Abstract

We derive five-dimensional super Yang-Mills theory from mass-deformed ABJM theory by expanding about $S^2$ for large Chern-Simons level $K$. We obtain the Yang-Mills coupling constant $g^2_{YM} = 4\pi^2 R/K$. If we consider $S^3/Z_K$ as a fiber bundle over $S^2$ then $R/K$ is the circumference of the fiber. The value on the coupling constant agrees with what one gets by compactifying M five-brane on that fiber. For this computation we take $R, K \to \infty$ while keeping $R/K$ at a fixed finite value. We also study mass deformed star-three-product BLG theory at $K = 1$ and $R \to \infty$. In that limit we obtain Lorentz covariant supersymmetry variations and gauge variations of a non-Abelian tensor multiplet.
1 Introduction

In M-theory it appears that three-manifolds take the role of two-manifolds in string theory. Three-algebra underlies the gauge structure of M2 brane theory [8], [9], [10] and the Nambu bracket, which is defined on a three-manifold, is one realization. The Nambu bracket does not close on any finite set of functions, except for the case of SO(4) three-algebra, the symmetry group of $S^3$. In all other cases we must include an infinite set of functions to close the algebra. The same is true for the Poisson bracket defined on a two-manifold, but here we have a deformation which is the star-commutator which can be used to obtain finite-dimensional Lie algebras.

ABJM theory appears as a strong candidate for the theory of M2 branes [2], [17], [3], [37], [5], [6]. But if we mass-deform ABJM theory, we can not see $S^3$ as a classical solution to the field equations, but only the $S^2$ base-manifold [13]. ABJM theory might describe all aspects of M2 branes correctly as a quantum theory. However it is hard to use ABJM theory to deconstruct the M5 brane wrapped on $S^3$ by fluctuation analysis, unless $S^3$ is a classical solution to the M2 brane theory. So we wish to seek a theory where $S^3$ is a classical solution. If $S^3$ is a solution to quantum ABJM theory, then perhaps one may think on this new theory as a quantum effective theory of ABJM theory.

In this paper we will derive the D4 brane Lagrangian from ABJM theory. More specifically we will obtain D4 wrapped on $S^2$ base-manifold of $S^3/\mathbb{Z}_K$. We will obtain the same answer including the right value on the Yang-Mills coupling constant [1],

$$g_{YM}^2 = 4\pi^2 \frac{R}{K}$$

as if we had dimensionally reduced M5 brane on the fiber of $S^3/\mathbb{Z}_K$. For this it is necessary to identify $K$ as the Chern-Simons level in ABJM theory. The emergence of the M5 brane coupling constant from ABJM theory is a remarkable result. It gives us hope that M5 brane physics can be extracted from ABJM theory.

In [13] a single D4 is obtained from ABJM theory. As this is a free theory we do not have a coupling constant. If nevertheless we would identify the overall factor multiplying the Lagrangian as a coupling constant, we would realize that a rescaling of the fields would change that overall constant. We can only determine the coupling constant if we require the gauge field to be normalized to satisfy the Dirac charge quantization condition. But this does not seem to be taken care of in [13], and we will not attempt this here either. Instead we will deconstruct non-Abelian D4 theory where we can safely identify $g_{YM}$.

In section 2 we study the three-algebra of functions on $S^3$ modulo an error of order $1/N$. In section 3 we write down the D4 brane action and Abelian M5 and relate their coupling constants. In section 4 we write down ABJM theory with its mass deformation. In section 5 we obtain the Bogomolnyi equation for supersymmetric vacua. In section 6 we deconstruct D4 from ABJM and obtain (1). In section 7 we discuss a Lagrangian for a selfdual three-form. In section 8 we partly derive from ABJM, and partly guess, the gauge variations and supersymmetry variations for what we believe is multiple M5 theory.

After the main part of this paper had been completed, a paper [30] appeared which appears to have conceptual overlaps with our work. Other recent works
2 Three-algebra

We can describe ABJM theory as a usual gauge theory. We have a Lie algebra associated with the gauge group, and the fields transform in certain representations of the gauge group. Three-algebra is not needed to describe ABJM theory. On the other hand it is possible to describe ABJM theory in the language of hermitian three-algebra [10], [11]. One realization of three-algebra is the Nambu three-bracket. This realization is hard to come by if one uses the Lie algebra language. To each three-algebra there is an associated ABJM theory. Only certain Lie algebras give ABJM theories [7], [11].

2.1 Three-product, Nambu-bracket and dimensional reduction

We will assume that we have some three-associative three-product,

\[(T^a T^b T^c) d T^d = T^a (T^b T^c d) T^e = T^a T^c (T^b d T^c)\],

where \(T^a\) denote elements in a three-algebra. We assume the existence of a conjugation, that is an operation \('*\'\) that squares to one. We denote by \(T^a = (T^a)^*\) the conjugate elements, and we thus have \((T^a)^* = (T^a)^{**} = T^a\). We define the three-bracket of three three-algebra elements \(T^a, T^b, T^c\) as

\[\begin{align*}
[T^a, T^b; T^c] &= T^a T^c T^b - T^b T^c T^a
\end{align*}\]  

which is a map from three three-algebra elements into a new three-algebra element. As seen from the definition, the three-bracket is complex anti-linear in its third entry. At this stage we do not make any assumption as to whether the conjugate elements \(T^a\) belong to the three-algebra or not. They may or may not be related to elements in the three-algebra. We do not include the conjugate elements as additional possibly independent elements in the set of three-algebra elements. That is the reason we choose to denote the three-bracket as \([T^a, T^b; T^c]\). This is the same notation as used in [36], but other notations for the same bracket has also appeared in the literature. It can be shown that (2) satisfies the hermitian fundamental identity

\[\begin{align*}
[[T^a, T^b; T^c], T^d; T^f] &= [T^a, [T^b, T^c; T^d; T^f]; T^e] + [T^a, [T^b, T^c; T^d; T^f]; T^e] \\
&- [T^a, [T^b, T^c; T^d; T^f]; T^e].
\end{align*}\]  

if conjugation acts on the three-product as

\[(T^a T^b)^* = T_b T^c T_a.\]

Then conjugation acts on the tree-bracket with a minus sign as

\[\begin{align*}
[T^a, T^b; T^c]^* &= -[T_a, T_b; T_c]
\end{align*}\]

\[\text{Note: The most commonly used notation appears to be } [T^a, T^b; T^c] \text{ for this three-bracket. This notation makes indices on the left and right-hand sides of equation (2) be positioned at the same level. This notation is confusing in this context though. First it is incompatible with assuming the bracket is complex anti-linear in the third entry. Second the notation makes the notion of three-algebra elements obscure as the } T_a \text{ are not elements of the algebra.}\]
We also find that the following Leibniz rule is obeyed,

\[
[T^a T^b, T^c; T^f] = [T^a, T^c; T^f] T^b T^a + T^a [T^c, T^f] T^b + T^a T_c [T^b, T^c; T^f].
\] (5)

Under certain circumstances the three-product realization of the three-bracket can also be expressed as

\[
[T^a, T^b; T^c] = T_c [T^a, T^b] + [T^a, T_c] T^b - [T^b, T_c] T^a
\] (6)

where a quantity like \([T^a, T^b] \) by itself is generically not defined as this involves only two elements, but quantities like \(T_c [T^a, T^b] \) may be well-defined in terms of the three-product. One realization of the three-product is by means of matrices. In this case \(T_a = (T^a)^\dagger\) is given by the hermitian conjugate matrix, and \([T^a, T^b] \) is a well-defined matrix commutator as long as we restrict ourselves to \(N \times N\) square matrices. If we assume generic \(N \times M\) matrices \(T^a\) and \(M \times N\) hermitian conjugate matrices \(T^a\), there is generically no notion of a product \(T^a T^b\) but we may still consider products of matrices on the alternating form \(T^a T^b T^c T^d \ldots\).

One solution to the fundamental identity (3) is provided by the Nambu bracket,

\[
\{T^a, T^b; T^c\} = \hbar \{T^a, T^b, T_c\},
\]

\[
\{T^a, T^b, T_c\} = \sqrt{g} \epsilon^{\alpha \beta} \partial_\alpha T^a \partial_\beta T^b \partial_\gamma T^c
\]

where

\[
\sigma^\alpha = (\sigma^m, \sigma^3)
\]

are three coordinates on \(\mathbb{R}^3\) with metric \(ds^2 = g_{\alpha \beta} d\sigma^\alpha d\sigma^\beta\) and the determinant of the metric is denoted as \(g\). We use the convention where \(\epsilon_{123} = 1\) and indices are rised by the inverse metric, \(g^{\alpha \beta}\).

If we restrict ourselves to functions on the form

\[
T^a = e^{i\sigma^3} \bar{T}^a (\sigma^m)
\]

the Nambu three-bracket reduces as

\[
[T^a, T^b; T^c] = -i\hbar \sqrt{G} \epsilon^{\alpha \beta} \partial_\alpha T^a \partial_\beta T^b \partial_\gamma T^c - \{T^b, T_c\} T^a - \{T^a, T_c\} T^b.
\] (8)

where \(\{T^a, T^b\} = \sqrt{g} \epsilon^{\alpha \beta} \partial_\alpha T^a \partial_\beta T^b\) and the determinant is denoted as \(G\). We will refer to this as 'dimensional reduction' of the Nambu bracket to Poisson brackets. We note that (8) is on the same form as (6). This is surprising since the Nambu bracket was defined with no mention of a three-product. The structure of the Nambu bracket appears to be quite different from the three-product structure of the three-bracket defined in (2).

Matrix commutator is mapped isomorphically into star-commutator. To lowest order in the purely imaginary two-dimensional non-commutativity parameter \(\epsilon\), we have

\[
[T^a, T^b] \cong \epsilon \{T^a, T^b\}.
\] (9)
where we have the matrix commutator in the left-hand side. We will let

$$\epsilon = -ih\sqrt{\frac{G}{g}}$$

(10)

where $h$ is the real three-dimensional non-commutativity parameter. The non-commutativity parameter $\epsilon$ must be purely imaginary, because only then we find the same minus signs on both sides of (9) under conjugation. On the LHS we get $[T^a, T^b] = -[T_a, T_b]$ because hermitian conjugation acts on a product of matrices as $(T^a T^b)^\dagger = T_b T_a$, on the RHS we get $(i\{T^a, T^b\})^* = -i\{T_a, T_b\}$ because of the factor of $i$. For the Poisson bracket conjugation acts just as expected, $\{T^a, T^b\}^* = \{T_a, T_b\}$. Likewise for the Nambu bracket we have $\{T^a, T^b, T^c\}^* = \{T_a, T_b, T_c\}$. Under dimensional reduction it is mapped into (8) divided by $h$. If we take the complex conjugate it appears like we would have $\{T^a, T^b, T^c\}^* = -\{T_a, T_b, T_c\}$, but that is not true. We are then forgetting that the form of the dimensionally reduced Poisson brackets in (8) depend on the phase factor $e^{i\sigma^3}$ in the $T^a$’s. Under complex conjugation of the Nambu bracket, these phase factors are conjugated into $e^{-i\sigma^3}$, whereby we get dimensionally reduced Poisson brackets with opposite sign. Taking this additional sign change into account, everything is consistent with assuming that $\{T^a, T^b, T^c\}^* = \{T_a, T_b, T_c\}$.

2.2 Star-three-product

There does not seem to exist a deformation of the Nambu bracket on the form

$$[T^a, T^b; T^c] = \hbar\{T^a, T^b, T^c\} + O(h^2)$$

which still satisfies the fundamental identity [31]. Only the linear term, which is the Nambu bracket, satisfies the fundamental identity. To go beyond this, we include additional terms at linear order, on the form

$$[T^a, T^b; T^c] = \hbar \left( \{T^a, T^b, T^c\} - \mathcal{T}_c \{T^a, T^b, \bullet\} + 2T^a \{T^b, T^c, \bullet\} \right) + O(h^2)$$

for certain coefficients $a$ and $b$. An associative star-three-product that gives a star-three-bracket on form above, was found in [15] and is given by

$$T^a \ast T_c \ast T^b(\sigma) = \lim_{\sigma \to \sigma' \to \sigma''} \exp \left\{ \frac{\hbar}{2} \frac{g}{\sqrt{G}} \frac{\sqrt{G}}{\sqrt{g}} \left( \frac{\partial_2 \partial_2' \partial_2'' + \partial''_2 \partial'_2 \partial_2'' - \partial''_2 \partial''_2 - \partial_2 \partial_2''}{\mathcal{T}^a(\sigma) \mathcal{T}^b(\sigma')} \right) \right\}$$

The outer derivatives $\partial_2''$ act on the next star-three-product by which this star-three-product may be three-multiplied, for example as $(T^a \ast T_c \ast T^b) \ast T_e \ast T^d$ but still we have some outer derivatives surviving, no matter how many nested star-three-products we consider. Due to these outer derivatives we do not get a closed three-algebra on any finite or infinite set of harmonic functions or any other basis of functions. We must extend the set of generators from functions $T^a$ to function-differential-operators on the form $\mathcal{T}_A = T^a \otimes D_A$. On a three-torus with euclidean metric, we may take generators as $\mathcal{T}_M = T^m \otimes D_M$ with

$$T^m = e^{im_x \sigma^1}.$$
where $m_\alpha$ and $M_\alpha$ are integers. We know how complex conjugation acts on $\mathcal{T}^m$. We shall of course have $(\mathcal{T}^m)^* = e^{-im\sigma}$. Slightly more tricky is the complex conjugation of $\mathcal{D}_M$. It should not be confused with hermitian conjugation with respect to the natural inner product on function space, where we have $\partial^\dagger = -\partial$. Here we rather shall take $\partial^\dagger = \partial$. As motivation for this, let us consider the complex conjugation of $\partial e^{im\sigma}$. This is given by $\partial e^{-im\sigma}$ and not by $(-\partial)e^{-im\sigma}$.

To summarize then, we shall have

$$
(T^m_M)^* \equiv T^{m^*}_M \equiv e^{-im\alpha} \otimes e^{-\frac{i}{2}M_\alpha \phi_{\text{out}}}.
$$

With these assignments we obtain

$$
T^m_M \ast T^p_p \ast T^n_n = e^{\frac{i}{2}\varphi(m,M,n,P)}T^{m+n-p}_{M+N-P} X_M^N,
$$

where

$$
\varphi(m,M,n,N,P) = \{m,n,p\} - M(n-p) - N(m-p) - P(m+n),
$$

$$
X_M^N = -\sqrt{g} e^{\frac{\alpha}{2}} \left( n_\alpha p_\alpha - m_\alpha p_\alpha + m_\beta n_\beta \right).
$$

Here

$$
\{m,n,p\} = -\frac{\hbar}{2} (e^{\frac{\alpha}{2}} m_\alpha n_\beta p_\gamma).
$$

If we choose $\hbar = \frac{4\pi}{N}$, then the structure constants will be invariant under $m_\alpha \to m_\alpha + N$. If we act with these outer derivatives only on functions $\mathcal{T}^m$ for $m = 0, \ldots, N-1$, then we see that $\mathcal{D}_M = \mathcal{D}_M \ast \mathcal{D}_{M+N}$. In that case, we have a finite-dimensional three-algebra generated by $\mathcal{T}^m_M$ where both $m_\alpha$ and $M_\alpha$ run over the finite set $\{0, \ldots, N-1\}$. Being finite-dimensional, it ought to have a matrix realization in the classification of [7].

