A hierarchy of sum-rules in out of equilibrium QFT

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Abstract

Generalising a result of classical mechanics an infinite set of conserved quantities can be found for the bare equations of motion describing the evolution of a scalar field in out of equilibrium quantum field theory, in the large $N$ approximation, with initial conditions corresponding to a thermal system of the free Hamiltonian. Using these new conserved quantities, sum-rules relating integrals over the mode-functions (momenta) can be derived. More, the corresponding renormalised quantities can also be computed out thus giving information about the evolution of the already known renormalised equations; finally it is also possible to write a renormalised version of the sum-rules.

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INTRODUCTION

Nowadays, out of equilibrium quantum field theory is believed to act as a powerful tool in order to understand systems within which the energy or particle density are high. This is typically the case for cosmological problems [1–3] such as inflation [4–8], for which the study of preheating [9,10] is now well understood with a self consistent treatment in out of equilibrium QFT. It is now understood how the particle production behaves [11,12], in particular what is the contribution of the quantum fluctuation in this process. The physics of heavy ion can also be studied [13] with this theory in particular the hadronisation process of the quark gluon plasma. The analysis of initial condition is also well understood [14,15].

During the last couple of years many impressive results have been found using accurate numerical computations with different approximations in the form of the equation of motion: one loop [16], Hartree, leading order in large $N$ expansion and even recently beyond the leading order. The variety was not only in the methods but also in the models and the physical problems that were studied: systems with fermions [17], electrodynamics [18,19], non-Abelian gauge fields [9]....

Although the scope of the theory is becoming wider and wider there only few analytical results compared to the numerical works. The new analytical result are mainly well controlled expansion [2], that have been occasionally re-summed [18,20]. Notwithstanding all those attempts exact results have seldom been found [21], only two sum-rules were found [22] in the case of a scalar field with initial conditions corresponding to a zero temperature bath, one has to mention that a generalisation for non zero temperature has been predicted using one loop computation [23].

The aim of the present paper is to prove the existence of a hierarchy of sum-rules that generalise the previous result. Using such sum-rules one should be able to find more on the system, since just using the first to sum-rules one was able to find out the asymptotical equation of state of the quantum system.

In order to get a progressive understanding the classical results will be summarised in the following section. Using those result it will be possible to construct in section I the bare conserved quantities for a scalar field in out of equilibrium field theory. Bare sum-rules expressed in terms of momenta, i.e. integrals over the modes and their derivates, will be explicited in section III. The analysis of the renormalisation process will be described in section IV for the conserved quantities; in section V it will be shown how the momenta and the sum-rules can be renormalised in a such way that the sum-rules still hold for renormalised quantities.

I. CLASSICAL OSCILLATOR IN RADIAL QUARTIC POTENTIAL

We just recall some results on the radial quartic potential, that have been studied by by Choodnovsky and Choodnovsky [24,25] and extended by Wojciechowski [26]. We shall give a brief review of the results of Grosse [27], we consider a N-dimensional system whose evolution is given from the following Hamiltonian:

$$H = \frac{1}{2} \sum_{i=1}^{N} (p_i^2 + r_k q_i^2) + \frac{1}{4} \left( \sum_{i=1}^{N} q_i^2 \right)^2,$$  (1.1)
where all the masses are different.

One can say that the $O(N)$ invariance is broken by different masses. Nonetheless the system remains integrable, since one can find a complete set of (independent) and commutating conserved quantities which can be written as:

$$K_i = \sum_{i\neq j} \frac{(q_ip_j - q_jp_i)^2}{r_j - r_i} + 2p_i^2 + 2r_iq_i^2 + q_i^2 \sum_{j=1}^{N} q_j^2$$  \hspace{1cm} (1.2)

It is straightforward to check the following:

$$\{ K_i, K_j \} = 0$$  \hspace{1cm} (1.3)

One should notice that, the Hamiltonian can easily be reconstructed from the $K_i$, one trivially has:

$$H = \frac{1}{4} \sum_{i=1}^{N} K_i$$  \hspace{1cm} (1.4)

II. BARE CONSERVED QUANTITIES IN OUT OF EQUILIBRIUM QFT

The main idea is to generalise the previous result to the infinite number of equation describing the motion of a scalar field in out of equilibrium quantum field theory. In the leading order in the large $N$ expansion the equations of motion read (for a detailed derivation one should refer to \[23,28,29,3\])

\[
\left( \frac{d^2}{dt^2} + M(t)^2 \right) \phi(t) = 0
\]

\[
\left( \frac{d^2}{dt^2} + k^2 + M(t)^2 \right) \varphi_k(t) = 0 ,
\]  \hspace{1cm} (2.1)

where the effective mass $M$ is defined through the bare fluctuations $\Sigma_B$ as:

$$M(t)^2 = m^2 + \frac{\lambda}{2} \phi(t)^2 + \frac{\lambda}{2} \Sigma(t)$$

$$\Sigma_B(t) = \frac{1}{4\pi^2} \int_0^\Lambda \coth \left( \frac{\beta W_k}{2} \right) |\varphi_k(t)|^2 k^2 dk$$

$$W_k = \sqrt{k^2 + M(0)^2} ,$$  \hspace{1cm} (2.2)

where $\Lambda$ denotes the cut-off.

