A partition theorem for a large dense linear order

M. Džamonja∗
School of Mathematics
University of East Anglia
Norwich, NR4 7TJ, UK

J.A. Larson† and W.J. Mitchell† ‡
Department of Mathematics
University of Florida-Gainesville
358 Little Hall, PO Box 118105
Gainesville, FL 32611–8105, USA

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Abstract

Let $\mathbb{Q}_\kappa = (Q, \leq_Q)$ be a strongly $\kappa$-dense linear order of size $\kappa$ for $\kappa$ a suitable cardinal. We prove, for $2 \leq m < \omega$, that there is a finite value $t^+_m$ such that the set $[\mathbb{Q}]^m$ of $m$-tuples from $\mathbb{Q}$ can be divided into $t^+_m$ many classes, such that whenever any of these classes $C$ is colored with $< \kappa$ many colors, there is a copy $\mathbb{Q}^*$ of $\mathbb{Q}_\kappa$ such that $[\mathbb{Q}^*]^m \cap C$ is monochromatic. As a consequence we obtain that whenever we color $[\mathbb{Q}_\kappa]^m$ with $< \kappa$ many colors, there is a copy of $\mathbb{Q}_\kappa$ all $m$-tuples from...

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which are colored in at most $t_m^+$ colors. In other words, the partition relation $\mathbb{Q}_\kappa \rightarrow (\mathbb{Q}_\kappa)^m_{<\kappa/r}$ holds for some finite $r = t_m^+$.

We show that $t_m^+$ is the minimal value with this property. We were not able to give a formula for $t_m^+$ but we can describe $t_m^+$ as the cardinality of a certain finite set of types. We also give an upper and a lower bound on its value and for $m = 2$ we obtain $t_2^+ = 2$, while for $m > 2$ we have $t_m^+ > t_m$, the $m$th tangent number. The paper also contains similar positive partition results about $\kappa$-Rado graphs.

A consequence of our work and some earlier results of Hajnal and Komjáth is that a theorem of Shelah known to follow from a large cardinal assumption in a generic extension, does not follow from any large cardinal assumption on its own.

1 Introduction

For an infinite cardinal $\kappa$, the set $\kappa^{>2}$, ordered by end-extension, $\subseteq$, is the complete binary tree on $\kappa$ with root the empty sequence, $\emptyset$. By $s \wedge t$ denote the meet of $s$ and $t$, namely the longest initial segment of both $s$ and $t$. Call two elements $s$ and $t$ of $\kappa^{>2}$ incomparable if neither is an end-extension of the other. The lexicographic order for us will be the partial order $\prec_{\text{lex}}$ on $\kappa^{>2}$ defined by $s \prec_{\text{lex}} t$ if $s$ and $t$ are incomparable and $(s \wedge t)^{\lhd \langle 0 \rangle} \subseteq s$ and $(s \wedge t)^{\lhd \langle 1 \rangle} \subseteq t$. We also define a linear order $\preceq_Q$ on $\kappa^{>2}$ by letting $s \preceq_Q t$ if and only if one of the following conditions holds: (1) $s = t$; (2) $t^{\lhd \langle 0 \rangle} \subseteq s$; (3) $s^{\lhd \langle 1 \rangle} \subseteq t$; or (4) $s$ and $t$ are incomparable and $s <_{\text{lex}} t$.

Let us recall the definition of a (strongly) $\kappa$-dense linear order: it is a linear order $\preceq$ on a set $L$ in which for every two subsets $A, B$ of $L$ of size less than $\kappa$ satisfying the property that for all $a \in A$ and $b \in B$ the relation $a < b$ holds, there is $c \in L$ such that $a < c < b$ for all $a \in A$ and $b \in B$. Since $A$ or $B$ may be taken empty here, the definition in particular implies that there are no endpoints in the order. The adjective ‘strongly’ is used to distinguish this type of ordering from the strictly weaker notion in which for any $a < b$ in $L$ one is required to have $\kappa$ many $c$ with $a < c < b$. Such orders were first studied by Felix Hausdorff in 1908 ([10]) and have been of continuous interest since. A recent paper on the subject, where one can also find a number of further references, is [4] by M. Dzamonja and Katherine Thompson, where they give a classification of $\kappa$-dense linear orders which are also $\kappa$-scattered. In this paper we shall only deal with strongly $\kappa$-dense linear orders so we shall omit the adjective ‘strongly’ from our notation.
It is easy to check that for regular $\kappa$ the structure $\mathbb{Q}_\kappa := (\kappa^{>2}, <_\mathbb{Q})$ is a $\kappa$-dense linear order (see 1.8). In case that $\kappa^{<\kappa} = \kappa$ then of course this order has cardinality $\kappa$. All $\kappa$ that we shall work with will satisfy this additional cardinal arithmetic assumption. It is well known and easily proved using a back-and-forth argument that in this case the $\kappa$-dense linear order of size $\kappa$ is unique up to isomorphism, and that it is a $\kappa$-saturated homogeneous model of the theory of a dense linear order with no endpoints. (In fact these properties are equivalent to $\kappa = \kappa^{<\kappa}$, as follows from Saharon Shelah’s classification theory, see [21]).

For $\kappa = \omega$, $\mathbb{Q}_\omega = \mathbb{Q}$ is a countable dense linear order with no endpoints, so it has the order type $\eta$ of the rationals. An unpublished result of Fred Galvin as quoted in [8] is that $\mathbb{Q} \rightarrow [\mathbb{Q}]^2_{<\omega, 2}$, see the notation below. Denis C. Devlin [3] proved that

$$\mathbb{Q} \rightarrow (\mathbb{Q})^{n}_{<\omega, t_n} \text{ and } \mathbb{Q} \nrightarrow (\mathbb{Q})^{n}_{<\omega, t_n - 1}$$

where the value of $t_n$ is the $n$-th tangent number. The notation here means that for every $n$, when one colors $[\mathbb{Q}]^n$ with finitely many colors (this is the role of the part $< \omega$ in the subscript), there is a copy $\mathbb{Q}^*$ of $\mathbb{Q}$ with the property that $[\mathbb{Q}^*]^n$ is colored in at most $t_n$ colors. At the same time, $t_n$ is the smallest number for which such a statement holds.

Tangent numbers may be computed using the power series $\tan(x) = \sum_1^\infty t_n \frac{x^{2n-1}}{(2n-1)!}$. Devlin’s proof used the language of category theory. A proof of this theorem using trees was sketched in in the Farah-Todorcevic book [25]; a complete proof was given by Vojkan Vuksanovic [27] in which he uses the special case for the complete binary tree $\omega^{>2}$ of Keith R. Milliken’s theorem (17) about weakly embedded subtrees. Since many of the notions used by Vuksanovic generalize to arbitrary infinite $\kappa$ in place of $\omega$, one may wonder if his proof may be used to obtain a partition theorem for $\kappa$-dense linear orders. Here we use some of these ideas along with new insights to get such a theorem. Our work was also inspired by the strong diagonalization of Norbert Sauer in [19], the approach to similarities in a triple paper by Claude Laflamme, Sauer and Vuksanovic [13], and the use of collapses both in the Shelah [22] version of the Halpern-Läuchli Theorem and in another paper of Vuksanovic [26].

One difficulty of the generalization was that the special case of Milliken’s theorem on binary trees was only known to be valid for $\omega^{>2}$, not for an arbitrary $\kappa$. In particular, suppose $\kappa$ is uncountable and $\prec$ is a well-ordering
of \( \kappa > 2 \) with the property that whenever \( s \) is shorter than \( t \) then also \( s < t \). Define a coloring of the height 2 complete binary trees strongly embedded in \( \kappa > 2 \) by \( g(S) = 0 \) if and only if the lexicographic order and the \( < \) -order agree on the leaves of \( S \). For any strongly embedded copy of \( \kappa > 2 \), this coloring on the binary trees generated by pairs of nodes on the \( \omega \)th level is essentially the Sierpinski partition, so has edges of both colors.

Milliken’s theorem for weakly embedded subtrees follows from his theorem for strongly embedded subtrees, and to prove it, he uses a generalization of the Halpern-Läuchli Theorem due independently to Richard Laver and David Pincus (see [17]). Here we are able to generalize part of D. Devlin’s theorem starting with a theorem of Shelah from [22], which is a generalization of the Halpern-Läuchli theorem [9]. We slightly improve Shelah’s Theorem to colorings of antichains rather than only level sets. Let us now state our main result for large dense linear orders.

**Theorem 1.1.** For every natural number \( m \) there is a value \( t^+_m < \omega \) such that for any cardinal \( \kappa \) which is measurable in the generic extension obtained by adding \( \lambda \) Cohen subsets of \( \kappa \), where \( \lambda \) is some cardinal satisfying \( \lambda \rightarrow (\kappa)^{2m}_2 \), the \( \kappa \)-dense linear order \( \mathbb{Q}_\kappa \) satisfies

\[
\mathbb{Q}_\kappa \rightarrow (\mathbb{Q}_\kappa)^m_{<\kappa, t^+_m} \quad \text{and} \quad \mathbb{Q}_\kappa \not\rightarrow (\mathbb{Q}_\kappa)^m_{<\kappa, t^+_m - 1}.
\]

In Theorem 3.15 the positive partition relation is shown to hold for \( t^+_m \) the number of sparse vip \( m \)-types (defined in Section 3). The sparse vip \( m \)-types are closely related to those unique \( m \)-element strongly diagonal subsets of \( 2^{m - 2} \geq 2 \) that are representatives of the “essential types” in [27]. If we close such a strongly diagonal set under initial segments and add a vip level order, we obtain a sparse vip \( m \)-type, and all sparse vip \( m \)-types are obtained in this way.

In Theorem 5.8 we show that the negative partition relation holds for the same value of \( t^+_m \). In Theorem 5.9 we show that for \( \kappa \) as in Theorem 1.1 there is a canonical partition \( \mathcal{C} = \{ C_0, C_1, \ldots, C_{t^+_m - 1} \} \) of \( [\mathbb{Q}_\kappa]^m \). That is, a partition whose classes are persistent and indivisible. We say \( C_j \) is persistent if for every \( \kappa \)-dense \( Q^* \subseteq \mathbb{Q} \) the set \( [Q^*]^m \cap C_j \) is non-empty. We say \( C_j \) is indivisible if for every coloring of \( C_j \) with fewer than \( \kappa \) many colors, there is a \( \kappa \)-dense subset \( Q^* \subseteq \mathbb{Q} \) on which \( [Q^*]^m \cap C_j \) is monochromatic.

Richard Rado [18] constructed a (strongly) universal countable graph in 1964. That is, he constructed a countable graph for which every countable graph is an induced subgraph. By a \( \kappa \)-Rado graph we mean a graph \( G \) of
size $\kappa$ with the property that for every two disjoint subsets $A$, $B$ of $G$, each of size $< \kappa$, there is $c \in G$ connected to all points of $A$ and no point of $B$. The existence of such $G$ follows from the assumption $\kappa^+ = \kappa$.

Note that a $\kappa$-Rado graph embeds every graph with at most $\kappa$ vertices. That is, it is universal for the family of graphs of size at most $\kappa$. For $\kappa = \omega$, this graph is also called the infinite random graph.

Interest in Rado graphs and the uncountable continues. Let $G_\omega$ denote the $\omega$-Rado graph. Recently Gregory Cherlin and Simon Thomas [2] have shown that for any infinite cardinals $\kappa \leq \lambda$, the assumption $\lambda \leq 2^\kappa$ is equivalent to the existence of a graph $G^*$ of size lambda which is elementarily equivalent to $G_\omega$ and which has a vertex whose set of neighbours has size $\kappa$.

Here is our main result on $\kappa$-Rado graphs.

**Theorem 1.2.** For every natural number $m$ there is a value $r_m^+ < \omega$ such that for any cardinal $\kappa$ which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda$ is some cardinal satisying $\lambda \to (\kappa)^m_\omega$, the $\kappa$-Rado graph $G_\kappa$

\[ \begin{align*}
G_\kappa & \to (G_\kappa)^m_{<\kappa, r_m^+} \quad \text{and} \quad G_\kappa \nrightarrow (G_\kappa)^m_{<\kappa, r_m^+-1}. 
\end{align*} \]

In Theorem 9.14 the positive partition relation is shown to hold for $r_m^+$ the number of vip $m$-types (defined in Section 3). The vip $m$-types are closely related to those unique $m$-element strongly diagonal subsets of $2^{m-2}$ that are representatives of the “essential types” used by Laflamme, Sauer and Vukasnovic in [13] and by Vukasnovic in [26] for the countable Rado graph. If we close such a strongly diagonal set under initial segments and add a vip level order, we obtain a vip $m$-type, and all vip $m$-types are obtained in this way.

In Theorem 10.9 we show that the negative partition relation holds for the same value of $r_m^+$.

It is not known if the large cardinal assumptions used in the proof of Shelah’s Theorem from [22] are optimal; in Section 8 we comment more on this as well as on the consistency strength of these requirements. We note that in conjunction with a result of András Hajnal and Péter Komjáth from [8] our theorem gives some necessary indestructibility conditions on $\kappa$ from Shelah’s Theorem. The paper also includes a section with a proof of the particular variant of Shelah’s Theorem that we need.

For the remainder of the paper an unattributed $m$ will mean a natural number with $2 \leq m$ and $\kappa$ a cardinal satisfying the hypotheses of Theorem
for some number $m$. In particular, $\kappa = \kappa^{<\kappa}$ and $\kappa$ is a strong limit.

The paper is organized as follows. In Section 2 we define the notion $\prec$-similarity, and state the special variant of Shelah’s Theorem that we will use.

In Section 3 we define the notions of diagonal, vip order and sparse vip $m$-type and use them together with our variant of Shelah’s Theorem to prove Theorem 3.15.

In Section 4 we show that for any $S \subseteq \kappa^{>2}$ with $(S, <_Q)$, one can build an almost perfect $\kappa$-dense subtree inside the tree of nodes of $\kappa^{>2}$ with a $\kappa$-dense set of extensions in $S$.

In Section 5 we use the construction techniques of Section 4 to prove Theorem 5.8 by showing all sparse vip $m$-types are embeddable in every sparse diagonal $\kappa$-dense subset.

In Section 6 we show the critical numbers $t^+_m$ for the $\kappa$-dense linear order are bounded below by $t_m$ and bounded above by $t_m(m-1)\prod_{i<m-1}(i!)^2$, and indicate how one may compute the value of $t^+_m$ recursively. We conclude the section with a table of small values of $t^+_m$ quoted from an upcoming paper by Jean Larson [14].

In Section 7 we prove the variant of Shelah’s Theorem that we use, formulating it using colorings of antichains. In Section 8 we comment on the necessity of the use of large cardinals in our theorem and in Shelah’s Theorem and the way that the results of this paper shed light on that question. We also give some open questions.

In Section 9 we use nuanced diagonalization to prove Theorem 9.14 giving the upper bound on the critical values $r^+_m$ for $\kappa$-Rado graphs.

In Section 10 we prove a reduction theorem for the translations to tree form of increasing embeddings of the $\kappa$-Rado graph $G_\kappa$: for every such translation with range $D$ there is always a particularly nice form of a diagonalization which has range a subset of $D$ and is itself the translation of an increasing embedding of $G_\kappa$ into itself. We use this reduction to prove Theorem 10.9 by showing every vip $m$-type is embeddable in the range of these particularly nice diagonalizations.

The remainder of this introduction is devoted to background information, including some definitions, notation, and the statements of some theorems that will be used as tools. For any cardinal $\lambda$, let $[A]^\lambda$ denote the collection of all subsets of $A$ of cardinality $\lambda$, and let $[A]^{<\lambda}$ denote the collection of all subsets of $A$ of cardinality less than $\lambda$.

For any tree $T = (T, \subseteq)$, a node $s$ is a leaf of $T$ or terminal node of $T$
if for all \( t \in T \setminus \{s\} \), one has \( s \not\subseteq t \). The notion of the meet of two nodes has already been defined, in the first paragraph of the Introduction. If \( s \) is a node of \( T \) and both \( s\langle 0 \rangle \) and \( s\langle 1 \rangle \) have extensions in \( T \), then we call \( s \) a splitting node of \( T \). For any subset \( S \subseteq T \), let \( S^\wedge \) denote the meet closure of \( S \), i.e. the set \( \{ u : u = s \land t \text{ for some } s, t \in S \} \). Note that \( S^\wedge \supseteq S \) and that \( S^\wedge \) is closed under meets and has a unique node of the smallest length.

**Definition 1.3.** For any tree \( T \) of sequences ordered by end extension, and any node \( s \in T \), define the set of immediate successors of \( s \) in \( T \) as

\[
\text{IS}(s,T) := \{ t \in T : s \subseteq t \land (\forall u)(s \subset u \subseteq t \implies u = t) \}.
\]

**Definition 1.4.** For any tree \( T \) of sequences under end extension, the \( \alpha \)th level of \( T \), in symbols \( T(\alpha) \), is the set of all nodes \( t \in T \) for which \( \alpha \) is the order type of the set of predecessors of \( t \), namely \( \{ s \in T : s \not\subset t \} \). For \( s \in T \) the length \( \lg(s) \) is defined to be \( \alpha \) if and only if \( s \in T(\alpha) \). Call \( T \) an \( \alpha \)-tree if each branch of \( T \) has order type \( \alpha \).

The above notion of an \( \alpha \)-tree in the case that \( \alpha \) is a cardinal differs from the usual notion of an \( \alpha \)-tree (see e.g. [12]), which is defined as a tree of height \( \alpha \) all of whose levels have size \( < \alpha \). We trust that no confusion will arise, but emphasize that our definition is given above. Note that for the complete binary tree \( T = \kappa>2 \), the levels of \( T \) are \( T(\alpha) = \alpha 2 \) for \( \alpha < \kappa \). Call \( A \subseteq T \) a level set if \( A \subseteq T(\alpha) \) for some \( \alpha \).

Milliken (see Definition 1.2 of [17]) defined a notion of strongly embedded tree which can be simplified in the case of a binary tree. The key idea is that there are splitting levels (see the range of \( h \) below); all nodes of a strongly embedded tree split at splitting levels as much as is possible, and no splitting occurs elsewhere. Another way to say this is that a strongly embedded subtree of height \( \alpha \) is a copy of \( ^{\alpha}>2 \) in \( ^{\kappa}>2 \) where levels are mapped to levels. Note that a strongly embedded subtree \( S \) of \( ^{\kappa}>2 \) is not required to be an induced subtree of \( ^{\kappa}>2 \), in the sense that it is not required that \( S \) is closed under \( \subseteq \).

**Definition 1.5.** A subset \( S \subseteq ^{\kappa}>2 \) is a strongly embedded subtree of \( ^{\kappa}>2 \) if

1. \((S, \subseteq)\) is an \( \alpha \)-tree for some \( \alpha \leq \kappa \);
2. \( S \) has a root and every non-maximal node \( s \in S \) has exactly one extension of \( s\langle 0 \rangle \) and one extension of \( s\langle 1 \rangle \) in \( \text{IS}(s,S) \);

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(3) there is a level assignment function \( h : \alpha \rightarrow \kappa \) which is a strictly increasing function such that for all \( \beta < \alpha \), \( S(\beta) \subseteq h(\beta)2 \);

(4) for any limit \( \beta < \alpha \), for any \( s \neq t \in S(\beta) \), there is \( \delta < \sup \{ h(\gamma) : \gamma < \beta \} \) such that \( s|\delta \neq t|\delta \).

Note of course that part (4) above is irrelevant for strongly embedded trees of height \( \leq \omega \), as originally studied by Milliken. As mentioned above, strongly embedded subtrees of \( \kappa>2 \) are copies of full binary trees of height \( \leq \kappa \) where levels are mapped to levels. The precise statement of this is the content of Lemma 1.7.

**Definition 1.6.** For any subsets \( S_0 \) and \( S_1 \) of \( \kappa>2 \), a function \( e : S_0 \rightarrow S_1 \) is a strong embedding if it is an injection with the following preservation properties:

1. (extension) \( s \subseteq t \) if and only if \( e(s) \subseteq e(t) \);

2. (level order) \( \lg(s) < \lg(t) \) if and only if \( \lg(e(s)) < \lg(e(t)) \) and \( \lg(s) = \lg(t) \) if and only if \( \lg(e(s)) = \lg(e(t)) \);

3. (passing number) if \( \lg(s) < \lg(t) \), then \( e(t)(\lg(e(s))) = t(\lg(s)) \).

**Lemma 1.7.** An \( \alpha \)-tree \( S \subseteq \kappa>2 \) is strongly embedded in \( \kappa>2 \) if and only if there is a strong embedding \( e : \alpha>2 \rightarrow \kappa>2 \) whose range is \( S \).

**Proof.** If \( e : \alpha>2 \rightarrow \kappa>2 \) is a strong embedding, then there is a level assignment function \( h : \kappa \rightarrow \kappa \) witnessing the strong embedding in that for all \( \beta < \alpha \), if \( s \in \beta2 \), then \( e(s) \in h(\beta)2 \) and the immediate successors of \( e(s) \) in the range of \( e \) are \( e(s^\sim(0)) \) and \( e(s^\sim(1)) \), which are both in \( h(\beta+1)2 \). Thus if \( e \) is a strong embedding its image is a strongly embedded tree.

For the other direction, suppose \( S \subseteq \kappa>2 \) is a strongly embedded \( \alpha \)-tree and \( h : \alpha \rightarrow \kappa \) is the level assignment function that witnesses it. We shall define \( e \) on \( \alpha>2 \) such that \( e \) maps \( \beta2 \) into \( h(\beta)2 \) for all \( \beta < \alpha \), as follows. Let \( e(\langle \rangle) \) be the root of \( S \). Given \( t \in \alpha>2 \) non-maximal and of height \( \beta \), let \( e(t^\sim(\langle l \rangle)) \) for \( l < 2 \) be the unique extension of \( e(t^\sim(\langle l \rangle)) \) in \( IS(e(t), S) \). For \( \beta < \alpha \) limit and \( t \in \beta2 \), note that \( \bigcup_{s \subseteq t} e(s) \) is an element of \( \kappa>2 \) and that it must have an extension in \( h(\beta)2 \), as \( S \) is an \( \alpha \)-tree. By requirement (4) in Lemma 1.5, this extension is unique and we choose it as \( e(t) \). Now \( e \) is a strong embedding whose image is \( S \).

\[ \square \]
Let us finish the section by proving the above mentioned fact that for regular \( \kappa \) the order \(<_Q \kappa >2\) is \( \kappa \)-dense.

**Lemma 1.8.** The order \((\kappa >2,<_Q)\) is \( \kappa \)-dense if and only if \( \kappa \) is a regular cardinal.

**Proof.** Suppose that \( \kappa \) is regular and \( A, B \subseteq \kappa >2 \) are both of size \( < \kappa \) and such that for all \( a \in A \) and \( b \in B \) we have \( a <_Q b \) (we write this as \( A <_Q B \)). Let \( B^* := \{ t \in \kappa >2 : (\exists b \in B)t^-\langle 0 \rangle \subseteq b \} \). If \( a \in A \) and \( t \in B^* \) then there is \( b \in B \) such that \( t^-\langle 0 \rangle \subseteq b \), in particular \( b <_Q t \). By the transitivity of \( <_Q \) we obtain \( a <_Q t \), and hence \( A <_Q B^* \).

Let \( \gamma \) be the minimal ordinal such that all elements of \( A \cup B \) have length \( < \gamma \), so by the regularity of \( \kappa \) we have \( \gamma < \kappa \). We define \( c \in \kappa >2 \) by defining \( c(\alpha) \) for \( \alpha < \gamma \) by recursion on \( \alpha \). If \( c|\alpha \in B \cup B^* \) we set \( c(\alpha) = 0 \), and we set \( c(\alpha) = 1 \) otherwise. We claim that \( A <_Q c <_Q B \). If \( b \in B \) then the length of \( b \) is less than the length of \( c \) and hence either \( b \subseteq c \) or \( b \) and \( c \) are incomparable. In the first case we have defined \( c \) so that \( b^-\langle 0 \rangle \subseteq c \). In the second case, if \( \alpha \) is the length of \( b \cap c \) and \( b(\alpha) = 0 \), then \( t := b|\alpha \in B^* \) so we have defined \( c(\alpha) = 0 \), contradicting the choice of \( \alpha \). So \( b(\alpha) = 1 \) and \( c(\alpha) = 0 \). In both cases we have \( c <_Q b \), and hence \( c <_Q B \).

If \( a \in A \) and \( a \subseteq c \) then we claim \( a^-\langle 1 \rangle \subseteq c \). Otherwise, since the length of \( a \) is strictly less than that of \( c \) we have \( a^-\langle 0 \rangle \subseteq c \), so \( a = c|\alpha \) must be a member of \( B \cup B^* \), a contradiction. If \( a \) and \( c \) are incomparable we can similarly show that \( a <_{\text{lex}} c \). In any case \( a <_Q c \) and we have proved that \( A <_Q c \). This proves that \( Q_\kappa \) is \( \kappa \)-dense.

Suppose now that \( \kappa \) is singular and \( \kappa_i \) for \( i < \text{cf}(\kappa) \) is an increasing sequence of limit ordinals with limit \( \kappa \), and with \( \kappa_0 = \kappa \). For all \( i \) let \( a_i \) be the sequence in \( \kappa >2 \) which is constantly equal to 1, and let \( A := \{ a_i : i < \text{cf}(\kappa) \} \). We then have \( |A| < \kappa \), yet we claim that there is no \( c \in \kappa >2 \) with \( A <_Q c \). Namely, let \( c \in \alpha >2 \) for some \( \alpha < \kappa \) and let \( i \) be such that \( \alpha \in [\kappa_i, \kappa_{i+1}) \). Then \( c^-\langle 1 \rangle \subseteq a_{i+2} \), so \( A <_Q c \) cannot hold. This proves that \( Q_\kappa \) is not \( \kappa \)-dense.

\( \square \)

An argument similar to the first part of the above proof is presented in the proof of Lemma 3.11.
2 Uniformization

In this section we state a particular variant of a theorem of Shelah. His theorem was stated as a generalization of the Laver-Pincus version of the Halpern-Läuchli Theorem, and the variant we state is a generalization of Milliken’s Ramsey theorem for finite weakly embedded subtrees.

Before we can state Shelah’s Theorem, we need the definition of two subsets being \( \prec \)-similar, where \( \prec \) is a level ordering of \( \kappa > 2 \):

Definition 2.1. Say that \( \prec \) is a level ordering of \( \kappa > 2 \) or alternatively that \( \prec \) is an ordering of the levels of \( \kappa > 2 \), if \( \prec \) extends the length order, i.e. \( \prec \) is a linear order of \( \kappa > 2 \) and \( \lg(s) < \lg(t) \) implies \( s \prec t \).

Milliken’s Theorem for weakly embedded subtrees requires us to recognize subtrees of the same embedding type. We will be counting the number of types of particularly nice trees at a later point in the paper, so it is convenient to relate these types to specific finite examples called similarity trees.

Definition 2.2. A similarity tree is a finite subtree of \( \omega > 2 \) closed under initial segments and such that every level contains at least one leaf (terminal) node or meet of leaf nodes (split point). An ordered similarity tree is a similarity tree \( t \) with an ordering \( \prec_t \) of its levels.

For a finite subset \( x \) of \( \kappa > 2 \) we define as \( \text{clp}(x) \) the subtree \( y \) of \( \omega > 2 \) that includes the root and is of minimal possible height such that there is a strong embedding from \( x^\wedge \) onto the closure \( z^\wedge \) of the set \( z \) of terminal nodes of \( y \). If \( \prec \) is a given order of \( \kappa > 2 \) then \( \prec_x \) is the order on \( \text{clp}(x) \) induced by the strong embedding from \( \text{clp}(x) \) to \( x \) and \( \prec \).

To illustrate the definition, consider a specific antichain:

\[
x = \{ \langle 0, 0, 0, 1 \rangle, \langle 0, 0, 1 \rangle, \langle 0, 1, 0, 0, 0, 0, 0, 1 \rangle \}.
\]

Further suppose that \( \prec \) is the following ordering of the levels: \( \langle 0, 0 \rangle \prec \langle 0, 1 \rangle; \langle 0, 0, 0 \rangle \prec \langle 0, 1, 0 \rangle \prec \langle 0, 0, 1 \rangle; \text{ and } \langle 0, 0, 0, 1 \rangle \prec \langle 0, 1, 0, 0 \rangle \). The corresponding similarity tree is \( \{ \text{clp}(x), \subseteq \} \) where

\[
\text{clp}(x) = \{ \emptyset, \langle 0 \rangle, \langle 1 \rangle, \langle 0, 0 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle, \langle 0, 0, 0 \rangle, \langle 1, 0, 0 \rangle, \langle 1, 0, 0, 0 \rangle \}.
\]

The induced ordering of the levels is \( \langle 0 \rangle \prec_x \langle 1 \rangle; \langle 0, 0 \rangle \prec_x \langle 1, 0 \rangle \prec_x \langle 0, 1 \rangle; \text{ and } \langle 0, 0, 1 \rangle \prec_x \langle 1, 0, 0 \rangle. \)
Definition 2.3. Suppose that \(\prec\) is an ordering of the levels of \(\kappa^+\). Two antichains \(x\) and \(y\) in \(\kappa^+\) are similar if \(\text{clp}(x) = \text{clp}(y)\), and \(\prec\)-similar if the ordering \(\prec\) induces the same ordering \(\prec_x = \prec_y\) on the collapsed trees. In this case we call \((\text{clp}(x), \prec_x)\) the ordered similarity type of \((x, \prec)\) and \((y, \prec)\).

We remark that if \(x\) and \(y\) are \(\prec\)-similar antichains, then \(\text{clp}(x)\) and \(\text{clp}(y)\) are subtrees of \(\kappa^+\) having the same weak embedding type, a notion which appears in various places in the literature, see e.g. [17] and [27]. The notion of \(\prec\)-similarity given above is a translation of the definition of similarity by Shelah in Definition 2.2 [22]. It also corresponds to the notion of strong similarity from [13], which coincides with the notion of similarity of that paper when restricted to strongly diagonal sets.

Theorem 2.5 below is a variant of Shelah’s Lemma 4.1 of [22] together with Conclusion 4.2 where it is called “\(\text{Pr}_{ht}^f(\kappa, m, \sigma)\) (even with (3))”. In fact, except for the assertion that \(e\) preserves \(\prec\) and that we are dealing with coloring of antichains rather than level sets (a level set is a subset of \(\alpha^+\) for some \(\alpha < \kappa\)), our Theorem 2.5 is equivalent to Shelah’s Conclusion 4.2 from that paper with option (3)(b). The fact that \(e\) preserves \(\prec\) implies that option (3)(a) is automatically satisfied as well. Lemma 7.1 which is the main lemma in the proof of Theorem 2.5 is equivalent to Shelah’s Lemma 4.1 with option (3)(a). In Shelah’s notation the superscript \(f\) of Lemma 4.1 indicates that one gets a strongly embedded tree \(T\), and the subscript \(ht\) means that the coloring restricted to \(m\)-element level sets of \(T\) is homogeneous, i.e. is constant on subsets whose collapses under the level assignment function of \(T\) are \(\prec\)-similar. The star orderings, \(\prec^*_\alpha\), in this option, are defined from the \(0\)-orderings, \(\prec^0\). We use \(\prec\) in place of \(\prec^0\), and will define the order of interest at a later point. We write \(e(\prec)\) for \(\{ (e(a), e(b)) : a \prec b \}\). Thus given a strong embedding \(e : \kappa^+ \rightarrow \kappa^+\), \(s \prec^*_\alpha t\) is translated by \(e(s) \prec e(t)\), which holds if and only if \(s \prec t\). We also note that the statement of the theorem as we give it only refers to a dense set of elements \(w\). A proof of Theorem 2.5 will be given in section 7.

The following notation is introduced for convenience.

Definition 2.4. Write \(\text{Cone}(w)\) for the set of all extensions \(t \supseteq w\) in \(\kappa^+\).

Theorem 2.5. [Shelah [22]] Suppose that \(m < \omega\) and \(\kappa\) is a cardinal which is measurable in the generic extension obtained by adding \(\lambda\) Cohen subsets of \(\kappa\), where \(\lambda \rightarrow (\kappa)^{2^m}_\lambda\). Then for any coloring \(d\) of the \(m\)-element antichains of \(\kappa^+\) into \(\sigma < \kappa\) colors, and any well-ordering \(\prec\) of the levels of \(\kappa^+\), there is a strong embedding \(e : \kappa^+ \rightarrow \kappa^+\) and a dense set of elements \(w\) such that
1. $e(s) \preceq e(t)$ for all $s \prec t$ from $\text{Cone}(w)$, and

2. $d(e[a]) = d(e[b])$ for all $\prec$-similar $m$-element antichains $a$ and $b$ of $\text{Cone}(w)$.

