Planar maps and continued fractions

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Abstract

We present an unexpected connection between two map enumeration problems. The first one consists in counting planar maps with a boundary of prescribed length. The second one consists in counting planar maps with two points at a prescribed distance. We show that, in the general class of maps with controlled face degrees, the solution for both problems is actually encoded into the same quantity, respectively via its power series expansion and its continued fraction expansion. We then use known techniques for tackling the first problem in order to solve the second. This novel viewpoint provides a constructive approach for computing the so-called distance-dependent two-point function of general planar maps. We prove and extend some previously predicted exact formulas, which we identify in terms of particular Schur functions.

1. Introduction

1.1. General introduction

Maps are fundamental objects of combinatorial theory, introduced by Tutte in the 60’s [1,2], which also appeared later as natural models for random surfaces in physics [3]. Many questions about maps boil down to enumeration problems, which in turn received a lot of attention from different communities. Different techniques of enumeration have been developed, for instance combinatorialists’ quadratic method [4], physicists’ matrix integrals [5] and more recently bijective coding by trees [6-9]. Most results deal with global enumeration problems, say counting families of maps with a control on their size and topology. The coding by trees allowed to address more refined questions about the distance in maps, say counting maps with a prescribed radius [10] or with marked pointed at prescribed distances [11]. This approach led to important applications in probability theory for the rigorous construction of scaling limits of large random maps [12].

One of the prominent global quantities, considered both in the recursive decomposition and the matrix integral approach, is the number of maps with one boundary,
with a control on both the total map size and the boundary length. More precisely, this information is captured by the moments, which are generating functions of maps with a fixed boundary length. These moments are determined by a closed system of equations, called either Tutte’s or loop equations respectively in the combinatorial or physical communities.

On the other hand, one of the simplest observables related to distance is the so-called two-point function, which is the number of maps with given size and having two marked vertices at fixed distance. In particular the dependence of this two-point function over the distance gives directly the average profile of random maps. The distance-dependent two-point function was first considered in [13] in the case of planar triangulations, where its scaling form was predicted via a “transfer-matrix” approach in the spirit of Tutte’s decomposition. An exact discrete expression for the two-point function was given in [11] for planar quadrangulations and more generally bipartite planar maps with a control on face degrees, whose scaling form agrees with [13]. This exact expression makes use of a coding of such maps by trees, leading to discrete recurrence equations whose solution was guessed.

The purpose of this paper is to present an unexpected yet remarkable connection between the two above notions showing that the information about the distance in maps is actually hidden in the global problem of enumerating maps with a boundary. More precisely, the moments and the distance-dependent two-point function constitute two possible expansions of the same quantity, the resolvent: the moments form its power series expansion while the two-point function encodes its continued fraction expansion. Using the standard theory of continued fractions this allows in particular to obtain explicit expressions for the two-point function via the known techniques for computing the moments. This program is carried out in detail in this paper. We begin by an overview of our main results in Sect. 1.2, where precise definitions for the moments and two-point function are given. The actual connection between them is established in Sect. 2 by use of the coding of maps by appropriate trees called mobiles. In Sect. 3 we derive an explicit expression for the moments in terms of weighted paths by various techniques: a direct non-constructive check, the solution of Tutte’s equation and a bijective construction. This third approach also gives rise to a one-parameter family of expressions for the moments forming so-called conserved quantities. Our expression for moments is used in Sect. 4 to derive explicit expressions for the two-point function in terms of Schur functions. The special case of bipartite maps, which was the scope of [11], is discussed in Sect. 5, while the particularly simple cases of triangulations and quadrangulations are addressed in Sect. 6. Concluding remarks and discussion are gathered in Sect. 7.

1.2. Overview of the main results

A planar map is a connected graph (possibly with loops and multiple edges) drawn on the sphere without edge crossings, and considered up to continuous deformation. It is made of vertices, edges and faces. The degree of a vertex or face is the number of edges incident to it (counted with multiplicity). A map is rooted if one of its edges is
distinguished and oriented, the root face being the face on the right of the root edge and the root degree being the degree of the root face.

In this paper, we generally consider enumeration problems for rooted planar maps subject to a control on face degrees, i.e. for each positive integer $k$ we prescribe the number of faces with degree $k$. In the language of generating functions, this is equivalent to attaching a weight $g_k$ to each face of degree $k$, where $(g_k)_{k \geq 1}$ is a sequence of indeterminates, so that the global weight of a map is the product of the weights of its faces. Setting $g_k = 0$ for all odd $k$ amounts to considering the planar maps which only have faces of even degree, which are the bipartite planar maps. Drastic simplifications occur in the bipartite case and these will be discussed in due course.

As a general preliminary remark, most quantities introduced in this paper will be formal power series in the $g_k$’s (with integer or rational coefficients) but, for the sake of concision, this will not be apparent on the notations – we shall write $X$ rather than $X(g_1, g_2, g_3, \ldots)$ for instance. If extra variables are involved, they shall be written explicitly.

The fundamental observation of this paper is a combinatorial identity between some a priori unrelated families of generating functions for rooted planar maps with the above weights. On the one hand, for each positive integer $n$, we consider the generating function $F_n$ for rooted planar maps with root degree $n$, which we call the $n$th moment. By convention we do not attach a weight $g_n$ to the root face and we set $F_0 = 1$. We may combine all values of $n$ into the generating function $F(z) = \sum_{n=0}^{\infty} F_n z^n$ where $z$ is an extra variable. This quantity essentially coincides with the planar “resolvent” encountered in the context of matrix integrals. On the other hand, again for each positive integer $n$, we consider the generating function for planar maps with two marked vertices at distance $n$, which we call the distance-dependent two-point function. To properly root these planar maps, we take as root an edge incident to and pointing away from one of the marked vertices. Then the endpoint of the root edge must be at distance $n - 1$, $n$ or $n + 1$ from the other marked vertex. We distinguish between these three cases and denote by $r_n$, $t_n$ and $r_{n+1}$ the respective generating function (clearly reversing the orientation of the root edge allows to identify the third case with the first one for $n \to n + 1$). We furthermore introduce the “cumulative” generating functions $R_n = \sum_{i=1}^{n} r_i$ and $T_n = \sum_{i=0}^{n} t_i$, corresponding to two marked vertices at distance less than or equal to $n$. For convenience, a conventional term 1 is also included in $r_1$. Then, our observation is that the sequences $(R_n)_{n \geq 1}$ and $(S_n)_{n \geq 0} = (\sqrt{T_n})_{n \geq 0}$ form the Jacobi type continued fraction (or J-fraction) expansion of $F(z)$, namely

$$F(z) = \sum_{n=0}^{\infty} F_n z^n = \frac{1}{1 - S_0 z - \frac{R_1 z^2}{1 - S_1 z - \frac{R_2 z^2}{1 - \ldots}}}.$$  \hspace{1cm} (1.1)

As discussed in detail in Sect. 2, this follows from the correspondence between planar maps and some labeled trees called mobiles [8], and the combinatorial theory of continued fractions [14,15]. Calling $F_n$ the $n$th moment is consistent with the usual
terminology of J-fractions \[16\]. A classical result states that the sequence \((R_n)_{n\geq 1}\) is closely related to the Hankel determinants of moments, namely

\[ R_n = \frac{H_n H_{n-2}}{H_{n-1}^2}, \quad H_n = \det_{0\leq i,j\leq n} F_{i+j} \]

(with the convention \(H_{-1} = 1\)) and similarly the sequence \((S_n)_{n\geq 0}\) is given by Hankel minors through

\[ \sum_{i=0}^{n} S_i = \tilde{H}_n \frac{H_n}{H_n}, \quad \tilde{H}_n = \det_{0\leq i,j\leq n} F_{i+j+\delta_{j,n}}. \]

In the specific context of planar maps, we derive in Sect. 3 a general formula for the moments \(F_n\) in terms of the weights \((g_k)_{k\geq 1}\). A very peculiar structure emerges when this formula is substituted into the determinants \(H_n\) and \(\tilde{H}_n\): after some simple manipulations we recognize instances of a classical identity for symplectic Schur functions, see Sect. 4 for details. Let us now state the important results of Sects. 3 and 4.

Our expression for the moments involves generating functions for three-step paths: these are lattice paths in the discrete Cartesian plane which consist of up-steps \((1,1)\), level-steps \((1,0)\) and down-steps \((1,-1)\). We denote by \(P(n; R, S)\) the generating function for three-step paths going from \((0,0)\) to \((n,0)\), where a weight \(S\) is attached to each level-step, a weight \(\sqrt{R}\) is attached to each up- or down-step and the global weight of a path is the product of the weights of its steps (clearly, this global weight involves an integer power of \(R\)). Furthermore, we say that a path is positive if it only visits vertices with non-negative ordinate, and we denote by \(P^+(n; R, S)\) the generating function for positive three-steps paths from \((0,0)\) to \((n,0)\), also known as Motzkin paths of length \(n\), with the same weights. We have the explicit expressions

\[ P(n; R, S) = \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} \frac{n!}{i!(i+1)!(n-2i)!} R^i S^{n-2i} \]

and

\[ P^+(n; R, S) = \sum_{i=0}^{\left\lfloor n/2 \right\rfloor} \frac{n!}{i!(i+1)!(n-2i)!} R^i S^{n-2i}. \]

Then, the general formula for the moments is

\[ F_n = \sum_{q=0}^{\infty} A_q P^+(n+q; R, S) \]

where the coefficients \(A_q\) are given by

\[ A_q = R \left( \delta_{q,0} - \sum_{k=q+2}^{\infty} g_k P(k-q-2; R, S) \right) \]

and where the step weights \(R, S\) are themselves power series in \((g_k)_{k\geq 1}\), which are implicitly determined by

\[ S = \sum_{k=1}^{\infty} g_k P(k-1; R, S), \quad R = 1 + \frac{1}{2} \sum_{k=1}^{\infty} g_k P(k; R, S) - \frac{S^2}{2}. \]
Now, substituting the expression (1.4) into the Hankel determinant $H_n$, elementary row and column manipulations, which amount to a natural decomposition of lattice paths, lead to

$$H_n = R^{\frac{n(n+1)}{2}} \det_{0 \leq i,j \leq n} (B_{i-j} - B_{i+j+2})$$

$$B_i = \sum_{q=0}^{\infty} A_q P_i(q; R, S)$$  \hspace{1cm} (1.7)

where $P_i(q; R, S)$ is the generating function for three-step paths from $(0,0)$ to $(q,i)$. Note that $P_{-i}(q; R, S) = P_i(q; R, S)$, hence $B_{-i} = B_i$. Similarly for $\tilde{H}_n$ we have

$$\tilde{H}_n = (n+1)SH_n + R^{\frac{n^2+n+1}{2}} \det_{0 \leq i,j \leq n} (B_{i-j-\delta_{j,n}} - B_{i+j+\delta_{j,n}+2})$$

(1.8)

At this stage, we fix a positive integer $p$ and consider maps whose faces have degree at most $p + 2$, i.e. we set $g_k = 0$ for $k > p + 2$. This implies, by the expression (1.3), that $A_q$ vanishes for $q > p$ and moreover, since $P_i(q; R, S) = 0$ for $|i| > q$, that $B_i$ vanishes for $i > p$. The matrices appearing in (1.7) and (1.8) are therefore band matrices. Their determinants can be identified, up to a normalization, with characters of the symplectic group $Sp_{2p}$ \cite{17}, also known as symplectic Schur functions. Following \cite{18}, we denote by $sp_{2p}(\lambda, \mathbf{x})$ the symplectic Schur function associated with the partition $\lambda$, which is a symmetric function of $2p$ variables $\mathbf{x} = (x_1, 1/x_1, x_2, 1/x_2, \ldots, x_p, 1/x_p)$. Here, these variables are the $2p$ roots of the “characteristic equation”

$$\sum_{n=-p}^{p} B_{n} x^n = 0$$  \hspace{1cm} (1.9)

(recall that $B_{-n} = B_n$, hence the equation is invariant under $x \to 1/x$). By the expressions (1.3) for $A_q$, (1.7) for $B_i$ and the identity $\sum_{q=-p}^{p} P_i(q; R, S) x^i = (\sqrt{R} x + S + \sqrt{R}/x)^q$, the characteristic equation reads explicitly

$$1 = \sum_{k=2}^{p+2} g_k \sum_{q=0}^{k-2} P(k-2-q; R, S) \left( \sqrt{R} x + S + \frac{\sqrt{R}}{x} \right)^q.$$  \hspace{1cm} (1.10)

The partitions associated with the determinants appearing in (1.7) and (1.8) are respectively the “rectangular” partition $\lambda_{p,n+1} = (n+1)^p$ made of $p$ parts of size $n+1$, and the “nearly-rectangular” partition $\tilde{\lambda}_{p,n+1} = (n+1)^{p-1} n$ made of $p-1$ parts of size $n+1$ plus one part of size $n$. In summary, we find that $H_n \propto sp_{2p}(\lambda_{p,n+1}, \mathbf{x})$ and $\tilde{H}_n - (n+1)SH_n \propto sp_{2p}(\tilde{\lambda}_{p,n+1}, \mathbf{x})$. Using the Weyl character formula for $sp_{2p}(\lambda, \mathbf{x})$, we end up with the following “nice” formulas for $R_n$ and $S_n$, which involve determinants of size $p$, independently of $n$:

$$R_n = R^{\frac{1}{2} \sum_{1 \leq i,j \leq p} (x_i^{n+1+j} - x_i^{-n-1-j})} \left( \frac{\det_{1 \leq i,j \leq p} (x_i^{n+1+j} - x_i^{-n-1-j})}{\det_{1 \leq i,j \leq p} (x_i^{n+j} - x_i^{-n-j})} \right)^2$$  \hspace{1cm} (1.11)
\[ S_n = S - \sqrt{R} \left( \frac{\det_{1 \leq i,j \leq p} (x_i^{n+1+j-\delta_{j,1}} - x_i^{n-1-j+\delta_{j,1}})}{\det_{1 \leq i,j \leq p} (x_i^{n+1+j} - x_i^{n-1-j})} - \frac{\det_{1 \leq i,j \leq p} (x_i^{n+j-\delta_{j,1}} - x_i^{-n-j+\delta_{j,1}})}{\det_{1 \leq i,j \leq p} (x_i^{n+j} - x_i^{-n-j})} \right). \]  

(1.12)

A number of simplifications occur in the bipartite case, discussed in detail in Sect. 5. Clearly \( F_n = 0 \) for \( n \) odd and the continued fraction expansion of \( F(z) \) naturally reduces to the Stieltjes type

\[ F(z) = \sum_{n=0}^{\infty} F_{2n} z^{2n} = \frac{1}{1 - \frac{R_1 z^2}{1 - \frac{R_2 z^2}{\ddots}}}. \]  

(1.13)

Consistently, the quantities \( S, \tilde{H}_n \) and \( S_n \) vanish while the Hankel determinants factorize as

\[ H_n = h_n^{(0)} \left[ \begin{array}{c} \delta \end{array} \right] h_n^{(1)} \left[ \begin{array}{c} n-1 \end{array} \right], \quad h_n^{(0)} = \det_{0 \leq i,j \leq n} F_{2i+2j}, \quad h_n^{(1)} = \det_{0 \leq i,j \leq n} F_{2i+2j+2}. \]  

(1.14)

This factorization property is also apparent at the level of Schur functions. Indeed, for bipartite maps whose faces have degree at most \( 2p+2 \), we have a characteristic equation of the form

\[ \sum_{n=-p}^{p} B_{2n} x^{2n} = 0 \]  

(1.15)

whose roots are of the form \( (x_1, 1/x_1, -x_1, -1/x_1, \ldots, x_p, 1/x_p, -x_p, -1/x_p) \), and \( \text{sp}_{4p}(\lambda, x) \) can be factorized as the product of two symmetric functions of the variables \( (x_1^2, 1/x_1^2, \ldots, x_p^2, 1/x_p^2) \) (a symplectic and an odd orthogonal Schur function). In the end, \( R_n \) can be expressed in terms of determinants of size \( p \) rather than \( 2p \), which matches the expression found in [11].

