Acceleration of particles and shells by Reissner-Nordström naked singularities

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Abstract

We explore the Reissner-Nordström naked singularities with a charge $Q$ larger than its mass $M$ from the perspective of the particle acceleration. We first consider a collision between two test particles following the radial geodesics in the Reissner-Nordström naked singular geometry. An initially radially ingoing particle turns back due to the repulsive effect of gravity in the vicinity of naked singularity. Such a particle then collides with another radially ingoing particle. We show that the center of mass energy of collision taking place at $r \approx M$ is unbound, in the limit where the charge transcends the mass by arbitrarily small amount $0 < 1 - M/Q \ll 1$. The acceleration process we described avoids fine tuning of the parameters of the particle geodesics for the unbound center of mass energy of collisions and the proper time required for the process is also finite. We show that the coordinate time as observed by the distant observer required for the trans-Plankian collisions to occur around the naked singularity with one solar mass is merely of the order of million years which is much smaller than the Hubble time. On the contrary, the time scale for collisions associated with extremal black hole in an analogous situation is many orders of magnitude larger than the age of the universe. We then study the collision of the neutral spherically symmetric shells made up of dust particles. In this case, it is possible to treat the situation by exactly taking into account the gravity due to the shells using Israel’s thin shell formalism, and thus this treatment allows us to go beyond the test particle approximation. The center of mass energy of collision of the shells is then calculated in a situation analogous to the test particle case and is shown to be bounded above. However, we find that the energy of a collision between two of constituent particles of the shells at the center of mass frame can exceed the Planck energy.

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I. INTRODUCTION

Since the terrestrial particles accelerators like Large Hadron Collider probe particle physics at the energy scales that are almost 15 orders of magnitude smaller than the Planck scale, it would interesting to investigate whether or not various naturally occurring high energy astrophysical phenomenon could shed light on the new physics at higher energy scales that remain unexplored. Stepping ahead towards this exciting possibility, an interesting proposal was made recently which suggests that the Kerr black holes could act as particle accelerators\cite{1}. It was shown that the two particles dropped in from infinity at rest, traveling along the timelike geodesics can collide and interact near the event horizon of a Kerr black hole with divergent center of mass energy, provided the black hole is close to being extremal and angular momentum of one of the particles takes a specific value of the orbital angular momentum. The possible astrophysical implications of this process around the event horizon of the central supermassive black hole in the context of annihilations of the dark matter particles accreted from the galactic halo were also investigated\cite{2}. This process of particle acceleration suffers from several drawbacks and limitations pointed out in\cite{3}. The angular momentum of one of the colliding particle must take a single fine tuned value. The proper time required for the particle with fine tuned angular momentum to reach the horizon and thus the time required for the collision to take place is infinite. The gravity produced by the colliding particles themselves was neglected. There were many investigations of this acceleration mechanism in the background of Kerr as well as many other black holes\cite{4}.

Two of present authors, PM and PSJ, studied and extended the particle acceleration mechanism to the Kerr naked singular geometries transcending Kerr bound by arbitrarily small amount $0 < a - 1 \ll 1$\cite{5}. We considered two different scenarios where the colliding particles follow a geodesic motion along the equatorial plane as well as along the axis of symmetry of the Kerr geometry. In the first case, the particles are released from infinity at rest in the equatorial plane. One of the initially infalling particle turns back as an outgoing particle due to its angular momentum. It then collides with another infalling particle around $r = 1$. We showed that the center of mass energy of collision between these two particles is arbitrarily large. The angular momentum of the colliding particles is required to be in a finite range as opposed to the single fine tuned value in case of Kerr black holes. Thus the extreme fine tuning of the angular momentum is avoided in such a collision.

The proper time required for such a collision to take place is also shown to be finite. In the second case, the particles are released from rest along the axis of symmetry, from large but finite distance. These particles have

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zero angular momentum. One of the particles initially falls in and then turns back due to the repulsive effect of gravity in the vicinity of a Kerr naked singularity. This particle then collides with an ingoing particle at \( z = 1 \). The center of mass energy of collision is arbitrarily large and the proper time required for the process to take place is finite. Thus two issues related to acceleration mechanism in Kerr black hole case, namely the fine tuning of the angular momentum and the infinite time required for the collision, are avoided in case of Kerr naked singularities.

The issue of the self-gravity of the point particles is difficult to deal in general. The accretion of the particles onto an astrophysical object can be expected to be more or less isotropic in many cases. Thus it would be interesting and more physical to study the motion and collisions of the shells of particles instead. The rigorous mathematical analysis of the shells would be very extremely difficult in the Kerr spacetime due to the lack of sufficient symmetry. By contrast, the motion and collision of the spherical shells would be exactly tractable in the spherically symmetric spacetimes following the Israel's thin shell formalism\(^6\). We first note that while no gravitational radiation is emitted by a perfectly spherical shell, the gravitational radiation per particle emitted by a quasishperical shell of particles will be significantly lower than the radiation emitted by a single particle\(^7\).

Thus it might be reasonable to ignore the gravitational radiation effects and focus entirely on the backreaction while dealing with the shells.

The acceleration of the particles around the extremal Reissner-Nordström black hole was studied in \(^8\), \(^10\). This process is mathematically similar to the acceleration process in Kerr geometry. The center of mass energy of collision near the horizon of the extremal Reissner-Nordström black hole, of the charged and uncharged particles is shown to be divergent. The collision of the charged and uncharged spherical shells was investigated in \(^11\). The dynamics of the shells when their gravity is ignored is same as that of the test particles. Whereas when the exact calculation is done taking into account the self-gravity effects, the center of mass energy turns out to be finite. Thus it was speculated that the center of mass energy of collision of particles around Kerr black hole might also turn out to be finite when the gravity due to the colliding particles is taken into account.

In this paper, we first describe the particle acceleration process in the background of Reissner-Nordström naked singularities. We show that the center of mass energy of collision between two uncharged particles, one of them initially ingoing and other one initially ingoing, but turning back due to the repulsive effect of gravity in the vicinity of naked singularity is arbitrarily large, when the collision happens around \( r \approx M \), provided that the deviation of the Reissner-Nordström charge from the mass is extremely small. We calculate the coordinate time as seen by the distant observer, associated with the ultra-high energy collisions for extremal black hole as well as for naked singularity. We show that the time scale associated with the trans-Plankian collisions around naked singularity with one solar mass is of the order of million years which is significantly smaller than the Hubble scale, whereas the timescale for the extremal black hole with the same mass as that of the naked singularity is fifteen orders of magnitude larger than the age of the universe. Thus collision process around black hole suffers from the inflating timescale problem while such issue is absent in case of the naked singularity. We then investigate the collision between two uncharged shells made up of dust particles, in a situation analogous to the particle collision, taking into account their gravity. We find that the center of mass energy of a collision between the shells is bounded above. However, the center of mass energy of a collision between two of constituent particles of the shells can exceed the Planck energy which might be a threshold value of the quantum gravity.

