Field theory of Skyrme lattices in quantum Hall ferromagnets

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We report the application of the nonlinear $\sigma$ model to study the multi-skyrmion problem in the quantum Hall ferromagnet system. We make use of a first-principle calculation to derive an analytical form for the inter-skyrmionic interaction to show that the ground state of the system can be described by a ferromagnet triangular Skyrmie lattice near $\nu = 1$ where skyrmions are extremely dilute and a continuous transition into antiferromagnet square lattice occurs by increasing the skyrmion density and therefore $|\nu - 1|$. Using these results we demonstrate that the transition for a triangular to a square lattice which was previously derived, using the Hartree-Fock method, can also be seen in the field theory picture. We investigate the possibility that the skyrmions bound in pair to make a bi-skyrmion triangular lattice when the Zeeman energy is extremely small. We show that the energy of a skyrmion with charge $Q$ is less than the energy of $Q$ skyrmions each with charge one when the short range interaction among them is considered. By taking the quantum fluctuations into account, we also argue the possibility of the existence of a superconductor-insulator and the non-zero temperature phase transitions.

I. INTRODUCTION

Recently, there have been a number of experimental and theoretical investigations of skyrmions in the integer and fractional quantum Hall effect (QHE) in 2-dimensional electron gas (2DEG). It has been reported\textsuperscript{5} that the microscopic calculations show the ground state of a two-dimensional electron system at the Landau level filling factor near $\nu = 1$ is a Skyrmie (soliton) lattice. Recently Brey \textit{et al.}\textsuperscript{6} and Côté \textit{et al.}\textsuperscript{7} have made use of the time-dependent random phase approximation (RPA) to show that the zero-temperature ground state of a multi-skyrmion system is a centered square lattice. In particular they have found the possibility of the structural phase transition into triangular Wigner crystal by changing the Zeeman energy and the Landau level filling factor. One may expect on the general grounds, these microscopic observations should be supported by other many body techniques, e.g., the appropriate field theory. It has been shown by Sondhi \textit{et al.}\textsuperscript{8} and Moon \textit{et al.}\textsuperscript{9} that the quantum Hall ferromagnet can be described by a generalized minimal nonlinear $\sigma$ (NL$\sigma$) model to add the Zeeman and Coulomb energy, as an alternative approach. The consistency of this minimal field theory with other microscopic calculations for a system of a single skyrmion has been reported numerically by Abolfath \textit{et al.}\textsuperscript{10}. Recently, Green \textit{et al.}\textsuperscript{11} have exploited a variational approach on the generalized NL$\sigma$-model to study the multi-skyrmion system. They have indicated that the ground state is closer to a hexagonal structure, i.e., a distorted centered square lattice. Surprisingly, they came to a quite different conclusion from those obtained by microscopic consideration. One naturally seeks the reasons for the reported discrepancies between the different approaches employed for the multi-skyrmion systems. The aim of this paper is to present a proper field theory picture which avoids such controvertial issues. By exploiting the minimal field theory of the quantum Hall NL$\sigma$-model\textsuperscript{12} we present a description for the field theory which is consistent with the microscopic picture. In this way, we present a \textit{first-principle} calculation to derive the analytical form for the inter-skyrmionic interaction.

To the best of our knowledge this type of calculation has not been reported before. In the past, several authors have used a phenomenological approach to guess the form of the interaction by using the known Hartree-Fock long wave length spin waves spectrum\textsuperscript{24,26}. In contrast to Green \textit{et al.}\textsuperscript{11}, we explicitly show that the ground state is a centered square lattice and/or triangular Wigner crystal, depending on the value of the Zeeman energy and/or Landau level filling factor. These results which were previously derived, using the Hartree-Fock method\textsuperscript{24,26} can also be seen in our field theory picture. We start from a proper superposition rule for the unit vectors of the NL$\sigma$-model to evaluate the correct analytical form for the inter-skyrmionic interaction. We therefore demonstrate that the field theoretic description of the NL$\sigma$-model does indeed lead to conclusion in agreement with microscopic Hartree-Fock calculations. We take advantage of the particle-hole symmetry\textsuperscript{24,26} (which is equivalent to skyrmion-antiskyrmion duality in the classical field theory) to study the skyrmions associated with holes ($\nu \leq 1$). We will show that within $\nu_{1c} < \nu < \nu_{2c}$, the ground state is a triangular lattice in order to reduce the Coulomb energy, since the effective interaction energy between skyrmions through the channel of the Zeeman term is negligible. Increasing the skyrmion density in such a way that $0 < \nu_2c < \nu < \nu_{1c}$, leads to the
formation of a centered square lattice. The square lattice is favored by the effective interaction made by the gradient and Zeeman energy. The effective interaction which is the attractive XY model allows the antiferromagnetic alignment of the skyrmion’s orientation. This can be anticipated by the frustration of the skyrmionic hedgehog fields within the triangular lattice which is the favorable orientation for the Coulomb energy. This leads to a structural phase transition due to varying the Landau level filling factor. The attractive XY interaction makes a tendency for collapsing the single skyrmions which are located in different positions. It favorably recombines the N-separated single skyrmions and forming a single skyrmion with topological charge $N$. The competition between the attractive XY interaction and the repulsive Coulomb interaction leads to stability of the Skyrme lattice of the single-skyrmions. We will discuss for any short range interaction, like the hard-core model, the strength of the repulsive interaction is not sufficient to make a stable lattice. In this case, the energy of a charge-$Q$ skyrmion is lower than any other configuration, hence the destruction of the lattice. We demonstrate in special situations, at extremely small values of the Zeeman energy, (depends on the filling factor) a single skyrmion square lattice (with anti-ferromagnets orientation) collapses to a triangular lattice of bi-skyrmions (charge-2 skyrmions). In the latter, the orientation of skyrmions is governed by the ferromagnets ordering. By taking the quantum fluctuations into account, we also argue the possibility of the ferromagnets ordering. By taking the quantum fluctuations into account, we also argue the possibility of the ferromagnets ordering.

\[ E_0[m] = \frac{\rho_s}{2} \int d^2r (\nabla m)^2, \quad (2a) \]
\[ E_z[m] = \frac{\tilde{g}}{2\pi \ell_0^2} \int d^2r \left[ 1 - m_z(r) \right], \quad (2b) \]
\[ E_c[m] = \frac{e^2}{2\epsilon} \int d^2r \int d^2r' \frac{\rho(r')\rho(r')}{|r - r'|}, \quad (2c) \]

