ON THE MODULI SPACE OF CERTAIN
SMOOTH CODIMENSION-ONE FOLIATIONS
OF THE 5-SPHERE BY COMPLEX SURFACES

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dedicated to Santiago López de Medrano on his 65\textsuperscript{th} birthday

Abstract. In this paper we first determine the set of all possible integrable almost
CR-structures on the smooth foliation of $S^5$ constructed in [M-V]. We give a specific
concrete model of each of these structures. We show that this set can be naturally
identified with $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$. We then adapt the classical notions of coarse and fine
moduli space to the case of a foliation by complex manifolds. We prove that the
previous set, identified with $\mathbb{C}^3$, defines a coarse moduli space for the foliation of
[M-V], but that it does not have a fine moduli space. Finally, using the same ideas
we prove that the standard Lawson foliation on the 5-sphere can be endowed with
almost CR-structures but none of these is integrable. This is a foliated analogue to
the examples of almost complex manifolds without complex structure.

0. Introduction

In [M-V] the authors have constructed a smooth foliation of the sphere $S^5$ by com-
plex surfaces i.e., an integrable and Levi-flat codimension-one almost CR-structure.
The underlying smooth foliation is a variation of that given by Blaine Lawson in
[La], however ours is topologically different. In particular, it contains exactly two
compact leaves.

The notion of foliation of a smooth manifold by complex manifolds can be seen
as a generalization of the notion of complex structure on a smooth manifold, which
appears as the case of codimension zero. The case of codimension-one is of par-
ticular interest: the smooth manifold has then to be of odd-dimension, so that it
is really the analogue in the odd-dimensional case of the existence of a complex
structure.

Keeping this in mind, it is clearly interesting to ask, in the spirit of Kodaira-
Spencer and Kuranishi, for a deformation theory of such foliations. This is what
we intend to do in this paper for the example constructed in [M-V]. To be more
precise, the purpose of this paper is threefold:

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(1) To determine the set of all possible integrable almost CR-structures on this foliation. This set turns out to be completely determined by the set of complex structures of the compact leaves. These leaves are primary Kodaira surfaces fibered over the same elliptic curve and we infer from this description that our set can be identified to $\mathbb{C} \times \mathbb{C} \times \mathbb{C}$. In particular, it is finite-dimensional. We also show that some of these structures admit non-trivial CR-automorphisms.

(2) To adapt the classical notions of coarse and fine moduli space to the case of a smooth foliation by complex manifolds. We prove that the previous set can be seen as a coarse moduli space for the foliation of [M-V], but that, due to the existence of non-trivial CR-automorphisms, this foliation does not have a fine moduli space.

(3) To prove that the standard Lawson foliation cannot be endowed with a CR-structure, but only with non-integrable almost CR-structures.

We emphasize that the set we consider is the set of integrable CR-structures on a fixed smooth foliation, modulo foliated CR-isomorphisms (see the precise Definition given in Section 2). In the same way, the notions of coarse and fine moduli space are defined on a fixed smooth foliation. This induces some subtleties. In particular, strictly speaking, there is not a unique standard Lawson foliation, but infinitely many. All are topologically isomorphic, but have non-conjugated contracting holonomies. The same is true for “the” foliation of [M-V]. This will be made precise in Section 1.

The first result shows that the CR-structure of our foliation is very rigid. In fact it is easy to construct examples of compact manifolds carrying a Levi-flat and integrable almost CR-structure whose set of integrable almost CR-structures is infinite-dimensional (see Section 2). On the other hand, the third result gives an example of a foliation which admits non-integrable almost CR-structures and all of whose leaves admit separately a complex structure but which cannot be endowed with an integrable almost CR-structure. This is the foliated analogue to the examples of almost complex surfaces without complex structure (see [B-H-P-V, IV.9]). Moreover it proves that the search for the existence of integrable and Levi-flat codimension-one almost CR-structures on a compact manifold cannot be reduced to local analytic questions such as solving a $\bar{\partial}$-problem along the leaves.

**Remark.** We would like to emphasize that, as far as we know, the foliation described in [M-V] (as well as the related examples of [M-V], Section 5) is the only known example of a smooth foliation by complex manifolds of complex dimension strictly greater than one on a compact manifold, which is not obtained by classical methods such as the one given by the orbits of a locally trivial smooth action of a complex Lie group, the natural product foliation on $M \times N$ where $M$ is foliated by Riemann surfaces and $N$ is a complex manifold, holomorphic fibrations, or trivial modifications of these examples such as cartesian products of known examples or pull-backs. Of course, it is very easy to give examples of foliations by complex manifolds on open manifolds (in fact even with Stein leaves). On the other hand, if a compact smooth manifold has an orientable smooth foliation by surfaces then, using a Riemannian metric and the existence of isothermal coordinates, we see that the foliation can be considered as a foliation by Riemann surfaces.
Although this paper obviously depends on the example given in [M-V] it can be read independently of that paper since it uses different methods and ideas. In this sense this paper is not just an addendum of [M-V].

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1. Preliminaries

In this Section, we fix some notations and recall some facts and constructions around foliations by complex manifolds. Most of this material comes from [M-V].

In this article, smooth means $C^\infty$-differentiable. We make use of the following notations.

(i) we denote by $\mathbb{D}$ the unit open disk of $\mathbb{C}$ and by $\overline{\mathbb{D}}$ its closure.
(ii) if $X$ is a complex object (e.g. complex manifold, foliation with complex leaves, etc), then $X^{\text{diff}}$ denotes the underlying compatible smooth object.
(iii) for $T_1,\ldots,T_n$ a set of automorphisms of a complex (respectively smooth) manifold $X$, we denote by $\langle T_1,\ldots,T_n \rangle$ the group generated by $T_1,\ldots,T_n$, and by $X/\langle T_1,\ldots,T_n \rangle$ the quotient space of $X$ by this group. We use this last notation only in the case where this quotient space is a complex (respectively smooth) manifold, i.e. when the action is free and totally discontinuous.
(iv) for a smooth manifold $M$, we denote by $\partial M$ the boundary of $M$ and by $\text{Int} M$ the interior part of $M$, namely the open manifold $M \setminus \partial M$.

We recall that, if $p : \tilde{M} \to M$ is a covering map of a smooth manifold $\tilde{M}$ with boundary onto a smooth manifold $M$ with boundary, then $p$ restricted to each component of $\partial \tilde{M}$ is a covering map onto its image, which is a component of $\partial M$.

1.1. Foliations by complex manifolds.

Recall that an almost CR-structure on a smooth manifold $V$ is the data of a subbundle $E$ of the tangent bundle $TV$ together with an operator $J : E \to E$ acting linearly on the fibers of $E$ and satisfying $J^2 = -\text{Id}$ on every fiber. We denote an almost CR manifold by $(V,J)$. In this paper every almost CR-structure is assumed to be smooth. Recall also that a CR-map between $(V,J)$ and $(V',J')$ is a smooth map $f$ from $V$ to $V'$ whose differential commutes with the almost CR-structures, that is $df \circ J \equiv J' \circ df$.

Remark. In our paper [M-V], an almost CR-structure is just called a CR-structure. We would like to thank Claude LeBrun, who pointed out to us that the terminology of almost CR-structure is more appropriate (since it is parallel to the notions of almost complex and complex structures).

We refer to our paper [M-V] and to [Tu] for the notions of integrability and Levi-flatness of an almost CR-structure. We just recall here the two definitions of [M-V] which are essential for this article.
**Definition.** Let $M$ be a smooth manifold of odd dimension and without boundary. A foliation by complex manifolds on $M$ is the data of a smooth codimension-one foliation $F$ of $M$ which is endowed with an integrable almost CR-structure whose corresponding distribution is the distribution tangent to the leaves of $F$.

**Remark.** An integrable almost CR-structure is also called a CR-structure.

**Remark.** Such an almost CR-structure is automatically Levi-flat by the Frobenius Theorem.

Equivalently, a foliation by complex manifolds $F$ of codimension-one on a smooth manifold $M$ of dimension $2n + 1$, can be defined by a foliated atlas

\[ \mathcal{A} = \{(U_i, \phi_i)_{i \in I} \mid \phi_i(U_i) \subset \mathbb{C}^n \times \mathbb{R} \cong \mathbb{R}^{2n+1}\} \]

such that the changes of charts

\[ (z, t) \in \phi_i(U_i \cap U_j) \mapsto \phi_j \circ \phi_i^{-1}(z, t) := (\xi_{ij}(z, t), \zeta_{ij}(t)) \in \phi_j(U_i \cap U_j) \]

are holomorphic in the tangential direction, i.e. the map $\xi_{ij}$ is holomorphic for fixed $t$.

Let $M$ be a smooth manifold with boundary. Let $N$ be the open manifold obtained by adding the collar $\partial M \times [0, 1)$ to $M$ equipped with the unique differentiable structure such that the natural inclusions of $\partial M \times [0, 1)$ and $M$ into $N$ are smooth embeddings (see [Hi, Chapter 8, Theorem 2.1]).

**Definition.** A (codimension-one) tame almost CR-structure on $M$ is the data of an almost CR-structure on the interior of $M$ and of an almost complex structure on the boundary $\partial M$ such that the following gluing condition is verified. Let $M \subset N$ and $\partial M \times [0, 1) \subset N$ be the natural embeddings. Then, the almost CR-structure on $M$ extends to a almost CR-structure on $N$ by considering on the collar the distribution tangent to the submanifolds $\partial M \times \{t\}$, $0 \leq t < 1$, and equipping this distribution with the natural almost complex structure inherited from $\partial M$.

A foliation by complex manifolds on $M$ is the data of a smooth foliation $F$ of $M$ of codimension-one which is endowed with an integrable tame almost CR-structure.

The same remark after the first definition is valid in this case.

**1.2. Lawson Foliation.** Let us now recall Lawson’s construction of a smooth codimension-one foliation of $S^5$. Let

\[ P : (z_1, z_2, z_3) \in \mathbb{C}^3 \mapsto z_1^3 + z_2^3 + z_3^3 \in \mathbb{C} \]

and let $V = P^{-1}(0)$. The manifold

\[ W := V \setminus \{(0, 0, 0)\} = \{(z_1, z_2, z_3) \neq 0 \mid z_1^3 + z_2^3 + z_3^3 = 0\} \]

intersects transversally the Euclidean unit sphere in the smooth compact manifold $K$. Moreover, it projects onto the projective space $\mathbb{P}^2$ as an elliptic curve $\mathbb{E}_\omega$ of modulus $\omega$. This curve admits an automorphism of order three, hence $\omega^3 = 1$. The canonical projection

\[ p : W \to \mathbb{E}_\omega \]
describes $W$ as a holomorphic principal $\mathbb{C}^*$-bundle over the elliptic curve $E_\omega$, with first Chern class equal to $-3$. By passing to the unit bundle, one has that $K$ is a principal circle-bundle over a torus with Euler class equal to $-3$ (see [Mi2, Lemma 7.1 and Lemma 7.2]).

Another way of seeing $K$ is the following. Let $A = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ then $K$ is the suspension of the unipotent isomorphism induced on the two-dimensional torus by the matrix $A$. Notice that, in this way, we define $K$ as a torus bundle over $S^1$, but not as a principal torus bundle. This subtlety will be important in the sequel.

The Milnor Fibration Theorem [Mi1, Theorem 4.8] shows that $S^5 \setminus K$ fibers over $S^1$ and describes $S^5$ as an open book in the sense of Winkelnkemper [Wi], with fibers diffeomorphic to $P^{-1}(z)$, $z \in \mathbb{C}^*$. On the other hand, $K$ is smoothly embedded in $S^5$ with trivial normal bundle and therefore has a closed tubular neighborhood diffeomorphic to $K \times \mathbb{B}$. Thus it follows that both a closed tubular neighborhood $\mathcal{N}$ of $K$ in $S^5$ and $\mathcal{E}$, the closure of its complement, fiber over the circle by fibrations which are also fibrations when restricted to the boundary. Hence they are smoothly foliated by using a standard Tourbillonnement Lemma:

**Lemma [La].** Let $M$ be a compact manifold with boundary $\partial M$, and suppose there exists a $C^\infty$-submersion $\psi : M \to S^1$ such that $\psi|_{\partial M}$ is a submersion of the boundary. Then there exists a codimension-one $C^\infty$-foliation of $M$.

Gluing carefully those pieces together by a diffeomorphism, Lawson obtained a smooth codimension-one foliation of $S^5$ with a unique compact leaf diffeomorphic to $K \times S^1$.

**Remark.** Notice that there is not a unique Lawson foliation, but infinitely many. Indeed, there are infinitely many ways of turbulizing, which give rise to non-conjugated holonomies of the compact leaf. Therefore, the standard Lawson foliation is unique topologically but certainly not differentiably. In the sequel, we will however still talk of the standard Lawson foliation. This is due to the fact that the result we prove (Theorem A) is independent of the choice of the turbulization.

### 1.3. A foliation of $S^5$ by complex surfaces.

Let us start with the following Lemma, whose proof is immediate. A CR-submersion is a smooth submersion between almost CR-manifolds which is also a CR-map.

**Lemma.** Let $(M, \mathcal{F})$ be a Levi-flat CR-structure, with or without boundary. Let $p : \hat{M} \to M$ be a smooth covering. Let $\hat{\mathcal{F}}$ be the foliation on $\hat{M}$ whose leaves are the connected components of the pull-back by $p$ of the leaves of $\mathcal{F}$. Then, there exists a unique smooth, integrable almost CR-structure on $(\hat{M}, \hat{\mathcal{F}})$ such that $p$ is a CR-submersion.

We call $\hat{\mathcal{F}}$ the induced pull-back foliation.

**Definition.** Let $(M, \mathcal{F})$ be a Levi-flat CR-structure. A covering map $\pi : \hat{M} \to M$ is called a product foliated covering if one of the two following statements is satisfied:
(i) if $\partial M$ is empty, then $\tilde{M}$ is diffeomorphic to $N \times \mathbb{R}$ (for $N$ a smooth manifold) and the induced pull-back foliation $\tilde{\mathcal{F}}$ has leaves diffeomorphic to $N \times \{\text{point}\}$.

(ii) if $\partial M$ is non-empty, then $\tilde{M}$ is diffeomorphic to $(N \times \mathbb{R}^+) \setminus (A \times \{0\})$ (for $N$ a smooth manifold and $A$ - possibly empty - analytic subset of $N$) and the foliation has leaves diffeomorphic to $N \times \{\text{point}\}$ in the interior and diffeomorphic to $(N \setminus A) \times \{0\}$ on the boundary.

Remark. In general, the complex structure on the leaves $N \times \{\text{point}\}$ depends on $t$ so a product foliated covering is not CR-isomorphic to a product.

Remark. An example of (ii) in the definition above is the solid cylinder $\mathbb{T} \times \mathbb{R} \cong (\mathbb{C} \times \mathbb{R}^+) \setminus \{(0,0)\}$ corresponding to the infinite cyclic covering of the solid torus $\mathbb{T} \times S^1$ foliated by the Reeb foliation, see [M-V, Lemma 2].

The detailed construction of the foliation of $S^5$ by complex surfaces can be found in [M-V]. We denote this foliation by $\mathcal{F}_C$. The idea is to endow suitable coverings of $\mathcal{N}$ and of the closure of its open complement in $S^5$ with trivial foliations by complex surfaces such that the covering transformations are CR-isomorphisms, i.e. to construct product foliated coverings as in the definition above (here we use the notation of 1.2). Then, taking the quotient, we obtain foliations by complex surfaces of $\mathcal{N}$ and of the closure of its open complement in $S^5$. Due to the tame condition, these two foliations can be glued together [M-V, Lemma 1].

We content ourselves with describing the two coverings and referring to [M-V] for more details.

Let $\lambda$ be a real such that $0 < \lambda < 1$.

Remark. This is not the same convention as in [M-V] where $\lambda$ is supposed to be strictly greater than one.

The following function

$$z \in \mathbb{C}^3 \mapsto \lambda \omega \cdot z \in \mathbb{C}^3$$

leaves $W$ invariant. The group generated by this transformation acts properly and discontinuously on $W$ and the quotient is a compact complex manifold diffeomorphic to $K \times S^1 \cong \partial \mathcal{N}$. Let us call it $S_\lambda$. We remark that it is a primary Kodaira surface.

Let

$$\tilde{X} = \mathbb{C}^* \times (\mathbb{C} \times [0, \infty) \setminus \{(0,0)\})$$

and let $\Gamma$ be the group generated by the commuting diffeomorphisms $T$ and $S$ defined as follows:

$$\forall (z, u, t) \in \tilde{X}, \quad T(z, u, t) = (z, \lambda \omega \cdot u, d(t))$$

and

$$S(z, u, t) = (\exp(2i\pi \omega) \cdot z, (\psi(z))^{-3} \cdot u, t)$$

where $d$ is a smooth diffeomorphism equal to $t$ when $t \leq 0$ and satisfying $d'(t) < 1$ when $t > 0$ and where $\psi$ is the automorphic factor of $W$ as $\mathbb{C}^*$-bundle over $E_\omega$.

Remark. This is not the same convention as in [M-V]. Indeed, it is linked to the previous remark, for taking $\lambda > 1$ (respectively $0 < \lambda < 1$) implies taking $d'(t) > 1$ for $t > 0$ (respectively $d'(t) < 1$), otherwise the previous action is not proper.
Let \( \mathcal{F}_t \) be the foliation whose leaves \( L_t \) are the level sets in \( \bar{X} \) of the projection on the third factor. These leaves are naturally complex manifolds biholomorphic to \( \mathbb{C}^* \times \mathbb{C} \) for \( t > 0 \) and \( \mathbb{C}^* \times \mathbb{C}^* \) for \( t = 0 \). The group \( \Gamma \) preserves the foliation \( \mathcal{F} \) and sends one leaf biholomorphically onto its image. Then it is proved in [M-V] that the quotient of \( \bar{X} \) by \( \Gamma \) is diffeomorphic to \( \mathcal{N} \) and thus provides a foliation by complex manifolds on this closed set. The boundary leaf is biholomorphic to \( S_\lambda \).

The other leaves are all biholomorphic to the line bundle over \( \mathbb{E}_\omega \) obtained from \( W \) by adding a zero section.

On the other hand, let \( g : \mathbb{C}^3 \times [-1, \infty) \to \mathbb{C} \) be the function defined by

\[
g((z_1, z_2, z_3), t) = z_1^3 + z_2^3 + z_3^3 - \phi(t)
\]

where \( \phi \) is a smooth function which is zero exactly on the non-positive real numbers.

Let \( \hat{g} = g^{-1}(\{0\}) \) and \( \Xi = \hat{g} \setminus (\{(0, 0, 0)\} \times [-1, \infty)) \). Then, \( \Xi \) is a smooth manifold and has a natural smooth foliation \( \mathcal{F}_e \) by complex manifolds whose leaves \( \{L_t\}_{t \in \mathbb{R}} \) are parametrized by projection onto the factor \([-1, \infty)\).

Let \( G : \Xi \to \Xi \) be the diffeomorphism given by

\[
G((z_1, z_2, z_3), t) = ((\lambda \omega \cdot z_1, \lambda \omega \cdot z_2, \lambda \omega \cdot z_3), h_\lambda(t))
\]

where \( h_\lambda \) is a smooth diffeomorphism whose fixed points are 0 and \((-\infty, -1]\). For good choices of \( \phi \) and \( h_\lambda \) which are specified in [M-V], the pair \( (\Xi, \mathcal{F}_e) \) is a covering of the closure of \( S^\omega \setminus \mathcal{N} \) with deck transformation group \( \Gamma \), generated by \( G \). The boundary is a leaf biholomorphic to \( S_\lambda \) and the gluing condition is verified. There is another compact leaf corresponding to \( \mathcal{F}_e \). For \( t = 0 \), it is also biholomorphic to \( S_\lambda \). These two compact leaves form the boundary of a collar whose interior leaves are all biholomorphic to \( W \). Finally, the other leaves are all biholomorphic to the affine cubic surface \( P^{-1}(1) \) of \( \mathbb{C}^3 \).

**Remark.** Observe that the second covering is not a product foliated covering since it has two topologically distinct leaves: \( W \) and \( P^{-1}(\{t\}) \). But it is a union of product foliated coverings. Indeed, the restriction of \( \Xi \) to \([-1, 0) \) is a product foliated covering, as well as its restriction to \((-1, 0)\) and to \([0, \infty)\).

**Remark.** The construction recalled above depends on the choices of the smooth functions \( d, \phi \) and \( h_\lambda \). Notice that \( d \) and \( h_\lambda \) define the holonomy of the compact leaves. As a consequence, if we construct such a foliation \( \mathcal{F} \) from \( d \) and \( h_\lambda \) and another one, say \( \mathcal{F}' \), from \( d' \) and \( h'_\lambda \) with the property that \( d' \) (or respectively \( h'_\lambda \)) is not smoothly conjugated to \( d \) (or respectively to \( h_\lambda \)), then \( \mathcal{F} \) and \( \mathcal{F}' \) are not smoothly isomorphic, although they are topologically isomorphic (such maps exist, see [Se]).

Nevertheless, the smooth type of the foliation is independent of the choice of the parameter \( \lambda \) in the following sense. Fix some \( \lambda \) and some smooth functions \( d \) and \( h_\lambda \) and call \( \mathcal{F}_\lambda \) the resulting foliation. Choose now \( 0 < \mu < 1 \) different from \( \lambda \). The function \( h_\lambda \) has the property that it coincides with the parabolic Möbius transformation \( t/(1 - 3(\log \lambda)t) \) near 0 (see [M-V, p. 925]). There exists a smooth function \( f : \mathbb{R} \to \mathbb{R} \) fixing 0 with the property that \( f \circ h_\lambda \circ f^{-1} \) coincides with the parabolic Möbius transformation \( t/(1 - 3(\log \mu)t) \). It is easy to check that this new diffeomorphism can be used as \( h_\mu \). As \( h_\mu \) is globally conjugated to \( h_\lambda \), the foliation \( \mathcal{F}_\mu \) obtained from the previous construction using the functions \( d \) and \( h_\mu \)
is smoothly isomorphic to $\mathcal{F}_\lambda$. However, they are not isomorphic as CR-structures, since different choices of $\lambda$ yield different complex structures on the compact leaves (see Section 5).

In the sequel, we still talk of the foliation of [M-V], since the results we prove (Theorems B, C, D) are independent of the particular choices of the functions $d$ and $h_\lambda$. It will be important however to keep in mind the independence of the foliation with respect to $\lambda$, as was indicated above.

2. Families of complex structures and locally trivial CR-fiber bundles

Let $(V,J)$ be a smooth almost CR-manifold and let $X$ be a smooth manifold. The following Definition is a reformulation in the language of almost CR-structures of Definition 1.1 of Kodaira-Spencer [K-S].

Definition. A smooth map $\pi : V \to X$ is a smooth family of complex structures or a smooth deformation family if

(i) It is a smooth submersion with compact fibers.
(ii) The almost CR-structure $J$ is integrable.
(iii) The almost CR-structure $J$ is Levi-flat and the associated smooth foliation is given by the level sets of $\pi$.

