Analytical representation for the problem of a loaded hollow layered cylinder taking into account the creep of its layers

T N Bobyleva¹ and A S Shamaev²,³

¹Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, 129337, Russia
²Laboratory of Mechanics of Controlled Systems, Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, 101-1, Pr. Vernadskogo, 119526, Moscow, Russian Federation
³Department of Differential Equations, Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, 119991, Russian Federation

E-mail: tatyana2211@outlook.com

Abstract. The paper deals with a cylindrical hollow rod with a layered structure along the radius. Layers consist of isotropic elastic creeping materials. An analytical solution is given for the problem of the stress-strain state of such rod. Tensile and bending stiffness can be presented in analytical form. An algorithm is proposed for constructing an analytical solution to the problem of various loading of a layered rod consisting of elastic-creeping layers in the presence of radial symmetry.

1. Introduction
Bar systems made of composite materials are widely used in modern construction. Asymptotic analysis of the equations of elasticity theory as they apply to such bar systems is undertaken in [1,2]. The averaging technique for partial differential equations [3-6] is used as major tool in this research. Heterogeneity is the cause of rod specific behavior during deformation. The method of asymptotic averaging turns such a material into a homogeneous one, which is described by averaged equations. Several examples of its application are given in [7,8]. The problem for thin periodically perforated rod is solved in the article [9]. In paper [10] authors studied periodic elastic rod-structures which are locally anisotropic. In order to find the effective behavior and approximate local behavior of such structures, one has to solve a finite number of boundary-value problems on one period of the rod-structure, namely the cell problem.

Usually the stress-strain state of composite materials changes over long time intervals. That is such materials have creep. This should be taken into account, since creep acts on the redistribution of stresses and in some cases can lead to an unacceptable increase in deformation. Therefore the problem of studying rheological properties is of current interest for construction practice. In [11-14] the foundations of the theory of the hereditary Boltzmann-Volterra mechanics are presented. This theory describes processes in which the state of a mechanical system depends on the entire history of actions performed on it. Averaged models describing the joint motion of the layers consisting of elastic and viscoelastic materials [15], and also of two creeping materials [16], were constructed. Examples of
approximate calculation of the stress tensor in layered elastic-creeping environments are presented in [17-21].

In this paper, we use the methods and results of the above studies to analyze stress-strain states of a rod composed of layers of elastic-creeping materials. The presence of radial symmetry and the assumption of creep kernels allow one to obtain analytical representations for the corresponding boundary value problems of the elastic-creeping model.

2. Problem specification and decision
A solution to the problem of a heterogeneous elastic rod with a periodic structure loaded at one end and fixed at the other end is proposed in [9]. An expression is obtained which, with high accuracy, represents the stress-strain state of the rod outside the vicinity of the fixing zone and the zone of application of external forces and moments. The thickness of the rod is assumed to be small, it is determined by a small parameter \( \varepsilon \). The representation for solving the problem of the stress-strain state of a bar given in [9] contains the matrix of functions \( N_1, N_2, N_3 \) and the vector function \( R \), for which there is no explicit analytical representation. Boundary-value problems for determining these functions are solved only numerically in the general case. This representation [9] with high (exponential) accuracy approximates the exact solution. The difference from the exact solution can be represented as boundary layers near the ends with the next estimates \( k_1 e^{-c_1 x}, k_2 e^{-c_2(L-x_1)} \), where \( k_1, k_2, c_1, c_2 \) are constants, \( L \) is rod length \( x_1 \) is variable along the axis of the rod.

In this paper, we consider a model of a composite consisting of elastic creeping materials. In this case, the stress-strain state of the rod will be determined by solving a system of integro-differential equations with corresponding boundary conditions. Compared with the formulation for the elastic model, the elastic-creeping model will contain an integral term of the convolution type with a kernel representing the sum of a finite number of exponential functions that decrease with increasing time. The simplest case is considered when the creep kernels contain only one exponential function, and for each of the phases, the kernel responsible for the shear creep effects is proportional to the kernel responsible for the bulk creep effects. In this case, the model contains only two exponents for each material, respectively, if we consider a composite consisting of two phases. The exact mathematical formulation of the boundary value problem will be given below. Here we note an important circumstance for us that after the Laplace transform with respect to the time variable, our problem will take the form similar to the problem considered in [9]. The coefficients of the resulting system will depend on the complex variable \( p \). After obtaining the analytical representation (if there is one), it is necessary to carry out the inverse Laplace transform. The case of the dependence of creep kernels on time in the form of the sum of exponential functions leads after applying the Laplace transform to the fractional rational functions of the complex variable \( p \). We show below that the analytical representation of the solution for such a problem will contain a dependence on the variable \( p \) in the form of a fractional rational function, the coefficients of which can be easily obtained using some algebraic transformations. Then to return to the original spatial-time variables, you need to perform the inverse Laplace transform from the complex variable \( p \) to the variable \( t \). For fractional rational functions, the inverse transformation can also be carried out analytically by decomposing fractional rational dependencies into simple fractions. We implement this procedure below for finding an analytical solution within the framework of an elastic-creeping model.