In this paper, we adopt the geometrized unit in which the speed of light and Newton’s gravitational constant are unity.

II. ACCELERATION OF PARTICLES BY REISSNER-NORDSTRÖM NAKED SINGULARITIES

A. Geometry of Reissner-Nordström spacetime

The Reissner-Nordström spacetime is a unique solution of Einstein equations under the assumptions of spherical symmetry, asymptotic flatness with the \( U(1) \) gauge field as a source of spacetime curvature. The line element of the Reissner-Nordström geometry in the spherical polar coordinates is given by

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

where

\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.
\]

The gauge field is given by

\[
A_\mu = \frac{Q}{r} \delta_\mu^t.
\]

This solution contains two parameters \( M \) and \( Q \), namely the mass and \( U(1) \) charge. In this paper, we assume that \( M \) and \( Q \) are positive,

\[
M > 0 \quad \text{and} \quad Q > 0.
\]

In the Reissner-Nordström spacetime, there is a spacetime singularity at \( r = 0 \). This singularity is timelike and thus is necessarily locally naked. The location of the horizon in the Reissner-Nordström spacetime is given by
a solution to the equation \( f(r) = 0 \). There are two roots to this quadratic equation given by

\[
r = r_{\pm} := M \pm \sqrt{M^2 - Q^2}.
\]  

(5)

There are two real positive roots to the equation if \( M > Q \). The larger root \( r = r_+ \) is the location of the event horizon and this spacetime corresponds to a spherically symmetric charged black hole. The smaller root \( r = r_- \) corresponds to the Cauchy horizon associated with the timelike singularity at \( r = 0 \). If \( M = Q \), there is only one positive root. In this case the black hole is known as the extremal black hole with a degenerate event horizon at \( r = M = Q \). In the case of \( M < Q \), there is no real root to the equation \( f(r) = 0 \). Thus, the event horizon is absent and the timelike singularity at \( r = 0 \) is exposed to the asymptotic observer at infinity. This configuration thus contains a globally visible naked singularity. We will investigate the last case in this paper from the perspective of particle acceleration.

Before proceeding further, it is worthwhile to mention that, the naked singularities are associated with pathological features like the breakdown of predictability and so on. That was precisely the reason why Penrose came up with the cosmic censorship conjecture abandoning the existence of naked singularities in our universe\(^{[11]}\). However there were recent developments in the framework in string theory, which suggests by means of the specific worked out examples, that the naked singularities might be resolved by high energy stringy modification to the classical general relativity \(^{[12]}\) and various pathological features disappear. This renders the classical naked singular solutions legal as long as one stays sufficiently away from high curvature region where quantum gravity would prevail.

## B. Motion of a test particle

We now study the motion of a point test particle following a timelike geodesic in the Reissner-Nordström spacetime. Let \( \mathbf{U} \) be the 4-velocity of the particle. Without loss of generality, we assume that the motion of the particle is confined to the equatorial plane \( \theta = \pi/2 \). All of the metric components \(^{[13]}\) are manifestly independent of time coordinate and azimuthal angular coordinate. This means that both of the time coordinate basis \( \partial / \partial t \) and azimuthal angular coordinate basis \( \partial / \partial \phi \) are Killing vectors. The following quantities are conserved along the geodesic of the particle

\[
E := -\mathbf{U} \left( \frac{\partial}{\partial t} \right) \quad \text{and} \quad L := \mathbf{U} \left( \frac{\partial}{\partial \phi} \right).
\]  

(6)

\( E \) can be interpreted as the conserved energy of the particle per unit mass and \( L \) can be interpreted as the conserved angular momentum of the particle per unit mass. Using these constants of motion and the normalization condition for 4-velocity of the particle, the components of the 4-velocity \( \mathbf{U} \) are written as

\[
U^t = \frac{E}{\tau}
\]

\[
U^r = \pm \sqrt{E^2 - f\left(1 + \frac{L^2}{r^2}\right)}
\]

\[
U^\theta = 0
\]

\[
U^\phi = \frac{L}{r^2}
\]

\( \pm \) stands for the radially outgoing and infalling particles respectively. The second one in the above equations can also be written in the following form

\[
\left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}} = E^2,
\]  

(8)

where \( \tau \) is the proper time of the particle, and

\[
V_{\text{eff}} = f \left(1 + \frac{L^2}{r^2}\right).
\]  

(9)

\( V_{\text{eff}} \) can be thought of as an effective potential.

For simplicity and from the perspective of the comparison to shell collision that would be discussed in the next section, we assume that the angular momentum of the particle is zero \( L = 0 \). This implies that the motion of the particle is purely radial. The effective potential now can be written as

\[
V_{\text{eff}} = f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}
\]  

(10)

The effective potential is plotted as a function of radius \( r \) in Fig. 1. For large values of radial coordinate \( r \rightarrow \infty \), we have \( V_{\text{eff}} \rightarrow 1 \). As one approaches the naked singularity \( r \rightarrow 0 \), effective potential blows up positively, i.e., \( V_{\text{eff}} \rightarrow \infty \). It always remains greater than zero and admits a minimum at \( r = r_{\text{min}} \) which is given by

\[
r_{\text{min}} = \frac{Q^2}{M},
\]  

(11)

and we have

\[
V_{\text{eff}}|_{r=r_{\text{min}}} = 1 - \frac{M^2}{Q^2}
\]  

(12)

Note that \( r_{\text{min}} \) coincides with the classical radius associated with an object of charge \( Q \) and mass \( M \). It is clear from the shape and slope of the effective potential curve that the gravity of the Reissner-Nordström naked singularity is attractive in the domain \( r_{\text{min}} < r < \infty \), from the classical radius all the way up to infinity. Whereas the gravity is repulsive in the region extending from the singularity to the classical radius \( 0 < r < r_{\text{min}} \). Similar behavior is also observed in case of other known examples of the stationary naked singularities\(^{[13]}\). An ingoing particle at initially speeds up up to the classical radius.
is positive asymtotic velocity of the particle as it reaches infinity $r \rightarrow \infty$. An allowed domain for the motion of a particle is depicted by a dashed line for each case of specific energy. It admits a minimum at the classical radius $r = Q^2/M$, depicted by ‘min’, where gravity changes its character from being attractive to repulsive in the close neighborhood of singularity. The ingoing particle thus gets reflected back as an outgoing particle close to singularity. The motion of a particle having energy $E = 0.8 < 1$ is bound and oscillates. The motion of a particle with energy $E = 1.1 > 1$ is unbound, has only one turning point. The motion of a particle with $E = 1$ is marginally bound, also has only one turning point. The potential energy curve asymptotes to the $E = 1$ as $r \rightarrow \infty$.

