Every Planar Graph with the Distance of $5^-$-Cycles at Least 3 from Each Other Is DP-3-Colorable

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Abstract: DP-coloring was introduced by Dvořák and Postle [J. Comb. Theory Ser. B 2018, 129, 38–54]. In this paper, we prove that every planar graph in which the $5^-$-cycles are at distance of at least 3 from each other is DP-3-colorable, which improves the result of Montassier et al. [Inform. Process. Lett. 2008, 107, 3–4] and Yin and Yu [Discret. Math. 2019, 342, 2333–2341].

Keywords: DP-coloring; planar graphs

1. Introduction

All graphs in this paper are finite, undirected, and simple. A planar graph is a graph that can be embedded into the plane. A plane graph is a particular embedding of a planar graph into the plane. We set a plane graph $G = (V, E, F)$, where $V$, $E$, and $F$ are sets of vertices, edges, and faces of $G$, respectively. Two faces are intersecting if they have a common vertex, and are adjacent if they have a common edge. A vertex is incident to a face if it is on the face. A vertex is adjacent to a face if it is not incident to the face but adjacent to a vertex on the face. For a face $f$, if the vertices on $f$ in a cyclic order are $v_1, v_2, \ldots, v_l$, then we write $f = [v_1, v_2, \ldots, v_l]$. The degree $d(x)$ of $x \in V$ is the number of edges incident with $x$. The degree $d(x)$ of $x \in F$ is the number of vertices incident with $x$. Let a $k$-vertex ($k^+$-vertex, $k^-$-vertex) be a vertex of degree $k$ (at least $k$, at most $k$), and a $k$-face ($k^+$-face, $k^-$-face) be a face of degree $k$ (at least $k$, at most $k$). The same notation will be applied to cycles. A $(l_1, l_2, \ldots, l_i)$-face is a $k$-face $f = [v_1v_2 \cdots v_k]$ with $d(v_i) = l_i$, respectively. A $(l_1, l_2)$-edge is an edge $e = v_1v_2$ with $d(v_1) = l_1$. Let $C$ be a cycle of a plane graph $G$, $|C|$ is the length of the cycle $C$. A triangle is a 3-cycle. An edge or a vertex of $G$ is triangular if it is on a triangle. A chord in a cycle $C$ is triangular if it splits the cycle $C$ into a triangle and a cycle of length $d(C) - 1$. We use $Int(C)$ and $Ext(C)$ to denote the sets of vertices located inside and outside of $C$, respectively, and put $Int(C) = G - Ext(C)$, $Ext(C) = G - Int(C)$. The cycle $C$ is called a separating cycle if $Int(C) \neq \emptyset$ and $Ext(C) \neq \emptyset$. A set of independent edges of $G$ is called a matching. Identifying vertices means merging the vertices into a single vertex.

The distance between two vertices $u$ and $v$ in $G$, denoted by $d_G(u, v)$, is the length (number of edges) of the shortest path between $u$ and $v$ in $G$. The distance between two cycles $C$ and $C'$ of $G$, denoted by $d(C, C')$, is defined as follows:

$$d(C, C') = \min \{d_G(u, v) : u \in V(C), v \in V(C')\}.$$
every $v \in V$ such that $\lambda(u) \neq \lambda(v)$ for every edge $uv \in E$. A graph $G$ is $k$-choosable if $G$ is $L$-colorable for every assignment $L$ with $|L(v)| \geq k$. The smallest $k$ such that $G$ is $k$-choosable is called the choice number of $G$ and is denoted by $\chi_l(G)$.

It is well known that the problem of deciding whether a planar graph is 3-colorable is NP-complete. This provides motivation to look for sufficient conditions for planar graphs to be 3-colorable. Grötzsch [1] showed that every planar graph without triangles is 3-colorable. In 1976, Steinberg [2] conjectured that every planar graph without 4-cycle and 5-cycle is 3-colorable. Havel [3] proposed the following condition for a planar graph to be 3-choosable.

**Theorem 1.** Every planar graph with the distance of 5−cycles at least 4 from each other is 3-choosable.

For ordinary coloring, the identification of vertices is involved in the reduction configurations. In list coloring, since different vertices may have different lists, it is not possible for one to use the identification of vertices. To overcome this difficulty, Dvořák and Postle [12] introduced DP-coloring (under the name correspondence coloring) as a generalization of list-coloring.

**Definition 1.** Let $G$ be a simple graph, and $L$ be a list assignment of $V(G)$. For each vertex $v \in V(G)$, let $L(v) = \{v\} \times L(v)$. For each edge $uv$ in $G$, let $M_{uv}$ be a partial matching between the sets $L_u$ and $L_v$, and let $M = \{M_{uv} : uv \in E(G)\}$, called the matching assignment. The matching assignment is called a $k$-matching assignment if $L(v) = \{k\}$ for each $v \in V(G)$.

**Definition 2.** A cover of $G$ is a graph $G_{L,M}$ (simply write $G$) satisfying the following two conditions:

1. the vertex set of $G$ is the disjoint union of $L_v$ for all $v \in V(G)$;
2. the edge set of $G$ is the matching assignment $M$.

Note that the induced subgraph $G[L_v]$ is an independent set for each vertex $v \in V(G)$.

**Definition 3.** Let $G$ be a simple graph, and $G$ be a cover of $G$. An $M$-coloring of $G$ is an independent set $I$ in $G$ such that $|I \cap L_v| = 1$ for each vertex $v \in V(G)$. The graph $G$ is DP-$k$-colorable if, for each $k$-list assignment $L$ and each matching assignment $M$ over $L$, it has an $M$-coloring. The minimum $k$ such that $G$ is DP-$k$-colorable is the DP-chromatic number of $G$, denoted by $\chi_{DP}(G)$.

Let $G_{L,M}$ be a cover of $G$. For a vertex $v \in V(G)$, if $c_1 \in L(v)$ and $c_2 \notin L(v)$, then consider the cover $G'_{L',M'}$ of $G$ such that $L'(v) = (L(v) \cup \{c_2\}) \setminus \{c_1\}$ and $L'(u) = L(u)$ for each $u \in V(G) \setminus \{v\}$. For each $e = (v,c_1)$ in $E(G_{L,M})$, $M'$ is obtained from $M$ by replacing the vertex $(v,c_1)$ by $(v,c_2)$.

