Cohomological BRST aspects of the massless tensor field with the mixed symmetry \((k, k)\)

C. C. Ciobircă, E. M. Cioroianu, S. O. Saliu
Faculty of Physics, University of Craiova
13 A. I. Cuza Str., Craiova 200585, Romania

September 13, 2018

Abstract

The main BRST cohomological properties of a free, massless tensor field that transforms in an irreducible representation of \(GL(D, \mathbb{R})\), corresponding to a rectangular, two-column Young diagram with \(k > 2\) rows are studied in detail. In particular, it is shown that any non-trivial co-cycle from the local BRST cohomology group \(H(s|d)\) can be taken to stop either at antighost number \((k + 1)\) or \(k\), its last component belonging to the cohomology of the exterior longitudinal derivative \(H(\gamma)\) and containing non-trivial elements from the (invariant) characteristic cohomology \(H^{\text{inv}}(\delta|d)\).

PACS number: 11.10.Ef

1 Introduction

An interesting class of field theories is represented by tensor fields in “exotic” representations of the Lorentz group, characterized by a mixed Young symmetry type \([1 2 3 4 5]\), which are known to appear in superstring theories, supergravities or supersymmetric high spin theories. This type
of models became of special interest lately due to the many desirable featured exhibited, like the dual formulation of field theories of spin two or higher \cite{7, 8, 9, 10, 11, 12}, the impossibility of consistent cross-interactions in the dual formulation of linearized gravity \cite{13} or a Lagrangian first-order approach \cite{14, 15} to some classes of massless or partially massive mixed symmetry-type tensor gauge fields, suggestively resembling to the tetrad formalism of General Relativity. A basic problem involving mixed symmetry-type tensor fields is the approach to their local BRST cohomology, since it is helpful at solving many Lagrangian and Hamiltonian aspects, like, for instance the determination of their consistent interactions \cite{16} with higher-spin gauge theories \cite{6, 18, 19, 20, 21, 22, 30, 31}. The present paper proposes the investigation of the basic cohomological ingredients involved in the structure of the co-cycles from the local BRST cohomology for a free, massless tensor gauge field $t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}$ that transforms in an irreducible representation of $GL(D, \mathbb{R})$, corresponding to a rectangular, two-column Young diagram with $k > 2$ rows.

In view of this, we firstly give the Lagrangian formulation of such a mixed symmetry tensor field from the general principle of gauge invariance and then systematically analyze this formulation in terms of the generalized differential complex \cite{24} $\Omega^2_2 (\mathcal{M})$ of tensor fields with mixed symmetries corresponding to a maximal sequence of Young diagrams with two columns, defined on a pseudo-Riemannian manifold $\mathcal{M}$ of dimension $D$. Secondly, we compute the associated free antifield-BRST symmetry $s$, which is found to split as the sum between the Koszul-Tate differential and the exterior longitudinal derivative only, $s = \delta + \gamma$. Thirdly, we pass to the cohomological approach to this model and prove the following results:

- the cohomology of the exterior longitudinal derivative $H(\gamma)$ is non-trivial only in pure ghost numbers of the type $kl$, with $l$ any non-negative integer;
- both the cohomologies of the exterior spacetime differential $d$ in the space of invariant polynomials and in $H(\gamma)$ are trivial in strictly positive antighost number and in form degree strictly less than $D$;
- there is no non-trivial descent for $H(\gamma|d)$ in strictly positive antighost number;
- the invariant characteristic cohomology $H^{\text{inv}}(\delta|d)$ is trivial in antighost numbers strictly greater than $(k + 1)$;
• any co-cycle from the local BRST cohomology \( H(s|d) \) of definite ghost number and in form degree \( D \) can be made to stop at a maximum value of the antighost number equal to either \( k \) or \((k + 1)\) by trivial redefinitions only;

• the non-trivial piece of highest antighost number from any such co-cycle can always be taken to belong to \( H(\gamma) \), with some coefficients that are non-trivial elements from \( H^{\text{inv}}(\delta|d) \).

The results contained in this paper can be used at the determination of the consistent couplings between the free, massless tensor field with the mixed symmetry \((k,k)\) and other matter and gauge fields.

2 Lagrangian formulation from the principle of gauge invariance

We consider a tensor field \( t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} \) that transforms in an irreducible representation of \( GL(D,\mathbb{R}) \), corresponding to a rectangular, two-column Young diagram with \( k > 2 \) rows

\[
t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} = \begin{pmatrix}
\mu_1 & \nu_1 \\
\vdots & \vdots \\
\mu_k & \nu_k
\end{pmatrix},
\]

or, in a shortened version, a tensor field with the mixed symmetry \((k,k)\). This means that \( t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} \) is separately antisymmetric in the first and respectively last \( k \) indices, is symmetric under the interchange between the two sets of indices

\[
t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} = t_{\nu_1 \cdots \nu_k|\mu_1 \cdots \mu_k},
\]

and satisfies the (algebraic) Bianchi I identity

\[
t_{[\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k]} = 0.
\]

Here and in the sequel the symbol \([\mu \cdots \nu]\) signifies the operation of complete antisymmetrization with respect to the indices between brackets, defined such as to include only the distinct terms for a tensor with given antisymmetry.
properties. For instance, the left-hand side of (3) contains precisely \((k + 1)\) terms

\[
\begin{align*}
t_{\mu_1 \cdots \mu_k | \nu_1 \nu_2 \cdots \nu_k} & \equiv t_{\mu_1 \cdots \mu_k | \nu_1 \nu_2 \cdots \nu_k} + (-)^k t_{\mu_2 \cdots \mu_k \nu_1 | \mu_1 \nu_2 \cdots \nu_k} \\ & + t_{\mu_3 \cdots \mu_k \nu_1 | \mu_2 \nu_2 \cdots \nu_k} + \cdots + (-)^k t_{\nu_1 \mu_1 \cdots \mu_{k-1} | \mu_k \nu_2 \cdots \nu_k}.
\end{align*}
\]

(4)

Assume that this tensor field is defined on a pseudo-Riemannian manifold \(\mathcal{M}\) of dimension \(D\), like, for instance, a Minkowski-flat spacetime of dimension \(D\), endowed with a metric tensor of ‘mostly plus’ signature \(\sigma_{\mu \nu} = (- + \cdots +)\). The various traces of this tensor field, to be denoted by 

\[
t_{\mu_1 \cdots \mu_k | \nu_1 \nu_2 \cdots \nu_k - m},
\]

(5)

plus the conventions

\[
f_{\mu_{m+1} \mu_m} \equiv f \text{ (scalar)}, \quad f_{\mu_m \mu_m} \equiv f \mu_m \text{ (vector)}.
\]

(6)

Obviously, each type of trace transforms in an irreducible representation of \(GL(D, \mathbb{R})\), corresponding to a rectangular, two-column Young diagram with \((k - m)\) rows

\[
t_{\mu_{m+1} \cdots \mu_k | \nu_{m+1} \nu_2 \cdots \nu_k} = \sigma^{\mu_1 \nu_1} \cdots \sigma^{\mu_{m+1} \nu_{m+1}} t_{\mu_1 \cdots \mu_k | \nu_1 \nu_2 \cdots \nu_k}, \quad m = 1, k,
\]

(7)

i.e., it is separately antisymmetric in the first and respectively last \((k - m)\) indices, is symmetric under the inter-change between the two sets of indices and satisfies the identity

\[
t_{\mu_{m+1} \cdots \mu_k | \nu_{m+1} \nu_2 \cdots \nu_k} \equiv 0,
\]

(8)

which results from (3) by some appropriate contractions.

We are interested in the Lagrangian description of a single, free, massless tensor field with this type of mixed symmetry, which is known to describe exotic spin-two particles for \(k \geq 2\). For \(k = 1\) we obtain nothing but the spin-two field in the linearized limit of General Relativity, known as the Pauli-Fierz theory [17], while for \(k = 2\) we recover the free, massless tensor field with the mixed symmetry of the Riemann tensor [5, 6]. The construction of the Lagrangian action for such a tensor field relies on the general principle
of gauge invariance, combined with the requirements of locality, Lorentz covariance, Poincaré invariance, zero mass and the natural assumptions that the field equations are linear in the field, second-order derivative and do not break the PT invariance. In view of all these, a natural point to start with is to stipulate the (infinitesimal) gauge invariance of the action such that to recover the linearized limit of diffeomorphisms for \( k = 1 \) and the gauge symmetry \([5, 6]\) of the free, massless tensor field with the mixed symmetry of the Riemann tensor for \( k = 2 \). The simplest way to achieve this is to ask that the Lagrangian action

\[
S_L [t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k}] = c_1 \int d^Dx \left( - \frac{1}{\nu_1 \ldots \nu_k} \frac{\nu_1 \ldots \nu_k - 1}{\nu_1 \ldots \nu_k} \right) \left( \partial_\mu t \right) \left( \partial_\nu t \right) +
\]

(9)

where we used the common notation \( f_{\mu} \equiv \partial_\mu f \). Indeed, for \( k = 1 \), \( t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k} \) becomes a symmetric two-tensor field, traditionally denoted by \( h_{\mu \nu} \) (the Pauli-Fierz field) and (9) takes the familiar form \( \delta \epsilon h_{\mu \nu} = \partial_\mu \epsilon_{\nu} + \partial_\nu \epsilon_{\mu} \), while for \( k = 2 \) one gets a tensor field \( t_{\mu \nu | \alpha \beta} \) with the mixed symmetry of the Riemann tensor, subject to the gauge transformations \( \delta \epsilon t_{\mu \nu | \alpha \beta} = \epsilon_{\mu \nu | \beta, \alpha} + \epsilon_{\alpha \beta | \nu, \mu} \). The mixed symmetry properties of the gauge parameters \( \epsilon_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k} \) follow from those of the left-hand side of (9) once we require that the mixed symmetry of \( t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k} \) is inherited by its gauge variation. As a consequence we find that \( \epsilon_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-1}} \) displays the mixed symmetry \( (k, k-1) \)

\[
\epsilon_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-1}} =
\begin{array}{ccc}
\mu_1 & \nu_1 \\
\vdots & \vdots \\
\nu_{k-1} & \\
\mu_k & \\
\end{array},
\]

(10)

so it is separately antisymmetric in the first \( k \) and respectively last \( (k-1) \) indices and satisfies the identity

\[
\epsilon_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-1}} = 0.
\]

(11)

The formula (11) has the role to enforce that \( \delta \epsilon t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k} \) satisfies a Bianchi I identity similar to that of the field itself, namely, (3). Taking into account the gauge transformations (9), we find that the most general form of a Lagrangian action that complies with all the above mentioned requirements is given by

\[
S_L [t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k}] = c_1 \int d^Dx \left( - \frac{1}{\nu_1 \ldots \nu_k} \frac{\nu_1 \ldots \nu_k - 1}{\nu_1 \ldots \nu_k} \right) \left( \partial_\mu t \right) \left( \partial_\nu t \right) +
\]

5
\[k-1\sum_{m=0}^{m} (-)^{m} \left( \frac{1}{2} \left( \partial_{\mu_1 \cdots \mu_{k-m}}^{\nu_1 \nu_2 \cdots \nu_{k-m}} \right) \left( \partial^{\nu_1 \nu_2 \cdots \nu_{k-m}}_{\mu_1 \cdots \mu_{k-m}} \right) + (k-m) \left( \partial^{\rho_{\mu_1 \cdots \mu_{k-m-1}}^{\nu_1 \nu_2 \cdots \nu_{k-m}}_{\rho_{\mu_1 \cdots \mu_{k-m-1}}^{\nu_1 \nu_2 \cdots \nu_{k-m}}}} \right) \left( \partial_{\nu_1 \nu_2 \cdots \nu_{k-m}}^{\mu_1 \cdots \mu_{k-m-1}} \right) \right) \]  

where \(kC_m\) represents the number of combinations of \(m\) objects drawn from \(k\) and \(c_1\) is a non-vanishing real constant. If we conveniently fix the value of this constant to 

\[c_1 = \frac{(-)^{k+1}}{k^2},\]  

then for \(k = 1, 2\) we are led precisely to the Lagrangian actions corresponding to the Pauli-Fierz field and respectively to the tensor field with the mixed symmetry of the Riemann tensor.

It can be checked that (9) is a generating set of gauge transformations for the action (12). This generating set is abelian and off-shell \((k-1)\)-order reducible. Indeed, if we make the transformation 

\[\epsilon_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_{k-1}} = \partial_{\mu_1} (1) \chi_{\mu_2 \cdots \mu_k|\nu_1 |\nu_2 \cdots \nu_{k-1}} + (-)^{k+1} 2 (1) \chi_{\mu_1 \cdots \mu_k|\nu_2 \cdots \nu_{k-1}, \nu_1},\]  

with \((1) \chi_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_{k-2}}\) an arbitrary tensor field on \(M\) displaying the mixed symmetry \((k, k-2)\), then the gauge transformations of the tensor field vanish identically 

\[\delta \epsilon_{(1) (\chi)} t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_{k}} = 0.\]  

Next, if we perform the transformation 

\[\chi_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_{k-2}} = \partial_{\mu_1} (2) \chi_{\mu_2 \cdots \mu_k|\nu_1 |\nu_2 \cdots \nu_{k-2}} + (-)^{k+1} 3 (2) \chi_{\mu_1 \cdots \mu_k|\nu_2 \cdots \nu_{k-2}, \nu_1},\]  

with \((2) \chi_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_{k-3}}\) an arbitrary tensor field on \(M\) that exhibits the mixed symmetry \((k, k-3)\), then we find that the gauge transformed parameters
strongly vanish

\[ \epsilon_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-1}} \left( \chi^{(1)} \left( \chi^{(2)} \right) \right) = 0. \]  \hfill (17)

Along a similar line it can be shown that if we perform the changes

\[ \chi^{(m)}_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} = \partial_{\mu_1} \chi^{(m+1)}_{\mu_2 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} + (-)^{k+1} (m + 2) \chi^{(m+1)}_{\mu_1 \cdots \mu_k | \nu_2 \cdots \nu_{k-m-1}, \nu_1}, \]  \hfill (18)

for \( 2 \leq m \leq k - 2 \), with \( \chi_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} \) some arbitrary tensor fields on \( \mathcal{M} \), with the mixed symmetry \( (k, k - m - 1) \) and \( \chi_{\mu_1 \cdots \mu_k | \nu_1 \nu_0} \equiv \chi_{\mu_1 \cdots \mu_k} \) a completely antisymmetric tensor (\( k \)-form field), then

\[ \chi^{(m-1)}_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m}} \left( \chi^{(m)} \left( \chi^{(m+1)} \right) \right) = 0, \quad 2 \leq m \leq k - 2, \]  \hfill (19)

while

\[ \chi^{(k-2)}_{\mu_1 \cdots \mu_k | \nu_1} \left( \chi^{(k-1)} \right) = 0 \]  \hfill (20)

if and only if

\[ \chi^{(k-1)}_{\mu_1 \cdots \mu_k} = 0. \]  \hfill (21)

The tensor fields \( \chi^{(m)}_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} \) will be called reducibility parameters of order \( m \). Excepting \( \chi_{\mu_1 \cdots \mu_k} \), which is completely antisymmetric, the reducibility parameters \( \chi^{(m)}_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} \) transform according to some irreducible representation of \( GL(D, \mathbb{R}) \), associated with the two-column Young diagrams

\[
\begin{array}{ccccccc}
\mu_1 & & & & & \nu_1 \\
& \vdots & & & & \vdots \\
\vdots & & \ddots & & \vdots & \vdots \\
\nu_{k-m-1} & & & & & \nu_{k-1} \\
& & & & \vdots & \\
\mu_k & & & & & & \\
\end{array}
\]  \hfill (22)
Hence, they are separately antisymmetric in the first \( k \) and respectively last \((k - m - 1)\) indices and satisfy the identities

\[
X^{(m)}_{[\mu_1 \cdots \mu_k | \nu_1 | \nu_2 \cdots \nu_{k-m-1}} = 0.
\]

(23)

In view of the reducibility structure exhibited by the generating set \( \mathcal{G} \) of gauge transformations, it follows that the spacetime dimension is subject to the condition

\[
D \geq 2k + 1,
\]

(24)

such that the \((k, k)\) tensor field has a non-negative number of physical degrees of freedom. [For \( D = 2k \) the Lagrangian action \( \mathcal{L} \) reduces to a full divergence, for \( D = 2k + 1 \) the \((k, k)\) tensor field has zero physical degrees of freedom, while for \( D > 2k + 1 \) it possesses strictly positive values of the number of physical degrees of freedom.]

