Phase transition of the monotonicity assumption in learning local average treatment effects

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Abstract

We consider the setting in which a strong binary instrument is available for a binary treatment. The traditional LATE approach assumes the monotonicity condition stating that there are no defiers (or compliers). Since this condition is not always obvious, we investigate the sensitivity and testability of this condition. In particular, we focus on the question: does a slight violation of monotonicity lead to a small problem or a big problem? We find a phase transition for the monotonicity condition. On one of the boundary of the phase transition, it is easy to learn the sign of LATE and on the other side of the boundary, it is impossible to learn the sign of LATE. Unfortunately, the impossible side of the phase transition includes data-generating processes under which the proportion of defiers tends to zero. This boundary of phase transition is explicitly characterized in the case of binary outcomes. Outside a special case, it is impossible to test whether the data-generating process is on the nice side of the boundary. However, in the special case that the non-compliance is almost one-sided, such a test is possible. We also provide simple alternatives to monotonicity.

1 Introduction

Instrumental variables (IV) regressions have been widely used to study treatment effects in economics and other disciplines. One important conceptual framework that justifies the causal interpretation of IV regressions is the local average treatment (LATE). In this paper, we consider a simple setting with no covariates and discuss the sensitivity of the key monotonicity assumption in the LATE framework.

We observe iid data \( \{(Y_i, D_i, Z_i)\}_{i=1}^n \), where \( Y_i = Y_i(1)D_i + Y_i(0)(1 - D_i) \), \( D_i = D_i(1)Z_i + D_i(0)(1 - Z_i) \) and \( Z_i, D_i(1), D_i(0) \in \{0, 1\} \). The treatment effect is \( Y_i(1) - Y_i(0) \). (Notice that this assumes that \( Z_i \) does not directly affect the potential outcomes: \( Y_i(z, d) = Y_i(d) \) for \( z, d \in \{0, 1\} \).) Throughout the paper, we maintain the assumption that the IV is strong (\( |\text{cov}(D_i, Z_i)| \geq C \) for a constant \( C > 0 \)) and the following exogeneity condition

Assumption 1. \( Z_i \) is independent of \( (Y_i(1), Y_i(0), D_i(1), D_i(0)) \).

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The typical IV regression exploits the moment condition \( EZ_i(Y_i - D_i \beta) = EZ_iE(Y_i - D_i \beta) \) (i.e., \( \text{cov}(Z_i, Y_i - D_i \beta) = 0 \)). This means\(^1\) that
\[
\beta = \frac{E(Y_i Z_i) - E(Y_i)E(Z_i)}{E(D_i Z_i) - E(D_i)E(Z_i)}.
\]

To describe the causal interpretation of \( \beta \), we categorize the population into four types depending on the value of \((D_i(1), D_i(0)) \in \{0, 1\} \times \{0, 1\}\). We introduce their definitions and their probability:
\[
\begin{align*}
P(D_i(1) = D_i(0) = 1) &= a \quad \text{always taker} \\
P(D_i(1) = 1, D_i(0) = 0) &= b \quad \text{complier} \\
P(D_i(1) = 0, D_i(0) = 1) &= c \quad \text{defier} \\
P(D_i(1) = 0, D_i(0) = 0) &= 1 - a - b - c \quad \text{never taker.}
\end{align*}
\]

As shown in Angrist et al. (1996),
\[
\beta = \frac{\mu_1 b - \mu_2 c}{b - c},
\]
where \( \mu_1 \) and \( \mu_2 \) are the local average treatment effects (LATE) for compliers and defiers, respectively:
\[
\begin{align*}
\mu_1 &= E(Y_i(1) - Y_i(0) \mid D_i(1) = 1, D_i(0) = 0) \\
\mu_2 &= E(Y_i(1) - Y_i(0) \mid D_i(1) = 0, D_i(0) = 1).
\end{align*}
\]

Since (2) is in general not a convex combination of \( \mu_1 \) and \( \mu_2 \), \( \beta \) typically does not have a causal interpretation without further assumptions. The classical assumption that makes \( \beta \) causally interpretable is the following monotonicity condition.

**Monotonicity condition:** either \( b = 0 \) or \( c = 0 \).

Clearly, under the monotonicity condition, (2) implies that \( \beta = \mu_1 \) or \( \beta = \mu_2 \), which can be summarized as \( \beta = E(Y_i(1) - Y_i(0) \mid D_i(1) \neq D_i(0)) \). Hence, \( \beta \) is interpreted as the average treatment effect on the sub-population for which \( D_i(1) \neq D_i(0) \).

As pointed out by Imbens (2014), perhaps the strongest justification of the monotonicity condition is when the instrument provides an incentive to choose the treatment or when the treatment is simply not an option without \( Z_i = 1 \). Outside these situations, the validity of the monotonicity condition is not always obvious. In this paper, we try to answer the following questions

- Suppose that the data can easily reject \( H_0 : \beta = 0 \). If the monotonicity is slightly violated (\( b \) is far from zero but \( c \) is close to zero), would this create a big problem or small problem for learning LATE?

- In applications with almost one-sided non-compliance \( P(D_i = 1 \mid Z_i = 0) \approx 0 \), should we worry about the interpretation of \( \beta \)?

- What are other options for learning LATE without the monotonicity condition?

\[^1\text{Under Assumption 1, } \beta \text{ can be written in other ways. For example, } \beta = \frac{E(Y_i | Z_i = 1) - E(Y_i | Z_i = 0)}{E(D_i | Z_i = 1) - E(D_i | Z_i = 0)}.\]
1.1 Background of the problem and summary of main results

Let us explain why (some of) these questions might be quite subtle and difficult although they seem to have an obvious answer at the first glance.

The majority of the paper focuses on the seemingly simple question of learning the sign of LATE (so we can answer the basic question of whether the treatment is beneficial or harmful). In particular, whether we can conclude that \( \mu_1 \) and \( \beta \) have the same sign when monotonicity is slightly violated (\( c \approx 0 \)). By rearranging (2), we have

\[
\mu_1 = \frac{c\mu_2 + (b - c)\beta}{b}.
\]

Suppose that \( \beta < 0 \). It is easy to see that \( \mu_1 < 0 \) (and thus has the same sign as \( \beta \)) if and only if \( \mu_2 < -(\beta/c)(b - c) \). Throughout the paper, we assume that \( P(|Y_i| \leq M) = 1 \) for a constant \( M > 0 \). Then the question of learning the sign of \( \mu_1 \) would seem straightforward. If \( c \to 0 \) and \( |\beta| \) and \( b \) are bounded below by a positive constant, then the threshold \( -(\beta/c)(b - c) \) tends to infinity. Since \( \mu_2 \) is bounded (due to the boundedness of \( Y_i \)), the condition of \( \mu_2 < -(\beta/c)(b - c) \) is asymptotically satisfied. Hence, the conclusion would be that no matter how slowly \( c \) goes to zero, it is asymptotically valid to conclude that \( \mu_1 \) and \( \beta \) have the same sign.

One subtly is whether modeling \( |\beta| \) as a quantity bounded below by a positive constant is an asymptotic framework that is empirically relevant. In many empirical studies, if we throw away half of the data and run the IV regression, we often do not find a statistically significant \( \beta \) anymore. Then it might be too strong to assume that \( |\beta| \) is of a much larger order of magnitude compared to the estimation noise. Moreover, statistically significance of \( \beta \) does not mean that \( |\beta| \) is bounded below by a positive constant; statistical significance is asymptotically guaranteed even if \( |\beta| \to 0 \) and \( \sqrt{n}|\beta| \to \infty \). To provide robust results that are empirically relevant, we shall allow \( |\beta| \to 0 \). In fact, the use of drifting sequences is the standard practice for establishing robust analysis in many areas of econometrics.\(^2\)

When \( |\beta| \) is allowed to tend to zero, the situation is less straightforward. When \( |b|, c \to 0 \),\(^3\) the threshold of \( -(\beta/c)(b - c) \) may or may not be tending to infinity, depending on the ratio \( |\beta|/c \). This paper tries to find out how worried we should be about \( c \to 0 \) (but \( c \neq 0 \)) in this case. It turns out that the answer depends on whether \( P(D_i = 1 | Z_i = 0) \) is close to zero or not. We now explain our findings. Let us try to construct a confidence set for the sign of \( \mu_1 \), i.e., a mapping from the data to a subset of \( \{-1, 0, 1\} \).

The case with \( P(D_i = 1 | Z_i = 0) \) being far away from zero is common, e.g., \( P(D_i = 1 | Z_i = 0) > 30\% \) in Angrist and Evans (1998). The question in this case is whether a slight violation of monotonicity is a big deal. From the discussion above, it is obvious that it is not a big deal if \( |\beta|/c \to \infty \). The natural way to proceed is to construct a test or a data-dependent check. If the test or data check suggests that monotonicity might be a problem, then use \( \{-1, 0, 1\} \) as the confidence set; if the test suggests otherwise, then use the more informative set \( \{-1\} \) (because \( \beta < 0 \)). This overall procedure has an answer in every situation, regardless

\(^2\)Examples include weak instruments (e.g., Staiger and Stock (1997)), local-to-unit-root process (e.g., Stock (1991)), estimation on the boundary (e.g., Andrews (1999)), model selection (e.g., Leeb and Pötscher (2005)), moment inequalities (e.g., Andrews and Guggenberger (2009)) and time series forecasting (e.g., Hirano and Wright (2017)) among others.

\(^3\)Under strong IV condition (say \( \text{cov}(D_i, Z_i) > 0 \), \( b \gtrsim \text{cov}(D_i, Z_i) \), which is bounded below by a positive constant.
of whether violation of monotonicity is a big deal. However, we show that if this procedure is robust (i.e., valid with or without monotonicity), then it must be uninformative (contains both −1 and 1) under monotonicity (c = 0). Notice that this is true no matter how sophisticated the test is. Therefore, although small enough violation of monotonicity (|β|/c → ∞) does not cause a problem, we cannot really check whether potential violation of monotonicity is small enough. As a result, if β is statistically significant and the violation of monotonicity tends to zero, this violation may or may not cause a problem, and we show that no data-dependent procedure is smart enough to find out (even after imposing constraints such as μ1 and μ2 having the same sign and both have magnitude at least |β| plus strong distributional restrictions such as Bernoulli).

Another common case is \( P(D_i = 1 \mid Z_i = 0) \approx 0 \). This is typical when the non-compliance is almost one-sided, e.g., \( P(D_i = 1 \mid Z_i = 0) < 2\% \) in the example of Job Training Partnership Act (JPTA). Of course, \( c \to 0 \) would still cause a problem if |β|/c → 0. However, since \( P(D_i = 1 \mid Z_i = 0) = a + c \geq c \), we can at least carve out a “safe” region based on the data. For example, if \( |β|/P(D_i = 1 \mid Z_i = 0) \to ∞ \), then |β|/c → ∞ and thus violation of monotonicity does not cause a problem. Notice that |β|/|P(D_i = 1 \mid Z_i = 0) → ∞ is testable since both |β| and \( P(D_i = 1 \mid Z_i = 0) \) can be learned from the data. In the case of binary outcomes, we provide a precise characterization of the “safe” region. It turns out that this “safe” region also highlights a sharp contrast. If the data-generating process is in the “safe” region, μ1 and β have the same sign; otherwise, the impossibility result from before holds.

