About Locality and the Relativity Principle Beyond Special Relativity

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Locality of interactions is an essential ingredient of Special Relativity. Recently, a new framework under the name of relative locality [1] has been proposed as a way to consider Planckian modifications of the relativistic dynamics of particles. We note in this paper that the loss of absolute locality is a general feature of theories beyond Special Relativity with an implementation of a relativity principle. We give an explicit construction of such an implementation and compare it both with the previously mentioned framework of relative locality and the so-called Doubly Special Relativity theories.

I. INTRODUCTION

Even if Special Relativity (SR) is at the basis of our present particle physics theories, in the last years a strong motivation to explore departures of the Lorentz or Poincaré invariance has emerged both from the theoretical and the phenomenological sides. The idea that Lorentz invariance might be only an approximate, low-energy, symmetry of Nature (as it happens with many other symmetries) suggests the presence of corrections (suppressed by a certain high-energy scale) to the standard relativistic dynamics (mainly through a modified dispersion relation and/or a modified energy-momentum conservation law).

In fact, modified dispersion relations have arisen in different approaches to quantum gravity such as: space-time foams [2], brane-world scenarios [3], string theory [1], loop quantum gravity [6], renormalization group of gravity [6], analog models of gravity in condensed matter [7], canonical non-commutative [8] and $\kappa$-Minkowski [9] spacetimes. Departures from Lorentz invariance have also been considered in several phenomenological contexts such as proton decay in grand unification theories [10], violations of the GZK cut-off [11], anomalies in the cut-off of photons from Markarian 501 [12], neutrino oscillations [13] or tests of an energy-dependence of the speed of light with measurements of gamma ray bursts [2].

There are two main possibilities in a theory going beyond SR: either the relativity principle is lost, which means there is a preferred reference frame (sometimes identified with that in which the cosmic microwave background is isotropic), or it is preserved and the Poincaré invariance of SR is just deformed. Concerning this last option it is important to stress that SR is not a consequence of only the relativity principle and the relativity of simultaneity, but also of the homogeneity of spacetime, the isotropy of space and some notion of causality, see [14]. We generally refer to the first case as a scenario of Lorentz invariance violation (LIV). A particular example of it is Kostelecky’s Standard Model Extension [15], in which Lorentz invariance is spontaneously broken at low energies. The second case, however, implies the existence of a relativity principle in the theory beyond SR. This was considered for the first time by G. Amelino-Camelia under the name of Doubly Special Relativity (DSR) [16].

Both scenarios have been mostly studied in momentum space, since a modified dispersion relation (the relation between energy and momentum of a free particle) is usually the main ingredient defining the extension or departure of SR. Interaction of particles may then be introduced through an energy-momentum conservation law that may or may not show corrections with respect to the usual additive law of SR (in the case of DSR, the conservation law must necessarily be modified in order to be consistent with the relativity principle).

The space-time structure corresponding to these modifications of SR is however less known. Several approaches to DSR suggest that it should contain noncommutative properties [17], although a commutative spacetime has also been considered to be compatible with DSR models [18]. This lack of a well-defined space-time picture has caused some difficulties concerning, for example, the correct expression for the velocity of a particle in terms of energy and momentum [19–22], something which is crucial in order to study some experimental implications of the theory.

In fact, recently a debate about the phenomenological consistency of relativistic theories beyond SR with an energy dependent velocity for photons has emerged [22]. The key point in the discussion is the consequences of a loss of locality in the theory. Actually, the issue of locality is also important in the case of a LIV scenario (that is, without a relativity principle), in which the properties of translations in the underlying spacetime and the Lorentz violating physics may affect the energy-momentum conservation law as in the case of DSR [22]. As we will see, there is an strict connection between the energy-conservation law and the space-time locality of the interactions.

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A byproduct of the above mentioned debate has been the introduction of a description of interactions in spacetime with an associated new notion of ‘relative locality’ in terms of a geometric interpretation of the departures from SR kinematics \[1\]. The proposal has been introduced in a rather general way and, in its present state, a relativity principle has not been implemented yet.

One of the main ingredients of the relative locality paper Ref. \[1\] is the emergence of spacetime from momentum space through a variational principle. We think this idea represents a very important contribution to the general problem of finding an appropriate space-time description in theories beyond SR. This essential ingredient will be reviewed in Section II. Then, we will use this mathematical construction to discuss the relationship between the notion of locality and the relativity principle in a general framework with just two basic elements: the dispersion relation, describing the propagation of free particles, and the energy-momentum conservation law, defining the interaction between these particles. As we will see, an additive conservation law describes an interaction which is local in spacetime, and non-linear corrections to this law causes the locality property to be lost for a general observer. This means that the only way to preserve absolute locality in a theory beyond SR is in a LIV scenario in which the dispersion relation is modified, but a linear conservation law can be imposed. In particular, since a theory beyond SR with a relative principle (such as DSR) requires a modification of both the dispersion relation and the conservation law, such a theory cannot be a local theory.

