Oblique repulsion in the nonnegative quadrant

Dominique Lépine
Université d’Orléans, FDP-MAPMO

January 29, 2014

Abstract

We consider the differential system \( \dot{x} = \alpha/x + \beta/y \), \( \dot{y} = \gamma/x + \delta/y \) in the nonnegative quadrant. Here \( \alpha \) and \( \delta \) are positive, \( \beta \) and \( \gamma \) are real constants. Under some condition on the constants there exists a unique global solution. The main difficulty is to prove uniqueness when starting at the corner of the quadrant.

1 Introduction.

We are interested in the question of existence and uniqueness of the solution \( u(t) = (x(t), y(t)) \) to the following integral system

\[
\begin{align*}
x(t) &= x + \alpha \int_{0}^{t} \frac{ds}{x(s)} + \beta \int_{0}^{t} \frac{ds}{y(s)} \\
y(t) &= y + \gamma \int_{0}^{t} \frac{ds}{x(s)} + \delta \int_{0}^{t} \frac{ds}{y(s)}
\end{align*}
\]

where \( x(\cdot) \) and \( y(\cdot) \) are continuous functions from \([0, \infty)\) to \([0, \infty)\) with the conditions

\[
\begin{align*}
\int_{0}^{t} \mathbb{1}_{\{x(s)=0\}} \frac{ds}{x(s)} &= 0 \\
\int_{0}^{t} \mathbb{1}_{\{x(s)>0\}} \frac{ds}{x(s)} &< \infty \\
\int_{0}^{t} \mathbb{1}_{\{y(s)=0\}} \frac{ds}{y(s)} &= 0 \\
\int_{0}^{t} \mathbb{1}_{\{y(s)>0\}} \frac{ds}{y(s)} &< \infty
\end{align*}
\]

for any \( t \geq 0 \). Here \( \alpha, \beta, \gamma \) and \( \delta \) are four real constants with \( \alpha > 0 \) and \( \delta > 0 \).

The system has a single singularity at each side of the nonnegative quadrant \( S = \{(x, y) : x \geq 0, y \geq 0\} \) and a double singularity at the corner \( 0 = (0, 0) \). We write \( S^\circ := S \setminus \{0\} \) for the punctured quadrant.

We will note \( \dot{z}(t) \) the derivative \( dx(t)/dt \). So the integral system (1) may be written as an initial-value problem

\[
\begin{align*}
\dot{x} &= \frac{\alpha}{x} + \frac{\beta}{y} \\
\dot{y} &= \frac{\gamma}{x} + \frac{\delta}{y}
\end{align*}
\]

with the initial condition \((x(0), y(0)) \in S\).

We first remark that if \( \beta < 0, \gamma < 0 \) and \( \alpha \delta < \beta \gamma \), there exist \( \lambda > 0 \) and \( \mu > 0 \) such that \( \lambda \alpha + \mu \gamma < 0 \) and \( \lambda \beta + \mu \delta < 0 \). Thus \( z(t) := \lambda x(t) + \mu y(t) \) is decreasing, \( \min (x(t), y(t)) \to 0 \) and \( \dot{z}(t) \to -\infty \) as \( t \to t_f \) where \( t_f < \infty \) and there is no solution. If \( \beta < 0, \gamma < 0 \) and \( \alpha \delta = \beta \gamma \), then \( v(t) := \alpha y(t) - \gamma x(t) \) remains equal to \( \alpha y - \gamma x \) and there is a unique solution \((x(t), y(t))\) that converges to \((\frac{\alpha y_0 - \alpha x_0}{\beta + \gamma}, \frac{\beta y_0 - \delta x_0}{\beta + \gamma})\) except if \((x, y) = 0\) in which case there is no solution.

