COMPLEX POWERS OF THE CONTACT LAPLACIAN AND THE BAUM-CONNES CONJECTURE FOR SU(n, 1).

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Abstract. This paper is an extended version of [Po3] where we point out and remedy a gap in the proof by Julg-Kasparov [JK] of the Baum-Connes conjecture for discrete subgroups of SU(n, 1). In particular, here we explain in details why the non-microlocality of the Heisenberg calculus prevents us from implementing into this framework the classical approach of Seeley to pseudodifferential complex powers, which was the main issue at stake in [Po3].

1. Introduction

A locally compact group \( \Gamma \) satisfies the Baum-Connes conjecture when the Baum-Connes assembly map \( \mu : K^\mathrm{top}_i(\Gamma) \to K_i(C^*_r(\Gamma)) \) is an isomorphism. Here \( K^\mathrm{top}_i \) denotes the geometric \( K \)-group of Baum-Connes (when \( \Gamma \) is torsion-free this the \( K \)-homology group \( K_i(B\Gamma) \) of the classifiant space \( B\Gamma \)) and \( K_i(C^*_r(\Gamma)) \) denotes the analytic \( K \)-group of the reduced \( C^* \)-algebra of \( \Gamma \). A stronger conjecture states that this holds with coefficients in any \( C^* \)-algebra acted on by \( \Gamma \). Since its statement in the early 80’s the Baum-Connes conjecture, with or without coefficients, has been shown for a variety of groups (see, e.g., [HK], [Ju], [Ka], [La], [MY]).

In this paper we are concerned with the proof by Julg-Kasparov [JK] of this conjecture for discrete subgroups of the complex Lorentz group SU(n, 1). The proof of Julg and Kasparov can be briefly summarized as follows.

First, the proof can be reduced to showing that Kasparov’s element \( \gamma_G \) is equal to 1 in the representation ring \( R(G) \), that is, if \( K \) is a maximal compact group of \( G \) the restriction map \( R(G) \to R(K) \) is an isomorphism. Second, the symmetric space \( G/K \) is a complex hyperbolic space and under the Siegel map it is biholomorphic to the unit ball \( B^{2n} \subset \mathbb{C}^n \) and its visual boundary is CR diffeomorphic to the unit sphere \( S^{2n-1} \) equipped its standard CR structure. Julg and Kasparov further showed that \( R(K) \) can be geometrically realized as \( KK_G(C(B^{2n}), \mathbb{C}) \), where \( C(B^{2n}) \) denotes the \( C^* \)-algebra of continuous functions on the closed unit ball \( B^{2n} \) and \( KK_G \) is the equivariant \( KK \) functor of Kasparov. Then they built a Fredholm module representing an element \( \delta \) in \( KK_G(C^0(B^{2n}), \mathbb{C}) \) which is mapped to \( \gamma \) in \( KK_G(\mathbb{C}, \mathbb{C}) = R(G) \) under the morphism induced by the map \( B^{2n} \to \{ \text{pt} \} \).

The construction of the element \( \delta \) in \( KK_G(C(B^{2n}), \mathbb{C}) \) involves in a crucial manner the contact complex of Rumin [Ru] on the unit sphere \( S^{2n-1} \) endowed with its standard contact structure. In the contact setting the main geometric operators are not elliptic and the relevant pseudodifferential calculus to deal with them is the Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta]. Then for constructing the Fredholm module representing \( \delta \) Julg and Kasparov proved that the complex
powers of the contact Laplacian are pseudodifferential operators in the Heisenberg calculus (see [JK Thm. 5.27]).

In [Se] Seeley settled a general procedure for obtaining complex powers of elliptic operators as pseudodifferential operators. Its approach relied on constructing an asymptotic resolvent in a suitable class of classical \( \Psi DO \)'s with parameter. Similarly, Julg and Kasparov constructed an asymptotic resolvent in a class of Heisenberg \( \Psi DO \)'s with parameter (see [JK Thm. 5.25]). We point out here that this class is not enough to allow us to carry out the rest of Seeley’s arguments. In fact, we further show that we cannot carry out at all Seeley’s approach in the setting of the Heisenberg calculus. Therefore, we have to rely on another approach to deal with the complex powers of the contact Laplacian.

This note is organized as follows. In Section 2 we recall the outline of Seeley’s approach to complex powers of elliptic operators. In Section 3 we recall the construction of the contact complex and the associated contact Laplacian. In Section 4 we recall the definition of the Heisenberg calculus and stress out its lack of microlocality. In Section 5 we explain the gap in the proof of Theorem 5.27 of [JK] and why the non-microlocality of the Heisenberg calculus prevents us from extending Seeley’s approach to the setting of the Heisenberg calculus.