Remark. The manifolds $V$ and $X$ can have boundary (for example, $X$ may be the closed interval $[0,1]$). In this case, we ask $\pi$ to be a submersion on Int $V$ and also on $\partial V$.

By Ehresmann’s Lemma [Eh], $\pi$ is a locally trivial smooth fiber bundle, therefore the fibers $\pi^{-1}(\{t\})$ are all diffeomorphic. Moreover, they are endowed with a complex structure obtained by restriction of $J$. We denote by $V_t$ the complex manifold corresponding to $\pi^{-1}(\{t\})$.

Definition. A smooth family of complex structures $\pi : V \to X$ is a locally trivial CR-fiber bundle if

(i) All the fibers $V_t$ are biholomorphic to a fixed complex manifold $V_0$.
(ii) There is an open covering $(U_\alpha)$ of $X$ and CR-isomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to V_0 \times U_\alpha$ (where the CR-structure of $V_0 \times U_\alpha$ is given by the complex tangent distribution to $V_0 \times \{x\}$ for $x$ varying in $U_\alpha$).
(iii) There are commutative diagrams

$$\begin{array}{ccc}
\pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & V_0 \times U_\alpha \\
\downarrow \pi & & \downarrow p \\
U_\alpha & \xrightarrow{Id} & U_\alpha
\end{array}$$

where $p$ is the natural projection.

The following Proposition is the CR-version of a classical result of Fischer and Grauert.

Proposition 1 (see [F-G]). A smooth family of complex structures $\pi : V \to X$ is a locally trivial CR-fiber bundle if and only if all the fibers $V_t$ are biholomorphic.
The proof of this result is rigorously identical to that given in [F-G]. It uses in an essential way Theorem 6.2 of [K-S].

In the particular case where \( X \) is diffeomorphic to an interval, then \( \pi \) is globally trivial [St, Theorem 11.4], that is the following diagram is commutative

\[
\begin{array}{ccc}
V & \longrightarrow & V_0 \times X \\
\downarrow \pi & & \downarrow p \\
X & \longrightarrow & X
\end{array}
\]

The following immediate Corollary will be used frequently in the sequel.

**Proposition 2.** Let \((M, F)\) be a Levi-flat CR-structure manifolds. Let \( \pi : \tilde{M} \rightarrow M \) be a product foliated covering without boundary. Assume that all the leaves of \( \tilde{M} \) are compact and are biholomorphic to a fixed compact complex manifold \( N \).

Then, \( \tilde{M} \) is CR-isomorphic to \( N \times \mathbb{R} \).

**Remark.** Except for very particular cases, we do not know if such a statement is true in the case with boundary, that is: if all the interior leaves of \( \tilde{M} \) are compact and are biholomorphic to a fixed compact complex manifold \( N \), and if the boundary leaf is biholomorphic to \( N \setminus A \) for \( A \) as in the Definition of a product foliated covering, then \( \tilde{M} \) is CR-isomorphic to \((N \times \mathbb{R}^+) \setminus (A \times \{0\})\).

If it was the case, many arguments in the sequel could be greatly simplified. The difficulty is to prove that the Levi-flat CR-structure on \( \tilde{M} \) can be extended to \( A \) on the boundary leaf; this of course can be done along the boundary leaf; but one has to check that this extension is smooth transversally to the leaves and this is not clear at all. The Example given below even shows that it could be false.

**Remark.** Of course, Proposition 2 is valid in the case of a product foliated covering with boundary if the set \( A \) is empty.

In the sequel, we will also consider smooth deformation families of non-compact manifolds. The only difference in the definition is that we have to impose that the family is smoothly trivial, since Ehresmann’s Lemma is false in the non-compact case. The notion of CR-triviality is then exactly the same as before.

One of the main technical difficulties in the sequel however is that statements such as Proposition 1 and 2 are false when the leaves are non-compact. Here is an Example showing how subtle is the situation in the non-compact case (the first author wants to thank Marco Brunella for explaining him this example).

**Example.** Consider the trivial foliation by Riemann spheres \( M = \mathbb{P}^1 \times [0, 1] \). Let \( s \) be a continuous section from \([0, 1]\) to \( M \) and let \( X \) be obtained from \( M \) by removing the image of this section. Then the natural projection map \( M \rightarrow [0, 1] \) is diffeomorphic to the trivial bundle \( \mathbb{R}^2 \times [0, 1] \rightarrow [0, 1] \). So is a deformation family of (non-compact) complex manifolds, each of them being a copy of the complex plane. Although the leaves are all biholomorphic, and although the family \( X \) can be compactified as the trivial deformation family \( M \), it is not CR-isomorphic to the product \( \mathbb{C} \times [0, 1] \).

Assume the contrary. Consider the CR-isomorphism

\[
X \xrightarrow{\cong} \mathbb{C} \times [0, 1]
\]
Let us denote by $i$ this map.

If we have a look at the following diagram

$$
\begin{array}{c}
X \\
\downarrow \text{natural inclusion}
\end{array}
\begin{array}{c}
\to C \times [0,1] \\
\downarrow \text{natural inclusion}
\end{array}
\begin{array}{c}
M = \mathbb{P}^1 \times [0,1] \\
\downarrow \phi
\end{array}
\begin{array}{c}
\to \mathbb{P}^1 \times [0,1]
\end{array}
$$

then we see that the bottom map $\phi$, which is a priori only defined outside the section $s$, extends as a CR-isomorphism of $\mathbb{P}^1 \times [0,1]$. Indeed, for fixed $t$, the map $\phi_t$ extends continuously and thus holomorphically to $\mathbb{P}^1$ by setting $\phi_t(s(t)) = \infty$. Hence $(\phi_t)$ is a family of rational maps of degree one of $\mathbb{P}^1$. Since this family is smooth in $t$ when restricted to $\mathbb{P}^1 \setminus s$, the coefficients of these rational maps are smooth and the family is smooth in $t$ on the whole $\mathbb{P}^1$. Now, this CR-isomorphism sends the continuous section $s$ to the smooth section $\infty \times [0,1]$. Contradiction.

### 3. Some examples of deformation families

In this Section, we first recall some basic facts about families of line bundles over elliptic curves, cf [Gu], [G-H]. The only part which is not classical (although it is an easy consequence of classical facts) is the Dumping Lemma.

Let $\alpha \in \mathbb{H}$ and let $n \in \mathbb{Z}$. The subset $\text{Pic}_n(E_\alpha)$ of the Picard group of the elliptic curve $E_\alpha$ is constituted by elements corresponding to line bundles of Chern number $n$. It has a natural structure of an elliptic curve [Gu, §7-8].

This structure of an elliptic curve makes a moduli space of $\text{Pic}_n(E_\alpha)$. This means in particular the following.

Let $\pi : (\mathcal{X}, J) \to [0, 1]$ be a smooth family of deformations of line bundles over $\mathbb{E}_\alpha$ of fixed topological degree $n$, that is (cf [K-S, §III.7])

(i) The map $\pi$ is a trivial smooth bundle, every fiber of $\pi$ is a line bundle of Chern number $n$ over $\mathbb{E}_\alpha$.

(ii) There is a commutative diagram

$$
\begin{array}{c}
(\mathcal{X}, J) \\
\downarrow p
\end{array}
\begin{array}{c}
\to \mathbb{E}_\alpha \times [0,1] \\
\downarrow \text{2nd projection}
\end{array}
\begin{array}{c}
\to [0,1]
\end{array}
$$

where the restriction of $p$ to a fiber of $\pi$ is the bundle projection.

(iii) The map $p : (\mathcal{X}, J) \to \mathbb{E}_\alpha \times [0,1]$ is a smooth fiber bundle with fiber $\mathbb{C}$ and structural group $\mathbb{C}^*$.

**Example.** Let $\pi : \tilde{M} \to M$ be a product foliated covering with surfaces as leaves. Assume that we have a smooth CR-embedding

$$i : \mathbb{E}_\alpha \times \mathbb{R} \to \tilde{M}$$

(or $i : \mathbb{E}_\alpha \times [0, \infty) \to \tilde{M}$ in the case of a product foliated covering with boundary).

We may choose locally defining functions for the submanifolds $E_t = i_!(\mathbb{E}_\alpha)$ (that
is local holomorphic functions on the leaf $L_t$ of $\tilde{M}$ whose zero set defines an open set of $E_t$ which depend smoothly on $t$. Therefore, we may choose for the normal bundles of $E_t$ in $L_t$ cocycles depending smoothly on $t$. Hence we may construct abstractly a family of deformations of line bundles over $E_\alpha$ of fixed topological degree $p : (X, J) \to E_\alpha \times [0, 1]$ such that $p^{-1}\{t\}$ is the normal bundle of $E_t$ in $L_t$.

Then the natural map

$$j : (X, J) \to \text{Pic}_n(E_\alpha)$$

sending a fiber $X_t = \pi^{-1}\{t\}$ onto the element of $\text{Pic}_n(E_\alpha)$ characterizing it as a line bundle is a smooth map. Indeed, in such a situation, we have a diagram of coverings

$$\begin{array}{ccc}
\mathbb{C} \times \mathbb{C} \times [0, 1] & \longrightarrow & (X, J) \\
\downarrow & & \downarrow p \\
\mathbb{C} \times [0, 1] & \longrightarrow & E_\alpha \times [0, 1]
\end{array}$$

so that we may locally choose for the bundles $p^{-1}(E_\alpha \times \{t\}) \to E_\alpha \times \{t\}$ a set of multipliers depending smoothly on $t$. This is enough to show the result (cf [G-H, §2.6]).

From this, we deduce easily the following Lemma.

**Dumping Lemma.** Let $\pi : (X, J) \to [0, 1]$ be a smooth family of deformations of line bundles over $E_\alpha$ of fixed topological degree $n$. Assume that, for every $t_0 \in [0, 1)$, there exists a sequence $(t_n)$ with 1 as limit such that $X_{t_n}$ is isomorphic (as a line bundle) to $X_{t_0}$.

Then all the fibers $X_t$ of $\pi$ are isomorphic.

**Remark.** Let $\pi_0 : X_0 \to E_\alpha$ and $\pi_1 : X_1 \to E_\alpha$ be two line bundles of fixed topological degree. Assume that they are biholomorphic as complex manifolds. Let $f$ be a biholomorphism between them. Notice that $f$ extends as a biholomorphism between the total spaces of the associated $\mathbb{P}^1$-bundles, say $X_0^c$ and $X_1^c$. Now, $f$ must preserve the Albanese varieties of $X_0^c$ and $X_1^c$, that is we have a commutative diagram

$$\begin{array}{ccc}
X_0^c & \xrightarrow{f} & X_1^c \\
\downarrow & & \downarrow \\
E_\alpha & \longrightarrow & E_\alpha
\end{array}$$

so $f$ maps biholomorphically each fiber of $\pi_0$ onto a fiber of $\pi_1$. It is then straightforward to check that $f$ must be a linear automorphism when restricted to a fiber if the bundles are holomorphically non-trivial. Hence, in the non-trivial case, the word isomorphic in the previous Lemma could be replaced by biholomorphic.

**Proof.** We consider the previously described smooth map:

$$j : (X, J) \to \text{Pic}_n(E_\alpha)$$

where $n$ is the common Chern number of the fibers of $X$. Let $t_0 \in [0, 1)$. By assumption, there exists $(t_n)$ with 1 as limit such that $j(X_{t_n}) = j(X_{t_0})$. Hence...
$$j(X_{t_0}) = j(X_1)$$ by continuity of $j$, that is the map $j$ is constant and all the fibers are isomorphic. $\square$

We finish this part with a short study of CR-suspensions, which give examples of deformation families of both compact and non-compact manifolds. We are mainly interested in knowing when two such families are CR-isomorphic.

(CR)-suspensions form a simple, but important case of product foliated coverings (with empty boundary). That is, given $L$ a complex manifold and $A$ a biholomorphism of $L$, form the smooth manifold $X_A = (L \times \mathbb{R})/\sim$ where the equivalence relation $\sim$ is given by

$$(z,t) \sim (w,s) \iff w = A^p(z), \quad s = t + p \quad \text{for some } p \in \mathbb{Z}.$$ 

Consider the trivial foliation of $L \times \mathbb{R}$ by complex leaves $L \times \{pt\}$. It is preserved under the equivalence relation and descends to a foliation by complex manifolds of $X_A$. Each leaf is biholomorphic to $L$ and the natural projection map

$$L \times \mathbb{R} \rightarrow X$$

is a foliated product covering.

Another way of describing the suspension is the following. The foliated manifold $X_A$ is obtained from $L \times [0,1]$ endowed with its trivial foliation by gluing $L \times \{0\}$ and $L \times \{1\}$ by $A$. If $(A_t)$ is a smooth isotopy of biholomorphisms of $L$ between $A = A_0$ and $A_1$, then the CR-map

$$(z,t) \in L \times [0,1] \mapsto (A_t \circ A_0^{-1}(z), t) \in L \times [0,1]$$

descends as a CR-isomorphism between $X_{A_0}$ and $X_{A_1}$.

Conversely, we have the following result.

**Proposition 3.** Let $\pi : \tilde{M} \rightarrow M$ be a product foliated covering with $\partial M$ empty. Assume that $\tilde{M}$ is CR-isomorphic to $L \times \mathbb{R}$ for some complex manifold $L$. Moreover, assume that the deck transformation group is isomorphic to $\mathbb{Z}$ and that it acts without fixing any leaf $L \times \{pt\}$.

Then, there exists a well-defined biholomorphism $A$ of $L$ (up to smooth isotopy) and $\tilde{M} \rightarrow M$ is CR-isomorphic as a covering space to $L \times \mathbb{R} \rightarrow X_A$ (where $X_A$ is the suspension of $L$ by $A$), i.e. the following diagram is commutative:

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{\text{CR-isomorphism}} & L \times \mathbb{R} \\
\downarrow \pi & & \downarrow \\
M & \xrightarrow{\text{CR-isomorphism}} & X
\end{array}$$

We call this biholomorphism the *monodromy* of the product foliated covering.

**Proof.** From the hypotheses, for any $t$, there exists $s > t$ such that a fundamental domain for the action is $L \times [t,s]$, once $L \times \{t\}$ is identified with $L \times \{s\}$ by $A_t$ (where $A(z,t) = (A_t(z), f(t))$ is a well-chosen generator of the deck transformation group). Hence $(M,\mathcal{F})$ is obtained as the suspension of $L$ by any $A_t$ for fixed $t$. $\square$

It should be noticed that there exist foliated product coverings which are not smoothly isomorphic to a trivial one. Here is an example.
Example. Let
\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \]
and let \( M \) be the manifold obtained as the suspension of a real 2-torus \( T \) by \( A \). The level sets of the suspension map \( M \to S^1 \) foliates \( M \) by copies of \( T \).

We claim that this foliation \( F \) can be turned into a foliation by elliptic curves. Indeed, choose any smooth path \( c \) in the upper half-plane \( \mathbb{H} \) such that \( c^{(n)}(1) = A \cdot c^{(n)}(0) \), i.e.
\[ c^{(n)}(1) = \frac{2 + c^{(n)}(0)}{1 + c^{(n)}(0)} \]
for all \( n \in \mathbb{N} \).

Consider then the action of \( \mathbb{Z}^3 \) onto \( \mathbb{C} \times \mathbb{R} \) given by
\[ ((p, q, r), (z, t)) \in \mathbb{Z}^3 \times (\mathbb{C} \times \mathbb{R}) \mapsto \left( \frac{1}{1 + c(0)} \right)^r (z + p + qc(t)), t + r \in \mathbb{C} \times \mathbb{R} . \]

where we extend \( c \) to \( \mathbb{R} \) by setting \( c(t) = A^{E(t)} \cdot c(t - E(t)) \) where \( E(t) \) is the integer part of \( t \).

The quotient space of \( \mathbb{C} \times \mathbb{R} \) by this action is exactly \( M \) and the trivial foliation by copies of \( \mathbb{C} \) descends to \( M \) and turns \( F \) into a foliation by elliptic curves as wanted. Identifying \( S^1 \) and \( \mathbb{R}/\mathbb{Z} \) and letting the brackets denote the class of a real in \( \mathbb{R}/\mathbb{Z} \), we have that the leaf over \([t]\) is the elliptic curve of modulus \( c(t) \).

Setting \( \tilde{M} = T \times \mathbb{R} \), we have a product foliated covering \( \tilde{M} \to M \). Since in this construction, the path \( c \) ends at a different point from its starting point, and thus cannot be constant, we observe that the leaves of \( \tilde{M} \) are not all biholomorphic. So this product foliated covering cannot be CR-trivial.

This is not due to the construction. We claim that there does not exist on \( (M, F) \) a complex structure such that \( \tilde{M} \) is CR-trivial. For if we could find such a structure, then by Proposition 3, the monodromy map
\[ A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \]
would be isotopic to an automorphism of the elliptic curve serving as complex leaf of the foliation. But no elliptic curve admits a biholomorphism smoothly isotopic to \( A \). Hence the result. Observe that, in fact, using Proposition 2, we have that \( (M, F) \) cannot be endowed with a complex structure whose leaves are all biholomorphic. Observe also that all this implies what was announced as an introduction to this Example: the CR-structure \( (M, F) \) is not smoothly isomorphic to a trivial CR-structure.

On the other hand, a uniformization result can be drawn from Proposition 3.

**Corollary 1.** Let \( \pi_i : (\tilde{M}, \tilde{F}_i) \to (M, F_i) \) (for \( i = 0, 1 \)) be two product foliated coverings satisfying the hypotheses of Proposition 3. Moreover, assume that:

(i) We have \( F_i^{\text{diff}} = F_j^{\text{diff}} \), \( \pi_i^{\text{diff}} = \pi_j^{\text{diff}} \).
(ii) The monodromies are equal (up to smooth isotopy of biholomorphisms).

Then \( (M, F_0) \) and \( (M, F_1) \) are CR-isomorphic.
Proof. Apply twice Proposition 3. It tells you that both \((M, \mathcal{F}_0)\) and \((M, \mathcal{F}_1)\) are CR-isomorphic to the suspension of the same complex manifold by the same map. Hence are CR-isomorphic. □

Let us now have a look to the non-empty boundary case. Let \(\pi : (\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F})\) be a product foliated covering with boundary. Assume that it is CR-trivial, i.e. that it is CR-isomorphic to \(L \times [0, \infty) \setminus A \times \{0\}\) for \(L\) a fixed complex manifold and \(A\) a subset of \(L\). Assume also that the deck transformation group is isomorphic to \(\mathbb{Z}\) and does not fix any leaf of the interior. This is analogous to the case of suspensions.

In the previous model, the action of the deck transformation group is generated by a map
\[
(z, t) \in L \times \mathbb{R}^+ \setminus A \times \{0\} \mapsto (T(z, t), d(t)) \in L \times \mathbb{R}^+ \setminus A \times \{0\}
\]
Since the interior of \(\tilde{M}\) is CR-trivial, it has a well-defined monodromy by Proposition 3. Notice that \(d\) has no positive fixed point since the deck transformation group does not fix any leaf of the interior.

We have

**Proposition 4.** Let \(\pi : (\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F})\) be a product foliated covering with boundary satisfying the hypotheses above. Assume moreover that the biholomorphism \(T_0(-) = T(-, 0)\) induced on the boundary leaf by a generator of the deck transformation group extends as a biholomorphism of \(L\).

Then there is a commutative diagram
\[
\begin{array}{ccc}
(M, \mathcal{F}) & \xrightarrow{\text{CR-isomorphism}} & L \times \mathbb{R}^+ \setminus A \times \{0\} \\
\pi \downarrow & & \downarrow \\
(\tilde{M}, \tilde{\mathcal{F}}) & \xrightarrow{\text{CR-isomorphism}} & X \\
\end{array}
\]
where \(X\) is CR-isomorphic to the quotient of \(L \times \mathbb{R}^+ \setminus A \times \{0\}\) by the group generated by
\[
(z, t) \in L \times \mathbb{R}^+ \setminus A \times \{0\} \mapsto (T_0(z), d(t)) \in L \times \mathbb{R}^+ \setminus A \times \{0\}
\]

Proof. Since \(T_0\) extends as a biholomorphism of \(L\) and since the monodromy of Int \(\tilde{M}\) is smoothly isotopic to \(T_0\), we may also choose as monodromy of this suspension the map \(T_0\). Observe that the CR-isomorphism sending Int \(\tilde{M}\) to \(L \times (0, \infty) \) and conjugating the generator \((z, t) \mapsto (T_1(z), d(t))\) to \((z, t) \mapsto (T_0(z), d(t))\) may be chosen to extend as the identity on the boundary. Indeed, this extension can be constructed as follows. Set
\[
W = (L \times \mathbb{R}^+ \setminus A \times \{0\}) \times [0, 1]
\]
and
\[
S : (z, t, s) \in W \mapsto (T_{st}(z), d(t), s) \in W.
\]

Then \(S\) generates a free and proper CR-action on \(W\) whose quotient \(V\) is diffeomorphic to \(M \times [0, 1]\) endowed with a Levi-flat CR-structure \(\mathcal{G}\) such that
Let Corollary 3. Indeed, this is true in greater generality and we have the following corollary.

Consider the smooth vector field

\[(z, t, s) \in W \quad \xi(z, t, s) = \frac{d}{ds}\bigg|_{s=s_0}(T_{st}(z), d(t), s)\].

It respects the foliation and is holomorphic along the leaves. It is also invariant by the action and descends as a vector field \(\zeta\) on \(V\) with the same properties with respect to \(\mathcal{G}\) this time. Seeing \(V\) as a closed subset of \(M \times \mathbb{R}\), we may extend \(\zeta\) as a vector field in \(M \times \mathbb{R}\) and take its flow. The time 1 flow sends CR-isomorphically the slice \((M \times \{0\}, \mathcal{G}|_{M \times \{0\}})\) to the slice \((M \times \{1\}, \mathcal{G}|_{M \times \{1\}})\). □

As in the case of suspensions, we draw a uniformization result.

**Corollary 2.** Let \(\pi_i : (\tilde{M}, \tilde{\mathcal{F}}) \to (M, \mathcal{F}_i)\) (for \(i = 0, 1\)) be two product foliated coverings satisfying the hypotheses of Proposition 4. Moreover, assume that:

(i) We have \(\mathcal{F}_0^{\text{diff}} = \mathcal{F}_1^{\text{diff}}, \pi_0^{\text{diff}} = \pi_1^{\text{diff}}\).

(ii) The interior leaf of \(\tilde{\mathcal{F}}_0\) is biholomorphic to the interior leaf of \(\tilde{\mathcal{F}}_1\).

(iii) The boundary leaf of \(\tilde{\mathcal{F}}_0\) is biholomorphic to the boundary leaf of \(\tilde{\mathcal{F}}_1\).

(iv) The biholomorphisms \(T_0\) and \(T_1\) induced on each boundary leaf by a generator of the deck transformation group are holomorphically conjugated.

Then \((M, \mathcal{F}_0)\) and \((M, \mathcal{F}_1)\) are CR-isomorphic.

