ON THE R-CIRCULANT MATRICES WITH THE
GENERALIZED BI-PERIODIC FIBONACCI NUMBERS

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Abstract. In this paper, we give upper and lower bounds for the spectral
norms of r-circulant matrices with the generalized bi-periodic Fibonacci
numbers. Moreover, we investigate the eigenvalues and determinants of
these matrices.

1. Introduction

The generalized bi-periodic Fibonacci sequence \( \{w_n\} := \{w_n(w_0, w_1; a, b)\} \)
with arbitrary initial conditions \(w_0\) and \(w_1\), is defined by the recurrence rela-
tion
\[
w_n = a^\zeta(n+1) b^\zeta(n) w_{n-1} + w_{n-2}, \quad n \geq 2
\]
where \(\zeta(n) = (1-(-1)^n)/2\), and \(a\) and \(b\) be nonzero real numbers [6]. Note that
\(\zeta(n)\) returns to 0 when \(n\) is even, and to 1 when \(n\) is odd. Several well-known
integer sequences are special cases of this sequence. For example, this sequence
is reduced to the bi-periodic Fibonacci sequence \(\{q_n\}\) for \(w_0 = 0, w_1 = 1,\)
and to the bi-periodic Lucas sequence \(\{p_n\}\) for \(w_0 = 2, w_1 = b\). We refer to
[3,6,13,17–20] for basic properties of these sequences and their generalizations.

For \(n > 0\), the Binet formula of the sequence \(\{w_n\}\) can be written as
\[
w_n = a^\zeta(n+1) \left(\frac{1}{ab}\right)^{\frac{n}{2}} (A \alpha^n - B \beta^n),
\]
where \(A = \frac{w_1 - (\beta/a)w_0}{\alpha - \beta}\) and \(B = \frac{w_1 - (\alpha/a)w_0}{\alpha - \beta}\) [17,19]. The numbers \(\alpha = \frac{ab + \sqrt{a^2b^2 + 4ab}}{2}\)
and \(\beta = \frac{ab - \sqrt{a^2b^2 + 4ab}}{2}\) are the roots of the polynomial
\(x^2 - abx - ab\),
and they satisfy the following equalities:
\(\alpha + \beta = ab, \quad \alpha - \beta = \sqrt{a^2b^2 + 4ab}, \quad \alpha\beta = -ab.\)
Thus the Binet formulas for the sequences \( \{q_n\} \) and \( \{p_n\} \) are given by

\[
q_n = \frac{a^{\xi(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \quad \text{and} \quad p_n = \frac{a^{-\xi(n)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left( \alpha^n + \beta^n \right).
\]

The generating function of \( \{w_n\} \) is given in [6, Theorem 8] as

\[
G(x) = \frac{w_0 + w_1 x + (aw_1 - (ab + 1)w_0) x^2 + (bw_0 - w_1) x^3}{1 - (ab + 2) x^2 + x^4}.
\]

Our goal is to calculate the Frobenius norm, and find upper and lower bounds on the spectral norm of \( r \)-circulant matrices whose entries are generalized bi-periodic Fibonacci numbers. To this purpose, we review the background material concerning the basic definitions and facts of \( r \)-circulant matrices and matrix norms in the rest of this section.

Let \( r \in \mathbb{C} - \{0\} \). An \( n \times n \) matrix \( C_r = [c_{ij}] \) with entries

\[
c_{ij} = \begin{cases} 
  c_{j-i}, & j \geq i \\
  rc_{n+i-j}, & j < i
\end{cases}
\]

is called an \( r \)-circulant matrix. In other words, \( C_r \) has the following form:

\[
C_r = \begin{bmatrix}
  c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
  rc_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
  rc_{n-2} & rc_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  rc_2 & rc_3 & rc_4 & \cdots & c_0 & c_1 \\
  rc_1 & rc_2 & rc_3 & \cdots & rc_{n-1} & c_0
\end{bmatrix}.
\]

For the simplicity, we denote \( C_r \) by \( \text{circ}_{r,n} \{c_0, c_1, \ldots, c_{n-1}\} \). Note that for \( r = 1 \), it reduces to circulant matrix \( C \). The eigenvalues of \( C_r \) are given as

\[
\lambda_j(C_r) = \sum_{k=0}^{n-1} c_k (\rho \omega^{-j})^k,
\]

where \( \rho \) is any \( n \)th root of \( r \), \( \omega \) is any \( n \)th root of unity, and \( j = 0, 1, \ldots, n - 1 \).

