ESTIMATES FOR MOMENTS OF GENERAL MEASURES ON CONVEX BODIES

SERGEY BOBKOV, BO’AZ KLARTAG, AND ALEXANDER KOLDOBSKY

Abstract. For \( p \geq 1, n \in \mathbb{N}, \) and an origin-symmetric convex body \( K \) in \( \mathbb{R}^n, \) let

\[
d_{\text{ovr}}(K, L^n_p) = \inf \left\{ \left( \frac{|D|}{|K|} \right)^{1/n} : K \subseteq D, \ D \in L^n_p \right\}
\]

be the outer volume ratio distance from \( K \) to the class \( L^n_p \) of the unit balls of \( n \)-dimensional subspaces of \( L^p. \) We prove that there exists an absolute constant \( c > 0 \) such that

\[
(0.1) \quad \frac{c \sqrt{n}}{\sqrt{p \log \log n}} \leq \sup K d_{\text{ovr}}(K, L^n_p) \leq \sqrt{n}.
\]

This result follows from a new slicing inequality for arbitrary measures, in the spirit of the slicing problem of Bourgain. Namely, there exists an absolute constant \( C > 0 \) so that for any \( p \geq 1, \) any \( n \in \mathbb{N}, \) any compact set \( K \subseteq \mathbb{R}^n \) of positive volume, and any Borel measurable function \( f \geq 0 \) on \( K, \)

\[
(0.2) \quad \int_K f(x) \, dx \leq C \sqrt{p} d_{\text{ovr}}(K, L^n_p) |K|^{1/n} \sup_{H} \int_{K \cap H} f(x) \, dx,
\]

where the supremum is taken over all affine hyperplanes \( H \) in \( \mathbb{R}^n. \) Combining (0.2) with a recent counterexample for the slicing problem with arbitrary measures from [9], we get the lower estimate from (0.1).

In turn, inequality (0.2) follows from an estimate for the \( p \)-th absolute moments of the function \( f \)

\[
\min_{\xi \in S^{n-1}} \int_K |(x, \xi)|^p f(x) \, dx \leq (Cp)^{p/2} d_{\text{ovr}}(K, L^n_p) |K|^{p/n} \int_K f(x) \, dx.
\]

Finally, we prove a result of the Busemann-Petty type for these moments.

1. Introduction

Suppose that \( K \subseteq \mathbb{R}^n (n \geq 1) \) is a centrally-symmetric convex set of volume one (i.e., \( K = -K \)). Given an even continuous probability density \( f : K \to [0, \infty), \) and \( p \geq 1, \) can we find a direction \( \xi \) such that the \( p \)-th absolute moment

\[
(1.1) \quad M_{K,f,p}(\xi) = \int_K |(x, \xi)|^p f(x) \, dx
\]

is smaller than a constant which does not depend on \( K \) and \( f? \) More precisely and in a more relaxed form, let \( \gamma(p, n) \) be the smallest number \( \gamma > 0 \) satisfying

\[
(1.2) \quad \min_{\xi \in S^{n-1}} M_{K,f,p}(\xi) \leq \gamma^p |K|^{p/n} \int_K f(x) \, dx
\]

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for all centrally-symmetric convex bodies $K \subseteq \mathbb{R}^n$ and all even continuous functions $f \geq 0$ on $K$. Here and below, we denote by $S^{n-1} = \{ \xi \in \mathbb{R}^n : |\xi| = 1 \}$ the Euclidean unit sphere centered at the origin, and $|K|$ stands for volume of appropriate dimension. (Note that the continuity property of $f$ in the definition of $\gamma$ is irrelevant and may easily be replaced by measurability.) As we will see, there is a two-sided bound on $\gamma(p,n)$.

**Theorem 1.1.** With some positive absolute constants $c$ and $C$, for any $p \geq 1$,

$$
\frac{c\sqrt{n}}{\sqrt{\log \log n}} \leq \gamma(p,n) \leq C\sqrt{pn}.
$$

To describe the way the upper bound is obtained, denote by $L^n_p$ the class of the unit balls of $n$-dimensional subspaces of $L_p$. Equivalently (see [11, p. 117]), $L^n_p$ is the class of all centrally-symmetric convex bodies $D$ in $\mathbb{R}^n$ such that there exists a finite Borel measure $\nu_D$ on $S^{n-1}$ satisfying

$$
\|x\|_D = \int_{S^{n-1}} |(x,\theta)|^p \ d\nu_D(\theta), \quad \forall x \in \mathbb{R}^n.
$$

Here $\|x\|_D = \inf\{a \geq 0 : x \in aD\}$ is the norm generated by $D$. Note that $L^n_1 = \Pi^n_*$ is the class of polar projection bodies which, in particular, contains the cross-polytopes; see [11, Ch.8] for details.

