UNIQUENESS OF GINZBURG-RALLIS MODELS: THE ARCHIMEDEAN CASE

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Abstract. In this paper, we prove the uniqueness of Ginzburg-Rallis models in the archimedean case. We also formulate a descent principle for mixed models.

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1. Introduction

In year 2000, D. Ginzburg and S. Rallis formulated a conjecture to characterize the nonvanishing of central values of partial exterior cube $L$-functions attached to irreducible cuspidal automorphic representations of $GL_6$ in terms of certain periods ([GR00]). This is analogous to the Jacquet conjecture for the triple product $L$-functions for $GL_2$.

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([HK04]) and to the more general Gross-Prasad conjecture for classical groups ([GP92], [GP94], [GJR03], [GJR05], and [GJR07]).

To be more precise, let $\mathbb{A}$ be the ring of adeles of a number field $k$. Fix a nontrivial unitary character $\psi_\mathbb{A}$ of $k\backslash\mathbb{A}$, and an arbitrary character $\chi_\mathbb{A}$ of $k^\times\backslash\mathbb{A}^\times$. For any quaternion algebra $D$ over $k$, denote $G_D := \text{GL}_3(D)$, and $S_D$ its subgroup consisting of elements of the form

$$
\begin{bmatrix}
  a & b & d \\
  0 & a & c \\
  0 & 0 & a 
\end{bmatrix}.
$$

Define a character $\chi_{S_D}$ of $S_D(\mathbb{A})$ by

$$
\chi_{S_D}\left(\begin{bmatrix}
  1 & b & d \\
  0 & 1 & c \\
  0 & 0 & 1 
\end{bmatrix} \cdot
\begin{bmatrix}
  a & 0 & 0 \\
  0 & a & 0 \\
  0 & 0 & a 
\end{bmatrix}\right) := \chi_{\mathbb{A}^\times}(\det(a)) \psi_\mathbb{A}(\text{tr}(b + c)),
$$

where $\text{det}$ and $\text{tr}$ stand for the reduced norm and the reduced trace, respectively.

Let $\varphi_D$ be an automorphic form on $G_D(k)\backslash G_D(\mathbb{A})$. The Ginzburg-Rallis period $\mathcal{P}_{\chi_{S_D}}(\varphi_D)$ of $\varphi_D$ is defined by the following integral

$$
\mathcal{P}_{\chi_{S_D}}(\varphi_D) := \int_{\mathbb{A}^\times S_D(k)\backslash S_D(\mathbb{A})} \varphi_D(s)(\chi_{S_D}(s))^{-1}ds,
$$

where $\mathbb{A}^\times$ is identified with the center of $G_D(\mathbb{A})$. The Ginzburg-Rallis conjecture can then be stated as follows.

**Conjecture 1.1.** (Ginzburg-Rallis, [GR00]) Let $\pi$ be an irreducible cuspidal automorphic representation of $\text{GL}_6(\mathbb{A})$ with central character $\chi_{\mathbb{A}^\times}^2$. For any quaternion algebra $D$ over $k$, denote by $\pi_D$ the generalized Jacquet-Langlands correspondence of $\pi$, which is either zero or an irreducible cuspidal automorphic representation of $G_D(\mathbb{A})$. Consider the irreducible representation $\Lambda^3 \otimes C^1$ of the $L$-group $\text{GL}_6(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$, where $\Lambda^3$ is the exterior cube product of the standard representation of $\text{GL}_6(\mathbb{C})$, and $C^1$ is the standard representation of $\text{GL}_1(\mathbb{C})$. The partial $L$-function $L^S(s, \pi \otimes \chi_{\mathbb{A}^\times}^{-1}, \Lambda^3 \otimes C^1)$ does not vanish at $s = \frac{1}{2}$ if and only if there exists a unique quaternion algebra $D$ such that

(a) the period $\mathcal{P}_{\chi_{S_D}}(\varphi_D)$ is nonzero for some $\varphi_D \in \pi_D$; and

(b) for any quaternion algebra $D'$ which is not isomorphic to $D$, the period $\mathcal{P}_{\chi_{S_D'}}(\varphi_{D'})$ is zero for every $\varphi_{D'} \in \pi_{D'}$.

We refer the reader to [GR00] and [GJ] for some partial results on the conjecture. Since the exterior cube $L$-functions are of symplectic
type, the central value \( L^S(\frac{1}{2}, \pi \otimes \chi_A^{-1}, \Lambda^3 \otimes \mathbb{C}^1) \) is expected to be of arithmetic significance.

It is clear that the uniqueness assertion is a central point in the statement of the Ginzburg-Rallis conjecture. It turns out that this delicate point can be precisely explained by the local theory of the corresponding Ginzburg-Rallis functionals on the local components of the automorphic representations. For local p-adic fields, such a local theory was considered in [J08]. In the following we consider the local theory for all local fields.

Let \( K \) be a local field of characteristic zero. Fix a nontrivial unitary character \( \psi_K \) of \( K \), and an arbitrary character \( \chi_K \times \) of \( K \times \). By abuse of notation, we do not distinguish an algebraic group over \( K \) with its \( K \) rational points. For any quaternion algebra \( D \) over \( K \), denote the group \( \text{GL}_3(D) \) and define its subgroup \( S_D \) as in the number field case. We also define the local analogy \( \chi_{S_D} \) of \( \chi_{S_D} \), by the same formula in terms of the characters \( \psi_K \) and \( \chi_{K \times} \).

If \( K \) is nonarchimedean, we let \( V_D \) be an irreducible smooth representation of \( G_D \), and if \( K \) is archimedean, let \( V_D \) be an irreducible Harish-Chandra smooth representation of \( G_D \). The notion of Harish-Chandra smooth representations will be explained in Section 12.

The first basic property to prove in the local theory of the Ginzburg-Rallis functionals is

**Conjecture 1.2.** The (local) Ginzburg-Rallis functionals on \( V_D \) is unique up to scalar, i.e. the space

\[ \text{Hom}_{S_D}(V_D, \mathbb{C}_{\chi_{S_D}}) \]

is at most one-dimensional. Here \( \mathbb{C}_{\chi_{S_D}} \) is the one dimensional representation of \( S_D \) given by the character \( \chi_{S_D} \).

This conjecture has been expected since the work [GR00]. In her PhD Thesis 2006, C.-F. Nien proved Conjecture 1.2 for all p-adic fields of characteristic zero ([N06]).

The following is a refinement of Conjecture 1.2 and is first stated in [J08].

**Conjecture 1.3.** Assume that \( D \) is split, and \( V_D \) is generic and unitarizable, with central character \( \chi_{K \times}^2 \).

(a) When \( K = \mathbb{C} \), we have \( \dim \text{Hom}_{S_D}(V_D, \mathbb{C}_{\chi_{S_D}}) = 1 \).

(b) When \( K \neq \mathbb{C} \), let \( D' \) be the unique quaternion division algebra over \( K \). Denote by \( V_{D'} \) the generalized Jacquet-Langlands correspondence of \( V_D \), which is either zero or an irreducible smooth
representation of $G_D'$. Then
\[
\dim \text{Hom}_{S_D}(V_D, \mathbb{C}_{\chi_{S_D}}) + \dim \text{Hom}_{S_{D'}}(V_{D'}, \mathbb{C}_{\chi_{S_{D'}}}) = 1.
\]

As in the case of Jacquet conjecture for the central values of the triple product $L$-functions for $\text{GL}_2$ ([HK04]), it is not difficult to see that Conjecture 1.3 implies the uniqueness assertion in Conjecture 1.1.

It is the objective of this paper to prove Conjecture 1.2, i.e. the local uniqueness for the Ginzburg-Rallis functionals, for all archimedean local fields (Theorem A below). This theorem combined with the local uniqueness for $p$-adic fields proved in [N06] establishes Conjecture 1.2 completely. Similar to the local uniqueness of the well-known Whittaker models, the local uniqueness of the Ginzburg-Rallis models has applications to the Rankin-Selberg method to study automorphic $L$-functions, namely to prove that certain global zeta integral can be expressed as an eulerian product of local zeta integrals. We refer the reader to a recent work of Ginzburg ([G06]) for some of the applications in this direction.

From now on, we will assume that $\mathbb{K}$ is the archimedean local field $\mathbb{R}$ or $\mathbb{C}$.

**Theorem A.** Let $V_D$ be an irreducible Harish-Chandra smooth representation of $G_D$. Then
\[
\dim \text{Hom}_{S_D}(V_D, \mathbb{C}_{\chi_{S_D}}) \leq 1.
\]

Note that the notion of Harish-Chandra smooth representations includes the requirement of moderate growth. This has the implication that
\[
\text{Hom}_{S_D}(V_D, \mathbb{C}_{\chi_{S_D}}) = 0,
\]
if one replaces the additive character $\psi_{\mathbb{K}}$ with one which is not unitary.

We introduce some notations. For any natural number $n$, denote by $\mathfrak{gl}_n(\mathbb{K})$ the space of $n \times n$ matrices with entries in $\mathbb{K}$. When the quaternion algebra $\mathbb{D}$ is split, we fix an identification of $\mathbb{D}$ with $\mathfrak{gl}_2(\mathbb{K})$, and then $G_\mathbb{D}$ is identified with $\text{GL}_3(\mathbb{K})$. For a square matrix $x$, if its entries are from $\mathbb{K}$, denote by $x^\tau$ its transpose. If $\mathbb{D}$ is not split and $x \in G_\mathbb{D} = \text{GL}_3(\mathbb{D})$, set
\[
x^\tau = \text{the transpose of } \bar{x},
\]
where “$-$” denotes the (element-wise) quaternionic conjugation.
Define the real trace form $\langle \cdot, \cdot \rangle_R$ on the Lie algebra $\mathfrak{gl}_3(D)$ of $G_D$ by

\begin{equation}
\langle x, y \rangle_R = \begin{cases} 
\text{the real part of the trace of } xy, & \text{if } D \text{ is split}, \\
\text{the reduced trace of } xy, & \text{otherwise}.
\end{cases}
\end{equation}

Denote by $\Delta_D$ the Casimir element with respect to $\langle \cdot, \cdot \rangle_R$, which is viewed as a bi-invariant differential operator on $G_D$.

We will see in Section 12 that by (a general form of) the Gelfand-Kazhdan criterion, Theorem A is implied by the following

**Theorem B.** Let $f$ be a tempered generalized function on $G_D$, which is an eigenvector of $\Delta_D$. If $f$ satisfies

\begin{equation}
f(sx) = f(xs^r) = \chi_{S_D}(s)f(x), \quad \text{for all } s \in S_D,
\end{equation}

then

\[ f(x) = f(x^r). \]

The notion of tempered generalized functions will be explained in Section 5. We remark that the equalities in the theorem are to be understood as equalities of generalized functions, and $f(sx)$ denotes the left translate of $f$ by $s^{-1}$. Similar notations apply throughout the article.

In order to prove Theorem B, we employ a descent argument based on an interesting observation of the relations between the Ginzburg-Rallis model and other (smaller) models. It is evident that this argument bears a strong resemblance to the well-known Harish-Chandra descent. See Definition 2.3, and Propositions B and C in Section 2.

Assume now that $D$ is split. Thus $G = \text{GL}_6$. (We drop the subscript $D$, and the coefficient field $K$ in all notations.) The first step of our descent argument reduces the proof of Theorem B to the following case. Take the maximal Levi subgroup $G_{4,2} := \text{GL}_4 \times \text{GL}_2$ of $G$ (or symmetrically $G_{2,4} := \text{GL}_2 \times \text{GL}_4$). Consider the intersection

\begin{equation}
S_{4,2} := G_{4,2} \cap S = (N_{2,2} \times \{I_2\}) \times \text{GL}_2^A,
\end{equation}

where $N_{2,2}$ is the unipotent radical of the standard maximal parabolic subgroup of $\text{GL}_4$ given by

\[ N_{2,2} := \left\{ n(x) = \begin{pmatrix} I_2 & x \\ 0 & I_2 \end{pmatrix} \in \text{GL}_4 \right\}, \]

and $\text{GL}_2^A \cong \text{GL}_2$ imbeds into $\text{GL}_4 \times \text{GL}_2$ diagonally. We expect that for any irreducible Harish-Chandra smooth representation $\pi$ of $G_{4,2}$,

\[ \dim \text{Hom}_{S_{4,2}}(\pi, \chi_{S_{4,2}}) \leq 1, \]

where $\chi_{S_{4,2}}$ is the characteristic function of $S_{4,2}$.
where $\chi_{S_{4,2}}$ is the restriction of the character $\chi_S$ to $S_{4,2}$. The reason for this expectation is that the case may be viewed as the local Gelfand-Graev model for the pair $(SO_6, SO_3)$, via the (incidental) identification of low rank algebraic groups. Note that the local uniqueness for this case is not known for either p-adic or the archimedean local fields. In the course of proof of Theorem B, we shall prove the local uniqueness for the pair $(G_{4,2}, S_{4,2})$ over archimedean local fields (in Section 13.3). (A similar argument should work for p-adic local fields.) This contains as a special case the local uniqueness of the Shalika model for $GL_4$.

We expect that a suitable generalization of the argument to work for the Shalika model of $GL_{2n}$, again over archimedean local fields. Note that for p-adic local fields, uniqueness of the Shalika model of $GL_{2n}$ was proved by Jacquet and Rallis in [JR96] (and later by Nien in [N07] by a different method).

Due to the afore-mentioned connection to the local uniqueness for the Gelfand-Graev pair $(SO_6, SO_3)$, we also expect our method to be relevant to a general Gelfand-Graev pair $(SO_m, SO_n)$ with $m > n$ and having different parity. Obviously the case of $(SO_m, SO_{m-1})$ is of particular significance. For p-adic local fields, this is the Multiplicity One Theorem proved by Aizenbud-Gourevitch-Rallis-Schiffmann [AGRS]. The archimedean case of the Multiplicity One Theorem is the subject of [SZ]. The general case of the Gelfand-Graev pair $(SO_m, SO_n)$ will be considered in a future work of the authors.

To examine the case $(G_{4,2}, S_{4,2})$, we perform a further descent (see Section 2 for details). Consider the maximal Levi subgroup

$$G_{2,2,2} := GL_2 \times GL_2 \times GL_2$$

of $G_{4,2}$ and the intersection

$$S_{2,2,2} = G_{2,2,2} \cap S_{4,2} = GL_2^\Delta,$$

which embeds diagonally into $G_{2,2,2}$. This is the well-known case of the trilinear model for $GL_2$. The local uniqueness was proved by D. Prasad ([P90]) for p-adic fields and by Loke ([L01]) for archimedean local fields. We will give a much simpler proof for this case using the theory of oscillator representations. The same proof works for p-adic local fields. The main ingredient is given in Section 6, and the local uniqueness is recorded in Section 13.1.