For details, we refer to [4, Lemma 4].

Let \( A = [a_{ij}] \) be an \( m \times n \) matrix. The Frobenius norm (also known as Hilbert-Schmidt norm or Schur norm) \( \|A\|_F \) of \( A \) is the square root of the sum of the squares of the absolute values of all entries of \( A \). That is,

\[
\|A\|_F := \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.
\]
Note that the Frobenius norm measures the size of a matrix. Another norm used in matrix applications is the spectral norm defined as
\[ \|A\|_2 := \sqrt{\lambda_{\text{max}}(A^*A)}, \]
where \(\lambda_{\text{max}}(A^*A)\) denotes the largest eigenvalue of \(A^*A\). Here, \(A^*\) is the conjugate transpose of \(A\). The following inequality by Zielke in [22] provides a relationship between Frobenius and spectral norms:
\[ \frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F \quad \text{and} \quad \|A\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2. \] (1.4)

Note that the Frobenius norm is an upper bound on the spectral norm. An important tool for finding bounds on the matrix norms is the Hadamard product. For \(m \times n\) matrices \(A = [a_{ij}]\) and \(B = [b_{ij}]\), the Hadamard product of \(A\) and \(B\) is defined as \(A \odot B := [a_{ij}b_{ij}]\). It is simply the entry wise multiplication of \(A\) and \(B\). In [10], Mathias proved that
\[ \|A \odot B\|_2 \leq \|A\|_2 \|B\|_2 \quad \text{and} \quad \|A \odot B\|_2 \leq r_1(A) c_1(B), \] (1.5)
where
\[ r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} \quad \text{and} \quad c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}. \]

Note that \(r_1(A)\) is the maximum row length norm of \(A\), and \(c_1(B)\) is the maximum column length norm of \(B\). See also [7].

There has been many works related to the properties of circulant matrices, \(r\)–circulant matrices and special type of circulant matrices with Fibonacci-like numbers. In particular, Solak [15, 16] obtained some bounds for the spectral norms of circulant matrices whose entries are Fibonacci and Lucas numbers. For the spectral norms of \(r\)–circulant matrices with Fibonacci and Lucas numbers, Shen and Cen [14] gave some bounds which generalized the results in [15]. Nalli and Sen [12] investigate the norms of circulant matrices with generalized Fibonacci numbers. Alptekin et al. [1] obtained the spectral norms and eigenvalues of circulant matrices whose entries are Horadam numbers. In [21], Yazlik and Taskara found upper and lower bounds on the norms of an \(r\)–circulant matrix with the generalized \(k\)–Horadam numbers. Also they established the determinant and eigenvalues of this matrix. For related works, we refer to [2, 5, 9, 11, 12]. Recently, Kome and Yazlik [8] have obtained some upper and lower bounds for the spectral norms of the \(r\)–circulant matrices whose entries are bi-periodic Fibonacci and Lucas numbers. Here, we examine upper and lower bounds for the spectral norms of \(r\)–circulant matrices with the generalized bi-periodic Fibonacci numbers. Also, we obtain the eigenvalues and determinants of these matrices.
2. Main results

Throughout this section we let \(a, b,\) and \(w_1\) to be positive integers and let \(w_0\) to be a nonnegative integer unless otherwise is stated.

