One-shot multi-sender decoupling and simultaneous decoding for the quantum MAC

Sayantan Chakraborty*  Aditya Nema*  Pranab Sen*

Abstract

In this work, we prove a novel one-shot ‘multi-sender’ decoupling theorem generalising Dupuis’ result. We start off with a multipartite quantum state, say on $A_1A_2R$, where $A_1$, $A_2$ are treated as the two ‘sender’ systems and $R$ is the reference system. We apply independent Haar random unitaries in tensor product on $A_1$ and $A_2$ and then send the resulting systems through a quantum channel. We want the channel output $B$ to be almost in tensor with the untouched reference $R$. Our main result shows that this is indeed the case if suitable entropic conditions are met. An immediate application of our main result is to obtain a one-shot simultaneous decoder for sending quantum information over a $k$-sender entanglement unassisted quantum multiple access channel (QMAC). The rate region achieved by this decoder is the natural one-shot quantum analogue of the pentagonal classical rate region. Assuming a simultaneous smoothing conjecture, this one-shot rate region approaches the optimal rate region of Yard et al. [YDH05] in the asymptotic iid limit. Our work is the first one to obtain a non-trivial simultaneous decoder for the QMAC with limited entanglement assistance in both one-shot and asymptotic iid settings; previous works used unlimited entanglement assistance.

1 Introduction

The paradigm of decoupling, that is the process of removing correlations between systems, has turned out to be a powerful and general technique for obtaining inner bounds for transmission of quantum information in quantum Shannon theory. Its importance can be seen by its role in obtaining coding strategies for sending quantum information over a quantum channel, one of the most basic tasks in quantum Shannon theory. Let $|\psi\rangle^{RA}$ be a pure state, where $R$ is the so-called reference system that will be untouched by all operations of our protocol. We want to isometrically encode the message system $A$ into a system $A'G$ and send $A'$ through a noisy quantum channel $N^{A'\rightarrow B}$ so that the receiver can decode the output $B$ to obtain a state close to $|\psi\rangle^{RA}$. Consider the Stinespring dilation of $N$, namely $U_{N}^{A'\rightarrow BE}$, where the system $E$ is treated as the purifying environment. Consider the global pure state $|\psi\rangle^{RBE}$. Suppose the following decoupling condition holds: $\psi^{RBE} \approx \psi^{R} \otimes \sigma^{EG}$ for some state $\sigma$ on $EG$. Let $I$ denote the identity superoperator. Then by Uhlmann’s theorem one can immediately conclude that there exists a decoding isometry $D^{B\rightarrow AF}$ such that

$$
(D^{B\rightarrow AF} \otimes I^{REG}) |\psi\rangle^{RBE} \approx |\psi\rangle^{RA} |\sigma\rangle^{EG},
$$

*School of Technology and System Science, Tata Institute of Fundamental Research, Mumbai, India. Email: {kings-bandz, aditya.nema30, pranab.sen.73}@gmail.com
where $|\sigma\rangle^{FEG}$ is a purification of $\sigma^{EG}$. Thus if a suitable isometric encoder of $A$ into $A'G$ can be found which satisfies the above decoupling condition, we can achieve quantum information transmission over a quantum channel without even constructing an explicit decoder. In other words, decoupling has allowed us to solve a quantum coding problem doing only half the work as compared to the classical setting!

The decoupling paradigm was used to obtain a protocol for the Fully Quantum Slepian Wolf (FQSW) problem [ADHW09], aka the mother protocol of quantum Shannon theory as it can in turn be used in a black-box fashion to obtain many other protocols for useful quantum information theoretic tasks in the asymptotic iid setting. An useful and powerful one-shot decoupling theorem, generalising many earlier decoupling constructions including that of [ADHW09], was obtained by Dupuis [Dup10]. We shall refer to this result henceforth as the single sender decoupling theorem. A high level description follows. Suppose Alice holds the $A$ register of a bipartite mixed state $\rho_{AR}$, where $R$ is the reference system. Alice applies a Haar random unitary $U_A$ followed by a completely positive (CP) map $T_{A \rightarrow E}$. Then if certain entropic conditions are met, the resulting state typically is close to the decoupled state $\omega_E \otimes \rho_R$ where $\omega_E$ is a state depending only on the channel $T$ and not on the state $\rho_{AR}$ nor on the unitary $U_A$.

The intuition described in the first paragraph of the introduction above can be made precise and indeed Dupuis’ used his single sender decoupling theorem to obtain nearly optimal one shot inner bounds for sending quantum information over a point to point quantum channel with limited entanglement assistance [Dup10], generalising an earlier result by Buscemi and Datta [BD10] for the same problem without entanglement assistance. In the asymptotic iid setting, Dupuis’ result recovers the well known regularised coherent information inner bound when there is no entanglement assistance [Lio97, Sho02, Dev05], and the well known mutual information inner bound when there is unlimited entanglment assistance [BSST02].

The main contribution of this work is the generalisation of the single sender decoupling theorem to the case of independent multiple senders. To be precise, we prove a theorem of the following kind. Consider the multipartite state $\rho^{A_1A_2...A_kR}$ where the users Alice$_1$, Alice$_2$ and so on only have access to their respective registers $A_1, A_2, \ldots, A_k$. Let Alice$_i$ apply a Haar random unitary $U_i$ to her register $A_i$ independently of the other Alice$_i$. After the individual unitaries are applied, a CP map $T^{A_1A_2...A_k \rightarrow E}$ is also applied. We show, if certain entropic conditions are met, that the resulting state typically is close to the decoupled state $\omega_E \otimes \rho_R$ where $\omega_E$ is a state depending only on the channel $T$ and not on the state $\rho^{AR}$ nor on the unitaries $U_i^{A_i}$.

We prove our multi-sender decoupling theorem by suitably extending Dupuis’ proof of his single sender decoupling theorem. As will become clear during the course of its proof, one of the main bottlenecks in proving such a theorem turns out to be defining and using the correct one-shot entropic quantities in order to bound the error in the protocol. We show that a modification of the conditional Rényi 2-entropy defined recently in [NS20] turns out to be the right quantity for our purpose. Our simultaneous decoding inner bound is thus stated in terms of a modified Rényi 2-coherent information and its smoothed version derived from the above quantity. A similar multi sender decoupling theorem was earlier proved by Dutil [Dut11]. However his formulation did not use the correct entropic quantity required to get strong bounds in applications e.g. inner bounds for quantum multiple access channels.

As an important application of our multi sender decoupling theorem, we consider the problem of proving inner bounds for the Quantum Multiple Access Channel (QMAC) with limited entanglement assistance in the one-shot setting i.e. when the channel is used only once. Previous
works only considered the QMAC in the asymptotic iid setting, either with no entanglement assistance \cite{YDH05} or with unlimited entanglement assistance \cite{HDW08}. In a very recent companion paper \cite{CNS21}, one shot inner bounds were shown for the QMAC with limited entanglement assistance which approach the previously known bounds in the asymptotic iid setting both without entanglement assistance as well as with unlimited entanglement assistance. However all these works use \textit{successive cancellation decoding} to obtain their inner bounds. Successive cancellation tends to have faster decoding strategies than \textit{simultaneous decoding}. However it also has some drawbacks like the difficulty of clock synchronisation that arises when used together with \textit{time sharing} in the asymptotic iid setting. This drawback can be eliminated by a technique called \textit{rate splitting} first developed for the classical asymptotic iid setting by \cite{GRUW01} and later extended to the one-shot quantum setting by \cite{CNS21}. However using rate splitting in the one shot setting brings a new feature which is aesthetically unappealing viz. the obtained inner bound is a subset of the familiar ‘pentagonal’ inner bound for the MAC. Note that the pentagonal inner bound holds both in the classical asymptotic iid setting \cite{Ahl71,Lia72}, as well as for transmitting classical information over a quantum MAC both with and without entanglement assistance \cite{Sen18b}. Proving a (super) pentagonal inner bound in the one shot setting requires simultaneous decoding.

Our multi sender decoupling theorem allows us, for the first time, to get a simultaneous decoder for sending quantum information over a QMAC with limited entanglement assistance. This allows us to obtain the (super) pentagonal rate region as shown in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Inner bound for the unassisted QMAC obtained by simultaneous decoding. The quantity $\tilde{I}_{2,\delta}(\cdot > \cdot)$ is the modified Rényi 2-coherent information and $\tilde{I}_{2,\delta}^\varepsilon(\cdot > \cdot)$ its smoothed version defined in Section 2. $O(\log \varepsilon)$ additive factors have been ignored in the figure.}
\end{figure}

Simultaneous decoders are an essential building block for obtaining the best inner bounds for several multiterminal channels in classical network information theory e.g. Marton’s inner bound with common message for the broadcast channel. It is expected that our simultaneous decoder for the QMAC will pave the way for similar results in quantum network information theory too.

A shortcoming of our results is that we are unable to show that our one shot inner bound for the unassisted QMAC recovers the optimal asymptotic iid result of \cite{YDH05}. To do so, one would require the existence of a single state which simultaneously smooths and (nearly) maximises all the three entropic quantities that arise in our simultaneous decoding inner bound pictured in Figure 1. The existence of such a state is a major open problem in quantum information theory and is known as the \textit{simultaneous smoothing conjecture}. The interested reader is referred to \cite{Sen18a}.
for more details.

In this paper we will be using a variant of the 2 entropy, defined by Nema and Sen in [NS20]. The interested reader is referred to [T om12] for an extremely comprehensive survey of these quantities and their properties. A naïve choice of the smoothed conditional Rényi 2-entropy does not work, for technical reasons that we will describe in Section 4.

1.1 Organisation of the Paper

The rest of the paper is organised as follows:

- In Section 2 we define the one-shot entropic quantities that we require to prove our theorems, along with the statements of useful facts about these quantities and other identities in general.
- In Section 4 we state and prove the multi sender decoupling theorem.
- In Section 5 we use the 2 sender version of the multi sender decoupling theorem to derive inner bounds for sending quantum information via the QMAC.
- We conclude with Section 6 by mentioning an immediate open problem which might be useful for other problems in quantum Shannon theory.

2 Preliminaries

2.1 Notation

All vector spaces considered the paper are finite dimensional inner product spaces, also called finite dimensional Hilbert spaces, over the complex field and denoted by $\mathcal{H}$. We use $|\mathcal{H}|$ to denote the dimension of a Hilbert space $\mathcal{H}$. Logarithms are all taken in base two. We tacitly assume that the ceiling is taken of any formula that provides dimension or value of $t$ in unitary $t$-design. The symbols $E$, $P$ denote expectation and probability respectively. The abbreviation "iid" is used to mean identically and independently distributed, which just means taking the tensor power of the identical copies of the underlying state. The notation ":=" is used to denote the definitions of the underlying mathematical quantities.

The notation $\mathcal{L}(A_1, A_2)$ denotes the Hilbert space of all linear operators from Hilbert space $A_1$ to Hilbert space $A_2$ with the inner product being the Hilbert-Schmidt inner product $\langle X, Y \rangle := \text{Tr}[X^\dagger Y]$. For the special case when $A_1 = A_2$ we use the phrase operator on $A_1$ and the symbol $\mathcal{L}(A_1)$. The symbol $I^A$ denotes the identity operator on vector space $A$. The matrix $\pi^A$ denotes the so-called completely mixed state on system $A$, i.e., $\pi^A := \frac{1}{|A|}$. We use the notation $U \circ A$ as a short hand to denote the conjugation of the operator $U$ on the operator $M$, that is, $U \cdot M := UMU^\dagger$.

The symbol $\rho$ usually denotes a quantum state, also called as a density matrix which is a Hermitian positive semidefinite matrix with unit trace, and $\mathcal{D}(\mathbb{C}^d)$ denotes the set of all $d \times d$ density matrices. The symbol $\text{Pos}(\mathbb{C}^d)$ denotes the set of all $d \times d$ positive semidefinite matrices, and the symbol $\mathbb{U}(d)$ denotes the set of all $d \times d$ unitary matrices with complex entries. For a positive semidefinite matrix $\sigma$, we use $\sigma^{-1}$ to denote the operator which is the orthogonal direct sum of the inverse of $\sigma$ on its support and the zero operator on the orthogonal complement of the support. This definition of $\sigma^{-1}$ is also known as the Moore-Penrose pseudoinverse. The symbol $|\psi\rangle$ denotes a
vector $\psi$ of unit Schatten 2-norm or the Frobenius norm, and $\langle \psi |$ denotes the corresponding linear functional. A pure quantum state is rank one density matrix. For brevity, a pure quantum state $|\psi\rangle \langle \psi |$ is denoted by $\psi$ to emphasise that it is a density matrix. For two Hermitian matrices $A, B$ of the same dimension, we use $A \geq B$ as a shorthand to imply that the matrix $A - B$ is positive semidefinite.

