ON A CURVATURE FLOW MODEL FOR EMBRYONIC EPIDERMAL WOUND HEALING

SHUHUI HE, GLEN WHEELER, VALENTINA-MIRA WHEELER

ABSTRACT. The paper studies a curvature flow linked to the physical phenomenon of wound closure. Under the flow we show that a closed, initially convex or close-to-convex curve shrinks to a round point in finite time. We also study the singularity, showing that the singularity profile after continuous rescaling is that of a circle. We additionally give a maximal time estimate, with an application to the classification of blowups.

CONTENTS

1. Introduction 1
2. Evolutions of length, area and curvature 5
3. Curvature estimates and finite time existence 8
4. Continuous rescaling 18
5. The non-convex case 21
6. A Lifespan theorem 24
7. Application of the Lifespan theorem to blowups 29
Appendix A. Estimates for the convex flow 30
Appendix B. Curvature estimate for the continuous rescaling 34
Acknowledgements 41
References 41

1. INTRODUCTION

Wound healing is a complex and essential biological process for an organism to repair damaged tissue and therefore survive the hostile external environment. Ideally, the injury site is to be replaced by tissue that has its original structure and functions. This is so called regenerative wound healing. Early research, such as [8, 17, 18], show that some eukaryotic organisms are able to perform regenerative healing throughout their life time, while humans only have this ability during prenatal development. In adulthood, wound healing typically generates a mass of non-functioning cells and structure which is referred to as scarring. This limitation of adult wound healing ability may lead to severe clinical consequence such as non-healing wounds, congestive heart failure and liver cirrhosis. Various existing models for wound healing focus on different aspects of the complicated adult wound healing process, see for example [9, 28, 29, 32].

Instead of studying the complicated biomechanical process, we focus on a much simpler setting: embryonic epidermal wound healing. This healing process is regenerative and has also

2000 Mathematics Subject Classification. 53C44 and 58J35.
Key words and phrases. curve shortening flow, free boundary conditions, biological membranes, geometric analysis.
been extensively studied. A current modeling technique is to study the change of shape of the wound under prescribed forces acting on the leading edge of the wound. We adapt the approach suggested in [2, 32], to describe the movement of the leading edge with a curvature term and a source term.

In our model, we treat the leading edge of a wound as a simple closed plane curve. This assumption is valid as in the embryonic wound healing setting, an epidermal wound typically contains very few layers of cells, and can be considered as a flat surface. We investigate the forces that contribute to the movement of the curve.

There are three forces acting on the leading edge that we take into consideration. The first force is actin cable contraction. When a wound is presented, actin is assembled to surround the wound opening to form a dense actin cable network. The contraction of the actin cable acts as a purse-string and contributes to a local force proportional to the curvature of the leading edge.

The second force comes from lamellipodial crawling. This is a biological response that cells on the leading edge extend protrusions into tissues within the wound and drag themselves forward, advancing into the wound. This closing force is locally constant and is in the direction normal to the edge.

The last force is epidermal tension, a force that comes from the pulling of surrounding cells, resisting the closure of the wound. This force is again acting in the normal direction and can be considered locally constant.

The velocity $V$ at a point on the leading edge is therefore

$$V(x) = \sigma_1 k(x) \nu(x) + (\sigma_L^2 - \sigma_E^2) \nu(x) = \sigma_1 k(x) \nu(x) + \sigma_2 \nu(x)$$

where $\sigma_1, \sigma_L^2, \sigma_E^2$ are constants corresponding to the three forces above, with $\sigma_1 > 0, \sigma_L^2, \sigma_E^2 \in \mathbb{R}$, $x$ is a position along the leading edge, $k(x)$ is the curvature of the leading edge at $x$ (it is positive for convex wounds), $\nu(x)$ is the inward-pointing unit normal to the leading edge at $x$, and $\sigma_2 := \sigma_L^2 - \sigma_E^2$.

Throughout this article we assume that:

$(\sigma_1 > 0)$ Physically this means that convex regions of the actin cable tend to contract inward, and concave regions tend to relax outward. This is reasonable, as elliptical regions are convex and if $\sigma_1 < 0$ then they would expand, which is not physical. Mathematically this is required for parabolicity and to therefore generate a unique solution from given initial data.

$(\sigma_2 > 0)$ This is a physical assumption amounting to the claim that lamellipodial crawling is a greater force than epidermal tension. Physically this is reasonable, for two reasons: One, a locally straight region of a closed wound (where $k = 0$) moves inward, if $\sigma_2 < 0$, then such regions would move outward; and two, if $\sigma_2 < 0$, then a circular wound of radius $-\frac{\sigma_1}{\sigma_2}$ would remain stationary under the flow and never heal.

Remark. While the above considerations justify the sign of $\sigma_1$ and $\sigma_2$, they do not address the fact that here we assume they are constant. This assumption is one of simplicity. In later work we will replace this with anisotropy and other more general hypotheses.

The velocity (1) gives rise to an evolution equation for embedded closed plane curves, a curvature flow, that is a modified version of the classical curve shortening flow. The curve shortening flow has velocity given by (1) with $\sigma_1 = 1$ and $\sigma_2 = 0$. The curve shortening flow has been extensively studied, see for example [13, 16, 20].

Let us now describe the mathematical model and state our main result. Let $\gamma_0 : S^1 \to \mathbb{R}^2$ be a plane embedded, closed curve describing an initial wound. Consider a family of closed, embedded curves $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ representing the leading edge of a wound as it evolves in
The problem (2) is a system of degenerate nonlinear parabolic PDE of second order on a compact domain. Local well-posedness for sufficiently smooth data follows by standard techniques.

In this paper, our focus is on global analysis for the flow. Our main result is as follows.

**Theorem 1.1.** Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a family of plane, smooth, embedded, closed curves evolving under the flow (2). Then the flow exists for at most finite time \( T < \infty \). If the initial curve \( \gamma_0 \) is convex, then the curves \( \gamma_t(\cdot) := \gamma(\cdot, t) \) contract exponentially fast in the smooth topology to a smooth round point as \( t \to T \).

Furthermore, if the initial curve \( \gamma_0 \) satisfies for some \( \alpha \in (0, 2) \)

\[
L_0 \| k_s \|_2^2 \bigg|_{t=0} \leq \frac{1}{196\sigma_1^2} \left( \sqrt{25\sigma_2^2 + \frac{14\sigma_1(2 - \alpha)}{T_{\text{max}}}} - 5\sigma_2 \right)^2,
\]

where \( T_{\text{max}} \) is an upper bound for the maximal time \( T \) of smooth existence that depends only on \( \gamma_0 \) (see Theorem 1.2), then there exists a \( t_0 \) such that for all \( t \in (t_0, T) \), \( \gamma(\cdot, t) \) is convex, and we have again smooth convergence to a round point as \( t \to T \).

Our method of proof is inspired by the existing literature. We use the integral estimate method of Gage-Hamilton \[13\] in Section 3 to show, essentially, that the flow may be smoothly extended so long as the enclosed area is bounded away from zero. This implies that the enclosed area must vanish at final time. An additional argument and convexity is required to show that length also vanishes at final time – a key step in arguments to come. The main technical difficulty is to identify the asymptotic shape of the flow. Indeed, there are conjectures \[10\] on non-local flows of a similar form that suspect the asymptotic shape is an ellipse or something more exotic.

A modified curve shortening flow with anisotropy is studied in \[5, 7\]. There, it is shown that some classes of curve shortening flows shrink convex curves to round points. In a later article the non-convex case is studied \[6\], however the condition (0.7) they place there rules out \( \sigma_2 = \text{const} \) unless \( \text{const} = 0 \). We use the ideas of Chou-Zhu for various estimates, especially for the a-priori curvature estimate on the rescaled flow. These appear in the Appendix. We thank the authors for the inspiration. We should also note that the Chou-Zhu curvature estimate uses in turn the ideas of Gage-Hamilton \[13\], in particular, using an estimate for the entropy to (eventually) bound the curvature.

We perform a natural rescaling that sets the final value of the enclosed area to \( \sigma_1 \pi \). The aforementioned curvature estimate allows us to extract a smooth limit from the rescaling. In order to identify the shape of this limit we use the monotonicity formula from Huisken \[21\], with a slight modification in order to apply it to our setting here.

Our partial result on the non-convex case is via integral estimate techniques. In Section 5, our key idea is to prove that a flow that is initially almost-convex can not admit a continuous rescaling (as in Section 5) that remains non-convex for all rescaled time. Then, once the flow is convex, arguments from Sections 2–4 and Appendices A, B in the paper apply to give smooth convergence of the rescaled flow to a round circle. Once written in terms of the original flow, the smallness condition relies upon an upper bound for maximal time of existence. This is elementary to derive, using either the avoidance principle for solutions to the flow, or the evolution of length (or area).
Much harder than the upper bound for maximal time is a lower bound. This has been classically interesting in the literature, and has impact on certain discrete rescalings around singularities, called blowups (see Theorem 1.3 below). For this we use a concentration-compactness alternative pioneered by Struwe \[34\]. Differences here abound: We use a product functional (length and curvature in \(L^2\)) which is scale invariant, the product functional may not be globally small regardless of initial data, and our flow can not exist globally. These features necessitate changes in the proof of the concentration-compactness alternative and the lower bound on maximal time. Key parts of these arguments are inspired by methods used by the second author in the study of curvature flow of higher-order and with the third author on curvature flow with free boundary \[27, 30, 31, 37, 38, 39\]. The maximal time estimate is:

**Theorem 1.2** (Lifespan theorem). Let \(\gamma : S^1 \times [0, T) \to \mathbb{R}^2\) be a non-convex solution to (2). There are constants \(\rho \in (0, 1), \varepsilon_1 > 0\), and \(c_0 < \infty\) such that

\[
\sup_{x \in \mathbb{R}^2} L_{B_\rho(x)} \int_{\gamma^{-1}(B_\rho(x))} k^2 \, ds \bigg|_{t=0} = \varepsilon(x) \leq \varepsilon_1
\]

implies that the maximal time \(T\) satisfies

\[
T_{\text{max}} := \frac{1}{2\pi \sigma_1 \sigma_2} \min\{L_0 \sigma_1, A_0 \sigma_2\} \geq T \geq \frac{1}{c_0} \rho^2 := T_{\text{min}},
\]

where \(L_0\) and \(A_0\) denote the initial length and enclosed area of the flow. We additionally have the estimate

\[
L_{B_\frac{x}{2}}(x) \int_{\gamma^{-1}(B_\frac{x}{2}(x))} k^2 \, ds \leq c \varepsilon_1 \quad \text{for} \quad 0 \leq t \leq T_{\text{min}}.
\]

Qualitatively, the lifespan theorem implies that a certain quantum of curvature concentrates along the flow in smaller and smaller intervals as a singularity develops. This has been used by, for example, Kuwert-Schätzle in their study of the Willmore flow of surfaces \[23, 22\]. In Section 7 we describe how it can be used to partially classify blowups of non-circular singular points for our flow here.

**Theorem 1.3** (Non-circular blowups). Let \(\gamma : S^1 \times [0, T) \to \mathbb{R}^2\) be a family of plane, smooth, immersed closed curves evolving under the flow (2). Suppose that \(\gamma_t\) do not contract to a round point as \(t \to T\). Then:

- \(\gamma_t\) is not convex for every \(t \in [0, T)\);
- For all \(t \in [0, T)\),

\[
L\|k_s\|_2^2(t) > \frac{1}{196 \sigma_1^2} \left( \frac{25 \sigma_2^2 + \frac{14 \sigma_1 (2 - \alpha)}{T_{\text{max}}} - 5 \sigma_2^2}{\sqrt{T_{\text{max}} - 5 \sigma_2^2}} \right)^2;
\]

- The discrete blowup \(\gamma^\infty\) exists, and has the following properties:
  - is an ancient non-convex solution to curve shortening flow, and
  - each component of the blowup is either non-compact, non-embedded, or both.

We conjecture that for some values of \(\sigma_1\) and \(\sigma_2\) there is non-preservation of embeddedness and non-round singular profiles. It is natural to guess that these singularities are both non-compact and non-embedded in the blowup.
2. EVOLUTIONS OF LENGTH, AREA AND CURVATURE

In this section and the rest of this article, latin subscripts are used to indicate partial derivatives unless otherwise stated.

Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ be a smooth family of closed, embedded, plane curves moving by the flow (2). The time derivative in (2) is taken along fixed values of the natural parameter $u \in S^1$. In order to study the intrinsic properties of the family of curves efficiently, we reparametrise the curves by arc-length, and denote the arc-length parameter by $s$. Note that the arc-length and the time derivatives do not commute. Indeed, we have

$$\partial_s = |\gamma_u|^{-1}\partial_u,$$

and

$$|\gamma_u|^{-1} : S^1 \times [0, T) \to \mathbb{R}$$

is not in general (in this setting) constant in $t$.

Let $\tau(u, t) = \gamma_u(u, t) / |\gamma_u(u, t)|$ be the unit tangent vector to $\gamma(u, t)$, and $\nu$ the inward pointing unit normal at $\gamma(u, t)$. By possibly changing orientation (mapping $u \mapsto -u$) we may assume that

$$\nu(u, t) = \text{rot}_{\pi/2}\tau(u, t)$$

where $\text{rot}_{\pi/2}(x, y) = (-y, x)$ is counter-clockwise rotation through $\pi/2$ radians.

The well-known Frenet-Serret equations are

$$\partial_s \tau = k\nu, \quad \partial_s \nu = -k\tau.$$ 

We wish to compute the evolution of geometric quantities under the flow (2). Let us first derive the commutator for $\partial_t$ and $\partial_s$.

**Lemma 2.1.** Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ be a solution to (2). The differential operators with respect to arc-length and time satisfy

$$\partial_t \partial_s - \partial_s \partial_t = kF \partial_s = k(\sigma_1 k + \sigma_2) \partial_s.$$

**Proof.** Using the Frenet-Serret equations, we compute

$$2|\gamma_u| |\gamma_u|_t = \partial_t |\gamma_u|^2 = 2 \langle \gamma_{ut}, \gamma_u \rangle = 2 \langle (F\nu)_u, \gamma_u \rangle$$

$$= 2|\gamma_u|^2 \langle F\nu + F\nu_s, \gamma_s \rangle$$

$$= 2|\gamma_u|^2 \langle -kF\tau, \tau \rangle$$

$$= -2kF|\gamma_u|^2.$$ 

Hence

$$(5) \quad |\gamma_u|_t = -kF|\gamma_u|,$$

and

$$\partial_t \partial_s = \partial_t \left(|\gamma_u|^{-1}\partial_u\right) = |\gamma_u|^{-1}\partial_u \partial_t - \sigma_1 |\gamma_u|^{-2}|\gamma_u|_t \partial_u$$

$$= \partial_s \partial_t + kF \partial_s = \partial_s \partial_t + k(\sigma_1 k + \sigma_2) \partial_s.$$

□

Let us compute the evolution of the unit tangent and unit normal with the commutator.

**Lemma 2.2.** Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ be a solution to (2). We have the following evolution equations

$$\tau_t = F_\nu \nu = \sigma_1 k_\nu \nu,$$

$$\nu_t = -F_\nu \tau = -\sigma_1 k_\nu \tau.$$
Proof. Applying Lemma 2.1 and the Frenet-Serret equations we obtain
\[ \tau_t = (\gamma_s)_t = \gamma_{ts} + kF \gamma_s = (F\nu)_s + kF \tau = F_s \nu - Fk \tau + kF \tau = F_s \nu. \]

Since \( \gamma \) is a plane curve, we have \( \langle \tau, \nu \rangle = 0 \) and \( \nu_t \) must be parallel to \( \tau \). We have
\[ \nu_t = \langle \tau, \nu_t \rangle \tau = -\langle \tau_t, \nu \rangle \tau = -F_s \tau. \]
\( \square \)

We may use the above lemmata to compute the evolution of length and the enclosed area of the curve.

Proposition 2.3 (Length evolution). Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a smooth family of curves evolving under the flow (2). The length \( L \) of the curve \( \gamma \) evolves according to
\[ L'(\gamma(\cdot, t)) = -\int_{\gamma} kF \, ds = -\sigma_1 \int_{\gamma} k^2 \, ds - 2\pi \omega \sigma_2, \]
where \( \omega = \int_{\gamma} k \, ds \) is the winding number of \( \gamma \).

Proof. Let us first note that
\[ \partial_t ds = \partial_t(|\gamma_u| \, du) = |\gamma_u| \, du = -kF \, ds. \]
By definition of the length of a curve and the fact that \( k = \theta_s \), where \( \theta \) is the angle made by \( \tau \) and a fixed vector, we compute
\[ L'(\gamma(\cdot, t)) = \int_{\gamma} \nu_{\cdot t} \, du = -\int_{\gamma} kF \, ds \]
\[ = -\int_{\gamma} k(\sigma_1 k + \sigma_2) \, ds = -\sigma_1 \int_{\gamma} k^2 \, ds - 2\pi \omega \sigma_2. \]
\( \square \)

Lemma 2.4 (Area evolution). Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a smooth family of curves evolving under the flow (2). The signed enclosed area \( A \) of the curve \( \gamma \) evolves according to
\[ A'(\gamma(\cdot, t)) = -\int_{\gamma} F \, ds = -2\pi \omega \sigma_1 - \sigma_2 L. \]

Proof. Note that the definition of signed enclosed area is
\[ A(\gamma(\cdot, t)) = -\frac{1}{2} \int_{\gamma} \langle \gamma, \nu \rangle \, ds. \]
Therefore, we see the rate of change of the area satisfies
\[ A'(\gamma(\cdot, t)) = \partial_t \left( -\frac{1}{2} \int_{\gamma} \langle \gamma, \nu \rangle \, ds \right) = -\frac{1}{2} \int_{\gamma} \langle \gamma_t, \nu \rangle + \langle \gamma, \nu_t \rangle - \langle \gamma, \nu \rangle kF \, ds \]
\[ = -\frac{1}{2} \int_{\gamma} \langle F\nu, \nu \rangle + \langle \gamma, -F_s \tau \rangle - \langle \gamma, F\tau_s \rangle \, ds \]
\[ = -\frac{1}{2} \int_{\gamma} F + \langle \gamma, -(F\tau)_s \rangle \, ds = -\frac{1}{2} \int_{\gamma} F \, ds - \frac{1}{2} \int_{\gamma} \langle \gamma, F\tau \rangle \, ds \]
\[= - \int_{\gamma} F \, ds = - \int_{\gamma} \sigma_1 k + \sigma_2 \, ds = - 2\pi \omega \sigma_1 - \sigma_2 L(\gamma(\cdot, t)) \]  

\[\square\]

**Lemma 2.5 (Energy for the flow).** Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a smooth family of curves evolving under the flow \((\gamma)\). Define the functional 
\[E(\gamma(\cdot, t)) = \sigma_1 L(\gamma(\cdot, t)) + \sigma_2 A(\gamma(\cdot, t))\]

Then 
\[E'(\gamma(\cdot, t)) = - \int_{\gamma} F^2 \, ds.\]

**Proof.** We calculate 
\[E'(\gamma(\cdot, t)) = \sigma_1 \left( - \sigma_1 \int_{\gamma} k^2 \, ds - 2\pi \omega \sigma_2 \right) + \sigma_2 \left( - 2\pi \omega \sigma_1 - \sigma_2 L(\gamma(\cdot, t)) \right)\]
\[= - \int_{\gamma} \sigma_1^2 k^2 + 2\sigma_1 \sigma_2 k + \sigma_2^2 \, ds = - \int_{\gamma} F^2 \, ds,\]

as required. \(\square\)

It follows immediately from the above lemmata and our assumptions that \(\sigma_1, \sigma_2 > 0\) that length and area are uniformly bounded along the flow. Lemma 2.5 additionally yields finite maximal existence time in the following sense.

