Color-singlet relativistic correction to inclusive $J/\psi$ production associated with light hadrons at $B$ factories

Yu Jia$^{1,2,3}$

$^1$Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100049, China
$^2$Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, Beijing 100049, China
$^3$Kavli Institute for Theoretical Physics China, Chinese Academy of Sciences, Beijing 100190, China

(Dated: June 18, 2010)

Abstract

We study the first-order relativistic correction to the associated production of $J/\psi$ with light hadrons at $B$ factory experiments at $\sqrt{s} = 10.58$ GeV, in the context of NRQCD factorization. We employ a strategy for NRQCD expansion that slightly deviates from the orthodox doctrine, in that the matching coefficients are not truly of “short-distance” nature, but explicitly depend upon physical kinematic variables rather than partonic ones. Our matching method, with validity guaranteed by the Gremm-Kapustin relation, is particularly suited for the inclusive quarkonium production and decay processes with involved kinematics, exemplified by the process $e^+e^- \rightarrow J/\psi + gg$ considered in this work. Despite some intrinsic ambiguity affiliated with the order-$v^2$ NRQCD matrix element, if we choose its value as what has been extracted from a recent Cornell-potential-model-based analysis, including the relative order-$v^2$ effect is found to increase the lowest-order prediction for the integrated $J/\psi$ cross section by about 30%, and exert a modest impact on $J/\psi$ energy, angular and polarization distributions except near the very upper end of the $J/\psi$ energy. The order-$v^2$ contribution to the energy spectrum becomes logarithmically divergent at the maximum of $J/\psi$ energy. A consistent analysis may require that these large end-point logarithms be resummed to all orders in $\alpha_s$.

PACS numbers: 12.38.-t, 12.38.Bx, 12.39.St, 13.66.Bc, 14.40.Pq
I. INTRODUCTION

Inclusive production of heavy quarkonium (especially $J/\psi$) in various high-energy collider experiments has long been an intriguing and interesting topic, to which a vast number of works have been devoted [1, 2]. To date the mainstream of theoretical investigations in this subject is based upon the nonrelativistic QCD (NRQCD) factorization approach, a formalism combining the effective-field-theory machinery together with the hard-scattering factorization [3]. In the context of NRQCD factorization, the inclusive quarkonium production rate can be expressed in a factorized form, that is, an infinite sum of products of the perturbatively calculable partonic cross sections and nonperturbative but universal NRQCD matrix elements. One great virtue of this approach is that its predictions can in principle be systematically improved. This approach systemizes, and, extends, the conventional color-singlet model (CSM). One striking, and, probably also disputable, ingredient of NRQCD factorization is the so-called color-octet mechanism, that a heavy quark-antiquark pair in a color-octet configuration created in a hard process, is presumed to have non-negligible probability to transition into a physical quarkonium state plus additional soft light hadrons. Historically, the rapid popularity gained by this novel mechanism is perhaps due to its economic explanation of the so-called ‘$\psi'$ surplus puzzle’ [4].

Although NRQCD factorization has enjoyed considerable successes in many inclusive quarkonium decay and production processes, it also faces some serious challenges. Most notably, the recent Fermilab Tevatron measurement for $J/\psi$ polarization at large $p_T$ seems to contradict with the benchmark predictions of the color-octet mechanism, i.e. the increasingly transverse polarization of the hadro-produced $J/\psi$ with increasing $p_T$ [5]. Moreover, there is also problem for $J/\psi$ production in $e^+e^-$ collision experiments. The color-octet mechanism also anticipates that an enhanced number of $J/\psi$ populate near the maximum energy region in $e^+e^-$ annihilation, but unfortunately, this quite distinct signature has also not been confirmed by recent $B$ factory experiments.

These acute discrepancies have triggered a great wave of theoretical efforts in recent years. Recent technical advancement makes it possible, for the first time, to compute the rather involved next-to-leading QCD corrections to $J/\psi$ hadroproduction in color-singlet channel [6–10], and its effects seem to be quite significant, i.e., to enhance the leading-order CSM contribution enormously. This may indicate that, the phenomenological impetus to including color-octet contribution seems not as indispensable as that in a decade ago, and the correct magnitudes of color-octet matrix elements might be considerably smaller than the old numbers extracted by implementing the LO CSM analysis only.

It is worth emphasizing that, the NRQCD factorization theorems for quarkonium production are only at a conjectural level, which have never been proven rigourously to hold to all orders in $\alpha_s$. Notwithstanding the great utility of the improvement on the short-distance coefficients, it is perhaps more urgent, from the theoretical perspective, to reexamine every assumption underlying the nonperturbative aspects of NRQCD factorization, especially for the color-octet mechanism. As one of the important progresses along this line, the validity of the factorization theorems at two-loop level for gluon-to-quarkonium fragmentation function has recently been established after some suitable refinement of the original color-octet NRQCD production operator [11]. There is also suspect about the applicability of

---

1 Now it becomes clear that in some case NRQCD factorization certainly will fail. For example, a novel phenomenon dubbed color transfer mechanism [12], was discovered in the production of $J/\psi$ comoving...
the NRQCD velocity-scaling rule to charmonium, in particular it was suggested that the spin-flip matrix element may play an important role for the hardronization of color-octet $c\bar{c}$ pair [13, 14]. A more serious problem is that since each NRQCD matrix element is a number instead of a distribution, so NRQCD factorization makes rather restrictive predictions to the various $J/\psi$ energy spectra. It has long been suggested that in certain kinematic region of quarkonium production, resummation of a class of enhanced nonperturbative effects is crucial to make reliable prediction, which effectively promotes the local NRQCD matrix element to a nonperturbative shape function [15, 16]. It is also worth noting that, there is also an ongoing endeavor to circumvent the velocity expansion framework of NRQCD, by introducing a more general set of fragmentation functions in conventional perturbative QCD (pQCD) factorization base to describe inclusive quarkonium production at large $p_t$ [17].

In recent years, $B$ factories also prove to be another active field for the study of charmonium production. The simplicity of the initial $e^+e^-$ state, together with the enormous integrated luminosity, make the theoretical analysis of the charmonium production process particularly clean and fertile. Some recent measurements at $B$ factories have also posed challenges to our understanding of charmonium production. One is the unexpectedly large cross sections for several exclusive double charmonium production processes in continuum $e^+e^-$ annihilation. For example, the cross section for producing $J/\psi + \eta_c$ was first measured by the BELLE collaboration [18], which turns out to be almost one order-of-magnitude larger than the leading order (LO) NRQCD predictions [19]. After various theoretical works from different angles, the consensus now is that after including the large QCD perturbative correction [20, 21], in combination with relativistic corrections, this disquieting discrepancy was claimed to be largely resolved within the context of NRQCD factorization [22, 23].

Another more perplexing observation arises from the inclusive production of $J/\psi$. The production of $J/\psi$ in association with extra charms, is found, quite counter-intuitively, to occur much more copiously than that in association with a non-charm final state. The fraction of number of events for $J/\psi$ plus charmed hadrons to that of the inclusive $J/\psi$ events, conventionally denoted $R_{cc}$, was first measured by BELLE collaboration to be $0.59^{+0.15}_{-0.13}\pm 0.12$ [18], later even shifted to $0.82\pm 0.15\pm 0.14$ [24]. This experimental values are in stark contrast to the leading-order (LO) NRQCD predictions to this ratio, which is only about 0.1 $^2$. Other theoretical approaches, e.g., the estimate based on quark-hadron duality hypothesis [32] and the color-evaporation model [33] also predict a quite small $R_{cc}$.

Once upon a time, the total $J/\psi$ production rate measured at $B$ factories appeared to be quite large, i.e., 2.5 pb measured by BABAR [34] and 1.5 pb [35] by BELLE, which seems to request a sizable color-octet contribution such as $e^+e^- \rightarrow c\bar{c}(^{1}S_0^{(8)},^{3}P_0^{(8)}) + g$. The color-octet effect for $J/\psi$ production in $e^+e^-$ annihilation was first investigated by Braaten and Chen [36] (see also [37], and for a very recent study of the NLO perturbative correction, see [38]). However, including this contribution will further dilute the ratio $R_{cc}$. Furthermore, an unusual signature of this mechanism is that an end-point peak is expected in the $J/\psi$ energy spectrum. Unfortunately, there is no experimental evidences for the existence of such a peak [34, 35]. To rescue the color-octet mechanism, later on the end-point Sudakov

with an additional heavy quark. In this case, soft color exchange between the comoving quark and the constitutes of $J/\psi$ may invalidate the NRQCD factorization at two-loop level.

$^2$ Note that the LO NRQCD predictions to the $J/\psi$ production associated with charmed or noncharmed hadrons are identical to the CSM predictions [25–31], i.e. to proceed through the parton processes $e^+e^- \rightarrow J/\psi + c\bar{c}$ and $e^+e^- \rightarrow J/\psi + gg$, respectively.
logarithms have been identified and resummed, together with introduction of a phenomenological shape function, one can show that the $J/\psi$ energy distribution can be smeared out in accordance with the data [39]. It is worth mentioning that absolute normalization of the color-octet contribution is not affected much by including these refinements, and by that time its contribution was assumed to predominate over the color-singlet contribution.

In the past couple of years, significant progresses toward resolving these puzzles have been made from both experimental and theoretical angles. From the experimental side, recently BELLE collaboration was able to precisely measure the cross sections for prompt $J/\psi$ production in association with charmed and non-charmed states separately [40]:

\[\sigma[e^+e^- \rightarrow J/\psi + X_{cc}] = 0.74 \pm 0.08^{+0.09}_{-0.08} \text{ pb}, \quad (1a)\]
\[\sigma[e^+e^- \rightarrow J/\psi + X_{\text{light}}] = 0.43 \pm 0.09 \pm 0.09 \text{ pb}. \quad (1b)\]

This new measurement has not subtracted the feeddown contribution from $\psi(2S)$.

The most important recent theoretical progress in this subject is perhaps the fulfillment of the NLO QCD corrections to both channels. It turns out that the inclusion of the NLO QCD correction significantly enhances the $J/\psi + c\bar{c}$ production rate [41, 42]. Recently, the next-to-leading (NLO) perturbative correction to $e^+e^- \rightarrow J/\psi + gg$ has also been conducted, which enhances the LO cross section by only about 20% [43, 44]. The significant enhancement to the former and the modest one to the latter is of help for the predicted $R_{cc}$ value to approach the measured one. When the feeddown effects are included, the rough agreement seems also to be achieved for both the associated $J/\psi$ production subprocesses, so the alarming discrepancies seem to be greatly alleviated.

The above analysis tends to indicate that, CSM alone seems sufficient to explain the data, and there seems no much room left for the color-octet contribution. As a result, the color-octet matrix element may be considerably smaller than what was used to be assumed when fitted from the Tevatron data. Nevertheless, it is still premature to assert that we already have satisfactory understanding of inclusive $J/\psi$ production at $B$ factory, because there is still one important component missing. That is, in compliance with the NRQCD power counting, one should also take the first-order relativistic correction into account, since its effect is parametrically more important than the color-octet contribution. This is particularly relevant for $e^+e^- \rightarrow J/\psi + X_{\text{light}}$ since the size of NLO QCD correction to $e^+e^- \rightarrow J/\psi + gg$ is mild. It is thus interesting to examine whether the relativistic correction brings in sizable effects to this process or not.

The main purpose of this work is to answer this question, that is, to calculate the first-order relativistic correction to the inclusive $J/\psi$ production rate in $e^+e^- \rightarrow J/\psi + X_{\text{light}}$ in NRQCD factorization. Concretely, we will be considering the process $e^+e^- \rightarrow J/\psi + gg$ at $O(\alpha_s^2)$. It turns out that this correction is comparable in magnitude with the NLO QCD correction, if not more important.

---

3 The end-point collinear logarithms in the color-singlet channel has been resummed, but its impact on the $J/\psi$ spectrum seems rather insignificant [45, 46].

4 However, although including NLO QCD correction helps to get the right answer for the inclusive production rate and energy spectrum of $J/\psi$ in $e^+e^- \rightarrow J/\psi + X_{cc}$, it was noted that even [44], it is still difficult to reproduce the measured $J/\psi$ polarization and angular distribution.

5 The relativistic correction to $e^+e^- \rightarrow J/\psi + X_{cc}$ has been calculated and was reported to be surprisingly small [22].
An experienced reader may agree that, calculations of QCD perturbative corrections can be guided by some standard and unambiguous procedure. By contrast, calculating relativistic corrections, unexaggeratedly speaking, seems often plagued with ambiguities and pitfalls \(^6\). The problem gets particularly acute, when the kinematics becomes involved, as in our case with three-body final states. It often occurs that, different results have been reported by different authors in calculating relativistic correction to the same process \(^7\). To the best of our knowledge, a simple and consistent recipe for calculating relativistic corrections for a generic quarkonium production process has not yet been explicitly given in literature. One of the major motifs of this work is also to fill this gap. We attempt to utilize a convenient yet slightly unconventional strategy to deduce the relativistic corrections, applicable to any inclusive quarkonium production (decay) process in color-singlet channel. Our approach is slightly different from the orthodox NRQCD doctrine, in that the matching coefficients are not truly of “short-distance” nature, for they explicitly depend upon physical kinematic variables rather than the partonic ones. However, we stress our method is still consistent, for its validity resting upon a rigorous relation in NRQCD, the Gremm-Kapustin relation \([53]\).

The rest of the paper is structured as follows. In section II, we state the NRQCD expansion formula relevant to this work and discuss the corresponding long-distance matrix elements of the color-singlet production operators. In section III, we outline our matching scheme that can be applied to any color-singlet quarkonium production process, and discuss its advantage over the more orthodox doctrine. In section IV, we give the differential expressions for the three-body phase space needed for the reaction \(e^+e^\rightarrow J/\psi + gg\). In section V, we present a detailed description on how to determine the short-distance coefficients through relative order \(v^2\) in \(e^+e^- \rightarrow J/\psi + X_{\text{light}}\) and how the physical predictions for the inclusive production rate of \(J/\psi\) in \(e^+e^-\) annihilation come out. In section VI, we apply our formulas to investigate the phenomenological impact of the order-\(v^2\) correction to the integrated production rate and the differential energy spectrum for the unpolarized \(J/\psi\), and the energy distribution of angular and polarization parameters at \(B\) factories. Finally we summarize our results in section VII. In Appendix A, we collect the analytic expressions for numerous types of differential cross sections of \(J/\psi\) in \(e^+e^-\) annihilation, at the leading order and next-to-leading order in \(v^2\). In Appendix B, we show that, it is possible to reexpress our predictions for the integrated \(J/\psi\) cross sections in a more orthodox form, that depending explicitly on the charm quark mass rather than the \(J/\psi\) mass.

II. NRQCD FACTORIZATION AND LONG-DISTANCE MATRIX ELEMENTS

According to the NRQCD factorization, the inclusive \(J/\psi\) production rate in \(e^+e^-\) collision can be schematically written as

\[
d\sigma[e^+e^- \rightarrow J/\psi + X] = \sum_n d\sigma[e^+e^- \rightarrow c\bar{c}(n) + X] \langle O_n^{J/\psi} \rangle.
\]  

\(^6\) See Ref. \([47]\) for an early discussion on one type of ambiguity affiliated with the normalization of particle states in relativistic correction calculations. See also \([48]\) for a related discussion.

\(^7\) For example, Ref. \([49]\) claims a disagreement with an earlier publication \([50]\) on the result of relativistic correction to photoproduction of \(J/\psi\). Likewise, a recent calculation of the relativistic corrections to the fragmentation function for the \(c\) quark to fragment into \(J/\psi\) \([51]\) also disagrees with an earlier work \([52]\).
The NRQCD expansion is organized by the velocity scaling of the vacuum matrix element of NRQCD 4-fermion operators, \( \langle O_n^{(J/\psi)} \rangle \), where \( n \) denotes the collective quantum numbers of the \( c\bar{c} \) pair created in the hard scattering. The \( d\sigma_n \) are canonically referred to as the process-dependent short-distance coefficients, which depend on all partonic kinematic variables for a given production process, but are insensitive to the long-distance aspects of the quarkonium state \( J/\psi \).

In this work we are only concerned with the color-singlet channel. The readers who are also interested in the color-octet contributions to this process can refer to Ref. [36, 37, 39]. For our purpose, the relevant 4-fermion color-singlet production operators are given by

\[
O_1^{(J/\psi)}(3S_1) = \frac{1}{2J+1} \chi^\dagger \sigma \sum_X |J/\psi + X\rangle \cdot \langle J/\psi + X| \psi^\dagger \sigma \chi,
\]

\[
P_1^{(J/\psi)}(3S_1) = \frac{1}{2m_c^2(2J+1)} \left[ \chi^\dagger \sigma \psi \sum_X |J/\psi + X\rangle \cdot \langle J/\psi + X| \psi^\dagger \sigma (-\frac{i}{2}\vec{D})^2 \chi + \text{H.c.} \right],
\]

where \( \psi \) and \( \chi \) are Pauli spinor fields for annihilating and a heavy quark, and creating a heavy antiquark, respectively, \( \sigma^i \) denotes the Pauli matrix, and \( \vec{D} \) is the spatial part of the antisymmetrical covariant derivative: \( \psi^\dagger \vec{D} \chi \equiv \psi^\dagger \vec{D} \chi - (\vec{D} \psi)^\dagger \chi \), in a form to preserve Galilean invariance. \( J = 1 \) denotes the total spin of \( J/\psi \), and \( X \) denotes additional light hadrons accompanied with \( J/\psi \) with net energy no larger than the ultraviolet cutoff of NRQCD. Note the sum for the intermediate states is extended not only over all additional light flavor states \( X \), but over \( 2J + 1 \) polarizations of the \( J/\psi \) as well.

