On the Classification of Finite Semigroups and RA-loops with the Hyperbolic Property

S. O. Juriaans\textsuperscript{a} A. C. Souza Filho\textsuperscript{b}

\textit{Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo, CEP 05315-970 - Brazil}

email addresses: \textsuperscript{a}ostanley@ime.usp.br \textsuperscript{b}calixto@ime.usp.br

Abstract

We classify the finite semigroups $S$, for which all $\mathbb{Z}$-orders $\Gamma$ of $\mathbb{Q}S$, the unit group $\mathcal{U}(\Gamma)$ is hyperbolic. We also classify the RA-loops $L$ for which the unit loop of its integral loop ring does not contain any free abelian subgroup of rank two.

1 Introduction

Initially we classify finite dimensional algebras $\mathcal{A}$ over $\mathbb{Q}$ such that if $\Gamma \subset \mathcal{A}$ is a $\mathbb{Z}$-order then $\mathcal{U}(\Gamma)$ is hyperbolic. If $\mathcal{A}$ is such an algebra, we say that $\mathcal{A}$ has the hyperbolic property.

In [7], are classified the semigroups $\Sigma$ with $\mathcal{U}(\mathbb{Z}\Sigma)$ finite. Therefore, for this class of semigroups the algebra $\mathbb{Q}\Sigma$ has the hyperbolic property.

First we classify the semisimple algebras $\mathbb{Q}S$ which are nilpotent free. If $\mathbb{Q}S$ has nilpotent elements there are two possibilities: either $S$ contains nilpotent elements, or not. In the latter case $S$ is a disjoint union of groups of certain types. In the former, $S$ is a union of groups and a subsemigroup of order five. We also give the structure of $S$ when $\mathbb{Q}S$ is non-semisimple and has the hyperbolic property. For the proofs of the results of section two see [5].

In the last section we classify the RA-loops $L$, such that, $\mathbb{Z}^2 \not\rightarrow \mathcal{U}(\mathbb{Z}L)$.

2 Semigroup Algebras

We will consider $\mathcal{A}$ a unitary finitely generated $\mathbb{Q}$-algebra and denote by $\mathcal{S}(\mathcal{A})$, respectively $J(\mathcal{A})$, the semisimple subalgebra, respectively the Jacobson radical, of $\mathcal{A}$ and by $E(\mathcal{A}) = \{E_1, \cdots, E_N\}, N \in \mathbb{Z}^+$, the set of the central primitive
idempotents of the semisimple algebra $S(A)$. A classical result of Wedderburn-Malcev states that
\[ A \cong S(A) \oplus J(A), \]
as a vector space. As a result, we have that $A$ is a artinian algebra and thus its radical is a nilpotent ideal. We denote $T_2(Q) := \begin{pmatrix} Q & Q \\ 0 & Q \end{pmatrix}$ the $2 \times 2$ upper triangular matrices over $Q$, with the usual matrix multiplication.

**Definition 2.1** Let $A$ be a finite dimensional algebra over $Q$ and $\Gamma$ be a $\mathbb{Z}$-order of $A$. If
\[ \mathbb{Z} \not\hookrightarrow U(\Gamma), \]
we say $A$ has the hyperbolic property.

**Theorem 2.2** Let $A$ be a finite dimensional $Q$-algebra. If $A_i$ is a simple epimorphic image of $A$, denote by $F_i$ a maximal subfield of $A_i$ and $\Gamma_i \subset A_i$ a $\mathbb{Z}$-order. The following conditions hold:

1. The algebra $A$ has the hyperbolic property, it is semisimple and it has no nilpotent element if, and only if,
\[ A = \oplus A_i, \]
whereof $A_i$ is a division ring and there exists at most one index $i_0$ such that $U(\Gamma_{i_0})$ is hyperbolic and infinite.