An edge $uv \in E(G)$ is straight in a $k$-matching assignment $M$ if every $(u,c_1)(v,c_2) \in E(M_{u,v})$ satisfies $c_1 = c_2$. One can construct a cover of any graph $G$ based on a list assignment for $G$, thus showing that list coloring is a special case of DP-coloring and, in particular, $\chi_{DP}(G) \geq \chi_l(G)$ for all graphs $G$. DP-coloring is quite different from list coloring—for example, Bernshteyn [13] showed that the DP-chromatic number of every graph $G$ with average degree $d$ is $\Omega(d/ \log d)$, while Alon [14] proved that $\chi_l(G) = \Omega(\log d)$ and the bound is sharp.
DP-coloring is a generalization of list-coloring. Dvořák and Postle [12] proved that every planar graph \( G \) without cycles of length from 4 to 8 is 3-choosable. They also noted that \( \chi_{DP}(G) \leq 3 \) if \( G \) is a planar graph without \( \{3, 4\} \)-cycles. Liu and Li [15] proved that every planar graph \( G \) without adjacent cycles of length at most 8 is 3-choosable.

Much attention was drawn to this new coloring; see, for example, [16–19]. Liu et al. [20,21] gave some sufficient conditions for a planar graph to be DP-3-colorable, and DP-4-colorable planar graphs can be found in [22–24]. Yin and Yu [25] gave the following condition for a planar graph to be DP-3-colorable:

**Theorem 2.** Every planar graph without \( \{4, 5\} \)-cycles and the distance of triangles at least 3 is DP-3-colorable.

In this paper, we improve the results in Theorems 1 and 2.

A 9-cycle \( C \) is bad if it is in a subgraph of \( G \) isomorphic to the graphs in Figure 1, and is the outer cycle of the subgraph. A 9-cycle is good if it is not bad.

![Figure 1. Bad 9-cycles. (a) One vertex in \( \text{Int}(C) \); (b) Three vertices are in \( \text{Int}(C) \).](image)

**Theorem 3.** Let \( G \) be a planar graph in which the \( 5^{-} \)-cycles are at distance of at least 3 from each other. Let \( C_0 \) be a \( 8^{-} \)-cycle or a good 9-cycle in \( G \). Then, each DP-3-coloring of \( C_0 \) can be extended to \( G \).

**Corollary 1.** Every planar graph in which the \( 5^{-} \)-cycles are at a distance of at least 3 from each other is DP-3-colorable (thus also 3-choosable).

**Proof.** Let \( G \) be a planar graph. Either \( G \) is \( 4^{-} \)-cycles free or it is not \( 4^{-} \)-cycles free. In the first case, as proved in Reference [12], \( G \) is DP-3-colorable. Thus, we only have to consider the case when \( G \) contains a \( 4^{-} \)-cycle. As proved in Reference [12], the \( 4^{-} \)-cycle can be precolored. Then, by Theorem 3, \( G \) is DP-3-colorable extended from the coloring of the \( 4^{-} \)-cycle when the \( 5^{-} \)-cycles are at distance of at least 3 from each other. \( \square \)

2. **Proof of Theorem 3**

We will prove Theorem 3 by reductio ad absurdum. Let’s start by a temporary assumption that the theorem is wrong. Then, there has to be a non-empty set of counterexamples to this theorem. Assume that \( G \) is a minimal (least number of vertices) counterexample to Theorem 3. Let \( C_0 \) be a \( 8^{-} \)-cycle or a good 9-cycle in \( G \).

**Lemma 1.** For each \( v \in V(G - C_0) \), \( d(v) \geq 3 \).

**Proof.** Let \( v \) be a \( 2^{-} \)-vertex in \( V(G - C_0) \). By the minimality of \( G \), each DP-3-coloring of \( C_0 \) can be extended to \( G - v \). Then, the coloring of \( G - v \) can be extended to \( G \) by selecting a color \( \phi(v) \) for \( v \) such that, for each neighbor \( u \) of \( v \), \( ((u, \phi(u))(v, \phi(v)) \not\in E(M_{uv})) \), a contradiction. \( \square \)

**Lemma 2.** There exist no separating \( 8^{-} \)-cycles or separating good 9-cycles.
Proof. First of all, we show that $C_0$ is not a separating cycle. Otherwise, if $C_0$ is a separating cycle, we may extend the coloring of $C_0$ to both $\text{Int}(C_0)$ and $\text{Ext}(C_0)$, respectively, and then combine them to get a coloring of $G$, a contradiction.

Let $C \neq C_0$ be a separating $8^-$-cycle or separating good 9-cycle in $G$. By the minimality of $G$, the coloring of $C_0$ can be extended to $\text{Ext}(C)$. Now that $C$ is colored, thus the coloring of $C$ can be extended to $\text{Int}(C)$ by the minimality of $G$ again. Combining the inside and outside of $C$, we have a coloring of $G$ extended from the coloring of $C_0$, a contradiction. □

Lemma 3. $C_0$ is the boundary of the out face of the embedding of $G$.

Proof. $C_0$ is not a separating cycle by Lemma 2. Thus, either $\text{Int}(C_0)$ or $\text{Ext}(C_0)$ is empty. Without loss of generality, we assume that $\text{Int}(C_0)$ is empty, and we can redraw the graph to make $\text{Ext}(C_0)$ empty instead. □

Lemma 4. If a 9$^-$-cycle $C$ in $G$ has an internal chord $e$, then $|C| \in \{7, 8, 9\}$ and either $e$ is triangular, or $|C| = 8$ and $e$ splits $C$ into a 4-cycle and a 6-cycle, or $|C| = 9$ and $e$ splits $C$ into a 4-cycle and a 7-cycle.

Proof. Due to fact that the cycles of lengths 3, 4, and 5 in $G$ are at a distance of at least 3 from each other, $C$ cannot have a chord if $|C| \leq 6$ and can have only a triangular one when $|C| = 7$. If $|C| = 8$ and $e$ is not triangular, then $e$ splits $C$ into a 4-cycle and a 6-cycle. If $|C| = 9$ and $e$ is not triangular, then $e$ splits $C$ into a 4-cycle and a 7-cycle, or $e$ splits $C$ into a 5-cycle and a 6-cycle. □

By Lemmas 3 and 4, if a bad 9-cycle $C$ (one type in Figure 1) is a subgraph in $G$, then $C$ must be induced.

Lemma 5. $C_0$ has no chord.

Proof. If $C_0$ contains a chord $e$, then $e$ is one of the types described in Lemma 4. By Lemma 2, $G$ has no separating 8$^-$-cycles. Thus, $G$ contains no other vertices and the coloring on $C_0$ is also a coloring of $G$, a contradiction. □

The following lemma from [21] provides a powerful tool to prove the reducibility.

