'], [\eta], [\eta']) \) is given by

\[ \int e^{2\pi i F([x], t), [\xi], [\xi'], [\eta], [\eta'])} a([x], t, [\xi], [\xi'], [\eta], [\eta']) dx dt. \]

Here, the phase function \( F([x], t, [\xi], [\xi'], [\eta], [\eta']) \) is

\[ \Phi(x, x_3, t, \xi, \xi_3) + \Phi(x, x_3, t, \xi', \xi'_3) - \Phi(x, x_3, t, \eta, \eta_3) + \Phi(x, x_3, t, \eta', \eta'_3) \]

and the amplitude \( a([x], t, [\xi], [\xi'], [\eta], [\eta']) \) is

\[ a_{k_1, \ell_1, \theta_1}^q ([x], t, [\xi]) a_{k_2, \ell_2, \theta_2}^q ([x], t, [\xi']) a_{k_1, \ell_1, \theta_1}^q ([x], t, [\eta]) a_{k_2, \ell_2, \theta_2}^q ([x], t, [\eta']) \]

where \( a_{k, \ell, \theta}^q \) is in (5.7).
7.2. Elliptic Conjugates in Torus × Torus. The symbol in (7.6) is the integral to be evaluated for the proof of (7.2) in Theorem 7.1. In view of (5.20) and (6.19), we recall that

\[ D_A = (t \partial_{x_3}, \partial_t, x_2 \partial_{x_3}, \partial_{x_2} - x_1 \partial_{x_3}) \]

From (5.20), (6.19) and (7.4) combined with (7.11) and (7.12), this condition is rewritten as

\[ \left( J(k_1, \ell_1), A_\theta J(k_1, \ell_1) \right) + O(2^{j/2}) \in S_m^2 \times S_n^2 + O(2^{j/2}), \]

\[ \left( J(k_2, \ell_2), A_\theta J(k_2, \ell_2) \right) + O(2^{j/2}) \in S_m^2 \times S_n^2 + O(2^{j/2}). \]

Then, we can expect that the integral in (7.6) is as small as \( 2^{-jM} \) for large \( M \gg 1 \), unless the application of the vector field \( D_A \) to the phase function \( F \) in (7.1) almost vanishes as

\[ D_A F = \left[ \left( (|\tilde{\xi}|, t\xi_3), U(\tilde{\xi}, t\xi_3) \right) + \left( (|\tilde{\xi}'|, t\xi_3'), U(\tilde{\xi}', t\xi_3') \right) \right] \]

\[ - \left[ \left( (|\tilde{\eta}|, t\eta_3), U(\tilde{\eta}, t\eta_3) \right) + \left( (|\tilde{\eta}'|, t\eta_3'), U(\tilde{\eta}', t\eta_3') \right) \right] = O(2^{j/2+2\epsilon j}). \]

In terms of (7.11) and (7.12), this condition is rewritten as

\[ D_A F = \left[ \left( J(k_1, \ell_1), A_\theta J(k_1, \ell_1) \right) + \left( J(k_1', \ell_1'), A_\theta J(k_1', \ell_1') \right) \right] \]

\[ - \left[ \left( J(k_2, \ell_2), A_\theta J(k_2, \ell_2) \right) + \left( J(k_2', \ell_2'), A_\theta J(k_2', \ell_2') \right) \right] = O(2^{j/2+2\epsilon j}). \]

This implies the following proposition.

**Proposition 7.1.** Recall that \( \mathcal{N} \) is the mapping the pair \((\xi, \xi') \in \mathbb{R}^2 \times \mathbb{R}^2 \) to its elliptic conjugate pair \( \mathcal{N}(\xi, \xi') \) associated with the ellipse \( \mathbb{E}(\xi, \xi') \) in the sense of (4.2). On the support of (7.3), (7.6) and (7.8), suppose that (7.14) and (7.12) hold. Then,

\[ \left( (|\tilde{\eta}|, \eta_3), (|\tilde{\eta}'|, \eta_3') \right) = \begin{cases} \left( (|\tilde{\xi}|, \xi_3), (|\tilde{\xi}'|, \xi_3') \right) + O(2^{j/2+2\epsilon j}) \text{ or} \\ \mathcal{N} \left( (|\tilde{\xi}|, \xi_3), (|\tilde{\xi}'|, \xi_3') \right) + O(2^{j/2+2\epsilon j} \right) \]
and

\[(7.16) \quad \left( U(\tilde{\eta}, t\eta_3), U(\tilde{\eta'}, t\eta'_3) \right) = \begin{cases} \left( U(\tilde{\xi}, t\xi_3), U(\tilde{\xi'}, t\xi'_3) \right) + O(2i/2+2j) \text{ or} \\ \mathcal{N}\left( U(\tilde{\xi}, t\xi_3), U(\tilde{\xi'}, t\xi'_3) \right) + O(2i/2+2j). \end{cases} \]

This is manifested in the definite size condition

\[(7.17) \quad \left( J(k_2, \ell_2), J(k'_2, \ell'_2) \right) = \begin{cases} \left( J(k_1, \ell_1), J(k'_1, \ell'_1) \right) + O(2i/2+2j) \text{ or} \\ \mathcal{N}(J(k_1, \ell_1), J(k'_1, \ell'_1)) + O(2i/2+2j). \end{cases} \]

and

\[(7.18) \quad (A_{\theta_2}J(k_2, \ell_2), A_{\theta'_2}J(k'_2, \ell'_2)) = \begin{cases} (A_{\theta_1}J(k_1, \ell_1), A_{\theta'_1}J(k'_1, \ell'_1)) + O(2i/2+2j) \text{ or} \\ \mathcal{N}(A_{\theta_1}J(k_1, \ell_1), A_{\theta'_1}J(k'_1, \ell'_1)) + O(2i/2+2j). \end{cases} \]

Proof. In (7.11) and (7.12), we have that

\[((|\tilde{\xi}|, t\xi_3), (|\tilde{\xi'}|, t\xi'_3)) \text{ and } ((|\tilde{\eta}|, t\eta_3), (|\tilde{\eta'}|, t\eta'_3)) \in S_m \times S_n + O(2i/2).\]

By combining this with \((|\tilde{\eta}|, t\eta_3) + (|\tilde{\eta'}|, t\eta'_3) = (|\tilde{\xi}|, t\xi_3) + (|\tilde{\xi'}|, t\xi'_3) + O(2i/2+2j)\) in (7.13), we apply Lemma 4.2 and Remark 4.2 to obtain (7.15). Next, in (7.11) and (7.12) we have that

\((U(\tilde{\xi}, t\xi_3), U(\tilde{\xi'}, t\xi'_3)) \text{ and } (U(\tilde{\eta}, t\eta_3), U(\tilde{\eta'}, t\eta'_3)) \in S_m \times S_n + O(2i/2).\)

From this and \(U(\tilde{\eta}, t\eta_3) + U(\tilde{\eta'}, t\eta'_3) = U(\tilde{\xi}, t\xi_3) + U(\tilde{\xi'}, t\xi'_3) + O(2i/2+2j)\) in (7.13), we apply Lemma 4.2 and Remark 4.2 to obtain (7.16). Inserting the size condition (7.11) and (7.12) into (7.15) and (7.16), we obtain (7.14) and (7.15).

\[\square\]

Definition 7.1. Proposition 7.1 leads us to define the following relation between two pairings in \(U_{m,n}^6\):

\[(k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1) \text{ and } (k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)\) into:

\[(7.19) \quad (k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2) \sim ((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)) \text{ if } (7.17) \text{ and } (7.18) \text{ hold.}\]

Lemma 7.1. The condition (7.19) is reflexive, in the sense that

\[((k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)) \sim ((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1))\]

implies that \(((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)) \sim ((k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)).\]

Proof. Suppose that \(((k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)) \sim ((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)).\) This implies that (7.17) and (7.18) hold. In view of (7.19) and Remark 4.1, we observe that \(\mathcal{N}\) is a reflection satisfying

\[\mathcal{N} \circ \mathcal{N}(J(k, \ell), J(k', \ell')) = (J(k, \ell), J(k', \ell')).\]
Hence, (7.17) can be switched with

$$(7.20) \quad (J(k_1, \ell_1), J(k_1', \ell_1')) = \begin{cases} (J(k_2, \ell_2), J(k_2', \ell_2')) + O(2^{j/2+2\epsilon_j}) & \text{or} \\ \mathcal{N}(J(k_2, \ell_2), J(k_2', \ell_2')) + O(2^{j/2+2\epsilon_j}) \end{cases}$$

Similarly, (7.18) can be switched with

$$(7.21) \quad (A_{\theta 0} J(k_1, \ell_1), A_{\theta 0}' J(k_1', \ell_1')) = \begin{cases} (A_{\theta 2} J(k_2, \ell_2), A_{\theta 2}' J(k_2', \ell_2')) + O(2^{j/2+2\epsilon_j}) & \text{or} \\ \mathcal{N}(A_{\theta 2} J(k_2, \ell_2), A_{\theta 2}' J(k_2', \ell_2')) + O(2^{j/2+2\epsilon_j}) \end{cases}$$

From (7.20) and (7.21), we have that $((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1')) \sim ((k_2, \ell_2, \theta_2), (k_2', \ell_2', \theta_2'))$. $\square$

### 7.3. Square Sum Estimates

**Lemma 7.2.** For each $((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1')) \in \mathbb{U}_{mn}^6$, let $\mathbb{U}_m^6((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1'))$ be given by

$$(7.22) \quad \left\{ ((k_2, \ell_2, \theta_2), (k_2', \ell_2', \theta_2')) \in \mathbb{U}_m^6 : ((k_2, \ell_2, \theta_2), (k_2', \ell_2', \theta_2')) \sim ((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1')) \right\}$$

Then,

$$(7.23) \quad \#(\mathbb{U}_m^6((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1'))) \leq C2^{20\epsilon_j}$$

**Proof of (7.23).** Fix $((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1'))$ in $\mathbb{U}_m^6$, and let $((k_2, \ell_2, \theta_2), (k_2', \ell_2', \theta_2')) \sim ((k_1, \ell_1, \theta_1), (k_1', \ell_1', \theta_1'))$.

1. Set $(u, v) = (J(k_1, \ell_1), J(k_1', \ell_1'))$ or $\mathcal{N}(J(k_1, \ell_1), J(k_1', \ell_1'))$. By (7.17), we observe that $((k_2, \ell_2), (k_2', \ell_2'))$ are contained in the set given by

$$\left\{ ((k_2, \ell_2), (k_2', \ell_2')) : d((J(k_2, \ell_2), J(k_2', \ell_2')), (u, v)) < 2^{j/2+2\epsilon_j} \right\},$$

whose cardinality is $O(2^{16\epsilon_j})$, because this is the number of possible quadruples of $(k_2, \ell_2, k_2', \ell_2')$:

$$|k_22^{j/2}, \ell_22^{j/2} - u| \leq 2^{j/2+2\epsilon_j} \text{ and } |(k_2', 2^{j/2}, \ell_2'2^{j/2}) - v| \leq 2^{j/2+2\epsilon_j}.$$ 

Hence, we may assume that $(J(k_2, \ell_2), J(k_2', \ell_2'))$ is a fixed pair with at most $2^{16\epsilon_j}$ different choices.

