Dark viscous fluid described by a unified equation of state in cosmology

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We generalize the ΛCDM model by introducing a unified EOS to describe the Universe contents modeled as dark viscous fluid, motivated by the fact that a single constant equation of state (EOS) \( p = -\rho_0 \) \((p_0 > 0)\) reproduces the ΛCDM model exactly. This EOS describes the perfect fluid term, the dissipative effect, and the cosmological constant in a unique framework and the Friedmann equations can be analytically solved. Especially, we find a relation between the EOS parameter and the renormalizable condition of a scalar field. We develop a completely numerical method to perform a \( \chi^2 \) minimization to constrain the parameters in a cosmological model directly from the Friedmann equations, and employ the SNe data with the parameter \( \mathcal{A} \) measured from the SDSS data to constrain our model. The result indicates that the dissipative effect is rather small in the late-time Universe.

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Introduction. The cosmological observations have provided increasingly convincing evidence that our Universe is undergoing a late-time accelerating expansion \[1\, 2\, 3\, 4\], and we live in a favored spatially flat Universe composed of approximately 4% baryonic matter, 22% dark matter and 74% dark energy. The simplest candidate for dark energy is the cosmological constant. Therefore, we suggest that a new cosmological model should be based on or can be reduced to the ΛCDM model naturally.

Time-dependent bulk viscosity \[9\], a linear EOS \[10, 11\], and the Hubble parameter dependent EOS \[12\] are considered in the study of the dark energy physics. The EOS approach is intensely studied in cosmology, partly such as scalar field model, exotic equation of state (EOS), modified gravity, and the inhomogeneous cosmology model. However, the available data sets in cosmology, especially the SNe Ia data \[5, 6, 7\], the SDSS data \[8\], and the three year WMAP data \[4\] all indicate that the ΛCDM model, which serves as a standard model in cosmology, is an excellent model to describe the cosmological evolution. Therefore, we suggest that a new cosmological model should be based on or can be reduced to the ΛCDM model naturally.

We consider the Friedmann-Robertson-Walker metric in the flat space geometry \((k=0)\) as the case favored by observational data

\[
ds^2 = -dt^2 + a(t)^2(dr^2 + r^2d\Omega^2),
\]

and assume that the cosmic fluid possesses a bulk viscosity \( \zeta \). The energy-momentum tensor is

\[
T_{\mu\nu} = \rho U_{\mu} U_{\nu} + (p - \zeta \theta) h_{\mu\nu},
\]

where in comoving coordinates \( U^\mu = (1, 0, 0, 0)\), \( h_{\mu\nu} = g_{\mu\nu} + U_{\mu} U_{\nu} \), and \( \theta = 3\dot{a}/a \) \[28\]. By defining the effective pressure as \( \tilde{p} = p - \zeta \theta \) and from the Einstein equation \( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \), where \( \kappa^2 = 8\pi G \), we obtain the Friedmann equations

\[
\frac{\ddot{a}}{a} = \kappa^2 \tilde{p}, \quad \ddot{a} = -\frac{\kappa^2}{3}(\rho + 3\tilde{p}).
\]

The conservation equation for energy, \( T^0_\nu = 0 \), yields

\[
\dot{\rho} + 3H (\rho + \tilde{p}) = 0,
\]

where \( H = \dot{a}/a \) is the Hubble parameter.

Physical meaning of each term. The EOS proposed in our previous work \[26\] is given by

\[
p = (\gamma - 1)\rho - \frac{2}{\sqrt{3}\kappa T_1} \sqrt{\tilde{p}} - \frac{2}{3\kappa^2 T_2^2}.
\]
The first term is the prefect fluid EOS, the second term describes the dissipative effect, and the third term corresponds to the cosmological constant. The dynamical equation of the scale factor $a(t)$ can be written as

$$\frac{\ddot{a}}{a} = -\frac{3\gamma - 2}{2} \frac{\dot{a}^2}{a^2} + \frac{1}{T_1 a} + \frac{1}{T_2}.$$  \hspace{1cm} (6)

The dimension of both $T_1$ and $T_2$ is [Time]. By concerning the initial conditions of $a(t_0) = a_0$ and $\theta(t_0) = \theta_0$, the analytical solution for $a(t)$ is given out in Ref. 29.