**Definition 1.** The \(r\)-circulant matrix \(W_r\) with the generalized bi-periodic Fibonacci numbers is defined as

\[
W_r : = \begin{bmatrix}
\left(\frac{\zeta}{2}\right)^{(0)} w_0 & \left(\frac{\zeta}{2}\right)^{(1)} w_1 & \left(\frac{\zeta}{2}\right)^{(2)} w_2 & \cdots & \left(\frac{\zeta}{2}\right)^{(n-1)} w_{n-1} \\
\left(\frac{a}{b}\right)^{n-1} w_{n-1} & \left(\frac{a}{b}\right)^{(0)} w_0 & \left(\frac{a}{b}\right)^{(1)} w_1 & \cdots & \left(\frac{a}{b}\right)^{(n-2)} w_{n-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\left(\frac{a}{b}\right)^{(1)} w_1 & \left(\frac{a}{b}\right)^{(2)} w_2 & \left(\frac{a}{b}\right)^{(3)} w_3 & \cdots & \left(\frac{a}{b}\right)^{(0)} w_0 \\
\end{bmatrix}
\]

In accord with the notational convention introduced in the previous section, we can write \(W_r\) as follows:

\[
W_r = \text{circ}_{r,n} \left[ \left(\frac{a}{b}\right)^{(0)} w_0, \left(\frac{a}{b}\right)^{(1)} w_1, \ldots, \left(\frac{a}{b}\right)^{(n-1)} w_{n-1} \right].
\]

**Lemma 1.** For \(n > 1\), we have

\[
\sum_{k=1}^{n} \left(\frac{a}{b}\right)^{(k)} w_k^2 = \frac{1}{b} (w_n w_{n+1} - w_0 w_1).
\]

**Proof.** Recall that the Binet formula of the sequence \(\{w_k\}\) is given by

\[
w_k = \frac{a^{\zeta(k+1)}}{(ab)^{\frac{k}{2}}} (A \alpha^k - B \beta^k).
\]

Since \(\zeta(n) + \zeta(n+1) = 1\) and \([n/2] + [(n + 1)/2] = n\), we have

\[
w_n w_{n+1} = \frac{a^{\zeta(n+1)+\zeta(n)}}{(ab)^{\frac{n}{2}} + \frac{n+1}{2}} (A \alpha^n - B \beta^n) (A \alpha^{n+1} - B \beta^{n+1})
\]

\[
= \frac{a}{(ab)^n} \left[ A^2 \alpha^{2n+1} - AB (\alpha \beta)^n (\alpha + \beta) + B^2 \beta^{2n+1} \right]
\]

\[
= \frac{a}{(ab)^n} \left[ A^2 \alpha^{2n+1} + B^2 \beta^{2n+1} - AB (-ab)^n (ab) \right]. \tag{2.1}
\]
On the other hand, we have

\[
w_k^2 = \frac{a^{2(\zeta(k+1))}}{(ab)^{2\lfloor \frac{k}{2} \rfloor}} \left[ A\alpha^k - B\beta^k \right]^2
\]

\[
= \frac{a^{2(\zeta(k+1))}}{(ab)^{2\lfloor \frac{k}{2} \rfloor}} \left[ A^2\alpha^{2k} + B^2\beta^{2k} - 2AB(\alpha\beta)^k \right].
\]

Now, if \( k \) is even, we get

\[
w_k^2 = \frac{a^2}{(ab)^k} \left[ A^2\alpha^{2k} + B^2\beta^{2k} - 2AB(\alpha\beta)^k \right],
\]

and if \( k \) is odd, we get

\[
w_k^2 = \frac{ab}{(ab)^k} \left[ A^2\alpha^{2k} + B^2\beta^{2k} - 2AB(\alpha\beta)^k \right].
\]

Since \( \alpha\beta = -ab \), and

\[
a^2 \left( \frac{b}{a} \right)^{\zeta(k)} = \begin{cases} a^2, & \text{if } k \text{ is even} \\ ab, & \text{if } k \text{ is odd} \end{cases}
\]

we can write

\[
w_k^2 = a^2 \left( \frac{b}{a} \right)^{\zeta(k)} \left[ A^2 \left( \frac{\alpha^2}{ab} \right)^k + B^2 \left( \frac{\beta^2}{ab} \right)^k - 2AB(-1)^k \right],
\]

or equivalently,

\[
a^{-2} \left( \frac{a}{b} \right)^{\zeta(k)} w_k^2 = A^2 \left( \frac{\alpha^2}{ab} \right)^k + B^2 \left( \frac{\beta^2}{ab} \right)^k - 2AB(-1)^k. \quad (2.2)
\]