**Proof.** Using Proposition 4, we have that \((M, \mathcal{F}_0)\) (respectively \((M, \mathcal{F}_1)\)) is CR-isomorphic to the quotient of \(L \times \mathbb{R}^+ \setminus \{0\}\) by the group generated by \((z, t) \mapsto (T_0(z), d(t))\) (respectively \((z, t) \mapsto (T_1(z), d(t))\)). By (iv), these maps are CR-conjugated. □

We add some important remarks and consequences.

**Remark.** Notice that the map \(d\) corresponds to the holonomy of the boundary leaf. It is thus a smooth invariant, that is \(\tilde{\mathcal{F}}_0^{\text{diff}} = \mathcal{F}_1^{\text{diff}}\) implies that the corresponding holonomies are smoothly conjugated and therefore can be assumed to be the same. In the sequel, since we deal with tame foliations, the function \(d\) will be tangent to the identity at 0. Nevertheless, it is important to notice that the previous Proposition is valid just assuming that \(d\) has no other fixed point that 0.

**Remark.** The tame condition is not stable under CR-isomorphisms. For example, consider as above the quotient of \(L \times \mathbb{R}^+ \setminus \{0\}\) by the group generated by \((z, t) \mapsto (T_t(z), d(t))\). If the function \(d\) is tangent to the identity at 0, but the function \(T\) is not (in \(t\)), then the product foliated covering could not be tame.

However, Proposition 4 shows that it is CR-isomorphic to the quotient of \(L \times \mathbb{R}^+ \setminus \{0\}\) by the group generated by \((z, t) \mapsto (T_0(z), d(t))\), which is obviously tame. Indeed, this is true in greater generality and we have the following corollary.

**Corollary 3.** Let \(\pi : (\hat{M}, \hat{\mathcal{F}}) \to (M, \mathcal{F})\) be a product foliated covering with boundary. Assume that the deck transformation group is generated by \(\mathbb{Z}\) and does not fix any interior leaf. Assume moreover that the holonomy of the boundary leaf is tangent to the identity. Then \((M, \mathcal{F})\) is CR-isomorphic to a tame foliation.
Proof. This is just a reparametrization argument. We may use the following model for \((\tilde{M}, \tilde{F})\):

\[ \tilde{M}, \tilde{F} = (L^{diff} \times [0, \infty) \setminus A^{diff} \times \{0\}, J) \]

with action generated by a CR-map:

\[ (z, t) \in L^{diff} \times \mathbb{R}^+ \setminus A^{diff} \times \{0\} \xrightarrow{T} (T_t(z), d(t)) \in L^{diff} \times \mathbb{R}^+ \setminus A^{diff} \times \{0\} \]

Let \(\theta\) be a diffeomorphism of \(\mathbb{R}^+\) flat at 0 (for example, \(\theta(t) = \exp(1/t)\) for \(t > 0\)). Call \(L_t\) the leaf corresponding to \(t\) and \(J_t\) the complex structure on \(L_t\). Perform the diffeomorphism

\[ (z, t) \in L^{diff} \times \mathbb{R}^+ \setminus A^{diff} \times \{0\} \xrightarrow{\Phi} (z, \theta^{-1}(t)) \in L^{diff} \times \mathbb{R}^+ \setminus A^{diff} \times \{0\} \]

and call \(J'\) the CR-structure on \(\tilde{M}\) which makes a CR-isomorphism of this diffeomorphism. Observe that

(i) For all \(t\), we have \(J'_t \equiv J_{\theta(t)}\).
(ii) The map \(T\) is conjugated through \(\Phi\) to the map \(S(z, t) = (T_{\theta(t)}(z), \theta^{-1} \circ d \circ \theta(t))\).

This shows that the quotient of \((\tilde{M}, J')\) by the action generated by \(S\), which is CR-isomorphic to \((M, F)\), is tame. \(\square\)

4. The compactification Lemma

Let us start by remarking that a non-compact leaf of \(\mathcal{F}_C\) has either one topological end (if the leaf is a smooth affine cubic or a line bundle over an elliptic curve) or two topological ends (if the leaf is a principal \(C^*\)-bundle over an elliptic curve). Here we represent ends of a non-compact leaf as descending sequences \(U_1 \supseteq U_2 \supseteq \cdots\) of open subsets of the leaf such that \(\partial U_i\) is compact and \(\bigcap_{i \geq 1} U_i = \emptyset\) where \(U_i\) and \(\partial\) are, respectively, the closure and boundary of \(U_i\) in the leaf with its topology as a manifold. The ends of the non-compact leaves “accumulate” respectively to one or two of the compact leaves. In other words, an end of a non-compact leaf spirals around a compact leaf, or, more precisely, \(\bigcap_{i \geq 1} \tilde{U}_i = \emptyset\), where \(\tilde{U}_0\) is a compact leaf and \(\tilde{U}_i\) is the closure of \(U_i\) in \(\mathbb{S}^5\). This imposes certain recurrence of the complex structure near the ends. Thus one may expect that the complex structure at infinity of the non-compact leaves is fixed by the complex structure of the compact leaves. The precise formulation of this rigidity property takes the form of the compactification Lemma stated below. In fact, this is the central idea of this paper. We need one more definition before stating this Lemma.

Definition. Let \(X\) be a complex manifold of dimension \(n\). Let \(H\) be a compact complex manifold of dimension \(n - 1\). Let \(E\) be an end of \(X\). Then we say that \(X\) admits a partial holomorphic compactification at \(E\)-infinity by adding \(H\) if there exists a structure of complex manifold on the disjoint union \(X \sqcup H\) such that

(i) The natural injections \(X \to X \sqcup H\) and \(H \to X \sqcup H\) are holomorphic.
(ii) The submanifold \(H\) of \(X \sqcup H\) is the limit set of \(E\).

The second point means the following: representing \(E\) by a sequence \(U_1 \supseteq U_2 \supseteq \cdots\) as before, we have that \(\bigcap_{i \geq 1} \tilde{U}_i = H\), where \(\tilde{U}_i\) denotes the closure of \(U_i\) in \(X \sqcup H\) equipped with the topology coming from its structure of a complex manifold.
Remark. If $X \sqcup H$ is compact, then this implies that $X$ has just one end and we say that $X$ admits a holomorphic compactification by adding $H$.

Here is an example. Let $H$ be a compact complex manifold and let $X$ be a principal $\mathbb{C}^*$-bundle over $H$. Then $X$ admits a partial holomorphic compactification at $0$-infinity by adding a zero section $H$. In this case $X \sqcup H$ is the associated line bundle over $H$.

Let $(M, \mathcal{F}_0)$ be a foliation by complex manifolds with $\partial M \neq \emptyset$. Let $L_0$ be a leaf of $\text{Int}(M)$ such that the limit set, $\cap_{i \geq 1} \bar{U}_i = C$ corresponding to a given end $U_1 \supseteq U_2 \supseteq \cdots$ of $L_0$ is a compact connected component $C$ of $\partial M$.

Now, let $(M, \mathcal{F}_1)$ be another foliation by complex leaves of $M$ such that

(i) The underlying smooth foliations are equal, that is $(\mathcal{F}_0)^{\text{diff}} = (\mathcal{F}_1)^{\text{diff}}$.

(ii) On $C$, the complex structures $J_1$ (respectively $J_0$) induced by $\mathcal{F}_1$ (respectively $\mathcal{F}_0$) agree, that is $(J_0)|_C \equiv (J_1)|_C$.

We want to compare $L_0$ to $L_1$ (the corresponding leaf of $\mathcal{F}_1$) as abstract complex manifolds. Since they are of course diffeomorphic, we want to compare their complex structures. A priori, there is no reason that they are biholomorphic. However, since the common limit of $L_0$ and $L_1$ is $C$, condition (ii) above means that these complex structures are in a sense close near $C$.

We are now in position to prove the Compactification Lemma. Roughly speaking, it states that, if $L_0$ can be compactified holomorphically at $C$-infinity, then so does $L_1$, since their complex structures are asymptotic near $C$.

**Compactification Lemma.** With the hypotheses above, $L_1$ admits a partial holomorphic compactification at $C$-infinity by adding a compact complex manifold $H$ if and only if $L_0$ admits a partial holomorphic compactification at $C$-infinity by adding a compact complex manifold $H$.

**Proof.** The statement is clearly symmetric, so assume that $L_0$ admits a partial holomorphic compactification at $C$-infinity by adding a compact complex manifold $H$.

For $x \in M$, let us denote by $J_0(x)$ (respectively $J_1(x)$) the almost complex operator of $\mathcal{F}_0$ at $x$ (respectively of $\mathcal{F}_1$). Let $L = (L_0)^{\text{diff}} = (L_1)^{\text{diff}}$ and $\mathcal{F} = (\mathcal{F}_0)^{\text{diff}} = (\mathcal{F}_1)^{\text{diff}}$. We consider them as smooth sections of the vector bundle $\text{End} (T\mathcal{F}) \to M$. These operators are in general not equal, but they agree on $C$. As they are smooth and as $C$ is compact, we deduce that, for every neighborhood of the zero section of $\text{End} (T\mathcal{F}) \to M$, there exists a neighborhood $V \subset M$ of $C$ such that $f(V) \subset W$, where $f$ is defined as the difference $J_0 - J_1$.

Looking at the injections $L \hookrightarrow M$ and $\text{End} (T\mathcal{F}) \hookrightarrow \text{End} (T\mathcal{F})$, this means that for every neighborhood of the zero section of $\text{End} (T\mathcal{F}) \to L$, there exists a neighborhood $V$ of the end $E$ of $L$ such that $f(V) \subset W$.

Consider now the inclusion diagram

$$
\begin{array}{ccc}
L & \hookrightarrow & L^c = L \sqcup H^{\text{diff}} \\
\downarrow f & & \downarrow \\
\text{End} (T\mathcal{F}) & \hookrightarrow & \text{End} (T\mathcal{F}^c)
\end{array}
$$
Setting a riemannian metric \( \| - \| \) on End \((TL^c)\), we have, using the compacity of \( H \), that, for all \( \epsilon > 0 \), there exists a neighborhood \( V_\epsilon \) of \( E \) in \( L \) such that
\[
\sup_{x \in V_\epsilon} \| J_0(x) - J_1(x) \| \leq \epsilon
\]
So \( f \) extends continuously as the zero section over \( H^{\text{diff}} \).

Now, assume that \( J_0 \) extends smoothly to \( L^c \). We claim that \( J_1 \) extends \textit{continuously} to \( L^c \) with \( J_0 \equiv J_1 \) on \( H^{\text{diff}} \).

Indeed, take local coordinates in a neighborhood \( U \) of \( x \in H^{\text{diff}} \). We assume that End \((TL^c)\) is trivial over \( U \), hence \( J_0 \) and \( J_1 \) are now functions with values in a euclidean space. We choose as norm \( \| - \| \) over \( U \) the euclidean norm. For \( \epsilon > 0 \) and for \( y \) sufficiently near \( x \), we have
\[
\| J_1(y) - J_0(x) \| \leq \| J_1(y) - J_0(y) \| + \| J_0(y) - J_0(x) \|
\leq \| J_1(y) - J_0(y) \| + \epsilon \quad \text{by continuity of } J_0
\leq 2\epsilon
\]
and the claim is proved.

This is however not enough; we would like to show that \( J_1 \) extends \textit{smoothly} to \( L^c \). To do this, we use the fact that our foliations by complex leaves are by definition tame. This means that we may extend \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) to \( W = M \cup C \times (0,1] \) by stating that the complex leaves of both \( \mathcal{F}_0 \) and \( \mathcal{F}_1 \) on \( C \times (0,1] \) are \( C \times \{t\} \). So the complex structures \( J_0 \) and \( J_1 \) may be assumed equal on the collar \( C \times [0,1] \).

We will now proceed by induction and repeat essentially the same argument in every \( k \)-jet bundle of sections of End \((TL^c)\) for \( k \geq 0 \).

We just do the case \( k = 1 \). Consider the map
\[
f_1 : x \in M \mapsto (x, j^1(f(x)) \in J^1(M, \text{End}(T\mathcal{F}))
\]
where \( J^1(M, \text{End}(T\mathcal{F})) \) is the bundle of 1-jets of sections of End \((T\mathcal{F})\) and \( j^1(f) \) is the 1-jet of \( f \).

Since \( J_0 \) is equal to \( J_1 \) not only on \( C \) but on a collar of the boundary, we have that \( j^1(f) \) is the zero section over \( C \).

As before, considering the restriction of the situation to \( L \) and seeing \( f_1 \) as a continuous map from \( L \) to \( J^1(L, \text{End}(TL)) \), we have that for every neighborhood of the zero section of \( J^1(L, \text{End}(TL)) \rightarrow L \), there exists a neighborhood \( V \) of the end \( E \) of \( L \) such that \( f_1(V) \subset W \).

Considering the inclusion of \( L \) into \( L^c \) and the corresponding inclusion for the jet bundles, and setting a riemannian metric \( \| - \|_1 \) on \( J^1(L^c, \text{End}(TL^c)) \), we have, using the compacity of \( H \), that, for all \( \epsilon > 0 \), there exists a neighborhood \( V_\epsilon \) of \( E \) in \( L \) such that
\[
\forall k \geq 0, \quad \sup_{x \in V_\epsilon} \| J_0(x) - J_1(x) \|_{1} \leq \epsilon
\]
So \( f_1 \) extends continuously as the zero section over \( H^{\text{diff}} \). Now, take local coordinates in a neighborhood \( U \) of \( x \in H^{\text{diff}} \). We assume that End \((TL^c)\) is trivial over \( U \), hence \( J_0 \) and \( J_1 \) are now functions with values in a euclidean space. We choose as norm \( \| - \|_1 \) over \( U \) the maximum at one point of the euclidean norms of the function and all its first-order derivatives. The same sequence of inequalities
as above but with the norm $\| - \|_1$ this time shows that $j^1(J_1)$ extends continuously on $L^c$ as $j^1(J_0)$ on $H^{diff}$. It is now easy to deduce that $J_1$ admits a $C^1$ extension to $L^c$ as $J_0$ on $H^{diff}$. By induction, this extension is in fact smooth.

Now the almost complex operator $J_1$ on the whole $L^c$ is automatically integrable, since it is on the open and dense subset $L \subset L^c$ so the Newlander-Nirenberg Theorem provides us with a complex structure. □

Remark. Observe that the tame condition can be replaced by the somewhat weaker condition: on $W = M \sqcup C \times (0, 1]$, both $J_0$ and $J_1$ extend in such a way that they are equal on the collar $C \times [0, 1]$. This allows to use the Compactification Lemma in some cases where the holonomy is not flat.

Remark. The Compactification Lemma compares two complex leaves of two different foliations as abstract complex manifolds. This implies that the compactification used may be arbitrary, i.e. does not depend on the foliations themselves. In particular, if $L_0$, as an abstract complex manifold, admits various partial holomorphic compactifications at $C$-infinity which are topologically distinct, the Lemma works for every compactification and $L_1$ will admit exactly the same number of partial holomorphic compactifications at $C$-infinity.

Here is an application of the Compactification Lemma. It shows how this Lemma can be used in some cases to determine the biholomorphism type of the interior leaves of a foliation by complex leaves. This type of argument will be used many times in the next Section.

**Example.** Consider the solid torus endowed with the classical Reeb foliation. This foliation may be turned into a foliation $\mathcal{F}_\tau$ by complex leaves, with boundary leaf biholomorphic to an arbitrary elliptic curve $E_\tau$ and with all interior leaves biholomorphic to $\mathbb{C}$ (see [M-V]).

We want to prove, using the Compactification Lemma, that there does not exist on this fixed smooth foliation a complex structure with at least one interior leaf biholomorphic to the unit disk. Let $\mathcal{F}$ be any complex structure on this fixed smooth foliation. The boundary leaf must be an elliptic curve, so this foliation agrees on the boundary with some $\mathcal{F}_\tau$.

Let $L$ be an interior leaf of $\mathcal{F}$ and let $L_\tau$ be the corresponding leaf of $\mathcal{F}_\tau$. These leaves have one end whose limit set is the boundary leaf $E_\tau$. Now, the leaf $L_\tau$ is biholomorphic to $\mathbb{C}$, so admits a holomorphic compactification as the Riemann sphere.

The Compactification Lemma tells us then that $L$ also admits a holomorphic one-point compactification. By the uniformization Theorem, this compactification is the Riemann sphere and $L$ is then a copy of $\mathbb{C}$. Hence, $\mathcal{F}$ does not have any leaf biholomorphic to a disk.

The Compactification Lemma compares two different foliations. However, it is possible to use it with just one foliation. We now explain this point.

Let $(M, \mathcal{F})$ be a foliation by complex manifolds with $\partial M \neq \emptyset$. Let $L$ be a leaf of $\text{Int}(M)$ such that the limit set $\bigcap_{i \geq 1} \mathcal{U}_i = C$ corresponding to a given end $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \cdots$ of $L$ is a compact connected component $C$ of $\partial M$. There exists a neighborhood $V$ of $C$ and a global submersion from $V$ to $C$ which is the identity on $C$. For example, we may use a collar of $C$ in $M$ and define
\[(z, t) \in C \times \mathbb{R}^+ \simeq V \mapsto z \in C .\]

Reducing \(V\) if necessary, we may assume that the previous submersion is a surjective local diffeomorphism when restricted to a leaf (intersected with \(V\)). In particular, \(L \cap V\) is locally diffeomorphic to \(C\). Call \((L \cap V)^{pb}\) the manifold diffeomorphic to \(L \cap V\) with the complex structure induced from that of \(C\) by pull-back and call \(\mathcal{F}^{pb}\) the corresponding complex structure on \(V\).

Notice that the complex structure of \((L \cap V)^{pb}\) is independent of the choice of the submersion. Indeed, for two different choices, the foliations \(\mathcal{F}^{pb}\) are CR-isomorphic since the submersions are isotopic by an isotopy which is the identity on \(C\).

**Corollary to Compactification Lemma.** Assume that \((L \cap V)^{pb}\) admits a partial holomorphic compactification at \(C\)-infinity by adding a compact complex manifold \(H\) of codimension-one. Then \(L\) admits a partial holomorphic compactification at \(C\)-infinity by adding \(H\).

**Proof.** It is enough to prove that the pull-back foliated complex structure \(\mathcal{F}^{pb}\) is a foliation by complex manifolds, that is that it is tame. The Compactification Lemma then applies. Now, since \(\mathcal{F}\) is tame by definition, we have that \((\mathcal{F})^{diff}\) is flat at the boundary (i.e. that the holonomies of the boundary components are smooth flat functions). This implies that the submersion used to define the pull-back foliation is flat at the boundary, hence the pull-back complex structure is tame. \(\square\)

In the sequel, by Compactification Lemma, we will always mean the Corollary to the Compactification Lemma. We will also use the Compactification Lemma to provide uniform compactification of a “tube” of leaves.

**Uniform Compactification Lemma.** In the same situation as in the previous Corollary, assume that \(L\) has trivial holonomy. Choose a closed transverse section \(s \simeq [-1, 1]\) to \(L\) and consider the tube of leaves \(\mathcal{L} \simeq_{diff} L^{diff} \times [-1, 1]\) intersecting \(s\). Assume that all the leaves in the tube accumulates uniformly onto \(C\). Assume also that \((L \cap V)^{pb}\) admits a partial holomorphic compactification at \(C\)-infinity by adding a compact complex manifold \(H\) of codimension-one. Then \(\mathcal{L}\) admits a partial CR compactification at \(C \times [-1, 1]\)-infinity by adding a compact CR manifold \(H \times [-1, 1]\).

We omit the exact Definition of partial CR compactification. It should be clear from the Definition of partial holomorphic compactification.

**Proof.** The proof is identical to the previous one. One just has to notice that the estimates work in a whole neighborhood of \(C\) in \(M\), so are valid not only for a single leaf but for a tube of leaves. \(\square\)

**Example.** We go back to the previous Example, where we show that a foliation by Riemann surfaces on a Reeb component has all interior leaves biholomorphic to \(\mathbb{C}\).

A construction of the smooth underlying foliation is as follows (cf [M-V], Lemma 2). Let

\[X = \mathbb{R}^2 \times [0, \infty) \setminus \{(0, 0, 0)\}\]

foliated by the level sets of the projection onto the second factor. Let

\[h : (x, y, t) \in X \mapsto (\alpha x - \beta y, \beta x + \alpha y, d(t)) \in X\]
where $0 < \alpha^2 + \beta^2 < 1$ and where $d$ is a smooth function of $\mathbb{R}$ into $\mathbb{R}^+$, which is a contracting diffeomorphism of $\mathbb{R}^+$ and which is the identity on $\mathbb{R}^-$. Then the quotient of $X$ by the group generated by $h$ is a Reeb component.

The foliations $\mathcal{F}_\tau$ above can be constructed by endowing $X$ with the trivial foliation $\mathbb{C} \times [0, \infty) \setminus \{(0, 0, 0)\}$.

Indeed, $h$ becomes multiplication by the complex number $\exp(2i\pi \tau) = \alpha + i\beta$ and is a biholomorphism of the leaves. The foliation descends to the foliation $\mathcal{F}_\tau$ previously described.

We claim that, given any foliation by Riemann surfaces on this Reeb component, it is CR-isomorphic in the interior to some $\mathcal{F}_\tau$. We now want to use the uniform Compactification Lemma to prove this assertion (compare the following argument with the Example at the end of Section 2). Endow the Reeb component with an arbitrary complex structure and endow the covering $X$ of the Reeb component by the pull-back complex structure. We already know that such a structure coincides on the boundary with some $\mathcal{F}_\tau$ and that all the interior leaves of $X$ are biholomorphic to $\mathbb{C}$. Now the uniform Compactification Lemma tells us that a tube $\mathcal{L}$ of interior leaves can be uniformly compactified as a closed manifold foliated by Riemann spheres. We may thus partially compactify $X$ as a product foliated covering $\tilde{X}$ with compact interior leaves. But then Proposition 2 implies that the interior $\tilde{X}$ is CR-isomorphic to the trivial family $\mathbb{P}^1 \times (0, \infty)$. Since the compactification is uniform, this CR-isomorphism sends $\text{Int} (\tilde{X} \setminus X)$ to a smooth section $s$ of $\mathbb{P}^1 \times (0, \infty)$. Since the automorphism group of $\mathbb{P}^1$ is transitive, there exists a CR-isomorphism of $\mathbb{P}^1 \times (0, \infty)$ sending $s$ onto the infinite section $\infty \times (0, \infty)$. Composing these isomorphisms, this gives a CR-isomorphism between $\mathbb{P}^1 \times (0, \infty)$ and $\text{Int} \tilde{X}$ sending $\mathbb{C} \times (0, \infty)$ onto $\text{Int} \tilde{X}$. We conclude by Corollary 1.

