Analysis of non-newtonian fluid with phase flow model

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Abstract

We considered a stagnation point of Non-Newtonian fluid with phase flow model over a stretching surface with slip conditions. Two types of the nanoparticle used, namely Cu and Al2O3 with base fluid H2O. Acceptable to theoretical study, the mathematical model has been constructed through flow assumptions. Partial differential equations are made by applying the boundary layer approximations on the momentum and energy equations. The suitable similarity transformations are applied to the partial equations which are converted into ordinary differential equations. These equations are solved by numerical scheme, namely bvp4c method. The involving physical parameters effect is shown by graphs and tables. Our work shows a good agreement with decay literature. The expressions $F''(0)$ and $-\theta'(0)$ achieve fewer values by hybrid nanofluid than that of nanofluid. Moreover $F''(0)$ and $-\theta'(0)$ increase for large values of dimensionless parameter ($N$) where as $F'(\xi)$ and $\theta(\xi).increase$ for large values of $\Phi_2$.

Key points: Hybrid nanofluid; Second grade fluid; Thermal slip; Stagnation flow: Numerically technique.

1. Introduction

The theoretical study of non-Newtonian fluids have achieved lots of importance due to its vast applications in science and technology. Because of complexity of non-Newtonian fluids, there is not a single fluid model exist in the literature which cover all the aspects of non-Newtonian fluid models have been proposed and studied. Due to the complexity of these models, the exact solutions in general are quite impossible. So the researchers have presented their special cases exact solution or analytical solution and numerical solutions. Rivlin and Eriksen [1] were proposed second grade fluid model early time. The simplest non-Newtonian
fluid model which exhibits many properties of the differential type fluids. Exact solutions of the second grade fluid by using the inverse method have been discussed by Labropulu [2]. Creeping flow of the second grade fluid by using the Lie group analysis have been discussed by Yürüsoy et al. [3]. Two dimensional Lagrangian flow of Euler and second grade fluid flow were discussed by Shkoller [4]. According to his study, he discussed a simple proof of global existence. Labropulu [5] was discussed the second grade fluid flow in the absence of body forces and thermal. Series solution of the second grade fluid flow over shrinking sheet under stagnation region was studied by Nadeem et al. [6]. Flow of Second grade micropolar fluids flows over stretching surface having a heat transfer has been studied by Mehmod et al. [7]. They also highlighted the effects of non orthogonal stagnation point flow. Akinbobola and Okoya [8] were analyzed the mixed effects of thermal conductivity and variable viscosity of second grade fluids at stretching the surface. In their study, the linear temperature function is inversely proportional viscosity while directly proportional to thermal conductivity. Majeed et al. [9] have been highlighted the effects of the second grade fluid flow at stretching cylinder having Dufour and Soret impacts numerically. They also considered the effects of thermal radiation with different aspects. It is seen that the impacts of Dufour and Soret on temperature find to be directly as increasing. Khan et al. [10] have been highlighted heat transfer and second grade fluid of axisymmetric at stretching sheet. Many investigators have explored the second grade fluid flow with different physical aspects (see Refs. [11-18]).

The attraction of stretching analysis have been forced to the researchers because lots of its applications in the fields of engineering namely, cooling of microelectronics, wire drawing, rapid spray, polymer extrusion, glass blowing, quenching in metal foundry etc. In the early time, the boundary layer flow at stretching surface theoretically has been analyzed by Crane [19]. Mekheimer et al. [20-21] discussed the flow behavior of Peristaltic at stretching the surface. Malvandi et al. [22] studied the nanofluid over shrinking/ stretching surface under stagnation point region. Many investigators have studied stretching surface having various assumptions [23-26].

Nanofluid is a fluid that combination of nanoparticle and base fluid. The word nanofluid introduced by Choi [27]. He introduced the nanofluid model and claimed that nanofluid achieved more heat transfer rate than that of simple fluid. This model satisfies the numerical and theatrical analysis. After this, several researchers have been focused on the heat transfer enhancement because of many applications in the fields of engineering, industrial and so on. Application of solar energy by using the nanofluid has been highlighted by Mahian et al. [28]. Heat transfer of nanofluid at rotating disk has been discussed by Turkyilmazoglu [29].
He applied the SCCCM on the boundary layer flow to discuss the flow of nanofluid. The flows of micropolar nanofluid at circular cylinder have been highlighted by Abba et al. [30]. Stagnation point flow of Maxwell nanofluid has discussed by Khan et al. [31]. Number of investigators have been studied the nanofluid model with different assumptions which reveals in Refs. [32-40].