An interesting phenomenon here is that in order to complete the proof for the case $(G_{4,2}, S_{4,2})$, one must also consider the descent to the maximal Levi subgroup

$$G_{3,1,2} := GL_3 \times GL_1 \times GL_2$$

of $G_{6,1,2}$.
of $G_{4,2} = \text{GL}_4 \times \text{GL}_2$. This case reduces essentially to the case $(\text{GL}_3, S_3)$, where

$$S_3 = \left\{ s((c, d), a) = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ 1 \\ a \end{pmatrix} \mid c, d \in \mathbb{K}, a \in \mathbb{K}^\times \right\}.$$  

The corresponding character of $S_3$ is given by

$$\chi_{S_3}(s((c, d), a)) := \chi_{\mathbb{K}^\times}(a)\psi(d).$$

It should come as no surprise that the pair $(\text{GL}_3, S_3)$ is a special case of the model introduced by Jacquet-Shalika ([JS90]) to construct the exterior square $L$-functions for $\text{GL}_{2n+1}$. The uniqueness for this case was not known for any $n$. In the course of our proof for Theorem B, we shall prove the local uniqueness for the pair $(\text{GL}_3, S_3)$ over archimedean local fields. (The $p$-adic case follows similarly.) The main ingredient is given in Section 7, in which we employ the notion of unipotently $\chi$-incompatibility (Definition 4.5) and the Fourier transform. The local uniqueness is recorded in Section 13.2.

We now describe the contents and the organization of this paper. In Section 2 and guided by our view on the desired results of descent, we formalize three concepts ($D$-incompatibility, $\chi$-incompatibility, complete descent) which ensure vanishing of certain spaces of generalized functions. Then we highlight three geometric properties (metrically proper-ness, unipotently $\chi$-incompatibility, $U\chi M$ property) which relate to the just mentioned concepts and which lie at the heart of our approach. The key Propositions (A, B, C, D) which lead to the proof of Theorem B are also highlighted. In Sections 3, 4, and 5, we review some generalities on differential operators, generalized and invariant generalized functions, basics of Nash manifolds and the associated notion of tempered-ness. We prove Proposition D in Sections 6 and 7, which cover the required results after descent. We prove Proposition C in Section 8, which allows us to do the decent from $G_{4,2}$ to $G_{2,2,2}$ and $G_{3,1,2}$, and Proposition B in Section 9, which allows us to do the decent from $G'$ (an open submanifold of $G$) to $G'_{4,2} = G_{4,2} \cap G'$ and $G'_{2,4} = G_{2,4} \cap G'$. Using an idea of Shalika, we prove Proposition A in Section 10, which allows us to focus our attention to $G'$ only. The complete proof of Theorem B will be given in Section 11. In Section 12, we derive Theorem A from Theorem B. Finally, in Section 13, we record local uniqueness of models occurring in the process of descent. We also give a simple proof of the local uniqueness of the Whittaker model based on the notion of unipotently $\chi$-incompatibility. Of course this
local uniqueness has been known for a long time since the fundamental work of Shalika in 1974 ([S74]).

It is the authors’ hope that the descent principle formulated in Definition 2.3 will serve as a useful guide in the study of mixed models, such as Shalika models, Gelfand-Graev models, Fourier-Jacobi models, and general degenerate Whittaker models ([MW87]).

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2. DESCENT AS METHOD OF PROOF

Let $M$ be a smooth manifold, and $Z$ be a locally closed subset of $M$. Denote $C^{-\infty}(M; Z)$ the space of all generalized functions $f$ on $U$ which are supported in $Z$, where $U$ is (any) open subset of $M$ containing $Z$ as a closed subset. For any differential operator $D$ on $M$, denote by $C^{-\infty}(M; Z; D)$ the space of all $f \in C^{-\infty}(M; Z)$ such that $Df = 0$.

**Definition 2.1.** A locally closed subset $Z$ of $M$ is called $D$-incompatible if $C^{-\infty}(M; Z; D) = 0$.

Assume furthermore that $M$ is a pseudo Riemannian manifold.

**Fact V1:** (see Lemma 3.5) Let $Z$ be a metrically proper submanifold of $M$, and let $D$ be a Laplacian type differential operator on $M$ (Definition 3.4). Then $Z$ is $D$-incompatible.

Now let $H$ be a Lie group acting smoothly on $M$. Fix a (continuous) character $\chi$ of $H$. Denote by $C^{-\infty}_\chi(M)$ the space of all generalized functions $f$ which are $\chi$-equivariant, i.e.,

$$f(hx) = \chi(h)f(x), \quad \text{for all } h \in H.$$  

When $Z$ is $H$ stable, denote by $C^{-\infty}_\chi(M; Z)$ the space of all $f \in C^{-\infty}(M; Z)$ which are $\chi$-equivariant. We shall use similar notations (such as $C^{-\infty}_\chi(M; D)$ and $C^{-\infty}_\chi(M; Z; D)$) without further explanation.

**Definition 2.2.** An $H$ stable locally closed subset $Z$ of $M$ is said to be $\chi$-incompatible if $C^{-\infty}_\chi(M; Z) = 0$.

**Fact V2:** (see Lemma 4.6) Let $Z$ be an $H$ stable submanifold of $M$ which is unipotently $\chi$-incompatible (Definition 4.5). Then $Z$ is $\chi$-incompatible.
We then define a new notion called $U_\chi M$ property, which is a synthesis of unipotently $\chi$-incompatibility and metrically proper-ness. See Definition 4.8.

**Fact V3:** (see Lemma 4.9) Fix a continuous character $\chi$ of $H$ and a Laplacian type differential operator $D$ on $M$. If $Z$ is an $H$ stable locally closed subset of $M$ having $U_\chi M$ property, then

$$C_\chi^{-\infty}(M; Z; D) = 0.$$ 

Motivated by our desire to prove the uniqueness of mixed models, we make the following definition.

**Definition 2.3.** A collection of pairs consisting of manifolds with group actions 

$$(H_1; M_1), (H_2; M_2), \cdots, (H_k; M_k)$$

is said to be a complete descent of $(H; M)$ if

(a) $M_i$ is a local $H$ slice of $M$ (Definition 4.2), $H_i$ is a closed subgroup of $H$ which leaves $M_i$ stable, $1 \leq i \leq k$, and

(b) the closed subset

$$Z := M \setminus (\bigcup_{i=1}^k H M_i)$$

of $M$ has $U_\chi M$ property.

We now return to Theorem B, for $G = GL_6$ (the split case). The non-split case, which is much simpler, will be discussed at the end of Section 11.

Let $H = S \times S$, which acts on $G$ by

$$(g_1, g_2)x = g_1 x g_2^\tau.$$ 

Define a character $\chi$ of $H$ by the external tensor product:

(2.1) 

$$\chi = \chi_S \otimes \chi_S.$$ 

For $x \in G$, define the rank matrix

$$R(x) = \begin{bmatrix} \text{rank}_{4 \times 4}(x) & \text{rank}_{4 \times 2}(x) \\ \text{rank}_{2 \times 4}(x) & \text{rank}_{2 \times 2}(x) \end{bmatrix},$$

where $\text{rank}_{i \times j}(x)$ is the rank of the lower right $i \times j$ block of $x$. Then $R(x)$ takes the following 21 possible values [N06]:

$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}. $$
For $R$ one of above, denote
\[(2.2)\]  
$$G_R = \{ x \in G \mid R(x) = R \}.$$  
Then
\[G = \bigsqcup_R G_R\]  
is the decomposition of $G$ into $P$-$P^r$ double cosets, where
\[P = \left\{ \begin{bmatrix} a_1 & b & d \\ 0 & a_2 & c \\ 0 & 0 & a_3 \end{bmatrix} \in G \right\}\]  
is a parabolic subgroup of $G$ containing $S$.

Define (the open)
\[G' = \{ x \in G \mid \text{rank}_{2\times 4}(x) = 2, \text{rank}_{2\times 2}(x) \geq 1, \text{rank}_{4\times 4}(x) \geq 3 \} .\]  
Then we have
\[G' = \bigsqcup G_R,\]  
where $R$ in the union runs through the following four matrices 
$$\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}.$$  
Set
\[(2.4)\]  
$$x_{\text{left}} = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{gl}_6(\mathbb{K}) \quad \text{and} \quad x_{\text{right}} = x_{\text{left}}^r.$$

Denote by $X_{\text{left}}$ the left invariant vector field on $G$ whose tangent vector at $x$ is $x x_{\text{left}}$, and by $X_{\text{right}}$ the right invariant vector field on $G$ whose tangent vector at $x$ is $x_{\text{right}} x$.

**Proposition A:** (see Propositions 10.1 and 10.8)

(1) For every $P$-$P^r$ double coset $G_R$ in $G \setminus G'$, either $X_{\text{left}}$ or $X_{\text{right}}$ is transversal to $G_R$.  

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(2) For any $\lambda \in \mathbb{C}$,

$$C_{\chi}^{-\infty}(G; G \setminus G'; \Delta - \lambda) = 0.$$ 

Proposition A allows us to replace $G$ by $G'$ in the statement of Theorem B. This simplifies the situation greatly.

Set

$$S_{4,2} = (\text{GL}_4 \times \text{GL}_2) \cap S.$$ 

Then the group

$$H_{4,2} = S_{4,2} \times S_{4,2},$$ 

stabilizes the manifold

$$G'_{4,2} = (\text{GL}_4 \times \text{GL}_2) \cap G'.$$

Let $\chi_{4,2} = \chi|_{H_{4,2}}$. Define $S_{2,4}, H_{2,4}, G'_{2,4}$ and $\chi_{2,4}$ similarly.

We shall always view $G$ as a pseudo Riemannian manifold, equipped with the bi-invariant metric whose value at the identity matrix is given by the real trace form (1.3).

**Proposition B:** The pairs $(H_{4,2}; G'_{4,2})$ and $(H_{2,4}; G'_{2,4})$ form a complete descent of $(H; G')$.

The essential point is that $G' \setminus (HG_{4,2}' \cup HG_{2,4}')$ is equal to

(2.5) 

$$Z_6 = G_R, \quad \text{with } R = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},$$

and we shall verify that $Z_6$ has $U_{\chi M}$ property (Section 9).

Set

$$M_2 = \text{GL}_2 \times \text{GL}_2 \times \{I_2\} \subset G',$$

which is stable under the subgroup

$$H_2 = \{(x, x^{-T}) \mid x \in \text{GL}_2\} \subset \text{GL}_2^\Delta \times \text{GL}_2^\Delta \subset H = S \times S.$$ 

Set

$$M_3 = \left\{ \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & x_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in G \mid x_{33} \neq x_{44} \right\} \subset G',$$ 

where
which is stable under the subgroup $H_3$ of $H$ consisting of elements of the form
\[
\begin{pmatrix}
  a & 0 & * & 0 & 0 & 0 \\
  0 & 1 & * & 0 & 0 & 0 \\
  0 & 0 & a & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & a & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
  a^{-1} & 0 & * & 0 & 0 & 0 \\
  0 & 1 & * & 0 & 0 & 0 \\
  0 & 0 & a^{-1} & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & a^{-1} & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

Similarly, set
\[
\begin{align*}
M_3^\circ &= \left\{ \begin{pmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & * & * & * & 0 \\
  0 & 0 & * & * & * & 0 \\
  0 & 0 & * & y_{33} & 0 \\
  0 & 0 & 0 & 0 & y_{44}
\end{pmatrix} \in G \mid y_{33} \neq y_{44} \right\} \subset G',
\end{align*}
\]

which is stable under the subgroup $H_3^\circ$ of $H$ consisting of elements of the form
\[
\begin{pmatrix}
  a & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & a & 0 & * & 0 \\
  0 & 0 & 0 & 1 & * & 0 \\
  0 & 0 & 0 & 0 & a & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
  a^{-1} & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & a^{-1} & 0 & * & 0 \\
  0 & 0 & 0 & 1 & * & 0 \\
  0 & 0 & 0 & 0 & a^{-1} & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Proposition C**: The pairs $(H_2; M_2)$ and $(H_3; M_3)$ form a complete descent of $(H_{4,2}; G'_{4,2})$. Similarly, the pairs $(H_2; M_2)$ and $(H_3^\circ; M_3^\circ)$ form a complete descent of $(H_{2,4}; G'_{2,4})$.

The essential point is that $G'_{4,2} \setminus (H_{4,2} M_2 \cup H_{4,2} M_3)$ is equal to (2.6)
\[
Z_4 = \left\{ \begin{pmatrix}
  a_{11} & a_{12} & 0 \\
  a_{21} & a_{22} & 0 \\
  0 & 0 & y
\end{pmatrix} \in G_{4,2} \mid y^{-1} a_{22} \text{ is nilpotent and nonzero} \right\} \subset G'_{4,2},
\]

and we shall verify that $Z_4$ has $U_{x_{4,2}} M$ property (Section 8).

As a consequence of Propositions B and C, we have

**Proposition C'**: (see Proposition 11.2) The pairs $(H_2; M_2)$, $(H_3; M_3)$ and $(H_3^\circ; M_3^\circ)$ form a complete descent of $(H, G')$. 

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The following proposition serves as the final ingredients towards the proof of Theorem B, and it will be proved in Sections 6 and 7. Not surprisingly the presentation is in reverse order (relative to the descent process).

**Proposition D:** Let \( i = 2, 3 \), and denote by \( \chi_i \) (or \( \chi_3^\circ \)) the restriction of \( \chi \) to \( H_i \) (or \( H_3^\circ \)). Then for all \( f \in C_{\chi_i}^\infty(M_i) \) (or \( C_{\chi_3^\circ}^\infty(M_3^\circ) \)) which is tempered,
\[
f(x^\tau) = f(x).
\]

The proof of Proposition D in Sections 6 and 7 relies on the following

**Proposition E:** Let \( E \) be a finite dimensional non-degenerate quadratic space over \( \mathbb{K} \), and let the orthogonal group \( O(E) \) act on \( E^k \) diagonally, where \( k \) is a positive integer. If \( k < \dim E \), and if a tempered generalized function \( f \) on \( E^k \) is \( \text{SO}(E) \)-invariant, then \( f \) is \( O(E) \)-invariant.

Proposition E may also be stated as that the determinant character of \( O(E) \) occurs first at \( k = \dim E \). This is a consequence of the theory of oscillator representations. See [HZ02, Corollary 2.5], for example.

Now Theorem B will follow from Propositions A, C’ and D. The details are given in Section 11.

### 3. Generalized functions and differential operators

Let \( M \) be a smooth manifold. Denote by \( D^\infty_0(M) \) the space of compactly supported smooth (complex) densities on \( M \), which is a complete locally convex topological vector space. By a generalized function on \( M \), we mean a continuous linear functional on \( D^\infty_0(M) \). The space \( C^\infty(M) \) of generalized functions is again a complete locally convex topological vector space, under the strong dual topology. As usual, the space \( C^\infty(M) \) of smooth functions is continuously embedded in \( C^\infty(M) \), with a dense image.

For \( k \in \mathbb{Z} \), denote by \( \text{DO}(M)_k \) the Fréchet space of differential operators on \( M \) of order at most \( k \), which by convention is 0 if \( k < 0 \). It is well-known that every differential operator \( D : C^\infty(M) \to C^\infty(M) \) may be continuously extended to \( D : C^\infty(M) \to C^\infty(M) \).

