Conformal Mass in AdS gravity

Dileep P. Jatkar$^1$*, Georgios Kofinas$^2$†, Olivera Miskovic$^3$‡, Rodrigo Olea$^4$§

$^1$Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Allahabad 211019, India
$^2$Research Group of Geometry, Dynamical Systems and Cosmology
Department of Information and Communication Systems Engineering
University of the Aegean, Karlovassi 83200, Samos, Greece
$^3$Instituto de Física, Pontificia Universidad Católica de Valparaíso,
Casilla 4059, Valparaíso, Chile
$^4$Departamento de Ciencias Físicas,
Universidad Andres Bello, República 220, Santiago, Chile

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Abstract

We show that the Ashtekar-Magnon-Das (AMD) mass and other conserved quantities are equivalent to the Kounterterm charges in the asymptotically AdS spacetimes that satisfy the Einstein equations, if we assume the same asymptotic fall-off behavior of the Weyl tensor as considered by AMD. This therefore implies that, in all dimensions, the conformal mass can be directly derived from the bulk action and the boundary terms, which are written in terms of the extrinsic curvature.

* dileep@hri.res.in
† gkofinas@aegean.gr
‡ olivera.miskovic@ucv.cl
§ rodrigo.olea@unab.cl
1 Introduction

The AdS/CFT correspondence [1] is an intriguing relation between the theory of gravity defined on the anti-de Sitter (AdS) space and the conformal field theory (CFT) defined on the boundary. This correspondence gives a concrete dictionary for the holographic description of the bulk gravity in terms of the boundary data of the conformal field theory [2, 3, 4]. This, to start with, involves matching the symmetries of the theories involved in the correspondence. In a diffeomorphism invariant theory, conserved quantities are written in terms of integrals over a surface of codimension 2, which is just a space-like slice of the space-time boundary. These asymptotic symmetries are identified with the Killing vector fields of the boundary space-time.

There are various definitions of these conserved quantities in the literature. For example, Ashtekar and Magnon showed that any conserved charge in asymptotically AdS (AAdS) gravity in four dimensions can be expressed as a surface integral of a quantity involving only the electric part of the asymptotic Weyl tensor and the asymptotic Killing field [5]. The result was generalized by Ashtekar and Das to higher dimensions [6]. This Ashtekar-Magnon-Das (AMD) formula for conserved charges in AAdS spacetimes was formally proved in Ref.[7] using the covariant phase space formalism of Wald [9]. In addition, there are approaches based on a linearized expansion around AdS background [10], or on the Hamiltonian formalism (e.g., Henneaux-Teitelboim [11]), or the holographic counterterm subtraction method of Henningson-Skenderis [12], developed also by Balasubramanian and Kraus in Ref.[13]. For a good comparison of various techniques we refer the reader to Ref.[7].

The AMD method is based on the Penrose’s conformal completion techniques [14] that bring the boundary at infinity of the D-dimensional AAdS space $\mathcal{M}$ to a finite distance. If the physical spacetime $\mathcal{M}$ is endowed with the metric $g_{\mu\nu}$ which obeys the Einstein equations, then its conformal completion consists of attaching the boundary, with topology $\mathbb{R} \times S^{D-2}$, by means of the conformal mapping

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}. \quad (1.1)$$

The unphysical spacetime $(\tilde{\mathcal{M}}, \tilde{g}_{\mu\nu})$ obtained in this manner has the smooth boundary $\partial \tilde{\mathcal{M}}$ set at a finite distance if the conformal factor $\Omega$ vanishes on $\partial \tilde{\mathcal{M}}$ and its derivative is finite, $\tilde{\nabla}_\mu \Omega = \tilde{n}_\mu \neq 0$. The previous set of conditions provides a definition of AAdS spacetime that will be useful to analyze the fall-off of physical quantities in the next section.

In particular, for asymptotic AdS spaces, which have time-like boundary, these conditions ensure that $\Omega$ can play the role of the radial coordinate in the neighborhood of $\partial \tilde{\mathcal{M}}$. The conformal factor $\Omega$ is then related to the Schwarzschild-like coordinate $r$ in the asymptotic region as $\Omega \sim 1/r$ and $\tilde{n}_\mu$ is the outward pointing normal to $\partial \tilde{\mathcal{M}}$.

The conserved quantities in the AMD approach are written in terms of the electric part of the Weyl tensor. The asymptotic behavior of the Weyl tensor is therefore important in determining the expressions for conserved charges. The fall-off behavior of the physical
Weyl tensor \( W_{\alpha\beta}^{\mu\nu} \) can be deduced as follows \([6]\). For the global AdS space, the Weyl tensor vanishes identically, \( W_{\alpha\beta}^{\mu\nu} = 0 \). Thus, a non-vacuum state with total mass \( M \) should satisfy \( W_{\alpha\beta}^{\mu\nu} \sim GM/r^{D-1} \) asymptotically, where the power factor \( D - 1 \) is determined purely by dimensional analysis. Using the fact that near the boundary the conformal factor falls off as \( \Omega \sim 1/r \), one can show that the fall-off behavior of the unphysical Weyl tensor is such that \( \Omega^{4-D} \tilde{W}_{\beta\nu\alpha\mu} \) is smooth on \( \tilde{\mathcal{M}} \) and it vanishes on the boundary \( \partial \tilde{\mathcal{M}} \). In fact, in Ref.\([7]\) it was proved that this fall-off property is always satisfied in the AAdS spacetimes.