2. The algebra $A$ has the hyperbolic property and it is semisimple with nilpotent elements if, and only if,
\[ A = (\oplus A_i) \oplus M_2(Q). \]

3. The algebra $A$ has the hyperbolic property and it is non semisimple with central radical if, and only if,
\[ A = (\oplus A_i) \oplus J. \]

4. The algebra $A$ has the hyperbolic property and it is non semisimple with non central radical if, and only if,
\[ A = (\oplus A_i) \oplus T_2(Q). \]

For each item above, $F_i$ is an imaginary quadratic field and $A_i$ is either an imaginary quadratic field or a totally definite quaternion algebra. Furthermore, every simple epimorphic image of $A$ in the direct sum is an ideal of $A$. 

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In what follows, $S$ denotes a finite semigroup, $\mathbb{Q}S$ denotes a unitary semigroup algebra over $\mathbb{Q}$, $\mathcal{M}^0(G; n, n; P)$ denotes the Rees semigroup with structural group $G$ and $P$ denotes an $n \times n$ sandwich matrix.

**Theorem 2.3** The algebra $\mathbb{Q}S$ has no nilpotent element and it has the hyperbolic property if, and only if, $S$ is an inverse semi-group and it admits a principal series, whose principal factors are isomorphic to groups $G$ and at most a unique $K$, listed below:

1. $G$ is an abelian group of exponent dividing 4 or 6;
2. $G$ is a hamiltonian 2-group;
3. $K \in \{C_5, C_8, C_{12}\}$.

**Theorem 2.4** Let $\mathbb{Q}S$ be an algebra with nilpotent elements. The algebra $\mathbb{Q}S$ is semisimple and it has the hyperbolic property if, and only if, $S$ admits a principal series whose principal factors are isomorphic to groups $G$ and a unique semigroup $K$, listed below:

1. $G$ is an abelian group of exponent dividing 4 or 6.
2. $G$ is a hamiltonian 2-group.
3. $K$ is a group of the set $\{S_3, D_4, Q_{12}, C_4 \rtimes C_4\}$.
4. $K$ is one of the Rees semigroups:
   \[ \mathcal{M}^0(\{1\}; 2, 2; I_d) = M \quad \text{or} \quad \mathcal{M}^0(\{1\}; 2, 2; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = M_{12}, \]

which is an ideal of $S$.

In particular, $S$ is the disjoint union of the groups $G$ and the semigroup $K$.

**Theorem 2.5** The algebra $\mathbb{Q}S$ is non semisimple and it has the hyperbolic property if, and only if, there exists a unique nilpotent element $j_0 \in S$, such that, the subsemigroup $\mathfrak{I} = \{\theta, j_0\}$ is an ideal of $S$, and $S \setminus \mathfrak{I}$ admits a principal series whose principal factors are isomorphic to abelian groups of exponent dividing 4 or 6, or a hamiltonian 2-group. In particular $S/\mathfrak{I}$ is the disjoint union of its maximal subgroups such that if $e_1 \in G_1$, and $e_N \in G_N$ are the respective group identity element, then $e_1 j_0 = j_0 e_N = j_0$. Writing
\[
e_1 = \sum E_{e_1} + E_1 + \lambda j_0, \lambda \in \mathbb{Q} \\
e_N = \sum E_{e_N} + E_N + \mu j_0, \mu \in \mathbb{Q}
\]

then only one of the following holds:
1. 

\[ e_1e_N = 0 \Leftrightarrow e_Ne_1 = e_3 = e_1 \quad \text{and} \quad \lambda + \mu = 0; \]

\[ T_2 \cong \{e_1, e_N, j_0, \theta\} \quad \text{is such that} \quad QT_2 \cong T_2(Q). \]

2. 

If \( e_Ne_1 \neq 0 \) then \( e_1e_N = e_Ne_1 = e_3 \) and \( \lambda + \mu = 0; \)

\[ T'_2 = \{e_1, e_2, e_3, j_0, \theta\} \quad \text{is a subsemigroup of} \quad S \quad \text{and} \quad QT'_2 \cong Q \oplus Q \oplus T_2(Q). \]

3. 

\[ e_Ne_1 = 0 \Leftrightarrow e_1e_N = j_0 \Leftrightarrow \lambda + \mu = 1; \]

\[ \hat{T}_2 = \{e_1, e_N, j_0, \theta\}, \quad \text{and} \quad Q\hat{T}_2 \cong T_2(Q). \]

The semigroups \( T_2, T'_2 \) and \( \hat{T}_2 \) are non isomorphic.

