AN ABSTRACT INTERPOLATION PROBLEM AND THE
EXTENSION THEORY OF ISOMETRIC OPERATORS*

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The algebraic structure of V.P. Potapov’s Fundamental Matrix Inequality (FMI) is discussed and its interpolation meaning is analyzed. Functional model spaces are involved. A general Abstract Interpolation Problem is formulated which seems to cover all the classical and recent problems in the field and the solution set of this problem is described using the Arov–Grossman formula. The extension theory of isometric operators is the proper language for treating interpolation problems of this type.

1. INTERPOLATION DATA AND EXAMPLES

Let \( \mathbb{L} \) and \( \mathbb{L}' \) be Hilbert spaces. We shall denote by \( B(\mathbb{L}, \mathbb{L}') \) the class of operator-valued functions which are holomorphic in the unit disk \(|\zeta| < 1\) and whose values are contractive operators from \( \mathbb{L} \) into \( \mathbb{L}' \). Let \( X \) be a linear space. We do not suppose that \( X \) is endowed with a topological structure. Let \( D \) be a sesquilinear form in \( X \) and let \( T \) be a linear operator on \( X \). Let \( E \) and \( M \) be linear operators from \( X \) into \( \mathbb{L} \) and \( \mathbb{L}' \), respectively.

We assume that the operators and the sesquilinear form are linked through the so-called Fundamental Identity (FI):

\[
D(x, y) - D(Tx, Ty) = \langle Ex, Ey \rangle_\mathbb{L} - \langle Mx, My \rangle_{\mathbb{L}'}.
\]

(FI)

The Fundamental Identity must be fulfilled for arbitrary \( x \) and \( y \) in \( X \). This framework arose from the study of a number of interpolation problems. We shall give some examples.

EXAMPLE 1. (The Nevanlinna-Pick Problem).

Interpolation Data: A sequence \( \{\zeta_k\}_{1 \leq k \leq \infty} \) of complex numbers \((|\zeta_k| < 1)\) and a sequence of contractive operators \( \{s_k\}_{1 \leq k \leq \infty} \), acting from \( \mathbb{L} \) into \( \mathbb{L}' \).

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A solution $s(\zeta)$ of this problem is an arbitrary holomorphic function from the class $B(\mathbb{L}, \mathbb{L}')$ which satisfies the interpolation conditions $s(\zeta_k) = s_k$. It is required to give a criteria for the existence of solutions and to describe the set of all solutions of the interpolation problem.

In this example, $X$ is the space of all infinite sequences whose entries are vectors from $\mathbb{L}$ such that only a finite number of them do not vanish:

$$x = \{\ell_1, \ell_2, \cdots, \ell_n, 0, \cdots, 0, \cdots\}.$$ 

The operators $T$, $E$, $M$ and the sesquilinear form $D$ are defined by the formulas

$$Tx = \{\zeta_1\ell_1, \zeta_2\ell_2, \cdots, \zeta_n\ell_n, 0, \cdots, 0, \cdots\},$$

$$Ex = \sum_{1 \leq k < \infty} \ell_k, \quad Mx = \sum_{1 \leq k < \infty} s_k\ell_k,$$

$$D(x, x) = \sum_{1 \leq j, k < \infty} \left\langle \frac{I - s_j^*s_k}{1 - \bar{s}_j \zeta k} \ell_k, \ell_j \right\rangle_{\mathbb{L}}.$$

The Fundamental Identity can be verified directly. The condition $D(x, x) \geq 0$ ($\forall x \in X$) is necessary and sufficient for the solvability of this problem.

EXAMPLE 2 (The Sarason Problem). Let $\theta$ be an inner function, let $\mathbb{K}_\theta := H^2 \ominus \theta H^2$, and let $P_\theta$ be the orthoprojection from $H^2$ onto $\mathbb{K}_\theta$. The interpolation data for this problem are the operator $T = P_\theta t|_{\mathbb{K}_\theta}$ i.e., the compressed shift on $\mathbb{K}_\theta$, and a contractive operator $W$ which commutes with $T$.

A solution of this problem is an arbitrary holomorphic function $w(\zeta)$ from the class $B(\mathbb{L}, \mathbb{L}')$ which satisfies the interpolation condition $W = P_\theta w|_{\mathbb{K}_\theta}$. It is required to describe the set of all solutions to this problem. The existence of solutions follows from the contractivity of $W$.

In this example $X$ coincides with $\mathbb{K}_\theta$, the operator $T$ is defined in the statement of the problem, $\mathbb{L} = \mathbb{L}' = \mathbb{C}$,

$$Ex \overset{\text{def}}{=} \langle x, e_\ast \rangle, \quad Mx \overset{\text{def}}{=} \langle Wx, e_\ast \rangle$$

and

$$D(x, x) \overset{\text{def}}{=} \langle (I - W^*W)x, x \rangle,$$

where $\langle , \rangle$ is the inner product in $\mathbb{K}_\theta$, the vector $e_\ast \in \mathbb{K}_\theta$ is defined by the formula $e_\ast = (\theta(t) - \theta(0))/t$. The Fundamental Identity is a consequence of the fact that

$$(I - T^*T)x = e_\ast \langle x, e_\ast \rangle.$$
for any $x \in \mathbb{K}_\theta$.

