Random Time Dynamical Systems

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Abstract

In this paper, we introduce the concept of random time changes in dynamical systems. The subordination principle may be applied to study the long time behavior of the random time systems. We show, under certain assumptions on the class of random time, that the subordinated system exhibits a slower time decay which is determined by the random time characteristics. Along the path asymptotic, a random time change is reflected in the new velocity of the resulting dynamics.

Keywords: Dynamical systems, random time change, inverse subordinator, long time behavior

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1 Introduction

The idea to consider stochastic processes with general random times is known at least from the classical book by Gikhman and Skorokhod [13]. In the case of Markov processes time changes by subordinators
has been already considered by Bochner in [6], showing that it gives again a Markov process, the so-called *Bochner subordinated Markov process*. A more interesting scenario is realized when analyzing the case of inverse subordinators. Indeed, after the time change, we fail to obtain a Markov process. Therefore, the study of such kind of processes becomes really challenging. From this perspective, let us recall the work by Montroll and Weiss, [28], where the authors consider the physically motivated case of random walks in random time. This seminal paper originated a wide research activity related to the study of Markov processes with inverse stable subordinators as random times changes, see the book [27] for a detailed review and historical comments.

It is worth mentioning that when we take into account the case of processes with random time change which are not subordinators or inverse subordinators, we can only rely on few results, the overall field having been less investigated.

Indeed, on the one hand, additional assumptions on the stable subordinator considerably reduce the set of time change processes we can count on, resulting in restrictive assumptions for possible applications. On the other hand, we find technical difficulties in handling general inverse subordinators. Such limitations can be overcome for certain sub-classes of inverse subordinators, see, e.g., [21, 22]. Let us underline that the random time change approach turns to be a very effective tools in modeling several physical systems, spanning from ecological to biological ones, see, e.g., [25] and references therein, also in view of additional applications.

There is a natural question concerning the use of a random time change not only in stochastic dynamics but also with respect to a wider class of dynamical problems. In this paper we focus on the analysis of latter task in the case of dynamical systems taking values in \( \mathbb{R}^d \). In particular, let \( X(t, x), t \geq 0, x \in \mathbb{R}^d \) be a dynamical system in \( \mathbb{R}^d \), starting from \( x \) at initial time, namely: \( X(0, x) = x \). Of course, such a system is also a deterministic Markov process. Given \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) we define
\[
u(t, x) := f(X(t, x)),
\]
hence obtaining a version of the Kolmogorov equation, called the *Liouville equation* within the theory of dynamical systems:
\[
\frac{\partial}{\partial t} u(t, x) = L u(t, x),
\]
\( L \) being the generator of a semigroup which results to be the solution of the Liouville equation, see, e.g., [11, 29, 32] for more details.

If \( E(t) \) is an inverse subordinator process (see Section 2 below for details and examples), then we may consider the time changed random dynamical systems
\[
Y(t) := X(E(t)).
\]
Our aim is to analyze the properties of \( Y(t) \) depending on those of the initial dynamical systems \( X(t) \). In particular, we can define
\[
v(t, x) := \mathbb{E}[f(Y(t, x))],
\]
then trying to compare the behavior \( u(t, x) \) and \( v(t, x) \) for a certain class of functions \( f \).

In what follows, we present the main problems which naturally appear studying random time changes in dynamical systems. Moreover, we provide solutions to these problems with respect to the examples collected in Section 2.

The rest of the paper is organized as follows. In Section 2 we present the classes of inverse subordinators and the associated general fractional derivatives. In Section 3 we study random time dynamical systems, also considering the simplest examples of them. Moreover, we also provide the first results when the random time is associated to the \( \alpha \)-stable subordinator. In Subsection 3.4 we consider a dynamical system as a deterministic Markov processes, also introducing the notion of potential and Green
measure of the dynamical system. In Subsection 3.5 we investigate the path transformation of a simple dynamical system by a random time. The last part of the work, namely Section 4, is devoted to the analysis of transport equations and random time changes in this important class of equations.

2 Random times

In what follows, we recall some preliminary definitions and results related to random times processes and subordinators. Let us start with the following fundamental definition

**Definition 2.1.** Let \((Ω, F, P)\) be a probability space. A random time is a process \(E : [0, +∞) \times Ω \to \mathbb{R}^+\) such that

(i) for a.e. \(ω \in Ω\), \(E(t, ω) \geq 0\) for all \(t \in [0, +∞)\),

(ii) for a.e. \(ω \in Ω\), \(E(0, ω) = 0\),

(iii) the function \(E(⋅, ω)\) is increasing and satisfies

\[
\lim_{t \to +∞} E(t, ω) = +∞.
\]

Concerning the concept of subordinators, we can introduce it as follows:

**Definition 2.2.** Let \((Ω, F, P)\) be a probability space. A process \(\{S(t), t ≥ 0\}\) is a subordinator if the following conditions are satisfied

(i) \(S(0) = 0\);

(ii) \(S(t + r) - S(t)\) has the same law of \(S(r)\) for all \(t, r > 0\);

(iii) if \((F_t)\), denotes the filtration generated by \((S(t))\), i.e. \(F_t = \sigma(\{S(r), r ≤ t\})\), then \(S(t+r) - S(t)\) is independent of \(F_t\) for all \(t, r > 0\);

(iv) \(t \to S(t)(ω)\) is almost surely right-continuous with left limits;

(v) \(t \to S(t)\) is almost surely an increasing function.

For the sake of completeness, let us note that the process \(S(⋅)\) is a Lévy process if it satisfies the conditions (i) – (iv), see, e.g., [3] for more details. Let \(S = \{S(t), t ≥ 0\}\) be Lévy process, then its Laplace transform can be written in terms of a Bernstein function (also known as Laplace exponent) \(Φ : [0, ∞) \to [0, ∞)\) by

\[
E[e^{-\lambda S(t)}] = e^{-\lambda Φ(\lambda)}, \quad \lambda ≥ 0.
\]

Moreover, the function \(Φ\) admits the representation

\[
Φ(\lambda) = \int_0^{+∞} (1 - e^{-\lambda τ}) dσ(τ), \quad (2.1)
\]

where the measure \(σ\), also called Lévy measure, has support in \([0, ∞)\) and fulfills

\[
\int_0^{+∞} (1 ∧ τ) dσ(τ) < ∞. \quad (2.2)
\]

Let \(σ\) be a Lévy measure, we define the associated kernel \(k\) as follows

\[
k : (0, ∞) \to (0, ∞), \quad (2.3)
\]
\[ t \mapsto k(t) := \sigma\left((t, \infty)\right). \]

Its Laplace transform is denoted by \( K \), and, for any \( \lambda \geq 0 \), one has
\[ K(\lambda) := \int_0^{\infty} e^{-\lambda t} k(t) \, dt. \]  

(2.4)

We note that the relation between the function \( K \) and the Laplace exponent \( \Phi \) is given by
\[ \Phi(\lambda) = \lambda K(\lambda), \quad \forall \lambda \geq 0. \]  

(2.5)

Throughout the paper we shall suppose that

**Hypotheses 2.1.** Le \( \Phi \) be a complete Bernstein function, that is, the Lévy measure \( \sigma \) is absolutely continuous with respect to the Lebesgue measure. The functions \( K \) and \( \Phi \) satisfy
\[ K(\lambda) \to \infty, \text{ as } \lambda \to 0; \quad K(\lambda) \to 0, \text{ as } \lambda \to \infty; \]
\[ \Phi(\lambda) \to 0, \text{ as } \lambda \to 0; \quad \Phi(\lambda) \to \infty, \text{ as } \lambda \to \infty. \]

**Example 2.1** (\( \alpha \)-stable subordinator). A classical example of a subordinator \( S \) is the so-called \( \alpha \)-stable process with index \( \alpha \in (0, 1) \). In particular, a subordinator is \( \alpha \)-stable if its Laplace exponent is
\[ \Phi(\lambda) = \lambda^\alpha = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^{\infty} (1 - e^{-\lambda \tau})^{1-\alpha} \tau^{1-\alpha} \, d\tau, \]

where \( \Gamma \) is the gamma function.