3. **Upper bound for dense linear orders**

In this section we prove a limitation of colors result for $\kappa$-dense linear orders using Shelah’s Theorem 2.5.

We turn to some ideas from work of Sauer, Laflamme and Vuksanovic and others for ways to guarantee that the $m$-element sets of our yet to be chosen transverse set have a minimal number of (unordered) weak embedding types. The notions of diagonal set and strongly diagonal set are used in [13], for example. Our definitions below are simplifications of those definitions to the special case of binary trees.

**Definition 3.1.** Call $A \subseteq \kappa^+2$ diagonal if it is an antichain (its elements are pairwise incomparable), and its meet closure, $A^\wedge$, is transverse. Call it strongly diagonal if, in addition, for all all $t \in A$ and all $s \in A^\wedge \setminus \{t\}$, the following implication holds:

$$(\lg(s) < \lg(t) \text{ and } t(\lg(s)) = 1) \implies s \subseteq t \text{ or } s \text{ has no extension in } A.$$  

**Lemma 3.2.** For any finite diagonal set $a \subseteq \kappa^+2$, its meet closure has $|a^\wedge| = 2|a| - 1$ elements and $\text{clp}(a) \subseteq 2|a| - 2 \geq 2$. Thus for positive $m < \omega$, the number of ordered similarity types of $m$-element diagonal sets is finite.

In the case of $\kappa$-dense linear orders, we are particularly interested in sparse diagonal sets.

**Definition 3.3.** Call $A \subseteq \kappa^+2$ sparse if for all $t \in A$ and $s \in A^\wedge \setminus \{t\}$ the following implication holds:

$$(\lg(s) < \lg(t) \text{ and } t(\lg(s)) = 1) \implies s \subseteq t.$$  

Notice that if $A$ is a sparse diagonal set, then it is a strongly diagonal set. However, a strongly diagonal subset need not be sparse.

We are interested in a special collection of level orders. We call them $D$-vip orders since the elements of $D$ are Very Important Points, with special roles to play in the orders, and these roles continue to be played even when finite subsets are collapsed.
Definition 3.4. Suppose $T$ is a subtree of $\kappa>2$ and $D \subseteq \kappa>2$. Call $\prec$ a \textit{pre-$D$-vip order} on $T \subseteq \kappa>2$ if $D$ is transverse and $\prec$ is a well-ordering of the levels of $T$ such that for every $d \in D$, $d$ is the $\prec$-least element of its level, $T(\lg(d))$, and for all $u, v \in T(\lg(d)) \setminus \{d\}$, $\prec$ satisfies the following condition:

1. if $d \land u \not\subseteq d \land v \neq d$, then $u \prec v$, and

If $D$ is diagonal, call $\prec$ a \textit{$D$-vip order} if it is a pre-$D^\land$-vip order which also satisfies the following condition for all $d \in D^\land$ and for all $u, v \in D \setminus \{d\}$:

2. if $d \land u = d \land v \neq d$ and $u(\lg(d)) < v(\lg(d))$, then $u \prec v$.

It is not difficult to construct a pre-$D$-vip order for a transverse set $D$.

Lemma 3.5. If $D \subseteq \kappa>2$ is transverse, then there is a pre-$D$-vip order of $\kappa>2$. If $D \subseteq \kappa>2$ is a sparse diagonal set and $\prec$ is a pre-$S$-vip order for some $S$ with $D^\land \subseteq S$, then $\prec$ is a $D$-vip order.

Proof. Let $\prec$ be any well-ordering of the levels of $\kappa>2$. Use recursion to define a pre-$D$-vip order $\prec$ by adjusting $\prec$ separately on each level which has an element of $D$.

Note that if $D \subseteq \kappa>2$ is a sparse diagonal set, and $\prec$ is a pre-$S$-vip order for some $S$ with $D^\land \subseteq S$, then the second clause never applies so $\prec$ is a $D$-vip order.

Now that we have at least some idea of the level order we will use, we define a family of ordered similarity types which is rich enough to have all the ordered similarity types of $m$-element subsets of sparse diagonal sets $D \subseteq \kappa>2$ for a $D$-vip level ordering $\prec$.

Definition 3.6. Call $\tau$ an \textit{$m$-type} if it is a downward closed subtree of $2m-2 \geq 2$ whose set $L$ of leaves is an $m$-element strongly diagonal set. Call $(\tau, \prec)$ a \textit{vip $m$-type} if $\tau$ is an $m$-type and $\prec$ is an $L$-vip order. If $L$ is sparse, then $\tau$ is called a \textit{sparse $m$-type} and $(\tau, \prec)$ is called a \textit{sparse vip $m$-type}.

Lemma 3.7. Assume $D \subseteq \kappa>2$ is a strongly diagonal set and $\prec$ is an ordering of the levels of $\kappa>2$ which is a $D$-vip order. Then for all $m$-element sets $x \subseteq D$, $(\clp(x), \prec_x)$ is a vip $m$-type, and if $D$ is sparse, then $(\clp(x), \prec_x)$ is a sparse vip $m$-type.
Proof. The collapse of any \( m \)-element strongly diagonal set \( x \) is a subtree closed under initial segments whose set of leaves, \( L \), is strongly diagonal. Moreover, since the order \( \prec \) is \( x \)-vip, it follows that \( \prec_x \) is \( L \)-vip. If, in addition, \( D \) is sparse, then \( \text{clp}(x) \) is also sparse, by definition of collapse. □

In order to apply Shelah’s Theorem, we shall need an ordering of the levels of \( \kappa>2 \) and a conveniently chosen subset of \( \kappa>2 \) the antichains of which will realize the smallest possible number of weak embedding types. Toward that end, we introduce cofinal transverse subsets of \( \kappa>2 \).

**Definition 3.8.** A subset \( S \subseteq \kappa>2 \) is cofinal above \( w \) if for all \( t \in \text{Cone}(w) \) there is some \( s \in S \) with \( t \subseteq s \). If \( w = \emptyset \), we say \( S \) is cofinal.

**Definition 3.9.** Call \( A \subseteq \kappa>2 \) transverse if distinct elements of \( A \) have different lengths.

By recursion one can construct a cofinal transverse subset of \( \kappa>2 \).

**Lemma 3.10.** There is a cofinal transverse subset of \( \kappa>2 \).

**Lemma 3.11.** If \( S \subseteq \kappa>2 \) is cofinal above \( w \) and transverse, then \( (S \cap \text{Cone}(w), \prec) \) is \( \kappa \)-dense.

**Proof.** Suppose \( A, B \subseteq S \cap \text{Cone}(w) \) are two disjoint subsets of size less than \( \kappa \) with \( a <_Q b \) for all \( a \in A \) and \( b \in B \). Use the fact that \( \text{Cone}(w) \) is \( \kappa \)-dense to find \( d \in \text{Cone}(w) \setminus (A \cup B) \) with \( a <_Q d <_Q b \) for all \( a \in A \) and \( b \in B \). Let \( \alpha \) be a limit ordinal larger than the length of any element of \( A \cup B \). Let \( d' \) be the extension of \( d \upharpoonright (1) \) by zeros of length \( \alpha \) and let \( c \) be an extension of \( d' \upharpoonright (0) \) in \( S \). Since \( d' \) and \( c \) are longer than any element of \( A \cup B \), they are not in \( A \) nor in \( B \). Then \( d <_Q c <_Q d' \) and \( d' <_Q b \) for all \( b \in B \). Thus \( c \) is the required witness showing \( (S, <_Q) \) is \( \kappa \)-dense. □

**Definition 3.12.** Assume \( z, w \in \kappa>2 \) and \( S \subseteq \kappa>2 \) is cofinal and transverse. Call \( g \) a sparse diagonalization of \( \kappa>2 \) into \( S \cap \text{Cone}(w) \) if \( g : \kappa>2 \to S \cap \text{Cone}(w) \) is an injective \( <_Q \)-preserving map such that \( D := g[\kappa>2] \) is a sparse diagonal subset, \( D^\land \subseteq S \cap \text{Cone}(w) \) and the following conditions hold:

1. for all three element diagonal sets \( \{ x, u, v \} \), if \( x \land u = x \land v \), then \( g(x) \land g(u) = g(x) \land g(v) \);
2. for all \( x, y, u, v \in \kappa>2 \), if \( \lg(x \land y) < \lg(u \land v) \), then \( \lg(g(x) \land g(y)) < \lg(g(u) \land g(v)) \);
3. for all sparse diagonal \( E \subseteq \kappa^{>2} \), \( \text{clp}(E) = \text{clp}(g[E]) \).

If \( w = \emptyset \), then we call \( g \) a sparse diagonalization of \( \kappa^{>2} \) into \( S \).

**Lemma 3.13.** [First Diagonalization Lemma] Assume \( \kappa = 2^{<\kappa}, w \in \kappa^{>2} \) and \( S \subseteq \kappa^{>2} \) is cofinal and transverse. Then there is a sparse diagonalization \( \varphi \) of \( \kappa^{>2} \) into \( S \cap \text{Cone}(w) \).

**Proof.** Let \( < \) be a well-ordering of the levels of \( \kappa^{>2} \). Let \( \langle t_\alpha : \alpha < \kappa \rangle \) list the elements of \( \kappa^{>2} \) in \( < \)-increasing order (recall that \( \kappa = 2^{<\kappa} \) is assumed).

Define functions \( \varphi_0, \varphi_1 \) and \( \varphi \) on \( \kappa^{>2} \) by recursion on \( \alpha \). To start the recursion, notice that \( t_0 = \emptyset \), and let \( \varphi_0(t_0) \) be an element of \( S \cap \text{Cone}(w) \) of minimal length.

If \( \varphi_0(t_\alpha) \) has been defined, let \( \varphi_1(t_\alpha) \) be an element of \( S \cap \text{Cone}(w) \) extending \( \varphi_0(t_\alpha \upharpoonright 1) \) and let \( \varphi(t_\alpha) \) be an element of \( S \cap \text{Cone}(w) \) extending \( \varphi_1(t_\alpha \upharpoonright 0) \). Note that \( \varphi(t_\alpha) \supseteq \varphi_1(t_\alpha) \supseteq \varphi_0(t_\alpha) \).

Suppose \( \varphi_0(t_\beta), \varphi_1(t_\beta) \) and \( \varphi(t_\beta) \) have been defined for all \( \beta < \alpha \), and \( t_\alpha = s_\alpha \upharpoonright \langle \delta \rangle \) for some \( \delta \). Let \( \gamma_\alpha \) be the least ordinal \( \gamma \) strictly greater than \( \lg(\varphi(t_\beta)) \) for all \( \beta < \alpha \). Then let \( \varphi_0(t_\alpha) \) be an element of \( S \cap \text{Cone}(w) \) extending the extension by zeros of \( \varphi_\delta(s_\alpha \upharpoonright \langle \delta \rangle) \) of length \( \gamma_\alpha \).

Suppose \( \varphi_0(t_\beta), \varphi_1(t_\beta) \) and \( \varphi(t_\beta) \) have been defined for all \( \beta < \alpha \), and \( \lg(t_\alpha) = \zeta \) is a limit ordinal. Let \( \varphi^-(t_\alpha) := \bigcup \{ \varphi_0(t_\alpha \upharpoonright \eta) : \eta < \zeta \} \). Let \( \gamma_\alpha \) be the least ordinal \( \gamma \) strictly greater than \( \lg(\varphi^-(t_\alpha)) \) and strictly greater than \( \lg(\varphi(t_\beta)) \) for all \( \beta < \alpha \). Let \( \varphi_0(t_\alpha) \) be an element of \( S \cap \text{Cone}(w) \) extending the extension by zeros of \( \varphi^-(t_\alpha) \) of length \( \gamma_\alpha \).

By construction, \( \varphi_0 \) and hence \( \varphi_1 \) and \( \varphi \) are injective. Moreover, the ranges of all three are subsets of \( S \cap \text{Cone}(w) \), so for \( D := \varphi[\kappa^{>2}] \), the meet closure satisfies \( D^\land \subseteq \text{Cone}(w) \). Below we shall show that \( D^\land \subseteq S \).

For all \( \alpha < \kappa \), one has \( \lg(\varphi_0(t_\alpha)) < \lg(\varphi_1(t_\alpha)) < \lg(\varphi(t_\alpha)) < \lg(\varphi_0(t_{\alpha+1})) \) and if \( \alpha \) is a limit then for all \( \beta < \alpha \) we have \( \lg(\varphi(t_\beta)) < \lg(\varphi(t_\alpha)) \). Thus if \( \alpha < \beta \), then \( \lg(\varphi(t_\alpha)) < \lg(\varphi(t_\beta)) \), so different elements of the union of the ranges of \( \varphi_0, \varphi_1 \) and \( \varphi \) have different lengths.

Consider two distinct elements, \( t_\alpha \prec_Q t_\beta \). By a case analysis below, we show that their images are incomparable, the \( \prec_Q \)-order between \( t_\alpha \) and \( t_\beta \) is preserved by \( \varphi \) and we show how to express the meet \( \varphi(t_\alpha) \land \varphi(t_\beta) \) as one of \( \varphi(t_\alpha \land t_\beta), \varphi_0(t_\beta) \) and \( \varphi_1(t_\alpha) \).

For the first case, suppose \( t_\alpha \) and \( t_\beta \) are incomparable. Then \( t_\alpha \prec_{\text{lex}} t_\beta \).

Let \( \gamma \) be such that \( t_\gamma = t_\alpha \land t_\beta \). Then \( t_\gamma \downarrow 0 \subseteq t_\alpha \) and \( t_\gamma \downarrow 1 \subseteq t_\beta \). From the
definition of \( \varphi_0, \varphi_1 \) and \( \varphi \), it follows that \( \varphi_0(t_\alpha \land t_\beta) = \varphi_0(t_\gamma) = \varphi(t_\alpha) \land \varphi(t_\beta) \) and \( \varphi(t_\alpha) <_{\text{lex}} \varphi(t_\beta) \), so \( t_\alpha <_Q t_\beta \) and \( \varphi(t_\alpha) <_Q \varphi(t_\beta) \).

For the second case, suppose \( t_\beta \subseteq t_\alpha \). Then, since \( t_\alpha \) and \( t_\beta \) are distinct and \( t_\alpha <_Q t_\beta \), it follows that \( t_\beta \setminus \langle 0 \rangle \subseteq t_\alpha \). Consequently \( t_\alpha \land t_\beta = t_\beta \), \( \varphi(t_\alpha) \land \varphi(t_\beta) = \varphi_0(t_\beta) \), and \( \varphi(t_\alpha) <_{\text{lex}} \varphi_1(t_\beta) \subseteq \varphi(t_\beta) \), so \( \varphi(t_\alpha) <_Q \varphi(t_\beta) \).

For the third case, suppose \( t_\alpha \subseteq t_\beta \). Then, since \( t_\alpha \) and \( t_\beta \) are distinct and \( t_\alpha <_Q t_\beta \), it follows that \( t_\alpha \setminus \langle 1 \rangle \subseteq t_\beta \). Consequently \( t_\alpha \land t_\beta = t_\alpha \), \( \varphi(t_\alpha) \land \varphi(t_\beta) = \varphi_1(t_\alpha) \). Since \( \varphi_1(t_\alpha) \setminus \langle 0 \rangle \subseteq \varphi(t_\alpha) \) and \( \varphi_1(t_\alpha) \setminus \langle 0 \rangle <_{\text{lex}} \varphi(t_\beta) \), it follows that \( \varphi(t_\alpha) <_Q \varphi(t_\beta) \).

By the above analysis, \( \varphi \) preserves the \( <_Q \)-order and sends distinct elements of \( ^{\kappa>2} \) to incomparable elements. Thus the image of \( \varphi \) is an antichain. Moreover, for \( \eta < \zeta \), the meet, \( \varphi(t_\eta) \land \varphi(t_\zeta) \), is one of \( \varphi_0(t_\eta \land t_\zeta) \) and \( \varphi_1(t_\eta \land t_\zeta) \), and the latter occurs only if \( t_\eta \setminus \langle 1 \rangle \subseteq t_\zeta \). Thus different elements of the meet closure of the image of \( \varphi \) have different lengths. It follows that the image of \( \varphi \) is diagonal. Moreover, by construction, the meet closure of the image \( D \) of \( \varphi \) is a subset of \( S \cap \text{Cone}(w) \).

To complete the proof of the lemma, we must show that the image \( D \) is sparse. First use induction on \( \beta < \kappa \) to show that for all \( \alpha < \beta \), the following two statements hold:

1. \( \varphi(t_\beta)(\lg(\varphi_0(t_\alpha))) = 0 \);

2. \( (\varphi(t_\beta)(\lg(\varphi(t_\alpha)))) = 1 \) or \( (\varphi(t_\beta)(\lg(\varphi(t_\alpha)))) = 1 \) if and only if \( (\varphi(t_\beta)(\lg(\varphi(t_\alpha)))) = 1 \) and \( (\varphi(t_\beta)(\lg(\varphi(t_\alpha)))) = 1 \) if and only if \( (t_\alpha \setminus \langle 1 \rangle) \subseteq t_\beta \) and \( \varphi_1(t_\alpha) \setminus \langle 1 \rangle \subseteq \varphi(t_\beta) \).

Next suppose \( s \) and \( t \) in the meet closure of \( \varphi[^{\kappa>2}] \) are such that \( \lg(t) > \lg(s) \) and \( t(\lg(s)) = 1 \). With loss of generality, we assume that \( t \) is in the image of \( \varphi \), since we know it has an extension in the image, and let \( t_\beta \) be such that \( t = \varphi(t_\beta) \). By construction, since \( t(\lg(s)) = 1 \), either \( s = \varphi_0(t_\alpha) \) or \( s = \varphi_1(t_\alpha) \) for some \( \alpha \leq \beta \). If \( \alpha < \beta \), then \( s \setminus \langle 1 \rangle \subseteq \varphi(t_\beta) \).

Lemma 3.14. Suppose \( \prec \) is ordering of the levels of \( ^{\kappa>2} \). If \( e : ^{\kappa>2} \rightarrow ^{\kappa>2} \) is a strong embedding which preserves \( \prec \) on \( \text{Cone}(w) \) and \( A \subseteq \text{Cone}(w) \) is an \( m \)-element sparse diagonal subset, then \( (\clp(A), \prec_A) = (\clp(e[A]), \prec_{e[A]}) \).
Proof. Under the hypotheses of the lemma, since \( e \) is a strong embedding, \( \text{clp}(e[A]) = \text{clp}(A) \). Since \( e \) preserves order on \( \text{Cone}(w) \), \( \prec e[w] = \prec A \). □

Theorem 3.15. Let \( m \geq 2 \) and suppose that \( \kappa \) is a cardinal which is measurable in the generic extension obtained by adding \( \lambda \) Cohen subsets of \( \kappa \), where \( \lambda \rightarrow (\kappa)^2 \). Then for \( t^+_m \) equal to the number ofvip \( m \)-types,

\[
Q_\kappa \rightarrow (Q^m_\kappa)_{<\kappa,t^+_m}.
\]

Proof. Suppose \( d : [\kappa>2]^m \rightarrow \mu \) is a fixed coloring for some \( \mu < \kappa \). Use Lemma 3.10 to find \( S \subseteq \kappa>2 \) cofinal and transverse. Lemma 3.3 to find a pre-\( S \)-vip order \( \prec \) of the levels of \( \kappa>2 \). Apply Shelah’s Theorem 2.5 to the restriction to antichains to obtain a strong embedding \( e \) and a node \( w \) such that \( e \) preserves \( \prec \) on \( \text{Cone}(w) \) and \( d \) is constant on \( m \)-element subsets of the same \( \prec \)-ordered similarity type.

Apply the First Diagonalization Lemma 3.13 to find a sparse diagonalization \( \varphi \) of \( \kappa>2 \) into \( S \cap \text{Cone}(w) \). Let \( D = \varphi[\kappa>2] \). Note that \( \prec \) is a \( D \)-vip order, since \( \prec \) is a pre-\( S \)-vip order. By Lemma 3.7 all \( \prec \)-ordered similarity types of \( m \)-element subsets of \( D \) are sparse vip \( m \)-types. Let \( Q = e[D] \). By Lemma 3.14 all \( \prec \)-ordered similarity types of \( m \)-element subsets of \( Q \) are sparse vip \( m \)-types.

Since \( \varphi \) is a sparse diagonalization and \( e \) a strong embedding, \( (D, <_Q) \) and \( (Q, <_Q) \) are both \( \kappa \)-dense. Since \( d \) is constant on \( m \)-element subsets of \( Q \) of the same \( \prec \)-ordered similarity type, it follows that \( d \) takes on no more colors than the number \( t^+_m \) of sparse vip \( m \)-types. □

4 An almost perfect subtree

In this section, in preparation for computing some small values of \( r^+_m \), we show that, for an arbitrary \( S \subseteq \kappa>2 \) with \( (S, <_Q) \) \( \kappa \)-dense, the set of all nodes \( w \) with a \( \kappa \)-dense set of extensions in \( S \) forms an almost perfect subtree (defined later in this section). We use the almost perfect subtree to construct a \( \kappa \)-dense diagonal subset of an arbitrary \( S \subseteq \kappa>2 \) with \( (S, <_Q) \) \( \kappa \)-dense.

Lemma 4.1. Suppose \( \kappa \) satisfies \( \kappa^{<\kappa} = \kappa \) (so \( \kappa \) is a regular cardinal), \( S \subseteq \kappa>2 \), and \( (S, <_Q) \) is a \( \kappa \)-dense linear order. For all \( w \in \kappa>2 \), the set \( S \cap \text{Cone}(w) \) is either empty, a singleton or \( \kappa \)-dense.
Proof. Fix \( w \in \kappa^2 \). If \( S \cap \text{Cone}(w) \) is empty or a singleton, there is nothing to prove. So suppose \( x \in S \) and \( y \in S \) are two different extensions of \( w \). Notice that \( w \subseteq x \land y \). Without loss of generality, assume \( x <_Q y \).

If \( x \) and \( y \) are incomparable and \( x <_Q z <_Q y \), then either \( x \sim (1) \subseteq z \) or \( y \sim (0) \subseteq z \) or \( x <_{\text{lex}} z <_{\text{lex}} y \). In all three cases \( w \subseteq z \).

If \( x \) and \( y \) are comparable, then either \( y \sim (0) \subseteq x \) or \( x \sim (1) \subseteq y \). Hence for any \( z \) with \( x <_Q z <_Q y \), one of \( x \) and \( y \) is a subset of \( z \). In either case \( w \subseteq z \).

Since \( S \) is \( \kappa \)-dense, it has a \( \kappa \)-dense subset \( S' \) with \( \{ x \} < S' < \{ y \} \). By the above two paragraphs, every element of \( S' \) is an extension of \( w \), so \( S \cap \text{Cone}(w) \) is \( \kappa \)-dense.

Definition 4.2. Suppose \( \kappa^\kappa = \kappa \), \( S \subseteq \kappa^2 \) and \((S,<_Q)\) is \( \kappa \)-dense. Let \( T(S) \) be the set of all \( t \in \kappa^2 \) for which \( S \cap \text{Cone}(t) \) is \( \kappa \)-dense.

We plan to show that \( T(S) \) is an almost perfect tree. The first step is to show that arbitrarily high above every node there is a densely splitting node.

Definition 4.3. Define \( W : \wp(\kappa^2) \to \wp(\kappa^2) \) by

\[
W(S) := \{ w \in S^\updownarrow : S \cap \text{Cone}(w\sim (0)) \text{ and } S \cap \text{Cone}(w\sim (1)) \text{ are } \kappa \text{-dense} \},
\]

and call the elements of \( W(S) \) densely splitting nodes of \( S \).

Lemma 4.4. Suppose \( \kappa^\kappa = \kappa \), \( S \subseteq \kappa^2 \) and \((S,<_Q)\) is a \( \kappa \)-dense linear order. Then \( W(S) \) is non-empty.

Proof. Assume toward a contradiction that for all \( u \in \kappa^2 \), one or both of \((S \cap \text{Cone}(u\sim (0)),<_Q)\) and \((S \cap \text{Cone}(u\sim (1)),<_Q)\) have cardinality less than \( \kappa \).

By Lemma 4.1 it follows that for all \( u \in \kappa^2 \), one or both of \( S \cap \text{Cone}(u\sim (0)) \) and \( S \cap \text{Cone}(u\sim (1)) \) have cardinality less than 2.

Define \( \langle t_\alpha : \alpha < \kappa \rangle \) by recursion and prove by induction that \( S \cap \text{Cone}(t_\alpha) \) is \( \kappa \)-dense for all \( \alpha \).

To start the recursion, let \( t_0 = \emptyset \). Then \( S \cap \text{Cone}(t_0) = S \) which is \( \kappa \)-dense.

If \( \alpha = \beta + 1 \) and \( S \cap \text{Cone}(t_\beta) \) is \( \kappa \)-dense, then let \( t_\alpha = t_\beta \sim (0) \) if \( S \cap \text{Cone}(t_\beta \sim (0)) \) is \( \kappa \)-dense, and \( t_\alpha = t_\beta \sim (1) \) otherwise. Since \( S \cap \text{Cone}(t_\beta) \subseteq (S \cap \text{Cone}(t_\beta \sim (0))) \cup (S \cap \text{Cone}(t_\beta \sim (1))) \), by Lemma 4.1 and a cardinality argument, \( S \cap \text{Cone}(t_\alpha) \) is \( \kappa \)-dense.
If \( \alpha \) is a limit ordinal, let \( t_\alpha = \bigcup \{ t_\beta : \beta < \alpha \} \). By assumption, for all \( \beta < \alpha \), if \( t_\beta \upharpoonright \langle \delta \rangle \not\subseteq t_{\beta+1} \), then \( S \cap \text{Cone}(t_\beta \upharpoonright \langle \delta \rangle) \) is not \( \kappa \)-dense, so has cardinality less than 2. It follows that \( |S \setminus \text{Cone}(t_\alpha)| < \kappa \). Hence by Lemma 4.4, \( S \cap \text{Cone}(t_\alpha) \) is \( \kappa \)-dense.

Note that \( b = \bigcup \{ t_\alpha : \alpha < \kappa \} \) is a branch through \( ^\omega 2 \). Every element \( s \) of \( S \) is either an initial segment of this branch or there is some \( \beta < \kappa \) such that \( s \cap t_{\beta+1} = t_\beta \). Recall our assumption that for all \( \beta < \alpha \), if \( t_\beta \upharpoonright \langle \delta \rangle \not\subseteq t_{\beta+1} \), then \( S \cap \text{Cone}(t_\beta \upharpoonright \langle \delta \rangle) \) is not \( \kappa \)-dense, so it has cardinality less than 2. It follows that \( |S| < \kappa \), contradicting the assumption that \( S \) is \( \kappa \)-dense. Thus the lemma follows.

**Lemma 4.5.** Suppose \( \kappa^{< \kappa} = \kappa \), \( S \subseteq ^\kappa 2 \) and \( (S, <_Q) \) is a \( \kappa \)-dense linear order. Then for every \( w \in \mathcal{W}(S) \), every \( \delta < 2 \), and every \( \alpha < \kappa \), there is \( u \in \mathcal{W}(S) \) such that \( w \upharpoonright \langle \delta \rangle \subseteq u \) and \( \log(u) \geq \alpha \).

**Proof.** Fix \( w \in \mathcal{W}(S) \), \( \delta < 2 \) and \( \alpha < \kappa \). Without loss of generality, assume \( \log(w \upharpoonright \langle \delta \rangle) < \alpha \). Since \( |S \cap \text{Cone}(w \upharpoonright \langle \delta \rangle)| = \kappa \), it follows that \( S \cap \text{Cone}(w \upharpoonright \langle \delta \rangle) \) is \( \kappa \)-dense by Lemma 4.4.

Since \( \kappa^{< \kappa} = \kappa \), the set of nodes of \( ^\omega 2 \) of length at most \( \alpha \) has cardinality less than \( \kappa \), but \( S \cap \text{Cone}(w \upharpoonright \langle \delta \rangle) \) has cardinality \( \kappa \) because it is \( \kappa \)-dense.

By the pigeonhole principle, there is some \( u_0 \in ^\alpha 2 \) which has \( \kappa \) many extensions in \( S \cap \text{Cone}(w \upharpoonright \langle \delta \rangle) \). It follows that \( w \upharpoonright \langle \delta \rangle \subseteq u_0 \) and that \( S \cap \text{Cone}(u_0) \) is \( \kappa \)-dense by Lemma 4.4. By Lemma 4.4 there is an extension \( u \supseteq u_0 \) in \( \mathcal{W}(S) \), so the lemma follows. \( \square \)

**Lemma 4.6.** Suppose \( \kappa^{< \kappa} = \kappa \), \( S \subseteq ^\kappa 2 \) and \( (S, <_Q) \) is \( \kappa \)-dense. Then \( (T(S), \subseteq) \) is a rooted tree, \( \mathcal{W}(S) \subseteq T(S) \), and for all \( t \in T(S) \), for all \( \alpha > \log(t) \), there is some \( r \in \mathcal{W}(S) \) with \( t \subseteq r \) and \( \log(r) \geq \alpha \).

**Proof.** Since \( T(S) \) is closed under initial segments, \( (T(S), \subseteq) \) is a rooted tree. By definition of \( \mathcal{W}(S) \), it is a subset of \( T(S) \). By Lemma 4.5, every element of \( T \) has extensions of arbitrarily large length which are densely splitting nodes. \( \square \)

To prove that \( T(S) \) is an almost perfect tree, we need to be able to prove that it has certain continuity properties at limit levels. Toward that end, we introduce the notion of a limit of densely splitting nodes of \( S \).

**Definition 4.7.** Suppose \( \kappa \) is a regular cardinal, \( S \subseteq ^\kappa 2 \), and \( (S, <_Q) \) is \( \kappa \)-dense. Say \( t \in ^\kappa 2 \) is a limit of densely splitting nodes of \( S \) if \( \log(t) \) is a
limit ordinal and for unboundedly many $\beta < \lg(t)$, $t \upharpoonright \beta$ is in $\mathcal{W}(S)$. If $t$ is a limit of densely splitting nodes of $S$, then say it is evenhanded in $S$ if for $\delta = 0, 1$, the set $\{ \beta < \lg(t) : t \upharpoonright \beta \in \mathcal{W}(S) \land t(\beta) = \delta \}$ is unbounded in $\lg(t)$.

**Lemma 4.8.** Suppose $\kappa^\kappa = \kappa$, $S \subseteq \kappa^{\kappa > 2}$ and $(S, <_Q)$ is a $\kappa$-dense linear order. Further suppose that $z$ is a limit of densely splitting nodes of $S$. If $z$ is evenhanded in $S$, then $z$ has an extension in $\mathcal{W}(S)$.

**Proof.** For each $\beta < \lg(z)$ with $z \upharpoonright \beta \in S$, let $g(\beta)$ be an element of $S$ extending $z \upharpoonright \beta \langle 1 - z(\beta) \rangle$. Let $A$ be the set of $g(\beta)$ for $\beta < \lg(z)$ with $z \upharpoonright \beta \in S$ and $z(\beta) = 1$, and let $B = \operatorname{ran}(g) \setminus A$.

Then $A <_Q \{ z \} <_Q B$.

**Claim 4.8.a.** If $w \in S$ and $A <_Q \{ w \} <_Q B$, then $z \subseteq w$.

**Proof.** Suppose $w \in S$ and $A <_Q \{ w \} <_Q B$.

Assume toward a contradiction that $\gamma := \lg(w \land z) < \lg(z)$. Use the fact that $z$ is evenhanded in $S$ to choose $\eta$ and $\theta$ strictly greater than $\gamma := \lg(w \land z)$ such that $z \upharpoonright \eta \in \mathcal{W}(S)$, $z(\eta) = 1$, $z \upharpoonright \theta \in \mathcal{W}(S)$ and $z(\theta) = 0$. Then $g(\eta) \in A$ and $g(\theta) \in B$, so $g(\eta) <_Q w <_Q g(\theta)$.