Recall we have the principal symbol map
\[
\sigma_k : \text{DO}(M)_k \to \Gamma^\infty(M, S^k(T(M) \otimes_\mathbb{R} \mathbb{C})),
\]

where $T(M)$ is the real tangent bundle of $M$, $S^k$ stands for the $k$-th symmetric power, and $\Gamma^\infty$ stands for smooth sections. The continuous linear map $\sigma_k$ is specified by the following rule:

$$\sigma_k(X_1X_2\cdots X_k)(x) = X_1(x)X_2(x)\cdots X_k(x),$$

and

$$\sigma_k|_{DO(M)_{k-1}} = 0,$$

for all $x \in M$ and for all smooth vector fields $X_1, X_2, \cdots, X_k$ on $M$.

Let $Z$ be a submanifold of $M$. Write

$$N_Z(M) = T(M)|_Z / T(Z)$$

for the normal bundle of $Z$ in $M$. Denote by

$$\sigma_{k,Z} : DO(M)_k \to \Gamma^\infty(Z, S^k(N_Z(M) \otimes \mathbb{C}))$$

the map formed by composing $\sigma_k$ with the restriction map to $Z$, and followed by the quotient map

$$\Gamma^\infty(Z, S^k(T(M)|_Z \otimes \mathbb{C})) \to \Gamma^\infty(Z, S^k(N_Z(M) \otimes \mathbb{C})).$$

**Definition 3.1.**

(a) A vector field $X$ on $M$ is said to be tangential to $Z$ if $X(z)$ is in the tangent space $T_z(Z)$ for all $z \in Z$, and transversal to $Z$ if $X(z) \notin T_z(Z)$ for all $z \in Z$; more generally

(b) a differential operator $D$ is said to be tangential to $Z$ if for every point $z \in Z$ there is an open neighbourhood $U_z$ in $M$ such that $D|_{U_z}$ is a finite sum of differential operators of the form $\varphi X_1X_2\cdots X_r$, where $\varphi$ is a smooth function on $U_z$, $r \geq 0$, and $X_1, X_2, \cdots, X_r$ are vector fields on $U_z$ which are tangential to $U_z \cap Z$. For $D \in DO(M)_k$, it is said to be transversal to $Z$ if $\sigma_{k,Z}(D)$ does not vanish at any point of $Z$.

The following lemma can be traced back to the work of Shalika [S74]. Its proof is based on a well-known result of L. Schwartz about local representation of a generalized function with support in a submanifold.

**Lemma 3.2.** Let $D_1$ be a differential operator on $M$ of order $k \geq 1$, which is transversal to $Z$, and let $D_2$ be a differential operator on $M$ which is tangential to $Z$. Then

$$C^{-\infty}(M; Z; D_1 + D_2) = 0.$$

We will highlight two special cases of Lemma 3.2. The first one was used by Shalika implicitly (c.f. proof of Proposition 2.10 in [S74]).

**Lemma 3.3.** Let $X$ be a vector field on $M$ which is transversal to $Z$, and let $D$ be a differential operator on $M$ which is tangential to $Z$. Then

$$C^{-\infty}(M; Z; X + D) = 0.$$
Now assume that $M$ is a pseudo Riemannian manifold, i.e., the tangent spaces are equipped with a smoothly varying family $\{\langle \cdot, \cdot \rangle_x : x \in M\}$ of nondegenerate symmetric bilinear forms.

**Definition 3.4.** (a) A submanifold $Z$ of $M$ is said to be metrically proper if for all $z \in Z$, the tangent space $T_z(Z)$ is contained in a proper nondegenerate subspace of $T_z(M)$.

(b) A differential operator $D \in DO(M)_2$ is said to be Laplacian type if for all $x \in M$, 
\[
\sigma_2(D)(x) = u_1 v_1 + u_2 v_2 + \cdots + u_m v_m,
\]
where $u_1, u_2, \cdots, u_m$ is a basis of the tangent space $T_x(M)$, and $v_1, v_2, \cdots, v_m$ is the dual basis in $T_x(M)$ with respect to $\langle \cdot, \cdot \rangle_x$.

Note that a Laplacian type differential operator is transversal to any metrically proper submanifold, from its very definition. The second special case of Lemma 3.2 that we will use is then the following

**Lemma 3.5.** Let $Z$ be a metrically proper submanifold of $M$, and let $D$ be a Laplacian type differential operator on $M$. Then
\[
C^{-\infty}(M; Z; D) = 0.
\]

4. **Invariant generalized functions**

Let $H$ be an arbitrary Lie group, acting smoothly on a manifold $M$. Fix a continuous character $\chi : H \to \mathbb{C}^\times$. The following result is a direct consequence of the Schwartz Kernel Theorem [S66]. It will be referred to in the sequel as the localization principle.

**Lemma 4.1.** Let $M'$ be another manifold with trivial $H$ action. Then there is a linear topological identification
\[
C_{\chi}^{-\infty}(M \times M') = C_{\chi}^{-\infty}(M) \hat{\otimes} C^{-\infty}(M').
\]

Here “$\hat{\otimes}$” stands for the projective tensor product of complete locally convex topological spaces, which coincides with the injective tensor product since the spaces involved are all nuclear. We refer the interested reader to [T67].

Let $\mathcal{M}$ be a submanifold of $M$ and denote
\[
\rho_{\mathcal{M}} : H \times \mathcal{M} \to M
\]
the action map.

**Definition 4.2.** (a) We say that $\mathcal{M}$ is a local $H$ slice of $M$ if $\rho_{\mathcal{M}}$ is a submersion, and an $H$ slice of $M$ if $\rho_{\mathcal{M}}$ is a surjective submersion.
(b) Given two submanifolds $\mathcal{Z} \subset \mathcal{M}$ of $M$, we say that $\mathcal{Z}$ is relatively $H$ stable in $\mathcal{M}$ if

$$\mathcal{M} \cap H\mathcal{Z} = \mathcal{Z}.$$ 

Note that the relative stable condition amounts to saying that $H \times \mathcal{Z}$ is a union of fibres of the action map $\rho_M$. The following lemma is elementary.

**Lemma 4.3.** Let $\mathcal{M}$ be an $H$ slice of $M$, and let $\mathcal{Z}$ be a relatively $H$ stable submanifold of $\mathcal{M}$. Then $Z = H\mathcal{Z}$ is a submanifold of $M$, and $\mathcal{Z}$ is an $H$ slice of $Z$. Furthermore if $\mathcal{Z}$ is closed in $\mathcal{M}$, then $Z$ is closed in $M$.

Now assume that $\mathcal{M}$ is an $H$ slice of $M$, and let $f \in C^{-\infty}_\chi(M)$. From the work of Harish-Chandra [HC64, Theorem 1] or Wallach [W88, Lemma 8.A.2.5], we know that the pull back map

$$(\rho_M)^* : C^\infty(M) \to C^\infty(H \times \mathcal{M})$$

extends continuously to

$$(\rho_M)^* : C^{-\infty}(M) \to C^{-\infty}(H \times \mathcal{M}).$$

(This holds for a submersion in general.) Let $H$ act on $H \times \mathcal{M}$ (by left multiplication on the first factor). Then $\rho_M$ is $H$ equivariant and therefore

$$(\rho_M)^*(f) \in C^{-\infty}_\chi(H \times \mathcal{M}).$$

By the localization principle,

$$C^{-\infty}_\chi(H \times \mathcal{M}) = C^{-\infty}_\chi(H) \hat{\otimes} C^{-\infty}(\mathcal{M}) = C\chi \hat{\otimes} C^{-\infty}(\mathcal{M}).$$

(See [W88], 8.A for the equality $C^{-\infty}_\chi(H) = C\chi$.) This implies that there is a unique generalized function on $\mathcal{M}$, which is called the restriction of $f$ along (the slice) $\mathcal{M}$, and is denoted by $f|_{\mathcal{M}}$, such that

$$(\rho_M)^*(f) = \chi \otimes f|_{\mathcal{M}}.$$ 

This leads to the following

**Lemma 4.4.** Assume that $\mathcal{M}$ is an $H$ slice of $M$. Let $H_{\mathcal{M}}$ be a closed subgroup of $H$ which leaves $\mathcal{M}$ stable, and set $\chi_{\mathcal{M}} = \chi|_{H_{\mathcal{M}}}$. Then restriction to $\mathcal{M}$ yields an injective linear map

$$C^{-\infty}_\chi(M) \hookrightarrow C^{-\infty}_\chi(\mathcal{M}).$$
Definition 4.5. An $H$ stable submanifold $Z$ of $M$ is said to be unipotently $\chi$-incompatible if for every $z_0 \in Z$, there is a local $H$ slice $\mathfrak{z}$ of $Z$, containing $z_0$, and a smooth function $\mathfrak{z} \to H$ such that the followings hold for all $z \in \mathfrak{z}$:

(a) $\chi(\phi(z)) \neq 1$,
(b) $\phi(z)z = z$, and
(c) the linear map

$$T_z(M)/T_z(Z) \to T_z(M)/T_z(Z)$$

induced by the action of $\phi(z)$ on $M$ is unipotent.

The following lemma will also be important for our later considerations.

Lemma 4.6. Let $Z$ be an $H$ stable submanifold of $M$ which is unipotently $\chi$-incompatible. Then $C^{-\infty}_\chi(M; Z) = 0$.

By using the filtration of the sheaf of generalized functions with support in a submanifold, Lemma 4.6 is implied by the following sublemma. The sublemma is essentially known, at least in the p-adic case. We give a proof for the sake of completeness.

Sublemma 4.7. Let $\mathfrak{z}$ be an $H$ slice of an $H$ manifold $Z$. Let $E$ be an $H$ equivariant smooth complex vector bundle over $Z$, of finite rank. Assume that there is a smooth map $\phi : \mathfrak{z} \to H$ such that for all $z \in \mathfrak{z}$,

(a) $\chi(\phi(z)) \neq 1$,
(b) $\phi(z)z = z$, and
(c) the linear map

$$\phi(z) : E_z \to E_z$$

is unipotent, where $E_z$ is the fibre of $E$ at $z$.

Then

$$\Gamma^{-\infty}_\chi(Z, E) = 0.$$

Here and as usual, $\Gamma^{-\infty}_\chi(Z, E)$ stands for the space of generalized sections $f$ of $E$ such that

$$f(hx) = \chi(h)h(f(x)), \quad \text{for all } h \in H.$$

Proof. Let $f \in \Gamma^{-\infty}_\chi(Z, E)$. As in the case of generalized functions, we define the pull back $\tilde{f} \in \Gamma^{-\infty}_\chi(H \times \mathfrak{z}, \tilde{E})$ of $f$ via the action map

$$\rho_3 : H \times \mathfrak{z} \to Z,$$

where $\tilde{E}$ is the pull back of $E$ via $\rho_3$. Note that the bundle $\tilde{E}|_{\{e\} \times \mathfrak{z}}$ is identified with $E|_{\mathfrak{z}}$. The restriction $\tilde{f} \in \Gamma^{-\infty}_\chi(\mathfrak{z}, E|_{\mathfrak{z}})$ of $f$ to $\mathfrak{z}$ is then
defined by the formula
\[ \tilde{f}(h, z) = \chi(h)h\tilde{f}(z). \]
It is straightforward to check that condition (b) implies
\[ \tilde{f}(h\phi(z), z) = \tilde{f}(h, z). \]
Namely
\[ \chi(h\phi(z))h\phi(z)\tilde{f}(z) = \chi(h)h\tilde{f}(z), \]
and so
\[ (\chi(\phi(z))\phi(z) - 1_{E_z})\tilde{f}(z) = 0, \]
where \( \phi(z) \) is viewed as a linear automorphism of \( E_z \), and \( 1_{E_z} \) is the identity map of \( E_z \). Conditions (a) and (c) imply that \( \chi(\phi(z))\phi(z) - 1_{E_z} \) is invertible on \( E_z \) and so
\[ \tilde{f}(z) = (\chi(\phi(z))\phi(z) - 1_{E_z})^{-1}(\chi(\phi(z))\phi(z) - 1_{E_z})\tilde{f}(z) = 0, \]
which implies that \( f = 0 \).

**Definition 4.8.** Let \( H \) be a Lie group acting smoothly on a pseudo Riemannian manifold \( M \). We say that an \( H \)-stable locally closed subset \( Z \) of \( M \) has \( U_\chi M \) property if there is a finite filtration
\[ Z = Z_0 \supset Z_1 \supset \cdots \supset Z_k \supset Z_{k+1} = \emptyset \]
of \( Z \) by \( H \)-stable closed subsets of \( Z \) such that each \( Z_i \setminus Z_{i+1} \) is a submanifold of \( M \) which is either unipotently \( \chi \) incompatible or metrically proper in \( M \).

The following lemma will be our basic tool, which is a combination of Lemma 3.5 and Lemma 4.6.

**Lemma 4.9.** Let \( H \) be a Lie group acting smoothly on a pseudo Riemannian manifold \( M \). Fix a continuous character \( \chi \) of \( H \) and a Laplacian type differential operator \( D \) on \( M \). If \( Z \) is an \( H \)-stable locally closed subset of \( M \) having \( U_\chi M \) property, then
\[ C^{-\infty}_\chi(M; Z; D) = 0. \]

5. Nash manifolds and tempered generalized functions

We begin with a review of basic concepts and properties of Nash manifolds and tempered generalized functions. Our main reference is [S87]. See also [C91, AG08].

Recall that the collection \( \mathcal{SA}_n \) of semialgebraic subsets of \( \mathbb{R}^n \) is the smallest set with the following properties:

(a) every element of \( \mathcal{SA}_n \) is a subset of \( \mathbb{R}^n; \)
(b) for every real polynomial function \( p \) on \( \mathbb{R}^n \), we have
\[
\{ x \in \mathbb{R}^n \mid p(x) > 0 \} \in \mathcal{S}\mathcal{A}_n;
\]

(c) \( \mathcal{S}\mathcal{A}_n \) is closed under the operation of taking intersection, and taking complement in \( \mathbb{R}^n \).

A Nash manifold of dimension \( n \) is a manifold \( M \), together with a collection \( \mathcal{N} \), whose members are called Nash charts, such that the followings hold:

(a) every Nash chart has the form \((\phi, U, U')\), where \( U \) is an open semialgebraic subset of \( \mathbb{R}^n \), \( U' \) is an open subset of \( M \), and \( \phi : U \to U' \) is a diffeomorphism;

(b) every two Nash charts \((\phi_1, U_1, U'_1)\) and \((\phi_2, U_2, U'_2)\) are Nash compatible, i.e., the graph of the diffeomorphism

\[
\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(U'_1 \cap U'_2) \to \phi_2^{-1}(U'_1 \cap U'_2)
\]

is semialgebraic;

(c) for every triple \((\phi, U, U')\) as in (a), if it is Nash compatible with all Nash charts, then itself is a Nash chart;

(d) there are finitely many Nash charts \((\phi_i, U_i, U'_i)\), \( i = 1, 2, \ldots, r \), such that

\[
M = U'_1 \cup U'_2 \cup \cdots \cup U'_r.
\]

A subset \( Z \) of \( M \) is called semialgebraic if

\[
\phi^{-1}(Z \cap U') \text{ is semialgebraic in } \mathbb{R}^n \text{ for all Nash chart } (\phi, U, U') \text{ of } M.
\]

A Nash manifold is either the empty set or a nonempty Nash manifold of dimension \( n \geq 0 \). A submanifold of a Nash manifold which is semialgebraic is called a Nash submanifold, which is automatically a Nash manifold. The product of two Nash manifolds is again a Nash manifold. A smooth map \( \phi : M_1 \to M_2 \) of Nash manifolds is called a Nash map if its graph is semialgebraic in \( M_1 \times M_2 \). (A Nash map always sends a semialgebraic set to a semialgebraic set.) A Nash function on a Nash manifold \( M \) is a Nash map from \( M \) to \( \mathbb{C} \), and a differential operator \( D \) on \( M \) is called Nash if \( D(f) \) is Nash for every Nash function \( f \) on every Nash open submanifold of \( M \).

