The Mukai pairing and integral transforms in Hochschild homology.

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Abstract

Let \( X \) be a smooth proper scheme over a field of characteristic 0. Following Shklyarov \([10]\), we construct a (non-degenerate) pairing on the Hochschild homology of \( \text{perf} (X) \), and hence, on the Hochschild homology of \( X \). On the other hand the Hochschild homology of \( X \) also has the Mukai pairing (see \([1]\)). If \( X \) is Calabi-Yau, this pairing arises from the action of the class of a genus 0 Riemann-surface with two incoming closed boundaries and no outgoing boundary in \( H_0(M_0(2,0)) \) on the algebra of closed states of a version of the B-Model on \( X \). We show that these pairings \"almost\" coincide. This is done via a different view of the construction of integral transforms in Hochschild homology that originally appeared in Caldararu’s work \([1]\). This is used to prove that the more \"natural\" construction of integral transforms in Hochschild homology by Shklyarov \([10]\) coincides with that of Caldararu \([1]\). These results give rise to a Hirzebruch Riemann-Roch theorem for the sheafification of the Dennis trace map.

Introduction.

Let \( X \) be a smooth proper scheme over a field \( \mathbb{K} \) of characteristic 0. Let \( \text{perf} (X) \) denote the DG-category of left bounded perfect injective complexes of \( \mathcal{O}_X \)-modules. There is a natural isomorphism of Hochschild homologies (see \([5]\) for instance)

\[
\text{HH}_\bullet (X) \simeq \text{HH}_\bullet (\text{perf} (X)).
\] (1)

If \( Y \) is any smooth proper scheme, an object \( \Phi \in \text{perf} (X \times Y) \) can be thought of as the kernel of an integral transform from \( \text{perf} (X) \) to \( \text{perf} (Y) \) (Section 8 of \([11]\)). This is a morphism from \( \text{perf} (X) \) to \( \text{perf} (Y) \) in the
homotopy category \( \text{Ho}(\text{dg-cat}) \) of dg-categories modulo quasi-equivalences. We will abuse notation and denote this by \( \Phi \) as well. It follows that \( \Phi \) induces a map \( \Phi_* : \text{HH}_\bullet(\text{perf}(X)) \to \text{HH}_\bullet(\text{perf}(Y)) \) and hence, by (1), a map

\[
\Phi_*^{\text{nat}} : \text{HH}_\bullet(X) \to \text{HH}_\bullet(Y).
\]

One also has (see [10]) a Kunneth quasiisomorphism \( \Phi \) of the form

\[
K : \text{HH}_\bullet(\text{perf}(X)) \otimes \text{HH}_\bullet(\text{perf}(Y)) \to \text{HH}_\bullet(\text{perf}(X \times Y)).
\]

Since \( X \) is smooth, the diagonal \( \Delta : X \to X \times X \) is a local complete intersection. Hence, \( \mathcal{O}_\Delta := R\Delta_! \mathcal{O}_X \) is a perfect complex on \( X \times X \) (see [11], Section 8). We will abuse notation and denote \( \mathcal{O}_\Delta \) thought of as the kernel of an integral transform from \( X \times X \) to \( \text{Spec} \mathcal{K} \) by \( \Delta \). One then has a pairing given by the composite map

\[
\begin{align*}
\text{HH}_\bullet(\text{perf}(X)) \otimes \text{HH}_\bullet(\text{perf}(X)) & \xrightarrow{\Phi_*^{\text{nat}}} \text{HH}_\bullet(Y) \\
\text{HH}_\bullet(\text{perf}(X \times X)) & \xrightarrow{\Delta_*} \text{HH}_\bullet(\text{perf}(\mathcal{K})) = \mathcal{K}.
\end{align*}
\]

Denote this pairing by \( \langle \ , \ \rangle_{\text{Shk}} \).

On the other hand, the work of A. Caldararu [1] constructs the following:

- A non-degenerate Mukai pairing

\[
\langle \ , \ \rangle_M : \text{HH}_\bullet(X) \otimes \text{HH}_\bullet(X) \to \mathcal{K}.
\]

- For each \( \Phi \in \text{Perf}(X \times Y) \) an "integral transform"

\[
\Phi^{\text{cal}}_* : \text{HH}_\bullet(X) \to \text{HH}_\bullet(Y).
\]

If \( X \) is Calabi-Yau, it has been argued implicitly by Caldararu [3] that \( \langle \ , \ \rangle_M \) is precisely the pairing on \( \text{HH}_\bullet(X) \) arising from the action (on \( \text{HH}_\bullet(X) \)) of the class of a genus 0 Riemann-surface with two incoming closed boundaries and no outgoing boundary in \( H_0(\mathcal{M}_0(2,0)) \). Let \( \vee : \text{HH}_\bullet(X) \to \text{HH}_\bullet(X) \) be the whose image under the Hochschild-Kostant-Rosenberg isomorphism is the involution on Hodge cohomology that acts on the direct summand \( H^q(X, \Omega^p_X) \) by multiplication by \((-1)^p\).
The ”natural” pairing and the Mukai pairing.

The main result of this note is as follows.

**Theorem 1.** Let $a, b \in \text{HH}_\bullet(X)$. Then,

$$\langle b^\vee, a \rangle_M = \langle a, b \rangle_{\text{Shk}}.$$  

If $X$ is a smooth proper quasi-compact scheme, the category $\text{perf}(X)$ is quasi-equivalent to $\text{perf}(A)$ for some DG-algebra $A$ (see [6],[11]). In this case, the pairing $\langle , \rangle_{\text{Shk}}$ on $\text{HH}_\bullet(X)$ is the pairing on $\text{HH}_\bullet(A)$ described in [10]. On the other hand, the Mukai pairing $\langle , \rangle_M$ has been explicitly computed at the level of Hodge cohomology in [8]. In an implicit form, this computation appeared earlier in [7]. Theorem 1 therefore, enables us to relate the familiar Riemann-Roch-Hirzebruch theorem for a proper scheme over $\mathbb{K}$ to the more abstract ”noncommutative” Riemann-Roch theorem in [10].

Further, if $X$ is Calabi-Yau, so is $A$. In this case Theorem 1 is very similar to Conjecture 6.2 in [10] for proper homologically smooth Calabi-Yau DG-algebras $A$ such that $\text{perf}(A)$ is quasi-equivalent to $\text{perf}(X)$ for some smooth proper quasi-compact scheme $X$. We make a remark about this in Section 2.3.

**Integral transforms in Hochschild homology.**

Let us outline how Theorem 1 is proven. It was stated and proven in [10] that if $\Phi \in \text{perf}(X \times Y)$, then $\Phi_{\text{nat}}^\ast$ is simply convolution with the Chern character of $\Phi$ with respect to the pairing $\langle , \rangle_{\text{Shk}}$. Besides [10], the reader may refer to Theorems 4 and 5 in this paper for the precise statement. We construct a map $\Phi_{\text{muk}}^\ast : \text{HH}_\bullet(X) \to \text{HH}_\bullet(Y)$ that is ”almost” convolution with the Chern character of $\Phi$ with respect to the Mukai pairing. We then proceed to prove that $\Phi_{\text{muk}}^\ast$ has all the ”good properties” one expects of an integral transform in Hochschild homology (Propositions 1, 2 and 3 of this paper). We recall that the integral transform from $\text{perf}(X)$ to $\text{perf}(X)$ arising out of the element $O_\Delta$ of $\text{perf}(X \times X)$ is the identity. It follows that $O_\Delta_{\text{nat}} = \text{id}$. Proposition 2, which says that $O_\Delta_{\text{muk}} = \text{id}$ as well, is then used to prove Theorem 1.
The fact that $\Phi_{\text{muk}}^*$ has all the "good properties" one expects of an integral transform in Hochschild homology is also exploited to prove the following theorem.