By definition of $g$, $z \upharpoonright \eta \subseteq g(\eta)$ and $z \upharpoonright \theta \subseteq g(\theta)$. Consequently $z \upharpoonright (\gamma + 1) \subseteq g(\eta) \land g(\theta)$, and $w \land g(\eta) = w \land z = w \land g(\theta)$. It follows that both $g(\eta)$ and $g(\theta)$ are in the same $<_Q$-relationship with $w$ as is $z$, which is not possible because either $w <_Q g(\eta)$ and $w <_Q g(\theta)$ contradicting the fact that $g(\eta) <_Q w <_Q g(\theta)$. Thus $\lg(w \land z) \geq \lg(z)$. In other words, $z \subseteq w$. \qed

Since $S$ is $\kappa$-dense, there is a $\kappa$-dense set $C \subseteq S$ with $A <_Q C <_Q B$. By the claim, $C \subseteq S \cap \operatorname{Cone}(z)$. It follows that $S \cap \operatorname{Cone}(z)$ is $\kappa$-dense, so by Lemma 4.3, $z$ has an extension in $\mathcal{W}(S)$. \qed

**Definition 4.9.** Suppose $\kappa^\kappa = \kappa$, $S \subseteq \kappa^{\kappa > 2}$ and $(S, <_Q)$ is $\kappa$-dense. Say $t \in \kappa^{\kappa > 2}$ favors $\delta$ above $s$ in $T(S)$ if $s \subseteq s \subseteq s \subseteq t$, either $t \in T(S)$ or $t$ is a limit of densely splitting nodes of $S$, and for all $r \in \mathcal{W}(S)$ with $s \subseteq r \subseteq t$, $t \neg \langle \delta \rangle \subseteq t$.

**Lemma 4.10.** Suppose $\kappa^\kappa = \kappa$, $S \subseteq \kappa^{\kappa > 2}$ and $(S, <_Q)$ is $\kappa$-dense. Further suppose $u_0 <_{\text{lex}} u_1$, $u_0$ favors $1$ above $u_0 \land u_1$ in $T(S)$, and $u_1$ favors $0$ above $u_0 \land u_1$ in $T(S)$. Then at least one of $u_0$ and $u_1$ is in $T(S)$.

**Proof.** If one of $u_0$ and $u_1$ is not a limit of densely splitting nodes of $S$, then it is in $T(S)$ by definition of favor. So assume both are limits of densely splitting nodes of $S$.\hfill 20
For each $r \in \mathcal{W}(S)$ with $u_0 \land u_1 \not\subseteq r \not\subseteq u_0$, let $r^- \in S$ be an extension of $r^-\langle 0 \rangle$, and let $A$ be the set of these elements. Similarly, for each $r \in \mathcal{W}(S)$ with $u_0 \land u_1 \not\subseteq r \not\subseteq u_1$, let $r^+ \in S$ be an extension of $r^\langle 1 \rangle$, and let $B$ be the set of these elements. Then $A \triangleleft_{\text{lex}} \{ u_0 \} <_Q \{ u_0 \land u_1 \} <_Q \{ u_1 \} \prec_{\text{lex}} B$.

**Claim 4.10.a.** If $A <_Q \{ w \} <_Q \{ u_0 \land u_1 \}$ and $w \neq u_0 \land u_1$, then $w$ extends $(u_0 \land u_1)^\langle 0 \rangle$ and either $w \subseteq u_0$ or $u_0 \subseteq w$ or $u_0 \prec_{\text{lex}} w$.

**Proof.** Suppose $w$ satisfies the hypotheses. Then by definition of $<_Q$, $(u_0 \land u_1)^\langle 0 \rangle \subseteq w$. Assume toward a contradiction that none of the three conclusions holds. Then $w <\text{lex} u_0$. Let $r \subseteq u_0$ be an element of $\mathcal{W}(S)$ with $u_0 \land u_1 \not\subseteq r$ and $\lg(r) > \lg(u \land w) + 1$. Then $w <_{\text{lex}} r$, so $w <_{\text{lex}} r^-$, contradicting $A <_Q \{ w \}$.

**Claim 4.10.b.** The set of all $w \in S$ such that $A <_Q \{ w \} <_Q \{ u_0 \land u_1 \}$, and $u_0 <_{\text{lex}} w$ has cardinality $< \kappa$.

**Proof.** Otherwise, by the pigeonhole principle and the previous claim, there is some $r \not\subseteq u_0$ with $(u_0 \land u_1)^\langle 0 \rangle \subseteq r$ such that the set of $w \in S$ with $u_0 <_{\text{lex}} w$ and $u_0 \land w = r$ has cardinality $\kappa$. It follows that $u_0(\lg(r)) = 0$, and, by Lemma 4.11, that $r^\langle 1 \rangle \in T(S)$. Thus $r \in \mathcal{W}(S)$ contradicts the assumption that $u_0$ favors 1 above $u_0 \land u_1$.

The proofs of the next two claims are similar to those above, so they are left to the reader.

**Claim 4.10.c.** If $\{ u_0 \land u_1 \} <_Q \{ w \} <_Q B$ and $w \neq u_0 \land u_1$, then $w$ extends $(u_0 \land u_1)^\langle 1 \rangle$ and either $w \subseteq u_1$ or $u_1 \subseteq w$ or $w <_{\text{lex}} u_1$.

**Claim 4.10.d.** The set of all $w \in S$ such that $A <_Q \{ w \} <_Q \{ u_0 \land u_1 \}$, and $u_0 <_{\text{lex}} w$ has cardinality $< \kappa$.

Let $C \subseteq S$ be a $\kappa$-dense subset with $A <_Q C <_Q B$. Since the inequalities $A <_Q \{ u_0 \land u_1 \} <_Q B$ hold, either $C^- := \{ c \in C : c <_Q u_0 \land u_1 \}$ or $C^+ := \{ c \in C : \text{ Cone}(u_0)^\langle 0 \rangle \subseteq C^- \}$ has cardinality $\kappa$. If $C^-$ has cardinality $\kappa$, then $S \cap \text{ Cone}(u_0)^\langle 0 \rangle$ has cardinality $\kappa$, so by Lemma 4.11, $u_0$ is in $T$. Similarly, if $C^+$ has cardinality $\kappa$, then $u_1$ is in $T$. Thus the lemma follows.

**Lemma 4.11.** Suppose $\kappa^{<\kappa} = \kappa$, $S \subseteq ^{<\mu} \kappa$ and $(S, <_Q)$ is $\kappa$-dense. For all $t \in T(S)$ there is a minimal extension of $t$ in $\mathcal{W}(S)$. That is, there is $r \in \mathcal{W}(S)$ such that $t \subseteq r$ and $r \subseteq w$ for all $w \in \mathcal{W}(S) \cap \text{ Cone}(t)$.
Proof. By Lemma 4.1, $W(S) \cap \text{Cone}(t)$ is non-empty. By definition of $W(S)$, the meet of two incomparable elements of it is also in the set. It follows that $W(S) \cap \text{Cone}(t)$ has an element of minimum length, and this element is the desired minimal extension. 

**Lemma 4.12.** Suppose $\kappa < \kappa = \kappa$, $S \subseteq \kappa > 2$ and $(S, <Q)$ is $\kappa$-dense. For all $x \in W(S)$ and all $\alpha > \lg(x)$, there is some extension $y$ of $x$ with $\lg(y) = \alpha$ such that $y$ favors one of $0, 1$ above $x$.

Proof. Fix $x$ in $W(S)$.

For as long as possible, define a $\subseteq$-increasing sequence $u_\alpha$ of extensions of $x^\langle 1 \rangle$ which favor 0 above $x$, with $\lg(u_\alpha) = \alpha$. To start the recursions with $\alpha = \lg(x) + 1$, let $u_\alpha = x^\langle 1 \rangle \in T(S)$. If $\alpha$ is a limit ordinal and $u_\beta$ has been defined for $\lg(x) < \beta < \alpha$, then let $u_\alpha = \bigcup \{ u_\beta : \lg(x) < \beta < \alpha \}$. If $\alpha = \beta + 1$, $u_\beta$ has been defined, and $u_\beta \in T(S)$, then let $u_\alpha = u_\beta^\langle 0 \rangle$ if $u_\beta \in W(S)$, and otherwise let $u_\alpha$ be the one-point extension of $u_\beta$ which is a subset of the extension of $u_\beta$ of minimal length in $W(S)$. If $\alpha = \beta + 1$, $u_\beta$ has been defined, and $u_\beta \notin T(S)$, then $\beta$ is a limit ordinal, $u_\beta$ is a limit of densely splitting nodes of $S$ and the recursion stops with its definition.

Also, for as long as possible, define a $\subseteq$-increasing sequence $v_\alpha$ of extensions of $x^\langle 0 \rangle$ which favor 1 above $x$, with $\lg(v_\alpha) = \alpha$. To start the recursion with $\alpha = \lg(x) + 1$, let $v_\alpha = x^\langle 0 \rangle \in T(S)$. If $\alpha$ is a limit ordinal and $v_\beta$ has been defined for $\lg(x) < \beta < \alpha$, then let $v_\alpha = \bigcup \{ v_\beta : \lg(x) < \beta < \alpha \}$. If $\alpha = \beta + 1$, $v_\beta$ has been defined, and $v_\beta \in T(S)$, then let $v_\alpha = v_\beta^\langle 0 \rangle$ if $v_\beta \in W(S)$, and otherwise let $v_\alpha$ be the one-point extension of $v_\beta$ which is a subset of the extension of $v_\beta$ of minimal length in $W(S)$. If $\alpha = \beta + 1$, $v_\beta$ has been defined, and $v_\beta \notin T(S)$, then $\beta$ is a limit ordinal, $v_\beta$ is a limit of densely splitting nodes of $S$ and the recursion stops with its definition.

By construction, for all $\alpha$ with $u_\alpha$ defined, $u_\alpha$ favors 0 above $x$ in $T(S)$. Similarly, for all $\alpha$ with $v_\alpha$ defined, $v_\alpha$ favors 1 above $x$ in $T(S)$. If one of the recursions continues for all $\alpha < \kappa$, the lemma follows.

Assume toward a contradiction that $u_\beta$ is defined but $u_{\beta+1}$ is not, and that $v_\beta$ is defined but $v_{\beta+1}$ is not. Then $u_\beta \notin T(S)$ and $v_\beta \notin T(S)$, contradicting Lemma 4.10.

**Definition 4.13.** A subset $T \subseteq \kappa > 2$ is an almost perfect tree if it is a rooted induced subtree of $\kappa > 2$ closed under initial segments such that the following conditions hold:

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1. for all $t \in T$, for all $\alpha < \kappa$, there is an extension $w \in T$ of $t$ of length at least $\alpha$ which is a densely splitting node (both $w \langle 0 \rangle$ and $w \langle 1 \rangle$ are in $T$);

2. for all $s \in \kappa > 2$, if $s$ is an evenhanded limit of densely splitting nodes of $T$, then $s$ is in $T$;

3. for all $s, t \in \kappa > 2$, if $\lg(s) = \lg(t)$, both $s$ and $t$ are limits of densely splitting nodes of $T$, and for some $x$, $s$ favors 0 above $x$ and $t$ favors 1 above $x$, then one of $s$ and $t$ is in $T$.

**Lemma 4.14.** Suppose $\kappa^\kappa = \kappa$, $S \subseteq \kappa > 2$ and $(S, <_Q)$ is $\kappa$-dense. Then $T(S)$ is almost perfect.

**Proof.** Apply Lemmas 4.6, 4.8 and 4.12.

**Lemma 4.15.** Suppose $\kappa^\kappa = \kappa$ and $T \subseteq \kappa > 2$ is almost perfect. Let $C$ be the set of all limit ordinals $\alpha > 0$ such that every node of $T$ of length less than $\alpha$ extends to a densely splitting node of $T$ of length less than $\alpha$. Then $C$ is closed unbounded in $\kappa$, and for all $\alpha \in C$, for all $u \in T \cap \alpha > 2$, the set $T_\alpha(u) := \{ t \in T : u \subseteq t \land \lg(t) = \alpha \}$ has cardinality at least $2^{\cf(\alpha)}$.

**Proof.** Use the definition of almost perfect to show that $C$ is non-empty and unbounded. It follows immediately from the definition of $C$ that it is closed.

Fix attention on $\alpha \in C$ and $u \in T \cap \alpha > 2$. Let $\lambda = \cf(\alpha)$, and suppose $\sigma \in \lambda^2$ is a sequence such that if $\eta = \theta + k < \lambda$ for $\theta$ is 0 or $\theta$ limit, and $k \equiv \delta \mod 3$ for $\delta < 2$, then $\sigma(\eta) = \delta$.

Define $\langle r_\sigma(\eta) : \eta < \lambda \rangle$ by recursion and show by induction that every element of it has length $< \alpha$.

To start the recursion, let $r_\sigma(0)$ be the minimal densely splitting node of $T$ extending $u$. It has length less than $\alpha$ by the definition of $C$.

Suppose $0 < \eta < \lambda$ and $r_\sigma$ has been defined on elements smaller than $\eta$. If $\eta = \zeta + 1$, let $r_\sigma(\eta)$ be a densely splitting node of $T$ which extends $r_\sigma(\zeta) \langle \sigma(\zeta) \rangle$ of length less than $\alpha$. Such a node exists by the definition of $C$.

If $\eta$ is a limit ordinal, then $r'_\sigma(\eta) := \bigcup \{ r_\sigma(\zeta) : \zeta < \eta \}$ is a limit of densely splitting nodes of $T$ and has length a limit ordinal less than $\alpha$ since $\eta < \lambda = \cf(\alpha)$. The properties of $\sigma$ guarantee that this union is evenhanded, so by the definition of an almost perfect tree, $r'_\sigma(\eta)$ is in $T$. Let $r_\sigma(\eta)$ be a densely
splitting node of length less than \( \alpha \) extending \( r'_\sigma(\eta) \), which exists by definition of \( C \).

This completes the definition of \( \langle r_\sigma(\eta) : \eta < \lambda \rangle \). Let \( s_\sigma \) be the union of this sequence. Then \( s_\sigma \) is a limit of densely splitting nodes in \( S \) and is evenhanded, so it is an element of \( T \). Since \( r_\sigma(0) \) is an extension of \( u \), so is \( s_\sigma \).

If \( s_\sigma \) has length \( \alpha \), then let \( t_\sigma = s_\sigma \) be this union, and notice that it is in \( T \). Otherwise let \( t_\sigma \) be an extension of \( s_\sigma \) of length \( \alpha \) in \( T \), which must exist by the definition of an almost perfect tree.

Notice that if \( \sigma, \tau \in \lambda^2 \) are two distinct sequences with the property that \( \eta = \theta + k < \lambda \) for \( \theta = 0 \) or \( \theta \) limit and \( k \equiv \delta \mod 3 \) for \( \delta < 2 \) implies \( \sigma(\eta) = \tau(\eta) = \delta \), then \( t_\sigma \neq t_\tau \).

Since no constraints have been placed on \( \sigma(\theta + k) \) for \( k \equiv 2 \mod 3 \), there are \( 2^\lambda \) sequences \( \sigma \in \lambda^2 \) with the special property described above. Thus the set \( T_\alpha(u) \) has cardinality at least \( 2^\lambda \), and the lemma follows.

**Lemma 4.16.** Suppose \( \kappa^{<\kappa} = \kappa \), \( S \subseteq \kappa^>2 \) and \( (S, <_q) \) is \( \kappa \)-dense. Let \( C(S) \) be the set of all limit ordinals \( \alpha > 0 \) such that every \( t \in T(S) \cap \alpha^>2 \) has proper extensions in both \( S \cap \alpha^>2 \) and \( W(S) \cap \alpha^>2 \). Then \( C(S) \) is closed unbounded in \( \kappa \).

**Proof.** Use the definition of \( T \) and Lemmas 4.4 and 4.4 to show that \( C(S) \) is non-empty and unbounded. It follows immediately from the definition of \( C(S) \) that it is closed.

**Lemma 4.17.** Suppose that \( \kappa \) is an inaccessible limit of inaccessible cardinals and \( S \) is a subset of \( \kappa^>2 \) with \( (S, <_q) \) \( \kappa \)-dense. Then there is a diagonal set \( D \subseteq S \) such that \( (D, <_q) \) is \( \kappa \)-dense.

**Proof.** Let \( < \) be a total order on \( \kappa^>2 \) satisfying \( \lg(s) < \lg(t) \implies s < t \). Define \( r : T(S) \to W(S) \) by setting \( r(t) \) to be the minimal extension of \( t \) in \( W(S) \). By Lemma 4.11 \( r \) is well-defined. Let \( s : T(S) \to S \) be such that \( s(t) \) is an extension of \( t \).

By recursion on \( < \), define functions \( \ell : \kappa^>2 \to \kappa \), \( f_0, f_1 : \kappa^>2 \to W(S) \), and \( g : \kappa^>2 \to S \) as follows.

To start the recursion, note that the \( < \)-least element is \( \emptyset \), define \( \ell(\emptyset) := 0 \), and let \( f_0(\emptyset) := r(\emptyset) \). If \( f_0(z) \) has been defined, let \( f_1(z) := r(f_0(z) \cap \langle 1 \rangle) \), and set \( g(z) := s(f_1(z) \cap \langle 0 \rangle) \).

To continue the recursion, suppose \( z \in \kappa^>2 \) has \( \lg(z) = \alpha \) and for all \( x < z \), both \( \ell(x) \) and \( f_0(x) \) have been defined, with \( f_1 \) and \( g \) defined from
them as above. Let $\ell(z)$ be the least ordinal greater than $\lg(g(x))$ for all $x \prec z$.

If $\alpha = \beta + 1$ is a successor and $z = y^\langle \delta \rangle$ for some $y$, then let $f_0(z)$ be an extension in $W(S)$ of $r(f_0(y)^\langle \delta \rangle)^{(1 - \delta)}$ of length greater than $\ell(z)$. Since $T(S)$ is almost perfect and $W(S)$ is the set of densely splitting nodes of $T(S)$, such a node exists.

If $\alpha > 0$ is a limit ordinal, then $f^-(z) := \bigcup \{ f(z|\beta) : \beta < \alpha \}$ has length a limit ordinal. Since $f^-(z)$ extends $f_0(z|\eta)$ for all $\eta < \alpha$, the definition of $f_0$ on nodes of successor length guarantees that $f^-(z)$ is a limit of densely splitting nodes of $S$ and evenhanded. Hence by Lemma 4.8, it has an extension in $W(S)$. It follows that $f^-(z)$ is in $T(S)$. Let $f_0(z)$ be an extension of $f^-(z)$ in $W(S)$ of length greater than $\ell(z)$.

Let $D$ be the range of $g$. Then $D$ is a subset of $S$, and the meet closure of $D$ is a subset of union of the ranges of $f_0$, $f_1$ and $g$. Since $f_0(x) \subsetneq f_1(x) \subsetneq g(x)$, the lengths of these sequences are strictly increasing. Also, by construction, if $x \prec z$, then $\lg(g(x)) < \lg(f_0(z))$. So different elements of the meet closure of $D$ have different lengths.

By induction, one can show that $f_0$ preserves $\subseteq$ and length order. By definition of $f_0$, $f_1$ and $g$, we have the following properties:

1. if $x$ and $y$ are incomparable with $x \ll y$, then $g(x) \land g(y) = f_0(x \land y)$ and $g(x) \ll g(y)$;
2. if $x^\langle 0 \rangle \subseteq y$, then $g(x) \land g(y) = f_0(x)$ and $g(y) \ll g(x)$;
3. if $x^\langle 1 \rangle \subseteq y$, then $g(x) \land g(y) = f_1(x)$ and $g(x) \ll g(y)$.

It follows that any two elements of $D$ are incomparable, that is, $D$ is an antichain. Hence $D$ is diagonal. By construction, $D$ is a subset of $S$. By the above three properties, $g$ preserves $\ll_Q$, so $D$ is $\kappa$-dense.

5 Lower bound for dense linear orders

In this section, we show that all sparse vip $m$-types can be embedded in any sparse diagonal set $D$ with $(D, \ll_Q) \kappa$-dense, and derive a lower bound result for $\mathbb{Q}_\kappa$.

**Definition 5.1.** Call an ordering $\prec$ of the levels of $^{\alpha>2}$ small if $(^{\alpha>2}, \prec)$ has order type $2^\alpha$ for each cardinal $\alpha < \kappa$. 

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Lemma 5.2. Suppose $\prec$ is a small ordering of the levels of $\kappa > 2$ and $\alpha < \kappa$ is a cardinal. For all $s$ with $\lg(s) < \alpha$, the interval $\{ t \in \alpha^2 : s \subseteq t \}$ is cofinal in $(\alpha^2, \prec)$.

Proof. For all $s$ with $\lg(s) < \alpha$, the interval $\{ t \in \alpha^2 : s \subseteq t \}$ has cardinality $2^\alpha$. Hence such an interval has order type $2^\alpha$ under $\prec$. \hfill \Box

Lemma 5.3. Suppose $\kappa$ is a limit cardinal, $\prec'$ is a small ordering of the levels of $\kappa > 2$ and $w \in \kappa > 2$. For all $n < \omega$ and orderings $\preceq$ of the levels of $n \geq 2$, there is an order preserving strong embedding $j$ of $(n \geq 2, \preceq)$ into $(\text{Cone}(w), \prec')$, i.e. $s < t$ implies $j(s) < j(t)$. Furthermore, $j$ may be chosen such that for all $s$, the length of $j(s)$ is a cardinal.

Proof. Use induction on $n$. For $n = 0$, there is only the empty sequence, and any embedding of this single point to a point $\text{Cone}(w)$ whose length is a cardinal works.

Suppose the lemma is true for $m$ and $\preceq$ is an ordering of the levels of $m \geq 2$ for $n = m + 1$. Let $j_0 : m \geq 2 \to \kappa > 2$ be an order preserving strong embedding obtained from the induction hypothesis. We define an extension $j$ of $j_0$ as follows. Let $\langle s_i : i < 2^{m+1} \rangle$ enumerate in increasing $\preceq$-order the nodes of $m + 1$. Pick a cardinal $\alpha$ larger than the level at which $m^2$ is embedded, and define by recursion on $i$ nodes $j(s_i) = t_i$ in $\alpha^2$ such that $j(s_i|m) \preceq \langle s_i(m) \rangle \subseteq t_i$ and $t_i \succ t_{i-1}$ if $i > 0$. Lemma 5.2 guarantees that this recursion is possible. \hfill \Box

Theorem 5.4. Suppose that $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \to (\kappa)_2^\kappa$. Further suppose $\prec$ is well ordering of the levels of $\kappa > 2$ and $\prec'$ is a small ordering of the levels of $\kappa > 2$. Then there is a strong embedding $e$ and a node $w$ such that $e$ preserves $\prec'$ and $\prec$ and $\prec'$ agree on all pairs in $e[\text{Cone}(w)]$.

Proof. Apply Shelah’s Theorem 2.5 to the coloring $d : [\kappa > 2]^2 \to 2$ defined using the Boolean value operator $\parallel \cdot \parallel$ by

$$d(\{s, t\}) = \parallel s \prec \text{lex} t \iff s \prec t \parallel$$

to get a strong embedding $e : \kappa > 2 \to \kappa > 2$ and an element $w \in \kappa > 2$ such that for all $s, t \in \text{Cone}(w)$ with $s \prec' t$ one has $e(s) \prec' e(t)$ and the color of $d(\{s, t\})$ depends only on the $\prec'$-ordered similarity type of the pair $\{s, t\}$. 26
If \( \lg(s) < \lg(t) \), then \( \lg(e(s)) < \lg(e(t)) \), so \( < \) and \( <' \) agree on \( e(s), e(t) \), since both are level orders. Similarly, if \( \lg(s) > \lg(t) \), then \( < \) and \( <' \) agree on \( e(s), e(t) \).

Next consider pairs \( \{s, t\} \) with \( \lg(s) = \lg(t) \), \( s <_{\text{lex}} t \) and \( s <' t \). Note that since Cohen forcing on \( \kappa \) does not add bounded subsets to \( \kappa \), our assumptions in particular imply that \( \kappa \) must be a limit cardinal. Suppose \( \alpha \) is a cardinal larger than \( \lg(w) \) but \( < \kappa \). Then \( e[\alpha^2 \cap \text{Cone}(w)] \subseteq \gamma^2 \) is infinite for some \( \gamma < \kappa \). Let \( \langle s_n \in \alpha^2 \cap \text{Cone}(w) : n < \omega \rangle \) be a sequence which is increasing in both the \( <_{\text{lex}} \) and \( <' \) orders. Since \( e \) is a strong embedding, the sequence \( \langle e(s_n) \in \gamma^2 : n < \omega \rangle \) is \( <_{\text{lex}} \)-increasing, and cannot be decreasing in \( < \). Since \( s <' t \) implies \( e(s) <' e(t) \), it follows that \( <' \) and \( < \) agree on pairs \( \{e(s), e(t)\} \) with \( \lg(s) = \lg(t) \), \( s <_{\text{lex}} t \) and \( s <' t \).

Finally consider pairs \( \{s, t\} \) with \( \lg(s) = \lg(t) \), \( t <_{\text{lex}} s \) and \( s <' t \). For \( \alpha \) as in the previous case, let \( \langle t_n \in \alpha^2 \cap \text{Cone}(w) : n < \omega \rangle \) be a sequence which is decreasing in the \( <_{\text{lex}} \) order and increasing in the \( <' \) order. Then \( \langle e(s_n) \in \gamma^2 : n < \omega \rangle \) is \( <_{\text{lex}} \)-decreasing, and cannot be decreasing in \( < \). Thus by an argument like that above, \( <' \) and \( < \) agree on pairs \( \{e(s), e(t)\} \) with \( \lg(s) = \lg(t) \), \( t <_{\text{lex}} s \) and \( s <' t \). \qed

For the following definition, recall the notation \( T(D) \) from Definition 4.2.

**Definition 5.5.** Suppose \( D \subseteq ^{\kappa^2}2 \) is diagonal and \((D, <_Q)\) is \( \kappa \)-dense. A function \( f : ^{\kappa^2}2 \to T(D) \) is a semi-strong embedding if \( f \) preserves extension and lexicographic order, maps levels to levels, and for every \( s \in ^{\kappa^2}2 \), there is some \( v \in D \) such that

\[
f(s) \subseteq f(s^\langle 0 \rangle) \land f(s^\langle 1 \rangle) \subseteq v
\]

and \( \lg(v) < \lg(f(s^\langle 0 \rangle)) \).

**Lemma 5.6.** Suppose \( \kappa \) is inaccessible and \( D \subseteq ^{\kappa^2}2 \) is diagonal and \((D, <_Q)\) is \( \kappa \)-dense. Then there is a semi-strong embedding \( f : ^{\kappa^2}2 \to T(D) \).

**Proof.** Let \( C \subseteq \kappa \) be the set of all limit \( \alpha > 0 \) such that for all \( t \in T(D) \cap \alpha 2 \), \( t \) is a limit of densely splitting points of \( T(D) \) and for all \( \beta < \alpha \), there is \( v \in T(D) \cap ^{\alpha^2}2 \) such that \( t \upharpoonright \beta \subseteq v \in D \). By Lemma 4.10, \( C \) is closed unbounded.

Define \( f \) on \( \alpha^2 \) by recursion on \( \alpha < \kappa \), using the assumption that \( \kappa \) is inaccessible. To start the recursion, let \( f(\emptyset) \) be an element of \( T(D) \) on the \( \gamma_0 \) level of \( ^{\kappa^2}2 \), where \( \gamma_0 \) is the least infinite cardinal in \( C \).
If \( \alpha \) is a limit and \( f \) has been defined on \( \alpha > 2 \), then let \( \gamma_\alpha \) be a cardinal in \( C \) greater than \( \text{sup} \beta < \alpha \gamma_\beta \), and for each \( s \in \alpha 2 \), let \( f(s) \) be an element of \( T(D) \cap \gamma_\alpha \) which is an evenhanded limit of densely splitting points of \( T(D) \) extending \( \bigcup \{ f(s|\beta) : \beta < \alpha \} \).

Suppose \( \alpha = \alpha' + 1 \) is a successor and \( f \) has been defined on \( \alpha > 2 \). Let \( \gamma_\alpha \) be a cardinal in \( C \) greater than \( \gamma_{\alpha'} \). Recall that by Lemma 4.11 every node of \( T(D) \) has a minimal extension to a densely splitting node (one in \( W(S) \)). Also, by Lemma 4.8 every evenhanded extension of a node of \( T(D) \) has an extension to a densely splitting node. For \( s = t^\frown \langle \delta \rangle \) and \( u_t \) the minimal length densely splitting node properly extending \( t \), let \( f(s) \) be an evenhanded extension of \( u_t^\frown \langle \delta \rangle \) of length \( \gamma_\alpha \).

Use induction to show that \( f \) preserves extension and lexicographic order. By its construction and the choice of \( C \), the remaining conditions are satisfied for all \( s \).

For the following definition, recall the notion of a sparse \( m \)-type from Definition 3.6.

**Theorem 5.7.** Suppose that \( \kappa \) is a cardinal which is measurable in the generic extension obtained by adding \( \lambda \) Cohen subsets of \( \kappa \), where \( \lambda \rightarrow (\kappa)^6 \). If \( D \subseteq \kappa^+ \) is a sparse diagonal set with \( (D, <_Q) \) \( \kappa \)-dense and \( \prec \) is a \( D \)-vip order of the levels of \( \kappa > 2 \), then every sparse vip \( m \)-type \((\tau, <)\) is realized as \( (\text{clp}(x), \prec_x) \) for some \( x \subseteq D \).

**Proof.** The assumptions imply that \( \kappa \) is inaccessible, so Lemma 5.6 applies. Let \( f : \kappa > 2 \rightarrow T(D) \) be a semi-strong embedding from Lemma 5.6. Define \( f^0 : \kappa > 2 \rightarrow D^\wedge \) by \( f^0(s) := f(s^\frown \langle 0 \rangle) \land f(s^\frown \langle 1 \rangle) \). Let \( f^1 : \kappa > 2 \rightarrow D \) be defined by \( f^1(s) = v \) where \( v \) is the minimal extension of \( f^0(s) \) in \( D \) as guaranteed by Definition 5.5.

For \( t \in \alpha 2 \) and \( i = 0, 1 \), define well-orderings \( \prec_i \) on \( \alpha 2 \) as follows: let \( \beta_i = \text{lg}(f^i(t)) \) and set \( s \prec_i \prec_0 s' \) if and only if \( f(s^\frown \langle 0 \rangle)[\beta_i < f(s^\frown \langle 0 \rangle)[\beta_i]. \)

Let \( \prec' \) be any small well-ordering of the levels of \( \kappa > 2 \). Call a triple \( \{ s, s', t \} \) local if \( \text{lg}(s) = \text{lg}(s') = \text{lg}(t), s \prec_{\text{lex}} s', t \prec' s, \) and \( t \prec' s' \), and \( s \land s' \not\subseteq t \). Let \( d \) be a coloring of the triples of \( \kappa > 2 \) defined as follows: if \( \{ s, s', t \} \) is not local, let \( d(\{ s, s', t \}) := (2, 2) \) and otherwise set

\[
d(\{ s, s', t \}) := (\| s \prec_0 s', s_0 \prec_t s', \| s \prec_1 s', s_1 \prec'_t s')\).
\]

Apply Shelah’s Theorem 2.5 to \( d \) and \( \prec' \) to obtain a strong embedding \( e : \kappa > 2 \rightarrow \kappa > 2 \) and a node \( w \) such that for triples from \( T := e[\text{Cone}(w)] \), the
coloring depends only on the $\prec'$-ordered similarity type of the triple. Then two local triples $\{s, s', t\}$ and $\{u, u', v\}$ of $T$ are colored the same if and only if

$$ s \prec_0^{0} s' \iff u \prec_0^{0} u' \quad \text{and} \quad s \prec_1^{1} s' \iff u \prec_1^{1} u'. $$

Hence for $t \in T$ and for both ordered similarity types of incomparable pairs of same length nodes from $T$, the orderings $\prec_0^a$ must always agree with one of $\prec'$ and its converse on $T$. Since $\prec$ is a well-ordering of the levels of $\kappa > 2$, the orderings $\prec_1^a$ are well-orderings of $\kappa > 2$, so they must agree with $\prec'$ in both cases.

Assume that $\tau$ has $n + 1$ leaves and let $L$ be the set of these leaves. Then $\tau$ is a subtree of $2n > 2$ and every level of $2n > 2$ has exactly one element of $L^\ell$. Extend $\prec$ defined on $\tau$ to $\prec^*$ defined on all of $2n > 2$ in such a way that the extension is still a $L^\ell$-vip order.

Apply Lemma 5.8 to get an order preserving strong embedding $j$ of $(2n > 2, \prec^*)$ into $(\kappa > 2, \prec')$.

Let $\langle t_\ell : \ell \leq 2n \rangle$ enumerate the elements of $L^\ell$ in increasing order of length. Note that $\ell = \log(t_\ell) = \ell$. For $\ell \leq 2n$, define $\beta_\ell := \log(f^i(e(j(t_\ell))))$ where $i = 0$ if $t_\ell \notin L$ and $i = 1$ if $t_\ell \in L$.

