Numerical evolutions of fields on the 2-sphere using a spectral method based on spin-weighted spherical harmonics

Florian Beyer, Boris Daszuta, Jörg Frauendiener and Ben Whale

Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9010, New Zealand

E-mail: fbeyer@maths.otago.ac.nz, dasbo917@student.otago.ac.nz, joergf@maths.otago.ac.nz and bwhale@maths.otago.ac.nz

Received 13 September 2013, revised 13 January 2014
Accepted for publication 24 February 2014
Published 14 March 2014

Abstract
Many applications in science call for the numerical simulation of systems on manifolds with spherical topology. Through the use of integer spin-weighted spherical harmonics, we present a method which allows for the implementation of arbitrary tensorial evolution equations. Our method combines two numerical techniques that were originally developed with different applications in mind. The first is Huffenberger and Wandelt’s spectral decomposition algorithm to perform the mapping from physical to spectral space. The second is the application of Luscombe and Luban’s method, to convert numerically divergent linear recursions into stable nonlinear recursions, to the calculation of reduced Wigner $d$-functions. We give a detailed discussion of the theory and numerical implementation of our algorithm. The properties of our method are investigated by solving the scalar and vectorial advection equation on the sphere, as well as the $2+1$ Maxwell equations on a deformed sphere.

Keywords: spin-weighted spherical harmonics, spectral methods, $\bar{\partial}$-formalism, numerical relativity
PACS numbers: 02.60.−x, 02.70.Hm, 04.25.D−

1. Introduction

We present a method for the numerical calculation of solutions to general hyperbolic partial differential equations (PDEs) over the sphere that combines several existing techniques in novel
ways. Our motivating interest is the analysis of gravitational radiation in a neighbourhood of infinity via Friedrich’s conformal field equations [12], where it is convenient to regard space-time as the product of a two-dimensional Lorentzian manifold and a Euclidean 2-sphere. The main difficulty for the numerical solution of problems in this geometric setting stems from the fact that $\mathbb{S}^2$ cannot be covered by a single chart. Hence, the coordinate description of fields inevitably breaks down. Spectral methods avoid most of the problems caused by this, as fields on $\mathbb{S}^2$ are expressed as a sum of functions that form well-defined bases on the sphere. Assuming that the properties of the functions in the sum are well understood, it is possible to avoid working directly with coordinate expressions that suffer from coordinate singularities. As an example, we mention that the recent work of [8, 26] in the context of Kerr space-time shows that indeed issues such as coordinate singularities and instability are avoided in such an approach, leading to accurate evolutions of dynamical equations. As a consequence, we use a spectral method in what follows.

To implement our spectral method, we choose to work with spin-weighted spherical harmonics (SWSHs) and the associated $\tilde{\delta}$ and $\tilde{\delta}'$ operators [22, 24]. SWSHs are a generalization of scalar, vector and tensor harmonics on the sphere [1, 34]. SWSHs form an orthonormal basis for $L^2(SU(2))$ [13] upon which the differential operators $\tilde{\delta}$ and $\tilde{\delta}'$ act by raising and lowering spin weight, respectively. In common with all spectral methods, this reduces the action of differential operators to algebraic manipulations. This is a property which we exploit to reduce PDEs to systems of coupled ODEs.

In order to apply SWSHs to a spectral evolution scheme, it is necessary to decompose arbitrary fields on $\mathbb{S}^2$ into a sum of SWSHs. Many methods to do this have been proposed, since such decompositions are important for the analysis of data over the sphere, e.g. [11, 14, 17, 20]. We choose Huffenberger and Wandelt’s method [15] (which is a modification of [9] and [19]) for three reasons. Firstly, the algorithm is theoretically exact if a minimum number of grid points are used. This is in contrast to several alternatives that are asymptotically exact in the limit of increasing numbers of grid points [2, 29, 32]. Secondly, the method can be applied simultaneously to functions of different spin weights. In our desired application, we will be working with several functions each with different spin weights [5]. As a consequence, Huffenberger and Wandelt’s method has reduced computational effort in comparison to a method which operates on each spin-weighted function separately. Thirdly, Huffenberger and Wandelt’s method via a clever mapping of the sphere into the 2-torus allows for fast Fourier transformations to be used, see also [3, 4, 21]. This is in contrast to spectral methods adapted to the non-periodic coordinate on $\mathbb{S}^2$, e.g. [11].

In order to calculate the values of SWSHs over $\mathbb{S}^2$, Huffenberger and Wandelt used a formula relating SWSHs to reduced Wigner $d$-functions evaluated at $\frac{\pi}{2}$. They implemented the calculation of these $d$-functions via the three term linear recursive relations given by Risbo and Trapani and Navaza [28, 35]. Trapani and Navaza’s scheme is both faster and more accurate than Risbo’s, but eventually becomes unstable. To cope with this, use a nonlinear scheme that is equivalent to the scheme proposed by Prézeau and Reinecke [25]. We build a hybrid linear/nonlinear recursion that avoids the numerical problems that both linear and nonlinear recursive schemes suffer from, see sections 3.1.3 and 4.2. A similar hybrid scheme has been proposed by Luscombe and Luban [18] for the calculation of $3j$ and $6j$-symbols. To the best of the authors’ knowledge, neither the use of a hybrid scheme for the calculation of reduced Wigner $d$-functions at $\frac{\pi}{2}$ nor the use of this method in SWSH decompositions of functions over $\mathbb{S}^2$ has been used before.

The $2 + 1$ Maxwell equations on a deformed sphere, section 5.2, have non-constant coefficients. We will therefore need to perform a SWSH decomposition of products of SWSHs. This requires the calculation of Clebsch–Gordan (CG) coefficients. To do this, we
use a hybrid linear/nonlinear recursion for $3j$-symbols originally presented by Luscome and Luban’s method [18] to Schulten and Gordan’s linear scheme [31]. As before, to the best of the author’s knowledge, this is the first time a hybrid linear/nonlinear scheme has been used for the calculation of CG coefficients. An alternative to the above explicit decomposition is to use spectral transformations directly to perform decompositions of products of SWSHs [3, 7] (the pseudo-spectral approach). We use this approach to check for accuracy.

It is our goal to provide a self-contained presentation, with consistent conventions, that may be readily adapted to general nonlinear hyperbolic PDEs using the outlined spectral method. Wherever possible, we present results for both integer and half-integer spin. This will provide a foundation for future extension to the half-integer spin case.

This paper is structured as follows. In section 2, we provide the geometric background appropriate for formulation of problems later in the work, in particular we review the construction of frames adapted for use with the \( \Theta \)-formalism. In section 2.2, we explicitly show how an arbitrary smooth tensor field may be represented in terms of spin-weighted quantities on \( S^2 \). In section 2.3, we introduce and review the properties of SWSHs. We describe in detail how products of SWSHs may be decomposed and give details about the symmetry properties of CG coefficients (Wigner 3\( j \)-symbols) appropriate for numerical use. In section 3.1, we briefly describe the spherical harmonic transformation of [15]. In section 3.2, we describe how we compute CG coefficients (Wigner 3\( j \)-symbols) numerically. In section 3.3, we demonstrate the property of spectral convergence for smooth test functions. In section 4.2, we demonstrate the instability of the recursive calculation of \( d_{mn}^l \) due to [35] and discuss how our hybrid scheme avoids this. In section 4.3, we contrast the pseudo-spectral and spectral approaches to the decomposition of products of SWSHs. In section 5.1, we construct the tensor advection equation in the \( \Theta \)-formalism, thus showing how the standard IVP for the scalar and vector advection equation on \( S^2 \) may be formulated; in section 5.1.1, we numerically solve this problem using our spectral method for test fields that lead to temporally periodic solutions. Exploiting this periodicity by comparing solutions at integer multiples of one period (i.e., stroboscopically) yields a method for performing convergence tests. In section 5.2, we construct the IVP for the 2\( +1 \) Maxwell equation where the spatial geometry is conformally related to \( S^2 \). In section 5.2.1, we numerically solve the problem, comparing spectral with pseudo-spectral methods.

2. Geometric preliminaries

2.1. The 2- and 3-spheres and the Hopf bundle

It is often useful to think of the manifold \( S^3 \) as the submanifold of \( \mathbb{R}^4 \) given by \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \). The Euler coordinates of \( S^3 \) can then be represented by

\[
\begin{align*}
  x_1 &= \cos \frac{\theta}{2} \cos \lambda_1, \\
  x_2 &= \cos \frac{\theta}{2} \sin \lambda_1, \\
  x_3 &= \sin \frac{\theta}{2} \cos \lambda_2, \\
  x_4 &= \sin \frac{\theta}{2} \sin \lambda_2,
\end{align*}
\]

where \( \theta \in (0, \pi) \) and \( \lambda_1, \lambda_2 \in (0, 2\pi) \). Clearly, these coordinates break down at \( \theta = 0 \) and \( \pi \).

For later convenience, we introduce coordinates \( (\theta, \rho, \phi) \) (which are also referred to as Euler coordinates) by

\[
\lambda_1 = (\rho + \phi)/2, \quad \lambda_2 = (\rho - \phi)/2.
\]

The set of complex unitary \( 2 \times 2 \)-matrices with unit determinant \( SU(2) \) endowed with the natural smooth manifold structure is diffeomorphic to \( S^3 \).
Here we again consider the following representation with respect to the Euler parametrization:

\[ L \] as follows. We define left and right translation maps:

\[ L, R : SU(2) \times SU(2) \rightarrow SU(2), \quad (u, v) \mapsto L_u(v) := uv, \quad (u, v) \mapsto R_u(v) := vu. \]

On any Lie group, the maps \( L_u \) and \( R_u \) are automorphisms for each element \( u \). Now, choose a basis of the tangent space at the unit element \( T_1(SU(2)) \) (i.e., a basis of the Lie algebra)

\[ \bar{Y}_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad \bar{Y}_2 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \bar{Y}_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \]

and define, for \( u \in SU(2) \),

\[ Y : u \mapsto (Y_u)_a := (L_u)_a(\bar{Y}_a), \quad Z : u \mapsto (Z_u)_a := (R_u)_a(\bar{Y}_a). \]

Clearly, \( (Y_1, Y_2, Y_3) \) is a smooth global frame on \( S^3 \) which is invariant under left translations while the frame \( (Z_1, Z_2, Z_3) \) is invariant under right translations. These fields have the following representation with respect to the Euler parametrization:

\[ Y_1 = -\sin \rho \partial_\phi - \cos \rho (\cot \theta \partial_\theta - \csc \theta \partial_\phi), \]
\[ Y_2 = -\cos \rho \partial_\theta + \sin \rho (\cot \theta \partial_\theta - \csc \theta \partial_\phi), \]
\[ Y_3 = \partial_\rho, \]
\[ Z_1 = -\sin \phi \partial_\phi + \cos \phi (\csc \theta \partial_\theta - \cot \theta \partial_\phi), \]
\[ Z_2 = -\cos \phi \partial_\theta - \sin \phi (\csc \theta \partial_\theta - \cot \theta \partial_\phi), \]
\[ Z_3 = -\partial_\phi. \]

One can show by direct Lie group arguments (or by using the coordinate representations of the fields) that

\[ [Y_1, Y_2] = Y_3, \quad [Y_2, Y_3] = Y_1, \quad [Y_3, Y_1] = Y_2, \]

similarly for the right-invariant fields, and

\[ [Y_k, Z_l] = 0, \quad \forall k, l = 1, 2, 3. \]

For later convenience, we define

\[ \partial := -(Y_2 + iY_1), \quad \partial' := -(Y_2 - iY_1), \]

which, as we shall see later, are closely related to the \( \partial \)-operators defined in [24]. We have

\[ [\partial, \partial'] = 2iY_3. \]

The Hopf map \( \pi : S^3 \rightarrow S^2 \) can be represented as

\[ (x_1, x_2, x_3, x_4) \mapsto (y_1, y_2, y_3) = (2(x_1x_3 + x_2x_4), 2(x_2x_3 - x_1x_4), x_1^2 + x_2^2 - x_3^2 - x_4^2) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta). \]

Here we again consider \( S^3 \) as being embedded into \( \mathbb{R}^4 \) by \( x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \), and the manifold \( S^2 \) is thought of being given by \( y_1^2 + y_2^2 + y_3^2 = 1 \) in \( \mathbb{R}^3 \). When we introduce standard polar coordinates on \( S^2 \), namely

\[ y_1 = \sin \theta \cos \phi, \quad y_2 = \sin \theta \sin \phi, \quad y_3 = \cos \theta, \]
then $\pi$ obtains the particularly simple representation

$$\pi : (\theta, \rho, \phi) \mapsto (\theta, \phi).$$

(2.2)

In particular, it becomes obvious that the push-forward of $Y_3$ to $S^2$ along $\pi$ vanishes. Indeed, $S^2$ is the principal bundle over $S^2$ with structure group $U(1)$ generated by $Y_3$ (whose integral curves are closed) and projection map $\pi$; this is the Hopf bundle.