**Theorem 2.**

$$\Phi_{\text{nat}}^* = \Phi_{\text{muk}}^* = \Phi_{\text{cal}}^*.$$  

In other words, the "good constructions" of integral transforms in Hochschild homology coincide.

**A Hirzebruch-Riemann-Roch for the sheafification of the Dennis trace map.**

We now mention another consequence of Theorems 1 and 2. Recall that we have an isomorphism of higher K groups

$$K_i(X) \simeq K_i(\text{perf}(X)).$$

For any DG-category $\mathcal{C}$, let $Z^0(\mathcal{C})$ denote the category such that

$$\text{Obj}(Z^0(\mathcal{C})) = \text{Obj}(\mathcal{C}) \quad \text{and} \quad \text{Hom}_{Z^0(\mathcal{C})}(M,N) = \text{Z}_0(\text{Hom}_\mathcal{C}(M,N))$$

$$\quad \forall \, M,N \in \text{Obj}(\mathcal{C}).$$

Here, $Z^0(\mathcal{C})$ is the space of 0-cocycles for any cochain complex $\mathcal{C}$. If $Z^0(\mathcal{C})$ is exact, one has a Dennis trace map

$$\text{Ch}_i^i : K_i(\mathcal{C}) \to \text{HH}_i(\mathcal{C})$$

(see [12]). This therefore, yields us a map

$$\text{Ch}_i^i : K_i(X) \to \text{HH}_i(\text{perf}(X)) \simeq \text{HH}_i(X).$$

This map is the "sheafification of the Dennis trace map" constructed in [13]. Let $I_{HKR} : \text{HH}_\bullet(X) \to \bigoplus_{p,q} H^p(X, \Omega^q_X)$ denote the Hochschild-Kostant-Rosenberg isomorphism. Let

$$\text{ch}_i^i : K_i(X) \to \bigoplus_j H^{j-i}(X, \Omega^j_X)$$

denote $I_{HKR} \circ \text{Ch}_i^i$. It was proven in [2] (Theorem 4.5) that $\text{ch}_i^0$ is the usual Chern character. We have the following generalization of the Hirzebruch Riemann-Roch theorem.
Theorem 3. Let $f : X \to Y$ be a smooth proper morphism between proper schemes $X$ and $Y$. Let $Z$ be a smooth quasi-compact separated scheme. Then,

$$(f \times \text{id})_*(\check{c}h^i(\alpha)\pi^*_X \text{td}(T_X)) = \check{c}h^i((f \times \text{id})_*(\alpha))\pi^*_Y \text{td}(T_Y)$$

for any $\alpha \in K_i(X \times Z)$.

Layout of this note.

Section 1 reviews some basic facts from D. Shklyarov’s work [10]. Section 2.1 recalls A. Caldararu’s construction of the Mukai pairing [1] and related results. In Section 2.2, we give an alternate construction of $\Phi_* : \text{HH}_*(X) \to \text{HH}_*(Y)$ for any $\Phi \in \text{perf}(X \times Y)$. We prove Theorem 1 and Theorem 2 in Section 2.2. Section 2.3 contains some remarks about what Theorem 1 means when $X$ is Calabi-Yau. Section 2.4 proves Theorem 3.

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1 The ”natural pairing” on the Hochschild homology of schemes.

This section primarily recalls material from D. Shklyarov’s work [10]. The term ”DG algebra” in this section shall refer to a proper homologically smooth DG-algebra unless explicitly stated otherwise.

1.1 Preliminary recollections.

Recall that a DG-algebra $A$ is proper if $\sum_n \dim H^n(A) < \infty$ and is homologically smooth if it is quasi isomorphic to a perfect $A^{op} \otimes A$-module. Here, $A^{op}$ denotes the opposite algebra of $A$. The term ”$A$-module” shall refer to a right $A$-module.
Recall that a $A$-module is said to be semi-free if it is obtained from a finite set of free $A$-modules after taking finitely many cones of degree 0 closed morphisms. A perfect $A$-module is a direct summand of a semi-free $A$-module. Let $\text{perf}(A)$ denote the DG-category of perfect $A$-modules. We recall the following facts from [10].

Fact 1: If $A$ is a DG-algebra, the natural embedding of the category with a unique object whose morphisms are given by $A$ into $\text{perf}(A)$ induces an isomorphism

$$\text{HH}_\bullet(A) \simeq \text{HH}_\bullet(\text{perf}(A))$$

Fact 2: If $A$ and $B$ are DG-algebras and $\Phi$ is a perfect $A^{\text{op}} \otimes B$-module, then $\Phi$ gives a (DG) functor

$$\Phi_* : \text{perf}(A) \to \text{perf}(B)$$

$$M \mapsto M \otimes_A \Phi.$$ 

$\Phi_*$ therefore induces a map

$$\Phi^\text{nat}_* : \text{HH}_\bullet(\text{perf}(A)) \to \text{HH}_\bullet(\text{perf}(B)).$$

Fact 3: Let $\Delta$ denote $A$ treated as a perfect $A^{\text{op}} \otimes A$-module in the natural way. Then, by Fact 2, we have a DG functor $\Delta_* : \text{perf}(A \otimes A^{\text{op}}) \to \text{perf}(K)$. Further, there is an isomorphism

$$K : \text{HH}_\bullet(\text{perf}(A)) \otimes \text{HH}_\bullet(\text{perf}(A^{\text{op}})) \to \text{HH}_\bullet(\text{perf}(A \otimes A^{\text{op}})).$$

The map $\Delta^\text{nat}_* \circ K : \text{HH}_\bullet(\text{perf}(A)) \otimes \text{HH}_\bullet(\text{perf}(A^{\text{op}})) \to \text{HH}_\bullet(\text{perf}(K)) = K$ therefore gives rise to a pairing

$$\langle \ , \ \rangle_{\text{Shk}} : \text{HH}_\bullet(A) \otimes \text{HH}_\bullet(A^{\text{op}}) \to K.$$

For any exact $K$-linear category $C$, let $K_0(C)$ denote the Grothendieck group of $C$. Recall from [10] that there is a Chern character

$$\text{Ch} : K_0(\text{perf}(A)) \to \text{HH}_0(\text{perf}(A)) \simeq \text{HH}_0(A).$$

Let $A$ and $B$ be DG-algebras. We abuse notation and denote the composite map

$$\text{HH}_\bullet(A) \otimes \text{HH}_\bullet(A^{\text{op}}) \otimes \text{HH}_\bullet(B) \xrightarrow{\langle \ , \ \rangle_{\text{Shk}} \otimes \text{id}} \text{HH}_\bullet(B)$$

by $\langle \ , \ \rangle_{\text{Shk}}$ itself. Identify $\text{HH}_\bullet(A^{\text{op}} \otimes B)$ with $\text{HH}_\bullet(A^{\text{op}}) \otimes \text{HH}_\bullet(B)$ via the inverse of the Kunneth isomorphism. If $\Phi \in \text{perf}(A^{\text{op}} \otimes B)$, the following theorem from [10] (Theorem 3.4 of [10]) says that $\Phi^\text{nat}_*$ is just "convolution with $\text{Ch}(\Phi)$".
Theorem 4.

\[ \Phi_{\text{nat}}^*(x) = \langle x, Ch(\Phi) \rangle_{\text{Shk}} \]

for any \( x \in HH_\bullet(A) \).