A Nash group is a group as well as a Nash manifold so that the group operations are Nash maps. A Nash action of a Nash group on a Nash Manifold is defined similarly.

We proceed to our discussion on the notion of tempered generalized functions on a Nash manifold. A smooth function \( f \) on a semialgebraic open subset \( U \) of \( \mathbb{R}^n \) is called a Schwartz function if \( D(f) \) is bounded for every Nash differential operator \( D \) on \( U \). Denote by \( \mathcal{S}(U) \) the Fréchet
space of Schwartz functions on $U$. Now let $M$ be a Nash manifold of dimension $n$. Pick a covering of $M$ by Nash charts $(\phi_i, U_i, U'_i)$, $i = 1, 2, \cdots, r$. By extending to zero outside $U'_i$, $\phi_i$ induces a continuous linear map

$$(\phi_i)_*: \mathcal{S}(U_i) \to C^\infty(M).$$

The space of Schwartz functions on $M$, denoted by $\mathcal{S}(M)$, is then defined to be the image of the map

$$\bigoplus(\phi_i)_*: \bigoplus_{i=1}^r \mathcal{S}(U_i) \to C^\infty(M),$$

equipped with the quotient topology of $\bigoplus_{i=1}^r \mathcal{S}(U_i)$. The space $\mathcal{S}(M)$ is independent of the covering we choose and it is a nuclear Fréchet space. One may similarly define the nuclear Fréchet space of Schwartz densities. Denote by $C^{-\xi}(M)$ its strong dual, whose members are called tempered generalized functions.

Now let $H$ be a Nash group, with a Nash action on a Nash manifold $M$. For a continuous character $\chi: H \to \mathbb{C}^\times$, we set

$$C_{\chi}^{-\xi}(M) = C_{\chi}^{-\infty}(M) \cap C^{-\xi}(M).$$

We record three elementary facts, which will be used in later sections.

**Lemma 5.1.** (a) Let $M'$ be another Nash manifold with trivial $H$ action. Then there is a linear topological identification

$$C_{\chi}^{-\xi}(M \times M') = C_{\chi}^{-\xi}(M) \hat{\otimes} C^{-\xi}(M').$$

(b) Let $\mathfrak{M}$ be a Nash $H$ slice of $M$, i.e., it is a Nash submanifold as well as an $H$ slice. Then

$$f|_{\mathfrak{M}} \in C^{-\xi}(\mathfrak{M}) \quad \text{for all } f \in C_{\chi}^{-\xi}(M).$$

Consequently we have an injection

$$C_{\chi}^{-\xi}(M) \hookrightarrow C_{\chi,\mathfrak{M}}^{-\xi}(\mathfrak{M}),$$

where $\chi_{\mathfrak{M}}$ is the restriction of $\chi$ to a closed subgroup $H_{\mathfrak{M}}$ of $H$ which leaves $\mathfrak{M}$ stable.

(c) If $C_{\chi}^{-\xi}(M) = 0$, then $C_{\chi}^{-\xi}(M') = 0$ for any Nash open submanifold $M'$ of the form $\phi^{-1}(N')$, where $\phi: M \to N$ is an $H$ equivariant Nash map, $N$ is a Nash manifold with trivial $H$ action, and $N'$ is a Nash open submanifold $N$.

We comment on the notion of unipotently $\chi$-incompatibility, in the context of this section.

A Nash group is said to be unipotent if it is Nash isomorphic to a connected closed subgroup of some $U_n$, where $U_n$ is the Nash group of unipotent upper triangular real matrices of size $n$. An element of
a Nash group $H$ is said to be (Nash) unipotent if it is contained in a unipotent Nash closed subgroup of $H$. We note that the general linear group $GL_n(K)$ is Nash and an element of $GL_n(K)$ is (Nash) unipotent if and only if it is unipotent in the usual sense, i.e., is a unipotent linear transformation.

We may then make the following definition. An $H$ stable submanifold $Z$ of $M$ is said to be (Nash) unipotently $\chi$-incompatible if for every point $z_0 \in Z$, there is a local $H$ slice $\mathfrak{Z}$ of $Z$, containing $z_0$, and a smooth map $\phi : \mathfrak{Z} \to H$ such that the following properties hold for all $z \in \mathfrak{Z}$:

(a) $\chi(\phi(z)) \neq 1$,
(b) $\phi(z)z = z$,
(c) $\phi(z)$ is (Nash) unipotent.

Note that the hypothesis of Nash action ensures that the map $T_z(M) \to T_z(M)$ induced by $\phi(z)$ is unipotent, for all $z \in \mathfrak{Z}$. This implies the unipotency condition in Definition 4.5.

6. The manifold $GL_2 \times GL_2$

Set

$$\tilde{H}_2 = \{1, \tau\} \ltimes GL_2(K),$$

where the semidirect product is given by the action

$$\tau(g) = g^{-\tau}.$$  

Denote by $\tilde{\chi}_2$ the character of $\tilde{H}_2$ such that

$$\tilde{\chi}_2|_{GL_2(K)} = 1 \quad \text{and} \quad \tilde{\chi}_2(\tau) = -1.$$  

It will be more convenient to work with $\tilde{H}_2$ than with $H_2$ (introduced in Section 2).

**Proposition 6.1.** Let $\tilde{H}_2$ act on $M_2 := GL_2(K) \times GL_2(K)$ by

$$g(x, y) = (gxg^{-1}, gyg^{-1}), \quad g \in GL_2(K),$$

and

$$\tau(x, y) = (x^\tau, y^\tau).$$

Then

$$C_{\tilde{\chi}_2}^{\xi}(M_2) = 0.$$
Proof. Using the same formula, we may extend the action of \( \tilde{H}_2 \) on \( \text{GL}_2(K) \times \text{GL}_2(K) \) to the larger space \( \mathfrak{gl}_2(K) \times \mathfrak{gl}_2(K) \). By part (c) of Lemma 5.1, it suffices to prove that

\[
C_{\chi_2}^\xi(\mathfrak{gl}_2(K) \times \mathfrak{gl}_2(K)) = 0.
\]

Identify \( K \) with the center of \( \mathfrak{gl}_2(K) \). We have

\[
\mathfrak{gl}_2(K) \times \mathfrak{gl}_2(K) = (\mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K)) \oplus (K \times K)
\]
as a \( K \) linear representation of \( \tilde{H}_2 \), where \( \tilde{H}_2 \) acts on \( K \times K \) trivially. By the localization principle, it suffices to prove that

\[
C_{\chi_2}^\xi(\mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K)) = 0.
\]

We view \( \mathfrak{sl}_2(K) \) as a three dimensional quadratic space under the trace form. Under this identification, the action of \( \tilde{H}_2 \) yields the diagonal action of \( O(\mathfrak{sl}_2(K)) \) on \( \mathfrak{sl}_2(K) \times \mathfrak{sl}_2(K) \), with \( \tilde{\chi}_2 \) corresponding to the determinant character. So the required vanishing result is a special case of Proposition E in Section 2. \( \square \)

7. The manifold \( \text{GL}_3 \times \text{GL}_1 \)

As in Section 6, we introduce a semidirect product group \( \tilde{H}_3 = \{1, \tau\} \ltimes H_3 \), where \( H_3 \) is given in Section 2.

Write

\[
L_3 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix} \mid a \in K^\times \right\},
\]
and

\[
N_3 = \left\{ \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid c, d \in K \right\}.
\]

Define

\[
\tilde{H}_3 = \{1, \tau\} \ltimes (L_3 \ltimes (N_3 \times N_3)),
\]

where the inner semidirect product is defined by the action

\[
l(g_1, g_2) = (lg_1l^{-1}, l^{-1}g_2l),
\]
and the outer semidirect product is defined by the action

\[
\tau(l, g_1, g_2) = (l^{-1}, g_2, g_1).
\]

Write

\[
\chi_{N_3} \left( \begin{bmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right) = \psi_K(d),
\]
which defines a character of $N_3$. Let $\tilde{\chi}_3$ be the character of $\tilde{H}_3$ such that
\[
\tilde{\chi}_3(l, g_1, g_2) = \chi_{N_3}(g_1)\chi_{N_3}(g_2), \quad (l, g_1, g_2) \in L_3 \times (N_3 \times N_3),
\]
and
\[
\tilde{\chi}_3(\tau) = -1.
\]

**Proposition 7.1.** Let $\tilde{H}_3$ act on
\[
M_3 = \{ (x, y) \in \text{GL}_3(\mathbb{K}) \times \mathbb{K}^\times \mid y \neq \text{the 3-3 entry of } x \}
\]
by
\[
(l, g_1, g_2)(x, y) = (lg_1xg_2^{\tau}l^{-1}, y)
\]
and
\[
\tau(x, y) = (x^{\tau}, y).
\]
Then
\[
C^-_{\tilde{\chi}_3}(M_3) = 0.
\]

**Remark:** As noted in Section 2, $M_3$ is a local $H$ slice of $G$, which is stable under $H_3$ and $\tau$.

**Proof.** First we note that the 3-3 entry of $x$ is invariant under $\tilde{H}_3$. Denote by $\text{GL}_3(\mathbb{K})'$ the set of matrices in $\text{GL}_3(\mathbb{K})$ whose 3-3 entry is not 1. Let $\tilde{H}_3$ act on $\text{GL}_3(\mathbb{K})' \times \mathbb{K}^\times$ by the same formula as its action on $M_3$. Then the map
\[
\text{GL}_3(\mathbb{K})' \times \mathbb{K}^\times \to M_3,
\]
\[
(x, y) \mapsto (yx, y)
\]
is an $\tilde{H}_3$ equivariant Nash diffeomorphism. Therefore
\[
C^-_{\tilde{\chi}_3}(M_3) \cong C^-_{\tilde{\chi}_3}(\text{GL}_3(\mathbb{K})' \times \mathbb{K}^\times).
\]
By the localization principle, it suffices to show that
\[
C^-_{\tilde{\chi}_3}(\text{GL}_3(\mathbb{K})') = 0.
\]
This will be implied by part (c) of Lemma 5.1 and Proposition 7.2 below. \square

The rest of this section is devoted to the proof of

**Proposition 7.2.** Let $\tilde{H}_3$ act on $\text{gl}_3(\mathbb{K})$ by
\[
(l, g_1, g_2)x = lg_1xg_2^{\tau}l^{-1}
\]
and
\[
\tau x = x^{\tau}.
\]
Then
\[
C^-_{\tilde{\chi}_3}(\text{gl}_3(\mathbb{K})) = 0.
\]
Write
\[ Z_{3,1} = \left\{ \begin{bmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix} \in \mathfrak{gl}_3(\mathbb{K}) \right\} \]

and
\[ Z_{3,2} = \left\{ \begin{bmatrix} * & * & a \\ * & a & 0 \\ a & * & 0 \end{bmatrix} \in \mathfrak{gl}_3(\mathbb{K}) \right\} . \]

**Lemma 7.3.** One has that \( C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K}) \setminus Z_{3,1}) = 0 \).

**Proof.** The Nash submanifold \( \mathfrak{gl}_2(\mathbb{K}) \times \mathbb{K}^\times \) is an \( \tilde{H}_3 \) slice of \( \mathfrak{gl}_3(\mathbb{K}) \setminus Z_{3,1} \), which is stable under the subgroup
\[ \tilde{H}_{3,1} = \{ 1, \tau \} \ltimes L_3 = \{ 1, \tau \} \ltimes \mathbb{K}^\times . \]

Denote by \( \tilde{\chi}_{3,1} \) the restriction of \( \tilde{\chi}_3 \) to \( \tilde{H}_{3,1} \). Then by restriction we have an injection
\[ C^{-\xi}_{\tilde{\chi}_3}(\mathfrak{gl}_3(\mathbb{K}) \setminus Z_{3,1}) \hookrightarrow C^{-\xi}_{\tilde{\chi}_{3,1}}(\mathfrak{gl}_2(\mathbb{K}) \times \mathbb{K}^\times) . \]

Now the lemma will follow by the localization principle, together with the following simple assertion: let \( \tilde{H}_{3,1} \) act on \( \mathbb{K} \times \mathbb{K} \) by
\[ a(x, y) = (ax, a^{-1}y), \quad a \in \mathbb{K}^\times \quad \text{and} \quad \tau(x, y) = (y, x), \]

then \( C^{-\xi}_{\tilde{\chi}_{3,1}}(\mathbb{K} \times \mathbb{K}) = 0 \).

We view \( \mathbb{K} \times \mathbb{K} \) as a split two dimensional quadratic space so that both \( \mathbb{K} \times \{ 0 \} \) and \( \{ 0 \} \times \mathbb{K} \) are isotropic. Then \( \tilde{H}_{3,1} \) is identified with the orthogonal group \( \text{O}(\mathbb{K} \times \mathbb{K}) \), with \( \tilde{\chi}_{3,1} \) corresponding to the determinant character. So this assertion is again a special case of Proposition E in Section 2.

**Lemma 7.4.** The \( \tilde{H}_3 \) stable manifold \( Z_{3,1} \setminus Z_{3,2} \) is unipotently \( \tilde{\chi}_3 \) incompatible.

**Proof.** For
\[ x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{bmatrix} \in Z_{3,1} \setminus Z_{3,2}, \]

write
\[ u(x, t) = \begin{bmatrix} 1 & 0 & x_{13}t \\ 0 & 1 & x_{23}t \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad v(x, t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx_{31} & tx_{32} & 1 \end{bmatrix}, \quad t \in \mathbb{K} . \]

Then
\[ u(x, t)x = xv(t, x), \quad \text{i.e.,} \quad (u(x, t), v(x, t)^{-\tau})x = x. \]
Since $x_{13} \neq x_{31}$,
\[ \tilde{\chi}_3(u(x, t), v(x, t)^{-\tau}) = \psi_K(x_{13}t - x_{31}t) \neq 1, \]
for a suitably chosen $t$. This shows that $Z_{3,1} \setminus Z_{3,2}$ is unipotently $\tilde{\chi}_3$ incompatible.

Lemma 7.3, Lemma 7.4 and Lemma 4.6 now imply the following

**Lemma 7.5.** Every generalized function in $C_{\tilde{\chi}_3}^{-\xi}(\mathfrak{gl}_3(K))$ is supported on $Z_{3,2}$.

We shall employ Fourier transform to finish the proof of Proposition 7.2.

