1/$N$-expansions in non-relativistic quantum mechanics

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An extensive number of numerical computations of energy 1/$N$ series using a recursive Taylor series method are presented in this paper. The series are computed to a high order of approximation and their behaviour on increasing the order of approximation is examined.

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I. INTRODUCTION

In chemistry and atomic physics, 1/$N$ (or 1/$D$) perturbation theory is very powerful in calculating energy eigenstates for many complicated systems[1–4]. The method is especially valuable when the applicability of Hartree-Fock methods is limited. In this study the behaviour of 1/$N$ expansions is investigated at a very high order of approximation. For the sake of simplicity only radial potentials with spherical symmetry will be considered. The $N$-dimensional non-relativistic Schrödinger equation for such potentials may be written as

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} + \frac{1}{2} \frac{N-1}{r} \frac{d}{dr} + \frac{l(l+N-2)}{2r^2} + V(r) \right] \phi(r) = E\phi(r)$$  \hspace{1cm} (1)

where $N$ is the number of spatial dimensions [5]. A brief review of this equation is presented in appendix B.

The concept underlying 1/$N$ expansion methods is the use of perturbation theory on the expandable parameter 1/$N$. A standard problem in perturbation theory would be a solvable Schrödinger equation plus a small perturbation, expressed as

$$H = H_0 + gV$$  \hspace{1cm} (2)

where $H_0$ is the solvable Hamiltonian and $gV$ is the small perturbation. In this notation $g$ is a real expansion coefficient. To obtain an approximation to the eigenstates of the perturbed Schrödinger equation one inserts a series for the wave function in powers of $g$ as

$$u = u^{(0)} + gu^{(1)} + g^2u^{(2)} + ...$$  \hspace{1cm} (3)

and a series in energy as

$$E = E^{(0)} + gE^{(1)} + g^2E^{(2)} + ...$$  \hspace{1cm} (4)

Finally, these series are solved for the unknown functions $u^{(r)}$ and the energy coefficients $E^{(r)}$.

For the standard problem this procedure provides an efficient way to obtain an estimate of the eigenstates and the energy of the physical system. Widespread application of this method is, however, limited since only problems with an expandable coefficient $g$ can be solved. The dimensional dependency of the Schrödinger equation provides us with a hidden expandable variable of the potential. Roughly speaking, enlarging the dimensions of a physical system turns the system “classical”, and in the limit of large dimensionality a particle in a potential is fixed at the minimum; in some sense, $\hbar \to 0$. A classical analog could be a damped pendulum. A differential equation describing this is e.g.:

$$m r \ddot{\theta} + b \dot{\theta} + mg\theta = 0$$  \hspace{1cm} (5)

where $m r \dot{\theta} \ll b \dot{\theta}$ (assuming small angles). In the case of a very heavily damped pendulum it is safe to disregard the $b m r$ term. This implies that the second-order differential equation reduces to the first-order differential equation

$$b \dot{\theta} + mg\theta = 0$$  \hspace{1cm} (6)

Solutions of a first-order differential equation are unable to oscillate. The pendulum thus seeks the fixed point of the potential.
An $N$-dimensional particle, where $N$ is very large, behaves somewhat like a pendulum in a highly viscous fluid. Its “damping term” (the first order differential term) in the Schrödinger equation is large, and hence its de Broglie wavelength becomes small. To leading order in $1/N$, the particle may be addressed by quasi-classical concepts. Accordingly, the fixed point of the potential provides a starting point for a power series expansion in $1/N$ for the energy eigenvalue and the wave function.

Most $1/N$-series provide accurate results even when only the first few terms are summed; it is thus more interesting to examine the behaviour of the series as the order of approximation increases. An interesting aspect of the $1/N$-method is that it is perturbatively exact in the sense that there are no explicit approximations in the method. Most potentials lead to apparently asymptotic series at large orders of approximation. The approach here is to calculate the $1/N$ series to a significant order (30 - 100 terms) for miscellaneous potentials and then explore the behaviour of the series. This is accomplished by employing a computer algebraic program. Round-off errors are also taken account of in the calculations.

II. THEORY

How to obtain the $1/N$-series

In order to obtain $1/N$-series for a given potential there are several ways to proceed. One procedure is to expand the wave function, the energy, and the potential in Taylor series [6]. This approach makes it easy to make an efficient computer algorithm. Before the Taylor series expansions are made, it is preferable to redefine the equations slightly.

First, the $N$-dimensional Schrödinger equation is transformed into

$$\left( -\frac{1}{2} \frac{d^2}{dr^2} + k^2 \left[ \frac{1}{2} \left( 1 - \frac{1}{2} \right) (1 - \frac{3}{2}) \right] + \hat{V}(r) \right) \psi(r) = E' \psi(r)$$

by redefining $\psi(r) = r^{(N-1)/2} \phi(r)$. Now, $k$ is defined by $k = N + 2l$. The energy eigenvalue is denoted by $E'$.

The position $r$ is then redefined in terms of $k$, $r = \sqrt{k} \rho$ or $yr = \rho$ where $y = \frac{1}{\sqrt{k}}$. The Schrödinger equation is subsequently recast into a differential equation in $\rho$, as

$$\left( -\frac{1}{2} \frac{d^2}{d\rho^2} + k^2 \left[ \frac{1}{2} \left( 1 - \frac{1}{2} \right) (1 - \frac{3}{2}) \right] + \hat{V}(\rho) \right) \psi(\rho) = E' k \psi(\rho)$$

If now $\hat{V}(r) = kV(\rho)$, then

$$\left( -\frac{1}{2} \frac{d^2}{d\rho^2} + k^2 \left[ \frac{1}{2} \left( 1 - \frac{1}{2} \right) (1 - \frac{3}{2}) \right] + V(\rho) \right) \psi(\rho) = E' k \psi(\rho)$$

In the large-$N$ limit one has an effective potential like

$$k \left[ \frac{1}{8 \rho^2} + V(\rho) \right]$$

Its minimum is the energy $k E^{(-2)}$ which matches the energy at the minimum $\rho_0$.

To ease the further derivations it is preferable to rescale distance as $x = \sqrt{k}(\rho - \rho_0)$ and define $\psi(\rho) = e^{U(x)}$ and

$$V_{eff}(x) = \frac{1}{8} \rho(x)^{-2} + V(\rho) - E^{(-2)}$$

Introducing $\psi$ in eq. (9) leads to a differential equation for $U(x)$.

$$-\frac{1}{2} \left[ \dot{U}(x) + \dot{U}(x)^2 \right] + kV_{eff}(x) + \left( \frac{1}{2} + \frac{3}{8} k^{-1} \right) \rho(x)^{-2} = E' - E^{(-2)} k = E$$

or

$$\dot{U}(x) + \dot{U}(x)^2 - 2W(x) + 2E = 0$$
where \( W(x) = kV_{\text{eff}}(x) + \left[ \frac{\pm \frac{1}{2} + k-1}{p(x)} \right] \). Now the actual Taylor series expansion may begin. Extending \( \dot{U}(x), E, W(x) \) in Taylor series in \( x \) and \( y \) one obtains.

\[
U(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ \frac{D_m x^{2m}}{2m} \right] y^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \left[ \frac{C_m x^{2m+1}}{2m+1} \right] y^{2n+1}
\]

(14)

\[
E = \sum_{n=0}^{\infty} E^{(n-1)} y^{2n}
\]

(15)

Coefficients of the type \( D^n_0 \) are defined to be zero. The series for the potential is extraordinary. For \( W(x) \) it turns out that for any power of \( x \), say \( k \ (k > 1) \), one has solely terms of \( y \) in powers \( k - 2, k \) and \( k + 2 \).