In Section III we will consider the simplest implementation of a relativity principle, based on the use of the choice of appropriate phase space coordinates, in a theory in which locality is lost. In Section IV we will discuss the physical meaning of the coordinates appearing in the model for the interaction of particles introduced in Section II and their relation to the notions of energy and momentum in a theory beyond SR. The well-defined space-time description of interaction that emerges from the present work, generalizing the SR image of interactions between particles as the crossing of space-time worldlines, will also allow us to give answers to some paradoxes that have been pointed out in the context of DSR theories related with the velocity transformation law under boosts. A comparison of the present approach with that of DSR theories and with the geometric interpretation of Ref. \[1\] will also be given in this Section. We will end with a short summary and some concluding remarks in Section V, mentioning different alternatives to the simplest implementation of the relativity principle considered in this work.

II. INTERACTION OF PARTICLES IN SPACETIME

The description of a multiparticle process in spacetime is based on a variational principle with an action

\[
S = \sum_{(a,+)} \int_{s_a}^\infty ds \left[ x_{-a} (i) (i) \mu , p_{(i)} (i) , p_{(i)} (i) \mu + N_{(i) a} (i) \left[ D(p_{(i)} (i) , a (i)) - m_{(i) a (i)}^2 \right] \right] \\
+ \sum_{(a,+), i} \int_{s_a}^\infty ds \left[ x_{a+} (i) (i) \mu , p_{(i)} (i) , p_{(i)} (i) \mu + N_{a+} (i) \left[ D(p_{(i)} (i) , a+ (i)) - m_{a+ (i)}^2 \right] \right] \\
+ \sum_{(a,b), i} \int_{s_a}^s ds \left[ x_{a,b} (i) (i) \mu , p_{(i)} (i) , p_{(i)} (i) \mu + N_{a,b} (i) \left[ D(p_{(i)} (i) , a,b (i)) - m_{a,b (i)}^2 \right] \right] \\
+ \sum_{b=0}^{\varepsilon^\mu_b} K_{\mu} (p_{(i)} (i) , p_{(j)} (i)) - K_{\mu} (p_{(i)} (i) , p_{(j)} (i)) \left( s_b \right). \tag{1}
\]

This action corresponds to a process with several (V) interactions, each one characterized by the index \( a = 1, 2, ..., V \) and parameters \( s_a \) in increasing order \( s_1 < s_2 < ... < s_V \). Phase space coordinates for incoming particles are \( (x_{-a} (i) (i) \mu , p_{(i)} (i) , a) \) and outgoing particle coordinates \( (x_{a+} (i) (i) \mu , p_{(i)} (i) , a+) \). The index \((- , a , i) \) \((a , + , i)\) refers to the incoming (outgoing) particle \( i \) in (from) the \( a \)-interaction. There are also internal particles with phase space coordinates \( (x_{a,b} (i) (i) \mu , p_{(i)} (i) , a,b) \) propagating between interactions \( a \) and \( b \). There is a function \( N \) for each particle to implement the reparametrization invariance for the worldlines. The function \( D(p) \) determines the energy of a free particle of mass \( m \) and momentum \( p \) as the (positive) solution for \( D(p) = m^2 \). There is also a constant vector \( \varepsilon^\mu_b \) to implement the energy-momentum conservation law in the \( b \)-interaction \( K_{\mu} (p_{(i)} (i) , p_{(j)} (i)) = K_{\mu} (p_{(i)} (i) , p_{(j)} (i), b) \). \( ^{1} \) The conservation law is

\(^{1} \) We are considering the natural form for a conservation law with a separation of incoming and outgoing momentum variables. More general cases are discussed in the last section of the paper.
defined by a set of functions, one for each integer \( n \), \( K : \mathcal{P}^n \rightarrow \mathcal{P} \) where \( \mathcal{P} \) is the four-momentum space of a particle and \( K_\mu \) in the action \( \mathcal{S} \) are the components of the image of these functions.

If one takes the addition for \( K \) and \( D(p) = p_0^2 - \vec{p}^2 \) then one has the description of particle local interactions in special relativity (SR). Other choices for \( D \) and \( K \) lead to generalizations of SR with modified dispersion relations for the particles and non-linear corrections in the energy-momentum conservation laws. Clearly, these corrections require at least one dimensionful scale which would be related to the Planck scale if these modifications come from quantum gravitational effects.

The action \( \mathcal{S} \), with one interaction, has been introduced in Ref. \[1\] in a discussion of the modification of the notion of locality in a generalization of SR based on a new understanding of the geometry of momentum space. The case of several interactions has also been discussed in a particular case in Ref. \[2\]. In these works the deviations from SR kinematics are interpreted as a consequence of different aspects of momentum space geometry with a metric geometry defined from the function \( D \) and non-linear corrections in conservation laws as a consequence of a connection defined from the function \( K \) for \( n = 2 \). This geometric interpretation of the generalization of SR assumes that the functions \( K \) with \( n > 2 \) are obtained by iteration from the case \( n = 2 \) but we do not see any physical reason for this additional assumption. From now on we will discuss the interaction of particles and the issue of locality without any reference to a (possible) geometric interpretation.