**Lemma 2.6 (**\(T < \infty\)**). Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a smooth family of curves evolving under the flow \((\gamma)\). Suppose that the winding number of the initial data is positive. Then \(T < \infty\).

**Proof.** We first note that (using Lemma 2.7 below) 
\[\frac{d}{dt} \int_{\gamma} k \, ds = \int_{\gamma} \sigma_1 k_{ss} + \sigma_1 k^3 + \sigma_2 k^2 - k^2 (\sigma_1 k + \sigma_2) \, ds = 0\]

so that the winding number is constant along the flow, and in particular remains positive.

Suppose \(T = \infty\). Then
\[\int_0^{t_i} \| F \|^2(t) \, dt = E(\gamma(\cdot, 0)) - E(\gamma(\cdot, t_i)) \leq E(\gamma(\cdot, 0)) < \infty,\]

so there exists a subsequence \(t_j \to \infty\) such that \(\| F \|^2(t_j) \searrow 0\). By smoothness and uniform boundedness of \(L\), this implies that there exists a sequence \(\varepsilon_j \searrow 0\) such that 
\[|\sigma_1 k(u, t_j) + \sigma_2| \leq \varepsilon_j.\]

That is, 
\[-\sigma_1^{-1}(\varepsilon_j + \sigma_2) \leq k(u, t_j) \leq \sigma_1^{-1}(\varepsilon_j - \sigma_2).\]

This means that for some sufficiently large \(j\), \(k(u, t_j)\) is uniformly negative. But this is impossible, since \(\int_{\gamma} k \, ds = 2\omega \pi > 0\). \(\square\)

One key property of the flow is the preservation of local convexity. For this, we first need the evolution of the curvature.
Lemma 2.7 (Curvature evolution). Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$ be a smooth family of curves evolving under the flow (2). The evolution of curvature is given by
\begin{equation}
k_t = \sigma_1 k_{ss} + \sigma_1 k^3 + \sigma_2 k^2.
\end{equation}

Proof. We use Lemma 2.1 and Lemma 2.2 to compute
\begin{align*}
k_t &= \langle \gamma_{ss} , \nu \rangle_t = \langle (\gamma_s)_t , \nu \rangle + \langle kF\gamma_{ss} , \nu \rangle + \langle k\nu , -F_s \tau \rangle \\
&= \langle (\gamma_s)s , k^2 F\nu , \nu \rangle = \sigma_1 k_{ss} + \sigma_1 k^3 + \sigma_2 k^2.
\end{align*}

Corollary 2.8 (Convexity preservation). Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$ be a smooth family of curves evolving under the flow (2). Suppose $k(u,0) \geq k_0$. Then $k(u,t) \geq k_0$.

Proof. Apply the minimum principle to (6). □

3. CURVATURE ESTIMATES AND FINITE TIME EXISTENCE

In this section we obtain estimates inspired by the classical work of Gage-Hamilton [13].

Let us start with reparametrising the family of curves by the tangent angle $\theta$, which is the angle between the tangent line and the $x$-axis. This is a convenient choice of parameter for the study of closed convex curves. We have the following relationship between the angle parameter $\theta$ and the arc-length parameter $s$.

Lemma 3.1. Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$ be a family of closed, embedded, convex plane curves parametrised by tangent angle $\theta$ under the flow (2). We have
\begin{align*}
\frac{\partial \theta}{\partial s} &= k, \\
\frac{\partial \theta}{\partial t} &= F_s = \sigma_1 k_s.
\end{align*}

Proof. For such a parametrisation, the unit tangent and unit normal of the curve has the expression $\tau(\theta) = (\cos \theta, \sin \theta)$ and $\nu(\theta) = (-\sin \theta, \cos \theta)$ respectively. We can therefore compute
\begin{align*}
\frac{\partial \tau}{\partial s} &= \frac{\partial \tau}{\partial \theta} \frac{\partial \theta}{\partial s} = (-\sin \theta, \cos \theta) \frac{\partial \theta}{\partial s} = \frac{\partial \theta}{\partial s} \nu; \\
\frac{\partial \tau}{\partial t} &= \frac{\partial \tau}{\partial \theta} \frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial t} \nu.
\end{align*}
The first equality of the lemma follows from comparing the top equation with the Fernet-Serret equation $\frac{\partial \tau}{\partial \theta} = k\nu$. The second equality is obtained by comparing the expression of $\frac{\partial \tau}{\partial \theta}$ to the corresponding one in Lemma 3.2. □

The space parameter $\theta$ does not commute with the time parameter $t$. However, we can reparametrise in time by $t'$ such that $(\theta, t')$ are independent. In the following, we reparametrise the curves from $(u, t)$ to $(\theta, t')$ and define the new differential operator $\partial_{t'}$ to be the time derivative taken along fixed $\theta$. We can write down the evolution equation for $k$ with parameters $\theta$ and $t'$ as the following.

Lemma 3.2. Let $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$ be a family of closed, embedded, convex plane curves under the flow (2). The evolution of curvature with respect to the parameters $(\theta, t')$ is given by
\begin{equation}
k_{t'} = k^2 (F_{\theta \theta} + F) = \sigma_1 k_{ss} + \sigma_1 k^3 + \sigma_2 k^2.
\end{equation}
Proof. This is a transformation of (6) via Lemma 3.1. Note that
\[ k_s = \frac{\partial k}{\partial \theta} \frac{\partial \theta}{\partial s} = k_\theta k, \quad k_{ss} = k \frac{\partial}{\partial \theta} \left( k \frac{\partial k}{\partial \theta} \right) = k k^2_\theta + k^2 k_\theta, \]
and
\[ k_t = \frac{\partial k}{\partial t} \frac{\partial t}{\partial \theta} + \frac{\partial k}{\partial \theta} \frac{\partial \theta}{\partial t} = \frac{\partial k}{\partial \theta} (\sigma_1 k_s) = k_t' + \sigma_1 kk^2_\theta. \]
Substituting these into (6), the result follows. □

For the rest of the paper, we abuse notation and replace \( t' \) by \( t \) for simplicity whenever the tangent angle is used as the space parameter.

Next, we present a result found in [13], which can be viewed as a version of the fundamental theorem of curves for simple closed convex plane curves. We provide a proof for the convenience of the reader.

**Lemma 3.3 (Gage-Hamilton [13]).** A positive \( 2\pi \) periodic function \( k(\theta) \) represents the curvature function of a simple closed strictly convex \( C^2 \) plane curve \( \gamma \) if and only if
\[
\int_0^{2\pi} \cos \theta \frac{k\theta}{k(\theta)} d\theta = \int_0^{2\pi} \sin \theta \frac{k\theta}{k(\theta)} d\theta = 0. 
\]

Proof. Let \( k : \mathbb{S}^1 \rightarrow \mathbb{R} \) be the curvature function of a unit speed curve. As the curve is closed, we must have
\[
0 = \int_0^L \tau ds = \int_0^L \frac{1}{k(\theta)} d\theta = \int_0^{2\pi} \left( \frac{\cos \theta}{k(\theta)}, \frac{\sin \theta}{k(\theta)} \right) d\theta.
\]
This proves one direction of the claim.

To see the other direction, suppose that \( k : \mathbb{S}^1 \rightarrow \mathbb{R} \) is a positive \( 2\pi \) periodic function satisfying (8). We claim
\[
\gamma(\theta) = \left( \int_0^\theta \frac{\cos \theta'}{k(\theta')} d\theta', \int_0^\theta \frac{\sin \theta'}{k(\theta')} d\theta' \right)
\]
represents the associated curve in the plane up to translation and rotation, i.e., isometries of \( \mathbb{R}^2 \).

Let
\[
x(\theta) = \int_0^\theta \frac{\cos \theta'}{k(\theta')} d\theta' \quad \text{and} \quad y(\theta) = \int_0^\theta \frac{\sin \theta'}{k(\theta')} d\theta'.
\]
As both \( \cos \theta' \) and \( k(\theta') \) are \( 2\pi \) periodic functions, we must have \( \frac{\cos \theta'}{k(\theta')} \) and hence \( x(\theta) \) also \( 2\pi \) periodic. Similarly, as \( \sin \theta' \) is \( 2\pi \) periodic as well, we conclude that \( y(\theta) \) must also be \( 2\pi \) periodic. Since the position vector is \( 2\pi \) periodic, the reconstructed curve \( \eta(\theta) = (x(\theta), y(\theta)) \) must be closed.

Let us compute the tangent vector \( \vec{T}(\theta) \) for \( \eta(\theta) = (x(\theta), y(\theta)) \) using (10),
\[
\vec{T}(\theta) := \eta_\theta = \left( \frac{\partial}{\partial \theta} \int_0^\theta \frac{\cos \theta'}{k(\theta')} d\theta', \frac{\partial}{\partial \theta} \int_0^\theta \frac{\sin \theta'}{k(\theta')} d\theta' \right) = \left( \frac{\cos \theta}{k(\theta)}, \frac{\sin \theta}{k(\theta)} \right).
\]
Hence $|\eta_\theta| = \frac{1}{k(\theta)}$ and the unit tangent $\tau(\theta) = (\cos \theta, \sin \theta)$. We also have the unit normal $\nu(\theta) = (-\sin \theta, \cos \theta)$. The curvature scalar of $\eta(\theta)$ can be computed via

$$k(\eta(\theta)) = \left\langle \frac{1}{|\eta_\theta|} \left( \frac{\eta_\theta}{|\eta_\theta|} \right)_\theta, \nu \right\rangle = (k(\theta)(-\sin \theta, \cos \theta), (-\sin \theta, \cos \theta)) = k(\theta).$$

Thus we conclude the function $k(\theta)$ represents the curvature function of $\eta(\theta) = (x(\theta), y(\theta))$. Hence $\gamma(\theta)$ as defined in (9) represents the same curve as $\eta(\theta)$ and we have proved the claim. $\square$

**Theorem 3.4.** The flow problem (2) is equivalent to the initial value PDE problem:

Find $k : S^1 \times [0,T) \to \mathbb{R}$ satisfying

1. $k \in C^{2+\alpha,1+\alpha}(S^1 \times [0,T - \varepsilon))$ for all $\varepsilon > 0$.
2. $k_3 = \sigma_3 k^2 k_{\theta\theta} + \sigma_1 k^3 + \sigma_2 k^2$.
3. $k(\theta, 0) = \psi(\theta)$ where $\psi \in C^{1+\alpha}(S^1)$ is strictly positive and satisfies

$$\int_0^{2\pi} \frac{\cos \theta}{\psi(\theta)} d\theta = \int_0^{2\pi} \frac{\sin \theta}{\psi(\theta)} d\theta = 0.$$

**Proof.** It is a direct consequence of Lemma 3.2 and Lemma 3.3 that a solution to (2) leads to a solution to the above initial value PDE system. On the other hand, given a solution to Theorem 3.4 part (2), we are able to re-construct the family of curves satisfying (2) up to translation and rotation. This is achieved using the formula (9). The initial condition of $k(\theta)$ ensures it can be viewed as a curvature function to an initial curve moving by the flow, and the initial curve is again constructed via (9). $\square$

The change of parametrisation does not affect the preservation of convexity (Corollary 2.8). We may phrase this in the context of Theorem 3.4 as follows.

**Lemma 3.5.** If $k : S^1 \times [0, T) \to \mathbb{R}$ satisfies the assumptions of Theorem 3.4 then $k_{\min}(t) = \inf\{k(\theta, t) | 0 \leq \theta \leq 2\pi\}$ is a nondecreasing function.

We next show that the curvature $k$ has a uniform bound if the area is uniformly bounded from below.

**Theorem 3.6 (Curvature bounds).** Suppose $k : S^1 \times [0, T) \to \mathbb{R}$ satisfies the assumptions of Theorem 3.4 and that the area enclosed by the associated curves is uniformly bounded away from zero. Then there exists constants $C_p$ depending only on $\sigma_1, \sigma_2, T$ and $\alpha(p)$ where

$$\alpha(p) = \sum_{j=0}^{P} \max \{\partial_\theta^j k(\theta, 0)\},$$

such that $\|k_{\theta\theta}\|_{\infty} \leq C_p$.

We will present a proof that follows the curve shortening case as in (14) with three steps to complete: the geometric estimate, the integral estimate and the pointwise estimate. The assumption of enclosed area bounded away from zero implies the length must also be strictly positive. These together provide a foundation for the geometric estimate and the rest of the argument.

In order to develop the geometric and integral estimates, we require the concept of the median curvature $k^*$:

$$k^* = \sup\{b : k(\theta) > b \text{ on some interval of length } \pi\}.$$
Proposition 3.7 (Geometric estimate [13]). Let $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a family of convex closed plane curve with curvature function $k : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$, corresponding enclosed area $A : [0, T) \to \mathbb{R}$ and length $L : [0, T) \to \mathbb{R}$. The relation $k^*(t) < L/A$ holds.

Proof. This result is independent of the flow. It was first proved in [13]. Here we give for the convenience of the reader an overview of the proof.

Given that the median curvature satisfies $k^*(t) > M$, the curve $\gamma$ restricted to an interval $(a, a + \pi)$ has curvature $k(\theta, t) > M$. This segment of the curve can be contained in a circle of radius $1/M$, and hence between two parallel lines whose distance is at least $2/M$ apart. This also implies the entire curve lies between these two parallel lines as the curve is convex. We further deduce that the convex curve can be contained in a rectangular box with width $2/M$ and length $L/2$. Comparing the enclosed area of $\gamma$ and the area of the box yields

$\quad A(\gamma) < A(\text{rectangular box}) = \frac{2L}{M} = \frac{L}{M}.$

Allowing $M$ to become arbitrarily close to $k^*$, we have $k^*(t) < L/A$ as required. \hfill \Box

The second step is the integral estimate under a bound on the median curvature.

Proposition 3.8 (Integral estimate). Suppose $k : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ satisfies the assumptions of Theorem 3.4 with $k(\theta, 0) > k_0 > 0$. Suppose for each $t \in [0, T)$, $k^*(t) < M < \infty$. Then we have the following uniform bound for the entropy along the flow:

$\quad \int_0^{2\pi} \log k(\theta, t) \, d\theta \leq \int_0^{2\pi} \log k(\theta, 0) \, d\theta + \left(2M + \frac{\sigma_2}{\sigma_1}\right) L(0) + 2\pi \sigma_1 M^2 T.$

Proof. Using the evolution of curvature and integration by parts we obtain

$\quad \frac{d}{dt} \int_0^{2\pi} \log k \, d\theta = \int_0^{2\pi} \frac{1}{k} \frac{\partial k}{\partial t} \, d\theta = \int_0^{2\pi} \sigma_1 k k_\theta \theta + \sigma_1 k^2 + \sigma_2 k \, d\theta$

$\quad = \sigma_1 \int_0^{2\pi} k^2 - k_\theta^2 \, d\theta + \sigma_2 \int_0^{2\pi} k \, d\theta.$

To estimate the first integral above, we see that for a fixed $t$, the space domain is comprised of two distinct subsets: the open set $U = \{ \theta | k(\theta, t) > k^*(t) \}$ and its complement set $V = \mathbb{S}^1 - U$. The definition of median curvature implies that the open set $U$ is a countable union of disjoint intervals $I_i$, each of length no bigger than $\pi$. We can apply the Wirtinger’s inequality to the function $k(\theta, t) - k^*(t)$ in the closure of each interval $I_i$ to see

$\quad \int_{I_i} (k - k^*)^2 \, d\theta \leq \int_{I_i} k_\theta^2 \, d\theta.$

Rearrange and noting that $k^*$ is positive by convexity, we have

$\quad \int_{I_i} k^2 - k_\theta^2 \, d\theta \leq 2k^* \int_{I_i} k \, d\theta - 2 \int_{I_i} (k^*)^2 \, d\theta \leq 2k^* \int_{I_i} k \, d\theta.$

Taking the union of the sets $I_i$, we obtain

$\quad \sigma_1 \int_{U} k^2 - k_\theta^2 \, d\theta \leq 2\sigma_1 k^* \int_{U} k \, d\theta \leq 2\sigma_1 k^* \int_0^{2\pi} k \, d\theta.$
On the complement set, we have $k(\theta, t) \leq k^*(t)$, hence
\[
\sigma_1 \int_V k^2 - k_0^2 \, d\theta \leq \sigma_1 \int_V k^2 \, d\theta \leq \sigma_1 \int_0^{2\pi} k^2 \, d\theta \leq 2\pi\sigma_1 (k^*)^2.
\]
We therefore get a bound on the whole interval
\[
\sigma_1 \int_0^{2\pi} k^2 - k_0^2 \, d\theta \leq 2\sigma_1 k^* \int_0^{2\pi} k \, d\theta + 2\pi\sigma_1 (k^*)^2.
\]
Recalling the evolution of length from Proposition 2.3 and $d\theta = k \, ds$, we obtain
\[
\frac{d}{dt} \int_0^{2\pi} \log k(\theta, t) \, d\theta \leq 2\sigma_1 k^* \int_0^{2\pi} k \, d\theta + 2\pi\sigma_1 (k^*)^2 + \sigma_2 \int_0^{2\pi} k \, d\theta
\leq \left(2k^* + \frac{\sigma_2}{\sigma_1}\right)(-L_t - 2\pi\sigma_2) + 2\pi\sigma_1 (k^*)^2.
\]
Note that here we used that $\omega = 1$, which follows immediately from convexity and embeddedness of the flow. Assume that $k^* < M$ and integrate to obtain
\[
\int_0^{2\pi} \log k(\theta, t) \, d\theta \leq \int_0^{2\pi} \log k(\theta, 0) \, d\theta + \left(2M + \frac{\sigma_2}{\sigma_1}\right)(L(0) - L(t) - 2\pi\sigma_2 t) + 2\pi\sigma_1 M^2 t
\]
for all $t < T$. 

Next, we shall upgrade the integral estimate to a pointwise estimate.

**Lemma 3.9.** Suppose $k : S^1 \times [0, T) \to \mathbb{R}$ satisfies the assumptions of Theorem 3.4 with $k(\theta, 0) > k_0 > 0$. The following estimate holds:
\[
\int_0^{2\pi} k_0^2 \, d\theta \leq \int_0^{2\pi} k_0^2 - k^2 \, d\theta \bigg|_{t=0} + \int_0^{2\pi} k^2 \, d\theta + \frac{\sigma_2^2}{\sigma_1} \int_0^t \int_0^{2\pi} k^2 \, d\theta \, dt.
\]

**Proof.** We compute
\[
\frac{d}{dt} \int_0^{2\pi} k^2 - k_0^2 \, d\theta = 2 \int_0^{2\pi} (k k_t - k_0 k_\theta) \, d\theta = 2 \int_0^{2\pi} (k_\theta k_t + k) \, d\theta
\]
\[
= 2 \int_0^{2\pi} (k_\theta + k) \left(\sigma_1 k^2 k_\theta + \sigma_1 k^3 + \sigma_2 k^2\right) \, d\theta
\]
\[
= 2\sigma_1 \int_0^{2\pi} (k_\theta + k)^2 k^2 \, d\theta + 2\sigma_2 \int_0^{2\pi} (k_\theta + k)^2 k^2 \, d\theta.
\]
We apply Cauchy’s inequality to obtain a lower bound for the last term with $\varepsilon > 0$ to be chosen:
\[
2\sigma_2 \int_0^{2\pi} (k_\theta + k) k^2 \, d\theta \geq -2\sigma_2 \int_0^{2\pi} |(k_\theta + k)| k |k| \, d\theta
\]
\[
\geq -2\sigma_2 \varepsilon \int_0^{2\pi} (k_\theta + k)^2 k^2 \, d\theta - \frac{2\sigma_2}{4\varepsilon} \int_0^{2\pi} k^2 \, d\theta.
\]
Choosing $\varepsilon = \sigma_1 / \sigma_2$, we obtain
\[
2\sigma_2 \int_0^{2\pi} (k_\theta + k) k^2 \, d\theta \geq -2\sigma_1 \int_0^{2\pi} (k_\theta + k)^2 k^2 \, d\theta - \frac{\sigma_2^2}{2\sigma_1} \int_0^{2\pi} k^2 \, d\theta.
\]
Hence
\[
\frac{d}{dt} \int_0^{2\pi} k_\theta^2 - k^2 \, d\theta \leq \frac{\sigma_1^2}{2\sigma_1} \int_0^{2\pi} k^2 \, d\theta.
\]
Integrating both sides in time, the conclusion follows.

\[\square\]

**Proposition 3.10** (Pointwise estimate). Suppose \( k : \mathbb{S}^1 \times [0, T) \to \mathbb{R} \) satisfies the assumptions of Theorem 3.4 with \( k(\theta, 0) > k_0 > 0 \). Suppose
\[
\int_0^{2\pi} \log k(\theta, t) \, d\theta \leq C_1.
\]
Then
\[
k_{\text{max}}(t) \leq 2e^{4(C_1 + 2|\log k_0|)(\sqrt{\frac{\pi}{\sigma_1}} + \|k_0\|)} + \left( \sqrt{\frac{\pi}{\sigma_1}} + \sigma_2 \sqrt{\frac{\pi T}{\sigma_1}} \right)^{-1} \left[ \int_0^{2\pi} k_0^2 - k^2 \, d\theta \right]_{t=0}.
\]
In particular \( k(\theta, t) \) is uniformly bounded on \( \mathbb{S}^1 \times [0, T) \).