The electric part of the unphysical Weyl tensor corresponds to the Weyl tensor projected to the boundary \( \partial \tilde{\mathcal{M}} \) that is parameterized by the local coordinates \( \tilde{x}^i \),

\[
\tilde{E}^i_j = \frac{1}{D-3} \Omega^{3-D} \tilde{W}^{i\mu}_j \tilde{n}_\mu \tilde{n}_\nu ,
\]

and it is trace-free and symmetric by definition. For the spacetimes satisfying the Einstein’s equations, the leading order boundary expression of the electric part of the unphysical Weyl tensor is also finite and divergenceless \([7]\). As a consequence, there exists a finite conserved charge \( \mathcal{H}[\xi] \) for every asymptotic symmetry \( \xi^i \) for a given choice of boundary conditions,

\[
\mathcal{H}[\xi] = -\frac{\ell}{8\pi G} \int_{\Sigma_\infty} d\tilde{\Sigma} \tilde{E}^i_j \xi^i \tilde{u}_j ,
\]

where the integral is on the spatial section \( \tilde{\Sigma}_\infty \) of the boundary and \( \ell \) is the AdS radius. Here, \( d\tilde{\Sigma} = \sqrt{\sigma} d^{D-2} \tilde{x} \) is the surface element of \( \tilde{\Sigma}_\infty \) and \( \tilde{u}_j \) is the unit timelike normal to \( \tilde{\Sigma}_\infty \).

For AAdS spacetimes, unphysical quantities correspond to regular ones at spatial infinity. In order to write down \( \mathcal{H}[\xi] \) in terms of tensors defined with the physical metric \( g_{\mu\nu} \), we need to rescale them as

\[
\begin{align*}
\tilde{n}_\mu &= \Omega n_\mu , \\
\tilde{u}_\mu &= \Omega u_\mu , \\
d\tilde{\Sigma} &= \Omega^{D-2} d\Sigma , \\
\tilde{\tilde{W}}^{\beta\nu}_{\alpha\mu} &= \Omega^{-2} W^{\beta\nu}_{\alpha\mu} .
\end{align*}
\]

The Killing vector is invariant under rescaling, as it is a part of the asymptotic conformal isometry, and hence, depends only on the conformal class of the boundary metric and not on any specific representative. This gives rise to

\[
\tilde{E}^i_j = \Omega^{1-D} E^i_j .
\]

These scaling properties are identical to those derived using dimensional analysis since \( \tilde{\Sigma}_\infty \) is a \((D-2)\)-dimensional surface and \( \tilde{W}^\alpha_{\beta\mu} = \tilde{W}^\alpha_{\beta\mu} \). Thus, it is easy to see that \( \tilde{E}^i_j d\tilde{\Sigma} \tilde{u}_j = E^i_j d\Sigma u_j \) and the conformal mass (and other conserved charges) are expressed as

\[
\mathcal{H}[\xi] = -\frac{\ell}{8\pi G} \int_{\Sigma_\infty} d\Sigma E^i_j \xi^i u_j .
\]
Since the Weyl tensor vanishes identically for the global AdS space, the charge in the AMD approach vanishes as well. This, in turn, implies that it does not include the vacuum energy of the AdS space. On the other hand, in the AdS/CFT approach, a vacuum energy is associated to odd-dimensional AAdS spacetimes, and it matches the Casimir energy in a holographically dual CFT. The expressions for conserved charges agree in both approaches up to finite terms [6, 7, 8]. The AMD formula for a conserved charge in AAdS spaces gives finite, that is regularized, quantity and moreover it takes a very simple form because it is expressed in terms of the electric part of the Weyl tensor only.

When the AMD formula for conserved charges is compared with that obtained using the holographic methods [13], or the holographic renormalization techniques [12], the result agrees, up to two caveats: (i) The conserved charge derived using the holographic methods always gives rise to the vacuum energy in odd dimensions, whereas as stated above, the AMD charge is designed in such a way that it vanishes in the pure AdS background, and hence, it does not contain the vacuum energy piece by construction, and (ii) The holographic methods contain additional finite terms. These kind of terms are scheme-dependent and do not contribute to the charge, after all, the physical quantity should be independent of choice of regularization.

We should emphasize here that the comparison between the AMD charge and the holographic charge has been done only up to five dimensions [6, 7, 8], partly because it becomes technically difficult to carry out this comparison in higher dimensions. On the other hand, the charge formula based on the Kounterterm method [15] is known in any dimension and for any Lovelock AdS gravity [16]. When the charges derived using the Kounterterm method are compared with those obtained from the holographic methods for some classes of Lovelock theories, it was shown [17] that, up to the finite counterterms, they are in agreement with each other including the vacuum energy. This is a consequence of the fact that, in AAdS spacetimes, the extrinsic curvature can be expanded in terms of intrinsic quantities of the boundary [8].

In this paper we show that the Kounterterm charges without vacuum energy are in exact agreement with the AMD charges, provided we consider the same asymptotic fall-off behavior for the Weyl tensor as considered by AMD. We hasten to point out here that while this behavior of the Weyl tensor was essentially assumed by AMD, it was put on firmer footing in [7]. Combining our results with those of [7, 17], we see that the Kounterterm charges are natural candidates to carry out comparison with both the AMD charges and charges obtained using the holographic methods. Although direct comparison of AMD and holographic methods is cumbersome, the Kounterterm method has the advantage that it can be compared with either of them with much less technical difficulty.
2  Kounterterm charges in Einstein-Hilbert AdS gravity

The Einstein-Hilbert AdS gravity in $D$ dimensions is described by the action

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^Dx \sqrt{-g} \left( R - 2\Lambda \right) + c_{D-1} \int_{\partial\mathcal{M}} d^{D-1}x B_{D-1}, \quad (2.1)$$

where $G$ is the gravitational constant in $D$ dimensions and $\Lambda = -(D-1)(D-2)/2\ell^2$ is the cosmological constant expressed in terms of the radius $\ell$ of AdS space. The scalar curvature $R$ is constructed from the Riemann curvature $R^{\mu\nu\alpha\beta}$ of the spacetime, and $B_{D-1}, c_{D-1}$ denote the boundary term and its coupling constant respectively, as we will discuss below.