The field equations resulting from the action \( \mathcal{L} \)

\[
\frac{\delta \mathcal{L}}{\delta t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} \equiv c_1 T^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \approx 0,
\]

(25)

involve the tensor \( T^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \), which is linear in the tensor field \( t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \), second-order in its derivatives and displays the mixed symmetry \((k, k)\). Its concrete expression reads as

\[
T^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \Box t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + (-)^k \left( \partial_{\mu} \partial_{[\nu_1} t^{\mu_2 \cdots \mu_k | \rho]}_{\nu_1 \cdots \nu_k} + \partial_{\nu_1} t^{\mu_2 \cdots \mu_k | \rho]}_{[\nu_1 \cdots \nu_k]} \right) + \sum_{m=1}^{k-1} \left( \frac{1}{m!} \delta^{[\mu_1}_{[\nu_1} \cdots \delta^{\mu_m}_{\nu_m} \left( \partial^\rho \partial_{\nu_{m+1} \cdots \nu_k} t^{\mu_{m+1} \cdots \mu_k | \rho]}_{\nu_{m+1} \cdots \nu_k} \delta_\rho^\lambda - m t^{\mu_{m+1} \cdots \mu_k | \rho]}_{\nu_{m+1} \cdots \nu_k} \right) \right) + \sum_{m=1}^{k-1} \left( \frac{1}{m!} \delta^{[\mu_1}_{[\nu_1} \cdots \delta^{\mu_m}_{\nu_m} \left( \partial_{\nu_{m+1} \cdots \nu_k} t^{\mu_{m+1} \cdots \mu_k | \rho]}_{\nu_{m+1} \cdots \nu_k} \right) + \frac{1}{m+1} \partial_{\nu_{m+1} \cdots \nu_k} t^{\mu_{m+1} \cdots \mu_k | \rho]}_{\nu_{m+1} \cdots \nu_k} \right) \right) \right).
\]

(26)

We notice that our antisymmetrization convention takes into account only the distinct terms. For instance, in

\[
\delta^{[\mu_1}_{[\nu_1} \cdots \delta^{\mu_m}_{\nu_m} t^{\mu_{m+1} \cdots \mu_k | \rho]}_{\nu_{m+1} \cdots \nu_k}.
\]
it is understood that there appear only \((k!/(k-m))^{2/m}\) terms. In a somehow abusive language we will name the components of this tensor the Euler-Lagrange (E.L.) derivatives of the action \((12)\). The various traces of \(T^{\mu_1\cdots\mu_k|\nu_1\cdots\nu_k}\) will be denoted by \((T^{\mu_{m+1}\cdots\mu_k|\nu_{m+1}\cdots\nu_k})_{m=1,k}\), being understood that they are defined in a manner similar to \((5)\). The gauge invariance of the Lagrangian action \((12)\) under the transformations \((9)\) is equivalent to the fact that the functions defining the field equations are not all independent, but rather obey the Noether identities

\[
\partial_\mu \frac{\delta S}{\delta t^{\mu_1\cdots\mu_k|\nu_1\cdots\nu_k}} \equiv c_1 \partial_\mu T^{\mu_{m+1}\cdots\mu_k-1|\nu_1\cdots\nu_k} = 0, \tag{27}
\]

while the presence of the reducibility shows that not all of the above Noether identities are independent. It can be checked that the functions \((26)\) defining the field equations, the gauge generators, as well as all the reducibility functions, satisfy the general regularity assumptions from \([23]\), such that the model under discussion is described by a normal gauge theory of Cauchy order equal to \((k+1)\).

3 Reconstruction of the Lagrangian formulation from the generalized 3-complex

3.1 Gauge invariance

This model describes a free gauge theory that can be interpreted in a consistent manner in terms of the generalized differential complex \([24]\) \(\Omega^2_2(M)\) of tensor fields with mixed symmetries corresponding to a maximal sequence of Young diagrams with two columns, defined on a pseudo-Riemannian manifold \(M\) of dimension \(D\). Let us denote by \(\bar{d}\) the associated operator (3-differential) that is third-order nilpotent, \(\bar{d}^3 = 0\), and by \(\Omega^p_2(M)\) the vector space spanned by the tensor fields from \(\Omega^2_2(M)\) with \(p\) entries. The action of \(\bar{d}\) on an element pertaining to \(\Omega^2_2(M)\) results in a tensor from \(\Omega^{p+1}_2(M)\) with one spacetime derivative, the action of \(d^2\) on a similar element leads to a tensor from \(\Omega^{p+2}_2(M)\) containing two spacetime derivatives, while the action of \(d^3\) on any such element identically vanishes. In brief, the generalized 3-complex \(\Omega^2_2(M)\) may suggestively be represented through the commutative
where the third-order nilpotency of $\bar{d}$ means that any vertical arrow followed by the closest higher diagonal arrow maps to zero, and the same with respect to any diagonal arrow followed by the closest higher horizontal one. Its bold part emphasizes the sequences that apply to the model under discussion: the first one governs the dynamics and indicates the presence of some gauge symmetry

\[
\begin{align*}
\Omega_2^{2k} &\xrightarrow{\bar{d}^2} \Omega_2^{2k+2} &\xrightarrow{\bar{d}} &\xrightarrow{} &\Omega_2^{2k+3} \\
t_{\mu_1\ldots\mu_k|\nu_1\ldots\nu_k} &\xrightarrow{} &F_{\mu_1\ldots\mu_k+1|\nu_1\ldots\nu_{k+1}} &\xrightarrow{} &\partial_{[\mu_1} F_{\mu_2\ldots\mu_{k+1}]}|\nu_1\ldots\nu_{k+1} = 0
\end{align*}
\]

(28)

while the second sequence solves the gauge symmetry

\[
\begin{align*}
\Omega_2^{2k-1} &\xrightarrow{\bar{d}^2} \Omega_2^{2k} &\xrightarrow{\bar{d}} &\xrightarrow{} &\Omega_2^{2k+2} \\
\epsilon_{\mu_1\ldots\mu_k|\nu_1\ldots\nu_{k-1}} &\xrightarrow{} &\delta_{\epsilon} t_{\mu_1\ldots\mu_k|\nu_1\ldots\nu_k} &\xrightarrow{} &\delta_{\epsilon} F_{\mu_1\ldots\mu_{k+1}|\nu_1\ldots\nu_{k+1}} = 0
\end{align*}
\]

(29)

Let us discuss the previous sequences. Starting from the tensor field $t_{\mu_1\ldots\mu_k|\nu_1\ldots\nu_k}$ from $\Omega_2^{2k}$, we can construct its curvature tensor $F_{\mu_1\ldots\mu_{k+1}|\nu_1\ldots\nu_{k+1}}$, defined via

\[
(\bar{d}^2 t)_{\mu_1\ldots\mu_{k+1}|\nu_1\ldots\nu_{k+1}} \sim F_{\mu_1\ldots\mu_{k+1}|\nu_1\ldots\nu_{k+1}} = \partial_{[\mu_1} t_{\mu_2\ldots\mu_{k+1}]}|\nu_2\ldots\nu_{k+1},\nu_1],
\]

(30)
which is second-order in the spacetime derivatives and belongs to $\Omega_2^{2k+2}$. Thus, the curvature tensor transforms in an irreducible representation of $\text{GL}(D, \mathbb{R})$ and exhibits the symmetries of the rectangular two-column Young diagram

$$F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} = \begin{pmatrix} \mu_1 & \nu_1 \\ \vdots & \vdots \\ \mu_{k+1} & \nu_{k+1} \end{pmatrix},$$

being separately antisymmetric in the first and respectively last $(k+1)$ indices, symmetric under the inter-change between the two sets of indices

$$F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} = F_{\nu_1 \cdots \nu_{k+1}|\mu_1 \cdots \mu_{k+1}},$$

and obeying the (algebraic) Bianchi I identity

$$F_{[\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}] \nu_2 \cdots \nu_{k+1} \equiv 0. $$

The action of $\bar{d}$ on $F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}}$ maps to zero

$$\left(\bar{d}^2 t\right)_{\mu_1 \cdots \mu_{k+2}|\nu_1 \cdots \nu_{k+1}} = \left(\bar{d}F\right)_{\mu_1 \cdots \mu_{k+2}|\nu_1 \cdots \nu_{k+1}} \sim \partial_{[\mu_1} F_{\mu_2 \cdots \mu_{k+2}|\nu_1 \cdots \nu_{k+1}} \equiv 0,$$

and represents nothing but the (differential) Bianchi II identity for the curvature tensor. Since the curvature tensor and its traces are the most general non-vanishing second-order derivative quantities in $\Omega_2 (\mathcal{M})$ constructed from $t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k}$, we expect that the E.L. derivatives of the action, completely rely on it. The formula (27) shows that the corresponding field equations cannot be all independent, but satisfy some Noether identities related to the Bianchi II identity of the curvature tensor. This already points out that the searched for free Lagrangian action must be invariant under a certain gauge symmetry. The second sequence, namely (29), gives the form of the gauge invariance. As the free field equations involve $F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}}$, it is natural to require that these are the most general gauge invariant quantities

$$\delta_\epsilon \left(\bar{d}^2 t\right)_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} \sim \delta_\epsilon F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} = 0. $$

This matter is immediately solved if we take

$$\left(\bar{d}\epsilon\right)_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} \sim \epsilon_{\mu_1 \cdots \mu_k|\nu_2 \cdots \nu_k} + \epsilon_{\nu_1 \cdots \nu_k|\mu_2 \cdots \mu_1} = \delta_\epsilon t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k},$$

\(11\)
where the gauge parameters $\epsilon_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-1}}$ pertain to $\Omega_2^{2k-1}$, because, on account of the third-order nilpotency of $\bar{d}$, we find that

$$\delta_{\mu} F_{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}} \sim (\bar{d}^3 \epsilon)_{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}} \equiv 0. \quad (37)$$

Clearly, the relation (36) coincides with the gauge transformations (9).

### 3.2 Lagrangian action

We complete our discussion by exemplifying the construction of the free field equations. Let us denote by $S[t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}]$ a free, second-order derivative action that is gauge invariant under (9), and by $\delta S^0 / \delta t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}$ its functional derivatives with respect to the fields, which are imposed to depend linearly on the undifferentiated curvature tensor. Then, as these functional derivatives must have the same mixed symmetry like $t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}$, it follows that they necessarily determine a tensor from $\Omega_2^{2k}$. The operations that can be performed with respect to the curvature tensor in order to reduce its number of indices without increasing its derivative order is to take its (multiple) traces

$$F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \sigma^{\mu_1 \nu_1} \cdots \sigma^{\mu_{k+1} \nu_{k+1}} F_{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}} \in \Omega_2^{2(k-m)}, \quad m = 0, k, \quad (38)$$

being understood that we maintain the conventions (9). By direct computation, from (30) we get that the various traces of the curvature tensor have the expressions

$$F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \Box t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + (-)^k \partial^\rho (\partial_{[\mu_1} t_{\mu_2 \cdots \mu_k] | \nu_1 \cdots \nu_k}$$

$$+ \partial_{[\nu_1} t_{\nu_2 \cdots \nu_k] | \mu_1 \cdots \mu_k} + \partial_{[\nu_1} t_{\mu_2 \cdots \mu_k] | \nu_2 \cdots \nu_k | \mu_1], \quad (39)$$

$$F_{\mu_1 \cdots \mu_{k-m} | \nu_1 \cdots \nu_{k-m}} = (m+1) \Box t_{\mu_1 \cdots \mu_{k-m} | \nu_1 \cdots \nu_{k-m} + \partial_{[\mu_1} t_{\mu_2 \cdots \mu_{k-m}] | \nu_2 \cdots \nu_{k-m} | \nu_1 \cdots \nu_{k-m}, \nu_1}$$

$$+ (-)^{k+m} (m+1) \partial^\rho (\partial_{[\mu_1} t_{\mu_2 \cdots \mu_{k-m}] | \nu_1 \cdots \nu_{k-m} + \partial_{[\nu_1} t_{\nu_2 \cdots \nu_{k-m}} | \nu_1 \cdots \mu_{k-m}])$$

$$- m (m+1) \partial^\rho \partial^\lambda t_{\mu_1 \cdots \mu_{k-m} | \nu_1 \cdots \nu_{k-m} \lambda}, \quad m = 0, k-1, \quad (40)$$

$$F = (k+1) \Box t - k (k+1) \partial^\rho \partial^\lambda t_{\rho | \lambda}, \quad (41)$$

where $t_{\rho | \lambda}$ is a symmetric two-tensor. The only (linear) combinations formed with these quantities that belong to $\Omega_2^{2k}$ are generated by $F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}$ and
\[
\left( M_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \right)_{m=1,k}, \text{ where}
\]
\[
\left( m \right)_{\mu_1 \cdots \mu_k} M_{\nu_1 \cdots \nu_k} \equiv \delta_{\nu_1}^{[\mu_1} \cdots \delta_{\nu_{m+1}}^{\mu_{m+1} \cdots \mu_k]} F_{\nu_{m+1} \cdots \nu_k], \quad m = 1, k, \tag{42}
\]
so in principle \( \delta S^L / \delta t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) can be written as a linear combination of these objects with coefficients that are real constants
\[
\frac{\delta S^L}{\delta t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} = c_1 F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + \sum_{m=1}^{k} c_{m+1} \left( m \right)_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} . \tag{43}
\]
According to our antisymmetrization convention, the right-hand side of (42) contains only the independent terms, in number of \( (k! / (k - m)!)^2 / m! \). However, the requirement that the above linear combination indeed stands for the functional derivatives of a sole functional restricts the parametrization of the functional derivatives by means of one constant only
\[
c_{m+1} = \frac{(-)^m}{(m+1)!} c_1, \quad m = 1, k, \tag{44}
\]
so we finally find that
\[
\frac{\delta S^L}{\delta t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} = c_1 \left( F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + \sum_{m=1}^{k} \frac{(-)^m}{(m+1)!} \left( m \right)_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \right) , \tag{45}
\]
where \( \left( m \right)_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) are defined in (42). By taking into account the formulas (39–41) we observe that from (45) we precisely recovery the Lagrangian action (12) together with the field equations (25). The E.L. derivatives (45) coincide with (26) up to the numerical factor \( c_1 \). This also allows us to identify the expression of \( T_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) from (25–26) in terms of the curvature tensor like
\[
T_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + \sum_{m=1}^{k} \frac{(-)^m}{(m+1)!} \left( m \right)_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} . \tag{46}
\]

3.3 Relationship with the curvature tensor

At this point, we can easily see the relationship of the field equations (25) and their Noether identities (27) with the curvature tensor (30) and accompanying Bianchi II identity (34). First, we observe that the field equations
are completely equivalent with the vanishing of the simple trace of the curvature tensor

\[ T_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \approx 0 \iff F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \approx 0. \]  

(47)

The direct statement holds due to the fact that \( T_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) is expressed only through \( F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) and its traces, such that its vanishing implies \( F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \approx 0 \). The converse implication holds because the vanishing of the components \( M_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) in the right-hand side of (46) is a simple consequence of \( F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \approx 0 \). Second, the Noether identities (27) are a direct consequence of the Bianchi II identity for the curvature tensor

\[ \partial_{[\mu_1} F_{\rho_2\nu_1 \cdots \nu_k]}|_{\lambda_1 \lambda_2 \mu_1 \cdots \mu_{k-1}} \equiv 0 \implies \partial^\mu T_{\mu_1 \cdots \mu_{k-1}| \nu_1 \cdots \nu_k} \equiv 0. \]  

(48)

Indeed, on the one hand the relation (46) yields

\[ \partial_{\mu} T^{\mu_1 \cdots \mu_{k-1}| \nu_1 \cdots \nu_k} = \partial_{\mu} F^{\mu_1 \cdots \mu_{k-1}| \nu_1 \cdots \nu_k} - \frac{1}{2} \partial_{\nu_1} F_{\nu_2 \cdots \nu_k}^{\mu_1 \cdots \mu_{k-1}} \]

\[ + \sum_{m=2}^{k} \left( \frac{1}{m!} \delta_{[\mu_1} \cdots \delta_{\nu_{m-1}] (\cdots (-)^{k+m} F_{\mu_{m+1} \cdots \mu_{k-1]}_{\nu_{m+1} \cdots \nu_k}]_{m} \right) \right). \]  

