Infinite Computations and the Generic Finite

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Abstract

This paper introduces the concept of a generic finite set and points out that a consistent and significant interpretation of the grossone, ① notation of Yaroslav D. Sergeyev is that ① takes the role of a generic natural number. This means that ① is not itself a natural number, yet it can be treated as one and used in the generic expression of finite sets and finite formulas, giving a new power to algebra and algorithms that embody this usage. In this view,

\[ \mathbb{N} = \{1, 2, 3, \ldots, ①-2, ①-1, ①\} \]

is not an infinite set, it is a symbolic structure representing a generic finite set. We further consider the concept of infinity in categories. An object \( A \) in a given category \( C \) is infinite relative to that category if and only if there is a injection \( J: A \rightarrow A \) in \( C \) that is not a surjection. In the category of sets this recovers the usual notion of infinity. In other categories, an object may be non-infinite (finite) while its underlying set (if it has one) is infinite. The computational methodology due to Yaroslav D. Sergeyev for executing numerical calculations with infinities and infinitesimals is considered from this categorical point of view.

Keywords: grossone, finite, infinite, generic finite, category

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1 Introduction

We consider the grossone, ①, formalism of Yaroslav Sergeyev [4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 13]. See also [15, 16] for other discussions of this structure and its

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applications. This paper begins in the stance that there are no completed infinite sets. We shall show that the grossone is then naturally interpreted as a generic finite natural number and that the Sergeyev natural number construction

\[ \mathbb{N} = \{1, 2, 3, \ldots, \mathbb{O} - 2, \mathbb{O} - 1, \mathbb{O} \} \]

can be seen as a generic finite set. By a generic finite set, I mean that \( \mathbb{N} \) represents the properties of an arbitrarily chosen finite set of integers taken from 1 to a maximal integer represented by \( \mathbb{O} \). This means that for any given finite realization of \( \mathbb{O} \), \( \mathbb{O} \) will be larger than all the other integers in that realization. In this sense, we can say that \( \mathbb{O} \) is larger than any particular integer. The grossone \( \mathbb{O} \) is a symbol representing the highest element in a generic finite set of natural numbers. Infinity is not the issue here. The issue is clarity of construction and the possibility of calculation with either limited or unlimited means.

The original intent for the grossone and the set \( \mathbb{N} = \{1, 2, 3, \ldots, \mathbb{O} - 2, \mathbb{O} - 1, \mathbb{O} \} \) was that \( \mathbb{N} \) should represent the infinity of the natural numbers in a way that makes the counting of infinity closer to computational issues than does the traditional Cantorian approach. In Sergeyev’s original approach, \( \mathbb{N} \) is understood to be an infinite set, but one does not use the usual property of a Cantorian infinite set that an infinite set is in 1–1 correspondence with a proper subset of itself. Rather, one regards special subsets such as the even integers as having their own measurement as parts of \( \mathbb{N} \). For example, in the original approach it is fixed that grossone \( \mathbb{O} \) is even, since it is postulated that the sets of even and odd numbers have the same number of elements, namely \( \mathbb{O} / 2 \). Analogously, \( \mathbb{O} / 3, \mathbb{O} / 4, \ldots, \mathbb{O} / n \) are integer for finite \( n \). In our approach, which in this sense is a relaxation of the Sergeyev original approach, we interpret the grossone as a generic natural number and so it can be interpreted to be either even or odd. We shall see, in Section 2 how this has specific computational consequences in the summation of series. Both interpretations are useful. The generic integer interpretation that we use here has the advantage that the assumptions made are just those that can be applied to finite sets of natural numbers. This will become apparent as we continue with more theory and examples.

We reflect here many of these issues in the interpretation of \( \mathbb{N} \) as a generic finite set. Our \( \mathbb{N} \) is not a set but rather a symbolic structure that stands for any finite set. As a symbolic structure \( \mathbb{N} \) is inductively defined so that any of the following list of symbols denotes \( \mathbb{N} \):

\[ \mathbb{N} = \{1, \cdots, \mathbb{O} \} \]
\[ \mathbb{N} = \{1, 2, \cdots, \mathbb{O} - 2, \mathbb{O} - 1, \mathbb{O} \} \]
\[ \mathbb{N} = \{1, 2, 3, \cdots, \mathbb{O} - 3, \mathbb{O} - 2, \mathbb{O} - 1, \mathbb{O} \} \]
\[ \mathbb{N} = \{1, 2, 3, 4 \cdots, \mathbb{O} - 3, \mathbb{O} - 2, \mathbb{O} - 1, \mathbb{O} \} \]
\[ \cdots \]
\[ \mathbb{N} = \{1, 2, 3, \cdots, n, \cdots, 1 - n + 1, \cdots 2, 1\} \]

