Cosmological Coleman-Weinberg Potentials and Inflation

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ABSTRACT

We consider an additional fine-tuning problem which afflicts scalar-driven models of inflation. The problem is that successful reheating requires the inflaton be coupled to ordinary matter, and quantum fluctuations of this matter induces Coleman-Weinberg potentials which are not Planck-suppressed. Unlike the flat space case, these potentials depend upon a still-unknown non-local functional of the metric which agrees with the Hubble parameter for de Sitter. Such a potential cannot be completely subtracted off by any local action. We numerically consider the effect of subtracting it off at the beginning of inflation in a simple model. For fermions the effect is to prevent inflation from ending unless the Yukawa coupling to the inflaton is so small as to endanger reheating. For gauge bosons the effect is to make inflation end almost instantly, again unless the gauge charge is unacceptably small.

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1 Introduction

The most recent results for the scalar spectral index $n_s$, and the limits on the tensor-to-scalar ratio $r$ \[1\], are still consistent with certain models of single scalar-driven inflation,

$$\mathcal{L} = \frac{R\sqrt{-g}}{16\pi G} - \frac{1}{2} \partial_{\mu}\varphi \partial_{\nu}\varphi g^{\mu\nu} \sqrt{-g} - V(\varphi)\sqrt{-g}.$$ \hspace{1cm} (1)

However, the allowed models suffer from severe fine-tuning problems associated with getting inflation to start and to avoid the loss of predictivity through the formation of a multiverse \[2\]. This has led to much controversy within the inflation community. \[3, 4, 5\].

The purpose of this paper is to study a different sort of fine-tuning problem which is associated with the necessity of coupling the inflaton to normal matter to make reheating efficient. It has long been known that the quantum fluctuations of such matter particles will induce Coleman-Weinberg corrections to the inflaton effective potential \[6\]. These corrections are dangerous for inflation because they are not Planck-suppressed.

Until recently the assumption was that cosmological Coleman-Weinberg potentials are simply local functions of the inflaton which could be subtracted at will. However, existing results (from scalars \[7\], from fermions \[8, 9\], and from gauge bosons \[10, 11\]) on de Sitter background show that the corrections actually take the form of the fourth power of the Hubble constant times a complicated function of the dimensionless ratio of the inflaton to the Hubble constant. Simple arguments show that these factors of the de Sitter Hubble parameter cannot be constant for evolving cosmologies, and are not even local functionals of the metric \[12\]. Of course this means that they cannot be completely subtracted.

In this paper we explore a possible partial subtraction scheme. Because cosmological Coleman-Weinberg potentials are only known for de Sitter we shall make an assumption about their form for a general homogeneous and isotropic geometry. The *Instantaneous Hubble Approximation* is that the general potential follows by replacing the de Sitter Hubble constant by the evolving Hubble parameter. Our scheme is to subtract the same term with the Hubble parameter evaluated at the initial time, so that the cancellation is perfect at the initial time. The appropriate modified Friedmann equations are derived in section 2. In section 3 we study the effects potentials induced by fermions and by gauge bosons. Section 4 presents our conclusions.
2 The Modified Friedmann Equations

The purpose of this section is to work out how the Friedmann equations change when the scalar potential is allowed to depend on the Hubble parameter, $V(\varphi) \rightarrow V(\varphi, H)$. Our technique exploits the famous theorem [13, 14] that specializing to a class of geometries before varying the action gives correct equations, even though it can miss constraints. The restriction to homogeneity and isotropy give the $ij$ Einstein equation and the scalar evolution equation, from which we infer the 00 equation. We then reduce these three equations to a dimensionless form.

We know the scalar potential model Lagrangian (1) for arbitrary metric and scalar field configurations $g_{\mu\nu}(t, \vec{x})$ and $\varphi(t, \vec{x})$. This makes it simple to vary the action first and then specialize to homogeneity and isotropy,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x}, \quad \varphi = \varphi_0(t).$$

The two nontrivial Einstein equations are the 00 and $ij$ components,

$$3H^2 = 8\pi G \left[ \frac{1}{2} \dot{\varphi}_0^2 + V(\varphi_0) \right],$$

$$-2\dot{H} - 3H^2 = 8\pi G \left[ \frac{1}{2} \dot{\varphi}_0^2 - V(\varphi_0) \right].$$

The scalar equation is,

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 + \frac{\partial V}{\partial \varphi_0} = 0.$$

Note the close relation which exists between the three equations,

$$\frac{d}{dt} \left[ \text{Eqn(3)} \right] + 3H \left[ \text{Eqn(3)} + \text{Eqn(4)} \right] = 8\pi G \dot{\varphi}_0 \left[ \text{Eqn(5)} \right].$$

