CONTINUOUS FAMILY OF EQUILIBRIA OF THE 3D AXISYMMETRIC RELATIVISTIC VLASOV-MAXWELL SYSTEM

KATHERINE ZHIYUAN ZHANG

ABSTRACT. We consider the relativistic Vlasov-Maxwell system (RVM) on a general axisymmetric spatial domain with perfect conducting boundary which reflects particles specularly, assuming axisymmetry in the problem. We construct continuous global parametric solution sets for the time-independent RVM. The solutions in these sets have arbitrarily large electromagnetic field and the particle density functions have the form \( f^\pm = \mu^\pm(e^\pm(x,v), p^\pm(x,v)) \), where \( e^\pm \) and \( p^\pm \) are the particle energy and angular momentum, respectively. In particular, for a certain class of examples, we show that the spectral stability changes as the parameter varies from 0 to \( \infty \).

1. Introduction

When the temperature is high or the density is low, the effect of collisions in a plasma becomes minor compared to the effect of the electromagnetic forces. Such a plasma is modeled by the relativistic Vlasov-Maxwell system (RVM). In many real world applications, the plasma is confined to a bounded region. A typical example is the tokamak, which is one of the main foci of research in fusion energy. A key direction of study is to construct the stationary solutions (equilibria) of the RVM system on a bounded domain with varied boundary conditions.

We consider the 3D relativistic Vlasov-Maxwell (RVM) system on an axisymmetric bounded domain \( \Omega \subset \mathbb{R}^3 \). We study a plasma with two species (ion and electron) with non-negative distribution functions \( f^\pm(t,x,v) \), where \( t \geq 0, x \in \Omega \subset \mathbb{R}^3, v \in \mathbb{R}^3 \). The system RVM is

\[
\begin{align*}
\partial_t f^\pm + \hat{v} \cdot \nabla_x f^\pm &\pm (E + \hat{v} \times B) \cdot \nabla_v f^\pm = 0, \\
\nabla_x \cdot E &\equiv \rho = \int_{\mathbb{R}^3} (f^+ - f^-)dv, \quad \nabla_x \cdot B = 0, \\
\partial_t E - \nabla_x \times B &= -j = -\int_{\mathbb{R}^3} \hat{v}(f^+ - f^-)dv, \quad \partial_t B + \nabla_x \times E = 0.
\end{align*}
\]

In this system, \( f^\pm(t,x,v) \geq 0 \) is the density distribution of ions (+) and electrons (−). We confine the plasma in a region \( \Omega \subset \mathbb{R}^3 \), so that \( x \in \Omega \) is the particle position. \( v \in \mathbb{R}^3 \) is the particle momentum, \( \langle v \rangle = \sqrt{1 + v^2} \) is the particle kinetic energy, and \( \hat{v} = v/\langle v \rangle \) is the particle velocity. Also, \( E \) is the electric field, \( B \) is the internal
magnetic field. Therefore ±(E +  \hat{v} \times B) is the total (internal) electromagnetic force. Moreover, the charge density \( \rho \) and the current density \( j \) are defined as

\[
\rho = \int_{\mathbb{R}^3} (f^+ - f^-) dv, \quad j = \int_{\mathbb{R}^3} \hat{v}(f^+ - f^-) dv.
\]

At the boundary we impose the specular condition (which means that \( f^\pm \) is even with respect to \( v_n = v \cdot e_n \) on \( \partial \Omega \), with \( e_n \) being the outward normal vector of \( \partial \Omega \) at \( x \)):

\[
f^\pm(t, x, v) = f^\pm(t, x, v - 2(v \cdot e_n(x))e_n(x)), e_n(x) \cdot v < 0, \quad \forall x \in \partial \Omega,
\]

as well as the perfect conductor boundary condition

\[
E(t, x) \times n(x) = 0, \quad B(t, x) \cdot n(x) = 0, \quad \forall x \in \partial \Omega.
\]

We assume axisymmetry in the problem. Therefore it is natural to use cylindrical coordinates here.

We aim to construct equilibrium solutions for the system (1.1) – (1.3) with the boundary conditions above. We consider equilibria with the form

\[
f^\pm = \mu^\pm(e^\pm(x, v), p^\pm(x, v))
\]

where \( e^\pm \) and \( p^\pm \) are the particle energy and angular momentum, respectively, and use a potential formulation together with the axisymmetry assumption of (1.1) – (1.3) so as to the equations satisfied by the equilibria can be reformulated as a coupled elliptic system (see (2.12) – (2.14) below). The resulting elliptic system is highly nonlinear on its right hand side.

Investigation for time-independent RVM on a domain with physical boundaries has been carried out over the years. Rein [20] studied weak stationary solutions of the RVM by variational methods. Batt & Fabian [2], Braasch [3], Poupaud [18], Sinitsyn & Dulov [24] constructed stationary solutions for boundary value problems making use of sub-solutions and super-solutions. In particular, Batt & Fabian [2] constructed a general family of large stationary solutions for boundary value problems when the domain is axially symmetric and does not touch the \( z \)-axis and Braasch [3] constructed large stationary solutions for problems with a specific type of axial symmetric domain. Poupaud [18] solves for large equilibria for general domains (not necessarily axially symmetric) with compact and connected boundary as well as general Dirichlet boundary conditions. Moreover, the lower dimensional models have also been considered. For example, Schaeffer [21] studied the stationary Vlasov-Maxwell equation (VM) in the 1.5D setting, when the spatial domain being the real line \( \mathbb{R} \), see also Guo [7].

Our strategy is to consider general parametrized family of particle density distribution function for the equilibria, and to apply the global implicit function theorem (Theorem 2.4) so as to obtain global solution sets that branch out from ”trivial” equilibrium solutions. The result is stated in Theorem 3.2. Similar methods have been applied in varied PDEs, see, for example, [4], [5], [6], [25]. For the Vlasov-Maxwell
equation (VM), Sidorov & Sinitsyn \[23\] considered a specific parametrized family of (single species) particle density distribution, and gave sufficient conditions for the existence of bifurcation points corresponding to distribution functions of a certain form.

In this paper, we consider a general parametrized family of particle density distribution \( f^\pm = \mu^\pm(e^\pm(x, v), p^\pm(x, v)) \), and construct a global continuum/loop of equilibria branching out from the trivial or neutral steady states for the RVM on a general axisymmetric spatial domain with perfect conducting boundary which reflects particles specularly, assuming axisymmetry in the problem. Moreover, in the single species case, with some additional constraints, the global continuum/loop of equilibria we constructed contains a locally analytic curve that is unbounded, see Theorem 3.3. Our approach enables us to observe the structure of the solution set, and reveal more information about the change of the electromagnetic potential (and therefore the electromagnetic field) as the density distribution function changes in different ways. In the case when the solution set is unbounded, we make use of the maximum principles and elliptic estimates to investigate size properties of the equilibria constructed in Theorem 3.2. In particular, we construct a continuum of equilibrium solutions with arbitrarily large electromagnetic field, see Proposition 4.1. It is an open question whether the solutions obtained in this paper are the same as the ones constructed in the historical literature \[20\], \[2\], \[3\], \[18\], \[24\], etc.

Our approach can also be applied in the case when an external magnetic field is present, see Remark 2.2. This type of problem is also considered in Weber \[22\], in which confined stationary solutions of the RVM in a long cylinder in a two and one-half dimensional setting are constructed. Moreover, Lin & Strauss \[11\] constructed confined small stationary solutions of the RVM in 3D (see the appendix of \[11\]). Such type of solutions are related to the mechanism of magnetic confinement, see, for example, \[14\], \[27\].

On the other hand, in plasma theory, an important goal is to study the stability properties of plasmas. In \[17\], the renowned Penrose’s sharp criterion on linear stability for a spatially homogeneous equilibrium of the Vlasov-Poisson system is derived. In \[7\], \[8\], \[11\], \[13\] and \[12\], the analysis of a spatially inhomogeneous equilibrium was carried out in domains without any spatial boundaries (i.e. whole space or periodic setting). A sharp criterion for spectral stability was given in \[12\], with some families of stable and unstable examples provided. The question of nonlinear stability is much more difficult, see, for example, \[13\]. An important topic is to understand the stability properties of a confined plasma. In \[9\], the confinement of a tokamak plasma is discussed using some fluid models and the role of different parts of the boundary are explored. For the microscopic model RVM, there are very few rigorous studies in bounded domains. In \[16\] and \[26\], in which the authors considered the case when the spatial domain is a torus or an axisymmetric bounded region (for example, a
tokamak), and toroidal or axisymmetric symmetry is assumed. A sharp criterion of spectral stability is obtained, thus reducing the problem of determining the linear stability to the positivity of a simpler self-adjoint operator $L^0$.

In this paper, we investigate the change of spectral stability property along a family of solutions constructed in Theorem 3.2 using the result obtained in [26]. Specifically, we track down the solutions along the solution set as the parameter $K$ changes from 0 to $\infty$, and show that the corresponding equilibrium solutions gradually become spectrally unstable as $K$ grows large. The details are given in Section 5, Proposition 5.2 and Theorem 5.4.

In addition, we provide explicit examples on which the results above can be applied. These examples cover cases when the density distribution function is parametrized as

$$\mu^{K,\pm}(e, p) = \mu^{0}(e, p) + K\mu^{\pm}(e, p)$$

or

$$\mu^{K,\pm}(e, p) = \mu^{0}(e, p) + \mu^{\pm}(e, Kp)$$

(where $K \in [0, \infty)$ is the parameter, $\mu^0$ and $\mu^\pm$ are fixed non-negative functions satisfying certain conditions) and generate unbounded solution sets of single-species equilibria. Solutions in these sets have arbitrarily large electromagnetic field (as $K$ gets large). In the example in Subsection 6.2, the equilibria turn from spectrally stable to unstable as the parameter $K$ changes from 0 to $\infty$.

The contents in the paper are arranged as follows. In Section 2, we set up the problem in cylindrical coordinates, including the coordinates and the symmetry assumptions. The description of the particle trajectories and the family of equilibria we consider in this paper are given, and relevant function spaces are introduced. The difficulty caused by the singularity at $r = 0$ is avoided by using the spaces $H^{k+}$ and $X$ (see (2.28) and (2.30)). We also state the local and global implicit function theorems, which are the key tools in this paper that provide the existence of the steady states. In Section 3 – 6, we consider the time-independent RVM with no external magnetic field. In Section 3, we consider a family of parametrized particle density distribution functions and construct an unbounded solution set/loop for the time-independent problem (2.12)–(2.14) with $A^0 = A^0_\varphi e_\varphi$, which gives a steady solution set for RVM. Furthermore, in the single species case, with additional constraints, the global continuum/loop of equilibria we constructed contains a locally analytic curve that is unbounded, yielding a global steady solution set which contains an unbounded analytic steady solution curve for RVM. Section 4 makes use of the maximum principles for elliptic equations and gives some properties of the equilibria constructed in Section 3. Theorem 3.2. It is then shown in Section 5 that by properly choosing ways of parametrization, one can create solution sets along which the linear stability changes. After that, in Section 6, we give two examples of parametrized families of particle density distribution functions and
apply the theory in Sections 3, 4, 5. Lastly, for the readers’ convenience, we provide the formulae for derivative operators in the cylindrical coordinates as well as the particle energy $e^{\pm}(x, v) = \langle v \rangle \pm \phi^0(r, z)$ and the particle angular momentum $p^{\pm}(x, v) = r(v_\varphi \pm A^0_\varphi(r, z))$ (or $p^{\pm}(x, v) = r(v_\varphi \pm A^0_\varphi(r, z) \pm A_{\varphi, ext}(r, z)$) when an external magnetic potential $A_{ext} = A_{\varphi, ext}(r, z)e_{\varphi}$ is present) in Appendix A and B, respectively.

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2. Set up

Let us introduce the basic settings of the stationary RVM problem. We consider the 3D relativistic Vlasov-Maxwell (RVM) system \[ (1.1) - (1.3) \] on the axisymmetric bounded domain $\Omega \subset \mathbb{R}^3$. We assume axisymmetry in the problem and use cylindrical coordinates $(r, \varphi, z)$. We consider a plasma constrained inside a closed region $\Omega$, which is a $C^1$ axisymmetric (with respect to the $z$-axis) domain in $\mathbb{R}^3$, i.e. rotational invariant around the $z$-axis. $\Omega$ can be viewed as a solid of revolution, determined as follows: Consider a counterclockwisely parametrized $C^1$ curve $C$ in the plane $\{ \varphi = 0 \}$, where $\beta$ is the arclength parameter:

\begin{align}
(2.1) \quad r &= \tilde{r}(\beta), \quad z = \tilde{z}(\beta), \quad \beta \in [0, 1], \quad \tilde{r} \geq 0 \text{ for all } \beta \in (0, 1), \quad \tilde{r}, \tilde{z} \in C^1(0, 1),
\end{align}

with either

\begin{align}
(2.2) \quad \tilde{r}(0) &= \tilde{r}(1) > 0, \quad \tilde{z}(0) = \tilde{z}(1), \quad \tilde{z}'(0) = \tilde{z}'(1),
\end{align}

or,

\begin{align}
(2.3) \quad \tilde{r}(0) &= \tilde{r}(1) = 0, \quad \tilde{z}'/\tilde{r}' = 0 \text{ at } \beta = 0, 1.
\end{align}

Rotating $C$ around the $z$-axis gives the boundary for the domain $\partial \Omega$. In all these cases, the boundary $\partial \Omega$ is $C^1$ smooth. We denote

\[ d := \sup_{x \in \Omega} r(x). \]

Let $e_r$, $e_\varphi$ and $e_z$ be the unit vectors in the cylindrical coordinate system (see Appendix A), and $e_n$, $e_{tg}$ to be the unit vectors in outward normal direction and tangential direction orthogonal to $e_\varphi$ on $\partial \Omega$, respectively. Then on $\partial \Omega$, the outward unit normal vector $e_n(x) = (\tilde{z}'e_r - \tilde{r}'e_z)/\sqrt{\tilde{z}'^2 + \tilde{r}'^2}$, and $e_{tg}(x) = (-\tilde{r}'e_r - \tilde{z}'e_z)/\sqrt{\tilde{z}'^2 + \tilde{r}'^2}$.

On the boundary, we assume the specular condition on the density function $f^{\pm}$ in case the particle hits the boundary. This means that $f^{\pm}$ is even with respect to $v_n = v \cdot e_n/|e_n|$ on $\partial \Omega$, i.e.

\begin{align}
(2.4) \quad f^{\pm}(t, x, v) = f^{\pm}(t, x, v - 2(v \cdot e_n(x))e_n(x)), \quad e_n(x) \cdot v < 0, \forall x \in \partial \Omega, \forall v \in \mathbb{R}^3.
\end{align}
We also put down the perfect conductor boundary condition on the electric and magnetic fields:

\( E(t, x) \times e_n(x) = 0, \ \mathbf{B}(t, x) \cdot e_n(x) = 0, \ \forall x \in \partial \Omega. \)

We also introduce the electric potential \( \phi \) and the magnetic potential \( \mathbf{A} \):

\[
E = -\nabla \phi - \partial_t \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A},
\]

and impose the Coulomb gauge

\[
\nabla \cdot \mathbf{A} = 0.
\]

**Remark 2.1.** The choices of \( \phi \) and \( \mathbf{A} \) are not unique. Actually the choice of \( \mathbf{A} \) can differ by the gradient of a harmonic function.

