Codes on Graphs: Models for Elementary Algebraic Topology and Statistical Physics

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Abstract—This paper is mainly a semi-tutorial introduction to elementary algebraic topology and its applications to Ising-type models of statistical physics, using graphical models of linear and group codes. It contains new material on systematic \((n, k)\) group codes and their information sets; normal realizations of homology and cohomology spaces; dual and hybrid models; and connections with system-theoretic concepts such as observability, controllability, and input/output realizations.

Index terms—Algebraic topology, graphical models, group codes, Ising models.

I. INTRODUCTION

Algebraic topology goes back to Kirchhoff’s circuit laws [3]; however, it is not very familiar to most engineers and scientists. A major purpose of this paper is to provide an introduction to elementary algebraic topology using graphical models that have arisen in coding theory—namely, normal realizations (NRs) [5] and normal factor graphs (NFGs) [14]—which turn out to be very well suited to this purpose.

This work was directly stimulated by that of Al-Bashabsheh and Vontobel [2], who as far as we know were the first to use NFGs to model algebraic topology spaces. They apply these models to computing partition functions of Ising-type models of statistical physics, which had been shown by Al-Bashabsheh and Mao [1] and Forney and Vontobel [9] to be nicely modeled by NFGs. Some differences in our approach are:

- We use NRs rather than NFGs to model the principal spaces of elementary algebraic topology.
- We focus on the group case rather than the field case, although we treat both.
- In particular, algebraic topology spaces are regarded as “systematic \((n, k)\) group codes.”

We make connections to system-theoretic notions such as observability, controllability and input/output (I/O) realizations.

Another stimulus was the work of Molkaraie et al. [15], [16], [19], who have used dual NFGs for Monte Carlo evaluations of partition functions of Ising-type models. Our work lays an algebraic foundation for such evaluations, and systematically presents alternative approaches to carrying them out.

In Section II, we develop the main results of elementary (one-dimensional) algebraic topology. We describe the topology of a graph \(G = (V, E)\) by its incidence matrix \(M\) [4], and show how this allows us to treat the group and vector space cases in a common framework.

We introduce the concept of a systematic \((n, k)\) group code over a group alphabet \(A\), generalizing a linear \((n, k)\) block code, and show that the principal spaces of elementary algebraic topology are systematic \((n, k)\) group codes. We model all of these spaces by normal realizations. Using system-theoretic concepts such as observability and controllability, we then reduce these realizations to nonredundant (observable and controllable) input/output (I/O) realizations.

We begin with cohomology (coboundary operators, etc.), which we regard as more basic than homology (boundary operators, etc.), and then obtain dual results using an elementary Adjoint Homomorphism Lemma, as well as normal realization duality. We exhibit “bases” of principal spaces that are based on cut sets and cycles of \(G\) in the primal and dual cases, respectively [4]. Finally, we give simple dual normal realizations that we believe capture the essences of the zeroth and first (co)homology spaces of \(G\).

In Section III, following [1], [2], [9], [19], we show how to model partition functions of an Ising-type (e.g., Ising or Potts) model by “edge-weighted normal factor graphs” based on normal realizations of algebraic-topology spaces as in Section II. Some of the I/O realizations of Section III are simpler than the straightforward normal realizations that have been used previously, and may be more suitable for simulations.
Since the partition function of such a model is just a number, it is equal to its Fourier transform, which is represented by the dual NFG, up to scale. As observed in [19], computations based on the dual NFG may be simpler, as in the case of a single-cycle graph, or may behave better at low temperatures. We generalize the well-known high-temperature expansion for Ising models (\(A = \mathbb{Z}_2\)) to Potts models (\(A = \mathbb{Z}_q\)) and to generalized Ising-type models whose spin alphabet \(A\) may be any finite abelian group.

In the presence of an external field, we show that a realization of the partition function using the dual NFG is generally more complicated (higher-dimensional) than using the primal NFG. To reduce this increased complexity, we suggest a novel hybrid model, with part in the primal domain, part in the dual domain, and a Fourier transform between them.

In Section IV following [2], we give an introduction to two-dimensional algebraic topology, using planar graphs to illustrate two-dimensional complexes. We introduce dual graphs, and show (as in [2, 19]) that there are in general four different ways to represent the partition function of an Ising-type model on a planar graph \(G\), involving either \(G\) or its dual graph \(\hat{G}\), and either the original interaction weights or their Fourier transforms (in which temperature is dualized).

In an Appendix, we give a very simple proof of the Normal Factor Graph Duality Theorem, including scale factors, and show how the scale factor must be modified when the NFG is based on a normal realization. Interestingly, this leads to an alternative proof of the Controllability Test of [3].

II. INTRODUCTION TO ALGEBRAIC TOPOLOGY

The concepts of elementary (one- and two-dimensional) algebraic topology are often phrased in scary mathematical jargon, but they actually involve only some elementary graph theory and linear algebra, or, more fundamentally, the algebra of abelian groups.

This section is a tutorial introduction to these concepts, with the following unusual features:

- we treat the field and group cases in a common setting;
- we give graphical models (normal realizations) of all important spaces;
- we begin with cochains and coboundary operators rather than chains and boundary operators; consequently, our primal model is the dual of the usual primal model, and vice versa;
- we use system-theoretic properties such as observability, controllability and I/O realizations.

A. Elementary graph theory

A finite undirected graph \(G = (V, E)\) is specified by a finite vertex set \(V\), a finite edge set \(E\), and a specification of which two vertices in \(V\) are incident on each edge \(e \in E\).

Rather than specifying the topology of \(G\) as usual by its adjacency matrix \(A\), indexed by \(V \times V\), we will use instead its incidence matrix \(M\), indexed by \(E \times V\), which is defined as follows [3].

We first give each edge \(e \in E\) an orientation, perhaps arbitrary. This orientation is merely a technical device to resolve ambiguities; we are still thinking of \(G\) as an undirected graph.

We then define \(M_{ev} = 1\) if \(v = h(e)\), the head of \(e\); \(M_{ev} = -1\) if \(v = t(e)\), the tail of \(e\); and \(M_{ev} = 0\) otherwise. (We assume that there are no self-loops; i.e., that \(h(e) \neq t(e)\).)

Each edge \(e \in E\) is thus associated with a \(\{0, \pm 1\}\)-valued vector \(M_e = (M_{ev}, v \in V)\), namely the \(e\)th row of \(M\), which has precisely two nonzero components, namely \(M_{eh(e)} = +1\) and \(M_{et(e)} = -1\). (As we will see, it makes no difference whether we take \(M_e\) or \(-M_e\) as the \(e\)th row of \(M\).)

Thus each of the \(|E|\) rows of the incidence matrix \(M\) has two nonzero values, namely \(\pm 1\). The number of nonzero values in the \(e\)th column \(M_e\) of \(M\) is the number of edges whose initial or final vertex is \(v\), namely the degree \(d_v\) of \(v\).

![Fig. 1. Graph \(G\) of Example 1.](image-url)

**Example 1.** Consider the directed graph \(G\) of Figure 1. \(G\) has \(|V| = 5\) vertices and \(|E| = 6\) edges. Its incidence matrix is

\[
M = \begin{bmatrix}
-1 & +1 & 0 & 0 & 0 \\
0 & -1 & +1 & 0 & 0 \\
-1 & 0 & 0 & 0 & +1 \\
0 & -1 & 0 & +1 & 0 \\
0 & 0 & -1 & +1 & 0 \\
0 & 0 & 0 & -1 & +1
\end{bmatrix}.
\]

The key graphical parameters of a finite graph \(G = (V, E)\) are:

- the number \(|V|\) of its vertices;
- the number \(|E|\) of its edges;
• the number \( \beta_0(\mathcal{G}) \) of its connected components, also called its zeroth Betti number;
• its cyclomatic number \( \beta_1(\mathcal{G}) = |E| - |V| + \beta_0(\mathcal{G}) \), also called its first Betti number.

For example, for our example graph \( \mathcal{G} \), we have \(|E| = 6\), \(|V| = 5\), and \( \beta_0(\mathcal{G}) = 1 \), so \( \beta_1(\mathcal{G}) = 6 - 5 + 1 = 2 \).

Evidently \( \beta_0(\mathcal{G}) \geq 0 \), with equality if and only if \( \mathcal{G} \) is the empty graph with \(|V| = |E| = 0\). We will assume that \( \mathcal{G} \) is nonempty; i.e., \( \beta_0(\mathcal{G}) \geq 1 \).

If \( \mathcal{G} = (V, E) \) has \( \beta_0(\mathcal{G}) \) connected components \( \mathcal{G}_i = (V_i, E_i) \), then \( V = \bigcup V_i \) and \( E = \bigcup E_i \). If the vertices and edges are ordered to reflect this partition, then the incidence matrix of \( \mathcal{G} \) is \( M = \text{diag}\{M_1, M_2, \ldots\} \), where \( M_i \) is the incidence matrix of \( \mathcal{G}_i \). The component graphs \( \mathcal{G}_i \) may then be analyzed independently. Therefore, without essential loss of generality, we will assume from now on that \( \mathcal{G} \) is connected; i.e., \( \beta_0(\mathcal{G}) = 1 \) and \( \beta_1(\mathcal{G}) = |E| - |V| + 1 \).

It is well known in elementary graph theory that if \( \mathcal{G} = (V, E) \) is connected, then \( \mathcal{G} \) contains a spanning tree \( T = (V, E_T) \), namely a cycle-free connected subgraph of \( \mathcal{G} \) that comprises all \(|V| \) vertices of \( \mathcal{G} \) and \(|E_T| = |V| - 1 \) of its edges. The number of edge deletions required to reduce \( \mathcal{G} \) to a spanning tree \( T \) of \( \mathcal{G} \) is thus \(|E| - |E_T| = |E| - |V| + 1 = \beta_1(\mathcal{G}) \), the cyclomatic number of \( \mathcal{G} \). It follows that \( \beta_1(\mathcal{G}) \geq 0 \), with equality if and only if \( \mathcal{G} \) is cycle-free. We will see that \( \beta_1(\mathcal{G}) \) measures the number of independent cycles in \( \mathcal{G} \).

For example, given the graph \( \mathcal{G} \) of Example 1, we may obtain a spanning tree \( T \subseteq \mathcal{G} \) by removing \( \beta_1(\mathcal{G}) = 2 \) edges; e.g., \( e_3 \) and \( e_5 \). Here \( \mathcal{G} \) contains three cycles, any two of which may be chosen as “independent;” the third cycle is then the “sum” of the other two. (These terms will be defined more satisfactorily later.)

B. Elements of algebraic topology

In algebraic topology, a finite graph \( \mathcal{G} = (V, E) \) is called a 1-dimensional complex. Its vertices \( v \in V \) are called its 0-dimensional objects, and its edges \( e \in E \) are called its 1-dimensional objects. As we shall see, its incidence matrix \( M \) is the matrix of a certain operator.

Let \( \mathcal{A} \) be some abelian group alphabet. The reader will not be misled if he or she assumes that \( \mathcal{A} \) is a field; however, we assert that all of our results hold if \( \mathcal{A} \) is any abelian group with a well-defined dual group (i.e., character group) \( \hat{\mathcal{A}} \); e.g., any finite abelian group. We will use notation that is appropriate when \( \mathcal{A} \) is a finite abelian group.

We consider the spaces \( C^0 = \mathcal{A}^V \) and \( C^1 = \mathcal{A}^E \) of all \( \mathcal{A} \)-valued functions defined on \( V \) and \( E \), respectively. In algebraic topology, the elements of \( \mathcal{A}^V \) and \( \mathcal{A}^E \) are called 0-cochains and 1-cochains, respectively. We will regard the elements \( x \in \mathcal{A}^V \) as sets \( x = (x_{v_1}, x_{v_2}, \ldots, x_{v_{|V|}}) \) of \(|V| \) elements of \( \mathcal{A} \) indexed by the elements of \( V \), and similarly the elements \( y \in \mathcal{A}^E \) as sets \( y = (y_{e_1}, y_{e_2}, \ldots, y_{e_{|E|}}) \) of \(|E| \) elements of \( \mathcal{A} \) indexed by the elements of \( E \). When \( x \) and \( y \) are used in conjunction with the incidence matrix \( M \), we may regard them as column vectors, and we will call them vertex vectors \( x \) and edge vectors \( y \), respectively.

In elementary algebraic topology, the coboundary operator \( d \) is defined as the homomorphism whose matrix is \( M \); i.e.,

\[
d : \mathcal{A}^V \to \mathcal{A}^E, \quad x \mapsto Mx.
\]

This definition makes sense because the “product” \( za \) is well-defined for all \( z \in \{0,1\}, a \in \mathcal{A} \), for any abelian group \( \mathcal{A} \); i.e., we regard \( M \) as an integer matrix. If \( \mathcal{A} \) contains a unit element \( 1 \) e.g., if \( \mathcal{A} \) is a field or a ring with unity— then \( M \) may alternatively be thought of as an \( \mathcal{A} \)-matrix.

The image of the coboundary operator \( d \) is called the coboundary space \( B^1 = \text{im} \, d \). Thus \( B^1 \) is the subspace/subgroup of \( \mathcal{A}^1 = \mathcal{A}^E \) that is generated by the elementary coboundary vectors \( \{M_0a | v \in V, a \in \mathcal{A}\} \).

If \( \mathcal{A} \) is a field, then \( B^1 \) is the column space of \( M \).

The kernel of \( d \) is called the zero-coboundary space \( Z^0 = \ker d \). We have immediately:

**Theorem 1(a) \((Z^0)\).** If \( \mathcal{G} = (V, E) \) is a connected graph, then its zero-coboundary space \( Z^0 = \ker d \) is the repetition code \( C = \{a, a, \ldots, a\} | a \in \mathcal{A} \} \subseteq \mathcal{A}^V \) over \( \mathcal{A} \).

**Proof:** The constraint \( Mx = 0 \) implies that for every edge \( e \in E \), \( M_e x = 0 \). Since the row vector \( M_e \) has precisely two nonzero values of opposite sign, this implies \( x_{h(e)} = x_{i(e)} \) for all \( e \in E \); i.e., the vertex values at the two ends of edge \( e \) must be equal. Since \( \mathcal{G} \) is connected, these edge constraints propagate throughout \( \mathcal{G} \), implying that all vertex values \( x_v \) must be equal.

A graphical illustration of this proof will be given in Section II-D; see Figure 5.

**Theorem 1(a) has the following corollary:**

2 More precisely, any group homomorphism \( \varphi : \mathcal{A}^V \to \mathcal{A}^E \) may be characterized by a matrix of group homomorphisms \( \{\varphi_{ev} : \mathcal{A} \to \mathcal{A}\} \). For the coboundary operator \( d \), the constituent homomorphisms \( d_{ev} \) are all zero, identity, or negative identity homomorphisms, as indicated by the entries \( M_{ev} \in \{0, \pm 1\} \) of the incidence matrix \( M \).
**Theorem 1(b) ($B^1$).** If $G = (V, E)$ is a connected graph, then its coboundary space $B^1 = \text{im} \ d$ is isomorphic to $\mathcal{A}^{|V| - 1}$.

**Proof:** By the fundamental theorem of homomorphisms, we have $B^1 = \text{im} \ d \cong C^0 / \ker d = C^0 / Z^0$.

For any $v \in V$, let $(C^0)_v$ denote the subset $\mathcal{A}^{|V| - 1}$ of $C^0 = \mathcal{A}^V$ such that $x_v = 0$. Evidently $(C^0)_v$ is isomorphic to $\mathcal{A}^{|V| - 1}$. Moreover, $(C^0)_v$ is a set of coset representatives for $C^0 / Z^0$, and the corresponding one-to-one map $(C^0)_v \leftrightarrow C^0 / Z^0$ is an isomorphism. Therefore $B^1 \cong C^0 / Z^0 \cong (C^0)_v \cong \mathcal{A}^{|V| - 1}$.

A graphical illustration of this proof will be given in Section [ ]

### C. Systematic $(n, k)$ group codes

In coding theory, if $A$ is a field, then $A^n$ is a vector space, and any subspace $C \subseteq A^n$ of dimension $k$ is called a linear $(n, k)$ code over $A$. Thus in the field case $Z^n = \ker d$ is a linear $|V| - 1$ block code over $A$, and $B^1 = \text{im} \ d$ is a linear $(|E|, |V| - 1)$ block code over $A$.

Moreover, every linear $(n, k)$ code $C$ has a systematic $k \times n$ generator matrix that contains a $k \times k$ identity matrix as a submatrix, whose column positions are said to be an *information set* of $C$; the remaining column positions are called a *check set*.