The cases of triangulations and quadrangulations are particularly simple as the roots of their characteristic equation are of the respective forms \( (x, 1/x) \) and \( (x, 1/x, -x, -1/x) \). Consequently, the two-point functions in these cases can be expressed in terms of a single quantity \( x \) solution of some algebraic equation. This is discussed in Sect. 6 and a simple interpretation via one-dimensional hard dimers is given.

### 2. From planar maps to continued fractions, via mobiles

The main purpose of this section is to establish that the generating functions for planar maps \( F_n, R_n \) and \( S_n \) defined in Sect. 1 are related by the J-fraction expansion (1.1). The plan is as follows. We recall some general results of the combinatorial theory of continued fractions (Sect. 2.1), before reviewing the coding of planar maps by mobiles (Sect. 2.2). We then interpret \( F_n, R_n \) and \( S_n \) as generating functions for mobiles and complete our proof (Sect. 2.3). Finally we mention a few other outcomes of our approach (Sect. 2.4).
2.1. Reminders of the combinatorial theory of continued fractions

In his seminal paper [14], Flajolet gave a combinatorial interpretation of continued fractions in terms of Motzkin paths. Let us recall a few important results of this theory. For the purposes of this subsection, \((R_m)_{m \geq 1}\) and \((S_m)_{m \geq 0}\) may denote arbitrary sequences of elements of a commutative ring. That is to say, we may temporarily forget about their map-related definition given in Sect. 1. We consider Motzkin paths, and more generally positive three-step paths as defined above, with the following ordinate-dependent weights: for all \(m\),

- each down-step of the form \((t, m) \to (t+1, m-1)\) receives a weight \(R_m\) \((m \geq 1)\)
- each level-step of the form \((t, m) \to (t+1, m)\) receives a weight \(S_m\) \((m \geq 0)\)
- each up-step receives a weight 1.

Note that attaching ordinate-dependent weights to up-steps would add essentially no generality. Let us now define \(F_n\) as the generating function for Motzkin paths of length \(n\) with ordinate-dependent weights (we also temporarily forget about the map-related definition of \(F_n\)) and, attaching a further weight \(z\) per step (of any type), \(F(z) = \sum_{n=0}^{\infty} F_n z^n\) the generating function for Motzkin paths of arbitrary length. Then, the Continued Fraction Theorem (as named in [15]) states that \(F(z)\) is given by the J-fraction (1.1).

In order to recall of its derivation, let us introduce a few notations which also prove useful later. Given arbitrary non-negative integers \(d\), \(d'\) and \(n\), we denote by \(Z_{d,d'}(n)\) the generating function for positive three-step paths from \((0, d)\) to \((n, d')\), with the above ordinate-dependent weights. Furthermore, we denote by \(Z_{d,d'}^+(n)\) the generating function for such paths restricted to have all their ordinates larger than or equal to \(\min(d, d')\) (instead of 0). Clearly \(F_n = Z_{0,0}(n) = Z_{0,0}^+(n)\). Now, for the case \(d = d'\) we have the relation

\[
\sum_{n \geq 0} Z_{d,d}^+(n) z^n = \frac{1}{1 - S_d z - R_{d+1} z^2 \sum_{n \geq 0} Z_{d+1,d+1}^+(n) z^n}
\]  

(2.1)

which simply translates the “arch decomposition”, namely by cutting a path contributing to the left-hand side at each occurrence of the ordinate \(d\), it is bijectively decomposed into a sequence of two types of objects: either (i) level-steps at ordinate \(d\), weighted by \(S_d\), or (ii) arches made of the concatenation of an up-step, a restricted path starting and ending at ordinate \(d+1\) and a down-step, with an overall weight \(R_{d+1} Z_{d+1,d+1}^+(n')\) for some \(n'\). By iterating (2.1) starting at \(d = 0\), we deduce (1.1). Note that the J-fraction is a well-defined power series in \(z\), because computing \(F_n\) only requires \(\lfloor n/2 \rfloor\) iterations (as Motzkin paths of length \(n\) reach at most the ordinate \(\lfloor n/2 \rfloor\)). If we iterate (2.1) starting at an arbitrary \(d\), we obtain

\[
\sum_{n \geq 0} Z_{d,d}^+(n) z^n = \frac{1}{1 - S_d z - \frac{R_{d+1} z^2}{1 - S_{d+1} z - \frac{R_{d+2} z^2}{1 - \cdots}}}
\]  

(2.2)

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where the right-hand side is called a \textit{truncation} of the fundamental fraction (1.1). Many other identities are known (e.g. for \( \sum_{n \geq 0} Z_{d,d'}^+ (n) z^n \) and \( \sum_{n \geq 0} Z_{d,d'} (n) z^n \) via “last-passages decompositions”) but we shall not need them here.

A particular case is when the weights do not depend on the ordinate, i.e. we set \( R_m = R \) for all \( m \geq 1 \) and \( S_m = S \) for all \( m \geq 0 \). Then, we recover the three-step path generating functions defined in Sect. 1, namely \( Z_{d,d'} (n) = R^{(d-d')/2} P_{d-d'} (n; R, S) \) and \( Z_{d,d'}^+ (n) = R^{(d-d')/2} P_{d'-d}^+ (n; R, S) \). Note that the \( R^{(d-d')/2} \) prefactor arises from the slightly different weighting convention. Most notably, we have

\[
\sum_{n \geq 0} P^+ (n; R, S) z^n = \frac{1}{1 - S z - \frac{R z^2}{1 - \cdots}}. \tag{2.3}
\]

Clearly this series is the solution of the quadratic equation \( X = 1 + S z X + R z^2 X^2 \). More generally, if we only assume that \( R_m \to R \) and \( S_m \to S \) for \( m \to \infty \) (in any sense of convergence), then obviously \( Z_{d,d} (n) \to P(n; R, S) \) and \( Z_{d,d'}^+ (n) \to P^+ (n; R, S) \) for \( d \to \infty \).

Let us now consider the inverse problem of determining the sequences \( (R_m)_{m \geq 1} \) and \( (S_m)_{m \geq 0} \) in the J-fraction expansion, knowing the power series expansion of \( F(z) \) i.e. the sequence \( (F_n)_{n \geq 0} \). It is a classical result that this problem is solved using Hankel determinants, and the solution is given by relations (1.2) and (1.3). See for instance the beautiful proof by Viennot [19] based on a combinatorial interpretation involving configurations of non-intersecting Motzkin paths.

2.2. Review of the coding of planar maps by mobiles

\begin{center}
\begin{tabular}{ll}
(a) & (b) \\
\includegraphics[width=0.4\textwidth]{fig1a.png} & \includegraphics[width=0.4\textwidth]{fig1b.png}
\end{tabular}
\end{center}

\textbf{Fig. 1}: The mobile construction applied (a) to an edge of type \((n, n-1)\) and (b) to an edge of type \((n, n)\). In (a) a mobile edge (thick line) connects an unlabeled vertex (big dot) to a labeled one. In (b) a bivalent flagged vertex (lozenge) is created and connected to both adjacent unlabeled vertices.
We now return to the context of planar maps and review their coding by mobiles presented in Ref. [8]. The coding applies to pointed maps, i.e. maps with a distinguished vertex called the origin, and works as follows: we start by labeling each vertex of the map by a non-negative integer equal to its graph distance from the origin and add a new unlabeled vertex at the center of each face. With the above labeling, any edge of the map is either of type \((n, n - 1)\), i.e. connects a vertex labeled \(n\) to a vertex labeled \(n - 1\) for some \(n \geq 1\), or of type \((n, n)\) for some \(n \geq 0\). For each edge of type \((n, n - 1)\), we draw a new edge connecting its incident vertex labeled \(n\) to the unlabeled vertex sitting at the center of the face on the right of the edge (oriented from \(n\) to \(n - 1\)). This procedure is displayed in Fig. 1-(a). For each edge of type \((n, n)\), we add in its middle a “flagged” vertex with flag \(n\) and connect it by two new edges to the two unlabeled vertices at the center of the two incident faces, as displayed in Fig. 1-(b). It was shown in [8] that the new edges form a tree which spans all the labeled vertices with label \(n \geq 1\) (i.e. all the original vertices of the map but its origin), as well as all the added unlabeled vertices and all the added flagged vertices.

![Diagram of labeled and flagged vertices with arrows indicating the direction of the labeling process.](image)

**Fig. 2:** An illustration of the property (P) around an unlabeled vertex.

The labels and flags satisfy the property:

(P) for each unlabeled vertex, the *clockwise* sequence of labels and flags on its adjacent vertices matches exactly the sequence of labels and flags obtained by attaching a label \(n\) to each down-step \((t, n) \rightarrow (t + 1, n - 1)\) and a flag \(n\) to each level-step \((t, n) \rightarrow (t + 1, n)\) of some three-step path with identical initial and final ordinates. Note that the three-step path in (P) is unique up to cyclic shifts, and that its length is given by the number of flags plus twice the number of labels. Returning to the original map, the sequence of ordinates in the three-step path simply reproduces the clockwise sequence of distances from the origin to the vertices around the face associated with the unlabeled vertex at hand, while the length of the path is nothing but the degree of this face (see Fig. 2).
In all generality, a mobile is defined as a plane tree with three types of vertices: unlabeled vertices, labeled vertices carrying an arbitrary integer label and flagged vertices carrying an arbitrary integer flag, with edges connecting only unlabeled vertices to labeled or flagged ones in such a way that the flagged vertices have degree 2 and property (P) holds around each unlabeled vertex. Note that shifting all labels and flags in a mobile by a fixed integer preserves property (P) so that the resulting object is still a mobile.

The above coding associates to each pointed map a mobile. As shown in [8], it is a bijection between pointed planar maps and mobiles satisfying the two extra requirements:

(R1) labels and flags are respectively positive and non-negative;
(R2) there is at least one label 1 or one flag 0.

Strictly speaking, the tree reduced to a single vertex labeled 1 satisfies (R1) and (R2), and corresponds to a conventional empty map.

Furthermore, the coding yields the following one-to-one correspondences between elements of a map and of its associated mobile. Map vertices at distance \( n \geq 1 \) from the origin in the map correspond to mobile vertices labeled \( n \). Map edges of type \((n, n)\), i.e. connecting two vertices at distance \( n \) from the origin, correspond to flagged vertices with flag \( n (n \geq 0) \) while map edges of type \((n, n-1)\) correspond to mobile edges incident to a labeled vertex with label \( n (n \geq 1) \). Faces of the map correspond to unlabeled vertices of the mobile and moreover, the clockwise sequence of distances from the origin of the vertices incident to a given face is directly read off the ordinates of the (unique cyclic) three-step path ensuring property (P) around the corresponding unlabeled vertex.

The mobile coding a pointed map is unrooted, i.e. it has no distinguished edge nor vertex. We define conventionally a rooted mobile as a mobile with a distinguished edge incident to a labeled vertex (the other incident vertex being necessarily unlabeled), whose label is then called the root label. The mobile reduced to a single labeled vertex is considered as a rooted mobile even if it has no edge. Regarding flagged vertices, we find more convenient to introduce the notion of half-mobile, which is defined exactly as a mobile except that it has one particular flagged vertex of degree 1 (the other flagged vertices being of degree 2 as before) whose flag is called the root flag. A mobile having a distinguished edge incident to a flagged vertex with flag \( n \) can clearly be seen as a pair of half-mobiles having the same root flag \( n \). Rooted mobiles and pairs of half-mobiles code naturally pointed rooted maps, namely maps with both a marked vertex (the origin) and a marked oriented edge (the root edge). More precisely, a pointed rooted map of type \((n \rightarrow n-1)\), i.e. whose root edge points from a vertex at distance \( n \) from the origin to a vertex at distance \( n-1 \), is coded bijectively by a rooted mobile satisfying (R1)-(R2) and having root label \( n \) (we distinguish the mobile edge associated with the root edge), while a pointed rooted map of type \((n \rightarrow n)\) is coded bijectively by a pair of half-mobiles with root flag \( n \) and whose union also satisfies (R1)-(R2) (we split the mobile at the flagged vertex associated with the root edge, whose orientation allows to distinguish the two half-mobiles). For the last remaining type \((n-1 \rightarrow n)\) we may simply reverse the root edge orientation. This encompasses all possible types of the
root edge, when \( n \) varies. Note that rooted maps can be seen as pointed rooted maps of type \((1 \rightarrow 0)\) or \((0 \rightarrow 0)\).

It is worth noting that a bijection also exists with mobiles where the conditions (R1) and (R2) are waived. It is obtained by now considering pointed rooted maps where the distance between the origin of the map and the starting point of the root edge is no longer fixed. Denoting by \( d \) the distance of the extremity of the root edge farthest from the origin, we now label each vertex by its distance from the origin minus \( d \) so that the pointed rooted map is either of type \((0 \rightarrow 0), (0 \rightarrow -1)\) or \((-1 \rightarrow 0)\). Applying the mobile construction rules of Fig. 1 to all edges of the map creates a mobile where the condition (P) still holds but where the conditions (R1) and (R2) have been waived. For type \((0 \rightarrow 0)\) we finally obtain a pair of half-mobiles with root flag 0, while for type \((0 \rightarrow -1)\) or \((-1 \rightarrow 0)\) we obtain a rooted mobile with root label 0 (for a proper bijection we need to adjoin a sign \( \pm 1 \) to the rooted mobile, in order to keep track of the orientation of the root edge).

2.3. Mobile interpretation of the generating functions \( F_n, R_n, S_n \)

We now want to show that (1.1) holds, when \( F_n, R_n \) and \( S_n \) are the generating functions for planar maps defined in Sect. 1. This requires identifying these quantities as generating functions for mobiles, and \( F_n \) will naturally appear as a sum over Motzkin paths with weights \( R_m \) per down-step starting at ordinate \( m \) and \( S_m \) per level-step at ordinate \( m \). As a general preliminary remark, note that the weight \( g_k \) per face of degree \( k \) in the map translates, in the mobile language, into a weight \( g_k \) per unlabeled vertex whose three-step path associated via (P) has length \( k \).