(49)

On the other hand, straightforward computation leads to

\[ \sigma^{\rho_1 \lambda_1} \sigma^{\rho_2 \lambda_2} \partial_{[\rho_1} F_{\rho_2\nu_1 \cdots \nu_k}|_{\lambda_1 \lambda_2 \mu_1 \cdots \mu_{k-1}} = \]

\[ -2 \left( \partial^\mu F_{\mu_1 \cdots \mu_{k-1}| \nu_1 \cdots \nu_k} - \frac{1}{2} \partial_{\nu_1} F_{\nu_2 \cdots \nu_k}^{\mu_1 \cdots \mu_{k-1}} \right). \]  

(50)

\[ \sigma^{\rho_1 \lambda_1} \sigma^{\rho_2 \lambda_2} \sigma^{\nu_1 \mu_1} \cdots \sigma^{\nu_{m-1} \mu_{m-1}} \partial_{[\rho_1} F_{\rho_2\nu_1 \cdots \nu_k}|_{\lambda_1 \lambda_2 \mu_1 \cdots \mu_{k-1}} = \]

\[ (-)^m (m + 1) \left( \partial^\mu F_{\mu_{m+1} \cdots \mu_{k-1}| \nu_{m+1} \cdots \nu_k} - \frac{1}{m + 1} \partial_{\nu_{m+1}} F_{\nu_{m+1} \cdots \nu_k}^{\mu_{m+1} \cdots \mu_{k-1}} \right). \]  

(51)

for all \( m = 2, k \). Thus, according to (50), we can state that the Bianchi II identity for the curvature tensor implies the identically vanishing of the right-hand side of (49), and hence enforces the Noether identities (27) for the action (12).
3.4 Generalized cohomology of the 3-complex

Next, we point out the relation between the generalized cohomology of the 3-complex $\Omega_2(M)$ and our model. The generalized cohomology of the 3-complex $\Omega_2(M)$ is given by the family of graded vector spaces $H_m(\bar{d}) = \text{Ker}(\bar{d}^m)/\text{Im}(\bar{d}^{3-m})$, with $m = 1, 2$. Each vector space $H_m(\bar{d})$ splits into the cohomology spaces $H^p_m(\Omega_2(M))$, defined like the equivalence classes of tensors from $\Omega^2_2(M)$ that are $\bar{d}^m$-closed, with any two such tensors that differ by a $\bar{d}^{3-m}$-exact element in the same equivalence class. The spaces $H^p_m$ are not empty in general, even if $M$ has a trivial topology. However, in the case where $M$ (assumed to be of dimension $D$) has the topology of $\mathbb{R}^D$, the generalized Poincaré lemma \cite{24} applied to our situation states that the generalized cohomology of the 3-differential $\bar{d}$ on tensors represented by rectangular diagrams with two columns is empty in the space $\Omega_2(\mathbb{R}^D)$ of maximal two-column tensors, $H^{2n}_{(m)}(\Omega_2(\mathbb{R}^D)) = 0$, for $1 \leq n \leq D - 1$ and $m = 1, 2$. In particular, for $n = k + 1$ and $m = 1$ we find that $H_{(1)}^{2k+2}(\Omega_2(\mathbb{R}^D)) = 0$ and thus, if the tensor $F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}}$ with the mixed symmetry \cite{31} of the curvature tensor is $\bar{d}$-closed, then it is also $\bar{d}^2$-exact. To put it otherwise, if this tensor satisfies the Bianchi II identity $\partial_{[\mu_1} F_{\mu_2 \cdots \mu_{k+2}]}|\nu_1 \cdots \nu_{k+1} \equiv 0$, then there exists an element $t_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k}$ with the mixed symmetry \cite{11}, with the help of which $F_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}}$ can precisely be written like in \cite{30}.

Finally, we observe that the formula \cite{46} relates the functions defining the free field equations \cite{25} to the curvature tensor by a generalized Hodge-duality. The generalized cohomology of $\bar{d}$ on $\Omega_2(M)$ when $M$ has the trivial topology of $\mathbb{R}^D$ together with this type of generalized Hodge-duality reveal many important features of the free model under study. For example, if $\bar{T}_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k}$ is a covariant tensor field with the mixed symmetry of the rectangular two-column Young diagram \cite{11} and satisfies the equation

$$\partial^\mu \bar{T}_{\mu_1 \cdots \mu_{k-1}|\nu_1 \cdots \nu_k} = 0,$$

then there exists a tensor $\bar{\Phi}_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} \in \Omega_2(\mathbb{R}^D)$ with the mixed symmetry of the curvature (of the rectangular Young diagram in \cite{31}), in terms of which

$$\bar{T}_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} = \partial^{\mu_{k+1}} \partial^{\nu_{k+1}} \bar{\Phi}_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} + c \sigma_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k},$$

with $c$ an arbitrary real constant and $\sigma_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k}$ being defined by the complete antisymmetrization of the product $\sigma_{\mu_1 \nu_1} \cdots \sigma_{\mu_k \nu_k}$ over the indices.
\{\mu_1, \cdots, \mu_k\}, which contains, according to our antisymmetrization convention, precisely \(k!\) terms. It is easy to check the above statement in connection with the functions (26) that define the field equations for the model under consideration. Indeed, direct computation provides \(c = 0\) and

\[ T_{\mu_1 \cdots \mu_k|\nu_1 \cdots \nu_k} = \partial^{\mu_{k+1}} \partial^{\nu_{k+1}} \Phi_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}}, \tag{54} \]

where

\[ \Phi_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}} = \sum_{m=0}^{k} \frac{(-)^m}{m!} \delta^{[\mu_1 \cdots \mu_m|\nu_1 \cdots \nu_m]}_{[\nu_{m+1} \cdots \nu_{k+1}]} \eta^{\mu_{m+1} \cdots \mu_{k+1}|\nu_{m+1} \cdots \nu_{k+1}} \delta^{\mu_{k+1}|\nu_{k+1}}, \tag{55} \]

such that the corresponding \(\Phi_{\mu_1 \cdots \mu_{k+1}|\nu_1 \cdots \nu_{k+1}}\) indeed displays the mixed symmetry (31) of the curvature tensor. We note that we employed the conventions (6), such that the element corresponding to \(m = 0\) in the right-hand side of (55) is

\[ t^{[\mu_1 \cdots \nu_k]}_{[\nu_1 \cdots \nu_k]} \delta^{\mu_{k+1}|\nu_{k+1}} \tag{56} \]

and it contains precisely \((k + 1)^2\) terms, while the element associated with \(m = k\) reads

\[ \frac{(-)^k}{k!} \delta_{[\nu_1 \cdots \nu_k]}^{[\mu_1 \cdots \mu_k|\nu_{m+1} \cdots \nu_{k+1}} \delta^{\mu_{k+1}|\nu_{k+1}}, \tag{57} \]

and it involves only \((k + 1)!\) terms. In general, the number of terms in

\[ \delta^{[\mu_1 \cdots \mu_m|\nu_1 \cdots \nu_m]}_{[\nu_{m+1} \cdots \nu_{k+1}} \delta^{\mu_{k+1}|\nu_{k+1}} \tag{58} \]

for a given value of \(m\) is equal to \(((k + 1)!/(k - m)!)^2/(m + 1)!\).

4 BRST symmetry

4.1 Construction of the differential BRST complex

In agreement with the general setting of the antibracket-antifield formalism, the construction of the BRST symmetry for the free theory under consideration starts with the identification of the BRST algebra on which the BRST differential \(s\) acts. The generators of the BRST algebra are of two kinds: fields/ghosts and antifields. The ghost spectrum for the model under study comprises the tensor fields

\[ \left( \begin{array}{c} \eta \vspace{0.1in} \end{array} \right) \left( \begin{array}{c} \mu_1 \cdots \mu_k|\nu_1 \cdots \nu_{k-m} \end{array} \right)_{m=0,k-1}, \tag{59} \]
with the Grassmann parities
\[ \varepsilon \left( \begin{array}{c} m \\ \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} \end{array} \right) = (m + 1) \mod 2, \; m = 0, k - 1. \] (60)

The fermionic ghosts \( \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-1}} \) are associated with the gauge parameters \( \varepsilon_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-1}} \) from the transformations (9), while the rest of the ghosts are due to the reducibility parameters of order \( m \) that appear in the relations (10) and (18). In order to make compatible the behaviour of the ghosts with that of the gauge and reducibility parameters, we ask that
\[ \left( \begin{array}{c} m \\ \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-1}} \end{array} \right) \]
for \( m = 0, k - 2 \) display the mixed symmetry \( (k, k - m - 1) \) of the two-column Young diagrams (10) and (22), so they are separately antisymmetric in the first \( k \) and respectively last \( (k - m - 1) \) indices and satisfy the identities
\[ \left( \begin{array}{c} m \\ \eta_{| \mu_1 \cdots \mu_k | \nu_2 \cdots \nu_{k-m-1}} \end{array} \right) \equiv 0, \; m = 0, k - 2 \] (61)
while the ghost for ghost \( \eta_{\mu_1 \cdots \mu_k} \) is completely antisymmetric. The antifield spectrum is organized into the antifields
\[ t^{*\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}, \left( \begin{array}{c} m \\ \eta^{*\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} \end{array} \right) \] (m = 0, k - 1) \] (62)
corresponding to the original tensor field and to the ghosts, of statistics opposite to that of the associated fields/ghosts
\[ \varepsilon \left( t^{*\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \right) = 1, \; \varepsilon \left( \begin{array}{c} m \\ \eta^{*\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}} \end{array} \right) = m \mod 2. \]

Obviously, the antifields exhibit the same mixed properties like the associated field/ghosts, so they are separately antisymmetric in the first \( k \) and respectively last \( k \) or \( (k - m - 1) \) indices and satisfy the identities
\[ t^{[\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \equiv 0, \; \left( \begin{array}{c} m \\ \eta^{[\mu_1 \cdots \mu_k | \nu_2 \cdots \nu_{k-m-1}} \end{array} \right) \equiv 0, \; m = 0, k - 2, \] (63)
while \( \eta^{(k-1)*\mu_1 \cdots \mu_k} \) is completely antisymmetric. In addition, \( t^{*\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) is symmetric under the inter-change between the two sets of indices
\[ t^{*\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = t^{*\nu_2 \cdots \nu_k | \mu_1 \cdots \mu_k}. \] (64)
We will denote the various traces of $t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}$ by
\[ t^{\mu_{m+1} \cdots \mu_k | \nu_{m+1} \nu_2 \cdots \nu_k} = \sigma_{\mu_1 \nu_1} \cdots \sigma_{\mu_m \nu_m} t^{\mu_1 \cdots \mu_k | \nu_1 \nu_2 \cdots \nu_k}, \quad m = 1, k, \] (65)
being understood that we maintain the conventions (6).

As both the gauge generators and reducibility functions for this model are field-independent, it follows that the associated BRST differential ($s^2 = 0$) splits into
\[ s = \delta + \gamma, \] (66)
where $\delta$ represents the Koszul-Tate differential ($\delta^2 = 0$), graded by the antighost number $\text{agh}$ ($\text{agh} (\delta) = -1$), while $\gamma$ stands for the exterior derivative along the gauge orbits and turns out to be a true differential ($\gamma^2 = 0$) that anticommutes with $\delta$ ($\delta \gamma + \gamma \delta = 0$), whose degree is named pure ghost number $\text{pgh}$ ($\text{pgh} (\gamma) = 1$). These two degrees do not interfere ($\text{agh} (\gamma) = 0$, $\text{pgh} (\delta) = 0$). The overall degree that grades the BRST differential is known as the ghost number ($\text{gh}$) and is defined like the difference between the pure ghost number and the antighost number, such that $\text{gh} (s) = \text{gh} (\delta) = \text{gh} (\gamma) = 1$. According to the standard rules of the BRST method, the corresponding degrees of the generators from the BRST complex are valued like
\[ \text{pgh} (t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}) = 0, \quad \text{pgh} \left( \frac{\eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}}}{\eta} \right) = m + 1, \] (67)
\[ \text{pgh} (t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}) = \text{pgh} \left( \frac{\eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}}}{\eta} \right) = 0, \] (68)
\[ \text{agh} (t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}) = \text{agh} \left( \frac{\eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}}}{\eta} \right) = 0, \] (69)
\[ \text{agh} (t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}) = 1, \quad \text{agh} \left( \frac{\eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}}}{\eta} \right) = m + 2, \] (70)
with $m = 0, k - 1$, while the actions of $\delta$ and $\gamma$ on them are given by
\[ \gamma t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \left( \begin{array}{c} 0 \\ \eta_{\mu_1 \cdots \mu_k | \nu_2 \cdots \nu_k, \nu_1} \end{array} \right) + \left( \begin{array}{c} 0 \\ \eta_{\nu_1 \cdots \nu_k | [\mu_2 \cdots \mu_k, \mu_1]} \end{array} \right), \] (71)
\[ \gamma \left( \frac{\eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}}}{\eta} \right) = \partial_1 \left( \frac{\eta_{\mu_2 \cdots \mu_k | \nu_1 \nu_2 \cdots \nu_{k-m-1}}}{\eta} \right) \quad \text{for} \quad m = 0, k - 2 \] (72)
\[\gamma \eta_{\mu_1 \cdots \mu_k} = 0,\]  
\[\gamma t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = 0, \quad \gamma (m) = 0, \quad m = 0, k - 1,\]  
\[\delta t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = 0, \quad \delta (m) \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k - m} = 0, \quad m = 0, k - 1,\]  
\[\delta t^{(m)}_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = -c_1 T^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k},\]  
\[\delta (m) \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k - m} = -2k \partial_\rho t^{(m)}_{\mu_1 \cdots \mu_k | \rho \nu_1 \cdots \nu_k - 1},\]  
\[(-)^{k-m} (k - m) (m + 2) \partial_\rho (m) \eta_{\mu_1 \cdots \mu_k | \rho \nu_1 \cdots \nu_k - m} = 0, \quad m = 1, k - 1,\]  
with \[T^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}\] resulting from (71) and both \(\delta\) and \(\gamma\) taken to act like right derivations.

The antifield-BRST differential is known to admit a canonical action in a structure named antibracket and defined by decreeing the fields/ghosts conjugated with the corresponding antifields, \(s^\ast = (\cdot, S)\), where \((\cdot, S)\) signifies the antibracket and \(S\) denotes the canonical generator of the BRST symmetry. It is a bosonic functional of ghost number zero involving both the field/ghost and antifield spectra, which obeys the classical master equation

\[(S, S) = 0.\]  

The classical master equation is equivalent with the second-order nilpotency of \(s\), \(s^2 = 0\), while its solution encodes the entire gauge structure of the associated theory. Taking into account the formulas (71–78), as well as the actions of \(\delta\) and \(\gamma\) in canonical form, we find that the complete solution to the master equation for the model under study reads as

\[S = S^L \left[ t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \right] + \int d^p x \left[ t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} (0) \eta_{\mu_1 \cdots \mu_k | \nu_2 \cdots \nu_k, \nu_1} \right]
+ \left[ \eta_{\nu_2 \cdots \nu_k | \mu_2 \cdots \mu_k, \mu_1} \right]
+ \sum_{m=0}^{k-2} \left[ (m) t^{\mu_1 \cdots \mu_k | \nu_2 \cdots \nu_{k-m-1}} (m+1) \eta_{\mu_2 \cdots \mu_k | \nu_{1} | \nu_2 \cdots \nu_{k-m-1}} \right]
+ \left[ (-)^{k+1} (m + 2) \eta_{\mu_1 \cdots \mu_k | \nu_2 \cdots \nu_{k-m-1}, \nu_1} \right].\]
The main ingredients of the antifield-BRST symmetry derived in this section will be useful in the sequel at the analysis of the BRST cohomology for the free, massless tensor field \((k, k)\).

