Geometric control of hybrid systems

Benoît Legat\textsuperscript{a}, Raphaël M. Jungers\textsuperscript{b}

\textsuperscript{a}LIDS, MIT, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA
\textsuperscript{b}ICTEAM, UCLouvain, 4 Av. G. Lemaître, 1348 Louvain-la-Neuve, Belgium

Abstract

In this paper, we present a geometric approach for computing controlled invariant sets for hybrid control systems. While the problem is well studied in the ellipsoidal case, this family is quite conservative for constrained or switched linear systems. We reformulate the invariance of a set as an inequality for its support function that is valid for any convex set. This produces novel algebraic conditions for the invariance of sets with polynomial or piecewise quadratic support function.

Keywords: Controller Synthesis; Set Invariance; LMIs; Scalable Methods.

1. Introduction

Computing controlled invariant sets is paramount in many applications \cite{6}. Indeed, the existence of a controlled invariant set is equivalent to the stabilizability\textsuperscript{1} of a control system \cite{28} and a (possibly nonlinear) stabilizable state feedback can be deduced from the controlled invariant set \cite{5}.

The stabilizability of a linear time-invariant (LTI) control system is equivalent to the stability of its uncontrollable subspace (which is readily accessible in its Controllability Form) \cite{30} Section 2.4. Indeed, the eigenvalues of its controllable subspace can be fixed to any value by a proper choice of linear state feedback. The resulting controlled system is stable hence an invariant ellipsoid can be determined by solving a system of linear equations \cite{19}. This

\textsuperscript{1}In the sense that the state variables can be controlled to remain bounded.

\textsuperscript{*}This paper extends our work on continuous-time controlled invariant sets presented at ADHS 2021 \cite{17} to hybrid systems. Corresponding author B. Legat.

Email addresses: blegat@mit.edu (Benoît Legat), raphael.jungers@uclouvain.be (Raphaël M. Jungers)
set is also controlled invariant for the control system. When a control system admits an ellipsoidal controlled invariant set, it is said to be quadratically stabilizable. When there exists a linear state feedback such that the resulting autonomous system admits an ellipsoidal invariant set, it is said to be quadratically stabilizable via linear control.

While the stabilizability of LTI control systems is equivalent to their quadratic stabilizability via linear control, it is no longer the case for uncertain or switched systems [23]. Furthermore, it is often desirable for constrained systems to find a controlled invariant set of maximal volume (or which is maximal in some direction [1]). For such problems, the method detailed above is not suitable as it does not take any volume consideration but more importantly, the maximal volume invariant set may not be an ellipsoid and may not be rendered stable via a linear control. For this reason, a Linear Matrix Inequality (LMI) was devised to encapsulate the controlled invariance of an ellipsoid via linear control [3 Section 7.2.2] and the conservativeness of the choice of linear control was analysed [28]. As the linearity of the control was found to be conservative for uncertain systems [23], the LMI (9) (or (8) for discrete-time) was found to encapsulate controlled invariance of an ellipsoid via any state-feedback [5].

Recent advances in control were enabled thanks to the introduction of new families of sets such as polynomial zonotopes [11, 12]. However, while the LMIs mentioned above have had a tremendous impact on control, the approach is limited to ellipsoids due to its algebraic nature. An attempt to generalize it to polynomials can be found in [24] but as detailed in [17, Section 2], it is quite conservative. The approach studied in [13] is complementary to our method as [13] computes outer bounds of the maximal controlled invariant sets while we compute actual controlled invariant sets (hence inner bounds to the maximal one).

In this paper, we reinterpret the controlled invariance in a geometric/behavioural framework, based on convex analysis, which allows us to formulate a general condition for the controlled invariance of arbitrary convex sets via any state-feedback in Theorem 1. While this condition reduces to (8) and (9) for the special case of ellipsoids, it provides a new method for computing convex controlled invariant sets with polynomial and piecewise quadratic support functions.

This paper generalizes [16], [15] and [17] into a framework for computing convex controlled invariant sets for linear hybrid control systems. In [16], the authors treat the particular case where the continuous dynamic at each
mode (see definition 1) is trivial, i.e., $\dot{x} = 0$. In [15], the authors extends [16] to piecewise semi-ellipsoids. In [17], the authors handle the particular case where there is only one mode and no transitions (see definition 1). While [16, 15] covers discrete-time systems and [17] covers continuous-time systems, we show in this paper that the two methods can be combined to compute controlled invariant sets for hybrid systems, exhibiting both discrete-time and continuous-time dynamics. Using the set programming framework [14], this compatibility can be understood as a consequence of the fact that the controlled invariance conditions require the sets to be represented with their support functions (see definition 6) both in discrete-time and continuous-time.

In Section 2, we show how to reduce the computation of controlled invariant sets for hybrid control systems to the computation of weakly invariant sets for hybrid algebraic systems. In Section 3, we develop a generic condition of control invariance for hybrid systems using our geometric approach. We particularize it for ellipsoids (resp. sets with polynomial and piecewise quadratic support functions) in Section 3.1 (resp. Section 3.2 and Section 3.3). We illustrate these new results with a numerical example in Section 4.