In general, let $E$ be a finite dimensional real vector space, equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_E$. The Fourier transform is a topological linear isomorphism
\[ \hat{\cdot} : \mathcal{S}(E) \to \mathcal{S}(E) \]
given by
\[ \hat{f}(x) = \int_E f(y) e^{-2\pi \sqrt{-1} \langle x, y \rangle_E} dy, \]
where $dy$ is the Lebesgue measure on $E$, normalized such that the volume of the cube
\[ \{ t_1v_1 + t_2v_2 + \cdots + c_r v_r \mid 0 \leq t_1, t_2, \cdots, t_r \leq 1 \} \]
is 1, for any orthogonal basis $v_1, v_2, \cdots, v_r$ of $E$ such that $\langle v_i, v_i \rangle_E = \pm 1$, $i = 1, 2, \cdots, r$. The Fourier transform extends continuously to a topological linear isomorphism
\[ \hat{\cdot} : C^{-\xi}(E) \to C^{-\xi}(E), \]
which is still called the Fourier transform.

We shall need the following lemma, which is a form of uncertainty principle. It can be easily deduced from Lemma 3.2. We shall present another proof, which is rather simple.

**Lemma 7.6.** Let $f \in C^{-\xi}(E)$. If $f$ and $\hat{f}$ have supports which are contained in a common nondegenerate proper subspace of $E$, then $f = 0$.

**Proof.** Write $F$ for the common nondegenerate proper subspace. We may choose an orthogonal basis $v_1, v_2, \cdots, v_r$ of $E$ such that $\langle v_i, v_i \rangle_E = \pm 1$, and $v_1, v_2, \cdots, v_k$ is an orthogonal basis of $F$, where $k < r$. Let $x_1, x_2, \ldots, x_r$ be the linear coordinates with respect to $v_1, v_2, \ldots, v_r$. 

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Due to tempered-ness, $f$ has a finite order as a distribution. The hypothesis about the supports of $f$ and $\hat{f}$ then implies that
\[ x_i^{\alpha_i} f = (\partial / \partial x_i)^{\beta_i} f = 0, \]
where $k < i \leq r$, and $\alpha_i, \beta_i$ are some positive integers. Thus $f$ generates a finite dimensional module under the Weyl algebra $W_{r-k}$, the algebra of linear differential operators with polynomial coefficients in $r-k$ variables. This is not possible unless $f = 0$. \hfill \Box

We continue with the proof of Proposition 7.2. Let $\mathfrak{g}_3(K)$ be equipped with the real trace from as in the Introduction and define the Fourier transform accordingly. Given $f \in C^\xi(\mathfrak{g}_3(K))$, it is easy to check that its Fourier transform $\hat{f} \in C^\xi(\mathfrak{g}_3(K))$ satisfies the followings:
\[
\begin{cases}
(a) \quad \hat{f}(lxl^{-1}) = \hat{f}(x), & l \in L_3, \\
(b) \quad \hat{f}(g_1^tg_2) = \chi_{N_3}(g_1)^{-1}\chi_{N_3}(g_2)^{-1}\hat{f}(x), & g_1, g_2 \in N_3, \text{ and,} \\
(c) \quad \hat{f}(x^\tau) = -\hat{f}(x).
\end{cases}
\]
Then as in Lemma 7.5, we conclude that $\hat{f}$ is supported on
\[
Z_{3,2}' = \left\{ \begin{bmatrix} 0 & * & a \\
* & * & * \\
a & * & * \end{bmatrix} \in \mathfrak{g}_3(K) \right\}.
\]
Therefore both $f$ and $\hat{f}$ are supported on the proper nondegenerate subspace
\[
Z_{3,2} + Z_{3,2}' = \left\{ \begin{bmatrix} * & * & a \\
* & * & * \\
a & * & * \end{bmatrix} \in \mathfrak{g}_3(K) \right\}.
\]
Lemma 7.6 then implies that $f = 0$. The proof of Proposition 7.2 is now complete.

8. A SUBMANIFOLD $Z_4$ OF $GL_4 \times GL_2$

As always, we equip $G = GL_6(K)$ with the bi-invariant pseudo Riemannian metric whose restriction to $T_e(G) = \mathfrak{g}_6(K)$ is the real trace form $\langle \cdot, \cdot \rangle_R$, given in (1.3).

As in the Introduction, write $G_{4,2} = GL_4(K) \times GL_2(K)$, which embeds into $G$ in the usual way. Then $G_{4,2}$ is a nondegenerate submanifold of $G$, with $T_e(M) = \mathfrak{g}_4(K) \times \mathfrak{g}_2(K)$. Thus $G_{4,2}$ is itself a pseudo Riemannian manifold.
As in the Introduction and Section 2, we denote

\[ S_{4,2} = (\text{GL}_4(\mathbb{K}) \times \text{GL}_2(\mathbb{K})) \cap S = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in G \right\}, \]

\[ H_{4,2} = S_{4,2} \times S_{4,2} \subset H = S \times S, \]

and the character \( \chi_{4,2} = \chi|_{H_{4,2}}. \)

Recall also the following \( H_{4,2} \) stable submanifold of \( G_{4,2} \):

\[ Z_{4} = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & y \end{bmatrix} \in G_{4,2} \mid y^{-1}a_{22} \text{ is nilpotent and nonzero} \right\}. \]

The main purpose of this section is to prove the following proposition. This will take a number of steps.

**Proposition 8.1.** The submanifold \( Z_{4} \) of \( G_{4,2} \) has \( U\chi_{4,2}M \) property.

Denote by \( \mathfrak{Z}_{4} \) all matrices in \( G_{4,2} \) of the form

\[ x = \begin{bmatrix} x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \]

**Lemma 8.2.** The submanifold \( \mathfrak{Z}_{4} \) is an \( H_{4,2} \) slice of \( Z_{4} \).

**Proof.** Let \( x \in Z_{4} \) be as in (8.1). Define

\[ \bar{\phi}(x) = \begin{bmatrix} a_{22} & 0 \\ 0 & y \end{bmatrix} \in \text{gl}_4(\mathbb{K}). \]

Note that \( \bar{\phi}(\mathfrak{Z}_{4}) \) consists of a single matrix

\[ \bar{x}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \]

The action of \( H_{4,2} \) on \( Z_{4} \) descents to a transitive action on the quotient manifold

\[ \bar{Z}_{4} = \{ \bar{\phi}(x) \mid x \in Z_{4} \}. \]

Therefore to show that the \( H_{4,2} \) equivariant action map

\[ \rho_{\mathfrak{Z}_{4}} : H_{4,2} \times \mathfrak{Z}_{4} \to Z_{4} \]
is a surjective submersion, it suffices to show the same for its restriction map

$$\tilde{\phi}^{-1}(\tilde{x}_0) \rightarrow \tilde{\phi}^{-1}(\tilde{x}_0).$$

Denote by $N_{4,2}$ the unipotent radical of $S_{4,2}$. Then

$$(N_{4,2} \times N_{4,2}) \times \mathfrak{Z}_4 \subset (\tilde{\phi} \circ \rho_{3_4})^{-1}(\tilde{x}_0),$$

and hence it suffices to show that the action map

$$(8.3)$$

$$ (N_{4,2} \times N_{4,2}) \times \mathfrak{Z}_4 \rightarrow \tilde{\phi}^{-1}(\tilde{x}_0)$$

is a surjective submersion.

Now let

$$x = \begin{bmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & 0 & 0 \\
  x_{21} & x_{22} & x_{23} & x_{24} & 0 & 0 \\
  x_{31} & x_{32} & 0 & 0 & 0 & 0 \\
  x_{41} & x_{42} & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \in \tilde{\phi}^{-1}(\tilde{x}_0).$$

Then $u(x)v(x) \in \mathfrak{Z}_4$, with

$$u(x) = \begin{bmatrix}
  1 & 0 & 0 & -x_{14} & 0 & 0 \\
  0 & 1 & 0 & -x_{24} & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

and

$$v(x) = \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 \\
  -x_{41} & -x_{42} & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},$$

which proves that the map (8.3) is surjective. One shows similarly that the differential of the map (8.3) is also surjective. \qed

Let

$$\mathfrak{Z}_{4,1} = \{ x \in \mathfrak{Z}_4 \text{ of the form } (8.2) \text{ with } x_{13} = x_{31} \}.$$ 

**Lemma 8.3.** The closed submanifold $\mathfrak{Z}_{4,1}$ is relatively $H_{4,2}$ stable in $\mathfrak{Z}_4$. 

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Proof. Let $x \in \mathcal{Z}_{4,1}$, and
\[
g = \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in S_{4,2}, \quad g' = \begin{bmatrix} a' & 0 & 0 \\ b' & a' & 0 \\ 0 & 0 & a' \end{bmatrix} \in S_{4,2}^r.
\]
We need to show that $gxg' \in \mathcal{Z}_{4,1}$, provided that $gxg' \in \mathcal{Z}_4$. The condition $gxg' \in \mathcal{Z}_4$ implies that
\[
a \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} a' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad a \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} a' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]
which is equivalent to
\[
a = \alpha \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \quad \text{and} \quad a' = \alpha^{-1} \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix},
\]
for some $\alpha \in \mathbb{K}^\times$ and $t \in \mathbb{K}$. It is now straightforward to check that $gxg' \in \mathcal{Z}_{4,1}$. \hfill \Box

Remark: In the sequel, we will skip the verification when we claim that a submanifold is relatively stable or is a slice with respect to a certain group action.

Write
\[
\mathcal{Z}_{4,1} = H_{4,2} \mathcal{Z}_{4,1},
\]
which is a closed submanifolds of $\mathcal{Z}_4$, by Lemma 4.3.

Lemma 8.4. The submanifold $\mathcal{Z}_4 \setminus \mathcal{Z}_{4,1}$ is unipotently $\chi_{4,2}$ incompatible.

Proof. The proof is similar to that of Lemma 7.4. Let $x \in \mathcal{Z}_4 \setminus \mathcal{Z}_{4,1}$ be as in (8.2) and write
\[
u(x,t) = \begin{bmatrix} 1 & 0 & x_{13}t & 0 & 0 & 0 \\ 0 & 1 & x_{23}t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]
and
\[
u(x,t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ tx_{31} & tx_{32} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\]
Then
\[ u(x, t)x = xv(t, x), \quad \text{i.e.,} \quad (u(x, t), v(x, t)^{-\tau})x = x. \]
Since \( x_{13} \neq x_{31} \),
\[ \chi_{4,2}(u(x, t), v(x, t)^{-\tau}) = \psi_{K}(x_{13}t - x_{31}t) \neq 1 \]
for a suitably chosen \( t \). This proves the lemma. \( \square \)

Write
\[ \mathfrak{Z}_{4,2} = \{ x \in \mathfrak{Z}_{4,1} \text{ of the form (8.2) with } x_{13} = x_{31} = 0 \}, \]
which is a relatively \( H_{4,2} \) stable closed submanifold of \( \mathfrak{Z}_{4,1} \). Therefore
\[ Z_{4,2} = H_{4,2}\mathfrak{Z}_{4,2} \]
is a closed submanifold of \( Z_{4,1} \).

**Lemma 8.5.** The submanifold \( Z_{4,1} \setminus Z_{4,2} \) is metrically proper in \( G_{4,2} \).

**Proof.** Denote by \( \mathfrak{Z}_{4,1} \) all matrices in \( G_{4,2} \) of the form
\[(8.4) \quad x = \begin{bmatrix} 0 & 0 & a & 0 & 0 & 0 \\ 0 & x_{22} & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & b \end{bmatrix}, \]
which forms an \( H_{4,2} \) slice of \( Z_{4,1} \setminus Z_{4,2} \).

Let \( x \) be as in (8.4). Then one checks that
\[ T_x(Z_{4,1}) = T_x(\mathfrak{Z}_{4,1}) + (\text{Lie}S_{4,2})x + x(\text{Lie}S_{4,2})^\top \subset \mathfrak{gl}_{4}(\mathbb{K})_{13=31} \times \mathfrak{gl}_{2}(\mathbb{K}), \]
where \( \mathfrak{gl}_{4}(\mathbb{K})_{13=31} \) is the set of matrices in \( \mathfrak{gl}_{4}(\mathbb{K}) \) whose \((1,3)\) entry equals its \((3,1)\) entry. We shall adopt similar notations in the sequel.

Let \( x' = e_{13} - e_{31} \in \mathfrak{gl}_{6}(\mathbb{K}) \),
where \( e_{ij} \) denotes the matrix unit with 1 at the \((i, j)\) entry and 0 elsewhere. Then
\[ \mathfrak{gl}_{4}(\mathbb{K})_{13=31} \times \mathfrak{gl}_{2}(\mathbb{K}) \subset (\mathbb{K}x')^\perp, \]
where \( \perp \) denotes the orthogonal complement with respect to the real trace form. Consequently,
\[ x^{-1}T_x(Z_{4,1}) \subset x^{-1}(\mathbb{K}x')^\perp = (\mathbb{K}xx')^\perp. \]

Note that
\[ xx' = a(e_{11} - e_{33}), \]
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which spans a nondegenerate $K$ subspace of $T_x(G_{4,2})$. This implies that $x^{-1}T_x(Z_{4,1})$ is contained in a proper nondegenerate subspace of $T_x(G_{4,2})$. Therefore by invariance of the metric, $T_x(Z_{4,1})$ is contained in a nondegenerate proper subspace of $T_x(G_{4,2})$, for any $x \in Z_{4,1} \setminus Z_{4,2}$. □

Denote by $Z_{4,2}'$ all matrices in $Z_{4,2}$ of the form

\begin{equation}
\begin{bmatrix}
x_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_{23} & 0 & 0 & 0 \\
0 & x_{32} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\end{equation}

which also forms an $H_{4,2}$ slice of $Z_{4,2}$.

Write

$Z_{4,3}^1 = \{ x \in Z_{4,2}' \text{ of the form (8.5) with } x_{23} = x_{32} \}$,

and

$Z_{4,3}^2 = \{ x \in Z_{4,2}' \text{ of the form (8.5) with } x_{23}x_{32} + x_{11} = 0 \}$.

They are both relatively $H_{4,2}$ stable closed submanifolds of $Z_{4,2}'$. Therefore both

$Z_{4,3}^1 = H_{4,2}Z_{4,3}^1$ and $Z_{4,3}^2 = H_{4,2}Z_{4,3}^2$

are closed submanifolds of $Z_{4,2}$.