### 4.2 Cohomology of the exterior longitudinal derivative and related matters

The main aim of this paper is to study of the local cohomology \(H(s|d)\) in form degree \(D\) \((D \geq 2k + 1)\). As it will be further seen, an indispensable ingredient in the computation of \(H(s|d)\) is the cohomology algebra of the exterior longitudinal derivative \((H(\gamma))\). It is defined by the equivalence classes of \(\gamma\)-closed non-integrated densities \(a\) of fields, ghosts, antifields and their spacetime derivatives, \(\gamma a = 0\), modulo \(\gamma\)-exact terms. If \(a \in H(\gamma)\) is \(\gamma\)-exact, \(a = \gamma b\), then \(a\) belongs to the class of the element zero and we call it \(\gamma\)-trivial. In other words, the solution to the equation \(\gamma a = 0\) is unique up to \(\gamma\)-trivial objects, \(a \rightarrow a + \gamma b\). The cohomology algebra \(H(\gamma)\) inherits a natural grading \(H(\gamma) = \bigoplus_{l \geq 0} H_l(\gamma)\), where \(l\) is the pure ghost number. Let \(a\) be an element of \(H(\gamma)\) with definite pure ghost number, antighost number and form degree (deg)

\[
\gamma a = 0, \quad \text{pgh}(a) = l \geq 0, \quad \text{agh}(a) = j \geq 0, \quad \text{deg}(a) = p \leq D. \quad (81)
\]

In the sequel we analyze the general form of \(a\) with the above properties with the help of the definitions (71–74).

The formula (74) shows that all the antifields \(\chi^\ast \Delta \equiv \left( t^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}, \left( \frac{(m)^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \cdots m - 1}}{n} \right)_{m=0,k-1} \right)\), and their spacetime derivatives belong (non-trivially) to \(H^0(\gamma)\). From (71) we observe that \(\gamma\) acts on the original tensor field through a gauge transformation \(\Delta\) with the gauge parameters replaced by the ghosts, such that the \(\gamma\)-closed quantities constructed out of the tensor gauge field \(t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}\) and its space-time derivatives are nothing but the gauge-invariant objects of the theory \(\Delta\). As it was discussed in Section 3, the only such objects are the curvature tensor \(F_{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}}\) and its derivatives, and thus they all belong to \(H^0(\gamma)\). With the help of the definitions in (72) one prove the following theorem.
Theorem 4.1 The cohomology spaces $H^l(\gamma)$ of the exterior longitudinal derivative in pure ghost number $1 \leq l \leq k - 1$ for the free, massless tensor field $t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k}$ are all vanishing

$$H^l(\gamma) = 0, \ 1 \leq l \leq k - 1.$$  

(83)

Proof The proof is purely computational and essentially relies on the fact that the most general $\gamma$-closed quantities that are linear in the ghosts $(m) \eta_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}$ are also $\gamma$-exact. Since the general case is intricate and yet not illuminating, we give below the detailed proof in the case $k = 3$. The ghost spectrum comprises the ghosts $\eta_{\mu \nu \rho | \alpha \beta}^{(0)}$, $\eta_{\mu \nu \rho | \alpha}^{(1)}$ and $\eta_{\mu \nu \rho}^{(2)}$, on which $\gamma$ acts through

$$\gamma t_{\mu \nu \rho | \alpha \beta \gamma} = (0) \eta_{\mu \nu \rho | \beta \gamma, \alpha} + (0) \eta_{\alpha \beta \gamma | [\nu \rho, \mu]},$$  

(84)

$$\gamma \eta_{\mu \nu \rho | \alpha \beta} = (1) \partial_\mu \eta_{\nu \rho | [\alpha | \beta]} + 2 (1) \eta_{\mu \nu \rho | [\beta, \alpha]},$$  

(85)

$$\gamma \eta_{\mu \nu \rho | \alpha} = (2) \partial_\mu \eta_{\nu \rho | \alpha} + 3 \partial_\alpha (2) \eta_{\mu \nu \rho},$$  

(86)

$$\gamma \eta_{\mu \nu \rho} = 0.$$  

(87)

Using the formula (83), we notice that there is no $\gamma$-closed linear combination of the undifferentiated ghosts of pure ghost number one $\eta_{\mu \nu \rho | \alpha \beta}^{(0)}$. Next, we analyze the presence of $\gamma$-closed linear combinations involving the first-order derivatives of the pure ghost number one ghosts. Taking into account the identity $\eta_{[\mu \nu \rho | \alpha \beta]}^{(0)} = 0$, it follows that the general expression of the linear combination involving the first-order derivatives of the pure ghost number one ghosts contains 30 real constants and reads as

$$A_{\mu_1 \ldots \mu_6} = \partial_{\mu_1} \left( k_1 (0) \eta_{\mu_2 \mu_4 \mu_6 | \mu_3 \mu_5} + k_2 (0) \eta_{\mu_3 \mu_4 \mu_6 | \mu_2 \mu_5} + k_3 (0) \eta_{\mu_2 \mu_5 \mu_6 | \mu_3 \mu_4} + k_4 (0) \eta_{\mu_3 \mu_5 \mu_6 | \mu_2 \mu_4} + k_5 (0) \eta_{\mu_4 \mu_5 \mu_6 | \mu_2 \mu_3} \right) + \partial_{\mu_2} \left( k_6 (0) \eta_{\mu_1 \mu_4 \mu_6 | \mu_3 \mu_5} + k_7 (0) \eta_{\mu_3 \mu_4 \mu_6 | \mu_1 \mu_5} + k_8 (0) \eta_{\mu_1 \mu_5 \mu_6 | \mu_3 \mu_4} + k_9 (0) \eta_{\mu_3 \mu_5 \mu_6 | \mu_1 \mu_4} + k_{10} (0) \eta_{\mu_4 \mu_5 \mu_6 | \mu_1 \mu_3} \right) + \partial_{\mu_3} \left( k_{11} (0) \eta_{\mu_1 \mu_4 \mu_6 | \mu_2 \mu_5} + k_{12} (0) \eta_{\mu_2 \mu_4 \mu_6 | \mu_1 \mu_5} \right)$$

(87)
Requiring that $\gamma A_{\mu_1 \cdots \mu_6} = 0$, we find a homogeneous algebraic system of 50 equations with 30 variables, whose rank is equal to 28. The solution to this system, expressed in terms of the two independent constants, taken for instance to be $k_{29}$ and $k_{30}$, is

\begin{align}
  k_1 & = k_2 = k_4 = k_6 = k_7 = k_{11} = 0, \\
  k_{12} & = k_{13} = k_{16} = k_{17} = k_{22} = k_{27} = 0, \\
  k_9 & = k_{14} = k_{21} = -k_3 = -k_8 = -k_{20} = k_{29} + k_{30}, \\
  k_5 & = k_{15} = k_{18} = k_{20} = k_{24} = -k_{29}, \\
  k_{10} & = k_{19} = k_{23} = k_{25} = -k_{28} = -k_{30}.
\end{align}

so we find that

\begin{equation}
  A_{\mu_1 \cdots \mu_6} = - ( (k_{29} + k_{30}) A_{\mu_1 \mu_3 \mu_4 | \mu_2 \mu_5 \mu_6} + k_{29} A_{\mu_1 \mu_2 \mu_3 | \mu_4 \mu_5 \mu_6} ),
\end{equation}

where $A_{\mu_1 \mu_2 \mu_3 | \mu_4 \mu_5 \mu_6}$ has the mixed symmetry $(3, 3)$ and is given by

\begin{equation}
  A_{\mu_1 \mu_2 \mu_3 | \mu_4 \mu_5 \mu_6} = (0) \eta_{\mu_1 \mu_2 \mu_3 | | \mu_5 \mu_6, \mu_4} + (0) \eta_{\mu_4 \mu_5 \mu_6 | | \mu_2 \mu_3, \mu_1}.
\end{equation}

It is easy to see from formula (84) that $A_{\mu_1 \mu_2 \mu_3 | \mu_4 \mu_5 \mu_6}$ is $\gamma$-exact

\begin{equation}
  A_{\mu_1 \mu_2 \mu_3 | \mu_4 \mu_5 \mu_6} = \gamma (t_{\mu_1 \mu_2 \mu_3 | \mu_4 \mu_5 \mu_6}),
\end{equation}

and thus it (and also $A_{\mu_1 \cdots \mu_6}$) must be discarded from $H^1(\gamma)$ as being trivial. Along the same line, one can prove that the only $\gamma$-closed combinations with
\( N \geq 2 \) spacetime derivatives of the ghosts \( (0)_{\mu\nu\rho|\alpha\beta} \) are actually polynomials with \((N - 1)\) derivatives in the elements \( A_{\mu_1\mu_2\mu_3|\mu_4\mu_5\mu_6} \), which, by means of (96), are \( \gamma \)-exact, and hence trivial in \( H^1(\gamma) \). In conclusion, there is no non-trivial object constructed out of the ghosts \( (0)_{\mu\nu\rho|\alpha\beta} \) and their derivatives in \( H^1(\gamma) \), which implies that

\[
H^1(\gamma) = 0 \quad \text{for} \quad k = 3,
\]

as there are no other ghosts of pure ghost number equal to one in the BRST complex.

With the help of the definition (86), we notice that there is no \( \gamma \)-closed linear combination of the undifferentiated ghosts of pure ghost number two \( (1)_{\mu\nu\rho|\alpha} \). Now, we pass to the determination of \( \gamma \)-closed linear combinations involving the first-order derivatives of the pure ghost number two ghosts. By means of the identity \( (1)_{\mu\nu\rho|\alpha} \equiv 0 \), we get that the general expression of the linear combination involving the first-order derivatives of the pure ghost number two ghosts contains 15 real constants and has the form

\[
B_{\mu_1\ldots\mu_5} = \partial_{\mu_1} \left( m_1 (1)_{\mu_2\mu_3\mu_4|\mu_5} + m_2 (1)_{\mu_2\mu_3\mu_5|\mu_4} + m_3 (1)_{\mu_2\mu_4\mu_5|\mu_3} \right)
+\partial_{\mu_2} \left( m_4 (1)_{\mu_1\mu_3\mu_4|\mu_5} + m_5 (1)_{\mu_1\mu_3\mu_5|\mu_4} + m_6 (1)_{\mu_1\mu_4\mu_5|\mu_3} \right)
+\partial_{\mu_3} \left( m_7 (1)_{\mu_1\mu_2\mu_4|\mu_5} + m_8 (1)_{\mu_1\mu_2\mu_5|\mu_4} + m_9 (1)_{\mu_1\mu_5\mu_5|\mu_2} \right)
+\partial_{\mu_4} \left( m_{10} (1)_{\mu_1\mu_2\mu_5|\mu_5} + m_{11} (1)_{\mu_1\mu_2\mu_5|\mu_4} + m_{12} (1)_{\mu_1\mu_3\mu_5|\mu_2} \right)
+\partial_{\mu_5} \left( m_{13} (1)_{\mu_1\mu_2\mu_3|\mu_4} + m_{14} (1)_{\mu_1\mu_2\mu_4|\mu_3} + m_{15} (1)_{\mu_1\mu_3\mu_4|\mu_2} \right),
\]

(98)

The requirement \( \gamma B_{\mu_1\ldots\mu_5} = 0 \) leads to a homogeneous algebraic system of 10 equations with 15 variables, whose rank is equal to 5. The solution to this system, expressed in terms of the five independent constants, taken for instance to be \( (m_i)_{i=11,15} \) is

\[
m_1 = - \frac{2m_{11}}{3} - \frac{m_{12}}{3} - \frac{m_{13}}{2} + \frac{5m_{14}}{6} + \frac{m_{15}}{6},
\]

(99)

\[
m_2 = m_{11} + \frac{m_{13}}{2} - \frac{m_{14}}{2} - \frac{m_{15}}{2},
\]

(100)
\[ m_3 = -m_{11} - m_{12} + m_{14} + m_{15}, \] (101)
\[ m_4 = \frac{2m_{11}}{3} + \frac{m_{12}}{3} + \frac{m_{13}}{2} - \frac{5m_{14}}{6} - \frac{7m_{15}}{6}, \] (102)
\[ m_5 = -m_{11} - m_{12} - \frac{m_{13}}{2} + \frac{m_{14}}{2} + \frac{m_{15}}{2}, \] (103)
\[ m_6 = m_{11} - m_{14}, \quad m_0 = m_{12} - m_{15}, \] (104)
\[ m_7 = \frac{m_{11}}{3} + \frac{2m_{12}}{3} - \frac{m_{13}}{2} - \frac{7m_{14}}{6} - \frac{5m_{15}}{6}, \] (105)
\[ m_8 = -m_{11} - m_{12} + \frac{m_{13}}{2} + \frac{m_{14}}{2} + \frac{m_{15}}{2}, \] (106)
\[ m_{10} = -\frac{m_{11}}{3} + \frac{m_{12}}{3} - m_{13} - \frac{m_{14}}{3} + \frac{m_{15}}{3}, \] (107)

which further yields

\[
B_{\mu_1 \cdots \mu_5} = \frac{1}{3} (2m_{14} - m_{11}) B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} + \frac{1}{6} (7m_{14} - 2m_{11}) B_{\mu_3 \mu_4 \mu_5 \mid \mu_1 \mu_2} \\
+ \frac{1}{6} (m_{15} - 2m_{12}) B_{\mu_1 \mu_2 \mu_3 \mid \mu_2 \mu_4} + \frac{1}{6} (m_{15} - 2m_{12}) B_{\mu_2 \mu_4 \mu_5 \mid \mu_1 \mu_3} \\
- \frac{m_{13}}{2} B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} - \frac{m_{14}}{2} B_{\mu_1 \mu_2 \mu_4 \mid \mu_3 \mu_5} - \frac{m_{15}}{2} B_{\mu_1 \mu_3 \mu_4 \mid \mu_2 \mu_5},
\] (108)

where \( B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} \) has the mixed symmetry \((3, 2)\) and reads as

\[
B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} = \partial_{[\mu_1} (1) \eta_{\mu_2 \mu_3] \mid \mu_4 \mu_5} + 2 (1) \eta_{\mu_1 \mu_2 \mu_3 \mid \mu_5 \mu_4].
\] (109)

On account of the definition (85) it results that \( B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} \) is \( \gamma \)-exact

\[
B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} = \gamma \left( (0) \eta_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} \right),
\] (110)

and thus it (as well as \( B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} \)) must be thrown out from \( H^2 (\gamma) \) as being trivial. In the meantime, it can be shown that the only \( \gamma \)-closed combinations with \( N \geq 2 \) spacetime derivatives of the ghosts \( \eta_{\mu_\nu \rho}^{(1)} \) are actually polynomials with \( (N - 1) \) derivatives in the elements \( B_{\mu_1 \mu_2 \mu_3 \mid \mu_4 \mu_5} \), which are \( \gamma \)-exact due to (110), and hence \( \gamma \)-trivial. This result allows us to state that there is no non-trivial object constructed out of the ghosts \( \eta_{\mu_\nu \rho}^{(1)} \) and their derivatives in \( H^2 (\gamma) \), so we have that

\[
H^2 (\gamma) = 0 \text{ for } k = 3,
\] (111)
on behalf of $\mathcal{M}$ and since there are no other ghosts of pure ghost number equal to two in the BRST complex.