We refer to this sharp contrast as a phase transition. On side of the boundary, learning the sign of μ1 is trivial, whereas it is impossible on the other side of the boundary. There is little or nothing in the middle. This is the case no matter whether \( P(D_i = 1 \mid Z_i = 0) \) is far away from or close to zero. The difference is that in the former case, it is impossible to find out on which side of the phase-transition bound the data-generating process is; the testability is possible in the latter case. In the former case, we still provide a precise characterization of the phase transition at least for binary outcomes because it is useful for robustness checks. For example, suppose that the boundary of the phase transition is 0.5% of defiers. Although it is impossible to check which side of the boundary the data-generating process is, it is still important to know that a mere 1% of defiers would put the data-generating process on the “dangerous” side of the phase-transition boundary.

We also outline other ways of learning LATE. We show that the magnitude of μ1 and μ2 is bounded below by |β| · γ, where γ can be consistently estimated and satisfies γ ∝ |cov(D_i, Z_i)|. We also show that imposing |μ1| ≥ |μ2| is enough to identify the sign of LATE. These results do not rely on monotonicity at all.

1.2 Related literature

The literature of IV regressions has a long history dating back to at least Wright (1928). An excellent review on this vast literature can be found in Imbens (2014). The framework of LATE was started by the seminal work of Imbens and Angrist (1994), Angrist et al. (1996) and Abadie (2003). Since then the LATE-type idea has also been explored in the study of quantile treatment effects, e.g., Abadie et al. (2002) and Wüthrich (2020). The framework of LATE fueled many empirical work ever since the early influential studies including Angrist (1991) and Angrist and Evans (1998). The nature of monotonicity condition has been discussed for decades, e.g., Robins (1989), Balke and Pearl (1995), Vytlacil
Table 1: Some estimates in two empirical studies

|               | \(P(D_i = 1)\) | \(P(D_i = 1 \mid Z_i = 1)\) | \(P(D_i = 1 \mid Z_i = 0)\) |
|---------------|-----------------|-----------------------------|-----------------------------|
| JPTA          | 0.6662          | 0.6228                      | 0.0112                      |
| Angrist and Evans (1998) | 0.5048          | 0.4105                      | 0.3557                      |

(2002) and Heckman and Vytlacil (2005). Since there is not always an obvious justification for the monotonicity condition, various specification tests and alternatives have been proposed, see Huber and Mellace (2015), Kitagawa (2015), Mourifié and Wan (2017), De Chaisemartin (2017) and Słoczyński (2020) among many others. Another interesting approach focuses on the partial identification of average treatment effects or other quantities under various restrictions, see Balke and Pearl (1997), Manski (2003), Swanson et al. (2018) and Machado et al. (2019) among many others.

2 Learning the sign of LATE

In the rest of the paper, we use the following notation. For \(x \in \mathbb{R}\), let

\[
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0.
\end{cases}
\]

In Table 1, we consider two empirical studies. In the JPTA study, the treatment \(D_i\) is job training and \(Z_i\) is the indicator of the randomized offer of training and the treat. In the example of Angrist and Evans (1998), we consider case with \(D_i\) being the indicator of being more than 2 children and \(Z_i\) being the indicator of same sex in the first two children.

We use these two studies to illustrate the two cases. In Angrist and Evans (1998), \(P(D_i = 1 \mid Z_i = 0)\) is not close to zero. Although the interpretation of the result is clear under the monotonicity condition (\(c = 0\)), what if we have a small proportion of defiers (\(c \approx 0\))? We consider this setting in Section 2.1. In the JPTA study, \(P(D_i = 1 \mid Z_i = 0)\) is close to zero, but does this mean that we do not need to worry? We provide analysis for this setting in Section 2.2. We state most of the theoretical results for \(\beta < 0\), but results for \(\beta > 0\) can be obtained analogously.

2.1 Small violation of monotonicity: \(P(D_i = 1 \mid Z_i = 0) \gg 0\)

For simplicity, we assume that the distribution of \(Z_i \in \{0, 1\}\) is known. We first introduce notations for the distribution of \((Y_i(1), Y_i(0), D_i(1), D_i(0))\). We specify the distribution of \((D_i(1), D_i(0))\) and then the conditional distribution of \((Y_i(1), Y_i(0)) \mid (D_i(1), D_i(0))\). The former is straight-forward; we simply use the same notation \(a, b, c\) as in (1). Define the conditional distribution

\[
H(y_1, y_0, d_1, d_0) = P(Y_i(1) \leq y_1 \text{ and } Y_i(0) \leq y_0 \mid D_i(1) = d_1, D_i(0) = d_0).
\]
Let $\theta = (a, b, c, H)$. Then the distribution of $(Z_i, Y_i(1), Y_i(0), D_i(1), D_i(0))$ is indexed by $\theta$. Let $P_{\theta}$ and $E_{\theta}$ denote the distribution and expectation under $\theta$, respectively. The following quantities can be written as a function of $\theta$:

- LATE for compliers: $\mu_1(\theta) = E_{\theta}(Y_i(1) - Y_i(0) \mid D_i(1) = 1, D_i(0) = 0)$
- LATE for defiers: $\mu_2(\theta) = E_{\theta}(Y_i(1) - Y_i(0) \mid D_i(1) = 0, D_i(0) = 1)$

From the data $W = \{(Y_i, D_i, Z_i)\}_{i=1}^n$, we can identify the following quantities:

- $k_1 = a + b = E(D_i \mid Z_i = 1)$
- $k_2 = a + c = E(D_i \mid Z_i = 0)$
- $\beta = [\mu_1(\theta)b - \mu_2(\theta)c]/(b - c)$.

Assuming that these three quantities are known, consider the following parameter space:

$$\Theta(\eta) = \left\{ \theta = (a, b, c, H) : a, b, c \in [0, 1], a + b + c \in [0, 1], a + b = k_1, a + c = k_2, \right.$$  

$$\left. \max_{d, z \in \{0, 1\}} P_{\theta}(|Y_i| \geq M \mid D_i = d, Z_i = z) = 0, \frac{\mu_1(\theta)b - \mu_2(\theta)c}{b - c} = \beta, \right.$$  

$$\left. |\mu_1(\theta)| \geq |\beta|, \ \text{sign}(\mu_1(\theta)) = \text{sign}(\mu_2(\theta)), \ 0 \leq c \leq \eta \right\},$$

where $M > 0$ is a constant.

Clearly, $\Theta(\eta)$ assumes a lot of structures that are typically unavailable in practice. In particular, it assumes that $P(D_i = 1 \mid Z_i = 1)$, $P(D_i = 1 \mid Z_i = 0)$ and the population IV regression coefficient $\beta$ are known. Moreover, it assumes that the LATE for the compliers and defiers has the same sign and that the magnitude of LATE for compliers is not too small. The only difficulty is that $c$ (proportion of defiers) might not be exactly zero and is allowed to be between 0 and a small tolerance level $\eta$. The point of this subsection is to show that even under these additional assumptions, allowing for a small $\eta$ makes it impossible to learn the sign of LATE. To make this point, we show that many data generating processes with no defiers and $\text{sign}(\mu_1) = \beta$ are observationally equivalent to those with a small proportion of defiers and $\text{sign}(\mu_1) \neq \beta$.

To formally state this, we define the following subset

$$\Theta_* = \{ \theta = (a, b, c, H) \in \Theta(\eta) : c = 0, Q_{1,\theta}(1 - \varepsilon_1) - Q_{2,\theta}(\varepsilon_1) > \varepsilon_2 \},$$

where $\varepsilon_1, \varepsilon_2 > 0$ are constants and $Q_{1,\theta}$ and $Q_{2,\theta}$ are the quantile functions of $Y_i \mid (D_i = 1, Z_i = 0)$ and $Y_i \mid (D_i = 0, Z_i = 1)$ under $P_{\theta}$, respectively; in other words, for any $\varepsilon \in (0, 1)$,

$$Q_{1,\theta}(\varepsilon) = \inf \{ t \in \mathbb{R} : P_{\theta}(Y_i \leq t \mid D_i = 1, Z_i = 0) \geq \varepsilon \}$$

and

$$Q_{2,\theta}(\varepsilon) = \inf \{ t \in \mathbb{R} : P_{\theta}(Y_i \leq t \mid D_i = 0, Z_i = 1) \geq \varepsilon \}.$$
Remark 1. The condition of $Q_{1,\theta}(1-\varepsilon_1) - Q_{2,\theta}(\varepsilon_1) > \varepsilon_2$ is not very restrictive. When $Y_i$ is binary in $\{0,1\}$, $Q_{1,\theta}(1-\varepsilon_1) - Q_{2,\theta}(\varepsilon_1) > \varepsilon_2$ holds if

$$E_\theta(Y_i \mid D_i = 1, Z_i = 0), E_\theta(Y_i \mid D_i = 0, Z_i = 1) \in (\varepsilon_1, 1-\varepsilon_1).$$

This seems to be reasonable since it might be a bit unrealistic to expect extreme situations with $E_\theta(Y_i \mid D_i = 1, Z_i = 0) \to 0$ or $E_\theta(Y_i \mid D_i = 0, Z_i = 1) \to 1$.

We now state the key observation.

Theorem 1. Let $M, \varepsilon_2 > 0$ and $\varepsilon_1 \in (0,1)$. Assume that $\beta < 0$, $0 < \eta < \varepsilon_1 \min\{k_2, 1 - k_1, k_1 - k_2\}$ and $3|\beta|/\eta < \varepsilon_2/(k_1 - k_2)$. Then for any $\theta \in \Theta$, there exists $\tilde{\theta} \in \Theta(\eta)$ such that

1. $P_\theta$ and $P_{\tilde{\theta}}$ imply the same distribution for the observed data $(Y_i, D_i, Z_i)$
2. $\mu_1(\theta) = \beta < 0$ and $\mu_1(\tilde{\theta}), \mu_2(\tilde{\theta}) > -\beta > 0$.

Theorem 1 provides the key insight on why lack of monotonicity creates difficult issues. For a small tolerance level $\eta$, as long as $|\beta|/\eta$ is not too large, a data-generating process with no defiers would look exactly like another data-generating process with a small proportion of defiers such that LATE has different signs under the two data-generating processes.