The initial conditions read

$$\varphi_k(0) = \frac{1}{\sqrt{W_k}}$$

$$\dot{\varphi}_k(0) = -i\sqrt{W_k}$$  \hspace{1cm} (2.3)
One should notice that since these conditions are smooth functions of \( k \), the mode functions \( \varphi_k(t) \) will also have the same property at all time \( t \). The latter definitions hold whenever the initial system corresponds to a thermal state of the free Hamiltonian, with a temperature of \( \beta^{-1} \). The zero temperature limit can be obtained by letting \( \beta \) go to \( +\infty \). Hitherto, it is always possible to write the equation of motion as:

\[
\left( \frac{d^2}{dt^2} + \mathcal{M}(t)^2 \right) \psi(t) = 0
\]

Where the effective mass is written as

\[
\mathcal{M}(t)^2 = m^2 + \psi^2(t)^2 + S_0(t)
\]

\[
S_0(t) = \int_0^\Lambda |\psi_k(t)|^2 dk
\] (2.5)

To get such an expression it is enough to introduce the following notations:

\[
\psi_\eta = \phi \sqrt{\frac{\lambda}{2}}
\]

\[
\psi_k = k \varphi_k \sqrt{\frac{\lambda}{8\pi^2}} \coth \frac{\beta W_k}{2}
\] (2.6)

It is also convenient to introduce:

\[
S = \psi_\eta^2 + S_0 = \psi_\eta^2 + \int_0^\Lambda |\psi_k(t)|^2 dk
\]

\[
r_k = m^2 + k^2
\]

\[
r_\eta = m^2
\] (2.7)

This quantity \( S \) appears as a general quadratic sum, this will play the same role than the sum \( \sum q_i^2 \) in the classical analysis. Nevertheless one should notice that now the quantities are complex and that the condensate \( \phi \) plays a particular role.

With these notations, the natural quantities that generalise those defined by equation (1.2) are:

\[
K_k = \int_0^\Lambda \frac{|\dot{\psi}_k \dot{\psi}_k' - \dot{\psi}_k \psi_k'|^2}{k'^2 - k^2} dk' - \frac{1}{k^2} |\psi_\eta \dot{\psi}_k - \dot{\psi}_\eta \psi_k|^2 + 2 \left( |\dot{\psi}_k|^2 + (m^2 + k^2)|\psi_k|^2 \right) + |\psi_k|^2 S
\]

\[
K_\eta = \int_0^\Lambda \frac{|\dot{\psi}_\eta \dot{\psi}_k - \dot{\psi}_\eta \psi_k|^2}{k^2} dk + 2 \left( \dot{\psi}_\eta^2 + m^2 \psi_\eta^2 \right) + \psi_\eta^2 S
\] (2.8)

In order to manipulate these quantities we have to know whether there are well defined. Let first study \( K_\eta \), the only tricky term is the \( 1/k^2 \) in the integral, but this in fact a false pole since \( \psi_k \) contains \( k \) factor. Since \( \varphi_k \) is regular \( K_\eta \) is well defined. The same can be said for the \( 1/k^2 \) term of \( K_k \). So for \( K_k \) the term that might be ill defined is the first integral which contains \( 1/(k'^2 - k^2) \), but since \( \psi_k(t) \) is a smooth function of \( k \). 


and $t$, then then ratio as finite value when $k'$ approaches $k$, hence the integral is well defined and so is $K_k$.

Now we shall prove that these quantities $K_k$ and $K_\eta$ do not depend on time. This can be directly done using the equation of motion for $\psi_\eta$ and the $\psi_k$. The main ingredient of the prove is the use of the following property:

$$\dot{S} = \psi_\eta \dot{\psi}_\eta + \int_0^\Lambda \left( \dot{\psi}_k \dot{\psi}_k^* + \psi_k \dot{\psi}_k^* \right)$$

(2.9)

Lets study $K_\eta$ simple algebraic computation using the equation of motion leads to:

$$\dot{K}_\eta = \int_0^\Lambda \frac{1}{k^2} \left( (\psi_\eta \ddot{\psi}_k - \dot{\psi}_\eta \dot{\psi}_k)(\psi_\eta \dot{\psi}_k - \dot{\psi}_\eta \dot{\psi}_k)^* + \text{c.c.} \right) dk$$

$$+ 4\psi_\eta (\ddot{\psi}_\eta + m^2 \psi_\eta) + 2\psi_\eta \ddot{\psi}_\eta S + \psi_\eta \dot{S}$$

$$= \int_0^\Lambda \frac{1}{k^2} \left( (\psi_\eta (m^2 + S) \psi_k - \psi_k (m^2 + k^2 S) \psi_\eta) (\psi_\eta \dot{\psi}_k - \dot{\psi}_\eta \dot{\psi}_k)^* + \text{c.c.} \right) dk$$

$$- 4\psi_\eta (\ddot{\psi}_\eta + m^2 \psi_\eta) + 2\psi_\eta \ddot{\psi}_\eta S + \psi_\eta \dot{S}$$

$$= \psi_\eta \int_0^\Lambda \left( \psi_k (\psi_\eta \dot{\psi}_k - \dot{\psi}_\eta \dot{\psi}_k)^* + \text{c.c.} \right) dk - 2\psi_\eta \ddot{\psi}_\eta S + \psi_\eta \dot{S}$$

$$= 2\psi_\eta \ddot{\psi}_\eta (S - \psi_\eta^2) - \psi_\eta^2 (\dot{S} - 2\psi_\eta \ddot{\psi}_\eta) - 2\psi_\eta \ddot{\psi}_\eta S + \psi_\eta \dot{S}$$

$$\dot{K}_\eta = 0$$

(2.10)

A similar computation proves that the $K_k$ are also time independent.