for any finite natural number \( n \). This means that there are infinitely many possible symbolic structures that indicate \( \mathbb{N} \). Each structure is finite as a symbolic structure. Depending upon the size of the set to which we shall refer, any and all of these symbols can be used for the reference. Since an integer \( n \) can occur and does occur in all such representatives past a certain point we say that \( n \) belongs to \( \mathbb{N} \) for all natural numbers \( n \). And we say that \( \xi \) can be greater than any given natural number \( n \). In this form we create a language in which speaking about \( \mathbb{N} \) is very similar to speaking about an infinite set, but \( \mathbb{N} \), being a symbol for a generic finite set, is not be interpreted as a Cantorian infinite set. The issues about subsets and \( 1 - 1 \) correspondences do not arise.

The paper is organized into sections 2, 3 and 4 devoted to this point of view about genericity. The style of writing for sections 2, 3 and 4 is sometimes polemic, as it is written in the form of a speaker who is convinced that only finite sets should be allowed in mathematics. Infinite sets are seen to be taken care of by the concept of generic finite set. We show how to apply the grossone formalism by first thinking about how a computation could be written finitely, and then how it may appear in the limit where the grossone \( \xi \) is itself regarded as an infinite number. Except for certain situations where we can consider limits as \( \xi \) becomes infinite, we do not use the original interpretations of the grossone as an infinite quantity. An example of a situation where one can shift from one interpretation to the other is in a summation of the form \( S = \Sigma_{n=1}^{\xi} F(n) \) where \( F \) is a function defined on the natural numbers. For example, consider

\[ \Sigma_{n=1}^{\xi} 2^n = 2^{\xi+1} - 1. \]

We can regard this as standing for all specific formulas of the type

\[ \Sigma_{n=1}^{N} 2^n = 2^{N+1} - 1 \]

where \( N \) is any natural number, or we can regard it as a generic summation with an generic result of \( 2^{\xi+1} - 1 \). The generic result can be regarded as formally similar to the infinite result that comes from taking \( \xi \) infinite. Both ways of thinking about the answer tell us how the summation behaves when the number of summands is \( \xi \) and \( \xi \) very large. Other examples will be discussed in the body of the paper.

Secton 5 is written from another point of view. In this section we take a categorical and relative point of view about “being infinite.” We accept that in bare set theory a set is infinite if it is in \( 1 - 1 \) correspondence with a proper subset of itself. But in another category we ask that this \( 1 - 1 \) correspondence be an injection in that category that is not a surjection. This means, for example, that a circle or a sphere in the topological category is not infinite (hence finite by definition) since there is no homeomorphism of a sphere to a proper subset of itself. We examine the model \( \mathbb{N} \) for the extended natural numbers from this point of view, and show that it
can be construed as set theoretically infinite, but topologically finite. We hope that these two points of view for interpretation will enrich the intriguing subject of the grossone and it uses.

2 The grossone, “Infinite” Series and The Three Dots

In Yaroslav Sergeyev’s theory of numeration, he considers a completion of the natural numbers

$$\mathcal{N} = \{1, 2, 3, \cdots \}$$

to a set that contains an infinite number, referred to as grossone. The completion is denoted by the notation $\mathbb{N}$ in the form

$$\mathbb{N} = \{1, 2, \ldots \ 1 - 2, \ 1 - 1, \ 1\}.$$  \hspace{1cm} (1)

The grossone, 1, behaves “just like” a very large integer, so that $\mathbb{N}$ as a set is conceived to have all of its properties analogous to those of a finite set of integers such as

$$\text{Set}(\mathbb{N}) = \{1, 2, \ldots \ N - 2, \ N - 1, \ N\}.$$  

While Sergeyev does not quite explicitly say that $\mathbb{N}$ and $\text{Set}(\mathbb{N})$ are logical twins just so long as $\mathbb{N}$ is very very large, this is the basic idea that we explore in this paper. For example, it should not be the case that $\mathbb{N}$ should appear infinite in the sense of having a 1 − 1 correspondence with a proper subset of itself. Some of the usual attempts obviously fail. For example if we try to map $\mathbb{N}$ to itself by the map $f(k) = k + 1$ then $f(1) = 1 + 1$ and so we need a larger domain. This mirrors the problem for the finite set into the infinite set $\mathbb{N}$.