Finally, we investigate the definition (73). It shows that the undifferentiated ghosts of pure ghost number equal to $k$, $\eta_{\mu_1 \cdots \mu_k}$, belong to $H(\gamma)$. The $\gamma$-closedness of $\eta_{\mu_1 \cdots \mu_k}$ further implies that all their derivatives are also $\gamma$-closed. Regarding their first-order derivatives, from the definition (72) for $m = k - 2$ we observe that their symmetric part is $\gamma$-exact

$$
\partial_{(\mu_1} \eta \eta_{\mu_2)\mu_3 \cdots \mu_{k+1}} \equiv \gamma \left( \frac{1}{k+1} \eta_{\mu_3 \cdots \mu_{k+1}(\mu_1|\mu_2)} \right),
$$

where $(\mu \nu \cdots)$ denotes plain symmetrization with respect to the indices between brackets without normalization factors, such that $\partial_{(\mu_1} \eta \eta_{\mu_2)\mu_3 \cdots \mu_{k+1}}$ will be removed from $H(\gamma)$. Meanwhile, their complete antisymmetric part $\partial_{[\mu_1} \eta \eta_{\mu_2 \cdots \mu_{k+1}]}$ is not $\gamma$-exact, and hence can be taken as a non-trivial representative of $H^k(\gamma)$. Actually, due to the relations

$$
\partial_{\mu_1} \eta_{\mu_2 \cdots \mu_{k+1}} = \frac{1}{k+1} \partial_{(\mu_1} \eta_{\mu_2)\mu_3 \cdots \mu_{k+1}} + \gamma \left( \frac{(-)^{k+1}}{k+1} \eta_{\mu_3 \cdots \mu_{k+1}|\mu_1} \right),
$$

$$
\frac{1}{2} \partial_{[\mu_1} \eta_{\mu_2] \mu_3 \cdots \mu_{k+1}} = \frac{1}{k+1} \partial_{(\mu_1} \eta_{\mu_2)\mu_3 \cdots \mu_{k+1}} + \gamma \left[ \frac{1}{k+1} \left( (-)^{k} \eta_{\mu_3 \cdots \mu_{k+1}|\mu_1} + \frac{1}{2} \eta_{\mu_3 \cdots \mu_{k+1} (\mu_1|\mu_2)} \right) \right],
$$

it is clear that $\partial_{\mu_1} \eta_{\mu_2 \cdots \mu_{k+1}}$ and $\frac{1}{2} \partial_{[\mu_1} \eta_{\mu_2] \mu_3 \cdots \mu_{k+1}}$ are in the same equivalence class from $H^k(\gamma)$

$$
\partial_{\mu_1} \eta_{\mu_2 \cdots \mu_{k+1}} \sim \frac{1}{2} \partial_{[\mu_1} \eta_{\mu_2] \mu_3 \cdots \mu_{k+1}} \sim \frac{1}{k+1} \partial_{[\mu_1} \eta_{\mu_2] \mu_3 \cdots \mu_{k+1}}.
$$

As a consequence, for $\mu_m \neq \mu_n$ with $m, n = 1, \cdots k + 1$ any of them can be used as a non-trivial representative of $H^k(\gamma)$. After some calculations, we
find that all the second-order derivatives of the ghosts \( (k-1) \eta_{\mu_1,\ldots,\mu_k} \) are \( \gamma \)-exact

\[
\partial_{\nu_1} \partial_{\nu_2} (k-1) \eta_{\mu_1,\ldots,\mu_k} = -\frac{1}{2k(k+1)} \gamma \left( \partial_{[\mu_1} (k-2) \eta_{\mu_2,\ldots,\mu_k]} (\nu_1|\nu_2) \right.
+ (-)^k (k+1) (k-2) \eta_{\mu_1,\ldots,\mu_k|(\nu_1,\nu_2)} \right),
\]

(116)

and so will be their higher-order derivatives, such that they all disappear from \( H(\gamma) \). In conclusion, the only non-trivial combinations in \( H(\gamma) \) constructed from the ghosts of pure ghost number equal to \( k \) are polynomials in \( (k-1) \eta_{\mu_1,\ldots,\mu_k} \) and \( \partial_{[\mu_1} (k-1) \eta_{\mu_2,\ldots,\mu_{k+1}]} \). Combining this result with the previous one on \( H^0(\gamma) \) being non-vanishing and with (83), we have actually proved that only the cohomological spaces \( H^{kl}(\gamma) \) with \( l \geq 0 \) are non-vanishing for the model under consideration or, equivalently, that

\[
H^{l'}(\gamma) = 0, \text{ for all } l' \neq kl.
\]

(117)

According to the results exposed so far, we can state that the general local solution to the equation (81) for \( pgh(a) = kl > 0 \) is, up to trivial, \( \gamma \)-exact contributions, of the type

\[
a = \sum_J \alpha_J \left( [\chi^{*\Delta}], [F_{\mu_1,\ldots,\mu_{k+1}|\nu_1,\ldots,\nu_{k+1}}] \right) e^J \left( (k-1) \eta_{\mu_1,\ldots,\mu_k}, \partial_{[\mu_1} (k-1) \eta_{\mu_2,\ldots,\mu_{k+1}]} \right),
\]

(118)

where the notation \( f([q]) \) means that the function \( f \) depends on the variable \( q \) and its subsequent derivatives up to a finite number. In the above, \( e^J \) are the elements of pure ghost number \( kl \) (and obviously of antighost number zero) of a basis in the space of polynomials in \( (k-1) \eta_{\mu_1,\ldots,\mu_k} \) and \( \partial_{[\mu_1} (k-1) \eta_{\mu_2,\ldots,\mu_{k+1}]} \)

\[
pgh(e^J) = kl > 0, \text{ agh}(e^J) = 0.
\]

(119)

The objects \( \alpha_J \) (obviously non-trivial in \( H^0(\gamma) \)) were taken to have a bounded number of derivatives, and therefore they are polynomials in the antifields \( \chi^{*\Delta} \), in the curvature tensor \( F_{\mu_1,\ldots,\mu_{k+1}|\nu_1,\ldots,\nu_{k+1}} \), as well as in their derivatives. In agreement with (81), they display the properties

\[
pgh(\alpha_J) = 0, \text{ agh}(\alpha_J) = j \geq 0, \text{ deg}(\alpha_J) = p \leq D.
\]

(120)
In the case \( l = 0 \), the general (non-trivial) local elements of \( H(\gamma) \) are precisely \( \alpha_J \left( [\chi^\Delta], [F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1}}] \right) \), which will be called “invariant polynomials” in what follows. At zero antighost number, the invariant polynomials are polynomials in the curvature tensor \( F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1}} \) and its derivatives.

In order to analyze the local cohomology \( H(s|d) \) we are going to need, besides \( H(\gamma) \), also the cohomology of the exterior spacetime differential \( H(d) \) in the space of invariant polynomials and other basic properties, which are addressed below.

**Theorem 4.2** The cohomology of \( d \) in form degree strictly less than \( D \) is trivial in the space of invariant polynomials with strictly positive antighost number. This means that the conditions

\[
\gamma \alpha = 0, \ d\alpha = 0, \ \text{agh}(\alpha) > 0, \ \deg \alpha < D, \ \alpha = \alpha \left( [\chi^\Delta], [F] \right), \tag{121}
\]

imply

\[
\alpha = d\beta, \tag{122}
\]

for some invariant polynomial \( \beta \left( [\chi^\Delta], [F] \right) \).

**Proof** In (121), the notation \( F \) signifies the curvature tensor, of components \( F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1}} \), and \( \chi^\Delta \) is explained in (82). Meanwhile, \( \deg \alpha \) is the form degree of \( \alpha \). In order to prove the theorem, we decompose \( d \) as

\[
d = d_0 + d_1, \tag{123}
\]

where \( d_1 \) acts on the antifields \( \chi^\Delta \) and their derivatives only, while \( d_0 \) acts on the curvature tensor and its derivatives

\[
d_0 = \partial^0_\mu dx^\mu, \ d_1 = \partial^1_\mu dx^\mu, \tag{124}
\]

with

\[
\partial^0_\mu = F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1},\mu} \frac{\partial}{\partial F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1}}} \partial + F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1},\mu\nu} \frac{\partial}{\partial F_{\mu_1\cdots\mu_{k+1}|\nu_1\cdots\nu_{k+1},\nu}} + \cdots, \tag{125}
\]

\[
\partial^1_\mu = \chi^\Delta,_{\mu} \frac{\partial^L}{\partial \chi^\Delta} + \chi^\Delta,_{\mu\nu} \frac{\partial^L}{\partial \chi^\Delta,_{\nu}} + \cdots. \tag{126}
\]
Obviously, \( d^2 = 0 \) on invariant polynomials is equivalent with the nilpotency and anticommutation of its components acting on invariant polynomials
\[
d_0^2 = 0 = d_1^2, \quad d_0 d_1 + d_1 d_0 = 0.
\] (127)
The action of \( d_0 \) on a given invariant polynomial with say \( l \) derivatives of \( F \) and \( m \) derivatives of \( \chi^\Delta \) results in an invariant polynomial with \((l + 1)\) derivatives of \( F \) and \( m \) derivatives of \( \chi^\Delta \), while the action of \( d_1 \) on the same object leads to an invariant polynomial with \( l \) derivatives of \( F \) and \((m + 1)\) derivatives of \( \chi^\Delta \). In particular, \( d_0 \) gives zero when acting on an invariant polynomial that does not involve the curvature or its derivatives, and the same is valid with respect to \( d_1 \) acting on an invariant polynomial that does not depend on any of the antifields or their derivatives. From (125–126) we observe that
\[
\text{agh} (d_0) = \text{agh} (d_1) = \text{agh} (d) = 0,
\] (128)
such that neither of them change the antighost number of the objects on which they act.

The antifields \( \chi^\Delta \) verify no relations between themselves and their derivatives, except the usual symmetry properties of the type \( \chi^\Delta_{\mu\nu} = \chi^\Delta_{\nu\mu} \), and accordingly will be named “foreground” fields. On the contrary, the derivatives of the components of the curvature tensor satisfy the Bianchi II identities (34), and in view of this we say that \( F_{\mu_1\cdots\mu_k|\nu_1\cdots\nu_{k+1}} \) are “background” fields. So, \( d_0 \) acts only on the background fields and their derivatives, while \( d_1 \) acts only on the foreground fields and their derivatives. According to the proposition on page 363 in [28], we have that the entire cohomology of \( d_1 \) in form degree strictly less than \( D \) is trivial in the space of invariant polynomials with strictly positive antighost number. This means that
\[
\alpha = \alpha \left( [\chi^\Delta], [F] \right), \quad \text{agh} (\alpha) = j > 0, \quad \deg (\alpha) = p < D, \quad d_1 \alpha = 0,
\] (129)
implies that
\[
\alpha = d_1 \beta,
\] (130)
with
\[
\beta = \beta \left( [\chi^\Delta], [F] \right), \quad \text{agh} (\beta) = j > 0, \quad \deg (\beta) = p - 1.
\] (131)
In particular, we have that if an invariant polynomial (of form degree \( p < D \) and with strictly positive antighost number) depending only on the undifferented antifields is \( d_1 \)-closed, then it vanishes
\[
(\bar{\alpha} = \bar{\alpha} \left( \chi^\Delta, [F] \right), \quad \text{agh} (\bar{\alpha}) > 0, \quad \deg (\bar{\alpha}) = p < D, \quad d_1 \bar{\alpha} = 0) \Rightarrow \bar{\alpha} = 0.
\] (132)
Only $d_0$ has non-trivial cohomology. For instance, any form depending only on the antifields and their derivatives is $d_0$-closed, but it is clearly not $d_0$-exact.

From now on, the proof is standard material and relies on decomposing $\alpha$ according to the number of derivatives of the antifields and on using the triviality of the cohomology of $d_1$ in form degree strictly less than $D$ in the space of invariant polynomials with strictly positive antighost numbers. For further details, see [30]. □

In form degree $D$ the Theorem 4.2 is replaced with: let $\alpha = \rho dx^0 \wedge \cdots \wedge dx^{D-1}$ be a $d$-exact invariant polynomial of form degree $D$ and of strictly positive antighost number, $\text{agh}(\alpha) = j > 0$, $\text{deg}(\alpha) = D$, $\alpha = d\beta$. Then, one can take the $(D - 1)$-form $\beta$ to be an invariant polynomial (of antighost number $j$). In dual notations, this means that if $\rho$ with $\text{agh}(\rho) = j > 0$ is an invariant polynomial whose Euler-Lagrange derivatives are all vanishing, $\rho = \partial_\mu j^\mu$, then $j^\mu$ can be taken to be also invariant. Theorem 4.2 can be generalized as follows.

**Theorem 4.3** The cohomology of $d$ computed in $H(\gamma)$ is trivial in form degree strictly less than $D$ and in strictly positive antighost number

$$H_{p,j}^0 (d, H(\gamma)) = 0, \quad j > 0, \quad p < D,$$

where $p$ is the form degree, $j$ is the antighost number and $g$ is the ghost number.

**Proof** The proof can be realized in a standard manner, like, for instance, in [30, 13]. It is however useful to mention that the operator $\bar{D}$ is defined in this case through the relations

$$D \alpha \left( \left[ X^S \right], [F] \right) = d \alpha \left( \left[ X^S \right], [F] \right),$$

$$\bar{D} \left( \eta_{\mu_1 \cdots \mu_k}^{(k-1)} \right) = \frac{1}{k+1} \partial_{[\alpha} \eta_{\mu_1 \cdots \mu_k]} dx^\alpha,$$

$$\bar{D} \left( \partial_{[\alpha} \eta_{\mu_1 \cdots \mu_k]}^{(k-1)} \right) = 0,$$

$$\bar{D} (\gamma b) = 0,$$

which is easily seen to be a differential in $H(\gamma)$, $\bar{D}^2 a = 0$ for any $a$ with $\gamma a = 0$. According to the relation (72) for $m = k - 2$, we have that

$$d \left( \eta_{\mu_1 \cdots \mu_k}^{(k-1)} \right) dx^\alpha = \frac{1}{k+1} \partial_{[\alpha} \eta_{\mu_1 \cdots \mu_k]}^{(k-1)} dx^\alpha,$$

29
Moreover, from (135–136) we observe that
\[ \bar{D} e^J = A^J_I e^I, \] (139)
for some constant matrix of elements \( A^J_I \), that involves \( dx^\alpha \), such that \( \bar{D} e^J \)
\[ da = \bar{D} a + \gamma \left( \sum J \alpha J \hat{e}^J \right), \] (142)
where \( \hat{e}^J \) depends in general on \( (k-1) \eta_{\mu_1 \cdots \mu_k} \partial_{[\alpha} (k-1) \eta_{\mu_1 \cdots \mu_k]} \) and \( [\eta_{(k-2) \mu_1 \cdots \mu_k}] \). Here, \( e^J \) are the elements with pure ghost number \( kl \) \((l > 0)\) of a basis in the
\[ \bar{D} = \bar{D}_0 + \bar{D}_1, \] (144)
defined through
\[ \bar{D}_0 \alpha (\chi^K, F) = \bar{D} \alpha (\chi^K, F) = da, \] (145)
such that the nilpotency of $\bar{D}$ is equivalent to the nilpotency and the anti-commutation of its components

$$\bar{D}^2 = 0 \Leftrightarrow (\bar{D}_0^2 = 0 = \bar{D}_1^2, \bar{D}_0 \bar{D}_1 + \bar{D}_1 \bar{D}_0 = 0).$$  \hspace{1cm} (153)$$

We reorganize the non-trivial elements $a \in H^{kl} (\gamma)$, with $l > 0$, like

$$a = a^{(0)} + a^{(1)} + \cdots + a^{(l)},$$  \hspace{1cm} (154)$$

where the piece $a^{(i)}$ contains $i$ antisymmetrized derivatives of the ghosts $\partial_{\alpha} \eta^{(k-1)}_{\mu_1 \cdots \mu_k}$ and $(l-i)$ undifferentiated ghosts $\eta^{(k-1)}_{\mu_1 \cdots \mu_k}$ and call $\bar{D}$-degree the number of factors of the type $\partial_{\alpha} \eta^{(k-1)}_{\mu_1 \cdots \mu_k}$. It is clear from (145–152) that the action of $\bar{D}_0$ on $a$ does not modify its $\bar{D}$-degree, while the action of $\bar{D}_1$ on the same element increases its $\bar{D}$-degree by one unit. From now the proof of the theorem follows in a close the manner the line from \[30, 13 \] and focuses on showing that

$$H^g_{p,j} (\bar{D}) = 0 \text{ for } g = kl - j, \; l, j > 0, \; p < D.$$  \hspace{1cm} (155)$$

In the meantime, we give below some common properties of $\gamma$ and $d$, which will be employed in subsection 4.4, namely

$$\gamma^2 = 0, \; d^2 = 0, \; \gamma d + d\gamma = 0, \; pgh (d) = 0, \; \deg (\gamma) = 0.$$  \hspace{1cm} (156)$$
\[
\sum \alpha_J \left( [\chi^* \Delta], [F] \right) e^J \left( \eta^{(k-1)}_{\mu_1 \cdots \mu_k}, \partial_{\mu_1} \eta^{(k-1)}_{\mu_2 \cdots \mu_{k+1}} \right)
\]
\[= \gamma \text{(something)} \iff \alpha_J = 0, \text{ for all } J, \quad (157)\]

\[
d\alpha_J \left( [\chi^* \Delta], [F] \right) = \alpha'_J \left( [\chi^* \Delta], [F] \right), \quad (158)\]

where
\[
\text{agh} (\alpha'_J) = \text{agh} (\alpha_J), \; \deg (\alpha'_J) = \deg (\alpha_J) + 1. \quad (159)\]

Theorem 4.3 is one of the main tools needed for the computation of \(H(s|d)\). In particular, it implies that there is no non-trivial descent for \(H(\gamma|d)\) in strictly positive antighost number.