$$\dot{K}_k = \int_0^\Lambda \frac{1}{k'^2 - k^2} \left( (\psi_k \ddot{\psi}_k' - \dot{\psi}_k' \dot{\psi}_k')(\psi_k \dot{\psi}_k' - \dot{\psi}_k' \dot{\psi}_k)^* + \text{c.c.} \right) dk'$$

$$\frac{1}{k^2} \left( (\psi_k \ddot{\psi}_k - \dot{\psi}_k \dot{\psi}_k)(\psi_k \dot{\psi}_k - \dot{\psi}_k \dot{\psi}_k)^* + \text{c.c.} \right)$$

$$+ 2 \left( \psi_k \dot{\psi}_k^* - \dot{\psi}_k \dot{\psi}_k^* \right) S + |\psi_k|^2 \dot{S}$$

$$= \int_0^\Lambda \frac{1}{k'^2 - k^2} \left( \psi_k \psi_k' (k^2 - k'^2)(\psi_k \dot{\psi}_k' - \dot{\psi}_k' \dot{\psi}_k)^* + \text{c.c.} \right) dk'$$

$$\frac{1}{k^2} \left( \psi_k \psi_k (-k^2)(\psi_k \dot{\psi}_k - \dot{\psi}_k \dot{\psi}_k)^* + \text{c.c.} \right)$$

$$+ 2 (-S \psi_k \dot{\psi}_k^* + \text{c.c.})$$

$$+ (\psi_k \dot{\psi}_k^* - \dot{\psi}_k \dot{\psi}_k^*) S + |\psi_k|^2 \dot{S}$$

$$= \int_0^\Lambda \left( \psi_k \psi_k' (\psi_k \dot{\psi}_k - \dot{\psi}_k \dot{\psi}_k)^* + \text{c.c.} \right) dk'$$

$$\psi_k (\psi_k \psi_k - \psi_k \dot{\psi}_k)^* + \text{c.c.}$$

$$+ 2 (-S \psi_k \dot{\psi}_k^* + \text{c.c.})$$

$$= (\psi_k \dot{\psi}_k^* + \text{c.c.})(S - \psi_\eta^2) - |\psi_k|^2 (\dot{S} - 2\psi_\eta \ddot{\psi}_\eta)$$

$$+ \psi_\eta^2 (\psi_k \dot{\psi}_k^* + \text{c.c.}) - 2\psi_\eta \ddot{\psi}_\eta |\psi_k|^2$$

$$- S (\psi_k \dot{\psi}_k^* + \text{c.c.}) + \dot{S} |\psi_k|^2$$

$$\dot{K}_k = 0$$

(2.11)
The final step is to express the conserved quantities in term of the physical modes, that is \( \phi \) and \( \varphi_k \); we get the following bare conserved quantities, the first is given from \( K_\eta \) and reads

\[
C_0, B = \frac{1}{2} \dot{\phi}^2 + \frac{m_B^2}{2} \phi^2 + \frac{\lambda_B}{8} \phi^4 + \frac{\lambda_B}{32 \pi^2} \int_0^\Lambda \coth \frac{\beta W_k}{2} \left[ |\dot{\phi}_k - \dot{\varphi}_k|^2 + k^2 |\varphi_k|^2 \phi^2 \right] dk ,
\]

and the other are a set \( C(k)_B \) given from the \( K_k \):

\[
C(k)_B = \frac{k^2}{2} \left( |\dot{\varphi}_k|^2 + (m_B^2 + k^2)|\varphi_k|^2 \right) + \frac{\lambda_B}{8} \left( k^2 |\varphi_k|^2 \phi^2 - |\dot{\phi}_k - \dot{\varphi}_k|^2 \right) + \frac{\lambda_B k^2}{32 \pi^2} \int_0^{k'} k'^2 \coth \frac{\beta W_{k'}}{2} \left[ \frac{|\varphi_k \dot{\varphi}_{k'} - \dot{\varphi}_k \varphi_{k'}|^2}{k'^2 - k^2} + |\varphi_k|^2 |\varphi_{k'}|^2 \right] dk' .
\]

One should notice, that in the last set of conserved quantities \( C(k = 0)_B \) is indeed trivial, since it is expressed in term of a Wronskian ( \( \phi \) and \( \varphi_0 \) are solution of the same second order differential equation),

\[
C(0)_B = - \frac{\lambda_B}{2} |\dot{\phi}_0 - \dot{\varphi}_0|^2 .
\]

### III. HIERARCHY OF BARE SUM-RULES

Although we have already found an infinite number of relations that can be considered as sum-rules, it is possible to go further and prove the existence of relations between several integrals over the modes.

The first sum-rule will be given from

\[
E_{0,B} = C_{0,B} + \frac{1}{4 \pi^2} \int_0^\Lambda C(k)_B \coth \frac{\beta W_k}{2} dk ,
\]

whereas the other will be derivated from

\[
E_{n,B} = \frac{1}{2 \pi^2} \int_0^\Lambda k^{2n} C(k)_B \coth \frac{\beta W_k}{2} dk , \quad n \geq 1 .
\]

All the quantities \( E_{n,B} \) are trivially conserved and a constant may be added without changing this property.

One can easily show that:

\[
E_{0,B} = \frac{1}{2} \dot{\phi}^2 + \frac{m_B^2}{2} \phi^2 + \frac{\lambda_B}{8} \left( \phi^4 - \Sigma_B^2 \right) + \frac{1}{8 \pi^2} \int_0^\Lambda k^2 \coth \frac{\beta W_k}{2} \left( |\dot{\varphi}_k|^2 + \omega_k^2 |\varphi_k|^2 \right) dk ,
\]

\[
\omega_k(t) = \sqrt{k^2 + M(t)^2} .
\]

This shows that \( E_{0,B} \) is nothing but the bare energy. This property was expected since it is a generalisation of equation (1.4).

The hierarchy of sum-rule will be given from the newly constructed conserved quantities
For $n$ greater than 1, hereafter we shall use the following notation to express some integrals over the modes, that are indeed momenta:

\[
\Sigma_{n,B} = \frac{1}{4\pi^2} \int_0^\Lambda k^{2n} |\varphi_k|^2 \coth \frac{\beta W_k}{2} dk
\]

\[
\Theta_{n,B} = \frac{1}{4\pi^2} \int_0^\Lambda k^{2n} |\dot{\varphi}_k|^2 \coth \frac{\beta W_k}{2} dk
\]

\[
\Xi_{n,B} = \frac{1}{8\pi^2} \int_0^\Lambda k^{2n} \left( \varphi_k \dot{\varphi}_k^* + \varphi_k^* \dot{\varphi}_k \right) \coth \frac{\beta W_k}{2} dk
\]

(3.4)

One should notice that $\Sigma_{1,B}$ is nothing but $\Sigma_B$.