Lemma 6. Let $k \geq 3$ and $H$ be a subgraph of $G$. If the vertices of $H$ can be ordered as $v_1, v_2, \ldots, v_l$ such that the following hold

(1) $v_1v_l \in E(G)$, and $v_1$ has no neighbor outside of $H$,
(2) $d(v_i) \leq k$ and $v_i$ has at least one neighbor in $G \setminus H$,
(3) for each $2 \leq i \leq l - 1$, $v_i$ has at most $k - 1$ neighbors in $G\{v_1, \ldots, v_{i-1}\} \cup (G \setminus H)$, then a DP-$k$-coloring of $G \setminus H$ can be extended to a DP-$k$-coloring of $G$.

A face in $G$ is internal if it contains no vertex of $C_0$ and a vertex in $G$ is internal if it is not incident to $C_0$. A 6-face $f$ in $G$ is bad if it is adjacent to a 5$^-$-face and a 6-face $f$ in $G$ is good if it is not bad.

Lemma 7. Let $f$ be an internal 6-face in $G$. If $f$ is a $(3, 3, 3, 3, 3, 3)$-face, then $f$ cannot be adjacent to an internal face $f_1$ with 5 or less vertices such that all vertices on $f_1$ are vertices with degree 3.

Proof. Let $f = [v_1v_2v_3v_4v_5v_6]$ be a $(3, 3, 3, 3, 3, 3)$-face and $f_1 = [v_1v_2\cdots v_i] (i \in \{3, 4, 5\})$, so that $v_1v_2$ is the common edge of $f$ and $f_1$, and all vertices on $f_1$ are 3-vertices. Order the vertices on $f$ and $f_1$ as $v_1, v_4, v_3, v_2, v_1, v_2, v_3, \cdots, v_i (i \in \{3, 4, 5\})$. Let $H$ be the set of vertices in the list. Since all vertices in $H$ are from the internal faces $f$ and $f_1$, no vertex in $C_0$ is going to be removed by such subtraction.
Because $G$ is a minimal counterexample, by Lemma 6, every DP-3-coloring of $G - H$ can be extended to $G$, a contradiction. \hfill \Box

Let $f$ be a $(3, 3, 3, 3, 3, 3)$-face adjacent to a 3-face $f'$. We call the vertex $v$ on $f'$ but not on $f$ the roof of $f$, and $f$ the base of $v$.

**Lemma 8.** Let $f$ be an internal 6-face in $G$ and $f_1$ be an internal face with 5 or less vertices which is adjacent to $f$. If $f_1$ has one 4-vertex, while the other vertices incident with $f_1$ are vertices with degree 3, then each of the following holds:

(a) $f$ cannot be adjacent to another face with five or less vertices;
(b) If $f$ is a $(3, 3, 3, 3, 3, 3)$-face such that $f$ and $f_1$ have a common $(3, 4)$-edge, then the other $(3, 4)$-edge of $f_1$ cannot be incident with another internal $(4, 3, 3, 3, 3, 3)$-face;
(c) If $f$ is a 6-face that all vertices on $f$ are vertices with degree 3, then $f_1$ cannot be adjacent to an internal $(4, 3, 3, 3, 3)$-face $f_2$ that $f_1$ and $f_2$ have a common $(3, 4)$-edge. This means that a 4-vertex incident with an internal $(4, 3, 3, 3, 3, 3)$-face is not a roof.

**Proof.**

(a) follows from the condition on the distance of 5-face.
(b) Let $f_1 = [v_1 v_2 \cdots v_i] (i \in \{3, 4, 5\})$, $d(v_2) = 4$ and $f = [v_1 v_2 w_1 w_2 w_3 w_4]$, so that $v_1 v_2$ is the common $(3, 4)$-edge of $f_1$ and $f$, and all other vertices incident with $f$ and $f_1$ are 3-vertices. Let $f_2 = [v_3 u_1 u_2 u_3 u_4 v_2]$ be the other $(3, 3, 3, 3, 3)$-face adjacent to $f_1$. Order the vertices on $f, f_1$ and $f_2$ as $v_2, u_4, u_3, u_2, u_1, v_3, \cdots, v_1, v_1, w_4, w_3, w_2, w_1 (i \in \{3, 4, 5\})$. Let $H$ be the set of vertices in the list. By Lemma 6, every DP-3-coloring of $G - H$ can be extended to $G$, a contradiction.

(c) Let $f_1 = [v_1 v_2 v_3 v_4 v_5 v_6] (i \in \{3, 4, 5\})$ and $f = [v_1 v_2 w_1 w_2 w_3 w_4]$, so that $v_1 v_2$ is the common $(3, 3)$-edge of $f_1$ and $f$. Let $f_2 = [v_j u_1 u_2 u_3 u_4 v_j + 1] (i \in \{2, \cdots, i - 1\})$ be the $(3, 3, 3, 3, 3)$-face adjacent to $f_1$, $d(v_j + 1) = 4$ and all other vertices on $f, f_1$ and $f_2$ are 3-vertices. If $j = 2$, then $u_1 = w_1$ and order the vertices on $f, f_1$ and $f_2$ as $v_1, v_j, \cdots, v_3, v_2, u_4, u_3, u_2, u_1 (w_1), w_2, w_3, w_4 (i \in \{3, 4, 5\})$. Let $H$ be the set of vertices in the list. By Lemma 6, every DP-3-coloring of $G - H$ can be extended to $G$, a contradiction. If $j > 2$, then order the vertices on $f, f_2$ and $f_2$ as $v_1, v_j, \cdots, v_{j - 1}, u_4, u_3, u_2, u_1, v_j, \cdots, v_2, w_1, w_2, w_3, w_4 (i \in \{3, 4, 5\})$. Let $H$ be the set of vertices in the list. By Lemma 6, every DP-3-coloring of $G - H$ can be extended to $G$, a contradiction. \hfill \Box

**Lemma 9.** Let $f = [v_1 v_2 v_3 v_4 v_5 v_6] (i \in \{1, 2, 3\})$ be an internal 6-face that is adjacent to an internal face with five or less vertices $f_1 = [v_1 v_2 w_1 \cdots w_i] (i \in \{1, 2, 3\})$. If all of the vertices on $f_1$ are vertices with degree 3, then $d(v_5) \geq 4$ or $d(v_6) \geq 4$.

**Proof.** We assume that $d(v_5) = d(v_6) = 3$, and we use $u$ to denote the neighbor of $w_1$ that is not on $f_1$. First, we may rename the lists of vertices in $\{w_1, v_2, v_1, v_3, v_4\}$ so that each edge in $\{uv_1, w_1 v_2, v_1 v_2, v_2 v_3, v_3 v_4\}$ is straight.