2. Seeley’s Approach to Complex Powers of Elliptic Operators

In this section we recall the main idea in the approach of Seeley [Se] to complex powers of elliptic operators (see also [GS, Sh]). To simplify the exposition we let \( M^n \) denote a compact manifold equipped with a smooth density \( \partial \). Let \( P : C^\infty(M) \to C^\infty(M) \) be a positive elliptic differential operator of order \( m \) with principal symbol \( p_m(x, \xi) > 0 \). Then for \( \Re s < 0 \) we can write:

\[
(1) \quad P^s = \frac{i}{2\pi} \int_{\Gamma_r} \lambda^s (P - \lambda)^{-1} d\lambda,
\]

\[
(2) \quad \Gamma_r = \{ \rho e^{i\theta}; \infty < \rho \leq r \} \cup \{ \rho e^{i\theta}; \theta \geq \theta - 2\pi \} \cup \{ \rho e^{-i\theta}; r \leq \rho \leq \infty \},
\]

where \( r \) is any real \( < \lambda_1(P) \) and \( \lambda_1(P) \) is the smallest non-zero eigenvalue of \( P \).

To show that the formula above defines a \( \Psi DO \), Seeley constructs an asymptotic resolvent \( Q(\lambda) \) as parametrix for \( P - \lambda \) in a suitable \( \Psi DO \) calculus with parameter. More precisely, let \( \Lambda \subset \mathbb{C} \setminus 0 \) be an open angular sector \( \theta < \theta' \) with \( 0 < \theta < \pi < \theta' < 2\pi \). In the sequel we will say that a subset \( \Theta \subset [R^n \times \mathbb{C}] \setminus 0 \) is \textit{conic} when for any \( t > 0 \) and any \( (\xi, \lambda) \in \Theta \) we have \( (t\xi, t^m\lambda) \in \Theta \). For instance, the subset \( \mathbb{R}^n \times \Lambda \subset \mathbb{R}^n \times \mathbb{C} \setminus 0 \) is conic.

Let \( U \subset \mathbb{R}^n \) be a local chart for \( M \). Then in \( U \) the asymptotic resolvent has a symbol of the form \( q(x, \xi; \lambda) \sim \sum_j q_{-m-j}(x, \xi; \lambda) \), where \( \sim \) is taken in a suitable sense (see [Se]) and there exists an open conic subset \( \Theta \subset [R^n \times \mathbb{C}] \setminus 0 \) containing \( \mathbb{R}^n \times \Lambda \) such that each symbol \( q_{-m-j}(x, \xi; \lambda) \) is smooth on \( U \times \Theta \) and satisfies

\[
(3) \quad q_{-m-j}(x, t\xi; t^m\lambda) = t^{-m-j} q(x, \xi; \lambda) \quad \forall t > 0.
\]

If \( p(x, \xi) = \sum_{j=0}^{m_j} p_{-m-j}(x, \xi) \) denotes the symbol of \( P \) in the local chart \( U \) then the symbol \( q(x, \xi; \lambda) \) satisfies

\[
1 \sim (p(x, \xi) - \lambda) q(x, \xi; \lambda) + \sum_{\alpha \neq 0} \frac{1}{\alpha!} \partial^\alpha p(x, \xi) D_\alpha^\alpha q(x, \xi; \lambda),
\]
from which we obtain

\( q_m(x, \xi; \lambda) = (p_m(x, \xi) - \lambda)^{-1}, \)  

(4)  

\( q_{m-j}(x, \xi; \lambda) = -(p_m(x, \xi) - \lambda)^{-1} \sum_{|\alpha|+k+t=j, l \neq j} \frac{1}{\alpha!} \partial_\xi^\alpha p_{m-k}(x, \xi) D_p^s q_{m-l}(x, \xi; \lambda). \)

(5)

Set \( \rho = \inf_{x \in U} \inf_{|\xi|=1} p(x, \xi). \) Possibly by shrinking \( U \) we may assume \( \rho > 0. \)

Let \( \Theta = [\mathbb{R}^n \times \Lambda] \cup \{(\xi; \lambda) \in \mathbb{R}^n \times \mathbb{C}; 0 \leq |\lambda| < \rho|\xi|^m \}. \) Then the formulas (4) and (5) show that each symbol \( q_{m-j}(x, \xi; \lambda) \) is well defined and smooth on \( U \times \Theta \) and is homogeneous in the sense of (4). Furthermore, it is analytic with respect to \( \lambda. \)

Therefore, for \( \Re s < 0 \) we define a smooth function on \( U \times (\mathbb{R}^n \setminus 0) \) by letting