Let us have a closer look to the situation of a product foliated covering $\pi: \tilde{M} \to M$ with boundary and deck transformation group isomorphic to $\mathbb{Z}$. It is diffeomorphic to $L^{\text{diff}} \times [0, \infty) \setminus A^{\text{diff}} \times \{0\}$ and a generator of the deck transformation group has the following form

$$(z, t) \in L^{\text{diff}} \times \mathbb{R}^+ \setminus A^{\text{diff}} \times \{0\} \mapsto (T(z, t), d(t)) \in L^{\text{diff}} \times \mathbb{R}^+ \setminus A^{\text{diff}} \times \{0\}$$

As usual, assume that $d$ has no positive fixed point. As a consequence of the uniform compactification Lemma, we have:

**Proposition 5.** Let $\pi: \tilde{M} \to M$ be a product foliated covering satisfying all the hypotheses described just above. Assume moreover that it is tame and that the boundary leaf $\partial \tilde{M}$ admits a partial holomorphic compactification at infinity by adding a compact curve $C$.

Then, the whole $\tilde{M}$ admits a uniform compactification at infinity by adding $C$; that is, setting

$$\tilde{M} = (L^{\text{diff}} \times [0, \infty) \setminus A^{\text{diff}} \times \{0\}, J)$$

then $J$ extends as a complex structure on $\tilde{M} \sqcup C \times \mathbb{R}^+$.

**Remark.** The boundary leaf $\partial \tilde{M}$ has two ends, being a $\mathbb{Z}$-covering of a compact manifold without boundary. On the other hand, it is diffeomorphic to $L^{\text{diff}} \setminus A^{\text{diff}}$ and every other leaf is diffeomorphic to $L^{\text{diff}}$. We implicitly assume that $L^{\text{diff}}$
has just one end, corresponding to the end of $\partial \tilde{M}$ called infinity in the statement of Proposition 5.

**Proof.** We take the same notations as in the proofs of the compactification Lemmas and consider $W = M \cup \partial M \times (-1, 0)$. Observe that the covering $\pi$ extends to a covering from $\tilde{W} = \tilde{M} \cup \partial \tilde{M} \times (-1, 0)$. Consider also the partially compactified space $\tilde{W} = \tilde{M} \cup C \times (-1, \infty)$. Fix $\alpha > 0$ and riemannian metrics on the bundles of $k$-jets of sections of $\text{End} (T\tilde{W})$. From the inequalities used in the proof of the Compactification Lemma, and taking the pull-back by the covering, we see that there exists, for all $\epsilon > 0$ and for all $k \in \mathbb{N}$, a neighborhood $\tilde{W}_{\epsilon, k}$ of the end at infinity of $L^{\text{diff}} \times [0, \alpha] \setminus A^{\text{diff}} \times \{0\}$ such that

$$\sup_{x \in \tilde{W}_{\epsilon, k}} \| \tilde{J}_0(x) - \tilde{J}^{pb}(x) \|_k \leq \epsilon$$

where $\tilde{J}_0$ (respectively $\tilde{J}^{pb}$) are the pull-back complex structures of $J_0$ (respectively $J^{pb}$).

By assumption, $\tilde{J}^{pb}$ extends at infinity on $C \times \{0\}$, and thus uniformly on $C \times [0, \alpha]$ since $(M^{\text{diff}}, \tilde{J}^{pb})$ is CR-trivial by definition. Now, the previous inequalities imply that this is also true for $\tilde{J}_0$ by arguing as in the proof of the Compactification Lemma. □

**Remark.** As above, observe that the tame condition can be replaced by the somewhat weaker condition: on $W = M \cup \partial M \times (0, 1]$, both $J_0$ and $J^{pb}$ extend in such a way that they are equal on the collar $\partial M \times [0, 1]$.

**Remark.** This Proposition can be considered as an extension result. In fact, we already know from the uniform compactification Lemma that the interior of $\tilde{M}$ admits the desired compactification. Hence Proposition 5 states that this compactification can be extended uniformly to the boundary leaf. This is done following the rough argument that the added curves $C$ in the interior of $\tilde{M}$ “converge” through the action of the deck transformation group onto the curve $C$ at the boundary. Hence the compactification is uniform.

To compare with, consider now the case of the other end of $\partial \tilde{M}$, i.e. assume that $\partial \tilde{M} = L \setminus A$. Then, as said before, it is not clear that the CR-structure of $\tilde{M}$ extends smoothly to $A$.

5. The Lawson foliation does not admit any integrable almost CR-structure

**Theorem A.** The Lawson foliation of $S^5$ can be endowed with a compatible almost CR-structure, however this structure can never be integrable.

**Proof.** Let us first prove that the Lawson foliation admits (non-integrable) Levi-flat CR-structures. Notice that the inclusion of $S^5$ as the unit Euclidean sphere of $\mathbb{C}^3$ defines the canonical integrable almost CR-structure on $S^5$. This structure is not Levi-flat since the corresponding distribution is the canonical contact structure of $S^5$. At $z \in S^5 \subset \mathbb{C}^3$, it is equal to

$$\{ w \in \mathbb{C}^3 \mid \langle z, w \rangle = 0 \}$$

where the angles denote the standard hermitian product of $\mathbb{C}^3$. 
This contact structure is orthogonal to the contact flow which gives the Hopf fibration of $S^5$ by circles. It is generated by the unit vector field

$$v : z \in S^5 \subset \mathbb{C}^3 \mapsto v(z) = iz.$$ 

It is enough to prove that the distribution $\mathcal{H}$ tangent to the Lawson foliation is homotopic to the distribution of this contact structure. This implies that $\mathcal{H}$, as an abstract vector bundle over $S^5$, is homotopic to a complex vector bundle. Therefore, it is a complex vector bundle by [St, Theorem 11.5], i.e. it has an almost CR-structure.

We claim that there are only two homotopy classes of unit vector fields over $S^5$ and that they are represented by $v$ from the one hand, and by $-v$ from the other hand. As a consequence, every 4-planes orientable distribution is homotopic to the distribution of the contact structure, once we take on it the "good" orientation. So admits an almost CR-structure.

Let us first prove that there are only two homotopy classes of unit vector fields over $S^5$. Indeed, this is exactly the number of homotopy classes of sections of the unit tangent bundle $U$ of $S^5$. Since it has a section (the previous vector field $v$), the homotopy sequence of the fibration splits at each step:

$$\pi_i(U) = \pi_i(S^5) \oplus \pi_i(S^4)$$

In particular, for $i = 5$, this gives

$$\pi_5(U) = \mathbb{Z} \oplus \mathbb{Z}_2$$

Given an element $j$ of $\pi_5(U)$, notice that its component in $\mathbb{Z}$ is the degree of the composition map

$$S^5 \xrightarrow{j} U \xrightarrow{\text{bundle projection}} S^5$$

Therefore, a homotopy class of sections of $U$ is exactly an element of $\pi_5(U)$ whose component in $\mathbb{Z}$ is one and the claim is proved.

To finish with, it is enough to prove that $v$ and $-v$, the two unit vector fields of $S^5$ we know, are not homotopic. Assume the contrary. Then the contact distribution of $S^5$ and the same distribution with the orientation reversed would be homotopic. Consider the previously described action of $S^1$ onto $S^5$. It leaves both distributions invariant. By [Br, Chapter VI, Theorem 3.1], we may assume that the homotopy between these two distributions is equivariant. So there is an equivariant oriented isomorphism between these two distributions [Br, Chapter II, Theorem 7.4]. Hence it descends to an oriented isomorphism between the tangent bundle of the complex projective space $\mathbb{P}^2$ and the tangent bundle of $\mathbb{P}^2$, the manifold obtained from $\mathbb{P}^2$ by reversing its orientation. Taking account of what we said above, this would imply that $\mathbb{P}^2$ has an almost complex structure. But this is known to be false, by use of Wu’s theorem characterizing homologically the existence of an almost complex structure on a real 4-manifold, see [B-H-P-V]. This finishes the proof.

Let us now prove that the almost CR-structures compatible with the Lawson foliation can never be integrable. Assume the contrary, i.e. assume the existence of such an integrable almost CR-structure. In fact, we will prove that it cannot exist.
on the interior part of the Lawson foliation (that is, with the notations of 1.2, the neighborhood \( \mathcal{N} \) of \( K \)). Recall that the boundary \( \partial \mathcal{N} \simeq K \times S^1 \) is a leaf and that the non-compact leaves of \( \mathcal{N} \) are all diffeomorphic to

\[
L = \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{D}
\]

and are equipped with a complex structure \( J \) by our assumption.

The proof will take the form of several Lemmas. We will describe some properties of \( J \) imposed by the topology of the foliation; these properties will lead to a contradiction. The outline of the proof is the following: in Lemmas 1 and 2, we describe the explicit biholomorphism type of the boundary leaf, using the Enriques-Kodaira classification. Then, in Lemmas 3 and 4, we prove that the non-compact leaves are all biholomorphic to the product of \( C \) by a fixed elliptic curve. We use the Compactification Lemma to obtain such a result. Finally, we exhibit an associated CR-trivial product foliated covering and compute a generator of its deck transformation group. This gives us an automorphism of the non-compact leaf.

The contradiction comes now from the fact that the previous biholomorphism is not compatible with the complex structure of a non-compact leaf; the map which should be a biholomorphism of the non-compact leaf does not belong to the automorphism group of the leaf for homological reason (its action on the homology groups is different from the action induced by any biholomorphism).

We first need to review Lawson’s construction with a little more care. We refer to [La] for details.

Recall the bundle map

\[
K \rightarrow S^1_1 \times S^1_2
\]

induced from the map \( W \rightarrow \mathbb{E}_\omega \) by passing to the associated unit bundle (see Section 1.2).

**Remark.** It will be important in the sequel to distinguish the different \( S^1 \)-factors, so we label them.

Define the map

\[
\pi : K \times S^3_3 \rightarrow S^1_1 \times S^1_2
\]

by composition of the natural projection and of the previous map. The important fact is that \( \pi \) defines \( K \times S^1 \) as a principal torus bundle over \( S^1 \times S^1 \).

**Remark.** Recall the other bundle map

\[
s : K \rightarrow S^1_1
\]

obtained from the description of \( K \) as a suspension. Then the map

\[
(s, \text{Id}) : K \times S^3_3 \rightarrow S^1_1 \times S^3_3
\]

is a smooth submersion with compact and connected fibers, so is a locally trivial smooth fiber bundle by Ehresmann’s Lemma; and it is not isomorphic to \( \pi \). But the key point for us is that this bundle is not principal, as previously remarked for the bundle map \( s \).
Starting with the map $\pi$, Lawson composes it with the natural projection onto one of the $S^1$-factors and extends it as a submersion
\[ p : K \times \overline{D}_4 \to S^1_1 \]

He foliates the interior $K \times D$ by the level sets of $p$. Notice that the choice of the projection does not matter. It is easy to prove that exchanging the factors of the base of $\pi$ lifts to a diffeomorphism of the total space $K \times S^1$ of $\pi$, hence the foliations obtained by the two different projections are diffeomorphic. Notice also that the two different projections come from projections $K \to S^1$ both of which define $K$ as the suspension of a torus by the matrix $A$. So we make the assumption that the image of $p$ is the same as the image of the suspension map $s$.

Consider now the following commutative diagram
\[
\begin{array}{ccc}
K \times \overline{D}_4 \setminus \{0\} & \xrightarrow{\sim} & K \times S^1_1 \times (0, 1] \\
\downarrow I_d & & \downarrow (p|_\partial, I_d) \\
K \times \overline{D}_4 \setminus \{0\} & \xrightarrow{\bar{p}} & S^1_1 \times (0, 1] \simeq \overline{D}_1 \setminus \{0\} \\
\downarrow I_d & & \downarrow \text{natural projection} \\
K \times \overline{D}_4 \setminus \{0\} & \xrightarrow{p} & S^1_1 \\
\end{array}
\]

where $\bar{p}$ is defined from the other arrows.

In this setting, the leaves of the foliation restricted to $K \times \overline{D}_4 \setminus \{0\}$ are also given by the inverse images
\[ \bar{p}^{-1}(\{\exp i\theta\} \times (0, 1]) \quad \theta \in \mathbb{R} \]

Now Lawson turbulizes this foliation by considering as leaves the inverse images $\bar{p}^{-1}(C_\theta)$, where $C_\theta$ spirals in the disk $S^1 \times (0, 1]$ as shown in the following picture.

![Spiral Foliation](image)

This turbulized foliation extends as a foliation of $K \times \overline{D}$ such that the boundary $K \times S^1$ is a leaf.

We may now start with the proof.
Lemma 1. The compact leaf \((K \times S^1, J)\) is a primary Kodaira surface, that is a principal holomorphic fiber bundle over an elliptic curve with fiber an elliptic curve.

We denote by \(S\) this complex compact surface, by \(E_\alpha\) the base space of the associated bundle map and by \(E_\beta\) the fibers of this map.

**Proof of Lemma 1.** From the construction of \(F_C\), we know that the smooth model of \(S\), that is \(K \times S^1\) admits a structure of a primary Kodaira surface. This implies that the Chern numbers \(c_1^2\) and \(c_2\) of this structure are zero and that the first Betti number is 3 (a fact that can also be easily recovered from the description of \(K\) as a suspension given in 1.2). As these numbers are topological invariants, they keep the same values for \(S\). On the other hand, the universal covering of \(K \times S^1\) is \(\mathbb{R}^4\), which implies, using for example the long exact sequence in homotopy of a fibration, that the second homotopy group of \(S\) is zero. Hence \(S\) is minimal. The Enriques-Kodaira classification ([B-H-P-V], p.244) shows that \(S\) is a primary Kodaira surface or a minimal properly elliptic surface. By [F-M, Theorem S3, (ii)], a smooth manifold cannot admit at the same time a complex structure of Kodaira dimension zero and another one of Kodaira dimension one. □

Lemma 2. The complex structure of \(S\) is compatible with \(\pi\), that is there exists a structure of elliptic curve \(E_\alpha\) on the base space of \(\pi\) such that \(\pi\) becomes a holomorphic principal fiber bundle from \(S\) to \(E_\alpha\).

**Proof of Lemma 2.** By Lemma 1, there exists a holomorphic principal elliptic fiber bundle \(S \to E_\alpha\), so that it is enough to prove that this bundle is smoothly isomorphic (that is isomorphic as \(C^\infty\) principal bundles) to the bundle \(\pi\): endowing \(K \times S^1\) with the complex structure obtained by pull-back by this isomorphism, it becomes a holomorphic principal elliptic bundle which is complex isomorphic to \(S \to E_\alpha\).

A principal elliptic fiber bundle over an elliptic curve \(E_\alpha\) is obtained from a \(C^*\)-principal bundle over \(E_\alpha\) by taking the quotient by a group acting as a complex homothety in the fibers. Notice that this description fits not only to the case of a holomorphic bundle but also to the case of a smooth bundle. Indeed a smooth principal elliptic bundle over \(E_\alpha\) can be thought of as a smooth bundle with complex fibers: fixing a structure of an elliptic curve on the fibers, it is preserved by the structural group which, by definition, consists only of translations. Then such an elliptic bundle is obtained from a smooth \(C^*\)-bundle (that is a smooth locally trivial bundle over \(E_\alpha\) with fiber \(C^*\) and structural group \(C^*)\) by taking the quotient by a group acting as a complex homothety in the fibers. In the sequel of the proof, by elliptic bundle (respectively \(C^*\)-bundle), we mean smooth ones.

To this \(C^*\)-bundle, we may associate its unit bundle, which is an oriented principal circle bundle over \(E_\alpha\). Two such elliptic fiber bundles are isomorphic if and only if the corresponding \(C^*\)-bundles are isomorphic and this occurs if and only if the associated oriented circle bundles are isomorphic. Finally this is the case if and only if the Euler numbers are equal.

On the other hand, from the description of \(K\) given in Section 1.2 and in [M-V], it is straightforward to check that, for \(n \in \mathbb{Z}\), the isomorphism class of circle bundles of Euler number \(n\) can be represented, as an oriented manifold, as the suspension of a real torus by the matrix

\[
A_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}
\]
From this, the first homology group of a circle bundle of Euler number \( n \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_{|n|} \). Hence if two such circle bundles are diffeomorphic as a manifold, then their Euler numbers are equal up to sign. In particular the Euler numbers of \( \pi \) and of \( \S \to E_\alpha \) differ at most by a sign. Now, changing the orientation of a circle bundle changes the sign of its Euler class, so we may assume that the Euler numbers of \( \pi \) and of \( \S \to E_\alpha \) are the same by choosing the “right” orientation on \( \pi \).

From all that precedes, it follows that \( \S \to E_\alpha \) is smoothly isomorphic to \( \pi \), and the Lemma is proved. □

**Lemma 3.** Let \( L \) be a complex non-compact leaf of \( \mathcal{N} \). Then \( L \) admits a holomorphic compactification as a ruled surface of genus 1 by adding an elliptic curve \( E_\beta \).

**Proof of Lemma 3.** As explained above, the Lawson foliation has leaves \( \tilde{p}^{-1}(C_\theta) \), where \( C_\theta \) is the curve of \( S^1 \times (0,1) \) previously drawn. Now, we may view the open set given by the intersection of a curve \( C_\theta \) with \( S^1 \times (1/2,1) \) as an open set of a \( \mathbb{Z} \)-covering of the circular boundary \( S^1 \times \{1\} \). From this, we infer that a non-compact leaf \( L^* \) of \( K \times S^1 \times (1/2,1) \) is an open set of the \( \mathbb{Z} \)-covering of \( K \times S^1 \) obtained by unrolling the circle \( S^1 \) following the diagram

\[
\begin{array}{ccc}
C_\theta \cap S^1 \times (1/2,1) & \to & S^1 \\
\downarrow p & & \uparrow p|_\theta \\
L^* & \to & K \times S^1
\end{array}
\]

Notice that \( L^* \) being the intersection of a leaf \( L \equiv \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{D} \) with \( K \times S^1 \times (1/2,1) \), it is diffeomorphic to \( \mathbb{R}^2/\mathbb{Z}^2 \times S^1 \times \mathbb{D} \setminus \{0\} \).

Let us pass now to the complex world. Then, putting on \( L^* \) its pull-back structure, we obtain an open set of a holomorphic \( \mathbb{Z} \)-covering satisfying the diagram (see [M-V, p.921–922] for the holomorphic triviality of the pull-back bundle)

\[
(L^*)^{pb} \subset C^* \times E_\beta \to \S
\]

\[
\begin{array}{ccc}
C^* & \to & E_\alpha \\
\downarrow \pi & & \downarrow \pi
\end{array}
\]

Since \( L^* \) is asymptotic to \( C^* \times E_\beta \) at \( \infty \), the compactification Lemma ensures us that \( L^* \) can be compactified by adding an elliptic curve \( E_\beta \) in two ways (exactly as it is the case for \( C^* \times E_\beta \)). We choose the compactification as follows (cf the remark following the Compactification Lemma)

\[
(L^*)^c = (\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{D} \setminus \{0\}) \cup (\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{D})
\]

so that the complex model \( (L,J) \) of a non-compact leaf of \( \mathcal{N} \) admits a holomorphic compactification

\[
L^c = (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}) \cup (\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{D}) = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^2
\]

equipped with an extension \( J^c \) of \( J \).

Using the Enriques-Kodaira classification, we immediately obtain that \( (L^c,J^c) \) is a ruled surface of genus 1 or an elliptic surface. Since \( L^c \) is diffeomorphic to the rational surface formed by the product of an elliptic curve by \( \mathbb{P}^1 \), we conclude from [F-M, Chapter II, Theorem S.3. (i)] that we are in the first case. □

Here comes the most difficult Lemma.
Lemma 4. The interior leaves are all biholomorphic to $\mathbb{C} \times \mathbb{E}_\beta$

Proof of Lemma 4. Recall the map (see 1.2)

$$A : [x, y] \in \mathbb{R}^2/\mathbb{Z}^2 \rightarrow [x - 3y, y] \in \mathbb{R}^2/\mathbb{Z}^2.$$ 

Consider now the covering

$$\begin{array}{ccc}
\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{D}_4 \setminus \{0\} & \xrightarrow{c} & K \times \mathbb{D}_4 \setminus \{0\} \\
\downarrow q & & \downarrow \bar{p} \\
\mathbb{R} \times (0, 1) & \rightarrow & \mathbb{D}_1 \setminus \{0\} \cong \mathbb{S}^1 \times (0, 1)
\end{array}$$

whose deck transformation group is generated by the map

$$(x, y, t, z) \in \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{D}_4 \setminus \{0\} \xrightarrow{T} (A[x, y], t + 1, z) \in \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{D}_4 \setminus \{0\}.$$ 

The pull-back by $c$ of the Lawson foliation restricted to $K \times \mathbb{D}_4 \setminus \{0\}$ is given by the inverse image by $q$ of the following foliation of the strip $\mathbb{R} \times (0, 1)$

\[\text{invariant by the horizontal translation } (t, s) \mapsto (t + 1, s).\]

As usual, we consider the pull-back complex structure on this foliation.

We know from the previous Lemmas that the boundary leaf is biholomorphic to $\mathbb{E}_\beta \times \mathbb{C}^*$. Notice that the leaves we consider have now two ends, since we restrict the foliation to $K \times \mathbb{D}_4 \setminus \{0\}$. We are interested in the end which accumulates onto the boundary. The restriction of the map $c$ of the previous covering to $q^{-1}(\mathbb{R} \times (1/2, 1))$ sends the foliation restricted to $q^{-1}(\mathbb{R} \times (1/2, 1))$ to the initial foliation with leaves $C_0 \cap (1/2, 1)$. In other words, we may make use of the Compactification Lemma on this covering to partially compactify the leaves as open subsets of ruled surfaces of genus 1 as in Lemma 3; but in this new context, we may also make use of the uniform Compactification Lemma to partially compactify a tube of leaves in a uniform way. Observe that the elliptic curve $\mathbb{E}_\beta$ we add has to be a section for topological reasons.

Remark. It should be pointed out that the covering we use here is not a product foliated covering. Nevertheless, this covering has a special property that we will use. Indeed, as said before, the leaves have two ends, but in the quotient only one end accumulates onto the boundary. And what is more important, this end accumulates onto the boundary not only in the quotient space but also in the total space of the covering. The important consequence is that we may assume that the
tube of leaves we compactify contains the boundary leaf, even if we are not under the hypotheses of Proposition 5.

We thus form a compactified tube of leaves $\Xi \simeq \mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{D} \times [0, 1]$ such that

(i) the leaf $X_1$ (boundary leaf) is $\mathbb{C} \times E_\beta$.
(ii) the other leaves are open dense subsets of ruled surfaces with a holomorphic section $E_\beta$.
(iii) there is a CR-injection

$$
\begin{array}{ccc}
E_\beta \times [0, 1] & \longrightarrow & \Xi \\
\downarrow & & \downarrow \\
[0, 1] & \longrightarrow & [0, 1]
\end{array}
$$

Notice that the last point is a direct consequence of the fact that the compactification is uniform. Consider now the normal bundles of $E_\beta$ in each fiber. Point (iii) implies that they fit into a deformation family $\mathcal{Y} \to [0, 1]$ of line bundles over $E_\beta$ (in the sense of Section 3), all of which are topologically trivial. Using the fact that $T$ sends a leaf of $X$ CR-isomorphically to another leaf which is closer from the boundary, we have that the family $\mathcal{Y}$ satisfies the hypothesis of the Dumping Lemma, with boundary fiber $\mathcal{Y}_1$ isomorphic to $\mathbb{C} \times E_\beta$.

