Quasi-periodic propagation in time of some classical/quantum systems: Nielsen’s conserved quantity and Floquet properties

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Abstract

We consider classical and quantum propagators for two different time intervals. If these propagators follow one another in a Fibonacci sequence we get a discrete quasi-periodic system. A theorem due to Nielsen provides a novel conserved quantity for this system. The Nielsen quantity controls the transition between commutative and non-commutative propagation in time. The quasi-periodically kicked oscillator, moreover, is dominated by quasi-periodic analogues of the Floquet theorem.

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1. Introduction

For a general description of time-dependent systems in quantum mechanics, we refer the reader to [3, 4, 8, 9, 19]. We consider the time evolution for a particular class of time-dependent classical/quantum systems that are periodic or quasi-periodic in time.

We restrict our attention to time evolutions, built piecewise from strings with classical symplectic propagators [1] \( g(t) \) in 2D phase space. These describe oscillators and dilatations, or positive and negative \( \delta \)-kicks. Quantum propagators are constructed as unitary representations \( S_{g(t)} \) of the classical linear canonical transformations for \( g(t) \) from the symplectic group \( Sp(2, \mathbb{R}) \).

We present Nielsen’s theorem on automorphisms of the free group \( F_2 \) with two generators and from it derive a conserved quantity in terms of the group commutator. We analyse the Floquet properties of classical and quantum propagators. We show the presence/absence of Floquet quantum numbers for a periodic system under the control of system parameters. For the periodically kicked classical oscillator we demonstrate the occurrence of Floquet bands and Floquet gaps.

For periodic systems built from two basic intervals, we construct Nielsen’s conserved quantity as a function of the system parameters. We extend the analysis to quasi-periodic strings. By building the strings through a Fibonacci automorphism we are able to survey their quasi-Floquet properties.

2. The Floquet theorem for linear differential equations

Theorem 1 (Floquet). For the real first-order periodic linear differential equation

\[
\frac{d}{dt} Y(t) = A(t) Y(t), \quad A(t) \text{ real},
\]

\[
A(t + T) = A(t), \quad T > 0,
\]

a fundamental system of two solutions can be written in the form

\[
Y(t) = \exp(\pm iQt) Y_0(t),
\]

\[
Y_0(t + T) = Y_0(t), \quad 0 \leq Q < \frac{2\pi}{T}.
\]
follows that the Floquet theorem requires complex solutions of the differential equation (1) for real \( A(t) \).

**Proof.** The discrete time translation \( T : A(t) \rightarrow A(t + T) \) is a symmetry operation of \( A(t) \). It generates the abelian group \( C_\infty \) with elements \( \pm nT, n = 0, 1, \ldots \). The unitary representations of this group have the form

\[
T \rightarrow \exp(i\Omega T),
\]

\[
D^\Omega(nT) = \exp(in\Omega T), \quad 0 \leq \Omega < \frac{2\pi}{T}.
\]

The orthogonality and completeness relation of \( C_\infty \) are given in appendix A.

Consider a solution of equation (2) that can be arranged so that it transforms according to an irreducible representation of \( C_\infty \),

\[
Y^\Omega(t) : Y^\Omega(t + T) = Y^\Omega(t) \exp(i\Omega T).
\]

From this solution, define the function

\[
Y^\Omega_0(t) := \exp(-i\Omega t) Y^\Omega(t).
\]

Then it easily follows that \( Y^\Omega_0(t) \) is periodic,

\[
Y^\Omega_0(t + T) = Y^\Omega_0(t).
\]

Since \( A(t) \) is real, the complex conjugate solution \( Y^-\Omega := Y^{\frac{-\Omega}{2}} \) is a second linearly independent solution with the complex conjugate Floquet factor. With \( \Omega = Q \) we obtain equation (2). \( \square \)

**Remark 1.** The Floquet theorem is mathematically equivalent to the Bloch theorem for systems with discrete position symmetry, which has the same abstract symmetry group \( C_\infty \).

**Remark 2.** We remark that the periodic system equation (1) for certain ranges of its parameters cannot have solutions with the Floquet property. We shall give examples of this situation in later sections.

3. The free group \( F_2 \) and its automorphisms

Refer to [18] for what follows. We also cite [7] as a useful reference for the basic concepts of free monoids.

**Definition 1.** The infinite free group \( F_2 \) consists of all words generated by concatenation from two non-commuting invertible generators \( \{y_1, y_2\} \). Its elements can be viewed as a tree formed by words of increasing length.