We use the cylindrical coordinates \((r, \varphi, z)\), and make the axisymmetry assumption:

\[
\phi, \ A_r, \ A_z, \ A_\varphi \text{ are independent of } \varphi.
\]

Therefore, \( f^\pm \) does not depend *explicitly* on \( \varphi \), although it might depend on it *implicitly* through the components of \( v \). The Maxwell system then becomes

\[
- \Delta \phi = \rho,
\]

\[
(\partial_t^2 - \Delta + \frac{1}{r^2})A_\varphi = j_\varphi,
\]

\[
(\partial_t^2 - \Delta)\tilde{A} + \partial_t \nabla \phi = \tilde{j}.
\]

Here \( \tilde{A} := A_r e_r + A_z e_z \).

We consider the equilibrium field, with the potential denoted by \( (\phi^0, A^0_\varphi, \tilde{A}^0) \), and for which the Maxwell equations become

\[
- \Delta \phi^0 = \rho^0,
\]

\[
(\Delta + \frac{1}{r^2})A^0_\varphi = j^0_\varphi,
\]

\[
(\Delta)\tilde{A}^0 = \tilde{j}^0.
\]

We consider an equilibrium that satisfies

\[
\mathbf{A}^0 = A^0_\varphi e_\varphi,
\]

which implies

\[
B^0_\varphi = 0.
\]

Under this constraint, \((2.12)-(2.14)\) become a system with reduced dimension:

\[
- \Delta \phi^0 = \rho^0,
\]
According to (2.3), we assume the following boundary conditions for the potentials, Section 4 in [26] for details:

(2.17) 
\[ \phi^0 = 0, \quad A^0_\varphi = 0, \quad A^0_{tg} = 0, \quad \tilde{z}' \partial_r A^0_n - \tilde{r}' \partial_z A^0_n + \frac{\tilde{z}'}{r} A^0_n = 0, \quad \forall x \in \partial \Omega. \]

Then the equilibrium field is

(2.18) 
\[ \mathbf{E}^0 = -\nabla \phi^0 = -\frac{\partial \phi^0}{\partial r} e_r - \frac{\partial \phi^0}{\partial z} e_z, \]

(2.19) 
\[ \mathbf{B}^0 = -\frac{\partial A^0_x}{\partial z} e_r + \frac{1}{r} \frac{\partial (r A^0_\varphi)}{\partial r} e_z. \]

The boundary condition (2.17) then becomes

(2.20) 
\[ \phi^0 = 0, \quad A^0_\varphi = 0, \quad \forall x \in \partial \Omega. \]

We define the particle trajectories as

(2.21) 
\[ X^\pm = \dot{V}^\pm, \quad \dot{V}^\pm = \pm \mathbf{E}^0(X^\pm) + \dot{V}^\pm \times \mathbf{B}^0(X^\pm) \]

with initial values \((X^\pm(0; x, v), V^\pm(0; x, v)) = (x, v)\). Each particle trajectory exists and preserves the axisymmetry up to the first time it meets the boundary. Let \(s_0\) be a time when the trajectory \(X^\pm(s_0^-; x, v)\) hits the boundary \(\partial \Omega\). Recall that \(v_n = v \cdot e_n = \frac{z' v_r - r' v_z}{\sqrt{z'^2 + r'^2}}, \quad v_{tg} = v \cdot e_{tg} = \frac{\tilde{z}' v_r - \tilde{r}' v_z}{\sqrt{\tilde{z}'^2 + \tilde{r}'^2}}\). For any given \((x, v)\) and \((X^\pm, V^\pm)\) with \(x\) and \(X^\pm\) on \(\partial \Omega\), we re-compose \(v\) and \(V^\pm\) into their \(n\)-component, \(tg\)-component and \(\varphi\)-component: \(v = v_n e_n + v_{tg} e_{tg} + v_\varphi e_\varphi\), \(V^\pm = V^\pm_n e_n + V^\pm_{tg} e_{tg} + V^\pm_\varphi e_\varphi\), and define

(2.22) 
\[ v_s = -v_n e_n + v_{tg} e_{tg} + v_\varphi e_\varphi, \quad V^\pm_s = -V^\pm_n e_n + V^\pm_{tg} e_{tg} + V^\pm_\varphi e_\varphi. \]

Thus from the specular boundary condition, the trajectory can be continued by the rule

(2.23) 
\((X^\pm(s_0^+; x, v), V^\pm(s_0^+; x, v)) = (X^\pm(s_0^-; x, v), V^\pm_s(s_0^-; x, v))\).

Furthermore, we assume the equilibrium has a particle density of the form \(f^{0, \pm}(x, v) = \mu^\pm(e^\pm(x, v), p^\pm(x, v))\), where

(2.24) 
\[ e^\pm(x, v) = \langle v \rangle \pm \phi^0(r, z), \quad p^\pm(x, v) = r(v_\varphi \pm A^0_\varphi(r, z)). \]

Here \(e^\pm\) and \(p^\pm\) are invariant along the particle trajectories under our assumptions (with specular boundary condition). (The proof of the invariance of \(e^\pm\) and \(p^\pm\) can be found in Appendix B.) We assume that \(\mu^\pm(e, p)\) are non-negative \(C^1\) functions which satisfy

(2.25) 
\[ |\mu^\pm(e, p)| + |\mu^\pm_p(e, p)| + |\mu^\pm_e(e, p)| \leq \frac{C\mu}{1 + |e|^\delta}, \quad \delta > 3. \]
Remark 2.2. Also our results can be applied to the RVM with an external magnetic field being applied. Consider an external magnetic potential
\[ A_{\text{ext}} = A_{\varphi,\text{ext}} e_{\varphi}, \]
where \( A_{\varphi,\text{ext}} \) is independent of \( \varphi \), only depends on \( r, z \).

Hence the corresponding external magnetic field satisfies \( B_{\varphi,\text{ext}} = 0 \). Again, we assume the equilibrium has a particle density of the form
\[ f^{0,\pm}(x, v) = \mu^{\pm}(e^{\pm}(x, v), p^{\pm}(x, v)), \]
where \( e^{\pm}(x, v) = \langle v \rangle^\pm + \phi^0(r, z) \) and \( p^{\pm}(x, v) = r \langle v \rangle^\pm + A_{\varphi}^0(r, z) \pm A_{\varphi,\text{ext}}(r, z) \).

(2.26) \( e^{\pm}(x, v) = \langle v \rangle^\pm + \phi^0(r, z) \), \( p^{\pm}(x, v) = r \langle v \rangle^\pm + A_{\varphi}^0(r, z) \pm A_{\varphi,\text{ext}}(r, z) \).

\( e^{\pm} \) and \( p^{\pm} \) are invariant along the particle trajectories
(2.27) \( \dot{X}^{\pm} = \dot{\varphi}^{\pm}, \dot{V}^{\pm} = \pm E_0^{\pm}(X^{\pm}) \pm \dot{\varphi}^{\pm} \times B_0^{\pm}(X^{\pm}) \pm \dot{\varphi}^{\pm} \times B_{\text{ext}}^{\pm}(X^{\pm}) \).

with all the other assumptions as mentioned in the previous case without an external magnetic field, see Appendix B.

We are going to solve the elliptic system (2.12) – (2.14) with the boundary conditions above as well as \( A_0^\varphi \equiv A_{\varphi,\text{ext}}^\varphi \). We use the letter \( \tau \) to denote the axisymmetric constraint for functions. Denote \( Y = L_2^{1,\tau}(\Omega) \), i.e. the weighted-\( L^2 \) space with weight \( 1/r^2 \). This weight gives some singularity at \( r = 0 \) if \( \Omega \) touches the \( z \)-axis. For this, we define
(2.28) \( H^{k\tau}(\Omega) := \{ g \in L^2(\Omega)| e^{i\varphi} g \in H^k(\Omega) \} \) \((k = 1, 2)\).

Then \( H^{2\tau}(\Omega) \subset Y \) because the identity
(2.29) \( -\Delta(e^{i\varphi}) = (-\Delta + \frac{1}{r^2})ge^{i\varphi} \)
holds for any \( \varphi \)-independent function \( g \). Indeed,
\[
-\Delta(e^{i\varphi}) = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} e^{i\varphi} \right) - \frac{\partial^2 g}{\partial z^2} e^{i\varphi} - \frac{1}{r^2} \frac{\partial^2 (e^{i\varphi})}{\partial \varphi^2} g
= -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial g}{\partial r} \right) e^{i\varphi} - \frac{\partial^2 g}{\partial z^2} e^{i\varphi} + \frac{1}{r^2} ge^{i\varphi}
= (-\Delta + \frac{1}{r^2})ge^{i\varphi}.
\]

We will use the following space regularly:
(2.30) \( \mathcal{X} := \{ g \in H^2(\Omega) : g \text{ independent of } \varphi, \exp(i\varphi)g \in H^2(\Omega), g|_{\partial\Omega} = 0 \} \subset H^{2\tau}(\Omega) \).

In the rest of the paper, \( A \lesssim B \ (A \gtrsim B) \) means \( A \leq CB \ (A \geq CB) \) for some constant \( C > 0 \) independent of \( K \).

Before we continue, let us state some local and global implicit function theorems, which are key ingredients in this paper. Readers can refer to [10], [3], [6], etc. for detailed discussions of these results.
Theorem 2.3. (Local Implicit Function Theorem) Let $X, Y, Z$ be (real) Banach spaces. Consider a mapping $F : U \times V \rightarrow Z$ with open sets $U \subset X$, $V \subset Y$, and the equation

$$F(x, y) = 0.$$  \hspace{1cm} (2.31)

Suppose the equation \((2.31)\) admits a solution $(x_0, y_0) \in U \times V$ such that the Fréchet derivative of $F$ with respect to $x$ at $(x_0, y_0)$ is bijective, i.e.

$$D_x F(x_0, y_0) : X \rightarrow Z$$

is bounded (continuous) with a bounded inverse.

Assume also that $F$ and $D_x F$ are continuous:

$$F \in C(U \times V, Z),$$  \hspace{1cm} (2.34)

$$D_x F \in C(U \times V, L(X, Z)).$$  \hspace{1cm} (2.35)

Here $L(X, Z)$ denotes the Banach space of bounded linear operators from $X$ to $Z$ endowed with the operator norm. Then there exists a neighbourhood $U_1 \times V_1$ in $U \times V$ of $(x_0, y_0)$ and a mapping $f : V_1 \rightarrow U_1 \subset X$, such that

$$f(y_0) = x_0,$$  \hspace{1cm} (2.36)

$$F(f(y), y) = 0$$

for all $y \in V_1$.

Furthermore, $f$ is continuous on $V_1$:

$$f \in C(V_1, X).$$  \hspace{1cm} (2.38)

Finally, every solution of \((2.31)\) in $U_1 \times V_1$ is of the form $(f(y), y)$.

Theorem 2.4. (Global Implicit Function Theorem) Let $X$ and $Z$ be (real) Banach spaces and $U$ be an open set in $X \times \mathbb{R}$. Consider a mapping $F : U \rightarrow Z$ that is $C^1$ in the Fréchet sense, and the equation

$$F(x, y) = 0$$  \hspace{1cm} (2.39)

with $x \in X$, $y \in \mathbb{R}$. Suppose the equation \((2.39)\) admits a solution $(x_0, y_0) \in U$ such that the Fréchet derivative of $F$ with respect to $x$ at $(x_0, y_0)$ is bijective, i.e.

$$D_x F(x_0, y_0) : X \rightarrow Z$$

is bounded (continuous) with a bounded inverse.

Assume also that the mapping $(x, y) \rightarrow F(x, y) - x$ is compact from $U$ to $X$. Let $S$ be the closure in $X \times \mathbb{R}$ of the solutions set $\{(x, y) : F(x, y) = 0\}$. Let $C$ be the connected component of $S$ to which $(x_0, y_0)$ belongs. Then one of the following three alternatives holds:

i) $C$ is unbounded in $X \times \mathbb{R}$;
ii) $C \setminus \{(x_0, y_0)\}$ is connected;
iii) $C \cap \partial U$ is not empty.

**Theorem 2.5. (Analytic Global Implicit Function Theorem)** Let $X$ and $Z$ be (real) Banach spaces, $I$ be an open interval (possibly unbounded) with $0 \in I$, and $U$ be an open set in $X$ with $0 \in \partial U$. Consider a mapping $F : U \times I \to Z$ that is real-analytic in the Fréchet sense, and the equation

$$F(x, y) = 0$$

with $x \in X$, $y \in I$. Assume that for any $(x, y) \in U \times I$ with $F(x, y) = 0$, the Fréchet derivative $D_x F(x, y) : X \to Z$ is Fredholm with index 0.

Suppose that there exists a continuous curve $C_{\text{loc}}$ of solution to (2.42), parametrized as

$$C_{\text{loc}} := \{(\tilde{x}(y), y) : 0 < y < y_*\} \subset F^{-1}(0)$$

for some $y_* > 0$ and continuous $\tilde{x} : (0, y_*) \to U$. If

$$\lim_{y \to 0^+} \tilde{x}(y) = 0 \in \partial U, \quad D_x F(\tilde{x}(y), y) : X \to Z$$

is invertible for all $y$,

then $C_{\text{loc}}$ is contained in a curve of solutions $C$, parametrized as

$$C := \{(x(s), y(s)) : 0 < s < +\infty\} \subset F^{-1}(0)$$

for some continuous function $s \mapsto (x(s), y(s)) \in U \times I$, $s \in (0, +\infty)$, with the following properties:

1) One of the following alternatives holds:

i) As $s \to +\infty$,

$$N(s) := \|x(s)\|_X + y(s) + \frac{1}{\text{dist}(x(s), \partial U)} + \frac{1}{\text{dist}(y(s), \partial I)} \to +\infty.$$ 

ii) There exists a sequence $s_n \to +\infty$ such that $\sup_n N(s_n) < +\infty$ but $\{x(s_n)\}$ has no subsequences converging in $X$;

2) Near each point $(x(s_0), y(s_0)) \in C$, we can reparametrize $C$ so that $s \mapsto (x(s), y(s))$ is real-analytic;

3) $(x(s), y(s)) \notin C_{\text{loc}}$ for $s$ sufficiently large.

We will also use the real value version of the following result from [19] Volume 1:

**Lemma 2.6. ([19] Volume 1, Theorem VI.14, analytic Fredholm theorem)**

Let $D$ be an open connected subset of $\mathbb{C}$. Let $f : D \to L(X, Z)$ be an analytic operator-valued function such that $f(z)$ is compact for each $z \in D$. Then one of the following two alternatives holds:

i) $(I - f(z))^{-1}$ exists for no $z \in D$;

ii) $(I - f(z))^{-1}$ exists for all $z \in D \setminus S$, where $S$ is a discrete subset of $D$. In this case, $(I - f(z))^{-1}$ is meromorphic in $D$, analytic in $D \setminus S$, the residues at he pole are
finite rank operators, and for \( z \in S \) the equation \( f(z)g = g \) has a non-zero solution in \( X \).