In the general case, as already noted, we cannot give an explicit analytical representation for the indicated auxiliary functions already within the framework of a purely elastic model. If, however, we additionally assume that the elastic characteristics of the cylinder depend only on the radius, then we can construct analytical representations for the function matrices \( N_1, N_2, N_3 \). This is done in this work. If we additionally assume that the torque \( M_1 \) applied to the end face of the rod is equal to zero, then the representation given in [9] will not contain the vector function \( R \), but only \( N_1, N_2, N_3 \). Therefore it will be possible to give an explicit analytical representation for the stress-strain state of the rod outside
the boundary zones in the vicinity of the ends which is performed with high (exponential in $\varepsilon$!) accuracy.

We present the formulation of the problem in the framework of an elastic-creeping model similar to [9], where the formulation of the problem is given for the elastic model. Let $B$ be a 1-periodic in the direction $x_i$, open, connected set in $\mathbb{R}^3$ with a smooth boundary. We assume that the origin belongs to B and that $|x_2| + |x_3| \leq C$ for all $x \in B$, here $x = (x_1, x_2, x_3)$. For a small $\varepsilon > 0$ let it be $B_\varepsilon = \{ x \in \mathbb{R}^3 : 0 < x_i < 1, \frac{x_i}{\varepsilon} \in B \}$. We suppose that $B_\varepsilon$ is a connected set. This set is the perforated rod.

Below we denote by $\partial B_\varepsilon$ the boundary of the set $B_\varepsilon$ and introduce the following sets:

$S^e_\varepsilon = \{ x \in \partial B_\varepsilon : x_i = 0 \}, S^i_\varepsilon = \{ x \in \partial B_\varepsilon : x_i = 1 \}, \Sigma_\varepsilon = \{ x \in \partial B_\varepsilon : 0 < x_i < 1 \}$. In the set $B_\varepsilon$ we consider the boundary value problem for the elasticity system [9], namely equations of instant equilibrium in $B_\varepsilon$

$$\frac{\partial}{\partial x_i} \left(A^I_{ij} \left(\frac{x}{\varepsilon}, t\right) \delta(t) + A^I_{ij} \left(\frac{x}{\varepsilon}, t\right)\right) * \frac{\partial}{\partial x_j} u_{ei} = 0$$ (1)

and the next boundary conditions: the side surface is free of loads, $S^e_\varepsilon$ is rigidly fixed, on $S^i_\varepsilon$ the load is set as follows

$$n_i \left(A^I_{ij} \left(\frac{x}{\varepsilon}, t\right) \delta(t) + A^I_{ij} \left(\frac{x}{\varepsilon}, t\right)\right) * \frac{\partial}{\partial x_j} u_{ei} \bigg|_{B^\varepsilon} = F \left(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right)$$ (2)

where $A^I_{ij} \left(\frac{x}{\varepsilon}, t\right)$ is an elasticity tensor with components $a^I_{ijkl} , 1 \leq i, j, k, l \leq 3$, which satisfies the standard symmetry and uniform ellipticity conditions, $A^I_{ij} \left(\frac{x}{\varepsilon}, t\right)$ is the creep tensor with components $a^I_{ijkl} , 1 \leq i, j, k, l \leq 3$, depending on $t$, namely $a^I_{ijkl} = d^I_{ijkl} e^{-\alpha^I_{ijkl} t}$, for each of the materials that make up the composite, $n(x) = (n_1(x), n_2(x), n_3(x))^T$ is the exterior normal on $\Sigma_\varepsilon$ ($T$ is transpose operation), and $u_{ei} = (u_{e1}, u_{e2}, u_{e3})^T$ is the vector of the displacement, $\delta(t)$ is Dirac-delta function, operation $A^I_{ij} \left(\frac{x}{\varepsilon}, t\right) \delta(t)$ is equivalent to multiplication by $A^I_{ij} \left(\frac{x}{\varepsilon}, t\right)$, $F \left(\frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right)$ is the field of external forces applied to the end face of the rod. The symbol * denotes the convolution operator:

$$\left( d_{ijkl}(x, t) * \frac{\partial u_{ij}}{\partial x_k} \right) = \int_0^t d_{ijkl}(x, t - \tau) \frac{\partial u_{ij}(x, \tau)}{\partial x_k} d\tau .$$ (3)