Going back to the Lawson foliation, all this means that an interior leaf is obtained from a ruled surface of genus 1 by removing a section with holomorphically trivial normal bundle. Therefore the ruled surface is a product and the interior leaves are biholomorphic to $\mathbb{C} \times E_\beta$. □

We are now in position to terminate the proof.

The open set $\text{Int } \mathcal{N}$ identifies with $K \times \mathbb{D}_4$. From 1.2, we have a $\mathbb{Z}$-covering

$$\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{D}_4 \longrightarrow K \times \mathbb{D}_4$$

whose deck transformations group is generated by

$$h : ([x, y], t, w) \longrightarrow (A[x, y], t + 1, w)$$

Consider the trivial foliation of $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{D}_4$ given by the level sets of the projection onto the $\mathbb{R}$-factor. This foliation is invariant by $h$ and, from what proceeds, descends on $K \times \mathbb{D}_4$ as the foliation used by Lawson before turbulization (recall that we made the assumption that the images of $p$ and of $s$ are the same). As this foliation is diffeomorphic to the final one, it is endowed with an integrable almost CR-structure by our assumption. We may take the pull-back of this structure and obtain thus an integrable almost CR-structure on $\mathbb{R}^2/\mathbb{Z}^2 \times \mathbb{R} \times \mathbb{D}_4$ such that the projection map is CR. In other words, it becomes a product foliated covering. Lemma 4 implies that the leaves of this covering are all biholomorphic to $\mathbb{C} \times E_\beta$. Moreover, since it is CR-isomorphic as a covering space to the covering used in the proof of Lemma 4 (in restriction to $\mathbb{D}_4 \setminus \{0\}$), the leaves can be compactified uniformly as $\mathbb{P}^1 \times E_\beta$. This compactified foliated covering is CR-trivial by Proposition 2. Since the compactification is uniform and since the automorphism group of $\mathbb{P}^1 \times E_\beta$ is transitive on the $\mathbb{P}_1$-factor, the initial product foliated covering is
CR-trivial. It follows now from Proposition 3 that the monodromy of this covering is well-defined as a biholomorphism of $C \times E_\beta$. Since the smooth monodromy is given by $(Id, A)$, this means that $C \times E_\beta$ should admit a biholomorphism smoothly isotopic to $(Id, A)$, and in particular which acts as the matrix $A$ on the first homology group $H_1(L, \mathbb{Z}) \simeq \mathbb{Z}^2$. However, it is an easy matter to check that every automorphism of $C \times E_\beta$ is isotopic to $(Id, S)$ where $S$ is an automorphism of $E_\beta$ and that this is not the case of $(Id, A)$. □

6. The set of integrable almost CR-structures of a smooth foliation

Let $(M, F^{diff})$ be the pair consisting of a smooth manifold $M$ without boundary and a smooth codimension-one foliation $F^{diff}$ on it. We call complex structure on $(M, F^{diff})$ the data of an integrable almost CR operator $J_{F}$ on the tangent bundle to the foliation $T F^{diff}$. Of course a complex structure corresponds to a foliation $F$ by complex manifolds of $M$ whose underlying smooth foliation is $F^{diff}$.

Two complex structures $J_F$ and $J_{F'}$ of $(M, F^{diff})$ are equivalent if there exists a foliated diffeomorphism of $(M, F^{diff})$ whose differential commutes with $J_F$ and $J_{F'}$. Two equivalent complex structures give rise to CR-isomorphic foliations by complex manifolds $F$ and $F'$.

Definition. We say that two complex structures $J_F$ and $J_{F'}$ are strongly equivalent if there exists a CR-isomorphism between $(M, J_F)$ and $(M, J_{F'})$ which does not exchange the leaves, i.e. which descends to the identity of the leaf space $M / F^{diff}$.

We are now in position to give the principal definitions of this Section. Assume from now on that $M$ is compact.

Definition. We call space of complex structures of $(M, F^{diff})$ the set of classes of strongly equivalent complex structures on $(M, F^{diff})$.

We denote it by $C(M, F^{diff})$.

Remark. In the same way, we may define a set $C(M, F^{top})$ by considering, modulo CR-isomorphisms which do not exchange the leaves, the set of classes of foliations by complex manifolds of $M$ homeomorphic to a fixed model $F^{top}$. We will see that it can be completely distinct from $C(M, F^{diff})$.

Remark. We could also define $C(M, F^{diff})$ (respectively $C(M, F^{top})$) as the set of classes of foliations by complex manifolds of $(M, F^{diff})$ modulo CR-isomorphisms, that is allowing also CR-isomorphisms which exchange the leaves. We will see in the second example of this Section that it may lead to different spaces.

Note that $C(M, F^{diff})$ can be empty even if each leaf of $F^{diff}$ can be endowed separately with a complex structure: by Theorem A, this is the case for the Lawson foliation.

Note that when non empty, $C(M, F^{diff})$ does not have in general a structure of a complex manifold; if it has, it may be finite or infinite dimensional.

We turn now to the definitions of deformation families and moduli spaces for foliations by complex manifolds (compare with [Su, p.138–139]).

Definition. Let $X$ be a complex manifold. Let $(V, J)$ be a smooth manifold endowed with a codimension-one integrable and Levi-flat almost CR-structure. Let $\pi : V \to X$ be a CR-submersion. Then, we say that $\pi$ is a deformation family of $(M, F^{diff})$ if, for every $x \in X$,
The manifold $\pi^{-1}(\{x\})$ is diffeomorphic to $M$.

The almost CR-structure $J$ of $V$ defines a codimension-one, integrable and Levi-flat almost CR-structure $J_x$ on the fiber $\pi^{-1}(\{x\})$.

There is a foliated diffeomorphism between $(V, F)$ and $M \times X$ endowed with the foliation $F_{\text{diff}}$ on each $M \times \{\text{pt}\}$ such that we have a diagram

\[
\begin{array}{ccc}
(V, F) & \xrightarrow{\approx} & M \times X \\
\downarrow \pi & & \downarrow \text{2nd projection} \\
X & \xrightarrow{id} & X
\end{array}
\]

In the case of a foliation having a compact leaf, there is a direct relationship between deformation families of the foliation and deformation families of the compact leaf.

**Proposition 6.** Let $\pi : V \to X$ be a deformation family of $(M, F_{\text{diff}})$. Let $L$ be a compact leaf of $F_{\text{diff}}$. Then $\pi$ induces a holomorphic deformation family $W \to X$ of $L$.

**Proof.** Consider the foliation by complex manifolds $G$ of $V$ induced by $J$. For $x$ varying in $X$, the union of the compact submanifolds of $\pi^{-1}(\{x\})$ corresponding to the compact leaf $L$ via point (iii) of the Definition is a leaf $W$ of $G$, and therefore a complex manifold. The restriction of $\pi$ to $W$ is a holomorphic submersion onto $X$ and defines thus a holomorphic deformation family of $L$. □

Given a deformation family $\pi : V \to X$ of $(M, F_{\text{diff}})$, there is a natural map $\alpha_\pi$ from $X$ to $\mathcal{C}(M, F_{\text{diff}})$: just define $\alpha_\pi(x)$ to be the point of $\mathcal{C}(M, F_{\text{diff}})$ corresponding to the CR-structure of $\pi^{-1}(\{x\})$.

**Definition.** Assume that $\mathcal{C}(M, F_{\text{diff}})$ can be endowed with the structure of a complex manifold $\mathcal{C}_C$. Let $i$ be the identification map between $\mathcal{C}(M, F_{\text{diff}})$ and $\mathcal{C}_C$. Then $\mathcal{C}_C$ is called a coarse moduli space if

(i) given any deformation family $\pi : V \to X$ of $(M, F_{\text{diff}})$, the natural map $i \circ \alpha_\pi$ is holomorphic as map into $\mathcal{C}_C$.

(ii) the pair $(\mathcal{C}_C, i)$ is unique up to composition with a biholomorphism of $\mathcal{C}_C$.

If a coarse moduli space exists, we denote it by $\mathcal{M}(M, F_{\text{diff}})$.

**Remark.** In the classical case of moduli spaces of compact complex manifolds, a coarse moduli space as well as the base of a deformation family are usually not assumed to be a manifold but only a complex analytic space (with some special properties). Here, we may easily modify our Definition of coarse moduli space in this way. However, the exact version of the singular deformation families in our case is not very clear: if we take in the definition of a deformation family a singular base $X$, then $(V, J)$ should also be singular. But $V$ is real, and cannot be taken real analytic but only smooth, since real analytic codimension-one foliations do not exist on a simply-connected compact manifold by a classical result of Haefliger [Hae].

Given a deformation family $\pi : V \to X$ and a holomorphic map between complex manifolds $f : Y \to X$, we may of course define a pull-back family $f^*\pi$. 

...
**Definition.** Assume that $(M, \mathcal{F}^{\text{diff}})$ has a coarse moduli space $\mathcal{M}(M, \mathcal{F}^{\text{diff}})$. Then it is a fine moduli space if there exists a deformation family $\Pi : W \to \mathcal{M}(M, \mathcal{F}^{\text{diff}})$ of $(M, \mathcal{F}^{\text{diff}})$ such that every deformation family $\pi : V \to X$ of $(M, \mathcal{F}^{\text{diff}})$ coincides with $\alpha_{\pi}^* \Pi$.

In the classical case of compact complex manifolds, it is too much to expect to have a fine moduli space and the corresponding Definition is not pertinent. Indeed, if a fine moduli space exists for a smooth compact manifold $X$, then every locally trivial holomorphic bundle with fiber diffeomorphic to $X$ has to be holomorphically trivial, since it defines a holomorphic family of deformations of the fiber which is the pull-back of a point in the moduli space. But this is very restrictive, and we do not know of any smooth compact manifold admitting a (non-empty) fine moduli space.

In our case, similarly, if a fine moduli space exists for $(M, \mathcal{F}^{\text{diff}})$, then every deformation family whose fibers are all biholomorphic must be trivial, that is biholomorphic to a product. For this reason, the examples we present in this article do not have a fine moduli space. This leads to the question.

**Question.** Does there exist a pair $(M, \mathcal{F}^{\text{diff}})$ which admits a (non-empty) fine moduli space?

Here are two examples.

**Example.** Consider the classical Reeb foliation $\mathcal{F}^{\text{Reeb}}$ of $S^3$ (see [Go]). Fix a Riemannian metric $\mu$ on $S^3$ and an orientation on $\mathcal{F}^{\text{Reeb}}$. The restriction of $\mu$ to a leaf $L$ defines a Riemannian metric on the oriented manifold $L$, that is a structure of Riemann surface on $L$. The foliation becomes in this way a foliation by Riemann surfaces. In particular $\mathcal{C}(S^3, \mathcal{F}^{\text{Reeb}})$ is not empty and there is a map between the set of conformal classes of Riemannian metrics on $S^3$ and $\mathcal{C}(S^3, \mathcal{F}^{\text{Reeb}})$. Conversely, fix a foliation by complex manifolds on $(S^3, \mathcal{F}^{\text{Reeb}})$. The integrable almost complex operator defines a conformal class of Riemannian metrics on each leaf and the property of transverse smoothness of the operator means that these conformal classes fit into a conformal class of Riemannian metrics on the whole $S^3$. Therefore this map is surjective.

Notice that the choice of the orientation of $\mathcal{F}^{\text{Reeb}}$ is not important. Indeed, let $J$ be an integrable almost CR-structure on $\mathcal{F}^{\text{Reeb}}$ respecting a fixed orientation. Then $-J$ defines an integrable almost CR-structure on $\mathcal{F}^{\text{Reeb}}$ respecting the other orientation. So the situation here is different from that of $S^5$.

The Reeb foliation is constituted by one compact leaf diffeomorphic to a real 2-torus and by non compact leaves, which are all diffeomorphic to $\mathbb{R}^2$. More precisely, the sphere $S^3$ is decomposed into the union of two solid tori $S^3 = S^1 \times \overline{D} \cup \overline{D} \times S^1$ each copy of these being endowed with a foliation (Reeb component) which can be described as follows (cf [M-V, Lemma 2]). Let

$$X = \mathbb{R}^2 \times [0, \infty) \setminus \{(0, 0, 0)\}$$

foliated by the level sets of the projection onto the second factor. Let

$$h : (x, y, t) \in X \mapsto (\alpha \cdot x, \beta \cdot y, d(t)) \in X$$
where $0 < \alpha^2 + \beta^2 < 1$ and where $d$ is a smooth function of $\mathbb{R}$ into $\mathbb{R}^+$, which is a contracting diffeomorphism of $\mathbb{R}^+$ and which is the identity on $\mathbb{R}^-$. Then the quotient of $X$ by the group generated by $h$ is a Reeb component.

Assume now that $(S^3, F^{Reeb})$ is endowed with a foliation by Riemann surfaces $\mathcal{F}$ and consider the induced integrable, Levi flat, almost CR-structure on each Reeb component. Then the previous covering $X$ becomes, by pull-back by the covering map, a product foliated covering. The compact leaf of $\mathcal{F}$ is an elliptic curve $\mathbb{E}_r$ and thus the boundary leaf of $X$ is $\mathbb{C}^*$. We claim that the non-compact complex leaves of $\mathcal{F}$ are all biholomorphic to $\mathbb{C}$. Indeed, by Corollary 3, we may assume that the previous structure is tame. Then the claim is proved in the first Example of Section 4.

Remark. Here is an alternative proof for this fact. Assume that there exists $t > 0$ such that $L_t$ is not biholomorphic to $\mathbb{C}$. By Riemann-Poincaré-Koebe Uniformization Theorem, it is thus biholomorphic to the open unit disk $\mathbb{D}$. In particular, there does not exist a sequence of holomorphic functions of the disk of radius $n$ into $L_t$ with derivatives in 0 bounded by below. Now, take an increasing sequence $\mathcal{D} = (D_n)$ of disks of radius $n$ in the boundary leaf of $X$ seen as $\mathbb{C}^*$. Note that $L_{t,n}(s)$ is biholomorphic to the unit disk for all $p$. As $d$ is contracting, this means that there exists a copy of $L_t$ as close to the boundary leaf as wanted. Note that the map

$$i_s : (x, y, 0) \in (\mathbb{R}^2 \times \{0\} \setminus \{(0, 0, 0)\}) \mapsto (x, y, s) \in L_s$$

injects smoothly the boundary leaf in the leaf $L_s$ for all $s \geq 0$. Using this map, we may embed smoothly the family $\mathcal{D}$ into $L_{t,n}(s)$ for all $p$ by embeddings with derivatives in 0 bounded by below. Although $i_s$ is not a priori holomorphic, as $i_0$ is the identity map of $\mathbb{C}^*$, then for $s$ sufficiently small, $i_s$ is quasi-conformal with a distortion factor which tends to one as $s$ tends to 0. As a consequence, the family $\mathcal{D}$ in $L_{t,n}(s)$ is a family of quasi-conformal disks with a distortion factor which tends to one when $p$ tends to $\infty$. Passing to the limit in $p$, we obtain holomorphic embeddings of each disk of $\mathcal{D}$ in $L_t \simeq \mathbb{D}$ with derivatives in 0 bounded by below. Contradiction.

Moreover, we know from the second Example of Section 4 that the interior part of the product foliated covering $X$ is biholomorphic to $\mathbb{C} \times (0, \infty)$.

We want now to prove that $X$ is CR-isomorphic to $\mathbb{C} \times \mathbb{R}^+ \setminus \{(0, 0)\}$. Let $A$ be an (open) annulus in $\partial X \simeq \mathbb{C}^*$. Observe that there exists a relatively compact open set $\mathcal{A}$, a real number $\epsilon$ and a map $\phi : \mathcal{A} \rightarrow A \times [0, \epsilon)$ such that

(i) The intersection of $\mathcal{A}$ with the leaf $X_t$ is a topological annulus $A_t$ (with $A_0 = A$) for $t < \epsilon$ and is empty for $t \geq \epsilon$.

(ii) The map $\phi$ is a CR-isomorphism.

This can be done by hand in this particular case, or may be deduced directly from [A-V, Proposition 2]. Let $D$ be the disk of $\mathbb{C}$ whose boundary is the exterior boundary of $A$. In the same way, let $D_t$ be the topological disk of $X_t$ whose boundary is the exterior boundary of $A_t$. Call $\mathcal{K}$ the union of $D \setminus \{0\}$ and $D_t$ for all $t$ in $(0, \epsilon)$. Note that the set $\mathcal{K}$ is open in $X$ and contains $\mathcal{A}$. Recall that we may uniformly compactify the family $X$ by adding one point at infinity to each leaf (cf Proposition 5). This gives a family $X^c$ whose interior leaves are all biholomorphic to $\mathbb{P}^1$ and whose boundary leaf is $\mathbb{C}$.
We perform now the following surgery on $X^c$: cut $K \setminus A$ and glue $D \times [0, \epsilon]$ along $A$ by $\phi$. We obtain in this way a new CR-manifold $\hat{X}^c$ such that 

(i) There is a CR-injection from $X^c \setminus K$ in $\hat{X}^c$.
(ii) The family $\hat{X}^c$ is a deformation family of $\mathbb{P}^1$ parametrized by $[0, \infty)$.

By Proposition 2, $\hat{X}^c$ is a trivial family. Hence there exists a CR-isomorphism $\psi$ between $X^c \setminus (K \setminus A)$ and $\mathbb{P}^1 \times [0, \infty) \setminus ((D \setminus A) \times [0, \epsilon))$. We may transfer the action through $\psi$. To be more precise, if, in a model $X = (\mathbb{R}^2 \times [0, \infty) \setminus \{(0,0)\}, J)$, the action defining the Reeb foliation is given by

$((z,t) \in \mathbb{R}^2 \times [0, \infty) \setminus \{(0,0)\} \mapsto (A_t(z), d(t)) \in \mathbb{R}^2 \times [0, \infty) \setminus \{(0,0)\})$

as usual, then let

$B_t = \psi_{d(t)} \circ A_t \circ \psi_t^{-1}$

Through the conjugation by $\psi$, the action on $\mathbb{P}^1 \times [0, \infty) \setminus ((D \setminus A) \times [0, \epsilon))$ is given by $(z,t) \mapsto (B_t(z), d(t))$ (where it is well-defined). Now, we may also see $(B_t)$ as a 1-parameter family of biholomorphisms defined in a neighborhood of $\infty$ in $\mathbb{P}^1$. Observe that these biholomorphisms fix $\infty$ and that this fixed point is contracting. Hence by Koenig's Theorem with parameters [Mi3, p.74-75], it may be linearized and the linearization map may be assumed smooth in the parameter. Hence, there exists a neighborhood $U$ of the infinite section $X^c \setminus X$ in $X^c$ and a CR-isomorphism $\chi$ sending $U$ to an open neighborhood of $\infty \times [0, \infty)$ in $\mathbb{P}^1 \times [0, \infty)$ which conjugates the previous action to a linear action

$((z,t) \in U' \subset \chi(U) \mapsto (\lambda(t) \cdot z, d(t)) \in U)$

where $(\lambda(t))_{t \in [0, \infty)}$ is a smooth family of complex numbers satisfying $0 < |\lambda(t)| < 1$, and $U'$ a well-chosen open set.

Since the action on $\mathbb{P}^1 \times [0, \infty) \setminus ((D \setminus A) \times [0, \epsilon))$ extends to $\mathbb{P}^1 \times [0, \infty)$, we may use it to extend the CR-isomorphism $\chi$ as a CR-isomorphism from $(X^{diff} \setminus \{0\} \times [0, \infty), J)$ to $\mathbb{C}^* \times [0, \infty)$ (we drop the infinite section; since the compactification is uniform and all the maps fix the section, this is not a problem), and finally from $X$ to $\mathbb{C} \times \mathbb{R}^+ \setminus \{(0,0)\}$ by Riemann's Theorem.

As a consequence of Corollary 2, two “complex” Reeb components are biholomorphic if and only if their compact leaves are biholomorphic (notice that the previously described CR-isomorphisms do not exchange the leaves). Therefore, to determine the set of almost CR-structures of $(S^3, F^{Reeb})$, we just have to determine which complex numbers can appear as modulus of the compact leaf.

We claim that any modulus $\tau$ can be obtained: to do this apply twice Lemma 2 of [M-V], once to obtain a complex Reeb component with boundary leaf biholomorphic to $E_\tau$ and then to obtain a complex Reeb component with boundary leaf biholomorphic to $E_{-\bar{\tau}}$. Glue together these two components via the biholomorphism switching meridians and parallels [M-V, Corollary of Lemma 1].

Observe that, if we set $F_\tau = (F^{Reeb}, J)$, then the "complex conjugate", that is $(F^{Reeb}, -J)$, is CR-isomorphic to $F_{-\bar{\tau}}$. This is because the complex conjugation on the leaves of the product foliated covering $\mathbb{C} \times \mathbb{R}^+ \setminus \{(0,0)\}$ descends to a biholomorphism between $F_\tau$ and $F_{-\bar{\tau}}$. 
As a conclusion of all that precedes, the set of complex structures of the Reeb foliation can be identified with $\mathbb{H}/\text{PSL}_2(\mathbb{Z}) \simeq \mathbb{C}$.

We claim that it is a coarse moduli space. To see this, take any deformation family $\pi : M \to X$ of $(\mathbb{S}^3, \mathcal{F}^{\text{Reeb}})$ and consider the natural map $\alpha_\pi : X \to \mathbb{C}$. Restricting $\pi$ to the union of the compact leaves of each fiber, we obtain a complex analytic deformation family of complex tori $\mathcal{E} \to X$ by Proposition 6. The map $\alpha_\pi$ can thus be seen as the modular function along the fibers of this family and is therefore holomorphic in the base space $X$. In other words, $(\mathbb{S}^3, \mathcal{F}^{\text{Reeb}})$ admits a coarse moduli space isomorphic to $\mathbb{C}$.

We also claim that there does not exist a fine moduli space. This is because the translations of the compact leaf extend to CR-isomorphisms of the foliation, [M-V, Lemma 2]. As a consequence, every holomorphic principal elliptic fiber bundle gives rise to a holomorphic family of deformations of $(\mathbb{S}^3, \mathcal{F}^{\text{Reeb}})$. Indeed, let $X \to B$ be such a bundle and let $E_\tau$ be its fiber. Let $(U_\alpha)_{\alpha \in A}$ be an open covering of $B$ and $g_{\alpha\beta} : U_\alpha \cap U_\beta \to E_\tau$ be a cocycle representing $X$. Then seeing now $E_\tau$ not as the group of translations of the fiber $E_\tau$ but as a subgroup of the group of CR-isomorphisms of $(\mathbb{S}^3, \mathcal{F}^{\text{Reeb}})$ endowed with the complex structure $\mathcal{F}_\tau$, we may construct from the cocycle $(g_{\alpha\beta})$ a holomorphic family of deformations of $(\mathbb{S}^3, \mathcal{F}^{\text{Reeb}})$ whose fibers are all isomorphic to $\mathcal{F}_\tau$. Assume that there exists a fine moduli space for $(\mathbb{S}^3, \mathcal{F}^{\text{Reeb}})$. As observed before, this implies that the previous family of deformations is CR-trivial. Then the bundle $X \to B$ has to be holomorphically trivial. But there exist principal holomorphic elliptic bundles which are not trivial, for example the surface $\mathcal{S}$ of this article. Contradiction.