For example, if $A$ is a field and $I$ is an index set of size $|I| = n$, then the repetition code $C_r = \mathcal{A}^I$ is a linear $(n, 1)$ code over $A$, the all-1 codeword is a systematic $1 \times n$ generator matrix, and for any $i \in I$ the singleton subset $\{i\} \subseteq I$ is an information set.

We now extend this nomenclature by defining *systematic $(n, k)$ group codes*. We will ultimately show that all of the important spaces of algebraic topology are systematic $(n, k)$ group codes.

**Definition (systematic $(n, k)$ group codes).** If $A$ is an abelian group and $I$ is an index set of size $|I| = n$, then a subgroup $C \subseteq A^I$ is a systematic $(n, k)$ group code over $A$. Moreover, such a code will be called a systematic $(n, k)$ group code *over $A$* if the projection of $C$ onto $A^{|J|}$ for some subset $J \subseteq I$ of size $|J| = k$ is an isomorphism to $A^k \cong A^k$. Any such subset $J$ will be called an information set of $C$, and its complement $\bar{J}$ a check set of $C$.

For example, if $A$ is a group, then the repetition code $C_r = \mathcal{A}^I$ is a systematic $(n, 1)$ group code over $A$, and for any $i \in I$ the singleton subset $\{i\}$ is an information set, since the projection of $C_r$ onto any coordinate is an isomorphism.

In the spirit of this definition, for any $(n, k)$ group code $C$ over an abelian group $A$, we may call $n$ its “length” and $k$ its “dimension,” even though, strictly speaking, the term “dimension” is inappropriate when $A$ is not a field. We may also call $A^k$ a “space” and $C \subseteq A^k$ a “subspace.”

For another example, a *zero-sum code* $C_+ = \mathcal{A}^I$ of length $n = |I|$ over $A$ is defined as

$$C_+ = \left\{ a \in A^I \mid \sum_{i \in I} a_i = 0 \right\}.$$  

$C_+$ is evidently a strict subgroup of $A^k$. Moreover, since we may choose any $n - 1$ elements $a_i$ of a codeword $a \in C_+$ freely from $A$, and then choose the remaining element so that $\sum a_i = 0$, it is evident that the projection of $C_+$ onto any subset $J \subseteq I$ of size $|J| = n - 1$ is $A^J \cong A^{n - 1}$. Thus a zero-sum code of length $n$ over $A$ is a systematic $(n, n - 1)$ group code over $A$, and every subset of $I$ of size $n - 1$ is an information set.

We now show that every systematic $(n, k)$ group code has an I/O map, which will allow us to represent all of our realizations as I/O behaviors.

**I/O Map Lemma.** If $C$ is a systematic $(n, k)$ group code over an abelian group $A$ with information set $J \subseteq I$ and check set $\bar{J} = I \setminus J$, then there exists a homomorphism $\varphi_C : A^J \to A^J$ such that

$$C = \left\{ (a_J, \varphi_C(a_J)) \mid a_J \in A^J \right\}.$$  

**Proof:** Write the codewords $c \in A^I$ as $c = (c_J, c_{\bar{J}})$, where $c_J$ and $c_{\bar{J}}$ are the projections of $c$ onto $A^J$ and $A^\bar{J}$, respectively. Then the required homomorphism $\varphi_C : A^J \to A^J$ exists, since there is an isomorphism $C \leftrightarrow C_J \cong A^J$, and the projection of $C$ onto $A^J$ is a homomorphism.

In coding theory, an encoder that maps an information sequence $a_J \in A^J$ to a check sequence $\varphi_C(a_J) \in A^J$ and transmits both as the codeword $(a_J, \varphi_C(a_J))$ is called a systematic encoder.

In system theory, a behavior of the form $\{(a_J, \varphi_C(a_J)) \mid a_J \in A^J\}$ is called an I/O behavior, where $a_J \in A^J$ is regarded as the input, and $\varphi_C(a_J) \in A^J$ as the output. In this context, we will call the homomorphism $\varphi_C : A^J \to A^J$ an I/O map.

To construct a "generator matrix" for a systematic $(n, k)$ group code $C$ with information set $J$, we observe that $C$ has a set of $k$ one-dimensional subcodes that may be regarded as a systematic "basis" of $C$, as follows. For each of the $k$ coordinates $i \in J$, consider the subcode
\( C_i \subseteq \mathcal{C} \) consisting of the codewords in \( \mathcal{C} \) that are all-zero in the remaining \( k-1 \) coordinates \( \mathcal{J} \setminus \{i\} \). By the isomorphism between \( \mathcal{A}^{\mathcal{J}} \) and \( \mathcal{C}, \mathcal{C}_i \) must be isomorphic to \( \mathcal{A} \) via projection onto the \( i \)th coordinate. Moreover, \( \mathcal{C} \) is evidently the direct sum of the subcodes \( \mathcal{C}_i \):

\[
\mathcal{C} = \bigoplus_{i \in \mathcal{J}} \mathcal{C}_i \cong \mathcal{A}^{\mathcal{J}}.
\]

For example, for the zero-sum code \( \mathcal{C}_+ \subseteq \mathcal{A}^2 \), any subset \( \{i\} \subseteq \mathcal{I} \) of size 1 is a check set, and the set of \( |\mathcal{I}| - 1 \) “one-dimensional” subcodes \( (\mathcal{C}_+)_{j \neq i} \), that consist of all codewords \( e \in \mathcal{C}_+ \) such that \( c_j = a, c_i = -a \), and all remaining coordinates are zero forms a “basis” for \( \mathcal{C}_+ \).

Finally, the support \( S \subseteq \mathcal{I} \) of an \((n,k)\) group code \( \mathcal{C} \subseteq \mathcal{A}^n \) is the set of indices \( i \in \mathcal{S} \) such that the projection \( \mathcal{C}_i \) of \( \mathcal{C} \) onto the \( i \)th coordinate is nontrivial. If \( \mathcal{C} \) has support \( S \) (or less), then we may say that \( \mathcal{C} \) is effectively a \((|S|,k)\) group code, and that its effective length is \(|S|\). For example, the subcodes \( \mathcal{C}_i \) defined in the previous paragraph are effectively \((n-k+1,1)\) group codes with supports \( \{i\} \cup \mathcal{J} \) (or less).

**D. Elementary normal realizations**

We will now construct normal realizations for the zero-coboundary space \( \mathbb{Z}^0 = \ker d \) and the coboundary space \( B^1 = \im d \) that will help to visualize and prove their properties.

In general, a normal realization [5] is a graphical model based on a graph \( G = (V,E,H) \), in which the vertices (or “nodes”) \( v \in V \) represent constraint codes \( \mathcal{C}_v \), the edges \( e \in E \) represent external variables, and the half-edges \( h \in H \) represent half-edge constraints. The constraint code \( \mathcal{C}_v \) is the set of all permissible (“valid”) values of the variables corresponding to the edges and half-edges that are incident on vertex \( v \). The set of all valid variable configurations—i.e., the configurations that are compatible with all constraints—is called the behavior \( \mathcal{B} \) of the realization, and the projection of the behavior onto the half-edge variables is called its external behavior \( \mathcal{C} \); alternatively, \( \mathcal{C} \) is called the code that is realized. If all constraint codes are linear, then \( \mathcal{B} \) and \( \mathcal{C} \) are vector spaces; if all constraint codes are abelian groups, then \( \mathcal{B} \) and \( \mathcal{C} \) are abelian groups.

For this paper, we need only elementary normal realizations, namely normal realizations that satisfy the following restrictions:

- All constraint codes are either repetition codes \( \mathcal{C}_+ \), which constrain all incident variables to be equal, or zero-sum codes \( \mathcal{C}_- \), which constrain the sum of all incident variables to be zero. Thus if the degree of vertex \( v \) is \( n \), then \( \mathcal{C}_v \) is either the \((n,1)\) repetition code \( \mathcal{C}_- \) over \( \mathcal{A} \), or the \((n,0)\) zero-sum code \( \mathcal{C}_+ \) over \( \mathcal{A} \).

We will indicate repetition constraints by the symbol \( \square \), and zero-sum constraints by the symbol \( \blacksquare \). We may also employ the following simple manipulations and special symbols:

- A repetition constraint of degree 2 may be simply replaced by an edge, since \( \square \yrightarrow \square \).
- A zero-sum constraint of degree 2 may be replaced by an edge with a small circle representing a sign inverter: \( \square \yrightarrow \bigcirc \). We will sometimes call this an inverting edge.
- A repetition constraint of degree 2 plus a sign inverter may be replaced by an inverting edge, since \( \square \yrightarrow \square \).
- A zero-sum constraint of degree 2 plus a sign inverter may be replaced simply by an edge, since \( \square \yrightarrow \square \).
- We may optionally put an arrow on an edge if we wish to indicate the direction of a cause-and-effect relationship, as we will illustrate shortly below.

Using normal realizations, we now wish to study the image and kernel of the coboundary operator \( d: \mathcal{A}^V \rightarrow \mathcal{A}^E, x \mapsto Mx \) of a graph \( G = (V,E) \) with incidence matrix \( M \). The input/output (I/O) behavior of \( d \) will be defined as \( W^{01} = \{(x,y) \in \mathcal{A}^V \times \mathcal{A}^E \mid y = Mx \} \), which is evidently a systematic \((|V|+|E|,|V|)\) group code over \( \mathcal{A} \) with information set \( V \).

Figure 2 shows an elementary normal realization of the I/O behavior \( W^{01} \) for our example graph \( G \). (We observe that the graph \( G = (V_G,E_G,H_G) \) of this normal realization actually has \(|V_G| = |V|+|E|\) vertices, \(|E_G| = 2|E|\) edges, and \(|H_G| = |V|+|E|\) half-edges.)

In Figure 2 a set \( x = \{x_v, v \in V\} \) of \(|V|\) external input variables is associated with the vertices \( v \in V \) of \( G \). Each vertex variable \( x_v \) is replicated \( d_v \) times via a repetition constraint, and passed on (with a sign inversion if \( v = h(e) \)) to a zero-sum constraint associated with one of the \( d_v \) adjacent edges \( e \in E(v) \), thus making the input to the zero-sum constraint \(-M_{e,v}x_v \). For each edge \( e \in E \), a zero-sum constraint on all the incident signed vertex variables and the output variable \( y_e \) at edge \( e \) enforces the constraint \( y_e + (-M_{e,v})x_v = 0 \); thus \( y = Mx \), as desired. Thus every constraint in the realization is realized as a little I/O behavior. Arrows
on all edges indicate the directions of these cause-and-effect relationships.

Now, to obtain a realization of the zero-coboundary space $Z^0 = \ker \partial = \{ x | Mx = 0 \}$, we constrain the external edge variables $y_e$ to equal zero, which simply removes them from the realization\(^3\). Also, since $\bigcap \{ A \} = \{ \}$, we may simply use the latter realization for each edge. Thus we obtain the extremely simple pictorial proof of Theorem 1(a).

Similarly, to get a realization of the coboundary space $B^1 = \text{im} \, \partial$, we simply remove the external vertex variables $x_v$ from Figure\(\[2\] while leaving the $d_e$ internal replica variables representing the values $x_v$. Thus we obtain the realization of $B^1$ shown in Figure\(\[4\].

\textbf{E. Partitions, cut sets, and bases for $B^1$}

We have seen in Theorem 1(b) that the coboundary space $B^1 = \text{im} \, \partial$ of a graph $G = (V, E)$ is a systematic ($|E|, |V| - 1$) group code over $A$. We will now show that for any spanning tree $T$ of $G$, $B^1$ has an information set corresponding to the $(|V| - 1)$-element edge set $E_T$ of $T$, and a corresponding “basis” consisting of certain cut-set vectors, to be defined below.

Let $P = V_1 \cup V_2$ be a nontrivial disjoint partition of the vertex set $V$ into two subsets, $V_1$ and $V_2 = \bar{V}_1$. $P$ induces a partition of the edge set $E$ into three subsets: a subset $E_1$ of edges whose ends are both in $V_1$, a subset $E_2$ of edges whose ends are both in $V_2$, and the remaining subset $E_P$ of edges that have one end in $V_1$ and one in $V_2$, which is called the cut set of the partition $P$.

We assume that the two subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are both connected.

We define a cut-set vector\(^4\) of the partition $P = V_1 \cup V_2$ as the image $y = Mx$ of any vertex vector $x = (x_1, x_2)$ such that $x_1$ is any element of the repetition code $C_e \subseteq A^{V_1}$ defined on the vertex subset $V_1$, and $x_2$ is the all-zero vector on the complementary subset $V_2$. Evidently $y \in B^1$.

\textbf{Cut-Set Lemma.} If $y \in A^E$ is a cut-set vector of a partition $P = V_1 \cup V_2$, then $y_e = 0$ if $e \in E_1$ or $e \in E_2$. Moreover, if $e \in E_P$, then $y_e = \pm a$ for some $a \in A$, where the sign depends on whether the edge $e$ goes from $V_1$ to $V_2$, or vice versa.

\(\text{In other words, } B^1 \text{ is the projection } (W^{01})_{|A^E} = \{ y \in A^E | \exists x : (x, y) \in W^{01} \} \text{ of } W^{01} \text{ onto } A^E.\)
Proof. The first statement follows from the fact that $x_1$ and $x_2$ are elements of the repetition codes over $V_1$ and $V_2$, respectively, and therefore are in the kernels of the coboundary operators $d_1 : \mathcal{A}^V \to \mathcal{A}^E$ and $d_2 : \mathcal{A}^V \to \mathcal{A}^E$ of the subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. The second statement follows from the fact that if $x_v = a$ for all $v \in V_1$ and $x_v = 0$ for all $v \in V_2$, and $e \in E_P$ connects a vertex in $V_1$ to a vertex in $V_2$, then $y_v = \pm a$, with the sign depending on whether the head or the tail of $e$ is in $V_1$.

To illustrate, in our example graph $G$, the edges $E_P = \{e_3, e_4, e_5\}$ form a cut set corresponding to the partition $P = \{v_1, v_2, v_3\} \cup \{v_4, v_5\}$ of $V$, as shown in Figure 5. Given an input vector $x$ that has the constant value $a$ on $V_1$ and $0$ on $V_2$, the coboundary operator $d$ produces an output vector $y = Mx$ that is all-zero on $E_1$ and $E_2$, and equal to either $a$ or $-a$ (depending on the edge orientation) on the cut set $E_P$.

For any nontrivial partition $P$ of $V$, the set of all cut-set vectors $y \in B^1$ as $x_1$ ranges through all elements of the repetition code $C_z \subseteq \mathcal{A}^V$ will be called the cut-set subspace $(B^1)_P \subseteq B^1$. Evidently $(B^1)_P$ is a one-dimensional linear or group code with support $E_P \subseteq E$.

We now find a set of $|V| - 1$ fundamental cut sets $E_P$ such that the corresponding one-dimensional cut-set subspaces $(B^1)_P$ generate the $\binom{|V| - 1}{1}$-dimensional space $B^1$; i.e., we find a “basis” for $B^1$ (where the quotes may be removed in the linear case). We start with any spanning tree $T = (V, E_T)$ of $G$, where $|E_T| = |V| - 1$. Since $T$ is a tree, every edge $e \in E_T$ is a cut set of $T$ that partitions the vertex set $V$ of $T$ into two connected subsets, say $V_1(e)$ and $V_2(e)$. The cut set $E_{P(e)}$ of $G$ that corresponds to the same partition $P(e)$ of the vertex set $V$ of $G$ will be defined as our $e$th fundamental cut set of $G$. The cut set $E_{P(e)}$ must include $e$, but cannot include any other edges in $E_T$, since their ends are either both in $V_1(e)$ or both in $V_2(e)$. Thus $E_{P(e)} \subseteq \{e\} \cup E_T$.

For example, for our example graph $G$, deleting the $\beta_1(G) = 2$ edges $e_3$ and $e_5$ yields a spanning tree $T$ with $E_T = \{e_3, e_5\}$. The respective cut sets in $(e_1) \cup E_T$, $(e_2) \cup E_T$, $(e_4) \cup E_T$, and $(e_5) \cup E_T$ are $(e_1, e_3, e_5, e_4, e_3, e_5)$ and $(e_6, e_3)$, as shown in Figure 6.

Theorem 2 (“basis” for $B^1$). For any graph $G$ and any spanning tree $T \subseteq G$ with cut set $E_T$, the $|V| - 1$ cut-set subspaces $\{(B^1)_{P(e)}, e \in E_T\}$ are a set of one-dimensional subspaces of $B^1$ that form a systematic “basis” for $B^1$. Thus $E_T$ is an information set for $B^1$.