The long range nature of the Coulomb interaction is crucial for stability of the Skyrme lattice. We will get back to this point in Sec. III.B. Here $\rho_s = e^2/(16\sqrt{\pi}\epsilon \ell_0)$ is the spin stiffness at $\nu = 1$ (assuming zero layer thickness for the 2DEG) which arises from the Coulomb exchange energy, $\epsilon$ is the background dielectric constant of the semiconductor, $\tilde{g} = ge^2/(2\epsilon \ell_0)$ is the Zeeman term, $g$ is the effective gyromagnetic ratio, and $\ell_0$ is the magnetic length. The charge density is given by
\[ \rho(r) = \frac{e^2}{2\epsilon} \epsilon_{\alpha\beta} m(\mathbf{r}) \cdot \partial_\alpha \mathbf{m}(\mathbf{r}) \times \partial_\beta \mathbf{m}(\mathbf{r}) \] which is equal to the filling factor times the topological O(3) spin texture density of the quantum Hall ferromagnet (QHF). The total skyrmion charge denoted by $Q$ is an integer-valued topological invariant. It can be determined by the integration upon the charge density and classified by homotopy group of a 2D-sphere respectively. The lowest energy skyrmion solution, $\tilde{m}(\mathbf{r})$, has to satisfy a non-linear differential equation which can be obtained by minimizing the energy in Eq.(1) with respect to $\mathbf{m}$

\[ \rho_s \left( -\nabla^2 + \tilde{m} \cdot \nabla \tilde{m} \right) \tilde{m}_\mu - \frac{\tilde{g}}{2\pi \ell_0^2} (\delta_{z\mu} - \tilde{m}_z \tilde{m}_\mu) - \frac{\nu}{4\pi} \epsilon_{\alpha\beta} \{ \partial_\alpha V(\mathbf{r}) \} (\tilde{m} \times \partial_\beta \tilde{m})_\mu = 0, \quad (3) \]

where $V(\mathbf{r})$ is the Hartree potential (the exchange potential resides in $\rho_s$)
\[ V(\mathbf{r}) = \frac{e^2}{\epsilon} \int d^2r' \frac{\tilde{\rho}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \quad (4) \]

and $\tilde{\rho}$ is the skyrmion charge density associated with the minimum energy solution, $\tilde{m}(\mathbf{r})$. The solutions of Eq.(3) can be classified by the skyrmion charge $Q = \int d^2r \rho(\mathbf{r})$. From now and for the sake of simplicity, we remove the tilde over the classical solution and denote it by $\mathbf{m}$. It is easy to find the following equation of motion of the optimal skyrmion by making use of the cross product of $\mathbf{m}$ upon Eq.(3)
\[ \partial_\alpha J_\alpha^m = \frac{\tilde{g}}{2\pi \ell_0^2} (\tilde{z} \times \mathbf{m})_\lambda, \quad (5) \]

where
\[ J_\alpha^{\lambda} = \rho_s (\mathbf{m} \times \partial_\alpha \mathbf{m})_\lambda - \frac{\nu}{4\pi} V(\mathbf{r}) \epsilon_{\alpha\beta} \partial_\beta m_\lambda. \quad (6) \]

One may immediately read off from Eq.(3) that $\partial_\alpha J_\alpha^m = 0$, i.e., a conserved iso-spin current. We may define the
ground state of QHF at $\nu = 1$ as a vacuum of skyrmionic spin textures where all spins are aligned along the magnetic field direction, i.e., the $\hat{z}$-axis. Note that in the absence of the Zeeman energy, the alignment of spins along an arbitrary axis occurs due to spontaneous global $O(3)$ symmetry breaking\(\ddagger\) hence the minimizing the electrons exchange energy. This is also the state of spins far from the center of the skyrmions. Note that the number of skyrmions (antiskyrmions) in the ground state is counted by $|N - N_0|$. Before we consider a lattice of skyrmions we need to have the correct shape of a single skyrmion. A single skyrmion is a topological optimal solution of Eq. (3) with $Q = 1$. For our purposes, it is convenient to parameterize the unit vector $\mathbf{m}$ by

$$ \mathbf{m} = (\varphi_x, \varphi_y, \sqrt{1 - \psi^2}), $$

where $\psi = \varphi_x + i\varphi_y$. Near the core of the skyrmion we do not expect that the shape of the skyrmion is influenced much by the magnetic field. However, for large distances from the core, the Zeeman energy becomes dominant and we need to consider its effect where the direction of the unit vector $\mathbf{m}$ is close to the vacuum, namely, $\hat{z}$-axis. Taking the limit of small $\psi$, we can expand the $\mathbf{m}$ up to quadratic order in the $\psi$. Putting this in Eq. (5) gives

$$ E[\psi] = \int d^2r \left( -\frac{\rho_s}{2} \nabla^2 \psi + \frac{\bar{g}}{4\pi\ell_0^2} \psi \right), $$

which leads to the equation of motion

$$ -\rho_s \nabla^2 \psi + \frac{\bar{g}}{2\pi\ell_0^2} \psi = 0. $$

Introducing $\kappa^2 = \bar{g}/(2\pi\ell_0^2\rho_s)$ we simply have the equation $-\nabla^2 \psi + \kappa^2 \psi = 0$. We are interested in the ‘vortex’ solution $\psi = 2\partial_z \Upsilon$, where $\partial_z = (\partial_x + i\partial_y)/2$ and $z = x + iy$. (If $\bar{g} = 0$, this would result in $\nabla^2 \Upsilon = 0$ with the solution $\Upsilon \propto \ln(r)$ and therefore $\psi \propto z/r^2$). Substituting this we find $-\nabla^2 \Upsilon + \kappa^2 \Upsilon = 0$ or $\Upsilon \propto e^{-\kappa r}/\sqrt{r}$ and therefore that $\psi \propto z e^{-\kappa r}/r^{3/2}$ far from the core. The most important of this part of course is the exponential (as opposed to, algebraic if $\bar{g} = 0$), fall-off.\(\ddagger\)

To check this, we solve Eq. (6) for a single-skyrmion. The results obtained from these numerical calculations shows $\theta(r) = |\psi(r)| = c\kappa K_1(\kappa r)/2\pi$ far from the core of skyrmion where $K_1(x) = \sqrt{\pi/(2x)}e^{-x}(1 + O(1/x))$ is the modified Bessel function. $c$ is a constant, can be obtained from the asymptotic form of $\theta(r)$

$$ c = 30.4. $$

The dynamics of a skyrmion spin texture in NL$\sigma$-model may be incorporated via the Wess-Zumino action.\(\ddagger\) The result of expansion for the single skyrmion’s Wess-Zumino term is

$$ S_{WZ} = \frac{\hbar}{2} \int_0^{\hbar} d\tau \int d^2r \mathbf{m} \cdot (\partial_\tau \mathbf{m} \times \partial_r \mathbf{m}) $$

$$ = \frac{\hbar}{8\pi\ell_0^2} \int_0^{\hbar} d\tau \int d^2r |\psi(r, \tau)| \partial_\tau \psi(r, \tau), $$

where we keep the quadratic terms in Eq. (11). At this level of approximation, the effective action may be obtained by Eq. (8) and Eq. (11) where the Lagrangian density is

$$ L = \bar{\psi}(r, \tau) \left( \frac{\hbar}{8\pi\ell_0^2} \frac{\partial}{\partial\tau} + \frac{\rho_s}{2} \nabla^2 - \frac{\bar{g}}{4\pi\ell_0^2} \right) \psi(r, \tau), $$

and $S = \int d\tau \int d^2r L$. The optimal solution of $\psi$ is then identical to the solution of the time dependent Schrödinger equation where the external potential is proportional to the Zeeman splitting factor. Then the single skyrmion behaves approximately like a quantum mechanical point particle far from its core. Eq. (12) describes the usual gapful ferromagnetic spin wave mode.

