Convex Symmetrization for Anisotropic Elliptic Equations with a lower order term
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Abstract - We use “generalized” version of total variation, coarea formulas, isoperimetric inequalities to obtain sharp estimates for solutions (and for their gradients) to anisotropic elliptic equations with a lower order term, comparing them with the solutions to the convex symmetrized ones.

Riassunto - In questa nota si usano le versioni “generalizzate” della variazione totale, delle formule di coarea e delle diseguaglianze isoperimetriche al fine di ottenere stime ottimali per le soluzioni (e per i loro gradienti) di equazioni ellittiche anisotrope con un termine di ordine inferiore, confrontandole con le soluzioni di quelle simmetrizzate convesse.

1 - INTRODUCTION

To reduce the complexity of a well defined class of problems, sometimes is possible to estimate the solutions by those of the corresponding symmetrized problem.

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In this way, we can deduce some information about the solutions to the generic problem using the solutions of the symmetrized one. So it is very significant to define an appropriate symmetrization.

By means of Schwarz (or spherical) symmetrization, it is possible to obtain comparison results for solutions to linear elliptic problems:

\[- \text{div}(A(x) \cdot \nabla u) = f \quad \text{in} \ \Omega, \quad u \in H^1_0(\Omega),\]  

(1)

where \( \nabla \) stands for the gradient operator, and \( A(x) \) is a measurable function such that

\[\langle A(x) \cdot \xi, \xi \rangle \geq |\xi|^2, \quad \forall \xi \in \mathbb{R}^n.\]  

(2)

If \( u \) is a solution to (1), then \( u^*(x) \leq v(x) \), where \( v \) solves

\[- \Delta v = f^* \quad \text{in} \ \Omega^*, \quad v \in H^1_0(\Omega^*),\]  

(3)

where \( f^* \) is the symmetrized function of \( f \) and \( \Omega^* \) is the ball centered in the origin such that \( |\Omega| = |\Omega^*| \). For example, in (Alvino et al., 1990) and in (Talenti, 1985), we find comparison results to elliptic operators of general form, that is with first and zero order terms, with different constraints on the coefficients of lower order terms. Further results can be found in (Alvino et al., 1990), (Alvino and Trombetti, 1979), (Betta and Mercaldo, 1991), (Ferone and Posteraro, 1992), for linear cases and (Betta et al., 1994), (Betta and Mercaldo, 1991) for non-linear cases. In (Alvino et al., 1997), section 4, we find a comparison result for solutions to problem

\[- \text{div}(a(x, u, \nabla u)) = f \quad \text{in} \ \Omega, \quad u \in H^1_0(\Omega)\]  

(4)

where

\[\langle a(x, \eta, \xi), \xi \rangle \geq H(\xi)^2 \quad \text{a.e.} \quad x \in \Omega, \quad \eta \in \mathbb{R}, \quad \xi \in \mathbb{R}^n,\]  

(5)

with \( H \) homogeneous convex function. The authors, using convex symmetrization, estimate a solution of (4) in terms of a function \( v \) that solves

\[- \Delta v = f^* \quad \text{in} \ \Omega^*, \quad v \in H^1_0(\Omega^*).\]  

(6)

In the present paper we consider a lower order term \( b(x, \nabla u) \) for (4), that is

\[- \text{div}(a(x, u, \nabla u)) + b(x, \nabla u) = f \quad \text{in} \ \Omega, \quad u \in H^1_0(\Omega)\]  

(7)

where \( a \) satisfies the ellipticity condition (5) and on \( b \) we assume that

\[|b(x, \xi)| \leq B(x)H(\xi)\]  

(8)
where $B(x)$ is an integrable function. Also in this case we use convex symmetrization, obtaining comparison results with solutions of the convexly symmetric problem

$$\begin{aligned}
\begin{cases}
-\text{div}(H(\nabla v)\nabla H(\nabla v)) + \tilde{b}(H_0(x))(\nabla H_0(x), \nabla H(\nabla v))H(\nabla v) = f^* \text{ in } \Omega^* \\
v \in H_0^1(\Omega^*),
\end{cases}
\end{aligned}$$

(9)

where $H_0$ is polar to $H$, $\tilde{b}$ is an auxiliary function related to $B$, $f^*$ is the convex rearrangement of $f$ with respect to $H$ and $\Omega^*$ is the set homothetic to $K_0 := \{x \in \mathbb{R}^n : H_0(x) \leq 1\}$.