**Example.** Let $\mathbb{T}^2$ denote the real 2-torus and let $X = \mathbb{T}^2 \times \mathbb{S}^1$. Consider on $X$ the smooth foliation $\mathcal{F}^{\text{diff}}$ by $\mathbb{T}^2$ given by the level sets of the projection onto the $\mathbb{S}^1$-factor. We claim that the set of integrable almost CR-structures of $(X, \mathcal{F}^{\text{diff}})$ is an infinite-dimensional Fréchet space.

To see that, recall the following construction of a complete deformation family for $\mathbb{T}^2$ (see [M-K, p.18]). Let $\mathbb{H}$ denote the Poincaré half-plane and consider the group of transformations of $\mathbb{H} \times \mathbb{C}$ given by

$$G = \{ (\tau, z) \in \mathbb{H} \times \mathbb{C} \mapsto (\tau, z + m + n\tau) \in \mathbb{H} \times \mathbb{C} \mid (m, n) \in \mathbb{Z}^2 \}.$$ 

This group acts freely and properly discontinuously on $\mathbb{H} \times \mathbb{C}$ so the quotient is a well-defined complex manifold $\mathcal{M}$. It can be checked that the natural projection $\mathcal{M} \to \mathbb{H}$ is a holomorphic submersion. This implies that $\mathcal{M}$ is diffeomorphic to $\mathbb{T}^2 \times \mathbb{R}^2$. The fiber over $\tau$ is the elliptic curve of modulus $\tau$.

Let $\alpha$ be a smooth map from $\mathbb{S}^1$ to $\mathbb{H}$. We can take the pull-back of the submersion $\mathcal{M} \to \mathbb{H}$ by $\alpha$. As this submersion is diffeomorphically trivial, the total space of the pull-back is diffeomorphic to $X$. 

$$\begin{CD}
X @>>> \mathcal{M} \\
@VVV \quad \downarrow \\
\mathbb{S}^1 @>\alpha>> \mathbb{H}
\end{CD}$$
This pull-back construction endows \((X, \mathcal{F}_{diff})\) with a compatible foliation by elliptic curves, let us denote it by \(\mathcal{F}_\alpha\).

Recall now that \(\mathbb{H}/\text{PSL}_2(\mathbb{Z}) \cong \mathbb{C}\) parametrizes the modulus of the elliptic curves. Let \(\pi\) be the natural projection from \(\mathbb{H}\) to \(\mathbb{H}/\text{PSL}_2(\mathbb{Z})\). Let \(V = \mathbb{T}^2 \times [0, 1]\) and let \(\mathcal{F}\) be a foliation by elliptic curves of \(V\) compatible with the natural projection onto \([0, 1]\). Then, there is a smooth map \(\bar{f}\) from \([0, 1]\) to \(\mathbb{H}/\text{PSL}_2(\mathbb{Z}) \cong \mathbb{C}\) given by the modular function on each fiber. And \(\bar{f}\) lifts to a smooth map \(f\) from \([0, 1]\) to \(\mathbb{H}\). Consider the pull-back foliation by elliptic curves on \(V\) defined as \(f^* \mathcal{F}\).

**Lemma.** The foliations \(\mathcal{F}\) and \(f^* \mathcal{M}\) are biholomorphic.

**Proof.** Since the family \(\mathcal{M} \to \mathbb{H}\) is complete, the foliations \(f^* \mathcal{M}\) and \(\mathcal{F}\) are locally biholomorphic. Hence \(f^* \mathcal{M}\) is locally biholomorphic to the quotient of \(\mathbb{C} \times [0, 1]\) by a smooth family of lattices \(\mathcal{L}_\alpha\). This implies that the universal covering of \(\mathcal{F}\) is a locally trivial CR bundle with fiber \(\mathbb{C}\) over the unit interval. But such a bundle is globally trivial and the previous biholomorphism is also global. \(\square\)

As a consequence of this Lemma, every foliation of \((X, \mathcal{F}_{diff})\) is biholomorphic to some \(\mathcal{F}_\alpha\). On the other hand, take two maps \(\alpha_1\) and \(\alpha_2\) from \(\mathbb{S}^1\) to \(\mathbb{H}\). If \(\mathcal{F}_{\alpha_1}\) is biholomorphic to \(\mathcal{F}_{\alpha_2}\), we have

\[\pi \circ \alpha_2 \equiv \pi \circ \alpha_1\]

Conversely, if this equality is satisfied, it lifts to a diffeomorphism \(f\) of \(X\) which maps the pull-back foliation \(\mathcal{F}_{\alpha_1}\) to the pull-back foliation \(\mathcal{F}_{\alpha_2}\) without exchanging the leaves. But, by the previous equality, the modulus of the leaf of \(\mathcal{F}_{\alpha_1}\) above \(z \in \mathbb{S}^1\) and the modulus of the leaf of \(\mathcal{F}_{\alpha_2}\) above \(z\) are the same and thus \(f\) is a biholomorphism.

Therefore \(\mathcal{C}(X, \mathcal{F}_{diff})\) can be identified with the space of smooth maps from \(\mathbb{S}^1\) to \(\mathbb{H}\), up to action of \(\text{PSL}_2(\mathbb{Z})\) on the target space. This is an infinite-dimensional Fréchet space [P-S]. Note that in this case there is no natural complex structure on this set of integrable almost CR-structures, and no coarse moduli space.

**Remark.** The space of smooth maps from \(\mathbb{S}^1\) to \(\mathbb{H}\), up to action of \(\text{PSL}_2(\mathbb{Z})\) on the target space is **different from** the loop space of \(\mathbb{H}/\text{PSL}_2(\mathbb{Z}) \cong \mathbb{C}\). In fact, a map from \(\mathbb{S}^1\) to \(\mathbb{H}/\text{PSL}_2(\mathbb{Z}) \cong \mathbb{C}\) lifts to a map from \([0, 1]\) to \(\mathbb{H}\) such that the images of \(0\) and \(1\) belong to the same \(\text{PSL}_2(\mathbb{Z})\)-orbit. By the pull-back construction, this defines a foliation by elliptic curves on the suspension of \(\mathbb{T}^2\) by some element of \(\text{PSL}_2(\mathbb{Z})\), and not on \(X\) (cf the Example after Proposition 3).

Note also that if we define \(\mathcal{C}(X, \mathcal{F}_{diff})\) as the set of foliations by complex manifolds of \(X\) diffeomorphic to \(\mathcal{F}_{diff}\) up to CR-isomorphisms, then it can be identified with the space of smooth maps from \(\mathbb{S}^1\) to \(\mathbb{H}\), up to action of \(\text{PSL}_2(\mathbb{Z})\) on \(\mathbb{H}\), and **up to reparametrization.**

**7. The set of integrable almost CR-structures of \(\mathcal{F}_{diff}\)**

Let \(X = K \times \mathbb{S}^1\). By Lemma 1, every complex structure on \(X\) is that of a primary Kodaira surface. It is obtained from a \(\mathbb{C}^*\)-bundle over an elliptic curve \(\mathbb{E}_\alpha\) of Chern number \(-3\) by taking the quotient of the fibers by a fixed homothety. Therefore, the set of pairwise non-biholomorphic complex structures on \(X\) - let us denote it by \(\mathcal{C}(X)\) - identifies naturally with

\[\{(\alpha, \beta, x) \mid \alpha \in \mathbb{H}/\text{PSL}_2(\mathbb{Z}), \beta \in \mathbb{H}/\text{PSL}_2(\mathbb{Z}), x \in \text{Pic}_{-3}(\mathbb{E}_\alpha)\}\]
where \( \text{Pic}_{-3}(E_{\alpha}) \) denotes the subset of the Picard group of \( E_{\alpha} \) constituted by elements corresponding to line bundles of Chern number \(-3\). Recall that it has a natural structure of an elliptic curve [Gu, §7-8].

**Remark.** The minus sign of the Chern number is not important. In fact, if \( X \rightarrow E_{\alpha} \) is an elliptic fibration of Chern number \(-3\), the automorphism \( z \rightarrow -z \) on the fibers sends \( X \) biholomorphically onto an elliptic fibration of Chern number \( 3 \). As a consequence, we could replace \( \text{Pic}_{-3}(E_{\alpha}) \) by \( \text{Pic}_3(E_{\alpha}) \) in the previous identification of \( C(X) \).

We will denote the corresponding manifolds by \( S(\alpha, \beta, x) \) and will say that such a complex \( X \) is of type \((\alpha, \beta, x)\). We denote by \( W(\alpha, x) \) the corresponding \( \mathbb{C}^* \)-bundle. By abuse of notation, \( \alpha \) (respectively \( \beta \)) will be considered as an element of \( \mathbb{H} \) and not only as a class of \( \mathbb{H}/\text{PSL}_2(\mathbb{Z}) \). Let \( \alpha \in \mathbb{H} \). Consider an embedding \( i_{\alpha} \) of \( E_{\alpha} \) into \( \mathbb{P}^2 \) as a cubic curve. To this embedding is associated the \( \mathbb{C}^* \)-bundle obtained by pull-back by \( i_{\alpha} \) of the bundle \( \mathbb{C}^3 \setminus \{(0,0,0)\} \rightarrow \mathbb{P}^2 \). This bundle is independent of the embedding. Indeed, two distinct embeddings are conjugated by an element of \( \text{PGL}_3(\mathbb{C}) \), which induces an isomorphism of the tautological bundles. We call this bundle the natural \( \mathbb{C}^* \)-bundle of \( E_{\alpha} \) and we denote it by \( \tilde{\alpha} \).

Recall (cf Section 1.3) that \( F_C \) is obtained by gluing a foliation of an open set \( \mathcal{N} \) and a foliation of its complement. We call interior part this set \( \mathcal{N} \) with its foliation and denote by \( S_1 \) the compact leaf which bounds it. On the other hand, the foliation of \( \mathbb{S}^5 \setminus \mathcal{N} \) contains another compact leaf. We denote it by \( S_2 \). The leaf \( S_2 \) separates \( \mathbb{S}^5 \setminus \mathcal{N} \) into an open set foliated by leaves diffeomorphic to \( W \) and an open set foliated by Milnor fibers. We call the first open set the collar and the second one the exterior part.

We turn now to the description of the set of integrable almost CR-structures of the foliation of \([M-V]\).

**Theorem B (Rigidity Theorem).**

(i) Let \( F \) be a foliation of \( \mathbb{S}^5 \) by complex surfaces diffeomorphic to \( F_C \). Then, the two compact leaves are of respective type \((\alpha, \beta, \tilde{\alpha})\) and \((\alpha, \beta', \tilde{\alpha})\), where \( \alpha, \beta \) and \( \beta' \) are any classes of \( \mathbb{H}/\text{PSL}_2(\mathbb{Z}) \).

(ii) Let \( F \) and \( F' \) be two foliations of \( \mathbb{S}^5 \) by complex surfaces diffeomorphic to \( F_C \). Then \( F \) and \( F' \) are biholomorphic if and only if the compact leaf \( S_1 \) of \( F \) is biholomorphic to the compact leaf \( S_1' \) of \( F' \) and the compact leaf \( S_2 \) of \( F \) is biholomorphic to the compact leaf \( S_2' \) of \( F' \).

The following Theorem is a realization result.

**Theorem C (Realization Theorem).** The set \( \mathcal{C}(\mathbb{S}^5, F_C^{\text{diff}}) \) identifies with

\[
\mathbb{H}/\text{PSL}_2(\mathbb{Z}) \times \mathbb{H}/\text{PSL}_2(\mathbb{Z}) \times \mathbb{H}/\text{PSL}_2(\mathbb{Z}) \simeq \mathbb{C} \times \mathbb{C} \times \mathbb{C}.
\]

A point of \( \mathcal{C}(\mathbb{S}^5, F_C^{\text{diff}}) \) is entirely determined by the images by the modular function of the moduli of the following elliptic curves:

(i) the common base of the two compact leaves.

(ii) the fiber of \( S_1 \).

(iii) the fiber of \( S_2 \).

In other words, the deformations of \( F_C \) are fixed by the deformations of the two compact leaves, and only three “parameters” among the six of \( \mathcal{C}(X) \times \mathcal{C}(X) \) may vary.
Remark. As noticed at the end of Section 1.3, the foliation of $\mathcal{F}_C^{\text{diff}}$ is not unique up to smooth isomorphisms, but only up to topological isomorphisms. As a consequence, it is easy to check that the set $\mathcal{C}(\mathbb{S}^5, \mathcal{F}_C^{\text{top}})$ is not finite dimensional.

Proof of Theorem B. Let $\mathcal{F}$ be a foliation of $\mathbb{S}^5$ by complex surfaces diffeomorphic to $\mathcal{F}_C$. 

Step 1: the interior part. We will prove that the non-compact leaves of the interior part are all biholomorphic and that their biholomorphism type is fixed by the type of $\mathcal{S}_1$. Let us denote by $(\alpha, \beta, x)$ the type of $\mathcal{S}_1$.

The general argument, which will be used here and in steps 2 and 3, is the following. Take a non-compact leaf $L$. As it spirals over $\mathcal{S}_1$, it (or an open set of it) admits a $C^\infty$-submersion over $\mathcal{S}_1$ (see Section 4). So we may define $L^{\text{pb}}$ and compare it to $L$. A priori they are different as complex manifolds. Now, we determine the exact biholomorphism type of $L^{\text{pb}}$ (notice that it depends only on the complex structure of $\mathcal{S}_1$) and show that it admits a holomorphic compactification. The compactification Lemma ensures us that $L$ admits also a holomorphic compactification. Then, making use of the Enriques-Kodaira classification, it is possible to give the biholomorphism type of this compactification and to prove that it is the same as the compactification of $L^{\text{pb}}$. Going back to $L$, we conclude that $L$ and $L^{\text{pb}}$ are biholomorphic and therefore that the complex structure of the non-compact leaves is fixed by the structure of the compact leaf.

Lemmas 1 and 2 are still valid. In particular, a non-compact leaf $L$ is diffeomorphic to a line bundle over $\mathbb{E}_{\alpha}$. Let $L^* \subset L$ be the complex manifold diffeomorphic to the associated $\mathbb{C}^*$-bundle. Let also $(L^*)^{pb}$ denote the associated pull-back complex structure. It follows from what precedes and from the construction that, topologically, $L^*$ is the $\mathbb{Z}$-covering of $\mathcal{S}_1$ obtained by unrolling the elliptic fibers so that 

$$(L^*)^{pb} = W_{(\alpha, x)}.$$ 

We may partially compactify $(L^*)^{pb}$ as the line bundle associated to $W_{(\alpha, x)}$ by adding an elliptic curve $\mathbb{E}_{\alpha}$. We denote this line bundle by $L_{(\alpha, x)}$. It is diffeomorphic to $L$. The compactification Lemma shows now that $L$ can be compactified by adding a copy of $\mathbb{E}_{\alpha}$ and is diffeomorphic to a $\mathbb{P}^1$-bundle over $\mathbb{E}_{\alpha}$. As usual, let $L^c$ be the corresponding compact surface.

Lemma 5. The compact surface $L^c$ is a ruled surface of genus one.

Proof. From the exact sequence in homotopy of the smooth $S^2$-fibration $L^c \to S^1 \times S^1$, we know immediately that the second homotopy group of $L^c$ is generated by the fiber and that the fundamental group of $L^c$ is $\mathbb{Z}^2$. Therefore $L^c$ is minimal with $b_1$ equal to two. Recall that the group of orientation preserving diffeomorphisms of the 2-sphere retracts onto $\text{SO}_2$ [Sm]; hence we may assume that our fiber bundle has structural group $\text{SO}_2$ and that it is the boundary of a disk bundle. Therefore $L^c$ is cobordant to zero and its Chern numbers and signature are zero. The Enriques-Kodaira classification proves thus that $L^c$ is a ruled surface of genus one, or a minimal properly elliptic surface or an hyperelliptic surface. This last case is impossible since the universal covering of an hyperelliptic surface is contractible whereas the universal covering of $L^c$ has non-zero second homotopy group.

By [B-H-P-V, Chapter IV, Theorem 2.7], we have $h^{1,0}(L^c) = 1$, i.e. there exists a global holomorphic 1-form on $L^c$. This implies that the Albanese torus is an elliptic curve $\mathbb{E}_\tau$ and the Albanese map is a surjective holomorphic map $\pi : L^c \to \mathbb{E}_\tau$. 

Assume now that $L^c$ is elliptic, that is that there exists a holomorphic map $p$ from $L^c$ to some Riemann surface $\Sigma$ whose generic fiber is an elliptic curve. Observe that the smooth vector fields of the base of the smooth fibration $L^c \to S^1 \times S^1$ can be lifted to smooth vector fields of $L^c$ by means of a connection. Hence the Euler characteristic of $L^c$ is zero, so this elliptic fibration is obtained from a torus bundle over $\Sigma$ by performing logarithmic transformations along some fibers (see [F-M, Chapter II, Proposition 7.2]). In particular, the fibers of $p$ are non-singular and form a family $\mathcal{F}$ of smooth elliptic curves which covers $L^c$.

Let $C \in \mathcal{F}$. The restriction of $\pi$ to $C$ is a holomorphic map between complex tori, that is a constant or a unramified covering of degree $d > 0$. By continuity of the degree of $\pi$ in the family $\mathcal{F}$, this does not depend on the curve $C$, that is one of the following statements is verified.

(i) For every $C \in \mathcal{F}$, the map $\pi$ restricted to $C$ is constant.
(ii) For every $C \in \mathcal{F}$, the map $\pi$ restricted to $C$ is a unramified covering of fixed degree $d$.

Assume (i). Then the fibers of $\pi$ contain elliptic curves, that is the generic fibers of $\pi$ are elliptic curves. In other words, $\pi$ is also an elliptic fibration, with base a complex torus. But this would imply that the second homotopy group of $L^c$ is zero and would give, as before, a contradiction. Indeed, let $i$ be a map from $S^2$ to $L^c$. The projection $\pi \circ i$ is homotopic to a constant since the target space is a torus. Now, we may define a connection on the elliptic fibration by taking a transverse field to the foliation it defines. This allows us to lift the previous homotopy to a homotopy between $i$ and a map whose image lies in a fiber. Since the fiber is a torus we may then homotope $i$ to a constant. The statement (ii) holds.

Remark. The same proof shows that, for any elliptic Seifert fibration $X \to B$ (that is with only multiple fibers), the $k$-th homotopy groups of $X$ and $B$ are equal (for $k > 1$).

As a consequence of (ii), the map $\pi$ is a holomorphic submersion, so, by Ehresmann’s Lemma, a locally trivial smooth fiber bundle. The exact sequence in homotopy of the bundle implies that the fiber is $\mathbb{P}^1$. The Theorem of Fischer-Grauert [F-G] implies that $\pi$ is a locally trivial holomorphic fiber bundle over $\mathbb{E}_\alpha$ with fiber $\mathbb{P}^1$, i.e. a ruled surface of genus one. □

A non-compact leaf $L$ is obtained from the ruled surface $L^c$ by removing an elliptic curve $E_\alpha$. This elliptic curve is a section of the $\mathbb{P}^1$-bundle since it is a section for its topological model and a holomorphic submanifold of $L^c$. In other words the ruled surface $L^c$ is a $\mathbb{P}$-bundle over $\mathbb{E}_\alpha$ with one holomorphic section, $\mathbb{P}$-bundle meaning locally trivial holomorphic bundle with fiber $\mathbb{P}^1$ and structural group $\text{PSL}_2(\mathbb{C})$; whereas $L$ is a $\mathbb{A}$-bundle over $\mathbb{E}_\alpha$, that is a locally trivial holomorphic bundle with fiber $\mathbb{C}$ and structural group $\text{Aff}(\mathbb{C})$, the one-dimensional affine group. We want to prove that $L$ is a line bundle over $\mathbb{E}_\alpha$. This is equivalent to showing that the $\mathbb{P}$-bundle $L^c$ admits another holomorphic section disjoint from the first one. Notice that $L$ is diffeomorphic to a line bundle of Chern number $-3$ over $\mathbb{E}_\alpha$. Consider an open covering $U_\alpha$ of $\mathbb{E}_\alpha$ and a cocycle

$$g_{\alpha \beta} : z \in U_\alpha \cap U_\beta \mapsto (w \to a(z)w + b(z)) \in \text{Aff}(\mathbb{C})$$
representing the affine bundle $L$. Then this cocycle is smoothly homotopic to the cocycle

$$z \in U_\alpha \cap U_\beta \mapsto a(z) \in \mathbb{C}^*$$

that is, the affine bundle is equivalent, as a smooth bundle, to the line bundle defined by the previous cocycle. Call this bundle the \textit{associated line bundle}. Notice that this line bundle has also Chern number $-3$.

The following result is a weak version of a Theorem of Atiyah.

\textbf{Theorem (see [At], Theorem 6.1).} Let $B$ be a $A$-bundle over an elliptic curve. If the degree of the associated line bundle is different from 0 and from $-1$, then $B$ is projectively equivalent to a $\mathbb{C}^*$-bundle.

Thanks to this Theorem, a non-compact leaf $L$ is a line bundle $L(\alpha, y)$. Nevertheless, step 1 is not yet finished, since it is not a priori clear that $y$ is independent of $L$ and that $y$ is equal to $x$. To prove this fact, we make use of the Dumping Lemma.

\textbf{Lemma 6.} Every non-compact leaf is biholomorphic to $L(\alpha, x)$.

\textbf{Proof of Lemma 6.} The argument used here will be referred hereafter as the \textit{dumping trick}. It uses of course the Dumping Lemma. Let

$$\tilde{X}^{\text{diff}} = \mathbb{R}^2 \setminus \{(0, 0)\} \times (\mathbb{R}^2 \times [0, \infty)) \setminus \{(0, 0, 0)\}$$

and define

$$T(x, y, u, v, t) = (x, y, (u, v) \cdot A, d(t))$$
$$S(x, y, u, v, t) = ((x, y) \cdot B, (u, v) \cdot (\Psi(x, y))^{-3}, t)$$

for $A$, $B$ and $\Psi$ matrices chosen so that the previous model forms exactly the smooth version of the complex $\mathbb{Z}_2$-covering used in [M-V, p.922] to foliate the interior part $N$ (see Section 1.3). Notice that $\text{Int } Y = \text{Int } (\tilde{X}^{\text{diff}})/(S)$ is diffeomorphic to $L^{\text{diff}} \times (0, \infty)$ and that $\partial Y$ is diffeomorphic to $W^{\text{diff}}$. Let $J(t)$ be the CR-structure on $\tilde{X}^{\text{diff}}$ coming from the foliation of $N$. The leaves are the level sets of the projection onto the $t$-factor. Let $J_0$ be the CR-structure induced on $Y$. Let $L_t$ be the complex leaf of $Y$ corresponding to the level set $t$. Remark that, for $t > 0$, the leaf $L_t$ is a line bundle $L(\alpha, x(t))$, whereas $L_0$ is $W(\alpha, x)$. Indeed, the $\langle T \rangle$-action onto $Y$ sends the leaf $L_t$ onto the leaf $L_{d(t)}$. As a consequence, there is no $\langle T \rangle$-action on a leaf $L_t$ for $t > 0$, thus the leaves of the interior of $N$ and the leaves of the interior of $Y$ are biholomorphic. Finally, the $\langle T \rangle$-action onto $L_0$ defines exactly the covering $W(\alpha, x) \rightarrow S_1$.