2. Set $(u, v) = (A_{\theta 0} J(k_1, \ell_1), A_{\theta 0}' J(k_1', \ell_1'))$ or $\mathcal{N}(A_{\theta 0} J(k_1, \ell_1), A_{\theta 0}' J(k_1', \ell_1'))$. As in (1), let $(x, y) = (J(k_2, \ell_2), J(k_2', \ell_2'))$. Then, by (7.18) the above $(\theta_2, \theta_2')$ are contained in

$$\left\{ (\theta_2, \theta_2') \in \mathbb{N} \times \mathbb{N} : d((A_{\theta 2} x, A_{\theta 2}' y), (u, v)) < 2^{j/2+2\epsilon_j} \right\},$$

whose cardinality is $O(2^{4j\epsilon_j})$. We claim that the left-hand side is the number of all possible pairs $(\theta_2, \theta_2')$ satisfying

$$|A_{\theta 2} x - u| \leq 2^{j/2+2\epsilon_j} \text{ and } |A_{\theta 2}' y - v| \leq 2^{j/2+2\epsilon_j} \text{ for fixed } (x, y) \text{ and } (u, v),$$

which is controlled by $C2^{4j\epsilon_j}$ from (7.22). This is true if det$(AE + (AE)^T) > 0$, which holds for the case $A = E$. 


The number of all possible choices \(((k_2, \ell_2, \theta_2), (k_2', \ell_2', \theta_2'))\) satisfying \((7.22)\) is at most \(O(2^{16j}) \times O(2^{4e})\) for the quadruples and pairs in (1) and (2). This proves \((7.23)\).

\[\square\]

**Proposition 7.2.** Recall the relation \(~\) in Definition \(7.1\) and the index set \((7.22)\). At every \((x, x_3, t)\), we have

\[
\sum_{(k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)} |T^q_{k_1, \ell_1, \theta_1}fT^q_{k'_1, \ell'_1, \theta'_1}f| \leq C 2^{20e} \left( \sum_{(k_1, \ell_1, \theta_1)} |T^q_{k_1, \ell_1, \theta_1}f|^2 \right) \left( \sum_{(k'_1, \ell'_1, \theta'_1)} |T^q_{k'_1, \ell'_1, \theta'_1}f|^2 \right).
\]

**Proof of Proposition 7.2** The left-hand side is bounded by

\[
\sum_{(k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)} |T^q_{k_1, \ell_1, \theta_1}fT^q_{k'_1, \ell'_1, \theta'_1}f|^2 + \sum_{(k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)} |T^q_{k_1, \ell_1, \theta_1}fT^q_{k'_1, \ell'_1, \theta'_1}f|^2.
\]

We invoke \((7.23)\) to obtain that the summation in \((7.24)\) is bounded by

\[
C 2^{20e} \sum_{(k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)} |T^q_{k_1, \ell_1, \theta_1}fT^q_{k'_1, \ell'_1, \theta'_1}f|^2.
\]

Because the summation in \((7.25)\) is taken over a finite set, we can change the order of summation. Combined with the fact that the relation \(~\) is reflexive in Lemma \(7.1\) we can write \((7.25)\) as

\[
\sum_{(k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)} |T^q_{k_2, \ell_2, \theta_2}fT^q_{k'_2, \ell'_2, \theta'_2}f|^2.
\]

By \((7.23)\) again, \((7.27)\) is less than \(C 2^{20e} \sum_{(k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)} |T^q_{k_2, \ell_2, \theta_2}fT^q_{k'_2, \ell'_2, \theta'_2}f|^2.\)

\[\square\]

**Proposition 7.3 (A good \(L^4\) estimate).** In view of \((7.3)\) and \((7.4)\), let

\[
((k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)) \text{ and } ((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1)) \in \mathbb{U}_{mn},
\]

And suppose that \(((k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)) \sim ((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1))\), which means that \(~\) in \((7.19)\) breaks down. Then, for the integral in \((7.23)\) and \((7.29)\) we have

\[
\int T^q_{k_1, \ell_1, \theta_1}f([x], t)T^q_{k'_1, \ell'_1, \theta'_1}f([x], t)T^q_{k_2, \ell_2, \theta_2}f([x], t)d[x]dt \leq 2^{-Mj} \|f\|_{L^4}^4.
\]

**Proof.** We compute the integral \(H((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1), (k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2))\) \((\xi, \xi', \eta, \eta', \xi_3, \eta_3, \eta_3', \eta_3')\) of \((7.6)\) as

\[
\int e^{2\pi i |F([x], t]|, (|\xi|, |\xi'|, |\eta|, |\eta'|)} a(x, t, |\xi|, |\xi'|, |\eta|, |\eta'|)d[x]dt
\]

Because \(((k_2, \ell_2, \theta_2), (k'_2, \ell'_2, \theta'_2)) \sim ((k_1, \ell_1, \theta_1), (k'_1, \ell'_1, \theta'_1))\), we observe that either \((7.15)\) or \((7.16)\) fails. Hence, this in the form of \((7.15)\) and \((7.16)\) implies (A) or (B) below:
We insert this into (7.5) with the support condition (5.11) to control (7.29) by

\[ \left( \left( |\eta|, t\eta_3 \right) + \left( |\eta'|, t\eta'_3 \right) \right) - \left( \left( |\xi|, t\xi_3 \right) + \left( |\xi'|, t\xi'_3 \right) \right) \geq c2^{j/2}2^{2cj}, \]

\[ \left( U \left( \eta, t\eta_3 \right) + U \left( \eta', t\eta'_3 \right) \right) - \left( U \left( \xi, t\xi_3 \right) + U \left( \xi', t\xi'_3 \right) \right) \geq c2^{j/2}2^{2cj}. \]

We have the following three size properties for the integrand of (7.6), which are similar to (6.12)–(6.14):

1. The (A) and (B) above in (7.13) and (7.14) yield a similar lower bound to (6.12):

\[ |D_AF(x, t, \xi, \xi', \eta, \eta', \eta'_3)| \geq 2^{j/2}2^{2cj}. \]

2. The derivative \( D_A \) of the amplitude \( D_A[a([x], t, [\xi], [\eta], [\eta'])] = O(2^{j/2+ej}) \) has a similar upper bound to (6.13). The derivatives \( \frac{|D_A|}{|D_AF|^2} \) have a similar upper bound \( 2^{-2cj} \) to (6.14).

3. We utilize \( D_A(e^{2\pi iF}) = 2\pi i e^{2\pi iF} \) to define \( D_A = D_A \cdot \left( \frac{|D_A|}{|D_AF|^2} \right) a \) and \( e^{2\pi iF} a = \int e^{2\pi iF} D(a) \).

Applying integration by parts (3) with (1)–(2) yields the same order of gain as in (6.15):

\[ D(a([x], t, [\xi], [\xi'], [\eta], [\eta'])) = O(2^{-ej}). \]

Repeating this \( N = [M/\epsilon] \) times yields the gain of \( D^N a = O(2^{-N\epsilon}) = O(2^{-(M+10)j}) \), so that

\[ |H(\xi, \xi_3, \xi', \eta, \eta_3, \eta', \eta'_3)| \lesssim \int |D^N a| dx d\eta \lesssim 2^{-(M+10)j}. \]

We insert this into (7.5) with the support condition (5.11) to control (7.29) by

\[ \int 2^{-j} f(\xi_3, \xi) \overline{f(\xi', \eta)} d\xi d\xi' \leq \int \left( \left| f(\eta, \eta_3) \right|^2 + \left| \overline{f(\eta', \eta'_3)} \right|^2 \right) \left( \left| f(\eta, \eta_3) \right|^2 + \left| \overline{f(\eta', \eta'_3)} \right|^2 \right) d\xi d\xi' d\eta d\eta' \]

One of the four terms in the middle line of (7.30) is \( 2^{-(M+10)j} \) times the product of

\[ \int \left| f(\eta, \eta_3) \right|^2 \eta d\eta_3 \approx \left\| f(\eta, \eta_3) \right\|_{L^2}^2 \left\| f(\eta, \eta_3) \right\|_{L^2} \]

and

\[ \int \psi \left( \frac{\xi'}{2^{j+ej+10}} \right) \psi \left( \frac{\xi'}{2^{j+ej+10}} \right) d\xi' \left( \frac{\eta'}{2^{j+ej+10}} \right) \psi \left( \frac{\eta'}{2^{j+ej+10}} \right) d\eta' \approx 2^{3(j+ej)}2^{3(j+ej)}. \]

The final line of (7.30) follows from the Schwartz inequality. □

**Proof of Theorem 7.1.** We can obtain (6.13) by applying Propositions 7.2 and 7.3 to the cases ∼ and ∼, respectively, in the summation (7.24). □
8. Vector-Valued Estimates

In this section, we shall show that for a certain vector-valued projection map \( f \to (P^q_{k, \ell, \theta} f)_{(k, \ell, \theta)} \), it holds that
\[
\left\| \left( \sum_{(q, k, \ell, \theta)} |T^q_{k, \ell, \theta} f|^2 \right)^{1/2} \right\|_{L^4} \lesssim \left( \sum_{(q, k, \ell, \theta)} \left\| T^q_{k, \ell, \theta} P^q_{k, \ell, \theta} f \right\|_{L^4} \right)^{1/2} \lesssim 2^{j/16} \left\| \sup_{q} \sum_{(k, \ell, \theta)} |P^q_{k, \ell, \theta} f|^2 \right\|_{L^4}. 
\]

For this purpose, we first decompose the phase space \( ([\eta + \eta_3 A(x), [q] \eta_3]) \) of \( f \) into small cubes of size \( 2^{j/2} \times 2^{j/2} \times 2^{j/2} \), according to the support of the amplitude for \( T^q_{k, \ell, \theta} \) in Theorem 7.1.

**Definition 8.1 (Cube Projection).** Fix \( [q] = q 2^{-j/2} \approx 1 \), where \( q \in U_1 = [2^{j/2}, 2^{j/2+1}] \cap \mathbb{Z} \). We define a projection operator to the cube of dimensions \( 2^{j/2} \times 2^{j/2} \times 2^{j/2} \) centered at \( 2^{j/2}(m_1, m_2, m_3) \) as
\[
P^q_{m_1, m_2, m_3} f(y, y_3) = \int e^{2\pi i [y \cdot \eta + y_3 \eta_3]} \psi \left( \frac{(\eta + \eta_3 A(y)) \cdot e_1 - m_1 2^{j/2}}{2^{j/2}} \right) \psi \left( \frac{(\eta + \eta_3 A(y)) \cdot e_2 - m_2 2^{j/2}}{2^{j/2}} \right)
\]
\[
\times \psi \left( \frac{|q| \eta_3 - m_3 2^{j/2}}{2^{j/2}} \right) \hat{f}(\eta, \eta_3) d\eta d\eta_3. 
\]

We apply the projections \( P^q_{m_1, m_2, m_3} \) to \( f_j \) in [5.11], where \( \hat{f}_j(\xi, \xi_3) \) is supported in \( \| (\xi, \xi_3) \| < 2^{j+\epsilon} \).