Finally define $\sigma : \tau \to T(D)$ by recursion on $\ell \leq 2n$. For $\ell = 0$, let $\sigma(0) = f^0(e(j(0)))$. For $\ell > 0$, consider three cases for elements of $\tau \cap \ell 2$. If $t_\ell \in L$, let $\sigma(t_\ell) = f^1(e(j(t_\ell)))$. If $t_\ell \notin L$, let $\sigma(t_\ell) = f^0(e(j(t_\ell)))$. Note that in both of these cases, $\beta_\ell = \log(\sigma(t_\ell))$. If $s \in \tau \setminus L^\ell$ has length $\ell$, then there is a unique immediate successor in $\tau$, $s \prec^\ell \langle 0 \rangle$. In this case, let $\sigma(s) = f^0(e(j(s)) - \langle 0 \rangle) \parallel \beta_\ell$. Since $j$ sends $\prec^*$-increasing pairs to $\prec'$-increasing pairs and $e$ is a $\prec'$ order preserving strong embedding, their composition sends $\prec^*$-increasing pairs to $\prec'$-increasing pairs. Since for $v_\ell = e(j(t_\ell)) \in T$, the order $\prec'$ agrees with $\prec^0_{v_\ell}$ on $T \cap \gamma 2$ where $\gamma = \log(v_\ell)$, it follows that $\sigma$ sends $\prec^*$-increasing pairs to $\prec$-increasing pairs. Since $f$ preserves extension and lexicographic order, $\sigma$ does as well. By construction $\sigma$ sends levels to levels, meets to meets (split nodes) and leaves to leaves (terminal nodes). Let $x = \sigma[L]$ be the image under $\sigma$ of the leaves of $\tau$. Then $(\text{clp}(x), \prec_x) = (\tau, \prec)$, as required.

**Theorem 5.8.** Let $m$ be a natural number and suppose that $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \rightarrow (\kappa)^2_m$. Then for $r = t_\lambda^\ell$ equal to the number of

29
sparse vip m-types, the $\kappa$-dense linear order $\mathbb{Q}_\kappa$ satisfies

$$\mathbb{Q}_\kappa \to (\mathbb{Q}_\kappa)^m_{<\omega,r-1}.$$  

Proof. Let $S \subseteq \kappa>2$ be a cofinal transverse subset obtained from Lemma \ref{3.10}. Let $\prec$ be a pre-$S$-vip order on $\kappa>2$ obtained from Lemma \ref{3.11}.

Let $\varphi: \kappa>2 \to S$ be a $\prec_Q$-preserving injection such that $D := \varphi[\kappa>2]$ is a sparse diagonal set with $D^\kappa \subseteq S$. Notice that $(D, \prec_Q)$ is $\kappa$-dense, since $\varphi$ is $\prec_Q$-preserving.

Let $(\tau_0, \prec_0), \ldots, (\tau_r, \prec_r)$ be an enumeration of the sparse vip $m$-types. Define $c: [\kappa>2]^m \to r$ by $c(a) = i$ where for $x := \varphi[a]$, $(\text{clp}(x), \prec_x) = (\tau_i, \prec_i)$.

Suppose $A \subseteq \kappa>2$ is a subset with $(A, \prec_Q)$ $\kappa$-dense and $i < r$. Since $\varphi$ is $\prec_Q$-preserving, its image $B := \varphi[A] \subseteq D$ is a sparse diagonal set with the property that $(B, \prec_Q)$ is $\kappa$-dense. Thus by Theorem \ref{5.7} the sparse vip $m$-type $(\tau_i, \prec_i)$ is realized as $(\text{clp}(x), \prec_x)$ for some $x \subseteq B$. Since $\varphi$ is injective, there is an $m$-element subset $u \subseteq A$ with $\varphi[u] = x$ and $c(u) = i$. Since $A$ and $i$ were arbitrary, in every $\kappa$-dense subset $A \subseteq \kappa>2$, every color $i$ is realized by some $m$-element subset. Therefore, the theorem follows.

Recall the definition of canonical partition introduced immediately after the statement of Theorem \ref{1.1}.

Theorem 5.9. Let $m$ be a natural number and suppose that $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \to (\kappa)^{<\omega,2}_\kappa$. For $t^+_m$ equal to the number of sparse vip $m$-types, there is a canonical partition of the $m$-element subsets of $\mathbb{Q}_\kappa = ([\kappa>2]^m, \prec_Q)$ into $t^+_m$ parts.

Proof. Use Lemma \ref{3.10} to find $S \subseteq \kappa>2$ cofinal and transverse. Use Lemma \ref{3.5} to find $\prec$ a pre-$S$-vip order on $\kappa>2$.

For all $w \in \kappa>2$, let $\varphi_w$ be a sparse diagonalization of $\kappa>2$ into $S \cap \text{Cone}(w)$. Use recursion on $\prec$ to define $\pi: \kappa>2 \to \kappa>2$ such that for all $t \in \kappa>2$, $\pi(t)$ is an extension of $t$ with $\text{Cone}(\pi(t))$ disjoint from the union over all $s < t$ of $\varphi_{\pi(s)}[\kappa>2]$. Since the order type of $\{ s \in \kappa>2 : s < t \}$ is less than $\kappa$ and each $\varphi_{\pi(s)}[\kappa>2]$ is a sparse diagonal subset of $S$, it is always possible to continue the recursion.

Define $h: \kappa>2 \to \kappa>2$ as follows. For $t$ with $\text{otp} \{ s \in \kappa>2 : s < t \} = \alpha$, $z_\alpha = \alpha \{0\}$, and $u \in \{ z_\alpha \} \cup \text{Cone}(z_\alpha \setminus \{1\})$, let $h(u) = \varphi_{\pi(t)}(u).$
Let \((\tau_0, <_0), (\tau_1, <_1), \ldots, (\tau_{r-1}, <_{r-1})\) enumerate the sparse vip \(m\)-types. Let \(C_0\) be the set of all \(m\)-element subsets \(A\) for which \((\text{clp}(h[A]), <_{h[A]})\) is either \((\tau_0, <_0)\) or not a sparse vip \(m\)-type. For positive \(j < r\), let \(C_j\) be the set of all \(m\)-element subsets \(A\) of \(\kappa^+\) for which \((\text{clp}(h[A]), <_{h[A]}) = (\tau_j, <_j)\). Then \(\mathcal{C} := \{C_0, C_1, \ldots, C_{r-1}\}\) is a partition of \([\kappa^+]^m\) into \(r\) sets.

To see that each class of \(\mathcal{C}\) is indivisible, suppose \(d : C_j \rightarrow \mu\) is a fixed coloring for some \(2 \leq \mu < \kappa\). Extend \(d\) to all of \([\kappa^+]^m\) by setting \(d(A) = 0\) if \(j > 0\) and \(A \notin C_j\) or by setting \(d(A) = 1\) if \(j = 0\) and \(A \notin C_j\). Apply Shelah’s Theorem 2.5 to the restriction to antichains to obtain a strong embedding \(e\) and a node \(w\) such that \(e\) preserves \(\prec\) on \(\text{Cone}(w)\) and \(d\) is constant on \(m\)-element subsets of the same \(\prec\)-ordered similarity type. Let \(\alpha\) be the order type of \(\{s \in [\kappa^+]^2 : s \prec w\}\). Then \(\text{Cone}(z_\alpha^{-}(1))\) is a \(\kappa\)-dense subset. Since \(h\) agrees with \(\varphi_{\pi(w)}\) on \(\text{Cone}(z_\alpha^{-}(1))\) and \(\varphi_{\pi(w)}\) is \(\prec_{\pi(w)}\)-preserving, \(D := h[\text{Cone}(z_\alpha^{-}(1))]\) is a \(\kappa\)-dense subset of \(\text{Cone}(\pi(w)) \subseteq \text{Cone}(w)\). Since \(\varphi_{\pi(w)}\) is a sparse diagonalization, the set \(D\) is a sparse diagonal set with \(D^h \subseteq S^\alpha \cap \text{Cone}(\pi(w))\). Thus by Lemma 3.7, all \(\prec\)-ordered similarity types of \(m\)-element subsets of \(D\) are sparse vip \(m\)-types. Let \(K := e[D]\). By Lemma 3.14 all \(\prec\)-ordered similarity types of \(m\)-element subsets of \(K\) are sparse vip \(m\)-types. It follows that \([K]^m \cap C_j\) is \(d\)-monochromatic. Hence each \(C_j\) is indivisible.

To see that each class of \(\mathcal{C}\) is persistent, suppose \(K \subseteq [\kappa^+]^m\) is \(\kappa\)-dense and \(j < r\). By Lemma 4.4 there is a node \(z^*\) in \(W(K)\), the set of densely splitting nodes of \(K\). Thus \(z^*^{-}(1)\) has a \(\kappa\)-dense set of extensions in \(K\). In other words, \(K \cap \text{Cone}(z^*)\) is \(\kappa\)-dense. Let \(z_\alpha\) be the longest initial segment of \(z^*^{-}(1)\) consisting only of zeros. Then \(\text{Cone}(z^*) \subseteq \text{Cone}(z_\alpha^{-}(1))\). Let \(t\) be the \(\alpha\)th element of \(\kappa^+\) in the \(\prec\) order. Since \(h\) agrees with \(\varphi_{\pi(t)}\) on \(\text{Cone}(z_\alpha^{-}(1))\), it follows that \(h[K \cap \text{Cone}(z^*)]\) is a sparse diagonal set with the property that \((h[K \cap \text{Cone}(z^*)], \prec_{Q})\) is \(\kappa\)-dense. Thus by Theorem 5.7, the sparse vip \(m\)-type \((\tau_j, <_j)\) is realized as \((\text{clp}(x), \prec_{x})\) for some \(x \subseteq h[K \cap \text{Cone}(z^*)]\). Since \(h\) is injective, there is an \(m\)-element subset \(u \subseteq K \cap \text{Cone}(z^*)\) with \(h[u] = x\), and \(u \in C_j\) as required.

### 6 Sparse vip types

In this section we give closed form upper and lower bounds for the number of sparse vip \(m\)-types that facilitate comparisons with D. Devlin’s theorem for \((\mathbb{Q}, <)\). We then describe a recursive procedure for computing the number...
of sparse vip $m$-types.

In Figure 1 we give a picture of a specific example of a sparse vip 5-type, which we will call $(\tau^*, <)$, so we can use it in later examples to illustrate a variety of definitions. To translate the figure into a representation in which each node is a sequence of 0’s and 1’s, note that the root is the empty sequence and only line segments with positive slope represent 1’s. We have circled the nodes in the meet closure of the set $L$ of leaves of $\tau^*$; these nodes are the designated elements for any ordering of the levels which makes $\tau^*$ a sparse vip 5-type. For only three pairs of nodes does the requirement that $(\tau^*, <)$ be a sparse vip 5-type fail to specify the order, and between each such pair we have indicated the order.

![Figure 1: Pictorial representation of $(\tau^*, <)$](image)

**Lemma 6.1.** If $\tau$ is a sparse $m$-type and $L$ is the set of its leaves, then $\tau = \{ x \upharpoonright i : x \in L \land i < 2m - 1 \}$.

**Proof.** By definition, $\tau$ is closed under initial segments. \hfill \Box

Before introducing a lemma on properties of the leaves of a sparse $m$-type, we list the leaves of $\tau^*$ (see Figure 1 on page 32) with the lexicographically least one at the top of the stack, and continuing in increasing order down the stack.

\[
\langle 0, 0, 0, 0 \rangle \\
\langle 0, 1, 0 \rangle \\
\langle 1, 0, 0, 0, 0, 0 \rangle \\
\langle 1, 0, 1, 0, 0, 0 \rangle
\]
Lemma 6.2. Suppose \( \tau \) is a sparse \( m \)-type whose set of leaves is \( D = \{ d_0, d_1, \ldots, d_{m-1} \} \) listed in \(<_{\text{lex}}\)-increasing order. Then

1. \( D^\wedge = D \cup \{ d_i \land d_{i+1} : i < m - 1 \} \);
2. \( d_0 \) is a sequence of all zeros;
3. if \( i < m - 1 \), then \( d_i(\lg(d_i \land d_{i+1})) = 0 \) and \( d_{i+1}(\lg(d_i \land d_{i+1})) = 1 \);
4. if \( i < m - 1 \), then for all \( p \) with \( \lg(d_i \land d_{i+1}) < p < \lg(d_{i+1}) \), \( d_{i+1}(p) = 0 \).

Proof. The first item follows from the fact that \( D \cup \{ d_i \land d_{i+1} : i < m - 1 \} \) is a subset of \( D^\wedge \) of size \( m + (m - 1) = 2m - 1 = |D^\wedge| \). Since \( D \) is diagonal, all its elements have different lengths from among 0, 1, \ldots, 2m – 2. Since \( D^\wedge \) has \( 2m - 1 \) elements, all of different lengths, it follows that for all \( p < 2m - 1 \), there is some \( x = x_p \in D^\wedge \) with \( \lg(x) = p \).

Now the second item holds, since for all \( p < \lg(d_0) \), \( d_0(p) = d_0(\lg(x_p)) = 0 \), either because \( x_p = d_i \land d_{i+1} \subseteq d_0 \) and \( d_0 <_{\text{lex}} d_{i+1} \) or because \( \tau \) is sparse.

Then the third item follows from the definition of \( \text{lex} \) order in a binary tree.

For the fourth item, fix attention on some \( i < m - 1 \). Notice that if \( \ell < i \) and \( d_\ell \land d_{\ell+1} \subseteq d_{i+1} \), then \( (d_\ell \land d_{\ell+1}) \land \langle 1 \rangle \subseteq d_i \land d_{i+1} \), since \( d_{\ell+1} <_{\text{lex}} d_i \).

Also, if \( i < \ell \) and \( d_\ell \land d_{\ell+1} \subseteq d_i \), then \( (d_\ell \land d_{\ell+1}) \land \langle 0 \rangle \subseteq d_i \), since \( d_i <_{\text{lex}} d_{\ell+1} \).

Now the fourth item follows from the previous two statements and the fact that \( \tau \) is sparse.

Next we associate with each sparse \( m \)-type a sequence which is characteristic.

Definition 6.3. Suppose \( \tau \) is a sparse \( m \)-type whose set of leaves is \( D = \{ d_0, d_1, \ldots, d_{m-1} \} \) listed in \(<_{\text{lex}}\)-increasing order. Define \( P(\tau) : (2m - 1) \to (2m - 1) \) by

\[
P(\tau)(k) = \begin{cases} 
\lg(d_i), & \text{if } k = 2i, \\
\lg(d_i \land d_{i+1}), & \text{otherwise}.
\end{cases}
\]

Lemma 6.4. If \( \tau \) and \( \tau' \) are sparse \( m \)-types and \( P(\tau) = P(\tau') \), then \( \tau = \tau' \).
Proof. Let \( D = L(\tau) \) and \( E = L(\tau') \) be the sets of leaves of \( \tau \) and \( \tau' \). List \( D \) and \( E \) in increasing lexicographic order as \( d_0, d_1, \ldots, d_{m-1} \) and \( e_0, e_1, \ldots, e_{m-1} \). Then for all \( i < m \), \( \lg(d_i) = \lg(e_i) \) by definition of \( P \). Also, for \( i < m - 1 \), \( \lg(d_i \land d_{i+1}) = \lg(e_i \land e_{i+1}) \). Use induction on \( i < m \) and the previous lemma to show \( d_i = e_i \). \( \square \)

**Definition 6.5.** A function \( P : (2^m - 1) \to (2^m - 1) \) is an **alternating permutation** if it is a permutation, for all even \( i < 2^m - 2 \), \( P(i) > P(i+1) \) and for all odd \( i < 2^m - 2 \), \( P(i) < P(i+1) \).

The study of alternating permutations and the alternating group dates back to André [1] in 1881, and continues to be an active area of investigation (see [16] for example).

**Lemma 6.6.** For all positive integers \( m \), \( P \) is a bijection between the collection of sparse \( m \)-types and the set of alternating permutations on \((2^m - 1)\).

Proof. Properties of meets guarantee that \( P(\tau) \) is an alternating permutation whenever \( \tau \) is a sparse \( m \)-type. By the previous lemma, \( P \) is one-to-one.

To see that it is onto, suppose \( p : (2^m - 1) \to (2^m - 1) \) is an alternating permutation. Let \( d_0 \) be the sequence of zeros of length \( p(0) \). Continue the recursion by defining \( d_1, d_2, \ldots, d_{m-1} \), by setting \( d_{i+1} \) to be the sequence of length \( p(2i + 2) \) which extends \((d_i | p(2i+1) \rangle \rangle (1) \) with all zeros. Since \( p \) is a permutation, all elements of \( D := \{ d_0, d_1, \ldots, d_{m-1} \} \) have different lengths, and \( D \) is an antichain listed in \( \langle \text{lex} \rangle \)-increasing order. Also by construction, \( \lg(d_j \land d_{j+1}) = p(2j + 1) \), so \( D \) is diagonal. By construction, \( d_0(p) = 0 \) for all \( p < \lg(d_0) \). Use induction on \( i \) to show that for all \( i < m - 1 \) and all \( p < \lg(d_{i+1}) \), if \( d_{i+1}(p) = 1 \), then for some \( \ell \leq i \), \( p = \lg(d_\ell \land d_{\ell+1}) \) and \( d_\ell \land d_{\ell+1} \subseteq d_{i+1} \). It follows that \( D \) is sparse. Let \( \tau = \clp(D) \). Then the set of leaves of \( \tau \) is \( D \) and \( \tau \) is a sparse \( m \)-type. By construction \( P(\tau) = p \). Since \( p \) was arbitrary, the mapping \( P \) is onto. \( \square \)

The above proof was inspired by the counting of Joyce trees (named by Ross Street [24]) in a paper by Joyce [11] applying category theory to physics. The meet closure of an \( m \)-element diagonal subset of \((2^m - 1) \rangle \rangle 2 \) is an example of a Joyce tree. They are counted up to a certain equivalence, and each equivalence class has exactly one example whose closure under initial segment is a sparse \( m \)-type. We also used ideas from a counting argument by Vuksanovic in [27].
**Lemma 6.7.** The number of alternating permutations on $2m - 1$ is $t_m$, where $t_m$ is the $m$-th tangent number, defined recursively by $t_1 = 1$ and

$$t_n = \sum_{i=1}^{n-1} \left( \frac{2n-2}{2i-1} \right) t_i t_{n-i}.$$ 

**Proof.** An exponential generating function for the sequence $a_n$, where $a_n$ is the number of alternating permutations on $n$, is $\sec(x) + \tan(x)$ (see Stanley’s *Enumerative Combinatorics I* [23]). Since the terms corresponding to $\sec(x)$ in this series have even powers and the terms corresponding to $\tan(x)$ have odd powers, an exponential generating function for $b_m$, where $b_m$ is the number of alternating permutations on $(2m - 1)$ is $\tan(x)$.

It is not difficult to show directly that the number of alternating permutations on $2m - 1$ satisfies the above recurrence. For notational convenience, let $K$ denote the set $2m - 1$. Given a $2i - 1$ element subset $I \subseteq K \setminus \{2i - 1\}$ and alternating permutations $p_0$ on $2i - 1$ and $p_1$ on $2(m - i) - 1 = 2m - 2i - 1$, use the unique order preserving maps $e_0$ from $(2i - 1)$ to $I$ and $e_1$ from $(2m - 2i - 1)$ to $K \setminus (I \cup \{2i - 1\})$ to $p = p_0 \circ e_0^{-1} \cup \{(2i - 1, 0)\} \cup p_1 \cup e_1^{-1}$.

(See [23] for details).

**Corollary 6.8.** The set of sparse $m$-types has cardinality $t_m$, where $t_m$ is the $m$-th tangent number, and $t_m \leq t_m^+$ where $t_m^+$ is the number of sparse vip $m$-types.

The next goal is to provide a closed form upper bound for the number of sparse vip $m$-types as a product of $t_m$ times a constant factor. We start by looking at the size of levels of $\text{clp}(A)$ for $A$ an $m$-element diagonal set.

**Definition 6.9.** Suppose $A \subseteq (2m-1)\geq 2$ is a diagonal set. Enumerate $A^\wedge$ in increasing order of length as $a_0, a_1, \ldots, a_{2m-2}$ and define

$$\ell_i(A) := i + 1 - 2|\{j < i : a_j \in A^\wedge \setminus A\}|.$$

Note that for $m$-element diagonal subsets of $(2m-1)\geq 2$, the lengths of the elements of the meet closure are $|a_0| = 0, |a_1| = 1, \ldots, |a_{2m-2}| = 2m - 2$.

**Lemma 6.10.** For $m \geq 2$, and any $m$-element diagonal subset $A \subseteq (2m-1)\geq 2$ whose meet closure is listed in increasing order as $a_0, a_1, \ldots, a_{2m-2}$, the cardinality of level $i$ of $\text{clp}(A)$ is

$$|\{a_j : i < 2m - 1\}| = \ell_i(A) = (2m - 1) - i - 2|\{j \geq i : a_j \in A^\wedge \setminus A\}|.$$

Moreover, if $i < m$, then $\ell_i(A) \leq i + 1$ and $\ell_{2m-2-i} \leq i + 1$. 

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Proof. Use induction to compute the size of $L_i := \{ a_j \mid |a_j| : j < 2m - 1 \}$ in two different ways. To start the inductions, note that for $i = 0$ and $i = 2m - 2$, $L_0 = \{a_0\}$ and $L_{2m-2} = \{a_{2m-2}\}$ are singletons, so $\ell_0(A) = 1$ and $\ell_{2m-2}(A) = 1$. Also, since $a_0$ is meet of two elements and $a_0, a_1$ are the shortest elements, we have $\ell_1(A) = |L_1| = 2$. Since $a_{2m-2}, a_{2m-3}$ are the longest elements, we have $L_{2m-3} = \{a_{2m-2}, a_{2m-3}\}$ is also a doubleton, thus $\ell_{2m-3}(A) = 2$. Counting up from $i = 0$, we get the values in the formula for $\ell_i(A)$, since the value for $|L_{i+1}|$ is one more than the value for $|L_i|$ if $a_i \in A^\wedge \setminus A$ and one less if $a_i \in A$. Thus for $0 < i < m$, $\ell_i(A) \leq \ell_{i-1}(A) + 1 \leq i + 1$. Counting down from $i = 2m - 2$, we get the values in the displayed formula, since the value for $|L_{i-1}|$ is one more than the value for $|L_i|$ if $a_{i-1} \in A$ and one less if $a_{i-1} \in A^\wedge \setminus A$. Thus for $0 < i < m$, $\ell_{2m-2-i}(A) \leq \ell_{2m-2-i-1}(A) + 1 \leq i + 1$.

The values $\ell_i(A) = i + 1$ and $\ell_{2m-2-i} = i + 1$ are achieved if $A$ is a comb whose leaves listed in increasing order as $a_0, a_1, \ldots, a_{m-1}$ satisfy $\lg(a_i) = m - 1 + i$ and for $i < m - 1$, $\lg(a_i \wedge a_{i+1}) = i$.

**Definition 6.11.** For any $m$-element diagonal subset $A \subseteq (2m-1)>2$, let $V(A)$ be the number of pairs $(\text{clp}(A), \prec)$ where $\prec$ is a $A^\wedge$-vip order of clp$(A)$.

**Lemma 6.12.** For any $m$-element diagonal subset $A \subseteq (2m-1)>2$, the number of $A$-vip orders of clp$(A)$ is bounded above:

$$V(A) \leq \prod_{0 < i < 2m - 2} (\ell_i(A) - 1)! \leq (m - 1)! \prod_{i < m - 1} (i!)^2.$$ 

**Proof.** Recall that every level of clp$(A)$ has an element of $A^\wedge$. For the first inequality, note that in every $A$-vip order, the element of $A^\wedge$ on each level is the least element of the level. The righthand side of the first inequality is the count of level orders that satisfy this constraint. For the second inequality, use the estimates of Lemma 6.10 and the fact that $0! = 1$.

**Lemma 6.13.** The number of sparse $\text{vip} \ r$-types at bounded above by $t^+_m \leq t_{m}(m - 1)! \prod_{i < m}(i!)^2$.

**Proof.** By Lemma 6.8, the number of sparse $m$-types is $t_m$, so the lemma follows from Lemma 6.12.

Using notions from finite combinatorics, such as reverse Raney sequences, and a fine analysis of sparse $m$-types, it is possible to calculate a closed form lower bound for the values of $t^+$. This is done in an upcoming paper [14] by J. Larson, one of the conclusions of which is
Figure 2: Some small values of $t_m^+$ and $t_m$.

**Theorem 6.14.** [Larson [14]] For all $m \geq 2$, $t_m + (2^{m-1})(-1 + \prod_{i<m} i!) \leq t_m^+$.

Figure 2 summarizes the calculation from [14] of values of $t_m^+$ for $m \leq 5$. A comparison with $t_m$ is also included.

## 7 A proof of Shelah’s Theorem

In this section we proof of Theorem 2.5 based on Shelah’s proof together with ideas from [20]. The major part of the proof is dedicated to the proof of Lemma 7.1 below; following this we show that this lemma implies Theorem 2.5.

**Lemma 7.1 (End Homogeneity).** Assume $m \geq 2$ and that $\kappa$ is measurable in the model obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \rightarrow (\kappa)_{2^m}^{\omega_m}$. Then for any well ordering $\prec$ of the levels of $\kappa^+2$ and coloring $d: \bigcup_{\alpha<\kappa}[\alpha^2]^m \rightarrow \sigma$ of the $m$-element level sets of $\kappa^+2$ with $\sigma < \kappa$ colors there is a strong embedding $e: \kappa^+2 \cong T \subseteq \kappa^+2$ such that whenever $s := (s_0, \ldots, s_{m-1}) \in [\alpha^2]^m$ and $\beta < \alpha$ are such that the members of $s|\beta := (s_0|\beta, \ldots, s_{m-1}|\beta)$ are distinct, and $s$ and $s|\beta$ are ordered the same way by $\prec$, then we have $d(e[s]) = d(e[s|\beta])$.

We actually use a slightly stronger version of Lemma 7.1 although we will indicate how this can be avoided at the cost of a slight strengthening of the hypothesis:

**Lemma 7.2.** Suppose that $\kappa$ and $m$ are as in Lemma 7.1 and that for each $\xi < \kappa$ we have a coloring $d_\xi$ of the $m$-element level sets of $\kappa^+2$ in fewer than $\kappa$ colors. Then there is a strong embedding $e$ such that whenever $s$ and $\beta$ are as in Lemma 7.1 we have $d_\xi(e[s]) = d_\xi(e[s|\beta])$ for each $\xi < \kappa$.

| $m$ | $t_m^+$ | $t_m$ |
|-----|---------|-------|
| 1   | 1       | 1     |
| 2   | 2       | 2     |
| 3   | 20      | 16    |
| 4   | 776     | 272   |
| 5   | 151, 184| 7936  |
We will give the proof of Lemma 7.1 and will indicate in footnotes how this should be modified to prove Lemma 7.2.

**Proof.** Let $m, \kappa, d$ and $\prec$ be as in Lemma 7.1 and let $P$ be the the forcing notion adding $\lambda$ many Cohen subsets of $\kappa$. Thus a condition in $P$ is a function $p$ with domain $\text{supp}(p) \in [\lambda]^{<\kappa}$ and with values $p(\nu) \in ^\kappa \{2\}$. We abuse notation by writing $p' \supseteq p$ if $p'$ is stronger than $p$, that is, if $\text{supp}(p') \supseteq \text{supp}(p)$ and $p' (\nu) \supseteq p(\nu)$ for each $\nu \in \text{supp}(p)$.

For $i < \lambda$ let $\eta_i$ be a name for the $i$th Cohen subset of $\kappa$, and let $D$ be a name for the $\kappa$-complete ultrafilter $D$ on $\kappa$ assumed to exist in the generic extension. For any $\xi < \kappa$ let $\phi(\xi) = \{ \eta_i : i < \kappa \} \in D$; we write $d(u)$ for this $\phi$ and let $d(u)$ be a name for $d(u)$.

If $W \subseteq \lambda$ then we write $P|W$ for $\{ p \in P : \text{supp}(p) \subseteq W \}$. If $H : W \rightarrow W'$ is an order preserving map between subsets of $\lambda$ then $h_H : P|W \rightarrow P|W'$ is the map defined by $h_H(p)(H(i)) = p(i)$, and if $\text{otp}(W) = \text{otp}(W')$ then we define $h_{W,W'} = h_H$ where $H : W \cong W'$ is the unique order preserving map.

**Claim 7.2.a.** There is a set $Z \in [\lambda]^{\kappa}$ and a function $W : [Z]^{\leq m} \rightarrow [\lambda]^{\leq \kappa}$ satisfying the following conditions:

1. If $u \in [Z]^{\leq m}$ then $u \subseteq W(u)$, and $P|W(u)$ contains a maximal antichain of conditions $p$ deciding the value of $d(u)$.

2. If $u, u' \in [Z]^{\leq m}$ and $|u| = |u'|$, then $\text{otp}(W(u)) = \text{otp}(W(u'))$, the function $h_{W(u),W(u')}$ maps $u$ to $u'$, and for all $p \in P|W(u)$ and $j < \sigma$, $p \Vdash d(u) = j \iff h_{W(u),W(u')}(p) \Vdash d(u') = j.$

3. If $u' \subseteq u \in [Z]^{\leq m}$ then $W(u') \subseteq W(u)$, and if $u, u' \in [Z]^{\leq m}$ then $W(u \cap u') = W(u) \cap W(u').$

**Proof.** Because $P$ has the $\kappa^+$-chain condition, there are sets $W'(u) \in [\lambda]^{\leq \kappa}$ such that clause 1 is satisfied when $W'$ is substituted for $W$. We can also arrange that $W'(u') \supseteq W'(u)$ whenever $u' \supseteq u$.

Now define an equivalence relation on $[\lambda]^{\leq 2m}$ as follows: two sets $u$ and $u'$ in $[\lambda]^{\leq 2m}$ are equivalent if they satisfy the following two conditions:

---

1For Lemma 7.2 this becomes $d_\xi(u)$ for each $\xi < \kappa$.

2For Lemma 7.2, $P|W(u)$ contains a maximal antichain of conditions deciding the value of $d_\xi(u)$ for each $\xi < \kappa$. 

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a. \(|u| = |u'|\), and if \(|u| \leq m\) then clause 2 above holds with \(W'(u)\) and \(W'(u')\) substituted for \(W(u)\) and \(W(u')\).

b. Suppose that \(u_1, u_2 \subseteq u\), with \(u_1, u_2 \in [\lambda]^{\leq m}\), and that \(u'_1\) and \(u'_2\) are the subsets of \(u'\) such that \((u, u_1, u_2, <) \cong (u', u'_1, u'_2, <)\). Then this isomorphism extends to an isomorphism

\[
(W'(u), W'(u_1), W'(u_2), <) \cong (W'(u'), W'(u'_1), W'(u'_2), <).
\]

There are only \(2^\kappa\) equivalence classes, and therefore the assumption that \(\lambda \to (\kappa)^{2m}\) implies that there is \(Z' \in [\lambda]^{\kappa}\) such that any pairs \(u, u' \in [Z']^k\) are equivalent for all \(k \leq 2m\). It follows that clauses 1 and 2 are satisfied when \(W'\) and \(Z'\) are substituted for \(W\) and \(Z\).

Now define

\[
W(u) := \bigcup \left\{ \bigcap_{v \in X} W'(v) : X \subseteq [Z']^{\leq m} \& \bigcap X \subseteq u \right\},
\]

and let \(Z = \{ \gamma_\nu : \nu < \kappa \}\) where \(\langle \gamma_\nu : \nu < \kappa \rangle\) is the increasing enumeration of \(Z'\). We claim that \(W\) and \(Z\) are as required.

First note that we only need to consider finite sets \(X\) in (2). To see this, suppose \(\bigcap X \subseteq u\), fix a set \(u_0 \in X\), and pick \(X' \subseteq X\) such that \(u_0 \in X'\) and for each \(\xi \in u_0 \setminus u\) there is a set \(u_\xi \in X'\) such that \(\xi \notin u_\xi\). Then \(\bigcap X' \subseteq u\), \(|X'| \leq 1 + |u_0| - |u| \leq 1 + m\), and \(\bigcap_{v \in X} W'(v) \supseteq \bigcap_{v \in X'} W'(v)\).