Note that Theorem 4 implies that \( \Phi_{\text{nat}}^* \) depends only on the image of \( \Phi \) in \( D(\text{perf} (A^{\text{op}} \otimes B)) \).

1.2 The natural pairing on the Hochschild homology of schemes.

In this subsection, whenever \( f : X \to Y \) is a morphism of schemes, \( f_*, f^* \) etc shall denote the corresponding derived functors. Let \( X \) be a quasicompact separated scheme over \( \mathbb{K} \). In this case, the (unbounded) derived category \( D_{\text{qcoh}}(X) \) of quasi-coherent \( \mathcal{O}_X \)-modules on \( X \) admits at least compact generator \( E \) (see [11]). This is a perfect complex of \( \mathcal{O}_X \)-modules. We recall the following facts.

**Fact 1:** For each compact generator \( E \) of \( D_{\text{qcoh}}(X) \) there one can choose a (proper if and only if \( X \) is proper) DG-algebra \( \mathcal{A}(E) \) such that \( \text{perf} (\mathcal{A}(E)) \) is quasi-equivalent to \( \text{perf} (X) \) (see [6],[11]).

**Fact 2:** Recall that if \( E \) is a compact generator of \( D_{\text{qcoh}}(X) \) and if \( F \) is a compact generator of \( D_{\text{qcoh}}(Y) \) then \( E \boxtimes F \) is a compact generator of \( D_{\text{qcoh}}(X \times_\mathbb{K} Y) \).

**Fact 3:** The \( \mathcal{A}(E) \) can be chosen so that

\[ \mathcal{A}(E \boxtimes F) = \mathcal{A}(E) \otimes \mathcal{A}(F) \]

whenever \( E \) and \( F \) are as in Fact 2 above.

**Fact 4:** If \( E \) is a compact generator of \( D_{\text{qcoh}}(X) \), so is the dual perfect complex \( E^\vee \). One can choose \( \mathcal{A}(E^\vee) \) to be \( \mathcal{A}(E)^{\text{op}} \). Hence, \( \text{perf} (\mathcal{A}(E)) \) is quasi-equivalent to \( \text{perf} (\mathcal{A}(E)^{\text{op}}) \).

From the quasi-equivalences \( \text{perf} (\mathcal{A}(E)) \simeq \text{perf} (X) \) and \( \text{perf} (\mathcal{A}(E)^{\text{op}}) \simeq \text{perf} (X) \), we obtain isomorphisms

\[ i : HH_\bullet(X) \simeq HH_\bullet(\mathcal{A}(E)) \]

\[ j : HH_\bullet(X) \simeq HH_\bullet(\mathcal{A}(E)^{\text{op}}) \]
For $X$ proper let $\langle \cdot, \cdot \rangle_{\text{Shk}}$ be the pairing on $\text{HH}_\bullet(X)$ such that

$$\langle a, b \rangle_{\text{Shk}} = \langle i(a), j(b) \rangle_{\text{Shk}}$$

for all $a, b \in \text{HH}_\bullet(X)$. Note that the RHS of the above equation has been defined in the previous subsection. We identify $\text{HH}_\bullet(X \times Y)$ with $\text{HH}_\bullet(X) \otimes \text{HH}_\bullet(Y)$ via the inverse of the Kunneth isomorphism. Recall from [11] that an element $\Phi$ of $\text{perf}(X \times Y)$ gives rise to an integral transform $\Phi$ from $\text{perf}(X)$ to $\text{perf}(Y)$. This is a morphism in $\text{Ho}(\text{dg-cat})$, the category of DG-categories modulo quasi-equivalences. The functor from $D(\text{perf}(X))$ to $D(\text{perf}(Y))$ induced by $\Phi$ is the functor

$$E \mapsto \pi_Y^*(\pi_X^* E \otimes^L \Phi).$$

$\Phi$ induces a map from $\text{HH}_\bullet(\text{perf}(X))$ to $\text{HH}_\bullet(\text{perf}(Y))$ and hence, a map from $\text{HH}_\bullet(X)$ to $\text{HH}_\bullet(Y)$ which we shall denote by $\Phi^\text{nat}_\bullet$. We now state the following consequence of Theorem 4. Like Theorem 4, Theorem 5 implies that $\Phi^\text{nat}_\bullet$ depends only on the image of $\Phi$ in $D(\text{perf}(X \times Y))$.

**Theorem 5.** For any $\Phi$ in $\text{perf}(X \times Y)$,

$$\Phi^\text{nat}_\bullet(x) = \langle x, Ch(\Phi) \rangle_{\text{Shk}} \in \text{HH}_\bullet(Y)$$

for all $x \in \text{HH}_\bullet(X)$.

**Sketch of proof of Theorem 5.** Theorem 5 is a direct consequence of Theorem 4 and the work of B. Toen [11]. Given two DG-categories $\mathcal{C}$ and $\mathcal{D}$, [11] constructs a DG-category $R\text{Hom}(\mathcal{C}, \mathcal{D})$. Let $X$ and $Y$ be quasi compact separated schemes over $K$. Let $E$ and $F$ be compact generators of $D_{qcoh}(X)$ and $D_{qcoh}(Y)$. Recall that in [11] it was shown that there is an identification

$$\beta : \text{perf}(\mathcal{A}(E)^\text{op} \otimes \mathcal{A}(F)) \to R\text{Hom}(\text{perf}(\mathcal{A}(E)), \text{perf}(\mathcal{A}(F)))$$

$$\Phi \mapsto M \mapsto M \otimes_A \Phi$$

in $\text{Ho}(\text{dg-cat})$. Similarly, there is an identification

$$\gamma : \text{perf}(X \times Y) \to R\text{Hom}(\text{perf}(X), \text{perf}(Y))$$

in $\text{Ho}(\text{dg-cat})$. If $\Phi$ is in $\text{perf}(X \times Y)$, $\gamma(\Phi)$ is the integral transform $\Phi$ from $\text{perf}(X)$ to $\text{perf}(Y)$ that we described before stating Theorem 5. We abuse notation and use $\eta$ to denote the quasi-equivalences $\text{perf}(\mathcal{A}(E)) \simeq \text{perf}(X), \text{perf}(\mathcal{A}(E)^\text{op} \otimes \mathcal{A}(F)) \simeq \text{perf}(X \times Y)$ and
RHom(\text{perf}(A(E)), \text{perf}(A(F))) \simeq \text{RHom}(\text{perf}(X), \text{perf}(Y)) described in [11].

It was shown in Section 8 of [11] that the following diagram commutes in $\text{Ho}(\text{dg-cat})$.

\[
\begin{array}{ccc}
\text{perf}(X \times Y) & \xrightarrow{\eta^{-1}} & \text{perf}(A(E)^{\text{op}} \otimes A(F)) \\
\downarrow{\gamma} & & \downarrow{\beta} \\
\text{RHom}(\text{perf}(X), \text{perf}(Y)) & \xrightarrow{\eta^{-1}} & \text{RHom}(\text{perf}(A(E)), \text{perf}(A(F)))
\end{array}
\]

Theorem 5 is then a direct consequence of Theorem 4 and the above commutative diagram.

