Summary: In this paper we study topological properties of maps constructed by Thimm’s trick with Guillemin and Sternberg’s action coordinates on a connected Hamiltonian $G$-manifold $M$. Since these maps only generate a Hamiltonian torus action on an open dense subset of $M$, convexity and fibre-connectedness of such maps does not follow immediately from Atiyah-Guillemin-Sternberg’s convexity theorem, even if $M$ is compact. The core contribution of this paper is to provide a simple argument circumventing this difficulty.

In the case where the map is constructed from a chain of subalgebras we prove that the image is given by a list of inequalities that can be computed explicitly in many examples. This generalizes the fact that the images of the classical Gelfand-Zeitlin systems on coadjoint orbits are Gelfand-Zeitlin polytopes. Moreover, we prove that if such a map generates a completely integrable torus action on an open dense subset of $M$, then all its fibres are smooth embedded submanifolds.

MSC:
53D20  Momentum maps; symplectic reduction

Keywords:
Hamiltonian $G$-manifold; Hamiltonian torus action; Gelfand-Zeitlin systems; coadjoint orbits

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