Reproducibility. The code used to obtain the results is published on codeocean [18]. The set programs are reformulated by SetProg [14] into a Sum-of-Squares program which is reformulated into a semidefinite program by SumOfSquares [29] which is solved by Mosek v8 [4] through MathOptInterface [? ].

2. Controlled invariant set

In this section we define hybrid control and algebraic systems as well as the notion of invariance that will be studied in this paper. We then show how the invariance relations between the two different classes of systems.

Definition 1. A Linear Hybrid Control System (HCS) is a system $S = (T, (A_q, B_q)_{q \in V}, (A_\sigma, B_\sigma)_{\sigma \in \Sigma}, (X_q, U_q)_{q \in V}, (U_\sigma)_{\sigma \in \Sigma})$ where $T = (V, \Sigma, \to)$, $V$ is a finite set of modes, $\Sigma$ is a finite set of signals and $\to \subseteq V \times \Sigma \times V$ is a set of transitions. We denote $(q, \sigma, q') \in \to$ by $q \to_\sigma q'$.

Given a mode $q \in V$, we denote the state dimension as $n_{q,x}$ and the input dimension as $n_{q,u}$. Given a signal $\sigma$, we denote the input dimension as $n_{\sigma,u}$. The set $X_q \subseteq \mathbb{R}^{n_{q,x}}$ is the safe set corresponding to mode $q$ and the sets $U_q \subseteq \mathbb{R}^{n_{q,u}}, U_\sigma \subseteq \mathbb{R}^{n_{\sigma,u}}$ are the sets of allowed inputs. For any mode $q$, we
have $A_q \in \mathbb{R}^{n_q \times n_q x}$, $B_q \in \mathbb{R}^{n_q x \times n_q u}$. For any transition $q \rightarrow q'$, we have $A_{q'} \in \mathbb{R}^{n_{q'} \times n_{q'} x}$, $B_{q'} \in \mathbb{R}^{n_{q'} x \times n_{q'} u}$.

A trajectory of $S$ is an increasing sequence of times $t_0 < t_1 < t_2 < \cdots < t_N < t_{N+1}$, transitions $q_0 \rightarrow \sigma_1 q_1 \rightarrow \sigma_2 \cdots \rightarrow \sigma_N q_N$, inputs $\bar{u}_k \in U_{\sigma_k}$ for $k \in [N]$, and trajectories $x_k : [t_k, t_{k+1}] \rightarrow X_q \in C^1$ and $u_k : [t_k, t_{k+1}] \rightarrow U_q$ for $k = 0, 1, \ldots, N$ satisfying:

\[ \forall k \in [N], \quad x_k(t_k) = A_{\sigma_k} x_{k-1}(t_k) + B_{\sigma_k} \bar{u}_k \]
\[ \forall k \in \{0, 1, \ldots, N\}, \forall t \in [t_k, t_{k+1}], \quad \dot{x}_k(t) = A_{q_k} x_k(t) + B_{q_k} u_k(t). \]

The hybrid system defined in definition 1 may be interpreted as a hybrid automaton [3] where the guard of each transition $q \rightarrow q'$ is $X_q$ or $\mathbb{R}^{n_q x}$. In this context, the discrete-time dynamical system $x^+ = A_{\sigma} x + B_{\sigma} u$ is commonly referred to as the reset map. We allow the state space of different modes to differ as our method naturally extends to different state spaces but the reader may consider them to have identical dimension for simplicity.

**Definition 2** (Controlled invariant sets for a HCS). Consider a HCS $S$ as defined in definition 1. We say that closed sets $S_q \subseteq X_q$ for $q \in V$ are controlled invariant for $S$ if

\[ \forall q \rightarrow q', \forall x \in S_q, \exists u \in U_q \text{ such that } A_{q'} x + B_{q'} u \in S_{q'} \quad (1) \]
\[ \forall q \in V, \forall x \in S_q, \exists u \in U_q \text{ such that } A_{q} x + B_{q} u \in T_{S_q}(x). \quad (2) \]

Equation (2) is commonly referred to as the Nagumo condition; see [6, Theorem 4.7]. In view of definition 2, the transitions are considered autonomous and not controlled; see details in [20, Section 1.1.3].

### 2.1. Linear Hybrid Algebraic System

In this section, we show the equivalence of the notion of invariance with another class of systems that directly models the geometric behaviours of the trajectories of a HCS with unconstrained input. The reduction of the computation of controlled invariant sets of HCS with constrained input to HCS of unconstrained input is detailed in [16, Section 2.2].