\textbf{Remark.} Instead of choosing a compact generator $E$ of $D_{\text{qcoh}}(X)$ and using the DG-algebra $A(E)$ to define $\langle \cdot, \cdot \rangle_{\text{Shk}}$ on $\text{HH}^\bullet(X)$, we could make do with any DG-algebra $A$ such that $\text{perf}(A)$ is quasi-equivalent to $\text{perf}(X)$.

\section{The Mukai pairing.}

\subsection{Some recollections.}

Let $X$ be a smooth proper scheme. Let $S_X$ denote the shifted line bundle on $X$ tensoring with which yields the Serre duality functor on the bounded derived category $D^b(X)$ of coherent $\mathcal{O}_X$-modules. If $f : X \to Y$ is a morphism of schemes, $f_\ast, f^\ast$ etc shall denote the corresponding derived functors in this section. Let $\Delta : X \to X \times X$ denote the diagonal embedding. Let $\Delta_!$ denote the left adjoint of $\Delta^!$. Let $\mathcal{O}_\Delta$ denote $\Delta_! \mathcal{O}_X$. Recall from [1] that there is an isomorphism

$$\text{HH}^\bullet(X) \simeq \text{RHom}_{X \times X}(\Delta_! \mathcal{O}_X, \mathcal{O}_\Delta).$$

Since $\Delta_! \mathcal{O}_X \simeq \Delta_! S_X^{-1}$, tensoring with $\pi_2^* S_X$ yields an isomorphism

$$D : \text{RHom}(\Delta_! \mathcal{O}_X, \Delta_* \mathcal{O}_X) \to \text{RHom}(\Delta_* \mathcal{O}_X, \Delta_* S_X).$$

\textbf{Definition.} The Mukai pairing $\langle \cdot, \cdot \rangle_M$ on $\text{HH}^\bullet(X)$ is the pairing

$$v \otimes w \rightsquigarrow \text{tr}_{X \times X}(D(v) \circ w).$$
where $\text{tr}_{X \times X}$ denotes the Serre duality trace on $X \times X$. The same pairing was constructed in the DG-algebra setup in [9].

Recall that the Hochschild-Kostant-Rosenberg map $I_{HKR}$ induces an isomorphism
\[ \text{HH}_i(X) \cong \oplus_{j} H^{j-i}(X, \Omega^j_X) \]
which we shall also denote by $I_{HKR}$. Let $f_x$ denote the linear functional on $\oplus_{p,q} H^p(X, \Omega^q_X)$ that coincides with the Serre duality trace on $H^n(X, \Omega^n_X)$ and vanishes on other direct summands. Let $\ast$ denote the involution on $\oplus_{p,q} H^p(X, \Omega^q_X)$ that acts on the summand $H^p(X, \Omega^q_X)$ by $(-1)^p$. The following result (implicitly in [7] and explicitly in [8]) computes $\langle , \rangle_M$ at the level of Hodge cohomology.

**Theorem 6.** For $a, b \in \text{HH}_\bullet(X)$,
\[ \langle a, b \rangle_M = \int_X I_{HKR}(a)^\ast I_{HKR}(b) td(T_X) . \]

### 2.2 Integral transforms in Hochschild homology.

Any $\Phi \in \text{perf}(X \times Y)$ yields an integral transform
\[ \Phi : \text{perf}(X) \rightarrow \text{perf}(Y) \]
as described in Section 1.2. Note that if $\Psi \in \text{perf}(Y \times Z)$, the image of the kernel of the integral transform $\Psi \circ \Phi$ in $\text{D} \circ \text{perf}(X \times Z)$ is precisely $\pi_{XZ}^\ast(\pi_{XY}^\ast \Phi \otimes \pi_{YZ}^\ast \Psi)$. A priori, there is more than one construction of the corresponding integral transform $\Phi_* : \text{HH}_\bullet(X) \rightarrow \text{HH}_\bullet(Y)$ such that

a. $(\Psi \circ \Phi)_* = \Psi_* \circ \Phi_* .

b. The following diagram commutes.
\[
\begin{array}{ccc}
\text{D(gerf}(X)) & \xrightarrow{\Phi} & \text{D(gerf}(Y)) \\
\downarrow \text{Ch} & & \downarrow \text{Ch} \\
\text{HH}_0(X) & \xrightarrow{\Phi^\ast} & \text{HH}_0(Y)
\end{array}
\]

For example, $\Phi^\ast_{\text{nat}}$ is seen to satisfy these properties without much difficulty. Another construction of $\Phi_*$ was given by A. Caldararu in [1]. Broadly speaking, one views $\text{HH}_\bullet(X)$ as an "ext of functors", $\text{Ext}(S_{\text{X}}^{-1}; \text{id})$. This can be done rigorously as in [3]. Let $\Phi^\ast$ be a left adjoint of $\Phi$. Then,
if $\alpha \in \text{Ext}(S_X^{-1}, \text{id})$, $\Phi_*(\alpha)$ is the following composite where the unlabeled arrows are adjunctions.

\[
\begin{array}{c}
S_Y^{-1} \\
\downarrow \\
\Phi \circ \Phi^Y \circ S_Y^{-1} \\
\downarrow \\
\Phi \circ S_X^{-1} \circ S_X \circ \Phi^Y \circ S_Y^{-1} \xrightarrow{\text{id} \circ \text{id} \circ \text{id} \circ \text{id}} \Phi \circ S_X \circ \Phi^Y \circ S_Y^{-1} \longrightarrow \text{id}_Y
\end{array}
\]

Theorem 6 enables us to give yet another construction of $\Phi_*$. This construction of $\Phi_*$ is motivated by Theorem 4, and plays a key role in relating the Mukai pairing to the natural pairing constructed in Section 1. In the rest of this section, the identification of $\text{HH}_*(X \times Y)$ with $\text{HH}_*(X) \otimes \text{HH}_*(Y)$ will be via the inverse of the Kunneth isomorphism. Recall that if $\Phi \in \text{perf}(X \times Y)$, the Chern character $\text{Ch}(\Phi) \in \text{HH}_*(X \times Y) \simeq \text{HH}_*(X) \otimes \text{HH}_*(Y)$ may be viewed as a $\mathbb{K}$-linear map from $\mathbb{K}$ to $\text{HH}_*(X) \otimes \text{HH}_*(Y)$.

Let $W : \text{HH}_*(X) \to \text{HH}_*(X)$ be the unique involution corresponding via $I_{HKR}$ to the involution $*$ on $\bigoplus_{p,q} H^p(X, \Omega^q_X)$ mentioned in the previous subsection.

**Construction.** We define $\Phi^\text{muk}_* : \text{HH}_*(X) \to \text{HH}_*(Y)$ to be the composite

\[
\begin{array}{c}
\text{HH}_*(X) \\
\downarrow \text{id} \otimes \text{Ch}(\Phi) \\
\text{HH}_*(X) \otimes^2 \otimes \text{HH}_*(Y) \\
\xrightarrow{W \otimes \text{id} \otimes \text{id}} \text{HH}_*(X) \otimes^2 \otimes \text{HH}_*(Y) \\
\xrightarrow{(\cdot)_Y \otimes \text{id}} \text{HH}_*(Y)
\end{array}
\]

**Proposition 1.** If $\Phi \in \text{perf}(X \times Y)$ and $\Psi \in \text{perf}(Y \times Z)$ then

\[
(\Psi \circ \Phi)^\text{muk}_* = \Psi^\text{muk}_* \circ \Phi^\text{muk}_*.
\]

**Proof.** We shall denote $\bigoplus p, qH^p(X, \Omega^q_X)$ by $H^\bullet(X)$. Recall (Theorem 4.5 in [2]) that for any smooth scheme $Z$, $I_{HKR} \circ \text{Ch} = \text{ch}$, the right hand side being the familiar Chern character map from $D(\text{perf}(Z))$ to $H^\bullet(Z)$.

