Asymmetric quantum convolutional codes

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Abstract

In this paper, we construct the first families of asymmetric quantum convolutional codes (AQCC)’s. These new AQCC’s are constructed by means of the CSS-type construction applied to suitable families of classical convolutional codes, which are also constructed here. The new codes have noncatastrophic generator matrices and they present great asymmetry. Since our constructions are performed algebraically, it is possible to derive several families of such codes and not only codes with specific parameters. Additionally, several different types of such codes are obtained.

Index Terms – convolutional codes, quantum convolutional codes

1 Introduction

Several works available in literature deal with constructions of quantum error-correcting codes (QECC, for short) and asymmetric quantum error-correcting codes (AQECC) [4, 5, 11, 13–16, 23, 28]. In contrast with this subject of research one has the theory of quantum convolutional codes [1–3, 7, 9, 17, 18, 24, 25, 31]. Ollivier and Tillich [24, 25] were the first to develop the stabilizer structure for these codes. Almeida and Palazzo Jr. constructed an [4, 1, 3] (memory \(m = 3\)) quantum convolutional code [1]. Grassl and Rötteler [8, 9] generated quantum convolutional codes as well as they provide algorithms to obtain non-catastrophic encoders. Forney, in a joint work with Guha and Grassl, constructed rate \((n - 2)/n\) quantum convolutional codes. Wilde and Brun [31] generated entanglement-assisted quantum convolutional coding.

An asymmetric quantum convolutional code (AQCC) is a quantum code defined over quantum channels where qudit-flip errors and phase-shift errors may have different probabilities. As it is well known, Steane [30] was the first who introduced the notion of asymmetric quantum errors. The parameters of an AQCC will be denoted by \([n, k, \mu^∗; \gamma, [d_z]/[d_x]]_q\), where \(n\) is the frame size, \(k\) is the number of logical qudits per frame, \(m\) is the memory, \([d_z]/[d_x]\) is the free distance corresponding to phase-shift (qudit-flip) errors and \(\gamma\) is the degree.

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of the code. The combined amplitude damping and dephasing channel (see [28]
the references therein) is a quantum channel whose probability of occurrence
of phase-shift errors is greater than the probability of occurrence of qudit-flip
errors.

In this paper we propose constructions of the first families of asymmetric
quantum convolutional codes. The constructions presented here are performed
algebraically. Our new AQCC’s have parameters:

- \([n, 2i - 4, \mu^*; 6, [d_z]_f/[d_x]_f]_q\),
  where \(q = 2^l, t \geq 4, n = q + 1, (d_z)_f \geq n - 2i - 1\) and \((d_x)_f \geq 3\), for all
  \(3 \leq i \leq \frac{q}{4} - 1\);

- \([n, 2i - 2t - 2, \mu^*; 6, [d_z]_f/[d_x]_f]_q\),
  where \(q = 2^l, t \geq 4, n = q + 1, t\) integer with \(1 \leq t \leq i - 2, 3 \leq i \leq \frac{q}{2}\);

- \([n, 2i - 2t - 2, \mu^*; 4, [d_z]_f/[d_x]_f]_q\),
  where \((d_z)_f \geq n - 2i - 1\) and \((d_x)_f \geq 2t + 3, q = 2^l, l \geq 4, n = q + 1, t\)
  integer with \(1 \leq t \leq i - 2, 2 \leq i \leq \frac{q}{2}\);

- \([n, 2i - 2t - 2, \mu^*; 4, [d_z]_f/[d_x]_f]_q\),
  where \((d_z)_f \geq 2t + 2, for all 1 \leq t \leq i - 1, \) with \(2 \leq i \leq \frac{q}{2} - 1\);

- \([n, 2i - 2t - 2, \mu^*; 6, [d_z]_f/[d_x]_f]_q\),
  where \(q = p^l, p\) is an odd prime, \(l \geq 2, n = q + 1, (d_z)_f \geq n - 2i\)
  \((d_x)_f \geq 2t + 2, for all 1 \leq t \leq i - 1\), where \(3 \leq i \leq \frac{q}{2} - 1\);

- \([n, 2i - 2t - 2, \mu^*; 4, [d_z]_f/[d_x]_f]_q\),
  where \(q = p^l, p\) is an odd prime, \(l \geq 2, n = q + 1, (d_z)_f \geq n - 2i\)
  \((d_x)_f \geq 2t + 2, for all 1 \leq t \leq i - 1\), with \(2 \leq i \leq \frac{q}{2} - 1\);

- \([n, 2i - 2t - 2, \mu^*; 3, [d_z]_f/[d_x]_f]_q\),
  where \(q \geq 8\) is a prime power \((d_z)_f \geq q - i - 1\) and \((d_x)_f \geq t + 2, for all
  1 \leq t \leq i - 2\), where \(3 \leq i \leq q - 3\);

- \([n, 2i - 2t - 2, \mu^*; 2, [d_z]_f/[d_x]_f]_q\),
  where \((d_z)_f \geq q - i + 1, (d_x)_f \geq t + 2, for all 1 \leq t \leq i - 1\), where
  \(2 \leq i \leq q - 3\);

- \([n, n - t - k - 2, \mu^*; 3, [d_z]_f/[d_x]_f]_q\),
  where \((d_z)_f \geq t + 2\) and \((d_x)_f \geq k + 1, where q \geq 5\) is a prime power,
  \(k \geq 1\) and \(n\) are integers such that \(5 \leq n \leq q\) and \(k \leq \frac{n - 4 \text{ and } t \text{ is an}}{5} + k - 2\);

- \([n, n - t - k - 1, \mu^*; 2, [d_z]_f/[d_x]_f]_q\),
  where \(q \geq 5\) is a prime power, \(k \geq 1, n \geq 5\) are integers such that \(n \leq q, k \leq n - 4, 1 \leq t \leq n - k - 1, (d_z)_f \geq t + 2\) and \((d_x)_f \geq k + 1\).