It then slows down due to the repulsive gravity and gets reflected back eventually. It then emerges as an outgoing particle.

If the conserved energy of the particle is less than unity $E < 1$ then the particle is bound, i.e., it oscillates back and forth in the radial domain $b_- \leq r \leq b_+$, where

$$b_\pm = \frac{M}{1-E^2} \left( 1 \pm \sqrt{1-\frac{Q^2}{M^2} \left( 1-E^2 \right)} \right).$$

In the case of $E = \sqrt{1-M^2/Q^2}$, $b_+$ is equal to $b_-$. This means that the particle stays stably at rest at the classical radius $r = Q^2/M$. If the conserved energy is identical to unity $E = 1$, then there is only one turning point given by $r = Q^2/2M$. In this case, the particle is at rest at infinity, and the motion of the particle is said to be marginally bound. In the case when energy is larger than unity $E > 1$, again there is only one turning point given by $r = b_-$ since $b_+$ is negative in this case. The asymptotic velocity of the particle as it reaches infinity is positive $U^r \rightarrow \sqrt{E^2-1}$. Such a particle trajectory is called the unbound one.

We should note that there is an important difference between the black hole case $M \geq Q$ and the naked singular case $M < Q$. In the case of the black hole $M \geq Q$, the radial motion cannot be restricted to only one asymptotically flat region. Since the inner turning point, $r = b_-$, is less than or equal to the radius of the Cauchy horizon $r = r_-$, the particle cannot return to the asymptotically flat region where it comes from. By contrast, in the case of the naked singularity $M > Q$, there is only one asymptotically flat region. Hereafter, we focus on the naked singular case $M > Q$.

### C. Collision of test particles

We now consider a collision between two particles moving along a radial geodesics i.e., $L = 0$, each with mass $m$ and conserved energy $E = 1$: Particles are assumed to be marginally bound, or in other words, they are released from rest from infinity. One could replace marginally bound particles by either unbound or bound particles. It does not change the conclusions. Let $U_1^\mu$ and $U_2^\mu$ be components of their 4-velocities with respect to the coordinate basis. We assume that one of the particles is initially ingoing particle which gradually slows down and eventually turns back as an outgoing particle due to the repulsive gravity in the vicinity of the naked singularity. Such a particle then collides with another ingoing particle at the radial coordinate $r$. By the assumption, $U_1^\mu$ and $U_2^\mu$ are given by

$$U_1^\mu = \left( \frac{1}{f}, \sqrt{1-f}, 0, 0 \right)$$
$$U_2^\mu = \left( \frac{1}{f}, -\sqrt{1-f}, 0, 0 \right)$$

The energy of a collision between two particles at the center of mass frame is then given by

$$E_{cm}^2 = 2m^2 (1 - g_{\mu\nu}U_1^\mu U_2^\nu) = \frac{4m^2}{f(r)} = \frac{4m^2}{V_{eff}},$$

where $g_{\mu\nu}$ is the metric tensor given in Eq. (1). It is seen from the above equation that the center of mass energy $E_{cm}$ of collision depends on the location for the collision, for given values of charge $Q$ and mass $M$. $E_{cm}$ takes maximum when the effective potential $V_{eff}$ takes minimum. The minimum of $V_{eff}$ is realized at the classical radius $r_{min} = Q^2/M$. If the collision takes place at $r = r_{min}$, $E_{cm}$ is given by

$$E_{cm,max}^2 = \frac{4m^2}{1-M^2/Q^2}$$

$E_{cm,max}$ depends on the ratio of mass to the charge of Reissner-Nordström spacetime. $E_{cm,max}$ is very large if the charge transcends the mass by infinitesimally small amount. Here, we introduce a parameter defined by

$$\epsilon := 1 - \frac{M}{Q}.$$

In the limit $\epsilon \rightarrow 0$, $tE_{cm,max}$ becomes infinite,

$$\lim_{\epsilon \rightarrow 0} \frac{E_{cm,max}^2}{\epsilon} = \frac{2m^2}{\epsilon} \rightarrow \infty.$$
The above equation implies that the energy of collision measured at the center of mass frame would be arbitrarily large. In case of the black hole, the divergence of center of energy in the collision has been demonstrated in near extremal or extremal geometries when the mass transcends the charge by arbitrarily small amount $\epsilon \rightarrow 0^-$. In this paper, we have shown the possibility of the indefinitely large center of mass energy in the naked singular geometry, which can be thought to be near extremal, with the charge transcending the mass by arbitrarily small amount $\epsilon \rightarrow 0^+$. 

D. Time scale of the collision

We now estimate the time scale associated with the ultra-high energy particle collisions in the Reissner-Nordström naked singular geometry as well as in the extremal black hole geometry and make a critical comparison. We compute the proper time in the reference frame attached to the colliding neutral particle, as well as the coordinate time measured by a distant static observer, required for the particle to reach the collision point $r = r_{\text{min}} = Q^2/M$ in the case of naked singularity, and the horizon $r = M$ in the extremal black hole case. The particle starts from a distant location with $r_1 > r_{\text{min}}$ and participates in the high energy collision.

In the extremal Reissner-Nordström black hole geometry with $Q = M$, the high energy collision between the particles takes place at a location extremely close to the event horizon. One of the colliding particles is charged and the other one is charge neutral. The charged particle experiences an outward repulsive electromagnetic force during its inward motion. For such a particle it turns out that $V_{\text{eff}} = V_{\text{eff}} = 0$ as it approaches the event horizon, as a consequence of which the proper time required for it to reach the horizon and participate in the high energy collision turns out to be infinite. The neutral particle, however, falls freely following a geodesic motion and reaches the event horizon in a finite proper time as we show later in this section. We also estimate the coordinate time as seen by the static observer at infinity, required for the neutral particle to participate in the high energy collision. We show that it diverges in the limit of approach to the horizon as it is an infinite blueshift/redshift surface and the timescale associated with the trans-Plankian collision is much larger than the age of the universe.