This sheds light on one of the most common problems in IV regressions. If we reject $H_0 : \beta = 0$ and have some arguments against the presence of defiers (e.g., $Z_i$ provides more information and thus encourages $D_i = 1$), can we reliably say that $\mu_1$ is non-zero and has the same sign as $\beta$? By Theorem 1, we see that a small proportion of defiers might be enough to invalidate the result. In the asymptotic framework, rejecting $H_0 : \beta = 0$ in large samples is almost guaranteed when $|\beta| \gg n^{-1/2}$. However, even if $\eta \to 0$ (the proportion of defiers is small), the observed data is indistinguishable from a distribution with $\mu_1 \neq \text{sign}(\beta)$ when $\eta \gg |\beta|$. Therefore, a slight violation of monotonicity creates a problem if $\eta \gg |\beta|$.

The natural question is whether or not we could check $\eta \gg |\beta|$ in the data. Unfortunately, the answer is no. We now show this using an adaptivity argument based on Theorem 1. We define a confidence set of $\mu_1$ to be any measurable function mapping the observed data $W_n$ to a subset of $\{-1,0,1\}$ with a guarantee on the coverage probability.

Corollary 1. Let $M, \varepsilon_1, \varepsilon_2, k_1, k_2 > 0$ be any fixed constants such that $k_1 - k_2 > 0$. Assume that $\beta < 0$, $\eta \to 0$ and $|\beta| \to 0$ such that $|\beta| \ll \eta$. Let $CS(W_n)$ be a confidence set for $\text{sign}(\mu_1(\theta))$ with validity over $\Theta(\eta)$, i.e.,

$$\liminf_{n \to \infty} \inf_{\theta \in \Theta(\eta)} P_\theta (\text{sign}(\mu_1(\theta)) \in CS(W_n)) \geq 1 - \alpha,$$

where $\alpha \in (0,1)$. Then

$$\liminf_{n \to \infty} \inf_{\theta \in \Theta(\eta)} P_\theta (\{-1,1\} \subset CS(W_n)) \geq 1 - 2\alpha.$$

The assumption of $k_2 = P(D_i = 1 \mid Z_i = 0)$ being fixed models the situation of $P(D_i = 1 \mid Z_i = 0)$ being far from zero. The strong IV condition corresponds to the requirement of $k_1 - k_2$ being fixed and positive. There are some important implications of Corollary 1. The following discussions are for $\eta \to 0$. The same argument obvious holds if $\eta = k_2$.

First, when $n^{-1/2} \ll |\beta| \ll \eta \ll 1$, the data allows us to distinguish $|\beta|$ from zero, but it is still impossible to consistently estimate the sign of $\mu_1$ over $\Theta(\eta)$. To see this, consider an
argument by contradiction. Suppose that there exists a consistent estimator, a function σ that
maps \( W_n \) to \{-1, 1, 0\} and \( \inf_{\theta \in \Theta(\eta)} P_{\theta}(\mu_1(\theta) = \sigma(W_n)) \geq 1 - o(1) \). In other words, \( \sigma(W_n) \)
is only one value in \{-1, 0, 1\}. Then we can set \( CS(W_n) = \{\sigma(W_n)\} \) and the assumption of
Corollary 1 holds with an arbitrary \( \alpha \), say \( \alpha = 0.05 \). The conclusion of Corollary 1 says that
with asymptotic probability at least 90%, \( CS(W_n) = \{\sigma(W_n)\} \) contains at least two elements,
which is impossible since by construction \( \{\sigma(W_n)\} \) is always a singleton. Hence, no consistent
estimator for \( \text{sign}(\mu_1(\theta)) \) exists on \( \Theta(\eta) \) when \( n^{-1/2} \ll |\beta| \ll \eta \ll 1 \).

Second, clever specification tests (for monotonicity or \( \eta \gg |\beta| \)) or other data-dependent
procedures might not be able to address the instability arising from a slight violation of the
monotonicity condition. One common purpose of specification tests is to allow us to handle the
problem based on the result of the tests. For example, when the test tells us the monotonicity
fails, we use a cautious set, say \{-1, 0, 1\}, as the confidence set for \( \text{sign}(\mu_1) \); when the test
tells us that the monotonicity holds, we use \( \{\text{sign}(\beta)\} \) as the confidence set for \( \text{sign}(\mu_1) \). Then
by Corollary 1, if this confidence set has uniform validity\(^4\) over \( \Theta(\eta) \), the confidence set must be
uninformative for \( \text{sign}(\mu_1) \) on the nice set \( \Theta_* \). If this confidence set does not have uniform
validity over \( \Theta(\eta) \), then one might question why we want to use a specification test in the first
place. Therefore, for the purpose of learning \( \text{sign}(\mu_1) \), even if we know that \( |\mu_1(\theta)| \gg n^{-1/2} \),
the monotonicity condition is not really testable even when the alternative is only a slight
violation of monotonicity \( (\eta \to 0) \).

Third, Corollary 1 implies a severe lack of adaptivity. It states that it is impossible to be
valid over the bigger set \( \Theta(\eta) \) while maintaining efficiency on the nice set \( \Theta_* \). Hence, requiring
validity over \( \Theta(\eta) \) necessarily causes loss of efficiency on \( \Theta_* \). Notice that the loss of efficiency is
not on some points in \( \Theta_* \). The efficiency loss occurs at every point in \( \Theta_* \); note that the
second inequality in Corollary 1 has \( \inf_{\theta \in \Theta_*} \) rather than \( \sup_{\theta \in \Theta_*} \). Therefore, the trade-off of
robustness and efficiency is quite stark.

The condition of \( \eta \gg |\beta| \) turns out to define the boundary of a “phase transition”. We have
seen that if we allow for \( \eta \gg |\beta| \), it is impossible to actually learn \( \text{sign}(\mu_1) \). On the other hand,
we can show that if \( \eta \ll |\beta| \), learning the sign of LATE is trivial: \( \text{sign}(\mu_1) = \text{sign}(\beta) \). To see
this, notice that \( |\mu_2(\theta)| \leq 2M \) (since \( P_{\theta}(|Y_i| \leq M) = 1 \)). Since \( \beta = (\mu_1(\theta) b - \mu_2(\theta) c)/(b - c) \),
\begin{align*}
\mu_1(\theta) &= \lambda \mu_2(\theta) + (1 - \lambda) \beta \\
&\leq 2M \lambda + (1 - \lambda) \beta,
\end{align*}
with \( \lambda = c/b \). Since \( \lambda = c/(k_1 - k_2 + c) \) and \( c \ll |\beta| \), we have that \( \lambda = o(|\beta|) \). This means that
\begin{align*}
\mu_1(\theta) &\leq o(|\beta|) + (1 - o(|\beta|)) \beta \\
&= \beta(1 + o(1)).
\end{align*}
By \( \beta < 0 \), we have \( \text{sign}(\mu_1(\theta)) = \text{sign}(\beta) \) asymptotically. We now summarize these results.

**Theorem 2 (Phase transition).** Let \( M, \varepsilon, \varepsilon_1, k_1, k_2 > 0 \) be any fixed constants such that
\( k_1 - k_2 > 0 \). Assume that \( \beta < 0, \eta \to 0 \) and \( |\beta| \to 0 \).

(1) If \( \eta \gg |\beta| \), then for any \( CS(W_n) \) satisfying
\begin{align*}
\liminf_{n \to \infty} \inf_{\theta \in \Theta(\eta)} P_{\theta}(\text{sign}(\mu_1(\theta)) \in CS(W_n)) \geq 1 - \alpha,
\end{align*}
with \( \alpha \in (0, 1) \), we have
\begin{align*}
\liminf_{n \to \infty} \inf_{\theta \in \Theta_*} P_{\theta}(\{-1, 1\} \subset CS(W_n)) \geq 1 - 2\alpha.
\end{align*}

\(^4\)One might wonder whether the requirement of uniform validity is too stringent. It turns out that a similar
result holds even if we replace uniform validity with pointwise validity.
(2) If \( \eta \ll |\beta| \), then
\[
\liminf_{n \to \infty} \inf_{\theta \in \Theta(\eta)} P_\theta(\text{sign}(\mu_1(\theta)) = \text{sign}(\beta)) = 1.
\]

By Theorem 2, the magnitude of \( |\beta| \) serves as the boundary (in rate) of phase transition. A slight violation may or may not be a huge problem depending on the order of magnitude of the violation \( \eta \). If \( |\beta| \ll \eta \), even imposing the extra condition of \( \text{sign}(\mu_1(\theta)) = \text{sign}(\mu_2(\theta)) \) does not help with learning the sign of \( \mu_1 \). In contrast, if \( |\beta| \gg \eta \), we can easily learn \( \text{sign}(\mu_1(\theta)) \) without assuming \( \text{sign}(\mu_1(\theta)) = \text{sign}(\mu_2(\theta)) \); the proof of the second part of Theorem 2 does not rely on this condition.

Since \( \Theta(\eta) \) allows for a large class of distributions, it is difficult to say much more than the rate. However, when the outcome variable is binary, we can precisely determine the boundary for the phase transition.

### 2.1.1 Exact boundary of the phase transition for binary outcomes

We define the counterparts of \( \Theta(\eta) \) and \( \Theta_* \) for the binary outcomes. Let

\[
\Theta_{binary}(\eta) = \left\{ \theta = (a, b, c, H) : a, b, c \in [0, 1], \ a + b + c \in [0, 1], \ a + b = k_1, \ a + c = k_2, \ P_\theta(Y_i = D_i = 1 \mid Z_i = 0), P_\theta(Y_i = D_i = 0 \mid Z_i = 1) \geq \varepsilon, \ P_\theta(Y_i \in \{0, 1\}) = 1, \ 0 \leq c \leq \eta \right\},
\]

where \( \varepsilon > 0 \) is a constant. The requirement that \( P_\theta(Y_i = D_i \mid Z_i) \) be bounded away from zero and one is mild in many applications.

**Theorem 3.** Let \( k_1, k_2, \varepsilon \in (0, 1) \) be given constants such that \( k_1 - k_2 > 0 \). Suppose that \( \eta \in [0, k_2] \) and \( \beta \) satisfy \( \beta < 0, \ |\beta| \to 0 \) and \( \eta \to 0 \).

(1) If \( \eta \geq |\beta|(k_1 - k_2) \), then there does not exist any estimator of \( \mu_1(\theta) \) that is consistent uniformly over \( \Theta_{binary}(\eta) \).

(2) If \( \eta < |\beta|(k_1 - k_2) \), then \( \text{sign}(\mu_1(\theta)) = \text{sign}(\beta) \) for any \( \theta \in \Theta_{binary}(\eta) \).

When the outcome variable is not binary, we can dichotomize it to binary variables. We can define the new outcome variable \( \tilde{Y}_i(1) = 1\{Y_i(1) \geq y\} \) and \( \tilde{Y}_i(0) = 1\{Y_i(0) \geq y\} \), where \( y \) is given. Then the treatment effect is how much the treatment changes the probability of \( Y_i \geq y \).