The paper is arranged as follows. In Section 2, we recall the concepts of
convolutional and quantum convolutional codes. In Section 3, we present the
contributions of this work, i.e., the first families of asymmetric quantum con-
volutional codes are constructed. In Section 4, we discuss the parameters of the
new codes and, in Section 5, the final remarks are drawn.
2 Background

Notation. Throughout this paper, $p$ denotes a prime number, $q$ is a prime power and $\mathbb{F}_q$ is a finite field with $q$ elements. The code length is denoted by $n$ and we always assume that $\gcd(q, n) = 1$. As usual, the multiplicative order of $q$ modulo $n$ is denoted by $l = \ord_d(q)$, and $\alpha$ is considered a primitive $n$-th root of unity in the extension field $\mathbb{F}_{q^l}$. The parameters of a linear block code over $\mathbb{F}_q$, of length $n$, dimension $k$ and minimum distance $d$, is denoted by $[n, k, d]_q$. Sometimes, we abuse the notation by writing $C = [n, k, d]_q$. If $C$ is a linear code then $C^\perp$ denotes its Euclidean dual.

2.1 Review of Convolutional Codes

Convolutional codes are extensively investigated in literature [6, 10, 12, 19, 26, 27, 29]. Recall that a polynomial encoder matrix $G(D) \in \mathbb{F}_q[D]^{k \times n}$ is called basic if $G(D)$ has a polynomial right inverse. A basic generator matrix is called reduced (or minimal, see [10, 29]) if the overall constraint length $\gamma = \sum_{i=1}^k \gamma_i$, where $\gamma_i = \max_{1 \leq j \leq n} \{\deg g_{ij}\}$, has the smallest value among all basic generator matrices. In this case, we say that $\gamma$ is the degree of the resulting code.

A rate $k/n$ convolutional code $C$ with parameters $(n, k; \gamma, m, d_f)_q$ is a submodule of $\mathbb{F}_q[D]_n$ generated by a reduced basic matrix $G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n}$, i.e., $C = \{u(D)G(D) | u(D) \in \mathbb{F}_q[D]^k\}$, where $n$ is the length, $k$ is the dimension, $\gamma = \sum_{i=1}^k \gamma_i$ is the degree, $m = \max_{1 \leq i \leq k} \{\gamma_i\}$ is the memory and $d_f = \min \{\text{wt}(v(D)) | v(D) \in C, v(D) \neq 0\}$ is the free distance of the code. In the above definition, the weight of an element $v(D) \in \mathbb{F}_q[D]^n$ is defined as $\text{wt}(v(D)) = \sum_{i=1}^n \text{wt}(v_i(D))$, where $\text{wt}(v_i(D))$ is the number of nonzero coefficients of $v_i(D)$. In the field of Laurent series $\mathbb{F}_q((D))$, whose elements are given by $u(D) = \sum_{i \geq 0} u_i D^i$, where $u_i \in \mathbb{F}_q$ and $u_i = 0$ for $i \leq r$, for some $r \in \mathbb{Z}$, we define the weight of $u(D)$ as $\text{wt}(u(D)) = \sum_{i \geq 0} \text{wt}(u_i)$. A generator matrix $G(D)$ is called catastrophic if there exists a $u(D) \in \mathbb{F}_q((D))^k$ of infinite Hamming weight such that $u(D)^k G(D)$ has finite Hamming weight.

The AQCC’s constructed in this paper have noncatastrophic generator matrices since the corresponding classical convolutional codes constructed here have basic (and reduced) generator matrices. The Euclidean inner product of two $n$-tuples $u(D) = \sum_i u_i D^i$ and $v(D) = \sum_j v_j D^j$ in $\mathbb{F}_q[D]^n$ is defined as $\langle u(D) | v(D) \rangle = \sum_{i=1}^n u_i \cdot v_i$. If $C$ is a convolutional code then the code $C^\perp = \{u(D) \in \mathbb{F}_q[D]^n | \langle u(D) | v(D) \rangle = 0 \text{ for all } v(D) \in C\}$ denotes its Euclidean dual.

Let $C \subseteq \mathbb{F}_q^n$ an $[n, k, d]_q$ block code with parity check matrix $H$. We split $H$
into $\mu + 1$ disjoint submatrices $H_i$ such that $H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_\mu \end{bmatrix}$, where each $H_i$ has $n$ columns, obtaining the polynomial matrix $G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2 + \ldots + \tilde{H}_\mu D^\mu$, where $\tilde{H}_i$, for all $1 \leq i \leq \mu$, are derived from the respective matrices $H_i$ by adding zero-rows at the bottom such that $\tilde{H}_i$ has $\kappa$ rows in total, where $\kappa$ is the maximal number of rows among the matrices $H_i$. The matrix $G(D)$ generates a convolutional code $V$.

Theorem 2.1 [2, Theorem 3] Let $C \subseteq F_q^n$ be a linear code with parameters $[n, k, d]_q$. Assume that $H \in F_q^{(n-k) \times n}$ is a parity check matrix for $C$ partitioned into submatrices $H_0, H_1, \ldots, H_\mu$ as above such that $\kappa = \text{rk} H_0$ and $\text{rk} H_i \leq \kappa$ for $1 \leq i \leq \mu$.

(a) The matrix $G(D)$ is a reduced basic generator matrix;
(b) If $d_f$ and $d_\perp$ denote the free distances of $V$ and $V^\perp$, respectively, $d_i$ denote the minimum distance of the code $C_i = \{v \in F_q^n \mid vH_i^\perp = 0\}$ and $d_\perp$ is the minimum distance of $C^\perp$, then one has $\min \{d_0 + d_\mu, d\} \leq d_f \leq d$ and $d_f \geq d_\perp$.

2.2 Review of quantum convolutional codes

In this subsection, we recall the concept of quantum convolutional code (QCC). For more details, the reader can consult [24, 25].

