A genuine reinterpretation of the Heisenberg’s ("uncertainty") relations

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Abstract

In spite of their popularity the Heisenberg’s ("uncertainty") Relations (HR) still generate controversies. The Traditional Interpretation of HR (TIHR) dominate our days science, although over the years a lot of its defects were signaled. These facts justify a reinvestigation of the questions connected with the interpretation / significance of HR. Here it is developped such a reinvestigation starting with a revaluation of the main elements of TIHR. So one finds that all the respective elements are troubled by insurmountable defects. Then it results the indubitable failure of TIHR and the necessity of its abandonment. Consequently the HR must be deprived of their quality of crucial physical formulae. Moreover the HR are shown to be nothing but simple fluctuations formulae with natural analogous in classical (non-quantum) physics. The description of the measuring uncertainties (traditionally associated with HR) is approached from a new informational perspective. The Planck’s constant $\hbar$ (also associated with HR) is revealed to have a significance of generic indicator for quantum stochasticity, similarly with the role of Boltzmann’s constant $k$ in respect with the thermal stochasticity. Some other adjacent questions are also briefly discussed in the end.
Motto: "uncertainty principle: it has to do with the uncertainty in predictions rather than the accuracy of measurement. I think in fact that the word "measurement" has been so abused in quantum mechanics that it would be good to avoid it altogether"

John S. Bell, 1985.

I. INTRODUCTION

The Heisenberg’s (or uncertainty) Relations (HR) have a large popularity, being frequently regarded as crucial formulae of physics or (Martens 1991) even as expression of ”the most important principle of the twentieth century physics”. Nevertheless today one knows (Bunge 1977) that HR ”are probably the most controverted formulae in the whole of the theoretical physics”. The controversies originate in the association of the (supposed special) characteristics of measurements at atomic scale with HR respectively with the foundation
and interpretation of quantum theory. The respective association was initiated and especially sophisticated within the Traditional (conventional or orthodox) Interpretation of HR (TIHR). Very often the TIHR is amalgamated with the so-called Copenhagen interpretation of quantum mechanics.

Elements of the alluded association were preserved one way or another in almost all investigations of HR subsequent to TIHR. It is notable that, in spite of their number and variety, the mentioned investigations have not yet solved in essence the controversies connected with TIHR. But, curiously, today, large classes of publications and scientists seem to omit (or even to ignore) discussions about the controversies and defects characterizing the TIHR. So, tacitly, in our days TIHR seems to remain a largely adopted doctrine which dominates the questions regarding the foundation and interpretation of quantum theory. For all that (Piron 1982) "the idea that there are defects in the foundations of orthodox quantum theory is unquestionable present in the conscience of many physicists".

No doubt, first of all, the above quoted idea regards questions connected with TIHR. Then the respective questions require further studies and probably new views. We believe that a promising strategy to satisfy such requirements is to develop an investigation guided by the goals presented under the following Points (P):

1. **P – 1.1**: From the vague multitude of sophisticated statements of TIHR to identify its main elements (hypotheses, arguments/motivations and assertions).
2. **P – 1.2**: To add together the significant defects of TIHR located in connection with the above mentioned elements.
3. **P – 1.3**: To examine the verity of the respective defects as well as their significance with respect to TIHR.
4. **P – 1.4**: To see if such an examination defends TIHR or irrefutably pleads against it.
5. **P – 1.5**: In the latter case to admit the failure of TIHR and to abandon it as an incorrect and useless doctrine.
6. **P – 1.6**: To see if HR are veritable physical formulae.
7. **P – 1.7**: To search for a genuine reinterpretation of HR.
P−1.8: To give a (first) evaluation of the direct consequences of the mentioned reinterpretation. ▲

P−1.9: To note a few remarks on some adjacent questions. ▲

In this paper we wish to develop an investigation of the HR problematic in the spirit of the above mentioned points P−1.1 — P−1.8. For such a purpose we will appeal to some elements (ideas and results) from our works published in last two decades (Dumitru 1974 a, 1974 b, 1977, 1980, 1984, 1987, 1988, 1989, 1991, 1993, 1996, 1999; Dumitru and Verriets 1995). But here we strive to incorporate the respective elements into a more argued and elaborated approach. Also we try to make our exposition as self-contained as possible so that the reader should find it sufficiently meaningful and persuasive without any important appeals to other texts.

Through the announced investigation we shall find that all the main elements of TIHR are affected by insurmountable defects. Therefore we shall reveal the indubitable failure of TIHR and the necessity of its abandonment. Then it directly follows that in fact HR do not have any significance connected with the (measuring) uncertainties. That is why in this paper for the respective relations we do not use the wide-spread denomination of "uncertainty relations".

A consequence of the above alluded revelations is the fact that HR must be deprived of their quality of crucial physical formulae. So we come in consonance with the guess (Dirac 1963) that: "uncertainty relations in their present form will not survive in the physics of future".

The failure of TIHR leaves open a conceptual space which firstly requires justified answers to the questions from P−1.6 and P−1.7. The respective answers must be incorporated in a concordant view about the subjects of the following points:

P−1.10: The genuine description of the measurements. ▲

P−1.11: The foundation and the interpretation of the actually known quantum theory. ▲
The above mentioned subjects were amalgamated by TIHR through a lot of assertions/assumptions which now appear as fallacious. That is why we suggest that an useful view to be built on a natural differentiation of the respective subjects.

In such a view the actual quantum theory must be considered as regarding only intrinsic properties of the entities (particles and fields) from the microworld. The aspects of the respective properties included in the theoretical version of HR refer to the stochastic characteristics of the considered entities. But note that stochastic attributes are specific also in the case some macroscopic physical systems (e.g. thermodynamical ones), characterized by a class of macroscopic formulae similar with HR. Also the Planck’s constant $\hbar$ (involved in quantum HR) proves itself to be similar with the Boltzmann’s constant $k$ (involved in the mentioned macroscopic formulae). Both mentioned constants appear as generic indicators of stochasticity.

In the spirit of the above suggested view the description of the measurements remains a question which is extrinsic as regards the properties of the considered physical systems. Also it must be additional and independent from the actually known branches of theoretical physics (including the quantum mechanics). The respective branches refer only to the intrinsic properties of the considered systems. Then the measurements appear as processes which supply out-coming (received) information/data about the intrinsic properties of the measured systems. So regarded the measurements can be described through some mathematical models. In such models the measuring uncertainties can be described by means of various estimators.

The above announced views about the HR problematic facilitate reconsiderations and (we think) nontrivial comments about some questions regarding the foundations of quantum mechanics.

For developing our exposition in the next sections we will quote directly only a restricted number of references. This because our goal is not to give an exhaustive review of the literature dealing with TIHR. The readers interested in such reviews are invited to consult the known monographical and bibliographical publications (e.g.: Jammer, 1966; De Witt and
Graham, 1971; Jammer, 1974; Nilson, 1976; Yanase et. al. 1978; Primas, 1981; Ballentine, 1982; Cramer, 1986; Dodonov and Man’ko, 1987; Martens, 1991; Braginski and Khalili, 1992; Omnes 1992,1994; Bush et. al., 1996).

II. THE MAIN ELEMENTS OF TIHR

In spite of its popularity, in its promoting literature, TIHR is reported rather as a vague multitude of sophisticated statements but not as a systematized ensemble of clearly defined main elements (hypotheses, arguments/motivations and assertions). However, from the respective publications there can be identified and sorted out such an ensemble which, in our opinion, can be presented as follows:

On the best authority (Heisenberg, 1977) today it is known that the TIHR story originates in the search of general answers to the primary questions mentioned under the following points:

\[ P_{-2.1} \]: Are all measurements affected by measuring uncertainties? ▲

\[ P_{-2.2} \]: How can the respective uncertainties be represented quantitatively in a mathematical scheme? ▲

In connection with \[ P_{-2.1} \], TIHR adopted the following hypotheses:

\[ P_{-2.3} \]: The measuring uncertainties are due to the perturbations of the measured system as a result of its interactions with the measuring instrument. ▲

\[ P_{-2.4} \]: In the case of macroscopic systems the mentioned perturbations can be made arbitrarily small and, consequently, always the corresponding uncertainties can be considered as negligible.

\[ P_{-2.5} \]: In the case of quantum systems (microparticles of atomic size) the alluded perturbations are essentially unavoidable and consequently for certain measurements (see below \[ P_{-2.12} \]) the corresponding uncertainties are non-negligible. ▲

In the shadow of the hypotheses mentioned in \[ P_{-2.4} \] and \[ P_{-2.5} \] the TIHR attention was limited only to the quantum cases. For approaching such cases with respect to \[ P_{-2.4} \]
TIHR restored to the following motivation resources:

**P − 2.6** : Analysis of some thought (gedanken) measuring experiments.▲

**P − 2.7** : Appeal to some theoretical formulae from the existing quantum mechanics.▲

The two resources were used in undisguised association. So, from the starting-point, in TIHR the questions regarding the description of the measurements, respectively the foundation and interpretation of the existing quantum theory, were explicitly amalgamated.