### III. SKYRMION INTERACTION

We can now consider the interaction between skyrmions by generalizing our linearized energy functional. For that one should note that for the optimal single skyrmion spin texture in Eq. (3) we may choose a particular orientation. In general the texture can be rotated without costing any energy. Since the system shows the $U(1)$ symmetry, any valid skyrmion spin texture can be obtained by rotating all spins about the $\hat{z}$-axis by angle $\chi$. Therefore the state, $\mathbf{m}' = \exp(i\chi I_z)\mathbf{m}$, is also an optimal solution of the Hamiltonian where $I_z = -i\partial_\chi/\partial\chi$ is the generator of the rotation along the $\hat{z}$-axis in the internal space. One may expect that it contributes to the Hamiltonian through $(\hbar^2/2\Lambda_0)(-i\partial/\partial\chi - \xi_0)^2$ where $\Lambda_0$ is the moment of inertia of the single skyrmion. This is the leading term of the total energy which is expanded with respect to the number of reversed spin, $\xi$. In general the dimensional analysis shows $E_z \propto \xi$ and $E_z \propto \sqrt{\xi}$. The optimal value of the number of reversed spin, $\xi_0 = E_z/(2\bar{g})$, can be evaluated by minimization of the total energy with respect to $\xi$

$$ \xi_0 = \frac{1}{4\pi\ell_0^2} \int d^2r [1 - m_z(r)], $$

where $m_z(r)$ is the optimal solution of Eq. (8) corresponding to the given Zeeman splitting factor. This leads to the optimal Coulomb and Zeeman energy and then the prediction of $E_z/\Lambda_0 = 2$ and hence $\Lambda_0 \equiv \hbar^2 (d^2 E/d\xi^2)\xi_0^{-1} = \hbar^2 E_z/(6\bar{g}^2)$. The validity of this simple dimensional analysis has been demonstrated numerically in Ref. 9. Note that $\xi$ is a quantized variable since $\chi$ is compact.

#### A. Square lattice

Our goal is now to calculate the interaction energy between skyrmions with different orientations. Following Piette et al.\(\ddagger\), we start with the conventional NL$\sigma$ model.
to find out the proper superposition rule for the hedgehog fields of different skyrmions. In the absence of the Zeeman and Coulomb energies, the energy functional is scale invariant and one may find the optimal solutions analytically. In this case, it can be shown that any analytic complex polynomial defines an optimal solution. Substituting Eq.(15)-Eq.(19) into Eq.(14) and expanding a complex polynomial defines an optimal solution.

\[
m(r) = \left( \frac{2w_x}{1 + |w|^2}, \frac{2w_y}{1 + |w|^2}, 1 - |w|^2 \right),
\]

where \( w = w_x + i w_y \) is any Q-sector analytical function. One may decompose \( w \) into a series of analytical functions each with \( Q = 1 \)

\[
w = \sum_{j=1}^{N} u_j.
\]

Any \( Q = 1 \) analytical function represent a single-skyrmion, then the number of skyrmions \( N \) is clearly the total skyrmions topological charge, i.e., \( N = Q \). Eq.(14) denotes a sequence of order parameters \( m_j(r) \) in configuration space. It is convenient to parameterize the \( m_j(r) \) by

\[
m_j(r) = \left( \sin \eta_j(r) \cos \zeta_j(r), \sin \eta_j(r) \sin \zeta_j(r), \cos \eta_j(r) \right),
\]

where \( \eta_j(r) \) and \( \zeta_j(r) \) are polar and azimuthal field variables associated with the \( j \)th skyrmion. One may define \( \zeta_j = \varphi - \chi_j \) where \( \varphi \) is the standard azimuthal angle, e.g., \( r = (r \cos \varphi, r \sin \varphi) \) and \( \chi_j \) is skyrmions orientation and measuring the deviation from the standard hedgehog fields. In Fig. 2 the in-plane relative orientation, \( \chi \), between two skyrmions with relative distance, \( R \), is shown. If we have a gas of skyrmions far away from each other, the total energy is invariant under variation of skyrmions orientation. For finite separation we expect a coupling between skyrmions due to different orientations. Here we consider a situation where \( m_i \) are localized and well separated. This is valid for dilute skyrmions in a quantum Hall system, e.g., \( \nu_{2e} \leq \nu < \nu_{1c} \). One may divide the configuration space (\( R^2 \)) into \( N \) regions such that \( u_i \) is significant in region \( i \) and small in others

\[
u_{ij} = \frac{m_{ij}}{1 + \hat{z} \cdot m_i}.
\]

Here \( \mu = (x, y) \) and

\[
m_j = (\varphi_j^x, \varphi_j^y, \sqrt{1 - \Phi_j \cdot \Phi_j}),
\]

where \( j \neq i \) and \( \Phi_j = (\varphi_j^x, \varphi_j^y, 0) \)

\[
u_{ij} = \frac{\varphi_j^i}{1 + \sqrt{1 - \Phi_j \cdot \Phi_j}}.
\]

Substituting Eq.(15)-Eq.(19) into Eq.(14) and expanding \( m_j \) up to \( \varphi_j \), we may find \( m \) in the \( i \)th region

\[
m = m_i + \Omega_i \times m_i + \frac{1}{2} \Omega_i \times (\Omega_i \times m_i) + \mathcal{O}(\Omega^3),
\]

here Eq.(20) describes an infinitesimal rotation of \( m_i \). About \( \Omega_i \) axis where

\[
\Omega_i = \frac{1}{2} m_i \times \{ (1 + \hat{z} \cdot m_j) \Phi_i^{\text{eff}} - (m_i \cdot \Phi_i^{\text{eff}}) \hat{z} \},
\]

and

\[
\Phi_i^{\text{eff}}(r) = \sum_{j \neq i} \Phi_j(r).
\]