The Hopf bundle can be identified with the bundle of orthonormal frames on $S^2$ with respect to any smooth Riemannian metric. An explicit construction in terms of the coordinates above can be done as follows. Let $U$ be an open subset of $S^2$; we assume that the poles $\theta = 0$, $\pi$ are outside of $U$ so that the representation of the Hopf map given by equation (2.2) is well defined and the Euler coordinates cover $\pi^{-1}(U)$. If we restrict to sufficiently small open subsets, this is no loss of generality, since for any sufficiently small choice of the open set $U$, we can always introduce the coordinates such that the poles are not in $U$. Let $(e_1^*, e_2^*)$ be a smooth orthonormal frame on $U$ and define the corresponding complex field

$$m^* := \frac{1}{\sqrt{2}} (e_1^* + ie_2^*).$$

We consider the action

$$m^* \mapsto m = e^{i\rho} m^*,$$

(2.3)

of $U(1)$ which is defined pointwise on $U$, i.e., the group parameter $\rho$ is a smooth function on $U$. Any specification of the function $\rho(\theta, \phi)$ therefore yields another smooth orthonormal frame on $U$ and can hence be interpreted as the smooth local section $U \rightarrow S^3$, $(\theta, \phi) \mapsto (\theta, \phi, \rho(\theta, \phi))$ in the bundle of orthonormal frames, or equivalently, in the Hopf bundle. Doing this for every open subset $U$ of $S^2$ (introducing coordinates so that the poles are not in $U$ as above), the full bundle of orthonormal frames can be recovered and can therefore be identified with the Hopf bundle. At every point $p$ of $U$, the fibre $\pi^{-1}(p)$ is the set of all orthonormal bases of $T_p(S^2)$.

2.2. Weighted quantities on the 2-sphere

Let $(\omega, \tilde{\omega})$ be the dual coframe of $(m, \tilde{m})$ and $(\omega^*, \tilde{\omega}^*)$ dual to $(m^*, \tilde{m}^*)$. Then, the above action of $U(1)$ implies

$$\omega = e^{-i\rho} \omega^*.$$

Now let an arbitrary smooth tensor field $T$ of type $(s_1 + s_2, r_1 + r_2)$ for integers $r_1, s_1, r_2, s_2 \geq 0$ be given on $U$, so that the function $v : U \rightarrow \mathbb{C}$ is defined by

$$v := T \bigg( \omega, \ldots, \omega, \tilde{\omega}, \ldots, \tilde{\omega}, m, \ldots, m, \tilde{m}, \ldots, \tilde{m} \bigg),$$

possibly after changing the order of the arguments of $T$. In principle, $v$ is a function on $U \subset S^2$.

But under rotations of the frame, it gives rise to a unique function on $\pi^{-1}(U)$ which changes along the fibre according to the transformation of the frame. This function on $S^3$ is denoted by the same symbol $v$ for simplicity. In particular, its dependence on the fibre coordinate $\rho$ is given by

$$v = e^{i(r_1 - r_2 - s_1 + s_2) \rho} T \bigg( \omega, \ldots, \omega, \omega^*, \ldots, \omega^*, m, \ldots, m, m^*, \ldots, m^* \bigg),$$

$$= e^{i(r_1 - r_2 - s_1 + s_2) \rho} v^*,$$

where we consider $v^*$ as independent of $\rho$ (because it is defined with respect to the reference frame $(m^*, \tilde{m}^*)$). We obtain
where \( s \) is the spin weight introduced in [24]. Hence, equation (2.1) becomes

\[
\partial \bar{Y}(\nu) = -2sv.
\]

In summary, every quantity \( v \) on \( U \subset S^2 \) of spin weight \( s \) can be lifted to a smooth function \( e^{i\nu} \cdot (v \circ \pi) \) on \( \pi^{-1}(U) \subset S^3 \) (which we denote by the same symbol from now). Vice versa, every such function on \( \pi^{-1}(U) \) pulls back to a function with spin weight \( s \) on \( U \) along a smooth section over \( U \). In the following, we will therefore often not distinguish between a function with spin weight \( s \) on \( S^2 \) and the corresponding function on \( S^3 \).

In the case of the 2-sphere with the standard round unit metric, we often consider the reference frame

\[
m^* := \frac{1}{\sqrt{2}} \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right),
\]

and choose \( U \) as the set of all points on \( S^2 \) without the two poles \( \vartheta = 0, \pi \). As the smooth local section, we choose \( \rho(\vartheta, \varphi) \equiv 0 \). Comparing this with the coordinate expressions above, we see that

\[
m^* = \pi^* \left( -(Y_2 + iY_1)\big|_{\rho=0} \right) / \sqrt{2} = \pi^* (\partial|_{\rho=0}) / \sqrt{2}.
\]

Therefore, if \( v \) is a function on \( U \) with spin weight \( s \) and \( \hat{v} \) (here, we exceptionally use two different symbols \( v \) and \( \hat{v} \)) the corresponding function on \( \pi^{-1}(U) \), then

\[
\partial|_{\rho=0}(\hat{v}) = \left( \partial_\theta - \frac{i}{\sin \theta} \partial_\phi \right) \hat{v} + i \cot \theta \partial_\rho \hat{v} \bigg|_{\rho=0} = (\sqrt{2}m^*(v) - s \cot \vartheta v) \circ \pi|_{\rho=0}.
\]

Under all these conditions, it makes sense therefore to simplify the notation and write

\[
\partial(v) = \sqrt{2}m^*(v) - sv \cot \vartheta,
\]

for a function on \( S^2 \) with spin weight \( s \). In the same way, we obtain

\[
\partial(v) = \sqrt{2}m^*(v) + sv \cot \vartheta.
\]

2.3. Spin-weighted spherical harmonics and decompositions via Clebsch–Gordan coefficients and Wigner 3j-symbols

In application of the Fourier–Galerkin (spectral) method to the solution of PDEs, products of SWSH will be encountered. This motivates the exploration of a convenient method of treating these product terms—which will result in the appearance of the CG series, calculation of which will be facilitated by the relation of coefficients in this series to the Wigner 3j-symbols.

We proceed by first stating coordinate expressions for the well-known Wigner D-matrices that form a basis for \( L^2(SU(2)) \), which will allow for the use of commonly encountered identities from the treatment of angular momentum in quantum mechanics [30].

The Euler parametrization of a rotation can be written in terms of the Euler coordinates \( \theta, \rho, \phi \) introduced above. According to [13, 30], we have

\[
D^{j}_{mn}(\rho, \theta, \phi) = e^{im\rho} d^{j}_{mn}(\theta) e^{in\phi},
\]

\[
d^{j}_{mn}(\theta) = \sum_{r=\max(0,m-n)}^{\min(l+m,l-n)} (-1)^{r-m+n} \frac{\sqrt{(l+m)!(l-m)!(l+n)!(l-n)!}}{r!(l+m-r)!(l-r-n)!(r-m+n)!} \times \cos^{2l-2r+m-n} \frac{\theta}{2} \sin^{2r-m+n} \frac{\theta}{2}.
\]
where \( l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) (or \( l \in \mathbb{N}_0 + 1/2 \) for spinorial quantities), \( m, n \in \mathbb{Z}, |m| \leq l, |n| \leq l \).

The quantity \( d_{lm}^m \) is the reduced Wigner matrix element and satisfies \( d_{lm}^m = \overline{d_{lm}^m} \) together with the indicial symmetry \( d_{lm}^m = (-1)^{m-l}d_{lm}^m \). Upon introduction of \((\theta, \varphi)\) on \( S^2 \) as above, we introduce\(^1\) the SWSHs as

\[
s_{Ylm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} Y_{lm}^{i}(0, \theta, \varphi),
\]

\[
= \sqrt{\frac{2l+1}{4\pi}} e^{im\varphi} d_{lm}^i(\theta).
\]

This fixes our convention to agree with \([24]\). From equations (2.8) and (2.10) we immediately observe a useful property the SWSH possess under complex conjugation

\[
s_{Ylm}(\theta, \varphi) = (-1)^{s-m} s_{Yl-m}(\theta, \varphi).
\]

For later convenience, we compare (see also equation (4.15.122)) of \([24]\) the algebraic action of the differential operators \( \partial, \partial' \) (cf equations (2.5) and (2.6)) and their explicit coordinatizations on \( \varsigma_{Ylm}(\theta, \varphi) \):

\[
\partial_{\varsigma} s_{Ylm}(\theta, \varphi) = (\sin \theta)^{-1}(\partial_\theta - i \csc \theta \partial_\varphi) [(\sin \theta)^{-s} s_{Ylm}(\theta, \varphi)]
\]

\[
= -\sqrt{(l-s)(l+s+1)}s_{Yl+1}(\theta, \varphi)
\]

\[
\partial'_{\varsigma} s_{Ylm}(\theta, \varphi) = (\sin \theta)^{-1}(\partial_\theta + i \csc \theta \partial_\varphi) [(\sin \theta)^{s} s_{Ylm}(\theta, \varphi)]
\]

\[
= \sqrt{(l+s)(l-s+1)}s_{Yl-1}(\theta, \varphi)
\]

\[
\Delta_{\varsigma} s_{Ylm}(\theta, \varphi) = \frac{1}{2} (\partial'_{\varsigma} \partial - \partial_{\varsigma} \partial') [s_{Ylm}(\theta, \varphi)] = (s^2 - l(l + 1))s_{Ylm}(\theta, \varphi).
\]

For later reference, we also restate the orthonormality relation

\[
\langle s_{Yl,m} | s_{Yl',m'} \rangle = \int_0^{2\pi} \int_0^\pi s_{Yl,m}(\theta, \varphi) s_{Yl',m'}(\theta, \varphi) \sin \theta \, d\theta \, d\varphi = \delta_{l,l'} \delta_{m,m'},
\]

which is directly inherited from the properties of the \( D \)-matrices. Observe that in equation (2.15), orthonormality holds for functions of the same spin weight. We now describe a closed-sum decomposition for products such as

\[
s_{Yl_1,m_1}(\theta, \varphi) s_{Yl_2,m_2}(\theta, \varphi) = \sqrt{\frac{2l_1+1}{4\pi}} D_{l_1,m_1}^{i}(0, \theta, \varphi) \sqrt{\frac{2l_2+1}{4\pi}} D_{l_2,m_2}^{j}(0, \theta, \varphi),
\]

which together with the action of the \( \partial, \partial' \) operators in equations (2.12) and (2.13) will form the basis of our spectral scheme. The decomposition we seek is the so-called CG series which in bra-ket notation is given by \([30]\)

\[
D_{m_1,n_1}^i(\rho, \theta, \phi) D_{m_2,n_2}^j(\rho, \theta, \phi) = \sum_{i,j} \langle i | l_1; m_1, n_1 | l_2; m_2 \rangle \langle j | l_1; m_2, n_2 | l_2; m_1 \rangle
\]

\[
\times \langle i | l_1; m_2, n_2 | l_2; m_1 \rangle \langle j | l_1; m_1, n_1 | l_2; m_2 \rangle \langle \Delta \rangle(\rho, \theta, \phi),
\]

\[
(2.17)
\]

where \( \Delta := \{\max(|l_1 - l_2|, |m_1 + m_2|, |n_1 + n_2|), \ldots, l_1 + l_2\} \). Note that each CG coefficient in the series is real, i.e., \( \langle i | \cdot \rangle \cdot \langle j | \cdot \rangle \cdot \langle \Delta \rangle \cdot \langle \Delta \rangle \in \mathbb{R} \).

