Mean values of local operators in highly excited Bethe states

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Abstract. We consider expectation values of local operators in (continuum) integrable models in a situation when the mean value is calculated in a single Bethe state with a large number of particles. We develop a form factor expansion for the thermodynamic limit of the mean value, which applies whenever the distribution of Bethe roots is given by smooth density functions. We present three applications of our general result: (i) in the framework of integrable quantum field theory (IQFT) we present a derivation of the LeClair–Mussardo formula for finite temperature one-point functions. We also extend the results to boundary operators in boundary field theories. (ii) We establish the LeClair–Mussardo formula for the non-relativistic 1D Bose gas in the framework of the algebraic Bethe ansatz (ABA). This way we obtain an alternative derivation of the results of Kormos et al for the (temperature dependent) local correlations using only the concepts of the ABA. (iii) In IQFT we consider the long-time limit of one-point functions after a certain type of global quench. It is shown that our general results imply the integral series found by Fioretti and Mussardo. We also discuss the generalized eigenstate thermalization hypothesis in the context of quantum quenches in integrable models. It is shown that a single mean value always takes the form of a thermodynamic average in a generalized Gibbs ensemble, although the relation to the conserved charges is rather indirect.

Keywords: correlation functions, form factors, integrable quantum field theory, quantum integrability (Bethe ansatz)

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1. Introduction and our main result

One of the central tasks of many-body quantum physics is the calculation of correlation functions. In addition to the ground state properties it is also important to consider correlations calculated in excited states. These quantities are relevant in (at least) two physical situations: they describe the finite temperature properties of the system and they can be used to calculate observables after a sudden quantum quench [1]. Besides calculating the space and time dependent correlations it is also important to consider the one-point functions, i.e. the mean values of local operators. Important examples include the (properly regularized) products of field operators, which can describe densities of conserved charges or certain amplitudes of inelastic many-body processes [2, 3].

In this paper we consider one-point functions in one-dimensional integrable models. These theories can be solved by the Bethe ansatz, which means that the spectrum and also the thermodynamic quantities can be determined exactly. Moreover, there are powerful methods available to obtain correlation functions. One of the most general methods is the so-called form factor approach, which consists of two main steps. The first step is the calculation of the matrix elements (form factors) of the local operators in the eigenstate basis of the Hamiltonian, which can then be used to obtain the correlation function as the sum of a spectral series.

In this work we elaborate on the form factor expansion for one-point functions, more specifically we consider mean values in highly excited states with a smooth distribution.
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of Bethe roots. We only consider theories with purely diagonal scattering, but otherwise the calculations are quite general. We consider both relativistic quantum field theories (QFT’s) and non-relativistic many-body systems in the framework of the algebraic Bethe ansatz. There are three main sources of motivation for the present work.

The first motivation comes from integrable QFT, where the form factors are obtained as solutions to the so-called form factor bootstrap program [4]–[7]. Originally the form factor approach was developed to obtain two-point functions at zero temperature, but it was hoped that the form factors could be used to describe finite temperature correlations too [8]–[12]. The formal Boltzmann averages are plagued with divergences, which arise from singularities of the form factors and different contributions to the partition function. It is fairly easy to show that the divergences cancel, but the determination of the left-over pieces is in general a demanding problem.

In [9] LeClair and Mussardo proposed compact integral formulas for the finite temperature correlations, which involve the (appropriately regularized) zero-temperature form factors and the occupation numbers as determined from the thermodynamic Bethe ansatz (TBA). In the case of two-point functions the results of [9] were questioned in [10,13]. On the other hand, it was proven in [10] that for one-point functions the LeClair–Mussardo (LM) formula is correct for operators describing densities of conserved quantities. Moreover, it was shown in [14] using a finite volume regularization that the LM series is valid for arbitrary operators up to the first three non-trivial orders. However, the methods of [14] can only be applied order-by-order, and the calculations become rather tedious for higher orders, therefore they are not suitable for an all-order-proof of the LM formula.

The second motivation for the present work comes from recently discovered relations between integrable QFT and non-relativistic models of the algebraic Bethe ansatz (ABA). In [15,16], a non-relativistic limit of the sinh–Gordon field theory was applied to obtain physical observables in the Lieb–Liniger model [17,18]. They performed the non-relativistic limit of the LM series for certain sinh–Gordon operators to arrive at the (temperature dependent) local correlations in the Bose gas. It was pointed out later in [19] that the non-relativistic limit not only applies to the physical observables, but also to the individual form factors entering the integral series. This means that the final results of [15,16] can be formulated using only the quantities calculated in the Bose gas, without actually referring to the sinh–Gordon model. Evidently, this suggests that the LeClair–Mussardo formalism is not at all limited to relativistic field theory and it could be applicable to other Bethe ansatz solvable models too. We stress that although there is a vast literature devoted to correlation functions in the Bose gas, the papers [15,16] were apparently the first ones to calculate the many-body local correlations using Bethe ansatz techniques.

It was a vital point of [15,16] to apply the LeClair–Mussardo formalism in the case of an arbitrary prescribed particle density, or in other words in the presence of a non-zero chemical potential $\mu$. Although this seems to be quite natural, we remark that neither the original derivation in [9], nor the partial proof of [14] applies in this case. Indeed, the calculation of [14] heavily relied on the fact that if $\mu = 0$, then the particle density $\rho = N/L$ can be expanded into a low-temperature expansion and that $\rho(T = 0) = 0$. On the other hand, the success of [15,16] suggests that it should be possible to prove
the LM series using a different approach, which should work at arbitrary \(\mu\), and both in relativistic QFT and in the algebraic Bethe ansatz.

The third motivation for the present work is the quench problem of Fioretto and Mussardo [20], where the authors consider the long-time limit of one-point functions after a sudden global quench. The resulting integral series is a generalization of the LM formula, where the statistical weight functions do not follow from thermodynamic quantities, but from the microscopic amplitudes of the quantum quench and a generalized TBA-like dressing procedure. Once again, the compact form of the final result suggests that it should be possible to derive it using alternative methods, bypassing the combinatorial difficulties encountered in [20].

In this paper we approach the above mentioned problems using a finite volume regularization for the mean values. We start from the Boltzmann average for thermal one-point functions in a finite volume \(L\), which can be written as

\[
\langle O \rangle_T = \frac{\sum_i \langle i | O | i \rangle L e^{-E_i/T}}{\sum_i e^{-E_i/T}} \tag{1.1}
\]

where the summation runs over a complete set of states. Note that contrary to the infinite volume case the above formula is perfectly well defined in finite volume. For simplicity we assume that there is only one particle type in the spectrum; the generalization to arbitrary diagonal scattering theories is straightforward. In this case the Bethe equations (the quantization conditions for the finite volume states \(|i\rangle_L\)) can be written as

\[
e^{ip(\theta_j)L} \prod_{k \neq j} S(\theta_j - \theta_k) = 1, \quad j = 1 \cdots N.
\]

Here \(S(\theta)\) is the elastic \(S\)-matrix, which is a pure phase in this case. The variables \(\theta_j\) are the rapidity parameters and one-particle momenta and energies are given by the functions \(p(\theta)\) and \(e(\theta)\). The results to be presented below apply both to relativistic and non-relativistic situations.

In [14] a low-temperature expansion of (1.1) was performed in massive integrable field theory. The derivation was built on the fact that if the chemical potential is zero then the contribution of the \(N\)-particle sector of the Fock space can be estimated as \((mL e^{-m/T})^N\), where \(L\) is the volume and \(m\) is the mass of the particle. Therefore, in the regime

\[
1 \ll mL \ll e^{m/T} \tag{1.2}
\]

the volume can be large enough to replace sums with appropriate integrals, and at the same time the relevant contributions in (1.1) would still come from sectors with a low number of particles. This was used in [14] to derive a systematic low-temperature expansion and the \(L \to \infty\) limit was taken at the end of the calculations. The same ideas were employed in the papers [21]–[24].

In the present work we pursue a complementary approach: we consider a very large volume such that there are a large number of particles present in the system. In this case it is expected that the thermal average will be dominated by those states in which the distribution of Bethe roots is described by the infinite volume TBA equations [25]–[27]. In fact the TBA yields densities which minimize the free energy functional for the partition function, and one can argue that the insertion of the local operator does not shift this saddle point solution. Moreover, one can show that in the thermodynamic limit it is sufficient to consider only one particular state and evaluate the mean value on this state.
Let us therefore introduce a ‘thermal state’ as

\[ |\Omega_L\rangle_L = |\theta_1, \ldots, \theta_N\rangle_L \quad L \langle \Omega | \Omega \rangle_L = 1, \]

where it is understood that the rapidities \( \theta_j \) satisfy the BA equations, the particle number is \( N = \lfloor \rho L \rfloor \) with \( \rho \) being the infinite volume particle density (\( \lfloor x \rfloor \) denotes the integer part of \( x \)), and the distribution of roots is given by the TBA equations (detailed definitions will be given below). Then the mean value (1.1) is given by the thermodynamic limit

\[ \langle O \rangle_T = \lim_{N,L \to \infty} \langle \Omega | O | \Omega \rangle_L. \quad (1.3) \]

In general the matrix elements with a large number of particles are complicated objects, and the evaluation of the thermodynamic limit can be a difficult problem [28]–[30]. However, in the case of mean values the situation is much simpler. In [14] it was shown that a diagonal matrix element is the sum of its properly defined disconnected terms, which can be expressed using form factors with a lower number of particles. Then the remaining task is to perform the thermodynamic limit for the disconnected terms.

It is useful to investigate the mean value in a more general situation. Let us consider a Bethe state \( |\psi\rangle_L \) in a large volume with \( N = \lfloor \rho L \rfloor \) particles. We assume that the distribution of Bethe roots is given by the functions \( \rho^o(\theta) \) and \( \rho^h(\theta) \), which describe the densities of occupied states and holes, respectively. They satisfy the constraint

\[ \rho^o(\theta) + \rho^h(\theta) = \frac{1}{2\pi} \varphi(\theta) + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \rho^o(\theta') \]

where \( \varphi = -i(d/d\theta) \log S(\theta) \) is the scattering kernel and \( p'(\theta) \) is the derivative of the one-particle momentum. The particle density is then given by the integral

\[ \rho = \int d\theta \rho^o(\theta). \]

We consider the thermodynamic limit of the mean value:

\[ \langle O \rangle = \lim_{N,L \to \infty} \langle \psi | O | \psi \rangle_L. \quad (1.5) \]

For the above limit we find our main result:

\[ \langle O \rangle = \sum_n \frac{1}{n!} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{j=1}^{n} f(\theta_j) \right) F^{O}_{2n,c}(\theta_1, \ldots, \theta_n). \quad (1.6) \]

Here \( F^{O}_{2n,c} \) are regularized evaluations of the diagonal form factors (to be defined in the main text) and the statistical weight functions are given by

\[ f(\theta) = \frac{\rho^o(\theta)}{\rho^o(\theta) + \rho^h(\theta)}. \quad (1.7) \]

The \( n \)th term in the integral series arises from the \( n \)-particle disconnected terms of the original matrix element (1.5). The form factors describe \( n \)-particle processes over the Fock vacuum and the only effect of the non-trivial background is the appearance of the weight functions \( f(\theta) \). Remarkably (1.6) holds both in massive relativistic field theories and in the non-relativistic 1D Bose gas, which is a gapless system. This will be proven in sections 2 and 3, respectively. In the latter case (1.6) can be considered as an (improved) \( 1/\gamma \) expansion, where \( \gamma \) is the dimensionless effective coupling of the Bose gas.
In the finite temperature problem one introduces the pseudo-energy function $\varepsilon(\theta)$ as

$$\frac{\rho^{(o)}(\theta)}{\rho^{(h)}(\theta)} = e^{-\varepsilon(\theta)},$$

where $\varepsilon(\theta)$ is determined by the TBA equation

$$T\varepsilon(\theta) = e(\theta) - \mu - T \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \log(1 + e^{-\varepsilon(\theta')}),$$

with $e(\theta)$ being the one-particle energy. Applying (1.6) in this case one obtains

$$\langle O \rangle_{T=0,\mu} = \sum_{m} \frac{1}{n!} \int_{-B}^{B} \frac{d\theta_1}{2\pi} \cdots \int_{-B}^{B} \frac{d\theta_n}{2\pi} \left( \prod_{j} \frac{1}{1 + e^{\varepsilon(\theta_j)}} \right) F_{2n,c}^{O}(\theta_1, \ldots, \theta_n),$$

which is just the LeClair–Mussardo series as originally proposed in [9]. Note that the general result (1.6) allows for an arbitrary chemical potential $\mu$. As a special case one can take the zero-temperature limit with a fixed particle density to obtain

$$\langle O \rangle_{T=0,\mu} = \sum_{k} \frac{1}{n!} \int_{-B}^{B} \frac{d\theta_1}{2\pi} \cdots \int_{-B}^{B} \frac{d\theta_n}{2\pi} F_{2n,c}^{O}(\theta_1, \ldots, \theta_n),$$

where $B$ is the Fermi-rapidity. This remarkably simple and intuitive result was written down for the first time in [15, 16]; it applies whenever the ground state with a fixed number of particles is unique.