**Definition 2.** An automorphism of \( F_2 \) is an invertible homomorphic map

\[
\phi : F_2 \rightarrow F_2,
\]

\[
\{y_1, y_2\} \rightarrow \{\phi_1(y_1, y_2), \phi_2(y_1, y_2)\}, \quad \{\phi_1, \phi_2\} \in F_2.
\]

and so preserves group multiplication and unit element. The set of all automorphisms of \( F_2 \) forms the infinite group \( \phi = \text{Aut}(F_2) \). This group, as was shown by Nielsen [18], is again finitely generated, with explicit generators given in [18]. Similar results hold true for the free group \( F_n \) and its group of automorphisms \( \text{Aut}(F_n) \). Homomorphic properties of finitely generated groups have the virtue that they can be verified by checking them only for the generators.

**Example 1.** The Fibonacci automorphism is defined by

\[
\phi_{\text{fib}} : \{y_1, y_2\} \rightarrow \{\phi_{\text{fib}}(y_1, y_2), \phi_{\text{fib}}(y_1, y_2)\}
\]

\[
= \{y_2y_1, y_1y_2\}, \quad \phi_{\text{fib}}^{-1} : \{y_1, y_2\} \rightarrow \{y_2y_1^{-1}, y_1\}.
\]

The discrete translation \( \phi_{\text{fib}} \) obeys (2) with \( \omega = 2\pi/5 \).

We shall use, in particular, iterated automorphisms as \( \phi_{\text{fib}}^n \). Then the words \( \phi_{\text{fib}}^n(y) \), with their length \( |w| \), defined [7] as the number of letters in \( w \) after reducing \( w \) by use of all relations of the type \( y^\pm 1 = e \), become

\[
\phi_{\text{fib}}^n(y_1): y_1 y_2 y_1 y_2 y_1 y_2 y_1 y_2 y_1 y_2 y_1 y_2 y_1 y_2 \cdots
\]

\[
|\phi_{\text{fib}}^n(y_1)|.) = 1 2 3 5 \cdots
\]

The word length \( |\phi_{\text{fib}}^n(y_1)| \) is the Fibonacci number \( n_j \)

\[
n_j : n_{j+2} = n_j + n_{j+1}, \quad n_1 = n_2 = 1.
\]

Notice the recursive structure of the images under \( \phi_{\text{fib}}^n \). Define \( \phi_{\text{fib}}(y_1) = w_1, \phi_{\text{fib}}(y_2) = w_2 \). Then it follows from equation (7) that \( \phi_{\text{fib}}^n(y_1) = w_2, \phi_{\text{fib}}^n(y_2) = (w_1w_2) \), and so we must, in step \( n \), simply concatenate the two words \( w_1, w_2 \) obtained in step \( n - 1 \).

**Theorem 2. (Nielsen) on \( \text{Aut}(F_2) \).** Consider the commutator \( K = y_1y_2y_1^{-1}y_2^{-1} \) in \( F_2 \) and a fixed automorphism \( \phi(F_2) \in \text{Aut}(F_2) \). Then the image of the commutator under \( \phi \) obeys ([18] theorem 3.9, p 165):

\[
\phi(K) = wK \pm 1 w^{-1}, \quad w \in F_2,
\]

where the sign \( \pm 1 \) and the element \( w \) depend on the chosen automorphism \( \phi \).

**Proof.** It suffices to prove the result for the finite set of generators of \( \text{Aut}(F_2) \) given by Nielsen [18].

So the commutator \( K \) is conserved, up to inversion and conjugation, under any automorphism of \( \text{Aut}(F_2) \). No counterpart of the Nielsen theorem 2 is known for \( \text{Aut}(F_n), n > 2 \). \( \square \)

**Example 2.** Evaluation of the image of the commutator under the particular Fibonacci automorphism \( \phi_{\text{fib}} \) yields

\[
\phi_{\text{fib}}(K) = y_1y_2y_1^{-1}y_2^{-1}, \quad \phi_{\text{fib}}(K) = y_2y_1y_2^{-1}y_1^{-1} = y_2y_1^{-1}y_2^{-1} = K^{-1}.
\]

The power of the Nielsen theorem appears through the notion of automorphisms induced on some group \( G \).