Define

\[
A_i(s_1, l_1, m_1; s_2, l_2, m_2) := \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi}} \frac{1}{\sqrt{2l_1+1}} \langle l_1, s_1; l_2, s_2 | l_1, l_2; l, (s_1 + s_2) \rangle
\]

\[
\times \langle l_1, m_1; l_2, m_2 | l_1, l_2; l, (m_1 + m_2) \rangle.
\]

\[
(2.18)
\]

\(^1\) This choice is standard and corresponds to a choice of smooth local section with \( \rho(\theta, \varphi) = 0 \), we assume this choice has been made henceforth, unless otherwise specified—see section 2.
Equation (2.18) together with equation (2.17) thus provides us with the following decomposition of equation (2.16):

$$s_1 Y_{l_1,m_1} (\vartheta, \varphi) s_2 Y_{l_2,m_2} (\vartheta, \varphi) = \sum_{\ell \in \Lambda'} \mathcal{A}_\ell (s_1, l_1, m_1; s_2, l_2, m_2) Y_{l_1+m_1} (\vartheta, \varphi),$$

(2.19)

where $\Lambda' := \{\max(|l_1 - l_2|, |s_1 + s_2|, |m_1 + m_2|), \ldots, l_1 + l_2\}$. Hence, the product of two SWSHs may be decomposed into a finite linear combination of SWSHs with spin weight equal to the sum of the original two spin weights. Spectral decomposition of evolution equations will also require the following identity:

$$\mathcal{I} = \int_0^{2\pi} \int_0^\pi s_1 Y_{l_1,m_1} (\vartheta, \varphi) s_2 Y_{l_2,m_2} (\vartheta, \varphi) \mathcal{A}_\ell (s_1, l_1, m_1; s_2, l_2, m_2) \delta_{l_1,0} \delta_{(m_1+m_2),m_3},$$

(2.20)

which may be obtained using equations (2.15) and (2.19).

In the interest of efficient numerical calculations, utilizing the relation of the CG coefficients to the Wigner 3j-symbols is prudent due to the convenient symmetry properties the latter possess [23]. We have

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^\dagger = \frac{(-1)^{j_1-j_2-m_3}}{\sqrt{2j_3+1}} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix},$$

(2.21)

the non-negative quantities $\{j_1, j_2, j_3\}$ are known as angular momentum numbers and may be integral or half-integral. The quantities $\{m_1, m_2, m_3\}$ are called the projective quantum numbers and are given by $m_3 = -j_3, -j_3 + 1, \ldots, j_3 - 1, j_3$, where $r = 1, 2, 3$. Three further constraints are placed on the $j_i$ and $m_i$:

WI) $J := j_1 + j_2 + j_3 \in \mathbb{N}_0$;

WII) $m_1 + m_2 + m_3 = 0$;

WIII) The triangle condition: $|j_r - j_s| \leq j_t \leq j_r + j_s$ where $r, s, t$ is any permutation of 1, 2, 3;

in the event these constraints fail to be satisfied the 3j-symbol is set to 0. The following symmetries will also be of use later.

(SI) invariance under permutation of columns

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^\text{cyclic} = (-1)^J \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix},$$

(SII) invariance under spatial inflection

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^\text{anticyclic} = (-1)^J \begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}.$$
From the symmetries of the 3\(j\)-symbols, we recover the following symmetries on the \(A\) of equation (2.19) that we will make use of later:

\[
A_l(s_1, l_1, m_1; s_2, l_2, m_2) = A_l(s_2, l_2, m_2; s_1, l_1, m_1), \\
A_l(s_1, l_1, m_1; s_2, l_2, m_2) = (-1)^{l_1+l_2-2(s_1+s_2)} A_l(-s_1, l_1, m_1; -s_2, l_2, m_2), \\
A_l(s_1, l_1, m_1; s_2, l_2, m_2) = (-1)^{l_1+l_2} A_l(s_1, l_1, -m_1; s_2, l_2, -m_2).
\]

When performing a decomposition such as in equation (2.19) numerically, it can be convenient to embed the above symmetries directly into the summation process. Furthermore, there also exist efficient storage schemes for 3\(j\)-symbols, utilizing symmetries such as (SI–SIII) [27]—this allows for precalculation of all required 3\(j\)-symbols and evaluating equation (2.19) in this manner may be more efficient under certain circumstances.

### 3. Numerical method

#### 3.1. Spectral transformation

In this section, we briefly describe the numerical implementation of Huffenberger and Wandelt’s spectral algorithm [15] that will allow for the decomposition of an integer spin-weighted function \(f \in L^2(SU(2))\) in terms of the SWSH of equation (2.9). Numerical calculations must be limited to a finite grid, hence the decomposition must be truncated at a maximal harmonic (band limit) \(L\). In terms of this band limit, the algorithm has \(\mathcal{O}(L^3)\) complexity which is achieved by exploiting a smooth periodic extension of the data to the 2-torus so that existing fast Fourier transform (FFT) methods can be used.

Consider a function \(s f \in L^2(SU(2))\). By the Peter–Weyl theorem for compact groups [33], we have

\[
\lim_{L \to \infty} \left\| \hat{f}(\theta, \phi, \rho) - \sum_{l=|s|}^{L} \sum_{m=-l}^{l} a_{lm} \sqrt{\frac{2l+1}{4\pi}} Y^l_{lm}(\rho, \theta, \phi) \right\|_2 = 0.
\]

Fixing the fibre coordinate as \(\rho = 0\) and performing the map equation (2.2), we then define \(s f(\theta, \varphi) := s f(\theta, \phi, \rho)\)|\(_{\rho=0}\) and arrive at the standard expansion

\[
s f(\theta, \varphi) = \lim_{L \to \infty} \sum_{l=|s|}^{L} \sum_{m=-l}^{l} a_{lm} Y^l_{lm}(\theta, \varphi).
\]  

(3.1)

Henceforth, we work with the band-limited expression

\[
s f(\theta, \varphi) = \sum_{l=|s|}^{L} \sum_{m=-l}^{l} a_{lm} Y^l_{lm}(\theta, \varphi),
\]

(3.2)

where it is assumed that the function being decomposed may be completely expressed by a finite linear combination of the basis functions.

#### 3.1.1. Forward transformation

We now describe the algorithm for evaluation of the forward transform \(F: s f \mapsto (a_{lm})\). As a first step, introduce the notation \(\Delta^l_{mn} := d^l_{mn}(\pi/2)\) which allows for the rewriting of equation (2.8) as [28]

\[
d^l_{mn}(\vartheta) = i^{-m-n} \sum_{q=-l}^{l} \Delta^l_{qm} e^{-iq\vartheta} \Delta^l_{qn},
\]

(3.3)
following from a factoring of rotations [35]. The details of how the $\Delta$ elements are calculated together with their symmetry properties are given in section 3.1.3.

Define the functional
\[
I_{m,n,}\varphi(\vartheta, \varphi) := \int_0^{2\pi} \int_0^\pi e^{-i m \vartheta} e^{-i n \varphi} s f(\vartheta, \varphi) \sin \vartheta \, d\vartheta \, d\varphi.
\] (3.4)

Equation (2.10) together with equations (3.1)–(3.4) lead to
\[
s_{\Delta l}^{m,n} = i^{l-m} \sqrt{\frac{2l+1}{4\pi}} \sum_{q=-l}^{l} \Delta l_{qn} I_{qm} \Delta q_{m}.
\]

We now wish to evaluate the expression for $I_{m,n}$. This may be done exactly by extension of the function $s f(\vartheta, \varphi)$ to the 2-torus $T^2$, which will permit the application of the standard 2D Fourier transform. Although this requires computation of points outside the domain of interest, the corresponding increase in the speed of performing the calculation (for large $L$) and favourable (spectral) convergence offered by this method compensate for the increased computational effort. Define the extended function
\[
s_{F}(\vartheta, \varphi) := \begin{cases} 
  s f(\vartheta, \varphi) & \vartheta \leq \pi \\
  s f(\pi - \vartheta, \varphi) & \vartheta > \pi,
\end{cases}
\]

where $\vartheta$ now takes values in $[0, 2\pi)$. Clearly this does not change the value of $I_{m,n}$ because extension of the function leaves its value unchanged within the domain of integration as defined in equation (3.4). The periodic extension is chosen by forming a linear combination of $s_{Ylm}(\vartheta, \varphi)$ and examining symmetries using the defining relations of equations (2.10) and (3.3). As the periodically extended function $s_{F}(\vartheta, \varphi)$ now possesses $2\pi$ periodicity in both arguments, it may be written as the two-dimensional, band-limited Fourier sum
\[
s_{F}(\vartheta, \varphi) = \sum_{k,n=-L}^{L} F_{kn} \exp(ik\vartheta) \exp(in\varphi).
\]

Substitution of this equation into equation (3.4) yields
\[
I_{m,n} = \sum_{k,n=-L}^{L} F_{kn} \left[ \int_0^{2\pi} \exp(i(n-m)\varphi) \, d\varphi \right] \left[ \int_0^\pi \exp(i(k-m')\vartheta) \sin \vartheta \, d\vartheta \right] (3.5a)
\]
\[
= 2\pi \sum_{k=-L}^{L} F_{kn} w(k-m'), (3.5b)
\]

where $w(p)$ for $p \in \mathbb{Z}$ is given by
\[
w(p) = \int_0^\pi \exp(ip\vartheta) \sin \vartheta \, d\vartheta = \begin{cases} 
  2/(1-p^2) & p \text{ even} \\
  0 & p \text{ odd}, \, p \neq \pm 1 \\
  \pm i\pi/2 & p = \pm 1.
\end{cases}
\]

Equation (3.5b) shows that $I_{m,n}$ is proportional to a discrete convolution in spectral space. By the convolution theorem, this implies that we may consider instead the inverse transform of $w(p)$ which maps the function back to its spatial representation $w_r$. Performing pointwise multiplication by $2\pi \cdot s_{F}$ and transforming the result will yield $I_{m,n}$. If the desired number of samples of the function $s_{F}(\vartheta, \varphi)$ over $\vartheta$ and $\varphi$ on $S^2$ is to be $N_\vartheta$ and $N_\varphi$, respectively, then for the number of samples for the extended function $s_{F}(\vartheta, \varphi)$, we take to be $N_{\vartheta}' = 2(N_\vartheta - 1)$ and $N_{\varphi}'$. The spatial sampling intervals are given by $\Delta \vartheta = \frac{2\pi}{N_{\vartheta}'}$ and $\Delta \varphi = \frac{2\pi}{N_{\varphi}'}$. Note that in order

\footnote{(B5) of [15] contains an error.}
to satisfy the Nyquist condition, we must take $N_\theta = 2(L + 2)$ and $N_\phi = 2(L + 2)$, where $L$ is the harmonic that the function $s f (\theta, \phi)$ is band limited to. With the stated sampling, the quadrature weights may be written as

$$w_r(q' \Delta \theta) = \sum_{p=-N_\phi/2}^{N_\phi/2 - 1} \exp(-ipq' \Delta \theta) w(p). \quad (3.6)$$

Upon performing a linear phase shift in equation (3.6), we finally arrive at the expression

$$I_{nm} = \frac{2\pi}{N_\theta N_\phi} \sum_{q=0}^{N_\phi - 1} \sum_{q'=0}^{N_\phi - 1} \exp(-imq' \Delta \phi) \exp(-imq \Delta \phi) w_r(q' \Delta \theta) F(q' \Delta \theta, q \Delta \phi),$$

which may be evaluated using a two-dimensional FFT. We note that if many transformations with the same band limits are to be performed, the weights can be pre-calculated.

Overall, the complexity of the outlined algorithm is $O(L^3)$. Two further linear improvements in execution speed are possible. The first reduces the total computation time of $s a_{lm}$ by a factor of 2. Equation (3.5) together with the symmetries of $\Delta^l_{mn}$ described in section 3.1.3 allows us for

$$\sum_{q'=l}^{l} \Delta^l_{qm} I_{qm} \Delta^l_{q(-s)} = \sum_{q=0}^{l} \Delta^l_{qm} K_{qm} \Delta^l_{q(-s)},$$

where

$$K_{qm} := \begin{cases} I_{qm} & \text{if } q = 0 \\ I_{qm} + (-1)^{m+s} I_{(-q)m} & \text{if } q > 0. \end{cases}$$

A second improvement (also by a factor of 2) is possible if the function $s f$ being analysed is real. Here the speedup is due to the FFT, where real input results in Hermitian output.

### 3.1.2. Backward transformation.

We now describe the algorithm for evaluation of the backward (inverse) transform $F^{-1} : (a_{lm}) \mapsto s f$. The backward spherical harmonic transform maps the expansion coefficients $s a_{lm}$, for $|s| \leq l \leq L$, to a function on $\mathbb{S}^2$. Because we are working with band-limited functions we can, at least in theory, perfectly reconstruct the original function. To this end, equation (3.2) must be evaluated. As the inverse transform does not contain integrals, issues of quadrature accuracy do not arise.

