HOMOLOGICAL PROJECTIVE DUALITY VIA VARIATION OF GEOMETRIC INVARIANT THEORY QUOTIENTS

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Abstract. We provide a geometric approach to constructing Lefschetz collections and Landau-Ginzburg Homological Projective Duals from a variation of Geometric Invariant Theory quotients. Additionally, we provide a description of the derived category of a degree \( d \) hypersurface fibration which recovers Kuznetsov’s result for quadric fibrations. Combining these two approaches yields homological projective duals for Veronese embeddings. We also extend the Homological Projective Duality framework to the relative setting for all of our results.

1. Introduction

A fundamental question in algebraic geometry is how invariants behave under passage to hyperplane sections. In his seminal work [Kuz07], Kuznetsov studied this question extensively for the bounded derived category of coherent sheaves on a projective variety and developed a deep homological manifestation of projective duality. He suitably titled this phenomenon, “Homological Projective Duality” (HPD).

The HPD setup is as follows. One starts with a smooth variety \( X \rightarrow \mathbb{P}(V) \), together with some homological data which is called a Lefschetz decomposition, and constructs a Homological Projective Dual; \( Y \rightarrow \mathbb{P}(V^*) \) together with a dual Lefschetz decomposition. This establishes a precise relationship between the derived categories of any dual complete linear sections \( X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp) \) and \( Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L) \); we call this result of Kuznetsov the “Fundamental Theorem of Homological Projective Duality.” [Kuz07, Theorem 6.3] (Theorem 2.4.10 below).

In this paper, we develop a robust geometric approach to constructing Homological Projective Duals as Landau-Ginzburg models. The idea, in the terminology of high-energy theoretical physics, is to pass to a gauged linear sigma model and “change phases” [HHP08, HTo07, DSh08, CDHPS10, Sha10]. In mathematical terms, this is first passing from a hypersurface to the total space of a line bundle [Isi12, Shi12], then varying Geometric Invariant Theory quotients (VGIT) to do a birational transformation to the total space of this line bundle [BFK12, H-L12, Kaw02, VdB04, Seg11, HW12, DSe12]. A nice consequence of our technique is that we can expand the Homological Projective Duality framework to the relative setting i.e. all our results are proven in the relative setting over a general smooth base variety.
Specifically, using the semi-orthogonal decompositions from [BFK12], we construct both Lefschetz collections and Homological Projective duals for a large class of quotient varieties. Our main application is to Veronese embeddings $\mathbb{P}(W) \to \mathbb{P}(S^dW)$ for $d \leq \dim W$. After recovering the natural Lefschetz decomposition in this case, we prove that the Landau-Ginzburg pair $(\mathbb{P}(S^dW^*)/\mathbb{G}_m, w)$, where the $\mathbb{G}_m$-action is by dilation on $W$ and $w$ is the universal degree $d$ polynomial, is a homological projective dual to the Veronese embedding. When $d > 2$, assuming a technical result (Conjecture 4.4.2), this Landau-Ginzburg pair is derived-equivalent to the pair $(\mathbb{P}(S^dW^*), A)$, where $A$ is a $\mathbb{Z}$-graded sheaf of $A_\infty$-algebras defined explicitly as

$$A = \text{Sym}(u\mathcal{O}_{\mathbb{P}(S^dW^*)}(1), u^{-1}\mathcal{O}_{\mathbb{P}(S^dW^*)}(-1)) \otimes \Lambda^*W^*,$$

where

$$\mu^d(1 \otimes v_i, \ldots, 1 \otimes v_i) = \frac{u}{d!} \frac{\partial^d w}{\partial x_{i_1} \ldots \partial x_{i_d}}$$

and $\mu^i = 0$ for $2 < i < d$. When $d = 2$, we recover the non-commutative variety from [Kuz05].

It should be noted that neither Kuznetsov’s precise definition of a homological projective dual nor his Fundamental Theorem are available at this level of generality. We instead construct non-commutative varieties which are weak homological projective duals and prove that the conclusions of the Fundamental Theorem hold directly in our setting (Theorem 3.1.3) (and in the relative setting).

Additionally, we provide a new description of the derived category of a degree $d$ hypersurface fibration, $p : X \to S$, extending the case $d = 2$ in [Kuz05]. Using the same method of starting with a variety, passing to a derived-equivalent gauged LG model using the results of [si12, Shi12], and performing VGIT, we prove a relative version of a theorem of Orlov [Orl09]. This compares the derived category of $p : X \to S$ to the derived category of a related gauged Landau-Ginzburg model. Then, following calculations in [Sei11, Dyc11, Efi12], we produce a local generator for the derived category of this gauged Landau-Ginzburg model and determine its derived endomorphism sheaf of dg-algebras $\mathcal{B}$ over the base $S$. A version of the homological perturbation lemma applies to show that, assuming the technical result (Conjecture 4.4.2) mentioned above, $\mathcal{B}$ can be replaced by a quasi-isomorphic sheaf $\mathcal{A}$ of minimal $A_\infty$-algebras. We also give formulas for the first $d$ higher multiplications of $\mathcal{A}$.

Homological Projective Duality was exhibited by Kuznetsov for the double Veronese embedding $\mathbb{P}(W) \hookrightarrow \mathbb{P}(S^2W)$.

In this case, Kuznetsov [Kuz05] proves that a homological projective dual is given by $Y = (\mathbb{P}(S^2W^*), \text{Cliff}_0)$, where $\text{Cliff}_0$ is a sheaf of even Clifford algebras. As a consequence, Kuznetsov recovers a theorem of Bondal and Orlov [BO95] relating the derived category of intersection of two even dimensional quadrics to the derived category of a hyperelliptic curve. Moreover, his Homological Projective Duality framework provides analogous descriptions for arbitrary intersections of quadrics as in [BO02].
In [Kuz06], Kuznetsov constructs the dual to the Grassmannian of two dimensional planes in a vector space $W$ of dimension 6 or 7 with respect to the Plücker embedding,
\[ \text{Gr}(2, W) \hookrightarrow \mathbb{P}(\Lambda^2 W). \]
In these cases, the homological projective dual is a non-commutative resolution of the classical projective dual: the Pfaffian variety. Among the many applications is a derived equivalence between two non-biratational Calabi-Yau varieties of dimension 3, originally studied by Rødland as an example in mirror symmetry [Rød00]. This derived equivalence was proven independently by Borisov and Căldăraru [BC09] who demonstrated that generic Grassmannian Calabi-Yau varieties can be realized as moduli spaces of curves on the dual Pfaffian Calabi-Yau. Homological Projective Duality for the Grassmannian $\text{Gr}(3, 6)$ was studied in [Del11].

A relative version of the 2-Veronese example was considered in [ABB11]; it was used to relate rationality questions to categorical representability. Another example of Homological Projective Duality is conjectured by Hosono and Takagi and supported by a proof of a derived equivalence between the corresponding linear sections [HTa11, HTa13a, HTa13b].

The main cases that this paper does not interpret in our larger framework are the Grassmannian and Hosono-Takagi examples [Kuz06, Del11, HTa11, HTa13a, HTa13b]. However, these examples do admit similar physical interpretations [DSh08, Hor11]. Thus, it is plausible that all known examples of HPD would fall within the scope of our methodological approach. The main issue is that the results of [BFK12] need to be expanded to handle the complexity of the VGIT theory which arises. Indeed, work of Addington, Donovan, and Segal, [ADS12] uses more complex GIT stratifications to understand the Grassmannian case, albeit in a slightly less general context than HPD.

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2. Background

2.1. Derived categories of LG models. Let $Q$ be a smooth and quasi-projective variety with the action of an affine algebraic group, $G$. Let $\mathcal{L}$ be an invertible $G$-equivariant sheaf on $Q$ and let $w \in H^0(Q, \mathcal{L})^G$ be a $G$-invariant section of $\mathcal{L}$. In this section, we recall the appropriate analog of the bounded derived category of coherent sheaves for a quadruple, $(Q, G, \mathcal{L}, w)$. Matrix factorization categories have been studied in [Eis80, Buc86, Orl04]. Building on these works, most of the ideas presented here are due to L. Positselski [Pos09, Pos11]. The authors generalize these ideas in [BDFIK12] to a setting which includes the material presented below.
Definition 2.1.1. A gauged Landau-Ginzburg model, or gauged LG model, is the quadruple, 
\((Q,G,L,w)\), with \(Q\), \(G\), \(L\), and \(w\) as above. We shall commonly denote a gauged LG model by the pair \([(Q/G),w]\).

To declutter the notation, given a quasi-coherent \(G\)-equivariant sheaf, \(\mathcal{E}\), we denote \(\mathcal{E} \otimes \mathcal{L}^n\) by \(\mathcal{E}(n)\). Given a morphism, \(f : \mathcal{E} \to \mathcal{F}\), we denote \(f \otimes \text{Id}_{\mathcal{L}^n}\) by \(f(n)\). Following Eisenbud, \[\text{Eis80}\], one gives the following definition.

Definition 2.1.2. A coherent factorization, or simply a factorization, of a gauged LG model, \([(Q/G),w]\), consists of a pair of coherent \(G\)-equivariant sheaves, \(E^{-1}\) and \(E^0\), and a pair of \(G\)-equivariant \(\mathcal{O}_Q\)-module homomorphisms,

\[
\phi_e^{-1} : \mathcal{E}^0(-1) \to \mathcal{E}^{-1} \\
\phi_e^0 : \mathcal{E}^{-1} \to \mathcal{E}^0
\]

such that the compositions, \(\phi_e^0 \circ \phi_e^{-1} : \mathcal{E}_0(-1) \to \mathcal{E}^0\) and \(\phi_e^{-1}(1) \circ \phi_e^0 : \mathcal{E}^{-1} \to \mathcal{E}^{-1}(1)\), are multiplication by \(w\). We shall often simply denote the factorization \((\mathcal{E}^{-1},\mathcal{E}^0,\phi_e^{-1},\phi_e^0)\) by \(\mathcal{E}\). The coherent \(G\)-equivariant sheaves, \(E^0\) and \(E^{-1}\), are called the components of the factorization, \(\mathcal{E}\).

A morphism of factorizations, \(g : \mathcal{E} \to \mathcal{F}\), is a pair of morphisms of coherent \(G\)-equivariant sheaves,

\[
g^{-1} : \mathcal{E}^{-1} \to \mathcal{F}^{-1} \\
g^0 : \mathcal{E}^0 \to \mathcal{F}^0,
\]

making the diagram,

\[
\begin{array}{ccc}
\mathcal{E}^0(-1) & \xrightarrow{\phi_e^0} & \mathcal{E}^{-1} \\
\mathcal{E}^0(-1) & \xrightarrow{\phi_e^{-1}} & \xrightarrow{\phi_e^0} & \mathcal{E}^{-1} \\
\mathcal{F}^0(-1) & \xrightarrow{\phi_f^0} & \mathcal{F}^{-1} \\
\mathcal{F}^0(-1) & \xrightarrow{\phi_f^{-1}} & \mathcal{F}^{-1} & \xrightarrow{\phi_f^0} & \mathcal{F}^0
\end{array}
\]

commute.

We let \(\text{coh}([Q/G],w)\) be the Abelian category of factorizations with coherent components.

There is an obvious notion of a chain homotopy between morphisms in \(\text{coh}([Q/G],w)\). Let \(g_1, g_2 : \mathcal{E} \to \mathcal{F}\) be two morphisms of factorizations. A homotopy between \(g_1\) and \(g_2\) is a pair of morphisms of quasi-coherent \(G\)-equivariant sheaves,

\[
h^{-1} : \mathcal{E}^{-1} \to \mathcal{F}^0(-1) \\
h^0 : \mathcal{E}^0 \to \mathcal{F}^{-1},
\]

such that

\[
g_1^{-1} - g_2^{-1} = h^0 \circ \phi_e^0 + \phi_f^{-1} \circ h^{-1} \\
g_1^0 - g_2^0 = h^{-1}(1) \circ \phi_e^{-1}(1) + \phi_f^0 \circ h^0.
\]
We let $K(\text{coh}[Q/G], w)$ be the corresponding homotopy category, the category whose objects are factorizations and whose morphisms are homotopy classes of morphisms.

There is a translation autoequivalence, $[1]$, defined as

$$E[1] := (E^0, E^{-1}(1), -\phi^{-1}_E, -\phi^{-1}_E(1)).$$

For any morphism, $g : E \to F$, there is a natural cone construction. We write, $C(g)$, for the resulting factorization. It is defined as

$$C(g) := \left( E^0 \oplus F^{-1}, E^{-1}(1) \oplus F^0, \begin{pmatrix} -\phi^0_E & 0 \\ g^{-1} & \phi^{-1}_F \end{pmatrix}, \begin{pmatrix} -\phi^{-1}_E(1) & 0 \\ g^0 & \phi^0_F \end{pmatrix} \right).$$

It is an easy exercise to see that translation and the cone construction induce the structure of a triangulated category on the homotopy category, $K(\text{coh}[Q/G], w)$.

We wish to derive $\text{coh}([Q/G], w)$, however, we lack a notion of quasi-isomorphism because our “complexes” lack cohomology. For the usual derived categories of sheaves, one can view localization by the class of quasi-isomorphisms as the Verdier quotient by acyclic objects. In [Pos09], Positselski defined the correct substitute in $\text{coh}([Q/G], w)$ for acyclic complexes. This notion of acyclic complexes and the corresponding quotient also appears in [Orl11].

Given a short exact sequence in $\text{coh}([Q/G], w)$,

$$0 \to E \overset{\alpha}{\to} F \overset{\beta}{\to} G \to 0,$$

the map, $\alpha$, induces a closed morphism, $\alpha' : E \to C(\beta)[1]$, in $\text{coh}([Q/G], w)$. The totalization of the exact sequence is the object $C(\alpha')$ in $\text{coh}([Q/G], w)$.

The following definitions give the correct analog for the derived category of sheaves for LG models, when $Q$ is smooth. These definitions are due to Positselski, see [Pos09, Pos11].

**Definition 2.1.3.** A factorization, $A$, is called **totally acyclic** if it lies in the smallest thick subcategory of $K(\text{coh}[Q/G], w)$ containing all totalizations of short exact sequences from $\text{coh}([Q/G], w)$. We let $\text{acycl}([Q/G], w)$ denote the thick subcategory of $K(\text{coh}[Q/G], w)$ consisting of totally acyclic factorizations.

The **absolute derived category of factorizations**, or the **derived category**, of the LG model $([Q/G], w)$, is the Verdier quotient,

$$D(\text{coh}[Q/G], w) := K(\text{coh}[Q/G], w)/\text{acycl}([Q/G], w).$$

Abusing terminology, we say that $E$ and $F$ are **quasi-isomorphic** factorizations if they are isomorphic in the absolute derived category.

Later in the paper we will also use the singularity category as an intermediary. We recall the definition.

**Definition 2.1.4.** Let $[Y/G]$ be a global quotient stack with $Y$ quasi-projective. The **category of singularities** of $[Y/G]$ is the Verdier quotient,

$$D_{\text{sg}}([Y/G]) := D^b(\text{coh}[Y/G])/\text{perf}([Y/G]),$$
of the bounded derived category of coherent sheaves by the thick subcategory of perfect complexes.

The following result, based on Koszul Duality, is referred to in the physics literature as the \( \sigma \)-model/Landau-Ginzburg-model correspondence for B-branes, arising from renormalization group flow. We sometimes refer to it briefly as the “\( \sigma \)-LG correspondence”.

**Theorem 2.1.5.** Let \( Y \) be the zero-scheme of a section \( s \in \Gamma(X, \mathcal{E}) \) of a locally-free sheaf of finite rank \( \mathcal{E} \) on a smooth variety \( X \). Assume that \( s \) is a regular section, i.e. \( \dim Y = \dim X - \text{rank } \mathcal{E} \). Then, there is an equivalence of triangulated categories

\[
D^b(\text{coh } Y) \cong D(\text{coh}[V(\mathcal{E})/\mathbb{G}_m], w)
\]

where \( w \) is the regular function determined by \( s \) under the natural isomorphism

\[
\Gamma(V(\mathcal{E}), \mathcal{O}) \cong \Gamma(X, \text{Sym } \mathcal{E})
\]

and \( \mathbb{G}_m \) acts by dilation on the fibers.

**Proof.** This is [Isi12, Theorem 3.6] or [Shi12, Theorem 3.4]. \( \square \)

**Corollary 2.1.6.** Let \( X \) be a smooth variety. Consider the trivial \( \mathbb{G}_m \)-action on \( X \) and let \( \chi \) be the identity character of \( \mathbb{G}_m \). For the Landau-Ginzburg model \( (X, \mathbb{G}_m, \mathcal{O}(\chi), 0) \), one has an equivalence of categories:

\[
D^b(\text{coh } X) \cong D(\text{coh}[X/\mathbb{G}_m], 0)
\]

**Proof.** This is a degenerate case of the above theorem where \( \mathcal{E} = 0 \). The statement can also be seen directly by observing that objects of \( D(\text{coh}[X/\mathbb{G}_m], 0) \) are the same as objects of \( D^b(\text{coh } X) \) grouped into even and odd homological grading. Namely, the grading on \( D(\text{coh}[X/\mathbb{G}_m], 0) \) induced by the \( \mathbb{G}_m \)-action is twice the homological grading on \( D^b(\text{coh } X) \). \( \square \)

**Remark 2.1.7.** We will also use the above corollary for global quotient stacks. The direct proof above can be seen to apply to factorizations in any Abelian category.

### 2.2. Semi-orthogonal decompositions.

In this section we provide background material on semi-orthogonal decompositions and record a few facts we will need later. Standard references are [Bon89, BK90, BO95].

**Definition 2.2.1.** Let \( \mathcal{A} \subseteq \mathcal{T} \) be a full triangulated subcategory. The **right orthogonal** \( \mathcal{A}^\perp \) to \( \mathcal{A} \) is the full subcategory of \( \mathcal{T} \) consisting of objects \( B \) such that \( \text{Hom}_\mathcal{T}(A, B) = 0 \) for any \( A \in \mathcal{A} \). The **left orthogonal** \( \mathcal{A}^\perp \mathcal{A} \) is is the full subcategory of \( \mathcal{T} \) consisting of objects \( B \) such that \( \text{Hom}_\mathcal{T}(B, A) = 0 \) for any \( A \in \mathcal{A} \).

The left and right orthodoxals are naturally triangulated subcategories.

**Definition 2.2.2.** A **weak semi-orthogonal decomposition** of a triangulated category, \( \mathcal{T} \), is a sequence of full triangulated subcategories, \( \mathcal{A}_1, \ldots, \mathcal{A}_m \), in \( \mathcal{T} \) such that \( \mathcal{A}_i \subseteq \mathcal{A}_j^\perp \) for \( i < j \) and, for every object \( T \in \mathcal{T} \), there exists a diagram:
where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We shall denote a weak semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$. If $\mathcal{A}_i$ are essential images of fully-faithful functors, $\Upsilon_i : \mathcal{A}_i \to \mathcal{T}$, we may also denote the weak semi-orthogonal decomposition by $\langle \Upsilon_1, \ldots, \Upsilon_m \rangle$.

**Lemma 2.2.3.** The assignments $T \mapsto T_i$ and $T \mapsto A_i$ appearing in the definition of a weak semi-orthogonal decomposition are unique and functorial.

**Proof.** This is standard, see e.g. [Kuz09, Lemma 2.4]. □

**Definition 2.2.4.** An additive category, $\mathcal{A}$, is idempotent complete if for every morphism $e : A \to A$ with $e^2 = e$ there is a splitting

$$A \cong \ker(e) \oplus \im(e).$$

We will need a few facts about idempotent completeness.

The idempotent completion of $\mathcal{A}$, denoted $\tilde{\mathcal{A}}$, is the additive category whose objects are pairs $(a, e)$ with $A \in \mathcal{A}$ and $e : A \to A$ idempotent. A morphism between $(A, e)$ and $(A', e')$ is a morphism $f : A \to A'$ in $\mathcal{A}$ such that $fe = e'f = f$.

**Theorem 2.2.5.** Let $\mathcal{T}$ be a triangulated category. The idempotent completion, $\tilde{\mathcal{T}}$, can be equipped uniquely with the structure of a triangulated category such that the natural inclusion $\mathcal{T} \to \tilde{\mathcal{T}}$ is exact. Moreover, triangles in $\tilde{\mathcal{T}}$ are exactly retracts of triangles in $\mathcal{T}$.

**Proof.** This is [BS01, Theorem 1.5, Theorem 1.12]. □

**Lemma 2.2.6.** Let $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a weak semi-orthogonal decomposition. Then, there is a weak semi-orthogonal decomposition $\tilde{\mathcal{T}} = \langle \tilde{\mathcal{A}}, \tilde{\mathcal{B}} \rangle$. Moreover, $\mathcal{T}$ is idempotent complete if and only if $\mathcal{A}$ and $\mathcal{B}$ are idempotent complete.