In the Reissner-Nordström naked singular geometry, the collision is between two charge neutral particles following a geodesic motion as they fall freely under the gravity. Both the conditions mentioned above in the last paragraph namely $V_{\text{eff}} = V_{\text{eff}} = 0$ are not satisfied simultaneously anywhere along the trajectory of either of the two particles. Thus the proper time required for the collision to take place for both the particles in their own frame is finite as we demonstrate later in this section. However, since it is necessary to have $f(r_{\text{min}}) \rightarrow 0^+$, for high energy collision to occur, which is precisely the condition for extremely large blueshift/redshift, one would expect that coordinate time as measured by the static observer at infinity would diverge. We show that for trans-Plankian collisions the coordinate time required is of the order of million years which is much smaller than the Hubble time.

For a particle moving along a radial geodesic with $E = 1$, from [5], we have

$$\frac{dt}{d\tau} = \frac{1}{f(r)}$$

$$\frac{dr}{d\tau} = \pm \sqrt{1 - f(r)}$$

where $\tau$ is a proper time and $\pm$ corresponds to radially outgoing and ingoing particles respectively.

1. Proper time

The proper time as measured in the reference frame attached to the particle when it travels from $r = r_1$ to $r = r_f$ can be obtained by integrating the (19) and is given by

$$\tau(r_1 \rightarrow r_f) = \pm \int_{r_1}^{r_f} \frac{1}{\sqrt{1 - f(r)}} dr$$

$$= \pm \frac{1}{3} \sqrt[2]{\frac{2}{M}} \left[ \left( r - \frac{Q^2}{2M} \right)^\frac{3}{2} \left( r + \frac{Q^2}{M} \right) \right]_{r_1}^{r_f},$$

where $\pm$ corresponds to the case where $r_f > r_1$ and $r_f < r_1$, i.e., when particle moves radially onwards and radially inwards respectively.

Extremal black hole

The proper time required for the neutral particle to reach horizon from the initial location $r = r_1$ using (20) is given by

$$\tau(r_1 \rightarrow M) = \frac{1}{3} \sqrt[2]{\frac{2}{M}} \left[ \left( r - \frac{M}{2} \right)^\frac{3}{2} (r + M) \right]_{r_1}^{r_f} - \frac{2}{3} M$$

which is clearly finite. The proper time required for the charged particle to reach the horizon however diverges as discussed earlier since its effective potential for the radial motion as well as its derivative goes to zero at the horizon.

Naked singularity

In the naked singularity case, one of the particles starts out as an ingoing particle at $r = r_{\text{in}}$, gets reflected back at $r = r_{\text{eff}} = Q^2/2M$ due to the repulsive effect of the naked singularity and arrives at the collision point $r = r_f$.
\[ r_{\text{min}} = Q^2/M \] as an outgoing particle. The proper time required in its rest frame from (20) is given by
\[ \tau_1 = \tau(r_i \to r_{\text{refl}}) + \tau(r_{\text{refl}} \to r_{\text{min}}) = \frac{2Q^3}{3M^2} \left( \frac{r_i - Q^2}{2M} \right)^{\frac{3}{2}} \left( r_i + \frac{Q^2}{M} \right) \] (22)

The second particle starts out at \( r = r_1 \) and reaches \( r_{\text{refl}} \) as an ingoing particle where it collides with the first particle. The proper time required in its rest frame is given by
\[ \tau_2 = \tau(r_i \to r_{\text{refl}}) = -\frac{2Q^3}{3M^2} \left( \frac{r_i - Q^2}{2M} \right)^{\frac{3}{2}} \left( r_i + \frac{Q^2}{M} \right) \] (23)

It is evident from (22), (23) that the proper time required for the collision is finite in the rest frame of both the particles.

2. Coordinate time

We now compute the coordinate time required for the collision as measured by the static distant observer in the extremal black hole and naked singularity cases. From (19) we get
\[ \frac{dr}{dt} = \pm f(r) \sqrt{1 - f(r)} \] (24)
The time observed by the distant observer as the particle moves from \( r = r_i \) to \( r_f \) can be obtained by integrating the equation above and is given by
\[ T(r_i \to r_f) = \pm \int_{r_i}^{r_f} \frac{1}{f(r) \sqrt{1 - f(r)}} dr = \pm [B(r_f) - B(r_i)] \] (25)
where \pm stands for the radially outgoing and radially ingoing particles with \( r_f > r_i \) and \( r_i < r_f \) respectively as stated earliar, and \( B(r) \) is the indefinite integral
\[ B(r) = \int_r^\infty \frac{dr}{f(r) \sqrt{1 - f(r)}}. \]

Extremal black hole

We now compute the coordinate time required for the neutral particle to reach the event horizon of the extremal Reissner-Nordström black hole. In this case the function \( B(r) \) is given by the expression
\[ B(r) = M \ln\left[ \frac{r + \sqrt{r^2 + 2rM - Q^2}}{r + \sqrt{r^2 - 2rM + Q^2}} \right] \]
\[ + \frac{(2M^2 - Q^2)}{\sqrt{Q^2 - M^2}} \arctan\left( \frac{\sqrt{2rM - Q^2 - M}}{\sqrt{Q^2 - M^2}} \right) \]
\[ + \frac{(2M^2 - Q^2)}{\sqrt{Q^2 - M^2}} \arctan\left( \frac{\sqrt{2rM - Q^2 + M}}{\sqrt{Q^2 - M^2}} \right) \]
\[ + \frac{\sqrt{2rM - Q^2}}{3M^2} (rM + Q^2 + 6M^2) \] (30)

Thus it follows from Eqs. (25) and (26) that the time required for the ingoing neutral particle to reach \( r_f \) diverges in the limit \( r_f \to M \) as
\[ T \simeq M \left( \frac{r_f - M}{M} \right)^{-\frac{3}{2}} \] (27)

The center of mass energy of collision \( E_{\text{cm}} \) between the charged and uncharged particles as a function of the collision location \( r_f \) varies as [10]
\[ E_{\text{cm}} \simeq \sqrt{2}m \left( \frac{r_f - M}{M} \right)^{-\frac{1}{2}} \] (28)

where \( m \) is the mass of each of the colliding particles.

It follows from Eqs. (27) and (28) that the time required for the neutral particle to participate in the collision at the radial location \( r = r_f \to M \) is thus given by
\[ T \simeq \frac{M}{2} \left( \frac{E_{\text{cm}}}{m} \right)^2 \]
\[ \simeq 1.3 \times 10^{25} \left( \frac{M}{M_{\odot}} \right) \left( \frac{E_{\text{cm}}}{E_{\text{pl}}} \right)^2 \left( \frac{m_p}{m} \right)^2 \text{ yr}, \] (29)

where \( M_{\odot} \) is the solar mass, \( E_{\text{pl}} \) is Planck energy and \( m_p \) is mass of the proton. The time required for the charged particle to reach the collision point will be even larger. Therefore the phenomenon of ultra-high energy collisions around charged black holes does not occur within the Hubble time scale and thus has no observable consequences whatsoever.