**Definition 3.** Induced automorphism: Consider a homomorphism \( h : F_2 \rightarrow G \) defined in terms of the generators
We shall consider in particular automorphisms induced on a discrete dynamical system on this conserved automorphisms and with increasing string length determine the corresponding automorphisms from $\text{Aut}(F_2)$. For an induced automorphism, the commutator induced in the group $G$ has additional significance since it measures the non-commutativity of the group. In case $K = e$ we have commutativity on the induced level, which allows us to rearrange the order of successive propagators.

Example 3. We shall consider in particular automorphisms induced on the symplectic group $Sp(2, R)$, the group of linear canonical transformations of a 1D classical Hamiltonian system. For $g \in Sp(2, R)$, we have the algebraic equivalence relation $g^{-1} = g$ and the trace relation $\text{tr}(g^{-1}) = \text{tr}(g)$. Note that the character is a class function. If these properties are combined with the Nielsen theorem, one finds:

**Proposition 1.** For any automorphism of the free group $F_2$ induced on $Sp(2, R)$, the half-trace of the commutator:

$$\frac{1}{2}\text{tr}(K) = \frac{1}{2}\text{tr}(g_1g_2g_1^{-1}g_2^{-1}).$$

is a quantity conserved under iterated automorphisms.

The condition $\frac{1}{2}\chi(K) = 1$ is a necessary but not a sufficient condition for $K = e$. So commutativity requires extra checking.

If by iterated automorphisms we can generate a string of arbitrary length, the conserved quantity equation (13) can be assigned to the infinite string.

If we choose the Fibonacci automorphism equation (8) and associate two matrix propagators with its two intervals, we generate a quasi-periodic system. We term such a system discrete quasi-periodic. Such a system forms a special class among general quasi-periodic systems [16].

In later sections, we shall associate with the classical system a quantum system. With a symplectic matrix $g$ we shall associate a representation by a unitary operator $S_g$ in the Hilbert space.

Since this association is a homomorphic map from symplectic matrices into propagators that represent linear canonical transformations, the Nielsen theorem then extends to a theorem in the Hilbert space.

4. Geometry and dynamics of systems of traces

The traces of the symplectic matrices $g \in Sp(2, R)$ obviously play an important part in the classical and quantum analysis. For a subclass of systems, the computation of these traces does not even require the computation of the underlying matrices [14]. The Fibonacci system example 1 belongs to this class. The trace invariant equation (13) geometrically becomes a cubic surface in three-space whose shape depends on the value of the invariant. The traces under repeated automorphisms and with increasing string length determine recursively a discrete dynamical system on this conserved surface. We sketch this approach: for the Fibonacci system equation (8), we define $y_1 := y_2$ and $g_3 := g_1g_2$. With the induced matrix automorphism written as

$$X^n, Y^n, Z^n := (\phi^0(g_1), \phi^0(g_2), \phi^n(g_3)),$$

and the matrix decomposition

$$g \in Sp(2, R), g = S(g) + A(g), \quad S = \frac{1}{2}\text{trace}(g)e,$$

one finds from equations (57)–(63) of [14] the recursion relations and the conserved quantity

$$S(X^{(n+1)}) = \begin{bmatrix} S(Y^n) \\ S(Z^n) \\ 2S(Y^n)S(Z^n) - S(X^n) \end{bmatrix}.$$

For an induced automorphism, the commutator induced on the symplectic group $Sp(n)$ is a quantity conserved under iterated automorphisms. For a subclass of systems, the computation of these traces play an important part in the classical and quantum analysis.

5. Quadratic Hamiltonians and propagators

The quantum propagators for quadratic Hamiltonians are given in [10, 11]. They are unitary integral operators $S_{g(t)}$, where $g(t)$ is a linear canonical transformation $g \in Sp(2m, R)$, which can be determined by exponentiation from the action of the Hamiltonian on the canonical position and momentum operators. So for quadratic Hamiltonians we have a very clear correspondence between classical and quantum propagation: the classical time evolution is given by the action of $g(t)$ on the initial canonical coordinates at time zero. The quantum propagator is given by the action of $S_{g(t)}$ on the initial state at time zero.

