Tidal properties of $D$-dimensional Tangherlini black holes

V P Vandeev* and A N Semenova

Petersburg Nuclear Physics Institute of National Research Centre “Kurchatov Institute”, Gatchina, Russia 188300

Received: 05 July 2022 / Accepted: 17 November 2022 / Published online: 25 December 2022

Abstract: The aim of this paper is to investigate tidal forces in multidimensional spherically symmetric spacetimes. We consider geodesic deviation equation in Schwarzschild–Tangherlini metric and its electrically charged version. We show that these equations can be solved explicitly for radial geodesics as quadratures in spaces of any dimension. In the cases of five, six and seven dimensional spaces, these solutions can be represented in terms of elliptic integrals. For spacetimes of higher dimension asymptotics of the solution are found instead. We established that the greater the dimension of space is, the stronger the tidal stretch along the radial direction in the vicinity of physical singularity is, whereas the tidal compression in direction transverse to the radial one, does not change in the leading order starting from a certain dimension. Also, in the case of non-radial geodesics, the presence of black hole electric charge does not affect the force of transverse compression in the leading order. Finally, for non-radial geodesics with nonzero angular momentum, the local properties of solutions of geodesic deviation equations in the vicinity of a singularity are studied.

Keywords: Black hole; Tidal force; Geodesic deviation equation; Multidimensional spacetime

1. Introduction

General relativity (GR) describes gravitational phenomena and large-scale structure of spacetime. One of the most interesting objects in the GR is black hole. Various black holes have intriguing properties worthy of thorough investigation, but in this paper we will focus our attention on the tidal properties of static spherically symmetric black holes. Paper [1] is the first work on tidal effects in Schwarzschild spacetime [2]. These ideas were developed further in [3]. In the work [4] the influence of tidal forces on the destruction of stars was considered. Finer effects associated with the presence of angular momentum of a test body moving in the gravitational field of a Schwarzschild black hole were investigated in [5]. The case of electrically charged Reissner–Nordström black holes [6, 7] was considered in [8]. Tidal force of black holes in the presence of a homogeneous cosmological constant [9] have been studied in [10]. In paper [11] it was shown how one can integrate the Jacobi equation in any spacetime admitting completely integrable geodesics.

The tidal properties of black holes surrounded by various matter were also studied studied in article [12] (focused on Kiselev black hole [13] surrounded by quintessence) and in article [14] (devoted to dirty black holes). The so-called regular black holes were also investigated in relation to tidal forces in [15, 16], despite the absence of a singularity. In papers [17, 18], naked singularities were considered. The study of geodesic deviation in the theories of modified gravity was carried out in the works [19, 20]. In the former, tidal effects are considered in the theory of holographic massive gravity; in the latter, the action of Einstein-Gauss-Bonnet is used instead. Ref. [21] discusses tidal effects created by dark matter halo. It is important to note that tidal acceleration can be studied not only in metrics of spherical symmetry, but also in axially symmetric spaces. Thus, tidal forces were studied in the Kerr metric [22] in [23]. In [24] it was studied tidal properties of Schwarzschild black hole surrounded with clouds of strings and quintessence.

There are works in which tidal acceleration is not an object of study, but a means of describing the formation of astrophysical jets, see [25–27].

Moreover, one can study how tidal forces affect a body moving on finite orbits, see [28–31]. These works are concerned with the mechanism of geodesic deviation in various gravitational fields.

Our main goal is to study the tidal forces of multidimensional Schwarzschild–Tangherlini black holes and...
their electrically charged generalization [32]. We investigate geodesic deviation equation, which is called the Jacobi equation in differential geometry. We are interested in how the rate of geodesic divergence depends on the spacetime dimension. Both radial geodesics and geodesics with non-zero angular momentum are considered. The article has the following structure. In Sect. 2, we define geodesic equations in $D$-dimensional spherically symmetric spacetime, in Sect. 3, we present geodesic deviation equation and find the spectrum of the tidal tensor, which allows us to diagonalize the equation by passing to the frame of reference of freely falling body. In Sect. 4, we present solutions of the equation of geodesic deviation at zero angular momentum for the spacetime dimensions that allow it. And for non-radial geodesics with nonzero angular momentum, we construct local solutions in the form of power series in the vicinity of physical singularity, using the Frobenius method for equations of the Fuch’s class. In Sect. 5, we summarize the results and outline ways of developing the study of tidal forces. The paper also contains two appendices with brief description of elliptic integrals used in the paper, Fuchs’ equations, and the method for constructing their solution.

We use the metric signature $(+, -, \ldots, -)$, greek indices $\alpha, \beta, \ldots$ take on the values $0, 1, \ldots, D - 1$, which correspond hyperspherical coordinates $t, r, \theta_1, \ldots, \theta_{D-2}$ and set the speed of light $c$ and Newtonian gravitational constant $G$ to 1 throughout this paper.

2. Geodesics in static spherically symmetric high-dimensional spacetimes

We consider spacetime with line element of $D$-dimensional static spherically symmetric metric given by

$$ ds^2 = g_{\mu \nu}dx^\mu dx^\nu = f(r)dr^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2_{D-2}, $$

(1)

where $d\Omega^2_{D-2}$ is

$$ d\Omega^2_{D-2} = d\theta_1^2 + \sum_{j=2}^{D-2} d\theta_j^2 \left( \prod_{k=1}^{j-1} \sin^2 \theta_k \right). $$

(2)

We will define the explicit form of the function $f(r)$ in the following sections. In this spacetime, we have the following geodesic equations

$$ u^0 = \frac{dr}{d\tau} = \frac{E}{f(r)}, $$

$$ (u^i)^2 = \left( \frac{dr}{d\tau} \right)^2 = E^2 - f(r) \left( \delta_1 + \frac{L^2}{r^2} \right), $$

$$ u^i = \frac{d\theta^j}{d\tau} = 0, \ j = 1, \ldots, D - 3, $$

$$ u^{D-2} = \frac{d\theta^{D-2}}{d\tau} = \frac{L}{r^2}, $$

(3–6)

where $\tau$ is affine parameter along the geodesic, $E$ and $L$ are energy and angular momentum of freely moving test body, and parameter $\delta_1$ defines the type of geodesic: $\delta_1 = 1$ corresponds to the timelike geodesic, $\delta_1 = 0$ corresponds to the lightlike geodesic, $\delta_1 = -1$ corresponds to the spacelike geodesic. And it should be noted that dynamics of the angular variables $\theta^j$ with $j = 1, \ldots, D - 3$ are trivial because in a spherically symmetric space movement occurs in one plane determined by the set of equalities $\theta_1 = \theta_2 = \cdots = \theta_{D-3} = \frac{\pi}{2}$. Azimuth variable dynamic $\theta^{D-2}$ is nontrivial because there is angular momentum $L$.