In the shortage of energy, most of the interest of the researchers has discussed numerically, analytically and experimentally to achieve more heat transfer rate than that of decay methods. Mostly used in the real life depends on the heat transfer and coolant. The word Hybrid nanofluid introduced which mean that a type of the fluid which consists a mixture of two different nanoparticles with base fluid. This is the extended version of nanofluid. These types of the fluids help them in the fields of industrial, science and engineering. When two nanosized particles are combined with base fluid which are enhanced the thermal conductivity of the fluid more than that of nanofluid and simple fluid. At the first time, Momin [41] was worked out on the experimental results of hybrid nanofluid with mixed convection. Analytical results find, out of the hybrid nanofluid by Suresh et al. [42]. Numerous investigators have been extended the Suresh et al. [42] work to find heat transfer rate with different assumptions. Stagnation flows of hybrid nanofluid at circular cylinder have been highlighted by Nadeem and Abbas [43]. Recently, a number of the researchers are discussed the hybrid nanofluid with different significant physical parameters which see Refs. [44-51].

The purpose of this analysis to investigate the effects of phase flow model with the second grade flow of fluid over a stretching surface are considered. Slip effects and hybrid nanofluid are considered in this analysis. This system has been converted into a system of an ordinary differential equations. The transformed system is solved through a numerical scheme, namely, BVP4C Method. The impacts of involving governing parameters are highlighted by graphs and tables. No one discussed before hybrid nanofluid with second grade.

2 Mathematical formulations

We considered stagnation point flow of second grade hybrid nanofluid with slip conditions over stretching surface. \( V \) and \( W \) are the velocity components in the direction of \( X - \) and \( Y - \) axis respectively. Stagnation point flow is considered like as \( V = aX \) and \( W = aY \) and \( a \) is stretching parameter see Refs. [52-56].

\[
\frac{\partial V}{\partial X} + \frac{\partial W}{\partial Y} = 0,
\]  

(1)
The equations (1-3) present continuity, momentum and energy equations respectively for the hybrid nanofluid. It is pertinent to mention here that usually the experimental data for both nanofluid and hybrid nanofluid for phase flow model are given only for Newtonian fluid. Here, we make a small substitution in such a way that if is replace with \( \frac{\mu_{nf}}{\alpha_{nf}} \), where \( \alpha \) is taken to be \( o(\delta^2) \). We see the order of approximation as \( o(V) = o(1) = o(X), \) \( o(Y) = o(\delta) = o(W) \). Now we considered \( \frac{\mu_{nf}}{\alpha_{nf}} \) in our assumptions for the mathematical point of view. This phenomenon may be exist as \( o(\delta^2) \). The physical properties are defined in the Tables [1-2]. \( \alpha_{nf}, N, \mu_{nf}, \rho_{nf}, \) and \( P \) are the thermal diffusivity, the dimensionless parameter viscosity, the density and the pressure respectively.

The boundary conditions are given as

\[
W = 0, \quad V = 0, \quad V = \omega_1 \left[ \frac{\partial V}{\partial Y} + \frac{\mu_{nf}}{\rho_{nf}} \left( \frac{N - 1}{N} \right) \left( \frac{\partial V}{\partial Y} + W \frac{\partial^2 V}{\partial Y^2} + V \frac{\partial^2 V}{\partial X \partial Y} \right) \right],
\]

\[
-k_{nf} \frac{\partial T}{\partial Y} = h_w (T_w - T) \quad \text{as} \quad Y \to 0, \quad T = T_\infty, \quad V = aX, \quad W = 0, \quad Y \to \infty.
\]

Where \( \omega, h_w, T_w \) and \( T_\infty \) are velocity slip parameter, temperature slip parameter, wall temperature and the ambient temperature respectively. The non-dimensional form the suitable similarity transformation has been presented as follows, below

\[
\zeta = \left( a / v_f \right)^{1/2} Y, \quad V = aXF'(\zeta), \quad T = (T_0 - T_w) \theta(\zeta) + T_w, \quad W = -(av_f)^{1/2} F(\zeta).
\]
Applying the above equations, then the equation (1) satisfy identically and equations (2) and (3) reduced into the following form

\[
(\frac{\mu_{\text{inj}}}{\rho_{\text{inj}}})F'''(\zeta) + (\frac{\mu_{\text{inj}}}{\rho_{\text{inj}}})(N - 1)(2F''(\zeta)F'(\zeta) + F''(\zeta)F''(\zeta) - F(\zeta)F'''(\zeta)) + 1 + F'(\zeta)F''(\zeta) - F'(\zeta)F'(\zeta) = 0, \\
\alpha_{\text{inj}} \theta''(\zeta) + F(\zeta)\theta'(\zeta) = 0.
\]