*Proof.* Preservation of convexity implies that \( k(\theta, t) > k_0 \) for all \( \theta \) and \( t \). Fix a time \( t_0 \) and a positive number \( V \), and consider the set \( I_V = \{ \theta : \log k(\theta, t_0) \geq V \} \) containing all points such that \( \log k(\theta, t_0) \) is not less than \( V \). We have
\[
C_1 \geq \int_0^{2\pi} \log k(\theta, t_0) \, d\theta = \int_{I_V} \log k(\theta, t_0) \, d\theta + \int_{\mathbb{S}^1 \setminus I_V} \log k(\theta, t_0) \, d\theta
\geq V \mu_L(I_V) + \log(k_0)\mu_L(\mathbb{S}^1 \setminus I_V)
\]
where \( \mu_L(\cdot) \) denotes the Lebesgue measure of a set. Rearrange to see that
\[
V \mu_L(I_V) \leq C_1 - \log(k_0)\mu_L(\mathbb{S}^1 \setminus I_V)
\leq C_1 + |\log(k_0)|\mu_L(\mathbb{S}^1 \setminus I_V)
\leq C_1 + |\log(k_0)|\mu_L(\mathbb{S}^1).
\]
Let \( C_2 = C_1 + |\log(k_0)|\mu_L(\mathbb{S}^1) \), then \( \mu_L(I_V) \leq \frac{C_2}{V} \). Fixing \( \delta = \frac{C_2}{V} \), we have \( k(\theta, t_0) \leq e^{\frac{C_2}{2V}} \) for all \( \theta \notin I_V \). We have for any \( \varphi \in \mathbb{S}^1 \), \( a \notin I_V \),
\[
k(\varphi) = k(a) + \int_a^\varphi k_\theta \, d\theta
\leq e^{\frac{C_2}{2V}} + \sqrt{\sigma_2 \left( \int_0^{2\pi} k_\theta^2 \, d\theta \right)}^{\frac{1}{2}}
\leq e^{\frac{C_2}{2V}} + \sqrt{\sigma_2 \left( \int_0^{2\pi} k^2 \, d\theta \right)}^{\frac{1}{2}} + \left( \frac{\sigma_1^2}{2\sigma_1} \int_0^{2\pi} k^2 \, d\theta \right)_{t=0}^{t=T} + \left( \frac{\sigma_1^2}{2\sigma_1} \int_0^{2\pi} k_\theta^2 - k^2 \, d\theta \right)_{t=0}^{t=T},
\]
by the previous Lemma. Thus, suppose \( k_{\text{max}} \) is the maximum value of \( k \), then
\[
k_{\text{max}} \leq e^{\frac{C_2}{2V}} + \sqrt{\frac{\pi T}{\sigma_1}} \sigma_2 k_{\text{max}} + \sqrt{\frac{\pi T}{\sigma_1}} \sigma_2 k_{\text{max}} + \left( \frac{\sigma_1^2}{2\sigma_1} \int_0^{2\pi} k_\theta^2 - k^2 \, d\theta \right)_{t=0}^{t=T}.
\]
Absorbing yields
\[ k_{\text{max}} \leq e_c^{2\pi} + \sqrt{\delta \int_0^{2\pi} k_0^2 - k^2 d\theta \bigg|_{t=0}}, \]

Choose finally \( \delta = \frac{1}{4} \left( \sqrt{2\pi + \sigma_2 \sqrt{\frac{2\pi}{\sigma_1}}} \right)^{-2} \), so that the above becomes
\[ k_{\text{max}} \leq 2e^{4C_2} \left( \sqrt{2\pi + \sigma_2 \sqrt{\frac{2\pi}{\sigma_1}}} \right)^2 + \left( \frac{\sqrt{2\pi + \sigma_2 \sqrt{\frac{\pi}{\sigma_1}}} - 1}{\sqrt{\int_0^{2\pi} k_0^2 - k^2 d\theta \bigg|_{t=0}}} \right), \]
as required. \( \square \)

In the rest of this section, we show that assuming \( k \) is bounded, we can find bounds for all higher derivatives of \( k \). Since we obtain the curvature bound so long as the area is positive, this implies that the flow continues to smoothly exist until it shrinks to a point. We begin with a series of lemmata.

**Lemma 3.11.** Suppose \( k : \mathbb{S}^1 \times [0,T) \to \mathbb{R} \) satisfies the assumptions of Theorem 3.4 with \( k(\theta,0) > k_0 > 0 \). If \( k(\theta,t) < k_{\text{max}} < \infty \), then
\[ |k_\theta(\theta,t)| \leq e^{2T(3\sigma_1 k_{\text{max}}^2 + 2\sigma_2 k_{\text{max}})} \max |k_\theta(\theta,0)|. \]

**Proof.** We use the maximum principle to prove that \( k_\theta \) grows at most exponentially, that is
\[ k_\theta(\theta,t) \leq e^{-2\alpha t} k_\theta(\theta,0), \]
for some negative constant \( \alpha \) on a finite time interval \([0,T)\).

We first compute
\[ k_{\theta\theta} = \sigma_1 \left(k^2 k_{\theta\theta\theta} + 2kk_{\theta\theta} + 3k^2 k_\theta \right) + 2\sigma_2 kk_\theta. \]

Let \( X = e^{\alpha t} k_\theta \), then the above equation can be rewritten as
\[ e^{-\alpha t} (X_t - \alpha X) = e^{-\alpha t} \left[ \sigma_1 \left(k^2 k_{\theta\theta\theta} + 2e^{-\alpha t} kXk_\theta + 3k^2 X \right) + 2\sigma_2 kX \right], \]
\[ X_t - \sigma_1 k^2 k_{\theta\theta} = 2e^{-\alpha t} \sigma_1 kXk_\theta + \left(3\sigma_1 k^2 + 2\sigma_2 k + \alpha \right) X. \]

Therefore,
\[ (\partial_t - \sigma_1 k^2 \partial_{\theta\theta}) X^2 = 2XX_t - \sigma_1 k^2(2X_\theta^2 + 2X X_\theta) \]
\[ = 2X(X_t - \sigma_1 k^2 X_\theta) - 2\sigma_1 k^2 X_\theta^2 \]
\[ = 2 \left(3\sigma_1 k^2 + 2\sigma_2 k + \alpha \right) X^2 + 2e^{-\alpha t} \sigma_1 kX(X_\theta) - 2\sigma_1 k^2 X_\theta^2 \]
\[ \leq 2 \left(3\sigma_1 k^2 + 2\sigma_2 k + \alpha \right) X^2 + 2e^{-\alpha t} \sigma_1 kX(X^2)_\theta. \]

Suppose \( k \) is bounded uniformly by \( k_{\text{max}} \). Then we can choose \( \alpha < -3\sigma_1 k_{\text{max}}^2 - 2\sigma_2 k_{\text{max}} \) so that the coefficient of \( X^2 \) is negative. Then, the conclusion follows by the maximum principle. \( \square \)

**Lemma 3.12.** Suppose \( k : \mathbb{S}^1 \times [0,T) \to \mathbb{R} \) satisfies the assumptions of Theorem 3.4 with \( k(\theta,0) > k_0 > 0 \). If \( k < k_{\text{max}} < \infty \) and \( |k_\theta| \leq C_5 \), then
\[ \int_0^{2\pi} k_{\theta\theta}^4 d\theta \leq \left( 2T \pi + \int_0^{2\pi} k_{\theta\theta}^4 d\theta \bigg|_{t=0} \right) e^{C_5^3 (36\sigma_1 + C_5^2 (81\sigma_1 k_{\text{max}}^2 + \sigma_2^2 |\sigma_1|^{-36})^2)}T. \]
Lemma 3.13. Suppose $k : S^1 \times [0, T) \to \mathbb{R}$ satisfies the assumptions of Theorem 3.4 with $k(\theta, 0) > k_0 > 0$. If $k < k_{\text{max}} < \infty$, $|k_\vartheta| \leq C_3$, and $|k_\varrho_\vartheta|_4^4 \leq C_4$ then
\[
\int_0^{2\pi} k^2_\varrho_\vartheta d\theta < \left( D_2 t + \int_0^{2\pi} k^2_\varrho_\vartheta d\theta \bigg|_{t=0} \right) e^{D_1 t},
\]
where $D_1 = 56\sigma_1 C_3^2$ and
\[
D_2 = 28\sigma_1 \pi \left[ 2 \left( C_4 + \frac{1}{8} k_0^{-4} C_3^5 \right) + \frac{9}{4} \left( C_4 + \frac{1}{4} k_{\text{max}}^4 \right) + 9C_3^4 + \sigma_2^2 \sigma_1^{-1} k_0^{-2} C_3^5 \right].
\]
Proof. In this proof we use \( k^{(4)} \) to denote the fourth derivative of \( k \). We apply integration by parts to find
\[
\frac{d}{dt} \int_0^{2\pi} k_{\theta\theta\theta}^2 d\theta = 2 \int_0^{2\pi} k_{\theta\theta\theta}(k_t)_{\theta\theta\theta} d\theta = -2 \int_0^{2\pi} k^{(4)}(k_t)_{\theta\theta} d\theta
\]
\[
= -2 \int_0^{2\pi} k^{(4)} (\sigma_1 k^2 k_{\theta\theta} + \sigma_1 k^3 + \sigma_2 k^2)_{\theta\theta} d\theta
\]
\[
= -2 \int_0^{2\pi} k^{(4)} (\sigma_1 k^2 k_{\theta\theta\theta} + 2\sigma_1 k k_{\theta\theta} + 3\sigma_1 k^2 k_{\theta} + 2\sigma_2 k k_{\theta})_{\theta} d\theta
\]
\[
= -2\sigma_1 \int_0^{2\pi} k^{(4)} \left( k^{(4)} \right)^2 + 4kk_{\theta}k_{\theta\theta\theta}k^{(4)} + 2k^2k_{\theta\theta}k^{(4)}
\]
\[
+ 3k^2k_{\theta\theta}k^{(4)} + 6k^2k^{(4)} d\theta - 4\sigma_2 \int_0^{2\pi} k^2 k^{(4)} + kk_{\theta\theta}k^{(4)} d\theta.
\]
Again, by applying Cauchy’s inequality, we can absorb all the other terms into the first term with some additional penalty terms. That is
\[
\frac{d}{dt} \int_0^{2\pi} k_{\theta\theta\theta}^2 d\theta \leq c_1 \int_0^{2\pi} k_{\theta\theta\theta}^2 d\theta + c_2 \int_0^{2\pi} k_{\theta\theta\theta}^4 d\theta + c_3 \int_0^{2\pi} \frac{k_{\theta\theta\theta}}{k^2} d\theta + c_4 \int_0^{2\pi} k^2 k_{\theta\theta\theta}^2 d\theta + c_5 \int_0^{2\pi} k_{\theta\theta\theta}^2 d\theta + c_6 \int_0^{2\pi} k_{\theta\theta}^2 d\theta.
\]
Estimating \( k_{\theta\theta}^2 \leq k_{\theta\theta\theta}^2 + \frac{1}{4} \) and invoking our hypotheses we find
\[
\frac{d}{dt} \int_0^{2\pi} k_{\theta\theta\theta}^2 d\theta \leq D_1 \int_0^{2\pi} k_{\theta\theta\theta}^2 d\theta + D_2,
\]
for universal \( D_1 \) and \( D_2 \). Applying Gronwall’s inequality, we recover (i). Following through on allowable constants we see that we may choose
\[
c_1 = 56\sigma_1, \quad c_2 = c_3 = 14\sigma_1, \quad c_4 = \frac{63}{2}\sigma_1, \quad c_5 = 126\sigma_1, \quad c_6 = c_7 = 14\frac{\sigma_2}{\sigma_1}.
\]
Then \( D_1 = c_1C_3^2 \) and
\[
D_2 = 2\pi \left[ c_2C_4 + c_3 \left( C_4 + \frac{1}{4}k_0^{-4}C_3^8 \right) + c_4 \left( C_4 + \frac{1}{4}k_{\text{max}}^4 \right) + c_5C_3^4 + c_6k_0^{-2}C_3^4 + c_7 \left( C_4 + \frac{1}{4} \right) \right].
\]
\[\square\]

Lemma 3.14. Suppose \( k : S^1 \times [0, T] \to \mathbb{R} \) satisfies the assumptions of Theorem 3.4 with \( k(\theta, 0) > k_0 > 0 \). If \( k < k_{\text{max}} < \infty \), then for all \( p \in \mathbb{N} \) we have
\[
|k_{\theta p}| \leq C(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(p)).
\]

Proof. We begin by collecting consequences of the assumed uniform curvature bound from above. First, Lemma 3.11 implies \( |k_{\theta}| \leq C(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(1)) \). Then Lemma 3.13 gives, by the fundamental theorem of calculus, \( |k_{\theta\theta}| \leq C(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(3)) \). To begin our general argument, we need to bound \( k_{\theta\theta\theta} \). We do this by the maximum principle.

Let us compute
\[
\frac{\partial}{\partial t} k_{\theta\theta\theta} = (k_t)_{\theta\theta\theta} = (\sigma_1 k^2 k_{\theta\theta} + \sigma_1 k^3 + \sigma_2 k^2)_{\theta\theta}
\]
\[
\begin{align*}
= \sigma_1 \left[ k^2k^{(4)} + 4kk_\theta k_{\theta\theta\theta} + 2kk_\theta^2k_{\theta\theta} + 2k_\theta^2k_{\theta\theta} + 3k^2k_{\theta\theta} + 6k_\theta^2 + \frac{2\sigma_2}{\sigma_1} (kk_\theta + k_\theta^2) \right] \\
= \sigma_1 \left[ k^2k^{(5)} + 6kk_\theta k^{(4)} + \left( 8kk_\theta + 6k_\theta^2 + 3k^2 + \frac{2\sigma_2}{\sigma_1} k \right) k_{\theta\theta\theta} \right] \\
+ \sigma_1 \left( 6kk_\theta^2 + 18kk_\theta k_{\theta\theta} + 6k_\theta^3 \right) + 6\sigma_2 k_\theta k_{\theta\theta}.
\end{align*}
\]

Since \( k, k_\theta \) and \( k_{\theta\theta} \) are all bounded on the finite time interval \([0, T)\), the second line of the last inequality above can be bounded by some constant \( D = D(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(3)) \), and the term in parentheses (the coefficient of \( k_{\theta\theta\theta} \)) may be bounded by a constant \( E = E(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(3)) \).

Letting \( Y = e^{\alpha t} k_{\theta\theta\theta} \), we can rewrite the above in the same way as we did in Lemma 3.11:

\[
e^{-\alpha t} (Y_t - \alpha Y) \leq e^{-\alpha t} \left[ k^2 Y_\theta + 6kk_\theta Y_\theta + EY \right] + D.
\]

Hence

\[
(\partial_t - \sigma_1 k^2 \partial_{\theta\theta})Y^2 = 2Y (Y_t - \sigma_1 k^2 Y_\theta) - 2\sigma_1 k^2 Y_\theta^2 \\
\leq 2\alpha Y^2 + 2\alpha_1 EY^2 + 6\sigma_1 (k_{\text{max}}) C(Y^2)_\theta + 2e^{\alpha t} DY.
\]

Suppose there exists a new maximum for \( Y^2 \) at the point \( (\theta_0, t_0) \). Note that we may assume \( Y^2(\theta_0, t_0) > 1 \) and \( \alpha < 0 \) so that

\[
(\partial_t - \sigma_1 k^2 \partial_{\theta\theta})Y^2 \leq Y^2 (2\alpha + 2\alpha_1 E + 2D) + 6\sigma_1 (k_{\text{max}}) C(Y^2)_\theta.
\]

Picking \( \alpha \) so negative that

\[
2\alpha + 2\sigma_1 E + 2D < 0
\]

and then noting that at \((\theta_0, t_0)\) we have \((Y^2)_\theta = 0\), we have \( (\partial_t - \sigma_1 k^2 \partial_{\theta\theta})Y^2 < 0 \), a contradiction. Hence, \( k^2_{\theta\theta\theta}(\theta, t) \leq e^{-2\alpha t} k_{\theta\theta\theta}(\theta, 0) \leq e^{-2\alpha t} \| k_{\theta\theta\theta} \|_\infty \) and so \( |k_{\theta\theta\theta}| \leq C(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(3)) \).

We now show that this implies all the higher derivatives of \( k \) are also bounded. The above bounds imply that if \( |k_{\theta\theta\theta}| \leq C(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(q)) \) for all \( q \in \{0, \ldots, p - 1\} \), the following evolution equation for \( k_{\theta\theta\theta} \) holds:

\[
(\partial_t - \sigma_1 k^2 \partial_{\theta\theta})k_{\theta\theta\theta} \leq 2p\sigma_1 kk_\theta (k_{\theta\theta\theta})_\theta + Ck_{\theta\theta\theta} + C.
\]

Using the substitution above and the assumed bounds, we see that \( |k_{\theta\theta\theta}| \leq C(\sigma_1, \sigma_2, T, k_{\text{max}}, \alpha(p)) \).

The claim follows by induction. \( \square \)

These estimates allow us to obtain qualitative information on the evolution of length and area up to final time.

**Theorem 3.15.** If \( \gamma(\cdot, 0) \) is convex, then \( A(\gamma(\cdot, t)) \rightarrow 0 \) and \( L(\gamma(\cdot, t)) \rightarrow 0 \) as \( t \searrow T \). There exists a final point \( \mathcal{O} \in \mathbb{R}^2 \) such that

\[
\gamma(S, t) \rightarrow \mathcal{O} \quad \text{as} \quad t \rightarrow T.
\]

Here convergence is understood with respect to the Hausdorff metric on \( \mathbb{R}^2 \).

**Proof.** Suppose that the area satisfies \( A(\gamma(\cdot, t)) \rightarrow \varepsilon > 0 \) as \( t \searrow T \). Note that Lemma 2.6 implies \( T < \infty \). Then Proposition 3.10 (note that here and throughout this proof we use convexity) yields a uniform estimate on the median curvature, which combined with Proposition 3.8 gives a uniform estimate on the entropy. If \( \gamma(\cdot, 0) \) is convex, then it remains so (see Corollary 2.8). Note that we know by Lemma 2.6 that \( T < \infty \). The uniform estimate on the entropy then by Proposition 3.10 becomes a uniform estimate on curvature depending only on \( \sigma_1, \sigma_2, T \) and the initial values of \( k \) and \( k_\theta \). Finally by Lemma 3.14 all derivatives of curvature are uniformly bounded. Recalling the
formulation of Theorem 3.4 this implies the extension of the solution beyond $T$ by a standard argument. Since $T$ is finite, this is a contradiction.