The Weyl tensor

$$W_{\alpha\beta\mu\nu} = R_{\alpha\beta\mu\nu} - \frac{1}{D-2} \left( \delta_{\alpha}^{\mu} R_{\beta}^{\nu} - \delta_{\beta}^{\mu} R_{\alpha}^{\nu} - \delta_{\alpha}^{\nu} R_{\beta}^{\mu} + \delta_{\beta}^{\nu} R_{\alpha}^{\mu} \right) + \frac{R}{(D-1)(D-2)} \delta^{[\mu\nu]}_{[\alpha\beta]} \quad (2.2)$$

vanishes for the global AdS space, which is a vacuum of the theory. (For definition of antisymmetrized Kronecker delta, see Appendix.) This can be made explicit by noting that the Einstein equation gives $R_{\mu\nu} = -(D-1)g_{\mu\nu}/\ell^2$ and $R = -D(D-1)/\ell^2$. Substituting this in the expression for the Weyl tensor, it can be written in terms of the AdS curvature,

$$W_{\mu\nu} = R_{\mu\nu} + \frac{1}{\ell^2} \delta^\alpha_{[\mu} \delta^\beta_{\nu]} \quad (2.3)$$

This expression clearly vanishes for pure AdS space.

We choose the radial foliation of the manifold $\mathcal{M}$ in the local coordinates $x^\mu = (r, x^i)$, which defines the outward-pointing unit normal $n_\mu = (n_r, n_i) = (N, \vec{0})$ to the boundary $\partial\mathcal{M}$. The boundary $\partial\mathcal{M}$ in these local coordinates is placed at constant $r$. The line element on $\mathcal{M}$ can then be written in terms of the Gaussian normal coordinates,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = N^2(r) \, dr^2 + h_{ij}(r, x) \, dx^i dx^j. \quad (2.4)$$

The geometry of the boundary $\partial\mathcal{M}$ is described by the boundary metric $h_{ij}$ at a fixed, large value of $r$, the intrinsic curvature $\mathcal{R}^i_{jkl}(h)$ and the extrinsic curvature

$$K_{ij} = -\frac{1}{2} \mathcal{L}_n h_{ij} = -\frac{1}{2N} h'_{ij}. \quad (2.5)$$

Here, the prime denotes derivative along the normal direction, which in the metric (2.4) corresponds to the radial derivative $\partial_r$. To determine the conserved charge we need to carry out surface integration of the conserved current on a codimension 2 surface in $\mathcal{M}$, which is an intersection of a time-like slice with the boundary. In order to pick this surface, we use a time-like foliation for the line element on $\partial\mathcal{M}$ with the coordinates $x^i = (t, y^m)$ in the form

$$h_{ij} \, dx^i dx^j = -\hat{N}^2(t) dt^2 + \sigma_{mn} (dy^m + \hat{N}^m dt)(dy^n + \hat{N}^n dt), \quad \sqrt{-h} = \hat{N} \sqrt{\sigma}. \quad (2.6)$$
The constant time hypersurfaces \( \Sigma_r \) are uniquely specified by the unit normal \( u_i = (u_t, u_m) = (-\hat{N}, \vec{0}) \) to them. The metric \( \sigma_{mn} \) describes the geometry of \( \Sigma_\infty \), i.e. the asymptotic boundary of spatial section at constant time. The conserved charges \( Q[\xi] \) of the theory, for a given set of asymptotic Killing vectors \( \{\xi\} \), are expressed as integrals over \( \Sigma_\infty \) (whose metric has been defined in Eq.(2.6)),

\[
Q[\xi] = \int_{\Sigma_\infty} d^{D-2}y \sqrt{\sigma} u_j \xi^i \left( q^j_i + q^{(0)i} \right). \tag{2.7}
\]

In general, the charge associated to the term \( q^j_i \) will give rise to the mass and the angular momentum of the black hole solution. In AdS gravity, because of the existence of black strings and planar black holes, the integration in Eq.(2.7) over a different surface (given by a different normal \( u_j \)) gives rise to, e.g., tension of a black string when the Killing vector is associated to translational invariance [18]. On the other hand, the \( q^{(0)i} \) dependent part of the conserved charge is related to the vacuum energy of AAdS spaces, which exists only in odd dimensions.