6. Hamiltonians and propagators for $n = 1$

In [10, 11], one finds for quadratic Hamiltonians the matrix analysis of propagators in various dimensions $m$. The classification of types of Hamiltonians for $n = 1, 2$ is discussed in detail in [21]. We briefly summarize the well-known results for $n = 1$. For $m = 1$, i.e. for one-dimensional position space, we have $g(t) \in Sp(2, R)$. This group is particularly transparent. Its classes admit a subdivision into three types, depending on the value of their half-trace $\frac{1}{2}\chi(g) := \frac{1}{2}\text{tr}(g)$. Since $\det(g) = 1$, the eigenvalues of $g$ are completely determined by the trace. Moreover, the trace is independent of symplectic similarity transformations, $\chi(qgg^{-1}) = \chi(g)$, and so allows us to determine the class of the symplectic matrix $g$. For $1: \frac{1}{2}|\chi(g)| < 1, g$ is symplectically equivalent to the dynamics generated by an oscillator Hamiltonian,

$$H = \frac{1}{2m} p^2 + \frac{m\omega^2}{2} X^2,$$
whereas for II: $\frac{1}{2} |\chi(X)| > 1$, it is equivalent to an inverted oscillator or a dilatation

$$ II: H = \kappa \frac{1}{2} (P X + X P) = \kappa \left( X P + \frac{\hbar}{2i} \right), \quad (18) $$

and for III: $\frac{1}{2} |\chi(X)| = 1$, it is, apart from the extra case $g = e, \frac{1}{2} |\chi(e)| = 1$, equivalent to a free motion,

$$ III: H = \frac{1}{2m} P^2. \quad (19) $$

All three Hamiltonians are Hermitian and time-independent. Hence their propagators $S_a(t)$ are unitary operators. For types I and III, the propagators can be written as integral operators according to Moshinsky and Quesne [20] in the Schrödinger representation or in the Bargmann space of analytic functions, compare [10].

**Type I.** Harmonic oscillator.

Rewrite the harmonic oscillator Hamiltonian equation (17) as

$$ H = \frac{1}{2} \omega \left[ \frac{1}{m \omega} p^2 + m \omega X^2 \right]. \quad (20) $$

The corresponding linear canonical transformation is obtained as

$$ g_\omega(t) = \begin{bmatrix} \sqrt{\omega} & 0 \\ 0 & \sqrt{\omega} \end{bmatrix} \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \times \begin{bmatrix} \sqrt{m \omega} & 0 \\ 0 & \sqrt{m \omega} \end{bmatrix} \begin{bmatrix} \cos(\omega t) & \frac{1}{m \omega} \sin(\omega t) \\ -(m \omega) \sin(\omega t) & \cos(\omega t) \end{bmatrix}. \quad (21) $$

Equation (21) admits the complex diagonalization

$$ \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} \sqrt{1/2} & i/\omega \exp(-i \omega t) \\ 1/\omega \exp(i \omega t) & -i/\omega \end{bmatrix} \begin{bmatrix} \sqrt{1/2} & i/\omega \exp(-i \omega t) \\ 1/\omega \exp(i \omega t) & -i/\omega \end{bmatrix}^{-1} = \begin{bmatrix} \sqrt{1/2} & i/\omega \exp(-i \omega t) \\ 1/\omega \exp(i \omega t) & -i/\omega \end{bmatrix}. \quad (22) $$

In the complex domain we can therefore pass to a pair of classical variables that propagate with $\exp(\mp i \omega t)$.

The corresponding quantum propagator in the Bargmann space of analytic functions is $[2, 10]$,

$$ \langle z' | S_\omega(t) | z \rangle = \exp(-i \omega t/2) \exp(\exp(i \omega t) z^* z). \quad (23) $$

The propagator in the Schrödinger representation is given in [10, pp 34–6].

**Type II.** The hyperbolic and dilatation Hamiltonian.

For type II the symplectic matrix may be taken as

$$ g(t) = \begin{bmatrix} \exp(\kappa t) & 0 \\ 0 & \exp(-\kappa t) \end{bmatrix}. \quad (24) $$

Note the correspondence to equation (22) for $\kappa \rightarrow -i \omega$.

Figure 1. Phase space representation. The harmonic and the hyperbolic column solutions I: equation (21) and II: equation (27) in phase space.

There is for type II no obvious integral operator representation. In this case we proceed directly from the Hamiltonian equation (18) and find

$$ \exp \left( -\frac{i}{\hbar} H \right) \psi(x) = \exp \left( -\frac{\kappa}{2} \right) \left( \exp \left( -\kappa t \frac{d}{dx} \right) \psi \right)(x) $$

$$ = \exp \left( -\frac{\kappa}{2} \right) \psi \left( \exp \left( -\kappa t \right) x \right). \quad (25) $$

This propagator is a time-dependent unitary dilatation. Applied to the ground state of an oscillator with frequency $\omega_0$, it transforms it into an oscillator ground state with frequency $\omega(t) = \omega_0 \exp(-2\kappa t)$. As a function of time, the uncertainty in position would increase, while the uncertainty of momentum would decrease.