**Proof.** The second statement is an immediate corollary of the first statement. So, assume that we have weak semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$. We have natural inclusions $\tilde{\mathcal{A}}, \tilde{\mathcal{B}} \to \tilde{\mathcal{T}}$. It is clear from the definition of the idempotent completion that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$. Let $(T, e)$ be an object of $\tilde{\mathcal{T}}$. Then, $T$ sits in a triangle

$$B \xrightarrow{\phi} T \xrightarrow{\psi} A \xrightarrow{\lambda} B[1].$$

There is a unique morphism of triangles
By Lemma 2.2.3, $e_a$ and $e_b$ are idempotents. Thus, the sequence

\[(B, e_b) \xrightarrow{\phi} (T, e) \xrightarrow{\psi} (A, e_a) \xrightarrow{\lambda} (B[1], e_b[1])\]

is a retract of the exact triangle

\[B \to T \to A \to B[1].\]

and is, by definition, an exact triangle. Thus, we satisfy the conditions of a weak semi-orthogonal decomposition. □

Closely related to the notion of a semi-orthogonal decomposition is the notion of a left/right admissible subcategory of a triangulated category.

**Definition 2.2.7.** Let $\alpha : A \to T$ be the inclusion of a full triangulated subcategory of $T$. The subcategory, $A$, is called **right admissible** if $\alpha$ has a right adjoint, denoted $\alpha^!$, and **left admissible** if $\alpha$ has a left adjoint, denoted $\alpha^*$. A full triangulated subcategory is called **admissible** if it is both right and left admissible.

**Definition 2.2.8.** A **semi-orthogonal decomposition** is a weak semi-orthogonal decomposition $\langle A_1, \ldots, A_m \rangle$ such that each $A_i$ is admissible. The notation is left unchanged.

### 2.3. Elementary wall-crossings

In this section, we review part of the relationship between variations of GIT quotients [Tha96, DH98] and derived categories, following [BFK12]. While consideration of the general theory was inspirational to our approach to Homological Projective Duality, it is sufficient for this paper to consider only the simplest types of variations of GIT quotients, namely elementary wall crossings.

Let $Q$ be a smooth, quasi-projective variety and let $G$ be a reductive linear algebraic group. Let

\[\sigma : G \times Q \to Q\]

denote an action of $G$ on $Q$. Recall that a **one-parameter subgroup**, $\lambda : \mathbb{G}_m \to G$, is an injective homomorphism of algebraic groups.

From $\lambda$, we can construct some subvarieties of $Q$. We let $Z_0^\lambda$ be a choice of connected component of the fixed locus of $\lambda$ on $Q$. Set

\[Z_\lambda := \{q \in Q \mid \lim_{t \to 0} \sigma(\lambda(t), q) \in Z_0^\lambda\} \subseteq Q.\]

The subvariety $Z_\lambda$ is called the **contracting locus** associated to $\lambda$ and $Z_0^\lambda$. If $G$ is Abelian, $Z_0^\lambda$ and $Z_\lambda$ are both $G$-invariant subvarieties. Otherwise, we must consider the orbits

\[S_\lambda := G \cdot Z_\lambda, S_0^\lambda := G \cdot Z_0^\lambda.\]
Also, let

\[ Q_\lambda := Q \setminus S_\lambda. \]

We will be interested in the case where \( S_\lambda \) is a smooth closed subvariety satisfying a certain condition. To state this condition we need the following group attached to any one-parameter subgroup

\[ P(\lambda) := \{ g \in G \mid \lim_{\alpha \to 0} \lambda(\alpha) g \lambda(\alpha)^{-1} \text{ exists} \}. \]

**Definition 2.3.1.** Assume \( Q \) is a smooth variety with a \( G \)-action. An elementary HKKN stratification of \( Q \) is a disjoint union

\[ \mathfrak{R} : Q = Q_\lambda \sqcup S_\lambda, \]

obtained from the choice of a one-parameter subgroup \( \lambda : \mathbb{G}_m \to G \), together with the choice of a connected component, denoted \( Z^0_\lambda \), of the fixed locus of \( \lambda \) such that

- \( S_\lambda \) is closed in \( X \).
- The morphism,

\[ \tau_\lambda : [(G \times Z_\lambda)/P(\lambda)] \to S_\lambda \]

\[ (g, z) \mapsto g \cdot z \]

is an isomorphism where \( p \in P(\lambda) \) acts by

\[ (p, (g, z)) \mapsto (gp^{-1}, p \cdot z). \]

We will need to attach an integer to an elementary HKKN stratification. We restrict the relative canonical bundle \( \omega_{S_\lambda/Q} \) to any fixed point \( q \in Z^0_\lambda \). This yields a one-dimensional vector space which is equivariant with respect to the action of \( \lambda \).

**Definition 2.3.2.** The weight of the stratum \( S_\lambda \) is the \( \lambda \)-weight of \( \omega_{S_\lambda/Q}|Z^0_\lambda \). It is denoted by \( t(\mathfrak{R}) \).

Furthermore, given a one parameter subgroup \( \lambda \) we may also consider its composition with inversion

\[ -\lambda(t) := \lambda(t^{-1}) = \lambda(t)^{-1}, \]

and ask whether this provides an HKKN stratification as well. This leads to the following definition.

**Definition 2.3.3.** An elementary wall-crossing, \( (\mathfrak{R}^+, \mathfrak{R}^-) \), is a pair of elementary HKKN stratifications,

\[ Q = Q_\lambda \sqcup S_\lambda, \]

\[ Q = Q_{\lambda^{-1}} \sqcup S_{\lambda^{-1}}. \]

We often let \( Q_+ := Q_\lambda \) and \( Q_- := Q_{-\lambda} \).

Let \( C(\lambda) \) denote the centralizer of the 1-parameter subgroup \( \lambda \). For an elementary wall-crossing set

\[ \mu = -t(\mathfrak{R}^+) + t(\mathfrak{R}^-). \]
Theorem 2.3.4. Let $Q$ be a smooth, quasi-projective variety equipped with the action of a reductive linear algebraic group, $G$. Let $w \in H^0(Q, \mathcal{L})^G$ be a $G$-invariant section of a $G$-invertible sheaf, $\mathcal{L}$, and assume that $\mathcal{L}$ has weight zero on $Z^0_\lambda$. Suppose we have an elementary wall-crossing, $(\mathcal{R}^+, \mathcal{R}^-)$,

$$Q = Q_+ \sqcup S_\lambda$$

and that $S^0_\lambda$ admits a $G$ invariant affine open cover. Fix any $d \in \mathbb{Z}$.

a) If $\mu > 0$, then there are fully-faithful functors,

$$\Phi^+_d : D(\text{coh}[Q_-/G], w|_{Q_-}) \to D(\text{coh}[Q_+/G], w|_{Q_+}),$$

and, for $-t(\mathcal{R}^-) + d \leq j \leq -t(\mathcal{R}^+) + d - 1$,

$$\Upsilon^+_j : D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)_{j} \to D(\text{coh}[Q_+/G], w|_{Q_+}),$$

and a semi-orthogonal decomposition,

$$D(\text{coh}[Q_+/G], w|_{Q_+}) = \langle \Upsilon^-_{-t(\mathcal{R}^-)+d}, \ldots, \Upsilon^-_{-t(\mathcal{R}^+)+d-1}, \Phi^+_d \rangle.$$

b) If $\mu = 0$, then there is an exact equivalence,

$$\Phi^+_d : D(\text{coh}[Q_-/G], w|_{Q_-}) \to D(\text{coh}[Q_+/G], w|_{Q_+}).$$

c) If $\mu < 0$, then there are fully-faithful functors,

$$\Phi^-_d : D(\text{coh}[Q_+/G], w|_{Q_+}) \to D(\text{coh}[Q_-/G], w|_{Q_-}),$$

and, for $-t(\mathcal{R}^+) + d \leq j \leq -t(\mathcal{R}^-) + d - 1$,

$$\Upsilon^-_j : D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)_{j} \to D(\text{coh}[Q_-/G], w|_{Q_-}),$$

and a semi-orthogonal decomposition,

$$D(\text{coh}[Q_-/G], w|_{Q_-}) = \langle \Upsilon^+_{-t(\mathcal{R}^+)+d}, \ldots, \Upsilon^+_{-t(\mathcal{R}^-)+d-1}, \Phi^-_d \rangle.$$

Proof. This is [BFK12 Theorem 3.5.2].

The categories, $D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)_{j}$, appearing in Theorem 2.3.4 are the full subcategories consisting of objects of $\lambda$-weight $j$ in $D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)$. For more details, we refer the reader to [BFK12]. In our situation, we will only need the conclusion of the following lemma. We set

$$Y_\lambda := [Z^0_\lambda/(C(\lambda)/\lambda)].$$

Lemma 2.3.5. We have an equivalence,

$$D(\text{coh}Y_\lambda, w_\lambda) \cong D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)_0.$$

Further, assume that there is a character, $\chi : C(\lambda) \to \mathbb{G}_m$, such that

$$\chi \circ \lambda(t) = t^l.$$ 

Then, twisting by $\chi$ provides an equivalence,

$$D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)_r \cong D(\text{coh}[Z^0_\lambda/C(\lambda)], w_\lambda)_{r+l},$$

for any $r \in \mathbb{Z}$. 

Proof. This is Lemma 3.4.4 of [BFK12]; we give the very simple and short proof here. A quasi-coherent sheaf on $Y_\lambda$ is a quasi-coherent $C(\lambda)$-equivariant sheaf on $Z_0^0$ for which $\lambda$ acts trivially, i.e. of $\lambda$-weight zero. For the latter statement just observe that twisting with $\chi$ is an autoequivalence of $D^b(\text{coh}[Z_0^0/C(\lambda)])$ which brings range to target and its inverse does the reverse. □

2.4. Homological Projective Duality. In this section, we provide an introduction to Homological Projective Duality (HPD) following [Kuz07].

Let $X$ be a smooth projective variety equipped with a morphism $f : X \to \mathbb{P}(V)$.

We have a canonical section of $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$ determined as follows. Under the natural isomorphism,

$$V^* \otimes V \cong \text{End}(V),$$

the identity map on $V$ corresponds to an element $u \in V \otimes V^*$. We define a section

$$\theta_V \in \Gamma(\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)) \cong V^* \otimes V$$

by taking the image of $u$ under the isomorphism above.

Let $\mathcal{O}_X(1)$ denote the pullback $f^*\mathcal{O}_{\mathbb{P}(V)}(1)$. We can also pull back $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$ to $X \times \mathbb{P}(V^*)$. Let $\theta_X$ denote the pull back of $\theta_V$.

**Definition 2.4.1.** The zero locus of $\theta_X$ is called the universal hyperplane section of $f$. It is denoted by $\mathcal{X}$.

The universal hyperplane section comes equipped with two natural morphisms,

$$p : \mathcal{X} \to X \text{ and } q : \mathcal{X} \to \mathbb{P}(V^*).$$

The fiber of $q$, $\mathcal{X}_H$, over $H \in \mathbb{P}(V^*)$ is exactly the hyperplane section of $X$ corresponding to $H$.

**Remark 2.4.2.** Recall that when $X$ is smooth, the projective dual to $X$ under the embedding $f$, is the closed subset

$$X^\vee := \{H \in \mathbb{P}(V^*) \mid X_H \text{ is singular}\},$$

with its reduced, induced scheme structure. Thus $X^\vee$ is the non-regular, i.e. critical, locus of $q : \mathcal{X} \to \mathbb{P}(V^*)$ in $\mathbb{P}(V^*)$.

Homological Projective Duality is a phenomenon that can be considered as a lifting of the notion of classical projective duality to non-commutative geometry. The starting data for HPD is a smooth variety, $X$, together with a map to a projective space, $f : X \to \mathbb{P}(V)$, and a special type of a semi-orthogonal decomposition called a Lefschetz decomposition. We now provide the setup to define a Lefschetz decomposition.

**Definition 2.4.3.** Let $B$ be an algebraic variety and $\mathcal{T}$ be a monoidal triangulated category. A $B$-linear structure on $\mathcal{T}$ is an exact monoidal functor

$$F : D^b(\text{coh } B) \to \mathcal{T}.$$
We will often use the above definition when $\mathcal{T}$ is a subcategory of $\text{D}^b(\text{coh} X)$ and $X$ is an $B$-scheme. The functor $F$ will implicitly be assumed to be the pullback functor.

**Definition 2.4.4.** An exact functor 

$$\Phi : \mathcal{T} \to \mathcal{T}'$$

between $S$-linear monoidal triangulated categories with respect to $F$ and $F'$ is called $B$-linear if there are bi-functorial isomorphisms 

$$\Phi(A \otimes F(T)) \cong \Phi(A) \otimes F'(T)$$

for any $T \in \mathcal{T}, B \in \text{D}^b(\text{coh} B)$.

Now let $B = \mathbb{P}(V)$ and consider a $\mathbb{P}(V)$-linear category $\mathcal{T}$ with respect to $F$. To simplify notation, denote by $(s)$ the functor of tensoring with $F(\mathcal{O}(s))$.

**Definition 2.4.5.** A Lefschetz decomposition of a $\mathbb{P}(V)$-linear category $\mathcal{T}$ is a semi-orthogonal decomposition of the form,

$$\mathcal{T} = \langle A_0, A_1(1), \ldots, A_i(i) \rangle,$$

where

$$0 \subset A_i \subset A_{i-1} \subset \ldots \subset A_1 \subset A_0 \subset \mathcal{T}$$

is a chain of admissible subcategories of $\mathcal{T}$ and $A_i(s)$ denotes the essential image of the category $A_i$ after application of the functor $(s)$.

**Definition 2.4.6.** A dual Lefschetz decomposition of a $\mathbb{P}(V^*)$-linear category $\mathcal{T}'$ is a semi-orthogonal decomposition of the form,

$$\mathcal{T}' = \langle B_{j-1}(1-j), B_{j-2}(2-j), \ldots, B_0 \rangle,$$

where

$$0 \subset B_{j-1} \subset B_{j-2} \subset \ldots \subset B_1 \subset B_0 \subset \mathcal{T}'$$

is a chain of admissible subcategories of $\mathcal{T}'$ and $B_i(s)$ denotes the essential image of the category $B_i$ after application of the functor $(s)$.

Now consider a morphism $f : X \to \mathbb{P}(V)$. The most important property of a Lefschetz decomposition is that it induces a semi-orthogonal decomposition on the derived category of any linear section of $X$.

**Proposition 2.4.7.** Consider a morphism $f : X \to \mathbb{P}(V)$ and a Lefschetz decomposition

$$\text{D}^b(\text{coh} X) = \langle A_0, A_1(1), \ldots, A_i(i) \rangle$$

with respect to $f^*$. Let $L \subseteq V^*$ be a linear subspace of dimension $r$ and

$$X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)$$

be a complete linear section of $X$ i.e. $\dim X_L = \dim X - \dim L$. There is a semi-orthogonal decomposition

$$\text{D}^b(\text{coh} X_L) = \langle C_L, A_r(r), \ldots, A_i(i) \rangle,$$

of $\text{D}^b(\text{coh} X_L)$ where the functor,

$$A_j(j) \to \text{D}^b(\text{coh} X_L),$$
is the composition,
\[ A_j(j) \to D^b(\text{coh } X) \to D^b(\text{coh } X_L) \]
of the inclusion and derived restriction to \( X_L \).

**Proof.** Let \( \delta : X_L \to X \) be the inclusion. The statement is equivalent to the fact that the restriction \( \delta^* : D^b(\text{coh } X) \to D^b(\text{coh } X_L) \) is fully-faithful on the full subcategory
\[ \langle A_r(r), \ldots, A_t(i) \rangle. \]

Let \( A_s \in A_s(s) \) and \( A_t \in A_t(t) \). Restrict the Koszul resolution on \( L \) to \( X \) to obtain an exact complex
\[ 0 \to \wedge^n L \otimes_k \mathcal{O}_X(-r) \to \cdots \to \mathcal{O}_X \to \mathcal{O}_{X_L} \to 0 \]
and tensor this complex with \( A_t \otimes_{\mathcal{O}_X} A_s^\vee \) to get
\[ 0 \to \wedge^n L \otimes_k A_t \otimes_{\mathcal{O}_X} A_s^\vee(-r) \to \cdots \to A_t \otimes_{\mathcal{O}_X} A_s^\vee \to \delta^* A_t \otimes_{\mathcal{O}_X} \delta^* A_s^\vee \to 0. \]

Applying global sections yields an exact sequence of hypercohomology
\[ 0 \to \wedge^n L \otimes_k \text{RHom}_X(A_t, A_s(-r)) \to \cdots \to \text{RHom}_X(A_t, A_s) \to \text{RHom}_{X_L}(\delta^* A_t, \delta^* A_s) \to 0. \tag{2.1} \]

Now, by definition of a Lefschetz decomposition \( \text{RHom}_{X_L}(A_t, A_s(-p)) = 0 \) if \( p \leq s \leq t \). Plugging into (2.1) we obtain
\[ \text{RHom}_{X_L}(\delta^* A_t, \delta^* A_s) \cong \begin{cases} \text{RHom}_X(A_t, A_s) & \text{if } r \leq s = t \\ 0 & \text{if } r \leq s < t \end{cases} \]
as desired. \( \square \)

A Lefschetz decomposition also induces a semi-orthogonal decomposition on the universal hyperplane section \( \mathcal{X} \) with respect to \( f \) and similarly on the family of linear sections over any \( L \subseteq V^* \),
\[ \mathcal{X}_L := \mathcal{X} \times_{\mathbb{P}(V^*)} \mathbb{P}(L). \]

Let \( \pi_L \) denote the natural map from \( \mathcal{X}_L \) to \( \mathbb{P}(L) \) and define \( A_k(k) \boxtimes D^b(\text{coh } \mathbb{P}(L)) \) to be the full triangulated subcategory of \( D^b(\text{coh } X \times \mathbb{P}(L)) \) generated by objects \( \mathcal{F} \boxtimes \mathcal{G} \), with \( \mathcal{F} \in A_k(k) \subset D^b(\text{coh } X) \) and \( \mathcal{G} \in D^b(\text{coh } \mathbb{P}(L)) \).

**Proposition 2.4.8.** For any Lefschetz decomposition,
\[ D^b(\text{coh } X) = \langle A_0, A_1(1), \ldots, A_t(i) \rangle, \]
af \( D^b(\text{coh } X) \) there is an associated semi-orthogonal decomposition,
\[ D^b(\text{coh } X_L) = \langle D_L, A_1(1) \boxtimes D^b(\text{coh } \mathbb{P}(L)), \ldots, A_t(i) \boxtimes D^b(\text{coh } \mathbb{P}(L)) \rangle \tag{2.2} \]
where \( D_L \) is defined as the right orthogonal to \( \langle A_1(1) \boxtimes D^b(\text{coh } \mathbb{P}(L)), \ldots, A_t(i) \boxtimes D^b(\text{coh } \mathbb{P}(L)) \rangle \).

**Proof.** Notice that we get a semi-orthogonal decomposition
\[ D^b(\text{coh } X \times \mathbb{P}(L)) = \langle A_0 \boxtimes D^b(\text{coh } \mathbb{P}(L)), \ldots, A_t(i) \boxtimes D^b(\text{coh } \mathbb{P}(L)) \rangle. \]
Now, consider \( X \times \mathbb{P}(L) \) with the Segre embedding and apply Proposition 2.4.7 to get the result. \( \square \)
The following is Definition 6.1 of [Kuz07].