\( p_{s,ms-j}(x, \xi) = \frac{i}{2\pi} \int_{\Gamma(\xi)} \lambda^s q_{m-j}(x, \xi; \lambda) d\lambda, \)

(6)

where \( \Gamma(\xi) \) is contour \( \Gamma_r \) in (2) with \( r = \frac{1}{2} \rho|\xi|^m. \) Moreover, one can check that \( p_{s,ms-j}(x, t\xi) = t^{ms-j} p_{s,ms-j}(x, \xi) \) for any \( t > 0, \) i.e., \( p_{s,ms-j}(x, \xi) \) is a homogeneous symbol of degree \( ms - j. \)

It can also be shown that on the chart \( U \) the operators \( P^s \) is a \( \Psi DO \) with symbol \( p_s \sim \sum_{j \geq 0} p_{s,ms-j}(x, \xi). \) This is true on any local chart so, as one can check that the Schwarz kernel of \( P^s \) is smooth off the diagonal of \( M \times M, \) it follows that \( P^s \) is a \( \Psi DO \) of order \( ms \) for \( \Re s < 0. \)

Furthermore, for \( k = 1, 2, \ldots \) and \( \Re s < k \) we have \( P^s = P^k \) \( P^{s-k} \). Here \( P^k \) is a differential operator of order \( k \) and as we have \( \Re s - k < 0 \) we know that \( P^{s-k} \) is a \( \Psi DO \) of order \( m(s-k). \)

Therefore, \( P^s \) is a \( \Psi DO \) of order \( ms \) for \( \Re s < k \) and any \( k = 1, 2, \ldots, \) i.e, \( P^s \) is a \( \Psi DO \) of order \( ms \) for \( \Re s < k \) and any \( s \in \mathbb{C}. \)

3. Ruin’s contact complex

Let \((M^{2n-1}, H)\) be an orientable contact manifold, so that there exists a global nonvanishing contact form \( \theta \) such that \( H = \ker \theta \) and \( d\theta|_{\Lambda} \) is nondegenerate. Let \( X_0 \) be the Reeb vector field of \( \theta \), so that \( \iota_{X_0} \theta = 1 \) and \( \iota_{X_0} d\theta = 0. \) In addition, we let \( J \) be a calibrated almost complex structure on \( H \), so that we have \( \delta \theta(X, JX) > 0 \) for any non-zero section of \( H \), and we endow \( TM \) with the Riemannian metric \( g_{0,J} = d\theta(\cdot, J\cdot) + \theta^2. \)

Observe that the splitting \( TM = H \oplus \mathbb{R}X_0 \) allows us to identify \( H^* \) with the annihilator of \( X_0 \) in \( T^*M \). More generally, by identifying \( \Lambda^k_c H^* \) with \( \ker \iota_{X_0} \) we get the splitting,

\( \Lambda^*_c TM = \bigoplus_{k=0}^{2n} \Lambda^k_c H^* \oplus \bigoplus_{k=0}^{2n} (\theta \wedge \Lambda^k_c H^*). \)

(7)

For any horizontal form \( \eta \in C^\infty(M, \Lambda^k_c H^* \) we can write \( d\eta = d\eta + \theta \wedge \mathcal{L}_{X_0} \eta \), where \( d\eta \) is the component of \( d\eta \) in \( \Lambda^k_c H^* \). This does not provide us with a complex, for we have \( d\theta \varepsilon = -\mathcal{L}_{X_0} \varepsilon(d\theta) = -\varepsilon(d\theta) \mathcal{L}_{X_0}, \) where \( \varepsilon(d\theta) \) denotes the exterior multiplication by \( d\theta. \)

The contact complex of Rumin [Ru] is an attempt to get a complex of horizontal differential forms by forcing the equalities \( d\theta^2 = 0 \) and \( (d\theta^2)^2 = 0. \)
A natural way to modify $d_b$ to get the equality $d^2_b = 0$ is to restrict $d_b$ to the subbundle $\Lambda^*_2 := \ker \varepsilon(d\theta) \cap \Lambda^*_C H^*$, since the latter is closed under $d_b$ and is annihilated by $d^2_b$.

Similarly, we get the equality $(d^2_b)^2 = 0$ by restricting $d^*_b$ to the subbundle $\Lambda^*_1 := \ker \iota(d\theta) \cap \Lambda^*_C H^*$ is injective for $k \leq n - 1$ and surjective for $k \geq n + 1$. This implies that $\Lambda^*_k = 0$ for $k \leq n$ and $\Lambda^*_k = 0$ for $k \geq n + 1$. Therefore, we only have two halves of complexes.