2. V.P. POTAPOV’S FUNDAMENTAL MATRIX INEQUALITY AND ITS TRANSFORMATION

We shall say that a function $s \in B(\mathbb{L}, \mathbb{L}')$ satisfies the V.P. Potapov’s Fundamental Matrix Inequality (the FMI) if for arbitrary $x \in \mathbb{X}$, $\ell \in \mathbb{L}$ and $|\zeta| < 1$,

$$\begin{bmatrix}
D((I - T\zeta)x, (I - T\zeta)x) & \langle \ell, (E - s^*(\zeta)M)x \rangle_{\mathbb{L}} \\
\langle (E - s^*(\zeta)M)x, \ell \rangle_{\mathbb{L}} & \langle \frac{\mathbb{I}_{\mathbb{L}} - s^*(\zeta)s(\zeta)}{1 - \zeta\zeta} \ell, \ell \rangle_{\mathbb{L}}
\end{bmatrix} \geq 0. \quad \text{(FMI)}$$

This inequality can be rewritten in another (equivalent) form:

$$\begin{bmatrix}
D((\zeta I - T)x, (\zeta I - T)x) & \langle \ell', (s(\zeta)E - M)x, \ell' \rangle_{\mathbb{L}'} \\
\langle (s(\zeta)E - M)x, \ell' \rangle_{\mathbb{L}'} & \langle \frac{\mathbb{I}_{\mathbb{L}'} - s(\zeta)s^*(\zeta)}{1 - \zeta\zeta} \ell', \ell' \rangle_{\mathbb{L}'}
\end{bmatrix} \geq 0 \quad \text{(FMI')}
$$

for arbitrary $x \in \mathbb{X}$, $\ell' \in \mathbb{L}'$ and $|\zeta| < 1$.

It will be shown, that the equivalence of the FMI and the FMI' is a consequence of the Fundamental Identity.

In this work we shall use several facts which we formulate here as propositions 1, 2 and 3 without proofs.

**PROPOSITION 1.** (The block-matrix lemma). Let $\mathbb{H}$ be a Hilbert space and let $A$ be a selfadjoint positive semidefinite ($A \geq 0$) operator acting on $\mathbb{H}$. Suppose that $h_0 \in \mathbb{H}$ and that there exists a constant $C \geq 0$ such that

$$|\langle h_0, h \rangle|^2 \leq C\|\sqrt{A}h\|^2, \quad \forall h \in D_{\sqrt{A}},$$

where $D_{\sqrt{A}}$ is the domain of the selfadjoint operator $\sqrt{A}$. Then there exists a unique vector $g_0 \in (\text{Ker}A)\perp \cap D_{\sqrt{A}}$ such that $\sqrt{A}g_0 = h_0$. Moreover,

$$\|g_0\|^2 \leq C.$$

We shall use the following notations:

$$g_0 = A^{-\frac{1}{2}}h_0 \quad \text{and} \quad \langle g_0, g_0 \rangle = \langle A^{-1}h_0, h_0 \rangle.$$

**REMARK.** The converse statement is obvious: If $A \geq 0$, $g_0 \in D_{\sqrt{A}}$ and

$$\|g_0\|^2 \leq C,$$
then
\[ |\langle \sqrt{A_0}, h \rangle|^2 \leq C \| \sqrt{A} h \|^2 \]
for arbitrary \( h \in D_{\sqrt{A}} \).

**PROPOSITION 2.** Let \( s \) be a contractive operator from \( \mathcal{L} \) into \( \mathcal{L}' \), then
\[ s(I_L - s^* s)^{[-\frac{1}{2}]} = (I_{L'} - s s^*)^{[-\frac{1}{2}]} s \]
and
\[ (I_L - s^* s)^{[-\frac{1}{2}]} s^* = s^*(I_{L'} - s s^*)^{[-\frac{1}{2}]} . \]

In particular, Proposition 2 contains the assertion that the domains of the operators on the right and left hand sides of these equalities coincide.

**PROPOSITION 3.** Let \( s \) be a contractive operator from \( \mathcal{L} \) into \( \mathcal{L}' \). Then the following statements are equivalent:

1. \( \ell' \oplus \ell \) belongs to the domain of the operator \( \begin{bmatrix} I_{L'} & s \\ s^* & I_L \end{bmatrix}^{[-\frac{1}{2}]} \)
2. \( \ell - s^* \ell' \) belongs to the domain of the operator \( (I_L - s^* s)^{[-\frac{1}{2}]} \)
3. \( \ell' - s \ell \) belongs to the domain of the operator \( (I_{L'} - s s^*)^{[-\frac{1}{2}]} \).

Moreover, if these properties are in force, then
\[
\langle \begin{bmatrix} I_{L'} & s \\ s^* & I_L \end{bmatrix}^{[-1]} (\ell' \oplus \ell), (\ell' \oplus \ell) \rangle_{L' \oplus L} = \langle \ell', \ell' \rangle_{L'} + \langle (I_{L'} - s^* s)^{[-1]}(\ell - s^* \ell'), (\ell - s^* \ell') \rangle_{L} = \langle \ell, \ell \rangle_{L} + \langle (I_{L'} - ss^*)^{[-1]}(\ell' - s \ell), (\ell' - s \ell) \rangle_{L'}. \]

The latter formulas imply the following:

**COROLLARY.** The inequalities
\[
\langle \begin{bmatrix} I_{L'} & s \\ s^* & I_L \end{bmatrix}^{[-1]} (\ell' \oplus \ell), (\ell' \oplus \ell) \rangle_{L' \oplus L} \geq \langle \ell, \ell \rangle_{L}
\]
and
\[
\langle \begin{bmatrix} I_{L'} & s \\ s^* & I_L \end{bmatrix}^{[-1]} (\ell' \oplus \ell), (\ell' \oplus \ell) \rangle_{L' \oplus L} \geq \langle \ell', \ell' \rangle_{L'}
\]
hold.

