On one-dimension quasilinear wave equations with
null conditions

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Abstract

In this paper, we show that one-dimension systems of quasilinear wave equations with null conditions admit global classical solutions for small initial data. This result extends Luli, Yang and Yu’s seminal work [G. K. Luli, S. Yang, P. Yu, On one-dimension semi-linear wave equations with null conditions, Adv. Math. 329 (2018) 174–188] from the semilinear case to the quasilinear case. Furthermore, we also prove that the global solution is asymptotically free in the energy sense. In order to achieve these goals, we will employ Luli, Yang and Yu’s weighted energy estimates with positive weights, introduce some space-time weighted energy estimates and pay some special attentions to the highest order energies, then use some suitable bootstrap process to close the argument.

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1 Introduction and main result

In [17], for Cauchy problems of semilinear wave equations with null conditions in one space dimension, Luli, Yang and Yu proved the global existence of classical solutions with small initial data (a former result can be found in [18]). This result can be viewed as a one-dimension and semilinear analogue of the pioneering works Klainerman [11] and Christodoulou [3] for the global existence of classical solutions for nonlinear wave equations with null conditions in three space dimensions, and of Alinhac [2] for the case of two space dimensions.

Global existence of small solutions to nonlinear wave equations with null conditions has been a subject under active investigation for the past four decades. The approach to understand the small data problem is based on the decay mechanism of linear waves. It is well-known that the decay rate is \((1 + t)^{-\frac{d}{2}}\), for \(d\)-dimension linear waves. Thus, when \(d \geq 4\), small-data-global-existence type theorems hold for generic quadratic nonlinearities, since the decay rate is integrable in time. See Klainerman [8, 10]. However, in \(\mathbb{R}^{1+3}\), the slower decay rate \((1 + t)^{-1}\) just barely fails to be integrable in time. Hence, the solution may blowup at finite time. See John [6]. In order to avoid the formation of singularities, Klainerman [9] introduced the celebrated null conditions, then for three dimensional quasilinear

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wave equations with quadratic nonlinearities satisfying null conditions, based on the decay rate for linear waves and some special cancelations provided by such structural conditions, Klainerman [11] and Christodoulou [3] independently proved the global existence theorems. As for $\mathbb{R}^{1+2}$, linear waves only admit decay rate $(1 + t)^{-1/2}$, which is far from integrable in time. Nevertheless, for a class of two dimensional quasilinear wave equations, based on such decay rate for linear waves and the null conditions imposed on quadratic and cubic nonlinearities, Alinhac [2] can show the global existence theorem. For more detailed history on small data theory of nonlinear wave equations and explanations on the concept of null conditions, we refer the reader to Luli, Yang and Yu [17].

As mentioned before, the proofs in [2], [3], [8], [10] and [11] in high space dimension case are all based on the decay mechanism of linear waves. However, in one space dimension case waves do not decay, and any nonlinear resonance (even arbitrarily high order) can lead to finite time blowup. Nevertheless, for one-dimension semilinear wave equations, Luli, Yang and Yu [17] can prove that small data still lead to global solutions if the null condition is satisfied. As pointed in [17], different from the high space dimension case, the mechanism for the global existence in one-dimension case is the interaction of waves with different speeds, which will lead to the decay of nonlinear terms. In order to display this mechanism, Luli, Yang and Yu developed a kind of weighted energy estimate with positive weights.

In this paper, we will consider the case for quasilinear wave equations, which arise naturally in many physical fields. For this purpose, instead of the standard Cartesian coordinates $(t, x)$, we will mainly use the null coordinates

\[ \xi = \frac{t + x}{2}, \quad \eta = \frac{t - x}{2}. \]  

(1.1)

We have the null vector fields

\[ \partial_\xi = \partial_t + \partial_x, \quad \partial_\eta = \partial_t - \partial_x, \]  

(1.2)

and also denote briefly $u_\xi = \partial_\xi u$ and $u_\eta = \partial_\eta u$.

It is well known that the one-dimension linear wave equation in the null coordinates $(\xi, \eta)$ can be written as $u_{\xi\eta} = 0$. We will treat some quasilinear perturbation of it. Consider the following one-dimension system of quasilinear wave equations

\[ u_{\xi\eta} = A_1(u, u_\xi, u_\eta)u_\xi u_\eta + A_2(u, u_\xi, u_\eta)u_{\xi\xi} + A_3(u, u_\xi, u_\eta)u_{\eta\eta} + F(u, u_\xi, u_\eta), \]  

(1.3)

where the unknown function $u = u(t, x) : \mathbb{R}^{1+1} \rightarrow \mathbb{R}^n$, for $i = 1, 2, 3$, $A_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n\times n}$ are given smooth and matrix valued functions, and $F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given smooth and vector valued function. Moreover, we will always assume that $A_i \ (i = 1, 2, 3)$ are symmetric.

We call that the system (1.3) satisfies the null condition, if near the origin in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$, it holds that

\[ A_1(u, u_\xi, u_\eta) = O(|u| + |u_\xi| + |u_\eta|), \]  

(1.4)

\[ A_2(u, u_\xi, u_\eta) = O(|u_\eta|), \]  

(1.5)

\[ A_3(u, u_\xi, u_\eta) = O(|u_\xi|), \]  

(1.6)

\[ F(u, u_\xi, u_\eta) = O(|u_\xi||u_\eta|). \]  

(1.7)
Inspired by Luli, Yang and Yu’s seminal result \cite{yang09} in the semilinear case, it is natural to conjecture that the Cauchy problem of one-dimension system of quasilinear wave equations (1.3) satisfying null conditions (1.4)–(1.7) admits a unique global classical solution for small initial data. The main aim of this paper is to verify this conjecture. Furthermore, we also want to clarify the asymptotic behavior of the global solution.

Now we introduce some vector fields and energies used in the following part of this paper. We will use

\[ Z = (\partial_\xi, \partial_\eta) \]  

(1.8)
as the commuting vector fields. For a multi-index \( a = (a_1, a_2) \), set

\[ Z^a = \partial^{a_1}_\xi \partial^{a_2}_\eta \]  

(1.9)
and \( |a| = a_1 + a_2 \).

Following Luli, Yang and Yu \cite{yang09}, we will use the following weighted energy with positive weights

\[ E_1(u(t)) = \|\langle \xi \rangle^{1+\delta} u_\xi \|^2_{L^2_x(\mathbb{R})} + \|\langle \eta \rangle^{1+\delta} u_\eta \|^2_{L^2_x(\mathbb{R})}. \]

(1.10)
where \( \langle \cdot \rangle = (1 + |\cdot|^2)^{1/2} \). Then we use

\[ E_2(u(t)) = \sum_{|a|=1} E_1(Z^a u(t)) \]  

(1.11)
to denote the second order energies. As for the third order (highest order) energies, we have to distinguish some “mixed derivatives” from others. Specifically speaking, we will employ the following third order weighted energies

\[ E_3(u(t)) = \sum_{|a|=2} \|\langle \xi \rangle^{1+\delta} Z^a u_\xi \|^2_{L^2_x(\mathbb{R})} + \|\langle \eta \rangle^{1+\delta} Z^a u_\eta \|^2_{L^2_x(\mathbb{R})} \]  

(1.12)
and

\[ \tilde{E}_3(u(t)) = \sum_{|a|=2} \|\langle \xi \rangle^{1+\delta} Z^a u_\xi \|^2_{L^2_x(\mathbb{R})} + \|\langle \eta \rangle^{1+\delta} Z^a u_\eta \|^2_{L^2_x(\mathbb{R})} \]  

(1.13)
We also set

\[ E(u(t)) = E_1(u(t)) + E_2(u(t)) + E_3(u(t)). \]  

(1.14)

Inspired by Alinhac \cite{alinhac01} and Lindblad and Rodnianski \cite{lindblad01}, based on (1.10), we further introduce the following space-time weighted energy

\[ \mathcal{E}_1(u(t)) = \int_0^t \|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} u_\xi \|^2_{L^2_x(\mathbb{R})} ds + \int_0^t \|\langle \xi \rangle^{-\frac{1+\delta}{2}} \langle \eta \rangle^{1+\delta} u_\eta \|^2_{L^2_x(\mathbb{R})} ds. \]