Therefore the effect of the other skyrmions on the specific skyrmion is the same as a single skyrmion with charge \( Q = 1 \) due to the effective linear field, \( \Phi_i^{\text{eff}} \). In order to study the effect of the Zeeman and Coulomb energy, we make the ansatz that the above superposition rule is valid for skyrmions far from each other even in the presence of the full interaction. This can be taken into account by Eq.(20) and evaluating the total energy of multi-skyrmion in QHF. It is obvious that the total energy can be divided into energies in separated regions. We assume that the interaction between the skyrmions is weak, hence \( \Omega_i = m_i \times \Phi_i^{\text{eff}} \). One may obtain easily the total energy by redoing the same calculation for all separated regions and sum over energies

\[
E[m] = \sum_{i=1}^{N} \left( E[m_i] + \int d^2r \Phi_i^{\text{eff}} \left( -\frac{\rho_s}{2} \nabla^2 + \frac{\tilde{g}}{4\pi \ell_0^2} \right) \psi_i^{\text{eff}} \right)
\]

\[+ E_i^{\text{eff}}[m], \tag{23} \]

where \( \psi_i^{\text{eff}} = \Phi_i^{\text{eff}} + i \Phi_i^{\text{eff}} \) is a complex field and \( E_i^{\text{eff}}[m] \) is the effective Coulomb interaction between skyrmions. Note that in the absence of the Coulomb term, the saddle point solution associated with the scalar field, \( \psi_i \), are vortices, i.e., \( -\nabla^2 \psi_i + \kappa^2 \psi_i = 0 \). One may divide the total energy, Eq.(23), into two parts, the self energy of skyrmions and the interaction energy which are designated by \( T \) and \( V \) respectively

\[
T[m] = \sum_{i=1}^{N} \left( E[m_i] + \int d^2r \sum_{j \neq i} \Phi_j \cdot \Phi_i \right) \psi_i^{\text{eff}} \tag{24} \}

The first term in Eq.(24) is the total self energy of the isolated skyrmions and the second term is the effect of their tail in other regions, i.e., the contribution of their energy in regions far from the core. Here we are interested to study the effect of interaction in a system of many skyrmions and their physical relevance. One may find the effective interaction between the skyrmions by making use of the Stokes theorem and neglecting the next nearest neighbor terms in Eq.(23) which are described by the terms like \( \int d^2r \sum_{j \neq i} \sum_{k \neq j} \Phi_j \cdot \Phi_k \) and \( \int d^2r \sum_{j \neq i} \sum_{k \neq j} \partial_{\phi} \Phi_j \cdot \partial_{\phi} \Phi_k \) then
$V_{\text{eff}}[\mathbf{m}] = E_{\text{eff}}^0[\mathbf{m}] + E_{\text{eff}}^C[\mathbf{m}]$, \hspace{1cm} (25)

and

$E_{\text{eff}}^0[\mathbf{m}] = \sum_{i=1}^N \int d^2r \partial_{\alpha} \{ J^{(i)}_{\alpha} \Omega_{\lambda} \}$

$= \frac{\rho^2}{2} \sum_{\langle ij \rangle} \int d^2r \mathbf{r} \cdot \{ -\nabla^2 + \kappa^2 \} \psi_i$, \hspace{1cm} (26)

where $J^{(i)}$ has been defined for the $i$th skyrmion and satisfies the Euler-Lagrange differential equations, Eq. (3) and Eq. (4). $E_{\text{eff}}^C[\mathbf{m}]$ is the contribution of the gradient and Zeeman energy to the effective interaction. It describes a system of interacting dipoles (see Fig. 3). Expanding the charge density of skyrmions, $\rho$, in terms of $\psi$, leads to the effective Coulomb interaction

$E_{\text{eff}}^C[\mathbf{m}] = \frac{e^2}{2\epsilon} \sum_{i \neq j} \int d^2r \int d^2r' \frac{\rho_i(r') \rho_j(r')}{|r-r'|}$

$+ \epsilon \mu \frac{\nu e^2}{4\pi \epsilon} \sum_{i \neq j} \{ \int d^2r \partial_\nu V_j(r) \partial_\nu \mathbf{m}_i \cdot \Omega_j$

$+ \int d^2r \partial_\nu V_j(r) \partial_\nu \mathbf{m}_j \cdot \Omega_i \} \}, \hspace{1cm} (27)$

where $\rho_i = \frac{\nu e^2}{4\pi \epsilon} \epsilon_{\alpha\beta} \epsilon_{\gamma\delta} [\partial_\alpha \mathbf{m}_i \times \partial_\beta \mathbf{m}_i]$. The first and second term in Eq. (27) are the Coulomb energy due to the monopole and dipole counterparts of skyrmions respectively. The former falls off like $R^{-1}$ whereas the latter falls as $R^{-2}$ where the distance between two skyrmions is denoted by $R$. Unlike the dense skyrmions in which the effect of the dipole term is crucial, the monopole term dominates for an extremely dilute (i.e., $\nu \simeq 1$) skyrmions system. Neglecting the Coulombic dipole term, one may evaluate the integrals in Eq. (26) by applying the techniques that were developed for a pair of skyrmions by Piette et al. [4]

$V_{\text{eff}}[\mathbf{m}] = \frac{e^2}{2\epsilon} \sum_{i \neq j} \int d^2r \int d^2r' \frac{\rho_i(r') \rho_j(r')}{|r-r'|}$

$+ \epsilon \mu \frac{\nu e^2}{4\pi \epsilon} \sum_{\langle ij \rangle} \cos(\chi_j - \chi_i) K_0(\kappa |\mathbf{R}_j - \mathbf{R}_i|)$, \hspace{1cm} (28)

where $\chi_j - \chi_i$ and $R_j - R_i$ describe the in-plane relative orientation and effective distance between skyrmion the $j$th and the $i$th. The first term in Eq. (28) is the electrostatic monopole term independent of the relative orientation. The normalization factor, $c = 30.4$, is calculated for a single skyrmion numerically. The second term in Eq. (28) describes a classical XY-model where the minimum energy configuration specifies the relative orientation corresponds to $\chi_j - \chi_i = \pi$. $K_0(r)$ is the modified Bessel function, i.e., the coupling between the site $i$ and $j$ decays exponentially. We see that there is an exponential decrease of the coupling between the $i$th and the $j$th skyrmions for $R \gg 1/\kappa$. We will show that an antiferromagnet ordering of the hedge-hog fields within centered square lattice is the minima of the effective potential, Eq. (28) in some circumstances although the Coulomb interaction in Eq. (27) is long range and the XY term contributes to the total energy as a short range interaction. The stability of the square lattice can be anticipated via generalizing the standard techniques for calculating the collective modes which has been developed by the Bonsall and Maradudin [4] for the multi-skyrmion system [4].