### 8.1. Integral Kernel of \( T^q_{k, \ell, \theta} P^q_{m_1, m_2, m_3} \)

Recall \( a^q_{k, \ell, \theta} \) in [5.7]. Then the composition of \( T^q_{k, \ell, \theta} \) and \( P^q_{m_1, m_2, m_3} \) is expressed as
\[
T^q_{k, \ell, \theta} P^q_{m_1, m_2, m_3} f(x, x_3, t) = \int a^q_{k, \ell, \theta}(x, x_3, t, \xi, \xi_3) [P^q_{m_1, m_2, m_3} f]^{\wedge}(\xi, \xi_3) d\xi d\xi_3.
\]

Insert \( [P^q_{m_1, m_2, m_3} f]^{\wedge}(\xi, \xi_3) \) of [8.1] into the above integral to rewrite it as
\[
T^q_{k, \ell, \theta} P^q_{m_1, m_2, m_3} f(x, x_3, t) = \int H^q_{k, \ell, \theta, m_1, m_2, m_3}(x, x_3, t, \eta, \xi_3) \hat{f}(\eta, \xi_3) d\eta d\xi_3
\]
where
\[
H^q_{k, \ell, \theta, m_1, m_2, m_3}(x, x_3, t, \eta, \xi_3) = \int_{\mathbb{R}^4} e^{2\pi i \Phi(x, x_3, t, \xi, \xi_3, y, \eta)} a^q_{k, \ell, \theta, m_1, m_2, m_3}(x, x_3, t, \xi, \xi_3, y, \eta) d\xi d\eta.
\]

The phase function of this is
\[
\Phi(x, x_3, t, \xi, \xi_3, y, \eta) = (x \cdot \xi + x_3 \xi_3) + t(\xi + \xi_3 A(x)) - (\xi - \eta) \cdot y,
\]
and the amplitude is
\[
a^q_{k, \ell, \theta, m_1, m_2, m_3}(x, x_3, t, \xi, \xi_3, y, \eta) = \psi(x, x_3) \psi \left( \frac{\xi + \xi_3 A(x)}{|\xi + \xi_3 A(x)|} - e([\theta]) \right) \psi \left( \frac{t - [q]}{2^{j/2}} \right)
\]
\[
\times \psi \left( \frac{|\xi + \xi_3 A(x)| - J(k)}{2^{j/2}} \right) \psi \left( \frac{|q| \xi_3 - J(\ell)}{2^{j/2}} \right)
\]
\[
\times \psi \left( \frac{\eta + \xi_3 A(y) - (m_1, m_2) 2^{j/2}}{2^{j/2}} \right) \psi \left( \frac{|q| \xi_3 - m_3 2^{j/2}}{2^{j/2}} \right).
\]
Lemma 8.1. The integral $H_{k,\ell,\theta,m_1,m_2}(x, x_3, t, \eta, \xi_3)$ in (8.5) is bounded by
\[
(8.5) \quad \psi \left( \frac{[q] \xi_3 - \beta 2^{j/2}}{2^{j/2}} \right) \int \frac{\left| \tilde{a}_{k,\ell,\theta,m_1,m_2,m_3}(x, x_3, t, \xi_3, \eta, y) \right|}{(|x - y + te(\theta)|^{2j/2} + 1)^N} \left( |\xi - \eta|^{2j/2 - \epsilon} + 1 \right)^N d\xi dy,
\]
where $N > 1$, and $\tilde{a}_{k,\ell,\theta,m_1,m_2,m_3}$ is a modification of $a_{k,\ell,\theta,m_1,m_2}$ in (8.4) with the same support.

Proof. We compute the derivatives of the phase function and amplitude in (8.4):
\[
\nabla_{x} \Phi(x, x_3, t, \xi_3, \eta, y) = (x - y + te \theta) + O(2^{-j/2})
\]
\[
\nabla_{y} \Phi(x, x_3, t, \xi_3, \eta, y) = \xi - \eta
\]
\[
\nabla_{x} a_{k,\ell,\theta,m_1,m_2,m_3}(x, x_3, t, \xi_3, \eta, y) = O(2^{-j/2})
\]
\[
\nabla_{y} a_{k,\ell,\theta,m_1,m_2,m_3}(x, x_3, t, \xi_3, \eta, y) = O(2^{j/2 + c_\epsilon}).
\]

Then, this enables us to apply the integration by parts in (8.3), which yields the desired result. \qed

8.2. Estimate of $\left\| T^{q}_{k,\ell,\theta} P_{m_1,m_2,m_3} f \right\|_{L^s}$. On the support of $a_{k,\ell,\theta,m_1,m_2,m_3}$ in (8.4), we have that
\[
(8.6) \quad \left( \xi + \xi_3 A(x) + t \xi_3 A^T \frac{[\xi + \xi_3 A(x)]}{[\xi + \xi_3 A(x)]} \right), [q] \xi_3 = (A_\theta J(k, \ell), J(\ell)) + O(2^{j/2}),
\]
\[
(8.7) \quad \left( \eta + \xi_3 A(y), [q] \xi_3 \right) = 2^{j/2}(m_1, m_2, m_3) + O(2^{j/2}).
\]

In (8.5), the main contribution occurs at $y = x + A(te(\theta)) + O(2^{j/2+\epsilon})$ and $\eta = \xi + O(2^{j/2+2\epsilon})$, namely
\[
\left( \eta + \xi_3 A(y), [q] \xi_3 \right) = \left( \xi + \xi_3 A(x) + t \xi_3 A(te(\theta)), [q] \xi_3 \right) + O(2^{j/2+2\epsilon}) = \left( \tilde{A}_\theta J(k, \ell), J(\ell) \right) + O(2^{j/2+2\epsilon})
\]
where
\[
\tilde{A}_\theta : = \left( e(\sqrt{\theta}) \right) \left( e(\theta) \right) \quad \text{whereas} \quad A_\theta = \left( e(\sqrt{\theta}) \right) A^T e(\theta) \quad \text{in (8.19)}.
\]

This leads us to expect $\left\| T^{q}_{k,\ell,\theta} P_{m_1,m_2,m_3} f \right\|_{L^s} \lesssim 2^{-M_j}$ whenever the support of (8.7) is away from
\[
\left( \eta + \xi_3 A(y), [q] \xi_3 \right) = \left( \tilde{A}_\theta J(k, \ell), J(\ell) \right) + O(2^{j/2+2\epsilon})
\]
which also lead us to select these effective points $2^{j/2}(m_1, m_2, m_3)$ of (8.7) as follows.

Definition 8.2. To each $(k, \ell, \theta) \in U_1 \times U_0 \times U_0$, we assign a grid-set of $\left( \tilde{A}_\theta J(k, \ell), J(\ell) \right) + O(2^{j/2+2\epsilon})$.
\[
(8.9) \quad Q_{k,\ell,\theta} = \left\{ 2^{j/2}(m_1, m_2, m_3) \in 2^{j/2}Z^3 : \left| 2^{j/2}(m_1, m_2, m_3) - \left( \tilde{A}_\theta J(k, \ell), J(\ell) \right) \right| < 2^{j/2+2\epsilon} \right\}.
\]

In (8.9), for a fixed $(k, \ell, \theta)$, there exists a constant $C > 0$ independent of $k, \ell, \theta$ such that
\[
\int (Q_{k,\ell,\theta}) \leq C^{2^{j+\epsilon}}.
\]
We define
\[
\mathcal{P}_{k,\ell,\theta}^q f = \sum_{(m_1, m_2, m_3) \in \mathcal{Q}_{k,\ell,\theta}} P_{m_1, m_2, m_3}^q f.
\]
Then,
\[
T_{k,\ell,\theta}^q f = T_{k,\ell,\theta}^q \mathcal{P}_{k,\ell,\theta}^q f + \sum_{(m_1, m_2, m_3) \notin \mathcal{Q}_{k,\ell,\theta}} P_{m_1, m_2, m_3}^q f.
\]

**Theorem 8.1.** For sufficiently large $M > 0$, we have that
\[
\| \left( \left( \sum_{(q, k, \ell, \theta) \in \mathcal{U}_1 \times \mathcal{U}_2} |T_{k,\ell,\theta}^q f|^2 \right)^{1/2} \right) \|_{L^4} \lesssim \| \left( \left( \sum_{(q, k, \ell, \theta) \in \mathcal{U}_1 \times \mathcal{U}_2} |T_{k,\ell,\theta}^q \mathcal{P}_{k,\ell,\theta}^q f|^2 \right)^{1/2} \right) \|_{L^4}
\]
with an error of $O(2^{-Mj} \| f \|_{L^4})$ on the right-hand side.

**Proof of (8.13).** By (8.12) with the cardinality $\sharp (\mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_0) \leq (2^{1/2} 2^{2j})^4$, it suffices to show that
\[
\left\| \sum_{(m_1, m_2, m_3) \notin \mathcal{Q}_{k,\ell,\theta}} P_{m_1, m_2, m_3}^q f \right\|_{L^4} \lesssim 2^{-Mj} \| f \|_{L^4}.
\]
We fix $(q, k, \ell, \theta) \in \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{U}_0$. Then, for $m = (m_1, m_2, m_3) \notin \mathcal{Q}_{k,\ell,\theta}$ it suffices to prove that
\[
\left\| \sum_{(m_1, m_2, m_3) \notin \mathcal{Q}_{k,\ell,\theta}} P_{m_1, m_2, m_3}^q f \right\|_{L^4} \lesssim 2^{-Mj} \| f \|_{L^4}
\]
for large $M > 0$.

The bottom part in (8.3) with $|(x, x_3), t| \lesssim 1$ and $|\xi|, |\xi_3| \lesssim 2^{1+j} \eta$ implies $|(y, y_3)| \lesssim 1$ and $|\eta| \lesssim 2^{j+2c_j}$. Combined with (8.7), this enables us to assume that $|(m_1, m_2, m_3)| \leq C 2^{j+r_j}$ when claiming (8.14). Thus, to prove (8.14) in the kernel of (8.3) above, it suffices to prove the pointwise estimate
\[
|H_{k,\ell,\theta,m_1,m_2}(x, x_3, t, \eta, \eta_3)| \lesssim 2^{-Mj}
\]
if $m = (m_1, m_2, m_3) \notin \mathcal{Q}_{k,\ell,\theta}$.