We are now ready to prove the equivalence of the FMI and the FMI'.
PROPOSITION 4. The FMI holds if and only if the FMI’ holds.

PROOF. In view of the block-matrix lemma, the FMI is equivalent to the following inequality

\[ D((I - T\bar{\zeta})x, (I - T\bar{\zeta})x) \]
\[ - \left\langle \left[ \frac{I_{L'} - s^*(\zeta)s(\zeta)}{1 - \zeta\bar{\zeta}} \right]^{[-1]} (E - s^*(\zeta)M)x, (E - s^*(\zeta)M)x \right\rangle \geq 0. \]

Using Proposition 3 we get

\[ D((I - T\bar{\zeta})x, (I - T\bar{\zeta})x) + (1 - \zeta\bar{\zeta})\{\langle Mx, Mx \rangle - \langle Ex, Ex \rangle\} \]
\[ - \left\langle \left[ \frac{I_{L} - s(\zeta)s^*(\zeta)}{1 - \zeta\bar{\zeta}} \right]^{[-1]} (s(\zeta)E - M)x, (s(\zeta)E - M)x \right\rangle \geq 0. \]

According to the Fundamental Identity

\[ \langle Mx, Mx \rangle - \langle Ex, Ex \rangle = D(Tx, Tx) - D(x, x). \]

Combining similar terms in the sesquilinear form \( D \) we obtain the inequality

\[ D((\zeta I - T)x, (\zeta I - T)x) - \left\langle \left[ \frac{I_{L'} - s(\zeta)s^*(\zeta)}{1 - \zeta\bar{\zeta}} \right]^{[-1]} (s(\zeta)E - M)x, (s(\zeta)E - M)x \right\rangle \geq 0, \]

which is equivalent to the FMI because of the block-matrix lemma. Hence Proposition 4 is proved.

Our aim now is to extract interpolation information from the FMI. We shall apply the method used earlier by one of the authors in [3].

We begin with an illustrative example: Let \( \zeta_0 \) be an eigenvalue of \( T \), let \( x_0 \) be the corresponding eigenvector, choose \( \zeta = \zeta_0 \) and \( x = x_0 \) in the FMI. Then

\[ D((\zeta_0 I - T)x_0, (\zeta_0 I - T)x_0) = 0, \]

and hence \( (s(\zeta_0)E - M)x_0 = 0 \), that is \( s(\zeta_0)Ex_0 = Mx_0 \). Thus, the value \( s(\zeta_0)Ex_0 \) is the same for any solution \( s(\zeta) \) of the FMI.

To formulate the theorem we need to define the Sz.-Nagy–Foias function space \( \mathbb{K}_s \) (see [4], [5]).

For \( s \in B(\mathbb{L}, \mathbb{L}') \) the space \( \mathbb{K}_s \) is the set of all vector-valued functions \( f = f_+ \oplus f_- \) which satisfy the following two conditions:

1) \( f_+(\zeta) \in H^2_+(\mathbb{L}') \) and \( f_-(\zeta) \in H^2_-(\mathbb{L}) \), where \( H^2_+(\mathbb{L}') \) and \( H^2_-(\mathbb{L}) \) are vector Hardy spaces of the disc \( |\zeta| \leq 1 \) with coefficients in the indicated space;

2) \( \int_T \left\langle \left[ \frac{I_{L'} s^*}{I_L} \right]^{[-1]} f, f \right\rangle dm < \infty \), where \( T \) is the unit circle and \( dm \) is normalized Lebesgue measure on it.
This integral defines an inner product in $K_s$. Sometimes it is convenient to use the following equivalent definition of the space $K_s$ as the set of all vector valued functions $f = f_+ \oplus f_-$ such that

1') $f_+(\zeta)$ is holomorphic in $|\zeta| < 1$, $f_-(\zeta)$ is anti-holomorphic in $|\zeta| < 1$, $f_-(0) = 0$,

2') $0 < r < 1 \sup_{|T_r|} \int \langle \left[ I_L \quad \frac{s}{s^*} \right]^{-1} f, f \rangle dm < \infty,$

where $T_r$ is the circle of radius $r$ centered at the origin and $dm$ is normalized Lebesgue measure on it.