Another form for class II is given by the hyperbolic Hamiltonian

$$ H = \frac{1}{2} \left( \frac{p^2}{m} - m \kappa^2 X^2 \right) \quad (26) $$

with the symplectic propagator

$$ g_\kappa(t) = \begin{bmatrix} \cosh(\kappa t) & (m \kappa)^{-1} \sinh(\kappa t) \\ (m \kappa) \sinh(\kappa t) & \cosh(\kappa t) \end{bmatrix}. \quad (27) $$

By choosing at time $t = 0$ the initial matrix

$$ h(0) = \begin{bmatrix} 1 & 0 \\ 0 & m \kappa \end{bmatrix}, \quad \det(h(0)) = m \kappa, \quad (28) $$

we determine two canonical column solutions given by

$$ \begin{bmatrix} X_1(t) \\ P_1(t) \end{bmatrix} = g_\kappa(t) h(0) = \begin{bmatrix} \cosh(\kappa t) & (m \kappa)^{-1} \sinh(\kappa t) \\ (m \kappa) \sinh(\kappa t) & \cosh(\kappa t) \end{bmatrix}. \quad (29) $$

Inserting the fundamental two solutions into the Hamiltonian equation (26), we find the two different energies

$$ E_1(0) = -\frac{m \kappa^2}{2}, \quad E_\Pi(0) = \frac{m \kappa^2}{2}. \quad (30) $$

The types I and II of propagation in phase space are shown in figure 1.

In appendix B, we give the relation between the propagator and the transfer matrix.
Type III. Free particle.

From the Hamiltonian

$$H = \frac{1}{2m} p^2,$$

one finds [10] the symplectic matrix

$$g(t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix},$$

and the propagator

$$\langle x' | S_{g(t)} | x \rangle = \exp(-i\pi/4) \sqrt{\frac{m}{2\pi\hbar}} \exp\left(i\frac{(x' - x)^2}{2t}\right).$$

Proposition 2. The types I, II and III together with the case $g = e$ exhaust, up to conjugation within $Sp(2, R)$, the symplectic propagators with quadratic Hamiltonians.

7. Quadratic piecewise time-independent Hamiltonians

We consider now a system whose Hamiltonian for $0 < t < T = T_1 + T_2$ is time independent on the two intervals. Assume that the propagation for the full interval $T$ is repeated periodically. Over the period we have then a propagation with the half-trace given by equation (42).

For $\frac{1}{2} |\chi| < 1$ we can write $\frac{1}{2} \chi = \cos(\Omega T)$ and determine $\Omega$, $0 < \Omega < 2\pi/T$. This value $\Omega$ is the Floquet index of a classical symplectic periodic system. The two Floquet factors $\exp(\pm i\Omega T)$ appear as the eigenvalues of the symplectic matrix $g_1(T_1)g_2(T_2)$ and determine the classical propagators $U(T)$. The Floquet index is completely determined by the parameters $(\omega_1, T_1, \omega_2, T_2)$ of the system. At multiples $nT$, $n = 1, 2, \ldots$ the classical propagator becomes $(U(T))^n$.

In the quantum propagation, $\Omega$ represents the overall frequency of an effective oscillator Hamiltonian. The propagator over the period $T$ from equation (23) becomes

$$\langle z' | S_{g(T)} | z \rangle = \exp(-i\Omega T/2) \exp(\exp(i\Omega T)z'z).$$

We emphasize that this result holds true only for $T = T_1 + T_2$, whereas at intermediate time the propagator is governed by $g_1(t)$ or $g_2(t)$, respectively. The classical Floquet parameter enters the quantum propagator in an essential way. The eigenstates $|N\rangle$ of the overall harmonic oscillator of frequency $\Omega$ from equation (34) are the monomials of degree $N$. These eigenstates under the discrete time translations transform according to

$$T : |N\rangle \rightarrow |N\rangle \exp(iN\Omega T).$$

This again generates an irreducible representation of the time translation group. By the unique decomposition $N\Omega = m2\pi/T + \Omega'$, $0 < \Omega' < 2\pi/T$, $m = 0, 1, 2, \ldots$, this representation becomes $D^{\Omega'}(NT) = D^{\Omega'}(T)$.

It follows that not the quantum propagator itself, but only the eigenstates in its decomposition transform irreducibly under $t \rightarrow t + T$.