**Definition 3.** A Linear Hybrid Algebraic System (HAS) is a system $S = (T, (C_q, E_q)_{q \in V}, (C_{\sigma}, E_{\sigma})_{\sigma \in \Sigma}, (X_q)_{q \in V})$ where $T = (V, \Sigma, \rightarrow)$, $V$ is a finite set of modes, $\Sigma$ is a finite set of signals and $\rightarrow \subseteq V \times \Sigma \times V$ is a set of transitions.
Given a mode \( q \in V \), we denote the state dimension as \( n_{q,x} \). The set \( \mathcal{X}_q \subseteq \mathbb{R}^{n_{q,x}} \) is the safe set corresponding to mode \( q \). For any mode \( q \), there exists a \( n_{q,p} \) such that, \( C_q \in \mathbb{R}^{n_{q,p} \times n_{q,x}} \), \( E_q \in \mathbb{R}^{n_{q,p} \times n_{q,x}} \). For any transition \( q \rightarrow q' \), there exists a \( n_{q,p} \) such that, \( C_q \in \mathbb{R}^{n_{q,p} \times n_{q,x}} \), \( E_q \in \mathbb{R}^{n_{q,p} \times n_{q,x}} \).

A trajectory of \( S \) is an increasing sequence of times \( t_0 < t_1 < t_2 < \cdots < t_N \), transitions \( q_0 \rightarrow q_1 \), \( q_1 \rightarrow q_2 \cdots \rightarrow q_N \), and trajectories \( x_k : [t_{k-1}, t_k] \rightarrow \mathcal{X}_{q_k} \in \mathcal{C}^1 \) for \( k = 0, 1, \ldots, N \) satisfying:

\[
\forall k \in [N], \quad E_{q_k} x_k(t_k) = C_{q_k} x_{k-1}(t_k) \\
\forall k \in \{0, 1, \ldots, N\}, \forall t \in [t_{k-1}, t_k], \quad E_{q_k} \dot{x}_k(t) = C_{q_k} x_k(t).
\]

**Definition 4** (Weakly invariant sets for a HAS). Consider a HAS \( S \) as defined in definition [3]. We say that closed sets \( \mathcal{S}_q \subseteq \mathcal{X}_q \) for \( q \in V \) are weakly invariant for \( S \) if

\[
\forall q \rightarrow q', \forall x \in \mathcal{S}_q, \quad C_q x \in E_q \mathcal{S}_{q'} \\
\forall q \in V, \forall x \in \mathcal{S}_q, \quad C_q x \in E_q T \mathcal{S}_q(x).
\]

We now show that the computation of controlled invariant sets for a HCS can be reduced to the computation of weakly invariant sets for a HAS.

**Lemma 1** ([17, Proposition 4]). Given a subset \( S \subseteq \mathbb{R}^r \) and matrices \( A \in \mathbb{R}^{r \times n} \), \( B \in \mathbb{R}^{r \times m} \), the following holds:

\[
AS + BR^m = \pi_{\text{Im}(B)^\perp}^{-1}\pi_{\text{Im}(B)^\perp} A\mathcal{S}
\]

where \( \pi_{\text{Im}(B)^\perp} \) is any orthogonal projection matrix onto the orthogonal subspace of \( \text{Im}(B) \) and \( \pi_{\text{Im}(B)^\perp}^{-1} \) is the preimage defined in eq. (A.1).

**Proof.** Given \( x \in \mathcal{S} \) and \( y \in \mathbb{R}^r \), we have \( y \in A\{x\} + B\mathbb{R}^m \) if and only if \( y - Ax \in \text{Im}(B) \). As \( \pi_{\text{Im}(B)^\perp} \) is orthogonal, its kernel is \( \text{Im}(B) \). Therefore \( y - Ax \in \text{Im}(B) \) is equivalent to \( \pi_{\text{Im}(B)^\perp} \pi_{\text{Im}(B)^\perp} x \).

The following proposition generalizes both [16, Proposition 2] and [17, Proposition 5].

**Proposition 1.** The sets \( \mathcal{S} = (\mathcal{S}_q)_{q \in V} \) are controlled invariant for the HCS \( S = \langle T, (A_q, B_q)_{q \in V}, (A_\sigma, B_\sigma)_{\sigma \in \Sigma}, (\mathcal{X}_q, \mathbb{R}^{n_{q,u}})_{q \in V}, (\mathbb{R}^{n_{q,u}})_{\sigma \in \Sigma} \rangle \) if and only if they are weakly invariant sets for the HAS

\[
\mathcal{S}' = \langle T, (\pi_{\text{Im}(B_q)^\perp} A_q, \pi_{\text{Im}(B_q)^\perp})_{q \in V}, (\pi_{\text{Im}(B_\sigma)^\perp} A_\sigma, \pi_{\text{Im}(B_\sigma)^\perp})_{\sigma \in \Sigma}, (\mathcal{X}_q)_{q \in V} \rangle.
\]

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Proof. By lemma \[1\] (1) is equivalent
\[
\pi_{\text{Im}(B_{\sigma})} A_{\sigma} x \in \pi_{\text{Im}(B_{\sigma})} S_{q}'.
\]
which is \[3\] for \(S'\).
Similarly, by lemma \[1\] (2) is equivalent
\[
\pi_{\text{Im}(B_{q})} A_{q} x \in \pi_{\text{Im}(B_{q})} T_{S_{q}}(x).
\]
which is \[4\] for \(S'\).