Therefore the area satisfies
\[ A(\gamma(t)) \to 0 \quad \text{as} \quad t \nearrow T. \]

This is the first conclusion of the theorem. The only possible limiting shapes for $\gamma(t)$ are straight line segments and points. As the flow is smooth and closed for all $t \in [0, T)$, any other possibility is not convex up to final time. If the limiting shape is a point, then we are finished.

If the limiting shape is a straight line segment, then
\[ \min_{\theta \in \mathbb{S}} k(\theta, t) \to 0 \quad \text{as} \quad t \nearrow T \]

But the maximum principle implies
\[ \min_{\theta \in \mathbb{S}} k(\theta, t) > 0. \]

This is a contradiction.

Therefore the only possible limiting shape is a point, and so $L(\gamma(t)) \to 0$ as required. □

4. CONTINUOUS RESCALING

We will now study a rescaling of the flow about the final point $O$. Our goal for the remainder of the section is to prove that in a weak sense a subsequence converges to a circle. There are essentially two steps to be completed. Firstly, we further develop our a-priori estimates on the speed, which then implies the entropy of the rescaled flow is bounded. Secondly, the entropy bound leads to a two way bound on the curvature and implies subconvergence of the family of curves.

Consider the scaling factor $\phi : [0, T) \to \mathbb{R}$ given by
\[ \phi(t) = (2T - 2t)^{-1/2} \]

and set the corresponding rescaled flow $\hat{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}^2$ to be
\[ \hat{\gamma}(\hat{\theta}, \hat{t}) = \phi(t(\hat{t})) \gamma(\phi(\hat{\theta}), t(\hat{t})) - O = \frac{1}{\sqrt{2T}} e^{\frac{t}{T}} [\gamma(\theta(\hat{\theta}), t(\hat{t})) - O], \]

where $O$ is the final point given by Theorem 3.15, and
\[ \theta(\hat{\theta}) = \hat{\theta}, \quad t(\hat{t}) = T(1 - e^{-2\hat{t}}), \quad \hat{t}(t) = -\frac{1}{2} \log \left(1 - \frac{t}{T}\right) \]

denote the rescaled time and space variables. Under the rescaling, we see that the flow (2) becomes
\[ \hat{\gamma}(\hat{\theta}, \hat{t}) = \phi'(t(\hat{t})) t'(\hat{t}) [\gamma(\theta(\hat{\theta}), t(\hat{t})) - O] + \phi(t(\hat{t}))(\partial_{\theta} \gamma)(\hat{\theta}, t(\hat{t})) t'(\hat{t}) \]
\[ = \phi(t(\hat{t})) [\gamma(\theta(\hat{\theta}), t(\hat{t})) - O] + \sqrt{2T} e^{-\frac{t}{T}} (\sigma_1 k(\hat{\theta}, t(\hat{t}))) + \sigma_2 \nu(\hat{\theta}, t(\hat{t})) \]
\[ = \hat{\gamma}(\hat{\theta}, \hat{t}) + (\sigma_1 k(\hat{\theta}, \hat{t}) + \sqrt{2T} e^{-\frac{t}{T}} \sigma_2) \hat{\nu}(\hat{\theta}, \hat{t}). \]

The motivation for choosing this particular rescaling is that we want the enclosed area $\hat{A}(\hat{t}) = \phi^2 A(t)$ of the normalised curve $\hat{\gamma}$ to approach a fixed value (in this case, $\sigma_1 \pi$) as $\hat{t}$ approaches infinity. To explain this we present the following fundamental lemma.

**Lemma 4.1.** Let $\hat{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}^2$ be a solution to the rescaled flow (12) with convex initial data. Then
\[ \hat{A}(\hat{\gamma}(\cdot, \hat{t})) \to \sigma_1 \pi \quad \text{as} \quad \hat{t} \nearrow \infty. \]
Proof. First note that we know \( A(\gamma(\cdot, t(\hat{t}))) \to 0 \) and \( L(\gamma(\cdot, t(\hat{t}))) \to 0 \) as \( \hat{t} \to \infty \) by Theorem 3.15. We calculate
\[
\lim_{\hat{t} \to \infty} \hat{A}(\gamma(\cdot, \hat{t})) = \lim_{t \to T} \frac{A(\gamma(\cdot, t))}{2T - 2t} = \lim_{t \to T} \frac{-2\pi \omega \sigma_1 - \sigma_2 L(\gamma(\cdot, t))}{-2}.
\]
Given that \( L \) vanishes at final time (and \( \omega = 1 \)), we conclude
\[
\lim_{\hat{t} \to \infty} \hat{A}(\gamma(\cdot, \hat{t})) = \sigma_1 \pi,
\]
as required. \( \square \)

Remark. In [14], the isoperimetric ratio is studied for curve shortening flow and used to prove that \( A(t) \to 0 \) implies \( L(t) \to 0 \). For the flow we study here this approach can not possibly work, since \( \sigma_2 \geq 0 \) and
\[
\left( \frac{L^2(t)}{A(t)} \right) = -2\sigma_1 \frac{L(t)}{A(t)} (\|k\|_2^2 - \frac{\pi}{2} \frac{L(t)}{A(t)}) + 2\sigma_2 \frac{L(t)}{A(t)} \left( \frac{1}{2} \frac{L^2(t)}{A(t)} - 2\pi \right).
\]
To estimate the first term one may use the inequality \( \|k\|_2^2 \geq \frac{4\pi^2}{T} \), which follows from the Poincaré inequality, combined with the isoperimetric inequality. The second term on the right is non-negative however and zero only for a circle (by the isoperimetric inequality).

We additionally note for interest that this rescaling and Theorem 3.15 give a bound on how fast \( \|k\|_2 \) blows up for the convex flow, effectively determining already the type of the singularity.

Lemma 4.2. Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a family of closed, embedded, convex plane curves evolving by the flow (2). Then
\[
\|k\|_2^2(t) = O \left( \frac{1}{\sqrt{T - t}} \right).
\]
Proof. First, note that
\[
\limsup_{t \to T} \hat{L}(\gamma(\cdot, \hat{t})) < \infty.
\]
This follows from the rescaled length estimate, Lemma B.4. We do not give the proof of Lemma B.4 here, but rather delay it until we analyse the rescaled entropy, in particular its eventual monotonicity. This is a crucial ingredient in the proof of Lemma B.4 and takes quite some work to establish.

Therefore
\[
\lim_{t \to T} \hat{L}(\gamma(\cdot, \hat{t})) = \lim_{t \to \infty} \phi(t(\hat{t})) L(\gamma(\cdot, t(\hat{t}))) = \lim_{t \to T} \frac{L(\gamma(\cdot, t))}{\sqrt{2T - 2t}} = \lim_{t \to T} \frac{-\sigma_1 \|k\|_2^2(t) - 2\pi \sigma_2}{- \sqrt{2T - 2t}} = \lim_{t \to T} \sigma_1 \|k\|_2^2(t) \sqrt{2T - 2t}.
\]
This proves the result. \( \square \)
Our main tool is Huisken’s monotonicity formula \cite{21}, with a minor modification so that it is suitable for our flow.

**Theorem 4.3.** Let $\hat{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}^2$ be a solution to the rescaled flow \cite{12}. Set

$$\rho(\hat{\gamma}) = e^{-\frac{|\hat{\gamma}|^2}{2\sigma_1}},$$

and

$$R(\hat{\gamma}) = \int_{\hat{\gamma}} \rho \, ds.$$

Then

$$R'(t) = - \int_{\hat{\gamma}} Q^2 \rho \, ds + \frac{T\sigma_2}{2\sigma_1} e^{-2i} \int_{\hat{\gamma}} \rho \, ds,$$

where

$$Q = \frac{\langle \hat{\gamma}, \hat{\nu} \rangle}{\sqrt{\sigma_1}} + \frac{\sqrt{2T} \sigma_2}{2\sqrt{\sigma_1}} e^{-i}.$$

**Proof.** We calculate

$$R'(t) = \int_{\hat{\gamma}} \left(1 - \sigma_1 k^2 - \sigma_2 k\sqrt{2T} e^{-i} - \sigma_1^{-1} \langle \hat{\gamma}, \hat{\nu} \rangle - (\sigma_1 k + \sigma_2 \sqrt{2T} e^{-i}) \hat{\nu} \right) \rho \, ds$$

$$= \int_{\hat{\gamma}} \left(1 - \sigma_1 k^2 - \sigma_2 k\sqrt{2T} e^{-i} - \sigma_1^{-1} |\hat{\gamma}|^2 - \sigma_1^{-1} \sigma_1 k \langle \hat{\gamma}, \hat{\nu} \rangle - \sigma_2 \sqrt{2T} e^{-i} \langle \hat{\gamma}, \hat{\nu} \rangle \right) \rho \, ds$$

$$= - \int_{\hat{\gamma}} \left(\frac{1}{\sqrt{\sigma_1}} \hat{\gamma} + \frac{\sqrt{2T} \sigma_2}{2\sqrt{\sigma_1}} e^{-i} \hat{\nu} \right)^2 \rho \, ds$$

$$+ \int_{\hat{\gamma}} \left(1 + \frac{2\sqrt{\sigma_1^{-1} \sigma_1 - \sigma_1^{-1} \sigma_1}}{\sqrt{2T} \sigma_2} e^{-i} \hat{\nu} \right) \rho \, ds + \frac{T\sigma_2}{2\sigma_1} e^{-2i} \int_{\hat{\gamma}} \rho \, ds.$$

Note that

$$\int_{\hat{\gamma}} (1 + \hat{\nu} \langle \hat{\gamma}, \hat{\nu} \rangle) \rho \, ds = \int_{\hat{\gamma}} \sigma_1^{-1} \langle \hat{\gamma}, \hat{\nu} \rangle^2 \rho \, ds,$$

and

$$\left(\frac{\hat{\gamma} - \langle \hat{\gamma}, \hat{\nu} \rangle \hat{\nu}}{\sqrt{\sigma_1}} + \frac{\sqrt{2T} \sigma_2}{2\sqrt{\sigma_1}} e^{-i} \hat{\nu} \right)^2$$

$$= \left(\frac{\hat{\gamma}}{\sqrt{\sigma_1}} + \frac{\sqrt{2T} \sigma_2}{2\sqrt{\sigma_1}} e^{-i} \hat{\nu} \right)^2 - \sigma_1^{-1} \langle \hat{\gamma}, \hat{\nu} \rangle^2.$$

This implies

$$R'(t) = - \int_{\hat{\gamma}} \left(\frac{\langle \hat{\gamma}, \hat{\nu} \rangle}{\sqrt{\sigma_1}} + \frac{\sqrt{2T} \sigma_2}{2\sqrt{\sigma_1}} e^{-i} \right)^2 \rho \, ds + \frac{T\sigma_2}{2\sigma_1} e^{-2i} \int_{\hat{\gamma}} \rho \, ds.$$
ON A CURVATURE FLOW MODEL FOR EMBRYONIC EPIDERMAL WOUND HEALING

\[ \int \frac{T\sigma_2^2}{2\sigma_1} e^{-2i} \int \rho \, ds , \]

as required. \[ \square \]

A uniform bound for curvature follows by ideas of Chou-Zhu, see Appendix B. This involves obtaining an a-priori estimate on the entropy (which also yields a length estimate, see Lemma B.4). For the convenience of the reader we state the main estimate (Theorem B.5) here.

**Theorem 4.4.** Let \( \hat{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow (12) with convex initial data. There exists a \( \hat{k}_1 \in [0, \infty) \) such that

\[ \hat{k}_{\text{max}}(\hat{t}) \leq \hat{k}_1 \]

for all \( \hat{t} \in [0, \infty) \).

We can now finish our proof.

**Theorem 4.5.** Let \( \hat{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow (12) with convex initial data. Then \( \hat{\gamma} \) converges smoothly in the \( C^\infty \) topology to a standard round circle.

**Proof.** Integrating equation (13) we find

\[ R(0) - R(\hat{t}_j) + \frac{T\sigma_2^2}{2\sigma_1} \int_0^{\hat{t}_j} e^{-2i} \int \rho \, ds \, dt = \int_0^{\hat{t}_j} \int \hat{\gamma} Q^2 \rho \, ds \, dt . \]

By the evolution equation, \( |\hat{\gamma}| \) converges, we let \( \{\hat{t}_j\} \) be this subsequence and conclude that

\[ \lim_{j \to \infty} \int_0^{\hat{t}_j} \int \hat{\gamma} Q^2 \rho \, ds \, dt \]

is bounded. This implies that there is a subsequence that we also call \( \{\hat{t}_j\} \) that along which we have

\[ \int_0^{\hat{t}_j} Q^2 \rho \, ds \to 0 . \]

Since we have a curvature bound and a length bound, the estimates from Section 3 apply to give uniform bounds on all derivatives of curvature. As in Huisken’s Proposition 3.4 of [21], we obtain convergence to a solution of

\[ \langle \gamma, \nu \rangle = -\sigma_1 k . \]

Abresch-Langer [1] have classified solutions to this equation. In particular, the only embedded solution is a circle. \( \square \)

### 5. THE NON-CONVEX CASE

Without strict convexity we may not use the \( \theta \) parametrisation. We briefly calculate evolution equations for the rescaled flow in terms of rescaled arclength, beginning with \( |\hat{\gamma}_u|^2 \):

\[ |\hat{\gamma}_u| = |\hat{\gamma}_u| \langle \partial_u (\hat{\gamma}_u), \hat{\tau} \rangle \]

\[ = |\hat{\gamma}_u| \left[ \left( \sigma_1 \hat{k} + \sqrt{2Te^{-i} \sigma_2} \right) \hat{\nu} + \hat{\gamma} \right] \langle \hat{\tau}, \hat{\tau} \rangle \]

\[ = |\hat{\gamma}_u| \left[ \left( -\hat{k} \left( \sigma_1 \hat{k} + \sqrt{2Te^{-i} \sigma_2} \right) \hat{\tau}, \hat{\tau} \right) + \langle \hat{\tau}, \hat{\tau} \rangle \right] \]

\[ = |\hat{\gamma}_u| \left[ -\hat{k} \left( \sigma_1 \hat{k} + \sqrt{2Te^{-i} \sigma_2} + 1 \right) \right]. \]
Therefore, the differential operators with respect to rescaled arc-length and rescaled time have the commutator relation
\[ \frac{\partial_t}{\partial s} = \frac{\partial_t}{\partial s} \left( |\gamma_a|^{-1} \partial_a \right) = |\gamma_a|^{-1} \partial_a \partial_t - |\gamma_a|^{-2} \gamma_0 \partial_a, \]
(15)
and
\[ = \partial_s \partial_t \left[ k \left( \sigma_1 \dot{k} + \sqrt{2T} e^{-i} \sigma_2 \right) - 1 \right] \partial_a. \]

We use the definition of the rescaling to calculate
\[ \hat{k}_i = \left( \frac{k}{\phi} \right) \frac{\partial}{\partial t} \left( k + \phi^{-1} \partial k \right) \frac{\partial}{\partial t} \]
(16)
\[ = (-\phi k + \phi^{-1} (\sigma_1 k_{ss} + \sigma_1 k^3 + \sigma_2 k^2)) \phi^{-2} \]
\[ = \hat{k}_i^2 \left( \sigma_1 \dot{k} + \sqrt{2T} e^{-i} \sigma_2 \right) - \dot{k} + \sigma_1 \dot{k}_s. \]

Then by (15) we have
\[ (\hat{k}_s)_i = \left[ k \left( \sigma_1 \dot{k} + \sqrt{2T} e^{-i} \sigma_2 \right) - 1 \right] \hat{k}_s + \partial_s (\hat{k}_i) \]
(17)
\[ = \sigma_1 (\hat{k}_s)_s + \left( 4 \sigma_1 k^2 + 3 \sqrt{2T} e^{-i} \sigma_2 k - 2 \right) \hat{k}_s \]
\[ = 2 \sigma_1 \int_{\gamma} \hat{k}_s \hat{k}_s \hat{d}s + 7 \sigma_1 \int_{\gamma} k \hat{k}_s \hat{k}_s \hat{d}s + 5 \sqrt{2T} e^{-i} \sigma_2 \int_{\gamma} k \hat{k}_s \hat{k}_s \hat{d}s - 3 \int_{\gamma} \hat{k}_s \hat{d}s. \]

The key evolution equation we require is for \( \|\dot{k}_s\|^2 \).

**Lemma 5.1.** Let \( \gamma : S^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow (12). There exist continuous functions \( \xi_1, \xi_2 \) such that
\[ \frac{d}{dt} \int \hat{k}_s^2 \hat{d}s = -2 \sigma_1 \int \hat{k}_s \hat{k}_s \hat{d}s + (1 + \alpha) \int \hat{k}_s \hat{d}s \]
\[ + \left( 7 \sigma_1 \hat{k}_s^2 (\xi_1(t)) + 5 \sqrt{2T} e^{-i} \sigma_2 \hat{k}_s (\xi_2(t)) - (2 - \alpha) \right) \int \hat{k}_s \hat{d}s, \]
for any \( \alpha \in \mathbb{R} \).

**Proof.** We calculate
\[ \frac{d}{dt} \int \hat{k}_s^2 \hat{d}s = \frac{d}{dt} \int \hat{k}_s \hat{k}_s |\gamma_a| \hat{d}u = \int \hat{k}_s \hat{k}_s |\gamma_a| \hat{d}u + \int \hat{k}_s ^2 |\gamma_a| \hat{d}u \]
\[ = \int \hat{k}_s \hat{k}_s |\gamma_a| \hat{d}u + \int \hat{k}_s ^2 |\gamma_a| \hat{d}u \]
\[ = 2 \sigma_1 \int \hat{k}_s \hat{k}_s \hat{d}s + 7 \sigma_1 \int \hat{k}_s \hat{k}_s \hat{d}s + 5 \sqrt{2T} e^{-i} \sigma_2 \int \hat{k}_s \hat{k}_s \hat{d}s - 3 \int \hat{k}_s \hat{d}s. \]

Using the mean value theorem for integrals (see the appendix in [33]) for example we have
\[ \frac{d}{dt} \int \hat{k}_s^2 \hat{d}s = -2 \sigma_1 \int \hat{k}_s \hat{k}_s \hat{d}s + 7 \sigma_1 \int \hat{k}_s \hat{k}_s \hat{d}s + 5 \sqrt{2T} e^{-i} \sigma_2 \int \hat{k}_s \hat{k}_s \hat{d}s - 3 \int \hat{k}_s \hat{d}s \]
\[ = -2 \sigma_1 \int \hat{k}_s \hat{k}_s \hat{d}s - 3 \int \hat{k}_s \hat{d}s \]
\[ + \left( 7 \sigma_1 \hat{k}_s^2 (\xi_1(t)) + 5 \sqrt{2T} e^{-i} \sigma_2 \hat{k}_s (\xi_2(t)) \right) \int \hat{k}_s \hat{d}s. \]

Splitting up the second integral on the right hand side yields the lemma. \( \square \)
We are now ready to prove our main result for this section.