To begin with, let us discuss the case of \( R_n \) (\( n \geq 1 \)). From our definition of Sect. 1.2 and the above discussion, \( R_n = \sum_{i=1}^{n} r_i \) enumerates pointed rooted maps of type \((i \rightarrow i - 1)\) for some \( i \) between 1 and \( n \), which are in bijection with rooted mobiles satisfying (R1)-(R2) and having a root label \( i \) between 1 and \( n \). These mobiles are in turn in bijection with rooted mobiles satisfying (R1) only and having root label \( n \). Indeed, we may transform the label \( i \) in the first family of mobiles into a label \( n \) by shifting all labels and flags of the mobile by \( n - i \). The condition (R1) is preserved by this non-negative shift but (R2) is no longer valid if \( i < n \). Conversely, assuming that (R1) is satisfied in a rooted mobile with root label \( n \geq 1 \), the minimal label among labeled vertices necessarily lies between 1 and \( n \), i.e. is of the form \( n - i + 1 \) for some \( i \) between 1 and \( n \). From the general condition (P) around each unlabeled vertex, we easily deduce that the minimal flag in the mobile is at least \( n - i \). Shifting all labels and flags by the (non-positive) quantity \( i - n \) preserves (R1) and restores (R2), while it transforms the root label \( n \) into an \( i \) between 1 and \( n \). To summarize, \( R_n \) may be identified as the generating function for rooted mobiles satisfying (R1) only and having root label \( n \). In this identification, the conventional term 1 added in \( r_1 \), hence in \( R_n \), accounts for the mobile reduced to a single labeled vertex with label \( n \).

As for \( T_n = \sum_{i=0}^{n} t_i \), it enumerates pointed rooted maps of type \((i \rightarrow i)\) for some \( i \) between 0 and \( n \), which are in bijection with pairs of half-mobiles with root flag \( i \) and whose union satisfies (R1)-(R2). By a similar argument as above, these are in
bijection with pairs of half-mobiles with root flag \( n \) satisfying (R1) only. Note that (R1) independently applies on each half-mobile. We arrive at \( T_n = (S_n)^2 \) where we identify directly \( S_n \) as the generating function for half-mobiles satisfying (R1) only and with root flag \( n \).

Fig. 3: The generating function \( F_n \) for rooted maps with root degree \( n \) is also that \( Z_{0,0}(n) \) for Motzkin paths of length \( n \) with ordinate-dependent weights.

Let us now come to the mobile interpretation of \( F_n \). From its definition of Sect. 1.2, it enumerates rooted maps with a root face of degree \( n \). Taking as origin of the map the origin of the root edge, we may look at the clockwise sequence of distances from this origin of the successive vertices incident to the root face in the map. For convenience, in the planar representation, we choose as external face the root face itself so that the clockwise orientation around it corresponds in practice to the counterclockwise orientation around the rest of the map. Starting from the origin, this sequence of distances forms a Motzkin path of length \( n \). The down-steps and the level-steps of the Motzkin path correspond respectively to the labeled vertices and the flagged vertices connected to the unlabeled vertex associated with the root face in the mobile. By removing this unlabeled vertex and all its incident edges, we obtain a sequence of rooted mobiles and half-mobiles (when removing an edge incident to a labeled vertex, we keep track of its position by distinguishing, say, the next edge incident to the same labeled vertex in clockwise direction). More precisely, on the Motzkin path, a down-step of the form \((t, m) \rightarrow (t+1, m-1)\) corresponds to a rooted mobile with root label \( m \) satisfying (R1) (possibly reduced to a single labeled vertex), while a level-step \((t, m) \rightarrow (t+1, m)\) corresponds to a half-mobile with root flag \( m \) also satisfying (R1). Conversely, starting from a Motzkin path and a sequence of rooted and half-mobiles so associated with its down- and level-steps, a complete mobile is immediately obtained by connecting the mobiles to a new unlabeled vertex, and properties (P)-(R1)-(R2) are satisfied by construction. This clearly forms a bijection. Translated in the language of generating functions, it says that \( F_n \) is equal to the generating function for Motzkin paths of length \( n \) with ordinate-dependent weights \( (R_m)_{m \geq 1} \) and \( (S_m)_{m \geq 0} \) (see Fig. 3), i.e. \( F_n = Z_{0,0}(n) \) with the notations of Sect. 2.1. Equation (1.1) follows from the general combinatorial theory of continued fractions.
To conclude this section, we mention that the continued fraction expansion may be directly interpreted at the level of maps, with no recourse to the coding by mobiles. It makes use of a particular decomposition of pointed rooted maps into slices, as explained in Appendix A.

2.4. Related results

Before proceeding to the next section, let us list a few other enumerative consequences of the coding of maps by mobiles. First, $R_n$ and $S_n$ satisfy recursive equations which translate decompositions of the corresponding mobiles. Indeed, let us consider a rooted (resp. half-) mobile with root label (resp. flag) $n$ satisfying (R1), not reduced to a single labeled vertex. We now decompose the mobile around the unlabeled vertex incident to the distinguished edge (resp. adjacent to the root flagged vertex). The three-step path associated with this unlabeled vertex has an arbitrary length $k \geq 1$ (thus leads to a weight $g_k$), is positive but does not necessarily attain 0. The distinguished edge yields a distinguished down-step (resp. level-step) starting from ordinate $n$. Hence all the other steps form a three-step path of length $k-1$ starting at ordinate $n-1$ (resp. $n$) and ending at ordinate $n$. As in the above discussion of $F_n$, each down- or level-step starting at an ordinate $m$ is associated with a rooted or half-mobile with root label or flag $m$ satisfying (R1) (except for the distinguished level-step when decomposing a half-mobile), and the decomposition is bijective. In the end, this translates into the relations

$$
\begin{align*}
R_n &= 1 + \sum_{k=1}^{\infty} g_k R_n Z_{n-1,n}(k-1) = 1 + \sum_{k=1}^{\infty} g_k Z_{n,n-1}(k-1) \\
S_n &= \sum_{k=1}^{\infty} g_k Z_{n,n}(k-1)
\end{align*}
$$

where 1 stands for the mobile reduced to a single vertex labeled $n$, and the $Z$’s are defined as in Sect. 2.1. In the second equation for $R_n$, we used the identity $Z_{n,n-1}(k-1) = R_n Z_{n-1,n}(k-1)$ obtained by reflecting the paths with respect to a vertical axis. Note that the $Z$’s depend implicitly on the sequences $(R_m)_{m \geq 1}$ and $(S_m)_{m \geq 0}$ hence relations (2.4) are of recursive nature. It is easily seen that they uniquely determine all $R_n$’s and $S_n$’s as power series in the $g_k$’s. These equations were already derived in [8] and also in [20] using a different coding of maps (the bipartite case was first discussed in [11] where an explicit solution was guessed).

Let us now briefly discuss the simpler case of mobiles in which condition (R1) is waived. By a simple shift of labels by $-n$, $R_n$ (resp. $S_n$) enumerates rooted mobiles (resp. half-mobiles) with root label (resp. flag) 0 and with their labels all strictly larger than $-n$ and their flags all larger than or equal to $-n$. Sending $n \to \infty$ amounts to waiving this lower bound. This implies that $R_n$ and $S_n$ converge for $n \to \infty$ in the sense of power series, i.e. each of their coefficients stabilizes. We denote by $R$ and $S$ their respective limits, which are nothing but generating functions for respectively
rooted mobiles with root label 0 and half-mobiles with root flag 0. By the discussion at the end of the Sect. 2.2, we find that \(2R + S^2\) is the generating function for pointed rooted maps. From (2.4) we deduce

\[
\begin{align*}
R &= 1 + \sum_{k=1}^{\infty} g_k \sqrt{R} P_{-1}(k - 1; R, S) \\
S &= \sum_{k=1}^{\infty} g_k P(k - 1; R, S)
\end{align*}
\]

with \(P_{-1}\) and \(P\) defined as in Sect. 1. Note that these relations are tantamount to (1.6) by virtue of the identity \(P(k; R, S) = SP(k - 1; R, S) + 2\sqrt{RP_{-1}(k - 1; R, S)}\). Note also that the \(k = 1\) term in the first line of (2.3) vanishes so that we may start the summation at \(k = 2\) for \(R\).

![Diagram](image_url)

**Fig. 4:** The generating function \(F_n^*\) for pointed rooted maps with root degree \(n\) is also that \(P(n; R, S)\) for three-step paths of length \(n\) starting and ending at ordinate 0, with ordinate-independent weights.

We finally consider pointed rooted maps whose root face has a prescribed degree \(n\), and denote by \(F_n^*\) the associated generating function (where the root face does not receive a weight \(g_n\)). \(F_n^*\) only differs from \(F_n\) by the extra marking of a vertex or, said otherwise, we no longer impose that the origin of the map be the starting point of the root edge. Denoting by \(d\) the distance from the origin of the map to the origin of the root edge, we now label each vertex by its distance from the origin of the map minus \(d\). Applying the mobile construction rules of Fig. 1, we obtain a mobile with a distinguished unlabeled vertex corresponding to the root face. Its associated three-step path codes the distances (minus \(d\)) from the origin to the map vertices incident to the root face. The origin of the root edge distinguishes a step starting at ordinate 0 on the path, therefore the path can be seen as a three-step path of length \(n\) which starts and ends at ordinate 0, but which is not necessarily positive. Decomposing around the distinguished unlabeled vertex, each down- or level-step of the path yields a rooted or half-mobile, whose root label or flag we may set to 0 by a suitable shift. This decomposition is
bijection, and translates into the relation (see Fig. 4)

\[ F_n^\bullet = P(n; R, S). \]  \hspace{1cm} (2.6)

This relation is analogous to the relation \( F_n = Z_{0,0}(n) \) established above, yet it is considerably simpler because it involves three-step paths with weights independent from the ordinate. In this sense, it is of the same nature as formula (1.4), which we shall now prove.

3. An explicit expression for the moments

We now turn to the derivation of formula (1.4) for the moments \( F_n \) as defined in Sect. 1. In contrast with the J-fraction expansion (1.1), which follows from a general scheme where planar maps and mobiles turn out to fit, formula (1.4) seems deeply related to the map structure. One of our purposes is to show that it naturally results from several approaches. First, we discuss a direct yet non-constructive check based on the comparison with formula (2.6) (Sect. 3.1). Then, we explain how formula (1.4) arises from the solution of Tutte’s equation, which is the original approach to enumerating maps (Sect. 3.2). Finally we present a bijective proof involving distance-dependent generating functions and mobiles (Sect. 3.3). Beyond our main purpose, this third approach yields a one-parameter family of expressions for \( F_n \), the so-called “conserved quantities”, interesting on their own.

3.1. Proof via pointed rooted maps

It has been noted on several occasions [21] that considering maps that are both pointed and rooted brings some simplifications. This is illustrated by formula (2.6) in our context. A natural idea is to deduce formula (1.4) from it. Because \( F_n^\bullet \) only differs from \( F_n \) by the extra marking of a vertex, there is a direct relation between these two quantities.

To make this relation explicit, it is most convenient to introduce an extra weight \( u \) per vertex. We denote by \( F_n^\bullet(u) \) and \( F_n(u) \) the correspondingly modified generating functions. (Note that \( u \) is actually a redundant parameter, as by a simple counting argument and Euler’s relation we have \( F_n(u) = u^{n/2+1} F_n(1) |_{g_k \to g_k u^{k/2-1}} \) and the same for \( F_n^\bullet \).) On the one hand, relation (2.6) immediately generalizes as

\[ F_n^\bullet(u) = u P(n; R(u), S(u)). \]  \hspace{1cm} (3.1)

where \( R(u) \) and \( S(u) \) enumerate mobiles and half-mobiles with an additional weight \( u \) per labeled vertex, and the extra factor \( u \) accounts for the origin. On the other hand, the extra marking of a vertex yields

\[ F_n^\bullet(u) = u \frac{d}{du} F_n(u). \]  \hspace{1cm} (3.2)
Therefore $F_n(u)$ is a primitive of $P(n; R(u), S(u))$ with respect to the variable $u$. It is shown in Appendix B that it has the explicit form

$$F_n(u) = \sum_{q=0}^{\infty} A_q(u) P^+(n+q; R(u), S(u))$$

(3.3)

with $A_q(u)$ defined as in (1.5) with $R, S$ replaced by $R(u), S(u)$. By specializing to $u = 1$ we recover the expression (1.4) for $F_n = F_n(1)$.

### 3.2. Tutte’s equation and its solution

The generating function $F_n$ for rooted planar maps with root degree $n$ is of utmost importance in Tutte’s original approach [2]. Indeed, the whole family $(F_n)_{n \geq 1}$ is uniquely determined as a power series in the variables $(g_k)_{k \geq 0}$ by the equation

$$F_n = n - 2 \sum_{i=0}^{n-2} F_i F_{n-2-i} + \sum_{k=1}^{\infty} g_k F_{n+k-2}$$

(3.4)

valid for all $n \geq 1$, with the convention $F_0 = 1$. This equation directly expresses that, removing the root edge of a planar map with root degree $n$, two situations may arise: either the map is split into two connected components, which can be seen as two rooted planar maps whose root degrees add up to $n - 2$ (possibly one of these maps is reduced to a single vertex and has root degree 0), or the map is not split, then its root degree is increased by $k - 2$ where $k$ is the degree of the face formerly on the left of the root edge. Passing to $F(z)$ we get Tutte’s equation [2]

$$F(z) = 1 + z^2 F(z)^2 + \sum_{k=1}^{\infty} g_k z^{2-k} \left( F(z) - \sum_{j=0}^{k-2} z^j F_j \right)$$

(3.5)

also known as loop equation in the context of matrix models.

The solution of Tutte’s equation (3.5) was obtained by Bender and Canfield [22] (with slight restrictions on the $g_k$), see also [23] for a more general formulation and [4] for the matrix model counterpart. These authors were ultimately interested in the series $F_2$ which, upon squeezing the bivalent root face, yields the “true” generating function for rooted planar maps. Our purpose here is to show that the general formula (1.4) for $F_n$ follows straightforwardly from their discussion, which we now recall briefly.

We first observe that Eq. (3.5) is quadratic in $F(z)$ (viewing the terms $\sum_{j=0}^{k-2} z^j F_j$ as constants), hence $F(z)$ is readily given by

$$F(z) = \frac{1}{2z^2} \left( 1 - \sum_{k=1}^{\infty} g_k z^{2-k} - \sqrt{\Delta(z)} \right)$$

(3.6)
where \( \Delta(z) = (1 - \sum_{k=1}^{\infty} g_k z^{2-k})^2 - 4z^2 \left(1 - \sum_{k=1}^{\infty} g_k \sum_{j=0}^{k-2} z^{2-k+j} F_j \right) \) is the discriminant. As shown in [22][23], \( \sqrt{\Delta(z)} \) has a factorization of the form

\[
\sqrt{\Delta(z)} = \Gamma(z-1) \sqrt{\kappa(z)}, \quad \Gamma(z-1) = \sum_{q=0}^{\infty} \gamma_q z^{-q}, \quad \kappa(z) = 1 + \kappa_1 z + \kappa_2 z^2
\]  

(3.7)

where the coefficients \( \gamma_q, \kappa_1, \kappa_2 \) are power series in the \( g_k \) to be determined. (In the actual proof, one must for a while assume that degrees are bounded, i.e. that \( g_k \) vanishes for all \( k \) larger than some fixed integer, which from (3.6) implies that \( \sqrt{\Delta(z)} \) does not contain arbitrarily large negative powers of \( z \). In this situation, the existence of a factorization follows from Brown’s theorem [24], and \( \Gamma(z^{-1}) \sqrt{\kappa(z)} \) has a well-defined expansion in \( z \).)