The expression set (3–6) forms a unit covariant $D$-velocity vector $u^\mu$ tangent to the geodesic.

3. Geodesic deviation equation

Below we consider geodesic deviation equation. As it is well known [33], the equation for the geodesic deviation vector $\xi^\mu$ is given by

$$ \frac{D^2 \xi^\mu}{d\tau^2} = R^\mu_{\nu\rho\sigma}u^\nu \xi^\rho \xi^\sigma, $$

(7)

where $\frac{D^2}{d\tau^2}$ is covariant derivative along the geodesic curve, $R^\mu_{\nu\rho\sigma}$ is Riemann curvature tensor of spacetime and $u^\nu$ is the unit vector of $D$-velocity tangent to the geodesic. $\xi^\mu$ connects two points on close geodesics which correspond to the same value of the affine parameter $\tau$.

The nonzero Riemann tensor components calculated by the metric (1) are
\[ R^3_{101} = -\frac{f''}{2f}, \]
\[ R^3_{202} = -\frac{rf'}{2}, \]
\[ R^3_{j0j} = -\frac{rf^j}{2} \prod_{k=1}^{i-2} \sin^2 \theta_k, \]
\[ R^3_{212} = -\frac{rf'}{2}, \]
\[ R^3_{ji} = (1-f) \prod_{k=1}^{i-2} \sin^2 \theta_k, \]
\[ R^3_{jj} = \left( 1 - f \right) \prod_{k=1}^{i-2} \sin^2 \theta_k, \]
\[ i = 2, \ldots, D - 2, \quad j = 3, \ldots, D - 1, \]

where prime means derivative with respect to radial variable \( r \).

As we can see that on the right-hand side of Eq. (7), there is a matrix \( P^\mu_{\nu} = R^\mu_{\nu \beta \gamma} u^\beta u^\gamma \), which is called tidal tensor, that has nonzero elements

\[ P^0_0 = \frac{\dot{r}^2 + \chi L^2 \sin^2 \theta}{r^2}, \]
\[ P^0_1 = -\frac{E \dot{r} f''}{2f}, \quad P^1_0 = \frac{E \dot{r} f''}{2}, \]
\[ P^1_1 = \frac{\chi L^2 \sin^2 \theta - E^2 f''}{r^2}, \]
\[ P^D_{D-1} = \frac{\chi EL}{r^2}, \quad P^D_{D-1} = -\frac{\chi EL \sin^2 \theta}{f}, \]
\[ P^D_{D-1} = \frac{\chi Lf}{r^2}, \quad P^D_{D-1} = -\frac{\chi Lf \sin^2 \theta}{f}, \]
\[ P^j_j = \frac{(f - 1)L^2 \sin^2 \theta}{r^4}, \quad P^D_{D-1} = -\chi \omega, \quad P^D_{D-1} = -\chi \omega, \]
\[ j = 2, \ldots, D - 2, \]

where

\[ \chi = \frac{f'}{2f}, \]
\[ \omega = \delta_1 + \frac{L^2}{r^2}, \]
\[ \dot{r} = u^1 \frac{dr}{d\tau} = \sqrt{E^2 - f(r) \left( \delta_1 + \frac{L^2}{r^2} \right)}. \]

Therefore, a parallelpropagated tetrad \( (D\text{-dimensional analog of 4-dimensional tetrads}, which we will call simply tetrads below) \) basis for the free fall frame of reference can be constructed. It has the form

\[ e^\mu_i = \frac{1}{\sqrt{1 + \frac{E \dot{r}}{r}}} \left( \begin{array}{c} E \frac{\dot{r}}{r}, \dot{r}, 0, L \end{array} \right), \]
\[ e^\mu_i = \frac{1}{\sqrt{1 + \frac{E \dot{r}}{r}}} \left( -\frac{\dot{r}}{r}, -E, 0, 0 \right), \]
\[ e^\mu_i = \left( 0, 0, \frac{1}{r}, 0 \right), \quad j = 1, \ldots, D - 3, \]
\[ e^\mu_0 = \frac{L}{r \sqrt{\delta_1^2 + \frac{L^2}{r^2}}} \left( \frac{E \frac{\dot{r}}{r}, \dot{r}, 0, \frac{\delta_1 + \frac{L^2}{r^2}}{r} \right). \]

These tetrads \( e^\mu_i \) satisfy normalization condition \( e^\mu_i e^\nu_j g_{\mu \nu} = \delta_{ij} \) with Minkowski metric \( g_{\mu \nu} = \text{diag}(1, -1, \ldots, -1) \). The geodesic deviation vector \( \xi^\mu \) can be substituted as

\[ \xi^\mu = e^\mu_i \xi^i. \]

Thus, the meaningful components of Eq. (7) in this frame of reference are

\[ \xi^0 = \left[ -\frac{f''}{2} \left( \delta_1 + \frac{L^2}{r^2} \right) + \frac{f'}{2r \frac{\dot{r}}{r}} \right] \xi^0, \]
\[ \xi^1 = \left[ -\frac{f'}{2r} \left( \delta_1 + \frac{L^2}{r^2} \right) + \frac{f - \frac{L^2}{r^2}}{2} \right] \xi^0, \]
\[ \xi^{D-2} = -\frac{\delta_1 f'}{2r^2} \xi^{D-2}, \]

where \( a = \theta_1, \ldots, \theta_{D-3} \) correspond to polar angular components, and temporary component is trivial \( \xi^0 = 0 \). These equations are the diagonal form of Eq. (7). It is worth noting that the dependence on all angular coordinates have disappeared from Eqs. (13–15) because there is no dynamics along the direction of the azimuthal angles \( \theta^i \) with \( i = 1, \ldots, D - 3 \) according to Eq. (5), and geodesics lie in the equatorial plane \( \theta_1 = \theta_2 = \cdots = \theta_{D-3} = \frac{\pi}{2} \). Also, it should be noted that without angular momentum \( L \) Eq. (14) and Eq. (15) coincide. The right parts of these equations describe tidal accelerations.