Consider the graph $G'$ obtained from $G - \{w_1, \cdots, w_i, v_1, v_2, v_3, v_4\} (i \in \{1, 2, 3\})$ by identifying $v_4$ and $u$. We claim that no new cycles of length from 3 to 5 multiple edges or loop are created. Otherwise, there is a path of length 1, 2, 3, 4 or 5 from $u$ to $v_4$ in $G$, which together with $v_1, v_2, v_3, v_4$ forms a cycle $C$, $5 \leq d(C) \leq 9$. Because $f_1$ is a 5-face, $C$ cannot be a 5-cycle.

- If $v_1$ and $v_6$ are in $Int(C)$, see Figure 2a, then $C$ is not a bad 9-cycle. Otherwise, since $v_1$ and $v_6$ are in $Int(C)$, $C$ must be the type (b) in Figure 1. Thus, $v_1$ and $v_6$ must be in a 3-cycle which is adjacent to $f$, a contradiction to Lemma 8(a). Thus, $C$ is a separating $(6, 7, 8)$-cycle or separating good 9-cycle, a contradiction to Lemma 2.
If \( v_1 \) and \( v_6 \) are not in \( \text{Int}(C) \), see Figure 2b. Due to \( d(v_3) = 3 \) and \( f \) being a 6-cycle, by Lemma 2 and 4, \( v_3 \) is incident with an edge \( e \) in \( \text{Int}(C) \). Either the other vertex incident to \( e \) is in the cycle \( C \), or it is not. If it is in the cycle \( C \), then edge \( e \) is its chord, \( 7 \leq d(C) \leq 9 \) and \( e \) must be incident with a 5-connected cycle \( C' \) by Lemma 4. The distance between \( f_1 \) and \( C' \) is at most 1, a contradiction. If the other vertex incident to \( e \) is not in cycle \( C \), then \( C \) is a separating cycle. If \( C \) is a bad 9-cycle, recall that \( e \) is incident \( v_3 \) and in \( \text{Int}(C) \), the \( v_3 \) must be incident with or adjacent to a 3-cycle. Thus, the distance between \( f_1 \) and the 3-cycle is at most 2, a contradiction. Thus, \( C \) must be a separating \{6,7,8\}-cycle or separating good 9-cycle, a contradiction to Lemma 2.

![Figure 2](image)

**Figure 2.** An internal 6-face \( f \) is adjacent to an internal 5-face \( f_1 \) that all vertices on \( f_1 \) are 3-vertices. \( v_1 \) and \( v_6 \) are in \( \text{Int}(C) \) (a) and not in \( \text{Int}(C) \) (b).

Now, we claim that there is no new chord in \( C_0 \) of \( G' \). Otherwise, \( u \) is on \( C_0 \) and \( v_4 \) is adjacent to a vertex \( v_4' \) which is on \( C_0 \). Then, there is a path between \( u \) and \( v_4' \) on \( C_0 \) with length at most four, which forms a \{5,6,7,8,9\}-cycle with \( w_1, v_2, v_3, v_4 \) in \( G \). Similar to the proof process above, this does not occur.

Because \( f_1 \) is a 5-connected cycle, the distance from \( v_4 \) to other 5-connected cycles is at least one and the distance from \( u \) to other 5-connected cycles is at least two. Thus, the 5-connected cycles are at a distance of at least 3 from each other in \( G' \). Since \( C_0 \) is not a bad 9-cycle in \( G \) and every bad 9-cycles in \( G \) is induced, \( C_0 \) is not a bad 9-cycle in \( G' \). Because \( C_0 \) still is the boundary of the out face of the embedding of \( G' \) and no new chord in \( C_0 \) is formed in \( G' \) and \( G \) is a minimal counterexample, the DP-3-coloring of \( C_0 \) can be extended to \( G' \). Now, we color \( v_4 \) and \( u \) with the color of the identified vertex and keep the colors of all vertices in \( \text{Int}(C) \). Since \( C_0 \) is on \( C_0 \) and the distance from \( v_3 \) is incident with an edge \( e \), then edge \( e \) is its chord, \( 7 \leq d(C) \leq 9 \) and \( e \) must be incident with a 5-connected cycle \( C' \) by Lemma 4. The distance between \( f_1 \) and \( C' \) is at most 1, a contradiction. If the other vertex incident to \( e \) is not in cycle \( C \), then \( C \) is a separating cycle. If \( C \) is a bad 9-cycle, recall that \( e \) is incident \( v_3 \) and in \( \text{Int}(C) \), the \( v_3 \) must be incident with or adjacent to a 3-cycle. Thus, the distance between \( f_1 \) and the 3-cycle is at most 2, a contradiction. Thus, \( C \) must be a separating \{6,7,8\}-cycle or separating good 9-cycle, a contradiction to Lemma 2.

**Lemma 10.** Let \( P = w_1, u_1, u_2, w_2, v_2, w_3, v_3 \) be a path in \( \text{Int}(C_0) \) and \( f = \{w'_1, w'_2, \ldots, w'_j\} \) (i \( \in \{3,4,5\}\) be an internal face with five or less vertices such that all vertices of \( f \) are vertices with degree 3, so that \( w_1, w_2, w_3, w_4, w_5, w_6 \in \text{E}(G) \). If \( d(w_1) = d(u_1) = d(u_2) = 3 \), then \( d(w_2) \geq 5 \). (In addition, similarly, if \( d(w_2) = d(v_1) = d(v_2) = 3 \), then \( d(w_2) \geq 5 \).

**Proof.** Assume that \( d(w_2) \leq 4 \). Since there is no separating 6-cycle and every 6-cycle has no chords by Lemma 2 and 4, the 6-cycle \( w_1 u_1 u_2 w_2 w_3 w_4 \) and \( w_2 v_2 v_3 w_3 w_4 \) are both 6-faces. If \( d(w_1) = 3 \), then, by Lemma 9, \( d(w_2) = 4 \). Let \( w_2' \) be the fourth neighbor of \( w_2 \). Since the cycles of length 3, 4, and 5 in \( G \) are at a distance of at least 3 from each other, \( w_2' \) is not one vertex in \( \{w_1, u_1, u_2, w_2, w_3, \ldots, w'_1, w'_2, w'_3, v_2, v_1\} \). We may rename the list of vertices in \( \{w_2, w'_2, w'_1, w'_3, v_3\} \) so that the edges \( \{w_2, w_2, w_2 w_2, w'_1, w'_2, w'_3, w'_3, w'_3, v_3\} \) are straight.