Note that vector-functions from $K_s$ have the following properties:

\[
\langle \left[ I_L \quad \frac{s}{s^*} \right]^{-1} f, f \rangle (rt) \to \langle \left[ I_L \quad \frac{s}{s^*} \right]^{-1} f, f \rangle (t),
\]

for almost all $t$, $|t| = 1$ (as $r \to 1$) and

\[
\int_{T_r} \langle \left[ I_L \quad \frac{s}{s^*} \right]^{-1} f, f \rangle (dm) \to \int_{T} \langle \left[ I_L \quad \frac{s}{s^*} \right]^{-1} f, f \rangle (dm),
\]

(as $r \to 1$). The interpolation sense of the FMI can be seen from the following:

THEOREM 1. Assume that the intersection of the spectrum of $T$ with the set $|\zeta| \neq 1$ consists of isolated points only, that the vector-valued functions $E((\zeta I - T)^{-1} x)$ and $M((\zeta I - T)^{-1} x)$ are holomorphic for $|\zeta| \neq 1$ (where they are defined) and the function $D(x, \frac{I + T \zeta}{I - T \zeta} x)$ is holomorphic in the disc $|\zeta| < 1$ (for all those $\zeta$ for which it is defined).

Then every holomorphic operator-valued function $s(\zeta)$ which is a solution of the FMI has the following properties: $F_s x \in K_s$ and

\[
\langle F_s x, F_s x \rangle_{K_s} \leq D(x, x), \quad (\forall \ x \in X),
\]

where

\[
F_s x \overset{def}{=} (F_s x)_+ \oplus (F_s x)_- \in K_s,
\]

\[
(F_s x)_+(\zeta) \overset{def}{=} (s(\zeta) E - M)(\zeta I - T)^{-1} x, \ |\zeta| < 1,
\]

\[
(F_s x)_-(\zeta) \overset{def}{=} \overline{\zeta}(E - s^*(\zeta) M)(I - T \overline{\zeta})^{-1} x, \ |\zeta| < 1.
\]

PROOF. Since $x \in X$ and $\ell \in L$ are chosen arbitrarily the FMI is equivalent to the inequality

\[
D((I - T \overline{\zeta}) x, (I - T \overline{\zeta}) x) + 2Re\langle (E - s^*(\zeta) M)x, \ell \rangle
+ \langle \frac{I_L - s^*(\zeta) s(\zeta)}{1 - \zeta \overline{\zeta}} \ell, \ell \rangle \geq 0. \quad (1)
\]
We now replace the vector $\ell$ in (1) by the vector $(\ell - E(I - T\zeta)x)$ and then, after multiplying the resulting expression through by $1 - \zeta \bar{\zeta}$, combine separately the terms which are quadratic with respect to $x$, the terms which are linear with respect to $x$ and the terms which are independent of $x$.

The quadratic term has the form

$$C_2 = (1 - \zeta \bar{\zeta})D((I - T\zeta)x, (I - T\zeta)x) - (1 - \zeta \bar{\zeta})(E(I - T\zeta)x, (E - s^*(\zeta)M)x)$$

$$= (1 - \zeta \bar{\zeta})(E - s^*(\zeta)M)x, E(I - T\zeta)x) + \langle (I_L - s^*(\zeta)s(\zeta))E(I - T\zeta)x, E(I - T\zeta)x \rangle .$$

The linear term is

$$C_1 = (1 - \zeta \bar{\zeta})(E - s^*(\zeta)M)x, \ell) - \langle (I - s^*(\zeta)s(\zeta))E(I - T\zeta)x, \ell \rangle .$$

The constant term is

$$C_0 = \langle (I_L - s^*(\zeta)s(\zeta))\ell, \ell \rangle .$$

In terms of this notation, inequality (1) can be expressed in the form

$$C_2 + 2\text{Re}C_1 + C_0 \geq 0 .$$

Using the arbitraryness of $x$ and $\ell$, we shall rewrite (2) in the form

$$\begin{bmatrix} C_2 & C_1 \\ C_1 & C_0 \end{bmatrix} \geq 0 .$$

From the obvious identity

$$(1 - \zeta \bar{\zeta})I = (I - T\zeta) - \bar{\zeta}(\zeta I - T),$$

we have

$$(1 - \zeta \bar{\zeta})(E - s^*(\zeta)M)x$$

$$= (E - s^*(\zeta)M)(I - T\zeta)x - \bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x .$$

It follows from (5) that

$$= (I_L - s^*(\zeta)s(\zeta))E(I - T\zeta)x - \bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x$$

$$C_1 = \langle s^*(\zeta)(s(\zeta)E - M)(I - T\zeta)x - \bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x, \ell \rangle .$$
Next, we transform the second term in \( C_2 \) with the help of (5) and the sum of the third and fourth terms with the help of (6) to obtain:

\[
C_2 = (1 - \zeta \zeta)D((I - T \zeta) x, (I - T \zeta) x)
\]

\[
- \langle E(I - T \zeta) x, (E - s^*(\zeta) M)(I - T \zeta) x - \zeta(E - s^*(\zeta) M)(\zeta I - T) x \rangle
\]

\[
- \langle s^*(\zeta)(s(\zeta) E - M)(I - T \zeta) x - \zeta(E - s^*(\zeta) M)(\zeta I - T) x, E(I - T \zeta) x \rangle
\]

\[
= (1 - \zeta \zeta)D((I - T \zeta) x, (I - T \zeta) x)
\]

\[
+ 2\text{Re} \langle E(I - T \zeta) x, \zeta(E - s^*(\zeta) M)(\zeta I - T) x \rangle
\]

\[
- \langle E(I - T \zeta) x, (E - s^*(\zeta) M)(I - T \zeta) x \rangle
\]

\[
- \langle s^*(\zeta)(s(\zeta) E - M)(I - T \zeta) x, E(I - T \zeta) x \rangle.
\]