Define the functional

$$J_{lm}[s a_{lm}] := i^{s-m} \sum_{l=|s|}^{L} \frac{2l+1}{4\pi} \Delta_{(-m)l}^{l} s a_{lm} \Delta_{(-m)l}^{l}. \quad (3.7)$$

Substitution of equation (2.10) together with equations (3.3) and (3.7) in equation (3.2) leads to

$$s f (\theta, \phi) = \sum_{m=-N_\phi}^{N_\phi} \sum_{n=-N_\phi}^{N_\phi} e^{i m \theta} e^{i n \phi} J_{mn}. \quad (3.8)$$

Evaluation of (3.8) results in $s f (\theta, \phi)$ sampled on $\mathbb{T}^2$. As we require the function on $\mathbb{S}^2$, we may truncate the output at $\theta = \pi$ discarding all data for $\theta > \pi$. Evaluating (3.8) scales as $O(L^3)$, just as when performing the forward transformation.

Taking into account the symmetries of the $\Delta^l_{mn}$ matrices provided by equations (3.9) leads to an analogous halving of the number of operations required for the evaluation of equation (3.8), as in the case of the forward transform. Similarly, a further speedup is possible if the input data to the FFT library are Hermitian.
3.1.3. Calculation of $\Delta$ elements. In this section, we follow [35] and briefly outline an efficient recursive method for computing the $\Delta_{mn}^l$ that appear upon decomposition of the Wigner $d$-matrices (equation (3.3)) when constructing the transformations in sections 3.1.1 and 3.1.2. It can be seen directly from equation (2.8) that the elements $\Delta_{mn}^l$ have the following symmetries:

$$\begin{align*}
\Delta_{l(-n)}^l &= (-1)^{l+n} \Delta_{mn}^l, \\
\Delta_{mn}^l &= (-1)^{l-m} \Delta_{mn}^l, \\
\Delta_{mn}^l &= (-1)^{n-m} \Delta_{mn}^l.
\end{align*}$$

(3.9)

where the $\Delta_{mn}^l$ are combinatorial expressions purely dependent on the choice of indices $l$, $m$ and $n$.

Suppose we require all possible $\Delta_{mn}^l$ up to a maximum $L$. Due to the symmetries of equations (3.9), only a subset of all allowable indices need be calculated. For each choice of $l$, we restrict the indices for which $\Delta_{mn}^l$ is calculated to the set $\{(m, n) \mid m \in \{0, 1, \ldots l\} \text{ and } n \in \{0, 1, \ldots l\}\}$.

We implement the Trapani–Navaza (TN) algorithm as follows:

(TN I) initialize $\Delta_{00}^l = 1$;

(TN II) iterate with

$$\Delta_{0l}^l = -\frac{\sqrt{2l-1}}{2l} \Delta_{0(l-1)}^{(l-1)};$$

(TN III) iterate with

$$\Delta_{ml}^l = \frac{l(2l-1)}{2(l+m)(l+m-1)} \Delta_{(m-1)(l-1)}^{(l-1)};$$

(TN IV) iterate with

$$\begin{align*}
\Delta_{mn}^l &= \begin{cases} 
2m l_{m(n+1)}^l &= \sqrt{\frac{2}{l-n}} \Delta_{mn+1}^{l} \\
2m l_{m(n+1)}^l &= \sqrt{\frac{2}{l+n+1}} \Delta_{mn+1}^{l} \\
\end{cases} 
\end{align*}$$

(3.10)

(TN V) use symmetries (i.e., Equations (3.9)) to find the remaining $\Delta_{mn}^l$.

Note that the $\Delta_{mn}^l$ may be viewed as having a square pyramidal lattice structure with TN I–IV being a calculation of an octant subset—TN V then allows for all values to be found: TN I corresponds to the apex $(0, 0, 0)$; TN II corresponds to a tangent of values, a descent from the apex through the points $(l, 0, l)$; TN III corresponds to calculation of a right-angle, triangular lattice of surface values $(l, m, l)$; TN IV corresponds to calculation of interior points $(l, m, n)$ constrained to lie in the set $\{(m, n) \mid m \in \{0, 1, \ldots l\} \text{ and } n \in \{0, 1, \ldots l\}\}$; TN V allows us for the recovery of all valid values of $\Delta_{mn}^l$ (those outside the octant subset but within the square pyramid) up to the chosen limit $L$.

One advantage of the TN algorithm is that it is well suited to parallelization. Performing steps TN I–III and retaining the results then allows for the trivially parallelizable TN IV to be performed on multiple threads as required. A disadvantage, however, is the instability for large values of $L$ (for our implementation $\geq 2595$). This issue, together with our proposal to correct for it without the loss of efficiency will be described in section 4.2.
3.2. Computation of Clebsch–Gordan Coefficients

In the solution of PDEs, product terms of SWSHs arise which may be decomposed as in equation (2.19). Two obvious paths are open to us: the first option is to implement a pseudospectral approach; extracting the requisite \( A_l \) by transforming two sets of appropriately seeded \( \sigma_{lm} \) coefficient sets to functions, pointwise multiplying then transforming back. We shall revisit this approach in section 5.2.1.

The second option, which we take, is direct computation of \( A_l \) factors by calculation of CG coefficients (or Wigner 3-\( j \)-symbols by equation (2.21)). To this end, we now describe an exact three-term linear recursive algorithm for calculating the Wigner 3-\( j \)-symbols due to [31]. The scheme we describe is numerically stable for ‘small’ (<100) values of \( j \) and \( m \). However, the possible occurrence of numerical loss of significance as well as overflow leads us to subsequently refine our approach. We apply the general conversion of a three-term linear recursion relation into a hybrid recursion relation, given by Luscombe and Luban [18], to the recursion given in [31]. Our implementation extends the mentioned schemes to cover calculation of 3-\( j \)-symbols of both integer and half-integer angular momentum numbers and projective quantum numbers.

Our aim is the simultaneous generation of

\[
    w(j_1) := \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}
\]

with fixed \( j_2, j_3, m_1, m_2, m_3 \) \hspace{1cm} (3.11)

for all \( j_1 \in \{j_{1\text{lin}}, \ldots, j_{1\text{max}}\} \) where \( j_{1\text{lin}} = \max\{|j_2 - j_3|, |m_1|\}, j_{1\text{max}} = j_2 + j_3 \), and where, in addition, \( j_1 \) is subject to the constraints discussed in section 2.3. Consider equation (3.11); if \( j_1 - 1, j_1, j_1 + 1 \) each provide a valid 3-\( j \)-symbol then the symbols may be connected via the following three-term linear recursion relation [31]:

\[
    j_l A(j_l + 1) w(j_l + 1) + B(j_l) w(j_l) + (j_l + 1) A(j_l) w(j_l - 1) = 0, \hspace{1cm} (3.12)
\]

where

\[
    A(j_l) := [j_l^2 - (j_2 - j_3)^2]^{1/2}[(j_2 + j_3 + 1)^2 - j_l^2]^{1/2}[j_l^2 - m_1^2]^{1/2}, \hspace{1cm} (3.13)
\]

\[
    B(j_l) := -(2j_l + 1)[j_2(j_2 + 1)m_1 - j_3(j_3 + 1)m_1 - j_l(j_l + 1)(m_3 - m_2)]. \hspace{1cm} (3.14)
\]

The normalization condition

\[
    \sum_{j_1 = j_{1\text{lin}}}^{j_{1\text{max}}} (2j_1 + 1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 = 1, \hspace{1cm} (3.15)
\]

together with the phase convention

\[
    \text{sign} \begin{pmatrix} j_{1\text{max}} & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_3 - j_2 - m_1} \hspace{1cm} (3.16)
\]

allows us for the determination of the family of 3-\( j \)-symbols of equation (3.11).

The range of \( j_1 \) in equation (3.11) is divided into a ‘classical’ and two complementary ‘non-classical’ regions. A classical region is defined as the set of \( \{j_1, j_2, j_3\} \) and \( \{m_1, m_2, m_3\} \) for which it is possible to construct a vector diagram which physically corresponds to the coupling of angular momentum—for more details, see [18]. For our purposes, it is sufficient to consider the non-classical region as values of \( \|w(j_1)\| \) that monotonically decrease as \( j_1 \to j_{1\text{lin}} \) and \( j_1 \to j_{1\text{max}} \). Consider figure 1 where we illustrate the typical functional form of \( w(j_1) \). In the classical region (denoted I), the amplitude of \( w(j_1) \) oscillates about 0. Within this region, \( w(j_1) \) may evaluate to 0 for specific choices of \( j_1 \). The left boundary of the classical region

\footnote{Terms involving \( j_1 \) that are outside the range of validity are set to 0 in the recursion relation.}
Figure 1. Typical functional form for a family of $3j$-symbols as specified by equation (3.11). Parameters: $j_2 = 90$, $j_3 = 60$; $m_1 = -15$, $m_2 = 70$, $m_3 = -55$. Classical region denoted by I, non-classical regions denoted by II. (Red stars) left boundary of classical region $j_{1\min} = 38$; right boundary of classical region $j_{1\max} = 79$. There are 26 orders of magnitude difference between the largest and smallest values of $w(j_1)$ in this family. Computed using hybrid algorithm.

is denoted by $j_{1\min}$, the right boundary is denoted by $j_{1\max}$. In the non-classical regions (denoted II), $|w(j_1)|$ monotonically decays to zero as the boundaries $j_{1\min}$ and $j_{1\max}$ are approached.

In order to achieve the numerical stability, the recursion relation of equation (3.12) that is used to generate the quantities $w(j_1)$ must be performed in the direction of increasing $|w(j_1)|$. More explicitly, if the desired solution of a recurrence relation such as equation (3.12) is monotonically decreasing (as is the case if one iterates from classical to non-classical region), then the other, linearly independent solution is monotonically increasing. Thus, numerical round-off in the calculation of a decaying solution of the recurrence relation triggers the growth of the unwanted, linearly independent, diverging solution [18]. This discussion implies that instead we should initialize equation (3.12) at the boundaries $j_{1\min} := \min(j_1)$ (left-to-right recursion) and $j_{1\max} := \max(j_1)$ (right-to-left recursion) and proceed towards the classical region I.

From equation (3.13), observe that $A(j_{1\max}) = 0$ and $A(j_{1\max} + 1) = 0$. The recursion relations at the boundaries are given by

$$B(j_{1\min})w(j_{1\min}) + j_{1\min}A(j_{1\min} + 1)w(j_{1\min} + 1) = 0;$$

$$B(j_{1\max})w(j_{1\max}) + (j_{1\max} + 1)A(j_{1\max})w(j_{1\max} - 1) = 0.$$  

Note that the specification of one initial parameter at each boundary is sufficient to start the recursion (equation (3.12)) in each direction.

As the initial choice of parameter for both the left and right recursions is arbitrary, the set of values generated by each recursion will be in error by factors $c_L$ and $c_R$ respectively; explicitly

$C_Lw(j_{1\min}); \ c_Lw(j_{1\min} + 1); \ \ldots; \ c_Lw(j_{1\max})$ (left-to-right recursion);

$C_Rw(j_{1\min}); \ c_Rw(j_{1\min} - 1); \ \ldots; \ c_Rw(j_{1\max})$ (right-to-left recursion),
where the recursion is terminated at a common, final \( j_1 \) value of \( j_{1\text{final}} \). The left and right recursions must match\(^5\) at \( j_{1\text{final}} \), which implies the constraint \( c_L w(j_{1\text{final}}) = c_R w(j_{1\text{final}}) \). We rescale the left recursion by \( c_R/c_L \) and determine \( c_R \) from the normalization condition (3.15). Finally, upon application of the phase condition (3.16), we have generated all valid \( w(j_1) \) as specified by (3.11).

The algorithm just discussed suffers both from numerical loss of significance as well as overflow. This is due to the large variation in \( |w(j_1)| \) for particular parameter choices; for example in the family of symbols shown in figure 1, there are 26 orders of magnitude difference between the largest and smallest values of \( |w(j_1)| \). The loss of significance/overflow is often mitigated by rescaling of iterates in the recursion. Alternatively, the use of a two-term nonlinear recursion may be employed [18]. To this end, we instead work with ratios of successive \( w(j_1) \) terms.