Throughout this work we assume that the tensors $A^I_{ij}(\xi), A^I_{ij}(\xi, t), A^I_{ij}(\xi, t)\,$ extended to the complement of $B$ by 0, is 1-periodic in the variable $\xi$, and possesses the following symmetry conditions

$$a^I_{ijkl}(S_1 \xi, t) = (-1)^{\delta_{1i} + \delta_{2j} + \delta_{3k} + \delta_{4l}} a^I_{ijkl}(\xi, t), a^I_{ijkl}(S_2 \xi, t) = (-1)^{\delta_{2i} + \delta_{3j} + \delta_{4k} + \delta_{1l}} a^I_{ijkl}(\xi, t), a^I_{ijkl}(S_3 \xi, t) = (-1)^{\delta_{3i} + \delta_{4j} + \delta_{1k} + \delta_{2l}} a^I_{ijkl}(\xi, t), a^I_{ijkl}(S_4 \xi, t) = (-1)^{\delta_{4i} + \delta_{1j} + \delta_{2k} + \delta_{3l}} a^I_{ijkl}(\xi, t),$$

where $S_1 \xi = (-\xi_1, -\xi_2, \xi_3)^T, S_2 \xi = (-\xi_1, \xi_2, -\xi_3)^T, S_3 \xi = (\xi_1, -\xi_2, -\xi_3)^T; \xi = (\xi_1, \xi_2, \xi_3) \equiv (\xi_1, \xi_2, \xi_3)$ is the variable inside area $B$. Similar symmetry relations must be satisfied for the components of the
tensor $a_{0,kl}^0$. The set $B$ is also invariant with respect to $S_2, S_3$. After applying the Laplace transform with respect to the variable $t$, problem (1) takes the form considered in [9]. All coefficients of the operators in this problem become dependent on the complex variable $p$ as a parameter. The latter circumstance enables us to apply the results of [9] for each fixed value of the complex parameter $p$, and problem will have a unique solution.

Further using the results [9], we describe this problem solution for $\varepsilon \to 0$. Then we apply the inverse Laplace transform to the obtained asymptotic, which will allow us to get the solution of the elastic-creeping problem with increasing time and for various types of loading at the end.

We begin by defining the resultant force and moment for the force $F\left(\frac{x_x}{\varepsilon}, \frac{x_y}{\varepsilon}\right)$ acting on the right base $S^e_1$ of the rod $B_e$. Namely, we set $\vec{W} = (\Phi_1, \Phi_2, \Phi_3, M_1, M_2, M_3)^T$ with

$$\Phi_i = \int_{S^e_i} F_i\left(\frac{x_x}{\varepsilon}, \frac{x_y}{\varepsilon}\right) dx_x dx_y, \quad i = 1, 2, 3, \quad M_i = \int_{S^e_i} \left(\frac{x_x}{\varepsilon} F_i\left(\frac{x_x}{\varepsilon}, \frac{x_y}{\varepsilon}\right) - \frac{x_y}{\varepsilon} F_i\left(\frac{x_x}{\varepsilon}, \frac{x_y}{\varepsilon}\right)\right) dx_x dx_y, \quad \text{(4)}$$

$$M_2 = \int_{S^e_2} x_x F_i\left(\frac{x_x}{\varepsilon}, \frac{x_y}{\varepsilon}\right) dx_x dx_y, \quad M_3 = \int_{S^e_3} x_y F_i\left(\frac{x_x}{\varepsilon}, \frac{x_y}{\varepsilon}\right) dx_x dx_y. \quad \text{(5)}$$