### 4. Solution of geodesic deviation equation

Using Eq. (4) left side of Eqs. (13–15) can be transformed into

\[ \frac{d}{d\tau} \frac{\xi^\mu}{\xi^0} = \left[ \left( \frac{E^2 - f \left( \delta_1 + \frac{L^2}{r^2} \right)}{\xi^0} \right)^2 - \frac{f'}{2r} \left( \delta_1 + \frac{L^2}{r^2} \right) - \frac{f L^2}{r^3} \right] \xi^\mu. \]
4.1. Radial geodesics

Radial geodesics are a special class of geodesics. The absence of angular momentum \( L \) Eqs. (13–15) simplifies

\[
(E^2 - \delta f')^n \frac{\delta f'}{2 \delta f'} \xi' + \frac{\delta f''}{2 \delta f'} \xi = 0, \tag{17}
\]

\[
(E^2 - \delta f') \xi'' - \frac{\delta f'}{2 \delta f'} \xi' + \frac{\delta f''}{2 \delta f'} \xi = 0, \tag{18}
\]

where \( j = \theta_1, \ldots, \theta_{D-2} \). These equations can be solved through quadratures

\[
\xi' = \sqrt{E^2 - \delta f} \left( A_r + B_r \int \frac{dr}{(E^2 - \delta f)^{\frac{1}{2}}}, \tag{19}\right)
\]

\[
\xi = r \left( C_j + N_j \int \frac{dr}{r^2 \sqrt{E^2 - \delta f}}, \tag{20}\right)
\]

where \( A_j, B_j, C_j, N_j \) are integration constants. These solutions have been known for a long time and have been repeatedly used in works [1, 8, 10, 12, 13, 19, 20]. It worth noting that lightlike geodesics with \( \delta_1 = 0 \) deviate in the same way in space of any dimension

\[
\xi'^{\mu} = A + Br \tag{21}
\]

for all spatial components \( \mu = r, \theta_1, \ldots, \theta_{D-2} \).

Previously, in a large number of papers expressions (19) and (20) were applied to 4-dimensional spacetimes. Below, where possible, we represent them in spaces of higher dimensions for timelike geodesics at \( \delta_1 = 1 \) as normal elliptic integrals.

4.1.1. D-dimensional Schwarzschild spacetimes

Electrically neutral Schwarzschild–Tangherlini metric in the form (1) has function \( f(r) \) as

\[
f(r) = 1 - \frac{\mu_D}{r^{D-3}}, \quad \mu_D = \frac{16\pi G_D M}{(D - 2)A_{D-2}}, \tag{22}\]

where \( A_{D-2} \) is the surface area of the unit \( S^{D-2} \) sphere given as

\[
A_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{23}\]

Expressions (19) and (20) in spaces of different dimensions are:

1. For 4-dimensional spacetime

\[
\xi' = A\xi_4 + B \left[ 3\mu_4 + r\psi^2 + \frac{3\mu_4}{\psi^2} \ln \left( \frac{\xi_4(r) - \psi}{\xi_4(r) + \psi} \right) \right], \tag{24}\]

\[
\xi^a = r \left( C + N\xi_4(r) \right), \tag{25}\]

where

\[
\psi = \sqrt{E^2 - 1}, \quad \xi_4(r) = \sqrt{E^2 - 1 + \frac{\mu_4}{r}}, \tag{26}\]

This solution was first obtained in work [1].

2. For 5-dimensional spacetime, solutions are expressed in elementary functions

\[
\xi' = A\xi_5 + B \left[ \frac{\xi_5(r)}{\psi^2} + \frac{3\mu_5}{\psi^2} \ln \left( \frac{\xi_5(r) - \psi}{\xi_5(r) + \psi} \right) \right], \tag{27}\]

\[
\xi^a = r \left[ C + N\ln \left( \xi_5(r) + \frac{\sqrt{\psi^2}}{r} \right) \right], \tag{28}\]

where

\[
\xi_5(r) = \sqrt{E^2 - 1 + \frac{\mu_5}{r^2}}. \tag{29}\]

3. For 6-dimensional spacetime

\[
\xi' = A\xi_6 + B \left[ 3\mu_6 + r\psi^2 + \frac{3\mu_6}{\psi^2} \right], \tag{30}\]

\[
\xi^a = r \left[ C + N\xi_6 \right]. \tag{31}\]

where

\[
\kappa = \frac{\sqrt{\mu_6}}{4}, \quad \psi = \sqrt{E^2 - 1}, \quad \xi_6(r) = \sqrt{E^2 - 1 + \frac{\mu_6}{r^3}}. \tag{32}\]

and functions \( J_0 \) and \( J_1 \) are Weierstrass elliptic integrals, which are defined in Appendix B, which have \( g_2 = 0 \) and \( g_3 = -\psi^2 \).

4. For 7-dimensional spacetime

\[
\xi' = B \left[ \frac{2r}{\psi} + \frac{3\mu_7}{\psi^2} \right] + \xi_7(r)A \tag{33}\]

\[
\xi^a = r \left[ C + NF \left( \frac{\omega \psi^2}{r}, i \right) \right], \tag{34}\]

where

\[
\omega = \sqrt{\frac{\mu_7}{E^2 - 1}}, \quad \xi_7(r) = \sqrt{E^2 - 1 + \frac{\mu_7}{r^4}}. \tag{35}\]

And functions \( E(x, k) \) and \( F(x, k) \) are Legendre elliptical integrals, which are defined in Appendix B, which have argument \( x = \omega \psi^2 / r \) and parameter \( k = i \).

5. For \( D \)-dimensional spacetime with \( D \geq 8 \), integrals in (19) and (20) are no longer elliptical. Therefore, below we demonstrate the asymptotic properties of these
solutions in the vicinity of a physical singularity and spatial infinity. At spatial infinity \( r \to \infty \), they are

\[
\zeta^r = Br + A + \frac{B r D (D - 1)}{2(D - 4)r^{D - 4}} + \frac{A \sigma_D}{2r^{D - 3}} - \frac{B \sigma_D^2 (D - 1)(D - 5)}{4(D - 4)(2D - 7)r^{2D - 7}} - \frac{A \sigma_D}{8r^{2D - 6}} + O(r^{10 - 3D}),
\]

\[
\zeta^a = Cr + N - \frac{N\sigma_D}{(2D - 4)r^{D - 3}} + \frac{3N\sigma_D^2}{(16D - 40)r^{2D - 6}} + O(r^9 - 3D),
\]

where \( \sigma_D = \frac{\mu_D}{r^D} \). The linear growth of all spatial geodesic deviation vector components at infinity is expected, because Schwarzschild–Tangherlini metric is asymptotically flat. Near physical singularity \( r \to 0 \), they are

\[
\zeta^r = \frac{A}{r^D} + \frac{A \sigma_D}{2} - \frac{B r D (D - 2)}{2(2D - 5)} + \frac{A r D (D - 3)}{8(3D - 11)} + O(r^{3D - 8}),
\]

\[
\zeta^a = Cr + N\frac{\sigma_D}{2D - 5} - \frac{N\sigma_D^2}{2(3D - 11)} + \frac{3N\sigma_D^3}{8(5D - 17)} + O(r^{3D - 8}),
\]

where \( \sigma_D = \frac{\mu_D}{r^D} \) and \( a = \theta_1, \ldots, \theta_{D - 2} \). It should be noted that the radial component in the vicinity of \( r = 0 \) increases indefinitely, but the angular components decrease. A large dimension \( D \) of spacetime corresponds to a more intensive growth of \( \zeta^r \).