**Definition 2.4.9.** Given \( f : X \to \mathbb{P}(V) \) and a Lefschetz decomposition \( \langle A_0, A_1(1), \ldots, A_i(i) \rangle \), a **Homological Projective Dual** \( Y \) is an algebraic variety together with a morphism \( g : Y \to \mathbb{P}(V^*) \) and a fully-faithful Fourier-Mukai transform \( \Phi_P \) with kernel \( P \in D^b(\text{coh } X \times \mathbb{P}(V^*)) \) which induces a semi-orthogonal decomposition

\[
D^b(\text{coh } X) = \langle \Phi_P(D^b(\text{coh } Y)), A_1(1) \boxtimes D^b(\text{coh } \mathbb{P}(V^*)), \ldots, A_i(i) \boxtimes D^b(\text{coh } \mathbb{P}(V^*)) \rangle.
\]

The Fundamental Theorem of HPD relates linear sections in \( X \) with respect to \( f \) to their dual linear sections of \( Y \) with respect to \( g \). Let \( N \) be the dimension of \( V \) and \( L \subset V^* \) be a linear subspace of dimension \( r \). Define,

\[
X_L := X \times_{\mathbb{P}(V)} \mathbb{P}(L^\perp)
\]

and

\[
Y_L := Y \times_{\mathbb{P}(V^*)} \mathbb{P}(L).
\]

**Theorem 2.4.10** (Fundamental Theorem of Homological Projective Duality). Let \( Y \to \mathbb{P}(V^*) \) be a homological projective dual to \( X \to \mathbb{P}(V) \) with respect to the Lefschetz decomposition \( \{A_i\} \) in the sense of Definition 2.4.9. With the notation above we have the following:

- **The category** \( D^b(\text{coh } Y) \) **admits a dual Lefschetz decomposition**

  \[
  D^b(\text{coh } Y) = \langle B_j(-j), \ldots, B_1(-1), B_0 \rangle
  \]

- **Assume that** \( X_L \) **and** \( Y_L \) **are complete linear sections**, i.e.

  \[
  \dim(X_L) = \dim(X) - r \text{ and } \dim(Y_L) = \dim(Y) + r - N.
  \]

  **Then there exist semi-orthogonal decompositions,**

  \[
  D^b(\text{coh } X_L) = \langle C_L, A_r(1), \ldots, A_i(i - r + 1) \rangle,
  \]

  **and,**

  \[
  D^b(\text{coh } Y_L) = \langle B_j(N - r - j - 1), \ldots, B_{N-r}(-1), C_L \rangle.
  \]

**Proof.** This is [Kuz07, Theorem 6.3].

**Remark 2.4.11.** Figure [1] is a useful representation of the pieces appearing in the semi-orthogonal decompositions in the theorem above. The boxes themselves represent what Kuznetsov calls primitive subcategories \( a_s := A_s/A_{s+1} \). The longer vertical line is placed at \( r \), the dimension of \( L \). The shaded boxes to the right of the long vertical line represent the terms of the perpendicular to \( C_L \) in \( D^b(\text{coh } X_L) \). The shaded boxes to the left of the vertical line represent the terms of the perpendicular to \( C_L \) in the derived category of the homological projective dual \( Y_L \). In the \( i \)th column, the category generated by the boxes below the staircase correspond to \( A_{i-1} \) and the category generated by the boxes above the staircase give \( B_{j-i+1} \).
Remark 2.4.12. Homological Projective Duality is a duality in the following sense. If \( Y \to \mathbb{P}(V^*) \) is a homological projective dual to \( X \to \mathbb{P}(V) \), then \( Y \to \mathbb{P}(V^*) \) has a dual Lefschetz decomposition \( D^b(\text{coh } Y) = \langle B_j(-j), \ldots, B_1(-1), B_0 \rangle \). By dualizing as in Theorem 7.3 in loc. cit., we get a Lefschetz decomposition \( D^b(\text{coh } Y) = \langle B_j^*(j), \ldots, B_1^*(1) \rangle \) for \( Y \to \mathbb{P}(V^*) \). With respect to this Lefschetz decomposition, Kuznetsov shows that \( X \to \mathbb{P}(V) \) is a homological projective dual to \( Y \to \mathbb{P}(V^*) \).

In our setting, Kuznetsov’s Fundamental Theorem of Homological Projective Duality does not apply. Instead, we are forced to prove a version of Kuznetsov’s Fundamental Theorem of Homological Projective Duality directly in the setup that we consider in this paper, stated below as Theorem 3.1.3. Theorems 2.3.4 and 2.1.5 are the central tools in our approach.

Let \( X \) be an \( S \)-scheme and \( \mathcal{E} \) be a locally-free coherent sheaf over \( S \). Let \( f : X \to \mathbb{P}_S(\mathcal{E}) \) be an \( S \)-morphism. We now consider Homological Projective Duality in the relative setting. This was already studied by Kuznetsov when \( \mathcal{E} \) is the trivial bundle \([\text{Kuz07}, \text{Theorem 6.27}]\) and, in the case of relative 2-Veronese embeddings, by Auel, Bernardara, and Bolognesi \([\text{ABB11}, \text{Theorem 1.13}]\).

The definition of a Lefschetz decomposition extends to the relative setting by replacing the projective space \( \mathbb{P}(V) \) by the projectivization \( \mathbb{P}_S(\mathcal{E}) \) and \( \mathcal{O}_{\mathbb{P}(V)}(1) \) by \( \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1) \).

Definition 2.4.13. Given an \( S \)-morphism \( f : X \to \mathbb{P}_S(\mathcal{E}) \) and a Lefschetz decomposition \( \langle \mathcal{A}_0, \mathcal{A}_1(1), \ldots, \mathcal{A}_i(i) \rangle \), a weak Homological Projective Dual \( Y \) relative to \( S \) is either

- a pair \( (Y_0, \mathcal{A}) \) where \( Y_0 \) is an \( S \)-scheme and \( \mathcal{A} \) is a sheaf of graded \( A_{\infty} \)-algebras, or
- a gauged Landau-Ginzburg model \( (\mathcal{Q}, G, \mathcal{L}, w) \)

The definition of \( A_\infty \)-algebras.

Together with an \( S \)-morphism \( g : (Y_0, \mathcal{A}) \to \mathbb{P}_S(\mathcal{E}^*) \) (or a \( (S, \mathbb{G}_m, \mathcal{O}, 0) \)-morphism of gauged LG models \( g : (\mathcal{Q}, G, \mathcal{L}, w) \to (\mathbb{P}_S(\mathcal{E}^*), \mathbb{G}_m, \mathcal{O}, 0) \)) such that there is a semi-orthogonal...
decomposition
\[ D^b(\text{coh} \mathcal{X}) = \langle \Phi, A_1(1) \boxtimes D^b(\text{coh} \mathbb{P}_S(\mathcal{E}^*)), \ldots, A_i(i) \boxtimes D^b(\text{coh} \mathbb{P}_S(\mathcal{E}^*)) \rangle, \]
where \( \Phi \) denotes the essential image of a fully-faithful \( \mathbb{P}_S(\mathcal{E}) \)-linear functor
\[ \Phi : D_{pe}(Y, \mathcal{A}) \to D^b(\text{coh} \mathcal{X}) \]
or
\[ \Phi : D(\text{coh}[Q/G], w) \to D^b(\text{coh} \mathcal{X}). \]

We may informally refer to any weak Homological Projective Dual \( Y \) that also satisfies a version of Theorem 2.4.10 as a Homological Projective Dual.

**Remark 2.4.14.** The difference, between Kuznetsov’s definition of Homological Projective Dual and the above definition of a weak Homological Projective Dual is in the assumption that the functor \( \Phi \) is given by a Fourier-Mukai kernel in the fiber product \( D^b(\text{coh} Y \times_{\mathbb{P}(V^*)} \mathcal{X}) \). Recent work by Ben-Zvi, Nadler and Preygel [B-ZNP13] shows that a Fourier-Mukai kernel in \( D^b(\text{coh} Y \times_{\mathbb{P}(V^*)} \mathcal{X}) \) for \( \Phi \) does exist when \( Y \) is a scheme and \( \mathbb{P}(V^*) \)-linearity is interpreted in a stronger, \( \infty \)-categorical sense.

**Remark 2.4.15.** In the relative setting, we will consider, instead of linear sections \( X_L \) and \( Y_L \), the fiber products \( X \times_{\mathbb{P}_S(\mathcal{E})} \mathbb{P}_S(\mathcal{W}) \) and \( Y \times_{\mathbb{P}_S(\mathcal{E}^*)} \mathbb{P}_S(\mathcal{V}) \) where \( \mathcal{V} \subset \mathcal{E}^* \) is a subbundle and \( \mathcal{W} \subset \mathcal{E} \) is the orthogonal subbundle.

### 3. Homological Projective Duality and VGIT

In this section we construct a weak homological projective dual to a GIT quotient provided we are also given the data of an elementary wall-crossing. We follow the notation of Section 2.3.

#### 3.1. Lefschetz decompositions and HPD from elementary wall crossings.

Let \( Q \) be a smooth quasi-projective variety equipped with the action of a reductive linear algebraic group \( G \) and a morphism
\[ p : [Q/G] \to S. \]

Let \( \lambda \) be a one-parameter subgroup of \( G \) which determines an elementary wall-crossing \((\mathfrak{R}^+, \mathfrak{R}^-)\)
\[ Q = Q_+ \sqcup S_\lambda \]
\[ Q = Q_- \sqcup S_{-\lambda}, \]
such that \( S^0_\lambda = G \cdot Z^0_\lambda \) admits a \( G \)-equivariant affine cover and \( S_\lambda \) has codimension at least 2. We let
\[ \mu = -t(\mathfrak{R}^+) + t(\mathfrak{R}^-) \]
and we assume that \( \mu \geq 0. \)

Assume that \( X := [Q_+/G] \) is a smooth and proper variety. Notice that \( X \) is an \( S \)-scheme by composing the inclusion with \( p \). We denote this map by
\[ g : X \to S. \]
Let $\mathcal{E}$ be a locally-free coherent sheaf of rank $N$ over $S$. One can consider the projective bundle

$$\mathbb{P}_S(\mathcal{E}) := [(V_S(\mathcal{E}) \setminus 0_{V_S(\mathcal{E})})/\mathbb{G}_m]$$

where $0_{V_S(\mathcal{E})}$ denotes the zero-section of $V_S(\mathcal{E})$. This bundle comes with a projection

$$\pi : \mathbb{P}_S(\mathcal{E}) \to S.$$ 

We denote the relative bundle by $\mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1)$.

Consider an $S$-morphism

$$f : X \to \mathbb{P}_S(\mathcal{E}).$$

We write

$$\mathcal{L} := f^* \mathcal{O}_{\mathbb{P}_S(\mathcal{E})}(1).$$

Now suppose that there exists a $G$-equivariant invertible sheaf $\mathcal{M}$ on $Q$ such that, as an invertible sheaf on $[Q/G]$, it restricts to $\mathcal{L}$ on $[Q_+/G]$.

$$\mathcal{M}|_{[Q_+/G]} \cong \mathcal{L}.$$ 

Furthermore, let $d > 0$ be the $\lambda$-weight of $\mathcal{M}$.

Recall that we have fully-faithful functors

$$\Upsilon_j^+ : \mathbb{D}^b(\text{coh}[Z_0^0/C(\lambda)])_j \to \mathbb{D}^b(\text{coh}X)$$

from Theorem 2.3.4 and Corollary 2.1.6. Therefore, when writing semi-orthogonal decompositions, we will denote the essential images of the functors $\Upsilon_j^+$ by $Z_j^+$. By Lemma 2.3.5 we see that

$$\mathbb{D}^b(\text{coh}[Z_0^0/C(\lambda)])_0 \cong \mathbb{D}(\text{coh}Y, w_{\lambda}),$$

where $Y := [Z_0^0/(C(\lambda)/\lambda)]$ and that tensoring by $\mathcal{L}$ induces an isomorphism between $Z_j^+$ and that of $Z_{n+d}^+$ for any $n \in \mathbb{Z}$.

When $\mu \geq 0$, the elementary wall crossing induces a semi-orthogonal decomposition on $\mathbb{D}(\text{coh}X)$, which is a Lefschetz decomposition when $X$ is considered together with the map $f$ to $\mathbb{P}_S(\mathcal{E})$. The fineness of the Lefschetz decomposition depends on the $\lambda$-weight $d$ of $\mathcal{M}$.

**Proposition 3.1.1.** If $\mu \geq 0$, there is a Lefschetz decomposition

$$\mathbb{D}^b(\text{coh}X) = \langle \mathcal{A}_0, \ldots, \mathcal{A}_i(i) \rangle$$

of $X$ with respect to $f$, where $i = \lceil \frac{\mu}{d} \rceil - 1$ and

$$\mathcal{A}_j = \begin{cases} 
\langle \mathbb{D}^b(\text{coh}[Q_-/G]), Z_0^+, \ldots, Z_{d-1}^+ \rangle & j = 0 \\
\langle Z_0^+, \ldots, Z_{d-1}^+ \rangle & 0 < j < \lceil \frac{\mu}{d} \rceil - 1 \\
\langle Z_0^+, \ldots, Z_{\mu-d(\lceil \frac{\mu}{d} \rceil - 1)}^+ \rangle & j = \lceil \frac{\mu}{d} \rceil - 1 
\end{cases}$$

**Proof.** Taking $d = -t(\mathfrak{h}^-)$ in Theorem 2.3.4 in combination with Corollary 2.1.6 gives a fully-faithful functor, $\Phi^+_t(\mathfrak{h}^-) : \mathbb{D}(\text{coh}[Q_-/G]) \to \mathbb{D}(\text{coh}X)$, and a weak semi-orthogonal decomposition

$$\mathbb{D}(\text{coh}X) = \langle Z_0^+, \ldots, Z_{\mu-1}^+, \mathcal{D} \rangle,$$ 

where $\mathcal{D}$ represents the essential image of the functor $\Phi^+_t(\mathfrak{h}^-)$. 
Since $X$ is smooth and proper, $D^b(\text{coh} X)$ is saturated [BV03, Corollary 3.1.5] and so is any weak semi-orthogonal component. By [BK90, Proposition 2.8], all the subcategories are fully admissible and we can mutate to get a new semi-orthogonal decomposition

$$D^b(\text{coh} X) = \langle D, Z^+_0, \ldots, Z^+_\mu \rangle.$$ 

We conclude the proof by noticing, as above, that tensoring by $L$ induces an isomorphism between $Z^+_{n}$ and that of $Z^+_{n+d}$ for any $n \in \mathbb{Z}$. 

Let $X_0$ be the incidence scheme in $\mathbb{P}_S(E) \times S \mathbb{P}_S(E^*)$ and let $X = X \times_{\mathbb{P}_S(E) \times S \mathbb{P}_S(E^*)} X_0$ be the relative universal hyperplane section of the $S$-morphism $f : X \to \mathbb{P}_S(E)$. 

We will now set up an elementary wall crossing for an action of $\tilde{G} = G \times \mathbb{G}_m \times \mathbb{G}_m$ on a space $U^1_{\tilde{G}}$, with a potential function $w$ such that

$$D^b(\text{coh} X) \cong D(\text{coh} [(U^1_{\tilde{G}})_+ / \tilde{G}], w).$$ 

The gauged Landau-Ginzburg model corresponding to the quotient $[(U^1_{\tilde{G}})_- / \tilde{G}]$ obtained from the elementary wall crossing will be our weak homological projective dual.

Let us define

$$U^1_{\tilde{G}} = V_Q(M) \times_S (V_S(E^*) \setminus 0_{V_S(E^*)})$$ 

with an action of $\tilde{G} = G \times \mathbb{G}_m \times \mathbb{G}_m$ which can be described by

$$\begin{array}{ccc}
G & g & 1 \\
\mathbb{G}_m & \alpha_1^{-1} & \alpha_1 \\
\mathbb{G}_m & \alpha_2 & 1
\end{array}$$

Here, $\alpha_1 \in \mathbb{G}_m$ and $\alpha_2 \in \mathbb{G}_m$ act by dilation on the fibers of the two respective bundles and the action of $G$ on $V_Q(M)$ is induced by the equivariant structure of $M$.

Let $\lambda_1$ be the one-parameter subgroup given by $\lambda_1(\alpha) = (\lambda(\alpha), 1, 1)$. The contracting locus for $\lambda_1$ is

$$S_{\lambda_1} = V_{S_{\lambda}}(M|_{S_{\lambda}}) \times_S (V_S(E^*) \setminus 0_{V_S(E^*)}),$$

while the contracting locus for $-\lambda_1$ is

$$S_{-\lambda_1} = 0_{V_{S_{-\lambda}}(M)} \times_S (V_S(E^*) \setminus 0_{V_S(E^*)}).$$

Therefore,

$$(U^1_{\tilde{G}})_+ = V_Q(M) \times_S (V_S(E^*) \setminus 0_{V_S(E^*)}) \subset U^1_{\tilde{G}}.$$

and, by definition,

$$(U^1_{\tilde{G}})_- = U^1_{\tilde{G}} \setminus S_{-\lambda_1}.$$ 

We will prove further below that $\lambda_1$ determines an elementary wall-crossing.

We now notice that the pull-back of the natural pairing

$$\theta_\mathcal{E} \in \Gamma(\mathbb{P}_S(E) \times S \mathbb{P}_S(E^*), \mathcal{O}_{\mathbb{P}_S(E) \times S \mathbb{P}_S(E^*)}(1, 1))$$
to $X \times_S \mathbb{P}(V^*)$, i.e. the section whose zero-scheme is $\mathcal{X}$, induces naturally a $G \times \mathbb{G}_m$-invariant function $w$ on $U_{\tilde{\mathcal{X}}_+}^1$. Indeed, as we have assumed that $S_\lambda$ had codimension at least two in $Q$, it follows that $S_{\lambda_1}$ has codimension at least two in $U_{\tilde{\mathcal{X}}_+}^1$, as well. This, together with the fact that $\mathcal{O}_{\mathbb{P}(\mathcal{E})} \times_{\mathbb{P}(\mathcal{E})^\ast} \mathcal{O}_{\mathbb{P}(\mathcal{E})^\ast}(1)$, gives equivalences

$$\Gamma([U_{\tilde{\mathcal{X}}_+}^1/(G \times \mathbb{G}_m)], \mathcal{O}) \cong \Gamma([U_{\tilde{\mathcal{X}}_+}^1/(G \times \mathbb{G}_m)], \mathcal{O})$$

$$\cong \Gamma(V \times \mathbb{P}(\mathcal{E})) (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathcal{E})^\ast}(1)), \mathcal{O})$$

$$\cong \Gamma(V \times \mathbb{P}(\mathcal{E}^\ast), \text{Sym} (\mathcal{L} \boxtimes \mathcal{O}_{\mathbb{P}(\mathcal{E})^\ast}(1)))$$.

Furthermore, $w$ is semi-invariant with respect to the action of the third component $\mathbb{G}_m$ of $\tilde{\mathcal{G}} = G \times \mathbb{G}_m \times \mathbb{G}_m$, with character $\beta(x, \alpha_1, \alpha_2) = \alpha_2$ of $\mathbb{G}_m$, i.e. homogeneous of degree 1 with respect to the final $\mathbb{G}_m$ component.

We are now ready to state:

**Theorem 3.1.2.** Let $Q$ be a smooth quasi-projective variety equipped with the action of a reductive algebraic group $G$ and a morphism $p : [Q/G] \to S$. Let $\lambda$ be a one-parameter subgroup of $G$ which determines an elementary wall-crossing $(\mathcal{K}^+, \mathcal{K}^-)$ such that $S_0^\lambda$ admits a $G$-equivariant affine cover and $S_\lambda$ has codimension at least 2 in $Q$. Assume that $X = [Q_+/G]$ is smooth and proper variety and let $f : X \to \mathbb{P}(\mathcal{E})$ be an $S$-morphism such that there exists a $G$-equivariant invertible sheaf $\mathcal{M}$ on $Q$ with $\mathcal{M}|_{[Q_+/G]} \cong f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

If $d \leq \mu$, the gauged Landau-Ginzburg model $((U_{\tilde{\mathcal{X}}_+}^1)_-, \tilde{\mathcal{G}}, \mathcal{O}(\beta), w)$ is a weak homological projective dual of $f$ with respect to the Lefschetz decomposition given by

$$\mathcal{A}_i = \begin{cases} 
(D^b(\text{coh}[Q_-/G]), Z_0^+, \ldots, Z_{d-1}^+) & i = 0 \\
(Z_0^+, \ldots, Z_{d-1}^+) & 0 < i < \lceil \frac{\mu}{d} \rceil - 1 \\
(Z_0^+, \ldots, Z_{\mu-d(\lceil \frac{\mu}{d} \rceil)-1}^+) & i = \lceil \frac{\mu}{d} \rceil - 1
\end{cases}$$

In particular, there is a semi-orthogonal decomposition

$$D^b(\text{coh} \mathcal{X}) = (D^b(\text{coh}[(U_{\tilde{\mathcal{X}}_+}^1)_-/\tilde{\mathcal{G}}], w), Z_d^+ \boxtimes D^b(\text{coh} \mathbb{P}(\mathcal{E}^\ast)), \ldots, Z_{\mu-1}^+ \boxtimes D^b(\text{coh} \mathbb{P}(\mathcal{E}^\ast)))$$,

where $Z_k^+$ are each equivalent to $D^b(\text{coh}[Z_{\lambda}^\ast/C(\lambda)]_k)$.

Further below we will give a proof of this theorem, based on Theorem 2.3.4 applied directly to the elementary wall-crossing given by $\lambda_1$ and Theorem 2.1.5, but we first state the other main result of this section.

For $\mathcal{V} \subset \mathcal{E}^\ast$ a subbundle, define

$$U_{\mathcal{V}}^1 := U_{\tilde{\mathcal{X}}_+}^1 \times_{\mathbb{P}(\mathcal{E})^\ast} \mathbb{P}(\mathcal{V}) = V_Q(\mathcal{M}) \times_{S} (V_S(\mathcal{V}) \mathcal{O}_{V_S(\mathcal{V})})$$

The action of $\tilde{\mathcal{G}}$ and $\lambda_1$ restrict to $U_{\mathcal{V}}^1$ to give an elementary wall-crossing and these structures are compatible with taking fibers. In particular

$$(U_{\mathcal{V}}^1)_- = ((U_{\tilde{\mathcal{X}}_+}^1)_-) \times_{\mathbb{P}(\mathcal{E}^\ast)} \mathbb{P}(\mathcal{V})$$.