In this case, the associated Lévy measure is given by \( d\sigma_\alpha(\tau) = \frac{\alpha}{\Gamma(1 - \alpha)} \tau^{-1+\alpha} \, d\tau \) and the corresponding kernel \( k_\alpha \) has the form
\[ k_\alpha(t) = g_{1-\alpha}(t) := \frac{t^{-\alpha}}{\Gamma(1 - \alpha)} t \geq 0, \]

with Laplace transform equal to \( K_\alpha(\lambda) = \lambda^{\alpha-1}, \text{ for } \lambda \geq 0. \)

**Example 2.2** (Gamma subordinator). The Gamma process \( Y^{(a,b)} \) with parameters \( a, b > 0 \) is another example of a subordinator with Laplace exponent
\[ \Phi_{(a,b)}(\lambda) = a \log \left( 1 + \frac{\lambda}{b} \right) = \int_0^{\infty} (1 - e^{-\lambda \tau}) a \tau^{-1} e^{-b \tau} \, d\tau, \]

the second equality being the Frullani integral. The associated Lévy measure is given by \( d\sigma_{(a,b)}(\tau) = a \tau^{-1} e^{-b \tau} \, d\tau \), with associated kernel equal to
\[ k_{(a,b)}(t) = a \Gamma(0, bt), \quad t > 0, \]

where
\[ \Gamma(\nu, z) := \int_z^{\infty} e^{-t} t^{\nu-1} \, dt \]

is the incomplete gamma function, see, e.g., [16, Section 8.3] for more details. Moreover, its Laplace transform is
\[ K_{(a,b)}(\lambda) = a \lambda^{-1} \log \left( 1 + \frac{\lambda}{b} \right), \quad \lambda > 0. \]
Example 2.3 (Truncated $\alpha$-stable subordinator). The truncated $\alpha$-stable subordinator, see [7] Example 2.1-(ii), $S_\delta, \delta > 0$, constitutes an example of a driftless $\alpha$-stable subordinator with Lévy measure given by
\[
d\sigma_\delta(\tau) := \frac{\alpha}{\Gamma(1-\alpha)} \tau^{-(1+\alpha)} \mathbb{1}_{(0,\delta]}(\tau) \, d\tau, \quad \delta > 0.
\]
The corresponding Laplace exponent is given by
\[
\Phi_\delta(\lambda) = \lambda \alpha \left(1 - \frac{\Gamma(-\alpha, \delta \lambda)}{\Gamma(-\alpha)}\right) + \frac{\delta - \alpha}{\Gamma(1-\alpha)},
\]
with associated kernel
\[
k_\delta(t) := \sigma_\delta((t, \infty)) = \frac{\mathbb{1}_{(0,\delta]}(t)}{\Gamma(1-\beta)} (t^{-\beta} - \delta^{-\beta}), \ t > 0.
\]

Example 2.4 (Sum of two alpha stable subordinators). Let $0 < \alpha < \beta < 1$ be given and let $S_{\alpha,\beta}(t)$, for $t \geq 0$, be the driftless subordinator with Laplace exponent given by
\[
\Phi_{\alpha,\beta}(\lambda) = \lambda \alpha + \lambda \beta.
\]
Then, by Example 2.7, we have that the corresponding Lévy measure $\sigma_{\alpha,\beta}$ is the sum of two Lévy measures. Indeed, it holds
\[
d\sigma_{\alpha,\beta}(\tau) = d\sigma_\alpha(\tau) + d\sigma_\alpha(\tau) = \frac{\alpha}{\Gamma(1-\alpha)} \tau^{-(1+\alpha)} \, d\tau + \frac{\beta}{\Gamma(1-\beta)} \tau^{-(1+\beta)} \, d\tau,
\]
implying that the associated kernel $k_{\alpha,\beta}$ reads as follow
\[
k_{\alpha,\beta}(t) := g_{1-\alpha}(t) + g_{1-\beta}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{t^{-\beta}}{\Gamma(1-\beta)}, \ t > 0,
\]
with associated Laplace transform given by
\[
\mathcal{K}_{\alpha,\beta}(\lambda) = \mathcal{K}_\alpha(\lambda) + \mathcal{K}_\beta(\lambda) = \lambda^{\alpha-1} + \lambda^{\beta-1}, \ \lambda > 0.
\]

Example 2.5 (Kernel with exponential weight). Taking $\gamma > 0$ and $0 < \alpha < 1$, let us consider the subordinator with Laplace exponent
\[
\Phi_\gamma(\lambda) := (\lambda + \gamma)^\alpha = \left(\frac{\lambda + \gamma}{\lambda}\right)^\alpha \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty (1-e^{-\lambda \tau}) \tau^{-(1+\alpha)} \, d\tau.
\]
Then the associated Lévy measure is given by
\[
d\sigma_\gamma(\tau) = \left(\frac{\lambda + \gamma}{\lambda}\right)^\alpha \frac{\alpha}{\Gamma(1-\alpha)} \tau^{-(1+\alpha)} \, d\tau,
\]
which implies a kernel $k_\gamma$ with exponential weight. In particular, we have
\[
k_\gamma(t) = g_{1-\alpha}(t) e^{-\gamma t} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} e^{-\gamma t}.
\]
The corresponding Laplace transform of $k_\gamma$ is then given by $\mathcal{K}_\gamma(\lambda) = \lambda^{-\alpha}(\lambda + \gamma)^\alpha, \ \lambda > 0.$


2.1 Inverse subordinators and general fractional derivatives

In this section we introduce the inverse subordinators and the corresponding general fractional derivatives.

**Definition 2.3.** Let \( S(\cdot) \) be a subordinator. We define \( E(\cdot) \) as the inverse process of \( S(\cdot) \), i.e.

\[
E(t) := \inf \{ r > 0 \mid S(r) > t \} = \sup \{ r \geq 0 \mid S(r) \leq t \} \quad \text{for all} \ t \in [0, +\infty).
\]

For any \( t \geq 0 \), we denote by \( G^t_k(\tau) := G^t(\tau) \), \( \tau \geq 0 \) the marginal density of \( E(t) \) or, equivalently

\[
G^t(\tau) \, d\tau = \frac{\partial}{\partial \tau} \mathbb{P}(E(t) \leq \tau) \, d\tau = \frac{\partial}{\partial \tau} \mathbb{P}(S(\tau) \geq t) \, d\tau = -\frac{\partial}{\partial \tau} \mathbb{P}(S(\tau) < t) \, d\tau.
\]

**Remark 2.1.** If \( S \) is the \( \alpha \)-stable process, \( \alpha \in (0, 1) \), then the inverse process \( E(t) \) has Laplace transform, see [5, Prop. 1(a)], given by

\[
E[\exp(-\lambda E(t))] = \int_0^\infty e^{-\lambda \tau} G^t(\tau) \, d\tau = \sum_{n=0}^{\infty} \frac{(-\lambda t^\alpha)^n}{n! (n\alpha + 1)} = E(\alpha)(-\lambda t^\alpha).
\]

By the asymptotic behavior of the Mittag-Leffler function \( E(\alpha) \), it follows that \( E[\exp(-\lambda E(t))] \sim C t^{-\alpha} \) as \( t \to \infty \). Using the properties of the Mittag-Leffler function \( E(\alpha) \), we can show that the density \( G^t(\tau) \) is given in terms of the Wright function \( W_{\mu,\nu} \), namely

\[
G^t(\tau) = t^{-\alpha} W_{-\alpha,1-\alpha}(\tau t^{-\alpha}),
\]

see [14] for more details.

For a general subordinator, the following lemma determines the \( t \)-Laplace transform of \( G^t(\tau) \), with \( k \) and \( K \) given in (2.3) and (2.4), respectively.

**Lemma 2.1.** The \( t \)-Laplace transform of the density \( G^t(\tau) \) is given by

\[
\int_0^\infty e^{-\lambda \tau} G^t(\tau) \, d\tau = K(\lambda) e^{-\tau \lambda K(\lambda)}.
\]

The double \((\tau,t)\)-Laplace transform of \( G^t(\tau) \) is

\[
\int_0^\infty \int_0^\infty e^{-p \tau} e^{-\lambda \tau} G^t(\tau) \, d\tau \, d\tau = \frac{K(\lambda)}{\lambda K(\lambda) + p}.
\]

**Proof.** For the proof see [19] or [31, Lemma 3.1] \( \square \)

Let us now recall the definition of General Fractional Derivative (GFD) associated to a kernel \( k \), see [19] and references therein for more details.