Let $X \in \mathcal{L}(A)$. The symbol $\text{Tr} X$ denotes the trace of operator $X$. Trace is a linear map from $\mathcal{L}(A)$ to $\mathcal{C}$. Let $A, B$ be two vector spaces. The partial trace $\text{Tr}_A [\cdot]$ obtained by tracing out $A$ is defined to be the unique linear map from $\mathcal{L}(A \otimes R)$ to $\mathcal{L}(R)$ satisfying $\text{Tr}_A [X \otimes Y] = (\text{Tr} X)Y$ for all operators $X \in \mathcal{L}(A), Y \in \mathcal{L}(R)$.

A linear map $T : \mathcal{M}_m \to \mathcal{M}_d$, that maps a linear operator to another linear operator is called a superoperator. A superoperator $T$ is said to be positive if it maps positive semidefinite operators to positive semidefinite operators, and completely positive if $T \otimes \mathbb{I}$ is a positive superoperator for all identity superoperators $\mathbb{I}$. A superoperator $T$ is said to be trace preserving if $\text{Tr}[T(X)] = \text{Tr}[X]$ for all $X \in \mathcal{M}_m$. Completely positive and Trace Preserving (abbreviated as CPTP) superoperators are called quantum operations or quantum channels. In this paper all the superoperators considered are completely positive and trace non-increasing superoperators, unless stated otherwise. The symbol $\mathbb{I}$ denotes the identity or the noiseless channel which does not alter the input at all or $\mathbb{I}(X) = X$, for all operators $X$.

The adjoint of a superoperator is defined with respect to the Hilbert-Schmidt inner product on matrices. If $T : \mathcal{M}_m \to \mathcal{M}_d$ is a superoperator, then its adjoint $T^\dagger : \mathcal{M}_d \to \mathcal{M}_m$ is a superoperator uniquely defined by the property that $\langle T^\dagger(A), B \rangle = \langle A, T(B) \rangle$ for all $A \in \mathcal{M}_d, B \in \mathcal{M}_m$.

### 2.2 Entropic Quantities

In this section we define the relevant entropic quantities used in the proof of our general multi-user decoupling theorem. We start with the conditional min entropy, followed by conditional 2-entropy, a variant of which will be used in most of our proofs.

**Definition 2.1** Let $0 \leq \epsilon < 1$. The $\epsilon$-smooth conditional min-entropy of $\rho^{AB}$ is defined as:

$$H_{\text{min}}^\epsilon(A|B)_\rho := \min_{\sigma^{AB} \in \text{Pos}(B)} \left\{ \text{Tr}(\gamma^B) : \gamma^B \in \text{Pos}(B), \sigma^{AB} \leq I^A \otimes \gamma^B, \|\sigma^{AB} - \rho^{AB}\|_1 \leq \epsilon \right\}.$$  

When $\epsilon = 0$, this is just $H_{\text{min}}(A|B)$ with $\sigma^{AB}$ replaced by $\rho^{AB}$.

**Definition 2.2** Let $0 \leq \epsilon < 1$. The $\epsilon$-smooth conditional Rényi 2-entropy for a bipartite positive semidefinite operator $\rho^{AR}$ on systems $A$ and $R$ is defined as:

$$H_2^\epsilon(A|R)_\rho := -2 \log \min_{\sigma^{AR} \in \text{Pos}(AR)} \left\{ \| (\omega^R \otimes I^A)^{-1/4} \sigma^{AR} (\omega^R \otimes I^A)^{-1/4} \|_2 \right\}.$$  

When $\epsilon = 0$, we simply refer to the above quantity as conditional Rényi 2-entropy and denote it by $H_2(A|R)_\rho$ and define $\tilde{\rho}^{AR} := (\omega^R \otimes I^A)^{-1/4} \rho^{AR} (\omega^R \otimes I^A)^{-1/4}$.

The advantage of working with smoothed conditional Rényi 2-entropy is that in the asymptotic iid limit it is appropriately bounded by conditional Shannon entropy, as mentioned in the following Fact 2.3.
Fact 2.3 Let $\epsilon > 0$. Then, $H_2^*(-A|B)_\rho \leq H(A|B)_\rho + 8\epsilon \log |A| + 2 + 2\log \epsilon^{-1}$. and $H_{\min}^c(A|B)_\rho \geq H(A|B) - 8\log |A| \times \sqrt{\log \frac{2}{\epsilon}}$.

The proof of the bounds on $H_{\min}$ can be found in [TCR09, Theorem 9] and for $H_2$ can be deduced by combining [TBH14, Equation 8] with [TCR09, Theorem 7, Lemma 2, Equation 33] and then applying the Alicki-Fannes inequality [AF04], respectively.

For our proofs we will be using a slightly modified version of the 2-entropy, where we fix the $\sigma^B$ to a special state instead of optimising over it. The justification for this definition is as follows:

1. This quantity is much more tractable than the optimised 2-entropy.
2. The smoothed version of this new quantity indeed approaches the conditional Shannon entropy in the asymptotic iid limit, as proved in [NS20].

Definition 2.4 $\delta$-Tilde Conditional 2-Entropy Given a state $\rho^{AB}$ on the registers $AB$ and $\delta \in (0,1)$ the $\delta$-Tilde conditional 2-entropy of $A$ given $B$ is defined as

$$H_{2,\delta}(A|B)_\rho := -\log \text{Tr}[(I^A \otimes \rho^B_\delta)^{-1/2}]$$

where $\rho^B_\delta$ is the positive semidefinite matrix that is obtained by zeroing out the smallest eigenvalues of $\rho^B$ that sum to less than or equal to $\delta$.

The smoothed variant of this quantity, as defined by Nema and Sen was shown in [NS20] to approach the Shannon conditional entropy in the asymptotic iid limit. We call this the $\epsilon$-smooth $\delta$-tilde conditional 2-entropy. In the interest of brevity, we will refer to this quantity simply as the smooth tilde 2-entropy from now on. We will require some additional definitions before introducing this quantity:

Definition 2.5 $\epsilon$-smooth Max Entropy Given a state $\rho^A$ and positive $\epsilon$, the $\epsilon$-smooth Max Entropy max entropy is given by

$$H_{\max}^\epsilon(A)_\rho := 2\log \min_{\rho^B \geq 0, \|\rho^B - \rho\|_1 \leq \epsilon} \text{Tr}[(\rho^B)^{-1}]$$

Definition 2.6 $\delta$-Tilde Max Entropy Given the state $\rho^A$, consider the state $\rho^A_\delta$ which is obtained by zeroing out the smallest eigenvalues of $\rho^A$ which sum to less than or equal to $\delta$. Then the $\delta$-Tilde Max Entropy is given by

$$H_{\max}^\delta(A)_\rho := \log \left\| (\rho^A_\delta)^{-1} \right\|_\infty$$

We are now ready to define the smooth tilde 2-entropy:

Definition 2.7 $\epsilon$-smooth $\delta$-Tilde Conditional 2-Entropy Given a state $\rho^{AB}$, consider the state $\rho^{AB}_{\epsilon,\delta}$ that is obtained by zeroing out those eigenvalues of $\rho^B$ which are smaller than $2^{-(1+\delta)H_{\max}^\epsilon(B)_\rho}$. Then, we define

$$H_{2,\delta}(A|B)_\rho := -\log \min_{0 \leq H_{\max}^\epsilon(B)_\rho \leq \rho^{AB}_{\epsilon,\delta}} \text{Tr}[(I^A \otimes \rho^{AB}_{\epsilon,\delta})^{-1/2}]$$

Fact 2.8 For $n \in \mathbb{N}$ and $\epsilon, \delta > 0$, given a quantum state $\rho^{AB}$ and its iid extension $\rho^{AB^\otimes n}$ the following holds

$$\lim_{\epsilon,\delta \to 0} \lim_{n \to \infty} \frac{H_{2,\delta}^\epsilon(A^n|B^n)_{\rho^n}}{n} \geq H(A|B)_{\rho}$$
2.3 Useful Facts

Fact 2.9 \textit{Wat18 NS20} Any superoperator $T^{A \rightarrow B}$ can be represented as:
\[
T^{A \rightarrow B}(M^A) = \text{Tr}_Z \{ V_{T}^{AC \rightarrow BZ} (M^A \otimes |0 \rangle \langle 0|)^C (W_{T}^{AC \rightarrow BZ})^\dagger \}
\]
where $V_T, W_T$ are operators that map vectors from $A \otimes C$ to vectors in $B \otimes Z$. Systems $C$ and $Z$ are considered as the input and output ancillary systems respectively, such that $|A| |C| = |B| |Z|$. Without loss of generality, $|C| \leq |B|$ and $|Z| \leq |A|$. Furthermore, in the following special cases $V_T, W_T$ have additional properties.

1. $T$ is completely positive if and only if $V_T = W_T$.
2. $T$ is trace preserving if and only if $V_T^{-1} = W_T^\dagger$. Thus, $T$ is completely positive and trace preserving if and only if $V_T = W_T$ and are unitary operators.
3. $T$ is completely positive and trace non-decreasing if and only if $V_T = W_T$ and $\| V_T \|_\infty \leq 1$.

Fact 2.10 \textit{Dup10} Swap Trick Given two linear operators $M$ and $N$ on the system $A$ and the swap operator $F_{AA'}$, where we denote by $A'$ an isomorphic copy of $A$, the following holds:
\[
\text{Tr}[MN] = \text{Tr}[(M \otimes N)F_{AA'}]
\]

Fact 2.11 \textit{Wat18} Uhlmann’s Theorem For quantum states $M$ and $N$ with purifications $|\phi\rangle_{XY}$ and $|\psi\rangle_{XZ}$ respectively (referring to the systems $Y$, $Z$ as purifying systems and systems $Y, Z$ need not be isomorphic). Then,
\[
F(M, N) = \max_{V: V^\dagger V = I^Y} | \langle \phi | V^\dagger | \psi \rangle |
\]
where the maximization is over all partial isometries $V (\equiv V^\dagger V = I^Y)$ from $Y$ to $Z$ with $\text{dim}(Z) \geq \text{dim}(Y)$.

Fact 2.12 Given a linear operator $M$ on $A^{\otimes 2}$ it holds that
\[
\mathbb{E}(M) = \int_{U \in U(A)} (U^{\otimes 2} \cdot M) dU = \alpha I_{AA'} + \beta F_{AA'}
\]
where $\alpha$ and $\beta$ are the solutions of the equations $\text{Tr}[M] = \alpha |A|^2 + \beta |A|$ and $\text{Tr}[FM] = \alpha |A| + \beta |A|^2$ and the integration is over the Haar measure over the Unitary group.

