Moment-Matching Conditions for Exponential Families with Conditioning or Hidden Data

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Abstract

Maximum likelihood learning with exponential families leads to moment-matching of the sufficient statistics, a classic result. This can be generalized to conditional exponential families and/or when there are hidden data. This document gives a first-principles explanation of these generalized moment-matching conditions, along with a self-contained derivation.

1 Overview

This document gives a self-contained introduction to maximum likelihood learning in normal or conditional exponential families, with or without hidden variables. The purpose is to show that the standard “moment matching” property extends to all of these cases, and that there is a mnemonic that makes it easy to remember all cases. All the results are summarized in Fig. 1 – the rest of this document derives these results and gives some discussion.

While self-contained, this document is not intended as a first or comprehensive introduction to exponential families [3, 2, 1]. There is little motivation and no examples. Rather, this is intended for a reader familiar with typical exponential families who wishes to understand how maximum likelihood works with hidden variables and/or conditioning.

1.1 Notation

We use sans-serif characters (e.g. x) to refer to random variables, and roman characters (e.g. x) to refer to specific values. Given a finite dataset, \( \bar{E} \) denotes empirical expectations. For example, given a dataset \( x^{(1)}, x^{(2)}, \cdots, x^{(n)} \), \( \bar{E}_x f(x) = \frac{1}{n} \sum_{i=1}^{n} f(x^{(i)}) \).

2 Exponential Family

The material in this section is all classic. We define an exponential family as

\[
p_\theta(x) = h(x) \exp \left( \theta^T T(x) - A(\theta) \right), \\
A(\theta) = \log \sum_x h(x) \exp \left( \theta^T T(x) \right).
\]

The objects are:

- \( x \) - the variable. For simplicity, we assume \( x \) takes values in a finite set, though the properties extend easily to the case where \( x \) is continuous.
- \( T(x) \) - the “sufficient statistics”. Abstractly, this is some vector-valued function that extracts important “features” of \( x \).
- \( \theta \) - parameter vector.
- \( h(x) \) - the scaling constant. This will often just be one.
- \( A(\theta) \) - the “cumulant” or “log-partition” function, which makes \( p_\theta \) sum to one.
Gradient of log-partition function. The key property that gives maximum-likelihood its elegance with exponential families is the gradient of $A$. This gradient is the expected value of the sufficient statistics, under the current parameters $\theta$. This is easy to show:

$$\frac{dA(\theta)}{d\theta} = \frac{d}{d\theta} \log \sum_x h(x) \exp (\theta^T T(x))$$

$$= \frac{d}{d\theta} \frac{\sum_x h(x) \exp (\theta^T T(x))}{\sum_x h(x) \exp (\theta^T T(x))}$$

$$= \sum_x h(x) \exp (\theta^T T(x)) \frac{\exp(A(\theta))}{T(x)}$$

$$= \sum_x h(x) \exp (\theta^T T(x) - A(\theta)) T(x)$$

$$= \sum_x p_\theta(x) T(x)$$

$$= \mathbb{E}_{p_\theta(x)} [T(x)]$$

Objective for a single datum. Suppose $x$ is a single datum. The log-likelihood of that datum is clearly

$$\log p_\theta(x) = \log h(x) + \theta^T T(x) - A(\theta).$$

Form of learning objective. Take a dataset $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$. In this case, it makes sense to maximize the mean log-likelihood

$$L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log p_\theta(x^{(i)}).$$

Likelihood of a dataset. Substituting the form of $p_\theta$, the learning objective becomes

$$L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log p_\theta(x^{(i)})$$

$$= \frac{1}{n} \sum_{i=1}^{n} \left( \log h(x^{(i)}) + \theta^T T(x^{(i)}) - A(\theta) \right)$$

Alternate form for the likelihood of a dataset. It is informative to write the learning objective using empirical expectations rather than explicit sums. If we do this, $L$ can be re-written as

$$L(\theta) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \log h(x^{(i)}) + \theta^T T(x^{(i)}) - A(\theta) \right)$$