For a (bounded) set $K$ in $\mathbb{R}^n$, define the quantity

$$
V(K, L^n_p) = \inf \{|D|^{1/n} : K \subseteq D, \ D \in L^n_p\}.
$$

If $K$ is measurable and has positive volume, we have the relation

$$
V(K, L^n_p) = d_{ovr}(K, L^n_p)|K|^{1/n},
$$

with

$$
d_{ovr}(K, L^n_p) = \inf \left\{ \left(\frac{|D|}{|K|}\right)^{1/n} : K \subseteq D, \ D \in L^n_p \right\}.
$$

For convex $K$, the latter may be interpreted as the outer volume ratio distance from $K$ to the class of unit balls of $n$-dimensional subspaces of $L_p$. The next body-wise estimates refine the upper bound in Theorem 1.1 in terms of the $d_{ovr}$-distance.

**Theorem 1.2.** Given a probability measure $\mu$ on $\mathbb{R}^n$ with a compact support $K$, for every $p \geq 1$,

$$
\min_{\xi \in S^{n-1}} \left( \int |(x,\xi)|^p \ d\mu(x) \right)^{1/p} \leq C\sqrt{p} V(K, L^n_p),
$$

where $C$ is an absolute constant. In particular, if $f$ is a non-negative continuous function on a compact set $K \subseteq \mathbb{R}^n$ of positive volume, then

$$
\min_{\xi \in S^{n-1}} M_{K,f,p}(\xi) \leq (Cp)^{p/2} d_{ovr}(K, L^n_p)^p |K|^{p/n} \int_K f(x) \ dx.
$$

In the class of centrally-symmetric convex bodies $K$ in $\mathbb{R}^n$, there is a dimensional bound $d_{ovr}(K, L^n_p) \leq \sqrt{n}$, which follows from John’s theorem and the fact that ellipsoids belong to $L^n_p$ for all $p \geq 1$ (see [9] and [11, Lemma 3.12]). Hence, the second upper bound of Theorem 1.2 is more accurate in comparison with the universal bound of Theorem 1.1.

Moreover, for several classes of centrally-symmetric convex bodies, it is known that the distance $d_{ovr}(K, L^n_p)$ is bounded by absolute constants. These classes
include duals of bodies with bounded volume ratio (see [14]) and the unit balls of normed spaces that embed in $L_q$, $1 \leq q < \infty$ (see [18, 15]). In the case $p = 1$, they also include all unconditional convex bodies [14]. The proofs in these papers estimate the distance from the class of intersection bodies, but the actual bodies used there (the Euclidean ball for $p > 1$ and the cross-polytope for $p = 1$) also belong to the classes $L^n_p$, so the same arguments work for $L^n_p$.

In order to prove the lower estimate of Theorem 1.1, we first establish the connection between question (1.1) and the slicing problem for arbitrary measures. The slicing problem of Bourgain [2, 3] asks whether $\sup_n L_n < \infty$, where $L_n$ is the minimal positive number $L$ such that, for any centrally-symmetric convex body $K \subseteq \mathbb{R}^n$,

$$|K| \leq L \max_{\xi \in S^{n-1}} |K \cap \xi^\perp| |K|^{1/n}.$$ 

Here, $\xi^\perp$ is the hyperplane in $\mathbb{R}^n$ passing through the origin and perpendicular to the vector $\xi$, and we write $|K \cap \xi^\perp|$ for the $(n-1)$-dimensional volume. Bourgain’s slicing problem is still unsolved. The best-to-date estimate $L_n \leq C_n^{1/4}$ was established by the second-named author [8], removing a logarithmic term from an earlier estimate by Bourgain [4].