**Theorem 5.2** (Eventual convexity). Let \( \dot{\gamma} : \mathbb{S}^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow (12). Set

\[
T_{\text{max}} := \frac{1}{2\pi \sigma_1 \sigma_2} \min \{ L_0 \sigma_1, A_0 \sigma_2 \},
\]

where \( L_0 \) and \( A_0 \) are the initial length and enclosed area of the original (not rescaled) flow. Suppose that the initial data of the original flow satisfies

\[
\sqrt{L_0} \| k_s \|_2 \big|_{t=0} \leq \frac{1}{14 \sigma_1} \left( 5 \sigma_1^2 \| \dot{k} \|_2^2 + 25 \sigma_1 (2 - \alpha) T_{\text{max}} - 5 \sigma_2 \right),
\]

for some \( \alpha \in (0, 2) \). Then the rescaled flow converges to a standard round circle as \( \hat{t} \to \infty \).

**Proof.** We make the following assumption:

\[
\text{(19)} \quad \text{For all} \hat{t} \text{ the function } \dot{k}(\cdot, \hat{t}) \text{ has a zero.}
\]

Our goal in this proof is to contradict (19). Assumption (19) implies the estimate

\[
\dot{k} \leq \int_{\tilde{\gamma}} |\dot{k}_s| \, ds.
\]

Therefore

\[
7 \sigma_1 \dot{k}^2 + 5 \sqrt{2T} e^{-i} \dot{k} \leq 7 \sigma_1 \tilde{L} \| \dot{k}_s \|_2^2 + 5 \sqrt{2T} \sigma_2 e^{-i} \sqrt{\tilde{L}} \| \dot{k}_s \|_2^2
\]

\[
\leq \sqrt{\tilde{L}} \| \dot{k}_s \|_2 \left( 7 \sigma_1 \sqrt{\tilde{L}} \| \dot{k}_s \|_2 + 5 \sigma_2 \sqrt{2T} e^{-i} \right).
\]

Setting \( x = \sqrt{\tilde{L}} \| \dot{k}_s \|_2 \) we require \( 7 \sigma_1 x^2 + 5 \sigma_2 e^{-i} \sqrt{2T} x < 2 - \alpha \). Since we have no control over how large \( \hat{t} \) is, we rephrase this as \( 7 \sigma_1 x^2 + 5 \sigma_2 \sqrt{2T} x < 2 - \alpha \). As \( x \geq 0 \), the condition we require is

\[
14 \sigma_1 \sqrt{\tilde{L}} \| \dot{k}_s \|_2 \leq \sqrt{50T \sigma_2^2 + 28 \sigma_1 (2 - \alpha) - 5 \sigma_2 \sqrt{2T}}.
\]

The function \( b \to -b + \sqrt{b^2 + 28 \sigma_1 (2 - \alpha)} \) is decreasing, and so our condition is

\[
14 \sigma_1 \sqrt{\tilde{L}} \| \dot{k}_s \|_2 \leq \sqrt{50T_{\text{max}} \sigma_2^2 + 28 \sigma_1 (2 - \alpha) - 5 \sigma_2 \sqrt{2T_{\text{max}}}}.
\]

Under this condition \( \| \dot{k}_s \|_2^2 \) is initially decreasing exponentially fast. However we need to satisfy (13) for positive \( \hat{t} \) so that exponential decay continues. This is difficult because of the rescaled length factor. Set

\[
\tilde{L}(\hat{t}) = e^{-\hat{t}} \hat{L}(\hat{t}).
\]

Then \( \tilde{L}' \leq 0 \), in particular,

\[
\tilde{L}(\hat{t}) \leq \tilde{L}(0) - \int_0^{\hat{t}} \sigma_1 \| \dot{k}_s \|_2^2 + 2 \pi \sigma_2 \sqrt{2T} e^{-\hat{t}} d\hat{t}.
\]

While the bound (18) is satisfied, we have

\[
\frac{d}{d\hat{t}} \left( \tilde{L} \int_{\tilde{\gamma}} \dot{k}_s^2 \, d\tilde{s} \right) \leq - (1 + \alpha) \tilde{L} \int_{\tilde{\gamma}} \dot{k}_s^2 \, d\tilde{s},
\]

or

\[
\frac{d}{d\hat{t}} \left( e^{(1 + \alpha)\hat{t}} \tilde{L} \int_{\tilde{\gamma}} \dot{k}_s^2 \, d\tilde{s} \right) \leq 0.
\]
This gives the decay estimate

$$\hat{L} \int \hat{k}_s^2 \, d\hat{s} \leq \hat{L}(0) \|\hat{k}\|_2^2(0) e^{-\alpha t}.$$ 

Now there are only two remaining possibilities. Either \(\|\hat{k}\|_2^2\) or \(\hat{L}\) converges to zero.

**Case 1.** \(\|\hat{k}\|_2^2\) converges to zero. This is a contradiction, since it would imply \(\hat{k}\) is asymptotic to a constant in \(L^2\), however as \(\hat{\gamma}\) is closed, this constant has to be strictly positive. But then for \(t\) sufficiently large the function \(\hat{k}\) has no zeroes; a contradiction with assumption (19).

**Case 2.** \(\hat{L}\) converges to zero. Note that \(L(t)\) also converges to zero, since \(L(t) = e^{-\hat{\gamma} t}\hat{L}(\hat{\gamma})\).

Then the calculation in Lemma 4.1 applies to show that \(\hat{A} \to \sigma_1 \pi > 0\). But this contradicts the isoperimetric inequality.

Therefore in any case we must have at some time strict convexity of the flow. After this time it remains so, and the theory in the first part of the paper applies to yield smooth convergence to a standard round circle. □

### 6. A Lifespan Theorem

In this section we estimate the maximal time from below. We follow the literature (primarily for higher-order flows, see the original work of Kuwert-Schätzle [23], which was later extended to other settings [3, 12, 13, 15, 16, 17, 18]). The idea for this kind of estimate can be traced back to work of Struwe on the harmonic map heat flow [34]. Each new setting requires different integral estimates; in the setting we study here, we use for the first time a product of two functionals (for scale-invariance reasons) and new arguments to contradict \(T = \lambda\). This is because the standard approach (that this implies the flow may be smoothly extended) may not be strictly true. Indeed, it is a reasonable possibility that the length would contract to zero so quickly that the quantity \(L\|k\|_2^2\) remains small while the solution shrinks to a point. In order to rule this out, we use an argument relying on isoperimetry and the continuous rescaling from Section 4.

We use the notation

\[
L_\phi = \int_\gamma \phi \, ds, \quad \text{and} \quad L_{B_\rho}(x) = \int_{\gamma^{-1}(B_\rho(x))} ds
\]

to denote localised length. In the equation above, \(\phi = \tilde{\phi} \circ \gamma\), with \(\tilde{\phi} : \mathbb{R}^2 \to [0, 1]\) a smooth function such that

\[\chi_{B_{\rho/2}(x)} \leq \tilde{\phi} \leq \chi_{B_\rho(x)}\]

and

\[|\phi_s| \leq \frac{c}{\rho}.
\]

To indicate that we are using \(\phi\) as above, we say simply that \(\phi\) is a smooth cutoff function on balls of radii \(\rho\) and \(\rho/2\).

We warm up with the following lemma, that contains a key integral estimate.

**Lemma 6.1.** Let \(\gamma : S^1 \times [0, T) \to \mathbb{R}^2\) be a solution to \(\dot{\gamma}\). Let \(T^* \in [0, T), \rho > 0\) be given. Suppose

\[
\sup_{t \in [0, T^*]} \sup_{x \in \mathbb{R}^2} L_{B_\rho}(x) \int_{\gamma^{-1}(B_\rho(x))} k^2 \, ds \leq \varepsilon \leq \frac{1}{16}.
\]
Suppose additionally that $\gamma$ is not convex on $[0, T^*]$, that is, that $k(\cdot, t) : S^1 \to \mathbb{R}$ has a zero for each $t \in [0, T^*]$. Then for all $t \in [0, T^*]$ we have

$$L_{\phi^t} \int_\gamma k^4 \phi^4 \, ds \leq \frac{1}{2} L_{\phi^t} \int_\gamma k_\gamma^2 \phi^4 \, ds + \frac{c\varepsilon^2}{\rho^2},$$

for an absolute constant $c > 0$, and where $\phi$ is a smooth cutoff function on balls of radii $\rho$ and $\rho/2$ centred anywhere.

**Proof.** For brevity let $\psi := \phi^4$. We calculate

$$L_\psi \int_\gamma k^4 \psi \, ds = (k^2 \phi^2)(\xi(t))L_\psi \int_\gamma k^2 \phi^2 \, ds$$

$$\leq 2L_\psi \int_\gamma k^2 \phi^2 \, ds + 2L_\psi \int_\gamma k^2 \phi \phi_s \, ds$$

$$\leq 2L_\psi \int_\gamma k^2 \phi^2 \, ds \left( \int_\gamma k \chi_{B(\rho)} \, ds \right) \left( \int_\gamma k \phi \, ds \right) + \frac{cL_\psi}{\rho} \int_\gamma k^2 \phi \, ds \int_\gamma k \phi \, ds$$

$$\leq \delta L_\psi \int_\gamma k^2 \phi \, ds + \frac{L_\psi}{\delta} \left( \int_\gamma k^2 \phi \, ds \right)^2 \int_\gamma k^2 \chi_{B(\rho)} \, ds$$

$$+ \frac{\delta \left( \int_\gamma k^2 \phi \, ds \right)^2}{\delta \phi^2} + \frac{c L_\psi^2}{\delta \phi^2} \left( \int_\gamma k^2 \phi \, ds \right)^2$$

$$\leq \delta L_\psi \int_\gamma k^2 \phi \, ds + \frac{L_\psi^2}{\delta} \int_\gamma k^4 \psi \, ds \int_\gamma k^2 \chi_{B(\rho)} \, ds$$

$$+ \frac{\delta L_\psi^2}{\delta} \int_\gamma k^4 \phi \, ds + \frac{c \varepsilon^2}{\delta \rho^2}$$

Let us take $\delta = \frac{1}{4}$ and assume $\varepsilon \leq \frac{1}{16}$; then absorb to find

$$L_\psi \int_\gamma k^4 \psi \, ds \leq \frac{1}{2} L_\psi \int_\gamma k^2 \phi \, ds + \frac{c \varepsilon^2}{\rho^2},$$

as required. \qed

Now we prove the key integral estimate that controls the concentration of curvature along the flow.

**Proposition 6.2.** Let $\gamma : S^1 \times [0, T) \to \mathbb{R}^2$ be a solution to (2). Let $T^* \in [0, T)$, $\rho > 0$ be given. Suppose that $\gamma$ is not convex on $[0, T^*]$, that is, that $k(\cdot, t) : S^1 \to \mathbb{R}$ has a zero for each $t \in [0, T^*]$. There exists an absolute constant $\varepsilon_0 > 0$ such that if

$$\sup_{t \in [0, T^*]} \sup_{x \in \mathbb{R}^2} L_{B(\rho)}(x) \int_{\gamma^{-1}(B(\rho)(x))} k^2 \, ds = \varepsilon \leq \varepsilon_0 \leq \frac{1}{16},$$

then

$$L_{\phi^t} \int_{\gamma^{-1}(B_2(x))} k^2 \, ds \leq \left( L_{\phi^t} \int_{\gamma^{-1}(B_2(x))} k^2 \, ds \right)_{t=0} + c_0 t (1 + \rho^{-2}) \varepsilon, \quad t \in [0, T^*],$$

for a constant $c_0 > 0$ depending only on $\sigma_1$ and $\sigma_2$. 
Proof. We again let \( \psi := \phi^3 \). Differentiating,

\[
\frac{d}{dt} \left( L_\psi \int_\gamma k^2 \psi \, ds \right) = \int_\gamma \frac{2}{3} k^2 \psi \, ds \left( \int_\gamma (-\sigma_1 k^2 - \sigma_3 k) \psi \, ds + 4 \int_\gamma \phi \phi^3 \, ds \right) \\
+ L_\psi \int_\gamma 4k^2 \phi^3 (D \phi \cdot \nu)(\sigma_1 k + \sigma_2) \, ds \\
+ L_\psi \int_\gamma (2k(\sigma_1 k^2 + k^2(\sigma_1 k + \sigma_2)) - \sigma_1 k^4 - \sigma_3 k^3) \psi \, ds \\
= -2L_\psi \sigma_1 \int_\gamma k^2 \psi \, ds - 8L_\psi \sigma_1 \int_\gamma k_s k \phi^3 \phi_s \, ds \\
- \sigma_1 \left( \int_\gamma \frac{2}{3} k^2 \psi \, ds \right)^2 - \sigma_2 \int_\gamma \frac{2}{3} k^2 \psi \, ds \int_\gamma k \psi \, ds \\
+ L_\psi \int_\gamma \sigma_1 k^4 + \sigma_3 k^3 \psi \, ds \\
+ 4L_\psi \int_\gamma \frac{2}{3} k^2 \phi^3 (D \phi \cdot \nu)(\sigma_1 k + \sigma_2) \, ds \\
+ 4 \int_\gamma \frac{2}{3} k^2 \psi \, ds \int_\gamma (D \phi \cdot \nu)(\sigma_1 k + \sigma_2) \phi^3 \, ds.
\]

We observe the estimates

\[
-8L_\psi \sigma_1 \int_\gamma k_s k \phi^3 \phi_s \, ds \leq 4\delta L_\psi \sigma_1 \int_\gamma k_s^2 \psi \, ds + \frac{4\sigma_1 c^2}{\rho^2 \delta} L_\psi \int_\gamma k^2 \phi^2 \, ds,
\]

\[
L_\psi \int_\gamma 4k^2 \phi^3 (D \phi \cdot \nu)(\sigma_1 k + \sigma_2) \, ds \leq \frac{\sigma_1}{\rho} L_\psi \int_\gamma k^3 \phi^3 \, ds + \frac{\sigma_2}{\rho} L_\psi \int_\gamma k^2 \phi^3 \, ds \\
\leq \frac{\sigma_1}{2} L_\psi \int_\gamma k^4 \psi \, ds + \frac{\sigma_1}{\rho^2} L_\psi \int_\gamma k^2 \phi^2 \, ds \\
+ \frac{\sigma_2}{\rho^2} L_\psi \int_\gamma k^2 \phi^2 \, ds + \sigma_2 L_\psi \int_\gamma k^2 \psi \, ds,
\]

\[
4 \int_\gamma \frac{2}{3} k^2 \psi \, ds \int_\gamma (D \phi \cdot \nu)(\sigma_1 k + \sigma_2) \phi^3 \, ds - \sigma_2 \int_\gamma \frac{2}{3} k^2 \psi \, ds \int_\gamma k \psi \, ds \\
\leq \left( \frac{\sigma_1}{\rho} + \sigma_2 \right) L_\phi^2 \left( \int_\gamma k^2 \psi \, ds \right)^{\frac{3}{2}} + \frac{\sigma_2}{\rho} L_\phi^3 \int_\gamma k^2 \psi \, ds.
\]
ON A CURVATURE FLOW MODEL FOR EMBRYONIC EPIDERMAL WOUND HEALING

\[
\leq \frac{\sigma_1}{4} \left( \int_{\gamma} k^2 \psi \, ds \right)^2 + c(\sigma_1 \rho^{-2} + \sigma_2 \rho^{-1} + \sigma_2 \rho^{-1})\epsilon \\
\leq \frac{\sigma_1}{4} L\psi \int_{\gamma} k^4 \psi \, ds + c(\sigma_1 \rho^{-2} + \sigma_2 \rho^{-1})\epsilon ,
\]

\[
L\psi \sigma_2 \int_{\gamma} k^3 \psi \, ds \leq \frac{\sigma_1}{4} L\psi \int_{\gamma} k^4 \psi \, ds + \frac{\sigma_2}{\sigma_1} L\psi \int_{\gamma} k^2 \psi \, ds ,
\]

\[
2L\psi \sigma_1 \int_{\gamma} k^4 \psi \, ds \leq L\psi \sigma_1 \int_{\gamma} k^2 \psi \, ds + \frac{c_0 \epsilon^2}{\rho^2}.
\]

The last estimate follows from Lemma 6.1. Therefore we have

\[
\frac{d}{dt} \left( L\psi \int_{\gamma} k^2 \psi \, ds \right) \leq -L\psi \sigma_1 \int_{\gamma} k^2 \psi \, ds - \sigma_1 \left( \int_{\gamma} k^2 \psi \, ds \right)^2 + c_0(1 + \rho^{-2})\epsilon ,
\]

where \( c_0 \) is a constant depending only on \( \sigma_1, \sigma_2 \) and \( \tilde{\phi} \). Integrating, we find

\[
L\psi \int_{\gamma} k^2 \psi \, ds + \int_0^t \left( L\psi \sigma_1 \int_{\gamma} k^2 \psi \, ds + \sigma_1 \left( \int_{\gamma} k^2 \psi \, ds \right)^2 \right) \, dt' \leq L\psi(0) \int_{\gamma} k^2 \psi \, ds \bigg|_{t=0} + c_0(1 + \rho^{-2})\epsilon .
\]

This estimate implies (21). □

**Theorem 6.3** (Lifespan theorem). Let \( \gamma : S^1 \times [0, T) \to \mathbb{R}^2 \) be a non-convex solution to (2). There are constants \( \rho \in (0, 1), \epsilon_1 > 0, \) and \( c_0 < \infty \) such that

\[
\sup_{x \in \mathbb{R}^2} L_{B_\rho(x)} \int_{\gamma^{-1}(B_\rho(x))} k^2 \, ds \bigg|_{t=0} = \epsilon(x) \leq \epsilon_1
\]

implies that the maximal time \( T \) satisfies

\[
T \geq \frac{1}{c_0} \rho^2 ,
\]

and we have the estimate

\[
L_{B_\rho(x)} \int_{\gamma^{-1}(B_\rho(x))} k^2 \, ds \leq c_1 \epsilon \quad \text{for} \quad 0 \leq t \leq \frac{1}{c_0} \rho^2 .
\]

**Proof.** We make the definition

\[
\eta(t) = \sup_{x \in \mathbb{R}^2} L_{B_\rho(x)} \int_{\gamma^{-1}(B_\rho(x))} k^2 \, ds .
\]

By covering \( B_\rho(x) \subset \mathbb{R}^2 \) with several translated copies of \( B_\rho \) there is a universal constant \( c_\eta \) such that

\[
\eta(t) \leq c_\eta \sup_{x \in \mathbb{R}^2} L_{B_\rho(x)} \int_{\gamma^{-1}(B_\rho(x))} k^2 \, ds .
\]

By short time existence the function \( \eta : [0, T) \to \mathbb{R} \) is continuous. We now define

\[
t_0 = \sup \{ 0 \leq t \leq \min(T, \lambda) : \eta(\tau) \leq \delta \ \text{for} \ 0 \leq \tau \leq t \},
\]

where \( \lambda, \delta \) are parameters to be specified later.
The proof continues in three steps.

(26) \( t_0 = \min(T, \lambda), \)

(27) \( t_0 = \lambda \implies \text{Lifespan theorem,} \)

(28) \( T \neq \infty \implies t_0 \neq T. \)

The three statements (26), (27), (28) together imply the Lifespan theorem. The argument is as follows: first notice that by (26) \( t_0 = \lambda \) or \( t_0 = T, \) and if \( t_0 = \lambda \) then by (27) we have the Lifespan theorem. Also notice that if \( t_0 = \infty \) then \( T = \infty \) and the Lifespan theorem follows from estimate (30) below (used to prove statement (27)). Therefore the only remaining case where the Lifespan theorem may fail to be true is when \( t_0 = T < \infty. \) But this is impossible by statement (28), so we are finished.