To get an idea how one may obtain an invariant trace form, it seems natural to try to up-lift the situation in two dimensions. In two dimensions, an invariant trace form can be constructed using the Leibniz rule

$$
[T^a, T^b, T^c] = [T^a, T^c] T^b + T^a [T^b, T^c]
$$

of a (star- or matrix-) commutator. Then we can take the trace

$$
\text{tr}([T^a T^b, T^c]) = \text{tr}([T^a, T^c] T^b) + \text{tr}(T^a [T^b, T^c])
$$

and by the cyclicity of the trace the LHS vanishes, and we obtain the invariance condition for the trace form $(T^a, T^b) = \text{tr}(T^a T^b) = \int T^a \ast T^b$. In three dimensions the analogous thing to consider would be a trace form of three elements,

$$
\langle T_M^a, T_N^b, T_P^c \rangle = \int \frac{d^3 \sigma}{(2\pi)^3} T_M^a \ast T_N^b \ast T_P^c.
$$

The invariance condition

$$
\langle [T_M^a, T_R^b; T_S^c], T_N^a \ast T_P^b \rangle + \langle T_M^a, [T_N^b, T_S^c; T_R^b], T_P^a \rangle + \langle T_M^a, T_N^b, [T_P^a, T_R^b; T_S^c] \rangle = 0
$$
follows from the Leibniz rule together with the fact that
\[
\langle [T^m_M, T^n_N, T^p_P] \rangle = 0
\]
The latter property follows from that the integrand, which is a three-bracket, is
total derivative.\footnote{We note that for three matrices we do not seem to have a corresponding property,
\[
\langle [T^a_c, T^b_c, T^c_c] \rangle = \text{tr}(T^a_c T^b_c - T^b_c T^a_c) \neq 0.
\] It now seems impossible to map our star-three-product isomorphically into matrices.}

But it is not desired to work with a trace form with three entries if we want
to use it in a Lagrangian. This is a first signal that something bad could be
going on. We could try and define an invariant inner product of two elements as
\[
\langle T^a_A, T^b_B \rangle = \langle T^a_A, T^b_B, 1 \rangle.
\]
But this inner product does not look very natural. Anyway it is invariant, so
let us elaborate on this inner product a bit more. Explicitly on the three-torus
we obtain
\[
\langle T^m_M, T^n_N \rangle = e^{i \frac{\hbar}{2} (M^\alpha n^\alpha - N^\alpha m^\alpha)} \delta_{m-n,0}.
\]
Let us now compute the inner product for an interaction term. We start by
computing
\[
\langle T^m_M, T^n_N : T^p_P, T^q_Q \rangle = e^{i \frac{\hbar}{2} (M^\alpha n^\alpha - N^\alpha m^\alpha)} \delta_{m-n,0}
\]
and through a cancellation, there is no piece surviving in the exponent which
is antisymmetric under exchange of the pair \(m, M\) with \(n, N\), such as \{m, n, p\}. These terms all cancel. This means that
\[
\langle [T^m_M, T^n_N]: T^p_P, T^q_Q \rangle = 0
\]
and consequently all interaction terms in ABJM theory that one would build
based on star-three-product using this (trivially) invariant inner product, will
vanish and we end up with just a free theory.

As we will see later, all is not lost however. Star-three-product formalism
is still a very useful tool to describe ABJM theory. But we will need to use a
different inner product, as will be presented in Eq (11). This inner product is
trace invariant if we restrict ourselves to generators on the form of Eq (7).

2.3 Truncation

If we assume that \(T^a = e^{i r^3} \tilde{T}^a\) (dimensional reduction), then we get [15]
\[
(T^a \ast T^c \ast T^b) \cdot 1 = T^a \ast_2 T^c \ast_2 T^b,
\]
\[
(T^a \ast T^b \ast T^c \ast T^d) \cdot 1 = T^a \ast_2 T^c \ast_2 T^b \ast_2 T^d.
\]
Here we indicate by acting on the constant function 1 that all differential op-
erators are dropped from the star-three-product. On the right-hand side we
have the two-dimensional star-products. We will refer to this procedure as ‘di-
 dimensional reduction’ plus ‘truncation’. By dimensional reduction we refer to
that we reduce the star-three-product to two consecutive star-products in two
dimensions. By truncation we refer to the action on 1. This is a consistent
truncation only after dimensional reduction, in the sense that we end up with
an associative star-product. If we do the truncation on the star-three-product
then we loose associativity. But associativity is recovered after dimensional
reduction. We have the following commuting diagram

\[
\begin{array}{ccc}
\text{star-three-product} & \rightarrow & \text{truncated star-three-product} \\
\downarrow & & \downarrow \\
\text{reduced star-three-product} & \rightarrow & \text{star-product}
\end{array}
\]

where horizontal arrows mean truncation and vertical arrows mean dimensional
reduction. Only the upper left and lower right corners correspond to associative
products. The other two corners do not. The reduced star-three-product refers
to what we get by dimensionally reducing the star-three-product without acting
on 1 to kill the derivatives. This dimensionally reduced product is not associative
in the two-dimensional sense. These two non-associative corners can be thought
of as intermediate steps between the associative three- and two-dimensional
star-products respectively.

The star-three-bracket is not totally antisymmetric due to the outer deriva-
tives. If we could kill these, by for instance acting on the constant function which
is equal to 1, then we obtain a totally antisymmetric truncated three-bracket,
which however in general will not satisfy the hermitian (or real) fundamental
identity.

If we assume that all three-algebra elements carry the same phase factor
along the fiber according to Eq (7), then the remarkable thing happens that the
truncated three-bracket which is totally antisymmetric three-bracket, does sat-
isfy the hermitian fundamental identity. The quick way to see this is by expand-
ing out the truncated three-bracket in terms of star-((two-))product commutators.
(This expansion is presented in Eq (8) to first order in the non-commutativity
parameter.) We can map the star-commutators to matrix commutators and by
expanding out these commutators, we realize that the three-bracket we have is
nothing but the standard ABJM three-bracket of matrices (that is, on the form
of Eq (2)) and of course this three-bracket satisfies the hermitian fundamental
identity. It is not totally antisymmetric from a two-dimensional view-point
where the three-algebra elements are functions \( \overline{T}^a \) living on a two-dimensional
space (or isomorphically matrices \( \overline{T}^a \)). The remarkable thing is that the same
three-bracket is totally antisymmetric from the three-dimensional view-point
where the elements are on the form of Eq (7), \( T^a = e^{i \sigma^3 \overline{T}^a} \). If we restrict
ourselves to such three-algebra generators only, then we can use another more
useful inner product which then is also invariant. This inner product is presented
below in Eq (11).

If on the other hand we allow for more general functions, say \( T^{a,p} = e^{ip \sigma^3 \overline{T}^a} \)
for some unspecified integers \( p \), then we do not recover the standard ABJM
three-bracket in two-dimensions (unless all the \( p \)'s are the same), and the fun-
damental identity is lost. Moreover, the inner product (11) is no longer invari-
ant. The truncated and totally antisymmetric three-bracket does in general not
satisfy the hermitian fundamental identity.
2.4 On the $3/2$-scaling

For Chern-Simons levels $K = 1, 2$ the three-bracket becomes totally antisymmetric by means of monopole operators which give rise to certain identities [38]. It was also speculated in [38] that these identities which give rise to the totally antisymmetric three-bracket, could reduce the degrees of freedom to give us the $3/2$-scaling behavior. One may now speculate that for $K = 1, 2$, we may still use the truncated three-bracket since this is indeed totally antisymmetric. One may speculate that truncation corresponds to taking into account the effect of monopole operators. If that is true, then we may think on the inner product as being on the form

$$
\langle T^a_A; T^b_B \rangle = \int d^3 \sigma \sqrt{g} (T^a_A \cdot 1)(T^B_b \cdot 1).
$$

where the notation $T^a_A \cdot 1$ can be thought of as the action of the three-algebra generator $T^a_A$ on the constant function which is equal to 1 everywhere. In other words, $T^a_A \cdot 1 = T^a_0$. We see that the inner product only depends on equivalence classes $[T^a_A]$ where we identify any two generators $T^a_A \sim T^b_B$ if $a = b$. This defines a gauge equivalence since the variation within the equivalence class does not affect the inner product. This inner product is not invariant, at least not in any obvious way and at a classical level. This is due to extra phase factors that comes in as a consequence of the truncated outer derivatives. It is still true that the magnitudes of $\langle [T^m_M; T^n_N; T^p_P] \rangle$ and $\langle [T^m_M; T^n_N; T^p_P] \rangle$ as calculated using truncated inner product, agree. But they differ by their different phase factors.

If we ignore this problem of understanding the trace invariance, and just use this truncated inner product, then we may obtain the $3/2$-scaling of number of degrees of freedom. The dimension of the moduli space corresponds to the number of solutions to the equation $\langle [T^m_M; T^n_N; T^p_P]; [T^m_M; T^n_N; T^p_P] \rangle = 0$ (which in turn comes from the sextic potential in ABJM/BLG theory). This equation has a maximal set of solutions given by $T^m_M$ with $m_3 = 0$ but with no restriction on $M$. We may refer to such a maximal set as a maximal set of three-commuting generators, or as a Cartan sub-three-algebra. The dimension of the Cartan sub-three-algebra is $N^2$ (times $N^3$ from the $M$-index). The total number of generators is $N^3$ (again times $N^3$ from the $M$-index). The generator $T^m_M$ is gauge equivalent with $T^m_0$ for $M = 0$. We now get the $N^{3/2}$ scaling of the number of gauge inequivalent generators, where $N = N^2$ is the dimension of gauge inequivalent vacuum configurations (i.e. the Cartan modulo gauge redundances).

The usual ABJM theory whose three-algebra is realized by matrices, can be obtained from ABJM theory whose three-brackets are realized by star-three-products. To this end we consider the limit $K$ large. Here $K$ is an orbifolding of the fiber-direction, when the three-manifold is viewed as a fiber bundle. In our three-torus example, if we choose $\sigma^3$ as the fiber direction, the orbifold identification will be

$$
\sigma^3 \sim \sigma^3 + \frac{2\pi}{K}.
$$

\footnote{It is possible that also star-three-products can be mapped into matrix multiplications, but we will not attempt to show this in this paper.}
In our finite truncation, we considered harmonics $T^m = e^{im_\sigma}$ in our finite truncation. Orbifolding restricts the possible set to those for which

$$m_3 = Km_3 + 1,$$

($m \in \mathbb{Z}$) which are those harmonics which obey the Bloch wave condition on the orbifolded three-torus [35],

$$T^m(\sigma^1, \sigma^2, \sigma^3 + 2\pi/K) = e^{i\frac{2\pi}{K}}T^m(\sigma^1, \sigma^2, \sigma^3).$$

Hence while $m_1, m_2 = 0, ..., N - 1$, we find that $m_3 = 0, ..., \lfloor \frac{N-1}{K} \rfloor$ where $[ \cdot ]$ denotes the integer part. So when $K = N$ or larger, we must choose $m_3 = 1$. For those 'large' values of $K = N, N + 1, ...$ the star-three-product reduces to a star-product on the base-manifold (the two-torus spanned by $\sigma^1$ and $\sigma^2$). This implies that the star-three-product theory becomes equivalent to ABJM theory when $K = N, N + 1, ...$.

Let us now compute the dimension of the Cartan $N$ and the number of three-algebra generators $D$ for generic $K$ and $N$. Let us ignore the $M^a$ indices from now on, these being just a gauge redundancy. Since $\partial_3 T^m \neq 0$ when orbifolding, we must choose Cartan generators with $m_3 \neq 0$. Let us choose the Cartan generators such that $m_{\underline{a}} = (m, 0, Kn + 1)$. Then we obtain [15]

$$N = N \left( \lfloor \frac{N-1}{K} \rfloor + 1 \right),$$

$$D = N^2 \left( \lfloor \frac{N-1}{K} \rfloor + 1 \right).$$

For large $N$ and large $K$ we then get

$$D = K^{1/2}N^{3/2}$$

which agrees with the result in [6] for large t Hooft coupling $\lambda = N/K$. When $\lambda < 1$ or in other words when $K = N$ or larger, the above formula breaks down and we get instead

$$D = N^2.$$

In [6] the number $N$ is the number that appears in the gauge group of ABJM theory as $U(N) \times U(N)$. But also $N$ in that approach coincides again with the dimension of the moduli space. This suggests that our effective theory which is characterized by an integer $N$, is the quantum effective theory of ABJM theory with gauge group characterized by the integer $N$. In usual matrix realization ABJM theory we consider fixed gauge group $U(N) \times U(N)$ and vary the Chern-Simons level $K$ say. The common wisdom is that $N$ counts the number of M2 branes, and we agree on that. However in star-three-product realization of ABJM theory it may appear more natural to keep $N$ fixed and vary $K$ say. But it is then that we get a quadratic type of behavior on the number of M2 branes as a function of $N$ for $K = 1, ..., N - 1$. On the other hand, for $K = N, N + 1, ...$ the integer numbers $N$ and $N$ coincide, and both count the number of M2 branes.
2.5 Tensor product of three-algebras

In order to deconstruct non-Abelian D4 and M5 brane theories, we need to consider a tensor product of two three-algebras. We thus consider a tensor product $A \otimes B$ of two three-algebras $A$ and $B$. The generators of $A \otimes B$ are

$$ T^{aa'} = T^a \otimes T^{a'} $$

where $T^a \in A$ and $T^{a'} \in B$. By repeated use of the abstractly defined three-bracket as in (2) where the generators are three-multiplied by some (yet unspecified) three-multiplication, we can express the tensor-product three-bracket in the form

$$ [T^{aa'}, T^{bb'}; T^{cc'}] = [T^a, T^b; T^c] \otimes T^{aa'} + T^{b'} T^a \otimes [T^{aa'} , T^{bb'}; T^{cc'}] \quad (13) $$

and we may further expand the three-bracket as

$$ [T^{aa'}, T^{bb'}; T^{cc'}] = [T^{aa'}, T^{b'}; T^{cc'}] - [T^{bb'}, T^{c'}; T^{aa'}] \quad (14) $$

This expression is still defined by use of three-multiplication among the three elements involved. So for instance a commutator of just generators would have been ill-defined, but here we have a multiplication by a third element so we may use three-multiplication to carry out these products.

One class of hermitian three-algebras has associated Lie algebras $U(N) \times U(N)$. These three-algebra generators also generate $U(N)$ Lie algebra [25]. The main example is $U(2) \times U(2)$ which has the same three-algebra as $SU(2) \times SU(2)$. As the associated three-algebra generators $T^a = (\sigma^i, i)$ generate $U(2)$ Lie algebra, their tensor products generate $U(4)$ Lie algebra. We now ask what is the corresponding three-algebra that these three-algebra generators generate? The answer is provided by [25]. The three-algebra is the one that comes with the associated Lie algebra $U(4) \times U(4)$. In general when we take the tensor product of $U(N_A) \times U(N_A)$ and $U(N_B) \times U(N_B)$ three-algebras, we get $U(N_A N_B) \times U(N_A N_B)$ three-algebra. A similar result appears to have been reached in [26]. It will be interesting to extend this to tensor products of any two ABJM gauge groups, and to understand what Lie algebra the three-algebra generate.

2.6 Three-algebra of a three-sphere

An example of a real three-algebra with complex generators is $SO(4)$ with three-algebra generators $T^i = (\frac{1}{2} \sigma^i, \frac{i}{2} I)$ where $\sigma^I \sigma^J = \delta^{IJ} + i \epsilon^{IJK} \sigma^K$. These generate the real three-algebra

$$ [T^i , T^j ; T^k] = -\frac{1}{2} \epsilon^{ijk} T^l. $$

There is no geometrical interpretation of the matrix realization of this algebra. If we realize the same algebra by the Nambu bracket on $S^3$, then the $T^i$ are real-valued coordinates in $\mathbb{R}^4$ describing the embedding of a round $S^3$. But this stands in conflict with the fact that $T^i$ as realized by matrices are not all hermitian. One may double the size of the matrices and consider hermitian $4 \times 4$ gamma matrices $\gamma^i$ whose off-diagonal blocks are $T^i$ and $T_i$ respectively. But these doubled matrices do not close into an algebra, but we rather get something
like \([\gamma^i, \gamma^j, \gamma^k] \sim \epsilon^{ijkl}\gamma^l\). The presence of \(\gamma_5\) means that this is not a closed algebra over the real (or complex) numbers.

These troubles may be an indication that we shall perhaps not try to realize \(S^3\) by matrices at all. Matrices are excellent tools to realize two-manifolds [19], but they do not seem to be suited for realizing three-manifolds. We may use matrices to realize a two-dimensional base-manifold of a three-dimensional fiber-bundle, but it seems we can not realize the whole three-manifold by matrices.