We claim that the semi-orthogonal decompositions in the statement of Kuznetsov’s Fundamental Theorem of Homological Projective Duality hold in our context.
Theorem 3.1.3. With the same assumptions as in Theorem 3.1.2 above, we have the following

- The derived category of the gauged Landau-Ginzburg model \(((U_1^1,-,\tilde{G},\mathcal{O}(\beta),w))\) admits a dual Lefschetz collection

\[ \text{D}^b(\text{coh}[((U_1^1,\mathcal{E}^*),\tilde{G})],w) = \langle B_{N-1}(-N+1),...,B_1(-1),B_0 \rangle \]

with

\[ B_i = \begin{cases} 
\langle \text{D}^b(\text{coh}[Q_-/G]),Z^+_0,...,Z^+_{d-1} \rangle & 0 \leq i \leq N - \lceil \frac{d}{2} \rceil - 1 \\
\langle \text{D}^b(\text{coh}[Q_-/G]),Z^+_{\mu+1-d(\lceil \frac{d}{2} \rceil-1)},...,Z^+_{d-1} \rangle & i = N - \lceil \frac{d}{2} \rceil \\
\langle \text{D}^b(\text{coh}[Q_-/G]) \rangle & N - \lceil \frac{d}{2} \rceil < i < N 
\end{cases} \]

- Let \( V \subset \mathcal{E}^* \) be a subbundle and \( W \) the orthogonal subbundle in \( \mathcal{E} \). Assume that \( X \times_{F_S(\mathcal{E})} \mathbb{P}_S(W) \) is a complete linear section, i.e.

\[ \dim(X \times_{F_S(\mathcal{E})} \mathbb{P}_S(W)) = \dim(X) - r. \]

Then there exist semi-orthogonal decompositions,

\[ \text{D}^b(\text{coh}X \times_{F_S(\mathcal{E})} \mathbb{P}_S(W)) = \langle C_V, A_r(1),...,A_i(i-r+1) \rangle, \]

and

\[ \text{D}(\text{coh}[(U_1^1,\mathcal{E}^*),\tilde{G}]),w) = \langle B_{N-1}(-r-N-2),...,B_{N-1-r}(-1),C_V \rangle. \]

Remark 3.1.4. The notation \( Z^+_j, C_V \) and \( \text{D}^b(\text{coh}[Q_-/G]) \) respectively, are used to illustrate that the corresponding categories are equivalent, even though they are embedded by different functors and in different categories. Similarly, the notation \( A_j \in \text{D}^b(\text{coh}X \times_{F_S(\mathcal{E})} \mathbb{P}_S(W)) \) (respectively \( B_i \in \text{D}(\text{coh}[(U_1^1,\mathcal{E}^*),\tilde{G}]),w) \)) represents the equivalent essential image of \( A_j \in \text{D}^b(\text{coh}X) \) (respectively \( B_i \in \text{D}(\text{coh}[(U_1^1,\mathcal{E}^*),\tilde{G}]),w) \)) under pullback.

Figure 2. A visual representation of the components appearing in the semi-orthogonal decompositions in Theorem 3.1.3
Remark 3.1.5. Figure 2 demonstrates the tabular representation of the Fundamental Theorem of Homological Projective Duality in the case of the above theorem. The long vertical line is placed at $r$, the dimension of $L$. The shaded boxes to the right of the vertical line represent the terms of the perpendicular to $C_Y$ in $D^b(\text{coh}_{\mathcal{P}_{S}(\mathcal{W})})$. The shaded boxes to the left of the vertical line represent the terms of the perpendicular to $C_Y$ in the derived category of the homological projective dual $D(\text{coh}_{\mathcal{U}_{1-V}})$. In the $i$th column, the category generated by the boxes below the staircase corresponds to $A_{i-1}$ and the category generated by the boxes above the staircase gives $B_{N-i+1}$. Comparing with Remark 2.4.11, one should notice that this picture breaks the subcategories into $Z_i$’s rather than the primitive subcategories, which are, in general, larger.

Before proving Theorems 3.1.2 and 3.1.3 we will set up a more complete picture of the various elementary wall-crossings that appear in the proofs. For each subbundle $V$ of $\mathcal{E}$, we will set up what is, in principle, a variation of GIT quotients problem (we will specify four different elementary wall crossings arising in such a setup), which interpolates between the corresponding linear sections of $X$, $\mathcal{X}$, the Landau-Ginzburg model $((U_1^1)_{-}, \tilde{G}, \mathcal{O}(\beta), w)$ which is the homological projective dual and the Landau-Ginzburg model whose derived category is equivalent to the category $C_Y$ in the statement of Theorem 3.1.3. The proofs will then follow from applying Theorem 2.3.4 to some of these wall-crossings.

Consider the variety

$$\tilde{Q}_V := V_{Q}(\mathcal{M} \oplus p^*\mathcal{V}) = V_{Q}(\mathcal{M}) \times_S V_S(\mathcal{V}),$$

equipped with a $G \times \mathbb{G}_m$-action. As $\mathcal{M}$ is a $G$-equivariant invertible sheaf on $Q$ we can take the induced $G$-action on $V_{Q}(\mathcal{M})$ and the trivial action on the other component $V_S(\mathcal{V})$. Meanwhile, we let $\mathbb{G}_m$ act with weight $-1$ on the fibers of $V_{Q}(\mathcal{M})$ and with weight $1$ on the fibers of $V_S(\mathcal{V})$. This describes the $G \times \mathbb{G}_m$-action. There is another $\mathbb{G}_m$-action which acts by dilation only on the fibers of $V_{Q}(\mathcal{M})$ giving us in total an action of $\tilde{G} = G \times \mathbb{G}_m \times \mathbb{G}_m$ but we will ignore this for now as it will not take part in the elementary wall crossings we consider. To describe these, let us consider four $G \times \mathbb{G}_m$-invariant open subsets

$$U_1^1 := \tilde{Q}_V \setminus (V_{Q}(\mathcal{M}) \times_S 0_{V_S(\mathcal{V}))}$$
$$U_2^2 := \tilde{Q}_V \setminus (S_{-\lambda} \times_S 0_{V_Q(\mathcal{M})} \times S_{-\lambda} \times_S V_{Q}(\mathcal{M}) \times_S 0_{V_S(\mathcal{V}))}$$
$$U_3^3 := \tilde{Q}_V \setminus (0_{V_Q(\mathcal{M})} \times_S V_S(\mathcal{V}))$$
$$U_4^4 := \tilde{Q}_V \setminus (V_{S_{\lambda}(\mathcal{M})} \times_S V_S(\mathcal{V}))$$

of $\tilde{Q}_V$, and one-parameter subgroups given by

$$\lambda_1(\alpha) := (\lambda(\alpha), 1)$$
$$\lambda_2(\alpha) := (1_G, \alpha)$$
$$\lambda_3(\alpha) := \lambda_1(\alpha) \lambda_2(\alpha)^d$$
$$\lambda_4(\alpha) := \lambda_2(\alpha)$$
Although what we do now is slightly more general than the GIT framework, it is convenient to assemble the four wall crossings and open sets into a GIT fan as seen in Figure 3. To clarify, in what follows, we set

\[ (U_i V)_{-} = (U_i V)_{+} \downarrow S_{\pm \lambda_i} \]

**Lemma 3.1.6.** There are equalities

\[ (U_1 V)_{-} = (U_1 V)_{+} = \tilde{Q} \backslash (0_{V, \mathcal{M}} \times S V_S(\mathcal{V}) \cup V_{S_{\lambda_i}}(\mathcal{M}) \times S 0_{V_S(\mathcal{V}))} \]

\[ (U_2 V)_{-} = (U_2 V)_{+} = \tilde{Q} \backslash (V_{Q}(\mathcal{M}) \times S 0_{V_S(\mathcal{V})} \cup 0_{V_{S_{\lambda_i}}(\mathcal{M})} \times S V_S(\mathcal{V})) \]

\[ (U_3 V)_{+} = (U_3 V)_{+} = \tilde{Q} \backslash (0_{V, \mathcal{M}} \times S V_S(\mathcal{V}) \cup V_{S_{\lambda_i}}(\mathcal{M}) \times S V_S(\mathcal{V})) \]

\[ (U_4 V)_{+} = (U_4 V)_{+} = \tilde{Q} \backslash (0_{V, \mathcal{M}} \times S V_S(\mathcal{V}) \cup V_{Q}(\mathcal{M}) \times S 0_{V_S(\mathcal{V}))} \]

**Proof.** This is easily checked. \( \square \)

**Lemma 3.1.7.** There are new elementary wall-crossings \((\mathcal{R}^+, \mathcal{R}^-)\) for \(1 \leq i \leq 4\).

\[ U_i^+ = (U_i^+)_{+} \cup S_{\lambda_i} \]

\[ U_i^- = (U_i^-)_{-} \cup S_{-\lambda_i} \]

with

\[ t(\mathcal{R}^+) = \begin{cases} t(\mathcal{R}^+) & \text{if } i = 1, 3 \\ \text{rk } \mathcal{V} & \text{if } i = 2, 4 \end{cases} \]

and

\[ t(\mathcal{R}^-) = \begin{cases} t(\mathcal{R}^-) - d & \text{if } i = 1 \\ t(\mathcal{R}^-) - d \cdot \text{rk } \mathcal{V} & \text{if } i = 3 \\ 1 & \text{if } i = 2, 4 \end{cases} \]
Proof. We treat the case where $i = 1$. The rest are similar. Denote by
\[ i_\pm : Q_\pm \to Q \]
the open immersions. Notice that
\[ U^1_i = (V_Q(i_+^*,M)) \times_S (V_S(V) \setminus 0_{V_S(V)}) \sqcup V_{S_1}(M|_{S_1}) \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
\[ U^1_i = (V_Q(M) \setminus 0_{V_{S-1}(M|_{S-1})}) \times_S (V_S(V) \setminus 0_{V_S(V)}) \sqcup 0_{V_{S-1}(M|_{S-1})} \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
We will verify that these are elementary HKKN stratifications.

As $S_{\pm \lambda}$ is closed by assumption, it is clear that
\[ S_{\lambda_1} = V_{S_\lambda}(M|_{S_\lambda}) \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
\[ S_{-\lambda_1} = 0_{V_{S-\lambda}(M|_{S-\lambda})} \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
is closed in $U^1_i$. Furthermore,
\[ Z_{\lambda_1} = V_{Z_{\lambda}}(M|_{Z_{\lambda}}) \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
\[ Z_{-\lambda_1} = 0_{V_{Z_{-\lambda}}(M|_{Z_{-\lambda}})} \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
By assumption
\[ \tau_{\lambda} : [G \times Z_{\lambda}/P(\lambda)] \to S_{\lambda} \]
is an isomorphism. Also
\[ P(\pm \lambda_1) = P(\pm \lambda) \times G_{m}. \]

It remains to check that the maps
\[ \tau_{\lambda_1} : [(G \times G_{m}) \times V_{Z_{\lambda}}(M|_{Z_{\lambda}}) \times_S (V_S(V) \setminus 0_{V_S(V)})/P(\lambda_1)] \to V_{S_{\lambda}}(M|_{S_{\lambda}}) \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
and
\[ \tau_{-\lambda_1} : [(G \times G_{m}) \times 0_{V_{Z_{-\lambda}}(M|_{Z_{-\lambda}})} \times_S (V_S(V) \setminus 0_{V_S(V)})/P(-\lambda_1)] \to 0_{V_{S_{-\lambda}}(M|_{S_{-\lambda}})} \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
are isomorphisms.

We will check this for the first map; the proof for the second one is similar. First, we can cancel the $G_{m}$ with the one appearing in $P(\lambda_1) = P(\lambda) \times G_{m}$ and look at the map
\[ [G \times V_{Z_{\lambda}}(M|_{Z_{\lambda}}) \times_S (V_S(V) \setminus 0_{V_S(V)})/P(\lambda)] \to V_{S_{\lambda}}(M|_{S_{\lambda}}) \times_S (V_S(V) \setminus 0_{V_S(V)}) \]
Now, we can forget the $(V_S(V) \setminus 0_{V_S(V)})$ on both sides, as $P(\lambda)$ acts trivially on this factor, and look at the map
\[ [G \times V_{Z_{\lambda}}(M|_{Z_{\lambda}})/P(\lambda)] \to V_{S_{\lambda}}(M|_{S_{\lambda}}) \]
or equivalently
\[ [V_{G \times Z_{\lambda}}(O_G \boxtimes M|_{G \times Z_{\lambda}})/P(\lambda)] \to V_{S_{\lambda}}(M|_{S_{\lambda}}). \]
We have an isomorphism, $\tau^\pm_\lambda M|_{S_{\lambda}} \cong O_G \boxtimes M|_{Z_{\lambda}}$. This induces the desired isomorphism on the corresponding geometric vector bundles. The computation of the $t(\mathfrak{h}^\pm_i)$ follows directly from the definitions. \qed
Remark 3.1.8. The fourth elementary wall crossing corresponding to $U^4_V$ and $\lambda_4$ is not used in the proofs which follow. However, it is interesting to note that this wall crossing can be used to prove the semi-orthogonal decompositions appearing in the Fundamental Theorem of Homological Projective Duality in the case where the Lefschetz collection is the trivial one with $\mathcal{A}_0 = D^b(\text{coh } X)$, which would give that

$$D^b(\text{coh } \mathcal{X}_L) = (D^b(\text{coh } X_L), D^b(\text{coh } X), \ldots, D^b(\text{coh } X)).$$

As we noted above, $X$, the relative universal hyperplane section of the $S$-morphism $f : X \to \mathbb{P}_S(\mathcal{E})$, is the zero locus of the pullback of the canonical section, $\theta_\mathcal{E} \in \Gamma(\mathbb{P}_S(\mathcal{E}) \times_S \mathbb{P}_S(\mathcal{E}^*), \mathcal{O}(1,1))$, to $X \times_S \mathbb{P}(V^*)$. Furthermore, we constructed a unique $G \times \mathbb{G}_m$-invariant function on $[\langle U^1_\mathcal{E} \rangle_+ + \widetilde{G}]$ that corresponds to $\theta_\mathcal{E}$. Since $\Gamma(\tilde{Q}_\mathcal{E}^*, \mathcal{O})^{G \times \mathbb{G}_m} \cong \Gamma([\langle U^1_\mathcal{E} \rangle_+ / G \times \mathbb{G}_m], \mathcal{O})$ we observe that there exists a unique $w$, a $G \times \mathbb{G}_m$-invariant function on $\tilde{Q}_\mathcal{E}^*$, corresponding to the canonical section $(\theta_\mathcal{E} : \mathcal{O} \to \mathcal{E} \times \mathcal{E}^*) \in \Gamma(S, \text{Sym}^1(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}^*)).$

The same is true for a subbundle $V$ and we will abuse notation by also writing this section as $w$ when $U^i_V$ is an open subset of $\tilde{Q}_V$ for general $V$ even though $w$, in general, depends on both $V$ and $1 \leq i \leq 4$.

Recall now, that we also have a fiber-wise $\mathbb{G}_m$-action on $\tilde{Q}_V$ which acts by dilation on the fibers of $V_Q(\mathcal{M})$ and trivially on the remaining fibers. This commutes with the $G \times \mathbb{G}_m$-action and hence can be inserted into all the elementary wall crossings of Lemma 3.1.7. Our total action on $\tilde{Q}_V$ is now by

$$\tilde{G} := G \times \mathbb{G}_m \times \mathbb{G}_m.$$

As before, since $w$ corresponds to an element of $\Gamma(S, \text{Sym}^1(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}^*))$ we see that it is invariant under the first two factors of $\tilde{G}$ and has weight one with respect to the third factor.

We are now ready to prove Theorems 3.1.2 and 3.1.3.

Proof of Theorem 3.1.2. We will prove the statement for any $V$ a subbundle of $\mathcal{E}^*$ and then, setting $V = \mathcal{E}^*$, we will obtain the desired result.

Consider the gauged Landau-Ginzburg model, $(U^3_V, \tilde{G}, \mathcal{O}(\beta), w)$ as above. By Theorem 2.1.5 there is an equivalence

$$D^b(\text{coh } \mathcal{X}_V) \cong D(\text{coh}[\langle U^1_\mathcal{E} \rangle_+ / \tilde{G}], w).$$

Consider the elementary wall-crossing $((\mathcal{R}^+)_1, (\mathcal{R}^-)_1)$ from Lemma 3.1.7. Since the new $\mathbb{G}_m$-action commutes with the $G \times \mathbb{G}_m$-action, it is also an elementary wall-crossing for the action of $\tilde{G}$.

Notice that since, by assumption, the $G$-action has weight $d > 0$ on the fibers of $V_Q(\mathcal{M})$, we can choose the following connected component of the fixed locus of $\lambda_1$:

$$Z^{0}_{\lambda_1} := (0_{V_\lambda Q(\mathcal{M}|_{\lambda_1}^2)} \times_S V_S(V)) \cap U^1_V$$
where $Z_\lambda^0$ is the connected component of the fixed locus chosen for $(\mathfrak{R}^+, \mathfrak{R}^-)$. Finally, inside $\tilde{G}$ we have,

$$C(\lambda_1) = C(\lambda) \times \mathbb{G}_m \times \mathbb{G}_m$$

and

$$[Z_{\lambda_1}^0/C(\lambda_1)] \cong [Z_{\lambda}^0/C(\lambda) \times \mathbb{G}_m] \times_S \mathbb{P}_S(\mathcal{V})$$

where the $\mathbb{G}_m$-action is trivial. Furthermore, for this choice,

$$S_{\lambda_1}^0 = V((\mathcal{M} \oplus \mathcal{V})|_{s\lambda}^0) \cap U^1_\mathcal{V}$$

which admits a $G$-invariant affine cover as we have assumed the existence of such for $S_{\lambda}^0$.

Therefore, we may apply Theorem 2.3.3 to obtain a weak semi-orthogonal decomposition

$$D^b(\text{coh} \mathcal{X} \times_{\mathbb{P}S(\mathcal{E})} \mathbb{P}_S(\mathcal{W})) = \langle Z_0^+ \boxtimes D^b(\text{coh} \mathbb{P}_S(\mathcal{V})), \ldots, Z_{\mu-1}^+ \boxtimes D^b(\text{coh} \mathbb{P}_S(\mathcal{V})), D(\text{coh}[U^1_{\mathcal{V}}]/\mathcal{G}), w) \rangle$$

As $\mathcal{X}$ is smooth and proper, by [BV03 Corollary 3.1.5] and [BK90 Proposition 2.8], we have a semi-orthogonal decomposition and can mutate to get a new semi-orthogonal decomposition

$$D^b(\text{coh} \mathcal{X}_\mathcal{Y}) = \langle D(\text{coh}[U^1_{\mathcal{V}}]/\mathcal{G}), w), Z_0^+ \boxtimes D^b(\text{coh} \mathbb{P}_S(\mathcal{V})), \ldots, Z_{\mu-1}^+ \boxtimes D^b(\text{coh} \mathbb{P}_S(\mathcal{V})) \rangle.$$ 

This identifies $\mathcal{D}_\mathcal{Y}$ with $D(\text{coh}[U^1_{\mathcal{V}}]/\mathcal{G}), w)$. Taking $\mathcal{V} = \mathcal{E}^*$, we get the desired semi-orthogonal decomposition.

The $\mathbb{P}_S(\mathcal{E}^*)$-linearity of the fully faithful functor $D(\text{coh}[U^1_{\mathcal{V}}]/\mathcal{G}), w) \to D^b(\text{coh} \mathcal{X})$ follows from the linearity of all the functors involved. 