As observed by Rumin [Ru], we get a full complex by connecting the two halves by means of the operator $D_{R,n} : C^\infty(M, \Lambda^*_n H^*) \to C^\infty(M, \Lambda^*_2 H^*)$ such that

$$D_{R,n} = \mathcal{L}_{X_\alpha} + d_{n,n-1} \varepsilon(d\theta)^{-1} d_b n_n,$$

where $\varepsilon(d\theta)^{-1}$ is the inverse of $\varepsilon(d\theta) : \Lambda^{n-1}_\mathbb{C} H^* \to \Lambda^{n+1}_{\mathbb{C}} H^*$. Notice that $D_{R,n}$ is a second order differential operator. This allows us to get the contact complex,

$$C^\infty(M) \xrightarrow{\partial_{b,n}} \ldots \xrightarrow{\partial_{b,1}} C^\infty(M, \Lambda^n) \xrightarrow{\partial_{b,0}^*} C^\infty(M, \Lambda^1) \ldots \xrightarrow{\partial_{b,2}^{n-1}} C^\infty(M, \Lambda^{2n}).$$

where $d_{R,k}$ agrees with $\pi_1 \circ d_b$ for $k = 0, \ldots, n - 1$ and with $d_{R,k} = d_b$ otherwise.

The contact Laplacian is defined as follows. In degree $k \neq n$ this is the differential operator $\Delta_{R,k} : C^\infty(M, \Lambda^k) \to C^\infty(M, \Lambda^k)$ such that

$$\Delta_{R,k} = \left\{ \begin{array}{ll} (n-k)d_{R,k-1} d_{R,k}^* + (k-n+1)d_{R,k+1}^* d_{R,k}, & k = 0, \ldots, n-1, \\ (k-n-1)d_{R,k-1}^* d_{R,k} + (k-n)d_{R,k+1}^* d_{R,k}, & k = n+1, \ldots, 2n. \end{array} \right.$$  

For $k = n$ we have the differential operators $\Delta_{R,n,j} : C^\infty(M, \Lambda^n) \to C^\infty(M, \Lambda^n)$, $j = 1, 2$, given by the formulas,

$$\Delta_{R,n,1} = (d_{R,n-1} d_{R,n}^* + D_{R,n}^* D_{R,n})^n, \quad \Delta_{R,n,2} = D_{R,n} D_{R,n}^* + (d_{R,n+1}^* d_{R,n}).$$

Observe that $\Delta_{R,k}, k \neq n$, is a differential operator order 2, whereas $\Delta_{R,n}$ is a differential operators of order 4. Moreover, Rumin [Ru] proved that in every degree the contact Laplacian is maximal hypoelliptic.

### 4. Heisenberg calculus

Let $(M^{2n-1}, H)$ be a contact manifold. In this setting the natural operators like the contact Laplacian are not elliptic, so the standard pseudodifferential calculus does not apply. The substitute is provided by the Heisenberg calculus of Beals-Greiner [BG] and Taylor [Ta]. The idea, which goes back to Elias Stein, is to construct a class of pseudodifferential operators, the $\Psi_H$DO’s, which at each point are modelled by left-convolutions operators on the Heisenberg group $\mathbb{H}^{2n-1}$. This motivated by the fact that in a suitable sense $\mathbb{H}^{2n-1}$ is tangent to a contact manifold at each of its points.

Locally the $\Psi_H$DO’s can be described as follows. Let $U \subset \mathbb{R}^{2n-1}$ be a local chart together with a frame $X_0, \ldots, X_{2n}$ such that $X_1, \ldots, X_{2n}$ span $H$. Such a chart is called a Heisenberg chart. Moreover, on $\mathbb{R}^{2n-1}$ we consider the dilations,

$$t \xi = (t^2 \xi_0, t \xi_1, \ldots, t \xi_{2n}), \quad \xi \in \mathbb{R}^{d+1}, \quad t > 0.$$

**Definition 4.1.** 1) $S_m(U \times \mathbb{R}^{2n+1})$, $m \in \mathbb{C}$, is the space of functions $p(x, \xi)$ in $C^\infty(U \times (\mathbb{R}^{2n-1} \setminus 0))$ such that $p(x, t \xi) = t^m p(x, \xi)$ for any $t > 0$. 

2) $S^m(U \times \mathbb{R}^{2n+1})$, $m \in \mathbb{C}$, consists of functions $p \in C^\infty(U \times \mathbb{R}^{2n+1})$ with an asymptotic expansion $p \sim \sum_{j \geq 0} p_{m-j}$, $p_k \in S_k(U \times \mathbb{R}^{2n+1})$, in the sense that, for any integer $N$ and for any compact $K \subset U$, we have

\begin{equation}
|\partial_x^\beta \partial_\xi^\gamma (p - \sum_{j < N} p_{m-j})(x,\xi)| \leq C_{\alpha\beta\gamma NK} \|\xi\|^{m-|\beta|-N}, \quad x \in K, \quad \|\xi\| \geq 1,
\end{equation}

where we have let $|\beta| = 2\beta_0 + \beta_1 + \ldots + \beta_n$ and $\|\xi\| = (\xi_0^2 + \xi_1^2 + \ldots + \xi_n^2)^{1/4}$.