We obtain the following estimates:

$$u^* \leq v$$

(10)

$$\int_\Omega H^q(\nabla u) \leq \int_{\Omega^*} H^q(\nabla v)$$

(11)

In the proof we use the generalized versions of total variation, coarea formulas and isoperimetric inequalities (see (Alvino et al., 1997), (Talenti, 1985)). We derive some differential inequalities for the rearrangement $u^*$ of the solution $u$ using Schwarz and Hardy inequalities and the properties of homogeneity and convexity of the function $H$. Finally we consider the case where $\tilde{b}$ is essentially bounded by a constant $\beta$; we can compare solutions of (7) with solutions to

$$\begin{aligned}
\begin{cases}
-\text{div}(H(\nabla v)\nabla H(\nabla v)) - \beta(\nabla H_0(x), \nabla H(\nabla v))H(\nabla v) = f^* \text{ in } \Omega^* \\
v \in H_0^1(\Omega^*)
\end{cases}
\end{aligned}$$

(12)

and we obtain the same estimates (10) and (11) of the preceding case. We refer to (Della Pietra and Gavitone) and (Della Pietra and Gavitone, 2013) for similar results under different assumptions on $b(x, \xi)$.

2 - PRELIMINARIES

2.1 - REARRANGEMENTS

Let $\Omega$ be a measurable and not negligible subset of $n$-dimensional euclidean space $\mathbb{R}^n$, let $u$ be a measurable map from $\Omega$ into $\mathbb{R}$. We define (see also (Talenti, 1976)):

the distribution function of $u$ as the map $\mu$ from $[0, \infty[ \to [0, \infty[$ such that

$$\mu(t) := |\{x \in \Omega : |u(x)| > t\}|$$

the decreasing rearrangement of $u$, denoted by $u^*$, as the map from $[0, \infty[ \to \mathbb{R}$ with
such that \( u^*(s) := \sup\{t > 0 : \mu(t) > s\} \);
the sferically decreasing rearrangement of \( u \), denoted by \( u^\# \), as the map from \([0, \infty]\) to \([0, \infty]\) such that \( u^\#(s) := \sup\{t > 0 : \mu(t) > \omega_n|x|^n\} \).
We denote by \( \Omega^\# \) the ball centered in the origin such that \( |\Omega^\#| = |\Omega| \).

2.2 - GAUGE

Let \( H : \mathbb{R}^n \to [0, \infty] \) be a \( C^1(\mathbb{R}^n \setminus \{0\}) \) convex function satisfying the homogeneity property:
\[
H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}.
\] (13)
Furthermore, assume that \( H \) satisfies
\[
\alpha|\xi| \leq H(\xi) \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^n,
\] (14)
for some positive constants \( \alpha \leq \beta \). We also assume that
\[
K = \{ x \in \mathbb{R}^n : H(x) \leq 1 \}
\] (15)
has measure \( |K| \) equal to the measure \( \omega_n \) of the unit sphere in \( \mathbb{R}^n \). Because of (13), this assumption is not restrictive. Sometimes we will say that \( H \) is the gauge of \( K \). If one defines the support function of \( K \) as:
\[
H_0(x) = \sup_{\xi \in K} \langle x, \xi \rangle.
\] (16)
Clearly \( H_0(x) \) itself is a gauge of the set:
\[
K_0 = \{ x \in \mathbb{R}^n : H_0(x) \leq 1 \},
\] (17)
we denote by \( \kappa_n \) the measure of \( K_0 \).
Let us observe that \( \nabla H_0(x) \) is, for a.e. \( x \), a vector normal to \( \partial K_0(x) \). Then the definition of \( H \) and \( H_0 \) gives (see (Rockafellar, 1970))
\[
H(\nabla H_0(x)) = \frac{\langle \nabla H_0(x), x \rangle}{H_0(x)} \quad \text{and} \quad H_0(\nabla H(x)) = \frac{\langle \nabla H(x), x \rangle}{H(x)}.
\] (18)
The homogeneity assumption (13) implies, by Euler’s Theorem, that
\[
H(\nabla H_0(x)) = 1 \quad \text{and} \quad H_0(\nabla H(x)) = 1.
\] (19)
It is useful to recall that by Euler Theorem we also have
\[
\nabla H(\nabla H_0(x)) = \frac{x}{H_0(x)}.
\] (20)
We define the \textit{(decreasing) convex rearrangement} of \(u\), denoted by \(u^*\), as the map such that \(u^*(x) = u^*(\kappa_n(H_0(x))^n)\). We denote by \(\Omega^*\) the set homothetic to \(K_0\) such that \(|\Omega^*| = |\Omega|\).