Now we suppose $M_1 = 0$, and the geometric shape and distribution of the elastic properties of the material in the periodicity cell depend only on the radius of the rod. Suppose also a rod is a hollow cylinder that consists of two (or more) layers of homogeneous isotropic elastic creeping materials. Then elements of matrices $A^0, A^r$ will be determined by Lame constants $\lambda, \mu$ and creep parameters $D_k(t), D_{ak}(t)$

$$a_{0,kl}^0 = \lambda \delta_i^0 \delta_{kh} + \mu(\delta_i^0 \delta_{jh} + \delta_i^0 \delta_{kh}, \quad d_{ijklh} = -(D_r(t) - \frac{1}{3} D_{sh}(t)) \delta_i^0 \delta_{kh} - \frac{1}{2} D_{sh}(t)(\delta_i^0 \delta_{jh} + \delta_i^0 \delta_{kh}). \quad \text{(6)}$$

Let us assume that the amplitude of the bulk creeping kernel $D_k(t)$ is proportional to the shear amplitude $D_{sh}(t)$ with a coefficient of proportionality $k_i$ for each layer that is: $(D_r)_i = k_i (D_{sh})$, $k_i = \text{const.}, \quad k > 0, (i = 1, 2)$. Further, $D_{ak}$ is denoted by $D$, and as we indicated above $D = de^{-\alpha t}$ for each layer also.

We introduce the notation $L = \lambda + 2\mu - \frac{d(k + 2)}{3p + \alpha}, \quad M = \mu - \frac{1}{2(p+\alpha)} d, \quad N = \lambda - \frac{d(k - 1)}{3p + \alpha}$.

After applying the Laplace transform, the matrices $\tilde{A}^r$ in a cylindrical coordinate system $(r, \theta, z)$ will have the following form

$$\tilde{A}^{11}(\frac{r}{\varepsilon}, p) = \begin{pmatrix} Lr^{-1} & 0 & 0 \\ 0 & Mr^{-1} & 0 \\ 0 & 0 & Nr^{-1} \end{pmatrix}, \quad \tilde{A}^{12}(\frac{r}{\varepsilon}, p) = \begin{pmatrix} 0 & Lr^{-2} & 0 \\ -Mr^{-2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{A}^{13}(\frac{r}{\varepsilon}, p) = \begin{pmatrix} 0 & 0 & N \\ 0 & 0 & 0 \\ M & 0 & 0 \end{pmatrix}$$

$$\tilde{A}^{22}(\frac{r}{\varepsilon}, p) = \begin{pmatrix} Mr^{-3} & 0 & 0 \\ 0 & Lr^{-3} & 0 \\ 0 & 0 & Mr^{-3} \end{pmatrix}, \quad \tilde{A}^{23}(\frac{r}{\varepsilon}, p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Lr^{-1} \\ 0 & Mr^{-1} & 0 \end{pmatrix}, \quad \tilde{A}^{33}(\frac{r}{\varepsilon}, p) = \begin{pmatrix} Mr^{-1} & 0 & 0 \\ 0 & Mr^{-1} & 0 \\ 0 & 0 & rL \end{pmatrix}.$$
\[ \tilde{A}^{12}(r, p) = \left( \tilde{A}^{12}(r, p) \right)^T, \tilde{A}^{13}(r, p) = \left( \tilde{A}^{13}(r, p) \right)^T, \tilde{A}^{23}(r, p) = \left( \tilde{A}^{23}(r, p) \right)^T. \] (7)

We have designated in these formulas \( \lambda = \lambda \left( \frac{r}{e} \right), \mu = \mu \left( \frac{r}{e} \right), d = d \left( \frac{r}{e} \right), \alpha = \alpha \left( \frac{r}{e} \right). \)

Further we define auxiliary function matrices \( \tilde{N}_1, \tilde{N}_2, \tilde{N}_3 \) as periodic in \( \xi \) solving the following boundary value problems in a region \( Q = \{ \xi \in B : 0 < \xi_1 < 1 \} \) with boundary conditions on the boundary \( \Sigma = \{ \xi \in \partial B : 0 < \xi_1 < 1 \} \) [9]:

\[
\frac{\partial}{\partial \xi_i} \left( \tilde{A}^i(\xi, p) \frac{\partial \tilde{N}_i}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} \tilde{A}^i(\xi, p), \quad \xi \in Q, \quad n_i \tilde{A}^i(\xi, p) \frac{\partial \tilde{N}_i}{\partial \xi_j} = -n_i \tilde{A}^{11}, \quad i = 1, 2, 3. \] (8)

\[
\frac{\partial}{\partial \xi_i} \left( \tilde{A}^i(\xi, p) \frac{\partial \tilde{N}_2}{\partial \xi_j} \right) = -\frac{\partial}{\partial \xi_i} \tilde{A}^i(\xi, p) \tilde{N}_1(\xi, p) + \tilde{A}^i(\xi, p) \frac{\partial \tilde{N}_2}{\partial \xi_j} + \tilde{A}^{11}(\xi, p) \tilde{H}_2, \quad \xi \in Q, \] (9)

\[
\frac{\partial}{\partial \xi_i} \left( \tilde{A}^i(\xi, p) \frac{\partial \tilde{N}_3}{\partial \xi_j} \right) = n_i \tilde{A}^{11} \tilde{N}_1, \] (10)