If $m \notin \mathcal{Q}_{k,\ell,\theta}$ in (8.3) with $|(m_1, m_2, m_3)| \leq C 2^{j+r_j}$, then
\[
|2^{1/2} (m_1, m_2, m_3) - (\tilde{A}_\theta J(k, \ell), J(\ell))| \geq 2^{2c_j} 2^{j/2}.
\]
By combining this with $|m_3 2^{1/2} - [q_3]| = O(2^{j/2})$ and $|[q_3] 2^{1/2} - \ell 2^{1/2}| = O(2^{j/2})$ in the support of (8.4), we have that
\[
|(m_1, m_2) 2^{1/2} - \tilde{A}_\theta J(k, \ell)| \geq 2^{1/2} 2^{c_j}.
\]
In view of (8.6) and (8.7) combined with $\tilde{A}_\theta J(k, \ell) = J(k) e(\theta) + J(\ell) A e(\theta)$, we have that
\[
|(m_1, m_2) 2^{1/2} - \tilde{A}_\theta J(k, \ell)| = |(\eta + \xi_3 A(y)) - (\xi + \xi_3 A(x) + t \xi_3 A e(\theta))| + O(2^{1/2})
\]
\[
= |(\eta - \xi) - \xi_3 (A(x) - A(y) + t A e(\theta)))| + O(2^{1/2}).
\]
Together with (8.16) and \(|\xi| \leq 2^{j+\epsilon}j\), this implies that at least one of the following two inequalities holds:

\[ |\eta - \xi| \geq 2^{2j}2^{j/2-10} \text{ or } |A(x) - A(y) + tAe([\theta])| \geq 2^{j+2j/2-10}. \]

We invoke these in (8.5) with \(N \gg M/\epsilon\) to obtain (8.15) whenever \(A\) is invertible. \(\square\)

### 8.3. Vector-Valued Estimates and Maximal Average of the Dual Part

For simplicity, we denote \((x, x_3) \in \mathbb{R}^2 \times \mathbb{R}\) by \([x]\) if there is no confusion. By Theorems 6.1 and 7.1 we have shown that

\[
\left( \sum_{q \in \mathcal{U}_1} \int_{[k, \ell, \theta]} |T_{k, \ell, \theta}^q f([x], t)|^4 \, d[x] dt \right)^{1/4} \lesssim 2^{2j2^{j/8}} \left( \sum_{q \in \mathcal{U}_1} \int_{[k, \ell, \theta]} |T_{k, \ell, \theta}^q f([x], t)|^2 \, d[x] dt \right)^{1/4},
\]

where \((k, \ell, \theta) \in \mathcal{U}_1 \times \mathcal{U}_0 \times \mathcal{U}_0\) in the summation. By Theorem 8.1 we have that

\[
\left( \int \left| \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q f([x], t) \right|^2 \, d[x] dt \right)^{1/4} \lesssim \left( \int \left| \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q \mathcal{P}_{k, \ell, \theta}^q f([x], t) \right|^2 \, d[x] dt \right)^{1/4} + 2^{-Mj} \|f\|_{L^4}.
\]

Furthermore, we can obtain the following vector-valued inequality.

**Theorem 8.2.** For all \(f \in L^4(\mathbb{R}^3)\), it holds that

\[
\left( \sum_{q \in \mathcal{U}_1} \int \left| \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q \mathcal{P}_{k, \ell, \theta}^q f([x], t) \right|^2 \, d[x] dt \right)^{1/4} \lesssim 2^{j/16+\epsilon} \left( \int \left( \sup_{q \in \mathcal{U}_1} \sum_{k, \ell, \theta} \mathcal{P}_{k, \ell, \theta}^q f(y, y_3) \right)^2 \, dy dy_3 \right)^{1/4}.
\]

**Proof of Theorem 8.2** By the disjointness of the supports of \(\psi \left(\frac{t-x}{2^{-j/2}}\right)\) with \([q] = 2^{j/2}q \approx 1\) for \(q \in \mathcal{U}_1\), we have that

\[
\sum_{q \in \mathcal{U}_1} \int \left| \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q \mathcal{P}_{k, \ell, \theta}^q f([x], t) \right|^2 \, d[x] dt = \int \left( \sum_{q \in \mathcal{U}_1} \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q \mathcal{P}_{k, \ell, \theta}^q f([x], t) \right)^2 \, d[x] dt.
\]

Then, there exists \(g\) such that \(\int |g([x], t)|^2 \, d[x] dt = 1\), satisfying

\[
\int \left| \sum_{q} \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q \mathcal{P}_{k, \ell, \theta}^q f([x], t) \right|^2 \, d[x] dt = \left( \int \sum_{q} \sum_{k, \ell, \theta} T_{k, \ell, \theta}^q \mathcal{P}_{k, \ell, \theta}^q f([x], t) \, g([x], t) \, d[x] dt \right)^2.
\]
By (8.14) and the Schwartz inequality, we have that
\[
\left| T_{k,\ell,\theta}^q \mathcal{P}_{k,\ell,\theta}^q f(x,t) \right|^2 = \int |H_{k,\ell,\theta}^q(x,x_3,t,y,y_3)|^{1/2} |H_{k,\ell,\theta}^q(x,x_3,t,y,y_3)|^{1/2} |\mathcal{P}_{k,\ell,\theta}^q f(y,y_3)| dy dy_3 \leq \int |H_{k,\ell,\theta}^q(x,x_3,t,y,y_3)| \cdot |\mathcal{P}_{k,\ell,\theta}^q f(y,y_3)|^2 dy dy_3 \leq \int |H_{k,\ell,\theta}^q(x,x_3,t,y,y_3)| \cdot |\mathcal{P}_{k,\ell,\theta}^q f(y,y_3)|^2 dy dy_3.
\]
(8.20)

By (5.13), the integral kernel $H_{k,\ell,\theta}^q(x,x_3,t,y,y_3)$ of $T_{k,\ell,\theta}^q$ is majorized by $H_0^q(x,x_3,t,y,y_3)$, which is given by
\[
\psi(x,x_3) \psi \left( \frac{t-[q]}{2^{-j/2}} \right) \varphi \left( \frac{x-y+\lfloor q \rfloor e([\theta]) - x_3 - y_3 + [q] A(x) \cdot e([\theta])}{2^{-j/2}} \right) 2^{3j/2},
\]
(8.21)
where $\varphi$ is a nonnegative Schwartz function and $[q] \approx 1$. By inserting (8.20) into the right-hand side of (8.19), we obtain
\[
\int \sum_{q \in \mathbb{U}_1} \sum_{k,\ell,\theta} |T_{k,\ell,\theta}^q \mathcal{P}_{k,\ell,\theta}^q f(x,x_3,t)|^2 g(x,x_3,t) dx dx_3 dt \leq \int \sum_{q \in \mathbb{U}_1} \sum_{k,\ell,\theta} |\mathcal{P}_{k,\ell,\theta}^q f(y,y_3)|^2 \left( \int H_0^q(x,x_3,t,y,y_3) g(x,x_3,t) dx dx_3 dt \right) dy dy_3.
\]
(8.22)
There exists a measurable function $(y,y_3) \rightarrow m(y,y_3) \in \mathbb{U}_1$ with $m$ depending on $f$ such that
\[
\sup_{q \in \mathbb{U}_1} \sum_{k,\ell,\theta} |\mathcal{P}_{k,\ell,\theta}^q f(y,y_3)|^2 = \sum_{k,\ell,\theta} |\mathcal{P}_{k,\ell,\theta}^{m(y,y_3)} f(y,y_3)|^2.
\]
Then, (8.22) is bounded by
\[
\int \sum_{k,\ell,\theta} |\mathcal{P}_{k,\ell,\theta}^{m(y,y_3)} f(y,y_3)|^2 \left( \int \sum_{q \in \mathbb{U}_1} H_0^q(x,x_3,t,y,y_3) g(x,x_3,t) dx dx_3 dt \right) dy dy_3 \leq \int \sum_{k,\ell,\theta} |\mathcal{P}_{k,\ell,\theta}^{m(y,y_3)} f(y,y_3)|^2 \left( \sup_{\theta} \int \sum_{q \in \mathbb{U}_1} H_0^q(x,x_3,t,y,y_3) g(x,x_3,t) dx dx_3 dt \right) dy dy_3.
\]
(8.23)
Because $|t-[q]| \leq 2^{-j/2}$ in each support of $\psi \left( \frac{t-[q]}{2^{-j/2}} \right)$ and $\varphi$ is rapidly deceasing, it holds that
\[
\sum_{q \in \mathbb{U}_1} H_0^q(x,x_3,t,y,y_3) \leq H_j(x,x_3,t,y,y_3,\theta),
\]
where
\[
H_j(x,x_3,t,y,y_3,\theta) = \chi(t) \psi(x,x_3) \varphi \left( \frac{x-y+\lfloor q \rfloor e([\theta]) - x_3 - y_3 + [q] A(x) \cdot e([\theta])}{2^{-j/2}} \right) 2^{3j/2}.
\]
We set the Nikodym-type maximal function $N_j$ as
\[
N_j(g)(y,y_3) = \sup_{\theta \in \mathbb{U}_0} \int H_j(x,x_3,t,y,y_3,\theta) g(x,x_3,t) dx dx_3 dt.
\]
Then, we can majorize \((S.25)\) by
\[
\int \sum_{k, l, \ell, \theta} |P_{k, l, \ell, \theta}^\nu f(y, y_3)|^2 N_j(g)(y, y_3) dy dy_3
\]
\[
\lesssim \left( \int \left( \sup_{\nu \in \Omega} \sum_{k, l, \ell, \theta} |P_{k, l, \ell, \theta}^\nu f(y, y_3)|^2 \right)^2 dy_3 \right)^{1/2} \left( \int |N_j(g)(y, y_3)|^2 dy_3 \right)^{1/2}.
\]
Thus, applying Proposition \(8.1\) below for \((8.25)\) yields \((8.17)\), proving Theorem \(8.2\). \hfill \Box

**Proposition 8.1.** Suppose that \(A\) is the skew-symmetric matrix \(E\). Then, the following boundness holds.

\[
(8.26) \quad \|N_j\|_{L^2(\mathbb{R})} \rightarrow L^2(\mathbb{R}) \lesssim 2^{j/8}
\]

**Proof.** We can replace the discrete function \(e(\theta)\) with \(\theta \in U_0\) by the continuous one \(e(\theta) = (\cos \theta, \sin \theta)\) with \(\theta \in [0, 2\pi]\) in \((S.21)\). Moreover, by switching the roles of \(x\) and \(y\), we can express \(N_j f(x, x_3)\) as

\[
(8.27) \quad \sup_{\theta \in [0, 2\pi]} \int \chi(t) \psi(x, x_3) \varphi \left( \frac{x - (y + te(\theta))}{2^{-j/2}}, \frac{x_3 - (y_3 + tE(x) \cdot e(\theta))}{2^{-j/2}} \right) 2^{3j/2} f(y, y_3, t) dy dy_3 dt
\]