3. Global continuation

In this section, we consider the system (2.15) – (2.16). Let us consider a parametrized family of functions \( \{ \mu_{K, \pm}(e, p) \}_{K \in [0, +\infty)} \), in which all the functions \( \mu_{K, \pm}(e, p) \) are non-negative \( C^1 \) functions that satisfy

\[
|\mu_{K, \pm}(e, p)| + |\mu_p^{K, \pm}(e, p)| + |\mu_e^{K, \pm}(e, p)| \leq \frac{C_{\mu}(K)}{1 + |e|^\delta}, \quad \delta > 3.
\]

Moreover, we assume the following:

**Assumption 3.1.** We assume the following:

1) \( \mu_{K, \pm}(e, p) \) depend on \( K \) in the \( C^1 \) sense, i.e. \( \frac{\partial \mu_{K, \pm}(e, p)}{\partial K} \in C^0 \). Moreover, \( \frac{\partial \mu_{K, \pm}(e, p)}{\partial K} |_{K=0} = 0 \).

2) \( \mu_0^{0, +}(e, p) = \mu_0^{0, -}(e, p) = M^0(e, p) \) for some non-negative function \( M^0(e, p) \) satisfying the decay assumption (3.1).

3) The matrix operator

\[
\begin{pmatrix}
\tilde{J}_{11} & \tilde{J}_{12} \\
\tilde{J}_{21} & \tilde{J}_{22}
\end{pmatrix} : X \times X \times \mathbb{R} \to H^2(\Omega) \times H^2(\Omega) \times \mathbb{R}
\]

is bounded with a bounded inverse. Here

\[
\begin{align*}
\tilde{J}_{11}(\delta u) &= \delta u - (-\Delta + \frac{1}{r^2})^{-1}\{f_1(r)\delta u\}, \\
\tilde{J}_{12}(\delta w) &= -(-\Delta + \frac{1}{r^2})^{-1}\{f_2(r)\delta w\}, \\
\tilde{J}_{21}(\delta u) &= \Delta^{-1}\{f_3(r)\delta u\}, \\
\tilde{J}_{22}(\delta w) &= \delta w + \gamma \Delta^{-1}\{f_4(r)\delta w\},
\end{align*}
\]

with

\[
\begin{align*}
f_1(r) &:= 2\int_{\mathbb{R}^3} r\hat{v}_\varphi M^0_p((v, rv_\varphi))dv, \\
f_2(r) &:= 2\int_{\mathbb{R}^3} \hat{v}_\varphi M^0_e((v, rv_\varphi))dv, \\
f_3(r) &:= 2\int_{\mathbb{R}^3} rM^0_p((v, rv_\varphi))dv, \\
f_4(r) &:= 2\int_{\mathbb{R}^3} M^0_e((v, rv_\varphi))dv.
\end{align*}
\]

By (2.15) – (2.16), we aim to solve the elliptic system

\[
-\Delta \phi^{K, 0} = \int_{\mathbb{R}^3} (\mu^{K, +}(e^{K, +}, p^{K, +}) - \mu^{K, -}(e^{K, -}, p^{K, -}))dv,
\]

(3.4)

\[
(\Delta - \frac{1}{r^2})A^{K, 0}_\varphi = \int_{\mathbb{R}^3} \hat{v}_\varphi (\mu^{K, +}(e^{K, +}, p^{K, +}) - \mu^{K, -}(e^{K, -}, p^{K, -}))dv,
\]

(3.5)
where
\[ e^{K,\pm} = \langle v \rangle \pm \phi^{K,0}(x), \quad p^{K,\pm} = r(v_\varphi \pm A^{K,0}_\varphi(x)). \]

It is obvious that \((\phi^{K,0}, A^{K,0}_\varphi, K) = (0, 0, 0)\) is a solution to (3.4) - (3.5). Starting from this trivial solution, we will construct a global solution set
\[
\{(\mu^{K,\pm}(e^{K,\pm}(x,v), p^{K,\pm}(x,v)), \phi^{K,0}(x), A^{K,0}_\varphi(x))\}_{K \in [0, +\infty)}
\]
for (3.4) - (3.5), therefore obtain a global solution curve for the time-independent RVM.

We use the Global Implicit Function Theorem 2.4 to obtain

**Theorem 3.2.** Let \( \Omega \) be a \( C^1 \) axisymmetric bounded domain as stated in Section 3. Let \( \mu^{K,\pm}(e, p) \) satisfy (3.1) as well as Assumption 3.7. Then there exists an unbounded continuous solution set or loop \( \mathcal{C} := \{(A^{K,0}_\varphi, \phi^{K,0}, K)\} \subset H^2(\Omega) \times H^2(\Omega) \times \mathbb{R} \) to the system (3.4) - (3.5) with \( e^{K,\pm} = \langle v \rangle \pm \phi^{K,0}(x), \quad p^{K,\pm} = r(v_\varphi \pm A^{K,0}_\varphi(x)), \) in which the solution \((0,0,0)\) is included.

**Proof.** We extend the definition of \( \mu^{K,\pm}(e, p) \) evenly with respect to \( K \) by taking
\[ \mu^{-K,\pm}(e, p) = \mu^{K,\pm}(e, p) \] for all \( K \geq 0. \) Recall
\[
\mathcal{X} := \{g \in H^2(\Omega) : g \text{ independent of } \varphi, \exp(i\varphi)g \in H^2(\Omega), \ g|_{\partial \Omega} = 0\}. \]

For any \( h \in L^2(\Omega), \) define \( \Delta^{-1}h \in H^2(\Omega) \times H^0_0(\Omega) \) to be the unique solution \( g \) to the elliptic problem with Dirichlet boundary condition
\[
\Delta g = h, \ g|_{\partial \Omega} = 0. \tag{3.6}
\]

Then \( \Delta^{-1} \) is a compact operator from \( L^2(\Omega) \) to \( L^2(\Omega) \) since the inclusion of \( H^1(\Omega) \) into \( L^2(\Omega) \) is compact. Also, for any \( h \in L^2(\Omega), \) we define \(( -\Delta + \frac{1}{r^2})^{-1}h \in H^2(\Omega) \times H^0_0(\Omega) \) to be the unique solution \( g \) to the elliptic problem with Dirichlet boundary condition
\[
(-\Delta + \frac{1}{r^2})g = h, \ g|_{\partial \Omega} = 0. \tag{3.7}
\]

To do this, we first solve the equation
\[ -\Delta \tilde{g} = he^{i\varphi}, \quad \tilde{g}|_{\partial \Omega} = 0, \]
then take
\[ (-\Delta + \frac{1}{r^2})^{-1}h = g := e^{-i\varphi} \tilde{g}. \]
Now \(( -\Delta + \frac{1}{r^2})^{-1} \) is a compact operator from \( L^2(\Omega) \) to \( L^2(\Omega) \) since the inclusion of \( H^1(\Omega) \) into \( L^2(\Omega) \) is compact.

We take \( X = L^2_0(\Omega) \times L^2_0(\Omega), \ Z = H^2_0(\Omega) \times H^2_0(\Omega) \) and \( U = \mathcal{X} \times \mathcal{X} \times \mathbb{R} \). Let
\[
G(u, w, K) := \left( u - (-\Delta + \frac{1}{r^2})^{-1} \left\{ \int_{\mathbb{R}^3} \hat{\varphi}(\mu^{K,+}(\langle v \rangle + w, r(v_\varphi + u)) - \mu^{K,-}(\langle v \rangle - w, r(v_\varphi - u)))dv \right\}, \right.
\]
\[
\left. w + \Delta^{-1} \left\{ \int_{\mathbb{R}^3} (\mu^{K,+}(\langle v \rangle + w, r(v_\varphi + u)) - \mu^{K,-}(\langle v \rangle - w, r(v_\varphi - u)))dv \right\} \right). \tag{3.8}
\]
Then $G$ is a mapping from $X \times \mathbb{R}$ to $Z$. Solving the system (3.4) – (3.5) is equivalent to solving the equation

(3.9) \[ G(u, w, K) = 0. \]

For each solution $(u, w, K)$ (if exists), $(u, w)$ is automatically in $\big( H^2(\Omega) \cap H^1_0(\Omega) \big)^2$. Consider the point $(u_0, w_0, K_0) = (0, 0, 0)$ in $X \times \mathbb{R}$. One can verify that $G(u_0, w_0, K_0) = 0$. We have: $G$ is a $C^1$ mapping form $X \times \mathbb{R}$ to $Z$, with

(3.10) \[ (D_{(u, w)} G)(u, w, K) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \]

where

(3.11) \[ J_{11}(\delta u) = \delta u - \left(-\Delta + \frac{1}{r^2}\right)^{-1} \left\{ \int_{\mathbb{R}^3} r \hat{v}_\varphi (\mu_0^K + (v + w, r(v_\varphi + u)) + \mu_0^K - (v) - w, r(v_\varphi - u)) \delta u dv \right\}, \]

(3.12) \[ J_{12}(\delta w) = -\left(-\Delta + \frac{1}{r^2}\right)^{-1} \left\{ \int_{\mathbb{R}^3} \hat{v}_\varphi (\mu_0^K + (v + w, r(v_\varphi + u)) + \mu_0^K - (v) - w, r(v_\varphi - u)) \delta w dv \right\}, \]

(3.13) \[ J_{21}(\delta u) = \Delta^{-1} \left\{ \int_{\mathbb{R}^3} r (\mu_0^K + (v + w, r(v_\varphi + u)) + \mu_0^K - (v) - w, r(v_\varphi - u)) \delta u dv \right\}, \]

(3.14) \[ J_{22}(\delta w) = \delta w + \Delta^{-1} \left\{ \int_{\mathbb{R}^3} (\mu_0^K + (v + w, r(v_\varphi + u)) + \mu_0^K - (v) - w, r(v_\varphi - u)) \delta w dv \right\}. \]

Hence we have

(3.15) \[ (D_{(u, w)} G)(u_0, w_0, K_0) = \begin{pmatrix} \tilde{J}_{11} & \tilde{J}_{12} \\ \tilde{J}_{21} & \tilde{J}_{22} \end{pmatrix}, \]

where

(3.16) \[ \tilde{J}_{11}(\delta u) = \delta u - 2\left(-\Delta + \frac{1}{r^2}\right)^{-1} \left\{ \int_{\mathbb{R}^3} r \hat{v}_\varphi (M_0^0, r v_\varphi) \delta u dv \right\} \]

\[ = \delta u - \left(-\Delta + \frac{1}{r^2}\right)^{-1} \left\{ f_1(r) \delta u \right\}, \]

(3.17) \[ \tilde{J}_{12}(\delta w) = -2\left(-\Delta + \frac{1}{r^2}\right)^{-1} \left\{ \int_{\mathbb{R}^3} \hat{v}_\varphi (M_0^0, r v_\varphi) \delta w dv \right\} \]

\[ = -\left(-\Delta + \frac{1}{r^2}\right)^{-1} \left\{ f_2(r) \delta w \right\}, \]

(3.18) \[ \tilde{J}_{21}(\delta u) = 2\Delta^{-1} \left\{ \int_{\mathbb{R}^3} r (M_0^0, r v_\varphi) \delta u dv \right\} \]

\[ = \Delta^{-1} \left\{ f_3(r) \delta u \right\}, \]
\[ (3.19) \]
\[
\tilde{J}_{22}(\delta w) = \delta w + 2\Delta^{-1}\left\{ \int_{\mathbb{R}^3} M^0_{\nu}(v, rv_{\varphi})\delta w dv \right\} \\
= \delta w + \Delta^{-1}\left\{ f_4(\delta) \right\}.
\]

Here
\[ f_1(r) := 2 \int_{\mathbb{R}^3} r\hat{v}_\varphi M^0_p((v), rv_{\varphi}) dv, \quad f_2(r) := 2 \int_{\mathbb{R}^3} \hat{v}_\varphi M^0_p((v), rv_{\varphi}) dv, \]
\[ f_3(r) := 2 \int_{\mathbb{R}^3} r M^0_p((v), rv_{\varphi}) dv, \quad f_4(r) := 2 \int_{\mathbb{R}^3} M^0_p((v), rv_{\varphi}) dv. \]

By Assumption 3.1, the operator \((D_{(u, w)}G)(u_0, w_0, K_0) = (D_{(u, w)}G)(0, 0, 0) : \mathcal{X} \times \mathcal{X} \to H^2(\Omega) \times H^2(\Omega)\) is bounded with a bounded inverse. Moreover, we have
\[ (3.20) \]
\[
(u, w, K) \mapsto G(u, w, K) - (u, w) = \left( -(-\Delta + \frac{1}{r^2})^{-1}\left\{ \int_{\mathbb{R}^3} \hat{v}_\varphi (\mu^{K,+}(v) + w, rv_{\varphi} + u) - \mu^{K,-}(v - w, rv_{\varphi} - u)) dv \right\} \right),
\]
is a compact operator from \(X \times \mathbb{R}\) to \(Z = X\). We can now apply the Global Implicit Function Theorem and obtain a solution continuum \(C\) for the equation \(G(u, w, K) = 0\), which is a solution continuum for the system (3.4) – (3.5). One of the three alternatives must hold. The third alternative does not hold since \(X \times \mathbb{R} = U\). The proof is complete.

\[ \square \]

In the single species case, under additional assumptions, we use the Analytic Global Implicit Function Theorem 2.5 to obtain:

**Theorem 3.3.** Let \(\Omega\) be a \(C^1\) axisymmetric bounded domain as stated in Section 2. Under the assumptions of Theorem 3.2 consider the (strict) single species case, that is, one of the following two cases hold
1) \(\mu^{0,+} = \mu^{K,-} \equiv 0\), with \(\mu^{K,+} > 0\) for all \(K > 0\);
2) \(\mu^{0,-} = \mu^{K,+} \equiv 0\), with \(\mu^{K,-} > 0\) for all \(K > 0\).

Let \(d = \sup_{x \in \Omega} r(x) < +\infty\), \(\mu^{K,\pm}\) satisfies that \(\mu^{K,\pm}\) is real-analytic with respect to \(K\), and \(\int_{\mathbb{R}^3} \mu^{K,\pm}(v + a, rv_{\varphi} + b) dv\), \(\int_{\mathbb{R}^3} \hat{v}_\varphi \mu^{K,\pm}(v + a, rv_{\varphi} + b) dv\) are real-analytic with respect to \(a\) and \(b\), i.e.
\[ (3.21) \]
\[
| \int_{\mathbb{R}^3} \partial^{k}_{\nu} \partial^{l}_{a} \mu^{K,\pm}(v + a, rv_{\varphi} + b) dv | \leq C^{k+l} k! l! , \]
\[
| \int_{\mathbb{R}^3} \hat{v}_\varphi \partial^{k}_{\nu} \partial^{l}_{a} \mu^{K,\pm}(v + a, rv_{\varphi} + b) dv | \leq C^{k+l} k! l! ,
\]
for some \(C = C(a, b, K) > 0\) and all \(k, l \in \mathbb{Z}_{\geq 0}\), \(a, b \in \mathbb{R}\), then the solution set \(C\) contains a locally analytic curve \(\mathcal{C}\) parametrized by \(s\), with the following holds: There
exists a sequence \( s_j \to +\infty \), such that
\[
\| (A^{K(s_j),0}, \phi^{K(s_j),0}) \|_{H^2 \times H^2} + K(s_j) \to +\infty \text{ as } j \to +\infty.
\]
Hence \( \tilde{C} \) is an unbounded solution curve.