For accuracy of the discussions in the following we shall use the term *variable* in order to denote a physical quantity which describes a specific property/characteristic of a physical system. With adequate delimitations the respective term will be used in both theoretical and experimental sense. In the former case it is connected with the theoretical modeling of system. In the latter case it is related with the data given by measurements about the system.

In connection with **P − 2.6** there was considered (Heisenberg, 1927, 1930) the case of simultaneous measurements of two (canonically) conjugated quantum variables A and B (such are coordinate q and momentum p or time t and energy E). The correspondingly Thought Experimental (TE) uncertainties are \( \Delta_{TE} A \) and \( \Delta_{TE} B \). They were found to be interconnected through the following A-B formula

\[
\Delta_{TE} A \cdot \Delta_{TE} B \approx \hbar
\]

(2.1)

where \( \hbar \) is the quantum Planck’s constant.

As regards **P − 2.7** firstly there was introduced (Heisenberg 1927, 1930) the following q-p theoretical formula:

\[
\Delta_\Psi q \cdot \Delta_\Psi p \geq \frac{\hbar}{2}
\]

(2.2)

(with equality only for Gaussian wave function \( \Psi \)). Afterwards, TIHR partisans replaced Eq. (2.2) by the more general A-B theoretical formula

\[
\Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle_\Psi \right|
\]

(2.3)
Here $[\hat{A}, \hat{B}]_-$ denotes the commutator of the quantum operators $\hat{A}$ and $\hat{B}$, with $[\hat{A}, \hat{B}]_- = \pm i\hbar$ in the case of conjugated variables. (For further details about the quantum notations, in actual usance version, see below the Sec. V).

Equations (2.1) - (2.2)/(2.3) were taken by TIHR as motivation supports. Based on the such supports TIHR partisans promoted a whole doctrine (vision). The main (essential) elements of the respective doctrine come down to the following points, grouped in pairs of Assertions (A) and Motivations (M):

**P – 2.8/A**: The quantities $\Delta_{TE} A$ and $\Delta_{\Psi} A$ from Eqs. (2.1) and (2.2)/(2.3) denoted by an unique symbol $\Delta A$, have an identical significance of measuring *uncertainty* for the quantum variable $A$.

**P – 2.8/M**: The above mentioned TIHR presumptions about $\Delta_{TE} A$ and the (formal) resemblance between Eqs. (2.1) and (2.2).

**P – 2.9/A**: Equations (2.1) and (2.2)/(2.3) admit the same generic interpretation of uncertainty relations for simultaneous measurements of the variables $A$ and $B$.

**P – 2.9/M**: The presumed significance for $\Delta_{TE} A$ and $\Delta_{TE} B$ from Eq. (2.1) and the resemblance between Eqs. (2.1) and (2.12)/(2.3).

**P – 2.10/A**: A solitary quantum variable $A$ can be measured without any uncertainty (with unlimited accuracy).

**P – 2.10/M**: For such a variable, considered independently from other variables, the Eqs. (2.1) - (2.3) do not impose a lower bound for the uncertainty $\Delta A$.

**P – 2.11/A**: Two commutable variables $A$ and $B$ can be measured simultaneously with arbitrarily small (even null) uncertainties $\Delta A$ and $\Delta B$.

**P – 2.11/M**: For such variables $[\hat{A}, \hat{B}]_- = 0$ and in Eq. (2.3) the product $\Delta A \cdot \Delta B$ has not a lower bound.

**P – 2.12/A**: Two non-commutable variables $A$ and $B$ can be measured simultaneously only with non-null and interdependent uncertainties $\Delta A$ and $\Delta B$.

**P – 2.12/M**: In such a case $[\hat{A}, \hat{B}]_- \neq 0$ and in Eq. (2.3) as well as in Eq. (2.1) the
The product $\Delta A \cdot \Delta B$ of the corresponding simultaneous uncertainties has as a lower bound a non-null quantity.\(^\text{\textbullet}\)

**P - 2.13/A**: The HR defined by Eqs. (2.1) - (2.3) (and named uncertainty relations) are typically quantum formulae and they have no similar in classical (non-quantum) physics.\(^\text{\textbullet}\)

**P - 2.13/M**: Presence of the quantum (Planck) constant $\hbar$ in Eqs. (2.1) - (2.3) and its absence in all known formulae of classical physics.\(^\text{\textbullet}\)

The above mentioned points can be regarded as main elements of TIHR. This because any piece of the variety of TIHR statements is obtained and advocated by means of some combinations of the respective elements.

Among the alluded pieces we mention here the ones regarding the mutual relations of quantum variables. TIHR adopted the idea:

**P - 2.14**: A variable exists / (can be defined) only when it is measurable with absolute accuracy (without uncertainty).\(^\text{\textbullet}\)

By combining this idea with **P - 2.11** and **P - 2.12** in the TIHR literature it is often promoted the statement:

**P - 2.15**: Two quantum variables are compatible respectively incompatible as their operators are respectively are not commutable. Consequently a complete description of a quantum system must be made in terms of a set of mutually compatible variables.\(^\text{\textbullet}\)

In the same literature one finds also the opinion that:

**P - 2.16**: Two incompatible variables (especially the canonically conjugated ones) are complementary (i.e. mutually exclusive) - similarly as in the complementarity relation for the corpuscular and wave characteristics of microparticles of atomic size.\(^\text{\textbullet}\)

### III. A FEW REMARKS ON TIHR HISTORY

TIHR was initiated by Heisenberg but later on it was developed and especially promoted by the Copenhagen School guided by N. Bohr. In a first stage TIHR had a relatively modest motivation, based only on the Eqs. (2.1) and (2.2). However it was largely accepted
in scientific and academic communities, partly due to the authority of its promoters. So, the establishing of TIHR as a doctrine started.

In a second stage TIHR partisans introduced a multitude of thought-experimental or theoretical formulae which resemble more or less the Eqs. (2.1)-(2.3). In spite of their (conceptual and/or mathematical) diversity the respective formulae were declared as *uncertainty relations* and their existence and interpretation were regarded as supports for an extended motivation of TIHR. So, for its partisans, TIHR was viewed as a well established and irrefutable doctrine. Such a view was widely promoted in leading publications (especially in textbooks).

In the meantime the alluded view was confronted with the notification of some defects of TIHR. But, as a rule, the respective notifications appeared disparately, sometimes in marginal publications and from non-leading authors. So the mentioned defects were not presented as a systematized ensemble and TIHR was criticized on certain points but not in its totality. An appreciation viewing somehow the alluded totality was noted altogether solitarily (Primas 1981). Referring to the post-Copenhagen interpretation of quantum mechanics it says: "Heisenberg’s uncertainty relations are no longer at the heart of the interpretation but are simple consequences of the basic mathematical formalism". Here one should remark that, as we know, such an appreciation has never been used in order to elucidate the shortcomings of the TIHR doctrine. Moreover it seems that, even in the our-days publications regarding the interpretation of quantum mechanics, the respective appreciation is not taken properly into account.

In the presented circumstances the TIHR partisans ignored or even denied the alluded defects. Such an attitude was sustained mainly by putting forward thought experiments and/or the authority of the mentioned partisans. But note that, in this way, for most of the cases, the notifications of TIHR defects were not really counteracted and the corresponding controversies were not elucidated. For all that TIHR survived over the decades with the appearance of an uncontroversial doctrine and became a veritable myth. Undoubted signs of the respective myth are present even today in publications (especially in textbooks) and
in the thinking of peoples (particularly in academic communities).