For example, in Angrist and Evans (1998), one outcome variable of interest \( Y_i \) is the number of weeks a person worked in a year. We can set \( y = 1 \) and ask how the treatment changes the probability of a person working for at least one week. Once we do this, we can estimate \( |\beta|(k_1 - k_2) \), the boundary of phase transition. The results are in Table 2. We see that any tolerance level of \( c \) above 0.52\% can cause a serious problem for the question of whether or not the LATE is negative. Hence, even if the proportion of defiers is known to be at most 1\%, it might not be obvious that we can safely conclude a negative LATE.

### 2.2 What about \( P(D_i = 1 \mid Z_i = 0) \approx 0? \)

In many studies, the absence of defiers is justified by the one-sidedness of non-compliance. For example, \( Z_i \in \{0, 1\} \) is the randomly assigned treatment and \( D_i \) is the actually treatment
Table 2: Estimating the phase-transition boundary using data in Angrist and Evans (1998)

| $\beta$ | $|\beta|(k_1 - k_2)$ |
|---------|------------------|
| -0.0950 | 0.0052           |

The outcome variable is whether or not a person worked for at least one week in the year.

status. When the compliance is not perfect (i.e., $P(Z_i = D_i) < 1$), the non-compliance is often one-sided: $P(D_i = 1 \mid Z_i = 1) < 1$ but $P(D_i = 1 \mid Z_i = 0) = 0$. However, we discuss a small violation to this ideal case $P(D_i = 1 \mid Z_i = 0) \approx 0$, see JPTA in Table 1 as an example.

Here, we provide a discussion for the case of $k_2 = P(D_i = 1 \mid Z_i = 0) \rightarrow 0$ in the case of binary outcomes. This is different from Theorem 3, which assumes that $k_2$ is bounded away from zero. Moreover, when $k_2 \rightarrow 0$, the natural choice of $\eta$ is $\eta = k_2$. To analyze this case, we consider the following parameter space

$$\Theta_{\text{binary,*}} = \left\{ \theta = (a, b, c, H) \in \Theta_{\text{binary}}(k_2) : P_\theta(Y_i = D_i = 1 \mid Z_i = 0) < |\beta|(k_1 - k_2) \right\},$$

where $\Theta_{\text{binary}}(\cdot)$ is defined in (3).

**Theorem 4.** Let $k_1, \varepsilon \in (0, 1)$ be given constants. Suppose that $k_2, |\beta| \rightarrow 0$ and $\beta < 0$.

1. there does not exist any estimator of $\mu_1(\theta)$ that is consistent over $\Theta_{\text{binary}}(k_2) \setminus \Theta_{\text{binary,*}}$.
2. $\text{sign}(\mu_1(\theta)) = \text{sign}(\beta)$ for any $\theta \in \Theta_{\text{binary,*}}$.

We notice that $P_\theta(Y_i = D_i = 1 \mid Z_i = 0) < |\beta|(k_1 - k_2)$ (the boundary in Theorem 4) is not the same as applying $\eta = k_2$ to Theorem 3. Applying $\eta = k_2$ to $\eta < |\beta|(k_1 - k_2)$ in Theorem 3 leads to $k_2 < |\beta|(k_1 - k_2)$. However, $P_\theta(Y_i = D_i = 1 \mid Z_i = 0) \leq k_2$. To see this, observe that $P_\theta(Y_i = D_i = 1 \mid Z_i = 0) = E_\theta(Y_i \mid D_i = 1 \mid Z_i = 0) \leq E_\theta(Y_i \mid D_i = 0 \mid Z_i = 0) = k_2$. What this means in practice is that Theorem 4 makes it easier to be on the “nice” side of the boundary; instead of requiring $|\beta|(k_1 - k_2)$ to be above $k_2$, we require it to be above $P_\theta(Y_i = D_i = 1 \mid Z_i = 0)$.

A more important implication of Theorem 4 is that it is possible to check whether or not we are on the nice side of the phase-transition boundary. The set $\Theta_{\text{binary,*}}$ is defined by the testable condition

$$P_\theta(Y_i = D_i = 1 \mid Z_i = 0) < |\beta|(k_1 - k_2).$$

We apply this to the JPTA study. We set the outcome variable to be $1\{\text{income < $50000}\}$. This means that we study the effect of treatment on the probability of earning less than $50000$. The results are in Table 3. Based on only the point estimates, the data-generating process is in $\Theta_{\text{binary,*}}$, which is the “nice” side of the phase-transition boundary.

### 3 Learning LATE without monotonicity: an alternative

There are already alternatives to the monotonicity condition in the literature. We add the following discussions.

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5We choose “less than” instead of “more than” to get a negative $\beta$. The interpretation is intuitively the same. Receiving treatment makes it less likely to earn less than $50000$ so it makes it more likely to earn at least $50000$. 

Table 3: Estimating the phase-transition boundary using data in JPTA

|                          | Point estimate |
|--------------------------|----------------|
| \( \beta \)             | -0.0363        |
| \( |\beta| (k_1 - k_2) \)  | 0.0222         |
| \( P_0(Y_i = 1, D_i = 1 \mid Z_i = 0) \) | 0.0157 |

The outcome variable is whether or not a person earns less than $50000 a year.

3.1 Learning the magnitude without any additional assumption

We now outline a lower bound for the magnitude of LATE.

**Theorem 5.** Let Assumption 1 hold. Assume \( \text{cov}(D_i, Z_i) \neq 0 \). Then

\[
\max\{|\mu_1|, |\mu_2|\} \geq |\beta| \cdot \gamma,
\]

where

\[
\gamma = \frac{|E(D_i \mid Z_i = 1) - E(D_i \mid Z_i = 0)|}{E(D_i \mid Z_i = 1) + E(D_i \mid Z_i = 0)}.
\]

Notice that \( \gamma \geq |\text{cov}(D_i, Z_i)| \). Therefore, in the case of strong instruments, the lower bound \( |\beta| \cdot \gamma \) is not too small compared to \( |\beta| \). From the proof, we can see that the lower bound is also tight in that the equality can hold (because the minimum in the proof can be achieved).

It is worth noting that the lower bound in Theorem 5 can be related to intent-to-treat (ITT) effects. We notice that

\[
|\beta| \cdot \gamma = \frac{|E(Y_i \mid Z_i = 1) - E(Y_i \mid Z_i = 0)|}{E(D_i \mid Z_i = 1) + E(D_i \mid Z_i = 0)} = \frac{|\text{ITT}|}{E(D_i \mid Z_i = 1) + E(D_i \mid Z_i = 0)}.
\]

Therefore, the lower bound satisfies \( |\beta| \cdot \gamma \geq |\text{ITT}|/2 \). Therefore, whenever we find that \( \beta \neq 0 \) or \( \text{ITT} \neq 0 \), it means that the treatment effect is not zero and we can use \( |\beta| \cdot \gamma \) as a lower bound.

3.2 Learning the sign: \( |\mu_1| \geq |\mu_2| \)

Learning the sign of LATE requires extra restrictions. This is inevitable; otherwise, monotonicity would be testable in Section 2.1.1. It turns out that simple restrictions such as \( |\mu_1| \geq |\mu_2| \) would suffice.

**Theorem 6.** Let Assumption 1 hold. Suppose that \( \beta \neq 0 \) and \( \text{cov}(D_i, Z_i) > 0 \). If \( |\mu_1| \geq |\mu_2| \), then \( \text{sign}(\mu_1) = \text{sign}(\beta) \).

We should notice that Theorem 6 imposes more than \( |\mu_1| \geq |\mu_2| \). The assumption of \( \text{cov}(D_i, Z_i) > 0 \) is not without loss of generality since the condition of \( |\mu_2| \geq |\mu_1| \) is not enough to identify the sign of LATE. Therefore, the result should be viewed in the context of the empirical application.
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A Proofs

A.1 Proofs of Theorem 1 and Corollary 1

We first state an auxiliary result. Since it is a result of elementary computations, the proof is omitted.

**Lemma 1.** Let $\theta = (a, b, c, H)$. Then $P_\theta(D_i = 1 \mid Z_i = 1) = a + b$, $P_\theta(D_i = 1 \mid Z_i = 0) = a + c$, and

\[
\begin{align*}
P_\theta(Y_i \leq y \mid D_i = 1, Z_i = 1) &= F_{1,1}(y) \frac{a}{a+b} + F_{1,0}(y) \frac{b}{a+b}, \\
P_\theta(Y_i \leq y \mid D_i = 1, Z_i = 0) &= F_{1,1}(y) \frac{a}{a+c} + F_{0,1}(y) \frac{c}{a+c}, \\
P_\theta(Y_i \leq y \mid D_i = 0, Z_i = 1) &= G_{0,1}(y) \frac{c}{1-a-b} + G_{0,0}(y) \frac{1-a-b-c}{1-a-b},
\end{align*}
\]
implies that

Moreover, $F_{d_1,d_0}(y) = H(y, \infty, d_1, d_0) = P_\theta(Y_i(1) \leq y \mid D_i(1) = d_1, D_i(0) = d_0)$ and $G_{d_1,d_0}(y) = H(\infty, y, d_1, d_0) = P_\theta(Y_i(0) \leq y \mid D_i(1) = d_1, D_i(0) = d_0)$.

**Proof of Theorem 1.** Fix an arbitrary $\theta = (a, b, c, H) \in \Theta_\ast$. Let $F_{d_1,d_0}$ and $G_{d_1,d_0}$ be the distribution implied by $H$; see Lemma 1. By the definition of $\Theta_\ast$, we have $a = k_2, b = k_1 - k_2$ and $c = 0$. The rest of the proof proceeds in three steps.

**Step 1:** define $\tilde{\theta} = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{H})$

We choose $\tilde{c} = \eta, \tilde{b} = k_1 - k_2 + \tilde{c}$ and $\tilde{a} = k_2 - \tilde{c}$. We choose $\tilde{H}(y_1, y_2, d_1, d_0) = \tilde{F}_{d_1,d_0}(y)\tilde{G}_{d_1,d_0}(y)$ with $\tilde{F}_{d_1,d_0}$ and $\tilde{G}_{d_1,d_0}$ chosen as follows.

Since $\varepsilon_2 > 3(k_1 - k_2)|\beta|/\eta$, there exists $\delta$ such that $0 < \delta < \varepsilon_2 - 3(k_1 - k_2)|\beta|/\eta$. Let $B_1 = Q_{1,\theta}(1 - \varepsilon_1) - \delta$. By $c = 0$, Lemma 1 implies that $P_\theta(Y_i(1) \leq y \mid D_i = 1, Z_i = 0) = F_{1,1}(y)$. Since $Q_{1,\theta} = \text{the quantile function of } Y_i \mid (D_i = 1, Z_i = 0)$, it is the quantile function of $F_{1,1}$. Since $B_1 < Q_{1,\theta}(1 - \varepsilon_1)$, we have $F_{1,1}(B_1) < 1 - \varepsilon_1$. By $\tilde{c}/k_2 = \eta/k_2 < \varepsilon_1$, we have $F_{1,1}(B_1) < 1 - \varepsilon_1 < 1 - \tilde{c}/k_2$. Define

$$\tilde{F}_{0,1}(y) = \frac{F_{1,1}(y) - F_{1,1}(B_1)}{1 - F_{1,1}(B_1)} \cdot 1\{y \geq B_1\}.$$ 

Clearly, $\tilde{F}_{0,1}$ is a legitimate cumulative distribution function (cdf) for a random variable with support in $[-M, M]$, i.e., non-decreasing and right-continuous.