Proof. The $|V| - 1$ cut-set subspaces $\{(B^1)_{P(e)}, e \in E_T\}$ are all independent, since their supports are completely disjoint on $E_T$. Since $B^1$ is $\binom{|V| - 1}{1}$-dimensional, it follows that every element of $B^1$ is a unique combination of elements of $\{(B^1)_{P(e)}, e \in E_T\}$.

Thus $B^1$ is sometimes called the cut space of $G$, and $\dim B^1 = |V| - 1$ the rank of $G$.

F. Nonredundant I/O realizations

We have seen that $Z^n$ and $B^1$ are systematic $(n, k)$ group codes, and we have identified their information sets. We will now construct I/O realizations with these information sets. Moreover, in order to obtain I/O realizations in which every edge may be labeled with a cause-and-effect arrow, we first analyze the unobservability (generator redundancy) and uncontrollability (constraint redundancy) properties of our previous realizations, and eliminate the corresponding redundancies.

As we have seen, $Z^n = \ker d$ is a $(|V|, 1)$ repetition code over $\mathcal{A}$, so the set $\{v\}$ comprising any single vertex $v \in V$ may be taken as an information set, and then the remaining vertices comprise the corresponding check set.

However, we can see that specifying an information set does not suffice to determine cause-and-effect relationships on the internal edges. The fundamental reason is that the realization of $Z^n$ in Figure 3 contains $\beta_1(G) = 2$ redundant edge constraints. Indeed, it is easy to see that it would suffice to propagate the value of any single input vertex variable through a spanning tree $T$ of $G$ in order to generate all other output vertex variables correctly;
moreover, in such a realization every edge in \( E_T \) would have a definite direction, namely the direction “away” from the input vertex. Such a nonredundant realization of \( Z^0 \) is illustrated in Figure 7 where we choose the information set \( \{ v_1 \} \) and \( E_T = \{ e_1, e_2, e_4, e_6 \} \).

Since the unobservable behavior \( \mathcal{B}^u \) of Figure 4 is a repetition code, it follows that adding the same element to \( \mathcal{A} \) to every vertex value does not change the output configuration in \( B^1 \). (In physics, this kind of unobservability is sometimes called “global symmetry.”)

To obtain an observable realization of \( B^1 \), we may therefore fix any internal vertex variable in Figure 4 to zero, breaking the global symmetry. (Again, this is an example of a “local reduction” as in [8].)

We illustrate such a realization in Figure 9 where we have chosen \( E_T = \{ e_1, e_2, e_4, e_6 \} \) as the information set, and fixed \( x_1 = 0 \). Such an I/O realization could be used in a simulation to generate all elements of \( B^1 \), by letting \( \{ y_1, y_2, y_4, y_6 \} \) run through all \(|\mathcal{A}|^4 \) possible configurations.

This argument is essentially the same as the algebraic argument used in our proof of Theorem 1(b), and Figure 9 may be regarded as an illustration of that proof.

G. Elements of duality theory

We now introduce some elementary duality theory for both the group and field cases. (See the Appendix for more on duality theory for abelian groups, focussing on Fourier transforms.)
We stipulate that \( A \) must have a well-defined **dual group** \( \hat{A} \), for which the inner product \( \langle \hat{a}, \hat{a} \rangle \) is well defined for all \( \hat{a} \in \hat{A}, a \in A \). If \( A \) is a field \( \mathbb{F} \), then \( \hat{A} \) may also be taken as \( \mathbb{F} \); if \( A \) is \( \mathbb{Z}_q \), then \( \hat{A} \) may also be taken as \( \mathbb{Z}_q \). More generally, if \( A \) is a finite abelian group, then \( \hat{A} \) has a well-defined dual group \( A \), called its **character group** that is isomorphic to \( A \). Again, we will assume that \( A \) is a finite abelian group (or finite field) in this paper.

Given a finite coordinate index set \( I \), the dual group to \( A^I \) is then \( \hat{A}^I \), with the inner product between \( \hat{a} \in \hat{A}^I \) and \( a \in A^I \) defined in standard coordinatewise fashion: \( \langle \hat{a}, a \rangle = \sum_{i \in I} \langle \hat{a}_i, a_i \rangle \). Two vectors \( \hat{a} \in \hat{A}^I \) and \( a \in A^I \) are **orthogonal** if \( \langle \hat{a}, a \rangle = 0 \). The dual code (or **orthogonal code**) \( C^\perp \) to a code group \( C \subseteq A^I \) is the set of all \( \hat{a} \in \hat{A}^I \) that are orthogonal to all \( a \in C \).

Given a group code \( C \subseteq A^I \) and an index subset \( J \subseteq I \) with complement \( J = I \setminus J \), the **projection** of \( C \) onto \( J \) is defined as \( C_{|J} = \{ a_J \in A^J \mid \exists a \in C \} \), and the **cross-section** of \( C \) on \( J \) is defined as \( C_{J} = \{ a_J \in A^J \mid (a_J, 0_{J^c}) \in C \} \). If \( C^\perp \) is the dual code to \( C \), then by projection/cross-section duality \( (C_{|J})^\perp = (C^\perp)_J \) and \( (C_{J})^\perp = (C^\perp)_{J^c} \).

Given a homomorphism \( \varphi : A^I \to A^J \), its **adjoint homomorphism** is defined as the unique homomorphism \( \hat{\varphi} : \hat{A}^J \to \hat{A}^I \) such that \( \varphi(\hat{a}) = \langle \hat{a}, \varphi(a) \rangle \) for all \( \hat{a} \in \hat{A}^J, a \in A^J \).

We define the I/O **code** of \( \varphi \) as \( W = \{ (a, \varphi(a)) \in A^J \times A^J \mid a \in A^I \} \), and its dual I/O code as \( W^\perp = \{ (-\varphi(\hat{a}), \hat{a}) \in \hat{A}^I \times A^J \mid \hat{a} \in \hat{A}^J \} \). Evidently \( W \) is essentially the I/O code of \( \varphi \), up to sign inversion and coordinate ordering.

Adjoint homomorphisms and their I/O codes have the following properties:

**Adjoint Homomorphism Lemma.** Given a homomorphism \( \varphi : A^I \to A^J \) and its adjoint homomorphism \( \hat{\varphi} : \hat{A}^J \to \hat{A}^I \):

(a) The I/O code \( W \) of \( \varphi \) is a systematic \( ([I] \cup [J], [I]) \) group code, with information set \( I \).

(b) The dual I/O code \( W^\perp \) is the orthogonal systematic \( ([I] \cup [J], [J]) \) group code \( W^\perp \) with information set \( J \).

(c) The kernel of \( \varphi \) and the image of \( \hat{\varphi} \) are orthogonal codes: \( \ker \varphi = \im \hat{\varphi} \).

(d) Similarly, \( \ker \hat{\varphi} = \im \varphi \).

**Proof:** (a) The projection of the I/O code \( W = \{ (a, \varphi(a)) \in A^J \times A^I \mid a \in A^I \} \) onto \( I \) is an isomorphism \( W \leftrightarrow A^I \) with image \( W_{|I} = A^I \).

(b) The projection of the dual I/O code \( W^\perp = \{ (-\varphi(\hat{a}), \hat{a}) \in \hat{A}^I \times A^J \mid \hat{a} \in \hat{A}^I \} \) onto \( J \) is an isomorphism \( W_{|J} \leftrightarrow \hat{A}^J \) with image \( W_{|J} = \hat{A}^J \).

Because the inner product is defined coordinatewise, the inner product between \( \langle -\hat{\varphi}(\hat{a}), \hat{a} \rangle \in W^\perp \) and \( (\hat{a}, \varphi(a)) \in W^\perp \) is \( \langle -\hat{\varphi}(\hat{a}), \hat{a} \rangle = 0 \) for all \( \hat{a} \in \hat{A}^I, a \in A^J \), by the definition of \( \hat{\varphi} \). Therefore \( W \) and \( W^\perp \) are orthogonal.

Indeed, they are dual codes, since \( \dim \hat{W} + \dim W = |J| + |I| \).

(c) The cross-section \( W_{|J} = \{ a \in A^J \mid (a, 0) \in W \} \) of \( W \) on \( A^J \) is precisely \( \ker \varphi \), whereas the projection \( (W^\perp)_{|J} = \{ b \in \hat{A}^I \mid (\hat{b}, \hat{a}) \in W^\perp \} \) of \( W^\perp \) on \( \hat{A}^I \) is precisely \( \im \hat{\varphi} \).

By projection/cross-section duality, \( W_{|J} = \ker \varphi \) and \( (W^\perp)_{|J} = \ker \hat{\varphi} \) are thus dual codes.

(d) **Mutatis mutandis**, \( W_{|J} = \im \varphi \) and \( (W^\perp)_{|J} = \ker \hat{\varphi} \).

In view of the I/O Map Lemma for systematic \((n, k)\) group codes, we thus have the following generalization of well-known results for the field case:

**Theorem 3 (Dual systematic group codes).** If \( C \subseteq A^I \) is a systematic \((n, k)\) code over \( A \), then its dual code \( C^\perp \subseteq A^I \) is a systematic \((n, n-k)\) code over the dual group \( \hat{A} \). Moreover, the information sets of \( C^\perp \) are the check sets of \( C \), and vice versa.

**Proof:** By the I/O Map Lemma, if \( C \) is a systematic \((n, k)\) group code with information set \( I \subseteq I \), then there exists a homomorphism \( \varphi_C : A^I \to A^J \) such that \( C \) is the I/O code of \( \varphi_C : A^J \to A^I \); i.e., \( C = \{ (a_J, \varphi_C(a_J)) \mid a_J \in A^J \} \). By the Adjoint Homomorphism Lemma, if \( \hat{\varphi}_C : \hat{A}^J \to \hat{A}^I \) is the adjoint homomorphism to \( \varphi_C \), then the dual I/O code \( \{ (-\hat{\varphi}_C(\hat{a}), \hat{a}) \in \hat{A}^I \times A^J \mid \hat{a} \in \hat{A}^J \} \) is the orthogonal code \( C^\perp \). Evidently \( C^\perp \) is a systematic \((n, n-k)\) group code over \( \hat{A} \) with an information set \( J \) that is equal to the complement of the information set \( J \) of \( C \).

For example, if \( C_{\text{n}} \subseteq A^I \) is an \((n, 1)\) repetition code over \( A \), then its dual code \( C_{\text{n}}^\perp \) is the \((n, n-1)\) zero-sum code \( C_{\text{n}}^\perp \subseteq A^I \). The information sets of \( C \) are the singleton sets \( \{ i \} \), whereas the information sets of \( C_{\text{n}}^\perp \) are the complementary sets \( \{ I \setminus \{ i \} \} \).

**H. Duality in elementary algebraic topology**

In algebraic topology, for historical reasons, the dual space to the space \( C^0 = A^V \) of 0-cochains is called the space of **0-chains**, denoted by \( C_0 = A^A \). Similarly, the dual \( A^F \) to the space \( C_1 = A^E \) of 1-cochains is called the space of **1-chains**, denoted by \( C_1 = A^E \).
\[ Z^0 = \ker d : ([V], 1) \rightarrow \mathcal{A}^V \]
\[ \mathcal{A}^V \overset{d}{\rightarrow} \mathcal{A}^E \]
\[ B_0 = \im \partial : ([V], |V| - 1) \rightarrow \mathcal{A}^V \]
\[ \mathcal{A}^V \overset{\partial}{\leftarrow} \mathcal{A}^E \]
\[ B_1 = \im d : ([E], |V| - 1) \rightarrow \mathcal{A}^E \]
\[ Z_1 = \ker \partial : ([E], |E| - |V| + 1) \rightarrow \mathcal{A}^E \]

Fig. 10. Duality of coboundary operator \( d : \mathcal{A}^V \rightarrow \mathcal{A}^E \) and boundary operator \( \partial : \mathcal{A}^E \rightarrow \mathcal{A}^V \).

We will regard the elements \( \hat{x} \in \mathcal{A}^V \) as sets \( \hat{x} = (\hat{x}_v_1, \hat{x}_v_2, \ldots, \hat{x}_v_{|V|}) \) of \([V]\) elements of \( \hat{A} \) indexed by the elements of \( V \), and similarly the elements \( \hat{y} \in \mathcal{A}^E \) as sets \( \hat{y} = (\hat{y}_e_1, \hat{y}_e_2, \ldots, \hat{y}_e_{|E|}) \) of \([E]\) elements of \( \hat{A} \) indexed by the elements of \( E \). When \( \hat{x} \) and \( \hat{y} \) are used in conjunction with the incidence matrix \( M \), we may regard them as row vectors, and call them dual vertex vectors \( \hat{x} \) and dual edge vectors \( \hat{y} \), respectively. We define inner products in standard componentwise fashion; e.g., for \( x \in \mathcal{A}^V \) and \( \hat{x} \in \mathcal{A}^V \), \( \langle \hat{x}, x \rangle = \sum_v \langle \hat{x}_v, x_v \rangle \).

The boundary operator \( \partial : \mathcal{A}^E \rightarrow \mathcal{A}^V \) is then defined as the homomorphism that maps \( \hat{y} \in \mathcal{A}^E \) to \( \partial(\hat{y}) = \hat{y}M \in \mathcal{A}^V \), where \( M \) is again the incidence matrix of \( \mathcal{G} \). Evidently, for any \( \hat{y} \in \mathcal{A}^E \), \( x \in \mathcal{A}^V \) we have
\[ \langle \partial(\hat{y}), x \rangle = \hat{y}Mx = \langle \hat{y}, d(x) \rangle. \]

It follows that:

**Boundary Operator Lemma.** The boundary operator \( \partial : \mathcal{A}^E \rightarrow \mathcal{A}^V \) is the adjoint homomorphism \( d \) to the coboundary operator \( d : \mathcal{A}^V \rightarrow \mathcal{A}^E \). \( \square \)

By the Adjoint Homomorphism Lemma, this implies that the kernel of \( \partial \), called the zero-boundary space \( Z_1 = \ker \partial \), is the orthogonal code \((B_1)^\perp\) to the image \( B_1 = \im d \) of \( d \); and conversely the image of \( \partial \), called the boundary space \( B_0 = \im \partial \), is the orthogonal code \((Z_0)^\perp\) to the kernel \( Z_0 = \ker d \) of \( d \). Therefore we have:

**Theorem 4** \((B_0, Z_1)\). For a connected graph \( \mathcal{G} = (V, E) \):

(a) The boundary space \( B_0 = \im \partial \) is a \((|V|, |V| - 1)\) zero-sum code \( C_+ \subseteq \mathcal{A}^V \).

(b) The zero-boundary space \( Z_1 = \ker \partial \) is a systematic \((|E|, \beta_1(\mathcal{G}))\) group code over \( \hat{A} \).

**Proof.** (a) By Theorem 1(a), \( Z_0 = \ker d \) is the \((|V|, 1)\) repetition code over \( \mathcal{A} \), so its dual code \( B_0 \) is the \((|V|, |V| - 1)\) zero-sum code over \( \mathcal{A} \).

(b) By Theorem 1(b), \( B_1 \subseteq \mathcal{A}^E \) is a systematic \((|E|, |E| - \beta_1(\mathcal{G}))\) group code over \( \mathcal{A} \), so its dual code \( Z_1 \) is a systematic \((|E|, \beta_1(\mathcal{G}))\) group code over \( \hat{A} \). \( \square \)

Figure 10 summarizes the duality relationships between the adjoint operators \( d : \mathcal{A}^V \rightarrow \mathcal{A}^E \) and \( \partial : \mathcal{A}^E \rightarrow \mathcal{A}^V \), and between their kernels and images.

I. Dual normal realizations

We will now construct normal realizations of the boundary space \( B_0 = \im \partial \) and the zero-boundary space \( Z_1 = \ker \partial \) of \( \mathcal{G} \) as dual realizations to our earlier realizations of the zero-coboundary space \( Z_0 = (B_0)^\perp \) and the coboundary space \( B_1 = (Z_1)^\perp \) of \( \mathcal{G} \), respectively.

In general, the dual realization to an elementary normal realization is obtained as follows:

- The variable alphabet \( \mathcal{A} \) is replaced by its dual alphabet \( \hat{A} \);
- Repetition constraints (\( \square \)) are replaced by zero-sum constraints (\( \square \)), and vice versa;
- Edges (\( \rightarrow \)) are replaced by inverting edges (\( \neg \rightarrow \)), and vice versa.

By the normal realization duality theorem 5, if the external behavior of the original normal realization is \( C \), then the external behavior of the dual normal realization is \( C^\perp \). (If \( \mathcal{A} \) is a finite abelian group, then this theorem is a corollary to the normal factor graph duality theorem; see Appendix.)