Fact 2.13 Given a positive semidefinite operator $\rho^{AB}$ on the system $AB$
\[
\frac{1}{|A|} \leq \frac{\text{Tr}[\rho^{AB^2}]}{\text{Tr}[\rho^{B^2}]} \leq |A|
\]

Fact 2.14 Let $M$ be any linear operator and $\sigma$ be a positive semidefinite operator on system $A$. Then
\[
\| M \|_1 \leq \sqrt{\text{Tr}[\sigma] \text{Tr}[\sigma^{-1/4} M \sigma^{-1/2} M^\dagger \sigma^{-1/4}]}
\]
and in particular, when $M$ is Hermitian
\[
\| M \|_1 \leq \sqrt{\text{Tr}[\sigma] \text{Tr}[\sigma^{-1/4} M \sigma^{-1/4}]}.
\]
3 Inner Bounds for the QMAC using the Multi Sender Decoupling Theorem

We will first consider the task of entanglement transmission. As before, we first consider the seemingly more general problem: We are given a QMAC \( N^{A'B'\rightarrow R^1} \) and two states pure states \( \psi^{AC_1,R_1} \) and \( \psi^{BC_2,R_2} \), with Alice holding the system \( A \), Bob the system \( B \) and Charlie the systems \( C_1,C_2 \). \( R_1,R_2 \) are the reference registers. Alice and Bob wish to send the registers \( A \) and \( B \) to Charlie through one use of the channel \( N \), such that at the end of the protocol, the state that Charlie holds is close to \( \psi^{AC_1,R_1}\psi^{BC_2,R_2} \). To do this, we must show the existence of encoders \( E_1 \) and \( E_2 \) and a decoder \( D \) such that

\[
\| D \circ N \circ (E \otimes E_2) (\psi \otimes \phi) - \psi \otimes \phi \|_1 \leq \epsilon
\]

We consider the complementary channel \( \tilde{N}^{A'B'\rightarrow E} \) and the randomized encoders \( E_1 \) and \( E_2 \) such that

\[
\tilde{N}^{A'B'\rightarrow E} \circ (E_1^{A\rightarrow A'} \otimes E_2^{B\rightarrow B'}) (\psi^{AR_1} \otimes \phi^{BR_2}) \approx \tilde{N}^{A'B'\rightarrow E} \circ (E_1^{A\rightarrow A'} \otimes E_2^{B\rightarrow B'}) (\psi^{A} \otimes \phi^{B}) \otimes (\psi^{R_1} \otimes \phi^{R_2})
\]

We will first fix a pure control state \( |\omega\rangle^{A''B''CE} := U^{A''B''\rightarrow CE} |\Omega\rangle^{A''A'} |\Delta\rangle^{B''B'} \). We consider randomized encoders \( E_1 \) and \( E_2 \), where the randomness is derived from independently picked unitaries \( U_1 \) and \( U_2 \), each of which is identically distributed with respect to the Haar measure. The single user decoupling theorem will clearly not work here, and hence we use our multisender decoupling theorem instead. Using that theorem, we show that there exist decoders which obey Eq. (1) as long as the following entropic inequalities are satisfied:

1. \(-\tilde{H}_{2,\delta}(A''|E) - \tilde{H}_{2,\delta}(A|R_1)\phi - \tilde{H}_{2,\delta}(B|R_2)\phi \leq \log \epsilon\)
2. \(-\tilde{H}_{2,\delta}(B''|E) - \tilde{H}_{2,\delta}(B|R_2)\phi \leq \log \epsilon\)
3. \(-\tilde{H}_{2,\delta}(A''|E) - \tilde{H}_{2,\delta}(A|R_1)\phi \leq \log \epsilon\)

One should note that to finish the argument, one still has to show the existence of two fixed isometric encoders \( V_1 \) and \( V_2 \) which perform almost as well as the randomized decoders. This argument however does not require the power of the multi sender decoupling theorem, a two separate applications of the single sender decoupling theorem provide a proof, with error estimates in terms of \( H_{\text{max}}(A) - \tilde{H}_{2,\delta}(A''|E) \) and \( H_{\text{max}}(B) - \tilde{H}_{2,\delta}(B''|E) \). The reader is referred to Theorem 5.1 for a detailed proof of the claims made above.

It is now easy to show inner bounds for the entanglement transmission for the QMAC from these bounds. We simply set \( A^{AC_1,R_1} \) to be \( \Phi^{R_1M_1} \otimes \Phi^{AC_1} \) and \( B^{BC_2,R_2} \) to be \( \Phi^{R_2M_2} \otimes \Phi^{BC_2} \). Recall that \( \Phi^{AC_1} \) and \( \Phi^{BC_2} \) represent pre-shared entanglement. So the inner bounds we derive are for partial entanglement assistance. For the unassisted inner bounds we simply set the systems \( A \) and \( B \) to be trivial. Please refer to Theorem 5.2 for details.

We note that it is possible to design an entanglement generation protocol for the QMAC as well using the multi sender decoupling theorem. The idea is as follows: we consider the control
state $|\omega\rangle^{ABCE} := U^{A'B' \rightarrow CE}_A |\Omega\rangle^{AA'} |\Delta\rangle^{BB'}$. Consider the projectors $\Pi_1^{A \rightarrow R_1}$ and $\Pi_2^{B \rightarrow R_2}$, of rank $|R_1|$ and $|R_2|$, where $R_1$ and $R_2$ are subspaces of $A$ and $B$ respectively. The idea is we hit the systems $A$ and $B$ with the operators $|\Delta\rangle_{R_1}^{A \rightarrow R_1} U^A$ and $|\Delta\rangle_{R_2}^{B \rightarrow R_2} U^B$ respectively, where $U^A$ and $U^B$ are Haar random unitaries. We want to show that, on average, the systems $R_1$ and $R_2$ are decoupled from the system $E$.

We note that, in contrast to the case of entanglement transmission, where the random unitary is a part of the encoder and acts on the system to be transmitted, in the case of entanglement generation, the averaging is actually done of the action of the random unitaries on the purifying system. Because of this, the case for entanglement generation seems potentially more challenging, as, instead on acting on two tensored subsystems ($\phi^{AR_1} \otimes \phi^{BR_2}$), the random unitaries act on a state which are entangled via the system $E$ viz. $\omega^{ABE}$.

Nonetheless, our multisender decoupling theorem is general enough to handle this case as well (see Theorem 4.2) and we are indeed able to show that on average the systems $R_1$ and $R_2$ are decoupled with $E$.

The proofs of the claims made above are included in the Appendix.

### 4 The Multi-Sender Decoupling Theorem

Before we move on to the general multi sender decoupling theorem it will be instructive to see the proof for the case of only 2 senders. The proof for more than 2 senders requires heavy notation. Hence we defer its exposition to a later section.

#### 4.1 A Warm-Up: The 2-Sender Decoupling Theorem

**Notation**: We will sometimes abbreviate the symbol for the unitary $U^A_i$ as $U_i$ to ease the notation. As before we will use the $'$ accent in conjunction with the name of a register to denote isomorphic copies of the original system, for e.g. $A$ and $A'$.

We will require the following lemma:

**Lemma 4.1** Given a linear operator $M$ on the space $A_1 A'_1 A_2 A'_2$ the following holds

$$
= \int \left( U_1^{\otimes 2} \otimes U_2^{\otimes 2} \right) \cdot M \, dU_1 dU_2 = \alpha_{00} \mathbb{I}^{A_1 A'_1} \otimes \mathbb{I}^{A_2 A'_2} + \alpha_{01} F^{A_1 A'_1} \otimes \mathbb{I}^{A_2 A'_2} + \alpha_{10} \mathbb{I}^{A_1 A'_1} \otimes F^{A_2 A'_2} + \alpha_{11} F^{A_1 A'_1} \otimes F^{A_2 A'_2}
$$

where the coefficients $\alpha_{ij}$ are given by

$$
\begin{bmatrix}
|A_1 A_2| & |A_2| & 1 & |A_2| \\
1 & |A_2| & 1 & 1
\end{bmatrix} \otimes
\begin{bmatrix}
|A_1| & 1 & 1 & |A_1| \\
1 & 1 & |A_1| & 1
\end{bmatrix}
\begin{bmatrix}
\alpha_{00} & \alpha_{01} \\
\alpha_{10} & \alpha_{11}
\end{bmatrix} =
\begin{bmatrix}
\text{Tr}[M] \\
\text{Tr}[F^{A_1 A'_1} \otimes \mathbb{I}^{A_2 A'_2} M] \\
\text{Tr}[\mathbb{I}^{A_1 A'_1} \otimes F^{A_2 A'_2} M] \\
\text{Tr}[F^{A_1 A'_1} \otimes F^{A_2 A'_2} M]
\end{bmatrix}
$$


Proof: The proof is an easy extension of Theorem 2.12 and the linearity of integration. We give it for completeness.

We expand $M^{A_1A_2A_3}$ in a Schmidt decomposition as: $M = \Sigma_i \left( X_i^{A_1A_2} \otimes (X_i')^{A_2A_3} \right)$. Then:

$$\int \left( \left[ U_1^{\otimes 2} \otimes U_2^{\otimes 2} \right] \cdot M \right) dU_1 dU_2 = \int \left( \Sigma_i \left[ U_1^{\otimes 2} \otimes U_2^{\otimes 2} \right] \cdot \left\{ \left[ X_i^{A_1A_2} \otimes (X_i')^{A_2A_3} \right] \right\} \right) dU_1 dU_2$$

$$= \Sigma_i \left[ \int \left( U_1^{\otimes 2} \cdot X_i^{A_1A_2} \right) \otimes \left[ U_2^{\otimes 2} \cdot (X_i')^{A_2A_3} \right] \right) dU_1 dU_2$$

$$= \Sigma_i \left[ \int \left( U_1^{\otimes 2} \cdot X_i^{A_1A_2} \right) dU_1 \otimes \int \left( U_2^{\otimes 2} \cdot (X_i')^{A_2A_3} \right) dU_2 \right]$$

$$= \Sigma_i \left[ \left( \alpha_i I^{A_1A_2} + \beta_i F^{A_1A_2} \right) \otimes \left( \alpha_i F^{A_2A_3} + \beta_i F^{A_2A_3} \right) \right]$$

where,

- (a) follows by Theorem 2.12
- (b) holds by the identity that $a_{00} := \Sigma_i a_{i\ell}, a_{i1} := \Sigma_i a_{i\ell}$ and $a_{11} := \Sigma_i a_{i\ell}$

In order to obtain the values of the coefficients $\{a_{0k}\}_{k=0}^1$, we use the identities $\text{Tr} \left[ \int U_1^{\otimes 2} \cdot X_i dU_1 \right] = \text{Tr}[X_i] = \alpha_i |A_1|^2 + \beta_i |A_1|$ and $\text{Tr} \left[ \int U_2^{\otimes 2} \cdot X_i dU_1 \right] = \text{Tr}[F X_i] = \alpha_i |A_1| + \beta_i |A_2|^2$ in equality (b) and solving the system of equations simultaneously, completes the proof. $\square$

We now state and prove the main technical tool of this work, the sender decoupling theorem:

**Theorem 4.2** Let $\rho^{A_1A_2R}$ be a density operator, $T^{A_1A_2\rightarrow E}$ be a CP map, and define $\omega^{A_1A_2E} := (I^{A_1A_2} \otimes \tilde{T})(\Phi^{A_1A_1} \otimes \Phi^{A_2A_2})$. For a given $\delta > 0$, we define $\sigma^E := \omega^E_\delta$ as the positive semidefinite matrix obtained by zeroing out the smallest eigenvalues of $\omega^E$ that sum to at most $\delta$. We define $\zeta := \rho^R_\delta$ similarly.

Define

1. $\tilde{T} := \sigma^{-1/4} \otimes T$

2. $\tilde{\rho}^{A_1A_2R} := \zeta^{-1/4} \otimes \rho^{A_1A_2AR}$

3. $\tilde{\omega}^{A_1A_2E} := \tilde{T}(\Phi^{A_1A_1} \otimes \Phi^{A_2A_2})$

Then,

$$\int \left\{ \left\| T \left( (U^{A_1} \otimes U^{A_2}) \cdot \rho^{A_1A_2R} \right) - \omega^E \otimes \rho^R \right\|_F^2 \right\} dU^{A_1} dU^{A_2}$$

$$\leq \left( \delta + \frac{|A_2|^2}{|A_2|^2 - 1} \left\{ \left\| \tilde{\rho}^{A_1A_2R} \right\|_2^2 \left\| \tilde{\omega}^{A_1E} \right\|_2^2 + \left\| \tilde{\rho}^{A_1A_2R} \right\|_2 \left\| \tilde{\omega}^{A_2E} \right\|_2 \right\} \right)^{1/2}$$

where $|A_2| > |A_1|$.