3. Computing controlled invariant sets

In this section we derive a characterization of the weak invariance of closed convex sets under the form of inequalities for their support functions. This section uses notions of convex geometry that are recalled in Appendix A.

The following theorem generalizes both [15, (27)] and [17, Theorem 7].

**Theorem 1.** Consider a HAS \(S\) as defined in definition \[3\]. Closed sets \(S_{q} \subseteq X_{q}\) for \(q \in V\) are weakly invariant for \(S\) if and only if
\[
\forall q \rightarrow_{\sigma} q', \forall y \in \mathbb{R}^{n_{\sigma},p}, \quad \delta^{*}(C_{\sigma}^{T} y | S_{q}) \leq \delta^{*}(E_{\sigma}^{T} y | S_{q}') 
\]
\[
\forall q \in V, \forall z \in \mathbb{R}^{n_{q},p}, \forall x \in F_{S}(E_{q}^{T} z), \quad (z, C_{q} x) \leq 0
\]
where \(F_{S}\) denotes the exposed face defined in definition \[9\].

**Proof.** We start by proving the equivalence between \[3\] and \[5\]. By proposition \[5\] eq. \[3\] is equivalent to
\[
\forall q \rightarrow_{\sigma} q', \forall y \in \mathbb{R}^{n_{\sigma},p}, \quad \delta^{*}(C_{\sigma} y | S_{q}) \leq \delta^{*}(E_{\sigma} y | S_{q}')
\]
which is equivalent to eq. \[5\] by proposition \[4\].
We now prove the equivalence between \[4\] and \[6\]. Given any mode \(q\), as \(S_{q}\) is convex, \(T_{S_{q}}(x)\) is a convex cone. By definition of the polar of a cone, \(x \in E_{q} T_{S_{q}}(x)\) if and only if \(\langle y, x \rangle \leq 0\) for all \(y \in [E_{q} T_{S_{q}}(x)]^{\circ}\). By Proposition \[3\] \([E_{q} T_{S_{q}}(x)]^{\circ} = E_{q}^{T} N_{S_{q}}(x)\). Therefore, the set \(S_{q}\) is weakly invariant if and only if
\[
\forall x \in \partial S_{q}, \forall z \in E_{q}^{T} N_{S_{q}}(x), (z, C_{q} x) \leq 0.
\]
By Proposition 2, we have
\[
\{ (x, z) \in \partial S_q \times \mathbb{R}^r \mid E^T z \in N_{S_q}(x) \} = \\
\{ (x, z) \in \partial S_q \times \mathbb{R}^r \mid x \in F_{S_q}(E_q^T z) \}.
\]

As we show in the remainder of this section, theorem 1 allows to reformulate the invariance as an inequality in terms of the support functions of the sets $S_q$. This is already the case of eq. (5) so it remains to reformulate eq. (6). As shown in the following theorem, this is possible in case the support function is differentiable. We generalize this result with a relaxed notion of differentiability in Theorem 3. The following theorem generalizes both [15, (27)] and [17, Theorem 8].

**Theorem 2.** Consider a HAS $S$ as defined in definition 3 and nonempty closed convex sets $S_q \subseteq X_q$ for $q \in V$ such that $\delta^*(\cdot|S_q)$ is differentiable for all $q \in V$. Then the sets are weakly invariant for $S$ if and only if
\[
\forall q \to q', \forall y \in \mathbb{R}^{n_{q,p}}, \delta^*(C_q^T y|S_q) \leq \delta^*(E_q^T y|S_{q'})
\]
\[
\forall q \in V, \forall z \in \mathbb{R}^{n_{q,p}}, \langle z, C_q \nabla \delta^*(E_q^T z|S_q) \rangle \leq 0.
\] (7)

**Proof.** By Proposition 6, $F_{S_q}(E_q^T z) = \{ \nabla \delta^*(E_q^T z|S_q) \}$ hence (6) is equivalent to (7). \qed

As theorem 2 formulates the invariance in terms of the support function of $S_q$, it allows to combine the invariance constraint with other set constraints that can be formulated in terms of support functions. Moreover, for an appropriate family of sets, also called template, the set program can be automatically rewritten into a convex program combining all constraints using set programming [14]. For this reason, we only focus on the invariance constraint and do not detail how to formulate the complete convex programs with the objective and all the constraints needed to obtain the results of Section 4 as these problems are decoupled.

### 3.1. Ellipsoidal controlled invariant set

In this section, we particularize Theorem 2 to the case of ellipsoids. Since the support function of an ellipsoid $E_Q$ is $\delta^*(y|E_Q) = \sqrt{y^T Q^{-1} y}$, we have the following corollary of Theorem 2 that generalizes both [16, Theorem 2] and [17, Corollary 9].