Naked singularity

We now discuss the timescale associated with the ultra-high energy collision around the Reissner-Nordström naked singularity. The function \( B(r) \) in this case is given by the following expression.
\[ B(r) = M \ln\left[ \frac{r - \sqrt{2rM - Q^2}}{r + \sqrt{2rM - Q^2}} \right] \]
\[ + \frac{(2M^2 - Q^2)}{\sqrt{Q^2 - M^2}} \arctan\left( \frac{\sqrt{2rM - Q^2 - M}}{\sqrt{Q^2 - M^2}} \right) \]
\[ + \frac{(2M^2 - Q^2)}{\sqrt{Q^2 - M^2}} \arctan\left( \frac{\sqrt{2rM - Q^2 + M}}{\sqrt{Q^2 - M^2}} \right) \]
\[ + \frac{\sqrt{2rM - Q^2}}{3M^2} (rM + Q^2 + 6M^2) \] (30)

The time required for the ingoing neutral particle starting at \( r = r_i \) to get reflected at \( r = r_{\text{refl}} = Q^2/2M \) as an outgoing particle and to reach the collision point
\[ r = r_{\text{min}} = Q^2/M \] in the limit \( Q \to M \), from Eqs. (25) and (30) is given by
\[ T_1 = T(r_i \to r_{\text{refl}}) + T(r_{\text{refl}} \to r_{\text{min}}) \approx \frac{3\pi}{2} \frac{M^2}{\sqrt{Q^2 - M^2}} \] (31)

Whereas the time required for the second ingoing neutral particle to reach \( r = r_{\text{min}} \) starting from \( r = r_i \), from Eqs. (24) and (30) is given by
\[ T_2 = T(r_i \to r_{\text{min}}) \approx \frac{\pi}{2} \frac{M^2}{\sqrt{Q^2 - M^2}} \] (32)

It is clear from (31), (32) that \( T_1 \) and \( T_2 \) diverge as \( 1/\sqrt{Q^2 - M^2} \) in the limit \( Q \to M \).

The center of mass energy of collision between two particles at \( r = r_{\text{min}} = Q^2/M \) in the Reissner-Nordström naked singularity case in the limit \( Q \to M \) is given by Eq. (16)
\[ E_{\text{cm}} \approx \frac{2mM}{\sqrt{Q^2 - M^2}} \] (33)

where \( m \) is the mass of each of the colliding particles.

From Eqs. (31), (32) and (33), the time scale associated with the collision is given by
\[ T \approx T_2 \approx \frac{T_1}{3} \approx \frac{\pi}{4} \frac{M}{m} \left( \frac{E_{\text{cm}}}{m} \right) \approx 2.32 \times 10^6 \left( \frac{M}{M_{\odot}} \right) \left( \frac{E_{\text{cm}}}{E_{\text{pl}}} \right) \left( \frac{m_p}{m} \right) \text{ yr} \] (34)

where as before \( M_{\odot} \) is mass of the sun, \( E_{\text{pl}} \) is the Planck energy and \( m_p \) is mass of the proton. We see that the time scale associated with the Planck scale collision of two neutrons around a solar mass naked singularity is merely of the order of million years which is \( 10^4 \) times smaller than the age of the universe.

This implies that the trans-Plankian collisions around naked singularities are conceivably and might be observable either in our galaxy or at very high cosmological redshifts. Furthermore if the particles continuously accrete from a distant location \( r = r_i > r_{\text{min}} \), in a steady state, the rate of occurrence of the collisions will be same as the accretion rate. Thus one could say that there is no inflating time-scale problem in the naked singular Reissner-Nordström spacetime while it does exists in the extremal black hole geometry.

III. ACCELERATION OF SHELLS BY REISSNÉR-NORDSTRÖM NAkED SINGULAR GEOMETRY

In this section, we discuss the validity of test particle approximation on the particle collision around Reissner-Nordström naked singular geometry. We should consider two type of “back reaction”, i.e., the effects of the gravitational radiation and the conservative self-force.

As long as we consider the radially moving particles, the effect of gravitational radiation does not change it to non-radial one. However, if the energy of the particle is released by the gravitational radiation, initially marginally bound particle will be bound. We denote the energies of the particles by \( E_1 \) and \( E_2 \). If the gravitational emission is negligible, \( E_1 \) and \( E_2 \) are constants of motion, but it might vary with time if the gravitational emission is not negligible. The 4-velocities of the particles are written in the form
\[ U_{1}^{\mu} = \left( \frac{E_1}{f}, \sqrt{E_1^2 - f}, 0, 0 \right), \quad U_{2}^{\mu} = \left( \frac{E_2}{f}, -\sqrt{E_2^2 - f}, 0, 0 \right). \] (35)

As before, we assume that the collision occurs at the minimum of \( f \), i.e., \( f = e(2 - \epsilon) \), where \( \epsilon \) has been defined by Eq. (17), and then the collision energy at the center of mass frame is given by
\[ E_{\text{cm}}^2 = \frac{2m^2}{f} \left[ f + E_1 E_2 + \sqrt{(E_1^2 - f)(E_2^2 - f)} \right] \] (37)

Since the \( r \)-components of \( U_{1}^{\mu} \) and \( U_{2}^{\mu} \) should be real, \( E_1 \) and \( E_2 \) should be larger than or equal to \( \sqrt{e(2 - \epsilon)} \). If \( E_1 \) and \( E_2 \) become several times \( \sqrt{e(2 - \epsilon)} \) by the emission of the gravitational radiation, the collision energy \( E_{\text{cm}} \) takes small value which is several times \( m \). However note that in this case the emitted gravitational radiation would be so large that the conserved energies, which were assumed to have unit value to begin with, drastically reduce to a value that is nearly equal to zero. It is beyond the scope of this paper to estimate how large is the amount of energies of the particles are released by the gravitational radiation, and hence, we cannot make any quantitative statement. If the colliding particles do not drastically loose the energies to a value close to zero and carry a descent fraction of the initial energies, the ultra-high energy collisions below the event horizon.

In this section, since to treat conservative self-force for the point particle is difficult, we study analogous system, i.e., the collision of spherical shells in the Reissner-Nordström naked singular geometry. It is also well justified on the physical grounds, since in a realistic situation the accretion of the matter onto a massive compact object would be more or less isotropic. Therefore the amount of gravitational radiation emitted per particle will be significantly reduced and its effect on the process of ultra-high energy collisions can be ignored to a very good approximation. Thus would suffice to
consider only the conservative self-force. The dynamics of the spherical thin shells is tractable exactly owing to the spherical symmetry of the system. Due to the gravity generated by the shells themselves, the equations describing the motion of shells are no longer the geodesic equations in the Reissner-Nordström spacetime.