But what set is this $\mathbb{N}$? How should we interpret its two appearances of the “three dots”? In the case of $\mathcal{N}$ the three dots refer to the Peano axioms for the (usual) natural numbers, assuring us that given a natural number $n$ then there is a successor to that number indicated by $n + 1$, and that $n + 1$ is never equal to $n$. The principle of mathematical induction is then used to characterize the set $\mathcal{N}$. If $M$ is any subset of $\mathcal{N}$ containing 1 and having the property that $n \in M$ implies that $(n + 1) \in M$, then $M = \mathcal{N}$.

At first glance one is inclined to guess that $\mathbb{N}$ consists in two copies of $\mathcal{N}$, one ascending

$$\{1, 2, 3, \cdots \},$$

and one descending

$$\{\cdots 1 - 3, 1 - 2, 1 - 1, 1\}.$$  

If we then take the union

$$\{1, 2, 3, \cdots \ 1 - 3, 1 - 2, 1 - 1, 1\},$$
is this \( \mathbb{N} \)? I submit that this is not the desired \( \mathbb{N} \). For one thing, if one starts counting upward 1, 2, \( \cdots \), one never leaves the left half of this union. And if one starts counting down, one never leaves the right half of the union. This is not in analogy to the large finite set, where either counting down or counting up will cover all of the territory. Also, there appears to be an injection of the union to itself that is not a surjection. Just add one to every element of the left half and subtract one from every element of the right half. All in all, we must search for a different model for \( \mathbb{N} \), or a different category for it to call its home.

I believe that the simplest interpretation for \( \mathbb{N} \) is that it is a generic finite set. This means that we interpret \( \mathbb{1} \) as a generic “large” natural number. \( \mathbb{1} \) is not any specific natural number unless we want to take it as such. Then since \( \mathbb{N} \) is a generically finite set, it has no 1−1 correspondence with a proper subset of itself. In fact, \( \mathbb{N} \) is not a set at all. It is a symbolic construct that represents the form of a finite set. It is not a set just as an algebraic variable \( x \), standing for a number, is not a number. As such, \( \mathbb{N} \) has no members, nor does it have any subsets. As a symbolic construct however, \( \mathbb{N} \) has some nice features. We all agree on the equalities

\[
\mathbb{N} = \{1, \cdots, \mathbb{1} - 1, \mathbb{1}\},
\]

\[
= \{1, 2, \cdots, \mathbb{1} - 2, \mathbb{1} - 1, \mathbb{1}\},
\]

\[
= \{1, 2, 3, \cdots, \mathbb{1} - 3, \mathbb{1} - 2, \mathbb{1} - 1, \mathbb{1}\},
\]

\[
= \cdots
\]

\[
= \{1, 2, 3, \cdots, n, \cdots, \mathbb{1} - n, \mathbb{1} - (n - 1), \cdots, \mathbb{1} - 3, \mathbb{1} - 2, \mathbb{1} - 1, \mathbb{1}\},
\]

for any specific natural number \( n \). This is the nature of \( \mathbb{N} \) as a symbolic construct. In this sense it gives the appearance of acting like an infinite set. This property of the notations that we use is an inheritance from the set theory notation where exactly the same symbolic phenomenon seems to indicate infinity where one only has the form of a series of numbers and the Peano property of knowing that for a given \( n \) there is an \( n + 1 \). In the usual notation for the set of natural numbers, we have

\[
\mathbb{N} = \{1, 2, 3, \cdots, n, \cdots\}
\]

for any specific natural number \( n \). Thus we can also regard \( \mathbb{N} \) as a generic symbol, but it is a generic symbol that is incomplete if it is to be viewed as representing a finite set. The symbolic construction for \( \mathbb{N} \) allows us to start anew and eliminate the notion of a completed infinity from our work.