To prove step 1, suppose it is false. This means that \( t_0 < \lambda. \) This implies that on \([0, t_0]\) we have \( \eta(t) \leq \delta, \) and \( \eta(t_0) = \delta. \) Setting \( \phi \) to be a cutoff function that is identically one on \( B_{\bar{2}}(x) \) and zero outside \( B_p(x), \) so that \( \phi = \phi \circ \gamma \) has the corresponding properties on the preimages of these balls under \( \gamma, \) Proposition 6.2 implies

\[
L_{B_{\bar{2}}(x)} \int_{\gamma^{-1}(B_{\bar{2}}(x))} k^2 \, ds \leq \left( L_{B_p(x)} \int_{\gamma^{-1}(B_p(x))} k^2 \, ds \right)_{t=0} + c_0(1 + \rho^{-2})t\varepsilon_1, \quad t \in [0, t_0].
\]

We use the assumption \( \rho \leq 1 \) to conclude

(29) \[
L_{B_{\bar{2}}(x)} \int_{\gamma^{-1}(B_{\bar{2}}(x))} k^2 \, ds \leq \left( L_{B_p(x)} \int_{\gamma^{-1}(B_p(x))} k^2 \, ds \right)_{t=0} + 2c_0\rho^{-2}t\varepsilon_1, \quad t \in [0, t_0].
\]

The aforementioned covering argument yields

(30) \[
L_{B_p(x)} \int_{\gamma^{-1}(B_p(x))} k^2 \, ds \leq c_q\varepsilon_1 + 2c_0c_0\rho^{-2}t\varepsilon_1.
\]

We choose \( \delta = 3c_q\varepsilon_1, \) and \( \varepsilon_1 \) small enough such that \( \delta \leq \varepsilon_0 \) where \( \varepsilon_0 \) is the constant from Proposition 6.2. Then, picking \( \lambda = \rho^2/c_0, \) the estimate (30) implies

\[
3c_q\varepsilon_1 = \eta(t) < 3c_q\varepsilon_1, \quad \text{for all } 0 \leq t \leq t_0,
\]

which is a contradiction.

We have also proved the second step (27). Observe that if \( t_0 = \lambda \) then by the definition (25) of \( t_0, \)

\[
T \geq \lambda = \rho^2/c_0,
\]

which is the lower bound for maximal time claimed by the Lifespan theorem. That is, we have proved if \( t_0 = \lambda, \) then the Lifespan theorem holds, which is the second step. The estimate (27) follows from \( \eta(t) \leq 3c_q\varepsilon_1 \) above, valid for \( t \in [0, t_0]. \)

We assume

\[
t_0 = T \neq \infty;
\]

since if \( T = \infty \) then the lower bound on \( T \) holds automatically and again the previous estimates imply the a-priori control (29) on \( \|k\|_{L^2(2^{-1}(B_{\bar{2}}(x)))}. \) Note also that we can safely assume \( T < \lambda, \) since otherwise we can apply step two to conclude the Lifespan theorem.

We now show that this can only lead to a contradiction. Suppose first that \( L \) is asymptotic to \( \mu > 0. \) Then by hypothesis \( \|k\|_{L^2}^2 \) is uniformly bounded and by a compactness theorem (see e.g. [4]) the flow has a \( C^{1, \alpha} \) limit as \( t \to T. \) Local existence and uniqueness for such initial data (by
using an argument analogous to Deckelnick [12] for example) we may extend the flow past $T$, a contradiction.

The only remaining possibility is that $L$ is asymptotic to zero. Note that the isoperimetric inequality implies that $A$ is also asymptotic to zero. Now consider the rescaled flow $\hat{\gamma}$ as in Section 5. Note that the proof of Lemma 4.1 works in this case without change, and implies that

$$
\lim_{t \to \infty} \hat{A} = \sigma_1 \pi.
$$

Since the area and length of $\gamma$ approach zero, there is a time $t^*$ such that for all $t > t^*$,

$$
L(\|k\|_2^2(t)) = \sup_{x \in \mathbb{R}^2} L_B_{\hat{\rho}}(x) \|k\|_{2,\gamma^{-1}(B_{\hat{\rho}}(x))}^2(t).
$$

Therefore, for $t > t^*$ we have the uniform estimate

$$
L(\|k\|_2^2) \leq \epsilon.
$$

We now calculate for the rescaled flow

$$
\lim_{t \to \infty} \hat{L}^2 = \lim_{t \to T} \frac{L^2}{2T - 2t} = \lim_{t \to T} -L(-\sigma_1 \|k\|_2^2 - 2\pi \sigma_2) = \sigma_1 \lim_{t \to T} L(\|k\|_2^2)
$$

so that $\hat{L}^2(\hat{t}) \leq \sigma_1 \epsilon + c_1(\hat{t})$, where $c_1(\hat{t}) \to 0$ as $\hat{t} \to \infty$. The isoperimetric inequality implies $\hat{L}^2 \geq 4\pi \hat{A}$, so that by (31) we have $\hat{L}^2(\hat{t}) \geq 4\sigma_1 \pi^2 - c_2(\hat{t})$, where $c_2(\hat{t}) \to 0$ as $\hat{t} \to \infty$. This is a contradiction for $\hat{t}$ sufficiently large, so long as $\epsilon < 4\pi^2$ (which is an absolute constant).

This establishes (28) and finishes the proof of the theorem. \( \square \)

7. APPLICATION OF THE LIFESPAN THEOREM TO BLOWUPS

In this section we outline how to prove Theorem 1.3. The key assumption is that

(32) Suppose that $\gamma_t$ do not contract to a round point as $t \nearrow T$.

The first dot point follows because if $\gamma_t$ were convex at any time, then Theorem 1.1 would apply with $\gamma_t$ as initial data and the flow would contract to a round point. This is in contradiction with (32).

The second dot point follows because if at any time that inequality is violated then the second part of Theorem 1.1 applies and we again have contraction to a round point.

For the third dot point, our idea is as follows. We use the Lifespan theorem 1.2 at each $t$ to rescale the solution, producing a sequence that converges to a limiting flow. This limiting flow is called a blowup. We calculate that it solves the curve shortening flow and is ancient. To this flow we apply a theorem of Daskalopoulos-Hamilton-Sesum [11], that gives our desired partial classification (the third dot point).

Let us define the critical radius

$$
r_t = \sup \left\{ \rho > 0 : \forall x \in \mathbb{R}^2, L_{B_{\rho}}(x) \int_{\gamma^{-1}(B_{\rho}(x))} k^2 \, ds \leq \epsilon_1 \right\}
$$

for $t \in [0, T)$. Note that

$$
L_{B_{r_t}}(x) \int_{\gamma^{-1}(B_{r_t}(x))} k^2 \, ds \leq \epsilon_1 \quad \forall x \in \mathbb{R}^2
$$
and that there exists for each \( t \in [0, T) \) an \( x_t \in \mathbb{R}^2 \) such that
\[
L_{B_{r_t}(x_t)} \int \gamma^{-1}(\gamma_t(x)) k^2 \, ds \geq \varepsilon_1.
\]
The Lifespan Theorem implies that (for \( T \) the maximal existence time)
\[
c_0^{-1}r_t^2 \leq T - t, \quad \text{for } 0 \leq t < T.
\]
In particular this implies \( r_t \to 0 \) as \( t \nearrow T \) (for \( T < \infty \)). Let us now introduce the notation \( \gamma_t(\cdot) := \gamma(\cdot, t) \). Consider the discrete rescaling
\[
\gamma_v^t := \frac{1}{r_t} (\gamma_{t+vT^2} - x_t)
\]
for \( 0 \leq v \leq c_0 \). Note that \( \gamma_v^t : \mathbb{S} \times [-r_t^{-2}t, r_t^{-2}(T-t)] \to \mathbb{R}^2 \), and
\begin{equation}
\label{eq:33}
\partial_v \gamma_v^t = (\sigma_1 r_t k_v^t + \sigma_2 r_t) \nu_{t+vT^2} = (\sigma_1 k_{\gamma_v^t} + \sigma_2 r_t) \nu_{\gamma_v^t}.
\end{equation}
Then
\[
L_{B_{r_t}(x)} \int (\gamma_v^t)^{-1}(B_{r_t}(x)) k_{\gamma_v^t}^2 \, ds_{\gamma_v^t} = L_{B_{r_t}(x)} \int (\gamma_v^t)^{-1}(B_{r_t}(x)) k_{\gamma_v^t}^2 \, ds_{\gamma_v^t} \leq c\varepsilon_1,
\]
for \( -r_t^{-2}T \leq v \leq c_0, \ 0 \leq t \leq T, \ x \in \mathbb{R}^2 \). A standard argument yields uniform estimates for all derivatives of curvature for the flows \( \gamma_v^t \) where \( v \leq c_0 \). Therefore a compactness theorem applies and we deduce the existence of subsequences \( t_j \to T \) such that the flows \( (\gamma_v^t_{t_j})_{v \in [-r_{t_j}^{-2}T, c_0]} \) converge after reparametrisation smoothly on compact subsets of \( \mathbb{R}^2 \) to a flow \( (\gamma_v^\infty)_{v \in (-\infty, c_0]} : \mathbb{S}^\infty \to \mathbb{R}^2 \), where \( \mathbb{S}^\infty \) is an open 1-manifold without boundary, possibly not connected. If \( \mathbb{S}^\infty \) contains a compact component, then \( \mathbb{S}^\infty \) is equal to this component, that is, has no further components.

We compute that the blowup \( \gamma^\infty \) satisfies for \( v \in (-\infty, c_0] \) the evolution equation
\[
\partial_v \gamma_v^\infty = \sigma_1 k_{\gamma_v^\infty} \nu_{\gamma_v^\infty}.
\]
The blowup is therefore an ancient non-convex solution to the curve shortening flow. Note that for \( v \in (-\infty, 0) \) curvature is uniformly bounded and we have \( ||k||_1 \leq c\varepsilon_1 \). If the blowup is embedded and compact, Daskalopoulos-Hamilton-Sesum \([11]\) implies that the blowup must be convex. Therefore each component of the blowup must be either noncompact, nonembedded, or both.

**Appendix A. Estimates for the Convex Flow**

This section follows ideas of Chou-Zhu for anisotropic flows by curvature. We produce the details here in our special case for the convenience of the reader.

We shall investigate the asymptotic behaviour of the flow with the key assumption being convexity. We claim that a family of closed embedded convex plane curves contract to a circular point as \( t \) approaches the final time \( T \). As seen in section 3 the curvature \( k \) tends to infinity as \( t \to T \) because the area vanishes. In order to control the blow up of curvature, we are going to rescale the curve in such a way that its enclosed area approaches a positive constant. The proof relies on a uniform bound on the rescaled curvature.

Let us introduce parametrisation by angle. We define the normal angle \( \vartheta \) to be the angle made by the inner normal of a curve \( \gamma \) and the positive \( x \)-axis. For a fixed time, the simple, closed, convex curve \( \gamma \) can be parametrised by normal angle \( \vartheta \) in such a way that \( \gamma = \gamma(\vartheta) \), with inner
Lemma A.1. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}$ be a closed embedded convex plane curve. The support function $h : \mathbb{S}^1 \to \mathbb{R}$ defined as $h(\gamma(\vartheta)) = (\gamma, -\nu) = (\gamma(\vartheta), (\cos \vartheta, \sin \vartheta))$ is related to the curvature $k(\vartheta)$ in such way
\[
h(\theta) + h_{\vartheta} \vartheta = \frac{1}{k}.\]

Let us define the width of a convex curve in the $(\cos \vartheta, \sin \vartheta)$ direction to be
\[
w(\vartheta) = h(\vartheta) + h(\vartheta + \pi).
\]

The next lemma relates the width to the entropy
\[
E(\gamma(\vartheta)) = \frac{1}{|\gamma|} \int_{\gamma} \log(k) \, d\vartheta.
\]

Lemma A.2. Let $\gamma : \mathbb{S}^1 \to \mathbb{R}$ be a closed, embedded, convex plane curve with support function $h : \mathbb{S}^1 \to \mathbb{R}$ as defined in Lemma A.1. There exists a positive constant $C$ such that for all $\vartheta \in \mathbb{S}^1$,
\[
w(\vartheta) \geq Ce^{-E(\gamma)}.
\]

Furthermore, if the entropy $E(\gamma)$ is uniformly bounded from above, then $w(\vartheta) > 0$.

Next, let us derive an a priori estimate on the speed $F(\theta, t)$ of the flow (2).

Proposition A.3. Let $\gamma : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ be a family of closed, embedded, convex plane curves evolving by the flow (2). Define
\[
M^2 = \sup_{\theta \in [0, 2\pi)} (F^2 + F^2_{\vartheta})(\theta, 0).
\]

The speed of the flow $F(\theta, t)$ satisfies:
\[
\sup_{\theta \in [0, 2\pi)} |F_{\vartheta}(\theta, t)| \leq M + \int_0^{2\pi} |F(\theta, t)| \, d\vartheta, \quad \text{and} \quad (35)
\]
\[
F_{\max}(t) \leq M_1 \left(1 + \int_0^{2\pi} |F(\theta, t)| \, d\vartheta\right), \quad (36)
\]
where $M_1 = \max\{2\pi M, 2\pi + (2\pi)^{-1}\}$, and
\[
F_{\max}(t) \leq 2F(\theta, t) + \frac{M}{2\pi}, \quad (37)
\]
for all $\theta$ satisfying $|\theta - \theta^*(t)| \leq \frac{1}{2\pi}$, where $\theta^*$ is such that $F(\theta, t) \leq F(\theta^*(t), t)$ for all $\theta \in [0, 2\pi]$.

The proof uses the following lemmata:

Lemma A.4. The speed of the flow $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ satisfies
\[
F_{\vartheta} = \sigma_1 k^2(F_{\vartheta \vartheta} + F). \quad (38)
\]

Proof. This follows by combining (1) with the definition of $F$ (recall (2)). \hfill \Box

Lemma A.5. Let $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}$ be the speed of the flow (2). Set $M^2$ as in (34). Then $(F^2 + F^2_{\vartheta})(\theta, t) > M^2$ implies $F_{\vartheta}(\theta, t) \geq 0$. \hfill
\textbf{Proof.} Fix a point \((\theta_0, t_0)\) for some \(t_0 > 0\), and let
\begin{equation}
B = (F^2 + F_\theta^2)^{1/2}(\theta_0, t_0).
\end{equation}
Assume that \(B > M\). We need to show that \(F_t(\theta_0, t_0) \geq 0\), which is equivalent to showing \((F_{\theta\theta} + F')|_{(\theta_0, t_0)} \geq 0\) (by Lemma [A.4]).

Pick \(\xi \in (-\pi, \pi)\) such that \(F(\theta_0, t_0) = B \cos \xi\) and \(F_\theta(\theta_0, t_0) = -B \sin \xi\). This is possible due to the differentiability of \(F\). Let us denote
\[ F^*(\theta) = B \cos(\theta - \theta_0 + \xi), \]
and consider the function
\[ G(\theta, t) = F(\theta, t) - F^*(\theta). \]
It is straightforward to see that \(G(\theta_0, t_0) = 0\) and \(G_\theta(\theta_0, t_0) = 0\), hence the point \((\theta_0, t_0)\) is a double root for \(G(\cdot, t_0)\).

Now we aim to show that, using the Sturmian oscillation theorem, this must be the only root for \(G\). Since \(F(\theta, t)\), and hence \(G(\theta, t)\), are \(2\pi\)-periodic functions, without loss of generality, we perform analysis on the interval \((\theta_0 - \xi - \pi, \theta_0 - \xi + \pi)\). As \(B > M\) and \((F^2 + F_\theta^2)\) is initially bounded by \(M^2\), we obtain
\[ G(\theta_0 - \xi, 0) = F(\theta_0 - \xi, 0) - B \leq \sup |F(\theta, 0)| - B < 0. \]
So on at least one interval, \(G\) is initially negative. At the end points \(\theta = \theta_0 - \xi \pm \pi\), we have
\[ G(\theta_0 - \xi \pm \pi, 0) = F(\theta_0 - \xi \pm \pi, 0) - F^*(\theta_0 - \xi \pm \pi, 0) = F(\theta_0 - \xi \pm \pi, 0) - B \cos(\pm \pi) \geq F(\theta_0 - \xi - 0) + B > 0. \]
Therefore, there is at least two open intervals on which \(G\) is initially positive. This implies that there are at least two roots for \(G\) at time \(t = 0\). Consider \(\theta_1\) in the interval \((\theta_0 - \xi - \pi, \theta_0 - \xi)\). We have \(\sin(\theta_1 - \theta_0 + \xi) < 0\). If \(\theta_1\) is an initial root of \(G\), (that is, \(G(\theta_1, 0) = 0\)) then \(F(\theta_1, 0) = F^*(\theta_1)\). We compute
\[ G_\theta(\theta_1, 0) = F_\theta(\theta_1, 0) + B \sin(\theta_1 - \theta_0 + \xi) < (B^2 - F^2(\theta_1, 0))^{1/2} + B \sin(\theta_1 - \theta_0 + \xi) \]
\[ = (B^2 - B^2 \cos^2(\theta_1 - \theta_0 + \xi))^{1/2} + B \sin(\theta_1 - \theta_0 + \xi) = 0. \]
Thus \(G\) is decreasing at every root on the interval \((\theta_0 - \xi - \pi, \theta_0 - \xi)\). This means that there can be only one simple root in the interval \((\theta_0 - \xi - \pi, \theta_0 - \xi)\). Indeed, if there were another root for \(G\) after the first, then at this root we must have \(G_\theta \geq 0\), which is impossible by the above calculation. A similar argument applies on \((\theta_0 - \xi, \theta_0 - \xi + \pi)\), yielding that at any root \(\theta_3\) in this interval we have \(G_\theta(\theta_3, 0) > 0\). We conclude that \(G\) has only two simple roots at \(t = 0\).

By Sturmian theory (see for example [13]), for \(G(\theta, t)\) satisfying the parabolic differential equation \(G_t = \sigma_t k^2 G_{\theta\theta}\), with two simple roots at initial time over its entire domain, there can be at most two simple roots (counted with multiplicity) at all later times \(t = t_0 > 0\). As \(G(\theta, t, 0)\) is a double root, it must be the only root for \(G(\cdot, t_0)\).

To show that \(G(\theta_0, t_0)\) is a minimum value for \(G(\cdot, t_0)\), we consider
\[ G\left(\theta_0 - \xi \pm \frac{\pi}{2}, t_0\right) = F\left(\theta_0 - \xi \pm \frac{\pi}{2}, t_0\right) - B \cos \left(\pm \frac{\pi}{2}\right) \]
\[ = F\left(\theta_0 - \xi \pm \frac{\pi}{2}, t_0\right). \]
Since \(F\) is initially positive by convexity and the assumption that \(\sigma_i \geq 0\), the maximum principle implies that \(F\) remains positive. Therefore \(G(\theta_0 - \xi \pm \frac{\pi}{2}, t_0) > 0\). As we have deduced from
the Sturmian theorem that \((\theta_0, t_0)\) is the only zero for \(G(\cdot, t_0)\), \(G(\cdot, t_0)\) is non-negative and so \(G(\theta_0, t_0)\) is a strict local minimum value for \(G(\cdot, t_0)\), as required. Therefore \(G(\theta_0, t_0) = (F(\theta_0) + F)(\theta_0, t_0)\) is non-negative. We have proved that

\[
(F(\theta_0) + F)\theta > M \implies (F(\theta_0) + F)(\theta_0, t_0) \geq 0
\]

which implies \(F_\theta(\theta_0, t_0) \geq 0\), as required. \(\square\)

We are now ready to prove Proposition A.5

**Proof of Proposition A.5** Let us fix a time \(t_0 > 0\) and let \(M^2\) be as in (34). Since estimate (35) is trivial for \(t = 0\), we may take

\[
t_0 = \sup\{t \in [0, T) : (35) \text{ holds}\}.
\]

Since \(\sigma_1, \sigma_2\) and \(k\) are non-negative, \(F\) may not vanish completely at any given time. Therefore by continuity of the flow \(t_0 > 0\).