Using these notations it is straightforward to get

\[
E_{n,B} = \Theta_{n+1,B} + m_B^2 \Sigma_{n+1,B} + \Sigma_{n+2,B}
\]

\[
+ \frac{\lambda_B}{4} \left( \phi^2 \Sigma_{n+1,B} - \dot{\phi}^2 \Theta_{n,B} - \dot{\phi}^2 \Sigma_{n,B} + 2\dot{\phi}\dot{\phi} \Xi_{n,B} + I_{n,B} + \Sigma_B \Sigma_{n+1,B} \right),
\]

where $I_{n,B}$ reads:

\[
I_{n,B} = \frac{1}{(4\pi^2)^2} \int_0^\Lambda |\varphi_k \dot{\varphi}_k' - \dot{\varphi}_k \varphi_k'|^2 \sqrt{\frac{k^{2n} k^{2n}}{k'^2 - k^2}} \coth \frac{\beta W_k}{2} \coth \frac{\beta W_{k'}}{2} dkdk'.
\]

(3.5)

It is possible to express $I_{n,B}$ as a sum of integrals $\Sigma_{p,B}$, $\Theta_{p,B}$ and $\Xi_{p,B}$, to achieve this one should use the trivial relation:

\[
x^n = (x - y)Q_n(x, y) + y^n
\]

\[
Q_n(x, y) = \sum_{p=0}^{n-1} x^p y^{n-1-p}
\]

(3.7)

Hence,

\[
I_{n,B} = \int_0^\Lambda \frac{k^{2n} k'^2}{4\pi^2} dk dk' |\varphi_k \dot{\varphi}_k' - \dot{\varphi}_k \varphi_k'|^2 Q_n(k^2, k'^2) + \frac{k'^2}{k'^2 - k^2} \coth \frac{\beta W_k}{2} \coth \frac{\beta W_{k'}}{2} \].

(3.8)

Swapping $k$ and $k'$ leads to:

\[
I_{n,B} = \frac{1}{2(4\pi^2)^2} \int_0^\Lambda |\varphi_k \dot{\varphi}_k' - \dot{\varphi}_k \varphi_k'|^2 Q_n(k^2, k'^2) k^2 k'^2 \coth \frac{\beta W_k}{2} \coth \frac{\beta W_{k'}}{2} dkdk'.
\]

(3.9)

Using the following Wronskian relation

\[
\varphi_k \dot{\varphi}_k' - \dot{\varphi}_k \varphi_k = 2i
\]

(3.10)

one gets:

\[
|\varphi_k \dot{\varphi}_k' - \dot{\varphi}_k \varphi_k| = |\dot{\varphi}_k|^2 |\varphi_k'|^2 + |\dot{\varphi}_k|^2 |\varphi_k|^2 + \frac{1}{2} (\dot{\varphi}_k \varphi_k^* + \dot{\varphi}_k^* \varphi_k + \dot{\varphi}_k^* \varphi_k^*) + 2
\]

(3.11)

The last 2 will just give an additive constant term to $E_{n,B}$ that can be dropped out, using the latter relation, one can write $I_{n,B}$ as
\[ I_{n,B} = \sum_{p=1}^{n} \left( \Sigma_{p,B} \Theta_{n+1-p,B} - \Xi_{p,B} \Xi_{n+1-p,B} \right) \]  

(3.12)

One finally get an infinite number of relations between the \( \Sigma_{p,B}, \Theta_{p,B} \) and \( \Xi_{p,B} \):

\[
E_{n,B} = \Theta_{n+1,B} + m_B^2 \Sigma_{n+1,B} + \Sigma_{n+2,B} \\
+ \frac{\lambda_B}{4} \left[ \phi^2 (\Sigma_{n+1,B} - \Theta_{n,B}) - \dot{\phi}^2 \Sigma_{n,B} + 2 \phi \dot{\phi} \Xi_{n,B} \\
+ \Sigma_B \Sigma_{n+1,B} + \sum_{p=1}^{n} \left( \Sigma_{p,B} \Theta_{n+1-p,B} - \Xi_{p,B} \Xi_{n+1-p,B} \right) \right], \quad n \geq 1.
\]

(3.13)

These equations can indeed be seen as a hierarchy of relations between integrals over the modes.

### IV. RENORMALISED CONSERVED QUANTITIES

It is has already been proven \[3,10,23\] that the equation of motion for a quantum scalar field can be written in term of renormalised quantities. Let recall here the main results that are given with more details in previous studies. There are several important points. First that since the short distance behaviour is not related to the out of equilibrium state of the system, the coupling constant is renormalised as in usual quantum field theory, that is:

\[
\lambda_B = \frac{\lambda_R}{1 - \frac{\lambda_R}{16\pi^2} \log \frac{\Lambda}{\mu}} \\
m_B^2 + \frac{\lambda_B}{16\pi^2} \Lambda^2 = m_R^2 \left( 1 + \frac{\lambda_B}{16\pi^2} \log \frac{\Lambda}{\mu} \right)
\]

(4.1)

Where \( \mu \) is the renormalisation scale and \( B \) and \( R \) subscripts denotes the bare and renormalised quantities.