Here, it is interesting to observe Heisenberg’s own attitude towards TIHR story. It is surprising to see in the afferent literature that, although he was the initiator of TIHR doctrine, Heisenberg was not involved in the subsequent history of the respective doctrine. So he did not develop mathematical generalizations or interpreting extensions/sophistications of the Eqs. (2.1)-(2.3). Also he did not participate in the controversies regarding the TIHR defects. Probably that was the reason why in one of his last publications on HR (Heisenberg, 1977) he did not refer to such developments and controversies but reminded only his thoughts connected with the beginning of the TIHR history. Can the alluded attitude be regarded as an evidence of the supposition that in fact Heisenberg was conscious of the insurmountable defects of TIHR? A pertinent answer to such a question is expected to be (eventually) known by the publication of all volumes of a planned monography (Mehra and Rechenberg, 1982) due, in part, to one of Heisenberg’s last collaborators.

With the Heisenberg case one discloses a particularity in the attitude of many scientists who promoted TIHR. As individuals, each of the respective scientists did not regard the TIHR as a whole doctrine but argued for only a few of its elements and ignored almost all of the defects. Often their considerations were amalgamated with ideas which do not pertain strictly to TIHR. That is why, probably, by the term *TIHR-partisans* it is more adequate to understand a fuzzy class of people rather than a rigorously delimited group of scientists.

Now looking back over the time, we believe that the verity and true significance of TIHR defects still remain open questions which require to be elucidated. Such a requirement implies the necessity of an argued and complete revaluation of TIHR. Then, there directly appears the need for a search of a genuine reinterpretation of HR. The alluded beliefs will guide our investigations in the following sections.
IV. STARTING CONSIDERATIONS OF TIHR

TIHR introduced its main elements presented in P − 2.8 — P − 2.13 by appealing to some starting considerations about the Eqs. (2.1)-(2.3). The appeals viewed the scientific achievements from the first years of quantum mechanics. Here, for a correct (re)evaluation of TIHR, it is the place to remember briefly the respective considerations.

Firstly it must be noted that the Eqs. (2.1) were introduced by using the wave characteristics of quantum microparticles. Consequently the quantum measurements were regarded by similitude with the optical ones. But in the mentioned years the performances of the optical measurements were restricted by the classical limitative criteria of resolution (due to Abbe and Rayleigh). Then TIHR promoted as starting consideration the following point:

\[ P − 4.1: \] The estimation of performances respectively of uncertainties for the quantum measurements must be done by using the alluded limitative criteria, transposed in quantum framework through de Broglie formula \( \lambda = h/p \) (\( \lambda = \) wave length).▲

By means of this consideration TIHR partisans obtained some relations similar with Eqs. (2.1), for all the thought-experiments promoted by them.

Referring to the Eqs. (2.2)-(2.3) the starting considerations promoted by TIHR can be resumed as follows. The state of a quantum microparticle is described by the wave function \( \Psi = \Psi(q) \) regarded as a vector in a Hilbert space (q denotes the set of specific orbital variables). In the respective vectorial space the scalar product \((\Psi_a, \Psi_b)\) of two functions \(\Psi_a\) and \(\Psi_b\) is given by

\[
(\Psi_a, \Psi_b) = \int_{\Omega_q} \Psi_a^* \Psi_a d\Omega_q
\]

\[ (4.1) \]

where \(\Psi_a^*\) = the complex conjugate of \(\Psi_a\), whereas \(\Omega_q\) and \(d\Omega_q\) denote the accessible respectively infinitesimal domains in q-space. A quantum variable \(A\) is described by the operator \(\hat{A}\) and its expected (mean) value \(\langle A \rangle_\Psi\) is defined by

\[
\langle A \rangle_\Psi = \left( \Psi, \hat{A}\Psi \right)
\]

\[ (4.2) \]

The quantity \(\Delta_\Psi A\) from Eqs. (2.2)-(2.3) is defined as follows:
\[ D_\Psi A = \left( \delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{A} \Psi \right) \quad ; \quad \delta_\Psi \hat{A} = \hat{A} - \langle A \rangle_\Psi \]

\[ \Delta_\Psi A = \sqrt{D_\Psi A} \quad (4.3) \]

Then for two variables \( A \) and \( B \) the following evident relation was appealed

\[ \left( \left( \alpha \delta_\Psi \hat{A} - i \delta_\Psi \hat{B} \right) \Psi, \left( \alpha \delta_\Psi \hat{A} - i \delta_\Psi \hat{B} \right) \Psi \right) \geq 0 \quad (4.4) \]

with \( \alpha \) an arbitrary and real parameter. In the TIHR literature this relation is transcribed into the formula

\[ \left( \Psi, \left( \alpha \delta_\Psi A + i \delta_\Psi \hat{B} \right) \left( \alpha \delta_\Psi \hat{A} - i \delta_\Psi \hat{B} \right) \Psi \right) \geq 0 \quad (4.5) \]

which is equivalent with the relation

\[ \alpha^2 (\Delta_\Psi A)^2 - \alpha \left( \left[ i \hat{A}, \hat{B} \right]_{-} \right)_\Psi + (\Delta_\Psi B)^2 \geq 0 \quad (4.6) \]

where \( \left[ \hat{A}, \hat{B} \right]_{-} = \hat{A} \hat{B} - \hat{B} \hat{A} \) is the commutator of the operators \( \hat{A} \) and \( \hat{B} \). As Eq. (4.6) is satisfied for any value of \( \alpha \) it directly results.

\[ (\Delta_\Psi A)^2 \cdot (\Delta_\Psi B)^2 \geq \frac{1}{4} \left( \left[ i \hat{A}, \hat{B} \right]_{-} \right)_\Psi^2 \quad (4.7) \]

or

\[ \Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left| \left[ \hat{A}, \hat{B} \right]_{-} \right|_\Psi \quad (4.8) \]

This latter formula is just the Eq. (2.3).

Now we point out here the following notable facts. The above mentioned starting considerations were founded on old scientific achievements. For all that, they are preserved and promoted in an unchanged form even in today’s literature (especially in textbooks).

**V. ULTERIOR SCIENTIFIC ACHIEVEMENTS**

For an up-to-date re-evaluation of TIHR the above mentioned starting considerations must be supplemented with elements regarding some ulterior scientific achievements. The
respective elements were reported in the decades after the debut of quantum mechanics but, surprisingly, even today they seem to have little popularity. Our alluded supplement regards the following things.

Firstly let us refer to the thought-experimental Eqs. (2.1) and, correspondingly, to the limitative criteria involved in the starting consideration mentioned in $P - 4.1$. In the last decades, in optical measurements, some super-resolution techniques have been achieved (Roychoudhuri, 1978; Scheer et. al., 1989; Croca et al., 1996). The performances of the respective techniques overstep the alluded limitative criteria. Then it seems to be possible that instead of $P - 4.1$ one should operate with the following up-to-date consideration:

**$P - 5.1$**: The accuracies and uncertainties of the quantum measurements can be estimated by transposition in adequate terms (by means of de Broglie formula $\lambda = h/p$) of the mentioned super-resolution performances.

Based on this consideration, it is easy to imagine some Super-Resolution Thought Experiments (SRTE). Then for the measurement of two variables $A$ and $B$ the corresponding uncertainties are $\Delta_{SRTE}A$ and $\Delta_{SRTE}B$. By rationing similarly as in the case of Eq. (2.1) one finds

$$\Delta_{SRTE}A \cdot \Delta_{SRTE}B < \hbar$$

This $SRTE$ - relation must be taken into account for an up-to-date re-evaluation of TIHR.

Now let us refer to some ulterior achievements connected with the theoretical Eqs. (2.2)-(2.3) or (4.8). The respective achievements regard mathematical generalizations of the Eq. (4.8). Note that there is known (Dodonov and Manˈko, 1987; Dumitru, 1988) a large variety of such generalized relations. But here we shall consider only a few of the respective relations which are of direct significance for the questions approached in this paper.