### 2.1 Even dimensions, \( D = 2n \)

We will now consider the boundary terms \( B_{D-1} \) and the couplings \( c_{D-1} \). In the even number of bulk dimensions, i.e. \( D = 2n \), the boundary term has the form [19]

\[
B_{2n-1} = 2n \sqrt{-h} \int_0^1 dt \delta_{[i_1 \ldots i_{2n-1}]}^{[j_1 \ldots j_{2n-1}]} K_{j_1}^{i_1} \left( \frac{1}{2} R_{j_2j_3}^{i_2i_3} - t^2 K_{j_2}^{i_2} K_{j_3}^{i_3} \right) \times
\]

\[
\ldots \times \left( \frac{1}{2} R_{j_2j_3 \ldots j_{2n-1}}^{i_2i_3 \ldots i_{2n-1}} - t^2 K_{j_2}^{i_2} K_{j_3}^{i_3} \ldots K_{j_{2n-1}}^{i_{2n-1}} \right), \tag{2.8}
\]

where \( R_{kl}^{ij} \) is the Riemann tensor constructed from the metric \( h_{ij} \), and \( K_{ij} \) is the extrinsic curvature defined in (2.5). The coupling constant, which is fixed by the variational principle, is

\[
c_{2n-1} = \frac{1}{16\pi G} \frac{(-1)^n \ell^{2n-2}}{n (2n - 2)!}. \tag{2.9}
\]

The charge density tensor \( q^j_i \) appearing in (2.7) is derived from (2.1) and (2.8),

\[
q^j_i = \frac{(-1)^n \ell^{2n-2}}{16\pi G (2n - 2)! \ell^{2n-2}} \delta_{[i_1 \ldots i_{2n-1}]}^{[j_1 \ldots j_{2n-1}]} K^{i_1}_{j_2j_3} \ldots R_{j_2j_3 \ldots j_{2n-1}}^{i_2i_3 \ldots i_{2n-1}} - \frac{(-1)^{n-1}}{\ell^{2n-2}} \delta_{[i_1j_2]}^{[i_1j_2]} \ldots \delta_{[j_2j_3]}^{[j_2j_3]} \ldots \delta_{[j_{2n-1}j_{2n-1}]}^{[j_{2n-1}j_{2n-1}]}, \tag{2.10}
\]

and \( q^{(0)i} = 0 \) for even dimensions. The expression of the charge density tensor can be put in a convenient form using the algebraic identity

\[
b^{n-1} - (-a)^{n-1} = (n - 1) (b + a) \int_0^1 du \left[u (b + a) - a \right]^{n-2}. \tag{2.11}
\]
This allows us to write an integral representation for the charge density tensor $q_i^j$ as

$$
q_i^j = \frac{(n - 1)(-1)^n \ell^{2n-2}}{16\pi G (2n - 2)!2^{n-2}} \delta_{[i_1\ldots i_{2n-1}]} \delta_{[j_1\ldots j_{2n-1}]} K_i^1 \left( R_{j_2j_3}^{i_2i_3} + \frac{1}{\ell^2} \delta_{[j_2j_3]}^{i_1j_1} \right) \times
$$

$$
\times \frac{1}{\ell^2} \left[ (1 - u) R_{j_4j_5}^{i_4i_5} - \frac{u}{\ell^2} \delta_{[j_4j_5]}^{i_4i_5} \right] \ldots \left[ (1 - u) R_{j_{2n-2}j_{2n-1}}^{i_{2n-2}i_{2n-1}} - \frac{u}{\ell^2} \delta_{[j_{2n-2}j_{2n-1}]}^{i_{2n-2}i_{2n-1}} \right].
$$

Let us now look at the radial dependence of the quantities appearing in the expression of the conserved charge. We will first look at the Weyl tensor, significance of which will become clear momentarily. The fall-off of the boundary components $W_{j_2j_3}^{i_2i_3}$ of the Weyl tensor (2.3) can be deduced as follows [6]. Recall that for the global AdS space the Weyl tensor vanishes identically and as a result we have $W_{j_2j_3}^{i_2i_3} = 0$. For a non-vacuum state with total mass $M$, the Weyl tensor will be nonvanishing and in general its asymptotic behavior will be $W_{j_2j_3}^{i_2i_3} \sim GM/r^s$, where $s$ is a number that will be determined using dimensional analysis. In the natural units, the dimensions are $[G] = (\text{length})^{D-2}$ and $[M] = (\text{length})^{-1}$. Since the Weyl tensor contains two derivatives of the metric, it is $[W] = (\text{length})^{-2}$. Combining these, we find $s = D - 1$. Thus, in the AMD method, the fall-off of the Weyl tensor is typically [7]

$$
W_{j_2j_3}^{i_2i_3} = O(1/r^{D-1}).
$$

Furthermore, using the asymptotic behavior of the metric (2.6) for AAdS spaces à la AMD, i.e. $g_{\mu\nu} \sim \Omega^2 \sim 1/r^2$, we can deduce that the radial dependence of the unit normal that generates the foliation (2.6) is

$$
u_t = O(r),
$$

and the radial dependence of the determinant of the metric on $\Sigma_\infty$ is

$$
\sqrt{\sigma} = O(r^{D-2}).
$$

These fall-off behaviors uniquely determine the behavior of the charge tensor. Notice that the expression of the charge tensor $q_i^j$ contains at least one power of the Weyl tensor. The $r$ dependence of terms in the conserved charge cancels out if only the leading terms in the expansion of the Riemann tensor $R_{kl}^{ij}$ and the extrinsic curvature $K_j^i$ are taken into account. The subleading terms in the expansion of the Riemann tensor $R_{kl}^{ij}$ and the extrinsic curvature $K_j^i$ do not contribute to the conserved charge. This corresponds to the following substitutions in Eq.(2.12),

$$
R_{kl}^{ij} = -\frac{1}{\ell^2} \delta_{[kl]}^{[ij]} + O(1/r^2), \quad K_j^i = -\frac{1}{\ell} \delta_j^i + O(1/r^2).
$$