By using the geometric sum formula, it can be seen that

\[
\sum_{k=1}^{n} \left( \frac{\alpha^2}{ab} \right)^k = \frac{\alpha^{2n+1}}{(ab)^{n+1}} \frac{\alpha}{ab} \quad \text{and} \quad \sum_{k=1}^{n} \left( \frac{\beta^2}{ab} \right)^k = \frac{\beta^{2n+1}}{(ab)^{n+1}} \frac{\beta}{ab},
\]

Now we take the summation of both sides of Equation \( 2.2 \) from 1 to \( k \):
\[
\sum_{k=1}^{n} a^{-2} \left( \frac{a}{b} \right)^{k} w_{k}^{2} = A^{2} \sum_{k=1}^{n} \left( \frac{a^{2}}{ab} \right)^{k} + B^{2} \sum_{k=1}^{n} \left( \frac{\beta^{2}}{ab} \right)^{k} - AB \sum_{k=1}^{n} 2 (-1)^{k} \\
= A^{2} \left[ \frac{a^{2n+1}}{(ab)^{n+1}} - \frac{\alpha}{ab} \right] + B^{2} \left[ \frac{\beta^{2n+1}}{(ab)^{n+1}} - \frac{\beta}{ab} \right] - AB \left[ (-1)^{n} - 1 \right] \\
= \frac{1}{ab} \left[ A^{2} \alpha^{2n+1} \left( \frac{1}{ab} \right)^{n} + B^{2} \beta^{2n+1} \left( \frac{1}{ab} \right)^{n} - A^{2} \alpha - B^{2} \beta - ABab \right] [\left( -1 \right)^{n} - 1] .
\]

By taking Equation 2.1 into account in the last line of the equation above, we get the desired result:

\[
\sum_{k=1}^{n} a^{-2} \left( \frac{a}{b} \right)^{k} w_{k}^{2} = \frac{1}{ab} \left( w_{n}w_{n+1} \right) + \frac{A^{2} \alpha}{ab} - \frac{B^{2} \beta}{ab} \\
= \frac{1}{ab} \left[ \frac{1}{a} \left( w_{n}w_{n+1} \right) + AB (ab) - \left( A^{2} \alpha + B^{2} \beta \right) \right] \\
= \frac{1}{ab} \left[ \frac{1}{a} \left( w_{n}w_{n+1} \right) - \frac{1}{a}w_{0}w_{1} \right] \\
= \frac{1}{a^{2}b} \left[ w_{n}w_{n+1} - w_{0}w_{1} \right] .
\]

\[\square\]

An immediate result of Lemma 1 is the following.

**Corollary 1.** For \( n > 0 \), the following equality holds:

\[
\sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^{k} w_{k}^{2} = \frac{1}{b} \left( w_{n}w_{n-1} - w_{0}w_{1} + bw_{0}^{2} \right) .
\]

**Remark 1.** If we take the initial conditions \( w_{0} = 0 \) and \( w_{1} = 1 \), we get the identity

\[
\sum_{k=1}^{n} \left( \frac{a}{b} \right)^{k} q_{k}^{2} = \frac{1}{b} q_{n}q_{n+1} ,
\]

given in [20]. Similarly, with the initial conditions \( w_{0} = 2 \) and \( w_{1} = b \), we get the identity

\[
\sum_{k=1}^{n} \left( \frac{a}{b} \right)^{k} p_{k}^{2} = \frac{1}{b} p_{n}p_{n+1} - 2 ,
\]

given in [18].
Now we are ready to provide lower and upper bounds on the spectral norm of an $r$–circulant matrix with generalized bi-periodic Fibonacci numbers

$$W_r = \text{circ}_{r,n} \left[ \left( \frac{a}{b} \right)^{\zeta(0)} w_0, \left( \frac{a}{b} \right)^{\zeta(1)} w_1, \ldots, \left( \frac{a}{b} \right)^{\zeta(n-1)} w_{n-1} \right].$$

But first, we calculate the Frobenius norm.