Next, for $j = 0,\ldots, 2n$ let $\sigma_j(x,\xi)$ denote the symbol (in the classical sense) of the vector field $\frac{1}{\sqrt{n}} x_j$ and set $\sigma = (\sigma_0,\ldots,\sigma_{2n})$. Then for $p \in S^m(U \times \mathbb{R}^{2n+1})$ we let $p(x, -iX)$ be the continuous linear operator from $C^\infty_c(U)$ to $C^\infty_c(U)$ such that

\begin{equation}
p(x, -iX)f(x) = (2\pi)^{- (d+1)} \int e^{ix \cdot \xi} p(x, \sigma(x, \xi)) \hat{f}(\xi) d\xi, \quad f \in C^\infty_c(U).
\end{equation}

**Definition 4.2.** $\Psi^m_H(U)$, $m \in \mathbb{C}$, consists of operators $P : C^\infty_c(U) \to C^\infty_c(U)$ which are of the form $P = p(x, -iX) + R$ for some $p \in S^m(U \times \mathbb{R}^{2n+1})$, called the symbol of $P$, and some smoothing operator $R$.

The class of $\Psi^m_H$DO’s is invariant under changes of Heisenberg chart (see [BG Sect. 16], [Po2 Appendix A]), so we may extend the definition of $\Psi^m_H$DO’s to an arbitrary Heisenberg manifold $(M, H)$ and let them act on sections of a vector bundle $\mathcal{E}$ over $M$. We let $\Psi^m_H(M, \mathcal{E})$ denote the class of $\Psi^m_H$DO’s of order $m$ on $M$ acting on sections of $\mathcal{E}$.

In addition, the Heisenberg calculus possesses a full symbolic calculus which allows us to construct parametrices for hypoelliptic operators. In the classical pseudodifferential calculus the relevant product at the level of homogeneous symbols is the pointwise product of functions which, under the Fourier transform, corresponds to the convolution on the Abelian group $\mathbb{R}^{2n-1}$.

Similarly, for homogeneous Heisenberg symbols the relevant product comes from the convolution on the Heisenberg group $H^{2n-1}$. More precisely, let $U \subset \mathbb{R}^{2n-1}$ be a Heisenberg chart as above and for $m \in \mathbb{C}$ let $S_m(\mathbb{R}^{2n-1})$ denotes the closed subspace of $C^\infty(\mathbb{R}^{2n-1} \setminus 0)$ consisting in functions $p(\xi)$ such that $p(t, \xi) = t^m p(\xi)$ for any $t > 0$. Then for each $x \in U$ we get a bilinear product

\begin{equation}
\ast^x : S_m(\mathbb{R}^{2n-1}) \times S_m(\mathbb{R}^{2n-1}) \to S_{m+m}(\mathbb{R}^{2n-1}).
\end{equation}

This product depends smoothly on $x$, so we may define the bilinear product,

\begin{equation}
\ast : S_m(U \times \mathbb{R}^{2n+1}) \times S_m(U \times \mathbb{R}^{2n+1}) \to S_{m+m}(U \times \mathbb{R}^{2n+1}),
\end{equation}

\begin{equation}
p_{m_1} \ast p_{m_2}(x, \xi) = [p_{m_1}(x, \cdot) \ast^x p_{m_2}(x, \cdot)](\xi), \quad p_{m_j} \in S_m(U \times \mathbb{R}^{2n+1}).
\end{equation}

Then we have:

**Proposition 4.1** ([BG Thm. 14.7]). For $j = 1, 2$ let $P_j \in \Psi^m_H(U)$ have symbol $p_j \sim \sum_{k \geq 0} p_{j, m_1, m_2-k}$ and assume that one of these operators is properly supported. Then the operator $P = P_1 P_2$ is a $\Psi_H$DO of order $m_1 + m_2$ and has symbol $p \sim \sum_{k \geq 0} p_{m_1+m_2-k}$, with

\begin{equation}
p_{m_1+m_2-k} = \sum_{k_1 + k_2 \leq k} \sum_{\alpha, \beta, \gamma, \delta} h_{\alpha\beta\gamma\delta}(D^k\xi p_{1, m_1-k_1}) \ast (\xi^\gamma \partial_\xi^\alpha \partial_\xi^\beta p_{2, m_2-k_2}),
\end{equation}
where \( \sum_{\alpha \beta \gamma \delta} \) denotes the sum over all the indices such that \( |\alpha| + |\beta| \leq |\gamma| + |\delta| = l \) and \( |\beta| = |\gamma| \), and the functions \( h_{\alpha \beta \gamma \delta}(x) \)'s are polynomials in the derivatives of the coefficients of the vector fields \( X_0, \ldots, X_d \).