Even with the replacement $H_{dS} \rightarrow H(t)$ in our de Sitter results for Coleman-Weinberg potentials we still do not know how the Lagrangian depends upon a general field configuration. What we know is its specialization to homogeneity and isotropy (2) before variation,

$$L = \frac{1}{2} a^3 \dot{\varphi}_0^2 - a^3V(\varphi_0, H) - \frac{6a^2H^2}{16\pi G}.$$

This might be thought to be a debilitating problem but it is not. We simply appeal to the theorem of Palais [13, 14] that all the equations arising from
such a specialized Lagrangian are at least correct, even though there may be
additional equations. The Euler-Lagrange equation for $\varphi_0(t)$ is identical to (5). The Euler-Lagrange equation for $a(t)$ follows from the derivatives of (7) with respect to $a$ and $\dot{a}$,

$$\frac{\partial L}{\partial a} = \frac{6a^2}{16\pi G} \left\{ 8\pi G \left[ \frac{1}{2} \varphi_0^2 - V(\varphi_0, H) + \frac{1}{3} H \frac{\partial V(\varphi_0, H)}{\partial H} \right] - H^2 \right\}, \quad (8)$$

$$\frac{\partial L}{\partial \dot{a}} = -\frac{6a^2}{16\pi G} \left\{ 8\pi G \left[ \frac{1}{3} \frac{\partial V(\varphi_0, H)}{\partial H} \right] + 2H \right\}. \quad (9)$$

Hence we arrive at the appropriate generalization of equation (4),

$$-2\dot{H} - 3H^2 = 8\pi G \left[ \frac{1}{2} \varphi_0^2 - V + H \frac{\partial V}{\partial H} + \frac{1}{3} \varphi_0 \frac{\partial^2 V}{\partial \varphi_0 \partial H} + \frac{1}{3} H \frac{\partial^2 V}{\partial H^2} \right]. \quad (10)$$

The homogeneous and isotropic Lagrangian (7) does not give us the generalization of equation (3). However, we can guess it, guided by three principles:

- The generalization must reduce to (3) when the potential has no dependence on $H$;
- The generalization must not involve either $\ddot{\varphi}_0$ or $\ddot{a}$; and
- Substituting the generalization for (3), and equation (10) for (4), in relation (6) should give the scalar evolution equation.

The desired generalization of (3) is easily seen to be,

$$3H^2 = 8\pi G \left[ \frac{1}{2} \varphi_0^2 + V - H \frac{\partial V}{\partial H} \right]. \quad (11)$$

Relations (5), (10) and (11) define how the scalar and the geometry of inflation evolve, but they are inconvenient because the scale of temporal variation changes dramatically over the course of inflation, and because the dependent variables are dimensionful. A more physically meaningful evolution parameter is the number of e-foldings since the beginning of inflation,

$$n \equiv \ln \left[ \frac{a(t)}{a(t_i)} \right] \implies \frac{d}{dt} = H \frac{d}{dn}, \quad \frac{d^2}{dt^2} = H^2 \left[ \frac{d^2}{dn^2} - \epsilon \frac{d}{dn} \right]. \quad (12)$$
The natural dimensionless fields and potential are,
\[ \phi(n) \equiv \sqrt{8\pi G} \varphi_0(t) \text{, } \chi(n) \equiv \sqrt{8\pi G} H(t) \text{, } U(\phi, \chi) \equiv (8\pi G)^2 V(\varphi_0, H). \]  
(13)

With these changes, the modified Friedmann equations (11) and (10) take the form,
\[ 3\chi^2 = \frac{1}{2} \chi^2 \phi'^2 + U - \chi \frac{\partial U}{\partial \chi}, \]  
(14)
\[ -2\chi\chi' - 3\chi^2 = \frac{1}{2} \chi^2 \phi'^2 - U + \chi \frac{\partial U}{\partial \chi} + \frac{1}{3} \chi \phi' \frac{\partial^2 U}{\partial \phi \partial \chi} + \frac{1}{3} \chi \phi \frac{\partial^2 U}{\partial \chi^2}. \]  
(15)

And the scalar evolution equation becomes,
\[ \phi'' + (3-\epsilon)\phi' + \frac{1}{\chi^2} \frac{\partial U}{\partial \phi} = 0, \]  
(16)
where the first slow roll parameter is,
\[ \epsilon(n) \equiv -\frac{\chi'}{\chi} = \frac{1}{2} \phi'^2 + \frac{\phi' \frac{\partial^2 U}{\partial \phi \partial \chi}}{1 + \frac{\phi'' \frac{\partial U}{\partial \chi}}{8 \chi}}. \]  
(17)