### 2.3 - Generalized Total Variation, Perimeter and Coarea Formula

It is possible to give the following definition of the total variation of a function \(u \in BV(\Omega)\) with respect to a gauge function \(H\) (see Amar and Bellettini, 1994):

\[
\int_{\Omega} |\nabla u|_H = \sup \left\{ \int_{\Omega} u \text{div} \varphi \, dx : \varphi \in C^1_0(\Omega; \mathbb{R}^n), H_0(\varphi) \leq 1 \right\}
\]

and the following "generalized" definition of perimeter of a set \(E\) with respect to \(H\):

\[
P_H(E; \Omega) = \int_{\Omega} |\nabla \chi_E|_H = \sup \left\{ \int_{\Omega} \text{div} \varphi \, dx : \varphi \in C^1_0(\Omega; \mathbb{R}^n), H_0(\varphi) \leq 1 \right\}
\]

These definitions yield to the coarea formula

\[
\int_{\Omega} |\nabla u|_H = \int_0^\infty P_H(\{u > s\}; \Omega) \, ds,
\]

and to the "generalized" isoperimetric inequality

\[
P_H(E; \mathbb{R}^n) \geq n \kappa_n^{1/n} |E|^{1-\frac{1}{n}}.
\]

We finally observe that if \(u \in W^{1,1}(\Omega)\) then

\[
\int_{\Omega} |\nabla u|_H = \int_{\Omega} H(\nabla u),
\]

and it holds

\[
- \frac{d}{dt} \int_{u > t} |\nabla u|_H \, dx = P_H(\{u > t\}; \Omega).
\]

### 2.4 - Preliminary Results

In this section we give three Lemmas, that are basic for our treatment.
Lemma 1. If \( u \) is any member of \( H^1_0(\Omega) \), then
\[
\frac{1}{n^2\kappa_n^2} \mu(t)^{\frac{n}{2} - 2} \left[ -\mu'(t) \right] \left[ -\frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right] \geq 1
\] (27)
for a.e. \( t \) such that \( 0 < t < \text{ess sup} |u| \).

Proof. For \( h > 0 \), Schwarz inequality gives
\[
\frac{1}{h} \int_{t < |u| \leq t+h} H(\nabla u) \leq \frac{1}{h} \left( \int_{t < |u| \leq t+h} dx \right)^\frac{1}{2} \left( \int_{t < |u| \leq t+h} H^2(\nabla u) \right)^\frac{1}{2} \] (28)
and
\[
\frac{1}{h} \int_{t < |u| \leq t+h} H(\nabla u) \leq \left( \frac{1}{h} (\mu(t) - \mu(t+h)) \right)^\frac{1}{2} \left( \frac{1}{h} \int_{t < |u| \leq t+h} H^2(\nabla u) \right)^\frac{1}{2} \]. (29)
Therefore, as \( h \to 0^+ \), we obtain
\[
- \frac{d}{dt} \int_{|u| > t} H(\nabla u) \leq (-\mu'(t))^{\frac{1}{2}} \left( -\frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right)^\frac{1}{2} \]. (30)
By (24) and (26), we have
\[
n\kappa_n^{1/n} \mu(t)^{1-\frac{n}{2}} \leq \left( -\mu'(t) \right)^{\frac{1}{2}} \left( -\frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right)^\frac{1}{2} . \] (31)
Then squaring and dividing by \( n^2\kappa_n^2 \mu(t)^{2-\frac{n}{2}} \), we obtain (27). \( \square \)