Let $B_2 = Q_{2,\theta}(\varepsilon_1)$. By $c = 0$, Lemma 1 implies that $P_\theta(Y_i \leq y \mid D_i = 0, Z_i = 1) = G_{0,0}(y)$. Since $Q_{2,\theta} = \text{the quantile function of } Y_i \mid (D_i = 0, Z_i = 1)$, it is the quantile function of $G_{0,0}$. Since $\tilde{c}/(1 - k_1) = \eta/(1 - k_1) < \varepsilon_1$, we have that $G_{0,0}(B_1) \geq \varepsilon_1 > \tilde{c}/(1 - k_1)$. Define

$$\tilde{G}_{0,1}(y) = \frac{G_{0,0}(y)}{G_{0,0}(B_2)} \cdot 1\{y \leq B_2\}.$$ 

Again, $\tilde{G}_{0,1}$ is a legitimate cdf for a random variable with support in $[-M, M]$. We then choose $\tilde{F}_{0,0}$ to be the cdf of any random variable with support in $[-M, M]$,

$$\tilde{F}_{1,1}(y) = \frac{k_2 F_{1,1}(y) - \tilde{c} \tilde{F}_{0,1}(y)}{k_2 - \tilde{c}}$$

and

$$\tilde{F}_{1,0}(y) = \frac{(k_1 - k_2) F_{1,0}(y) + \tilde{c} \tilde{F}_{0,1}(y)}{\tilde{b}}.$$ 

Finally, we choose $\tilde{G}_{1,1}$ to be the cdf of any random variable with support in $[-M, M]$, $\tilde{G}_{0,1}(y) = G_{0,0}(y)$,

$$\tilde{G}_{0,0}(y) = \frac{(1 - k_1) G_{0,0}(y) - \tilde{c} \tilde{G}_{0,1}(y)}{1 - k_1 - \tilde{c}}$$

and

$$\tilde{G}_{1,0}(y) = \frac{(k_1 - k_2) G_{1,0}(y) + \tilde{c} \tilde{G}_{0,1}(y)}{\tilde{b}}.$$
Clearly, $\tilde{F}_{1,0}$ and $\tilde{G}_{1,0}$ are legitimate cdf’s since each is a convex combination of cdf’s (due to $k_1 - k_2, \tilde{c} > 0$ and $\tilde{b} = k_1 - k_2 + \tilde{c}$). We now check $\tilde{F}_{1,1}$. By the definition of $\tilde{F}_{0,1}$, we have

$$\tilde{F}_{1,1}(y) = \begin{cases} \frac{k_2}{k_2 - \tilde{c}} F_{1,1}(y) & \text{if } y < B_1 \\ \frac{1}{k_2 - \tilde{c}} \left( k_2 - \frac{\tilde{c}}{1 - F_{1,1}(B_1)} \right) \cdot F_{1,1}(y) + \frac{\tilde{c}}{1 - F_{1,1}(B_1)} F_{1,1}(B_1) & \text{if } y \geq B_1. \end{cases}$$

We first observe $\tilde{F}_{1,1}(M) = 1$. We also observe that $\tilde{F}_{1,1}(B_1) = \frac{k_2}{k_2 - \tilde{c}} F_{1,1}(B_1) \geq \frac{k_2}{k_2 - \tilde{c}} \lim_{y \uparrow B_1} F_{1,1}(B_1) = \lim_{y \uparrow B_1} \tilde{F}_{1,1}(B_1)$. Moreover, $\tilde{F}_{1,1}$ is clearly non-decreasing and right-continuous on $[\Theta, B_1)$. It is also right-continuous on $[B_1, M]$. Since $F_{1,1}(B_1) < 1 - \tilde{c}/k_2$, we have that $k_2 - \frac{\tilde{c}}{1 - F_{1,1}(B_1)} > 0$ and $\tilde{F}_{1,1}$ is non-decreasing on $[B_1, M]$. Therefore, $\tilde{F}_{1,1}$ is a legitimate cdf for a distribution supported inside $[-M, M]$.

Finally, we check $\tilde{G}_{0,0}$. We observe

$$\tilde{G}_{0,0}(y) = \begin{cases} \frac{1}{1 - k_1 - \tilde{c}} \cdot \left( 1 - k_1 - \frac{\tilde{c}}{\tilde{G}_{0,0}(B_2)} \right) \cdot G_{0,0}(y) & \text{if } y \leq B_2 \\ \frac{1 - k_1 - \tilde{c}}{1 - k_1 - \tilde{c}} \cdot G_{0,0}(y) & \text{if } y > B_2. \end{cases}$$

Since $G_{0,0}(B_2) \geq \tilde{c}/(1 - k_1)$, we have that $1 - k_1 - \frac{\tilde{c}}{\tilde{G}_{0,0}(B_2)} \geq 0$ and $\tilde{G}_{0,0}$ is non-decreasing on $[-M, B_2]$. The rest of the argument is analogous to that for $\tilde{F}_{1,1}$. This concludes that $\tilde{G}_{0,0}$ is a legitimate cdf for a distribution supported inside $[-M, M]$. Therefore, we have proved that $\{\tilde{F}_{d,1,d_0} : d, d_0 \in \{0, 1\}\}$ and $\{\tilde{G}_{d,1,d_0} : d, d_0 \in \{0, 1\}\}$ are cdf’s for a distribution with support in $[-M, M]$.

**Step 2:** show that $\tilde{\theta} \in \Theta(\eta)$.

We clearly have $\tilde{a} + \tilde{b} = k_1, \tilde{a} + \tilde{c} = k_2, \tilde{c} \in [0, \eta], \tilde{a}, \tilde{b}, \tilde{c} \in [0, 1]$ and $\tilde{a} + \tilde{b} + \tilde{c} \in [0, 1]$.

We now show $P_{\tilde{\theta}}(Y_i \leq M \mid D_i, Z_i) = 1$. By Lemma 1 (see Step 3), the distribution of $Y_i \mid (D_i = d, Z_i = z)$ is a mixture of $\{\tilde{F}_{d,1,d_0} : d, d_0 \in \{0, 1\}\}$ and $\{\tilde{G}_{d,1,d_0} : d, d_0 \in \{0, 1\}\}$. By Step 1, the support of $Y_i \mid (D_i = d, Z_i = z)$ is in $[-M, M]$ for any $d, z \in \{0, 1\}$.

It remains to check $[\mu_1(\tilde{\theta}) \tilde{b} - \mu_2(\tilde{\theta}) \tilde{c}] / (\tilde{b} - \tilde{c}) = \beta$. We notice that $\mu_1(\tilde{\theta}) = \beta$ since $c = 0$. Therefore,

$$\beta = \mu_1(\tilde{\theta}) = E_{\tilde{\theta}}(Y_i(1) \mid D_i(1) = 1, D_i(0) = 0) - E_{\tilde{\theta}}(Y_i(0) \mid D_i(1) = 1, D_i(0) = 0) = \int y dF_{1,0}(y) - \int y dG_{1,0}(y). \quad (4)$$

We now observe

$$\mu_2(\tilde{\theta}) = E_{\tilde{\theta}}(Y_i(1) \mid D_i(1) = 0, D_i(0) = 1) - E_{\tilde{\theta}}(Y_i(0) \mid D_i(1) = 0, D_i(0) = 1) = \int y dF_{0,1}(y) - \int y dG_{0,1}(y). \quad (5)$$
Similarly,
\begin{align*}
\mu_1(\tilde{\theta}) &= E_{\tilde{\theta}}(Y_1(1) \mid D_1(1) = 1, D_1(0) = 0) - E_{\tilde{\theta}}(Y_1(0) \mid D_1(1) = 1, D_1(0) = 0) \\
&= \int yd\tilde{F}_{1,0}(y) - \int yd\tilde{G}_{1,0}(y) \\
&= \left(\frac{k_1 - k_2}{b} \int ydF_{1,0}(y) + \frac{\tilde{c}}{b} \int yd\tilde{F}_{0,1}(y)\right) - \left(\frac{k_1 - k_2}{b} \int ydG_{1,0}(y) + \frac{\tilde{c}}{b} \int yd\tilde{G}_{0,1}(y)\right) \\
&= \frac{k_1 - k_2}{b} \left(\int ydF_{1,0}(y) - \int ydG_{1,0}(y)\right) + \frac{\tilde{c}}{b} \left(\int yd\tilde{F}_{0,1}(y) - \int yd\tilde{G}_{0,1}(y)\right) \\
&= \frac{k_1 - k_2}{b} \beta + \frac{\tilde{c}}{b} \mu_2(\tilde{\theta}),
\end{align*}

where (i) follows by the definitions of $\tilde{F}_{1,0}$ and $\tilde{G}_{1,0}$ and (ii) follows by (4) and (5).

By (5) and (6) as well as $\tilde{b} - \tilde{c} = k_1 - k_2$, we have
\begin{align*}
\frac{\mu_1(\tilde{\theta})\tilde{b} - \mu_2(\tilde{\theta})\tilde{c}}{\tilde{b} - \tilde{c}} &= \frac{\mu_1(\tilde{\theta})\tilde{b} - \mu_2(\tilde{\theta})\tilde{c}}{k_1 - k_2} = \frac{(k_1 - k_2)\beta}{k_1 - k_2} = \beta.
\end{align*}

It remains to show that $\mu_1(\tilde{\theta}) > -\beta$ and $\mu_2(\tilde{\theta}) > -\beta$. First we show $\mu_1(\tilde{\theta}) > -\beta$. By (6), we only need to verify $\tilde{c}\mu_2(\tilde{\theta}) > -(k_1 - k_2 + \tilde{b})\beta$. By (5), it suffices to verify
\begin{align*}
\int yd\tilde{F}_{0,1}(y) - \int yd\tilde{G}_{0,1}(y) > - \frac{(k_1 - k_2 + \tilde{b})}\beta = - \frac{[2(k_1 - k_2) + \eta]\beta}{\eta}.
\end{align*}

Since $\eta < k_1 - k_2$, it is enough to check
\begin{align*}
\int yd\tilde{F}_{0,1}(y) - \int yd\tilde{G}_{0,1}(y) > - \frac{3(k_1 - k_2)\beta}{\eta}.
\end{align*}

Notice that $\tilde{F}_{0,1}$ is the cdf of a random variable taking values in $[B_1, M]$. Thus, $\int yd\tilde{F}_{0,1}(y) \geq B_1$. Similarly, $\tilde{G}_{0,1}$ is the cdf of a random variable taking values in $[-M, B_2]$, which means that $\int yd\tilde{G}_{0,1}(y) \leq B_2$. It follows that
\begin{align*}
\int yd\tilde{F}_{0,1}(y) - \int yd\tilde{G}_{0,1}(y) \geq B_1 - B_2 = Q_{1,\theta}(1 - \varepsilon_1) - \delta - Q_{2,\theta}(\varepsilon_1) \geq \varepsilon_2 - \delta.
\end{align*}

By $\delta < \varepsilon_2 - 3(k_1 - k_2)\beta/\eta$, (7) follows. Hence, we have proved $\mu_1(\tilde{\theta}) > -\beta$.