Finally, note that the leading slow roll approximations for the scalar and tensor power spectra take the form,
\[ \Delta_R^2(n) \approx \frac{1}{8\pi^2} \times \frac{\chi^2(n)}{\epsilon(n)} \text{, } \Delta_T^2(n) \approx \frac{1}{8\pi^2} \times 16\chi^2(n). \]  
(18)

3 The Fate of the \( m^2\varphi^2 \) Model

It is useful to study what Coleman-Weinberg corrections do to the familiar \( V = \frac{1}{2} m^2 \varphi^2 \) model, even though that model is no longer consistent with the data. In the slow roll approximation the evolution of the dimensionless scalar and the first slow roll parameter are independent of the mass term,
\[ \text{Slow Roll } \implies \phi(n) \simeq \sqrt{\phi^2(0) - 4n} \text{, } \epsilon(n) \simeq \frac{2}{\phi^2(0) - 4n}. \]  
(19)

To make inflation last about 100 e-foldings (without the Coleman-Weinberg correction) we choose the initial conditions,
\[ \phi(0) = 20 \text{, } \phi'(0) = -\frac{1}{10}. \]  
(20)
We will continue using these conditions after the Coleman-Weinberg potential is added, with the initial value of $\chi$ chosen to obey equation (14). We parameterize the mass in terms of a constant $k$ which is chosen to make the amplitude of the scalar power spectrum agree with observation [1] (again, without the Coleman-Weinberg correction),

$$m^2 \equiv \frac{k^2}{8\pi G}, \quad \frac{(202k)^2}{96\pi^2} \simeq 2 \times 10^{-9}. \quad (21)$$

This defines the classical model which is being corrected. We first consider an inflaton which is Yukawa-coupled to a fermion, then we consider a charged inflaton which is coupled to a gauge boson. In each case the Coleman-Weinberg potential has disastrous consequences unless the coupling constant is so small as to preclude efficient reheating.

### 3.1 Inflaton Yukawa-Coupled to Fermions

If the Yukawa coupling constant is $\lambda$, and we subtract the quantum correction at $n = 0$, the dimensionless potential is,

$$U(\phi, \chi) = \frac{1}{2} k^2 \phi^2 - \frac{\chi^4}{8\pi^2} f\left(\frac{\lambda \phi}{\chi}\right) + \frac{\chi^4(0)}{8\pi^2} f\left(\frac{\lambda \phi}{\chi(0)}\right). \quad (22)$$

Here the scalar dependent part of the Coleman-Weinberg potential is [8, 9],

$$f(z) = 2\gamma z^2 - \left[\zeta(3) - \gamma\right] z^4 + 2 \int_0^z dx (x + x^3) \left[\psi(1 + ix) + \psi(1 - ix)\right], \quad (23)$$

where $\psi(x) \equiv \frac{d}{dx} \ln[\Gamma(x)]$ is the digamma function. The small value of $k^2 \sim 4 \times 10^{-11}$ needed to reproduce the scalar amplitude (21) means that the quantum corrections tend to overwhelm the classical term in (22), unless the Yukawa coupling is chosen to be very small. With order one values of $\lambda$ there is no evolution at all. This is because the middle term of (22) decreases relative the the final term as a function of $\chi$. Hence a putative decrease in $\chi$ would actually increase $U(\phi, \chi)$, which is inconsistent with equation (14), unless the classical term dominates the two quantum corrections.

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1. The tensor-to-scalar ratio of $r \approx 0.16$ does not agree with observation [1], which is why this model is disfavored. However, it is very simple, and well known, and the robustness of our results does not justify employing a more viable model.
Figure 1: Plots of the dimensionless scalar $\phi(n)$ (on the left), the dimensionless Hubble parameter $\chi(n)$ (middle) and the first slow roll parameter $\epsilon(n)$ (on the right) for the quantum-corrected model \((22)\) with Yukawa coupling $\lambda = 5 \times 10^{-4}$. Even with this minuscule value of $\lambda$ the geometry approaches de Sitter at a reduced scale.

We did not start to see evolution until values of about $\lambda \sim 10^{-3}$. Figure 1 shows the result for $\lambda = 5 \times 10^{-4}$. Although the model evolves noticeably for the first 100 e-foldings, there are considerable deviations from the classical result. These deviations become extreme at late times, for which the figure shows that the quantum-corrected model approaches de Sitter expansion at a reduced Hubble parameter.

Figure 2: Results from the classical model $U = \frac{1}{2}k^2\phi^2$ (in blue) versus the quantum corrected model \((22)\) (in red) assuming the inflaton is Yukawa-coupled to a fermion. We show the dimensionless scalar $\phi(n)$ (left hand graph), the dimensionless Hubble parameter $\chi(n)$ (middle graph), and the first slow roll parameter $\epsilon(n)$ (right hand graph). The value of the Yukawa coupling was chosen to be $\lambda = 1.15 \times 10^{-4}$.