Define the nonlinear left-to-right recursion

\[
    s(j_1) := \frac{w(j_1)}{w(j_1 + 1)} = \frac{-j_1 A(j_1 + 1)}{B(j_1) + (j_1 + 1)A(j_1)s(j_1 - 1)}, \quad j_1 \geq j_{1\text{max}} + 1, \tag{3.17}
\]

and the nonlinear right-to-left recursion

\[
    r(j_1) := \frac{w(j_1)}{w(j_1 - 1)} = \frac{-j_1 A(j_1)}{B(j_1) + j_1A(j_1 + 1)r(j_1 + 1)}, \quad j_1 \leq j_{1\text{max}} - 1. \tag{3.18}
\]

Since \( A(j_{1\text{max}}) = 0 \) and \( A(j_{1\text{max}} + 1) = 0 \), the initial values

\[
    s(j_{1\text{max}} + 1) = \frac{-(j_{1\text{max}} + 1)A(j_{1\text{max}} + 2)}{B(j_{1\text{max}} + 1)},
\]

\[
    r(j_{1\text{max}} - 1) = \frac{-j_{1\text{max}} A(j_{1\text{max}} - 1)}{B(j_{1\text{max}} - 1)},
\]

are known. The numerical advantage of making the transformations of equations (3.17) and (3.18) is that \( s(j_1) \) and \( r(j_1) \) maintain values of order unity throughout the recursion (i.e., at each iterate), thus avoiding significance/overflow issues. A disadvantage introduced by the nonlinear scheme is that equations (3.17) and (3.18) are poorly defined for values of \( j_1 \) where \( w(j_1) = 0 \). However, as this occurs only in the classical region I, this motivates the consideration of a hybrid scheme—a combination of nonlinear (in the nonclassical regions II) and linear (in the classical region I).

In order to split recursion schemes between the two methods, the location of the left \( j_{1L} \) and right \( j_{1R} \) boundaries must be known. The precise choice of \( j_{1L} \) and \( j_{1R} \) is not crucial; the essential point is that nonlinear iteration is terminated near the boundary, such that no zero values of \( w(j_1) \) in the classical region are encountered. As \( |w(j_1)| \) monotonically decreases in the nonclassical regions and is known to be non-zero there, a simple algorithmic method of determining the aforementioned values is to begin iteration with equations (3.17) and (3.18), comparing the magnitude of two consecutive iterates until a local maximum in \( |w(j_1)| \) is achieved.

The coefficients as determined by the nonlinear scheme in the nonclassical region are hence computed using

\[
    w(j_{1L} - k) = w(j_{1L}) \prod_{p=1}^{k} s(j_{1L} - p), \quad 1 \leq k \leq j_{1L} - j_{1\text{max}},
\]

\[
    w(j_{1R} + k) = w(j_{1R}) \prod_{p=1}^{k} r(j_{1R} + p), \quad 1 \leq k \leq j_{1\text{max}} - j_{1R}.
\]

\(^5\) In order to achieve a robust implementation, one performs left and right recursions until several common \( j_1 \) values are achieved. This allows us for the avoidance of singular terms in the classical region.
Once the coefficients in the nonclassical regions are known, we may perform iteration using the three-term linear recurrence relation in the classical region using \(\{w(j_{1\mid -1}), w(j_{1\mid 0}), w(j_{1\mid +1})\}\) as initial values in equation (3.12) for the left and right recursion, respectively. The left and right hybrid recursion schemes must match at a common \(j_{\text{max}}\), upon appropriate rescaling and normalization (equation (3.15)), we finally apply the phase convention of equation (3.16). This procedure allows us for the determination of an entire family of \(w(j_{1})\) as stated in equation (3.11).

If the coefficients are required at several stages of a calculation, it is generally more efficient to perform a recursive pre-computation, exploiting the symmetries (SI–SIII) (see section 2.3) in order to minimize memory usage. We implement this approach in our code where feasible, for details see [27].

In order to check the outlined algorithms, we perform comparisons with a symbolic calculation [23, 27]. Symbolic calculations are slow. Being of arbitrary accuracy, they serve, however, as an excellent check. To verify our scheme, all valid SU(2) 3j-symbols were symbolically calculated up to a maximum entry of \(j_{1} = 10\). Figure 2 shows the results of this procedure—excellent accuracy (machine precision) is achieved.

4. Pseudo-spectral transformations: tests and comparison

In this section, we describe a simple method for checking the associated numerical error of our implementation of the spectral transformation outlined in section 3.1. Furthermore, we check that well-known exponential decay properties of the magnitude of the representation of a function in spectral space are satisfied. Although for the solution of a PDE, band limits \(J_{\text{max}}\) that probe the border of stability of the TN recursive scheme (see section 3.1.3) may be difficult
to achieve, we future-proof our work by proposing a simple linear/nonlinear hybrid scheme in analogy to section 3.2 that restores stability. We then briefly contrast a pseudo-spectral approach with the full spectral scheme.

4.1. Spectral transformation—coefficient decay and $\Delta$-stability

As a preliminary check that the implementation of the spectral transformation is sound, we proceed as follows. First, the generation of a complex-valued function from a pseudo-random linear combination of the basis elements $s_{Ylm}(\vartheta, \varphi)$ is made. The random weights of the aforementioned linear combination are denoted by $\tilde{a}_{lm}$. Taking the inverse transformation of $\tilde{a}_{lm}$ allows us for the construction of the spatial representation of the function. Performing the forward transformation in order to reconstruct the original random weights (reconstructed weights denoted $\tilde{\alpha}_{lm}$) then allows us for a comparison between $\tilde{a}_{lm}$ and $\tilde{\alpha}_{lm}$.

By first separating the real and imaginary parts of each $\tilde{a}_{lm}$ as $\tilde{a}_{lm} = \Re[\tilde{a}_{lm}] + i \Im[\tilde{a}_{lm}]$, we generate both $\Re[\tilde{a}_{lm}]$ and $\Im[\tilde{a}_{lm}]$ by sampling from the continuous uniform random distribution on the interval $[-1, 1]$.

Although sampling random data as above provide us with a simple and robust diagnostic on our implementation, it is also informative to examine coefficient decay for specific test functions that are known to possess specific properties with respect to their spectral representation.

We now inspect how the magnitude of coefficients of a function decay with increasing band limit. In order to represent this decay in a convenient manner, define the averaged coefficient $A_l := \langle |a_{lm}| \rangle_m := \sum_m |a_{lm}|/(2l + 1)$. (4.1)

We now introduce the following smooth test functions:

$$2f(\vartheta, \varphi) := 1.12Y_{4,1}(\vartheta, \varphi) - 3.32Y_{7,-6}(\vartheta, \varphi),$$ (4.2)

$$3ht(\vartheta, \varphi) := 2f(\vartheta, \varphi),$$ (4.3)

$$2g(\vartheta, \varphi) := 2f(\vartheta, \varphi) \exp\left(-[0.70Y_{3,-2}(\vartheta, \varphi) + 2.0Y_{11,-9}(\vartheta, \varphi)]^2\right),$$ (4.4)

$$0k(\vartheta, \varphi) := \exp(-0Y_{1,-1}(\vartheta, \varphi) + 0Y_{1,1}(\vartheta, \varphi)),$$ (4.5)

$$-2g(\vartheta, \varphi) := -\frac{1}{256}2Y_{25,-9}(\vartheta, \varphi) \exp(9.30Y_{20,-2}(\vartheta, \varphi) - 22.50Y_{15,3}(\vartheta, \varphi)),$$ (4.6)

$$-4r(\vartheta, \varphi) := -\frac{1}{600}2q^2(\vartheta, \varphi) \exp(-50Y_{1,0}(\vartheta, \varphi)),$$ (4.7)

the spatial representation of which is initially constructed numerically. Performing the forward transformation yields the spectral representation, which together with equation (4.1) yields the averaged coefficient decay which is shown in figure 3(a). Equations (4.2) and (4.3) are clearly comprised of a finite linear combination of SWSH and indeed this fact is reflected by their complete capture at the band limits tested. As equation (4.2) is completely captured
Figure 3. (a) Average magnitude of coefficients for fixed $l$ in the spectral representation of the functions in equations (4.2)–(4.5). Cyan ’⋄’: $s = 2$ transformation of equation (4.2) at a band limit of $L = 64$. Green ’◦’: $s = 2$ transformation of equation (4.2) at a band limit of $L = 1024$. Magenta ’+’: $s = 8$ transformation of equation (4.3) at a band limit of $L = 512$. Blue ’∗’: $s = 2$ transformation of equation (4.4) at a band limit of $L = 512$. Black ’□’: $s = 0$ transformation of equation (4.5) at a band limit of $L = 32$. Note: the difference between coefficient magnitudes for transformations of equation (4.2) at $L = 64$ and $L = 1024$ is solely due to the additional number of arithmetic operations required when performing a transform at a higher band limit $L$.

(b) (Left to right) blue ’∗’: $s = 2$ transformation of equation (4.4) at a band limit of $L = 512$. Cyan line: linear fit of $l = (259, 260, \ldots, 405), \log(A_l)$. Green ’◦’: $s = −2$ transformation of equation (4.6) at a band limit of $L = 1024$. Magenta line: linear fit of $l = (408, 409, \ldots, 611), \log(A_l)$. Red ’⋄’: $s = −4$ transformation of equation (4.7) at a band limit of $L = 1024$. Black line: linear fit of $l = (660, 661, \ldots, 890), \log(A_l)$. See the text for discussion.

4.2. Wigner $\Delta$-stability

One aspect of the TN algorithm (briefly mentioned in section 3.1.3) is that when transformations with a high band limit are required, instability may arise. Figure 4(a) shows that instability occurs at $L \simeq 2595$; the situation is analogous to that when encountered whilst performing calculations of the $3j$-symbols—hence, we proceed analogously. In order to ameliorate the issue, we construct a new nonlinear recursion relation.
Recall TN I–IV of section 3.1.3. Fix \( l \) and \( m \). Observe that in the region \( m^2 + n^2 \geq l^2 \), we are performing recursion with a three-term linear relation, furthermore, in this region, we have values of \( \Delta^l_{mn} \) that monotonically increase in magnitude when \( n \) is decreased from \( l \). This leads to instability of the TN algorithm. In order to avoid this, we apply Luscombe and Luban’s general method [18], for the construction of a hybrid linear/nonlinear recursion to Trapani and Navaza’s linear recursive scheme.

To rewrite the three-term linear recursion relation stated in equation (3.10) as a two-term nonlinear recursion relation, define the ratio \( r^l_{mn} := \Delta^l_{mn} / \Delta^l_{m(n-1)} \). This results in

\[
r^l_{m(n+1)} = \left( \frac{2m}{\sqrt{(l-n)(l+n+1)}} \right) - r^l_{m(n+2)} \left( \frac{(l-n+1)(l+n+2)}{(l-n)(l+n+1)} \right)^{-1}. \]

To make use of this equation, we require a term \( \Delta^l_{mn} \) within the range where nonlinear recursion is performed—such that ratios may be converted to absolute values. While edge values with \( n = l \) (for fixed \( m \)) may be used to calculate explicit terms from ratios, for reasons of numerical accuracy, it is advantageous to instead perform three-term iteration from \( n = m \) to the edge of the region for which values are computed by the nonlinear scheme (see the dashed red arc in figure 4(b)). Hence, we now write a new three-term relation for a fixed \( n \) connecting three different values of \( m \)—this is accomplished using the indicial symmetry given by the third of equation (3.9)

\[
\Delta^l_{(m+2)n} = \frac{2n}{\sqrt{(l-m-1)(l+m+2)}} \Delta^l_{mn} - \frac{(l-m)(l+m+1)}{(l-m-1)(l+m+2)} \Delta^l_{(m+1)n}. \]
4.3. A comparison of pseudo-spectral and spectral schemes

Consider two, smooth spin-weighted functions \( s_1 f, s_2 g \in L^2(S^2) \). Both \( s_1 f \) and \( s_2 g \) may be decomposed according to equation 3.2, in terms of SWSHs of spin weight \( s_1 \) and \( s_2 \), respectively, for some choice of band limit \( L \). As we will see in section 5.1.1, product terms such as \( s_1 f, s_2 g \) and hence products of the form \( s_1 Y_{lm_1}; s_2 Y_{lm_2} \) will need to be expanded in terms of a linear combination of \( s_1 + s_2 \). We have already discussed how such expansions may be performed in a completely spectral scheme in section 2.3; however, when formulating the solution of nonlinear PDEs in particular, multiple-sum terms can quickly become cumbersome without automatic code generation or some level of abstraction.