This result allows us to carry out the classical parametrix construction, provided we can invert the principal symbol with respect to the product \( \ast \). This may be difficult in practice because this is not anymore the pointwise product of functions, but this can be completely determined in terms of a representation theoretic criterion, the so-called Rockland condition (see [Po2]). For instance, in every degree the contact Laplacian satisfies this condition and admits a parametrix in the Heisenberg calculus (see [JK, Sect. 3.5]).

Finally, it should be stressed out that in [10–14] the \( x \)-values of \( p_{m_1} \ast p_{m_2} \) depend only on that of \( p_{m_1} \) and \( p_{m_2} \), i.e., \( p_{m_1} \ast p_{m_2}(x,.) \) depends only on \( x \) and \( p_{m_1}(x,.), p_{m_2}(x,.). \) However, for any \( (x,\xi) \in U \times (\mathbb{R}^{2n-1} \setminus 0) \) the value of \( p_{m_1} \ast p_{m_2}(x,\xi) \) depends on all the values of \( p_{m_1}(x,\xi') \) and \( p_{m_2}(x,\xi') \) as \( \xi' \) ranges over \( \mathbb{R}^{2n-1} \setminus 0 \). Thus the Heisenberg calculus is local, but is not microlocal.

5. Complex powers of the contact Laplacian

Let \( S^{2n-1} \subset \mathbb{C}^n \) be the open angular sector \( \theta \) such that \( \theta = \sum_{j=0}^{n} \tau_j d\bar{z}_j \). Since \( H \) is invariant under the multiplication by \( i \) the complex structure of \( \mathbb{C}^{n+1} \) induces a complex structure \( J \) on \( H \). This is a calibrated complex structure on \( H \) and we endow \( TM \) with the Riemannian metric \( g_{\theta,J} = d\theta + \theta^2 \).

Let \( \Delta_R \) be the contact Laplacian of \( S^{2n-1} \). In the proof of the Baum-Connes conjecture with coefficients for discrete subgroups of \( SU(n,1) \) in [JK] the following result is needed.

**Theorem 5.1 ([JK] Thm. 5.27)**. Let \( s \in \mathbb{C} \). Then:

1) The operator \( \Delta_{R,k}^s \), \( k \neq n \), is a \( \Psi \) DO of order \( 2s \);

2) The operator \( \Delta_{R,nj}^s \), \( j = 1,2 \), is a \( \Psi \) DO of order \( 4s \).

We would like to point out a gap in the proof of this result in [JK]. For simplicity we will explain it in degree \( k = n \), but the argument works in degree \( k = n \) as well. The idea in the proof in [JK] is to carry out Seeley’s approach in the Heisenberg setting. As in [12] for \( \Re s < 0 \) we have

\[
\Delta_{R,k}^s = \frac{i}{2\pi} \int_{\Gamma} \lambda^s (\Delta_{R,k} - \lambda)^{-1} d\lambda,
\]

where \( \Gamma \) is as in [2]. Let \( \Lambda = \mathbb{C} \setminus 0 \) be an open angular sector \( \theta < \arg \lambda < \theta' \) with \( 0 < \theta < \pi < \theta' < 2\pi \). Then Julg and Kasparov showed that the resolvent \( (\Delta_{R,k} - \lambda)^{-1} \) belongs to a class \( \Psi^{-2}(S^{2n-1}, \Lambda; \Lambda) \) of \( \Psi \) DO’s parametrized by \( \Lambda \) (see [JK] Thm. 5.25)). In particular, in a Heisenberg chart \( U \subset \mathbb{R}^{2n-1} \) the resolvent \( (\Delta_{R,k} - \lambda)^{-1} \) has a symbol \( q(x,\xi;\lambda) \) in \( C^\infty(U \times \mathbb{R}^{2n+1} \times \Lambda) \) with an expansion \( q(x,\xi;\lambda) \sim \sum_{j \geq 0} q_{-2-j}(x,\xi;\lambda) \), where \( \sim \) is taken in a suitable sense and \( q_{-2-j}(x,\xi;\lambda) \in C^\infty(U \times (\mathbb{R}^{2n-1} \setminus 0) \times \Lambda) \) is such that for any \( t > 0 \) we have \( q_{-2-j}(x,t\xi;t^2\lambda) = t^{-2-j}q_{-2-j}(x,\xi;\lambda) \).

