Quantum Dynamics of Topological Singularities:
Feynman’s Influence Functional Approach

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Abstract

Starting from the microscopic theory of Bardeen-Cooper-Schrieffer (BCS) for the fermionic superfluids, we show that the vortex dynamics can be followed naturally by extending Feynman’s influence functional approach to incorporate the transverse force. There is a striking mutual independence of the transverse and longitudinal influences: The former has the topological origin and is insensitive to details, while the latter corresponds to the well-known damping kernel depending on details.

I. INTRODUCTION

Vortices in superconductors and He3 superfluids are topological excitations. They determine the global properties such as the stability of the supercurrent carrying states, and have been under intensive theoretical and experimental studies since earlier sixties [1]. The detailed microscopic BCS type theory for fermionic superfluids is successful and well defined. The derivation of vortex dynamics is, however, non-trivial and less certain. Here we present a path integral derivation of vortex dynamics following the line of geometric methods [2,3]. The advantage of the present method is that in the effective vortex Lagrangian the separation of geometric and dissipative contributions is
natural. The important results here are that the transverse force agrees with the one obtained by the Berry phase method [2] and by the total force-force correlation function method [3], and is insensitive to details. The friction is determined by the spectral function of the Hamiltonian, sensitive to details. The large transverse force has been found recently by a direct measurement [4], in consistent with the prediction in Ref’s.[2,3].

II. INFLUENCE FUNCTIONAL APPROACH

To find the effective vortex action, we begin with the standard BCS Lagrangian for s-wave pairing in the imaginary time representation. Here the unwanted fermionic degrees of freedoms will be integrated. We will only consider neutral superfluids here, but the coupling to electromagnetic fields does no affect our main results. More detailed analysis will be published elsewhere. [5] The model BCS Lagrangian is

\[ L_{BCS} = \sum_\sigma \psi_\sigma^\dagger(x,\tau) \left( \hbar \partial_\tau - \mu_F - \frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \psi_\sigma(x,\tau) \]

\[ -g \psi_\uparrow^\dagger(x,\tau) \psi_\downarrow^\dagger(x,\tau) \psi_\uparrow(x,\tau) \psi_\downarrow(x,\tau) , \]

where \( \psi_\sigma \) describes electrons with spin \( \sigma = (\uparrow, \downarrow) \), \( \mu_F \) the chemical potential determined by the electron number density, \( V(x) \) the impurity potential, and \( x = (x, y, z) \). A vortex at \( x_v \) has been implicitly assumed. The partition function is

\[ Z = \int \mathcal{D}\{x_v, \psi_\sigma^\dagger, \psi\} \times \exp \left\{ -\frac{1}{\hbar} \int_0^{\beta} d\tau \int d^3x L_{BCS} \right\} , \]

with \( \beta = 1/k_B T \) , and \( d^3x = dx dy dz \). Next, we first perform the Bogoliubov transformation by introducing the auxiliary fields \( \{\Delta^*, \Delta\} \), integrate over the electron fields \( \psi_\sigma^\dagger \) and \( \psi_\sigma \), then over the auxiliary(pair) fields under the meanfield approximation. The partition function for the vortex is
\[
Z = \int \mathcal{D}\{x_v\} \exp \left\{ -\frac{S_{\text{eff}}}{\hbar} \right\},
\]
with the effective vortex action
\[
\frac{S_{\text{eff}}}{\hbar} = - Tr \ln G^{-1} - \frac{1}{\hbar g} \int_{0}^{\beta} d\tau \int d^3x |\Delta|^2,
\]
where \( Tr \) includes internal and space-time indices, and the Nambu-Gor'kov Green's function \( G \) defined by
\[
(h\partial_{\tau} + \mathcal{H})G(x,\tau;x',\tau') = \delta(\tau - \tau')\delta^3(x - x'),
\]
which together with the BCS gap equation, or the self-consistent equation,
\[
\Delta(x,\tau) = -g <\psi_\downarrow(x,\tau)\psi_\uparrow(x,\tau) >.
\]
Here the Hamiltonian is defined as
\[
\mathcal{H}(\Delta,\Delta^*) = \begin{pmatrix}
H & \Delta \\
\Delta^* & -H^*
\end{pmatrix},
\]
with \( H = -(\hbar^2/2m)\nabla^2 - \mu_F + V(x) \).