Condition at optimum. Now, suppose that one has found $\theta$ that maximizes $L$. Then, it must be true that the gradient of $L$ is zero. Thus, it must be true that

$$0 = \frac{dL(\theta)}{d\theta}$$

$$= \frac{d}{d\theta} \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \log h(x^{(i)}) + \theta^T T(x^{(i)}) - A(\theta) \right)$$

$$= \lim_{n \to \infty} \frac{d}{d\theta} \frac{1}{n} \sum_{i=1}^{n} \left( \log h(x^{(i)}) + \theta^T T(x^{(i)}) - A(\theta) \right)$$

$$= \lim_{n \to \infty} \left( \mathbb{E}_{x^{(i)}} [T(x^{(i)})] - \mathbb{E}_{p_\theta(x)} [T(x)] \right).$$

Thus, if $\theta$ are the maximum-likelihood parameters, it must be true that

$$\lim_{n \to \infty} \mathbb{E}_{x^{(i)}} [T(x^{(i)})] = \mathbb{E}_{p_\theta(x)} [T(x)].$$

These are the classic “moment-matching” conditions of maximum-likelihood.
3 Conditional Exponential Family

A conditional exponential family can be written as

\[
p_{\theta}(y|x) = h(x, y) \exp \left( \theta^\top T(x, y) - A(x) \right),
\]

(2)

\[
A(x, \theta) = \log \sum_y h(x, y) \exp \theta^\top T(x, y).
\]

One could derive this by defining a joint exponential family over the joint space \((x, y)\) and then conditioning it. However, it is best to think of it as a new model definition. This emphasizes that \(p_{\theta}(x)\) is not even defined.

**Gradient of log-partition function.** In conditional exponential families, the gradient of \(A\) becomes the conditional expected value of the sufficient statistics, under the current parameters \(\theta\).

\[
\frac{dA(x, \theta)}{d\theta} = \frac{d}{d\theta} \log \sum_y h(x, y) \exp \left( \theta^\top T(x, y) \right)
\]

\[
= \frac{\frac{d}{d\theta} \sum_y h(x, y) \exp \left( \theta^\top T(x, y) \right)}{\sum_y h(x, y) \exp \left( \theta^\top T(x, y) \right)}
\]

\[
= \sum_y h(x, y) \exp \left( \theta^\top T(x, y) \right) T(x, y)
\]

\[
= \sum_y p_{\theta}(y|x) T(x, y)
\]

\[
= \mathbb{E}_{p_{\theta}(y|x)} \left[ T(x, y) \right]
\]

(3)

**Objective for a single datum.** Suppose \((x, y)\) is a single datum. Since \(p_{\theta}(x)\) is not defined, it doesn’t even make sense to talk about \(p_{\theta}(x, y)\). Instead, the natural object is the conditional log-likelihood. This is

\[
\log p_{\theta}(y|x) = \log h(x, y) + \theta^\top T(x, y) - A(x, \theta).
\]

**Form of learning objective.** Take a dataset \((x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})\). The mean conditional log-likelihood is

\[
L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log p_{\theta} \left( y^{(i)} | x^{(i)} \right).
\]

**Likelihood of a dataset.** Substituting the form of \(p_{\theta}\), the learning objective becomes

\[
L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log p_{\theta} \left( y^{(i)} | x^{(i)} \right)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left( \log h(x^{(i)}, y^{(i)}) + \theta^\top T(x^{(i)}, y^{(i)}) - A(x^{(i)}, \theta) \right).
\]