Then we consider a quantum microparticle for which the orbital state and variables are described by the wave function $\Psi$ respectively by the operators $\hat{A}_k (k = 1, 2, \ldots, n)$. With the same significance of notations as in Eqs. (4.1)-(4.4), we can define the correlations:

$$C_\Psi (A_j A_k) = \left( \delta_\Psi \hat{A}_j \Psi, \delta_\Psi \hat{A}_k \Psi \right)$$
\[ D_{\Psi} A_j = C_{\Psi} (A_j A_j), \quad \Delta_{\Psi} A_j = \sqrt{D_{\Psi} A_j} \] (5.2)

If \( \alpha_k (k = 1, 2, \ldots, n) \) are a set of arbitrary and complex parameters we can write the evident relation

\[
\left( \sum_{k=1}^{n} \alpha_k \delta_{\Psi} \hat{A}_k \Psi, \sum_{l=1}^{n} \alpha_l \delta_{\Psi} \hat{A}_l \Psi \right) \geq 0
\] (5.3)

which can be transcribed directly as:

\[
\sum_{k} \sum_{l} \alpha_k^* \alpha_l \left( \delta_{\Psi} \hat{A}_k \Psi, \delta_{\Psi} \hat{A}_l \Psi \right) \geq 0
\] (5.4)

The quantities \( \left( \delta_{\Psi} \hat{A}_k \Psi, \delta_{\Psi} \hat{A}_l \Psi \right) (k; l = 1, 2, \ldots, n) \) represents the correlation matrix of the set of variables \( A_k \). It is obvious that

\[
\left( \delta_{\Psi} \hat{A}_k \Psi, \delta_{\Psi} \hat{A}_l \Psi \right)^* = \left( \delta_{\Psi} \hat{A}_l \Psi, \delta_{\Psi} \hat{A}_k \Psi \right)
\] (5.5)

i.e. the correlation matrix is Hermitian. Equation (5.4) shows that the respective matrix is also non-negative definite. Then from the matrix algebra (see Korn and Korn, 1968) it results

\[
\text{det} \left[ \left( \delta_{\Psi} \hat{A}_k \Psi, \delta_{\Psi} \hat{A}_l \Psi \right) \right] \geq 0 \] (5.6 CR)

where \( \text{det} [a_{kl}] \) denotes the determinant with the elements \( a_{kl} \). Here, and in the following notations, the index CR added to the number of a formula shows the belonging of the respective formula to a general family of similar correlation relations (CR).

For two operators \( \hat{A}_1 = \hat{A} \) and \( \hat{A}_2 = \hat{B} \) from Eq. (5.6) one obtains

\[
\left( \delta_{\Psi} \hat{A} \Psi, \delta_{\Psi} \hat{A} \Psi \right) \cdot \left( \delta_{\Psi} \hat{B} \Psi, \delta_{\Psi} \hat{B} \Psi \right) \geq \left| \left( \delta_{\Psi} \hat{A} \Psi, \delta_{\Psi} \hat{B} \Psi \right) \right|^2
\] (5.7 CR)

If the two operators satisfy the conditions

\[
\left( \hat{A}_k \Psi, \hat{A}_l \Psi \right) = \left( \Psi, \hat{A}_k \hat{A}_l \Psi \right) (k = 1, 2; l = 1, 2)
\] (5.8)

equation (5.6) gives directly
\[ \Delta_\Psi A \cdot \Delta_\Psi B \geq \left| \langle \delta_\Psi \hat{A} \cdot \delta_\Psi \hat{B} \rangle_\Psi \right| \]  \hspace{1cm} (5.9\hspace{0.5em}CR)

When Eq. (5.8) is satisfied we have also

\[ \langle \delta_\Psi \hat{A} \delta_\Psi \hat{B} \rangle_\Psi = \frac{1}{2} \left\langle [\delta_\Psi \hat{A}, \delta_\Psi \hat{B}]_+ \right\rangle_\Psi - \frac{i}{2} \left\langle i [\hat{A}, \hat{B}] \right\rangle_\Psi \]  \hspace{1cm} (5.10)

where \([\hat{A}, \hat{B}]_\pm = \hat{A}\hat{B} \pm \hat{B}\hat{A}\) (i.e. the anticommutator respectively commutator of \(\hat{A}\) and \(\hat{B}\)) while \(\left\langle [\delta_\Psi \hat{A}, \delta_\Psi \hat{B}]_+ \right\rangle_\Psi\) and \(\left\langle i [\hat{A}, \hat{B}] \right\rangle_\Psi\) are real quantities. Then Eq. (5.9) can be transcribed as

\[ \Delta_\Psi A \cdot \Delta_\Psi B \geq \sqrt{\frac{1}{4} \left\langle [\delta_\Psi \hat{A}, \delta_\Psi \hat{B}]_+ \right\rangle_\Psi^2 + \frac{1}{4} \left\langle i [\hat{A}, \hat{B}] \right\rangle_\Psi^2} \]  \hspace{1cm} (5.11\hspace{0.5em}CR)

This formula implies the following two less restrictive relations

\[ \Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left\langle [\delta_\Psi \hat{A}, \delta_\Psi \hat{B}]_+ \right\rangle_\Psi \]  \hspace{1cm} (5.12\hspace{0.5em}CR)

and

\[ \Delta_\Psi A \cdot \Delta_\Psi B \geq \frac{1}{2} \left| \left\langle [\hat{A}, \hat{B}]_- \right\rangle_\Psi \right| \]  \hspace{1cm} (5.13\hspace{0.5em}CR)

One can see that the latter relation is exactly the theoretical version from Eqs.(2.3)/(4.8) of HR.

Note now that there are situations when the Eq. (5.8) is satisfied for \(k = l\) but not for \(k \neq l\). In such situations from Eq. (5.7) instead of Eq. (5.9) one obtains the relation

\[ \Delta_\Psi A \cdot \Delta_\Psi B \geq \left| \langle \delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{B} \Psi \rangle \right| \]  \hspace{1cm} (5.14\hspace{0.5em}CR)

From the above presented considerations it results that the true generalizations of the theoretical HR given by Eqs. (2.2)/(2.3)/(4.8)/(5.13) are exactly the Eqs. (5.7) and (5.6).

The above discussed relations refer to the orbital variables of a quantum microparticle. But such a microparticle has also spin variables characterized by similar relations. So if for an electron the spin state and the spin variables are described by the spinor \(\chi\) (spin wave function) respectively by the matrices (operators) \(\hat{A}_j\), the alluded relations can be
introduced as follows. With the usual notations (see Bransden and Joachain, 1994) the expected values are \( \langle A_j \rangle_\chi = \chi^+ A_j \chi \) while the correlations \( C_\chi (A_j A_e) \), dispersions \( D_\chi A_j \), and standard deviation \( \Delta_\chi A_j \) are given by

\[
C_\chi (A_j A_i) = \left( \delta_\chi \hat{A}_j \chi \right)^\dagger \cdot \left( \delta_\chi \hat{A}_i \chi \right), \quad \delta_\chi \hat{A}_j = \hat{A}_j - \langle A_j \rangle_\chi
\]

\[
D_\chi A_j = C_\chi (A_j A_j), \quad \Delta_\chi A_j = \sqrt{D_\chi A_j}
\]

Similarly to the orbital Eqs. (5.3)-(5.6) it is easy to see that the spin-correlations \( C_\chi (A_j A_i) \) satisfy the relation

\[
\det \left[ C_\chi (A_j A_i) \right] \geq 0 \tag{5.16}_{CR}
\]

For two variables \( A_1 = A \) and \( A_2 = B \), which satisfy conditions similar to (5.8), from (5.16) one obtains

\[
\Delta_\chi A \cdot \Delta_\chi B \geq \left| \left\langle \delta_\chi \hat{A} \delta_\chi \hat{B} \right\rangle_\chi \right| \tag{5.17}_{CR}
\]

\[
\Delta_\chi A \cdot \Delta_\chi B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle_\chi \right| \tag{5.18}_{CR}
\]

Equations (5.18) and (5.16) are nothing but spin similars of orbital HR and of their generalizations given by Eqs. (5.13) respectively (5.6).

Theoretical versions of HR (as well as their generalizations) imply, for the variables of the quantum microparticles, a lot of probabilistic parameters such as: expected/mean values, dispersions, standard deviations and correlations. This means that the respective variables have stochastic (or random) characteristics. Then there follows directly the question: are there similars of HR for other physical systems, different from quantum microparticles, which have also variables with stochastic characteristics? The answer to the mentioned questions is affirmative and it regards macroscopic systems studied in both classical and quantum statistical physics. Here we shall illustrate the respective answer by taking over some ideas from our earlier works (Dumitru, 1974a, 1977, 1988, 1993).
Firstly let us refer to a macroscopic system, consisting of a large number of microparticles, considered in a thermodynamic equilibrium state. In the framework of classical (nonquantum) statistical physics such a system can be approached in terms of: (a) phenomenological (quasithermodynamic) fluctuations theory respectively (b) classical statistical mechanics. In the mentioned approaches the state of the system is described (Landau and Lifchitz, 1984; Zubarev, 1971; Ruppeiner, 1995) by the distribution function \( w = w(x) \). The variable \( x \) denotes in the two cases: (a) the set of independent macroscopic variables of the system as a whole, respectively (b) the phase space coordinates of the microparticles constituting the system. In both cases a specific variable \( A \) characterizing the system is a real stochastic (random) quantity with a continuous spectrum of values which depends on \( x \), i.e. \( A = A(x) \). The mean (or expected) value of \( A(x) \) is given by

\[
\langle A \rangle_w = \int_{\Omega_x} A(x) \, w(x) \, d\Omega_x
\]

where \( \Omega_x \) and \( d\Omega_x \) denote the accessible respectively infiniesimal domains in the \( x \)-space.