Let us consider the (typically infinite) set of local conserved quantities $Q_i$ with $i = 1, 2, \ldots, \infty$. The first two members of the series can be chosen as the particle number and the energy. If the one-particle eigenvalues are given by the functions $q_i(\theta)$ then the macroscopic values of the charges are given by the integrals

$$Q_i = L \int d\theta \rho^{(o)}(\theta) q_i(\theta).$$

Assuming that there is a one-to-one correspondence between the root densities and the conserved charges (i.e. the relation (1.11) can be inverted) the main result (1.6) can be interpreted as a ‘generalized eigenstate thermalization hypothesis’ [31]: the mean value always takes the form of a thermodynamic average and the weight functions only depend on the macroscopic value of the conserved charges.

The remainder of the paper will be devoted to the proofs and applications of our general formula (1.6). In section 2 we establish (1.6) by performing the thermodynamic limit of the mean value in relativistic QFT. In section 2.2 we extend the results to boundary operators in boundary QFT and prove the corresponding LeClair–Mussardo formula proposed in [32]. In section 3 we apply our formalism to the Lieb–Liniger model in the framework of the algebraic Bethe ansatz. In section 4 we explain how to apply our general formula to quench problems. We re-derive the results of Fioretto and Mussardo [20] and we also discuss more general quench situations. Section 5 includes our conclusions, and a number of technical details are presented in the appendices.

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2. Form factors and expectation values

Let us consider a massive integrable QFT. The basic object of the theory is the factorized $S$-matrix \[33,34\]. For simplicity we assume that there is only one particle type in the spectrum with no internal degrees of freedom. In this case the $S$-matrix is a pure phase:

$$S(\theta_i - \theta_j) = e^{i\vartheta(\theta_i - \theta_j)}.$$ 

One-particle energy and momentum are given by the functions

$$e(\theta) = m \cosh \theta \quad \text{and} \quad p(\theta) = m \sinh \theta.$$ 

In infinite volume the Hilbert space is spanned by the asymptotic states $|\theta_1, \ldots, \theta_N\rangle$.

Their energy and momentum (and higher conserved charges) can be calculated additively.

Consider a local operator $O(x, t)$. The form factors of $O$ are defined as \[5\]

$$F_{O, M,N}^{\theta_1', \ldots, \theta_M'}(\theta_1', \ldots, \theta_N') = \langle \theta_1', \ldots, \theta_M' | O | \theta_1, \ldots, \theta_N \rangle.$$ (2.1)

With the help of the crossing relations

$$F_{O, M,N}^{\theta_1', \ldots, \theta_M'}(\theta_1', \ldots, \theta_N') = F_{O, M-1,N+1}^{\theta_1', \ldots, \theta_M'}(\theta_1', \ldots, \theta_M' + i\pi, \theta_1, \ldots, \theta_N)$$

$$+ \sum_{k=1}^{N} 2\pi \delta(\theta_M' - \theta_k) \prod_{l=1}^{k-1} S(\theta_l - \theta_k) F_{M-1,N-1}^{\theta_1', \ldots, \theta_M'}(\theta_1', \ldots, \theta_k-1, \theta_k+1, \ldots, \theta_N)$$

$$\times (\theta_1', \ldots, \theta_M' - 1| \theta_1, \ldots, \theta_{k-1}, \theta_{k+1}, \ldots, \theta_N)$$ (2.2)

all form factors can be expressed in terms of the elementary form factors

$$F_{N}^{O}(\theta_1, \ldots, \theta_N) = \langle 0 | O(0,0) | \theta_1, \ldots, \theta_N \rangle$$ (2.3)

which satisfy the following equations:

(I) Lorentz transformation:

$$F_{N}^{O}(\theta_1 + \Lambda, \theta_2 + \Lambda, \ldots, \theta_N + \Lambda) = \exp(s_O \Lambda) F_{N}^{O}(\theta_1, \theta_2, \ldots, \theta_N)$$ (2.4)

where $s_O$ denotes the Lorentz spin of the operator $O$.

(II) Exchange:

$$F_{N}^{O}(\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_N) = S(\theta_k - \theta_{k+1}) F_{N}^{O}(\theta_1, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_N).$$ (2.5)

(III) Cyclic permutation:

$$F_{N}^{O}(\theta_1 + 2i\pi, \theta_2, \ldots, \theta_N) = F_{N}^{O}(\theta_2, \ldots, \theta_N, \theta_1).$$ (2.6)

(IV) Kinematical singularity

$$-i \text{Res}_{\theta = \theta'} F_{N+2}^{O}(\theta + i\pi, \theta', \theta_1, \ldots, \theta_N) = \left(1 - \prod_{k=1}^{N} S(\theta' - \theta_k)\right) F_{N+1}^{O}(\theta_1, \ldots, \theta_N).$$ (2.7)
There is also a further equation related to bound states which we do not need in the sequel.

In order to derive a regularized form factor expansion for the mean values we consider the theory in a finite volume \( L \). Assuming periodic boundary conditions the Bethe equations are written as

\[
e^{p_j L} \prod_{k \neq j} S(\theta_j - \theta_k) = 1, \quad j = 1 \cdots N,
\]

where \( p_j = p(\theta_j) \). In the logarithmic form:

\[
Q_j = p_j L + \sum_{k \neq j} \vartheta(\theta_j - \theta_k) = 2\pi I_j \quad j = 1 \cdots N, \quad I_j \in \mathbb{Z}.
\]

(2.8)

The multi-particle energies and momenta are calculated additively:

\[
P_N = \sum_j p(\theta_j) \quad E_N = \sum_j e(\theta_j).
\]

In the following we denote the finite volume states as

\[
|\theta_1, \ldots, \theta_N\rangle_L
\]

where it is understood that the rapidities solve equation (2.8).

It is a very important task to determine the finite volume form factors in terms of the infinite volume quantities. In the case of relativistic QFT this problem was solved in [35,14]. In a generic case the result reads

\[
\langle \theta'_1, \ldots, \theta'_M | \mathcal{O} | \theta_1, \ldots, \theta_N \rangle_L = \frac{F_{O,N,M}(\theta'_1, \ldots, \theta'_M | \theta_1, \ldots, \theta_N)}{\sqrt{\rho_M(\theta'_1, \ldots, \theta'_M) \rho_N(\theta_1, \ldots, \theta_N)}} + \mathcal{O}(e^{-\mu L}). \quad (2.9)
\]

The quantities \( \rho_N \) and \( \rho_M \) are the multi-particle density of states and they are given by

\[
\rho_N(\theta_1, \ldots, \theta_N) = \det J^{ij}, \quad J^{ij} = \frac{\partial Q_j}{\partial \theta_i}.
\]

(2.10)

The error exponent \( \mu \) is universal in the sense that it only depends on the analytic properties of the \( S \)-matrix and not on the particular form factor in question. Equation (2.9) is very natural in the algebraic Bethe ansatz approach for models with \( sl(2) \) or \( U_q(sl(2)) \) symmetry [36]. In these models (2.9) is exact and \( \rho_N \) is just the Gaudin determinant, which describes the norm of the Bethe wavefunction.

The relation (2.9) is valid when there are no kinematical poles present due to colliding rapidities. It was pointed out in [14] that there are only two situations when rapidities do coincide in a finite volume. One possibility is to have zero-momentum particles in parity symmetric states; these matrix elements will not be needed here, therefore we omit the details which can be found in [14]. The other case is the problem of diagonal matrix elements (expectation values), which will be discussed below. Note that the diagonal limit in (2.9) is ill defined due to the multiple kinematical poles prescribed by (2.7).

In the following we restate the relevant results of [14]. Consider the diagonal form factor of \( N \) particles in infinite volume:

\[
F_{O,2N}^{\mathcal{O}}(\theta_1 + \varepsilon_1, \ldots, \theta_N + \varepsilon_N | \theta_N, \ldots, \theta_1).
\]

(2.11)
Here the singularities have been shifted off by the infinitesimal quantities $\varepsilon_i$. It was proven in [14] that there exists a finite limit when all $\varepsilon_i$ go to zero simultaneously with their ratios fixed. Moreover, there are two special evaluation schemes which respect the physical requirement that the diagonal form factors should not depend on the order of the rapidities. First of all, one can consider the symmetric limit

\[
F_{O_{2N,s}}(\theta_1, \ldots, \theta_N) \equiv \lim_{\varepsilon \to 0} F_{O_{2N}}(\theta_1 + \varepsilon, \ldots, \theta_N + \varepsilon | \theta_N, \ldots, \theta_1).
\]

(2.12)

On the other hand, one can also consider the connected part of the diagonal form factor, which is defined to be the contribution to (2.11) which contains no singular factors of the form $\varepsilon_i/\varepsilon_j$ and products thereof:

\[
F_{O_{2N,c}}(\theta_1, \ldots, \theta_N) \equiv \text{(finite part of) } F_{O_{2N}}(\theta_1 + \varepsilon_1, \ldots, \theta_N + \varepsilon_N | \theta_N, \ldots, \theta_1).
\]

(2.13)

The general structure of the singularities in (2.11) was worked out in [14]. For future use we recall the relation between the symmetric and connected evaluation schemes. First we introduce the necessary notations.

Let us take $n$ vertices labeled by the numbers $1, 2, \ldots, n$ and let $G$ be the set of the directed graphs $G_i$ with the following properties:

- $G_i$ is tree-like.
- For each vertex there is at most one outgoing edge.

For an edge going from $i$ to $j$ we use the notation $E_{ij}$.

**Theorem 2.1.** The function $F_{O_{2N,s}}(\theta_1, \ldots, \theta_N)$ can be evaluated as a sum over all graphs in $G$, where the contribution of a graph $G_i$ is given by the following two rules:

- Let $A_i = \{a_1, a_2, \ldots, a_m\}$ be the set of vertices from which there are no outgoing edges in $G_i$. The form factor associated with $G_i$ is $F_{O_{2m,c}}(\theta_{a_1}, \theta_{a_2}, \ldots, \theta_{a_m})$.

(2.14)

- For each edge $E_{jk}$ the form factor above has to be multiplied by $\varphi(\theta_j - \theta_k)$.

This theorem was proven in [14] using the kinematical singularity axiom (2.7).