From the vanishing of the coefficients of the variations \( \delta x^\mu \), \( \delta p_\mu \) for \( s \neq s_a \), \( \delta N \) and \( \delta z^\mu \) in the variation of the action we have the equations

\[
\dot{p}_{(i)\mu}^{-a} = \dot{p}_{(i)\mu}^{a,b} = \dot{p}_{(i)\mu} = 0
\]

\[
\dot{x}_{a,+}^{(i)\mu} = N_{a,+}^{(i)\mu}\frac{\partial D(p_{(i)}^{a,+})}{\partial p_{(i)\mu}} \quad \dot{x}_{a,+}^{(i)\mu} = N_{a,+}^{(i)\mu}\frac{\partial D(p_{(i)}^{a,+})}{\partial p_{(i)\mu}} \quad \dot{x}_{a,b}^{(i)\mu} = N_{a,b}^{(i)\mu}\frac{\partial D(p_{(i)}^{a,b})}{\partial p_{(i)\mu}}
\]

\[
D(p_{(i)}^{a,-}) = m_{-a}^{i2} \quad D(p_{(i)}^{a,+}) = m_{a,+}^{i2} \quad D(p_{(i)}^{a,b}) = m_{a,b}^{i2}
\]

\[
K_\mu(p_{(i)}^{b,c}, p_{(i)}^{b,+}) = K_\mu(p_{(i)}^{b,-}, p_{(i)}^{a,b}) \quad b = 1, 2, ..., V
\]

i.e., the constancy of momentum, the relation between the four velocity and momentum, the dispersion relation and the energy-momentum conservation law at each interaction.

The vanishing of the coefficient of \( \delta p_\mu \) for \( s = s_a \) gives the conditions

\[
x_{a,+}^{(i)\mu}(s_a) = z_a^\nu \frac{\partial K_\nu(p_{(i)}^{a,+}, p_{(k)}^{a,c})}{\partial p_{(i)\mu}} \quad x_{a,+}^{(i)\mu}(s_a) = z_a^\nu \frac{\partial K_\nu(p_{(i)}^{a,+}, p_{(k)}^{a,c})}{\partial p_{(i)\mu}}
\]

which give the position of incoming and outgoing particles at the interaction. When \( K \) deviates from the addition one has different particles at different points (non-local interactions) as a consequence of the non-linear corrections to the energy-momentum conservation laws. These conditions, together with the equation which gives the four-velocity in terms of the momentum, fix the worldlines that describe the propagation of the incoming particles before the interaction and the outgoing particles after the interaction:

\[
x_{a,+}^{(i)\mu}(s) = z_a^\nu \frac{\partial K_\nu(p_{(i)}^{a,+}, p_{(k)}^{a,c})}{\partial p_{(i)\mu}} + (s - s_a) N_{a,+}^{(i)} \frac{\partial D(p_{(i)}^{a,+})}{\partial p_{(i)\mu}}
\]

\[
x_{a,+}^{(i)\mu}(s) = z_a^\nu \frac{\partial K_\nu(p_{(i)}^{a,+}, p_{(k)}^{a,c})}{\partial p_{(i)\mu}} + (s - s_a) N_{a,+}^{(i)} \frac{\partial D(p_{(i)}^{a,+})}{\partial p_{(i)\mu}}
\]

\[2\] As usual, we make use of the reparametrization invariance of worldlines to choose \( s \) such that \( N \) is constant.
For the coordinates of the particles propagating between interactions $a$ and $b$ one has two conditions from the vanishing of the coefficients of $\delta p_{a,b}^{(i)\mu}$ at $s = s_a$ and $s = s_b$

\begin{align}
    x_{a,b}^{(i)\mu}(s_a) &= z_a \frac{\partial K_\nu(p_{a,c}^{(i)},p_{a,+}^{(i)},s_a)}{\partial p_{a,b}^{(i)\mu}} \quad x_{a,b}^{(i)\mu}(s_b) = z_b \frac{\partial K_\nu(p_{a,b}^{(i)},p_{b,+}^{(i)},s_b)}{\partial p_{a,b}^{(i)\mu}} .
\end{align}