**The recursion formulas**

Using the Taylor series technique, series expansions for the energy and the wave function have to be found by recursion. The easiest way to do this is to find universal recursion formulas [6] for the required coefficients and then use a computer to recursively calculate each coefficient one by one to a preferred number of terms. For the ground state, the recursion relations are established by the requirement that

\[
\dot{U}(x) + \dot{U}(x)^2 - 2W(x) + 2E = 0
\]

(16)

or

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{n} [(2m + 1)D^n_{m+1} x^{2m}] y^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} [(2m)C^n_{m} x^{2m-1}] y^{2n+1}
\]

\[
+ \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \left[ D^n_{m} x^{2m-1} \right] y^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \left[ C^n_{m} x^{2m} \right] y^{2n+1} \right]^2 + 2 \sum_{n=0}^{\infty} E^{(n-1)} y^{2n}
\]

\[
- 2 \left[ W^0_0 + W^0_2 x^2 + (W^1_1 x + W^1_3 x^3) y + \sum_{n=1}^{\infty} (W^2_{2n-2} x^{2n-2} + W^2_{2n} x^{2n} + W^2_{2n+2} x^{2n+2}) y^{2n}
\]

\[
+ \sum_{n=1}^{\infty} (W^2_{2n-1} x^{2n-1} + W^2_{2n+1} x^{2n+1} + W^2_{2n+3} x^{2n+3}) y^{2n+1} \right] = 0
\]

(17)

From these equations it is apparent that

\[
D^0_1 = - \sqrt{2W^0_2}
\]

(18)

and

\[
C^n_{n+2} = D^n_{n+2} = 0
\]

(19)

Likewise, for the coefficients \( D^n_m \) it follows that

\[
D^n_m = - \frac{1}{2D^1_1} \left[ -2W^2_{2m} + (2m + 1)D^n_{m+1} + \sum_{i=1}^{n-1} \sum_{j=1}^{i+1} D^i_j D^{n-i}_{m+1-j} + \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} C^i_j C^{n-i}_{m-j} \right]
\]

(20)

For the coefficients \( C^n_m \) one obtains

\[
C^n_m = - \frac{1}{2D^1_1} \left[ -2W^2_n + 2(m + 1)C^n_{m+1} + 2 \sum_{i=1}^{n} \sum_{j=1}^{i+1} D^i_j C^{n-i}_{m+1-j} \right]
\]

(21)
And for the energy one obtains, after some algebra,
\[ E^{(n-1)} = \frac{1}{2} \left[ -D_1^n + 2W_0^{2n} - \sum_{i=0}^{n-1} C_i^n C_0^{n-i-1} \right] \] (22)

All coefficients of the type \( C_j^n \), where \( j \) is negative, or \( D_j^n \), where \( j \) is zero or negative, are assumed to be zero in the summation.

Obvious extensions of this technique to include excited states are possible. We can always write
\[ \psi = (xy - A(y)) e^{U(x)} \] (23)
for the first excited state, and
\[ \psi = (x^2y^2 + xyC(y) + B(y)) e^{U(x)} \] (24)
for the second excited state, etc. Here, \( A(y), B(y) \) and \( C(y) \) are series expansions in \( y \) for the nodes in \( \psi \).

\[ A(y) = \sum_{n=1}^{\infty} a_n y^{2n} \quad B(y) = \sum_{n=1}^{\infty} b_n y^{2n} \quad C(y) = \sum_{n=1}^{\infty} c_n y^{2n} \] (25)

Subsequently, a differential equation for \( U(x) \) is obtained for the first excited state.
\[ (xy - A) \left( \ddot{U}(x) + \dot{U}(x)^2 - 2W(x) + 2E \right) + 2y\dot{U}(x) = 0 \] (26)

Expressing
\[ \ddot{U}(x) + \dot{U}(x)^2 - 2W(x) + 2E \] (27)
in a new Taylor series
\[ T + S = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ T_{m,2^n} x^{2m} \right] y^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ S_{m,2^n+1} x^{2m+1} \right] y^{2n+1} \] (28)
the previously obtained universal recursion formulas for the ground state are again useful. The recursion formulas for \( T \) and \( S \) are arrived at using the relation
\[ (xy - A) (T + S) + 2y\dot{U}(x) = 0 \] (29)
or
\[ \left[ xy - \sum_{n=1}^{\infty} a_n y^{2n} \right] \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ T_{m,2^n} x^{2m} \right] y^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[ S_{m,2^n+1} x^{2m+1} \right] y^{2n+1} \right] + 2y \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \left[ D_{m,2^{n-1}} x^{2m+1} \right] y^{2n+1} + \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \left[ C_{m,2^{n+1}} x^{2m+1} \right] y^{2n+1} = 0 \] (30)

One obtains
\[ T_m^n = \sum_{k=1}^{n-m} a_k S_{m-k}^{n-k} - 2D_{m+1}^n \] (31)
and
\[ S_m^n = \sum_{k=1}^{n-m} a_k T_{m+1}^{n-k+1} - 2C_{m+1}^n \] (32)

\[ a_n = \frac{1}{T_0} \left[ 2C_0^{n-1} - \sum_{k=1}^{n-1} a_k T_0^{n-k} \right] \] (33)
Once again

$$D_1^0 = -\sqrt{2W_2^0}, \quad C_{n+2}^n = D_{n+2}^n = 0$$  \hspace{1cm} (34)$$

$$D_m^n = -\frac{1}{2D_1^0} \left[ -T_m^n - 2W_m 2^{n+1} + (2m + 1)D_m^{n+1} + \sum_{i=1}^{n} \sum_{j=1}^{i+1} D_i D_{m+1-i-j}^{n+1} + \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} C_i C_{m+1-i-j}^{n+1} \right]$$  \hspace{1cm} (35)$$

$$C_m^n = -\frac{1}{2D_1^0} \left[ -S_m^n - 2W_m 2^{n+1} + 2(m + 1)C_m^{n+1} + 2 \sum_{i=1}^{n} \sum_{j=1}^{i+1} D_i C_{m+1-i-j}^{n+1} \right]$$  \hspace{1cm} (36)$$

$$E^{(n-1)} = \frac{1}{2} \left[ T_0^n - D_1^n + 2W_0 2^{n-1} - \sum_{i=0}^{n-1} C_0 C_{n-i}^{n-1} \right]$$  \hspace{1cm} (37)$$

Successively for the second excited state it is found that

$$(x^2 y^2 + C x y + B) \left( \ddot{U}(x) + \dot{U}(x)^2 - 2W(x) + 2E \right) + 2y^2 + 2 \left( 2x y^2 + y C \right) \dot{U}(x) = 0$$  \hspace{1cm} (38)$$

Note that there is a misprint in [6], in that $2x^2 y$ should read $2x y^2$. Once again

$$x^2 y^2 + x y \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_n y^{2n} \left[ \sum_{n=0}^{\infty} \sum_{m=0}^{n} [T^n_m x^{2m}] y^{2n} + \sum_{n=0}^{\infty} \sum_{m=0}^{n} [S^n_m x^{2m+1}] y^{2n+1} \right] \hspace{1cm} (39)$$

Now

$$T_m^n = -\sum_{k=1}^{n-m} \left[ 2c_k C_{m+1}^{n-k} + b_k T_{m+1}^{n-k+1} + c_k S_{m}^{n-k} \right] - 4D_m^n$$  \hspace{1cm} (40)$$

$$S_m^n = -\sum_{k=1}^{n-m} \left[ 2c_k D_{m+2}^{n-k+1} + b_k S_{m+1}^{n-k+1} + c_k T_{m+1}^{n-k+1} \right] - 4C_m^n$$  \hspace{1cm} (41)$$

$$b_n = -\frac{1}{T_0^n} \left[ \sum_{k=1}^{n-1} \left[ 2c_k C_{0}^{n-k-1} + b_k T_{0}^{n-k} \right] + 2\delta_{n,1} \right]$$  \hspace{1cm} (42)$$

$$c_n = -\frac{1}{T_0^n + 2D_1^n} \left[ \sum_{k=1}^{n} \left[ 2D_1^n T_0^{n-k} \right] c_k + \sum_{k=1}^{n} b_k S_0^{n-k} + 4C_0^{n-1} \right]$$  \hspace{1cm} (43)$$

Employing the Taylor expansion technique the sign of the $D_1^0$ coefficient has to be decided. The two signs lead to distinct dissimilar solutions. The choice leading to finite energy eigenvalues is the negative solution, and this is the one we must choose.