**Lemma 2.** The Frobenius norm of $W_r$ is given by

$$\|W_r\|_F = \sqrt{\sum_{k=0}^{n-1} \left( n + k \left( |r|^2 - 1 \right) \right) \left( \frac{a}{b} \right)^{\zeta(k)} w_k^2}.$$  

**Proof.** By using Lemma 1 and Corollary 1 it is clear that

$$\|W_r\|_F = \sqrt{\sum_{k=0}^{n-1} \left( n - k \left( \frac{a}{b} \right)^{\zeta(k)} w_k^2 + \sum_{k=1}^{n-1} k |r|^2 \left( \frac{a}{b} \right)^{\zeta(k)} w_k^2 \right.}$$

$$= \sqrt{\sum_{k=0}^{n-1} \left( n + k \left( |r|^2 - 1 \right) \right) \left( \frac{a}{b} \right)^{\zeta(k)} w_k^2}.$$  

□

**Theorem 1.** Let $\Delta := w_{n-1} w_n - w_0 w_1 + bw_0^2$. Then the following inequalities hold for the $r$–circulant matrix $W_r$:

(i) If $|r| \geq 1$, then

$$\sqrt{\frac{\Delta}{b}} \leq \|W_r\|_2 \leq \sqrt{(n-1) |r|^2 + 1} \frac{\Delta}{b}.$$

(ii) If $|r| < 1$, then

$$|r| \sqrt{\frac{\Delta}{b}} \leq \|W_r\|_2 \leq \sqrt{n} \frac{\Delta}{b}.$$

**Proof.** (i) Let $|r| \geq 1$. From Corollary 1 and Lemma 2 we have

$$\|W_r\|_F \geq \sqrt{\sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^{\zeta(k)} w_k^2} = \sqrt{n} \frac{\Delta}{b}.$$  

Therefore, we can write

$$\frac{1}{\sqrt{n}} \|W_r\|_F \geq \sqrt{\frac{\Delta}{b}},$$

From the inequality (1.2), we obtain

$$\sqrt{\frac{\Delta}{b}} \leq \|W_r\|_2.$$
In order to provide an upper bound for the spectral norm of $W_r$, we consider the following matrices:

$$U = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 \\
r & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
r & r & \cdots & 1 & 1 \\
r & r & \cdots & r & 1
\end{bmatrix},$$

$$W = \begin{bmatrix}
\left(\frac{a}{b}\right) \zeta(0) w_0 & \left(\frac{a}{b}\right) \zeta(1) w_1 & \left(\frac{a}{b}\right) \zeta(2) w_2 & \cdots & \left(\frac{a}{b}\right) \zeta(n-1) w_{n-1} \\
\left(\frac{a}{b}\right) \zeta(n-1) w_{n-1} & \left(\frac{a}{b}\right) \zeta(0) w_0 & \left(\frac{a}{b}\right) \zeta(1) w_1 & \cdots & \left(\frac{a}{b}\right) \zeta(n-2) w_{n-2} \\
\left(\frac{a}{b}\right) \zeta(n-2) w_{n-2} & \left(\frac{a}{b}\right) \zeta(0) w_0 & \left(\frac{a}{b}\right) \zeta(1) w_1 & \cdots & \left(\frac{a}{b}\right) \zeta(n-3) w_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left(\frac{a}{b}\right) \zeta(1) w_1 & \left(\frac{a}{b}\right) \zeta(2) w_2 & \left(\frac{a}{b}\right) \zeta(3) w_3 & \cdots & \left(\frac{a}{b}\right) \zeta(0) w_0
\end{bmatrix}. $$

Note that $W = \text{circ}_{1,n} \left[ \left(\frac{a}{b}\right) \zeta(0) w_0, \left(\frac{a}{b}\right) \zeta(1) w_1, \ldots, \left(\frac{a}{b}\right) \zeta(n-1) w_{n-1} \right]$. It is clear that $W_r = U \circ W$ where $\circ$ denotes the Hadamard product. Now we calculate the maximum row length norm $r_1(U)$ of $U$, and maximum column length norm $c_1(W)$ of $W$. Since $|r| \geq 1$, we have

$$r_1(U) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |u_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |u_{nj}|^2} = \sqrt{(n-1)|r|^2 + 1},$$

and

$$c_1(W) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |w_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} \left(\frac{a}{b}\right) \zeta(k) w_k^2} = \sqrt{\Delta \frac{b}{a}}. 
$$