Lemma 2. \[
\int_E |f| \leq \int_0^{|E|} f^*(s) \, ds \] (32)
for any measurable set \( E \).

This Lemma is a special case of a theorem by Hardy and Littlewood (see Hardy et al., 1964), Theorem 378).

Lemma 3. If \( \varphi \) is bounded and
\[
\varphi(t) \leq \int_t^{+\infty} K(s)\varphi(s) \, ds + \psi(t) \] (33)
for a.e. \( t > 0 \), then
\[
\varphi(t) \leq \int_t^{+\infty} \exp \left( \int_t^s K(r) \, dr \right) (-d\psi(s))
\]
for a.e. \( t > 0 \). Here \( K \) is any nonnegative integrable function, \( \psi \) has bounded variation and vanishes at \(+\infty\).

Lemma[3] is a generalization of Gronwall’s lemma.

3 - MAIN RESULT

In this section we discuss the main result of the paper. It consists in showing that a solution to (7) can be compared in term of a solution to (9), where the function \( \tilde{b} \) is known as a pseudo rearrangement of \( B(x) \). It can be defined as
\[
\tilde{b} \left( \left( \frac{s}{\kappa_n} \right)^{\frac{1}{2}} \right) = \left( \frac{d}{ds} \int_{|u| > u^*} B^2(x) \right)^{\frac{1}{2}},
\]
(35)
We refer to (Alvino and Trombetti, 1978)and (Talenti, 1985)for further details.

**Theorem 1.** Let \( u \in H^1_0(\Omega) \) be a solution to the problem
\[
\begin{cases}
-\operatorname{div}(a(x,u,\nabla u)) + b(x,\nabla u) = f & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega
\end{cases}
\]
(36)
where \( a(x,\eta,\xi) \equiv \{a_i(x,\eta,\xi)\}_{i=1,...,n} \) are Carathéodory functions satisfying
\[
\langle a(x,\eta,\xi),\xi \rangle \geq (H(\xi))^2 \quad \text{a.e.} \quad x \in \Omega, \quad \eta \in \mathbb{R}, \quad \xi \in \mathbb{R}^n.
\]
(37)
and \( b(x,\xi) \) is such that:
\[
|b(x,\xi)| \leq B(x)H(\xi),
\]
(38)
where \( B \in L^k(\Omega) \), with \( k > n \). We assume further that \( f \in L^{\frac{2n}{n+2}}(\Omega) \) if \( n \geq 3 \);
\( f \in L^p(\Omega), \ p > 1 \), if \( n = 2 \);
\( H : \mathbb{R}^n \to [0,\infty[ \) is a convex function satisfying (13)-(14).
Then
\[
u^* \leq v
\]
(39)
\[
\int_\Omega H^q(\nabla u) \leq \int_{\Omega^*} H^q(\nabla v)
\]
(40)
with \(0 < q \leq 2\), and

\[
v(x) = \int_{H_0(x)} \left( \frac{\kappa_n}{\pi n} \right)^{1/n} \frac{1}{t^{n-1}} \exp \left( \int_t^r \tilde{b}(r')dr' \right) f^*(\kappa_n r^n) r^{n-1} dr.
\]

(41)

where \(\tilde{b}\) is defined as in (35).