The slicing problem for arbitrary measures was introduced in [12] and considered in [13, 14, 15, 5, 9]. In analogy with the original problem, for a centrally-symmetric convex body $K \subseteq \mathbb{R}^n$, let $S_{n,K}$ be the smallest positive number $S$ satisfying

$$\int_K f(x) \, dx \leq S \max_{\xi \in S^{n-1}} \int_{K \cap \xi^\perp} f(x) \, dx \, |K|^{1/2}$$

for all even continuous functions $f \geq 0$ in $\mathbb{R}^n$ (where $dx$ on the right-hand side refers to the Lebesgue measure on the corresponding affine subspace of $\mathbb{R}^n$). It was proved in [13] that $S_n = \sup_{K \subseteq \mathbb{R}^n} S_{n,K} \leq 2\sqrt{n}$.

However, for many classes of bodies, including intersection bodies [12] and unconditional convex bodies [14], the quantity $S_{n,K}$ turns out to be bounded by an absolute constant. In particular, if $K$ is the unit ball of an $n$-dimensional subspace of $L_p$, $p > 2$, then $S_{n,K} \leq C\sqrt{p}$ with some absolute constant $C$; see [14]. These results are implied by the following estimate proved in [14]:

**Theorem 1.3.** ([14]) For any centrally-symmetric star body $K \subseteq \mathbb{R}^n$ and any even continuous non-negative function $f$ on $K$,

$$\int_K f(x) \, dx \leq 2 \, d_{\text{ovr}}(K, \mathcal{I}_n) \max_{\xi \in S^{n-1}} \int_{K \cap \xi^\perp} f(x) \, dx \, |K|^{1/n},$$

where $d_{\text{ovr}}(K, \mathcal{I}_n)$ is the outer volume ratio distance from $K$ to the class $\mathcal{I}_n$ of intersection bodies in $\mathbb{R}^n$.

The class of intersection bodies $\mathcal{I}_n$ was introduced by Lutwak [17]; it can be defined as the closure in the radial metric of radial sums of ellipsoids centered at the origin in $\mathbb{R}^n$.

On the other hand, it was shown in [9] that in general the constants $S_n$ are of the order $\sqrt{n}$, up to a doubly-logarithmic term.
Theorem 1.4. ([9]) For any \( n \geq 3 \), there exists a centrally-symmetric convex body \( T \subseteq \mathbb{R}^n \) and an even, continuous probability density \( f : T \to [0, \infty) \) such that, for any affine hyperplane \( H \subseteq \mathbb{R}^n \),

\[
\int_{T \cap H} f(x) \, dx \leq C \sqrt{\frac{\log \log n}{n}} |T|^{-1/n},
\]

where \( C > 0 \) is a universal constant.

The connection between (1.2) and the slicing inequality for arbitrary measures (1.5) is as follows.

Lemma 1.5. Given a Borel measurable function \( f \geq 0 \) on \( \mathbb{R}^n \), for any \( \xi \in S^{n-1} \) and \( p > 0 \),

\[
2^p (p + 1) \left( \sup_{s \in \mathbb{R}} \int_{(x, \xi) = s} f(x) \, dx \right)^p \int |(x, \xi)|^p f(x) \, dx \geq \left( \int f(x) \, dx \right)^{p+1}.
\]

If \( f \) is defined on a set \( K \) in \( \mathbb{R}^n \), we then have

\[
2^p (p + 1) \left( \sup_{s \in \mathbb{R}} \int_{K \cap \{(x, \xi) = s\}} f(x) \, dx \right)^p M_{K,f,p}(\xi) \geq \left( \int_{K} f(x) \, dx \right)^{p+1}.
\]

The lower bound in Theorem 1.1 thus follows, by combining the above inequality with (1.2) and Theorem 1.4.

Corollary 1.6. With some positive absolute constants \( c \) and \( C \), for every \( p \geq 1 \),

\[
\frac{c\sqrt{n}}{\sqrt{\log \log n}} \leq S_n \leq C \gamma(p, n).
\]

Lemma 1.5, in conjunction with Theorem 1.2, leads to a new slicing inequality. In the case of volume, where \( f \equiv 1 \), this inequality was established earlier by Ball [1] for \( p = 1 \) and by Milman [18] for arbitrary \( p \).