Using the above quantities, we obtain

$$\|W_r\|_2 = \|U \circ W\|_2 \leq r_1(U) c_1(W) = \sqrt{(n-1)|r|^2 + 1} \frac{\Delta}{b}. $$

(ii) Let $|r| < 1$. Suppose $k$ is an integer with $0 \leq k \leq n-1$. Since $|r|^2 - 1 < 0$, the minimum of $n + k(|r|^2 - 1)$ is achieved when $k = n - 1$. So, for $k = n - 1$ we have $n + k(|r|^2 - 1) = n|r|^2 - |r|^2 + 1 \geq n|r|^2$. Then it follows that

$$n + k(|r|^2 - 1) \geq n|r|^2.$$
for each \( k \) with \( 0 \leq k \leq n - 1 \). Therefore, we can write

\[
\| W_r \|_F = \sqrt{\sum_{k=0}^{n-1} \left( n + k \left( |r|^2 - 1 \right) \right) \left( \frac{a}{b} \right)^k w_k^2}
\]

\[
\geq \sqrt{\sum_{k=0}^{n-1} n |r|^2 \left( \frac{a}{b} \right)^k w_k^2}.
\]

Then it follows that

\[
\frac{1}{\sqrt{n}} \| W_r \|_F \geq |r| \sqrt{\frac{\Delta}{b}}.
\]

Using the inequality (1.4) again, we get

\[
\| W_r \|_2 \geq |r| \sqrt{\frac{\Delta}{b}}.
\]

In order to provide an upper bound for the spectral norm of \( W_r \), we consider the matrices \( U \) and \( V \) defined above. Note that

\[
r_1(U) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |u_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |u_{1j}|^2} = \sqrt{n},
\]

and

\[
c_1(W) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |w_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^k w_k^2} = \sqrt{\frac{\Delta}{b}}.
\]

In conclusion,

\[
\| W_r \|_2 = \| U \circ W \|_2 \leq r_1(U) c_1(W) = \sqrt{\frac{\Delta}{b}}.
\]

**Remark 2.** From Theorem [1], we have the following results:

If we take \( w_0 = 0, w_1 = 1 \), then we get the upper and lower bounds for \( r \)–circulant matrix with bi-periodic Fibonacci numbers as:

\[
\sqrt{\frac{q_n - 1}{b}} \leq \| W_r \|_2 \leq \sqrt{(n - 1) |r|^2 + 1} \frac{q_n - 1}{b}, |r| \geq 1
\]

\[
|r| \sqrt{\frac{q_n - 1}{b}} \leq \| W_r \|_2 \leq n \frac{q_n - 1}{b}, |r| < 1.
\]
If we take \( w_0 = 2, w_1 = b \), then we get the upper and lower bounds for \( r \)-circulant matrix with bi-periodic Lucas numbers as:

\[
\sqrt{\frac{p_n-1p_n}{b} + 2} \leq \|W_r\|_2 \leq \sqrt{(n-1)|r|^2 + 1} \left( \frac{p_n-1p_n}{b} + 2 \right), |r| \geq 1 \\
|r| \sqrt{\frac{p_n-1p_n}{b} + 2} \leq \|W_r\|_2 \leq \sqrt{n \left( \frac{p_n-1p_n}{b} + 2 \right)}, |r| < 1.
\]

If we take \( a = b = 1 \), then we get the upper and lower bounds for \( r \)-circulant matrix with generalized Fibonacci numbers as:

\[
\sqrt{w_{n-1}w_n - w_0w_1 + w_0^2} \leq \|W_r\|_2 \leq \sqrt{(n-1)|r|^2 + 1} (w_{n-1}w_n - w_0w_1 + w), |r| \geq 1 \\
|r| \sqrt{w_{n-1}w_n - w_0w_1 + w} \leq \|W_r\|_2 \leq \sqrt{n(w_{n-1}w_n - w_0w_1 + w)}, |r| < 1.
\]