**Definition 2.4.** Let \( S \) be a subordinator and the kernel \( k \in L^1_{\text{loc}}(\mathbb{R}^+) \) given in (2.3). We define a differential-convolution operator by

\[
(D^{(k)}_t u)(t) = \frac{d}{dt} \int_0^t k(t - \tau) u(\tau) \, d\tau - k(t) u(0), \quad t > 0.
\]

**Remark 2.2.** The operator \( D^{(k)}_t \) is also known as Generalized Fractional Derivative.
Example 2.6 (Distributed order derivative). Consider the kernel \( k \) defined by

\[
k(t) := \int_0^1 g_\alpha(t) \, d\alpha = \int_0^1 \frac{\alpha^{\alpha-1}}{\Gamma(\alpha)} \, d\alpha, \quad t > 0.
\]

Then it is easy to see that

\[
K(\lambda) = \int_0^\infty e^{-\lambda t} k(t) \, dt = \frac{\lambda - 1}{\lambda \log(\lambda)}, \quad \lambda > 0.
\]

The corresponding differential-convolution operator \( D^{(k)}_t \) is called distributed order derivative, see, e.g., [1, 10, 15, 17, 18, 26] for more details and applications.

We conclude this section with a result that will be useful later on, starting by recalling the following definition.

Definition 2.5. Given the functions \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \), we say that \( f \) and \( g \) are asymptotically equivalent at infinity, and denote \( f \sim g \) as \( x \to +\infty \), if

\[
\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1.
\]

Moreover, we say that \( f \) is slowly varying if

\[
\lim_{x \to +\infty} \frac{f(\lambda x)}{f(x)} = 1, \quad \text{for any } \lambda > 0.
\]

For more details on slowly varying functions, we refer the interested reader to, e.g., [12, 30].

Lemma 2.2. Suppose hypotheses 2.7 are satisfied, and that the subordinator \( S(t) \), along with its inverse \( E(t) \), \( t \geq 0 \), are such that

\[
K(\lambda) \sim \lambda^{-\gamma} Q \left( \frac{1}{\lambda} \right), \quad \lambda \to 0,
\]

where \( 0 \leq \gamma \leq 1 \) and \( Q(\cdot) \) is a slowly varying function. Moreover, define

\[
A(t, z) := \int_0^\infty e^{-z \tau} G_t(\tau) \, d\tau, \quad t > 0, \quad z > 0.
\]

Then it holds

\[
A(t, z) \sim \frac{1}{z} \frac{t^{\gamma-1}}{\Gamma(\gamma)} Q(t), \quad t \to \infty.
\]

Proof. For the proof see [20] Theorem 4.3. \( \square \)

Remark 2.3. We point out that the condition (2.11) on the Laplace transform of the kernel \( k \) is satisfied by all Examples 2.1–2.5 and 2.6 stated above. The case of Example 2.4 is easily checked as

\[
K(\lambda) = \lambda^\alpha + \lambda^\beta = \lambda^{(1-\alpha)}(1 + \lambda^{-(\alpha-\beta)}) = \lambda^{-\gamma} Q \left( \frac{1}{\lambda} \right),
\]

where \( \gamma = 1 - \alpha > 0 \) and \( Q(t) = 1 + t^{\alpha-\beta} \) is a slowly varying function.
2.2 Compound Poisson Process

A significant example of subordinator is given by the Compound Poisson Process (CPP). Roughly speaking, a CPP is a jump (stochastic) process, whose both jumps size and the number of them, are independent random variables. First, we define a random process \(N(t)\) modeling the number of jumps that occurred in given time interval \([0, t]\), \(t > 0\).

**Definition 2.6.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. A random process \(N : [0, +\infty) \times \Omega \to \mathbb{N} \cup \{0\}\) is a discrete Poisson process with rate \(\lambda > 0\) if it satisfies the following properties

(i) \(N(0) = 0\);

(ii) \(\forall t, s \in [0, +\infty)\) such that \(t > s\) one has that \(N(t) - N(s)\) is independent of \(N(s)\);

(iii) the random variable \(N(t) - N(s)\) has a Poisson distribution with parameter \(\lambda(t - s)\).

**Remark 2.4.** The property (iii) implies that \(N(\cdot, \omega)\) is increasing for almost all \(\omega \in \Omega\), namely

\[
\mathbb{P}\{\omega \in \Omega : N(\cdot, \omega) \text{ is not increasing}\} = 0.
\]

Moreover

\[
\lim_{t \to +\infty} N(t, \omega) = +\infty, \text{ for a.a. } \omega \in \Omega.
\]

A CPP is defined as follows.

**Definition 2.7.** Let \(N(\cdot)\) be a discrete Poisson process with rate \(\lambda\), then \(S(\cdot)\) is said to be a CPP of rate \(\lambda > 0\) if it admits the following representation

\[
S(t) = \sum_{i=1}^{N(t)} R_i,
\]

where \(\{R_i\}\) are non-zero and non-negative i.i.d. random variables independent of \(N(\cdot)\).

It is straightforward to note that a CPP is also a random time process.

**Remark 2.5.** The random variables \(R_i\) represent the jumps of the process \(S\), while \(N(t)\) is the number of jumps occurred in \([0, t]\). Moreover, for each \(\omega \in \Omega\), \(N(\cdot, \omega)\) is represented by an increasing step function.

It is well known that the moment generating function (or Laplace transform) of a CPP of parameter \(\lambda\) is given by

\[
L_{S(t)}(s) := \mathbb{E}[e^{sS(t)}] = e^{\lambda(t) - 1},
\]

\(L_{R(\cdot)}\) being the moment generating function of the random variables \(R_i\). This results holds true for all \(s\) in the domain of \(L_{R}\).

According to the definitions given in \((2.1)-(2.5)\), for a CPP we have:

- the Laplace exponent is given by

\[
\Phi(s) = \lambda(1 - L_{R(-s)}) = \lambda \mathbb{E}[1 - e^{-sR}] = \lambda \int_0^{+\infty} (1 - e^{-sr}) dF_R(r),
\]

where \(F_R(\cdot)\) is the cumulative distribution function of \(R\);
the associated Lévy measure and kernel are respectively given by
\[ \sigma((a,b)) = \lambda P(a \leq R \leq b), \quad k(t) = \sigma((t, +\infty)) = \lambda P(R \geq t), \]
while the Laplace transform of \( k \) reads as follow
\[ K(s) = \Phi(s) = \lambda (1 - L R(-s)) s. \]
Let us note that in this latter case Hypotheses 2.1 are not satisfied, since \( \Phi(s) \to \lambda \) when \( s \to +\infty \).

### 3 Random time dynamical systems

#### 3.1 Dynamical systems and Liouville equations

There is a natural question concerning the use of a random time change not only in stochastic dynamics, but more generally in an ample class of dynamical problems. In what follows, we shall focus the attention on the analysis of the random time change approach for dynamical systems taking values in \( \mathbb{R}^d \).

Let \( X(t, x), t \geq 0 \) be a dynamical system in \( \mathbb{R}^d \) such that \( X(0, x) = x \in \mathbb{R}^d \). Such a system is also a deterministic Markov process. Therefore, given \( f : \mathbb{R}^d \to \mathbb{R} \), and defining
\[ u(t, x) := f(X(t, x)), \]
we have a version of the Kolmogorov equation, which is nothing but the Liouville equation within the theory of dynamical systems. Indeed,
\[ u_t(t, x) = Lu(t, x), \quad (3.1) \]
where \( L \) is the generator of the semigroup solution of the Liouville equation, see, e.g., [11, 29, 32].

#### 3.2 Random time changes and fractional Liouville equations

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Let \( X(t, x), t \geq 0 \), be a dynamical system in \( \mathbb{R}^d \) starting at time \( t = 0 \) from \( x \in \mathbb{R}^d \). Given an inverse subordinator process \( E(\cdot) \), we consider the time changed random dynamical systems
\[ Y(t, \omega; x) = X(E(t, \omega); x), \quad t \in [0, +\infty), x \in \mathbb{R}^d, \omega \in \Omega. \]

For a suitable \( f : \mathbb{R}^d \to \mathbb{R} \) we define
\[ v(t, x) := E[f(Y(t; x))], \quad (3.2) \]
where, without loss of generality, with \( E(t) \) and \( Y(t; x) \) we shortly refer to \( E(t, \cdot; x) \), resp. to \( Y(t, \cdot; x) \). As pointed out in, e.g., [7, 31], \( v(t, x) \) solves an evolution equation with the generator \( L \), with generalized fractional derivative (see (2.9)), i.e.
\[ D_t^{(k)} v(\cdot, x)(t) = Lv(t, x). \quad (3.3) \]
Let \( u(t, x) \) be the solution to (3.1) with the same generator \( L \) in (3.3). Under quite general assumptions there is an essentially obvious relation between these evolutions
\[ v(t, x) = \int_0^\infty u(\tau, x) G(t) d\tau, \quad (3.4) \]
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\( G_t(\tau) \) being the density of \( E(t) \), as defined in Section 2.1.