The uniform compactification Lemma tells us that we may compactify $\text{Int } Y$ as a deformation family of $\mathbb{P}^1$-bundles by adding $\mathbb{E}_\alpha \times (0, \infty)$ smoothly in $t$. Moreover, by Proposition 5, this uniform compactification can be extended to the boundary leaf. Consider now the union of the normal bundles of this smooth family of elliptic curves. We obtain a smooth deformation family of line bundles (in the sense of Section 3) over $\mathbb{E}_\alpha$ of fixed topological degree. Now this family satisfies the hypotheses of the Dumping Lemma, hence all these bundles are isomorphic. Since these bundles are indeed isomorphic to the bundle $L_0$, and since $L_0 = L(\alpha, x)$, we have that every $L_t$ is biholomorphic to $L(\alpha, x)$. □

Step 2 : the collar. The argument is very similar to that of the interior part. A non-compact leaf $L$ is diffeomorphic to $W$, that is to a $\mathbb{C}^*$-bundle over an elliptic
curve. Such a leaf has two ends, corresponding to the two ends of $\mathbb{C}^*$, its fiber. It spirals over $\mathcal{S}_1$ at one end and over $\mathcal{S}_2$ at the other end. Therefore $L^{pb}_1$ (respectively $L^{pb}_2$) is the $\mathbb{Z}$-covering of $\mathcal{S}_1$ (respectively of $\mathcal{S}_2$) obtained by unrolling the fibers. Denote by $(\alpha', \beta', x')$ the type of $\mathcal{S}_2$.

As in step 1, the use of the compactification Lemma on both ends ensures us that $L$ admits a holomorphic compactification by adding two elliptic curves $E_\alpha$ and $E_{\alpha'}$. This compactification $L^c$ is diffeomorphic to a ruled surface of genus one. By Lemma 5, it is a ruled surface of genus one and the two elliptic curves are holomorphic sections of this bundle, hence $\alpha = \alpha'$. Besides, $L$ is biholomorphic to some $W(\tilde{\alpha}, \tilde{x})$. Notice that, in this case, we do not need the result of Atiyah, since we know that $L^c$ is a ruled surface with two disjoint holomorphic sections.

Finally, we use the dumping trick of Lemma 6 for the product foliated covering described in [M-V, p.925–926] to prove that each non-compact leaf is in fact biholomorphic to $W(\alpha, x)$. The smooth model of this covering can be described as follows. Let

$$Z = W^{diff} \times [-1, 0] \subset \mathbb{R}^6 \times [-1, 0]$$

and let

$$g(x, t) = ((x_1, x_2) \cdot A, (x_3, x_4) \cdot A, (x_5, x_6) \cdot A, h(t))$$

where $A$ is a fixed matrix and where $h$ is a smooth diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ described in [M-V, p.925] whose only fixed points in $[-1, 0]$ are $-1$ and $0$.

Set $Z_0 = W^{diff} \times [-1, 0]$ and $Z_1 = W^{diff} \times (-1, 0]$. Then $Z_0 \to Z_0/\langle g \rangle$ and $Z_1 \to Z_1/\langle g \rangle$ are product foliated coverings of the collar minus $\mathcal{S}_1$ or minus $\mathcal{S}_2$ to which we apply the dumping trick. As a consequence, $L = W(\alpha, x) = W(\alpha, x')$ and $x = x'$. The two compact leaves differ only by the modulus of the fibers.

**Step 3 : the exterior part.** We consider the product foliated covering of the exterior part coming from the Milnor fibration. More precisely, setting

$$Y = P^{-1}(\mathbb{R}^+) \setminus \{(0, 0, 0)\} \subset \mathbb{R}^6$$

and considering $Y$ as a smooth manifold with boundary, we have a $\mathbb{Z}$-covering

$$s : Y \to S^5 \setminus \text{Int} (\mathcal{N})$$

satisfying the following commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{s} & S^5 \setminus \text{Int} (\mathcal{N}) \\
\downarrow^p & & \downarrow \\
\mathbb{R}^+ & \xrightarrow{\mathbb{Z}-\text{covering}} & S^1
\end{array}
$$

The deck transformation group is generated by

$$x \in Y \mapsto ((x_1, x_2) \cdot A, (x_3, x_4) \cdot A, (x_5, x_6) \cdot A)$$

where $A$ is the same fixed matrix as in step 2 (to be more precise, it is the real matrix form of the complex number $\lambda \omega$, see [M-V, p.924]).

**Remark.** It is important to keep in mind that this product foliated covering is not tame.
We may thus equip $Y$ with the pull-back CR-structure induced from this covering. The associated smooth foliation is given by the level sets of $P$ (cf [M-V, p.923–924]).

Let $L_t$ be the complex leaf diffeomorphic to $P^{-1}(\{t\})$. Assume $t \neq 0$. Then $s(L_t)$ is still biholomorphic to $L_t$ and descends as a leaf in $S^2 \setminus \mathcal{N}$ whose unique end spirals over $S_2$. There exists thus a closed set $F_t$ in $L_t$ such that $L_t \setminus F_t$ is diffeomorphic to a covering of $S_2 \simeq K \times S^1$. Indeed, by [Mi1, Lemma 6.1 and Theorem 5.11], it is diffeomorphic to $K \times \mathbb{R}$. This implies that $(L_t \setminus F_t)^{pb}$ is biholomorphic to $W$. By use of the compactification Lemma, we may thus compactify $L_t$ by adding an elliptic curve $\mathbb{E}_a$ and obtain a compact surface $L'_t$.

Remark. The use of the Compactification Lemma is possible even if the covering is not tame because it satisfies the weaker condition detailed in the remark after the Compactification Lemma.

On the other hand, $L_t$ is diffeomorphic to the affine cubic surface $P^{-1}(\{t\})$ of $\mathbb{C}^3$, so that $L'_t$ is diffeomorphic to a cubic surface in $\mathbb{P}^3$.

**Lemma 7.** The compact surface $L'_t$ is biholomorphic to the blow-up of $\mathbb{P}^2$ at 6 points or to the blow-up of the Hirzebruch surface $\mathbb{F}_2$ at 5 points.

**Proof.** Since $L'_t$ is diffeomorphic to a cubic surface, it is diffeomorphic to $\mathbb{P}^2_{\mathbb{R}}(6)\mathbb{P}^2_{\mathbb{R}}$, the blow-up of $\mathbb{P}^2$ at 6 points. We infer from this description that $L'_t$ has at most 6 disjoint embedded rational curves with self-intersection $-1$. The minimal model of $L'_t$, let us denote it by $X$, satisfies

$$c_1^2(X) \geq 3 \quad c_2(X) \leq 9 \quad \tau \geq -5$$

where $c_1$ and $c_2$ denote the Chern numbers of $X$ and $\tau$ the signature.

The Enriques-Kodaira classification tells us that $X$ is a rational surface or that $X$ is of general type. This last case is impossible, due to the following deep result of Friedmann and Qin.

**Theorem [F-Q].** A surface of general type cannot be diffeomorphic to a rational surface.

Therefore, $X$ is rational. There are two cases:

(i) the surface $X$ is $\mathbb{P}^2$ and $L'_t$ is the blow-up of $\mathbb{P}^2$ at 6 points.

(ii) the surface $X$ is a Hirzebruch surface $\mathbb{F}_a$ and $L'_t$ is the blow-up of $\mathbb{F}_a$ at 5 points.

Assume (ii). Then, since $\mathbb{F}_1$ is not minimal and since the blow-up of $\mathbb{P}^2$ at six points and the blow-up of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ at five points are biholomorphic, we assume that $a \geq 2$.

Assume that $a = 2r$ is even. Then $\mathbb{F}_a$ is diffeomorphic to $S^2 \times S^2$, hence a basis of topological cycles of dimension 2 of $\mathbb{F}_a$ is given by $C_1$ diffeomorphic to $\{pt\} \times S^2$ and $C_2$ diffeomorphic to $S^2 \times \{pt\}$. On the other hand, we may assume that $C_1$ is a fibre of the bundle $\mathbb{F}_a \to \mathbb{P}^1$ and that a basis of analytical cycles on $\mathbb{F}_a$ is given by $C_1$ and by its zero section $\tilde{S}$ homologous to $C_2 - rC_1$. Consider the blow-up map $p : L'_t \to \mathbb{F}_a$. A basis of analytic cycles of dimension 2 of $L'_t$ is then given by $S$, the pull-back by $p$ of $\tilde{S}$, by $E_1, \ldots, E_5$, the exceptional divisors, and by $F$ the pull-back by $p$ of a fiber of $\mathbb{F}_a$, not containing any of the blown up points. Let $k$ be the number of points to blow up which belongs to $p(S)$. Assume that $E_1, \ldots,$
$E_k$ are the corresponding exceptional divisors. Notice that $k$ may take any value between 0 and 5. One obtains easily the following table of intersection numbers.

\[
\begin{align*}
S \cdot S &= -a - k, \quad S \cdot F = 1, \quad S \cdot E_i = 1 \quad (i \leq k), \quad S \cdot E_i = 0 \quad (i > k) \\
F \cdot S &= 1, \quad F \cdot F = 0, \quad F \cdot E_i = 0 \quad (\text{for all } i)
\end{align*}
\]

We know that $\mathcal{L}_c$ contains an elliptic curve $E_\alpha$ such that the pair $(\mathcal{L}_c, E_\alpha)$ is diffeomorphic to the pair constituted by a non-singular cubic surface of $\mathbb{P}^3$ and by its section at infinity. Using this model, we see that $\mathcal{L}_c$ is diffeomorphic to the blow-up of $\mathbb{P}^2$ at six points. Then, another basis of topological cycles for $\mathcal{L}_c$ is given by $H$ diffeomorphic to a line in $\mathbb{P}^2$ not intersecting the exceptional divisors and to $D_1, \ldots, D_6$ diffeomorphic to the 6 exceptional divisors. In this basis, we also have that (see [Ha, V.4.8])

\[
E_\alpha = 3H - D_1 - \cdots - D_6
\]

where $=$ stands here and in the sequel of the proof for “is homologous”.

On the other hand, one checks easily that we may assume that

\[
D_1 = C_1 - E_1, \quad D_2 = C_2 - E_1, \quad H = C_1 + C_2 - E_1, \quad E_i = D_{i+1} \quad (\text{for } 2 \leq i \leq 5)
\]

where we still call $C_1$ and $C_2$ in $\mathcal{L}_c$ the pull-back by $p$ of $C_1$ and $C_2$ in $\mathbb{P}_a$.

Now, the elliptic curve $E_\alpha$ must have non-negative intersection with the basis of analytic cycles of $\mathcal{L}_c$. Taking into account that

\[
F = C_1 \quad S = C_2 - rC_1 - E_1 - \cdots - E_k
\]

this gives the following inequality

\[
E_\alpha \cdot S = 2 - 2r - k \geq 0
\]

Hence $r = 0$ or $r = 1$ and $k = 0$. But $r = 0$ means $a = 0$, a case that we have already excluded. So we have $r = 1$ and $k = 0$, i.e. $\mathcal{L}_c$ is the blow-up of $\mathbb{P}_2$ at five points not belonging to the zero-section $S$.

Assume now that $a = 2r + 1$ is odd. Exactly the same line of arguments leads to the inequality

\[
2 - (2r + 1) - k \geq 0
\]

hence $r = 0$ and $k = 0$. So $a = 1$, a case that we have already excluded. □

We would like to thank Lucy Moser for the following observation.

Remark. There really exists such an elliptic curve in the blow-up of $\mathbb{P}_2$ at five points, so we cannot exclude this case at this stage. Indeed, consider the curve $z^2y^2 - x^4 - z^4$ in $\mathbb{P}^3$. This is an elliptic curve with a unique singularity: a tacnode at infinity. The resolution of this singularity requires two successive blow-ups. In this way, we obtain a non-singular elliptic curve in the blow-up of $\mathbb{P}_1$ at one point belonging to the zero section. But this is the same as the blow-up of $\mathbb{P}_2$ at a point not belonging to the zero section this time. Observe that, in this last description, the elliptic curve does not intersect the exceptional divisor. Hence we may blow down to $\mathbb{P}_2$ keeping the non-singular elliptic curve. Finally blow-up this curve at five points.

We want to conclude that $\mathcal{L}_c$ is the blow-up of $\mathbb{P}_2$ at six points, and even more that it embeds in $\mathbb{P}^3$ as a non-singular cubic surface. We first need to collect one more fact about this surface.
Lemma 8. The surface $L_t^c$ admits an automorphism of order three.

Proof. Consider the product foliated covering $\text{Int} \ Y \to (0, \infty)$. By the uniform Compactification Lemma, the compactification of the leaves $L_t$ as $L_t^c$ can be assumed uniform. This gives a deformation family $\tilde{Y}$ over $[0, \infty)$ whose interior has compact leaves. For each $t > 0$, the corresponding leaf is $L_t^c$ and contains an elliptic curve $E_t$ biholomorphic to $E_\alpha$.

The deck transformation group identifies leaves of $\text{Int} \ Y$. In the presentation given at the beginning of step 3, the leaves $L_t$ and $L_{f(t)} = L_{\lambda^3 t}$ are CR-isomorphic and it extends as a CR-isomorphism between $L_t^c$ and $L_{f(t)}^c$. On the other hand, $\text{Int} \ Y$ is diffeomorphic to the suspension of $(L_t^c)^\text{diff}$ by a diffeomorphism of order three (this is the monodromy of the Milnor’s fibration used in the initial construction of the foliation $F_\mathbb{C}$). More precisely, on the smooth model, representing $L_t$ as the affine cubic surface $P - 1(t)$ for fixed $t$, the monodromy in restriction to $L_t$ is given by the map

$$(z_1, z_2, z_3) \in P - 1(t) \mapsto (\omega \cdot z_1, \omega \cdot z_2, \omega \cdot z_3) \in P - 1(t)$$

which extends to the projectivization of $P - 1(t)$ as the identity on the elliptic curve.

We want to prove that this diffeomorphism of order three is in fact an automorphism of order three of each leaf. This could be easily deduced from Proposition 3 if we knew that all the leaves are biholomorphic. But here we follow the inverse way: we want to use the existence of such an automorphism to deduce that all leaves of $\text{Int} \ Y$ are biholomorphic. To achieve our goal, we first need a better model for $\text{Int} \ Y$: we will construct a CR-map from $\text{Int} \ Y$ to $\mathbb{P}^3$ which is an embedding when restricted to an open and dense subset of any leaf $L_t^c$.

To avoid cumbersome repetitions in the sequel, we say that a leaf $L_t^c$ is of type (i) if it is the blow-up of $\mathbb{P}^2$ at 6 points, and of type (ii) if it is the blow-up of $F_2$ at five points. We thus a priori have in our deformation family $\text{Int} \ Y$ leaves of type (i) and leaves of type (ii).

From the computations of intersection numbers held in Lemma 7, it follows that in case (i), the six points belong to a cubic curve of $\mathbb{P}^2$, whereas, in case (ii), the five points belong the elliptic curve of $F_2$ described in the previous remark. In both cases, it follows that it is in the linear system of the anticanonical divisor $-K_t$ of $L_t^c$. This family of divisors gives rise to a family of line bundles over $L_t^c$ that we still call $-K_t$ or $E_t$.

We claim that, for all $t$, we have

$$\dim H^0(L_t^c, \mathcal{O}(E_t)) = 4.$$  

(we owe this computation to Laurent Bonavero).

Consider the short exact sequence [B-H-P-V, p.62]

$$0 \to \mathcal{O}_{L_t^c} \to \mathcal{O}_{L_t^c}(E_t) \to \mathcal{O}_{E_t}(E_t) \to 0.$$  

Since $H^1(L_t^c, \mathcal{O}_{L_t^c})$ is zero by 1-connectedness of $L_t^c$, and since $H^2(L_t^c, \mathcal{O}_{L_t^c})$ is zero by Serre duality, we obtain from the long exact sequence

$$H^1(L_t^c, \mathcal{O}_{L_t^c}(E_t)) = H^1(E_t, \mathcal{O}(E_t))$$  

and by Serre duality on $E_t$,
\[
H^1(L_t^c, \mathcal{O}_{L_t^c}(E_t)) = H^0(E_t, \mathcal{O}(-E_t))
\]
But $-E_t^2$ is $-3$ (cf the proof of Lemma 7) hence this last group is zero. On the other hand, we have also that $H^2(L_t^c, \mathcal{O}_{L_t^c}(E_t))$ is zero by Serre duality. Finally, by Riemann-Roch formula [G-H, p.472],
\[
\dim H^0(L_t^c, \mathcal{O}(E_t)) = \frac{E_t \cdot E_t - E_t \cdot K_t}{2} + \chi(L_t^c) = \frac{E_t \cdot E_t + E_t \cdot E_t}{2} + 1 = 4
\]
and we are done.

By [K-S, Theorem 2.1], it follows from the claim that there exists a smooth (in $t$) family $\Sigma(t) = (\sigma_1(t), \ldots, \sigma_4(t))$ of holomorphic sections of $-K_t$ which form a basis of $H^0(L_t^c, \mathcal{O}(E_t))$ for each $t$. Remark that, when $L_t^c$ is of type (i), then $E_t$ is ample by Nakai’s criterion; and when it is of type (ii), it is ample outside the zero section. In both cases, $\Sigma$ defines a CR map from $\text{Int} \ Y$ to $\mathbb{P}^3$ which is an embedding for fixed $t$ when restricted to an open and dense subset of $L_t^c$.

Now, fix $t > 0$. Recall that $L_t^c$ and $L_{f(t)}^c$ are biholomorphic. We notice that since the biholomorphism between these two manifolds comes from a biholomorphism between $L_t$ and $L_{f(t)}$ it sends $E_t$ onto $E_{f(t)}$. Hence $\Sigma(L_t^c)$ and $\Sigma(L_{f(t)}^c)$ are holomorphic basis of sections of the same divisor, hence their image in $\mathbb{P}^3$ are isomorphic and differ from a global automorphism of $\mathbb{P}^3$, let us call it $A$. Choose a smooth path of automorphisms $(A_s)$ between $A_t = A$ and $A_{f(t)} = Id$.

We now replace the family $\Sigma(s)$ for $s \in [t, f(t)]$ by $T(s) = A_s \Sigma(s)$. It has the property that it maps an open set of the deformation family $\text{Int} \ Y$ in $\mathbb{P}^3$ in such a way that $T(t)$ and $T(f(t))$ are equal. Observe that this implies that the CR-isomorphism of order three is in fact isotopic to a biholomorphism of $T(t)$. Since $T(t)$ is an embedding of an open and dense subset of $L_t^c$, we conclude that $L_t^c$ admits an automorphism isotopic to a self-map of order three.

Let us call $\phi^3_t$ this automorphism. Then $\phi^3_t$ is isotopic to the identity. But there is no non-zero holomorphic vector fields on $L_t^c$ (this is proved in [K, p.225] for $L_t^c$ of type (i), but the same arguments work for $L_t^c$ of type (ii)). Hence $\phi^3_t$ is the identity and $L_t^c$ admits an automorphism of order three. $\square$

We may finally state.

**Lemma 9.** Every surface $L_t^c$ embeds as a non-singular cubic surface $S_t$ of $\mathbb{P}^3$ and the restriction of the embedding $L_t^c \to S_t \subset \mathbb{P}^3$ embeds $\mathbb{E}_\alpha$ as a hyperplane section $H$ of $S_t$.

**Proof.** We know that there exists an automorphism $\phi_t$ of order three on $L_t^c$ whose fixed point set is an elliptic curve $\mathbb{E}_\alpha$. Assume that $L_t^c$ is of type (ii). Then this automorphism sends the five exceptional divisors of $L_t^c$ onto five disjoint rational curves with self-intersection $-1$. So it projects onto an automorphism $\psi_t$ of order three of $\mathbb{F}_2$. Now $\psi_t$ preserves the unique ruling of $\mathbb{F}_2$ and induces thus an automorphism $\chi_t$ of order three of $\mathbb{P}^1$, the base of the bundle $\mathbb{F}_2 \to \mathbb{P}^1$. A straightforward computation shows that $\chi_t$ is conjugated to the identity, the multiplication by $\omega$ or the multiplication by $\omega^2$ on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. Such a map has two fixed points (0 and $\infty$) or the whole $\mathbb{P}^1$ of fixed points.
Now the fixed points of $\psi_t$ are exactly the fixed points of its restriction to the fibers of $F_2 \to \mathbb{P}^1$ which are fixed by $\chi_t$. And, as before, the restriction to a fiber is conjugated to the identity, the multiplication by $\omega$ or the multiplication by $\omega^2$ on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. We infer from this description that the fixed point set of $\psi_t$ is four distinct points, or two points and a disjoint $\mathbb{P}^1$ or two disjoint $\mathbb{P}^1$ or the whole $F_2$. In particular, if $\psi_t$ is not the identity, this set is not connected. Blowing-up at five points and lifting $\psi_t$ will not give an automorphism with an elliptic curve as fixed point set. Contradiction. The surface $L^c_t$ is the blow-up of $\mathbb{P}^2$ at six points.

To conclude that $L^c_t$ is a non-singular cubic surface $S_t$ of $\mathbb{P}^2$, it is enough to prove that the six points are in general position.

Assume the contrary. The automorphism $\phi_t$ of $L^c_t$ defines an automorphism $\psi_t$ of order three of $\mathbb{P}^2$ with an elliptic curve $C$ as fixed point set. The six points must belong to $C$. Assume first that three of them lie on the same projective line $D$. Then $\psi_t$ leaves $D$ globally invariant and fixes three distinct points of $D$. This implies that the whole $D$ is fixed. Contradiction with the fact that the fixed point set of $\psi_t$ is an elliptic curve. Assume now that the six points lie on the same conic. Then $\psi_t$ leaves this conic globally invariant and fixes six distinct points of it. This implies that the whole conic is fixed. Once again, contradiction. The surface $L^c_t$ is a non-singular cubic surface $S_t$ of $\mathbb{P}^2$.

We would like to thank J. Kollar for the following proof.

The curve $E_0$ is homologous to a hyperplane section of $S_t$. By [Ha, V.4.8], the linear class of a divisor of a cubic surface is entirely determined by its intersection numbers with the 6 exceptional curves and with a hyperplane section. Therefore $E_0$ embeds in $S_t$ as a divisor which is linearly equivalent to the hyperplane section. Since the surface is embedded in $\mathbb{P}^3$ by its hyperplane class, a divisor linearly equivalent to the hyperplane section is a hyperplane section. □

As a consequence, $L_t$ is an affine cubic surface.