The vacuum expectation values of these NRQCD production operators are genuinely nonperturbative objects, whose exact values are even difficult to ascertain from the powerful nonperturbative tools such as lattice QCD, mainly owing to the obstacle in implementing those asymptotic states containing \( X \). Fortunately, in practice one can always invoke the so-called vacuum saturation approximation (VSA) for the color-singlet channel, which is accurate up to an error of relative order \( v^4 \), to link these NRQCD operator matrix elements with the more familiar Schrödinger wave functions at the origin for the \( J/\psi \):

\[
\langle O_1^{(J/\psi)} \rangle \approx \langle J/\psi |O_1^{(3S_1)_{\text{BBL}}} |J/\psi \rangle \approx |\langle 0 | \chi^\dagger \sigma \psi |J/\psi \rangle |^2 = 2N_c \psi_{J/\psi}^2(0),
\]

\[
\langle P_1^{(J/\psi)} \rangle \approx \langle J/\psi |P_1^{(3S_1)_{\text{BBL}}} |J/\psi \rangle \approx \text{Re} \left[ \langle J/\psi |\psi^\dagger \chi |0 \rangle \cdot \langle 0 | \chi^\dagger \sigma (-\frac{i}{2}\vec{D})^2 \chi |J/\psi \rangle \right]
\]

\[
= -2N_c \text{Re} \left[ \psi_{J/\psi}^*(0)\nabla^2 \psi_{J/\psi}(0) \right].
\]

Under this approximation, the vacuum matrix element for \( J/\psi \) production can be approximated by the square of vacuum-to-\( J/\psi \) matrix element in NRQCD, and further by the corresponding decay matrix element for the \( J/\psi \) state.

The leading-order color-singlet \( J/\psi \) production matrix element, \( \langle O_1^{(J/\psi)} \rangle \), can be identified with the familiar wave function at the origin for \( J/\psi \), \( \psi_{J/\psi}(0) \). This quantity can be

---

8 In this work, we find it convenient to choose a different normalization for the \( J/\psi \) production operators \( O_1^{(J/\psi)} \) and \( P_1^{(J/\psi)} \) other than the standard ones introduced by Bodwin, Braaten and Lepage (BBL) [3], i.e., we define \( O_1^{(J/\psi)} = \frac{1}{2J+1}O_1^{(J/\psi)_{\text{BBL}}} \) and \( P_1^{(J/\psi)} = \frac{1}{(2J+1)m_c^2} P_1^{(J/\psi)_{\text{BBL}}} \). Note the prefactor \( 1/m_c^2 \) normalizes the operator \( P_1^{(J/\psi)} \) such as to carry the same mass dimension as \( O_1^{(J/\psi)} \).
determined by several means, e.g., from lattice simulation, or from phenomenological quark potential models, or directly from the measured leptonic width of \( J/\psi \).

The determination of the relative order-\( v^2 \) production matrix element, \( \langle P_{j/\psi}^1 \rangle \), turns to be more problematic. In Coulomb gauge, the gauge field piece in this matrix element is suppressed relative to the ordinary derivative piece. By VSA, it seems intuitive to interpret \( \langle P_{j/\psi}^1 \rangle \) as product of \( \psi_{J/\psi}(0) \) and the second derivative of the wave function at the origin, \( \nabla^2 \psi_{J/\psi}(0) \). Nevertheless, one should be cautioned that such a naive interpretation is obscure. This is because the bare NRQCD matrix element contains linear ultraviolet divergence, hence it needs to be regularized and renormalized, thus depending on the cutoff of the NRQCD lagrangian (An overbar put above the wave function is to remind this). There seems no direct way to directly infer this matrix element from phenomenological potential model. Nevertheless, it is the NRQCD effective theory framework that endows this nonperturbative quantity a meaningful definition.

For later use, it is convenient to introduce the dimensionless ratio of the vacuum matrix elements of the following NRQCD operators:

\[
\langle v^2 \rangle_{J/\psi} = \frac{\langle P_{j/\psi}^1 \rangle}{\langle O_{j/\psi}^1 \rangle} \approx \frac{\langle J/\psi(\lambda) | \psi^\dagger(-i\frac{1}{2} \nabla)^2 \sigma \cdot \epsilon(\lambda) \chi | 0 \rangle}{m_c^2 \langle J/\psi(\lambda) | \psi^\dagger \sigma \cdot \epsilon(\lambda) \chi | 0 \rangle}. \tag{5}
\]

This quantity, characterizing the typical size of relativistic correction for \( J/\psi \), is supposedly around 0.3. Note that its value is independent of the \( J/\psi \) helicity \( \lambda \) in above equation.

The vacuum-to-quarkonium relativistic correction matrix element has been measured by lattice QCD, though the uncertainty is quite large. There is an interesting relation, first derived by Gremm and Kapustin (G-K) [53], which derives from the equation of motion of NRQCD, and expresses the relativistic correction NRQCD matrix element in terms of the LO NRQCD matrix element, physical \( J/\psi \) mass and the charm quark mass:

\[
\frac{M_{J/\psi}}{2m_c} = 1 + \frac{1}{2} \langle v^2 \rangle_{J/\psi} + O(v^4), \tag{6}
\]

In NRQCD, the quark mass parameter is most naturally identified with the quark pole mass. Unfortunately, due to the intrinsic ambiguity of the charm quark pole mass, this relation cannot be utilized to nail down the precise value of \( \langle v^2 \rangle_{J/\psi} \). It can not be precluded that, the actual value of this quantity might be quite far from the naive expectation, 0.3. Since it is a subtracted quantity, it will be perfectly consistent if \( \langle v^2 \rangle_{J/\psi} \) turns to vanish or become even negative \(^9\).

During the era antecedent the inception of the NRQCD approach, many authors preferred to using the binding energy, i.e. \( \epsilon \equiv M_{J/\psi} - 2m_c \) to parameterize the contribution of relativistic corrections (for example, see [50, 55]). With the aid of the G-K relation (6), all those old results can be readily translated into the modern form, with relativistic correction designated by the NRQCD operator matrix elements.

\(^9\) It is worth mentioning that, recently there have been claims that this quantity can be quite accurately extracted from the Cornell-potential-model-based analysis, \( \langle v^2 \rangle_{J/\psi} = 0.225^{+0.106}_{-0.088} \) [54].
III. PERTURBATIVE MATCHING STRATEGY

The central ingredient of the NRQCD factorization formula is to deduce the NRQCD short-distance coefficients. The procedure of determining these coefficients are usually referred to as matching. The idea is rather straightforward, since these short-distance coefficients are in principle insensitive to the long-distance nonperturbative physics, therefore one may replace $J/\psi$ by a free $c\bar{c}$ pair carrying the quantum number of $^3S_1^{(1)}$, then both sides of Eq. (2), including the NRQCD matrix elements in the right side, can be accessed entirely in perturbation theory. Matching both sides, one then be able to extract the desired short-distance coefficients $d\hat{\sigma}_n$.

Let us take $e^+e^- \rightarrow J/\psi + X$ as an explicit example to illustrate the problem faced for the matching calculation beyond the lowest order in $v^2$. Schematically, one can express the corresponding perturbative matching formula for (2) as

$$
\sum_X (2\pi)^4 \delta^4(K-P-k_X) \left| \mathcal{M} \left[ e^+e^- \rightarrow c\bar{c}(^3S_1^{(1)}, P, \lambda) + X \right] \right|^2 = \sum_n d\hat{\sigma}_n(P, \lambda) \langle O_{c\bar{c}}^n \rangle, \quad (7)
$$

where the flux factor associated with the single-inclusive cross section has been suppressed for simplicity. $K$ stands for the sum of momenta of the colliding electron and positron, i.e., the 4-momentum of the virtual photon into which the electron and the positron annihilate. The sum in the left side is extended over the spins of all the additional partonic states $X$, as well as over the phase space integration affiliated with $X$.

It is clear from (7) that, to identify the matching coefficients, one needs to expand the inclusive production rate for perturbative $c\bar{c}(^3S_1^{(1)})$ pair systematically in the small relative momentum between $c$ and $\bar{c}$, $q$. This procedure entails two essential ingredients, one is to expand the matrix element squared in powers of $q$, the other is to expand the phase space integrals accordingly. The former operation is more or less standard, but the latter potentially cause some problems for the orthodox matching method, as will be reviewed in section III A. The main trouble is that, in the standard matching calculation, it is $m_c$, instead of the physical $J/\psi$ mass, $M_{J/\psi}$ that should enter the NRQCD short-distance coefficients. The orthodox method then requires that the phase space integral be expanded around a fictitious $c\bar{c}(^3S_1^{(1)})$ state of mass $2m_c$. Technically, such an expansion of the phase space integral at differential level is not easy to realize. More importantly, this operation leads to some inevitably unsatisfactory feature in predicting the differential $J/\psi$ spectrum: when approximating $J/\psi$ mass by $2m_c$, the incorrect kinematics causes the spectrum somewhat distorted, which may become particularly problematic near the phase space boundary $^{10}$.

In section III B, we will elaborate on the matching method adopted in this work. Motivated by the aforementioned shortcoming of the orthodox matching strategy, we attempt to circumvent the most difficult part arising from expanding the phase space integral. The key point is that we choose to use physical kinematics instead of the partonic one, and the

---

10 If one is content to knowing only the total cross section, this orthodox method should be straightforward and does not cause any problem. For instance, for simpler reactions such as $gg \rightarrow \eta_c$, exclusive double charmonium production $e^+e^- \rightarrow J/\psi + \eta_c$, or inclusive quarkonium decays $J/\psi \rightarrow ggg$, it is possible to first work out the phase space integration analytically, then expanding the resulting integrated partonic production/decay rates in powers of $q$ about a fictitious charmonium of mass $2m_c$. However, in the case of more involved kinematics, it is usually not feasible to acquire the integrated rate in a closed form.
invariant mass of the $c\bar{c}$ pair appearing in the matching calculation is taken as $M_{J/\psi}$. As we shall see, this brings in great technical simplifications. As a result, we can perform the matching at the level of the matrix element squared, instead of at the level of the production rate. Although our matching method somewhat deviates from the ordinary tenet in that the “short-distance” coefficients now explicitly depend on $M_{J/\psi}$, it is nevertheless still theoretically consistent, thanks to the G-K relation (6).

A. Orthodox matching strategy motivating the shape-function method

In this subsection we review what the standard NRQCD matching strategy would look like. Historically, this method antecedes, and, motivates, the so-called shape function method [15]. The orthodox doctrine of NRQCD matching is common in any effective field theory, in that the short-distance coefficients should depend only on the parton kinematics, thus on the quark mass, and there is no way for quarkonium mass, which inevitably entails the long-distance hadronization effect, to enter into them.

Let $c$ and $\bar{c}$ that evolve to $J/\psi$ in (7) have momenta $p$ and $\bar{p}$. Both $c$ and $\bar{c}$ are supposed to be on-shell, and their momenta can be decomposed as

$$p = \frac{1}{2} \hat{P} + q_1, \quad \bar{p} = \frac{1}{2} \hat{P} + q_2. \tag{8a}$$

Here the “total” momentum $\hat{P}$, which is deliberately chosen to satisfy $\sqrt{\hat{P}^2} = 2m_c$, should be distinguished from the true total momentum of the pair, $P = p + \bar{p}$, with invariant mass of $2E_q$, where $E_q = \sqrt{m_c^2 + q^2}$ is the energy of the $c$ or the $\bar{c}$ in the rest frame of the $c\bar{c}$ pair. In the rest frame, these 4-momenta have following explicit assignments: $\hat{P}^\mu = (2m_c, 0)$, $q_1^\mu = (E_q - m_c, q)$, $q_2^\mu = (E_q - m_c, -q)$, respectively. In the laboratory frame, it is understood that a suitable boost along the moving direction of the $c\bar{c}$ pair is imposed on all the 4-vectors.

The purpose of introducing $\hat{P}$ is that one needs to expand the phase space integral around a fictitious $c\bar{c}$ pair of invariant mass $2m_c$, which serves as the basis momentum. Concretely, the constrained phase space measure for the partonic process of (7) is

$$d\Pi = \frac{d^3 \hat{P}}{(2\pi)^3 2\hat{p}^0} \prod_i \frac{d^3 k_i}{(2\pi)^3 2k_i^0} (2\pi)^4 \delta(4)(K - \sum_i k_i - \hat{P} - (q_1 + q_2)). \tag{9}$$

where $K$ is the momentum of the virtual photon, and $k_i$ represents the additional partons in $X$. Note $q_i$ inside the $\delta$-function are understood to be subject to an appropriate Lorentz boost from the rest frame of $P$ to the laboratory frame.

The squared quark amplitude can then be folded with the phase space measure (9) to obtain the partonic cross section. The cross section needs to be expanded in the small momenta $q_i$, and powers of momentum can be identified with derivatives acting on the heavy quark fields according to NRQCD factorization. Factors of relative momentum $q_1 - q_2$ (in the rest frame of the $c\bar{c}$ pair) typically arise from expanding the amplitude, which can be identified with the $\psi^\dagger(i\hat{D})\chi$. Furthermore, in expanding the $\delta$-function in phase space measure (9), one typically encounters a different type of factor, the center-of-mass (cms)
type momentum $q_1^0 + q_2^0$ (in the rest frame of the $c\bar{c}$ pair), which can be identified with a total time derivative $iD_0(\psi^\dagger\chi)$.

As noted in Ref. [15] (see also [16]), these cms-derivative operators, though nominally of higher-order than the relative momentum operators in NRQCD expansion, can be of dynamical significance near the kinematic endpoint of quarkonium spectrum. Upon expansion of the $\delta$-function in $q_1 + q_2$, the resulting power series in $v^2$ make increasingly singular contributions near the boundary of partonic phase space, which signals that NRQCD expansion may break down near the endpoint region. Fortunately, it has been shown that such enhanced kinematic contribution due to these cms relativistic corrections can be resummed, whose effects are then encoded in the universal nonperturbative shape function [15].

The shape function is of greatest utility to improve the predictions for inclusive quarkonium production in the color-octet channel [15, 16, 39]. Nevertheless, it can also play a nontrivial role even for the color-singlet channel, which is relevant to our case. It turns out that the resulting series from expanding the $\delta$-function in (9) can be exactly resummed without introducing any new nonperturbative parameter other than the quarkonium mass. Its sole effect is to account for the difference between quark and quarkonium mass, consequently shift the partonic boundary of phase space to the hadronic one. The remarkable effect can be easily understood. The cms-momentum factor $q_1^0 + q_2^0$ equals $2E_q - 2m_c$ in the rest frame of $P$. When identified with the total time derivative $i\partial_0(\psi^\dagger\chi)$, this operator can convert to the binding energy $\epsilon = M_{J/\psi} - 2m_c$ when sandwiched between the vacuum and the physical $J/\psi$ states, thus helping to recover the hadronic kinematics. Not surprisingly, the underlying reason is nothing but the G-K relation.

The role played by the color-singlet shape function seems to strongly suggest that, the inconvenience brought in by the orthodox matching method, i.e., the procedure of expanding and reassembling of the phase-space $\delta$-function, may be avoidable. As will be elaborated in more detail in next subsection, one may just remain the physical kinematics intact in (9) throughout the matching computation. Lastly, we mention the fact that, somewhat ironically, there seems no any practical calculation of the complete first-order relativistic correction for the inclusive quarkonium production process that is based on this orthodox matching tenet.

B. Matching strategy adopted in this work

In light of the complication inherent in the orthodox matching method, in this section we are going to describe a different matching strategy, which is suitable for any inclusive quarkonium production (decay) process in the color-singlet channel. We will take the reaction $e^-e^+ \rightarrow J/\psi + gg$ as an explicit example to illustrate our method.

When assigning the momenta of $c$ and $\bar{c}$ in perturbative matching, the separation between “total” and “relative” momenta is just a matter of taste, by no means unique. Here we will choose a different one from that in section III A. The momenta of the on-shell $c$ and $\bar{c}$ can be decomposed in the following form:

\begin{align}
p &= \frac{1}{2}P + q, \quad \text{(10a)} \\
\bar{p} &= \frac{1}{2}P - q. \quad \text{(10b)}
\end{align}

We stress here $P$ represents the true total momentum of the pair, $P = p + \bar{p}$, with invariant
mass of $2E_q$, and now $P$ and $q$ are chosen to be orthogonal: $P \cdot q = 0$, in contrast to the choice made in (8). In the rest frame of the $c\bar{c}$ pair, the explicit components of the momenta are $P^\mu = (2E_q, 0)$, $q^\mu = (0, q)$, $p^\mu = (E_q, q)$, and $\bar{p}^\mu = (E_q, -q)$, respectively. In the laboratory frame, one has $P^\mu = (P^0, \mathbf{P}) = (\sqrt{\mathbf{P}^2 + 4E_q^2}, \mathbf{P})$ and appropriate Lorentz boost is understood to be imposed on any other 4-vector. It is worth mentioning that, it is this form of momentum assignment, rather than (8), that has been practically used in most calculations of relativistic corrections [22, 51, 56–59].