Now note that condition (2) can be extended to arbitrary finite sets: If \(\{ u_i : i < k \}\) and \(\{ u'_i : i < k \}\) are subsets of \([Z']^{<\omega}\) such that

\[
\left( \bigcup_{i < k} u_i, u_0, \ldots, u_{k-1}, < \right) \cong \left( \bigcup_{i < k} u'_i, u'_0, \ldots, u'_{k-1}, < \right) \tag{3}
\]

then

\[
\left( \bigcup_{i < k} W'(u_i), W'(u_0), \ldots, W'(u_{k-1}), < \right)
\]

\[
\cong \left( \bigcup_{i < k} W'(u'_i), W'(u'_0), \ldots, W'(u'_{k-1}), < \right). \tag{4}
\]
To see this, note that condition 3 implies that $\text{otp}(W'_u) = \text{otp}(W'_v)$ whenever $u, u' \in [Z]^s$ for some $s \leq 2m$. Let $W'(u_i) = \{ \gamma_{i,\nu} : \nu < \xi_i \}$ and $W'(u'_i) = \{ \gamma'_{i,\nu} : \nu < \xi_i \}$ be the increasing enumerations. It follows that $\bigcup_{i<k} W'(u_i) = \{ \gamma_{i,\nu} : i < k \land \nu < \xi_i \}$; and for each $i, i' < k$ and $\nu < \xi_i$ and $\nu' < \xi_{i'}$ we have $\gamma_{i,\nu} = \gamma'_{i',\nu'} \iff \gamma'_{i,\nu} = \gamma'_{i',\nu'}$, and $\gamma_{i,\nu} < \gamma_{i',\nu'} \iff \gamma'_{i,\nu} < \gamma'_{i',\nu'}$. This easily implies 4.

A consequence of this is that the intersection $\bigcap_{v \in X} W'(v)$ in 2 does not depend on $X$, but only on the isomorphism type of $(\bigcup X, u, u_0, \ldots, u_{k-1})$ where $X = \{ u_i : i < k \}$. The proof of this proceeds by induction on the size of the set of ordinals in $\bigcup X$ on which two sets $X$ and $X'$ with the same isomorphism type differ. Let $\xi \in \bigcup X$ and $\xi' \in \bigcup X'$ be the least corresponding pair of ordinals which differ between $X$ and $X'$. Assuming without loss of generality that $\xi < \xi'$, let $X''$ be the set obtained by replacing $\xi$ by $\xi'$ in $X'$. Then $\bigcap_{v \in X} W'(v) = \bigcap_{v \in X''} W'(v)$ by the induction hypothesis. To see that $\bigcap_{v \in X'} W'(v) = \bigcap_{v \in X''} W'(v)$, pick some $v_0 \in X'$ with $\xi \notin v_0$. Then $\bigcap_{v \in X'} W'(v) \subseteq W'(v_0)$, and the members of $W'(v_0)$ are not moved in the isomorphism 4 between $\bigcup_{v \in X'} W'(v)$ and $\bigcup_{v \in X''} W'(v)$.

This implies that the union in 2 can be taken to be finite, since there are only finitely many such isomorphism classes, and hence $|W(u)| \leq \kappa$. If $u, u' \in [Z]^s$ for some $s \leq m$ then it follows that $\text{otp}(W(u)) = \text{otp}(W(u'))$, since the fact that $Z$ contains only limit members of $Z'$ implies that for each set $X$ contributing an intersection to $W(u)$ there is an isomorphic set $X'$ contributing an intersection to $W(u')$. Furthermore this isomorphism between $W(u)$ and $W(u')$ preserves the isomorphism between $W'(u)$ and $W'(u')$, so the equivalence 4 in clause 2 of Claim \ref{zeta2a} holds for $W$ as well as for $W'$. Finally, the definition of $W(u)$ easily implies clause 3 of Claim \ref{zeta2a}.

We are now ready to construct the promised strong embedding $e$. For sets $u \in [Z]^{\leq m}$ define $W^*(u) = W(u) \setminus \bigcup_{u' \subseteq u} W(u')$.

For $\alpha < \kappa$ let $R_\alpha$ be the set of one to one functions $s$ with $\text{dom}(s) \in [nZ]^{\leq m}$ and $\text{ran}(s) \subseteq Z$. If $s \in R_\alpha$, $\beta < \alpha$, and the members of $\{ x|\beta : x \in \text{dom}(s) \}$ are all distinct then we will abuse notation by writing $s|\beta$ for the function $s' \in R_\beta$ with $\text{dom}(s') = \{ x|\beta : x \in \text{dom}(s) \}$ defined by $s'(x|\beta) = s(x)$. The sets $R_\alpha$ include the empty function $\emptyset$ as a member, and we will abuse the notation by taking the function $\emptyset \in R_\alpha$ to be different from $\emptyset \in R_\beta$ whenever $\alpha \neq \beta$.

We use recursion on $\alpha < \kappa$ to define an ordinal $\zeta_\alpha < \kappa$ and a map
e|α2: α2 → ζα2, along with conditions ps ∈ P|W*(ran(s)) for each s ∈ Rα. Several times in the following construction we will use the observation that this, together with the fact from Claim 7.2.a that W(u) ∩ W(u′) = W(u ∩ u′) for all u, u′ ∈ [Z]≤m, implies that ps and p′$s$ are compatible whenever s ∪ s′ is a function. Hence \( \bigcup \{ p_s : s \in R_\alpha \land s \subseteq \tau \} \) is a condition for any one to one function \( \tau \) with \( \text{dom}(\tau) \subseteq \alpha^2 \) and \( \text{ran}(\tau) \subset Z \). In particular this is true for all \( \tau \in R_\alpha \).

The construction will satisfy the following induction hypotheses:

1. If \( g : \text{ran}(s) \to Z \) preserves order, then \( p_g \circ s = h_{W^*(\text{ran}(s)), W^*(\text{ran}(g))}(p_s) \).
2. If \( \beta < \alpha \) and the sets \( \{ x \restriction \beta : x \in \text{dom}(s) \} \) are distinct then \( p_s = p_s \restriction \beta \).
3. \( \bigcup_{t \leq s} p_t \models d(\text{ran}(s)) = d(e[\text{dom}(s)]) \) whenever s is an order isomorphism between \( (\text{dom}(s), \prec) \) and \( (\text{ran}(s), \in) \). \(^3\)
4. If \( \text{dom}(s) = \{ x \} \) and \( s(x) = \eta \in Z \) then \( p_s(\eta) = e(x) \).

Clause 1 asserts that \( p_s \) depends, up to isomorphism, only on \( \text{dom}(s) \) and the order which \( s \) induces on that set.

Clause 3 will imply the required end-homogeneity, since if \( \beta < \alpha \) then \( \bigcup_{t \leq s} p_t \) forces that \( d(\text{ran}(s)) = d(\text{ran}(s \restriction \beta)) \), and consequently \( d(e[\text{dom}(s)]) = d(e[\text{dom}(s \restriction \beta)]) \) whenever \( \prec \) orders \( \text{dom}(s) \) and \( \text{dom}(s \restriction \beta) \) alike.

For \( n \) small enough that \( 2^n < m \) we can define \( \zeta_n = n \), along with \( e(s) = s \) and \( p_s = \emptyset \) for all \( s \in R_n \). The construction for \( \alpha \) with \( 2^\alpha \geq m \) consists of three steps. The first step defines conditions \( \bar{p}_s \) which will satisfy clauses 1 and 2 (with \( \bar{p}_s \) instead of \( p_s \)) and satisfies clause 3 to the extent that

\[
(\forall s \in R_\alpha)(\exists j_s < \sigma) \bigcup_{t \leq s} \bar{p}_t \models d(\text{ran}(s)) = j_s. \tag{5}
\]

The second step will define \( \zeta_\alpha \) and define \( e|\alpha^2: \alpha^2 \to \zeta\alpha^2 \) so that if the ordering given to \( \text{dom}(s) \) by \( s \) agrees with the given ordering \( \prec \) then \( d(e[\text{dom}(s)]) \) has the value \( j_s \) determined in the first step. The final step consists of setting \( p_s = \bar{p}_s \) except for the adjustments necessary to satisfy clause 4.

The first step is divided into two cases, depending on whether or not \( \alpha \) is a limit ordinal.

\(^3\)For Lemma 7.2 this should hold for \( d_\xi \) for all \( \xi < \alpha \).
Case 1: \( \alpha \) is a limit ordinal. Let \( \bar{\zeta} := \sup_{\beta < \alpha} \zeta_\beta \). For \( x \in {}^\omega 2 \) let \( \bar{e}(x) := \bigcup_{\beta < \alpha} (e(x|\beta)), \) and for \( s \in R_\alpha, \) let
\[
\bar{p}_s := \bigcup \{ p_{s|\gamma} : \beta < \alpha \& \{ x|\beta : x \in \text{dom}(s) \} \text{ are pairwise distinct} \}.
\]
Note that the induction hypothesis implies that \( \bar{p}_s \in P|W^*(\text{ran}(s)) \). Furthermore \( \bar{p}_s \) satisfies clause 2 and by the induction hypothesis this implies that \( \bar{p}_s \) satisfies 5.

Case 2: \( \alpha \) is a successor ordinal. Let \( \alpha = \beta + 1, \) set \( \bar{\zeta} = \zeta_\beta + 1, \) and set \( \bar{e}(x) = e(x|\beta)^\frown (x(\beta)) \in {}^2 2 \) for each \( x \in {}^\omega 2 \). For each \( s \in R_\alpha \) such that the members of \( \{ x|\gamma : x \in \text{dom}(s) \} \) are distinct set \( p^0_s = p_{s|\beta} \). If the members of \( \{ x|\gamma : x \in \text{dom}(s) \} \) are not distinct then set \( p^0_s = \emptyset \).

Let \( \langle s_i : i < \gamma \rangle \) enumerate the set of \( s \in R_\alpha \) such that \( \text{ran}(s) \) is an initial segment of \( Z \) of length \( m \). Thus for every \( s \in R_\alpha \) there is a unique ordinal \( i < \gamma \) and order preserving map \( g \) such that \( s = g \circ s_i \). Now define \( p_i^s \) for each \( i \leq \gamma \) and \( s \in R_\alpha \) by recursion on \( i \leq \gamma \): If \( i \) is a limit ordinal then \( p_i^s = \bigcup_{\nu < i} p_{s,\nu}^\nu \). For a successor ordinal \( i + 1 \), suppose that \( p_i^s \) is defined for all \( s \in R_\alpha \) and set \( p^s := \bigcup_{t \subseteq s_i} p_{t,i}^s \). Now choose \( q \supseteq q' \) in \( W(\text{ran}(s_i)) \) so that \( q \) decides the value\(^4\) of \( d(\text{ran}(s_i)) \), and for each \( t \subseteq s_i \) set \( p_t^{i+1} = q(W^*(\text{ran}(t))) \).

To finish up, define \( p_{\text{got}}^{i+1} = h_{W^*(\text{ran}(s_i)), W^*(\text{ran}(g))}(p_t) \), as in clause 1 for all \( t \subseteq s_i \) and all order preserving functions \( g : \text{ran}(t) \to Z \); and set \( p^s_{i+1} = p^s_i \) for all other \( s \in R_\alpha \).

Now complete the first step of the construction by setting \( \bar{p}_s = p^s_\gamma \).

For the second step of the construction let \( \tau : {}^\omega 2 \to Z \) be a one to one map which preserves the given ordering \( < \) on \( {}^\omega 2 \). Such a map exists since \( |{}^\omega 2| < \kappa = |Z| \). Now set \( q := \bigcup \{ p_{\tau|y} : y \in [{}^\omega 2]^{\leq m} \} \). This is a condition, and \( q \) decides the value of \( d(\tau(y)) \) for each \( y \in [{}^\omega 2]^m \).

Now, since \( \mathcal{D} \) is forced to be a \( \kappa \)-complete ultrafilter in the generic extension, there is a condition \( q' \supseteq q \) and an ordinal \( \xi \geq \bar{\zeta} \) such that
\[
q' \models d(\{ \eta_{\nu} \mid \nu \in \tau(y) \}) = d(\tau(y))^5
\]
for each \( y \in [{}^\omega 2]^m \). We can assume without loss of generality that for each \( x \in {}^\omega 2 \), we have \( \text{dom}(q'(\tau(x))) \supseteq \xi \). Set \( \zeta_\alpha = \xi \) and complete the definition.

\(^4\)For Lemma \( \square \) \( q \) decides \( d_\xi(\text{ran}(s_i)) \) for each \( \xi < \alpha \).
\(^5\)For Lemma \( \square \) this should hold for \( d_\xi \) for each \( \xi < \kappa \).
of $e^\alpha 2$ by setting $e(x) = q'(\tau(x))|\xi$. Finally complete the definition of $p_s$ by setting $p_s = \bar{p}_s$ unless $s = \tau|\{x\}$ for some $x$, in which case define $p_s \supseteq \bar{p}_s$ by setting $p_s(\tau(x)) = q'(\tau(x))|\xi + 1$.

This completes the definition of the embedding $e$, and hence of the proof of the End Homogeneity Lemma 7.1. 

**Corollary 7.3.** Suppose that $<$ and $\{<\xi; \xi < \sigma\}$ are well orderings of $\kappa > 2$. Then there is a strong embedding $e: \kappa > 2 \to \kappa > 2$ such that for a dense set of nodes $t \in \kappa > 2$, we have $s_0 \prec s_1 \iff e(s_0) <_\xi e(s_1)$ for all $s_0, s_1 \supseteq t$ and $\xi < \sigma$.

**Proof.** Define $d: \bigcup_{\alpha < \kappa} [\alpha 2]^2 \to \sigma 2$ by setting $d(s_0, s_1)(\xi)$ equal to the boolean value $|| s_0 <_{\text{lex}} s_1 \iff s_0 <_\xi s_1 ||$.

Apply Lemma 7.4 to get a strong embedding $e: \kappa > 2 \to \kappa > 2$ so that if $s_0 <_{\text{lex}} s_1$ are in $\alpha 2$ then $d(e(s_0), e(s_1))$ depends only on whether or not $s_0 \prec s_1$. We want to find, for any given node $t$, a node $t' \supseteq t$ such that $d(e(s_0), e(s_1))(\xi) = || s_0 < s_1 ||$ for all $s_0, s_1 \in \alpha 2$ for $\alpha > \lg(t)$ such that $s_0 <_{\text{lex}} s_1$, $t' \subseteq s_0 \wedge s_1$. It will be sufficient to show that we can do this for any $\xi < \sigma$ and for one of the two cases $s_0 \prec s_1$ or $s_1 \prec s_0$. We will do it for $s_0 \prec s_1$, and will indicate the change for $s_1 \prec s_0$.

Suppose, for the sake of contradiction, that for every $t' \supseteq t$ there is a level set $\{s_0, s_1\}$ such that $s_0 <_{\text{lex}} s_1$, $t' \subseteq s_0 \wedge s_1$ and $s_0 \prec s_1$ but $e(s_0) >_\xi e(s_1)$. By the end-homogeneity the same will be true of any $s'_0 \prec s'_1$ such that $s'_0 \supseteq s_0$.

Define an infinite sequence of pairs $\langle (u_n, v_n) : n < \omega \rangle$ of nodes as follows: Set $v_0 = t$. If $v_n$ is defined then let $u_n, v_{n+1}$ be nodes of the same length, extending $v_n$, such that $u_n <_{\text{lex}} v_{n+1}$ and $e(s_0) >_\xi e(s_1)$ for all $s_0 \supseteq u_n$ and $s_1 \supseteq v_{n+1}$ such that $s_0 \prec s_1$.

Now fix $\alpha < \kappa$ such that $\alpha > \lg(u_n)$ for each $n < \omega$, and choose $w_n \supseteq u_n$ for each $n < \omega$. The nodes $w_n$ have the properties that $w_n <_{\text{lex}} w_{n'}$ for each $n < n'$, and $e(w_n) >_\xi e(w_{n'})$ for each $n < n'$ such that $w_n < w_{n'}$. Now apply Ramsey’s theorem to get an infinite subset $X \subseteq \omega$ such that the boolean values $|| w_n < w_{n'} ||$ are constant for $\{n, n'\} \subseteq [X]^2$. Since $<$ is a well order we must have $w_n < w_{n'}$ for all $n < n'$ in $X$, but then $\langle e(w_n) : n \in X \rangle$ is an infinite descending $<_\xi$-sequence, contradicting the assumption that $<_\xi$ is a well order.

The proof for the case $s_1 \prec s_0$ is the same except that the pair $\{u_n, v_{n+1}\}$ satisfies $v_{n+1} <_{\text{lex}} u_n$ for $s_0 \supseteq u_n$ and $s_1 \supseteq v_{n+1}$. 

\[\square\]
Corollary 7.4. Suppose that \( \kappa, m, \prec, \) and \( d_\xi \) are as in Lemma 7.2. Then there is a strongly embedded tree \( T \subseteq \kappa>2 \) of height \( \kappa \) such that \( d_\xi(s) = d_\xi(s') \) whenever \( s \) and \( s' \) are \( \prec \)-similar members of \( [\alpha_2 \cap T]^m \) and \( \xi \) is smaller than the number of split levels of \( T \) below \( \alpha \).

Proof. Let \( e : \kappa>2 \to \kappa>2 \) be the strong embedding given by Lemma 7.2. By Corollary 7.3 there is a strong embedding \( e' : \kappa>2 \to \kappa>2 \) and a \( t \in \kappa>2 \) such that, for all \( s, s' \supseteq t \) we have \( e'(s) \prec e'(s') \iff e'(s)[\prec] e'(s') \iff s e^{-1}[\prec] s' \). Then \( T = e' e[\kappa>2] \) is as required.

Lemma 7.5. Suppose that \( \kappa \) is a cardinal which is measurable in the generic extension obtained by adding \( \lambda \) Cohen subsets of \( \kappa \), where \( \lambda \to (\kappa)^{2m}_2 \). Then for any coloring \( d \) of the \( m \)-element antichains of \( \kappa>2 \) into \( \sigma < \kappa \) colors, and any well-ordering \( \prec \) of the levels of \( \kappa>2 \), there is a strong embedding \( e : \kappa>2 \cong T \subseteq \kappa>2 \) such that \( d(a) = d(b) \) for all \( \prec \)-similar \( m \)-element antichains \( a \) and \( b \) of \( T \).

First we show that this lemma implies Theorem 2.5:

Proof of Theorem 2.5 from Lemma 7.5. Let \( e : \kappa>2 \to \kappa>2 \) satisfy the conclusion of Lemma 7.5. By Corollary 7.3 there is a strong embedding \( e' : \kappa>2 \to \kappa>2 \) and a dense set of nodes \( w \in \kappa>2 \) such that \( s \prec t \) if and only if \( e(s) \prec e(t) \) for all \( s, t \in \bigcup_{\alpha<\kappa} \cap \text{Cone}(w) \). The composition \( e' \circ e \) satisfies the conclusion of Theorem 2.5.

Proof of Lemma 7.5. The maximum height of a similarity tree with \( m \) terminal nodes is equal to the number of meets plus the number of terminal nodes, which is \( (m - 1) + m = 2m - 1 \). We will define a sequence \( T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{2m-2} \subseteq T_{2m-1} = \kappa>2 \) of strongly embedded subtrees, using a reverse recursion on \( k < 2m - 1 \) starting with \( T_{2m-1} = \kappa>2 \).

For \( k < 2m - 1 \) we assume as a recursion hypothesis that \( T_{k+1} \) has the following homogeneity property:

Suppose that \( x, y \in [T_{k+1}]^m \) are \( \prec \)-similar antichains, with the common collapse \( \text{clp}(x, \prec) = \text{clp}(y, \prec) = (t, \prec_t) \), and suppose that \( i|^{k+2} = j|^{k+2} \) where \( i : (t, \prec_t) \cong (x^\lambda, \prec) \) and \( j : (t, \prec_t) \cong (y^\lambda, \prec) \) are the maps witnessing this similarity. Then \( d(x) = d(y) \).

Note that this puts no constraint on \( T_{2m-1} \), and it implies that \( T_0 \) satisfies the conclusion of Lemma 7.5.
For each $\alpha < \kappa$, each antichain $a \in [T_{k+1} \cap \alpha^2]^\le m$, and each ordered similarity tree $(t, \prec_t)$ consider the following coloring $d_{k,a,t}$ of the $\le m$-element level sets $x \in [T_{k+1} \cap \alpha^2]^\le m$: If $x \cup a$ is an antichain, and $\text{clp}(x \cup a) = (t \cap i^\ge 2, \prec_t)$, then $d_{k,a,t}(x) = d(y)$ where $y \in [T_{k+1}]^m$ is any antichain such that $x = \{ \nu|\alpha : \nu \in y \}$. If $x \cup a$ is not an antichain, if $\text{clp}(x \cup a) \neq (t \cap i^\ge 2, \prec_t)$, or if no such antichain $y$ exists then set $d_{k,a,t}(x) = 0$. Lemma 7.5 implies that there is a strongly embedded subtree $T_k \subset T_{k+1}$ on which $d_{k,a,t}(x) = d_{k,a,t}(\{ \nu \cap \beta : \nu \in x \})$ whenever the nodes $\nu \cap \beta$ for $\nu \in x$ are distinct, and the sets $x$ and $\{ \nu \cap \beta : \nu \in x \}$ are both in $T_k$ and are ordered the same way by $\prec$. Thus $T_k$ satisfies the recursion hypothesis.

We note that the proof of Theorem 2.5 does not require that the ultrafilter $\mathcal{D}$ be normal, and hence is valid for $\kappa = \omega$. This is essentially Harrington’s proof of the Halpern-Läuchli theorem, which may be found in the Farah-Todorcevic book [25]. Harrington’s proof served as a starting point for Shelah in [22].

8 Further remarks on partition theorems

There are a number of remaining open questions suggested by the results presented so far so we comment on some of them.

**Question 8.1.** Is Theorem 2.5 (or Lemma 7.1) consistent with the GCH? Is the conclusion of Theorem 2.5 compatible with $\text{L}$, namely can there be an uncountable cardinal $\kappa$ in $\text{L}$ which satisfies that conclusion?

Note that the hypothesis of lemma 7.1 implies that the GCH fails at almost every $\alpha < \kappa$. Indeed for almost every $\alpha < \kappa$ the power set of $\alpha$ in $V$ is a generic extension obtained by adding $\lambda_\alpha$ Cohen subsets of $\alpha$ to some ground model $M_\alpha$, where $M_\alpha \models \lambda_\alpha \rightarrow (\alpha)^{2^m}_{(2^m)_{M_\alpha}}$, since this is true in the generic extension of $V$ obtained by adding $\lambda$ Cohen subsets to $\kappa$, and this extension does not add bounded subsets to $\kappa$.

The latter part of the question was asked by Michael Hrušák. See below for an argument showing that such a $\kappa$ must be weakly compact.

**Question 8.2.** Can Theorem 2.5 be strengthened to require that $e[\prec] \subseteq \prec$?

It would be sufficient to prove this for level sets in $[\kappa^2]^2$, as this would imply the corresponding modification of Corollary 7.3. A positive solution would be likely to also give a positive answer to the next question:
Question 8.3. Is the relation $\kappa \rightarrow \str^2 \sigma \leftarrow \omega$ consistent? That is, is it consistent that for any $d : [\kappa^2 \sigma] < \omega \rightarrow \sigma$ for some $\sigma < \kappa$, there is a strongly embedded tree $T$ such that $d[[T]^m]$ is finite for each $m < \omega$?

Shelah observes that the corresponding variation of Lemma 7.1 follows from the assumption that $\kappa$ is measurable after adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \rightarrow (\kappa^2 < \omega)$. This may be easily seen by examining the proof of Lemma 7.1 given here, which requires only minor changes. However the top-down proof of Theorem 2.5 from Lemma 7.1 given here does not work with the superscript $< \omega$.

It should be noted that in spite of the use of the ultrafilter $D$ in the construction, the set of splitting levels of the homogeneous strongly embedded subtree $T$ is not in any sense a member of $D$; indeed it is far from being even stationary. This follows from the fact that the set of splitting levels of $T$ need not contain any fixed points, even for $m = 1$, which in turn follows from the observation that $\alpha^2$ contains only $\alpha$ many strongly embedded subtrees, each of which has $2^\alpha$ many branches. Thus we can color, for each cardinal $\alpha$, the members of $\alpha^2$ with $2^\alpha$ colors in such a way that any strongly embedded subtree of $\alpha^2$ contains branches with every color.

Question 8.4. What is the large cardinal strength of the conclusion of Theorem 2.5?

It is easy to see that a supercompact cardinal whose supercompactness was made indestructible to $(< \kappa)$-directed closed forcing using Laver’s method [15], satisfies the assumptions of the theorem. A much better upper bound is available. Namely, work of Gitik in [7] (mentioned also as [6] in [22]), together the Erdős-Rado theorem, implies that a model satisfying the hypothesis of Theorem 2.5 can be constructed by forcing over a model of $\text{GCH} + o(\kappa) = \kappa^{+2m+2}$.

On the other hand, it is easy to see that Theorem 2.5 implies that $\kappa$ is weakly compact. To show that $\kappa \rightarrow (\kappa)^2$, let the function $g : [\kappa]^2 \rightarrow 2$ be given and define a coloring $h$ of the two element antichains in $\kappa^2$ by $h(\{s, t\}) = g(\{lg(s), lg(s \wedge t)\})$ if $lg(s) = lg(t)$ and $h(\{s, t\}) = 0$ otherwise. If $T$ is the strongly embedded tree whose existence is guaranteed by Shelah’s Theorem 2.5 then there are two possible $\prec$-similarity classes of 2-element level sets, one in which the lexicographic order of the pair agrees with the $\prec$-order and the other in which these two orders disagree. By a Sierpinski argument, at the $\omega$th level of $T$, there are pairs of both classes with meets
on the same level. Since the coloring of these level sets depends only on the lengths of the pair and their meet, both \( \prec \)-similarity classes receive the same color. It follows that \( g \) is monochromatic on pairs from the set of \( \alpha \) for which \( \alpha \) is a splitting level of \( T \).

As a consequence of Theorem 1.1 and an earlier result of Hajnal and Komjáth \([8]\) we obtain the following theorem which shows that the conclusion of Theorem 2.5 does not follow from any large cardinal hypothesis. This suggests that the use of an ultrafilter in a generic extension, as opposed to one in \( V \), is a necessary part of the proof.

**Theorem 8.5.** The conclusion of Theorem 2.5 does not follow from any large cardinal hypothesis on \( \kappa \).

**Proof.** Hajnal and Komjáth \([8]\) show that there is a forcing of size \( \aleph_1 \) which adds an order type \( \theta \) of size \( \aleph_1 \) with the property that \( \psi \nrightarrow [\theta]_{\omega_1}^2 \) for every order type \( \psi \), regardless of its size. That is, there is a coloring of the pairs of \( \psi \) into \( \aleph_1 \) many colors such that every suborder of type \( \theta \) gets all the colors. The conclusion of our Theorem 1.1 states, in contrast, that the ordering \( Q_\kappa \) has, for any coloring of its pairs, a subset of the full order type \( Q_\kappa \) which gets only finitely many colors. This subset contains subsets of every order type of size less than \( \kappa \), since \( Q_\kappa \) is a \((< \kappa)\)-universal linear order. Hence Theorem 1.1 cannot hold in the Hajnal-Komjáth extension, and so Shelah’s theorem from which it is derived, cannot either.

### 9 Upper bound for \( \kappa \)-Rado graphs

In this section we prove a limitation of colors result for \( \kappa \)-Rado graphs orders using Shelah’s Theorem 2.5. By a \( \kappa \)-Rado graph we mean a graph \( G \) of size \( \kappa \) with the property that for every two disjoint subsets \( A, B \) of \( G \), each of size \( < \kappa \), there is \( c \in G \) connected to all points of \( A \) and no point of \( B \). The existence of such \( G \) follows from the assumption \( \kappa^{< \kappa} = \kappa \) is regular.

In order to apply Shelah’s Theorem, we need a method of embedding \( \kappa \)-Rado graphs into \( \kappa > 2 \) and a well-ordering of the levels of \( \kappa > 2 \) that is compatible with that embedding. We observe that any \( \kappa \)-Rado graph is isomorphic to one whose universe is \( \kappa \), and generalize the approach used by Erdős, Hajnal and Pósa \([4]\) to embed such a graph into \( \kappa > 2 \).
Definition 9.1. Given a $\kappa$-Rado graph $G = (\kappa, E)$, the tree embedding of $G$ into $^{\kappa>2}$ is the function $\sigma_G : \kappa \to ^{\kappa>2}$ defined by $\sigma_G(0) = \emptyset$, and for $\alpha > 0$, $\sigma_G(\alpha) : \alpha \to 2$ is defined by $\sigma_G(\alpha)(\beta) = 1$ if and only if $\{ \alpha, \beta \} \in E$.

Lemma 9.2. For any $\kappa$-Rado graph $G = (\kappa, E)$, the range of the tree embedding $\sigma$ of $G$ into $^{\kappa>2}$ is a cofinal transverse subset of $^{\kappa>2}$.

Proof. By definition of $\sigma$, for all $\alpha < \kappa$, $\lg(\sigma(\alpha)) = \alpha$, so the range of $\sigma$ is transverse.

To see that the range is cofinal, suppose $s \in ^{\kappa>2}$. Let $\alpha = \lg(s)$ and let $A$ be the set of all $\beta < \lg(s)$ with $s(\beta) = 1$. Since $G$ is a $\kappa$-Rado graph, there is an element $\gamma > \alpha$ such that $\{ \beta, \gamma \} \in E$ for all $\beta \in A$ and $\{ \beta, \gamma \} \notin E$ for all $\beta \in \alpha \setminus A$. It follows that $s \subseteq \sigma(\gamma)$. Thus $\sigma[\kappa]$ is cofinal in $^{\kappa>2}$. \hfill \Box

Our next goal in this section is a translation of questions about isomorphisms of $\kappa$-Rado graphs into themselves to questions about $^{\kappa>2}$. Toward that end, we define passing number preserving maps. This notion was used in the proof of the limitation of colors result by Laflamme, Sauer and Vuksanovic [13] for the countable Rado graph, which is also known as the (infinite) random graph.

Definition 9.3. For $s, t \in ^{\kappa>2}$ with $|t| > |s|$, call $t(|s|)$ the passing number of $t$ at $s$. Call a function $f : ^{\kappa>2} \to ^{\kappa>2}$ passing number preserving or a pnp map if it preserves

1. length order: $\lg(s) < \lg(t)$ implies $\lg(f(s)) < \lg(f(t))$; and
2. passing numbers: $\lg(s) < \lg(t)$ implies $f(t)(\lg(f(s))) = t(\lg(s))$.

The first lemma states that any induced subgraph of a $\kappa$-Rado graph $G = (\kappa, E)$ has an induced subgraph which is isomorphic to the $\kappa$-Rado graph by an isomorphism that also preserves $\prec$.

Lemma 9.4. For any cardinal $\kappa$ with $\kappa^{<\kappa} = \kappa$, any $\kappa$-Rado graph $G = (\kappa, E)$ and any $H \subseteq \kappa$ with $G \cong (H, E|H)$ there is a $\prec$-increasing map $g : \kappa \to H$ with $G \cong (g[\kappa], E|g[\kappa])$.

Proof. Fix attention on a particular $\kappa$-Rado graph $G = (\kappa, E)$ and a specific induced subgraph $(H, E|H)$ isomorphic to $G$. Let $h : \kappa \to H$ be the isomorphism.
Since \( \kappa = \kappa^{\aleph_0} \), by the mapping extension property, for any \( \gamma < \kappa \) and any subset \( A \subseteq \gamma \), there are cofinally many \( \zeta \) with \( \{ \delta < \gamma : \{ \delta, \zeta \} \in E \} = A \).

Define \( z : \kappa \to \kappa \) and \( g : \kappa \to H \) by recursion. Let \( z(\emptyset) = \emptyset \) and \( g(\emptyset) = h(\emptyset) \). Suppose \( z|\alpha \) and \( g|\alpha \) have been defined such that \( z \) is increasing, for all \( \beta < \alpha \), \( g(\beta) = h(z(\beta)) \), and \( g|\alpha \) is an increasing isomorphism of \( (\alpha, E|\alpha) \) into \( (H, E\restriction H) \). Let \( \gamma > \alpha \) be so large that if \( h(\eta) < \sup \{ \lg(g(\beta)) + 1 : \beta < \alpha \} \), then \( \gamma > \eta \). Let \( A_\alpha := \{ z(\beta) : \beta < \alpha \wedge \{ \beta, \alpha \} \in E \} \). Let \( z(\alpha) \geq \gamma \) be such that \( \{ \delta < \gamma : \{ \delta, \zeta \} \in E \} = A_\alpha \). Let \( g(\alpha) = h(z(\alpha)) \). Since \( h \) is an isomorphism, a pair \( \{ h(z(\beta)), h(z(\alpha)) \} \) is in \( E \) if and only if the pair \( \{ z(\beta), z(\alpha) \} \) is in \( E \). It follows that \( \{ \beta < \alpha : \{ g(\beta), g(\alpha) \} \in E \} = A_\alpha \). Therefore by induction, \( g \) is the desired increasing isomorphism into \( H \). \( \square \)

**Lemma 9.5.** Suppose \( G = (\kappa, E) \) is a \( \kappa \)-Rado graph with tree embedding \( \sigma \) and \( S = \sigma[\kappa] \). For any \(-\)-increasing map \( g : \kappa \to \kappa \) with \( G \cong (g[\kappa], E[g[\kappa]]) \), the composition \( \sigma \circ g \circ \sigma^{-1} : S \to S \) is a pnp map.