**Alternate form for the likelihood of a dataset.** Re-written in terms of empirical expectations, the learning objective is

\[
L(\theta) = \mathbb{E}_{x, y} \left[ \log h(x, y) + \theta^\top T(x, y) - A(x) \right].
\]
**Condition at optimum.** Now, suppose that one has found $\theta$ that maximizes $L$. Then, it must be true that the gradient of $L$ is zero. Thus, it must be true that

$$0 = \frac{dL(\theta)}{d\theta} = \frac{d}{d\theta} \mathbb{E}_x \left[ \log h(x, y) + \theta^\top T(x, y) - A(\theta) \right]$$

$$= \mathbb{E}_x \left[ T(x, y) \right] - \frac{d}{d\theta} \mathbb{E}_x [A(\theta)]$$

$$= \mathbb{E}_x \left[ T(x, y) \right] - \mathbb{E}_x \mathbb{E}_{p(x|y)} \left[ T(x, y) \right].$$

Thus, if $\theta$ are the maximum-likelihood parameters, it must be true that

$$\mathbb{E}_{x,y} \left[ T(x, y) \right] = \mathbb{E}_{x \sim p(x|y)} \mathbb{E}_{y \sim y(x)} \left[ T(x, y) \right]. \tag{4}$$

This generalizes the moment-matching conditions from Eq. (1). Intuitively, we still have a “data expectation on one side” and a “model expectation on the other side”. However, the model does not define a distribution over $x$. Intuitively, Eq. (4) “fills in” the model expectation on the right-hand side using the data.

### 4 Exponential Family with Hidden Variables

Take an exponential family jointly over $(x, u)$

$$p_\theta(x, u) = h(x, u) \exp \left( \theta^\top T(x, u) - A(\theta) \right)$$

$$A(\theta) = \log \sum_{x,u} h(x, u) \exp \theta^\top T(x, u)$$

**Marginal distribution.** What is the marginal distribution of $p_\theta$ over $x$? It’s not hard to show that

$$p_\theta(x) = \sum_u p_\theta(x, u)$$

$$= \sum_u h(x, u) \exp \left( \theta^\top T(x, u) - A(\theta) \right)$$

$$= \exp \left( \log \sum_u h(x, u) \exp \left( \theta^\top T(x, u) - A(\theta) \right) \right)$$

$$= \exp \left( \log \sum_u h(x, u) \exp \left( \theta^\top T(x, u) \right) - A(\theta) \right) - A(\theta)$$

$$= \exp \left( A(x, \theta) - A(\theta) \right),$$

where we define

$$A(x, \theta) = \log \sum_u h(x, u) \exp \left( \theta^\top T(x, u) \right).$$

One should think of the extra argument of $x$ in $A(x, \theta)$ as meaning that $x$ remains fixed in the sum defining this new log-partition function.

**Gradient of log-partition function.** By the same logic as in Section 2 the gradient of $A(\theta)$ is

$$\frac{dA(\theta)}{d\theta} = \mathbb{E}_{p_\theta(x,u)} [T(x, u)].$$
Meanwhile, the gradient of \( A(x, \theta) \) is
\[
\frac{dA(x, \theta)}{d\theta} = \frac{d}{d\theta} \log \sum_u h(x, u) \exp (\theta^T T(x, u))
\]
\[
= \frac{\# \sum_u h(x, u) \exp (\theta^T T(x, u))}{\sum_u h(x, u) \exp (\theta^T T(x, u))} \exp(A(x, \theta))
\]
\[
= \sum_u h(x, u) \exp (\theta^T T(x, u)) T(x, u)
\]
\[
= \mathbb{E}_{p_\theta(u|x)} [T(x, u)]
\]
Notice here that \( x \) is a fixed value, not a random variable.

**Objective for a single datum.** Suppose \( x \) is a single datum, with the corresponding \( u \) left unobserved. It is impossible to evaluate \( p_\theta(x, u) \). A natural alternative is the marginal log-likelihood. This is
\[
\log p_\theta(x) = A(x, \theta) - A(\theta).
\]

**Form of learning objective.** Take a dataset \((x^{(1)}), \cdots, (x^{(n)})\). The mean marginal log-likelihood is
\[
L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log p_\theta \left( x^{(i)} \right).
\]

**Likelihood of a dataset.** Substituting the form of \( p_\theta(x) \), the learning objective becomes
\[
L(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log p_\theta \left( x^{(i)} \right)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \left( A(x^{(i)}, \theta) - A(\theta) \right).
\]