4.1.2. \( D \)-dimensional Reissner–Nordström spacetimes

Electrically charged Reissner–Nordström–Tangherlini metric in the form (1) has function \( f(r) \) as

\[
f(r) = 1 - \frac{\mu_D}{r^{D - 3}} + \frac{q_D^2}{r^{2(D - 3)}},
\]

where \( q_D \) is

\[
q_D^2 = \frac{8\pi G_D Q^2}{(D - 2)(D - 3)A_{D - 2}},
\]

and \( \mu_D \) and \( A_{D - 2} \) are defined in Eqs. (22) and (23), respectively. General expressions (19) and (20) in spaces of different dimensions take the form:

1. For 4-dimensional spacetime

\[
\zeta^r = \zeta_4(r) \left[ A - \frac{3B \mu_4}{(E^2 - 1)^2} \text{Arth} \left( \frac{(E^2 - 1) + \frac{q_4}{r}}{2} \right) \right] + B \left[ 2r + \frac{\mu_4 (\mu_4^2 + 5)}{E^2 - 1} - \frac{2q_4 (\mu_4^2 + 2)}{r(E^2 - 1)} \right],
\]

\[
\zeta^a = r \left( C + N \arcsin \left( \frac{\Psi \left[ \mu_4 - \frac{2q_4}{r} \right] }{r} \right) \right),
\]

where

\[
\Psi^2 = \left( 4q_4^2 E^2 - 4q_4^2 + \mu_4^2 \right)^{-1},
\]

\[
\zeta_4(r) = \sqrt{E^2 - 1 + \frac{\mu_4}{r} - \frac{q_4^2}{r^2}}.
\]

These solutions were first obtained and well studied in work [8].

2. For 5-dimensional spacetime

\[
\zeta^r = A \zeta_5(r) - \frac{B \zeta_5^2(r)r}{\sqrt{E^2 - 1}} + \frac{b B \zeta_5^2}{E^2 - 1} \times
\]

\[
\left[ F(x, k) - E(x, k) + \frac{\Pi(x, k, 1) - b^4 \Pi(x, k, ib^{-2})}{1 + b^4} \right],
\]

\[
\zeta^a = r(C + NF(x, k)),
\]

where

\[
x = \frac{\beta}{br}, \quad k = ib^2, \quad \beta = \sqrt{\frac{q_5^2}{E^2 - 1}},
\]

\[
b^2 = \frac{\mu_5}{2q_5 \sqrt{E^2 - 1}} + \frac{1}{1 + \frac{\mu_5^2}{4q_5^2 (E^2 - 1)}}.
\]

\[
\zeta_5(r) = \sqrt{E^2 - 1 + \frac{\mu_5}{r^2} - \frac{q_5^2}{r^4}},
\]

and \( E(x, k), F(x, k) \) and \( \Pi(x, k, c) \) also defined in Appendix B.

3. For \( D \)-dimensional spacetime with \( D \geq 6 \), integrals in (19) and (20) are not elliptical. Therefore, below we demonstrate the asymptotic properties of these solutions in the vicinity of a physical singularity and spatial infinity. At spatial infinity \( r \to \infty \), they are

\[
\zeta^r = Br + A + \frac{B r D (D - 1)}{2(D - 4)r^{D - 4}} + \frac{A \sigma_D}{2r^{D - 3}} - \frac{B \sigma_D^2 (D - 1)(D - 5)}{4(D - 4)(2D - 7)r^{2D - 7}} - \frac{A \sigma_D}{8r^{2D - 6}} + O(r^{10 - 3D}),
\]

where \( \sigma_D = \frac{\mu_D}{r^D} \).
The character of (48) and (49) coincides with the electrically neutral case (36) and (37), respectively. Near physical singularity \( r \to 0 \), they are

\[
\xi^r = \frac{A}{r^{D-3}} - \frac{N \sigma_D}{2h_D} \frac{A(\sigma_D^2 + 4h_D)r^{D-3}}{8h_D^2} - \frac{N(3\sigma_D^2 + 4h_D)r^{D-9}}{24D - 8} + O(r^{D-12}),
\]

and

\[
\xi^\alpha = Cr + N - \frac{N\sigma_D}{(2D - 4)r^{D-3}} + \frac{N(3\sigma_D^2 + 4w_D)}{(16D - 40)r^{2D-6}} + O(r^{9-3D}),
\]

where \( \sigma_D = \frac{\mu_0}{E - \frac{1}{2}} \) and \( w_D = \frac{\sigma_0}{E - \frac{1}{2}} \). The presence of black hole electric charge, as expected, did not change spatial components behavior at infinity, because Reissner–Nordström–Tangherlini metric is also asymptotically flat. Therefore, the character of (48) and (49) coincides with the electrically neutral case (36) and (37), respectively. Near physical singularity \( r \to 0 \), they are

\[
\xi^r = \frac{A}{r^{D-3}} - \frac{N \sigma_D}{2h_D} \frac{A(\sigma_D^2 + 4h_D)r^{D-3}}{8h_D^2} - \frac{N(3\sigma_D^2 + 4h_D)r^{D-9}}{24D - 8} + O(r^{D-12}),
\]

and

\[
\xi^\alpha = Cr + N - \frac{N\sigma_D}{(2D - 4)r^{D-3}} + \frac{N(3\sigma_D^2 + 4w_D)}{(16D - 40)r^{2D-6}} + O(r^{9-3D}),
\]

where \( \sigma_D = \frac{\mu_0}{E - \frac{1}{2}} \), \( h_D = \frac{\sigma_0}{E - \frac{1}{2}} \) and \( a = 0, \ldots, \theta_{D-2} \). Compared with (38) and (39), the case of a charged black hole shows us that the presence of a charge does not change angular components behavior (51), but enhances the tidal stretch along the radial direction (50).