Since $\mu_2(\tilde{\theta}) = \int yd\tilde{F}_{0,1}(y) - \int yd\tilde{G}_{0,1}(y)$ and $\eta \leq k_1 - k_2$, (7) implies that
\begin{align*}
\mu_2(\tilde{\theta}) > -3\beta > -\beta.
\end{align*}

**Step 3:** show that the observed data $W_n$ has the same distribution under $\theta$ and $\tilde{\theta}$.

By Lemma 1, the distribution of $Y_i \mid (D_i, Z_i)$ under $\theta$ is given by
\begin{align*}
P_{\theta}(Y_i \leq y \mid D_i = 1, Z_i = 1) &= F_{1,1}(y) \frac{k_2}{k_1} + F_{1,0}(y) \frac{k_1 - k_2}{k_1}, \\
P_{\theta}(Y_i \leq y \mid D_i = 1, Z_i = 0) &= F_{1,1}(y), \\
P_{\theta}(Y_i \leq y \mid D_i = 0, Z_i = 1) &= G_{0,0}(y),
\end{align*}

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Proof of Corollary 1. Throughout the proof, we assume that $n$ is large enough so we have $0 < \eta < \varepsilon_1 \min \{k_2, 1 - k_1, k_1 - k_2\}$ and $|\beta|/\eta < \varepsilon_2/(k_1 - k_2)$. Fix an arbitrary $\theta_0 \in \Theta_\ast$. Notice that

\[
P_{\theta_0}(\{-1, 1\} \subset CS(W_n)) = P_{\theta_0}\left(\{-1 \in CS(W_n)\} \cap \{1 \in CS(W_n)\}\right)
= 1 - P_{\theta_0}\left(\{-1 \notin CS(W_n)\} \cup \{1 \notin CS(W_n)\}\right)
\geq 1 - P_{\theta_0}(\{-1 \notin CS(W_n)\}) - P_{\theta_0}(1 \notin CS(W_n))
= P_{\theta_0}(-1 \in CS(W_n)) + P_{\theta_0}(1 \in CS(W_n)) - 1. \tag{8}
\]

By Theorem 1, there exists $\tilde{\theta} \in \Theta(\eta)$ such that $\mu_1(\tilde{\theta}) > 0$ and $W_n$ has the same distribution under $P_{\theta_0}$ and $P_{\tilde{\theta}}$. This means that

\[
P_{\theta_0}(1 \in CS(W_n)) = P_{\tilde{\theta}}(1 \in CS(W_n)) = P_{\tilde{\theta}}(\text{sign}(\mu_1(\tilde{\theta})) \in CS(W_n)).
\]

Moreover, since $\mu_1(\theta_0) = \beta < 0$, we have

\[
P_{\theta_0}(-1 \in CS(W_n)) = P_{\theta_0}(\text{sign}(\mu_1(\theta_0)) \in CS(W_n)).
\]
Therefore,
\[ P_{\theta_0} (\{1, -1\} \subset CS(W_n)) \geq P_{\theta_0} (\text{sign}(\mu_1(\theta_0)) \in CS(W_n)) + P_\theta (\text{sign}(\mu_1(\tilde{\theta})) \in CS(W_n)) - 1 \]
\[
\overset{(i)}{\geq} 2 \cdot \inf_{\theta \in \Theta(\eta)} P_\theta (\text{sign}(\mu_1(\theta)) \in CS(W_n)) - 1,
\]
where (i) follows by \( \theta_0, \tilde{\theta} \in \Theta(\eta) \). Since the above bound holds for an arbitrary \( \theta_0 \in \Theta_\ast \) and the right-hand side does not depend on \( \theta_0 \), we can take an infimum over \( \theta_0 \), obtaining
\[
\inf_{\theta_0 \in \Theta_\ast} P_{\theta_0} (\{1, -1\} \subset CS(W_n)) \geq 2 \cdot \inf_{\theta \in \Theta(\eta)} P_\theta (\text{sign}(\mu_1(\theta)) \in CS(W_n)) - 1.
\]
Now we take \( \lim \inf \) on both sides and use \( \lim \inf_{n \to \infty} \inf_{\theta \in \Theta(\eta)} P_\theta (\text{sign}(\mu_1(\theta)) \in CS(W_n)) \geq 1 - \alpha \). The desired result follows.

\[ \square \]

A.2 Proof of Theorem 3

We start with two auxiliary results.

**Lemma 2.** Let \( \theta = (a,b,c,H) \) satisfy \( P_\theta(Y_i \in \{0,1\}) = 1 \). Assume that \( P_\theta(Y_i = D_i = 1 \mid Z_i = 0), P_\theta(Y_i = D_i = 0 \mid Z_i = 1) \), \( k_2 \) and \( k_1 - k_2 \) are bounded below by a positive constant. If \( \eta \in [0,k_2] \) and \( \beta \) satisfy \( \beta < 0, c = 0, |\beta| \to 0, \eta \to 0 \) and \( \beta(k_1 - k_2) + \eta \geq 0 \), then for large enough \( n \) there exists \( \tilde{\theta} = (\tilde{a},\tilde{b},\tilde{c},\tilde{H}) \) such that (1) \( P_{\tilde{\theta}} \) and \( P_\theta \) imply the same distribution for the observed data \( (Y_i,D_i,Z_i) \) and (2) \( \mu_1(\tilde{\theta}) \geq 0 \) and \( \tilde{c} \leq \eta \).

**Proof.** We follow a similar argument as in the proof of Theorem 1. Since \( Y_i \in \{0,1\} \), we can simplify \( H \) to 8 numbers as follows. We parametrize \( Y_i(d) \mid (D_i(1),D_i(0)) \) as \( E(Y_i(1) \mid D_i(1) = d_1, D_i(0) = d_0) = r_{d_1,d_0} \) and \( E(Y_i(0) \mid D_i(1) = d_1, D_i(0) = d_0) = t_{d_1,d_0} \), where \( d_1, d_0 \in \{0,1\} \).

Fix \( \theta = (a,b,c,r,t) \) satisfying the following:

- \( c = 0 \).
- \( \beta = r_{1,0} - t_{1,0} < 0 \).
- \( k_1 = P_\theta(D_i = 1 \mid Z_i = 1) = a + b = k_2 + b \)
- \( k_2 = P_\theta(D_i = 1 \mid Z_i = 0) = a + c = a \).

Similar to Lemma 1, we observe that
\[
\rho_{1,1} := E_\theta(Y_i \mid D_i = 1, Z_i = 1) = r_{1,1} \frac{k_2}{k_1} + r_{1,0} \frac{k_1 - k_2}{k_1},
\]
\[
\rho_{1,0} := E_\theta(Y_i \mid D_i = 1, Z_i = 0) = r_{1,1},
\]
\[
\rho_{0,1} := E_\theta(Y_i \mid D_i = 0, Z_i = 1) = t_{0,0},
\]
\[
\rho_{0,0} := E_\theta(Y_i \mid D_i = 0, Z_i = 0) = t_{1,0} \frac{k_1 - k_2}{1 - k_2} + t_{0,0} \frac{1 - k_1}{1 - k_2}.
\]

Since \( \beta = r_{1,0} - t_{1,0} \), we can write \( \beta = (\rho_{1,1}k_1 - \rho_{1,0}k_2 - \rho_{0,0}(1 - k_2) + \rho_{0,1}(1 - k_1))/(k_1 - k_2) \). This means that we can eliminate \( \rho_{0,0} \) from future calculations by observing
\[
\rho_{0,0} = \frac{\rho_{1,1}k_1 - \rho_{1,0}k_2 + \rho_{0,1}(1 - k_1) - \beta(k_1 - k_2)}{1 - k_2}.
\]
We observe that
\[
\rho_{1,0}k_2 = P_\theta(Y_i = 1 \mid D_i = 1, Z_i = 0) \cdot P_\theta(D_i = 1 \mid Z_i = 0) \\
= P_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0)
\] (9)
and similarly
\[
(1 - t_{0,0})(1 - k_1) = P_\theta(Y_i = 0 \mid D_i = 0, Z_i = 1) \cdot P_\theta(D_i = 0 \mid Z_i = 1) \\
= P_\theta(Y_i = 0, D_i = 0 \mid Z_i = 1). \tag{10}
\]

We now construct \(\tilde{\theta} = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{r}, \tilde{t})\). We set \(\tilde{b} = k_1 - k_2 + \tilde{c}\) and \(\tilde{a} = k_2 - \tilde{c}\), where \(\tilde{c} = \min\{k_2, \eta\}\). Similar to Lemma 1, we observe

\[
E_{\tilde{\theta}}(Y_i \mid D_i = 1, Z_i = 1) = \tilde{r}_{1,1} \frac{\tilde{a}}{\tilde{a} + \tilde{b}} + \tilde{r}_{1,0} \frac{\tilde{b}}{\tilde{a} + \tilde{b}} = \tilde{r}_{1,1} \frac{k_2 - \tilde{c}}{k_1} + \tilde{r}_{1,0} \frac{k_1 - k_2 + \tilde{c}}{k_1},
\]
\[
E_{\tilde{\theta}}(Y_i \mid D_i = 1, Z_i = 0) = \tilde{r}_{1,1} \frac{\tilde{a} + \tilde{c}}{\tilde{a} + \tilde{b}} + \tilde{r}_{0,1} \frac{\tilde{c}}{\tilde{a} + \tilde{b}} = \tilde{r}_{1,1} \frac{k_2 - \tilde{c}}{k_2} + \tilde{r}_{0,1} \frac{\tilde{c}}{k_2},
\]
\[
E_{\tilde{\theta}}(Y_i \mid D_i = 0, Z_i = 1) = \tilde{t}_{0,1} \frac{\tilde{c}}{1 - \tilde{a} - \tilde{b}} + \tilde{t}_{0,0} \frac{1 - \tilde{a} - \tilde{b} - \tilde{c}}{1 - \tilde{a} - \tilde{b}} = \tilde{t}_{0,1} \frac{\tilde{c}}{1 - k_1} + \tilde{t}_{0,0} \frac{1 - k_1 - \tilde{c}}{1 - k_1},
\]
\[
E_{\tilde{\theta}}(Y_i \mid D_i = 0, Z_i = 0) = \tilde{t}_{1,0} \frac{\tilde{b}}{1 - \tilde{a} - \tilde{c}} + \tilde{t}_{0,0} \frac{1 - \tilde{a} - \tilde{b} - \tilde{c}}{1 - \tilde{a} - \tilde{c}} = \tilde{t}_{1,0} \frac{k_1 - k_2 + \tilde{c}}{1 - k_2} + \tilde{t}_{0,0} \frac{1 - k_1 - \tilde{c}}{1 - k_2}.
\]