III. THE RESULTS

Exploiting a computer algebraic algorithm programmed in MapleV [*], all coefficients were calculated to a selected order of approximation by recursion.
The energy estimates obtained using the $1/N$ method were compared with strict analytical or accurate numerical results from different sources [6–10]. In Table I, the results of the series for the harmonic oscillator and the Coulomb potential are displayed. The approximation to the appropriate energy eigenvalues is extremely good, indicating that in this case the Taylor expansion procedure is exceedingly reliable. The $1/N$ outcome for two additional potentials is also shown in the Table I. The results are excellent, but the energy sum starts to diverge at some point in the approximation (around 29th order or so), suggesting that the approximation is correct to about 6-7 digits. To illustrate the behaviour graphically, some plots of the partial sums of the energy approximation versus the order of approximation have been made for these potentials (see figures 1, 2 and 3).

Some potentials, also dealt with in [6, 9], are recomputed here to a much higher order of approximation (see Tables II, III and IV). The results are very reasonable, and the high order of approximation make the results nearly exact. In particular, it is observed that the high-$l$ states approximate the correct energy quite well. In Table IV for the 2nd excited states, only series which did not begin to diverge to around 30th order of approximation are presented. Most of the approximated partial energy sums for the 1st and 2nd excited states diverged at low $l$ in an oscillatory manner around the correct energy eigenvalue. A high-$l$ quantum number entirely cancelled the divergence, leading to a seemingly convergent series. As seen from the Tables III and IV the results for the harmonic oscillator and the Coulomb potential are excellent and correspond exactly to the analytically known energy eigenvalues.

The surveyed energy series are frequently seen to oscillate around a certain energy value when plotted against the order of approximation. The exact energy eigenvalue is found to be around this energy. The series begins to diverge at some order of approximation but still varies around the same definite energy. The oscillating series are often seen to fluctuate not termwise, but with two terms or more above the specific energy value and then two terms or more below, etc.

Some convention is required in order to discuss the results when this behaviour is seen. In order to discover the best energy eigenvalue, the last (first) term above and first (last) term below (or opposite) the center of oscillation with least variation across it is found. The result presented in the Tables is then these two terms. For a normal oscillating series, the two terms with least variation across the center of oscillation are presented. In Tables II, III and IV the normal oscillating behaviour is observed. The order of approximation of the partial sums employed is noted in the Tables. For the convergent series, only the highest order term is presented.

To examine the oscillations more carefully another potential also discussed in [6, 9], namely the linear potential $2\pi r$, will be analysed. (the curious value of the constant is chosen such that the results in this paper can be directly compared with those in [6, 9]) The results are presented in Table V. For this potential the energy eigenvalue is calculated to about the 29th order using the $1/N$ method for the first three excited states. The results show partial energy sums for the ground, the 1st and 2nd excited state seemingly swinging around the correct energy eigenvalue. Some graphs showing the oscillations of the partial energy sums are presented (see figures 4, 5 and 6).

Different radial potentials for which “exact” results from [9] are used, are shown in Tables VI. For most of these, satisfactory agreement with exact values is obtained. Many of them are not termwise oscillating series. The order of approximation employed gives a hint about the convergence of the expansion. The $r^k$ potentials where $k > 3$ are seen to diverge very quickly. An empirical formulation of this principle would be “the higher the power of the potential, the poorer the convergence of the resulting $1/N$ energy series.”

In order to investigate the applicability of the $1/N$ procedure for potentials with analytically precise eigenvalues, the method is now applied in the case of fabricated potentials where the potential is set up from the Schrödinger equation (see appendix A). These examples illustrate the reliability of $1/N$ progressions in the case of explicit, analytically known solutions and are hence interesting for our purposes. The results in Table VII are obtained using potentials calculated from a selected eigenfunction. Most of the constructed potentials are seen in particular to have a not termwise oscillatory behaviour.

Miscellaneous “quark” potentials and a particular double-well potential have also been developed in $1/N$ series (see Tables VIII and IX). The double-well potential has also been handled in [10] using ordinary perturbation calculations.

These results are not quite as good as those encountered for the other potentials, but the Taylor method still yields a fairly good approximation to the proper eigenvalue.

The double-well potential $((x^2-R^2)^2)/8R^2$, where $R$ is a constant) is in fact a rather peculiar potential because in a normal perturbation calculation one has to add powers of small terms proportional to $\sim \exp(-4R^2)$ in order to take into account quantum-mechanical tunnelling [10] in the perturbation series. Such tunnelling considerations have not been included here, but clearly for certain potentials such terms proportional to e.g. $\exp(-aN)$ should have an effect on the approximated energy eigenvalues. Maybe that is why the result for the double-well potential deviates slightly from the correct energy eigenvalue. This should of course be investigated in more details before using the Taylor expansion method more generally. To demonstrate the behaviour of some of the approximations, plots of the partial energy sums versus the order of convergence for some of the potentials are presented (see figures 7, 8, 9, 10, 11 and 12).

Most series are generated to about the 29th order of approximation with a numerical precision of about 100 digits.
In fact, it is feasible to generate the series to the 100th (or higher) order of approximation with a numerical precision of about 1000 digits. Further, it is conceivable to calculate the 1/N-series exactly using calculations with fractions and fractional roots to about the 100th order of approximation. The most important potentials are the harmonic oscillator, the Coulomb potential, and with the constructed potentials examples where analytic exact solutions are available.

Shanks resummed sums could have been used to resum the series. In a Shanks resummation Shanks extrapolants are calculated from the partial sums, as

\[ S_n = \frac{P_{n+1} \cdot P_{n-1} - P_n^2}{P_{n+1} + P_{n-1} - 2P_n} \]

where \( S_n \) is the \( n \)th Shanks extrapolant and \( P_n \) the \( n \)th partial sum. See [11]. In this paper it has been chosen not to do so because this question is an auxiliary aspect of the technique.

IV. DISCUSSION

The major aspect of the long term behaviour of the 1/N-series is of course their apparently asymptotic behaviour. As demonstrated, many 1/N series match the exact energy satisfactorily when only a certain number of terms are summed, but as further terms are added the sum diverges oscillatingly to infinity. The plots presented clearly show this behaviour with some minor differences. Convergent 1/N series are also seen, such as the Coulomb and the harmonic oscillator potentials series, together with the high-\( l \) states 1/N series (i.e. 1/k). This behaviour is very similar to that of asymptotic series. Most perturbation series are actually asymptotic. An asymptotic series is, briefly stated, a series which begins to converge towards a finite value, but which in the long run diverges. Mathematically, an asymptotic power series in \( \frac{1}{x} \) (here \( x \) plays the rule of \( N \)) is a series for which

\[ \lim_{|x| \to \infty} x^n \left[ f(x) - \sum_{r=0}^{n} \frac{a_r}{x^r} \right] \to 0 \]  

for all zero and positive \( n \), (Poincaré’s definition) [12] or [13] where \( f(x) \) is a function and the sum is a partial sum for the asymptotic power series of \( f(x) \).