Remark. The proof that $L^c_t$ is a non-singular cubic surface could be shortened if we knew that the CR-structure of the family $\text{Int } \bar{Y}$ extends smoothly to the singular point of $\partial Y \simeq P^{-1}(\{0\}) \setminus \{0\}$. Since it also extends at infinity by Proposition 5, we would obtain a compactified deformation family whose boundary $L^c_0$ is a singular cubic surface of $\mathbb{P}^3$. Taking into account that $E_0 \subset L^c_0$ is very ample with space of holomorphic sections of dimension 4, it would follow directly from the claim proved in Lemma 8 and from a result of Schneider [Sc, p.174] (which generalizes [K-S, Theorem 2.1] used above) that the deformation family embeds in $\mathbb{P}^3$ near the boundary leaf. This would imply that every $L^c_t$ for small $t$ (and thus for all $t$) embeds in $\mathbb{P}^3$ through the linear system $E_t$ and satisfies the conclusion of Lemma 9.

We have now to prove that the complex structure on $L_t$ does not depend on $t$.

Let us say that $S_t$ has equation $P_t = 0$ in $\mathbb{P}^3$. The automorphism of order three of $L^c_t$ defines an automorphism of order three of $S_t$ which is the identity on the hyperplane section at infinity $H$. By [G-H, p.178], such an automorphism is the restriction to $S_t$ of an automorphism of $\mathbb{P}^3$. Straightforward computations show that it is projectively conjugated to

$$
\begin{pmatrix}
\omega & 0 \\
0 & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\omega^2 & 0 \\
0 & 1
\end{pmatrix}
$$

or

$$
\begin{pmatrix}
\omega & \omega^2 \\
0 & 1
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\omega^2 & \omega^2 \\
0 & 1
\end{pmatrix}
$$
Therefore, the cubic surface $S_t$ has the form

$$P_t = Q_t(x, y, z) + a_t w^3 = 0$$

where $Q_t$ is a homogeneous polynomial of degree 3 and $a_t$ a non-zero constant. Remark that we may assume that $a_t$ is one, making a change of coordinate in $w$ if necessary. It follows that this equation is entirely determined by the modulus of the elliptic curve at infinity. But this curve is $E_\alpha$ for all $t$. Therefore, $Q_t$ is a cubic equation of $E_\alpha$ and all the non-compact leaves are biholomorphic.

Moreover, the deformation family $\text{Int} \ Y \to (0, \infty)$ is CR-isomorphic to

$$(z, t) \in P_{h^{-1}}(t) \times (0, \infty) \longmapsto t \in (0, \infty)$$

Consider once again the compactified deformation family $\text{Int} \ \bar{Y} \to (0, \infty)$. As said before, we have a smooth injection of $E_\alpha \times (0, \infty)$ in $\text{Int} \ \bar{Y}$ by the uniform compactification Lemma. And it extends smoothly in 0 by Proposition 5. Take the union of the normal bundles of this family of elliptic curves. This gives a deformation family of line bundles over $E_\alpha$ (in the sense of Section 3) of fixed topological degree. It satisfies the hypotheses of the Dumping Lemma, hence they are all isomorphic. Of course, we already know that, except in 0, all these bundles are biholomorphic to the normal bundle of the hyperplane section $H$ in $S$, that is biholomorphic to the natural bundle associated to $E_\alpha$. We know now that it is also the case at 0. Since the (non-compactified) boundary leaf is $W(\alpha, x)$, we conclude that $x$ is in fact $\check{\alpha}$, the natural $\mathbb{C}^*$-bundle associated to $E_\alpha$. This finishes the third step.

Let us sum up the first three steps. Let $F$ be a foliation by complex surfaces diffeomorphic to $F_C$. Then

(i) the compact leaves $S_1$ and $S_2$ are of respective type $(\alpha, \beta, \check{\alpha})$ and $(\alpha, \beta', \check{\alpha})$.
(ii) the non-compact leaves of the interior part (respectively of the exterior part, of the collar) are all biholomorphic. We may thus talk of the non-compact leaf of the interior part (respectively of the exterior part, of the collar).
(iii) if $F'$ is another such foliation with $S_1'$ biholomorphic to $S_1$, and $S_2'$ biholomorphic to $S_2$, then the non-compact leaf of the interior part of $F'$ (respectively of the exterior part, of the collar) is biholomorphic to the non-compact leaf of the interior part of $F$ (respectively of the exterior part, of the collar).

To finish with the proof of Theorem B, we have to prove that, under the assumptions of (iii), there really exists a biholomorphism between $F$ and $F'$.

Step 4: construction of the biholomorphism. Consider now the exterior part of $F$, without the compact leaf $S_2$. Let us call it $E$. It admits a product foliated covering $Y$ fibered over $(0, \infty)$. As shown before, we may uniformly compactify the total space of this covering into a family of deformations of cubic surfaces by adding the same elliptic curve to each fiber. All these cubics are biholomorphic and the family is parametrized by $(0, \infty)$. It follows from Proposition 2 that this compactified covering is biholomorphic to a product $S \times \mathbb{R}$, for $S$ the fixed cubic surface described in the previous step. Therefore the product foliated covering $C$
is biholomorphic to \( A \times \mathbb{R} \), where \( A \) is the affine part of \( S \). We obviously have a commutative diagram of CR-maps

\[
\begin{array}{ccc}
C & \longrightarrow & A \times \mathbb{R} \\
\downarrow & & \downarrow \\
\mathcal{E} & \longrightarrow & \mathcal{E}
\end{array}
\]

i.e. \( \mathcal{E} \) is biholomorphic to a fixed model. This allows us to construct a biholomorphism \( \phi \) between \( \mathcal{E} \) (corresponding to \( \mathcal{F} \)) and \( \mathcal{E}' \) (corresponding to \( \mathcal{F}' \)).

We need an extension Lemma.

**Lemma 10.** The CR-isomorphism \( \phi \) extends as a biholomorphism between \( S_2 \) and \( S'_2 \).

**Proof.** Consider the compactified deformation family \( f : \bar{Y} \to [0, \infty) \) coming from \( \mathcal{F} \). The interior leaves are biholomorphic to a fixed cubic surface \( S \) say \( P_h(x, y, z, w) = Q_h(x, y, z) + w^3 = 0 \) in \( \mathbb{P}^3 \).

As a consequence, the interior \( \text{Int} \bar{Y} \) can be embedded in \( \mathbb{P}^3 \) as the family

\[
t \in (0, \infty) \quad \quad Q_h(x, y, z, w) + tw^3 = 0
\]

The deck transformation group of this compactified product foliated covering is generated by

\[
[x, y, z, w] \in \mathbb{P}^3 \mapsto [\omega \cdot x, \omega \cdot y, \omega \cdot z, \lambda^{-1}w] \in \mathbb{P}^3
\]

It follows from this description that the complex structure on the whole family (that is including \( t = 0 \)) is uniquely determined by the complex structure of one leaf. In fact, it is obtained by stretching the complex structure of \( S \) through the previous automorphism. The effect of this transformation is to make bigger and bigger a neighborhood of the infinite hyperplane section \( E_\alpha \) and has at end-point \( (t = 0) \) the normal bundle of \( E_\alpha \) in \( S \). This description characterizes uniquely the deformation family up to CR-isomorphism. It follows that the previous embedding extends to \( t = 0 \). Now, the same is true from the family \( \bar{Y}' \) coming from \( \mathcal{F}' \), hence the CR-isomorphism \( \phi \) extends to the boundary. \( \square \)

We claim now that the product foliated covering with boundary \( Z_1 \) of step 2 is CR-trivial (recall that we already know that its interior is CR-trivial). Since the boundary fiber \( \partial Z_1 = W \) is pseudo-convex, this foliated covering, as a deformation family, is pseudo-trivial by [A-V, Proposition 2]. This means that, given a relatively compact open set \( A \in \partial Z_1 \), there exists a relatively compact open set \( A \) in \( Z_1 \) and a CR-isomorphism between \( A \) and \( A \times [0, \epsilon) \) for some \( \epsilon > 0 \).

Consider \( W \) as a \( \mathbb{C}^* \) principal bundle over \( E_\alpha \) and let \( L \) be the associated line bundle. Let \( D \) be a tubular neighborhood of the zero section of \( L \) in \( L \). We may take a holomorphic disk bundle over \( E_\alpha \) as \( D \). Let \( A \) be an annulus bundle included in \( D \) (that is the difference between \( D \) and a smaller disk bundle included in \( D \)). It is a relatively compact open subset of \( W \). Use the property of pseudo-triviality with this \( A \). We obtain a relatively compact open set \( A \) in \( Z_1 \) and a CR-isomorphism
\( \chi \) sending \( \mathcal{A} \) to \( A \times [0, \epsilon) \subset L \times [0, \epsilon) \). Recall that the leaves of the interior of \( Z_1 \) may be compactified as copies of \( L \) by adding a zero section (this is also true at infinity, but we do not need this here). Call \( Z_1^c \) the corresponding family. Remark that \( \mathcal{A} \) intersects each fiber of each leaf of \( Z_1^c \) in an annulus. Let \( \mathcal{K} \) be the open set of \( Z_1^c \) whose intersection with a fiber of a leaf of \( Z_1^c \) is the corresponding disk (or punctured disk if the leaf is the boundary leaf). So \( \mathcal{K} \) intersects each interior leaf \( L_0 \) as a disk bundle \( K_t \) over \( \mathbb{E}_\alpha \), and the boundary leaf \( W \) as a punctured disk bundle \( (K_0)^* \).

We claim that \( \chi \) extends as a CR-isomorphism from \( \mathcal{K} \) to \( D \times [0, \epsilon) \). The extension is made in two steps. First, fix \( t \) and consider \( (K_t)^* \) (\( K_t \) minus the zero section). Since \( L \) has negative first Chern number, this means that the zero section of \( L \) admits a strictly pseudo-convex neighborhood. Hence we may assume that the boundary of \( K_t \) is strictly pseudo-convex. This allows to extend \( \chi \) to \( (K_t)^* \) (indeed, \( (K_t)^* \) admits a Stein completion by one point, which is obtained by blowing down the zero section of \( K_t \); and every CR-function on the strictly pseudo-convex boundary of \( K_t \) extends to this Stein completion). Notice that the extension is unique and is defined using the Bochner-Martinelli kernel. As a consequence, this extension is smooth in the parameter \( t \). Now, this extension must fix each fiber of \( (K_t)^* \), since it fixes them in restriction to \( A_t \). This implies that \( \chi \) is locally bounded near the zero-section of each \( (K_t) \) hence extends to \( K_t \) by Riemann’s Theorem. So finally, we obtain a CR-injection of \( \mathcal{K} \) into \( L \times [0, \epsilon) \). But this obviously means that the complex structure of the boundary leaf \( W \) extends to \( L \) in the family (that is in such a way that this extension is smooth in the transverse parameter). Using this, and recalling that we may also uniformly compactify all the leaves of \( Z_1^c \) at infinity by adding an elliptic curve, we CR-embed \( Z_1^c \) and thus \( Z_1 \) in a deformation family of \( E \), the compact ruled surface associated to \( L \). But by Proposition 2, such a family is CR-trivial, so, taking into account that the compactifications are uniform, we conclude that \( Z_1 \) is a CR-trivial product foliated covering.

Notice that the same line of arguments proves that \( Z_0 \) as well as the product foliated covering of step 1 (with its boundary) are CR-trivial.

But now, it follows from Corollary 2 that \( \phi \) extends to the product foliated covering \( Z_1 \) of step 2; and then as a biholomorphism between \( S^1 \) and \( S_0^1 \), also from Corollary 2 applied to the product foliated covering \( Z_0 \). And finally, applying once again Corollary 2, but this time on the product foliated covering of the interior part (with boundary), \( \phi \) extends to a global biholomorphism between \( \mathcal{F} \) and \( \mathcal{F}' \).

This concludes the proof of Theorem B. \( \Box \)

We pass now to the proof of Theorem C.

Proof of Theorem C. To prove Theorem C, the only way to realize a foliation by complex surfaces diffeomorphic to \( \mathcal{F}_C \) for every triple \( (J(\alpha), J(\beta), J(\beta')) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C} \). Here is the construction. Fix such a triple and fix \( \alpha, \beta \) and \( \beta' \) in \( \mathbb{H} \). The foliation of \( \mathcal{F}_C \) restricted to the interior part can easily be adapted: take for \( P \) a homogeneous polynomial such that the projectivization of the set \( P^{-1}\{0\} \setminus \{(0,0,0)\} \) is \( \mathbb{E}_\alpha \), and take as new definitions of \( T \) and \( S \) (compare with [M-V, p.922–923])

\[
T(z, u, t) = (z, \lambda \omega \cdot u, d(t))
\]

\[
S(z, u, t) = (z, \exp(2i\pi \alpha) \cdot \psi(z)^{-1} \cdot u, t)
\]

where \( \psi \) is the automorphism factor associated to the \( \mathbb{C}^* \)-bundle

\[
P^{-1}\{0\} \setminus \{(0,0,0)\} \rightarrow \mathbb{E}_\alpha
\]
and where $\lambda \omega = \exp(2i\pi \beta)$.

**Remark.** With this convention, we have $|\lambda| < 1$. Notice that it is different from the convention of [M-V], where we have $\lambda \omega = -\exp(2i\pi \beta)$, hence $|\lambda| > 1$.

Proceed now exactly as in [M-V, p.922-923].

We modify now the foliation of the exterior part and of the collar.

Set $\lambda' \omega = \exp(2i\pi \beta')$. Let $\mu$ and $\theta$ be smooth real maps satisfying

(i) For $t \leq -1$, we have $\mu(t) \exp(i\theta(t)) = \lambda$.
(ii) For $t \geq 0$, we have $\mu(t) \exp(i\theta(t)) = \lambda'$.

Notice that $\mu(0) = |\lambda'| < 1$. Let $h: \mathbb{R} \to \mathbb{R}$ be a diffeomorphism satisfying

(i) $h(t) = t$ for $t \leq -1$.
(ii) $h$ coincides with the map $t \mapsto \frac{t}{3t+1}$ on $[0, \infty)$.
(iii) The only fixed points of $h$ are 0 and $(-\infty, -1]$.

Such a map is constructed in [M-V, p.925]. Define

$$
\phi: t \in \mathbb{R} \mapsto \begin{cases} 
\mu(0)^{1/t} \exp(i\theta(0)/t) & \text{if } t > 0 \\
0 & \text{else.}
\end{cases}
$$

Notice that $\phi$ is $C^\infty$. Let

$$
g: (z, t) \in \mathbb{C}^3 \times \mathbb{R} \mapsto P(z) - \phi(t) \in \mathbb{C}.
$$

Set

$$
\Xi = g^{-1}(\{0\}) \setminus \{(0, 0, 0) \times \mathbb{R}\}
$$

and

$$
L_t = \{(z, t) \in \Xi \mid P(z) = \phi(t)\}.
$$

Finally, define

$$
G: (z, t) \in \Xi \mapsto (\mu(t) \exp(i\theta(t))\omega \cdot z, h(t)) \in \Xi.
$$

Then $G$ sends the leaf $L_s$ onto the leaf $L_{h(s)}$ and the quotient of $\Xi$ by $\langle G \rangle$ gives a foliation by complex surfaces of $S^5 \setminus N$, as in [M-V]. There are two compact leaves and it is straightforward to see that they are primary Kodaira surfaces of type $(\alpha, \beta, \tilde{\alpha})$ and $(\alpha, \beta', \tilde{\alpha})$.

**Lemma 11.** Consider the foliation by complex surfaces of $S^5$ described just above. Then, it is diffeomorphic to $F_C$.

**Proof.** Due to the minor changes in the definitions of $T$ and $S$, the new foliation of the interior part is clearly diffeomorphic to $F_C$ restricted to $\text{Int}(N)$.

On the other hand, it follows from the previous proof that:

(i) The (open) exterior part fibers over the circle with same fiber (up to diffeomorphism) and same monodromy (given by multiplication by $\omega$ on $P^{-1}(1)$) as $F_C$.
(ii) The collar is diffeomorphic to $W^{diffeo} \times (0, 1)$.

We deduce from these two facts that the new foliation is homeomorphic to $F_C$ from one hand, and from the other hand diffeomorphic to it outside the two compact leaves. To finish with, it is enough to check that the holonomies of the two compact leaves are conjugated. But, in the two cases, they are given by the functions $d$ and $h$ (see also the remark at the end of Section 1). □

This completes the proof of Theorem C. □
8. The moduli space of $\mathcal{F}^{\text{diff}}_C$

**Theorem D.**

(i) The set $\mathcal{C}(S^5, \mathcal{F}^{\text{diff}}_C)$ identified with $\mathbb{C}^3$ is a coarse moduli space for $(S^5, \mathcal{F}^{\text{diff}}_C)$.

(ii) There does not exist a fine moduli space for $(S^5, \mathcal{F}^{\text{diff}}_C)$.

**Proof.** The proof is very similar to that given in Section 5 for the Reeb foliation.

Take any deformation family $\pi : M \rightarrow X$ of $(S^5, \mathcal{F}^{\text{diff}}_C)$ and consider the natural map $\alpha : X \rightarrow \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. The three components of this map may be thought of as the modular function on the common base of the compact leaves, on a fixed fiber of the first compact leaf and on a fixed fiber of the second compact leaf. Indeed, such a deformation family $\mathcal{M}_1 \rightarrow X$ (respectively $\mathcal{M}_2 \rightarrow X$) induces a deformation family $\mathcal{D}_1$ (respectively $\mathcal{D}_2$) of the first compact leaf $S^4_1$ (respectively the second compact leaf $S^4_2$) by Proposition 6. On the other hand, $\mathcal{D}_1 \rightarrow X$ (respectively $\mathcal{D}_2 \rightarrow X$) is a locally trivial smooth fiber bundle with structural group $\text{Diff}^+(S^4_1)$ (respectively $\text{Diff}^+(S^4_2)$ - the + meaning orientation-preserving).

Now, by Lemma 2, an orientation-preserving diffeomorphism of $S^4_1$ (respectively $S^4_2$) is in fact a bundle isomorphism. Therefore, $\mathcal{D}_1 \rightarrow X$ (respectively $\mathcal{D}_2 \rightarrow X$) induces a deformation family of the base of $S^4_1$ (respectively $S^4_2$). This allows us to identify $\alpha$ to a triple of modular functions and to conclude that $\mathbb{C}^3$ is a coarse moduli space for $(S^5, \mathcal{F}^{\text{diff}}_C)$.

Assume now that there exists a fine moduli space. Then every deformation family of $(S^5, \mathcal{F}^{\text{diff}}_C)$ with all fibers biholomorphic to $\mathcal{F}_C$ is biholomorphic to a product. By restriction to the base of a fixed compact leaf, such a deformation family yields a family of complex structures on the smooth torus with all leaves biholomorphic to $E_\omega$. Of course, this family has to be holomorphically trivial. Now, there exist non holomorphically trivial families of complex structures with all leaves biholomorphic to $E_\omega$. This is due to the existence of a non-trivial automorphism $\alpha$ of order three on $E_\omega$. For example, one may glue two copies of $\mathbb{C} \times E_\omega$ along $\mathbb{C}^* \times E_\omega$ with

$$h : (z, w) \in \mathbb{C}^* \times E_\omega \mapsto (1/z, \alpha(w)) \in \mathbb{C}^* \times E_\omega$$

to obtain such an example. Assume now that $\alpha$ extends to a biholomorphism $A$ of $\mathcal{F}_C$. Then, in the same way, glue two copies of $\mathbb{C} \times S^5$ endowed with the foliation $\mathcal{F}_C$ on each $S^5$ along $\mathbb{C}^* \times S^5$ by

$$H : (z, w) \in \mathbb{C}^* \times S^5 \mapsto (1/z, A(w)) \in \mathbb{C}^* \times S^5.$$ 

This gives a deformation family of $(S^5, \mathcal{F}^{\text{diff}}_C)$ with all fibers biholomorphic to $\mathcal{F}_C$ and such that its restriction to the base of a fixed compact leaf is non-trivial. Contradiction.

To finish with, it is thus enough to prove the following Lemma.

**Lemma 12.** The foliation $\mathcal{F}_C$ admits a biholomorphism $A$ of order three which extends an automorphism of order three $\alpha$ on the the base of the compact leaves.

**Proof.** We just have to define this automorphism on the product foliated coverings used to foliate $\mathcal{N}$ and $S^5 \setminus \mathcal{N}$. We use the notations of [M-V]. Start with the automorphism

$$\alpha : [z_1, z_2, z_3] \in E_\omega = \{ [z] \in \mathbb{P}^2 \mid z_1^3 + z_2^3 + z_3^3 = 0 \} \mapsto [z_1, \omega \cdot z_2, \omega^2 \cdot z_3] \in E_\omega.$$
Consider first the product foliated covering $\tilde{X}/\langle S \rangle$. By Proposition 2, it admits a compactification biholomorphic to $L^c \times [0, \infty)$, where $L^c$ is the compactification of $W$ as a ruled surface. Therefore, $\tilde{X}/\langle S \rangle$ is biholomorphic to $L \times [0, \infty) \setminus \{s_0 \times 0\}$, where $s_0$ denotes the zero section of the line bundle $L$. By definition of $L$, the automorphism $\alpha$ extends to an automorphism of order three of $L$ and of $W = L \setminus s_0$. Therefore, we may define a biholomorphism of order three on $\tilde{X}/\langle S \rangle$. It is easy to see that it commutes with $T$ and thus descends to a biholomorphism of order three of $(N, F_C)$. For the exterior, just take the biholomorphism

$$(z_1, z_2, z_3, t) \in C^3 \times \mathbb{R} \mapsto (z_1, \omega \cdot z_2, \omega^2 \cdot z_3, t) \in C^3 \times \mathbb{R}$$

and verify that it preserves $\Xi$ and commutes with $G$. It is now straightforward to check that this biholomorphism glue with the previous one and define $A$. This completes the proof of Theorem D. □

9. Concluding remarks

In this paper, we do not speak about the foliated analogue of Kodaira-Spencer theory of small deformations of compact complex manifolds (see [K-S], [M-K]). As noticed in Section 5, the notion of fine moduli space is not a good concept in the theory of compact complex manifolds. The right one is that of versal deformation space, that is of local moduli space. One may thus expect that the same is true for foliations by complex manifolds.

Let $F$ be a foliation by complex manifolds on $X$. Define $\Theta_F$ as the sheaf of germs of smooth vector fields on $X$ which are tangent to $F$ and holomorphic when restricted to any leaf of $F$. As in the classical case, the cohomology group $H^1(X, \Theta_F)$ should describe the infinitesimal deformations of $(X, F)$. Of course, it can be infinite-dimensional as it is the case for the second example of Section 5. Nevertheless, in the case it is finite-dimensional, it is quite possible that the classical Theorems could be adapted with essentially the same proof. For example, the following result should be true: if $H^2(X, \Theta_F) = 0$, then there exists a germ of deformation family of $(X, F_{\text{diff}})$ of dimension $H^1(X, \Theta_F)$ centered in $(X, F)$ which contains all “sufficiently small” deformations of $(X, F)$. It should be interesting to develop such a theory and to apply it to deformations of CR-fiber bundles and to deformations of foliations given by suspension.
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