Consider the graph \( G' \) obtained from \( G \) by identifying \( w_3 \) and \( w'_2 \). Since the cycles of length 3, 4, and 5 in \( G \) are a distance of at least 3 from each
other, the distance between \( w_3 \) and \( w_3'' \) in \( G' \) is at least 2 and the distance between \( v_1 \) and \( w_3'' \) in \( G' \) is at least 4. We claim that no new cycles of length from 2 to 5 are created. Otherwise, there is a path of length 2 to 5 from \( w_3 \) and \( w_3'' \) in \( G \), which together with \( w_2, w_2', w_2'' \) forms a \( \{6, 7, 8, 9\} \)-cycle \( C \). If \( u_1 \) is in \( \text{Int}(C) \), then \( C \) is not a bad 9-cycle. Otherwise, since \( u_1 \) is in \( \text{Int}(C) \), \( u_1, u_2, w_1 \) and \( w_1' \) are in \( \text{Int}(C) \). Then, \( C \) is not isomorphic to a bad 9-cycle in Figure 1. Thus, \( C \) is a separating \( \{6, 7, 8\} \)-cycle or separating good 9-cycle, a contradiction to Lemma 2. Let each vertex \( v \in V(G) \) be the charge of \( x \in V \cup F \) after the discharging procedure. To lead to a contradiction, we shall prove that \( \mu^*(x) \geq 0 \) for all \( x \in V \cup F \setminus \{f_0\} \) and \( \mu^*(f_0) > 0 \).

For shortness, let \( f_k = \{f : f \text{ is a k-face and } V(f) \cap C_0 \neq \emptyset\} \).

The discharging rules:

(R1): If \( v \) is an internal 4-face and incident with a 3-face \( f \), then \( v \) gives \( \frac{1}{2} \) to its incident 3-face, \( \frac{1}{2} \) to its base when \( v \) has a base and \( \frac{1}{2} \) to its incident \( \{3, 3, 3, 3, 4^+\}-face \) that is adjacent to \( f \).
Lemma 11. Every vertex \( v \) in \( G \) has nonnegative final charge.

**Proof.** By (R3), the outer face \( f_0 \) get \( \mu(v) \) from each \( v \in C_0 \) whether \( \mu(v) \) is positive or negative, each vertex on \( C_0 \) has final charge 0. Thus, we assume that \( v \) is an internal vertex of \( G \), then \( d(v) \geq 3 \) by Lemma 1. If \( d(v) = 3 \), then \( \mu^*(v) = 0 \).

If \( d(v) = 4 \). If \( v \) is on a 5-face \( f \), then it is not on or adjacent to other 5-faces. If \( d(f) = 3 \), by Lemma 8 (b) and (c), \( v \) cannot be on two \((3, 3, 3, 3, 4)\)-faces which are adjacent to \( f \) at the same time, and \( v \) cannot be a root and on a \((3, 3, 3, 3, 4)\)-face at the same time, \( v \) gives \( \frac{5}{2} \) to the 3-face, and at most \( \frac{1}{2} \) to 6-faces by (R1) (1). Thus, \( \mu^*(v) \geq 2d(v) - 6 - \frac{3}{2} - \frac{1}{2} \geq 0 \). If \( d(f) = 4 \), then \( v \) gives 2 to the 4-face by (R1) (3). Thus, \( \mu^*(v) \geq 2d(v) - 6 - 2 \geq 0 \). If \( d(f) = 5 \), then \( v \) gives 1 to the 5-face by (R1) (4). Thus, \( \mu^*(v) \geq 2d(v) - 6 - 1 \geq 0 \). Now, assume that \( v \) is adjacent to a 5-face \( f \) that all vertices on \( f \) are 3-vertices, then it is not on or adjacent to other 5-faces. Thus, by (R1) (5), \( v \) gives 1 to the 5-face, and \( \frac{1}{2} \) to each other incident 6-faces that are not adjacent to the 5-face. Thus, \( \mu^*(v) \geq 2d(v) - 6 - \frac{1}{2} \times 2 = 0 \). Finally, assume that \( v \) is not on a 5-face or adjacent to a 5-face \( f \) that all vertices on \( f \) are 3-vertices, then, by (R1) (7), its final charge is \( \mu^*(v) \geq 2d(v) - 6 - \frac{1}{2} \times (d(v) - 4) = 0 \).

If \( d(v) = k \geq 5 \). Because of this, the cycle of lengths \( 3, 4, \) and \( 5 \) are at a distance of at least 3 from each other. If \( v \) is on a 5-face \( f \). If \( d(f) = 3 \), then, by (R1) (2), \( v \) gives \( \frac{5}{2} \) to the 3-face, and \( \frac{1}{2} \) to its base or incident \((3, 3, 3, 3, 4)\)-faces that is adjacent to \( f \). Thus, \( \mu^*(v) \geq 2d(v) - 6 - \frac{3}{2} - \frac{1}{2} \times 3 \geq 0 \). If \( d(f) = 4 \), then, by (R1) (3) \( v \) gives 2 to the 4-face. Thus, \( \mu^*(v) \geq 2d(v) - 6 - 2 \geq 0 \). If \( d(f) = 5 \), then, by (R1) (4) \( v \) gives 1 to the 5-face. Thus, \( \mu^*(v) \geq 2d(v) - 6 - 1 \geq 0 \). If \( v \) is adjacent to a 5-face \( f \) that all vertices on \( f \) are 3-vertices. Thus, by (R1) (6), \( v \) gives at most 2 to the 5-face, and \( \frac{1}{2} \) to each other incident 6-faces that are not adjacent to the 5-face. Hence, \( \mu^*(v) \geq 2d(v) - 6 - \frac{1}{2} \times (d(v) - 2) = \frac{3}{2} \times (d(v) - \frac{14}{3}) \geq 0 \). Finally, assume that \( v \) is not on a 5-face or adjacent to a 5-face \( f \) that all vertices on \( f \) are 3-vertices, then, by (R1) (7), its final charge is \( \mu^*(v) \geq 2d(v) - 6 - \frac{1}{2} \times (d(v) - 4) = \frac{3}{2} \times (d(v) - 4) > 0 \).