From now on we will make the following hypothesis:

\[
(H) \quad \max (\beta, \gamma) \geq 0 \quad \text{or} \quad \beta \gamma < \alpha \delta.
\]
This is equivalent to the existence of \( \lambda \geq 0 \) and \( \mu \geq 0 \) such that \( \lambda \alpha + \mu \gamma > 0 \) and \( \lambda \beta + \mu \delta > 0 \). This last formulation amounts to saying that the matrix

\[
A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

is an S-matrix in the terminology of [1]. In the sequel, we fix a pair \((\lambda, \mu)\) with \( \lambda > 0, \mu > 0 \), such that \( \lambda \alpha + \mu \gamma > 0 \) and \( \lambda \beta + \mu \delta > 0 \).

The aim of this note is to prove the following result.

**Theorem 1** Under condition \((H)\), there exists a unique solution to (1) for any starting point \((x, y) \in S\).

### 2 Some preliminary lemmata.

We begin with a comparison lemma.

**Lemma 2** Let \( x_1 \) and \( x_2 \) be nonnegative continuous functions on \([0, \infty)\) which are solutions to the system

\[
\begin{align*}
x_1(t) &= v_1(t) + \alpha \int_0^t \frac{ds}{x_1(s)} \\
x_2(t) &= v_2(t) + \alpha \int_0^t \frac{ds}{x_2(s)}
\end{align*}
\]

where \( \alpha > 0, v_1 \) and \( v_2 \) are continuous functions such that \( 0 \leq v_1(0) \leq v_2(0) \), and \( v_2 - v_1 \) is nondecreasing. Then \( x_1 \leq x_2 \) on \([0, \infty)\).

Proof. Assume there exists \( t > 0 \) such that \( x_1(t) > x_2(t) \). Set

\[
\tau := \max \{ s \leq t : x_1(s) \leq x_2(s) \}.
\]

Then

\[
x_2(t) - x_1(t) = x_2(\tau) - x_1(\tau) + (v_2(t) - v_1(t)) - (v_2(\tau) - v_1(\tau)) + \alpha \int_\tau^t \left( \frac{1}{x_2(s)} - \frac{1}{x_1(s)} \right) \frac{ds}{x_2(s)} \geq 0,
\]

a contradiction. \( \blacksquare \)

**Lemma 3** Let the system

\[
\begin{align*}
\dot{x} &= \frac{\alpha}{x} + \phi(x, z) \\
\dot{z} &= \psi(x, z)
\end{align*}
\]

with \( x(0) = x_0 \geq 0, z(0) = z_0 \in \mathbb{R}, \alpha > 0, \phi \) and \( \psi \) two Lipschitz functions on \( \mathbb{R}_+ \times \mathbb{R} \), and \( |\phi| \leq c \) for some \( c < \infty \). Then there exists a unique solution to (4). Moreover, for this solution, \( x(t) > 0 \) for any \( t > 0 \).

Proof. Assume first \( x_0 > 0 \). Then the system (4) is Lipschitz on \([\min \{ x_0, \frac{\alpha}{c} \}, \infty) \times \mathbb{R}\) and the solution does not step out of this domain, so there is a unique global solution. When \( x_0 = 0 \), we let \( w_0(t) = 0, z_0(t) = z_0 \) and for \( n \geq 1 \)

\[
\begin{align*}
w_n(t) &= 2at + 2 \int_0^t \sqrt{w_{n-1}(s)} \phi(\sqrt{w_{n-1}(s)}, z_{n-1}(s)) ds \\
z_n(t) &= z_0 + \int_0^t \psi(\sqrt{w_{n-1}(s)}, z_{n-1}(s)) ds.
\end{align*}
\]
Let $M > 0$ and assume $|w_{n-1}(t)| \leq M$ on some interval $[0, T]$. Then, for $0 \leq t \leq T$,

$$|w_n(t)| \leq T(2\alpha + 2c\sqrt{M})$$

and this is again $\leq M$ for $T$ small enough. We also have $|z_n(t)| \leq M'$ for any $n \geq 0$ for $T$ small enough. Equicontinuity of $(w_n, z_n)_{n \geq 0}$ is easily verified and from the Arzela-Ascoli theorem it follows there exists a subsequence $(w_{n_k}, z_{n_k})$ converging on $[0, T]$ to a solution $(w, z)$ of the system

$$\dot{w} = 2\alpha + 2\sqrt{w} \phi(\sqrt{w}, z)$$

$$\dot{z} = \psi(\sqrt{w}, z)$$

with the initial conditions $w(0) = 0$, $z(0) = z_0$. For small $T$, $\dot{w} > 0$ on $[0, T]$. Set now $x(t) = \sqrt{w(t)}$. Then $(x, z)$ is a solution to (4) on $[0, T]$ with $x(T) > 0$. We may extend the solution to $[0, \infty)$ by using the above result with $x_0 > 0$.