Here, matrices \( \tilde{H}_2, \tilde{H}_3 \) depending on the complex parameter \( p \) are selected from the solvability condition for problems (9) and (10). The equation for \( \tilde{N}_1 \) should contain \( \tilde{H}_1 \), but calculations show \( \tilde{H}_1 = 0 \).

Next, we determine according to the results of [9] the function \( v \left( x, \frac{x}{e}, p \right) \), which describes the stress-strain state of the cylinder

\[ v \left( x, \frac{x}{e}, p \right) = \frac{c_3 x}{v_1 x_1 + c_4 x_1} + \frac{c_3}{v_2 x_1 + c_4 x_1} + \frac{x}{v_2 x_1 + c_4 x_1} + \frac{x}{2v_2 + 6c_4 x_1} + \frac{x}{6c_3}. \]
\[ \alpha_{11} = \int_{\xi_1} L_{11} \left( 0, \xi \right) d\xi, \quad \alpha_{22} = 6\varepsilon \int_{\xi_1} P_{22} \left( 0, \xi \right) d\xi + 6\varepsilon^2 \int_{\xi_1} S_{22} \left( 0, \xi \right) d\xi, \quad \alpha_{24} = 2\varepsilon \int_{\xi_1} P_{24} \left( 0, \xi \right) d\xi, \]

\[ \alpha_{33} = 6\varepsilon \int_{\xi_1} P_{33} \left( 0, \xi \right) d\xi + 6\varepsilon^2 \int_{\xi_1} S_{33} \left( 0, \xi \right) d\xi, \quad \alpha_{35} = 2\varepsilon \int_{\xi_1} P_{35} \left( 0, \xi \right) d\xi, \]

\[ \alpha_{55} = 2\varepsilon \int_{\xi_1} \left( P_{13} \left( 0, \xi \right) \xi_3 - P_{33} \left( 0, \xi \right) \xi_3 \right) d\xi, \]

\[ \alpha_{62} = 6\varepsilon \int_{\xi_1} \left( P_{12} \left( 0, \xi \right) \xi_2 - P_{22} \left( 0, \xi \right) \xi_2 \right) d\xi + 6\varepsilon^2 \int_{\xi_1} \left( S_{12} \left( 0, \xi \right) \xi_2 - S_{22} \left( 0, \xi \right) \xi_2 \right) d\xi, \]

\[ L_{ij} = \begin{cases} A_k \left( \frac{\partial \tilde{N}_1}{\partial \xi_k} + \delta_{ik} \tilde{E} \right) \\ A_k \left( \frac{\partial \tilde{N}_2}{\partial \xi_k} + \delta_{ik} \tilde{N}_1 \right) \\ A_k \left( \frac{\partial \tilde{N}_3}{\partial \xi_k} + \delta_{ik} \tilde{N}_2 \right) \end{cases} \]

\[ P_{ij} = \begin{cases} A_k \left( \frac{\partial \tilde{N}_1}{\partial \xi_k} + \delta_{ik} \tilde{E} \right) \\ A_k \left( \frac{\partial \tilde{N}_2}{\partial \xi_k} + \delta_{ik} \tilde{N}_1 \right) \\ A_k \left( \frac{\partial \tilde{N}_3}{\partial \xi_k} + \delta_{ik} \tilde{N}_2 \right) \end{cases} \]

\[ S_{ij} = \begin{cases} A_k \left( \frac{\partial \tilde{N}_1}{\partial \xi_k} + \delta_{ik} \tilde{E} \right) \\ A_k \left( \frac{\partial \tilde{N}_2}{\partial \xi_k} + \delta_{ik} \tilde{N}_1 \right) \\ A_k \left( \frac{\partial \tilde{N}_3}{\partial \xi_k} + \delta_{ik} \tilde{N}_2 \right) \end{cases} \]

We use our assumption that the Lamé constants and creep parameters depend only on the variable \( r \).