With \(A = E\) in \((S.27)\), where \(E(x_1, x_2) = (-x_2, x_1)\), we can switch \(E(y)\) and \(E(x)\) in the third component of \(\varphi\) in \((S.27)\), because the fact that \(y = x - te(\theta) + O(2^{-j/2})\) in the first two components implies that \(E(y) \cdot e(\theta) = E(x) \cdot e(\theta) + O(2^{-j/2})\) as \(E(e(\theta)) \cdot e(\theta) = 0\). Here, we can also write \(\psi(x, x_3)\) owing to the support condition \(\psi(y, y_3)\) which is now absorbed into \(f\). Combined with the Fourier inversion formula, this yields that the above integral can be written as

\[
\int e^{2\pi i [\xi \cdot (x - (y + te(\theta))) + \xi_3 (x_3 - (y_3 + tE(x) \cdot e(\theta)))]} \chi(t) \psi(x, x_3) \hat{\varphi} \left( \frac{\xi}{2^{j/2}}, \frac{\xi_3}{2^{j/2}} \right) d\xi d\xi_3 f(y, y_3, t) dy dy_3 dt
\]

\[
= \int e^{2\pi i [\xi \cdot (x + te(\theta)) + \xi_3 (x_3 + tE(x) \cdot e(\theta))]} \chi(t) \psi(x, x_3) \hat{\varphi} \left( \frac{\xi}{2^{j/2}}, \frac{\xi_3}{2^{j/2}} \right) \hat{f}(\xi, \xi_3, t) d\xi d\xi_3 dt.
\]

Then the \(\theta\)-derivative is given by

\[
\int e^{2\pi i [\xi \cdot (x + te(\theta)) + \xi_3 (x_3 + tE(x) \cdot e(\theta))]} 2\pi i (\xi + \xi_3 E(x)) \cdot e'(\theta) \chi(t) \psi(x, x_3) \hat{\varphi} \left( \frac{\xi}{2^{j/2}}, \frac{\xi_3}{2^{j/2}} \right) \hat{f}(\xi, \xi_3, t) d\xi d\xi_3 dt.
\]

We may assume that \(|\xi|, |\xi_3| \leq 2^{j/2}\) and replace \(\hat{\varphi}\) by the product of \(\psi's\) defined on \(\mathbb{R}^2\) and \(\mathbb{R}^1\).

Moreover, from the Sobolev imbedding inequality \(\sup_{\theta \in [0, 2\pi]} |F_\theta(\cdot)| \lesssim \|\partial_\theta^{1/2} F_\theta(\cdot)\|_{L^2([0, 2\pi])}\), it suffices to treat the oscillatory integral operator \(M f(x, x_3, \theta)\) defined by

\[
\int e^{2\pi i [\xi \cdot (x + te(\theta)) + \xi_3 (x_3 + tE(x) \cdot e(\theta))]} \chi(t) \psi(\theta) \psi(x, x_3) \psi \left( \frac{\xi}{2^{j/2}} \right) \psi \left( \frac{\xi_3}{2^{j/2}} \right) \times (2^{j/2})^{1/2} \psi \left( \frac{\xi + \xi_3 E(x)}{2^{j/2}} \right) \hat{f}(\xi, \xi_3, t) d\xi d\xi_3 dt
\]
and prove that \( \|M\|_{L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)} \lesssim 2^{1/8} \). To this end, we apply the Plancherel theorem on the third variables \( \xi_3 \) and \( x_3 \). Then, we observe that \( \|M\|_{L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)} \lesssim 2^{1/8} \) is equivalent to
\[
\|T^{\xi_3}\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim 2^{1/8} \text{ uniformly in } \xi_3,
\]
where
\[
T^{\xi_3} f(x, \theta) = (2^{j/2})^{1/2} \psi \left( \frac{\xi_3}{2^{j/2}} \right) \int e^{2\pi i \xi_3 (x + \langle \theta, e(x) \rangle)} \chi(t) \psi(\theta) \psi(x) \psi \left( \frac{\xi}{2^{j/2}} \right) \times \psi \left( \frac{(\xi + E(x)) \cdot e'(\theta)}{2^{k/2}} \right) \hat{f}(\xi, t) d\xi dt.
\]
It suffices to deal with the case that \( |\xi_3| \approx 2^{j/2-k} \) for \( k \geq 0 \). By applying the change of variable \( \xi \to \xi_3 \xi \) in \( \mathbb{R}^2 \) together with \( \|\xi_3 f(\xi_3, t)\|_{L^2} = \|f(\cdot, t)\|_{L^2} \), we can work with
\[
T^{\xi_3} f(x, \theta) = (2^{j/2})^{1/2} \xi_3 2^{k/2} \psi (2^{k} x) \int e^{2\pi i \xi_3 2^{k} \xi (x + \langle \theta, e(x) \rangle + t E(x) \cdot e(\theta))} \chi(t) \psi(\theta) \psi(x) \psi(\xi) \psi(t) \times \psi \left( (\xi + E(x)) \cdot e'(\theta) \right) \hat{f}(\xi, 2^{k} t) \chi(2^{k} t) d\xi dt,
\]
where we used \( \psi(2^{k} x) = \psi(2^{k} x) \psi(x) \) and \( \chi(2^{k} t) = \chi(2^{k} t) \psi(t) \). Write \( 2\pi 2^{2k} \xi_3 \) in the above exponent as \( \lambda \) and write the compactly supported cutoff function as
\[
\psi(x, \theta, \xi, t) = \psi(x) \psi(\theta) \psi(\xi) \psi(t) \psi \left( (\xi + E(x)) \cdot e'(\theta) \right),
\]
and define
\[
T^\lambda f(x, \theta) = \int e^{i \lambda \langle x - \xi + E(x) - E(\theta) \rangle} \psi(x, \theta, \xi, t) f(\xi, t) d\xi dt.
\]
We assume that \( \|T^\lambda\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim \lambda^{-1} \lambda^{-1/4} \), which will be demonstrated in Lemma 8.2 below. Then \( \lambda = 2\pi 2^{2k} \xi_3 \) with \( \xi_3 = 2^{j/2-k} \) in (8.29) implies that
\[
\|T^{\xi_3}\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim \left[ (2^{j/2})^{1/2} \xi_3 2^{k/2} \right] \left[ (2^{2k} \xi_3) \right]^{-1/4} = 2^{j/8} 2^{-2k+4},
\]
which in turn implies (8.28). \( \square \)

**Lemma 8.2.** Consider the operators \( T^\lambda \) defined in (8.30) where \( \psi \in C_0^\infty (\mathbb{R}^3 \times \mathbb{R}^3) \). Then
\[
\|T^\lambda\|_{L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)} \lesssim \lambda^{-1} \lambda^{-1/4}.
\]
Theorem 2.1 of [4] yields that if one of the projections (say \(\pi\)) admits a fold singularity at \((x_0, y_0)\), then
\[
\text{Proof of (8.31).}
\]
\[
T^\lambda f(x) = \int e^{i\lambda\Phi(x,y)}\psi(x,y)f(y)dy \quad \text{where the smooth function } \psi \text{ is supported near } (x_0, y_0).
\]
(8.32) 
\[
\|T^\lambda\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} = O(\lambda^{(d-1)/2-1/4}).
\]
Let \(h(x, y) = \det(\Phi''_{xy}(x, y))\) and \(h(x_0, y_0) = 0\). We verify that \(\pi_L\) has a fold singularity at \((x_0, y_0)\) by setting \(\delta = 1\) for all left kernel field \(V_L = \sum_{j=1}^d b_j \partial_{y_j}\) with \((b_j) \in Ker(\Phi''_{xy}(x_0, y_0))\).

To standardize the notation, we set the variables \(x_3 = \theta, y_1 = \xi_1, y_2 = \xi_2\) and \(y_3 = t\) in (8.30), and rewrite our operator \(T^\lambda\) in (8.30) as
\[
T^\lambda f(x_1, x_2, x_3) = \int_{y \in \mathbb{R}^d} e^{i\lambda[x_1y_1 + x_2y_2 + \epsilon(x_3)(y_1, y_2)y_3 + (-x_2, x_1)\epsilon(x_3)y_3]}\psi(x, y)f(y)dy
\]
where \(x = (x_1, x_2, x_3)\) and \(y = (y_1, y_2, y_3)\). Then, the mixed Hessian matrix of the phase function \(\Phi\) for \(T^\lambda\) is given by
\[
\Phi''_{xy}(x, y) = \begin{pmatrix} 1 & 0 & \sin x_3 \\ 0 & 1 & -\cos x_3 \\ -\sin x_3 & \cos x_3 & y_3 \end{pmatrix}
\]
with the determinant \(h(x, y) = \det(\Phi''_{xy})\) given by
\[
h(x, y) = y_3 + (-x_2, x_1) + (y_1, y_2) \cdot (-\sin x_3, \cos x_3).
\]
Along the points \(\{h(x, y) = 0\}\) with \(\text{rank}(\Phi''_{xy}) = 2\), we find the kernel field \(V_L = -\sin x_3 \partial_{y_1} + (\cos x_3) \partial_{y_2} + \partial_{y_3}\). Then a straightforward computation yields that \(V_L h(x, y) = 2\). This implies that the projection \(\pi_L : (x, y) \to (\Phi_x(x, y))\) has a fold singularity (namely \(\pi_L : C \to T^*(\mathbb{R}^d)\) is a submersion with a fold) at each point of \(\{h(x, y) = 0\}\). Therefore, we can apply (8.32), so that
\[
\|T^\lambda\|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \lesssim \lambda^{-(d-1)/2}\lambda^{-1/4},
\]
which is the desired bound of (8.31) for \(d = 3\).

Note that \(V_R = (y_3 \sin x_3) \partial_{x_1} + (-y_3 \cos x_3) \partial_{x_2} + \partial_{x_3}\) and \(V_R h = ((-x_2, x_1) + (y_1, y_2)) \cdot (\cos x_3, \sin x_3)\) can vanish where \(h(x, y) = 0\). This implies our fold singularity is only one-sided (not two-sided).

Remark 8.1. The Nikodym maximal operator in (8.27) is represented as the composition of the maximal operator
\[
N_{cone} f(x, x_3) := \sup_{\theta \in [0, 2\pi]} \int f(x-(y+te(\theta)), x_3 - E(x)-(y+te(\theta)), t) |\psi(y_1/\delta)/\delta| |\psi(y_2/\delta)/\delta| \psi(t) dy_1 dy_2 dt
\]
for $\delta = 2^{-j/2}$ and the one dimensional Hardy–Littlewood maximal operator along the third direction. We regard $N_{\text{cone}} f(x, x_3)$ as the cone-type-modification of the maximal average

$$N_{\text{horizon}}^\delta f(x, x_3) = \int f(x - (y + t e(\theta)), x_3 - E(x) \cdot (y + t e(\theta))) \psi(y_1/\delta) / \delta \psi(y_2/\delta) / \delta \psi(t) dy_1 dy_2 dt,$$

over the $\delta \times \delta \times 1$ tubes restricted on the horizontal plane of the Heisenberg group. In [50], it is shown that $\delta^{-1/4} \lesssim \|N_{\text{horizon}}^\delta\|_{L^2(\mathbb{H}^1)} \lesssim \delta^{-1/4} \log(1/\delta)$ where $\delta^{-1/4}$ coincides with $2^{j/8}$ in (8.20) for $\delta = 2^{-j/2}$.