We now prove uniqueness. Let $(x, z)$ and $(x', z')$ be two solutions of (4). Then

$$(x(t) - x'(t))^2 + (z(t) - z'(t))^2$$

$$= 2\alpha \int_0^t (x(s) - x'(s)) \left( \frac{1}{x(s)} - \frac{1}{x'(s)} \right) ds + 2 \int_0^t (x(s) - x'(s))(\phi(x(s), z(s)) - \phi(x'(s), z'(s))) ds$$

$$+ 2 \int_0^t (z(s) - z'(s))(\psi(x(s), z(s)) - \psi(x'(s), z'(s))) ds$$

$$\leq 4L \int_0^t ((x(s) - x'(s))^2 + (z(s) - z'(s))^2) ds$$

where $L$ is the Lipschitz constant of $\phi$ and $\psi$. Uniqueness follows from Gronwall’s inequality.

$\blacksquare$

**Lemma 4** Let $u(\cdot) = (x(\cdot), y(\cdot))$ be a solution to (1) and let $\nu = (\lambda, \mu)$. Then the function $z(\cdot) := \nu \cdot u(\cdot) = \lambda x(\cdot) + \mu y(\cdot)$ is increasing on $[0, \infty)$ and we have $u(t) \in S^o$ for any $t > 0$.

Proof. Recall that condition (H) is in force. We easily check that $\dot{z}(t)$ is positive. $\blacksquare$

### 3 Existence. Case $x = 0, y = 0$.

There is an explicit solution to (1) when the starting point is the corner.

**Proposition 5** There is a solution to (1) with initial condition $0$ given by

$$x(t) = c \sqrt{t}$$

$$y(t) = d \sqrt{t}$$

where

$$c = (2\alpha + \frac{\beta}{\delta} (\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta})^{1/2}$$

$$d = (2\delta + \frac{\alpha}{\gamma} (\gamma - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta})^{1/2}.$$ 

Proof. Writing down $x(t) = c \sqrt{t}$ and $y(t) = d \sqrt{t}$ we have to solve the system

$$\frac{c}{\sqrt{t}} = \frac{\alpha}{c} + \frac{\beta}{\delta}$$

$$\frac{d}{\sqrt{t}} = \frac{\gamma - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2\alpha}.$$ 

We first compute

$$\frac{d}{c} = \frac{\gamma - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2\alpha}.$$ 

(8)
and then obtain (7) provided that
\[ C = 2\alpha + \frac{\beta}{\alpha}(\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta}) \]
\[ D = 2\delta + \frac{\gamma}{\alpha}(\gamma - \beta + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta}) \]
are positive. If \( \beta \geq 0 \), \( C \) is clearly positive. This is also true if \( \beta < 0 \) and \( \beta\gamma < \alpha\delta \) since \( C \) may be written
\[ C = \frac{4\alpha(\alpha\delta - \beta\gamma)}{2\alpha\delta - \beta\gamma + \beta^2 - \beta\sqrt{4(\alpha\delta - \beta\gamma) + (\beta + \gamma)^2}}. \]
The proof for \( D \) is similar.

Uniqueness in this case is more involved and will be treated in the last section. We only remark for the moment that the system (3) with \( \alpha = \delta = 0 \), \( \beta > 0 \), \( \gamma > 0 \) and initial value \( 0 \) has a one-parameter family of solutions.