The second point is that all the modes are not modified since the effective mass \( M \) is not changed:

\[
M_B(t) = M_R(t) \\
m_B^2 + \frac{\lambda_B}{2} \phi(t)^2 + \frac{\lambda_B}{2} \Sigma_B(t) = m_R^2 + \frac{\lambda_R}{2} \phi(t)^2 + \frac{\lambda_R}{2} \Sigma_R(t)
\]

(4.2)

Using these notations, the renormalised equations remains formally the same

\[
\left( \frac{d^2}{dt^2} + M(t)^2 \right) \phi(t) = 0 \\
\left( \frac{d^2}{dt^2} + k^2 + M(t)^2 \right) \varphi_k(t) = 0 \\
M(t)^2 = m_R^2 + \frac{\lambda_R}{2} \phi(t)^2 + \frac{\lambda_R}{2} \Sigma_R(t)
\]

(4.3)

just the fluctuations \( \Sigma \) are renormalised as:
\[
\Sigma_R(t) = \frac{1}{4\pi^2} \int_0^\infty k^2 \left[ \coth \left( \frac{\beta W_k}{2} \right) |\varphi_k(t)|^2 - \frac{1}{k} + \frac{\Theta(k - \mu)}{2k^3} \mathcal{M}(t)^2 \right] dk
\]

\[
W_k = \sqrt{k^2 + \mathcal{M}(0)^2}
\]  \hspace{1cm} (4.4)

With the unchanged initial conditions:

\[
\varphi_k(0) = \frac{1}{\sqrt{W_k}}
\]
\[
\dot{\varphi}_k(0) = -i \sqrt{W_k}
\]  \hspace{1cm} (4.5)

The last point concerns the behaviour of \(|\varphi_k|^2\) and \(|\dot{\varphi}_k|^2\) for large momenta \(k\), that can be found using a WKB expansion, in term of the effective potential \(v(t)\), these read:

\[
|\varphi_k|^2 = \frac{1}{k} + \frac{v}{2k^3} + \frac{3v^2 - \ddot{v}}{8k^5} + \mathcal{O}(\frac{1}{k^7})
\]
\[
|\dot{\varphi}_k|^2 = k - \frac{v}{2k} + \dot{v} - \nu^2 + \mathcal{O}(\frac{1}{k^5})
\]
\[
v(t) = -\mathcal{M}(t)^2
\]  \hspace{1cm} (4.6)

Now in order to construct renormalised conserved quantities, we start with the previously found quantities, which are expressed in terms of the bare coupling constants and fluctuations.

A direct computation leads to:

\[
\left(1 - \frac{\lambda_R}{16\pi^2} \log \frac{\Lambda}{\mu} \right) C_{0,B} = \frac{1}{2} \dot{\phi}^2 + \frac{m_R^2}{2} \phi^2 + \frac{\lambda_R}{8} \phi^4 - \frac{\lambda_R}{32\pi^2} \int_0^\Lambda \left[ 2k\phi^2 + \frac{\Theta(k - \mu)}{k} \dot{\phi}^2 \right] dk
\]
\[
+ \frac{\lambda_R}{32\pi^2} \int_0^\Lambda \left[ |\dot{\varphi}_k - \dot{\varphi}_k|^2 + k^2 \phi^2 |\varphi_k|^2 \right] \coth \frac{\beta W_k}{2} \frac{dk}{dk}
\]
\[
= \frac{1}{2} \dot{\phi}^2 + \frac{m_R^2}{2} \phi^2 + \frac{\lambda_R}{8} \phi^4
\]
\[
+ \frac{\lambda_R}{16\pi^2} \int_0^\Lambda \left[ |\dot{\varphi}_k - \dot{\varphi}_k|^2 + k^2 \phi^2 |\varphi_k|^2 \right] \frac{dk}{e^{\beta W_k} - 1}
\]
\[
+ \frac{\lambda_R}{32\pi^2} \varphi_k \dot{\varphi}_k^2 \int_0^\Lambda \left[ |\varphi_k|^2 + k^2 |\varphi_k|^2 - 2k \right] dk
\]
\[
+ \frac{\lambda_R}{32\pi^2} \dot{\varphi}_k \varphi_k^2 \int_0^\Lambda \left[ |\varphi_k|^2 - \frac{\Theta(k - \mu)}{k} \right] \frac{dk}{dk}
\]
\[
- \frac{\lambda_R}{32\pi^2} \varphi_k \dot{\varphi}_k^2 \int_0^\Lambda \left[ |\varphi_k|^2 + \dot{\varphi}_k^2 + \varphi_k^2 \right] \frac{dk}{dk} \quad (4.7)
\]

The integral has been split into four parts each of which having, as it will be shown, a finite limit when \(\Lambda\) goes to infinity. The explicit temperature dependence has been put into the first integral. The latter is convergent since the mode functions have, in the worse case, a polynomial behaviour which is washed out by the exponential factor.

Only the last three integrals have to studied with care. This is done using equations (4.6), which give the asymptotic behaviour of the modes for large \(k\):
\[ |\dot{\varphi}_k|^2 + k^2|\varphi_k|^2 - 2k = \frac{v^2}{4k^3} + \mathcal{O}\left(\frac{1}{k^6}\right) \]
\[ |\varphi_k|^2 - \frac{1}{k} = \frac{v}{2k^3} + \mathcal{O}\left(\frac{1}{k^6}\right) \]  
(4.8)

This proves the convergence of the two first integrals, the last one is analysed computing:

\[ |\varphi_k\dot{\varphi}_k^*=| = |\varphi_k|^2|\dot{\varphi}_k^*|^2 = \left(1 + \frac{v}{2k^2} + \frac{3v^2}{8k^4}\right) \left(1 - \frac{v}{2k^2} + \frac{v^2}{8k^4}\right) + \mathcal{O}\left(\frac{1}{k^6}\right) \]
\[ = 1 + O\left(\frac{1}{k^6}\right) \]
\[ = (\text{Re}\varphi_k\dot{\varphi}_k^*)^2 + (\text{Im}\varphi_k\dot{\varphi}_k^*)^2 \]  
(4.9)

The Wronskian relation (3.10) can be rewritten as

\[ (\text{Im}\varphi_k\dot{\varphi}_k^*) = 1. \]  
(4.10)

This leads directly to

\[ (\text{Re}\varphi_k\dot{\varphi}_k^*) = \varphi_k\dot{\varphi}_k^* + \dot{\varphi}_k\varphi_k^* = O\left(\frac{1}{k^3}\right) \]  
(4.11)

which proves the convergence of the last integral.