Dimensionless form of the boundary conditions is presented below by using the equation (4) and equation (5)

\[
F'(\zeta) = \frac{\lambda}{3}\frac{\mu_{\text{inj}}}{\rho_{\text{inj}}}(\frac{N - 1}{N})F'(\zeta), \quad F(\zeta) = 0, \quad \theta(\zeta) - 1 = (\frac{k_{\text{inj}}}{k_f})\gamma \theta'(\zeta), \quad \text{as} \quad \zeta \to 0, \\
\theta(\zeta) = 0, \quad F'(\zeta) = 1, \quad \zeta \to \infty.
\]

The non dimensional parameters namely, thermal slip, velocity slip and non dimensional \(\gamma\), \(\lambda\) and \(\xi\). The \(Nu_X\) is defined as

\[
Nu_X = -\left(\frac{k_{\text{inj}}}{k_f}\right)(X / (T_w - T_\infty))(\frac{\partial T}{\partial Y})_{Y=0},
\]

and the skin friction coefficient

\[
C_f = \left[\frac{\partial V}{\partial Y} + \frac{\mu_{\text{inj}}}{\rho_{\text{inj}}}(\frac{N - 1}{N})\left\{\frac{\partial V}{\partial X} \frac{\partial V}{\partial Y} + W \frac{\partial^2 V}{\partial Y^2} + V \frac{\partial^2 V}{\partial Y \partial X}\right\}\right]_{Y=0},
\]

\[
Re_X = \frac{aX^2}{v_f}\]

is the local Reynolds number.

### 3 Solution procedure

In this analysis, we considered second grade hybrid nanofluid over stretching surface under the stagnation point region. The mathematical model has been constructed through flow assumptions. Partial differential equations are made by applying the boundary layer approximations on the Navier Stokes equations. The suitable similarity transformations are applied to the partial equations which are converted into ordinary differential equations. These equations are solved by numerical scheme, namely BVP4C method. We have to find the solution of the above system. We take three assumptions like as

- If \(K = \left(\frac{\mu_{\text{hnf}}}{\rho_{\text{hnf}}}\right)\left(\frac{N-1}{N}\right)\) and \(\Phi_1 = 0 = \Phi_2\) while \(N \in [0, 1]\) then this system becomes Second grade fluid model.
- If \(K = \left(\frac{\mu_{\text{hnf}}}{\rho_{\text{hnf}}}\right)\left(\frac{N-1}{N}\right)\) and \(\Phi_1 = 0 = \Phi_2\) while \(N \in [0, 1]\) then this system becomes a Viscoelastic fluid model. It is seen that velocity profile gains boundary layer thickness for
large values of velocity slip parameter which reveals in Fig. 1. When the thermal slip increases, which shows decline the curve of temperature profile as see in Fig. 1. Table 3 reveals the comparative results of our present results with Ariel [52]. It is seen that our present results found to be good agreement with Ariel [52].

• If $N = 1$ and rest of the physical parameters are fixed then this Model becomes Newtonian fluids model. We find the comparative present results with decay results. Table 4 shows the comparison present results of $F''(0)$ for different values of $\lambda$ with Bachok al et. [54] and Wang [53]. It is found to be good agreement with Bachok al et. [54] and Wang [53] when $\Phi_1 = \Phi_2 = 0$.

Table 5 reveals the influence of velocity slip $\lambda$ and nanoparticle concentration $\Phi_2$ on the $\frac{Nu_X}{\sqrt{Re_X}}$ and $\frac{c_f}{\sqrt{Re_X}}$. Our work found to be good agreement with Yacob et al. [35] and Bachok et al. [54] which see in Table 5. Table 6 shows the effects of $\lambda$ on the $F''(0)$. It is observed that the comparison with our work and Bachok et al. [54] and Malvandi et al. [55] is found to be good agreement. Let us we considered phase flow with second grade fluid on the assumption of mathematically $O(\delta^2) = \left(\frac{\mu_{hnf}}{\rho_{hnf}}\right)\left(\frac{N-1}{N}\right) = \left(\frac{\alpha}{\rho_{hnf}}\right)$. Our supposition found to be good agreement with existence literature while the order of approximation also satisfied.