Notice that the symbol \( q(x,\xi;\lambda) \) does not make sense for \( \lambda > 0 \), because only angular sectors contained in \( \mathbb{C} \setminus [0,\infty) \) are considered in [JK] Thm. 5.25]. However,
we have to allow the homogeneous components of the symbol of \((\Delta_{R,k} - \lambda)^{-1}\) to be defined for some \((x, \xi; \lambda) \in U \times (\mathbb{R}^{2n-1} \times 0) \times \mathbb{C}\) with \(\lambda > 0\) because the contour \(\Gamma\) in \([19]\) always crosses the positive real axis.

This leads us to a discrepancy in \([19, p. 128, line 9 from bottom]\) when the authors claim that their Theorem 5.25 insures us that the resolvent \((\Delta_{R,k} - \lambda)^{-1}\) belongs to a class \(\Psi^2_{H}(S^{2n-1}, \Lambda^k; \Lambda)\) for some subset \(\Lambda\) containing the contour \(\Gamma\). In particular, the authors cannot claim that, in a Heisenberg chart \(U \subset \mathbb{R}^{2n-1}\), the symbol of \(\Delta_{R,k}\) is given by the formula,

\[
p_\ast(x, \xi) = \frac{i}{2\pi} \int_{\Gamma} \lambda^s q(x, \xi; \lambda) d\lambda,
\]

where \(q(x, \xi; \lambda)\) denotes the symbol of \((\Delta_{R,k} - \lambda)^{-1}\), since the homogeneous components of the latter symbol don’t make sense for \(\lambda > 0\).

In fact, we cannot implement at all Seeley’s approach to the setting of the Heisenberg calculus. More precisely, to carry out Seeley’s approach we have to construct an asymptotic resolvent for \(\Delta_{R,k}\) in a class of \(\Psi_H DO's\) with parameter associated to an angular sector \(\Lambda\) as above and given in a local Heisenberg chart \(U \subset \mathbb{R}^{2n-1}\) by parametric symbols, \(q(x, \xi; \lambda) \sim \sum_{j \geq 0} q_{m-j}(x, \xi; \lambda)\), where \(\sim\) is taken in a suitable sense and there exists an open conic subset \(\Theta \subset (\mathbb{R}^n \times \mathbb{C}) \setminus 0\) containing \(\mathbb{R}^n \times \Lambda\) such that each symbol \(q_{m-j}(x, \xi; \lambda)\) is smooth on \(U \times \Theta\) and satisfies \(q_{m-j}(x, t\xi; t^2\lambda) = t^{-2-j}q(x, \xi; \lambda)\) for any \(t > 0\).

If we let \(p(x, \xi) = \sum p_{2-j}(x, \xi)\) be the symbol of \(\Delta_{R,k}\) in the Heisenberg chart, then by Proposition \(4.1\) we have

\[
(20) \quad 1 \sim \sum_{j \geq 0} \sum_{k+l \leq j} \sum_{\alpha, \beta, \gamma, \delta} h_{\alpha, \beta, \gamma, \delta}(x)(D^\delta_x p_{2-k}) \ast (\xi^\gamma \partial_x^\alpha \partial^\beta_x q_{2-l})(x, \xi; \lambda),
\]

where the notation is the same as in \([19]\). Therefore, we get

\[
(21) \quad 1 \sim \sum_{j \geq 0} \sum_{k+l \leq j} \sum_{\alpha, \beta, \gamma, \delta} h_{\alpha, \beta, \gamma, \delta}(x)(D^\delta_x p_{2-k}) \ast (\xi^\gamma \partial_x^\alpha \partial^\beta_x q_{2-l})(x, \xi; \lambda),
\]

where \((p_2 - \lambda)^{*-1}\) denotes the inverse of \(p_2 - \lambda\) with respect to the product \(*\).

If \(q_1(x, \xi; \lambda)\) and \(q_2(x, \xi; \lambda)\) are two homogeneous Heisenberg symbols with parameter then the product \(q_1\) and \(q_2\) should be defined as

\[
(22) \quad q_1 \ast q_2(x, \xi; \lambda) = [q_1(x, ; \lambda) \ast^\xi q_2(x, ; \lambda)](\xi).
\]

As mentioned at the end of Section \(4\) the definition of \([q_1(x, ; \lambda) \ast^\xi q_2(x, ; \lambda)](\xi)\) depends on all the values of \(q_1(x, \xi'; \lambda)\) and \(q_2(x, \xi'; \lambda)\) as \(\xi'\) ranges over \(\mathbb{R}^{2n-1} \setminus 0\). For a parameter \(\lambda > 0\) the symbols \(q_1(x, \xi; \lambda)\) and \(q_2(x, \xi; \lambda)\) are only defined for \(\xi\) in \(\{\xi; (x, \xi; \lambda) \in \Theta\}\) which does not agree with \(\mathbb{R}^{2n-1} \setminus 0\), so we cannot define \(q_1 \ast q_2(x, \xi; \lambda)\) for \(\lambda > 0\). Therefore, the formula \(23\) does not make sense for \(\lambda > 0\).