An alternative to the above is the pseudo-spectral method. Suppose we initially have the coefficients \( s_1 \tilde{f}_{l_1 m_1} \) and \( s_2 \tilde{g}_{l_2 m_2} \) representing the functions \( s_1 f \) and \( s_2 g \) sampled (or initially seeded) at some band limit \( L \). The coefficients \( s_1 + s_2 \tilde{a}_{lm} \) of the associated product \( s_1 \tilde{f}; s_2 \tilde{g} \) can be calculated by performing the transformations
\[
\mathcal{F}^{-1}_1 : \{ s_1 \tilde{f}_{l_1 m_1} \} \mapsto s_1 \tilde{f}, \quad \mathcal{F}^{-1}_2 : \{ s_2 \tilde{g}_{l_2 m_2} \} \mapsto s_2 \tilde{g};
\]
subsequently taking the pointwise product and transforming
\[
\mathcal{F}^{-1}_1 : \{ s_1 \tilde{f} \tilde{g} \} \mapsto \{ (s_1 + s_2) \tilde{a}_{lm} \},
\]
we find an approximation to an expansion utilizing equation (2.19) directly. In order for the pseudo-spectral and spectral to coincide (to numerical tolerance) at some band limit \( L \), we have found that it is best to choose a band limit for the pseudo-spectral scheme \( \frac{2}{3} L_{PS} \approx L \) then truncating the constructed coefficients at a band limit of \( L \)—this is the so-called ‘Orszag 2/3 rule’ [6] which aids in the suppression of spurious aliasing. We emphasize that this method also easily allows one to take into account the action of the \( \tilde{\partial}, \tilde{\partial} \) operators on functions by embedding their action as multiplication (see equations (2.12) and (2.13)) in coefficient space, together with taking account of their spin raising and lowering properties when transformations are performed.

5. Numerical investigations of geometric initial value problems on the 2-sphere

Let \( M := \mathbb{R} \times S^2 \), and \( t : M \to \mathbb{R} \) be the smooth time function given by \( t(t, p) = t \) for all \( t \in \mathbb{R} \) and \( p \in \mathbb{S}^2 \) with non-vanishing gradient. For each \( p \in \mathbb{S}^2 \), we consider the curve \( \gamma_p : \mathbb{R} \to M, t \mapsto (t, p) \), and the corresponding smooth tangent vector field \( T = \gamma_p' \); in particular, we have \( T(\tau) = 1 \). Because of this, we can introduce coordinates \( (t, \vartheta, \varphi) \) on \( M \), where \( (\vartheta, \varphi) \) are standard polar coordinates on \( \mathbb{S}^2 \) and where \( T = \partial_t \). Let the level sets of \( t \) be denoted by
\[
\Sigma_t := \{ t \} \times \mathbb{S}^2 \simeq \mathbb{S}^2, \quad t \in \mathbb{R}.
\]
Any such subset is a smooth embedded submanifold of \( M \). We make the same assumptions as in the last paragraph of section 2.2—in particular, \( U \) is a dense open subset of \( \mathbb{S}^2 \)—and we choose a smooth complex vector field\(^6\) \( m \) on \( \mathbb{R} \times U \) which is tangent to \( \Sigma_t \) at each \( t \) and which satisfies \( m(t) = 0 \). Let us suppose that \( \{ b_0, b_1, b_2 \} := (T, m, \bar{m}) \) is a smooth frame almost everywhere on \( M \). We let \( (\alpha^0, \alpha^1, \alpha^2) \) be the corresponding dual frame and hence deduce that \( \alpha^0 = dt \). The following notion of spin weight based on frame transformations of the form
\[
T \mapsto T, \quad m \mapsto e^{i\varphi} m,
\]
6 At this stage, we make no further assumptions about \( m \); in particular, \( m \) should not be confused with the field in equation (2.3). We shall choose \( m \) specifically in the two applications discussed below. Notice also that since there is no metric defined on \( M \) so far, we do also not assume a normalization for \( m \) yet.
where $\rho$ can be an arbitrary smooth function on $M$, is useful for the following discussion. Any quantity $h$ on $M$, which behaves like $h \mapsto e^{i\rho}h$ under this transformation, is said to have spin weight $s$. For instance, the frame vector $T$ has spin weight $0$, $m$ has spin weight $1$ and $\overline{m}$ has spin weight $-1$.

Of particular importance for the following discussions are commutators of the frame fields

$$C^s_{jk} := \langle \alpha^s, [b_j, b_k] \rangle.$$ 

The assumptions above yield

$$C^0_{12} = 0,$$

and all commutators can be computed from the following ones:

$$C^0_{01} =: \kappa_1, \quad C^2_{01} =: \kappa_2, \quad C^1_{01} =: \mu_0 + L_0, \quad C^2_{12} =: \mu_1 + L_1.$$

The functions $\kappa_1, \kappa_2, \mu_0, \mu_1, L_0$ and $L_1$ have the following transformation behaviour under the frame transformations above:

$$\kappa_s \mapsto e^{i\rho}\kappa_s, \quad \mu_s \mapsto e^{i\rho}\mu_s,$$

for $s = 0, 1, 2$.

i.e., these are functions have a well-defined spin weight $s$, while

$$L_0 = iT(\rho), \quad L_1 = -im(\rho)e^\rho,$$

and hence do not have a spin weight. Note that in particular, $L_0 = L_1 = 0$ for $\rho \equiv 0$, i.e., for the reference frame.

5.1. Tensor advection equation

We want to start with the following advection problem. Let $V$ be a given smooth vector field on $M$ with $V(\tau) = 1$. Pick a smooth $(r, 0)$-tensor field $N_r$ in a neighbourhood of $\Sigma_0$. We consider the initial value problem

$$L_V N = 0, \quad N|_{t=0} = N_r|_{t=0} = 0. \tag{5.1}$$

for an unknown $(r, 0)$-tensor field $N$ on $M$.

Since $V$ is smooth, it generates a flow on $M$ which maps each point $p$ of $M$ to a point on the integral curve of $V$ through $p$. Due to the condition $V(\tau) = 1$, it follows that $V$ has a non-vanishing ‘time component’ in the direction of $T$ and, in general, a non-vanishing ‘spatial component’ tangential to $\Sigma$. Solutions $N$ of (5.1) are invariant under this flow generated by $V$ and are therefore advected in time along the ‘spatial component’ of $V$. We note that in the case $r = 0$, (5.1) reduces to the standard scalar advection equation. Moreover, we observe that this initial value problem does not require the specification of a metric on $M$.

In order to bring equation (5.1) into a form for which our formalism applies, we choose the frame $(b_0, b_1, b_2) = (T, m, \overline{m})$ above with $T = \partial_1$ and $m = m^r$ given by equation (2.4). Hence, the dual frame is $(\alpha^0, \alpha^1, \alpha^2) = (n, \sigma, \overline{\sigma})$ with

$$n = d\tau, \quad \sigma = \frac{1}{\sqrt{2}}(d\vartheta + i \sin \vartheta d\varphi).$$

With this it follows that

$$\kappa_1 = \kappa_2 = \mu_0 = 0, \quad \mu_1 = -\frac{1}{2\sqrt{2}} \cot \vartheta. \tag{5.2}$$

Since the vector field $V$ has the property $V(\tau) = 1$, there exists a smooth complex function $\xi$, so that

$$V = T + \sqrt{2}m \xi + \sqrt{2} \overline{\xi}.$$ \tag{5.3}
Note that $\xi$ has spin weight $-1$ (and hence $\bar{\xi}$ spin weight 1). For simplicity, we assume that $\xi$ is independent of $t$.

We restrict to the scalar and vectorial advection problems now, i.e., to $r = 0$ and $r = 1$. In the scalar case, $N$ is a function with spin weight zero, hence equation (5.1) translates to

$$\partial_t N = -\xi \partial_N - \bar{\xi} \partial_{\bar{N}} - \xi \partial_\sigma - \bar{\xi} \partial_{\bar{\sigma}} - \xi \partial_\sigma - \bar{\xi} \partial_{\bar{\sigma}} = 0,$$

using (2.5) and (2.6) for $s = 0$. We check easily that each term in this equation is of spin weight 0. In general, consistency of the spin weights of the terms in an equation is a good indication that it has been derived correctly.

In the vectorial case, we decompose the vector $N$ as

$$N = \sigma \xi T + \sqrt{2} \psi \eta m + \sqrt{2} \zeta \eta \bar{m},$$

where $\eta$ is of spin weight $s$ and where $\psi \eta = -\bar{\psi} \eta$. Projecting (5.1) onto $n$, $\sigma$ and $\bar{\sigma}$, and using (5.2), (5.3) and (5.5) and the fact that $L_0$ and $L_1$ vanish for the reference frame, we find

$$\partial_t (\sigma N) + \sqrt{2} \xi \partial_m (\sigma N) + \sqrt{2} \bar{\xi} \partial_{\bar{m}} (\sigma N) = 0,$$

$$\partial_t (-\psi \eta \xi m) + \sqrt{2} \psi \partial_m (\psi \eta \xi m) + \sqrt{2} \bar{\psi} \partial_{\bar{m}} (\psi \eta \xi m) = 0,$$

$$\partial_t (\bar{\sigma} \eta \xi \bar{m}) + \sqrt{2} \bar{\sigma} \partial_m (\bar{\sigma} \eta \xi \bar{m}) + \sqrt{2} \bar{\xi} \partial_{\bar{m}} (\bar{\sigma} \eta \xi \bar{m}) = 0.$$

Using the relations (2.5) and (2.6) to write

$$m_{\langle s | f \rangle} = \frac{1}{\sqrt{2}} (\bar{\partial}_{\langle s | f \rangle} + i s f \partial \vartheta), \quad \bar{m}_{\langle s | f \rangle} = \frac{1}{\sqrt{2}} (\bar{\partial}_{\langle s | f \rangle} - i s f \partial \vartheta),$$

for any quantity $f$ of spin weight $s$, we find

$$\partial_t (\sigma N) = -\xi \partial_t (\sigma N) - \bar{\xi} \partial_t (\sigma N),$$

$$\partial_t (-\psi \eta \xi m) = -\xi \partial_t (-\psi \eta \xi m) - \bar{\xi} \partial_t (-\psi \eta \xi m) + \bar{\partial} (\psi \eta \xi m) + \bar{\partial} (\bar{\psi} \eta \xi m),$$

$$\partial_t (\bar{\sigma} \eta \xi \bar{m}) = -\xi \partial_t (\bar{\sigma} \eta \xi \bar{m}) - \bar{\xi} \partial_t (\bar{\sigma} \eta \xi \bar{m}) + \bar{\partial} (\bar{\sigma} \eta \xi \bar{m}) + \bar{\partial} (\bar{\sigma} \eta \xi \bar{m}).$$

We realize that all terms are of a well-defined and consistent spin-weight and that all formally singular terms (i.e., those proportional to $\cot \vartheta$) disappear (as one expects). The first of equation (5.6) is of the same form as (5.4) and is decoupled from the other two. The third equation is the complex conjugate of the second one.

### 5.1.1. Spectral decomposition and numerical results.

In this section, we solve (5.4) and (5.6) together with appropriate initial data by application of the Fourier–Galerkin method. We will choose advecting fields $\{ -\bar{\psi} \eta : \xi , \bar{\xi} \eta : \bar{\xi} \}$ comprised of linear combinations of axial rotations. This choice is particularly amenable to analysis of numerical error and stability since initial data that is advected by such fields will undergo a time evolution that must periodically return to its initial state—it is this periodic behaviour that we exploit for our numerical tests.

Recall that the generators of rotations in $\mathbb{R}^3$ are given by

$$X = -\sin \varphi \partial_\theta - \cot \vartheta \cos \varphi \partial_\varphi , \quad Y = \cos \varphi \partial_\theta - \cot \vartheta \sin \varphi \partial_\varphi , \quad Z = \partial_\varphi,$$

respectively. If we choose $V$ to be one of these generators and then decompose $V$ as in (5.3), we find

$$-i \bar{\xi} X = -i \sqrt{\frac{2\pi}{3}} (Y_{1,-1} - i Y_{1,1}), \quad i \bar{\xi} \bar{X} = -i \sqrt{\frac{2\pi}{3}} (Y_{1,1} - i Y_{1,-1});$$

$$-i \xi Y = \sqrt{\frac{2\pi}{3}} (Y_{1,-1} + i Y_{1,1}), \quad i \xi \bar{Y} = \sqrt{\frac{2\pi}{3}} (Y_{1,1} - i Y_{1,-1});$$

$$-i \xi Z = -2i \sqrt{\frac{\pi}{3}} Y_{1,0}, \quad i \xi \bar{Z} = -2i \sqrt{\frac{\pi}{3}} Y_{1,0}.$$
It is convenient to further normalize the advecting fields \(s_Nlm := \sqrt{2\pi} s_Nlm\), such that the
final state of the fields being advected will again coincide with the choice of the initial field
configuration after a temporal period \(T = 1\).