(1.15)
Denote the second order space-time weighted energies by

\[ \mathcal{E}_2(u(t)) = \sum_{|a|=1} \mathcal{E}_1(Z^a u(t)). \]  

(1.16)
Corresponding to (1.12) and (1.13), we will use the following third order space-time weighted energies

\[ E_3(u(t)) = \sum_{|a|=2 \atop a_1 \neq 0} \int_0^t \langle \eta \rangle^{-\frac{1+a}{2}} \langle \xi \rangle^{1+\delta} Z^a u_\xi \|_{L^2_x(\mathbb{R})}^2 \, ds \]

\[ + \sum_{|a|=2 \atop a_2 \neq 0} \int_0^t \langle \xi \rangle^{-\frac{1+a}{2}} \langle \eta \rangle^{1+\delta} Z^a u_\eta \|_{L^2_x(\mathbb{R})}^2 \, ds \]  

(1.17)

and

\[ \tilde{E}_3(u(t)) = \sum_{|a|=2 \atop a_1 = 0} \int_0^t \langle \eta \rangle^{-\frac{1+a}{2}} \langle \xi \rangle^{1+\delta} Z^a u_\xi \|_{L^2_x(\mathbb{R})}^2 \, ds + \sum_{|a|=2 \atop a_2 = 0} \int_0^t \langle \xi \rangle^{-\frac{1+a}{2}} \langle \eta \rangle^{1+\delta} Z^a u_\eta \|_{L^2_x(\mathbb{R})}^2 \, ds \]

\[ = \int_0^t \langle \eta \rangle^{-\frac{1+a}{2}} \langle \xi \rangle^{1+\delta} u_{\eta\eta\xi} \|_{L^2_x(\mathbb{R})}^2 \, ds + \int_0^t \langle \xi \rangle^{-\frac{1+a}{2}} \langle \eta \rangle^{1+\delta} u_{x\xi\eta} \|_{L^2_x(\mathbb{R})}^2 \, ds. \]  

(1.18)

We also use the notation

\[ E(u(t)) = E_1(u(t)) + E_2(u(t)) + E_3(u(t)). \]  

(1.19)

The reason why the third order energies \( \tilde{E}_3(u(t)) \) and \( \tilde{E}_3(u(t)) \) concerning some “mixed derivatives” need to be considered separately is that in our weighted energy estimates, they are not compatible with the null structure of the quasilinear part. In other words, if we treat them by weighted energy estimates, after integrating by parts, some terms in the quasilinear part will be uncontrollable. Thus, they can not be estimated via weighted energy estimates. This fact is also a key difference between the semilinear case and the quasilinear case. Fortunately, by using the system (1.3) directly, we find that \( \tilde{E}_3(u(t)) \) and \( \tilde{E}_3(u(t)) \) can be controlled by \( E(u(t)) \) and \( E(u(t)) \), respectively (see Lemma 2.4). This observation is a key point in our treatment for the quasilinear part of the system.

In order to describe the asymptotic behavior of the global solution, now we introduce some concepts on it. We say that a function \( u = u(t,x) \in C(\mathbb{R}^+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}^+; L^2(\mathbb{R})) \) is asymptotically free in the energy sense, if there is \( (v_0, v_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) such that

\[ \lim_{t \to +\infty} \left( \| u_\xi - v_\xi \|_{L^2_x(\mathbb{R})}^2 + \| u_\eta - v_\eta \|_{L^2_x(\mathbb{R})}^2 \right) = 0, \]  

(1.20)

where \( v \in C(\mathbb{R}^+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}^+; L^2(\mathbb{R})) \) is the unique global solution to the Cauchy problem of homogeneous linear wave equations

\[ v_{\xi\eta} = 0 \]  

(1.21)

with initial data

\[ t = 0 : \ v = v_0, \ v_t = v_1. \]  

(1.22)

Consider the Cauchy problem for the system (1.3) with initial data

\[ t = 0 : \ u = u_0, \ u_t = u_1. \]  

(1.23)

Our main result is the following
**Theorem 1.1.** Assume that the system (1.3) satisfies the null condition. Then for all \(0 < \delta < 1\), there exists a positive constant \(\varepsilon_0\) such that for any \(0 < \varepsilon \leq \varepsilon_0\), if

\[
\sum_{i=0}^{2} (\langle x \rangle^{1 + \delta} \| \partial_x^2 u_0 \|_{L^2_x(R)} + \| \langle x \rangle^{1 + \delta} \partial_x u_1 \|_{L^2_x(R)}) \leq \varepsilon,
\]

then the Cauchy problem (1.3)–(1.23) admits a unique global classical solution \(u\). Moreover, the global solution \(u\) is asymptotically free in the energy sense.

**Remark 1.1.** For general one-dimension quasilinear wave equations with quadratic nonlinearity

\[ u_{\xi\eta} = Q(u, u_\xi, u_\eta, u_{\xi\xi}, u_{\xi\eta}, u_{\eta\eta}), \]

Li, Yu and Zhou [12] showed the classical solution only admits some lifespan of order \(\varepsilon^{-1/2}\). Note that the null condition can enhance the lifespan from such short time order to infinity.

**Remark 1.2.** A small-data-global-existence type result for one-dimension quasilinear hyperbolic systems of diagonal form using the energy method is established in Zha [20]. But we should note that the system (1.3) can not be reduced to this case in general. Some small-data-global-existence type results using the characteristic approach can be found in Dai and Kong [4], Li, Zhou and Kong [13], Wong [19], and Zhou [21], etc. While, for some large-data-global-existence results, we refer the reader to, for example, Abbrescia and Wong [1], Gu [5], Liu and Liu [15], and Liu and Qu [16].

The outline of this paper is as follows. In Section 2, some necessary tools used to prove Theorem 1.1 are introduced. Section 3 is devoted to the proof of Theorem 1.1.

## 2 Preliminaries

First, by the fundamental theorem of calculus, chain rule and Leibniz’s rule, it is easy to get the following two lemmas.

**Lemma 2.1.** Assume that \(A_1 = A_1(u, u_\xi, u_\eta), A_2 = A_2(u, u_\xi, u_\eta), A_3 = A_3(u, u_\xi, u_\eta)\) and \(F = F(u, u_\xi, u_\eta)\) satisfy (1.4), (1.5), (1.6) and (1.7), respectively, and

\[ |u| + |u_\xi| + |u_\eta| \leq \nu_0. \]

Then we have

\[
|\partial_\xi A_1| \leq C(|u_\xi| + |u_{\xi\xi}| + |u_{\xi\eta}|), \quad |\partial_\eta A_1| \leq C(|u_\eta| + |u_{\xi\eta}| + |u_{\eta\eta}|),
\]

\[
|\partial_\xi A_2| \leq C|u_\xi| + C|u_\eta|(|u_\xi| + |u_{\xi\xi}|), \quad |\partial_\eta A_2| \leq C(|u_\eta| + |u_{\xi\eta}| + |u_{\eta\eta}|),
\]

\[
|\partial_\xi A_3| \leq C(|u_\xi| + |u_{\xi\xi}| + |u_{\xi\eta}|), \quad |\partial_\eta A_3| \leq C|u_\xi|(|u_\eta| + |u_{\eta\eta}|),
\]

\[
|\partial_\xi F| \leq C(|u_\xi||u_\xi\eta| + |u_\eta||u_{\xi\xi}| + |u_\xi||u_\eta|), \quad |\partial_\eta F| \leq C(|u_\eta||u_\xi\eta| + |u_\xi||u_{\eta\eta}| + |u_\xi||u_\eta|),
\]

where \(C = C(\nu_0)\) is a constant depending on \(\nu_0\).
Lemma 2.2. Assume that $A_1 = A_1(u, u_\xi, u_\eta), A_2 = A_2(u, u_\xi, u_\eta), A_3 = A_3(u, u_\xi, u_\eta)$ and $F = F(u, u_\xi, u_\eta)$ satisfy (1.4), (1.5), (1.6) and (1.7), respectively, and

$$|u| + |u_\xi| + |u_\eta| + |Zu_\xi| + |Zu_\eta| \leq \nu_1. \quad (2.7)$$

Then for any multi-index $a, |a| = 1$, it holds that

$$|Z^a A_1| \leq C(|u_\xi| + |u_\eta| + |Z^a u_\xi| + |Z^a u_\eta|), \quad (2.8)$$

$$|Z^a A_2| \leq C(|u_\eta| + |Z^a u_\eta|), \quad (2.9)$$

$$|Z^a A_3| \leq C(|u_\xi| + |Z^a u_\xi|), \quad (2.10)$$

$$|Z^a F| \leq C(|u_\xi| + u_\xi) + |Z^a u_\xi||u_\eta| + |Z^a u_\eta||u_\xi|). \quad (2.11)$$