Proof: We will start by using Theorem 2.14 with the weighting matrix $(\sigma^E \otimes \zeta^R)$. First observe that by definition $\text{Tr}[\sigma^E \otimes \zeta^R] \leq 1$. Then,

$$\int \left\{ \left\| T \left( (U^{A_1} \otimes U^{A_2}) \cdot \rho^{A_1A_2R} \right) - \omega^E \otimes \rho^R \right\|_F^2 \right\} \leq \sqrt{\text{Tr} \left[ \left( \tilde{T} \left( (U^{A_1} \otimes U^{A_2}) \cdot \tilde{\rho}^{A_1A_2R} \right) - \tilde{\omega}^E \otimes \tilde{\rho}^R \right) \right] \right)^2}$$
Next, opening up the square and using the fact that the integral and $\tilde{T}$ commute we see that,

\[
\int \text{Tr} \left[ (\tilde{T}\{(U^{A_1} \otimes U^{A_2}) \cdot \tilde{\rho}^{A_1 A_2 R} - \tilde{\omega}^E \otimes \tilde{\rho}^R \})^2 \right]
\]

\[
= \int \text{Tr} \left[ (\tilde{T}\{(U^{A_1} \otimes U^{A_2}) \cdot \tilde{\rho}^{A_1 A_2 R} \})^2 \right] - 2 \text{Tr} \left[ \tilde{T}\{ \int (U^{A_1} \otimes U^{A_2}) \cdot \tilde{\rho}^{A_1 A_2 R} dU_1 dU_2 \}(\tilde{\omega}^E \otimes \tilde{\rho}^R) \right]
\]

\[
+ \text{Tr} \left[ (\tilde{\omega}^E \otimes \tilde{\rho}^R)^2 \right]
\]

\[
= \int \text{Tr} \left[ \{ \tilde{T}\{(U^{A_1} \otimes U^{A_2}) \cdot \tilde{\rho}^{A_1 A_2 R} \})^2 \right] - \text{Tr} \left[ (\tilde{\omega}^E \otimes \tilde{\rho}^R)^2 \right]
\]

By standard manipulations, using [Theorem 2.10] and the definition of adjoint of an operator, it follows that

\[
\int \text{Tr} \left[ \{ \tilde{T}\{(U^{A_1} \otimes U^{A_2}) \cdot \tilde{\rho}^{A_1 A_2 R} \})^2 \right] dU_1 dU_2
\]

\[
= \int \text{Tr} \left[ (\tilde{\rho}^{A_1 A_2 R})^2 \left\{ \left( (U_1^{+ \otimes 2} \otimes U_2^{+ \otimes 2}) \cdot \tilde{T}^{+ \otimes 2} (EE') \right) \otimes F^{RR'} \right\} \right] dU_1 dU_2
\]

\[
= \text{Tr} \left[ (\tilde{\rho}^{A_1 A_2 R})^2 \left\{ \int \left( (U_1^{+ \otimes 2} \otimes U_2^{+ \otimes 2}) \cdot \tilde{T}^{+ \otimes 2} (EE') \right) dU_1 dU_2 \otimes F^{RR'} \right\} \right]
\]

We will now use [Theorem 4.1] by plugging in the matrix $\tilde{T}^{+ \otimes 2} (EE')$ into $M$. The first step is to compute the entries of the vector on the R.H.S. in the matrix equation in [Theorem 4.1]. We will demonstrate one such computation, the rest follow along similar lines.
Computing \( \text{Tr} \left[ (F^{A_1A'_1} \otimes F^{A_2A'_2}) \, \hat{T}^{+ \otimes 2} (F^{EE'}) \right] \):

\[
\begin{align*}
\text{Tr} \left[ (F^{A_1A'_1} \otimes F^{A_2A'_2}) \, \hat{T}^{+ \otimes 2} (F^{EE'}) \right] & = \text{Tr} \left[ \hat{T}^{\otimes 2} (F^{A_1A'_1} \otimes F^{A_2A'_2}) \, F^{EE'} \right] \\
& = \text{Tr} \left[ \hat{T}^{\otimes 2} (F^{A_1A'_1} (I^{A_1} \otimes I^{A'_1}) \otimes F^{A_2A'_2} (I^{A_2} \otimes I^{A'_2})) \, F^{EE'} \right] \\
& \overset{a}{=} |A_1|^2 |A_2|^2 \text{Tr} \left[ \hat{T}^{\otimes 2} (F^{A_1A'_1} (\text{Tr}_{A_1} (\Phi^{A_1A'_1}) \otimes \text{Tr}_{A'_1} (\Phi^{A'_1A'_1}))) \right. \\
& \left. \times F^{A_2A'_2} (\text{Tr}_{A_2} (\Phi^{A_2A'_2}) \otimes \text{Tr}_{A'_2} (\Phi^{A'_2A'_2})) \right] \, F^{EE'} \\
& = |A_1|^2 |A_2|^2 \text{Tr} \left[ \text{Tr}_{A_1,A'_1,A_2,A'_2} \left\{ (\hat{T}^{\otimes 2} \otimes I^{A_1A'_1A_2A'_2}) \left( (F^{A_1A'_1} \otimes I^{A_1A'_1}) (\Phi^{A_1A_1A'_1A'_1}) \right) \right. \\
& \left. \times \left( (F^{A_2A'_2} \otimes I^{A_2A'_2}) (\Phi^{A_2A_2A'_2A'_2}) \right) \right\} \, F^{EE'} \right] \\
& \overset{b}{=} |A_1|^2 |A_2|^2 \text{Tr} \left[ \left\{ (\hat{T}^{\otimes 2} \otimes I^{A_1A'_1A_2A'_2}) \left( (I^{A_1A'_1} \otimes (F^T) \, \hat{A}_1 \hat{A}'_1) (\Phi^{A_1A_1A'_1A'_1}) \right) \right. \\
& \left. \times \left( (I^{A_2A'_2} \otimes (F^T) \, \hat{A}_2 \hat{A}'_2) (\Phi^{A_2A_2A'_2A'_2}) \right) \right\} \, (F^{EE'} \otimes I^{A_1A'_1A_2A'_2}) \right] \\
& = |A_1|^2 |A_2|^2 \text{Tr} \left[ \left\{ (\hat{T}^{\otimes 2} \otimes I^{A_1A'_1A_2A'_2}) (\Phi^{A_1A_1A_2A'_1A'_1}) \right. \\
& \times \left( (F^{EE'} \otimes (F^T) \, \hat{A}_1 \hat{A}'_1) \otimes (F^T) \, \hat{A}_2 \hat{A}'_2) \right) \right. \\
& \left. \times (F^{EE'} \otimes (F^T) \, \hat{A}_1 \hat{A}'_1) \otimes (F^T) \, \hat{A}_2 \hat{A}'_2) \right] \\
& \overset{c}{=} |A_1|^2 |A_2|^2 \left\| \omega^{A_1A_1A_2A_2} \right\|_2^2
\end{align*}
\]

where,

- (a) follows by defining systems \( A_1 \cong \hat{A}'_1 \), \( A_2 \cong \hat{A}'_2 \), \( \Phi^{A_1\hat{A}_1} \), \( \Phi^{A'_1A'_1} \), \( \Phi^{A_2\hat{A}_2} \), \( \Phi^{A'_2A'_2} \) are the maximally entangled states and the fact that \( I^{A_m} = |A_1| \text{Tr}_{A_m} (\Phi^{A_m\hat{A}_m}) \) for \( m = \{1, 2\} \);
• (b) follows from the fact that for maximally entangled states, say $\Phi^{AA'}$ and any operator $M^A$ it holds that $(M^A \otimes I^{A'})\Phi^{AA'} = (I^A \otimes (M^T)^{A'})\Phi^{AA'}$ with the identification of the systems as $A = A_1A'_1, A' = \hat{A}_1\hat{A}'_1$ and the operator $M$ as the swap operator $F$; and

• (c) follows from the [Theorem 2.10](#) and the observation that $F^T$ is equivalent to $F$.

Using similar arguments it can be shown that

1. $\text{Tr} \left[ \mathcal{F}_t \otimes \mathcal{F}_t \left( F^{EE'} \right) \right] = |A_1|^2 |A_2|^2 \|\hat{\omega}^E\|^2_2$

2. $\text{Tr} \left[ \mathcal{F}_t A_1^{A_1'} \otimes I^{A_2^{A_2'}} \mathcal{F}_t \otimes \mathcal{F}_t \left( F^{EE'} \right) \right] = |A_1|^2 |A_2|^2 \|\hat{\omega}^{\hat{A}_1E}\|^2_2$

3. $\text{Tr} \left[ I^{A_1^{A_1'}} \otimes \mathcal{F}_t A_2^{A_2'} \mathcal{F}_t \otimes \mathcal{F}_t \left( F^{EE'} \right) \right] = |A_1|^2 |A_2|^2 \|\hat{\omega}^{\hat{A}_2E}\|^2_2$

Finally, to get meaningful bounds we need to bound the values of $a_{00}, a_{01}, a_{10}$ and $a_{11}$. To do this we first invert the matrix in [Theorem 4.1](#) and observe the following:

\[
\begin{bmatrix}
    a_{00} \\
    a_{01} \\
    a_{10} \\
    a_{11}
\end{bmatrix} = \frac{|A_1 A_2|}{(|A_1|^2 - 1)(|A_2|^2 - 1)} \begin{bmatrix}
    |A_1| & -|A_2| & -|A_1| & 1 \\
    -|A_2| & |A_1| & |A_1| & -|A_2| \\
    -|A_1| & 1 & -|A_1| & -|A_2| \\
    1 & -|A_1| & -|A_2| & |A_1| & |A_2|
\end{bmatrix} \begin{bmatrix}
    \|\hat{\omega}^E\|^2_2 \\
    \|\hat{\omega}^{\hat{A}_1E}\|^2_2 \\
    \|\hat{\omega}^{\hat{A}_2E}\|^2_2 \\
    \|\hat{\omega}^{\hat{A}_1A_2E}\|^2_2
\end{bmatrix}
\]

(3)

\[
\begin{bmatrix}
    a_{00} \\
    a_{01} \\
    a_{10} \\
    a_{11}
\end{bmatrix} = \frac{|A_1 A_2|}{(|A_1|^2 - 1)(|A_2|^2 - 1)} \begin{bmatrix}
    |A_1| & -|A_2| & -|A_1| & 1 \\
    -|A_2| & |A_1| & |A_1| & -|A_2| \\
    -|A_1| & 1 & -|A_1| & -|A_2| \\
    1 & -|A_1| & -|A_2| & |A_1| & |A_2|
\end{bmatrix} \begin{bmatrix}
    \|\hat{\omega}^E\|^2_2 \\
    \|\hat{\omega}^{\hat{A}_1E}\|^2_2 \\
    \|\hat{\omega}^{\hat{A}_2E}\|^2_2 \\
    \|\hat{\omega}^{\hat{A}_1A_2E}\|^2_2
\end{bmatrix}
\]

(4)

Bounding $a_{00}$

Consider the quantity

\[ I := |A_1| |A_2| \left( \|\hat{\omega}^E\|^2_2 - |A_2| \|\hat{\omega}^{\hat{A}_1E}\|^2_2 - |A_1| \|\hat{\omega}^{\hat{A}_2E}\|^2_2 + \|\hat{\omega}^{\hat{A}_1A_2E}\|^2_2 \right) \]

By [Theorem 2.13](#) we have that

1. $\left\| \hat{\omega}^{\hat{A}_1A_2E} \right\|^2_2 \leq |A_1| \left\| \hat{\omega}^{\hat{A}_2E} \right\|^2_2$

2. $\left\| \hat{\omega}^{\hat{A}_1E} \right\|^2_2 \geq \frac{1}{|A_1|} \left\| \hat{\omega}^E \right\|^2_2$
The above inequalities imply that
\[ I \leq \frac{|A_2| \cdot (|A_1|^2 - 1)}{|A_1|} \|\tilde{\omega}^E\|_2^2 \]

Thus on solving the system of Equations 3 and the bound on \( I \) further implies that
\[ \alpha_{00} = \frac{|A_1| |A_2|}{(|A_1|^2 - 1)(|A_2|^2 - 1)} I \]
\[ \leq \frac{|A_2|^2}{|A_2|^2 - 1} \|\tilde{\omega}^E\|_2^2 \]

**Bounding \( \alpha_{01} \)**

Define
\[ \mathbf{II} := \begin{bmatrix} -|A_2| & |A_1| |A_2| & 1 & -|A_1| \end{bmatrix} \begin{bmatrix} \|\tilde{\omega}^E\|_2^2 \\ \|\tilde{\omega}^{A_1 E}\|_2^2 \\ \|\tilde{\omega}^{A_2 E}\|_2^2 \\ \|\tilde{\omega}^{A_1 A_2 E}\|_2^2 \end{bmatrix} \]

By [Theorem 2.13](#) we have \( \|\tilde{\omega}^{A_1 E}\|_2^2 \leq |A_1| \|\tilde{\omega}^{A_1 A_2 E}\|_2^2 \) and \( \|\tilde{\omega}^{A_2 E}\|_2^2 \geq |A_1|^{-1} \|\tilde{\omega}^{A_2 E}\|_2^2 \) which implies that
\[ \alpha_{01} = \frac{|A_1| |A_2|}{(|A_1|^2 - 1)(|A_2|^2 - 1)} \mathbf{II} \]
\[ \leq \frac{|A_2|^2}{|A_2|^2 - 1} \|\tilde{\omega}^{A_1 E}\|_2^2 \]