Let us now explain how to compute practically \( \kappa(z) \) and \( \Gamma(z^{-1}) \). Following the approach of [7], we introduce the power series \( R \) and \( S \) defined by

\[
S = -\frac{\kappa_1}{2}, \quad R = \frac{\kappa_1^2 - 4\kappa_2}{16}, \quad \text{i.e.} \quad \kappa(z) = (1 - Sz)^2 - 4Rz^2.
\]  

(3.8)

This choice is particularly convenient because the series expansions of \( \sqrt{\kappa(z)} \) and \( 1/\sqrt{\kappa(z)} \) are related to the generating functions \( P(n; R, S) \) and \( P^+(n; R, S) \) for three-step paths defined in Sect. 1. Namely, it is elementary to check that

\[
\sqrt{\kappa(z)} = 1 - Sz - 2Rz^2 \sum_{n=0}^{\infty} P^+(n; R, S) z^n, \quad \frac{1}{\sqrt{\kappa(z)}} = \sum_{n=0}^{\infty} P(n; R, S) z^n.
\]  

(3.9)

Now, substituting (3.7) in (3.6), extracting the coefficient of \( z^n \) for \( n \geq 0 \) and using the first relation (3.9), we obtain

\[
F_n = R \sum_{q=0}^{\infty} \gamma_q P^+(n + q; R, S)
\]  

(3.10)

which coincides with (1.4) up to the identification \( A_q = \gamma_q R \). It remains to determine the unknowns \( \gamma_q, R, S \). For this it is convenient to rewrite (3.6) as

\[
\sum_{q=0}^{\infty} \gamma_q z^{-q-2} = \frac{z^{-2} - \sum_{k=1}^{\infty} g_k z^{-k} - 2F(z)}{\sqrt{\kappa(z)}}
\]  

(3.11)

and recall that \( F(z) \) only contains nonnegative powers of \( z \), with \( F_0 = 1 \). Using then the second relation (3.9), extracting the coefficient of \( z^{-q-2} \) for \( q \geq 0 \) leads directly to (1.3) while extracting the coefficients of \( z^{-1} \) and \( z^0 \) yields nothing but (1.4). This establishes the general formula for \( F_n \) announced in Sect. 1.
Fig. 5: The generating function $F_{n:d}$ for pointed rooted maps with root degree $n$, where the origin $v$ of the root edge is one of the vertices closest to the origin $v_0$ of the map among those incident to the root face, with a distance $d(v_0, v)$ less than $d$, is also that $Z^+_{d,d}(n)$ for three-step paths of length $n$ starting at, ending at, and restricted to stay above ordinate $d$, with ordinate-dependent weights.

3.3. Bijective derivation

Let us finally present a last derivation of the expression (1.4) for the moments $F_n$. It is based on a bijective decomposition of pointed rooted maps keeping track of some distances between the origin and vertices incident to the root face. Passing to mobiles, this gives rise to a one-parameter family of expressions for $F_n$ (the so-called conserved quantities), interesting on their own, and encompassing (1.4)-(1.5) as a limit case.

We first introduce the generating function $f_{n:i}$ for pointed rooted maps whose root face has degree $n$ and whose origin is both at distance $i$ from the origin of the root edge and at distance $i$ from the boundary of the root face, defined as the set of vertices incident to this root face. In other words, we demand that the origin of the root edge be one of the vertices closest to the origin of the map among all vertices incident to the root face. For $i = 0$, the origin of the map is forced to be the origin of the root edge, so that we have simply $f_{n:0} = F_n$. We also introduce $F_{n:d} = \sum_{i=0}^{d} f_{n:i}$ enumerating maps where the above distance $i$ is less than or equal to some fixed value $d$. Clearly, we have for any $d \geq 0$ the equality

$$F_n = F_{n:d} - \sum_{i=1}^{d} f_{n:i}. \tag{3.12}$$

The two terms in the r.h.s may be evaluated in the mobile language as follows: to evaluate $F_{n:d}$, we look at the sequence of distances from the origin of the vertices around the root face. Starting from the origin of the root edge, we have a three-step path from $(0, i)$ to $(n, i)$ for some $i$ between 0 and $d$, with all its ordinates larger than or equal to $i$. This path defines a sequence of labels and flags for the labeled and flagged vertices around the unlabeled vertex associated with the root face and the full mobile
is obtained by attaching a rooted mobile to each of these labeled vertices and a half-mobile to each of these flagged vertices. The entire mobile must satisfy (R1) and (R2) but, by the same argument as in Sect. 2.3, we may transform it into a mobile satisfying (R1) only by shifting all labels by $d - i$. The attached rooted mobiles (respectively half-mobiles) are now counted by $R_m$ (respectively $S_m$) with indices corresponding to the down- (respectively level-) steps of a shifted three-step path from $(0, d)$ to $(n, d)$ which never dips below $d$. To summarize, $F_{n,d}$ is the generating function of three-step paths from $(0, d)$ to $(n, d)$, staying above $d$, and with a weight $R_m$ per down-step $(t, m) \to (t + 1, m - 1)$ and $S_m$ per level-step $(t, m) \to (t + 1, m)$ (see Fig. 5 for an illustration). With the notations of Sect. 2.1, our first result may be stated as:

$$F_{n,d} = Z_{d,d}(n).$$  \hspace{1cm} (3.13)

\[ \text{Fig. 6: A schematic explanation for the bijection between maps in } f_{n;i} \text{ for } i \geq 1 \text{ and maps in } \mathcal{M}(n; i, k, j) \text{ with arbitrary } k \geq 3 \text{ and } j \geq 1 \text{ (see text).} \]

As for the subtracted term in (3.12), it concerns maps where the origin of the root edge, which we from now on denote by $v$, is at distance $i \geq 1$ from the origin of the map denoted by $v_0$. Then there exist vertices adjacent to $v$ and at distance $i - 1$ from $v_0$, hence edges of the type $(i, i - 1)$ incident to $v$. Let us pick the leftmost such edge (ordering them from the root edge, see Fig. 6) and denote by $v'$ its endpoint. Note that $v'$ cannot lie on the boundary of the root face as, otherwise, the distance from $v_0$ to the boundary would be strictly less than $i$. Therefore, the face on the left of the leftmost edge from $v$ to $v'$ cannot be the root face but is instead a regular face of degree $k \geq 3$ weighted by $g_k$ (the degree cannot be 2 since we picked the leftmost edge from $v$ to $v'$,
nor 1 since it is of type \((i - 1, i)\)). We may now merge this face with the root face into a new external face with larger degree \(n + k\) by simply splitting the vertex \(v\) into two vertices \(v_1\) and \(v_2\) as shown in Fig. 6. Looking again at the sequence of distances along this larger boundary starting from the new vertex \(v\), the first \(n + 1\) vertices correspond to vertices of the former boundary and their distance to the external face is now a path from \((0, 0, k)\) around the unlabeled vertex associated with the external face. Now a path from \((0, 0, k)\) around the unlabeled vertex associated with the external face is now a path from \((0, 0, k)\) around the unlabeled vertex associated with the external face.

Looking again at the sequence of distances along this larger boundary starting from the new vertex \(v\), the first \(n + 1\) vertices correspond to vertices of the former boundary and their distance to \(v_0\) is necessarily larger than or equal to what it was before since the splitting did not create new adjacencies but suppressed some. Moreover, the new vertex \(v\) is still at distance \(i\) from \(v_0\) but \(v\) is necessarily at a distance strictly larger than \(i\). This is again because we chose the leftmost occurrence of an edge \((i, i - 1)\) adjacent to \(v\). The sequence of distances, when going from \(v\) to \(v\) around the new external face, now forms a three-step path from \((0, i)\) to \((n, i + j)\) for some \(j \geq 1\), that never dips below \(i\). The following distances (going from \(v\) to \(v')\) then form a three-step path from \((n, i + j)\) to \((n + k - 1, i - 1)\) with no imposed lower bound (apart from having non-negative ordinates), as displayed in Fig. 6. To summarize, we end up with a pointed rooted map, whose root face has degree \(n + k\) and whose sequence of vertex distances around it defines a three-step path made of three parts: a path from \((0, i)\) to \((n, i + j)\) for some \(j \geq 1\), that never dips below \(i\), a three-step path from \((n, i + j)\) to \((n + k - 1, i - 1)\) that never dips below 0 and a final up-step \((n + k - 1, i - 1)\) to \((n + k, i)\). We call \(M(n; i, k, j)\) the family of such maps.

If conversely, we start from a map in \(M(n; i, k, j)\) for fixed \(n \geq 0\) and \(i \geq 1\) and arbitrary \(k \geq 3\) and \(j \geq 1\), gluing the first vertex \(v\) of the boundary to the \(n\)-th one \(v\) into a single vertex \(v\) will split the external face into a face of degree \(k\) and a new external face of degree \(n\). The new vertex \(v\) will be at distance \(i\) from the origin and all the vertices on the new boundary will be at distance larger than or equal to \(i\). Indeed, the only possible way to reduce the distance from the origin after gluing is to use a path which goes through \(v\) but such a path has a length larger than or equal to \(i\) so the distance cannot be reduced to less than \(i\). Finally, all the vertices adjacent to \(v\) were at a distance larger than or equal to \(i + j - 1 \geq i\) hence their distance after gluing remains \(\geq i\) so that the leftmost edge \((i, i - 1)\) incident to \(v\) after gluing is precisely the edge from \(v\) to the vertex \(v'\) formerly preceding \(v\) along the external face. In other words, the splitting and gluing transformations are inverse of one-another and provide a bijection between maps enumerated by \(f_{n;i}\) for some fixed \(n \geq 0\) and \(i \geq 1\) and maps in \(M(n; i, k, j)\) with arbitrary \(k \geq 3\) and \(j \geq 1\). These latter maps may themselves be transformed into mobiles satisfying (R1) and (R2). As before, the set of all values \(i = 1, \ldots, d\) is well captured in a single family of mobiles satisfying (R1) only upon shifting the labels and flags by \(d - i\). In these mobiles, the three-step path ensuring (P) around the unlabeled vertex associated with the external face is now a path from \((0, 0, k)\) to \((n + k, d)\) made of three parts: a first part of length \(n\) from \((0, 0, k)\) to \((n, d + j)\) for some \(j \geq 1\) that never dips below \(d\), enumerated by \(Z_{d,d+j}^+(n)\), a second part of length \(k - 1\) from \((n, d + j)\) to \((n + k - 1, d - 1)\) that never dips below 0, enumerated by \(Z_{d+j,d-1}(k - 1)\), and a final up step from \((n + k - 1, d - 1)\) to \((n + k, d)\), with weight
1. This is summarized in the equality:

\[ \sum_{i=1}^{d} f_{n;i} = \sum_{k \geq 3} g_k \sum_{j \geq 1} Z_{d,d+j}^+(n)Z_{d+j,d-1}(k - 1). \] (3.14)

From (3.13) and (3.14), we deduce eventually the expression for \( F_n \):

\[ F_n = Z_{d,d}^+(n) - \sum_{k \geq 3} g_k \sum_{j \geq 1} Z_{d,d+j}^+(n)Z_{d+j,d-1}(k - 1) \] (3.15)

valid for any \( d \geq 0 \). An important outcome of this result is that the quantity on the r.h.s of Eq. (3.15) is a *conserved quantity*, i.e. it is independent of \( d \). This conservation property was already proved in [25] in the restricted case of bipartite maps via completely different and much more involved arguments, with no direct combinatorial interpretation. Taking \( d = 0 \), we have in particular the equality

\[ F_n = Z_{d,d}^+(n) \] (3.16)

valid for all \( d \geq 0 \). In particular, by letting \( d \to \infty \) and recalling from Sect. 2.1 that \( Z_{d,d}(n) \to P(n; R, S) \) and \( Z_{d,d}^+(n) \to P^+(n; R, S) \), we arrive at the simplest formula

\[ F_n = P^+(n; R, S)R - \sum_{k \geq 2} g_k \sum_{q \geq 0} P^+(n + q; R, S)R P(k - 2 - q; R, S) \] (3.17)

valid for all \( d \geq 0 \). In particular, by letting \( d \to \infty \) and recalling from Sect. 2.1 that \( Z_{d,d}(n) \to P(n; R, S) \) and \( Z_{d,d}^+(n) \to P^+(n; R, S) \), we arrive at the simplest formula

\[ F_n = P^+(n; R, S)R - \sum_{k \geq 2} g_k \sum_{q \geq 0} P^+(n + q; R, S)R P(k - 2 - q; R, S) \] (3.18)

which is precisely the desired expression (1.4)–(1.5).
The generating function \( \sum_{i=0}^{\infty} f_{n;i} \) for pointed rooted maps with root degree \( n \), where the origin \( v \) of the root edge is one of the vertices closest to the origin \( v_0 \) of the map among those incident to the root face, is also that \( P^+(n; R, S) \) for Motzkin paths of length \( n \) with ordinate-independent weights.

in the first place label the vertices of the maps counted by \( f_{n;i} \) by their distance to the origin minus \( i \) and consider values of \( i \) no longer bounded by \( d \). We end up with mobiles where the conditions (R1) and (R2) are waived and whose sequence of labels and flags around the unlabeled vertex associated with the root face match the down- and level-steps of a three-step path from \((0, 0)\) to \((n, 0)\) that never dips below 0. In the absence of the conditions (R1) and (R2), each down- (respectively level-) step is now weighted by \( R \) (respectively \( S \)) so that we arrive directly at

\[
\sum_{i=0}^{\infty} f_{n;i} = P^+(n; R, S).
\] (3.19)

This formula is illustrated in Fig. 7. If we now consider the case of arbitrary \( i \geq 1 \) and use the above bijection of Fig. 6 with configurations of rooted maps in \( \mathcal{M}(n; i, k, j) \) for arbitrary \( k \geq 3 \) and \( j \geq 1 \), we arrive by again shifting the labels in these maps by \( i \) to mobiles counted directly by

\[
\sum_{i=1}^{\infty} f_{n;i} = \sum_{k \geq 3} g_k \sum_{j \geq 1} \sqrt{R}P^+_{j}(n; R, S)P_{j-1}(k-1; R, S)
\] (3.20)

where, for an arbitrary integer \( m \), \( P_m(q; R, S) \) denotes as before the generating function for arbitrary three-step paths from \((0, 0)\) to \((q, m)\) with a weight \( \sqrt{R} \) per up- or down-step and \( S \) per level-step and, for \( m \geq 0 \), \( P_m^+(n; R, S) \) restricts the enumeration to paths whose ordinates all stay above 0. Using the identity (2.5) for \( R \) we arrive directly at

\[
F_n = P^+(n; R, S)R - \sum_{k \geq 2} g_k \sum_{j \geq 0} \sqrt{R}P^+_{j}(n; R, S)P_{j-1}(k-1; R, S)
\] (3.21)

upon denoting, in the r.h.s of the second line, by \( n + q + 1 \) the abscissa of the first occurrence of the ordinate \(-1\) in the paths enumerated in the r.h.s of the first line.
4. Study of the Hankel determinants

Having established the formula (1.4) for $F_n$, we now study its consequences on the Hankel determinants $H_n = \det_{0 \leq i,j \leq n} F_{i+j}$ and minors $\tilde{H}_n = \det_{0 \leq i,j \leq n} F_{i+j+\delta_{i,n}}$. We first establish formulas (1.7) and (1.8) via a natural path decomposition which translates into matrix identities (Sect. 4.1). We then identify the resulting determinants with symplectic Schur functions (Sect. 4.2).