**B. Recombination of the skyrmions**

One may consider the static and dynamical properties of a skyrmion lattice using the interaction that we have derived in the previous section

$V_{\text{eff}} = \sum_{\langle ij \rangle} V_0(|\mathbf{R}_i - \mathbf{R}_j|)$

$+ \sum_{\langle ij \rangle} J(|\mathbf{R}_i - \mathbf{R}_j|) \cos(\chi_i - \chi_j)$. \hspace{1cm} (29)

In our model $V_0(R)$ is the electrostatic monopole interaction and $J(R) = \frac{\epsilon^2 \mu}{4\pi \epsilon} K_0(\kappa R)$. In Fig. 3, J is depicted as a function of the Landau level filling factor for a given Zeeman energy. The sign of $V_{\text{eff}}$ specifies the global minima of the energy functional. For instance, if $V_{\text{eff}}$ is positive then $E(Q = N) > NE(Q = 1)$ and the system can be composed by seperated single skyrmions. The validity of this inequality has been investigated recently by Lilliehök et al. [5] when $N > 2$. The case of $N = 2$ is marginal and will be discussed in the next subsection. The ionization energy of the lattice per skyrmion is identified by $E(Q = 1)$ where $R \to \infty$. Here $E(Q)$ is the optimized energy of an individual Q-skyrmion (the self energy of the skyrmion). One may immediately notice that the XY term in Eq. (28) favors forming a Q-skyrmion ($Q = N$) by recombining the $N$ single skyrmions. Note that the sign of this term in the antiferromagnet ordering is negative, therefore smaller separation between skyrmions, $R$, is favorable. Such a combination costs the Coulomb energy. Since the Coulomb interaction is long range and the XY term contributes to the total energy as a short range interaction, and the order of the Coulomb energy and the short range XY interaction with respect to the number of particles are $O(N^2)$ and $O(N)$ respectively, then the stability of the lattice at the limit of large $N$ as a local minima is guaranteed (the collapse of the lattice costs too much energy). For large $N$, an antiferromagnet ordering between the single skyrmions within a square lattice can be the minima of the interaction energy. Conversely, for the short range interactions, namely the hard-core model

$E_c[\mathbf{m}] \propto \frac{1}{2} \int d^2r (\partial_x \mathbf{m}(r) \times \partial_y \mathbf{m}(r))^2$, \hspace{1cm} (30)
we obtain \( E(Q = N) < NE(Q = 1) \), i.e., a charge-
Q skyrmion is the global minima, amongst all other configurations. \( ^{11} \) These models are convenient for the
nuclear physics, to describe the combination of the individual
nucleons to make a stable nuclei with the same
baryonic number. Here the baryonic number is ana-
logues of the skyrmions charge. This configuration which is
favorable for the short range strong interaction be-
tween the nucleons, can be destroyed by the long range
Coulomb interaction. Therefore the long range nature of
the Coulomb interaction is crucial for the stability of
the lattice. Varying the functional form of the inter-
action term, in \( E_c \), changes the optimized value of the
self energies for different models of skyrmions and subse-
quently changes the effective potential among them. A
cross over between a collapsed phase (no long range or-
der phase) into a Skyrmie lattice occurs due to changing
the functional form of interaction, from short range to a
long range. This cross over can be identified by the sign
of the energy difference (the skyrmions self energy dif-
fERENCE) between the different topological spin config-
urations, \( E(Q) \) where \( Q = Q_1 + Q_2 \). We end
up this subsection with a comment. The results of our
calculation show \( V_{\text{eff}} \) is negligible as \( \tilde{g} \) tends to zero. For
the zero Zeeman energy the Belavin-Polyakov solutions
are the well known analytical scale invariant solutions,
i.e., any analytical function possess the same energy and
neither the skyrmion positions nor the skyrmion size are
given. \( ^{12} \) In these circumstances, all of the possible
Q-sector configurations of a multi-skyrmion system are
degenerate, e.g., the energy of the \( N(= \sum Q_i) \) single
skyrmions is equivalent to the charged-\( N \) skyrmion. The
separated skyrmions do not feel each other, hence there
will be no effective potential among them. \( V_{\text{eff}} \) is non-
zero if \( \tilde{g} \neq 0 \).

C. Triangular lattice of the bi-skyrmions

An estimate based on the total energy of a pair of
skyrmions with the long range Coulomb interaction,
shows that the energy of a skyrmion with charge two is
lower than the energy of two skyrmions each with charge
one when \( \tilde{g} < \tilde{g}_e (= 5.3 \times 10^{-6} e^2 /\ell_0) \). \( ^{13} \) For small values
of the Zeeman energy a pair of skyrmions is prefered.
Moreover, a bi-skyrmion (charge-2 skyrmion) is stable to
accretion of further charge. One may expect a Skyrme
lattice is governed via the formation of the bi-skyrmions
when \( \tilde{g} < \tilde{g}_e \). The inter-skyrmionic interaction between
the bi-skyrmions can be obtained by the same calcu-
lation which has been introduced in Sec. \( \text{III} \). In this
case, the relation between the hedge-hog fields of the
bi-skyrmions relative to the standard hedge-hog fields
can be defined by \( \xi_j = 2\varphi - \chi_j \). Similar to the case
of the single-skyrmion lattice, we should have the correct
shape of a charge-2 skyrmion to obtain the effective po-
tential. We have solved Eq.\( (3) \), using the finite-difference
method to find out the asymptotic behavior of a circular
symmetric bi-skyrmion. Far from its core we find

\[
\theta(r) = \frac{c\tilde{g}^2}{2\pi} K_2(\kappa r)
\]

where \( K_2(x) = \sqrt{\pi/(2x)} e^{-x}(1 + O(1/x)) \) is the modified
bessel function. Our numerical calculation gives \( c = 79 \).
It is straightforward to show that the interaction between
the bi-skyrmions can be obtained by:

\[
V_{\text{eff}}[\mathbf{m}] = \frac{2e^2}{\epsilon} \sum_{i \neq j} \int d^2r \int d^2r' \frac{\rho_i(r) \rho_j(r')}{|r-r'|}
- \frac{c \tilde{g}^2}{8\pi^3} \sum_{<ij>} \cos(\chi_j - \chi_i) K_0(\kappa |R_j - R_i|).
\]