**Bounding \( \alpha_{10} \)**

Define
\[ \mathbf{III} := \begin{bmatrix} -|A_1| & 1 & |A_1| |A_2| & -|A_2| \end{bmatrix} \begin{bmatrix} \|\tilde{\omega}^E\|_2^2 \\ \|\tilde{\omega}^{A_1 E}\|_2^2 \\ \|\tilde{\omega}^{A_2 E}\|_2^2 \\ \|\tilde{\omega}^{A_1 A_2 E}\|_2^2 \end{bmatrix} \]

By [Theorem 2.13](#) we have \( \|\tilde{\omega}^{A_2 E}\|_2^2 \leq |A_1| \|\tilde{\omega}^{A_1 A_2 E}\|_2^2 \) and \( \|\tilde{\omega}^E\|_2^2 \geq |A_1|^{-1} \|\tilde{\omega}^{A_1 E}\|_2^2 \), which im-
plies that

\[
\alpha_{10} = \frac{|A_1||A_2|}{(|A_1|^2 - 1)(|A_2|^2 - 1)} III
\]

\[
\leq \frac{|A_2|^2}{|A_2|^2 - 1} \|\hat{\omega}_{A_2 E}\|_2^{2}
\]

**Bounding \(\alpha_{11}\)**

Define

\[
IV = \begin{bmatrix}
1 & -|A_1| & -|A_2| & |A_1||A_2| \\
|A_1| & 1 & -|A_2| & |A_1||A_2| \\
|A_2| & -|A_1| & 1 & |A_1||A_2| \\
|A_1||A_2| & |A_1||A_2| & |A_1||A_2| & 1
\end{bmatrix}
\]

By [Theorem 2.13] we have \(\|\hat{\omega}_{A_2 E}\|_2^2 \geq |A_1|^{-1} \|\hat{\omega}_{A_1 A_2 E}\|_2^2\) and \(\|\hat{\omega}_E\|_2^2 \leq |A_1| \|\hat{\omega}_{A_1 E}\|_2^2\), which implies that

\[
\alpha_{11} = \frac{|A_1||A_2|}{(|A_1|^2 - 1)(|A_2|^2 - 1)} IV
\]

\[
\leq \frac{|A_2|^2}{|A_2|^2 - 1} \|\hat{\omega}_{A_1 A_2 E}\|_2^2
\]

Collating all these bounds and the fact that systems \(\hat{A}_i\) can be relabelled as \(A'_i\) for \(i = \{1, 2\}\) we see that

\[
\int \text{Tr} \left[ (\mathcal{T} \{ (U^{A_1} \otimes U^{A_2}) \cdot \hat{\rho}_{A_1 A_2 R} \})^2 \right] dU_1 dU_2
\]

\[
\leq \frac{1}{|A_2|^2 - 1} \left[ \|\hat{\rho}^R\|_2^2 \|\hat{\omega}_E\|_2^2 + \|\hat{\rho}_{A_2^R}\|_2^2 \|\hat{\omega}_{A_2^R E}\|_2^2 + \|\hat{\rho}_{A_1 R}\|_2^2 \|\hat{\omega}_{A_1 R E}\|_2^2 + \|\hat{\rho}_{A_1 A_2 R}\|_2^2 \|\hat{\omega}_{A_1 A_2 R E}\|_2^2 \right]
\]

Plugging this bound into the main expression of the theorem we find that

\[
\int \left\| \mathcal{T} (U^{A_1} \otimes U^{A_2} \cdot \hat{\rho}_{A_1 A_2 R}) - \omega^E \otimes \rho^R \right\|_1
\]

\[
\leq \frac{1}{|A_2|^2 - 1} \left[ \|\hat{\rho}^R\|_2^2 \|\hat{\omega}_E\|_2^2 + \frac{|A_2|^2}{|A_2|^2 - 1} \left[ \|\hat{\rho}_{A_2^R}\|_2^2 \|\hat{\omega}_{A_2^R E}\|_2^2 + \|\hat{\rho}_{A_1 R}\|_2^2 \|\hat{\omega}_{A_1 R E}\|_2^2 + \|\hat{\rho}_{A_1 A_2 R}\|_2^2 \|\hat{\omega}_{A_1 A_2 R E}\|_2^2 \right] \right]
\]

By the choice of the weighting matrices \(\sigma^E\) and \(\zeta^R\) in the statement of the theorem, \(\|\hat{\rho}^R\|_2^2 \|\hat{\omega}_E\|_2^2 \leq 1\), for instance \(\sigma^E\) is the matrix obtained by zeroing out the smallest eigen values that sum up to \(\delta\) and hence \(\|\hat{\omega}_E\|_2^2 \leq 1\) and similarly choosing \(\zeta^R\) by curtailing the marginal density operator \(\rho^R\) and finally choosing \(|A_2|\) large such that the first term is less than \(\delta\). This concludes the proof. \(\Box\)

We now state a corollary that will be useful in stating our coding theorems:
Corollary 4.3 Given the same conditions as in Theorem 4.2 the following holds
\[ \int \left\| \mathcal{T} \{ (U_{\text{RAND}}^1 \otimes \cdots \otimes U_{\text{RAND}}^k) \cdot \rho^{A_0 \cdots A_{k-1} R} \} - \omega^E \otimes \rho^R \right\|_1 \, dU_0 \cdots dU_{k-1} \leq \left( \delta + 2 \cdot 2^{-H_{2,\delta}(A_1|R) - H_{2,\delta}(A_2|E)} + 2 \cdot 2^{-H_{2,\delta}(A_1 A_2|E)} \right) \]

Proof: The proof is easy, since the term \( \frac{|A_2|^2}{|A_2| - 1} \leq 2 \) for all large \( |A_2| \). The rest follows trivially by the definition of \( \tilde{H}_{2,\delta}(\cdot|\cdot) \).

4.2 The Theorem for Multiple Senders

We now generalise the tensor product decoupling theorem for \( k > 2 \) senders.

Theorem 4.4 (Generalised \( k \) sender Tensor Product Decoupling Theorem) Let \( \rho^{A_0 \cdots A_{k-1} R} \) be a density operator that can be thought of as an entangled state between \( k \) senders with each sender denoted by \( \{ A_i \}_{i=0}^{k-1} \) and a reference \( R \). Let \( \mathcal{T}^{A_0 \cdots A_{k-1} \to E} \) be a CP map, and define \( \omega^{A_0 \cdots A_{k-1} E} := (I \otimes \mathcal{T}) \cdot (\otimes_i \Phi^{A_i A_i}) \), the Choi state for the superoperator \( \mathcal{T} \). Then,
\[ \int \left\| \mathcal{T} \{ (\otimes_i U_{\text{RAND}}^i) \cdot \rho^{A_0 \cdots A_{k-1} R} \} - \omega^E \otimes \rho^R \right\|_1 \, dU_0 \cdots dU_{k-1} \leq \left( \frac{|A_1 \cdots A_{k-1}|^2}{(|A_{k-1}|^2 - 1) \ldots (|A_1|^2 - 1)} - 1 \right) \cdot \frac{\| \omega^E \|^2_2 \cdot \| \rho^R \|^2_2}{2} + \frac{|A_0 \cdots A_{k-1}|^2}{(|A_{k-1}|^2 - 1) \ldots (|A_0|^2 - 1)} \sum_{b \neq 0} \frac{\| \omega^{A_b E} \|^2_2}{2} \left( \frac{\| \rho^{A_b R} \|^2_2}{2} \cdot 2^k + \frac{\| \rho^R \|^2_2}{2} \right) \]

where we assume that \( |A_0| \) is the smallest among the dimensions of the registers and the indices \( b \in \{0, 1, \ldots, k-1\} \) are represented as bit strings of length \( k \).

Proof: For brevity of notation, let \( A_{[k]} := A_{k-1,k-2,\ldots,1,0} \) denote the system representing the joint state of all the \( k \) senders with dimension \( |A_{[k]}| := \prod_{i=0}^{k-1} |A_i| \). We will use the same definitions of \( \tilde{T}, \tilde{\omega} \) and \( \tilde{\rho} \) as in Theorem 4.2, that is, to represent that \( \cdot \) denotes conjugation of the underlying operator by appropriate weighting matrices arising due to Fact 2.14. We begin with the application of Fact 2.14 as follows:
\[ \int \left\| \mathcal{T} \{ (\otimes_i U_{\text{RAND}}^i) \cdot \rho^{A_0 \cdots A_{k-1} R} \} - \omega^E \otimes \rho^R \right\|_1 \, dU_0 \cdots dU_{k-1} \leq \sqrt{\text{Tr} \left[ (\mathcal{T} \{ (\otimes_i U_{\text{RAND}}^i) \cdot \rho^{A_0 \cdots A_{k-1} R} \} - \omega^E \otimes \rho^R)^2 \right]} \, dU_0 \cdots dU_{k-1} \]

Recall that
\[ \int \text{Tr} \left[ (\mathcal{T} \{ (\otimes_i U_{\text{RAND}}^i) \cdot \rho^{A_0 \cdots A_{k-1} R} \} - \omega^E \otimes \rho^R)^2 \right] \, dU_0 \cdots dU_{k-1} = \int \text{Tr} \left[ \left( \mathcal{T} \{ (\otimes_i U_{\text{RAND}}^i) \cdot \rho^{A_0 \cdots A_{k-1} R} \} \right)^2 \right] \, d(\otimes_i U_i) - \text{Tr}[(\omega^E)^2] \cdot \text{Tr}[(\rho^R)^2] \]
Again, using standard arguments from Fact 2.10 and manipulation similar to Equation 2 in Theorem 4.2 we see that:

\[
\int \text{Tr} \left[ \left( \bigotimes_i U_i^{A_i} \cdot \tilde{\rho}^{A_0 \ldots A_{k-1}} \right)^2 \right] d( \bigotimes_i U_i ) = \text{Tr} \left[ \left( \tilde{\rho}^{A_0 \ldots A_{k-1}} \right)^{\otimes 2} \left( \bigotimes_i (U_i^A \otimes U_i^{A'}) \cdot M^{A_i A_i'} \otimes F^{R'R'} \right) \right]
\]

(5)

where \( M^{A_i A_i'} := (\tilde{\rho}^A)^{\otimes 2} (F^{EE'}) \) and the expectation is taken over independent choice of Haar random unitaries \( \{U_i\}_{i=0}^{k-1} \).

From Theorem 4.1 we have:

\[
\mathbb{E}_{U_0^{A_0}, \ldots, U_{k-1}^{A_{k-1}}} \left[ \bigotimes_i (U_i^A \otimes U_i^{A'}) \cdot M^{A_i A_i'} \right] = \mathbb{E}_{U_0^{A_0}, \ldots, U_{k-1}^{A_{k-1}}} \left[ \bigotimes_i (U_i^A \otimes U_i^{A'}) \cdot M^{A_i A_i'} \right]
\]

(6)

\[
\sum_{a := a_{k-1} \ldots a_0 = 0^k} \alpha_a \bigotimes_i (F^{A_i A_i'})^{a_i}
\]

(7)

where in the last equality we represent the indices \( \{a\}_{a=0}^{k-1} \) in binary as \( a_{k-1}, \ldots, a_0 \) for each \( a_i \in \{0, 1\} \). To evaluate the coefficients \( \alpha_a \) we again apply Lemma 4.1 with the following equalities:

\[
\text{Tr}(M^{A_i A_i'}) = \text{Tr} \left( \mathbb{E}_{i=0}^{k-1} U_i^A \left[ \bigotimes_{i=0}^{k-1} (U_i^A \otimes U_i^{A'}) \circ M^{A_i A_i'} \right] \right)
\]

(8)

and

\[
\text{Tr} \left( \bigotimes_{i=0}^{k-1} (F^{A_i A_i'})^{a_i} \cdot (M^{A_i A_i'}) \right) = \text{Tr} \left( \bigotimes_{i=0}^{k-1} (F^{A_i A_i'})^{a_i} \cdot \mathbb{E}_{i=0}^{k-1} U_i^A \left[ \bigotimes_{i=0}^{k-1} (U_i^A \otimes U_i^{A'}) \circ M^{A_i A_i'} \right] \right).
\]

(9)

This gives the matrix equation

\[
K \cdot \begin{bmatrix} \alpha_{0^k} \\ \vdots \\ \alpha_a \\ \vdots \\ \alpha_{1^k} \end{bmatrix} = |A_{|k}|^2 \begin{bmatrix} \|\tilde{\omega}^{A_{k-1} \ldots A_0} \|_2^2 \\ \vdots \end{bmatrix}
\]

(11)

where, for the bit string \( b := b_{k-1} \ldots b_0 \) we define:

\[
\tilde{\omega}^{A_{k-1} \ldots A_0} \cdot b = \tilde{\omega}^{A_{k-1} \ldots A_0} \cdot E
\]

17
The matrix $K$ is a $2^k \times 2^k$ matrix with rows indexed by bit vector $a \in \{0, 1\}^k$ and columns indexed by the bit vector $b \in \{0, 1\}^k$ and is obtained from Eq. (8) and Eq. (9) with entries-1

$$
(K)_{b,a} = |A_{|k|}^{|k-1|} \prod_{i=0}^{k-1} A_i^{(b_i \oplus a_i)},
$$

where $\oplus$ denotes the bit-wise XOR. This is not hard to see as the $i$-th term in the product is the $(b_i, a_i)$ term of the $i$-th $2 \times 2$ matrix in the tensor product in Eq. (11) which is exactly $|A_i|^{b_i \oplus a_i}$.