### 4.2. Nonradial geodesics

We now turn to the study of Eqs. (13–15) in the presence of nonzero angular momentum \( L \). In this case these equations cannot be solved in quadratures, so we will look for local solutions in the form of generalized power series using the Frobenius method (briefly described in Appendix A).

Equations (13–15) for timelike geodesics \( \delta_1 = 1 \) can be rewritten as

\[
\xi^{\alpha''} + P(r)\xi^{\alpha'} + Q(r)\xi^{\alpha} = 0.
\]

where

\[
P(r) = -\frac{\ell^2}{E^2 - f(1 + \frac{\mu}{r})},
\]

\[
Q(r) = \frac{\ell^2}{E^2 - f(1 + \frac{\mu}{r})},
\]

and

\[
Q_{\alpha}(r) = \frac{\ell^2}{E^2 - f(1 + \frac{\mu}{r})},
\]

\[
Q_{\alpha\beta}(r) = \frac{\ell^2}{E^2 - f(1 + \frac{\mu}{r})},
\]

and indices \( i, j \) take all spatial values \( r, a, \theta_{D-2} \). It is easy to see that the coefficients \( P(r) \) and \( Q(r) \) of Eq. (52) are rational functions. We are interested in the deviation geodesic vector \( \xi^\alpha \) behavior in the vicinity of physical singularity \( r = 0 \). We will explore the equation for all spatial components in both metrics (22) and (40).

#### 4.2.1. D-dimensional Schwarzschild spacetimes

Coefficients (53) of differential Eq. (52) in multidimensional Schwarzschild spacetime are

\[
P(r) = -\frac{1}{2r'};
\]

\[
\mu_0L^2(D - 1) + \mu_1(D - 3)r^2 - 2L^2r^{D-3},
\]

\[
\mu_0L^2 + \mu_1r^2 - L^2r^{D-3} + (E^2 - 1)r^{D-1}.
\]

\[
Q(r) = -\frac{\mu_0(D - 3)}{2r^2},
\]

\[
\frac{L^2(D - 1) + (D - 2)r^2}{\mu_0L^2 + \mu_1r^2 - L^2r^{D-3} + (E^2 - 1)r^{D-1}},
\]

\[
\frac{L^2(D - 1) + (D - 3)r^2}{\mu_0L^2 + \mu_1r^2 - L^2r^{D-3} + (E^2 - 1)r^{D-1}},
\]

\[
\frac{\mu_0(D - 3)}{2[\mu_0L^2 + \mu_1r^2 - L^2r^{D-3} + (E^2 - 1)r^{D-1}]}.
\]

where \( a = 0, \ldots, \theta_{D-3} \).

Consider all components of Eq. (52):

- The radial equation uses coefficients (54a) and (54b)

\[
\xi^{\alpha''} + P(r)\xi^{\alpha'} + Q(r)\xi^{\alpha} = 0.
\]

To investigate the properties of a given equation solution in the vicinity of a physical singularity \( r = 0 \) using the Frobenius method, one need to expand the coefficients in a Laurent series

\[
P(r) = -\frac{D - 1}{2r} + O(1),
\]

\[
Q(r) = -\frac{(D - 1)(D - 3)}{2r^2} + O(r^{-1}).
\]

Therefore, the equation for the leading powers of linearly independent solutions takes the form
leading order the behavior of the polar components $n$ physical singularity can be found using the initial data. In the vicinity of a physical singularity, the deviation of geodesics along the radial direction. It can be seen that with $L 
eq 0$, Eq. (58b) in leading order coincides with (38).
- The polar equations use coefficients (54a) and (54c) $a = \theta_1, \ldots, \theta_{D-3}$. Similarly, we use the Laurent series expansion of the coefficient $P(r)$ found in (56a) and Laurent series for the coefficient $Q_\nu(r)$ $Q_\nu(r) = \frac{D-1}{2} + O(r^{-1})$.

Therefore, the equation for the leading powers of linearly independent solutions takes the form $\zeta (\zeta - 1) - \left( \frac{D-1}{2} \right) \zeta + \frac{D-1}{2} = 0$. Its solutions are $\zeta_1 = \frac{D-1}{2}, \zeta_2 = 1$. The difference of these roots is not integer, so two local linearly independent solutions can be represented as

$$\xi_1^n(r) = r^{\frac{D-1}{2}} \sum_{k=0}^{\infty} c_k r^k,$$

$$\xi_2^n(r) = \sum_{k=0}^{\infty} d_k r^{k+1},$$

where $a = \theta_1, \ldots, \theta_{D-3}$ and the set of coefficients $c_k$ and $d_k$ can be found using the initial data. Around $r = 0$ for any $D$, the first solution is negligible compared to the second, because it has negative powers of the radial variable $r$. Thus, the dimension of spacetime affects the deviation of geodesics along the radial direction. It can be seen that with $L 
eq 0$, Eq. (58b) in leading order coincides with (38).
- The azimuthal equation uses coefficients (54a) and (54d) and its solutions are represented as $Q_\nu(r)$ in leading order.

Similarly, we use the Laurent series expansion of the coefficient $P(r)$ found in (56a) and Laurent series for the coefficient $Q_\nu(r)$ $Q_\nu(r) = \frac{D-3}{2L^2} + O(r)$. It can be seen that the coefficient at $r^{-2}$ is absent. Therefore, the equation for the leading powers of linearly independent solutions takes the form $\zeta (\zeta - 1) - \left( \frac{D-1}{2} \right) \zeta = 0$. Its solutions are $\zeta_1 = \frac{D-1}{2}, \zeta_2 = 0$. The difference of these roots is not integer, so two local linearly independent solutions are represented as

$$\xi_1^{\nu}(r) = r^{\frac{D-1}{2}} \sum_{k=0}^{\infty} c_k r^k,$$

$$\xi_2^{\nu}(r) = \sum_{k=0}^{\infty} d_k r^{k+1},$$

where the set of coefficients $c_k$ and $d_k$ can be found using the initial data. Around $r = 0$ for any $D$ the first solution is negligible compared to the second, therefore the azimuth component of the geodesic deviation vector is constant. And this component behavior differs from (49) at $a = \theta_{D-2}$. This means that the azimuthal component $\xi_\nu^{\theta_{D-2}}$ depends on $L$. And along this direction, a freely falling body does not experience tidal compression because $\xi_\nu^{\theta_{D-2}} \propto \text{const}$. A common property of the geodesic deviation vector transverse components $\xi_\nu^\alpha r, a = \theta_1, \ldots, \theta_{D-3}$ and $\xi_\nu^{\theta_{D-2}} \propto \text{const}$ is that its behavior near a physical singularity $r = 0$ is independent on the spacetime dimension, in contrast to longitudinal component $\xi_\nu^r \propto \frac{1}{\sqrt{\nu^{D-7}}}$. 4.2.2. $D$-dimensional Reissner–Nordström spacetimes Coefficients (53) of differential Eq. (52) in multidimensional Reissner–Nordström spacetime are $P(r) = -\frac{D-3}{2r}$.