### 4 Results and discussions

The purpose of this analysis to show the effect of involving physical parameters on temperature profile and velocity profile are discussed. The involving physical parameters are nanoparticle concentration of aluminium oxide ($\Phi_1$), nanoparticle concentration of copper ($\Phi_2$), velocity slip parameter ($\lambda$), dimensionless parameter ($N$) and thermal slip parameter ($\gamma$). The nanoparticle concentration of aluminium oxide ($\Phi_1 = 0.1$) is taken to be fixed in overall study. The range of the physical parameters are considered as $0.005 \leq \Phi_2 \leq 0.09$, $0.0 \leq \gamma \leq 0.5$, $0.0 \leq \lambda \leq 0.5$ and dimensionless parameter $N \notin [0, 1]$. The effects of physical are highlighted through graphs and tables. The comparative analysis of $Cu – Al_2O_3/H_2O$ and $Cu/H_2O$ on the $F'(\xi)$ and $\theta(\xi)$ are highlighted in Figs. (2-3). It is noted that $F'(\xi)$ enhances for increasing in the $\Phi_2$. Momentum boundary layer thickness achieves by $Cu – Al_2O_3/H_2O$ higher than that of $Cu/H_2O$. Fig. (3) shows the impacts of $\Phi_2$ on $\theta(\xi)$. It is noted that $\theta(\xi)$ achieves larger when values of $\Phi_2$ increases. Thermal boundary layer thickness gains higher by $Cu – Al_2O_3/H_2O$ than that of $Cu/H_2O$. Figs. (4-5) reveals the
effects of $\Phi_2$ on the $\theta(\xi)$ and $F'(\xi)$. It is seen that $\theta(\xi)$ and $F'(\xi)$ increase with increasing in $\Phi_2$ due to show high resistance to the fluid velocity. Fig. (6) shows the impact of thermal slip on $\theta(\xi)$ and velocity slip on velocity profile. It is observed that $F'(\xi)$ increases with increasing in velocity slip parameter which $\theta(\xi)$ shows decline the curve when enhance thermal slip parameter. Fig. (6) reveals the effects of dimensionless parameter on $\theta(\xi)$ and $F'(\xi)$. It is seen that $F'(\xi)$ increases with increasing in dimensionless parameter ($N$) and $\theta(\xi)$ reduced for higher values of dimensionless parameter which reveals in Fig. 7. Momentum boundary layer thickness increases and thermal boundary layer reduces for large values of dimensionless parameter. Table 7 reveals the impacts of different involving physical parameters $\Phi_2$, $\gamma$, $\lambda$ and $N$ on $\theta'(0)$ and $F''(0)$. We also worked out on the comparative study like $Cu - Al_2O_3/H_2O$ and Cu/H_2O. We observed that $F''(0)$ increases with increasing in $\Phi_2$. In case of $Cu - Al_2O_3/H_2O$, the values of $F''(0)$ less than that of $Cu/H_2O$. The $\theta'(0)$ reduces for higher values of $\Phi_2$ while $Cu - Al_2O_3/H_2O$ gains less values of $\theta'(0)$ than that of $Cu/H_2O$. Thermal slip shows the effects on $\theta'(0)$ which reveals in Table 7. It is seen that thermal slip increases with decreasing the values of $\theta'(0)$. $Cu - Al_2O_3/H_2O$ gains less values of $\theta'(0)$ than that of $Cu/H_2O$. Velocity slip effects reveal on the $F''(0)$ and $\theta'(0)$ in Table 7. The velocity slip increases with increasing $\theta'(0)$ while decreasing $F'(0)$ respectively. It is observed that surprisingly $Cu - Al_2O_3/H_2O$ gains less values than that of $Cu/H_2O$. The dimensionless parameter increases with increasing in $\theta'(0)$ and $F''(0)$. It is also seen that surprisingly $Cu - Al_2O_3/H_2O$ gains less values than that of $Cu/H_2O$.

5 Final remarks

We considered a stagnation point flow of second grade hybrid nanofluid over stretching surface with slip conditions. Involving physical parameters are highlighted through tables and graphs. We find out some significant results surprisingly which mention below.

- $F''(0)$ and $-\theta'(0)$ achieves less values by $Cu - Al_2O_3/H_2O$ than that of $Cu/H_2O$.
- $F''(0)$ and $-\theta'(0)$ for both cases increases for large values of dimensionless parameter ($N$).
- $F'(\xi)$ and $\theta(\xi)$ increase for large values of $\Phi_2$.
- Our work shows a good agreement with decay literatures.