Also note that The RHS of Eq. (11) comes from the fact that

$$
\text{Tr} \left[ \bigotimes_i \left( F^{A_i} A_i' \right)^{b_i} M \right] = \left( \prod_{i \in \{0,1,...,k-1\}} |A_i| \right)^2 \| \tilde{\omega} A^b E \|_2^2
$$

This leads to the following representation of $K$:

$$
K = |A_{|k-1|}||A_{|k-2|}||...||A_0| \left( \begin{array}{c} \frac{1}{1} \\ A_{|k-1|} \end{array} \right) \otimes \left( \begin{array}{c} \frac{1}{1} \\ A_{|k-2|} \end{array} \right) \otimes \ldots \otimes \left( \begin{array}{c} \frac{1}{1} \\ A_0 \end{array} \right)
$$

From Eq. (13) we note that:

$$
K^{-1} = \frac{|A_{|k|}|}{(|A_{|k-1|}|^2 - 1) \ldots (|A_0|^2 - 1)} \bigotimes_{i \in \{k-1, k-2, \ldots, 0\}} \left( \begin{array}{c} A_i \\ -1 \end{array} \right)
$$

Then coupled with Eq. (12) and Eq. (11) Eq. (14) implies that

$$
(K^{-1})_{a,b} = \frac{|A_{|k|}|}{(|A_{|k-1|}|^2 - 1) \ldots (|A_0|^2 - 1)} \prod_{i \in \{k-1, k-2, \ldots, 0\}} |A_i|^{b_i \oplus a_i} (-1)^{a_i \oplus b_i}
$$

This directly implies that

$$
\alpha_a = \frac{|A_{|k|}|}{(|A_{|k-1|}|^2 - 1) \ldots (|A_0|^2 - 1)} \sum_{b=0}^{k} \left( \prod_{i \in \{k-1, k-2, \ldots, 0\}} |A_i|^{b_i \oplus a_i} (-1)^{a_i \oplus b_i} \right) \left\| \tilde{\omega} A^b E \right\|_2^2
$$

We will differentiate between the following two cases. Define $c := a \oplus b$.

**Case 1:** $a = 0^k$

and let without loss of generality $A_0$ be the register with the smallest dimension. Then,

$$
\alpha_a = \frac{|A_{|k-1|} \ldots A_0|^2}{(|A_{|k-1|}|^2 - 1) \ldots (|A_0|^2 - 1)} \left( \left\| \tilde{\omega} E \right\|_2^2 - \frac{\left\| \tilde{\omega} A E \right\|_2^2}{|A_0|^2} \right)
$$

$$
+ \frac{|A_{|k-1|} \ldots A_0|}{(|A_{|k-1|}|^2 - 1) \ldots (|A_0|^2 - 1)} \sum_{c_i := c_k, \ldots, c_1 \neq 0^{k-1}} \left[ \left( \prod_{i \neq 0} |A_i^{c_i}| (-1)^{c_i} \right) A_0 \left\| \tilde{\omega} A^{c} E \right\|_2^2 \right.
$$

$$
- \left( \prod_{i \neq 0} |A_i^{c_i}| (-1)^{c_i} \right) \left\| \tilde{\omega} A^{c} A_0 E \right\|_2^2 \right)
$$

(16)
For \( c' \) with odd parity and \( \textbf{Theorem 2.13} \) the term inside the summation is:

\[
\left( \prod_{i \neq 0} |A_i^{c_i}| \right) \left[ \left\| \hat{\alpha} A^{c'} A_0 E \right\|_2^2 - |A_0| \left\| \hat{\alpha} A^{c'} E \right\|_2^2 \right] \leq 0
\]  

(17)

For \( c' \) with even parity and again \( \textbf{Theorem 2.13} \) we have that:

\[
\left( \prod_{i \neq 0} |A_i^{c_i}| \right) \left[ |A_0| \left\| \hat{\alpha} A^{c'} E \right\|_2^2 - \left\| \hat{\alpha} A^{c'} A_0 E \right\|_2^2 \right] \\
\leq \left( \prod_{i \neq 0} |A_i^{c_i}| \right) \cdot \left( \frac{|A_0|^2 - 1}{|A_0|} \right) \cdot \left\| \hat{\alpha} A^{c'} E \right\|_2^2
\]  

(18)

Substituting Eq. (17) and Eq. (18) in equation Eq. (16) and using the bound \( \textbf{Theorem 2.13} \) for \( \| \hat{\alpha} A_{0} E \|_2^2 \geq \| \hat{\alpha}^E \|_2^2 / |A_0| \), we get:

\[
\alpha_{0^k} \leq \left\| \hat{\alpha}^E \right\|_2^2 \cdot \frac{|A_{[k]}|^2}{(|A_{k-1}|^2 - 1 \ldots) (|A_0|^2 - 1) \left[ 1 - \frac{1}{|A_0|^2} \right]} \\
+ \sum_{c' \neq 0^{k-1}} \left\| \hat{\alpha}^{A^{c'} E} \right\|_2^2 \cdot \frac{|A_1 \ldots A_{k-1}|^2}{(|A_{k-1}|^2 - 1 \ldots) (|A_1|^2 - 1)} \left[ |A_0| - \frac{1}{|A_0|} \right] \\
= \left\| \hat{\alpha}^E \right\|_2^2 \cdot \frac{|A_{1} \ldots A_{k-1}|^2}{(|A_{k-1}|^2 - 1 \ldots) (|A_1|^2 - 1)} \\
+ \sum_{c' \neq 0^{k}, c_0 = 0} \left\| \hat{\alpha}^{A^{c'} E} \right\|_2^2 \cdot \frac{\prod_{i \neq 0} |A_i^{c_i}| + 1}{(|A_{k-1}|^2 - 1 \ldots) (|A_1|^2 - 1)}
\]  

(19)

\textbf{Case 2:} \( a \neq 0^k \)

Firstly observe that, given a fixed \( a \in \{0,1\}^k \), and \( c = a \oplus b \) for some \( b \in \{0,1\}^k \),

\[
\left( \prod_{i} |A_i^{c_i}| \right) \cdot \left\| \hat{\alpha} A^{a \oplus c E} \right\|_2^2 \leq \left( \prod_{i} |A_i| \right) \cdot \left\| \hat{\alpha} A^{c E} \right\|_2^2
\]  

(20)

This is easy to verify on a case by case basis, by considering any fixed index \( i \in [k] \) and iterating through all possible values of the tuple \( (a_i, c_i) \). The above identity holds in each of these four possible cases, which is seen either directly or by invoking \( \textbf{Theorem 2.13} \) as the case demands. Then we simply bound the value of \( a_a \) as follows:

\[
a_a \leq \frac{|A_{[k]}|^2}{(|A_{k-1}|^2 - 1 \ldots) (|A_0|^2 - 1)} \left\| \hat{\alpha} A^{a E} \right\|_2^2 \\
+ \frac{|A_{[k]}|^2}{(|A_{k-1}|^2 - 1 \ldots) (|A_0|^2 - 1)} \sum_{b \neq 0} \left( \prod_{i} |A_i^{c_i}| \right) \cdot \left\| \hat{\alpha} A^{a \oplus c E} \right\|_2^2 \\
\leq \frac{|A_{[k]}|^2}{(|A_{k-1}|^2 - 1 \ldots) (|A_0|^2 - 1)} \left\| \hat{\alpha} A^{a E} \right\|_2^2 \cdot 2^k
\]  

(21)
where in the first inequality we upper bound every term by its absolute values and use [Eq. (20)] Finally we collate the estimates for $a_\alpha$ for the two different cases from [Eq. (19)] and [Eq. (21)] and substitute these values in [Eq. (6)] to get:

$$\mathbb{E}_{U_i \cdots U_i} [\bigotimes_{i=0}^{k-1} (U_i^+ A_i \otimes U_i^+ A_i')] \cdot M_{A_i|A_i}$$

$$\leq \left( \frac{|(A_{k-1} \cdots A_1)|^2}{(A_{k-1}^2 - 1) \cdots (A_1^2 - 1)} \| \hat{\omega} E \|_2^2 \right) (I_{A_i|A_i})$$

$$+ \left( \sum_{b \neq 0} \frac{|A_i|^2}{(|A_{k-1}|^2 - 1) \cdots (|A_0|^2 - 1)} \| \hat{\omega} A_i E \|_2^2 \cdot 2^k \right) \otimes (F_{A_i|A_i})$$

By substituting [Eq. (22)] in [Eq. (5)] we get:

$$\mathbb{E}_{U_0 \cdots U_{k-1}} \text{Tr} \left[ \left( \bigotimes_i U_{i \text{RAND}}^+ \right) \cdot \hat{\rho}^{A_0 \cdots A_{k-1} R} - \hat{\omega} E \otimes \hat{\rho}^R \right]^2$$

$$\leq \left( \frac{|A_1 \cdots A_{k-1}|^2}{(|A_{k-1}|^2 - 1) \cdots (|A_1|^2 - 1)} \right) \| \hat{\omega} E \|_2^2 \cdot \| \hat{\rho}^R \|_2^2$$

$$+ \left( \frac{|A_0 \cdots A_{k-1}|^2}{(|A_{k-1}|^2 - 1) \cdots (|A_0|^2 - 1)} \sum_{b \neq 0} \| \hat{\omega} A_i E \|_2^2 \right) \| \hat{\rho}^{A_i E} \|_2^2 \cdot 2^k + \| \hat{\rho}^R \|_2^2$$

This concludes the proof. □

**Remark 4.5** Just as in [Theorem 4.2] we can justify that $\| \hat{\omega} E \|_2^2 \| \hat{\rho}^R \|_2^2$ is much smaller than the second term in [Theorem 4.4] above. For this, recall $\hat{\omega} E := (\omega'' E) - \frac{1}{2} E (\omega'' E) - \frac{1}{2} E$, with $(\omega'' E)$ as the operator defined by zeroing out the smallest eigen values of $\omega E$ that sum up to $\delta$. Thus, $\| \hat{\omega} E \|_2^2 \leq 1$.

Similarly, define $\hat{\omega} R$ as the operator obtained by zeroing out those eigen values of $\rho^R$ that sum to $\delta$. Thus, $\| \hat{\omega} R \|_2^2 \leq 1$. Hence, $\| \hat{\omega} E \|_2^2 \| \hat{\rho}^R \|_2^2 \leq 1$. Thus the term with $\| \hat{\omega} E \|_2^2 \| \hat{\rho}^R \|_2^2$ can be neglected in the multi-user tensor product decoupling theorem above. The second term involving $\| \hat{\omega} E \|_2^2 \times \| \hat{\rho}^E \|_2^2$ serves as the entropic quantity that gives the rate region for reliable communication for QMAC.

## 5 The Multiple Access Channel

In this section we will use the results of the previous section to derive coding theorems, in the one shot regime for the Quantum Multiple Access Channel. The task we will consider is the following: Given a quantum multiple access channel $\mathcal{N}^{A' B' \rightarrow C}$ with Alice and Bob as senders and Charlie as receiver, consider two pure states $\psi^{A_C R_1}$ and $\phi^{B_C R_2}$ where $A$ and $B$ are registers that belong to Alice and Bob respectively, $C_1$ and $C_2$ belong to Charlie and $R_1$ and $R_2$ are purifying systems. The task is for Alice and Bob to send their shares of these states to Charlie by a single use of
the channel. To do this we will show the existence of encoding isometries $U_{ALICE}$ and $V_{BOB}$ and a decoding CPTP map $\mathcal{D}$ such that

$$\left\| \mathcal{D} \circ \mathcal{N}(U_{ALICE} \otimes V_{BOB} \cdot \psi^{A_1 R_1} \otimes \phi^{B_2 R_2}) - \psi^{A_1 R_1} \otimes \phi^{B_2 R_2} \right\|_1 \leq \epsilon$$

for some small $\epsilon$. The strategy we follow closely resembles the one for the point to point channel.