Suppose that \(t_0 < T\). Let \(t_1 = t_0 + \epsilon\) where \(\epsilon \in (0, T - t_0)\) is a small parameter. If \((F(\theta_0) + F)(\theta_1, t_1) \leq M^2\), then obviously \(\sqrt{F(\theta_1, t_1)} \leq M\) and \(\max(t_1)\) is also bounded by \(M\). This is (stronger than) the estimate in (35). This contradicts the definition of \(t_0\).

Therefore it must be the case that \((F(\theta_0) + F)(\theta_1, t_1) > M^2\) for some \(\theta \in [0, 2\pi]\). This implies that there exists a first \(\theta_1 \in [0, 2\pi]\) such that \((F(\theta_1, t_1) = M^2\) and \((F(\theta_1, t_1) < M^2\) for \(\theta \in [0, \theta_1]\). There exists a maximal interval \((\theta_1, \theta_2)\) such that \((F(\theta_1, t_1) > M^2\) for \(\theta \in (\theta_1, \theta_2)\). Then by Lemma A.5

\[
F_\theta(\theta, t_1) \geq 0 \quad \text{for all } \theta \in (\theta_1, \theta_2).
\]

Note that by continuity of \(F_\theta\), for \(\epsilon\) sufficiently small it can not be the case that \(\theta_1 = 0\) and \(\theta_2 = 2\pi\). Indeed, we have that \(|F_\theta(\theta, t_0)| \leq M + J_0^{2\pi} |F(\theta, t_0)| d\theta\) with equality achieved at one or more isolated points.

Note that \(F_\theta\) may not achieve equality in this estimate on any open interval, otherwise by analyticity of the flow it is constant on \([0, 2\pi]\) and then \(F_\theta(\theta, t_0) = 0\), which means \(k\) is constant. As the flow is closed, \(k\) must be itself a constant, so that \(\gamma(\cdot, t_0)\) is a circle. But then \(F_\theta = 0\), so \(F_\theta\) did not achieve equality after all. In fact, \(\theta_2(\epsilon) \searrow \theta_1(\epsilon)\) as \(\epsilon \searrow 0\), so clearly we may assume that \(\theta_2 - \theta_1 < 2\pi\) by taking \(\epsilon\) sufficiently small.

This means that \((F(\theta_1, t_1) = M^2\), in particular that \((F(\theta_2, t_1) \leq M^2\). Using Lemma A.5 we find

\[
\int_0^{\theta_2} (F(\theta, t_1) d\theta \geq 0,
\]

where \(\theta \in (\theta_1, \theta_2)\). Rearranging yields

\[
|F_\theta(\theta, t_1)| \leq |F_\theta(\theta_2, t_1) + \int_\theta^{\theta_2} F(\theta, t_1) d\theta|
\]

\[
\leq M + \int_0^{2\pi} |F(\theta, t_1)| d\theta.
\]

Now by assumption we have \(F(\theta, t_1) \leq M^2\) for all \(\theta \in [0, \theta_1]\). Combining this with the above we have

\[
\sup_{\theta \in [0, \theta_2]} |F_\theta(\theta, t_1)| \leq M + \int_0^{\theta_2} |F(\theta, t_1)| d\theta.
\]
Finally, the above argument holds on all maximal intervals \((\theta_1, \theta_2)\) where the estimate \(F^2 + F_\theta^2 \leq M^2\) fails. Arguing as above for each of these intervals, we obtain the estimate

\[
\sup_{\theta \in [0,2\pi]} |F_\theta(t_1)| \leq M + \int_0^{2\pi} |F(\theta, t_1)| \, d\theta.
\]

This implies that (35) holds for \(t = t_1 > t_0\), which is a contradiction. Therefore \(t_0 = T\).

Now we show the remaining estimates in Proposition A.3. By fundamental theorem of calculus,

\[
F_{\max}(t) \leq \frac{1}{2\pi} \left| \int_0^{2\pi} F(\theta, t) \, d\theta \right| + \int_0^{2\pi} |F_\theta(\theta, t)| \, d\theta
\]

\[
\leq \frac{1}{2\pi} \int_0^{2\pi} |F(\theta, t)| \, d\theta + 2\pi \left( M + \int_0^{2\pi} |F(\theta, t)| \, d\theta \right)
\]

\[
\leq M_1 \left( 1 + \int_0^{2\pi} |F(\theta, t)| \, d\theta \right)
\]

where \(M_1 = \max\{2\pi M, 2\pi + (2\pi)^{-1}\}\).

Let \(\theta^*\) be such that \(F_{\max}(t) = F(\theta^*, t)\). Let \(\theta \in (0, 2\pi)\) be such that \(|\theta - \theta^*| \leq \frac{1}{4\pi}\). The fundamental theorem of calculus implies

\[
F_{\max}(t) = F(\theta, t) + \int_{\theta}^{\theta^*} F_\theta(\theta, t) \, d\theta
\]

\[
\leq F(\theta, t) + |\theta^* - \theta| \sup_{\theta \in [0,2\pi]} |F_\theta(\theta, t)|
\]

\[
\leq F(\theta, t) + |\theta^* - \theta| \left( M + \int_0^{2\pi} |F(\theta, t)| \, d\theta \right)
\]

\[
\leq F(\theta, t) + \frac{1}{4\pi} (M + 2\pi F_{\max}(t))
\]

Making \(F_{\max}(t)\) the subject, we obtain

\[
F_{\max}(t) \leq 2F(\theta, t) + \frac{M}{2\pi}
\]

as desired. □

**APPENDIX B. CURVATURE Estimate FOR THE CONTINUOUS RESCALING**

In this section we carry through details of the calculations from Chou-Zhu in the special case of our family of flows here, for the convenience of the reader. Briefly, once we have a curvature bound in hand, we will obtain convergence of the rescaled flow to a limit (using the same argument as Huisken [21]), and then this limit must satisfy \(Q = 0\) by Theorem 4.3.

In order to obtain pointwise information on the blowup of the curvature, we use the assumption \(\sigma_1 \geq 1\) together with the following argument.

Our goal is to prove an a-priori estimate for the rescaled curvature \(\hat{k}\). Although the above lemma gives useful information on the blowup rate for the \(L^2\)-norm of curvature, it does not give any information at all on the pointwise blowup of \(k\). In the next lemma, we attack this problem, first showing that its blowup rate is at worst subexponential in rescaled time.
Lemma B.1. Let \( \gamma : S^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow \( (12) \) with convex initial data. The maximum of rescaled curvature \( \hat{k}_{\text{max}}(\hat{t}) \) satisfies
\[
\lim_{\hat{t} \to \infty} e^{-\hat{t}} \hat{k}_{\text{max}}(\hat{t}) = 0.
\]

Proof. Note first that
\[
t_\epsilon = 2T e^{-2\hat{t}} = 2T - 2T(1 - e^{-2\hat{t}}) = (2T - 2t) = \phi^{-2}.
\]

We now differentiate length \( L(t(\hat{t})) \) with respect to the normalised time variable \( \hat{t} \). Using Proposition 2.3 and the above, we compute
\[
L_i = L_i t_\epsilon = \left( -\sigma_1 \int_\gamma k^2 ds - 2\pi \sigma_2 \right) e^{-2}\phi^2
\]
\[
= -2T e^{-2\hat{t}} \int_{S^1} k d\theta - 4T e^{-2\pi} \sigma_2.
\]

By Theorem 3.15, the length vanishes at final time \( t = T, \) and we have \( \lim_{\hat{t} \to \infty} L(t(\hat{t})) = 0. \) Integrating \( (41) \) with respect to \( \hat{t} \) yields
\[
-L(0) = \int_0^\infty L_i d\hat{t} = -2T \int_0^\infty \int_0^{2\pi} \sigma_1 k e^{-2\hat{t}} d\theta d\hat{t} - 4\pi T \int_0^\infty \sigma_2 e^{-2\hat{t}} d\hat{t}.
\]
That is,
\[
\int_0^\infty \int_0^{2\pi} F e^{-2\hat{t}} d\theta d\hat{t} = \frac{L(0)}{2T}.
\]

Integrating the estimate (42), we find
\[
\int_0^\infty F_{\text{max}} e^{-2\hat{t}} d\hat{t} \leq \int_0^\infty e^{-2\hat{t}} M_1 \left( 1 + \int_0^{2\pi} |F| d\theta \right) d\hat{t}
\]
where \( M_1 \) is as in Proposition A.3. Convexity implies \( F \geq 0 \) and so, using (42), we see that
\[
\int_0^\infty F_{\text{max}} e^{-2\hat{t}} d\hat{t} \leq \frac{M_1}{2} + \frac{L(0)}{2T}.
\]

We know that the curvature \( k(t(\hat{t})) \) blows up as \( \hat{t} \to \infty \). This implies \( F(\theta, \hat{t}) = \sigma_1 k + \sigma_2 \) can not have a uniform upper bound. In particular, it is eventually strictly larger than the constant \( M^2 \) (see (44)). Call this time \( \hat{t}_* \). Lemma A.3 implies that \( F_{\hat{t}}(\theta, \hat{t}) \geq 0 \) on an open interval \( [\hat{t}_*, \hat{t}_* + \varepsilon) \). Now, as \( F \) is non-decreasing along this interval, we see that \( F \) remains strictly larger than \( M^2 \) and indeed may never decrease again. Taking \( \varepsilon \) to be maximal, we find that \( F_{\hat{t}} \geq 0 \) on \( [\hat{t}_*, \infty) \).

Since \( \sigma_1 > 0 \), this means that \( k \) is monotone increasing for all \( \hat{t} > \hat{t}_* \). In particular the maximum of \( F \) is monotone increasing. The \( L^1 \) in rescaled time bound (43), yields
\[
(F_{\text{max}} e^{-2\hat{t}}(\hat{t}_j) \to 0
\]
along a subsequence \( \{\hat{t}_j\}_{j=1}^\infty, \hat{t}_j \to \infty. \) Since \( F \) is monotone, the limit is independent of subsequence and we conclude
\[
0 = \lim_{\hat{t} \to \infty} F_{\text{max}} e^{-2\hat{t}} = \lim_{\hat{t} \to \infty} (\sigma_1 k_{\text{max}}(\hat{t}) + \sigma_2)e^{-2\hat{t}} = \lim_{\hat{t} \to \infty} \sigma_1 e^{-\hat{t} \hat{k}_{\text{max}}(\hat{t})},
\]
as required. \( \square \)
Proposition B.2. Let \( \hat{\gamma} : \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{R}^2 \) be a solution to the rescaled flow (12) with convex initial data. There exists a positive constant \( \hat{t}_0 \) such that
\[
\hat{f}(\hat{t}) := \int_0^{2\pi} u d\hat{\theta}
\]
is non-positive for all \( \hat{t} > \hat{t}_0 \).

Proof. First, we calculate
\[
\hat{k}_i = \left( \frac{k}{\phi} \right) = \left( \frac{\partial(\phi^{-1})}{\partial \theta} k + \phi^{-1} \frac{\partial k}{\partial \theta} \right) \frac{\partial t}{\partial \theta}
= -\hat{k} + \sigma_1 \hat{k}^2 \left( \hat{k}_{\hat{\theta}} + \hat{k} \right) + \sqrt{2T} e^{-i} \sigma_2 \hat{k}^2.
\]

Using (45), we compute the evolution of the entropy
\[
\hat{E}(\hat{t}) := E(\hat{\gamma}(\hat{\theta}, \hat{t})) = \frac{1}{2\pi} \int_0^{2\pi} \log \hat{k} d\hat{\theta}
\]
in terms of \( \hat{f} \) and an error term:
\[
2\pi \hat{E}_t = \int_0^{2\pi} \frac{\hat{k}_i d\hat{\theta}}{\hat{k}} = \int_0^{2\pi} -1 + \sigma_1 \hat{k} \left( \hat{k}_{\hat{\theta}} + \hat{k} \right) + \sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}
= \int_0^{2\pi} -1 + \sigma_1 \hat{k} \left( \hat{k}_{\hat{\theta}} + \hat{k} \right) - 2\sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta} + \int_0^{2\pi} 3\sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}
= \int_0^{2\pi} u d\hat{\theta} + \int_0^{2\pi} 3\sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}
\]
(47)
\[
= f + \int_0^{2\pi} 3\sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}.
\]

We shall compute the evolution of \( \hat{f}(\hat{t}) \). We first note the following equality, that follows easily from integration by parts:
\[
f(\hat{t}) = \int_0^{2\pi} -1 - \sigma_1 \hat{k}_i^2 - \sigma_1 \hat{k}^2 - 2\sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}.
\]

Recall that the angle parameter \( \hat{\theta} \) is chosen to commute with the time parameter \( \hat{t} \). We use this and integration by parts to compute the evolution of \( \hat{f}(\hat{t}) \):
\[
f'(\hat{t}) = \int_0^{2\pi} -2\sigma_1 \hat{k} \hat{k}_{\hat{\theta}} + 2\sigma_1 \hat{k} \hat{k}_i + 2\sqrt{2T} e^{-i} \sigma_2 \hat{k} - 2\sqrt{2T} e^{-i} \sigma_2 \hat{k}_i d\hat{\theta}
= 2 \int_0^{2\pi} \left[ \sigma_1 \left( \hat{k}_{\hat{\theta}} + \hat{k} \right) - \sqrt{2T} e^{-i} \sigma_2 \right] \hat{k}_i + \sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}.
\]
Expanding out the expression of \( \hat{k}_i \) using (45), we find
\[
f'(\hat{t}) = 2 \int_0^{2\pi} \left[ \sigma_1 \left( \hat{k}_{\hat{\theta}} + \hat{k} \right) - \sqrt{2T} e^{-i} \sigma_2 \right] \hat{k}_i + \sqrt{2T} e^{-i} \sigma_2 \hat{k} d\hat{\theta}.
\]
Comparing the integrand in (48) with
\[
\left[ -\hat{k} + \sigma_1 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right) + \sqrt{2T} e^{-i} \sigma_2 \hat{k}^2 \right] \ d\theta
\]
\[+ 2 \int_0^{2\pi} \sqrt{2T} e^{-i} \sigma_2 \hat{k} \ d\theta \]
\[= 2 \int_0^{2\pi} -\sigma_1 \hat{k} \left( \hat{k}_{\theta\theta} + \hat{k} \right) + \sigma_1 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right)^2 \]
\[+ 2\sqrt{2T} e^{-i} \sigma_2 \hat{k} - 2T e^{-2i} \sigma_2 \hat{k}^2 . \]

We wish to rewrite the above in terms of \( u \) and \( u^2 \). We note that
\[
u^2 = 1 + \sigma_1^2 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right)^2 + 8T e^{-2i} \sigma_2 \hat{k}^2
\]
\[-2\sigma_1 \hat{k} \left( \hat{k}_{\theta\theta} + \hat{k} \right) + 4\sqrt{2T} e^{-i} \sigma_2 \hat{k} - 4\sqrt{2T} e^{-i} \sigma_2 \sigma_1 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right) . \]

Comparing the integrand in (48) with \( u \) and \( u^2 \), we obtain
\[
f'(i) = 2 \int_0^{2\pi} u^2 \ d\theta + 2 \int_0^{2\pi} -1 - 2\sqrt{2T} e^{-i} \sigma_2 \hat{k} + \sigma_1 \hat{k} \left( \hat{k}_{\theta\theta} + \hat{k} \right)
\]
\[-10T e^{-2i} \sigma_2 \hat{k}^2 + 4\sqrt{2T} e^{-i} \sigma_2 \sigma_1 \hat{k} \left( \hat{k}_{\theta\theta} + \hat{k} \right) \ d\theta
\]
\[= 2 \int_0^{2\pi} u^2 \ d\theta + 2 \int_0^{2\pi} u \ d\theta
\]
\[+ 2 \int_0^{2\pi} -10T e^{-2i} \sigma_2 \hat{k}^2 + 4\sqrt{2T} e^{-i} \sigma_2 \sigma_1 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right) \ d\theta . \]

We wish to absorb the last term of (49) into the other terms. First, we calculate
\[
2 \int_0^{2\pi} 4\sqrt{2T} e^{-i} \sigma_2 \sigma_1 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right) \ d\theta
\]
\[= \int_0^{2\pi} 8\sqrt{2T} \sigma_2 e^{-i} \left( \hat{k}_u + \hat{k} + 2\sqrt{2T} \sigma_2 e^{-i} \hat{k}^2 \right) \ d\theta
\]
\[= \int_0^{2\pi} 8\sqrt{2T} \sigma_2 e^{-i} \hat{k}_u \ d\theta + \int_0^{2\pi} 8\sqrt{2T} \sigma_2 e^{-i} \hat{k} \ d\theta
\]
\[+ \int_0^{2\pi} 32T \sigma_2^2 e^{-2i} \hat{k}^2 \ d\theta . \]

We then apply the inequality \( ab \geq -\frac{1}{4} a^2 - b^2 \) to the RHS above, yielding
\[
2 \int_0^{2\pi} 4\sqrt{2T} e^{-i} \sigma_2 \sigma_1 \hat{k}^2 \left( \hat{k}_{\theta\theta} + \hat{k} \right) \ d\theta
\]
\[\geq - \int_0^{2\pi} \frac{1}{4} 64(2T) \sigma_2^2 e^{-2i} \hat{k}^2 \ d\theta - \int_0^{2\pi} u^2 \ d\theta + \int_0^{2\pi} 8\sqrt{2T} \sigma_2 e^{-i} \hat{k} \ d\theta
\]
\[+ \int_0^{2\pi} 32T \sigma_2^2 e^{-2i} \hat{k}^2 \ d\theta
\]
\[= - \int_0^{2\pi} u^2 \ d\theta + \int_0^{2\pi} 8\sqrt{2T} \sigma_2 e^{-i} \hat{k} \ d\theta . \]
Combining with (49), we have

\[ f'(\hat{t}) \geq \int_0^{2\pi} u^2 \, d\theta + 2 \int_0^{2\pi} u \, d\theta - 20T\sigma^2 \int_0^{2\pi} e^{-2i\hat{t}^2} \, d\theta + 2\sqrt{2T\sigma^2} \int_0^{2\pi} 4e^{-i\hat{k}} \, d\theta. \]  

(50)

It follows from Lemma B.1 that for every \( \delta > 0 \), there exists a \( \hat{t}_\delta \) such that for every \( \hat{t} > \hat{t}_\delta \),

\[ e^{-i\hat{t}_{\max}(\hat{t})} \leq \delta. \]  

(51)

Let us briefly analyse the latter two integrals on the RHS of (50). For \( \hat{t} > \hat{t}_\delta \), we have

\[ -20T\sigma^2 \int_0^{2\pi} e^{-2i\hat{t}^2} \, d\theta + 2\sqrt{2T\sigma^2} \int_0^{2\pi} 4e^{-i\hat{k}} \, d\theta = 2\sqrt{T\sigma^2} \int_0^{2\pi} e^{-i\hat{k}(\sqrt{2}(4) - 10\sqrt{T\sigma^2})} \, d\theta \geq 0 \]

if

\[ \sqrt{2}(4) - 10\sqrt{T\sigma^2} \leq 0, \]

which is satisfied so long as \( e^{-i\hat{k}} \leq \frac{\sqrt{2}(4)}{10\sqrt{T\sigma^2}} \). Therefore, if we choose \( \delta = \frac{\sqrt{2}(4)}{10\sqrt{T\sigma^2}} \), by (51) we have for \( \hat{t} > \hat{t}_\delta \)

\[ f'(\hat{t}) \geq \int_0^{2\pi} u^2 \, d\theta + 2 \int_0^{2\pi} u \, d\theta. \]  

(52)

Using Hölder's inequality, we conclude that for all \( \hat{t} > \hat{t}_\delta \) (with \( \delta \) chosen as above),

\[ f'(\hat{t}) \geq \int_0^{2\pi} u^2 \, d\theta + 2 \int_0^{2\pi} u \, d\theta \geq \frac{1}{2\pi} \left( \int_0^{2\pi} u \, d\theta \right)^2 + 2 \int_0^{2\pi} u \, d\theta \geq \frac{1}{2\pi} f^2(\hat{t}) + 2f(\hat{t}). \]  

(53)

We are now ready to finish the proof. Our proof is by contradiction. Fix a positive constant \( \delta \). We denote \( \hat{t}_\delta \) to be the value such that (53) is valid for all \( \hat{t} > \hat{t}_\delta \). Let \( c > 0 \) be arbitrary. Suppose that there exists \( \hat{t}_1 > \hat{t}_\delta \) such that \( f(\hat{t}_1) \geq c > 0 \). Thus, we have from (53) that

\[ f'(\hat{t}_1) \geq \frac{1}{2\pi} f^2(\hat{t}_1) + 2f(\hat{t}_1) > 0. \]  

(54)

This implies

\[ f(\hat{t}) > c > 0, \text{ for all } \hat{t} > \hat{t}_1. \]

Hence the inequality (54) holds for all \( \hat{t} > \hat{t}_1 \). We can rearrange (54) and integrate over time to see

\[ \int_{\hat{t}_1}^i \frac{f'(\hat{t})}{f^2(\hat{t})} \, d\hat{t} \geq \int_{\hat{t}_1}^i \frac{1}{2\pi} \, d\hat{t}, \]

or

\[ -\frac{1}{f(t)} \geq \frac{1}{2\pi} (\hat{t} - \hat{t}_1) - \frac{1}{f(\hat{t}_1)}. \]
The positivity of \( f(\hat{t}) \) implies

\[
f(\hat{t}) \geq \frac{f(\hat{t}_1)}{1 - f(\hat{t}_1)\frac{1}{2\pi}(\hat{t} - \hat{t}_1)}.
\]

Let us take the sequence \( \{\hat{t}_i\} \) such that \( \hat{t}_i \to \hat{t}^* \), where \( \hat{t}^* = \hat{t}_1 + \frac{1}{f(\hat{t}_1)\frac{1}{2\pi}} \) is a finite number. It is clear that \( f(\hat{t}_i) \) blows up as \( \hat{t}_i \) tends to \( \hat{t}^* \). But, \( \hat{t}^* \) is a finite time and as the flow is smooth, the quantity \( f(\hat{t}^*) \) is bounded. This is a contradiction.