Since

\[
\langle s^*(\zeta)(s(\zeta) E - M)(I - T \zeta) x, E(I - T \zeta) x \rangle
\]

\[
= \langle (s(\zeta) E - M)(I - T \zeta) x, s(\zeta) E(I - T \zeta) x \rangle
\]

and

\[
\langle E(I - T \zeta) x, (E - s^*(\zeta) M)(I - T \zeta) x \rangle
\]

\[
= \langle E(I - T \zeta) x, E(I - T \zeta) x \rangle - \langle E(I - T \zeta) x, s^*(\zeta) M(I - T \zeta) x \rangle
\]

\[
= \langle E(I - T \zeta) x, E(I - T \zeta) x \rangle - \langle s(\zeta) E(I - T \zeta) x, M(I - T \zeta) x \rangle
\]

\[
= \langle E(I - T \zeta) x, E(I - T \zeta) x \rangle - \langle M(I - T \zeta) x, M(I - T \zeta) x \rangle
\]

\[
- \langle (s(\zeta) E - M)(I - T \zeta) x, M(I - T \zeta) x \rangle,
\]

the expression (8) for \( C_2 \) takes the form

\[
C_2 = (1 - \zeta \zeta)D((I - T \zeta) x, (I - T \zeta) x)
\]

\[
+ 2\text{Re} \langle E(I - T \zeta) x, \zeta(E - s^*(\zeta) M)(\zeta I - T) x \rangle
\]

\[
- \langle E(I - T \zeta) x, E(I - T \zeta) x \rangle + \langle M(I - T \zeta) x, M(I - T \zeta) x \rangle
\]

\[
- \langle s(\zeta) E - M)(I - T \zeta) x, (s(\zeta) E - M)(I - T \zeta) x \rangle.
\]

Using the Fundamental Identity and grouping similar terms in the quadratic form \( D \) we can reexpress (9) as

\[
C_2 = \text{Re} D((T + \zeta I)(I - T \zeta) x, (T - \zeta I)(I - T \zeta) x)
\]

\[
+ 2\text{Re} \langle E(I - T \zeta) x, \zeta(E - s^*(\zeta) M)(\zeta I - T) x \rangle
\]

\[
- \langle (s(\zeta) E - M)(I - T \zeta) x, (s(\zeta) E - M)(I - T \zeta) x \rangle.
\]
Next, upon taking into account expression (7) for $C_1$ and block-matrix lemma, one can transform inequality (3) to the following equivalent inequality:

$$C_2 \geq \| (I_L - s^*(\zeta)s(\zeta))^{[1/2]} [s^*(\zeta)(s(\zeta)E - M)(I - T\bar{\zeta})x - \bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x] \|^2.$$  

(11)

Then, upon substituting formula (10) for $C_2$ into (11) and using Proposition 3 we obtain

$$ReD((T + \zeta I)(I - T\bar{\zeta})x, (T - \zeta I)(I - T\bar{\zeta})x)$$

$$+ 2Re\langle E(I - T\bar{\zeta})x, \bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x \rangle$$

(12)

$$- \left[ \begin{array}{cc}
I_L' & s(\zeta) \\
s^*(\zeta) & I_L
\end{array} \right]^{-1} \left[ \begin{array}{c}
(s(\zeta)E - M)(I - T\bar{\zeta})x \\
\bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x
\end{array} \right] \cdot \left[ \begin{array}{c}
(s(\zeta)E - M)(I - T\bar{\zeta})x \\
\bar{\zeta}(E - s^*(\zeta)M)(\zeta I - T)x
\end{array} \right] \geq 0.$$  

The inequality (12) can be considered as the final form of the transformed FMI. We emphasize that all the transformations are based on identities and do not use any spectral properties of the operator $T$.

The left hand side of the inequality (12) admits a dual representation. It can be obtained from expression (12) by regrouping the entries in the first two terms and invoking the identity

$$\frac{1}{2} D((T + \zeta I)y_1, (I - T\bar{\zeta})y_2) - \zeta \langle Ey_1, (E - s^*(\zeta)M)y_2 \rangle$$

$$= \frac{1}{2} D((T - \zeta I)y_1, (I + T\bar{\zeta})y_2) + \zeta \langle (s(\zeta)E - M)y_1, M y_2 \rangle,$$

(13)

which follows directly from the FI for arbitrary $y_1$ and $y_2 \in X$. Inserting $y_1 = (I - T\bar{\zeta})x$ and $y_2 = (T - \zeta I)y$ into (13), we obtain

$$\frac{1}{2} D((T + \zeta I)(I - T\bar{\zeta})x, (T - \zeta I)(I - T\bar{\zeta})y)$$

$$+ \zeta \langle E(I - T\bar{\zeta})x, (E - s^*(\zeta)M)(\zeta I - T)y \rangle$$

$$= \frac{1}{2} D((I - T\bar{\zeta})(\zeta I - T)x, (I + T\bar{\zeta})(\zeta I - T)y)$$

$$- \zeta \langle (s(\zeta)E - M)(I - T\bar{\zeta})x, M(\zeta I - T)y \rangle.$$  

(14)

for arbitrary $x, y \in X$. Substituting (14) into (12), we obtain

$$ReD((I - T\bar{\zeta})(\zeta I - T)x, (I + T\bar{\zeta})(\zeta I - T)x)$$

$$- 2Re\zeta \langle (s(\zeta)E - M)(I - T\bar{\zeta})x, M(\zeta I - T)y \rangle.$$  