**Theorem 5.1** Let $\psi^{A_1 R_1}$ and $\phi^{B_2 R_2}$ be pure states and the registers $C_1$ and $C_2$ are held by Charlie. $\mathcal{N}^{A'B'\rightarrow C}$ is a CPTP map. $\omega^{A'B'C'E} := \mathcal{N}(\Omega^{A''A'} \otimes \Delta^{B''B'})$ where $\Omega^{A''A'}$ and $\Delta^{B''B'}$ are pure states and $|A''| = |A'|$ and $|B''| = |B'|$. Then there exist encoding isometries $U_{ALICE}$ and $V_{BOB}$ and a decoding CPTP $\mathcal{D}^{C_1 C_2 \rightarrow B_1 C_2}$ such that

$$\left\| \mathcal{D} \circ \mathcal{N}(U_{ALICE} \otimes V_{BOB} \cdot \psi^{A_1 R_1} \otimes \phi^{B_2 R_2}) - \psi^{A_1 R_1} \otimes \phi^{B_2 R_2} \right\|_1 \leq 2\sqrt{\delta_4}$$

where, for some $\delta, \delta_1, \delta_2, \delta_3, \delta_4 > 0$, we have:

$$\delta + 2\delta_2 + 2\delta_3 + 2\delta_4 = 2\delta_1$$

**Proof:** Let the states $|\Omega\rangle^{A'A'}$ and $|\Delta\rangle^{B'B'}$ to be copies of the original $|\Omega\rangle^{A''A'}$ and $|\Delta\rangle^{A''A'}$ states, where $|A| = |A''|$ and $|B| = |B''|$. Define

$$\mathcal{T}^{A'B'}(\rho) := |A''B''\rangle \langle D^{A} \otimes op_{A ightarrow A'}(|\Omega\rangle) \otimes op_{B ightarrow B'}(|\Delta\rangle) : \rho$$

Firstly, observe that:

$$\omega^{A''B'E} = \text{Tr}_C[\omega^{A''B'C'E}]$$

(23)

$$= \mathcal{N}^{A'B'E}(\Omega^{A''A'} \otimes \Delta^{B''B'})$$

(24)

$$= \mathcal{T} \otimes \text{I}^{B''B''}(\hat{\Phi}^{A''A''} \otimes \hat{\Phi}^{B''B''})$$

(25)

Let $W_1^{A ightarrow A'}$ and $W_2^{B ightarrow B'}$ be two isometries. Then the tensor product decoupling theorem then implies:

$$\int \left\| \mathcal{T}((U_{\text{RAND}}^{A'} W_1 \otimes V_{\text{RAND}}^{B'} W_2) : (\psi^{A R_1} \otimes \phi^{B R_2})) - \omega^{E} \otimes \psi^{R_1} \otimes \phi^{R_2} \right\|_1 dUdV$$

$$\leq \sqrt{\delta + 2\delta_2 + 2\delta_3 + 2\delta_4} = \delta_1$$

Now we show the existence of two isometries $U_{ALICE}$ and $V_{BOB}$ which approximately emulate the action of the operators $\sqrt{|A''\rangle \langle A'|} U_{\text{RAND}}^{A'} W_1$ and $\sqrt{|B''\rangle \langle B'|} V_{\text{RAND}}^{B'} W_2$. To that end define the maps:
1. $\mathcal{E}^A \rightarrow G(\rho) := |A''| \text{Tr}[\text{op}_{\hat{A} \rightarrow A'}(\Omega) \cdot \rho]$ 
2. $\mathcal{F}^B \rightarrow G(\rho) := |B''| \text{Tr}[\text{op}_{\hat{B} \rightarrow B'}(\Delta) \cdot \rho]$ 

where $G$ and $G'$ are one dimensional systems. Then, using the vanilla (non smooth) decoupling theorem twice we get 

1. 

$$\int \|I^{C_1 R_1} \otimes \mathcal{E}(U_{\text{RAND}} W_1 \cdot \text{ψ}^{AC_1 R_1}) - \text{ψ}^{C_1 R_1}\|_1 \, dU \leq 2\frac{1}{\max(A)} \cdot \frac{1}{\Delta H_{2,2}(A'' \omega) = \delta_2}$$ 

2. 

$$\int \|I^{C_2 R_2} \otimes \mathcal{F}(V_{\text{RAND}} W_2 \cdot \text{φ}^{BC_2 R_2}) - \text{φ}^{C_2 R_2}\|_1 \, dV \leq 2\frac{1}{\max(B)} \cdot \frac{1}{\Delta H_{2,2}(B'' \omega) = \delta_3}$$ 

where we have used the facts that $\mathcal{E}^{(\hat{A}^{A''})} = \omega^{A''}$ and $\mathcal{F}^{(\hat{B}^{B''})} = \omega^{B''}$. 

Consider the random variables defined as follows: 

1. $X := \|\mathcal{T}(U_{\text{RAND}} W_1 \otimes V_{\text{RAND}} W_2) \cdot (\text{ψ}^{AR_1} \otimes \text{φ}^{BR_2}) - \omega^{E} \otimes \text{ψ}^{R_1} \otimes \text{φ}^{R_2}\|_1$ 

2. $Y := \|\mathcal{E}(U_{\text{RAND}} W_1 \cdot \text{ψ}^{AC_1 R_1}) - \text{ψ}^{C_1 R_1}\|_1$ 

3. $Z := \|\mathcal{F}(V_{\text{RAND}} W_2 \cdot \text{φ}^{BC_2 R_2}) - \text{φ}^{C_2 R_2}\|_1$ 

and the following events: 

1. $E_1 := \{X \geq 4\delta_1\}$ 

2. $E_2 := \{Y \geq 4\delta_2\}$ 

3. $E_3 := \{Z \geq 4\delta_3\}$ 

Now by Markov’s inequality (for instance 

$$\Pr[E_2] \leq \left[ \int \|I^{C_1 R_1} \otimes \mathcal{E}(U_{\text{RAND}} W_1 \cdot \text{ψ}^{AC_1 R_1}) - \text{ψ}^{C_1 R_1}\|_1 \, dU \right] \leq \frac{1}{4\delta_2}$$ 

and union bound for events $E_1, E_2, E_3$ we get 

$$\Pr[E_1 \cap E_2 \cap E_3] > 0$$ 

which implies that there exists fixed pair of unitaries $U_{\text{RAND}}^A$ and $V_{\text{RAND}}^B$ which satisfy the event $E_1 \cap E_2 \cap E_3$. Fix such a pair of unitaries. Then, from Uhlmann’s theorem we see that Fact 2, III there exist isometries $U^{A \rightarrow A'}_{\text{ALICE}}$ and $V^{B \rightarrow B'}_{\text{BOB}}$ such that: 

$$\|A''|\text{op}_{\hat{A} \rightarrow A'}(\Omega) U^{\hat{A}}_{\text{FIXED}} W_1 \cdot \text{ψ}^{AC_1 R_1} - U^{A \rightarrow A'}_{\text{ALICE}} \cdot \text{ψ}^{AC_1 R_1}\|_1 \leq 4\sqrt{\delta_2} \quad (26)$$

$$\|B''|\text{op}_{\hat{B} \rightarrow B'}(\Delta) V^{\hat{B}}_{\text{FIXED}} W_2 \cdot \text{φ}^{BC_2 R_2} - V^{B \rightarrow B'}_{\text{BOB}} \cdot \text{φ}^{BC_2 R_2}\|_1 \leq 4\sqrt{\delta_3} \quad (27)$$

Define $\text{Tr}[B''|\text{op}_{\hat{B} \rightarrow B'}(\Delta) V^{\hat{B}}_{\text{FIXED}} W_2 \cdot \text{φ}^{BC_2 R_2}] := c_0$. Since trace is a quantum operation, from the equations above we see that 

$$|c_0 - 1| \leq 4\sqrt{\delta_3}$$
This gives:

$$
\left\| A''B''(\text{op}_{A\rightarrow A'}(\Omega)\mu^A_{\text{FIXED}}, W_1 \otimes \text{op}_{B\rightarrow B'}(\Delta)\nu^B_{\text{FIXED}}, W_2) \cdot (\phi^{A^1R_1} \otimes \phi^{B^2C_2}) - U_{\text{ALICE}} \otimes V_{\text{BOB}} \cdot \phi^{A^1C_1} \otimes \phi^{B^2C_2} \right\|_1
$$

(28)

$$
\leq \left\| B''(\text{op}_{B\rightarrow B'}(\Delta)\nu^B_{\text{FIXED}}, W_2) \cdot \phi^{B^2C_2} \right\|_1 \times \left\| A''(\text{op}_{A\rightarrow A'}(\Omega)\mu^A_{\text{FIXED}}, W_1) \cdot \phi^{A^1C_1} - U_{\text{ALICE}} \cdot \phi^{A^1C_1} \right\|_1
$$

(29)

$$
+ \left\| U_{\text{ALICE}} \cdot \phi^{A^1C_1} \right\|_1 \times \left\| B''(\text{op}_{B\rightarrow B'}(\Delta)\nu^B_{\text{FIXED}}, W_2) \cdot \phi^{B^2C_2} - \nu^B_{\text{BOB}} \cdot \phi^{B^2C_2} \right\|_1
$$

(30)

$$
\leq (1 + 4\sqrt{\delta_3}) \times (4\sqrt{\delta_2} + 4\sqrt{\delta_3})
$$

(31)

where we bound $\delta_0$ by $(1 + 4\sqrt{\delta_2})$. Finally, we use the triangle inequality and the monotonicity of 1-norm under a quantum operation (which is partial trace over $C_1C_2$ followed by $\mathcal{N}$) to obtain:

$$
\left\| \mathcal{N}(U_{\text{ALICE}} \otimes V_{\text{BOB}} \cdot \psi^{A^1R_1} \otimes \phi^{B^2C_2}) - \lambda^E \otimes \psi^{R_1} \otimes \phi^{R_2} \right\|_1 \leq \delta_1 + 4\sqrt{\delta_2} + 4\sqrt{\delta_3} + 12\sqrt{\delta_2\delta_3}
$$

(32)

$$
= \delta_4
$$

(33)

We conclude by invoking Uhlmann’s theorem Fact 2.11 again for the last inequality to prove the exists a decoder $D_{C_1C_2\rightarrow E}\tilde{A}\tilde{B}C_1C_2$ such that:

$$
\left\| DU_{\mathcal{N}}(U_{\text{ALICE}} \otimes V_{\text{BOB}} \cdot \psi^{A^1C_1} \otimes \phi^{B^2C_2}) - \lambda^E \otimes \psi^{A^1C_1} \otimes \phi^{B^2C_2} \right\|_1 \leq 2\sqrt{\delta_4}
$$

where $\lambda^E$ is some purification of $\psi^E$.

\[QED\]

We are now ready to state our coding theorem. The task is to use the QMAC to send arbitrary states, tensored across the registers belonging to Alice and Bob, with high fidelity to Charlie. It was shown in [KW03] that this task is equivalent to sending one half of two maximally entangled states (one belonging to Alice and one to Bob) across the channel. For a more general setting where there may be entanglement assistance, we reformulate in the language of Theorem 5.1.

We set the states $\psi^{A^1C_1}$ and $\phi^{B^2C_2}$ as $\Phi^{R_1M_1} \otimes \Phi^{\tilde{A}C_1}$ and $\Phi^{R_2M_2} \otimes \Phi^{\tilde{B}C_2}$ respectively. Here the registers $M_1\tilde{A}$ and $M_2\tilde{B}$ play the roles of $A$ and $B$. For a given $\epsilon > 0$ we say the the rate quadruple $(Q_A, E_A, Q_B, E_B)$ is $\epsilon$-achievable if there exist encoding isometries $U_{\text{ALICE}}$, $V_{\text{BOB}}$ and decoding CPTP $\mathcal{D}$, with $|M_1| = 2Q_A$, $|\tilde{A}| = 2E_A$, $|M_2| = 2Q_B$ and $|\tilde{B}| = 2E_B$, such that

$$
\left\| \mathcal{D} \circ \mathcal{N}(U_{\text{ALICE}} \otimes V_{\text{BOB}} \cdot \psi^{A^1C_1} \otimes \phi^{B^2C_2}) - \psi^{A^1C_1} \otimes \phi^{B^2C_2} \right\|_1 \leq \epsilon
$$

The rate pair $(Q_A, Q_B)$ is achievable for entangled assisted transmission if there exist $E_A, E_B > 0$ such that $(Q_A, Q_B, E_A, E_B)$ is $\epsilon$-achievable. The pair $(Q_A, Q_B)$ is achievable for unassisted transmission of $(Q_A, Q_B, 0, 0)$ is $\epsilon$-achievable. The one shot capacity region is the union of all achievable points $(Q_A, Q_B)$ for a fixed $\epsilon$, over all controlling states $\omega$ as defined in Theorem 5.1.