**Alternate form for the likelihood of a dataset.** Re-written in terms of empirical expectations, the learning objective is
\[
L(\theta) = \mathbb{E}_x [A(x, \theta) - A(x)].
\]

**Condition at optimum.** Now, suppose that one has found \( \theta \) that maximizes \( L \). Then, it must be true that the gradient of \( L \) is zero. Thus, it must be true that
\[
0 = \frac{dL(\theta)}{d\theta}
\]
\[
= \frac{d}{d\theta} \mathbb{E}_x [A(x, \theta) - A(\theta)]
\]
\[
= \mathbb{E}_x \left[ \frac{d}{d\theta} A(x, \theta) \right] - \frac{d}{d\theta} A(\theta)
\]
\[
= \mathbb{E}_x \mathbb{E}_{p_\theta(u|x)} [T(x, u)] - \mathbb{E}_{p_\theta(x,u)} [T(x, u)].
\]

Thus, if \( \theta \) are the maximum-likelihood parameters, it must be true that
\[
\mathbb{E}_{p_\theta(u|x)} [T(x, u)] = \mathbb{E}_{p_\theta(x,u)} [T(x, u)]. \tag{5}
\]

The again generalizes the moment-matching conditions from Eq. (1). Intuitively, we still have a “data expectation on one side” and a “model expectation on the other side”. However, the data does not contain values for \( u \). Intuitively, Eq. (5) “fills in” the data expectation on the left-hand side using the model.

This is in contrast to the conditions for the conditional exponential family seen in Eq. (4). There, it was the model that was missing data, which was “filled in” by the data on the right-hand side.
5 Conditional Exponential Family with Hidden Variables

Take an exponential family jointly over \((y, u)\) but conditional on \(x\). This is essentially the same as Section 3, just with \(y\) transformed to \((y, u)\).

\[
\begin{align*}
p_\theta(y, u|x) &= h(x, y, u) \exp (\theta^\top T(x, y, u) - A(x, \theta)) \\
A(x, \theta) &= \log \sum_{y, u} h(x, y, u) \exp \theta^\top T(x, y, u).
\end{align*}
\]

**Marginal distribution.** What is the marginal distribution of \(p_\theta\) over \(y\) given \(x\)? This is

\[
\begin{align*}
p_\theta(y|x) &= \sum_u p_\theta(y, u|x) \\
&= \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u) - A(x, \theta)) \\
&= \exp \left( \log \left( \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u) - A(x, \theta)) \right) \right) \\
&= \exp \left( \log \left( \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u)) \right) - A(x, \theta) \right) \\
&= \exp (A(x, y, \theta) - A(x, \theta)),
\end{align*}
\]

where we define

\[
A(x, \theta) = \log \left( \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u)) \right).
\]

One should think of the extra argument of \(x\) in \(A(x, \theta)\) as meaning that \(x\) remains fixed in the sum defining this new log-partition function.

**Conditional distribution.** We will also need the conditional \(p_\theta(u|y, x)\). This is

\[
\begin{align*}
p_\theta(u|y, x) &= \frac{p_\theta(y, u|x)}{p_\theta(y|x)} \\
&= \frac{h(x, y, u) \exp (\theta^\top T(x, y, u) - A(x, \theta))}{\exp (A(x, y, \theta) - A(x, \theta))} \\
&= h(x, y, u) \exp (\theta^\top T(x, y, u) - A(x, y, \theta)).
\end{align*}
\]

(This formula does not appear in the cheatsheet at the start of this document.)