For $\frac{1}{2} |\chi| > 1$ the situation changes. By writing $\frac{1}{2} |\chi| = \cosh(\kappa T)$ we determine an effective classical dilatation parameter $\kappa$. The eigenvalues $\exp(\pm \kappa T)$ represent this dilatation over one period. The corresponding classical propagation does not represent a Floquet propagation since the eigenvalues of the symplectic matrix do not have absolute value 1. They form a non-unitary real representation of the time translation group. The $n$th powers of these two factors now approach 0 and $\infty$, and respectively.

The unitary quantum propagator over a period $T$ from equation (25) becomes

$$\exp\left(-\kappa T \frac{1}{\hbar} H\right) \psi(x)$$

$$= \exp\left(-\kappa \frac{T}{2}\right) \left(\exp\left(-\kappa T \frac{1}{\hbar} \frac{d}{dx}\right) \psi(x)\right)$$

$$= \exp\left(-\kappa \frac{T}{2}\right) \psi(\exp(-\kappa T)x).$$

Again, this propagator is valid only at time $T = T_1 + T_2$. For intermediate time the propagator is governed by one of the oscillator matrices $g_1(t)$ or $g_2(t)$, respectively. The quantum propagator over the periods $T$ stays unitary, but for $n \rightarrow \infty$ when acting on a state transforms it up to normalization into a $\delta$-distribution of sharp effective momentum/position.

Remark 3. The results on time-periodic systems from piecewise quadratic Hamiltonians resemble the analysis of position-periodic Kronig–Penney systems [12] with piecewise constant potentials. In the latter systems, the potentials play the role of the frequencies, and the Bloch $k$-label plays the role of the Floquet index. Real values of the Bloch parameter yield a band index, whereas imaginary values indicate exponentially increasing or decreasing solutions of the Schrödinger equation, which are excluded as they belong to band gaps.

Example 4. For the system of section 8 in case $|\frac{1}{2} \chi| < 1$, we obtain an overall Floquet state with $\Omega$ determined by $\frac{1}{2} \chi = \cos(\Omega T)$. Each Floquet state is an analogue of a Bloch state of a band model in $x$-space.

8. Oscillators and $\delta$-kicks

The transfer and propagator matrix methods apply to sequences of time intervals.

Example 5. The classical square well for the time interval $t$ has the type I oscillator propagator

$$g_0(t) = \begin{pmatrix} \cos(\omega t) & (m\omega)^{-1} \sin(\omega t) \\ -(m\omega) \sin(ka) & \cos(\omega t) \end{pmatrix}. \tag{37}$$

As the initial values for the two fundamental column solutions $I$ and $II$ we take

$$h(0) = \begin{pmatrix} 1 & 0 \\ 0 & m\omega \end{pmatrix}. \tag{38}$$

Then the energies computed by insertion into the harmonic oscillator Hamiltonian become

$$E_1 = E_{II} = \frac{m\omega^2}{2}. \tag{39}$$

Example 6. The square tunnel for the time interval $t$ has the repulsive oscillator type $II$ propagator equation (27).
The energy of the two fundamental solutions are
\[ E_1 = -\frac{mk^2}{2}, \quad E_2 = \frac{mk^2}{2}. \]

Following what is done in space in [13, 14] we introduce, w.r.t. time, negative and positive $\delta$-kicks. They are obtained as limits of square well and square tunnel intervals of finite length. In the limits $m\omega^2 T \to u'$, $\omega T \to 0$ and $mk^2 T \to u'$, $\kappa T \to 0$ with $u$ being the strength parameter, we get from equations (37) and (38), respectively, the symplectic transfer matrices for negative/positive $\delta$-kicks in the form
\[ g_{\pm u'} = \begin{bmatrix} 1 & 0 \\ \mp u' & 1 \end{bmatrix}. \quad (40) \]

9. The periodically kicked oscillator

We employ here the transfer matrix, see equation (B.1), with $u' = mu$,
\[ M(t) = \lambda_i = \begin{bmatrix} \cos(\omega T_{\pm}) & \omega^{-1} \sin(\omega T_{\pm}) \\ -\omega \sin(\omega T_{\pm}) & \cos(\omega T_{\pm}) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}, \quad i = 1, 2, \quad (41) \]

with positive $\delta$-kicks. The half-trace becomes
\[ \frac{1}{2} \chi = \frac{1}{2} \text{Tr}(M(T)) = \cos(\omega T) + \frac{u}{2\omega} \sin(\omega T). \quad (42) \]