Remark 3.4. Under the assumptions of Theorem 3.3, the result above implies that the set \( C \) obtained in Theorem 3.2 is unbounded.

Remark 3.5. If \( d = \sup_{x \in \Omega} r(x) < +\infty \), \( \mu^{K, \pm} \) satisfy
\[
\partial^k \partial^l \mu^{K, \pm}(e, p) \leq \frac{C}{1 + |e|^\delta} k! l!
\]
for all \( k, l \in \mathbb{Z}_{\geq 0} \) and some \( C = C(K) > 0 \), then (3.21) holds.

Proof. Following some argument in the proof for Theorem 3.2, we can apply the Local Implicit Function Theorem 2.5 and obtain a continuous local solution curve \( C_{loc} \).

We consider the single species case with \( \mu^{K, -} \equiv 0 \) and let \( I = \mathbb{R}, U = \{ (u, w) \in X \times X : w > 0 \text{ in } \Omega \} \), then \( (u_0, w_0) = (0, 0) \in \partial U \) (here we use the single species assumption). If \( d = \sup_{x \in \Omega} r(x) < +\infty \), \( \mu^{K, \pm} \) satisfies (3.21) for some \( C = C(a, b, K) > 0 \) and all \( k, l \in \mathbb{Z}_{\geq 0} \) and \( a, b \in \mathbb{R} \), then
\[
| \partial^k \partial^l \int_{\mathbb{R}^3} \mu^{K, \pm}(\langle v \rangle + a, r(v_e + b)) dv | = | r^l \int_{\mathbb{R}^3} \partial^k \partial^l \mu^{K, \pm}(\langle v \rangle + a, r(v_e + b)) dv | \leq C^{k+l} k! l!,
\]
\[
| \partial^k \partial^l \int_{\mathbb{R}^3} \partial v_e \mu^{K, \pm}(\langle v \rangle + a, r(v_e + b)) dv | = | \int_{\mathbb{R}^3} \partial v_e \partial^k \partial^l \mu^{K, \pm}(\langle v \rangle + a, r(v_e + b)) dv | \leq C^{k+l} k! l!
\]
Hence \( G \) is real-analytic in the Fréchet sense. Also, the Fréchet derivative \( D_{(u, w)} G(u, w, K) : X \to Z \) is Fredholm with index 0 since the operators \( (-\Delta + \frac{1}{r^2})^{-1} \) and \( \Delta^{-1} \) are compact. Moreover, by Lemma 2.6 we have that \( D_{(u, w)} G(u, w, K) : X \to Z \) is invertible for all the \( (u, w) \)'s in the local solution curve \( C_{loc} \). We apply the Analytic Global Implicit Function Theorem 2.5 to obtain that the solution set \( C \) contains a locally analytic curve \( \tilde{C} \).

We rule out possibility ii) in Theorem 2.5. Let \( \{ s_n \} \) be a sequence such that \( s_n \to +\infty \) such that sup \( n N(s_n) < +\infty \), then
\[
N(s) = \|(u(s), w(s))\|_X + K(s) + \frac{1}{\text{dist}((u(s), w(s)), \partial U)} + \frac{1}{\text{dist}(K(s), \partial I)} \to +\infty
\]
as \( s \to +\infty \). It is not possible that \( \frac{1}{\text{dist}(K(s), \partial U)} \to +\infty \) since \( I = \mathbb{R} \). Hence there must hold
\[
\|(u(s), w(s))\|_{H^2 \times H^2} + K(s) + \frac{1}{\text{dist}((u(s), w(s)), \partial U)} \to +\infty
\]
as \( s \to +\infty \).

If \( \|(u(s), w(s))\|_{H^2 \times H^2} + K(s) \to +\infty \) as \( s \to +\infty \), then the analytic curve \( \tilde{C} \) is unbounded. On the other hand, if \( \|(u(s), w(s))\|_{H^2 \times H^2} + K(s) \) stays bounded as \( s \to +\infty \). Consider any sequence \( \{s_j\}_{j=1}^{\infty} \) with \( s_j \to +\infty \) as \( j \to \infty \). By compactness, there exists a subsequence still denoted by \( \{s_j\}_{j=1}^{\infty} \), such that

\[
(3.26) \quad (u(s_j), w(s_j), K(s_j)) \to (u_1, w_1, K_1) \text{ in } H^2 \times H^2 \times \mathbb{R} \text{ for some } (u_1, w_1, K_1).
\]

Moreover, we have \( w_1 \in \partial U \), therefore there exists some \( x_0 \in \Omega \), such that \( w_1(x_0) = 0 \), and \( w_1 \geq 0 \). By strong maximum principle, we have \( w_1 \equiv 0 \). Plugging this into (3.4) \(- (3.5)\), and noticing that \( \mu^{0,+} = \mu^{K,-} \equiv 0 \), \( \mu^{K,+} > 0 \) for all \( K > 0 \). Hence we must have \( K = 1 \), and therefore \( u_1 \equiv 0 \). By basis analysis fact, we have
\[
(3.27) \quad \lim_{s \to +\infty} (u(s), w(s), K(s)) = (0, 0, 0).
\]

By the uniqueness in the Local Implicit Function Theorem \([2,3]\), the curve \( \tilde{C} \) must rejoin itself at some \( s \in (0, +\infty) \), which violates Theorem \([2,5]\). A contradiction. Hence \( \|(u(s), w(s))\|_{H^2 \times H^2} + K(s) \) cannot stay bounded when \( s \to +\infty \). Combining these arguments together, we conclude that there exists a sequence \( s_j \to +\infty \), such that
\[
(3.28) \quad \|(A^{K(s_j)}_{\varphi}, \phi^{K(s_j)}_{0})\|_{H^2 \times H^2} + K(s_j) \to +\infty \text{ as } j \to +\infty.
\]

holds.

Similar argument as above can be carried out for the other single species case \( \mu^{K,+} \equiv 0 \). The proof is complete. \( \square \)

4. Properties of the solution set

In this section, we consider the case when the set \( C \) obtained in Theorem \([3,2]\) is unbounded, and show the following

**Proposition 4.1.** Let \( \Omega \) be a \( C^1 \) axisymmetric bounded domain as stated in Section \([2]\). Assume that the set \( C \) obtained in Theorem \([3,2]\) is unbounded, and for all \( (e, p) \in \mathbb{R}^2 \),
\[
\mu^{K,+}(e, p) + \mu^{K,-}(e, p) \leq m(K)(1 + |e|^{\beta})^{-1},
\]
\[
(4.1) \quad \int_{\mathbb{R}^3} [\mu^{K,+}((v) + a, r(v) + b)] - \mu^{K,-}((v) - a, r(v) - b)]dv \geq N(a, b, K).
\]

Here \( m(K) \) and \( N(a, b, K) \) are positive functions, and the constant \( \beta > 3 \). \( m(K) \) satisfies \( m(0) = 0 \), \( m(K) \to +\infty \) as \( K \to +\infty \). \( N(a, b, K) \) is some function satisfies
that when $K$ is large enough, $N(a,b,K)$ is positive and monotonically decreasing in $a$ and $b$. Moreover, we assume that for each $(a,b) \in \mathbb{R}^2$,

$$N(a,b,K) \to +\infty \text{ when } K \to +\infty.$$  \hfill (4.2)

Then we have

1) \hfill (4.3) \hfill 
\[ \sup_{\{ (A^K_0, \phi^K_0, K) \in C \}} \| (\phi^K_0, A^K_0) \|_{H^2(\Omega)} = +\infty. \]

2) In particular, in the case that $\sup_{\{ (A^K_0, \phi^K_0, K) \in C \}} K = +\infty$ below holds, we have:

$$\text{As } K \to \infty, \| (\phi^K_0, A^K_0) \|_{L^\infty(\Omega)} \to +\infty.$$  \hfill (4.4)

3) If $\mu^K_{-} \equiv 0$, then

$$\sup_{\{ (A^K_0, \phi^K_0, K) \in C \}} K = +\infty.$$  \hfill (4.5)

Remark 4.2. We can switch the role of ions and electrons (+ and −), and the same results hold.

Remark 4.3. In general, without the single species assumption, it is not certain if (4.5) holds, that is, it is not clear if the parameter $K$ stays bounded in the set $C$.

Before proving the proposition, let us introduce the following tool lemmas:

**Lemma 4.4.** Let $\Omega$ be a bounded domain in $\mathbb{R}^N$. Assume $u \in H^2(\Omega)$ satisfies

$$-\Delta u \geq \alpha(x)\zeta(u), \text{ for } x \in \Omega,$$

$$u \geq 0, \text{ for } x \in \partial \Omega,$$

where $\alpha$ is a non-negative function and $\zeta : [0, a_\zeta) \to [0, +\infty) \ (0 < a_\zeta \leq +\infty)$ is a non-decreasing $C^1$ function such that $\zeta(0) > 0$. Let $F(t) := \int_0^t \frac{d s}{\zeta(s)}$, $d_{\Omega}(x) := \text{dist}(x, \partial \Omega)$. Then

$$F(u(x)) \geq \alpha_x(d_{\Omega}(x)) \frac{d_{\Omega}(x)^2}{2N} \text{ for all } x \in \Omega,$$

and equivalently (since $F(t)$ is a non-decreasing function)

$$u(x) \geq F^{-1}(\alpha_x(d_{\Omega}(x))) \frac{d_{\Omega}(x)^2}{2N} \text{ for all } x \in \Omega.$$  \hfill (4.8)

Here $\alpha_x(r) := \inf_{y \in B_r(x)} \alpha(y)$, $B_r(x) = \{ y : |y - x| < r \}$.

**Proof.** The proof can be easily adapted from Theorem 2.1 in [1] by approximating $H^2$ functions using $C^2$ functions. \hfill \square
Lemma 4.5. For each \( x \in \Omega, \delta > 3 \), there holds

\[
(4.9) \quad \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle \pm \phi^0(x)|^\delta + |r(v_\varphi \pm A^0_\varphi(x))|^\delta} dv \leq (2 + 2^\delta |\phi^0(x)|^\delta) \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle|^\delta} dv,
\]

\[
(4.10) \quad \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle \pm \phi^0(x)|^\delta + |r(v_\varphi \pm A^0_\varphi(x))|^\delta} dv \geq \frac{1}{2^\delta + 2^\delta |\phi^0(x)|^\delta} \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle|^\delta + r^\delta |v_\varphi|^\delta} dv.
\]

Here \( C_\delta > 0 \) is a constant that only depends on \( \delta \).

Proof. Firstly, let us prove (4.9). For any \( x \in \Omega \), the integral \( \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle|^\delta} dv \) is convergent because \( \delta > 3 \). For any \( x \), it suffices to prove

\[
(4.11) \quad 1 + |\langle v \rangle|^\delta \leq (2 + 2^\delta |\phi^0(x)|^\delta)(1 + |\langle v \rangle \pm \phi^0(x)|^\delta),
\]

which implies

\[
(4.12) \quad 1 + |\langle v \rangle|^\delta \leq (2 + 2^\delta |\phi^0(x)|^\delta)(1 + |\langle v \rangle \pm \phi^0(x)|^\delta + |r(v_\varphi \pm A^0_\varphi(x))|^\delta).
\]

Let \( D_1 := \{ v \in \mathbb{R}^3 : |\langle v \rangle \pm \phi^0(x)| > \frac{1}{2} \langle v \rangle \} \), \( D_2 := \{ v \in \mathbb{R}^3 : |\langle v \rangle \pm \phi^0(x)| \leq \frac{1}{2} \langle v \rangle \} \). We will show that the claim holds true on both sets. Indeed, on \( D_1 \),

\[
(4.13) \quad 1 + |\langle v \rangle|^\delta \leq 2(1 + |\langle v \rangle \pm \phi^0(x)|^\delta).
\]

On \( D_2 \), note that \( |\langle v \rangle \pm \phi^0(x)| \leq \frac{1}{2} \langle v \rangle \) implies \( 1 \leq |\langle v \rangle| \leq 2|\phi^0(x)| \). Therefore

\[
(4.14) \quad 1 + |\langle v \rangle|^\delta \leq 1 + 2^\delta |\phi^0(x)|^\delta.
\]

(4.9) is verified.

Next, we prove (4.10). It suffices to show (4.15)

\[
1 + |\langle v \rangle \pm \phi^0(x)|^\delta + |r(v_\varphi \pm A^0_\varphi(x))|^\delta \leq (2^\delta + 2^\delta |\phi^0(x)|^\delta + 2^\delta r^\delta |A^0_\varphi(x)|^\delta)(1 + |\langle v \rangle|^\delta + r^\delta |v_\varphi|^\delta).
\]

Indeed, the left hand side of the inequality (4.15) satisfies

\[
1 + |\langle v \rangle \pm \phi^0(x)|^\delta + |r(v_\varphi \pm A^0_\varphi(x))|^\delta \leq 1 + 2^\delta |\langle v \rangle|^\delta + 2^\delta |\phi^0(x)|^\delta + 2^\delta r^\delta |v_\varphi|^\delta + 2^\delta r^\delta |A^0_\varphi(x)|^\delta,
\]

and the right hand side of (4.15) satisfies

\[
(2^\delta + 2^\delta |\phi^0(x)|^\delta + 2^\delta r^\delta |A^0_\varphi(x)|^\delta)(1 + |\langle v \rangle|^\delta + r^\delta |v_\varphi|^\delta) \geq 1 + 2^\delta |\langle v \rangle|^\delta + 2^\delta |\phi^0(x)|^\delta + 2^\delta r^\delta |v_\varphi|^\delta + 2^\delta r^\delta |A^0_\varphi(x)|^\delta.
\]

The two estimates above then gives (4.15), and hence (4.10) is proved. The proof of the lemma is complete.

\[ \Box \]

Denote

\[
C_{\text{int},1} := \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle|^\delta} dv, \quad C_{\text{int},2} := \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle|^\delta + r^\delta |v_\varphi|^\delta} dv.
\]

The proof of Proposition 6.2 is given by the following three lemmas:
Lemma 4.6. Assume that assumption (4.1) of Proposition 4.1 holds. Then we have

\[ \sup \{ (A_{K,0}^{K,0}, \phi_{K,0}) \} \in C \| (\phi_{K,0}^{K,0}, A_{\phi_{K,0}}^{K,0}) \|_{H^2(\Omega)} = +\infty. \]  

Proof. Assume \( \sup \{ (A_{K,0}^{K,0}, \phi_{K,0}) \} \in C K = +\infty. \) We are going to show

\[ \sup \{ (A_{K,0}^{K,0}, \phi_{K,0}) \} \in C \| (\phi_{K,0}^{K,0}, A_{\phi_{K,0}}^{K,0}) \|_{L^\infty(\Omega)} \leq c(K) \leq C_0 \]

for some constant \( C_0 > 0 \) independent of \( K \). By the assumption (4.1) of Proposition 4.1, we then have

\[ \int_{\mathbb{R}^3} [\mu^{K,+}(\langle \nu \rangle + \phi_{K,0}^{K,0}(x), r(\nu_{\phi} + A_{\phi_{K,0}}^{K,0}(x))) - \mu^{K,-}(\langle \nu \rangle - \phi_{K,0}^{K,0}(x), r(\nu_{\phi} - A_{\phi_{K,0}}^{K,0}(x)))] dv \]

\[ \geq N(C_0, C_0, K) \]

for some positive continuous function \( N(a, b, K) \), and \( N(C_0, C_0, K) \) goes to \( +\infty \) as \( K \to +\infty \) for any fixed \( C_0 \).