**Lemma 12.** Every face other than \( f_0 \) in \( G \) has a nonnegative final charge.

**Proof.** Let \( d(f) = 3 \). If \( f \) contains some vertices of \( C_0 \), then \( f \) gets 3 from \( f_0 \) by (R3), so \( \mu^*(f) = 0 \). Let \( f \) be an internal face. If \( f \) contains at least two \( 4^* \)-vertices, then, by (R1) (1) and (2), \( f \) gets \( \frac{5}{2} \) from each of the incident \( 4^* \)-vertices, thus \( \mu^*(f) \geq d(f) - 6 + \frac{3}{2} \times 2 = 0 \). If \( f \) is incident with exactly one
4\textsuperscript{+}-vertex, then \( f \) gets \( \frac{3}{2} \) from the incident 4\textsuperscript{+}-vertex by (R1) \( 1 \) and \( 2 \), and gets \( \frac{1}{2} \) from each of the adjacent 6\textsuperscript{−}-face by (R2), thus \( \mu^+(f) \geq d(f) - 6 + \frac{3}{2} \times 2 = 0 \). Now, we assume that \( f = [v_1v_2'v_3'] \) is an internal (3, 3, 3)-face. Let \( v_1v_1', v_2v_2', v_3v_3' \in E(G) \) and let \( f_1, f_2, f_3 \) be the three adjacent faces of \( f \) so that \( f_1 \) contains \( v_1v_1', v_2v_2', v_2'v_2 \) and \( f_2 \) contains \( v_2v_2', v_2'v_3', v_3'v_3 \). If \( f_1, f_2, f_3 \) are three 7\textsuperscript{+} or non-internal 6-faces, then they sent 1 to \( f \) by (R2) and \( \mu^+(f) \geq d(f) - 6 + 3 = 0 \). Now, let \( f \) be adjacent to an internal 6-face, say \( f_1 \), then \( f \) is adjacent to at least one 4\textsuperscript{+}-vertex (say \( v_2 \)), which is incident with \( f_1 \) by Lemma 9.

- If both \( f_2 \) and \( f_3 \) are internal 6-faces, then one of \( \{v_1v_3\} \) is a 4\textsuperscript{+}-vertex by Lemma 9. By Lemma 10, either all of \( \{v_1, v_2, v_3\} \) are 4-vertices, or one of them is a 5\textsuperscript{−}-vertex, or one of them (say \( v_1 \)) is a 3-vertex and the other two are 4-vertices, in which case both \( f_1 \) and \( f_3 \) contain 4\textsuperscript{−}-vertices which are not adjacent to \( f \). Thus, \( \mu^+(f) \geq d(f) - 6 + \min\{3 \times 1, 2 + 1, 2 + \frac{1}{2} \times 2\} = 0 \) by (R1) \( 5, 6 \) and (R2).

- If both \( f_2 \) and \( f_3 \) are 7\textsuperscript{−} or non-internal 6-faces, then, by (R1) \( 5, 6 \) and (R2), \( \mu^+(f) \geq d(f) - 6 + 1 + 1 = 0 \).

Thus, we may assume that one of \( \{f_2, f_3\} \) is an internal 6-face and the other is a non-internal 6-face or a 7\textsuperscript{−}-face. If \( f_3 \) is an internal 6-face, then one of \( \{v_1, v_3\} \) is a 4\textsuperscript{+}-vertex by Lemma 9. Thus, \( f \) gets 2 from the two adjacent 4\textsuperscript{−}-vertices by (R1) \( 5 \) and \( 6 \), and \( f \) gets 1 from \( f_2 \) by (R2). Thus, \( \mu^+(f) \geq d(f) - 6 + 1 + 1 + 1 = 0 \). Thus, we may assume that \( f_2 \) is an internal 6-face and \( f_3 \) is a 7\textsuperscript{−} or non-internal 6-face. If both \( v_1 \) and \( v_3 \) are 3-vertices and \( d(v_2) \geq 4 \), by Lemma 10, both \( f_1 \) and \( f_2 \) contain at least one 4\textsuperscript{−}-vertex that is not adjacent to \( f \). Thus, by (R2), \( f \) gets \( \frac{1}{2} \times 2 \) from \( f_1 \) and \( f_2 \), gets 1 from \( f_3 \), and gets 1 from \( v_2 \). Thus, \( \mu^+(f) \geq d(f) - 6 + 1 + 1 + 1 = 0 \). If \( d(v_2) = 4 \) and one vertex of \( \{v_1, v_3\} \) is a 4\textsuperscript{+}-vertex, then, by (R1) \( 5 \), \( f \) gets 1 from \( v_2 \) and the 4\textsuperscript{−}-vertex of \( \{v_1, v_3\} \), and by (R2) \( f \) gets 1 from \( f_3 \). Thus, \( \mu^+(f) \geq d(f) - 6 + 1 + 1 + 1 = 0 \). If \( d(v_2) \geq 5 \), then by (R1) \( 6 \), \( f \) gets 2 from \( v_2 \) and by (R2) \( f \) gets 1 from \( f_3 \). Thus, \( \mu^+(f) \geq d(f) - 6 + 2 + 1 = 0 \).

Let \( d(f) = 4 \). If \( f \) contains some vertices of \( C_0 \), then \( f \) gets 2 from \( f_0 \) by (R3), so \( \mu^+(f) = 0 \). Let \( f \) be an internal face. If \( f \) contains a 4\textsuperscript{−}-vertex, then, by (R1) \( 3 \), \( f \) gets 2 from each of the incident 4\textsuperscript{−}-vertices, thus \( \mu^+(f) \geq d(f) - 6 + 2 = 0 \). Now, we assume that \( f = [v_1v_2'v_3'v_4'] \) is an internal (3,3,3,3)-face. Let \( v_1v_1', v_2v_2', v_3v_3', v_4v_4' \in E(G) \) and let \( f_1, f_2, f_3, f_4 \) be the four adjacent faces of \( f \). If two faces of \( \{f_1, f_2, f_3, f_4\} \) are 7\textsuperscript{−} or non-internal 6-faces, then they sent 1 to \( f \) by (R2) and \( \mu^+(f) \geq d(f) - 6 + 2 = 0 \). Thus, we may assume that it is adjacent to three or four internal 6-faces. By Lemma 9, \( f \) is adjacent to at least two 4\textsuperscript{−}-vertices, so, by (R1) \( 5 \) and \( 6 \), \( f \) gets 1 from each of the two 4\textsuperscript{−}-vertices. Thus, \( \mu^+(f) \geq d(f) - 6 + 2 = 0 \).

Let \( d(f) = 5 \). If \( f \) contains some vertices of \( C_0 \), then \( f \) gets 1 from \( f_0 \) by (R3), so \( \mu^+(f) = 0 \). Let \( f \) be an internal face. If \( f \) contains a 4\textsuperscript{−}-vertex, then, by (R1) \( 4 \), \( f \) gets 1 from each of the incident 4\textsuperscript{−}-vertices, thus \( \mu^+(f) \geq d(f) - 6 + 2 = 0 \). Now, we assume that \( f = [v_1v_2'v_3'v_4'v_5'] \) is an internal (3,3,3,3,3)-face. Let \( v_1v_1', v_2v_2', v_3v_3', v_4v_4', v_5v_5' \in E(G) \) and let \( f_1, f_2, f_3, f_4, f_5 \) be the five adjacent faces of \( f \). If one face of \( \{f_1, f_2, f_3, f_4\} \) is 7\textsuperscript{−} or non-internal 6-faces, then they sent 1 to \( f \) by (R2) and \( \mu^+(f) \geq d(f) - 6 + 1 = 0 \). Thus, we may assume that it is adjacent to five internal 6-faces. By Lemma 9, \( f \) is adjacent to at least three 4\textsuperscript{−}-vertices, so, by (R1) \( 5 \) and \( 6 \), \( f \) gets 1 from each of the three 4\textsuperscript{−}-vertices. Thus, \( \mu^+(f) \geq d(f) - 6 + 3 > 0 \).