We observe that the effective Coulomb interaction in
the multi bi-skyrmionic system is a factor of 4 of the
single-skyrmionic lattice. Moreover, the sign of the XY
term is negative, rendering the possibly ferromagnet state
(\( \chi_i = \chi_j \)) within the triangular lattice for small values
of the Zeeman splitting factor. This result needs more
comments. We have seen in Sec. \( \text{III} \) that the effective
potential for the interaction between two seperated
single-skyrmions is similar to a pair of classical dipoles,
i.e., a single-skyrmion behaves as a source of the dipole
fields. On the other hand, the interaction form of a sys-
tem of coupled bi-skyrmions acts as a pair of quadrupole
fields (a charge-2 skyrmion is a source of quadrupole
fields). This is shown in Fig. \( \text{III} \). The minimum energy
configuration is therefore ferromagnet alignment of the
quadrapoles. This can explain why the negative sign in
the front of the bi-skyrmionic XY effective potential does
exist. However, the rest functional form of the interac-
tion between the charge-Q skyrmions is Q-independent.
This stems from the behavior of the asymptotic form of
our linear field theory\( ^{13} \) which has been implemented
to derive the effective interaction among the skyrmions.
Note that the leading XY-term in \( \text{III} \) is the quadrupole-
quadrupole interaction, where the XY dipole-dipole term
is absent. This is originated from the non-trivial topolog-
ical structure of the charged-2 skyrmions. The prediction
of our classical field theory, therefore, rendering a struc-
tural phase transition, single-skyrmionic square lattice
into bi-skyrmionic triangular lattice at \( \tilde{g}_e \). We expect
a first order phase transition occurs at \( \tilde{g}_e \) where a latent
heat is needed. However, one may expect this phase tran-
sition can be affected by other mechanisms, like skyrmion
scattering from the impurities and the quantum disor-
dering. Since the energy differences needed for observ-
ing this structural phase transition are very small, due
to extremely small Zeeman energy, then one expects the
quantum fluctuations which are present at zero temper-
ature destroys the long range-order of the bi-skyrmion
lattice. This requires further investigations, one should
include such processes to the present formalism in or-
der to obtain a realistic phase diagram. The results of
our calculation for the stability of the bi-skyrmion lattice against the quantum fluctuations is still on the way and will be presented elsewhere. Note that the sign of the XY-term in Eq. (28) is therefore negligible compared to the effective Coulomb energy. This implies that for a lower density of the single skyrmions the Coulomb interaction determines the lattice structure and leading to a triangular lattice, assuming $\tilde{g} > \tilde{g}_c$. However, for a more dense system one may expect a different stable structural symmetry, as we have mentioned before. Unlike the case where the competition between the monopole Coulomb term and the gradient dipole terms is crucial to the stability of the square lattice, the electrostatic monopole interaction is the dominant term to determine the structure of the lattice for an extremely dilute $(\nu \sim 1)$ skyrmions, where the Coulomb interaction is long range and the XY term contributes to the total energy as a short range interaction. In this situation a specific skyrmion moves in a background which is being made by the vacuum of the other skyrmions. Since $u_i$ which has been defined by Eq. (17), is zero everywhere but in the $i$th region, then $\rho(r) = \sum_i \rho_i(r)$. Clearly the total charge is the summation upon the individual skyrmion’s charge, i.e., $\nu_{tot} = \sum_i Q_i$ where $Q_i$ is the topological charge of localized skyrmion in that region. In this case, the interaction between separated skyrmions is independent of the relative orientation. The skyrmions may be considered by point particles where $\rho_i(r) = \nu_0 \delta (r - R_i)$ and

$$V_{\text{eff}} = \frac{e^2}{2\epsilon} \sum_{i \neq j}^{N} \int d^2 r \int d^2 r' \frac{\rho_i(r) \rho_j(r')}{|r - r'|} = \frac{\nu^2 e^2}{2\epsilon} \sum_{i \neq j}^{N} \frac{1}{|R_i - R_j|}. \tag{33}$$

Therefore the ground state is being described by Eq. (33) is clearly a triangular lattice, independent of the skyrmion orientation, i.e., a Wigner crystal. This is in agreement with the field theoretic result of Green et al. and the microscopic pictures. One may find the critical filling factor $\nu_{tc}$ where a transition into square lattice takes place. We define a domain of separation for which the square lattice is valid, e.g., $\nu_{tc} < \nu < \nu_{tc}$ or $\kappa^{-1} < R < R_0$. Here the length scale cut off is denoted by $R_0$, beyond that a phase transition to a triangular lattice occurs. Note that we have made an approximation associated with $R > \kappa^{-1}$ to demonstrate the existence of the square lattice. We may expect the approximation, namely the linear superposition rule, Eq. (20), fails for a high density skyrmions. The crossover between low and high density of skyrmions occurs at $\kappa R \sim 1$ and leading to the lower critical filling factor $\nu_{tc}$, i.e., $\nu_{tc} = 1 - \tilde{g}/2 \rho_0$. A comparison between the ground state energy per skyrmion for the square lattice and for the triangular lattice yields an estimate for the critical filling factor, $\nu_{tc}$, the square-into-triangular transition point at $\tilde{g}(> \tilde{g}_c)$. The result obtained from this calculation leads to a condition for the stability of the square lattice

$$\frac{3}{4} \delta \nu |1 - \nu|^{-1/4} \exp \left\{ -6.333 \frac{\delta \nu}{|1 - \nu|} \right\} \geq 2 \times 10^{-4}. \tag{34}$$

Here $\delta \nu = \tilde{g}/(e^2/\epsilon \hbar)$. Using the typical Zeeman energy, $\tilde{g} = 0.015 e^2/\epsilon \hbar$ for GaAs and taking advantage of the skyrmion-antiskyrmion duality, we obtain $|\nu_{tc} - 1| \sim 0.02$ and $|\nu_{tc} - 1| \sim 0.3$. Our estimate for the structural phase transition at $\nu_{tc}$ is a factor of 10 smaller than the microscopic Hartree-Fock result, although it shows qualitative agreement with that microscopic picture.

