Colouring of graphs by HB colour matrix algorithm method

A A Bhang, H R Bhapkar
MIT ADT University, Pune, Maharashtra, India-412201.

E-mail: archana.bhange17@gmail.com

Abstract. Graph Colouring is a chief element in graph theory with tremendous applicability in computer science like data mining, clustering, networking, image segmentation etc. And a variety of implementations in aircraft scheduling, register allocation, sudoku, mobile networking, etc. Various algorithms were contrived for vertex colouring. This paper defines the HB colour matrix method and its kinds. There are three types of such matrices, Vertex HB colour matrix (VHBCM), Edge HB Colour matrix (EHBCM), and Region HB colour matrix (RHBCM). Also, the HB colour matrix algorithm is developed using a special assignment method, which gives the chromatic number of the given graph. Further, the algorithm is used to develop the python program, giving time complexity O(n) and space complexity O(n^2). Also, the output of the python program for some standard graphs is calculated. The Similar algorithm can be developed for edge and face colouring of the graphs. Colouring of the graph further can be extended to perfect colouring.

1. Introduction
Graph theory is swiftly operating into the mainstream of mathematics mainly because of its implementations in various domains which comprise networking and coding theory for electrical engineering, algorithms or computations for computer science and operations research for timetabling and scheduling[3]. Graph colouring is a core concept of graph theory and is applied in many actual life applications like organizing jobs, Air traffic management, computer network security, Map colouring and networking of GSM mobile phones, mechanized channel allocation for mini wireless LAN (local area networks), traffic signal assignments, etc. [2]. A graph H which can be represented in a plane such that its edges intersect each other at extreme points only is known as a planar graph[5]. Colouring the components of graphs using the minimum number of colours is graph colouring. The colouring of nodes of a connected graph H such that adjoining nodes should get non-identical colours is a vertex colouring of H[3]. The chromatic number of a graph H is defined as the minimum number of colours required to colour the nodes of H so that adjoining nodes will get separate colours and is indicated by \(\chi(H)\)[3]. If graph H is a P chromatic graph then it's chromatic number \(\chi(H) = P\) [5]. And we cannot use more than four colours to colour any planar graph is stated in four colour theorem.

The graph colouring problem can be solved mainly using three types of methods heuristic methods, Metaheuristic methods, and hybrid methods. Among the heuristic method, DSATUR and RLF are most famous due to their calculation speed but their results are not satisfactory. Among metaheuristic methods, the local search method is popular due to its results and the population-based method gives good results for big graphs, whereas hybrid methods are popular because of their execution time. The classical vertex colouring algorithms are improved to get exact and heuristic algorithms for the problem. And the computational results for well-performing algorithms are mentioned by Enrico Malaguti and
Paolo Toth, Philippe Galinier et al. formulated heuristics like the greedy method, local search, population-based search for better and satisfactory results. Murat Aslan et al. tested heuristics algorithms like FF, LDF, Welsh and Powel, IDO, RLF, and DSATUR on benchmark graphs mentioned by DIMACS, and have demonstrated that DSATUR and RLF give better results for graph colouring. Welsh-Powel algorithm gives the best approximations on register allocation and RLF works better on Stanford queen graphs. A Shukla et al. proposed a list-based approach for graph colouring problem solutions efficiently without adding extra constraints. Manouchehr Zaker introduced a new colouring methodology which is a combination of the methodologies of Grundy colouring and b-colouring which satisfies an additional property[5]. Corwin Sinnamon proposed three algorithms for colouring of edges viz. Greedy Euler Colour, Euler Colour, and Random Euler Colour. Also, stated their time complexities for some simple graphs[1]. A new colouring algorithm named as R- Colouring was introduced by H Rafat by comparing elements of adjacency matrix[3]. K A Santoso et al. worked on a graph colouring algorithm using adjacency matrix. Here in the adjacency matrix row sum and column sum are calculated and using their comparison colours to the graph is given[4]. This paper is organized as below. Section 2 includes the HB colour matrix method, it’s different types, HB colour matrix algorithm for graph coloring and it’s illustration. In section 3, the output of a python program for certain standard graphs is given and a table containing some standard graphs and their chromatic number using the HB colour matrix method is given. Section 4 is a discussion includes comparison of our method over greedy method and explained it for Heawood graph. Section 5 states the conclusion of the paper while section 6 is an acknowledgement.