For general data, equation (3.2) gives
\[
s_Nlm(\theta, \varphi) = \sum_{l=-L}^{L} \sum_{m=-l}^{l} s_Nlm Y_{lm}(\theta, \varphi),
\]
\[
\dot{s}_N(\theta, \varphi; t) = \sum_{l=-L}^{L} \sum_{m=-l}^{l} \dot{s}_Nlm(t) Y_{lm}(\theta, \varphi),
\]
for advecting fields and the fields being advected, respectively. The time dependence of the
solution is carried by the expansion coefficients \(\dot{s}_Nlm(t)\). Equations (2.12), (2.13), (2.15),
(2.20), (5.4) and (5.7) lead to the spectral representation of the scalar advection problem
\[
0 \dot{N}_{lm}(t) = \sum_{l_a, l_b=1}^{L} \sum_{m_a, m_b=-l_b}^{l_b} \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] A_l(-1, 1, m_a; 1, l_b, m - m_a)
\]
\[
\times 0 \dot{N}_{l_b, m-m_a}(t) (-1)^{l_a+l_b} sY_{l_a,m} \right] \right) \right),
\]
where \(\left[ \delta_{l_a, m_a} \right] \) is a Boolean function—equal to 1 when the inequality is satisfied and 0
when it is not. We introduce such functions to make explicit the distinct limits of summation
of the terms in equation (5.8).

For the vector advection problem, the spectral representation of the spin 0 component of
the system is again given by equation (5.8). Due to the condition \(s_N = -1s\) and equation (2.11),
we have \(s_Nlm = -1s_{l, m-l}\), hence in order to completely specify the system, only the spin \(-1\)
equation is required:
\[
-i \dot{N}_{lm}(t) = \sum_{l_a=1}^{L} \sum_{m_a=-l_a}^{l_a} \sum_{l_b=1}^{L} \sum_{m_b=-l_b}^{l_b} \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] (-1) l_a l_a m_a m_a (-1)^{l_a+l_b} sY_{l_a,m} \right] \right) \right),
\]
\[
\times \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] A_l(-1, 1, m_a; 1, l_b, m - m_a)
\]
\[
+ \sqrt{(l_b - 1)(l_b + 2)} \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] A_l(-1, 1, m_a; -2, l_b, m - m_a) \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] A_l(-1, 1, m_a; -2, l_b, m - m_a)
\}
\]
\[
+ \sqrt{l_b(l_b+1)} \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] A_l(-1, 1, m_a; -2, l_b, m - m_a) \left[ \delta_{l_a, l_b} \delta_{m_a, m_b} \right] \left[ \sqrt{l_b(l_b+1)} \right] A_l(-1, 1, m_a; -2, l_b, m - m_a)
\}
\]

Figure 5 shows the results of numerical convergence tests performed using an explicit
fourth-order Runge–Kutta (RK) method in time for a variety of time steps \(N_a\) and band limits
\(L\). We show the absolute value of the maximum difference in coefficient space \(\dot{\varepsilon}_{abs}\), between
initial configurations of test fields and their final configurations upon advection for five periods
\(T\). Excellent agreement with the expected fourth-order convergence is displayed. We find that
in both scalar and vector cases, the error associated with the temporal discretization dominates
that of the spatial scheme—this was verified by constructing a semilog plot of data generated
for various values of \(L\) versus \(\dot{\varepsilon}_{abs}(5T)\) at fixed \(N_a\), where we found a horizontal line. This is
not unexpected as we are advecting smooth fields by smooth fields and expect an exponential
convergence for the spatial sampling (see section 4.1), whereas only fourth-order convergence
in time is provided by the RK method. Note that here and in what follows when performing
numerical expansions as in equations (2.17) and (2.19), all terms with \(l > L\) are discarded.
5.2. The $2 + 1$-vacuum Maxwell equations on the 2-sphere

As a second application, we study the initial value problem of the $2 + 1$-vacuum Maxwell equations on $M$, i.e., the equations

$$dF = 0, \quad \delta F = *d*F = 0,$$

for the electromagnetic 2-form $F$ on $M$, where $d$ is the exterior derivative and $*$ is the Hodge dual associated with a Lorentzian metric $g$ on $M$; we assume the signature $(+, -, -)$. In abstract index notation, these equations can also be written as

$$\nabla_{[\mu} F_{\nu\rho]} = 0, \quad \nabla^\mu F_{\mu\nu} = 0.$$

Let $(e_0, e_1, e_2)$ be an orthonormal frame with respect to $g$ and choose the frame $(b_0, b_1, b_2) = (T, m, \bar{m})$ by $e_0 = b_0 = T$ and $m = (e_1 - i e_2)/\sqrt{2}$. This implies for example $g(T, T) = 1$, $g(m, m) = -1$ and $g(m, T) = 0$. Let $(w^0, w^1, w^2)$ be the coframe dual to $(e_0, e_1, e_2)$. If $(a^0, a^1, a^2) = (n, \sigma, \bar{\sigma})$ is the coframe dual to $(T, m, \bar{m})$, then

$$w^0 = n = d\tau, \quad \sigma = \frac{1}{\sqrt{2}}(w^1 + i w^2).$$
Since any smooth metric on $S^2$ is conformal to the standard metric on the unit sphere, a general smooth metric on $M$ is described by a smooth real\(^7\) strictly positive function $f : M \to \mathbb{R}$ of spin weight zero, so that
\[ m = fm^*, \]
where $m^*$ is given by equation (2.4). In the special case $f \equiv 1$, one obtains the geometry of the standard round unit sphere. Physically, our system can therefore be interpreted as vacuum electrodynamics in a universe of two-dimensional spatial spherical topology whose geometry is described by the function $f$. Note that we allow $f$ to depend on time. As before, we assume $T = \partial_t$. Then, it follows
\[
\kappa_1 = \kappa_2 = 0, \quad \mu_0 = \frac{\partial_t f}{f}, \quad \mu_1 = m^*(f) - \frac{1}{\sqrt{2}} f \cot \vartheta.
\]
In $2+1$ dimensions, the 2-form $F$ has three independent components which we write as
\[
F = E_1 \omega^1 \wedge \omega^0 + E_2 \omega^2 \wedge \omega^0 + B \omega^1 \wedge \omega^2.
\]
Hence $E_1$ and $E_2$ can be interpreted as the two components of the electric field and $B$ as the component of the magnetic field. In fact, we can introduce a purely spatial ‘electric’ 1-form:
\[
E = - E_1 \omega^0 + E_2 \omega^0,
\]
so that the 1-form $*F$ becomes
\[
*F = E - B \omega^0.
\]
For the following, it is useful to define the three complex Maxwell scalars as
\[
\begin{align*}
_{1}F & := F(T, m) = - \frac{1}{\sqrt{2}} (E_1 - i E_2), \\
_{0}F & := i F(m, \overline{m}) = -B,
\end{align*}
\]
where $\sigma F$ has spin weight $\sigma$. These fields satisfy the reality conditions
\[
_{0}F = \overline{_{0}F}, \quad _{-1}F = \overline{_{-1}F}.
\]
We find that the electric 1-form $E$ can be written as
\[
E = - _{i}F \sigma + _{-i}F \overline{\sigma}.
\]
The first Maxwell equation $dF = 0$ is equivalent to $\delta * F = 0$ and hence to
\[
\delta (Bn) = \delta E.
\]
This equation corresponds symbolically to the Maxwell equation ‘$\dot{B} = \text{div } E$’. Using the same arguments as for the advection problem, we arrive at the following evolution equation for the magnetic field:
\[
\partial_t 0F = \frac{i}{\sqrt{2}} (_{1}F \sigma f - _{-1}F \overline{\sigma} f + f \sigma _{-1}F - \overline{f} \sigma' _{-1}F) + 2_{0}F \frac{\partial_t f}{f}.
\]
(5.11)
The second Maxwell equation $\delta F = 0$ is equivalent to $d*F = 0$ and hence to
\[
* d (Bn) = * dE.
\]
This corresponds to three equations. The $n$-component is the constraint
\[
0 = (*dE)(F) = \frac{1}{\sqrt{2}} (\partial f _{-1}F - f \partial' _{-1}F + \overline{f} \partial _{-1}F - f \overline{\partial'} _{-1}F) =: \frac{C}{\sqrt{2}}.
\]
(5.12)
\footnote{We could also assume that $f$ is complex. Its phase, however, would generate nothing but a rotation of the frame and would hence not contribute to the metric.}
The $\sigma$- and $\bar{\sigma}$-components yield evolution equations for the components of the electric field

$$\partial_t - 1 F = - \frac{1}{\sqrt{2}} i f \partial_0 F + \frac{\partial f}{f} F,$$

(5.13)

$$\partial_t 1 F = \frac{1}{\sqrt{2}} i f \partial_0 F + \frac{\partial f}{f} F.$$

(5.14)

We see explicitly that all terms in the equations are non-singular and of consistent spin weight. Moreover, when we write the $\partial_0$-operator in a coordinate basis, we see that the evolution system is symmetric hyperbolic and hence gives rise to a well-posed initial value problem. It remains to show that the constraint $C = 0$ propagates under the evolution. For this, we derive the evolution equation for the constraint violation quantity $C$ given in equation (5.12). We take the time derivative of equation (5.12) and use the evolution equations of $-1 F$, $0 F$, $1 F$. Then, it is straightforward to find that

$$\partial_t C = 2 \frac{\partial f}{f} C.$$

(5.15)

It follows that if the initial data satisfy the constraint, i.e., $C = 0$ at the initial time, then $C \equiv 0$ for all times and hence the constraints will be satisfied, up to machine precision, during the whole evolution. Furthermore, as can easily be seen from the evolution equations, the reality conditions of equation (5.9) are preserved during the whole evolution, provided they are fulfilled at the initial time.

Note that $0 = (\ast d E)(T)$ is equivalent to $d E(e_1, e_2) = 0$. Since $E$ is purely spatial this is equivalent to $\tilde{d} E = 0$, where $\tilde{d}$ is the purely spatial exterior derivative on the initial hyper-surface. Because $\mathbb{S}^2$ is simply connected, it follows that every solution $E$ of the constraint is of the form

$$E = \tilde{d} \Phi = \frac{f}{\sqrt{2}} \{ (\tilde{\partial} \Phi) \sigma + (\tilde{\partial}^\dagger \Phi) \bar{\sigma} \},$$

(5.16)

where $\Phi$ is an arbitrary smooth scalar function of spin weight zero on the initial hyper-surface. Comparison of equation (5.16) with equation (5.10) yields

$$-1 F = - i \frac{f}{\sqrt{2}} \tilde{\partial} \Phi, \quad 1 F = i \frac{f}{\sqrt{2}} \tilde{\partial}^\dagger \Phi.$$

(5.17)

As the constraint equation (5.12) is independent of $0 F = -B$, the magnetic field may be prescribed freely, subject only to reality conditions.

5.2.1. Numerical results. Our goal is now the construction of numerical solutions by means of spectral decompositions of the dynamical equations governing the Maxwell system. In order to check the numerical solutions thus constructed, we examine the preservation of the constraints associated with the system. In order to further ensure that our implementation is accurate and robust, we compare our calculation to a pseudo-spectral implementation based on discussion of section 4.3. We proceed in two stages. We consider the case where $\mathbb{S}^2$ is deformed at the initial time and examine how a simple choice of initial data evolves with this fixed geometry. We then allow for a time-dependent change of the geometry, again examining how our solution for the fields develops with time.

As the method is entirely analogous to that of section 5.1.1, we do not explicitly state our expansions of equations (5.11), (5.13), (5.14), (5.17) and (5.12) here. However, it is worth pointing out that due to (5.9) and (2.11), we again have

$$0 F_{i,m} = 0 F_{i,-m} (-1)^m, \quad 1 F_{i,m} = -1 F_{i,-m} (-1)^{1-m},$$

26
which implies that only a subset of the full dynamical system needs to be evolved, the rest may be extracted by these symmetries. We have also found it convenient to re-expand terms such as \((\partial_t f)/f\) by defining an auxiliary function

\[ g(\phi, \psi; t) := \partial_t (\ln(f(\phi, \psi; t))) = \frac{\partial f}{f} = \sum_{l=0}^{L} \sum_{m=-l}^{l} \beta_{l,m}(t), sY_{l,m}(\phi, \psi), \]

rather than dealing with \(f\) directly.