**Remark 1.** The function in (41) is convexly symmetric, in the sense that \(v(x) = v^*(x)\). Indeed the function

\[
v^*(s) = \int_0^s \frac{1}{t^{n-1}} dt \int_0^t \exp \left( \int_t^r \tilde{b}(r')dr' \right) f^*(r) dr
\]

(42)

is decreasing and \(v(x) = v^*(\kappa_n(H_0(x))^n)\). We observe that \(v(x)\) is a solution in \(H_0^1(\Omega^*)\) to the problem

\[
\begin{cases}
-\text{div}(H(\nabla v) \nabla H(\nabla v)) - \tilde{b}(H_0(x)) \langle \nabla H_0(\nabla v), \nabla H(\nabla v) \rangle = f^* \text{ in } \Omega^* \\
v = 0 \text{ on } \partial \Omega^*.
\end{cases}
\]

(43)

In fact, if we define \(\rho = H_0(x)\) and we look for a solution such that \(v(\rho) = v(H_0(x))\), we obtain

\[
\begin{align*}
\nabla v &= v'(\rho) \nabla H_0(x), \\
H(\nabla v) &= -v'(\rho) H(\nabla H_0(x)) = -v'(\rho), \\
\nabla H(\nabla v) &= \nabla H(v'(\rho) \nabla H_0(x)) = \nabla H(\nabla H_0(x)) = \frac{x}{H_0(x)}.
\end{align*}
\]

(44-46)

A direct computation gives

\[
-\text{div}(H(\nabla v) \nabla H(\nabla v)) - \tilde{b}(H_0(x)) \langle \nabla H_0(\nabla v), \nabla H(\nabla v) \rangle = -v''(\rho) - \frac{n}{\rho} v'(\rho) + \tilde{b}(H_0(x)) v'(\rho).
\]

(47)

Using (41), we can write:

\[
v(\rho) = \int_\rho^\infty \left( \frac{\kappa_n}{\pi n} \right)^{1/n} \frac{1}{t^{n-1}} dt \int_0^t \exp \left( \int_\rho^{r'} g(r')dr' \right) f^*(\kappa_n r'^n) r'^{n-1} dr
\]

(48)

and we have:

\[
- v''(\rho) - \frac{n}{\rho} v'(\rho) + \tilde{b}(H_0(x)) v'(\rho) = f^*(\rho).
\]

(49)
Collecting (48) and (49) we obtain that the function in (41) solves (43).

**Remark 2.** We can compute \( \int_{\Omega^*} H_q(\nabla v). \) By (45) we have

\[
[H(\nabla v(x))]^q = [v'(\rho)]^q = \left[ -\frac{1}{\rho^{n-1}} \int_0^\rho \exp \left( \int_r^\rho \tilde{b}(r')dr' \right) f^*(\kappa_n r^n) r^{n-1} dr \right]^q
\]

(50)

where \( \rho = H_0(x). \) An integration by the substitution \( s = \kappa_n r^n \) gives

\[
[H(\nabla v(x))]^q = \left[ -\frac{1}{n \kappa_n \rho^{n-1}} \int_0^{\kappa_n \rho^n} \exp \left( \int_0^\rho \frac{1}{\kappa_n} \tilde{b}(r')dr' \right) f^*(s) ds \right]^q,
\]

(51)

dependently, by an integration on \( \Omega^* \), we have

\[
\int_{\Omega^*} [H(\nabla v(x))]^q
\]

\[
= \int_0^{[\Omega]} \left[ -\frac{1}{n \kappa_n \rho^{n-1}} \int_0^{\kappa_n \rho^n} \exp \left( \int_0^\rho \frac{1}{\kappa_n} \tilde{b}(r')dr' \right) f^*(s) ds \right]^q d\rho.
\]

(52)

Hence, by the substitution \( \tau = \kappa_n \rho^n \), we have

\[
\int_{\Omega^*} [H(\nabla v(x))]^q
\]

\[
= \int_0^{[\Omega]} \left[ -\frac{1}{n \kappa_n^{1/n} \tau^{1/n-1}} \int_0^\tau \exp \left( \int_0^\tau \frac{1}{\kappa_n} \tilde{b}(r')dr' \right) f^*(s) ds \right]^q d\tau.
\]