We start with the dual I/O behavior \( W_{10} = \{(-\hat{y}M, \hat{y}) \mid \hat{y} \in \mathcal{A}^E \} \) of the boundary operator \( \partial \), which by the Adjoint Homomorphism Lemma is the dual code to the I/O behavior \( W_{01} = \{(x, Mx) \mid x \in \mathcal{A}^V \} \) of the coboundary operator \( d \); i.e., \( W_{10} = (W_{01})^\perp \). Thus \( W_{10} \) is a systematic \((|V| + |E|, |E|)\) group code over \( \hat{A} \) with information set \( E \).

We may thus obtain an elementary normal realization of \( W_{10} \) by dualizing our earlier realization of \( W_{01} \) in Figure 2 which results in the realization of Figure 11. Again, we have included arrows on all edges to indicate cause-and-effect relationships; note that all arrows are now reversed.

In Figure 11 a set \( \hat{y} = \{\hat{y}_e, e \in E\} \) of \(|E|\) external input variables is associated with the edges \( e \in E \) of \( \mathcal{G} \). Each edge variable \( \hat{y}_e \) is replicated twice via a repetition constraint of degree 3. Each replica is passed on (with a sign inversion if \( v = t(e) \)) to the zero-sum constraint associated with one of the two vertices \( h(e), t(e) \), thus making \( \hat{y}_e M_{ew} \) the input to this zero-sum constraint. At each vertex \( v \in V \) of \( \mathcal{G} \), the zero-sum constraint on
all these incident variables plus an output variable \( \hat{x}_v \) enforces the constraint \( \hat{x}_v = -(\hat{y}M)_v \); thus \( \hat{x} = -\hat{y}M \), as desired.

To get a realization of the boundary space \( B_0 = \text{im } \partial = \{ yM \mid y \in \mathcal{A}^E \} \), we may simply remove the external variables \( \hat{y}_e \) in Figure 11 while leaving the two internal replica variables representing the external variables. Since \( \overrightarrow{\text{---}} = \overrightarrow{---} \), we may simply use the latter realization for each edge. Thus we obtain the simple realization of \( B_0 \) shown in Figure 12 (Strictly, this realization realizes \( -B_0 = \{ -\hat{y}M \} \), but since \( B_0 \) is an abelian group, we have \( -B_0 = B_0 \).) Alternatively, since \( B_0 = (Z^0)^\perp \), we may obtain the realization of Figure 12 by dualizing the Figure 3 realization of \( Z^0 \).

Similarly, to get a realization of the zero-boundary space \( Z_1 = \ker \partial = (B^1)^\perp \), we may either zero the external vertex variables \( \hat{x}_v \) in Figure 11 or else dualize the realization of \( B^1 \) in Figure 3. By either method, we obtain the realization of \( Z_1 \) shown in Figure 13.

\section*{J. Cycles and bases for \( Z_1 \)}

By Theorem 4(b), \( Z_1 \) is a \(|E|, \beta_1(G)\) group code over \( \mathcal{A} \). We now show that \( Z_1 \) is generated by a certain set of \( \beta_1(G) \) cycle vectors. Thus \( Z_1 \) is often called the cycle space of \( G \), and \( \beta_1(G) \) the nullity of \( G \). Historically, this seems to have been the starting point of algebraic topology.

Given an edge \( e \in E \) of a graph \( G \) with head and tail vertices \( h(e) \) and \( t(e) \), we define \( -e \) as the reverse edge with head and tail vertices \( t(e) \) and \( h(e) \), respectively. A cycle of a graph \( G \) is then a simple closed path \( e = (\pm e_1, \ldots, \pm e_n) \), consisting of a set of \( n \) edges or reversed edges such that \( h(e_i) = t(e_{i+1}) \) for all \( i \in [1, n] \), where \( e_{n+1} = e_1 \), and no edge or vertex is repeated.

We then define a cycle vector \( [4] \) corresponding to a cycle \( e = (\pm e_1, \ldots, \pm e_n) \) as any edge vector \( \hat{y} \in \mathcal{A}^E \) such that, for some \( \bar{a} \in \mathcal{A} \), \( \hat{y}_e = \bar{a} \) if \( +e \) is an element of \( e \), \( \hat{y}_e = -\bar{a} \) if \( -e \) is an element of \( e \), and \( \hat{y}_e = 0 \) otherwise. The \(|A| \) cycle vectors corresponding to a given cycle \( e \) evidently form a one-dimensional cycle code \( C(e) \subseteq \mathcal{A}^E \) that is effectively a repetition code with support \( e \).

For example, our example graph \( G \) has a cycle \( e = (e_1, e_4, e_6, -e_3) \) of length \( n = 4 \). The corresponding cycle code is \( C(e) = \{ (\bar{a}, 0, -\bar{a}, \bar{a}) \mid \bar{a} \in \mathcal{A} \} \).

It is easy to see that if \( y \in \mathcal{A}_E \) is any cycle vector, then \( \partial(y) = yM = 0 \); i.e., every cycle vector is in \( Z_1 = \ker \partial \). For example, we show in Figure 14(a) that if the cycle vector \( (\bar{a}, 0, -\bar{a}, \bar{a}) \) is the input to the I/O realization of Figure 11 then the output vector is \( yM = 0 \). Therefore:

\textbf{Cycle Code Lemma.} For every cycle \( e \) of a connected graph \( G \), the cycle code \( C(e) \) is a one-dimensional subcode of \( Z_1 \) with support \( e \).

We will now show that \( Z_1 \) has a “basis” consisting of \( \beta_1(G) \) cycle codes \( C(e) \).

We start with any spanning tree \( T = (V, E_T) \) of \( G \).

We define \( E_T = E \setminus E_T \) as the complement of \( E_T \), so \( |E_T| = |E| - |V| + 1 = \beta_1(G) \). For each edge \( e \in E_T \), there exists a fundamental cycle \( e_c \) of \( G \) comprising \( e \).
and the path connecting its final vertex to its initial vertex in \( T \). Since the support of the cycle code \( C(e_e) \) is \( e_e \subseteq \{ e \} \cup E_T \), it follows that the \( \beta_1(G) \) cycle codes \( C(e_e) \) are independent, since their supports are completely disjoint on \( E_T \). Thus we have proved:

**Theorem 5** ("basis" for \( Z_1 \)). For any graph \( G \) and any spanning tree \( T \subseteq G \), \( E_T \) is an information set for \( Z_1 \), and the \( \beta_1(G) = |E| - |V| + 1 \) cycle codes \( C(e_e), e \in E_T, \) are a set of one-dimensional subspaces of \( Z_1 \) that form a systematic "basis" for \( Z_1 \).

For example, for our example graph \( G \), deleting the \( \beta_1(G) = 2 \) edges \( e_3 \) and \( e_5 \) yields a spanning tree \( T \). The respective fundamental cycles in \( \{ e_3 \} \cup E_T \) and \( \{ e_5 \} \cup E_T \) are \( e_3 = (e_1, e_4, e_6, -e_3) \) and \( e_5 = (e_2, -e_5, -e_4) \). Thus \( C(e_3) = \{ (\hat{a}, 0, -\hat{a}, \hat{a}, 0) \mid \hat{a} \in A \} \) and \( C(e_5) = \{ (0, \hat{a}, 0, -\hat{a}, -\hat{a}) \mid \hat{a} \in A \} \), as shown in Figure 14.

We can now make precise our earlier statement that \( \beta_1(G) \) is the number of independent cycles in \( G \). Using a spanning tree \( T \) of \( G \), we have identified \( \beta_1(G) \) fundamental cycles \( e_e, e \in E_T, \) such that \( Z_1 \) is generated by the cycle codes \( C(e_e) \). Thus for any cycle \( e \), the cycle code \( C(e) \subseteq Z_1 \) is generated by the cycle codes \( C(e_e) \). For example, our example graph \( G \) has one other cycle, whose cycle vectors \( \{ (\hat{a}, -\hat{a}, -\hat{a}, \hat{a}) \mid \hat{a} \in A \} \) are sums of the corresponding vectors of \( C(e_3) \) and \( C(e_5) \).

**K. Dual nonredundant input/output realizations**

As we have seen, the dual of an I/O realization is an I/O realization of the dual code with the complementary information set. For example, the dual to the I/O realization of \( Z^0 \) in Figure 7 is the I/O realization of the dual zero-sum code \( B_0 = (Z^0)^\perp \) shown in Figure 15 which is also controllable and observable.

Alternatively, to obtain this I/O realization, we could have started with the realization of \( B_0 \) in Figure 12. The unobservable behavior \( B^u \) of this realization, shown in Figure 16 has dimension \( \dim B^u = \beta_1(G) = 2 \); i.e., there are \( \beta_1(G) = 2 \) internal degrees of freedom, corresponding to cycles in \( Z_1 = \ker \partial \) that do not affect the vertex vector \( \hat{x} \) (sometimes called "local symmetries"). Thus we could have obtained Figure 15 by using two local reductions to break these local symmetries.

It is shown in [7], [8] that if a realization has \( \dim B^u \) "degrees of unobservability," then its dual realization has \( \dim B^u \) "degrees of uncontrollability." Thus the fact that the realization of \( B_0 \) in Figure 12 has \( \beta_1(G) = 2 \) degrees of unobservability alternatively follows from the fact that the realization of \( Z^0 \) in Figure 3 has \( \beta_1(G) = 2 \) degrees of uncontrollability. (We have found that it is usually easiest to determine the controllability properties of a realization from the observability properties of its dual.)
Finally, the dual to the I/O realization of $B^1$ in Figure 9 is a controllable and observable I/O realization of the dual code $Z_1 = \ker \partial$, which is a systematic $(|E|, \beta_1(G))$ group code over $\mathcal{A}$, using the complementary information set $E_T$, as shown in Figure 17.

Alternatively, we could have started with the realization of $Z_1$ in Figure 13. As we have just discussed, since this realization is the dual to the unobservable realization of $B^1$ in Figure 4 with $\dim \mathfrak{S}_n = 1$, it must have one “degree of uncontrollability.” The reader may verify that the realization of Figure 13 continues to function correctly if any single zero-sum constraint (18) is removed. For example, Figure 17 is Figure 13 with the top left constraint removed.

We summarize the observability and controllability properties of the realizations of $Z^0, B^1, Z_1$ and $B_0$ in Figures 8, 4, 13 and 12 in Table I (where “deg. unc.” denotes “degrees of uncontrollability”).

| space            | dimension | realization | observable? | $\mathfrak{S}_n$ | controllable? | deg. unc. |
|------------------|-----------|-------------|-------------|-----------------|---------------|----------|
| $Z^0 = \ker d$   |           | $\beta_0(G)$ | Fig. 3       | yes             | $\{0\}$      | no       | $\beta_1(G)$ |
| $B^1 = \im d$   |           | $|E| - \beta_1(G)$ | Fig. 5       | no              | $\mathfrak{S}_4$ | yes     | 0        |
| $Z_1 = \ker \partial$ |         | $\beta_1(G)$ | Fig. 13      | yes             | $\{0\}$      | no       | $\beta_0(G)$ |
| $B_0 = \im \partial$ |         | $|V| - \beta_0(G)$ | Fig. 12      | no              | $\mathfrak{S}_5$ | yes     | 0        |

**TABLE I**

Observability and controllability properties of realizations of $Z^0, B^1, Z_1$ and $B_0$.

L. Homology spaces

In algebraic topology, homology spaces are quotient spaces (resp. quotient groups) whose dimensions (resp. “free ranks”) are topological invariants of complexes. We will not delve deeply into this topic, but for future reference we give the homology spaces for the elementary one-dimensional complexes that we have studied in this section, and we exhibit nice dual realizations for them.

For a graph $G = (V, E)$, the zeroth cohomology space is defined as $H^0 = Z^0$. We have seen that if $G$ is connected, then $\dim Z^0 = 1$, so $\dim H^0 = 1$ for all connected graphs.

The zeroth homology space of $G$ is defined as the quotient space $H_0 = C_0 / B_0$. Since $B_0 = (Z^0)^{\perp} = (H^0)^{\perp}$, $H_0$ is the dual space to $H^0$, and thus has the same dimension, namely $\dim H_0 = 1$, again for all connected graphs.

More generally, it is easy to see that if $G$ is not connected, but rather consists of $\beta_0(G) > 1$ connected components, then $H^0 = Z^0$ consists of the direct sum of $\beta_0(G)$ independent repetition codes, one defined on each of the $\beta_0(G)$ components of $G$ (see the discussion in connection with Figure 3). Thus in general $\dim H^0 = \beta_0(G)$. Dually, $B_0 = (Z^0)^{\perp}$ consists of the direct sum of $\beta_0(G)$ independent zero-sum codes, one defined on each of the $\beta_0(G)$ components of $G$, so in general $\dim B_0 = |V| - \beta_0(G)$, and $\dim H_0 = \beta_0(G)$.

For a graph $G$, the first homology space of $G$ is defined as $H_1 = Z_1$. We have seen that if $G$ is connected, then $\dim Z_1 = \beta_1(G)$, so $\dim H_1 = \beta_1(G)$ for all connected graphs. Also, the first cohomology space of $G$ is defined as $H^1 = C^1 / B^1$. Since $B^1 = (Z_1)^{\perp} = (H_1)^{\perp}$, $H^1$ is the dual space to $H_1$, and thus has the same dimension, namely $\dim H^1 = \beta_1(G)$, for all connected graphs.

More generally, it is easy to see that if $G$ consists of $\beta_0(G) > 1$ connected components, then $H_1 = Z_1$ consists of the direct sum of $\beta_0(G)$ independent codes of dimensions $\beta_1(G_i)$, one defined on each of the $\beta_0(G)$ components $G_i = (V_i, E_i)$ of $G$. Thus in general $\dim H_1 = \sum_i \beta_1(G_i) = \sum_i (|E_i| - |V_i| + 1) = |E| - |V| + \beta_0(G) = \beta_1(G)$. Dually, $B^1 = (Z_1)^{\perp}$ consists of the direct sum of $\beta_0(G)$ independent codes of dimensions $|E_i| - \beta_1(G_i)$, one defined on each component $G_i$ of $G$, so in general $\dim B^1 = |E| - \beta_1(G)$, and $\dim H^1 = \beta_1(G)$.

The properties of these homology spaces are elegantly captured by the unobservable behaviors $\mathfrak{S}_n$ of Figures 8 and 13 respectively, which we recapitulate in Figure 18. We note that these realizations are each others’ duals.

It seems to us that the simple, elegant and dual realizations of Figure 18 represent the Platonic essences of the zeroth cohomology space $H^0$ and the first homology space $H_1$ of a graph $G$. 

Fig. 17. I/O realization of $Z_1 = \ker \partial$, with information set $\{\hat{y}_3, \hat{y}_5\}$. 

Fig. 18.
III. ISING-TYPE MODELS

We now show how to extend these graphical models of algebraic topology spaces to realize partition functions of Ising and related models of statistical physics.

A. Ising models and Ising-type models

In statistical physics, an Ising model is defined as follows:

- Particles are associated with the vertex set \( V \) of a connected graph \( G = (V, E) \).
- Each particle may be in one of two states (often called “spins”). The state space is \( A = \mathbb{Z}_2 \), and the state of the particle at the \( v \)th vertex is denoted by \( x_v \in A \). The configuration space is thus \( A^V \).
- Two particles \( v, v' \in V \) interact directly only if they are joined by an edge \( e \in E \); i.e., if \( (v, v') = (h(e), t(e)) \), or vice versa, where \( (h(e), t(e)) \) are the two vertices at the ends of edge \( e \). Their interaction energy \( E_e(x_v, x_{v'}) \) is \(-J_e\) if \( x_v = x_{v'} \) and \( J_e \) if \( x_v \neq x_{v'} \), where \( J_e \) is the interaction strength for edge \( e \). In other words, \( E_e(x_v, x_{v'}) = -(-1)^{x_v-x_{v'}} J_e = -(-1)^{x_{h(e)}-x_{t(e)}} J_e \).
- The probability of a configuration \( x \in A^V \) is then given by the Boltzmann distribution

\[
p(x) = \frac{1}{Z} \prod_{e \in E} \exp \left( \beta J_e (-1)^{x_{h(e)}-x_{t(e)}} \right),
\]

where \( \beta \) is the inverse temperature, and the partition function \( Z \) is

\[
Z = \sum_{x \in A^V} \prod_{e \in E} \exp \left( \beta J_e (-1)^{x_{h(e)}-x_{t(e)}} \right).
\]

Evidently the probability of a configuration \( x \in A^V \) depends only on the corresponding edge configuration \( y(x) = d(x) = Mx \in A^E \). Thus

\[
p(x) = \frac{1}{Z} \prod_{e \in E} \exp \left( \beta J_e (-1)^{y_e(x)} \right),
\]

where \( y_e(x) = M_e x = x_{h(e)} - x_{t(e)} \). Moreover,

\[
Z = \sum_{x \in A^V} \prod_{e \in E} e^{\beta J_e (-1)^{y_e(x)}} = |A| \sum_{y \in B^1} \prod_{e \in E} e^{\beta J_e (-1)^{y_e}},
\]

where \( B^1 = \text{im} \ d \), and \(|A| = |\ker d| \) is the number of configurations in \( A^V \) that map to each \( y \in B^1 \).