Having in mind the analysis of the random time change influence on the asymptotic properties of \( v(t, x) \), we may suppose that the latter formula gives all necessary technical equipments. Unfortunately, the situation is essentially more complicated. In fact, the knowledge we have of the properties characterizing the density \( G_t(\tau) \) is, in general, very poor. The aim of this section is to describe a class of subordinators for which we may obtain information about the time asymptotic of the generalized fractional dynamics.

### 3.3 First examples

We consider the simplest evolution equation in \( \mathbb{R}^d \)

\[
dX(t) = v dt \in \mathbb{R}^d, \quad X(0) = x_0 \in \mathbb{R}^d,
\]

with corresponding dynamics given by

\[
X(t) = x_0 + vt, \quad t \geq 0.
\]

Without loss of generality, let us assume that \( x_0 = 0 \). Then, we take \( f(x) = e^{-\alpha |x|}, \alpha > 0 \). Hence, the corresponding solution to the Liouville equation is

\[
u(t, x) = e^{-\alpha t|v|}, \quad t \geq 0.
\]

**Proposition 3.1.** Assume that the assumptions of Lemma 2.2 are satisfied. Then

\[
v(t, x) \sim \frac{1}{\alpha |v| \Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.
\]

**Proof.** From the explicit form of the solution \( u(t, x) \), and using both (3.4) and Lemma 2.2 we obtain

\[
v(t, x) \sim \frac{1}{\alpha |v| \Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.
\]

In particular, for the \( \alpha \)-stable subordinator considered in Example 2.1, we obtain

\[
u(t, x) \sim C t^{-\alpha}, \quad t \to \infty.
\]

For \( d = 1 \) consider the dynamics

\[
\beta dX(t) = \frac{1}{X^{\beta-1}(t)} dt, \quad \beta \geq 1,
\]

then the solution is given by

\[
X(t) = (t + C)^{1/\beta}.
\]

Considering the function \( f(x) = \exp(-a|x|^\beta), a > 0 \), and supposing that the assumptions of Lemma 2.2 are satisfied, then, exploiting the explicit form of the solution \( u(t, x) \), we have that the long time behavior of the subordination \( v(t, x) \) is given by

\[
v(t, x) \sim \frac{e^{-aC}}{a \Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.
\]

In particular, choosing the density \( G_t(\tau) \) of the inverse subordinator \( E(t) \) as in the Example 2.4 we obtain

\[
v(t, x) \sim C t^{-\alpha}(1 + t^{\alpha-\beta}) \sim C t^{-\alpha}, \quad t \to \infty.
\]
3.4 Green measures

The notion of potential is a classical topic within the theory of Markov processes, see, e.g., [4]. Recently, it has been proposed the concept of Green measure as a representation of potentials in an integral form, see [23]. The modification of these concepts for time changed Markov processes was investigated in [24]. Considering a dynamical system as a deterministic Markov processes, we have the possibility to study the notion of potential and Green measure in this context.

According with the above stated framework, given a function \( f : \mathbb{R}^d \to \mathbb{R} \), we consider the solution to the Cauchy problem

\[
\begin{aligned}
  u_t(t, x) &= Lu(t, x), \\
  u(0, x) &= f(x),
\end{aligned}
\]

obtaining

\[ u(t, x) = (e^{tL}f)(x). \]

Then, by defining the potential for the function \( f \) as

\[ U(f, x) := \int_0^\infty u(t, x) \, dt = \int_0^\infty (e^{tL}f)(x) \, dt = -(L^{-1}f)(x), \quad x \in \mathbb{R}^d, \]

the existence of \( U(f, x) \) is not clear at all. Indeed, it depends on the class of functions \( f \) and the Liouville generator \( L \). Assuming the existence of \( U(f, x) \) we aim at obtaining an integral representation

\[ U(f, x) = \int_{\mathbb{R}^d} f(y) \, d\mu^x(y), \quad (3.5) \]

\( \mu^x \) being a random measure on \( \mathbb{R}^d \), that we will call the Green measure of our dynamical system. As in the case of Markov processes, the definition of the potential is easy to introduce but difficult to analyse for each particular model. Moreover, on the base of specific examples, we may assume that the potentials are well defined for special classes of functions \( f \). Nevertheless, we can not expect the existence of an associated Green measure.

As already seen, after a random time change we will have the subordinated solution \( v(t, x) \) for the fractional equation, see equation (3.3). Then we can try to re-define the potential

\[ V(f, x) := \int_0^\infty v(t, x) \, dt, \quad x \in \mathbb{R}^d, \]

which turns to be divergent for general random times. Indeed, by the subordination formula (3.4) and the Fubini theorem, we have

\[ V(f, x) = \int_0^\infty \int_0^\infty u(\tau, x)G_t(\tau) \, d\tau \, dt = \int_0^\infty u(\tau, x) \left( \int_0^\infty G_t(\tau) \, d\tau \right) \, dt \, d\tau, \]

where the inner integral is not convergent because of the equality (2.7) together with the Hypotheses 2.1. To overcome this difficulty we may use the notion of renormalized potential. In particular, inspired by the time change of Markov processes (see [24] for details), we define the renormalized potential

\[ V_r(f, x) := \lim_{t \to \infty} \frac{1}{N(t)} \int_0^t v(s, x) \, ds, \quad t \geq 0, \quad (3.6) \]

where \( N(t) \) is defined by \( N(t) := \int_0^t k(s) \, ds \). Then by assuming the existence of \( U(f, x) \), we have

\[ V_r(f, x) = \int_0^\infty u(t, x) \, dt. \]
3.5 Path transformations

Let us now investigate how the trajectories of dynamical systems transform under random times. According to what seen above, we consider the Liouville equation for
\[ u(t, x) := f(X(t, x)), \quad t \geq 0, \ x \in \mathbb{R}^d, \]
that is,
\[ u_t(t, x) = Lu(t, x), \quad u(0, x) = f(x), \]
\(L\) being the generator of a semigroup. In addition, let \(E(t), \ t \geq 0,\) be the inverse subordinator process. Then we can consider the time changed random dynamical systems
\[ Y(t, x) = X(E(t), x), \quad t \geq 0, \ x \in \mathbb{R}^d, \]
where, without loss of generality, \(E(t),\) resp. \(Y(t; x),\) shortly refer to \(E(t, \cdot),\) resp. to \(Y(t, \cdot; x).\)
Defining
\[ v(t, x) := \mathbb{E}[f(Y(t, x))], \]
by the subordination formula, we have
\[ v(t, x) = \int_0^\infty u(\tau, x)G_t(\tau) \, d\tau. \]
Considering the vector-function \(f : \mathbb{R}^d \to \mathbb{R}\) defined as
\[ f(x) = x, \]
we have that the average trajectories of \(Y(t, x)\) read as follow
\[ \mathbb{E}[Y(t, x)] = \int_0^\infty X(\tau, x)G_t(\tau) \, d\tau. \]
Then considering the dynamical system of Section 3.3 namely \(X(t, x) = vt,\) we obtain
\[ \mathbb{E}[Y(t, x)] = v \int_0^\infty \tau G_t(\tau) \, d\tau. \]
Therefore, we need to know the first moment of the density \(G_t.\) Considering the case of the inverse \(\alpha\)-stable subordinator stated in Example 2.1 we have
\[ \int_0^\infty \tau G_t(\tau) \, d\tau = Ct^{\alpha}. \]
Therefore, the asymptotic of the time changed trajectory will be slower (proportional to \(t^{\alpha}\)) instead of initial linear \(vt\) motion. In a forthcoming paper we will study in detail these results for other classes of inverse subordinators.

4 Random time transport equations

Let \(b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d, \ d \geq 1,\) be a bounded continuous vector field. We consider the following dynamical system
\[
\begin{cases}
  dX(t; x) = b(X(t; x)) \, dt, \\
  X(0; x) = x,
\end{cases}
\]
(4.1)
with starting point $x \in \mathbb{R}^d$, at initial time $t = 0$. Let us consider a bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$ and define
\[ u(t, x) := f(X(t; x)), \]
(4.2)
where $X$ is defined by (4.1).

In what follows, we prove that $u$ is a classical solution of the first-order parabolic equation, provided $b$ and $f$ are regular enough.