As in the proof of Theorem 1, the distribution of \((D_i, Z_i)\) is the same under \(P_\theta\) and under \(P_\tilde{\theta}\). The distribution of \(Y_i \mid (D_i, Z_i)\) is also identical under \(P_\theta\) and \(P_\tilde{\theta}\) if the conditional mean \(E(Y_i \mid D_i, Z_i)\) matches all the four cases of \((D_i, Z_i) \in \{0, 1\} \times \{0, 1\}\). This means that
\[
r_{1,1}k_2 + r_{1,0}(k_1 - k_2) = \tilde{r}_{1,1}(k_2 - \tilde{c}) + \tilde{r}_{1,0}(k_1 - k_2 + \tilde{c})
\]
\[
r_{1,1}k_2 = \tilde{r}_{1,1}(k_2 - \tilde{c}) + \tilde{r}_{0,1}\tilde{c}
\]
\[
t_{0,0}(1 - k_2) = \tilde{t}_{0,1}\tilde{c} + \tilde{t}_{0,0}(1 - k_2 - \tilde{c})
\]
\[
t_{1,0}(k_1 - k_2) + t_{0,0}(1 - k_1) = \tilde{t}_{1,0}(k_1 - k_2 + \tilde{c}) + \tilde{t}_{0,0}(1 - k_1 - \tilde{c}).
\]

Once we impose the constraints of \(\tilde{r}_{d_1,d_0}, \tilde{t}_{d_1,d_0} \in [0, 1]\), we have that
\[
\tilde{r}_{1,1} = \frac{r_{1,1}k_2 - \tilde{r}_{0,1}\tilde{c}}{k_2 - \tilde{c}}
\]
\[
\tilde{r}_{1,0} = \frac{r_{1,0}(k_1 - k_2) + \tilde{r}_{0,1}\tilde{c}}{k_1 - k_2 + \tilde{c}}
\]
\[
\tilde{t}_{0,0} = \frac{t_{0,0}(1 - k_2) - \tilde{t}_{0,1}\tilde{c}}{1 - k_2 - \tilde{c}}
\]
\[
\tilde{t}_{1,0} = \frac{t_{1,0}(k_1 - k_2) + \tilde{t}_{0,1}\tilde{c}}{k_1 - k_2 + \tilde{c}}.
\]
as well as
\[
\max \left\{ 0, 1 + \frac{(\rho_1,0 - 1)k_2}{\hat{c}} \right\} \leq \bar{r}_{0,1} \leq \min \left\{ 1, \frac{\rho_1,0k_2}{\hat{c}} \right\}
\] (11)
and
\[
\max \left\{ 0, 1 + \frac{(t_0,0 - 1)(1 - k_2)}{\hat{c}} \right\} \leq \bar{t}_{0,1} \leq \min \left\{ 1, \frac{t_0,0(1 - k_2)}{\hat{c}} \right\}.
\] (12)

Since \( \hat{c} = o(1) \), we have \( k_2 + \hat{c} \leq 1 \). By this and \( \hat{c} \leq k_2 \), the above inequalities can hold. We set \( \bar{r}_{0,1} = \min\{1, \rho_1,0k_2/\hat{c}\} \) and \( \bar{t}_{0,1} = \max\{0, 1 + (t_0,0 - 1)(1 - k_1)/\hat{c}\} \). Then

\[
\mu_1(\bar{\theta}) = \bar{r}_{1,0} - \bar{t}_{1,0} = \frac{r_{1,0}(k_1 - k_2) + \bar{r}_{0,1}\hat{c} - t_{1,0}(k_1 - k_2) + \bar{t}_{0,1}\hat{c}}{k_1 - k_2 + \hat{c}} = \frac{(r_{1,0} - t_{1,0})(k_1 - k_2) + (\bar{r}_{0,1} - \bar{t}_{0,1})\hat{c}}{k_1 - k_2 + \hat{c}} = \beta(k_1 - k_2) + (\bar{r}_{0,1} - \bar{t}_{0,1})\hat{c}.
\]

It only remains to show that
\[
\beta(k_1 - k_2) + (\bar{r}_{0,1} - \bar{t}_{0,1})\hat{c} \geq 0.
\]

By the definitions of \( \bar{r}_{0,1} \) and \( \bar{t}_{0,1} \), we need to show that
\[
\beta(k_1 - k_2) + \min\{\hat{c}, \rho_1,0k_2\} - \max\{0, \hat{c} + (t_0,0 - 1)(1 - k_1)\} \geq 0.
\] (13)

By (9) and (10), the assumptions imply that \( \rho_1,0k_2 \) and \( (1 - t_0,0)/(1 - k_1) \) are bounded below by a positive constant. For large \( n \), \( \hat{c} = \min\{k_2, \eta\} = \eta \to 0 \). Then for large enough \( n \), \( \min\{\hat{c}, \rho_1,0k_2\} = \eta \) and \( \max\{0, \hat{c} + (t_0,0 - 1)(1 - k_1)\} = 0 \). Thus, (13) becomes \( \beta(k_1 - k_2) + \eta \geq 0 \), which is assumed to be true. The proof is complete. \( \square \)

**Lemma 3.** Let \( \theta = (a, b, c, H) \) satisfy \( P_\theta(Y_i \in \{0, 1\}) = 1 \). Assume that \( P_\theta(Y_i = D_i = 1 \mid Z_i = 0), P_\theta(Y_i = D_i = 0 \mid Z_i = 1), k_2 \) and \( k_1 - k_2 \) are bounded below by a positive constant. If \( \eta \in [0, k_2] \) and \( \beta \) satisfy \( \beta < 0, c \leq \eta, |\beta| \to 0, \eta \to 0 \) and \( \beta(k_1 - k_2) + \eta < 0 \), then \( \mu_1(\theta) < 0 \).

**Proof.** Recall that \( \beta = \frac{\mu_1(\theta) - b - \mu_2(\theta) - c}{b - c} \). Therefore,
\[
\mu_1(\theta) = \frac{c}{b} \mu_2(\theta) + \left( 1 - \frac{c}{b} \right) \beta.
\]

Since \( \beta < 0 \), it suffices to show that \( c \mu_2(\theta) + (b - c)\beta < 0 \). Since \( b - c = k_1 - k_2 \), we need to show that \( c \mu_2(\theta) < -\beta(k_1 - k_2) \). Since \( c \leq \eta \) and \( \mu_2(\theta) \in \{0, 1\} \), we have \( c \mu_2 \leq \eta \). Hence, \( c \mu_2(\theta) < -\beta(k_1 - k_2) + \eta \to 0 \). The proof is complete. \( \square \)

**Proof of Theorem 3.** Part (2) follows by Lemma 2 and follow the same argument as the proof of Corollary 1. Therefore, when \( \beta(k_1 - k_2) + \eta \geq 0 \), if a confidence set \( CS(W_n) \) satisfies
\[
\liminf_{n \to \infty} \inf_{\theta \in \Theta_{binary}(\eta)} P_\theta(\mu_1(\theta) \in CS(W_n)) \geq 1 - \alpha
\]
for \( \alpha \in (0, 1) \), then
\[
\liminf_{n \to \infty} \inf_{\theta \in \Theta_{binary}(0)} P_\theta([-1, 0] \subset CS(W_n)) \geq 1 - \alpha.
\]
Suppose that a consistent estimator exists, i.e., $\rho(W_n) \in \{-1, 0, 1\}$ and
\[
\liminf_{n \to \infty} \inf_{\theta \in \Theta_{\text{binary}}(\eta)} \mathbb{P}_\theta(\mu_1(\theta) \in \{\rho(W_n)\}) = 1 - o(1) \quad \text{since } \Theta_{\text{binary}}(0) \subset \Theta_{\text{binary}}(\eta),
\]
it follows that
\[
\liminf_{n \to \infty} \inf_{\theta \in \Theta_{\text{binary}}(\eta)} \mathbb{P}_\theta(\mu_1(\theta) \in \{\rho(W_n)\}) = 1 - o(1).
\]

However, this contradicts $\liminf_{n \to \infty} \inf_{\theta \in \Theta_{\text{binary}}(\eta)} \mathbb{P}_\theta(\{-1, 0\} \subset \{\rho(W_n)\}) \geq 1 - \alpha$ since $\{\rho(W_n)\}$ is a singleton. The proof is complete.  \hfill \square

### A.3 Proof of Theorem 4

**Lemma 4.** Let $\theta = (a, b, c, H)$ satisfy $\mathbb{P}_\theta(Y_i \in \{0, 1\}) = 1$. Assume that $\mathbb{P}_\theta(Y_i = D_i = 0 \mid Z_i = 1)$ and $k_1 - k_2$ are bounded below by a positive constant. If $c = 0$, $k_2 = o(1)$ and $\beta = o(1)$ satisfy $\beta < 0$ and $\beta(k_1 - k_2) + \mathbb{P}_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0) \geq 0$, then for large enough $n$ there exists $\bar{\theta} = (\bar{a}, \bar{b}, \bar{c}, \bar{H})$ such that (1) $\mathbb{P}_\theta$ and $\mathbb{P}_{\bar{\theta}}$ imply the same distribution for the observed data $(Y_i, D_i, Z_i)$ and (2) $\mu_1(\bar{\theta}) \geq 0$ and $\bar{c} \leq k_2$.