And for any multi-index $a, |a| = 2$, it holds that

$$|Z^a A_1| \leq C \sum_{|b| \leq 2} (|Z^b u_\xi| + |Z^b u_\eta|), \quad (2.12)$$

$$|Z^a A_2| \leq C \sum_{|b| \leq 2} |Z^b u_\eta| + C|u_\eta| \sum_{|b| \leq 2} |Z^b u_\xi|, \quad (2.13)$$

$$|Z^a A_3| \leq C \sum_{|b| \leq 2} |Z^b u_\xi| + C|u_\xi| \sum_{|b| \leq 2} |Z^b u_\eta|, \quad (2.14)$$

$$|Z^a F| \leq C \sum_{|b|+|c| \leq 2} |Z^b u_\xi||Z^c u_\eta|. \quad (2.15)$$

Here $C = C(\nu_1)$ is a constant depending on $\nu_1$.

The following pointwise estimates will be used frequently in the next section.

Lemma 2.3. Let $u$ be a smooth function with sufficient decay at the spatial infinity. Then it holds that

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq CE_1^{1/2}(u(t)), \quad (2.16)$$

$$\|\langle \xi \rangle^{1+\delta} u_\xi\|_{L^\infty_\nu(\mathbb{R})} + \|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_\nu(\mathbb{R})} \leq CE_1^{1/2}(u(t)) + CE_2^{1/2}(u(t)), \quad (2.17)$$

$$\|\langle \xi \rangle^{1+\delta} (u_\xi + u_\eta)\|_{L^\infty_\nu(\mathbb{R})} + \|\langle \eta \rangle^{1+\delta} (u_\eta + u_\xi)\|_{L^\infty_\nu(\mathbb{R})} \leq CE_1^{1/2}(u(t)) + CE_3^{1/2}(u(t)), \quad (2.18)$$

and

$$\|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} u_\xi\|_{L^2 L^\infty_\nu} + \|\langle \xi \rangle^{-\frac{1+\delta}{2}} \langle \eta \rangle^{1+\delta} u_\eta\|_{L^2 L^\infty_\nu} \leq CE_1^{1/2}(u(t)) + CE_2^{1/2}(u(t)), \quad (2.19)$$

$$\|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} (u_\xi + u_\eta)\|_{L^2 L^\infty_\nu} + \|\langle \xi \rangle^{-\frac{1+\delta}{2}} \langle \eta \rangle^{1+\delta} (u_\eta + u_\xi)\|_{L^2 L^\infty_\nu} \leq CE_1^{1/2}(u(t)) + CE_3^{1/2}(u(t)). \quad (2.20)$$

Proof. It follows from the fundamental theorem of calculus and Hölder inequality that

$$\|u(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq \|u_\xi(t, \cdot)\|_{L^1(\mathbb{R})} \leq \|u_\xi\|_{L^1(\mathbb{R})} + \|u_\eta\|_{L^1(\mathbb{R})} \leq \|\langle \xi \rangle^{-1+\delta} \|L^2_\nu(\mathbb{R})\| \|\langle \xi \rangle^{1+\delta} u_\xi\|_{L^2_\nu(\mathbb{R})} + \|\langle \eta \rangle^{-1+\delta} \|L^2_\nu(\mathbb{R})\| \|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^2_\nu(\mathbb{R})} \leq CE_1^{1/2}(u(t)). \quad (2.21)$$
While (2.17), (2.18), (2.19) and (2.20) can be proved by Sobolev embedding \( H^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \) and the following fact
\[
|\partial_x \langle \xi \rangle^{1+\delta} | \leq C \langle \xi \rangle^{1+\delta}, \quad |\partial_x (\langle \eta \rangle - \frac{\langle \xi \rangle^{1+\delta}}{\langle \xi \rangle^{1+\delta}}) | \leq C \langle \eta \rangle - \frac{\langle \xi \rangle^{1+\delta}}{\langle \xi \rangle^{1+\delta}} \langle \eta \rangle^{1+\delta},
\]
\[
|\partial_x (\langle \eta \rangle^{1+\delta} | \leq C \langle \eta \rangle^{1+\delta}, \quad |\partial_x (\langle \xi \rangle - \frac{\langle \eta \rangle^{1+\delta}}{\langle \xi \rangle^{1+\delta}}) | \leq C \langle \xi \rangle - \frac{\langle \eta \rangle^{1+\delta}}{\langle \xi \rangle^{1+\delta}} \langle \eta \rangle^{1+\delta}. \tag{2.22}
\]
\[
|\partial_x (\langle \xi \rangle^{1+\delta} | \leq C \langle \xi \rangle^{1+\delta}, \quad |\partial_x (\langle \eta \rangle - \frac{\langle \xi \rangle^{1+\delta}}{\langle \xi \rangle^{1+\delta}}) | \leq C \langle \eta \rangle - \frac{\langle \xi \rangle^{1+\delta}}{\langle \xi \rangle^{1+\delta}} \langle \eta \rangle^{1+\delta}. \tag{2.23}
\]

The following lemma will play a key role in the proof of global existence part of Theorem 1.1.

**Lemma 2.4.** Let \( u \) be a classical solution to the system (1.3) satisfying null conditions (1.4)–(1.7). Assume that
\[
\varepsilon_1 = \sup_{0 \leq s \leq t} E^{1/2}(u(s))
\]
is sufficiently small. Then we have
\[
\sup_{0 \leq s \leq t} \tilde{E}_3(u(s)) \leq C \sup_{0 \leq s \leq t} E(u(s)) \tag{2.25}
\]
and
\[
\tilde{E}_3(u(t)) \leq CE(u(t)). \tag{2.26}
\]

**Proof.** From the system (1.3), we have
\[
\begin{align*}
\eta \xi & = A_1 \eta \xi + A_2 \eta \xi + A_3 \eta \eta + \partial_\eta A_1 \eta \xi + \partial_\eta A_2 \eta \xi + \partial_\eta A_3 \eta \eta + \partial_\eta F, \tag{2.27} \\
\xi \eta & = A_1 \xi \eta + A_2 \xi \eta + A_3 \xi \eta + \partial_\xi A_1 \xi \eta + \partial_\xi A_2 \xi \eta + \partial_\xi A_3 \xi \eta + \partial_\xi F. \tag{2.28}
\end{align*}
\]
We first estimate \( \tilde{E}_3(u(t)) \). It follows from (2.27), (2.16), Lemma 2.1 and Sobolev embedding \( H^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \) that
\[
\begin{align*}
\| \langle \xi \rangle^{1+\delta} \eta \xi \|_{L^2_2(\mathbb{R})} & \leq C \|
\end{align*}
\]
Similarly, by (2.28), (2.16), Lemma 2.1 and Sobolev embedding \( H^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R}) \), we can also obtain
\[
\| \langle \eta \rangle^{1+\delta} \eta \xi \|_{L^2_2(\mathbb{R})} \leq CE^{1/2}(u(t)) \| \langle \eta \rangle^{1+\delta} \eta \xi \|_{L^2_2(\mathbb{R})} + C(E(u(t)) + E^{3/2}(u(t))). \tag{2.30}
\]
Thus, thanks to (2.29) and (2.30), we have

\[ \widetilde{E}_3^{1/2}(u(t)) \leq C E^{1/2}(u(t)) \widetilde{E}_3^{1/2}(u(t)) + C (E(u(t)) + E^{3/2}(u(t))) \]

\[ \leq C \varepsilon_1 \widetilde{E}_3^{1/2}(u(t)) + C (\varepsilon_1 + \varepsilon_1^2) E^{1/2}(u(t)). \]  

(2.31)

If \( \varepsilon_1 \) is sufficiently small, we can get (2.25).