**Lemma 8.6.** The manifold $Z_{4,2} \setminus (Z_{4,3}^1 \cup Z_{4,3}^2)$ is unipotently $\chi_{4,2}$ incompatible.

**Proof.** Let $x \in Z_{4,2}' \setminus (Z_{4,3}^1 \cup Z_{4,3}^2)$ be as in (8.5). Set

\[ u(x, t) = \begin{bmatrix} 1 & 0 & x_{11} & x_{32}^{-1}t & 0 & 0 & 0 \\
t & 1 & 0 & x_{23}t & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & t & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & t & 1 \end{bmatrix}, \]

and

\[ v(x, t) = \begin{bmatrix} 1 & t & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
x_{11} & x_{23}^{-1}t & 0 & 1 & t & 0 \\
0 & x_{32}t & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & t \\
0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]
Then \( u(x, t)x = xv(x, t) \) and
\[
\chi_{4,2}(u(x, t), v(x, t)^{-\tau}) = \psi_{x}(x_{32}^{-1} - x_{23}^{-1}l(x_{11} + x_{23}x_{32})) \neq 1
\]
for a suitably chosen \( t \). The lemma follows.

\[\square\]

**Lemma 8.7.** The submanifold \( Z_{1,3}^{1,3} \) is metrically proper in \( G_{4,2} \).

**Proof.** Let \( x \in Z_{4,3}^{1,3} \)' as in (8.5). Write \( a = x_{23} = x_{32} \) and
\[
x' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a \\
0 & 0 & 0 & 0 & -a & 0
\end{bmatrix}.
\]

Then one checks that
\[
T_{x}(Z_{4,3}^{1,3}) = T_{x}(Z_{4,3}^{1,3}') + (\text{Lie}S_{4,2})x + x(\text{Lie}S_{4,2})^\tau \subset (Kx')^\perp.
\]
Note that
\[
x'x = a(\text{diag}(0, 1, -1, 0, 1, -1)),
\]
which spans a nondegenerate \( K \) subspace of \( T_{e}(G_{4,2}) \). Here and as usual, \( \text{diag} \) represents a diagonal matrix (with the obvious diagonal entries). The lemma follows, as in the proof of Lemma 8.5.

\[\square\]

**Lemma 8.8.** The submanifold \( Z_{1,3}^{2,3} \) is metrically proper in \( G_{4,2} \).

**Proof.** Let \( x \in Z_{4,3}^{1,3} \)' as in (8.5). Write
\[
x' = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_{23} & 0 & 0 & 0 \\
0 & x_{32} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_{23}x_{32} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]

Then one checks that
\[
T_{x}(Z_{4,3}^{2,3}) = T_{x}(Z_{4,3}^{2,3}') + (\text{Lie}S_{4,2})x + x(\text{Lie}S_{4,2})^\tau \subset (Kx')^\perp,
\]
and
\[
x'x = x_{23}x_{32}(\text{diag}(-1, 1, 1, -1, 0, 0)).
\]
The lemma follows, as before.

\[\square\]

We now consider the \( H_{4,2} \) stable filtration
\[
Z_{4} \supset Z_{4,1} \supset Z_{4,2} \supset Z_{4,3}^{1,3} \cup Z_{4,3}^{2,3} \supset Z_{4,3}^{1} \supset \emptyset.
\]
In view of the proceeding lemmas, the proof of Proposition 8.1 is complete.

9. A SUBMANIFOLD \( Z_6 \) OF \( \text{GL}_6 \)

Recall (from Section 2)

\( Z_6 = G_R, \) with \( R = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}. \)

Clearly \( Z_6 \) is an \( H = S \times S \) stable submanifold of \( G \), as with each \( G_R \).

The purpose of this section is to prove the following proposition. Again it will take a number of steps.

**Proposition 9.1.** The submanifold \( Z_6 \) of \( G \) has \( U_\chi M \) property.

Denote by \( 3_6 \) all matrices in \( Z_6 \) of the form

\[
(9.2) \quad x = \begin{bmatrix} \ast & \ast & \ast & \ast & x_{15} & 0 \\ \ast & \ast & \ast & \ast & x_{25} & 0 \\ \ast & \ast & 0 & 0 & x_{35} & 0 \\ \ast & \ast & 0 & 0 & x_{45} & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
\]

which forms an \( H \) slice of \( Z_6 \). Write

\[
3_{6,1} = \{ x \in 3_6 \text{ of the form (9.2) with } x_{35} = x_{53} \},
\]

and

\[
3_{6,2} = \{ x \in 3_6 \text{ of the form (9.2) with } x_{35} = x_{53} = 0 \}.
\]

They are both relatively \( H \) stable closed submanifolds of \( 3_6 \). Therefore both

\[
Z_{6,1} = H3_{6,1} \quad \text{and} \quad Z_{6,2} = H3_{6,2}
\]

are closed submanifolds of \( Z_6 \).

**Lemma 9.2.** The submanifold \( Z_6 \setminus Z_{6,1} \) is unipotently \( \chi \) incompatible.

**Proof.** Let \( x \in 3_6 \setminus 3_{6,1} \) be as in (9.2). Write

\[
u(x, t) = \begin{bmatrix} 1 & 0 & 0 & 0 & x_{15}t & 0 \\ 0 & 1 & 0 & 0 & x_{25}t & 0 \\ 0 & 0 & 1 & 0 & x_{35}t & 0 \\ 0 & 0 & 0 & 1 & x_{45}t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},
\]
and
\[ v(x, t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ tx_51 & tx_52 & tx_53 & tx_54 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \]

Then \( u(x, t) x = xv(x, t) \), and the lemma follows, as before. \( \square \)

**Lemma 9.3.** The submanifold \( Z_{6,1} \setminus Z_{6,2} \) is metrically proper in \( G \).

**Proof.** Every element of \( Z_{6,1} \setminus Z_{6,2} \) is in the same \( H \)-orbit as an element of the form
\[ x = \begin{bmatrix} * & * & 0 & * & 0 & 0 \\ * & * & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \]

Fix such an \( x \). Then
\[ T_x(Z_{6,1}) = T_x(Z_{6,1}) + \text{Lie}(S)x + x\text{Lie}(S^r) \subset \mathfrak{g}l_6(\mathbb{K}_{35=53}) = (\mathbb{K}x')^\perp, \]
where \( x' = e_{35} - e_{53} \). Now \( x'x = a(e_{33} - e_{55}) \) and we finish the proof, as before. \( \square \)

Denote by \( Z'_{6,2} \) all matrices in \( Z_{6,2} \) of the form
\[
(9.3) \quad x = \begin{bmatrix}
    x_{11} & x_{12} & x_{13} & 0 & 0 & 0 \\
    x_{21} & x_{22} & x_{23} & 0 & 0 & 0 \\
    x_{31} & x_{32} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]
which also forms an \( H \) slice of \( Z_{6,2} \). Set
\[ Z'_{6,3} = \{ x \in Z'_{6,2} \text{ of the form (9.3) with } x_{13} = x_{31} \}, \]
and
\[ Z'_{6,4} = \{ x \in Z'_{6,2} \text{ of the form (9.3) with } x_{13} = x_{31} = 0 \}. \]

They are both relatively \( H \) stable closed submanifolds of \( Z'_{6,2} \). Therefore both
\[ Z_{6,3} = HZ'_{6,3} \text{ and } Z_{6,4} = HZ'_{6,4} \]
are closed submanifolds of \( Z_{6,2} \).

**Lemma 9.4.** The manifold \( Z_{6,2} \setminus Z_{6,3} \) is unipotently \( \chi \) incompatible.
Proof. This is identical to the proof of Lemma 7.4. □

Lemma 9.5. The submanifold $Z_{6,3} \setminus Z_{6,4}$ is metrically proper in $G$.

Proof. Every matrix in $Z'_{6,3} \setminus Z'_{6,4}$ is in the same $H$ orbit as a matrix of the form

$$ x = \begin{bmatrix}
0 & 0 & a & 0 & 0 & 0 \\
0 & x_{22} & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}. $$

Fix such an $x$. Then

$$ T_x(Z_{6,3}) = T_x(Z'_{6,3}) + \text{Lie}(S)x + x\text{Lie}(S^\tau) \subset \mathfrak{gl}_6(\mathbb{K})_{13=31} = (\mathbb{K}x')^{'}, $$

where $x' = e_{13} - e_{31}$. Now $x'x = a(e_{11} - e_{33})$ and we finish the proof, as before. □

Denote by $Z''_{6,4}$ all matrices in $Z'_{6,4}$ of the form

$$ x = \begin{bmatrix}
x_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & x_{23} & 0 & 0 & 0 & 0 \\
0 & x_{32} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}, $$

which also forms an $H$ slice of $Z_{6,4}$. Set

$$ Z_{6,5} = \{ x \in Z''_{6,4} \text{ of the form (9.4) with } x_{23} = x_{32} \}, $$

which is a relatively $H$ stable closed submanifolds of $Z''_{6,4}$. Therefore

$$ Z_{6,5} = HZ''_{6,5} $$

are closed submanifolds of $Z_{6,4}$.

Lemma 9.6. The manifold $Z_{6,4} \setminus Z_{6,5}$ is unipotently $\chi$ incompatible.

Proof. Let $x \in Z''_{6,4} \setminus Z''_{6,5}$ be as in (9.4). Set

$$ u(x, t) = \begin{bmatrix}
1 & 0 & x_{11}x_{32}^{-1}t & 0 & 0 & 0 \\
t & 1 & 0 & 0 & x_{23}t & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & t & 1 & 0 & t \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & t & 1 \\
\end{bmatrix}. $$
and
\[ v(x, t) = \begin{bmatrix}
1 & t & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
x_{23}^{-1}t x_{11} & 0 & 1 & t & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & tx_{32} & 0 & 0 & 1 & t \\
0 & 0 & 0 & t & 0 & 1
\end{bmatrix}. \]

Then
\[ u(x, t) = \begin{bmatrix}
x_{11} & x_{11}t & 0 & 0 & 0 & 0 \\
t x_{11} & 0 & x_{23} & x_{23}t & 0 & 0 \\
0 & x_{32} & 0 & 0 & 0 & 0 \\
0 & tx_{32} & 0 & 0 & 1 & t \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 1
\end{bmatrix} = xv(x, t) \]
and the lemma follows, as before. \(\square\)

**Lemma 9.7.** The submanifold \(Z_{6,5}\) is metrically proper in \(G\).

**Proof.** Let \(x \in Z'_{6,5}\) be as in (9.4), with \(x_{23} = x_{32} = a\). Write

\[ x' = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a & 0 & 0 \\
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \]

Then
\[ T_x(Z_{6,5}) = T_x(Z_{6,5}) + \text{Lie}(S)x + x\text{Lie}(S^*) \subset (\mathbb{K}x')^\perp, \]
and \(x'x = a(\text{diag}(0, 1, -1, -1, 1, 0))\). The lemma follows, as before. \(\square\)

We now consider the \(H\) stable filtration
\[ Z_6 \supset Z_{6,1} \supset Z_{6,2} \supset Z_{6,3} \supset Z_{6,4} \supset Z_{6,5} \supset \emptyset. \]
In view of the proceeding lemmas, the proof of Proposition 9.1 is complete.

10. Small submanifolds of \(GL_6\)

Recall the open subset \(G'\) of \(G\), defined in Equation (2.3). The methods we have developed thus far allow us to deal with tempered generalized functions in \(G'\). We expect that for the rest of \(G_N\)'s in \(G \setminus G'\), all of them have \(U_xM\) property, just as in the case of \(Z_6\) (which
is $G_R$ with $R = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$. If this were done, we would have sufficient preparation to attempt a proof of Theorem B. We shall not take this route. Instead we will give another method, which significantly shortens the required argument.

In the sequel, we will always assume that $R$ is one of the 21 rank matrices in Section 2. Recall the vector fields $X_{\text{left}}$ and $X_{\text{right}}$, defined in Equation (2.4).

The first purpose of this section is to prove the following proposition. We shall divide it into a number of lemmas (Lemmas 10.2, 10.4, 10.5, Lemma 10.6).

**Proposition 10.1.** Assume that $R$ is not one of the followings:

- $\begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$
- $\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$
- $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$
- $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.

Then either $X_{\text{left}}$ or $X_{\text{right}}$ is transversal to the double coset $G_R$.

**Lemma 10.2.** If the lower right entry of $R$ is zero, then $X_{\text{left}}$ is transversal to $G_R$.

**Proof.** Assume that there is a $x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & 0 \end{bmatrix} \in G_R$ such that

$$X_{\text{left}}(x) \in T_x(G_R),$$

i.e.,

$$\begin{bmatrix} 0 & x_{11} & x_{12} \\ 0 & x_{21} & x_{22} \\ 0 & x_{31} & x_{32} \end{bmatrix} \in \text{Lie}(P)x + x\text{Lie}(P^T).$$

Note that the lower right $2 \times 2$ block of every element of $\text{Lie}(P)x + x\text{Lie}(P^T)$ is 0. Therefore $x_{32} = 0$, which further implies that the lower right $2 \times 4$ block of every element of $\text{Lie}(P)x + x\text{Lie}(P^T)$ is 0. Therefore $x_{31} = 0$. This contradicts the fact that $x$ is invertible. □

The following lemma provides a technical simplification.

**Lemma 10.3.** Let $x, y$ be two matrices in $G_R$ such that $PxS^r = PyS^r$. Then

$$X_{\text{left}}(x) \in T_x(G_R) \quad \text{if and only if} \quad X_{\text{left}}(y) \in T_y(G_R).$$
Proof. Write
\[ y = pxq, \quad p \in P, q \in S^\tau, \]
and assume that \( X_{\text{left}}(x) \in T_x(G_R), \) i.e.,
\[ xx_{\text{left}} \in \text{Lie}(P)x + x\text{Lie}(P^\tau). \]
One easily checks that
\[ x_{\text{left}}q - qx_{\text{left}} \in \text{Lie}(P^\tau). \]
Therefore
\[ yx_{\text{left}} = pxqx_{\text{left}} \]
\[ \in pxx_{\text{left}}q + px\text{Lie}(P^\tau) \]
\[ \subset p(\text{Lie}(P)x + x\text{Lie}(P^\tau))q + px\text{Lie}(P^\tau) \]
\[ = \text{Lie}(P)y + y\text{Lie}(P^\tau). \]
The last equality holds because
\[ p\text{Lie}(P) = \text{Lie}(P)p = \text{Lie}(P), \quad q\text{Lie}(P^\tau) = \text{Lie}(P^\tau)q = \text{Lie}(P^\tau). \]

\[ \square \]

**Lemma 10.4.** If the second row of \( R \) is \([1 \ 1]\), then \( X_{\text{left}} \) is transversal to \( G_R \).

Proof. Let \( R \) be as in the lemma. Then every matrix in \( G_R \) is in the same \( P-S^\tau \) double coset with a matrix of the form
\[ x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\
 x_{21} & x_{22} & x_{23} \\
 x_{31} & 0 & \delta_2 \end{bmatrix} \in G_R, \]
where
\[ \delta_2 = \begin{bmatrix} 0 & 0 \\
 0 & 1 \end{bmatrix}. \]
Assume that \( X_{\text{left}}(x) \in T_x(G_R), \) i.e.,
\[ \begin{bmatrix} 0 & x_{11} & x_{12} \\
 0 & x_{21} & x_{22} \\
 0 & x_{31} & 0 \end{bmatrix} \in \text{Lie}(P)x + x\text{Lie}(P^\tau). \]
Note that the middle 2 \( \times \) 2 block of the last two rows of every matrix in \( \text{Lie}(P)x + x\text{Lie}(P^\tau) \) has the form
\[ \delta_2u, \quad u \in \mathfrak{gl}_2(\mathbb{K}). \]
This implies that the first row of \( x_{31} \) is zero, and consequently, the fifth row of \( x \) is zero, which contradicts the fact that \( x \) is invertible. \( \square \)
Similarly, we have

**Lemma 10.5.** If the second column of \( R \) is \(
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\), then \( X_{\text{right}} \) is transversal to \( G_R \).