Hence the renormalised conserved quantity \( C_{0,R} \) reads

\[ C_{0,R} = \frac{1}{2} \dot{\phi}^2 + \frac{m_R^2}{2} \phi^2 + \frac{\lambda_R}{8} \phi^4 \]
\[ + \frac{\lambda_R}{32\pi^2} \int_0^\infty \left[ \left( |\dot{\varphi}_k - \dot{\varphi}_k k^2 + k^2 \phi^2 |\varphi_k|^2 \right) \coth \frac{\beta W_k}{2} - 2k\phi^2 - \frac{\Theta(k - \mu)}{k} \phi^2 \right] dk \]  
(4.12)

Now it is possible to do about the same work with \( C_{n,B} \), a similar computation leads to:

\[ \left(1 - \frac{\lambda_R}{16\pi^2} \log \frac{\Lambda}{\mu}\right) C(k)_B = \frac{k^2}{2} \left( |\dot{\varphi}_k|^2 + (m_R^2 + k^2)|\varphi_k|^2 \right) + \frac{\lambda_R}{8} \left( k^2|\varphi_k|^2\phi^2 - |\dot{\varphi}_k - \dot{\varphi}_k|^2 \right) \]
\[ + \frac{\lambda_Rk^2}{16\pi^2} \int_0^\Lambda \frac{k^2}{k^2 - \kappa^2} \left[ \frac{|\varphi_k\dot{\varphi}_k' - \dot{\varphi}_k\varphi_k'|^2}{k^2 - k^2} + |\varphi_k|^2|\varphi_k'|^2 \right] dk' \]
\[ + \frac{\lambda_Rk^2}{32\pi^2} \int_0^\Lambda \left[ k^2 \left( |\varphi_k\dot{\varphi}_k' - \dot{\varphi}_k\varphi_k'|^2 + |\varphi_k|^2|\varphi_k'|^2 \right) - 2k'|\varphi_k|^2 \right] dk' \]
\[ - \frac{\lambda_Rk^2}{32\pi^2} \int_0^\Lambda \frac{\Theta(k - \mu)}{\kappa'} \left( |\dot{\varphi}_k|^2 + k^2|\varphi_k|^2 \right) dk' \]  
(4.13)

However, as it shall be shown it is not yet possible to let \( \Lambda \) go to infinity in this last expression.

There are three integrals that may diverge:
\[
\int_\Lambda \left( \frac{k'^2}{k'^2 - k^2} |\phi_{k'}|^2 + k'^2 |\varphi_{k'}|^2 - 2k' - \frac{k^2}{k'} \right) dk'
\]
\[
\int_\Lambda \left( \frac{k'^2}{k'^2 - k^2} |\varphi_{k'}|^2 - \frac{1}{k'} \right) dk'
\]
\[
\int_\Lambda \frac{k'^2}{k'^2 - k^2} \left( \varphi^*_k \phi^*_k \phi_k \varphi_{k'} + \phi_k \phi^*_k \phi^*_k \varphi^*_k \right) dk'
\]
(4.14)

Since we are interested in the behaviour for large \( k' \) the lower bound of the integral has been omitted, and is indeed assumed to be greater the \( k \) and \( \mu \), that is avoiding the possible divergences for \( k' \) going to \( k \) that are known not to exist.

The expansions already used previously proves that the integrants of the to first integrals are such that
\[
\frac{k'^2}{k'^2 - k^2} |\phi_{k'}|^2 + k'^2 |\varphi_{k'}|^2 - 2k' - \frac{k^2}{k'} = O\left( \frac{1}{k'^3} \right)
\]
\[
\frac{k'^2}{k'^2 - k^2} |\varphi_{k'}|^2 - \frac{1}{k'} = O\left( \frac{1}{k'^3} \right)
\]
(4.15)

This means that the two first integrals have a finite value for large \( \Lambda \).

For that last integral we shall again use the Wronskian relation (3.10) as in equation (3.11):
\[
\varphi^*_k \phi^*_k \phi_k \varphi_{k'} + \varphi_k \phi^*_k \phi^*_k \varphi^*_k = 2 + \frac{1}{2} (\hat{\phi}_k \hat{\varphi}_k + \varphi_k \hat{\phi}_k)(\hat{\phi}_{k'} \varphi^*_k + \varphi_{k'} \hat{\varphi}_{k'})
\]
(4.16)

So the last equation can be rewritten as:
\[
\int_\Lambda \frac{k'^2}{k'^2 - k^2} \left( \varphi^*_k \phi^*_k \phi_k \varphi_{k'} + \phi_k \phi^*_k \phi^*_k \varphi^*_k \right) dk'
\]
\[
= \frac{1}{2} \int_\Lambda \frac{k'^2}{k'^2 - k^2} \left( \varphi^*_k \phi^*_k + \varphi_k \hat{\phi}_k \right) \left( \varphi_{k'} \varphi^*_k + \varphi_{k'} \hat{\varphi}_{k'} \right) dk' + \int_\Lambda \frac{2k^2}{k'^2 - k^2} dk' + \int_\Lambda 2 dk',
\]
(4.17)

where just the last integral will give rise to a divergence, but this integral is indeed a constant so can be dropped without adding a time dependent quantity.