**Proposition 4.1.** Let $b$ and $f$ be bounded continuous functions. Then, $u$, defined by (4.2), is a classical solution of the following first-order parabolic equation
\[
\begin{cases}
  u_t(t, x) = b(x) \cdot \nabla u(t, x), \forall (t, x) \in (0, +\infty) \times \mathbb{R}^d, \\
  u(0, x) = f(x), \forall x \in \mathbb{R}^d.
\end{cases}
\]
(4.3)

**Remark 4.1.** In case of non-autonomous drift, i.e. $b = b(t, x)$, the equation (4.3) fails to be true, as shown by the following counter-example. If $d = 1$ and
\[ f(x) = x, \quad b(t, x) = t + x, \]
the solution of (4.1) is $u(t, x) = X(t; x) = (1 + x)e^t - t - 1$. Then, one has
\[ u_t(t, x) = (1 + x)e^t - 1, \quad b(t, x)u_x(t, x) = (t + x)e^t, \]
and the equation (4.3) is not satisfied.

**Proof of Proposition 4.1.** By assumptions on $b$ and $f$, we have
\[
\begin{align*}
  u_t(t, x) &= \nabla f(X(t; x)) \cdot b(X(t; x)), \\
  b(x) \cdot \nabla u(t, x) &= b(x) \cdot (\nabla f(X(t; x))\text{Jac}_x(X(t; x))).
\end{align*}
\]
To prove (4.3), we have to check
\[ b(X(t; x)) = \text{Jac}_x(X(t; x))b(x), \]
equivalently
\[ \varphi(t, x) := b(X(t; x)) - \text{Jac}_x(X(t; x))b(x) = 0 \quad \forall t \geq 0, x \in \mathbb{R}^d. \]
Computing the time derivative of $\varphi$, we obtain
\[ \varphi_t(t, x) = \text{Jac}_b b(X(t; x))b(X(t; x)) - \partial_t \text{Jac}_x(X(t; x))b(x). \]
By differentiating equation (4.1) with respect to $x$, we have
\[
\begin{align*}
  \partial_t \text{Jac}_x(X(t; x)) &= \text{Jac}_b b(X(t; x))\text{Jac}_x(X(t; x)), \\
  \text{Jac}_x(X(0; x)) &= I_{d \times d},
\end{align*}
\]
and the derivative $\varphi_t$ becomes
\[ \varphi_t(t, x) = \text{Jac}_b b(X(t; x))(b(X(t; x)) - \text{Jac}_x(X(t; x))b(x)) = \text{Jac}_b (X(t; x))\varphi(t, x). \]
Thus, $\varphi$ satisfies the following ODE in time
\[
\begin{cases}
  \varphi_t(t, x) = \alpha(t, x)\varphi(t, x), \\
  \varphi(0, x) = 0,
\end{cases}
\]
(4.4)
where $\alpha(t, x) := \text{Jac}_b (X(t; x))$. Since $b$ is smooth then $\alpha$ is locally Lipschitz, and (4.4) admits the unique solution $\varphi = 0$, completing the proof.

Let us note that if $b$ and $f$ are continuous functions, previous computations fail to be true, and the equation (4.3) has to be understood in the *viscosity sense*, see below.
4.1 Viscosity solutions

For the sake of completeness, let us introduce some notations that we will use throughout this section. We indicate with $\text{USC}([0, +\infty) \times \mathbb{R}^d)$ the space of upper semicontinuous functions on $[0, +\infty) \times \mathbb{R}^d$, while we use $\text{LSC}([0, +\infty) \times \mathbb{R}^d)$ for the space of lower semicontinuous functions on $[0, +\infty) \times \mathbb{R}^d$. Given a function $g : \mathbb{R}^d \to \mathbb{R}$, we say that

- $g$ satisfies the Hölder condition if the following holds
  $$|g(x) - g(y)| \leq C|x - y|^\beta,$$
  for $0 < \beta \leq 1$, $C > 0$.
- $g$ is a Lipschitz function with sublinear growth if
  $$|g(x)| \leq C(1 + |x|^\beta), \quad |g(x) - g(y)| \leq C|x - y|,$$
  for $C > 0$, $0 < \theta < 1$;
- $g$ is a continuous decreasing function if $\forall x, y \in \mathbb{R}$ one has
  $$\langle g(x) - g(y), x - y \rangle \leq 0$$

Let $u : (0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$ be a continuous function. We define $D^{1,+}u(t_0, x_0)$ the superdifferential of $u$ at $(t_0, x_0)$, i.e., the set of all points $(a, p) \in \mathbb{R} \times \mathbb{R}^d$ such that for $(t, x) \to (t_0, x_0)$ we have
$$u(t, x) \leq u(t_0, x_0) + a(t - t_0) + \langle p, x - x_0 \rangle + o(|t - t_0| + |x - x_0|).$$
Moreover, we define $D^{-1}u(t_0, x_0)$ the subdifferential of $u$ at $(t_0, x_0)$, i.e., the set of all points $(a, p) \in \mathbb{R} \times \mathbb{R}^d$ such that for $(t, x) \to (t_0, x_0)$ we have
$$u(t, x) \geq u(t_0, x_0) + a(t - t_0) + \langle p, x - x_0 \rangle + o(|t - t_0| + |x - x_0|).$$
Furthermore, we recall the definition of viscosity solutions given in [9].

**Definition 4.1.**

(i) Let $u \in \text{USC}([0, +\infty) \times \mathbb{R}^d)$ and be a bounded function from above. We say that $u$ is a viscosity subsolution of (4.3) in $(0, +\infty) \times \mathbb{R}^d$ if
$$\begin{cases}
\varphi_t(t, x) - b(x) \cdot \nabla \varphi(t, x) \leq 0, \\
u(0, x) \leq f(x),
\end{cases}$$
for any $\varphi \in C^1(\mathbb{R}^{d+1})$ such that $\varphi - u$ has a (strict) minimum value at $(t, x) \in (0, +\infty) \times \mathbb{R}^d$.

(ii) Let $u \in \text{LSC}([0, +\infty) \times \mathbb{R}^d)$ and be a bounded function from below. We say that $u$ is a viscosity supersolution of (4.3) in $(0, +\infty) \times \mathbb{R}^d$ if
$$\begin{cases}
\varphi_t(t, x) - b(x) \cdot \nabla \varphi(t, x) \geq 0, \\
u(0, x) \geq f(x),
\end{cases}$$
for any $\varphi \in C^1(\mathbb{R}^{d+1})$ such that $\varphi - u$ has a (strict) maximum value at $(t, x) \in (0, +\infty) \times \mathbb{R}^d$.

(iii) Let $u$ be a bounded continuous function. We say that $u$ is a viscosity solution of (4.3) in $(0, +\infty) \times \mathbb{R}^d$ if it is both a subsolution and supersolution.

**Remark 4.2.** As it is well know, Def. 4.1 can be expressed in terms of subdifferential and superdifferential, i.e.,
$$a - b(x) \cdot p \leq 0 \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^d, \forall (a, p) \in D^{1,+}u(t, x),$$
$$a - b(x) \cdot p \geq 0 \quad \forall (t, x) \in (0, +\infty) \times \mathbb{R}^d, \forall (a, p) \in D^{-1}u(t, x).$$
4.2 Existence and uniqueness results

To prove both existence and uniqueness of a solution $u$ as in (4.2), we need to show the following

**Theorem 4.1 (Comparison Principle).** Let $u$, resp. $v$, be a subsolution, resp. a supersolution, of (4.3). Suppose that:

1. $f$ is bounded and satisfies the Hölder condition;
2. at least one of the following conditions is satisfied
   
   (i) $b$ is a Lipschitz function with sublinear growth,
   (ii) $b$ is a continuous decreasing function.

Then $u(t, x) \leq v(t, x)$ for all $(t, x) \in [0, +\infty) \times \mathbb{R}^d$.

**Remark 4.3.** The proof is based on the ideas developed in [8]. However, the parabolic case in the unbounded domain $\mathbb{R}^d$ was not developed there. For the convenience of the reader we include the proof of Theorem 4.1 within the Appendix 5.

**Theorem 4.2.** Suppose that $f$ and $b$ satisfies hypotheses of Thorem 4.1. Then there exists a unique viscosity solution $u$ of the problem (4.3). Moreover, $u$ admits the following representation formula

$$u(t, x) = f(X(t; x)),$$

where $X(t; x)$ solves (4.1).

**Proof.** The uniqueness directly follows from Theorem 4.1. Concerning the existence, if $f$ and $b$ are smooth function, then the existence and the representation formula have been already proved in Proposition 4.1.