(53)

**Theorem 2.** Let \( u \in H_0^1(\Omega) \) be a solution to problem (36) under the assumption (37). Furthermore we suppose that (38) holds with

\[
||B||_{L^\infty(\Omega)} = \beta \leq \infty;
\]

(54)

\( f \in L^{\frac{2n}{n+2}}(\Omega) \) if \( n \geq 3; f \in L^p(\Omega), p > 1, \) if \( n = 2; H : \mathbb{R}^n \to [0, \infty[ \) is a convex function satisfying (13)-(14).

Then (39) and (40) holds with

\[
v(x) = \int_{H_0(x)} \left[ \frac{1}{t^{n-1}} dr \int_0^t e^{\beta(r-t)} f^*(\kappa_n r^n) r^{n-1} dr. \right]
\]

(55)
Remark 3. The function $v(x)$ in (55) is a solution in $H^1_0(\Omega^*)$ to the problem
\[
\begin{aligned}
&-\text{div}(H(\nabla v)\nabla H(\nabla v)) - \beta\langle \nabla H_0(x), \nabla H(\nabla v) \rangle H(\nabla v) = f^* \quad \text{in } \Omega^* \\
v = 0 \quad \text{on } \partial \Omega^*.
\end{aligned}
\tag{56}
\]

The proof of Theorem 2 is similar to that of Theorem 1 and it can be obtained from it considering the function $B(x)$ as a constant.

4 - PROOF OF THEOREM 1

Let us start by proving a preliminary result about the function $\tilde{b}$ (see (Talenti, 1985)).

Lemma 4. If $\tilde{b}$ is defined by (35), then
\[
\left( -\frac{d}{dt} \int_{|u|>t} B^2(x) \right)^{\frac{1}{2}} = \sqrt{-\mu'(t)} \tilde{b} \left( \left( \frac{\mu(t)}{\kappa} \right)^{\frac{1}{n}} \right)
\tag{57}
\]
and
\[
-\frac{d}{dt} \int_{|u|>t} B(x) H(\nabla u) \leq \left( -\frac{d}{dt} \int_{|u|>t} B(x) \right)^{\frac{1}{2}} \left( -\frac{d}{dt} \int_{|u|>t} H^2(\nabla u) \right)^{\frac{1}{2}}
\tag{58}
\]
for almost every $t \in [0, \text{ess sup}_\Omega |u|]$.

Proof. Let $p(t)$ and $q(s)$ be the integrals of $B(x)$ over $\{|u|>t\}$ and $\{|u|>u^*(s)\}$ respectively, hence $p'(t) = q'(\mu(t))\mu'(t)$ for almost every $t \in [0, \text{ess sup}_\Omega u]$. So equality (57) is proved.

By Hölder inequality, we have
\[
-\frac{d}{dt} \int_{|u|>t} B(x) H(\nabla u) \leq \left( -\frac{d}{dt} \int_{|u|>t} B(x) \right)^{\frac{1}{2}} \left( -\frac{d}{dt} \int_{|u|>t} H^2(\nabla u) \right)^{\frac{1}{2}},
\tag{59}
\]
by (57) we obtain
\[
-\frac{d}{dt} \int_{|u|>t} B(x) H(\nabla u) \leq \sqrt{-\mu'(t)} \tilde{b} \left( \left( \frac{\mu(t)}{\kappa} \right)^{\frac{1}{n}} \right) \left( -\frac{d}{dt} \int_{|u|>t} H^2(\nabla u) \right)^{\frac{1}{2}},
\tag{60}
\]
hence, by Lemma 1,

$$-\frac{d}{dt} \int_{|u|>t} B(x)H(\nabla u)$$

$$\leq -\mu'(t)\frac{\mu(t)^{-1}}{n\kappa_n^{1/n}} \tilde{b} \left( \frac{\mu(t)}{\kappa_n} \right) \left( -\frac{d}{dt} \int_{|u|>t} H^2(\nabla u) \right),$$

that is equal to the right-hand side of (58).