**Proof.** We repeat all the arguments in the proof of Lemma 2 including Equation (13), which we repeat here:
\[
\beta(k_1 - k_2) + \min\{\bar{c}, \rho_1, k_2\} - \max\{0, \bar{c} + (t_0, 0) - 1, (1 - k_1)\} \geq 0.
\]
We inherit all the notations from the proof of Lemma 2. The only difference is that $\bar{c} = \rho_1, k_2$ (rather than $\bar{c} = \min\{\eta, k_2\}$). As argued in (10) in the proof of Lemma 2, $\beta = o(1)$ satisfies $\beta < 0$ and $\beta(k_1 - k_2) + \mathbb{P}_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0) \geq 0$, which by assumption is bounded below by a positive constant. Since $\bar{c} \leq k_2 \to 0$, the above display for large enough $n$ becomes
\[
\beta(k_1 - k_2) + \min\{\bar{c}, \rho_1, k_2\} \geq 0. \quad \text{(14)}
\]

By $\bar{c} = \rho_1, k_2$, this becomes $\beta(k_1 - k_2) + \rho_1, k_2 \geq 0$. As argued in (9) in the proof of Lemma 2, $\rho_1, k_2 = \mathbb{P}_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0)$. Thus, $\beta(k_1 - k_2) + \rho_1, k_2 \geq 0$ follows by assumption. The proof is complete.  \hfill \square

**Lemma 5.** Let $\theta = (a, b, c, H)$ satisfy $\mathbb{P}_\theta(Y_i \in \{0, 1\}) = 1$. Assume that $\mathbb{P}_\theta(Y_i = D_i = 0 \mid Z_i = 1)$ and $k_1 - k_2$ are bounded below by a positive constant. If $c \leq k_2 = o(1)$ and $\beta = o(1)$ satisfy $\beta < 0$ and $\beta(k_1 - k_2) + \mathbb{P}_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0) < 0$, then $\mu_1(\theta) < 0$.

**Proof.** Recall that $\beta = \frac{\mu_1(\theta)-b-c}{b-c}$, which means that $\mu_1(\theta) = \frac{\bar{c}}{\bar{c}} \mu_2(\theta) + (1 - \bar{c}) \beta$. Since $\beta < 0$, it suffices to show that $c \mu_2(\theta) + (b - c)\beta < 0$. Since $b - c = k_1 - k_2$, we need to show that
\[
c \mu_2(\theta) < -\beta(k_1 - k_2).
\]
Following a similar computation as in the proof of Lemma 2, we have
\[
E_\theta(Y_i \mid D_i = 1, Z_i = 0) = r_{1,1} \frac{k_2 - c}{k_2} + r_{0,1} \frac{c}{k_2},
\]
where $r_{1,1} = \mathbb{E}_\theta(Y_i(1) \mid D_i(1) = 1, D_i(0) = 1)$ and $r_{0,1} = \mathbb{E}_\theta(Y_i(1) \mid D_i(1) = 0, D_i(0) = 1)$.

Since $P_\theta(D_i = 1 \mid Z_i = 0) = k_2$, we have
\[
P_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0) = P_\theta(Y_i = 1 \mid D_i = 1, Z_i = 0) \cdot P_\theta(D_i = 1 \mid Z_i = 0)
= r_{1,1} (k_2 - c) + r_{0,1} c \geq r_{0,1} c \geq \mu_2(\theta)c,
\]

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where (i) follows by the fact that \( \mu_2(\theta) = r_{0,1} - E_\theta(Y_i(0) \mid D_i(1) = 0, D_i(0) = 1) \leq r_{0,1} \). By the assumption of \( \beta(k_1 - k_2) + P_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0) < 0 \), we have

\[
\beta(k_1 - k_2) + c\mu_2(\theta) \leq \beta(k_1 - k_2) + P_\theta(Y_i = 1, D_i = 1 \mid Z_i = 0) < 0.
\]

This proves (15). The proof is complete. \( \square \)

**Proof of Theorem 4.** The proof follows the same argument as the proof of Theorem 3, except that Lemmas 2 and 3 is replaced by Lemmas 4 and 5. \( \square \)

### A.4 Proof of Theorems 5 and 6

**Proof of Theorem 5.** Without loss of generality, we assume \( \text{cov}(D_i, Z_i) > 0 \). (This is because we can swap the 0-1 labels for \( Z_i \) to obtain \( \text{cov}(D_i, Z_i) > 0 \) and such a swap does not change \( \max\{\lambda_1, \lambda_2\} \).)

The case for \( \text{cov}(D_i, Z_i) < 0 \) follows by swap \( Z_i \) and \( 1 - Z_i \). We recall the notation of \( k_1 = E(D_i \mid Z_i = 1) = E(D_i(1)) = a + b \) and \( k_2 = E(D_i \mid Z_i = 0) = E(D_i(0)) = a + c \). We observe that \( \text{cov}(D_i, Z_i) > 0 \) implies that \( E(D_i \mid Z_i = 1) > E(D_i \mid Z_i = 0) \), which means \( k_1 - k_2 = b - c > 0 \). We notice that \( \max\{\lambda_1, \lambda_2\} = \max\{\mu_1, -\mu_1, \mu_2, -\mu_2\} \).

By \( \beta = (\mu_1 b - \mu_2 c)/(b - c) \), we have

\[
\mu_1 = \lambda \mu_2 + (1 - \lambda) \beta,
\]

where \( \lambda = c/b \). This means that \( \max\{\lambda_1, \lambda_2\} = f(\mu_2, \lambda) \), where

\[
f(\mu_2, \lambda) := \max \{ \lambda \mu_2 + (1 - \lambda) \beta, \ -\lambda \mu_2 - (1 - \lambda) \beta, \ \mu_2, -\mu_2 \}.
\]

Recall that \( k_1 = a + b, k_2 = a + c \) and \( b - c = k_1 - k_2 > 0 \). Thus, \( \lambda = c/(k_1 - k_2 + c) \) with \( c \in [0, k_2/k_1] \subset [0, 1] \). We now find

\[
\min_{\mu_2 \in \mathbb{R}, \lambda \in [0, k_2/k_1]} f(\mu_2, \lambda).
\]

For any \( \lambda > 0 \), we notice that \( \min_{\mu_2 \in \mathbb{R}} f(\mu_2, \lambda) \) must occur at a point such that two of the four components of \( f(\mu_2, \lambda) \) are equal; otherwise, we can change \( \mu_2 \) slightly to lower the largest component even further to lower \( f(\mu_2, \lambda) \). Therefore, \( \mu_2(\lambda) \in \text{arg min}_{\mu_2 \in \mathbb{R}} f(\mu_2, \lambda) \) if and only if at least one of the following is true:

- \( \lambda \mu_2(\lambda) + (1 - \lambda) \beta = \mu_2(\lambda) \) (i.e., \( \mu_2(\lambda) = \beta \))
- \( \lambda \mu_2(\lambda) + (1 - \lambda) \beta = -\mu_2(\lambda) \) (i.e., \( \mu_2(\lambda) = -\beta(1 - \lambda)/(1 + \lambda) \))
- \( \mu_2(\lambda) = -\mu_2(\lambda) \) (i.e., \( \mu_2(\lambda) = 0 \))
- \( \lambda \mu_2(\lambda) + (1 - \lambda) \beta = -\lambda \mu_2(\lambda) - (1 - \lambda) \beta \) (i.e., \( \mu_2(\lambda) = (1 - \lambda^{-1}) \beta \)).

We plug these four values of \( \mu_2(\lambda) \) into \( f(\mu_2, \lambda) \) and take the minimum of the four values of \( f(\mu_2, \lambda) \), obtaining that for \( \lambda > 0 \),

\[
\min_{\mu_2 \in \mathbb{R}} f(\mu_2, \lambda) = |\beta| \cdot \min \left\{ 1, |1 - \lambda|, \frac{|1 - \lambda|}{1 + \lambda}, \frac{|1 - \lambda|}{\lambda} \right\}.
\]
Since $\lambda \in [0, 1)$, it follows that for $\lambda > 0$,

$$\min_{\mu_2 \in \mathbb{R}} f(\mu_2, \lambda) = |\beta| \cdot \frac{|1 - \lambda|}{1 + \lambda}.$$  

Now we take the infimum over $\lambda \in (0, k_2/k_1]$, obtaining

$$\inf_{\mu_2 \in \mathbb{R}, \lambda \in (0, k_2/k_1]} f(\mu_2, \lambda) = \inf_{\lambda \in (0, k_2/k_1]} |\beta| \cdot \frac{|1 - \lambda|}{1 + \lambda} = |\beta| \cdot \frac{k_1 - k_2}{k_1 + k_2}.$$  

We observe that $\min_{\mu_2 \in \mathbb{R}} f(\mu_2, 0) = |\beta|$. Thus,

$$\min_{\mu_2 \in \mathbb{R}, \lambda \in [0, k_2/k_1]} f(\mu_2, \lambda) = |\beta| \cdot \frac{k_1 - k_2}{k_1 + k_2}.$$  

Therefore, we have proved that for any $\mu_2 \in \mathbb{R}$ and for any $\lambda \in [0, k_2/k_1]$,

$$\max\{|\mu_1|, |\mu_2|\} \geq |\beta| \cdot \frac{k_1 - k_2}{k_1 + k_2}.$$  

The proof is complete. \hfill \square

**Proof of Theorem 6.** We recall that $\beta = (\mu_1 b - \mu_2 c)/(b - c)$, which means

$$\mu_1 = (c/b) \mu_2 + (1 - c/b) \beta. \quad (16)$$

We observe that $\text{cov}(D_i, Z_i) > 0$ implies that $E(D_i \mid Z_i = 1) > E(D_i \mid Z_i = 0)$, which means $b - c > 0$. We consider two cases: (A) $\beta > 0$ and (B) $\beta < 0$.

**Step 1:** prove the result in the case of $\beta > 0$.

We proceed by contradiction. Suppose that $|\mu_1| \geq |\mu_2|$ and $\mu_1 \leq 0$. By (16), this means $c\mu_2 + (b - c)\beta \leq 0$, which can be written as $\mu_2 \leq -(b - c)\beta/c$. Since $b - c > 0$ and $\beta > 0$, this means that $\mu_2 \leq 0$. Since $\mu_1, \mu_2 \leq 0$ and $|\mu_1| \geq |\mu_2|$, we have $\mu_1 \leq \mu_2$. By (16), this means

$$(c/b) \mu_2 + (1 - c/b) \beta \leq \mu_2.$$  

Using $b - c$, we obtain $\mu_2 \geq \beta$. Since $\beta > 0$, we have $\mu_2 > 0$. However, this contradicts $\mu_2 \leq 0$. Therefore, we have proved $\mu_1 > 0$.

**Step 2:** prove the result in the case of $\beta < 0$.

The argument is analogous. We state the argument here for completeness. Suppose that $|\mu_1| \geq |\mu_2|$ and $\mu_1 \geq 0$. By (16), this means $c\mu_2 + (b - c)\beta \geq 0$, which can be written as $\mu_2 \geq -(b - c)\beta/c$. Since $b - c > 0$ and $\beta < 0$, this means that $\mu_2 \geq 0$. Since $\mu_1, \mu_2 \geq 0$ and $|\mu_1| \geq |\mu_2|$, we have $\mu_1 \geq \mu_2$. By (16), this means

$$(c/b) \mu_2 + (1 - c/b) \beta \geq \mu_2.$$  

Using $b - c$, we obtain $\mu_2 \leq \beta$. Since $\beta < 0$, we have $\mu_2 < 0$. However, this contradicts $\mu_2 \geq 0$. Therefore, we have proved $\mu_1 < 0$.

Therefore, we have proved that in both cases, $\text{sign}(\mu_1) = \text{sign}(\beta)$. \hfill \square