In the general case, let $\{f_n\}_n$ and $\{b_n\}_n$ be two approximating sequences of smooth functions, locally uniformly converging to $f$ and $b$. Without loss of generality, we can suppose that $b_n$ and $f_n$ are respectively locally and globally bounded uniformly in $n$, and satisfy the same assumptions of $f$ and $b$ with Hölder and Lipschitz constants bounded uniformly in $n$.

Let $X_n$ be the solution of (4.1) associated to $b_n$. Let $u_n$ be the solution of (4.3) with $f_n$ and fix $K \subset \subset [0, +\infty) \times \mathbb{R}^d$. Since $b_n$ and $f_n$ are locally bounded uniformly in $n$, we have that for all $(t, x) \in K$

$$|X_n(t; x)| \leq |x| + \int_0^t |b_n(X_n(s; x))| \, ds \leq C_K,$$

$$|u_n(t, x)| = |f_n(X_n(t; x))| \leq C_K,$$

for a positive constant $C_K$.

Now we want to prove the Lipschitz bounds for $X_n$ and the Hölder bounds for $u_n$. Using the estimate in (4.6), and recalling that $X_n$ is solution of (4.1) we have

$$|X_n(t, x) - X_n(s, x)| \leq \int_s^t |b_n(X_n(r; x))| \, dr \leq C_K|t - s|$$

for any $(t, x), (s, x) \in K$ and $t \geq s$.

Whereas for all $(t, x)$ and $(t, y)$ belong to $K$, one has

$$|X_n(t, x) - X_n(t, y)|^2 \leq |x - y|^2 + \int_0^t (b_n(X_n(r, x)) - b_n(X_n(r, y))) \cdot (X_n(r, x) - X_n(r, y)) \, dr.$$
If \( b_n \) satisfies (i), then we have
\[
|X_n(t, x) - X_n(t, y)|^2 \leq |x - y|^2 + C \int_0^t |X_n(r, x) - X_n(r, y)|^2 \, dr.
\]
By Gronwall’s lemma, we have
\[
|X_n(t, x) - X_n(t, y)|^2 \leq |x - y|^2 e^{Ct} \implies |X_n(t, x) - X_n(t, y)| \leq C_K |x - y|,
\]
therefore obtaining Lipschitz bounds for \( X_n \).

Concerning the Hölder bounds for \( n \), and passing to the limit for (4.3). Indeed, we consider a smooth function \( \phi \).

Then, by combining above estimates, we have
\[
|u(t, x) - u(t, y)| \leq C |x - y|,
\]
where \( C \) is a positive constant.

Then, by combining above estimates, we have
\[
|X_n(t, x) - X_n(t, s)| \leq C_K (|t - s| + |x - y|), \quad \forall (t, x), (s, y) \in K. \tag{4.8}
\]

Concerning the Hölder bounds for \( u_n \), by the Hölder bound on \( f_n \), together with the estimate \( 4.8 \), we have
\[
|u_n(t, x) - u_n(t, s)| \leq C_1 |X_n(t, x) - X_n(s, y)|^{\beta} \leq C_K (|t - s|^{\beta} + |x - y|^{\beta}). \tag{4.9}
\]

Therefore, \( \{u_n\}_n \) and \( \{X_n\}_n \) are uniformly bounded and uniformly equicontinuous in all compact sets \( K \subset \subset [0, \infty) \times \mathbb{R} \). Using the Ascoli-Arzela’s theorem, we obtain
\[
\exists u, X \quad \text{s.t.} \quad u_n \to u, \quad X_n \to X \quad \text{locally uniformly}.
\]

Recalling that (4.5) holds true for \( u_n \), and by the uniform convergence of \( f_n, X_n \) and \( u_n \), we obtain that \( u \) admits the representation formula (4.5). Therefore, we are left to prove that \( u \) is a solution of (4.3). Indeed, we consider a smooth function \( \phi \) such that \( \phi - u \) achieves his (strict) minimum in \( (t, x) \in (0, +\infty) \times \mathbb{R}^d \). Perturbing \( \phi \) with a smooth function \( \psi \) as follow:

- \( \phi(s, y) = \psi(s, y) \) for any \((s, y) \in B_1(t, x)\);
- \( \lim_{s \to +\infty} \psi(s, y) = \lim_{|y| \to +\infty} \psi(s, y) = +\infty; \)
- \( \psi - u \) achieves his (strict) minimum in \((t, x)\).

and recalling that \( u_n \to u \) uniformly, we derive that \( \psi - u_n \) admits minimum, i.e.
\[
\min_{(0, +\infty) \times \mathbb{R}^d} (\psi - u_n) = \psi(t_n, x_n) - u_n(t_n, x_n).
\]

Since \( \psi \) is coercive and \( u_n \) is bounded, there exists a compact subset \( S \) of \((0, +\infty) \times \mathbb{R}^d \) such that the sequence \( \{(t_n, x_n)\}_n \subset S \). Then, up to subsequences, we have that \( (t_n, x_n) \to (\bar{t}, \bar{x}) \), for a certain \((\bar{t}, \bar{x}) \in S \). But since \((t_n, x_n)\) is the minimum for \( \psi - u_n \), we have
\[
\psi(s, y) - u_n(s, y) \geq \psi(t_n, x_n) - u_n(t_n, x_n), \forall (s, y) \in (0, +\infty) \times \mathbb{R}^d,
\]
and passing to the limit for \( n \to +\infty \), we obtain
\[
\psi(s, y) - u(s, y) \geq \psi(\bar{t}, \bar{x}) - u(\bar{t}, \bar{x}),
\]
meaning that \((\bar{t}, \bar{x})\) is a global minimum for \(\psi - u\), and so \((\bar{t}, \bar{x}) = (t, x)\). Using the definition of viscosity subsolution of \((4.3)\) for \(u_n\), we have
\[
\psi_t(t_n, x_n) - b(x_n) \cdot \nabla \psi(t_n, x_n) \leq 0.
\]
Again, passing to the limit and taking into account that \(\psi\) and \(\phi\) coincide in \(B_1(t, x)\), we get
\[
\phi_t(t, x) - b(x) \cdot \nabla \phi(t, x) \leq 0,
\]
which proves that \(u\) is a subsolution of \((4.3)\). Analogously, we can show that \(u\) is also a supersolution, hence completing the proof. \(\square\)

4.3 Asymptotic of viscosity solutions

In this section we study the asymptotic behaviour of \(u\), starting by studying the asymptotic behaviour of \(X\).

**Proposition 4.2.** Let \(b\) be a continuous function satisfying hypotheses of Theorem \(4.7\). Suppose that \(X : (0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d\) is a solution of \((4.1)\), then the following holds true.

- If \(b\) satisfies the following
  \[
  b(x) \cdot (x - x_0) < 0 \tag{4.10}
  \]
  for some \(x_0 \in \mathbb{R}^d\) and \(x \in \mathbb{R}^d\), then \(x_0\) is a globally asymptotically stable equilibrium point.

- If \(b\) is such that
  \[
  b(x) \cdot (x - x_0) \leq -c|x - x_0|^2, \tag{4.11}
  \]
  for some positive constant \(c\), then \(X\) satisfies
  \[
  |X(t; x) - x_0| \leq |x - x_0|e^{-ct} \tag{4.12}
  \]
  for all \(t \in (0, +\infty), x \in \mathbb{R}^d\).

**Proof.** The proof is a standard application of both Lyapunov’s Theorem and Gronwall’s lemma. Suppose that \(b\) satisfies \((4.10)\), then evaluating \((4.10)\) in \(x = x_0 + re_i, r \in \mathbb{R}, i = 1, \ldots, d\), we get
\[
re_i b(x_0 + re_i) < 0 \quad \forall r \in \mathbb{R} \implies b(x_0 + re_i) = 0 \quad \forall i \implies b(x_0) = 0,
\]
which implies that \(x_0\) is an equilibrium point. Moreover
\[
V(x) = |x - x_0|^2,
\]
is a strict Lyapunov function, hence \(x_0\) is a globally asymptotically stable equilibrium point. Suppose that \((4.11)\) holds true. Using the fundamental theorem of calculus, we have
\[
|X(t; x) - x_0|^2 = |x - x_0|^2 + 2 \int_0^t b(X(s; x)) \cdot (X(s; x) - x_0) \, ds
\]
\[
\leq |x - x_0|^2 - 2c \int_0^t |X(s; x) - x_0|^2 \, ds,
\]
for some positive constant \(c\) and by Gronwall’s lemma we get
\[
|X(t; x) - x_0|^2 \leq |x - x_0|^2 e^{-2ct},
\]
implying
\[
|X(t; x) - x_0| \leq |x - x_0|e^{-ct}.
\]
Proposition 4.2 implies the following convergence result for the function $u$.