A quantum convolutional code is defined by means of its stabilizer which is a subgroup of the infinite version of the Pauli group, consisting of tensor products of generalized Pauli matrices acting on a semi-infinite stream of qudits. The stabilizer can be defined by a stabilizer matrix of the form $S = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_\mu \end{bmatrix}$ partitioned into submatrices $H_0, H_1, \ldots, H_\mu$ as above such that $\kappa = \text{rk} H_0$ and $\text{rk} H_i \leq \kappa$ for $1 \leq i \leq \mu$. The stabilizer can be defined by $S = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_\mu \end{bmatrix}$partitioned into submatrices $H_0, H_1, \ldots, H_\mu$ as above such that $\kappa = \text{rk} H_0$ and $\text{rk} H_i \leq \kappa$ for $1 \leq i \leq \mu$. If $\gamma$ has the smallest value among all basic generator matrices then $\gamma$ is the degree of the code. The memory $m$ of $Q$ is defined as $m = \max_{1 \leq i \leq n-k, 1 \leq j \leq n} \{\max \{\deg X_{ij}(D), \deg Z_{ij}(D)\}\}$, and the free distance is defined analogously as to classical convolutional codes, e.g., the minimum of the weights of nonzero codewords in $Q$. Then $Q$ is a rate $k/n$ code with parameters $[n, k, m; \gamma, d_f]_q$, where $n$ is the frame size, $k$ is the number of logical qudits per frame, $m$ is the memory, $\gamma$ is the degree and $d_f$ is the free distance of the code.

On the other hand, a quantum convolutional code can also be described in terms of a semi-infinite stabilizer matrix $S$ with entries in $F_q \times F_q$ in the following way. If $S(D) = \sum_{i=0}^{m} G_i D^i$, where each matrix $G_i$ for all $i = 0, \ldots, m$,
is a matrix of size \((n - k) \times n\), then the semi-infinite matrix is defined as

\[
S = \begin{bmatrix}
G_0 & G_1 & \ldots & G_m & 0 & \ldots & \ldots & \\
0 & G_0 & G_1 & \ldots & G_m & 0 & \ldots & \\
0 & 0 & G_0 & G_1 & \ldots & G_m & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & \\
\end{bmatrix}.
\]

Let us recall the well known CSS-like construction:

**Theorem 2.2** [5, 9] (CSS-like Construction) Let \(C_1\) and \(C_2\) be two classical convolutional codes with parameters \((n, k_1)\) and \((n, n - k_2)\), respectively, such that \(C_2^\perp \subset C_1\). The stabilizer matrix is given by

\[
\begin{pmatrix}
H_2(D) & 0 \\
0 & H_1(D)
\end{pmatrix} \in \mathbb{F}_q[D]^{(n-k_1+k_2)\times 2n},
\]

where \(H_1(D)\) and \(H_2(D)\) denote parity check matrices of \(C_1\) and \(C_2\), respectively. Then there exists an \([n, K = k_1 - k_2, (d_z)_f/(d_x)_f]_q\) convolutional stabilizer code, where \((d_x)_f = \min\{\text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp)\}\) and \((d_z)_f = \max\{\text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp)\}\).

**Remark 2.3** To avoid stress of notation, we assume throughout this paper that if \((d_x)_f > (d_z)_f\), then the values are changed.

### 3 Asymmetric quantum convolutional codes

In this section we present the contributions of this paper. As it was said previously, we construct the first families of AQCC’s by means of algebraic methods. More specifically, we construct reduced basic generator matrices for two classical convolutional codes \(V_1\) and \(V_2\), where \(V_2 \subset V_1\), in order to apply the CSS-type construction. This section is divided in three subsections, which contain three distinct code constructions.

#### 3.1 Construction I

The first construction method presented here is described in Theorem 3.1:

**Theorem 3.1** (General Construction) Let \(q\) be a prime power and \(n\) be a positive integer. Then there exist asymmetric quantum convolutional codes with parameters \([n, k, \mu^*; \gamma, (d_z)_f/(d_z)_f]_q\), where \(k, \gamma\) and bounds for \((d_z)_f\) and \((d_z)_f\) are computed.
Proof: Consider a set of \( m < n \) linearly independent (LI) vectors \( \mathbf{v}_i \in F_q^m \) given by \( \mathbf{v}_i = \{v_{i1}, v_{i2}, \ldots, v_{im}\} \), where \( i = 1, 2, \ldots, m \). Let

\[
H = \begin{bmatrix}
H_0 \\
H_0' \\
H_1 \\
H_1' \\
\vdots \\
H_\mu \\
H_\mu'
\end{bmatrix}
\]

be the matrix whose rows are the vectors \( \mathbf{v}_i, \ i = 1, 2, \ldots, m \). The matrices \( H_0, H_0', H_1, H_1', \ldots, H_\mu, H_\mu' \), are mutually disjoint. The matrices \( H_i \) for all \( i = 0, 1, \ldots, \mu \) are chosen in such a way that \( \operatorname{rk} H_i = \operatorname{rk} H_j \), for all \( i, j = 0, 1, \ldots, \mu \).

In order to compute the degree of the convolutional code constructed in the sequence, we assume that \( H_0 \geq H_1 \geq \ldots \geq H_\mu \).