**Lemma 10.6.** If
\[
R = \begin{bmatrix}
2 & 2 \\
2 & 2
\end{bmatrix},
\]
then \( X_{\text{left}} \) is transversal to \( G_R \).

**Proof.** Every matrix in \( G_R \) is in the same \( P-S^\tau \) double coset with a matrix of the form
\[
x = \begin{bmatrix}
x_{11} & x_{12} & 0 \\
x_{21} & 0 & 0 \\
0 & 0 & 1
\end{bmatrix} \in G_R.
\]
Assume that
\[
X_{\text{left}}(x) \in T_x(G_R),
\]
i.e.,
\[
\begin{bmatrix}
0 & x_{11} & x_{12} \\
0 & x_{21} & 0 \\
0 & 0 & 0
\end{bmatrix} \in \text{Lie}(P)x + x\text{Lie}(P^\tau).
\]
Note that the central \( 2 \times 2 \) block of every matrix in \( \text{Lie}(P)x + x\text{Lie}(P^\tau) \) is zero, which implies that \( x_{21} = 0 \). This contradicts the fact that \( x \) is invertible. \( \square \)

We next explore consequences of the transversality of \( X_{\text{left}} \) or \( X_{\text{right}} \).

**Lemma 10.7.** Let \( \Delta \) be the Casimir element of \( G \), as in the Introduction. Then there exists a nonzero number \( c \), an element \( \lambda \in \mathbb{K}^\times \), and a differential operator \( D_{\text{left}} \) on \( G \), which is tangential to every \( P-P^\tau \) double coset of \( G \), such that
\[
\Delta f = (cX_{\text{left}}(\lambda) + D_{\text{left}})f
\]
for all \( f \in C^\infty(G) \). Here \( X_{\text{left}}(\lambda) \) is the left invariant vector field on \( G \) whose tangent vector at \( x \in G \) is \( \lambda x x_{\text{left}} \), and \( x_{\text{left}} \) is given in Equation (2.4). The same is true if one replaces “left” by “right” everywhere.

**Proof.** The Lie algebra \( \mathfrak{g} \) of \( G \) has a decomposition
\[
\mathfrak{g} = \mathfrak{n} + \mathfrak{l} + \mathfrak{n}^\tau,
\]
where \( \mathfrak{n} \) is the Lie algebra of the unipotent radical \( N \) of \( P \), and \( \mathfrak{l} \) is the Lie algebra of the Levi factor \( \text{GL}_2(\mathbb{K}) \times \text{GL}_2(\mathbb{K}) \times \text{GL}_2(\mathbb{K}) \). Recall that
\( \mathfrak{g} \) is equipped with the real trace form. Let \( X_1, X_2, \ldots, X_r \) be a basis of \( \mathfrak{n} \), and write
\[
\Delta_1 = X_1X'_1 + X_2X'_2 + \cdots + X_rX'_r \in U(\mathfrak{g}),
\]
where \( X'_1, X'_2, \ldots, X'_r \) is the dual basis of \( X_1, X_2, \ldots, X_r \) in \( \mathfrak{n}^\tau \). Note that \( \Delta_1 \) is independent of the choice of basis of \( \mathfrak{n} \). We identify elements of \( U(\mathfrak{g}) \) with left invariant (real) differential operators on \( G \) as usual. It is then easy to see that
\[
(10.1) \quad \Delta - 2\Delta_1 \in U(\mathfrak{l}).
\]
Let \( d\chi_S \) be the differential of the character \( \chi_S \). Write
\[
\chi_{n^\tau}(X) = d\chi_S(-X^\tau), \quad X \in \mathfrak{n}^\tau,
\]
which defines a character of \( \mathfrak{n}^\tau \). Then every generalized function \( f \in C_{\chi}^{-\infty}(G) \) satisfies
\[
Xf = -\chi_{n^\tau}(X)f, \quad \text{for all } X \in \mathfrak{n}^\tau.
\]
Now choose \( X_1 \) to be perpendicular to the kernel of \( \chi_{n^\tau} \). This is unique up to a multiple in \( \mathbb{R} \times \mathbb{K}^\times \), and has the form \( X_{\text{left}}(\lambda) \) for some \( \lambda \in \mathbb{K}^\times \). This choice of \( X_1 \) also implies that
\[
\chi_{n^\tau}(X'_2) = \chi_{n^\tau}(X'_3) = \cdots = \chi_{n^\tau}(X'_r) = 0,
\]
and \( \chi_{n^\tau}(X'_1) \) is a nonzero number.
Therefore
\[
(10.2) \quad \Delta_1 f = -\chi_{n^\tau}(X'_1)X_{\text{left}}(\lambda)f \quad \text{for all } f \in C_{\chi}^{-\infty}(G).
\]
Equations (10.1) and (10.2) will now imply the lemma, in view of the fact that a differential operator in \( U(\mathfrak{l}) \) is tangential to every \( P-P^\tau \) double coset. \( \square \)

We now prove the following

**Proposition 10.8.** Let \( f \in C_{\chi}^{-\infty}(G) \). If \( f \) is an eigenvector of \( \Delta \), and \( f \) vanishes on \( G' \), then \( f = 0 \).

**Proof.** Take a sequence
\[
G' = G_4^{\text{open}} \subset G_5^{\text{open}} \subset \cdots \subset G_{20}^{\text{open}} \subset G_{21}^{\text{open}} = G,
\]
of open subsets of \( G \) so that every difference \( G_i^{\text{open}} \setminus G_{i-1}^{\text{open}} \) is a \( P-P^\tau \) double coset, \( i = 5, 6, \ldots, 21 \). Denote by \( f_i \) the restriction of \( f \) to \( G_i^{\text{open}} \). We shall use induction to show that all \( f_i \)'s are zero. Thus assume that \( f_{i-1} = 0 \).
By Proposition 10.1, either $X_{\text{left}}$ or $X_{\text{right}}$ is transversal to $G_i^{\text{open}} \setminus G_{i-1}^{\text{open}}$. Without loss of generality assume that $X_{\text{left}}$ is transversal to $G_i^{\text{open}} \setminus G_{i-1}^{\text{open}}$. Lemma 10.7 implies that 

$$(X_{\text{left}}(\lambda) + D)f_i = 0,$$

where $D$ is a differential operator on $G_i^{\text{open}}$ which is tangential to $G_i^{\text{open}} \setminus G_{i-1}^{\text{open}}$. It is clear that $X_{\text{left}}$ is transversal to $G_i^{\text{open}} \setminus G_{i-1}^{\text{open}}$ will imply the same for $X_{\text{left}}(\lambda)$. Invoking Lemma 3.3, we see that $f_i = 0$. □

11. PROOF OF THEOREM B

We are in the setting of Section 2. We will first examine the case where the quaternion algebra $\mathbb{D}$ is split, namely $G = \text{GL}_6(\mathbb{K})$. We start with the following

Lemma 11.1.  (a) If $\mathcal{Z}$ is a unipotently $\chi_{4,2}$-incompatible $H_{4,2}$ stable submanifold of $G_{4,2}$, then $Z = H\mathcal{Z}$ is a unipotently $\chi$-incompatible submanifold of $G$.

(b) If $\mathcal{Z}$ is a metrically proper $H_{4,2}$ stable submanifold of $G_{4,2}$, then $Z = H\mathcal{Z}$ is a metrically proper submanifold of $G$.

Proof. Part (a) is clear. For Part (b), we note

$$Z = H\mathcal{Z} = U_{4,2}\mathcal{Z}U_{4,2}^T,$$

where

$$U_{4,2} = \left\{ \begin{bmatrix} I_2 & 0 & d \\ 0 & I_2 & c \\ 0 & 0 & I_2 \end{bmatrix} \mid c, d \in \mathfrak{gl}_2(\mathbb{K}) \right\}.$$

By invariance of the metric, we only need to show that $Z$ is metrically proper at every point $z \in \mathcal{Z}$, i.e., the tangent space $T_z(Z)$ is contained in a nondegenerate proper subspace of $T_z(G)$.

First we assume that $z$ is the identity matrix $e$. Then

$$T_e(Z) = T_e(\mathcal{Z}) + (\text{Lie}(U_{4,2}) + \text{Lie}(U_{4,2}^T))$$

is metrically proper since

$$T_e(\mathcal{Z})$$

is metrically proper in $T_e(\text{GL}_4(\mathbb{K}) \times \text{GL}_2(\mathbb{K}))$, and

$$T_e(G) = T_e(\text{GL}_4(\mathbb{K}) \times \text{GL}_2(\mathbb{K})) \oplus (\text{Lie}(U_{4,2}) + \text{Lie}(U_{4,2}^T))$$

is an orthogonal decomposition.

Now let $z \in \mathcal{Z}$. Note that

$$z^{-1}Z = U_{4,2}(z^{-1}\mathcal{Z})U_{4,2}^T,$$

where
and
\[ z^{-1}Z \] is metrically proper in \( \text{GL}_4(K) \times \text{GL}_2(K) \).
Therefore the above argument implies that \( z^{-1}Z \) is metrically proper at \( e \). Using the left multiplication by \( z \)
\[ l_z : (G, z^{-1}Z, e) \to (G, Z, z), \]
we conclude that \( Z \) is metrically proper at \( z \). \qed

Recall the open submanifold \( G' \) of \( G \). Recall also the submanifolds \( M_2 \subset G'_{4,2} \cap G'_{2,4}, M_3 \subset G'_{4,2} \) and \( M_3^o \subset G'_{2,4} \).

Let
\[ G'' = HM_2 \cup HM_3 \cup HM_3^o \subset G'. \]

**Proposition 11.2.** The pairs \( (H_2; M_2), (H_3; M_3) \) and \( (H_3^o; M_3^o) \) form a complete descent of \( (H; G') \). Consequently if \( f \in C^{-\infty}(G') \) is an eigenvector of \( \Delta \), and \( f \) vanishes on \( G'' \), then \( f = 0 \).

**Proof.** It is easy to check that \( M_2, M_3 \) and \( M_3^o \) are local \( H \) slices of \( G' \). Similarly, \( G' \setminus G'' = Z_6 \sqcup HZ_4 \sqcup HW_4 \);
\( Z_6 \) is closed in \( G' \setminus G'' \);
Both \( HZ_4 \) and \( HW_4 \) are closed in \( HZ_4 \sqcup HW_4 \).
By Proposition 9.1, the submanifold \( Z_6 \) has \( U \chi M \) property. By Proposition 8.1 and Lemma 11.1, the submanifold \( HZ_4 \) has \( U \chi M \) property. Similarly, \( HW_4 \) also has \( U \chi M \) property. Therefore the \( H \) stable closed subset \( Z_6 \sqcup HZ_4 \sqcup HW_4 \) of \( G' \) has \( U \chi M \) property. The assertion follows. \qed

Now set
\[ \tilde{H} = \{ 1, \tau \} \rtimes H = \{ 1, \tau \} \rtimes (S \times S), \]
where the semidirect product is defined by the action
\[ \tau(g_1, g_2) = (g_2, g_1), \quad g_1, g_2 \in S. \]
Extend \( \chi \) to a character \( \tilde{\chi} \) of \( \tilde{H} \) by requiring
\[ \tilde{\chi}(\tau) = -1, \]
and extend the action on \( G \) of \( H \) to \( \tilde{H} \) by requiring
\[ \tau x = x^\tau. \]
Proposition 11.3. One has that $C^{-\xi}(G'') = 0$.

Proof. As a direct consequence of Lemma 5.1 (b) and Proposition 6.1, we have

$$C^{-\xi}(HM_2) = 0.$$  

Similarly, Lemma 5.1 (b) and Proposition 7.1 imply that

$$C^{-\xi}(HM_3) = 0,$$

and likewise,

$$C^{-\xi}(HM^0_3) = 0.$$  

The proposition follows from the above three vanishing results. $\square$

We are now ready to prove Theorem B for the split case. Let $f$ be as in the theorem. Write

$$f^\tau(x) = f(x^\tau).$$

Then $f^\tau$ still satisfies (1.4), which implies that

$$f - f^\tau \in C^{-\xi}(G).$$

From Proposition 11.3, we know that $f - f^\tau = 0$ on $G''$. Note that $\tau$ commutes with the differential operator $\Delta$ on $G$. So $f^\tau$ is an eigenvector of $\Delta$, with the same eigenvalue as that of $f$. Therefore $f - f^\tau$ is again an eigenvector of $\Delta$. Proposition 11.2 implies that $f - f^\tau = 0$ on $G'$. By Proposition 10.8, we finally conclude that

$$f - f^\tau = 0.$$  

In the rest of the section, we sketch the proof of Theorem B for the case $D = \mathbb{H}$ (the real quaternion division algebra), which is much simpler than the split case of $GL_6(K)$. As in the split case, define a parabolic subgroup $P_H$ containing $S_H$ and the rank matrix $R(x)$ (for $x \in G_H$) in the obvious way. Then $R(x)$ takes the following 6 possible values:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

which gives rise to 6 $P-P'$ double cosets $\{G_{\mathbb{H},R}\}$.

Let $f$ be as in the theorem. If we replace $GL_6(K)$ by $\mathbb{H}^\times$, the analog of Proposition 6.1 still holds. This will imply that $f - f^\tau$ vanishes on $G_{\mathbb{H},R_{\text{open}}}$. As in the split case, we define a left invariant vector field $X_{\text{left}}$
on $G_H$ using $x_{\text{left}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}l(H)$. Then as in Section 10, one checks that $X_{\text{left}}$ is transversal to every double coset $G_{H,R}$ for $R \neq R_{\text{open}}$. We conclude as in the split case that $f - f^\tau = 0$.

**Remarks:**

(a) Theorem B in fact holds without the tempered-ness condition on $f$. But we shall not prove or exploit this fact.

(b) We also expect Theorem B to hold without the assumption that $f$ is an eigenvector of $\Delta_D$.

12. **Proof of Theorem A**

The argument of this section is standard, and it works for a more general real reductive group $G$.

By a representation of $G$, we mean a continuous linear action of $G$ on a complete, locally convex, Hausdorff complex topological vector space. We say that a representation $V$ of $G$ is a Harish-Chandra smooth representation if it is Fréchet, smooth, of moderate growth, admissible and $Z(\mathfrak{g}_C)$ finite. Here and as usual, $Z(\mathfrak{g}_C)$ is the center of the universal enveloping algebra $U(\mathfrak{g}_C)$ of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. The reader may consult [C89, S85, W92] for more details about Harish-Chandra smooth representations.

Let $V_1$ and $V_2$ be two Harish-Chandra smooth representations of $G$. We say that they are contragredient to each other if there exists a nondegenerate continuous $G$ invariant bilinear form $\langle \cdot, \cdot \rangle : V_1 \times V_2 \to \mathbb{C}$.