If the 1/N series are asymptotic, the Coulomb and harmonic oscillator potentials would then just be special cases in which the divergence is extremely slow. Comparing the eigenfunctions for the Coulomb and harmonic oscillator potentials, they are both of the type \( \exp(r^k) \), with \( k = 1 \) and \( k = 2 \), respectively. As a numerical experiment in this paper, eigenfunctions of the type \( \exp(r^k) \) are used to find the corresponding potential with eigenvalue fixed at \( E = 1 \) (see appendix A). These potentials are then expanded using the 1/N method, (see Table VII). Examining this table one notices that the nearer \( k \) is to 1 or 2, the later the divergence of the resulting series, suggesting that the point of divergence is related to \( k \). For instance, with \( k \) equal to 1.2 or 1.15, the convergence is good to 25-30 orders or so.

One possible hint could be the radius of convergence for the Taylor series of the \( U(x) \) function, but further investigations examining the 1/N series for, e.g., \( U(x) = \cosh(x) \) have shown that there is no simple relationship between the radius of convergence of the Taylor series of the \( U(x) \) function and the point of divergence of its corresponding 1/N series. Furthermore, for \( k = 3 \) the resulting series is divergent, like the series for \( k \) between 1 and 2. In a more comprehensive treatment this should be inspected more carefully, and the apparent relation between \( k \) and the order of convergence of the resulting 1/N series surveyed.

The series reported in this article are derived to a very high numerical accuracy. Strong evidence that numerical round-off errors have not affected the results have been presented. By calculating results for different power-law potentials it has been found that results obtained “exactly” (by fractions and fractional roots) and results obtained with an exactness of 1000 digits and 100 digits, respectively, are nearly identical, and that differences occur only at about the 60th-70th digit.

The purpose of this article has been to consider the convergence of quantum mechanical 1/N series in detail. Many of the energy series presented are extremely close to the exact energy eigenvalues. The results clearly demonstrate that 1/N methods can be used to obtain approximate energy eigenvalues for different physical and chemical potentials.

The method has proved to yield fine results not only for the ground state but also for the 1st and 2nd excited states. For the harmonic oscillator and Coulomb potentials the results are extremely accurate.

The question of the convergence of the series has been settled, and much evidence points towards the 1/N series being asymptotic. Some of the series are so rapidly convergent at low orders that it will be very difficult to see that they diverge at all.
It has been established that numerical round-off errors are completely insignificant here. This is a very important point because it demonstrates that the divergence problem is a feature of the method itself and not a product of the computer arithmetic calculations.

V. ACKNOWLEDGEMENTS

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APPENDIX A: HOW TO CONSTRUCT THE POTENTIAL FROM A GIVEN GROUND-STATE WAVEFUNCTION

If \( \phi(r) = \exp(\psi(r)) \) is an eigenfunction of

\[
\left[ -\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{2} \frac{N-1}{r} \frac{d}{dr} + V(r) \right] \phi(r) = E\phi(r) \tag{A1}
\]

with eigenvalue \( E \), then the potential must be

\[
V(r) = -\frac{1}{2} \frac{\partial^2 \psi(r)}{\partial r^2} + \frac{1}{2} \frac{\partial \psi(r)}{\partial r} + \frac{1}{r} \frac{\partial \psi(r)}{\partial r} + E \tag{A2}
\]

APPENDIX B: THE SCHRODINGER EQUATION IN N DIMENSIONS

In this appendix we shall see how to derive the most important equation in this paper, namely the radial Schrödinger equation in \( N \) dimensions (see also [5]).

The Laplace operator in polar coordinates can be written [5] as

\[
\nabla^2 = \frac{1}{\hbar} \sum_{j=0}^{N-1} \frac{\partial}{\partial \theta_j} \left( \frac{\hbar}{h_j} \frac{\partial}{\partial \theta_j} \right) \tag{B1}
\]

where \( \theta_0 = r, h = \prod_{j=0}^{N-1} h_j \) and \( h_i^2 = \sum_{j=1}^{N} \left( \frac{\partial x_j}{\partial \theta_i} \right)^2 \).

The expressions for the spatial coordinates in \( N \) dimensions are given by

\[
\begin{align*}
x_1 &= r \cos \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{N-1} \\
x_2 &= r \sin \theta_1 \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{N-1} \\
&\vdots \\
x_i &= r \cos \theta_{i-1} \sin \theta_i \sin \theta_{i+1} \sin \theta_{N-1} \\
&\vdots \\
x_N &= r \cos \theta_{N-1}
\end{align*}
\tag{B2}
\]

for \( N \geq 3, (x_1 = r \cos \theta_1, x_2 = r \sin \theta_1; N = 2) \) and where \( 0 \leq r < \infty; 0 \leq \theta_1 \leq 2\pi; 0 \leq \theta_i \leq \pi; 2 \leq i \leq N - 1 \) and \( r^2 = \sum_{i=1}^{N} x_i^2 \).

The functions \( h_j \) are given by the following expressions

\[
\begin{align*}
h_0 &= 1 \\
h_1 &= r \sin \theta_2 \sin \theta_3 \cdots \sin \theta_{N-1} \\
&\vdots \\
h_j &= r \sin \theta_{j+1} \sin \theta_{j+2} \cdots \sin \theta_{N-1} \\
&\vdots \\
h_{N-1} &= r
\end{align*}
\tag{B3}
\]

Inserting the expression for \( h_j \) in equation (B1) leads to

\[
\nabla^2 = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} r^{N-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{i=1}^{N-2} \frac{1}{\prod_{j=i+1}^{N-1} \sin^2 \theta_j} \left[ \frac{1}{\sin^{i-1} \theta_i} \frac{\partial}{\partial \theta_i} \sin^{i-1} \theta_i \frac{\partial}{\partial \theta_i} \right] \\
+ \frac{1}{r^2} \left[ \frac{1}{\sin^{N-2} \theta_{N-1}} \frac{\partial}{\partial \theta_{N-1}} \sin^{N-2} \theta_{N-1} \frac{\partial}{\partial \theta_{N-1}} \right] \tag{B4}
\]
In general, the Hamiltonian can be written ($m = 1 = \hbar$)

$$\hat{H} = -\frac{1}{2} \nabla^2 = \hat{p}^2 \hat{F} + \frac{\hat{L}^2}{2r^2}$$  \hspace{1cm} (B5)$$

separating angular momentum and radial momentum. Hence, from equation (B5) and equation (B4) one can define total angular momentum operators as

$$\hat{L}_1^2 = -\frac{\partial^2}{\partial \theta_1^2}$$
$$\hat{L}_2^2 = -\left[ \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \sin \theta_2 \frac{\partial}{\partial \theta_2} - \hat{L}_1^2 \right]$$
$$\vdots$$
$$\hat{L}_i^2 = -\left[ \frac{1}{\sin \theta_i} \frac{\partial}{\partial \theta_i} \sin \theta_i \frac{\partial}{\partial \theta_i} - \hat{L}_{i-1}^2 \right]$$
$$\vdots$$
$$\hat{L}_{N-1}^2 = -\left[ \frac{1}{\sin \theta_{N-1}} \frac{\partial}{\partial \theta_{N-1}} \sin \theta_{N-1} \frac{\partial}{\partial \theta_{N-1}} - \hat{L}_{N-2}^2 \right]$$  \hspace{1cm} (B6)$$

In three dimensions, $\hat{L}_1^2$ is simply $\hat{L}_z^2$, and $\hat{L}_2^2$ is $\hat{L}^2$. Also, it is seen that $\hat{L}_i^2$ has the same expression independently of the number of spatial dimensions.

Equation (B4) may now be recast as

$$\nabla^2 = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} r^{N-1} \frac{\partial}{\partial r} - \hat{L}_{N-1}^2$$  \hspace{1cm} (B7)$$

The eigenvalues of the operator $\hat{L}_{N-1}^2$ will subsequently be determined.