Then, in the case of such a macroscopic system, for a set \( A_j (j = 1, 2, ..., n) \) of specific variables, the thermal fluctuations are described by the correlations \( C_w (A_j A_l) \), dispersions \( D_w A_j \) and standard deviations \( \Delta_w A_j \) given by

\[
C_w (A_j A_l) = \langle \delta_w A_j \delta_w A_l \rangle_w , \quad \delta_w A_j = A_j - \langle A_j \rangle_w
\]

\[
D_w A_j = C_w (A_j A_j) , \quad \Delta_w A_j = \sqrt{D_w A_j}
\]

By relations similar to Eqs.(5.3)-(5.5) it is easy to see that the correlations \( \langle \delta_w A_j \delta_w A_l \rangle \) are the elements of a non-negative real matrix. Then similarly with Eq. (5.6) one can write

\[
\det \left[ \langle \delta_w A_j \delta_w A_l \rangle_w \right] \geq 0
\]

(5.21 \( \text{CR} \))

Particularly for two variables \( A_1 = A \) and \( A_2 = B \) one obtains

\[
D_w A \cdot D_w B \geq \langle \delta_w A \cdot \delta_w B \rangle_w^2
\]

(5.22 \( \text{CR} \))
\[ \Delta_w A \cdot \Delta_w B \geq |\langle \delta_w A \cdot \delta_w B \rangle_w| \]  
\hspace{1cm} (5.23_{CR})

Equations (5.20)-(5.22) can be called \textit{thermal correlation relations}. Some examples of such relations are given below in Sec. VI. K. (see also Dumitru, 1974a, 1988, 1993).

One observes that Eqs. (5.23) and (5.21) are the macroscopic similars of microscopic HR and of their generalizations defined by Eqs. (2.3)/(4.8)/(5.13) respectively (5.6). Here it must be noted that there are also other macroscopic similars of HR, namely relations from the framework of quantum statistical mechanics. Such relations can be obtained as follows: The state respectively the specific variables of a macroscopic system in the mentioned framework are described by the statistical operator (density matrix) \( \hat{\rho} \) respectively by the operators \( \hat{A}_j \) \( (j = 1, 2, ..., n) \). With the expected values defined as \( \langle A_j \rangle_\rho = T_r \left( \hat{A}_j \hat{\rho} \right) \) the macroscopic correlations \( C_\rho (A_j A_e) \), dispersions \( D_\rho A_j \) and standard deviations \( \Delta_\rho A_j \) are given by

\[ C_\rho (A_j A_e) = \left\langle \delta_\rho \hat{A}_j \cdot \delta_\rho \hat{A}_l \right\rangle_\rho, \quad \delta_\rho \hat{A}_j = \hat{A}_j - \langle A_j \rangle_\rho \]

\[ D_\rho A_j = C_\rho (A_j A_j), \quad \Delta_\rho A_j = \sqrt{D_\rho A_j} \]  
\hspace{1cm} (5.24)

In sufficiently general circumstances (among them the most important being some conditions similar with Eq.(5.8)) the quantities from Eqs.(5.24) satisfy the relations

\[ \det \left[ \left\langle \delta_\rho \hat{A}_j \delta_\rho \hat{A}_l \right\rangle_\rho \right] \geq 0 \]  
\hspace{1cm} (5.25)

\[ D_\rho A \cdot D_\rho B \geq |\langle \delta_\rho A \cdot \delta_\rho B \rangle_\rho|^2 \]  
\hspace{1cm} (5.26_{CR})

\[ \Delta_\rho A \cdot \Delta_\rho B \geq |\langle \delta_\rho A \cdot \delta_\rho B \rangle_\rho| \]  
\hspace{1cm} (5.27_{CR})

with \( \hat{A} = \hat{A}_1, \ \hat{B} = \hat{A}_2 \). From Eq.(5.27) one obtains also the following truncated (less restrictive) relations

\[ \Delta_\rho A \cdot \Delta_\rho B \geq \frac{1}{2} \left\langle \left[ \delta_\rho \hat{A}, \delta_\rho \hat{B} \right]_+ \right\rangle_\rho \]  
\hspace{1cm} (5.28_{CR})
\[ \Delta_\rho A \cdot \Delta_\rho B \geq \frac{1}{2} \left| \left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle \right| \rho \]  

(5.29 CR)

The latter relation is exactly a macroscopic similar of HR defined by Eqs. (2.3)/(4.8)/(5.13).

The above discussed relations are unitemporal in the sense that the implied probabilistic parameters (correlations, dispersions, standard deviations) of the stochastic variables \( A_j \) are considered for the same moment of time. But it easy to see that similar relations can be written if the mentioned parameters are taken into account for different time moments. So, if the orbital quantum state of a microparticle is described by the time \( t \) dependent wave function \( \Psi (q, t) \), instead of unitemporal Eq. (5.14) one can write the following bitemporal relation:

\[ \Delta_{\Psi_1} A \cdot \Delta_{\Psi_2} B \geq \left| \left\langle \delta_{\Psi_1} \hat{A} \Psi_1, \delta_{\Psi_2} \hat{B} \Psi_2 \right\rangle \right| \]

(5.30)

where \( \Psi_1 = \Psi (q, t_1) \) and \( \Psi_2 = \Psi (q, t_2) \) with \( t_1 \neq t_2 \).

Another well-known way of introducing the theoretical HR for orbital variables is based on Fourier analysis as follows. Let be \( f(x) \) a continuous and quadratically integrable function in the range \( x \in (-\infty, \infty) \). Then its Fourier transforms \( \tilde{f}(k) \) is defined by:

\[ \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \]  

(5.31)

If \( |f(x)|^2 \) and \( |\tilde{f}(k)|^2 \) are normalized to unity by using the Parseval formula one can write

\[ \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk = 1 \]  

(5.32)

Then \( |f(x)|^2 \) and \( |\tilde{f}(k)|^2 \) can be interpreted as probability densities for the variables \( x \) respectively \( k \). Consequently the mean (expected) values of functions like \( A(x) \) or \( B(k) \) are given by

\[ \langle A(x) \rangle = \int_{-\infty}^{\infty} A(x) |f(x)|^2 dx \]  

(5.33)

\[ \langle B(k) \rangle = \int_{-\infty}^{\infty} B(k) |\tilde{f}(k)|^2 dk \]  

(5.34)
With the above mentioned elements one can demonstrate (De Bruijn, 1967) the following relation
\[ \langle (x-a)^2 \rangle \cdot \langle (k-b)^2 \rangle \geq \frac{1}{4} \] (5.35)
with \( a \) and \( b \) as two arbitrary constants.

One observes that from Eq. (5.35) it results directly the HR defined by Eq. (2.2), for the Cartesian coordinate \( x \) and momentum \( p \). For such a result one must take \( f(x) \) as the wave function \( \Psi(x) \) and respectively \( a = \langle x \rangle, b = \langle k \rangle \) and \( k = p/h \).

If \( f(x) \) is periodic in \( x \), with the period \( \Lambda \), or is defined in the range \( x \in [x_0, \Lambda] \) with the same values on the boundaries, it satisfies the relation
\[ f(x_0) = f(x_0 + \Lambda) \] (5.36)
Then instead of \( \tilde{f}(k) \) from Eq. (5.31) we have the Fourier coefficients
\[ \tilde{f}_n = \frac{1}{\sqrt{\Lambda}} \int_{x_0}^{x_0+\Lambda} f(x) e^{-i k_n x} \, dx \] (5.37)
with \( k_n = n \cdot 2\pi/\Lambda \) and \( n = 0, \pm 1, \pm 2, \ldots \). Similarly with Eq. (5.33) we can take:
\[ \int_{x_0}^{x_0+\Lambda} |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |\tilde{f}_n|^2 = 1 \] (5.38)
This means that \( |f(x)|^2 \) can be interpreted as probability density for the continuous variable \( x \) while \( |\tilde{f}_n|^2 \) signify the probabilities associated with the discrete variable \( k_n \). In such a case instead of Eqs. (5.33) and (5.34) one must write
\[ \langle A(x) \rangle = \int_{x_0}^{x_0+\Lambda} A(x) |f(x)|^2 \, dx \] (5.39)
\[ \langle B(k) \rangle = \sum_{n=-\infty}^{\infty} B(k_n) |\tilde{f}_n|^2, \quad k_n = n \frac{2\pi}{\Lambda} \] (5.40)
and instead of Eq. (5.35) one obtains
\[ \langle (x-a)^2 \rangle \cdot \langle (k-b)^2 \rangle = \frac{1}{4} |(\Lambda f(x_0) - 1)|^2 \] (5.41\text{CR})
This latter formula is applicable in some cases for the variables azimuthal angle \( \varphi \) and angular momentum \( L_z \) (see below the Sec. VI.F). In such cases \( f(x) \) is the periodic wave function \( \Psi(\varphi) \) and respectively \( a = \langle x \rangle, b = \langle k \rangle \) with \( x = \varphi, k = L_z / \hbar \) and \( \Lambda = 2\pi \).