**Corollary 4.1** If \(a\) with
\[
\text{agh} (a) = j > 0, \; \text{gh} (a) = g \geq -j, \; \deg (a) = p \leq D, \quad (160)\]
satisfies the equation
\[
\gamma a + db = 0, \quad (161)\]
where
\[
\text{agh} (b) = j > 0, \; \text{gh} (b) = g + 1 > -j, \; \deg (b) = p - 1 < D, \quad (162)\]
then one can always redefine \(a\)
\[
a \to a' = a + d\nu, \quad (163)\]
so that
\[
\gamma a' = 0. \quad (164)\]

**Proof** The proof can be done in a standard fashion, like, for instance, in [30]. Meanwhile, it is worth noticing that the “current” \(b\) from (161) has the expression
\[
b = -\gamma \nu + df, \quad (165)\]
with \(\gamma f \neq 0\) in general. ■
4.3 Local cohomology of the Koszul-Tate differential

The second essential ingredient in the analysis of the local cohomology $H (s|d)$ is the local cohomology of the Koszul-Tate differential in pure ghost number zero and in strictly positive antighost numbers, $H (\delta|d)$, also known as the characteristic cohomology. We recall that the local cohomology $H (\delta|d)$ is completely trivial at both strictly positive antighost and pure ghost numbers (for instance, see [25], Theorem 5.4 and [27]). An element $\alpha$ with the properties

$$\text{agh} (\alpha) > 0, \text{pgh} (\alpha) = 0,$$

is said to belong to $H (\delta|d)$ if and only if it is $\delta$ closed modulo $d$

$$\delta \alpha = dc, \text{pgh} (c) = 0.$$  \hspace{1cm} (167)

If $\alpha \in H (\delta|d)$ is a $\delta$-boundary modulo $d$

$$\alpha = \delta b + dc, \text{pgh} (\alpha) = \text{pgh} (b) = \text{pgh} (c) = 0, \text{agh} (\alpha) = \text{agh} (c) > 0,$$  \hspace{1cm} (168)

we will call it trivial in $H (\delta|d)$. The solution to the equation (167) is thus unique up to trivial objects, $\alpha \rightarrow \alpha + \delta b + dc$. The local cohomology $H (\delta|d)$ inherits a natural grading in terms of the antighost number, such that from now on we will denote by $H_j (\delta|d)$ the local cohomology of $\delta$ in antighost number $j$. As we have discussed in Section 2 the free model under study is a normal gauge theory of Cauchy order equal to $(k + 1)$. Using the general results from [25] (also see [13] and [26, 29]), one can state that the local cohomology of the Koszul-Tate differential at pure ghost number zero is trivial in antighost numbers strictly greater than its Cauchy order

$$H_j (\delta|d) = 0, j > k + 1.$$  \hspace{1cm} (169)

The final tool needed for the calculation of $H (s|d)$ is the local cohomology of the Koszul-Tate differential in the space of invariant polynomials, $H^{\text{inv}} (\delta|d)$, also called the invariant characteristic cohomology. It is defined via an equation similar to (167), but with $\alpha$ and $c$ replaced by invariant polynomials. Along the same line, the notion of trivial element from $H^{\text{inv}} (\delta|d)$ is revealed by (168) up to the precaution that both $b$ and $c$ must be invariant polynomials. It appears the natural question if the result (169) is still valid in the space of invariant polynomials. The answer is affirmative

$$H_j^{\text{inv}} (\delta|d) = 0, j > k + 1$$  \hspace{1cm} (170)

and is proved below, in Theorem 4.4. Actually, we prove that if \( \alpha_j \) is trivial in \( H_j(\delta|d) \), then it can be taken to be trivial also in \( H^{\text{inv}}_j(\delta|d) \). We consider only the case \( j \geq k + 1 \) since our main scope is to argue the triviality of \( H^{\text{inv}}_j(\delta|d) \) in antghost number strictly greater than \((k + 1)\). First, we prove the following lemma.

**Lemma 4.1** Let \( \alpha \) be a \( \delta \)-exact invariant polynomial

\[
\alpha = \delta \beta. \tag{171}
\]

Then, \( \beta \) can also be taken to be an invariant polynomial.

**Proof** Let \( v \) be a function of \([\chi^*\Delta]\) and \([t_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k]\). The dependence of \( v \) on \([t_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k]\) can be reorganized as a dependence on the curvature and its derivatives, \([F]\), and on

\[
\tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k = \{ t_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k, \partial t_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k, \cdots \}, \tag{172}
\]

where \( \tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k \) are not \( \gamma \)-invariant. If \( v \) is \( \gamma \)-invariant, then it does not involve \( \tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k \), i.e., \( v = v|_{\tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k = 0} \), so we have by hypothesis that

\[
\alpha = \alpha|_{\tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k = 0}. \tag{173}
\]

On the other hand, \( \beta \) depends in general on \([\chi^*\Delta]\), \([F]\) and \( \tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k \). Making \( \tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k = 0 \) in (171), using (172) and taking into account the fact that \( \delta \) commutes with the operation of setting \( \tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k \) equal to zero, we find that

\[
\alpha = \delta \left( \beta|_{\tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k = 0} \right), \tag{174}
\]

with \( \beta|_{\tilde{t}_{\mu_1\cdots\mu_k}\nu_1\cdots\nu_k = 0} \) invariant. This proves the lemma. \( \blacksquare \)

Now, we have the necessary tools for proving the next theorem.

**Theorem 4.4** Let \( \alpha_j^p \) be an invariant polynomial with \( \deg(\alpha_j^p) = p \) and \( \text{agh}(\alpha_j^p) = j \), which is \( \delta \)-exact modulo \( d \)

\[
\alpha_j^p = \delta \lambda_j^{p+1} + d \lambda_j^{p-1}, \quad j \geq k + 1. \tag{175}
\]

Then, we can choose \( \lambda_j^{p+1} \) and \( \lambda_j^{p-1} \) to be invariant polynomials.
Proof Initially, by successively acting with $d$ and $\delta$ on (175) (see, for instance [30, 13]) we obtain the tower of equations

$$\alpha^D_{j+D-p} = \delta \lambda^D_{j+D-p+1} + d \lambda^{D-1}_{j+D-p},$$

$$\vdots$$

$$\alpha^{p+1}_{j+1} = \delta \lambda^{p+1}_{j+2} + d \lambda^p_{j+1},$$

$$\alpha^p_j = \delta \lambda^p_{j+1} + d \lambda^{p-1}_{j},$$

$$\alpha^{p-1}_{j-1} = \delta \lambda^{p-1}_{j} + d \lambda^{p-2}_{j-1},$$

$$\vdots$$

$$\alpha^0_{j-p} = \delta \lambda^0_{j-p+1} \text{ or } \alpha^{p-j+k+1}_{k+1} = \delta \lambda^{p-j+k+1}_{k+2} + d \lambda^{p-j+k}_3.$$  \hfill (176)

All the $\alpha$'s in the descent (176) are invariant. Using the general line from [30, 13] it can be shown that if one of the $\lambda$'s in (176) is invariant, then all the other $\lambda$'s can be taken to be also invariant.

If $j \geq D + k + 1$ (and hence $j - p \geq k + 1$), the last equation from the descent (176) for $p = D$ reads as

$$\alpha^0_{j-D} = \delta \lambda^0_{j-D+1}.$$  \hfill (177)

Using Lemma 4.1 it results that $\lambda^0_{j-D+1}$ can be taken to be invariant, such that the above arguments lead to the conclusion that all the $\lambda$’s from the descent can be chosen invariant. As a consequence, in the first equation from the descent in this situation, namely, $\alpha^D_j = \delta \lambda^D_{j+1} + d \lambda^{D-1}_j$, we have that both $\lambda^D_{j+1}$ and $\lambda^{D-1}_j$ are invariant. Therefore, the theorem is true in form degree $D$ and in all antighost numbers $j \geq D + k + 1$, so it remains to be proved that it holds in form degree $D$ and in all antighost numbers $k + 1 \leq j < D + k + 1$. This is done below.

In the sequel we consider the case $p = D$ and $k + 1 \leq j < D + k + 1$. The top equation from (176), written in dual notations, takes the form

$$\alpha_j = \delta \lambda_{j+1} + \partial_{\mu} \lambda^\mu_j, \quad k + 1 \leq j < D + k + 1.$$  \hfill (178)
On the other hand, we can express $\alpha_j$ in terms of its E.L. derivatives by means of the homotopy formula

$$
\alpha_j = \partial_\mu \epsilon_j^\mu + \int_0^1 d\tau \left( \sum_{m=0}^{k-1} \frac{\delta^R \alpha_j}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} (\tau) \left( \frac{\delta^R \alpha_j}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} + \frac{\delta^R \alpha_j}{\delta t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k}} (\tau) t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k} \right) \right), \quad \text{(179)}
$$

where

$$
\frac{\delta^R \alpha_j}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} (\tau) = \frac{\delta^R \alpha_j}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} \left( \tau \left[ t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k} , \tau \left[ \chi^* \Delta \right] \right] \right), \quad \text{(180)}
$$

and similarly for the other terms. Denoting the E.L. derivatives of $\lambda_{j+1}$ by

$$
\frac{\delta^R \lambda_{j+1}}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} = G_{j-m-1}, \quad m = 0, k-1, \quad \text{(181)}
$$

$$
\frac{\delta^R \lambda_{j+1}}{\delta t_{\mu_1 \ldots \mu_k | \nu_1 \ldots \nu_k}} = G_j, \quad \text{(182)}
$$

and using (178) we find after some computation that the E.L. derivatives of $\alpha_j$ are given by

$$
\frac{\delta^R \alpha_j}{\delta \eta_{(k-1)^* \mu_1 \ldots \mu_k}} = (-)^{k-1} \delta \frac{\delta^R \alpha_j}{\delta \eta_{(k-1)^* \mu_1 \ldots \mu_k}} G_{j-k}, \quad \text{(184)}
$$

$$
\frac{\delta^R \alpha_j}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} = (-)^m \left( \delta \frac{\delta^R \alpha_j}{\delta \eta_{(m)^* \mu_1 \ldots \mu_k | \nu_1 \ldots \nu_{k-m-1}}} G_{j-m-1} \right) - \left( \delta \frac{\delta^R \alpha_j}{\delta t_{(m+1)^* \mu_2 \ldots \mu_k | \nu_2 \ldots \nu_{k-m-1}}} G_{j-m-2} \right) + (-)^{k+1} (m+2) \frac{\delta^R \alpha_j}{\delta \eta_{(m+1)^* \mu_1 \ldots \mu_k | \nu_2 \ldots \nu_{k-m-1}, \nu_1}}, \quad m = 0, k-2, \quad \text{(185)}
$$
\[
\frac{\delta R \alpha_j}{\delta t_{\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k}} = -\delta (k)_{\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} \frac{\partial}{\partial \alpha_j} + G_{j-1}^{(0) \mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} + G_{j-1}^{(0) \mu_1 \cdots \mu_k, \mu_1}^{\nu_1 \cdots \nu_k, \nu_1} \tag{186}
\]

\[
\frac{\delta R \alpha_j}{\delta t_{\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k}} = \delta L_{j+1}^{\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} + c_1 \frac{\partial}{\partial \alpha_{k+1}} G_{j}^{(k)\mu_{m+1} \cdots \mu_k}^{\nu_1 \cdots \nu_k} \tag{187}
\]

In the above, \( G_j^{(k)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} \) has the same mixed symmetry like the curvature tensor \( F_{\mu_1 \cdots \mu_{k+1}}^{\nu_1 \cdots \nu_k} \)

\[
G_j^{(k)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} = \sum_{m=0}^{k} \left(-\right)^m \frac{\partial}{\partial \alpha_m} G_j^{(k)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} \delta_{\mu_1}^{\mu_{m+1}} \cdots \delta_{\nu_1}^{\nu_{m+1}} \delta_{\nu_k}^{\nu_k} \delta_{\nu_k}^{\nu_{k+1}} \tag{188}
\]

and \( G_j^{(k)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} \) denote the traces of \( G_j^{(k)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_k} \) appearing in the right-hand side of the formulas (186, 187).

As the E.L. derivatives of an invariant quantity are also invariant, the equation in (184) together with Lemma 4.1 (as \( j - k > 0 \)) lead to

\[
\frac{\delta R \alpha_j}{\delta \eta^{(k-1)\mu_1 \cdots \mu_k}} = \left(-\right)^{k-1} \delta (k-1)_{\mu_1 \cdots \mu_k}^{(k-1)\mu_1 \cdots \mu_k} \tag{190}
\]

with \( G_{j-k}^{(k-1)\mu_1 \cdots \mu_k} \) invariant. Following a similar reasoning, we find that

\[
\frac{\delta R \alpha_j}{\delta \eta^{(m)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_{k-1}}} = \left(-\right)^m \delta (m)_{\mu_1 \cdots \mu_k}^{\mu_1 \cdots \mu_k} + G_{j-m-1}^{(m+1)\mu_2 \cdots \mu_k}^{\nu_2 \cdots \nu_{k-1}} + (\cdots) \frac{\partial}{\partial \alpha_{j-m-2}} G_{j-m-2}^{(m+1)\mu_1 \cdots \mu_k}^{\nu_1 \cdots \nu_{k-1}} \tag{191}
\]

\[\]

37
\[
\frac{\delta R_{\alpha_j}}{\delta t^*_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} = -\delta G_j + \frac{(k)_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}}{\delta t^*_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} + \frac{(0)_{\mu_1 \cdots \mu_k | [\nu_2 \cdots \nu_k, \nu_1]}}{\delta t^*_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} + \frac{G_{j-1}}{\delta t^*_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} ,
\]

(192)

\[
\frac{\delta R_{\alpha_j}}{\delta t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} = \delta \bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} - c_1 \partial_{\nu_{k+1}} \partial_{\nu_{k+1}} G_j \frac{(k)_{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}}}{\delta t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} ,
\]

(193)

where all the bar quantities are invariant. Since \( \alpha_j \) is invariant, it depends on \( t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) only through the curvature and its derivatives, such that

\[
\frac{\delta R_{\alpha_j}}{\delta t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}} = \partial_{\nu_{k+1}} \partial_{\nu_{k+1}} \Delta_j^{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}} ,
\]

(194)

where \( \Delta_j \) has the mixed symmetry of the curvature tensor. The part from \((k)\) involving \( G_j \) has a form similar to that of the right-hand side of (194). Then, \( \bar{L}_{j+1} \) must be expressed in the same manner, i.e.,

\[
\delta \bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \partial_{\nu_{k+1}} \partial_{\nu_{k+1}} \Omega_j^{\mu_1 \cdots \mu_{k+1} | \nu_1 \cdots \nu_{k+1}} ,
\]

(195)

for some \( \Omega_j \) with the mixed symmetry of the curvature. The equation (195) shows that for some given indices \( \nu_1 \cdots \nu_k \), the object \( \bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) belongs to \( H_{j+1}^{D-k} (\delta |d) \). As \( H_{j+1}^{D-k} (\delta |d) \simeq H_{j+2}^{D-k+1} (\delta |d) \simeq \cdots \simeq H_{j+k+1}^{D} (\delta |d) \) (see [25], Theorem 8.1) and \( H_{j+k+1}^{D} (\delta |d) \simeq 0 \), the equation (195) implies that

\[
\bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \delta R_{j+2}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + \partial_{\rho} U_{j+1}^{\rho \mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} ,
\]

(196)

where \( R_{j+2} \) is separately antisymmetric in the indices \( \{ \mu_1 \cdots \mu_k \} \) and \( \{ \nu_1 \cdots \nu_k \} \), and \( U_{j+1} \) is antisymmetric in \( \{ \rho \mu_1 \cdots \mu_k \} \), as well as in \( \{ \nu_1 \cdots \nu_k \} \).