**Gradient of log-partition function.** By the same logic as in Section 3 the gradient of \(A(\theta)\) is

\[
\frac{dA(x, \theta)}{d\theta} = \mathbb{E}_{p_\theta(y, u|x)} [T(x, y, u)].
\]

Meanwhile, the gradient of \(A(x, \theta)\) is...
\[
\frac{dA(x, y, \theta)}{d\theta} = \frac{d}{d\theta} \log \left( \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u)) \right)
= \frac{d}{d\theta} \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u))
= \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u)) \frac{\partial}{\partial \theta}(\exp(A(x, y, \theta)))
= \sum_u h(x, y, u) \exp (\theta^\top T(x, y, u) - A(x, y, \theta)) T(x, y, u)
= \sum_u p_\theta(u|y, x) T(x, y, u)
= E_{p_\theta(u|x, y)}[T(x, y, u)]
\]

Notice here that \(x\) and \(y\) a fixed values.

**Objective for a single datum.** Suppose \((x, y)\) is a single datum, with the corresponding \(u\) left unobserved. The natural objective is the "marginal conditional" log likelihood \(p_\theta(y|x)\). (\(u\) is marginalized out, while \(x\) is conditioned upon.) This objective is

\[
\log p_\theta(y|x) = A(x, y, \theta) - A(x, \theta).
\]

**Form of learning objective.** Take a dataset \((x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(1)})\). The mean marginal conditional log-likelihood is

\[
L(\theta) = \frac{1}{n} \sum_{i=1}^n \log p_\theta(y^{(i)}|x^{(i)}).
\]

**Likelihood of a dataset.** Substituting the form of \(p_\theta(x)\), the learning objective becomes

\[
L(\theta) = \frac{1}{n} \sum_{i=1}^n \log p_\theta(y^{(i)}|x^{(i)})
= \frac{1}{n} \sum_{i=1}^n \left( A(x^{(i)}, y^{(i)}, \theta) - A(x^{(i)}, \theta) \right).
\]

**Alternate form for the likelihood of a dataset.** Re-written in terms of empirical expectations, the learning objective is

\[
L(\theta) = \hat{E}_x [A(x, y, \theta) - A(x, \theta)].
\]

**Condition at optimum.** Now, suppose that one has found \(\theta\) that maximizes \(L\). Then, it must be true that the gradient of \(L\) is zero. Thus, it must be true that

\[
0 = \frac{dL(\theta)}{d\theta}
= \frac{d}{d\theta} \hat{E}_{x,y} [A(x, y, \theta) - A(x, \theta)]
= \hat{E}_{x,y} \left[ \frac{d}{d\theta} A(x, y, \theta) \right] - \hat{E}_x \left[ \frac{d}{d\theta} A(x, \theta) \right]
= \hat{E}_{x,y} E_{p_\theta(u|x, y)} [T(x, y, u)] - \hat{E}_x E_{p_\theta(y|u,x)} [T(x, y, u)].
\]

Thus, if \(\theta\) are the maximum-likelihood parameters, it must be true that

\[
\hat{E}_{x,y} E_{p_\theta(u|x, y)} [T(x, y, u)] = \hat{E}_x E_{p_\theta(y|u,x)} [T(x, y, u)].
\]
The again generalizes the moment-matching conditions from Eq. (1), but in both of the ways that Eq. (4) and Eq. (5) did. Intuitively, we still have a “data expectation on one side” and a “model expectation on the other side”. However, the data does not contain values for $u$. Intuitively, Eq. (6) “fills in” the data expectation on the left-hand side using the model. Moreover, the model does not define $p_\theta(x)$. The data “fills in” that on the right-hand side.

6 Mnemonic

We propose the following mnemonic to remember the final moment matching optimality conditions:

- Put “as much data expectation” $\hat{E}[T(\cdot)]$ on the left hand side as possible.
- Put “as much model expectation” $E_{p_\theta}[T(\cdot)]$ on the right hand side as possible.
- If variables are “missing” from either the model or the data, use the other to “fill them in”.
- At maximum likelihood, the two expectations must be equal.

We could also write this (very informally) as

\[
\hat{E}_x \left[ T(x) \right] = E_{p_\theta} \left[ T(x) \right].
\]

The idea is to start with this equation, then “fill in” the left-hand side with expectations over $p_\theta$ and the right-hand side with empirical expectations as necessary in order for the condition to make sense. This is easiest to see through looking at each of the cases.