4.1. Path decomposition

We first establish the formulas (1.7) and (1.8). We start with the identity

$$P^+_{k-l}(q;R,S) - P^+_{k+l+2}(q;R,S)$$

obtained by decomposing the paths of length $i+j+q$ in $P^+(i+j+q;R,S)$ into a first path of length $i$ ending at the ordinate $k$, a second path of length $q$ ending at the ordinate $\ell$ and a third path of length $j$ back to ordinate 0 (see Fig. 8 for an illustration).

The medial part must have non-negative ordinates only, hence we must subtract from $P^+_{k-\ell}(q;R,S)$ the configurations which dip below 0. By a classical reflection argument,
those configurations are in one-to-one correspondence with paths from ordinate \( -k-2 \)
to ordinate \( \ell \), enumerated by \( P_{k+\ell+2}(q; R, S) \).

Using now formula (1.4), we deduce the matrix identity:

\[
\mathbf{H} = \mathbf{T}^T \cdot \mathbf{B} \cdot \mathbf{T}
\]  

(4.2)

where \( \mathbf{H} \), \( \mathbf{T} \) and \( \mathbf{B} \) are the semi-infinite matrices

\[
\mathbf{H} \equiv (F_{i+j})_{i,j \geq 0}, \quad \mathbf{T} \equiv (P_{i}^+(j; R, S))_{i,j \geq 0}, \quad \mathbf{B} \equiv (B_{i-j} - B_{i+j+2})_{i,j \geq 0},
\]

(4.3)

with \( B_i \) defined as in (1.4) and \( \mathbf{T}^T \) denoting the transpose of \( \mathbf{T} \). Note that \( \mathbf{T} \) is upper triangular, hence the matrix product (4.2) involves only a finite sum for each matrix element of \( \mathbf{H} \).

For a semi-infinite matrix \( \mathbf{M} = (M_{i,j})_{i,j \geq 0} \), let us denote by \( \mathbf{M}_n \) its submatrix obtained by keeping only the row and column indices between 0 and \( n \). Note that \( \mathbf{M}_n \) has size \( (n+1) \times (n+1) \). We also denote by \( \mathbf{M}_n^{(i)} \), \( \mathbf{M}_n^{(j)} \) and \( \mathbf{M}_n^{(i,j)} \) the submatrices obtained by further removing respectively the row with index \( i \), the column with index \( j \) and both row \( i \) and column \( j \).

We have, by definition, \( H_n = \det \mathbf{H}_n \). Because \( \mathbf{T} \) is upper triangular, the identity (4.2) restricts safely to

\[
\mathbf{H}_n = \mathbf{T}_n^T \cdot \mathbf{B}_n \cdot \mathbf{T}_n
\]  

(4.4)

which yields

\[
H_n = (\det \mathbf{T}_n)^2 \det \mathbf{B}_n.
\]  

(4.5)

Noting that \( T_{ii} = P_i^+(i) = R^i/2 \), the triangular determinant is readily computed, establishing formula (1.7).

As for \( \tilde{H}_n \equiv \det \mathbf{H}_{n+1}^{(n+1:n)} \), we have the restriction

\[
\mathbf{H}_{n+1}^{(n+1:n)} = \mathbf{T}_n^T \cdot \mathbf{B}_{n+1}^{(n+1:i)} \cdot \mathbf{T}_{n+1}^{(i:n)}
\]  

(4.6)

and the Cauchy-Binet formula yields

\[
\tilde{H}_n = \det \mathbf{T}_n \sum_{i=0}^{n+1} \det \mathbf{B}_{n+1}^{(n+1:i)} \det \mathbf{T}_{n+1}^{(i:n)}.
\]  

(4.7)

Note that \( \mathbf{T}_{n+1}^{(i:n)} \) is still triangular and, unless \( i = n \) or \( n+1 \), it has at least one zero on the diagonal. Therefore only those two terms are non-zero in (1.7) and, as \( T_{n,n+1} = P^+_n(n+1) = (n+1)R^{n/2}S \), the triangular determinants are again readily computed. Finally we have \( \det \mathbf{B}_{n+1}^{(n+1:n+1)} = \det \mathbf{B}_n = H_n/R^{n(n+1)/2} \), and \( \det \mathbf{B}_{n+1}^{(n+1:n)} \) is nothing but the determinant in formula (1.8), which is now established.

To conclude this subsection, let us mention that using the decomposition (1.2) of the Hankel matrix \( \mathbf{H} \) for determinant evaluation amounts to doing elementary row and column manipulations, as stated in Sect. 1.
4.2. Determinants and minors of the matrix $B$ identified as Schur functions

We now turn to the study of the quantities $\det B_n$ and $\det B_{n+1;n}$ encountered above, which are minors of the semi-infinite matrix $B$. This matrix is remarkably the difference between the symmetric Toeplitz matrix $(B_{i-j})_{i,j \geq 0}$ and the Hankel matrix $(B_{i+j+2})_{i,j \geq 0}$, strongly suggesting possible determinantal identities. The Laurent series $\sum_{i=-\infty}^{\infty} B_i x^i$ is known as the symbol of the Toeplitz matrix.

To proceed further with our analysis, we shall assume that the map degrees are bounded, namely we fix a positive integer $p$ and consider maps whose faces have degree at most $p+2$. As already mentioned above, this amounts to setting $g_k = 0$ for $k > p+2$, so that we only deal with power series in the finite set of variables $g_1, \ldots, g_{p+2}$. Then $B_i$ vanishes for $|i| > p$ (and $B_p \neq 0$) so that $B$ is a band matrix. The symbol is a symmetric Laurent polynomial of degree $p$ in $x$, which thus admits $2p$ roots of the form $x = (x_1, 1/x_1, x_2, 1/x_2, \ldots, x_p, 1/x_p)$. These are precisely the solutions of Eq. (1.9) (which by analogy with the “perturbative” approach of [11] we call the characteristic equation).

Up to the overall factor $B_p = -g_{p+2}R^{(p+2)/2}$, we may view $x$ as a parametrization of the $B$’s, namely

$$B_j = (-1)^{p+j} B_p e_{p+j}(x)$$

where $e_i(x)$ is the $i$th elementary symmetric function of $x$, defined for instance via

$$\prod_{i=1}^{p} \left(1 + x_i t \right) \left(1 + \frac{t}{x_i} \right) = \sum_{i=-\infty}^{\infty} e_i(x) t^i.$$  

(4.8)

Note that $e_{p-j}(x) = e_{p+j}(x)$ and $e_i(x) = 0$ for $i > 2p$ or $i < 0$. It follows that $\det B_n$ and $\det B_{n+1;n}$ are symmetric functions of $x$, which we may hopefully identify.

Let us first quickly describe a naive strategy, based on finding the zeroes of the symmetric functions at hand. We observe that, for $x$ a root of the symbol, the semi-infinite vector $((x^{i+1} - x^{-i-1})/(x - x^{-1}))_{i \geq 0}$ is in the kernel of $B$. Its restriction to the indices $\{0, \ldots, n\}$ is however generally not in the kernel of $B_n$ nor $B_{n+1;n}$. We may attempt to find a vector in the kernel of $B_n$ (resp. $B_{n+1;n}$) by taking a linear combination over the $p$ independent roots of the symbol. Doing so, we find that we have to satisfy $p$ “boundary conditions” which are linear equations for the $p$ coefficients of the linear combination. A non-trivial solution exists if the associated $p \times p$ determinant vanishes, hence this determinant (which is again a symmetric function of $x$) divides $\det B_n$ (resp. $\det B_{n+1;n}$). By comparing degrees, we find that they only differ by a constant. In the end, we have expressions for $\det B_n$ and $\det B_{n+1;n}$ in terms of $p \times p$ determinants. It turns out that these are nothing but the determinants present in the Weyl character formula for the symplectic group $Sp_{2p}$, written down below. What we have just encountered are classical formulas in representation theory.

Let us now be more educated and directly identify $\det B_n$ and $\det B_{n+1;n}$ as instances of the general “symplectic e-formula” [17,18]

$$sp_{2p}(\lambda, x) = \det_{1 \leq i, j \leq m} \left( e_{\lambda'_{j-i}}(x) - e_{\lambda'_{j+i}}(x) \right).$$

(4.10)
Comparing with (4.8) and recalling that $e_p(x) = e_{p-1}(x)$ for all $q$, we obtain the general formula

$$
\det_{0 \leq i, j \leq n} (B_{i-j-\mu_j} - B_{i+j+\mu_j+2}) = (-1)^{|\lambda|} B_p^{n+1} \text{sp}_{2p}(\lambda, x)
$$

(4.11)

where $\mu_j = p - \lambda'_j + 1$, $m = n + 1$ and $|\lambda|$ denotes the sum of $\lambda$ (equal to that of $\lambda'$). It is now immediate to identify $\det B_n$ with the case $\mu_j = 0$, hence with the “rectangular” partition $\lambda'_{p,n+1} = p^{n+1}$ made of $n + 1$ parts of size $p$, conjugate to the partition $\lambda_{p,n+1} = (n + 1)^p$ with $p$ parts of size $n + 1$. Similarly $\det B_n^{(n+1,n)}$ corresponds to the case $\mu_j = \delta_{j,n}$, hence to the “nearly-rectangular” partition $\tilde{\lambda}'_{p,n+1} = p^n(p-1)$ made of $n$ parts of size $p$ and one part of size $p-1$, conjugate to the partition $\tilde{\lambda}_{p,n+1} = (n+1)^{p-1}n$ with $p-1$ parts of size $n+1$ and one part of size $n$. We end up with the compact formula for the Hankel determinants

$$
H_n = (-1)^{p(n+1)} B_p^{n+1} R^{\frac{n(n+1)}{2}} \text{sp}_{2p}(\lambda_{p,n+1}, x)
$$

(4.12)

from which, using (1.3), we deduce

$$
R_n = R \frac{\text{sp}_{2p}(\lambda_{p,n+1}, x) \text{sp}_{2p}(\lambda_{p,n-1}, x)}{\text{sp}_{2p}(\lambda_{p,n}, x)^2}.
$$

(4.13)

Similarly, for the Hankel minors we have

$$
\tilde{H}_n - (n + 1) S H_n = (-1)^{p(n+1)+1} B_p^{n+1} R^{\frac{n^2+n+1}{2}} \text{sp}_{2p}(\tilde{\lambda}_{p,n+1}, x)
$$

(4.14)

from which, using (1.3), we deduce

$$
S_n = S - \sqrt{R} \left( \frac{\text{sp}_{2p}(\tilde{\lambda}_{p,n+1}, x)}{\text{sp}_{2p}(\lambda_{p,n+1}, x)} - \frac{\text{sp}_{2p}(\lambda_{p,n}, x)}{\text{sp}_{2p}(\lambda_{p,n}, x)} \right).
$$

(4.15)

Beside the above $e$-formula, other expressions are known for the symplectic Schur function $\text{sp}_{2p}$, namely the “$h$-formula” and the Weyl character formula. Both involve determinants of size $p$, independently of the variable $n$ in $H_n$, $\tilde{H}_n$, $R_n$ and $S_n$. On the one hand, the $h$-formula involves the complete symmetric function $h_i(x)$, defined for instance via

$$
\prod_{i=1}^p \frac{1}{1-x_i t} \frac{1}{1-\frac{t}{x_i}} = \sum_{i=0}^\infty h_i(x) t^i,
$$

(4.16)

and reads [17][18]

$$
\text{sp}_{2p}(\lambda, x) = \det_{1 \leq i, j \leq p} (h_{\lambda_j-j+1}(x); h_{\lambda_j-j+i}(x) + h_{\lambda_j-j-i+2}(x))
$$

(4.17)
where \((a_j; a_{i,j})\) denotes the matrix with elements \(a_j\) in the first row and elements \(a_{i,j}\) in the rows \(i > 1\). It immediately allows to rewrite the r.h.s of (4.12) and (4.14) as determinants of size \(p\). On the other hand, the Weyl character formula reads \cite{17}

\[
\text{sp}_{2p}(\lambda, x) = \frac{\det_{1 \leq i,j \leq p} (x_i^{\lambda_{p+1-j}+j} - x_i^{-\lambda_{p+1-j}-j})}{\det_{1 \leq i,j \leq p} (x_i^j - x_i^{-j})}
\]

(4.18)

and involves the ratio of two determinants of size \(p\). However, the denominators cancel in the expressions (4.13) for \(R_n\) and (4.15) for \(S_n\), which establishes the announced nice formulas (1.11) and (1.12).

5. The special case of maps with even face degrees

As mentioned in Sect. 1, some simplifications occur when we restrict our analysis to bipartite maps. The distance from a given origin in the map changes parity between adjacent vertices and consequently, there are no edges of type \((n, n)\), leading to mobiles without flagged vertices. At the level of generating function, this implies that \(S\) and \(S_n\) for all \(n\) vanish so that the continued fraction (1.11) becomes of the Stieltjes type (1.13), consistent with the property that \(F_n = 0\) for \(n\) odd as the root face must have even degree in a bipartite map. The Hankel determinants factorize as

\[
H_{2n} = h_n^{(0)} h_{n-1}^{(1)}, \quad H_{2n+1} = h_n^{(0)} h_n^{(1)}
\]

(5.1)

where

\[
h_n^{(0)} = \det_{0 \leq i,j \leq n} F_{2i+2j}, \quad h_n^{(1)} = \det_{0 \leq i,j \leq n} F_{2i+2j+2}.
\]

(5.2)

Using the same path decomposition as in Sect. 4.1, we may write

\[
h_n^{(0)} = R^{n(n+1)} \det_{0 \leq i,j \leq n} (\hat{B}_{i-j} - \hat{B}_{i+j+1})
\]

\[
h_n^{(1)} = R^{(n+1)^2} \det_{0 \leq i,j \leq n} (\hat{B}_{i-j} - \hat{B}_{i+j+2}).
\]

(5.3)

where we introduce the notation

\[
\hat{B}_i = B_{2i}
\]

(5.4)

for all integers \(i\) (note that \(B_{2i+1} = 0\) and \(\hat{B}_{-i} = \hat{B}_i\)). Let us now turn to the case of maps with maximal degree \(2p + 2\), for some \(p \geq 1\), so that \(\hat{B}_i\) vanishes for \(|i| > p\). Writing the characteristic equation as

\[
\sum_{i=-p}^{p} \hat{B}_i y^i = 0, \quad y \equiv x^2,
\]

(5.5)
whose \(2p\) roots are gathered in the \(2p\)-uple \(y \equiv (y_1, 1/y_1, \ldots, y_p, 1/y_p)\), we now have the identification

\[
\hat{B}_i = (-1)^{p+i} B_p e_{p+j}(y)
\]  

(5.6)

in terms of the elementary symmetric function \(e_j(y)\). This yields

\[
h_n^{(0)} = (-1)^{p(n+1)} (\hat{B}_p)^{n+1} R^{(n+1)} \det_{1 \leq i, j \leq n+1} \left( e_{p-j+i}(y) + e_{p-i-j+1}(y) \right)
\]

\[
= (-1)^{p(n+1)} (\hat{B}_p)^{n+1} R^{(n+1)} \sigma_{2p+1}(\lambda_{p,n+1}, y)
\]

\[
h_n^{(1)} = (-1)^{p(n+1)} (\hat{B}_p)^{n+1} R^{(n+1)} \det_{1 \leq i, j \leq n+1} \left( e_{p-j+i}(y) - e_{p-i-j}(y) \right)
\]

\[
= (-1)^{p(n+1)} (\hat{B}_p)^{n+1} R^{(n+1)} \sigma_{2p}(\lambda_{p,n+1}, y)
\]  