For large \( K \), we apply Lemma 4.4 to (3.4) with Dirichlet boundary condition, taking \( \alpha(x) \equiv 1, \zeta \equiv N(C_0, C_0) \). We obtain, for all \( x \in \Omega \),

\[ F(\phi_{K,0}^{K,0}(x)) \geq \frac{d_\Omega(x)^2}{6}, \]

with

\[ F(\phi_{K,0}^{K,0}(x)) = \int_0^{\phi_{K,0}^{K,0}(x)} \frac{1}{N(C_0, C_0, K)} ds. \]

Hence

\[ \frac{\phi_{K,0}^{K,0}(x)}{N(C_0, C_0, K)} \geq \frac{d_\Omega(x)^2}{6}. \]

Let us pick some \( x \) with \( d_\Omega(x) > 0 \). Letting \( K \to +\infty \), we arrive at a contradiction to the assumption \( \lim_{K \to \infty} N(C_0, C_0, K) = +\infty \). Therefore there must hold

\[ \sup \{ (A_{K,0}^{K,0}, \phi_{K,0}) \} \in C \| (\phi_{K,0}^{K,0}, A_{\phi_{K,0}}^{K,0}) \|_{H^2(\Omega)} = +\infty. \]

In the first case, we must have

\[ \sup \{ (A_{K,0}^{K,0}, \phi_{K,0}) \} \in C \| (\phi_{K,0}^{K,0}, A_{\phi_{K,0}}^{K,0}) \|_{H^2(\Omega)} = +\infty. \]
since the set $\mathcal{C}$ is unbounded. Therefore in either case (4.16) holds.

**Lemma 4.7.** Let all the assumptions in Proposition 4.1 and in particular the assumption in Proposition 4.1 2) hold. We have, as $K \to \infty$,

$$\|(\phi^{K,0}, A^{K,0}_\varphi)\|_{L^\infty} \to +\infty. \tag{4.23}$$

**Proof.** Denote

$$\sup_{\{(A^{K,0}_\varphi, \phi^{K,0}) \in \mathcal{C}\}} \|(\phi^{K,0}, A^{K,0}_\varphi)\|_{L^\infty} := c_\infty(K).$$

Let $B_0$ be a maximum inscribed ball of $\Omega$ and $R$ be its radius, then $d_\Omega(x) = R$. Let $B_1$ be the ball which has the same center as $B_0$ and half the radius as it, that is, the radius of $B_1$ is $\frac{1}{2}R$.

In the same way as in Lemma 4.6, we have

$$\|(\phi^{K,0}, A^{K,0}_\varphi)\|_{L^\infty} \geq N(c_\infty(K), c_\infty(K), K). \tag{4.24}$$

Assume that $c_\infty(K)$ stays bounded when $K \to +\infty$, i.e. $c_\infty(K) \leq C_{\infty}$ for some constant $C_{\infty} > 0$ independent of $K$, then $N(c_\infty(K), c_\infty(K), K) \geq N_0$ for some $N_0 > 0$ when $K \to \infty$.

For large $K$, we apply Lemma 4.4 to $-\Delta \phi^{K,0}(x) \geq \alpha(x)\zeta(u)$ with Dirichlet boundary condition and $\alpha(x) \equiv 1$, $\zeta \equiv N(c_\infty(K), c_\infty(K), K)$. We obtain that for all $x \in \Omega$,

$$F(\phi^{K,0}(x)) \geq \frac{d_\Omega(x)^2}{6}. \tag{4.25}$$

Here

$$F(\phi^{K,0}(x)) = \int_0^{\phi^{K,0}(x)} N_0^{-1} ds.$$  

Hence

$$|\phi^{K,0}(x)| N_0^{-1} \geq \frac{d_\Omega(x)^2}{6}, \tag{4.26}$$

Let us pick some $x \in B_1$. Taking $K \to +\infty$ yields a contradiction to our assumption. Hence there must hold

$$c_\infty(K) \to +\infty \text{ as } K \to +\infty. \tag{4.27}$$

**Lemma 4.8.** Let all the assumptions in Proposition 4.1 and in particular the assumption in Proposition 4.1 3) hold. Then we have

$$\sup_{\{(A^{K,0}_\varphi, \phi^{K,0}) \in \mathcal{C}\}} K = +\infty. \tag{4.27}$$
Proof. We have $\mu^{K,+} \geq 0$, $\mu^{K,-} = 0$, hence the solution $\phi^{K,0}$ to the elliptic system (3.4) – (3.5) must be non-negative. Suppose $\sup_{\{(A_{\varphi}^{K,0}, \phi^{K,0}, K)\} \in \mathcal{C}} |K| < +\infty$. Then there must hold
\[
\sup_{\{(A_{\varphi}^{K,0}, \phi^{K,0}, K)\} \in \mathcal{C}} \| (\phi^{K,0}, A_{\varphi}^{K,0}) \|_{H^2(\Omega)} = +\infty.
\]
We denote $\sup_{\{(A_{\varphi}^{K,0}, \phi^{K,0}, K)\} \in \mathcal{C}} K := \kappa$.

By the standard elliptic estimates and that $\|g\|_{L^2(\Omega)} \lesssim \|g\|_{L^\infty(\Omega)}$ (since $\Omega$ is bounded), as well as Lemma 4.5, we have
\[
\| (\phi^{K,0}, A_{\varphi}^{K,0}) \|_{H^2(\Omega)} \lesssim \| \int_{\mathbb{R}^3} \mu^{K,+} (\langle v \rangle + \phi^{K,0}(x), r(v\varphi + A_{\varphi}^{K,0}(x))) dv \|_{L^2(\Omega)}
\]
\[
\lesssim \| \int_{\mathbb{R}^3} \mu^{K,+} (\langle v \rangle + \phi^{K,0}(x), r(v\varphi + A_{\varphi}^{K,0}(x))) dv \|_{L^\infty(\Omega)}
\]
\[
\lesssim m(\kappa) \| \int_{\mathbb{R}^3} \frac{1}{1 + |\langle v \rangle + \phi^{K,0}(x)|^\delta} dv \|_{L^\infty(\Omega)}
\]
\[
\leq m(\kappa) \| \int_{\mathbb{R}^3} \frac{1}{1 + \langle v \rangle^\delta} dv \|_{L^\infty(\Omega)}
\]
\[
\lesssim m(\kappa).
\]
Taking the supremum over $\mathcal{C}$ and using the assumption $\sup_{\{(A_{\varphi}^{K,0}, \phi^{K,0}, K)\} \in \mathcal{C}} K < +\infty$, we arrive at a contradiction to $\sup_{\{(A_{\varphi}^{K,0}, \phi^{K,0}, K)\} \in \mathcal{C}} \| (\phi^{K,0}, A_{\varphi}^{K,0}) \|_{H^2(\Omega)} = +\infty$. Hence there must hold
\[
\sup_{\{(A_{\varphi}^{K,0}, \phi^{K,0}, K)\} \in \mathcal{C}} K = +\infty.
\]
\[
\square
\]

Notice that we can switch the role of ions and electrons (+ and −), and the same results hold. In the case of electrons ($\mu^{K,-} \geq 0$, $\mu^{K,+} = 0$), $\phi^{K,0}(x) \leq 0$, $e^- = \langle v \rangle - \phi^{K,0}(x) \geq \langle v \rangle$.

5. Spectral Stability in a Single Species Case

In this section, we consider a single species plasma and study the change of spectral stability for the solutions $(\phi^{K,0}, A_{\varphi}^{K,0})$ in the set $\mathcal{C}$ under Assumption 5.3 below. Let us first state the following theorem on spectral stability of equilibria in [26] (notice that the $\mu^{\pm}$ in Theorem 5.1 below is different from the $\mu^{\pm}$ in all other places in the paper):

**Theorem 5.1.** Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2. Let $\mathcal{H}^\pm$ be the space of functions of $x \in \Omega$ and $v \in \mathbb{R}^3$ with the norm
\[
\| g(x, v) \|_{\mathcal{H}^\pm} := \left( \int_{\Omega} \int_{\mathbb{R}^3} \mu^\pm(e^\pm, p^\pm) |g(x, v)|^2 dxdv \right)^{1/2}.
\]
Denote $D^\pm = v \cdot \nabla_x \pm (E^0 + \hat{v} \times B^0) \cdot \nabla_v$, and $\mathcal{P}^\pm$ be the orthogonal projection on the kernel of $D^\pm$ in the space $\mathcal{H}^\pm$. Define

$$A_0^h = \Delta h + \sum_{\pm} \int_{\mathbb{R}^3} \mu_\varphi^\pm (1 - \mathcal{P}^\pm) hdv,$$

$$A_2^h = (-\Delta + \frac{1}{r^2}) h - \sum_{\pm} \int_{\mathbb{R}^3} \hat{v}_\varphi \big( \mu_\varphi^\pm rh + \mu_\varphi^\pm \mathcal{P}^\pm(\hat{v}_\varphi h) \big) dv,$$

$$B^h = -\sum_{\pm} \int_{\mathbb{R}^3} \hat{v}_\varphi \mu_\varphi^\pm (1 - \mathcal{P}^\pm) hdv.$$

Let $(f_0^\pm, E^0, B^0)$ be an equilibrium of the relativistic Vlasov-Maxwell system satisfying $f_0^\pm(x, v) = \mu_\varphi^\pm (e^\pm, p^\pm) \geq 0$ and $\mu_\varphi^\pm (e, p) < 0$, $|\mu_\varphi^\pm| = |\mu_\varphi^\pm| \leq \frac{C_0}{1 + |e|}$ with $\delta > 3$, $\mu^\pm \in C^1$, $\varphi_0 \in C(\Omega)$, $A_0^\varphi \in C(\Omega)$. Then the operator

$$L^0 = A_2^0 - B^0 (A_1^0)^{-1} (B^0)^*$$

on $\mathcal{X}$ is self-adjoint. Also, we have
1) If $L^0 \geq 0$, there exists no growing mode of the linearized RVM equation.
2) Any growing mode, if it exists, must be purely growing, i.e. the exponent $\lambda$ of instability must be a real number.
3) If $L^0 \not\geq 0$, there exists a growing mode of the linearized Vlasov equation and the linearized Maxwell system with the boundary conditions.
4) The operator $A_1^0$ is negative definite, and hence $B^0 (A_1^0)^{-1} (B^0)^*$ is negative definite.

Therefore spectral instability only occurs from the operator $A_2^0$. To be precise, $L^0$ is nonnegative definite if $A_1^0$ is nonnegative definite.

Moreover, the result in the theorem holds for the single species case (either $\mu^+ \equiv 0$ or $\mu^- \equiv 0$).

We replace $\mu^\pm$ in the operators by $\mu^{K,\pm}$ and define

$$D^{K,\pm} = v \cdot \nabla_x \pm (\mathbf{E}^{K,0} + \hat{v} \times \mathbf{B}^{K,0}) \cdot \nabla_v.$$

Here $\mathbf{E}^{K,0} = -\nabla \phi^{K,0}$, $\mathbf{B}^{K,0} = \nabla \times \mathbf{A}^{K,0}$, $\mathbf{A}^{K,0} = A_\varphi^{K,0} e_\varphi$. Let $\mathcal{H}^{K,\pm}$ be the space of functions of $x \in \Omega$ and $v \in \mathbb{R}^3$ with the norm

$$\|g(x, v)\|_{\mathcal{H}^{K,\pm}} := \left( \int_{\Omega} \int_{\mathbb{R}^3} |(\mu^{K,\pm})_e (e^\pm, p^\pm)| g^2(x, v) dxdv \right)^{1/2}$$

and $\mathcal{P}^{K,\pm}$ be the orthogonal projection from $\mathcal{H}^{K,\pm}$ onto $\ker D^{K,\pm}$. In analogy with (5.2), (5.3) and (5.4), we define

$$A_1^{K,\pm} := \Delta h + \sum_{\pm} \int_{\mathbb{R}^3} (\mu^{K,\pm})_e (1 - \mathcal{P}^{K,\pm}) hdv.$$
EQUILIBRIA OF THE 3D AXISYMMETRIC RVM SYSTEM\[23\]

\[(5.9)\]
\[A^{0,K}_2 h := (-\Delta + \frac{1}{r^2})h - \sum_{\pm} \int_{\mathbb{R}^3} \hat{v}_\varphi ((\mu^{K,\pm})_p rh + (\mu^{K,\pm})_e \mathcal{P}^{K,\pm}(\hat{v}_\varphi h)) dv\]

\[(5.10)\]
\[B^{0,K}_0, K h := -\sum_{\pm} \int_{\mathbb{R}^3} \hat{v}_\varphi (\mu^{K,\pm})_e (1 - \mathcal{P}^{K,\pm}) h dv\]

\[(5.11)\]
\[\mathcal{L}^{0,K} := A^{0,K}_2 - B^{0,K}_0 (A^{0,K}_1)^{-1} (\mathcal{B}^{0,K})^* .\]

Let us first consider the case when \(K\) is small. We give the following observation:

**Proposition 5.2.** Let \(\Omega\) be a \(C^1\) axisymmetric bounded domain as stated in Section \(2\). Let \(\mu^{K,\pm}\) satisfy

\[(\mu^{0,\pm})_p \equiv 0, \quad (\mu^{0,\pm})_e < 0.\]

then for \(K = K_0 = 0\), the equilibrium \((\mu^{0,\pm}, \phi^{0,0}, A^{0,0}_\varphi) = (\mu^{0,\pm}, \phi^{0,0}, 0)\) is spectrally stable. Hence by continuity we have, when \(K\) is close to 0, the equilibrium \((\mu^{K,\pm}, \phi^{K,0}, A^{K,0}_\varphi)\) is spectrally stable. Moreover, the result in the theorem holds for the single species case (either \(\mu^+ \equiv 0\) or \(\mu^- \equiv 0\)).