We end this section with a notification regarding the relations expressed by Eqs.: (5.6), (5.9), (5.11)-(5.14), (5.16)-(5.18), (5.21)-(5.23), (5.25)-(5.29), (5.30), (5.35) and (5.41). From a mathematical viewpoint all the respective relations refer to variables with stochastic characteristics. Also, by their mathematical significances, they belong to the same family of similar formulae which can be called correlation relations (CR). That is why we added the index CR to the numbers of all the respective relations.

VI. DEFECTS OF TIHR

With the above mentioned facts now we can proceed to present the defects of TIHR. Note that the respective defects appear not as a systematized ensemble but rather as a dispersed set of (relatively) distinct cases. That is why our approach aims not at a precisely motivated order of exposition. We mostly wish to show that taken together the set of the alluded defects irrefutably incriminate all the main elements of TIHR reviewed in Sec. II. Then our presentation includes the defects revealed in the following sub-sections:

A. Groundlessness of the term ”uncertainty”

Through the assertion \( P - 2.8 / A \) of TIHR the thought respectively theoretical quantities \( \Delta_{TE}A \) and \( \Delta_{\varphi}A \) from HR are termed measuring uncertainties. But the respective term appears as groundless if it is regarded comparatively with a lot of facts which we present here.

Firstly note that a minute examination of all thought experiments referred in connection with Eq. (2.1) does not justify the mentioned term for one of the implied quantities \( \Delta_{TE}A \) and \( \Delta_{TE}B \). So \( \Delta_{TE}p \) (in the coordinate q -momentum p case) and \( \Delta_{TE}E \) (in the time t - energy E case) represent the jumps of the respective variable from an initial value (before
the measurements) to a final value (after the measurements). Then it results that $\Delta_{TEP}$ and $\Delta_{TE}E$ can not be regarded as uncertainties (i.e. measuring parameters) with respect to the measured state which is the initial one. This because (Albertson, 1963): “it seems essential to the notion of a measurements that it answer a question about the given situation existing before measurement. Whether the measurement leaves the measured system unchanged or brings about a new and different state of that system is a second and independent question”.

The remaining quantities $\Delta_{TEq}$ and $\Delta_{TEt}$ from Eq. (2.1) are also in the situation of infringing the term “measuring uncertainties” in the sense attributed by TIHR. The respective situation is generated by the same facts which will be presented below in the Sec. VI.B.

As regards the theoretical quantity $\Delta_{\Psi}A$ the following observations must be taken into account. $\Delta_{\Psi}A$ depends only on the theoretical model (wave function $\Psi$) of the considered microparticle but not on the characteristics of the measurements on the respective microparticle. Particularly note that the value of $\Delta_{\Psi}A$ can be modified only by changing the microparticle state (i.e. its wave function $\Psi$). Comparatively the measuring uncertainties can be modified by improving (or worsening) the performances of the experimental techniques, even if the state of the measured microparticle remains unchanged.

In connection with the term "uncertainty" it is here the place to point out also the following remarks. In quantum mechanics a variable $A$ is described by an adequate operator $\hat{A}$ which (Gudder, 1979) is a generalized stochastic (or random) quantity. The probabilistic (stochastic) characteristics of the considered microparticle are incorporated in its wave function $\Psi$. Then the expected value $\langle A \rangle_\Psi$ and the standard deviation $\Delta_{\Psi}A$ appears as quantities which are exclusively of intrinsic nature for the respective microparticle. In such a situation a measurements must consist of a statistical sampling but not of a solitary detection (determination). The respective sampling give an output-set of data on the recorder of the measuring instrument. From the mentioned data one obtains the out-mean $\langle A \rangle_{OUT}$ and out-deviation $\Delta_{OUT}A = \sqrt{\langle (A - \langle A \rangle_{OUT})^2 \rangle_{OUT}}$. Then it results that in fact the measuring uncertainties must be described by means of the differences $\langle A \rangle_{OUT} - \langle A \rangle_\Psi$ and $\Delta_{OUT}A - \Delta_{\Psi}A$ but not through the quantity $\Delta_{\Psi}A$. (For other comments about the measurements regarded as here
see below the Sec. IX).

The above mentioned facts prove the groundlessness of the term "uncertainty" in connection with the quantities $\Delta_{TE}A$ and $\Delta \varphi A$. But such a proof must be reported as a defect of TIHR.

*Observation:* Sometimes, particularly in old texts, the quantities $\Delta_{TE}A$ and $\Delta \varphi A$ are termed "indeterminacy" of the quantum variable $A$. If such a term is viewed to denote the "non-deterministic" or "random" character of $A$ it can be accepted for a natural interpretation of $\Delta \varphi A$. So the HR given by Eqs. (2.3)/(4.8)/(5.13) and their generalizations from Eq. (5.6) appear to be proper for a denomination of "indeterminacy relations". But then it seems strange for the respective relations to be considered as "crucial and fundamental formulae" (as in TIHR conception). This because in non-quantum branches of probabilistic sciences an entirely similar "indeterminacy relation" is regarded only as a modest, and, by no means a fundamental formula. The alluded non-quantum "indeterminacy relation" expresses simply the fact that the correlation coefficient $(\gamma_{AB} = \langle \delta A \cdot \delta B \rangle / \Delta A \cdot \Delta B)$ takes values within the range $[-1, 1]$. (Schilling, 1972; Gellert *et.al.*, 1975). One can see that such a non-quantum relation is also Eq. (5.23).

**B. The nature and the performances of the referred experiments**

As it was shown in Sec. II one of the major supports of TIHR is the reference to experiments of "thought" nature. In such a reference, by means of "thought motivations", the results (ideas) from known real experiments were transplanted in a new context. But such a transplantation seems to be inadequate for a true scientific acceptance, mainly if the new context is practically and conceptually different from the old one. Also it is known that the alluded acceptance must be founded only on two pieces of resistance: (a) concrete data from the real and specially designed experiments, and (b) correct rigorous mathematical (logical) reasonings.

One must add another less known observation about the nature of the experiments re-
ferred by TIHR. Practically all the respective experiments are of "thought"-type and (Jamm-
er, 1974, p.81) there are not known any real experiments capable of attesting (verifying) the TIHR with a convincing accuracy.

The above presented facts reveal the uselessness respectively the incorrectness of the main elements $P_{-2.6}$ and $P_{-2.8}$ of TIHR. This means that by their existence the mentioned facts evidentiate a defect of TIHR.

The thought experiments referred by TIHR operate with the limitative criteria included in $P_{-4.1}$ and consequently with Eq. (2.1). But, as it was mentioned in Sec. V, today there are known real experiments with super-resolutions which overstep the respective criteria. Then it is easy to imagine some Super-Resolution-Thought-Experiments (SRTE) for which instead of Eq. (2.1) is satisfied the SRTE relation given by Eq. (5.1). One observes now directly that the existence of the respective SRTE relation incriminates the assertion $P_{-2.12/A}$ of TIHR. Such an incrimination must be reported also as a defect of TIHR.

C. Inaccuracy of the referred theoretical formulae

Among the main supports of TIHR we find the theoretical Eqs. (2.3)/(4.8). But, mathematically, the respective formula is incompletely accurate because it fails if the condition expressed by Eq. (5.8) is not satisfied. The complete accuracy is given by the more general Eqs. (5.7) and (5.14).