The greatest advantage of this kind of momentum decomposition is that, there is no need to expand the total momentum $P$ of the $c\bar{c}$ pair around a basis momentum with invariant mass of $2m_c$, and we will leave phase space measure intact by assuming the $c\bar{c}$ pair with an invariant mass $2E_q$. We argue by this way the relativistic effects in phase space measure are automatically incorporated, at least to relative order $v^2$. We will come back to the connection between the factor $E_q$ and $M_{J/\psi}$ in nonrelativistic expansion later in this subsection.

Even though we are coping with inclusive $J/\psi$ production process, but insofar as the color-singlet channel is concerned, it is not necessarily be committed to the cross section level at the very beginning. In fact, since we no longer need worry about the complication from the phase space integral, it seems legitimate to invoke the NRQCD factorization at the amplitude level, To this end, we need only retain those operator matrix elements that form the corresponding NRQCD factorization formula. In our case, the amplitude for producing $J/\psi$ in the right side of (11). Howev er, in the right-hand side of Eq. (11), the $J/\psi$ state appearing in the NRQCD matrix elements conventionally assumes the nonrelativistic normalization. To compensate this difference, one must insert a factor $\sqrt{2M_{J/\psi}}$ in the right side of (11).

Squaring both sides of (11), summing over the final-state spin/col ours as well as averaging upon the initial-state spins, the matrix element squared reads

$$
\sum |\mathcal{M}[J/\psi(\lambda) + gg]|^2 = 2M_{J/\psi}\langle \mathcal{O}_1^{J/\psi}\sum \{ |F_0|^2 + 2 \text{Re}[F_0F_2^*] \langle v^2\rangle_{J/\psi} + \cdots \} ,
$$

where the VSA has been invoked and $\langle \mathcal{O}_1^{J/\psi}\rangle$ has been given in (4a), and $\langle v^2\rangle_{J/\psi}$ defined in (5). The symbol $\sum$ indicates the suitable color-spin summation/average.

To determine the coefficients $|F_0|^2$ and $F_0F_2^* + \text{H.c.}$, we follow the moral that these short-distance coefficients are insensitive to the long-distance confinement effect, so one can replace the physical $J/\psi$ state by a free $c\bar{c}$ pair of quantum number $^3S_1^{(1)}$, by which the NRQCD operator matrix elements can be perturbatively calculated. The short-distance coefficients $F_0(\lambda)$ can then be read off by comparing the full QCD amplitude for producing $c\bar{c}(^3S_1^{(1)})$ and the corresponding NRQCD factorization formula. In our case, the amplitude for producing
a $c\bar{c}(3S_1^{(1)})$ pair associated with two gluons is

$$M[cc(3S_1^{(1)}, \lambda) + gg] = F_0(\lambda)\langle cc(3S_1)|\psi^\dagger \sigma \cdot \epsilon|0\rangle + \frac{F_2(\lambda)}{m_c^2} \langle cc(3S_1)|\psi^\dagger \sigma \cdot \epsilon(-\frac{i}{2}\bar{D})^2|0\rangle$$

$$= \sqrt{2N_c}2E_q \left[ F_0(\lambda) + \frac{F_2(\lambda)}{m_c^2} \frac{q^2}{m_c^2} \right].$$

(13)

In Eq. (13), we use relativistic normalization for the $c$ and $\bar{c}$ states in the computation of the full QCD amplitude and in the computation of the NRQCD matrix elements. Consequently, a factor $2E_q$ appears in the second expression of Eq. (13). An additional factor $\sqrt{2N_c}$ arises from the spin and color factors of the NRQCD matrix elements. From (13), it is straightforward to extract the short-distance coefficients $F_0(\lambda)$:

$$F_0(\lambda) = \frac{M[cc(3S_1^{(1)}, \lambda) + gg]}{\sqrt{2N_c}2E_q} \bigg|_{q^2=0},$$

(14a)

$$F_2(\lambda) = \frac{m_c^2}{q^2} \left( \frac{M[cc(3S_1^{(1)}, \lambda) + gg]}{\sqrt{2N_c}2E_q} - F_0(\lambda) \right) \bigg|_{q^2=0}.$$ (14b)

The LO coefficient $F_0$ can be obtained by putting $q \to 0$ in the amplitude and equating $E_q$ and $m_c$. To deduce the coefficient $F_2$, it is understood that one has to first expand the amplitude to the first order in $q^2$ prior to taking the $q \to 0$ limit. Consequently, it is necessary to distinguish between $E_q$ and $m_c$.

Although the expression of $F_0$ can be unequivocally determined, it is not without ambiguity to deduce the coefficient $F_2$. This is because, determination of this relativistic correction coefficient crucially hinges on which quantity is chosen to be expanded around in powers of $q^2$ in the quark amplitude.

Obviously, those terms that contain explicit $q^2$ factor should contribute to $F_2$. Besides these terms, in the matching procedure adopted by most authors, one usually often includes relativistic effects implicit in all the expressions that contain the factor $E_q$, where $E_q$ is always expanded around $m_c$ in power series of $q^2$, i.e. $E_q = m_c + \frac{q^2}{2m_c} + \mathcal{O}(q^4)$. By collecting all the sources proportional to $q^2$, one is then able to deduce the coefficient $F_2$ according to (14b).

In this work, we find it much more advantageous to take a somewhat different route. Aside from retaining those relativistic correction contributions that contain $q^2$ explicitly, we choose to expand every occurrence of $m_c$ in the amplitude in term of $q^2/E_q^2$, while keeping $E_q$ intact:

$$m_c = E_q - \frac{q^2}{2E_q} + \mathcal{O}(q^4).$$

(15)

Somewhat nonstandard as it seems, but as we will see shortly, by choosing this way of expansion, we circumvent the most difficult task, i.e., expanding the three-body phase space integral. This procedure turns out to be the simplest in practice, especially when contrasted

---

11 Throughout this work, the bold-faced symbols, such as $q$, if not otherwise stated, are exclusively referring to the spatial vectors defined in the rest frame of the $c\bar{c}(P)$ pair, whereas the italic symbols, such as $q$, are reserved for the covariant 4-vectors, often presumed in the laboratory frame.
with the orthodox matching method outlined in section III A. This will be exemplified by more concrete examples in section V.

In (14b), we have refrained from expressing $F_2$ by taking the second-order derivatives of the quark amplitude over $q$, as frequently adopted in many works [22, 51, 56, 58]. The reason is that we try to avoid potential ambiguity associated with this operation, which usually happens when one performs the standard expansion around $m_c$. The recipe given in (14b) is unambiguous and simple provided that $E_q$ is kept fixed. The expression obtained this way are connected to the standard one through reshuffling some terms between $F_0$ and $F_2$.

Squaring the matrix element (13), summing over final-state polarizations and averaging upon the initial-state spins, we obtain

$$
\sum |\mathcal{M}[c\bar{c}(3S_1, \lambda) + gg]|^2 = 4E_q^2 \sum \{ |F_0(\lambda)|^2 \langle \mathcal{O}^{c\bar{c}}_1 \rangle + 2 \text{Re}[F_0 F_2^* \langle \mathcal{P}^{c\bar{c}}_1 \rangle] + \cdots \} = 4E_q^2 (2N_c) \sum \left\{ |F_0(\lambda)|^2 + 2 \text{Re}[F_0 F_2^*] \frac{q^2}{m_c^2} + \cdots \right\}.
$$

(16)

The matrix elements $\langle \mathcal{O}^{c\bar{c}}_1 \rangle$ and $\langle \mathcal{P}^{c\bar{c}}_1 \rangle$ denote the vacuum expectation values of the production operators for producing the free $c\bar{c}(3S_1^{(1)})$ state, which are given by

$$
\langle \mathcal{O}^{c\bar{c}}_1 \rangle = 2N_c,
$$

(17a)

$$
\langle \mathcal{P}^{c\bar{c}}_1 \rangle = \frac{q^2}{m_c^2} \langle \mathcal{O}^{c\bar{c}}_1 \rangle,
$$

(17b)

where the factor of $2N_c$ in the right side of Eq. (17a) arises from the spin and color normalization factors for free $c\bar{c}$ states. Comparing both sides of (16), one may directly deduce the short-distance coefficients $|F_0|^2$ and $\text{Re}[F_0 F_2^*]$.

In passing it may be worth reminding that, during the polarization sum/average procedure, new factors of $E_q$ will be unavoidably regenerated in the squared amplitude. Evidently, such factors can arise from summing the polarization states of $c\bar{c}(3S_1^{(1)})$ state. In the standard way of expansion, one needs re-expand these occurring $E_q$ factors once and more, and keeping reshuffling the corresponding terms from the LO matrix element squared to the relativistic correction piece. Fortunately, since we keep $E_q$ fixed in our approach, no any extra labor needs to be invested for such complication. This comprises another attractive trait of our expansion strategy.

Substituting the short-distance coefficients (14) to (12), or directly converting the quark amplitude squared (16) to (12), after some straightforward algebra, one then obtains the desired squared matrix element for producing $J/\psi$ plus light hadrons.

There arises one immediate question. Since $m_c$ has been eliminated in favor of $E_q$ in the physical matrix element squared, it is necessary to specify which value of $E_q$ should be taken, in order to make concrete predictions. If there were no rationale to restrict its value, our approach would just yield ad hoc predictions and lack attractiveness.

Fortunately, the answer to this question is definite, i.e., theoretical consistency requires that $E_q$ can be fixed in an unambiguous manner. To see this, let us first make the following observation:

$$
2E_q = 2m_c \left( 1 + \frac{q^2}{2m_c^2} + O(q^4) \right),
$$

(18a)

$$
M_{J/\psi} = 2m_c \left( 1 + \frac{1}{2}(q^2)_{J/\psi} + O(v^4) \right),
$$

(18b)
where (18a) comes from simple nonrelativistic kinematics, the inverse relation of (15). 
Eq. (18b) is nothing but the G-K relation (6).

The similarity between (18a) and (18b) strongly suggests that, $E_q$ appearing everywhere in the short-distance coefficients in (12) can be replaced by $M_{J/\psi}/2$. Naively, the entering of $J/\psi$ mass into short-distance coefficients seems to be a nuisance, which is against the doctrine of the EFT. Nevertheless, this procedure is valid, at least to the present accuracy of order $v^2$, thanks to the G-K relation 12.

Identification of $2E_q$ with $M_{J/\psi}$, in conjunction with our nonstandard expansion (15), turn out to have great advantages. By this way, the relativistic effects in phase space integrals are automatically incorporated. In some sense, our approach fulfills the role of the color-singlet shape-function by promoting the partonic kinematics to hadronic kinematics, but not necessarily restricted to the region of the maximum $J/\psi$ energy. In addition, since the mass of $J/\psi$ is known rather precisely, it is better to choose it as the input parameter than the ambiguously defined charm quark mass. To summarize, our method greatly simplifies the efforts required for the matching calculation, by reducing the task of matching the cross section to matching the amplitude squared.

\[ \text{IV. THREE-BODY PHASE SPACE FOR } e^+e^- \rightarrow c\bar{c}(3S_1) + gg \]

One integral part of the matching procedure is to consistently incorporate the relativistic effects inherent in the phase space integration. As was explained in section III B, owing to the virtue of our matching approach, no special care needs to be paid to the phase space integral, provided that we identify the invariant mass of $c\bar{c}$ pair, $2E_q$, to be $M_{J/\psi}$. For the process $e^-(l_1) + e^+(l_2) \rightarrow \gamma^*(K) \rightarrow c\bar{c}(3S_1^{(1)}, P) + g(k_1) + g(k_2)$ considered in this work, the energy-momentum conservation requires $K = l_1 + l_2 = P + k_1 + k_2$. The electron and gluon are treated to be massless, and $P^2 = M^2_{J/\psi}$. We will evaluate the three-body phase space $d\Pi_3$ in the $e^+e^-$ center-of-momentum (laboratory) frame.

The center-of-mass energy squared is defined by $s \equiv K^2$. It is also convenient to define a dimensionless ratio $r \equiv M^2_{J/\psi}/s$. The differential three-body phase space can be expressed as follows:

\[
\int d\Pi_3 = \int \frac{d^3P}{(2\pi)^3 2P^0} \frac{d^3k_1}{(2\pi)^3 2k_1^0} \frac{d^3k_2}{(2\pi)^3 2k_2^0} (2\pi)^4 \delta^{(4)}(K - P - k_1 - k_2)
= \frac{s}{2(4\pi)^4} \int_{2\sqrt{s}}^{1+r} dz \int_0^1 d\cos \theta \int_{x_1^1}^{x_1^+} dx_1 \int_0^{2\pi} d\Phi_1^*.
\]

(19)

It is worth noting that, in contrast to (9), the advantage of our matching method is there is no need to expand $P$ around the momentum of a fictitious particle with mass of $2m_c$. For notational simplicity, we have introduced three dimensionless variables, $z \equiv \frac{2P \cdot K}{K^2} = \frac{2P^0}{\sqrt{s}}$, $x_1 \equiv \frac{2k_1 \cdot K}{K^2} = \frac{2k_1^0}{\sqrt{s}}$, $x_2 \equiv \frac{2k_2 \cdot K}{K^2} = \frac{2k_2^0}{\sqrt{s}}$, to characterize the fractional energies carried by the $c\bar{c}(3S_1^{(1)})$ pair, the gluon 1, and gluon 2 in the laboratory frame. Only two of these three variables are independent, since they are subject to the constraint from energy conservation:

\[ \text{12 It is worth noting that this substitution has also been adopted in a recent investigation of the relativistic corrections to the exclusive charmonium production process } e^+e^- \rightarrow J/\psi + \eta_c. [23] \]
\(x_1 + x_2 + z = 2\). In the following, we will eliminate \(x_2\) everywhere in favor of \(z\) and \(x_1\) as the integration variables.

We use \((\theta, \phi)\) to denote the polar and azimuthal angles of the outgoing \(J/\psi\) momentum with respect to the moving direction of \(e^-\) in the laboratory frame. We have suppressed \(d\phi\) in the integration measure since it has been trivially integrated over due to the axial symmetry of the reaction under consideration. It is convenient to introduce a set of auxiliary solid-angle variables \((\Theta_1^*, \Phi_1^*)\) as the polar and azimuthal angles of the moving direction of the gluon 1 in a rotated coordinate system relative to the laboratory frame, where the \(J/\psi\) moves along the new \(+\hat{z}\) axis. Given the energy fractions \(z\) and \(x_1\), four-momentum conservation uniquely constrains the polar angle \(\Theta_1^*\):

\[
\cos \Theta_1^* = \frac{2(1 + r - z) - x_1(2 - z)}{x_1 \sqrt{z^2 - 4r}}. \tag{20}
\]

The main advantage of choosing these integration variables as given in (19), is that each of them has an intuitive interpretation and the respective integration boundaries are rather simple. This is in contrast with the set of variables employed in Ref. [22, 28, 59].

The integration boundaries for \(z\) have been explicitly labeled in (19), and those for \(x_1\) can be easily inferred:

\[
x_1^\pm = \frac{2 - z \pm \sqrt{z^2 - 4r}}{2}. \tag{21}
\]

For the suppressed variable \(x_2\), the boundaries would be the exactly same as \(x_1\).

If one concentrates only on the energy spectrum of (un)polarized \(J/\psi\), disregarding its angular distribution, one may take a shortcut– by starting with the simpler process \(\gamma^* \rightarrow J/\psi + gg\), then converting the differential decay rate to the \(J/\psi\) differential production cross section [25]. In such case, since there is no preferred orientation in space, two angular variables, \(\cos \theta\) and \(\Phi_1^*\), can be trivially integrated over in (19), consequently one is left with only two dimensionless energy variables in the three-body-phase-space measure:

\[
\int d\Pi_3 = \frac{s}{2(4\pi)^3} \int_{2\sqrt{r}}^{1+r} dz \int_{x_1^-}^{x_1^+} dx_1. \tag{22}
\]

In some situation, it is desirable to know the analytic expression for the integrated \(J/\psi\) production rate. To this purpose, it seems more advantageous to choose a different order to perform the phase-space integration:

\[
\int d\Pi_3 = \frac{s}{2(4\pi)^3} \int_0^{1-r} dx_1 \int_{1-x_1+\frac{r}{x_1}}^{1+r} dz. \tag{23}
\]

As an intermediate byproduct, one can deduce the gluon energy spectrum once performing the integration over \(z\). Phenomenologically, knowing this is not so meaningful, since it cannot be directly linked with a physical observable. However, as a calculational device, choosing this particular order for phase-space integration leads to considerable technical simplicity, because the integration boundaries in (23) are far simpler than those in (22). As a consequence, by this way one can readily deduce the \(J/\psi\) total production rate in a closed form, which is otherwise rather difficult to achieve if one starts from (22).
V. OUTLINE OF MATCHING CALCULATIONS FOR $e^+e^- \rightarrow c\bar{c}(3S_1) + gg$

In this section, we present a detailed description on how the short-distance coefficients through relative order-$v^2$ can be determined via our matching procedure, concretizing the method put forward in section III B. We also illustrate how the various types of predictions for the inclusive $J/\psi$ production rate in $e^+e^-$ annihilation emerge.

In order to deduce the intended short-distance coefficients, one needs to consider the parton process $e^+e^- \rightarrow c\bar{c}(3S_1, P, \lambda) + gg$, with one typical lowest-order diagram shown in Fig. 1. The calculation is expedited by the covariant projection technique developed by Bodwin and Petrelli [57], which helps to readily project out the amplitude for the $c\bar{c}$ pair being in the color-singlet spin-triplet state $^{13}$.