In cylindrical coordinates, the first column of the matrix \( \tilde{N}_1 \) with components \( n_{11}, n_{12}, n_{13} \) satisfies the following equations and boundary conditions (we omit the dependence on the parameter \( p \) for brevity)

\[ \frac{d}{dr} \left( \frac{L}{r} \frac{d\left( r n_{11} \right)}{dr} \right) = -\frac{1}{r} \frac{dL}{dr}, \quad \left[ L \frac{dn_{11}}{dr} + N n_{11} \right]_{r=r_0} = -L \left| \frac{d}{dr} \right|_{r=r_0}, \]  

\[ \frac{d}{dr} \left( \frac{L}{r} \frac{d\left( r n_{21} \right)}{dr} \right) = -\frac{1}{r} \frac{dL}{dr}, \quad \left[ M \frac{dn_{21}}{dr} - M n_{21} \right]_{r=r_0} = -L \left| \frac{d}{dr} \right|_{r=r_0}, \]  

\[ \frac{d}{dr} \left( Mr \frac{dn_{31}}{dr} \right) = -\frac{1}{r} \frac{d\left( N r \right)}{dr}, \quad \left[ M \frac{dn_{31}}{dr} \right]_{r=r_0} = -N \left| \frac{d}{dr} \right|_{r=r_0}. \]

In (12) - (14) \( r_0, R_0 \) are the inner and outer radii of the hollow cylindrical rod. Similar equations and boundary conditions hold for the other two columns of the matrix \( \tilde{N}_1 \) with components \( (n_{12}, n_{22}, n_{32}) \), \( (n_{13}, n_{23}, n_{33}) \).

Note that the right-hand sides of equations (12) – (14) depend only on the variable \( r \) because they have the form of divergences from vectors whose components in the projections onto the coordinate lines of the cylindrical coordinate system also depend only on the variable \( r \). Equations for matrices \( \tilde{N}_2, \tilde{N}_3 \) will have a similar form. The dependence of the right-hand sides only on the variable \( r \) is related to their divergent form, which has already been noted with respect to the matrix \( \tilde{N}_1 \). We also note that systems (8) - (10) of dimension three decompose into separate scalar equations. That greatly simplifies their analysis and allows one to obtain analytical representations for solutions of the corresponding boundary value problems. Indeed, consider the equation and boundary conditions

\[ \frac{d}{dr} \left( A(r) \frac{dw}{dr} \right) = \frac{d}{dr} F_1(r) + F_2(r), \quad \left[ \frac{dw}{dr} + u(r)w - g(r) \right]_{r=r_0} = 0. \]
Let the operator $\mathcal{I}$ is the operator of “taking the antiderivative”: $$(\mathcal{I} f)(r) = \int_0^r f(\tau) d\tau.$$ Put

$$\Phi(r) = \frac{1}{A(r)} \left[ F_1(r) + \mathcal{I} \left( \frac{F_2(r)}{A(r)} \right) \right], \quad X(r) = \frac{1}{A(r)} + a(r) \mathcal{I} \left( \frac{1}{A(r)} \right), \quad Z(r) = g(r) - \Phi(r) - a(r) \mathcal{I} \left( \frac{\Phi(r)}{A(r)} \right).$$

$c_1, c_2$ there is a solution to the following linear system of the second order

$$\begin{cases}
\hat{c}_1 X(r_1) + c_2 a(r_1) = Z(r_1) \\
\hat{c}_1 X(R_0) + c_2 a(R_0) = Z(R_0)
\end{cases} \tag{16}$$

Then we have

$$\begin{cases}
w(r) = \mathcal{I} [\Phi(r)] + \hat{c}_1 \mathcal{I} \left( \frac{1}{A(r)} \right) + \hat{c}_2 \\
dw = \Phi(r) + \frac{\hat{c}_1}{A(r)}
\end{cases} \tag{17}$$

It is easy to show that the construction of solutions of these systems with the corresponding boundary conditions at a fixed value of the parameter $p$ can be reduced to the construction of the solution of the boundary value problem (17). As already noted, under our assumptions, the solutions of systems (8) - (10) obtained in Laplace images depend on the complex parameter $p$ as fractional rational functions; therefore, the inverse Laplace transform is easily carried out in an analytical form.

The analytical expressions constructed in this way make it possible to calculate the magnitude of the creeping deformations of the rod under long-term loads at its end face and, in particular, to compare the magnitude of the indicated strain under longitudinal and transverse loading at the end face with the same force fields with the same absolute value.

These results can be used to select the parameters of the complex layered structure of the rod in order to minimize the magnitude of creeping deformations.

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