Let us illustrate further how this works in our \(SO(4)\) example. Let us define

\[
G^a = T^a + iT^b, \\
G^1 = T^1 + iT^2, \\
G^2 = T^3 + iT^4.
\]

These \(G^a\) are nothing but the GRVV matrices [16], [18] for rank \(N = 2\). But instead of four independent hermitian matrices, we just have three independent since \(G^1\) is hermitian, so we only have \(G^1, G^2\) and \(G_2\) as independent. We have the three-algebra (GRVV algebra)

\[
[G^a, G^b; G^c] = -2\delta_{cd}G^d
\]

But since the matrix realization of this algebra only has three independent real generators it misses out some part of the \(S^3\) geometry. In fact, and as we will explain shortly, the matrices only realize the fuzzy \(S^2\) base-manifold of the \(S^3\) viewed as a circle bundle [13].

If we expand the three-bracket as (6) we see that the GRVV algebra reduces to an oscillator algebra,

\[
[G^a, G^b] = 0, \\
[G^a, G_b] = \delta^n_b
\]

This in turn implies that we have an \(SU(2)\) algebra induced from the GRVV algebra, generated by

\[
J_I = G^a(\sigma_I)_a^b G_b.
\]

However the oscillator algebra only has the infinite-dimensional matrix realization. For a single oscillator this matrix is \((G^1)_{mn} = \sqrt{\delta_{m,n-1}}\).

The GRVV algebra also has finite-dimensional matrix realizations. For generic \(N\), the GRVV matrices, as obtained independently by the authors of [16] and [18] respectively, and which were further studied in [17], are given by

\[
G^1 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \sqrt{1} & \cdots & 0 \\
0 & 0 & \cdots & \sqrt{N-1} \\
0 & \sqrt{N-1} & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix},
\]

\[
G^2 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & \sqrt{1} & \cdots & 0 \\
0 & 0 & \cdots & \sqrt{N-1} \\
0 & \sqrt{N-1} & \cdots & 0 \\
0 & 0 & \cdots & 0
\end{pmatrix}.
\]

Again \(G^1\) is hermitian. We also have the radius constraints

\[
G^a G_a = (N-1)\mathbb{I},
\]
\[ G_a G^a = N (I - E) \]

where \( I = \text{diag}(1,1,\ldots,1) \) and \( E = \text{diag}(1,0,\ldots,0) \). The \( G^a \) realize the above \( SO(4) \) three-algebra for any \( N \).

Let us consider four real coordinates \( T_i \) in euclidean \( \mathbb{R}^4 \) describing the embedding of a round \( S^3 \). These are thus subject to the \( SO(4) \) three-algebra and the radius constraint,

\[
\{ T^i, T^j, T^k \} = \frac{1}{R} \epsilon_{ijkl} T^l, \quad T^i T^i = R^2
\]

To make the relation with the above \( SO(4) \) algebra transparent, we define complex coordinates

\[
G^1 = T^1 + i T^2, \quad G^2 = T^3 + i T^4
\]

in terms of which we have

\[
\{ G^a, G^b, G^c \} = \frac{4}{R} \delta^{ab} G^d, \quad G^a G^a = R^2
\]

In order to relate the functions \( G^a \) with the matrices \( G^a \), we define a three-bracket

\[
[G^a, G^b; G^c] = \hbar \{ G^a, G^b, G^c \}.
\]

Then

\[
[G^a, G^b; G^c] = \frac{4\hbar}{R} \delta^{ab} G^d, \quad G^a G^a = \frac{R^2}{\sqrt{N-1}} G^a
\]

We will keep \( R \) fixed.

If we define an equivalence

\[ G^a \sim e^{i\psi} G^a \]

then we have the isomorphism between such equivalence classes and matrices,

\[ [G^a] \cong \frac{R}{\sqrt{N-1}} G^a \]

if we take

\[ \hbar = -\frac{R^3}{2(N-1)}, \quad (15) \]

We see that \( \hbar \to 0 \) as \( N \to \infty \). We get a classical sphere in the large \( N \) limit as expected. The reason we must consider equivalence classes is that the three-algebra is invariant under \( G^a \to e^{i\psi} G^a \).

We may parametrize the embedding coordinates \( G^a \) as

\[ G^1 = \frac{R}{\sqrt{2}} \sqrt{1 + \cos \theta} e^{i(\phi + \psi)}, \]

\[ G^2 = \frac{R}{\sqrt{2}} \sqrt{1 - \cos \theta} e^{i(\phi - \psi)}, \]

\[ G^3 = \frac{R}{\sqrt{2}} \sqrt{1 - \sin \theta} e^{i(\phi - \psi)}, \]

\[ G^4 = \frac{R}{\sqrt{2}} \sqrt{1 + \sin \theta} e^{i(\phi + \psi)}. \]
\[ G^2 = \frac{R}{\sqrt{2}} \sqrt{1 - \cos \theta} e^{i\psi} \]  

(16)

This parametrization makes the fiber-bundle structure manifest. It also enable a simple description of \( S^3/\mathbb{Z}_K \). We just make the identification \( \psi \sim \psi + \frac{2\pi}{K} \).

The length of the fiber is \( 2\pi R/K \). Locally there is no difference between \( S^3 \) and \( S^3/\mathbb{Z}_K \). We will define

\[ G^a = e^{i\psi} \tilde{G}^a. \]

Then \( \tilde{G}^a \) is a representative in \([G^a]\).

Let us define

\[ g^a = \frac{R}{\sqrt{N - 1}} G^a \]

We then have the isomorphism

\[ g^a \cong \tilde{G}^a. \]

If we normalize the trace form as \( \langle g^a, g^b \rangle = \langle \tilde{G}^a, \tilde{G}^b \rangle = \delta^a_b \), we have the realizations

\[ \langle g^a, g^b \rangle = \frac{2}{R^2 N} \text{tr}(g^a g_b), \]

\[ \langle \tilde{G}^a, \tilde{G}^b \rangle = \frac{1}{2\pi R^2} \int d\theta d\varphi \sin \theta \tilde{G}^a \tilde{G}^b. \]

and we can make the identification

\[ \frac{1}{N} \text{tr} \cong \frac{1}{\pi R^2} \int d\theta d\varphi \left( \frac{R}{2} \right)^2 \sin \theta. \]  

(17)

If we define the Hopf projection

\[ X^I = G^a (\sigma^I)_a^b G^b \]

where \( \sigma^I \) are the Pauli matrices, or explicitly

\[ X^1 = G^1 G_2 + G^2 G_1, \]
\[ X^2 = -iG^1 G_2 + iG^2 G_1, \]
\[ X^3 = G^1 G_1 - G^2 G_2 \]

then we get

\[ X^1 = R^2 \sin \theta \cos \varphi, \]
\[ X^2 = -R^2 \sin \theta \sin \varphi, \]
\[ X^3 = R^2 \cos \theta. \]

By using the Fierz identity

\[ (\sigma^I)_a^b (\sigma^I)_c^d = 2\delta_a^c \delta_b^d - \delta_a^b \delta_c^d \]

we find that

\[ \frac{1}{4R^2} dX^I dX^I = dG^a d\bar{G}_a - R^2 (d\psi + A_\varphi d\varphi)^2 \]
More explicitly we get
\[
\frac{1}{4R^2} dX^I dX^I = \frac{R^2}{4} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\]
\[
dG^a dG_a = \frac{R^2}{4} \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) + R^2 \left( d\psi + A_\varphi d\varphi \right)^2
\]
where
\[
A_\varphi = \frac{1}{2} (1 + \cos \theta)
\]
is the gauge field of a magnetic monopole of unit one. Its field strength is
\[
F_{\varphi\theta} = \frac{1}{2} \sin \theta
\]
and the integral over \( S^2 \) is
\[
\int d\varphi d\theta F_{\varphi\theta} = 2\pi.
\]
We will denote the coordinates by \( \sigma^m = (\sigma^m, \psi) = (\theta, \varphi, \psi) \) and the metric tensors by \( G_{mn} \) on \( S^2 \) and \( g_{\alpha\beta} \) on \( S^3 \). Their square root determinants are given by
\[
\sqrt{g} = \frac{R^3}{4} \sin \theta,
\]
\[
\sqrt{G} = \left( \frac{R}{2} \right)^2 \sin \theta. \quad (18)
\]
The radius of \( S^2 \) is \( \frac{R}{2} \), the radius of the fiber is \( R \) which is the same as the radius of \( S^3 \). This can also be confirmed by a computation of the volume of \( S^3 \),
\[
\int_0^{2\pi} d\psi \int_0^\pi d\theta \int_0^{2\pi} d\varphi \frac{R^3}{4} \sin \theta = (2\pi R) \left( 4\pi \left( \frac{R}{2} \right)^2 \right)
\]
The result is \( 2\pi^2 R^3 \) as is the volume of \( S^3 \) with radius \( R \), but it can be computed as the length of the fiber which is \( 2\pi R \) irrespectively of at which point on the base-manifold it is evaluated, times the volume of the base-manifold, which equals the volume of \( S^2 \) of radius \( \frac{R}{2} \).

2.7 Three-algebra basis on a fuzzy three-sphere

We would like to consider the algebra of functions on \( S^3 \). If we consider the infinite set of functions we can three-multiply together \( G^a \) and \( G_a \) by using the usual multiplication of two functions iteratively. If we want to consider a finite truncation we must use star-three-multiplication [15]. The important properties are associativity and that upon dimensional reduction it is mapped isomorphically into matrix multiplication. We may then consider functions
\[
\mathcal{T}^a = \left( G^{a_1} * G_{b_1} * G^{a_2} * ... * G^{a_{r-1}} * G_{b_{r-1}} * G^{a_r} \right) \cdot 1
\]
These all have a trivial dependence on the fiber given by the phase factor $e^{i\psi}$.
We may write
\[ T^\alpha (\theta, \varphi) = e^{i\psi} \overline{T^\alpha (\theta, \varphi)}. \]
Since these functions all have the same trivial dependence on the fiber, it is clear that these do not constitute a basis of all functions on $S^3$. In fact, these $T^\alpha$ already have the dimensional reduced form (7) and shall be associated with the $S^2$ base manifold. Since we thus consider both truncation and dimensional reduction, we may replace all star-three-products with two-dimensional star products for free. We may then turn to the isomorphic matrix realization where we have the basis elements
\[ T^{\vec{a}} = \sum_{r=1}^{N-1} G^{a_1} G^{a_2} \cdots G^{a_r} G^{b_{r-1}} G^{a_r} \]
for $r = 1, \ldots, N - 1$. This set is finite so we may count how many they are. We may express any element generated by this basis as a linear combination
\[ M = c_a G^a + c_{a_1 a_2}^{b_1} G^{a_1} G^{a_2} + \cdots + c_{a_1 \cdots a_{l-1}}^{b_1 \cdots b_l} G^{a_1} G^{a_2} \cdots G^{a_{l-1}} G^{a_l} + \cdots + c_{a_1 \cdots a_{N-1}}^{b_1 \cdots b_{N-2}} G^{a_1} G^{a_2} \cdots G^{a_{N-2}} G^{a_{N-1}}. \]  
We ask how many independent components $c_{a_1 \cdots a_{l-1}}^{b_1 \cdots b_{l-1}}$ we have. For commuting functions we find symmetrized indices $a_1, \ldots, a_l$ as well as $b_1, \ldots, b_{l-1}$. Hence we have $(l + 1)l$ components. But these are not all independent because of the sphere constraint $G^a G_a = R^2$. For matrices we only need to consider down traces since up traces are related to down traces by the sphere equation. For commuting functions down and up traces are the same, by just commuting the functions. We then need to remove the number of components in a down trace say. There are $l(l - 1)$ such components. We are left with $2l$ independent components in $c_{a_1 \cdots a_{l-1}}^{b_1 \cdots b_{l-1}}$. Summing them up we get
\[ \sum_{l=1}^{N-1} 2l = (N-1)N \]
components in total. We have $N \times (N-1)$ matrices, and indeed we may truncate the GRVV matrices to size $N \times (N-1)$ and then their conjugates will be of size $(N-1) \times N$. Then it is easy to see that $T^{\vec{a}}$ is an $N \times (N-1)$ matrix, and any $N \times (N-1)$ matrix can be expressed as a linear combination $c_{\vec{a}} T^{\vec{a}}$.

The GRVV matrices have real entries (though they are not hermitian). We may then restrict ourselves to real coefficients $c_{\vec{a}}$ and to $N \times (N-1)$ matrices with real entries. It is now clear that $T^{\vec{a}}$ (which equals the transpose of $T^{\vec{a}}$), is an $(N-1) \times N$ matrix. Ignoring the mismatch of $O(1/N)$, we may expand this matrix in the basis of $N \times (N-1)$ matrices as
\[ T^{\vec{a}} = \kappa_{\vec{a} \vec{b}} T^{\vec{b}}. \]
Clearly $T^{\vec{a}}$ must generate the three-algebra which is associated to $U(N) \times U(N)$ gauge group (modulo the $O(1/N)$ mismatch).

We have a basis of functions on a fuzzy $S^2$ (ignoring $1/N$ mismatch) corresponding to spherical harmonics with $\ell = 0, \ldots, N-1$. A basis for functions on the fuzzy $S^3$ is
\[ T^{\vec{a}} = e^{im\psi} \overline{T^{\vec{a}}}. \]
where \( m \) ranges over \([-N - 1, N - 1]\), that is \( m \) takes \( 2N - 1 \) different values. In particular when \( N = 2 \) we have the basis elements \( e^{\pm i\psi G^a} \) and \( e^{\pm i\psi \tilde{G}^a} \) which correspond to the complete set of embedding coordinates \( G^a \) and \( \tilde{G}^a \) of \( S^3 \) in \( \mathbb{R}^4 \).

The number of M2 branes should be given by the dimension of the moduli space. These are the three-commuting functions. One maximal set of three-commuting functions is given by \( \tilde{T} \). Since these only depend on two coordinates, these have vanishing three-bracket (and in particular vanishing Nambu bracket). The number of degrees of freedom should on the other hand be proportional to the dimension of the span of \( T_m \) where the matter fields are valued. The former is \( \mathcal{N} = N(N - 1) \), the latter is \( \mathcal{D} = N(N - 1)(2N - 1) \). Orbifolding the three-sphere restricts \( m \) to take \( [(2N - 1)/K] \) different values. When \( K = 2N \) or larger, we find \( \mathcal{D} \propto N^2 \).

### 3 The D4 brane

The theory of multiple M5’s should reduce to the theory of multiple D4’s upon compactification on a circle. We may also derive D4 from M2 by fluctuation analysis about a nontrivial vacuum solution. By so identifying these two D4 brane theories, we will be able to derive the M5 coupling constant (which is uniquely fixed) directly from the M2 brane theory.