6.1 Exponential Family

For a normal exponential family $p_\theta(x)$ and standard likelihood with a dataset $x^{(1)} \cdots x^{(n)}$ the heuristic gives

\[
\hat{E}_x \left[ T(x) \right] = E_{p_\theta(x)} \left[ T(x) \right].
\]

In this case, no further changes are needed.

6.2 Conditional Exponential Family

For a conditional exponential family $p_\theta(y|x)$ and conditional likelihood with a dataset $(x^{(1)}, y^{(1)}) \cdots (x^{(n)}, y^{(n)})$ the heuristic gives:

\[
\hat{E}_{x,y} \left[ T(x, y) \right] = E_{p_\theta(x,y)} \left[ T(x, y) \right].
\]

This is in quotes because it is not correct and is not a real equation. The problem is that the right-hand side is meaningless since $p_\theta$ does not define an expectation over $x$. Instead, we must fill in with the data. This gives the correct condition

\[
\hat{E}_{x,y} \left[ T(x, y) \right] = \hat{E}_x E_{p_\theta(y|x)} \left[ T(x, y) \right].
\]

6.3 Exponential Family with Hidden Variables

Now, take an exponential family $p_\theta(x, u)$ that is over $x$ and $u$ together but where $u$ is hidden in the dataset $x^{(1)}, \cdots, x^{(n)}$. If maximizing the marginal likelihood over $x$, then, the heuristic gives the condition

\[
\hat{E}_{x,u} \left[ T(x, u) \right] = E_{p_\theta(x,u)} \left[ T(x, u) \right].
\]
Again, this is in quotes because it is not correct and not a real equation. The problem now is the left-hand side is meaningless since we do not have data for $u$. The solution is to fill in using the model. This gives the correct condition

$$\hat{E}_x \mathbb{E}_{p_\theta(u|x)} [ T(x, u) ] = \mathbb{E}_{p_\theta(x,u)} [ T(x, u) ].$$

### 6.4 Conditional Exponential Family with Hidden Variables

Finally take a conditional exponential family $p_\theta(y, u|x)$ where $u$ is hidden in the dataset $(x^{(1)}, y^{(1)}), \ldots, (x^{(n)}, y^{(n)})$. The objective is maximize the marginal conditional likelihood (marginalizing out $u$) and conditioning on $x$. The heuristic gives the (again, not correct) condition

$$\hat{E}_{x,y,u} [ T(x, y, u) ] = \mathbb{E}_{p_\theta(x,y,u)} [ T(x, y, u) ].$$

There are two issues. Firstly, the left-hand side is incorrect because the data does not provide $u$. Secondly, the right-hand side is incorrect because the model does not define $p_\theta(x)$. If we fix both of these, we get the correct condition.

$$\hat{E}_{x,y} \mathbb{E}_{p_\theta(u|x,y)} [ T(x, y, u) ] = \mathbb{E}_{x} \mathbb{E}_{p_\theta(y,u|x)} [ T(x, y, u) ].$$