Theorem 1.7. Let \( f \geq 0 \) be a Borel measurable function on a compact set \( K \subseteq \mathbb{R}^n \) of positive volume. Then, for any \( p > 2 \),

\[
\int_{K} f(x) \, dx \leq C \sqrt{p} \, d_{\text{ovr}}(K, L_p^n) \, |K|^{1/n} \sup_{H} \int_{K \cap H} f(x) \, dx,
\]

where the supremum is taken over all affine hyperplanes \( H \) in \( \mathbb{R}^n \), and \( C \) is an absolute constant.

Theorem 1.7 also holds for \( 1 \leq p \leq 2 \), but in this case it is weaker than Theorem 1.3 because the unit ball of every finite dimensional subspace of \( L_p \), \( 0 < p \leq 2 \), is an intersection body; see [10]. However, for \( p > 2 \) the unit balls of subspaces of \( L_p \) are not necessarily intersection bodies. For example the unit balls of \( \ell_p^n \) are not intersection bodies if \( p > 2, n \geq 5 \); see [11, Th. 4.13]. So the result of Theorem 1.7 is new for \( p > 2 \), and generalizes the estimate from [15] in the case where \( K \) itself belongs to the class \( L_p^n \).

Theorem 1.7 gives another reason to estimate the outer volume ratio distance \( d_{\text{ovr}}(K, L_p^n) \) from an arbitrary symmetric convex body to the class of unit balls of subspaces of \( L_p \). As mentioned before,

\[
d_{\text{ovr}}(K, L_p^n) \leq \sqrt{n},
\]
uniformly over all centrally-symmetric convex bodies $K$ in $\mathbb{R}^n$. Surprisingly, the corresponding lower estimates seem to be missing in the literature. Combining Theorems 1.7 and 1.4, we get a lower estimate which shows that $\sqrt{n}$ is optimal up to a doubly-logarithmic term with respect to the dimension $n$ and a term depending on the power $p$ only.

**Corollary 1.8.** There exists a centrally-symmetric convex body $T \subseteq \mathbb{R}^n$ such that

$$d_{ovr}(T, L^n_p) \geq c \frac{\sqrt{n}}{\sqrt{p \log \log n}}$$

for every $p \geq 1$, where $c > 0$ is a universal constant.

We end the Introduction with a comparison result for the quantities $M_{K,f,p}(\xi)$.

For $p \geq 1$, introduce the Banach-Mazur distance

$$d_{BM}(M, L^n_p) = \inf \{a \geq 1 : \exists D \in L^n_p \text{ such that } D \subset M \subset aD\}$$

from a star body $M$ in $\mathbb{R}^n$ to the class $L^n_p$. Recall that $L^n_p$ is invariant with respect to linear transformations. By John’s theorem, if $M$ is origin-symmetric and convex, then $d_{BM}(M, L^n_p) \leq \sqrt{n}$. We prove the following:

**Theorem 1.9.** Let $K$ and $M$ be origin-symmetric star bodies in $\mathbb{R}^n$, and let $f \geq 0$ be an even continuous function on $\mathbb{R}^n$. Given $p \geq 1$, suppose that for every $\xi \in S^{n-1}$

$$\int_K |(x, \xi)|^p f(x) \, dx \leq \int_M |(x, \xi)|^p f(x) \, dx. \tag{1.7}$$

Then

$$\int_K f(x) \, dx \leq d_{BM}^p(M, L^n_p) \int_M f(x) \, dx.$$

This result is in the spirit of the isomorphic Busemann-Petty problem for arbitrary measures proved in [16]: with the same notations, if

$$\int_{K \cap \xi^\perp} f(x) \, dx \leq \int_{M \cap \xi^\perp} f(x) \, dx, \quad \forall \xi \in S^{n-1},$$

then

$$\int_K f(x) \, dx \leq d_{BM}(K, L_n) \int_M f(x) \, dx.$$

We refer the reader to [11, Ch.5] for more about the Busemann-Petty problem.

Throughout this paper, we write $a \sim b$ when $ca \leq b \leq Ca$ for some absolute constants $c, C$. A convex body $K$ in $\mathbb{R}^n$ is a compact, convex set with a non-empty interior. The standard scalar product between $x, y \in \mathbb{R}^n$ is denoted by $(x, y)$ and the Euclidean norm of $x \in \mathbb{R}^n$ by $|x|$. We write $\log$ for the natural logarithm.