**Proof.** Let \( f := \sigma \circ g \circ \sigma^{-1} \) for some \(-\)-increasing isomorphism of \( G \) into itself. Suppose \( s, t \in S \) and \( \beta := \lg(s) < \lg(t) = \alpha \). Then \( \sigma^{-1}(s) = \beta \) and \( \sigma^{-1}(t) = \alpha \). Since \( g \) is \(-\)-increasing, \( g(\beta) < g(\alpha) \). Hence \( \lg(f(s)) = g(\beta) < g(\alpha) = \lg(f(t)) \). Moreover, \( t(\lg(s)) = t(\beta) = 1 \) if and only if \( \{ \beta, \alpha \} \in E \). Since \( g \) is an isomorphism, \( t(\lg(s)) = 1 \) if and only if \( \{ g(\beta), g(\alpha) \} \in E \). By definition of tree embedding, it follows that \( t(\lg(s)) = 1 \) if and only if \( f(t)(\lg(s)) = 1 \). Thus \( f \) is a pnp map from \( S \) into \( S \). \( \square \)

By much the same reasoning, one can show the converse.

**Theorem 9.6.** [Translation Theorem] Suppose \( G = (\kappa, E) \) is a \( \kappa \)-Rado graph with tree embedding \( \sigma \) and \( S = \sigma[\kappa] \). For any pnp map \( f : S \to S \), the composition \( g := \sigma^{-1} \circ f \circ \sigma : \kappa \to \kappa \) is an \(-\)-increasing map with \( G \cong (g[\kappa], E[g[\kappa]]) \).

**Proof.** Let \( g := \sigma^{-1} \circ f \circ \sigma \) for some pnp map \( f : S \to S \). Suppose \( \beta < \alpha < \kappa \). Then \( \lg(\sigma(\beta)) = \beta < \alpha = \lg(\sigma(\alpha)) \). Since \( f \) is a pnp map, \( \sigma^{-1} \circ f \circ \sigma(\beta) = \lg(f(\sigma(\beta))) < \lg(f(\sigma(\alpha))) = \sigma^{-1} \circ f \circ \sigma(\alpha) \), so \( g \) is a \(-\)-increasing map.

By the definition of the tree embedding, \( \{ \beta, \alpha \} \) is an edge of \( G \) if and only if \( \sigma(\alpha)(\lg(\sigma(\beta))) = 1 \). Since \( f \) is a pnp map, it follows that \( \{ \beta, \alpha \} \) is an edge of \( G \) if and only if \( f(\sigma(\alpha))(\lg(f(\sigma(\beta)))) = 1 \). Apply the definition of tree embedding to \( g(\alpha) = \sigma^{-1}(f(\sigma(\alpha))) \) and \( g(\beta) = \sigma^{-1}(f(\sigma(\beta))) \), to see that \( \{ \beta, \alpha \} \) is an edge of \( G \) if and only if \( \{ g(\beta), g(\alpha) \} \) is an edge.

Thus \( g \) is a \(-\)-increasing isomorphism of \( G \) into itself. \( \square \)
The next definition identifies sufficient conditions for a map to carry a strongly diagonal set to one of the same \( m \)-type.

**Definition 9.7.** Call a map \( f: \kappa^+ \to \kappa^+ \) **polite** if it satisfies the following conditions for all \( x, y, u, v \):

1. (preservation of lexicographic order) if \( x \) and \( y \) are incomparable and \( x \leq_{\text{lex}} y \), then \( f(x) \) and \( f(y) \) are incomparable and \( f(x) <_{\text{lex}} y \);

2. (meet regularity) if \( \{x, u, v\} \) is diagonal and \( x \land u = x \land v \), then \( f(x) \land f(u) = f(x) \land f(v) \);

3. (preservation of meet length order) if \( \lg(x \land y) < \lg(u \land v) \), then \( \lg(f(x) \land f(y)) < \lg(f(u) \land f(v)) \).

Call it **polite to strongly diagonal sets** if it is a pnp map which satisfies the above conditions for all \( x, y, u, v \) with \( \{x, y, u, v\} \) a strongly diagonal set.

The next lemma follows immediately from the above definition.

**Lemma 9.8.** Strong embeddings are polite and polite to strongly diagonal sets. The collection of polite embeddings is closed under composition as is the collection of embeddings polite to strongly diagonal sets.

**Lemma 9.9.** Suppose \( \phi: \kappa^+ \to \kappa^+ \) is a map which is polite to strongly diagonal sets and whose image is a strongly diagonal set. Then for any strongly diagonal set \( A \), \( \text{clp}(A) = \text{clp}(\phi[A]) \) and there is a pnp map \( \bar{\phi}: A^\land \to (\phi[A])^\land \) such that for all \( x, y \) in \( A \), \( \bar{\phi}(x \land y) = \phi(x) \land \phi(y) \).

**Proof.** Fix a strongly diagonal set \( A \). Let \( \bar{\phi}(a) = \phi(a) \) for \( a \in A \) and let \( \bar{\phi}(a \land b) = \phi(a) \land \phi(b) \) for \( a, b \in A \). The proof of the lemma for \( A \) proceeds by a series of claims.

**Claim 9.9.a.** The map \( \bar{\phi} \) is well-defined.

**Proof.** We must show that for all \( x, y, u, v \in A \), if \( x \land y = u \land v \), then \( \phi(x) \land \phi(y) = \phi(u) \land \phi(v) \). If \( x = y \) or \( u = v \), then \( x = y = u = v \) since \( A \) is diagonal. In this case the claim follows immediately, so assume \( x \neq y \) and \( u \neq v \). If \( \{x, y\} \cap \{u, v\} \) is non-empty, then the claim follows from the assumption of meet regularity. So assume \( \{x, y\} \cap \{u, v\} \) is empty. Since \( \{x, y, u\} \) is a three element diagonal set and \( x \land y \) is an initial segment of all three elements, either \( x \land y = x \land u \) or \( x \land y = y \land u \). Hence by meet regularity, either \( \phi(x) \land \phi(y) = \phi(x) \land \phi(u) = \phi(u) \land \phi(v) \) or \( \phi(x) \land \phi(y) = \phi(y) \land \phi(u) = \phi(u) \land \phi(v) \), and the claim follows. \( \square \)
Claim 9.9.b. The map $\phi$ preserves length order.

Proof. Assume $s, t \in A^\wedge$ satisfy $\lg(s) < \lg(t)$. Let $x, y \in A$ be such that $s = x \wedge y$ and $u, v \in A$ be such that $t = u \wedge v$. By preservation of meet length order, since $\lg(x \wedge y) < \lg(u \wedge v)$, it follows that $\lg(\phi(s)) = \lg(\phi(x) \wedge \phi(y)) < \lg(\phi(u) \wedge \phi(v)) = \lg(\phi(t))$. □

Claim 9.9.c. The map $\phi$ is a pnp map such that for all $x, y$ in $A$, $\phi(s \wedge t) = \phi(s) \wedge \phi(t)$.

Proof. By the definition of $\phi$ and the previous claims, it is enough to show that $\phi$ preserves passing numbers. Suppose $s$ and $t$ are in $A^\wedge$ and $\lg(s) < \lg(t)$. Since any element $t'$ of the closure of $A$ is either in $A$ or has a proper extension $t$ in $A$ with $\phi(t') \subseteq \phi(t) = \phi(t)$, we may assume without loss of generality that $t$ is in $A$. If $s \in A$, then the conclusion follows since $\phi$ is a pnp map.

So suppose $s = x \wedge y$ for $x$ and $y$ distinct elements of $A$. Consider $s \wedge t$. Either it has the same length as $s$ or it is shorter.

If $\lg(s \wedge t) < \lg(s)$, then $\lg(\phi(s \wedge t)) < \lg(\phi(s))$ by the previous claim. In this case, $\phi(t)(\lg(\phi(s))) = 0 = t(\lg(s))$, since $A$ and its image under $\phi$ are both strongly diagonal.

If $\lg(s \wedge t) = \lg(s)$, then $s \subseteq t$. Let $w \in A$ be such that $s = t \wedge w$. Then the value of $t(\lg(s))$ and $\phi(t)(\lg(\phi(s)))$ are determined by the lexicographic order of the pairs $t, w$ and $\phi(t), \phi(w)$. Since $\phi$ preserves lexicographic order, $\phi(t)(\lg(\phi(s))) = t(\lg(s))$, as required. □

Claim 9.9.d. $\clp(A) = \clp(\phi[A])$.

Proof. Enumerate $A^\wedge$ in increasing order of length as $\langle a_\alpha : \alpha < \mu \rangle$ for some $\mu < \kappa$. Let $B = \phi[A]$. Then $B$ is a strongly diagonal set since it is a subset of a strongly diagonal set. Enumerate $B^\wedge$ in increasing order of length as $\langle b_\beta : \beta < \nu \rangle$ for some $\nu < \kappa$.

Since $\phi$ is a pnp map that carries $A^\wedge$ onto $B^\wedge$, it is a bijection from $A^\wedge$ to $B^\wedge$. Since the order type of $\{a \in A^\wedge : \lg(a) < \lg(a_\alpha)\}$ is $\alpha$ and the order type of $\{b \in B^\wedge : \lg(b) < \lg(b_\beta)\}$ is $\beta$, it follows that $\phi(a_\alpha) = b_\beta$ and $\mu = \nu$.

For $\alpha < \mu$, let $A_\alpha := \{a : \lg(a_\alpha) : a \in A\}$ and $B_\alpha := \{b : \lg(b_\alpha) : b \in B\}$. Let $A_\mu = A$ and $B_\mu = B$. Use induction to prove that for all positive $\alpha \leq \mu$, $\clp(A_\alpha) = \clp(B_\alpha)$. To start the induction, observe that $|A_0| = |B_0| = 1$, so $\clp(A_0) = \{\emptyset\} = \clp(B_0)$. For the limit case, assume $\alpha$ is a limit ordinal and for all $\beta < \alpha$, $\clp(A_\beta) = \clp(B_\beta)$. In this case, $\clp(A_\alpha) = \clp(B_\alpha)$, since
\( \text{clp}(A_\alpha) = \bigcup \{ \text{clp}(A_\beta) : \beta < \alpha \} \) and \( \text{clp}(B_\alpha) = \bigcup \{ \text{clp}(B_\beta) : \beta < \alpha \} \). For the successor case, assume \( \alpha = \beta + 1 \) and \( \text{clp}(A_\beta) = \text{clp}(B_\beta) \). Let \( \sigma = \sigma_\alpha \) be the increasing enumeration of \( \{ \lg(a_\gamma) : \gamma < \alpha \} \) and let \( \tau = \tau_\alpha \) be the increasing enumeration of \( \{ \lg(b_\gamma) : \gamma < \alpha \} \). Then \( \text{clp}(A_\alpha) \) is the similarity tree whose set of leaves is \( \{ a \circ \sigma : a \in A_\alpha \} \) and \( \text{clp}(B_\alpha) \) is the similarity tree whose set of leaves is \( \{ b \circ \tau : b \in B_\alpha \} \). Moreover \( \text{clp}(A_\beta) \) is the similarity tree whose set of leaves is \( \{ a \circ (\sigma \upharpoonright \beta) : a \in A_\beta \} \) and \( \text{clp}(B_\beta) \) is the similarity tree whose set of leaves is \( \{ b \circ (\tau \upharpoonright \beta) : b \in B_\beta \} \). For \( a \in A_\alpha \setminus A_\beta \), \( a \circ \sigma(\beta) \) is the passing number of \( a \) at \( \lg(a_\beta) \). Since \( \phi \) preserves passing numbers and carries elements of \( A \setminus A_\beta \) to elements of \( B \setminus B_\beta \), it follows that \( \text{clp}(A_\alpha) = \text{clp}(B_\alpha) \) in this case as well. Therefore by induction, \( \text{clp}(A) = \text{clp}(A^\wedge) \) and \( \text{clp}(B) = \text{clp}(B^\wedge) \), the claim follows.

Since \( A \) was an arbitrary strongly diagonal set, by the last two claims, the lemma follows.

Our next goal is to prove the existence of a \textit{pnp diagonalization}.

**Definition 9.10.** Suppose \( w \in \kappa^+ \) and \( S \subseteq \kappa^+ \). Call \( f \) a \textit{pnp diagonalization} into \( S \cap \text{Cone}(w) \) if \( f \) is a polite injective \( <_Q \)-preserving pnp map whose range is a strongly diagonal subset \( D \) with \( D^\wedge \subseteq S \cap \text{Cone}(w) \). Call \( f \) a \textit{pnp diagonalization} if it is a pnp diagonalization into \( \kappa^+ \cap \text{Cone}(\emptyset) \).

An extra quality we desire for our diagonalization is \textit{level harmony}, which will be used in the section on lower bounds.

**Definition 9.11.** Suppose \( f : \kappa^+ \rightarrow \kappa^+ \) is an injective map. Define \( \hat{f} : \kappa^+ \rightarrow \kappa^+ \) by \( \hat{f}(s) = f(s^\wedge \langle 0 \rangle) \land f(s^\wedge \langle 1 \rangle) \). The function \( f \) has level harmony if \( \hat{f} \) is an extension and \( <_{\text{lex}} \)-order preserving map such that for all \( s, t \in \kappa^+ \), the following conditions hold:

1. \( \hat{f}(s) \not\subseteq f(s) \);
2. \( \lg(s) < \lg(t) \) implies \( \lg(f(s)) < \lg(f(t)) \);
3. \( \lg(s) = \lg(t) \) implies \( \lg(\hat{f}(s)) < \lg(f(t)) \).

**Lemma 9.12.** [Second diagonalization lemma] Suppose \( w \in \kappa^+ \) and that \( S \subseteq \kappa^+ \) is cofinal and transverse. Then there is a pnp diagonalization into \( S \cap \text{Cone}(w) \) which has level harmony.
Proof. Our plan is to approach the problem in pieces by using recursion to define three functions, $\varphi_0, \varphi_1, \varphi : \kappa^+ \to S$ so that $\varphi$ is the desired diagonalization, $\hat{\varphi} = \varphi_0$, $\varphi_1(t)$ is the minimal extension in $S$ of $\varphi_0(\langle 1 \rangle)$, $\varphi(t) \wedge \varphi(t \langle 1 \rangle) = \varphi_1(t)$, and $\varphi(t) <_{\text{lex}} \varphi(t \langle 1 \rangle)$.

For notational convenience, if $\varphi$ has been defined on $\alpha^2$, then we let $\ell_0(\alpha)$ be the least $\theta$ such that $\lg(\varphi(t)) < \theta$ for all $t \in \alpha^2$. Also, if $\varphi_1$ has been defined on $\alpha^2$, then we let $\ell_1(\alpha)$ be the least $\theta$ such that $\lg(\varphi_1(t)) < \theta$ for all $t \in \alpha^2$.

Let $\prec$ be a well-ordering of the levels of $\kappa^+$. We use recursion on $\alpha < \kappa$ to define the restrictions to $\alpha_2$ of $\varphi_0, \varphi_1$ and $\varphi$ so that the following properties hold:

1. extension and lexicographic order:
   
   (a) the restriction of $\varphi_0$ to $\alpha^2$ is extension and $<_{\text{lex}}$-order preserving;
   (b) for all $s \in \alpha^2$, $\varphi_1(s)$ is the minimal extension in $S \cap \text{Cone}(w)$ of $\varphi_0(s) \langle 1 \rangle$;
   (c) for all $s \in \alpha^2$, $\varphi(s)$ is an extension in $S \cap \text{Cone}(w)$ of $\varphi_1(s) \langle 0 \rangle$;
   (d) for all $s \in \alpha^2$, $\varphi_0(s) \langle 0 \rangle$ is an extension of $\varphi_0(s) \langle 1 \rangle$ and $\varphi_0(s) \langle 1 \rangle$ is an extension of $\varphi_1(s) \langle 1 \rangle$;

2. length order:
   
   (a) for all $t \in \alpha^2$ and $s \in \alpha^2$, if $s \prec t$, then
      $\ell_0(\lg(s)) \leq \lg(\varphi_0(s)) < \lg(\varphi_0(t))$ and
      $\ell_1(\lg(s)) \leq \lg(\varphi(s)) < \lg(\varphi(t))$;

3. passing number:
   
   (a) for all $t \in \alpha^2$ and $s \in \alpha^2$, if $s \prec t$ and $s \not\subseteq t$, then
      $\varphi_0(t)(\lg(\varphi_0(s))) = 0$ and $\varphi_0(t)(\lg(\varphi_1(s))) = 0$;
   (b) for all $t \in \alpha^2$ and $s \in \alpha^2$, if $s \prec t$ and $s \not\subseteq t$, then
      $\varphi(t)(\lg(\varphi_0(s))) = 0$ and $\varphi(t)(\lg(\varphi_1(s))) = 0$;
   (c) for all $t \in \alpha^2$ and $s \in \alpha^2$, $\varphi_0(t)(\lg(\varphi(s))) = t(\lg(s))$;
   (d) for all $s,t \in \alpha^2$, if $s \prec t$, then $\varphi(t)(\lg(\varphi(s))) = 0$.

Suppose $\alpha < \kappa$ is arbitrary and for all $\beta < \alpha$, the restrictions to $\beta^2$ of $\varphi_0, \varphi_1$ and $\varphi$ have been defined. To maintain length order, we first define $\varphi_0$.
and \(\phi_1\) by recursion on \(<\) restricted to level \(\alpha^2\). So suppose \(\lg(t) = \alpha\) and for all \(s < t\), \(\phi_0\) and \(\phi_1(s)\) have been defined.

Use extension and \(<_{\text{lex}}\)-order properties to identify an element \(\phi_0^-(t)\) of which \(\phi_0(t)\) is to be an extension. If \(\alpha = 0\), set \(\phi_0^-(t) = \emptyset\). If \(\alpha\) is a limit ordinal, let \(\phi_0^-(t) = \bigcup \{ \phi_0(t|\beta) : \beta < \alpha \}\). If \(\alpha\) is a successor ordinal and \(t = t^- \cup \delta\), let \(\phi_0^-(t) = \phi_0(t^-) \cup \delta\).

Next determine an ordinal \(\gamma_0(t)\) sufficiently large that if \(\phi_0(t)\) is at least that length, it will satisfy the length order property. If \(t\) is the \(<\)-least element of length \(\alpha\), let \(\gamma_0(t) = \ell_0(\alpha)\). If \(t\) has a \(<\)-immediate predecessor \(t'\) of length \(\alpha\), let \(\gamma_0(t) = \lg(\phi_1(t')) + 1\). If \(t\) is a \(<\)-limit of elements of length \(\alpha\), then let \(\gamma_0(t)\) be the supremum of \(\lg(\phi_1(s)) + 1\) for \(s\) of length \(\alpha\) with \(s < t\).

Next define an extension \(\phi_0^n(t)\) of \(\phi_0^-(t)\) of length \(\gamma_0(t)\) so that the passing number properties are satisfied by \(\phi_0^n(t)\) and \(\phi_1^0(t)\) have been defined. Let \(\gamma_0(0) = 0\) and \(\phi_0^+(t) = \emptyset\). If \(\alpha > 0\) is a limit ordinal, then by induction on \(\beta < \alpha\), \(\ell_0(\beta)\) is an increasing sequence. Moreover, the limit of this sequence is the length of \(\phi_0^0(t)\). It follows that \(\phi_0^-(t)\) satisfies the passing number properties for \(s \in \alpha^2\). Let \(\varphi^+(t)\) be the sequence extending \(\phi_0^-(t)\) by zeros, as needed, to a length of \(\gamma_0(t)\). If \(\alpha\) is a successor ordinal and \(t = t^- \cup \delta\), then let \(\varphi^+(t)\) be the extension of \(\phi_0^-\) of length \(\gamma_0^0(t)\) such that for all \(\eta\) with \(\lg(\varphi_0^0(t)) \leq \eta < \gamma_0(t)\), \(\varphi^+(\eta) = \delta\) if \(\eta = \lg(\phi(s))\) for some \(s\) with \(\lg(s) + 1 = \alpha\), and \(\varphi^+(\eta) = 0\) otherwise.

Next let \(\varphi_0(t)\) be an extension in \(S \cap \text{Cone}(w)\) of \(\varphi_0^+(t)\) and let \(\varphi_1(t)\) be an extension in \(S \cap \text{Cone}(w)\) of \(\varphi_0(t^-)\) as required by the extension and lexicographic order properties. The careful reader may now check that the various properties hold for the restrictions of \(\varphi_0\) and \(\varphi_1\) to \(\alpha^2\).

Use a similar process to define the restriction of \(\varphi\) to \(\alpha^2\) by recursion on \(<\) restricted to \(\alpha^2\). Suppose that \(\lg(t) = \alpha\) and for all \(s < t\), \(\varphi(s)\) has been defined. Let \(\varphi^-(t) = \varphi_1(t)^- \cup \emptyset\).

If \(t\) is the \(<\)-least element of length \(\alpha\), let \(\gamma_1(t) = \ell_1(\alpha)\). If \(t\) has a \(<\)-immediate predecessor \(t'\) of length \(\alpha\), let \(\gamma_1(t) = \lg(\varphi_1(t')) + 1\). If \(t\) is a \(<\)-limit of elements of length \(\alpha\), then let \(\gamma_1(t)\) be the supremum of \(\lg(\varphi(s)) + 1\) for \(s\) of length \(\alpha\) with \(s < t\).

Next define an extension \(\varphi^+(t)\) of \(\varphi^-(t)\) of length \(\gamma_1(t)\) so that the passing number properties are satisfied by \(\varphi^+(t)\). If \(\alpha = 0\), there are no passing number properties that need be checked, and we set \(\varphi^+(t) = \varphi^-(t)\). If \(\alpha > 0\), then let \(\varphi^+(t)\) be the extension by zeros of \(\varphi^-(t)\) of length \(\gamma_1(t)\). Since \(\varphi_0(t)\) and \(\varphi_1(t)\) satisfy the passing numbers properties, it follows that \(\varphi^+(t)\) does as well, since all passing numbers longer than \(\lg(\varphi^-(t))\) will be zero.
Finally let $\varphi(t)$ be an extension in $S \cap \text{Cone}(w)$ of $\varphi^+(t)$. The careful reader may now check that the various properties hold for the restriction of $\varphi$ to $\alpha^2$.

This completes the recursive construction of $\varphi_0$, $\varphi_1$ and $\varphi$. By induction, the various properties hold for all $\alpha < \kappa$.

Thus $\varphi_0 = \hat{\varphi}$ is extension and $<_{\text{lex}}$-order preserving, and by the length order property, different elements of the union of the ranges of $\varphi_0$, $\varphi_1$ and $\varphi$ have different lengths. It also follows that these three maps are injective. Moreover the union of their ranges is a subset of $S \cap \text{Cone}(w)$. By the extension and lexicographic order properties and the length order property, $\varphi$ has level harmony.

By the passing number properties, $\varphi$ is a pnp map. By the extension and lexicographic order properties, $\varphi$ preserves $<_{Q}$-order.

By the extension and lexicographic order properties, $\varphi$ carries incomparable elements into incomparable elements and preserves $<_{\text{lex}}$-order. By the length order property, $\varphi$ preserves meet length order. Since $\hat{\varphi} = \varphi_0$ preserves extension, $\varphi$ satisfies meet regularity. Thus $\varphi$ is polite.

From the extension and lexicographic order properties, it follows that the meet closure of $D := \text{ran}(\varphi)$ is the union of the ranges of $\varphi$, $\varphi_0$ and $\varphi_1$ and all elements of the range of $\varphi$ are incomparable. Hence $D$ is an antichain and $D^\wedge \subseteq S \cap \text{Cone}(w)$ is transverse, so $D$ is diagonal. Note that passing numbers of 1 were introduced only to keep $\varphi_0$ extension and $<_{\text{lex}}$-order preserving, to ensure $\varphi(t) <_{\text{lex}} \varphi_1(t)$ so that $<_{Q}$-order is preserved, and to ensure $\varphi$ is a pnp map. It follows that $D^\wedge$ is strongly diagonal.

Therefore, $\varphi$ is the required pnp diagonalization into $S \cap \text{Cone}(w)$ with level harmony.

**Lemma 9.13.** Suppose $S \subseteq {\kappa}^+ > 2$ is cofinal and transverse. Then there are a diagonal set $D$, maps $(\varphi_t : t \in {\kappa}^+ > 2)$ and a pre-$S$-vip order $\prec$ such that for all $t \in {\kappa}^+ > 2$, the following conditions hold:

1. $\varphi_t$ is a pnp diagonalization with level harmony into $S \cap \text{Cone}(t)$;
2. the meet closure of the set $D_t := \text{ran}(\varphi_t)$ is a subset of $S$ disjoint from $D^\wedge_s$ for all $s \prec t$; and
3. $\prec$ is a $D_t$-vip order.

**Proof.** Use Lemma 3.5 to find $\prec^*$, a pre-$S$-vip order on $\kappa > 2$. By Lemma 3.11 $S$ is dense.
Apply the Second Diagonalization Lemma \[9.12\] to each \( t \in {}^\kappa 2 \) to obtain \( \varphi^*_t \), a pnp diagonalization into \( S \cap \text{Cone}(t) \) which has level harmony. Use recursion on \( \prec \) to define \( \pi : {}^\kappa 2 \to {}^\kappa 2, \langle \varphi^*_t : t \in {}^\kappa 2 \rangle \) and \( \langle D_t : t \in {}^\kappa 2 \rangle \) such that for all \( t \in {}^\kappa 2, D_t := \text{ran}(\varphi^*_t) \), \( \pi(t) \) is an extension of \( t \) with \( \text{Cone}(\pi(t)) \) disjoint from the union over all \( s \prec t \) of \( D^*_s \). Since the order type of \( \{ s \in {}^\kappa 2 : s \prec t \} \) is less than \( \kappa \) and each \( D_s \) is a strongly diagonal set whose meet closure is a of \( S \), it is always possible to continue the recursion.

Use induction on the recursive construction to show that the meet closures of the sets \( D_t \) are disjoint.

Let \( D = \bigcup \{ D^*_t : t \in {}^\kappa 2 \} \). Then \( D \) is transverse since it is a subset of \( S \) and \( S \) is transverse. Let \( \prec \) agree with \( \prec^* \) on all pairs from different levels, and use recursion on \( \alpha < \kappa \) to define \( \prec \) from \( \prec^* \) as follows. If there is no element of \( D \) in \( {}^\alpha 2 \) then \( \prec \) and \( \prec^* \) agree on \( {}^\alpha 2 \).

So suppose \( d \in D^*_t \) and \( \lg(d) = \alpha \). For each \( \beta < \alpha \), let \( C(\beta) \) be the set of all \( x \in {}^\alpha 2 \) such that \( x|\beta = d|\beta \) and \( x(\beta) \neq d(\beta) \). Since \( \prec^* \) is a pre-S-vip order, if \( \beta < \gamma < \alpha \), then \( C(\beta) \prec^* C(\gamma) \) in the sense that for every element \( x \) of \( C(\beta) \) and \( y \) of \( C(\gamma) \), one has \( x \prec^* y \). Use the fact that \( D_t \) is a diagonal set to partition \( {}^\alpha 2 = \{ d \} \cup A_\alpha(0) \cup A_\alpha(1) \cup A_\alpha(2) \) into disjoint pieces where for \( \delta < 2 \), \( A_\alpha(\delta) := \{ u|\alpha : u \in D_\delta \land u(\alpha) = \delta \} \). Let the restriction of \( \prec \) to \( {}^\alpha 2 \) be such that for each \( \beta < \alpha \),

\[
C(\beta) \cap A_\alpha(0) \prec C(\beta) \cap A_\alpha(1) \prec C(\beta) \cap A_\alpha(2)
\]

and otherwise \( \prec \) agrees with \( \prec^* \). Then the restriction of \( \prec \) to \( {}^\alpha 2 \) is a well-order, since the restriction of \( \prec^* \) is and because \( \prec^* \) is a pre-S-vip order.

Since the restriction of \( \prec \) to each level is a well-order, it follows that \( \prec \) is a well-ordering of the levels of \( {}^\kappa 2 \). For each \( t \in {}^\kappa 2 \), since \( \prec^* \) is a pre-S-vip order, it follows that \( \prec \) is a pre-\( D_t \)-vip order, so by construction, \( \prec \) is a \( D_t \)-vip order.

**Theorem 9.14.** Let \( m \geq 2 \) and suppose that \( \kappa \) is a cardinal which is measurable in the generic extension obtained by adding \( \lambda \) Cohen subsets of \( \kappa \), where \( \lambda \to (\kappa)_2^{2^m} \). Then for \( r^+_m \) equal to the number of vip \( m \)-types, any \( \kappa \)-Rado graph \( G = (\kappa, E) \) satisfies

\[
G \to (\mathcal{G})^{r^+_m}_{<\kappa,r^+_m}.
\]

**Proof.** Let \( \sigma : \kappa \to {}^\kappa 2 \) be the tree embedding and set \( S = \text{ran} \sigma \). Then by Lemma \[9.12\] \( S \) is cofinal and transverse. Apply Lemma \[9.13\] to obtain a
pre-$S$-vip order $\prec$ and a sequence $\langle \varphi_t : t \in {}^\kappa \omega \rangle$ such that for all $t \in {}^\kappa \omega$, the three listed properties of the lemma hold.

Fix a coloring $c : [\kappa]^m \to \mu$ where $\mu < \kappa$. Define $d : [{}^\kappa \omega]^m \to \mu$ by $d(z) = c(\sigma^{-1}[\varphi_0[z]])$.

Apply Shelah’s Theorem 2.5 to the restriction of $d$ to $m$-element antichains to obtain a strong embedding $e$ and a node $w$ such that $e$ preserves $\prec$ on $\text{Cone}(w)$ and $d(e[a]) = d(e[b])$ for all $\prec$-similar $m$-element antichains $a$ and $b$ of $\text{Cone}(w)$.

Now $D_w := \text{ran}(\varphi_w)$ is a strongly diagonal set and $\prec$ is $D_w$-vip. Let $D = e[\varphi_w[S]]$. Since $\varphi_w[S]$ is a subset of $D_w$, $\varphi_w[S]$ is a strongly diagonal set since $D_w$ is. Also, $\prec$ is $\varphi_w[S]$-vip level order on the downwards closure under initial segments of $\varphi_w[S]$, since $\prec$ is a $D_w$-vip level order. Since $e$ is a strong embedding, $D$ is a strongly diagonal set. Since $e$ preserves $\prec$ on $\text{Cone}(w)$ and $\varphi_w[S] \subseteq D_w \subseteq \text{Cone}(w)$, it follows that the restriction of $\prec$ to the downward closure of $D$ is a $D$-vip order. Hence for all $x \in [D]^m$, the ordered similarity type $(\text{clp}(x), \prec_x)$ is a vip $m$-type.

Finally let $K := \sigma^{-1}[\varphi_0[D]] = \sigma^{-1} \circ \varphi_0 \circ e \circ \varphi_w \circ \sigma[\kappa]$. Note that $D = \varphi_0^{-1}[\sigma[K]]$, so for all $m$-element subsets $u$ of $K$, the image, $x = \varphi_0^{-1}[\sigma[K]][u] \subseteq D$, is a strongly diagonal set, and $(\text{clp}(x), \prec_x)$ is a vip $m$-type.

Since $\varphi_0$, $e$ and $\varphi_w$ are all pnp maps, so is their composition. By the First Translation Theorem 9.6 this mapping is an isomorphism of $G$ into itself.