Now we are in a position to state one of the central results of [14], which is the following relation between diagonal form factors in infinite and finite volume:

\[
\langle \theta_1, \ldots, \theta_N | \mathcal{O} | \theta_1, \ldots, \theta_N \rangle_L = \frac{1}{\rho_N(\theta_1, \ldots, \theta_N)} \sum_{\{\theta_+\} \cup \{\theta_\} \cup \{\theta_\}} F_{O_{2n,s}}^{\mathcal{O}}(\{\theta_\}) \rho_{N-n}(\{\theta_+\}).
\]

(2.15)

The sum runs over all possible bipartite partitions. Similar expansions are known for mean values in the algebraic Bethe ansatz literature [37]–[39], [18]. In that context the symmetric evaluation is called the ‘irreducible part’ of the form factor. More details about the relation of our expansions to those in the ABA are presented in section 3.

A very important result of [14] is that (2.15) can be expressed alternatively in terms of the connected evaluation of the diagonal form factors. To restate this result we introduce a restricted Gaudin determinant as follows. For a given bipartite partition

\[
\{\theta_1, \ldots, \theta_N\} = \{\theta_+\} \cup \{\theta_-\}
\]

\[
|\{\theta_+\}| = N - n \quad \text{and} \quad |\{\theta_-\}| = n
\]
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we define the restricted determinant
\[
\bar{\rho}_{N-n}(\{\theta_+\}|\{\theta_-\}) = \det \mathcal{J}_+
\]  
(2.16)

where \( \mathcal{J}_+ \) is the sub-matrix of \( \mathcal{J} \) corresponding to the particles in the set \( \{\theta_+\} \). Note that \( \bar{\rho}_{N-n}(\{\theta_+\}|\{\theta_-\}) \) still contains information about the complementary set of rapidities \( \{\theta_-\} \), and it should not be confused with the density \( \rho_{N-n}(\{\theta_+\}) \), which depends only on the rapidities \( \{\theta_+\} \).

With these notations, the alternative expression for the expectation value reads
\[
\langle O|\theta_1,\ldots,\theta_N\rangle_L = \frac{1}{\rho_N(\theta_1,\ldots,\theta_N)} \sum_{\{\theta_+\}\cup\{\theta_-\}} F_{2n,c}^O(\{\theta_-\}) \bar{\rho}_{N-n}(\{\theta_+\}|\{\theta_-\}).
\]
(2.17)

The equivalence of (2.15) and (2.17) is proven in theorem 2 of [14]. We wish to note that the two relations are expected to be exact to all orders in \( 1/L \), but there are residual finite size effects of order \( \mathcal{O}(e^{-\mu L}) \). The general structure of exponential corrections to form factors is not yet known, however it is expected that the growth of the coefficients of the \( \mathcal{O}(e^{-\mu L}) \) terms with any \( \mu' \geq \mu \) is only polynomial in \( N \). Therefore it is safe to neglect them in taking the thermodynamic limit of (2.15) and (2.17).

2.1. Evaluation of the thermodynamic limit

We consider the diagonal matrix element
\[
\langle O|_{N,L} = \langle \theta_1,\ldots,\theta_N|O|\theta_1,\ldots,\theta_N\rangle_L
\]
(2.18)

with a large number of particles in a large volume \( L \). The idea is to evaluate expressions (2.15) and (2.17) in the \( L \to \infty \) limit assuming that the distribution of roots is given by the smooth functions \( \rho^{(o)}(\theta) \) and \( \rho^{(h)}(\theta) \). They describe the density of occupied roots and holes, respectively. They satisfy the constraint
\[
\rho^{(o)}(\theta) + \rho^{(h)}(\theta) = \frac{p'(\theta)}{2\pi} + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \rho^{(o)}(\theta'),
\]
(2.19)

which follows from the thermodynamic limit of the Bethe equations. The kernel in (2.19) is given by \( \varphi = -i(d/d\theta) \log S(\theta) \) and \( p(\theta) \) is the one-particle momentum. The total particle density is given by the integral
\[
\rho = \frac{N}{L} = \int d\theta \rho^{(o)}(\theta).
\]

Let us introduce the quantities
\[
C_{n,L} = \sum_{|\{\theta_-\}|=n} F_{2n,c}^O(\{\theta_-\}) \bar{\rho}_{N-n}(\{\theta_+\}|\{\theta_-\}) \rho_{N}(\theta_1,\ldots,\theta_N)
\]
\[
D_{n,L} = \sum_{|\{\theta_-\}|=n} F_{2n,s}^O(\{\theta_-\}) \rho_{N-n}(\{\theta_+\}) \rho_{N}(\theta_1,\ldots,\theta_N)
\]

such that
\[
\langle O\rangle_{N,L} = \sum_{n=0}^{N} C_{n,L} = \sum_{n=0}^{N} D_{n,L}.
\]

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It will be shown that both $C_{n,L}$ and $D_{n,L}$ possess a well-defined $L \to \infty$ limit for arbitrary $n$. This way the expectation value is expressed as

$$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} D_n$$

where

$$C_n = \lim_{L \to \infty} C_{n,L} \quad D_n = \lim_{L \to \infty} D_{n,L}.$$ 

We assume that performing the summation over $n$ and taking the thermodynamic limit can be exchanged. Note that the total number of $C_{n,L}$ and $D_{n,L}$ depends on $L$, because $n \leq \lfloor \rho L \rfloor$. However this is not a serious problem, because for any $n$ there will be a large enough volume with $N > n$, so that $C_{n,L}$ will appear in the sum; moreover it will have a finite limit as $L$ is sent further to infinity.

The $n = 0$ term reproduces the infinite volume vacuum expectation value:

$$C_{0,L} = D_{0,L} = \langle 0 | \mathcal{O} | 0 \rangle.$$ 

In the present approach this vacuum expectation value is interpreted as the maximally disconnected part of the diagonal form factor.

In the case of $n = 1$ the quantities $C_{1,L}$ and $D_{1,L}$ are expressed as a single sum over $\theta_j \in \{\theta_1, \ldots, \theta_N\}$:

$$C_{1,L} = \sum_{\theta_j} F_{2k,c}(\theta_j) \frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N | \theta_j)}{\rho_N(\theta_1, \ldots, \theta_N)}$$

$$D_{1,L} = \sum_{\theta_j} F_{2k,s}(\theta_j) \frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N)}{\rho_N(\theta_1, \ldots, \theta_N)}.$$ 

Let us assume that the ratios of the determinants have a well-defined $L \to \infty$ limit while keeping $\theta_j$ fixed. Then the summation over $\theta_j$ can be expressed as an integral:

$$\sum_{\theta_j} \to \int d\theta \rho^{(o)}(\theta)L.$$ 

The ratios of the determinants scale as $1/L$, therefore $C_{1,L}$ and $D_{1,L}$ behave as $\mathcal{O}(L^0)$. This is expected in order to have a well-defined $L \to \infty$ limit.

The limiting values of the ratios of the determinants can be determined using the techniques of [38, 18]. In appendix A it is shown that

$$\lim \frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N | \theta_j)}{\rho_N(\theta_1, \ldots, \theta_N)} = \frac{1}{2\pi L(\rho^{(o)}(\theta_j) + \rho^{(h)}(\theta_j))}$$

$$\lim \frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N)}{\rho_N(\theta_1, \ldots, \theta_N)} = \frac{1}{2\pi L(\rho^{(o)}(\theta_j) + \rho^{(h)}(\theta_j))} \omega(\theta_j).$$ 

Here

$$\omega(\theta) = \exp \left( - \int \frac{d\theta'}{2\pi} f(\theta') \varphi(\theta - \theta') \right)$$
and the weight function $f(\theta)$ is defined as

$$f(\theta) = \frac{\rho^{(o)}(\theta)}{\rho^{(o)}(\theta) + \rho^{(h)}(\theta)}.$$ 

Putting everything together one obtains

$$C_1 = \lim_{L \to \infty} C_{1,L} = \int \frac{d\theta}{2\pi} f(\theta) F^{c}_{2n,c}(\theta),$$  

$$D_1 = \lim_{L \to \infty} D_{1,L} = \int \frac{d\theta}{2\pi} f(\theta) \omega(\theta) F^{s}_{2n,s}(\theta).$$  

One can repeat this line of reasoning for the case when a fixed ($n > 1$) number of rapidities are chosen to be in the subset \{\theta_\cdot\}, which are to be substituted into a diagonal 2n-particle form factors. It is easy to see that in the thermodynamic limit there will be no complications, in particular the application of the Pauli principle for the summation over \{\theta_\cdot\} only causes corrections of $\mathcal{O}(1/L)$. Therefore one can immediately write down the general result

$$C_n = \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{j=1}^{n} f(\theta_j) \right) F^{c}_{2n,c}(\theta_1, \ldots, \theta_n)$$  

and

$$D_n = \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{j=1}^{n} f(\theta_j) \omega(\theta_j) \right) F^{s}_{2n,s}(\theta_1, \ldots, \theta_n).$$

The expectation value of $\mathcal{O}$ is then given by

$$\langle \mathcal{O} \rangle = \sum_n \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{j=1}^{n} f(\theta_j) \right) F^{c}_{2n,c}(\theta_1, \ldots, \theta_n)$$

$$= \sum_k \frac{1}{n!} \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \left( \prod_{j=1}^{n} f(\theta_j) \omega(\theta_j) \right) F^{s}_{2n,s}(\theta_1, \ldots, \theta_n).$$

The two different expressions represent a rearrangement of terms, similar to the equivalence of (2.15) and (2.17). We wish to note that (2.23) is the main result of this work.

Both series in (2.23) are expected to be convergent. We are not able to present a rigorous proof, because the behavior of the form factors is in general not known. However, it is generally assumed that there exists a $K \in \mathbb{R}^+$ such that

$$|F^{c}_{n+m}(\theta'_1, \ldots, \theta'_n|\theta_1, \ldots, \theta_m)| < K^{n+m} \quad \text{for} \quad \theta_j, \theta'_j \in \mathbb{R}.\quad (2.24)$$

This implies that both functions $F^{c}_{2n,s}$ and $F^{c}_{2n,c}$ are bounded in the sense of (2.24) and the series (2.23) are convergent.
2.2. Boundary operators in boundary field theories

It is possible to extend the previous calculations to relativistic boundary QFT’s [40,41]. Consider an operator $\mathcal{O}(t)$ localized at the boundary of a half-infinite system. The boundary is supposed to be integrable; the scattering off the boundary is described by the elastic reflection factor $R(\theta)$. Asymptotic states can be defined similarly as in the bulk case; the only difference is that for incoming (outgoing) states all the rapidities are positive (negative). Form factors of $\mathcal{O}$ are then defined as

$$G^\mathcal{O}_N(\theta_1, \ldots, \theta_N) = \langle 0 | \mathcal{O} | \theta_1, \ldots, \theta_N \rangle.$$  

The analytic properties of the functions $G^\mathcal{O}_N$ (also called boundary form factor equations) were derived in [42]. For the sake of brevity we do not cite them here.

We will be interested in the mean values of the boundary operator $\mathcal{O}$. We start with a finite system of size $L$ with two integrable boundaries $a, b$ with elastic reflection factors $R_{a,b}(\theta)$; the operator $\mathcal{O}$ lives on the boundary $a$. The Bethe equations read

$$e^{iQ^B_B} \equiv e^{2i\varphi^L a R_a(\theta_j) R_b(\theta_j) \prod_{k \neq j} S(\theta_k - \theta_j) S(\theta_k + \theta_j)} = 1, \quad \theta_j > 0.$$  

The rapidities are restricted to take only positive values. The $N$-particle density of states is obtained as

$$\rho^B_N(\{\theta\}) = \det \mathcal{J}^B, \quad \mathcal{J}^B_{ij} = \frac{\partial Q^B_B}{\partial \theta_j}.$$  

Analogously to (2.16) we define a restricted density for a given partition $\{\theta\} = \{\theta_+\} \cup \{\theta_-\}$ as the sub-determinant belonging to the subset $\{\theta_+\}$:

$$\bar{\rho}^B_N(\{\theta_+\}, \{\theta_-\}) = \det \mathcal{J}^B_+.$$  

The relation between finite volume and infinite volume form factors was worked out in [43]. In the generic case with no coinciding rapidities they take the same form as (2.9) with the obvious replacements $F^\mathcal{O}_N \rightarrow G^\mathcal{O}_N$ and $\rho_N \rightarrow \rho^B_N$. For diagonal form factors one obtains

$$\langle \theta_1, \ldots, \theta_N | \mathcal{O} | \theta_1, \ldots, \theta_N \rangle_L = \frac{1}{\rho^B_N(\{\theta_+\}, \{\theta_-\})} \sum_{\{\theta_+\} \cup \{\theta_-\}} G^\mathcal{O}_N(\{\theta_-\}) \bar{\rho}^B_{N-n}(\{\theta_+\} \cup \{\theta_-\}), \quad (2.25)$$

where $G^\mathcal{O}_{2n,c}$ are the connected parts of the diagonal form factors defined analogously to (2.13). Clearly, (2.25) is a simple generalization of the relation in the bulk (2.17). It is also possible to define the symmetric evaluation of the boundary form factors as in (2.12), but the naive generalization of (2.15) does not hold in the boundary case, as explained in detail in [43].