4 Angular behavior.

We are now in a position to study the behavior of \( \frac{y(t)}{x(t)} \). For any \( u = (x, y) \in S^o \) we set
\[ \theta(u) = \arctan \frac{y}{x}. \]
We also set
\[ u_* = (x_*, y_*) := \left( \frac{c}{\lambda c + \mu d}, \frac{d}{\lambda c + \mu d} \right). \]

Proposition 6 Let \( u(.) \) be a solution to (1) starting at \( u = (x, y) \in S^o \). Then for any \( t > 0 \)
1.
\[ \frac{d\theta(u(t))}{dt} > 0 \quad \text{and} \quad \theta(u(t)) < \theta(u_*) \quad \text{if} \quad \theta(u) < \theta(u_*), \]
\[ \frac{d\theta(u(t))}{dt} = 0 \quad \text{and} \quad \theta(u(t)) = \theta(u_*) \quad \text{if} \quad \theta(u) = \theta(u_*), \]
\[ \frac{d\theta(u(t))}{dt} < 0 \quad \text{and} \quad \theta(u(t)) > \theta(u_*) \quad \text{if} \quad \theta(u) > \theta(u_*). \]

2.
\[ x(t) \geq \min \left\{ x, c\frac{\lambda x + \mu y}{\lambda c + \mu d} \right\} \]
\[ y(t) \geq \min \left\{ y, d\frac{\lambda x + \mu y}{\lambda c + \mu d} \right\}. \]

Proof. From Lemma 4 we know that \( u(t) \in S^o \) for any \( t \geq 0 \).

1. We compute
\[ \frac{d\theta(u(t))}{dt} = \frac{1}{x^2(t) + y^2(t)} \left( \frac{d}{c} \frac{y(t)}{x(t)} \right) \left[ \alpha + \frac{x(t)(\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha\delta})}{2y(t)} \right] \]
and the conclusion follows.

2. Let \( a, b \in S^o \) with \( 0 \leq \theta(a) < \theta(u_*), \theta(u_*) < \theta(b) \leq \frac{\pi}{2} \) and let \( l > 0 \). We set
\[ A = \{ v \in S^o : \theta(a) \leq \theta(v) \leq \theta(u_*) \} \]
\[ B = \{ v \in S^o : \theta(b) \geq \theta(v) \geq \theta(u_*) \}. \]
It follows from above that any solution starting from \( A \) stays in \( A \), and the same is true for \( B \). If \( u \in A \),
\[
x(t) \geq -\frac{\mu}{\lambda}y(t) + (x + \frac{\mu}{\lambda}y) \geq -\frac{\mu d}{\lambda c} x(t) + x + \frac{\mu}{\lambda}y
\]
and therefore
\[
x(t) \geq \frac{\lambda x + \mu y}{\lambda c + \mu d}.
\]
If \( u \in B \),
\[
x(t) \geq \frac{x}{y}y(t) \geq \frac{x}{y}(-\frac{\lambda}{\mu}x(t) + \frac{\lambda x + \mu y}{\mu})
\]
and therefore
\[
x(t) \geq x.
\]
Same estimations for \( y(t) \).

\[ \square \]

**Corollary 7** Let \( u(\cdot) \) be a solution to (1). Then
\[
\lim_{t \to \infty} \theta(u(t)) = \theta(u_*), \quad \text{i.e.} \quad \lim_{t \to \infty} \frac{y(t)}{x(t)} = \frac{d}{c}.
\]

Proof. If \( u = (x, y) \in S^0 \), this is an easy consequence of (10). If \( u = (0, 0) \) we may apply Lemma 4 and then (10).

\[ \square \]

**5 Existence and uniqueness. Case \( \lambda > 0, \mu > 0 \).**

**Proposition 8** There exists a unique solution \( u(\cdot) \) to (1) starting at \( u = (x, y) \) with \( x > 0, y > 0 \). It satisfies \( x(t) > 0, y(t) > 0 \) for any \( t \geq 0 \).