**Proof.** By Theorem \(5.1\), it suffices for us to show that \(A^{0,0}_2\) is nonnegative definite. For \(K = K_0 = 0\), the operator \(A^{0,0}_2\) corresponding to the equilibrium \((\mu^{0,\pm}, \phi^{0,0}, A^{0,0}_\varphi) = (\mu^{0,\pm}, \phi^{0,0}, 0)\) are described as

\[(5.12)\]
\[A^{0,0}_2 h = (-\Delta + \frac{1}{r^2})h - \sum_{\pm} \int_{\mathbb{R}^3} \hat{v}_\varphi (\mu^{0,\pm})_e \mathcal{P}^{0,\pm}(\hat{v}_\varphi h) dv.\]

Using the Dirichlet boundary condition, we have

\[\langle A^{0,0}_2 h, h \rangle_{L^2} = \int_{\Omega} (|\nabla h|^2 + \frac{1}{r^2} |h|^2) dx + \sum_{\pm} \|\mathcal{P}^{0,\pm}(\hat{v}_\varphi h)\|^2_{H^{1,\pm}} \geq 0.\]

The proof of the proposition is complete. (The proof for the single species case is similar. ) \(\square\)

From now on in this section, we assume

**Assumption 5.3.**

\[\mu^{K,-} \equiv 0 \text{ for all } K, \]
\[\mu^{K,+}_e < 0 \text{ for all } K, \]
\[p \mu^{K,+}_p(e, p) \geq C'_\mu K^{1+m-\epsilon} |p|^{1-\epsilon} \nu(e) \text{ when } K \text{ is large enough}, \]

\[(5.13)\]
\[\mu^{K,+}_e(e, p) \leq K^m \frac{C'_\mu}{1 + |e|^\delta + K^\delta |p|^\delta} \text{ when } K \text{ is large enough}, \]
\[\mu^{K,+}_p(e, p) \leq K^m \frac{C'_\mu}{1 + |e|^\delta + K^\delta |p|^\delta} \text{ when } K \text{ is large enough}, \]
\[\mu^{K,+}_p(e, p) \leq K^{1+m} \frac{C'_\mu}{1 + |e|^\delta} \text{ when } K \text{ is large enough}.\]
Here $\nu(e)$ is some positive function satisfying
\begin{equation}
(5.14) \quad \nu(e) \geq C_\nu \exp(-e)
\end{equation}
and the constants $\epsilon, \delta, m$ and $C_\nu$ satisfy $\epsilon > 0, \delta > 4, m \in (-1, 1), C_\nu > 0, \epsilon < 1 - |m|$(hence $\epsilon + m < 1, \epsilon < 1 + m$).

**Theorem 5.4.** Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2, with $\inf_{x \in \Omega} r(x) > 0$. Recall $d := \sup_{x \in \Omega} r(x)$ and assume $d > 1$. Let all the assumptions in Theorem 3.2, Theorem 3.3, Proposition 4.1, in particular Proposition 4.1 3), hold (so we have $\sup_{(\phi^{K,0}, \phi^{K,0}, K)} K = +\infty$). Let Assumption 5.3 hold. Then when $K$ is large enough, then the equilibrium $(\mu^{K,+}(e, p), \mu^{K,-}(e, p) \equiv 0, \phi^{K,0}, A^{K,0}_\varphi)$ is spectrally unstable. More precisely, suppose there exists some $h \in X$ (normalized in such a way that $\int_{\Omega} (|\nabla h|^2 + \frac{1}{r^2} |h|^2)dx = 1$), and $K$ is so large that
\begin{equation}
(5.15) \quad 1 - H_1 C_1 (K)^{1+m-\epsilon} + 120 \cdot 2^\delta \pi^2 b^2 (H_1 + H_2) C_2^2 (K)^{2m} + 2^4 H_2 C_2 C_\alpha (K)^m + 256 \pi^2 c_P (K)^2 H_2 (K)^{2m} < 0 ,
\end{equation}
holds, where
\begin{align*}
H_1 &= \int_{r \geq 1} r |h|^2 dx , \\
H_2 &= \|h\|_{L^2}^2 , \\
C_1 &= 2^{-1-\epsilon/2} b^{-\epsilon} , \\
C_2 &= \frac{8}{3} \pi + \frac{4}{\delta - 1} \pi^2 ,
\end{align*}
and $c_P$ is the square of the Poincaré constant of $\Omega, C_\nu$ is as defined in (5.14). Then the equilibrium $(\mu^{K,+}(e, p), \mu^{K,-}(e, p) \equiv 0, \phi^{K,0}, A^{K,0}_\varphi)$ is spectrally unstable.

**Remark 5.5.** The condition $p \mu^{K,+}_p(e, p) \geq C_\nu (K)^{1+m-\epsilon} |p| (p) - \epsilon \nu(e)$ together with that $\mu^{K,+}_p(e, p) \leq (K)^{1+m} \frac{C_\nu}{1+|p|^\epsilon}$ implies that $|\nu(e)|$ is controlled by $\frac{|p|^\epsilon}{1+|p|^\epsilon}$ with $\delta > 4$, which ensures the integrals in the proof are convergent. (This is where we need $\delta > 4$. Notice that $\epsilon \leq 1$.)

**Remark 5.6.** We can switch the role of ions and electrons (+ and −), and the same results hold.

**Remark 5.7.** In fact, we only need $\mu^{K,-} \equiv 0$ to hold for $K$ large enough. Having $\mu^{K,-} \equiv 0$ in Assumption 5.3 is mainly for convenience in description. Moreover, we can extend the result for the case when $\mu^{K,-} \equiv 0$ for $K$ large enough does not necessarily hold, with the constraint on the constant $m \in (-1, 1)$ replaced by $m \in (-1, 0)$. The proof is similar.

Now we discuss the proof of Theorem 5.4.

In the same way as in the proofs of Lemma 8.2, 8.3 and 8.4 in [26], we can obtain
Lemma 5.8. 1) We have

\[ \|A^{0,K}_1\|_{L^2 \to L^2} \geq c_P^{-1}. \]

Hence for all \( \tilde{g} \in \mathcal{X} \), there holds

\[ \|\left\langle (A^{0,K}_1)^{-1}\tilde{g}, \tilde{g} \right\rangle\|_{L^2} \leq c_P\|\tilde{g}\|^2_{L^2}. \]

Here \( c_P \) is the square of the Poincaré constant of \( \Omega \).

2) For all \( K \geq 1 \),

\[ \| (B^{0,K}_0)^*h \|_{L^2 \to L^2} \leq 8\sqrt{2\pi}C\mu K^m. \]

3) For any \( h \in \mathcal{X} \), there holds

\[ \left\langle (A^{0,K}_1)^{-1}(B^{0,K}_0)^*h, (B^{0,K}_0)^*h \right\rangle_{L^2} \leq c_P\| (B^{0,K}_0)^*h \|^2_{L^2} \]

\[ \leq 2c_P \cdot 128\pi^2 C^2 \mu^2 K^{2m}\|h\|^2_{L^2} \]

\[ = 256\pi^2 c_P C^2 \mu^2 H K^{2m}. \]

We will also need the following

Lemma 5.9. Assume that \( \inf_{x \in \Omega} r(x) > 0 \). A solution \((\phi^{K,0}_\varphi, A^{K,0}_\varphi)\) to the coupled elliptic system with the settings in this section satisfies

\[ \|(\phi^{K,0}_\varphi, A^{K,0}_\varphi)\|_{L^\infty} \lesssim K^{-1+m}, \]

where the constant in \( \lesssim \) is independent of \( K \).

Proof. By standard elliptic estimates, the assumptions \( \mu^{K,-} = 0 \) and the decay assumption on \( \mu^{K,+} \), it suffices for us to estimate

\[ \int_{\mathbb{R}^3} K^m \frac{1}{1 + |\langle v \rangle + \phi^{K,0}_\varphi(x)|^\delta + |Kr(v_\varphi + A^{K,0}_\varphi(x))|^\delta} dv. \]

(For \( A^{K,0}_\varphi \), we make use of the identity \(-\Delta (ge^{i\varphi}) = (-\Delta + \frac{1}{r^2})ge^{i\varphi} \), which holds if \( g \) is cylindrically symmetric, to get rid of the term \( \frac{1}{r}A^{K,0}_\varphi \) in the elliptic equation.)

We make a change of variable \( K(v_\varphi + A^{K,0}_\varphi(x)) \mapsto v_\varphi \) and compute

\[ \int_{\mathbb{R}^3} K^m \frac{1}{1 + |\langle v \rangle + \phi^{K,0}_\varphi(x)|^\delta + |Kr(v_\varphi + A^{K,0}_\varphi(x))|^\delta} dv \]

\[ = \int_{\mathbb{R}^3} K^m \frac{1}{1 + |\langle \tilde{v} \rangle + \phi^{K,0}_\varphi(x)|^\delta + |Kr(v_\varphi + A^{K,0}_\varphi(x))|^\delta} d\tilde{v}dv_\varphi \]

\[ = \int_{\mathbb{R}^3} K^{-1+m} \frac{1}{1 + |\sqrt{1 + |\tilde{v}|^2 + |v_\varphi|^2 + |A^{K,0}_\varphi(x)|^2} + |\phi^{K,0}_\varphi(x)|^\delta + |rv_\varphi|^\delta} d\tilde{v}dv_\varphi \]

\[ \leq \int_{\mathbb{R}^3} K^{-1+m} \frac{1}{1 + |\sqrt{1 + |\tilde{v}|^2 + |\phi^{K,0}_\varphi(x)|^\delta + |rv_\varphi|^\delta}} d\tilde{v}dv_\varphi. \]
By the maximum principle, \( \phi^{K,0}(x) \geq 0 \). Thus the last line is bounded from above as
\[
\int_{\mathbb{R}^3} K^{-1+m} \frac{1}{1 + |v|^2 + \phi^{K,0}(x) |z|^2 + |r v_r|^2} d\tilde{v} dv_x \\
\leq K^{-1+m} \int_{\mathbb{R}^3} \frac{1}{1 + (\tilde{v})^2 + |r v_r|^2} dv_x \\
\lesssim K^{-1+m}.
\]
Here in the last line we have used \( \inf_{x \in \Omega} r(x) > 0 \). \((5.19)\) now follows from standard elliptic estimates. \( \square \)

Notice that we can switch the role of ions and electrons (+ and −), and the same results hold. In the case of electrons \( (\mu^{K,-} \geq 0, \mu^{K,+} = 0), \phi^{K,0}(x) \leq 0, e^− = \langle \nu \rangle - \phi^{K,0}(x) \geq \langle \nu \rangle \).

Now we are ready to prove Proposition \([5.4] \).

**Proof.** By the assumptions in Proposition \([4.1] \), in particular the assumption in Proposition \([4.3] \), we have \( \sup_{\{(A^{K,0}_{\nu}, \phi^{K,0}, K)\in \mathcal{C} \}} K = +\infty \).

For \( h \in \mathcal{X} \), we first take care of the term \( \langle A^{K,h}_2, h \rangle_{L^2} \). For simplicity, we normalize \( h \) such that \( \int_{\Omega} (|\nabla h|^2 + \frac{1}{r^2} |h|^2) dx = 1 \). Recall that \( e^{K,\pm} = \langle \nu \rangle \pm \phi^{K,0}(x), p^{K,\pm} = r(\nu \pm A^{K,0}_\nu(x)) \).

We observe
\[
\langle A^{K,h}_2, h \rangle_{L^2} = 1 - \int_{\Omega} \int_{\mathbb{R}^3} \langle \nu \rangle^{-1} p^{K,+} p^{K,+} (e^{K,+}, p^{K,+}) |h|^2 dv dx \\
+ \int_{\Omega} \int_{\mathbb{R}^3} \frac{p^{K,+} (e^{K,+}, p^{K,+})}{\langle \nu \rangle} r A^{K,0}_\nu |h|^2 dv dx + \|P^{K,+}(\nu, h)\|^2_{H^{K,+}} \\
:= 1 + I + II + III.
\]
We are going to show that when \( K \geq 1 \) is large enough, the term \( I \) dominates all the others and remains negative. We compute
\[
I \leq -C'_\mu \int_{\Omega} \int_{\mathbb{R}^3} \langle \nu \rangle^{-1} K^{1+m-\epsilon} |p^{K,+}| (p^{K,+})^{-\epsilon} |e^{K,+}| dv |h|^2 dx \\
\leq -C'_\mu K^{1+m-\epsilon} \int_{\Omega} \int_{|p^{K,+}(x,v)| > 1} \langle \nu \rangle^{-1} |p^{K,+}|^{1-\epsilon} \cdot 2^{-\epsilon/2} |e^{K,+}| dv |h|^2 dx \\
\leq -C''_\mu K^{1+m-\epsilon} M,
\]
where
\[
M = 2^{-\epsilon/2} \inf_{r(x) \geq 1, x \in \Omega, r(x) \geq 1} \left[ \int \int_{\{|v| \in \mathbb{R}^3, |p^{K,+}(x,v)| > 1\}} \langle \nu \rangle^{-1} |v \nu + A^{K,0}_\nu (r, z)|^{1-\epsilon} |e^{K,+}| dv \right] r^{-\epsilon} |h|^2 dx.
\]
Noting that $\|\phi^{K,0}, A^{K,0}_\varphi\|_{L^\infty} \lesssim K^{-1+m} \leq 1/2$ when $K$ is large, we have

\begin{equation}
(5.23) \quad M = 2^{-\varepsilon/2} \inf_{r(x) \geq 1} \int_{x \in \Omega, r(x) \geq 1} \left( \int_{\{v \in \mathbb{R}^3, |v| > r(x)\}} \langle v \rangle^{-1} |v_\varphi + A^{K,0}_\varphi(r, z)|^{-1} \nu(e^{K,0}) dv \right) r^{-\varepsilon} |h|^2 dx \geq 2^{-\varepsilon/2} \int_{r(x) \geq 1} \inf_{x \in \Omega, r(x) \geq 1} \left( \int_{\{v \in \mathbb{R}^3, |v| > 2 + \frac{r(x)}{\mu}\}} \langle v \rangle^{-1} \frac{1}{r(x)^{1-\varepsilon}} \nu(\langle v \rangle + \phi^{K,0}) dv \right) r^{-\varepsilon} |h|^2 dx \geq 2^{-\varepsilon/2} \int_{r(x) \geq 1} \inf_{x \in \Omega, r(x) \geq 1} \left( \int_{\{v \in \mathbb{R}^3, |v| > 3\}} \langle v \rangle^{-1} \nu(\langle v \rangle + \phi^{K,0}) dv \right) r^{-\varepsilon} |h|^2 dx \geq 2^{-\varepsilon/2} C_\varphi \cdot \frac{1}{2} \int_{r(x) \geq 1} r^{-\varepsilon} |h|^2 dx \geq 2^{-\varepsilon/2} \cdot \frac{1}{2} d^{-\varepsilon} C_\varphi \int_{r(x) \geq 1} r |h|^2 dx,
\end{equation}

where $d = \sup_{x \in \Omega} r(x) > 1$. Hence

$$I \leq -H_1 C_1 C_\varphi C_\mu K^{1+m-\varepsilon},$$

where $H_1 = \int_{r \geq 1} r |h|^2 dx$, $C_1 = 2^{-1-\varepsilon/2} b^{-\varepsilon}$. From Remark 1, we have $|\nu(e^{K,0})| \leq \frac{\langle p^{K,0} \rangle}{\mathcal{C}(1+\langle e^{K,0} \rangle^\delta)}$ with $\delta > 4$, which ensures that the integral $\int_{\mathbb{R}^3} \langle v \rangle^{-1} \nu_1^2 \nu(e^{K,0}) dv$ is convergent. In the end we obtain $I = O(K^{1+m-\varepsilon})$.