**Theorem 2.** The eigenvalues of \( W_r \) are

\[
\lambda_j(W_r) = \left( \frac{\alpha}{\beta} \right)^{\frac{j}{2}} rw_n - w_0 + \rho \omega^{-j} \left( \frac{\alpha}{\beta} \right)^{\frac{j}{2}} rw_{n-1} + \left( \frac{\alpha}{\beta} \right)^{\frac{j}{2}} (bw_0 - w_1)
\]

where \( \rho \) is any \( n \)th root of unity, \( \omega \) is any \( n \)th root of unity, and \( j = 0, 1, \ldots, n-1 \).

**Proof.** From \([1,3]\), we have

\[
\lambda_j(W_r) = \sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^{\frac{j}{2}} w_k \rho^k \omega^{-kj}
\]

\[
= Aa \sum_{k=0}^{n-1} \left( \frac{\alpha \rho \omega^{-j}}{(ab)^{\frac{j}{2}}} \right)^k - Ba \sum_{k=0}^{n-1} \left( \frac{\beta \rho \omega^{-j}}{(ab)^{\frac{j}{2}}} \right)^k
\]

\[
= \frac{a}{(ab)^{\frac{j+1}{2}}} \left( A \frac{\alpha^n - (ab)^{\frac{j}{2}}}{\alpha \rho \omega^{-j} - (ab)^{\frac{j}{2}}} - B \frac{\beta^n - (ab)^{\frac{j}{2}}}{\beta \rho \omega^{-j} - (ab)^{\frac{j}{2}}} \right)
\]

\[
= \frac{a}{(ab)^{\frac{j+1}{2}}} \left( \alpha \rho \omega^{-j} - (ab)^{\frac{j}{2}} \right) \left( \beta \rho \omega^{-j} - (ab)^{\frac{j}{2}} \right) \left( r \rho \omega^{-j} (\alpha \beta) \left( A \alpha^{n-1} - B \beta^{n-1} \right) \\
- r (ab)^{\frac{j}{2}} (A \alpha^n - B \beta^n) - \rho \omega^{-j} (ab)^{\frac{j}{2}} (A \beta - B \alpha) + (ab)^{\frac{j+1}{2}} (A - B) \right).
\]

If \( k \) is even, we obtain

\[
\lambda_j(W_r) = \frac{rw_n - w_0 + \left( \frac{a}{b} \right)^{\frac{j}{2}} \rho \omega^{-j} (rw_{n-1} + (bw_0 - w_1))}{\rho^2 \omega^{-2j} + (ab)^{\frac{j}{2}} \rho \omega^{-j} - 1}.
\]
ON THE R-CIRCULANT MATRICES WITH THE GENERALIZED BI-PERIODIC FIBONACCI NUMBERS

If \( k \) is odd, we get

\[
\lambda_j(W_r) = \frac{\left(\frac{\rho}{\omega}\right)^{\frac{1}{2}} r w_n - w_0 + \rho \omega^{-j} \left(r w_{n-1} + \left(\frac{\rho}{\omega}\right)^{\frac{1}{2}} (b w_0 - w_1)\right)}{\rho^2 \omega^{-2j} + (ab)^{\frac{1}{2}} \rho \omega^{-j} - 1}.
\]

\[\square\]

Theorem 3. The determinant of \( W_r \) is

\[
\det(W_r) = \frac{\left(w_0 - \left(\frac{\rho}{\omega}\right)^{\frac{\zeta(n)}{2}} r w_n\right)^n - r \left(w_0 - \left(\frac{\rho}{\omega}\right)^{\frac{\zeta(n+1)}{2}} r w_{n-1} + \left(\frac{\rho}{\omega}\right)^{\frac{1}{2}} (b w_0 - w_1)\right)^n}{1 - \left(\frac{\rho}{\omega}\right)^{\frac{\zeta(n)}{2}} p_n r + (-1)^n r^2}.
\]

Proof. Since \( \det(W_r) = \prod_{j=0}^{n-1} \lambda_j(W_r) \), we get the desired result. \[\square\]

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