(5.7)

where, as before, \(\lambda_{p,n+1} = (n+1)^p\) is the partition with \(p\) parts of size \(n+1\). Here we used again \((4.10)\) to identify the second determinant with a symplectic Schur function of \(y\), while the first determinant is now recognized as an instance of the general formula

\[
\sigma_{2p+1}(\lambda, y) = \det_{1 \leq i,j \leq m} \left( e_{\lambda'_j-j+1}(\bar{y}) ; e_{\lambda'_j-j+i}(\bar{y}) + e_{\lambda'_j-j-i+2}(\bar{y}) \right) = \det_{1 \leq i,j \leq m} \left( e_{\lambda'_j-j+i}(y) + e_{\lambda'_j-j-i+1}(y) \right)
\]  

(5.8)

where \(\sigma_{2p+1}(\lambda, y)\) stands for the odd-orthogonal Schur function associated with the partition \(\lambda\) (having at most \(p\) parts), \(\lambda'\) denoting its conjugate partition (having at most \(m\) parts), and where \(\bar{y}\) denotes the \((2p+1)\)-uple \((y_1, 1/y_1, \ldots, y_p, 1/y_p, 1)\) with an additional 1 term. The first identity may be found in \([18]\) and it implies the second one by elementary manipulations on the determinant. Again, the odd-orthogonal Schur function admits a simple Weyl formula, namely:

\[
\sigma_{2p+1}(\lambda, y) = \frac{\det_{1 \leq i,j \leq p} \left( y_i^{\lambda_{p+1-j+1}+j-\frac{1}{2}} - y_i^{-\lambda_{p+1-j-j+1}+\frac{1}{2}} \right)}{\det_{1 \leq i,j \leq p} \left( y_i^{-\frac{1}{2}} - y_i^{-j+\frac{3}{2}} \right)}
\]  

(5.9)

Alternatively, the factorization \((5.1)\) of \(H_n\), with \(h_n^{(0)}\) and \(h_n^{(1)}\) expressed directly as in \((5.7)\), may be obtained right away from the general formula \((4.12)\) (with \(p\) replaced by \(2p\)) for \(H_n\) upon using the identity

\[
\sigma_{4p}(\lambda_{2p,n+1}; x \cup -x) = (-1)^{p(n+1)} \sigma_{2p}(\lambda_{p,\left\lfloor \frac{n+1}{2} \right\rfloor}, y) \sigma_{2p+1}(\lambda_{p,\left\lfloor \frac{n+2}{2} \right\rfloor}, y)
\]  

(5.10)

where \(x \cup -x = (x_1, 1/x_1, \ldots, x_p, 1/x_p, -x_1, 1/x_1, \ldots, -x_p, 1/x_p)\) and where \(y = (y_1, 1/y_1, \ldots, y_p, 1/y_p)\) with \(y_i = (x_i)^2\). This identity is easily proved via elementary determinant manipulations in the Weyl formula for each of these Schur functions.
The final “nice” formula for \( R_n \) becomes

\[
R_n = R \frac{\det_{1 \leq i, j \leq p} \left( y_i^{\frac{p}{2} + j + \frac{1}{2}} - y_i^{\frac{-p}{2} - j + \frac{1}{2}} \right) \det_{1 \leq i, j \leq p} \left( y_i^{\frac{p}{2} + j - 1} - y_i^{\frac{-p}{2} - j + 1} \right)}{\det_{1 \leq i, j \leq p} \left( y_i^{\frac{p}{2} + j} - y_i^{\frac{-p}{2} - j} \right) \det_{1 \leq i, j \leq p} \left( y_i^{\frac{p}{2} + j - \frac{1}{2}} - y_i^{\frac{-p}{2} - j + \frac{1}{2}} \right)}
\]  

(5.11)

which matches formula (5.8) of Ref. [11], given there without a complete proof. To make the identification complete, we have to rewrite the characteristic equation (1.10) as

\[
1 = \sum_{k=1}^{p+1} g_{2k} \sum_{q=0}^{k-1} P(2k - 2 - 2q; R, 0) \left( \sqrt{R} x + \frac{\sqrt{R}}{x} \right)^{2q}
\]

\[
= \sum_{k=1}^{p+1} g_{2k} R^{k-1} \sum_{q=0}^{k-1} \binom{2k - 2 - 2q}{k - q - 1} \left( x + \frac{1}{x} \right)^{2q}
\]

(5.12)

\[
= \sum_{k=1}^{p+1} g_{2k} R^{k-1} \sum_{m=0}^{k-1} \binom{2k - 1}{k - m - 1} \sum_{j=-m}^{m} y^j,
\]

with \( y = x^2 \). This is precisely the form found in [11]. The third step in (5.12) follows from the identity

\[
\sum_{q=|j|}^{k-1} \binom{2k - 2q - 2}{k - q - 1} \binom{2q}{q + j} = \sum_{m=|j|}^{k-1} \binom{2k - 1}{k - m - 1}
\]  

(5.13)

which itself follows from the more obvious identity

\[
\sum_{q=|j|}^{k-1} \binom{2k - 2q - 2}{k - q - 1} \left[ \binom{2q}{q + |j|} - \binom{2q}{q + |j| + 1} \right] = \binom{2k - 1}{k - |j| - 1}
\]  

(5.14)

obtained for instance by enumerating paths with \( \pm 1 \) steps of length \( 2k - 1 \) starting at height 0 and ending at height \( 2|j| + 1 \) in two manners: either directly (r.h.s) or by decomposing at the last passage at height 0 (l.h.s).

6. Triangulations and quadrangulations

We now turn to the particularly simple cases of triangulations and quadrangulations, where the expressions (1.11), (1.12) (for triangulations) and (5.11) (for quadrangulations) involve only determinants of size 1. They admit an elementary combinatorial interpretation in terms of one-dimensional hard dimers, as we shall see.
6.1. Triangulations

The case of triangulations, characterized by $g_k = g_3 \delta_{k,3}$ is particularly simple. In this case, the only non-vanishing $A_q$ coefficients are $A_0$ and $A_1$, given from (1.3) by

$$A_0 = R - g_3 R S, \quad A_1 = -g_3 R$$  \hspace{1cm} (6.1)

with $R$ and $S$ given implicitly from (1.6) by

$$S = g_3 (S^2 + 2R), \quad R = 1 + \frac{g_3}{2} (S^3 + 6g_3 RS) - \frac{S^2}{2} = 1 + 2g_3 R S.$$  \hspace{1cm} (6.2)

From (1.7), the non-vanishing $B_i$ coefficients are

$$B_0 = A_0 + A_1 S = 1 - 2g_3 R S = 1$$

$$B_1 = B_{-1} = A_1 \sqrt{R} = -g_3 R^{3/2}$$  \hspace{1cm} (6.3)
while the characteristic equation (1.9) reads
\[ 1 - g_3 R^{3/2} \left( x + \frac{1}{x} \right) = 0 . \] (6.4)

Formulas (1.11) and (1.12) for the two-point functions reduce to
\[
R_n = R \frac{(x^{n+2} - x^{-n-2})(x^n - x^{-n})}{(x^{n+1} - x^{-n-1})^2},
\]
\[
S_n = S - \sqrt{R} \frac{x^{n+1} - x^{-n-1} - x^n - x^{-n}}{x^{n+2} - x^{-n-2} - x^{n+1} - x^{-n-1}}.
\] (6.5)

To recover the expressions given in [20], we set
\[ y = x^2, \] (6.6)
which is solution of
\[ y + \frac{1}{y} + 2 = \frac{1}{g_3^2 R^3}, \] (6.7)
and obtain eventually
\[
R_n = R \frac{(1 - y^n)(1 - y^{n+2})}{(1 - y^{n+1})^2},
\]
\[
S_n = S - g_3 R^2 y^n \frac{(1 - y)(1 - y^2)}{(1 - y^{n+1})(1 - y^{n+2})}.
\] (6.8)

Interestingly, these formulas admit a combinatorial interpretation in terms of one-dimensional hard-dimers. Indeed, for triangulations, formula (1.4) for the generating function \( F_n \) reduces to
\[
F_n = A_0 P^+(n; R, S) + A_1 P^+(n + 1; R, S)
\] (6.9)
so that \( F_{i+j} \) may be interpreted as enumerating three-step paths with non-negative ordinates from, say, \((-i,0)\) to \((j + 1,0)\) with a weight \( \sqrt{R} \) per up- or down-step and \( S \) per level-step, except for steps in a central strip of width 1 (i.e. the strip between abscissas 0 and 1) which receive instead weights incorporating the \( A_q \) factors, namely \( A_1 \times \sqrt{R} = -g_3 R^{3/2} \) for the up- and down-steps, and \( A_0 \times 1 + A_1 \times S = R - 2g_3 RS = 1 \) for the level-steps (see Fig. 9). Note that these weights are nothing but \( B_1 \) and \( B_0 \) respectively, as might be seen by specializing the discussion of Sect. 4.1. Alternatively, the paths may be viewed as oriented paths on a graph drawn from a square grid in the upper-half plane by keeping the horizontal sides and the diagonals of the squares, with the two diagonals in each square viewed as passing on top of each other with no vertex at their crossing point in the plane. The graph is implicitly endowed with some orientation from left to right so that the paths always have strictly increasing abscissas. On such
an oriented graph, we may apply the so-called LGV (Lindström-Gessel-Viennot) lemma [26], which states that
\[ H_n = \det_{0 \leq i, j \leq n} F_{i+j} \]
enumerates configurations made of \( n + 1 \) paths connecting the set of points \((I_j)_{j=0,\ldots,n}\) with coordinates \((-j,0)\) to the set of points \((O_j)_{j=0,\ldots,n}\) with coordinates \((j+1,0)\) with no intersections at vertices of the graph. In addition to the above weights for up-, down- and level-steps, each configuration receives a \( \pm 1 \) factor equal to the signature of the permutation \( \sigma \) of \( \{0, \ldots, n\} \) characterizing its connections (namely \( I_j \) is connected to \( O_{\sigma(j)} \)). Now it is easy to see that, for such configurations of non-intersecting paths, the part of the path outside the central strip is entirely fixed to be a set of straight lines going up on the left of the strip and straight lines going down on its right, contributing an overall factor \( R_n^{n(n+1)/2} \) (see Fig. 10). The only freedom comes from the possibility of crossings (in the plane) along the two diagonals of a given square in the strip (since these diagonals do not intersect on the graph). Moreover, two such crossings cannot take place on two neighboring squares in the strip. An acceptable crossing configuration may therefore be viewed as a hard dimer configuration on the segment \([0,n]\), as shown in Fig. 10. Each dimer receives a weight \( W = -(g_3 R^{3/2})^2 = -g_3^2 R^3 \) with a minus sign for each crossing, equal to the signature of a transposition component in the permutation \( \sigma \). In conclusion, the Hankel determinant \( H_n \) is simply given by
\[ H_n = R^{n(n+1)/2} Z_{\text{hard dimers on } [0,n]} \] (6.10)
where \( Z_{\text{hard dimers on } [0,n]} \) is the generating function for hard dimers on the segment \([0,n]\) with weight \( W \) per dimer. Now it is a classical result that
\[ Z_{\text{hard dimers on } [0,n]} = \frac{1}{(1+y)^{n+1}} \frac{1 - y^{n+2}}{1 - y} \] (6.11)
with the parametrization
\[ W = - \frac{1}{y + \frac{1}{y} + 2} \] (6.12)
Since \( W = -g_3^2 R^3 \), \( y \) is precisely the solution of (6.7).

As for the Hankel minor \( \tilde{H}_n \), it is easily identified in the hard dimer language as (see Fig. 11 for an illustration)
\[ \tilde{H}_n = (n+1) S R^{n(n+1)/2} Z_{\text{hard dimers on } [0,n]} - g_3 R^{3/2} R^{n^2+n+1} Z_{\text{hard dimers on } [0,n-1]} \] (6.13)
Expressions (6.8) follow immediately via (1.2) and (1.3).

6.2. Quadrangulations

Let us now come to the case of quadrangulations, characterized by \( g_k = g_4 \delta_{k,4} \). We have \( S = 0 \) in this case so that only paths of even length contribute to \( P \) or \( P^+ \). The only non-vanishing \( A_q \) coefficients are \( A_0 \) and \( A_2 \), from (1.5), given by
\[ A_0 = R - 2g_4 R^2 = 1 + g_4 R^2, \quad A_2 = -g_4 R \] (6.14)
with $R$ now given, from (1.6), by

$$R = 1 + 3g_4 R^2.$$  (6.15)

As quadrangulations belong to the class of maps with even face degrees, we use the results of Sect. 5. The non-vanishing $\hat{B}_i$'s are

$$\hat{B}_0 = 1 - g_4 R^2, \quad \hat{B}_1 = \hat{B}_{-1} = -g_4 R^2$$  (6.16)

so that the characteristic equation reads

$$(1 - g_4 R^2) - g_4 R^2 \left(y + \frac{1}{y}\right) = 0.$$  (6.17)

By (5.11), the two-point function reads

$$R_n = R \frac{(1 - y^n)(1 - y^{n+3})}{(1 - y^{n+1})(1 - y^{n+2})},$$  (6.18)
Fig. 12: Interpretation of formula (6.20) for $F_{2i+2j}$ in the case of quadrangulations. The weights per step are different (and allow for a level-step) in the central strip as shown.

Fig. 13: Correspondence between configurations of non-intersecting paths on the graph of Fig. 12 and hard dimers on a segment of length $2n + 1$.

which matches Eq. (4.10) of [11] (up to the change of notation $y \to x$ and $n \to n + 1$).