Let $a \in \text{HH}_*(X)$. Note that $\text{HH}_0(X \times Y) \simeq \bigoplus_i \text{HH}_i(X) \otimes \text{HH}_{-i}(Y)$. Hence,

\[
\text{Ch}(\Phi) = \sum_i \sum_{\lambda(i) \in I_i} \alpha_{\lambda(i)} \otimes \beta_{\lambda(i)}
\]
for some index sets $I_i$ and $\alpha_{\lambda(i)} \in \text{HH}_i(X)$ and $\beta_{\lambda(i)} \in \text{HH}_{-i}(Y)$. By Theorem 6 and the construction of $\Phi^muk_*$,

$$IHKR(\Phi^muk_*(a)) = \sum_i \sum_{\lambda(i) \in I_i} \left( \int_X IHKR(a)IHKR(\alpha_{\lambda(i)})td(T_X) \right)IHKR(\beta_{\lambda(i)}) \, .$$

(3)

Now suppose that

$$\text{Ch}(\Psi) = \sum_j \sum_{\mu(j) \in J_j} \gamma_{\mu(j)} \otimes \delta_{\mu(j)}$$

for some index sets $J_j$ and $\gamma_{\mu(j)} \in \text{HH}_j(Y)$ and $\delta_{\mu(j)} \in \text{HH}_{-j}(Z)$. Then, by (3),

$$IHKR(\Psi^muk_* \circ \Phi^muk_*(a)) = \sum_{i,j} \sum_{\lambda(i) \in I_i, \mu(j) \in J_j} IHKR(\delta_{\mu(j)})(\int_X IHKR(a)IHKR(\alpha_{\lambda(i)})td(T_X))(\int_Y IHKR(\beta_{\lambda(i)})IHKR(\gamma_{\mu(j)})td(T_Y)) \, .$$

Recall that $\Psi \circ \Phi = \pi_{XZ\ast}(\pi^*_Y \Psi \otimes \pi^*_{XY} \Phi)$ The desired proposition will follow from (3) if we can show that

$$\text{ch}(\Psi \circ \Phi) = \sum_{i,j} \sum_{\lambda(i) \in I_i, \mu(j) \in J_j} (\int_X IHKR(\beta_{\lambda(i)})IHKR(\gamma_{\mu(j)})td(T_Y))\alpha_{\lambda(i)} \otimes \delta_{\mu(j)} \, .$$

(4)

Recall that after identifying $H^\ast(X \times Y)$ with $H^\ast(X) \otimes H^\ast(Y)$, $\pi^*_Y$ gets identified with $\int_X \otimes \text{id}$. Also, $\pi^*_Y$ is identified with the map $a \sim 1 \otimes a$ from $H^\ast(Y)$ to $H^\ast(X \times Y)$. With this in mind, (4) can be rewritten as,

$$\text{ch}(\Psi \circ \Phi) = \pi_{XZ\ast}(\text{ch}(\pi^*_Y(\Phi))).\text{ch}(\pi^*_Y \Psi).\pi^*_Y \text{td}(T_Y)) \, .$$

This follows directly from the Riemann-Roch-Hirzebruch theorem applied to the map $\pi_{XZ} : X \times Y \times Z \to X \times Z$.

Let $\mathcal{O}_\Delta = \Delta_\ast \mathcal{O}_X$ be treated as the kernel of an integral transform from $X$ to $X$. Then,

**Proposition 2.**

$$\mathcal{O}_{\Delta_\ast muk} = \text{id} \, .$$
Proof. Since $\mathcal{O}_\Delta$ is the kernel of the identity, $\mathcal{O}_\Delta \circ \mathcal{O}_\Delta = \mathcal{O}_\Delta$. By Proposition 1, $\mathcal{O}_{\Delta^s}^{\mu k}$ is an idempotent endomorphism of $\text{HH}_\bullet(X)$. To prove that it is the identity, it suffices to show that it is surjective.

For this, note that $\mathcal{O}_{\Delta^s}^{\mu k} = \text{id}$. By theorem 4,

$$\text{Ch}(\mathcal{O}_\Delta) = \sum_i \sum_k e_{i,k} \otimes f_{i,k}$$

where the $e_{i,k}$ form a basis of $\text{HH}_i(X)$ and the $f_{i,k}$ form a basis of $\text{HH}_{-i}(X)$ such that

$$\langle f_{i,k}, e_{i,l} \rangle_{\text{Shk}} = \delta_{k,l}.$$  

The $\delta$ on the right hand side of the above equation is the Kronecker delta.

Let $W$ be the involution on $\text{HH}_\bullet(X)$ which we defined earlier before constructing $\Phi_\mu^{\mu k}$. It follows that if $x \in \text{HH}_i(X)$, then

$$\mathcal{O}_{\Delta^s}^{\mu k}(x) = \sum_k \langle W(x), e_{-i,k} \rangle_M f_{-i,k}.$$  

Recall from [1] that the pairing $\langle \cdot, \cdot \rangle_M$ is non-degenerate. Moreover, $W$ is an involution on $\text{HH}_\bullet(X)$. Since the $e_{-i,k}$ form a basis of $\text{HH}_{-i}(X)$, there exist elements $x_k$ in $\text{HH}_i(X)$ such that

$$\langle W(x_l), e_{-i,k} \rangle_M = \delta_{kl}.$$  

Clearly,

$$\mathcal{O}_{\Delta^s}^{\mu k}(x_k) = f_{-i,k}.$$  

This proves that $\mathcal{O}_{\Delta^s}^{\mu k}$ is surjective, as was desired. \hfill \Box

We are now ready to prove Theorem 1.

**Proof of Theorem 1.**

Proof. This follows almost immediately from the fact that $\mathcal{O}_{\Delta^s}^{\mu k} = \mathcal{O}_{\Delta^s}^{\mu k} = \text{id} : \text{HH}_\bullet(X) \rightarrow \text{HH}_\bullet(X)$. Since $\mathcal{O}_{\Delta^s}^{\mu k} = \text{id}$,

$$\langle f_{-i,k}, e_{-i,l} \rangle_{\text{Shk}} = \delta_{k,l}.$$  

On the other hand since $\mathcal{O}_{\Delta^s}^{\mu k} = \text{id}$ by Proposition 2,

$$\langle W(f_{-i,k}), e_{-i,l} \rangle_M = \delta_{k,l}.$$
It follows from the $\mathbb{K}$ bi-linearity of the pairings

$$(a, b) \rightsquigarrow \langle a, b \rangle_{\text{Shk}}$$

$$(a, b) \rightsquigarrow \langle W(a), b \rangle_M$$

that

$$\langle a, b \rangle_{\text{Shk}} = \langle W(a), b \rangle_M.$$  \hspace{1cm} (5)

Recall that $\vee$ denotes the involution on $\text{HH}^* (X)$ corresponding via $\text{I}^H \text{K} \text{R}$ to the involution on $\text{H}^* (X)$ which acts on the direct summand $\text{H}^q (X, \Omega^p_X)$ by multiplication by $(-1)^p$. Now, $\text{I}^H \text{K} \text{R} (a)^* \cup \text{I}^H \text{K} \text{R} (b) = \text{I}^H \text{K} \text{R} (b) \cup \text{I}^H \text{K} \text{R} (a^\vee)$ in $\text{H}^* (X)$. Hence, Theorem 6 may be rewritten to say that

$$\langle a, b \rangle_M = \int_X \text{I}^H \text{K} \text{R} (b) \text{I}^H \text{K} \text{R} (a^\vee) \text{td}(T_X).$$

By (5),

$$\langle a, b \rangle_{\text{Shk}} = \int_X \text{I}^H \text{K} \text{R} (a) \text{I}^H \text{K} \text{R} (b) \text{td}(T_X) = \langle b^\vee, a \rangle_M.$$

This proves Theorem 1.