#### 3.1 Single M5 reduced to single D4

Let us start by a single M5 which has \((2,0)\) supersymmetry in six dimensions. We assume eleven-dimensional gamma matrices \( \Gamma_M \) and \( \Gamma_I \) where \( M = 0, 1, 2, 3, 4, 5 \) and \( I = 6, 7, 8, 9, 10 \). We define \( \Gamma = \Gamma_{012345} \) and assume a supersymmetry parameter subject to

\[
\Gamma \omega = \omega
\]

The fermions in \((2,0)\) tensor multiplet are now subject to

\[
\Gamma \chi = -\chi
\]

We have the supersymmetry variations

\[
\delta \phi^I = i\bar{\omega} \Gamma^I \chi,
\]

\[
\delta B_{MN} = i\bar{\omega} \Gamma_{MN} \chi,
\]

\[
\delta \chi = \frac{1}{12} \Gamma^{MNP} \omega H_{MNP} + \Gamma^M \Gamma_{I} \omega \partial_M \phi^I
\]

and reduce on a circle, which means split \( M = (\mu, \psi) \) and ignore derivatives with respect to \( \psi \). We then get (for a precise definition of the dimensionally reduced fields, we refer to eq (20))

\[
\delta \phi^I = i\bar{\omega} \Gamma^I \chi,
\]

\[
\delta A_\mu = i\bar{\omega} \Gamma_\mu \Gamma \psi \chi,
\]

\[
\delta \chi = \Gamma^\mu \Gamma_{I} \omega \partial_\mu \phi^I + \frac{1}{2} \Gamma^{\mu\nu} \Gamma \psi \omega F_{\mu\nu}.
\]

but still we have the eleven-dimensional spinor quantities which are subject to unusual chirality conditions. From ten-dimensional point, \( \Gamma \psi \) is the natural
chirality matrix with respect to which we shall define chiralities of our spinors. Hence we like to use supersymmetry parameter \( \epsilon \) and spinor field \( \psi \) subject to chirality conditions

\[
\Gamma_\psi \epsilon = \epsilon,
\Gamma_\psi \psi = -\psi.
\]

Such spinors can be related to the previous ones by a unitary rotation

\[
\omega = U \epsilon,
\chi = U \psi
\]

where

\[
U = \frac{1}{\sqrt{2}}(1 + \sigma),
\sigma = \Gamma_{01234}.
\]

Here we have the property \( \sigma^2 = -1 \) and \( \sigma^\dagger = -\sigma \). If we also make a field redefinition

\[
\xi = \sigma \psi
\]

which changes the chirality condition as

\[
\Gamma_\psi \xi = \xi
\]

then in terms of these redefined spinors we have

\[
\begin{align*}
\delta \phi^I &= i\bar{\epsilon} \Gamma^I \xi,
\delta A_\mu &= i\bar{\epsilon} \Gamma_\mu \xi,
\delta \xi &= \frac{1}{2} \Gamma^{\mu\nu} \epsilon F_{\mu\nu} + \Gamma_\mu \epsilon \partial_\mu \phi^I.
\end{align*}
\]

Here we have been able to completely eliminate all \( \Gamma_\psi \) using the chirality condition of \( \xi \). Still we work with gamma matrices that anti-commute

\[
\{ \Gamma^I, \Gamma_\mu \} = 0.
\]

3.2 Multiple D4

Now we have obtained exactly the supersymmetry variations that we would derive if we reduce ten-dimensional Abelian super Yang-Mills to five dimensions. But of course it would be no more difficult to reduce non-Abelian super Yang-Mills. If we do, then we get the supersymmetry variations

\[
\begin{align*}
\delta \phi^I &= i\bar{\epsilon} \Gamma^I \xi,
\delta A_\mu &= i\bar{\epsilon} \Gamma_\mu \xi,
\delta \xi &= \frac{1}{2} \Gamma^{\mu\nu} \epsilon F_{\mu\nu} + \Gamma_\mu \epsilon D_\mu \phi^I + \frac{1}{2} \epsilon[\phi^I, \phi^J].
\end{align*}
\]

The challenge now is to see whether these variations can be derived from a proposed non-Abelian (2,0) theory.
The super Yang-Mills Lagrangian is given by

\[
\frac{1}{g_{YM}^2} \text{tr} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D^\mu \phi^I D_\mu \phi^I + \frac{1}{4} [\phi^I, \phi^J]^2 \right).
\]

where

\[
D^\mu \phi^I = \partial^\mu \phi^I + i [A^\mu, \phi^I],
\]
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i [A_\mu, A_\nu].
\]

Alternatively we have

\[
\text{tr} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} D^\mu \phi^I D_\mu \phi^I + g_{YM}^2 \frac{1}{4} [\phi^I, \phi^J]^2 \right).
\]

where

\[
D^\mu \phi^I = \partial^\mu \phi^I + ig_{YM} [A^\mu, \phi^I],
\]
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_{YM} [A_\mu, A_\nu].
\]

These two formulations are related by the field rescaling

\[
\phi^I \rightarrow g_{YM} \phi^I, \quad A_\mu \rightarrow g_{YM} A_\mu.
\]

### 3.3 The Yang-Mills coupling constant from M5

If we start with M five-brane action \[^{5}\]

\[
\frac{1}{4\pi} \int d^6 x \left( -\frac{1}{12} H^{MNP} H_{MNP} - \partial_M \phi^I \partial^M \phi^I \right)
\]

and dimensionally reduce it on a circle with circumference \(2\pi R\), define new dimensionally reduced fields as

\[
A_\mu = 2\pi R B_\mu \psi, \\
\phi^I = 2\pi R \phi^I
\]

then we get

\[
\frac{1}{4\pi^2 R} \int d^5 x \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} \partial^\mu \phi^I \partial_\mu \phi^I \right).
\]

We read off the Yang-Mills coupling constant

\[
g_{YM}^2 = 4\pi^2 R.
\]

\[^{5}\text{Here the spacetime index runs over } M = 0, 1, ..., 5 \text{ and } H_{MNP} \text{ is not selfdual. However the antiselfdual part decouples and is not part of the } (2, 0) \text{ tensor multiplet and does not occur in the supersymmetry variations. The decoupling of the antiselfdual piece of } H \text{ gets clear when we couple this action to a background } C \text{ field as in } [33] \text{ where only the selfdual piece of } H \text{ couples to } C. \text{ The decoupling of the antiselfdual piece is best understood by carrying out a holomorphic factorization of the partition function } [33], [34]. \text{ Of course the partition function for the M five-brane is just a holomorphic factor, and the action for the nonselfdual two form is a tool to obtain the M five-brane partition function via holomorphic factorization. Keeping holomorphic factorization in mind, it seems fair to say that this action can be used to describe a single M five brane.}\]
The coupling becomes infinite as \( R \to \infty \). It has been argued that the strong coupling limit of five-dimensional super Yang-Mills might be six-dimensional \((2,0)\) theory \([1], [28], [29]\).

Of course in this example the theory is free and we may not interpret \( g_{YM} \) as a coupling constant. But we can get non-Abelian super Yang-Mills theory essentially by replacing \( \partial_\mu \) by \( \partial_\mu + [A_\mu, \bullet] \) (and adding some more interaction terms). Doing that, we conclude that \( g_{YM} \) is the coupling constant. Also it is important to note that the Dirac charge quantization condition is inherited from six dimensions as

\[
\int d\psi \wedge dX^\mu \wedge dX^\nu H_{\mu \nu \psi} = \int dX^\mu \wedge dX^\nu F_{\mu \nu}
\]

as\(^6 \psi \in [0, 2\pi R] \) and the \( X^\mu \) are coordinates describing the embedding of some two-manifold in five dimensions.

4 Star-three-product BLG theory

It was noted in \([35]\), that BLG theory\(^7\) on an orbifold is equivalent with ABJM theory. Here we will extend that idea to our star-three-bracket. The star-three-bracket

\[
[T^a, T^b; T^c] = T^a * T^c * T^b - T^b * T^c * T^a
\]

is not totally antisymmetric. But the truncated three-bracket

\[
[T^a, T^b, T^c] \equiv [T^a, T^b; T^c] \cdot 1
\]

is indeed totally antisymmetric \([15]\), and this we indicate by using comma to separate the three entries, instead of a semi-colon. In the process we must also change the third entry to its conjugate element. To leading order this totally antisymmetric bracket is equal to the Nambu bracket,

\[
[T^a, T^b, T^c] = \hbar \{T^a, T^b, T^c\} + O(\hbar^2).
\]

Moreover, by restricting ourselves to generators of the form of Eq (7), this totally antisymmetric three-bracket will satisfy the hermitian fundamental identity. Since the three-bracket is already truncated, we can safely use the inner product (11) and this will be invariant on the restricted set of generators which are on the form of Eq (7).

The total antisymmetry implies that the three-algebra is real, since we can not separate \( T_a \) from \( T^a \) as these are related by antisymmetry of the three-bracket. This justifies using the notation \([T^a, T^b, T^c]\) for this real and totally

---

\(^6\)Here we work in a convention where \( \psi \) has mass dimension \(-1\) so that \( H_{\mu \nu \psi} \) will have the usual engineering mass dimension 3.

\(^7\)Usually BLG theory is thought of as ABJM theory with gauge group \( SO(4) \). But BLG theory is more than that. We also have BLG theories for the Nambu bracket. Here we extend this considerably by using truncated star-three-product bracket in place of the Nambu bracket, which enable us to use BLG for finite-dimensional three-algebras whose generators are on the form of Eq (7). These are not really new theories. They are the good old ABJM theories, though expressed from a three-dimensional vantage point. We could also just stick to the usual formulation of ABJM theory.
antisymmetric three-bracket. Real in the sense that this bracket is complex conjugate linear in all three entries. It does not mean that all the $T^a$ have to be real. It only means that $T^a$ must be expressible as a linear combination of the $T^a$, and we can surely find a linear combination of generators so they all become real.

Now since three-brackets in this ABJM Lagrangian are totally antisymmetric (from the three-dimensional viewpoint), we can rewrite the ABJM Lagrangian as a BLG theory [9], [8] with manifest $SO(8)$ symmetry. This is true for any Chern-Simons level $K$, and $SO(8)$ is only broken explicitly when $K > 2$ by the orbifold identification $\mathbb{C}^4/\mathbb{Z}_K$.

This means that supersymmetry enhancement in ABJM theory becomes a triviality when using the star-three-product formalism. If we use matrix real-ization of the gauge group $U(N)_K \times U(N)_{-K}$, we must use monopole operators to understand the supersymmetry enhancement for $K = 1, 2$ [37], [38].

There are some subtleties with applying the star-three-product formalism in ABJM and BLG theory though. First of all, we need a three-manifold on which we can define the star-three-product One way of generating such a three-manifold, is by mass deforming ABJM theory, so that it possesses a two-sphere vacuum solution. This two-sphere can then be interpreted as the base-manifold of $S^3/\mathbb{Z}_K$. But also, the identification of $K$ with the Chern-Simons level is not entirely obvious, even though it appears that Chern-Simons level $K$ corresponds M2 branes probing the $\mathbb{C}^4/\mathbb{Z}_K$ orbifold singularity [2]. The intersection of this orbifold and $S^3$ is of course the orbifolded $S^3/\mathbb{Z}_K$. The second subtlety is the isomorphism from functions being star-three-multiplied, to matrices and the gauge group $U(N) \times U(N)$. As we saw, we could only generate $N \times (N - 1)$ bifundamental matrices by multiplying together an odd number of GRVV matrices, and maybe that should be taken seriously as saying that the gauge group we get from $S^3/\mathbb{Z}_K$ really corresponds to $U(N) \times U(N - 1)$ rather than $U(N) \times U(N)$. This remains to be analyzed in more detail. In this paper we just ignore this discrepancy, and treat it like a $1/N$ correction.

Eventhough we may define our star-three-product over $S^3$, we can not really probe the $S^3$ structure as we are confined to the subset of three-algebra generators which are on the form of Eq (7). To really see the $S^3$ structure we would need to extend this to a complete set of functions on $S^3$, but this is not clear how to do and will presumably require the use of monopole operators.

By using triality of $SO(8)$ in BLG theory we may take the eight scalar fields $X^\alpha$ to transform as an $SO(8)$ Weyl spinor, the eight spinors $\psi_{\dot{\alpha}}$ transform as a Weyl cospinor, and the eight supersymmetry parameters $\epsilon^I$ as vector. We assume a real three-algebra with totally antisymmetric three-bracket, but may use a complex basis for the three-algebra generators. We define the covariant derivative as

$$D_\mu X^\alpha = \partial_\mu X^\alpha + [X^\alpha, T^c, T_d]A^d _\mu .$$

With these assignments, we have the $\mathcal{N} = 8$ supersymmetry variations

$$\delta X^\alpha = -i\epsilon_\beta \Gamma^{\beta \dot{\alpha}} ,$$

$$\delta \psi_{\dot{\alpha}} = \gamma^\mu \epsilon^I D_\mu X^\alpha \Gamma_{I \alpha \alpha} - \frac{1}{6} \Gamma_{K \alpha \beta} \Gamma^{I J \beta} \Gamma^{J \delta \alpha} \epsilon^K [X^\alpha, X^\delta, X_\gamma] ,$$

$$\delta A_\mu = i\epsilon_\beta \Gamma^{I \dot{\alpha} \beta} [\epsilon, \psi_{\dot{\alpha}}, X_\beta] .$$

21
Here
\[ \Gamma_I = \begin{pmatrix} 0 & \Gamma_{I\alpha\beta} \\ \Gamma_{I\alpha\beta} & 0 \end{pmatrix} \]
are $SO(8)$ gamma matrices, and $\gamma_\mu$ are $SO(1,2)$ gamma matrices.

If we decompose the matter fields as
\[ X^\alpha = \begin{pmatrix} Z^A \\ Z_A \end{pmatrix} \]
and
\[ \psi_\alpha = \begin{pmatrix} \psi^A \\ -\psi_A \end{pmatrix} \]
and let $X_\alpha = (X^\alpha)^*$ and $Z_A = (Z^A)^*$, one may check that the $N = 8$ supersymmetry variations reduce to $N = 6$ variations of ABJM type
\[
\begin{align*}
\delta Z^A &= -i\epsilon^{AB}\psi_B, \\
\delta\psi_A &= \gamma^\mu\epsilon_{AB}D_\mu Z^B - \epsilon_{AB}[Z^B, Z^C, Z_C] - \epsilon_{BC}[Z^B, Z^C, Z_A], \\
\delta A_\mu &= i\epsilon_{AB}\gamma_\mu[\psi_A, Z_B] - i\epsilon^{AB}\gamma_\mu[\psi, Z_B].
\end{align*}
\]
We have used scalar fields $X^\alpha$. Valued in three-algebra these have components
\[ X^\alpha = \begin{pmatrix} Z_a^AT^a_a \\ Z_{Aa}^AT^a_a \end{pmatrix}. \]
If the three-algebra is real, we include $T^a_a = (T^a)^*$ as generators of the three-algebra. But we also have stressed that $T^a_a$ are not independent but can rather be expressed as a linear combination of $T^a_a$,
\[ T^a_a = \kappa_{ab}T^b. \]
We may then write the scalar fields in the form
\[ X^\alpha = \begin{pmatrix} Z_a^AT^a_a \\ Z_{Aa}^AT^a_a \end{pmatrix}. \]
where $Z_{Aa} = \kappa_{ab}Z^b_a$. However this latter form is more subtle as it is no longer manifest in this notation that $Z_{Aa}T^a_a$ is the complex conjugate of $Z^a_aT^a_a$. This fact is very important as it means that we do not double the field content. The components of $X^\alpha$ are complex, but we do not double the number of independent components. We note that $\kappa_{ab}$ in star-three-product ABJM theory appears to play a role similar to that of a monopole operator in usual ABJM theory.