An intriguing feature of the extended ΛCDM model is that it possesses both physical significance and mathematical exact solutions. Form the physical point of view, Eq. (6) naturally contains the dissipative process in the cosmological evolution. If we set the EOS as $p = p_0$ and the bulk viscosity coefficient $\zeta$ is constant, the first term describes the dark matter, the last term ($T_2$ term) describes the dark energy, and the middle term ($T_1$ term) describes the dissipative effects probably coursed by the interaction between the dark matter and dark energy. In Refs. 29, 30, 31, 32, 33, 34, the viscosity in cosmology has been studied in various aspects. The qualitative analysis of Eq. (6) can be easily obtained if we assume that $H$ is always decreasing during the evolution of the Universe. The three terms in the right-hand side of Eq. (6) are proportional to $H^2$, $H^3$, and $H^0$, respectively, therefore, the three terms dominate alternatively during the cosmological evolution and it approaches to a de Sitter Universe finally. Actually, we can see that each term in the right-hand side of Eq. (6) accounts for the time-dependent bulk viscosity or the variable cosmological constant.

**Unified description of dark matter and dark energy.** The ΛCDM model is based on the $H$-$z$ relation

$$H(z)^2 = H_0^2[\Omega_m(1 + z)^3 + 1 - \Omega_m],$$  \hspace{1cm} (7)

where $z = a_0/a - 1$ is the redshift. We find that for a single constant EOS $p = p_0$ ($p_0 > 0$), the $H$-$z$ solution from the Friedmann equations without viscosity is

$$H(z)^2 = H_0^2 \left[ \left( 1 - \frac{\kappa^2 p_0}{3H_0^2} \right) (1 + z)^3 + \frac{\kappa^2 p_0}{3H_0^2} \right],$$  \hspace{1cm} (8)

which exactly possesses the same form of Eq. (7), with $\Omega_m = 1 - \frac{\kappa^2 p_0}{3H_0^2}$. In the ΛCDM model, the Universe contains two fluids, i.e., the dark matter and dark energy, for which the EOS are $p = 0$ and $p = -\rho$, respectively. In our case, a single EOS unifies the dark matter and dark energy modeled as dark viscous fluid, which is consistent with the cosmological principle. However, it does not necessarily mean that the nature of the dark matter and dark energy is the same. The Chaplygin gas model $p = -A/\rho$ 35 also serves as a unified model of dark matter and dark energy, but it cannot reduce to Eq. (7) exactly. As a special case of Eq. (6), a linear EOS of the dark fluid is studied in Ref. 10, and the dark fluid is also studied by other approaches, such as Ref. 30, 37.

Our motivation is to find a more general EOS, which possesses as many as possible physical meanings and the Friedmann equations can be exactly solved, as the following picture shows.

$$\begin{cases}
  p = 0 & \text{(CDM)} \\
  p = -\rho & \text{(A)}
\end{cases} \Leftrightarrow p = -p_0 \rightarrow \text{Eq. (6)}$$

Luckily we obtain one which is just Eq. (6), and gives rise to four implying unifications summarized at the end of this article. Based on this EOS, we establish a cosmological model, called the extended ΛCDM model.

**Variable cosmological constant model.** It turns out that the Friedmann equations combined with the renormalization equation which determines the variable cosmological constant can be reduced to the same form of Eq. (6) 22.

**Scalar field model.** The authors of Refs. 22, 24 give a general method to obtain the potential of a scalar model. We have found that the potential of the corresponding scalar model is

$$V(\varphi) = C_1 e^{\varphi^2} + C_2 e^{\varphi^2/2} + C_3$$  \hspace{1cm} (9)

if $\gamma \neq 0$ 27. However, we missed an important case, $\gamma = 0$. In this case, the EOS is

$$p = -\rho - \frac{2}{\sqrt{3\kappa T_1}} \sqrt{\rho} - \frac{2}{3\kappa^2 T_2^2}.$$  \hspace{1cm} (10)

Using the same method, which is also outlined in Ref. 27, we obtain the potential of the corresponding scalar field

$$V(\varphi) = \frac{3\kappa^2}{64T_1^2} \varphi^4 - \frac{3\kappa}{4\sqrt{2T_1T_2}} \varphi^3 + \left( \frac{3}{2T_2^2} - \frac{1}{8T_1^2} \right) \varphi^2 + \frac{1}{\sqrt{2\kappa T_1 T_2}} \varphi^2 - \frac{1}{\kappa^2 T_2^2}.$$  \hspace{1cm} (11)

As a special case, if the bulk viscosity vanishes, $p = -\rho - p_0$. The potential of the corresponding scalar field is $V(\varphi) = \kappa^2 p_0 \varphi^2$ by neglecting the constant term. In general, Eq. (11) is a non-renormalizable potential, however, if the coefficient before $\rho$ is precisely equal to $-1$, we obtain a renormalizable field. Moreover, the $\sqrt{\rho}$ term in the EOS gives a contribution of $\varphi^4$ term in the scalar field. This property of such scalar field was missed in our previous work. We think that there is a profound relation between the renormalizability of the scalar field and that the EOS parameter of the vacuum is precisely equal to $-1$.