One of these conditions can be used to determine the worldline of the particle propagating between the two interactions

\begin{align}
    x_{a,b}^{(i)\mu}(s) &= z_a \frac{\partial K_\nu(p_{a,c}^{(i)},p_{a,+}^{(i)},s_a)}{\partial p_{a,b}^{(i)\mu}} + (s - s_a) N_{a,b}^{(i)} \frac{\partial D(p_{a,b}^{(i)})}{\partial p_{a,b}^{(i)\mu}}
\end{align}

but there is one extra condition

\begin{align}
    z_b \frac{\partial K_\nu(p_{a,b}^{(i)},p_{b,+}^{(i)},s_a)}{\partial p_{a,b}^{(i)\mu}} - z_a \frac{\partial K_\nu(p_{a,c}^{(i)},p_{a,+}^{(i)},s_a)}{\partial p_{a,b}^{(i)\mu}} = (s_b - s_a) N_{a,b}^{(i)} \frac{\partial D(p_{a,b}^{(i)})}{\partial p_{a,b}^{(i)\mu}}
\end{align}

to be taken into account when one identifies the general solution of the variational principle based on the action $\mathcal{I}$. Any choice of momenta compatible with the dispersion relations (4) and the conservation laws (5) and any choice of coordinates $z_a$ for the interactions compatible with the conditions (11), which can also restrict the possible choice of momenta, define a solution for the space-time description of the multiparticle process. The general solution can be obtained by considering a set of $V - 1$ internal lines such that they connect all of the $V$ vertices (this is known as a “minimal tree”). One can first use $V - 1$ of the constraints (5) to fix the internal momenta of the minimal tree in terms of the external momenta and the $I - V + 1$ remaining internal momenta outside the minimal tree. Next one can take the $V - 1$ constraints in Eq. (11) corresponding to the momenta in the minimal tree as a set of linear equations for $V - 1$ of the coordinates $z_a$ whose solution gives these coordinates in terms of one remaining coordinate, for example $z_1$, and the momenta. The remaining $I - V + 1$ constraints in Eq. (11) can then be used to determine the internal momenta outside the minimal tree as a function of $z_1$ and the external momenta. Finally there is one constraint in Eq. (10) which gives a condition for the external momenta.

The fact that there is a solution for each choice of $z_1$ implies the translational invariance of the model: in fact, given a solution, one can obtain a 4-parameter family of equivalent solutions, given by

\begin{align}
    z_a' = z_a + \delta z_a ,
\end{align}

where the $V$ parameters $\delta z_a$ satisfy $(V - 1)$ independent equations,

\begin{align}
    \delta z_b \frac{\partial K_\nu(p_{a,b}^{(i)},p_{b,+}^{(i)},s_a)}{\partial p_{a,b}^{(i)\mu}} - \delta z_a \frac{\partial K_\nu(p_{a,c}^{(i)},p_{a,+}^{(i)},s_a)}{\partial p_{a,b}^{(i)\mu}} = 0 .
\end{align}

The above equations determine all but one of the $\delta z_a$, for example $\delta z_1$: there are 4 independent translations.

The transformations (12) connecting solutions with different choices for $\delta z_1$ reflect the absence of any reference to a point in the space-time description of the multiparticle process. These solutions correspond to the description of the process by different observers related by translations. Translations are defined then as transformations on the set of worldlines of the particles participating in the process. The transformation of each worldline depends on the momenta of different particles and on the proper time intervals $s_b - s_a$. This nontrivial realization of translations, in contrast with a space-time translation of each of the points of the different worldlines, is a direct consequence of the nonlinear corrections in the energy-momentum conservation laws. There is an observer for which $z_1 = 0$ and then all the incoming and outgoing particles in one of the interactions are at the origin of the coordinates of spacetime. This is the reason why one refers to this framework as a replacement of absolute locality by relative locality $\mathcal{I}$. Any physical process always involves more than one interaction (in the simplest case one has an interaction in the production of a particle and an interaction in its detection) and then one can say that in the presence of non-linear corrections in the energy-momentum conservation laws one has real, physical consequences of the loss of absolute locality. However in order to have observable consequences of the loss of absolute locality one requires a process with interactions separated by very long distances $\mathcal{I}$.

III. RELATIVITY PRINCIPLE

In the case of SR ($D(p) = p_\mu^2$ and $K([p]) = \sum p$) there are, together with translations, also transformations among solutions with different incoming and outgoing momenta. Given a solution $([p], z)$ of the variational principle
for the action $I$ in SR, a combination ($\{p', z\}$) with

$$p'_\mu = \Lambda_{\mu}^{'\nu} p_\nu$$

$$z'^\mu = \eta^{\mu\nu} \Lambda_{\nu}^{'\rho} \eta_{\rho\nu} z^\nu = \Lambda_{\mu}^{'\rho} z^\rho$$

(14)

will also be a solution if $\Lambda_{\mu}^{'\nu}$ is a Lorentz transformation (where we used that $\eta^{\rho\sigma} = \eta^{\mu\nu} \Lambda_{\nu}^{'\rho} \Lambda_{\rho}^{'\sigma} = (\Lambda^T \eta \Lambda)^{\rho\sigma}$).