If we interpret \( \mathbb{N} \) as some particular finite set, then it has \( N \) members where \( N \) is some specific finite natural number. It can have either an odd or an even number of members. It is subject to permutations just as any finite set is so subject. The availability of such interpretations makes our viewpoint different from the original interpretation of the grossone. In the original interpretation, \( \mathbb{1} \) is always infinite.
In our interpretation ① as a symbolic entity is not a finite natural number. It stands for any top value in some finite set, but not any particular value. For this reason we can do arithmetic with ① and say that ① is taken to be greater than \(n\) for any standard natural number \(n\) and that \(① - 1 < ① < ① + 1 < 2①\).

*In the generic context, for a statement about ① to be true, it must be true for all sufficiently large substitutions of natural numbers for ①.* For example \(① > 2^{100}\) is true because \(N > 2^{100}\) is true for all natural numbers \(N\) that are larger than \(2^{100}\). Some statements are true for all numbers. Thus \(2① > ①\) is true since \(2^N > N\) for all natural numbers \(N\). In this way, we can use the grossone as an infinite number when we wish.

Series of the form

\[
\sum_{k=1}^{①} a_k,
\]

where \(a_k\) are any real or complex numbers or algebraic expressions, are well defined. For example, we can write

\[
S = 1 + x + x^2 + \cdots + x^{①},
\]

and this is a generic finite sum in the form of a geometric series. We can operate algebraically on such sums. For example,

\[
xS = x(1 + x + x^2 + \cdots + x^{①}) = x + x^2 + \cdots + x^{①} + x^{①+1},
\]

whence

\[
(1 - x)S = 1 - x^{①+1};
\]

and so, for \(x \neq 1\)

\[
S = \frac{1 - x^{①+1}}{1 - x}.
\]

At this point we can note the effects of taking ① even or odd, or the effect of taking a limit as ① becomes arbitrarily large. For example if the absolute value of \(x\) is less than 1, then \(x^{①+1}\) becomes arbitrarily small as ① becomes arbitrarily large. Generically, we can regard ① as an infinite number if we wish. In that case we have that \(S\) is infinitesimally close to \(\frac{1}{1-x}\), but the expression \(S = \frac{①}{① - x} + 1\) is a more accurate rendition of the actual situation. Working with generic formalism allows us to dispense with limits in many cases and adds detail that can be sometimes ignored in the usual category of working with limits.

Note in the last example, that if we take \(x = -1\), then

\[
S = \frac{1 - (-1)^{①+1}}{2} = \frac{1 + (-1)^①}{2}.
\]
If we take $\mathcal{O}$ even, then $S = 1$. If we take $\mathcal{O}$ to be odd, then $S = 0$. We see that $(-1)^{\mathcal{O}}$ is a well-defined symbolic value that can be either positive or negative, depending on its instantiation as a number. This is in direct contrast to Sergeyev’s usage for the grossone, where $\mathcal{O}$ is taken to be even and so $S$ takes only the value 1. Our interpretation reflects the fact that the corresponding finite series oscillate between 0 and 1.

Here is another example. Let

$$E(x) = \left(1 + \frac{x}{\mathcal{O}}\right)^{\mathcal{O}}.$$  

This is of course a rendition of $(1 + x/N)^N$ for large generic $N$. We can apply the binomial theorem to conclude that

$$E(x) = 1 + \frac{\mathcal{O}}{1!} \left(\frac{x}{\mathcal{O}}\right) + \frac{\mathcal{O}(\mathcal{O} - 1)}{2!} \left(\frac{x}{\mathcal{O}}\right)^2 + \cdots + \frac{\mathcal{O}(\mathcal{O} - 1)(\mathcal{O} - 2) \cdots (1)}{\mathcal{O}!} \left(\frac{x}{\mathcal{O}}\right)^{\mathcal{O}}.$$ 

Thus

$$E(x) = 1 + \frac{x}{1!} + \frac{\mathcal{O}(\mathcal{O} - 1)}{\mathcal{O}!} \frac{x^2}{2!} + \frac{\mathcal{O}(\mathcal{O} - 1)(\mathcal{O} - 2)}{\mathcal{O}^2} \frac{x^3}{3!} + \cdots + \left(\frac{x}{\mathcal{O}}\right)^{\mathcal{O}}.$$ 

From this expression we can see the limit structure that leads to the usual series formula for

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and we see the luxury in the exact formula for $E(x)$. By not taking the limit and examining the exact structure of the formula for large generic $\mathcal{O}$, we see more and can write down more exact approximations for specific values $N$ substituted for the generic $\mathcal{O}$.