$$\mu_d D^{D-3} \left( r^2 + \frac{(D-1)L^2}{D-3} \right) - 2q_d^2 \left( r^2 + \frac{(D-2)L^2}{D-3} \right) \left( E^2 - 1 \right) \frac{r^{2D-4} - L^2 r^{2D-6} + (\mu_d D^{D-3} - q_d^2) (r^2 + L^2)}{L^2 - 1}.$$

(67a)
The radial equation uses coefficients (67a) and (67b). 

\[ Q(r) = -D - 3 \frac{2D^3}{2} \]

\[
\mu_D r^{D-3} [(D-2) r^2 + L^2 (D-1)] - 4q_D^2 [(D - D) r^2 + (D - 2) L^2] \\
(E^2 - 1) r^{2D-4} - L^2 r^{2D-6} + \mu_D D^{2D-3} - q_D^2 (r^2 + L^2)^2,
\]

(67b)

\[ Q_{\alpha}(r) = \frac{D - 3}{2} \]

\[
\mu_D r^{D-3} - 2q_D^2 \\
(E^2 - 1) r^{2D-4} - L^2 r^{2D-6} + \mu_D D^{2D-3} - q_D^2 (r^2 + L^2)^2,
\]

(67c)

\[ Q_{0 \alpha}(r) = \frac{D - 3}{2} \]

\[
\mu_D r^{D-3} - 2q_D^2 \\
(E^2 - 1) r^{2D-4} - L^2 r^{2D-6} + \mu_D D^{2D-3} - q_D^2 (r^2 + L^2)^2,
\]

(67d)

where \( \alpha = \theta_1, \ldots \theta_{D-3} \).

Consider all components of Eq. (52):

- The radial equation uses coefficients (67a) and (67b).

\[ \zeta'' + P(r) \zeta'' + Q_{\alpha}(r) \zeta = 0. \]

(68)

To investigate the properties of a given equation solution in the vicinity of a physical singularity \( r = 0 \) using the Frobenius method, we expand the coefficients in a Laurent series

\[ P(r) = -D - 2 \frac{2D^3}{r} + O(1), \]

(69a)

\[ Q_{\alpha}(r) = -2(D - 3)(D - 2) \frac{2D^3}{r^2} + O(r^{-1}). \]

(69b)

Therefore, the equation for the leading powers of linearly independent solutions takes the form

\[ \zeta'(\zeta - 1) - \zeta(D - 2) - 2(D - 3)(D - 2) = 0. \]

(70)

Its solutions are \( \zeta_1 = 2D - 4, \zeta_2 = 3 - D \). The difference of these roots is integer, so the larger of the roots determines the leading power of the first solution, a logarithmic term may appear in the second linearly independent solution

\[ \zeta_{\alpha} = r^{2D-4} \sum_{k=0}^{\infty} c_k r^k, \]

(71a)

\[ \zeta_{\alpha} = r^{3-D} \sum_{k=0}^{\infty} d_k r^k + A \zeta_{\alpha}(r) \ln(r), \]

(71b)

where the set of coefficients \( c_k, d_k \) and \( A \) can be found using the initial data. It should be noted that for any \( D \geq 4 \) in the vicinity of the singularity, \( 2D - 4 \) is positive, and hence the product \( r^{2D-4} \ln(r) \) tends to zero, so the leading term is proportional to \( r^{3-D} \) that goes to infinity for any dimension. This result is similar to (50).

- The polar equations use coefficients (67a) and (67c).

\[ \zeta'' + P(r) \zeta'' + Q_{\alpha}(r) \zeta = 0, \]

(72)

where \( \alpha = \theta_1, \ldots \theta_{D-3} \). Similarly, we use the Laurent series expansion of the coefficient \( P(r) \) found in (69a) and Laurent series for the coefficient \( Q_{\alpha}(r) \)

\[ Q_{\alpha}(r) = \frac{D - 2}{r^2} + O(r^{-1}). \]

(73)

Therefore, the equation for the leading powers of linearly independent solutions takes the form

\[ \zeta'(\zeta - 1) - \zeta(D - 2) + (D - 2) = 0. \]

(74)

Its solutions are \( \zeta_1 = D - 2, \zeta_2 = 1 \). The difference of these roots is integer, so the larger of the roots determines the leading power of the first solution, a logarithmic term may appear in the second linearly independent solution

\[ \zeta_{\alpha} = r^{D-2} \sum_{k=0}^{\infty} c_k r^k, \]

(75a)

\[ \zeta_{\alpha} = \sum_{k=0}^{\infty} d_k r^k + A \zeta_{\alpha}(r) \ln(r), \]

(75b)

where \( \alpha = \theta_1, \ldots \theta_{D-3} \), the set of coefficients \( c_k, d_k \) and \( A \) can be found using the initial data. It should be noted that for any \( D \geq 4 \) in the vicinity of the singularity, \( D - 2 \) is bigger then 1, and hence the product \( r^{D-2} \ln(r) \) tends to zero faster than \( r \), so the leading term is proportional to \( r \). This linear growth of polar geodesic deviation vector components coincides in the leading order with Eq. (51) at \( \alpha = \theta_1, \ldots \theta_{D-3} \).