We are interested in the thermodynamic limit of (2.25) along the lines of section 2.1. Using the techniques of appendix A it can be shown that the behavior of the ratios of determinants is the same as in the bulk. For example

$$\lim \frac{\bar{\rho}^B_N(\{\theta\} \cup \{\hat{\theta}_j\}, \{\theta_+\}, \{\theta_-\})}{\rho^B_N(\{\theta\}, \{\theta_+\}, \{\theta_-\})} = \frac{1}{2\pi L (\rho^o(\theta) + \rho^b(\theta))}.$$  

\(^1\) Form factors of boundary operators are denoted by $G^\mathcal{O}_N$ to distinguish them from the form factors of bulk operators.
where $\rho^{(o)}(\theta)$ and $\rho^{(h)}(\theta)$ describe the densities of occupied roots and holes, respectively. Therefore, the thermodynamic limit of (2.25) results in the series
\[
\langle \mathcal{O} \rangle = \sum_{n} \frac{1}{n!} \int_{0}^{\infty} \frac{d\theta_{1}}{2\pi} \cdots \int_{0}^{\infty} \frac{d\theta_{n}}{2\pi} \left( \prod_{j=1}^{n} f(\theta_{j}) \right) G_{2n,c}^{o}(\theta_{1}, \ldots, \theta_{n}),
\]
(2.26)
where
\[
f(\theta) = \frac{\rho^{(o)}(\theta)}{\rho^{(o)}(\theta) + \rho^{(h)}(\theta)}.
\]

In the case of finite temperature expectation values one can show that the distribution of roots will be given by the same functions as in the periodic case, as expected by thermodynamic arguments. Therefore
\[
\langle \mathcal{O} \rangle_{T,\mu} = \sum_{n} \frac{1}{n!} \int_{0}^{\infty} \frac{d\theta_{1}}{2\pi} \cdots \int_{0}^{\infty} \frac{d\theta_{n}}{2\pi} \left( \prod_{j=1}^{n} \frac{1}{1 + \exp(\theta_{j})} \right) G_{2n,c}^{o}(\theta_{1}, \ldots, \theta_{n}),
\]
(2.27)
where $\varepsilon(\theta)$ is the solution of the TBA with periodic boundary conditions (1.8). The series (2.27) was proposed in [32] and it was proven there up to third non-trivial order using the techniques of [14]. The all-orders proof together with the general formula (2.26) is a new result of this work.

3. Application to the 1D Bose gas

In this section we consider the 1D interacting Bose gas, also known as the Lieb–Liniger (LL) model [17, 18]. The second quantized form of the Hamiltonian in volume $L$ with periodic boundary conditions is given by
\[
H_{LL} = \int_{0}^{L} dx \left( \partial_{x} \Psi \partial_{x} \Psi^{\dagger} + c \Psi^{\dagger} \Psi \Psi^{\dagger} \Psi \right).
\]
(3.1)

Here $\Psi(x, t)$ and $\Psi^{\dagger}(x, t)$ are canonical non-relativistic Bose fields satisfying
\[
[\Psi(x, t), \Psi^{\dagger}(y, t)] = \delta(x - y),
\]
(3.2)
and $c$ is the coupling constant. We used the conventions $m = 1/2$ and $\hbar = 1$. The Fock vacuum is defined as
\[
\Psi \tilde{0} = 0, \quad \langle 0 | \Psi^{\dagger} = 0.
\]

The eigenstates of the Hamiltonian (3.1) can be constructed using the Bethe ansatz. The scattering of the particles is described by the two-particle $S$-matrix
\[
S_{LL} = \frac{\lambda - ic}{\lambda + ic},
\]
where $\lambda$ is the non-relativistic rapidity variable and multi-particle energies and momenta are given by
\[
E_{N} = \sum_{j} \lambda_{j}^{2} \quad P_{N} = \sum_{j} \lambda_{j}.
\]
Mean values of local operators in highly excited Bethe states

The algebraic Bethe ansatz (ABA) provides a framework to calculate form factors and correlation functions. However, before turning to the ABA solution we present the results of the papers [15, 16], where a non-relativistic limit of the sinh–Gordon theory was performed to obtain physical quantities in the LL model. They considered the temperature dependent expectation values

$$\langle O_k \rangle \equiv \langle \Psi_k^\dagger \Psi_k \rangle = g_k(T, c) \rho$$

where $\rho = N/L$ is the particle density. The dimensionless quantities $g_k$ are important for the phenomenology of the Bose gas, for example $g_3$ describes the recombination rate of the gas [2, 3].

The main result of [15, 16] is the integral series

$$\langle O_k \rangle = \sum_{N=k}^{\infty} \frac{1}{N!} \int \frac{d\lambda_1}{2\pi} \cdots \frac{d\lambda_N}{2\pi} \left( \prod_{j=1}^{N} \frac{1}{1 + e^{\varepsilon(\lambda_j)}} \right) F_{2N,c}^k(\lambda_1, \ldots, \lambda_N)$$

where $\varepsilon(\lambda)$ is the solution of the non-relativistic TBA equation

$$T \varepsilon(\theta) = \lambda^2 - \mu - T \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \log(1 + e^{-\varepsilon(\theta')}),$$

where

$$\varphi(\lambda) = -i \frac{\partial}{\partial \lambda} \log S_{\text{LL}} = \frac{2c}{c^2 + \lambda^2}.$$  

The functions $F_{2N,c}^k$ appearing in (3.3) were derived by the non-relativistic limit of certain form factors in the sinh–Gordon model. The first few examples were given explicitly in [16]. They vanish for $N < k$ and the asymptotic behavior at $c \to \infty$ is given by

$$F_{2N,c}^k \sim c^{-(k(k-2)+N)}.$$  

In the following we re-establish the series (3.3) using the methods of the algebraic Bethe ansatz [37–39], [18]. Also, we clarify the connection between the functions $F_{2N,c}^k$ above and the form factors calculated from the ABA.

In the ABA approach the central object is the monodromy matrix, which can be written as

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}.$$  

The commutation relations satisfied by the entries can be expressed in a compact form as

$$R(\lambda, \mu) T(\lambda) \otimes T(\mu) = T(\mu) \otimes T(\lambda) R(\lambda, \mu).$$

Here $R(\lambda, \mu)$ the R-matrix is of XXX-type:

$$R(\lambda, \mu) = \begin{pmatrix} f(\lambda, \mu) & 0 & 0 & 0 \\ 0 & g(\lambda, \mu) & 1 & 0 \\ 0 & 1 & g(\lambda, \mu) & 0 \\ 0 & 0 & 0 & f(\lambda, \mu) \end{pmatrix}$$

where

$$f(\lambda, \mu) = \frac{\lambda - \mu + ic}{\lambda - \mu}, \quad g(\lambda, \mu) = \frac{ic}{\lambda - \mu}.$$
The vacuum eigenvalues of the operators $A(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are
\[ C(\lambda)|0\rangle = 0 \quad A(\lambda)|0\rangle = a(\lambda)|0\rangle \quad D(\lambda)|0\rangle = d(\lambda)|0\rangle. \]

For definiteness we specify
\[ a(\lambda) = e^{-i\lambda L/2} \quad d(\lambda) = e^{i\lambda L/2}. \]

The function $l(\lambda) = a(\lambda)/d(\lambda) = e^{-i\lambda L}$ will be used extensively.

The Bethe states are then defined as the two sets of rapidities are solutions to the Bethe equations. In this case the variables $\lambda_j$ only depends on $2N$ variables given by $f^J_j = l(\lambda^C_j), f^K_j = l(\lambda^K_j)$. Note that $M^k_{2N} = 0$ if $k > N$.

The ‘on-shell’ matrix element or ‘form factor’ is defined as a special case of (3.9) when the two sets of rapidities are solutions to the Bethe equations. In this case the variables $l_j$ can be expressed in terms of the functions $f(\lambda_i, \lambda_j)$ and one obtains the function
\[ \mathbb{F}^k_{2N}(\{\lambda^C\}_N|\{\lambda^K\}_N) = M^k_{2N}(\{\lambda^C\}_N, \{\lambda^K\}_N, \{f^C\}_N, \{f^K\}_N) \]
which only depends on $2N$ variables.

It can be shown that the form factor has first order poles in the case of colliding rapidities. The residue at $\lambda^C_N \to \lambda^K_N$ is given by [39, 44]
\begin{align*}
\mathbb{F}^k_{2N}(\{\lambda^C\}_N|\{\lambda^K\}_N)_{\lambda^C_N \to \lambda^K_N} & \to \frac{i c}{\lambda^C_N - \lambda^K_N} \left( \prod_{j=1}^{N-1} f^K_j f^K_N - \prod_{j=1}^{N-1} f^K_j f^K_N \right) \\
& \times \mathbb{F}^k_{2N-2}(\{\lambda^C\}_{N-1}|\{\lambda^K\}_{N-1}).
\end{align*}
Mean values of local operators in highly excited Bethe states

Here $f_{B,C}^N = f(\lambda_B^C - \lambda_N^B)$. Other cases $\lambda_C^j \rightarrow \lambda_B^j$ follow from the symmetry properties of the form factor. Note that equation (3.10) does not depend on $k$, i.e. the singularity properties are the same for all $O_k$. Moreover, it can be shown that an analogous equation holds for the operators

$$O_{k,l} = (\Psi^\dagger)_k \Psi_l,$$

where $k, l \in \mathbb{N}$. In appendix B we present a general proof of (3.10).

Note that (3.10) has essentially the same structure as the kinematic singularity axiom (2.7) in relativistic QFT. There are two differences which are related to different normalizations of the Bethe vectors; this was pointed out in [19]. First of all, the norm of the states (3.7) is equal to

$$\langle \lambda_1, \ldots, \lambda_N | \lambda_1, \ldots, \lambda_N \rangle = \rho_N(\lambda_1, \ldots, \lambda_N).$$

Here $\rho_N(\lambda_1, \ldots, \lambda_N)$ is the Gaudin determinant:

$$\rho_N(\lambda_1, \ldots, \lambda_N) = \det J, \quad J_{jk} = \frac{\partial Q_j}{\partial \lambda_k}.$$

On the other hand, in the relativistic situation the norm of the finite volume states is given simply by the density of states $\rho_N$, as is implicitly assumed in (2.9).