Another example will show even more. Let

$$\mathcal{P} = \{p_1, p_2, \ldots, p_{\mathcal{O} - 1}, p_{\mathcal{O}}\}$$

denote a list of all the prime numbers up to a generic natural number $\mathcal{O}$. Define an analogue of the Riemann Zeta function via

$$Z(s) = \prod_{n=1}^{N(\mathcal{O})} \frac{1}{1 - \frac{1}{p_n^s}} = \sum_{n \in N(\mathcal{O})} \frac{1}{n^s}.$$ 

Here $N(\mathcal{O})$ denotes all natural numbers that can be constructed as products of prime powers from the set of primes $\mathcal{P}$. Note that there is no limit to the size of elements of $N(\mathcal{O})$. If we wish to keep bounds on that, we will have to introduce further notation. Even though $Z(s)$ is a finite product, it produces natural numbers of arbitrarily high size. Thus a better finite zeta function will be given by

$$ZZ(s) = \prod_{k=1}^{\mathcal{O}} (1 + p_k^{-s} + p_k^{-2s} + \cdots + p_k^{-(\mathcal{O}s)}).$$
Then

\[ ZZ(s) = \sum_{n \in N(\overline{1}, \overline{1})} \frac{1}{n^s} \]

where \( N(\overline{1}, \overline{1}) \) denotes the generically finite set of natural numbers whose prime factorization is from the first \( \overline{1} \) primes, and whose prime powers are no higher than \( \overline{1} \). For a particular finite instantiation of \( \overline{1} \) this zeta function, being a finite sum, can be computed for any complex number \( s \). This truncation of the usual limit version of the zeta function allows new computational investigations of its properties.

But the reader may ask, does not the construction of \( \mathbb{N} \) connote an infinite set if we regard the grossone \( \overline{1} \) as standing for an infinite evaluation, larger than any natural number? Well, dear reader, what will you have? Should we not be able, in this putative infinity, to count down from \( \overline{1} \) by successive subtractions to some number that we found by counting up from 1 by successive additions? This is true of any finite set. Yet if this is true then it must be that for some natural numbers \( n, m \) we have

\[ n = \overline{1} - m \]

and so we find that

\[ n + m = \overline{1} \]

and so the set \( \mathbb{N} \) would be neither generic, nor infinite, but simply one of the multitude of finite sets, in fact, the one of the form

\[ \{1, 2, 3, \ldots, n + m - 2, n + m - 1, n + m\} \].

Once again,

\[ \mathbb{N} = \{1, 2, 3, \ldots, \overline{1} - 2, \overline{1} - 1, \overline{1}\} \]

is not a set at all, but the form of a generic finite set. It is a symbol for the finite set structure, not a set at all. But this gives us freedom to regard this symbolic structure as something like a set, and in it, we indeed cannot countdown from the grossone to a finite natural number.

### 3 Infinity and the Generic Finite

In many aspects of mathematics there is no need for any infinite set. It is sufficient to have the concept of a generic finite set. For this we need the notion of a generic natural number. Thus one speaks of the set of natural numbers from 1 to \( n \), \( \{1, 2, 3, \ldots, n - 2, n - 1, n\} \), and one usually conceives this as referring to some, as yet unspecified number \( n \). One writes a formula such as

\[ 1 + 2 + \cdots + n = n(n + 1)/2, \]
and it is regarded as true for any specific number \( n \). The concept of a generic \( n \) requires a shift of attention, but no actual change in the formalism of handling finite sets and finite series. To write

\[
1 + 2 + \cdots + \infty = \infty(\infty + 1)/2
\]

is conceptually different from the above formula with \( n \). In the formula with \( n \) we are referring to some specific natural number \( n \). In the formula with grossone, we are indicating a form that is true when \( \infty \) is replaced by a specific integer \( n \), and we are also indicating the behaviour of a corresponding limit or infinite sum.

Once we take a notation for a generic natural number as with Sergeyev’s grossone, we are lead to use the concept of genericity. In this view \( \infty \) is symbolic, not infinite, but can be regarded as indefinitely large. We can regard it as larger than any given number that is named. This is a way of thinking instantiated by our Transfer Principle that Any statement \( P(\infty) \), using \( \infty \), is true if there is a natural number \( N \) such that \( P(n) \) is a true statement about finite natural numbers \( n \) for all \( n > N \). This principle provides a transfer statement that allows us to apply the grossone in many particular situations. The principle and its consequences are distinct from the Sergeyev use of the grossone, and constitute a relaxation of that usage.