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### References

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|                             | Exponential Family          | Conditional Exponential Family | Exponential Family with Hidden Variables | Conditional Exponential Family with Hidden Variables |
|-----------------------------|-----------------------------|-------------------------------|----------------------------------------|---------------------------------------------------|
| **Definition**              | $p_θ(x) = h(x) \exp \left( θ^T T(x) - A(θ) \right)$ | $p_θ(y|x) = h(x, y) \exp \left( θ^T T(x, y) - A(x, θ) \right)$ | $p_θ(x, u) = h(x, u) \exp \left( θ^T T(x, u) - A(x, θ) \right)$ | $p_θ(y|u,x) = h(x, y, u) \exp \left( θ^T T(x, y, u) - A(x, θ) \right)$ |
| **Log-partition function**  | $A(θ) = \log \sum_x h(x) \exp θ^T T(x)$ | $A(x, θ) = \log \sum_y h(x, y) \exp θ^T T(x, y)$ | $A(θ) = \log \sum_{x,u} h(x, u) \exp θ^T T(x, u)$ | $A(x, θ) = \log \sum_{y,u} h(x, y, u) \exp θ^T T(x, y, u)$ |
| **Gradient of log-partition function** | $\frac{dA(θ)}{dθ} = \mathbb{E}_{p_θ(x)}[T(x)]$ | $\frac{dA(x, θ)}{dθ} = \mathbb{E}_{p_θ(y|x)}[T(x, y)]$ | $\frac{dA(θ)}{dθ} = \mathbb{E}_{p_θ(x,u)}[T(x, u)]$ | $\frac{dA(x, θ)}{dθ} = \mathbb{E}_{p_θ(y,u|x)}[T(x, y, u)]$ |
| **Objective for a single datum** | $\log p_θ(x) = \log h(x) + θ^T T(x) - A(θ)$ | $\log p_θ(y|x) = h(x, y) + θ^T T(x, y) - A(x, θ)$ | $\log p_θ(x) = \log h(x, u) + A(x, θ) - A(θ)$ | $\log p_θ(y|x) = A(x, y, θ) - A(x, θ)$ |
| **Form of learning objective** | $L(θ) = \frac{1}{n} \sum_{i=1}^{n} \log p_θ(x^{(i)}|θ)$ | $L(θ) = \frac{1}{n} \sum_{i=1}^{n} \log p_θ(y^{(i)}|x^{(i)})$ | $L(θ) = \frac{1}{N} \sum_{n=1}^{N} \log p_θ(x^{(i)})$ | $L(θ) = \frac{1}{n} \sum_{i=1}^{n} \log p_θ(y^{(i)}|x^{(i)})$ |
| **Alternative form** | $L(θ) = \frac{1}{n} \sum_{i=1}^{n} θ^T T(x^{(i)}) - A(θ)$ | $L(θ) = \frac{1}{n} \sum_{i=1}^{n} \left( h(x^{(i)}, y^{(i)}) + θ^T T(x^{(i)}, y^{(i)}) - A(x^{(i)}, θ) \right)$ | $L(θ) = \frac{1}{N} \sum_{n=1}^{N} \left( A(x^{(i)}, θ) - A(θ) \right)$ | $L(θ) = \frac{1}{n} \sum_{i=1}^{n} \left( A(x^{(i)}, y^{(i)}, θ) - A(x^{(i)}, θ) \right)$ |
| **Alternative form** | $L(θ) = \frac{2}{x} \mathbb{E}_x \left[ θ^T T(x) \right] - A(θ)$ | $L(θ) = \frac{2}{x} \mathbb{E}_{x,y} \left[ \log h(x,y) + θ^T T(x,y) \right] - \frac{2}{x} \mathbb{E}_x \left[ A(x, θ) \right]$ | $L(θ) = \frac{2}{x} \mathbb{E}_{x,y} \left[ A(x, θ) \right] - A(θ)$ | $L(θ) = \frac{2}{x} \mathbb{E}_{x,y} \left[ A(x, y, θ) - A(x, θ) \right]$ |
| **Condition at optimum** | $\frac{2}{x} \mathbb{E}_x [T(x)] = \mathbb{E}_{p_θ(x)} [T(x)]$ | $\frac{2}{x} \mathbb{E}_{x,y} [T(x,y)] = \frac{2}{x} \mathbb{E}_{p_θ(y|x)} [T(x,y)]$ | $\frac{2}{x} \mathbb{E}_{x,u} [T(x,u)] = \mathbb{E}_{p_θ(x,u)} [T(x,u)]$ | $\frac{2}{x} \mathbb{E}_{x,y,u} [T(x,y,u)] = \mathbb{E}_{p_θ(y,u|x)} [T(x,y,u)]$ |
| **Note**                   | $p_θ(x)$ undefined & irrelevant | $p_θ(u|x)$ defined & relevant | $p_θ(x)$ undefined & irrelevant | $p_θ(u|x, y)$ defined and relevant |

Figure 1: A summary of all moment-matching conditions.