Let \( H \) be a parity check matrix of a linear block code \( C = [n, k, d]_q \), where \( k = n - m \). Consider the linear block code \( C^* = [n, k^*, d^*]_q \) with parity check matrix

\[
H^* = \begin{bmatrix}
H_0' \\
H_1' \\
\vdots \\
H_\mu'
\end{bmatrix}
\]

Next, we construct a matrix \( G_1(D) \) as follows:

\[
G_1(D) = \left[ \begin{array}{c} H_0 \\ -H_0' \end{array} \right] + \left[ \begin{array}{c} H_1 \\ -H_1' \end{array} \right] D + \left[ \begin{array}{c} H_2 \\ -H_2' \end{array} \right] D^2 + \ldots + \left[ \begin{array}{c} H_\mu \\ -H_\mu' \end{array} \right] D^\mu.
\]

We can assume without loss of generality (w.l.o.g.) that \( H_0' \) has full rank and \( \operatorname{rk} H_i' \geq \operatorname{rk} H_j' \), for all \( i = 1, \ldots, \mu \). The matrices \( H_i' \) for all \( 1 \leq i \leq \mu \), are obtained from the respective matrices \( H_i' \) by adding zero-rows at the bottom such that \( \tilde{H}_i \) has \( \operatorname{rk} H_i' \) rows in total. Further, let us consider the submatrices \( G_0(D) \) and \( G_2(D) \) of \( G_1(D) \), given, respectively, by

\[
G_0(D) = H_0 + H_1 D + H_2 D^2 + \ldots + H_\mu D^\mu
\]

and

\[
G_2(D) = H_0' + \tilde{H}_1' D + \tilde{H}_2' D^2 + \ldots + \tilde{H}_\mu' D^\mu.
\]

We know that \( G_1(D) \in F_q[D]^{\kappa \times n} \), i.e., \( G_1(D) \) has full rank \( \kappa = \operatorname{rk} H_0 + \operatorname{rk} H_0' \). \( G_2(D) \) has full rank \( k' = \operatorname{rk} H_0' \). From construction, it follows that \( G_1(D) \) and \( G_2(D) \) are reduced basic generator matrices of convolutional codes \( V_1 \) and \( V_2 \), respectively. Both convolutional codes have memory \( \mu \). Proceeding similarly as
in the proof of [2, Theorem 3], the free distance \((d_1)_f\) of the convolutional code \(V_1\) and the free distance \((d_1)^\perp_f\) of its Euclidean dual \(V_1^\perp\) satisfy \(\min\{D_0 + D_\mu, d\} \leq (d_1)^\perp_f \leq d\) and \((d_1)_f \geq d^\perp\), where \(D_0\) is the minimum distance of the code with parity check matrix \[
\begin{bmatrix}
H_0 \\
H_\mu
\end{bmatrix}
\] and \(D_\mu\) is the minimum distance of the code with parity check matrix \[
\begin{bmatrix}
- & - \\
H_0' \\
H_\mu'
\end{bmatrix}
\]. Similarly, the free distance \((d_2)_f\) of \(V_2\) and the free distance \((d_2)^\perp_f\) of \(V_2^\perp\) satisfy \(\min\{d'_0 + d'_\mu, d^*\} \leq (d_2)_f \leq d^*\) and \((d_2)_f \geq (d^*^\perp)_f\), where \(d'_0\) is the minimum distance of the code \(C_0\) with parity check matrix \(H_0'\) and \(d'_\mu\) is the minimum distance of the code with parity check matrix \(H_\mu'\). The degree \(\gamma_2\) of \(V_2\) equals \(\gamma_2 = \mu(\text{rk} H_\mu') + \sum_{i=1}^{\mu-1} (\mu - i)[\text{rk} H_\mu'(\mu - i) - \text{rk} H_\mu'(\mu - i + 1)];\) the code \(V_2^\perp\) also has degree \(\gamma_2\). On the other hand, the degree \(\gamma_1\) of \(V_1\) is equal to \(\gamma_1 = \mu(\text{rk} H_\mu + \text{rk} H_\mu') + \sum_{i=1}^{\mu-1} (\mu - i)[\text{rk} H_\mu'(\mu - i) - \text{rk} H_\mu'(\mu - i + 1)];\) \(V_1^\perp\) also has degree \(\gamma_1\).

We know that \(V_2 \subset V_1\). The corresponding CSS-type code derived from \(V_1\) and \(V_2\) has frame size \(n, k = \text{rk} H_0\) logical qudits per frame, degree \(\gamma = \gamma_1 + \gamma_2\), \((d_x)_f \geq (d_1)_f \geq d^\perp\) and \((d_z)_f \geq (d_2)_f \geq (d^*^\perp)_f\), where \(\min\{d'_0 + d'_\mu, d^*\} \leq (d_2)_f \leq d^*\). Thus one can get an \(\lfloor n, \text{rk} H_0, \mu^* \gamma_1 + \gamma_2, (d_x)_f / (d_z)_f \rfloor_q\) AQCC. If \(H_1(D)\) is a generator matrix of the code \(V_1^\perp\) then a stabilizer matrix of our AQCC is given by

\[
\begin{pmatrix}
G_2(D) & 0 \\
0 & H_1(D)
\end{pmatrix}.
\]

Other variant of this construction can be obtained by considering a CSS-type code derived from the pair of classical convolutional codes \(V_1^\perp \subset V_2^\perp\). The proof is complete. \(\square\)

### 3.2 Construction II

Let \(q\) be a prime power and \(n\) a positive integer such that \(\gcd(q, n) = 1\). Let \(\alpha\) be a primitive \(n\)-th root of unity in some extension field. Recall that a cyclic code \(C\) of length \(n\) over \(\mathbb{F}_q\) is a Bose-Chaudhuri-Hocquenghem (BCH) code with designed distance \(\delta\) if, for some integer \(b \geq 0\), we have \(g(x) = \text{l.c.m.}\{M^{(b)}(x), M^{(b+1)}(x), \ldots, M^{(b+\delta-2)}(x)\}\), i.e., \(g(x)\) is the monic polynomial of smallest degree over \(\mathbb{F}_q\) having \(\alpha^b, \alpha^{b+1}, \ldots, \alpha^{b+\delta-2}\) as zeros. Therefore, \(c \in C\) if and only if \(c(\alpha^b) = c(\alpha^{b+1}) = \ldots = c(\alpha^{b+\delta-2}) = 0\). Thus the code has a string of \(\delta - 1\) consecutive powers of \(\alpha\) as zeros. It is well known that the minimum distance of a BCH code is greater than or equal to its designed...
distance $\delta$. A parity check matrix for $C$ is given by

$$H_{b,b} = \begin{bmatrix}
1 & a^b & a^{2b} & \cdots & a^{(n-1)b} \\
1 & a^{(b+1)} & a^{2(b+1)} & \cdots & a^{(n-1)(b+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & a^{(b+\delta-2)} & a^{2(b+\delta-2)} & \cdots & a^{(n-1)(b+\delta-2)}
\end{bmatrix},$$

where each entry is replaced by the corresponding column of $l$ elements from $\mathbb{F}_q$, where $l = \text{ord}_n(q)$, and then removing any linearly dependent rows. The rows of the resulting matrix over $\mathbb{F}_q$ are the parity checks satisfied by $C$.