This substitution in (2.12) completely eliminates its dependence on the parameter $u$, and as a result, the integration over $u$ in Eq.(2.12) becomes trivial,

$$
q_i^j = \frac{(n - 1)(-1)^n \ell^{2n-2}}{16\pi G (2n - 2)!2^{n-2}} \delta_{[i_1\ldots i_{2n-1}]} \delta_{[j_1\ldots j_{2n-1}]} \left( -\frac{1}{\ell^2} \delta_i^1 \right) W_{j_2j_3}^{i_2i_3} \times
$$

$$
\times \int_0^1 du \left( -\frac{1}{\ell^2} \right)^{n-2} \delta_{[j_4j_5]}^{i_4i_5} \ldots \delta_{[j_{2n-2}j_{2n-1}]}^{i_{2n-2}i_{2n-1}} + O(1/r^{D+1}),
$$

(2.17)
and we obtain for the charge density tensor,

\[ q^j_i = -\frac{(n - 1)\ell}{16\pi G (2n - 2)!2^n} \frac{1}{2^n} \delta_{[i_2 \ldots i_{2n-1}]}^{[j_2 \ldots j_{2n-1}]} W_{j_2j_3}^{i_2i_3} \delta_{[i_4 \ldots i_{2n-1}]}^{[j_4 \ldots j_{2n-1}]} \cdots \delta_{[i_{2n-2} i_{2n-1}]}^{[j_{2n-2} j_{2n-1}]} + \mathcal{O}(1/r^{D+1}). \]  (2.18)

Taking into account the fall-off behavior of the Weyl tensor, it is clear that the asymptotic radial dependence of \( q^j_i \) goes as \( 1/r^{D-1} \). This behavior of \( q^j_i \) gives rise to the finite conserved charge (2.7), as expected.

Using multiplicative properties of antisymmetrized Kronecker deltas we can contract them in (2.18) to get a simple expression for the charge tensor,

\[ q^j_i = -\frac{\ell}{32\pi G (2n - 3)} \delta_{[i_2 i_3]}^{[j_2 j_3]} W_{j_2j_3}^{i_2i_3} \]  (2.19)

where we have neglected the \( \mathcal{O}(1/r^{D+1}) \) terms that fall off rapidly at the boundary and hence will give vanishing contribution to the charge. We will now use the fact that the Weyl tensor is traceless, that is \( W_{\mu\alpha}^{\mu\beta} = 0 \), which can be used to write down relations between the components of the Weyl tensor,

\[ W_{kr}^{kr} = 0, \quad W_{ri}^{ri} + W_{kj}^{ki} = 0. \]  (2.20)

Taking trace of the second equation, we also find \( W_{ki}^{ki} = 0 \). As a consequence of these relations we further simplify the expression of \( q^j_i \). It is easy to see that the first term in Eq.(2.19) vanishes, and the second term can be rewritten in terms of \( W_{ri}^{rij} \),

\[ q^j_i = -\frac{\ell}{8\pi G (2n - 3)} W_{ri}^{rij}. \]  (2.21)

On the other hand, the electric part of the Weyl tensor \( E^j_i \), defined in terms of the normal \( n_\mu \) to the boundary, reads

\[ E^j_i = \frac{1}{D-3} W_{ij}^{\mu\nu} n^\mu n^\nu. \]  (2.22)

The Kounterterm charge density tensor can now be written in terms the electric part of the Weyl tensor

\[ q^j_i = -\frac{\ell}{8\pi G} E^j_i. \]  (2.23)

Substituting this in the conserved charge formula we get

\[ Q[\xi] = -\frac{\ell}{8\pi G} \int_{\Sigma_{\infty}} d^{D-2}y \sqrt{\sigma} E^j_i \xi^i u_j = -\frac{\ell}{8\pi G} \int_{\Sigma_{\infty}} d\Sigma E^j_i \xi^i u_j, \]  (2.24)

where

\[ d\Sigma = d^{D-2}y \sqrt{\sigma}. \]  (2.25)

Therefore, the conserved quantities \( Q[\xi] \) coming from Kounterterms regularization in \( D = 2n \) are the same as Eq.(1.6), that is the Ashtekar-Magnon-Das formula (1.3). Thus, we conclude

\[ \mathcal{H}[\xi] = Q[\xi]. \]  (2.26)

In four dimensions, the equivalence between the conformal mass and the Kounterterm charge (2.12) was established using the asymptotic expansion of the metric in Ref.[20].
2.2 Odd dimensions, $D = 2n + 1$

The boundary term that regularizes the Einstein-Hilbert AdS gravity in odd dimensions is given in terms of two parametric integrations as [15]

$$B_{2n} = 2n\sqrt{-h} \int_{0}^{1} du \int_{0}^{u} ds \delta_{[i_{1}\cdots i_{2n}]}^{[j_{1}\cdots j_{2n}]} K_{j_{1}}^{i_{1}} \delta_{j_{2}}^{i_{2}} \left( \frac{1}{2} R_{j_{3}j_{4}}^{i_{3}i_{4}} - u^{2} K_{j_{3}}^{i_{3}} K_{j_{4}}^{i_{4}} + \frac{s^{2}}{\ell_{\text{eff}}^2} \delta_{j_{3}}^{i_{3}} \delta_{j_{4}}^{i_{4}} \right) \times$$

$$\cdots \times \left( \frac{1}{2} R_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} - u^{2} K_{j_{2n-1}}^{i_{2n-1}} K_{j_{2n}}^{i_{2n}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{2n-1}}^{i_{2n-1}} \delta_{j_{2n}}^{i_{2n}} \right), \quad (2.27)$$

where the corresponding coupling constant, obtained from the variational principle, has the form

$$c_{2n} = -\frac{1}{16\pi G n(2n-1)!} \left[ \int_{0}^{1} du \left( u^{2} - 1 \right)^{n-1} \right]^{-1}. \quad (2.28)$$

This integral representation will be useful later.