D. Triangular lattice of the single-skyrmions

Obviously, the skyrmion density may be controlled via the Landau level filling factor. For larger separations, the XY term in Eq. (28) is therefore negligible compared to the effective Coulomb energy. This implies that for a lower density of the single skyrmions the Coulomb interaction determines the lattice structure and leading to a triangular lattice, assuming $\tilde{g} > \tilde{g}_c$. However, for a more dense system one may expect a different stable structural symmetry, as we have mentioned before. Unlike the case where the competition between the monopole Coulomb term and the gradient dipole terms is crucial to the stability of the square lattice, the electrostatic monopole interaction is the dominant term to determine the structure of the lattice for an extremely dilute $(\nu \sim 1)$ skyrmions, where the Coulomb interaction is long range and the XY term contributes to the total energy as a short range interaction. In this situation a specific skyrmion moves in a background which is being made by the vacuum of the other skyrmions. Since $u_i$ which has been defined by Eq. (17), is zero everywhere but in the $i$th region, then $\rho(r) = \sum_i \rho_i(r)$. Clearly the total charge is the summation upon the individual skyrmion’s charge, i.e., $Q_{tot} = \sum_i Q_i$ where $Q_i$ is the topological charge of localized skyrmion in that region. In this case, the interaction between separated skyrmions is independent of the relative orientation. The skyrmions may be considered by point particles where $\rho_i(r) = \nu_0 \delta (r - R_i)$ and

$$V_{\text{eff}} = \frac{e^2}{2\epsilon} \sum_{i \neq j}^{N} \int d^2 r \int d^2 r' \frac{\rho_i(r) \rho_j(r')}{|r - r'|} = \frac{\nu^2 e^2}{2\epsilon} \sum_{i \neq j}^{N} \frac{1}{|R_i - R_j|}. \tag{33}$$

Therefore the ground state is being described by Eq. (33) is clearly a triangular lattice, independent of the skyrmion orientation, i.e., a Wigner crystal. This is in agreement with the field theoretic result of Green et al. and the microscopic pictures. One may find the critical filling factor $\nu_{tc}$ where a transition into square lattice takes place. We define a domain of separation for which the square lattice is valid, e.g., $\nu_{tc} < \nu < \nu_{tc}$ or $\kappa^{-1} < R < R_0$. Here the length scale cut off is denoted by $R_0$, beyond that a phase transition to a triangular lattice occurs. Note that we have made an approximation associated with $R > \kappa^{-1}$ to demonstrate the existence of the square lattice. We may expect the approximation, namely the linear superposition rule, Eq. (20), fails for a high density skyrmions. The crossover between low and high density of skyrmions occurs at $\kappa R \sim 1$ and leading to the lower critical filling factor $\nu_{tc}$, i.e., $\nu_{tc} = 1 - \tilde{g}/2 \rho_0$. A comparison between the ground state energy per skyrmion for the square lattice and for the triangular lattice yields an estimate for the critical filling factor, $\nu_{tc}$, the square-into-triangular transition point at $\tilde{g}(> \tilde{g}_c)$. The result obtained from this calculation leads to a condition for the stability of the square lattice

$$\frac{3}{4} \delta \nu |1 - \nu|^{-1/4} \exp \left\{ -6.333 \frac{\delta \nu}{|1 - \nu|} \right\} \geq 2 \times 10^{-4}. \tag{34}$$

Here $\delta \nu = \tilde{g}/(e^2/\epsilon \hbar)$. Using the typical Zeeman energy, $\tilde{g} = 0.015 e^2/\epsilon \hbar$ for GaAs and taking advantage of the skyrmion-antiskyrmion duality, we obtain $|\nu_{tc} - 1| \sim 0.02$ and $|\nu_{tc} - 1| \sim 0.3$. Our estimate for the structural phase transition at $\nu_{tc}$ is a factor of 10 smaller than the microscopic Hartree-Fock result, although it shows qualitative agreement with that microscopic picture.

E. Quantum phases

The appropriate quantum mechanical spectrum of the Skyrmion lattice in our model can be obtained through Eq. (28) and the Wess-Zumino action

$$\left\{ \sum_{i=1}^{N} \left( -\frac{\hbar^2}{2m} \chi_i + \chi_i \partial_j \chi_i - \chi_i \partial_j \right)^{2} + \sum_{i \neq j} V_0(|R_i - R_j|) \right\} \Psi(\chi; \tau)$$

$$+ \sum_{<ij>} J(|R_i - R_j|) \cos(\chi_i - \chi_j) \Psi(\chi; \tau)$$

$$= -\frac{\hbar}{8\pi} \partial_r \Psi(\chi; \tau), \tag{35}$$

where $\Psi(\chi; \tau) \equiv (\chi_1, \chi_2, \chi_3, ..., \chi_N)$ and $\chi_i \equiv (R_i \cos \varphi_i, R_i \sin \varphi_i)$. $\varphi_i$ is the standard azimuthal angle, indicating the $i$th skyrmion. $A_0 = \frac{\hbar^2}{3} / (3\tilde{g})$ is the moment of inertia of a single skyrmion, and $\xi_0 = E_z / (2\tilde{g})$. Obviously, Eq. (35) describes a quantum rotor problem for $\chi$-degree of freedom. This equation which has been obtained from a first-principle calculation is consistent with the result of Ref. 27, and 28 which has been estimated phenomenologically via the fitting the known Hartree-Fock long wave length spin waves spectrum. The ground state described by Eq. (35) for skyrmions orientation is a quantum antiferromagnet where the U(1) symmetry is broken spontaneously. This model has been used to describe the superconductor-insulator (SCI) quantum phase transition in granular superconductors and Josephson junction arrays, and predicts presumably a continuous quantum phase transition in QHF. The appropriate
parameters which control the SCI phase transition at zero temperature is the Zeeman splitting ($\tilde{g}$) and the Landau level filling factor. Using the empirical data for practical systems ($\tilde{g} = 0.015e^2/\ell\ell_0 \sim 2K$) shows the XY coupling constant, $J = (c^2\tilde{g}/4\pi^2)K_0(\sqrt{\tilde{g}/2\pi\ell^2\rho_\perp})$, is a fraction of the $U = (h^2/2\Lambda_0) \sim 0.4K$ within $\nu \gtrsim \nu_c$. It leads to the prediction of the Mott-insulating phase for this system where we assume that the position of skyrmions are fixed, e.g., they are crystalized. Here $E_z = 0.214e^2/\ell\ell_0$ is the Zeeman energy associated with the optimal single skyrmion solution corresponding to $\tilde{g} = 0.015e^2/\ell\ell_0$, is evaluated by solving Eq. (3), for a single skyrmion. The details of this calculation have been presented elsewhere. We also expect that the system go through superconducting phase for $J$ greater than $U$ since the skyrmions carrying the electrical charges. In Fig. 3, $J$ and $U$ are depicted in terms of the Landau level filling factor for a given Zeeman energy. One may see that at $\nu = 1$ the XY-interaction between the skyrmions is negligible. As it is shown in Fig. 3 there are two symmetric critical point around $\nu = 1$. This symmetry is due to the duality of the quantum Hall skyrmion-antiskyrmion at $\nu = 1$. The nature of this phase transition could be anticipated via the bosonic Hubbard model of the quantum rotors. Its critical behavior can be characterized via a map onto a $2 + 1$D classical XY model. Note that $\ell_0$ is a continuous variable. It changes continuously with the size of skyrmions and/or the Zeeman energy. For the half integer value of $\ell_0$ where $2\ell_0$ is odd, our model leads to the Mott-insulating phase about $\nu = 1$ for the quantum Hall liquid where the classical ordered phase is destroyed by the quantum fluctuations. Within the quantum rotors model, the skyrmions are analogous to the vortices. This is a dual picture for the quantum Hall state, where the Mott-insulating gap of the bosons and the super-current of the vortices are equivalent to the quantum Hall gap and the quantum Hall current, respectively, consistent with the microscopic picture of Ref. 27. The energy gap of the Mott-insulating phase, $\Delta \equiv 2U(=6\tilde{g}^2/E_z)$, is the analogous of the long wave length ($k \to 0$) energy gap of the magnons. Magnons are the appropriate collective modes of the quantum Hall liquid corresponding to a single flipped spin, propagating through the system. The numerical study for the single skyrmions shows that $E_z \propto (\ell^2/\ell\ell_0)^{2/3}\tilde{g}^{1/3}$ as $\tilde{g} \to 0$. This leads to the prediction of vanishing of the Mott-insulating gap as $\Delta \to \tilde{g}^{5/3}$. We estimate $\Delta \sim 1K$ at $\tilde{g} = 0.015e^2/\ell\ell_0$. Therefore for the zero Zeeman splitting ($U = J = 0$), the effective Hamiltonian is reduced to the exchange Coulomb interactions among the electrons. In this case, the gapless Goldstone modes corresponding to the spontaneous global $SU(2)$ symmetry breaking (spin waves) are responsible for the incompressibility of the quantum Hall liquid. The stability of the quantum Hall state can be investigated via evaluating the dispersion curve of the magnons.