We may consider the triality map $(I, \alpha, \dot{\beta}) \mapsto (\dot{\alpha}, I, \beta)$. This maps the two half-gamma matrices
\[ \Gamma_{I\alpha\dot{\beta}}, \quad \Gamma^{I\dot{\alpha}\beta} \]
into
\[ \Gamma_{I\beta}\dot{\alpha}, \quad \Gamma^{I\beta\alpha} \]
if we define a cyclic symmetry $\Gamma_{I_{\alpha\beta}} = \Gamma_{\beta I_{\alpha}} = \Gamma_{\alpha I_{\beta}}$. We then find the supersymmetry variations

\[
\delta X^I = -i\bar{\epsilon}_\alpha \Gamma^I_{\dot{\alpha}\beta} \psi^\beta,
\]

\[
\delta \psi_\alpha = \gamma^\mu \Gamma_{I\alpha\dot{\alpha}} \epsilon^\dot{\alpha} D_\mu X^I - \frac{1}{6} (\Gamma_K \Gamma^I \Gamma_J)_{\alpha\dot{\alpha}} \epsilon^\dot{\alpha} [X^I, X^K, X_J],
\]

\[
\delta A_\mu = i\bar{\epsilon}_\alpha \gamma_\mu \Gamma^I_{\alpha\dot{\alpha}} [\bullet, \psi_\alpha, X_I].
\]

To compare with BLG theory as originally formulated in [9], we convert to eleven dimensional gamma matrices

\[
\Gamma^\mu = \gamma^\mu \otimes \Gamma,
\]

\[
\Gamma_I = 1 \otimes \Gamma_I
\]

Along with this, we also declare that $X_I = X^I$. We assume chirality conditions

\[
\Gamma \psi = -\psi,
\]

\[
\Gamma \epsilon = \epsilon.
\]

If we also shift the sign of the fermion $\psi \rightarrow -\psi$, then in terms of these gamma matrices the $\mathcal{N} = 8$ supersymmetry variations read

\[
\delta X^I = i\bar{\epsilon} \Gamma_I\psi,
\]

\[
\delta \psi = \Gamma^\mu \Gamma_I \epsilon D_\mu X^I + \frac{1}{6} \Gamma_{IJK} [X^I, X^J, X^K],
\]

\[
\delta A_\mu = i\bar{\epsilon} \Gamma^\mu \Gamma_I [\bullet, X_I, \psi].
\]

where we have noted the identity

\[
\Gamma_I \Gamma_J \Gamma_K - \Gamma_K \Gamma_J \Gamma_I = 2\Gamma_{IJK}.
\]

The gauge variations are most clearly expressed in Lie algebra language as

\[
\delta X^I = \Lambda (X^I),
\]

\[
\delta A_\mu = -D_\mu \Lambda.
\]

Here gauge covariant derivatives are defined as

\[
D_\mu X^I = \partial_\mu X^I + A_\mu (X^I),
\]

\[
D_\mu \Lambda = \partial_\mu \Lambda + [A_\mu, \Lambda].
\]

In three-algebra notation we have

\[
\Lambda (X^I) = [X^I, T^c; T^d] \Lambda^d_c = \tilde{\Lambda}^a_b X^I_b T^a_c,
\]

where, if we let $[T^b, T^c; T^d] = f^b_{\phantom{b}d} a T^a_c$,

\[
\tilde{\Lambda}^b_a = f^b_{\phantom{b}d} a \Lambda^d_c.
\]
4.1 Mass deformation

We can mass deform these variations and still keep the \( N = 8 \) supersymmetry while breaking \( SO(8) \) down to \( SO(4) \times SO(4) \) [39], [40]. The mass deformation amounts to adding the term to the supersymmetry variation of the fermions

\[
\delta \psi_\alpha = m X^I \Gamma_4(4) \alpha^\beta \Gamma_{I\beta\gamma} \epsilon^\gamma
\]

Here

\[
\Gamma_4 = \Gamma_{1234}.
\]

In the trial version this reads

\[
\delta \psi_\dot{\alpha} = m X^\alpha \Gamma_I \alpha^\beta \Gamma_4(4) \beta \epsilon^I.
\]

and this we can also rewrite in the form of ABJM theory.

\[
\delta \psi_A = m \epsilon_{BC} G_B^C Z^C
\]

in ABJM theory. Here the matrix \( G_B^C \) will be defined in eq’s (24), (25).

4.2 The mass deformed BLG Lagrangian

The Chern-Simons term is unaffected by \( SO(8) \) triality and will thus always look the same. It is given by

\[
\mathcal{L}_{CS} = \frac{1}{2} \epsilon^{\mu \nu \lambda} \left( \langle T^b, [T^c, T^d, T^e] \rangle A_\mu^c A_\lambda^d A_\nu^e - \frac{1}{2} \langle [T^a, T^c, T^d], [T^f, T^b, T^e] \rangle A_\mu^a A_\nu^b A_\rho^c A_\lambda^d A_\sigma^e \right).
\] (23)

In trial version of general BLG theory, the matter Lagrangian and mass deformation terms are given by

\[
\mathcal{L}_{kin} + \mathcal{L}_V + \mathcal{L}_Yakawa = -\frac{1}{2} \langle D_\mu X^\alpha, D^\mu X^\alpha \rangle - \frac{1}{12} \langle [X^\alpha, X^\beta; X^\gamma], [X^\alpha, X^\beta; X^\gamma] \rangle + \frac{i}{2} \langle \tilde{\psi}_\dot{\alpha}, \gamma^\mu D_\mu \psi_\alpha \rangle - \frac{i}{4} \langle X^\alpha, [X^\beta, \tilde{\psi}_\dot{\beta}; \psi_\dot{\alpha}] \rangle \Gamma_{I\alpha\beta} \Gamma^{I\dot{\beta}\dot{\alpha}},
\]

\[
\mathcal{L}_m + \mathcal{L}_{flux} = -\frac{m^2}{2} \langle X^\alpha, X^\alpha \rangle + \frac{m}{48} \Gamma_{I\alpha\beta} \Gamma^{I\dot{\beta}\dot{\alpha}} \Gamma_{K\gamma\dot{\delta}} \Gamma_{K^\gamma\dot{\delta}} G_{\gamma\dot{\delta}} X^\alpha X^\gamma X_{\dot{\delta}} X_{\dot{\alpha}}
\]

where

\[
G = \frac{1}{2} (\Gamma_{1234} + \Gamma_{\dot{1}\dot{2}\dot{3}\dot{4}}).
\] (24)

In the original BLG theory we have

\[
\mathcal{L}_m + \mathcal{L}_{flux} = -\frac{m^2}{2} \langle X^I, X^I \rangle + \frac{m}{6} \left( \epsilon_{ijkl} \langle X^i, [X^j, X^k, X^l] \rangle + \epsilon_{ijkl} \langle X^i, [X^j, X^k, X^l] \rangle \right).
\]
4.3 Rewriting in the form of ABJM

We split \( X^\alpha \) into Weyl components \( Z^A \) and \( Z_A \) and we get

\[
\mathcal{L}_{\text{kin}} = -\frac{1}{2} \langle D_\mu X^\alpha, D^\mu X^\alpha \rangle = -\langle D_\mu Z^A, D^\mu Z^A \rangle.
\]

Using the fundamental identity we may also show that

\[
\mathcal{L}_V = -\frac{1}{12} \langle [X^\alpha, X^\beta; X^\gamma], [X^\alpha, X^\beta; X^\gamma] \rangle
\]

\[
= -\frac{2}{3} \left( \langle [Z^A, Z^B; Z^C], [Z^A, Z^B; Z^C] \rangle - \frac{1}{2} \langle [Z^C, Z^A; Z^B], [Z^B, Z^A; Z^C] \rangle \right).
\]

By some work\(^8\) one can show that the Yukawa type coupling can be recast in the ABJM form as well,

\[
\mathcal{L}_{\text{Yukawa}} = -i \frac{1}{4} \langle X^\alpha, [X^\beta, \bar{\psi}_\alpha ; \psi_\beta] \rangle \Gamma^I \gamma^\beta
\]

\[
= \left( -\frac{1}{2} \epsilon^{ABCD} Z^a \bar{\psi}_{BB} \psi_{CC} Z^d - \frac{1}{2} \epsilon^{ABCD} \bar{\psi}^A a Z_{BB} Z_{CC} \psi^D d + i Z^a \bar{\psi}_{BB} Z^C \psi^B d - \frac{i}{2} Z^a \bar{\psi}_{BB} Z^B \psi^A d \right) f^{bcda}.
\]

The Chern-Simons term is unchanged as it carries no \( SO(8) \) indices. We may also reduce the mass term to corresponding terms in ABJM theory. If we let

\[
G^\alpha_\beta = \begin{pmatrix} G^A_B & 0 \\ 0 & G^B_A \end{pmatrix}
\]

we get

\[
\mathcal{L}_m + \mathcal{L}_{\text{flux}} = -m^2 \langle Z^A, Z^A \rangle + 2m G^B_A \langle Z^A, [Z^B, Z^C; Z^C] \rangle
\]

Notice that the second term equals the two terms obtained in [20] upon expanding the three-bracket in a matrix realization.

5 Maximally supersymmetric vacuum

The static Lagrangian can be expressed as a perfect square,

\[
\mathcal{L} = -\langle W^{AB}_C, W^{AB}_C \rangle
\]

where

\[
W^{AB}_C = \delta^{[A}_C [Z^B, Z^C] + [Z^A, Z^B; Z^C] + m G^{[A}_C Z^B].
\]

This shows that the energy is bounded from below by zero, and that it is zero if and only if

\[
W^{AB}_C = 0.
\]

\(^8\)We may realize the \( SO(8) \) gamma matrices as \( \Gamma^I = (\Sigma^M \otimes \sigma^1, 1 \otimes \sigma^2, \Sigma \otimes \sigma^1) \) for which we have the Fierz identities \( (\Gamma^I)^A_B (\Gamma^K)^C_D = 2 \delta^I_K \delta^A_D, \Sigma_{MAB} \Sigma_{MCD} = -2 \epsilon_{ABCD} \) and \( \Sigma_{MAB} \Sigma_{MCD} = -4 \delta^{[CD}_{AB} \). Here \( I = (M, X) \) is split as \( 8 \rightarrow 6 + 2 \). Indices \( A, B, \ldots \) are 4 Weyl of \( SO(6) \) whereas \( \Sigma \) denotes the chirality matrix.
Any space-time independent solution where the gauge field vanishes, and fermions vanish, is a maximally supersymmetric vacuum since the supersymmetry variation is given by
\[ \delta \psi^A = \epsilon^{AB} W^B_C. \]
One particularly simple solution is obtained by taking \( Z^A = T^A \)
where
\[
T^A = \begin{pmatrix} T_a \\ T_\tilde{a} \end{pmatrix}.
\]
and \( T_\tilde{a} = 0 \), and
\[
[T^a, T^b; T^c] = -m \left( \delta^a_c T^b - \delta^b_c T^a \right).
\]
This equation can be solved by taking \( T^a \propto G^a \) which describe \( S^3 \) with radius
\[
R = \sqrt{(N-1)m} \quad (27)
\]
We will keep the radius \( R \) fixed. The mass parameter \( m \) will not play any role for us, but is just a tool we use to obtain the desired geometry characterized by \( R \). The radius characterizes the geometry of the M5 brane, and should be kept fixed as \( N \) may be taken to infinity.

The background shall satisfy the vacuum equation (26) and be valued in the three-algebra
\[
T^A = T^A_\tilde{b} T^\tilde{b} \otimes \mathcal{T}^\tilde{\mathcal{G}}.
\]
One solution is to take
\[
T^A = T_b^A G_b \otimes 1
\]
where \( T_b^A \propto \delta_b^A \) (for \( A = a \)) and \( G^b \) is given by (16). However, this is an element in the \( S^3/\mathbb{Z}_K \)-three-algebra only for \( K = 1 \) where the function 1 can be expressed as a linear combination of \( T^\tilde{a} \). For generic \( K \) we require
\[
T^A(\psi + 2\pi) = e^{2\pi i K} T^A(\psi).
\]
Dimensional reduction amounts to pick only the zero mode\(^9\)
\[
T^\tilde{a} = e^{i\psi} \tilde{T}^\tilde{a}.
\]
We will assume the same orbifolding for algebra \( \mathcal{B} \), so three-algebra generators in \( A \otimes \mathcal{B} \) will be on the form
\[
T^{\tilde{a} \tilde{a}^*} = e^{i\psi} \otimes e^{i\psi'} \tilde{T}^{\tilde{a} \tilde{a}^*}
\]
We want the resulting theory to dimensionally reduce to super Yang-Mills theory when we take large \( K \). This means that we can not allow for a three-algebra to survive dimensional reduction. A natural way of reducing three-algebra \( \mathcal{B} \) to a Lie algebra is by demanding orbifolding by \( \mathbb{Z}_K \). Assume we took a different discrete group \( \mathbb{Z}_L \) instead, and defined algebra \( \mathcal{B} \) on \( S^3/\mathbb{Z}_L \). We would then require that
\[
\frac{2\pi m}{K} + \frac{2\pi n}{L} = \frac{2\pi p}{K}
\]
\( ^9 \)The name zero mode refers to that we take \( m = 0 \) in Eq (12), an assumption that strictly speaking can be justified only when \( K \) is larger than \( N \). Since \( N \) is taken to infinity, we must take \( K \) to infinity as well, such that the ratio \( N/K > 1 \).
As we run through all the integers \( m, n \), the integer \( p \) must also run through all integers. But this is possible only if \( L = K \).

We will take the background to be given by

\[
\mathcal{T}^A = T^A_b \mathcal{G}^b e^{i\psi} \otimes e^{i\psi'}
\]  

The most general ansatz we can make that satisfies the vacuum equation (26) is on the form \( T^A_{bb'} \mathcal{G}^b \otimes \mathcal{T}^{b'} \) where \( \mathcal{T}^{b'} \) depends on any two (out of three) coordinates, which makes the three-bracket \( [\mathcal{T}^{b'}, \mathcal{T}^{c'}, \mathcal{T}^{d'}] \) vanishing, which is a necessary condition for the vacuum equation to be satisfied. When we dimensionally reduce we want the background to be commuting so we will also demand that \( [\tilde{T}^a, \tilde{T}^b] = 0 \). This implies that \( \tilde{T}^a \) can only depend on one coordinate, and this one coordinate must be \( \psi' \) for the orbifolding condition to be satisfied, and we end up with (28)

However the function \( e^{i\psi'} \) does not correspond to a three-algebra element for finite \( N \). It corresponds to the unit \( N \times N \) matrix, but the three-algebra consists of \( N \times (N - 1) \) matrices. We thus have an error of order \( 1/N \) in all our calculations.

\section{D4 from ABJM}

We will now expand the star-three-product ABJM or star-three-product BLG Lagrangian (which formulation we use is just a matter of taste as they are the same) around the supersymmetric vacuum \( S^3/\mathbb{Z}_K \) and take the limit \( K \to \infty \) and \( R \to \infty \) while keeping \( R/K \) finite. We can also work with the usual matrix realized ABJM theory and expand this theory about the \( S^2 \) base manifold and get exactly the same result. This is so because for \( K > N \), star-three-product ABJM becomes isomorphic to usual ABJM. In the present case \( N = N_A N_B \) where \( N_A \) is the number of D2 branes and \( N_B \) is the number of D4 branes.\(^\text{10}\)

We will eventually take \( N_A \to \infty \) but it seems plausible that we may take the limit \( K \to \infty \) first, or in other words always secure that \( K > N \). For the purpose of deriving D4 we would then only need usual ABJM theory. However to derive the M5, or more generally, to consider cases where \( K < N \), we must use star-three-product BLG theory.

We begin this section by studying the Higgs mechanism in abstract ABJM theory. By abstract, we mean that we keep the realization of the three-algebra unspecified. Working at this abstract level has the advantage that we can apply the same equations to all kind of realizations later on. Next we will expand the resulting abstract Lagrangian in fluctuation fields and derive the full non-Abelian five-dimensional super Yang-Mills Lagrangian and identify the super Yang-Mills coupling constant.

\subsection{The Higgs mechanism}

The vacuum we have found has non-vanishing vacuum expectations values of the scalar fields,

\[
Z^A = T^A + Y^A
\]

\(^\text{10}\)We count the number of D-branes rather than M-branes.
where $T^A$ is the vacuum expectation value, and $Y^A$ are fluctuations. The Higgs mechanism, by which is meant the derivation of an effective action by expanding about a vacuum expectation value, can be studied as a separate problem by itself [27]. The Higgs mechanism does not have to be related with the deconstruction of $D4$, but arises naturally in deconstruction of $D4$.

For non-degenerate situations (meaning square matrices), hermitian three-algebra generators can be divided into two sets by extracting one generator, let us denote that one as $T^\#$. Then the remaining generators $T^a$, with $a \neq \#$ can be assumed to be hermitian. It follows from the fundamental identity, that the bracket

$$[T^a, T^b] = [T^a, T^b; T^\#]$$

is a Lie bracket. For example, for $SO(4)$ we can take $T^a$ to be the Pauli sigma matrices (which are hermitian Lie algebra generators), and $T^\# = iI$.

If we realize the inner product by matrices, we will use the normalization

$$\langle T^a, T^b \rangle = \frac{1}{N} \text{tr}(T^a T_b).$$

If we use star-product the inner product is

$$\langle T^a, T^b \rangle = \frac{1}{\pi R^2} \int d^2\sigma \sqrt{G} T^a * T_b.$$

Here the star-product is superfluous since all higher order terms amount to terms that are total derivatives.