We now test the following initial conditions:

\[
\begin{align*}
0\phi_{2,0} &= 1; \\
0\Phi_{1,1} &= i; \quad 0\Phi_{1,-1} = i; \\
0f_{0,0} &= 10\sqrt{\pi}; \quad 0f_{2,0} = 1; \quad 0f_{4,3} = 2i; \quad 0f_{4,-3} = 2i
\end{align*}
\]  

(5.18)

with all other values set to zero. This corresponds to a static deformation of \(S^2\), where at \(t = 0\), the deformation is chosen and fixed for all later times. We calculate the solution numerically making use of the spectral and pseudo-spectral methods together with the embedded RK5(4)7M algorithm of [10]. This last choice of integrator allows us for the local error estimation and hence adaptive control of step size in time. In order to verify that our implementation is consistent, we check that the constraints are satisfied. This is done by (pseudo)spectral decomposition of equation (5.12) in a similar manner to the preceding equations of this section. The results of this are shown for both spectral (figure 6(a)) and pseudo-spectral (figure 6(b)) implementations; we find excellent agreement between both methods.

Figure 6. Value of the constraints (equation (5.12)) associated with solutions of the Maxwell system for static deformation of \(S^2\) subject to the initial conditions of equation (5.18) using adaptive RK5(4)7M as described in the text. Red: \(L = 8\), ‘*’; blue: \(L = 16\), ‘o’; green: \(L = 32\), ‘□’; magenta: \(L = 64\), ‘<’.

Observe that constraints are well-preserved during the entire course of the solution—with the spectral (a) and pseudo-spectral (b) methods providing comparable accuracy. At \(L = 32\), we find that we have approximately saturated the convergence in \(L\), that is, further increasing the band limit will only marginally increase solution accuracy and a law of diminishing returns applies cf figure 7.
The relative error in the energy $\varepsilon_r(t)$ as defined in equation (5.20) for numerical solution of the Maxwell system subject to the initial conditions of equation (5.18). Red: $L = 8$, ‘$\ast$’; blue: $L = 16$, ‘$\circ$’; green: $L = 32$, ‘$\Box$’; magenta: $L = 64$, ‘$\triangle$’. In both solutions, constructed via spectral (a) and pseudo-spectral (b) methods, respectively, we find that oscillations in $\varepsilon_r(t)$ are observed for $L \in \{8, 16\}$. We ascribe the aforementioned oscillations to the numerical scheme and not to any physical property of the Maxwell system as for $L \in \{32, 64\}$, these spurious features disappear. It is important to note that even though oscillatory behaviour in $\varepsilon_r(t)$ is observed, it is bounded over the interval $t$ shown.

Checking other invariants of the system can give further insight into the performance of our numerical scheme. Consider the energy as a function of time

$$
E(t) = \frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi (F_0 F_0 + 2 F_{-1} F_1) \sin \theta d\theta d\varphi.
$$

Due to equation (5.9), we can conclude that equation (5.19) is the integral of a positive definite quadratic form (since $(\sin \theta)/f^2 \geq 0$ for $\theta \in [0, \pi]$) and hence $E(t) \geq 0$ for all $t$. For time-independent $f$, $\partial_t$ is a Killing vector and hence by Noether’s theorem if $t_0$ is the initial time, then we have $E(t_0) = E(t)$. That is, under a static deformation, energy is conserved. In order to examine any deviations in $E$ that may occur due to numerical error, it is convenient to further define relative error in the energy via

$$
\varepsilon_r(t) := \left| \frac{E(t_0) - E(t)}{E(t)} \right|,
$$

which serves as a measure of the failure of energy conservation. We show the value of $\varepsilon_r$ in figure 7 for the solution of the Maxwell system with the initial conditions of equation (5.18), again both spectral and pseudo-spectral methods perform with consistent accuracy affirming our intuition about energy conservation for the system in the case of static deformation.

We now allow a time-dependent deformation of $S^2$ using the pseudo-spectral method. The direction $\partial_t$ is no longer a Killing vector and energy expression of equation (5.19) will no longer be conserved; however, we still have $E(t) \geq 0$ for all $t$. For initial conditions, we take

$$
\partial_2 t_0 = 1; \quad \partial_1 t_1 = 1; \quad \partial_1 t_{-1} = 1;
$$
\[
\begin{align*}
\alpha g_{0,0} &= 10\sqrt{\pi}; & \alpha g_{2,0} &= 1; & \alpha g_{4,3} &= 2i; & \alpha g_{4,3} &= 2i; \\
\alpha h_{0,0} &= 12\sqrt{\pi}; & \alpha h_{2,0} &= 1; & \alpha h_{8,-1} &= 2i; & \alpha h_{8,1} &= 2i,
\end{align*}
\]

where

\[
\alpha f(t, \vartheta, \phi) = \frac{1}{t_c}(t_c - t)\alpha g(\vartheta, \phi) + t \alpha h(\vartheta, \phi), \quad t_c := 2f,
\]

governs the time-dependent deformation of \(S^2\) over an interval \(t \in [0, t_f]\). We show the solution at selected times, together with the geometric picture in figure 8. As \(f\) is now time dependent, the constraint propagation equation (5.15) suggests that numerical violations of the constraints grow with time. Convergence of \(C\) to a well-defined value with increasing band limit \(L\) is still
expected and we find that a higher band limit $L$ is required (see figure 9(a)) in contrast to the static (no deformation) case. Finally, we note that as anticipated, $E(t) \geq 0$ for $t \in [0, t_1]$ (see figure 9(b)).

6. Conclusion

In this work, we have presented a method for evolving tensorial equations on manifolds with spherical topology. It is based on the use of the spin-weighted spherical harmonics, a class of functions on the sphere which is closely related to irreducible representations of $SU(2)$. We have demonstrated that our method exhibits the accuracy and rapid convergence to solutions that is expected from spectral methods.

Of course, this method is not limited to a single sphere. Instead, our main application will be in systems defined on a manifold with spatial topology $\mathbb{R} \times S^2$, where the tensor fields are split into various pieces intrinsic to the sphere factor and depending on a ‘radial’ coordinate corresponding to the $\mathbb{R}$ factor. This kind of topology occurs naturally in the description of the global structure of space-times in general relativity.

This method can be further generalized to half-integer spin. This will allow us to solve spinorial equations, such as the Dirac equation or Weyl’s equation for a (massless) neutrino on space-times with spheroidal components as discussed above.

Acknowledgments

This research was partly funded by the Marsden Fund of the Royal Society of New Zealand under contract number UOO0922.
References

[1] Adams J C and Swarztrauber P N 1997 Spherepack 2.0: a model development facility NCAR
www2.cisl.ucar.edu/resources/legacy/spherepack

[2] Alpert B K and Rokhlin V 1991 A fast algorithm for the evaluation of Legendre expansions SIAM
J. Sci. Stat. Comput. 12 158–79

[3] Bartnik R A and Norton A H 2000 Einstein equations in the null quasi-spherical gauge: III.
Numerical algorithms SIAM J. Sci. Comput. 22 917–50

[4] Beyer F 2009 A spectral solver for evolution problems with spatial S3-topology J. Comput.
Phys. 228 6496–513

[5] Beyer F, Douli G, Frauendiener J and Whale B 2012 Numerical space-times near space-like and
null infinity. The spin–2 system on Minkowski space Class. Quantum Grav. 29 245015

[6] Boyd J P 2001 Chebyshev and Fourier Spectral Methods 2nd edn (New York: Dover)

[7] Brügmann B 2013 A pseudospectral matrix method for time-dependent tensor fields on a spherical
shell J. Comput. Phys. 235 216–40

[8] Csizmadia P, László A and Rácz I 2013 On the use of multipole expansion in time evolution
of nonlinear dynamical systems and some surprises related to superradiance Class. Quantum
Grav. 30 015010

[9] Dilts G A 1985 Computation of spherical harmonic expansion coefficients via FFT’s J. Comput.
Phys. 57 439–53

[10] Dormand J R and Prince P J 1980 A family of embedded Runge–Kutta formulae J. Comput. Appl.
Math. 6 19–26

[11] Doroshkevich A G, Naselsky P D, Verkhodanov O V, Novikov D I, Turchaninov V I, Novikov I D,
Christensen P R and Chiang 2005 Gauss–Legendre sky pixelization (GLESP) for CMB maps
Int. J. Mod. Phys. D 14 275–90

[12] Friedlich H 1998 Gravitational fields near space-like and null infinity J. Geom. Phys. 24 83–163

[13] Goldberg J N, Macfarlane A J, Newman E T, Rohrlich F and Sudarshan E C G 1967 Spin-s
spherical harmonics and J. Math. Phys. 8 2155

[14] Górski K M, Hivon E, Banday A J, Wandelt B D, Hansen F K, Reinecke M and Bartelmann M 2005
HEALPix: a framework for high-resolution discretization and fast analysis of data distributed
on the sphere Astrophys. J. 622 759–71

[15] Huffenberger K M and Wandelt B D 2010 Fast and exact spin-s spherical harmonic transforms
Astrophys. J. Suppl. Ser. 189 255–60

[16] Katznelson Y 2004 An Introduction to Harmonic Analysis 3rd edn (Cambridge: Cambridge
University Press)

[17] Kostelec P J and Rockmore D N 2008 FFTs on the rotation group J. Fourier Anal. Appl. 14 145–79

[18] Luscombe J H and Luban M 1998 Simplified recursive algorithm for Wigner 3 j and 6 j symbols
Phys. Rev. E 57 7274–7

[19] McEwen J D 2011 Fast exact (but unstable) spin spherical harmonic transforms All Results J. Phys.
1 1–48

[20] McEwen J D and Wiaux Y 2011 A novel sampling theorem on the sphere IEEE Trans. Signal
Process. 59 5876–87

[21] Muciaccia P F, Natoli P and Vittorio N 1997 Fast spherical harmonic analysis: a quick algorithm for
generating and/or inverting full-sky, high-resolution cosmic microwave background anisotropy
maps Astrophys. J. Lett. 488 L63

[22] Newman E T and Penrose R 1966 Note on the Bondi–Metzner–Sachs group J. Math. Phys.
7 863–70

[23] Olver F W J, Lozier D W, Boisvert R F and Clark C W 2010 NIST Handbook of Mathematical
Functions (Cambridge: Cambridge University Press)

[24] Penrose R and Rindler W 1984 Two-spinor Calculus and Relativistic Fields Spinors and Space-Time
vol 1 (Cambridge: Cambridge University Press)

[25] Prézeau G and Reinecke M 2010 Algorithm for the evaluation of reduced Wigner matrices
Astrophys. J. Suppl. Ser. 190 267

[26] Rácz I and Tóth G Zs 2011 Numerical investigation of the late-time Kerr tails Class. Quantum
Grav. 28 195003

[27] Rasch J and Yu A C H 2003 Efficient storage scheme for precalculated Wigner 3 j, 6 j and Gaunt
coefficients SIAM J. Sci. Comput. 25 1416–28

[28] Risbo T 1996 Fourier transform summation of Legendre series and D-functions J. Geod. 70 383–96
[29] Rokhlin V and Tygert M 2006 Fast algorithms for spherical harmonic expansions SIAM J. Sci. Comput. 27 1903–28
[30] Sakurai J J and Tuan S F 1994 Modern Quantum Mechanics (Reading, MA: Addison-Wesley)
[31] Schulten K and Gordon R G 1975 Exact recursive evaluation of 3j- and 6j-coefficients for quantum-mechanical coupling of angular momenta J. Math. Phys. 16 1961–70
[32] Suda R and Takami M 2002 A fast spherical harmonics transform algorithm Math. Comput. 71 703–15
[33] Sugiura M 1990 Unitary Representations and Harmonic Analysis: An Introduction (Amsterdam: North-Holland)
[34] Thorne K S 1980 Multipole expansions of gravitational radiation Rev. Mod. Phys. 52 299–339
[35] Trapani S and Navaza J 2006 Calculation of spherical harmonics and Wigner d functions by FFT. applications to fast rotational matching in molecular replacement and implementation into AMoRe Acta Cryst. A 62 262–9