Our matching method will be exemplified by the following three subsections. In section VA and section VB, where we are only interested in the energy spectra of unpolarized and longitudinally-polarized $J/\psi$, we take the shortcut by considering the simpler process $\gamma^* \rightarrow c\bar{c}(3S_1, P, \lambda) + gg$; in section VC, where we are also interested in the angular as well as the energy distributions of $J/\psi$, we work with the full process $e^+e^- \rightarrow c\bar{c}(3S_1, P, \lambda) + gg$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Lowest-order Feynman diagrams for $e^+e^- \rightarrow J/\psi + gg$.}
\end{figure}

A. Matching calculation for $\gamma^* \rightarrow c\bar{c}(3S_1) + gg$

We start with the tree-level quark amplitude $\gamma^*(K) \rightarrow c(p)\bar{c}(\bar{p}) + g(k_1) + g(k_2)$, with the momenta of $c$ and $\bar{c}$ defined in (10) $^{14}$. The amplitude can be written as

$$\bar{u}(p)\mathcal{A}v(\bar{p}) = \text{Tr} [v(\bar{p})\bar{u}(p)\mathcal{A}].$$

$^{13}$ An alternative approach, the threshold expansion method [60], is also valid to deduce the NRQCD matching coefficients. This method has been utilized to investigate the order-$v^2$ relativistic correction to $J/\psi$ photoproduction at HERA [49]. However, for the process with involved kinematics like ours, this method, which requires to extensively deal with the algebra of two-component spinors, seems not as convenient as the covariant projection technique outlined in [57].

$^{14}$ The calculation presented in this subsection is somewhat similar to that for the process $g^* \rightarrow J/\psi + gg$, from which the relativistic correction to the gluon-to-$J/\psi$ color-singlet fragmentation function can be extracted [56].
Here $\mathcal{A}$ is a Dirac-color-space matrix, which reads

$$\mathcal{A} = (e_c e g_s^2) T^a T^b \otimes \gamma^* (g; k_1) \frac{1}{\not{p} + \not{k}_1 - m_c} \gamma (\gamma^*; K) \frac{1}{-\not{p} - \not{k}_2 - m_c} \gamma^* (g; k_2) + 5 \text{ perms}, \quad (25)$$

where $g_s$ signifies the QCD coupling strength, and $e e_c$ denotes the electric charge of the charm quark ($e_c = \frac{2}{3}$). $a, b$ denote the color indices of the two gluons, $\epsilon (\gamma^*; K), \epsilon^* (g; k_1)$ and $\epsilon^* (g; k_2)$ represent the polarization vectors of the decaying virtual photon, gluon 1 and 2, respectively.

One of the important sources of relativistic corrections stem from expanding the quark propagators. Apart from retaining the factor $q$ in the numerator of the propagator, one needs also expand its denominator to the quadratic order in $q$. Taking the diagram shown in Fig. 1 as an example, two propagators there are expanded to be

$$\frac{1}{(p + k_1)^2 - m_c^2} = \frac{1}{P \cdot k_1} - \frac{2q \cdot k_1}{(P \cdot k_1)^2} + \frac{4(q \cdot k_1)^2}{(P \cdot k_1)^3} + O(q^3), \quad (26a)$$

$$\frac{1}{(\not{p} + \not{k}_2)^2 - m_c^2} = \frac{1}{P \cdot k_2} + \frac{2q \cdot k_2}{(P \cdot k_2)^2} + \frac{4(q \cdot k_2)^2}{(P \cdot k_2)^3} + O(q^3). \quad (26b)$$

To proceed, we need project the amplitude (24) onto the spin-triplet color-singlet $c(p)\bar{c}(\bar{p})$ state, by replacing the $v(\not{p})\bar{u}(\not{p})$ with a suitable projection matrix. The projector that is valid to all orders in $q$ for the spin-triplet color-singlet channel, denoted by $\Lambda_3^{(1)} (p, \bar{p}, \lambda)$ ($\lambda$ characterizes the polarization of this spin-triplet $c\bar{c}$ pair), assumes the particular form [57]:

$$\Lambda_3^{(1)} (p, \bar{p}, \lambda) = \frac{1}{4 \sqrt{2} E_q (E_q + m_c)} (\not{p} - m_c) \gamma^* (\lambda) (P + 2E_q) (\not{p} + m_c) \otimes \frac{1}{\sqrt{N_c}}, \quad (27)$$

where $1_c$ is the unit matrix in the fundamental representation of the color $SU(3)$ group, and the spin-polarization vector $\epsilon^* (\lambda)$ satisfies $P \cdot \epsilon^* (\lambda) = 0$. The above spin projector is derived by assuming the relativistic normalization convention for Dirac spinor: $\bar{u}^{(r)} u^{(s)} = 2m_\delta_{rs}$ and $u^{(r)\dagger} u^{(s)} = 2E_q \delta_{rs}$. Applying this spin-color projector to (24), we obtain

$$\mathcal{M}_{cc} (P, q, \lambda; k_1, k_2) = \text{Tr} \{ \mathcal{A} \lambda_3^{(1)} (p, \bar{p}, \lambda) \}, \quad (28)$$

where the trace acts on both Dirac and color spaces. $\mathcal{M}_{cc} (P, q, \lambda; k_1, k_2)$ can be interpreted as the amplitude for producing a color-singlet, spin-triplet $c\bar{c}$ pair in association with two gluons.

We have emphasized that our method differs from the conventional tenet of matching. Rather than Taylor-expanding $E_q$ around $m_c$ in $q^2/m_c^2$ everywhere in $\mathcal{M}_{cc} (P, q, \lambda; k_1, k_2)$, we choose to expand $m_c$ around $E_q$ in powers of $q^2/E_q$ using (15). It is worth reminding that we should not forget to trade $m_c$ for $E_q$ that appears in the denominator of the projector (27):

$$\frac{1}{E_q + m_c} = \frac{1}{2E_q} \left( 1 + \frac{1}{4} \frac{q^2}{E_q^2} + O(q^4) \right). \quad (29)$$

It is convenient, at this stage, to truncate the amplitude $\mathcal{M}_{cc}$ such that all the terms in it are at most quadratic in $q$.

In the amplitude (28), the $c\bar{c}$ pair is warranted to be in the spin-triplet, but not necessarily in the $S$-wave orbital-angular-momentum state. To project out the $S$-wave amplitude, one
needs average the amplitude $\mathcal{M}_{cc}$ over all the direction of the relative momentum $q$ in the rest frame of $cc(P)$ pair. This literal angular averaging procedure can help to acquire a specific class of relativistic corrections to all orders in $v$. However it is feasible only for few processes with very simple kinematics [23, 57]. For the process at hand, this procedure would become extremely cumbersome, if not impossible. Fortunately, to the intended $O(v^2)$ accuracy, one can utilize a standard trick to project out the $S$-wave part. Now we already have the amplitude truncated up to two powers of $q$. Terms that contain no powers of $q$, or contain explicitly the Lorentz scalar $q^2$ (which can be translated into $-q^2$), already yield a pure $S$-wave contribution; for those terms containing the tensor $q^\mu q^\nu$ contracted with other 4-vectors, we can make the following substitution to extract the $S$-wave piece 15:

$$q^\mu q^\nu \rightarrow \frac{q^2}{3} \Pi^{\mu\nu}(P),$$

where

$$\Pi^{\mu\nu}(P) \equiv -g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2}.$$

Following all these steps, we then obtain the desired $^{3}S_{1}$ piece of the amplitude, $\mathcal{M}(^{3}S_{1}, P, \lambda; k_1, k_2)$, accurate through the order $a_2^2$. Following what has been elaborated in section III.B, at this stage it is legitimate to replace $E_q$ everywhere by $M_{J/\psi}/2$ in $\mathcal{M}(^{3}S_{1}, P, \lambda; k_1, k_2)$.

It is now the time to deduce the desired “short-distance” coefficients $F_0$ and $F_2$, following the recipes given in (14a) and (14b). After this is done, we need square these coefficients and perform the corresponding spin-color sum/average:

$$\sum |F_0|^2 = \frac{1}{3} \Pi_{\mu\nu}(K) \Pi_{\rho\sigma}(P) (-g_{\alpha\alpha'})(-g_{\beta\beta'}) \mathcal{F}_0^{\mu; \rho\alpha\beta} \mathcal{F}_0^{* \mu'; \rho'\alpha'\beta'},$$

$$2 \sum \Re [F_0 F_2^*] = \frac{1}{3} \Pi_{\mu\nu}(K) \Pi_{\rho\sigma}(P) (-g_{\alpha\alpha'})(-g_{\beta\beta'}) 2 \Re \mathcal{F}_0^{\mu; \rho\alpha\beta} \mathcal{F}_2^{* \mu'; \rho'\alpha'\beta'},$$

where the amputated “short-distance” coefficients $\mathcal{F}_i$ ($i = 0, 2$) are defined through

$$F_i = \mathcal{F}_i^{\mu; \rho\alpha\beta} \epsilon_\mu(\gamma^*; K) \epsilon_\rho^{\ast}(^{3}S_1; P, \lambda) \epsilon_\alpha^{\ast}(g; k_1) \epsilon_\beta^{\ast}(g; k_2).$$

For simplicity, we have suppressed the color indices, implicitly contained in $\mathcal{F}_i$, which reads $\text{Tr}(T^a T^b)/\sqrt{N_c}$. In Eqs. (32), we have summed over polarization and color of the $cc(3S_1^{(1)})$ pair and two gluons, and averaged upon three spin states of the virtual photon (note the prefactor $\frac{1}{3}$). Polarization sum for two gluon states has been taken into account by the metric tensors $-g_{\alpha\alpha'}$ and $-g_{\beta\beta'}$, and that for the virtual photon and $J/\psi$ by the polarization tensor $\Pi_{\mu\nu}(K)$ and $\Pi_{\rho\sigma}(P)$.

Note this $S$-wave projection operation will regenerate factors of $E_q$ in the denominator of the amplitude. Obviously it will not cause any trouble for our strategy of matching. In contrast, in the orthodox matching recipe, one is forced to reexpand those terms containing these newly-generated $E_q$ factors, and reshuffling the corresponding terms from the “leading-order” piece to the relativistic correction piece. This further exhibits the merit of our method.
According to equation (12), we then obtain the squared matrix element for $\gamma^* \rightarrow J/\psi + gg$. Including the 3-body phase space measure (22), we can obtain the differential decay rate for unpolarized $J/\psi$:

$$\frac{d\Gamma[\gamma^* \rightarrow J/\psi + gg]}{dzdx_1} = \frac{1}{2!} \frac{\sqrt{s}}{4(4\pi)^3} \sum |\mathcal{M}[\gamma^* \rightarrow J/\psi + gg]|^2,$$

where we have included a statistical factor of $\frac{1}{2!}$ to account for the indistinguishability of two gluons.

To convert the differential decay rate in Eq. (34) into $J/\psi$ production cross section, one can use the formula [25]

$$\frac{d\sigma[e^+e^- \rightarrow J/\psi + gg]}{dzdx_1} = \frac{4\pi\alpha}{s^{3/2}} \frac{d\Gamma[\gamma^* \rightarrow J/\psi + gg]}{dzdx_1}.$$  

(35)

It is not difficult to integrate over the fractional energy of gluon 1, $x_1$, with the integration boundaries specified in (21) to acquire the energy spectrum for unpolarized $J/\psi$.

**B. Matching calculation for $\gamma^* \rightarrow c\bar{c}(^3S_1, \lambda = 0) + gg$**

It is also interesting to know how the polarization information of $J/\psi$ varies with its energy. To accomplish this, in addition to the differential energy spectrum for unpolarized $J/\psi$ as given in section VA, we also need know that for the longitudinally-polarized $J/\psi$. In this section we outline how the corresponding matching calculation is carried out.

The “short-distance” coefficients $F_0$ and $F_2$ can be obtained following the same procedure as outlined in section VA. Nevertheless, the longitudinal polarization vector of $J/\psi$ can be explicitly substituted by

$$\epsilon^*_L(\mu; P, \lambda = 0) = \frac{P_0}{2E_q |\mathbf{P}|} \frac{2E_q}{\sqrt{s}} \mathbf{P} K^\mu,$$

(36)

which satisfies $\epsilon_L \cdot P = 0$ and $\epsilon_L \cdot \epsilon^*_L = -1$.

We then square these coefficients and perform the corresponding spin-color sum/average:

$$\sum |F_0|^2 = \frac{1}{3} \Pi_{\rho'\rho}(K) \epsilon^*_L(\rho; S_1) \epsilon_L(\rho'; S_1) (-g_{\alpha\alpha'})(-g_{\beta\beta'}) F_0^{\mu;\rho\rho'} F_0^{\mu';\rho'\rho'},$$

$$2 \sum \text{Re} [F_0 F^*_2] = \frac{1}{3} \Pi_{\rho'\rho}(K) \epsilon^*_L(\rho; S_1) \epsilon_L(\rho'; S_1) (-g_{\alpha\alpha'})(-g_{\beta\beta'}) 2 \text{Re}[F_0^{\mu;\rho\rho'} F_2^{\mu';\rho'\rho'\rho'}].$$

(37a)

(37b)

Here the amputated “short-distance” coefficients $F_i$ are the same as what appear in Eqs. (32). Obviously the sum over polarizations need not act on the $c\bar{c}(^3S_1, \lambda = 0)$ state.

Apart from replacing $E_q$ everywhere with $M_{j/\psi}/2$, we also need substitute $P^0 = \frac{\sqrt{s}}{2} z$ and $|\mathbf{P}| = \frac{\sqrt{s}}{2} \sqrt{z^2 - 4r}$ for $\epsilon^*_L$ in above expressions. We then follow (12) to obtain the squared matrix element for $\gamma^* \rightarrow J/\psi_L + gg$. With this expression at hand, one can use (34) to infer the differential decay rate from a virtual photon, subsequently use (35) to deduce the corresponding differential cross section for producing the longitudinal-polarized $J/\psi$ in $e^+e^-$ annihilation.
C. Matching calculation for $e^+e^- \to c\bar{c}(3S_1) + gg$

In section VA and VB, we have resorted to a shortcut by considering the production rate of a $J/\psi + gg$ from a virtual photon decay, since the sole purpose is to deduce the energy spectrum of unpolarized or longitudinally-polarized $J/\psi$ in $e^+e^-$ annihilation. In this subsection, we are interested in knowing the angular-energy double differential distribution of unpolarized $J/\psi$. To this end, it is compulsory to begin with the full process $e^-(l_1)e^+(l_2) \to J/\psi(P) + g(k_1)g(k_2)$. We use $l_1$ and $l_2$ to signify the momenta of $e^-$ and $e^+$, respectively, and $l_1 + l_2 = K$.

The main result derived in section VA can be directly transplanted here. In particular, the “short-distance” coefficients $F_0$ and $F_2$ for $\gamma^* \to J/\psi + gg$, determined there by employing (14a) and (14b), only need undergo some slight modifications to meet our purpose. That is, one needs replace the polarization vector of the virtual photon by a $e^+e^-$ bispinor and insert a photon propagator and a QED coupling. These slight changes are embodied in squaring these coefficients and performing the corresponding spin-color sum/average:

$$
\sum |F_0|^2 = \frac{1}{4} L_{\mu\nu} \epsilon^2 \Pi_{\rho\sigma} (P) (-g_{\alpha\alpha'})(-g_{\beta\beta'}) F_0^{\mu; \rho\alpha\beta} F_0^{* \nu; \rho'\alpha'\beta'}, \tag{38a}
$$

$$
2 \sum \text{Re}[F_0 F_2^*] = \frac{1}{4} L_{\mu\nu} \epsilon^2 \Pi_{\rho\sigma} (P) (-g_{\alpha\alpha'})(-g_{\beta\beta'}) 2 \text{Re}[F_0^{\mu; \rho\alpha\beta} F_2^{* \nu; \rho'\alpha'\beta'}]. \tag{38b}
$$

Here the amputated “short-distance” coefficients $F_i$ are the same as what appear in Eqs. (32). The factor $1/4$ represents the average over polarizations of the initial $e^-e^+$ state, and $L_{\mu\nu}$ denotes the leptonic tensor:

$$
L_{\mu\nu} = \sum_{s,r} \left[ \bar{v}(l_2; r) \gamma^0 u(l_1; s) \right] \left[ \bar{u}(l_1; s) \gamma_{\mu\nu} v(l_2; r) \right] = 4 \left( l_{\mu 1} l_{\nu 2} + l_{\nu 1} l_{\mu 2} - l_1 \cdot l_2 g_{\mu\nu} \right), \tag{39}
$$

where the sum is extended over all possible polarization states of the electron and positron.