**Theorem 4.3.** Suppose that $b$ satisfies the hypotheses of Proposition 4.2 and let $f$ be a function such that the assumptions of Theorem 4.1 hold true. Then, the solution $u$ of (4.3) satisfies

$$|u(t, x) - f(x_0)| \leq C|x - x_0|^\beta e^{-c \beta t} \text{ for all } t \in (0, +\infty), \ x \in \mathbb{R}^d,$$

where $c$ and $C$ are positive constants, $0 < \beta \leq 1$ and $x_0 \in \mathbb{R}^d$.

**Proof.** By the Hölder assumption on $f$, recalling the representation (4.5) and (4.12), we have

$$|u(t, x) - f(x_0)| = |f(X(t; x)) - f(x_0)| \leq C|X(t; x) - x_0|^\beta \leq C|x - x_0|^\beta e^{-c \beta t},$$

and the proof is complete.

### 4.4 Random time viscosity solution

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $E(\cdot)$ be an inverse subordinator process. Given the solutions $u$, resp. $X$ of (4.3), resp. (4.1), we can consider the random time dynamic

$$Y(t, \omega; x) = X(E(t, \omega); x), \quad t \in [0, +\infty), \ x \in \mathbb{R}^d, \ \omega \in \Omega,$$

with corresponding function $v$

$$v(t, x) = \mathbb{E}[u(E(t), x)] = \mathbb{E}[f(Y(t; x))],$$

(4.14)

where, without loss of generality, with $E(t)$, resp. $Y(t; x)$, we shortly refer to $E(t, \cdot)$, resp. to $Y(t, \cdot; x)$. According to what we have already observed in Section 3, $v$ is the solution to an evolution equation with the same generator $L$.

#### 4.4.1 General classes of random times

In the case of dissipative dynamic, we can easily obtain a general estimate for the function $v$.

**Proposition 4.3.** Assume that the assumptions of Lemma 2.2 and Theorem 4.3 are satisfied, with $f(x_0) = 0$. Then

$$|v(t, x)| \leq C_x \frac{1}{\alpha|v|\Gamma(\gamma)} t^{\gamma-1} Q(t), \quad t \to \infty.$$

**Proof.** From (4.13) we have

$$|u(t, x)| \leq C|x - x_0|^\beta e^{-c \beta t}.$$

Hence, using (3.4) we get

$$|v(t, x)| \leq C_x \int_0^\infty e^{-c \beta t} G_t(\tau) \ d\tau,$$

and by Lemma 2.2 we conclude the proof.

**Remark 4.4.** For the $\alpha$-stable subordinator considered in Example 2.1, we obtain $v(t, x) \sim C t^{-\alpha}$, $C$ is a constant. Therefore, starting with a solution $u(t, x)$ with exponential decay after subordination we observe a polynomial decay with the order defined by the random time characteristics.

In the next Subsection 4.4.2 we analyse the behavior of the subordination under a Compound Poisson Process (CPP).
The case of inverse Poisson process

In the case of a CPP, we cannot apply Lemma 2.2 since hypotheses 2.1 are not satisfied. Therefore, we need to change our approach.

**Theorem 4.4.** Suppose that the assumptions in Proposition 4.2 and Theorem 4.1 hold. Let $S(\cdot)$ be a CPP with rate $\lambda$. Assume that the jumps $R_i$ have a finite moment of order $\alpha > 0$. Then the function $v$ satisfies

$$|v(t,x) - f(x_0)| \leq \frac{C}{t^\alpha} |x - x_0|^{\beta},$$

where $x_0 \in \mathbb{R}^d$, $C$ is a positive constant and $0 < \beta \leq 1$. Moreover, if there exists $\delta > 0$ such that $\mathbb{E} \left[e^{\delta R}\right] < +\infty$, then $v$ satisfies

$$|v(t,x) - f(x_0)| \leq C|x - x_0|^{\beta} e^{-\eta t},$$

with $\eta > 0$.

**Proof.** By Theorem (4.3) we get

$$|v(t,x) - f(x_0)| \leq C|x - x_0|^{\beta} \mathbb{E} \left[e^{-c\beta E(t)}\right]. \tag{4.15}$$

In order to estimate the average term, we can argue as follows:

$$\mathbb{E} \left[e^{-c\beta E(t)}\right] = \int_0^{+\infty} \mathbb{P} \left(e^{-c\beta E(t)} \geq r\right) dr = \int_0^1 \mathbb{P} \left(E(t) \leq -\frac{\ln r}{c\beta}\right) dr = \int_0^1 \mathbb{P} \left(S\left(-\frac{\ln r}{c\beta}\right) \geq t\right) dr,$$

since $-\ln r < 0$ when $r > 1$ and $\mathbb{P}(E(t) \leq z) = \mathbb{P}(S(z) \geq t)$ for all $t, z > 0$. To compute the last integral, we separately study the quantity $\mathbb{P}(S(z) \geq t)$, for $t, z \geq 0$. We have

$$\mathbb{P}(S(z) \geq t) = \sum_{k=0}^{+\infty} \mathbb{P}(S(z) \geq t|N(z) = k)\mathbb{P}(N(z) = k) = \sum_{k=1}^{+\infty} \mathbb{P} \left(\sum_{i=1}^k R_i \geq t\right) e^{-\lambda z} \left(\frac{\lambda z}{k!}\right)^k,$$

which allows us to derive

$$\mathbb{E} \left[e^{-c\beta E(t)}\right] = \int_0^1 \sum_{k=1}^{+\infty} \mathbb{P} \left(\sum_{i=1}^k R_i \geq t\right) e^{\frac{\lambda \ln s}{c\beta}} \left(\frac{\lambda \ln s}{c\beta}\right)^k \frac{1}{k!} ds = \sum_{k=1}^{+\infty} \frac{1}{k!} \left(\frac{\lambda}{c\beta}\right)^k \mathbb{P} \left(\sum_{i=1}^k R_i \geq t\right) \int_0^1 e^{-\frac{\ln y}{c\beta} - \frac{\lambda}{c\beta} y} y^k dy. \tag{4.16}$$

where we have used the change of variable $y = -\ln s$. Exploiting the density of a $\Gamma\left(k + 1, \frac{\lambda + c\beta}{c\beta}\right)$ function, we know that the latter integral equals

$$\int_0^1 e^{-\frac{\ln y}{c\beta} - \frac{\lambda}{c\beta} y} y^k dy = k! \left(\frac{c\beta}{\lambda + c\beta}\right)^{k+1}.$$

Then, exploiting above estimates, we can rewrite (4.15) as follows

$$|v(t,x) - f(x_0)| \leq \frac{Cc\beta}{\lambda + c\beta} |x - x_0|^\beta \sum_{k=1}^{+\infty} \left[ \mathbb{P} \left(\sum_{i=1}^k R_i \geq t\right) \left(\frac{\lambda}{\lambda + c\beta}\right)^k \right]. \tag{4.17}$$
By assumption on $R_i$, and using Markov’s inequality, we get
\[
P\left( \sum_{i=1}^{k} R_i \geq t \right) \leq \frac{\mathbb{E} \left[ \left( \sum_{i=1}^{k} R_i \right)^{\alpha} \right]}{t^{\alpha}} \leq \frac{1}{t^{\alpha}} k^{\alpha+1} \mathbb{E}[R^\alpha]. \tag{4.18}
\]
Replacing (4.18) in (4.17), and recalling that the series converge, we have
\[
|v(t, x) - f(x_0)| \leq C t^{\alpha} |x - x_0|^\beta + \sum_{k=1}^{+\infty} \left( k^{\alpha+1} \left( \frac{\lambda}{\lambda + c\beta} \right)^k \right) \leq C t^{\alpha} |x - x_0|^\beta,
\]
where $C$ is a positive constant. Analogously, if $\mathbb{E}[e^{\delta R}] < +\infty$ for a certain $\delta > 0$, fixing $0 < \eta < \delta$, we can apply again Markov’s inequality, with the function $\phi(x) = e^{\eta x}$, to get
\[
P\left( \sum_{i=1}^{k} R_i \geq t \right) \leq e^{-\eta t} \mathbb{E} \left[ e^{\eta \sum_{i=1}^{k} R_i} \right] = e^{-\eta t} \mathbb{E} \left[ e^{\eta R} \right]^{k},
\]
implying that (4.17) becomes
\[
|v(t, x) - f(x_0)| \leq C e^{-\eta t} |x - x_0|^\beta \sum_{k=1}^{+\infty} \left( \frac{\lambda \mathbb{E}[e^{\eta R}]}{\lambda + c\beta} \right)^k = C e^{-\eta t} |x - x_0|^\beta,
\]
where we choose $\eta$ such that
\[
\frac{\lambda \mathbb{E}[e^{\eta R}]}{\lambda + c\beta} < 1,
\]
which completes the proof.