Only total angular momentum operators have so far been addressed, but as in three dimensions, angular momentum components also exist. They are written as

$$\hat{L}_{ij} = -\hat{L}_{ji} = \hat{r}_i \hat{p}_j - \hat{r}_j \hat{p}_i \quad 1 \leq i \leq j - 1; \quad 2 \leq j \leq N$$  \hspace{1cm} (B8)$$

where $\hat{p}_j$ is defined as

$$\hat{p}_j = -i \sum_{k=0}^{N-1} \left( \frac{1}{h_k} \frac{\partial \hat{x}_j}{\partial \theta_k} \right) \frac{\partial}{\partial \theta_k}$$  \hspace{1cm} (B9)$$

Considering the potential combinations, there are $\frac{N(N-1)}{2}$ distinct (not just differing by a sign) operators of the pattern $\hat{L}_{ij}$. The generalized angular momentum operators can then be obtained in polar coordinates from equations (B2), (B9) and (B3). The results, i.e. the operators $\hat{L}_{ij}$, are then as follows

$$\hat{L}_{12} = -i \frac{\partial}{\partial \theta_1}$$
$$\hat{L}_{13} = -i \left[ \sin \theta_1 \cot \theta_2 \frac{\partial}{\partial \theta_1} \cos \theta_1 \frac{\partial}{\partial \theta_2} \right]$$
$$\hat{L}_{23} = +i \cos \theta_1 \cot \theta_2 \frac{\partial}{\partial \theta_1} + \sin \theta_1 \frac{\partial}{\partial \theta_2}$$
$$\hat{L}_{1j} = -i \left[ \frac{\sin \theta_1 \cot \theta_{j-1}}{\prod_{k=2}^{j-2} \sin \theta_k} \frac{\partial}{\partial \theta_1} - \sum_{k=2}^{j-2} \cot \theta_k \cos \theta_1 \cos \theta_{j-1} \left( \frac{\prod_{h=k+1}^{j-1} \sin \theta_h}{\prod_{h=k+1}^{j-2} \sin \theta_h} \right) \frac{\partial}{\partial \theta_k} \right]$$
$$+ i \left[ \cos \theta_1 \prod_{k=2}^{j-2} \sin \theta_k \frac{\partial}{\partial \theta_{j-1}} \right], \quad 4 \leq j \leq N$$
\[\hat{L}_{2j} = +i \left[ \frac{\cos \theta_j \cot \theta_{j-1}}{\prod_{k=2}^{j-2} \sin \theta_k} \frac{\partial}{\partial \theta_1} + \sum_{k=2}^{j-2} \cot \theta_k \sin \theta_1 \cos \theta_{j-1} \left( \frac{\prod_{k=2}^{j-1} \sin \theta_k}{\prod_{h=k+1}^{j} \sin \theta_h} \right) \frac{\partial}{\partial \theta_k} \right] \]

\[+ i \left[ \sin \theta_j \prod_{k=2}^{j-2} \sin \theta_k \frac{\partial}{\partial \theta_{j-1}} \right], \quad 4 \leq j \leq N \]

\[\hat{L}_{ij} = -i \left[ \frac{\sin \theta_{j-1} \cot \theta_{j-1}}{\prod_{k=1}^{j-2} \sin \theta_k} \frac{\partial}{\partial \theta_1} - \sum_{k=1}^{j-2} \cot \theta_k \sin \theta_1 \cos \theta_{j-1} \left( \frac{\prod_{k=1}^{j-1} \sin \theta_k}{\prod_{h=k+1}^{j} \sin \theta_h} \right) \frac{\partial}{\partial \theta_k} \right] \]

\[+ i \left[ \cos \theta_{j-1} \prod_{k=1}^{j-2} \sin \theta_k \frac{\partial}{\partial \theta_{j-1}} \right], \quad 3 \leq i \leq j - 2, \quad 5 \leq j \leq N \]

\[\hat{L}_{ij} = -i \left[ \sin \theta_{j-2} \cot \theta_{j-1} \frac{\partial}{\partial \theta_{j-2}} - \cos \theta_{j-2} \frac{\partial}{\partial \theta_{j-1}} \right], \quad 4 \leq j \leq N \] (B10)

for \(N \geq 3, (h_0 = 1, h_1 = r; N = 2) [5]\). For \(N = 3\), it is noted that \(\hat{L}_{12}\) is simply \(\hat{L}_z\), \(\hat{L}_{13}\) is \(-\hat{L}_y\) and that \(\hat{L}_{23}\) is \(\hat{L}_x\). As equation (B6) shows \(\hat{L}_1^2, \hat{L}_2^2, \ldots, \hat{L}_{N-1}^2\) clearly commute because they depend on distinct angles, meaning that they have simultaneous eigenfunctions. Since \(\hat{L}_2^2\) is a sum of squares of Hermitian operators of the form \((\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i)\), it is known that the eigenvalues of \(\hat{L}_2^2\) will be real and non-negative. Furthermore, the eigenfunctions are orthogonal because they are eigenfunctions of Hermitian operators. Assuming a proper normalisation, they may be written as

\[Y (\lambda_{N-1}, \ldots, \lambda_1) = Y_{\lambda_{N-1}, \ldots, \lambda_1} (\theta_1, \ldots, \theta_{N-1}) = \prod_{i=1}^{N-1} \Theta_i (\lambda_i, \lambda_{i-1})\] (B11)

where \(\Theta_i (\lambda_i, \lambda_{i-1})\) is a function only of \(\theta_i\), and \(\Theta_1 (\lambda_1, \lambda_0) \equiv \Theta_1 (\lambda_1)\) and \(\lambda_i\) is the associated eigenvalue of \(\hat{L}_i^2\). Clearly, \(\Theta_i (\lambda_i, \lambda_{i-1})\) will satisfy

\[\hat{L}_i^2 \Theta_i (\lambda_i) = \lambda_i \Theta_i (\lambda_i)\]

\[\hat{L}_i^2 (\lambda_{i-1}) \Theta_i (\lambda_i, \lambda_{i-1}) = \lambda_i \Theta_i (\lambda_i, \lambda_{i-1})\]

\[\left( \hat{L}_i^2 (\lambda_{i-1}) - \hat{L}_{i-1}^2 (\lambda_{i-2}) \right) \Theta_{i-1} (\lambda_{i-1}, \lambda_{i-2}) \Theta_i (\lambda_i, \lambda_{i-1}) \]

\[= (\lambda_i - \lambda_{i-1}) \Theta_{i-1} (\lambda_{i-1}, \lambda_{i-2}) \Theta_i (\lambda_i, \lambda_{i-1})\] (B12)

for \(2 \leq i \leq N - 1\). Since

\[\hat{L}_i^2 (\lambda_{i-1}) - \hat{L}_{i-1}^2 (\lambda_{i-2}) = \sum_{j=1}^{i} (\hat{x}_j \hat{p}_{i+1} - \hat{x}_{i+1} \hat{p}_j)^2\] (B13)

the eigenvalues of \(\left( \hat{L}_i^2 (\lambda_{i-1}) - \hat{L}_{i-1}^2 (\lambda_{i-2}) \right)\) = \(\lambda_i - \lambda_{i-1}\) must be non-negative and real simply because the sum is the diagonal matrix elements of the Hermitian operator \((\hat{x}_j \hat{p}_{i+1} - \hat{x}_{i+1} \hat{p}_j)^2\) [5]. Clearly

\[\lambda_i \geq \lambda_{i-1}\] (B14)

which implies that

\[\lambda_{N-1} \geq \lambda_{N-2} \geq \cdots \geq \lambda_1 \geq 0\] (B15)

because \(\lambda_1 = m^2 \geq 0\). At \(N = 3\), \(\lambda_1 = \ell_1^2 = m^2\) and \(\lambda_2 = \ell_2 (\ell_2 + 1) = \ell (\ell + 1) = \ell (\ell + 2 - 1)\). This suggests that in \((i - 1)\) dimensions

\[\lambda_i = \ell_i (\ell_i + i - 1)\] (B16)

Although the value of \(\ell_i\) is unknown, a sensible guess would be \(\ell_i \in \mathbb{N}_0\) since it should be legitimate for \(N = 3\). By the principle of induction it is feasible to prove this proposition by proving that if

\[\hat{L}_{i-1}^2 (\lambda_{i-2}) \Theta_{i-1} (\lambda_{i-1}, \lambda_{i-2}) = \lambda_{i-1} (\lambda_{i-1} + i - 2) \Theta_{i-1} (\lambda_{i-1}, \lambda_{i-2})\] (B17)
where \( l_{i-1} \in \mathbb{N}_0 \),

then

\[
\hat{L}_i^2 (l_{i-1}) \Theta_i (l_i, l_{i-1}) = l_i (l_i + i - 1) \Theta_i (l_i, l_{i-1})
\]  

(B18)

\( l_i \in \mathbb{N}_0 \), and for a given \( l_i = 0, 1, 2,.. \) one has \( l_{i-1} = 0, 1, 2,.., l_i \).