The following result establishes conditions in which it is possible to construct AQCC’s derived from BCH codes.

**Theorem 3.2** Let $q = 2^t$, where $t \geq 4$ and consider that $n = q + 1$ and $a = 2^t/2$. Then there exists an AQCC with parameters $[(n,2i-4,\mu^*,6,[d_z]_f/[d_x]_f)]_q$, where $(d_z)_f \geq n - 2i - 1$ and $(d_x)_f \geq 3$, for all $3 \leq i \leq a - 1$.

**Proof:** Consider the parity check $\mathbb{F}_q$-matrix of the BCH code $C$ given by

$$H = \begin{bmatrix}
1 & a^a & \cdots & \cdots & a^{(n-1)a} \\
1 & a^{(a-1)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
1 & a^{(a-i)} & a^{2(a-i)} & \cdots & a^{(n-1)(a-i)}
\end{bmatrix},$$

whose entries are expanded with respect to some $\mathbb{F}_q$-basis $B$ of $\mathbb{F}_q^2$, after removing the linearly dependent rows. It is easy to see that $C$ is a MDS code with parameters $[n,n - 2i - 2,2i + 3]_q$. Its dual $C^\perp$ is also a MDS code with parameters $[n,2i + 2,n - 2i - 1]_q$.

Next, we construct a classical convolutional code $V_1$ generated by the reduced basic matrices

$$G_1(D) = \begin{bmatrix}
1 & a^{(a-i+2)} & a^{2(a-i+2)} & \cdots & a^{(n-1)(a-i+2)} \\
1 & a^a & \cdots & \cdots & a^{(n-1)a} \\
1 & a^{(a-1)} & \cdots & \cdots & a^{(n-1)(a-1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & a^{(a-i+3)} & a^{2(a-i+3)} & \cdots & a^{(n-1)(a-i+3)} \\
1 & a^{(a-i+1)} & a^{2(a-i+1)} & \cdots & a^{(n-1)(a-i+1)} \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix} + D$$
and
\[
G_2(D) = \begin{bmatrix}
1 & \alpha^{(a-i+2)} & \alpha^{2(a-i+2)} & \ldots & \alpha^{(n-1)(a-i+2)} \\
1 & \alpha^{(a-i+1)} & \alpha^{2(a-i+1)} & \ldots & \alpha^{(n-1)(a-i+1)}
\end{bmatrix} + \\
\begin{bmatrix}
1 & \alpha^{(a-i+1)} & \alpha^{2(a-i+1)} & \ldots & \alpha^{(n-1)(a-i+1)} \\
1 & \alpha^{(a-i+2)} & \alpha^{2(a-i+2)} & \ldots & \alpha^{(n-1)(a-i+2)}
\end{bmatrix}D
\]

The code \(V_1\), generated by \(G_1(D)\), is a unit memory code of dimension \(k_1 = 2(i-1)\) and degree \(\gamma_1 = 4\); \(V_1\) is an \((n, 2[i-1], 4; 1, \mu_1^+, \mu_2^+; [d_1]_f, [d_2]_f)\) code. Its Euclidean dual code \(V_1^\perp\) has parameters \((n, n-2[i-1], 4; \mu_1^+, \mu_2^+; [d_1]_f, [d_2]_f)\). From construction, it follows that \(V_2 \subset V_1\), so \(V_1^\perp \subset V_2^\perp\). Consider the stabilizer matrix given by
\[
\begin{pmatrix}
H_1(D) & 0 \\
0 & G_2(D)
\end{pmatrix},
\]
where \(H_1(D)\) is a parity check matrix of the code \(V_1^\perp\). The corresponding CSS-type code has \(K = 2i - 4\), \(\gamma = 6\), \((d_z)_f \geq n - 2i - 1\) and \((d_x)_f \geq 3\). Thus there exists an \([n, 2i - 4, \mu^+; 6, [d_z]_f, [d_x]_f]_q\) AQCC. \(\square\)

Theorem 3.2 can be generalized as follows:

**Theorem 3.3** Let \(q = 2^l\), where \(l \geq 4\) and consider that \(n = q + 1\) and \(a = \frac{q}{2}\). Then there exist AQCC’s with parameters

\(a)\) \([n, 2i - 2t - 2, \mu^+; \mu^+; 6, [d_z]_f, [d_x]_f]_q\), where \((d_z)_f \geq n - 2i - 1\) and \((d_x)_f \geq 2t + 3\), \(i\) and \(t\) are positive integers such that \(1 \leq t \leq i - 2\) and \(3 \leq i \leq a - 1\);

\(b)\) \([n, 2i - 2t, \mu^+; 4, [d_z]_f, [d_x]_f]_q\), where \((d_z)_f \geq n - 2i - 1\) and \((d_x)_f \geq 2t + 3\), \(i\) and \(t\) are positive integers such that \(1 \leq t \leq i - 1\) and \(2 \leq i \leq a - 1\).