More generally, we define an Ising-type model as a statistical model based on a graph \( G = (V, E) \) in which

- Particles are associated with the vertex set \( V \) and have states \( x_v \in A \), where the state space \( A \) may be any finite abelian group;
- The probability of a configuration \( x \in A^V \) depends only on the corresponding edge configuration \( y(x) = d(x) = Mx \in A^E \), and is given by

\[
p(x) = \frac{1}{Z} \prod_{e \in E} f_e (y_e(x))
\]

for some set of edge-weighting functions \( \{f_e(y_e), e \in E\} \), where

\[
Z = \sum_{x \in A^V} \prod_{e \in E} f_e (y_e(x)) = |A| \sum_{y \in B^1} \prod_{e \in E} f_e (y_e).
\]

For example, a Potts model is an Ising-type model in which \( A = \mathbb{Z}_q \) for \( q > 2 \), and \( f_e(0) = \exp(\beta J_e) \), while \( f_e(y_e) = 1 \) for \( y_e \neq 0 \). Much broader generalizations are evidently possible, but we do not know to what extent they may have been studied in statistical physics.

The usual object of study is the partition function \( Z \) as a function of inverse temperature \( \beta \). In an Ising model, as \( \beta \to 0 \) (i.e., in the high-temperature limit), the weight \( w_e(x) = \exp(\beta J_e (-1)^{y_e(x)}) \) tends to 1, independent of \( y_e(x) \), so \( Z \to |A|^{|V|} \) and \( p(x) \to |A|^{-|V|} \) for all \( x \in A^V \); i.e., in physical terms, the interaction between neighboring vertices disappears. On the other hand, as \( \beta \to \infty \) (i.e., in the low-temperature limit), and if \( J_e > 0 \) (the ferromagnetic case), then the weight \( w_e(x) \) is large if \( y_e = 0 \) \( (x_{h(e)} = x_{t(e)}) \) and small if \( y_e \neq 0 \) \( (x_{h(e)} \neq x_{t(e)}) \), which tends to force these two values to agree. If \( \beta \to \infty \) and if \( J_e < 0 \) (the antiferromagnetic case), then the weight \( w_e(x) \) becomes small if \( y_e = 0 \),
\((x_{h(e)} = x_{t(e)})\) and large if \(y_e \neq 0\) \((x_{h(e)} \neq x_{t(e)})\), which tends to force these two values to disagree.

B. From normal realizations to edge-weighted NFGs

Following [19, 2], we will now show how the partition function of an Ising-type model may be represented by a normal factor graph (NFG), and in particular by an edge-weighted NFG (EWNFG).

Normal factor graphs [14, 1, 9] build on the concepts of normal realizations [5] and factor graphs [13]. NFGs are used to model functions that can be represented as real- or complex-valued sums of products, such as partition functions. (For more on partition functions of NFGs, see [9].)

The semantics of NFGs are similar to those of normal realizations. Again, a normal factor graph is a graphical model based on a graph \(G = (V, E, H)\), in which the edges \(e \in E\) represent internal variables \(s_e \in A_e\), and the half-edges \(h \in H\) represent external variables \(a_h \in A_h\). However, the vertices \(v \in V\) now represent complex-valued functions \(f_v(s_v, a_v)\) of the values \((s_v, a_v)\) of all variables that correspond to the edges and half-edges that are incident on vertex \(v\).

We will assume that all variable alphabets are finite abelian groups. The internal and external configuration spaces are then \(A_E = \prod_{e \in V} A_e\) and \(A_H = \prod_{h \in H} A_h\), respectively. The NFG then represents the sum of products

\[
Z(a) = \sum_{s \in A_E} \prod_{e \in V} f_v(s_v, a_v),
\]

called the partition function (or “exterior function” [1], [2], or “partition sum” [2]) of the NFG.

NFGs generalize normal realizations in the following sense. Given a normal realization based on \(G = (V, E, H)\), if each vertex constraint code \(C_v\) is replaced by its indicator function \(\delta_{C_v}\) (i.e., \(\delta_{C_v}(s_v, a_v) = 1\) if \((s_v, a_v) \in C_v\), else \(\delta_{C_v}(s_v, a_v) = 0\)), then

\[
Z(a) = \sum_{s \in A_E} \delta_{\mathcal{B}}(s, a),
\]

where \(\delta_{\mathcal{B}}(s, a)\) is the indicator function of the behavior \(\mathcal{B} = \{(s, a) \mid \text{all constraints satisfied}\}\) of the normal realization. Thus \(Z(a) > 0\) if and only if \(a\) is in the external behavior \(\mathcal{C}\), which is the projection \(\mathcal{C} = \mathcal{B}|_{A_H}\) of \(\mathcal{B}\) onto the external configuration space \(A_H\).

If all constraint codes \(C_v\) are group codes, then the behavior and the external behavior are group codes \(\mathcal{B} \subseteq A_E \times A_H\) and \(\mathcal{C} \subseteq A_H\), respectively. Moreover, if \(\mathcal{B}\) is finite, then by the group property the number of elements of \(\mathcal{B}\) that map to each element of \(\mathcal{C}\) is the same, namely \(|\mathcal{B}^u| = |\mathcal{B}|/|\mathcal{C}|\), where

\[
\mathcal{B}^u = \mathcal{B}|_{A_E} = \{(s, 0) \in \mathcal{B} \mid s \in A_E\}
\]
is the unobservable behavior of the realization. Therefore \(Z(a) = |\mathcal{B}^u| \delta_{\mathcal{C}}(a)\).

In summary:

**Theorem 6 (normal realization as an NFG).** If all alphabets are finite abelian groups, then a normal realization with external behavior \(\mathcal{C}\) and unobservable behavior \(\mathcal{B}^u\) may be interpreted as a normal factor graph whose partition function is \(Z(a) = |\mathcal{B}^u| \delta_{\mathcal{C}}(a)\).

Next, we extend this definition as follows. An edge-weighted NFG consists of a normal realization of a group code \(\mathcal{C}\) as above, in which all internal functions \(\{f_v, v \in V\}\) are indicator functions \(\delta_{C_v}\) of group codes \(C_v\), plus a set \(\{f_h, h \in H\}\) of edge-weighting functions \(f_h\) attached to each external half-edge \(h \in H\) of the normal realization. The resulting NFG has no external variables, and its partition function is evidently the complex number

\[
Z = \sum_{a \in A_H} Z(a)f(a) = |\mathcal{B}^u| \sum_{a \in \mathcal{C}} f(a),
\]

where \(\mathcal{B}^u\) is the unobservable behavior of the normal realization, and \(f(a) = \prod_{h \in H} f_h(a_h)\).

As [2], [19] have observed, the partition functions of Ising-type models are naturally represented by such edge-weighted NFGs. In particular, the partition function of an Ising-type model may be represented as an EWNFG based on a normal realization of the coboundary space \(B^1 = \text{im } d\) of a graph \(G = (V, E)\) over the finite group alphabet \(A\), and an appropriate set of edge-weighting functions \(\{f_v(y_v), e \in E\}\). The partition function of such a model is thus

\[
Z = |\mathcal{B}^u| \sum_{y \in B^1} f(y) = |\mathcal{B}^u| \sum_{y \in B^1} \prod_{e \in E} f_v(y_e),
\]

where \(|\mathcal{B}^u|\) is the size of the unobservable behavior of the normal realization.

For example, Figure [19] shows the EWNFGs derived from the normal realizations of \(B^1 = \text{im } d\) of Figures [4] and [9] respectively (recall that Figure [9] is an I/O realization using the information set \(y_T = \{y_1, y_2, y_4, y_6\}\)).

We recall that for the normal realization underlying Figure [19(a)], we have \(|\mathcal{B}^u| = |A|\), whereas for that of Figure [19(b)], \(|\mathcal{B}^u| = 1\). Thus the partition function of the EWNFG of Figure [19(a)] is \(|A| \sum_{y \in B^1} f(y) = Z\), whereas the partition function of Figure [19(b)] is \(|A| \sum_{y \in B^1} f(y) = Z/|A|\). The two EWNFGs are thus equivalent up to a scale factor of \(|A|\).
Notice that such EWNGFs alternatively model the communications scenario in which the possible transmitted sequences are codewords $y$ from the linear code $B^1$, and the relative (unscaled) likelihood of each possible symbol $y_e \in A$ is $f_e(y_e) = \delta_{y_e 1}$. The dimension of $B^1$ is $\dim B^1 = |E| - \beta_1(G) = |V| - 1$, the number of edges in a spanning tree $T$ of $G$. Thus $Z$ may be computed as in (7) by choosing a spanning tree $T \subseteq G$, letting $y_T$ run freely through $A^{E_T}$, extending each $y_T$ to the corresponding $y \in B^1$ via a linear transformation, computing $f_e(y) = \prod_{e \in E} f_e(y_e)$, summing these contributions, and finally multiplying by $|A|$. If we use an I/O realization of $B_1$ as in Figure 9 then the extension of each $y_T$ to the corresponding $y \in B^1$ occurs "automatically" by propagation through the I/O realization.

Molkaraie [15], [16] has proposed an importance-sampling algorithm to estimate the partition function $Z$, using the information set $y_T$. The idea is to choose a series of samples $y_T \in A^{E_T}$ according to an auxiliary probability distribution $p(y_T)$, extend each such $y_T$ to the corresponding $y \in B^1$, and then compute $f_e(y)$ for each sample. For the auxiliary probability distribution, he proposes $p(y_T) = f_e(y_T)/Z_T$, where

$$f_e(y_T) = \prod_{e \in E_T} f_e(y_e); \quad Z_T = \sum_{y_T \in A^{E_T}} f_e(y_T).$$

Again, an I/O realization of $B_1$ could be used to compute this extension of $y_T$ to $y$ "automatically."

\footnote{Indeed, it was the recognition of this analogy by Sourlas \cite{Sourlas} in 1989 that led to the first connections between coding theory and the Ising models of statistical physics.}

\footnote{Hakimi and Bredeson \cite{Hakimi} conducted an early investigation into whether graphical binary linear codes such as $B^1$ could be useful for data communications.}

C. Dual realizations

Molkaraie and Loeliger \cite{Molkaraie} observed that it is sometimes easier to compute or estimate the partition function $Z$ of an EWNGF by using the dual EWNGF. We now revisit this observation in the context of this paper.

The normal factor graph duality theorem (NFGDT) \cite{DualityTheorem}. \cite{DualityTheorem}. \cite{DualityTheorem} says that the dual of an NFG whose partition function is $Z$ is an NFG whose partition function is the Fourier transform $Z$ of $Z$, up to a certain scale factor. In the Appendix, we give the simplest proof we know of this very powerful and general result, using a simple Edge Replacement Lemma.

The dual of an EWNGF based on a normal realization of some group or linear code $C$ and edge-weighting functions $\{f_e, e \in E\}$ is evidently an EWNGF based on the dual normal realization with the Fourier-transformed edge weighting functions $\{f_e, e \in E\}$.

Since the partition function of an EWNGF is a complex number $Z$, its Fourier transform is simply $Z = Z^*$; i.e., the Fourier transform of a number (namely, a complex-valued function of no variables) is that number.

In the Appendix, we consider interpreting a dual normal realization as an NFG. Considering all relevant scale factors, we show that

**Theorem 7** (Dual normal realization as an NFG). The dual of a finite abelian group normal realization with behavior $B$ and external behavior $C$ may be interpreted as an NFG with partition function $|B||A_E||C_V|^{-1} \delta_{C^\perp}(\hat{a})$, where $|A_E| = \prod_v |A_v|$ and $C_V = \prod_v |C_v|$. □

As discussed in the Appendix, Theorems 6 and 7 imply that:

(a) The external behavior of the dual normal realization is $C^\perp$. Thus, when all alphabets are finite abelian groups, we obtain the normal realization duality theorem (NRDT) as a corollary.
Figure 19(a) is 

\[ G \]

Figure 20(a) is 

\[ |A| \]

function, by letting 

\[ |A| \]

the scale factor must be 

\[ (This \ result \ was \ derived \ previously \ by \ Molkaraie \ [17].) \]

\[ Z_f \] of the dual edge-weighted NFG as in Figure 20(a) is 

\[ Fig. \ 20. \ Dual \ EWNFGs \ for \ scaled \ Fourier \ transform \ of \ partition \ function \ of \ Ising-type \ model, \ based \ on \ normal \ realizations \ of \ Z_k = \ker \delta \] in 

(a) Figure 13(b) (b) Figure 17

(b) The size of the unobservable behavior of the dual normal realization is \[ |2B| = |2B| |A| \bar{E} | |C_V|^{-1}. \] Thus the dual NR is observable and the NR is controllable if and only if \[ |2B| |A| \bar{E} | |C_V|^{-1} = 1. \] Interestingly, this yields the controllability test of [8] for the finite abelian group case.

For Ising-type models, dualization of a normal realization of \( B^1 = \text{im} \, d \) as in Figure 4 or 9 gives a normal realization of \( Z_1 = \ker \delta = (B^1)^{-1} \) as in Figure 13 or Figure 17. Correspondingly, dualization of the corresponding edge-weighted NFGs in Figure 19 yields the dual EWNFGs shown in Figure 20.

We recall that Figure 17 is an I/O realization using the information set \( y^T = \{y_3, y_5\} \). The dual partition function may be computed or sampled like the primal function, by letting \( y^T \) range freely on the complement \( E_T \) of the edge set \( E_T \) of any spanning tree \( T \) of \( G \). Thus if we choose the same tree \( T \) for the primal and dual realizations, then the two information sets are complements of each other.

In the Appendix, we compute the scale factor for dual Ising-type models based on a graph \( G = (V, E) \). If the partition function of the primal edge-weighted NFG as in Figure 19(a) is \( Z \), then \( \bar{Z} = Z \), and the partition function of the dual edge-weighted NFG as in Figure 20(a) is

\[ |A|^{|E|}{|V|} \bar{Z} = |A|^{\delta \beta (G) - 1} \bar{Z}. \]

(This result was derived previously by Molkaraie [17].)

Since the scale factor is independent of edge weights, one way of computing it is to calculate partition functions for the particular case where \( f_c(y_c) = 1 \) for all \( y_c \in A \) and for all \( e \in E \); then all configurations in Figure 19(a) have weight 1, so \( Z = |A|^{\bar{V}} \). Now \( f_c(y_c) = |A|^{\delta(y_c)} \), so only the all-zero configuration contributes to \( Z \), with weight \( |A| \) for each edge \( e \in E \), so the partition function of Figure 20(a) is \( |A|^{|E|} \). Since \( |A|^{\bar{V}} = |A|^{|E|}{|V|} \), times \( Z = |A|^{\bar{V}} \), the scale factor must be \( |A|^{(|E| - |V|)} \).

Dualization turns hard constraints into soft constraints, and vice versa. For example, an interaction weight function \( f_e(y_e) \) represents a strict (equality) constraint if \( f_e(y_e) \propto \delta_{\{0\}}(y_e) \), for then only configurations \( y \) with \( y_e = 0 \) contribute to the partition function. On the other hand, \( f_e(y_e) \) represents no constraint if \( f_e(y_e) \propto 1 \) for all \( y_e \), for then it makes the same contribution for every configuration \( y \). Since the dual code to \{0\} is the universe code \( A \), the Fourier transform of a strict-constraint function is a no-constraint function, and vice versa. Similarly, in an Ising-type model, the Fourier transform of a low-temperature interaction weight function \( f_e(y_e) \) is a high-temperature weight function, and vice versa. For this reason, an expression for \( \bar{Z} \) is sometimes called a high-temperature expansion of \( Z \). High-temperature constraints are softer than low-temperature constraints, and have fewer long-range correlations, so convergence of Monte Carlo estimates is faster and less random [19].

D. Example: Single-cycle graph

Following [19], we now give an example of dual realizations of a partition function \( Z \) and its Fourier transform \( \bar{Z} \) on a single-cycle graph. In statistical physics, a single-cycle graph arises in a one-dimensional (1D) Ising-type model with periodic boundary conditions.