**Proof:**

Using (B6), \( \hat{L}_i^2 (l_{i-1}) \) may be transformed into

\[
\hat{L}_i^2 (l_{i-1}) = - \left[ \frac{\partial^2}{\partial \theta_i^2} + (i - 1) \cot \theta_i \frac{\partial}{\partial \theta_i} - \frac{l_i - 1 (l_i + i - 2)}{\sin^2 \theta_i} \right]
\]  

(B19)

If ladder operators are defined as

\[
\hat{L}_i^+ (l_{i-1}) = \frac{\partial}{\partial \theta_i} - l_i \cot \theta_i
\]  

(B20)

\[
\hat{L}_i^- (l_{i-1}) = - \frac{\partial}{\partial \theta_i} - (l_i + i - 2) \cot \theta_i
\]  

(B20)

then \( \hat{L}_i^2 (l_{i-1}) \) can be written as

\[
\hat{L}_i^2 (l_{i-1}) = \hat{L}_i^+ (l_{i-1} - 1) \hat{L}_i^- (l_{i-1}) + (l_i + i - 2) (l_i - 1)
\]  

(B21)

By multiplying (B21) from right and left with \( \hat{L}_i^\pm (l_{i-1}) \) one finds

\[
\hat{L}_i^2 (l_{i-1} + 1) \hat{L}_i^- (l_{i-1}) = \hat{L}_i^+ (l_{i-1}) \hat{L}_i^2 (l_{i-1})
\]  

(B22)

\[
\hat{L}_i^2 (l_{i-1} - 1) \hat{L}_i^- (l_{i-1}) = \hat{L}_i^- (l_{i-1}) \hat{L}_i^2 (l_{i-1})
\]  

(B22)

If now (B22) operates on \( \Theta_i (\lambda_i, l_{i-1}) \) it leads to

\[
\hat{L}_i^2 (l_{i-1} + 1) \hat{L}_i^+ (l_{i-1}) \Theta_i (\lambda_i, l_{i-1}) = \lambda_i \hat{L}_i^+ (l_{i-1}) \Theta_i (\lambda_i, l_{i-1})
\]  

(B23)

\[
\hat{L}_i^2 (l_{i-1} - 1) \hat{L}_i^- (l_{i-1}) \Theta_i (\lambda_i, l_{i-1}) = \lambda_i \hat{L}_i^- (l_{i-1}) \Theta_i (\lambda_i, l_{i-1})
\]  

(B23)

where \( A, B \) are normalizing factors. It turns out that \( B (\lambda_i, l_{i-1} + 1) = A^* (\lambda_i, l_{i-1}) \) \([5]\).

Thus, from (B21)

\[
\hat{L}_i^2 (l_{i-1}) \Theta_i (\lambda_i, l_{i-1}) = \lambda_i \Theta_i (\lambda_i, l_{i-1})
\]  

(B24)

As a consequence of (B15), i.e. \( \lambda_i \geq 0 \), there must exist a maximum value for \( l_{i-1} \), say \( l_i \), for a given \( \lambda_i \), such that

\[
\hat{L}_i^+ (l_i) \Theta_i (\lambda_i, l_i) = 0
\]  

(B25)

where \( l_i \in \mathbb{N}_0 \), because \( l_{i-1} \in \mathbb{N}_0 \). This leads to

\[
\lambda_i = l_i (l_i + i - 1)
\]  

(B26)

where \( l_i \) is a positive integer or zero.

To see what the possible values of \( l_{i-1} \) are, given \( l_i \), one notes that clearly \( l_{i-1} = l_i \) is plausible and that (B23) implies that

\[
\hat{L}_i^- (l_i) \Theta_i (\lambda_i, l_i)
\]  

(B27)

is an eigenfunction of \( \hat{L}_i^2 (l_{i-1}) \) with eigenvalue \( \lambda_i \). Hence, \( l_{i-1} = l_i - 1 \) is viable. Continuing this argument and introducing \( l_{i-1} = l_i - 1 \) into (B23), one discovers that \( \hat{L}_i^2 (l_i - 2) \) has the eigenvalue \( \lambda_i \). Consequently, another
feasible value for $l_{i-1}$ is $l_i - 2$. Since 0 is the lowest probable value of $l_{i-1}$ by presumption and one diminish $l_{i-1}$ one by one, it is unquestionably seen that the possible values of $l_{i-1}$ are $l_i, l_i - 1, l_i - 2, ..., 0$ for $l_i \in \mathbb{N}_0$.

From (B15) it is demonstrated that $\lambda_{N-1}$, and accordingly $l_{N-1}$, have no upper boundaries. It thus follows that $l_{N-1} = 0, 1, 2, ..., \infty$ and in addition it is seen that $l_{N-1} = 0, 1, 2, ..., l_1$ for $l_i \in \mathbb{N}_0$.

From (B15) it is demonstrated that $\lambda_{N-1}$, and accordingly $l_{N-1}$, have no upper boundaries. It thus follows that $l_{N-1} = 0, 1, 2, ..., \infty$ and in addition it is seen that $l_{N-1} = 0, 1, 2, ..., l_1$ for $l_i \in \mathbb{N}_0$.

Equation (B11) may now be written as

$$Y(l_{N-1}, \ldots, l_1) = \prod_{i=1}^{N-1} \Theta_i(l_i, l_{i-1})$$

and

$$\hat{L}_i^2 Y(l_{N-1}, \ldots, l_1) = l_i (l_i + i - 1) Y(l_{N-1}, \ldots, l_1)$$

Substituting $i = N - 1$ into (8), one finally arrives at

$$\left[ -\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{2} \frac{N - 1}{r} \frac{d}{dr} + \frac{l(l + N - 2)}{2r^2} + V(r) \right] \phi(r) = E\phi(r)$$

as the $N$-dimensional radial Schrödinger equation.

[4] MapleV, V. 3.0, Maple and MapleV are trademarks of Waterloo Maple Software
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FIG. 1: Long-term behaviour of the 1/N-energy series for the Coulomb potential. The dotted line represents the exact energy eigenvalue.

FIG. 2: Caption as for figure 1 for the harmonic oscillator potential.
FIG. 3: Long-term behaviour for an analytic solvable supersymmetric potential. Pot: $r^2 + \frac{r^2}{1+gr^2}$, $t = 0.1$ and $g = 0.1$. Angular momentum zero. [8]

FIG. 4: Graph of the energy series for the ground state of the linear potential: $2^r$ with zero angular momentum: $l = 0$. It is seen that the convergence for the first 10-20 terms is excellent. The series starts to diverge oscillatingly at about the 25th order of approximation.
FIG. 5: Caption as for figure 4 for the 1st excited state.