3 The hyperbolicity of the RA-loop loop units

In this section we classify the RA-loops \( L \) such that \( \mathbb{Z}^2 \not\cong U(\mathbb{Z}L) \), the loop of units of \( \mathbb{Z}L \). A loop \( L \) is a nonempty set, with a closed binary operation \( \cdot \), such that the equation \( a \cdot b = c \) has a unique \( b \in L \) when \( a, c \in L \) are known, and a unique solution \( a \in L \) when \( b, c \in L \) are know, and with a two-side identity element 1. Denoting by \( [x, y, z] = (xy)z - x(yz) \), recall that a ring \( A \) is alternative if \( [x, x, y] = [y, x, x] = 0 \), for every \( x, y \in A \). An RA-loop is a loop whose loop ring \( RL \) over some commutative, associative and unitary ring \( R \) of characteristic not equal to 2 is alternative, but not associative. The basic reference is \([10]\).

For a theoretical group property \( P \), a group \( G \) is virtually \( P \) if it has a subgroup of finite index, \( H \) say, with property \( P \).

**Theorem 3.1** ([12], Theorem 3.3.6) Let \( L \) be a RA-loop. \( U(\mathbb{Z}L) \) has the hyperbolic property if, and only if, \( L \) is a finite loop or a loop whose center is virtually cyclic, the torsion subloop \( T(L) \) of \( L \) is such that, if \( T(L) \) is a group, then it is an abelian group of exponent dividing 4 or 6 or a hamiltonian 2-group whose subgroups are all normal in \( L \) and if \( T(L) \) is a loop then it is a hamiltonian Moufang 2-loop whose subloops are all normal in \( L \). In this conditions we also have that \( U_1(\mathbb{Z}L) = L \).

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References

[1] L. G. X. de Barros, S. O. Juriaans, Units in Alternative Integral Loop Rings, Result. Math., 31 (1997), 266-281.
[2] A. Borel, H. Chandra, Arithmetic Subgroups of Algebraic Groups, Annals of Mathematics, 75(3), 1962.
[3] A. Dooms, E. Jespers, Generators for a subgroup of finite index in the unit group of an integral semigroup ring, J. Group Theory 7(2004), 543-553.
[4] M. Gromov, Hyperbolic Groups, in Essays in Group Theory, M. S. R. I. publ. 8, Springer, 1987, 75-263.
[5] E. Iwaki, S. O. Juriaans, A. C. Souza Filho, Hyperbolicity of Semigroup Algebras, www.arxiv.org.
[6] E. Jespers, Free Normal Complements and the Unit Group of Integral Group Rings, Proceedings of the American Mathematical Society, vol 122, number 1, 1994.
[7] E. Jespers, D. Wang, Units of Integral Semigroup Rings, Journal of Algebra, vol 181, pages 395-413, 1996.
[8] S. O. Juriaans, I. B. S. Passi, D. Prasad, Hyperbolic Unit Groups, Proceedings of the American Mathematical Society, vol 133(2), 2005, pages 415-423.
[9] A. H. Clifford, G. B. Preston, The Algebraic Theory of Semigroups, American Mathematical Society, Mathematical Surveys number 7, Rhode Island, 1961.
[10] E. G. Goodaire, E. Jespers, F. C. Polcino Milies, Alternative Loop Rings, Elsevier, Oxford, 1996.
[11] J. Okniński, Semigroup Algebras, Pure and Applied Mathematics, Dekker, USA, 1991.
[12] A.C. Souza Filho, Sobre uma Classificação dos Anéis de Inteiros, dos Semigrupos Finitos e dos RA-Loops com a Propriedade Hiperbólica (On a Classification of the Integral Rings, Finite Semigroups and RA-Loops with the Hyperbolic Property), PhD. Thesis, IME-USP, São Paulo, 2006, 108 pages.