The other issue is related to the exchange of rapidities in the Bethe state. The $B(\lambda)$ operators commute with each other, therefore the states (3.7) are totally symmetric with respect to the exchange of rapidities. This property also applies to the finite volume matrix elements in relativistic QFT, as it was emphasized in [35]. On the other hand, in relativistic QFT the infinite volume form factors satisfy the exchange property (2.5), which follows from the Faddeev–Zamolodchikov algebra. In the present case the $S$-matrix is a pure phase, therefore one can introduce the phase structure by multiplying with factors of $\sqrt{S(\lambda)}$. With a slight abuse of notation the differences between the normalization conventions can be summarized as [19]

$$|\lambda_1, \ldots, \lambda_N\rangle_{ABA} \sim c^{N/2} \left( \prod_{j<k} f_{jk} \right) |\lambda_1, \ldots, \lambda_N\rangle_{QFT}. \quad (3.12)$$

In order to make contact with the form factors appearing in the series (3.3) we define

$$F_{2N}^k(\{\lambda_C\}_N | \{\lambda_B\}_N) = \left( c^N \prod_{j<k} f_{jk}^B f_{jk}^C \right)^{-1} \mathbb{F}_{2N}^k(\{\lambda_C\}_N | \{\lambda_B\}_N). \quad (3.13)$$

The functions $F_{2N}^k$ satisfy the exchange property (2.5) and the non-relativistic version of the kinematical singularity axiom (2.7).

Now we are in a position to define the diagonal limits of the non-relativistic form factors. In accordance with (2.12) and (2.13) we introduce

$$F_{2N,s}^k(\lambda_1, \ldots, \lambda_N) \equiv \lim_{\varepsilon \to 0} F_{2N}^k(\lambda_1 + \varepsilon, \ldots, \lambda_N + \varepsilon | \lambda_N, \ldots, \lambda_1) \quad (3.14)$$

$$F_{2N,c}^k(\lambda_1, \ldots, \lambda_N) \equiv \text{finite part of} \ F_{2N}^k(\lambda_1 + \varepsilon_1, \ldots, \lambda_N + \varepsilon_N | \lambda_N, \ldots, \lambda_1). \quad (3.15)$$

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The properties of these functions are the same as in the relativistic case. Both are completely symmetric in their variables and the relation between $F_{2N,s}^k$ and $F_{2N,c}^k$ is given by theorem 2.1. We remark that the normalization in (3.13) commutes with the diagonal limit, because the extra factors do not introduce any new poles.

Now we turn our attention to the mean values of the operators $O_k$. One can define the diagonal limit of (3.9) as

$$M_N^k(\{\lambda\}_N, \{l\}_N, \{z\}_N) = \lim_{\lambda_j^C \to \lambda_j^B} M_N^k(\{\lambda^C\}_N, \{\lambda^B\}_N, \{l^C\}_N, \{l^B\}_N).$$

Formally it will depend on $3N$ independent variables $\{\lambda\}_N$, $\{l\}_N$ and $\{z\}_N$, where $z(\lambda) = \frac{d}{d\lambda} \log l(\lambda)$.

The expectation value is then given by the special case of (3.16) when the rapidities satisfy the Bethe equations; the remaining task is to express it in terms of the form factors defined in (3.14) and (3.15). The result for the normalized expectation value is given by the following formula:

$$\langle O_k \rangle_N = \frac{1}{\rho_N(\lambda_1, \ldots, \lambda_N)} \sum_{\lambda\cup (\lambda)} F_{2n,s}^k(\{\lambda\}) \rho_{N-n}(\{\lambda\})$$

$$= \frac{1}{\rho_N(\lambda_1, \ldots, \lambda_N)} \sum_{\lambda\cup (\lambda)} F_{2n,c}^k(\{\lambda\}) \bar{\rho}_{N-n}(\{\lambda\}).$$

(3.17)

Clearly, the above formulas are the non-relativistic versions of (2.15) and (2.17). The equivalence between the first and the second line can be proven with the non-relativistic version of theorem 2 in [14]. Although (3.17) could be proven by performing a non-relativistic limit along the lines of [15,16], we felt it worthwhile to provide a derivation using only the techniques of the ABA. This is presented in appendix C.

Performing the thermodynamic limit of (3.17) along the lines of section 2.1 one readily derives the integral series (3.3) and also the alternative form

$$\langle O_k \rangle = \sum_N \frac{1}{N!} \int \frac{d\lambda_1}{2\pi} \cdots \frac{d\lambda_N}{2\pi} \left( \prod_{j=1}^N \frac{\omega(\lambda_j)}{1 + e^{\epsilon(\lambda_j)}} \right) F_{2N,s}^k(\lambda_1, \ldots, \lambda_N)$$

(3.18)

where

$$\omega(\lambda) = \exp \left( - \int \frac{d\lambda'}{2\pi} \frac{1}{1 + e^{\epsilon(\lambda')}} \varphi(\lambda - \lambda') \right).$$

Equation (3.18) is a new result of this work. It is reminiscent of the integral series for the two-point functions derived in [37]–[39]. In fact these works consider the thermodynamic expectation value

$$\langle e^{\alpha Q(x)} \rangle, \quad \alpha \in \mathbb{C},$$

where

$$Q(x) = \int_0^x \Psi^\dagger(x) \Psi(x).$$

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Expectation values of (powers of) $Q(x)$ are then obtained as
\[ \frac{\partial}{\partial \alpha} e^{\alpha Q(x)} \bigg|_{\alpha=0} = Q(x), \quad \frac{\partial^2}{\partial \alpha^2} e^{\alpha Q(x)} \bigg|_{\alpha=0} = Q^2(x). \quad (3.19) \]
The two-point functions are obtained further by differentiating w.r.t. $x$. For example the current–current correlator is given by
\[ \langle J(x)J(0) \rangle = -\frac{1}{2} \frac{d^2}{dx^2} Q^2(x), \quad J(x) = \Psi^\dagger(x) \Psi(x). \quad (3.20) \]
In [37,38] a form factor expansion was derived for the above correlator. The resulting series has similar properties to our formula (3.18), in particular the weight functions are exactly the same. On the other hand, the two-point function is a more complicated object because its $x$-dependence, and the integral series in [37,38] involve a non-trivial dressing procedure for the momenta of the multi-particle excitations.

It seems natural that our series (3.18) could be obtained by taking the (properly regularized) $x \to 0$ limit of the expectation values of $Q^k(x)$. In appendix E we show how this limit works for the finite volume matrix elements (3.17) in the case of $k = 2$ and $N = 2$. In principle the $x \to 0$ limit could be performed for arbitrary $k$ and $N$, then the thermodynamic limit would follow in a straightforward way. However, we do not pursue this problem here, because the intention of this work was to derive the results for the one-point functions in a direct way.

In order to evaluate the series (3.3)–(3.18) one has to determine the form factors $F_{2N,c}^k$ and $F_{2N,s}^k$. The first few cases were derived in [15,16] using the non-relativistic limit of the sinh–Gordon form factors. In principle this procedure can be performed for any $k$ and $N$, moreover it is easy to implement it with symbolic manipulation programs. It was demonstrated in [15,16] that in the strong coupling regime it is sufficient to consider only the first few terms in the integral series. However the general forms of $F_{2N,c}^k$ and $F_{2N,s}^k$ are not known, therefore at present the exact result (3.3)–(3.18) can be considered a formal expansion.

In appendix D we derive the form factors for $k = 1, 2$ and arbitrary $N$; this gives an independent confirmation of the results of [15,16] in the cases $k = 1, 2$ and $N = 1–3$. On the other hand, the general results are not known for $k \geq 3$. One possibility is to solve the recursive equations (3.10) or to employ the techniques of [45,46] to obtain determinant formulas for the off-diagonal form factors; the diagonal limit should be taken afterward. However, this problem is beyond the scope of the present work.

4. Application to quench problems

In [20] the authors considered the real-time evolution of the expectation value of a local operator in integrable QFT after a certain type of quench, which changes the Hamiltonian from $H_0$ to $H$, where $H$ (possibly also $H_0$) is considered to be integrable. The main assumption is that the initial state of the system (which is the ground state of $H_0$) can be expanded in the multi-particle basis of the integrable Hamiltonian $H$ in the form of a boundary state [41]. In the simplest case with only one particle type in the spectrum the corresponding expression is
\[ |B\rangle = \exp \left( \int \frac{d\theta}{4\pi} K(\theta) A(-\theta) A(\theta) \right) |0\rangle, \]
doi:10.1088/1742-5468/2011/01/P01011
where $K(\theta)$ is an arbitrary function satisfying $K(\theta) = S(2\theta)K(-\theta)$. The time evolution of an expectation value is then given by

$$\langle \mathcal{O}(0, t) \rangle = \langle B | e^{i H t} \mathcal{O}(0, 0) e^{-i H t} | B \rangle.$$  

The main idea is that in the $t \to \infty$ limit (and taking a proper time average) only the diagonal matrix elements contribute and the off-diagonal ones can be neglected. The authors then arrive at the final expression

$$\langle \mathcal{O} \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle \mathcal{O}(0, t) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \prod_i \left\{ \int \frac{d\theta_i}{2\pi} \frac{|G(\theta_i)|^2}{1 + |G(\theta_i)|^2} \right\} F_{2n,c}(\theta_1, \ldots, \theta_n), \quad (4.1)$$

where the weight function is given by the solution of the TBA-like equation

$$- \log |G(\theta)|^2 = - \log |G_0(\theta)|^2 - \int \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \log (1 + |G(\theta')|^2). \quad (4.2)$$

The source term is given by

$$G_0 = e^{-2m_{\tau_0} \cosh \theta} K(\theta),$$

where $\tau_0$ is a UV cut-off, which can be interpreted as the extrapolation length known in the theory of critical boundary phenomena [47]. From a formal point of view a finite $\tau_0$ is needed to have a normalizable boundary state.

Here we re-derive equation (4.1) from our general result (1.6). It will be shown that (1.6) is applicable with the weight function

$$f(\theta) = \frac{|G(\theta)|^2}{1 + |G(\theta)|^2}.$$  

First we consider a large volume $L$. In finite volume the boundary state is given by [48]

$$|B\rangle_L = \sum_{N=0}^{\infty} \sum_{\theta_1, \ldots, \theta_N} \mathcal{N}(\theta_1, \ldots, \theta_N) K(\theta_1) \cdots K(\theta_N)|-\theta_1, \ldots, -\theta_N, \theta_N\rangle_L.$$  

Here the summation runs over all parity symmetric configurations, where the set \{-\theta_1, \theta_1, \ldots, -\theta_N, \theta_N\} is assumed to satisfy the $2N$-particle Bethe equations and we require $\theta_i > \theta_j$ for $i > j$. The additional normalization factors $\mathcal{N}(\theta_1, \ldots, \theta_N)$ are specific to the finite volume situation. They are given by

$$\mathcal{N}(\theta_1, \ldots, \theta_N) = \sqrt{\rho_{2N}(-\theta_1, \theta_1, \ldots, -\theta_N, \theta_N)} \rho_N(\theta_1, \ldots, \theta_N),$$

where $\rho_{2N}$ is the usual $2N$-particle density of states, and $\rho_N$ is the constrained density in the space of rapidity pairs [48]. It can be shown that in the thermodynamic limit $\mathcal{N}$ will converge to a finite value [49], and it will drop out from the calculation of the expectation value.

The boundary state is normalizable with any finite $\tau_0$ and the (time averaged) expectation value is given by

$$\langle \mathcal{O} \rangle = \lim_{L \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \frac{L\langle B | e^{i H_L t} \mathcal{O}(0, 0) e^{-i H_L t} | B \rangle_L}{L \langle B | B \rangle_L}.$$  

We assume that in the large-time limit it is sufficient to consider only the diagonal matrix elements. In this case the calculation boils down to a simple statistical average:

$$\langle \mathcal{O} \rangle = \lim_{L \to \infty} \frac{\sum_i |w_i|^2 \langle i | \mathcal{O} | i \rangle_L}{\sum_i |w_i|^2}.$$  

$$\quad (4.4)$$
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where for simplicity we denoted

$$|i\rangle_L = |\theta_1, \theta_1, \ldots, -\theta_N, \theta_N\rangle_L$$

$$w_i = \mathcal{N}(\theta_1, \ldots, \theta_N)K(\theta_1) \cdots K(\theta_N).$$

This statistical average (4.4) can be considered as a grand-canonical ensemble for particle pairs \((-\theta, \theta)\). In fact, one can identify the weights as

$$|G_0(\theta)|^2 \sim e^{-E(\theta)}$$

where we set \(T = 1\) and \(E(\theta)\) is interpreted as a bare excitation energy:

$$E(\theta) = 4m\tau_0 \cosh \theta - \log |K(\theta)|^2.$$  (4.5)

For any regular \(K(\theta)\) this bare energy has the typical features of a kinetic term: it is a regular function for a small \(\theta\), whereas for large \(|\theta|\) it goes to infinity.