Proof of Theorem 3.1.3: We will first use the wall crossing $((\mathfrak{R}^+)_2, (\mathfrak{R}^-)_2)$ as in Lemma 3.1.7. Consider $U^3_{\mathcal{V}}$ together with the (restriction of the) function $w$ and the additional $\mathbb{G}_m$-action scaling the fibers of $V_Q(\mathcal{M})$ (as before). All together we have a $\tilde{G}$-action on $U^3_{\mathcal{V}}$. As the additional $\mathbb{G}_m$-action commutes, the elementary wall crossing $((\mathfrak{R}^+)_2, (\mathfrak{R}^-)_2)$ persists. The fixed locus of $\lambda_2$ is

$$Z_{\lambda_2}^0 = 0_{V_{\mathcal{X}}^0(\mathcal{M}|_{\mathcal{X}_0})} \times_S 0_{V_S(\mathcal{V})} \cong Q_-$$

Furthermore $C(\lambda_3) \cong G \times \mathbb{G}_m$ acts with the usual $G$-action on $Q_-$ and with a trivial $\mathbb{G}_m$-action. Also notice that we have a character

$$\sigma : \tilde{G} \to \mathbb{G}_m$$

given by projection onto the second factor and $\sigma \circ \lambda_2 = \text{Id}$. We now apply Theorem 2.3.4, Lemma 2.3.5, and Corollary 2.1.6 see also Remark 2.1.7. For notational purposes, we denote the essential image(s) of the embedding of the category $D^b(\text{coh}[Q_-/\mathcal{G}])(\mathcal{O}(\sigma^j))$ by $\mathcal{Y}_j$. We obtain a semi-orthogonal decomposition

$$D(\text{coh}[U^3_{\mathcal{V}}]/\mathcal{G}), w) = \langle D(\text{coh}[U^3_{\mathcal{V}}]/\mathcal{G}), w), \mathcal{Y}_0, \ldots, \mathcal{Y}_{r-2} \rangle$$

(3.3)

For the elementary wall crossing $((\mathfrak{R}^+)_3, (\mathfrak{R}^-)_3)$, consider $U^3_{\mathcal{V}}$ together with the function $w$ and again with an additional $\mathbb{G}_m$-action scaling the fibers of $V_Q(\mathcal{M})$. All together we have a $\tilde{G}$-action on $U^3_{\mathcal{V}}$. As the additional $\mathbb{G}_m$-action commutes, the elementary wall crossing of Lemma 3.1.7 remains valid. The fixed locus of $\lambda_3$ is

$$Z_{\lambda_3}^0 = V_{Z_3}(\mathcal{M}|_{Z_3}) \times_S 0_{V_S(\mathcal{V})}.$$
and since \( C(\lambda_3) = C(\lambda) \times \mathbb{G}_m \times \mathbb{G}_m \) we may cancel the fibers of this line bundle with the first \( \mathbb{G}_m \)-action to obtain an isomorphism
\[
[Z^0_{\lambda_3}/C(\lambda_3)] \cong [Z^0_{\lambda}/C(\lambda) \times \mathbb{G}_m]
\]

We apply Theorem \[2.3.4\] and Corollary \[2.1.6\] (see also Remark \[2.1.7\]) to two cases. When \( \mu > \dr \) we obtain a semi-orthogonal decomposition
\[
D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w) = \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), Z^0_\mu, \ldots, Z^\mu_{-\dr-1} \rangle. \tag{3.4}
\]
When \( \mu \leq \dr \) we obtain a semi-orthogonal decomposition
\[
D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w) = \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), Z^-_\mu, \ldots, Z^-_{dr-\mu-1} \rangle, \tag{3.5}
\]
where, as before, we denote by \( Z^-_k \) the essential images of the fully faithful functors \( \Upsilon^-_k \).

Therefore, in the case \( \mu \leq \dr \) we have
\[
D(\text{coh}([U^1_{\mathcal{V}}]/\tilde{G}], w) = D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w))
\]
\[
= \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), \mathcal{Y}_0, \ldots, \mathcal{Y}_{r-2} \rangle
\]
\[
= \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), \mathcal{Y}_0, \ldots, \mathcal{Y}_{r-2} \rangle
\]
\[
= \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), \mathcal{Y}_0, \ldots, \mathcal{Y}_{r-2}, Z^-_\mu, \ldots, Z^-_{dr-\mu-1} \rangle
\]
\[
= \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), \mathcal{Y}_0, \ldots, Z^-_{dr-\mu-1}, \mathcal{Y}_{r-2}, Z^-_{dr-\mu-1}, \ldots, Z^-_{dr-\mu-1} \rangle
\]
\[
= \langle D^b(\text{coh} X \times_{FS(E)} P_S(W)), \mathcal{Y}_0, \ldots, Z^-_{dr-\mu-1}, \mathcal{Y}_{r-2}, Z^-_{dr-\mu-1}, \ldots, Z^-_{dr-\mu-1} \rangle \tag{3.6}
\]
where the first line is \[3.2\], the second is \[3.3\], the third is \[3.1\], the fourth is \[3.5\], the fifth comes from mutating and pairing (starting with the last) \( \mathcal{Y}_r \) with the last \( d-1 \) of the \( Z^-_k \)'s until we exhaust the \( Z^-_k \)'s, the sixth is Theorem \[2.1.5\] using the fact that \( X \times_{FS(E)} P_S(W) \) is a complete linear section, and the last comes from equivalences between the essential images \( Z^+_j \) of \( \Upsilon^+_j \) in \( D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w)) \) and the essential images \( Z^-_j \) of \( \Upsilon^-_j \) in \( D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w)) \).

In the case where, \( \dr < \mu \), as before, we still have a semi-orthogonal decomposition
\[
D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w)) = \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), \mathcal{Y}_0, \ldots, \mathcal{Y}_{r-2} \rangle \tag{3.7}
\]
\[
= \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), \mathcal{Y}_0, \ldots, \mathcal{Y}_{r-2} \rangle
\]
and a decomposition
\[
D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w)) = \langle D(\text{coh}([U^3_{\mathcal{V}}]/\tilde{G}], w), Z^+_0, \ldots, Z^+_{dr-\mu-1} \rangle. \tag{3.8}
\]
from \[3.4\].

We can now proceed to the proof of the statements in the theorem. Setting \( \mathcal{V} = \mathcal{E}^* \) and noticing that in this case \( X \times_{FS(E)} P_S(W) = \emptyset \), we obtain the dual Lefschetz decomposition
\[
D^b(\text{coh}([U^1_{\mathcal{V}}]/\tilde{G}], w)) = \langle \mathcal{B}_{N-1}(-N+1), \ldots, \mathcal{B}_1(-1), \mathcal{B}_0 \rangle
\]
where
\[ B_i = \begin{cases} 
\langle \text{D}^b(\text{coh}[Q-/G]), Z_0^+, \ldots, Z_{d-1}^+ \rangle & 0 \leq i \leq N - \lceil \frac{\mu}{d} \rceil - 1 \\
\langle \text{D}^b(\text{coh}[Q-/G]), Z_{\mu+1-d(\lceil \frac{\mu}{d} \rceil - 1)}^+, \ldots, Z_{d-1}^+ \rangle & i = N - \lceil \frac{\mu}{d} \rceil \\
\langle \text{D}^b(\text{coh}[Q-/G]) \rangle & N - \lceil \frac{\mu}{d} \rceil < i < N 
\end{cases} \]

Equation 3.6, with \( C_V = \text{D}^b(\text{coh}\langle U^3 / \tilde{G}, w \rangle) \), amounts to the statement of the theorem in the case where \( \mu \leq dr \) and equations 3.7 and 3.8, when setting \( C_V = \text{D}(\text{coh}[U^3 / \tilde{G}], w) \), amount to the statement of the theorem in the case where \( dr < \mu \).

\[ \square \]

**Remark 3.1.9.** From the proof of the theorem we see that when \( dr < \mu \), the category \( C_V \) has an interpretation as \( \text{D}(\text{coh}[U^3 / \tilde{G}], w) \). On the other hand when \( \mu \leq dr \) we have \( C_V \cong \text{D}(\text{coh} X \times \mathbb{P}(\mathcal{E}), \mathcal{O}(\mathcal{E} \times \mathbb{P}(\mathcal{E}^*) \langle 1 \rangle, w) \).

**Remark 3.1.10.** Notice that in the special case that \( Q_- = \emptyset \), \( B_i = 0 \) for \( i > N - \lceil \frac{\mu}{d} \rceil \). In other words, the Lefschetz collection has fewer terms and a simpler form. Furthermore, the homological projective dual simplifies to

\[ (Q \times \mathbb{P}(\mathcal{E}^*), G, M \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E}^*) \langle 1 \rangle}, w) \]

with \( G \) acting as usual on \( Q \) and trivially on the factor \( \mathbb{P}(\mathcal{E}^*) \). In addition, \( (U^3)^- = (U^3)^- \) and the linear sections of the homological projective dual can be directly compared to \( X \times \mathbb{P}(\mathcal{E}) \mathbb{P}(\mathcal{W}) \). As a GIT fan, this is displayed in Figure 4. The examples below will have \( Q_- = \emptyset \).

![Figure 4. GIT fan relating categories appearing in HPD in the case \( Q_- = \emptyset \)](image-url)

**Remark 3.1.11.** Dropping the assumption that \( X \times_{\mathbb{P}(\mathcal{E})} \mathbb{P}(\mathcal{W}) \) is a complete linear section the theorem above continues to hold if we replace \( X \times_{\mathbb{P}(\mathcal{E})} \mathbb{P}(\mathcal{W}) \) either by the gauged Landau-Ginzburg model \( (U^3, \tilde{G}, \mathcal{O}(\beta), w) \) or equivalently by the derived fiber product \( X \times_{\mathbb{P}(\mathcal{E})} \mathbb{P}(\mathcal{W}) \) (see Remark 4.7 of [Isi12]).
3.2. A first example: Projective Bundles. In this section we provide an elementary and explicit example of Homological Projective Duality using the results of the previous section. The results presented here were first proved in [Kuz07].

Let \( P \) be a locally-free coherent sheaf on \( B \) with, \[ V := H^0(P)^* \neq 0. \]

For the projective bundle, \( \pi : \mathbb{P}_B(P) \to B \), the relative invertible sheaf, \( \mathcal{O}_P(P)(1) \), provides a map, \( j : \mathbb{P}_B(P) \to \mathbb{P}(V) \).

With the notation as in the previous section, we set
\[ Q = V_B(P), \]
\[ G = \mathbb{G}_m, \]
\[ \lambda(\alpha) = \alpha^{-1}, \]
\[ M = \mathcal{O}(\chi), \]
\[ S = \text{Spec } k, \]
where \( \mathbb{G}_m \) acts by fiber-wise dilation and \( \chi(\alpha) = \alpha \). It follows that
\[ [Q_+/G] = \mathbb{P}_B(P), \]
\[ [Q_-/G] = \emptyset, \]
\[ \mu = \text{rk } P, \]
\[ d = 1, \]
\[ \mathcal{A}_s = \pi^* D^b(\text{coh } B) \text{ for } 0 \leq s < \mu \]

By Remark 3.1.10 the weak homological projective dual reduces to
\[ (V_B(P) \times_k \mathbb{P}(V^*), \mathbb{G}_m, \mathcal{O}(\chi) \boxtimes \mathcal{O}(1), w) \]
with \( \mathbb{G}_m \) acting fiber-wise with weight 1. This is isomorphic to
\[ (V_B \times_k \mathbb{P}(V^*) \mathcal{P} \boxtimes \mathcal{O}(1)), \mathbb{G}_m, \mathcal{O}(\chi), w) \]
where \( \mathbb{G}_m \) acts fiber-wise with weight 1. Therefore, in this case, we can do more. Namely, we may apply Theorem 2.1.5 to see that
\[ D(\text{coh}[V_B \times_k \mathbb{P}(V^*) \mathcal{P} \boxtimes \mathcal{O}(1)] / \mathbb{G}_m, w) \cong D^b(\text{coh } Z(w)) \]
where \( Z(w) \) is the zero locus of \( w \) in \( B \times_k \mathbb{P}(V^*) \). Furthermore, by definition, \( Z(w) \) can be described as the set
\[ Z(w) = \{(b, s) | s(b) = 0 \} \subseteq B \times_k \mathbb{P}(V^*). \]

Remark 3.2.1. This is precisely the homological projective dual obtained by Kuznetsov in [Kuz07]. Also notice, as observed in [Kuz07] Lemma 8.1], that
\[ Z(w) \cong \mathbb{P}_B(P^\perp) \]
where \( P^\perp \) is the locally-free coherent sheaf defined as the kernel of the evaluation map
\[ V^* \otimes \mathcal{O}_B \to \mathcal{P}. \]
Remark 3.2.2. If we project down to \( \mathbb{P}(V^*) \) then the fiber over \( s \in V^* = H^0(B, \mathcal{P}) \) is precisely the vanishing of \( s \). In particular, the image is the set of degenerate sections of \( \mathcal{P} \). When \( \text{rk} \, \mathcal{P} = \dim B + 1 \), this is precisely the projective dual of \( \mathbb{P}_B(\mathcal{P}) \) (see Theorem 3.11 in [GKZ94]). However, unlike the usual projective dual, the homological projective dual is smooth.

4. Derived categories of degree d hypersurface fibrations

4.1. An aside: Relative version of Orlov’s theorem. We start by stating a well-known theorem of Orlov.

Theorem 4.1.1. Let \( X \) be a hypersurface of degree \( d \) given as the zero-scheme of \( w \in \Gamma(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) \). Let \( N = \dim V \).

a) If \( d < N \), then there is a semi-orthogonal decomposition
\[
\text{D}^b(\text{coh } X) = \langle \mathcal{O}_Y(d - N + 1), \ldots, \mathcal{O}_Y, D(\text{coh } \mathbb{A}^N / \mathbb{G}_m, w) \rangle.
\]
b) If \( d = N \), there is an equivalence
\[
\text{D}^b(\text{coh } X) \cong D(\text{coh } \mathbb{A}^N / \mathbb{G}_m, w)
\]
c) If \( d > N \), there is a semi-orthogonal decomposition
\[
D(\text{coh } \mathbb{A}^N / \mathbb{G}_m, w) = \langle k(N - d + 1), \ldots, k, \text{D}^b(\text{coh } X) \rangle.
\]

Proof. This is [Orl09, Theorem 2.13].

We will generalize this statement and the complete intersection version of it (also in [Orl09]) to families of complete intersections over a base. To this end, let \( S \) be a smooth, connected variety, \( \mathcal{E} \) be a locally-free coherent sheaf of rank \( n \) on \( S \) and let \( \mathcal{L}_i \) be invertible sheaves on \( S \), for \( 1 \leq i \leq c \). Let \( \mathcal{U} = \bigoplus_i \mathcal{L}_i \). Let
\[
Q := V_S(\mathcal{E} \oplus \mathcal{U}).
\]

Let \( q : \mathbb{P}(\mathcal{E}) \to S \) be the projection. Choose sections \( s_i \in \Gamma(S, S^d_i \mathcal{E} \otimes \mathcal{L}_i) \), let \( \tilde{s}_i \) be the corresponding sections in \( \Gamma(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d_i) \otimes q^* \mathcal{L}_i) \) and let \( X \) be the zero locus of \( (\tilde{s}_1, \ldots, \tilde{s}_c) \) in \( \mathbb{P}(\mathcal{E}) \). Let \( w_i \) be the associated regular functions on \( Q \). Let \( w = \sum_i w_i \). Consider the \( \mathbb{G}_m^2 \)-action on \( Q \) given by
\[
\sigma : \mathbb{G}_m^2 \times Q \to Q
\]
\[
(\alpha_1, \alpha_2, (e, \oplus_i p_i, s)) \mapsto (\alpha_1^{-1} e, \oplus_i \alpha_1^{d_i} \alpha_2^{-1} p_i, s).
\]
The function \( w \) becomes invariant with respect to the \( \mathbb{G}_m \)-action given by the one-parameter subgroup \( \lambda(\alpha) = (\alpha, 1) \). It is semi-invariant of weight \( 1 \) for the other \( \mathbb{G}_m \)-action.

The one-parameter subgroup \( \lambda \) induces an elementary wall crossing. To see this, first observe that the fixed locus, \( Z_\lambda^0 \), is the zero section, \( 0_Q \), of \( Q \). We have
\[
S_\lambda = Z_\lambda = 0_{V_S(\mathcal{E})} \times_S V_S(\mathcal{U})
\]
\[
S_{-\lambda} = Z_{-\lambda} = V_S(\mathcal{E}) \times_S 0_{V_S(\mathcal{U})}.
\]
Both are closed. The condition involving $P(\lambda)$ is trivial when the ambient group is Abelian.

We then have
\[
Q_+ = (V_S(\mathcal{E}) \setminus 0_{V_S(\mathcal{E})}) \times S V_S(\mathcal{U})
\]
\[
Q_- = V_S(\mathcal{E}) \times S (V_S(\mathcal{U}) \setminus 0_{V_S(\mathcal{U})}).
\]
And, $\mu = \text{rank } \mathcal{E} - \sum d_i$. Note that $C(\lambda) = \mathbb{G}_m^2$ so $C(\lambda)/\lambda \cong \mathbb{G}_m$. By Corollary 2.1.6 we have an equivalence
\[
D(\text{coh}[Z^0_\lambda/(C(\lambda)/\lambda)], w_\lambda) = D(\text{coh}[S/\mathbb{G}_m], 0) \cong D^b(\text{coh } S).
\]
In this example, recall from [BFK12, Section 3.4] that $\Upsilon^\pm$ is the functor
\[
i^*_\pm \circ (j^*\pm) \circ \pi^*_\pm : D(\text{coh}[Z^0_\lambda/(C(\lambda)], w_\lambda)_l \to D(\text{coh}[Q^\pm/\mathbb{G}_m^2], w).
\]
where
\[
S \xleftarrow{\pi^\pm} V_S(\mathcal{E}) \xrightarrow{j^\pm} Q \xleftarrow{i^\pm} Q _-
\]
are projections and inclusions. Let $\pi : Q \to \mathbb{P}^n(\mathcal{E})$ be the projection. The pullback, $\pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(1)$ has weight 1 with respect to $\lambda$. Therefore, we have equivalences between the essential images
\[
\text{EssIm } \Upsilon^+_0 \otimes \pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(l) \cong \text{EssIm } \Upsilon^+_l
\]
\[
\text{EssIm } \Upsilon^-_0 \otimes \pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(-l) \cong \text{EssIm } \Upsilon^-_l.
\]
Applying Theorem 2.3.4 we have the following statements.

- If $\mu = \text{rank } \mathcal{E} - \sum d_i > 0$, there is a semi-orthogonal decomposition
  \[
  D(\text{coh}[Q^+/\mathbb{G}_m^2], w) \cong (\Upsilon^+_0 : D(\text{coh}[S/\mathbb{G}_m], 0), \Upsilon^+_0 D(\text{coh}[S/\mathbb{G}_m], 0) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(1), \ldots,
  \Upsilon^+_0 D(\text{coh}[S/\mathbb{G}_m], 0) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(\mu - 1), \Phi^+ D(\text{coh}[Q^-/\mathbb{G}_m^2], w)).
  \]

- If $\mu = \text{rank } \mathcal{E} - \sum d_i = 0$, there is an equivalence
  \[
  D(\text{coh}[Q^+/\mathbb{G}_m^2], w) \cong D(\text{coh}[Q^-/\mathbb{G}_m^2], w).
  \]

- If $\mu = \text{rank } \mathcal{E} - \sum d_i < 0$, there is a semi-orthogonal decomposition
  \[
  (\Upsilon^-_0 D(\text{coh}[S/\mathbb{G}_m], 0), \Upsilon^-_0 D(\text{coh}[S/\mathbb{G}_m], 0) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(-1), \ldots,
  \Upsilon^-_0 D(\text{coh}[S/\mathbb{G}_m], 0) \otimes \pi^*\mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(\mu + 1), \Phi^- D(\text{coh}[Q^+/\mathbb{G}_m^2], w)) \cong D(\text{coh}[Q^-/\mathbb{G}_m^2], w).
  \]

There is an isomorphism
\[
[Q^+/\lambda] \cong V_{\mathbb{P}^n(\mathcal{E})} \left( \bigoplus_i \mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(d_i) \otimes \mathcal{O}_{\mathbb{P}^n(\mathcal{E})} q^*\mathcal{L}_i \right)
\]
under which $w$ corresponds to the regular function determined by $\bigoplus i \tilde{s}_i$. Assuming further that $\bigoplus i \tilde{s}_i$ is a regular section (which implies that $X$ is of pure codimension $c$) and applying Theorem 2.1.5 we have an equivalence
\[
\Psi : D(\text{coh}[Q^+/\mathbb{G}_m^2], w) \cong D(\text{coh}[V_{\mathbb{P}^n(\mathcal{E})} \left( \bigoplus_i \mathcal{O}_{\mathbb{P}^n(\mathcal{E})}(d_i) \otimes \mathcal{O}_{\mathbb{P}^n(\mathcal{E})} q^*\mathcal{L}_i \right)/\mathbb{G}_m], w) \cong D^b(\text{coh } X).
\]
Under this equivalence, we have an isomorphism of functors
\[ \Psi \circ (\Upsilon^+_0 \otimes \pi^* \mathcal{O}_P^E(l)) \cong q^* \otimes \mathcal{O}_X(l) : \mathcal{D}^b(\text{coh} S) \to \mathcal{D}^b(\text{coh} X) \]
where \( q : X \subset \mathbb{P}(E) \to S \) also denotes the projection from \( X \). Thus, we get the following

**Proposition 4.1.2.** With the assumptions above, we have the following

- If \( \mu = \text{rank} \mathcal{E} - \sum d_i > 0 \), there is a semi-orthogonal decomposition
  \[ \mathcal{D}^b(\text{coh} X) \cong \langle q^* \mathcal{D}^b(\text{coh} S), \ldots, q^* \mathcal{D}^b(\text{coh} S) \otimes \mathcal{O}_X(\mu - 1), \Phi^+ \mathcal{D}(\text{coh}[Q_+ / G^2_m], w) \rangle. \]
- If \( \mu = \text{rank} \mathcal{E} - \sum d_i = 0 \), there is an equivalence
  \[ \mathcal{D}^b(\text{coh} X) \cong \mathcal{D}(\text{coh}[Q_+ / G^2_m], w). \]
- If \( \mu = \text{rank} \mathcal{E} - \sum d_i < 0 \), there is a semi-orthogonal decomposition
  \[ \langle \Upsilon^-_0 \mathcal{D}(\text{coh}[S/G^m_m], 0), \Upsilon^-_0 \mathcal{D}(\text{coh}[S/G^m_m], 0) \otimes \pi^* \mathcal{O}_P^E(-1), \ldots, \Upsilon^-_0 \mathcal{D}(\text{coh}[S/G^m_m], 0) \otimes \pi^* \mathcal{O}_P^E(\mu + 1), \Phi^- \mathcal{D}^b(\text{coh} X) \rangle \cong \mathcal{D}(\text{coh}[Q_+ / G^2_m], w). \]

**Proof.** This follows from Theorem 2.3.4 and Theorem 2.1.3 as discussed above. \( \square \)

In the case when \( X \) is a family of degree \( d \) hypersurfaces, \( \mathcal{U} \) is an invertible sheaf and we can write
\[ [Q_+ / G^2_m] = [V_S^d(E) \times_S (V_S(\mathcal{U}) \setminus 0) / G^2_m] \cong [V_S(E) / G_m]. \]
where the action of \( G_m \) is by fiber-wise dilation. In the coming sections, we will restrict our attention only to this case. We record this in the following corollary.