In the general case one can always make use of canonical transformations in order to study the properties of the solutions of the variational principle. We will consider a function $D(p)$ in the modified dispersion relation such that it is possible to find new momentum variables $\pi_\mu$, defined by a set of non-linear functions $F_\mu$ through $\pi_\mu \equiv F_\mu(p)$, so that

$$\pi_0^2 - \vec{\pi}^2 = [F_0(p)]^2 - \sum_{i=1}^{3} [F_i(p)]^2 = D(p).$$

(15)

The energy-momentum conservation laws in terms of the new momentum variables require to consider

$$K_{\mu}(\{F^{-1}(\pi)\}) \equiv K_{\mu}(\{\pi\}).$$

(16)

In order to identify transformations among solutions of the variational principle it is necessary to identify transformations of the momentum variables compatible with the restrictions on those variables from the dispersion relation and the conservation laws. The dispersion relation is invariant under a Lorentz transformation

$$\pi'^{(i)}_{\mu} = \Lambda_{\mu}^{'\nu} \pi^{(i)}_{\nu}$$

(17)

of the new momentum variables. If the conservation laws are such that

$$K_{\mu}(\{\pi\}) = \sum_{i=1}^{N} f_i(\{\pi\}) \pi_{\mu}^{(i)}$$

(18)

with $f_i(\{\pi'\}) = f_i(\{\pi\})$, i.e. a function of the $N(N+1)/2$ Lorentz invariant combinations of the $N$ momenta $\{\pi\}$, then the conservation laws will also be invariant under the transformation (17) and then $\{\{\pi'\}, \pi'\}$ with

$$p'_\mu = F^{-1}_\mu(\pi')$$

$$z'^\mu = \eta^{\mu\rho} \Lambda_{\rho}^{'\sigma} \eta_{\sigma\nu} z^\nu = \Lambda_{\mu}^{'\rho} z^\rho$$

(19)

will be a solution of the variational problem if $\Lambda_{\mu}^{'\nu}$ is a Lorentz transformation and if $\{\{\pi\}, \pi\}$ is a solution. In this case we have a space-time description of a multiparticle process going beyond SR and compatible with a relativity principle.

Together with the new momentum variables one can introduce new space-time coordinates $\xi$ through

$$x^\mu = \xi^\nu \frac{\partial F_\nu}{\partial p_\mu}$$

(20)

such that $x^\mu \hat{p}_\mu = \xi^\nu \hat{p}_\nu$ (i.e., to complete a canonical transformation). The variational problem can be solved in terms of the variables $\xi$, $\pi$ and the solution for the worldlines of the different particles will be

$$\xi_{-a}^{(i)\mu}(s) = z'^\mu a \frac{\partial K_{\nu}(\pi_{(j)} - a, \pi_{(k)} + c)}{\partial \pi_{-a}^{(i)\mu}} + (s - s_a) N_{-a}^{(i)} 2\eta^{\mu\nu} \pi_{(i)\nu}$$

(21)

$$\xi_{a+}^{(i)\mu}(s) = z'^\mu a \frac{\partial K_{\nu}(\pi_{(j)} + c, \pi_{(k)} + a)}{\partial \pi_{a}^{(i)\mu}} + (s - s_a) N_{a+}^{(i)} 2\eta^{\mu\nu} \pi_{(i)\nu}$$

(22)

$$\xi_{a,b}^{(i)\mu}(s) = z'^\mu a \frac{\partial K_{\nu}(\pi_{(j)} + c, \pi_{(k)} + a)}{\partial \pi_{a}^{(i)\mu}} + (s - s_a) N_{a,b}^{(i)} 2\eta^{\mu\nu} \pi_{(i)\nu}.$$  

(23)

When the conservation laws (18) are chosen in a way compatible with the relativity principle then one has that the Lorentz transformations connecting different solutions of the variational principle act on the worldlines as

$$\xi^{\mu}(s) = \eta^{\mu\nu} \Lambda_{\nu} \eta_{\sigma\rho} \xi^{\rho}(s) = \Lambda_{\mu}^{\rho} \xi^{\rho}(s)$$

(24)
which is the Lorentz transformation law of worldlines in special relativity. This transformation of worldlines can be reexpressed in terms of the original space-time coordinates by using Eq. (20).

The simplest example of a conservation law compatible with a relativity principle corresponds to the case of two particles \( (N = 2) \) and one has

\[
K_\mu (\pi (1), \pi (2)) = f(\pi (11), \pi (12), \pi (22)) \pi_\mu (1) + f(\pi (22), \pi (12), \pi (11)) \pi_\mu (2) \tag{25}
\]

with \( \pi (ij) = \eta^{\mu \nu} \pi_\mu (i) \pi_\nu (j) \). The general form of the conservation law is determined by a function \( f \) of three variables. If one assumes that the conservation laws with more than two particles are obtained from the case \( N = 2 \) by iteration then the function \( f \) determines all conservation laws. It has been shown \(^1\) that in this case it is possible to define a connection on momentum space related to \( f \); the implementation of the relativity principle as discussed in this Section leads to a geometry of momentum space which is fixed by the functions \( D \) and \( f \). In the general case the conservation law with more than two particles involves new independent functions and there is no geometrical interpretation of the generalization of SR.