We deal with the situation that is analogous to the scenario described in the previous section, in order to draw a parallel to and compare with the test particle case. We assume that the deviation of the charge from the mass associated with the naked singularity is vanishingly small.

**A. Junction conditions**

We first describe the procedure to deal with the thin shells with taking into account their gravity. We basically follow notation and convention of Ref. [8]. A trajectory of a shell that is being considered here is a timelike hypersurface with a thin surface layer of matter in the four dimensional ambient spacetime manifold: we denote it by $\Sigma$. Then, a shell $S$ means an intersection between $\Sigma$ and a spacelike hypersurface with constant time coordinate chosen appropriately. Since the finite amount of energy exists within the infinitesimally thin layer, the energy-momentum tensor is infinite, but it is possible to define it as a distribution.

The geometry of a hypersurface can be described by specifying a three dimensional metric $h_{ab}$ within it (also known as the induced metric) and an extrinsic curvature $K_{ab}$, which is a three dimensional tensor describing how the hypersurface is embedded in the ambient spacetime. Even if the trajectory of the shell is a singular hypersurface, we assume that the metric of four dimensional spacetime is everywhere continuous. Thus, the induced metric $h_{ab}$ of the shell is assumed to be continuous. By contrast, the extrinsic curvature $K_{ab}$ of the shell may be discontinuous across the shell due to the distributional energy-momentum tensor on the shell. $\Sigma$ separates the spacetime in two regions $\mathcal{V}_1$ and $\mathcal{V}_2$. The coordinates defined in these two regions is denoted by $x^a_J$ ($J = 1, 2$), whereas the coordinates within $\Sigma$ is denoted by $y^a$ ($a = 0, 1, 2$). Although the metric is continuous, the components of it may not be continuous, since the coordinate systems may be discontinuous at $\Sigma$.

The projection operator from the four dimensional ambient spacetime to $\Sigma$ is given by

$$e^\mu_J = \frac{\partial x^\mu_J}{\partial y^a}.$$  \hspace{1cm} (38)

Here, the index $J$ of the projection operator indicates side of $\Sigma$ on which the quantity is defined, $\mathcal{V}_1$ or $\mathcal{V}_2$. Denoting the components of the metric by $g_{J\mu\nu}$, the induced metric on the hypersurface is given by

$$h_{ab} = g_{J\mu\nu}e^\mu_J e^\nu_J.$$  \hspace{1cm} (39)

The extrinsic curvature of the shell is given by

$$K_{ab} = e^\mu_J e^\nu_J n_{J\mu\nu}.$$  \hspace{1cm} (40)

where $n_{J\mu\nu}$ denotes a component of the covariant derivative of the unit normal vector $n_J$, to $\Sigma$, which is directed from $\mathcal{V}_1$ to $\mathcal{V}_2$. Here, note that the unit normal vector to $\Sigma$ is unique, since the metric tensor is everywhere continuous.

By denoting the components of the energy momentum tensor of the shell by $T^\mu_{\nu J}$, it is given by the following form

$$T^\mu_{\nu J} = \delta(\lambda) S_{ab} e^\mu_J e^\nu_J,$$  \hspace{1cm} (41)

where $S_{ab}$ is a three dimensional tensor defined over $\Sigma$ shell, which is called the surface energy-momentum tensor, $\delta(\lambda)$ is Dirac delta function, and $\lambda$ is the Gaussian normal coordinate which is equal to zero on $\Sigma$.

The junction condition is given in the form of the condition on the discontinuity of the extrinsic curvature of $\Sigma$ as follows;

$$K_{2ab} - K_{1ab} = -8\pi \left( S_{ab} - \frac{1}{2} h_{ab} \delta_{\nu}^c \right).$$  \hspace{1cm} (42)

**B. Motion of a neutral dust shell**

We now consider a case where the both regions $\mathcal{V}_1$ and $\mathcal{V}_2$ are the Reissner-Nordström and the shell $S$ is spherically symmetric and made up of charge neutral dust. Due to the charge neutrality of the shell, the charge parameters in the both regions are identical, and we denote it by $Q$. By contrast, the value of the mass parameter would be different in two regions (see Fig. 2).

We use the coordinate systems $x^a_J = (t_J, r, \theta, \phi)$ and $x^a_2 = (t_2, r, \theta, \phi)$ in the region $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Note that the time coordinate is not continuous, whereas $r$, $\theta$ and $\phi$ are everywhere continuous. The metric in these two regions can be written as

$$ds^2 = -f_J(r)dt_J^2 + \frac{1}{f_J(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (43)

where

$$f_J(x) = 1 - \frac{2M_J}{x} + \frac{Q^2}{x^2}.$$  \hspace{1cm} (44)

The Misner-Sharp mass of the shell is defined by

$$\mu := M_2 - M_1,$$  \hspace{1cm} (45)

and we assume $\mu > 0$. The positivity of $\mu$ naturally introduce a picture that $\mathcal{V}_1$ is the inside of the shell $S$, whereas $\mathcal{V}_2$ is the outside.

We use the coordinates $(\tau, \theta, \phi)$ on $\Sigma$, where $\tau$ is taken to be the proper time for an observer comoving with the shell. The intrinsic metric of the shell is then written as

$$ds_{\Sigma}^2 = -d\tau^2 + R(\tau)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right).$$  \hspace{1cm} (46)
The proper time of the shell can parametrize the trajectory of the shell, i.e.,
\[ t_J = T_J(\tau) \quad \text{and} \quad r = R(\tau). \] (47)

The projection operator is then given by
\[ e^{J}_\tau = \left( \hat{T}_J, \hat{R}, 0, 0 \right), \] (48)
\[ e^{J}_\theta = \left( 0, 0, 1, 0 \right), \] (49)
\[ e^{J}_\phi = \left( 0, 0, 0, 1 \right). \] (50)

The induced metric is given by
\[ ds^2_\Sigma = g_{\mu\nu} e^\mu_{J} e^\nu_{J} d\theta^a d\phi^b \]
\[ = \left[ -f_J(R)\hat{T}_J^2 + \frac{1}{f_J(R)} \hat{R}^2 \right] d\tau^2 + R(\tau)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \] (51)

where a dot represents a derivative with respect to \( \tau \). Equations (48) and (51) imply
\[ f_J(R)\hat{T}_J = \sqrt{\hat{R}^2 + f_J(R)} =: \beta_J(R, \hat{R}). \] (52)

Equation (52) implies that the time coordinate in the regions \( \Sigma_1 \) and \( \Sigma_2 \) necessarily have to be different.