$I$ vanishes if $A^{K,0}_\varphi = 0$. If $A^{K,0}_\varphi \neq 0$, we estimate, using Lemma 5.9

\begin{equation}
(5.24) \quad II \leq 20\pi b^2 C_\mu (H_1 + H_2) \|A^{K,0}_\varphi\|_{L^\infty} \sup_x \int_{\mathbb{R}^3} \langle v \rangle^{-1} |\mu^{K,0}_\varphi(e^{K,0}, p^{K,0})| dv \leq 20\pi b^2 C_\mu (H_1 + H_2) C_\mu K^{1+m-\varepsilon} \cdot K^{1+m} \sup_x \int_{\mathbb{R}^3} \frac{1}{\langle v \rangle (1+\langle e^{K,0} \rangle^\delta)} dv \leq 20\pi b^2 C_\mu (H_1 + H_2) C_\mu K^{2m} \sup_x \int_{\mathbb{R}^3} \frac{1}{\langle v \rangle (1+\langle \nu \rangle + \phi^{K,0}(x))\delta} dv (1 + K^{(-1+m)\delta}) \leq 40 \cdot 2^\delta \pi b^2 (H_1 + H_2) C_\mu^2 K^{2m} \int_{\mathbb{R}^3} \frac{1}{1 + \langle v \rangle^\delta} dv \leq 40 \cdot 2^\delta \pi b^2 (H_1 + H_2) C_\mu^2 K^{2m} \cdot 3\pi \leq 120 \cdot 2^\delta \pi b^2 (H_1 + H_2) C_\mu^2 K^{2m},
\end{equation}

Here $H_2 = \int_{\Omega} |h|^2 dx = \|h\|_{L^2}^2$. Therefore the term $II$ is $O(K^{2m})$. 
For $III = \| \mathcal{P}^{K,+}(\hat{v}_\varphi h) \|^2_{H^{K,+}}$, we estimate

\begin{align}
III = \| \mathcal{P}^{K,+}(\hat{v}_\varphi h) \|^2_{H^{K,+}} & \leq \int_\Omega \sup_v |\mathcal{P}^{K,+}(\hat{v}_\varphi h)|^2 dx \sup_v \left( \int_{\mathbb{R}^3} |\mu^{e,K,+} + (e^{K,+},p^{K,+})| dv \right) \\
& \leq K^m H^2 \sup_v \left( \int_{\mathbb{R}^3} \frac{C^2_{\mu}}{1 + |\langle v \rangle + \phi^{K,0}(x)|^\delta} dv \right) \\
& \leq 2^\delta H^2 C^2_\mu \cdot 2K^m \int_{\mathbb{R}^3} \frac{1}{1 + \langle v \rangle^\delta} dv \\
& \leq 2^\delta H^2 C^2_\mu K^m.
\end{align}

Here $C_2 = \frac{8}{3}\pi + \frac{4}{\delta-1}\pi^2$. Therefore $III = O(K^m)$.

Combining together all the estimates, as well as Lemma 5.8, we conclude

\begin{align}
(\mathcal{L}^{0,K,h},h)_{L^2} & \leq 1 - H_1 C_1 C_2 C^\prime_\mu K^{1-m-\epsilon} + 120 \cdot 2^\delta \pi^2 b^2 (H_1 + H_2) C_2^2 K^{2m} + 2^\delta H_2 C_2 C_\mu K^m \\
& + 256\pi^2 c_p C^2_\mu H^2 K^{2m}.
\end{align}

Hence $(\mu^{K,\pm},\phi^{K,0},A^{K,0}_\varphi)$ is spectrally unstable if $C_\mu$, $K$ and $h$ satisfies (5.15). \hfill \square

6. Special cases

6.1. Special case 1. Let us take a special case of Theorem 3.2. We take some $\mu^\pm(e,p), \mu^0(e,p)$ which are non-negative $C^1$ functions that satisfy

\begin{align}
|\mu^0(e,p)| + |\mu^0_p(e,p)| + |\mu^0_e(e,p)| & \leq \frac{C^2_\mu}{1 + |e|^\delta}, \quad \delta > 3, \\
|\mu^\pm(e,p)| + |\mu^p\pm(e,p)| + |\mu^e\pm(e,p)| & \leq \frac{C^2_\mu}{1 + |e|^\delta}, \quad \delta > 3,
\end{align}

and that $\mu^0(e,p)$ is even in $p$. Let $\gamma \geq 0$ be a temporarily fixed parameter. For any $K \in [0, +\infty)$, consider the equilibrium $\mu^{K,\pm}$ given by

\begin{align}
\mu^{K,+}(e,p) & = \gamma \mu^0(e,p) + a^+(K) \mu^+(e,p), \\
\mu^{K,-}(e,p) & = \gamma \mu^0(e,p) + a^-(K) \mu^-(e,p).
\end{align}

Here $a^\pm(K)$ are nonnegative $C^1$ functions of $K$ which satisfies $a^+(0) = a^-(0) = 0$, and there exists no $K \neq 0$ such that $a^+(K) = a^-(K) = 0$. Moreover, we assume that $(a^+)\gamma'(0) = (a^-)\gamma'(0) = 0$. It is obvious that for each $\gamma \in \mathbb{R}$, $(\phi^{K,0},A^{K,0}_\varphi,K) = (0,0,0)$ is a solution to (3.4) – (3.5).

**Theorem 6.1.** Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2. Let $\mu^{K,\pm}$ satisfy (6.2) with (6.1), and that the conditions above for $a^\pm(K)$ hold. There exists a discrete subset $S_\gamma$ of $[0, +\infty)$, such that $0 \in S_\gamma$, and for all $\gamma \in [0, +\infty) \setminus S_\gamma$,
the following holds:

There exists an unbounded continuous solution set or loop \( C := \{ (A_{\varphi}^{K,0}, \phi^{K,0}, K) \} \subset H^2(\Omega) \times H^2(\Omega) \times \mathbb{R} \) to the system (3.4) – (3.5) (which includes the parameter \( \gamma \)) with \( e^{K,\pm} = \langle v \rangle \pm \phi^{K,0}(x) \), \( p^{K,\pm} = r(v_\varphi \pm A_{\varphi}^{K,0}(x)) \), in which the solution \((0,0,0)\) is included.

Moreover, consider the (strict) single species case (either \( \mu^{0,+} = \mu^{K,-} \equiv 0 \), with \( \mu^{K,+} > 0 \) for all \( K > 0 \), or \( \mu^{0,-} = \mu^{K,+} \equiv 0 \), with \( \mu^{K,-} > 0 \) for all \( K > 0 \)), and let \( d = \sup_{x \in \Omega} r(x) < +\infty \), \( a^\pm(K) \) are analytic functions, \( \mu^0 \) and \( \mu^\pm \) satisfy

\[ |\partial_k^l \mu^0(e,p)| \leq \frac{C}{1 + |e|^\delta} k! l!, \]

(6.3)

\[ |\partial_k^l \mu^\pm(e,p)| \leq \frac{C}{1 + |e|^\delta} k! l! \]

for all \( k, l \in \mathbb{Z}_{\geq 0} \) and some \( C > 0 \), \( \delta > 3 \), then (3.21) holds, and hence the solution set \( C \) contains a locally analytic curve \( \tilde{C} \) parametrized by \( s \). Moreover, \( \tilde{C} \) is unbounded in the sense that there exists a sequence \( s_j \to +\infty \), such that (3.22) holds.

**Proof.** By Theorem 3.2, we only need to show that there exists a discrete subset \( S_\gamma \) of \([0, +\infty)\), such that for all \( \gamma \in [0, +\infty) \setminus S_\gamma \), \( \mu^{K,\pm} \) satisfy Assumption 3.1. It suffices to verify 4) in Assumption 3.1 since all others are straightforward (in particular, \( (a^+)'(0) = (a^-)'(0) = 0 \) implies \( \partial \mu^{K,\pm}(e,p)|_{K=0} = 0 \)). For this purpose, we use the real value version of Lemma 2.6 to analyze the matrix operator in 4) in Assumption 3.1.

Notice that the matrix operator in 4) in Assumption 3.1 now can be written as

(6.4)

\[
\begin{pmatrix}
\tilde{J}_{11} & \tilde{J}_{12} \\
\tilde{J}_{21} & \tilde{J}_{22}
\end{pmatrix} : \mathcal{X} \times \mathcal{X} \times \mathbb{R} \to H^2(\Omega) \times H^2(\Omega) \times \mathbb{R}
\]

where

\[
\tilde{J}_{11}(\delta u) = \delta u - \gamma(-\Delta + 1/r^2)^{-1}\{f_1(r)\delta u\},
\]

(6.5)

\[
\tilde{J}_{12}(\delta w) = -\gamma(-\Delta + 1/r^2)^{-1}\{f_2(r)\delta w\},
\]

\[
\tilde{J}_{21}(\delta u) = \gamma \Delta^{-1}\{f_3(r)\delta u\},
\]

\[
\tilde{J}_{22}(\delta w) = \delta w + \gamma \Delta^{-1}\{f_4(r)\delta w\},
\]

and

\[
f_1(r) := 2 \int_{\mathbb{R}^3} r \hat{v}_\varphi \mu^0_e(\langle v \rangle, rv_\varphi) dv, \quad f_2(r) := 2 \int_{\mathbb{R}^3} \hat{v}_\varphi \mu^0_e(\langle v \rangle, rv_\varphi) dv,
\]

\[
f_3(r) := 2 \int_{\mathbb{R}^3} r \mu^0_e(\langle v \rangle, rv_\varphi) dv, \quad f_4(r) := 2 \int_{\mathbb{R}^3} \mu^0_e(\langle v \rangle, rv_\varphi) dv.
\]

This matrix operator is bounded with a bounded inverse when \( \gamma = 0 \). We apply Lemma 2.6 to this matrix operator with the varying parameter \( \gamma \) playing the role of
z in the lemma. We obtain that there exists a discrete set $S_\gamma \subset [0, +\infty)$ such that for all $\gamma \in [0, +\infty) \setminus S_\gamma$, this matrix operator is bounded with a bounded inverse. This completes the proof of the first part of the theorem.

Moreover, if $d = \sup_{x \in \Omega} r(x) < +\infty$, $a^\pm(K)$ are analytic functions, $\mu^0$ and $\mu^\pm$ satisfy (6.3) for all $k, l \in \mathbb{Z}_{\geq 0}$ and some $C > 0$, then

\[
(6.6) \quad \int_{\mathbb{R}^3} |\partial_1^k \partial_2^l \mu^0(\langle v \rangle + a, r(v_\varphi + b))| dv \leq \int_{\mathbb{R}^3} \frac{C}{1 + |\langle v \rangle + a|^\delta + |r(v_\varphi + b)|^\delta} dv \leq C_1^{k+l} k! l! ,
\]

\[
|\int_{\mathbb{R}^3} \hat{\partial}_\varphi \partial_1^k \partial_2^l \mu^0(\langle v \rangle + a, r(v_\varphi + b))| dv \leq \int_{\mathbb{R}^3} \frac{C}{1 + |\langle v \rangle + a|^\delta + |r(v_\varphi + b)|^\delta} dv \leq C_1^{k+l} k! l! ,
\]

\[
|\int_{\mathbb{R}^3} \partial_1^k \partial_2^l \mu^\pm(\langle v \rangle + a, r(v_\varphi + b))| dv \leq \int_{\mathbb{R}^3} \frac{C}{1 + |\langle v \rangle + a|^\delta + |r(v_\varphi + b)|^\delta} dv \leq C_2^{k+l} k! l! ,
\]

\[
|\int_{\mathbb{R}^3} \hat{\partial}_\varphi \partial_1^k \partial_2^l \mu^\pm(\langle v \rangle + a, r(v_\varphi + b))| dv \leq \int_{\mathbb{R}^3} \frac{C}{1 + |\langle v \rangle + a|^\delta + |r(v_\varphi + b)|^\delta} dv \leq C_2^{k+l} k! l! ,
\]

hold for all $k, l \in \mathbb{Z}_{\geq 0}$, $a, b \in \mathbb{R}$, $C_1(a, b) > 0$, $C_2(a, b) > 0$. Taking $C(a, b, K) = \gamma C_1(a, b) + (a_+(K) + a_-(K))C_2(a, b)$, then (3.21) holds. Therefore the solution set $\mathcal{C}$ contains a locally analytic curve by Theorem 3.3. The rest of the proof is the same as the argument in the proof of Theorem 3.3.

\[
\square
\]

Let us consider the single species case and assume furthermore

\[
(6.7) \quad a^+(K) \to +\infty \text{ as } K \to \infty, \quad a^-(K) \equiv 0,
\]

and

\[
(6.8) \quad \mu^+(e, p) \geq C_{\mu^+} (1 + |e|^{\alpha} + |p|^{\alpha})^{-1}, \quad \mu^0(e, p) \equiv 0
\]

for all $(e, p) \in \mathbb{R}^2$ and some constants $\alpha > 3, C_{\mu^+} > 0$.

Using Proposition 4.1, we derive

**Proposition 6.2.** Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2. Then with the assumptions (6.7) and (6.8), the set $\mathcal{C}$ obtained in Theorem 6.1 is unbounded. Moreover, there holds:

\[
(6.9) \quad \sup_{\{(A_{K,0}^L, A_{\varphi,0}^L) \in \mathcal{C}\}} K = +\infty,
\]

and

\[
(6.10) \quad \text{As } K \to \infty, \quad \|((\phi_{K,0}^L, A_{\varphi,0}^L))_{L^\infty} \to +\infty.
\]

**Proof.** Applying Theorem 6.1, we conclude that the set $\mathcal{C}$ obtained in Theorem 6.1 is unbounded. It then suffices to verify the conditions in Proposition 4.1 and the only
non-trivial one is (4.1). Indeed, by Lemma 4.5, we have
\[
\int_{\mathbb{R}^3} [\mu^{K,+}(v) + a, r(v_\varphi + b)] - [\mu^{K,-}(v) - a, r(v_\varphi - b)] dv
\]
(6.11)
\[
= \int_{\mathbb{R}^3} (a^+(K)\mu^+)(v) + a, r(v_\varphi + b)) \geq 2^{-\delta}a^+(K)C_{\text{int},2}C_{\mu^+}(1 + |a|^\alpha + d^n|b|^\alpha)^{-1}
\]
\[
=: N(a, b, K).
\]
Then when $K$ is large enough, $N(a, b, K)$ is positive and monotonically decreasing in $a$ and $b$. Moreover, for each $(e, p) \in \mathbb{R}^2$,
(6.12)
\[
N(e, p, K) \to +\infty \text{ when } K \to +\infty.
\]
(4.1) is verified. Therefore we can apply Proposition 4.1 and obtain the desired results. \qed

**Remark 6.3.** We can switch the role of ions and electrons (+ and −), and the same results in this subsection hold.

6.2. **Special case 2.** Let us consider another another special case of Theorem 3.2 other than the one in Section 6.1 (Theorem 6.1). We take some $\mu^{\pm}(e, p)$, $\mu^{0,\pm}(e, p)$ which are non-negative $C^1$ functions that satisfy (6.1), and $\mu^{0}(e, p)$ is even in $p$. Let $\gamma \geq 0$ be a temporarily fixed parameter. For any $K \in [0, +\infty)$, consider the equilibrium $\mu^{K,\pm}$ given by
(6.13)
\[
\mu^{K,+}(e, p) = \gamma \mu^0(e, Kp) + a^+(K)\mu^+(e, Kp),
\]
\[
\mu^{K,-}(e, p) = \gamma \mu^0(e, Kp) + a^-(K)\mu^-(e, Kp).
\]
Here $a^{\pm}(K)$ are nonnegative functions of $K$ which satisfies $a^+(0) = a^-(0) = 0$, and there exists no $K \neq 0$ such that $a^+(K) = a^-(K) = 0$. Moreover, we assume that $(a^+)'(0) + a^+(0)pp^+(e, 0) = (-a^-)'(0) - a^-(0)pp^-(e, 0) = 0$. It is obvious that for each $\gamma \in \mathbb{R}$, $(\phi^{K,0}, A^{K,0}, K) = (0, 0, 0)$ is a solution to (3.1) – (3.5).