One can now apply standard arguments to look for those configurations which dominate the average (4.4). Repeating all the steps which lead to the Boundary TBA equations [50] it is then a standard exercise to arrive at (4.2). In fact, (4.2) is identical to the BTBA equation of [50] with the substitution \(R = \tau_0\).

With this we have shown that the distribution of roots is governed by (4.2), therefore (4.1) follows from our general formula (1.6). Let us make some further remarks about this result.

The most remarkable property of (4.1) was already pointed out in [20]: although the boundary states only include rapidity pairs, the form factors in the final result represent processes with single particle excitations. This can be understood by recalling how we derived our main formulas in section 2.1. There it was shown that the individual contributions of the LeClair–Mussardo formula arise from disconnected terms of a matrix element with a large number of particles. Moreover, the thermodynamic limit of the mean value does not depend on the details of the finite volume state, but only on the distribution of roots. Therefore it is completely irrelevant that the actual state only involves particle pairs. The distribution of roots is always parity symmetric, therefore the final result is completely consistent with the structure of the boundary state.

To conclude this section we discuss the implications of the main result (2.23) for more general quench situations. Consider the time evolution generated by an integrable Hamiltonian \(H\) from an initial state \(\Psi_0\), which does not have the form of a boundary state. Neglecting possible complications due to degeneracies, the infinite-time limit of local observables will be given by the diagonal ensemble

$$\langle \mathcal{O} \rangle = \frac{\sum_{\alpha} |c_{\alpha}|^2 \langle \alpha | \mathcal{O} | \alpha \rangle_L}{\sum_{\alpha} |c_{\alpha}|^2},$$  (4.6)

where the summation runs over the eigenstates of \(H\) and

$$c_{\alpha} = \langle \alpha | \Psi_0 \rangle.$$  

Typically the overlaps are not known, therefore it is very hard to determine which states will have an important effect on the average (4.6). On the other hand, in the thermodynamic limit it is natural to assume that the dominant states will have a smooth distribution of roots.

Consider such a state \(|\alpha\rangle\) and the mean value \(\langle \alpha | \mathcal{O} | \alpha \rangle_L\). According to (2.23) it always takes the form of a thermal average with the weight functions determined by the root densities. Assuming that there is a one-to-one correspondence between the distributions
and the infinite set of conserved charges, this means that the mean value only depends on the macroscopic value of the conserved quantities and not on the details of the state $\alpha$. This result is a generalized form of the ‘eigenstate thermalization hypothesis’ [51, 52] appropriate to integrable theories [31]. Assuming further that the dominant states in (4.6) will have the same set of macroscopic conserved charges, we conclude that the average (4.6) can be substituted by a single mean value $\langle \alpha | O | \alpha \rangle_L$ and then the integral series (2.23) applies. This result can be interpreted as a generalized Gibbs ensemble (GGE), as proposed in [53], although the relation to the conserved charges is rather indirect.

In order to apply these results in other quench situations the assumptions about the root distributions have to be justified. More specifically it needs to be checked whether all relevant states in (4.6) can be described by a single distribution. This can be a quite challenging problem. We wish to stress that our approach only applies to the thermodynamic limit with the prescription (4.3), and it is not clear how to obtain finite size effects or observables at large but not infinite times with the present methods.

5. Conclusions

In this paper we studied mean values of local operators in Bethe states with a large number of particles and a smooth distribution of roots. The main result is formula (2.23); it can be considered as a generalization of the LeClair–Mussardo formula originally proposed in [9]. We showed that it applies both to relativistic field theory and the non-relativistic Lieb–Liniger model. The individual terms in the series arise from disconnected pieces of the diagonal matrix element; the $m$th term represents $m$-particle processes over the Fock vacuum.

Our results are analogous to the expansion for the two-point functions in the Bose gas derived in the papers [37]–[39]. The finite volume formulas (3.17) are a new result of this work and they serve as a starting point for the thermodynamic limit. The functions $F_{2m,s}^k$ in the first line of (3.17) (the symmetric evaluations of the diagonal form factors) are analogous to the ‘irreducible parts’ of the correlators known in the ABA literature. On the other hand, the expansion in the second line of (3.17) employs the functions $F_{2m,c}^k$ which are the connected parts of the infinite volume diagonal form factors. Relativistic field theory motivated the use of the connected parts and they seem to be new in the context of the algebraic Bethe ansatz.

It would be interesting to extend the present approach to non-diagonal scattering theories, which correspond to nested Bethe ansatz systems. In integrable field theory the infinite volume form factors are explicitly known in a number of models [5, 54], but the relation to the finite volume form factors has not yet been determined. The first step would be to write down the corresponding generalization of (2.15) for the finite volume mean values. If such an expression is found, then the thermodynamic limit could yield a LeClair–Mussardo formula for non-diagonal scattering theories. On the other hand, it is not clear how much can be achieved in non-relativistic (nested) Bethe ansatz systems. At present the only available results concern the spectrum (including the thermodynamics [55]) or

In the problem of Fioretto and Mussardo expectation values at finite times can be obtained using the methods of [48]. However, so far only the first few terms have been calculated explicitly and the general all-orders result is not yet available.

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the norms of Bethe wavefunctions \[56,57\], and it is not clear how to obtain the form factors. In section 4 we showed that our general result also applies to quantum quench problems and we provided an independent derivation of the results of Fioretto and Mussardo \[20\]. In the context of quench problems our expansion (2.23) can be interpreted as a proof of a ‘Generalized Eigenstate Thermalization Hypothesis’. It would be interesting to check whether the assumption of the single dominant root distribution holds in other quench situations.

In this work we only considered the one-point functions of local operators. It is very natural to suspect that similar results hold for multi-point correlation functions, however our methods do not apply in that case. Concerning the non-relativistic Bose gas it can be shown along the lines of \[37\]–\[39\] that equal-time two-point functions in excited states only depend on the distribution of roots, however there is a non-trivial dressing involved for the intermediate state momenta. It is not clear how to obtain similar results in the relativistic setting. In the case of finite temperature two-point functions (with zero chemical potential) the first few terms of the form factor expansion have been derived recently in \[24\], but it is not yet clear whether the series can be re-summed to a compact expression.

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Appendix A. Ratios of Gaudin determinants in the thermodynamic limit

In this appendix we consider the thermodynamic limit of the ratios

\[
\frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N)}{\rho_N(\theta_1, \ldots, \theta_N)} \quad \text{and} \quad \frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N)}{\rho_N(\theta_1, \ldots, \theta_N)}.
\]

The determinants above are defined in the main text.

As a first step we write the matrix \(J\) defined in (2.10) as

\[
J = G \Theta \quad \text{where} \quad \Theta_{ij} = \delta_{ij} \vartheta_j, \quad G_{ij} = \delta_{ij} - \frac{\varphi(\theta_{jk})}{\vartheta_j},
\]

\[
\vartheta_j = mL \cosh \theta_j + \sum_{i=1}^{N} \varphi(\theta_{jk}).
\]

With this notation

\[
\rho_N(\theta_1, \ldots, \theta_N) = \det G_N \det \Theta_N
\]

and also

\[
\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N) = \det G_{N-1} \det \Theta_{N-1}.
\]
The matrix $\Theta$ is diagonal, therefore the factorization property also applies to the sub-determinant:

$$\bar{\rho}_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N|\theta_j) \det \bar{G}_{N-1} \det \bar{\Theta}_{N-1}$$

where $\bar{G}_{N-1}$ is obtained from $G_{N-1}$ simply by erasing the row and column corresponding to $\theta_j$.

First we consider the ratio

$$\bar{\rho}_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N|\theta_j) = \frac{\det \bar{\Theta}_{N-1} \det \bar{G}_{N-1}}{\det \Theta_N \det G_N} = \frac{1}{\vartheta_j} \det G_{N-1},$$

In the $L \to \infty$ limit we have from (2.19)

$$\vartheta_j \to 2\pi L \rho(\theta_j).$$

The elements of $G_N$ can be written asymptotically as

$$G_{ij} = \delta_{ij} - \frac{1}{L} \varphi(\theta_{jk}) \rho(\theta_j).$$

The limit of $\det G_N$ is given by the Fredholm determinant

$$\det \left( \hat{1} - \frac{1}{2\pi} \hat{K} \right),$$

where

$$(\hat{K}(f))(x) = \int \frac{dy}{2\pi} \varphi(x - y) f(y).$$

Intuitively it is clear that this determinant should not change if we erase one ‘discretization point’ given by $\theta_j$. This would then result in

$$\frac{\det \bar{G}_{N-1}}{\det G_N} = 1 + \mathcal{O}\left( \frac{1}{L} \right).$$

(A.1)

However, we can be more precise about this. Let us decompose $\det G_N$ as a single sum of sub-determinants along the column $j$:

$$\det G_N = \left(1 - \varphi(0) \frac{1}{L \rho(\theta_j)}\right) \det \bar{G}_{N-1} - \sum_{\substack{i=1 \atop i \neq j}}^{N} \frac{\varphi(\theta_{ji})}{L \rho(\theta_i)} \det G_{N-1}^{(i)},$$

There are $\mathcal{O}(L)$ terms in the sum on the right-hand side, however the sum itself is still of $\mathcal{O}(1/L)$ because the sub-determinants $\det G_{N-1}^{(i)}$ are $\mathcal{O}(1/L)$ too. This proves (A.1) and for the ratio in question we obtain

$$\lim \frac{\bar{\rho}_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N|\theta_j)}{\rho_N(\theta_1, \ldots, \theta_N)} = \frac{1}{2\pi L \rho(\theta_j)}.$$

The ratio

$$\frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N)}{\rho_N(\theta_1, \ldots, \theta_N)} = \frac{\det \Theta_{N-1} \det G_{N-1}}{\det \Theta_N \det G_N}$$

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is more involved because $\Theta_{N-1}$ and $G_{N-1}$ are not restrictions of $\Theta_N$ and $G_N$; rather we have to subtract all contributions associated with $\theta_j$. Following the corresponding calculation in [38] we write

$$\det \Theta_N = \theta_j \det \Theta_{N-1} \prod_{l \neq j}^{N} \frac{\vartheta_l}{\vartheta_l - \varphi(\theta_{jl})}.$$

In the thermodynamic limit we have

$$\frac{\det \Theta_{N-1}}{\det \Theta_N} = \frac{1}{L\rho(\theta_j)} \prod_{l \neq j}^{N} \left( 1 - \frac{\varphi(\theta_{jl})}{L\rho(\theta_l)} \right) \to \frac{1}{2\pi L\rho(\theta_j)} \omega(\theta_j),$$

where we introduced the function

$$\omega(\theta) = \exp \left( - \int \frac{d\theta'}{2\pi} f(\theta') \varphi(\theta - \theta') \right).$$

It is not hard to convince ourselves that

$$\frac{\det G_{N-1}}{\det G_N} = 1 + \mathcal{O}\left( \frac{1}{L} \right). \quad (A.2)$$

Putting everything together

$$\lim \frac{\rho_{N-1}(\theta_1, \ldots, \hat{\theta}_j, \ldots, \theta_N)}{\rho_N(\theta_1, \ldots, \theta_N)} = \frac{\omega(\theta_j)}{2\pi L\rho(\theta_j)}.$$

Appendix B. Form factors in the 1D Bose gas

Here we present a detailed derivation of the singularity property (3.10). Consider the scalar product of two arbitrary states (not necessarily eigenvectors):

$$\langle 0 | \prod_{j=1}^{N} \mathbb{C}(\lambda_j^C) \prod_{k=1}^{N} \mathbb{B}(\lambda_k^B) | 0 \rangle.$$

The scalar product has a simple pole as $\lambda_N^C \to \lambda_N^B$:

$$\langle 0 | \prod_{j=1}^{N} \mathbb{C}(\lambda_j^C) \prod_{k=1}^{N} \mathbb{B}(\lambda_k^B) | 0 \rangle|_{\lambda_N^C \to \lambda_N^B} \to \frac{ic}{\lambda_N^C - \lambda_N^B} \left( \prod_{j=1}^{N-1} f_{Nj}^B f_{Nj}^C \right) \times \langle 0 | \prod_{j=1}^{N-1} \mathbb{C}(\lambda_j^C) \prod_{k=1}^{N-1} \mathbb{B}(\lambda_k^B) | 0 \rangle_{\text{mod}}. \quad (B.1)$$

Here the scalar product on the rhs has to be calculated with the modified vacuum eigenvalues

$$a_{\text{mod}}(\lambda) = a(\lambda)f(\lambda, \lambda_N) \quad d_{\text{mod}}(\lambda) = d(\lambda)f(\lambda_N, \lambda).$$

It should be noted that the remaining rapidities $\{\lambda_1, \ldots, \lambda_{N-1}\}$ satisfy the modified Bethe equations with $l_{\text{mod}} = a_{\text{mod}}/d_{\text{mod}}$. 