Figure 2 compares the quantum-corrected model (in red) with the classical results (in blue) for the even smaller Yukawa coupling of $\lambda = 1.15 \times 10^{-4}$. Although the two models seem to track for about 100 e-foldings, inflation ends in the classical model whereas the quantum corrected model again ap-
proaches de Sitter. There is not much point in considering even smaller Yukawa couplings because they would make reheating inefficient.

### 3.2 Charged Inflaton Coupled to Gauge Bosons

The quantum-corrected dimensionless potential for a charged inflaton (with charge $q$) is,

$$U(\phi, \chi) = k^2 \phi^* \phi + \frac{3\chi^4}{8\pi^2} f\left(\frac{q^2 \phi^* \phi}{\chi^2}\right) - \frac{3\chi^4(0)}{8\pi^2} f\left(\frac{q^2 \phi^* \phi}{\chi^2(0)}\right).$$  \hspace{1cm} (24)

The function $f(z)$ appropriate for a gauge boson is \cite{10, 11},

$$f(z) = -(1-2\gamma)z - \left(\frac{3}{2} - \gamma\right)z^2$$

$$+ \int_0^z dx \left(1+x\right) \left[\psi\left(\frac{3}{2} + \frac{1}{2} \sqrt{1-8x}\right) + \psi\left(\frac{3}{2} - \frac{1}{2} \sqrt{1-8x}\right)\right].$$  \hspace{1cm} (25)

Of course a bosonic quantum correction adds to the vacuum energy, which makes the result opposite to that for fermions. For order one values of the inflaton charge $q$ the two quantum corrections totally dominate the classical term and inflation ends almost instantly. Making inflation last for 60 e-foldings requires the minuscule value of $q^2 = 5.5 \times 10^{-6} e^2$, the effects of which are shown in Figure 3. Even with this charge there are noticeable deviations from the classical model, in particular, a much more sudden end to inflation.

Figure 3: Results from the classical model $U = \frac{1}{2}k^2\phi^2$ (in blue) versus the quantum corrected model (24) (in red) assuming a charged inflaton (with charge $q^2 = 5.5 \times 10^{-6} e^2$) is minimally coupled to vector bosons. The left hand graph shows the scalar $\phi(n)$, the middle graph gives the dimensionless Hubble parameter $\chi(n)$, and the right hand graph depicts the first slow roll parameter $\epsilon(n)$.  

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4 Discussion

Scalar-driven inflation suffers from many fine-tuning problems. These are exacerbated by the need to couple the inflaton to normal matter in order to make reheating efficient. Quantum fluctuations of normal matter quanta induce cosmological Coleman-Weinberg potentials which are not Planck-suppressed and, for de Sitter, depend in complicated ways on the dimensionless ratio of the square of the coupling constant times the inflaton over the Hubble parameter. Although exact results do not exist for more general backgrounds, it is possible to show that the factors of \( H^2 \) are not generally constant, nor even local functionals of the metric. The absence of locality restricts the extent to which these corrections can be subtracted off. The purpose of this paper was to study the consequences to inflation under two assumptions:

1. The de Sitter Hubble constant is replaced by the evolving Hubble parameter \( H(t) \) in the cosmological Coleman-Weinberg potentials; and

2. The potentials are completed subtracted at the beginning of inflation with the de Sitter Hubble constant replaced by the initial value of the Hubble parameter.

In section 2 we derived the appropriate generalizations to the Friedmann equations, and we cast the formalism in terms of dimensionless variables evolved with respect to the number of e-foldings from inflation. In section 3 we numerically evolved the \( m^2 \phi^2 \) model, assuming first that the inflaton is Yukawa-coupled to a fermion and then that a charged inflaton is minimally coupled to a gauge boson. The results were uniformly catastrophic. For the case of fermions inflation never ends unless the Yukawa coupling is chosen so small as to preclude efficient reheating. For bosons the quantum-corrected effective potential causes inflation to end almost instantly, again unless the charge is chosen to be minuscule.

These results are completely unacceptable for scalar-driven inflation. It is worth investigating subtractions based on replacing the factors of \( H^2 \) by \( \frac{1}{12}R \). That replacement would be perfect for the de Sitter approximation to the Coleman-Weinberg potential, but there is still a difference between any local subtraction and the nonlocal Coleman-Weinberg potential it attempts to cancel. To study this difference we would need a more refined analysis of the nonlocal Coleman-Weinberg potential. In particular, what is a generally
applicable approximation for the de Sitter factors of “$H^2$”? Attempting to answer this question seems worthwhile in view of the crippling potential problem to the viability the current study has exposed.

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