**Theorem 6.4.** Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2. Let $\mu^{K,\pm}$ satisfy (6.13) with (6.1), and that the conditions above for $a^+(K)$ hold. There exists a discrete subset $S_\gamma$ of $[0, +\infty)$, such that $0 \in S_\gamma$, and for all $\gamma \in [0, +\infty) \setminus S_\gamma$, the following holds:

There exists an unbounded continuous solution set or loop $\mathcal{C} := \{(A^{K,0}_\varphi, \phi^{K,0}, K)\} \subset H^2(\Omega) \times H^2(\Omega) \times \mathbb{R}$ to the system (3.4) – (3.5) (which includes the parameter $\gamma$) with $e^{K,\pm} = (v) + \phi^{K,0}(x)$, $p^{K,\pm} = r(v_\varphi + A^{K,0}_\varphi(x))$, in which the solution $(0, 0, 0)$ is included. Moreover, consider the single species case ($\gamma \mu^0(e, p) \equiv 0$, together with $a^{-}(K)\mu^{-}(e, p) \equiv 0$ or $a^{+}(K)\mu^{+}(e, p) \equiv 0$ and let $d = \sup_{x \in \Omega} r(x) < +\infty$, $a^{\pm}(K)$ are analytic functions, $\mu^0$ and $\mu^{\pm}$ satisfy (6.3) for all $k, l \in \mathbb{Z}_{\geq 0}$ and some $C > 0$, then (3.21) holds,
and hence the solution set $\mathcal{C}$ contains a locally analytic curve $\tilde{\mathcal{C}}$. Moreover, $\tilde{\mathcal{C}}$ is unbounded in the sense that (3.22) holds.

**Proof.** By Theorem 3.2, we only need to show that there exists a discrete subset $S_\gamma$ of $[0, +\infty)$, such that for all $\gamma \in [0, +\infty) \setminus S_\gamma$, $\mu^{K, \pm}$ satisfy Assumption 3.1. It suffices to verify 4) in Assumption 3.1 since all others are straightforward (in particular, $(a^+)'(0)\mu^+(e, 0) + a^+(0)\mu^-_p(e, 0) = (a^-)'(0)\mu^-(e, 0) + a^-(0)\mu^-_p(e, 0) = 0$ implies $\frac{\partial \mu^{K, \pm}(e, p)}{\partial K}{|}_{K=0} = 0$).

Notice that the matrix operator in 4) in Assumption 3.1 now can be written as

$$
\begin{pmatrix}
\tilde{J}_{11} & \tilde{J}_{12} \\
\tilde{J}_{21} & \tilde{J}_{22}
\end{pmatrix} : \mathcal{X} \times \mathcal{X} \times \mathbb{R} \to H^2(\Omega) \times H^2(\Omega) \times \mathbb{R}
$$

where

$$
\begin{align*}
\tilde{J}_{11}(\delta u) &= \delta u, \\
\tilde{J}_{12}(\delta w) &= -\gamma(-\Delta + \frac{1}{r^2})^{-1}\{f_2(r)\delta w\}, \\
\tilde{J}_{21}(\delta u) &= 0, \\
\tilde{J}_{22}(\delta w) &= \delta w + \gamma\Delta^{-1}\{f_4(r)\delta w\},
\end{align*}
$$

and

$$
\begin{align*}
f_2(r) &= 2\int_{\mathbb{R}^3} \hat{\psi}_\phi \mu_0^0(\langle v \rangle, rv\phi) dv, \\
f_4(r) &= 2\int_{\mathbb{R}^3} \mu_0^0(\langle v \rangle, 0) dv.
\end{align*}
$$

This matrix operator is bounded with a bounded inverse when $\gamma = 0$. We apply Lemma 2.6 to this matrix operator with the varying parameter $\gamma$ playing the role of $z$ in the lemma. We obtain that there exists a discrete set $S_\gamma \subset [0, +\infty)$ such that for all $\gamma \in [0, +\infty) \setminus S_\gamma$, this matrix operator is bounded with a bounded inverse. This completes the proof of the first part of the theorem. The analytic part can be shown in the same way as in Theorem 6.1. □

Let us consider the single species case and assume furthermore

$$
(a^+)(K) \to +\infty \text{ as } K \to \infty, \quad a^-(K) \equiv 0,
$$

and

$$
\mu^+(e, p) \geq C_{\mu^+}(1 + |e|^\alpha + |p|^\alpha)^{-1}, \quad \mu^0(e, p) \equiv 0
$$

for all $(e, p) \in \mathbb{R}^2$ and some constants $\alpha > 3$, $C_{\mu^+} > 0$.

In the same way as in the proof of Proposition 6.2, we obtain, using Proposition 4.1.
Proposition 6.5. Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2. Then with the assumptions (6.16) and (6.17), the set $C$ obtained in Theorem 6.4 is unbounded. Moreover, there holds:

\[(6.18) \sup_{(A^{K,0}_\phi,0)\in C} K = +\infty,\]

and

\[(6.19) \text{as } K \to \infty, \| (\phi^{K,0}_\phi, A^{K,0}_\phi) \|_{L^\infty} \to +\infty.\]

Proof. The proof is very similar to the one for Proposition 6.2 so we omit it. 

Next, we consider the spectral stability of the solutions $\phi^{K,0}_\phi, A^{K,0}_\phi)$. We first have the following observation for small $K$:

Proposition 6.6. Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2. Let $\mu_0^+, \mu_0^- < 0$, then for $K = K_0 = 0$, the equilibrium $(\mu_0^+, \phi^{0,0}_\phi, A^{0,0}_\phi) = (\mu_0^+, \phi^{0,0}_\phi, 0)$ is spectrally stable. Hence by continuity we have, when $K$ is close to 0 (depends on $\gamma, a^\pm, \mu_0^\pm$, $\mu_0^\pm$), the equilibrium $(\mu^{K,0}_K, \phi^{K,0}_\phi, A^{K,0}_\phi)$ is spectrally stable.

Proof. The conclusion directly follows from Proposition 5.2. 

Now, let us assume furthermore the following conditions and discuss the spectral stability of the solutions $(\phi^{K,0}_\phi, A^{K,0}_\phi)$ for large $K$:

Assumption 6.7.

\[\mu_e^+ < 0\]

and

\[a^+(K) \sim K^m\]

for some $m \in (-1, 1)$ when $K \to +\infty$.

Hence we have only ions in the system.

Theorem 6.8. Let $\Omega$ be a $C^1$ axisymmetric bounded domain as stated in Section 2, with $\inf_{x \in \Omega} r(x) > 0$. Let $\mu_K^+$ satisfy (6.13) and the decay assumption (6.1), with $\delta > 3$ replaced by $\delta > 4$. Assume that (6.16), (6.17) and Assumption 6.7 hold. Recall $d := \sup_{x \in \Omega} r(x)$ and assume $d > 1$. Let $\epsilon$ be a positive constant such that $\epsilon < 1 - |m|$, so $\epsilon + m < 1$, $\epsilon < 1 - m$. Let $\mu^+$ be such that

\[(6.20) p \mu^+_p(e, p) \geq C'_\mu |p|\langle p \rangle^{-\epsilon} \nu(e)\]

for some positive function $\nu(e)$, with

\[(6.21) \nu(e) \geq C_\nu \exp(-e)\]

for some constant $C_\nu > 0$. Then if $K$ is large enough, then the equilibrium $(\mu^{K,0}_K(e, p), \mu^{K,-}(e, p) \equiv 0, \phi^{K,0}_\phi, A^{K,0}_\phi)$ is spectrally unstable.
Remark 6.9. The condition $p\mu_p^+ \geq C_\mu'|p\langle p \rangle^{-\gamma}v(e)$ together with that $|\mu_p^+(e,p)| \leq \frac{C_\mu|v|}{1+|e|^{\gamma}}$ (assumed throughout all discussion) implies that $|v(e)| \leq \frac{\langle p \rangle^\gamma}{1+|e|^{\gamma}}$ with $\gamma > 4$, which ensures the integrals in the proof are convergent. (This is where we need $\delta > 4$. Notice that $\epsilon \leq 1$.)

Proof. It is easy to verify that the assumptions in Theorem 5.4, that is, Assumption 5.3 hold. In particular, $p\mu_p^+(e,p) \geq C_\mu' |p\langle p \rangle^{-\gamma}v(e)$ together with that $\mu^{K,+}(e^{K,+}, p^{K,+}) = \gamma \mu^0(e^{K,+}, Kp^{K,+}) + a^+(K)\mu^0(e^{K,+}, Kp^{K,+}) \sim K^\delta \mu^0(e^{K,+}, Kp^{K,+})$ for large $K$ imply $p\mu_p^{K,+}(e,p) \geq C_\mu'K^{1+m-\epsilon}|p\langle p \rangle^{-\gamma}v(e)$ for large $K$. Therefore we can directly apply Theorem 5.4 and obtain Theorem 6.8. \hfill \Box

Remark 6.10. We can switch the role of ions and electrons (+ and −), and the same results in this subsection hold.

Appendix A. Derivative Operators in the Cylindrical Coordinates

For the readers’ convenience, we provide the information of the derivatives under the cylindrical coordinates. Using $(x_1, x_2, x_3)$ to denote the Cartesian coordinates and $(r, \varphi, z)$ to be the cylindrical coordinates, we have

$$ x_1 = r \cos \varphi, \quad x_2 = r \sin \varphi, \quad x_3 = z $$

and therefore

$$ e_r = (\cos \varphi, \sin \varphi, 0), \quad e_\varphi = (-\sin \varphi, \cos \varphi, 0), \quad e_z = (0, 0, 1). $$

Hence for any function $f(r, \varphi, z)$ and any vector field $A(r, \varphi, z)$, we have

$$ \nabla_x f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \varphi} e_\varphi + \frac{\partial f}{\partial z} e_z, $$

$$ \Delta_x f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, $$

$$ \nabla_x \cdot A = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}, $$

$$ \nabla_x \times A = \left( \frac{1}{r} \frac{\partial A_\varphi}{\partial z} - \frac{\partial A_z}{\partial \varphi} \right) e_r + \left( \frac{\partial A_z}{\partial r} - \frac{\partial A_r}{\partial z} \right) e_\varphi + \frac{1}{r} \left( \frac{\partial (r A_\varphi)}{\partial r} - \frac{\partial A_r}{\partial \varphi} \right) e_z. $$

Appendix B. Key Invariants

In this section, we prove the invariance of $e^\pm(x, v) = \langle v \rangle \pm \phi^0(r, z)$ and $p^\pm(x, v) = r(v_x \pm A^0_x(r, z))$ (or $p^\pm(x, v) = r(v_x \pm A^0_x(r, z) \pm A_{\varphi,ext}(r, z))$ when an external magnetic potential $A_{ext} = A_{\varphi,ext}(r, z)e_\varphi$ is present) along the particle trajectories given in Section 2

$$ (B.1) \quad \dot{X}^\pm = \dot{V}^\pm, \quad \dot{V}^\pm = \pm \dot{E}^0(X^\pm) \pm \dot{V}^\pm \times \dot{B}^0(X^\pm). $$
(Or
\[ \dot{X}^\pm = \dot{V}^\pm, \quad \dot{V}^\pm = \pm E^0(X^\pm) \pm \dot{V}^\pm \times B^0(X^\pm) \pm \dot{V}^\pm \times B_{\text{ext}}(X^\pm) \]
when the external magnetic potential is present.)

We only compute the + case for simplicity. The − case is similar. In fact, along the particle trajectories, we have
\[ \dot{e} = \dot{V} \cdot \dot{X} + \dot{X} \cdot \nabla \phi^0 \]
\[ = \dot{V} \cdot (E^0 + \dot{V} \times B^0) + \dot{V} \cdot \nabla \phi^0 \]
\[ = \dot{V} \cdot (\phi^0 + \dot{V} \times B^0) + \dot{V} \cdot \nabla \phi^0 \]
\[ = 0, \]

\[ \dot{p} = \dot{X} \cdot \nabla X p + \dot{V} \cdot \nabla V p \]
\[ = \dot{X} \cdot \nabla X (r v_\varphi + A^0_\varphi (r, z)) + \dot{V} \cdot \nabla V (r (v_\varphi + A^0_\varphi (r, z))) \]
\[ = \dot{X} \cdot \nabla X (r v_\varphi) + \dot{X} \cdot \nabla X r A^0_\varphi + \dot{V} \cdot r \nabla V v_\varphi \]
\[ = \dot{X} \cdot \nabla X (r v_\varphi) + \dot{X} \cdot A^0_\varphi e_r + \dot{X} \cdot r \frac{\partial A^0_\varphi}{\partial z} e_z + \dot{X} \cdot r \frac{\partial A^0_\varphi}{\partial \varphi} e_\varphi \]
\[ + r (-\nabla \phi^0 + \dot{X} \times \frac{\partial A^0_\varphi}{\partial z} e_r + \dot{X} \times \frac{1}{r} A^0_\varphi e_z + \dot{X} \times \frac{\partial A^0_\varphi}{\partial \varphi} e_\varphi) \cdot e_\varphi \]
\[ = \dot{X} \cdot \nabla X (r v_\varphi) + \dot{X} \cdot A^0_\varphi e_r + \dot{X} \cdot r \frac{\partial A^0_\varphi}{\partial z} e_z - r \frac{\partial A^0_\varphi}{\partial z} \dot{X} e_z - A^0_\varphi \dot{X} e_\varphi - r \frac{\partial A^0_\varphi}{\partial \varphi} \dot{X} e_\varphi \]
\[ = \dot{X} \cdot \nabla X (r v_\varphi). \]

(Here \( r, \varphi, z, v_r, v_\varphi, v_z \) are components of \( X \) and \( V \).) In the computation above, we used that \( \nabla \phi^0 \) does not have \( e_\varphi \)-component. We evaluate \( \dot{X} \cdot \nabla X (r v_\varphi) \) using the Cartesian coordinates:
\[ \dot{X} \cdot \nabla X (r v_\varphi) = \dot{X} \cdot \nabla X (r V \cdot (-\frac{X_2}{r}, \frac{X_1}{r}, 0)) \]
\[ = \dot{X} \cdot \nabla X (-X_2 V_1 + X_1 V_2) \]
\[ = \frac{1}{\langle V \rangle} (V_1, V_2, V_3) \cdot (V_2, -V_1, 0). \]
\[ = 0 \]

Hence \( \dot{p} = 0 \). Moreover, one can show in a similar way that \( p(x, v) = r (v_\varphi + A^0_\varphi (r, z) + A_{\varphi, \text{ext}} (r, z)) \) is invariant along the particle trajectories satisfying the ODE \( \dot{X} = \dot{V} \), \( \dot{V} = E^0(X) + \dot{V} \times B^0(X) + \dot{V} \times B_{\text{ext}}(X) \). The invariance along the particle trajectories is now verified.
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Courant Institute of Mathematical Sciences, New York University