After letting \( \Lambda \) go to infinity the renormalised \( C(k)_R \) read:
\[
C(k)_R = \frac{k^2}{2} \left( |\phi_k|^2 + (m^2 + k^2) |\varphi_k|^2 \right) + \frac{\lambda_R}{8} \left( k^2 |\varphi_k|^2 \phi^2 - |\phi_k| |\phi_{k'} - \hat{\phi}_{k'}|^2 \right)
\]
\[
+ \frac{\lambda_R k^2}{32 \pi^2} \int_0^\infty \left[ k^2 \coth \left( \frac{\beta W_k}{2} \right) \left( \frac{|\varphi_k \phi_{k'} - \hat{\varphi}_k \phi_{k'}|^2}{k'^2 - k^2} + |\varphi_k|^2 |\varphi_{k'}|^2 \right) - 2 - 2k' |\varphi_k|^2 - \frac{\Theta(k' - \mu)}{k'} \left( |\varphi_k|^2 + k^2 |\varphi_k|^2 \right) \right] dk'
\]
(4.18)

V. RENORMALISED SUM-RULES AND MOMENTA

The aim of the present section is to generalises the sum-rules.
\[ E_{n,B} = \Theta_{n+1,B} + m_B^2 \Sigma_{n+1,B} + \Sigma_{n+2,B} \]
\[ + \frac{\lambda_B}{4} \left[ \phi^2 (\Sigma_{n+1,B} - \Theta_{n,B}) - \dot{\phi}^2 \Sigma_{n,B} + 2\phi \dot{\phi} \Xi_{n,B} \right] + \Sigma_B \Sigma_{n+1,B} + \sum_{p=1}^{n} \left( \Sigma_{p,B} \Theta_{n+1-p,B} - \Xi_{p,B} \Xi_{n+1-p,B} \right) \]
\[ , \quad n \geq 1 \quad (5.1) \]

with renormalised version of the momenta.

The pivotal relations that shall be used are the transcription of the equation of motion in term of the momenta, those read as:

\[ \dot{\Theta}_n + 2\mathcal{M}^2 \Xi_n + 2\Xi_{n+1} = 0 \]
\[ \dot{\Sigma}_n = 2\Xi_n \]
\[ \dot{\Xi}_n + \Sigma_{n+1} + \mathcal{M}^2 \Sigma_n = \Theta_n \quad (5.2) \]

This set of equations are true for bare quantities and we generalise them for regularised and renormalised momenta.

In order, to simplify the discussion the following notation is introduce: the sum-rules are written as:

\[ E_n = \Theta_{n+1} + \Sigma_{n+2} + \mathcal{M}^2 \Sigma_{n+1} + F_n, \quad n \geq 1 \quad (5.3) \]

where \( F_n \) is a functional depending only on \( \Sigma_k, \Theta_k \) and \( \Xi_k \) for \( k \) at most equal to \( n \); a similar relation is also true for the energy \( (E_0) \):

\[ 2E_0 = \Theta_1 + \Sigma_2 + \mathcal{M}^2 \Sigma_1 + F_0, \quad (5.4) \]

where \( F_0 \) depend on \( \Sigma_1, \phi \) and \( \dot{\phi} \).

The prove is inductive, we shall construct all the renormalised momenta order per order and thus generalising the sum-rules. Using the standard expansion \((4.6)\), one is able to renormalise the first momenta:

\[ \Sigma_{0,R} = \frac{1}{4\pi^2} \int_0^\infty \left| \phi_k \right|^2 \coth \frac{\beta W_k}{2} - \frac{1}{k} \Theta(k - \mu) \right) dk , \]
\[ \Theta_{0,R} = \frac{1}{4\pi^2} \int_0^\infty \left( \frac{\beta W_k}{2} - k + \frac{v}{k} \right) \Theta(k - \mu) \right) dk . \quad (5.5) \]

We straightforwardly construct \( \Xi_{0,R} \) and \( \Sigma_{1,R} \) using the equation of motion for the momenta:

\[ \Xi_{0,R} = \frac{1}{2} \dot{\Sigma}_{0,R} , \]
\[ \Sigma_{1,R} = \Theta_{0,R} - (\Xi_{0,R} + \mathcal{M}^2 \Sigma_{0,R}) , \quad (5.6) \]

one finds that

\[ \Xi_{0,R} = \frac{1}{8\pi^2} \int_0^\infty \left( \phi_k \hat{\phi}_k \hat{\phi}_k \phi_k \right) \coth \frac{\beta W_k}{d} dk , \]
\[ \Sigma_R = \Sigma_{1,R} = \frac{1}{4\pi^2} \int_0^\infty \left( k^2 |\phi_k|^2 \coth \frac{\beta W_k}{2} - k - \frac{v}{2k} \Theta(k - \mu) \right) dk . \quad (5.7) \]
One should notice that this last expression is the usual form for the fluctuations. Whenever one wants to make a finite change in the renormalisation of the fluctuation $\Sigma$, one should also make a change in the momenta $\Theta_{0,R}$ and for the renormalised mass $m^2_{R}$; for instance a common change is:

$$m^2_{R} \rightarrow m^2_{R} + \frac{\lambda_{R}}{2} \Sigma_{1,R}(0) ,$$
$$\Sigma_{1,R} \rightarrow \Sigma_{1,R} - \Sigma_{1,R}(0) ,$$
$$\Theta_{0,R} \rightarrow \Theta_{0,R} - \Sigma_{1,R}(0) .$$  \hspace{1cm} (5.8)

The key point is that this transformation do not modify the equations of motion (5.2).