IV. PHYSICAL INTERPRETATION AND RELATION WITH OTHER PROPOSALS

What is the physical meaning of the momentum coordinates \( p_\mu \) and the space-time coordinates \( x^\mu \) used in the description of the multiple-particle process? The energy-momentum of a particle has to be determined through a measurement process, which gives certain values \( p_\mu \). This is done using apparatuses that are calibrated in such a way that they measure the special relativistic energy and momentum in the limit where corrections to SR can be neglected. However, when these corrections are taken into account, we do not know exactly what our apparatus is measuring and in fact different apparatuses may give different results.

In the simplest kinematic generalization of SR one considers a modification of the dispersion relation at very high energies but the usual additive energy-momentum conservation laws. This is included in the general framework based on the action \( \Pi \) with \( K \) additive but a nontrivial function \( D \). In this case one has a violation of Lorentz invariance but locality is preserved.

When one goes beyond SR in a way compatible with the relativity principle then one has the action \( \Pi \) with

\[
D(p) = [F_0 (p)]^2 - \sum_{i=1}^{3} [F_i (p)]^2 \tag{26}
\]

and a nonlinear energy-momentum conservation law fixed by

\[
K_\mu (\{p\}) = K_\mu (\{F(p)\}) \tag{27}
\]

defined in terms of the \( K_\mu \) introduced in Eq. (18). Absolute locality is lost in this case owing to the nonlinearity of \( K_\mu (\{p\}) \), which arises in principle from two sources: the functions \( F \) and \( K \). The implementation of the relativity principle and in particular the generalized boost transformation of energy-momentum is fixed by the nonlinear mapping in momentum space \( F \). Different results for different measurement apparatuses would correspond to a dependence of the mapping \( F \) on the choice of the energy-momentum measurement apparatus. The coordinates \( \pi_\mu = F_\mu (p) \) are familiar in the context of generalizations of the relativity principle (DSR) compatible with an observer-independent high-energy scale \(^2\) \(^3\). They are considered as a useful tool in DSR under the name of auxiliary (or classical) energy-momentum variables. However, in canonical implementations of DSR the auxiliary variables are normally considered to compose additively \(^3\) and in this case all the departures from SR including the loss of absolute locality would be due to the difference between the results of energy-momentum measurements (the \( p_\mu \) variables) and the auxiliary variables \( \pi_\mu \). Different choices for the mapping \( F \) depending on a new energy scale (to be identified with the Planck mass if one assumes a gravitational origin for the generalization of SR) can reproduce the different versions of DSR. Nevertheless, in the general case one has nonlinear corrections in the energy-momentum conservation laws both from the mapping \( F \) and from \( K_\mu \) in Eq. (18). This gives a new perspective on DSR with a different realization of spacetime and invariance under translations. In fact the auxiliary space-time coordinates \( \xi^\mu \) corresponding to the auxiliary energy-momentum variables \( \pi_\mu \) share an important property with the spacetime of SR: it is possible to speak of their transformed space-time coordinates under a Lorentz boost (see Eq. (24)). This is not so, however, with respect

\(^3\) The fact that this does not need to be necessarily the case was indicated for the first time in Ref. \(^3\).
to translations, as it can be seen from Eqs. (21)-(23) (recall that a translation has been defined as \( z_1' \equiv z_1 + \epsilon \), where \( z_1 \) are the coordinates of one of the interactions). If one takes the measured energy-momentum \( p_\mu \) and the corresponding spacetime \( x^\mu \) then one has to consider also boost transformations on phase space.

Some paradoxes that have been pointed out in the context of DSR can be solved in the present perspective of generalizations of special relativity. Specifically, it has been indicated that the ‘natural’ definition of velocity in DSR, as in Hamiltonian mechanics with a canonical phase-space, \( v_g \equiv \partial E/\partial p_0 \) (which also allows one to interpret it as the group velocity of wave packets representing point particles in specific DSR frameworks [22]), poses some interpretation problems of the velocity as the parameter of Lorentz transformations (because the relation between relative velocity and rapidity depends on the mass of the particle), and with the principle of relativity (two particles which interact in a reference frame (RF) may never meet, as seen from another RF) [20, 21]. A different relation between the velocity and the energy-momentum variables arises in the context of free point particles propagating in \( \kappa \)-Minkowski spacetime, in such a way that \( v \neq v_g \) satisfies the usual relativistic composition law. This avoids the previous paradoxes at the cost of giving up the relation between wave packets and point particles in these theories [22].