Recall from [1] that the integral transform from $\text{D} (\text{perf} (Y))$ to $\text{D} (\text{perf} (X))$ due to $\text{R} \text{H} \text{om} (\Phi, \mathcal{O}_{X \times Y}) \otimes \omega X S_X$ is the right adjoint of that from $\text{D} (\text{perf} (X))$ to $\text{D} (\text{perf} (Y))$ due to $\Phi$. Let $\Phi^!$ denote $\text{R} \text{H} \text{om} (\Phi, \mathcal{O}_{X \times Y}) \otimes \omega X S_X$. We also have the following proposition, which shows that $\Phi^!_{\text{muk}}$ is a "good candidate" for the integral transform on Hochschild homology defined by $\Phi$.

**Proposition 3.** If $x \in \text{HH}^* (X)$ and $y \in \text{HH}^* (Y)$, then

$$\langle \Phi^!_{\text{muk}} (x), y \rangle_M = \langle x, \Phi^!_{\text{muk}} (y) \rangle_M.$$

**Proof.** The notation used in this proof is as in the proof of Proposition 1. Assume that after identifying $\text{HH}^* (X \times Y)$ with $\text{HH}^* (X) \otimes \text{HH}^* (Y)$ (via the inverse of the Kunneth map),

$$\text{Ch}(\Phi) = \sum_i \sum_{\lambda(i) \in I_i} \alpha_{\lambda(i)} \otimes \beta_{\lambda(i)}$$

for some index sets $I_i$ and $\alpha_{\lambda(i)} \in \text{HH}_i (X)$ and $\beta_{\lambda(i)} \in \text{HH}_{-i} (Y)$. Then, by Theorem 6 and (3),

$$\langle \Phi^!_{\text{muk}} (x), y \rangle_M = \int_X \text{I}^H \text{K} \text{R} (b) \text{I}^H \text{K} \text{R} (a^\vee) \text{td}(T_X) = \langle b^\vee, a \rangle_M.$$
\[
\sum_i \sum_{\lambda(i) \in I_i} \left( \int_X I_{HKR}(x) I_{HKR}(\alpha_{\lambda(i)}) \text{td}(T_X) \right) \left( \int_Y I_{HKR}(\beta_{\lambda(i)})^* I_{HKR}(y) \text{td}(T_Y) \right).
\]

Note that \( \text{Ch}(\Phi^1) = \sum_i \sum_{\lambda(i) \in I_i} (-1)^i W(\beta_{\lambda(i)}) \otimes [W(\alpha_{\lambda(i)}) \text{Ch}(S_X)]. \) The \((-1)^i\) comes from the fact that the composite

\[
\text{HH}_\bullet(X) \otimes \text{HH}_\bullet(Y) \xrightarrow{K} \text{HH}_\bullet(X \times Y) \xrightarrow{K^{-1}} \text{HH}_\bullet(Y) \otimes \text{HH}_\bullet(X)
\]

is the signed map swapping factors. It follows from Theorem 6 and (3) that

\[
\langle x, \Phi^1 \rangle_M = \sum_i \sum_{\lambda(i) \in I_i} \left( \int_Y I_{HKR}(\alpha_{\lambda(i)})^* I_{HKR}(y) \text{td}(T_Y) \right) \left( \int_X I_{HKR}(\beta_{\lambda(i)}) \text{td}(T_X) \right).
\]

Now, if \( n \) is the dimension of \( X \), \( \text{ch}(S_X) = (-1)^n \text{ch}(\Omega^n_X) \). Also, \( \text{td}(T_X) \text{ch}(\Omega^n_X) = \text{td}(T_X)^* \) (see [2]). It follows that

\[
I_{HKR}(x)^* I_{HKR}(\alpha_{\lambda(i)})^* \text{ch}(S_X) \text{td}(T_X) = (-1)^n (I_{HKR}(x) I_{HKR}(\alpha_{\lambda(i)}) \text{td}(T_X))^*.
\]

Hence,

\[
\int_X I_{HKR}(x)^* I_{HKR}(\alpha_{\lambda(i)})^* \text{ch}(S_X) \text{td}(T_X) = \int_X I_{HKR}(x) I_{HKR}(\alpha_{\lambda(i)}) \text{td}(T_X)
\]

This proves the desired proposition.

\[\square\]

Note that Proposition 1 and Proposition 3 parallel Theorems 5.3 and 7.3 respectively in [1]. However, since we use the Riemann-Roch theorem for (proper) projections to prove Proposition 1, the construction of \( \Phi^\text{muk}_* \) by itself does not amount to a self-contained construction of integral transforms in Hochschild homology at this stage. However, it helps prove Theorem 1, which in turn leads to Theorem 2, showing that all three constructions of integral transforms in Hochschild homology coincide. In particular, it tells us that the integral transform constructed by A. Caldararu [1] coincides with the more "natural" construction of the integral transform constructed by D. Shklyarov [10].

Let \( \Phi \in \text{perf}(X \times Y) \). Denote the integral transform \( \Phi_* : \text{HH}_\bullet(X) \to \text{HH}_\bullet(Y) \) constructed by A. Caldararu [1] and described briefly earlier in this section by \( \Phi^\text{cal}_* \).
Proof of Theorem 2.

Proof. That $\Phi^\text{muk}_* = \Phi^\text{nat}_*$ is an immediate consequence of Theorem 1 and Theorem 5. We therefore need to show that $\Phi^\text{muk}_* = \Phi^\text{cal}_*$. For this, we will follow D. Shklyarov and imitate the proof of Theorem 4 (Theorem 3.4 in [10]) in [10]).

Step 1: Recall that if $\Phi \in \text{perf} (X \times Y)$ and $\Phi' \in \text{perf} (X' \times Y')$, $\Phi \boxtimes \Phi' \in \text{perf} (X \times X' \times Y \times Y')$. We then have integral transforms in Hochschild homology

$$\Phi^\text{muk}_* : \text{HH}_*(X) \to \text{HH}_*(Y), \Phi'^\text{muk}_* : \text{HH}_*(X') \to \text{HH}_*(Y')$$

$$(\Phi \boxtimes \Phi')^\text{muk}_* : \text{HH}_*(X \times X') \to \text{HH}_*(Y \times Y') .$$

Identify $\text{HH}_*(X \times X')$ and $\text{HH}_*(Y \times Y')$ with $\text{HH}_*(X) \otimes \text{HH}_*(X')$ and $\text{HH}_*(Y) \otimes \text{HH}_*(Y')$ respectively via the inverse of the relevant Kunneth isomorphisms. It follows from the construction of $\Phi^\text{muk}_*$ that

$$(\Phi \boxtimes \Phi')^\text{muk}_* = \Phi^\text{muk}_* \otimes \Phi'^\text{muk}_* .$$