9. Littlewood–Paley Inequality Associated with Cubes

In view of Theorem [8.2] to complete the proof of Main Theorem 1, it remains to verify the following perturbed version of the Littlewood–Paley inequality.

**Theorem 9.1.** Let $P_{k,\ell,\theta}^q f = \sum_{(m_1,m_2,m_3) \in Q_{k,\ell,\theta}} P_{m_1,m_2,m_3}^q f$ in (8.11). Then it holds that

$$\left( \int \left( \sup_{q \in U_1} \sum_{k,\ell,\theta} |P_{k,\ell,\theta}^q f(x,y_3)|^2 \right)^{2/4} dy dy_3 \right)^{1/4} \leq 2^{2cj} \|f\|_{L^1(\mathbb{R}^3)}.$$

9.1. Uniform Distribution of Cubes $Q_{k,\ell,\theta}$

**Proposition 9.1.** [Non-overlapping Property of Cubes $Q_{k,\ell,\theta}$ in (8.4)] Recall the cubes $Q_{k,\ell,\theta}$ consisting of at most $2^{6\epsilon j}$ number of grids in (8.11). Let $A = E$. Then, there exists $C > 0$ independent of $k,\ell,\theta$ such that

$$Q_{k,\ell,\theta} \cap Q_{k',\ell',\theta'} = \emptyset \quad \text{if} \quad |(k, \ell, \theta) - (k', \ell', \theta')| \geq C 2^{2\epsilon j}.$$

This implies that

$$\sum_{k,\ell,\theta} |P_{k,\ell,\theta}^q f(x, x_3)|^2 \lesssim 2^{6\epsilon j} \sum_{m_1,m_2,m_3 \in \mathbb{Z}^3} |P_{m_1,m_2,m_3}^q f(x, x_3)|^2,$$

where $P_{m_1,m_2,m_3}^q$ is the projection defined in (8.1).

**Proof.** We observe that $Q_{k,\ell,\theta}$ (depending on $q$) is contained in the cube with dimensions $2^{j/2+2\epsilon j} \times 2^{j/2+2\epsilon j} \times 2^{j/2+2\epsilon j}$ centered at the fixed point $\tilde{A}_0 J(k,\ell), J(\ell) \in \mathbb{R}^3$, where $\tilde{A}_0$ is defined in (8.8). If $|\ell - \ell'| \geq 2^{2\epsilon j}$, then $Q_{k,\ell,\theta} \cap Q_{k',\ell',\theta'} = \emptyset$. Thus, it suffices to consider the case that $\ell = \ell'$. Fix $\ell$ (thus fixing the last component of the center of $Q_{k,\ell,\theta}$), and consider the map $F: \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(k, \theta) := \tilde{A}_0 J(k, \ell) = \left( e(\theta) \quad A e(\theta) \right) J(k, \ell) = J(k) e(\theta) + J(\ell) A e(\theta),$$

where $A$ is a linear transformation.
where \( J(k) = k2^{j/2} \) and \( e([\theta]) = \begin{pmatrix} \cos 2^{-j/2}\theta \\ \sin 2^{-j/2}\theta \end{pmatrix} \). Regard \( k, \theta \) as continuous variables, and compute the Jacobian matrix:

\[
D_{k, \theta}F = (F'_k(k, \theta), F'_\theta(k, \theta)) = \begin{pmatrix} 2^{j/2}e([\theta]) & 2^{-j/2}(J(k)I + J(\ell)A)e([\theta]) \end{pmatrix}.
\]

Let \( A = E \). Then using \( \langle Ee([\theta]), Ae([\theta]) \rangle = 0 \), we obtain that

\[
\det(D_{k, \theta}F) = \begin{pmatrix} 2^{j/2}Ee([\theta]), 2^{-j/2}(J(k)I + J(\ell)A)e([\theta]) \end{pmatrix} = J(k) + J(\ell) \langle Ee([\theta]), Ae([\theta]) \rangle = J(k) \approx 2^j.
\]

Hence, \( \|D_{k, \theta}F\| \geq 2^{j/2} \). Under the assumptions that \( \ell = \ell' \) and \( C2^{2j} \leq \|(k, \theta) - (k', \theta')\| \), we have that

\[
|F(k, \theta) - F(k', \theta')| \geq \|D_{k, \theta}F\| \|(k, \theta) - (k', \theta')\| \geq C2^{-10}2^{j/2}2^{2cj}.
\]

This combined with (8.9) for \( C \geq 2^{100} \) implies (9.2). We note that the one-to-one correspondence of \( F \) was already justified in (4.31). Finally, (9.3) follows from the non-overlapping property (9.2) together with (8.9).

### 9.2. Littlewood–Paley Inequality

By Proposition 9.1 to prove Theorem 9.1 it suffices to prove the following Proposition.

**Proposition 9.2.** Let \( P^q_{m_1, m_2, m_3} \) be the projection in (9.1), and let \( p \geq 2 \). Then, for all \( f \in S(\mathbb{R}^3) \) we have that

\[
(9.4) \quad \left( \int \left( \sup_{q \in U_1} \sum_{m = (m_1, m_2, m_3) \in \mathbb{Z}^3} |P^q_{m_1, m_2, m_3}f(x, x_3)|^2 \right)^{p/2} dx dx_3 \right)^{1/p} \lesssim 2^{2j} \|f\|_{L^p(\mathbb{R}^3)}.
\]

Our proof of Proposition 9.2 is based on the following lemma.

**Lemma 9.1.** Let \( 1 \leq [q] \leq 2 \) and \( m \in \mathbb{Z} \). For a Schwartz function \( f \in S(\mathbb{R}) \), denote

\[
[P^q_m]f(x) = \int_{\mathbb{R}} \psi([q]_\xi - m) \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.
\]

Let \( \hat{\psi} \) be the Euclidean Fourier transform of \( \psi \) in \( \mathbb{R} \). Then, for a fixed \( x \in \mathbb{R}^1 \), it holds that,

\[
(9.5) \quad \sum_{m \in \mathbb{Z}} |[P^q_m]f(x)|^2 = \int_{\theta \in [0, 1]} \left| \sum_{m \in \mathbb{Z}} f(x + [q](\theta + m)) \hat{\psi}(\theta + m) \right|^2 d\theta,
\]

where

\[
\sum_{m \in \mathbb{Z}} [P^q_m]f(x)e^{2\pi i \imath m \theta} = \sum_{m \in \mathbb{Z}} f(x + [q](\theta + m)) \hat{\psi}((\theta + m)).
\]
Proof of (9.5). Because $\psi ([q] \xi - m)$ is rapidly decreasing in $m$, the Poisson summation formula yields that for $\theta \in [0,1]$,
\begin{equation}
\sum_{m \in \mathbb{Z}} [P_m]^q f(x)e^{2\pi i m \theta} := \sum_{m \in \mathbb{Z}} \left( \int \psi ([q] \xi - m) \hat{f}(\xi) e^{2\pi i \xi x} d\xi \right) e^{2\pi i m \theta}
= \sum_{m \in \mathbb{Z}} f(x + [q] (\theta + m)) \hat{\psi}(\theta + m).
\end{equation}
(9.6)

Fix $x$. With $\| [P_m]^q f(x) \| \lesssim (1 + |m|)^{-N}$, the Parseval identity for $\theta \to \sum_{m \in \mathbb{Z}} [P_m]^q f(x)e^{2\pi i m \theta}$ implies (9.5).

\[\square\]

Scheme of Proof for Proposition 9.2 Let $[q] = 2^{-j/2}q \approx 1$ with $q \in U_1$ in (5.6). For $x \in \mathbb{R}^2$ and $(m_1, m_2, m_3) \in \mathbb{Z}^3$, we set
\[P_{m_1,m_2}^{12} f(x, x_3) = \int e^{2\pi i [x_3 + x_3] \xi} \prod_{\nu=1}^2 \psi (\xi_\nu + \xi_3 A(x) \cdot e_\nu - m_\nu 2^{j/2}) \hat{f}(\xi_3) d\xi d\xi_3 \]
\[[P_{m_3}^3]^q f(x, x_3) = \int e^{2\pi i [x_3 + x_3] \xi} \psi (\frac{|q| \xi_3 - m_3 2^{j/2}}{2^{j/2}}) \hat{f}(\xi_3) d\xi d\xi_3. \]

From (5.1), we have that $P_{m_1,m_2,m_3}^q f(x, x_3) = [P_{m_3}^3]^q (P_{m_1,m_2}^{12} f)(x, x_3)$. By applying the change of variable $x_3 \to 2^{-j/2}x_3$, we obtain
\[\left( \int \left( \sup_{q \in U_1} \sum_{m=(m_1,m_2,m_3) \in \mathbb{Z}^3} \left| [P_{m_3}^3]^q (P_{m_1,m_2}^{12} f)(x, x_3) \right|^2 \right)^{p/2} dxdx_3 \right)^{1/p} \]
\[= \left( \int \left( \sup_{q \in U_1} \sum_{m=(m_1,m_2,m_3) \in \mathbb{Z}^3} \left| (2^{j/2})^{1/p} [P_{m_3}^3]^q (P_{m_1,m_2}^{12} f)(x, 2^{-j/2} x_3) \right|^2 \right)^{p/2} dxdx_3 \right)^{1/p}. \]

Note that
\[ (2^{-j/2})^{1/p} [P_{m_3}^3]^q (P_{m_1,m_2}^{12} f)(x, 2^{-j/2} x_3) = [P_{m_3,0}^3]^q ([P_{m_1,m_2}^{12} f]_{2^{j/2}})(x, x_3) \]
where
\[ [P_{m_1,m_2}^{12} f]_{2^{j/2}}(x, x_3) = (2^{-j/2})^{1/p} P_{m_1,m_2}^{12} f(x, 2^{-j/2} x_3) \]
and
\[ [P_{m_3,0}^3]^q f(x, x_3) = \int e^{2\pi i [x_3 + x_3] \xi} \psi (\xi) \xi_3 - m_3) \hat{f}(\xi, \xi_3) d\xi d\xi_3. \]
Thus, to prove (9.4), we shall prove the following two inequalities:

\[
\left(\int \left( \sup_{q \in U_1} \sum_{m = (m_1, m_2, m_3)} \left\| P_{m_3, 0} \right\|^q \left( \left\| P_{m_1 m_2}^{12} f \right\|_{2/2}^{2/2}(x, x_3) \right)^2 \right)^{p/2} dx \right)^{1/p} \\
\lesssim 2^{2j}\varepsilon \left( \int \left( \sum_{(m_1, m_2) \in \mathbb{Z}^2} \left\| P_{m_1 m_2}^{12} f \right\|_{2/2}^{2/2}(x, x_3) \right)^{p/2} dx \right)^{1/p}
\]