A single-cycle graph \( G \) of length \( n \) has \( n \) vertices \( v_i, i \in \mathbb{Z}_n \), and \( n \) edges \( e_i, i \in \mathbb{Z}_n \), such that edge \( e_i \) connects vertices \( v_i \) and \( v_{i+1} \) (with index arithmetic in \( \mathbb{Z}_n \), so \( v_{n-1} \) connects \( v_{n-1} \) and \( v_0 \)).

Figure 21 shows an NFG with partition function \( Z \) for a 1D Ising-type model of length \( n \) over \( A = \mathbb{Z}_q \) with periodic boundary conditions. The realization is based on an image realization of \( B^1 = \text{im} \, d \), which is the \((n, n - 1)\) zero-sum code over \( \mathbb{Z}_q \).
Since $Z^0 = \ker d$ is the $(n, 1)$ repetition code over $A$, every codeword of $B^1 = \text{im } d$ is the image of $|A|$ different vertex vectors $x + a 1, a \in A$. Thus, if we like, we may fix any vertex variable to zero, say $x_0 = 0$, without affecting the partition function, up to a scale factor of $|A|$. We may also replace each repetition constraint of degree 2 plus its neighboring sign inverter by an inverting edge, since $\circ \underline{\circ} = \circ \underline{\circ}$. This results in the observable I/O realization of Figure 22 whose underlying graph $T$, a spanning tree of $G$, is a cycle-free chain graph. This shows that a 1D Ising-type model defined on $T$ with fixed boundary conditions (i.e., $x_0 = x_n = 0$) has the same partition function (up to a scale factor of $|A|$) as the same model with periodic boundary conditions (i.e., $x_0 = x_n$).

Figure 23 shows the dual NFG, whose partition function is (up to scale) the Fourier transform $Z$ of the partition function $Z$ of Figure 21. The underlying normal realization is a dual I/O realization of $Z_1 = \ker \partial$, the $(n, 1)$ repetition code over $\hat{A}$. Thus $\hat{Z} \propto \sum_{\hat{y} \in \hat{A}} \prod_{i} \hat{f}_i(\hat{y})$. Computing the partition function of the dual NFG is clearly much easier than computing $Z$.

E. Ising-type models with an external field

An Ising-type model may be generalized to include an external field as follows [15]. In addition to the interaction energies $E_v(x_v, x_{v'})$, there is an external field energy $E_e(x_v)$ for each particle $v \in V$. The external field weight function is defined as $g_e(x_v) = \exp(-\beta E_e(x_v))$, where $\beta > 0$ is again the inverse temperature. The probability of a configuration $x \in \mathcal{A}^V$ is then given by the Boltzmann distribution

$$p(x) = \frac{1}{Z} \left[ \prod_{v \in E} f_e(x_{h(e)} - x_{t(e)}) \right] \left[ \prod_{v \in V} g_e(x_v) \right],$$

where the partition function is now defined as

$$Z = \sum_{x \in \mathcal{A}^V} \left[ \prod_{v \in E} f_e(x_{h(e)} - x_{t(e)}) \right] \left[ \prod_{v \in V} g_e(x_v) \right].$$

This partition function may be realized by starting with a realization of $\hat{Z}$ as in Figure 19(a), and attaching to each vertex $v \in V$ a function vertex representing the external field weight function $g_e(x_v)$, as shown in Figure 24. The resulting EWNFG has no external variables, and evidently realizes the partition function of the model including an external field.
For the special case of the Ising model ($\mathcal{A} = \mathbb{Z}_2$) with a constant external field, there are well-known explicit formulas for this high-temperature expansion [20], [12].

The above expression generalizes these high-temperature expansions to general finite abelian group alphabets $\mathcal{A}$ and non-constant external fields $H_v$.

However, since $|E| = |V| - 1 + \beta_1(\mathcal{G})$, the dual I/O behavior will be higher-dimensional than the primal I/O behavior if $\beta_1(\mathcal{G}) > 1$; i.e., if $\mathcal{G}$ has more than one cycle. Thus with an external field it will generally be more complex to compute the partition function of the dual EWNFG than that of the primal EWNFG. We will now explore methods of reducing this increased complexity.

### F. Alternative and hybrid I/O realizations

We see that whereas in the primal domain the presence of an external field adds only one dimension to the realization of $Z$ (the difference between the dimensions of $B_1$ and $W_01$), in the dual domain it adds $|V| - 1$ dimensions (the difference between the dimensions of $Z_1$ and $W_{10}$). We now suggest alternative realizations of the partition function $Z$ and its Fourier transform $\mathcal{Z}$ for Ising-type models with an external field, with the objective of simplifying their calculation.

As discussed above, $Z$ is the partition function of an EWNFG based on a normal realization of the I/O behavior $W_01 = \{(x, d(x)) | x \in \mathcal{A}^V\}$ of the coboundary operator $d$, as illustrated in Figure 2 where $W_01$ is a linear ($|E| + |V|, |V|$) code over $\mathcal{A}$.

We have seen that the vertex vector $x \in \mathcal{A}^V$ is an obvious information set for $W_01$. However, we will now show that for any $v \in V$ and any edge set $E_T$ such that $T$ is a spanning tree of $\mathcal{G}$, $(x_v, y_T) \in \mathcal{A}^{(v)} \times \mathcal{A}^{E_T}$ is an information set for $W_01$. Moreover, we will give an I/O realization of $W_01$ based on this information set.

We recall that the projection of $W_{01}$ onto the edge configuration space $\mathcal{A}^E$ is $B_1 = \ker d$, which is an $(|E|, |E| - \beta_1(\mathcal{G})) = (|E|, |V| - 1)$ systematic group code with information set $E_T$ for any spanning tree $T \subseteq \mathcal{G}$. Hence for any $(x, y) \in W_{01}$, the edge vector $y$ may be realized by taking $y_T$ as the input to an I/O realization of $B_1$, such as the realization illustrated in Figure 25 the output is then the unique $y_T$ such that $y = (y_T, y_T) \in B_1$.

Now we may extend an I/O realization of $B_1$ such as Figure 25 to obtain a vertex vector $x(y_T)$ as another output, thus obtaining a pair $(x(y_T), y) \in W_01$. Note that we have shown that any single vertex variable $x_v$ in Figure 25 can be fixed to 0 (i.e., $x_v(y_T) = 0$, where $v \in V$ may be chosen arbitrarily), so $(x_v(y_T), y) \in (W_{01})_{v}$. Moreover, the set of all pairs $(x, y) \in W_01$ that have edge vector $y$ is the coset $(x + Z^0, y)$ of $Z^0 \times \{0\}$ that contains $(x, y)$, where $Z^0 = \ker d$ is the $(|E|, 1)$ repetition code over $\mathcal{A}$.

It follows that $W_{01}$ may be generated by the normal realization shown in Figure 26. On the left, a single input vertex variable $x_v \in \mathcal{A}^{(v)}$ generates the unique vertex vector $x_v(x_v) = (x_v, \ldots, x_v) \in \mathcal{A}^{V \setminus \{v\}}$ such that $(x_v, x_v(x_v)) \in Z_0$. On the right, an input edge vector $y_T \in \mathcal{A}^{E_T}$ generates the unique edge vector $y_T(y_T) \in \mathcal{A}^{E \setminus E_T}$ such that $(y_T, y_T(y_T)) \in B_1$, and an associated vertex vector $x_v(y_T) \in \mathcal{A}^{V \setminus \{v\}}$ such that $(x_v(y_T), y_T(y_T)) \in W_01$, where $x_v(y_T) = (0, x_v(y_T)) \in \mathcal{A}^V$. The final output pair is $(x_v(x_v, y_T), y_T(y_T))$, where $x_v(x_v, y_T) = x_v(x_v) + x_v(y_T)$, which together with the input pair $(x_v, y_T)$ give the unique pair $(x, y) \in W_01$ that is consistent with the inputs $(x_v, y_T) \in \mathcal{A}^{(v)} \times \mathcal{A}^{E_T}$.

![Diagram](image-url)
which may be realized by extending the I/O realization of $Z_1 = \ker \partial$ in Figure 17, which uses $\hat{y}_T$ as an information set, in a dual manner to the extension of Figure 29 discussed above, which involves adding $x_\hat{v}$ as a second information set. Finally, the remainder of the realization is dualized by replacing the zero-sum constraint by an equality constraint, and changing the directions of the arrows. Thus we obtain the realization of Figure 27.

These normal realizations may be used to realize scaled versions of $Z$ and $\hat{Z}$ by attaching edge weights. For example, the dual EWNFG of Figure 28 realizes a scaled version of $\hat{Z}$ by attaching edge weights $\hat{g}(x)$ and $\hat{f}(\hat{y})$ to components of $\hat{x}$ and $\hat{y}$, respectively.

A hybrid alternative that may be attractive because of the simplicity of the repetition code $Z^0$ is shown in Figure 29. Here, using the Edge Replacement Lemma (see Appendix), the left (vertex) side is realized in the primal domain, and the right (edge) side is realized in the dual domain, with a connection via a Fourier transform function (plus sign inverter) between the $(|V| - 1)$-dimensional primal and dual vertex vectors $x_\hat{v}$ and $\hat{x}_\hat{v}$. (In this figure, the scale factors of the Edge Replacement Lemma have been omitted.)

Fig. 27. I/O realization of the dual I/O behavior $W_{10} = \{-\partial(\hat{y}), \hat{y}\}$ with information set $(x_\hat{v}, \hat{y}_T)$.

Fig. 28. Dual EWNFG for Ising-type model.

Fig. 29. Hybrid NFG for Ising-type model.

In this hybrid realization, the left side is 1-dimensional, the right side is $\beta_1(\mathcal{G})$-dimensional, and the two are linked by $|V| - 1$ edge variables, which require a $(|V| - 1)$-dimensional Fourier transform over $\mathcal{A}$. For many alphabets $\mathcal{A}$—e.g., $\mathbb{Z}_2$—“fast Fourier transform” algorithms may be used. Overall, such a hybrid realization may therefore yield a significant reduction in complexity.

IV. TWO-DIMENSIONAL ALGEBRAIC TOPOLOGY

We now consider graphs that have well-defined faces. We will primarily consider connected planar graphs with no self-loops and no dangling edges, which is the simplest case.

A. Two-dimensional complexes

In elementary algebraic topology [3], a graph $\mathcal{G} = (V, E)$ plus a set $F$ of faces is called a two-dimensional complex $\mathcal{G}^+ = (V, E, F)$. A face $f \in F$ of $\mathcal{G}^+$ is called a two-dimensional object.

Faces are defined according to the topological space on which $\mathcal{G}$ is imagined to be drawn. For instance, a planar graph is one that can be drawn on a plane without any edges crossing. The interior faces of $\mathcal{G}$ are then the areas of the plane that are bounded by the edges of $\mathcal{G}$. We will assume that $\mathcal{G}$ has no “dangling” degree-1 vertices, then every edge of $\mathcal{G}$ bounds either two interior faces of $\mathcal{G}$, if it is an interior edge, or one interior face of $\mathcal{G}$, if it is an exterior edge.

It is easy to see that in general the number of interior faces of a planar graph is $|F| = \beta_1(\mathcal{G})$. For example, as a planar graph, our example graph $\mathcal{G}$ in Figure 11 has two interior faces, one interior edge, and five exterior edges.

Vontobel [23] has pointed out that the “Hamiltonian NFGs” in [22] Fig. 4) are also hybrid realizations, in that they include both primal and dual parts, with Legendre transforms between them.

10The “no dangling vertices” assumption is the dual to the “no self-loops” assumption.
Alternatively, a planar graph $\mathcal{G}$ may be considered to be drawn on a sphere; then its “exterior face” is also taken to be a face, and every edge of $\mathcal{G}$ bounds precisely two faces of $\mathcal{G}$. In this case, $|F| = \beta_1(\mathcal{G}) + 1 = |E| - |V| + 2$. For example, as a planar graph on a sphere, our example graph $\mathcal{G}$ has $|V| = 5, |E| = 6$, and $|F| = 3$.

In algebraic topology, the vector space $C^2 = A^F$ of column vectors over $A$ indexed by $A$ is the space of 2-cochains of $\mathcal{G}$. Again, we will call such vectors face vectors, and write them as column vectors $z$ in matrix expressions. The dual vector space $C_2 = \hat{A}^F$ is called the space of dual face vectors, and use the standard inner product. In general, the boundary space $B$ of a face $f$ is a set $E(f) \subseteq E$ of directed edges. Each face $f \in F$ is assumed to have an orientation, perhaps arbitrary; then each edge $e \in E(f)$ is given a sign $\alpha_{fe} = 1$ if the face $f$ and edge orientations are aligned, or $\alpha_{fe} = -1$ otherwise. If $e \notin E(f)$, then $\alpha_{fe} = 0$. The $\{0, \pm 1\}$-valued matrix

$$M_2 = \{\alpha_{fe} \mid f \in F, e \in E\}$$

will be called the second incidence matrix of $\mathcal{G}$. (Hereafter, our original incidence matrix $M$ will be denoted $M_1$ and called the first incidence matrix of $\mathcal{G}$.) Since an edge of $\mathcal{G}^+$ is incident on two faces if it is an interior edge or on one face if it is an exterior edge, each of the $|E|$ columns of $M_2$ has one or two nonzero values. The number of nonzero values in the $f$th row is the degree $\delta_f$ of the face $f$, i.e., the number of edges bounding $f$.

The second boundary operator is defined as the homomorphism $\partial_2 : \hat{A}^F \to \hat{A}^E, z \mapsto zM_2$; i.e., the map whose matrix is the second incidence matrix $M_2$ of $\mathcal{G}$. Its kernel $Z_2 = \ker \partial_2$ is called the second zero-boundary space of $\mathcal{G}^+$, and its image $B_1 = \text{im} \partial_2$ is the second boundary space of $\mathcal{G}^+$.

For example, our example graph $\mathcal{G}$ of Figure 1 may be taken as a planar graph $\mathcal{G}^+$ drawn on a plane, with two interior faces. If each face is given a clockwise orientation, then its second incidence matrix is

$$M_2 = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \end{bmatrix}.$$ 

Note that only one edge is interior, and that $Z_2 = \ker \partial_2$ is trivial, for this graph and in general for planar graphs drawn on planes. Thus $\dim B_1 = 2$. In general, $\dim B_1 = |F| = \beta_1(\mathcal{G})$.

However, if we consider our example graph to be drawn on a sphere, then $\mathcal{G}^+$ has three faces, and all edges become interior edges. If the “exterior face” is given a counterclockwise orientation, then

$$M_2 = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & -1 & -1 \end{bmatrix}. $$

Since all edges are now interior, each column of $M_2$ now has precisely 2 nonzero values $\pm 1$. Moreover, since the sum of the three rows is $0 \in \hat{A}^E$, the second boundary operator $\partial_2$ now has a nontrivial kernel $Z_2$ of dimension 1. Therefore $\dim Z_2 = 1$ in this case and, by similar arguments, for planar graphs in general. However, the image $B_1$ remains unchanged, regardless of whether we take $\mathcal{G}^+$ as a graph on a plane or on a sphere. Thus its dimension remains $\dim B_1 = 2$ for this example, or $\dim B_1 = \beta_1(\mathcal{G})$ for planar graphs in general.

Notice that the first two rows of $M_2$ are the edge vectors $\hat{y}(p_1), \hat{y}(p_2)$ corresponding to the two cycles $p_1, p_2$ that bound the two interior faces of $\mathcal{G}^+$, and therefore are elements of the zero-boundary space $Z_1$, the kernel $Z_1$ of our original first boundary operator $\partial_1 : \hat{A}^E \to \hat{A}^V, \hat{y} \mapsto \hat{y}M_1$. It follows that $B_1 = Z_1$ in this example, and, by similar arguments, for planar graphs in general.

Dually, the image of our original first coboundary operator $d_1 : \hat{A}^E \to \hat{A}^F, x \mapsto M_1x$ will continue to be denoted as $B^1$, and its kernel as $Z^0$. Since $d_1$ is the adjoint homomorphism to $\partial_1$, $B^1 = (Z_1)^\perp$ and $Z^0 = (B_0)^\perp$. Similarly, we define the second coboundary operator as the adjoint homomorphism to $\partial_2$—i.e., the operator $d_2 : \hat{A}^F \to \hat{A}^G, y \mapsto M_2y$. By the Adjoint Homomorphism Lemma, the kernel $Z^1$ and image $B^2$ of $d_2$ are then $Z^1 = (B_1)^\perp$ and $B^2 = (Z_2)^\perp$.

### B. Homology spaces

In general, it is straightforward to show that in any two-dimensional complex, the second boundary space $B_1$ is a subspace of the first zero-boundary space $Z_1 = \ker \partial_1$, by showing that every row of $M_2$ is an edge vector $\hat{y}(p)$ corresponding to a cycle $p$. Thus for a general two-dimensional complex we have $B_1 \subseteq Z_1$. The first homology space is then defined as $H_1 = Z_1/B_1$.