The unit normal vector \( n_{J\mu} \) to \( \Sigma \) is given by
\[ n_{J\mu} = \left( -\hat{R}, \hat{T}_J, 0, 0 \right) = \left( -\hat{R}, \frac{\beta_J}{f_J}, 0, 0 \right). \] (53)

Substituting the above expression into Eq. (48), we find that the non-vanishing components of the extrinsic curvature are given by
\[ K^\tau_J = \frac{\beta_J}{R}, \quad K^\theta_J = K^\phi_J = \frac{\beta_J}{R}. \] (54)

The energy-momentum tensor of the dust within the thin shell is given by
\[ T^{\mu\nu}_J = \sigma(\tau) \delta(\lambda) U^\mu_J U^\nu_J \] (55)

where \( \sigma \) is the surface density and \( U^\mu_J \) is a component of the 4-velocity of the dust, which is equivalent to \( e^\mu_{J\tau} \).

We assume that \( \sigma \) is non-negative. Comparing the above equation with Eq. (11), we get
\[ S^{ab} = \sigma u^a u^b, \] (56)

where \( u^a \) is the 3-velocity of the dust within the shell, whose components are given by
\[ u^a = (1, 0, 0). \] (57)

By using Eqs. (54) and (56), we now write down Eq. (42) and obtain
\[ -\sigma = \frac{\beta_2 - \beta_1}{4\pi R} \] (58)
\[ 0 = \frac{\beta_2 - \beta_1}{R} + \frac{\beta_2 - \beta_1}{R} \] (59)

Equations (58) and (59) taken together give
\[ 4\pi R^2 \sigma = m = \text{constant}. \] (60)

The constant \( m \) is interpreted as the proper mass of the shell, and we get an energy equation for the shell as follows
\[ \hat{R}^2 = \frac{1}{m^2} \left( \mu + \frac{m^2}{2R} \right)^2 - f_1(R) \]
\[ = \frac{1}{m^2} \left( \mu - \frac{m^2}{2R} \right)^2 - f_2(R) \] (61)

As in the case of the test particle, let us introduce the following effective potential
\[ V_{\text{eff}} = 1 - \frac{2\langle M \rangle}{R} + \frac{Q^2}{R^2} - \left( \frac{m}{2R} \right)^2, \] (62)

where
\[ \langle M \rangle = \frac{M_1 + M_2}{2}. \] (63)

Then, Eq. (61) is written in the very similar form to Eq. (8) for the test particle as
\[ \left( \frac{d\hat{R}}{d\tau} \right)^2 + V_{\text{eff}} = E^2, \] (64)

where \( E = \mu/m \) is the energy of the shell per unit proper mass.

We depict the effective potential \( V_{\text{eff}} \) for the shell as a function of \( \hat{R} \) in Figs. 3-6. First, we consider the case of \( m < 2Q \). For \( E < 1 \) (see Fig. 3), the motion of the shell is restricted within the domain \( B_- \leq \hat{R} \leq B_+ \), where
\[ B_{\pm} = \frac{\langle M \rangle}{1 - E^2} \left( 1 \pm \sqrt{1 - \frac{Q^2 - m^2/4}{\langle M \rangle^2} (1 - E^2)} \right). \] (65)
The effective potential $V_{\text{eff}}$ minus the square of specific energy $E^2$ of the shell is depicted for the case of $m < 2Q$ and $E < 1$. The allowed domain for the motion of the shell is specified by the dashed line.

In the case of $E = \sqrt{1 - (M^2/4)}/(M)$, the particle stays stably at rest at the radius $R = (Q^2 - m^2/4)/(M)$. For $E \geq 1$ (see Fig. 4), the allowed domain for the motion is $R \geq B_+$. Initially outgoing shell monotonically approaches to infinity, $R \to \infty$, whereas initially ingoing shell turns to be outgoing. These behaviors are basically the same as that of the test particle. The repulsive nature of the charged singularity halts the collapse of the shell.

By contrast, in the case of $m \geq 2Q$, the allowed domain for the motion is $R \leq B_+$ for $E < 1$ (see Fig. 5) and thus the shell with $E < 1$ necessarily collapses to the singularity at $r = 0$. In the case of $E \geq 1$ (see Fig. 6), whole domain is allowed for the motion of the shell; the initially outgoing shell goes to infinity, whereas the initially ingoing shell collapses to the singularity at $r = 0$. The self-gravity of the shell overcomes the repulsive gravity of the charged singularity.

We rewrite the effective potential in the form

$$V_{\text{eff}}(r) = f_2(r) + E^2 - \left(\frac{E - m^2r}{2M}\right)^2. \quad (66)$$

From the above equation, we have

$$\dot{R}^2 = -V_{\text{eff}}(R_+) + E^2 = \left(\frac{E - m^2r}{2M}\right)^2 \geq 0, \quad (67)$$

where $R_+$ is a larger root of $f_2(R_+) = 0$. Thus, in the case of $M_2 \geq Q$, an ingoing shell necessarily enters into the black hole and goes to the another asymptotically flat region. By contrast, in the case of $M_2 < Q$, there is only one asymptotically flat region, and hence even in the case of the ingoing shell, the shell remains in this asymptotically flat region as long as it does not hit the spacetime singularity at $r = 0$. The situation is similar to the case of a radially moving test particle.

### C. Collision of charge neutral shells

Now we describe the process of acceleration and collision of charge neutral shells whose motion has been considered in the preceding section. We consider two concentric spherical thin shells $S_1$ and $S_2$. These shells divide the spacetime into three regions each of which is denoted by $V_j$ ($J = 1, 2, 3$): $S_1$ faces $V_1$ and $V_2$, whereas $S_2$ faces $V_2$ and $V_3$ (see Fig. 7). The metric in the three regions would be given by Reissner-Nordström geometry.
The shells are assumed to be electrically neutral and thus marginally bound, i.e., $\mu > 0$. Using the normalization condition for 4-velocity $U_j \cdot U_j = -1$, we obtain the time components of the 4-velocity with respect to the coordinate basis in the domain $\mathcal{V}_2$,

$$
\dot{R}_j = \frac{1}{f_2(R_j)} \sqrt{\dot{R}_j^2 + f_2(R_j)}.
$$

The radial components of 4-velocities of both $S_1$ and $S_2$ would go to zero at infinity by the assumption of $\mu = m$. By carefully taking a limit $E \to 1$ for $B_-$ in Eq. (65), we have the turning points $R = B_-$ for $S_1$ and $R = B_-$ for $S_2$, where

$$
B_1^- = \frac{Q^2 - m^2/4}{2M + m} \approx \frac{Q^2}{2M} \approx \frac{M}{2},
$$

$$
B_2^- = \frac{Q^2 - m^2/4}{2(M + m)} \approx \frac{Q^2}{2M} \approx \frac{M}{2}.
$$

Both the turning points are the same order, but $B_1^- > B_2^-$. We consider a situation where the inner shell $S_1$ starts off at infinity as an outgoing shell; $S_1$ then turns back at $R \approx M/2$ and emerges as an outgoing shell and collides with the outer ingoing shell $S_2$ at $R \approx Q^2/M \approx M$. This situation is exactly analogous to the situation encountered in the case of test particles in Sec. II.