The main difficulty to define velocity in DSR theories is that, being formulated in momentum space, they miss a clear space-time picture that should give an unequivocal answer. In the present work, however, one can explicitly calculate the velocity as

\[
v^j = \frac{\dot{x}^j}{\sqrt{\dot{x}^0}} = \frac{\partial D/\partial p_j}{\partial D/\partial p_0}_{D(p)=m^2} = \frac{dE}{dp_j}_{E=p_0(\vec{p}, m^2)}, \tag{28}
\]

so that indeed \( v = v_g \). If one forgets the difference between the auxiliary energy-momentum variables (\( \pi_\mu \)) and the results of the energy-momentum measurement (\( p_\mu \)) one would have a velocity

\[
v^j = \frac{\dot{x}^j}{\sqrt{\dot{x}^0}} = \frac{dE}{dp_j}_{E=p_0(\vec{p}, m^2)} = \pi_\mu^j, \tag{29}
\]

and then a composition law for velocities which is exactly that of SR, so that no paradoxes arise. If however one uses the physical phase-space variables \( (x^\mu, p_\mu) \), then the transformation of the velocity of a particle between two inertial observers will indeed depend on the mass of the particle [20]. Nevertheless, there is no contradiction with the relativity principle. The argument saying that two particles of different masses which are at relative rest for one inertial observer will indeed depend on the mass of the particle [20]. Nevertheless, there is no contradiction with the usual relativistic composition law.

The proposal \( v = v_g \) also produces an energy dependent velocity of photons in certain DSR models, whose consistency with (macroscopic) experimental observations has been recently questioned and debated [23]. Neglecting the difference between auxiliary energy-momentum variables and the measured energy-momentum, the velocity of photons is independent of the energy. This is also the case if \( F \) is such that the model has a special relativistic velocity transformation law. In any case the real issue is not the energy dependence or independence of the velocity of photons but the consequences of the absolute locality breaking, and there is a consistency with experimental observations.
because all of them correspond to an observer whose origin is not far away from the observations corresponding to interactions, which are well approximated by an absolutely local interaction model.

The possibility that the measurement process could define different ‘physical’ momenta depending on the apparatus is connected with the freedom in the choice of ‘calorimeters’ that has been introduced in Refs. [1, 25, 27], in which the proposed generalization of SR has a reading in terms of a geometry of the momentum space.

A geometry in momentum space can always be introduced if one considers a generalized energy-momentum conservation law which can be derived from the conservation law for a two particle system. Taking auxiliary variables \( \pi \) as coordinates in momentum space, a connection can be obtained from the algebra associated to the nonlinear conservation law \( K_\mu((\pi)) \) in Eq. (25) following the general procedure introduced in Ref. [1]. An internal law \( \oplus \) is defined in momentum space by

\[
\left[ \pi^{(1)} \oplus \pi^{(2)} \right]_\mu = K_\mu(\pi^{(1)}, \pi^{(2)})
\]

and the inverse \( \ominus \pi \) by

\[
K_\mu(\pi, \ominus \pi) = 0
\]

where it is necessary to assume that \( f(x, 0, 0) = f(0, 0, x) = 1 \) so that the origin in momentum space is the identity of the composition law. In the case of a function \( f \) symmetric under exchange of its first and third arguments, the inverse takes the simple form \( [\ominus \pi]_\mu = -\pi_\mu \) and the result for the connection can be written in a compact form,

\[
\Gamma^\alpha_\gamma(\pi) = - \left[ f(\pi^2, -\pi^2, \pi^2) f(0, 0, 0) \right]^2 \left[ \left( \delta^\beta_\gamma \pi^\alpha + \delta^\alpha_\gamma \pi^\beta \right) \partial_2 f(\pi^2, 0, 0) + 2\eta^{\beta\gamma} \pi_\alpha \partial_3 f(\pi^2, 0, 0) + \pi_\alpha \delta^{\beta\gamma} \pi^\gamma \partial_2 \partial_3 f(\pi^2, 0, 0) \right],
\]

where \( \partial_2 \) (\( \partial_3 \)) denotes the partial derivative of the function \( f \) of three variables in Eq. (25) with respect to its second (third) argument. It is clear that one does not have a torsion as a consequence of the commutativity of the algebra associated to the conservation law defined by \( K_\mu \) in Eq. (25).

If one uses the measured energy-momentum \( p_\mu \) then the internal law in Eq. (32) takes the form

\[
\left[ p^{(1)} \oplus p^{(2)} \right]_\mu = \tilde{K}_\mu(p^{(1)}, p^{(2)})
\]

with

\[
\tilde{K}_\mu(\{p\}) = F^{-1}_\mu(K(\{F(p)\})) = F^{-1}_\mu(K(\{p\}))
\]

where in the second equality we have used Eq. (27). The connection in the coordinates \( p_\mu \) is

\[
\tilde{\Gamma}^{\nu\mu}(p) = \left[ \frac{\partial F_\alpha}{\partial p_\mu} \frac{\partial F_\beta}{\partial p_\nu} \Gamma^{\beta\gamma}_\alpha (F(p)) + \frac{\partial^2 F_\gamma}{\partial p_\mu \partial p_\nu} \right] \frac{\partial F^{-1}_\rho}{\partial p_\gamma} (F(p)).
\]

We have included the expression for the connection in Eq. (37) to make explicit the geometric reformulation of the implementation of the relativity principle proposed in Section III and its relation to the proposal of relative locality [1] but we do not think that this is relevant for an analysis of the phenomenological consequences of the generalization of SR proposed in this work.