2. HB colour matrix method

Let G(V, E) be any graph. A HB colour matrix of a graph G is defined as C = (C_{ij})_{p \times p}, where
\begin{align*}
C_{ij} = \infty, & \text{ if labels of row } R_i \text{ and column } C_j \text{ have different colours} \\
C_{ij} = 0, & \text{ if labels of row } R_i \text{ and column } C_j \text{ have the same colour.}
\end{align*}
The rows or columns of this matrix are labelled by using vertices or edges or a region or any other property of the corresponding graph. HB colour matrix has only two elements, either 0 or \infty. There are different kinds of HB colour matrices which are given below.

2.1. Vertex HB colour matrix

Let G be a graph with n nodes or vertices say v₁, v₂, v₃, …, vₙ. A Vertex HB Colour matrix of a graph G is denoted as C(V) and is defined as
C(V) = (C_{ij})_{n \times n}, where
\begin{align*}
C_{ij} = \infty, & \text{ if } v_i \text{ and } v_j \text{ are adjacent vertices} \\
C_{ij} = 0, & \text{ if } v_i \text{ and } v_j \text{ are not adjacent vertices.}
\end{align*}

2.1.1. Properties of Vertex HB colour matrix

- Every Vertex HB Colour Matrix (VHBCM) is a symmetric matrix. All diagonal elements of this matrix are \infty.
- The number of zeros in each column or row is equivalent to the number of vertices that are non-adjacent to the corresponding vertex.
- The number of \infty in every column or row is equal to the aᵢ₊₁, where aᵢ is the number vertices adjacent to the corresponding vertex.
- If all elements of a row are \infty, then the corresponding vertex or node is adjacent to all remaining vertices or nodes of that graph.
- If all elements of a row are zeros except the diagonal element, then the corresponding vertex is not adjacent to all remaining vertices of that graph. Such vertex is either a null vertex or a vertex with loops only.
- If a VHBCM with n vertices contains all zeros except diagonal elements then the corresponding graph is either a Null graph or a disconnected graph. Such a graph is one colourable.
Theorem 2.1 If a VHBCM with \( n \) vertices contains all \( \infty \) then the corresponding graph is the complete graph on \( n \) vertices (\( K_n \)).

**Proof** Let \( G \) be a graph with \( n \) vertices whose VHBCM contains \( \infty \) everywhere. Hence every vertex of graph \( G \) is adjacent to remaining all vertices. Therefore \( G \) is a complete graph on \( n \) vertices.

Theorem 2.2 The VHBCM of a wheel graph with \( p \) vertices contains a row with all elements \( \infty \).

**Proof** Let \( G \) be a wheel graph with \( p \) vertices. Therefore there exist a vertex \( v \) in \( G \) which will be adjoint to the rest of the vertices, hence the corresponding row contains all \( \infty \).

2.1.2. Algorithm of HB colour matrix method for the vertex colouring of graphs. Let \( G \) be any graph with \( n \) vertices \( v_1, v_2, v_3, \ldots, v_n \). The following is an algorithm for the vertex colouring of any graph.

**Step 1:** Write the HB colour matrix \( C(V) \) of the given graph \( G \). Make assignments only in the upper triangular form of \( C(V) \) and denote it by \( H \).