Substituting Eqs. (38) into (12), we then obtain the color-spin averaged/summed matrix element squared for the process $e^+e^- \to J/\psi + gg$. Including the 3-body phase space measure (19) and the flux factor, we can obtain the differential production rate for unpolarized $J/\psi$:

$$
\frac{d\sigma[e^+e^- \to J/\psi + gg]}{dz \, d\cos \theta \, dx_1 \, d\Phi_1^*} = \frac{1}{2!} \frac{1}{4(4\pi)^4} \sum |\mathcal{M}[e^+e^- \to J/\psi + gg]|^2. \tag{40}
$$

In the squared matrix elements, all the scalar products can be expressed in terms of $z$, $\theta$, $x_1$, and one additional angular variable, $\theta_1$, which represents the angle between the 3-momentum of gluon 1 and the beam direction in the laboratory frame. This polar angle is connected to $\theta$, $\Theta_1^*$ and $\Phi_1^*$ through

$$
\cos \theta_1 = \cos \theta \cos \Theta_1^* - \sin \theta \sin \Theta_1^* \cos \Phi_1^*, \tag{41}
$$

where $\cos \Theta_1^*$ is uniquely determined when $z$ and $x_1$ are given, as indicated in (20).

As first elaborated in [28], general consideration based on Lorentz invariance, parity and gauge invariance demands that for inclusive $J/\psi$ production in $e^+e^-$ annihilation, the double differential distribution must bear the following form:

$$
\frac{d\sigma[e^+e^- \to J/\psi + X]}{dz \, d\cos \theta} = S(z)[1 + A(z) \cos^2 \theta]. \tag{42}
$$
It is interesting to note that, after the suitable reduction, the squared matrix element for this reaction can depend upon the polar angles only through rather limited combinations—1, $\cos^2 \theta$, $\cos \theta \cos \Theta_1$, and $\cos^2 \theta_1$, respectively. Therefore, to arrive at the expression indicated in (42), suffices it to know the following integrations over $d\Phi_1$:

$$
\int_0^{2\pi} d\Phi_1^* = 2\pi, \quad (43a)
$$

$$
\int_0^{2\pi} d\Phi_1^* \cos \theta_1 = 2\pi \cos \theta \cos \Theta_1^*, \quad (43b)
$$

$$
\int_0^{2\pi} d\Phi_1^* \cos \theta_2^1 = 2\pi \left[ \frac{1}{4} (1 + \cos^2 \theta) + \left( \frac{1}{4} - \frac{1}{2} \cos^2 \Theta_1^* \right) (1 - 3 \cos^2 \theta) \right]. \quad (43c)
$$

Thus we are reassured that only the zeroth and second powers of $\cos \theta$ are allowed to appear in the double differential distribution for $J/\psi$ production, in conformity to (42). It may be also worth pointing out that, one great simplification can be made insofar as only the differential energy spectrum of unpolarized $J/\psi$ is concerned. In this case, the second term inside the square bracket in (43c) can be discarded, since its contribution vanishes upon integration over $\theta$.

**VI. INCLUSIVE $J/\psi$ DISTRIBUTIONS AT $B$ FACTORY**

In this section, we report our results of order-$v^2$ relativistic correction to inclusive $J/\psi$ production associated with non-$c \bar{c}$ states at $B$ factory at $\sqrt{s} = 10.58$ GeV. We investigate its impact on the integrated cross sections, and various types of distributions of $J/\psi$ at $B$ factory, which is found to be modest.

Very recently the order-$v^2$ correction to $e^+e^- \rightarrow J/\psi gg$ at $B$ factory has also been studied by He, Fan and Chao [59], who have found similar magnitude of the relativistic correction to the integrated cross section for unpolarized $J/\psi$. Since none of the differential distributions for energy, angular and polarization of $J/\psi$ have been explicitly given in [59], it is not possible at this stage to make a detailed comparison between our results and theirs. Nevertheless, these authors chose to expand $E_q$ around $m_c$ in powers of $q^2$ in the amplitude, which is opposite to the strategy employed in this work. Most notably, it seems that relativistic correction effects associated with the three-body phase space has been neglected in [59], thus the exact agreement between our results and theirs will not be expected.

**A. Choice of input parameters**

To make concrete predictions, we need specify various input parameters, in particular the corresponding NRQCD production matrix elements. The LO NRQCD matrix element $\langle \mathcal{O}_{J/\psi}^1 \rangle$, together with some specific combination of coupling constants and mass scales will be frequently encountered in many expressions for $J/\psi$ production rate. For notational compactness, it is thus convenient to lump them into a single factor:

$$
\tilde{\sigma}_0 = \frac{256 \pi (e_c \alpha_s) \alpha_s^2}{27 M_{J/\psi} s^2} \langle \mathcal{O}_{J/\psi}^1 \rangle. \quad (44)
$$
We take \( M_{J/\psi} = 3.097 \text{ GeV}, \sqrt{s} = 10.58 \text{ GeV} \) at \( B \) factory energy, thus fix \( r \equiv M_{J/\psi}^2/s = 0.0857 \). For the NRQCD matrix elements, we quote the values extracted from the recent Cornell-potential-model-based analysis [54]:

\[
\langle O_{i}^{J/\psi} \rangle = 0.440^{+0.067}_{-0.055} \text{ GeV}^3, \quad (45a)
\]
\[
\langle v^2 \rangle_{J/\psi} = 0.225^{+0.106}_{-0.088}. \quad (45b)
\]

Note the uncertainties affiliated with each NRQCD matrix elements are quite sizable. With resort to the G-K relation (18b), one finds that this value for \( \langle v^2 \rangle_{J/\psi} \) corresponds to the charm quark pole mass \( m_c = 1.39 \pm 0.06 \text{ GeV} \).

Targeting at a better accuracy, we also including the running effect in the electromagnetic coupling, i.e., we take the fine structure constant to be \( \alpha(\sqrt{s}) = 1/130.9 \), rather than the commonly used 1/137 [23]. For the strong coupling constant, we take the central value \( \alpha_s \) equal to 0.21, corresponding to choosing the renormalization scale \( \mu \) at \( \sqrt{s}/2 \). The corresponding uncertainty is estimated by varying this coupling between 0.17 to 0.26, obtained by sliding the renormalization scale \( \mu \) between \( \sqrt{s} \) and \( \sqrt{s}/4 \) [23].

With all these parameters specified, we find

\[
\bar{\sigma}_0 = 0.150^{+0.115}_{-0.064} \text{ pb}. \quad (46)
\]

The attached error comes from the uncertainties of \( \alpha_s \) and of the NRQCD matrix element \( \langle O_1^{J/\psi} \rangle \).

**B. The integrated production rate for \( J/\psi \) associated with light hadrons**

The lowest-order NRQCD predictions to the \( J/\psi \) associated production rate have been available for a long while [25, 28–31]. For convenience of the reader, we collect in Appendix A the expressions for the energy distribution of (un)polarized \( J/\psi \), at LO as well as at NLO in \( v^2 \).

One may attempt to directly integrate the differential \( J/\psi \) spectrum over the entire \( J/\psi \) energy range to deduce the integrated \( J/\psi \) cross section. Unfortunately, it seems rather difficult to obtain the analytic expression by this way, even for the LO cross section. Fortunately, to this purpose, it is much more advantageous to carry out the 3-body phase space integration in a route as specified in (23), where the corresponding integration boundaries become simpler. After some straightforward calculations, we find that the LO integrated cross section for the unpolarized \( J/\psi \) can be put in the following compact form:

\[
\sigma^{(0)}[J/\psi + X_{\text{light}}] = \bar{\sigma}_0 \left\{ \frac{2 - r - 12r^2 + 8r^3}{2(1-r)^2} \arctanh^2 \sqrt{1-r} + \frac{4 - 9r + 8r^2}{(1-r)^{3/2}} \arctanh \sqrt{1-r} + \frac{5 - 14r + 3r^2}{2(1-r)^2} \ln r - \frac{9(1-2r + 2r^2)}{2(1-r)} \right\}. \quad (47)
\]

We note that the analytic expression for \( \sigma^{(0)} \) has already been available recently [44]. Our expression is in agreement with equation (2) in [44], but appears to be simpler.\(^\text{16}\)

\(^\text{16}\) This can be mainly attributed to the fact that a pair of dilogarithms appearing in their formula can actually be transformed away, by exploiting a sequence of identities about dilogarithms.
It is interesting to examine the asymptotic behavior of (47) in the high energy limit $\sqrt{s} \gg M_{J/\psi}$:

$$\sigma^{(0)}[J/\psi + X_{\text{light}}] = \tilde{\sigma}_0 \left[ \frac{1}{4} \ln^2 r + \left( \frac{1}{2} - \ln 2 \right) \ln r + \ln^2 2 + 4 \ln 2 - \frac{9}{2} + O(r \ln^2 r) \right].$$

(48)

Beside the power-law scaling contained in $\tilde{\sigma}_0 (\propto 1/s^2)$, the asymptotic behavior of the total cross section is dominated by the double logarithm term. This expression is superficially analogous to the NLO perturbative correction to the exclusive double-charmonium production process $e^+e^- \rightarrow J/\psi + \eta_c$, which also exhibits a double logarithm scaling [21].

It is also of some interest to examine the opposite limit $r \rightarrow 1$, in which the $J/\psi$ is produced just above the kinematic threshold. The total cross section in this limit vanishes as $\tilde{\sigma}_0 (1 - r)^3 + O((1 - r)^2)$, which may reflect that gluon radiation off the heavy quark is greatly damped in very restricted phase space.

Substituting $r = 0.0857$ and the value of $\tilde{\sigma}_0$ given in (46) into (47), or equivalently, numerically integrating the spectrum (A1) over the entire $J/\psi$ energy, we find the LO prediction to the total cross section for $J/\psi$ associated with non-$c\bar{c}$ states at $B$ factory is

$$\sigma^{(0)}[J/\psi + X_{\text{light}}] = 0.200^{+0.153}_{-0.085} \text{ pb}. \quad (49)$$

The error is solely due to the uncertainty in $\tilde{\sigma}_0$.

We now turn to the order-$v^2$ contribution to the integrated cross section for producing the unpolarized $J/\psi$. Unlike in the LO case, the corresponding analytical expression is too complicated to be presented here, and we are content with providing numerical result only. Taking the value of $\tilde{\sigma}_0$ from (46), together with the ratio of the NRQCD matrix elements $\langle v^2 \rangle_{J/\psi}$ in (45b), integrating the order-$v^2$ correction to the spectrum (A2) over the entire $J/\psi$ energy range, we find

$$\sigma^{(2)}[J/\psi + X_{\text{light}}] = 0.061^{+0.097}_{-0.040} \text{ pb}, \quad (50)$$

where the attached error is due to the uncertainties in $\tilde{\sigma}_0$ and in $\langle v^2 \rangle_{J/\psi}$. It is clear to see that for the central value of the predictions, the inclusion of the order-$v^2$ correction enhances the LO result by about 30%. This seems in conformity to the naive expectation about the size of relativistic correction for charmonium system. It is interesting to note that, the central value of the relative order-$v^2$ contribution seems even slightly larger than the recently-computed NLO perturbative correction, which enhances the LO result by about 20% [43, 44]. But fairly speaking, the effects of both types of corrections are not significant.

The sum of (49) and (50) turns to be

$$\left( \sigma^{(0)} + \sigma^{(2)} \right)[J/\psi + X_{\text{light}}] = 0.261^{+0.250}_{-0.125} \text{ pb}. \quad (51)$$

Compared with the latest BELLE measurements for prompt $J/\psi$ production rate associated with light hadrons, (1b), we find rough agreement between (51) and the data with large uncertainty. This agreement could be even more satisfactory if further including the NLO perturbative correction and the feeddown contribution from higher charmonium states.

In light of this rough agreement achieved by the color-singlet contribution alone, one important question is to ask how much room is left for the color-octet contribution to inclusive $J/\psi$ production at $B$ factory. It seems fair to state that earlier estimates of its
contribution [36, 37, 39] may turn out to be overly optimistic. Nevertheless, we would like to caution that, our predictions, both the LO one in (49), and the NLO one in (50), are subject to large theoretical uncertainty, so we are unable to draw any firm conclusion about the actual size of the color-octet contribution.

C. Energy spectrum of unpolarized $J/\psi$

![Energy spectrum of unpolarized $J/\psi$](image)

FIG. 2: The energy spectra of the unpolarized $J/\psi$ (left panel) and longitudinally polarized $J/\psi$ (right panel) associated with light hadrons at the energy of $B$ factory. The dot-dashed curve represents $d\sigma^{(0)}/dz$, the dashed curve represents $d\sigma^{(2)}/dz$, and the solid curve represents their sum. For simplicity, in all the figures in this work, we have taken only the central values of the input parameters and not drawn the error band.

Aside from the total production rate of $J/\psi$, it is also useful to look closely into the differential observable. As a matter of fact, $B$ factory experiments have already measured various types of $J/\psi$ distributions. In this subsection we investigate the effect of first-order relativistic correction for the energy distribution of unpolarized $J/\psi$. In Figure 2, we display the energy spectrum of unpolarized $J/\psi$ at $B$-factory energy, including both the LO result and the first-order relativistic correction.

As one can tell from Figure 2, the LO distribution admits a finite limit when the $J/\psi$ energy approaches its maximum:

$$
\left. \frac{d\sigma^{(0)}}{dz} \left[ e^+ e^- \rightarrow J/\psi + gg \right] \right|_{z \rightarrow 1+r} \rightarrow \tilde{\sigma}_0 \frac{1 + 2r}{1 - r}. \tag{52}
$$

A novel feature of relative order-$v^2$ contribution is that, as can be clearly seen in Figure 2, the spectrum has a sharp rise near the very upper end of the $J/\psi$ spectrum. After some straightforward manipulation on the analytic expression of order-$v^2$ contribution, which is recorded in equation (A2), we find the following limiting value near the endpoint:

$$
\left. \frac{d\sigma^{(2)}}{dz} \left[ e^+ e^- \rightarrow J/\psi + gg \right] \right|_{z \rightarrow 1+r} \rightarrow \tilde{\sigma}_0 \frac{\langle v^2 \rangle_{J/\psi}}{3} \times \
9 - 23r - 10r^2 - 12r(1 + r) \ln \left[ \frac{r(1+r-z)}{(1-r)^2} \right] \frac{(1-r)^2}{(1-r)^2}. \tag{53}
$$
Clearly the endpoint singularity is of the form \( \ln(1 + r - z) \).

The logarithmic divergence near the endpoint is not something new. It simply signals the breakdown of the NRQCD expansion near the kinematic boundary, as a result we should no longer trust our prediction in this region. Recall that for the NLO perturbative correction to the same process, the logarithmic singularity of \( \ln(1 + r - z) \) is also expected to appear near the maximum of \( J/\psi \) energy [45, 46]. However, it is worth mentioning that, the \( \ln(1 + r - z) \) has rather different origin for both types of corrections. For the NLO perturbative correction, the \( \ln(1 + r - z) \) term should be attributed to the collinear singularity associated with the gluonic jet recoiling against \( J/\psi \). The reason is that, at LO in \( v \), the soft gluon cannot resolve the color-singlet \( c\bar{c} \) pair (color-transparency), as a result the net contributions from soft gluons cancel, so the logarithm can be only of the collinear origin \(^{17}\). However, for the contribution from relativistic correction, this endpoint singularity comes from the region where one of the recoiling gluon becomes soft. Since we have gone beyond the LO in \( v \), the color-singlet \( c\bar{c} \) pair could still develop a nonzero color dipole, therefore it may strongly interact with the soft gluons. Therefore it is natural to identify this resulting \( \ln(1 + r - z) \) with the soft origin. It is interesting to ask whether the method presented in [45, 46], which combine NRQCD and the soft-collinear effective theory, can be generalized to resum those types of logarithm in (53) to all orders in \( \alpha_s \), to render the \( J/\psi \) energy spectrum well-behaved near the end point region.

Note this endpoint singularity is integrable, therefore we are still able to obtain a finite order-\( v^2 \) correction to the integrated cross section (see (50)). This is similar to quarkonium semi-inclusive radiative decay \( J/\psi \to \gamma + X \), where the order-\( v^2 \) correction to the photon spectrum also develops an integrable endpoint singularity. Nevertheless in that case, at relative order \( v^4 \), the photon spectrum would develop a linear infrared divergence near the end point, which results in a logarithmic divergence for the integrated decay rate [57]. It is the color-octet mechanism that should be invoked to tame this infrared divergence. In our case, we expect the exactly same pattern will occur. That is, at \( O(v^4) \), the \( J/\psi \) energy spectrum would develop a linear endpoint singularity, consequently the integrated cross section would contain a logarithmic infrared divergence, which must in turn be cured by including the color-octet contribution.

D. Polarization distribution of \( J/\psi \)

Babar and Belle collaborations can also determine the polarization of \( J/\psi \) as a function of its energy by measuring the muons’ angular distribution from \( J/\psi \to \mu^+\mu^- \). The commonly used polarization parameter is defined by

\[
\alpha(z) = \frac{d\sigma/dz - 3d\sigma_L/dz}{d\sigma/dz + d\sigma_L/dz},
\]

(54)

where \( d\sigma_L/dz \) signifies the differential cross section for producing a longitudinally-polarized \( J/\psi \). \( \alpha = 1 \) and \(-1 \) correspond to 100\% transversely- and longitudinally-polarized, whereas \( \alpha = 0 \) corresponds to 100\% unpolarized.

\(^{17}\) It seems enlightening to contrast the single collinear logarithm associated with the color-singlet channel at LO in \( v \) with the Sudakov double logarithm associated with the color-octet channel [39].
To deduce the function $\alpha(z)$, it is necessary to know the expression for $d\sigma_L/dz$. The analytical expressions for this distribution, at both LO and NLO in $v^2$, have been given in Appendix A. Moreover, both the LO and NLO contributions to the energy spectrum for the longitudinally-polarized $J/\psi$ at $B$-factory is shown in Figure 2.