The Noether charge is written as an integral over the sum of two tensors as in Eq. (2.7). The term $q_{i}^{j(0)i}$ in Eq. (2.7) no longer vanishes in odd dimensions. The first part takes the form

$$q_{i}^{j} = \frac{1}{16\pi G (2n-1)!2^{n-2}} \delta_{[i_{1}i_{2}\cdots i_{2n}]}^{[j_{1}j_{2}\cdots j_{2n}]} K_{j_{1}}^{i_{1}} \delta_{j_{2}}^{i_{2}} \left[ \delta_{j_{3}j_{4}}^{i_{3}i_{4}} \cdots \delta_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} \right]$$

$$+ 16\pi G (2n-1)! nc_{2n} \int_{0}^{1} du \left( R_{j_{3}j_{4}}^{i_{3}i_{4}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{3}j_{4}}^{i_{3}i_{4}} \right) \cdots \left( R_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} \right). \quad (2.29)$$

The second part, namely $q_{(0)i}^{j}$, of the charge represents a covariant formula for the vacuum energy for any asymptotically AdS spacetime

$$q_{(0)i}^{j} = nc_{2n} \int_{0}^{1} du \left[ \delta_{[i_{1}i_{2}\cdots i_{2n}]}^{[j_{1}j_{2}\cdots j_{2n}]} \left( K_{j_{1}}^{i_{1}} \delta_{j_{2}}^{i_{2}} + K_{j_{1}}^{i_{1}} \delta_{j_{2}}^{i_{2}} \right) \left( \frac{1}{2} R_{j_{3}j_{4}}^{i_{3}i_{4}} - u^{2} K_{j_{3}}^{i_{3}} K_{j_{4}}^{i_{4}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{3}j_{4}}^{i_{3}i_{4}} \right) \times \right.$$\n
$$\cdots \times \left( \frac{1}{2} R_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} - u^{2} K_{j_{2n-1}}^{i_{2n-1}} K_{j_{2n}}^{i_{2n}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} \right) \right). \quad (2.30)$$

The latter term $q_{(0)i}^{j}$, which corresponds to the vacuum energy of the asymptotically AdS space, is not considered in the AMD method. We will therefore restrict our current discussion to the charge (2.29) and ignore the term (2.30).

In order to perform a comparison, we take the charge (2.29) and express it as

$$q_{i}^{j} = \frac{nc_{2n}}{2^{n-2}} \int_{0}^{1} du \left[ \delta_{[i_{1}i_{2}\cdots i_{2n}]}^{[j_{1}j_{2}\cdots j_{2n}]} K_{j_{1}}^{i_{1}} \delta_{j_{2}}^{i_{2}} \left( R_{j_{3}j_{4}}^{i_{3}i_{4}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{3}j_{4}}^{i_{3}i_{4}} \right) \cdots \left( R_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} + \frac{u^{2}}{\ell_{\text{eff}}^2} \delta_{j_{2n-1}j_{2n}}^{i_{2n-1}i_{2n}} \right) \right]$$

$$- \left( \frac{u^{2} - 1}{\ell_{\text{eff}}^2} \right)^{n-1} \delta_{[i_{1}i_{2}\cdots i_{2n}]}^{[j_{1}j_{2}\cdots j_{2n}]} \cdots \delta_{[j_{2n-1}j_{2n}]}^{[i_{2n-1}i_{2n}]} \right), \quad (2.31)$$
with the explicit form (2.28) of the constant $c_{2n}$.

As in the even-dimensional case, we can show that the charge is factorizable over $\left(R + \frac{1}{\ell^2} \delta^{[2]}\right)$, by employing the formula (2.11). To do this, we introduce another integration parameter $s$ and choose the coefficients $b = R + \frac{u^2}{\ell^2} \delta^{[2]}$ and $a = \frac{1-u^2}{\ell^2} \delta^{[2]}$. Using this integral representation we can rewrite the charge density tensor (2.31) as

$$
q_i^j = n(n-1) \frac{c_{2n}}{2^{n-2}} \frac{\delta^{[jj_2\cdots jj_{2n}]}}{[i_1i_2\cdots i_{2n}]} K_{i_1}^{i_2} \left( R_{i_3j_4}^{i_3j_4} + \frac{1}{\ell^2} \delta^{[i_3i_4]} \right) \times
$$

$$
\int_0^1 du \int_0^1 ds \left[ s \left( R_{j_3j_6}^{j_3j_6} + \frac{1}{\ell^2} \delta^{[i_7i_6]} \right) + \frac{u^2-1}{\ell^2} \delta^{[i_7i_6]} \right] \times \cdots \\
\cdots \times \left[ s \left( R_{j_2n-1j_2n}^{j_2n-1j_2n} + \frac{1}{\ell^2} \delta^{[i_{2n-1}i_{2n}]} \right) + \frac{u^2-1}{\ell^2} \delta^{[i_{2n-1}i_{2n}]} \right].
$$

(2.32)

Notice that, in three dimensions, $q_i^j$ is zero, what is a reflection of the fact that the Weyl tensor vanishes identically. Therefore, the mass of AAdS black holes in this case comes from Eq.(2.30).