**Corollary 4.1.3.** Let \( s \in \Gamma(S, \text{Sym}^d \mathcal{E} \otimes \mathcal{L}) \) and let \( X \subset \mathbb{P}(E) \) be the associated degree \( d \) hypersurface fibration over \( S \), with structure map, \( q : \mathbb{P}(E) \to S \).

- If \( \mu = \text{rank} \mathcal{E} - d > 0 \), there is a semi-orthogonal decomposition
  \[ \mathcal{D}^b(\text{coh} X) = \langle q^* \mathcal{D}^b(\text{coh} S), \ldots, q^* \mathcal{D}^b(\text{coh} S) \otimes \mathcal{O}_X(\mu - 1), \mathcal{D}(\text{coh}[V_S(E)/G_m], w) \rangle. \]
- If \( \mu = \text{rank} \mathcal{E} - d = 0 \), there is an equivalence
  \[ \mathcal{D}^b(\text{coh} X) \cong \mathcal{D}(\text{coh}[V_S(E)/G_m], w). \]
- If \( \mu = \text{rank} \mathcal{E} - d < 0 \), there is a semi-orthogonal decomposition
  \[ \langle \Upsilon^-_0 \mathcal{D}(\text{coh}[S/G^m_m], 0), \Upsilon^-_0 \mathcal{D}(\text{coh}[S/G^m_m], 0) \otimes \pi^* \mathcal{O}_P^E(-1), \ldots, \Upsilon^-_0 \mathcal{D}(\text{coh}[S/G^m_m], 0) \otimes \pi^* \mathcal{O}_P^E(\mu + 1), \mathcal{D}^b(\text{coh} X) \rangle \cong \mathcal{D}(\text{coh}[V_S(E)/G_m], w). \]

**Proof.** This is a special case of Proposition 4.1.2. \( \square \)
4.2. A local generator and Morita theory. In this section, we continue within the setting presented in Section 4.1 with \( c = 1 \) and show that the gauged LG model \( ([V_S(\mathcal{E})/\mathbb{G}_m], w) \) is derived-equivalent to the pair \((S, \mathcal{B}_w)\) where \( \mathcal{B}_w \) is an equivariant sheaf of dg-algebras. However, it will be more convenient to work with the isomorphic gauged LG model \( ([Q_-/\mathbb{G}_m^2], w) = \left( (V(\mathcal{U}) \setminus 0_{V_S(\mathcal{U})}) \times_S V(\mathcal{E})/\mathbb{G}_m^2, w \right) \). The sheaf \( \mathcal{B}_w \) will be the derived equivariant endomorphism sheaf of algebras of a ‘local generator’ \( \mathcal{G} \) of the derived category of this latter LG model.

Throughout this section, we will make the further assumption that the subvariety defined by \( w = 0 \) in \( \mathbb{P}_S(\mathcal{E}) \) is smooth.

Let us recall our setup. We work over a base \( S \) which is a smooth connected variety and \( \mathcal{E} \) is a locally-free sheaf of \( S \). Meanwhile, we have specialized to the case where \( \mathcal{U} \) is an invertible sheaf on \( S \). We have a \( \mathbb{G}_m^2 \)-action on

\[
Q = V_S(\mathcal{E}) \times_S V_S(\mathcal{U}),
\]

given by

\[
\sigma : \mathbb{G}_m^2 \times Q \to Q, \quad (\alpha_1, \alpha_2, (e, p, s)) \mapsto (\alpha_1^{-1} e, \alpha_1^d \alpha_2^{-1} p, s).
\]

We also have a projection \( q : \mathbb{P}(\mathcal{E}) \to S \) and have denoted by \( X \), which is assumed to be smooth, the zero locus in \( \mathbb{P}(\mathcal{E}) \) of a section of \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d_i) \otimes_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}} q^* \mathcal{U} \) corresponding to the regular function \( w \) on \( Q \).

Recall that

\[
Q_- = V_S(\mathcal{E}) \times_S (V_S(\mathcal{U}) \setminus 0_{V_S(\mathcal{U})}).
\]

We will replace the category \( \text{D}(\text{coh}[Q_-/\mathbb{G}_m^2], w) \) by the derived category of a \( \mathbb{G}_m \)-equivariant sheaf of dg-algebras over \( S \). Let \( \pi : Q \to S \) denote the projection. We shall also denote the projection, \( \pi : Q_- \to S \). Recall that

\[
\pi_* \mathcal{O}_Q \cong \text{Sym}(\mathcal{E}) \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U})
\]

and define

\[
\mathcal{R} := \pi_* \mathcal{O}_{Q_-} \cong \text{Sym}(\mathcal{E}) \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}).
\]

Since \( \pi \) is affine, we have an equivalence of \( \text{D}(\text{coh}[Q_-/\mathbb{G}_m^2], w) \) with \( \text{D}(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \), the category of \( \mathbb{Z}^2 \)-graded coherent factorizations over \( \mathcal{R} \). From now on we will be working in this latter category.

Let \( Y = 0_{V_S(\mathcal{E})} \times_S (V_S(\mathcal{U}) \setminus 0_{V_S(\mathcal{U})}) \subset Q_- \). We have the object

\[
0 \longleftarrow 0 \longrightarrow \mathcal{O}_Y
\]

of \( \text{D}(\text{coh}[Q_-/\mathbb{G}_m^2], w) \) which corresponds to the object

\[
\mathcal{G} := (0, \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}), 0, 0)
\]

in \( \text{D}(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \).
Proposition 4.2.1. The objects \( \{ G \otimes_{O_S} L \} \), with \( L \) invertible \( \mathbb{G}_m \)-equivariant sheaves on \( S \), generate \( D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \).

Proof. Let \( C \) be the zero locus of \( w \) inside the relative spectrum \( \text{Spec} \mathcal{R} \). Then, there is an essentially surjective functor,

\[
\Upsilon : D^G_{\text{sg}}(C) \to D(\text{coh}[Q_-/\mathbb{G}_m], w) \cong D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w),
\]

by [BFK11, Proposition 3.64]. The category, \( D^G_{\text{sg}}(C) \), is generated by objects pushed forward from the singular locus of \( C \) by [BFK11, Corollary 4.14]. Since we have assumed that \( X \) is smooth, the singular locus of \( C \) is exactly \( Y = \text{Spec} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \). The \( \mathbb{G}_m \)-equivariant derived category of \( Y \) is equivalent to the \( \mathbb{G}_m \)-equivariant derived category of \( S \) under pullback via the projection

\[
\rho : Y \to S.
\]

The composition of pullback to \( D^b(\text{coh} Y) \), push-forward to \( D^G_{\text{sg}}(C) \) and \( \Upsilon \), is essentially surjective and maps an invertible equivariant sheaf \( L \) on \( S \) to \( G \otimes_{O_S} L \). Therefore it suffices to show that the derived category of \( [\mathbb{S}/\mathbb{G}_m] \) is generated by invertible sheaves.

Define

\[
B_w := \bigoplus_{i \in \mathbb{Z}} \mathcal{R}\mathcal{H}om_{\mathcal{R}, w, \mathbb{Z}^2}(G(i, 0), G),
\]

and the functor,

\[
F : D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \to D(\text{Mod}_{\mathbb{Z}} B_w)
\]

\[
\mathcal{E} \mapsto \bigoplus_{i \in \mathbb{Z}} \mathcal{R}\mathcal{H}om_{\mathcal{R}, w, \mathbb{Z}^2}(G(i, 0), \mathcal{E}).
\]

Proposition 4.2.2. The functor, \( F \), is fully-faithful onto its image.

Proof. We first check that \( F \) induces natural quasi-isomorphisms of chain complexes

\[
\mathcal{R}\mathcal{H}om_{\mathcal{R}, w, \mathbb{Z}^2}(G, \mathcal{E}) \to \mathcal{R}\mathcal{H}om_{B_w, \mathbb{Z}}(F(G), F(\mathcal{E}))
\]

for any object \( \mathcal{E} \in D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \). Since \( B_w := F(G) \), we have

\[
\mathcal{R}\mathcal{H}om_{B_w, \mathbb{Z}}(F(G), F(\mathcal{E})) = \mathcal{H}om_{B_w, \mathbb{Z}}(B_w, F(\mathcal{E}))
\]

\[
\cong (F(\mathcal{E}))_0 := \left( \bigoplus_{i \in \mathbb{Z}} \mathcal{R}\mathcal{H}om_{\mathcal{R}, w, \mathbb{Z}^2}(G(i, 0), \mathcal{E}) \right)_0 = \mathcal{R}\mathcal{H}om_{\mathcal{R}, w, \mathbb{Z}^2}(G, \mathcal{E}).
\]
Therefore,
\[ \mathcal{R} \text{Hom}_{\mathcal{R}, w, \mathbb{Z}^2}(\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{L}, \mathcal{E}) \cong \mathcal{R} \text{Hom}_{\mathcal{R}, w, \mathbb{Z}^2}(\mathcal{G}, \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{-1}) \]
\[ \cong \mathcal{R} \text{Hom}_{\mathcal{B}_w, \mathbb{Z}}(F(\mathcal{G}), F(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{L}^{-1})) \]
\[ \cong \mathcal{R} \text{Hom}_{\mathcal{B}_w, \mathbb{Z}}(F(\mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{L}), F(\mathcal{E})), \]
where the last step follows from the fact that \( F \) commutes with \( \bullet \otimes_{\mathcal{O}_S} \mathcal{L} \).

Applying \( \mathcal{R} \Gamma \) shows that \( F \) is fully-faithful on the smallest thick subcategory of \( D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \) containing all the objects \( \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{L} \) for any equivariant invertible sheaf \( \mathcal{L} \) on \( S \). By Proposition 4.2.1, this is all of \( D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \). \( \square \)

**Lemma 4.2.3.** The category, \( D(\text{Mod}_Z \mathcal{B}_w) \), is a compactly-generated triangulated category. Moreover, the set of objects \( \{ \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \} \) is a set of compact generators.

**Proof.** The category, \( D(\text{Mod}_Z \mathcal{B}_w) \), admits all small coproducts so we only need to find a set of compact objects whose right orthogonal is zero. It is clear that each \( \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \) is compact. The proof of the generation statement is identical to the last part of the proof of Proposition 4.2.1. For a \( \mathcal{B}_w \)-module \( \mathcal{M} \) such that \( \mathcal{H}^n(\mathcal{M}) \neq 0 \), there is an equivariant invertible sheaf \( \mathcal{L} \) such that there is a section \( \mu \in \Gamma(S, \ker d^i_{\mathcal{M}} \otimes_{\mathcal{O}_S} \mathcal{L}^{-1})_{\mathbb{G}_m} \). This gives a non-zero morphism \( \mathcal{B}_w \rightarrow \mathcal{M} \otimes_{\mathcal{O}_S} \mathcal{L}^{-1}[n] \) by sending 1 to \( \mu \) and hence a non-zero morphism \( \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \rightarrow \mathcal{M}[n] \). Thus, \( \{ \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \} \perp = 0 \). \( \square \)

**Definition 4.2.4.** A \( \mathcal{B}_w \)-module \( \mathcal{M} \) is **perfect** if it lies in the smallest thick subcategory classically generated by the \( \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \). The category \( D_{\text{pe}}(\text{Mod}_Z \mathcal{B}_w) \) is the full subcategory of perfect objects in \( D(\text{Mod}_Z \mathcal{B}_w) \).

**Lemma 4.2.5.** The compact objects of \( D(\text{Mod}_Z \mathcal{B}_w) \) and the perfect objects coincide.

**Proof.** Combining Lemma 4.2.3 \( \text{[Nee92, Lemma 1.7]} \) and \( \text{[Nee92, Theorem 2.1]} \) proves that the subcategory of all compact objects lies in the smallest thick subcategory containing all objects of the form \( \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \) for invertible equivariant sheaves \( \mathcal{L} \). The other inclusion is by definition. \( \square \)

**Proposition 4.2.6.** The essential image of \( F \) is the subcategory of perfect modules.

**Proof.** Since by Proposition 4.2.1 \( D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \) is generated by objects of the form \( \mathcal{G} \otimes_{\mathcal{O}_S} \mathcal{L} \) and \( F \) is fully-faithful by Proposition 4.2.2, the essential image of \( F \) is dense in the smallest thick subcategory containing the set of objects \( \{ \mathcal{B}_w \otimes_{\mathcal{O}_S} \mathcal{L} \} \) for \( \mathcal{L} \) invertible. By Lemma 4.2.5 this is the subcategory of perfect modules. Finally, by Corollary 4.1.3 and Lemma 2.2.6 \( D(\text{mod}_{\mathbb{Z}^2} \mathcal{R}, w) \) is idempotent complete thus the essential image is also thick. \( \square \)

We have thus proved the following:

**Proposition 4.2.7.** There is an equivalence
\[ D(\text{coh}[Q_-, \mathbb{G}_m^2], w) \cong D_{\text{pe}}(\text{Mod}_Z \mathcal{B}_w) \]
We now calculate the endomorphism sheaf of dg-algebras of \( \mathcal{G} \) and obtain a more explicit description of \( B_w \). To this end, we will replace \( G \) with a quasi-isomorphic locally-free factorization in \( D(\text{mod}_{\mathbb{Z}} R, w) \).

Define a factorization by

\[
\mathcal{F}^{-1} := \bigoplus_{r=2s+1} \Lambda^r(\mathcal{E}) \otimes_{\mathcal{O}_S} \mathcal{R}(r, 0)
\]

\[
\mathcal{F}^{0} := \bigoplus_{r=2s} \Lambda^r(\mathcal{E}) \otimes_{\mathcal{O}_S} \mathcal{R}(r, 0)
\]

with differential

\[
\delta_{\mathcal{F}} = d_{\text{Koszul}} + \gamma \wedge \bullet.
\]

The Koszul differential \( d_{\text{Koszul}} \) is given by the composition

\[
\Lambda^i \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1})(i, 0) \rightarrow \Lambda^i \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}^* \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1})(i, 0)
\]

\[
\rightarrow \Lambda^{i-1} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1})(i-1, 0)
\]

where the first map is induced by the map \( \mathcal{O}_S \rightarrow \mathcal{E}^* \otimes_{\mathcal{O}_S} \mathcal{E} \) corresponding to the identity element in \( \mathcal{E} \text{nd}_{\mathcal{O}_S}(\mathcal{E}) \), while the second one is induced by contraction \( \Lambda^i \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}^* \rightarrow \Lambda^{i-1} \mathcal{E} \) and multiplication in \( \text{Sym} \mathcal{E} \).

The relative 1-form \( \gamma \) is defined as

\[
\gamma := \frac{1}{d} dw \in \Lambda^1 \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}^{d-1} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}^1 \mathcal{U}
\]

where \( dw \) is the relative algebraic deRham differential of \( w \).

In local coordinates, where the \( x_i \) are a basis of \( \mathcal{E} \), \( \frac{\partial}{\partial x_i} \) is the corresponding basis of 1-forms in \( \Lambda(\mathcal{E}^*) \), and \( dx_i \) is the corresponding basis of 1-forms in \( \Lambda(\mathcal{E}) \), we have \( \delta_{\mathcal{F}} = i_\eta + \gamma \wedge \bullet \), where

\[
\eta = \sum x_i \frac{\partial}{\partial x_i} \text{ and } \gamma = \frac{u}{d} \sum \frac{\partial w}{\partial x_i} dx_i.
\]

**Lemma 4.2.8.** The factorization \( \mathcal{F} \) is quasi-isomorphic to the factorization \( \mathcal{G} = (0, \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}), 0, 0) \).

**Proof.** This is a direct consequence of [BDFIK12, Theorem 3.9]. \( \square \)

Since the equivalences above respect the grading shifts, we can now use the graded endomorphism algebra \( \bigoplus_{i \in \mathbb{Z}} \mathcal{H} \text{om}_{\mathcal{R}}(\mathcal{F}(i, 0), \mathcal{F}) \), which we still denote by \( B_w \), to calculate \( \bigoplus_{i \in \mathbb{Z}} R \mathcal{H} \text{om}_{\mathcal{R}}(\mathcal{F}(i, 0), \mathcal{G}) \).

Note that as an \( \mathcal{R} := \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \)-module, we can write \( B_w \) as

\[
B_w = \Lambda^* \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \otimes_{\mathcal{O}_S} \Lambda^* \mathcal{E}^*.
\]
Under this description, a basic local section \((\beta \otimes f \otimes \theta)\), denoted briefly as \((\beta, f, \theta)\), for \(\beta \in \Lambda^\bullet(\mathcal{E})\), \(f \in \text{Sym}(\mathcal{E}) \otimes_{\mathcal{O}_S} \text{Sym}(U, U^{-1})\) and \(\theta \in \Lambda^\bullet(\mathcal{E}^*)\), corresponds to the endomorphism of \(\mathcal{F}\) that acts on basic local sections of \(\mathcal{F}\) by

\[
(\beta, f, \theta) (\beta', f', \theta') = \langle \theta, \beta' \rangle \beta, ff', \theta',
\]

where the pairing \(\langle \theta, \beta' \rangle\) is the natural pairing between \(\Lambda^\bullet(\mathcal{E}^*)\) and \(\Lambda^\bullet(\mathcal{E})\) (in particular, the pairing is 0 unless \(\theta\) and \(\beta'\) live in the same wedge power, so this pairing is different from the contraction pairing).

The sections of the sheaf \(\mathcal{B}_w\) have a dg algebra structure with differential \(\delta\) induced by \(\delta_F\) and product structure, induced by composition of the endomorphisms, given by

\[
m(b, b') := bb' := (\langle \theta, \beta' \rangle \beta, ff', \theta'),\tag{4.1}
\]

for \(b = (\beta, f, \theta)\) and \(b' = (\beta', f', \theta')\) basic local sections as above.

Finally, note that there are two gradings on \(\mathcal{B}_w\), the internal \(\mathbb{Z}\)-grading and the cohomological grading. Sections of \(\Lambda^1\mathcal{E}\) have internal degree \(-1\) and cohomological degree \(-1\), sections of \(\Lambda^1\mathcal{E}^*\) have the opposite gradings; whereas sections of \(U\) have internal degree \(d\) and cohomological degree \(2\) and sections of \(\text{Sym} \mathcal{E}\) have internal degree \(-1\) and cohomological degree \(0\).

4.3. **Transferring to an \(A_\infty\)-structure**. We will be working with sheaves of \(A_\infty\)-algebras over \(S\) and modules over them. These will have an “internal” \(\mathbb{Z}\)-grading in addition to the usual cohomological grading on such objects. We require that the restriction maps for these sheaves are always strict morphisms of \(A_\infty\)-algebras and modules.

In this section we will prove the following theorem:

**Theorem 4.3.1.** There exists a sheaf of graded \(A_\infty\) \(\text{Sym}(U, U^{-1})\)-algebras \((\mathcal{A}, \mu^*)\) with

\[
\mathcal{A} = \text{Sym}(U, U^{-1}) \otimes_{\mathcal{O}_S} \Lambda^\bullet \mathcal{E}^*,
\]

such that there is an \(A_\infty\)-quasi-isomorphism

\[
\mathcal{A} \to \mathcal{B}_w.
\]

Moreover,

\(i\) If \(d = 2\), \(\mathcal{A}\) is a sheaf of Clifford algebras. The Clifford relations are given in local coordinates, with \(x_i\) a local basis of \(\mathcal{E}\) and \(v_i\) the dual basis, by

\[
\mu^2(1 \otimes v_i, 1 \otimes v_j) + \mu^2(1 \otimes v_j, 1 \otimes v_i) = \frac{\partial^2 w}{\partial x_i \partial x_j},
\]

\(ii\) If \(d > 2\), \(\mathcal{A}\) is a minimal \(A_\infty\) algebra with the following properties:

- The multiplication \(\mu^2\) is given by the usual wedge product on \(\mathcal{A}\) induced from \(\Lambda^\bullet(\mathcal{E}^*)\).
- \(\mu^k = 0\) for \(3 \leq k \leq d - 1\) and, in local coordinates, we have

\[
\mu^d(1 \otimes v_i, \ldots, 1 \otimes v_i) = \frac{1}{d!} \frac{\partial^d w}{\partial x_i \ldots \partial x_i}.
\]
where the \( v_{ij} \) are not necessarily distinct.