- The azimuthal equation uses coefficients (67a) and (67d).

\[ \zeta'' + P(r) \zeta'' + Q_{\theta_{\alpha}}(r) \zeta = 0. \]

(76)

Similarly, we use the Laurent series expansion of the coefficient \( P(r) \) found in (69a) and Laurent series for the coefficient \( Q_{\theta_{\alpha}}(r) \)

\[ Q_{0 \alpha}(r) = \frac{D - 3}{L} + O(r). \]

(77)

It can be seen that the coefficient at \( r^{-2} \) is absent. Therefore, the equation for the leading powers of linearly independent solutions takes the form

\[ \zeta'(\zeta - 1) - \zeta(D - 2) = 0. \]

(78)

Its solutions are \( \zeta_1 = D - 1, \zeta_2 = 0 \). The difference of these roots is integer, so the larger of the roots determines the leading power of the first solution, a logarithmic term may appear in the second linearly independent solution
Thus, we presented new solutions of the geodesic deviation
Legendre elliptic integrals (33) and (34), respectively.
while in six- and seven-dimensional spacetime solutions
warzschild–Tangherlini metric is asymptotically flat. On
the dimension
D
grow linearly with respect to
infinity, all components of the geodesic deviation vector
quadratures into power series. As expected, at spatial
positions in five-dimensional spacetime (27) for radial geodesic
in (22). It allows to express solutions in elementary func-
tions started with the Schwarzschild–Tangherlini metric defined
(20) for radial geodesics with
L
¼
rD
using the initial data. It should be noted that for any
D
≥ 4 in the vicinity of the singularity, D − 1 is bigger than zero,
and hence the product
rD−1 ln(r)
is infinitesimal at
r → 0,
so the leading term is proportional to constant. As in the
electrically neutral case, the behavior of the azimuthal component
ξ\(_{0-2}^{\theta,}(r)\) changed with the appearance of the angular momentum from (51) to (79b).

A common property of the transverse components of
gedesic deviation vector is that its behavior near a
physical singularity does not depend on the spacetime
dimension \(D\) (\(ξ^\phi \propto r, a = 0, 1, \ldots, \theta_{D-3}\) and \(ξ^{0\theta-2} \propto \text{const}\)),
in contrast to longitudinal component \(ξ^r \propto \frac{1}{r^{D-2}}\).

5. Conclusions

In this paper, we consider the geodesic deviation equation
(7) in the D-dimensional spherically symmetric Tangher-
lini spacetime. Firstly, we showed that a parallelpropagated
tetrad can be constructed. It describes the transition to a
reference frame where the equation acquires a digonal form
(11). This allowed us to obtain expressions (13)–(15) for all
spatial components of the tidal force. In Sect. (4), we
solved the geodesic deviation equations with respect to the
radial variable. We showed that the equations are inte-
rated in quadratures for any spacetime dimension (19) and
(20) for radial geodesics with \(L = 0\).

Secondly, we considered several particular metrics. We
started with the Schwarzschild–Tangherlini metric defined
in (22). It allows to express solutions in elementary func-
tions in five-dimensional spacetime (27) for radial geodesic
deviation vector component and (28) for angular ones,
while in six- and seven-dimensional spacetime solutions
are expressed in Weierstrass elliptic integrals (30), (31) and
Legendre elliptic integrals (33) and (34), respectively.
Thus, we presented new solutions of the geodesic deviation
equation in multidimensional Schwarzschild spacetimes
\(D = 5, 6, 7\) while the previously known solution was for
the 4-dimensional case in Ref. [1]. In the case of higher
dimensions, we expanded the solutions expressed in
quadratures into power series. As expected, at spatial
infinity, all components of the geodesic deviation vector
grow linearly with respect to \(r\) (36) and (37), regardless of
the dimension \(D\). This is due to the fact that the Sch-
warzschild–Tangherlini metric is asymptotically flat. On
the contrary, in the vicinity of a physical singularity, there
is a dependence on \(r\) of geodesic deviation vector behavior.
Thus for \(r \to 0\) radial component \(ξ^r \propto \frac{1}{\sqrt{r}}\), and angular
components \(ξ^a \propto r, a = 0, 1, \ldots, \theta_{D-2}\). That is, as we
approach the physical singularity, \(ξ^r\) grows indefinitely,
and all \(ξ^a\) shrinks to zero. Therefore, we can conclude that
test bodies moving in a gravitational field will experience
stretching along the radial direction in higher-dimensional
spaces. Tidal compression does not depend on the dimen-
sion of space in transverse angular directions, but the
number of these components is determined by \(D\), therefore,
it is determined by the number of directions along which
the test body is comprissed.

Next, we concentrate on Reissner–Nordström–
Tangherlini metric defined by (40). Explicit solution in 4-
dimensional spacetime was described in [8]. We have
succeeded in obtaining solutions of the geodesic deviation
equation in five-dimensional spacetime in terms of
Legendre elliptic integrals (45) and (46). For dimensions
greater than five, we present the solutions as power series.
Thus, at spatial infinity, we once again obtain a linear
growth of all deviation vector components (48) and (49),
because the presence of an electric charge has not change
the asymptotic flatness of spacetime. But in the vicinity of
a physical singularity \(r = 0\) geodesic deviation vector
behavior depends on spacetime dimension. So, according
to (50) radial component \(ξ^r \propto \frac{1}{\sqrt{r}}\), and, according to (51),
angular components \(ξ^a \propto r, a = 0, 1, \ldots, \theta_{D-2}\). This
allowed us to notice that presence of electrical charge
changed the intensity of the longitudinal tension in the
leading order, but that it did not change the strength of
the transverse compression.

In Sect. (4.2), we generalized the solutions found in Ref.
[5] by solving the geodesic deviation equations (52) in the
presence of angular momentum \(L \neq 0\) using the Frobenius
method for equations of the Fuchs class, since these
equations have no solutions in quadratures. This allowed us
to find the behavior of solutions in the vicinity of the origin
\(r = 0\) in the leading order, taking into account the differ-
ence between the polar and azimuthal components in the
presence of nonzero angular momentum. So in the
Schwarzschild–Tangherlini metric according to (58b), (62b),
(66b) it was found
\(ξ^r \propto \frac{1}{\sqrt{r}}, ξ^a \propto r, ξ^{0\theta-2} \propto \text{const}, a = 0, 1, \ldots, \theta_{D-3}\).
That is, in the leading order, only the behavior of the azimuthal
\(ξ^{0\theta-2}\) component of the geodesic deviation vector has
changed, while the behavior of the radial \(ξ^r\) and all polar
\(ξ^a, a = 0, 1, \ldots, \theta_{D-3}\) components did not change. If a black
hole has an electric charge, in accordance with (71b),
(75b), (79b) it was found
\(ξ^r \propto \frac{1}{\sqrt{r}}, ξ^a \propto r, ξ^{0\theta-2} \propto \text{const}, a = 0, 1, \ldots, \theta_{D-3}\). Thus,
in both considered Tangerlini metrics, we found that the angular geodesic deviation vector components do not depend on the spacetime dimension in the leading order on $r$ in the vicinity of the black hole singularity. And only the radial components grow indefinitely as the radial variable $r$ decreases. This is in complete agreement with the conclusions of the work [5], where local solutions of Eq. (52) depending on $L$ were considered for the first time (only for the case $D = 4$). Therefore, a test body freely falling along a non-radial geodesic undergoes unlimited stretching along the radial direction, and compression along all transverse directions, except for the azimuthal one. In the present case, the intensity of radial stretching in the leading order depends on the presence of a black hole charge and spacetime dimension, while the intensity of transverse compression does not depend on the presence of a charge or on spacetime dimension.