Now, we prove the theorem in the case \( k+1 \leq j < D+k+1 \) by induction. This is, we assume that the theorem is valid in antighost number \( j + k + 1 \) and in form degree \( D \), and show that it holds in antighost number \( j \) and in form degree \( D \). In agreement with the induction hypothesis, \( R_{j+2} \) and \( U_{j+1} \) can be assumed to be invariant. On the other hand, \( \bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) must verify the mixed symmetry of the tensor field \( t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \) with respect to the given values \( \{ \nu_1 \cdots \nu_k \} \), i.e., \( \bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = 0 \), which further implies that

\[
\delta R_{j+2}^{\mu_1 \cdots \mu_{k-1} \mu_k | \nu_1 \cdots \nu_k} + \partial_{\rho} U_{j+1}^{\rho \mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = 0 .
\]

(197)
Acting with \( \delta \) on (197), we obtain
\[
\delta U_{j+1}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] = 0,
\]
such that
\[
\delta U_{j+1}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] \equiv \partial_j V_j^{\gamma \rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k], \tag{198}
\]
where \( V_j^{\gamma \rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] \) is separately antisymmetric in \( \{\gamma \rho_{j+1} \cdots \mu_{k-1}\} \) and \( \{\mu_k \nu_1 \cdots \nu_k\} \) (the double bar \(||\) signifies that in general this tensor neither satisfies the identity \( V_j^{\gamma \rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] \equiv 0 \nor is symmetric under the interchange of the two sets of indices). The equation (198) shows that for some fixed indices \( \{\mu_k \nu_1 \cdots \nu_k\} \), \( U_{j+1}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] \) pertains to \( H_{j+1}^{D-k} (\delta|d) \), that is finally found isomorphic to \( H_{j+k+1}^{D-k} (\delta|d) \simeq 0 \), so \( U_{j+1}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] \) is trivial
\[
U_{j+1}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] = \delta W_{j+2}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] + \partial_j S_j^{\gamma \rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k], \tag{199}
\]
with \( W_{j+2} \) antisymmetric in both \( \{\rho_{j+1} \cdots \mu_{k-1}\} \) and \( \{\mu_k \nu_1 \cdots \nu_k\} \) and \( S_{j+1} \) separately antisymmetric in \( \{\gamma \rho_{j+1} \cdots \mu_{k-1}\} \) and \( \{\mu_k \nu_1 \cdots \nu_k\} \). Using again the induction hypothesis, we can assume that \( W_{j+2} \) and \( S_{j+1} \) are invariant. In order to reconstruct \( \alpha_j \) through the homotopy formula (178), we need to compute \( \delta L_{j+1}^{\mu_{j+1}}[\nu_1 \cdots \nu_k] \) by means of formula (196), so eventually we need to calculate \( \partial \rho U_{j+1}^{\rho_{j+1} \cdots \mu_{k-1}}[\mu_k|\nu_1 \cdots \nu_k] \). In this respect we use the equation (199) and the identity (that holds only for a tensor that is separately antisymmetric in its first \((k+1)\) and respectively in its last \( k \) indices, which does not have to satisfy any further identity)
\[
U_{j+1}^{\rho_{j+1} \cdots \mu_k}[\nu_1 \cdots \nu_k] = \frac{(-1)^k}{k+1} \sum_{m=0}^k \left( \frac{1}{kC_m} U_{j+1}^{\nu_1 \cdots \nu_m \mu_{m+1} \cdots \mu_k}[\rho|\mu_1 \cdots \mu_m \nu_{m+1} \cdots \nu_k] \right), \tag{200}
\]
where two further antisymmetrization should be performed, one over each underlined group of indices, i.e., \( \{\nu_1 \cdots \nu_k\} \) and \( \{\mu_1 \cdots \mu_k\} \). On behalf of the relations (196) (200), after some computation we obtain that
\[
\partial \rho U_{j+1}^{\rho_{j+1} \cdots \mu_k}[\nu_1 \cdots \nu_k] = \delta W_{j+2}^{\mu_{j+2}}[\nu_1 \cdots \nu_k] + \partial_j \delta S_{j+1}^{\mu_{j+1} \cdots \mu_k}[\nu_1 \cdots \nu_k \gamma], \tag{201}
\]
where
\[
\delta S_{j+1}^{\mu_{j+1} \cdots \mu_k}[\nu_1 \cdots \nu_k \gamma] = \frac{(-1)^k}{k+1} \sum_{m=0}^k \left( \sum_{i=m}^{[\frac{k}{2}]} \left( \sum_{i=m}^{[\frac{k}{2}]-1} \frac{1}{kC_i} \right) (-1)^{i+m} \right)
\]
\[ + \frac{1}{2\varepsilon_{k+1}} \left( \frac{1}{C} \right)^{m+\left[ \frac{k}{2} \right]} \left( S_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \gamma} \right. \]
\[ + S_{j+1}^{\mu_1 \cdots \mu_m | \nu_{m+1} \cdots \nu_k \gamma} \left[ | \nu_1 \cdots \nu_m | \mu_{m+1} \cdots \mu_k \right) \right] , \tag{202} \]

and \( \varepsilon_{k+1} \) is defined via
\[ \varepsilon_{k+1} = (k+1) \mod 2. \tag{203} \]

On account of (202) it is now obvious that
\[ \tilde{S}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \tilde{S}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k}, \tag{204} \]
in the sense that it indeed displays the mixed symmetry of the curvature tensor (it is separately antisymmetric in the indices \( \{ \mu_1 \cdots \mu_k \} \) and \( \{ \nu_1 \cdots \nu_k \} \), but also symmetric under the interchange of these two sets of indices, although it does not verify in general the Bianchi I identity for a \( (k+1, k+1) \) tensor). The tensor \( \tilde{S}_{j+1} \) is invariant as \( S_{j+1} \) is also invariant. Inserting (201) in (196) and employing (204) it results that
\[ \bar{L}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \delta \tilde{R}_{j+2}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + \partial_{\rho} \partial_{\gamma} \tilde{S}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \gamma}. \tag{205} \]

With the help of (190, 193) and (203), the formula (179) becomes
\[ \alpha_j = \delta \left[ \int_0^1 d\tau \left( \sum_{j=0}^m G_{j-m-1}^{(m) \mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1} \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m-1}}} \right. \]
\[ + \tilde{G}_j^{(k) \mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \]
\[ + \left( \partial_{\rho} \partial_{\gamma} \tilde{S}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \gamma} \right) t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \right] + \partial_{\mu} \sigma_{j}^{\mu}. \tag{206} \]

The last term in the argument of \( \delta \) can be written in the form
\[ \left( \partial_{\rho} \partial_{\gamma} \tilde{S}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \gamma} \right) t_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} = \]
\[ \frac{1}{(k+1)^2} \tilde{S}_{j+1}^{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \gamma} F_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k \gamma} + \partial_{\rho} \phi_{j+1}^{\mu}, \tag{207} \]
so finally we arrive at

\[ \alpha_j = \delta \int_0^1 d\tau \left( \sum_{j=0}^{m} G_{j-m-1}^{(m)} \xi_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m}} \eta_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_{k-m}} + G_j^{(k)} \xi_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} + \frac{1}{(k+1)^2} \tilde{G}_{j+1}^{(k)} \xi_{\mu_1 \cdots \mu_k | \nu_1 \cdots \nu_k} \right) + \partial_{\mu} \psi^\mu_j. \] 

We observe that all the terms from the integrand are invariant. In order to prove that the current \( \psi^\mu_j \) can also be taken invariant, we switch (208) to the original form notation

\[ \alpha_j^D = \delta \lambda_{j+1}^D + d\lambda_j^{D-1}, \] 

(209)

(where \( \lambda_{j+1}^{D-1} \) is dual to \( \psi^\mu_j \)). As \( \alpha_j^D \) is by assumption invariant and we have shown that \( \lambda_{j+1}^{D-1} \) can be taken invariant, (209) becomes

\[ \beta_j^D = d\lambda_j^{D-1}. \] 

(210)

It states that the invariant polynomial \( \beta_j^D = \alpha_j^D - \delta \lambda_{j+1}^D \), of form degree \( D \) and of strictly positive antighost number, is \( d \)-exact. Then, in agreement with the Theorem 4.2 in form degree \( D \) (see the paragraph following this theorem), we can take \( \lambda_{j+1}^{D-1} \) (or, which is the same, \( \psi^\mu_j \)) to be invariant.

In conclusion, the induction hypothesis for antighost number \( (j + k + 1) \) and form degree \( D \) leads to the same property for antighost number \( j \) and form degree \( D \), which proves the theorem for all \( j \geq k + 1 \) since we have shown that it holds for \( j \geq D + k + 1 \).

The most important consequence of the last theorem is the validity of the result (170) on the triviality of \( H_{inv}^{(\delta|d)} \) in antighost number strictly greater than \( (k + 1) \).

4.4 Local cohomology of the BRST differential

Now, we have all the necessary tools for the study of the local cohomology \( H^{(s|d)} \) in form degree \( D \) \((D \geq 2k+1)\). We will show that it is always possible to remove the components of antighost number strictly greater than \( (k + 1) \) from any co-cycle of \( H^*_D \) \((s|d) \) in form degree \( D \) only by trivial redefinitions.
We consider a co-cycle from $H^g_D(s|d)$, $sa + db = 0$, with $\text{deg}(a) = D$, $\text{gh}(a) = g$, $\text{deg}(b) = D - 1$, $\text{gh}(b) = g + 1$. Trivial redefinitions of $a$ and $b$ mean the simultaneous transformations $a \to a + sc + de$ and $b \to b + df + se$. We expand $a$ and $b$ according to the antighost number and ask that $a_0$ is local, such that each expansion stops at some finite antighost number $s_{\text{gh}}$, $a = \sum_{j=0}^I a_j, b = \sum_{j=0}^M b_j$, $\text{agh}(a_j) = j = \text{agh}(b_j)$. Due to (65), the equation $sa + db = 0$ is equivalent to the tower of equations

$$
\delta a_1 + \gamma a_0 + db_0 = 0,
$$

$$
\vdots
$$

$$
\delta a_I + \gamma a_{I-1} + db_{I-1} = 0,
$$

$$
\vdots
$$

The form of the last equation depends on the values of $I$ and $M$, but we can assume, without loss of generality, that $M = I - 1$. Indeed, if $M > I - 1$, the last $(M - I)$ equations read as $db_j = 0$, $I < j < M$, which imply that $b_j = df_j$, $\text{deg}(f_j) = D - 2$. We can thus absorb all the pieces $(df_j)_{I < j \leq M}$ in a trivial redefinition of $b$, such that the new “current” stops at antighost number $I$. Accordingly, the bottom equation becomes $\gamma a_I + db_I = 0$, so the Corollary 4.1 ensures that we can make a redefinition $a_I \to a_I - d\rho_I$ such that $\gamma (a_I - d\rho_I) = 0$. Meanwhile, the same corollary (see the formula (162)) leads to $b_I = dg_I + \gamma \rho_I$, where $\text{deg}(\rho_I) = D - 1$, $\text{deg}(g_I) = D - 2$, $\text{agh}(\rho_I) = \text{agh}(g_I) = I$, $\text{gh}(\rho_I) = g$, $\text{gh}(g_I) = g + 1$. Then, it follows that we can make the trivial redefinitions $a \to a - d\rho_I$ and $b \to b - dg_I - s\rho_I$, such that the new “current” stops at antighost number $(I - 1)$, while the last component of the co-cycle from $H^g_D(s|d)$ is $\gamma$-closed.

In consequence, we obtained the equation $sa + db = 0$, with

$$
a = \sum_{j=0}^I a_j, \quad b = \sum_{j=0}^{I-1} b_j,
$$

(211)

where $\text{agh}(a_j) = j$ for $0 < j < I$ and $\text{agh}(b_j) = j$ for $0 < j < I - 1$. All $a_j$ are $D$-forms of ghost number $g$ and all $b_j$ are $(D - 1)$-forms of ghost number $(g + 1)$, with $\text{pgh}(a_j) = g + j$ for $0 < j < I$ and $\text{pgh}(b_j) = g + j + 1$ for $0 < j < I - 1$. The equation $sa + db = 0$ is now equivalent with the tower of equations (where some $(b_j)_{0 \leq j \leq I-1}$ could vanish)

$$
\delta a_1 + \gamma a_0 + db_0 = 0,
$$

(212)

42
\[ \delta a_{j+1} + \gamma a_j + db_j = 0, \]  
\[ \vdots \]  
\[ \delta a_I + \gamma a_{I-1} + db_{I-1} = 0, \]  
\[ \gamma a_I = 0. \]  

Next, we show that we can eliminate all the terms \((a_j)_{j>k+1}\) and \((b_j)_{j>k}\) from the expansions (211) by trivial redefinitions only.

Assuming that \(a\) stops at a value \(L' \neq kL\) of the pure ghost number, \(g + I = L'\), the bottom equation, (215), yields \(a_I \in H^{L'}(\gamma)\). Then, in agreement with the result (117), \(a_I = \gamma \bar{a}_I\), where \(\text{agh} (\bar{a}_I) = I\), \(\text{pgh} (\bar{a}_I) = g + L' - 1\) and \(\text{deg} (\bar{a}_I) = D\). Consequently, we can make the trivial redefinition \(a \rightarrow a - s \bar{a}_I\), whose decomposition stops at antighost number \((I - 1)\), such that the bottom equation corresponding to the redefined co-cycle of \(H_D^g (s|d)\) takes the form \(\gamma a_{I-1} + db_{I-1} = 0\). Now, we apply again the Corollary 4.1 and replace it with the equation \(\gamma a_{I-1} = 0\), such that the new “current” can be made to end at antighost number \((I - 2)\), \(b = \sum_{j=0}^{I-2} b_j\). In conclusion, if \(g + I = L' \neq kL\), we can always remove the last components \(a_I\) and \(b_{I-1}\) from a co-cycle \(a \in H_D^g (s|d)\) and its corresponding “current” by trivial redefinitions only.

We can thus assume, without loss of generality, that any co-cycle \(a\) from \(H_D^g (s|d)\) can be taken to stop at a value \(I\) of the antighost number such that \(g + I = kL\), \(a = \sum_{j=0}^{I} a_j\), \(b = \sum_{j=0}^{I-1} b_j\). We consider that \(I > k + 1\). The last equation from the system equivalent with \(sa + db = 0\) takes the form (215), with \(\text{pgh} (a_I) = g + I = kL\), so \(a_I \in H^{kL} (\gamma)\). In agreement with the general results on \(H (\gamma)\) (see Subsection 4.2) it follows that

\[ a_I = a_I^{(0)} + \cdots + a_I^{(L)} + \gamma \bar{a}_I, \]  

where \(a_I^{(i)} = \sum_{J} \alpha_{J,i} e^J, \ i = 0, \cdots, L\).