**Corollary 1.** Consider a HAS $S$ as defined in definition 3 and positive semidefinite matrices $Q_q$ such that the ellipsoid $E Q_q \subseteq X_q$ for $q \in V$. Then the sets are weakly invariant for $S$ if and only if

\[
\forall q \rightarrow_q q', C_\sigma Q_q^{-1} C_\sigma^T \preceq E_\sigma Q_q^{-1} E_\sigma^T \tag{8}
\]

\[
\forall q \in V, C_q Q_q^{-1} E_q^T + E_q Q_q^{-1} C_q \preceq 0. \tag{9}
\]

Observe that for the trivial case $\text{Im}(B_q) = \mathbb{R}^{n_q,x}$ for some node $q$, proposition 1 produces a HAS with $n_{q,p} = 0$ hence the LMI (9) will be trivially satisfied for any $Q_q^{-1}$, which is expected. The same applies for (8) in case $\text{Im}(B_\sigma) = \mathbb{R}^{n_\sigma,x}$ for some $\sigma$.

**3.2. Polynomial controlled invariant set**

In this section, we derive the algebraic condition for the controlled invariance of a set with polynomial support function. This template is referred to as polyset; see [14, Section 1.5.3]. The following corollary generalizes both [16, Theorem 5] and [17, Corollary 10].

**Corollary 2.** Consider a HAS $S$ as defined in definition 3, convex homogeneous nonnegative polynomials $(p_q(x))_{q \in V}$ of degree $2d$ and the sets $S_q$ defined by the support function $\delta^*(y|S_q) = p_q(y)^{1/2d}$ for $q \in V$. Suppose that $S_q \subseteq X_q$ for all $q \in V$. Then the sets are weakly invariant for $S$ if and only if

\[
\forall q \rightarrow_q q', \forall y \in \mathbb{R}^{n_\sigma,p}, p_q(C_\sigma^T y) \leq p_q(E_\sigma^T y) \tag{10}
\]

\[
\forall q \in V, \forall z \in \mathbb{R}^{n_q,p}, z^T C_q \nabla p_q(E_q^T z) \leq 0. \tag{11}
\]

**Proof.** We have

\[
\nabla \delta^*(y|S_q) = \frac{1}{p_q(y)^{1-\frac{1}{2d}}} \nabla p_q(y).
\]

If $p(y)$ is identically zero, this is trivially satisfied. Otherwise, $p_q(y)^{1-\frac{1}{2d}}$ is nonnegative and is zero in an algebraic variety of dimension $n - 1$ at most. Therefore, (7) is equivalent to (11).

The conditions (10) and (11) require the nonnegativity of a multivariate polynomial. While verifying the nonnegativity of a polynomial is co-NP-hard, a sufficient condition can be obtained via the standard Sum-of-Squares

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2 A polynomial is **homogeneous** if all its monomials have the same total degree
programming framework; see [7]. Moreover, the theorem requires the convexity of the polynomials $p_q$. It is shown in [2] that the convexity or quasi-convexity of a multivariate polynomial of degree at least four is NP-hard to decide. However, the convexity constraint can be replaced by the tractable SOS-convexity constraint which is a sufficient condition for convexity [2].

3.3. Piecewise semi-ellipsoidal controlled invariant set

In [10], the authors study the computation of piecewise quadratic Lyapunov functions for continuous-time autonomous piecewise affine systems. In [15], the authors present a convex programming approach to compute piecewise semi-ellipsoidal controlled invariant sets for discrete-time control systems. A similar approach is developed in [17] for continuous-time control system. In this section, we combine the two approaches into a condition for hybrid systems using Theorem 1. We recall [15, Definition 2] below.

**Definition 5.** A polyhedral conic partition of $\mathbb{R}^n$ is a set of $m$ polyhedral cones $(P_i)_{i=1}^m$ with nonempty interior such that for all $i \neq j$, $\dim(P_i \cap P_j) < n$ and $\cup_{i=1}^m P_i = \mathbb{R}^n$.

A polyhedral conic partition defines the full-dimensional faces of a complete fan, as defined in [31, Section 7].

A piecewise semi-ellipsoid has a support function of the form

$$\delta^*(y|S) = \sqrt{y^T Q_i y}, \quad y \in P_i, \quad i = 1, \ldots, m$$

where $(P_i)_{i=1}^m$ is a polyhedral conic partition. The support function additionally has to satisfy [15] (2) and (3)] to ensure its continuity and convexity. The following theorem generalizes both [15] (27) and [17] Theorem 12.