The mentioned incompleteness seems to be regarded as entirely unnatural by the TIHR partisans - e.g. when they consider the case of the variables angular momentum $\hat{L}_z$ and azimuthal angle $\varphi$ (see also below Sec. VI.F.). So, instead of the failing Eqs. (2.1)/(4.8), in order to preserve at any price the elements like $P_{-2.12}$ and $P_{-2.13}$, the respective partisans use a strange lot of unnaturally "adjusted" relations. But one can easily see that the natural attitude in the alluded case is to refer to the Eqs. (5.7)/(5.14) or, equivalently, to Eq. (5.41). Of course that in the discussed case the respective equations degenerate into trivial equality 0=0 which is evidently in contradiction with the elements $P_{-2.12}$ and
P – 2.13 of TIHR.

Then one finds that one way or another TIHR is incompatible with the absolute accuracy of the referred theoretical equations: This fact constitutes a notable defect of TIHR

D. Solitary variables

By $P – 2.10/A; M$ TIHR states that in a measurement for a solitary variable the quantity $\Delta A$ can be taken as unlimitedly small. But if $\Delta A$ is identified with $\Delta_\Psi A$ the respective quantity has a precisely defined value (dependent on the wave function $\Psi$ describing the state of the considered microparticle) which can not be diminished in a measurement. Then it results that $P – 2.10/A; M$ are incorrect. The respective results points another defect of TIHR.

E. Commutable variables

The main idea of TIHR about the commutable variables is asserted in $P – 2.11/A; M$. It is based on the fact that in Eq. (2.3) the product of the corresponding quantities $\Delta_\Psi A$ and $\Delta_\Psi B$ has not a non-null lower bound. But besides the Eq. (2.3), the respective product satisfies also the Eq. (5.12) where the term $\left| \left[ \delta_\Psi \hat{A}, \delta_\Psi \hat{B} \right]_+ \right\rangle_{\Psi}$ can be a non-null quantity. In this respect we quote the following example (Dumitru, 1988):

Let be a quantum microparticle moving in a two-dimensional potential well, characterized by the potential energy $V=0$ for $0 < x < a$, $0 < y < b$ and $V=\infty$ otherwise. The corresponding wave functions are

$$\Psi_{n_1n_2} = \frac{2}{\sqrt{ab}} \sin \left( \frac{n_1 \pi}{a} x \right) \sin \left( \frac{n_2 \pi}{b} y \right) (n_1, n_2 = 1, 2, 3, \ldots) \quad (6.1)$$

As two commutable variables $A$ and $B$ with $\left| \left[ \delta_\Psi \hat{A}, \delta_\Psi \hat{B} \right]_+ \right\rangle_{\Psi} \neq 0$ we consider the Cartesian coordinates $x'$ and $y'$ given by

$$\hat{A} = x' = x \cos \varphi + y \sin \varphi$$
\[ \hat{B} = y' = x \sin \varphi - y \cos \varphi \]  
(6.2)

with \( 0 < \varphi < \frac{\pi}{2} \). For the case pointed by Eqs. (6.1) and (6.2) one obtains:

\[ \Delta \varphi A = \left[ \frac{a^2}{12} \left( 1 - \frac{6}{\pi^2 n_1^2} \right) \cos^2 \varphi + \frac{b^2}{12} \left( 1 - \frac{6}{\pi^2 n_2^2} \right) \sin^2 \varphi \right]^{1/2} \]

\[ \Delta \varphi B = \left[ \frac{a^2}{12} \left( 1 - \frac{6}{\pi^2 n_1^2} \right) \sin^2 \varphi + \frac{b^2}{12} \left( 1 - \frac{6}{\pi^2 n_2^2} \right)^2 \cos^2 \varphi \right]^{1/2} \]  
(6.3)

\[ \left| \langle \delta \varphi A, \delta \varphi B \rangle \right| = \left[ 2 \sin 2\varphi \left( \frac{a^2 - b^2}{12} - \frac{6}{\pi^2} \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \right) \right]^{1/2} \]

Then it results that in the mentioned case the Eq. (5.12) is satisfied with non-null quantity in the right-hand term. But such a result shows that the main idea of TIHR about the commutable variables is doubtful. So we find another defect of TIHR.

F. The \( \ell_z - \varphi \) case

The situation of the variables \( \ell_z \) and \( \varphi \) (\( z \)-component of angular momentum and azimuthal angle) represents one of the most controverted case in connection with TIHR.

Firstly, it was noted that the respective variables are canonically conjugated their quantum operators being \( \hat{\ell}_z = -i\hbar \frac{\partial}{\partial \varphi} \) and \( \hat{\varphi} = \varphi \cdot . \) Then it was supposed that for \( \ell_z - \varphi \) case the TIHR must be applicable similarly as for other pairs of conjugated variables-e.g. for \( p_x \) and \( x \) (Cartesian momentum and coordinate). Consequently it was expected to have the following ordinary \( \ell_z - \varphi \) theoretical HR

\[ \Delta \varphi \ell_z \cdot \Delta \varphi \varphi \geq \frac{\hbar}{2} \]  
(6.4)

Also one knows attempts (Kompaneyets, 1966) for introducing a thought-experimental \( \ell_z - \varphi \) relation of the form

\[ \Delta_{TE} \ell_z \cdot \Delta_{TE} \varphi \approx \hbar \]  
(6.5)
which is similar with the \( p_x - x \) version of Eq. (2.1). But note that in fact Eq. (6.5) is only an unmasked conversion of Eq. (2.1).

Secondly, for the \( L_z - \varphi \) case the inapplicability of TIHR in its usual form (presented in Sec. II) was remarked. Such remarks were signaled in a lot of debating works (Judge, 1963; Judge and Lewis, 1963; Judge, 1964; Bouten et al, 1965; Evett and Mahmoud, 1965; Krauss, 1965, 1968; Carruthers and Nietto, 1968; Levy-Leblond, 1976; Harris and Strauss, 1978; Hasse, 1980; Holevo, 1981; Yamada, 1982; Galitski et al., 1985). The mentioned inapplicability regards mainly from the HR Eq. (6.4) referred to some known systems such as: an atomic electron, a rotator or a bead shifting on a ring. Note that by their properties all the respective systems are \( \varphi \)-circular (or azimuthally finite). This means that for them: (i) \( \varphi \in [0, 2\pi] \), (ii) the states with \( \varphi = 0 \) and \( \varphi = 2\pi \) coincide and (iii) the states with \( \varphi \geq 0 \) and \( \varphi \leq 2\pi \) are closely adjacent. Moreover for the mentioned systems one considers only the states which are nondegenerate in respect with \( L_z \). This means that each of such state correspond to a distinct (eigen-) value of \( L_z \). The alluded states are described by the wave functions:

\[
\Psi_m(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \quad (m = 0, \pm 1, \pm 2, \ldots)
\]

Then, for the demarcated states, one finds \( \Delta \varphi L_z = 0, \Delta \varphi = \pi / \sqrt{3} \) and (6.4) gives the absurd results \( 0 \geq \hbar / 2 \). Such a result drives TIHR in a evident deadlock.

For avoiding the alluded deadlock the THR partisans advocated the following idea: In the \( L_z - \varphi \) case the theoretical HR must not have the ordinary form of Eq. (6.4) but an “adjusted version”, concordant with TIHR vision. Thus the following lot of “adjusted \( L_z - \varphi \) HR” was invented:

\[
\frac{\Delta \varphi L_z \cdot \Delta \varphi}{1 - 3 (\Delta \varphi / \pi)^2} \geq 0.16\hbar
\]

\[
\frac{(\Delta \varphi L_z)^2 \cdot (\Delta \varphi)^2}{1 - (\Delta \varphi)^2} \geq \frac{\hbar^2}{4}
\]

\[
(\Delta \varphi L_z)^2 + \left(\frac{\hbar \omega}{2}\right)^2 (\Delta \varphi)^2 \geq \frac{\hbar^2}{2} \left[ \frac{9}{\pi^2} + \frac{\omega^2}{\pi^2} \right]^{1/2} - \frac{3}{\pi^2}
\]
\[
\frac{\Delta_{\psi} L_z \cdot \Delta_{\psi} \varphi}{1 - 3 \left(\frac{\Delta_{\psi} \varphi}{\pi}\right)^2} \geq \frac{\hbar^2}{3} \left(\frac{V_{\text{min}}}{V_{\text{max}}}\right) \tag{6.10}
\]

\[
(\Delta_{\psi} L_z)^2 \cdot \langle \left(\delta_{\psi} f (\varphi)\right)^2 \rangle_{\psi} \geq \frac{\hbar^2}{4} \left(\frac{df}{d\varphi}\right)_{\psi}^2 \tag{6.11}
\]

\[
\Delta_{\psi} L_z \cdot \Delta_{\psi} \varphi_1 \geq \frac{\hbar}{2} \tag{6.12}
\]

\[
\Delta_{\psi} L_z \cdot \Delta_{\psi} \varphi \geq \frac{\hbar}{2} \left|\langle E (\varphi)\rangle_{\psi}\right| \tag{6.13}
\]

\[
\Delta_{\psi} L_z \cdot \Delta_{\psi} \varphi \geq \frac{\hbar}{2} \left|1 - 2\pi |\Psi (2\pi)|^2\right| \tag{6.14}
\]