5 Appendix: proof of Theorem 4.1

We note that, with the change of variable $\tilde{u}(t, x) = e^{-\lambda t} u(t, x)$, the system (4.3) is equivalent to the following one
\[
\begin{cases}
u_t(t, x) - b(x) \cdot \nabla u(t, x) + \lambda u(t, x) = 0, \\
u(0, x) = f(x).
\end{cases}
\tag{5.1}
\]
therefore, to show the validity of the comparison principle, we will work with the problem (5.1), for a certain $\lambda \gg 1$ to be later chosen.

Let $u$, resp. $v$, be a subsolution, resp. a supersolution, of (5.1). Arguing by contradiction, let us assume that there exists $(s, z) \in [0, +\infty) \times \mathbb{R}^d$ such that $u(s, z) - v(s, z) = \delta > 0$. For $\alpha, \nu > 0$, we consider
\[
u_t(t, x) - v(t, y) - \frac{\alpha}{2} |x - y|^2 - \frac{1}{\alpha} |x|^2 - \nu t.
\tag{5.2}
\]
Due to the coercive term $\frac{1}{\alpha} |x|^2$ and the boundedness of $u$ and $v$, the (5.2) achieves a maximum.

We denote the maximum by $M$ and one of its maximum points with $(\bar{t}, \bar{x}, \bar{y}) \in [0, +\infty) \times \mathbb{R}^d$. Hence, for $\nu$ sufficiently small and $\alpha$ sufficiently large, we have
\[
u_t(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) - \frac{\alpha}{2} |\bar{x} - \bar{y}|^2 - \frac{1}{\alpha} |\bar{x}|^2 - \nu \bar{t} \geq u(s, z) - v(s, z) - \frac{1}{\alpha} |\bar{z}|^2 - \nu s \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}. \tag{5.3}
\]
Moreover, thanks to the boundedness of $u$ and $v$, it holds
\[
\frac{1}{\alpha} |\bar{x}|^2 + \frac{1}{2} |\bar{x} - \bar{y}|^2 \leq u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y}) \leq C \implies \lim_{\alpha \to \infty} |\bar{x} - \bar{y}| = \lim_{\alpha \to \infty} \frac{|\bar{x}|^{1+\theta}}{\alpha} = 0. \tag{5.4}
\]

**Case 1** If $\bar{t} = 0$, one has
\[
\frac{\delta}{2} \leq u(s, z) - v(s, z) - \frac{1}{\alpha} |z|^2 - \nu s \leq u(0, \bar{x}) - v(0, \bar{y}) - \frac{\alpha}{2} |\bar{x} - \bar{y}|^2 - \frac{1}{\alpha} |\bar{x}|^2.
\]
Since $f$ is Hölder continuous, $u(0, \cdot) \leq f(\cdot) \leq v(0, \cdot)$ and $|\bar{x} - \bar{y}| \to 0$, we have that for $\alpha$ sufficiently large
\[
u(0, \bar{x}) - v(0, \bar{y}) \leq \frac{\delta}{3} \implies u(0, \bar{x}) - v(0, \bar{y}) - \frac{\alpha}{2} |\bar{x} - \bar{y}|^2 - \frac{1}{\alpha} |\bar{x}|^2 \leq \frac{\delta}{3},
\]
which leads to a contradiction.

**Case 2:** Suppose that $\bar{t}$ belongs to $\in (0, +\infty)$. Using [8, Theorem 8.3.] with the following choices
\[
u_1(t, x) = u(t, x), \quad u_2(t, y) = -v(t, y), \quad \phi(t, x, y) = \frac{\alpha}{2} |x - y|^2 + \frac{1}{\alpha} |x|^2 + \nu t
\]
then there exist $a, c \in \mathbb{R}$ such that $a + c = \nu$ and
\[(a, \nabla_x \phi(\bar{t}, \bar{x}, \bar{y})) \in \overline{D^{1,\nu} u(\bar{t}, \bar{x})}, \quad (-c, -\nabla_y \phi(\bar{t}, \bar{x}, \bar{y})) \in \overline{D^{1,\omega} v(\bar{t}, \bar{y})}.
\]
From now on, we will omit the dependence on $(\bar{t}, \bar{x}, \bar{y})$ for the function $\phi$. Since $u$ is a subsolution, while $v$ is a supersolution, of (5.1), then we have
\[
\begin{align*}
\nu + \lambda (u(\bar{t}, \bar{x}) - v(\bar{t}, \bar{y})) &\leq b(\bar{x}) \cdot \nabla_x \phi + b(\bar{y}) \cdot \nabla_y \phi. \tag{5.5}
\end{align*}
\]
The first term in the left-hand side is non-negative, so we can ignore it. For the second term, using (5.3), we get
\[
\frac{\lambda \delta}{2} + \frac{\alpha}{2} |\bar{x} - \bar{y}|^2 \leq b(\bar{x}) \cdot \nabla_x \phi + b(\bar{y}) \cdot \nabla_y \phi.
\]
To estimate the right-hand side term, we first compute the derivatives of $\phi$
\[
\nabla_x \phi = \alpha (\bar{x} - \bar{y}) + \frac{2}{\alpha} \bar{x}, \quad \nabla_y \phi = -\alpha (\bar{x} - \bar{y}),
\]
therefore we have
\[
b(\bar{x}) \cdot \nabla_x \phi + b(\bar{y}) \cdot \nabla_y \phi \leq \alpha (b(\bar{x}) - b(\bar{y})) \cdot (\bar{x} - \bar{y}) + \frac{2}{\alpha} b(\bar{x}) \cdot \bar{x}.
\]
If $b$ satisfies the condition $(i)$, using (5.4) we get
\[
\alpha (b(\bar{x}) - b(\bar{y})) \cdot (\bar{x} - \bar{y}) + \frac{2}{\alpha} b(\bar{x}) \cdot \bar{x} \leq C \alpha |x - y|^2 + \omega(\alpha),
\]
21
where $\omega(\alpha)$ is a quantity which tend to 0 when $\alpha \to +\infty$.

On the other hand, if $b$ satisfies the condition $(ii)$ we get

$$\alpha(b(\bar{x}) - b(\bar{y})) \cdot (\bar{x} - \bar{y}) \leq 0, \quad b(\bar{x}) \cdot \bar{x} \leq b(0) \cdot \bar{x},$$

hence

$$\alpha(b(\bar{x}) - b(\bar{y})) \cdot (\bar{x} - \bar{y}) + \frac{2}{\alpha} b(\bar{x}) \cdot \bar{x} \leq \omega(\alpha).$$

In both cases we obtain

$$b(\bar{x}) \cdot \nabla x \phi + b(\bar{y}) \cdot \nabla y \phi \leq C\alpha |\bar{x} - \bar{y}|^2 + \omega(\alpha).$$

Using the above estimates in (5.5), we get

$$\frac{\lambda \delta}{2} + \frac{\lambda \alpha}{2} |\bar{x} - \bar{y}|^2 \leq C\alpha |\bar{x} - \bar{y}|^2 + \omega(\alpha),$$

which again leads to a contradiction for $\alpha$ sufficiently small and $\lambda$ sufficiently large. Since $u$ and $v$ are, respectively, a subsolution and a supersolution of (5.1) for $\lambda \geq 0$, then $e^{\lambda t} u$, resp. $e^{\lambda t} v$, is a subsolution, resp. a supersolution, of (4.3). Therefore, $e^{(\lambda - \mu)t} u$, resp. $e^{(\lambda - \mu)t} v$, is a subsolution, resp. a supersolution, of (5.1) with $\lambda$ replaced by $\mu$. Then, considering $\mu$ large enough we have

$$e^{(\lambda - \mu)t} u \leq e^{(\lambda - \mu)t} v \implies u \leq v,$$

completing the proof.

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