For example, if we are working with a computer that is limited to a specific size of natural number, then from the outside, as theorists, we can easily say that \( \infty \) will be greater than any number allowed in that computational domain. We can think of \( N = \{1, 2, 3, \ldots, \infty - 2, \infty - 1, \infty\} \) as a set that is larger than any specific finite set we care to name, but it is still generically finite and does not partake of the Cantorian property of being in 1−1 correspondence with a proper subset of itself. It is like a finite set, but it is not any particular finite set.

The concept of a generic set is different from the concept of a set in the same way that the variable \( x \) is different from a specific number in elementary algebra. A generic set such as \( N \) does not have a cardinality in the sense of Cantor. It is not in 1−1 correspondence with any specific finite set, but if a specific natural number \( n \) is given for \( \infty \), then the resulting set is of cardinality \( n \). This is precisely analogous to the situation with an algebraic \( x \) that in itself has no numerical value, but any substitution of a number for \( x \) results in the specific value of that number. Just as we endow algebraic expressions with the same properties as the numbers that they abstract, we endow generic sets with the properties of the finite sets which they stand for.

Our point in this paper has been that the symbolic constructions of Yaroslav Sergeyev can be regarded as generic finite sets. Does this preclude an infinite interpretation? In the case of a series such as

\[
S = 1 + x + x^2 + \cdots + x^{\infty} = \frac{1 - x^{\infty+1}}{1 - x}
\]
(x not equal to 1), we would like to give the interpretation that if $\mathbb{1}$ is an infinite integer, and $|x| < 1$, then $S$ is infinitesimally close to $1/(1-x)$. The problem with this is the same as the corresponding problem in the calculus. An integral is a limit of finite summations. We are led to imagine (by the Leibniz \[1,2\] notation for example) that the integral

$$\int_{-\infty}^{\infty} e^{-x^2/2} \, dx$$

is an uncountably infinite sum of infinitesimal terms of the form $e^{-x^2/2} \, dx$. It takes the language of non-standard analysis \[3\] to make formal sense out of this statement. We can, using the grossone formalism, go in the other direction and articulate that integral as a generic finite sum. In order to do this we have to make a choice of method of integration and then write the formula in generic fashion. Thus we can write for

$$\int_{-\infty}^{\infty} f(x) \, dx$$

the finite generic expression

$$\sum_{k=-\mathbb{1}}^{\mathbb{1}} \frac{f(k\mathbb{1})}{\mathbb{1}} \cdot \frac{1}{\mathbb{1}}.$$ 

We can even write

$$\int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \lim_{\mathbb{1} \to \infty} \sum_{k=-\mathbb{1}}^{\mathbb{1}} E\left(\frac{k^2}{2\mathbb{1}}\right) \frac{1}{\mathbb{1}}$$

where $E(x)$ is our finite version of the exponential function from the previous section. Here $\lim F(\mathbb{1}) = \lim_{n \to \infty} F(n)$ where the limit is the classical limit and $n$ runs over the classical natural numbers. We cannot write a limit as $\mathbb{1}$ approaches something since $\mathbb{1}$ is generic and does not approach anything other than itself.

Let us write $a \doteq b$ to mean that $a$ and $b$ differ by an infinitesimal amount (in the sense of the discussion above) or, equivalently, that $\lim a = \lim b$. Thus

$$\frac{1 - x^{\mathbb{1}+1}}{1 - x} \doteq \frac{1}{1 - x}$$

when $|x| < 1$.

Here is yet another example. We have the classical formula

$$e^{i\theta} = \cos(\theta) + isin(\theta).$$

This is often seen as a consequence of the series formula for $e^x$. We can write the equation

$$\left(1 + \frac{i\theta}{\mathbb{1}}\right)^{\mathbb{1}} \doteq \cos(\theta) + isin(\theta).$$
In particular, the well-known formula $e^{i\pi} = -1$ becomes the following generic limit formula

$$(1 + \frac{i\pi}{1})^1 \doteq -1.$$ 

In fact, we can, since this formula refers to properties of all formulas where $1$ is replaced by a specific integer, take roots and solve for $\pi$ as follows.