Now we will estimate \( \tilde{E}_3(u(t)) \). From (2.28), (2.16), Lemma 2.1 and Sobolev embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \), we can see

\[
\| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta u_{\xi \xi \xi} \|_{L^2_x} \\
\leq C \| A_1 \|_{L^\infty_x} \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta u_{\xi \xi \eta} \|_{L^2_x} + C \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta A_2 \|_{L^2_x} \| u_{\xi \xi} \|_{L^\infty_x} \| L^2_x \\
+ C \| A_3 \|_{L^\infty_x} \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta u_{\xi \eta \eta} \|_{L^2_x} + C \| \partial_t A_1 \|_{L^\infty_x} \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta u_{\eta \eta} \|_{L^2_x} \\
+ C \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta \partial_t \xi \|_{L^2_x} + C \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta u_{\eta \eta} \|_{L^2_x} \\
+ \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta A \|_{L^2_x} \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \| (\xi) - \frac{1+\delta}{2} \langle \eta \rangle 1+\delta u_{\eta \eta} \|_{L^2_x} \\
+ C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} E(u(s)) \right) E^{1/2}(u(t)).
\]

(2.32)

Similarly, by (2.27), (2.16), Lemma 2.1 and Sobolev embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \), we can also get

\[
\| (\eta) - \frac{1+\delta}{2} (\xi)^{1+\delta} u_{\eta \eta \xi} \|_{L^2_x} \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \| (\eta) - \frac{1+\delta}{2} (\xi)^{1+\delta} u_{\eta \eta \xi} \|_{L^2_x} \\
+ C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} E(u(s)) \right) E^{1/2}(u(t)).
\]

(2.33)

By (2.32) and (2.33) we can obtain

\[
\tilde{E}_3^{1/2}(u(t)) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \tilde{E}_3^{1/2}(u(t)) + C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} E(u(s)) \right) E^{1/2}(u(t)) \\
\leq C \varepsilon_1 \tilde{E}_3^{1/2}(u(t)) + C (\varepsilon_1 + \varepsilon_1^2) E^{1/2}(u(t)).
\]

(2.34)

If \( \varepsilon_1 \) is sufficiently small, we can get (2.26).

The following lemma will be used in the proof of asymptotic behavior part of Theorem 1.1. For the proof of this lemma, we refer the reader to Lemma 6.12 in Katayama [7], where high space dimension case is also considered.
Lemma 2.5. If $G \in L^1(\mathbb{R}^+; L^2(\mathbb{R}))$, i.e.,

$$\int_0^{+\infty} \|G(t, \cdot)\|_{L^2(\mathbb{R})} dt < +\infty,$$

(2.35)

Then the global solution to

$$u_{\xi \eta} = G$$

(2.36)

is asymptotically free in the energy sense.

Finally, for the convenience, we will introduce some notations concerning the weight functions used in the weighted energy estimates in the next section. Fix $0 < \delta < 1$. Set

$$\phi(x) = \langle x \rangle^{2+2\delta}.$$  

(2.37)

It is easy to verify that

$$|\phi'(x)| \leq 4 \langle x \rangle^{1+2\delta}.$$  

(2.38)

Let

$$q(x) = \int_{-\infty}^{x} \langle \rho \rangle^{-(1+\delta)} d\rho$$

(2.39)

and

$$\psi(x) = e^{-q(x)}.$$  

(2.40)

We can verify that

$$\psi'(x) = -\psi(x) \langle x \rangle^{-(1+\delta)}.$$  

(2.41)

We note that there exists a positive constant $c$ such that

$$c^{-1} \leq \psi(x) \leq c,$$

(2.42)

thus it holds that

$$c^{-1} \langle x \rangle^{-(1+\delta)} \leq -\psi'(x) \leq c\langle x \rangle^{-(1+\delta)}.$$  

(2.43)

3 Proof of Theorem 1.1

Now we will prove Theorem 1.1. We first prove the global existence part of Theorem 1.1 by some bootstrap argument. Assume that $u$ is a classical solution to the Cauchy problem (1.3)–(1.23). We will show that there exist positive constants $\varepsilon_0$ and $A$ such that

$$\sup_{0 \leq s \leq t} E(u(s)) + E(u(t)) \leq A^2 \varepsilon^2$$

(3.1)

under the assumption

$$\sup_{0 \leq s \leq t} E(u(s)) + E(u(t)) \leq 4A^2 \varepsilon^2,$$

(3.2)

where $0 < \varepsilon \leq \varepsilon_0$. Then based on estimate (3.1) and Lemma 2.5, the proof of asymptotic behavior part of Theorem 1.1 can also be given.
3.1 Low order energy estimates

We first estimate \( \sup_{0 \leq s \leq t} E_2(u(s)) \) and \( E_2(u(t)) \). \( \sup_{0 \leq s \leq t} E_1(u(s)) \) and \( E_1(u(t)) \) can be estimated similarly.

For any multi-index \( \alpha = (\alpha_1, \alpha_2), |\alpha| = 1 \), note that \( Z^\alpha u \) satisfies

\[
Z^\alpha u_{\xi \eta} = A_1 Z^\alpha u_{\xi \eta} + A_2 Z^\alpha u_{\xi} + A_3 Z^\alpha u_{\eta} + Z^\alpha A_1 u_{\xi} + Z^\alpha A_2 u_{\eta} + Z^\alpha A_3 u_{\eta} + Z^\alpha F. \tag{3.3}
\]

Multiply \( 2\psi(\eta)\phi(\xi)Z^\alpha u_{\xi}^T \) on both sides of (3.3). Noting the symmetry of \( A_1 \), by Leibniz’s rule we have

\[
(\psi(\eta)\phi(\xi)|Z^\alpha u_{\xi}|^2)_\eta - \psi'(\eta)\phi(\xi)|Z^\alpha u_{\xi}|^2 = (\psi(\eta)\phi(\xi)Z^\alpha u_{\xi}^T A_1 Z^\alpha u_{\xi})_\eta - \psi'(\eta)\phi(\xi)Z^\alpha u_{\xi}^T A_1 Z^\alpha u_{\xi}
- \psi(\eta)\phi(\xi)Z^\alpha u_{\xi}^T \partial_\eta A_1 Z^\alpha u_{\xi} + 2\psi(\eta)\phi(\xi)Z^\alpha u_{\xi}^T G_a, \tag{3.4}
\]

where

\[
G_a = A_2 Z^\alpha u_{\xi} + A_3 Z^\alpha u_{\eta} + Z^\alpha A_1 u_{\xi} + Z^\alpha A_2 u_{\eta} + Z^\alpha A_3 u_{\eta} + Z^\alpha F. \tag{3.5}
\]

Similarly, multiply \( 2\psi(\eta)\phi(\xi)Z^\alpha u_{\eta}^T \) on both sides of (1.3). The symmetry of \( A_1 \) and Leibniz’s rule also imply

\[
(\psi(\eta)\phi(\xi)|Z^\alpha u_{\eta}|^2)_\xi - \psi'(\xi)\phi(\eta)|Z^\alpha u_{\eta}|^2 = (\psi(\xi)\phi(\eta)Z^\alpha u_{\eta}^T A_1 Z^\alpha u_{\eta})_\xi - \psi'(\xi)\phi(\eta)Z^\alpha u_{\eta}^T A_1 Z^\alpha u_{\eta}
- \psi(\xi)\phi(\eta)Z^\alpha u_{\eta}^T \partial_\xi A_1 Z^\alpha u_{\eta} + 2\psi(\xi)\phi(\eta)Z^\alpha u_{\eta}^T G_a. \tag{3.6}
\]