**Proof:** We only show Item \(a)\), since Item \(b)\) is similar. The notation and the matrix \(H\) is the same as in the proof of Theorem 3.2. We split \(H\) into disjoint submatrices in order to construct a reduced basic generator matrix \(G_1(D)\) of
the code $V_1$, given by

$$G_1(D) = \begin{bmatrix}
1 & \alpha^{a-t+1} & \alpha^{2a-t+1} & \cdots & \alpha^{(n-1)a-t+1} \\
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a} \\
1 & \alpha^{a-1} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
1 & \alpha^{a-(t-1)} & \alpha^{2a-(t-1)} & \cdots & \alpha^{(n-1)a-(t-1)} \\
- & - & - & - & - \\
1 & \alpha^{a-(t+1)} & \alpha^{2a-(t+1)} & \cdots & \alpha^{(n-1)a-(t+1)} \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
1 & \alpha^{a-(i-1)} & \alpha^{2a-(i-1)} & \cdots & \alpha^{(n-1)a-(i-1)} \\
1 & \alpha^{a-i} & \cdots & \cdots & \alpha^{(n-1)(a-i)} \\
1 & \alpha^{a-t} & \cdots & \cdots & \alpha^{(n-1)(a-t)} \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + D,$$

Let $V_2$ be the convolutional code generated by the reduced basic matrix $G_2(D)$

$$G_2(D) = \begin{bmatrix}
1 & \alpha^a & \cdots & \cdots & \alpha^{(n-1)a} \\
1 & \alpha^{a-1} & \cdots & \cdots & \alpha^{(n-1)(a-1)} \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
1 & \alpha^{a-(t-1)} & \alpha^{2a-(t-1)} & \cdots & \alpha^{(n-1)a-(t-1)} \\
1 & \alpha^{a-t} & \cdots & \cdots & \alpha^{(n-1)(a-t)} \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - \\
0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} + D,$$

It is easy to see that the code $V_1$ has parameters $(n, 2i - 2, 4; [d_1]_f \geq n - 2i - 1)_q$ and $V_1^\perp$ has parameters $(n, n - 2i + 2, 4; \mu^{\perp, [d_1]_f})_q$. The code $V_2$, generated by $G_2(D)$, is an $(n, 2t, 2; 1, [d_2]_f)_q$ code, so $V_2^\perp$ has parameters $(n, n - 2t, 2; \mu^{\perp, [d_2]_f})_q$ code, so $V_2^\perp \subset V_1^\perp$. Since $V_2 \subset V_1$, it follows that $V_1^\perp \subset V_2^\perp$. Thus there exists an $(n, 2i - 2t, 2, \mu^*; 6, [d_2]_f/(d_2)_f)_q$ AQCC, where $(d_2)_f \geq n - 2i - 1$ and $(d_2)_f \geq 2t + 3$.

**Example 3.1** Applying Theorem 3.3, one can get AQCC’s with parameters

$$[(17, 6, \mu^*; 6, [d_2]_f \geq 6/[d_2]_f \geq 5)]_{16}, \quad [(17, 8, \mu^*; 4, [d_2]_f \geq 6/[d_2]_f \geq 5)]_{16},$$
Theorem 3.4 Assume that \( q = p^j \), where \( p \) is an odd prime and \( l \geq 2 \). Consider that \( n = q + 1 \) and \( a = \frac{q}{p} \). Then there exist AQCC’s with parameters

\[
\begin{align*}
& a) \left( (n, 2i - 2t - 2, \mu^*; 6, [d_z]_f/[d_x]_f)_q \right), \text{ where } (d_z)_f \geq n - 2i \text{ and } (d_x)_f \geq 2t + 2, \\
& \quad \text{for all } 1 \leq t \leq i - 2, \text{ where } 3 \leq i \leq a - 1; \\
& b) \left( (n, 2i - 2t, \mu^*; 4, [d_z]_f/[d_x]_f)_q \right), \text{ where } (d_z)_f \geq n - 2i \text{ and } (d_x)_f \geq 2t + 2, \\
& \quad \text{for all } 1 \leq t \leq i - 1, \text{ where } 2 \leq i \leq a - 1.
\end{align*}
\]

Proof: Analogous to that of Theorem 3.3. \( \square \)

Remark 3.5 Note that the constructions presented in Theorems 3.2 and 3.3 can be generalized to multi-memory codes in a straightforward way, i.e., the unit-memory matrices \( G_1(D) \) and \( G_2(D) \) can be easily converted into multi-memory matrices as well.

3.3 Construction III

In this subsection we are interested in constructing AQCC’s derived from Reed-Solomon (RS) and generalized Reed-Solomon (GRS) codes. We first deal with RS codes. Recall that a RS code over \( \mathbb{F}_q \) is a BCH code, of length \( n = q - 1 \), with parameters \( [n, n - d + 1, d]_q \), where \( 2 \leq d \leq n \). A parity check matrix of a RS code is given by

\[
H_{\delta,b} = \\
\begin{bmatrix}
1 & \alpha^b & \alpha^{2b} & \cdots & \alpha^{(n-1)b} \\
1 & \alpha^{(b+1)} & \alpha^{2(b+1)} & \cdots & \alpha^{(n-1)(b+1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha^{(b+d-2)} & \cdots & \cdots & \alpha^{(n-1)(b+d-2)}
\end{bmatrix},
\]

whose entries are in \( \mathbb{F}_q \).

In Theorem 3.6 presented in the following, we construct AQCC’s derived from RS codes:

Theorem 3.6 Assume that \( q \geq 8 \) is a prime power. Then there exist AQCC’s with parameters

\[
\begin{align*}
& a) \left( (q - 1, i - t - 1, \mu^*; 3, [d_z]_f/[d_x]_f)_q \right), \text{ where } (d_z)_f \geq q - i - 1, (d_x)_f \geq t + 2, \\
& \quad \text{for all } 1 \leq t \leq i - 2, \text{ where } 3 \leq i \leq q - 3; \\
& b) \left( (q - 1, i - t, \mu^*; 2, [d_z]_f/[d_x]_f)_q \right), \text{ where } (d_z)_f \geq q - i - 1, (d_x)_f \geq t + 2, \\
& \quad \text{for all } 1 \leq t \leq i - 1, \text{ where } 2 \leq i \leq q - 3.
\end{align*}
\]
is well known that GRS elements of GRS codes are GRS elements of 

so on.