Proof. We now assume \( \theta(a) > 0 \) and \( \theta(b) < \frac{\gamma}{\alpha} \) in (11). Let \( l > 0 \) and \( \nu = (\lambda, \mu) \). We set \( L := \{ v \in S^0 : v \cdot v \geq l \} \). From Lemma 4 and Proposition 6 we know that any solution starting from \( A \cap L \) stays in \( A \cap L \), and the same is true for \( B \cap L \). As the system is Lipschitz in \( A \cap L \) and in \( B \cap L \), there is a unique global solution to (1) in both cases.

\[ \square \]

**6 Existence and uniqueness. Case \( \lambda = 0, \mu > 0 \).**

**Proposition 9** There exists a unique solution \( u(\cdot) \) to (1) starting at \( u = (x, y) \) with \( x = 0, y > 0 \). It satisfies \( x(t) > 0, y(t) > 0 \) for any \( t > 0 \).

Proof. Let \( \varepsilon \in (0, \frac{md}{\lambda c + \mu d}) \). We define on \( \mathbb{R}_+ \times \mathbb{R} \)
\[
\psi_\varepsilon(x, z) := \frac{1}{\max(\gamma x + z, \alpha \varepsilon)}.
\]
We apply Lemma 3 to obtain a unique solution \( x_\varepsilon(\cdot), z_\varepsilon(\cdot) \) to
\[
x_\varepsilon(t) = \alpha \int_0^t \frac{ds}{x_\varepsilon(s)} + \alpha^m \beta \int_0^t \psi_\varepsilon(x_\varepsilon(s), z_\varepsilon(s))ds
\]
\[
z_\varepsilon(t) = \alpha y + \alpha (\alpha \delta - \beta \gamma) \int_0^t \psi_\varepsilon(x_\varepsilon(s), z_\varepsilon(s))ds.
\]
(12)
Let
\[ y_\varepsilon(t) = \frac{1}{\alpha}(\gamma x_\varepsilon(t) + z_\varepsilon(t)) \]
\[ \tau_y(\varepsilon) = \inf\{ t > 0 : y_\varepsilon(t) < \varepsilon \}. \]

On the interval \([0, \tau_y(\varepsilon)]\), \((x_\varepsilon(\cdot), y_\varepsilon(\cdot))\) is the unique solution to (1). From (9) we know that \(y_\varepsilon(t) > \varepsilon\) on this interval. Thus \(\tau_y(\varepsilon) = \infty\) and \((x(\cdot), y(\cdot)) := (x_\varepsilon(\cdot), y_\varepsilon(\cdot))\) is the unique global solution to (1).

### 7 Path behavior.

Let us note \(u(t, u_0)\) the solution to (1) starting at \(u_0 \in S^o\). Using Gronwall’s inequality as in the proof of uniqueness, it is easily seen that for any \(t > 0\) the solution \(u(t, u_0)\) continuously depends on the initial condition \(u_0\). It has the Scaling Property:

\[ \text{(SC)} \quad u(r^2 t, u_0) = ru(t, u_0) \]

for any \(r > 0, t \geq 0, u_0 \in S^o\). Using Lemma 4 we also note that any solution \(u(\cdot)\) to (1) has the Semi-group Property:

\[ \text{(SG)} \quad u(s + t) = u(t, u(s)) \]

for any \(s > 0\) and \(t \geq 0\). With Proposition 8 and Proposition 9 this entails that \(x(t) > 0\) and \(y(t) > 0\) for any \(t > 0\). We now set for any \(r > 0\):

\[ L_r := \{ u = (x, y) : x > 0, y > 0, \nu. u = r \} \]
\[ L^r_r := \{ u = (x, y) : x \geq 0, y \geq 0, \nu. u = r \} \]

**Lemma 10** Let \(u(\cdot)\) be a solution to (1) starting at \(u_0 \in S\) with \(\nu. u_0 \leq r\). We set

\[ \tau(r) := \inf\{ t \geq 0 : \nu. u(t) = r \}. \]

Then

\[ \tau(r) \leq \frac{r^2}{2[\lambda(\lambda \alpha + \mu \gamma) + \mu(\lambda \beta + \mu \delta)]}. \]

Proof. Set

\[ z(t) := \nu. u(t). \]