Integrating on \([0, t] \times \mathbb{R}\) on both sides of (3.4) and (3.6), we conclude from the fundamental theorem of calculus that

\[
\int_{\mathbb{R}} \left( e_2(t, x) + \tilde{e}_2(t, x) \right) dx + \int_0^t \int_{\mathbb{R}} p_2(s, x) dx ds
= \int_{\mathbb{R}} \left( e_2(0, x) + \tilde{e}_2(0, x) \right) dx + \int_0^t \int_{\mathbb{R}} q_2(s, x) dx ds
+ 2 \int_0^t \int_{\mathbb{R}} \psi(\eta)\phi(\xi)Z^\alpha u_{\xi}^T G_a dx ds + 2 \int_0^t \int_{\mathbb{R}} \psi(\xi)\phi(\eta)Z^\alpha u_{\eta}^T G_a dx ds, \tag{3.7}
\]

where

\[
e_2 = \psi(\eta)\phi(\xi)|Z^\alpha u_{\xi}|^2 + \psi(\xi)\phi(\eta)|Z^\alpha u_{\eta}|^2, \tag{3.8}
\]

\[
\tilde{e}_2 = -\psi(\eta)\phi(\xi)Z^\alpha u_{\xi}^T A_1 Z^\alpha u_{\xi} - \psi(\xi)\phi(\eta)Z^\alpha u_{\eta}^T A_1 Z^\alpha u_{\eta}, \tag{3.9}
\]

\[
p_2 = -\psi'(\eta)\phi(\xi)|Z^\alpha u_{\xi}|^2 - \psi'(\xi)\phi(\eta)|Z^\alpha u_{\eta}|^2 \tag{3.10}
\]

and

\[
q_2 = -\psi'(\eta)\phi(\xi)Z^\alpha u_{\xi}^T A_1 Z^\alpha u_{\xi} - \psi(\eta)\phi(\xi)Z^\alpha u_{\eta}^T \partial_\eta A_1 Z^\alpha u_{\xi}
- \psi'(\xi)\phi(\eta)Z^\alpha u_{\eta}^T A_1 Z^\alpha u_{\eta} - \psi(\xi)\phi(\eta)Z^\alpha u_{\eta}^T \partial_\xi A_1 Z^\alpha u_{\eta}. \tag{3.11}
\]
In view of (2.37) and (2.42), we can see
\begin{equation}
  c^{-1} e_2(t, x) \leq |\langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 + |\langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2| \leq c e_2(t, x).
\end{equation}

It follows from (2.37), (2.42), (1.4), (2.16) and Sobolev embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \) that
\begin{align*}
  |\tilde{e}_2(t, x)| &\leq C|\langle \xi \rangle^{2+2\delta} Z^a u_{\xi}^T A_1 Z^a u_{\xi} + C|\langle \eta \rangle^{2+2\delta} Z^a u_{\eta}^T A_1 Z^a u_{\eta}| \\
&\leq C|\langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 (|u| + |u_{\xi}| + |u_{\eta}|) + C|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2 (|u| + |u_{\xi}| + |u_{\eta}|) \\
&\leq C (E_1^{1/2}(u(t)) + E_2^{1/2}(u(t))) \left( |\langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 + |\langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2 \right) \\
&\leq C E_1^{1/2}(u(t)) \left( |\langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 + |\langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2 \right)^2.
\end{align*}

Thus, for small solutions, we can get
\begin{equation}
  (2c)^{-1} \left( e_2(t, x) + \tilde{e}_2(t, x) \right) \leq |\langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 + |\langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2 | \leq 2c \left( e_2(t, x) + \tilde{e}_2(t, x) \right).
\end{equation}

From (2.37) and (2.43), it follows that
\begin{equation}
  c^{-1} p_2(t, x) \leq |\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 + |\langle \xi \rangle^{-\frac{1+\delta}{2}} \langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2 | \leq c p_2(t, x).
\end{equation}

According to (2.37), (2.42), (2.43), (1.4) and Lemma 2.1, we have
\begin{align*}
  |q_2(t, x)| &\leq C|\langle \xi \rangle^{-1+\delta} \langle \xi \rangle^{2+2\delta} Z^a u_{\xi}^2 (|u| + |u_{\xi}| + |u_{\eta}|) + C|\langle \xi \rangle^{2+2\delta} Z^a u_{\xi}^2 (|u_{\xi}| + |Z u_{\eta}|) \\
&+ C|\langle \xi \rangle^{1+\delta} \langle \eta \rangle^{2+2\delta} Z^a u_{\eta}^2 (|u| + |u_{\xi}| + |u_{\eta}|) + C|\langle \eta \rangle^{2+2\delta} Z^a u_{\eta}^2 (|u| + |u_{\xi}| + |u_{\eta}|) \\
&\leq C|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 (|u| + |u_{\xi}| + |u_{\eta}|) + C|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^a u_{\xi}^2 (|u| + |u_{\xi}| + |u_{\eta}|) \\
&+ C|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^a u_{\eta}^2 (|u_{\xi}| + |u_{\xi}| + |u_{\eta}|) + C|\langle \xi \rangle^{-\frac{1+\delta}{2}} \langle \eta \rangle^{1+\delta} Z^a u_{\eta}^2 (|u_{\xi}| + |u_{\xi}| + |u_{\eta}|).
\end{align*}

By (3.5), (1.5), (1.6) and Lemma 2.2, we can obtain
\begin{equation}
  |G_\alpha| \leq C|Z^a u_{\xi}| |u_{\eta}| + C|Z^a u_{\eta}| |u_{\xi}| + C(|u_{\xi}| + |u_{\eta}| + |Z^a u_{\xi}| + |Z^a u_{\eta}|) |u_{\xi}| \\
+ C(|u_{\xi}| + |Z^a u_{\eta}|) |u_{\xi}| + C(|u_{\xi}| + |Z^a u_{\xi}|) |u_{\xi}| \\
+ C(|u_{\xi}| + |Z^a u_{\xi}|) |u_{\xi}| + C|Z^a u_{\eta}| |u_{\xi}|.
\end{equation}

Thanks to (3.7), (3.14) and (3.15), we see
\begin{align*}
  \sup_{0 \leq s \leq t} E_2(u(s)) + E_2(u(t)) &\leq C E_2(u(0)) + C \sum_{|\alpha|=1} \int_0^t \|q_2(s, \cdot)\|_{L^1(\mathbb{R})} ds \\
&+ C \sum_{|\alpha|=1} \int_0^t \|\langle \xi \rangle^{2+2\delta} Z^a u_{\xi} G_\alpha\|_{L^1(\mathbb{R})} ds + C \sum_{|\alpha|=1} \int_0^t \|\langle \eta \rangle^{2+2\delta} Z^a u_{\eta} G_\alpha\|_{L^1(\mathbb{R})} ds.
\end{align*}

It follows from (3.16), Hölder inequality, Lemma 2.3, Sobolev embedding \( H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \)
and Lemma 2.4 that

\[
\int_0^t \|q_2(s, \cdot)\|_{L^1(\mathbb{R})} ds \\
\leq C \|\langle \eta \rangle^{1+\delta} \xi\|_{L^2_x} (\|u\|_{L^\infty_x} + \|u_\xi\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\xi}\|_{L^2_x} (\|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_x} + \|\langle \eta \rangle^{1+\delta} Z u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\xi}\|_{L^2_x} (\|u\|_{L^\infty_x} + \|u_\xi\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\xi}\|_{L^2_x} (\|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_x} + \|\langle \eta \rangle^{1+\delta} Z u_\eta\|_{L^\infty_x}) \\
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \overline{E}_3^{1/2}(u(s)) \right) \mathcal{E}_2(u(t)) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}_2(u(t)). \tag{3.19}
\]

By Hölder inequality, it is easy to see that

\[
\int_0^t \|\langle \xi \rangle^{2+\delta} Z^a u_{\xi} G_a\|_{L^1(\mathbb{R})} ds \\
\leq C \|\langle \eta \rangle^{1+\delta} \xi\|_{L^2_x} (\|u\|_{L^\infty_x} + \|u_\xi\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}\|_{L^2_x} (\|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_x} + \|\langle \eta \rangle^{1+\delta} Z u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}\|_{L^2_x} (\|u\|_{L^\infty_x} + \|u_\xi\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}\|_{L^2_x} (\|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_x} + \|\langle \eta \rangle^{1+\delta} Z u_\eta\|_{L^\infty_x}) \\
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \overline{E}_3^{1/2}(u(s)) \right) \mathcal{E}_3^{1/2}(u(t)) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}_3^{1/2}(u(t)). \tag{3.20}
\]