(12')
Finally, we shall use the spectral properties of operator $T$. Substituting the vector $(I - T\zeta)^{-1}(\zeta I - T)^{-1}x$ in place of $x$ in (12') we obtain

$$\text{Re}D(x, \frac{I + T\zeta}{I - T\zeta} x) - 2\text{Re}\zeta \langle (s(\zeta)E - M)(\zeta I - T)^{-1}x, M(I - T\zeta)^{-1}x \rangle - \| \begin{bmatrix} I_{\mathbb{L}'} & s(\zeta) \end{bmatrix}^{[-1/2]} \begin{bmatrix} (s(\zeta)E - M)(\zeta I - T)^{-1}x \\ \zeta(E - s^*(\zeta)M)(I - T\zeta)^{-1}x \end{bmatrix} \|^{2} \geq 0. \tag{15}$$

Let us now recall the notation:

$$(F_s x)_+(\zeta) \overset{\text{def}}{=} (s(\zeta)E - M)(\zeta I - T)^{-1}x,$$

$$(F_s x)_-(\zeta) \overset{\text{def}}{=} \zeta(E - s^*(\zeta)M)(I - T\zeta)^{-1}x,$$

$$F_s x \overset{\text{def}}{=} (F_s x)_+ \oplus (F_s x)_-,$$

and define

$$P_\zeta(x, y) = \frac{1}{2} D(x, \frac{I + T\zeta}{I - T\zeta} y) - \zeta \langle (F_s x)_+(\zeta), M(I - T\zeta)^{-1}y \rangle. \tag{16}$$

Then, in view of (14), we also have

$$P_\zeta(x, y) = \frac{1}{2} D(\frac{T + \zeta I}{T - \zeta I} x, y) + \langle E(\zeta I - T)^{-1}x, (F_s y)_-(\zeta) \rangle. \tag{16'}$$

In terms of these notations inequality (15) can be expressed in the following form:

$$P_\zeta(x, x) + P^*_\zeta(x, x) - \left\langle \begin{bmatrix} I_{\mathbb{L}'} & s(\zeta) \end{bmatrix}^{[-1]} (F_s x)(\zeta), (F_s x)(\zeta) \right\rangle \geq 0. \tag{17}$$

By assumption, the function $P_\zeta(x, x)$ is holomorphic everywhere in $|\zeta| < 1$ with the possible exception of a set of isolated points. It follows from (17) that the real part of the function $P_\zeta(x, x)$ is nonnegative. Hence, all the singularities of this function in the disk $|\zeta| < 1$ are removable. Furthermore, in view of (17) and the corollary to Proposition 3, the functions $(F_s x)_\pm(\zeta)$ possess a harmonic majorant:

$$P_\zeta(x, x) + P^*_\zeta(x, x) \geq \|(F_s x)_\pm(\zeta)\|^2.$$
This implies that these functions are in $H^2_r(\mathbb{L}')$ and $H^2_r(\mathbb{L})$ respectively. In particular, all their singularities are removable. Moreover, it can be seen directly from the definition that $(F_s x)_-(0) = 0$.

It follows from formula (16) and from the regularity of the functions $(F_s x)(\zeta)$ on $|\zeta| < 1$ that

$$P_0(x, y) = \frac{1}{2} D(x, y).$$

Integrating the inequality (17) over the circle $\mathbb{T}_r$ of radius $r$ centered at the origin with respect to normalized Lebesgue measure $dm(\zeta) = \frac{1}{2\pi} \frac{d\zeta}{\zeta}$ we get

$$\int_{\mathbb{T}_r} \left[ \begin{array}{c} I_{L'} \\ s(\zeta) \end{array} \right]^{[-1]} (F_s x)(\zeta) (F_s x)(\zeta) dm(\zeta) \leq \int_{\mathbb{T}_r} [P_0(x, x) + P_0(x, x)] dm(\zeta)$$

$$= P_0(x, x) + P_0(x, x) = D(x, x).$$

Thus $F_s x \in \mathbb{K}_s$ and $(F_s x, F_s x)_{\mathbb{K}_s} \leq D(x, x)$. The theorem is proved.

The following proposition shows how the action of the operator $T$ (which was introduced in Theorem 1) is changed by the transformation $F_s$ which acts from the space $X$ into the space $\mathbb{K}_s$.

**PROPOSITION 5.** Let $T$ and $F_s$ be the same as in Theorem 1, then

$$(F_s T x)(t) \overset{a.e.}{=} t(F_s x)(t) - \left[ \begin{array}{c} I_{L'} \\ s(t) \end{array} \right] \left[ \begin{array}{c} -M x \\ E x \end{array} \right], \quad (|t| = 1) \quad (18)$$

**PROOF.** This follows from the definition of $F_s$ by a straightforward calculation.

**REMARK.** If the spectral condition for $T$ which was formulated in Theorem 1 is satisfied, then the transformation $F_s$ from $X$ in $\mathbb{K}_s$ is defined uniquely by the relation (18).

The following proposition is a converse of Theorem 1.