Let us first investigate the integrated cross section for producing a longitudinally polarized $J/\psi$. As in the unpolarized case discussed in Section VI B, if one carries out the 3-body phase-space integration following the order specified in (23), the LO integrated cross section for the longitudinally-polarized $J/\psi$ can also be put in a closed form:

$$
\sigma_L^{(0)}[J/\psi(\lambda = 0) + \lambda] = \tilde{\sigma}_0 \left\{ \frac{4 - 2r - 4r^2 - 3r^3 + 3r^4}{4(1 - r)^2} \right. \\
- \frac{4 - 4r + r^2 - 3r^3}{2(1 - r)^{3/2}} \arctanh \sqrt{1 - r} + \frac{2 - 10r + 7r^2 - 6r^3 + 3r^4}{4(1 - r)^2} \ln r \\
+ \frac{\sqrt{r}(6 - r + 3r^2)}{4} \left( \text{Li}_2(\sqrt{r}) - \text{Li}_2(-\sqrt{r}) - \arctanh \sqrt{r} \ln r \right) \\
+ \frac{\pi^2}{16} \left( 2 - 12\sqrt{r} + 3r + 2r^{3/2} + 3r^2 - 6r^{3/2} \right) + \frac{8 - 19r + 11r^2 - 6r^3}{4(1 - r)} \right\},
$$

(55)

where $\text{Li}_2$ stands for the dilogarithm. This analytic expression has not been known previously. A nontrivial check of the correctness of this result is to examine its threshold behavior. In the limit $r \to 1$, $\sigma_L^{(0)}$ approaches zero as $\tilde{\sigma}_0 \frac{1 - r}{\gamma^2} + O((1 - r)^2)$, as expected, one-third of that for the polarization-summed case.

It is of interest to ascertain the asymptotic behavior of $\sigma_L^{(0)}$ in the high energy limit $\sqrt{s} \gg M_{J/\psi}$:

$$
\sigma_L^{(0)}[J/\psi(\lambda = 0) + \lambda] = \tilde{\sigma}_0 \left[ \frac{1}{4} \ln^2 r + \left( \frac{3}{2} - \ln 2 \right) \ln r + \ln^2 2 - 2 \ln 2 + 2 + \frac{\pi^2}{8} + O(\sqrt{r}) \right].
$$

(56)

Note the leading double logarithm appearing here has the same coefficient as in the polarization-summed case, (48)\(^{18}\). One can then readily infer the asymptotic behavior of the transversely-polarized $J/\psi$ production rate: $\sigma_T^{(0)} \equiv \sigma^{(0)} - \sigma_L^{(0)} \to \tilde{\sigma}_0 \left[ \ln \frac{1}{\sqrt{r}} + 6 \ln 2 - \frac{13}{2} - \frac{\pi^2}{8} \right]$, only exhibiting a single-logarithm scaling.

Substituting $r = 0.0857$ into (55) and using the value of $\tilde{\sigma}_0$ given in (46), or straightforwardly integrating the spectrum (A3) over the entire $J/\psi$ energy numerically, we find the LO prediction to the integrated rate for producing longitudinally-polarized $J/\psi$ in association with non-$c\bar{c}$ states at $B$ factory to be

$$
\sigma_L^{(0)}[J/\psi + \lambda] = 0.128^{+0.098}_{-0.054} \text{ pb}.
$$

(57)

The error originates solely from the uncertainty in $\tilde{\sigma}_0$. According to (54), and using the central value of (49) and (57), we find the $\alpha = -0.56$ averaged over the entire $J/\psi$ energy range.

\(^{18}\) In contrast to (48), the leading correction to this asymptotic expression is of relative order $1/\sqrt{s}$, instead of $1/s$.  

26
For the order-$v^2$ contribution to $\sigma_L$, the corresponding analytic expression is too involved, if not impossible, to deduce, so we are content with providing numerical result only. Using $\langle v^2 \rangle_{J/\psi}$ as given in (45b), integrating the order-$v^2$ correction (A4) over the entire $J/\psi$ energy range, we get

$$\sigma_L^{(2)}[J/\psi + X_{light}] = 0.037^{+0.059}_{-0.024} \text{ pb.}$$  \hspace{1cm} (58)

The attached error comes from the uncertainties in $\bar{\sigma}_0$ and $\langle v^2 \rangle_{J/\psi}$. For the central values of the predictions, inclusion of the order-$v^2$ correction enhances the LO cross section by about 29%, which has a very similar magnitude of enhancement as for the unpolarized $J/\psi$. This is again in accordance with the naive expectation about the size of relativistic correction for charmonium system.

The sum of (57) and (58) is

$$\left(\sigma_L^{(0)} + \sigma_L^{(2)}\right)[J/\psi + X_{light}] = 0.165^{+0.157}_{-0.076} \text{ pb.}$$  \hspace{1cm} (59)

Including the order-$v^2$ correction, the central value of the average polarization variable $\alpha$ shifts from $-0.56$ to $-0.55$. Hence the relativistic correction has a rather minor effect in changing the polarization of $J/\psi$.

Now let us examine the differential distribution $d\sigma_L^{(0)}/dz$. As can be seen from Figure 2, or can be directly inferred from (A3), the LO distribution has a finite limit when the energy of the longitudinally-polarized $J/\psi$ approaches its maximum:

$$\left.\frac{d\sigma_L^{(0)}}{dz} \right|_{z \to 1+r} \rightarrow \bar{\sigma}_0 \frac{1}{1 - r}. \hspace{1cm} (60)$$

From (52) and (60), it is ready to see that, at the endpoint $z = 1 + r$, the LO polarization variable, $\alpha^{(0)}$, approaches the constant $-\frac{1+r}{1+r}$.

As can be seen in Fig. 2, the order-$v^2$ correction to the energy spectrum of the longitudinally polarized $J/\psi$ also diverges logarithmically near the upper end. After some manipulation on equation (A4), we find the following limiting behavior:

$$\left.\frac{d\sigma_L^{(2)}}{dz} \right|_{z \to 1+r} \rightarrow \bar{\sigma}_0 \frac{\langle v^2 \rangle_{J/\psi}}{3} \times \frac{5 - 17r + 4r^2 - 8r \ln \left[ \frac{r(1+r-z)}{(1-r)^2} \right]}{(1-r)^2}. \hspace{1cm} (61)$$

The situation very much resembles that for the unpolarized $J/\psi$. One can refer to the paragraphs after (53) for similar discussions.

In Figure 3, we also display how the polarization parameter $\alpha$ varies with the $J/\psi$ energy. Clearly, the inclusion of relativistic correction seems to have a minor impact in most of the region of $z$, except increasing it modestly near the upper end.

### E. Angular-Energy distribution of $J/\psi$

Experimentally it is also possible to measure the production rate for $J/\psi$ in $e^+e^-$ annihilation that is differential in $\cos \theta$, the cosine of the angle between the momentum of $J/\psi$ and
the incident $e^-$ beam in the laboratory frame. It is thus theoretically interesting to study the differential angular distribution of $J/\psi$. As pointed out in (42), for inclusive $J/\psi$ production in $e^+e^-$ annihilation, general consideration constrains the double differential distribution of the following form [28]:

$$\frac{d\sigma^{(i)}}{dz d\cos\theta}[e^+e^- \to J/\psi + X] = S^{(i)}(z)\left[1 + A^{(i)}(z)\cos^2\theta\right],$$  \hspace{1cm} (62)

where $A(z)$ is a angular parameter that satisfies $|A(z)| \leq 1$.

The analytic expressions at LO in $v$, $S^{(0)}(z)$ and $A^{(0)}(z)$ have been known long ago. The closed forms of the order-$v^2$ contributions, $S^{(2)}(z)$ and $A^{(2)}(z)$, are derived in this work for the first time. For completeness, we reproduce all of them in Appendix A. From Eqs. (A6) and (A7), one finds that the LO double differential spectrum admits a finite limit near the upper end [36]:

$$\frac{d\sigma^{(0)}}{dz d\cos\theta}[e^+e^- \to J/\psi + gg] \bigg|_{z\to1+r} \rightarrow \frac{3\sigma_0}{4} \left(\frac{1+r}{1-r} - \cos^2\theta\right),$$  \hspace{1cm} (63)

which implies that at the endpoint $z = 1 + r$, $A^{(0)} = -\frac{1+r}{1+r}$.

In Figure 4, we display the angular function $A(z)$ at energy of the $B$ factory, $\sqrt{s} = 10.58$ GeV. Both the LO prediction and that including the first-order relativistic correction are shown. Note that the correct $A(z)$ incorporating the order-$v^2$ effect is given by [28]

$$A^{v^2}(z) = \frac{S^{(0)}(z)A^{(0)}(z) + S^{(2)}(z)A^{(2)}(z)}{S^{(0)}(z) + S^{(2)}(z)}.$$  \hspace{1cm} (64)

As can be seen in Figure 4, including the first-order relativistic correction seems to have modest effect, which only slightly softens the angular distribution near the upper end.

Next let us inspect the end-point behavior of the functions $S^{(2)}(z)$ and $A^{(2)}(z)$. After some straightforward algebra from Eqs. (A8) and (A9), we find the following limiting behaviors

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3.png}
\caption{Profile of the $J/\psi$ polarization parameter $\alpha(z)$ in associated production with light hadrons at $\sqrt{s} = 10.58$ GeV. The dot-dashed curve represents the leading order prediction $\alpha^{(0)}$, whereas the solid curve represents the corresponding one including the $O(v^2)$ effect.}
\end{figure}
FIG. 4: Profile of the $J/\psi$ angular distribution parameter $A(z)$ in associated production with light hadrons at $\sqrt{s} = 10.58$ GeV. The dot-dashed curve represents the leading-order prediction $A^{(0)}$, whereas the solid curve represents $A^{v^2}(z)$ defined in (64), which has included the order-$v^2$ effect.

when $z$ approaches its maximum:

$$S^{(2)}(z) \rightarrow \frac{\hat{\sigma}_0 \langle v^2 \rangle_{J/\psi}}{4} \frac{5 - 16r - 5r^2 - 2r(5 + 3r) \ln \left[ \frac{r(1+r-z)}{(1-r)^2} \right]}{(1-r)^2},$$

$$(65a)$$

$$S^{(2)}(z)A^{(2)}(z) \rightarrow \frac{\hat{\sigma}_0 \langle v^2 \rangle_{J/\psi}}{4} \frac{3 + 5r + 6r \ln \left[ \frac{r(1+r-z)}{(1-r)^2} \right]}{1 - r}.$$  

$$(65b)$$

These emerging logarithmic divergences near the endpoint simply reflects that the differential cross section diverges in that region. However, according to Eq. (64), the angular distribution $A^{v^2}(z)$, defined as a ratio, still remains a finite and smooth function near the very upper end.

It is worth mentioning that BELLE collaboration has recently measured the average angular variable for the $J/\psi$ production in association with noncharmful states, $\bar{A}_{\exp} = 5.2^{+0.6}_{-2.4}(0.3)$ [40]. Theoretically, it is straightforward to define the corresponding $\bar{A}$ by integrating (62) over $z$:

$$\frac{d\sigma^{(i)}}{d\cos \theta} [e^+e^- \rightarrow J/\psi + X_{\text{tight}}] = \bar{S}^{(i)} \left[ 1 + \bar{A}^{(i)} \cos^2 \theta \right].$$

$$(66)$$

From Eqs. (A6) and (A7), and inserting $r = 0.0857$, we find the LO NRQCD prediction is $\bar{A}^{(0)} = -0.037$. Notwithstanding the large experimental uncertainty, this prediction is in apparent disagreement with the BELLE measurement, even the sign is opposite. Subsequent studies reveal that including the NLO perturbative correction does not help to resolve this discrepancy [42].

One may naturally wonder whether implementing the relativistic correction will bring the NRQCD prediction closer to the data or not. In analogy with (64), we introduce a new average angular variable that incorporates the $O(v^2)$ effect:

$$\bar{A}^{v^2} = \frac{S^{(0)}A^{(0)}(z) + S^{(2)}A^{(2)}(z)}{S^{(0)} + S^{(2)}}.$$  

$$(67)$$
Starting from Eqs. (A8) and (A9), and adopting the central value of $\langle v^2 \rangle_{J/\psi}$ tabulated in (45b), we then find $\mathcal{A} v^2 = 0.0011$, which now has the same sign as the measured value, though still differs considerably in the absolute magnitude. Therefore, including NLO relativistic correction (plus perturbative correction) seems not to be sufficient to explain the data. It remains to be a challenge how to correctly account for the measured angular distribution in the $J/\psi + X_{\text{light}}$ channel within the NRQCD factorization framework.

VII. DISCUSSION AND SUMMARY

In this work, we have introduced a somewhat heterodox NRQCD matching strategy, which is particularly suitable for calculating the relativistic correction to (inclusive) quarkonium production and decay processes with involved kinematics in the color-singlet channel. The great advantage of our approach over the orthodox matching strategy is that, it can take into account the relativistic correction effect in the phase space integration with much ease, thanks to the Gremm-Kapustin relation. As a nontrivial application of this method, we have systematically investigated the relative order-$v^2$ correction to the inclusive $J/\psi$ production associated with light hadrons at $B$ factories. We have found that it can modestly enhance the lowest-order NRQCD prediction for the integrated $J/\psi$ cross section, about 30% if we choose the relativistic correction matrix element as specified in [54]. We find its impact on the $J/\psi$ polarization and angular distributions is quite minor. The magnitude of the order-$v^2$ correction seems to be comparable with that of the respective NLO perturbative correction. We would like to caution that, our predictions of the order-$v^2$ correction are likely subject to large theoretical uncertainty. In particular, some intrinsic uncertainty related to the relative order-$v^2$ NRQCD matrix element seems to restrict our ability to make precise predictions for the relativistic correction. Since the corrections computed in this work has only a modest effect, we feel unable to draw any sharp conclusion, especially for the actual size of the color-octet contribution.

Of the special theoretical interest, is the logarithmic divergence near the upper end of the $J/\psi$ spectrum found in this work for the order-$v^2$ contribution. It is desirable to extend the theoretical framework developed in [39, 45, 46] beyond the LO in $v$, to see whether such type of soft endpoint logarithms can be resummed to all orders in $\alpha_s$, to render the $J/\psi$ energy spectrum well-behaved in the end point region.

Another interesting direction is to incorporate the order-$v^4$ correction to the process considered in this work. We expect that the perturbative matching approach described in this work, after some straightforward extension, is well-suited to achieve this goal. At $O(v^4)$, the $J/\psi$ energy spectrum is expected to exhibit a linear divergence near the upper end point, and consequently, the integrated cross section will be logarithmically divergent. It will be interesting to see how the color-octet contribution from the $^3P_j^{(8)}$ NRQCD production operator is explicitly put into work to tame this infrared divergence.

Acknowledgments

First I wish to thank Jian-Xiong Wang for his inquiry in spring of 2008 that stimulated me to initiate this research, and for many informative exchanges concerning $J/\psi$ production in various collision experiments. It is also a pleasure to acknowledge Bin Gong, Adam Leibovich, Jian-Wei Qiu and Guo-Huai Zhu for valuable communications on related topics.
I would also like to take this opportunity to thank KITPC at Beijing for hosting an enjoyable program entitled *Effective Field Theories in Particle and Nuclear physics* (Aug. 3–Sep. 11, 2009), during which part of this manuscript was written. This research was supported in part by the National Natural Science Foundation of China under grants No. 10875130, 10935012, and by the Project of Knowledge Innovation Program (PKIP) of Chinese Academy of Sciences, Grant No. KJCX2.YW.W10.

**Appendix A: Miscellaneous formulas for inclusive $J/\psi$ production associated with light hadrons in $e^+e^-$ annihilation**

In this section, we collect the analytic expressions for various types of distributions for $J/\psi$ associated production with light hadrons in $e^+e^-$ annihilation. Each type of differential cross section is understood to contain two parts: $d\sigma = d\sigma^{(0)} + d\sigma^{(2)}$, which represent the leading order contribution, and the contribution of relative order-$v^2$, respectively. We emphasize that it is the physical $J/\psi$ mass, rather than the charm quark mass, that enters into the formulas of each part.