**Theorem 3.** Consider a HAS $S$ as defined in definition 3, polyhedral conic partitions $(P_{q,i})_{i=1}^{m_q}$ and nonempty closed convex sets $(S_q)_{q \in V}$ defined by the support function

$$\delta^*(y|S_q) = f_{q,i}(y), \quad y \in P_{q,i}, \quad i = 1, \ldots, m_q.$$ 

Suppose that $S_q \subseteq \mathcal{X}_q$ for all $q \in V$. The sets $S_q$ are weakly invariant for $S$ if and only if

$$\forall q \to q', \forall i \in [m_q], \forall j \in [m_{q'}],$$

$$\forall y \in C_q^{-T} P_{q,i} \cap E_{q'}^{-T} P_{q',j}, \quad f_{q,i}(C_q^T y) \leq f_{q',j}(E_{q'}^T y) \quad (13)$$

$$\forall q \in V, \forall i \in [m_q], \forall z \in E_{q,i}^{-T} P_{q,i}, \quad \langle z, C_q \nabla f_{q,i}(E_q^T z) \rangle \leq 0. \quad (14)$$
Proof. If \( y \in C^{\top}_{q,i} P_{q,i} \cap E^{\top}_{q,j} P_{q,j} \), then \( \delta^*(C^{\top}_{q,i} y|S_q) = f_{q,i}(C^{\top}_{q,i} y) \) and \( \delta^*(E^{\top}_{q,j} y|S_{q}) = f_{q,j}(E^{\top}_{q,j} y) \) hence (5) is reformulated as (13).

We now prove the equivalence between (6) and (14). Consider a mode \( q \in V \). Given \( z \in \mathbb{R}^{n_{q,p}} \) such that \( E^{\top}_{q} z \) is in the intersection of the boundary of \( S_q \) and the interior of \( P_{q,i} \), the support function is differentiable at \( E^{\top}_{q} z \) hence, by Proposition 6 \( F_S(E^{\top}_{q} z) = \{ \nabla f_{q,i}(E^{\top}_{q} z) \} \). The condition (6) is therefore reformulated as (14).

Given a subset \( I \) of \( \{ 1, \ldots, m \} \) and \( z \in \mathbb{R}^{n_{q,p}} \) such that \( E^{\top}_{q} z \) is in the intersection of the boundary of \( S_q \) and \( \cap_{i \in I} P_{q,i} \), \( F_S(E^{\top}_{q} z) \) is the convex hull of \( \nabla \delta^*(E^{\top}_{q} z|S_q) \) for all \( i \in I \). For any convex combination (i.e., nonnegative numbers summing to 1) \( (\lambda_i)_{i \in I} \), (14) implies that
\[
\langle z, C_q \sum_{i \in I} \lambda_i \nabla f_{q,i}(E^{\top}_{q} z) \rangle = \sum_{i \in I} \lambda_i \langle z, C_q \nabla f_{q,i}(E^{\top}_{q} z) \rangle \leq 0.
\]

The following corollary generalizes both \[15\] and \[17\].

**Corollary 3.** Consider a HAS \( S \) as defined in definition \[3\] and piecewise semi-ellipsoids \( S_q \subseteq X_q \) for \( q \in V \). The sets are weakly invariant for \( S \) if and only if
\[
\forall q \rightarrow_q q', \forall i \in [m_q], \forall j \in [m_{q'}],
\forall y \in C^{\top}_{q,i} P_{q,i} \cap E^{\top}_{q,j} P_{q,j}, y^{\top} E_q Q_{q,j} E^{\top}_q y \leq y^{\top} E_q Q_{q,j} E^{\top}_q y \quad (15)
\forall q \in V, \forall i \in [m_q], \forall z \in E^{\top}_{q} P_{q,i}, \quad z^{\top} C_q Q_{q,i}^{-1} E^{\top}_q z + z^{\top} E_q Q_{q,i}^{-1} C_q z \leq 0. \quad (16)
\]

The conditions (15) and (16) amount to verifying the positive semidefiniteness of a quadratic form when restricted to a polyhedral cone. When this cone is the positive orthant, this is called the **copositivity** which is co-NP-complete to decide \[21\]. However, a sufficient LMI is given in \[15\] and a necessary and sufficient condition is given by a hierarchy of Sum-of-Squares programs \[22\]. We use the sufficient LMI in the numerical example of section 4.
4. Numerical example

This example considers the HCS with one mode of continuous-time dynamics:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= u(t)
\end{align*}
\]

with state constraint \(x \in [-1, 1]^2\) and input constraint \(u \in [-1, 1]\) and the following transition from the only mode to itself:

\[
\begin{align*}
x_1^+ &= -x_1 + u/8 \\
x_2^+ &= x_2 - u/8
\end{align*}
\]

with state constraint \(x \in [-1, 1]^2\) and input constraint \(u \in [-1, 1]\).

The union of controlled invariant sets is controlled invariant. Moreover, by linearity and convexity of the constraint sets, the convex hull of the unions of controlled invariant sets is controlled invariant. Therefore, there exists a maximal controlled invariant set, i.e., a controlled invariant set in which all controlled invariant sets are included, for any family that is closed under union (resp. convex hull); it is the union (resp. convex hull) of all controlled invariant sets included in \([-1, 1]^2\).

For this simple planar system, the maximal controlled invariant set can be obtained by hand. We represent it in yellow in Figure 1 and Figure 2.