As

\[ z(t) = \nu. u_0 + [\lambda(\lambda \alpha + \mu \gamma) + \mu(\lambda \beta + \mu \delta)] \int_0^t \frac{ds}{z(s)} + \int_0^t f(s) ds \]

with

\[ f(s) = \mu(\lambda \alpha + \mu \gamma) \frac{y(s)}{x(s)} + \lambda(\lambda \beta + \mu \delta) \frac{x(s)}{y(s)} > 0 \]

for \(s > 0\), it follows from Lemma 2 that \(z(t) \geq w(t)\) where

\[ w(t) = \nu. u_0 + [\lambda(\lambda \alpha + \mu \gamma) + \mu(\lambda \beta + \mu \delta)] \int_0^t \frac{ds}{w(s)}, \]

and then

\[ z^2(t) \geq w^2(t) = 2[\lambda(\lambda \alpha + \mu \gamma) + \mu(\lambda \beta + \mu \delta)] t + (\nu. u_0)^2. \]
The conclusion follows.

We now define $q : L_1 \to L_1$ by

$$q(u_1) = \frac{1}{2} u(\tau(2), u_1)$$

where

$$\tau(2) = \inf \{ t \geq 0 : \nu. u(t, u_1) = 2 \}$$

is finite from the above Lemma. Let now $r > 0$ and $u \in L_r$. From (SC), the geometric paths in $S$ of $u(., u)$ and $ru(., \frac{u}{r})$ are identical. Therefore

$$u(\tau(2r), u) = ru(\tau(2), \frac{u}{r})$$

where in this equality $\tau(2r)$ is relative to $u(., u)$ and $\tau(2)$ is relative to $u(., \frac{u}{r})$. Thus

$$q(\frac{u}{r}) = \frac{1}{2r} u(\tau(2r), u).$$

Iterating and using (SG), we get for any $n \geq 1$

$$q^n(\frac{u}{r}) = \frac{1}{2^{n}r} u(\tau(2^n r), u).$$

Proof. From Proposition 6 we know that $q$ has a unique invariant point $u_*$. We consider the solution $u(t, u_1) = (x(t), y(t))$ on the time interval $[0, \tau(2)]$, where $\tau(2)$ was defined in (13). We first assume that

$$\frac{y_1}{x_1} < \frac{y_*}{x_*} = \frac{d}{c}.$$ 

We note for further use that

$$0 \leq y_1 < y_* < \frac{1}{\mu}.$$ 

We set

$$u_2 = (x_2, y_2) := u_1 + \frac{(\alpha y_1 + \beta x_1, \gamma y_1 + \delta x_1)}{\lambda (\alpha y_1 + \beta x_1) + \mu (\gamma y_1 + \delta x_1)}.$$ 

Then, $2u_*, u_* + u_1$ and $u_2 \in L_2$. Setting for $z \in [0, \infty]$

$$h(z) = \frac{\gamma z + \delta}{\alpha z + \beta}$$

we compute

$$\frac{dh}{dz}(z) = \frac{\beta \gamma - \alpha \delta}{(\alpha z + \beta)^2}.$$ 