FIG. 6: Caption identical as for figure 4 and 5 for the 2nd excited state.
FIG. 7: Energy series for the ground state of the potential $r^2$ with zero angular momentum as a function of the order of approximation. As seen the convergence of the series is good up to a certain point, but at about the 20th order of approximation it starts to diverge. The series is seen to be oscillatory.

FIG. 8: Caption as for figure 7 for the potential $r^5$. Now the divergence is much faster, and the series diverges even before it has reached its exact value (the power effect).
FIG. 9: Caption as for figure 7 for the ln($r$) potential.

FIG. 10: The ground state of the quark potential $\frac{r}{(2.34)^2} - \frac{52}{1000}$ with zero angular momentum is seen to diverge at approximately the 20th order of approximation.
FIG. 11: Caption as for figure 10 for the quark potential $r^2 + r^4$. The divergence just starts much faster than in figure 10.

FIG. 12: Energy series for the ground state of the double-well potential $\frac{l^2 - R^2}{m^2}, R = 4$ with zero angular momentum. It is seen that the $1/N$ series converges to about the 20th order of approximation. It is also seen that oscillatory divergence does not occur for this potential.
TABLE I: Table showing the convergence of the $1/N$ series for 4 different potentials ($2m = 1 = \hbar$), except for the Coulomb potential in which ($m = 1 = \hbar$). The exact energy for the last potential is calculated in [8] with a supersymmetric method.

| Order of approximation | $r^2$ | $-\frac{1}{r}$ | $r^2 + \frac{r^2}{1 + \frac{1}{r}}$ | $r^2 + \frac{100}{1 + \frac{1}{r}} \cdot r^2$ |
|-------------------------|-------|----------------|--------------------------------|----------------------------------|
|                         | 2.1213203433 | -0.22222222222 | 2.2154999 | 2.07828211 |
|                         | 2.9526299690 | -0.49108367627 | 3.0712441 | 2.89354432 |
|                         | 3.0036770689 | -0.49993508216 | 3.1238941 | 2.94360413 |
|                         | 2.9997496105 | -0.49999961670 | 3.1200487 | 2.93982608 |
|                         | 3.0000000000 | -0.50000000000 | 3.1200752 | 2.93999831 |
|                         | 3.0000000000 | -0.50000000000 | diverges | diverges |

Exact results for the eigenenergies

| $E_{\text{exact}} [7]$ | 3.120081 [8] | 2.94 $E_{\text{exact}}$ super sym. [8] |

TABLE II: Results for the ground state energy ($n = 0$), $N = 3$, for different potentials ($m = \hbar = 1$). The “exact” results refer to [6, 9]; the calculated results are referred to as $E_{\text{Taylor}}$.

| Potential | $l$ | $E_{\text{shifted method}}$ [9] | $E_{\text{Taylor}}$ | $E_{\text{exact num}}$ [6, 9] | Used order |
|-----------|----|--------------------------------|----------------|-------------------------------|-----------|
| $-2^{1/2} r^{-0.2}$ | 0 | -2.68601 | -2.685882 | -2.686 | 29 |
| $-2^{1/2} r^{-0.2}$ | 1 | -2.34494 | -2.344946 | -2.345 | 29 |
| $-2^{1/2} r^{-0.2}$ | 3 | -2.02906 | -2.029065 | -2.029 | 29 |
| $-2^{0.8} r^{-0.8}$ | 0 | -1.21870 | -1.218693 | -1.218 | 29 |
| $-2^{0.8} r^{-0.8}$ | 1 | -0.50044 | -0.5004397 | -0.500 | 29 |
| $-2^{0.8} r^{-0.8}$ | 2 | -0.29470 | -0.2946959 | -0.295 | 29 |
| $-2^{0.8} r^{-0.8}$ | 3 | -0.20191 | -0.2019137 | -0.202 | 29 |

TABLE III: Results for the 1st excited state ($n = 1$), $N = 3$, for different potentials ($m = \hbar = 1$). The “exact” results refer to [6, 7, 9]; the calculated results are referred to as $E_{\text{Taylor}}$.

| Potential | $l$ | $E_{\text{shifted method}}$ [9] | $E_{\text{Taylor}}$ | $E_{\text{exact num}}$ [6, 9] | Used order |
|-----------|----|--------------------------------|----------------|-------------------------------|-----------|
| $-2^{1/2} r^{-0.2}$ | 0 | -2.25483 | -2.253515 - (-2.25314) | -2.253 | 16-17 |
| $-2^{1/2} r^{-0.2}$ | 2 | -1.99015 | -1.990056 | -1.990 | 29 |
| $-2^{1/2} r^{-0.2}$ | 3 | -1.90491 | -1.904867 | -1.905 | 29 |
| $-2^{0.8} r^{-0.8}$ | 0 | -0.46282 | -0.462291 | -0.462 | 22 |
| $-2^{0.8} r^{-0.8}$ | 1 | -0.28071 | -0.280648 | -0.281 | 29 |
| $-2^{0.8} r^{-0.8}$ | 2 | -0.19493 | -0.194912 | -0.195 | 29 |
| $-2^{0.8} r^{-0.8}$ | 3 | -0.14635 | -0.146342 | -0.146 | 29 |

| Potential | $l$ | $E_{\text{Taylor}}$ | $E_{\text{exact}}$ [7] | Used order |
|-----------|----|----------------|----------------|-----------|
| $-\frac{1}{r}$ | 0 | -0.1250000 | -0.125 $E_{\text{exact}}$ | 29 |
| $-\frac{1}{r}$ | 1 | -0.0555556 | -0.05 $E_{\text{exact}}$ | 29 |
| $-\frac{1}{r}$ | 2 | -0.0312500 | -0.03125 $E_{\text{exact}}$ | 29 |
| $-\frac{1}{r}$ | 3 | -0.0200000 | -0.02 $E_{\text{exact}}$ | 29 |
| $-\frac{1}{r}$ | 4 | -0.01388889 | -0.013 $E_{\text{exact}}$ | 29 |
| $\frac{1}{r}$ | 0 | 3.5000000 | 3.5 $E_{\text{exact}}$ | 29 |
| $\frac{1}{r}$ | 1 | 4.5000000 | 4.5 $E_{\text{exact}}$ | 29 |
| $\frac{1}{r}$ | 2 | 5.5000000 | 5.5 $E_{\text{exact}}$ | 29 |
| $\frac{1}{r}$ | 3 | 6.5000000 | 6.5 $E_{\text{exact}}$ | 29 |
| $\frac{1}{r}$ | 4 | 7.5000000 | 7.5 $E_{\text{exact}}$ | 29 |
TABLE IV: Results for the 2nd excited state ($n = 2$), $N = 3$, for different potentials ($m = \hbar = 1$). The “exact” results refer to [6, 7, 9]; the calculated results are referred to as $E_{\text{Taylor}}$.

| Potential | $t$ | $E_{\text{shifted method}}$ | $E_{\text{Taylor}}$ | $E_{\text{exact num}}$ | Used order |
|-----------|-----|-----------------------------|---------------------|------------------------|------------|
| $-2^{0.7}r^{-0.2}$ | 1   | -1.95147                    | -1.95072            | -1.951                   | 29         |
| $-2^{0.7}r^{-0.2}$ | 2   | -1.87535                    | -1.87503            | -1.875                   | 29         |
| $-2^{0.7}r^{-0.2}$ | 3   | -1.81266                    | -1.81250            | -1.812                   | 29         |
| $-2^{0.8}r^{-0.8}$ | 1   | -0.18745                    | -0.187318 - (-0.187320) | -0.187                  | 28-29      |
| $-2^{0.8}r^{-0.8}$ | 2   | -0.14202                    | -0.141980           | -0.142                   | 29         |
| $-2^{0.8}r^{-0.8}$ | 3   | -0.11280                    | -0.112788           | -0.113                   | 29         |