We claim that $c$ is constant on $m$-element subsets of $K$ whose images under $\varphi_0^{-1} \circ \sigma$ are $\prec$-similar. Consider two such $m$-element subsets $u, v$ of $K$. Let $a'$ and $b'$ be the subsets of $\kappa$ for which $\sigma^{-1} \circ \varphi_0 \circ e \circ \varphi_w \circ \sigma[a'] = u$ and $\sigma^{-1} \circ \varphi_0 \circ e \circ \varphi_w \circ \sigma[b'] = v$. Let $a = \varphi_w[\sigma[a']]$ and $b = \varphi_w[\sigma[b']]$. Then $a$ and $b$ are $m$-element strongly diagonal subsets of $\text{Cone}(w)$. Notice that $u = \sigma^{-1}[\varphi_0[e[a]]]$ and $v = \sigma^{-1}[\varphi_0[e[b]]]$. Thus $e[a]$ and $e[b]$ are $\prec$-similar. Since $e$ is a strong embedding which preserves $\prec$ on $\text{Cone}(w)$, it follows that $a$ and $b$ are $\prec$-similar. By the application of Shelah’s Theorem above, $d(e[a]) = d(e[b])$. From the definition of $d$ it follows that $c(u) = c(\sigma^{-1}[\varphi_0[e[a]]]) = d(e[a])$ and $c(v) = c(\sigma^{-1}[\varphi_0[e[b]]]) = d(e[b])$. Consequently, $c(u) = c(v)$.

Since the image under $\varphi_0^{-1} \circ \sigma$ of any $m$-element subset of $K$ is similar to a vip $m$-type and any two $\prec$-similar subsets receive the same color from $c$, the $m$-element subsets of $K$ are colored with at most $t^+_m$ colors, so the theorem follows. \qed
10 Lower bounds for Rado graphs

The computation of lower bounds for Rado graphs is a bit more complicated than the computation for $\kappa$-dense linear orders. We reduce the problem by showing for suitable $\kappa$ that if $D \subseteq \kappa^>2$ is the range of a pnp diagonalization with level harmony and $\prec$ is a $D$-vip level order, then every vip $m$-type is realized as $(\text{clp}(x), \prec_x)$ for some $x \subseteq D$. This theorem is the companion to Theorem 5.7. Its proof uses a pnp diagonalization with level harmony in place of a semi-strong embedding.

Theorem 10.1. Suppose that $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \rightarrow (\kappa)^6_{\geq \kappa}$. Further suppose $f : (\kappa^>2 \rightarrow \kappa^>2$ is a pnp diagonalization with level harmony, $D := f[\kappa^>2]$, and $\prec$ is a $D$-vip order of the levels of $\kappa^>2$. Then every vip $m$-type $(\tau, \prec)$ is realized as $(\text{clp}(x), \prec_x)$ for some $x \subseteq D$.

Proof. For $t \in \alpha^2$, $i = 0, 1$ and $\delta = 0, 1$, define well-orderings $\prec^i,\delta_t$ on $\alpha^2$ as follows. Let $\beta^i,\delta_t = \lg(f(t))$ and set $\beta^1_t = \lg(f(t))$. Then $s \prec^i,\delta_t s'$ if and only if $f(s(\delta))|\beta^i_t \prec f(s'(\delta))|\beta^i_t$.

Let $\prec'$ be any small well-ordering of the levels of $\kappa^>2$. Call a triple $\{s, s', t\}$ local if $\lg(s) = \lg(s') = \lg(t)$, $s <_{\text{lex}} s'$, $t \prec' s$, and $t <' s'$, and $s \land s' \not\subseteq t$.

Let $d$ be a coloring of the triples of $\kappa^>2$ defined as follows: if $\{s, s', t\}$ is not local, let $d^i,\delta_{\{s, s', t\}} := 2$ and otherwise set $d^i,\delta_{\{s, s', t\}} := \|s \prec' s' \iff s \prec^i,\delta_t s'\|$. For $b = \{s, s', t\} \in [\alpha^2]^3$, define $d(b) := (d^0.0(b), d^1.0(b), d^0.1(b))$.

Apply Shelah’s Theorem 2.5 to $d$ and $\prec'$ to obtain a strong embedding $e : \kappa^>2 \rightarrow \kappa^>2$ and an element $w$ so that for triples from $T := e[\text{Cone}(w)]$, the coloring depends only on the $\prec'$-ordered similarity type of the triple.

Then two local triples $\{s, s', t\}$ and $\{u, u', v\}$ of $T$ are colored the same if and only if for all $i = 0, 1$ and $\delta = 0, 1$,

$s \prec^i,\delta_t s' \iff u \prec^i,\delta_u u'$.

Thus for $t \in T$, the orderings $\prec^i,\delta_t$ must always agree with one of $\prec'$ and its converse on $T$ on pairs $\{s, s'\} \subseteq T$ with $\{s, s', t\}$ local and $s \prec' s'$. Similarly, they must always agree with one of $\prec'$ and its converse on $T$ on pairs $\{s, s'\} \subseteq T$ with $\{s, s', t\}$ local and $s' \prec' s$. Since $\prec$ is a well-order, all of the orderings $\prec^i,\delta_t$ are also well-orders. Thus they always agree with $\prec'$.
Let \((\tau, \prec)\) be an arbitrary vip \(m\)-type. Let \(L\) be the set of leaves of \(\tau\). Then \(\tau\) is a subtree of \(2^{m-2} \geq 2\) and every level of \(2^{m-2} \geq 2\) has exactly one element of \(L^\wedge\). Extend \(\prec\) defined on \(\tau\) to \(\prec^*\) defined on all of \(2^n \geq 2\) in such a way that the extension is still a \(L^\wedge\)-vip order.

Apply Lemma \ref{lem:order-preserving-strong-embedding} to get an order preserving strong embedding \(j\) of \((2^{m-2} \geq 2, \prec^*)\) into \((\text{Cone}(w), \prec')\).

Let \(\langle t_\ell : \ell \leq 2m - 2 \rangle\) enumerate the elements of \(L^\wedge\) in increasing order of length. Note that \(\log(t_\ell) = \ell\). For \(\ell \leq 2m - 2\), define

\[
\beta_\ell := \begin{cases} 
\log(\hat{f}(e(j(t_\ell)))) & \text{if } t_\ell \notin L, \\
\log(f(e(j(t_\ell)))) & \text{if } t_\ell \in L.
\end{cases}
\]

Finally define \(\rho : \tau \to \kappa^* \geq 2\) by recursion on \(\ell \leq 2m - 2\). For \(\ell = 0\), let \(\rho(\emptyset) = \hat{f}(e(j(\emptyset)))\). For \(\ell > 0\), consider three cases for elements of \(\tau \cap \ell 2\).

If \(t_\ell \in L\), let \(\rho(t_\ell) = f(e(j(t_\ell)))\). If \(t_\ell \notin L\), let \(\rho(t_\ell) = \hat{f}(e(j(t_\ell)))\). Note that in both these cases, \(\beta_\ell = \log(\rho(t_\ell))\). If \(s \in \tau \setminus L^\wedge\) has length \(\ell\), then there is a unique immediate successor in \(\tau\), \(s^\wedge \langle \delta \rangle\). In this case, let \(\rho(s) = f(e(j(s))\setminus \langle \delta \rangle)\beta_\ell\).

Since \(j\) sends \(\prec^*\)-increasing pairs to \(\prec'\)-increasing pairs and \(e\) is a \(\prec'\)-order preserving strong embedding, their composition sends \(\prec^*\)-increasing pairs to \(\prec'\)-increasing pairs. Since for \(v_\ell = e(j(t_\ell)) \in T\), the order \(\prec'\) agrees with \(\prec^*\) on \(T \cap \gamma 2\) where \(\gamma = \log(v_\ell)\), it follows that \(\rho\) sends \(\prec^*\)-increasing pairs to \(\prec\)-increasing pairs.

Since \(\hat{f}\) preserves extension and lexicographic order and \(\hat{f}(s) \subseteq f(s)\), \(\rho\) preserves extension and lexicographic order. By construction \(\rho\) sends levels to levels, meets to meets and leaves to leaves. Let \(x = \rho[L]\) be the image under \(\rho\) of the leaves of \(\tau\). By construction, \(x \subseteq \text{ran}(f)\). Also \((\text{clp}(x), \prec_x) = (\tau, \prec)\), as required.

Since \((\tau, \prec)\) was arbitrary, the theorem follows.

We would like to define a coloring of the \(m\)-tuples of \(\kappa\) using \(t^+_m\) colors such that for any \(H \subseteq \kappa\) with \((H, E[H])\) isomorphic to our Rado graph \(\mathbb{G}\), every color is the color of some \(m\)-tuple from \(H\).

By Lemma \ref{lem:isomorphic-to-rado}, there is a \(\prec\)-increasing map \(h : \kappa \to H\) such that the graph \((h[\kappa], E[h[\kappa]])\) is isomorphic to \(\mathbb{G}\). By Lemma \ref{lem:composition-preserves-isomorphism} \(g = \sigma \circ h \circ \sigma^{-1}\) is a pup map. By the previous theorem, every vip \(m\)-type can be realized in the range of a pup diagonalization with level harmony. By the Translation Theorem, a pup map gives rise to an induced subgraph of the Rado graph which is isomorphic to the whole graph.
Our plan is to define a coloring \( c \) using \( t^+_m \) as follows. Apply Lemma 9.13 to obtain a pre-\( S \)-vip order \( \prec \) and a sequence \( \langle \varphi_t : t \in ^* \omega \rangle \) such that for all \( t \in ^* \omega \), the three listed properties of the lemma hold. In particular, \( \varphi_0 \) is a pnp diagonalization into \( S \) with level harmony, \( D_0 := \varphi_0[^* \omega] \) is a strongly diagonal set and \( \prec \) is a \( D \)-vip order. Enumerate the vip \( m \)-types as \( \langle (\tau_j, \preceq_j) : j < t^+_m \rangle \); define \( d \) on \( m \)-element subsets of \( ^* \omega \) by \( d(x) = j \) if \( (\text{clp}(\varphi_0[x]), \prec\text{clp}(\varphi_0[x])) = (\tau_j, \preceq_j) \); and set \( c(b) = d(\sigma[b]) \).

Then for every isomorphic copy \( H \) of the Rado graph, the associated pnp map \( \varphi_0 \circ g \) described above has the property that different colors of \( m \)-element subsets of \( K \) are distinguished by the ordered similarity types of elements of the range of \( \varphi_0 \circ g \). Thus it is enough to show there is a pnp diagonalization \( f \) with level harmony whose range is a subset of \( \varphi_0 \circ g \), and use Theorem 10.1 with \( f \) to show that all the colors appear in every isomorphic copy.

We show that for any pnp map \( g \) with range a strongly diagonal set there is a pnp diagonalization \( f \) with level harmony with \( \text{ran}(f) \subseteq \text{ran}(g) \) in Theorem 10.7 below. We work by successive approximation, using the next lemma.

**Lemma 10.2.** Let \( D \) be the collection of pnp maps whose domain is \( ^* \omega \) and whose range is a strongly diagonal subset of \( ^* \omega \). Then \( D \) is closed under composition as are the following subfamilies:

1. maps in \( D \) with meet regularity;
2. maps in \( D \) with meet regularity that preserve lexicographic order of incomparable pairs;
3. maps in \( D \) that are polite on strongly diagonal sets;
4. maps in \( D \) that are polite.

Moreover, if \( g \) is in one of these families and \( e \) is a strong embedding, then \( e \circ g \) is also in that family.

**Proof.** Apply Lemma 9.8 and use the definitions of terms to see that the various families are closed under composition. Since strong embeddings preserve all the properties in question, and carry strongly diagonal sets to strongly diagonal sets, their composition with any function in one of the various families is also in that family. \( \square \)
Lemma 10.3. Suppose \( w \in {}^{<\omega} 2 \) and \( S \) is cofinal and transverse. Further suppose \( f \) is polite to strongly diagonal sets and has range a strongly diagonal set \( D \) with \( D^\downarrow \subseteq S \cap \text{Cone}(w) \) and \( g \) is a pnp diagonalization which has level harmony. Then \( f \circ g \) is a pnp diagonalization into \( S \cap \text{Cone}(w) \) which has level harmony.

Proof. First notice that \( f \) and \( g \) are in \( D \), so by Lemma 10.2, their composition is a polite pnp map whose range is a strongly diagonal set. Since \( <_q \) agrees with the lexicographic order on incomparable pairs, \( g \) preserves \( <_q \) and sends all pairs to incomparable pairs, the composition \( f \circ g \) preserves \( <_q \).
Since \( g \) is injective with range a strongly diagonal set and \( f \) is a pnp map, the composition is injective. Thus, since the range of \( f \circ g \) is a subset of the range of \( f \), the composition, \( f \circ g \), is a pnp diagonalization into \( S \cap \text{Cone}(w) \).

For notational convenience, let \( h = f \circ g \). Then \( h(s) := h(s^{<0}) \land h(s^{<1}) = f(\langle g(s^{<0}) \rangle) \land f(\langle g(s^{<1}) \rangle) \).

Claim 10.3.a. For any \( s \in {}^{<\omega} 2 \), \( \hat{h}(s) = h(s^{<1} - \delta) \land h(s) \subseteq h(\delta) \), where \( \delta = g(s)(\hat{g}(s)) \).

Proof. Since \( g \) has level harmony, \( \hat{g}(s) \subseteq g(s) \). Set \( \delta := g(s)(\lg(\hat{g}(s)) \).
Since \( s^{<0} \lt_{\text{lex}} s^{<1} \) and \( g \) preserves lexicographic order, it follows that \( g(s^{<0}) \lt_{\text{lex}} g(s^{<1}) \), so \( g(s^{<0})(\lg(\hat{g}(s)) = \delta) \).

Since \( f \) is polite, by meet regularity, \( f(g(s^{<1} - \delta))) \land f(g(s^{<\delta})) = f(g(s^{<1} - \delta)) \land f(g(s)) \). That is, \( \hat{h}(s) = h(s^{<1} - \delta) \land h(s) \).

Claim 10.3.b. The function \( \hat{h} \) preserves lexicographic order.

Proof. Suppose \( s \lt_{\text{lex}} t \). Since \( g \) has level harmony, \( \hat{g}(s) \lt_{\text{lex}} \hat{g}(t), \hat{g}(s) \subseteq g(s) \) and \( \hat{g}(t) \subseteq g(t) \). Hence \( \hat{g}(s) \land \hat{g}(t) = g(s) \land g(t) \). Thus \( \lg(g(s) \land g(t)) < \lg(g(s^{<0}) \land g(s^{<0})) \) and \( \lg(g(s) \land g(t)) < \lg(g(t^{<0}) \land g(t^{<0})) \). Since \( f \) preserves meet length order, it follows that \( \lg(h(s) \land h(t)) < \lg(h(s)) \) and \( \lg(h(s) \land h(t)) < \lg(\hat{h}(t)) \).

Since \( f \) preserves lexicographic order, \( h(s) \lt_{\text{lex}} h(t) \). By the Claim 10.3.a, \( \hat{h}(s) \subseteq h(s) \) and \( \hat{h}(t) \subseteq h(t) \). Thus \( \hat{h}(s) \lt_{\text{lex}} \hat{h}(t) \).

Claim 10.3.c. For all \( s \) and \( t \), \( \lg(s) < \lg(t) \) implies \( \lg(h(s)) < \lg(\hat{h}(t)) \).

Proof. Suppose \( \lg(s) < \lg(t) \). Then \( \lg(g(s)) < \lg(\hat{g}(t)) = \lg(g(s^{<0})) \land g(s^{<1})) \). Since \( g(s) \land g(s) = g(s) \), by preservation of meet length order by \( f \), it follows that \( \lg(h(s) < \lg(\hat{h}(t)) \).
Claim 10.3.d. For all $s$ and $t$, $\lg(s) = \lg(t)$ implies $\lg(\hat{h}(s)) < \lg(h(t))$.

Proof. Suppose $\lg(s) = \lg(t)$. Then $\lg(\hat{g}(s)) < \lg(g(t))$. Argue as in the previous claim: by preservation of meet length order by $f$, it follows that $\lg(\hat{h}(s)) < \lg(h(t))$. \hfill \Box

Claim 10.3.e. The function $\hat{h}$ preserves extension.

Proof. Suppose $s \subseteq t$. Then $\hat{g}(s) \subseteq \hat{g}(t)$, since $g$ has level harmony. Since $f$ satisfies preservation of meet length order, $\lg(\hat{h}(s)) < \lg(\hat{g}(s))$. By Claim 10.3.a, $h(t) \subseteq h(t)$, so $\lg(\hat{h}(s)) < \lg(h(t))$. Thus to show $h(s) \subseteq h(t)$, it is enough to show $\hat{h}(s) \subseteq h(t)$.

If $h(t)(\lg(\hat{h}(s))) = 1$, then $\hat{h}(s) \subseteq h(t)$, since the range of $h$ is strongly diagonal. If $t$ is one of $s \langle 0 \rangle$ and $s \langle 1 \rangle$, then $\hat{h}(s) \subseteq h(t)$ by definition of $\hat{h}(s)$.

So suppose $h(t)(\lg(\hat{h}(s))) = 0$ and $\lg(t) > \lg(s) + 1$. Since $g$ has level harmony, $\hat{g}(s) \nsubseteq g(s)$ and $\hat{g}(t) \nsubseteq g(t)$. By Claim 10.3.a, $\hat{h}(s) = h(t \langle 1 - \delta \rangle) \wedge h(s)$ where $\delta = g(\lg(\hat{g}(s)))$. For notational convenience, let $\alpha = \lg(\hat{g}(s))$. For the first subcase, suppose $g(t)(\alpha) \neq g(s)(\alpha)$. Then $g(t)(\alpha) = g(s \langle 1 - \delta \rangle)(\alpha)$, so $g(s) \wedge g(s \langle 1 - \delta \rangle) = g(s) \wedge g(t)$. Since the range of $g$ is strongly diagonal and $f$ is polite, it follows that $f(g(s)) \wedge f(g(s \langle 1 - \delta \rangle)) = f(g(s)) \wedge f(g(t))$, so $\hat{h}(s) = h(s) \wedge h(t) \subseteq h(t)$. For the second subcase, in which $g(t)(\alpha) = g(s)(\alpha)$, interchange the roles of $g(s)$ and $g(s \langle 1 - \delta \rangle)$. The parallel argument concludes with the inclusion $\hat{h}(s) = h(s \langle 1 - \delta \rangle) \wedge h(t) \subseteq h(t)$. \hfill \Box

Now the lemma follows from the claims. \hfill \Box

The next lemma gives an inequality, for pnp maps, which compares lengths of meets of images with lengths of images of meets.

Lemma 10.4. Suppose $g : \kappa \rightarrow \kappa$ is a pnp map and $\{x, u, v\}$ is a three element strongly diagonal set with $x \wedge u = x \wedge v \nsubseteq u \wedge v$. Then $\lg(g(x) \wedge g(u)) \leq \lg(g(x \wedge u))$ and $\lg(g(x) \wedge g(v)) \leq \lg(g(x \wedge u))$.

Proof. Let $\alpha := \lg(u \wedge x) = \lg(u \wedge v)$ and set $\beta := \lg(g(x \wedge u)) = \lg(g(x \wedge v)$. Since $g$ is a pnp map, $\beta < \lg(g(x))$ and $g(x)(\beta) = u(\alpha)$. Similarly, $\beta < \lg(g(u))$ and $g(u)(\beta) = v(\alpha)$. Also, $\beta < \lg(g(v))$ and $g(v)(\beta) = v(\alpha)$.

Since $x \wedge u = x \wedge v$, it follows that $x(\alpha) \neq u(\alpha) = v(\alpha)$. Consequently, $g(x)(\beta) \neq g(u)(\beta) = g(v)(\beta)$. Thus $\lg(g(x) \wedge g(u)) \leq \beta$ and $\lg(g(x) \wedge g(v)) \leq \beta$, so the lemma follows. \hfill \Box
Next we show how to use Shelah’s Theorem 2.5 to obtain a pnp map with meet regularity from a pnp map whose range is a strongly diagonal set.

**Lemma 10.5.** Suppose that $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \rightarrow (\kappa)^{\lambda}_2$. Further suppose $g : \kappa^+ \rightarrow \kappa^+$ is a pnp map whose range is a strongly diagonal set. Then there is a pnp map $f$ which satisfies meet regularity and whose range is a subset of the range of $g$.

**Proof.** Apply Lemma 9.13 to obtain a pre-$S$-vip order $\prec$ and a sequence $\langle \varphi_t : t \in \kappa^+ \rangle$ such that for all $t \in \kappa^+$, the three listed properties of the lemma hold. Say a three element subset $\{b_0, b_1, b_2\}$ listed in increasing lexicographic order witnesses meet regularity for $g$ if it is diagonal and either $(b_0 \wedge b_1 = b_0 \wedge b_2$ and $g(b_0) \wedge g(b_1) = g(b_0) \wedge g(b_2))$ or $(b_2 \wedge b_0 = b_2 \wedge b_1$ and $g(b_2) \wedge g(b_0) = g(b_2) \wedge g(b_1))$. Say a three element subset $\{b_0, b_1, b_2\}$ listed in increasing lexicographic order refutes meet regularity for $g$ if it is diagonal, $b_0 \wedge b_1 = b_0 \wedge b_2$ implies $g(b_0) \wedge g(b_1) \neq g(b_0) \wedge g(b_2)$ and $b_2 \wedge b_0 = b_2 \wedge b_1$ implies $g(b_2) \wedge g(b_0) \neq g(b_2) \wedge g(b_1)$.

Define a coloring $d'$ on three element subsets of $\kappa^+$ by $d'(x) = 0$ if $x$ is strongly diagonal and witnesses meet regularity for $g$, $d'(x) = 1$ if $x$ is strongly diagonal and refutes meet regularity for $g$ and $d'(x) = 2$ otherwise. Apply Shelah’s Theorem 2.5 to $d'$ and $\prec$ to obtain a strong embedding $e$ and $w \in \kappa^+$ such that $e$ preserves $\prec$ on Cone($w$) and for all three element subsets $x$ of $T = e[Cone(w)]$, the value of $d'(x)$ depends only on the $\prec$-ordered similarity type of $x$.

Let $\psi = \varphi_w$. Then $\psi$ is a pnp diagonalization into $S \cap Cone(w)$. We claim that $f := g \circ e \circ \psi$ is a pnp map which satisfies meet regularity and whose range is a subset of the range of $g$.

Since $e \circ \psi$ is a pnp map which witnesses meet regularity and whose range is a strongly diagonal set, it is enough to show that every strongly diagonal subset $x \subseteq T$ witnesses meet regularity for $g$.

Let $(\tau, \prec)$ be an arbitrary vip $m$-type. Enumerate the leaves of $\tau$ in increasing order of length as $a_0, a_1, a_2$. We must show that $(\tau, \prec)$ witnesses meet regularity for $g$.

**Claim 10.5.a.** If $\log(a_0) = 1$, then every $x \subseteq T$ with ordered similarity type $(\tau, \prec)$ witnesses meet regularity for $g$.

**Proof.** Notice that since that since $\log(a_0) = 1$ and $\tau$ has an elements of the
meet closure of the leaves of lengths 0, 1, 2, 3, 4, we must have \( \lg(a_0) = 1 < \lg(a_1 \land a_2) \).

Let \( \nu \) be a cardinal larger than \( \lg(g(e(\psi(a_0)))) \). For \( \alpha < \nu \), let \( b_\alpha \) be the sequence of length \( 2\alpha + 3 \) if \( \alpha < \omega \) and of length \( \gamma + 2n + 1 \) if \( \alpha = \gamma + n \) for some limit ordinal \( \gamma \geq \omega \) and \( n < \omega \) such that

\[
\beta(\eta) = \begin{cases} 
  a_1(0), & \text{if } \eta = 0, \\
  a_1(1), & \text{if } \eta = 1, \\
  a_2(2), & \text{if } 0 < \eta < \lg(b_\alpha) - 1 \text{ even}, \\
  a_1(2), & \text{if } \eta = \lg(b_\alpha) - 1, \\
  a_2(3), & \text{otherwise.}
\end{cases}
\]

Note that the length of \( b_\alpha \) is always odd and at least 3, and that for odd \( \eta \geq 3 \), \( b_\alpha(\eta) = a_2(3) \). For \( \beta < \alpha \), by construction, \( b_\beta \land b_\alpha = b_\beta | (\lg(b_\beta) - 1) \).

Using the above calculations, the careful reader may check that \( f \) or \( x \eta \geq \eta \leftarrow \eta + 0.5 \) for each \( x \), by construction, \( s_\beta \land b_\alpha = b_\beta | (\lg(b_\beta) - 1) \).

By Lemma 9.9, since \( e \circ \psi \) is a polite pnp map, we have \( \clp(x(\beta, \alpha)) = \clp(\{a_0, b_\beta, b_\alpha\}) = \tau \). Since there is only one vip order on \( \tau \), it follows that \( (\clp(x(\beta, \alpha)), <) = (\tau, <) \).

Since each \( x(\beta, \alpha) \) is a strongly diagonal set, it either witnesses or refutes meet regularity for \( g \). Since each \( x(\beta, \alpha) \) is a subset of \( T \), either they all witness or all refute meet regularity for \( g \).

Since \( e \circ \psi \) satisfies meet regularity, \( t_0 \land s_\beta = t_0 \land s_\alpha \). Suppose each \( x(\beta, \alpha) \) refutes meet regularity. Then \( g(t_0) \land g(s_\beta) \neq g(t_0) \land g(s_\alpha) \) for \( \beta < \alpha < \nu \). Since \( \nu \) is a cardinal larger than \( \lg(g(e(\psi(a_0)))) \), by the Pigeonhole Principle, there are \( \beta_0 < a_0 \) with \( g(t_0) \land g(s_\beta) = g(t_0) \land g(s_\alpha) \). This contradiction shows that each \( x(\beta, \alpha) \) witnesses meet regularity, so by choice of \( d' \), \( e \) and \( w \), the claim follows.

Claim 10.5.b. If \( \lg(a_0 \land a_1) = 1 \), then every \( x \subseteq T \) with ordered similarity type \((\tau, <)\) witnesses meet regularity for \( g \).

Proof. Notice that since that since \( \lg(a_0 \land a_1) = 1 \) and \( \tau \) has an elements of the meet closure of the leaves of lengths 0, 1, 2, 3, 4, we must have \( \lg(a_0 \land a_1) = 1 < \lg(a_0) < \lg(a_1) < \lg(a_2) \). As in the previous case, there is only one vip level order on \( \tau \).
Let $z = e(\psi(a_0))$ and let $\nu = (2^{\lg(g(z))})^+$. We proceed much as in the previous case, except we start by constructing $\{b_\alpha : \alpha < \nu\}$ and $\{c_\alpha : \alpha < \nu\}$ such that for all $\beta < \alpha$, $c_\alpha \land b_\beta = \emptyset = c_\alpha \land b_\alpha$. Then we set $t_\alpha = e(\psi(c_\alpha))$, $s_\alpha = e(\psi(b_\alpha))$, and $x(\beta, \alpha) = \{s_\beta, s_\alpha, t_\alpha\}$. In the previous case we had only $t_0$, so in this case we will use the Pigeonhole Principle to select a collection of $\alpha$'s for which the behavior of $g$ on $t_\alpha$ is sufficiently uniform to reduce this case to one like the previous one.

For $\alpha = \gamma + n$, where $\gamma = 0$ or $\gamma$ limit, let $b_\alpha$ be the sequence of length $\gamma + 2n + 2$ such that

$$b_\alpha(\eta) = \begin{cases} a_1(0), & \text{if } \eta = 0, \\ a_1(1), & \text{if } \eta < \lg(b_\alpha) - 1 \text{ odd,} \\ a_1(2), & \text{if } 0 < \eta < \lg(b_\alpha) - 1 \text{ even,} \\ a_0(1), & \text{if } \eta = \lg(b_\alpha) - 1. \end{cases}$$

Note that the length of $b_\alpha$ is always even and its Cantor normal form has an even finite part that is at least 2. Also, for positive even $\eta$, $b_\alpha(\eta) = a_1(2)$. For $\beta < \alpha$, by construction, $b_\beta \land b_\alpha = b_\beta \land (\lg(b_\beta) - 1)$. For $\alpha < \eta$, let $c_\alpha$ be the sequence of length $\lg(b_\alpha) + 1$ such that $c_\alpha(0) = a_2(0)$, $c_\alpha(1) = a_2(1)$, $c_\alpha(\lg(b_\alpha)) = a_2(3)$, and for all $\eta$ with $1 < \eta < \lg(b_\alpha)$, $c_\alpha(\eta) = a_2(2)$. Then for all $\alpha, \beta < \nu$, $b_\beta \land c_\alpha = \emptyset$. Using these definitions and calculations, the careful reader may check that for $\beta < \alpha$, one has $\clp(\{b_\beta, b_\alpha, c_\alpha\}) = \tau$.

Since sol $\psi$ is a polite pnp map, by Lemma 9.9 we have $\clp(x(\beta, \alpha)) = \clp(\{b_\beta, b_\alpha, c_\alpha\}) = \tau$. Since there is only one vip order on $\tau$, it follows that $\clp(x(\beta, \alpha)), <) = (\tau, \prec)$. Since $e \circ \psi$ satisfies meet regularity, $t_\alpha \land s_\beta = t_\alpha \land s_\alpha$ for all $\beta < \alpha < \nu$.

By Lemma 10.3, $\lg(g(t_\alpha) \land g(s_\beta)) \leq \lg(g(\psi(c_\alpha \land b_\alpha))) = \lg(g(z))$. Similarly, $\lg(g(t_\alpha) \land g(s_\beta)) \leq \lg(g(z))$.

Set $\gamma = \lg(g(z))$. Apply the Pigeonhole Principle, to find a sequence $p$ of length $\gamma$ so that the collection $A = A_\nu = \{\alpha : g(t_\alpha) = \gamma = p\}$ has cardinality $\nu$.

Since each $x(\beta, \alpha)$ is a strongly diagonal set, it either witnesses or refutes meet regularity for $g$. Since each $x(\beta, \alpha)$ is a subset of $T$, either they all witness or all refute meet regularity for $g$. Suppose each $x(\beta, \alpha)$ refutes meet regularity. Then $g(t_\alpha) \land g(s_\beta) \neq g(t_\alpha) \land g(s_\alpha)$ for $\beta < \alpha < \nu$. 

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In particular, for all $\beta < \alpha$ from $A$, one has
\[
g(t_\beta) \wedge g(s_\beta) = p \wedge g(s_\beta) = g(t_\alpha) \wedge g(s_\beta) \neq g(t_\alpha) \wedge g(s_\alpha).
\]
Now we have a contradiction that $\{ g(t_\alpha) \wedge g(s_\alpha) : \alpha \in A \}$ is a set of $\nu$ distinct initial segments of $p$ and $\lg(p) = \gamma < \nu$. This contradiction shows that each $x(\beta, \alpha)$ witnesses meet regularity, so by choice of $d'$, $e$ and $w$, the claim follows. \qed

**Claim 10.5.c.** If $\lg(a_0 \wedge a_2) = 1$, then every $x \subseteq T$ with ordered similarity type $(\tau, \prec)$ witnesses meet regularity for $g$.

**Proof.** Notice that since that since $\lg(a_0 \wedge a_2) = 1$ and $\tau$ has an elements of the meet closure of the leaves of lengths 0, 1, 2, 3, 4, we must have $\lg(a_0 \wedge a_2) = 1 < \lg(a_0) < \lg(a_1) < \lg(a_2)$. As in the previous case, there is only one vip level order on $\tau$.

Let $z = e(\psi(a_0))$ and let $\nu = (2^{\lfloor \lg(|z|) \rfloor})^+$. We proceed much as in the previous case, except we start by constructing $\{ b_\beta : \beta < \nu \}$ and $\{ c_\beta : \alpha < \nu \}$ such that for all $\beta < \alpha$, $c_\beta \wedge b_\beta = \emptyset = c_\beta \wedge b_\alpha$. Then we set $t_\beta = e(\psi(c_\beta))$, $s_\beta = e(\psi(b_\beta))$, and $x(\beta, \alpha) = \{ s_\beta, t_\beta, s_\alpha \}$.

For $\beta = \gamma + n$, where $\gamma = 0$ or $\gamma$ limit, let $b_\beta$ be the sequence of length $\gamma + 4n + 2$ such that for $\eta = \zeta + 4k + \ell$ where $\zeta = 0$ or $\zeta$ limit, $k < \omega$ and $\ell < 4$,
\[
b_\beta(\eta) = \begin{cases} a_2(\ell), & \text{if } \eta < \lg(b_\beta) - 1, \\ a_0(1), & \text{if } \eta = \lg(b_\beta) - 1. \end{cases}
\]
Note that the length of $b_\beta$ is always even and its Cantor normal form has an even finite part that is at least 2. For $\beta < \alpha$, by construction, $b_\beta \wedge b_\alpha = b_\beta(\lg(b_\beta) - 1)$, and $b_\alpha(\lg(b_\alpha) - 1) = a_2(1)$. For $\beta < \nu$, let $c_\beta$ be the sequence of length $\lg(b_\beta) + 1$ such that $c_\beta(0) = a_1(0)$, $c_\beta(\lg(b_\beta) - 1) = a_1(1)$, $c_\beta(\lg(b_\beta)) = a_1(2)$, and for all $\eta$ with $0 < \eta < \lg(b_\beta) - 1$, $c_\beta(\eta) = 0$. Then for all $\beta \leq \alpha < \nu$, $b_\beta \wedge c_\alpha = \emptyset$. Using these definitions and calculations, the careful reader may check that for $\beta < \alpha$, one has $\clp(\{ b_\beta, c_\beta, c_\alpha \}) = \tau$.