All \(\alpha_{J,i}\) are invariant polynomials, with

\[ \text{agh} (\alpha_{J,i}) = I, \ \text{deg} (\alpha_{J,i}) = D, \]

\[ \text{agh} (\bar{a}_I) = I, \ \text{deg} (\bar{a}_I) = D. \]
are the elements of pure ghost number $kL$ of a basis in the space of polynomials in $(k-1)\eta_{\mu_1 \cdots \mu_k}$ and $\partial_{[\alpha_1} \eta_{\mu_1 \cdots \mu_k]}$ with the $\bar{D}$-degree equal to $i$. Applying $\gamma$ on (214) and using (215) together with the properties (156) we find that

$$-d(\gamma b_{I-1}) = 0,$$

such that the triviality of the cohomology of $d$ implies that

$$\gamma b_{I-1} + dc_{I-1} = 0,$$

(220)

where $\text{agh} (c_{I-1}) = I-1$, $\text{pgh} (c_{I-1}) = kL+1$, $\text{deg} (c_{I-1}) = D-2$. From the Corollary 4.1 it follows (as $I > k+1$ and $k \geq 2$ by assumption, so $I-1 > 0$) that we can make a trivial redefinition such that (220) is replaced with the equation

$$\gamma b_{I-1} = 0.$$  

(221)

In agreement with (221), $b_{I-1}$ belongs to $H^{kL}(\gamma)$, so we can take

$$b_{I-1} = b_{I-1}^{(0)} + \cdots + b_{I-1}^{(L)} + \gamma \bar{b}_{I-1},$$

(222)

where

$$b_{I-1}^{(i)} = \sum_j \beta_{J,i} e^{J,i}, \ i = 0, \cdots, L.$$  

(223)

All $\beta_{J,i}$ are invariant polynomials, with

$$\text{agh} (\beta_{J,i}) = I-1, \ \text{deg} (\beta_{J,i}) = D-1,$$

(224)

and $e^{J,i}$ are the elements (219) with the $\bar{D}$-degree equal to $i$. Inserting (216–217) and (222–223) in (214) and employing the relation (142) for $b_{I-1} \in H^{kL}(\gamma)$, we get that

$$\sum_{i=0}^{L} \sum_j \left[ \pm (\delta \alpha_{J,i} + \bar{D} \beta_{J,i}) e^{J,i} + \beta_{J,i} \bar{D} e^{J,i} \right] = \gamma \left( -a_{I-1} - \bar{b}_{I-1} + \delta \bar{a}_{I} + d \bar{b}_{I-1} \right),$$

(225)
where $\hat{b}_{I-1}$ comes from $db_{I-1} = \hat{D}b_{I-1} + \gamma \hat{b}_{I-1}$. As $\delta \alpha_{J,i}$ and $\hat{D} \beta_{J,i} = d \beta_{J,i}$ are invariant polynomials, while $\hat{D}e^{J,i} = \sum_{J'} A_{J',+1}^{J,i} e^{J',i+1}$ (see the formula (139)), the property (157) ensures that the left-hand side of (225) must vanish

$$
\sum_{i=0}^{L} \sum_{J} \left[ \pm (\delta \alpha_{J,i} + \hat{D} \beta_{J,i}) e^{J,i} + \beta_{J,i} \hat{D} e^{J,i} \right] = 0. 
$$

(226)

Using the decomposition (144) and the definitions (145-152), the projection of the equation (226) on the various values of the $\hat{D}$-degree becomes equivalent with the equations

$$
0 : \delta \alpha_{J,0} + d \beta_{J,0} = 0, 
$$

(227)

$$
1 : \pm (\delta \alpha_{J,1} + d \beta_{J,1}) + \beta_{J,0} A_{J,1}^{J',0} = 0, 
$$

(228)

$$
\vdots
$$

(229)

$$
L : \pm (\delta \alpha_{J,L} + d \beta_{J,L}) + \beta_{J,L-1} A_{J,L}^{J',L-1} = 0, 
$$

(228)

while the equation (226) projected on the value $(L + 1)$ of the $\hat{D}$-degree is automatically satisfied, $\hat{D} e^{J,L} = 0$ due to the relation (136) and as $e^{J,L}$ contains $L$ factors of the type $\partial_{\mu_1} \eta_{\mu_2 \cdots \mu_{k+1}}$.

From (227) we read that for all $J$ the invariant polynomials $\alpha_{J,0}$ belong to $H^D_I (\delta | d)$. Thus, as we assumed that $I > k+1$ and we know that $H^D_I (\delta | d) = 0$ for $I > k+1$, we deduce that all $\alpha_{J,0}$ are trivial

$$
\alpha_{J,0} = \delta \lambda^{D}_{I+1,J,0} + d \lambda^{D-1}_{I,J,0},
$$

(230)

where all $\lambda^{D}_{I+1,J,0}$ are $D$-forms of antighost number $(I + 1)$ and all $\lambda^{D-1}_{I,J,0}$ are $(D - 1)$ forms of antighost number $I$. Applying the result of the Theorem 4.4 we have that all $\lambda^{D}_{I+1,J,0}$ and $\lambda^{D-1}_{I+1,J,0}$ can be taken to be invariant polynomials, so all $\alpha_{J,0}$ are in fact trivial in $H^D_{I+1} (\delta | d)$. Replacing (230) in (227) and using $\delta^2 = 0$ together with $d \delta + d \delta = 0$, we obtain that $d (-\delta \lambda^{D-1}_{I,J,0} + \beta_{J,0}) = 0$. As $\lambda^{D-1}_{I,J,0}$ and $\beta_{J,0}$ are invariant polynomials of strictly positive antighost number and of form degree $(D - 1)$, by Theorem 4.2 it follows that $-\delta \lambda^{D-1}_{I,J,0} + \beta_{J,0} = d \lambda^{D-2}_{I-1,J,0}$, where $\lambda^{D-2}_{I-1,J,0}$ are also invariant polynomials for all $J$, with $\text{agh} (\lambda^{D-2}_{I-1,J,0}) = I - 1$ and $\text{deg} (\lambda^{D-2}_{I-1,J,0}) = D - 2$, so

$$
\beta_{J,0} = \delta \lambda^{D-1}_{I,J,0} + d \lambda^{D-2}_{I-1,J,0}.
$$

(231)
From (230), we have that

\[
^{(0)}a_I = \sum_j (\delta \lambda^{D}_{I+1,J,0} + d \lambda^{D-1}_{I,J,0}) e^{J,0}
\]

\[
= \pm s \left( \sum_j \lambda^{D}_{I+1,J,0} e^{J,0} \right) \pm d \left( \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} \right) \pm \sum_j (\lambda^{D-1}_{I,J,0} d e^{J,0}).
\]

As \(de^{J,0} = \sum_{J'} A^{J,0}_{I,J'} e^{\sigma'_J} + \gamma e^{J,0}\) and \(\gamma \lambda^{D-1}_{I,J,0} = 0\), we find that

\[
^{(0)}a_I = \pm s \left( \sum_j \lambda^{D}_{I+1,J,0} e^{J,0} \right) \pm d \left( \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} \right)
\]

\[
\mp \gamma \left( \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} \right) \mp \sum_{J,J'} \left( \lambda^{D-1}_{I,J,0} A^{J,0}_{I,J'} e^{J',1} \right).
\]  

Similarly, relying on (231) we deduce that

\[
^{(0)}b_{I-1} = \pm s \left( \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} \right) \pm d \left( \sum_j \lambda^{D-2}_{I-1,J,0} e^{J,0} \right)
\]

\[
\mp \gamma \left( \sum_j \lambda^{D-2}_{I-1,J,0} e^{J,0} \right) \mp \sum_{J,J'} \left( \lambda^{D-2}_{I-1,J,0} A^{J,0}_{I,J'} e^{J',1} \right).
\]  

If we perform the trivial redefinitions

\[
a'_I = a_I \mp s \left( \sum_j \lambda^{D}_{I+1,J,0} e^{J,0} \right) \mp d \left( \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} \right),
\]

\[
b'_{I-1} = b_{I-1} \mp s \left( \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} \right) \mp d \left( \sum_j \lambda^{D-2}_{I-1,J,0} e^{J,0} \right),
\]

and meanwhile partially fix \(\bar{a}_I\) and \(\bar{b}_{I-1}\) from (216) and respectively (222) to

\[
\bar{a}_I = \pm \sum_j \lambda^{D-1}_{I,J,0} e^{J,0} + \cdots,
\]

\[
\bar{b}_{I-1} = \pm \sum_j \lambda^{D-2}_{I-1,J,0} e^{J,0} + \cdots.
\]
then (233) ensure that the lowest value of the $\bar{D}$-degree in the decompositions of $a'$ and $b'_{I-1}$ is equal to one. In conclusion, under the hypothesis that $I > k + 1$, we annihilated all the pieces from $a_I$ and $b_{I-1}$ with the $\bar{D}$-degree equal to zero by trivial redefinitions only. We can then successively remove the terms of higher $\bar{D}$-degree from $a_I$ and $b_{I-1}$ by a similar procedure (and also the residual $\gamma$-exact terms by conveniently fixing the pieces “...” from $\tilde{a}_I$ and $\tilde{b}_{I-1}$) until we completely discard $a_I$ and $b_{I-1}$. Next, we pass to a co-cycle $a$ from $H_{\bar{D}}(s|d)$ that ends at the value $(I - 1)$ of the antighost number, and hence $g + I - 1 \neq kl$, so we can apply the arguments preceding the equation (216) and remove both $a_{I-1}$ and $b_{I-2}$. This procedure can be continued until we reach antighost number $(k + 1)$. If $g + k + 1$ is $kl$ we cannot go down and discard $a_{k+1}$ and $b_k$, since both $H^{g+k+1}_g(\gamma)$ and $H^{D_{\text{inv}}}_{k+1}(\delta|d)$ are non-trivial. However, if $g + k + 1 \neq kl$, then $H^{g+k+1}_g(\gamma) = 0$, so we can go one step lower and remove $a_{k+1}$ and $b_k$. In conclusion, we can take, without loss of generality

\begin{align}
a &= a_0 + \cdots + a_{k+1}, \quad b = b_0 + \cdots + b_k, \quad \text{if} \quad g + k + 1 = kl, \quad (239) \\
a &= a_0 + \cdots + a_k, \quad b = b_0 + \cdots + b_{k-1}, \quad \text{if} \quad g + k + 1 \neq kl, \quad (240)
\end{align}

in the equation $sa + db = 0$, where $\text{gh}(a) = g$. Furthermore, the last terms can be assumed to involve only non-trivial elements from $H^{\text{inv}}_d(\delta|d)$.

5 Conclusion

To conclude with, in this paper we have used some specific cohomological techniques, based on the Lagrangian BRST differential, to show that every non-trivial co-cycle from the local BRST cohomology in form degree $D$ for a free, massless tensor field $t_{\mu_1\cdots\mu_k|^1\cdots|^k}$ that transforms in an irreducible representation of $GL(D, \mathbb{R})$, corresponding to a rectangular, two-column Young diagram with $k > 2$ rows, can be taken to stop at antighost number $k$ or $(k + 1)$, its last component belonging to $H(\gamma)$ and containing only non-trivial elements from $H^{\text{inv}}_d(\delta|d)$. This result is based on various cohomological properties involving the exterior longitudinal derivative, the Koszul-Tate differential, as well as the exterior spacetime differential, which have been proved in detail. The results contained in this paper are important from the perspective of constructing consistent interactions that involve this type of mixed symmetry tensor field since it is known that the first-order deformation of the
solution to the master equation is a co-cycle of the local BRST cohomology $H_0^D(s|d)$ in form degree $D$ and in ghost number zero.

References

[1] T. Curtright, Generalized gauge fields, *Phys. Lett.* **B165** (1985), 304; T. Curtright, P. G. O. Freund, Massive dual fields, *Nucl. Phys.* **B172** (1980), 413.

[2] C. S. Aulakh, I. G. Koh, S. Ouvry, Higher spin fields with mixed symmetry, *Phys. Lett.* **B173** (1986), 284.

[3] J. M. Labastida, T. R. Morris, Massless mixed symmetry bosonic free fields, *Phys. Lett.* **B180** (1986), 101; J. M. Labastida, Massless particles in arbitrary representations of the Lorentz group, *Nucl. Phys.* **B322** (1989), 185.

[4] C. Burdik, A. Pashnev, M. Tsulaia, On the mixed symmetry irreducible representations of the Poincaré group in the BRST approach, *Mod. Phys. Lett.* **A16** (2001), 731.

[5] Yu. M. Zinoviev, On massive mixed symmetry tensor fields in Minkowski space and (A)dS, hep-th/0211233.

[6] C. Bizdadea, C. C. Ciobirca, E. M. Cioroianu, S. O. Saliu, S. C. Sararu, Interactions of a massless tensor field with the mixed symmetry of the Riemann tensor. No-go results, hep-th/0306154.

[7] C. M. Hull, Duality in gravity and higher spin gauge fields, *JHEP* **0109** (2001), 027.

[8] X. Bekaert, N. Boulanger, Tensor gauge fields in arbitrary representations of $GL(D,\mathbb{R})$: duality & Poincaré lemma, hep-th/0208058.

[9] X. Bekaert, N. Boulanger, Massless spin-two field S-duality, *Class. Quant. Grav.* **20** (2003), S417; On geometric equations and duality for free higher spins, *Phys. Lett.* **B561** (2003), 183.

[10] H. Casini, R. Montemayor, L. F. Urrutia, Duality for symmetric second rank tensors. II. The linearized gravitational field, *Phys.Rev.* **D68** (2003), 065011.
[11] N. Boulanger, S. Cnockaert, M. Henneaux, A note on spin-s duality, *JHEP* **0306** (2003), 060.

[12] P. de Medeiros, C. Hull, Exotic tensor gauge theory and duality, *Commun. Math. Phys.* **235** (2003), 255.

[13] X. Bekaert, N. Boulanger, M. Henneaux, Consistent deformations of dual formulations of linearized gravity: A no-go result, *Phys. Rev.* **D67** (2003), 044010.

[14] Yu. M. Zinoviev, First order formalism for mixed symmetry tensor fields, [hep-th/0304067](hep-th/0304067).

[15] Yu. M. Zinoviev, First order formalism for massive mixed symmetry tensor fields in Minkowski and \((A)dS\) spaces, [hep-th/0306292](hep-th/0306292).

[16] G. Barnich, M. Henneaux, Consistent couplings between fields with a gauge freedom and deformations of the master equation, *Phys. Lett. B* **311** (1993), 123.

[17] W. Pauli, M. Fierz, On relativistic field equations of particles with arbitrary spin in an electromagnetic field, Helv. Phys. Acta **12** (1939) 297; M. Fierz, W. Pauli, On relativistic wave equations for particles of arbitrary spin in an electromagnetic field, Proc. Roy. Soc. Lond. **A173** (1939) 211.

[18] A. K. Bengtsson, I. Bengtsson, L. Brink, Cubic interaction terms for arbitrarily extended supermultiplets, *Nucl. Phys.* **B227** (1983), 41.

[19] M. A. Vasiliev, Cubic interactions of bosonic higher spin gauge fields in \(AdS(5)\), *Nucl. Phys.* **B616** (2001), 106; Erratum-ibid. **B652** (2003), 407.

[20] E. Sezgin, P. Sundell, \(7-D\) bosonic higher spin theory: symmetry algebra and linearized constraints, *Nucl. Phys.* **B634** (2002), 120.

[21] D. Francia, A. Sagnotti, Free geometric equations for higher spins, *Phys. Lett.* **B543** (2002), 303.

[22] C. Bizdadea, C. C. Ciobirca, E. M. Cioroianu, I. Negru, S. O. Salii, S. C. Sararu, Interactions of a single massless tensor field with the mixed symmetry \((3,1)\). No-go results, *JHEP* **0310** (2003), 019.
[23] G. Barnich, M. Henneaux, Comments on Unitarity in the Antifield Formalism, *Mod. Phys. Lett.* **A7** (1992), 2703.

[24] M. Dubois-Violette, M. Henneaux, Generalized cohomology for irreducible tensor fields of mixed Young symmetry type, Lett. Math. Phys. **49** (1999) 245; Tensor fields of mixed Young symmetry type and $N$ complexes, *Commun. Math. Phys.* **226** (2002) 393.

[25] G. Barnich, F. Brandt, M. Henneaux, Local BRST cohomology in the antifield formalism. I. General theorems, *Commun. Math. Phys.* **174** (1995), 57; Local BRST cohomology in gauge theories, *Phys. Rept.* **338** (2000), 439.

[26] N. Boulanger, T. Damour, L. Gualtieri, M. Henneaux, Inconsistency of interacting multi-graviton theories, *Nucl. Phys.* **B597** (2001), 127.

[27] M. Henneaux, Space-time locality of the BRST formalism, *Commun. Math. Phys.* **140** (1991), 1.

[28] M. Dubois-Violette, M. Henneaux, M. Talon, C. M. Viallet, Some results on local cohomologies in field theories, *Phys. Lett.* **B267** (1991), 81.

[29] G. Barnich, F. Brandt, M. Henneaux, Local BRST cohomology in the antifield formalism. II. Application to Yang-Mills theory, *Commun. Math. Phys.* **174** (1995), 93.

[30] C. Bizdadea, C. C. Ciobirca, E. M. Cioroianu, S. O. Saliu, S. C. Sararu, BRST cohomological results on the massless tensor field with the mixed symmetry of the Riemann tensor, [hep-th/0402099](https://arxiv.org/abs/hep-th/0402099).

[31] N. Boulanger, S. Cnockaert, Consistent deformations of [p,p]-type gauge field theories, *hep-th/hep-th/0402180.*