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Effects of Cu/H₂O and Cu−Al₂O₃/H₂O on F′(ξ) and θ(ξ).
Effects of $\Phi_2$ on $F'(\xi)$ and $\theta(\xi)$.

Effects of $N$, $\gamma$ and $\lambda$ on $F'(\xi)$ and $\theta(\xi)$. 
Table 1: Physical properties nanofluid and hybrid nanofluid of thermodynamics.

| Properties     | Nanofluid                                                                 | Hybrid nanofluid                                                                 |
|----------------|---------------------------------------------------------------------------|--------------------------------------------------------------------------------|
| Density        | \( \rho_{nf} = \Phi \rho_s + (1 - \Phi)\rho_f \)                         | \( \rho_{nf} = \Phi_2 \rho_{s_2} + [(\Phi_1 \rho_{s_1} + \rho_f (1 - \Phi_2) (1 - \Phi_1))] \) |
| Viscosity      | \( \mu_{nf} = \frac{\mu_f}{(1-\Phi)^{2.5}} \)                           | \( \mu_{h,nf} = \frac{\mu_f}{(1-\Phi_1)^{2.5} (1-\Phi_2)^{2.5}} \)           |
| Heat capacity  | \((\rho C_p)_{nf} = \Phi (\rho C_p)_s + (1 - \Phi)(\rho C_p)_f\)          | \((\rho C_p)_{h,nf} = \Phi_2 \rho (\rho C_p)_{s_2} + [(\rho C_p)_f (1 - \Phi_2) (1 - \Phi_1)] + \Phi_1 (\rho C_p)_{s_1}\) |
| Thermal        | \( \frac{\kappa_{nf}}{\kappa_f} = \frac{\kappa_s - (n-1)\Phi (\kappa_f - \kappa_s) + (n-1)\kappa_f}{\kappa_s + \Phi(\kappa_f - \kappa_s) + (n-1)\kappa_f} \) | \( \frac{\kappa_{nf}}{\kappa_f} = \frac{k_{s_2} + (n-1)\Phi (k_{f_2} - k_{s_2})}{\kappa_s + (n-1)\Phi (k_{f_2} + k_{s_2})} \), where  
| conductivity   |                                                                           | \( \kappa_{bf} = \frac{k_{s_2} + (n-1)\Phi (k_{f_2} - k_{s_2})}{k_{s_1} + (n-1)\Phi (k_{f_1} - k_{s_1})} \) |

Table 2: Thermo-physical properties.

| Thermo-physical properties | Base fluid (H₂O) | Al₂O₃ | Cu |
|---------------------------|-------------------|-------|----|
| \( \rho \) (kg/m³)        | 997.1             | 3970  | 8933 |
| \( k \) (W/mK)            | 0.613             | 40    | 400 |
| \( C_p \) (j/kg)K         | 4179              | 765   | 385 |

Table 3: Numerical results of [52] compared with present results.

| \( N - 1 \) \( N \) | Present solution | Approximate solution [52] | Exact solution [52] |
|-----------------------|------------------|----------------------------|---------------------|
|                       | \( F''(0) \)     | \( F''(0) \)               | \( F''(0) \)        |
| 0.00                  | 1.232479         | 1.232588                   | 1.224745            |
| 0.05                  | 1.169785         | 1.179830                   | 1.185498            |
| 0.1                   | 1.121512         | 1.134114                   | 1.149241            |
| 0.2                   | 1.078543         | 1.058131                   | 1.084652            |
| 0.3                   | 1.019854         | 0.996844                   | 1.028992            |
| 0.4                   | 0.965843         | 0.945869                   | 0.980581            |
| 0.5                   | 0.923564         | 0.902500                   | 0.938083            |
| 1.0                   | 0.778532         | 0.752766                   | 0.784465            |
| 2.0                   | 0.609856         | 0.596769                   | 0.618347            |
| 3.0                   | 0.517703         | 0.510703                   | 0.526235            |
| 4.0                   | 0.460396         | 0.453968                   | 0.465812            |
| 5.0                   | 0.413285         | 0.412885                   | 0.422308            |
| 6.0                   | 0.379865         | 0.381336                   | 0.389071            |
| 7.0                   | 0.353241         | 0.356110                   | 0.362613            |
| 8.0                   | 0.340521         | 0.335335                   | 0.340905            |
| 10.0                  | 0.306571         | 0.302882                   | 0.307093            |
| 20.0                  | 0.221324         | 0.218554                   | 0.220316            |
| 50.0                  | 0.141241         | 0.140077                   | 0.140579            |
| 100.0                 | 0.099854         | 0.099515                   | 0.099701            |
Table 4: Numerical results of [53] and [54] compared with present results with $\Phi_1 = \Phi_2 = 0$.