$$(1 + \frac{i\pi}{1})^1 \doteq -1$$

$$1 + \frac{i\pi}{1} \doteq (-1)^\frac{1}{1}$$

$$\frac{i\pi}{1} \doteq (-1)^\frac{1}{1} - 1$$

$$i\pi \doteq 1((-1)^\frac{1}{1} - 1)$$

$$\pi \doteq 1((-1)^\frac{1}{1} - 1)$$

One can verify that this last limit formula is indeed a correct limit formula for $\pi$. Well-known limit formulas appear from this formula when we replace $1$ by $2$ in it. We then have

$$\pi \doteq \frac{2^1((-1)^\frac{1}{2} - 1)}{1}$$

In this version the finite versions are of the form

$$2^N((-1)^\frac{1}{2} - 1)$$

and the limit formula is

$$\pi = \lim_{N \to \infty} \frac{2^N((-1)^\frac{1}{2} - 1)}{1}$$

$$\pi = \lim_{N \to \infty} 2^N \text{Imag}((-1)^\frac{1}{2}),$$

where Im denotes the imaginary part of the given complex number. Here we have successive square roots of $-1$ and we can use the formula

$$\sqrt{a+bi} = \sqrt{(1+a)/2} + i\sqrt{(1-a)/2}$$

when $a^2 + b^2 = 1$. From this it is easy to derive the famous formula of Viète:

$$\pi = \frac{2}{\sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{2} + \cdots}}}}$$

The grossone notation and the notion of generic finite sets allows us to write this derivation in a concise and precise manner.
4 Limits, Ordinals and Further Relaxations

In articulating the notion of the generic finite we have examined an interpretation of the grossone extension of the natural numbers of Yaroslav Sergeyev as a generic finite set. This works because the grossone extension puts a cap, the grossone $\infty$, at the top of its model of the natural numbers and so formally resembles our way of thinking about a finite set such as $\{1, 2, 3\}$ where there is a least element, a greatest element, and a size for the set in the numerical sense of counting or ordinality. By working with this correspondence we may end up with evaluations of size that are different than the Sergeyev theory. As case in point is the finite sets of even natural numbers.

$\{2\}$,
$\{2, 4\}$,
$\{2, 4, 6\}$,
$\cdots$

The generic set of even natural numbers is

$$E = \{2, 4, 6, \cdots 2\infty\}.$$

In Sergeyev’s language the size of this set would be $\infty/2$, since it comprises half of the natural numbers. We have not discussed size, but we can do this in our way by the transfer principle of the previous section. The size of a specific finite set of even numbers

$$\{2, 4, 6, \cdots 2n\}$$

is the number of elements of the set, which is $n$ in this case. Thus we assign $\infty$ as the size for $E$ above. This moves our theory in the direction of Cantorian counting for sets even though we have not invoked a notion of $1 - 1$ correspondence for infinite sets. The point of this example is to show that there are are natural differences between the generic finite set approach and the original approach to the grossone.

There are other relationships of the generic finite set concept and standard set theory. For example, in standard set theory we take as representative ordinals

$$0 = \{\},$$
$$1 = \{0\},$$
$$2 = \{0, 1\},$$
$$3 = \{0, 1, 2\},$$

and so the generic finite ordinal is

$$\mathcal{G} = \{0, 1, \cdots, \infty\}.$$
This should be held in contrast to the first infinite ordinal
\[ \omega = \{0, 1, 2, \cdots \}. \]

The generic finite ordinal has \( 1 + 1 \) members and so we could name it
\[ 1 + 1 = \{0, 1, \cdots, 1\} \]
in the tradition of making ordinals. However, ordinals made in this generic fashion are not well-ordered since there is no end to the descending sequence \( 1, 1 - 1, 1 - 2, \cdots \). This means that a theory of ordinals based on generic finite sets will have a character of its own. This will be the subject of a separate paper.

5 Grossone and Infinity Relative to a Category

In this section, we take a different approach to the grossone. We assume here the existence of infinite sets and the usual terminology of point set topology. With that we discuss, by using categories, relative notions of infinity. After all, a circle is not homeomorphic to any proper subset of itself. Therefore the point set for the circle, uncountable in pure set theory, is \emph{finite} in the category of topological spaces! We formalize this notion below and indicate how it can be interfaced with the grossone.