Again, we may give a combinatorial hard-dimer interpretation to these results as follows. The generating function $F_n$ vanishes for $n$ odd and we have

$$F_{2n} = A_0 P^+(2n; R, 0) + A_2 P^+(2n + 2; R, 0) = A_0 R^n \text{cat}_n + A_2 R^{n+1} \text{cat}_{n+1}$$

where $\text{cat}_n = \binom{2n}{n}/(n + 1)$ are the Catalan numbers. In particular $F_{2i+2j}$ may be interpreted as enumerating paths from, say $(-2i, 0)$ to $(2j + 2, 0)$ on the graph made of the restriction in the upper half-plane made of a tilted square grid, completed by horizontal segments in the central strip between abscissas 0 and 2 (see Fig. 12). Again the graph is implicitly oriented from left to right so that paths have increasing abscissas.
Each up- or down-step of the path receives a factor $\sqrt{R}$ except those in the central strip which receive instead a weight $\sqrt{R} \sqrt{A} = i \sqrt{g R}$, while the horizontal paths receive a weight $A_0 = 1 + g_4 R^2$. Using again the LGV lemma [26], the quantity $h_n^{(0)} = \det_{0 \leq i,j \leq n} F_{2i+2j}$ enumerates sets of $n+1$ non-intersecting paths from points $(I_j)_{j=0,\ldots,n}$ with coordinates $(-2j, 0)$ to points $(O_j)_{j=0,\ldots,n}$ with coordinates $(2j + 2, 0)$. Again these paths have fixed ascending and descending parts on both sides of the central strip, contributing an overall factor $R^{n(n+1)}$ and the only freedom comes from the central strip where the $j$-th path (numbered 0 to $n$ from bottom to top) connects $(0, 2j)$ to $(2j, 2)$ either via a horizontal step with weight $1 + g_4 R^2$, or by a two-step sequence passing either by $(1, 2j - 1)$ (if $j \geq 1$) or by $(1, 2j - 1)$, with a total weight $(i \sqrt{g R})^2 = g_4 R^2$. Since the paths are non-intersecting, these (up or down) two-step sequences cannot be adjacent in the strip and their vertical positions define a hard dimer configuration in $[0, 2n + 1]$ (see Fig. 13). Extracting an overall factor $(1 + g_4 R^2)^{n+1}$, each dimer receives a weight

$$W = -\frac{g_4 R^2}{1 + g_4 R^2}.$$  

(6.20)

so that we have

$$h_n^{(0)} = R^{n(n+1)} (1 + g_4 R^2)^{n+1} Z_{\text{hard dimers on } [0, 2n+1]}.$$  

(6.21)

![Fig. 14: Interpretation of formula (6.20) for $F_{2i+2j+2}$ in the case of quadrangulations. Note that the central strip is shifted by one unit with respect to Fig. 12.](image)

As for $h_n^{(1)} = \det_{0 \leq i,j \leq n} F_{2i+2j+2}$, it enumerates sets of $n+1$ non-intersecting paths from points $(I_j)_{j=0,\ldots,n}$ with coordinates $(-2j, 0)$ to points $(O_j)_{j=0,\ldots,n}$ with coordinates $(2j + 4, 0)$ with now a strip between abscissas 1 and 3 (see Fig. 14), easily transformed into a hard dimer configuration in $[0, 2n + 2]$ (see Fig. 15). We deduce from this picture the formula

$$h_n^{(1)} = R^{(n+1)^2} (1 + g_4 R^2)^{n+1} Z_{\text{hard dimers on } [0, 2n+2]}.$$  

(6.22)
Using the explicit form (6.11), this leads eventually to

\[
R_{2n+1} = \frac{h_{n-1}^{(1)}}{h_{n-1}^{(0)}} = R \frac{(1 - y^{2n+4})(1 - y^{2n+1})}{(1 - y^{2n+2})(1 - y^{2n+3})}
\]

\[
R_{2n} = \frac{h_{n-2}^{(0)}}{h_{n-2}^{(1)}} = R \frac{(1 - y^{2n+3})(1 - y^{2n})}{(1 - y^{2n+1})(1 - y^{2n+2})}
\]

where \(y\) is precisely the solution of (6.17). The above equations are summarized into (6.18).

7. Conclusion and discussion

To conclude, we discuss further connections of our results raising some open questions. Sect. 7.1 is devoted to the manifestations in our context of the intimate relation between continued fractions and orthogonal polynomials. Sect. 7.2 makes the connection with the matrix integral approach.

7.1. Convergents and orthogonal polynomials

There is a deep connection between continued fractions and orthogonal polynomials. Let us indeed consider the family of polynomials \((q_n(z))_{n \geq 0}\) defined by

\[
q_n(z) = \frac{1}{H_{n-1}} \det_{0 \leq i,j \leq n} \left( F_{i+j} ; z^j \right)
\]

where \((a_{i,j} ; a_j)\) denotes the matrix with elements \(a_j\) in the last row and elements \(a_{i,j}\) in all previous rows. It is a classical exercise to check that \(q_n(z)\) is a monic polynomial.
of degree \( n \), that the family is orthogonal with respect to the scalar product defined by
\[
\langle z^n, z^m \rangle = F_{n+m}, \quad \text{with} \quad \langle q_n(z), q_m(z) \rangle = \delta_{n,m} H_n/H_{n-1},
\]
and that it satisfies Favard’s three-term recurrence
\[
z q_n(z) = q_{n+1}(z) + S_n q_n(z) + R_n q_{n-1}(z) \quad (7.2)
\]
with the initial data \( q_0(z) = 1, \quad q_{-1}(z) = 0 \). The connection with continued fractions comes from the fact that the reciprocal of \( q_n(z) \) appears as the denominator of the \( n \)th convergent of the J-fraction (1.1), namely
\[
\tilde{p}_n(z) \quad \text{and} \quad \tilde{q}_n(z) \quad \text{also appear within an expression for the} \quad \text{n}th \quad \text{truncation} \quad (2.2), \quad \text{see for instance} \quad [14,15].
\]
In the context of maps, the \( d \)th convergent of (1.1) may be interpreted as the generating functions for rooted maps where every vertex incident to the root face is at a distance lesser than or equal to \( d \) from the origin of the root edge. As for the \( d \)th truncation (2.2), its interpretation is given in Sect. 3.3 via (3.13) as a generating function for pointed rooted maps with a control both on the root degree and on the distance from the origin to the root face. Truncations are known to admit a simple expression in the case of quadrangulations [27], namely
\[
\frac{1}{1 - \frac{R_{d+1} z^2}{1 - \ldots}} = \tilde{W}_1 - \frac{1}{\tilde{W}_1 - 1} \frac{1 - y^{d+1}}{1 - y^{d+1} - y^{d+2}} \quad (7.4)
\]
with \( R \) and \( y \) as in Sect. 6.2 and \( \tilde{W} = (1 - \sqrt{1 - 4Rz^2})/(2Rz^2) \). We may wonder if this expression generalizes to arbitrary \( g_k \)’s. We provide in Appendix C a general expression for the orthogonal polynomials \( q_n(z) \). Unfortunately we lack an expression for the \( \tilde{p}_n(z) \), as they do not admit a formula similar to (7.1).

### 7.2. Connection with matrix integrals

Another classical and fruitful approach to map enumeration problems is via matrix integrals. Let us now comment informally on its connection with our present results.

Matrix integrals give a simple expression for the all-genus generating function
\[
F_n[N] \equiv \sum_{h \geq 0} N^{-2h} F_n^{(h)} \quad (7.5)
\]
where $F_n^{(h)}$ denotes the generating function for rooted maps of genus $h$ with a root face of degree $n$, with face weights $(g_k)_{k \geq 1}$ as before. In particular, we have $F_n = F_n^{(0)} = \lim_{N \to \infty} F_n[N]$. We have the matrix integral representation

$$F_n[N] = \frac{\int dM \, \text{Tr} (M^n) \exp(-N \text{Tr} V(M))}{N \int dM \, \exp(-N \text{Tr} V(M))}$$

(7.6)

where $dM$ denotes the Lebesgue (translation-invariant) measure over the space of $N \times N$ hermitian matrices and

$$V(x) \equiv x^2 - \sum_{k \geq 1} g_k x^k.$$  

(7.7)

The terms $g_k x^k$ act as a “perturbation” of the quadratic potential $x^2/2$ corresponding to the well-known Gaussian Unitary Ensemble. In all rigor, expression (7.6) must be understood as a power series in the $g_k$’s and $N$, whose precise definition is beyond the scope of this section.

We now briefly discuss the usual approaches for studying (7.6), beside the loop equations already mentioned in Sect. 3.2. The original “physical” approach is the so-called saddle-point or steepest descent method [5]. It consists in remarking that the integrands in (7.6) only depend on the eigenvalues of $M$, and observing that for large $N$ the dominant contribution comes the “equilibrium” continuous distribution of eigenvalues. It yields

$$F_n = \int d\lambda \, \lambda^n \rho(\lambda)$$

(7.8)

where $\rho(\lambda)$ denotes the density of eigenvalues. Therefore, in the saddle-point picture our moments $F_n$ are precisely those associated with the spectral measure.

This has to be contrasted with the usual method of orthogonal polynomials in random matrix theory [3]. There, we consider the family of polynomials $(q_i^{(N)}(\lambda))_{i \geq 0}$, with $q_i^{(N)}(\lambda)$ monic of degree $i$, orthogonal with respect to the scalar product defined by $(\lambda^n, \lambda^m) = \int d\lambda \lambda^{n+m} \exp(-N V(\lambda))$. In other words, these orthogonal polynomials are defined with respect to the “$N = 1$” eigenvalue density while those of (7.1) are defined with respect to the “$N \to \infty$” eigenvalue density. Favard’s theorem states that we still have a three-term recurrence

$$\lambda q_i^{(N)}(\lambda) = q_{i+1}^{(N)}(\lambda) + S_i^{(N)} q_i^{(N)}(\lambda) + R_i^{(N)} q_{i-1}^{(N)}(\lambda)$$

(7.9)

where $(S_i^{(N)})_{i \geq 0}$ and $(R_i^{(N)})_{i \geq 1}$ (with $R_0^{(N)} = 0$) are associated with the scalar product at hand, for instance they may be expressed via Hankel determinants of the moments of the measure $d\lambda \exp(-N V(\lambda))$. Now, because of the specific form of the scalar product $(\cdot, \cdot)$, we also have the relations [3]

$$R_i^{(N)} = \frac{i}{N} + \sum_{k \geq 2} g_k Z_{i,i-1}^{(N)} (k-1)$$

$$S_i^{(N)} = \sum_{k \geq 1} g_k Z_{i,i}^{(N)} (k-1).$$

(7.10)
Here $Z_{{i,j}^{(N)}}(k)$ denotes the generating function of three-step paths from $(0, i)$ to $(k, j)$ with a weight $R_m^{(N)}$ per down-step $(t, m) \to (t + 1, m - 1)$ and a weight $S_m^{(N)}$ per level-step $(t, m) \to (t + 1, m)$. Remarkably enough, these equations look very similar to (2.4), and allow to identify $R_i^{(N)}$ and $S_i^{(N)}$ with generating functions for mobiles (respectively half-mobiles) as defined in this paper with, however, a somewhat mysterious and non-conventional weight $(m/N)$ for labeled vertices with label $m$. In terms of orthogonal polynomials, $F_{n}[N]$ is given by

$$ F_{n}[N] = \frac{1}{N} \sum_{i=0}^{N-1} \left[ \frac{\lambda^n q^{(N)}_{i}(\lambda)}{q^{(N)}_{i}(\lambda)} \right] = \frac{1}{N} \sum_{i=0}^{N-1} Z_{{i,i}\lambda}^{(N)}(n). \quad (7.11) $$

We may further connect the relations (7.10) to known mobile generating functions by considering the limit $N \to \infty$. In this limit, the quantity $u = i/N$ is treated as a continuous variable, and $R_i^{(N)}$ and $S_n^{(N)}$ become at leading order in $1/N$ smooth functions $R(u)$ and $S(u)$. Then, in this limit (7.10) yields

$$ R(u) = u + \sum_{k \geq 2} g_k \sqrt{R(u)P_{-1}(k-1; R(u), S(u))} \quad (7.12) $$

$$ S(u) = \sum_{k \geq 1} g_k P(k-1; R(u), S(u)) $$

involving paths which receive homogeneous weights $R(u)$ and $S(u)$, irrespectively of the ordinates. Setting $u = 1$, we recover precisely (2.5). For general $u$, $R(u)$ and $S(u)$ are generating functions for mobiles and half-mobiles with an extra weight $u$ per labeled vertex, as encountered in Sect. 3.1 and Appendix B. In the limit $N \to \infty$, the sum in (7.11) becomes an integral, leading to the expression

$$ F_{n} = \int_{0}^{1} du P(n; R(u), S(u)) \quad (7.13) $$

which is consistent with the discussion of Sect. 3.1.

For finite $N$, a natural question is whether the unconventional weight of labeled vertices, as well as the form (7.11) of the all-genus generating function may be given a direct combinatorial interpretation. This could open the way to get explicit formulas for discrete distance-dependent two-point functions in maps of higher genus.

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Fig. 16: The decomposition (a) of a pointed rooted map into slices by cutting it along all leftmost geodesic paths emerging from vertices incident to the root face. Slices are of two types: those (b) with a basal edge of type \((m, m)\), counted by \(S_m\), and those (c) with a basal edge of type \((m, m-1)\), counted by \(R_m\). Note that edges of type \((m-1, m)\) delimit empty slices. The actual depth of a slice may be less than \(m\) since its two boundaries may merge before the origin \((s_i\) and \(r_i\) counting the slices with depth \(i\)).

Appendix A. Decomposition of maps into slices

So far we used mobiles as the natural framework to introduce continued fractions in map enumeration problems. Still, the explicit recourse to mobiles is not \textit{stricto sensu} necessary to prove the basic continued fraction expansion (1.1) and more generally the interpretation (3.13) of truncations (2.2), when the weights \(R_m\) and \(S_m\) have their original interpretation of Sect. 1 as map generating functions. This relation can be understood alternatively as the result of some appropriate decomposition of maps which can be defined without reference to the underlying mobile structure even though, in practice, the two are intimately related.

We start again with pointed rooted maps enumerated by \(f_{n,d}\), i.e. maps with a root face of length \(n\), with the origin of the root edge at distance \(d\) from the origin of the map and with all the other vertices incident to the root face at a distance larger than or equal to \(d\). As before, we choose the root face for external face and the clockwise sequence of distances of its incident vertices defines a three-step path of length \(n\) with endpoints at ordinate \(d\), and which stays above \(d\). For each vertex incident to the root face, we draw the \textit{leftmost geodesic} path from this vertex to the origin of the map.
Cutting along these paths decomposes the map into a number of connected domains that we call slices (see Fig. 16). These slices are maps whose boundary is made of one basal edge originally incident to the root face and of two leftmost geodesic paths merging at some apex (possibly different from the former origin of the map). Note that the leftmost geodesics may themselves contain edges originally incident to the root face. The basal edge of a non-empty slice is an edge originally of type \((m, m)\) or of type \((m, m - 1)\) (counterclockwise). Indeed, the edges of type \((m - 1, m)\) give rise to empty slices as they lie on the leftmost geodesic path emerging from their endpoint at distance \(m\). Non-empty slices are therefore associated only with the down- and level-steps of the three-step path above. Since two consecutive leftmost geodesics may merge before reaching the origin of the map, the actual length of the leftmost geodesics is reduced in the slice by some value \(m - i\) corresponding to the length of their common part. This leads us to associate a weight \(S_m = \sum_{i=0}^{m} s_i\) (respectively \(R_m = \sum_{i=1}^{m} r_i\)) to each level- (respectively down-) step starting at ordinate \(m\) in the three-step path, where \(s_i\) (respectively \(r_i\)) are the generating functions for slices of depth \(i\), i.e. with a boundary made of a basal edge and two leftmost geodesics of the same length \(i\) (respectively of length \(i - 1\) and \(i\)) (see Fig. 16). Here again, we have to add to \(r_1\), hence to \(R_m\), a conventional factor 1 which accounts for the case where an edge of type \((m, m - 1)\) incident to the root face would be the boundary of an empty slice, which happens when its endpoint at distance \(m - 1\) is the only vertex at distance \(m - 1\) from the origin adjacent to its endpoint at distance \(m\). On the contrary, each edge of type \((m, m)\) incident to the root face is the basal edge of a non-empty slice. We therefore recover precisely the generating function \(Z_{d,d}^+(n)\) defined in Sect. 2.1, which, from the above decomposition procedure, enumerates all maps in \(f_{n;d}\) as well as all maps in \(f_{n;i}\) for \(i < d\), since the concatenation of slices produces maps whose origin may remain at a distance less than \(d\). We end up with the desired relation \(F_{n;d} = Z_{d,d}^+(n)\), but now with a different interpretation for the weights \(S_m\) and \(R_m\) as slice generating functions. It remains to show that these new definitions match their former definition of Sect. 1.2 in terms of map generating functions.