For a two-dimensional complex $\mathcal{G}^+$ based on a planar graph $\mathcal{G}$, we have seen that $B_1 = Z_1$, so for our example planar graph and in general, we have $\dim H_1 = 0$, whether we take $\mathcal{G}^+$ as a graph on a plane or on a sphere.

In a general two-dimensional complex, the first cohomology space is defined as $H^1 = Z^1/B^1 = (B_1)^\perp/(Z_1)^\perp$, which from linear algebra is the dual space to $H_1 = Z_1/B_1$. Thus for a planar graph $\dim H^1 = \dim H_1 = 0$; i.e., the image $B^1$ of $d_1$ is equal to the kernel $Z^1$ of $d_2$. 
In this context, the second homology space $H_2$ of $G^+$ is defined as $Z_2$. We have seen that $\dim Z_2 = 0$ if $G$ is a planar graph defined on a plane, but $\dim Z_2 = 1$ if $G$ is defined on a sphere. The second cohomology space is defined as $H^2 = C^2/B^2 = C^2/(Z_2)\perp$, which is the dual space to $H_2 = Z_2$; thus $\dim H^2 = \dim H_2$.

The dimensions of the homology or cohomology spaces $H_0, H_1, H_2$ are thus $(1, 0, 0)$ if $G$ is regarded as a planar graph on a plane, or $(1, 0, 1)$ if $G$ is regarded as a planar graph on a sphere. We see that these dimensions are universal for all graphs of these respective types [3].

Two-dimensional complexes become more interesting when $H_1 = Z_1/B_1$ is nontrivial; i.e., when there exist cycle vectors in $Z_1 = \ker \partial_1$ that are not second boundary vectors in $B_1 = \im \partial_2$. For example, as shown in [2], if $G$ is a 2-dimensional square lattice graph drawn on a torus, then $\dim H_1 = 2$, since there are two independent cycle vectors that do not bound faces.

C. Normal realizations

Figure 30(a) depicts a normal realization of $Z_2 = \ker \partial_2$ when we view $G^+$ as being drawn on a sphere, so there is an exterior face $f_0$ as well as the two interior faces $f_1, f_2$. The three equality constraints correspond to the face variables $z_0, z_1, z_2$, and the six zero-sum constraints correspond to the edge variables $y_1, \ldots, y_6$, which are all set to 0 in this kernel realization. One of the two incident face variables to each edge constraint is negated, according to the entries $\alpha_{f_e}$ of $M_2$.

Figure 30(b) depicts a simplified realization of $Z_2 = \ker \partial_2$. Note the resemblance of this realization to that of $Z^0 = \ker \partial_1$ in Figure 3 or 7.

Similarly, Figure 31 depicts the dual realization of $B^2 = \im \partial_2$, which resembles that of $B_0 = \im \partial_1$ in Figure 12 or 15. We now explain these resemblances.

D. Dual graphs

Let $G = (V, E)$ be a connected planar graph, let $G^+ = (V, E, F)$ be the associated 2-dimensional complex when $G$ is drawn on a sphere, and let $M_1$ and $M_2$ be the first and second incidence matrices of $G^+$. The number of faces is then $|F| = \beta_1(G) + 1 = |E| - |V| + 2$, including the exterior face. Assuming that $G$ has no dangling vertices, every edge is incident on two distinct faces, a right face $r(e) \in F$ and a left face $l(e) \in F$. (For our purposes, it does not matter which is which.)

Each row $(M_1)_e$ of $M_1$ then has two nonzero values, $(M_1)_{eh(e)} = +1$ and $(M_1)_{et(e)} = -1$. Similarly, each column $(M_2)_e$ of $M_2$ has two nonzero values, $(M_2)_{er(e)} = +1$ and $(M_2)_{et(e)} = -1$.

It is thus natural to define the dual graph $\hat{G} = (F, E)$ as the planar graph with the same edge set $E$, but with vertex and face sets interchanged, so $\hat{G}^+ = (F, E, V)$ has incidence matrices $\hat{M}_1 = (M_2)^T$ and $\hat{M}_2 = (M_1)^T$ equal to the transposes of the original incidence matrices. Thus $\beta_1(\hat{G}) = |E| - |F| + 1 = |V| - 1 = |E| - \beta_1(G)$.

Geometrically, we may construct this dual graph by putting vertices of $\hat{G}$ inside each face of $G$ (including the exterior face), and putting edges between two such vertices if and only if the corresponding faces share an edge, as illustrated for our example graph in Figure 32.

The similarity of this dual graph $\hat{G}$ to the graphs of Figures 30 and 31 is no accident. For $\hat{G}^+ = (F, E, V)$, the first boundary operator $\partial_1 : \hat{A}^E \rightarrow \hat{A}^F$ has matrix $\hat{M}_1 = \boxed{\hat{M}_1}$.
(M\_2)\^T, and thus maps the row vector \( \hat{y} \in \hat{A}^E \) to \( \hat{z} = M_2\hat{y} \in \hat{A}^F \). This map is evidently the same up to transposition as that of the second coboundary operator \( d_2 : A^E \to A^F \) of \( G^+ \), which maps the column vector \( y \in A^E \) to \( z = M_2y \). Similarly, \( \hat{d}_2 : \hat{A}^V \to \hat{A}^E \) is the same map up to transposition as \( d_1 : A^V \to A^E \), and the first and second coboundary operators, \( \hat{d}_1 : \hat{A}^F \to \hat{A}^E \) and \( d_2 : A^F \to A^E \), are the same up to transposition as \( \hat{d}_2 \) and \( d_1 \), respectively.

Thus the boundary and zero-boundary spaces for \( G^+ \) are simply the transposes of those spaces for \( G^+ \): \( \hat{B}_0 = (B^2)^T, \hat{Z}_1 = (Z^1)^T, \hat{B}_1 = (B^3)^T, \hat{Z}_2 = (Z^2)^T, \hat{B}_1^\perp = (B_0)^T, \hat{Z}_1^\perp = (Z_2)^T, \hat{B}_2 = (B_0)^T, \hat{Z}_1 = (Z_1)^T \).

Figure 30 may thus be seen as realizing either \( Z_2 = \ker \hat{d}_2 \) for \( G^+ \), or \( Z^0 = \ker \hat{d}_1 \) for \( G^+ \). Similarly, Figure 31 may be seen as realizing either \( B^2 = \im \hat{d}_2 \) for \( G^+ \), or \( B_0 = \im \hat{d}_1 \) for \( G^+ \).

Using the dual graph \( \hat{G} \), we may also realize \( Z^1 = \ker d_2 \) and \( B_1 = \im d_2 \) for \( \hat{G}^+ \) as \( \hat{B}_1 = \im \hat{d}_1 \) for \( \hat{G} \). For example, Figures 33(a) and 33(b) show realizations of \( Z^1 \) and \( B_1 \) for our example graph \( \hat{G}^+ \) as realizations of \( \hat{Z}_1 \) and \( \hat{B}_1 \) for its dual graph \( \hat{G} \) as in Figures 13 or 4. Again, these are dual realizations; i.e., \( Z^1 = (B_1)^\perp \).

However, for a two-dimensional complex \( \hat{G}^+ \) based on a planar graph \( G \), we also have \( Z^1 = B^1 = \im d_1 \) and \( B_1 = Z_1 = \ker d_1 \); therefore we may alternatively realize \( Z^1 \) as an image realization based on \( G \) as in Figure 4 or \( B_1 \) as a kernel realization based on \( G \) as in Figure 13. Since \( \hat{G} \) has \( |E| \) edges and \( |V| = |E| - \beta_1(G) + 1 \) vertices, whereas \( \hat{G} \) has \( |E| \) edges and \( \beta_1(\hat{G}) + 1 \) vertices, the representation based on \( \hat{G} \) (resp. \( \hat{G} \)) will in general be simpler if \( \beta_1(G) > |E|/2 \) (resp. \( \beta_1(G) < |E|/2 \)).

E. Realizations of partition functions for planar graphs

Let \( G^+ = (V, E, F) \) be a two-dimensional complex based on a connected planar graph \( G \). Then we have seen that the partition function \( Z(G) \) of an Ising-type model based on \( G \) with interaction weight functions \( \{ f_e(a) \mid e \in E, a \in A \} \) may be represented (up to scale) as the partition function of an EWNFG consisting of a realization of its first coboundary space \( B^1 = \im d_1 \) with edge weights \( \{ f_e(a) \} \). We recall that \( \dim B^1 = |V| - 1 - |E| - \beta_1(G) \).

As we have seen, with a planar graph we have \( B^1 = \im d_1 \). Moreover, \( Z^1 \) may be realized as \( \hat{Z}_1 = \ker \hat{d}_1 \) for the dual graph \( \hat{G} \). Hence \( \hat{Z}(\hat{G}) \) may also be represented (up to scale) by a realization of the first zero-boundary space \( \hat{Z}_1 = \ker \hat{d}_1 \) of \( \hat{G} \) with edge weights \( \{ f_e(a) \} \).

Furthermore, the Fourier transform \( \hat{Z}(\hat{G}) \) of \( Z(\hat{G}) \), which as we have seen is equal to \( Z(\hat{G}) \) up to scale, may be represented (up to scale) by a kernel realization of its first zero-boundary space \( \hat{Z}_1 = \ker \hat{d}_1 \), with Fourier-transformed edge weights \( \{ f_e(a) \} \). We have \( \dim \hat{Z}_1 = \beta_1(\hat{G}) \), which can be less than \( \dim B^1 = |V| - 1 - |E| - \beta_1(G) \), as we have seen in Figures 21 and 23. The Fourier transform will in general convert a low-temperature to a high-temperature model, and vice versa.

If \( G \) is a planar graph, then \( Z_1 = B_1 \), and \( B_1 \) may be realized as \( \hat{B}_1 = \im \hat{d}_1 \) for the dual graph \( \hat{G} \). Thus \( \hat{Z}(\hat{G}) \) may also be represented (up to scale) by an image realization of the first coboundary space \( \hat{B}^1 = \im \hat{d}_1 \) of \( \hat{G} \), with Fourier-transformed edge weights \( \{ f_e(a) \} \).

Table 1 summarizes these four possible representations.

Al-Bashabsheh and Vontobel [2] show that the partition function \( Z(G) \) of an EWNFG based on a connected planar graph \( G \) and edge weights \( \{ f_e(a) \} \) is equal up to scale to the partition function of an EWNFG based on the dual graph \( \hat{G} \) with dual edge weights \( \{ f_{-e}(-a) \} \). This follows from equating (up to scale) the first and fourth lines of Table 1.

For example, let \( G \) be a single-cycle graph of length \( N \), which is a connected planar graph with \( |V| = |E| = N \) and \( \beta_1(G) = 1 \). Its dual graph \( \hat{G} \) thus has only \( |V| = 2 \) vertices, while \( \beta_1(\hat{G}) = N - 1 \). Figure 21 shows a representation of the partition function \( Z(\hat{G}) \) using an image realization of the \( (N, N - 1) \) linear code \( B^1 = \im d_1 \) on the graph \( \hat{G} \), and Figure 23 shows a representation of its Fourier transform \( \hat{Z}(\hat{G}) \) (up to scale) using a kernel realization of the \( (N, 1) \) code \( \hat{Z}_1 = \ker \hat{d}_1 \).
on $\mathcal{G}$. We now see that $Z(\mathcal{G})$ could alternatively be represented (up to scale) by a kernel realization of $Z(\hat{\mathcal{G}})$ using the $(N, N-1)$ code $\hat{Z}_1 = \ker \hat{\partial}_1$ with edge weights $\{f_e(a)\}$ on the dual graph $\hat{\mathcal{G}}$, or alternatively by an image realization of $\hat{Z}(\hat{\mathcal{G}})$ using the $(N, 1)$ code $\hat{B}_1 = \im \hat{d}_1$ with edge weights $\{\hat{f}_e(a)\}$ on $\hat{\mathcal{G}}$.

V. Conclusion

In this paper, we have presented an introduction to elementary algebraic topology using normal realizations, and, following [2], [19], we have shown how such realizations may be used for calculating partition functions of Ising-type models. Indeed, using dual realizations and dual graphs, we have given multiple alternative ways of representing such partition functions, summarized in Section IV-E.

While Molkaraie et al. [15], [16], [18], [19], have successfully exploited such alternatives in Monte Carlo simulations using importance sampling, much more could be done. In particular, in the presence of external fields, the hybrid models suggested in Section III-F should be explored further.

Our results are very general; in particular, they apply for any finite abelian group alphabet $\mathcal{A}$. Thus one could explore Ising-type models with group alphabets more general than $\mathbb{Z}_q$; however, we have no idea whether such models would be of interest to statistical physicists.

For the field of codes on graphs, this development suggests exploring graphical models that are inspired more by algebraic topology than by traditional system-theory models (e.g., trellises, tail-biting trellises, kernel and image representations). For example, we have recently found a simple and elegant “2-state” elementary normal realization of the $(8, 4, 4)$ first-order Reed-Muller code on a 3-cube graph, shown in Figure 34. Are there similarly “nice” realizations of more complex codes?

---

**TABLE II**

| realizes | space | dimension | realization type | graph | no. vertices | edge wts. |
|----------|-------|-----------|------------------|-------|--------------|-----------|
| $Z(\mathcal{G})$ | $B^1 = \im d_1$ | $|E| - \beta_1(\mathcal{G})$ | Fig. [4] or [9] | $\mathcal{G}$ | $|E| - \beta_1(\mathcal{G}) + 1$ | $\{f_e(a)\}$ |
| $Z(\hat{\mathcal{G}})$ | $\hat{Z}_1 = \ker \hat{\partial}_1$ | $|E| - \beta_1(\hat{\mathcal{G}})$ | Fig. [15] or [17] | $\hat{\mathcal{G}}$ | $\beta_1(\hat{\mathcal{G}}) + 1$ | $\{\hat{f}_e(a)\}$ |
| $\hat{Z}(\hat{\mathcal{G}})$ | $\hat{Z}_1 = \ker \hat{\partial}_1$ | $\beta_1(\hat{\mathcal{G}})$ | Fig. [15] or [17] | $\hat{\mathcal{G}}$ | $|E| - \beta_1(\hat{\mathcal{G}}) + 1$ | $\{f_e(a)\}$ |
| $\hat{Z}(\mathcal{G})$ | $\hat{B}_1 = \im \hat{d}_1$ | $\beta_1(\mathcal{G})$ | Fig. [4] or [9] | $\mathcal{G}$ | $\beta_1(\mathcal{G}) + 1$ | $\{\hat{f}_e(a)\}$ |

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APPENDIX: THE NFG DUALITY THEOREM

The normal factor graph duality theorem (NFGDT) \cite{1, 6, 9} is the key duality result for normal factor graphs (NFGs) over finite abelian groups. It shows that the partition function of a dual NFG is equal to the Fourier transform of the partition function of the primal NFG, up to a scale factor which was shown in \cite{1} to be $|A|/|A'|$ (see below for terminology and notation).

In this appendix we give the simplest proof we know of this result, and extend it to situations in which the NFG is based on a normal realization \cite{5}, specifically, to NFGs that are based purely on normal realizations, and to edge-weighted NFGs (EWNFGs), as discussed in the main text. We compute the appropriate scale factors for these cases.

A.1 Fourier transforms over finite abelian groups

Given an additive finite abelian group $A$, its dual group (or character group) $\hat{A}$ may be defined as the set of all homomorphisms $\hat{a} : A \rightarrow \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}/\mathbb{Z}$ is the additive group of real numbers modulo 1. It is well known that $\hat{A}$ is a finite abelian group that is isomorphic to $A$. Also, the dual group to $\hat{A}$ is $A$, where $a : \hat{A} \rightarrow \mathbb{R}/\mathbb{Z}$ is defined by $a(\hat{a}) = \hat{a}(a)$.

For $\hat{a}, a \in \hat{A}$, we may define the “inner product” (pairing) $\langle \hat{a}, a \rangle = \hat{a}(a) = a(\hat{a})$. The usual inner product properties—e.g., $\langle \hat{a}, 0 \rangle = 0$, $\langle \hat{a}, -a \rangle = -\langle \hat{a}, a \rangle$, $\langle \hat{a}, a + b \rangle = \langle \hat{a}, a \rangle + \langle \hat{a}, b \rangle$—then follow from the properties of homomorphisms.