**PROPOSITION 6.** Let $\mathbb{L}$, $\mathbb{L}'$, $X$, $T$, $D$, $E$ and $M$ be the objects occurring(*) in Theorem 1, let $s \in B(\mathbb{L}, \mathbb{L}')$ and let $(F_s x)(t)$ be a family of functions in the variable $t \in \mathbb{T}$ which depends linearly on $x \in X$ and is defined by the following (generically implicit) formula:

i) $$(F_s T x)(t) \overset{a.e.}{=} t(F_s x)(t) - \left[ \begin{array}{c} I_{L'} \\ s(t) \end{array} \right] \left[ \begin{array}{c} -M x \\ E x \end{array} \right], \quad (|t| = 1).$$

(*) We do not impose any spectral conditions on $T$ here.
Assume further that

\[ F_s x \in \mathbb{K}_s, \quad \forall \ x \in X \]

and

\[ \langle F_s x, F_s x \rangle_{\mathbb{K}_s} \leq D(x, x), \quad \forall \ x \in X. \]

Then \( s(\zeta) \) is a solution of the FMI.

**PROOF.** Fix a point \( \zeta \) with \(|\zeta| < 1\) and consider the pair of vectors

\[ (F_s x)(t) \quad \text{and} \quad \left[ \begin{array}{cc} \frac{I_{L'}}{s^*(t)} & \frac{I_{L'}}{I_{L}} \frac{t'}{1 - t\zeta} \end{array} \right] ; \quad \ell' \in \mathbb{L}' , \]

both of which belong to \( \mathbb{K}_s \). Then, upon calculating all pairwise scalar products (in \( \mathbb{K}_s \)) formed from them and writing out the nonnegativity condition for the Gram matrix we obtain

\[
\begin{bmatrix}
\langle F_s x, F_s x \rangle_{\mathbb{K}_s} & \langle (F_s x)_+(\zeta), \ell' \rangle \\
\langle (F_s x)_+(\zeta), \ell' \rangle & \langle \frac{I_{L'} - s(\zeta)s^*(\zeta)}{1 - s(\zeta)} \ell', \ell' \rangle
\end{bmatrix} \geq 0 . \tag{19}
\]

Substituting the vector \( (\zeta I - T)x \) in place of the vector \( x \) in (19) and invoking the analytic continuation of \( F_s x \) (which is defined on the boundary in i)) and the linearity of \( F_s x \) in \( x \), we have

\[ (F_s(\zeta I - T)x)_+(\zeta) = (s(\zeta)E - M)x , \tag{20} \]

and, in view of iii),

\[ \langle F_s(\zeta I - T)x, F_s(\zeta I - T)x \rangle_{\mathbb{K}_s} \leq D((\zeta I - T)x, (\zeta I - T)x) . \tag{21} \]

Inserting (20) and (21) into (19) we obtain the FMI'.

The FMI can be obtained analogously by considering the Gram matrix of the pair of vectors

\[ (F_s x)(t) \quad \text{and} \quad \zeta \left[ \begin{array}{cc} I_{L'} & s(t) \\
- s(t)^* & I_{L} \end{array} \right] \frac{t}{t - \zeta} , \quad \ell \in \mathbb{L} . \]

**3. THE ABSTRACT INTERPOLATION PROBLEM**

Let \( \mathbb{X} \) be a linear space, \( T \) a linear operator on \( \mathbb{X} \), \( D \) a nonnegative sesquilinear form in \( \mathbb{X} \) and let \( E \) and \( M \) be linear operators from \( \mathbb{X} \) into the Hilbert spaces \( \mathbb{L} \) and \( \mathbb{L}' \), respectively. Suppose, moreover, that the identity

\[ D(x, x) - D(Tx, Tx) = \langle Ex, Ex \rangle_{\mathbb{L}} - \langle Mx, Mx \rangle_{\mathbb{L}'} \]
is satisfied.

Let \((F_s x)(t)\) be a family of functions in variable \(t \in \mathbb{T}\) which depends linearly on \(x \in \mathbb{X}\) and is defined by the following (generically implicit) formula:

\[
\text{i) } (F_s T x)(t) = t(F_s x)(t) - \begin{bmatrix} I_{\mathbb{L}'} & s(t) \\ s^*(t) & I_{\mathbb{L}} \end{bmatrix} \begin{bmatrix} -M x \\ E x \end{bmatrix}, \quad (|t| = 1).
\]

The operator–valued function \(s(\zeta) \in B(\mathbb{L}, \mathbb{L}')\) which is holomorphic in the disc \(|\zeta| < 1\) is said to be a solution of the Abstract Interpolation Problem if

\[
\text{ii) } F_s x \in K_s, \quad \forall x \in \mathbb{X}
\]

and

\[
\text{iii) } (F_s x, F_s x)_{K_s} \leq D(x, x), \quad \forall x \in \mathbb{X}.
\]

Our objective is to describe all the solutions of the Abstract Interpolation Problem. *

The remaining discussion depends on the paper [6]. To study the problem in question, we need some objects connected with the given spaces and operators. Let \(Dx\) denotes the conjugate linear functional defined by the formula

\[
Dx(y) \overset{\text{def}}{=} D(x, y).
\]

The scalar product is naturally defined on the set \(\{D x\}_{x \in \mathbb{X}}\) by the rule

\[
\langle D x_1, D x_2 \rangle \overset{\text{def}}{=} D(x_1, x_2).
\]