Example 3.2. By means of Theorem 3.6, one can construct AQCC’s with parameters \([10, 4, \mu^*; 3, [d_z]_f \geq 4/[d_x]_f \geq 3]_{11}, [(10, 5, \mu^*; 3, [d_z]_f \geq 3/[d_x]_f \geq 3)]_{11}, [(10, 2, \mu^*; 3, [d_z]_f \geq 6/[d_x]_f \geq 3)]_{11}, [(10, 1, \mu^*; 3, [d_z]_f \geq 6/[d_x]_f \geq 4)]_{11}, and so on.

Let us recall the definition of GRS codes. Let \(n\) be an integer such that \(1 \leq n \leq q\), and choose an \(n\)-tuple \(\zeta = (\zeta_0, ..., \zeta_{n-1})\) of distinct elements of \(\mathbb{F}_q\). Assume that \(v = (v_0, ..., v_{n-1})\) is an \(n\)-tuple of nonzero (not necessary distinct) elements of \(\mathbb{F}_q\). For any integer \(k, 1 \leq k \leq n\), consider the set of polynomials of degree less than \(k\), in \(\mathbb{F}_q[x]\), denoted by \(\mathcal{P}_k\). Then we define the GRS codes as \(\text{GRS}_k(\zeta, v) = \{(v_0 f(\zeta_0), v_1 f(\zeta_1), ..., v_{n-1} f(\zeta_{n-1})) | f \in \mathcal{P}_k\}\). It is well known that \(\text{GRS}_k(\zeta, v)\) is a MDS code with parameters \([n, k, n-k+1]_q\). The (Euclidean) dual \(\text{GRS}_k^D(\zeta, v)\) of \(\text{GRS}_k(\zeta, v)\) is also a GRS code and \(\text{GRS}_k^D(\zeta, w) = \text{GRS}_{n-k}(\zeta, v)\) for some \(n\)-tuple \(w = (w_0, ..., w_{n-1})\) of nonzero elements of \(\mathbb{F}_q\). A generator matrix of \(\text{GRS}_k(\zeta, v)\) is given by

\[
G = \begin{bmatrix}
    v_0 & v_1 & \cdots & v_{n-1} \\
v_0 \zeta_0 & v_1 \zeta_1 & \cdots & v_{n-1} \zeta_{n-1} \\
v_0 \zeta_0^2 & v_1 \zeta_1^2 & \cdots & v_{n-1} \zeta_{n-1}^2 \\
\vdots & \vdots & \ddots & \vdots \\
v_0 \zeta_0^{k-1} & v_1 \zeta_1^{k-1} & \cdots & v_{n-1} \zeta_{n-1}^{k-1}
\end{bmatrix}
\]
a parity check matrix of $\text{GRS}_k(\zeta, \nu)$ is

$$H = \begin{bmatrix}
  w_0 & w_1 & \cdots & w_{n-1} \\
  w_0\zeta & w_1\zeta & \cdots & w_{n-1}\zeta^{n-1} \\
  w_0\zeta^2 & w_1\zeta^2 & \cdots & w_{n-1}\zeta^{2(n-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  w_0\zeta^{n-k-1} & w_1\zeta^{n-k-1} & \cdots & w_{n-1}\zeta^{(n-k-1)(n-1)}
\end{bmatrix}.$$

In the next result, we construct new AQCC’s derived from GRS codes.

**Theorem 3.7** Let $q \geq 5$ be a prime power. Assume that $k \geq 1$ and $n \geq 5$ are integers such that $n \leq q$ and $k \leq n - 4$. Choose an $n$-tuple $\zeta = (\zeta_0, \ldots, \zeta_{n-1})$ of distinct elements of $\mathbb{F}_q$ and an $n$-tuple $\nu = (v_0, \ldots, v_{n-1})$ of nonzero elements of $\mathbb{F}_q$. Then there exists an $[(n, n - t - k - 2, \mu^*; 3, [d_z] / [d_x] \mu)_q]$ AQCC, where $(d_z)_f \geq t + 2$ and $(d_x)_f \geq k + 1$, $1 \leq t \leq n - k - 2$.

**Proof:** Let

$$H = \begin{bmatrix}
  w_0 & w_1 & \cdots & w_{n-1} \\
  w_0\zeta & w_1\zeta & \cdots & w_{n-1}\zeta^{n-1} \\
  w_0\zeta^2 & w_1\zeta^2 & \cdots & w_{n-1}\zeta^{2(n-1)} \\
  \vdots & \vdots & \ddots & \vdots \\
  w_0\zeta^{n-k-1} & w_1\zeta^{n-k-1} & \cdots & w_{n-1}\zeta^{(n-k-1)(n-1)}
\end{bmatrix}.$$

be a parity check matrix of an $\text{GRS}_k(\zeta, \nu)$ code. We split $H$ to form polynomial matrices $G_1(D)$ and $G_2(D)$ of codes $V_1$ and $V_2$, respectively, as follows:

$$G_1(D) = \begin{bmatrix}
  w_0^{n-k-3} & w_1^{n-k-3} & \cdots & w_{n-1}\zeta^{(n-k-3)(n-1)} \\
  w_0 & w_1 & \cdots & w_{n-1}\zeta^{n-1} \\
  w_0\zeta & w_1\zeta & \cdots & w_{n-1}\zeta^{n-1} \\
  \vdots & \vdots & \ddots & \vdots \\
  w_0\zeta^{t-1} & w_1\zeta^{t-1} & \cdots & w_{n-1}\zeta^{(t-1)(n-1)} \\
  w_0\zeta^t & w_1\zeta^t & \cdots & w_{n-1}\zeta^t \\
  \vdots & \vdots & \ddots & \vdots \\
  w_0\zeta^{n-k-2} & w_1\zeta^{n-k-2} & \cdots & w_{n-1}\zeta^{(n-k-2)(n-1)}
\end{bmatrix} + D$$