In the geometric context one considers observables which can be expressed in terms of intrinsic geometric properties [22]. The interpretation we are proposing is however less restrictive. It is possible to find special cases of observables whose expression in terms of the coordinates do not depend on the choice of coordinates, and in the particular case when there is a geometrical interpretation for the generalization of SR these observables will have a geometrical meaning. Their values will be independent of the choice of the energy-momentum measurement apparatus. But surely this is not a necessary condition that an observable should fulfil: the energy-momentum in SR (and its generalization) is a counterexample. In this sense the difference between our interpretation of variables and observables and that of Refs. [1, 22, 27] is similar to the one existing between Special Relativity (which is specifically contained in our proposal) and General Relativity.

In the discussion of the phenomenological implications of a modification of SR one requires to establish a correspondence among observables. In general one finds different possible candidates (different measurement apparatuses)

\[4 \text{ Or, in geometric language, if one changes coordinates in momentum space.} \]
for a given observable in SR. This ambiguity puts limitations on the predictability of the general framework used to discuss generalizations of SR. The possibility to find in some cases a candidate with a geometrical meaning does not eliminate the limitations on the predictability. The selection of this candidate is a choice among different observables. An application of these arguments in the example of the energy-dependence of the time delay of different signals from sources at very long distances will be presented elsewhere [28].

V. CONCLUDING REMARKS

We have presented a framework describing the interactions of particles in spacetime in a theory beyond SR. We have seen that a relativity principle (with a generalization of both translations and Lorentz transformations) may be implemented in the theory, which inevitably causes a loss of absolute locality. The central idea of describing interactions in spacetime as a generalization of the crossing of worldlines in SR through a variational principle was introduced in Ref. [1]. We describe however a more general framework which does not necessarily include the geometric interpretation of the formalism defined in Ref. [1] (for which a detailed discussion of the implementation of a relativity principle is still missing). We have also considered the physical interpretation of the variables appearing in the variational problem and the similarities and differences with DSR. In particular, some paradoxes concerning the definition of velocity in DSR find their natural solution within the present work. Phenomenological implications of the present way to introduce interactions beyond SR, such as time-delay calculations, will be presented elsewhere [28].

At this point we would like to make some comments on the generality of the implementation of the relativity principle based on the functions $K_\mu$ in Eq. (15). When imposing $K_\mu(\{\pi\}^\Lambda) = \Lambda_\mu^{\nu} K_\nu(\{\pi\})$, we have implicitly assumed that under a Lorentz transformation

$$\{\pi\}^\Lambda = \{\pi^\Lambda\}, \quad \text{(38)}$$

i.e., we have assumed that under a Lorentz transformation the set of momenta of different particles transform independently. But we could consider more general cases where one has directly a realization of Lorentz transformations on the set of momenta of the particles in each interaction. In fact we have seen that a translation requires to consider all the worldlines of the particles in each interaction and one does not have a realization of translations acting independently on each worldline. It could be interesting to explore these alternatives to the implementation of the relativity principle and their possible relation with a noncommutativity in spacetime and nontrivial implementations of boost transformations for multiparticle states.

There is still another possibility to try to implement a relativity principle in a generalization of SR. As discussed in Ref. [1], the idea of relative locality has a natural realization within a perspective based on a geometry of the momentum space. From this point of view it is natural to try to implement the relativity principle through the isometries of this geometry. In this way one does not need to make a choice of energy-momentum measurement apparatus as a starting point for the implementation of the relativity principle. It could be interesting to see the relation between this geometric implementation of the relativity principle and the simplest implementation based on the functions $K_\mu$ in Eq. (15) or the alternatives with a Lorentz transformation on the whole set of momenta of the particles participating in each interaction.

Another remark on the generality of the proposal for a generalization of SR based on the action (1) is that we could have considered more general forms of the energy-momentum conservation laws. From the algebra associated to the composition of two momenta it is possible to define the inverse of a given momentum. One possibility to implement the conservation laws is to consider the vanishing of the momentum obtained by successive composition of the momenta of the incoming particles and the inverse of the momenta of the outgoing particles. Another possibility, the one we have considered in our proposal, is based on the identification of an initial momentum as a result of successive compositions of the momenta of the incoming particles and a final momentum resulting from the composition of the outgoing particles. The conservation laws amount to the equality of the initial and final momentum or, equivalently, to the vanishing of the composition of the initial momentum and the inverse of the final momentum.

As a final remark, all the discussion in this work is based on a classical model for multiparticle interactions based on the action (1). It seems also interesting to explore the possibility to include quantum effects by going beyond the variational principle and considering a sum over paths weighted with the exponential of the action (1) with just one interaction.\(^5\) This could correspond to a first-quantized approach instead of a field theoretical approach to the generalized quantum theory.

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\(^5\) It is not necessary to include several interactions in the action; the contribution of processes with several interactions is included in the expansion of the exponential.
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