By (3.17), Lemma 2.3 and Lemma 2.4, we can get

\[
\|\langle \eta \rangle^{1+\delta} \xi\|_{L^2_x} (\|u\|_{L^\infty_x} + \|u_\xi\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}\|_{L^2_x} (\|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_x} + \|\langle \eta \rangle^{1+\delta} Z u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}\|_{L^2_x} (\|u\|_{L^\infty_x} + \|u_\xi\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \|\langle \eta \rangle^{1+\delta} Z^a u_{\eta}\|_{L^2_x} (\|\langle \eta \rangle^{1+\delta} u_\eta\|_{L^\infty_x} + \|\langle \eta \rangle^{1+\delta} Z u_\eta\|_{L^\infty_x}) \\
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \overline{E}_3^{1/2}(u(s)) \right) \mathcal{E}_3^{1/2}(u(t)) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}_3^{1/2}(u(t)). \tag{3.21}
\]

It follows from (3.20) and (3.21) that

\[
\int_0^t \|\langle \xi \rangle^{2+\delta} Z^a u_{\xi} G_a\|_{L^1(\mathbb{R})} ds \leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}_2(u(t)). \tag{3.22}
\]

Similarly, we can also get

\[
\int_0^t \|\langle \eta \rangle^{2+\delta} Z^a u_{\eta} G_a\|_{L^1(\mathbb{R})} ds \leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}_3(u(t)). \tag{3.23}
\]
Consequently, the combination of (3.18), (3.19), (3.22) and (3.23) implies
\[
\sup_{0 \leq s \leq t} E_2(u(s)) + \mathcal{E}_2(u(t)) \leq C E_2(u(0)) + C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}(u(t)).
\] (3.24)

By a similar (but simpler) argument, we can also show
\[
\sup_{0 \leq s \leq t} E_1(u(s)) + \mathcal{E}_1(u(t)) \leq C E_1(u(0)) + C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}(u(t)).
\] (3.25)

### 3.2 High order energy estimates

Now we will estimate \( \sup_{0 \leq s \leq t} E_3(u(s)) \) and \( \mathcal{E}_3(u(t)) \).

For any multi-index \( a = (a_1, a_2) \), \( |a| = 2, a_1 \neq 0 \), Leibniz’s rule gives
\[
Z^a u_{\xi a} = A_1 Z^a u_{\xi a} + + A_2 Z^a u_{\xi a} + A_3 Z^a u_{\eta a} + H_a,
\] (3.26)
where
\[
H_a = \sum_{c+d=a, \ c \neq 0} \lambda_{cd} Z^c A_1 Z^d u_{\xi a} + \sum_{c+d=a, \ c \neq 0} \lambda_{cd} Z^c A_2 Z^d u_{\xi a} + \sum_{c+d=a, \ c \neq 0} \lambda_{cd} Z^c A_3 Z^d u_{\eta a} + Z^d F,
\] (3.27)
\( \lambda_{cd} \) are some constants. Multiply \( 2\psi(\eta)\phi(\xi) Z^a u_\xi^T \) on both sides of (3.26). Noting the symmetry of \( A_1, A_2 \) and \( A_3 \), by Leibniz’s rule we can get
\[
(\psi(\eta)\phi(\xi)|Z^a u_\xi|^2)_\eta - \psi'(\eta)\phi(\xi)|Z^a u_\xi|^2
= (\psi(\eta)\phi(\xi) Z^a u_\xi A_1 Z^a u_\xi)_\eta - \psi'(\eta)\phi(\xi) Z^a u_\xi A_1 Z^a u_\xi - (\psi(\eta)\phi(\xi) Z^a u_\xi A_2 Z^a u_\xi)_\xi - \psi'(\eta)\phi(\xi) Z^a u_\xi A_2 Z^a u_\xi - (\psi(\eta)\phi(\xi) Z^a u_\xi A_3 Z^a u_\xi)_\eta - \psi'(\eta)\phi(\xi) Z^a u_\xi A_3 Z^a u_\xi
- (\psi(\eta)\phi(\xi) Z^a u_\eta A_3 Z^a u_\eta)_\xi + (\psi(\eta)\phi(\xi) Z^a u_\eta A_3 Z^a u_\eta)_\eta + \psi(\eta)\phi(\xi) Z^a u_\eta A_3 Z^a u_\eta + 2\psi(\eta)\phi(\xi) Z^a u_\eta H_a.
\] (3.28)

Integrating on \([0, t] \times \mathbb{R}\) on both sides of (3.28), by the fundamental theorem of calculus, we have
\[
\int_\mathbb{R} e_3(t, x) dx + \int_0^t \int_\mathbb{R} p_3(s, x) dx ds
= \int_\mathbb{R} (e_3(0, x) + \tilde{e}_3(0, x)) dx - \int_\mathbb{R} \tilde{e}_3(t, x) dx \int_0^t \int_\mathbb{R} q_3(s, x) dx ds
+ 2 \int_0^t \int_\mathbb{R} \psi(\eta)\phi(\xi) Z^a u_\xi^T H_a dx ds,
\] (3.29)
where
\[
e_3 = \psi(\eta)\phi(\xi)|Z^a u_\xi|^2, \quad p_3 = -\psi'(\eta)\phi(\xi)|Z^a u_\xi|^2,
\] (3.30)
\[
\tilde{e}_3 = -\psi(\eta)\phi(\xi) Z^a u_\xi A_1 Z^a u_\xi - \psi(\eta)\phi(\xi) Z^a u_\xi A_2 Z^a u_\xi
- 2\psi(\eta)\phi(\xi) Z^a u_\xi A_3 Z^a u_\eta + \psi(\eta)\phi(\xi) Z^a u_\eta A_3 Z^a u_\eta
\] (3.31)
and
\[
q_3 = -\psi'(\eta)\phi(\xi)Z^a u^T_\xi A_1 Z^a u_\xi - \psi(\eta)\phi(\xi)Z^a u^T_\xi \partial_\eta A_1 Z^a u_\xi \\
- \psi(\eta)\phi'(\xi)Z^a u^T_\xi A_2 Z^a u_\xi - \psi(\eta)\phi(\xi)Z^a u^T_\xi \partial_\xi A_2 Z^a u_\xi \\
- 2\psi'(\eta)\phi(\xi)Z^a u^T_\xi A_3 Z^a u_\eta - 2\psi(\eta)\phi(\xi)Z^a u^T_\xi \partial_\eta A_3 Z^a u_\eta \\
+ \psi(\eta)\phi'(\xi)Z^a u^T_\eta A_3 Z^a u_\eta + \psi(\eta)\phi(\xi)Z^a u^T_\eta \partial_\xi A_3 Z^a u_\eta.
\] (3.32)

In view of (2.37) and (2.42), we have
\[
ce^{-1}e_3(t, x) \leq |\langle \xi \rangle|^{1+\delta} Z^a u_\xi|^2 \leq ce_3(t, x).
\] (3.33)

By (2.37) and (2.43), we can obtain
\[
ce^{-1}p_3(t, x) \leq |\langle \eta \rangle|^{1+\delta} Z^a u_\xi|^2 \leq cp_3(t, x).
\] (3.34)