The energy of two shells at “the center of mass frame” was defined in [10] in a following way by generalizing the definition of the center of mass energy of the particles. In case of the particle collisions, in order to compute the center of mass energy, one goes to the orthonormal frame in which the spatial components of the total momentum of the two particles is zero. The time component yields the center of mass energy. While dealing with the collision event of the shells, the center of mass frame was defined to be an orthonormal frame in which the energy flux along the spatial direction is zero and the center of mass energy is defined analogously. We obtain for the shells

$$
E_{cm}^2 = 2m^2 \left(1 - U_1 \cdot U_2\right)
$$

We can compute $U_1 \cdot U_2$ by using their components with respect to the coordinate basis in the region $\mathcal{V}_2$, and we have the center of mass energy of collision at any given value of $R$ as

\[
(\frac{1}{2} + m \dot{R} \mu) - f_{j+1}(R_j) = \frac{1}{2} - f_j(R_j) \tag{70}
\]

where $f_j$ are identical to Eq. (44), and $\pm$ stands for outgoing and ingoing shell, respectively.
The circumferential radius at the the minimum of $f_2$ is $R = Q^2/M_2$, and let us consider the collision there. From Eqs. (68) and (69), we have

$$E_{cm} = 2m^2\left[1 - \frac{1}{f_2} \left(\hat{R}_1 \hat{R}_2 - \sqrt{(\hat{R}_1^2 + f_2)(\hat{R}_2^2 + f_2)}\right)\right]. \tag{74}$$

We can see from the above equation that as in the case of test particles, the energy of two spherical shells at the center of mass frame can be arbitrarily large. Here, we assume that a shell is composed of $N$ particles each of which has a mass $\delta m = m/N$. The center of mass energy $E_p$ of a collision between two of constituent particles is given by

$$E_p^2 = \frac{\delta m^2}{m^2}E_{cm}^2. \tag{77}$$

Using the above equation and Eq. (70), the collision energy at $R = Q^2/M_2$ with $0 < \epsilon \ll 1$ is given by

$$E_p^2 \simeq \frac{\epsilon^2}{(2 - \hat{\mu})\epsilon}. \tag{78}$$

The above equation seems to imply that the center of mass energy can be indefinitely large. However, in order that the description by a spherical shell is valid, the number of particles $N$ should be much larger than unity, i.e.,

$$N = \frac{m}{\delta m} = \frac{Q\hat{\mu}}{2\delta m\epsilon} = \frac{M\hat{\mu}}{2\hat{\mu}m(1 - \epsilon)} \gg 1, \tag{79}$$

or, by assuming $\delta m/M \ll 1$,

$$\epsilon \gg \frac{2\delta m}{M\hat{\mu}}. \tag{80}$$

Due to this constraint, we have

$$E_p \ll \sqrt{\frac{2\hat{\mu}}{2 - \hat{\mu}}} \delta m M \ll \sqrt{2\delta m M}$$

$$= 4.58 \times 10^{28} \left(\frac{\delta m}{m_p}\right)^{\frac{1}{2}} \left(\frac{M}{M_\odot}\right)^{\frac{1}{2}} \text{GeV}. \tag{81}$$

The above equation implies that if $M$ is of order of the solar mass $M_\odot = 1.99 \times 10^{30}$kg, the collision energy $E_p$ between particles at the center of mass frame can exceed Planck scale $m_{pl} = \sqrt{\hbar c/2G} = 2.16 \times 10^{19}$GeV even if $\delta m$ is the order of the proton mass $m_p = 0.938$GeV.

**IV. CONCLUSIONS**

In this paper, we studied the particle and shell acceleration by Reissner-Nordström naked singularities. The phenomenon of particle acceleration and collision with extremely large energy at the center of mass frame was previously studied and explored in the background of extremal and near extremal black holes. We extended this result to the near extremal naked singularities. We showed that there are significant qualitative differences in the particle acceleration mechanism between black holes and naked singularities. In case of black, the particle collision between ingoing particles should be considered, and in order to achieve large collision energy at the center of mass frame, fine tuning of parameters is necessary, and further the proper time of one of two particles required for such a collision is very long. On the contrary, in case of naked singularity, it is possible to consider a collision between ingoing and outgoing particles, since due to the absence of the event horizon and the repulsive gravity effects near singularity, initially ingoing particle turns back as an outgoing particle. This fact eliminates the necessity of the fine tuning of some parameters and also the required proper time required for such a collision need not be so long.

We also calculate the coordinate time as seen by the observer at infinity required for the ultra-high energy collisions to occur for extremal black hole as well as naked singular geometry. We show that the time required for the Planck-scale collisions around naked singularity is of the order of million years which is much smaller than the age of the universe. Whereas the time scale in extremal black hole case in the analogous process is many orders of magnitude larger than the Hubble time. Therefore the high energy collisions occurring around the naked singularities, subject to their existence will be observable. Rate of occurrence of the collisions will be same as the rate of the accretion of the matter in a steady state. On the contrary, in the black hole case high energy collisions would not occur within the Hubble time and thus would have no observational consequences.

Particles participating in the collision are assumed to be test particles following the geodesics on the background geometry. The effects of gravity generated by the particles are ignored. Thus, to study whether or not the phenomenon of divergence of center of mass energy survives, we studied the collision between the concentric spherical shells. The gravity of the shells is taken into account in an exact calculation, and the energy of collision between shells at “the center of mass frame” is computed in a situation analogous to the test particle case.
It is shown that, in this case, due to the condition that the outermost region is described by the over-charged RN spacetime, the center of mass energy of a collision between two of the constituent particles of the shells is bounded above. However, if the mass of the central naked singularity is order of the solar mass, and if the mass of a constituent particle of the shells is order of the proton mass, the upper bound exceeds $10^{28}$ GeV which is much larger than the Planck scale.

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