1. **Energy distribution of unpolarized $J/\psi$**

   The energy spectrum of unpolarized $J/\psi$ at LO in $v$ reads:

   $$
   \frac{d\sigma^{(0)}}{dz} [e^+e^- \to J/\psi + gg] = \frac{\hat{\sigma}_0}{(2 - z)^2(2 - z)^3} \times \left\{(z - 2r)\sqrt{z^2 - 4r} \left[4(1 + 5r + 7r^2 + 4r^3) - 12(1 + r)(1 + 2r)z + (13 + 14r)z^2 - 4z^3\right] + 4(1 + r - z)\left[2r(1 - r)(1 + 8r + 4r^2) - 2r(5 - 2r - 6r^2)z + (1 + r - 5r^2)z^2\right] \ln \left(\frac{z - 2r + \sqrt{z^2 - 4r}}{z - 2r - \sqrt{z^2 - 4r}}\right)\right\},
   $$

   (A1)

   where the quantity $\hat{\sigma}_0$ has been defined in (44). This expression agrees with the result given in [25], but differs from Ref. [28–30] by an overall constant.
The first-order relativistic correction to the energy spectrum of unpolarized \( J/\psi \) reads:

\[
\frac{d\sigma^{(2)}}{dz} [e^+e^- \to J/\psi + gg] = \frac{1}{3} \frac{\langle v^2 \rangle_{J/\psi}}{(2 - z)^4(z - 2r)^5} \times \left\{ (z - 2r)\sqrt{z^2 - 4r} \left[ 64r(3 + 11r - 2r^2 - 4r^3 - 20r^4 - 15r^5) \right. \right.
\]

\[
- 32r(22 + 30r - 21r^2 - 91r^3 - 89r^4 - 7r^5)z
\]

\[
- 16(1 - 48r - 9r^2 + 171r^3 + 213r^4 + 35r^5)z^2
\]

\[
+ 16(4 - 22r + 66r^2 + 133r^3 + 35r^4)z^3
\]

\[
- 4(18 + 15r + 170r^2 + 70r^3)z^4
\]

\[
+ 4(11 + 15r + 16r^2)z^5 - (11 - 2r)z^6
\]

\[
+ 4 \left[ 32r^2(1 + r)(3 + 9r - 6r^2 + 9r^3 + 14r^4 + 15r^5) \right.
\]

\[
- 16r^2(24 + 50r + 21r^2 + 124r^3 + 228r^4 + 126r^5 + 7r^6)z
\]

\[
- 8r(3 - 63r - 103r^2 - 278r^3 - 697r^4 - 149r^5 - 49r^6)z^2
\]

\[
+ 8r(9 - 50r - 186r^2 - 549r^3 - 477r^4 - 77r^5)z^3
\]

\[
+ 2(2 - 37r + 248r^2 + 993r^3 + 1122r^4 + 272r^5)z^4
\]

\[
- 4(2 - 2r + 124r^2 + 192r^3 + 71r^4)z^5
\]

\[
+ (7 + 26r + 140r^2 + 87r^3)z^6
\]

\[
\left. \left. - (3 + 2r + 14r^2)z^7 \right\} \ln \left( \frac{z - 2r + \sqrt{z^2 - 4r}}{z - 2r - \sqrt{z^2 - 4r}} \right) \right\}, \quad (A2)
\]

where \( \langle v^2 \rangle_{J/\psi} \) has been introduced in (5).

2. Energy distribution of longitudinally-polarized \( J/\psi \)

The energy spectrum of the longitudinally-polarized \( J/\psi \) at LO in \( v \) reads:

\[
\frac{d\sigma^{(0)}}{dz} [e^+e^- \to J/\psi(\lambda = 0) + gg] = \frac{1}{(2 - z)^2(z - 2r)^3(z^2 - 4r)} \left\{ (z - 2r)\sqrt{z^2 - 4r} \right.
\]

\[
- 8r^2(9 + 9r + r^2 + 3r^3) + 16r(2 + 8r + r^2 + 3r^3)z
\]

\[
- 4(1 + 10r + 3r^2 + 8r^3)z^2 + 4(1 - r + 2r^2)z^3 + z^4 \bigg] \right.
\]

\[
+ 4(1 + r - z) \left[ -4r^3(1 - r)(9 + 2r + 3r^2) + 8r^2(2 - 4r - 3r^3)z
\]

\[
- 2r(1 - 16r - 2r^2 - 9r^3)z^2 - 2r(5 + 3r + 3r^2)z^3
\]

\[
+ (1 + r + r^2)z^4 \right\} \ln \left( \frac{z - 2r + \sqrt{z^2 - 4r}}{z - 2r - \sqrt{z^2 - 4r}} \right) \right\}. \quad (A3)
\]
The order-$v^2$ correction to the energy spectrum of the longitudinally-polarized $J/\psi$ is

\[
\frac{d\sigma_L^{(2)}}{dz} [e^+e^- \rightarrow J/\psi(\lambda = 0) + gg] = \frac{\sigma_0}{3} \frac{1}{(2 - z)^4(z - 2r)^5(z^2 - 4r)} \times \left\{ \left. \left( z - 2r \right) \sqrt{z^2 - 4r} \right[ -128r^3(15 + 4r + 5r^2 + r^3 - 4r^4 - 15r^5) \right.ight.
\]
\[
+ 64r^2(6 + 62r + 30r^2 + 29r^3 + 21r^4 - 101r^5 - 9r^6)z
\]
\[
- 32r^2(6 + 122r + 153r^2 + 215r^3 - 247r^4 - 57r^5)z^2
\]
\[
- 16r(2 + 6r - 275r^2 - 571r^3 + 2233r^4 + 141r^5)z^3
\]
\[
- 16(1 + 7r + 51r^2 + 358r^3 + 16r^4 - 83r^5)z^4
\]
\[
+ 8(6 + 45r + 205r^2 + 113r^3 - 41r^4)z^5
\]
\[
- 4(14 + 91r + 76r^2 + 3r^3)z^6
\]
\[
+ 4(8 + 15r + 5r^2)z^7 - (3 + 4r)z^8 \right]\] + 4 \left[ -64r^4(15 + 9r + 3r^2 - 2r^3 - r^4 + 9r^5 + 15r^6) \right.
\]
\[
+ 32r^3(6 + 74r + 5r^2 - 38r^2 + 26r^5 + 134r^6 + 9r^7)z
\]
\[
- 16r^4(108 - 9r - 166r^2 - 218r^3 + 478r^4 + 75r^5)z^2
\]
\[
- 8r^2(6 + 116r + 71r^2 + 492r^3 + 784r^4 - 840r^5 - 261r^6)z^3
\]
\[
- 8r(1 - 21r - 195r^2 - 437r^3 - 947r^4 + 308r^5 + 243r^6)z^4
\]
\[
- 4r(2 + 63r + 497r^2 + 1253r^3 + 133r^4 - 252r^5)z^5
\]
\[
+ 2(2 + 21r + 189r^2 + 964r^3 + 447r^4 - 127r^5)z^6
\]
\[
- 4(2 + 16r + 91r^2 + 86r^3 - r^4)z^7
\]
\[
+ (7 + 46r + 48r^2 + 11r^3)z^8
\]
\[
- (3 + 3r + r^2)z^9 \right] \right\}. \quad \text{(A4)}
\]

3. Angular-energy distribution for unpolarized $J/\psi$

The doubly differential angular-energy distribution of unpolarized $J/\psi$ can be parameterized in the following form:

\[
\frac{d\sigma^{(i)}}{dz \cos \theta} [e^+e^- \rightarrow J/\psi + gg] = S^{(i)}(z) \left[ 1 + A^{(i)}(z) \cos^2 \theta \right], \quad \text{(A5)}
\]

where the superscript $i = 0, 2$ represents the leading-order and first-order contributions in relativistic expansion.
The corresponding functions at LO in $v$ read:

$$S^{(0)}(z) = \frac{3 \sigma_0}{4} \left( \frac{1}{(2-z)^2(z-2r)^3(z^2-4r)} \right) \times \left\{ (z-2r)\sqrt{z^2-4r} \left[ -4r(1+r)(3+12r+13r^2) ight] \\
+ 32r(1+r)(1+3r)z + 4(1-7r-12r^2+2r^3)z^2 \\
+ 4(2+r+3r^2)z^3 + 7(1+r)z^4 - 2z^5 \right\} \\
- 2(1+r-z) \left[ 4r^2(1-r)(3+24r+13r^2) - 8r^2(7-3r-12r^2)z \right] \\
+ 2r(1-10r-27r^2+4r^3)z^2 + 2r(7+7r-6r^2)z^3 \\
- (1-r)(1+5r)z^4 \right\} \ln \left( \frac{z-2r+\sqrt{z^2-4r}}{z-2r-\sqrt{z^2-4r}} \right), \quad (A6)$$

and

$$S^{(0)}(z)A^{(0)}(z) = \frac{3 \sigma_0}{4} \left( \frac{1}{(2-z)^2(z-2r)^3(z^2-4r)} \right) \times \left\{ (z-2r)\sqrt{z^2-4r} \left[ 4r(1+5r+19r^2+7r^3) \\
- 96r^2(1+r)z - 4(1-5r-22r^2-2r^3)z^2 \\
+ 4r(7+3r)z^3 + (5+7r)z^4 - 2z^5 \right] \\
+ 2(1+r-z) \left[ 4r^2(1+7r)(1-r^2) - 8r^2(1+3r)(1-4r)z \right] \\
- 2r(1+10r+57r^2+4r^3)z^2 + 2r(1+29r+6r^2)z^3 \\
+ (1-8r-5r^2)z^4 \right\} \ln \left( \frac{z-2r+\sqrt{z^2-4r}}{z-2r-\sqrt{z^2-4r}} \right). \quad (A7)$$

Note these expressions are exactly twice smaller than Eqs. (A1a) and (A1b) in Ref. [28].
At the relative order $v^2$, the corresponding functions $S(z)$ and $A(z)$ are

\[
S^{(2)}(z) = \frac{\bar{\sigma}_0 \langle v^2 \rangle_{J/\psi}}{4} \frac{1}{(2-z)^4(z-2r)^3(z^2-4r)} \\
\times \left\{ (z-2r)\sqrt{z^2-4r} \left[ -64r^2(9 + 35r - 8r^2 - 14r^3 - 57r^4 - 45r^5) \right. \right.
+ 32r^2(64 + 84r - 89r^2 - 269r^3 - 275r^4 - 19r^5)z \\
+ 16r(13 - 69r + 61r^2 + 557r^3 + 638r^4 + 64r^5)z^2 \\
- 8r(110 + 161r + 559r^2 + 725r^3 + 3r^4 - 14r^5)z^3 \\
+ 4r(287 + 451r + 383r^2 - 251r^3 - 70r^4)z^4 \\
+ (2 - 87r - 29r^2 + 109r^3 + 35r^4)z^5 \\
- 4(6 - 32r + 75r^2 + 35r^3)z^6 \\
+ 2(9 + 9r + 16r^2)z^7 - (5 - r)z^8 \left. \right\}
+ 2 \left[ -64r^3(9 + 38r + 13r^2 + 18r^3 + 53r^4 + 80r^5 + 45r^6) \right. \\
+ 32r^3(70 + 166r + 123r^2 + 340r^3 + 620r^4 + 390r^5 + 19r^6)z \\
+ 16r^2(19 - 140r - 430r^2 - 922r^3 - 1927r^4 - 1410r^5 - 102r^6)z^2 \\
- 8r^2(124 - 219r - 1406r^2 - 3154r^3 - 2596r^4 - 139r^5 + 14r^6)z^3 \\
+ 4r(18 - 193r + 840r^2 + 2976r^3 + 2400r^4 - 327r^5 - 98r^6)z^4 \\
+ 8r(29 + 28r + 355r^2 + 181r^3 - 352r^4 - 77r^5)z^5 \\
+ 2(2 - 141r - 131r^2 + 293r^3 + 1041r^4 + 272r^5)z^6 \\
- 4(2 - 34r + 71r^2 + 196r^3 + 71r^4)z^7 \\
+ (7 - 5r + 147r^2 + 87r^3)z^8 \\
- (3 + r + 14r^2)z^9 \left\} \ln \left( \frac{z - 2r + \sqrt{z^2 - 4r}}{z - 2r - \sqrt{z^2 - 4r}} \right) \right\},
\]
and

\[
S^{(2)}(z)A^{(2)}(z) = \frac{\bar{\sigma}_{0}(v^2)_{J/\psi}}{4} \frac{1}{(2-z)^4(z-2r)^3(z^2-4r)}
\times \left\{ (z-2r)\sqrt{z^2-4r} \left[ 64r^2(3 + 17r - 8r^2 - 10r^3 - 11r^4 - 15r^5) \\
+ 32r^2(16 + 12r - 99r^2 - 79r^3 - 113r^4 - r^5)z \\
+ 16r(7 + 89r + 271r^2 + 335r^3 + 370r^4 + 32r^5)z^2 \\
+ 8(90 + 595r + 789r^2 + 775r^3 + 161r^4 + 14r^5)z^3 \\
+ 4(1 + r)(8 + 325r + 836r^2 + 321r^3 + 70r^4)z^4 \\
+ 8(10 + 129r + 291r^2 + 141r^3 + 35r^4)z^5 \\
- 4(18 + 104r + 119r^2 + 35r^3)z^6 \\
+ 2(1 + r)(17 + 16r)z^7 - (7 - r)z^8 \right] \left\{ 64r^3(3 + 18r + 15r^2 + 30r^3 - 25r^4 + 8r^5 + 15r^6) \\
+ 32r^3(18 + 98r + 201r^2 + 289r^3 + 36r^4 + 162r^5 + r^6)z \\
+ 16r^2(9 - 12r - 490r^2 - 566r^3 - 389r^4 - 710r^5 - 34r^6)z^2 \\
+ 8r^2(36 - 257r - 1410r^2 - 1670r^3 - 1980r^4 - 193r^5 - 14r^6)z^3 \\
+ 4r(14 + 221r + 1360r^2 + 3208r^3 + 3800r^4 + 595r^5 + 98r^6)z^4 \\
+ 8r(35 + 300r + 817r^2 + 1203r^3 + 284r^4 + 77r^5)z^5 \\
+ 2(2 + 219r + 1177r^2 + 1973r^3 + 669r^4 + 272r^5)z^6 \\
+ 4(2 + 82r + 275r^2 + 124r^3 + 71r^4)z^7 \\
+ (7 + 119r + 119r^2 + 87r^3)z^8 \\
- (3 + 5r + 14r^2)z^9 \right\} \ln \left( \frac{z - 2r + \sqrt{z^2 - 4r}}{z - 2r - \sqrt{z^2 - 4r}} \right) \right\}. \tag{A9}
\]

Integrating Eq. (A5) over the polar angle \( \theta \) from 0 to \( \pi \), we arrive at the following identity:

\[
\frac{d\sigma^{(i)}}{dz} [e^+e^- \to J/\psi + gg] = 2S^{(i)}(z) \left[ 1 + \frac{1}{3}A^{(i)}(z) \right], \tag{A10}
\]

where \( d\sigma^{(i)}/dz \) represent the energy distributions for unpolarized \( J/\psi \), which have been given in Eqs. (A1) and (A2). This relation can serve as a consistency check of our results. We have explicitly verified that our expressions obey this relation for both \( i = 0 \) and 2.

**Appendix B: Equivalence between our matching method and the “orthodox” one**

It is curious to ask, whether the inclusive \( J/\psi \) production rate derived from our matching procedure, can be translated into a more orthodox form, that is, everything is expressed in terms of charm quark mass rather than the charmonium mass. That corresponds to what would be obtained from a literal matching method. As we shall see, this is possible only for the integrated cross section. And we like to stress, there should be no any theoretical ambiguity and confusion for the relativistic correction contribution at this level.
To better orientate ourselves, let us begin with a one-dimensional toy integral:

$$
\int_{g(x,y)} f(x,y) \, dt \, W(t, y) = \int_{g(x,y)}^f f(x,0) \, dt \, W(t, 0) + y \left\{ \int_{g(x,y)}^f f(x,0) \, dt \, W'_y(t, 0) + W(f(x,0), 0) f'_y(x,0) - W(g(x,0), 0) g'_y(x,0) \right\} + O(y^2),
$$

(B1)

where we have assumed the integrand $W$ is regular at the end points of the integral and used the shorthand $f'_y(x,0) \equiv \partial f(x, y)/\partial y|_{y=0}$. The final result of the integral in the left-hand side will be a function of $x$ and $y$. Here $y$ is assumed to be a small constant, and it is assumed that both the integrand $W$ and the integration boundaries $f, g$ depend on $y$. Our goal is to reexpress the original integral in a Taylor-series in $y$. Since in many situations, the closed form for such integral is presumably not available, or at least difficult to obtain, it is thus desirable to find a general numerical recipe to accomplish this expansion.

In the right-hand side of (B1), we give the intended answer for this expansion through the linear order in $y$. The leading term is obtained by neglecting $y$ simultaneously in the integrand and integration boundaries. The coefficients of order $y$ come from either expanding the integrand or taking into account the correction to the integration boundaries.

The goal is to reexpress our “leading-order” cross sections in terms of a new series including the first-order relativistic correction, with all the occurrences of $M_{J/\psi}$ replaced by $2m_c$ in a consistent way. Note both the matrix element squared and the boundaries of the phase space integral depend on $v^2$ implicitly through $M_{J/\psi}$. Clearly, $v^2$ is the counterpart of $y$ in (B1) that acts as the small expansion parameter. One can utilize (B1) to work out the desired expanded form.

For concreteness, we take the unpolarized $J/\psi$ energy distribution as an example. The LO energy spectrum of $J/\psi$ derived from our matching method has been given in Eq. (A1):

$$
\frac{d\sigma^{(0)}}{dz} \left[ e^+e^- \to J/\psi + gg \right]
= \frac{256\pi (e_\sigma \alpha_s)^2}{27 M_{J/\psi} s^2} \langle O_1^{J/\psi} \rangle \frac{1}{(2-z)^2(z-2r)^3}
\times \left\{ (z-2r)\sqrt{z^2-4r} \left[ 4(1+5r+7r^2+4r^3) 
\right.ight.
\left. - 12(1+r)(1+2r)z + (13+14r)z^2 - 4z^3 \right]
\left. + 4(1+r-z) \left[ 2r(1-r)(1+8r+4r^2) - 2r(5-2r-6r^2)z \right. \right.
\left. + (1+r-5r^2)z^2 \right] \ln \left( \frac{z-2r+\sqrt{z^2-4r}}{z-2r-\sqrt{z^2-4r}} \right) \right\}.
$$

(B2)

For clarity, here we abandon the use of the abbreviation $\tilde{\sigma}_0$ and supply the complete expression of the prefactor.