For $n > 1$, using (2.3), we can rewrite (2.32) in terms of the Weyl tensor. When we utilize the fall-off behavior of the Weyl tensor given in Eq.(2.13), we are essentially forced to consider only the leading-order terms in the asymptotic expansion of the curvature tensors, that is Eq.(2.16). We can therefore express the charge density tensor (2.31) as

$$
q_i^j = n(n-1) \frac{c_{2n}}{2^{n-2}} \frac{\delta^{[jj_2\cdots jj_{2n}]}}{[i_1i_2\cdots i_{2n}]} \left( -\frac{1}{\ell} \delta_i^{i_1} \right) \delta_j^{i_2} W_{i_3j_4}^{i_3j_4} \times
$$

$$
\int_0^1 du \int_0^1 ds \left( \frac{u^2-1}{\ell^2} \right)^{n-2} \delta^{[i_7i_6]} \cdots \delta^{[i_{2n-1}i_{2n}]} + \mathcal{O}(1/r^{D+1}).
$$

(2.33)

Since the integrand in (2.33) is independent of $s$, we can carry out the trivial integration over the parameter $s$, and the charge is rewritten as

$$
q_i^j = \frac{(n-1)\ell}{16\pi G (2n-1)! 2^{n-2}} \frac{\int_0^1 du \int_0^1 du \left( u^2-1 \right)^{n-2} \delta^{[jj_2\cdots jj_{2n-1}]} W_{j_2j_3}^{j_2j_3} \delta^{[i_7i_6]} \cdots \delta^{[i_{2n-1}i_{2n-1}]} }{\int_0^1 du \int_0^1 du \left( u^2-1 \right)^{n-1}},
$$

(2.34)

where we again omit writing the $\mathcal{O}(1/r^{D+1})$ terms that vanish on the boundary.

Using the properties of product of antisymmetrized Kronecker deltas listed in the Appendix and the relation

$$
\frac{1}{\int_0^1 du \left( u^2-1 \right)^{n-2}} \int_0^1 du \left( u^2-1 \right)^{n-2} = -\frac{2n-1}{2(n-1)},
$$

(2.35)
we can further simplify the formula to
\[
q^i_j = -\frac{\ell}{32\pi G (2n-2)!} (2n-3)! \delta_{[i_2i_3]}^{[j_2j_3]} W^{i_2i_3}_{j_2j_3}. \tag{2.36}
\]

The above formula (2.36) is equivalent to
\[
q^i_j = -\frac{\ell}{32\pi G (2n-2)} \delta_{[i_2i_3]}^{[j_2j_3]} W^{i_2i_3}_{j_2j_3}
= \frac{\ell}{8\pi G (2n-2)} W^{kj}_{ki}. \tag{2.37}
\]

This form of the charge density tensor is equivalent to that in even dimensions. Hence it can be again written in terms of the electric part of the Weyl tensor as
\[
q^i_j = -\frac{\ell}{8\pi G} E^j_i. \tag{2.38}
\]

Thus, we find that in the odd number of dimensions too the conformal mass and the mass derived from the Kounterterms are related in a similar way, namely,
\[
\mathcal{H}[\xi] = Q[\xi]. \tag{2.39}
\]

This proves that the conserved quantities, derived from the action supplemented with Kounterterms, are in exact agreement with the AMD charges (1.3) in any dimension.

## 3 Conclusions

We have provided an explicit comparison between conformal mass in AAdS gravity and Kounterterm charges in all dimensions.

The agreement between these different notions of conserved quantities in AdS gravity seems to indicate that most of the holographic information of AAdS spacetimes is encoded in the electric part of the Weyl tensor. The only difference with respect to holographic charges appears in the odd-dimensional case, where there is a piece that gives rise to the vacuum energy.

AMD charges are obtained from the asymptotic resolution of the bulk field equations to the relevant order in the conformal factor $\Omega$, such that the addition of boundary terms to the action does not play any role in their derivation. The proof given here links the definition of conformal mass to the addition of extrinsic counterterms in all dimensions, and the fact that the variation of the total action is factorizable over the Weyl tensor.

The Kounterterm regularization can be extended to the Einstein-Gauss-Bonnet gravity for all values of the Gauss-Bonnet coupling, where two AdS branches can be defined. The charge formulae for this theory can be factorized in a similar fashion as in Eqs.(2.12) and
(2.32) for the Einstein gravity, but the relation (2.3) does not hold in that case. However, it can be seen that Eq. (2.22) is proportional to the boundary Weyl tensor in Einstein-Hilbert gravity. Thus, one can use the fact that, at a linearized level, charge expressions for Einstein-Gauss-Bonnet are proportional to the boundary Weyl tensor to work out a conformal mass definition in this gravity theory.

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A Conventions

We will list here our notation and conventions used in the main text. The antisymmetrized Kronecker delta is defined as

\[
\delta_{[\mu_1 \cdots \mu_p]}^{[\nu_1 \cdots \nu_m]} := \left| \begin{array}{cccc}
\delta_{\mu_1}^{\nu_1} & \delta_{\mu_1}^{\nu_2} & \cdots & \delta_{\mu_1}^{\nu_m} \\
\delta_{\mu_2}^{\nu_1} & \delta_{\mu_2}^{\nu_2} & \cdots & \delta_{\mu_2}^{\nu_m} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{\mu_m}^{\nu_1} & \delta_{\mu_m}^{\nu_2} & \cdots & \delta_{\mu_m}^{\nu_m} \\
\end{array} \right|. \tag{A.1}
\]