F. Melting of the Skyrmé lattice

In the following subsection, we implement the theory of elasticity to study the non-zero temperature phase transitions, and the zero-temperature quantum fluctuations. The universality class of the phases associated with the Hamiltonian, Eq. (32), can be changed when $T > 0$. The Skyrmé lattice can be anticipated via the long-range order of the position of the skyrmions as well as the orientational long-range order of their hedge-hog fields. These long-range orders can be destroyed at finite temperature. One may show the fluctuations of the skyrmion orientation fields diverge logarithmically, $\langle \chi^2 \rangle \sim T\ln(L/R)$, where $L$ is the system size. This characterize the quasi-long range order of the skyrmions orientation fields on the $2D$-plane. Forming a pair of the vortex-antivortex from the hedge-hog fields, leads to a finite temperature Kosterlitz-Thouless phase transition ($T_{KT}$), assuming that skyrmions are crystalized. We may predict a critical temperature for the KT phase transition based on the classical 2D XY-model. Our calculation for the classical 2D nearest neighbor XY-model on the square lattice, shows that the phase transition occurs around $T_c = 0.9J \sim 4K$ at $\nu = 0.8$ ($B_0 \sim 7\ell$). One may also expect the 2D-Skyrmé lattice melts (at $T = T_m$) via the Kosterlitz-Thouless-Halperin-Nelson-Young (KTHNY) mechanism. In this case the disorder phase can be emerged via mediating a pair of dislocations. Similarly, the fluctuations of displacement fields diverge logarithmically, $\langle R^2 \rangle \sim T\ln(L/R)$. This exhibits a finite temperature quasi-long range positional order. Our classical field theoretic results for the $T_{KT}$ and $T_m$ are shown in Fig. 4. The details of these calculations will be presented elsewhere. The numerical calculation shows that the KT phase transition can not account for this system except for small $|1 - \nu|$ ($\nu > \nu_c$) and/or large $\tilde{g}$ where $J < k_BT_m$, as depicted in Fig. 4. In contrast to the case of small Zeeman energy where the KTHNY is responsible for destroying the translational long range order (and therefore melting the Skyrmé lattice), the orientational long-range order of the hedge-hog fields can be destroyed through the KT transition when the Zeeman energy is large. However, more accurate calculation is needed to obtain the melting temperature of the Skyrmé lattice. For this, we perform a Monte Carlo simulation. The results will be presented elsewhere. By taking the zero point quantum fluctuations into account, the Hamiltonian Eq. (35) leads to $\langle R^2 \rangle \sim 1 - \nu|a|^2/5$ at $T = 0$, where $a$ is the lattice spacing. Using the Lindemann criterion, we observe that the zero point quantum fluctuations destroys the long range order of the Skyrmé lattice when $|1 - \nu| \geq 0.5$. This is the high density limit of the skyrmions where our linear field theory may fail.
IV. CONCLUSION

In this paper we presented the analytical form of the inter-skyrmionic interaction via a first-principle calculation within the classical field theory. The effective Hamiltonian demonstrated that for extremely dilute skyrmions, the ground state is a triangular lattice to minimize the Coulomb repulsion. However at higher densities, a square lattice forms to optimize the spin gradient and Zeeman energies. This work has provided a possibility that a triangular to a square lattice phase transition which has been previously derived, using the Hartree-Fock method, can also be revealed by the field theory picture. In contrast to Ref. 11, we show that the field theoretic description of the NLσ-model does indeed lead to quantitative conclusion in agreement with microscopic Hartree-Fock calculations. We also argued the possibility of the superconducting-insulator phase transition. We have shown that within some special short range interactions the level-crossing between different kind of skyrmions may occur to destabilize the Skyrme lattice, and a single-skyrmion with the same topological charge can be formed. We have shown that for extremely small Zeeman energy, skyrmions bound in pair to make a triangular lattice with the ferromagnets orientation. The possibility of the Kosterlitz-Thouless phase transition as well as the Kosterlitz-Thouless-Halperin-Nelson-Young phase transition has been investigated.

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FIG. 2. Minima configuration of the system of a coupled single- (a) and bi-skyrmions (b) are shown. The single (bi) skyrmion act as a source of the classical dipole (quadrupole) fields, far from its core.

FIG. 3. The hopping term of the Bosonic Hubbard model, $J$, and the charging energy $U$ as a function of the Landau level filling factor are depicted for a typical Zeeman splitting factor.

FIG. 4. The prediction of the classical field theory for the disordering temperature as a function of the Landau level filling factor is shown for $\tilde{g} = 0.015e^2/\epsilon\ell_0$ and $\tilde{g} = 0.1e^2/\epsilon\ell_0$ (the inset). In contrast to the case of small Zeeman energy where the mechanism of the KTHNY is responsible for the melting of the Skyrme lattice, the orientaional long-range order of the hedge-hog fields can be destroyed through the KT transition for the large Zeeman energies.
a) \[ \chi_1 \] \hspace{1cm} \[ \chi_2 \]

b) \[ \chi_1 \] \hspace{1cm} \[ \chi_2 \]