where $D$ is a $n \times n$ matrix with all entries equal to 1.
and

\[
G_2(D) = \begin{bmatrix}
w_0 & w_1 & \cdots & w_{n-1} \\
w_0\zeta & w_1\zeta & \cdots & w_{n-1}\zeta_{n-1} \\
w_0\zeta^2 & w_1\zeta^2 & \cdots & w_{n-1}\zeta_{n-1}^2 \\
\vdots & \vdots & \ddots & \vdots \\
w_0\zeta^{t-1} & w_1\zeta^{t-1} & \cdots & w_{n-1}\zeta_{n-1}^{t-1} \\
w_0\zeta^t & w_1\zeta^t & \cdots & w_{n-1}\zeta_{n-1}^t \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} + D,
\]

where \( w = (w_0, \ldots, w_{n-1}) \) is a vector such that \( \text{GRS}_k^q(\zeta, w) = \text{GRS}_{n-k}(\zeta, v) \). The code \( V_1 \) has parameters \((n, n-k-2, 2; 1, [d_1]_f \geq k+1)_q\) and \( V_1^\perp \) has parameters \((n, k+2, 2; \mu_1^*, [d_1]_f^\perp)_q\). Similarly, \( V_2 \) is an \((n, t, 1; 1, [d_1]_f)_q\) code and \( V_2^\perp \) is an \((n, n-t, 1; \mu_2^*, [d_1]_f^\perp \geq t+2)_q\) code. Then there exists an \([(n, n-t-k-2, \mu^*; 3, [d_z]_f/[d_x]_f)]_q\) code, where \((d_z) \geq t+2\) and \((d_x) \geq k+1\).

\[\square\]

**Example 3.3** From Theorem 3.7, we can construct AQCC’s with parameters

\[[(5,1,\mu^*;3,[d_z]_f \geq 5/[d_x]_f \geq 2)]_5, [(7,1,\mu^*;3,[d_z]_f \geq 4/[d_x]_f \geq 3)]_7, [(8,1,\mu^*;3,[d_z]_f \geq 5/[d_x]_f \geq 3)]_8, [(17,7,\mu^*;3,[d_z]_f \geq 7/[d_x]_f \geq 4)]_17, [(17,7,\mu^*;3,[d_z]_f \geq 6/[d_x]_f \geq 5)]_17, [(17,6,\mu^*;3,[d_z]_f \geq 7/[d_x]_f \geq 5)]_17, [(17,4,\mu^*;3,[d_z]_f \geq 9/[d_x]_f \geq 5)]_17 \]

and so on.

**Theorem 3.8** Let \( q \geq 5 \) be a prime power. Assume that \( k \geq 1 \) and \( n \geq 5 \) are integers such that \( n \leq q \) and \( k \leq n-4 \). Choose an \( n \)-tuple \( \zeta = (\zeta_0, \ldots, \zeta_{n-1}) \) of distinct elements of \( \mathbb{F}_q \) and an \( n \)-tuple \( v = (v_0, \ldots, v_{n-1}) \) of nonzero elements of \( \mathbb{F}_q \). Then an \([(n, n-t-k-1, \mu^*; 2, [d_z]_f/[d_x]_f)]_q\) AQCC, where \((d_z) \geq t+2\), \((d_x) \geq k+1\) and \( 1 \leq t \leq n-k-1 \) can be constructed.

**Proof:** Similar to that of Theorem 3.7. \[\square\]

**Example 3.4** From Theorem 3.8, we obtain AQCC’s with parameters

\[[(5,1,\mu^*;2,[d_z]_f \geq 4/[d_x]_f \geq 2)]_5, [(7,2,\mu^*;2,[d_z]_f \geq 4/[d_x]_f \geq 3)]_7, [(7,2,\mu^*;2,[d_z]_f \geq 5/[d_x]_f \geq 2)]_7, [(7,1,\mu^*;2,[d_z]_f \geq 5/[d_x]_f \geq 3)]_7.\]

**4 Code Comparison**

Let \( C \) be an \([(n, k, m; \gamma, d_f)]_q\) quantum convolutional code. The code \( C \) is said to be pure if does not exist errors of weight less than \( d_f \) in the stabilizer of \( C \). Recall the quantum generalized Singleton bound (GQSB) for QCC’s:
Theorem 4.1 [3] (Quantum Singleton bound) The free distance of an \([n, k, m; \gamma, d_f]_q\) Fq2-linear pure convolutional stabilizer code is bounded by
\[
d_f \leq \frac{n-k}{2} \left\lceil \frac{2\gamma}{n+k} \right\rceil + 1 + \gamma.
\]

The parameters of our AQCC's are given by \([(n, k, m; \gamma, [d_z]/[d_x])_q]\), where
\[
(d_x)_f = \min\{\text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp)\}
\]
and
\[
(d_z)_f = \max\{\text{wt}(C_1 \setminus C_2^\perp), \text{wt}(C_2 \setminus C_1^\perp)\}.
\]

In this context, if one puts the constraint of pure codes, the free distance \((d_x)_f\)
with respect to qudit-flip errors satisfies the GQSB. However, much research
remains to be done in the area of AQCC's. In fact, there is no bound for the
respective free distances nor relationships among the parameters of AQCC's.
Other impossibility of our comparison, is the fact that our codes have param-
ters quite distinct of the QCC's available in literature. This occurs since our
work is the first one with respect to AQCC's. This area of research needs much
investigation, since it was introduced recently (in 2003; see [24]). Additionally,
even in the case of constructions of good QCC's, only few works are displayed
in literature [3, 7, 17, 18].

5 Summary

We have constructed the first families of asymmetric quantum convolutional
codes available in literature. These new AQCC's are derived from suitable
families of classical convolutional codes with good parameters, which were also
constructed in this paper. Our codes have great asymmetry. Additionally, great
variety of distinct types of codes have also been presented. However, much
work remains to be done in order to find bounds for AQCC's as well as for the
development of such area of research.

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