**Step 2:** Select vertex \( v \) i.e. the first row. Find \( (v_i, v_j) = 0 \), for the smallest \( j \), \( j=2,3,4,\ldots,n \). If the smallest \( j \) is the \( k \) then assign the same colour say \( C_1 \) to \( v_1 \) and \( v_k \).

i) If \( \exists \) the smallest \( r \) such that \( (v_1, v_{k+r}) = 0 \) and \( (v_k, v_{k+r}) = 0 \), then assign the same colour to \( v_k \) and \( v_k+r \) i.e. \( C_1 \) colour. If \( \exists \) smallest \( s \) such that \( (v_1, v_{k+r+s}) = 0 \) then check labels of \( (v_k, v_{k+r+s}) \) and \( (v_k+r, v_{k+r+s}) \).

a) If one of them is \( \infty \) then cross \((v_1, v_{k+r+s}) = 0\).

b) If all are zero then assign colour \( C \) to \( v_{k+r+s} \). Continue in this way for all elements of row 1.

ii) If \( (v_1, v_{k+r}) = \infty \) and \( (v_k, v_{k+r}) = 0 \) then cross zero or if \( (v_1, v_{k+r}) = 0 \) and \( (v_k, v_{k+r}) = \infty \) then cross the zero of the corresponding place.

iii) If \( (v_1, v_{k+r}) = \infty \) and \( (v_k, v_{k+r}) = \infty \) then \( v_1, v_{k+r} \) should have different colours.

**Step 3:** Apply the same procedure for the second row vertex \( v_2 \), then \( v_3, v_4, \ldots, v_n \).

**Step 4:** If all zeros get assigned or crossed then check whether we colour all vertices or not. If vertices are remaining then assign a different colour to these vertices.

2.1.3. Illustration of VHBCM algorithm. Consider a graph \( G \) with five vertices viz. 1, 2, 3, 4, 5 , seven edges \( x, y, z, p, q, r, s \) and four regions \( A, B, C, D \) (Figure 1).

![Figure 1. Graph G](image1)

![Figure 2. Vertex colouring of graph G](image2)

The vertex HB colour matrix is created from graph \( G \) (figure 3(a)). The upper triangular of the HB matrix is formed as it's a symmetric matrix (figure 3(b)). Following the steps of HB colour matrix algorithm, an assignment is made at first zero of the first row i.e. cell (1,3) and strike out column 3 (figure 3(c)), hence vertex 1 and 3 will have the same colour.

![Figure 3. Implementation of VHBCM algorithm to graph G](image3)
Further, there is no more zero in the first row so move to row 2 and search for the first zero, here it is at cell (2,5) so make assignment at (2,5) and strike out the column 5 (figure 3(d)). This denotes vertex 2 and 5 will have the same colour. As row 4 is having all ∞ so assign the third colour to it. In VHBCM, colour assignments are, colour C1(Red) to vertices 1 and 3, colour C2(Green) to vertices 2 and 5, and colour C3(yellow) to vertex 4 (figure 2). Hence the vertex chromatic number of graph G is 3.

2.2. Edge HB colour matrix

Let G (V, E) be a graph with m number of edges say e1, e2, e3, . . . e m. An Edge HB Colour Matrix (EHBCM) of a graph G is denoted by C(E) and defined as C(E) = (Cij), mXm, where

\[ C_{ij} = \infty, \text{ if } e_i \text{ is adjoint to } e_j \text{ in graph } G, \]
\[ C_{ii} = \infty, \text{ if } i \text{ is equal to } j \text{ and } \]
\[ C_{ij} = 0, \text{ if } e_i \text{ is not adjoint to } e_j \text{ in the graph } G. \]

**Theorem 2.3** If an EHBCM is a diagonal matrix then all components of G are either K1 or K2 or vertex with loops.

**Proof** If EHBCM is a diagonal matrix then all non-diagonal elements are zeros, which means either edges are not adjacent to each other or vertices are isolated. Therefore no two edges of G are adjacent. So all components of G are either K1 or K2 or vertex with loops only.

**Theorem 2.4** If all elements of C(E) are ∞s then the corresponding graph is the star graph or cycle graph Cn.

**Proof** Consider an EHBCM of graph G with all elements ∞s. So every edge is adjacent to every other edge of graph G, Hence G is either a star graph or cycle graph Cn.