Similarly, we have integral transforms in Hochschild homology

$$\Phi^\text{cal}_* : \text{HH}_*(X) \to \text{HH}_*(Y), \Phi'^\text{cal}_* : \text{HH}_*(X') \to \text{HH}_*(Y')$$

$$(\Phi \boxtimes \Phi')^\text{cal}_* : \text{HH}_*(X \times X') \to \text{HH}_*(Y \times Y') .$$

It can be verified without much difficulty (see [16], Lemma 2.1 for instance) that

$$(\Phi \boxtimes \Phi')^\text{cal}_* = \Phi^\text{cal}_* \otimes \Phi'^\text{cal}_* .$$

Step 2: Note that $\Phi \in \text{perf} (X \times Y)$ may also be thought of as the kernel of an integral transform from $\text{Spec } \mathbb{K}$ to $X \times Y$. We will denote $\Phi$ thought of in this manner by $\Phi_{\text{pt-}\to X \times Y}$. Let $\Delta$ denote $\mathcal{O}_\Delta$ thought of as the kernel of an integral transform from $X \times X$ to $\text{Spec } \mathbb{K}$. Also identify $\text{HH}_*(X)$ with $\text{HH}_*(X) \otimes \text{HH}_*(\text{Spec } \mathbb{K})$ via the map $y \mapsto y \otimes 1$. Then,

$$\Phi = \Delta \circ (\mathcal{O}_\Delta \boxtimes \Phi_{\text{pt-}\to X \times Y})$$

$$\implies \Phi^\text{muk}_* = \Delta^\text{muk}_* \circ (\mathcal{O}_\Delta \boxtimes \Phi_{\text{pt-}\to X \times Y})^\text{muk}_* = \Delta^\text{muk}_* \circ (\mathcal{O}_\Delta^\text{muk}_* \boxtimes (\Phi_{\text{pt-}\to X \times Y})^\text{muk}_*(1))$$

$$\Phi^\text{cal}_* = \Delta^\text{cal}_* \circ (\mathcal{O}_\Delta \boxtimes \Phi_{\text{pt-}\to X \times Y})^\text{cal}_* = \Delta^\text{cal}_* \circ (\mathcal{O}_\Delta^\text{cal}_* \boxtimes (\Phi_{\text{pt-}\to X \times Y})^\text{cal}_*(1))$$

Now, by Proposition 2,

$$\mathcal{O}_\Delta^\text{cal}_* = \mathcal{O}_\Delta^\text{muk}_* = \text{id} .$$
Also, \((\Phi_{pt\to X\times Y})^\text{cal}_*(1) = \text{Ch}(\Phi)\) by Definition 6.1 in [1] and Theorem 4.5 in [2]. \((\Phi_{pt\to X\times Y})^\text{muk}_*(1) = \text{Ch}(\Phi)\) by the construction of \((\Phi_{pt\to X\times Y})^\text{muk}_*(1)\). We therefore need to show that

\[
\Delta^\text{muk}_* = \Delta^\text{cal}_* : \text{HH}_\bullet(X \times X) \to \text{HH}_\bullet(\text{Spec } \mathbb{K}) = \mathbb{K}.
\]

With the above identification of \(\text{HH}_\bullet(\text{Spec } \mathbb{K})\) with \(\mathbb{K}\), for any \(x \in \text{HH}_\bullet(\text{Spec } \mathbb{K})\),

\[
x = \langle x, 1 \rangle_M.
\]

Let \(\Delta^!\) denote \(\text{RHom}(\Delta, \mathcal{O}_{X\times X}) \otimes^L S_{X\times X}\). If \(\alpha \in \text{HH}_\bullet(X \times X)\),

\[
\langle \Delta^\text{muk}_*(\alpha), 1 \rangle_M = \langle \alpha, \Delta^\text{muk}_!(1) \rangle_M
\]

by Proposition 3. By Theorem 7.3 in [1],

\[
\langle \Delta^\text{cal}_*(\alpha), 1 \rangle_M = \langle \alpha, \Delta^\text{cal}_!(1) \rangle_M.
\]

Now, \(\Delta^\text{cal}_!(1) = \text{Ch}(\Delta^!)\) by Definition 6.1 in [1] and Theorem 4.5 in [2]. \(\Delta^\text{muk}_!(1) = \text{Ch}(\Delta^!\) by the construction of \(\Delta^\text{muk}_*\). This yields the desired theorem.

\[\Box\]

### 2.3 When \(X\) is Calabi-Yau.

In such a situation, \(\text{D}^b(X)\) can be be thought of as the category of open states of the B-Model on \(X\) (see [3]). The corresponding algebra of closed states is the Hochschild cohomology \(\text{HH}^\bullet(\text{perf } (X)) \simeq \text{HH}^\bullet(X)\). As \(X\) is Calabi-Yau, there is an identification

\[
\text{HH}^\bullet(X) \simeq \text{HH}_\bullet(X).
\]

The Mukai pairing constructed by A. Caldararu in [1] on \(\text{HH}_\bullet(X)\) then gives a pairing on \(\text{HH}^\bullet(X)\). Moreover, for any \(\mathcal{E} \in \text{D}^b(X)\), there are natural maps

\[
\iota^\mathcal{E} : \text{Hom}_{\text{D}^b(X)}(\mathcal{E}, \mathcal{E}) \to \text{HH}^\bullet(X)
\]

\[
\iota_\mathcal{E} : \text{HH}^\bullet(X) \to \text{Hom}_{\text{D}^b(X)}(\mathcal{E}, \mathcal{E})
\]

as constructed in [3]. The Cardy condition verifies that this data gives a topological quantum field theory. Of course, the Mukai pairing in this case is the pairing obtained by the action of the class of a genus 0 Riemann-surface with two incoming closed boundaries and no outgoing boundary in \(H_0(\mathcal{M}_0(2,0))\) on \(\text{HH}_\bullet(X)\), the action coming from the fact that \(\text{HH}^\bullet(X)\)
with Mukai pairing is a "good" algebra of closed states as verified by the Cardy condition.

On the other hand, [4] gives the category of open states of the B-Model on $X$ as an $A_\infty$ enrichment of $D^b(X)$. The closed TCFT one associates with this category has homology

$$\text{HH}_\bullet(X) \simeq \text{HH}^\bullet(X).$$

This is also equipped with a pairing coming out of the action of the class of a genus 0 Riemann-surface with two incoming closed boundaries and no outgoing boundary in $H_0(M_0(2,0))$ on the homology of the closed TCFT one constructs in [4] from the B-Model. Whether these pairings coincide is however, not clear currently.

Theorem 1 is similar Conjecture 6.2 in [10] for Calabi-Yau algebras $A$ such that perf $(A)$ is quasi-equivalent to perf $(X)$ for some quasi-compact separated smooth scheme $X$.

2.4 Proof of Theorem 3.

The sheafification of the Dennis trace map. Let us briefly recall how the sheafification of the Dennis trace map is constructed. The material we are recalling is from [12],[13],[14] and [15]. Let $X$ be a smooth quasicompact separated scheme. As in Section 1.2, choose a compact generator $E$ of $D_{qcoh}(X)$ and a DG-algebra $A(E)$ such that perf $(A(E))$ is quasi-equivalent to perf $(X)$. Let $Z^0(\text{perf}(A(E)))$ be the exact category whose objects are those of perf $(A(E))$ such that

$$\text{Hom}_{Z^0(\text{perf}(A(E)))}(M, N) = Z^0(\text{Hom}_{\text{perf}(A(E))})(M, N).$$

As pointed out by B. Keller in [14], using the Waldhausen structure of $Z^0(\text{perf}(A(E)))$, we can construct a Dennis trace map

$$\text{Dtr} : K_i(X) \simeq K_i(Z^0(\text{perf}(A(E)))) \to \text{HH}_{i,\text{McC}}(Z^0(\text{perf}(A(E)))) \forall i \geq 0.$$ 

Here, $\text{HH}_{i,\text{McC}}$ is the Hochschild homology constructed by R. McCarthy in [15]. As Keller further points out in [14], there is a natural transformation

$$\text{HH}_{i,\text{McC}}(Z^0(\text{perf}(A(E)))) \to \text{HH}_i(Z^0(\text{perf}(A(E)))).$$ 

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Further, we also have a natural transformation
\[ HH_i(Z^0(\text{perf}(A(E)))) \to HH_i(\text{perf}(A(E))) \].