## 2. Proofs

In this section we prove Theorem 1.2, Lemma 1.5 and Theorem 1.9. The other results of this paper will follow as explained in the Introduction.

Given a compact set $K \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$, put

$$\|x\|_K = \min\{a \geq 0 : x \in aK\},$$

if $x \in aK$ for some $a \geq 0$, and $\|x\|_K = \infty$ in the other case. For star bodies, it represents the usual Minkowski functional associated with $K$. 
Proof of Theorem 1.2. Let $D \subseteq \mathbb{R}^n$ be the unit ball of an $n$-dimensional subspace of $L_p$, so that the relation (1.3) holds for some measure $\nu_D$ on the unit sphere $S^{n-1}$. Then, integrating the inequality

$$
\min_{\theta \in S^{n-1}} \int_K |(x, \theta)|^p \, d\mu(x) \leq \int_K |(x, \xi)|^p \, d\mu(x) \quad (\xi \in S^{n-1})
$$

over the variable $\xi$ with respect to $\nu_D$, we get the relation

$$
\nu_D(S^{n-1}) \min_{\theta \in S^{n-1}} \int_K |(x, \theta)|^p \, d\mu(x) \leq \int_K \|x\|_D^p \, d\mu(x).
$$

In the case $K \subseteq D$, we have $\|x\|_D \leq \|x\|_K \leq 1$ on $K$, so that the last integral does not exceed $\mu(K) = 1$, and thus

$$
(2.1) \quad \nu_D(S^{n-1}) \min_{\theta \in S^{n-1}} \int_K |(x, \theta)|^p \, d\mu(x) \leq 1.
$$

In order to estimate the left-hand side of (2.1) from below, we represent the value $\nu_D(S^{n-1})$ as the integral $\int_{S^{n-1}} |x|^p \, d\nu_D(x)$ and apply the well-known formula

$$
|x|^p = \frac{\Gamma \left( \frac{p+1}{2} \right)}{2 \pi^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right)} \int_{S^{n-1}} |(x, \theta)|^p \, d\theta, \quad x \in \mathbb{R}^n
$$

(see for example [11] Lemma 3.12). Using (2.3), this yields the representation

$$
\nu_D(S^{n-1}) = \frac{\Gamma \left( \frac{p+1}{2} \right)}{2 \pi^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right)} \int_{S^{n-1}} |(x, \theta)|^p \, d\nu_D(x)
$$

$$
= \frac{\Gamma \left( \frac{p+1}{2} \right)}{2 \pi^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right)} \int_{S^{n-1}} \|\theta\|_D^p \, d\theta.
$$

The last integral may be related to the volume of $D$, by using the polar formula for the volume of $D$,

$$
n |D| = \int_{S^{n-1}} \|\theta\|_D^n \, d\theta = \int_{S^{n-1}} \|\theta\|_D^n \, d\sigma_{n-1}(\theta),
$$

where $\sigma_{n-1}$ denotes the normalized Lebesgue measure on $S^{n-1}$ and $s_{n-1} = \frac{(n \pi)^{n/2}}{\Gamma \left( \frac{n}{2} \right)}$ is its $(n-1)$-dimensional volume. Namely, by Jensen’s inequality, we have

$$
\int \|\theta\|_D^n \, d\sigma_{n-1}(\theta) \geq \left( \int \|\theta\|_D^n \, d\sigma_{n-1}(\theta) \right)^{\frac{n}{n-1}},
$$

or equivalently

$$
\int \|\theta\|_D^n \, d\theta \geq s_{n-1}^{\frac{n}{n-1}} (n |D|)^{-\frac{n}{n-1}}.
$$

Thus,

$$
\nu_D(S^{n-1}) \geq \frac{\Gamma \left( \frac{p+1}{2} \right)}{2 \pi^{\frac{n-1}{2}} \Gamma \left( \frac{n+1}{2} \right)} \frac{s_{n-1}^{\frac{n+1}{2}}}{s_{n-1}^{\frac{n}{2}}} n |D|^{-\frac{n}{2}}
$$

$$
= \frac{\sqrt{n}}{\pi} \frac{\Gamma \left( \frac{p+1}{2} \right)}{\Gamma \left( \frac{p+1}{2} \right)} \frac{s_{n-1}^{\frac{n}{2}}}{s_{n-1}^{\frac{n}{2}}} \frac{n |D|}{n |D|} \geq \frac{c^p}{\Gamma \left( \frac{p+1}{2} \right)} |D|^{-\frac{n}{2}},
$$