2.2.1. Algorithm of HB colour matrix method for the edge colouring of graphs. Let H be any graph with m number of edges e1, e2, e3, . . . em. An edge HB colour matrix of graph H is of order m.

By using analogy of the Vertex HB algorithm and replacing vertices by edges, resultant algorithm can be generated.

2.2.2. Illustration of an EHBCM algorithm. An edge chromatic number calculation of graph G (figure 1) using EHBCM algorithm is shown in figure 4. In EHBCM, colour assignments are, colour C1 (Red) to edges x and z, colour C2 (yellow) to edges y and q, colour C3 (green) to edges p and s, colour C4 (blue) to edge r. Thus the edge chromatic number of G is 4 (Figure 5).

2.3. Region HB colour matrix

Consider any planar graph H with regions or faces say F1, F2, F3, . . . Fr. A region HB Colour Matrix (RHBCM) of graph H is indicated by C(F) and defined as C(F) = (Cij), rXr, where

\[ C_{ij} = \infty, \text{ if } F_i \text{ is adjoint to } F_j, \]
\[ C_{ii} = \infty, \text{ if } i = j \text{ and } \]
\[ C_{ij} = 0, \text{ if } F_i \text{ is not adjacent to } F_j \text{ in graph } H. \]

2.3.1. Properties of RHBCM:
• Every RHBCM is a symmetric matrix of size r.
• If H is a null or zero graph then it's RHBCM is of order 1.
• RHBCM of non-planar graph does not exist.

Theorem 2.5 If all elements of RHBCM of planar graph H are $\infty$ then H is the perfect HB map.

Proof Let H be any planar graph with regions say $F_1, F_2, F_3, \ldots F_r$ where $r \leq 4$. If all elements of RHBCM of graph H are $\infty$, then all regions of H are adjoin to each other. So each region of H is the pivot region. Hence H must be the perfect HB map.

Theorem 2.6 If H is a planar graph then RHBCM of H does not involve 5 or more rows with all $\infty$s.

Proof Let H be any planar graph. Assume that RHBCM of H involves 5 or more rows with all elements as $\infty$s. Therefore, there are 5 or more regions that are adjacent to each other, which contradicts the assumption that H is a planar graph. The contrapositive of this statement is "If RHBCM of H involves 5 or more rows with all elements as $\infty$s then the corresponding graph is not planar".

Theorem 2.7 If H is a planar graph then the RHBCM of H does not involve a square submatrix of order 5.

Proof By generalizing, the logic of the theorem 2.6, we will get the proof.

Theorem 2.8 The RHBCM of maximal planar graph H, involves exactly four $\infty$ in each row or column.

Proof Let H be any maximal planar graph. Therefore every region of H is a triangle. There are $2(n-2)$ regions in H. Each triangle is adjoin to exactly three identical regions of H. So there are three $\infty$ in each row or column and one more $\infty$ at the diagonal place. Hence the proof.

Theorem 2.9 If two planar maps G and H are *isomorphic then their RHBCM varies only by the permutation of columns or rows.

Proof Suppose two planar maps G and H are *isomorphic. So, there is the bijective mapping from the regions of G to the respective regions of H and it keeps closeness and non-closeness of regions. Hence the RHBCM of these matrices varies by permutation of columns or rows.

2.3.2. Algorithm of HB colour matrix method for the region colouring of graphs. Let H be any graph with r edges $F_1, F_2, F_3, \ldots F_r$. The region HB colour matrix of H is of order r. Using analogy of the Vertex HB colour matrix algorithm and replacing vertices by regions will give HB colour matrix algorithm for the region colouring of graphs.

2.3.3. Illustration of RHBCM algorithm. Calculation of the region chromatic number of graph G (figure 1) is shown in figure 6.

In RHBCM, assign colour C1(Red) to regions B and D, colour C2 (Blue) to region C, and colour C3(Green) to region A (figure 7). Hence the region chromatic number of G is 3.