It follows from (2.37), (2.42), (1.4), (1.5) and (1.6) that
\[
|\xi_3(t, x)| \leq C |\langle \xi \rangle|^{2+2\delta} [Z^a u^T_\xi A_1 Z^a u_\xi] + C |\langle \xi \rangle|^{2+2\delta} [Z^a u^T_\xi A_2 Z^a u_\xi] \\
+ C |\langle \xi \rangle|^{2+2\delta} [Z^a u^T_\xi A_3 Z^a u_\eta] + C |\langle \xi \rangle|^{2+2\delta} [Z^a u^T_\eta A_3 Z^a u_\eta] \\
\leq C |\langle \xi \rangle|^{1+\delta} Z^a u_\xi^2(|u| + |u_\xi| + |u_\eta|) \\
+ C |\langle \xi \rangle|^{1+\delta} Z^a u_\xi||\langle \xi \rangle|^{1+\delta} Z^a u_\eta||\langle \xi \rangle|^{1+\delta} Z^a u_\eta| \\
\leq C |\langle \xi \rangle|^{1+\delta} Z^a u_\xi^2(|u| + |u_\xi| + |u_\eta|) + C \sum_{|\xi| \geq 2} |\langle \xi \rangle|^{1+\delta} Z^b u_\xi||\langle \xi \rangle|^{1+\delta} Z^a u_\eta|.
\] (3.35)

Here we also use the following simple but important fact: for the multi-index \(a = (a_1, a_2), |a| = 2, a_1 \neq 0,\) it holds that
\[
|\langle \xi \rangle|^{1+\delta} Z^a u_\eta = |\langle \xi \rangle|^{1+\delta} \partial_{\xi_1}^a \partial_{\eta_2} a_2 u_\eta \leq \sum_{|\xi| \geq 2} |\langle \xi \rangle|^{1+\delta} Z^b u_\xi|.
\] (3.36)

According to (1.4), (1.5), (1.6), (2.37), (2.38), (2.42), (2.43), Lemma 2.2 and (3.36), we can get
\[
|q_3(t, x)| \leq C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{2+2\delta} [Z^a u_\xi]^2(|u| + |u_\xi| + |u_\eta|) + C |\langle \xi \rangle|^{2+2\delta} [Z^a u_\xi]^2(|u_\eta| + |Z u_\eta|) \\
+ C |\langle \xi \rangle|^{2+2\delta} [Z^a u_\xi](|u_\xi| + |Z u_\xi|)[Z^a u_\eta] + C |\langle \xi \rangle|^{2+2\delta} [Z^a u_\eta]^2(|u_\xi| + |Z u_\xi|) \\
\leq C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^a u_\xi^2(|u| + |u_\xi| + |u_\eta|) \\
+ C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^a u_\xi^2(|\langle \eta \rangle|^{1+\delta} u_\eta| + |\langle \eta \rangle|^{1+\delta} Z u_\eta|) \\
+ C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^a u_\eta||\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} (|u_\xi| + |Z u_\xi|)| |\langle \eta \rangle|^{1+\delta} Z^a u_\eta| \\
+ C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^a u_\eta||\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} (|u_\xi| + |Z u_\xi|)| |\langle \eta \rangle|^{1+\delta} Z^a u_\eta| \\
\leq C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^a u_\xi^2(|u| + |u_\xi| + |u_\eta|) \\
+ C |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^a u_\xi^2(|\langle \eta \rangle|^{1+\delta} u_\eta| + |\langle \eta \rangle|^{1+\delta} Z u_\eta|) \\
+ C \sum_{|\xi| \geq 2} |\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} Z^b u_\xi||\langle \eta \rangle|^{-(1+\delta)} |\langle \xi \rangle|^{1+\delta} (|u_\xi| + |Z u_\xi|)| |\langle \eta \rangle|^{1+\delta} Z^a u_\eta|.
\] (3.37)
Lemma 2.2 implies
\[
|H_a| \leq C \sum_{|b| \leq 2} |Z^b u_{\xi}| (|u_\eta| + |u_{\eta\eta}| + |u_\eta| |u_{\xi\xi}|) \\
+ C \sum_{|b| \leq 2} |Z^b u_\eta| (|u_\xi| + |u_{\xi\xi}| + |u_\xi| |u_{\eta\eta}|) \\
+ C \sum_{|b| + |c| \leq 2} |Z^b u_\xi| |Z^c u_\eta|. 
\] (3.38)

From (3.29), (3.33), (3.34), we obtain
\[
\sup_{0 \leq s \leq t} \sum_{|\alpha| = 2 \atop \alpha_1 \neq 0} \sum_{|\beta| = 2 \atop \beta_1 \neq 0} \|\langle \xi \rangle^{1+\delta} Z^\alpha u_{\xi} \|^2_{L^2_x} + \sum_{|\alpha| = 2 \atop \alpha_1 \neq 0} \|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^\alpha u_{\xi} \|^2_{L^2_x}
\leq CE_3(u(0)) + C \sup_{0 \leq s \leq t} \sum_{|\alpha| = 2 \atop \alpha_1 \neq 0} \|\tilde{c}_3(s, \cdot)\|_{L^1(\mathbb{R})} + C \sum_{|\alpha| = 2 \atop \alpha_1 \neq 0} \int_0^t \|q_3(s, \cdot)\|_{L^1(\mathbb{R})} ds \\
+ C \sum_{|\alpha| = 2 \atop \alpha_1 \neq 0} \int_0^t \|\langle \xi \rangle^{2+2\delta} Z^\alpha u_{\xi} H_a\|_{L^1_x(\mathbb{R})} ds. 
\] (3.39)

By (3.35), Hölder inequality, (2.16), Sobolev embedding \(H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})\) and Lemma 2.4, we have
\[
\|\tilde{c}_3(t, \cdot)\|_{L^1(\mathbb{R})} \leq C\|\langle \xi \rangle^{1+\delta} Z^\alpha u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \sum_{|\beta| = 2} \|\langle \xi \rangle^{1+\delta} Z^\beta u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \sum_{|\beta| = 2} \|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^\beta u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \sum_{|\beta| = 2} \|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^\beta u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \tilde{E}_3^{1/2}(u(s)) \right) \left( \mathcal{E}(u(t)) + \tilde{E}_3(u(t)) \right) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}(u(t)). 
\] (3.40)

Thanks to (3.37), Hölder inequality, Lemma 2.3 and Lemma 2.4, we get
\[
\int_0^t \|q_3(s, \cdot)\|_{L^1(\mathbb{R})} ds \\
\leq C\|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^\alpha u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \sum_{|\beta| = 2} \|\langle \xi \rangle^{1+\delta} Z^\beta u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
+ C \sum_{|\beta| = 2} \|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^\beta u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x} + \|u_\eta\|_{L^\infty_x}) \\
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \tilde{E}_3^{1/2}(u(s)) \right) \left( \mathcal{E}(u(t)) + \tilde{E}_3(u(t)) \right) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}(u(t)). 
\] (3.41)

Hölder inequality implies
\[
\int_0^t \|\langle \xi \rangle^{2+2\delta} Z^\alpha u_{\xi} H_a\|_{L^1_x(\mathbb{R})} ds \\
\leq C\|\langle \eta \rangle^{-\frac{1+\delta}{2}} \langle \xi \rangle^{1+\delta} Z^\alpha u_{\xi} \|^2_{L^2_x} (\|u\|_{L^\infty_x} + \|u_{\xi}\|_{L^\infty_x}) \\
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \tilde{E}_3^{1/2}(u(s)) \right) \left( \mathcal{E}(u(t)) + \tilde{E}_3(u(t)) \right) \\
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) \mathcal{E}(u(t)). 
\] (3.42)
Via (3.38), Lemma 2.3 and Lemma 2.4, we can see
\[
\| \langle \eta \rangle^{1+\delta} \xi H_\eta \|_{L^2_x} \leq C \sum_{|b| \leq 2} \| \langle \eta \rangle^{-\frac{1+\delta}{2}} (\xi)^{1+\delta} Z^b u_\xi \|_{L^2_{x,t}} \| \langle \eta \rangle^{1+\delta} (|u_\eta| + |u_\eta| + |u_\eta| + |u_\eta| + |u_\eta|) \|_{L^\infty_t} 
+ C \sum_{|b| \leq 2} \| \langle \eta \rangle^{1+\delta} Z^b u_\eta \|_{L^\infty_t L^2_x} \| \langle \eta \rangle^{-\frac{1+\delta}{2}} (\xi)^{1+\delta} (|u_\xi| + |u_\xi| + |u_\xi| + |u_\xi| + |u_\xi|) \|_{L^2_t L^\infty_x} 
+ C \sum_{|b| + |c| \leq 2} \| \langle \eta \rangle^{-\frac{1+\delta}{2}} (\xi)^{1+\delta} Z^b u_\xi \|_{L^2_{x,t}} \| \langle \eta \rangle^{1+\delta} Z^c u_\eta \|_{L^\infty_t L^2_x} 
+ C \sum_{|b| + |c| \leq 2} \| \langle \eta \rangle^{-\frac{1+\delta}{2}} (\xi)^{1+\delta} Z^b u_\xi \|_{L^2_{x,t}} \| \langle \eta \rangle^{1+\delta} Z^c u_\eta \|_{L^\infty_t L^2_x} 
\leq C \left( \sup_{0 \leq s \leq t} E^{1/2}(u(s)) + \sup_{0 \leq s \leq t} \tilde{E}_3^{1/2}(u(s)) \right) (E^{1/2}(u(t)) + \tilde{E}_3^{1/2}(u(t))) 
\leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) E^{1/2}(u(t)). \tag{3.43}
\]