Potential | $t$ | $E_{\text{Taylor}}$ | $E_{\text{exact}}$ | Used order |
|-----------|-----|---------------------|-------------------|------------|
| $-1^{-2}$ | 0   | diverges            | -0.03125000       | 29         |
| $-1^{-1}$ | 1   | -0.03125000         | -0.03125$\text{exact}$ | 29         |
| $-1^{-1}$ | 2   | -0.02000000         | -0.02$\text{exact}$ | 29         |
| $-1^{-1}$ | 3   | -0.01388889         | -0.013$\text{exact}$ | 29         |
| $-1^{-1}$ | 4   | -0.01020408         | $-\frac{1}{2}n_{\text{exact}} \approx -0.01020408$ | 29         |
| $\frac{1}{2}^{-2}$ | 0   | 5.500000000         | 5.5$\text{exact}$  | 29         |
| $\frac{1}{2}^{-2}$ | 1   | 6.500000000         | 6.5$\text{exact}$  | 29         |
| $\frac{1}{2}^{-2}$ | 2   | 7.500000000         | 7.5$\text{exact}$  | 29         |
| $\frac{1}{2}^{-2}$ | 3   | 8.500000000         | 8.5$\text{exact}$  | 29         |
| $\frac{1}{2}^{-2}$ | 4   | 9.500000000         | 9.5$\text{exact}$  | 29         |

TABLE V: Table showing partial energy sums for the potential: $2^{2}r$. Results for the first three excited states are shown ($m = 1 = \hbar$).

| Ground-state energy | 1st excited state energy | 2nd excited state energy | Order of approximation |
|---------------------|--------------------------|--------------------------|------------------------|
| 9.35240            | 16.35180                  | 22.70413                 | 10                     |
| 9.35229            | 16.35175                  | 21.52097                 | 15                     |
| 9.35153            | 16.35274                  | 22.62047                 | 20                     |
| 9.38745            | 16.44861                  | 21.61422                 | 25                     |
| 9.54245            | 16.35286                  | 22.45952                 | 26                     |
| 9.24170            | 15.68014                  | 21.13008                 | 27                     |
| 7.95750            | 15.92972                  | 23.45280                 | 28                     |
| 9.26333            | 21.51171                  | 25.79375                 | 29                     |

Exact energies [6, 9]

| 9.35243$\text{exact}$ | 16.3518$\text{exact}$ | 22.08224$\text{exact}$ | $\sim$ |

Shifted method [9]

| 9.35243            | 16.32636                  | 22.02319                 | $\sim$ |
TABLE VI: Results for the ground state energy \((l = n = 0), N = 3\), for different potentials \((2m = h = 1)\). For the “exact” results, reference is made to [9]; \(E_{\text{Taylor}}\) refers to the calculated results.

| Potential      | \(E_{\text{shifted method}} [9]\) | \(E_{\text{Taylor}}\) | \(E_{\text{exact}}\) | \(\text{Used order}\) |
|----------------|-------------------------------------|------------------------|-----------------------|------------------------|
| \(-r^{1.5}\)   | -0.29888                            | -0.29880 - (-0.29931)  | -0.29609              | 27-28                  |
| \(-r^{1.25}\)  | -0.22035                            | -0.22038               | -0.22029              | 29                     |
| \(r^{0.15}\)   | 1.32795                             | 1.32781 - 1.32797      | 1.32795               | 15-16                  |
| \(r^{0.5}\)    | 1.83411                             | 1.83361 - 1.83287      | 1.83339               | 13-14                  |
| \(r^{0.75}\)   | 2.10815                             | 2.10769 - 2.11223      | 2.10814               | 15-16                  |
| \(r^{1.5}\)    | 2.70806                             | 2.70780 - 2.71299      | 2.70809               | 14-15                  |
| \(r^{3}\)      | 3.45111                             | 3.46173 - 3.43840      | 3.45056               | 11-12                  |
| \(r^{4}\)      | 3.80139                             | 3.81467 - 3.37768      | 3.79967               | 7-8                    |
| \(r^{5}\)      | 4.09146                             | 4.19331 - 4.03767      | 4.08916               | 6-7                    |
| \(\ln(r)\)     | 1.04436                             | 1.04457 - 1.04414      | 1.0443                | 13-14                  |

TABLE VII: Results for the ground state energy with \((l = n = 0)\) and \(N = 3\), for different constructed potentials with \((2m = h = 1)\). The exact energies are chosen to be 1; \(E_{\text{Taylor}}\) refers to the calculated results.

| Eigenfunction | \(E_{\text{Taylor}}\) | \(\text{Used order}\) |
|---------------|------------------------|------------------------|
| \(\exp(-r^{0.05})\) | 1.0000 - 0.99917       | 12-13                  |
| \(\exp(-r^{0.05})\) | 1.00005 - 0.99939      | 29-30                  |
| \(\exp(-r^{0.05})\) | 1.00003 - 0.999998     | 23-24                  |
| \(\exp(-r^{0.05})\) | 1.0000 - 0.99999       | 24-25                  |
| \(\exp(-r^{0.05})\) | 1.0000 - 1.00000       | 24-25                  |
| \(\exp(-r^{1.05})\) | 1.0001 - 0.99985       | 18-19                  |
| \(\exp(-r^{1.05})\) | 1.0002 - 0.99975       | 18-19                  |
| \(\exp(-r^{1.05})\) | 1.0004 - 0.99960       | 18-19                  |
| \(\exp(-r^{1.05})\) | 1.0004 - 0.99953       | 12-13                  |
| \(\exp(-r^{1.05})\) | 1.0005 - 0.99958       | 12-13                  |
| \(\exp(-r^{1.05})\) | 1.0006 - 0.99960       | 12-13                  |
| \(\exp(-r^{1.05})\) | 1.0008 - 0.99960       | 12-13                  |
| \(\exp(-r^{1.05})\) | 1.0002 - 0.99996       | 15-16                  |
| \(\exp(-r^{1.05})\) | 1.0005 - 0.99990       | 15-16                  |
| \(\exp(-r^{1.05})\) | 1.0010 - 0.99973       | 15-16                  |
| \(\exp(-r^{1.05})\) | 1.0015 - 0.99944       | 14-15                  |
| \(\exp(-r^{1.05})\) | 1.0019 - 0.99963       | 14-15                  |
| \(\exp(-r^{1.05})\) | 1.0016 - 0.99968       | 14-15                  |

TABLE VIII: Results for the ground-state energy \((l = n = 0)\), \(N = 3\), for different potentials \((2m = h = 1)\). For the “exact” results, reference is made to [9], again \(E_{\text{Taylor}}\) refers to the calculated results.

| Potential      | \(E_{\text{shifted method}} [9]\) | \(E_{\text{Taylor}}\) | \(E_{\text{exact}}\) | \(\text{Used order}\) |
|----------------|-------------------------------------|------------------------|-----------------------|------------------------|
| \(6.8698r^{0.1} - 8.064\) | -0.31914                           | -0.31936 - (-0.31883)  | -0.31917              | 11-10                  |
| \(r^{2.31} - 0.52r^{1/2}\) | 0.47800                            | 0.48049 - 0.47990      | 0.47811               | 10-9                   |
| \(r^{2} + r^{4}\) | 4.65061                             | 4.66212 - 4.64267      | 4.64881               | 11-10                  |
TABLE IX: Double-well potential: \( \frac{(r^2 - R^2)^2}{8R^2} \), \( R = 4 \). Results for the ground-state energy \((l = n = 0), \ (m = 1 = h), \ N = 3\).

| \( E_{\text{num}} \) [10] | \( E_{\text{Taylor}} \) | \( E_{\text{perturbation}} \) [10] | Used order |
|-------------------------|----------------|----------------|----------|
| 0.483053433             | 0.483018 - 0.483015 | 0.483053390 | 13-12    |