It is easily seen (Fig. 17) that, starting from a slice counted by \(r_i\), completing the leftmost geodesic of length \(i - 1\) by the basal edge (of type \((i - 1, i)\)) and gluing it with the other leftmost geodesic of length \(i\) produces bijectively a pointed map with a marked edge of type \((i, i - 1)\) with respect to the origin of the map. This marked edge is promoted as root edge. Note that, after gluing, we are apparently left with a marked geodesic but it is by construction the leftmost geodesic starting with the root edge (oriented from \(i\) to \(i - 1\)) and ending at the origin. It is therefore uniquely determined and may be erased without loss of information. This bijection guarantees that our new definition of \(r_i\) (and \(R_m\)) matches the definition of Sect. 1. As for \(S_n\), we see (Fig. 17) that, upon gluing two slices in \(S_n\) (respectively in \(s_i\) and \(s_j\) for some \(i \leq n\) and \(j \leq n\)), we get a pointed map with a marked edge of type \((m, m)\) for some \(m \leq n\) (with \(m = \max(i, j)\)). This marked edge is oriented with, say, the first slice on its right, and promoted as root edge. Again, we have after gluing two marked leftmost geodesics (starting respectively with the root edge and the reversed root edge) which
Fig. 17: Gluing two slices in $s_i$ and $s_j$ and identifying pairwise boundary edges creates a pointed rooted map of type $(m \to m)$ with $m = \max(i, j)$, implying $S_n^2 = T_n$ with $T_n$ defined as in Sect. 1. Identifying pairwise boundary edges of a slice in $r_i$ creates a pointed rooted map of type $(i \to i - 1)$. 

may be erased without loss of information. Hence $S_n^2 = T_n$ with $T_n$ defined as in Sect. 1.

Appendix B. Proof of (3.3)

Let us start by showing the following identity, valid for any $n \geq 0$ and $k \geq 2$

$$
\frac{\partial}{\partial R} \left( R \sum_{q=0}^{k-2} P^+(n + q; R, S) P(k - 2 - q; R, S) \right) = P(n; R, S) \frac{\partial}{\partial R} \left( \sqrt{R} P_{-1}(k - 1; R, S) \right)
$$

$$
+ \sqrt{R} P_{-1}(n; R, S) \frac{\partial}{\partial R} P(k - 1; R, S). 
$$

(B.1)
At this stage $R$ and $S$ are arbitrary coefficients. Attaching a weight $x^ny^{k-2}$ and summing over $n \geq 0$ and $k \geq 2$, the r.h.s produces the combination

$$
\pi(x) \frac{\partial}{\partial R} \left( \frac{\pi_1(y)}{y} \right) + \pi_1(x) \frac{\partial}{\partial R} \left( \frac{\pi(y) - 1}{y} \right),
$$

where we introduced the generating functions

$$
\pi(z) \equiv \sum_{n \geq 0} P(n; R, S) z^n = \frac{1}{\sqrt{\kappa(z)}}
$$

$$
\pi_{-1}(z) \equiv \sum_{n \geq 0} \sqrt{R} P_{-1}(n; R, S) z^n = z \frac{1 - Sz - \sqrt{\kappa(z)}}{2Rz^2 \sqrt{\kappa(z)}}
$$

with $\kappa(z)$ as in (3.8). The last formula comes from the identification $\pi_{-1}(z) = z\pi^+(z)\pi(z)$, where

$$
\pi^+(z) \equiv \sum_{n \geq 0} P^+(n; R, S) z^n = \frac{1 - Sz - \sqrt{\kappa(z)}}{2Rz^2}.
$$

As for the l.h.s in (B.1), setting $s = n + q$ and $t = k - 2 - q$, we have to compute

$$
\sum_{n \geq 0} \sum_{k \geq 2} \sum_{q = 0}^{k-2} P^+(n + q; R, S) P(k - 2 - q; R, S) x^n y^{k-2}
$$

$$
= \sum_{s \geq 0} \sum_{t \geq 0} \sum_{q = 0}^{s} x^s y^t \left( \frac{y}{x} \right)^q P^+(s; R, S) P(t; R, S)
$$

$$
= \sum_{s \geq 0} \sum_{t \geq 0} \frac{x^{s+1} - y^{s+1}}{x - y} y^t P^+(s; R, S) P(t; R, S)
$$

$$
= \frac{x\pi^+(x) - y\pi^+(y)}{x - y} \pi(y).
$$

Proving (B.1) therefore reduces to checking the relation

$$
\frac{\partial}{\partial R} \left( R \frac{x\pi^+(x) - y\pi^+(y)}{x - y} \pi(y) \right) = \pi(x) \frac{\partial}{\partial R} \left( \frac{\pi_1(y)}{y} \right) + \pi_1(x) \frac{\partial}{\partial R} \left( \frac{\pi(y) - 1}{y} \right),
$$

which, knowing the explicit forms of all the involved generating functions, is a straightforward task.

Multiplying (B.1) by $\delta_{k,2} - g_k$, summing over $k \geq 2$ and exchanging the sums over $k$ and $q$ leads to

$$
\frac{\partial}{\partial R} \sum_{q = 0}^{\infty} A_q P^+(n + q; R, S) = P(n; R, S) \frac{\partial}{\partial R} u + \sqrt{R} P_{-1}(n; R, S) \frac{\partial}{\partial R} v
$$

(43)
where we introduced

\[
    u \equiv R - \sum_{k \geq 2} g_k \sqrt{R} P_{-1}(k-1; R, S)
\]

\[
    v \equiv S - \sum_{k \geq 1} g_k P(k-1; R, S)
\]

and where \(A_q\) is defined as in (1.5). Note that we added for convenience a trivial constant term \(g_1\) in the definition of \(v\), which disappears after derivation with respect to \(R\).

Similarly, one can easily prove

\[
    \frac{\partial}{\partial S} \left( R \sum_{q=0}^{k-2} P^+(n + q; R, S) P(k - 2 - q; R, S) \right) = P(n; R, S) \frac{\partial}{\partial S} \left( \sqrt{R} P_{-1}(k-1; R, S) \right) + \sqrt{R} P_{-1}(n; R, S) \frac{\partial}{\partial S} P(k-1; R, S)
\]

by checking the identity

\[
    \frac{\partial}{\partial S} \left( R \frac{x \pi^+(x) - y \pi^+(y)}{x - y} \pi(y) \right) = \pi(x) \frac{\partial}{\partial S} \left( \frac{\pi_1(y)}{y} \right) + \pi_1(x) \frac{\partial}{\partial S} \left( \frac{\pi(y) - 1}{y} \right)
\]

This implies the relation

\[
    \frac{\partial}{\partial S} \sum_{q=0}^{\infty} A_q P^+(n + q; R, S) = P(n; R, S) \frac{\partial}{\partial S} u + \sqrt{R} P_{-1}(n; R, S) \frac{\partial}{\partial S} v
\]

with \(u\) and \(v\) as above. Equations (B.8) and (B.11) may be rewritten in terms of differential forms

\[
    d \left( \sum_{q=0}^{\infty} A_q P^+(n + q; R, S) \right) = P(n; R, S) du + \sqrt{R} P_{-1}(n; R, S) dv
\]

When \(v = 0\), Eqs. (B.8) match precisely the equations satisfied by the mobile generating functions \(R(u)\) and \(S(u)\) with a weight \(u\) per labeled vertex. Therefore, by restricting (B.12) to the line \(v = 0\), we deduce

\[
    \frac{d}{du} \sum_{q=0}^{\infty} A_q(u) P^+(n + q; R(u), S(u)) = P(n; R(u), S(u))
\]

with \(A_q(u)\) defined as in (1.3) with \(R, S\) replaced by \(R(u), S(u)\). This establishes Eq. (3.3) upon integrating with respect to \(u\) and noting that both sides of (3.3) vanish at \(u = 0\) (\(A_q(0) = 0\) since \(R(0) = 0\) and \(F_n(0) = 0\) from its map definition).
Appendix C. An expression for the orthogonal polynomials

Using the notations of Sect. 4, we have \( q_n(z) = \frac{1}{H_{n-1}} \det \left( H_n^{(n)} \right) \) where \( z_n \) is the row vector \( (z^j)_{0 \leq j \leq n} \). Inspired by (1.2), we may write

\[
\left( H_n^{(n)} \right) = \begin{pmatrix} T_n^{-1} & 0 \\ 0 & 1 \end{pmatrix} \cdot \left( z_n \cdot T_n^{-1} \right) \cdot T_n. \tag{C.1}
\]

By Proposition 11 of [14], the inverse matrix of \( T = (P_i^+(j; R, S))_{i,j \geq 0} \) is given by the matrix of coefficients of orthogonal polynomials associated with the continued fraction (2.3). The \( j \)th such polynomial is \( U_j((z - S)R^{-1/2}) \), where \( U_j(z) \) denotes the \( j \)th Chebyshev polynomial of the second kind (defined for instance via \( U_j(2 \cos \theta) \sin \theta = \sin(j + 1)\theta \)). Hence we have \( z_n \cdot T_n^{-1} = (U_j((z - S)R^{-1/2}))_{0 \leq j \leq n} \).

Passing to determinants in (C.1), expanding the middle r.h.s determinant over the last row and using the known expressions for \( \det T_n \) and \( H_n \), we obtain

\[
q_n(z) = R^{n/2} \sum_{m=0}^{n} (-1)^{m+n} U_m \left( \frac{z - S}{\sqrt{R}} \right) \frac{\det B_{n,m}^{(n,m)}}{\det B_{n-1}}
\]

\[
= R^{n/2} \sum_{m=0}^{n} U_m \left( \frac{z - S}{\sqrt{R}} \right) \frac{\sp_{2p}(\lambda_{p,n}^{(m)}, \chi)}{\sp_{2p}(\lambda_{p,n}, \chi)}
\]

\[
\tag{C.2}
\]

where \( \lambda_{p,n}^{(m)} = n^{p-1}m \) denotes the partition with \( p - 1 \) parts of size \( n \) and one part of size \( m \). The \( h \)- or Weyl formulas allow to further reexpress \( q_n(z) \) in terms of \( p \times p \) determinants, where only one row depends on \( z \).
References

[1] W.T. Tutte, *A Census of Planar Maps*, Canad. J. of Math. **15** (1963) 249-271.
[2] W.T. Tutte, *On the enumeration of planar maps*, Bull. Amer. Math. Soc. **74** (1968) 64-74.
[3] See for instance: P. Di Francesco, P. Ginsparg and J. Zinn–Justin, *2D Gravity and Random Matrices*, Physics Reports **254** (1995) 1-131, and references therein.
[4] See for instance Sect. 2.9 of: I.P. Goulden and D.M. Jackson, *Combinatorial Enumeration*, John Wiley & Sons, New York, 1983, republished by Dover, New York, 2004, and references therein.
[5] E. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Planar Diagrams*, Comm. Math. Phys. **59** (1978) 35-51.
[6] G. Schaeffer, *Conjugaison d’arbres et cartes combinatoires aléatoires*, PhD Thesis, Université Bordeaux I (1998).
[7] J. Bouttier, P. Di Francesco and E. Guitter, *Census of planar maps: from the one-matrix model solution to a combinatorial proof*, Nucl. Phys. **B645[PM]** (2002) 477-499, [arXiv:cond-mat/0207682](http://arxiv.org/abs/cond-mat/0207682).
[8] J. Bouttier, P. Di Francesco and E. Guitter. *Planar maps as labeled mobiles*, Elec. Jour. of Combinatorics **11** (2004) R69, [arXiv:math.CO/0405099](http://arxiv.org/abs/math.CO/0405099).
[9] J. Bouttier, *Physique statistique des surfaces aléatoires et combinatoire bijective des cartes planaires*, PhD Thesis, Université Paris 6 (2005).
[10] P. Chassaing and G. Schaeffer, *Random Planar Lattices and Integrated SuperBrownian Excursion*, Probability Theory and Related Fields **128**(2) (2004) 161-212, [arXiv:math.CO/0205229](http://arxiv.org/abs/math.CO/0205229).
[11] J. Bouttier, P. Di Francesco and E. Guitter, *Geodesic distance in planar graphs*, Nucl. Phys. **B663[FS]** (2003) 535-567, [arXiv:cond-mat/0303272](http://arxiv.org/abs/cond-mat/0303272).
[12] See for instance: G. Miermont, *Random maps and their scaling limits*, in C. Bandt, P. Mörters, M. Zähle (Eds.), Proceedings of the conference Fractal Geometry and Stochastics IV, Greifswald (2008), Progress in Probability, Vol. **61**, 197-224, Bikhäuser (2009), and references therein.
[13] J. Ambjörn and Y. Watabiki, *Scaling in quantum gravity*, Nucl.Phys. **B445** (1995) 129-144, [arXiv:hep-th/9501049](http://arxiv.org/abs/hep-th/9501049).
[14] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete Mathematics 32 (1980), 125–161. Reprinted in the 35th Special Anniversary Issue of Discrete Mathematics, Vol. **306**, Issue 10-11, Pages 992-1021 (2006).
[15] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University Press, 2009, section V.4.
16. H.S. Wall, *Analytic Theory of Continued Fractions*, D. Van Nostrand, New York, 1948, reprinted by Chelsea, New York, 1973.

17. W. Fulton and J. Harris, *Representation Theory*, Springer-Verlag, New York, 1991.

18. M. Fulmek and C. Krattenthaler, *Lattice path proofs for determinant formulas for symplectic and orthogonal characters*, J. Combin. Theory Ser. A 77 (1997), 3-50.

19. X.G. Viennot, *Une théorie combinatoire des polynômes orthogonaux*, Lecture Notes UQAM, 217p., Publication du LACIM, Université du Québec à Montréal (1984), reîd. 1991.

20. P. Di Francesco, *Geodesic Distance in Planar Graphs: An Integrable Approach*, The Ramanujan Journal 10 (2005) 153-186, [arXiv:math/0506543](http://arxiv.org/abs/math.CO).

21. G. Chapuy, M. Marcus and G. Schaeffer, *A bijection for rooted maps on orientable surfaces*, SIAM J. Discrete Math. 23(3) (2009) 1587-1611, [arXiv:0712.3649](http://arxiv.org/abs/math.CO).

22. E.A. Bender and E.R. Canfield, *The number of degree-restricted rooted maps on the sphere*, SIAM J. Discrete Math. 7 (1994) 9-15.

23. M. Bousquet-Mélou and A. Jehanne, *Polynomial equations with one catalytic variable, algebraic series and map enumeration*, J. Combin. Theory Ser. B 96 (2006) 623-672.

24. W.G. Brown, *On the existence of square roots in certain rings of power series*, Math. Ann. 158 (1965) 82-89.

25. P. Di Francesco and E. Guitter, *Integrability of graph combinatorics via random walks and heaps of dimers*, J. Stat. Mech. (2005) P09001, [arXiv:math/0506542](http://arxiv.org/abs/math.CO).

26. See for instance: I.M. Gessel and X.G. Viennot, *Binomial determinants, paths and hook length formulae*, Adv. in Math. 58 (1985) 300-321; I.M. Gessel and X.G. Viennot, *Determinants, Paths, and Plane Partitions*, preprint (1989), available for instance at [http://people.brandeis.edu/~gessel/](http://people.brandeis.edu/~gessel/).

27. J. Bouttier and E. Guitter, *Distance statistics in quadrangulations with a boundary, or with a self-avoiding loop*, J. Phys. A: Math. Theor. 42 (2009) 465208, [arXiv:0906.4892](http://arxiv.org/abs/math-ph).