Obviously this inner product is well-defined. Let us denote by \(K\) the completion of the space \(\{D x\}_{x \in \mathbb{X}}\) with respect to the inner product introduced above. Then, \(K\) is a Hilbert space. The Fundamental Identity enables us to define an isometric operator from the space \(K \oplus \mathbb{L}\) into the space \(K \oplus \mathbb{L}'\). Let us define an operator \(V\) by the formula

\[
V(DT x \oplus E x) \overset{\text{def}}{=} Dx \oplus M x.
\]

The domain \((D_V)\) of the operator \(V\) is the closure in \(K \oplus \mathbb{L}\) of the set of all the vectors \(DT x \oplus E x\), the range \(\Delta_V\) is the closure in \(K \oplus \mathbb{L}'\) of the set of all the vectors \(D x \oplus M x\).

* For some choices of data there exists a unique linear mapping \(F_s\) from the space \(\mathbb{X}\) into the space \(K_s\) with properties i)–iii) for any solution \(s(\zeta)\) of the Abstract Interpolation Problem, for some other data there might be many mappings for the same solution \(s(\zeta)\). In any case all these mappings \(F_s\) can be described along with the description of the solutions (see [7], [12]).
Let $\mathbb{H}$ be a Hilbert space and let $U$ be a unitary operator from $\mathbb{H} \oplus \mathbb{L}$ onto $\mathbb{H} \oplus \mathbb{L}'$. Following the paper [6] we define the scattering function $s(\zeta)$ of $U$ with respect to the spaces $\mathbb{L}$ and $\mathbb{L}'$ in the following way:

$$s(\zeta) = P_{\mathbb{L}'}U(I_{\mathbb{H} \oplus \mathbb{L}} - \zeta P_{\mathbb{H}}U)^{-1}l.$$ 

Consider also the functional representation of the space $\mathbb{H}$ which is defined by formula

$$Gh = (Gh)_+ \oplus (Gh)_-,$$

where

$$(Gh)_+(\zeta) = P_{\mathbb{L}'}U(I_{\mathbb{H} \oplus \mathbb{L}} - \zeta P_{\mathbb{H}}U)^{-1}h$$

and

$$(Gh)_-(\zeta) = \overline{\zeta}P_{\mathbb{L}}U^*(I_{\mathbb{H} \oplus \mathbb{L}'} - \overline{\zeta}P_{\mathbb{H}}U^*)^{-1}h.$$ \hspace{1cm} (\|\zeta\| < 1)

**PROPOSITION 7.** $G$ maps $\mathbb{H}$ into $\mathbb{K}_s$, and

$$\|Gh\|_{\mathbb{K}_s} \leq \|h\|_{\mathbb{H}}.$$ 

The following statement yields a connection between all the solutions of the abstract interpolation problem and all the scattering functions of the unitary extensions of the isometry $V$ (see (22)) with respect to the spaces $\mathbb{L}$ and $\mathbb{L}'$.

**PROPOSITION 8.** Let $\mathbb{H} \supset \mathbb{K}$, let $U$ be unitary operator from $\mathbb{H} \oplus \mathbb{L}$ onto $\mathbb{H} \oplus \mathbb{L}'$ which extends $V$ and let $s(\zeta)$ be the scattering function of $U$ with respect to the spaces $\mathbb{L}$ and $\mathbb{L}'$. Then the functional transformation $F_s$

$$F_s x \overset{\text{def}}{=} GDx \quad (x \in X)$$

has property (18), i.e.,

$$F_s T x = tF_s x - \begin{bmatrix} I_{\mathbb{L}'} & s \\ s^* & I_{\mathbb{L}} \end{bmatrix} \begin{bmatrix} -Mx \\ Ex \end{bmatrix}.$$ 

**COROLLARY.** The scattering function of any unitary extension of an isometry $V$ is a solution of the abstract interpolation problem.

**PROPOSITION 9.** Let $s(\zeta) \in B(\mathbb{L}, \mathbb{L}')$, and let $F_s$ be a mapping from $\mathbb{X}$ into $\mathbb{K}_s$ which satisfies the conditions i)–iii) of the abstract interpolation problem. Then $s(\zeta)$ is the scattering matrix of a unitary extension of the isometry $V$ with respect to the spaces $\mathbb{L}$ and $\mathbb{L}'$. 
COROLLARY. The set of all the solutions of the abstract interpolation problem admits the following description

\[ s(\zeta) = s_{12}(\zeta) + s_{11}(\zeta)\varepsilon(\zeta)[I - s_{21}(\zeta)\varepsilon(\zeta)]^{-1}s_{22}(\zeta), \ |\zeta| < 1, \]

where

\[ S(\zeta) = \begin{bmatrix} s_{11}(\zeta) & s_{12}(\zeta) \\ s_{21}(\zeta) & s_{22}(\zeta) \end{bmatrix} \]

is the scattering matrix of the isometry \( V \), \([6]\) and \( \varepsilon(\zeta) \) is an arbitrary holomorphic contractive operator-valued function which acts from \( \mathbb{M}_V = (\mathbb{K} \oplus \mathbb{L}) \ominus D_V \) into \( \mathbb{N}_V = (\mathbb{K} \oplus \mathbb{L}') \ominus \Delta_V \).

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