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Equation (B.1) is valid for arbitrary vacuum eigenvalues \( a(\lambda) \) and \( d(\lambda) \). In the physical case the residue vanishes because

\[
\frac{ic}{\lambda^c_N - \lambda^b_N} (l^c_N - l^b_N) \to cl_N z_N
\]

where we defined

\[
z(\lambda) = i \frac{\partial}{\partial \lambda} \log l(\lambda).
\]

In the Bose gas one has \( z(\lambda) = L \), however it is useful to leave \( z(\lambda) \) unspecified.

In the diagonal case, when \( \lambda_j^c \to \lambda_j^b \) for every \( j \) the scalar product (the norm) depends explicitly on the variables \( l_j = l(\lambda_j) \) and \( z_j = z(\lambda_j) \). The dependence on \( z_N \) is linear and one obtains from (B.1)

\[
\frac{\partial}{\partial z_N} \langle 0 | \prod_{j=1}^{N} \mathbb{C}(\lambda_j^c) \prod_{k=1}^{N} \mathbb{B}(\lambda_k^b) | 0 \rangle = cl_N \left( \prod_{j=1}^{N-1} f_{N-j}^b f_{N-j}^c \right) \times \langle 0 | \prod_{j=1}^{N-1} \mathbb{C}(\lambda_j^c) \prod_{k=1}^{N-1} \mathbb{B}(\lambda_k^b) | 0 \rangle_{\text{mod}}.
\]

(B.2)

Equation (B.2) was used in [58] to prove the norm formula (3.11).

Let us consider the action of the field operator \( \Psi = \Psi(0) \) on Bethe states. It is given by [39]

\[
\Psi \prod_{k=1}^{N} \mathbb{B}(\lambda_k) | 0 \rangle = -i \sqrt{c} \sum_{k=1}^{N} l(\lambda_k) \left( \prod_{m=1}^{N} f(\lambda_k, \lambda_m) \right) \prod_{m=1}^{N} \mathbb{C}(\lambda_m) | 0 \rangle.
\]

(B.3)

We also need the action of \( \Psi^\dagger \):

\[
\langle 0 | \prod_{k=1}^{N} \mathbb{C}(\lambda_k) \Psi^\dagger = i \sqrt{c} \sum_{k=1}^{N} \left( \prod_{m=1}^{N} f(\lambda_m, \lambda_k) \right) \langle 0 | \prod_{m=1}^{N} \mathbb{C}(\lambda_m).
\]

(B.4)

Note that contrary to (B.3) the function \( l(\lambda) \) is not present in the pre-factor in (B.4). This is due to the normalization of the operators \( \mathbb{B}(\lambda) \) and \( \mathbb{C}(\lambda) \).

We also need the multiple action of the field operators, which is given by

\[
\Psi^k \prod_{j=1}^{N} \mathbb{B}(\lambda_j) | 0 \rangle = -(m!) c \sum_{\{\lambda^+\}\cup\{\lambda^-\}} \left( \prod_{j=1}^{m} l(\lambda_j^+) \right) \left( \prod_{j=1}^{N-m} f(\lambda_j^+, \lambda_j^-) \right) \prod_{o=1}^{N-m} \mathbb{B}(\lambda_o^-) | 0 \rangle.
\]

(B.5)

Here the summation runs over all bipartite partitions \( \{\lambda\}_N = \{\lambda^+\}_m \cup \{\lambda^-\}_{N-m} \) and we made use of the identity

\[
\sum_{\sigma} \left( \prod_{i<j} f(\lambda_{\sigma_i}, \lambda_{\sigma_j}) \right) = m!,
\]

where the summation runs over all permutations \( \sigma \in S_m \). The generalization of (B.4) follows straightforwardly.

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Here we consider the diagonal limit

$$M_N^k(\{\lambda^C\}_N, \{\lambda^B\}_N, \{l^C\}_N, \{l^B\}_N) = \langle 0 | \prod_{j=1}^N C(\lambda_j^C)(\Psi^\dagger)^k \Psi^k \prod_{k=1}^N \mathcal{B}(\lambda_k^B) | 0 \rangle$$

follow from (B.5) and the basic formula (B.1). The residue at $\lambda_j^C \to \lambda_j^B$ is given by

$$M_N^k(\{\lambda^C\}_N, \{\lambda^B\}_N, \{l^C\}_N, \{l^B\}_N)|_{\lambda_j^C \to \lambda_j^B} \to \frac{ic}{\lambda_j^C - \lambda_j^B} (l^C_N - l^B_N) \left( \prod_{j=1}^{N-1} f_{nj}^C f_{nj}^B \right)$$

\times M_{N-1}^k(\{\lambda^C\}_{N-1}, \{\lambda^B\}_{N-1}, \{l^C\}_{{mod}_{N-1}}, \{l^B\}_{{mod}_{N-1}}). \quad (B.6)

The modified $l$-functions are given by

$$l_{mod}(\lambda) = l(\lambda) \frac{f(\lambda, \lambda_N)}{f(\lambda, \lambda)}.$$

(B.7)

The form factors $F_N^k$ are obtained from $M_N^k$ in the case when two vectors are solutions to the Bethe equations and the variables $l_j$ are expressed in terms of the functions $f(\lambda_i, \lambda_j)$. The analytic properties of the matrix elements then follow from (B.6):

$$F_N^k(\{\lambda^C\}_N, \{\lambda^B\}_N)|_{\lambda_j^C \to \lambda_j^B} \to \frac{ic}{\lambda_j^C - \lambda_j^B} \left( \prod_{j=1}^{N-1} f_{nj}^C f_{nj}^B - \prod_{j=1}^{N-1} f_{nj}^B f_{nj}^C \right)$$

\times $F_{N-1}^k(\{\lambda^C\}_{N-1}, \{\lambda^B\}_{N-1}). \quad (B.8)$

Here we relied on the fact that the rules for the modification of $l_{\mod}^B$ on the rhs of (B.6) automatically produce the $N - 1$-particle form factor.

**Appendix C. Mean values in the 1D Bose gas**

Here we consider the diagonal limit

$$M_N^k(\{\lambda\}_N, \{l\}_N, \{z\}_N) = \lim_{\lambda_j^C \to \lambda_j^B} M_{2N}^k(\{\lambda^C\}_N, \{\lambda^B\}_N, \{l^C\}_N, \{l^B\}_N),$$

which depends on $3N$ independent variables $\{\lambda\}_N, \{l\}_N$ and $\{z\}_N$. The dependence on $z_N$ is linear and it is given by the residue (B.6):

$$\frac{\partial}{\partial z_N} M_N^k(\{\lambda\}_N, \{l\}_N, \{z\}_N) = \partial_N \left( \prod_{j=1}^{N-1} f_{nj}^C f_{nj}^B \right)$$

\times $M_{N-1}^k(\{\lambda\}_{N-1}, \{l\}_{{mod}_{N-1}}, \{z\}_{{mod}_{N-1}}). \quad (C.1)$

The modification rule for the variables $z_j$ follows from (B.7) and is given by

$$z_{\mod}(\lambda) = z(\lambda) + \varphi(\lambda - \lambda_N). \quad (C.2)$$

The ‘on-shell’ mean value

$$\langle \mathcal{O}_k \rangle_N (\{\lambda\}_N, \{z\}_N) = M_N^k(\{\lambda\}_N, \{l\}_N, \{z\}_N)$$

is a function of $2N$ parameters $\{\lambda\}_N$ and $\{z\}_N$. The dependence on $z_N$ is linear and the

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The function $F$ solves the Bethe equations:

$$\langle O \rangle_N = \lim_{\varepsilon \to 0} M^k_{2N}(\{\lambda \}^B, \{\lambda \}^B, \{\lambda \}^B, \{\lambda \}^B).$$

On the other hand, the form factor is obtained by

$$F^k_{2N,s}(\{\lambda \}^B) = \lim_{\varepsilon \to 0} F^k_{2N}(\{\lambda \}^B + \varepsilon, \{\lambda \}^B).$$

where the Bethe equations are satisfied. This means that $\{\tilde{l}^B\}_N$ and $\{l^B\}_N$ will be given by the appropriate products of $S$-matrices. However, a constant shift in the rapidities yields

$$\{\tilde{l}^B\}_N = \{l^B\}_N.$$

This proves the theorem. \hfill \Box

For any partition $\{\lambda\}_N = \{\lambda^+\}_n \cup \{\lambda^-\}_{N-n}$ we define the function

$$S_N(\{\lambda^+\}_n, \{\lambda^-\}_{N-n}, \{\nu\}_N) = c^N \left( \prod_{j=1}^n \prod_{k=1}^{N-n} |f(\lambda^+ - \lambda^0)|^2 \right) \left( \prod_{1 \leq j < k \leq n} |f(\lambda^- - \lambda^-)|^2 \right)$$

$$\times \rho_{N-n}(\{\lambda^-\}_{N-n}, \{\nu\}_N).$$

The dependence on $z_N$ is given by

$$\frac{\partial}{\partial z_N} S_N(\{\lambda^+\}_n, \{\lambda^-\}_{N-n}, \{\nu\}_N)$$

$$= \begin{cases} 
  c \left( \prod_{j=1}^{N-1} f_{Nj} f_{Nj} \right) & S_{N-n}(\{\lambda^-\}_{N-n}, \{z_{mod}\}_N) \quad \text{if } \lambda_N \in \{\lambda^-\}_{N-n} \\
  0 & \text{if } \lambda_N \in \{\lambda^+\}_n.
\end{cases} \quad (C.6)$$
Now we are in a position to prove the main theorem of this appendix.

**Theorem C.2.** The mean value can be expressed as

\[ \langle O_k \rangle_N = \sum_{\{\lambda^+\} \cup \{\lambda^-\}} F^k_{n,s}(\{\lambda^+\}) S_{N-n}(\{\lambda^-\}) \]  

where the summation is over the bipartite partitions of the rapidities into two subsets \( \{\lambda\}_N = \{\lambda^+\}_n \cup \{\lambda^-\}_{N-n} \).

**Proof.** The proof is given by induction over \( N \). The form factor vanishes for \( N < k \), therefore the first member will be at \( N = k \). In this case there is no dependence on any of the \( z_j \), therefore (C.7) is satisfied. Now let us assume that (C.7) is proven for \( N < M \). We prove that it is valid for \( N = M \).