**Remark 4.3.2.** Local calculations have been provided in the above statement as they are more explicit and easier to state. We will also give formulas for the global \( \mu^k \) later in Lemma 4.3.12 and Proposition 4.3.15 below.

To prove Theorem 4.3.1, we first observe that \( A \) is the cohomology sheaf of algebras of a different \( \mathbb{Z} \)-graded dg-algebra \( B \), which is the same as \( B_w \) except that it has a modified differential and is easily seen to be formal. The strategy is to use the homological perturbation lemma to obtain an \( A_{\infty} \) structure on \( A = H^*(B) \) that makes it quasi-isomorphic to \( B_w \).

To this end, let us consider the pair \((F, d_{\text{Koszul}})\) instead of \((F, \delta_F)\) where we had \( \delta_F = d_{\text{Koszul}} + \gamma \wedge \bullet \). Let \( B \) be the endomorphism dg algebra of \((F, d_{\text{Koszul}})\). As we had for \( B_w \), we have

\[
B = \Lambda^* \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \otimes_{\mathcal{O}_S} \Lambda^* E^*,
\]

with the same product structure described in the previous section, but with a differential now induced by \( d_{\text{Koszul}} \), different from the differential of \( B_w \).

**Definition 4.3.3.** We denote the differential induced by \( d_{\text{Koszul}} \) by \( \partial : B \rightarrow B \) and differential of \( B_w \) by \( \overline{\partial} \) in keeping with standing notation for homological perturbation.

**Lemma 4.3.4.** The sheaf of dg-algebras, \((B, d_{\text{Koszul}})\), is formal with

\[
A := H^*(B) \cong \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \otimes_{\mathcal{O}_S} \Lambda^* \mathcal{E}^*.
\]

**Proof.** The factorization, \( F \), after forgetting the differential \( \gamma \wedge \bullet \) becomes a chain complex quasi-isomorphic to \( \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \). Since \( F \) is locally-free, we have a quasi-isomorphism

\[
B = \bigoplus_i \mathcal{H}om_{\mathcal{R},w \mathbb{Z}^2}(F(i,0), F) \cong \bigoplus_i \mathcal{H}om_{\mathcal{R},w \mathbb{Z}^2}(F(i,0), \text{Sym}(\mathcal{U}, \mathcal{U}^{-1})).
\]

The latter chain complex is formal with cohomology exactly as claimed. \( \square \)

Note that sections of \( \mathcal{U} \) have internal degree \( d \) and cohomological degree 2 whereas sections of \( \Lambda^1 \mathcal{E}^* \) have internal degree 1 and cohomological degree 1.

We now define the maps which will allow us to transfer the dg-structure on \( B \) to the \( A_{\infty} \)-structure on \( A \).

We define \( p : B \rightarrow A \) to be the projection by the ideal generated by \( \text{Sym} \mathcal{E} \) and \( \Lambda^* \mathcal{E} \). Note that this is not a map of sheaves of algebras, but a map of sheaves of chain complexes.

Next, we will define a map \( i : A \rightarrow B \). In order to do that, for each \( a \in \mathbb{Z}_{\geq 0} \), consider the composition \( \alpha_a \) of the maps

\[
\begin{align*}
\Lambda^a \mathcal{E} & \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \otimes_{\mathcal{O}_S} \Lambda^* \mathcal{E}^* \\
& \rightarrow \Lambda^a \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \otimes_{\mathcal{O}_S} (\Lambda^1 \mathcal{E} \otimes_{\mathcal{O}_S} \Lambda^1 \mathcal{E}^*) \otimes_{\mathcal{O}_S} \Lambda^* \mathcal{E}^* \\
& \rightarrow \Lambda^{a+1} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym} \mathcal{E} \otimes_{\mathcal{O}_S} \text{Sym}(\mathcal{U}, \mathcal{U}^{-1}) \otimes_{\mathcal{O}_S} \Lambda^* \mathcal{E}^*,
\end{align*}
\]
where the first map is induced by the map $\mathcal{O}_S \to \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{E}^*$ corresponding to the identity in $\mathcal{E}nd_{\mathcal{O}_S}(\mathcal{E})$, while the second one is induced by the wedge product. We define $i_0 : A \to B$ to be the obvious inclusion and $i_k : A \to B$ by

$$i_k = \frac{1}{k!} \alpha_{k-1} \circ \alpha_{k-2} \circ ... \circ \alpha_0 \circ i_0.$$ 

We can now define $i$ by

$$i = \sum_{k \geq 0} i_k.$$

Lastly, we want to define a homotopy between $ip$ and 1. We first define $h_0 : B \to B$ to take $(\beta, f, \theta)$ to $(df \wedge \beta, \theta)$ for basic local sections $(\beta, f, \theta)$ of $B$. Here, $d$ is the deRham differential. One can then define

$$h_k = \frac{1}{s(s+1) \ldots (s+k)} \alpha_{k-1} \circ \alpha_{k-2} \circ ... \circ \alpha_0 \circ h_0,$$

where $s = \text{deg } f + \text{deg } \beta$. The homotopy $h : B \to B$ is defined to be

$$h = \sum_{k \geq 0} h_k.$$

**Lemma 4.3.5.** The morphisms, $h, i, p$, satisfy:

- $i : A \to B$ is an algebra homomorphism.
- $pi = 1$
- $h^2 = ph = hi = 0$
- $ip - 1 = d_{Koszul} h + h d_{Koszul}$.

**Proof.** This is a straightforward, but tedious, computation. It is suppressed. \qed

**Proposition 4.3.6.** There exists an $A_\infty$ structure, $\mu$, on $A$ and a quasi-isomorphism $f : (A, \mu) \to B_w$.

**Proof.** Lemmas [4.3.4] and [4.3.5] guarantee that we can apply homological perturbation, as in [KS01], Section 2.4 in [Cra04], or [Mar04], which provides the desired $\mu$ and $f$. \qed

The general formulas for the higher products of Proposition [4.3.6] on $A$ can be described as sums over ribbon trees with one root and $d$ leaves such that the valency of any vertex is either 2 or 3.

**Definition 4.3.7.** [KS01, IK04] A ribbon tree is a tree $T$ with a collection of vertices, a collection of semi-infinite edges $\{e_0, \ldots, e_n\}$ and a collection of finite edges such that:

(a) Each semi-infinite edge is incident to a single vertex.
(b) Each finite edge is incident to exactly two vertices.
(c) The planar structure of $T$ has the semi-infinite edges $e_0, \ldots, e_n$ arranged in clockwise order. The edge $e_0$ is called the root and $e_1, \ldots, e_n$ are called the leaves.
Each such tree $T$ with $k$-leaves determines a term $\mu^k_T$ in the higher product $\mu^k$ on $\mathcal{A}$. Orienting $T$ from the leaves to the root, we can explicitly describe the composition of maps that define $\mu^k_T$ as follows:

- For all incoming edges we have the map $i : \mathcal{A} \to \mathcal{B}$
- For the outgoing edges we have the map $p : \mathcal{B} \to \mathcal{A}$
- For all finite edges we have the map $h : \mathcal{B} \to \mathcal{B}$
- For a bivalent vertex we have the map $(\tilde{\partial} - \partial) : \mathcal{B} \to \mathcal{B}$
- For a trivalent vertex we have the multiplication on $\mathcal{B}$

\[
\mu^2_T = p(m(i(\bullet), i(\bullet))) \\
\mu^2_T = p(m(i(\bullet), (h \circ (\tilde{\partial} - \partial) \circ i)(\bullet)))
\]

**Figure 5.** Examples of ribbon trees with one and two vertices.

**Figure 6.** Ribbon trees contributing to the differential on $\mathcal{A}$

The higher products are given by

\[
\mu^k = \sum \mu^k_T, \quad (4.2)
\]

where the sum is taken over all ribbon trees with $k$-leaves.
Corollary 4.4.4 above is the first part of Theorem 4.3.1. Using the explicit description of the $\mu$ above, we will now verify the properties of the $A_\infty$-structure on $\mathcal{A}$. In the process of doing so, we will make some calculations in local coordinates. More precisely, consider any affine open $U \subset S$ where $\mathcal{E}$ and $\mathcal{U}$ are trivial and take $\{x_i\}$ to be a basis of $\mathcal{E}|_U$ and $u$ in $\mathcal{U}$. We will denote the corresponding basis of $\Lambda^1 E$ by $\{dx_i\}$ and the dual basis of $\Lambda^1 E^*$ by $\{\frac{\partial}{\partial x_i}\}$ or by $\{v_i\}$.

In these local coordinates, the formulas above can be rewritten as follows:

- $i : \mathcal{A} \to \mathcal{B}$ is given locally by
  \[
i(r, \theta) = \sum_{k \geq 0} \sum_{j_1 < \ldots < j_k} (dx_{j_1} \wedge \ldots \wedge dx_{j_k}, r, \frac{\partial}{\partial x_{j_k}} \wedge \ldots \wedge \frac{\partial}{\partial x_{j_1}} \wedge \theta).
\] (4.3)

- $h : \mathcal{B} \to \mathcal{B}$ is given locally by
  \[
h(\beta, f, \theta) = \sum_{k \geq 0} s(s + 1) \ldots(s + k) \sum_{j_1 < \ldots < j_k} (df \wedge \beta \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_k}, \frac{\partial}{\partial x_{j_k}} \wedge \ldots \wedge \frac{\partial}{\partial x_{j_1}} \wedge \theta),
\]
  where $s = \deg f + \deg \beta$ (when $f \beta$ is a constant, $h$ takes the element $(\beta, f, \theta)$ to zero).

- $\overline{\partial} - \partial$ is given locally by:
  \[
  (\overline{\partial} - \partial)((\beta, f, \theta)) = (\gamma \wedge \beta, uf, \theta) + (-1)^{\deg \beta} \sum_k (\beta, \frac{\partial w}{\partial x_k}uf, i_{dx_k} \theta).
\] (4.5)

**Remark 4.3.8.** In the arguments that follow, we will make use of a $\mathbb{Z}^2$-grading on $\mathcal{B}$ different from the internal and cohomological gradings we have considered so far. This is only for the purposes of the arguments below and will help us in simplifying the computation of the $A_\infty$ products on $\mathcal{A}$. The two $\mathbb{Z}$-gradings consist of:

- The $f$-degree on $\mathcal{B}$ is a $\mathbb{Z}$-grading which comes from considering $\text{Sym}(\mathcal{E})$ with its natural grading and the other factors of the tensor product in degree zero.
- The $\beta$-degree on $\mathcal{B}$ is a $\mathbb{Z}$-grading which comes from considering $\Lambda(\mathcal{E})$ with its natural grading and the other factors of the tensor product in degree zero.

**Lemma 4.3.9.** The $f$-degree and $\beta$-degree have the following properties:

- The $f$-degree of $h$ is $-1$.
- The $f$-degree of $\overline{\partial} - \partial$ is $d - 1$.
- For any $b, b' \in \mathcal{B}$ homogeneous in $f$-degrees, the $f$-degree of $m(b, b')$ is the sum of the $f$-degrees of $b$ and $b'$.
- For any $b \in \mathcal{B}$ homogeneous of $\beta$-degree $s$, $h(b) \in \mathcal{B}_{>s,\beta}$, where $\mathcal{B}_{>s,\beta}$ denotes the set of all elements of $\beta$-degree strictly larger than $s$.
- For any $b \in \mathcal{B}$ homogeneous of $\beta$-degree $s$, $(\overline{\partial} - \partial)(b) \in \mathcal{B}_{\geq s,\beta}$.
- For any $b \in \mathcal{B}$ homogeneous of $\beta$-degree $s$ and any $b' \in \mathcal{B}$, $m(b, b')$ is homogeneous of $\beta$-degree $s$.

**Proof.** This follows immediately from the definitions. \(\square\)
Remark 4.3.10. The number of trees contributing to each $A_\infty$ product is finite. Indeed, any tree containing a long enough chain will not contribute to the summation because both $h$ and $\partial - \partial$ increase the $\beta$-degree and all elements of positive $\beta$-degree are in the kernel of $p$.

Lemma 4.3.11. The sheaf of graded $A_\infty$-algebras $\mathcal{A}$ is minimal, i.e., the differential $\mu^1$ on $\mathcal{A}$ is trivial.

Proof. Consider the trees in Figure 6. By the tree summing formula (4.2), the differential is given by

$$\mu^1(a) = p(\partial - \partial)i(a) + p(\partial - \partial)h(\partial - \partial)i(a) + p(\partial - \partial)h(\partial - \partial)h(\partial - \partial)i(a) + \ldots.$$ 

Since $\partial - \partial$ has $f$-degree $d - 1$, and $p$ kills everything of positive $f$-degree we have $p(\partial - \partial) = 0$ (the $f$-degree is an $\mathbb{N}$-grading).

Lemma 4.3.12. The multiplicative structure $\mu^2$ on $\mathcal{A}$ can be described as follows:

(i) If $d > 2$, $\mu^2$ is induced by the ribbon tree with two leaves and a single trivalent vertex. In particular, the multiplication is given by the usual wedge product on $\mathcal{A}$ induced from the wedge product on $\Lambda^* (E^*)$.

(ii) If $d = 2$, $\mu^2$ is induced by the ribbon tree described in (i) plus the ribbon tree with two leaves and one bivalent vertex connected to the second leaf. In particular, the multiplication satisfies the Clifford relations, given locally by

$$\mu^2(1 \otimes v_i, 1 \otimes v_j) + \mu^2(1 \otimes v_j, 1 \otimes v_i) = \frac{\partial^2 w}{\partial x_i \partial x_j},$$

and globally by

$$\mu^2(1 \otimes s_1, 1 \otimes s_2) + \mu^2(1 \otimes s_2, 1 \otimes s_1) = \frac{1}{2} \partial (d w \ldots s_1 \ldots s_2),$$

where $d$ is the deRham differential and $s_1, s_2$ are sections of $\Lambda^1 E^*$.

Proof. Let $a_i = (r_i, v_i)$ and $a_j = (r_j, v_j)$ be in $\mathcal{A}$ with $v_i, v_j \in \{ \partial \partial x_1, \ldots, \partial \partial x_n \}$.

(i) Any tree $T$ contributing to the summation formula for $\mu^2$ will have exactly one trivalent vertex (since it has two leaves). Moreover, if we let $m$ be the number of bivalent vertices, we see that both $h$ and $(\partial - \partial)$ appear $m$ times in $T$. However, by Lemma 4.3.9, $h$ has
Moreover, as before, the only part of we obtain is the one we had in (i). Since the \( \beta \)-degree terms, after multiplication, will be sent to 0 via \( p \). Thus, without loss of generality, we may assume \( i(a_i) = (1, r_i, v_i) \). By the definition of the product structure on \( B \) the only part of \( i(a_j) \) contributing to the product is the \( \beta \)-degree 1 component (all the others will vanish when multiplied with \( i(a_i) \)). More precisely, we can assume \( i(a_j) = (v_i^*, r_j, v_i \wedge v_j) \). Thus \( a_i a_j = (r_i r_j, v_i \wedge v_j) \).

(ii) Again, any tree \( T \) contributing to the summation formula will have exactly one trivalent vertex since it has two leaves. Moreover, if we let \( m \) be the number of bivalent vertices, then \( h \) and \( (\overline{\partial} - \partial) \) appear \( m \) times in \( T \).

Since \( p(\overline{\partial} - \partial) = 0 \), the last vertex (the one connected to the root) must have valency 3. We have

\[
p(m(h(-), -)) = 0,
\]

as, by Lemma 4.3.9, \( h(b) \in B_{>0, \beta} \) for any \( b \), and \( m(B_{>0, \beta}, B) \subseteq B_{>0, \beta} \). It follows that the last vertex must be connected to the first leaf. Thus,

\[
\mu^2_T(a_i, a_j) = p(m(i(a_i), P(a_j)),
\]

where \( P \) is the remaining part of the operator which is attached to the ribbon tree with the edges connecting the leaf and the root to the last vertex removed and \( h \) replacing \( p \). Now, as we saw in part (i), without any loss of generality, we can assume that \( i(a_i) = (1, r_i, v_i) \). Moreover, as before, the only part of \( i(a_j) \) contributing to \( m \) is the \( \beta \)-degree 1 component. Since the \( \beta \)-degree of \( h \) is \( \geq 1 \) it follows that \( m \) has to be 0 or 1. If \( m = 0 \) then the tree that we obtain is the one we had in (i). If \( m = 1 \) we obtain the tree with one trivalent vertex and one bivalent one connected to the second leaf. (c.f. Figure 7)

To compute the contribution of the latter tree we first note that only the \( \beta \)-degree 1 part of the output of \( P \) will contribute to the operator induced by \( T \). Now, the operator \( P \) is just \( h(\overline{\partial} - \partial)i \) so, since \( h \) already has \( \beta \)-degree \( > 0 \) we can assume, with loss of generality, that \( i(a_j) = (1, r_j, v_j) \) and that only the \( \beta \)-degree 0 part of \( (\overline{\partial} - \partial) \) will contribute to the operator. Last, but not least, we also see that only the \( \beta \)-degree 1 term in \( h \) will contribute to the sum.

We now compute \( P(a_j) \). We have

\[
P(a_j) = h((\overline{\partial} - \partial)(i(a_j))) = h(\sum_k (1, \frac{\partial w}{\partial x_k} r_j, i_{d_{x_k}} v_j))
= h(1, \frac{\partial w}{\partial x_j} r_j, 1) = \frac{1}{2} \sum_k (d_{x_k}, \frac{\partial^2 w}{\partial x_k \partial x_j} r_j, 1).
\]  \label{eq:4.6}
Therefore, the total contribution of $T$ is,

$$
\mu^2_T(a_i, a_j) = p(m((1, r_i, v_i), \frac{1}{2} \sum_k (dx_k, \frac{\partial^2 w}{\partial x_k \partial x_j} r_j, 1) = p(\frac{1}{2} r_i r_j \frac{\partial^2 w}{\partial x_i \partial x_j}) = \frac{1}{2} r_i r_j \frac{\partial^2 w}{\partial x_i \partial x_j}.
$$

Thus, we have proved that $\mu^2(a_i, a_j) = (r_i r_j, v_i \wedge v_j) + (\frac{1}{2} r_i r_j) \frac{\partial^2 w}{\partial x_i \partial x_j}$ and this is a Clifford multiplication on $A$. Similarly, we have that $\mu^2(a_j, a_i) = (r_i r_j, v_j \wedge v_i) + (\frac{1}{2} r_i r_j) \frac{\partial^2 w}{\partial x_i \partial x_j}$ and therefore $\mu^2(a_i, a_j) + \mu^2(a_j, a_i) = r_i r_j \frac{\partial^2 w}{\partial x_i \partial x_j}$ which gives the Clifford algebra structure on $A$.

The global calculation follows directly from this local version. \hfill \Box

The following proposition follows a similar argument to those in [Sei11]:

**Proposition 4.3.13.** The $A_\infty$-structure on $A$ coming from the tree summation formula agrees with the trivial $A_\infty$-structure up to order $d - 1$.

**Proof.** We want to show that any tree with $k$ leaves, for $k < d$, contributes a trivial operator to the summation and therefore $\mu^k = 0$ for $k < d$.

Consider now the $A_\infty$ product, $\mu^k$ for $k > 2$. Note that for $k > 2$, any tree with no bivalent vertex does not contribute to the summation as the term $h(i(a)i(a')) = h(i(aa')) = 0$ necessarily appears as the output of a trivalent vertex and thus the operator induced by such a tree would be 0. Therefore, the number $m$ of bivalent vertices, is at least 1.

By Lemma 4.3.9, $h$ has $f$-degree $-1$, while $\bar{\partial} - \partial$ has $f$-degree $d - 1$. Since $T$ has $k$-leaves, it has exactly $k - 1$ trivalent vertices. Therefore $h$ appears in the operator at most $k - 2 + m$ times while $\bar{\partial} - \partial$ appears $m$-times. Since $m$ preserves the $f$-degree, it follows that before applying $p$, the operator will have $f$-degree $m(d - 1) - (k - 2 + m)$. Therefore if

$$
m(d - 1) - (k - 2 + m) > 0
$$

then the operator vanishes.