**Mathematical features.** In the mathematical aspect, the transformation $\varphi = a^{\gamma/2}$ reduces Eq. (6) to a linear differential equation of $\chi(t)$

$$\ddot{\chi} - \frac{1}{T_1} \dot{\chi} - \frac{3\gamma}{2T_2} \chi = 0,$$  \hspace{1cm} (12)

which can be solved easily. The variable $\chi$ serves as a rescaled scale factor and behaves like the amplitude of
a damping harmonic oscillator. The $T_1$ term is just the damping term. The equation which determines the evolution of the Hubble parameter, $\dot{H} = -\frac{3\gamma}{2} H^2 + \frac{1}{T} H + \frac{1}{T^2}$, has possessed a form invariance for $H \to H + \delta H$.

Supernovae constraints. The observations of the SNe Ia have provided the direct evidence for the cosmic accelerating expansion for our current Universe. Any model attempting to explain the acceleration mechanism should be consistent with the SNe Ia data implying results, as a basic requirement. We have found the viscosity without cosmological constant possesses a $(1+z)^{3/2}$ contribution \cite{27}, which seems to be an interpolation between the matter $(1+z)^3$ and the $\Lambda$-term $(1+z)^\alpha$. The method of the data fitting is illustrated in Refs. \cite{40, 41}, in which the explicit solution $H(z)$ is required. We develop a completely numerical method to perform a $\chi^2$ minimization to fit the optimized values of the parameters in a cosmological model directly from the Friedmann equations, without knowing the $H$-$z$ relation. Define a new function

$$F(z) = \int_0^z \frac{dz}{E(z)}, \quad (13)$$

where $E(z) = H(z)/H_0$ is the dimensionless Hubble parameter. The relations implied by Eq. (13)

$$E(z) = F'(z)^{-1}, \quad E'(z) = -F''(z)F'(z)^{-2} \quad (14)$$

can transform an equation for $H(z)$ to another one for $F(z)$, then one solves $F(z)$ numerically and obtains the luminosity distance $d_L = (c/H_0)(1+z)F(z)$. This is a general numerical method and it can be applied if only the dynamical equations determining the scale factor is known.

The $\chi^2$ is calculated from

$$\chi^2 = \sum_{i=1}^n \left( \frac{\mu_{obs}(z_i) - M' - 5 \log_{10} D_{L,th}(z_i; c_o)}{\sigma_{obs}(z_i)} \right)^2$$

$$+ \left( \frac{A - 0.469}{0.017} \right)^2, \quad (15)$$

where $M'$ is a free parameter related to the Hubble constant and $D_{L,th}(z_i; c_o)$ is the theoretical prediction for the dimensionless luminosity distance of a SNe Ia at a particular distance, for a given model with parameters $c_o$. The parameter $A$ is defined in Ref. \cite{3}. Here $\Omega_m = 1 - \frac{\gamma}{T^2 H_0^2}$ is used in our model and we take $\gamma = 1$ as in the $\Lambda$CDM model. We will consider the $\Lambda$CDM model for comparison and perform a best-fit analysis with the minimization of the $\chi^2$, with respect to $M'$, $T_1 H_0$, and $T_2 H_0$. We employ the 157 gold data, the SNLS data, and the 182 SNe data compiled by Riess et al., recently combined with the parameter $A$ to constrain the parameters and plot the $T_1$-$T_2$ relation in Fig. [1] and Fig. [2]. From the results, we see that the $T_1$ term is made less than 10% contributions to that of the $T_2$ term on $2\sigma$ C.L. If we adopt the interpretation of viscosity of our model, the fitting result shows that the dissipative effect is rather small, as we expect that the additional term is a small correction to the $\Lambda$CDM model.

Discussion. The approach of the unified EOS considered in this paper have enabled us to describe the Universe contents and related to several fundamental issues in cosmological evolution from a united viewpoint. We have extended the $\Lambda$CDM model into a more general framework by introducing this unified EOS. (i) This EOS describes the perfect fluid term, the dissipative effect and the cosmological constant in a unique equation. (ii) This general EOS unifies the dark matter and the dark energy as a single dark viscous fluid and can be exactly reduced to the $\Lambda$CDM model as a special case. (iii) The variable cosmological constant model is mathematically equiva-
lent to the form by using this EOS. (iv) We also find a scalar field that is equivalent to this EOS, moreover, the renormalizable condition of the scalar field requires the coefficient before $\rho$ is precisely equal to $-1$. Thus, it is very interesting that concerning on the bulk viscosity, modified EOS, variable cosmological constant model, and scalar field model can be described in one general dynamical equation which determines the scale factor. In this sense, our model has unified the exact solutions of several models. The viewpoint of modified EOS is rather phenomenological, however, we have showed that it is strongly related to some fundamental concepts of cosmology. The incoming data sets will give more constraints to the modified EOS approach in cosmology.

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