**Proof of Theorem 1** Suppose $u$ is a weak solution of problem (36), then

$$\int_{\Omega} \langle a(x, u, \nabla u), \nabla \varphi \rangle + \int_{\Omega} b(x, \nabla u) \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in H^1_0(\Omega).$$

For $h > 0$, $t > 0$, let $\varphi$ be the following test function

$$\varphi_h(x) = \begin{cases} h, & \text{if } |u| > t + h \\ |u| - t, & \text{if } t < |u| \leq t + h \\ 0, & \text{if } |u| \leq t, \end{cases}$$

then

$$\nabla \varphi_h(x) = \begin{cases} 0, & \text{if } |u| > t + h \\ \nabla_i u, & \text{if } t < |u| \leq t + h \\ 0, & \text{if } |u| \leq t. \end{cases}$$

Inserting this test function in (62), we have

$$\int_{t<|u|\leq t+h} \langle a(x, u, \nabla u), \nabla \varphi \rangle + \int_{|u|>t+h} b(x, \nabla u) \varphi = \int_{|u|>t+h} f \varphi + \int_{t<|u|\leq t+h} (f - b(x, \nabla u))(|u| - t) \text{sgn } u. \quad (65)$$

The last term is smaller than $\int_{t<|u|\leq t+h} (f - b(x, \nabla u))(|u| - t)$ and, by hypothesis (37) and (38), we have

$$\int_{t<|u|\leq t+h} H^2(\nabla u) - h \int_{|u|>t+h} B(x)H(\nabla u) \leq \int_{|u|>t+h} fh + \int_{t<|u|\leq t+h} (f - b(x, \nabla u))(|u| - t). \quad (66)$$

Dividing each term by $h$, as $h \to 0^+$, (66) becomes

$$-\frac{d}{dt} \int_{|u|>t} H^2(\nabla u) - \int_{|u|>t} B(x)H(\nabla u) \leq \int_{|u|>t} f,$$
and, by Lemma 2,
\[- \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) - \int_{|u| > t} B(x) H(\nabla u) \leq \int_0^{\mu(t)} f^*(s) ds. \tag{68}\]

Now, we can write
\[\int_{|u| > t} B(x) H(\nabla u) = \int_t^{+\infty} \left( - \frac{d}{ds} \int_{|u| > s} B(x) H(\nabla u) \right) ds \tag{69}\]
and hence, by Lemma 4, we have
\[\int_{|u| > t} B(x) H(\nabla u) \leq \int_t^{+\infty} \left( - \frac{d}{dt} \int_0^{\mu(t)} b(r') dr \right) \left( - \frac{d}{ds} \int_{|u| > s} H^2(\nabla u) \right) ds. \tag{70}\]

Inserting (70) in (68) we obtain
\[- \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \leq \int_0^{\mu(t)} f^*(s) ds + \int_0^{\mu(t)} \exp \left( - \int_0^{\mu(t)} \frac{b(r')}{\mu} dr' \right) [-d\psi(s) ds]. \tag{71}\]

Now we can use Lemma 3 with \( \varphi(t) = - \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \). We have
\[- \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \leq \int_t^{+\infty} \exp \left( \int_t^s \frac{b(r')}{\mu} dr' \right) [-d\psi(s) ds], \tag{72}\]
where \( \psi(s) = \int_0^{\mu(s)} f^*(\xi) d\xi \).