**Theorem 5.2** Given a quantum multiple access channel $\mathcal{N}^{A''B''\rightarrow C}$ and fixed $\delta > 0$, define $\epsilon := \delta_4$ where $\delta_4$ is as defined in Theorem 5.1. Let $\Omega^{A''B''}$ and $\Delta^{B''C''}$ be pure states and $\omega^{A''B''C''} := \mathcal{U}_\mathcal{N}(\Omega \otimes \Delta)$. Then the
rate quadruple \((Q_A, E_A, Q_B, E_B)\) is \(\epsilon\)-achievable for quantum transmission with rate limited entanglement assistance through \(\mathcal{N}\) if

\[
Q_A - E_A + Q_B - E_B < H_{2,\delta}(A''B''|E)_{\omega} + 2\log(1 - \delta)
\]

and

\[
Q_A - E_A < H_{2,\delta}(A''|E)_{\omega} + \log(1 - \delta)
\]

\[
Q_B - E_B < H_{2,\delta}(B''|E)_{\omega} + \log(1 - \delta)
\]

\[
Q_A + E_A < H_{2,\delta}(A')_{\omega}
\]

\[
Q_B + E_B < H_{2,\delta}(B')_{\omega}
\]

**Proof:** The proof is essentially an application of [Theorem 5.1](#). First, set the states \(\psi^\tilde{A} C_1 R_1 = \Phi^R_1 M_1 \otimes \Phi^\tilde{A} C_1\) and \(\psi^\tilde{B} M_2 R_2 = \Phi^R_2 M_2 \otimes \Phi^\tilde{B} C_2\) where the registers \(\tilde{A} M_1\) and \(\tilde{B} M_2\) are placeholders for the registers \(A\) and \(B\) in [Theorem 5.1](#). Then, invoking [Theorem 5.1](#) for the channel \(\mathcal{N}\) with controlling state \(\omega\), we see that there exist encoding isometries \(U_{\text{ALICE}}\), \(V_{\text{BOB}}\) and decoding CPTP \(D\) such that

\[
\|D \circ \mathcal{N}(U_{\text{ALICE}} \otimes V_{\text{BOB}} \cdot \psi^\tilde{A} C_1 R_1 \otimes \psi^\tilde{B} M_2 R_2) - \psi^\tilde{A} C_1 R_1 \otimes \psi^\tilde{B} M_2 R_2\|_1 \leq \epsilon
\]

, where \(\epsilon = \delta_4\) and

\[
\delta + 2^{-H_{2,\delta}(A' | E)_{\omega}} + 2^{-H_{2,\delta}(A'' | E)_{\omega}} + 2^{-H_{2,\delta}(B' | E)_{\omega}} + 2^{-H_{2,\delta}(B'' | E)_{\omega}} < \delta_2
\]

\[
2^\frac{1}{2} H_{\text{max}}(A) - \frac{1}{2} H_{2,\delta}(A')_{\omega} = \delta_2
\]

\[
2^\frac{1}{2} H_{\text{max}}(B) - \frac{1}{2} H_{2,\delta}(B')_{\omega} = \delta_3
\]

\[
\delta_1 + 4\sqrt{\delta_2} + 4\sqrt{\delta_3} + 12\sqrt{\delta_2 \delta_3} = \delta_4
\]

Observe that

\[
H_{2,\delta}(\tilde{A} M_1 | R_1)_{\phi, \epsilon} \geq -Q_A + E_A + \log(1 - \delta)
\]

\[
H_{2,\delta}(\tilde{B} M_2 | R_2)_{\phi, \epsilon} \geq -Q_B + E_B + \log(1 - \delta)
\]

\[
H_{\text{max}}(\tilde{A} M_1)_{\phi, \epsilon} \leq Q_A + E_A
\]

\[
H_{\text{max}}(\tilde{B} M_2)_{\phi, \epsilon} \leq Q_B + E_B
\]

Plugging in these estimates in the expressions for \(\delta_1, \delta_2, \delta_3\) such that \(\delta_4 = O(\sqrt{\delta})\) we conclude that the statement of the theorem is true. \(\square\)

### 6 Conclusion and Open Problems

In this paper, we have proven a decoupling theorem which involves multiple random unitaries, in tensor product with each other, chosen independently from the Haar measure as the decoupling unitary and the QMAC channel as the superoperator that results in decoupling the reference system of the input states and the channel environment, when expectation is taken over these unitaries in tensor product. The unitaries in tensor product can be thought of as independent encoders and the decoupling is achieved as a decoding step. The analysis of the error rate in our decoupling theorem leads to the characterization of an achievable rate region. We then proceed...
to evaluate the asymptotic iid limit of our rate region. However, we cannot recover the asymptotic iid rate region of Yard et al. in [YDH05]. The reason being an immediate open problem, that is to find an optimising state that simultaneously smooths the three different conditional Rényi 2-entropies mentioned in Theorem 5.2.

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We present inner bounds for the QMAC for the task of entanglement generation.

The following is an easy corollary of Theorem 4.2:

**Corollary A.1** Given orthogonal projectors $\Pi_{A_1 \rightarrow E_1}$ and $\Pi_{A_2 \rightarrow E_2}$, define the map

$$T^{A_1 A_2 \rightarrow E_1 E_2} := \frac{|A_1|}{|E_1|} \cdot \frac{|A_2|}{|E_2|} \cdot \Pi_{A_1 \rightarrow E_1} \otimes \Pi_{A_2 \rightarrow E_2}$$
Then given the density operator $\rho^{A_1A_2R}$ the following holds:

$$\int \left\| T(U_{\text{RAND}}^{A_1} \otimes U_{\text{RAND}}^{A_2} \cdot \rho^{A_1A_2R}) - \pi^{E_1} \otimes \pi^{E_2} \otimes \rho^R \right\|_1 dU^{A_1} dU^{A_2} \leq \left( \delta + 2 \cdot |E_1|2^{-\frac{H_2(\rho^{E_1} \otimes \rho^{E_2} \otimes \rho^R})}{\pi^{E_1} \otimes \pi^{E_2} \otimes \rho^R} + |E_2|2^{-\frac{H_2(\rho^{E_1} \otimes \rho^{E_2} \otimes \rho^R})}{\pi^{E_1} \otimes \pi^{E_2} \otimes \rho^R} \right)^{1/2}$$

**Proof:** First, observe that

$$\text{Tr}_{A_1'} A_2' T(\Phi^{A_1} \otimes \Phi^{A_2}) = \pi^{E_1} \otimes \pi^{E_2}$$

Then, define

$$\tilde{\bar{\omega}}^{A_1' A_2 E_1 E_2} := \sigma^{E_1 - 1/4} \otimes \sigma^{E_2 - 1/4} \cdot T(\Phi^{A_1} \otimes \Phi^{A_2})$$

where $\sigma^{E_1} := \pi^{E_1}$ and $\sigma^{E_2} := \pi^{E_2}$. Then note that:

1. $\tilde{\bar{\omega}}^{E_1} = \frac{1}{\sqrt{|E_1|}}$ and $\tilde{\bar{\omega}}^{E_2} = \frac{2}{|E_2|}$. Then note that:

2. $\tilde{\bar{\omega}}^{A_1' E_1 E_2} = \tilde{\bar{\omega}}^{A_1' E_2}$ and $\tilde{\bar{\omega}}^{A_2' E_1 E_2} = \tilde{\bar{\omega}}^{A_2' E_2}$. It is easy to see that $\text{Tr}[\tilde{\bar{\omega}}^{A_1' E_1^2} = |E_1|$. since

$$\text{Tr}[\tilde{\bar{\omega}}^{A_1' E_1^2}] = \left( \sqrt{|E_1|} \cdot \frac{|A_1|}{|E_1|} \right)^2 \cdot \text{Tr}[\Pi_1 |\Phi\rangle ^{A_1 A_1'} \langle \Phi| \Pi_1 |\Phi\rangle ^{A_1 A_1'} |\Phi\rangle |\Pi_1\rangle]$$

Similarly one can show that $\text{Tr}[\tilde{\bar{\omega}}^{A_2' E_2^2}] = |E_2|$. We conclude by noting that

$$\|\tilde{\bar{\omega}}^{A_1' E_1 E_2}\|_2^2 = \|\tilde{\bar{\omega}}^{A_1' E_1}\|_2^2 \cdot \text{Tr}[\Pi_1 |\Phi\rangle ^{A_1 A_1'} \langle \Phi| \Pi_1 |\Phi\rangle ^{A_1 A_1'} |\Phi\rangle |\Pi_1\rangle]$$

and similarly $\|\tilde{\bar{\omega}}^{A_2' E_2 E_2}\|_2^2 = |E_2|$. This completes the proof.

To get a channel coding theorem we will use Theorem A.1 but with some overloading of notation. Consider the quantum multiple access channel $N^{A' B' \rightarrow C}$ with isometric extension $U^{A' B' \rightarrow CE}$ where we use the register $E$ to mean the environment. Consider the controlling state $|\omega\rangle^{ABCE} := U^{A' B' \rightarrow CE} |\Omega\rangle ^{AA'} |\Delta\rangle ^{BB'}$, where $|\Omega\rangle ^{AA'}$ and $|\Delta\rangle ^{BB'}$ are arbitrary pure states. Then the following theorem holds:

**Theorem A.2** Given $\delta$ as in Theorem 4.4 and a positive $\epsilon$, the rates $(m_{\text{ALICE}}, n_{\text{BOB}})$ for entanglement generation over the channel $N^{A' B' \rightarrow E}$ are achievable whenever

$$m_{\text{ALICE}} < H_2(\rho^{A|E}) + \log \epsilon$$

$$n_{\text{BOB}} < H_2(\rho^{B|E}) + \log \epsilon$$

Then $m_{\text{ALICE}} + n_{\text{BOB}} < H_2(\rho^{AB|E}) + \log \epsilon$ with error $\sqrt{\delta + 6\epsilon}$.

**Proof:** We simply relabel terms from Theorem A.1

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1. Registers: $A \leftarrow A_1$, $B \leftarrow B_1$ and $E \leftarrow R$.

2. Registers: $R_1 \leftarrow E_1$ and $R_2 \leftarrow E_2$.

3. State: $\omega^{ABE} \leftarrow \rho^{A_1A_2R}$.

4. $|R_1| = 2^n_{\text{ALICE}}$ and $|R_2| = 2^n_{\text{BOB}}$.

Define

$$\mathcal{T}_{AB \rightarrow R_1R_2} := \frac{|A|}{|R_1|} \cdot \frac{|B|}{|R_2|} \cdot \Pi_1^{A \rightarrow R_1} \otimes \Pi_2^{B \rightarrow R_2}$$

Then applying Theorem A.1 we see that:

$$\int \left\| \mathcal{T}(U^A_{\text{RAND}} \otimes U^B_{\text{RAND}} \cdot \sigma^{ABE}) - \pi^{R_1} \otimes \pi^{R_2} \otimes \rho^E \right\|_1 dU^A dU^B \leq \left( \delta + 2 \cdot |R_1| 2^{-\beta_2(\delta|AB|E)} + |R_2| 2^{-\beta_2(\delta|B|E)} + |R_1 R_2| 2, \delta^{-\beta_2(\delta|AB|E)} \right)^{1/2}$$

Finally, the above equation implies that there exist fixed unitaries $U^A_{\text{FIXED}}$ and $U^B_{\text{FIXED}}$ such that the above inequality still holds. To conclude, by using the usual argument of applying Uhlmann’s theorem to the purifying register $C$ and requiring that every term inside the curly braces be $< \epsilon$ we conclude the proof.

**Remark A.3** We can emulate the $\mathcal{T}$ operation in the usual way by picking unitaries independently from using two 2-designs instead of two Haar random unitaries. This reduces the required amount of shared randomness necessary to implement the $\mathcal{T}$ operation from infinite to the log of product of the cardinality of the designs. Classical communication is necessary however so that Alice and Bob can let Charlie know which code they are using. □