From Proposition 6 we know that for any $t \in [0, \tau(2)]$

$$\frac{y(t)}{x(t)} < \frac{d}{c}.$$ 

Proposition 11 There exists $k \in (0, 1)$ such that for any $u_1 \in L_1$

$$|q(u_1) - u_*| \leq k |u_1 - u_*|.$$
When \( \alpha \delta > \beta \gamma \), it follows from (15) and (16) that
\[
\frac{\dot{y}(t)}{\dot{x}(t)} = h(y(t)) > h\left(\frac{d}{c}\right) = \frac{d}{c}
\]
and then \( 2q(u_1) \) belongs to the open interval \( (2u_*, u_* + u_1) \) on \( L_2 \). Therefore,
\[
|2q(u_1) - 2u_*| < |u_1 - u_*|.
\]
When \( \alpha \delta = \beta \gamma \), the path of the solution is a straight half-line with slope \( \frac{d}{c} \) and
\[
|2q(u_1) - 2u_*| = |u_1 - u_*|.
\]
When \( \alpha \delta < \beta \gamma \), \( \frac{\dot{y}(t)}{\dot{x}(t)} \) is increasing on \([0, \tau(2)]\) and then
\[
\frac{\gamma y_1 + \delta x_1}{\alpha y_1 + \beta x_1} \leq \frac{\dot{y}(t)}{\dot{x}(t)} < \frac{d}{c}
\]
As a result, \( 2q(u_1) \) belongs to the open interval \((u_* + u_1, u_2)\) on \( L_2 \). Moreover, using the relation \( \lambda x_1 + \mu y_1 = 1 \) twice, we get
\[
2x_1 - x(\tau(2)) > 2x_1 - x_2 = x_1 - \frac{\alpha y_1 + \beta x_1}{\alpha \lambda x_1 + \beta \lambda x_1 + \gamma y_1 + \delta x_1} \lambda(\alpha y_1 + \beta x_1 + \gamma y_1 + \delta x_1) - \alpha y_1 - \beta x_1 = \frac{\alpha \mu}{\lambda(\alpha y_1 + \beta x_1 + \gamma y_1 + \delta x_1)}(x_1 \frac{d}{c} - y_1)(y_1 + \frac{\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2x} x_1).
\]
In the same way,
\[
y(\tau(2)) - 2y_1 > y_2 - 2y_1 = \frac{\alpha (\lambda x + \mu y)}{c(\lambda(\alpha y_1 + \beta x_1 + \gamma y_1 + \delta x_1))}(y_1 - y_1)(y_1 + \frac{\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta}}{2x} x_1).
\]
Setting
\[
k_1 = \frac{\lambda \mu (\beta - \gamma + \sqrt{(\beta - \gamma)^2 + 4\alpha \delta})}{4(\lambda (\alpha x + \mu y) + \mu (\lambda x + \mu y))} > 0
\]
we obtain
\[
|2q(u_1) - 2u_*| > 2k_1 |u_1 - u_*|
\]
and then
\[
|q(u_1) - u_*| = |u_1 - u_*| - |q(u_1) - u_1| < (1 - k_1) |u_1 - u_*|.
\]
If now \( \frac{y_1}{x_1} > \frac{d}{c} \), in the same way there exists \( k_2 > 0 \) such that
\[
|q(u_1) - u_*| < (1 - k_2) |u_1 - u_*|
\]
We may take \( k = 1 - \min(k_1, k_2) \).
8 Uniqueness. Case $x = 0, y = 0$.

Existence was proven in Section 3. We may now conclude the proof of Theorem 1.

Proposition 12 The solution given by (6) is the unique solution to (1) starting at 0.

Proof. Let $u(.)$ be a solution to (1) starting at 0. For any $n \geq 1$ and $s > 0$,

$$u(\tau(s)) = u(\tau(s), u(\tau(s2^{-n})))$$

where $\tau(s)$ in the l.h.s. is relative to $u(.)$ and $\tau(s)$ in the r.h.s. is relative to $u(., u(\tau(s2^{-n})))$.

We may apply (14) with $r = s2^{-n}$ and $u = u(\tau(s2^{-n}))$. We obtain

$$\frac{u(\tau(s))}{s} = q^n \left( \frac{u(\tau(s2^{-n}))}{s2^{-n}} \right).$$

From Proposition 11 (or directly from (10) it follows that the r.h.s. converges to $u_*$ as $n \to \infty$.

Thus for any $s > 0$

$$\frac{u(\tau(s))}{s} = u_*$$

and this implies

$$\frac{y(\tau(s))}{x(\tau(s))} = \frac{d}{c}.$$

From Lemma 10 we know that $\tau$ is one-to-one from $[0, \infty)$ to $[0, \infty)$, and thus for any $t > 0$,

$$\frac{y(t)}{x(t)} = \frac{d}{c}.$$

Going back to the system (1) we conclude that

$$x(t) = c\sqrt{t},$$

$$y(t) = d\sqrt{t}.$$  ■

References

[1] Fiedler P. and Pták V., *Some generalizations of positive definiteness and monotonicity*, Numerische Mathematik, 9 (1966), 163-172.