| $\lambda$ | Present results | Bachok al et [54] | Wang [53] |
|-----------|-----------------|-------------------|-----------|
|           | $F''(0)$        | $F''(0)$          | $F''(0)$  |
| 0         | 1.225684        | 1.232588          | 1.232588  |
| 0.5       | 0.712358        | 0.713295          | 0.71330   |
| 1.0       | 0               | 0                 | 0         |
| 2.0       | 1.225684        | -1.887307         | -1.88731  |

Table 5: Numerical results of [54] and [56] compared with present results with $\Phi_1 = 0$.

| Cu/H$_2$O | Present results | Bachok al et. [54] | Yacob et al. [56] |
|-----------|-----------------|---------------------|------------------|
|           | $C_f$          | $\theta'$          | $C_f$          |
|           | Nu$_{X}$       | Nu$_{X}$           | Nu$_{X}$       |
|           | $\sqrt{Re_X}$ | $\sqrt{Re_X}$     | $\sqrt{Re_X}$  |
| $\Phi_2$  | $\lambda$      |                     |                 |
| 0.1       | 0.0            | 1.7968              | 1.4043          | 1.8843          | 1.4043          |
| 0.2       | 0.5            | 2.4589              | 1.6421          | 2.6226          | 1.6692          |
| 0.1       | 0.0            | 1.0795              | 1.7895          | 1.0904          | 1.8724          |
| 0.2       | 0.5            | 1.5004              | 2.0987          | 1.5177          | 2.1577          |

Table 6: Comparative results of [55] and [54] with present results $\Phi_2 = 0.1$ and $\Phi_1 = 0$.

| $\lambda$ | Bachok et al.[54] | Malvandi et al. [55] | Present results |
|-----------|--------------------|-----------------------|-----------------|
|           | $F''(0)$          | $F''(0)$              | $F''(0)$        |
| 0.0       | 1.4479777         | 1.449777471           | 1.3578          |
| 0.5       | 0.8379404         | 0.837940401           | 0.8238          |
| 1.0       | 0                 | 0                     | 0               |
| 1.5       | -                 | -1.026658507          | -1.0252         |
| 2.0       | -2.217106         | -2.217105947          | -2.1987         |

Table 7: Skin frictions and Nusselt numbers of Cu – Al$_2$O$_3$/H$_2$O and Cu/H$_2$O.

| Cu – Al$_2$O$_3$/H$_2$O | Cu/H$_2$O |
|-------------------------|-----------|
| $\Phi_2$                | $\gamma$ | $\lambda$ | N | $F''(0)$ | $-\theta'(0)$ | $F''(0)$ | $-\theta'(0)$ |
| 0.005                   | 0.3      | 0.03      | 2.0| 0.787087 | 0.925585    | 0.854221 | 1.0599       |
| 0.02                   | 0.809948 | 0.903288  | 0.885582 | 1.03756 |
| 0.04                   | 0.838365 | 0.87378  | 0.923625 | 1.007111 |
| 0.06                   | 0.86466  | 0.844716 | 0.957992 | 0.976308 |
| 0.08                   | 0.888998 | 0.816229 | 0.989151 | 0.945545 |
|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 0.04 | 0.1 | 0.838365 | 1.03848 | 0.923625 | 1.16947 |
| 0.2 | 0.838365 | 0.949037 | 0.923625 | 1.08223 |
| 0.3 | 0.838365 | 0.87378 | 0.923625 | 1.00711 |
| 0.4 | 0.838365 | 0.809582 | 0.923625 | 0.941738 |
| 0.3 | 0.1 | 1.0672 | 0.802870 | 1.13178 | 0.925748 |
| 0.2 | 0.937379 | 0.842992 | 1.01565 | 0.970982 |
| 0.3 | 0.838365 | 0.87378 | 0.923625 | 1.00711 |
| 0.4 | 0.759557 | 0.898241 | 0.848211 | 1.03673 |
| 0.3 | 0.0 | 0.322344 | 0.648421 | 0.348645 | 0.734637 |
| 2.0 | 0.838365 | 0.87378 | 0.923625 | 1.00711 |