Using, similar technics one easily finds that:

$$\Xi_{1,R} = \frac{1}{2} \dot{\Sigma}_{1,R} = \frac{1}{8\pi^2} \int_{0}^{\infty} \left( k^2 \left( \varphi_{k} \dot{\varphi}_{k} \dot{\varphi}_{k}^{*} \right) \coth \frac{\beta W_{k}}{2} + \frac{i}{4k} \Theta(k - \mu) \right) dk .$$  \hspace{1cm} (5.9)

In order to find $\Theta_{1,R}$ and $\Sigma_{2,R}$ one should use the following relations together

$$2E_{0} = \Theta_{1} + \Sigma_{2} + M^2 \Sigma_{1} + F_{0} ,$$
$$\dot{\Xi}_{1} + \Sigma_{2} + M^2 \Sigma_{1} = \Theta_{1} .$$  \hspace{1cm} (5.10)

Since the energy is a constant it can be renormalised, with a simple subtraction:

$$E_{0,R} = E_{0,R}(t_{0}) + E_{0,B} - E_{0,B}(t_{0})$$  \hspace{1cm} (5.11)

Moreover $F_{0}$ can be renormalised since it is expressed in term of renormalisable quantities. So we juste have to solve a set of two linear equations in $\Theta_{1,R}$ and $\Sigma_{2,R}$.

Now one can generalised this to all the momenta by induction. Let suppose that one already has renormalised all the momenta up to $\Sigma_{n+1}, \Xi_{n}$ and $\Theta_{n}$. On easily get:

$$\Xi_{n+1,R} = \frac{1}{2} \dot{\Sigma}_{n+1,R}$$  \hspace{1cm} (5.12)

To find $\Theta_{n+1}$ and $\Sigma_{n+2}$, one uses the fact that $E_{n}$ is a constant so can be renormalised, so one have to solve a set of to linear equations for renormalised quantities.

$$E_{n} = \Theta_{n+1} + \Sigma_{n+1} + M^2 \Sigma_{n+2} + F_{n} ,$$
$$\dot{\Xi}_{n+1} + \Sigma_{n+2} + M^2 \Sigma_{n+1} = \Theta_{n+1} ,$$  \hspace{1cm} (5.13)

where $F_{n}$ is a function of already renormalised quantities. Using this method one is able to both get a renormalisation procedure for the momenta and conserve the form of the sum-rule, that can be written in terms of renormalised quantities:

$$E_{n,R} = \Theta_{n+1,R} + m^2_{R} \Sigma_{n+1,R} + \Sigma_{n+2,R} + \frac{\lambda_{R}}{4} \left[ \varphi^2 (\Sigma_{n+1,R} - \Theta_{n,R}) - \dot{\varphi}^2 \Sigma_{n,R} + 2 \dot{\varphi} \dot{\varphi} \Xi_{n,R} + \Sigma_{R} \Sigma_{n+1,R} + \sum_{p=1}^{n} (\Sigma_{p,R} \Theta_{n+1-p,R} - \Xi_{p,R} \Xi_{n+1-p,R}) \right] , \ n \geq 1 ,$$  \hspace{1cm} (5.14)
OUTLINE

The main result of the paper was to prove the existence of a hierarchy of sum-rules which were given in term of the mode functions of a scalar field in out of equilibrium quantum field theory.

The bare and renormalised conserved quantities read

\[
C_{0,B} = \frac{1}{2} \dot{\phi}^2 + \frac{m_B^2}{2} \phi^2 + \frac{\lambda_B}{8} \phi^4 + \frac{\lambda_B}{32\pi^2} \int_0^\Lambda \coth \frac{\beta W_k}{2} \left[ |\phi_k - \dot{\phi} \varphi_k|^2 + k^2 |\varphi_k|^2 \phi^2 \right] dk,
\]

\[
C(k)_B = \frac{k^2}{2} \left( |\dot{\phi}_k|^2 + (m_B^2 + k^2) |\varphi_k|^2 \right) + \frac{\lambda_B}{8} \left( k^2 |\varphi_k|^2 \phi^2 - |\phi_k - \dot{\phi} \varphi_k|^2 \right)
+ \frac{\lambda_B k^2}{32\pi^2} \int_0^\Lambda k'^2 \coth \frac{\beta W_{k'}}{2} \left[ \frac{|\varphi_{k'} - \dot{\phi} \varphi_{k'}|^2}{k'^2 - k^2} + |\varphi_k|^2 |\varphi_{k'}|^2 \right] dk'.
\] (5.15)

and

\[
C_{0,R} = \frac{1}{2} \dot{\phi}^2 + \frac{m_R^2}{2} \phi^2 + \frac{\lambda_R}{8} \phi^4
+ \frac{\lambda_R}{32\pi^2} \int_0^\infty \left[ \left( |\phi_k - \dot{\phi} \varphi_k|^2 + k^2 \phi^2 |\varphi_k|^2 \right) \coth \frac{\beta W_k}{2} - 2k\phi^2 - \Theta(k - \mu) \frac{k^2 \phi^2}{k} \right] dk,
\]

\[
C(k)_R = \frac{k^2}{2} \left( |\dot{\phi}_k|^2 + (m_R^2 + k^2) |\varphi_k|^2 \right) + \frac{\lambda_R}{8} \left( k^2 |\varphi_k|^2 \phi^2 - |\phi_k - \dot{\phi} \varphi_k|^2 \right)
+ \frac{\lambda_R k^2}{32\pi^2} \int_0^\infty \left[ k'^2 \coth \frac{\beta W_{k'}}{2} \left( \frac{|\varphi_{k'} - \dot{\phi} \varphi_{k'}|^2}{k'^2 - k^2} + |\varphi_k|^2 |\varphi_{k'}|^2 \right) \right.
- 2 - 2k'|\varphi_k|^2 - \frac{\Theta(k' - \mu)}{k'} \left( |\varphi_k|^2 + k^2 |\varphi_k|^2 \right) \left] \right. \left. dk' \right),
\] (5.16)

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