3. Computational Results
We have derived a python program using HB colour matrix algorithm and executed for a few standard graphs. The time complexity of the program is O(n) and space complexity is O(n).

3.1. Python program output
The vertex chromatic number outputs for Moser Spindle, Petersen Graph, Bidiakis Cube and Desargues graph are shown in figure 8, 9, 10 and 11 respectively.
3.2. Chromatic number for few more standard graphs using HBCM algorithm
Chromatic number by HB colour matrix method applied on some standard graphs are stated in Table 1.

| Sr. No. | Name Of the Graph          | Chromatic Number By HB Colour Matrix Method |
|---------|----------------------------|---------------------------------------------|
| 1       | Bidiakis Cube              | 3                                           |
| 2       | Bull Graph                 | 3                                           |
| 3       | chvatal Graph              | 4                                           |
| 4       | Moser Spindle              | 4                                           |
| 5       | Franklin Graph             | 2                                           |
| 6       | Frucht Graph               | 3                                           |
| 7       | Goldner–Harary Graph       | 4                                           |
| 8       | Golomb Graph               | 4                                           |
| 9       | Grotzsch Graph             | 4                                           |
| 10      | Mobius-Kantor graph        | 2                                           |
| 11      | Desargues Graph            | 2                                           |
| 12      | Herschel Graph             | 2                                           |
| 13      | Wanger Graph               | 3                                           |
| 14      | Petersen Graph             | 3                                           |
| 15      | Heawood Graph              | 2                                           |

4. Discussion
There are a variety of methods discovered for graph colouring problems so far. The greedy colouring method is widely used method among them and is the base for all graph colouring methods developed
so far. In greedy the vertex is coloured with the lowest value that doesn’t emerge among previously coloured neighbors. The greedy method is giving better approximation for certain graphs. It failing in some cases where it needs to colour the non-adjacent vertices using previously used colour. Also, they are time consuming for graphs with a large number of vertices as there is lots of comparison and tie breaking situations.

Though our method is giving same results as greedy, it executes with minimum number of operations. Also, HB colour matrix method is giving precise chromatic number for all planar graphs. The algorithm is operating on binary matrix, it is easily readable by computers making calculations efficiently in lesser time. This special assignment method for calculation of chromatic number is new and unique in graph colouring theory.

In our method if we change the labelling of the vertices it is assigning different colours to vertices in each case. But ultimately it is giving unique chromatic number to the particular graph in each labelling cases. The table below shows the execution of greedy method and HB colour matrix method to calculate chromatic number for Heawood graph (figure 12). It shows that greedy method needs to execute 12 times while HB colour matrix method gives result in 4 steps.

| Table 2. Chromatic number by greedy method vs HB colour matrix method for Heawood graph |
|-----------------------------------------------|
| Heawood graph | Greedy algorithm method | HB colour matrix method |
| 1) Label the vertices. | 1) Label the vertices. |
| 2) Pick any random vertex (suppose vertex 1) and colour it with first colour. | 2) Construct HB colour matrix. |
| 3) Consider any vertex out of remaining 13 vertices and colour it with the least numbered colour that has not been used on any previously coloured vertices adjacent to it. If all previously used colours appear on vertices adjacent to the picked vertex, assign a new colour to it. | 3) Give input of HB colour matrix to python program. |
| 4) Repeat the step 3, 12 times. | 4) Result of chromatic number as output. |

Figure 12. Heawood Graph

5. Conclusion
In this paper, we proposed a new and unique method for graph colouring named HB colour matrix method. The results of VHBCM, EHBCM, and RHBCM are proved. The HB colour matrix algorithm is presented using a special assignment. The proposed method is verified and numerical examples are carried out using the algorithm stated in the paper. The output of the python program for a few standard graphs are given in the paper. Also, the results of the HB colour matrix for some standard graphs are presented. This graph colouring method can be implemented for a total colouring of the graphs.

6. Acknowledgement
We are thankful to the whole team of ICDMS 2020 for organizing the conference. We are also greatful to reviewers for valuable suggestions.
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