where $c > 0$ is an absolute constant. Here we used the well-known asymptotic relation $\sqrt{n} s_{n-1}^{\frac{n}{2}} \rightarrow c_0$ as $n \rightarrow \infty$, for some absolute $c_0 > 0$, as well as the estimate $\Gamma \left( \frac{p+1}{2} \right) / \Gamma \left( \frac{n}{4} \right) \geq (cn)^{p/2}$. 
Applying this lower estimate on the left-hand side of (2.1), we get
\[
\min_{\theta \in S^{n-1}} \int_K |(x, \theta)|^p \, d\mu(x) \leq C^p \Gamma\left(\frac{p+1}{2}\right) |D|^\frac{1}{p}.
\]
It remains to take the minimum over all admissible $D$ and note that $\Gamma\left(\frac{p+1}{2}\right)^{1/p} \leq c\sqrt{p}$ for $p \geq 1$.

To prove Lemma 1.5, we need the following simple assertion.

**Lemma 2.1.** Given a measurable function $g : \mathbb{R} \to [0, 1]$, the function
\[
q \mapsto \left(\frac{q+1}{2} \int_{-\infty}^{\infty} |t|^q g(t) \, dt\right)^{\frac{1}{p+1}}
\]
is non-decreasing on $(-1, \infty)$.

**Proof.** The standard argument is similar to the one used in the proof of Lemma 2.4 in [7]. Given $-1 < q < p$, let $A > 0$ be defined by
\[
\int_{-\infty}^{\infty} |t|^q g(t) \, dt = \int_{-A}^{A} |t|^q \, dt = \frac{2}{q+1} A^{q+1}.
\]
Using $|t|^p \leq A^{p-q} |t|^q \ (|t| \leq A)$ and $|t|^p \geq A^{p-q} |t|^q \ (|t| \geq A)$, together with the assumption $0 \leq g \leq 1$, we then have
\[
\int_{|t| \leq A} (1 - g(t)) |t|^p \, dt - \int_{|t| > A} g(t) |t|^p \, dt \leq A^{p-q} \left( \int_{|t| \leq A} (1 - g(t)) |t|^q \, dt - \int_{|t| > A} g(t) |t|^q \, dt \right) = 0.
\]
Hence
\[
\int_{-\infty}^{\infty} g(t) |t|^p \, dt \geq \int_{-A}^{A} |t|^p \, dt = \frac{2}{p+1} A^{p+1},
\]
that is,
\[
\left(\frac{p+1}{2} \int_{-\infty}^{\infty} g(t) |t|^p \, dt\right)^{\frac{1}{p+1}} \geq A = \left(\frac{q+1}{2} \int_{-\infty}^{\infty} g(t) |t|^q \, dt\right)^{\frac{1}{q+1}}.
\]

**Proof of Lemma 1.5.** One may assume that $f$ is integrable. For $t \in \mathbb{R}$, introduce the hyperplanes $H_t = \{(x, \xi) = t\}$. Since $f$ is Borel measurable on $\mathbb{R}^n$, the function
\[
g(t) = \frac{\int_{H_t} f(x) \, dx}{\sup_s \int_{H_s} f(x) \, dx}
\]
is Borel measurable on the line and satisfies $\|g\|_\infty = 1$. By Fubini’s theorem,
\[
\int_{-\infty}^{\infty} |t|^p g(t) \, dt = \int \int_{(x, \xi) = t} |(x, \xi)|^p f(x) \, dx \, d\mu(x),
\]
\[
\int_{-\infty}^{\infty} g(t) \, dt = \int \frac{f(x) \, dx}{\sup_s \int_{H_s} f(x) \, dx}.
\]
Applying Lemma 2.1 to the function \( g \) with \( q = 0 \) and \( p \), we get
\[
\frac{1}{2} \int_{-\infty}^{\infty} g(t) \, dt \leq \left( \frac{p + 1}{2} \int_{-\infty}^{\infty} |t|^p g(t) \, dt \right)^{\frac{1}{p+1}},
\]
which in our case becomes
\[
\left( \int f(x) \, dx \right)^{p+1} \leq (p + 1) \left( 2 \sup_s \int_{H_s} f(x) \, dx \right)^p \int |(x, \xi)|^p f(x) \, dx.
\]

Proof of Theorem 1.9. Let \( D \in L^n_p \) be such that the distance \( d_{ovr}(M, L^n_p) \) is almost realized, i.e., for small \( \delta > 0 \), suppose that \( D \subseteq M \subseteq (1+\delta) d_{BM}(M, L^n_p) \).