Let \( d(f) = 6 \). If \( f \) contains vertices of \( C_0 \) or \( f \) is not adjacent to an internal 3-face, then, by (R2) and (R3), \( \mu^+(f) = 0 \). Now, we assume that \( f \) is an internal 6-face that is adjacent to an internal 3-face \( f' = [v_1v_2v_3] \) with the common edge \( v_1v_2 \) and \( d(v_1) \leq d(v_2) \).

- If \( d(v_1) = d(v_2) \geq 4 \), then \( f \) gives nothing to \( f' \), so \( \mu^+(f) = 0 \).
- If \( d(v_1) = 3 \) and \( d(v_2) \geq 5 \), then, by (R1) \( 2 \) and (R2), \( f \) gets \( \frac{1}{2} \) from \( v_2 \) and gives \( \frac{1}{2} \) to \( f' \). Thus, \( \mu^+(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0 \).
- If \( d(v_1) = d(v_2) = 3 \). Now, we assume that \( d(v_3) = 4 \). If \( f \) has a 4\textsuperscript{−}-vertex, then, by (R1) \( 2 \) and (R2) \( f \) gets \( \frac{1}{2} \) from the 4\textsuperscript{−}-vertex and gives \( \frac{1}{2} \) to \( f' \). Thus, \( \mu^+(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0 \). If \( f \) is
a (3, 3, 3, 3, 3, 3)-face, then, by (R1) ①, ② and (R2) \( f \) gets \( \frac{1}{2} \) from \( v_3 \), and gives \( \frac{1}{2} \) to \( f' \). Thus, 
\[ \mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0. \]
If \( d(v_3) = 3 \), by Lemma 7 \( f \) has a 4*-vertex. By (R2), \( f' \) gets \( \frac{1}{2} \) from \( f \) when \( f \) contains a 4*-vertex that is not adjacent to the 3-face, in which case, by (R1) ⑦ \( f \) gets \( \frac{1}{2} \) from the 4*-vertex, so \( \mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0. \)

- If \( d(v_1) = 3 \) and \( d(v_2) = 4 \). If \( f \) is an internal (4, 3, 3, 3, 3, 3)-face, then \( f \) gets \( \frac{1}{2} \) from \( v_2 \) by (R1) ①, or else \( f \) contains another 4*-vertex, then, by (R1) ⑦ \( f \) gets \( \frac{1}{2} \) from the 4*-vertex. Thus, 
\[ \mu^*(f) \geq d(f) - 6 + \frac{1}{2} - \frac{1}{2} = 0. \]

If \( d(f) \geq 7 \). Since the cycles of lengths 3, 4, and 5 in \( G \) are a distance of at least 3 from each other, \( f \) is adjacent to at most \( \left\lfloor \frac{d(f)}{4} \right\rfloor \) 3-faces. Thus, 
\[ \mu^*(f) \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor \geq 0 \] by (R2). \( \square \)

We call a bad 6-face \( f \) in \( F_6 \) Special if \( f \) is adjacent to one internal 5-face.

Lemma 13. The final charge of \( f_0 \) is positive.

Proof. Assume that \( \mu^*(f_0) \leq 0 \). Let \( E(C_0, G - C_0) \) be the set of edges between \( C_0 \) and \( G - C_0 \). Let \( e' \) be the number of edges in \( E(C_0, G - C_0) \) that is not on a 5*-face and \( x \) be the charges \( f_0 \) receives by (R2). Let \( x_5 = |F_5|, x_4 = |F_4|, x_3 = |F_3| \) and \( x_6 \) be the number of special 6-faces. By Lemma 5, \( C_0 \) has no chord, each 5*-face in \( F_3, F_4 \) and \( F_5 \) contains at least two edges in \( E(C_0, G - C_0) \). By (R2) and (R3), the final charge of \( f_0 \) is
\[
\mu^*(f_0) = d(f_0) + 6 + \sum_{v \in C_0} (2d(v) - 6) - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x
\]
\[
= d(f_0) + 6 + \sum_{v \in C_0} 2(d(v) - 2) - 2d(C_0) - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x
\]
\[
= 6 - d(f_0) + 2|E(C_0, G - C_0)| - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x
\]
\[
\geq 6 - d(f_0) + 4\ell_3 + 4\ell_4 + 4\ell_5 + 2e' - 3\ell_3 - 2\ell_4 - \ell_5 - \ell_6 + x
\]
\[
= 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + 2e' - \ell_6 + x
\]
where the equality follows from that each 5*-face in \( F_3, F_4 \) and \( F_5 \) contains two edges in \( E(C_0, G - C_0) \).

Note that, for each special 6-face \( f \), no edge in \( E(C_0, G - C_0) \cap E(f) \) is on a 5*-face, then \( e' \geq \ell_6 \). When \( e' = \ell_6 \neq 0 \), \( \mu^*(f_0) \geq 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + \ell_6 + x \), then \( \ell_3 = \ell_4 = \ell_5 = \ell_6 = 0 \) and \( 9 \geq d(f_0) \geq 6 + \ell_6 + x \). If \( e' = \ell_6 = 1 \), then \( 9 \geq d(f_0) \geq 7 \). Thus, there is only a 7*-face adjacent to \( C_0 \), a contradiction to \( \ell_6 = 1 \). If \( e' = \ell_6 = 2 \), then \( 9 \geq d(f_0) \geq 8 + x \) and \( x \leq 1 \). Since a special 6-face shares at most four vertices with \( C_0 \), so \( C_0 \) is adjacent to a 9*-face \( f \) that contains at least four consecutive 2-vertices on \( C_0 \). Thus, there is only a 6-face adjacent to \( C_0 \), a contradiction to \( \ell_6 = 2 \). If \( e' = \ell_6 \geq 3 \), then \( d(f_0) = 9 \) and \( x = 0 \) and \( e' = \ell_6 = 3 \), in which case, we have a bad 9-cycle as in Figure 1. Thus, we may assume that \( e' \geq \ell_6 + 1 \). Thus,
\[
\mu^*(f_0) \geq 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + 2e' - \ell_6 + x
\]
\[
\geq 6 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + \ell_6 + 2 + x
\]
\[
= 8 - d(f_0) + \ell_3 + 2\ell_4 + 3\ell_5 + \ell_6 + x
\]

Since \( \mu^*(f_0) \leq 0 \), \( d(f_0) \geq 8 \). Thus, if \( \ell_6 = 1 \), then \( d(f_0) = 9 \), \( x = \ell_3 = \ell_4 = \ell_5 = \ell_6 = 0 \) and \( e' = 2 \). Since the 6-face shares at most four vertices with \( C_0 \), \( C_0 \) is adjacent to a 10*-face \( f \) that contains at least five consecutive 2-vertices on \( C_0 \). Thus, by (R2), \( x \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor > 0 \), a contradiction.