The rest of the proof in this case parallels that of the previous one and the details are left to the reader. \qed

Since $S$ is cofinal and transverse, its intersection with $\Cone(p_1)$ is cofinal above $p_1$ and transverse. Thus by Lemma 3.11, $(S \cap \Cone(w), <_Q)$ is $\kappa$-dense. Then $U := T(S \cap \Cone(w))$ is an almost perfect tree by Lemma 4.16.

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**Claim 10.5.d.** If \( \lg(a_1 \land a_2) = 1 \) and \( p_0, p_1 \) are incomparable elements of \( U \) such that \( p_0 \wedge p_1 \) is a densely splitting point of \( S \cap \Cone(w) \) and \( p_0 <_{\lex} p_1 \) if and only if \( a_0 <_{\lex} a_1 \), then there are \( c \in U \cap \Cone(p_0) \) and \( b, b' \in U \cap \Cone(p_1) \) with \( \max(\lg(p_0), \lg(p_1)) < \lg(b \wedge b') < \lg(b) < \lg(b') \) such that \( \{ c, b, b' \} \), \( \triangleleft \) has the same ordered similarity type as \( (\tau, \triangleleft) \).

**Proof.** By Lemma 4.14, \( U \) is almost perfect. Use that fact to find a densely splitting node \( q_1 \) of \( U \) extending \( p_1 \) of length longer than \( \max(\lg(p_0), \lg(p_1)) \).

Let \( C(S \cap \Cone(w)) \) be the set of all limit ordinals \( \alpha > 0 \) such that every \( t \in U \cap \alpha \geq 2 \) has proper extensions in both \( S \cap \alpha \geq 2 \) and \( \mathcal{W}(S) \cap \alpha \geq 2 \) By Lemma 4.16, \( C(S \cap \Cone(w)) \) is closed and unbounded in \( \kappa \).

Let \( \lambda \in C(S \cap \Cone(w)) \) be a regular cardinal greater than \( \lg(q_1) \). Use the fact that \( U \) is almost perfect to find an extension \( c \) of \( p_0 \) in \( S \) of length greater than \( \lambda \).

Define \( h : \lambda \geq 2 \rightarrow U \cap \Cone(q_1) \) by recursion as follows and prove by induction that for all \( s \in \lambda^2 \), \( h(s) \) is a densely splitting point of \( U \cap \Cone(q_1) \cap \lambda^2 \). To start the recursion, let \( h(\emptyset) = q_1 \). If \( h(s) \) has been defined and \( \delta < 2 \), let \( h_0(s \triangledown \{ \delta \}) \) be an extension of \( h(s) \triangledown \{ \delta \} \) of length less than \( \lambda \) which is a densely splitting node of \( S \cap \Cone(w) \) and let \( h(s \triangledown \{ \delta \}) \) be an extension of \( h_0(s \triangledown (1 - \delta)) \) of length less than \( \lambda \) which is a densely splitting node of \( S \cap \Cone(w) \). If \( \gamma < \lambda \) is a limit ordinal and \( h(s \rhd \beta) \) has been defined for all \( \beta < \gamma \), let \( h(s) \) be an extension of \( U \{ h(s) \rhd \beta : \beta < \gamma \} \) of length less than \( \lambda \) which is a densely splitting node of \( S \cap \Cone(w) \). If \( h(s \rhd \beta) \) has been defined for all \( \beta < \lambda \), let \( h(s) \) be \( U \{ h(s) \rhd \beta : \beta < \lambda \} \). Since \( U \) is an almost perfect tree and for each \( s \) of limit length \( \gamma \), the node \( U \{ h(s) \rhd \beta : \beta < \gamma \} \) is an even-handed limit of densely splitting points, the recursion is well-defined.

Set \( A := h[\lambda^2] \). Since \( \lambda \) is regular, it follows that \( A \subseteq \lambda^2 \). Also, \( |A| = 2^\lambda \).

Let \( \xi = \lg(c) \) and let \( h^* : A \rightarrow U \cap \xi \geq 2 \) be an injection with \( s \subseteq h^*(s) \) for all \( s \in A \). Such an injection exists since \( U \) is almost perfect. Define \( \prec^* \) on \( A \) by \( s \prec^* t \) if and only if \( h^*(s) < h^*(t) \). Notice that \( \prec^* \) is a well-order, since \( \prec \) is a well order.

Since \( h \) preserves lexicographic order, we can find \( r_0 \in A \) such that both \( \{ s \in A : s \prec_{\lex} r_0 \} \) and \( \{ t \in A : r_0 \prec_{\lex} t \} \) have cardinality \( 2^\lambda \). Thus there are \( s_0 \) and \( t_0 \) with \( s_0 \prec_{\lex} r_0 \prec_{\lex} t_0 \) and \( r_0 \prec^* s_0, r_0 \prec^* t_0 \). That is there are pairs from \( A \) where the lexicographic order and the \( \prec^* \)-order agree and where they disagree.

Choose \( r_1, r_2 \in A \) such that \( r_1 \prec_{\lex} r_2 \) if and only if \( a_1 \prec_{\lex} a_2 \) and \( r_1 \prec^* r_2 \) if and only if \( a_1|2 \prec a_2|2 \).
Use the fact that \( U \) is almost perfect to find an element \( b \) of \( S \) extending \( h^*(a_1) \) of length greater than \( \xi = \log(c) \) and to find an element \( b' \) of \( S \) extending \( h^*(a_2) \) of length greater than \( \log(b) \). Then \( \{c, b, b'\} \) has the same ordered similarity type as \( (\tau, \prec) \).

\[\text{Claim 10.5.e. If } \log(a_1 \land a_2) = 1, \text{ then every } x \subseteq T \text{ with ordered similarity type } (\tau, \prec) \text{ witnesses meet regularity for } g.\]

**Proof.** Notice that since that since \( \log(a_1 \land a_2) = 1 \) and \( \tau \) has an elements of the meet closure of the leaves of lengths \( 0, 1, 2, 3, 4 \), we must have \( \log(a_1 \land a_2) = 1 < \log(a_0) < \log(a_1) < \log(a_2) \).

The proof of this claim shares similarities with the proofs of Claims 10.5.a, 10.5.c and 10.5.b once the construction of the family of sets \( x(\beta, \alpha) \) has been completed.

Let \( p_0 \) and \( p_1 \) be incomparable elements of \( U \) such that \( p_0 \land p_1 \) is a densely splitting node of \( S \cap \text{Cone}(w) \) and \( p_0 \preceq_\text{lex} p_1 \) if and only if \( a_0 \preceq_\text{lex} a_1 \).

Set \( z = e(p_0 \land p_1) \) and let \( \nu = (2^{\log(g(z))})^+ \).

Define by recursion sequences \( \{p_{\delta, \alpha} : \alpha < \nu \} \) for \( \delta < 2 \), \( \{c_\alpha : \alpha < \nu \} \) and \( \{b_\alpha : \alpha < \nu \} \) and \( \{b'_\alpha : \alpha < \nu \} \) as follows. To start the recursion, let \( p_{0,0} = p_0 \) and \( p_{1,0} = p_1 \). If \( p_{0,\alpha} \) and \( p_{1,\alpha} \) have been defined with \( p_0 \subseteq p_{0,\alpha} \) and \( p_1 \subseteq p_{1,\alpha} \), then apply Claim 10.5.a to \( p_{0,\alpha} \) and \( p_{1,\alpha} \) to obtain \( c_\alpha \in S \cap \text{Cone}(p_{0,\alpha}) \) and \( b_\alpha, b'_\alpha \in S \cap \text{Cone}(p_{1,\alpha}) \) such that \( \{c_\alpha, b_\alpha, b'_\alpha\} \) has the same ordered similarity type as \( (\tau, \prec) \) and the following inequalities hold:

\[
\max(lg(p_{0,\alpha}), lg(p_{1,\alpha})) < \log(b_\alpha \land b'_\alpha) < \log(b_\alpha) < \log(b'_\alpha).
\]

Note that \( c_\alpha \land b_\alpha = p_0 \land p_1 \).

If \( \alpha > 0 \) and \( p_{0,\beta}, p_{1,\beta}, c_\beta, b_\beta \) and \( b'_\beta \) have all been defined for \( \beta < \alpha \), then set \( p_{0,\alpha} := \cup \{c_\beta : \beta < \alpha \} \) and \( p_{1,\alpha} := \cup \{b'_\beta : \beta < \alpha \} \).

By construction, the sequence \( S = \{b_\alpha : \alpha < \nu \} \) is increasing in length. Fix attention on a specific pair \( \beta < \alpha < \nu \). Since \( s_\alpha \in \text{Cone}(p_{1,\alpha}) \), it follows from the definition of \( p_{1,\alpha} \) that \( b'_\beta \subseteq s_\alpha \). Thus \( \{c_\beta, b_\beta, b'_\beta\} \) and \( \{c_\beta, b_\beta, b'_\beta\} \) have the same \( \prec \)-ordered similarity type, namely \( (\tau, \prec) \).

Furthermore, \( c_\beta \land b_\beta = p_0 \land p_1 \) and \( c_\beta \land b_\alpha = p_0 \land p_1 \).

For \( \alpha < \nu \), let \( t_\alpha = e(c_\alpha) \) and let \( s_\alpha = e(b_\alpha) \). For \( \beta < \alpha < \nu \), let \( x(\beta, \alpha) = \{t_\beta, s_\beta, s_\alpha\} \). Since \( e \) preserves meets, it follows that \( e(c_\beta) \land e(b_\beta) = e(p_0 \land p_1) = z \) and \( e(c_\beta) \land e(b_\alpha) = e(p_0 \land p_1) = z \). Since \( e \) is a strong embedding which preserves \( \prec \) on \( \text{Cone}(w) \), and \( x(\beta, \alpha) \subseteq \text{Cone}(w) \), it follows
that \((x(\beta, \alpha), \prec)\) and \(\{c_\beta, b_\beta, b_\alpha\}, \prec\) have the same \(\prec\)-ordered similarity type, namely \((\tau, \preceq)\).

The rest of the proof in this case parallels those of Claims 10.5.c and 10.5.b and the details are left to the reader. \(\square\)

By Claims 10.5.a, 10.5.c, 10.5.b and 10.5.e, we have shown that \((\tau, \preceq)\) witnesses meet regularity for \(g\). Since \((\tau, \preceq)\) was an arbitrary vip \(m\)-type, every such type witness meet regularity for \(g\). Thus \(f = g \circ e \circ \psi\) is a pnp map with meet regularity whose range is a subset of the range of \(g\). \(\square\)

**Lemma 10.6.** Suppose that \(\kappa\) is a cardinal which is measurable in the generic extension obtained by adding \(\lambda\) Cohen subsets of \(\kappa\), where \(\lambda \rightarrow (\kappa)^{\kappa} \rightarrow \kappa^+\). Further suppose \(g : \kappa^{>2} \rightarrow \kappa^{>2}\) is a pnp map which satisfies meet regularity and whose range is a strongly diagonal set. Then there is a pnp map \(f\) which satisfies meet regularity and preserves lexicographic order and whose range is a subset of the range of \(g\).

**Proof.** Apply Lemma 9.13 to obtain a pre-\(S\)-vip order \(\prec\) and a sequence \(\langle \varphi_t : t \in \kappa^{>2} \rangle\) such that for all \(t \in \kappa^{>2}\), the three listed properties of the lemma hold. Let \(\tau_0\) be the 2-type whose leaves are \(\langle 0 \rangle\) and \(\langle 1, 0 \rangle\). Let \(\tau_0\) be the 2-type whose leaves are \(\langle 1 \rangle\) and \(\langle 0, 0 \rangle\). Let \(\tau_2\) be the 2-type whose leaves are \(\langle 0 \rangle\) and \(\langle 1, 1 \rangle\). Let \(\tau_3\) be the 2-type whose leaves are \(\langle 1 \rangle\) and \(\langle 0, 1 \rangle\). Then for all \(x \in [\kappa^{>2}]^2\), the set \(\text{clp}(x)\) is one of \(\tau_0, \tau_1, \tau_2\) and \(\tau_3\).

Define a coloring \(d''\) on pairs from \(\kappa^{>2}\) by \(d''(x) = j\) where \(\text{clp}(g[x]) = \tau_j\).

Apply Shelah’s Theorem 2.3 to \(d''\) and \(\prec\) to obtain a strong embedding \(e\) and \(w \in \kappa^{>2}\) such that \(e\) preserves \(\prec\) on \(\text{Cone}(w)\) and for all pairs \(x\) of \(T = e[\text{Cone}(w)]\), the value of \(d''(x)\) depends only on the \(\prec\)-ordered similarity type of \(x\).

Let \(\psi = \varphi_w\). Then \(\psi\) is a pnp diagonalization into \(S \cap \text{Cone}(w)\). We claim that \(f := g \circ e \circ \psi \circ g \circ e \circ \psi\) is a pnp map which satisfies meet regularity and preserves lexicographic order. Clearly the range of \(f\) is a subset of the range of \(g\).

Since \(e \circ \psi\) is a pnp map which witnesses meet regularity and whose range is a strongly diagonal set, by Lemma 10.5 the functions \(g \circ e \circ \psi\) and \(f\) satisfy meet regularity.

Since \(g\) is a pnp map, for any pair \(x\) from \(e[\text{Cone}(w)]\) with \(\text{clp}(x) = \tau_0\) or \(\text{clp}(x) = \tau_1\), we must have \(\text{clp}(g(x)) = \tau_0\) or \(\text{clp}(g(x)) = \tau_1\). Similarly, for any pair \(x\) from \(e[\text{cone}(w)]\) with \(\text{clp}(x) = \tau_1\) or \(\text{clp}(x) = \tau_2\), we must have \(\text{clp}(g(x)) = \tau_1\) or \(\text{clp}(g(x)) = \tau_2\).
Say $g$ sends $i$ to $j$ if for all incomparable pairs $x$ from $e[\text{Cone}(w)]$ with $\text{clp}(x) = \tau_i$ one has $\text{clp}(g[x]) = \tau_j$.

**Claim 10.6.a.** Either (a) $g$ sends 0 to 1 and 1 to 0, or (b) $g$ sends 0 to 0 and 1 to 1.

*Proof.* Consider $a = \langle 0, 0, 0, 0 \rangle$, $b = \langle 0, 1 \rangle$ and $c = \langle 1, 0, 0 \rangle$. Then $\{a, b, c\}$ is diagonal, and $c \land a = \emptyset = c \land b$.

Now $\text{clp}(\{a, c\}) = \tau_1$ and $\text{clp}(\{b, c\}) = \tau_0$. Let $s = e(\psi(a))$, $t = e(\psi(b))$ and $u = e(\psi(c))$. Set $x := \{s, u\}$ and $y = \{t, u\}$. Since $e \circ \psi$ is a polite pnp map, by Lemma 9.9, we have $\text{clp}(x) = \tau_1$ and $\text{clp}(y) = \tau_0$.

Since $g \circ e \circ \psi$ satisfies meet regularity, $g(u) \land g(s) = g(u) \land g(t)$. That is, $g(s)$ and $g(t)$ are on the same side of $g(u)$. Let $\gamma = \lg(g(u) \land g(s))$. Then $g(s)(\gamma) = g(t)(\gamma) = 1 - g(u)(\gamma)$. If $g(s)(\gamma) = 0$ then $\text{clp}(g[x]) = \tau_1$ and $\text{clp}(g[y]) = \tau_0$, so $g$ sends 1 to 1 and 0 to 0. Otherwise $g(s)(\gamma) = 1$. In this case $\text{clp}(g[x]) = \tau_0$ and $\text{clp}(g[y]) = \tau_1$, so $g$ sends 1 to 0 and 0 to 1. \qed

**Claim 10.6.b.** Either (a) $g$ sends 2 to 3 and 3 to 2, or (b) $g$ sends 2 to 2 and 3 to 3.

*Proof.* The proof parallels that of the previous claim using $a = \langle 1, 1, 1, 1 \rangle$, $b = \langle 1, 0 \rangle$ and $c = \langle 0, 0, 1 \rangle$. The details are left to the reader. \qed

Notice that $e \circ \psi$ sends pairs from $\kappa^2$ to incomparable pairs from $\text{Cone}(w)$. For incomparable pairs, $e \circ \psi$ preserves the similarity type by Lemma 9.9. Hence by the above claims, for any incomparable pair $z$ from $\kappa^2$ with $\text{clp}(z) = \tau_i$, we have $\text{clp}(f[z]) = \tau_i$. Therefore $f$ is a pnp map which satisfies meet regularity and preserves lexicographic order and has range a subset of the range of $g$. \qed

**Theorem 10.7.** Suppose that $\kappa$ is a cardinal which is measurable in the generic extension obtained by adding $\lambda$ Cohen subsets of $\kappa$, where $\lambda \to (\kappa)^{\lambda}_2$. If $g : \kappa^2 \rightarrow \kappa^2$ is a pnp map whose range is a strongly diagonal set whose meet closure is a subset of $S = \sigma[\kappa]$, then there is a pnp diagonalization $f : \kappa^2 \rightarrow \text{ran}(g)$ into $S$ with level harmony.

*Proof.* Apply Lemma 9.13 to obtain a pre-$S$-vip order $\prec$ and a sequence $\langle \varphi_t : t \in \kappa^2 \rangle$ such that for all $t \in \kappa^2$, the three listed properties of the lemma hold. Apply Lemma 10.5 to $g$ to obtain a pnp map $g_0$ which satisfies meet regularity and whose range is strongly diagonal subset of the range of $g$. Apply Lemma 10.6 to $g_0$ to obtain a pnp map $h$ which satisfies meet...
regularity and preserves lexicographic order and whose range is a strongly diagonal subset of \( \text{ran}(g_0) \subseteq \text{ran}(g) \).

Define a coloring \( d^* \) on three element antichains \( b = \{ b_0, b_1, b_2 \} \) listed in increasing order of length as follows: \( d^*(b) = 0 \) if \( \lg(b_0) < \lg(b_1 \land b_2) \) and \( \lg(h(b_0)) < \lg(h(b_1) \land h(b_2)) \); \( d^*(b) = 1 \) if \( \lg(b_0) < \lg(b_1 \land b_2) \) and \( \lg(h(b_0)) \geq \lg(h(b_1) \land h(b_2)) \); and \( d^*(b) = 2 \) otherwise.

Apply Shelah’s Theorem \( \ref{shelah} \) to \( d^* \) and \( \ll \) to obtain a strong embedding \( e \) and \( w \in {}^2 \mathbb{N} \) such that \( e \) preserves \( \ll \) on \( \text{Cone}(w) \) and for all pairs \( x \) of \( T = e[\text{Cone}(w)] \), the value of \( d^*(x) \) depends only on the \( \ll \)-ordered similarity type of \( x \).

Let \( \psi = \varphi_w \). Then \( \psi \) is a pnp diagonalization into \( S \cap \text{Cone}(w) \) with level harmony. We claim that \( f^* := h \circ e \circ \psi \) is a pnp map which satisfies meet regularity and preserves lexicographic order. Clearly the range of \( f^* \) is a subset of the range of \( g \).

Since \( e \circ \psi \) is a pnp map which witnesses meet regularity and preserves lexicographic order, and whose range is a strongly diagonal set, by Lemma \( \ref{10.6} \) the function \( f^* \) satisfies meet regularity and preserves lexicographic order.

Our next goal is to prove \( f^* \) also preserves meet length order. The claim below is a preliminary step toward that goal.

**Claim 10.7.a.** If \( b = \{ b_0, b_1, b_2 \} \subseteq e[\text{Cone}(w)] \) is a strongly diagonal set listed in increasing order of length and \( \lg(b_0) < \lg(b_1 \land b_2) \), then \( d^*(b) = 0 \) and \( \lg(h(b_0)) < \lg(h(b_1) \land h(b_2)) \).

**Proof.** Assume toward a contradiction that \( b = \{ b_0, b_1, b_2 \} \subseteq e[\text{Cone}(w)] \) is a strongly diagonal set listed in increasing order of length with \( \lg(b_0) < \lg(b_1 \land b_2) \) and \( d^*(b) \neq 0 \). Then \( d^*(b) = 1 \). Let \( \tau = \text{clp}(b) \) and list the elements of \( a \) in increasing order of length as \( \tau = \{ a_0, a_1, a_2 \} \). Then \( \lg(a_0) \).

Let \( z = e(\psi(a_0)) \) and let \( \nu \) be an uncountable cardinal greater than \( 2^{\lg(h(z))} \). Using the proof of Claim \( \ref{10.5.a} \) as a guide, construct a sequence \( \langle b_\alpha : \alpha < \nu \rangle \) such that for all \( \alpha < \nu \), \( a_0 \land b_\alpha = \emptyset \) and for all \( \beta < \alpha < \nu \), \( \text{clp}(\{ a_0, b_\beta, b_\alpha \}) = \tau \) and \( \lg(b_\beta) < \lg(b_\alpha) \). Set \( s_\alpha := e(\psi(b_\alpha)) \) for all \( \alpha < \nu \). For \( \beta < \alpha < \nu \), let \( x(\beta, \alpha) := \{ z, s_\beta, s_\alpha \} \). Thus for all \( \beta < \alpha < \nu \), \( \text{clp}(x(\beta, \alpha)) = \tau \).

There is only one vip order \( \ll \) on \( \tau \). Since \( \psi \) is a pnp diagonalization into \( S \cap \text{Cone}(w) \), \( \ll \) is a pre-\( S \)-vip order, and \( e \) preserves \( \ll \) on \( \text{Cone}(w) \), it follows that \( (\text{clp}(x(\beta, \alpha), \ll_{x(\beta, \alpha)})) = (\tau, \ll) \), since both are vip 3-types and \( \text{clp}(x(\beta, \alpha)) = \tau \).
Since \( d^* \) takes the same value on all triples from \( e[Cone(u)] \) of the same \( \prec \)-ordered similarity type, it follows that \( \lg(h(z)) \geq \lg(h(s_\beta) \wedge h(s_\alpha)) \) for all \( \beta < \alpha < \nu \). Since the range of \( h \) is a strongly diagonal set, the inequalities must be strict.

Since \( \nu \) is an uncountable cardinal greater than \( 2^{\lfloor \lg(h(z)) \rfloor} \), by the Pigeonhole Principle, there are \( \beta < \gamma \) with \( h(t_\beta) \lfloor \lg(h(z)) \rfloor = h(t_\gamma) \lfloor \lg(h(z)) \rfloor \). Thus we have reached the contradiction that \( h(t_\beta) \wedge h(t_\gamma) \) must have length short and greater than or equal to \( \lg(h(z)) \). Thus the claim follows.

**Claim 10.7.b.** If \( \lg(x \wedge y) < \lg(u \wedge v) \) and \( |\{x, y, u, v\}| = 3 \), then \( \lg(f'(x) \wedge f'(y)) < \lg(f'(u) \wedge f'(v)) \).

**Proof.** If \( x = y \), then the claim follows from Claim 10.7.a. So assume \( x \neq y \). Then one of \( x \) and \( y \) is either \( u \) or \( v \), so \( x \wedge y \subseteq u \wedge v \). Let \( z \) be the unique one of \( x \) and \( y \) which is not in \( \{u, v\} \). Then \( z \wedge u = z \wedge v = x \wedge y \). By meet regularity, \( f'(z) \wedge f'(u) = f'(z) \wedge f'(v) = f'(x) \wedge f'(y) \). Let \( \beta = \lg(f'(x) \wedge f'(y)) \). Since \( f' \) preserves lexicographic order, \( f'(z)(\beta) \neq f'(u)(\beta) = f'(v)(\beta) \). It follows that \( \lg(f'(u) \wedge f'(v)) > \beta = \lg(f'(x) \wedge f'(y)) \).

**Claim 10.7.c.** If \( x \wedge y \not\subseteq u \wedge v \) and \( |\{x, y, u, v\}| = 4 \), then \( \lg(f'(x) \wedge f'(y)) < \lg(f'(u) \wedge f'(v)) \).

**Proof.** If \( x \wedge y = x \) or \( x \wedge y = y \), then the claim follows from Claim 10.7.a. So assume \( x \neq x \wedge y \neq y \). Since \( x \wedge y \) is a proper initial segment of \( u \wedge v \), either \( (x \wedge y)^-\{0\} \subseteq u \wedge v \) or \( (x \wedge y)^-\{1\} \subseteq u \wedge v \). Consequently, either \( x \wedge y = x \wedge u = x \wedge v \) or \( x \wedge y = y \wedge u = y \wedge v \). In the first case, the claim follows from Claim 10.7.b applied to \( \{x, u, v\} \) and in the second case, it follows from Claim 10.7.b applied to \( \{y, u, v\} \).

**Claim 10.7.d.** If \( \lg(x \wedge y) < \lg(u \wedge v) \), \( x \wedge v \not\subseteq u \wedge v \) and \( |\{x, u, v\}| = 4 \), then \( \lg(f'(x) \wedge f'(y)) < \lg(f'(u) \wedge f'(v)) \).

**Proof.** Since \( \lg(x \wedge y) < \lg(u \wedge v) \) and \( x \wedge v \not\subseteq u \wedge v \), it follows that \( x \wedge y \) and \( u \wedge v \) are incomparable. By Lemma 10.4, \( \lg(f'(x) \wedge f'(y)) \leq \lg(f'(x \wedge y)) \).

If one of \( u \) and \( v \) is an initial segment of the other. Since \( e \circ \psi \) is a pnp map, \( e(\psi(x \wedge y)) \) is shorter than both \( e(\psi(u)) \) and \( e(\psi(v)) \). If neither is an initial segment of the other, the \( e(\psi(x \wedge y)) \) is shorter than both \( e(\psi(u)) \) and \( e(\psi(v)) \) since \( e \circ \psi \) preserves meet length order. Thus by Claim 10.7.a applied to \( \{e(\psi(x \wedge y)), e(\psi(u)), e(\psi(v))\} \), \( \lg(f'(x \wedge y)) < \lg f'(u) \wedge \lg(f'(v)) \) and the claim follows.
Claim 10.7.e. The function \( f' \) preserves meet length order.

**Proof.** Let \( x, y, u, v \) be arbitrary with \( \log(x \land y) < \log(u \land v) \). Consider the set \( \{x, y, u, v\} \). It must have at least two elements and at most four. If it has three or four elements, then \( \log(f'(x) \land f'(y)) < \log(f'(u) \land f'(v)) \) by one of Claims 10.7.b, 10.7.c and 10.7.d. If it has two elements, say \( z \) and \( w \) with \( \log(z) < \log(w) \), then there are only three possible meets, listed here in increasing order of length: \( z \land w \), \( z \land z \) and \( w \land w \). Since \( f' \) is a pnp map, \( \log(f'(z)) < \log(f'(w)) \). Since the range of \( f' \) is strongly diagonal set, \( f' \) is incomparable, so \( f'(z) \land f'(w) \) is a proper initial segment of both \( f'(z) \) and \( f'(w) \). Thus \( f' \) preserves meet length order.

Since \( f' \) satisfies meet regularity and preserves both lexicographic order and meet length order, it is polite. We saw above that \( f' \) is a pnp map whose range is a subset of the range of \( g \). Recall that \( \psi \) is a pnp diagonalization with level harmony into \( S \cap \text{Cone}(w) \). Thus by Lemma 10.3, the function \( f = g \circ f' \) is a pnp diagonalization into \( S \) with \( \text{ran}(f) \subseteq \text{ran}(g) \subseteq S \).

Note that for \( m = 2 \), there are four \( m \)-types and each of them admits a single vip order. Any copy of an uncountable Rado \( G \) has an induced subgraph which is a countable Rado graph. Since Laflamme, Sauer and Vuksanovic [13] have shown that these four types must appear in translations of every induced subgraph of the countable Rado graph which is itself isomorphic to the countable Rado graph, it follows from their work that \( G_\kappa \not\rightarrow (G_\kappa)^2_{<\omega, r_2^m - 1} \).

For larger values of \( m \), we use Shelah’s Theorem.

**Question 10.8.** Suppose \( \kappa \) is an uncountable cardinal with \( \kappa^< \kappa = \kappa \), \( G_\kappa \) is a \( \kappa \)-Rado graph and \( 2 < m < \omega \). Does \( G \not\rightarrow (G)_m^{<\omega, r_m^+ - 1} \)? That is, does the lower bound hold even when \( \kappa \) does not satisfy the hypothesis of Shelah’s Theorem?

**Theorem 10.9.** Let \( m \geq 3 \) be a natural number and suppose that \( \kappa \) is a cardinal which is measurable in the generic extension obtained by adding \( \lambda \) Cohen subsets of \( \kappa \), where \( \lambda \rightarrow (\kappa)^6_{2^\kappa} \). Further suppose \( G_\kappa \) is a \( \kappa \)-Rado graph. Then for \( r_m^+ \) equal to the number of vip \( m \)-types, \( G_\kappa \) satisfies

\[
G_\kappa \not\rightarrow (G_\kappa)^m_{<\omega, r_m^+ - 1}.
\]
$m$ & $r_m^+$ & $r_m$
--- & --- & ---
1 & 1 & 1
2 & 4 & 4
3 & 128 & 112
4 & 26,368 & 12,352
5 & 41,932,288 & 4,437,760

Figure 3: Some small values of $r_m^+$ and $r_m$.

Proof. Apply Lemma 9.13 to obtain a pre-$S$-vip order $\prec$ and a sequence $\langle \varphi_t : t \in \kappa > 2 \rangle$ such that for all $t \in \kappa > 2$, the three listed properties of the lemma hold. Enumerate the vip $m$-types: $(\tau_0, \prec_0), (\tau_1, \prec_1), \ldots, (\tau_{r-1}, \prec_{r-1})$.

Define a coloring $d : [\kappa > 2]^m \to r$ by $d(x) = i$ if $(\text{clp}(\varphi_0[x]), \prec_{\varphi_0[x]}) = (\tau_i, \prec_i)$. Then $c : [\kappa]^m \to r$ defined by $c(b) = d(\sigma[b])$ is a coloring of the $m$-element subsets of $\kappa$ with $r$ colors.

To prove the theorem, we must show that every isomorphic copy of $G_\kappa$ contains sets of every color. Toward that end, let $K \subseteq \kappa$ be an arbitrary set such that $(K, E|_K) \cong (\kappa, E)$ and let $\rho : \kappa \to K$ be the isomorphism and let $j < r$ be arbitrary. By Lemmas 9.3 and 9.6, we may assume that $\sigma \circ \rho \circ \sigma^{-1}$ is a pnp map with domain $S$. Hence $g := \varphi_0 \circ \sigma \circ \rho \circ \sigma^{-1} \circ \varphi_0$ is a pnp map with domain $\kappa > 2$ whose range is a strongly diagonal subset of $S$ and $\text{ran}(g)^\prec \subseteq S$.

Apply Theorem 10.7 to $g$ obtain a pnp diagonalization $f : \kappa > 2 \to \text{ran}(g)$ into $S$ with level harmony. Since $\prec$ is a pre-$S$-vip order, it is also a $D$-vip order for $D = \text{ran}(f)$.

By Theorem 10.1 applied to $f$ and $D$, there is some $y \subseteq D$ such that $(\tau_j, \prec_j) = (\text{clp}(y), \prec_y)$. Recall that $\text{ran}(f) \subseteq \text{ran}(g)$. Let $a$ be such that $g[a] = x$. Then $b := \rho[\sigma^{-1}[\varphi_0[a]]]$ is well-defined and by definition of $\sigma$ and $\rho$, we can see that $b \subseteq K$. Let $x = \sigma[b]$. Then $\varphi_0[x] = g[a] = y$ by definition of $g$ and choice of $a$. So $d(x) = j$, by definition of $d$. Thus $c(b) = d(\sigma[b]) = j$. Since $K$ and $j$ were arbitrary, every isomorphic copy of $G_\kappa$ contains sets of every color.

Figure 3 summarizes the calculation from [14] of values of $r_m^+$ for $m \leq 5$. A comparison with $r_m$, the number of $m$-types, is also included, where $r_m$ is the critical value for finite colorings of $m$-tuples of the countable Rado graph.
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