We investigate the \( z_j \) dependence of (C.7). According to the assumption of induction it follows from (C.3) and (C.6) that the \( z_j \) dependence of the lhs and rhs in (C.7) coincides. Therefore we consider the value at \( z_j = 0 \). At this point the norm functions the rhs of (C.7) vanish and one is left with

\[ F^k_{M,s}(\{\lambda\}). \]  

(D.8)

It was proven in Theorem 1 that this form factor coincides with the irreducible part of the mean value. Thus (C.7) is proven.

The normalized expression of (3.17) follows from (C.7) by dividing with the norm (3.11).

**Appendix D. The form factors \( F^1_{2N,s} \) and \( F^2_{2N,s} \)**

In this section we derive explicit expressions for the functions \( F^k_{2N,s} \) and \( F^k_{2N,c} \), for \( N \in \mathbb{N} \) and \( k = 1, 2 \).

In the first case the operator in question is \( O_1 = \Psi^\dagger \Psi \), which describes the total particle density. Therefore its mean value is given by

\[ \langle \Psi^\dagger \Psi \rangle_N = \frac{N}{L} = \frac{1}{\rho_N(\{\lambda\})} \frac{N \rho_N(\{\lambda\})}{L}. \]  

(D.1)

On the other hand theorem C.2 gives

\[ \langle \Psi^\dagger \Psi \rangle_N = \frac{1}{\rho_N(\{\lambda\})} \sum_{\{\lambda^+\} \cup \{\lambda^-\}} F^1_{2k,s}(\{\lambda^-\}) \rho_{N-k}(\{\lambda^+\}). \]  

(D.2)

Comparing (D.1) and (D.2) one can inductively deduce the form factors \( F^1_{2N} \). The simplest way is to apply theorem C.1, which yields

\[ F^1_{2N,s}(\{\lambda\}) = \frac{N \rho_N(\{\lambda\})}{L} \bigg|_{L=0}. \]

Let us introduce the matrix \( J \), which is given by \( \rho_N \) at \( L = 0 \):

\[ J_{ij} = \delta_{ij} \left( \sum_{k=1}^{n} \varphi_{ik} \right) - \varphi_{ij}. \]
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It is easy to see that

$$\frac{\rho_N(\{\lambda\})}{L} \bigg|_{L=0} = \sum_{i=1}^{N} \bar{J}_{ii}$$

where the elements of $\bar{J}$ are the corresponding minors of $J$. The matrix $J$ has the property that the sums of its rows and the sums of its columns give zero, therefore its minors coincide. Therefore

$$F_{2N,s}^1(\{\lambda\}_N) = N^2 \bar{J}_{11}.$$ 

In graph theory the matrix $J$ is the generalized Laplacian matrix of $G_n$ (the complete graph with $n$ edges), where for each vertex from node $i$ to $j$ one associates the ‘weight’ $\varphi_{ij}$. Then according to Kirchhoff’s theorem the minor $\bar{J}_{11}$ gives the enumeration of the spanning trees of $G_n$:

$$\bar{J}_{11} = \mathcal{G}_N(\theta_1, \ldots, \theta_N),$$

where

$$\mathcal{G}_N(\theta_1, \ldots, \theta_N) = \sum_{G \in ST} \left( \prod_{\alpha \in V_G} \varphi(\alpha) \right).$$

The sum runs over all spanning trees and the product runs over the vertices $\alpha$ of a given spanning tree, and

$$\varphi(\alpha) = \varphi_{ij}$$

for a vertex $\alpha$ going from edge $i$ to $j$. Putting everything together

$$F_{2N,s}^1(\{\lambda\}_N) = N^2 \sum_{G \in ST} \left( \prod_{\alpha \in V_G} \varphi(\alpha) \right).$$

By the reverse application of theorem 2.1 one can deduce that

$$F_{2N,c}^1(\lambda_1, \ldots, \lambda_N) = \sum_{\sigma} \varphi(\lambda_{\sigma_1} - \lambda_{\sigma_2})\varphi(\lambda_{\sigma_2} - \lambda_{\sigma_3})\cdots\varphi(\lambda_{\sigma_{N-1}} - \lambda_{\sigma_N}). \tag{D.3}$$

Here the summation runs over all permutations $\sigma \in S_N$.

In the case of the operator $\mathcal{O}_2 = \Psi^\dagger \Psi^\dagger \Psi \Psi$ one makes use of the fact that it is just the interaction term in the Hamiltonian. Then the Hellmann–Feynman theorem gives [59]

$$\langle \Psi^\dagger \Psi^\dagger \Psi \Psi \rangle_N = \frac{1}{L} \frac{\partial E(L, c)}{\partial c}.$$ 

Here $E(L, c)$ is the total energy of the $N$-particle state for fixed momentum quantum numbers. It follows from scaling arguments that

$$E(L, c) = \frac{h(Lc)}{L^2},$$

where $h(x)$ is a dimensionless function. Therefore

$$\frac{1}{L} \frac{\partial E}{\partial c} = \frac{1}{c} \left( \frac{\partial E}{\partial L} + \frac{2E}{L} \right).$$

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The energy is given by
\[ E = \sum_i \lambda_i^2 \] and
\[ \frac{\partial E}{\partial L} = 2\lambda_i \frac{\partial \lambda_i}{\partial L}. \]

The derivative with respect to \( L \) can be calculated from the Bethe equations:
\[ \lambda_i + J_{ij} \frac{\partial \lambda_j}{\partial L} = 0. \]

Therefore
\[ \langle \Psi^\dagger \Psi^\dagger \Psi \Psi \rangle_N = \frac{1}{\rho_N} \frac{2}{c} \left[ \lambda_i \left( -J_{ij} + \delta_{ij} \rho_N L \right) \lambda_j \right]. \]

Making use of theorem C.1 one arrives at
\[ F_{2N,c}(\theta_1, \ldots, \theta_N) = \frac{2}{c} G_N(\theta_1, \ldots, \theta_N) \times \left( \sum_{i<j} (\lambda_i - \lambda_j)^2 \right). \]

Theorem 2.1 then yields
\[ F_{2N,c}(\lambda_1, \ldots, \lambda_N) = \frac{1}{c} \sum_{\sigma} \varphi(\lambda_{\sigma_1} - \lambda_{\sigma_2}) \varphi(\lambda_{\sigma_2} - \lambda_{\sigma_3}) \cdots \varphi(\lambda_{\sigma_{N-1}} - \lambda_{\sigma_N})(\lambda_{\sigma_1} - \lambda_{\sigma_N})^2. \]

The summation runs over all permutations \( \sigma \in S_N \). The results (D.3) and (D.4) are in agreement with the formulas of [16] presented for \( N = 1–3 \).

Appendix E. The \( x \to 0 \) limit of the non-local correlation function

In this appendix we demonstrate how to re-derive the expectation values \( \langle O_k \rangle \) from the previously available results for non-local correlation functions [38]. We only consider \( O_2 = \Psi^\dagger \Psi^\dagger \Psi \Psi \), which is related to the operator \( Q^2(x) \). The normal ordering of the field operators yields:
\[ Q^2(x) - Q(x) = \int_0^x \int_0^x dx_1 dx_2 \Psi^\dagger(x_1) \Psi^\dagger(x_2) \Psi(x_1) \Psi(x_2). \]

Taking the limit \( x \to 0 \) one gets:
\[ \Psi^\dagger \Psi^\dagger \Psi \Psi = \lim_{x \to 0} \frac{Q^2(x) - Q(x)}{x^2}. \]

Therefore, in order to obtain the expectation value of \( O_2 = \Psi^\dagger \Psi^\dagger \Psi \Psi \) one has to pick the \( O(x^2) \) term from the expansion of \( Q^2(x) - Q(x) \). We perform this for the expectation values with a finite number of particles. The relevant result reads (equation (3.32) from [38])
\[ \langle (Q(x))^2 \rangle_N = \frac{1}{\rho_N(\{\lambda\})} \left[ \langle (Q(x))^2 \rangle_0^0 + \sum_{\{\lambda_+\} \cup \{\lambda_-\}} I_{n,N}(\{\lambda_-\}, \{\lambda_+\}, \{\lambda_+\}, \{\lambda_-\}) \rho_{N-n}(\{\lambda_+\}) \right]. \]

Here the summation runs over the partitions with \(|\{\lambda_-\}| = n \) with \( n = 2 \cdots N \). The first term is given by the ‘irreducible part of the identity operator’ (equation (3.28) in [38])
\[ \langle (Q(x))^2 \rangle_0^0 = \sum_{\{\lambda_+\} \cup \{\lambda_-\}} n^2 \rho_n^2(\{\lambda_x\}) \rho_{N-n}^2(\{\lambda_+\}). \]
Here \( \rho_n^x(\{\lambda_x\}) \) and \( \rho_{N-n}^y(\{\lambda_y\}) \) are the Gaudin determinants of a system with volume parameters \( x \) and \( y = L - x \), respectively. The quantities \( I_{n,N}(\{\lambda_x\}_n, \{\lambda_y\}_{N-n}) \) are the ‘irreducible parts’ of the operator \( Q^2 \). They have a non-trivial dependence on \( x \). In the case of \( Q(x) \) the ‘irreducible parts’ are zero and the mean value can be expressed as (equation (3.23) in [38])

\[
\langle Q(x) \rangle_N = \frac{1}{\rho_N(\{\lambda\})} \sum_{\{\lambda_x\}_n \cup \{\lambda_y\}_{N-n}} n\rho_n^x(\{\lambda_x\})\rho_{N-n}^y(\{\lambda_y\}).
\] (E.4)

In the following we only consider the first non-trivial case \( N = 2 \). Equations (E.2), (E.3) and (E.4) yield

\[
\langle (Q(x))^2 - Q(x) \rangle = \frac{1}{\rho_2(\lambda_1, \lambda_2)}(2\rho_2^x(\lambda_1, \lambda_2) + I_2(\lambda_1, \lambda_2)).
\]

Here

\[
\rho_2^x(\lambda_1, \lambda_2) = x(x + 2\varphi_{12})
\]

and \( I_2 \) is given by 8.11–8.12 in [37]:

\[
I_2 = -\frac{2}{\lambda_{12}^2} \left[ (e^{-ix\lambda_{12}} - 1)\frac{\lambda_{12} + ic}{\lambda_{12} - ic} + (e^{ix\lambda_{12}} - 1)\frac{\lambda_{12} - ic}{\lambda_{12} + ic} \right].
\]

Expanding in \( x \) and keeping only the first two terms one has

\[
I_2 = -4x\varphi_{12} + x^2\left( \frac{2\varphi_{12}\lambda_{12}^2}{c} - 2 \right) + \cdots.
\]

Putting everything together one observes that the \( O(x) \) terms indeed cancel and one is left with

\[
\langle (Q(x))^2 - Q(x) \rangle = \frac{x^2}{\rho_2(\lambda_1, \lambda_2)} \frac{2\varphi_{12}\lambda_{12}^2}{c}.
\]

Therefore

\[
\langle \Psi^\dagger \Psi^\dagger \Psi \Psi \rangle = \frac{1}{\rho_2(\lambda_1, \lambda_2)} \frac{2\varphi_{12}\lambda_{12}^2}{c}.
\]

Comparing with (3.17) we conclude that

\[
F_{2,s}^2 = 0, \quad F_{4,s}^2 = \frac{2\varphi_{12}\lambda_{12}^2}{c}.
\]

The above formulas are in complete agreement with our result (D.4).

Note added in proof. While this article was being finished there appeared an article [60] where the LeClair–Mussardo formalism was applied to the Super Tonks–Girardeau gas, which is a highly excited (stable) state of the Lieb–Liniger model with repulsive coupling. Our proof of the LeClair–Mussardo series thus directly applies to the calculations of [60].

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