We recall that a category \( C \) is a collection of objects and morphisms where a morphism is associated to two objects and is usually written as \( f : A \rightarrow B \) where \( A \) and \( B \) are objects. Note that the morphism \( f \) provides a directed arrow from \( A \) to \( B \). Without further axioms the concept of a category is the same as the concept of a directed multi-graph. The axioms for a category are as follows:

1. Given morphisms \( f : A \rightarrow B \) and \( g : B \rightarrow C \), there is a well-defined morphism called the \emph{composition of \( f \) and \( g \)} and denoted
\[ g \circ f : A \rightarrow B. \]
The object \( A \) is called the \emph{domain of \( f \)} and the object \( B \) is called the \emph{codomain or range of \( f \)}.

2. Every object \( A \) has a unique \emph{identity morphism}
\[ 1_A : A \rightarrow A \]
such that for any \( f : A \rightarrow B \), \( f \circ 1_A = f \) and for any \( g : B \rightarrow A \), \( 1_A \circ g = g \).

3. If \( f : A \rightarrow B \), \( g : B \rightarrow C \) and \( h : C \rightarrow D \), then
\[ h \circ (g \circ f) = (h \circ g) \circ f. \]
Thus composition of morphisms is associative.
If \( C \) and \( C' \) are categories, then we say that a functor from \( C \) to \( C' \), denoted \( F : C \rightarrow C' \), is a function that takes objects to objects and morphisms to morphisms such that if \( f : A \rightarrow B \) is a morphism in \( C \), then \( F(f) : F(A) \rightarrow F(B) \) is a morphism in \( C' \). Furthermore, we require of a functor that identity morphisms are carried to identity morphisms, and that compositions are taken to compositions in the sense that \( F(f \circ g) = F(f) \circ F(g) \) for all compositions \( f \circ g \) in \( C \).

In this paper we will use categories whose objects are sets and whose morphisms are maps of these sets with whatever extra structure is demanded by that category. Then there is a forgetful functor \( FG : C \rightarrow Sets \) obtained by just taking the objects as sets and the morphisms as maps of sets, ignoring the extra structure. In such categories, a morphism \( f : A \rightarrow B \) is said to be injective if \( FG(f) \) is injective in \( Sets \), and \( f \) is said to be surjective if \( FG(f) \) is surjective in \( Sets \).

In \( Sets \) one says that a set \( A \) is infinite if there exists an injection \( i : A \rightarrow A \) that is not a surjection. If every injection of \( A \) to itself is a surjection, we say that the set \( A \) is finite. We relativize this notion to other categories. If \( C \) is a set-based category, we say that an object \( A \) of \( C \) is finite (in \( C \)) if every injection of \( A \) to itself in the category \( C \) is surjective. For example consider the category \( Top \) of topological spaces. We see at once that the circle \( S^1 = \{ (x,y) | x^2 + y^2 = 1 \} \)

where \( x, y \) are real numbers, is finite in this category since the circle is not homeomorphic to any proper subset of itself. Thus, while the circle consists in infinitely many points when looked at under the forgetful functor to set theory, in the topological category the circle is finite. We shall see that this point of view on finite and infinite is very useful in sorting out how we deal with mathematical objects in many situations.

We now combine this topological point of view on finiteness with the grossone. In Figure 1 we show an embedding of a union of two sets of points in the form

\[
\mathbb{N} = \{1,2,3,\ldots, G-3, G-2, G-1, G\}.
\]

We regard \( \mathbb{N} \) as embedded in a circle with two points removed. The two vertical marks on the circle in the figure denote the removed points. The circle with two points removed partakes of the subspace topology from the Euclidean plane in which it is embedded. We work in the category of orientation preserving homeomorphisms of the deleted circle \( S \) that map the the intervals \([i,i+1]\) and \([G-i,G-i+1]\) to themselves. In the categorical sense, \( S \) is finite and there can be no such homeomorphisms that take \( \mathbb{N} \) to a proper subset of itself. In fact, in this model, every such homeomorphism is the identity map when restricted to \( \mathbb{N} \).
Figure 1: Grossone Circle

Figure 2: Grossone Graphs
In Figure 2 we give another example of how to topologically make an infinite set finite. We have labeled a set of graphs with the “elements” of \( \mathbb{N} \). None of these graphs are homeomorphic (we take the nodes of the graphs to be disks and the edges to be topological intervals) and so a homeomorphism of the entire collection must take each graph to itself. There is no topological injection of the collection of graphs to a subcollection of itself.

We produce this model to suggest some ways to view \( \mathbb{N} \) as infinite and yet finite without paradox. I am sure that other models will emerge in relation to applications of these ideas.

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