As proposition 1 requires the input to be unconstrained, it cannot be applied to this system directly. We follow the approach detailed in [16, Section 2.2] to reduce the computation of controlled invariant sets for this system to a system with uncontrolled input. In this example, it corresponds to the projection onto the first two dimensions of controlled invariant sets for the following lifted system:

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= x_3(t) \\
\dot{x}_3(t) &= u(t)
\end{align*}
\]

with state constraint \(x \in [-1, 1]^3\); with a first transition to a temporary mode:

\[
\begin{align*}
x_1^+ &= x_1 \\
x_2^+ &= x_2 \\
x_3^+ &= u
\end{align*}
\]
with state constraint \( x \in [-1, 1]^3 \) and unconstrained input; and a second transition back to the original mode:

\[
\begin{align*}
  x_1^+ &= -x_1 + x_3/8 \\
  x_2^+ &= x_2 - x_3/8 \\
  x_3^+ &= u.
\end{align*}
\]

Note that the input \( u \) chosen in the first transition is the input that will be used for the reset map and the input \( u \) chosen for the second jump is the input that will be used for the state \( x_3 \) of the continuous-time system.

As shown in proposition \([1]\) a set is controlled invariant for this system if and only if it is weakly invariant for the algebraic system

\[
\begin{align*}
  \dot{x}_1(t) &= x_2(t) \\
  \dot{x}_2(t) &= x_3(t)
\end{align*}
\]

with state constraint \( x \in [-1, 1]^3 \); with a first transition to a temporary mode:

\[
\begin{align*}
  x_1^+ &= x_1 \\
  x_2^+ &= x_2
\end{align*}
\]

with state constraint \( x \in [-1, 1]^3 \) and a second transition back to the original mode:

\[
\begin{align*}
  x_1^+ &= -x_1 + x_3/8 \\
  x_2^+ &= x_2 - x_3/8.
\end{align*}
\]

We represent the state set \([-1, 1]^2\) and its polar in green in Figure \([1]\) and Figure \([2]\).

While the maximal invariant set is well defined, it is not the case anymore when we restrict the set to belong to the family of ellipsoids, polysets or piece-wise semi-ellipsoids for a fixed polyhedral conic partition as these families are not invariant under union nor convex hull. The objective used to determine which invariant set is selected depends on the particular application. Let \( \mathcal{D} \) be the convex hull of \( \{(-1 + \sqrt{3}, -1 + \sqrt{3}), (-1/2, 1), (-1, 3/4), (1 - \sqrt{3}, 1 - \sqrt{3}), (1/2, -1), (1, -3/4)\} \). For this example, we maximize \( \gamma \) such that \( \gamma \mathcal{D} \) is included in the projection of the invariant set onto the first two dimensions. We represent \( \gamma \mathcal{D} \) in red in Figure \([1]\) and Figure \([2]\).
For the ellipsoidal template considered in Section 3.1, the optimal solution is shown in Figure 1 as ellipsoids corresponds to polysets of degree 2. The optimal objective value is $\gamma \approx 0.894$.

For the polyset template considered in Section 3.2, the optimal solution are represented in Figure 1. The optimal objective value for degree 4 (resp. 6 and 8) is $\gamma \approx 0.896$. (resp. $\gamma \approx 0.93$ and $\gamma \approx 0.96$).

For the piecewise semi-ellipsoidal template, we consider as polyhedral conic partitions the face fan \([31, \text{Example } 7.2]\), i.e., the conic hull of each facet, of the polytope with extreme points

$$\left(\cos(\alpha) \cos(\beta), \sin(\alpha) \cos(\beta), \sin(\beta)\right)$$

(17)

where $\alpha = 0, 2\pi/m_1, 4\pi/m_1, \ldots, 2(m_1-1)\pi/m_1$ and $\beta = -\pi/2, \ldots, -2\pi/(m_2-1), -\pi/(m_2-1), 0, \pi/(m_2-1), 2\pi/(m_2-1), \ldots, \pi/2$.

The optimal objective value for $m = (4, 3)$ (resp. (8, 5), (16, 7)) is $\gamma \approx 0.894$ (resp. $\gamma \approx 0.92$, $\gamma \approx 0.94$). The corresponding optimal solution is shown in Figure 2.

5. Conclusion

We proved a condition for controlled invariance of convex sets for a hybrid control system based on their support functions. We particularized the condition for three templates: ellipsoids, polysets and piecewise semi-ellipsoids. In the ellipsoidal case, it combines known LMIs for discrete-time and continuous-time systems. In the polyset case, it provides a condition significantly less conservative than \([24]\). Indeed, our condition is equivalent to invariance by Corollary 2 and, as shown in \([17, \text{Section } 2]\), \([24]\) is quite conservative. In the piecewise semi-ellipsoidal case, it provides the first convex programming approach for the controlled invariance of hybrid control systems to the best of our knowledge.

As future work, we aim to apply this framework to other families such as the piecewise polysets defined in \([14]\). Moreover, instead of considering a uniform discretization of the hypersphere as in \([17]\), a more adaptive methods could be considered. The sensitivity information provided by the dual solution of the optimization program could for instance determine which pieces of the partition should be refined.