Integrating both sides of (1.7) over \( \xi \in S^{n-1} \) with respect to the measure \( \nu_D \) from (1.3), we get
\[
\int_K \|x\|^p_D f(x) \, dx \leq \int_M \|x\|^p_D f(x) \, dx.
\]
Equivalently, using the integrals in spherical coordinates, we have
\[
0 \leq \int_{S^{n-1}} \|\theta\|^p_D \left( \int_{\|\theta\|^{-1}_K} r^{n+p-1} f(r\theta) \, dr \right) d\theta = \int_{S^{n-1}} \|\theta\|^p_M I(\theta) \, d\theta,
\]
where
\[
I(\theta) = \|\theta\|^p_M \int_{\|\theta\|^{-1}_K} r^{n+p-1} f(r\theta) \, dr.
\]

For \( \theta \in S^{n-1} \) such that \( \|\theta\|_K \geq \|\theta\|_M \), the latter quantity is non-negative, and one may proceed by writing
\[
I(\theta) = \int_{\|\theta\|^{-1}_K} \left( \|\theta\|^p_M - r^{-p} \right) r^{n+p-1} f(r\theta) \, dr + \int_{\|\theta\|^{-1}_M} \|\theta\|^p_M r^{n-1} f(r\theta) \, dr \leq \int_{\|\theta\|^{-1}_K} r^{n-1} f(r\theta) \, dr.
\]
But, in the case \( \|\theta\|_K \leq \|\theta\|_M \), we have
\[
-I(\theta) = \|\theta\|^p_M \int_{\|\theta\|^{-1}_M} r^p r^{n-1} f(r\theta) \, dr \geq \int_{\|\theta\|^{-1}_M} r^{n-1} f(r\theta) \, dr,
\]
which is the same upper bound on \( I(\theta) \) as before. Thus,
\[
0 \leq \int_{S^{n-1}} \|\theta\|^p_D \left( \int_{\|\theta\|^{-1}_K} r^{n-1} f(r\theta) \, dr \right) d\theta,
\]
that is,
\[
\int_{S^{n-1}} \|\theta\|^p_D \left( \int_0^{\|\theta\|^{-1}_K} r^{n-1} f(r\theta) \, dr \right) d\theta \leq \int_{S^{n-1}} \|\theta\|^p_M \left( \int_0^{\|\theta\|^{-1}_M} r^{n-1} f(r\theta) \, dr \right) d\theta.
\]
Now, by the choice of \( D \),
\[
\|\theta\|_M \leq \|\theta\|_D \leq (1+\delta) d_{BM}(M, L^n_p) \|\theta\|_M.
\]
for every $\theta \in S^{n-1}$. Hence

$$\int_K f(x) \, dx = \int_{S^{n-1}} \left( \int_0^{\|\theta\|^{-1}_K} r^{n-1} f(r\theta) \, dr \right) \, d\theta$$

$$\leq \int_{S^{n-1}} \frac{\|\theta\|^p_D}{\|\theta\|^p_M} \left( \int_0^{\|\theta\|^{-1}_K} r^{n-1} f(r\theta) \, dr \right) \, d\theta$$

$$\leq \int_{S^{n-1}} \frac{\|\theta\|^p_D}{\|\theta\|^p_M} \left( \int_0^{\|\theta\|^{-1}_M} r^{n-1} f(r\theta) \, dr \right) \, d\theta$$

$$\leq (1 + \delta) d^p_{BM}(M, L^n_p) \int_{S^{n-1}} \left( \int_0^{\|\theta\|^{-1}_M} r^{n-1} f(r\theta) \, dr \right) \, d\theta$$

$$= (1 + \delta) d^p_{BM}(M, L^n_p) \int_M f(x) \, dx.$$ 

Sending $\delta$ to zero, we get the result. \qed

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