The abstractly defined ABJM Lagrangian then, is given by

$$\mathcal{L} = \mathcal{L}_{CS} + \frac{KN}{2\pi} \left\{ -\langle D_\mu Z^A, D^\mu Z^A \rangle - V(Z) \right\}$$

$$\mathcal{L}_{CS} = \frac{KN}{2\pi} \left\{ \frac{1}{2} \epsilon^{\mu \nu \lambda} \left( \langle T^b, [T^c, T^d; T^a] \rangle A^b_{\mu} A^c_{\nu} A^d_{\lambda} + \frac{1}{3} \langle [T^a, T^c; T^d], [T^f, T^b; T^e] \rangle A_\mu^b A_{\nu}^c A_{\lambda}^d A_{\sigma}^f \right) \right\}$$

where

$$V(Z) = \frac{2}{3} \left( \langle [Z^A, Z^B; Z^C], [Z^A, Z^B; Z^C] \rangle - \frac{1}{2} \langle [Z^C, Z^A, Z^B], [Z^B, Z^A, Z^C] \rangle \right).$$

The gauge covariant derivative is given by

$$D_\mu Z^A = \partial_\mu Z^A + [Z^A, T^c; T^d] A^d_{\mu}.$$

We may expand out the three-bracket and we have

$$D_\mu Z^A = \partial_\mu Z^A + Z^A A^R_{\mu} - A^L_{\mu} Z^A.$$

where

$$A^R_{\mu} = A^d_{\mu} T_d T^c,$$

$$A^L_{\mu} = A^d_{\mu} T^c T_d.$$
Then the Chern-Simons term can be written
\[ \frac{KN}{4\pi} \left( A^R dA^R + \frac{2}{3} (A^R)^3 - A^L dA^L - \frac{2}{3} (A^L)^3 \right) \]

In order to study the Higgs mechanism, we define
\[
a_\mu = A_{\mu d} [T_c, T_d], \\
b_\mu = A_{\mu d} \{T_c, T_d\}.
\]

In terms of these gauge potentials, we have
\[
D_\mu Z^A = \partial_\mu Z^A + \frac{1}{2} [Z^A, b_\mu] - \frac{1}{2} \{Z^A, a_\mu\}
\]

When we expand about a Higgs vacuum expectation value as
\[ Z^A = T^A + Y^A \]

it is natural to define a covariant derivative as
\[
D_\mu Y^A = \partial_\mu Y^A + \frac{1}{2} [Y^A, b_\mu]
\]

and we have
\[ D_\mu Z^A = D_\mu Y^A - T^A a_\mu \]

The Lagrangian reads
\[
\mathcal{L} = \frac{KN}{2\pi} \left( -\frac{1}{2} ag + \frac{1}{12} a^3 \right) + \frac{KN}{2\pi} \left( - D_\mu Y^A, D^\mu Y^A \right) + \left( a_\mu D^\mu (T^A Y_A + T_A Y^A) \right) - T^A T_A \langle a_\mu a^\mu \rangle - V(T + Z) \]

where
\[ g = db - \frac{1}{2} b^2 \]

For later convenience, we make the replacement
\[ Z^A \to \frac{1}{\sqrt{-\hbar}} Z^A \]

which amounts to \( T^A \to \frac{1}{\sqrt{-\hbar}} T^A \) and \( Y^A \to \frac{1}{\sqrt{-\hbar}} Y^A \). We recall that \(-\hbar > 0\).

We now get
\[
\mathcal{L} = \frac{KN}{2\pi \hbar} \left( - \frac{\hbar}{4} \epsilon^{\mu\nu\lambda} \left( a_\mu g_{\nu\lambda} + \frac{1}{3} a_\mu a_\nu a_\lambda \right) \right)
\]

\[ {\text{11}} \]

Here we lend the bracket from the three-algebra and just remove the comma. We write it as (●) with no comma. In our two examples this bracket is given by either tr or \( f \). However in a more abstract setting a comma could be desired. It should be possible to re-introduce a comma in this bracket and hence promote it to a trace form. But this will then be a trace form on the Lie algebra associated with the three-algebra. This is natural since the gauge field takes values in the Lie algebra.
\[ + \langle D_\mu Y^A, D^\mu Y^A \rangle - \langle a_\mu D^\mu (T^A Y_A + T_A Y^A) \rangle + T^A T_A \langle a_\mu a^\mu \rangle + \frac{1}{\hbar^2} V(T + Y) \] 

After the rescaling by \(1/\sqrt{-\hbar}\), we declare that
\[ T^A T_A = \frac{R^2}{2}. \]

We see that \(a_\mu\) is auxiliary and can be integrated out. If \(\mathcal{L} = a_\mu V^\mu + \beta a_\mu a^\mu\), then integrating out \(a_\mu\) amounts to 
\[ \mathcal{L} = -\frac{1}{8\pi} V^\mu V^\mu. \]

Here this gives us
\[ \mathcal{L} = \frac{KN\hbar}{32\pi R^2} \langle g_{\mu\nu} g^{\mu\nu} \rangle + \frac{KN}{2\pi\hbar} \left( \langle D_\mu Y^A D^\mu Y_A \rangle - \frac{1}{2R^2} \langle (T^A D_\mu Y_A + T_A D_\mu Y^A)^2 \rangle \right) \]

If we define
\[ \bar{Y}^A = Y^A - \frac{1}{R^2} T^A (T_B Y^B + T^B Y_B), \] (29)

we have the identity
\[ D_\mu \bar{Y}^A D^\mu \bar{Y}_A = D_\mu Y^A D^\mu Y_A - \frac{1}{2R^2} (T^A D_\mu Y_A + T_A D_\mu Y^A)^2 \]

and hence
\[ \mathcal{L} = \frac{KN\hbar}{32\pi R^2} \langle g_{\mu\nu} g^{\mu\nu} \rangle + \frac{KN}{2\pi\hbar} \left( \langle D_\mu \bar{Y}^A D^\mu \bar{Y}_A \rangle + \frac{1}{\hbar^2} V(T + Y) \right) \]

We rescale \(b_\mu = 2b_\mu\), and use (17) for the inner product, and insert the value for \(\hbar\) given by (15) and we get
\[ \mathcal{L} = - \frac{K}{16\pi^2 R^2} G_{\mu\nu} G^{\mu\nu} - \frac{KN^2}{\pi^2 R^2} D_\mu \bar{Y}^A D^\mu \bar{Y}_A \]

where
\[ D_\mu \bar{Y}^A = \partial_\mu \bar{Y}^A + [\bar{Y}^A, B_\mu]. \]

We have dropped the integration over the two-sphere and now view \(\mathcal{L}\) as a five-dimensional Lagrangian. If we consider a tensor product \(\mathcal{A} \otimes \mathcal{B}\) of three-algebras, then we evaluate only the inner product associated to algebra \(\mathcal{A}\), and the inner product associated with algebra \(\mathcal{B}\) remains. This inner product is suppressed in the Lagrangian above, but could have been displayed as \(\langle \bullet \rangle_{\mathcal{B}}\).

If we have a tensor product of three-algebras \(\mathcal{A}\) and \(\mathcal{B}\) associated with gauge groups \(U(N_A) \times U(N_A)\) and \(U(N_B) \times U(N_B)\), then we have the three-algebra \(\mathcal{A} \otimes \mathcal{B}\) associated with gauge group \(U(N) \times U(N)\) where \(N = N_A N_B\). The remaining inner product of algebra \(\mathcal{B}\) will therefore be isomorphic to the unit normalized trace,
\[ \langle \bullet \rangle_{\mathcal{B}} = \text{tr}. \]

For later use, we define a gauge field \(A_\mu\) as
\[ B_\mu = -\frac{\lambda}{\epsilon} A_\mu. \]
The parameter $\epsilon$ is given by (10) and explicitly by (31). We now have reached the final form of our Lagrangian, that will be our starting point for performing fluctuation analysis,

$$
\text{tr} \left[ \frac{K}{4\pi^2 R} \left( \frac{\lambda}{\epsilon} \right)^2 \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) - \frac{KN^2}{\pi^4 R^8} \left( D_\mu \tilde{Y}^A D^{\mu} \tilde{Y}_A + \frac{1}{\hbar^2} V(T + Z) \right) \right]
$$

(30)

Here

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{\lambda}{\epsilon} [A_\mu, A_\nu],$$

$$D_\mu \tilde{Y}^A = \partial_\mu \tilde{Y}^A + \frac{\lambda}{\epsilon} [A_\mu, \tilde{Y}^A].$$

### 6.2 Fluctuation analysis

For the purpose of deconstruction of D4, we need to reconsider the above Higgs mechanism. Let us choose the three-algebra basis as

$$T^{aa'} = T^a \otimes T^{a'}.$$  

and let us expand

$$DZ^A \equiv dZ^A + [Z^A, B]$$

where

$$B = A^{dd'}_{cc'} \frac{1}{2} \left( T^{cc'} T_{dd'} + T_{dd'} T^{cc'} \right),$$

$$Z^A = Z^A_{bb'} T^{bb'}.$$

We expand

$$[T^{bb'}, T_{dd'} T^{cc'}] = \epsilon \{ T^b, T_d T^c \} T^{bb'} T_d T^{cc'} + T_d T^{cc'} T^{bb'} [T^b, T_d T^c].$$

Here

$$\epsilon = -\frac{i\hbar}{R}$$

(31)

and the bracket is the Poisson bracket that should not be confused with an anticommutator. We now see that

$$DZ^A = dZ^A + [Z^A, B] + \epsilon \{ Z^A, B \}.$$  

Expanding about the Higgs vacuum, we keep the following terms,

$$DZ^A = dY^A + [Y^A, B] + \epsilon \{ T^A, B \}$$

$$= DY^A + \epsilon \{ T^A, B \}. \quad (32)$$

Our previous analysis of the Higgs mechanism must be modified by the addition of the last term, in the deconstruction of D4.

We may define real coordinates $X^M = (X^A, X^{A+4})$ as

$$Z^A = \frac{1}{\sqrt{2}} (X^A + iX^{A+4})$$
and we have

\[ Z^A Z_A + Z_A Z^A = R^2 = X^M X^M. \]

We split \( M = (m, I) \) and we have the metric

\[ ds^2 = dX^M dX^M = G_{mn} d\sigma^m d\sigma^n + (d\psi + A)^2 + dR^2 + \delta_{IJ} dX^I dX^J \]

where the first two terms correspond to the metric on the three-sphere and \( G_{mn} \) is the metric on the two-sphere. We now have

\[
\begin{align*}
\partial_{(m} Z^A \partial_{n)} Z_A &= \frac{1}{2} G_{mn}, \\
\partial_{(I} Z^A \partial_{J)} Z_A &= \frac{1}{2} \delta_{IJ}.
\end{align*}
\]

(33)

Things get more transparent if we use the notion of fake BLG theory. Then we consider eight-component scalar fields, and fluctuations

\[
Y^\alpha = \begin{pmatrix} Y^A \\ Y_A \end{pmatrix}
\]

We have a projection, corresponding to (29),

\[
\tilde{Y}^\alpha = \left( \delta^\alpha_\beta - \frac{1}{R^2} T^\alpha T_\beta \right) Y^\beta
\]

We will decompose the fluctuation part into transverse and tangential parts,

\[
\tilde{Y}^\alpha = Y^m \partial_m T^\alpha + Y^I \partial_I T^\alpha.
\]

Here \( \sigma^m \) are coordinates on \( S^2 \) and \( T^I \) are coordinates transverse to \( S^2 \) and to \( R \). It means that one of the directions labeled by \( I = 1, \ldots, 5 \) must in fact be along the fiber of \( S^3 \) as it can not be a radial direction that is projected out by the Higgs mechanism. We relate the fluctuations \( Y^m \) and \( Y^I \) to a gauge field \( A_m \) and five scalar fields \( \phi^I \) on D4 according to

\[
\begin{align*}
Y^m &= \lambda \sqrt{G} \epsilon^{mn} A_n \\
Y^I &= \lambda \phi^I.
\end{align*}
\]

(34) \hspace{1cm} (35)

The constant \( \lambda \) is determined by relating Dirac charge quantization of the gauge field, with the winding number of the reparametrization, characterized by large fluctuations \( Y^m \). However, for the purpose of determining the super Yang-Mills coupling constant, \( \lambda \) does not play any role since we can always make any kind of field redefinition and in particular we can make any field rescaling. This does not change the Yang-Mills coupling constant. So we will not need the actual value of \( \lambda \) since it will cancel out in the computation of the Yang-Mills coupling constant. We have put the computation of \( \lambda \) in Appendix B, as it may come to use in future studies.

6.3 The kinetic term

Let us first consider the kinetic term for the matter fields. We first expand the covariant derivative. From (32) we get

\[
D_\mu Z^A = \lambda \left( \sqrt{G} \epsilon^{mn} F_{\mu n} \partial_m T^A + D_\mu \phi^I \partial_I T^A \right).
\]
where
\[
D_\mu \phi^I = \partial_\mu \phi^I + \frac{\lambda}{\epsilon} [A_\mu, \phi^I],
\]
\[
F_{\mu n} = \partial_\mu A_n - \partial_n A_\mu + \frac{\lambda}{\epsilon} [A_\mu, A_n].
\]
The kinetic term becomes
\[
- \langle D_\mu Z^A, D^\mu Z^A \rangle = \lambda^2 \left( -\frac{1}{2} \langle F_{\mu n}, F^{\mu n} \rangle - \frac{1}{2} \langle D_\mu \phi^I, D^\mu \phi^I \rangle \right).
\]
(36)

Here we have used the metric components (33).

6.4 The sextic potential plus flux term

6.4.1 Quadratic order
At quadratic order we need to combine the contribution from the sextic potential and the flux term to get
\[
\lambda^2 R^2 \epsilon |\epsilon|^2 \mu \left( -\frac{1}{4} \langle f_{mn}, f^{mn} \rangle - \frac{1}{2} \langle \partial_m \phi^I, \partial^m \phi^I \rangle \right)
\]
where
\[
f_{mn} = \partial_m A_n - \partial_n A_m.
\]

6.4.2 Cubic order
At cubic order and higher, the flux term drops to zero as we take \( N \) large, as can be inferred from Eq (27). We thus only need to expand the sextic potential. We choose to work with the BLG theory sextic potential which is more convenient than the ABJM sextic potential. We first expand
\[
\langle [T^\alpha, T^\beta; Y^\gamma], [T^\alpha, Y^\beta; Y^\gamma] \rangle
\]
and then we note that there are 12 terms that give the same contribution so it will be sufficient to just compute this term and multiply the result by 12. We assume that
\[
\partial_\psi Y^\gamma = 0,
\]
\[
T^\alpha T_\alpha = R^2,
\]
where \( \psi \) parametrizes the fiber on \( S^3/{\mathbb Z}_K \). We expand
\[
[T^\alpha, T^\beta; Y^\gamma] = [T^\alpha, Y_\gamma] T^\beta - [T^\beta, Y_\gamma] T^\alpha + Y_\gamma [T^\alpha, T^\beta],
\]
\[
[T^\alpha, Y^\beta; Y^\gamma] = [T^\alpha, Y_\gamma] Y^\beta - [Y^\beta, Y_\gamma] T^\alpha + Y_\gamma [T^\alpha, T^\beta]
\]
If \( Y_\gamma \) is star-three-multiplied with a commutator reduced to the \( S^2 \) base-manifold, we will necessarily need to act by a \( \psi \)-derivative on \( Y_\gamma \) which kills the whole term. As for the second expansion, we are interested in only the non-Abelian part as the Abelian part will vanish in the large \( R \) limit. In the above expansions we thus keep the terms
\[
[T^\alpha, T^\beta; Y^\gamma] = [T^\alpha, Y_\gamma] T^\beta - [T^\beta, Y_\gamma] T^\alpha,
\]

33
and we get
\[
\langle [T^\alpha, Y^\beta], [T^\alpha, Y^\gamma] \rangle = R^2 \langle [T^\beta, Y_\gamma], [Y^\beta, Y_\gamma] \rangle
\]
\[
= R^2 \epsilon \sqrt{G} \epsilon^{mn} \langle \partial_m Y^\beta \partial_n Y_\gamma, [Y^\beta, Y_\gamma] \rangle.
\]

We next insert the expansion Eq (34) and we get
\[
= R^2 \epsilon \sqrt{G} \epsilon^{mn} G_{mp} \langle \partial_m Y_\gamma, [Y^p, Y_\gamma] \rangle
\]
\[
= R^2 \text{Re} G^{mn} \langle \partial_m Y_\gamma, [A_n, Y_\gamma] \rangle
\]
\[
= \lambda^3 R^2 \epsilon \left( G^{mn} G^{pq} \langle \partial_m A_p, [A_n, A_q] \rangle + G^{mn} \langle \partial_m \phi^I, [A_n, \phi^I] \rangle \right)
\]

In the last step we have noted a cancelation of two terms
\[
\sqrt{G} \epsilon^{mn} \langle A_n, [A_m, Y^R] \rangle - \sqrt{G} \epsilon^{mn} \langle Y^R, [A_n, A_m] \rangle
\]
which cancel by trace invariance \( \langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle \) and cyclicity \( \langle X, Y \rangle = \langle Y, X \rangle \).