Using the substitution \( \rho = \mu(s) \) and \( \sigma = \mu(r) \), we obtain
\[- \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \leq \int_0^{\mu(t)} \exp \left( \int_0^{\mu(t)} \frac{b(\rho)}{\mu} d\rho \right) f^*(\sigma) d\sigma. \tag{73}\]
Inequality (73) and Lemma 1 give

\[
1 \leq \frac{1}{n^2 \kappa_n^{2/n}} \mu(t)^{\frac{2}{n} - 2}(-\mu'(t)) \int_0^{\mu(t)} \exp \left( \int_{\frac{\sigma}{\kappa_n}} \frac{1}{n} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma. \tag{74}
\]

for a.e. \( t \in [0, \text{ess sup}|u|] \), then integration of both sides with respect to \( t \) over the interval \([0, u^*(s)]\) yields

\[
u^*(s) \leq \left| \int_{\Omega} t^{\frac{2}{n} - 2} \int_0^t \exp \left( \int_{\frac{\sigma}{\kappa_n}} \frac{1}{n} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \right|, \tag{75}
\]

From formula (41), we learn that \( v^*(s) \) is the right-hand side of (75), so (39) is satisfied.

In order to prove (40), we observe that Hölder inequality gives

\[
\frac{1}{h} \int_{t < |u| \leq t + h} H^q(\nabla u) \leq \left( \frac{1}{h} \int_{t < |u| \leq t + h} dx \right)^{1 - \frac{q}{2}} \left( \frac{1}{h} \int_{t < |u| \leq t + h} H^2(\nabla u) \right)^{\frac{q}{2}}, \tag{76}
\]

and hence, for \( t \to 0^+ \),

\[
- \frac{d}{dt} \int_{|u| > t} H^q(\nabla u) \leq (-\mu'(t))^{1 - \frac{q}{2}} \left( - \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right)^{\frac{q}{2}}, \tag{77}
\]

provided that \( 0 < q \leq 2 \). Lemma 1 gives

\[
\left[ - \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right]^{\frac{1}{2}} \leq \frac{1}{n^{1/n} \mu(t)^{\frac{1}{n} - 1}(-\mu'(t))^{\frac{1}{2}}} \left[ - \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right], \tag{78}
\]

hence by inequality (73)

\[
\left[ - \frac{d}{dt} \int_{|u| > t} H^2(\nabla u) \right]^{\frac{1}{2}} \leq \frac{1}{n^{1/n} \mu(t)^{\frac{1}{n} - 1}(-\mu'(t))^{\frac{1}{2}}} \int_0^{\mu(t)} \exp \left( \int_{\frac{\sigma}{\kappa_n}} \frac{1}{n} \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma. \tag{79}
\]
Coupling (79) with (77)

\[- \frac{d}{dt} \int_{|u|>t} H^q(\nabla u) \]

\[\leq (\mu'(t))^{1-q/2} \left[ \left( \frac{d}{dt} \int_{|u|>t} H^2(\nabla u) \right)^{\frac{1}{q}} \right]^q \]

\[\leq (\mu'(t))^{1-q/2} \left[ \frac{1}{n\kappa_n^{(1/n)}} \mu(t)^{\frac{1}{n} - 1} \int_0^{\mu(t)} \exp \left( \int_{\left(\frac{\mu(t)}{n\kappa_n}\right)^{\frac{1}{n}}}^1 \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \right]^q. \]

Consequently

\[\int_\Omega H^q(\nabla u) \]

\[\leq \int_0^{[\Omega]} -\mu'(t) \]

\[\leq \frac{1}{n\kappa_n^{(1/n)}} \int_0^{\mu(t)} \exp \left( \int_{\left(\frac{\mu(t)}{n\kappa_n}\right)^{\frac{1}{n}}}^1 \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \]

\[dt. \]

and hence, by the substitution \( \tau = \mu(t) \),

\[\int_\Omega H^q(\nabla u) \]

\[\leq \int_0^{[\Omega]} \left[ \frac{1}{n\kappa_n^{(1/n)}} \tau^{\frac{1}{n} - 1} \int_0^\tau \exp \left( \int_{\left(\frac{\tau}{n\kappa_n}\right)^{\frac{1}{n}}}^1 \tilde{b}(\rho) d\rho \right) f^*(\sigma) d\sigma \right]^q \]

\[d\tau = \int_{\Omega^*} H^q(\nabla v), \]

so the theorem is proved.
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