All this shows that the non-microlocality of the Heisenberg calculus prevents us from implementing Seeley’s approach into the setting of the Heisenberg calculus.

Therefore, we have to rely on other approaches to deal with complex powers within the framework of this calculus. This is done for instance in \([19, Sect. 5.3]\) for a large class of operators, including the contact Laplacian on any compact
contact manifold. This allows us to complete the proof in [JK] of the Baum-Connes conjecture with coefficients for discrete subgroups of $SU(n,1)$.

The approach in [Po2] is based on combining Mellin’s formula for the complex powers with the pseudodifferential representation of the heat kernel of [BGS]. Furthermore, the above-mentioned issues with considering Heisenberg ΨDO’s with parameter associated to homogeneous symbols with parameter disappear when we substitute the latter by almost homogeneous symbols with parameter (see [Po1], [Po4]). This provides us with an alternative approach to deal with complex power within the framework of the Heisenberg calculus.

It is also possible to construct complex powers in this setting of the Heisenberg by means of a pseudodifferential representation of the resolvent. Instead of using homogeneous symbols with parameters as above, we can use almost homogeneous symbols with parameter. This is more suitable for domains containing contour $\Gamma$ as in [2] and [19] and allows us to resolve all the issues raised above. This is done for sublaplacians in [Po1] and for more general operators in [Po4].

REFERENCES

[BG] Beals, R.; Greiner, P.C.: Calculus on Heisenberg manifolds. Annals of Mathematics Studies, vol. 119. Princeton University Press, Princeton, NJ, 1988.

[BGS] Beals, R.; Greiner, P.C.; Stanton, N.K.: The heat equation on a CR manifold. J. Differential Geom. 20 (1984), no. 2, 343–387.

[GS] G. Grubb, R.T. Seeley: Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems. Invent. Math. 121 (1995), no. 3, 481–529.

[HK] Higson, N.; Kasparov, G.: $E$-theory and $KK$-theory for groups which act properly and isometrically on Hilbert space. Invent. Math. 144 (2001), no. 1, 23–74.

[Ju] Julg, P.: La conjecture de Baum-Connes à coefficients pour le groupe $\text{Sp}(n,1)$. C. R. Math. Acad. Sci. Paris 334 (2002), no. 7, 533–538.

[JK] Julg, P.; Kasparov, G.: Operator $K$-theory for the group $SU(n,1)$. J. Reine Angew. Math. 463 (1995), 99–152.

[Ka] Kasparov, G.: Lorentz groups: $K$-theory of unitary representations and crossed products. Dokl. Akad. Nauk SSSR 275 (1984), no. 3, 541–545.

[La] Lafforgue V.: $K$-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. Invent. Math. 149 (2002), 1–95.

[MY] Mineyev, I.; Yu, G.: The Baum-Connes conjecture for hyperbolic groups. Invent. Math. 149 (2002), no. 1, 97–122.

[Po1] Ponge, R.: Calcul hypoelliptique sur les variétés de Heisenberg, résidu non commutatif et géométrie pseudo-hermitienne. PhD Thesis, Univ. Paris-Sud (Orsay), Dec. 2000.

[Po2] Ponge, R.: Heisenberg calculus and spectral theory of hypoelliptic operators on Heisenberg manifolds. E-print, arXiv, Sep. 05, 138 pages. To appear in Mem. Amer. Math. Soc..

[Po3] Ponge, R.: Comment on: “Operator $K$-theory for the group $SU(n,1)$” by P. Julg and G. Kasparov.” E-print, arXiv, Jan. 06, 2 pages.

[Po4] Ponge, R.: Functional calculus and spectral asymptotics for hypoelliptic operators on Heisenberg manifolds. A resolvent approach. In preparation.

[Ru] Rumin, M.: Formes différentielles sur les variétés de contact. J. Differential Geom. 39 (1994), no.2, 281–330.

[Se] Seeley, R.T. Complex powers of an elliptic operator. Proc. Sympos. Pure Math., Vol. X, pp. 288–307, AMS, 1967.

[Sh] Shubin, M. Pseudodifferential operators and spectral theory. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin-New York, 1987.

[Ta] Taylor, M.E.: Noncommutative microlocal analysis. I. Mem. Amer. Math. Soc. 52 (1984), no. 313.

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