Therefore, for every \( c > 0 \),

\[
f(\hat{t}) < c \quad \text{for all} \quad \hat{t} > \hat{t}_\delta.
\]

But this can only be the case if \( f(\hat{t}) \leq 0 \) for all \( \hat{t} > \hat{t}_\delta \). This finishes the proof. \( \square \)

Next we show that the rescaled entropy \( \hat{E}(\hat{t}) \) is uniformly bounded for all \( \hat{t} \in [0, \infty) \).

**Proposition B.3.** Let \( \gamma : S^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow (12) with convex initial data. Then for all \( \hat{t} \in [0, \infty) \) the rescaled entropy satisfies

\[
\hat{E}(\hat{t}) \leq C,
\]

where

\[
C = \max \left\{ \sup_{\hat{t} \in [0, \hat{t}_0]} \{ \hat{E}(\hat{t}) \}, \hat{E}(\hat{t}_0) + \frac{3\sigma_2 L(0)}{2\pi\sigma_1} - \frac{3T\sigma_2^2}{\sigma_1} \right\}.
\]

Here \( \hat{t}_0 \) is as in the statement of Proposition B.2.

**Proof.** The statement is immediate for \( \hat{t} \leq \hat{t}_0 \) by definition of \( C \). Note that the original flow remains uniformly strictly convex, and the curvature uniformly bounded on any compact subinterval of \([0, T]\), and so the supremum \( \sup_{\hat{t} \in [0, \hat{t}_0]} \{ \hat{E}(\hat{t}) \} \) is finite.

It remains to deal with the case of \( \hat{t} > \hat{t}_0 \). Here, we use Proposition B.2 to estimate the evolution of the entropy (47) by

\[
2\pi\hat{E}_\hat{t} \leq \int_0^{2\pi} 3\sqrt{2T} e^{-\hat{t}_k} \sigma_2 \hat{k} d\hat{\theta}.
\]

Note that

\[
\hat{k} e^{-\hat{t}} = \sqrt{2T} e^{-2\hat{t}_k} \hat{k} = \sqrt{2T} e^{-2\hat{t}} (\sigma_1^{-1} F - \sigma_1^{-1} \sigma_2)
\]

Integrating (56) and using the estimate (12), we obtain

\[
\hat{E}(\hat{t}) - E(\hat{t}_0) \leq \frac{3T\sigma_2}{\sigma_1 \pi} \int_0^\hat{t} \int_0^{2\pi} F e^{-2\hat{t}} d\hat{\theta} d\hat{t} - \frac{3T\sigma_2^2}{\sigma_1 \pi} \int_0^\hat{t} \int_0^{2\pi} e^{-2\hat{t}} d\hat{\theta} d\hat{t}
\]

\[
\leq \frac{3\sigma_2 L(0)}{2\pi\sigma_1} - \frac{3T\sigma_2^2}{\sigma_1}.
\]

The proof is completed. \( \square \)

The entropy bound gives many things, including a uniform estimate on rescaled length.

**Lemma B.4.** Let \( \gamma : S^1 \times [0, \infty) \to \mathbb{R}^2 \) be a solution to the rescaled flow (12) with convex initial data. There exists an \( \hat{L}_0 \in [0, \infty) \) depending only on the constant \( C \) in Proposition B.3 and initial area of the rescaled flow such that

\[
\hat{L}(\hat{t}) \leq \hat{L}_0
\]

for all \( \hat{t} \in [0, \infty) \).
Proof. Note that we have a uniform bound on the entropy by Proposition \[A.3\] and so Lemma \[A.2\] provides a uniform positive lower bound on the inradius of the rescaled flow, depending only on the constants in the entropy bound. As the area of the rescaled curves approach a constant (Lemma \[A.1\]), the maximum diameter of the family of curves must be bounded from above by a constant depending only on initial area and \(\sigma_1\). Since the curve \(\gamma\) (before rescaling) is uniformly convex up until the final time (Theorem \[3.6\]), the rescaled curve \(\hat{\gamma}\) must also be convex. As a result the rescaled length is bounded by \(\pi\) times the diameter of \(\hat{\gamma}\), and the lemma follows. \(\square\)

We finally prove a uniform estimate for rescaled curvature.

**Theorem B.5.** Let \(\hat{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}^2\) be a solution to the rescaled flow \([12]\) with convex initial data. There exists a \(\hat{k}_1 \in [0, \infty)\) such that

\[
\hat{k}_{\max}(\hat{t}) \leq \hat{k}_1
\]

for all \(\hat{t} \in [0, \infty)\).

**Proof.** We prove this by contradiction. Recall the speed estimate \([37]\) in Proposition \[A.3\]. For a fixed time \(t\) and a small neighbourhood \(|\theta - \hat{\theta}^*| \leq 1/4\pi\) around where the spatial maxima of curvature occurs, \(\hat{\theta}^*\) at \(t\), we have

\[
\sigma_1 k_{\max}(t) < F_{\max}(t) \leq 2(\sigma_1 k(\theta, t) + \sigma_2) + \frac{M}{2\pi}.
\]

Apply the rescaling to obtain

\[
\sigma_1 \hat{k}_{\max}(\hat{t}) \leq 2\sigma_1 \hat{k}(\hat{\theta}, \hat{t}) + \left(2\sigma_2 + \frac{M}{2\pi}\right) \sqrt{2T} e^{-\hat{t}},
\]

and rearrange to see

\[
\hat{k}(\hat{\theta}, \hat{t}) \geq \frac{1}{2} \left(\hat{k}_{\max} - C_1 e^{-\hat{t}}\right),
\]

where \(C_1 = \frac{1}{\sigma_1} (2\sigma_2 + \frac{M}{2\pi}) \sqrt{2T}\). Thus

\[
\log \hat{k}(\hat{\theta}, \hat{t}) \geq \log \left(\hat{k}_{\max}(\hat{t}) - C_1 e^{-\hat{t}}\right) - \log(2).
\]

Integrating yields

\[
\int_{|\hat{\theta} - \hat{\theta}^*| \leq \frac{1}{4\pi}} \log \hat{k}(\hat{\theta}, \hat{t}) \, d\hat{\theta} \geq \int_{|\hat{\theta} - \hat{\theta}^*| \leq \frac{1}{4\pi}} \log \left(\hat{k}_{\max}(\hat{t}) - C_1 e^{-\hat{t}}\right) - \log(2) \, d\hat{\theta}.
\]

Now suppose that \(\hat{k}\) is unbounded. The only way this can happen is asymptotically at infinity. Thus there exists a sequence of times \(\{\hat{t}_j\}\) with \(\hat{t}_j \to \infty\), such that \(\hat{k}_{\max}(\hat{t}_j) \to \infty\). We may assume without loss of generality that \(\hat{t}_1 > \hat{t}_0\) (where \(\hat{t}_0\) is as in Proposition \[B.2\]) and that \(\hat{t}_{j+1} > \hat{t}_j\).

For each \(\hat{t}_j > \hat{t}_0\), the entropy estimate \([57]\) implies

\[
E(\hat{t}_0) + \frac{3\sigma_2 L(0)}{2\pi \sigma_1} - \frac{3T \sigma_1^2}{\sigma_1} \geq E(\hat{t}_j) = \frac{1}{2\pi} \int_{S^1} \log \hat{k}(\hat{\theta}, \hat{t}_j) \, d\hat{\theta}.
\]

Let us fix \(\hat{t} = \hat{t}_j\), and partition of the space domain in the following way:

\[S^1 = \left\{ \hat{\theta} - \hat{\theta}^* \leq \frac{1}{4\pi} \right\} \cup \left\{ \hat{k} < 1 \right\} \cup \left\{ \hat{k} \geq 1 \right\} \cup \left\{ \hat{k} = 1 \right\} \cup \left\{ \hat{\theta} - \hat{\theta}^* \leq \frac{1}{4\pi} \right\} \cup \left\{ \hat{\theta} - \hat{\theta}^* \leq \frac{1}{4\pi} \right\}.
\]
We estimate $E(\hat{t}_j)$ on each of these disjoint sets separately. The estimate on the first set is given by (58). On the second set, we note that $0 > k \log k \geq -e^{-1}$ for $k \in (0, 1)$, hence

\begin{align*}
\int_{\{k<1\} \setminus \{||\hat{\theta} - \hat{\theta}^*| \leq \frac{1}{4\pi}\}} \log \hat{k}(\hat{\theta}, \hat{t}_j) \, d\hat{\theta} &\geq \int_{\{k<1\} \setminus \{||\hat{\theta} - \hat{\theta}^*| \leq \frac{1}{4\pi}\}} \hat{k}(\hat{s}, \hat{t}_j) \log \hat{k}(\hat{s}, \hat{t}_j) \, d\hat{s} \\
&\geq \int_{\{k<1\} \setminus \{||\hat{\theta} - \hat{\theta}^*| \leq \frac{1}{4\pi}\}} -\frac{1}{e} \, d\hat{s} \\
&\geq -\frac{1}{e} \hat{L}_0
\end{align*}

(60)

where $\hat{L}_0$ is the uniform bound on rescaled length from Lemma B.4.

Let us now consider the third set. We note that on this set $\log \hat{k} \geq 0$, and so trivially

\begin{align*}
\int_{\{k\geq1\} \setminus \{||\hat{\theta} - \hat{\theta}^*| \leq \frac{1}{4\pi}\}} \log \hat{k}(\hat{\theta}, \hat{t}_j) \, d\hat{\theta} &\geq 0.
\end{align*}

Combining (58), (59), (60) and the above, we obtain

\begin{align*}
E(\hat{t}_0) + \frac{3\sigma_2 L(0)}{2\pi \sigma_1} - \frac{3T \sigma_2^2}{\sigma_1} \geq \frac{1}{8\pi^2} \left[ \log \left( \hat{k}_{\text{max}}(\hat{t}_j) - C_1 e^{-\hat{t}_j} \right) - \log(2) \right] - \frac{1}{2e\pi} \hat{L}_0.
\end{align*}

While the left hand side is a finite constant, the right hand side tends to infinity as $\hat{t}_j \to \infty$, which is a contradiction. We deduce that the normalised curvature $\hat{k}(\hat{\theta}, \hat{t})$ must be uniformly bounded from above.

\[\Box\]

ACKNOWLEDGEMENTS

The first author is supported by an IPRS Scholarship at University of Wollongong. The last two authors gratefully acknowledge the support of ARC grant DP150100375.

REFERENCES

[1] Uwe Abresch, Joel Langer, et al. The normalized curve shortening flow and homothetic solutions. Journal of Differential Geometry, 23(2):175–196, 1986.
[2] L. Almeida, P. Bagnenerini, A. Habbal, S. Noselli, and F. Serman. Tissue repair modeling. In Singularities in nonlinear evolution phenomena and applications, volume 9, pages 27–46. Edizioni Della Normale, 2008.
[3] Yann Bernard, Glen Wheeler, and Valentina-Mira Wheeler. Concentration-compactness and finite-time singularities for chen’s flow. arXiv preprint arXiv:1706.01707, 2017.
[4] Patrick Breuning. Immersions with bounded second fundamental form. The Journal of Geometric Analysis, 25(2):1344–1386, 2015.
[5] K. S. Chou and X.-P. Zhu. The curve shortening problem, CRC Press, 2001.
[6] Kai-Seng Chou and Xi-Ping Zhu. A convexity theorem for a class of anisotropic flows of plane curves. Indiana University mathematics journal, pages 139–154, 1999.
[7] Kai-Seng Chou, Xi-Ping Zhu, et al. Anisotropic flows for convex plane curves. Duke mathematical journal, 97(3):579–619, 1999.
[8] A.S. Colwell, M.T. Longaker, and H.P. Lorenz. Fetal wound healing. Frontiers in bioscience: a journal and virtual library, 8:1240–8, 2003.
[9] P.D. Dale, J.A. Sherratt, and P.K. Maini. A mathematical model for collagen fibre formation during foetal and adult dermal wound healing. Proceedings of the Royal Society of London B: Biological Sciences, 263(1370):653–660, 1996.
[10] Michael C Dallaston and Scott W McCue. A curve shortening flow rule for closed embedded plane curves with a prescribed rate of change in enclosed area. Proceedings of the Royal Society A, 472(2185):20150629, 2016.
[11] Panagiota Daskalopoulos, Richard Hamilton, Natasa Sesum, et al. Classification of compact ancient solutions to the curve shortening flow. *Journal of Differential Geometry*, 84(3):455–464, 2010.

[12] Klaus Deckelnick. Weak solutions of the curve shortening flow. *Calculus of Variations and Partial Differential Equations*, 5(6):489–510, 1997.

[13] M. Gage and R.S. Hamilton. The heat equation shrinking convex plane curves. *J. Differential Geom.*, 23(1):69–96, 1986.

[14] Michael E Gage. Curve shortening makes convex curves circular. *Inventiones mathematicae*, 76(2):357–364, 1984.

[15] Victor A Galaktionov. Geometric Sturmian theory of nonlinear parabolic equations and applications. CRC Press, 2004.

[16] M. Grayson. The heat equation shrinks embedded plane curves to round points. *J. Differential Geom.*, 26:285–314, 1987.

[17] G. C. Gurtner, M. J. Callaghan, and M. T. Longaker. Progress and potential for regenerative medicine. *Annu. Rev. Med.*, 58:299–312, 2007.

[18] G.C. Gurtner, S. Werner, Y. Barrandon, and M.T. Longaker. Wound repair and regeneration. *Nature*, 453(7193):314–321, 2008.

[19] S. He. *Curvature flows in wound healing*. PhD thesis, University of Wollongong, Australia, 2018.

[20] G. Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[21] G. Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[22] E. Kuwert and R. Schatzle. The Willmore flow with small initial energy. *J. Differential Geom.*, 57(3):409–441, 2001.

[23] Ernst Kuwert and Reiner Schätzle. Gradient flow for the Willmore functional. *Communications in Analysis and Geometry*, 10(2):307–340, 2002.

[24] James McCoy, Scott Parkins, and Glen Wheeler. The geometric triharmonic heat flow of immersed surfaces near spheres. *Nonlinear Analysis*, 161:44–86, 2017.

[25] James McCoy and Glen Wheeler. Finite time singularities for the locally constrained Willmore flow of surfaces. *arXiv preprint arXiv:1201.4541*, 2012.

[26] James McCoy, Glen Wheeler, and Graham Williams. Lifespan theorem for constrained surface diffusion flows. *Mathematische Zeitschrift*, 269(1):147–178, 2011.

[27] James McCoy, Glen Wheeler, and Yuhan Wu. A sixth order flow of plane curves with boundary conditions. *To appear in MATRIX Annals*, 2017.

[28] S. McDougall, J. Dallon, J. Sherratt, and P. Maini. Fibroblast migration and collagen deposition during dermal wound healing: mathematical modelling and clinical implications. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 364(1843):1385–1405, 2006.

[29] L. Olsen, J.A. Sherratt, P.K. Maini, and F. Arnold. A mathematical model for the capillary endothelial cell-extracellular matrix interactions in wound-healing angiogenesis. *Mathematical Medicine and Biology: A Journal of the IMA*, 14(4):261–281, 1997.

[30] Scott Parkins and Glen Wheeler. The polyharmonic heat flow of closed plane curves. *Journal of Mathematical Analysis and Applications*, 439(2):608–633, 2016.

[31] Scott Parkins and Glen Wheeler. The anisotropic polyharmonic curve flow for closed plane curves. *arXiv preprint arXiv:1706.02045*, 2017.

[32] Andrea Ravasio, Ibrahim Cheddadi, Tianchi Chen, Telmo Pereira, Hui Ting Ong, Cristina Bertocchi, Agusti Brugues, Antonio Jacinto, Alexandre J Kabla, Yusuke Toyama, et al. Gap geometry dictates epithelial closure efficiency. *Nature communications*, 6, 2015.

[33] Joseph L Shomberg. Exponential decay results for semilinear parabolic PDE with $C^0$ potentials: A “mean value” approach. *Differential Equations and Dynamical Systems*, pages 1–16, 2014.

[34] Michael Struwe. On the evolution of harmonic mappings of riemannian surfaces. *Commentarii Mathematici Helvetici*, 60(1):558–581, 1985.

[35] Glen Wheeler. Lifespan theorem for simple constrained surface diffusion flows. *Journal of Mathematical Analysis and Applications*, 375(2):685–698, 2011.

[36] Glen Wheeler. Surface diffusion flow near spheres. *Calculus of Variations and Partial Differential Equations*, 44(1):131–151, 2012.

[37] Glen Wheeler. On the curve diffusion flow of closed plane curves. *Annali di Matematica Pura ed Applicata*, 192(5):931–950, 2013.

[38] Glen Wheeler. Global analysis of the generalised Helfrich flow of closed curves immersed in $\mathbb{R}^n$. *Transactions of the American Mathematical Society*, 367(4):2263–2300, 2015.
[39] Glen Wheeler and Valentina-Mira Wheeler. Curve diffusion and straightening flows on parallel lines. *arXiv preprint arXiv:1703.10711*, 2017.

Shuhui He, Institute for Mathematics and its Applications, University of Wollongong, Northfields Avenue, Wollongong, NSW, 2522, Australia, email: sh807@uowmail.edu.au

Glen Wheeler, Institute for Mathematics and its Applications, University of Wollongong, Northfields Avenue, Wollongong, NSW, 2522, Australia, email: glenw@uow.edu.au

Valentina-Mira Wheeler, Institute for Mathematics and its Applications, University of Wollongong, Northfields Avenue, Wollongong, NSW, 2522, Australia, email: vwheeler@uow.edu.au