In accordance with (B1), we may reexpress the integrated cross section of (B2) as a sum of the following three terms, each of which now depends on the charm quark mass rather than the $J/\psi$ mass:

$$
\int_{\sqrt{\tau}}^{1+\r} dz \frac{d\sigma^{(0)}}{dz} = \int_{\sqrt{\tau_0}}^{1+\r_0} dz \frac{d\tilde{\sigma}^{(0)}}{dz} + \int_{\sqrt{\tau_0}}^{r_0} dz \frac{d\tilde{\sigma}^{(2a)}}{dz} + \tilde{\sigma}^{(2b)} + O(v^4\sigma).
$$

(B3)
where \( r_0 \equiv \frac{4m_c^2}{s} \). Upon expanding (B2), we need replacing every occurrence of \( M_{J/\psi} \) with the combination of \( m_c \) and \( \langle v^2 \rangle_{J/\psi} \) through the G-K relation (18b):

\[
\frac{1}{M_{J/\psi}} = \frac{1}{2m_c} \left( 1 - \frac{1}{2} \langle v^2 \rangle_{J/\psi} + O(v^4) \right). \tag{B4b}
\]

In the resulting new expression, we only need retain those terms at most of order \( v^2 \).

The first term in the right side of (B3) constitutes the leading contribution, the second one comes from the expansion of the integrand, and the third one arises from the correction due to integration boundaries. Their explicit expressions are

\[
\begin{align*}
\frac{d\tilde{\sigma}^{(0)}}{dz} &= \frac{d\sigma^{(0)}}{dz} \bigg|_{M_{J/\psi} \to 2m_c, r \to r_0}, \tag{B5a} \\
\frac{d\tilde{\sigma}^{(2a)}}{dz} &= \frac{64\pi(e_c\alpha_s)^2}{27m_c s^2} \langle P_{J/\psi}^{I} \rangle \frac{1}{(2-z)^2(z-2r)^4\sqrt{z^2-4r}} \\
&\times \left\{ (z-2r) \left[ -32r^2(5+17r-4r^2-12r^3) \\
&- 16r(1-17r-8r^2+32r^3)z \\
&+ 8r(15+7r+17r^2-12r^3)z^2 \\
&- 4(3+47r+11r^2-32r^3)z^3 \\
&+ 2(10+51r-14r^2)z^4 - 13(1+2r)z^5 + 4z^6 \right] \\
&+ 4\sqrt{z^2-4r} \left[ 4r^2(5+24r+3r^2+8r^3+12r^4) \\
&+ 2r(1-36r-45r^2-32r^3-56r^4)z \\
&+ 2r(1+30r+36r^2+53r^3)z^2 \\
&- (1+2r+34r^2+55r^3)z^3 \\
&+ (1-r+15r^2)z^4 \right] \ln \left( \frac{z-2r+\sqrt{z^2-4r}}{z-2r-\sqrt{z^2-4r}} \right) \right\} \bigg|_{r \to r_0}, \tag{B5b} \\
\tilde{\sigma}^{(2b)} &= \frac{512\pi(e_c\alpha_s)^2 m_c}{27s^3} \langle P_{J/\psi}^{I} \rangle \frac{1+2r}{1-r} \bigg|_{r \to r_0}, \tag{B5c}
\end{align*}
\]

where \( \langle P_{J/\psi}^{I} \rangle \) is given in (4b). Needless to say, the new LO term is exactly of the same functional form as the old one in (B2), except \( M_{J/\psi} \) everywhere replaced by \( 2m_c \). For the newly generated relativistic correction pieces \( \tilde{\sigma}^{(2a)} \) and \( \tilde{\sigma}^{(2b)} \), one does not need to carefully distinguish \( r_0 \) and \( r \) in them, since the induced error would be of order \( v^4 \), which is beyond the intended accuracy of this work.

All these three terms, in combination with (A2), the genuine \( O(v^2) \) contribution in our matching approach \(^{19}\), constitute an alternative but equally valid prediction to the integrated \( J/\psi \) cross section that is accurate at relative order \( v^2 \). Since the expression for the integrated \( J/\psi \) production rate, when everything is expressed in term of \( m_c \), has no any ambiguity.

\(^{19}\) Note we can carelessly replace \( M_{J/\psi} \) by \( 2m_c \) in (A2), and, in the corresponding phase-space integral boundaries, since the induced error would be \( O(v^4) \).
through $O(v^2)$, it can be used to check the correctness of the calculation performed in an “orthodox” matching method (e.g., see [59]).

To clearly see how (B3) works, we can take advantage of our analytic knowledge for the integrated $J/\psi$ cross section at $O(v^0)$. Directly Taylor expanding (47) around $r = r_0$ to first order in $r - r_0$, we find

$$
\tilde{\sigma}^{(2a)} + \tilde{\sigma}^{(2b)} = -\frac{128\pi (\epsilon_c \alpha_s)^2}{27m_c s^2} \langle P_1^{J/\psi} \rangle \left\{ \frac{2 - 9r + 39r^2 - 28r^3 + 8r^4}{4(1 - r)^3} \right\}
\times \arctanh^2 \sqrt{1 - r} + \frac{3 - 4r - 9r^2 + 4r^3}{(1 - r)^{5/2}} \arctanh \sqrt{1 - r}
+ \frac{5 - 11r + 33r^2 - 3r^3}{4(1 - r)^3} \ln r - \frac{11 - 19r - 46r^2 + 18r^3}{4(1 - r)^2}
\right\}. \quad (B6)
$$

We have numerically compared (B6) with the sum of (B5b) and (B5c) upon integration over the full range of $z$, and indeed found exact agreement.

We have also numerically checked that, both sides of (B3), assuringly, do agree with each other at the integrated level, up to an error of order $v^4\,^{20}$.

Finally, it might be worth mentioning that, the differential distribution $d\tilde{\sigma}^{(2a)}/dz$ diverges at both upper and lower ends of $z$ (albeit being the integrable singularities):

$$
\left. \frac{d\tilde{\sigma}^{(2a)}}{dz} \right|_{z \to 2\sqrt{r}} \rightarrow -\frac{64\pi (\epsilon_c \alpha_s)^2}{27m_c s^2} \langle P_1^{J/\psi} \rangle \frac{4 - 8\sqrt{r} + 7r}{(1 - \sqrt{r})^{2}\sqrt{z^2 - 4r}}.
$$

$$
\left. \frac{d\tilde{\sigma}^{(2a)}}{dz} \right|_{z \to 1+r} \rightarrow -\frac{64\pi (\epsilon_c \alpha_s)^2}{27m_c s^2} \langle P_1^{J/\psi} \rangle
\times \left(1 - 3r\right) \left(1 + 11r + 6r^2\right) + 8r (1 - 3r - r^2) \ln \left[ \frac{r(1+r-z)}{(1-r)^2} \right]
\right\}. \quad (B7b)
$$

These artificial end-point singularities affiliated with the relativistic correction to $J/\psi$ energy distributions, especially the one appearing at the lower end, are clearly at odds with one's expectation and certainly not favored by the data. This may signal that, even if feasible, it is not of much benefit to perform the NRQCD matching in a strictly orthodox ansatz. Instead the matching method described in this work seems much more satisfactory.

[1] For a comprehensive, but a slightly outdated review on quarkonium production, see N. Brambilla et al. [Quarkonium Working Group], arXiv:hep-ph/0412158.

[2] For a latest review, see G. T. Bodwin, _Quarkonium Production and Decay: NRQCD Confronts Experiment_, talk given at the KITPC-EFT-2009 program. The content can be downloaded at the following URL: http://www.kitpc.ac.cn/program.jsp?id=PE20090720&i=sched.

[3] G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D 51, 1125 (1995) [Erratum-ibid. D 55, 5853 (1997)] [arXiv:hep-ph/9407339].

---

\(^{20}\) For $J/\psi$ production at $B$ factory, the relativistic correction stemming from expanding the phase space boundaries, $\tilde{\sigma}^{(2b)}$, makes negligible contribution due to the additional suppression by $m_c^2/s$. 

39
[4] E. Braaten and S. Fleming, Phys. Rev. Lett. 74, 3327 (1995) [arXiv:hep-ph/9411365].
[5] P. L. Cho and M. B. Wise, Phys. Lett. B 346, 129 (1995) [arXiv:hep-ph/9411303].
[6] J. M. Campbell, F. Maltoni and F. Tramontano, Phys. Rev. Lett. 98, 252002 (2007) [arXiv:hep-ph/0703113].
[7] P. Artoisenet, J. P. Lansberg and F. Maltoni, Phys. Lett. B 653, 60 (2007) [arXiv:hep-ph/0703129].
[8] B. Gong and J. X. Wang, Phys. Rev. Lett. 100, 232001 (2008) [arXiv:0802.3727 [hep-ph]].
[9] B. Gong and J. X. Wang, Phys. Rev. D 78, 074011 (2008) [arXiv:0805.2469 [hep-ph]].
[10] B. Gong, X. Q. Li and J. X. Wang, Phys. Lett. B 673, 197 (2009) [arXiv:0805.4751 [hep-ph]].
[11] G. C. Nayak, J. W. Qiu and G. Sterman, Phys. Lett. B 613, 45 (2005) [arXiv:hep-ph/0501235]; Phys. Rev. D 72, 114012 (2005) [arXiv:hep-ph/0509021]; Phys. Rev. D 74, 074007 (2006) [arXiv:hep-ph/0608066].
[12] G. C. Nayak, J. W. Qiu and G. Sterman, Phys. Rev. Lett. 99, 212001 (2007) [arXiv:0707.2973 [hep-ph]]; Phys. Rev. D 77, 034022 (2008) [arXiv:0711.3476 [hep-ph]].
[13] K. Y. Liu, J. P. Ma and X. G. Wu, Phys. Lett. B 645, 180 (2007) [arXiv:hep-ph/0601215].
[14] M. Beneke, I. Z. Rothstein, and M. B. Wise, Phys. Lett. B 408, 373 (1997) [arXiv:hep-ph/9705286].
[15] M. Beneke, I. Z. Rothstein, and M. B. Wise, Phys. Lett. B 408, 373 (1997) [arXiv:hep-ph/9705286].
[16] T. Mannel and G. A. Schuler, Z. Phys. C 67, 159 (1995) [arXiv:hep-ph/9410333]; T. Mannel and S. Wolf, arXiv:hep-ph/9701324; I. Z. Rothstein and M. B. Wise, Phys. Lett. B 402, 346 (1997) [arXiv:hep-ph/9701404]; M. Beneke, G. A. Schuler and S. Wolf, Phys. Rev. D 62, 034004 (2000) [arXiv:hep-ph/0001062].
[17] Z. B. Kang, J.-W. Qiu and G. Sterman, in preparation. The preliminary content can be found in the talk delivered by J.-W. Qiu at the KITPC-EFT-2009 program, PQCD factorization for heavy quarkonium production, http://www.kitpc.ac.cn/program.jsp?id=PE20090720&i=sched.
[18] K. Abe et al. [Belle Collaboration], Phys. Rev. Lett. 89, 142001 (2002) [arXiv:hep-ex/0205104].
[19] E. Braaten and J. Lee, Phys. Rev. D 67, 054007 (2003) [Erratum-ibid. D 72, 099901 (2005)] [arXiv:hep-ph/0211085]; K. Y. Liu, Z. G. He and K. T. Chao, Phys. Lett. B 557, 45 (2003) [arXiv:hep-ph/0211181].
[20] Y. J. Zhang, Y. J. Gao and K. T. Chao, Phys. Rev. Lett. 96, 092001 (2006) [arXiv:hep-ph/0506076].
[21] B. Gong and J. X. Wang, Phys. Rev. D 77, 054028 (2008) [arXiv:0712.4220 [hep-ph]].
[22] Z. G. He, Y. Fan and K. T. Chao, Phys. Rev. D 75, 074011 (2007) [arXiv:hep-ph/0702239].
[23] G. T. Bodwin, J. Lee and C. Yu, Phys. Rev. D 77, 094018 (2008) [arXiv:0710.0995 [hep-ph]].
[24] T. V. Uglov, Eur. Phys. J. C 33, S235 (2004).
[25] W. Y. Keung, Phys. Rev. D 23, 2072 (1981).
[26] J. H. Kuhn and H. Schneider, Z. Phys. C 11, 263 (1981); Phys. Rev. D 24, 2996 (1981).
[27] V. M. Driesen, J. H. Kuhn and E. Mirkes, Phys. Rev. D 49, 3197 (1994).
[28] P. L. Cho and A. K. Leibovich, Phys. Rev. D 54, 6690 (1996) [arXiv:hep-ph/9606229].
[29] F. Yuan, C. F. Qiao and K. T. Chao, Phys. Rev. D 56, 321 (1997) [arXiv:hep-ph/9703438].
[30] S. Baek, P. Ko, J. Lee and H. S. Song, J. Korean Phys. Soc. 33, 97 (1998) [arXiv:hep-ph/9804455].
[31] K. Hagiwara, E. Kou, Z. H. Lin, C. F. Qiao and G. H. Zhu, Phys. Rev. D 70, 034013 (2004)
[arXiv:hep-ph/0401246].
[32] A. V. Berezhnoy and A. K. Likhoded, Phys. Atom. Nucl. 67, 757 (2004) [Yad. Fiz. 67, 778 (2004)] [arXiv:hep-ph/0303145].
[33] D. Kang, J. W. Lee, J. Lee, T. Kim and P. Ko, Phys. Rev. D 71, 094019 (2005) [arXiv:hep-ph/0412381].
[34] B. Aubert et al. [BABAR Collaboration], Phys. Rev. Lett. 87, 162002 (2001) [arXiv:hep-ex/0106044].
[35] K. Abe et al. [BELLE Collaboration], Phys. Rev. Lett. 88, 052001 (2002) [arXiv:hep-ex/0110012].
[36] E. Braaten and Y. Q. Chen, Phys. Rev. Lett. 76, 730 (1996) [arXiv:hep-ph/9508373].
[37] F. Yuan, C. F. Qiao and K. T. Chao, Phys. Rev. D 56, 1663 (1997) [arXiv:hep-ph/9701361].
[38] Y. J. Zhang, Y. Q. Ma, K. Wang and K. T. Chao, arXiv:0911.2166 [hep-ph].
[39] S. Fleming, A. K. Leibovich and T. Mehen, Phys. Rev. D 68, 094011 (2003) [arXiv:hep-ph/0306139].
[40] P. Pakhlov, arXiv:0901.2775 [hep-ex].
[41] Y. J. Zhang and K. T. Chao, Phys. Rev. Lett. 98, 092003 (2007) [arXiv:hep-ph/0611086].
[42] B. Gong and J. X. Wang, arXiv:0904.1103 [hep-ph].
[43] Y. Q. Ma, Y. J. Zhang and K. T. Chao, Phys. Rev. Lett. 102, 162002 (2009) [arXiv:0812.5106 [hep-ph]].
[44] B. Gong and J. X. Wang, Phys. Rev. Lett. 102, 162003 (2009) [arXiv:0901.0117 [hep-ph]].
[45] Z. H. Lin and G. h. Zhu, Phys. Lett. B 597, 382 (2004) [arXiv:hep-ph/0406121].
[46] A. K. Leibovich and X. Liu, Phys. Rev. D 76, 034005 (2007) [arXiv:0705.3230 [hep-ph]].
[47] I. Maksymyk, arXiv:hep-ph/9710291.
[48] E. Braaten and Y. Q. Chen, Phys. Rev. D 57, 4236 (1998) [Erratum-ibid. D 59, 079901 (1999)] [arXiv:hep-ph/9710357].
[49] C. B. Paranavitane, B. H. J. McKellar and J. P. Ma, Phys. Rev. D 61, 114502 (2000).
[50] H. Jung, D. Krucker, C. Greub and D. Wyler, Z. Phys. C 60, 721 (1993).
[51] W. L. Sang, L. F. Yang and Y. Q. Chen, Phys. Rev. D 80, 014013 (2009).
[52] A. P. Martynenko, Phys. Rev. D 72, 074022 (2005) [arXiv:hep-ph/0506324].
[53] M. Gremm and A. Kapustin, Phys. Lett. B 407, 323 (1997) [arXiv:hep-ph/9701353].
[54] G. T. Bodwin, H. S. Chung, D. Kang, J. Lee and C. Yu, Phys. Rev. D 77, 094017 (2008) [arXiv:0710.0994 [hep-ph]].
[55] W. Y. Keung and I. J. Muzinich, Phys. Rev. D 27, 1518 (1983).
[56] G. T. Bodwin and J. Lee, Phys. Rev. D 69, 054003 (2004) [arXiv:hep-ph/0308016].
[57] G. T. Bodwin and A. Petrille, Phys. Rev. D 66, 094011 (2002) [arXiv:hep-ph/0205210].
[58] Y. Fan, Y. Q. Ma and K. T. Chao, Phys. Rev. D 79, 114009 (2009) [arXiv:0904.4025 [hep-ph]].
[59] Z. G. He, Y. Fan and K. T. Chao, arXiv:0910.3636 [hep-ph].
[60] E. Braaten and Y. Q. Chen, Phys. Rev. D 54, 3216 (1996) [arXiv:hep-ph/9604237].