Finally, our definition of hybrid control system (definition 1) does not support encoding a guard that would restrict the possible transitions depend-
Figure 1: In blue are the solution for polysets of different degrees. The degrees from top to bottom are respectively 2, 4, 6 and 8. The green set is the safe set $[-1, 1]^2$, the yellow set is the maximal controlled invariant set and the red set is $\gamma D$. The sets are represented in the primal space in left figures and in polar space in the right figures.
Figure 2: In blue are the solution for piecewise semi-ellipsoids for two different polyhedral conic partitions. The partitions from top to bottom are as described in (17) with $m = (4, 3)$ (resp. $(8, 5), (16, 7)$). The green set is the safe set $[-1, 1]^2$, the yellow set is the maximal controlled invariant set and the red set is $\gamma D$. The sets are represented in the primal space in left figures and in polar space in the right figures.

Integrating this additional feature to the framework would allow the method to handle any hybrid automaton with linear continuous-time dynamic at each mode and linear reset maps.

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**Acknowledgements**

The first author is a post-doctoral fellow of the Belgian American Educational Foundation. His work is partially supported by the National Science Foundation under Grant No. OAC-1835443. The second author is a FNRS honorary Research Associate. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement No 864017 - L2C.
RJ is also supported by the Innoviris Foundation and the FNRS (Chist-Era Druid-net)

Appendix A. Convex geometry

**Definition 6** ([26, p. 28]). Consider a convex set $S$. The *support function* of $S$ is defined as

$$
\delta^*(y|S) = \sup_{x \in S} \langle y, x \rangle.
$$

**Definition 7** (Polar set). For any convex set $S$ the polar of $S$, denoted $S^\circ$, is defined as

$$
S^\circ = \{ y \mid \delta^*(y|S) \leq 1 \}.
$$

We define the *tangent cone* as follows [6, Definition 4.6].

**Definition 8** (Tangent cone). Given a closed convex set $S$ and a distance function $d(S, x)$, the *tangent cone* to $S$ at $x$ is defined as follows:

$$
T_S(x) = \left\{ y \mid \lim_{\tau \to 0} \frac{d(S, x + \tau y)}{\tau} = 0 \right\}
$$

where the distance is defined as

$$
d(S, x) = \inf_{y \in S} \| x - y \|
$$

where $\| \cdot \|$ is a norm. The tangent cone is a convex cone and is independent of the norm used.

For a convex set $S$, the *normal cone* is the polar of the tangent cone $N_S(x) = T_S^\circ(x)$.

The *exposed face* (also called the *support set*, e.g., in [27, Section 1.7.1]) is defined as follows [9, Definition 3.1.3].

**Definition 9** (Exposed face). Consider a nonempty closed convex set $S$. Given a vector $y \neq 0$, the *exposed face* of $S$ associated to $y$ is

$$
F_S(y) = \{ x \in S \mid \langle x, y \rangle = \delta^*(y|S) \}.
$$

The exposed faces and normal cones are related by the following property [9, Proposition C.3.1.4].
Proposition 2. Consider a nonempty closed convex set $S$. For any $x \in S$ and nonzero vector $y$, $x \in F_S(y)$ if and only if $y \in N_S(x)$.

Given a set $S$ and a matrix $A$, let $A^{-\top}$ denote the preimage:

$$A^{-\top} S \triangleq \{ x \mid A^\top x \in S \}. \quad (A.1)$$

Proposition 3 ([26, Corollary 16.3.2]).
For any convex set $S$ and linear map $A$,

$$(AS)^\circ = A^{-\top} S^\circ.$$

where $S^\circ$ denotes the polar of the set $S$.

Proposition 4 ([25, Corollary 11.24(c)] or [26, Corollary 16.3.1]). Given a matrix $A \in \mathbb{R}^{n_1 \times n_2}$ and a nonempty closed convex set $S \subseteq \mathbb{R}^{n_2}$, for all $y \in \mathbb{R}^{n_1}$, the following holds:

$$\delta^*((y|AS)) = \delta^*((A^\top y|S)). \quad (A.2)$$

Proposition 5 ([26, Corollary 13.1.1]). Consider two nonempty closed convex subsets $S_1, S_2 \subseteq \mathbb{R}^n$. The inclusion $S_1 \subseteq S_2$ is equivalent to the inequality

$$\delta^*(x|S_1) \leq \delta^*(x|S_2) \text{ for all } x \in \mathbb{R}^n.$$

When the support function is differentiable at a given point, $F_S$ is a singleton and may be directly obtained using the following result:

Proposition 6 ([26, Corollary 25.1.2]).
Given a nonempty closed convex set $S$, if $\delta^*(y|S)$ is differentiable at $y$ then $F_S(y) = \{ \nabla \delta^*(y|S) \}$.

In fact, for nonempty compact convex sets, the differentiability at $y$ is even a necessary and sufficient conditions for the uniqueness of $F_S(y)$ [27, Corollary 1.7.3].