\subsection*{6.4.3 Quartic order}

At quartic order we have three terms of the type
\[
\langle [T^\alpha, Y^\beta], [T^\alpha, Y^\beta], Y^\gamma \rangle
\]
\[
= R^2 \langle [Y^\beta, Y_\gamma], [Y^\beta, Y_\gamma] \rangle
\]
\[
= \lambda^4 R^2 \left( G^{mnpq} \langle [A_m, A_n], [A_p, A_q] \rangle + 2G^{mn} \langle [A_m, \phi^I], [A_n, \phi^I] \rangle \right.
\]
\[
\quad + \left. \langle [\phi^I, \phi^I], [\phi^I, \phi^I] \rangle \right)
\]

\subsection*{6.5 Summarizing}

We shall multiply the cubic term by \(-\frac{12}{45 \pi^2 R^2} \frac{K N^2}{\pi^2 R^2}\) and the quartic term by \(-\frac{3}{12 \pi^2 R^2} \frac{K N^2}{\pi^2 R^2}\) as we infer from (30), and the combinatorics give us the factors of 12 and 3 respectively. Discarding the common factor of \( \frac{K N^2}{\pi^2 R^2} \), we have
\[
- \frac{R^2 \lambda^2}{2 \hbar^2} G^{mn} \left( |\epsilon|^2 \langle \partial_m \phi^I, \partial_n \phi^I \rangle + \lambda (e - e^*) \langle \partial_m \phi^I, [A_n, \phi^I] \rangle + \lambda^2 \langle [A_m, \phi^I], [A_n, \phi^I] \rangle \right)
\]
plus
\[
- \frac{R^2 \lambda^2}{4 \hbar^2} G^{mnpq} \left( |\epsilon|^2 \langle f_{mn}, f_{pq} \rangle + \frac{1}{2} \lambda (e - e^*) \langle \partial_m A_n, [A_p, A_q] \rangle + \lambda^2 \langle [A_m, A_n], [A_p, A_q] \rangle \right)
\]

The various terms combine into covariant expressions. Re-instating the common factor of \( \frac{K N^2}{\pi^2 R^2} \), we get
\[
\frac{K N^2}{\pi^2 R^2} \lambda^2 \left( - \frac{1}{4} G^{mnpq} \langle f_{mn}, f_{pq} \rangle - \frac{1}{2} G^{mn} \langle D_m Y^I, D_n Y^I \rangle \right)
\]

34
where

\[ D_m Y^I = \partial_m Y^I + \frac{\lambda}{\epsilon} [A_m, Y^I], \]
\[ F_{mn} = \partial_m A_n - \partial_n A_m + \frac{\lambda}{\epsilon} [A_m, A_n]. \]

To read off the Yang-Mills coupling constant, we may make a field redefinition that removes the factor \( \frac{\lambda}{\epsilon} \) from the covariant derivative and the gauge field strength, and puts \( \left( \frac{|\epsilon|}{\lambda} \right)^2 \) as an overall factor. Then we get the total overall factor as

\[ \frac{K N^2}{\pi^2 R^5} \lambda^2 \left( \frac{|\epsilon|}{\lambda} \right)^2 = \frac{K}{4\pi^2 R} \]

from which we read off the Yang-Mills coupling as

\[ g_{YM}^2 = 4\pi^2 R K. \]

We notice that the first term in (30) also comes with the overall factor \( \frac{K}{4\pi^2 R} \) after we have multiplied it by \( \left( \frac{|\epsilon|}{\lambda} \right)^2 \). Likewise we find this factor in the kinetic term (36). We have now derived the full non-Abelian super Yang-Mills Lagrangian from ABJM theory.

The compactification radius is \( \frac{R}{K} \) if we put M five-brane on the orbifold \( S^3/\mathbb{Z}_K \). We thus see that we can derive the selfdual coupling constant of M5 brane directly from the M2 brane. We think this is a quite remarkable discovery. It gives us hope that it might be possible to derive M5 brane physics from ABJM theory.

There are higher order non-Abelian terms induced from ABJM theory, that we did not bother to compute. These are \( 1/R \)-suppressed, and may be neglected for small \( g_{YM} \). We also did not bother to compute the \( 1/R \) correction terms that arise because we consider SYM on \( \mathbb{R}^{1,2} \times S^2 \) rather than on \( \mathbb{R}^{1,4} \). In the Abelian case, these terms were considered in [13]. Here we considered the flat space limit with both \( K, R \rightarrow \infty \) while \( g_{YM} \) is kept fixed.

When we derived the five-dimensional super Yang-Mills Lagrangian, we naturally ended up with star-commutators rather than matrix commutators. But these are isomorpic. For instance in the relation

\[ [T^a, T^b; T^c] = T^c [T^a, T^b] + [T^a', T^c'] T^b' - [T^b', T^c'] T^a' \]

If we use star-three-product, then upon dimensional reduction these star-commutators are given by

\[ [T^a, T^b] = \epsilon_B \{T^a, T^b\} + O(\epsilon_B^2) \]

But upon dimensional reduction we may also map functions into matrices. In that case we have

\[ [T^a, T^b; T^c] = T^c [T^a, T^b] + [T^a', T^c'] T^b' - [T^b', T^c'] T^a' \]

and now the commutators on the right-hand side are usual matrix commutators, so we have the isomorphism

\[ [T^a, T^b] \cong [T^a', T^b'] \]
7 Lagrangian for a selfdual three-form

We can not write down the action of a selfdual three-form in six dimensions. From the M2 we rather get \([22]\) (at quadratic order there is no essential difference between Abelian and non-Abelian M5)

\[
\mathcal{L} = -\frac{1}{4} H_{\mu\alpha\beta} H^{\mu\alpha\beta} - \frac{1}{12} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} - \frac{1}{2} \sqrt{g} \epsilon^{\alpha\beta\gamma} \epsilon_{\mu\nu\lambda} \partial_\beta B_{\mu\gamma} \partial_\nu B_{\lambda\alpha}. \tag{37}
\]

where \(\mu = 0, 1, 2\) and \(\alpha = 3, 4, 5\). That is, we break \(SO(1, 5) \rightarrow SO(1, 2) \times SO(3)\). To better understand how to interpret this Lagrangian for the selfdual gauge field we compute its Hamiltonian and compare with the Hamiltonian of a non-chiral gauge field. Let us start with the Lagrangian of a non-chiral gauge field on \(\mathbb{R} \times M_5\),

\[
\mathcal{L} = -\frac{1}{12} H_{MNP} H^{MNP}
\]

Let us split the vector index as \(M = (0, m)\) and compute the conjugate momenta

\[
E^{mn} = \frac{\partial \mathcal{L}}{\partial \partial_0 B_{mn}} = -\frac{1}{2} H^{0mn}
\]

and the Hamiltonian

\[
\mathcal{H} = \frac{1}{4} H^{0mn} H^{0}_{mn} + \frac{1}{12} H^{mnp} H_{mnp}
\]

If we split the field strength into selfdual parts,

\[
*H^\pm = \pm H^\pm
\]

we get

\[
\mathcal{H} = \frac{1}{12} \epsilon^{0mnpr} (-H^+_0 H^+_{pr} + H^-_{0mn} H^-_{pqr}). \tag{38}
\]

Let us now repeat the same steps, but with the Lagrangian (37). We split \(\mu = (0, i)\) and compute the conjugate momenta

\[
E^{ij} = 0,
E^{i\alpha} = -\frac{1}{2} \epsilon^{\alpha\beta\gamma} \epsilon^{i0} \partial_\beta B_{j\gamma},
E^{\alpha\beta} = -\frac{1}{2} H^{0\alpha\beta}
\]

and the Hamiltonian

\[
\mathcal{H} = -\frac{1}{4} H^{0\alpha\beta} H_{\alpha\beta} + \frac{1}{4} H^{i\alpha\beta} H_{i\alpha\beta} + \frac{1}{12} H^{\alpha\beta\gamma} H_{\alpha\beta\gamma}
\]

Let us assume this field strength is already selfdual. Then we can also write this Hamiltonian as

\[
\mathcal{H} = \epsilon^{\alpha\beta\gamma i 0} \left( \frac{1}{4} H^+_{\alpha\beta\gamma} H^+_{0ij} - \frac{1}{2} H^+_{\alpha\beta\gamma} H^+_{ij0} + \frac{1}{12} H^+_{\alpha\beta\gamma} H^+_{0ij} \right)
\]

\[
= \frac{1}{12} \epsilon^{mnpqr 0} H^+_{0mn} H^+_{pqr}
\]

which agrees with the chiral part of the non-chiral Hamiltonian (38).
8  Multiple M5

Let us choose \( K = 1 \) for simplicity, and consider expanding mass deformed star-three-product BLG theory about \( S^3 \). We have not settled the issue as to whether we may relax the form of Eq (7) and admit a complete set of functions on \( S^3 \). We can rigorously just work with ABJM theory or any equivalent formulation thereof. We thus can not in any way rigorously deconstruct a six-dimensional theory from BLG/ABJM theory at this stage. Let us suppose that we have overcome this obstacle by means of monopole operators say, and have understood how to do this deconstruction. For example by using our conjectural truncated inner product (11). We may then ask what can be the most general interaction terms in the M5 Lagrangian? When we expand BLG theory, we get terms like

\[
\tilde{K} \left( (DX)^2 + X^6 \right) = \tilde{K} \left( (DY)^2 + T^4Y^2 + T^3Y^3 + T^2Y^4 + TY^5 + Y^6 \right)
\]

where \( X \) and \( T \) have length dimensions \( L^{-\frac{3}{2}} \), and \( \tilde{K} \) is some unknown overall coupling coefficient of star-three-product theory. Now let us rescale the field \( Y \) into a field \( \phi \) with length dimension \( L^{-2} \). The field \( \phi \) represents one of the scalar fields or a two-form gauge potential in the M5 brane world volume. The M5 brane Lagrangian is now

\[
(D\phi)^2 + c^3\phi^3 + c^4\phi^4 + c^5\phi^5 + c^6\phi^6
\]

where \([c_3] = L^0, [c_4] = L^2, [c_5] = L^4, [c_6] = L^6\). The only length parameter available comes from the radius \( R \). In the decompactification limit \( R \to \infty \) causing the interaction terms to blow up to infinity. The only interaction term that could stay finite is on the form \( c_3\phi^3 \). We may drop all interactions and just keep gauge interactions. Indeed this is interesting starting point since it can teach us about the gauge structure of multiple M5. We assume the three-algebra generators are real. We expand the eight real scalar fields \( X^I \) about the \( S^3 \) background, that we write as \( T^I \), and define fluctuation fields according to

\[
\begin{align*}
X^I &= T^I + Y^I, \\
Y^I &= Y^\alpha \partial_\alpha T^I + Y^J \partial_J T^I, \\
Y^\alpha &= \frac{1}{2} \sqrt{g} \epsilon^{\alpha \beta \gamma} B_{\beta \gamma}, \\
B_{\alpha \mu} &= T^c \partial_\alpha T_d A^d_{\mu} c, \\
A_{\mu} &= T^c T_d A^d_{\mu} c,
\end{align*}
\]

and

\[
\begin{align*}
\Lambda_\gamma &= -T^c \partial_\gamma T_d \Lambda^d_c, \\
\Lambda &= T^c T_d \Lambda^d_c.
\end{align*}
\]

The easiest thing to deconstruct are the gauge variations. From BLG gauge variation (21) we get

\[
\begin{align*}
\delta B_{\alpha \beta} &= \partial_\alpha \Lambda_\beta - \partial_\beta \Lambda_\alpha + [B_{\alpha \beta}, \Lambda], \\
\delta Y^I &= [Y^I, \Lambda].
\end{align*}
\]

and from (22) we get

\[
\begin{align*}
\delta B_{2 \mu} &= \partial_2 A_\mu - D_\mu \Lambda_2 + [B_{2 \mu}, \Lambda],
\end{align*}
\]
\[\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda].\]

We should also be able to formulate the gauge variations in an \(SO(1, 5)\) covariant way. We define the gauge covariant derivative as

\[D_M = \partial_M + [\cdot, A_M].\]

It is now not hard to see that the following variations.

\[
\begin{align*}
\delta B_{MN} &= D_M \Lambda_N - D_N \Lambda_M + [\Lambda, B_{MN}], \\
\delta A_M &= \partial_M \Lambda + [\Lambda, A_M]
\end{align*}
\]

constitute the \(SO(1, 5)\) covariant counterpart of the above gauge variations. These gauge variations are consistent with assigning \(Y^I, B_{MN}, \Lambda_M\) to be three-algebra valued, and \(A_M, \Lambda\) to be Lie algebra valued where the Lie algebra is the one that is associated to the three-algebra. Perhaps a bit surprising that the two-form shall be three-algebra valued as this is a gauge field. With this assignment we can also show that these gauge variations close according to

\[
[\delta_{\Lambda'}, \delta_{\Lambda}] = \delta_{\Lambda''}
\]

with

\[
\begin{align*}
\Lambda'' &= [\Lambda, \Lambda'], \\
\Lambda''_M &= [\Lambda, \Lambda'_M] + [\Lambda_M, \Lambda'].
\end{align*}
\]

To show this one may use the generalized Jacobi identities [8] of a three-algebra. One may also assume all the fields are Lie algebra valued. But this seems to be physically incorrect, or at least does not seem to give a theory that can describe the M5 branes. Anyway, gauge symmetry and supersymmetry works out well so at this stage we can not explain why we should not assume all fields be Lie algebra valued. Such a theory can not be obtained from the M2 brane by fluctuation analysis though.

Closure of these gauge variations is highly non-trivial. We may in particular notice that the variation of \(B_{MN}\) contains a term \([\Lambda, B_{MN}]\) which does not look gauge covariant. The familiar situation is that a variation of a connection one-form is a gauge covariant quantity (and this is indeed also the case here), but with a non-Abelian two-form something much more subtle is apparently going on.

If we just have gauge interactions, we may easily write down supersymmetry variations

\[
\begin{align*}
\delta Y^I &= i\bar{\omega} \Gamma^I \chi, \\
\delta B_{MN} &= i\bar{\omega} \Gamma_{MN} \chi, \\
\delta A_M &= 0, \\
\delta \chi &= \frac{1}{12} \Gamma^{MNP} \omega H_{MNP} + \Gamma^M \Gamma_I \omega D_M \phi^I
\end{align*}
\]

where

\[
H_{MNP} = D_M B_{NP} + D_P B_{MN} + D_N B_{PM}.
\]
As we demonstrate in the Appendix A, these supersymmetry variations close up to a gauge variation of precisely the proposed form (39), on the equations of motion

\[ F_{MN} = 0, \]
\[ H_{QMN} + \frac{1}{6} \epsilon_{QMNRS} H_{RST} = 0, \]
\[ \Gamma^M D_M \chi = 0. \]

In [21] an additional field \( C^M \) was introduced in order to admit more general interaction terms and a clear relation with the theory of D4. No field such as \( C^M \) arises in the fluctuation analysis of M2. However \( A_M \) and \( B_{MN} \) both arise naturally as fluctuation fields. By solving for the equation of motion for \( C^M \) in [21], it was found that the theory becomes equivalent with D4. Our proposal for the theory of multiple M5 is somewhat related with [21]. If we put \( C^M = 0 \) in [21] we arrive at the above supersymmetry variations.