This expression has the period $\frac{2\pi}{\omega}$. We find
\[ \frac{1}{2} \chi = 0 : \cot(\omega T_0) = -\frac{u}{2\omega}. \quad (43) \]

The solutions $T_0$ of this equation exist in each interval $2m\pi \leqslant \omega T \leqslant 2(m+1)\pi$, $m = 0, \pm 1, \pm 2, \ldots$ and mark the centers of Floquet bands as functions of $\kappa$. The equations
\[ \frac{1}{2} \chi = +1 : \tan\left(\frac{\omega T_{\pm}}{2}\right) = \frac{u}{2\omega}, \quad (44) \]
\[ \frac{1}{2} \chi = -1 : \cot\left(\frac{\omega T_{\pm}}{2}\right) = -\frac{u}{2\omega} \]
determine for any such band its edges $T_{\pm}$ as functions of $\kappa$. Note that these equations have in $\omega T_0$ the period $4\pi$. In between two Floquet bands, there are gaps. In these gaps we have $\frac{1}{2} |\chi| > 1$ and so solutions of type II.

Proposition 3. (the Floquet spectrum of the periodically kicked oscillator) The periodically kicked oscillator has a Floquet band spectrum. Each interval $2m\pi \leqslant \omega T \leqslant 2(m+1)\pi$, $m = 0, 1, 2, \ldots$ carries two bands. The centers $T_0$ from equation (43) are repeated periodically, the edges being given by equation 44. The bands are separated by gaps that admit only solutions of type II.

Example 7. A simple example is provided by the choice $\frac{u}{2\omega} = 1$. Then the half-trace $\frac{1}{2} \chi$ equation (42) with $\beta := \omega T$ becomes
\[ \frac{1}{2} \chi = \cos(\beta) + \sin(\beta) = \sqrt{2} \cos(\beta - \pi/4). \quad (45) \]

Figure 2. Floquet band structure for the periodically kicked oscillator. Shown is the half-trace $\frac{1}{2} \chi$ equation (42) as a function of $\omega T$ for the parameter $\frac{u}{2\omega} = 1$ of equation (45). Bands and gaps alternate periodically and both have the width $\Delta \beta = \frac{\pi}{2}$.

The band edges $\beta^+$, $\beta^-$ and the band centers $\beta^0$ are located at
\[ \beta^+ : 0 + 2\pi m_1, \quad \frac{\pi}{2} + 2\pi m_2, \quad \beta^- : \pi + 2\pi m_3, \quad \frac{3\pi}{2} + 2\pi m_4, \quad \beta^0 : \frac{\pi}{4} + \pi m_5, \quad m_j = 0, \pm 1, \pm 2, \ldots. \quad (46) \]

Bands and gaps alternate with the same width $\frac{\pi}{2}$, as shown in figure 2.

10. The quasi-periodically kicked oscillator

For the properties of systems with a general quasi-periodic Hamiltonian we refer the reader to [5]. The Nielsen theorem is valid for all discrete quasi-periodic systems of Fibonacci type, but its significance can already be seen from a simple example. Here we study a specific discrete quasi-periodic system of Fibonacci type in time, in analogy to a system in space from [12]. Following [6], it would be worth studying the spectral function for this discrete quasi-periodic system.

For two intervals (1, 2) that form a quasi-periodic Fibonacci sequence, we take the matrix propagators in the form
\[ \lambda_i = \begin{bmatrix} \cos(\omega T_i) & \omega^{-1} \sin(\omega T_i) \\ -\omega \sin(\omega T_i) & \cos(\omega T_i) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix}, \quad i = 1, 2. \quad (47) \]

Then we compute, expanded in powers of $\frac{u}{\omega}$, the products
\[ \lambda_1 \lambda_2 = \begin{bmatrix} \cos(\omega (T_1 + T_2)) & \omega^{-1} \sin(\omega (T_1 + T_2)) \\ -\omega \sin(\omega (T_1 + T_2)) & \cos(\omega (T_1 + T_2)) \end{bmatrix} \]
\[ + \left(\frac{u}{\omega}\right)^2 \begin{bmatrix} \sin(\omega T_1) \sin(\omega T_2) & 0 \\ \omega \cos(\omega T_1) \sin(\omega T_2) & 0 \end{bmatrix} \]
\[ \lambda_2 \lambda_1 = \begin{bmatrix} \cos(\omega (T_1 + T_2)) & -\omega^{-1} \sin(\omega (T_1 + T_2)) \\ \omega \sin(\omega (T_1 + T_2)) & \cos(\omega (T_1 + T_2)) \end{bmatrix} \]
\[ + \left(\frac{u}{\omega}\right)^2 \begin{bmatrix} \sin(\omega T_1) \sin(\omega T_2) & 0 \\ \omega \cos(\omega T_1) \sin(\omega T_2) & 0 \end{bmatrix} \]
\[ (\lambda_2 \lambda_1)^{-1} = \begin{bmatrix} \cos(\omega (T_1 + T_2)) & -\omega^{-1} \sin(\omega (T_1 + T_2)) \\ \omega \sin(\omega (T_1 + T_2)) & \cos(\omega (T_1 + T_2)) \end{bmatrix} \]
The invariant $K$ is the transformed inverse. This does not affect the trace, which becomes an invariant under the automorphism.