Therefore, we may assume that \( \ell_6 = 0 \), and \( 9 \geq d(f_0) \geq 6 + \ell_3 + 2\ell_4 + 3\ell_5 + 2e' + x \), so \( e' \leq 1 \). Let \( e' = 1 \), it follows that \( \ell_3 \leq 1, \ell_4 = 0 \) and \( \ell_5 = 0 \).

- If \( \ell_3 = 1 \), then \( d(f_0) = 9 \) and \( x = 0 \). Since \( C_0 \) is not a bad 9-cycle, \( C_0 \) is adjacent to a 7*-face \( f \) and \( f \) is adjacent to the 3-face. Thus, by (R2), \( f \) gives at least 1 to \( f_0 \), that is, \( x \geq 1 \), a contradiction.
• If \( \ell_3 = 0 \), then \( d(f_0) \geq 8 \) and \( x \leq 1 \). Note that \( C_0 \) is adjacent to a \( 8^+ \)-face \( f \) that contains \( d(C_0) - 1 \) consecutive 2-vertices on \( C_0 \), thus, by (R2), \( f \) gives at least \( x \geq d(f) - 6 - \left\lceil \frac{d(f) - d(f_0)}{4} \right\rceil \geq 2 \) to \( f_0 \), a contradiction to \( x \leq 1 \).

Finally, let \( e' = 0 \), then \( \ell_3 + 2\ell_4 + 3\ell_5 + x \leq d(f_0) - 6 \), and each edge in \( E(C_0, G - C_0) \) is on a \( 5^- \)-face. Note that we may assume that \( \ell_3 + \ell_4 + \ell_5 \geq 1 \); otherwise, \( G = C_0 \), so \( d(f_0) \geq 7 \).

Because of that, the cycle of lengths 3, 4, and 5 are a distance of at least 3 from each other, and, by Lemma 1, \( d(v) \geq 3 \) for each \( v \in G - C_0 \), so there must be a \( 8^+ \)-face \( f \) adjacent to the \( 5^- \)-face and \( C_0 \). The \( 8^+ \)-face do not give charge to at least one \( 5^- \)-face, so \( x \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor \geq 1 \). It follows that \( \ell_3 \leq 2 \), \( \ell_4 \leq 1 \) and \( \ell_5 = 0 \).

• If \( \ell_3 = 2 \), then \( \ell_4 = 0 \), \( x = 1 \) and \( d(f_0) = 9 \). There must be a \( 8^+ \)-face \( f \) adjacent to the two \( 3 \)-faces and \( C_0 \). The \( 8^+ \)-face does not give charge to the two \( 3 \)-faces, so \( x \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor \geq 2 \), a contradiction to \( x = 1 \).

• If \( \ell_3 = 1 \), then \( \ell_4 = 0 \), \( 1 \leq x \leq 2 \) and \( 8 \leq d(f_0) \leq 9 \). If \( d(f_0) = 8 \), then \( x = 1 \). In this case, \( C_0 \) is adjacent to a \( 9^+ \)-face \( f \) that contains at least six consecutive 2-vertices on \( C_0 \); thus, by (R3), \( f \) gives at least \( x \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor \geq 2 \) to \( f_0 \), a contradiction to \( x = 1 \). If \( d(f_0) = 9 \), then \( x = 2 \). In this case, \( C_0 \) is adjacent to a \( 10^+ \)-face \( f \) that contains at least seven consecutive 2-vertices on \( C_0 \); thus, by (R3), \( f \) gives at least \( x \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor \geq 3 \) to \( f_0 \), a contradiction to \( x = 2 \).

• If \( \ell_3 = 0 \), then \( \ell_4 = 1 \), \( x = 1 \) and \( d(f_0) = 9 \). \( C_0 \) is adjacent to a \( 9^+ \)-face \( f \) that contains at least six consecutive 2-vertices on \( C_0 \); thus, by (R3), \( f \) gives at least \( x \geq d(f) - 6 - \left\lfloor \frac{d(f)}{4} \right\rfloor \geq 2 \) to \( f_0 \), a contradiction to \( x = 1 \).

\( \square \)

**Proof of Theorem 3.** By Lemmas 11–13, \( \sum_{x \in V \cup F} \mu^x(x) > 0 \), a contradiction to \( \sum_{x \in V \cup F} \mu(x) = 0 \). Thus, the counterexample can’t exist, which, in turn, would show that the theorem is true for all cases. \( \square \)

3. Conclusions

The coloring theory of graphs plays a very important role in combinatorial optimization, computer theory, allocation of wireless communication channels, network data transmission, and so on. For example, the efficient design of airline schedules, the design of computer coding programs, etc.

It is well known that the problem of deciding whether a planar graph is 3-colorable is NP-complete. The Three Color Problem is very much alive, replete with an assortment of established results and an abundance of open problems. In addition, 3-coloring is the first significant graph coloring problem and, on the plane, the only unqualified graph coloring problem remaining. DP-coloring is one generalization of list coloring, which is a stronger version of proper coloring. It is very difficult to determine whether a graph is DP-colorable, or even whether a planar graph is DP-3-colorable.

It is unknown if Theorem 3 is most possible in the sense that there exists a planar graph with \( 5^- \)-cycles that are a distance of at least 2 from each other is not DP-3-colorable.

Let \( d_1 \) denote the least integer \( k \) such that every planar graph with \( 5^- \)-cycles are at distance from each other of at least \( k \) is DP-3-colorable.

**Problem 1.** What is the exact value of \( d_1 \)?

Let \( d_2 \) denote the least integer \( k \) such that every planar graph with \( 5^- \)-cycles are at distance from each other of at least \( k \) is 3-choosable.

**Problem 2.** What is the exact value of \( d_2 \)?

It follows from Theorem 3 that \( d_1 \leq 3 \) and \( d_2 \leq 3 \).
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