In \(d\) dimensions, a contraction of \(k \leq p\) indices in the Kronecker delta of rank \(p\) produces a delta of rank \(p - k\),

\[
\delta_{[\mu_1 \cdots \mu_k \cdots \mu_p]}^{[\nu_1 \cdots \nu_k \cdots \nu_p]} \delta_{\nu_1}^{\mu_1} \cdots \delta_{\nu_k}^{\mu_k} = \frac{(d - p + k)!}{(d - p)!} \delta_{[\mu_{k+1} \cdots \mu_p]}^{[\nu_{k+1} \cdots \nu_p]}. \tag{A.2}
\]

We work with the Gaussian normal coordinates in which the metric takes the form

\[
ds^2 = g_{\mu \nu} dx^\mu dx^\nu = N^2(r) dr^2 + h_{ij}(r, x) dx^i dx^j. \tag{A.3}
\]

The Riemann curvature tensor is defined as

\[
R^\mu_{\nu \lambda \rho} = \partial_\lambda \Gamma^\mu_{\nu \rho} - \partial_\rho \Gamma^\mu_{\nu \lambda} + \Gamma^\mu_{\sigma \lambda} \Gamma^\sigma_{\nu \rho} - \Gamma^\mu_{\sigma \rho} \Gamma^\sigma_{\nu \lambda}, \tag{A.4}
\]

and the extrinsic curvature of the boundary \(\partial \mathcal{M}\) (with the boundary metric \(h_{ij}\)) reads

\[
K_{ij} = -\frac{1}{2} \mathcal{L}_n h_{ij} = -\frac{1}{2N} h'_{ij}. \tag{A.5}
\]
The outward-pointing unit normal of $\partial \mathcal{M}$ is the space-like unit vector, $g_{\mu \nu} n^\mu n^\nu = 1$, with the components $n_\mu = (n_r, n_i) = (N, \vec{0})$.

The line element on $\partial \mathcal{M}$ has the ADM form
\[ h_{ij} \, dx^i dx^j = -\hat{N}^2(t) \, dt^2 + \sigma_{mn} (dy^m + \hat{N}^m dt)(dy^n + \hat{N}^n dt). \]  \hspace{1cm} (A.6)

The unit normal of the surface $\Sigma_r$ (endowed with the metric $\sigma_{mn}$) is the time-like unit vector, $h_{ij} u^i u^j = -1$, with components $u_i = (u_t, u_m) = (-\hat{N}, \vec{0})$.

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2 (1998) 231 [arXiv:hep-th/9711200].

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory”, Phys. Lett. B 428 (1998) 105 [arXiv:hep-th/9802109].

[3] E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. 2 (1998) 253 [arXiv:hep-th/9802150].

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity”, Phys. Rept. 323, (2000), 183 [arXiv:hep-th/9905111].

[5] A. Ashtekar and A. Magnon, “Asymptotically anti-de Sitter space-times”, Class. Quant. Grav. 1 (1984) L39.

[6] A. Ashtekar and S. Das, “Asymptotically Anti-de Sitter space-times: Conserved quantities”, Class. Quant. Grav. 17 (2000) L17 [arXiv:hep-th/9911230].

[7] S. Hollands, A. Ishibashi and D. Marolf, “Comparison between various notions of conserved charges in asymptotically AdS-spacetimes”, Class. Quant. Grav. 22 (2005) 2881 [arXiv:hep-th/0503045].

[8] I. Papadimitriou and K. Skenderis, “Thermodynamics of asymptotically locally AdS spacetimes”, JHEP 0508 (2005) 004 [arXiv:hep-th/0505190].

[9] R. M. Wald and A. Zoupas, “A General definition of 'conserved quantities' in general relativity and other theories of gravity”, Phys. Rev. D 61 (2000) 084027 [arXiv:gr-qc/9911095].

[10] L. F. Abbott and S. Deser, “Stability of Gravity with a Cosmological Constant”, Nucl. Phys. B 195 (1982) 76.

[11] M. Henneaux and C. Teitelboim, “Asymptotically Anti-De Sitter Spaces”, Commun. Math. Phys. 98 (1985) 391.
[12] M. Henningson and K. Skenderis, “The Holographic Weyl anomaly”, *JHEP* **9807** (1998) 023 [arXiv:hep-th/9806087].

[13] V. Balasubramanian and P. Kraus, “A Stress tensor for Anti-de Sitter gravity”, *Commun. Math. Phys.* **208** (1999) 413 [arXiv:hep-th/9902121].

[14] R. Penrose, “Asymptotic properties of fields and space-times”, *Phys. Rev. Lett.* **10** (1963) 66; “Zero rest mass fields including gravitation: Asymptotic behavior”, *Proc. Roy. Soc. Lond. A* **284** (1965) 159.

[15] R. Olea, “Regularization of odd-dimensional AdS gravity: Counterterms”, *JHEP* **0704** (2007) 073 [arXiv:hep-th/0610230].

[16] G. Kofinas and R. Olea, “Universal regularization prescription for Lovelock AdS gravity”, *Phys. Rev. D* **74** (2006) 084035 [arXiv:hep-th/0606253].

[17] O. Misković and R. Olea, “Counterterms in Dimensionally Continued AdS Gravity”, *JHEP* **0710** (2007) 028 [arXiv:0706.4460[hep-th]].

[18] J. H. Traschen and D. Fox, “Tension perturbations of black brane space-times,” *Class. Quant. Grav.* **21** (2004) 289 [arXiv:gr-qc/0103106].

[19] R. Olea, “Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes”, *JHEP* **0506** (2005) 023 [arXiv:hep-th/0504233].

[20] O. Miskovic and R. Olea, “Topological regularization and self-duality in four-dimensional anti-de Sitter gravity”, *Phys. Rev. D* **79** (2009) 124020 [arXiv:0902.2082[hep-th]].