In [21] it was claimed that the theory with \( C^M = 0 \) corresponds to non-interacting M5 branes. We may move the branes transverse to each other with no energy cost since there is no scalar potential. That alone does not mean the branes are non-interacting. If the branes are non-interacting then also small fluctuations on one brane should not affect the other brane. This may not be true if there are non-Abelian gauge interactions. If we separate two M5 branes by giving an expectation value to a scalar field, then we induce a Higgs mass to the gauge field and to all the other fluctuation fields. If we are able to find a selfdual string soliton, its tension should be determined by the separation of the M5 branes. By scattering elementary fluctuation quanta against the string, it should produce a wave that goes out to the other M5 brane, and the two M5 branes would be interacting.

While it appears that having only gauge interactions give consistent supersymmetry variations that close on-shell, it also appears that these equations of motion can not follow from an action. To this end it seems one needs to introduce a selfdual auxiliary three-form in order to be able to write down a Chern-Simons type of action for the connection one-form \( A_M \). This is work in progress.

**Acknowledgements**

I have discussed this work with Takao Suyama and Soo-Jong Rey. This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) through the Center for Quantum Spacetime(CQeST) of Sogang University with grant number 2005-0049409.
A Closing M5 supersymmetry

Here we demonstrate closure of the supersymmetry variations
\[
\begin{align*}
\delta Y^I &= i \bar{\omega} \Gamma^I \chi, \\
\delta B_{MN} &= i \bar{\omega} \Gamma_{MN} \chi, \\
\delta A_M &= 0, \\
\delta \chi &= \frac{1}{12} \Gamma^{MNP} \omega H_{MNP} + \Gamma^M \Gamma_I \omega D_M \phi^I.
\end{align*}
\]

We define
\[
\Gamma = \Gamma_{012345},
\epsilon_{012345} = 1
\]
and assume that
\[
\Gamma \omega = -\omega, \quad \Gamma \chi = \chi.
\]

Using the gamma matrix identities
\[
\{ \Gamma_{MN}, \Gamma^{PQR} \} = -12 \delta_{M}^{[P} \delta_{N}^{Q]} + 2 \epsilon_{SMN}^{PQR} \Gamma^S \Gamma, \\
\Gamma^{MNP} \Gamma_Q \Gamma_{MN} = -24 \delta_Q^M + 4 \Gamma_Q \Gamma^P
\]
and for \( \Gamma \omega = -\omega \) and \( \Gamma \rho = -\rho \) we have the Fierz identity
\[
\omega \bar{\rho} - \rho \bar{\omega} = \left( -\frac{1}{16} (\bar{\rho} \Gamma_M \omega) \Gamma^M + \frac{1}{16} (\bar{\rho} \Gamma_M \Gamma_A \omega) \Gamma^M \Gamma_A \right) (1 + \Gamma)
\]
\[
- \frac{1}{192} (\bar{\rho} \Gamma_{MNP} \Gamma_{AB} \omega) \Gamma^{MNP} \Gamma^{AB}
\]
we get
\[
\begin{align*}
[\delta_{\rho}, \delta_{\bar{\omega}}] Y^I &= -2 i \bar{\omega} \Gamma^Q \rho D_Q Y^I, \\
[\delta_{\rho}, \delta_{\bar{\omega}}] B_{MN} &= -2 i \bar{\omega} \Gamma^Q \rho D_Q B_{MN}, \\
[\delta_{\rho}, \delta_{\bar{\omega}}] A_M &= 0.
\end{align*}
\]

Using gamma matrix relations
\[
\Gamma_{MN} \Gamma^{PQR} + \Gamma^{PQR} \Gamma_{MN} = -12 \delta_{M}^{[P} \delta_{N}^{Q]} + 2 \epsilon_{MN}^{PQR} \Gamma^S, \\
\Gamma_{MNPQR} = \epsilon_{MNPQRS} \Gamma^S
\]
and
\[
\begin{align*}
\Gamma^{MNP} \Gamma^Q \Gamma_{MN} &= -24 \eta^{PQ} + 4 \Gamma^Q \Gamma^P, \\
\Gamma^{MNP} \Gamma^{QRS} \Gamma_{MN} &= 4 \left( 3 \eta^{P[Q,RS]} + 3 \eta^{P[Q} \Gamma^{RS]} + \Gamma^{PQRS} \right), \\
\Gamma^{PQRS} + \Gamma^{QRS} \Gamma^P &= 3 \eta^{P[Q,RS]}, \\
\Gamma^M \Gamma^{QRS} + \Gamma^{QRS} \Gamma^M &= 6 \eta^{M[Q,RS]}
\end{align*}
\]
we find explicitly
\[
\begin{align*}
[\delta_{\rho}, \delta_{\bar{\omega}}] B_{MN} &= -2 i \bar{\omega} \Gamma^T \rho T B_{MN} \\
&\quad + 2 D_{[M} A_{N]} + [\Lambda, B_{MN}]
\end{align*}
\]
\[ [\delta_\rho, \delta_\omega] A_M = -2i\bar{\omega} \Gamma^T \rho \partial_T A_M + D_M \Lambda + 2i\bar{\omega} \Gamma^T \rho F_{TM}, \]
\[ [\delta_\rho, \delta_\omega] \phi^I = -2i\bar{\omega} \Gamma^T \rho \partial_T \phi^I + [\Lambda, \phi^I], \]
\[ [\delta_\rho, \delta_\omega] \chi = -2i\bar{\omega} \Gamma^T \rho \partial_T \chi + [\Lambda, \chi] + \frac{3i}{4} (\omega \Gamma^N \rho) \Gamma_N (\Gamma^M D_M \chi) + \frac{i}{2} (\bar{\rho} \Gamma^N \Gamma^A \omega) \Gamma_N \Gamma_A (\Gamma^M D_M \chi) \]

where

\[ \Lambda_N = -i\bar{\omega} \Gamma^T \rho B_{NT} - 2i\bar{\omega} \Gamma_N \Gamma_A \phi^I, \]
\[ \Lambda = 2i\bar{\omega} \Gamma^T \rho A_T. \]

Interestingly we have closure up to a gauge variation of precisely the form (39), provided we go on-shell where we have the equations of motion

\[ F_{MN} = 0, \]
\[ \Gamma^M D_M \chi = 0 \]
\[ H_{QMN} + \frac{1}{6} \epsilon_{QMN RST} H^{RST} = 0. \]
B Quantization condition for the fluctuation fields

Since the gauge field is associated with a Dirac quantization condition, which in the Abelian case reads

$$\int \frac{F}{2\pi} \in \mathbb{Z}$$

we should find that its dual field $Y^m$ is also subject to a quantization condition.

Let us assume that the relation

$$Y^\alpha = Y^m \partial_m T^\alpha$$

holds for any finite $Y^m$. Let us map our complex coordinates into three real coordinates, so that

$$T^iT^i = R^2$$

describes $S^2$ embedded in flat euclidean three-dimensional space.

We are now particularly interested in coordinate transformations that are not continuously connected with the identity. On a circle such transformations are characterized by a winding number, which can be any integer number. The winding number of the coordinate transformation

$$\varphi \mapsto \varphi' = \varphi + Y\varphi$$

is computed intrinsically by the integral

$$w = \int \frac{d\varphi'}{2\pi} = 1 + \int \frac{dY\varphi}{2\pi}.$$  

We can express the same thing extrinsically by the integral

$$w = \int \frac{1}{2\pi R^2} \epsilon_{ij} X^i dX^j$$

where

$$X^i = T^i + Y^i$$

is a coordinate transformation that respects the constraint $T^iT^i = R^2 = X^iX^i$.

We get back the intrinsic integral if we let

$$T^1 + iT^2 = Re^{iy}\varphi,$$
$$X^1 + iX^2 = Re^{iy}\varphi'.$$

There is a third way of expressing the winding number. Let us define

$$Y^i = Y^\varphi \partial_\varphi T^i$$

If we then expand out the extrinsic integral we can compute the variation of the winding number (that is, $\delta w = w - 1$ where $w = 1$ in the original configuration) as

$$\delta w = \int \frac{1}{2\pi R^2} \epsilon_{ij} \left( Y^i dT^j + T^i dY^j + Y^i dY^j \right)$$
\[
= \int \frac{d\varphi}{2\pi R^2} \epsilon_{ij} T^i dT^j \partial_\varphi Y^\varphi
\]

To obtain this result we have used

\[
T^i \partial_\varphi T^i = 0, \quad \frac{\partial^2 T^i}{\partial \varphi^2} = -T^i
\]
as is explicit from the parametrization above.

### B.1 Two-sphere

This can be generalized to \( S^n \) for any integer \( n \). For \( S^2 \) we compute the winding number by the extrinsic integral

\[
\delta w = \int \frac{1}{8\pi R^3} \epsilon_{ijk} X^i dX^j \wedge dX^k.
\]

Defining the variation

\[
X^i = T^i + Y^i, \quad Y^i = Y^m \partial_m T^i
\]
we get, by doing the same steps as we did above for \( S^1 \),

\[
\delta w = \int \frac{1}{8\pi R^3} \epsilon_{ijk} T^i dT^j \wedge dT^k D_m Y^m.
\]

To get here we have used

\[
D_m \partial_n T^i = -\frac{1}{R^2} G_{mn} T^i.
\]

We can also express this integral as

\[
\delta w = \frac{1}{4\pi} \int d\varphi \wedge d\theta \sin \theta D_m Y^m
\]

Let us denote the metric of the unit sphere as \( \hat{G}_{mn} \). Then we have

\[
\delta w = \int d^2 \sigma \sqrt{G} D_m Y^m.
\]

We dualize \( Y^m \) according to Eq (34). We then get

\[
\delta w = \frac{\lambda}{4\pi R^2} \int F.
\]

Both sides are integer quantized, and this fixes

\[
\lambda = 2R^2. \quad (40)
\]
B.2 Three-sphere

Let us do one more example, that perhaps can be for future use to derive the M5 brane coupling directly from M2 without taking the detour via five-dimensional super Yang-Mills. Let us consider $S^3$ and dualize the fluctuations as

$$Y^\alpha = \lambda \sqrt{g} \epsilon^{\beta \gamma} B_{\beta \gamma}. \quad (41)$$

The integer winding number is given by

$$w = \int \frac{1}{12 \pi^2 R^4} \epsilon_{ijkl} T^i dT^j \wedge dT^k \wedge dT^l$$

and for the difference that we define via the fluctuation field as

$$Y^i = T^i + Y^\alpha \partial_\alpha T^i$$

we get

$$\delta w = \int \frac{1}{12 \pi^2 R^4} \epsilon_{ijkl} T^i dT^j \wedge dT^k \wedge dT^l D_\alpha Y^\alpha$$

or in terms of local coordinates,

$$\delta w = \frac{1}{2 \pi R^3} \int d^3 \sigma \sqrt{g} D_\alpha Y^\alpha$$

Inserting the ansatz, we get

$$\delta w = \frac{\lambda}{2 \pi R^3} \int H$$

and we see that we must choose

$$\lambda = \frac{\pi R^3}{2} \quad (42)$$

to be compatible with the Dirac charge quantization condition $\int H \in 2\pi \mathbb{Z}$.

B.3 Three-sphere as a fiber bundle over two-sphere

Let us establish that (42) and (40) are related by

$$\int d\psi B_{m\psi} = A_m$$

in dimensional reduction where we put $\partial_\psi = 0$ and we thus have

$$2\pi B_{m\psi} = A_m.$$ 

Inserting this into (41) we get

$$Y^m = \frac{2 \pi R^3}{4} \sqrt{g} \epsilon^{mn\psi} B_{n\psi}$$

$$= \frac{R^2}{2} \sqrt{G} \epsilon^{mn} A_n$$

$$= \frac{2 \pi R^3}{4} \sqrt{G} \epsilon^{mn} A_n.$$  

Then we must recall that the radius on the base $S^2$ is $\frac{R}{\sqrt{2}}$. We see that we reproduce the result for the two-sphere in Eq (34), (40).
References

[1] D. Tong, arXiv:hep-th/0509216.
[2] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, arXiv:0806.1218 [hep-th].
[3] S. Kim, Nucl. Phys. B 821, 241 (2009) [arXiv:0903.4172 [hep-th]].
[4] D. Bashkirov and A. Kapustin, arXiv:1007.4861 [hep-th].
[5] A. Kapustin, B. Willett and I. Yaakov, JHEP 1010, 013 (2010) [arXiv:1003.5964 [hep-th]].
[6] N. Drukker, M. Marino and P. Putrov, arXiv:1007.3837 [hep-th].
[7] S. Kim, Nucl. Phys. B 821, 241 (2009) [arXiv:0903.4172 [hep-th]].
[8] D. Bashkirov and A. Kapustin, arXiv:1007.4861 [hep-th].
[9] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, arXiv:0806.1218 [hep-th].
[10] A. Kapustin, B. Willett and I. Yaakov, JHEP 1010, 013 (2010) [arXiv:1003.5964 [hep-th]].
[11] N. Drukker, M. Marino and P. Putrov, arXiv:1007.3837 [hep-th].
[12] S. Kim, Nucl. Phys. B 821, 241 (2009) [arXiv:0903.4172 [hep-th]].
[13] D. Bashkirov and A. Kapustin, arXiv:1007.4861 [hep-th].
[14] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, arXiv:0806.1218 [hep-th].
[15] A. Kapustin, B. Willett and I. Yaakov, JHEP 1010, 013 (2010) [arXiv:1003.5964 [hep-th]].
[16] N. Drukker, M. Marino and P. Putrov, arXiv:1007.3837 [hep-th].
[17] S. Kim, Nucl. Phys. B 821, 241 (2009) [arXiv:0903.4172 [hep-th]].
[18] D. Bashkirov and A. Kapustin, arXiv:1007.4861 [hep-th].
[19] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, arXiv:0806.1218 [hep-th].
[20] A. Kapustin, B. Willett and I. Yaakov, JHEP 1010, 013 (2010) [arXiv:1003.5964 [hep-th]].
[21] N. Drukker, M. Marino and P. Putrov, arXiv:1007.3837 [hep-th].
[22] S. Kim, Nucl. Phys. B 821, 241 (2009) [arXiv:0903.4172 [hep-th]].
[23] D. Bashkirov and A. Kapustin, arXiv:1007.4861 [hep-th].
[24] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, arXiv:0806.1218 [hep-th].
[25] A. Kapustin, B. Willett and I. Yaakov, JHEP 1010, 013 (2010) [arXiv:1003.5964 [hep-th]].
[26] Jakob Palmkvist, private communication.

[27] S. Mukhi and C. Papageorgakis, JHEP **0805**, 085 (2008) [arXiv:0803.3218 [hep-th]].

[28] N. Lambert, C. Papageorgakis and M. Schmidt-Sommerfeld, arXiv:1012.2882 [hep-th].

[29] M. R. Douglas, arXiv:1012.2880 [hep-th].

[30] S. Terashima and F. Yagi, arXiv:1012.3961 [hep-th].

[31] C. H. Chen, P. M. Ho and T. Takimi, JHEP **1003**, 104 (2010) [arXiv:1001.3244 [hep-th]].

[32] M. Henningson, Commun. Math. Phys. **257** (2005) 291 [arXiv:hep-th/0405056].

[33] E. Witten, J. Geom. Phys. **22**, 103 (1997) [arXiv:hep-th/9610234].

[34] M. Henningson, B. E. W. Nilsson and P. Salomonson, JHEP **9909**, 008 (1999) [arXiv:hep-th/9908107].

[35] N. Kim, Phys. Rev. D **81** (2010) 086006 [arXiv:0807.1349 [hep-th]].

[36] S. Cherkis, V. Dotsenko and C. Saemann, Phys. Rev. D **79**, 086002 (2009) [arXiv:0812.3127 [hep-th]].

[37] D. Bashkirov and A. Kapustin, arXiv:1007.4861 [hep-th].

[38] A. Gustavsson and S. J. Rey, arXiv:0906.3568 [hep-th].

[39] J. Gomis, A. J. Salim and F. Passerini, JHEP **0808**, 002 (2008) [arXiv:0804.2186 [hep-th]].

[40] K. Hosomichi, K. M. Lee and S. Lee, Phys. Rev. D **78**, 066015 (2008) [arXiv:0804.2519 [hep-th]].