(9.9)

and

\[
\left( \int \left( \sum_{(m_1, m_2) \in \mathbb{Z}^2} \left\| P_{m_1 m_2}^{12} f \right\|_{2/2}^{2/2}(x, x_3) \right)^{p/2} dx \right)^{1/p} \lesssim \|f\|_{L^p(\mathbb{R}^3)}.
\]

(9.10)

Proof of (9.9). We set \(F_{m_1 m_2} = \left\| P_{m_1 m_2}^{12} f \right\|_{2/2}\) in (9.9). For \(r > 0\), let

\[
A_r f(x, x_3) = \frac{1}{2r} \int_{-r}^{r} f(x, x_3 + y_3) dy_3.
\]

Then, it suffices to prove that

\[
\sup_{q \in U_1} \sum_{m = (m_1, m_2, m_3)} \left\| P_{m_3, 0} \right\|^q \left( F_{m_1 m_2} \right)(x, x_3) \right|^2 \lesssim 2^{2j}\varepsilon A_{2^{j+3}r} \left( \sum_{(m_1, m_2) \in \mathbb{Z}^2} \left\| F_{m_1 m_2} \right\|^2 \right) (x, x_3)
\]

(9.11)

\[
+ \sum_{|m_3| \geq 2^{j+3}} \frac{1}{|m_3|} A_{5|m_3|} \left( \sum_{(m_1, m_2) \in \mathbb{Z}^2} \left\| F_{m_1 m_2} \right\|^2 \right)(x, x_3)
\]

because the \(L^{p/2}\) norm on the right-hand side of (9.12) yields (9.9). To show (9.12), we remove the dependence of \(q\) by expressing \(\sum_{m = (m_1, m_2, m_3)} \left\| P_{m_3, 0} \right\|^q \left( F_{m_1 m_2} \right)(x, x_3) \right|^2\) as the \(\theta\) average in Lemma 9.1. By applying (9.5) in Lemma 9.1 for the third component below and (9.8), we obtain

\[
\sum_{m = (m_1, m_2, m_3)} \left\| P_{m_3, 0} \right\|^q \left( F_{m_1 m_2} \right)(x, x_3) \right|^2 = \sum_{(m_1, m_2) \in \mathbb{Z}^2} \int_{\theta_3 \in [0, 1]} \left| \sum_{m_3 \in \mathbb{Z}} \left( F_{m_1 m_2} \right)(x_1, x_2, x_3 + \left| q \right| (\theta_3 + m_3)) \right|^{2} d\theta_3.
\]

(9.13)

Next, we split the \(d\theta_3\)-integral in (9.13) into the following local and global sums over \(m_3\):

\[
L_q(x, x_3) := \int_{\theta_3 \in [0, 1]} \left| \sum_{|m_3| \leq 2^j} \left( F_{m_1 m_2} \right)(x_1, x_2, x_3 + \left| q \right| (\theta_3 + m_3)) \right|^{2} d\theta_3
\]

\[
G_q(x, x_3) := \int_{\theta_3 \in [0, 1]} \left| \sum_{|m_3| > 2^j} \left( F_{m_1 m_2} \right)(x_1, x_2, x_3 + \left| q \right| (\theta_3 + m_3)) \right|^{2} d\theta_3.
\]
First, we treat the case with $L_q(x, x_3)$. By applying the Schwartz inequality for the sum $\sum_{|m| \leq 2^j}$, we obtain

$$L_q(x, x_3) \leq 2^{2j} \sum_{|m| \leq 2^j} \int_{\theta_3 \in [0, 1]} |F_{m_1 m_2}(x_1, x_2, x_3 + \theta_3 m_3) \hat{\psi}(\theta_3 + m_3)|^2 d\theta_3.$$ 

In the above integral, $|\theta_3 m_3| \leq 2^{3j + 3}$ because $|m_3| \leq 2^j$ and $0 \leq \theta_3 \leq 1$. We apply the change of variable $[q](\theta_3 + m_3) = y_3$ for the $d\theta_3$ integral with the Jacobian $[q] \approx 1$ to obtain

$$\int_{\theta_3 \in [0, 1]} |F_{m_1 m_2}(x_1, x_2, x_3 + [q](\theta_3 + m_3)) \hat{\psi}(\theta_3 + m_3)|^2 d\theta_3 \leq \frac{2}{2} \int_{y_3 \in \mathbb{R}} |F_{m_1 m_2}(x_1, x_2, x_3 + y_3)|^2 dy_3 \leq 2^{3j + 4} A_{2^j + 3}(|F_{m_1 m_2}|^2)(x, x_3),$$

where $A_{2^j + 3}$ is defined in (9.11). This implies that

(9.14) 

$$L_q(x, x_3) \leq C 2^{2j} A_{2^j + 3}(|F_{m_1 m_2}|^2)(x, x_3) \text{ with } C \text{ independent of } q.$$ 

We now consider the case of $G_q(x, x_3)$. We apply the Schwartz inequality for $\sum_{m_3 > 2^j}$ in $G_q(x, x_3)$ utilizing the decay $|\hat{\psi}(\theta_3 + m_3)| \leq C/|m_3|^{\nu_0}$ to obtain,

$$G_q(x, x_3) \leq \sum_{|m_3| > 2^j} \frac{C}{|m_3|^{\nu_0}} \int_{\theta_3 \in [0, 1]} |F_{m_1 m_2}(x_1, x_2, x_3 + [q](\theta_3 + m_3))|^2 |\hat{\psi}(\theta_3 + m_3)| d\theta_3.$$ 

Next, we apply the change of variable $[q](\theta_3 + m_3) = y_3$ for the $d\theta_3$ integral. Then, as $|[q](\theta_3 + m_3)| \leq 5|m_3|$, we obtain

$$\int_{\theta_3 \in [0, 1]} |F_{m_1 m_2}(x_1, x_2, x_3 + [q](\theta_3 + m_3))|^2 d\theta_3 \leq 2^{10}|m_3| A_{5|m_3|}(|F_{m_1 m_2}|^2)(x, x_3).$$

This implies that

(9.15) 

$$G_q(x, x_3) \leq C \sum_{|m_3| > 2^j} \frac{1}{|m_3|^{\nu_0}} A_{5|m_3|}(|F_{m_1 m_2}|^2)(x, x_3) \text{ with } C \text{ independent of } q.$$ 

Then we apply (9.14) and (9.15) in (9.13) to obtain (9.12). \hfill \Box

**Proof of (9.10).** Owing to the $L^p$ norm invariance in the scaling (9.7), it suffices to show that

(9.16) 

$$\left\| \left( \sum_{m = (m_1, m_2) \in \mathbb{Z}^2} |P_{m_1 m_2}^{12} f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^p(\mathbb{R}^3)}.$$ 

By rescaling $(x, x_3) \to (2^{-j/2}x_1, 2^{-j/2}x_2, 2^{-j}x_3)$ and $(\xi, \xi_3) \to (2^{j/2}\xi_1, 2^{j/2}\xi_2, 2^j\xi_3)$, to prove (9.16), it suffices to consider the case $j = 0$ and $P_{m_1 m_2}^{12}$ is given by

(9.17) 

$$P_{m_1 m_2}^{12} f(x, x_3) = \int e^{2\pi i [x \cdot \xi_3 + \xi_3 \xi_3]} \prod_{i=1}^2 \psi(\xi_\nu + \xi_3 A(x) \cdot e_\nu - m_\nu) \hat{f}(\xi, \xi_3) d\xi d\xi_3.$$
As \( \hat{f} \) is rapidly decreasing, we observe that

\[
|P_{m_1m_2}^{12}f(x, x_3)| \leq C(x, x_3)(1 + |(m_1, m_2)|)^{-N}.
\]

Thus, we can apply (9.6) to the function with \( \theta = (\theta_1, \theta_2) \in [0, 1]^2 \) given by

(9.18) \[
W_{\theta}f(x, x_3) = \sum_{m = (m_1, m_2) \in \mathbb{Z}^2} P_{m_1m_2}^{12}f(x, x_3)e^{2\pi i (m_1\theta_1 + m_2\theta_2)}.
\]

When applying (9.6) to the double sum in (9.18), we replace \( \hat{\psi} \) with \( \hat{\psi}(\cdot + \xi_3 A(x) \cdot e_\nu) \) for \( \nu = 1 \) and \( \nu = 2 \) in (9.17). Then we obtain that

\[
W_{\theta}f(x, x_3) = \sum_{m \in \mathbb{Z}^2} f(x_1 + \theta_1 + m_1, x_2 + \theta_2 + m_2, x_3 + A(x) \cdot (\theta_1 + m_1, \theta_2 + m_2)) \prod_{\nu=1}^2 \hat{\psi}(\theta_\nu + m_\nu).
\]

As for the Parseval identity in Lemma 9.1, we have for a fixed \((x, x_3) \in \mathbb{R}^3\) that

(9.20) \[
\left( \sum_{m \in \mathbb{Z}^2} |P_{m_1m_2}^{12}f(x, x_3)|^2 \right)^{1/2} = \left( \int_{\theta \in [0, 1]^2} |W_{\theta}f(x, x_3)|^2 d\theta_1 d\theta_2 \right)^{1/2}.
\]

Taking the \( L^p \) norm of (9.20) and changing the order of integrals for \( L^p(\mathbb{R}^3) \) and \( L^2(d\theta) \), we obtain

\[
\left\| \left( \sum_{m \in \mathbb{Z}^2} |P_{m_1m_2}^{12}f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^3)} \lesssim \left( \int_{\theta \in [0, 1]^2} \|W_{\theta}f\|^2_{L^p(\mathbb{R}^3)} d\theta \right)^{1/2} \lesssim \|f\|_{L^p(\mathbb{R}^3)},
\]

as \( \|W_{\theta}f\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)} \) because \( |\hat{\psi}(\theta_1 + m_1)\hat{\psi}(\theta_2 + m_2)| \lesssim (1 + |(m_1, m_2)|)^{-10} \) in (9.19). \( \square \)

9.3. Concluding Remark. If \( A \neq E \), the critical region of (6.19) neither forms a \( C_{2 \times 2} \) cone nor a torus \( S_n \times S_n \). However, whenever \( \det((EA) + (EA)^T) > 0 \), it holds that for \( J(k, \ell) \in S_m \),

\[
\left( J(k, \ell), A_{\theta} J(k, \ell) \right) \in \mathcal{B}(S_m) := \bigcup_{u \in S_m} \{ u \} \times \mathcal{E}(u) \text{ where } \mathcal{E}(u) \text{ is an ellipse.}
\]

The set \( \mathcal{B}(S_m) \) is an elliptic bundle with the base \( S_m \) rather than a torus \( S_m^2 \). This structure enables us to modify all the estimates in this paper except Proposition 9.1. It would be interesting to obtain the \( L^p \) boundeness of the circular maximal function \( M_4^A \) under the assumption that \( \det((EA) + (EA)^T) > 0 \), equivalently \( A \) has only complex eigenvalues.