We assume that the vortex is confined to move in a small regime around a point at \( x_0 \), which allows a small parameter expansion in terms of the difference between the vortex position \( x_v \) and \( x_0 \). We look for the long time behavior of vortex dynamics under this small parameter expansion. For the meanfield value of the order parameter, this expansion is
\[
\Delta(x,\tau,x_v) = \left( 1 + \delta x_v(\tau) \cdot \nabla_0 + \frac{1}{2}(\delta x_v(\tau) \cdot \nabla_0)^2 \right) \Delta_0(x,x_0).
\]
Here \( \delta x_v = x_v - x_0 \). In Eq. (8) we have used the fact that when \( x_v = x_0 \) the vortex is static. The effective vortex action to the same order is, after dropping a constant term and assuming the rotational symmetry under the impurity average, a straightforward calculation leads to the following effective vortex action
\[ S_{\text{eff}} = \frac{1}{2} \int_0^{\hbar \beta} d\tau \left[ K |\delta x_v(\tau)|^2 + \int_0^{\tau} d\tau' F_{\parallel}(\tau - \tau') |\delta x_v(\tau) - \delta x_v(\tau')|^2 \right. \\
\left. + \int_0^{\hbar \beta} d\tau' F_{\perp}(\tau - \tau')(\delta x_v(\tau) \times \delta x_v(\tau')) \cdot \hat{z} \right] , \tag{9} \]

with the spring constant in the effective potential,
\[ K = \frac{1}{g} \int d^3x |\nabla_0 \Delta_0^*(x, x_0)|^2 - \int_0^\infty d\omega J(\omega) \omega , \tag{10} \]

the damping kernel,
\[ F_{\parallel}(\tau) = \frac{1}{\pi} \int_0^\infty d\omega J(\omega) \frac{\cosh \left[ \omega \left( \frac{\hbar \beta}{2} - |\tau| \right) \right]}{\sinh \left[ \omega \left( \frac{\hbar \beta}{2} \right) \right]} , \tag{11} \]

and the transverse kernel, in the long time limit, in terms of the virtual transitions,
\[ F_{\perp}(\tau) = -\partial_\tau \delta(\tau) \sum_{k,k'} \int d^3x \int d^3x' \hbar(f_k - f_{k'}) \\\n\frac{1}{2} \hat{z} \cdot \left( \Psi_k^\dagger(x') \nabla_0 \Psi_k(x') \times \nabla_0 \Psi_{k'}^\dagger(x) \Psi_k(x) \right) , \tag{12} \]

or in terms of the contribution from each state,
\[ F_{\perp}(\tau) = -\partial_\tau \delta(\tau) \sum_k \int d^3x \hbar \hat{z} \cdot (f_k \nabla_0 u_k^*(x) \times \nabla_0 u_k(x) \\\n-(1 - f_k) \nabla_0 v_k^*(x) \times \nabla_0 v_k(x)) . \tag{13} \]

with the Fermi distribution function \( f_k = 1/(1 + e^{E_k}) \), and the spectral function
\[ J(\omega) = \frac{\pi}{2} \sum_{k,k'} \delta(\hbar \omega - |E_k - E_{k'}|) |f_k - f_{k'}| \times \\
\left| \int d^3x \Psi_k^\dagger(x) \nabla_0 \mathcal{H}_0 \Psi_{k'}(x) \right|^2 . \tag{14} \]

In Eqs.(12-14), the wavefunctions \( \left\{ \Psi_k(x) = \begin{pmatrix} u_k(x) \\ v_k(x) \end{pmatrix} \right\} \) are the eigenfunctions of the Hamiltonian \( \mathcal{H}_0 = \mathcal{H}(\Delta_0, \Delta_0^*) \), determined by the stationary Schrödinger equation, the Bogoliubov-de Gennes equation,
\[ \mathcal{H}_0 \Psi_k(x) = E_k \Psi_k(x) . \tag{15} \]
III. DISCUSSIONS

The influence functional approach has been successfully implemented in recent studies of quantum dissipative dynamics [3], as also manifested in the present conference proceedings. However, only the longitudinal response has been considered in those systems. The above microscopic derivation of vortex dynamics demonstrates that the transverse force can be obtained from this approach, too. Therefore we have extended the influence functional approach.

Another interesting feature of the transverse response is its insensitivity to details, which is most transparent from Eq.(13): In the one-body density matrix both the electron number density and the phase $\theta(x - x_v)$ are all insensitive to details, if the localization effect caused by impurities is negligible. Here the phase $\theta$ is defined through the order parameter $\Delta(x, \tau, x_v) \rightarrow |\Delta|e^{iq\theta(x-x_v)}$, with $q = \pm 1$ describing the vorticity along the $\hat{z}$ direction and $\theta(x) = \arctan(y/x)$.

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