The half-trace of the commutator therefore becomes the invariant

$$I = \frac{1}{2} \text{Tr}(K)$$

$$= \frac{1}{2} \text{Tr}(K^{-1})$$

$$= 1 + \frac{1}{2} \left( \frac{u}{\omega} \right)^2 (\sin(\omega(T_1 - T_2))^2).$$

Of interest are the points where the invariant takes the value $I = 1$. These are given from equation (50) by

$$\sin(\omega(T_1 - T_2)) = 0, \quad (51)$$

$$\omega(T_1 - T_2) = m\pi, \quad m = 0, \pm 1, \pm 2, \ldots. \quad (52)$$

For the Fibonacci case $T_1 = T$, $T_2 = \tau T$, we find from equations (50) and (51)

$$\omega(\tau - 1)T = m\pi, \quad m = 0, \pm 1, \pm 2, \ldots. \quad (53)$$
Figure 4. The quasi-periodic Floquet theorem: for multiples of a value $ωT$ with $I(ωT) = 1$, hence commutative, and moreover, in the overlap of the two Floquet bands, all phase space propagators fall on an ellipse equivalent to a harmonic oscillator.

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Appendix A. Irreps of the discrete translation group

We give here the orthogonality and completeness relations for the discrete translation group $C_\infty$ as discussed in [15]. In the present case, they apply to the Floquet states.

The orthogonality relation is

$$D^\Omega(nT) = \exp(2\pi i \Omega n T), \quad \Omega \in BZ = \{0, 2\pi / T\}, \quad n \in \{0, \pm 1, \pm 2, \ldots\}: \quad (A.1)$$

$$\times \sum_n D^\Omega(nT)D^{\Omega'}(nT) = |BZ| \delta(\Omega - \Omega').$$

Equation (A.1) yields an orthogonality relation between evolutions belonging to two different Floquet indices. It does not imply an orthogonality between pairs with the same Floquet index from different Floquet bands, as they are found for example in section 10. The completeness relation is

$$\int_{\Omega \in BZ} D^\Omega(nT)D^{\Omega}(mT)d\Omega = |BZ|\delta_{n,m}. \quad (A.2)$$

Appendix B. Classical propagator and transfer matrix

We denote the matrix that propagates the canonical pair of position and momentum $(X(t), P(t))$ as the classical (symplectic) propagator $g(t) \in Sp(2, R)$, det$(g(t)) = 1$. We denote the matrix that propagates a fundamental pair of position and velocity $(X(t), \dot{X}(t))$ as the transfer matrix $M(t)$. Again we may choose det$(M(t)) = 1$. For a Hamiltonian with standard kinetic energy, we have the relation $P(t) = m \dot{X}(t)$. From this we get the conjugation relation

$$M(t) = \left[ \begin{array}{cc} \sqrt{m} & 0 \\ 0 & \frac{1}{\sqrt{m}} \end{array} \right] g(t) \left[ \begin{array}{cc} \frac{1}{\sqrt{m}} & 0 \\ 0 & \sqrt{m} \end{array} \right]. \quad (B.1)$$

between the symplectic propagator $g(t)$ and the transfer matrix $M(t)$. From the relation equation (B.1) it follows that for a sequence of strings both their symplectic propagators and their transfer matrices are multiplied in the same order.

The first and second columns of the propagator matrix $g(t)$ each yield a fundamental canonical pair $(I, II)$ of solutions and trajectories for the Hamiltonian equations of motion. For quadratic Hamiltonians, the Hamiltonian equations of motion are linear. Then the most general classical solution is a linear superposition of fundamental solutions. The superposition changes the classical energy, which therefore must be recalculated for any superposition.

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