10. Appendix

Proof of Lemma 9.3. For each \( \alpha > 0 \), we set two functions \( f_0(x) = f(x)\chi_{[0, \delta_0]}(|f(x)|) \) and \( f_1(x) = f(x)\chi_{[\delta_0, \infty]}(|f(x)|) \), each depending on \( \alpha \). We will determine \( \delta \) later. For each \( i = 0, 1 \), we use the
definition of $L^{p_1,\infty}$ norm with \((8.3)\) and \((3.11)\) to obtain that

$$\alpha^{p_0} \sum_k 2^k \|\{x : |U_k f_0(x)| > \alpha\}\| \lesssim \|U(f_0)\|_{L^{p_0}(\mathbb{R}^3)} \lesssim C_0^{p_0} \int_{|f(x)| < \alpha} |f(x)|^{p_0} dx,$$

$$\alpha^{p_1} \sum_k 2^k \|\{x : |U_k f_1(x)| > \alpha\}\| \lesssim \|U(f_1)\|_{L^{p_1}(\mathbb{R}^3)} \lesssim C_1^{p_1} \int_{|f(x)| \geq \alpha} |f(x)|^{p_1} dx.$$  

We insert these into the third line below:

$$\|U(f)\|_{L^p(\mathbb{R}^3)} = p \sum_k \int \alpha^{p-1} 2^k \|\{x : |U_k f(x)| > \alpha\}\| d\alpha$$

$$\leq p \sum_k \int \alpha^{p-1} 2^k \left( \|\{x : |U_k f_0(x)| > \alpha/2\}\| + \|\{x : |U_k f_1(x)| > \alpha/2\}\| \right) d\alpha$$

$$\lesssim \int C_0^{p_0} \alpha^{p-1} \int_{|f(x)| < \alpha} |f(x)|^{p_0} d\alpha d\alpha + \int C_1^{p_1} \alpha^{p-1} \int_{|f(x)| \geq \alpha} |f(x)|^{p_1} d\alpha d\alpha$$

$$\lesssim (C_0^{p_0} \delta^{p_0-p} + C_1^{p_1} \delta^{p_1-p}) \|f\|_{L^p} \lesssim C_0^{1-\theta} C_1^{\theta} \|f\|_{L^p}$$

where we choose $\delta$ satisfying $C_0^{p_0} \delta^{p_0-p} = C_1^{p_1} \delta^{p_1-p}$ ($p_1 < p < p_0$).

\[\Box\]

**Lemma 4.3 [Darboux Equation]** Let $f \in \mathcal{S}(\mathbb{R}^3)$. The circular average $u$ given by $u(x, x_3, t) = A_{S^1(A)}(f)(x, x_3, t)$ in \((4.3)\) with $A = E$ is the solution of the initial value problem of

$$(X_1^2 + X_2^2 - (tX_3)^2) u = \left( \frac{\partial}{\partial t} \right)^2 u$$

for $u(x, x_3, 0) = f(x, x_3)$ and $u_t(x, x_3, 0) = 0$.

**Proof of Lemma 4.3** For $X_1, X_2$ and $X_3$ in \((4.10)\), we have that

$$X_1^2 = \left( \partial/\partial x_1 \right)^2 - 2x_2(\partial/\partial x_1)(\partial/\partial x_3) + x_2^2(\partial/\partial x_3)^2$$

$$X_2^2 = \left( \partial/\partial x_2 \right)^2 + 2x_1(\partial/\partial x_2)(\partial/\partial x_3) + x_1^2(\partial/\partial x_3)^2$$

$$tX_3^2 = t^2(\partial/\partial x_3)^2.$$

Let $J_0(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{i r \sin \theta} d\theta$ be the Bessel function, and let $m(\rho_1, \rho_2) = \frac{1}{2\pi} \int_0^{2\pi} e^{i (\rho_1 \rho_2 - \cos \theta, \sin \theta)} d\theta$. Then, $m$ is a radial function $m(\rho) = J_0(|\rho|)$ for $\rho = (\rho_1, \rho_2)$ with the Bessel function $J_0$, satisfying the properties:

$$J_0'(r) + \frac{J_0(r)}{r} = -J_0(r).$$

By \((2.3)\) with $A = E$, we have that

$$A_{S^1(A)}(f)(x, x_3, t) = \int e^{2\pi i (\xi_1 x_1 + \xi_3 x_3)} m(2\pi t(\xi + \xi_3 A(x))) \tilde{f}(\xi, \xi_3) d\xi d\xi_3$$

$$= \int e^{2\pi i (\xi_1 x_1 + \xi_3 x_3)} \frac{1}{2\pi} \int_0^{2\pi} e^{2\pi i (\xi_3 A(x)) \cdot (\cos \theta, \sin \theta)} d\theta \tilde{f}(\xi, \xi_3) d\xi d\xi_3$$

$$= \int e^{2\pi i (\xi_1 x_1 + \xi_3 x_3)} J_0(2\pi t(\xi + \xi_3 A(x))) \tilde{f}(\xi, \xi_3) d\xi d\xi_3.$$

(10.2)
For $A = E$, we set the phase function $\Phi(x, x_3, \xi, \xi_3, \theta, t) = \xi - x + \xi_3 x_3 + t(\xi_1 - x_2 \xi_3, \xi_2 + x_1 \xi_3) - (\cos \theta, \sin \theta)$, and apply the vector fields $X_1^2, X_2^2$, and $(tX_3)^2$ in (10.1) to the phase function:

(10.3) \[ X_1^2 \Phi(x, x_3, \xi, \xi_3, \theta, t) = \left( (\xi_1 + 3\xi_3 t \sin \theta)^2 - 2x_2(\xi_1 + 3\xi_3 t \sin \theta)\xi_3 + x_2^2\xi_3^2 \right), \]

(10.4) \[ X_2^2 \Phi(x, x_3, \xi, \xi_3, \theta, t) = \left( (\xi_2 - 3\xi_3 t \cos \theta)^2 + 2x_1(\xi_2 - 3\xi_3 t \cos \theta)\xi_3 + x_1^2\xi_3^2 \right), \]

(10.5) \[ (tX_3)^2 \Phi(x, x_3, \xi, \xi_3, \theta, t) = (\xi_3)^2. \]

Then, $(X_1^2 + X_2^2 - (tX_3)^2) A_{S^1(\mathcal{A})}(f)(x, x_3, t)$ is given by

(10.6) \[ \frac{(2\pi)^2}{2\pi} \int_{\mathbb{R}^2} \int_{0}^{2\pi} \left( (X_1^2 + X_2^2 - (tX_3)^2) \Phi(x, x_3, \xi, \xi_3, \theta, t) \right) e^{2\pi i \Phi(x, x_3, \xi, \xi_3, \theta, t)} d\theta d\xi d\xi_3. \]

Next, we apply (10.3) to (10.5) to write the above amplitude $(X_1^2 + X_2^2 - (tX_3)^2) \Phi(x, x_3, \xi, \xi_3, \theta, t)$ as

$\xi_1^2 + \xi_2^2 + 2\xi_3(x_1 \xi_2 - x_2 \xi_1) + (x_1^2 + x_2^2)\xi_3^2 + 2\xi_3t((\xi_1 - 2x_3 \xi_3) \sin \theta - (\xi_2 + x_1 \xi_3) \cos \theta)

= |\xi + \xi_3 A(x)|^2 + 2\xi_3t \left( (\xi_1 - 2x_3 \xi_3) \sin \theta - (\xi_2 + x_1 \xi_3) \cos \theta \right).$

From (P1) of the above and the properties of $J_0$, we have that

\[ \int e^{2\pi i t(\xi_1 - 2x_3 \xi_3) \sin \theta, (\xi_2 + x_1 \xi_3) \cos \theta} d\theta = 0. \]

So, there remains $|\xi + \xi_3 A(x)|^2$ in the integral (10.6), namely $(X_1^2 + X_2^2 - (tX_3)^2) A_{S^1(\mathcal{A})}(f)(x, x_3, t)$ is

(10.7) \[ (2\pi)^2 \int_{\mathbb{R}^2} \frac{1}{2\pi} \int_{0}^{2\pi} |\xi + \xi_3 A(x)|^2 e^{2\pi i \Phi(x, x_3, \xi, \xi_3, \theta, t)} d\theta d\xi d\xi_3. \]

We express this in terms of the Bessel function of (10.2) as

\[ (X_1^2 + X_2^2 - (tX_3)^2) A_{S^1(\mathcal{A})}(f)(x, x_3, t) \]

\[ = - \int_{\mathbb{R}^2} e^{2\pi i(\xi \cdot x + \xi_3 x_3)} (2\pi)(\xi + \xi_3 A(x))^2 J_0(2\pi t|\xi + \xi_3 A(x)|) f(\xi, \xi_3) d\xi d\xi_3. \]

On the other hand, by applying (P2) above we compute \((\frac{\partial}{\partial t})^2 + \frac{4}{t^2} + \frac{4}{t} \frac{\partial}{\partial t}) J_0(2\pi t|\xi + \xi_3 A(x)|)\) as

\[ (2\pi t|\xi + \xi_3 A(x)|)^2 J_0''(2\pi t|\xi + \xi_3 A(x)|) + \frac{2\pi|\xi + \xi_3 A(x)|}{t} J_0'(2\pi t|\xi + \xi_3 A(x)|) \]

\[ = (2\pi t|\xi + \xi_3 A(x)|)^2 \left( J_0''(2\pi t|\xi + \xi_3 A(x)|) + \frac{J_0'(2\pi t|\xi + \xi_3 A(x)|)}{2\pi t|\xi + \xi_3 A(x)|} \right) \]

\[ = -(2\pi t|\xi + \xi_3 A(x)|)^2 J_0(2\pi t|\xi + \xi_3 A(x)|). \]
We insert this in the place of \( J_0(2\pi t\abs{\xi + \xi_3 A(x)}) \) in (10.2) and obtain that
\[
\left( \frac{\partial}{\partial t} \right)^2 + \frac{1}{t} \frac{\partial}{\partial t} \right) A_{S^1(A)}(f)(x, x_3, t)
= -\int e^{2\pi i (\xi \cdot x + \xi_3 x_3)} (2\pi \abs{\xi + \xi_3 A(x)})^2 J_0(2\pi t\abs{\xi + \xi_3 A(x)}) \hat{f}(\xi, \xi_3) d\xi d\xi_3.
\]
This coincides with (10.7). \( \square \)

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