Tidal forces are an interesting and not yet fully understood phenomenon of gravitational interaction. There are a large number of works devoted to the study of tidal forces near black holes or separately to the geometric properties of the geodesic deviation equation. However, there is still a huge number of unsolved problems related to tidal forces. For example, no one has yet considered the tidal effects on spacetime dimension, while the intensity of transverse compression does not depend on the presence of a charge or on spacetime dimension.

Elliptic integrals

The simplest indefinite elliptic integral is the expression

$$\int R(z, y(z)) \, dz,$$

(85)

where $y(z) = \sqrt{a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4}$ and $R(z, y)$ is rational function of two variables. Any elliptic integral can be brought to the form

$$\int R_1(z) \, dz + \int \frac{R_2(z)}{y(z)} \, dz,$$  

(86)

where $R_1(z)$ and $R_2(z)$ are rational functions of one variable. Linear fractional transformation

$$z = \frac{ax + b}{cx + d}$$

(87)

can turn the second term of (86) into

$$\int \frac{\hat{R}_2(x)}{\hat{y}(x)} \, dx,$$  

(88)

where $\hat{R}_2(x)$ is another rational function, and $\hat{y}(x)$ is reduced to Weierstrass form

$$\hat{y}(x) = \sqrt{4x^3 - g_2 x - g_3},$$

(89)

or Legendre form

$$\hat{y}(x) = \sqrt{(1-x^2)(1-k^2x^2)}.$$  

(90)

As a result, expression (88) can be represented as a sum of integrals of three kinds.

- In Weierstrass form integrals of three kinds are
Tidal properties of $D$-dimensional

\[
J_0(g_2, g_3) = \int \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}, \quad (91a)
\]

\[
J_1(g_2, g_3) = \int \frac{xdx}{\sqrt{4x^3 - g_2x - g_3}}, \quad (91b)
\]

\[
H(x|g_2, g_3) = \int \frac{dx}{(x-c)\sqrt{4x^3 - g_2x - g_3}}. \quad (91c)
\]

- In Legendre form integrals of three kinds are

\[
F(x, k) = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad (92a)
\]

\[
E(x, k) = \int \frac{1-k^2x^2}{1-x^2}dx, \quad (92b)
\]

\[
\Pi(x, k, c) = \int \frac{dx}{(1-cx^2)(1-x^2)(1-k^2x^2)}. \quad (92c)
\]

Where $c$ is some pole of the function $\tilde{K}_2(x)$, which can be complex.

**Acknowledgements** We would like to express our gratitude to Yuri Viktorovich Pavlov and Polina Igorevna Kakin for meaningful discussions and useful advice.

**References**

[1] H Fush *Ann. Physik* **495** 231 (1983).
[2] K Schwarzschild and B Sitzungsbesichte *Phys. Math. Klasse* 189 (1916).
[3] S L Bazanski and P J Jaranowski *Math. Phys.*** **30** 1794 (1989).
[4] J P Luminet and J A Marck *Mon. Not. R. Astron. Soc.* **212** 57 (1985).
[5] V P Vandeve and A N Semenova *Eur. Phys. J. C*** **137** 185 (2022).
[6] H Reissner *Ann. Physik* **50** 106 (1916).
[7] G Nordström *Proc. Kon. Ned. Akad. Wet.* **20** 1238 (1918).
[8] L C B Crispino et al *Eur. Phys. J. C*** **76** 168 (2016).
[9] F Kottler *Ann. Physik* **56** 401 (1918).
[10] V P Vandeve and A N Semenova *Eur. Phys. J. C*** **81** 610 (2021).
[11] M Cariglia, T Houri, P Krtouš et al *Eur. Phys. J. C*** **78** 661 (2018).
[12] M U Shahzad and A Jawad *Eur. Phys. J. C*** **77** 372 (2017).
[13] V Kiselev *Class. Quant. Grav.* **20** 1187 (2003).
[14] H C D Lima Junior, M M Correa, C F B Macedo et al *Eur. Phys. J. C*** **82** 479 (2022).
[15] M Sharif and S Sadiq *JETP* **153** 232 (2018).
[16] H C Lima and L C Crispino *Int. J. Mod. Phys. D*** **29** 2041014 (2020).
[17] A Goel, R Rma, P Roy and T Sarkar *Phys. Rev. D*** **91** 104029 (2015).
[18] S Madan and P Bambahaniya, https://doi.org/10.48550/ arXiv.2201.13163.
[19] Soon-Tae Hong et al *Phys. Lett. B*** **881** 135967 (2020).
[20] Jing Li et al *Eur. Phys. J. C*** **81** 590 (2021).
[21] J Liu, S Chen and J Jing *Chin. Phys. C*** **46** 105104 (2022).
[22] R P Kerr *Phys. Rev. Lett.* **11** 237 (1963).
[23] H C Lima and L C Crispino *Eur. Phys. J. Plus*** **135** 334 (2020).
[24] R Uniyal *Eur. Phys. J. C*** **82** 567 (2022).
[25] C Chicone, B Mashhoon and B Punsly *Int. J. Mod. Phys. D*** **13** 945 (2004).
[26] C Chicone and B Mashhoon *Ann. Phys.* **14** 290 (2005).
[27] D Bini, C Chicone and B Mashhoon *Phys. Rev. D*** **95** 104029 (2017).
[28] M F Shirokov *Gen. Rel. Grav.* **4** 131 (1973).
[29] A Nduka *Gen. Rel. Grav.* **8** 347 (1977).
[30] E Y Melikumova et al *Sov. Phys. J.*** **33** 349 (1990).
[31] D Philipp, V Perlick, C Lammerzahl and K Deshpande *IEEE Metrol. Aerosp. (MetroAeroSp.)* **2015** 198 (2015).
[32] F R Tangherlini *Nuovo Cim.* **27** 636 (1963).
[33] C W Misner, K S Thorne and J A Wheeler *Gravitation* (San Francisco: W.H. Freeman) (1973).
[34] D E Hodgkinson *Gen. Rel. Grav.* **3** 351 (1972).

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.