It follows from (3.42) and (3.43) that
\[
\int_0^t \| \langle \xi \rangle^{2+2\delta} Z^a u_\xi H_\xi \|_{L^1_x(B)} ds \leq C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) E(u(t)). \tag{3.44}
\]

We conclude from (3.39), (3.40), (3.41) and (3.44) that
\[
\sup_{0 \leq s \leq t} \sum_{|a|=2, a_1 \neq 0} \| \langle \eta \rangle^{1+\delta} Z^a u_\eta \|_{L^2_{x,t}}^2 + \sum_{|a|=2, a_1 \neq 0} \| \langle \eta \rangle^{-\frac{1+\delta}{2}} (\xi)^{1+\delta} Z^a u_\xi \|_{L^2_{x,t}}^2 
\leq C E_3(u(0)) + C \sup_{0 \leq s \leq t} E^{3/2}(u(s)) + C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) E(u(t)). \tag{3.45}
\]

Noting the duality between \( \xi \) and \( \eta \), by a similar way, we can also get
\[
\sup_{0 \leq s \leq t} \sum_{|a|=2, a_2 \neq 0} \| \langle \eta \rangle^{1+\delta} Z^a u_\eta \|_{L^2_{x,t}}^2 + \sum_{|a|=2, a_2 \neq 0} \| \langle \xi \rangle^{-\frac{1+\delta}{2}} (\eta)^{1+\delta} Z^a u_\eta \|_{L^2_{x,t}}^2 
\leq C E_3(u(0)) + C \sup_{0 \leq s \leq t} E^{3/2}(u(s)) + C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) E(u(t)). \tag{3.46}
\]

Accordingly, we have obtained
\[
\sup_{0 \leq s \leq t} E_3(u(s)) + \mathcal{E}_3(u(t)) 
\leq C E_3(u(0)) + C \sup_{0 \leq s \leq t} E^{3/2}(u(s)) + C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) E(u(t)). \tag{3.47}
\]

### 3.3 Global existence

Noting (3.24), (3.25) and (3.47), we have
\[
\sup_{0 \leq s \leq t} E(u(s)) + \mathcal{E}(u(t)) 
\leq C E(u(0)) + C \sup_{0 \leq s \leq t} E^{3/2}(u(s)) + C \sup_{0 \leq s \leq t} E^{1/2}(u(s)) E(u(t)). \tag{3.48}
\]
Under the assumption (3.2), we have

$$\sup_{0 \leq s \leq t} E(u(s)) + \mathcal{E}(u(t)) \leq C_1 \varepsilon^2 + 8C_1 A^3 \varepsilon^3.$$  \hspace{1cm} (3.49)

Assume that

$$E(u(0)) \leq \widetilde{C}_1 \varepsilon^2.$$  \hspace{1cm} (3.50)

Taking \(A^2 = 4 \max \{C_1, \widetilde{C}_1\}\) and \(\varepsilon_0\) so small that

$$16C_1 A \varepsilon_0 \leq 1,$$  \hspace{1cm} (3.51)

for any \(\varepsilon\) with \(0 < \varepsilon \leq \varepsilon_0\), we have

$$\sup_{0 \leq s \leq t} E(u(s)) + \mathcal{E}(u(t)) \leq A^2 \varepsilon^2.$$  \hspace{1cm} (3.52)

This completes the proof of global existence part of Theorem 1.1.

### 3.4 Asymptotic behavior

In view of Lemma 2.5, in order to prove that the global solution \(u\) to the system (1.3) is asymptotically free in the energy sense, we only need to show

$$\int_0^{+\infty} \|A_1 u_{\xi\eta}\|_{L^2_x(\mathbb{R})} + \|A_2 u_{\xi\xi}\|_{L^2_x(\mathbb{R})} + \|A_3 u_{\eta\eta}\|_{L^2_x(\mathbb{R})} + \|F\|_{L^2_x(\mathbb{R})} dt < +\infty.$$  \hspace{1cm} (3.53)

First, by (1.4), (1.5), (1.6), Lemma 2.3, Lemma 2.4 and (3.52), we have pointwise estimates

$$|A_1 u_{\xi\eta}| + |A_2 u_{\xi\xi}| + |A_3 u_{\eta\eta}| + |F| \leq C(|u||u_{\xi\eta}| + |u_{\xi\xi}| + |u_{\eta\eta}|) + C(|u||u_{\xi\xi}| + |u_{\eta\eta}|) \leq CE^{1/2}(u(t))|u_{\xi\eta}| + C\langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)} E^{1/2}(u(t)) (E^{1/2}(u(t)) + \widetilde{E}_3^{1/2}(u(t)))$$

$$\leq CE^{1/2}(u(t))|u_{\xi\eta}| + C\langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)} E(u(t)) \leq CA \varepsilon|u_{\xi\eta}| + CA^2 \varepsilon^2 \langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)}.$$  \hspace{1cm} (3.54)

In view of the system (1.3), we also have

$$|u_{\xi\eta}| \leq |A_1 u_{\xi\eta}| + |A_2 u_{\xi\xi}| + |A_3 u_{\eta\eta}| + |F| \leq CA \varepsilon|u_{\xi\eta}| + CA^2 \varepsilon^2 \langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)}.$$  \hspace{1cm} (3.55)

If \(\varepsilon\) is sufficiently small, it holds that

$$|u_{\xi\eta}| \leq CA^2 \varepsilon^2 \langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)}.$$  \hspace{1cm} (3.56)

It follows from (3.54) and (3.56) that

$$|A_1 u_{\xi\eta}| + |A_2 u_{\xi\xi}| + |A_3 u_{\eta\eta}| + |F| \leq CA^2 \varepsilon^2 \langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)}.$$  \hspace{1cm} (3.57)

Noting that

$$\langle \xi\rangle^{-(1+\delta)}\langle \eta\rangle^{-(1+\delta)} \leq C(t + x)^{-(1+\delta)}(t - x)^{-(1+\delta)} \leq C(t + |x|)^{-(1+\delta)}|t - |x||^{-(1+\delta)} \leq C(t)^{-(1+\delta)}|t - |x||^{-(1+\delta)},$$  \hspace{1cm} (3.58)
we have
\[ \| \langle \xi \rangle^{-1+\delta} \langle \eta \rangle^{-1+\delta} \|_{L^2_x(\mathbb{R})} \leq C \| (t-|x|)^{-1+\delta} \|_{L^2_x(\mathbb{R})} \leq C(t)^{-1+\delta}. \] (3.59)

The combination of (3.57) and (3.59) gives
\[ \| A_1 u_{\xi \eta} \|_{L^2_x(\mathbb{R})} + \| A_2 u_{\xi \xi} \|_{L^2_x(\mathbb{R})} + \| A_3 u_{\eta \eta} \|_{L^2_x(\mathbb{R})} + \| F \|_{L^2_x(\mathbb{R})} \leq CA^2 \varepsilon^2 (t)^{-1+\delta}. \] (3.60)

(3.53) is just a consequence of (3.60).

This completes the proof of asymptotic behavior part of Theorem 1.1.

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