If $V_1$ and $V_2$ are contragredient to each other, then $V_1$ is irreducible if and only if $V_2$ is.

Let $S_1$ and $S_2$ be two closed subgroups of $G$, with continuous characters (not necessarily unitary)

$$\chi_{S_i} : S_i \to \mathbb{C}^\times, \quad i = 1, 2.$$ 

Let $\tau$ be a continuous anti-automorphism of $G$ (not necessarily an anti-involution).

The following is a generalization of the usual Gelfand-Kazhdan criterion. See [SZ08] for a detailed proof. Recall that $U(\mathfrak{g}_C)^G$ is identified with the space of bi-invariant differential operators on $G$, as usual.

**Proposition 12.1.** Assume that for every $f \in C^{-\xi}(G)$ which is an eigenvector of $U(\mathfrak{g}_C)^G$, the conditions

$$f(sx) = \chi_{S_1}(s)f(x), \quad s \in S_1,$$

where $\chi_{S_1}$ is the continuous character of $S_1$. Then $f \in C^\infty(G)$.

Proof: We can write $f$ in the form $f = \sum a_k G_{H_k}$, where $a_k$ are complex numbers and $G_{H_k}$ are the irreducible Harish-Chandra smooth representations of $G$.

Using the contragredient $f^\tau$, we see that $f^\tau(sx) = \chi_{S_1}(s)f^\tau(x), \quad s \in S_1$. Then $a_k G_{H_k}^\tau = \chi_{S_1}(s)a_k G_{H_k}$, which gives $f^\tau = \sum a_k G_{H_k^\tau}$.

By the uniqueness of the decomposition, we conclude that $f = f^\tau$. Therefore, $f \in C^\infty(G)$. \hfill $\blacksquare$
and
\[ f(xs) = \chi_{S_2}(s)^{-1} f(x), \quad s \in S_2 \]
imply that
\[ f(x^\tau) = f(x). \]
Then for any two irreducible Harish-Chandra smooth representations \( V_1 \) and \( V_2 \) of \( G \) which are contragredient to each other, one has that
\[ \dim \text{Hom}_{S_1}(V_1, \mathbb{C}_{\chi_{S_1}}) \dim \text{Hom}_{S_2}(V_2, \mathbb{C}_{\chi_{S_2}}) \leq 1. \]

Now we finish the proof of Theorem A. Assume that \( V_1 = V \) is an irreducible Harish-Chandra smooth representation of \( G \). Define the irreducible Harish-Chandra smooth representation \( V_2 \) of \( G \) as follows. The representation \( V_2 \) equals to \( V \) as a topological vector space, and the action \( \rho_2 \) of \( G \) on \( V_2 \) is given by
\[ \rho_2(g)v = \rho_1(g^{-\tau})v, \quad g \in G, v \in V, \]
where \( \rho_1 \) is the action of \( G \) on \( V_1 \). Using character theory and the fact that \( g \) is always conjugate to \( g^\tau \), we conclude that \( V_1 \) and \( V_2 \) are contragredient to each other [AGS07, Theorem 2.4.2]. Now let
\[ S_1 = S, \quad S_2 = S^\tau, \quad \chi_{S_1} = \chi_S, \]
and
\[ \chi_{S_2}(g) = \chi_{S}(g^{-\tau}), \quad g \in S_2. \]
Theorem B says that the assumption of Proposition 12.1 is satisfied, and so
\[ \dim \text{Hom}_{S_1}(V_1, \mathbb{C}_{\chi_{S_1}}) \dim \text{Hom}_{S_2}(V_2, \mathbb{C}_{\chi_{S_2}}) \leq 1. \]
Note that by the identification \( V_1 = V_2 = V \) as well as the explicit actions, we have
\[ \text{Hom}_{S_1}(V_1, \mathbb{C}_{\chi_{S_1}}) = \text{Hom}_{S_2}(V_2, \mathbb{C}_{\chi_{S_2}}) = \text{Hom}_{S}(V, \mathbb{C}_{\chi_{S}}). \]
Hence
\[ \dim \text{Hom}_{S}(V, \mathbb{C}_{\chi_{S}}) \leq 1, \]
and the proof is complete.

### 13. Some consequences

#### 13.1. Uniqueness of trilinear forms

The following theorem is proved in [L01] (in an exhaustive approach), and its p-adic analog was proved much earlier in [P90, Theorem 1.1].
**Theorem 13.1.** Let $V$ be an irreducible Harish-Chandra smooth representation of $GL_2(\mathbb{K}) \times GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$. Then

$$\dim \text{Hom}_{GL_2(\mathbb{K})}(V, \mathbb{C}_\chi) \leq 1.$$ 

Here we view $GL_2(\mathbb{K})$ as the diagonal subgroup of $GL_2(\mathbb{K}) \times GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$, and $\chi = \chi_K \circ \text{det}$ is a character of $GL_2(\mathbb{K})$.

**Proof.** By the Gelfand-Kazhdan criterion, one just needs to show the following: let $GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$ act on $G_{2,2,2} = GL_2(\mathbb{K}) \times GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$ by

$$(g_1, g_2)(x, y, z) = (g_1 x g_2^\tau, g_1 y g_2^\tau, g_1 z g_2^\tau), \quad g_1, g_2 \in GL_2(\mathbb{K}).$$

Denote by $\chi_{2,2}$ the character of $GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$ given by

$$\chi_{2,2}(g_1, g_2) = \chi_K \circ \text{det}(g_1) \chi_K \circ \text{det}(g_2), \quad g_1, g_2 \in GL_2(\mathbb{K}).$$

Then for all $f \in C^{-\xi}_{\chi_{2,2}}(G_{2,2,2})$, we have

$$f(x^\tau, y^\tau, z^\tau) = f(x, y, z).$$

To show the above, we observe that $M_2 = GL_2(\mathbb{K}) \times GL_2(\mathbb{K}) \times \{I_2\}$ is a $GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$ slice of $G_{2,2,2}$, which is stable under $H_2 = \{(x, x^\tau) \mid x \in GL_2(\mathbb{K})\} \subset GL_2(\mathbb{K}) \times GL_2(\mathbb{K})$ and $\tau$. The result then follows from Lemma 5.1 (b) and Proposition 6.1. $\Box$

As noted near the end of Section 11, if we replace $GL_2(\mathbb{K})$ by $\mathbb{H}^\times$, the analog of Proposition 6.1 still holds (again by using Proposition E of Section 2). Thus the analog of Theorem 13.1 for $\mathbb{H}^\times$ holds. Of course this is well-known and easier.

**13.2. Uniqueness of the Jacquet-Shalika model for $GL_3(\mathbb{K})$**

Let $L_3$ and $N_3$ be the subgroups of $GL_3(\mathbb{K})$, as in Section 7. Write $S_3 = L_3 N_3$, and

$$\chi_{S_3}(\begin{bmatrix} 1 & 0 & d \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{bmatrix}) = \chi_K \psi_K(d),$$

which defines a character of $S_3$.

**Theorem 13.2.** Let $V$ be an irreducible Harish-Chandra smooth representation of $GL_3(\mathbb{K})$. Then

$$\dim \text{Hom}_{S_3}(V, \mathbb{C}_{\chi_{S_3}}) \leq 1.$$
Proof. As a corollary of Proposition 7.2, we know that if \( f \in C^{-\xi}(\operatorname{GL}_3(K)) \) satisfies
\[
f(sx) = f(xs^\tau) = \chi_{S_3}(s)f(x), \quad \text{for all } s \in S_3,
\]
then
\[
f(x^\tau) = f(x).
\]
The theorem then follows, as in Section 12. \( \square \)

We remark that the p-adic analog of Theorem 13.2 holds true, as the same proof goes through.

Remark: By inducing the character \( \chi_{S_3} \) to a Hisenberg group, one may obtain uniqueness of the Fourier-Jacobi model for \( \operatorname{GL}_3(K) \).

13.3. Uniqueness of a certain model for \( \operatorname{GL}_4(K) \times \operatorname{GL}_2(K) \)

Recall from the Introduction:
\[
S_{4,2} = (\operatorname{GL}_4(K) \times \operatorname{GL}_2(K)) \cap S = \left\{ \begin{bmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \in G \right\},
\]
and \( \chi_{S_{4,2}} = \chi_S|_{S_{4,2}} \).

Theorem 13.3. Let \( V \) be an irreducible Harish-Chandra smooth representation of \( \operatorname{GL}_4(K) \times \operatorname{GL}_2(K) \). Then
\[
\dim \hom_{S_{4,2}}(V, \mathbb{C}\chi_{S_{4,2}}) \leq 1.
\]

Proof. Denote by \( \Delta_{4,2} \) the Casimir operator on \( \operatorname{GL}_4(K) \times \operatorname{GL}_2(K) \) associated to the real trace form. Arguing as in Section 12, we will just need to show that, if \( f \in C^{-\xi}(\operatorname{GL}_4(K) \times \operatorname{GL}_2(K)) \) is an eigenvector of \( \Delta_{4,2} \), and if
\[
f(sx) = f(xs^\tau) = \chi_{S_{4,2}}(s)f(x), \quad \text{for all } s \in S_{4,2},
\]
then
\[
f(x^\tau) = f(x).
\]
To conclude the above, we further assume that \( f(x^\tau) = -f(x) \). We need to show that \( f = 0 \).

Denote
\[
C_{4,2} = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & y \end{bmatrix} \in \operatorname{GL}_4(K) \times \operatorname{GL}_2(K) \mid y^{-1}a_{22} \text{ is nilpotent} \right\}.
\]
This is the union of $Z_4$ (in Section 8) and $Z'_4$, where

$$Z'_4 = \left\{ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & y \end{bmatrix} \in \text{GL}_4(\mathbb{K}) \times \text{GL}_2(\mathbb{K}) \right\}.$$

By using Proposition 6.1 and Proposition 7.1, we first show that $f$ is supported on $C_{4,2}$. Proposition 8.1 further implies that $f$ can only be supported on $Z'_4$.

Now set

$$x_{4,\text{left}} = \begin{bmatrix} 0 & I_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{gl}_4(\mathbb{K}) \times \mathfrak{gl}_2(\mathbb{K}),$$

and denote by $X_{4,\text{left}}$ the left invariant vector field on $\text{GL}_4(\mathbb{K}) \times \text{GL}_2(\mathbb{K})$ whose tangent vector at $x \in G$ is $xx_{4,\text{left}}$. As in Section 10, one checks that $X_{4,\text{left}}$ is transversal to $Z'_4$. We may then conclude that $f = 0$, as in Section 11. □

13.4. Uniqueness of Whittaker models

Let $G$ be a quasisplit connected reductive algebraic group defined over $\mathbb{R}$. Let $B$ be a Borel subgroup of $G$, with unipotent radical $N$. Let

$$\chi_N : N(\mathbb{R}) \to \mathbb{C}^\times$$

be a generic unitary character. The meaning of “generic” will be explained later in the proof.

The following theorem is fundamental and well-known. For $G = \text{GL}_n$, this is a celebrated result of Shalika [S74]. A proof in general can be found in [CHM00, Theorem 9.2]. We shall give a short proof based on the notion of unipotent $\chi$ incompatibility.

**Theorem 13.4.** Let $V$ be an irreducible Harish-Chandra smooth representation of $G(\mathbb{R})$. Then

$$\dim \text{Hom}_{N(\mathbb{R})}(V, \mathbb{C}_{\chi_N}) \leq 1.$$

**Proof.** Define a Harish-Chandra distributional representation to be the strong dual of a Harish-Chandra smooth representation. The current theorem can then be reformulated as follows: the space

$$U^{\chi_N^{-1}} = \{ u \in U \mid gu = \chi_N^{-1}(g)u \text{ for all } g \in N(\mathbb{R}) \}$$

is at most one dimensional for every irreducible Harish-Chandra distributional representation $U$ of $G(\mathbb{R})$. □
Let $\bar{B}$ be a Borel subgroup opposite to $B$, with unipotent radical $\bar{N}$. Then $T = B \cap \bar{B}$ is a maximal torus. Let
\[ \chi_T : T(\mathbb{R}) \to \mathbb{C}^\times \]
bear an arbitrary character. Then
\[ U(\chi_T) = \{ f \in C^\infty(G(\mathbb{R})) \mid f(t\bar{n}x) = \chi_T(t)f(x) \quad \text{for all } t \in T(\mathbb{R}), \bar{n} \in \bar{N}(\mathbb{R}) \} \]
is the distributional version of nonunitary principal series (Harish-Chandra) presentations. By Casselman’s subrepresentation theorem (in the category of Harish-Chandra distributional representations), it suffices to show that
\[ \dim U(\chi_T)^{\mathcal{N}_1} \leq 1, \quad \text{for any } \chi_T. \]  

Let
\[ H_G = \bar{B}(\mathbb{R}) \times N(\mathbb{R}), \]
which acts on $G(\mathbb{R})$ by
\[ (\bar{b}, n)x = \bar{b}xn^{-1}. \]
Write
\[ \chi_G(t\bar{n}, n) = \chi_T(t)\chi_N(n), \]
which defines a character of $H_G$. Then (13.1) is equivalent to
\[ \dim C_{\chi_G}^\infty(G(\mathbb{R})) \leq 1. \]

Let $W$ be the Weyl group of $G(\mathbb{R})$. We have the Bruhat decomposition
\[ G(\mathbb{R}) = \bigsqcup_{w \in W} G_w, \quad \text{with } G_w = \bar{B}(\mathbb{R})wN(\mathbb{R}). \]
From this we form a $H_G$ stable filtration
\[ \emptyset = G^0 \subset G^1 \subset G^2 \subset \cdots \subset G^r = G(\mathbb{R}) \]
of $G(\mathbb{R})$ by open subsets, with $G^1 = \bar{B}(\mathbb{R})N(\mathbb{R})$ and every difference $G^i \setminus G^{i-1}$ a Bruhat cell $G_w$, for $i \geq 2$.

Clearly (13.2) is implied by the following two assertions:
\[ \dim C_{\chi_G}^\infty(G^1) = 1; \]  
and
\[ \text{if } f \in C_{\chi_G}^\infty(G^i) \text{ vanishes on } G^{i-1}, \text{ then } f = 0, \]
for $i \geq 2$. The equality (13.3) is clear as $G^1 = \bar{B}(\mathbb{R})N(\mathbb{R})$. For (13.4), we write
\[ G^i \setminus G^{i-1} = G_w, \quad \text{with } w \text{ a non-identity element of } W. \]
The genericity means that $\chi_N$ has nontrivial restriction to $N(\mathbb{R}) \cap w^{-1}(N(\mathbb{R}))w$. Pick
\[ n = w^{-1}n w \in N(\mathbb{R}) \cap w^{-1}(N(\mathbb{R}))w \]
so that $\chi_N(n) \neq 1$. Then $(\bar{n}, n) \in H_G$ satisfies
\[ (\bar{n}, n)w = w, \quad \text{and} \quad \chi_G(\bar{n}, n) = \chi_N(n) \neq 1. \]
Consequently, $G_w$ is unipotently $\chi_G$ incompatible. Now (13.4) follows from Lemma 4.6.

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