If $f(a)$ is any complex-valued function $f : A \rightarrow \mathbb{C}$, then its Fourier transform $\hat{f}(\hat{a})$ is the complex-valued function $\hat{f} : \hat{A} \rightarrow \mathbb{C}$ defined by

$$\hat{f}(\hat{a}) = \sum_{a \in A} f(a)e^{2\pi i \langle \hat{a}, a \rangle}.$$  

The matrix $F = \{e^{2\pi i \langle \hat{a}, a \rangle} | a \in A, \hat{a} \in \hat{A}\}$ is called the Fourier transform matrix over $A$. We will regard $F$ as a function of two variables with alphabets $A$ and $\hat{A}$.

If $\hat{g}(\hat{a})$ is any complex-valued function $\hat{g} : \hat{A} \rightarrow \mathbb{C}$, then its inverse Fourier transform $g(a)$ is the complex-valued function $g : A \rightarrow \mathbb{C}$ defined by

$$g(a) = |A|^{-1} \sum_{\hat{a} \in \hat{A}} \hat{g}(\hat{a})e^{-2\pi i \langle \hat{a}, a \rangle}.$$  

The matrix $F^{-1} = \{|A|^{-1}e^{-2\pi i \langle \hat{a}, a \rangle} | \hat{a} \in \hat{A}, a \in A\}$ is called the inverse Fourier transform matrix over $A$, and will also be regarded as a function of two variables with alphabets $\hat{A}$ and $A$.

We may verify that $F^{-1}$ is in fact the inverse of $F$ by using the basic orthogonality relation $|A|^{-1} \sum_{\hat{a} \in \hat{A}} e^{2\pi i \langle \hat{a}, a \rangle} = \delta_a$.

Given a subgroup $C \subseteq A$, the orthogonal subgroup $C^\perp \subseteq \hat{A}$ is the set $\{\hat{a} \in \hat{A} | \langle \hat{a}, a \rangle = 0, \forall a \in C\}$. The orthogonal subgroup to $C^\perp$ is $C$, and $|C||C^\perp| = |A|$. It is well known, and easy to prove, that the Fourier transform of the indicator function $\delta_C$ of $C$ is the scaled indicator function $|C|\delta_{C^\perp}$ of $C^\perp$. The above orthogonality relation follows from the special case in which $C = \{0\}$ and $C^\perp = \hat{A}$.

Finally, if $f(a)$ is a function of multiple variables $a = \{a_i | i \in I\}$, then its Fourier transform $\hat{f}(\hat{a})$ is obtained by taking the Fourier transform of each variable separately, since $\langle \hat{a}, a \rangle = \sum_{i} \langle \hat{a}_i, a_i \rangle$; i.e., the Fourier transform is separable, as illustrated in Figure \ref{fig:35}.

A.2 Normal factor graphs

A normal factor graph (NFG) is based on a graph $G = (V, E, H)$ consisting of a set of vertices indexed by a vertex index set $V$, a set of edges indexed by an edge index set $E$, and a set of half-edges indexed by a half-edge index set $H$. With each edge $e \in E$ we associate an internal (state) variable $s_e$ whose alphabet is a finite abelian group denoted by $A_e$, and with each half-edge we associate an external variable $a_h$ whose alphabet is a finite abelian group denoted by $A_h$. The internal variable configuration space is defined as the Cartesian product $A_E = \prod_{e \in E} A_e$.

With each vertex $v \in V$ we associate a complex-valued function $f_v(s_v, a_v)$ of the variables corresponding to the edges and half-edges that are incident on vertex $v$. The partition function (or “exterior function” \cite{1}) of the NFG is then defined as the following function of its external variables:

$$Z(a) = \sum_{s \in A_E} \prod_{v \in V} f_v(s_v, a_v).$$

The dual normal factor graph to an NFG as defined above is based on the same graph $G = (V, E, H)$, but with the following replacements:

\footnote{The simple and lovely proof goes as follows: (a) obvious for $\hat{a} \in C^\perp$, since $\langle \hat{a}, a \rangle = 0$ for all $a \in C$; (b) if $\hat{a} \notin C^\perp$, then $\langle \hat{a}, b \rangle \neq 0$ for some $b \in C$; for this $b$, we have $e^{2\pi i \langle \hat{a}, b \rangle} = \sum_{a \in C} e^{2\pi i \langle \hat{a}, a \rangle} = \delta_{C + b}$, since $C + b = C$; but since $e^{2\pi i \langle \hat{a}, b \rangle} \neq 1$, this equation can hold only if $\sum_{a \in C} e^{2\pi i \langle \hat{a}, a \rangle} = 0$.}

\[\begin{aligned}
\hat{A}_1 & \quad \hat{A}_2 \quad \hat{A}_3 \\
\hat{A}_1 & \quad \hat{A}_2 \quad \hat{A}_3 \\
\hat{A}_1 & \quad \hat{A}_2 \quad \hat{A}_3 \\
\hat{A}_1 & \quad \hat{A}_2 \quad \hat{A}_3 \\
\end{aligned}\]

Fig. 35. The Fourier transform is separable.
• For each edge \( e \in E \), the variable \( s_e \in A_e \) is replaced by a dual variable \( \hat{s}_e \in \hat{A}_e \), where \( \hat{A}_e \) denotes the dual group to \( A_e \).
• For each half-edge \( h \in H \), the variable \( a_h \in A_h \) is replaced by a dual variable \( \hat{a}_h \in \hat{A}_h \).
• For each vertex \( v \in V \), the function \( f_v(s_v, a_v) \) is replaced by its Fourier transform \( \hat{f}_v(\hat{s}_v, \hat{a}_v) \).
• Lastly, each edge is replaced by a sign-inverting edge.

The normal factor graph duality theorem (NFGDT) says that the partition function of the dual NFG is the Fourier transform of the partition function of the primal NFG, up to a scale factor that will be determined shortly. The key to the proof of the NFGDT is the following Edge Replacement Lemma (an example of what is called a “holographic transformation” in [1], [6], [9]):

**Edge Replacement Lemma.** In any NFG, any edge representing a variable whose alphabet is a finite abelian group \( A \) may be replaced by a cascade of \( F \), a sign inverter, and \( \hat{F} \), plus a disconnected node \( \hat{A} \) that contributes a scale factor of \( |A|^{-1} \) to the partition function, without changing the partition function.

**Proof:** By the basic orthogonality relation given above, we have \( |A|^{-1} \sum_{\hat{A}} e^{2\pi i (\hat{a}, a)} e^{-2\pi i (\hat{a}, a')} = \delta_{a'a} \).

This lemma is illustrated by Figure 36.

Fig. 36. Edge Replacement Lemma: an edge with alphabet \( A \) may be replaced by a cascade of \( F \), a sign inverter, and \( \hat{F} \), plus a disconnected node representing a scale factor \( |A|^{-1} \).

The NFGDT then follows:

**Theorem (NFG duality).** If \( G = (V, E, H) \) is a normal factor graph with functions \( \{f_v | v \in V\} \), internal variable alphabets \( \{A_e | e \in E\} \), and external variable alphabets \( \{A_h | h \in H\} \) whose partition function is \( Z(a) \), then the dual normal factor graph with functions \( \{\hat{f}_v | v \in V\} \), internal variable alphabets \( \{\hat{A}_e | e \in E\} \), and external variable alphabets \( \{\hat{A}_h | h \in H\} \), plus sign inverters inserted in each edge has partition function \( |A_E| \hat{Z}(\hat{a}) \), where \( \hat{Z}(\hat{a}) \) is the Fourier transform of \( Z(a) \), and the scale factor is \( |A_E| = \prod_E |A_e| \).

**Proof:** If we have an NFG whose partition function is \( Z(a) \), then by definition and by separability the Fourier transform \( \hat{Z}(\hat{a}) \) is the partition function of the NFG that results when each external half-edge, representing a variable with alphabet \( A_h, h \in H, \) is replaced by \( -\hat{F}\hat{F} \), namely a Fourier transform from \( A_h \) to the dual external variable alphabet \( \hat{A}_h \). By the Edge Replacement Lemma, if we then replace each edge \( e \in E \), representing an internal variable with alphabet \( A_e \), by \( \hat{F}\hat{F} \), omitting the scale factor \( |A_e|^{-1} \), then the resulting NFG has Fourier transform \( |A_E| \hat{Z}(\hat{a}) \), where \( |A_E| = \prod_E |A_e| \). Now each function node \( f_v, v \in V \), in the NFG is surrounded by edge and half-edge segments of the form \( -\hat{F}\hat{F} \). By the definition of the Fourier transform and by separability, each such node and its surrounding Fourier transform functions realizes the Fourier-transformed function \( \hat{f}_v \).

The scale factor \( |A_E| \) is thus the product of all internal variable alphabet (state space) sizes \( |A_e| \), as in the version of the NFGDT derived in [1].

### A.3 Scale factors of code indicator functions

In Section III-B we consider interpreting a normal realization as an NFG. Edges and half-edges represent the same internal and external variables, but a constraint code \( C_e \) is now interpreted as its indicator function \( \delta_{C_e} \).

If the normal realization has external behavior \( C \) and unobservable behavior \( B^u \), then we conclude that the partition function of the resulting NFG is as follows:

**Theorem 6 (Normal realization as an NFG).** If all alphabets are finite abelian groups, then a normal realization with external behavior \( C \) and unobservable behavior \( B^u \) may be interpreted as a normal factor graph whose partition function is \( Z(a) = |B^u|\delta_C(a) \).

By the NFG duality theorem, the dual NFG to the NFG of Theorem 6 has partition function

\[
|A_E| \hat{Z}(\hat{a}) = |A_E| |B^u| |C| \delta_{\hat{C}}(\hat{a}).
\]

The vertex functions of this dual NFG are the Fourier transforms of the primal vertex functions \( f_v = \delta_{C_v} \), and therefore are of the form \( \hat{f}_v = |C_v| \delta_{\hat{C}_v} \). The NFG of the dual NR of the NR of Theorem 6 is the same as this dual NFG, except that its function nodes are of the form \( \delta_{\hat{C}_v} \); therefore its partition function is

\[
|A_E| |C| \delta_{\hat{C}}(\hat{a}) = |A_E| |B^u| |C| \delta_{\hat{C}}(\hat{a}) = |A_E| |B^u| |C| \delta_{\hat{C}}(\hat{a}) = |A_E| |B^u| |C| \delta_{\hat{C}}(\hat{a})
\]

Therefore, the scale factor is \( |A_E| = \prod_E |A_e| \), and \( C_v = \prod_V |C_v| \). In summary:

**Theorem 7 (Dual normal realization as an NFG).** The dual of a finite abelian group normal realization with behavior \( B^u \) and external behavior \( C \) may be interpreted as an NFG with partition function \( |B^u| |A_E| |C| \delta_{\hat{C}}(\hat{a}) \), where \( |A_E| = \prod_E |A_e| \) and \( C_v = \prod_V |C_v| \).

Now if we denote the external behavior of the dual normal realization as \( \hat{C} \) and its unobservable behavior as...
\( \hat{B}^u \), then, from Theorem 6, its partition function as an NFG is \(|B^u|\delta_C(\hat{a})\). We thus conclude that:

(a) The external behavior \( \hat{C} \) of the dual normal realization is \( C^\perp \). Thus, when all alphabets are finite abelian groups, we obtain the normal realization duality theorem as a corollary.

(b) The size \(|B^u|\) of the unobservable behavior of the dual normal realization is \(|B||A_E||C_V|^{-1}\).

There is an interesting connection between result (b) and the notions of observability and controllability of normal realizations defined in [8]. A finite abelian group normal realization with unobservable behavior \( B^u \) is observable if and only if \(|B^u| = 1\), and \( \dim B^u \) measures its “degree of unobservability.” A finite abelian group normal realization whose dual has unobservable behavior \( \hat{B}^u \) is controllable if and only if \(|\hat{B}^u| = 1\), and \( \dim \hat{B}^u \) measures its “degree of uncontrollability.” Thus as a further corollary we have the controllability test of [8]:

**Corollary (Controllability test [8])**. Given a finite abelian group normal realization with behavior \( B \), total constraint size \(|C_V| = \prod_V |C_v|\), and total state space size \(|A_E| = \prod_E |A_e|\), the unobservable behavior \( B^u \) of the dual normal realization has size \(|\hat{B}^u| = |B||A_E||C_V|^{-1} \geq 1\). Thus the realization is controllable if and only if \(|B| = |C_V|/|A_E|\).

This controllability test may be understood as follows. If all edges are removed from the realization, then its behavior is simply \( C_V = \prod_V C_v \), the Cartesian product of the behaviors \( C_v \) of each of its disconnected nodes. If we reinsert the edge constraints, each of which is a degree-2 equality constraint between two variables with a common alphabet \( A_e \), then each such constraint will reduce the size of the behavior by a factor of \(|A_e|\), provided that it is independent of all previous constraints. Thus \(|B^u| \geq |C_V|/|A_E|\), with equality if and only if all constraints are independent.

### A.4 Scale factors for edge-weighted NFGs

An edge-weighted NFG consists of an NFG based on a normal realization of a linear or group code \( C \) as above, in which all internal functions \( f_e \) are indicator functions \( \delta_{C_e} \) of linear or group codes \( C_e \), plus edge-weighting functions \( f_h \) attached to each external half-edge \( h \in H \) of the normal realization. The resulting NFG has no external variables, and its partition function is the constant \( Z = |B^u|\sum_{\hat{a} \in C^\perp} f(\hat{a}) \), where \( B^u \) is the unobservable behavior of the normal realization as above, and \( f(\hat{a}) = \prod_H f_h(a_h) \).

As noted above, if \( C \) is a linear or group code, then the Fourier transform of \( \delta_C(s) \) is the scaled indicator function \( |C|\delta_{C^\perp}(\hat{s}) \). Thus if we construct the “dual” edge-weighted NFG by replacing every node function \( \delta_{C_e}(s) \) simply by \( \delta_{C_e}(\hat{s}) \) rather than by \( |C_e|\delta_{C^\perp}(\hat{s}) \), then the partition function will be reduced by a scale factor of \(|C^\perp|^{-1} = \prod_V |C_v|^{-1}\).

Thus we obtain the following corollary of the NFG duality theorem:

**Theorem 8 (Edge-weighted NFG duality).** Given an edge-weighted NFG based on a graph \( G = (V, E, H) \) with internal functions \( \{\delta_{C_e} \mid v \in V\} \), internal variable alphabets \( \{A_e \mid e \in E\} \), external variable alphabets \( \{A_h \mid h \in H\} \), and edge-weighting functions \( \{f_h \mid h \in H\} \) that realizes a partition function \( Z = |B^u|\sum_{\hat{a} \in C^\perp} f(\hat{a}) \), then the “dual edge-weighted NFG” with internal functions \( \{\delta_{C^\perp_e} \mid v \in V\} \), internal variable alphabets \( \{\hat{A}_e \mid e \in E\} \), external variable alphabets \( \{\hat{A}_h \mid h \in H\} \), and external weighting functions \( \{\hat{f}_h \mid h \in H\} \), and sign inverters inserted in each edge realizes the partition function

\[
\frac{|A_E|}{|C_V|} Z = |B^u|\sum_{\hat{a} \in C^\perp} \hat{f}(\hat{a}),
\]

where \(|A_E| = \prod_E |A_e|\), \(|C_V| = \prod_v |C_v|\), and \(|B^u| = |B||A_E||C_V|^{-1}\).

For example, let us consider an Ising-type model based on a graph \( \hat{G} = (\hat{V}, \hat{E}) \), with variable alphabet \( \hat{A} \) and edge-weighting functions \( \{f_e(y_e) \mid e \in E\} \), such as the edge-weighted NFG shown in Figure 19(a). The resulting edge-weighted NFG \( G = (V_G, E_G) \) actually has \(|V_G| = |V| + 2|E|\) vertices, consisting of \(|V|\) equality functions, \(|E|\) zero-sum functions, and \(|E|\) edge-weighting functions; \(|E_G| = 3|E|\) edges, each representing an internal variable with alphabet \( \hat{A} \); and no half-edges, so its partition function is a constant \( Z \).

If we wish to compute the partition function of the dual edge-weighted NFG as shown in Figure 20(a) from the partition function \( Z \) of the primal edge-weighted NFG, then we need to adjust \( Z \) as follows. Since the NFG has \(|E_G| = 3|E|\) edges, we must multiply by the scale factor \(|A_E| = |A|^{3|E|}\). Since the primal NFG has \(|V|\) equality functions and \(|E|\) zero-sum weight functions of degree 3 with total dimension \( 2|E|\), we need to divide by \(|C_V| = |A|^{|V|+2|E|}\), Therefore the partition function of the dual EWNGF is \( |A|^{|E|}Z \).

Footnote 8 gives an alternative derivation.
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