In summary, in order to get a non-trivial contribution from trees with $m$ bivalent vertices one must have

$$
1 \leq m \leq \frac{k - 2}{d - 2}.
$$

Thus, $\mu^k$ can be non-trivial only when $d \leq k$. \hfill \Box

**Lemma 4.3.14.** Any tree providing a nontrivial contribution to $\mu^d$ has exactly 1 bivalent vertex.

**Proof.** Equation (4.7) for $k = d$ tells us that the number of bivalent vertices must be 1. \hfill \Box

The following proposition completes the proof of Theorem 4.3.1.
Proposition 4.3.15. For \( d \geq 3 \), the \( d \)-th higher multiplication on \( A \) can be described locally by

\[
\mu^d(r_1 \otimes v_{i_1}, \ldots, r_d \otimes v_{i_d}) = \frac{1}{d!} \frac{\partial^d w}{\partial x_{i_1} \ldots \partial x_{i_d}} r_1 \ldots r_d,
\]

where \( v_{i_j} \) are not necessarily distinct elements of the basis \( \{ v_i = \frac{\partial}{\partial x_i} \} \); and globally by

\[
\mu^d(1 \otimes s_1, \ldots, 1 \otimes s_d) = \frac{1}{d!}(d(d(w \uparrow s_1) \uparrow s_2) \ldots \uparrow s_d).
\]

Proof. We consider \( a_j = (r_j, v_{i_j}) \) for \( j = 1, \ldots, d \) and \( v_{i_j} = \frac{\partial}{\partial x_{i_j}} \). Before calculating \( \mu^d(a_1, \ldots, a_d) \) we first note that there is only one tree contributing to the summation formula (Figure 8).

Indeed, let \( T \) be any tree that contributes to the summation formula for \( \mu^d(a_1, \ldots, a_d) \). By Lemma 4.3.14, \( T \) has only 1 bivalent vertex. On the other hand, since \( T \) has \( d \) leaves, it has exactly \( d - 1 \) trivalent vertices. Moreover, the bivalent vertex has to be connected to one of the last two leaves since otherwise there is at least one \( h \) appearing before \( \overline{a} - \partial \) and that means the \( f \)-degree of the output of the operator given by \( T \) (before applying \( p \)) has \( f \)-degree greater than or equal to 1 and thus it lies in the kernel of \( p \).

Therefore, the vertex connected to the root has valency 3. Furthermore, arguing as in Lemma 4.3.12 it follows that this last vertex is connected to a leaf. Thus,

\[
\mu_T^d(a_1, \ldots, a_d) = p(m(i(a_1)), P(a_2, \ldots, a_d)),
\]

where \( P \) is the remaining part of the operator which, as before, is attached to the ribbon tree with the edges connecting the leaf and the root to the last vertex removed and \( h \) replacing \( p \). In summary, we have established that the last/left-most vertex appears as in Figure 8. Moreover, we note that, without any loss of generality, we can assume \( i(a_1) = (1, r_1, x_{i_1}) \).

This follows from the same argument as above, when we calculated the multiplication on \( A \). This also forces \( P(a_2, \ldots, a_d) \) to have \( \beta \)-degree 1 (or, more precisely, all other terms in \( P(a_2, \ldots, a_d) \) will vanish when multiplied with \( i(a_1) \)).

We now argue that the one bivalent vertex cannot be connected to the penultimate leaf since, in this case, the operator \( P \) would have \( \beta \)-degree \( \geq 2 \) and thus the operator induced by \( T \) would be 0. This follows inductively by tracing the \( \beta \)-degree of the operator induced by the tree we are considering. We thus conclude that there is only one tree contributing
to the summation formula giving $\mu^d$ and we can calculate its contribution using a similar calculation as in Lemma 4.3.12 which yields the desired result.

The formula for the global version follows directly from this local version.

4.4. **The derived category and a technical assumption.** Recall that $\mathcal{A}$ and $A_\infty \mathcal{A}$-modules are assumed to have strict restriction morphisms. We start with the following definition:

**Definition 4.4.1.** For a sheaf of graded $A_\infty$ algebras $A$, let $D_{pe}(\text{Mod}_{\infty,Z} A)$ be the smallest thick triangulated subcategory of the derived category of strictly unital graded $A_\infty \mathcal{A}$-modules containing all modules of the form $A \otimes_{\mathcal{O}_S} \mathcal{L}$ for graded invertible sheaves $\mathcal{L}$ on $S$. If we wish to emphasize the underlying variety, we will also denote this category by $D_{pe}(\text{Mod}_{\infty,Z} (S,A))$.

Recall that for a sheaf of graded dg $\mathcal{O}_S$-algebras $B$, $D_{pe}(\text{Mod}_Z B)$ is the full subcategory of the derived category of dg $\mathcal{B}$-modules, classically generated by objects of the form $B_w \otimes_{\mathcal{O}_S} \mathcal{L}$.

The following conjecture is believed to be true, however a proof is beyond the scope of this paper.

**Conjecture 4.4.2.** If a sheaf $B$ of $\mathbb{Z}$-graded differential-graded $\mathcal{O}_S$-algebras is graded-quasi-isomorphic to a sheaf $A$ of $\mathbb{Z}$-graded $A_\infty \mathcal{O}_S$-algebras, then there is a $\mathbb{P}(S)$-linear equivalence $D_{pe}(\text{Mod}_{\infty,Z} A) \cong D_{pe}(\text{Mod}_Z B)$.

**Remark 4.4.3.** When $S = \text{Spec } k$, this conjecture is known by the results in [LH03], Section 2.4.2; in which case, the category of $A_\infty$-modules has a model structure where the weak equivalences are quasi-isomorphisms and every object is fibrant and cofibrant. This makes considerations of derived functors more straightforward. For the general case, one would need to define the appropriate model structures and prove the derived-equivalences induced by the diagram in the proof of Lemma 2.4.2.3. in loc. cit..

In the notation of Section 4.3, we have the following:

**Corollary 4.4.4.** Assuming Conjecture 4.4.2, the quasi-isomorphism, $f$, induces an equivalence,

$$D_{pe}(\text{Mod}_{\infty,Z} A) \cong D_{pe}(\text{Mod}_Z B_w).$$

In particular,

$$D(\text{coh}[Q_-/ \mathbb{G}^2], w) \cong D_{pe}(\text{Mod}_{\infty,Z} \mathcal{A}).$$

**Proof.** The first statement follows from Theorem 4.3.1 and Conjecture 4.4.2. The second is then an immediate consequence of Propositions 4.2.2 and 4.2.6. □

**Remark 4.4.5.** Assuming Conjecture 4.4.2, the results of Section 4.2 and Corollary 4.4.4 could have alternatively been obtained (and generalized) by proving a relative version of Baranovsky’s BGG correspondence for projective complete intersections [Bar05].
5. Homological Projective Duality for d-th degree Veronese embeddings

We will now apply the results of the previous two sections to construct a homological projective dual to the degree $d$ Veronese embedding. In view of potential applications, we will do this in the relative setting. Then, if $d = 2$, we will recover Kuznetsov’s construction for degree two Veronese embeddings [Kuz05] (when $S$ is a point) and the relative version in [ABB11].

Let $S$ be a smooth, connected variety and $\mathcal{P}$ be a locally-free coherent sheaf on $S$. We consider the relative degree $d$ Veronese embedding for $d > 0$,

$$g_d : \mathbb{P}_S(\mathcal{P}) \to \mathbb{P}_S(S^d\mathcal{P}).$$

Notice that $g_d^*(\mathcal{O}_{\mathbb{P}(S^d\mathcal{P})}(1)) \cong \mathcal{O}_{\mathbb{P}(\mathcal{P})}(d)$. Consider the Lefschetz decomposition

$$D^b(\text{coh } \mathbb{P}_S(\mathcal{P})) = \langle A_0, \ldots, A_i(i) \rangle$$

where the subcategories $A_j$ are defined to be

$A_0 = \ldots = A_{i-1} = \langle p^* D^b(\text{coh } S), \ldots, p^* D^b(\text{coh } S)(d - 1) \rangle$

$$A_i = \langle p^* D^b(\text{coh } S), \ldots, p^* D^b(\text{coh } S)(k - 1) \rangle$$

where $k = \text{rk } \mathcal{P} - d(\lfloor \frac{\text{rk } \mathcal{P}}{d} \rfloor - 1)$.

We will first consider $\mathbb{P}_S(\mathcal{P})$ as a quotient and use the results of Section 3. Let us take $Q = V_S(\mathcal{P})$ and consider the $G = \mathbb{G}_m$-action given by fiber-wise dilation. Take the character given by $\chi(\alpha) = \alpha^d$ and the invertible sheaf $\mathcal{M} = \mathcal{O}(\chi)$ on $Q$. Taking the one-parameter subgroup $\lambda : \mathbb{G}_m \to \mathbb{G}_m$ given by $\lambda(\alpha) = \alpha^{-1}$, we see that we have an elementary wall crossing with

$$S_\lambda = 0_{V_S(\mathcal{P})},$$

$$S_{-\lambda} = V_S(\mathcal{P}).$$

We get that

$$[Q_+/G] = \mathbb{P}_S(\mathcal{P}),$$

$$[Q_-/G] = \emptyset$$

$$\mu = \text{rk } \mathcal{P} - d,$$

$$d = d,$$

where $d$ is the weight of the $\lambda$-action on $\mathcal{M}$. This shows that $\mathcal{M}$ induces the morphism $g_d : \mathbb{P}(\mathcal{P}) \to \mathbb{P}(S^d\mathcal{P}^*)$.

Using Proposition 3.1.1, we recover the Lefschetz decomposition with

$$A_0 = \ldots = A_{i-1} = \langle p^* D^b(\text{coh } S), \ldots, p^* D^b(\text{coh } S)(d - 1) \rangle,$$

$$A_i = \langle p^* D^b(\text{coh } S), \ldots, p^* D^b(\text{coh } S)(k - 1) \rangle.$$

The universal degree $d$ polynomial $w$ is given by

$$w := (g_d \times 1)^* \theta \in \Gamma(\mathbb{P}_S(\mathcal{P})) \times \mathbb{P}_S(S^d\mathcal{P}^*), \mathcal{O}_{\mathbb{P}_S(\mathcal{P})}(d) \otimes \mathcal{O}_{\mathbb{P}_S(S^d\mathcal{P}^*)(1)},$$
where \( \theta \) is the tautological section in \( \Gamma(\mathbb{P}(S^d\mathcal{P}) \times_S \mathbb{P}(S^d\mathcal{P}^*)/\mathbb{G}_m, O_{\mathbb{P}(S^d\mathcal{P})}(1) \otimes O_{\mathbb{P}(S^d\mathcal{P}^*)}(1)) \). The zero locus \( w \) in \( \mathbb{P}(\mathcal{P}) \times_S \mathbb{P}(S^d\mathcal{P}^*) \) is the universal hyperplane section \( \mathcal{X} \) of \( \mathbb{P}(\mathcal{P}) \) with respect to the embedding \( g_d \).

We have thus constructed a Landau-Ginzburg model which is a homological projective dual.

**Theorem 5.1.** The gauged Landau-Ginzburg model \( (\mathbb{P}(\mathcal{P}) \times_S \mathbb{P}(S^d\mathcal{P}^*)/\mathbb{G}_m, w) \) is a weak homological projective dual to \( \mathbb{P}(\mathcal{P}) \) with respect to the embedding \( g_d \) and the Lefschetz decomposition constructed above.

Moreover, we have:

- The derived category of the Landau-Ginzburg model \( (\mathbb{P}(\mathcal{P}) \times_S \mathbb{P}(S^d\mathcal{P}^*)/\mathbb{G}_m, w) \) admits a dual Lefschetz collection

\[
\text{D}(\text{coh}[\mathbb{P}(\mathcal{P}) \times_S \mathbb{P}(S^d\mathcal{P}^*)/\mathbb{G}_m], w) = \langle B_1(-j), \ldots, B_1(-1) \rangle
\]

- Let \( V \subset S^d\mathcal{P}^* \) be a subbundle and \( W \) the orthogonal subbundle in \( S^d\mathcal{P} \). Assume that \( \mathbb{P}(\mathcal{P}) \times_{\mathbb{P}(S^d\mathcal{P})} \mathbb{P}(W) \) is a complete linear section (not necessarily smooth). Then, there exist semi-orthogonal decompositions,

\[
\text{D}^b(\text{coh}[\mathbb{P}(\mathcal{P}) \times_{\mathbb{P}(S^d\mathcal{P})} \mathbb{P}(W)]) = \langle C_V, A_r(1), \ldots, A_i(i-r+1) \rangle,
\]

and,

\[
\text{D}(\text{coh}[\mathbb{P}(\mathcal{P}) \times_S \mathbb{P}(V)/\mathbb{G}_m], w) = \langle B_0(-r-N-2), \ldots, B_N(-1) \rangle, C_V \rangle.
\]

**Proof.** By directly applying Theorems 3.1.2 and 3.1.3 to the elementary wall crossing described above, and simplifying as described in Remark 3.1.10, we get the desired result.

For the first part of the theorem, we can alternatively consider \( \mathcal{X} \) as a degree \( d \) hypersurface fibration over \( \mathbb{P}(S^d\mathcal{P}^*) \) and use our relative version of Orlov’s theorem (Corollary 4.1.3) with \( S = \mathbb{P}(S^d\mathcal{P}^*), \mathcal{E} = \pi^*\mathcal{P} \) and \( \mathcal{U} = O_{\mathbb{P}(S^d\mathcal{P}^*)}(1) \) to get the decomposition

\[
\text{D}^b(\text{coh} \mathcal{X}) = \langle \text{D}(\text{coh}[\mathbb{P}(S^d\mathcal{P}^*)/\mathbb{G}_m], w), A_1(1) \otimes \text{D}^b(\text{coh} \mathbb{P}(S^d\mathcal{P}^*)), \ldots, A_i(i) \otimes \text{D}^b(\text{coh} \mathbb{P}(S^d\mathcal{P}^*)) \rangle.
\]

Observing that \( V_{\mathbb{P}(S^d\mathcal{P}^*)}(\pi^*\mathcal{P}) \cong V_S(\mathcal{P}) \times_S \mathbb{P}(S^d\mathcal{P}) \), we obtain the required semi-orthogonal decomposition. \( \square \)

Using this result, we can state the following:

**Theorem 5.2.** Let \( S \) be a smooth, connected variety and \( \mathcal{P} \) be a locally-free coherent sheaf over \( S \). Let \( d \geq 3 \) and \( d \leq \text{rank} \mathcal{P} \). Assuming Conjecture 4.4.2 there exists a sheaf of minimal \( A_{\infty} \)-algebras \( (\mathcal{A}, \mu) \) on \( \mathbb{P}(S^d\mathcal{P}^*) \) with

\[
\mathcal{A} = \text{Sym}(\mu^* O_{\mathbb{P}(S^d\mathcal{P}^*)}(1), \mu^{-1} O_{\mathbb{P}(S^d\mathcal{P}^*)}(-1)) \otimes \Lambda^* \mathcal{P}^*.
\]

and
With the same assumptions as in Theorem 5.2 above, we have the following

Theorem 5.3. With the same assumptions as in Theorem 5.2 above, we have the following

(i) The perfect derived category of the non-commutative variety \((\mathbb{P}_S(S^d\mathcal{P}^*), \mathcal{A})\) admits a dual Lefschetz collection

\[
D_{pe}(\text{Mod}_{\mathbb{Z}} (\mathbb{P}_S(S^d\mathcal{P}^*), \mathcal{A})) = (\mathcal{B}_j(-j), \ldots, \mathcal{B}_1(-1), \mathcal{B}_0)
\]

(ii) Let \(V \subset S^d\mathcal{P}^*\) be a subbundle and \(W\) the orthogonal subbundle in \(S^d\mathcal{P}\). Assume that \(\mathbb{P}_S(\mathcal{P}) \times_{\mathbb{P}_S(S^d\mathcal{P})} \mathbb{P}_S(W)\) is a smooth, complete linear section, i.e.

\[
\dim(\mathbb{P}_S(\mathcal{P}) \times_{\mathbb{P}_S(S^d\mathcal{P})} \mathbb{P}_S(W)) = \dim(\mathbb{P}_S(\mathcal{P})) - r.
\]

Then, there exist semi-orthogonal decompositions:

\[
D^b(\text{coh}\mathbb{P}_S(\mathcal{P}) \times_{\mathbb{P}_S(S^d\mathcal{P})} \mathbb{P}_S(W)) = (D_{pe}(\text{Mod}_{\mathbb{Z}} (\mathbb{P}_S(V), \mathcal{A}|_{\mathbb{P}_S(V)})), \mathcal{A}_r(1), \ldots, \mathcal{A}_i(i - r + 1)),
\]

if \(\text{rank}(V) \leq \frac{\text{rank} \mathcal{E}}{d}\) or

\[
D_{pe}(\text{Mod}_{\mathbb{Z}} (\mathbb{P}_S(V), \mathcal{A}|_{\mathbb{P}_S(V)})) = (\mathcal{B}_j(j), \ldots, \mathcal{B}_k(-k), D^b(\text{coh}\mathbb{P}_S(\mathcal{P}) \times_{\mathbb{P}_S(S^d\mathcal{P})} \mathbb{P}_S(W))),
\]

where \(k = \text{rank} \mathcal{E} - \text{rank} V - 1\), if \(\text{rank}(V) \geq \frac{\text{rank} \mathcal{E}}{d}\)

Proof of Theorems 5.2 and 5.3: We start with Proposition 5.1 and use the results in Sections 4.2 and 4.3 and to prove that, for any subbundle \(V \subset S^d\mathcal{P}^*\), the triangulated category \(D(\text{coh}[V\mathbb{P}_S(\mathcal{P}) \times_{\mathbb{P}_S(S^d\mathcal{P})} \mathbb{P}_S(V)/\mathbb{G}_m], w)\) is \(\mathbb{P}(S^d\mathcal{P}^*)\)-linearly equivalent to \(D_{pe}(\text{Mod}_{\mathbb{Z}} (\mathbb{P}_S(V), \mathcal{A}|_{\mathbb{P}_S(V)}))\).

Indeed, the results of Section 4.2 applied with \(S = \mathbb{P}_S(\mathcal{P}), \mathcal{E} = (\pi^*\mathcal{P})|_V, \mathcal{U} = O_{\mathbb{P}_S(V)}(1)\) and the same \(w\), combined with Theorem 4.3.1 give the equivalences. Notice that the assumption of smoothness of the linear sections in the second part of Theorem 5.3 is required for Proposition 4.2.1 to hold in this case. All the properties of \(\mathcal{A}\) follow from Theorem 4.3.1.

Remark 5.4. As an intermediate result, the conclusions of Theorem 5.2 and Theorem 5.3 hold when the non-commutative variety \((\mathbb{P}_S(S^d\mathcal{P}^*), \mathcal{A})\) is replaced by \((\mathbb{P}_S(S^d\mathcal{P}^*), \mathcal{B}_w)\) without the need for Conjecture 4.4.2.

Corollary 5.5. For any linear subspace \(L \subset S^dV^*\) such that the corresponding intersection of degree \(d\) hypersurfaces in \(\mathbb{P}(V)\) is smooth and complete (i.e. it has dimension \(\dim(\mathbb{P}(V)) - \dim(L)\)), we have
• If \( \dim(L) \leq n/d \), then there exists a semi-orthogonal decomposition:
\[
\mathcal{D}^b(\text{coh } X_L) = \langle \mathcal{D}_{pe}(\text{Mod}_{\infty, Z}(\mathbb{P}(L), \mathcal{A}|_L)), \mathcal{A}_r, \ldots, \mathcal{A}_i \rangle.
\]

• If \( \dim(L) \geq n/d \), then there exists a semi-orthogonal decomposition:
\[
\mathcal{D}_{pe}(\text{Mod}_{\infty, Z}(\mathbb{P}(L), \mathcal{A}|_L)) = \langle \mathcal{B}_j(-j), \ldots, \mathcal{B}_{\text{dim}(V) - \text{dim}(L) - 1}(-\dim(V) + \dim(L) + 1), \mathcal{D}^b(\text{coh } X_L) \rangle.
\]

Proof. This follows from setting \( S \) equal to \( \text{Spec } k \) in Theorem 5.3. □

Remark 5.6. If \( d = 2 \) then, as we have noticed in the previous section, \( \mathcal{A} \) is actually a sheaf of Clifford algebras, therefore by [Ric10], we get an equivalence
\[
\mathcal{D}_{pe}(\text{mod}(\mathbb{P}(S^2 V^*, \mathcal{B}_w))) \cong \mathcal{D}_{pe}(\text{mod}(\mathbb{P}(S^d V^*, \mathcal{A})),
\]
without the need for Conjecture 4.4.2. Now, using Proposition 3.7 in [Kuz05], one has
\[
\mathcal{D}_{pe}(\text{mod}(\mathbb{P}(S^2 V^*, \mathcal{A}))) \cong \mathcal{D}^b(\text{mod}(\mathbb{P}(S^d V^*, \mathcal{B}_0)),
\]
where \( \mathcal{B}_0 \) is the sheaf of even Clifford algebras defined in [Kuz05]. This recovers the homological projective dual in [Kuz05]. Similarly, for a smooth complete intersection of quadrics, we get the same description as in loc. cit. using Corollary 5.5. The relative versions in [Kuz05] and [ABB11] follow similarly.

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