The obvious compositions then give us a map
\[ \text{Ch}^i : K_i(X) \simeq K_i(Z^0(\text{perf}(A(E)))) \to HH_i(\text{perf}(A(E))) \simeq HH_i(X) \].

Let \( Y \) be a smooth quasicompact separated scheme. Let \( F \) and \( A(F) \) be as in Section 1.2. Let \( \Psi \in \text{perf} (A(F)_{\text{op}} \otimes A(E)) \). The following proposition, analogous to Theorem 7.1 of [1], says that the sheafification of the Dennis trace map is "functorial".

**Proposition 4.** The following diagram commutes.

\[
\begin{array}{ccc}
K_i(Z^0(\text{perf}(A(E)))) & \xrightarrow{\Psi^*} & K_i(Z^0(\text{perf}(A(F)))) \\
\downarrow \text{Ch}^i & & \downarrow \text{Ch}^i \\
HH_i(\text{perf}(A(E))) & \xrightarrow{\Psi_{\text{nat}}^*} & HH_i(\text{perf}(A(F)))
\end{array}
\]

**Proof.** This proposition will follow easily once we verify that \( \Psi: Z^0(\text{perf}(A(E))) \to Z^0(\text{perf}(A(F))) \) preserves cofibrations and weak equivalences. By [12], the weak equivalences in \( Z^0(\text{perf}(A)) \) for any DG-algebra \( A \) are quasiisomorphisms. The cofibrations in \( Z^0(\text{perf}(A)) \) are morphisms of \( A \)-modules that admit retractions as morphisms of graded \( A \)-modules. That \( \Psi \) preserves cofibrations follows without difficulty from the fact that \( \Psi: \text{perf}(A(E)) \to \text{perf}(A(F)) \) is a DG-functor. That \( \Psi \) preserves weak equivalences follows from the fact that perfect modules are homotopically projective (see Proposition 2.5 of [10]).

**Proof of Theorem 3.** We warn the reader that in the proof that follows, \( X \) and \( Y \) denote proper smooth quasicompact separated schemes.

**Proof. Step 1:** Let \( \Phi \in \text{perf}(X \times Y). \) The first step is to note that even though \( Z \) is not necessarily proper, the kernel \( \Phi \otimes \mathcal{O}_{\Delta Z} \in \text{perf}(X \times Z \times Y \times Z) \) induces an integral transform from \( \text{perf}(X \times Z) \) to \( \text{perf}(Y \times Z) \). This follows from the fact that if \( E \) and \( F \) are compact generators of \( D_{\text{coh}}(X) \) and \( D_{\text{coh}}(Z) \) respectively, the compact generator \( E \boxtimes F := \pi_Y^* E \otimes \pi_Z^* F \) of \( D_{\text{coh}}(X \times Z) \) is mapped by the integral transform with kernel \( \Phi \otimes \mathcal{O}_{\Delta Z} \) to the perfect complex \( \pi_Y^*(\Phi \otimes_{\mathcal{O}} \pi_X^* E) \boxtimes F \).
Also, after identifying $\text{HH}_\bullet(X \times Z)$ and $\text{HH}_\bullet(Y \times Z)$ with $\text{HH}_\bullet(X) \otimes \text{HH}_\bullet(Z)$ and $\text{HH}_\bullet(Y) \otimes \text{HH}_\bullet(Z)$ respectively via the inverse of the relevant Kunneth isomorphisms,

$$(\Phi \boxtimes \mathcal{O}_{\Delta_Z})^\text{nat}_* = \Phi^\text{nat}_* \otimes \text{id} : \text{HH}_\bullet(X) \otimes \text{HH}_\bullet(Z) \to \text{HH}_\bullet(Y) \otimes \text{HH}_\bullet(Z).$$

This follows from the facts that $\mathcal{O}_{\Delta_Z}^\text{nat}_* = \text{id}$ and from Proposition 2.11 of [10].

**Step 2:** By the Proposition 4, the following diagram commutes.

$$
\begin{array}{ccc}
K_i(\text{perf } (X \times Z)) & \xrightarrow{(\Phi \boxtimes \mathcal{O}_{\Delta_Z})_*} & K_i(\text{perf } (Y \times Z)) \\
\downarrow \text{Ch}^i & & \downarrow \text{Ch}^i \\
\text{HH}_i(X \times Z) & \xrightarrow{(\Phi \boxtimes \mathcal{O}_{\Delta_Z})^\text{nat}_*} & \text{HH}_i(Y \times Z)
\end{array}
$$

(7)

After identifying $\text{HH}_\bullet(X \times Z)$ and $\text{HH}_\bullet(Y \times Z)$ with $\text{HH}_\bullet(X) \otimes \text{HH}_\bullet(Z)$ and $\text{HH}_\bullet(Y) \otimes \text{HH}_\bullet(Z)$ respectively via the inverse of the relevant Kunneth isomorphisms, we have the following commutative diagram by (7) and (8).

$$
\begin{array}{ccc}
K_i(\text{perf } (X \times Z)) & \xrightarrow{(\Phi \boxtimes \mathcal{O}_{\Delta_Z})_*} & K_i(\text{perf } (Y \times Z)) \\
\downarrow \text{Ch}^i & & \downarrow \text{Ch}^i \\
\oplus_{p+q=i} \text{HH}_p(X) \otimes \text{HH}_q(Z) & \xrightarrow{\Phi^\text{nat}_* \otimes \text{id}} & \oplus_{p+q=i} \text{HH}_p(Y) \otimes \text{HH}_q(Z)
\end{array}
$$

(8)

Now, it follows from Theorem 1 and Theorem 3 that $\Phi^\text{muk}_* = \Phi^\text{nat}_*$. Hence, by (8) and Proposition 3,

$$
\langle , \rangle_M \otimes \text{id}_{\text{HH}_\bullet(Z)}(f^*y \otimes \text{Ch}^i(\alpha)) = \langle , \rangle_M \otimes \text{id}_{\text{HH}_\bullet(Z)}(y \otimes (\text{id} \times f)_* \text{Ch}^i(\alpha))
$$

(9)

for any $\alpha \in K_i(Z \times X)$, $y \in \text{HH}_\bullet(Y)$. By Theorem 4, (9) can be rewritten to say that

$$
\int_X I_{HKR}(f^*(y))^* \text{ch}^i(\alpha) \text{td}(T_X) = \int_Y I_{HKR}(y)^* \text{ch}^i((f \times \text{id})_* \alpha) \text{td}(T_Y)
$$

as elements of $H^*(Z)$. The desired theorem now follows from the facts that $f^*$ commutes with $I_{HKR}$ (see Theorem 7 of [7]) and commutes with the involution $\ast$. 

\[\square\]
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