In Eq. (6.9) \(\omega\) is a real nonnegative parameter. In Eq. (6.10) \(V_{\text{min}}\) and \(V_{\text{max}}\) represents the minimum respectively the maximum values of \(V (\theta) = \int_{-\pi}^{\pi} |\Psi (\varphi + \theta)|^2 d\varphi\), where \(\theta \in [-\pi, \pi]\). In Eq. (6.11) \(f (\varphi)\) is a real periodic function of \(\varphi\) e.g. \(f (\varphi) = \sin \varphi\) or \(f (\varphi) = \cos \varphi\) and \(\delta_{\psi} f = f - \langle f \rangle_{\psi}\). In Eq. (6.12) \(\varphi_1 = \varphi + 2\pi N\), \(\Delta_{\psi} \varphi_1 = [2\pi^2 \left(\frac{1}{12} + N^2 - N_1^2 + N - N_1\right)]^{1/2}\) and \(N, N_1=\) two arbitrary integer numbers with \(N \neq N_1\). In Eq. (6.13) \(E(\varphi)\) is a complicated expression of \(\varphi\).

Connected with the Eqs. (6.7)-(6.14) and the afferent TIHR debates the following facts are easily observed. From a subjective view, in TIHR literature, none of the Eqs. (6.7)-(6.14) is unanimously accepted as the true version for theoretical \(L_z - \varphi\) HR. In a objective view the Eqs. (6.7)-(6.14) appear as a set of completely dissimilar formulae. This because they are not mutually equivalent and each of them is applicable only in particular and different ”circumstances”. Moreover it is doubtful that in the cases of Eqs. (6.7)-(6.13) the respective ”circumstances” should have in fact real physical significances. Another aspect from an objective view is the fact that the Eqs. (6.7)-(6.13) have no correct support in the natural (non-adjusted) mathematical formalism of quantum mechanics. Only Eq. (6.14) has such a support through the Eq. (5.41). The alluded observations evince clearly the persistence of TIHR deadlock as regards the \(L_z - \varphi\) case. But in spite of the mentioned evidence, in our days almost all of the publications seem to cultivate the belief that the
problems of $L_z - \varphi$ case are solved by the adjusted Eqs. (6.7)-(6.14). In the TIHR literature the mentioned belief is often associated with a more inciting opinion. According to the respective opinion the ordinary theoretical HR expressed by Eq. (6.4) is incorrect for any physical system and, consequently, the respective relation must be prohibited. Curiously, through the alluded association, TIHR partisans seem to ignore the thought experimental Eq. (6.5). But note that the simple removal of the respective ignorance can not solve the TIHR deadlock regarding the $L_z - \varphi$ case. Moreover, such a removal is detrimental for TIHR because the Eq. (6.5) is only a conversion of Eq. (2.1) which is an unjustified relation (see Sec. VI.B)

As regards the TIHR attitude towards the Eq. (6.4) there is another curious ignorance/omission. In the afferent literature it is omitted any discussion on the $L_z$ - degenerate states of the circular systems. Such a state is associated with a set of eigenvalue of $\hat{L}_Z$ and it is described by a linear superpositions of eigenfunctions of $\hat{L}_Z$. As example can be taken the state of a free rigid rotator with a given energy $E_l = \hbar^2 l(l + 1)/2J$ ($l =$ orbital quantum number, $J =$ moment of inertia). The respective state is described by the wave function

$$\Psi_l(\theta, \varphi) = \sum_{m=-l}^{l} C_m Y_{lm}(\theta, \varphi)$$

(6.15)

where $Y_{lm}(\theta)$ are the spherical functions, $l$ and $m$ denote the orbital respectively magnetic quantum numbers while $C_m$ are arbitrary complex constants which satisfy the condition $\sum_m |C_m|^2 = 1$. In respect with the wavefunction given in Eq. (6.15) for the operators $\hat{L}_Z = -i\hbar\frac{\partial}{\partial \varphi}$ and $\hat{\varphi} = \varphi$ one obtains

$$(\Delta_\varphi L_Z)^2 = \sum_m |C_m|^2 \hbar^2 m^2 - \left( \sum_m |C_m|^2 \hbar m \right)^2$$

(6.16)

$$(\Delta_\varphi \varphi)^2 = \sum_m \sum_{m'} C_m^* C_{m'} \langle Y_{lm}, \varphi Y_{lm} \rangle$$

$$- \left[ \sum_m \sum_{m'} C_m^* C_{m'} \langle Y_{lm}, \varphi Y_{lm} \rangle \right]^2$$

(6.17)
With the expressions $\Delta_\Psi L_Z$ and $\Delta_\Psi \varphi$ given by Eqs. (6.16) and (6.17) it is possible that the HR from Eq. (6.4) to be satisfied (For more details see below the discussions about the Eq. (8.3) in Sec. VIII). But surprisingly such a possibility was not examined by TIHR partisans which persevere to opine that Eq. (6.4) must be prohibited as incorrect in respect with any physical situation. We think that such an attitude has to be considered as a defect of TIHR doctrine.

Contrary to the TIHR partisans opinion about the HR given by Eq. (6.4) it is easy to see (Dumitru, 1988, 1991), that the respective relation remains rigorously valid at least in the case of one quantum system. The respective system is a Quantum Torsion Pendulum (QTP) oscillating around the $z$-axis. Such a QTP is completely analogous with the well-known (recti)linear oscillator.

The states of the QTP are described by the wave functions:

$$\Psi_n (\varphi) = \Psi_n (\xi) = \left( \frac{J \omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\xi^2/2} H_n (\xi)$$

with $\xi = \varphi \cdot \sqrt{J \omega / \hbar}$, $\varphi = $azimuthal angle, $J =$moment of inertia, $\omega =$angular frequency, $n = 0, 1, 2, \ldots =$the oscillation quantum number and $H_n (\xi) = (-1)^n \left( e^{\xi^2} \right) \cdot \left( d^n e^{-\xi^2} / d\xi^n \right)$ are the Hermite polynomials.. By its properties, QTP is $\varphi$-torsional ($\varphi -$non-circular or azimuthally infinite). This means that for it: (i) $\varphi \in (-\infty, \infty)$, (ii) the states with $\varphi = 0$ and $\varphi = 2\pi$ do not coincide and (iii) the states with $\varphi \gtrless 0$ and $\varphi \lesssim 2\pi$ are not closely adjacent.

In the case of QTP for the variables $L_z$ and $\varphi$, described by the operators $\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}$ and $\hat{\varphi} = \varphi$, one obtains the expression:

$$\Delta_\Psi L_z \sqrt{\hbar J \omega \left( n + \frac{1}{2} \right) ^2}, \quad \Delta_\Psi \varphi = \sqrt{\frac{\hbar J \omega}{J \omega} \left( n + \frac{1}{2} \right)}$$

With these expressions one finds that for QTP the $L_z - \varphi$ theoretical HR is satisfied in the ordinary/common form given by Eq. (6.4). So the existence of QTP example invalidate the above mentioned opinion of TIHR partisans about the HR from Eq. (6.4).The alluded invalidation makes even deeper the deadlock of TIHR with respect to the $L_z - \varphi$ case.

All the above mentioned deadlooks of TIHR-doctrine in connection with the pair $L_z - \varphi$ must be reported as indubitable defects of the respective doctrine.
G. The N-Φ case

Another case which drove TIHR in deadlock is that of pair N-Φ (number-phase) connected with the study of quantum oscillator. N represents the quantum oscillation number, described by the operator \( \hat{N} = \hat{a}^{+} \hat{a} \) (where \( \hat{a}^{+} \) and \( \hat{a} \) are the known ladder operators) while Φ is taken as variable conjugated with N. Often, if the oscillator is considered as radiative, N and Φ are regarded as number respectively phase of the radiated particles (photons or phonons). In the Φ - representation we have

\[ \hat{N} = i \frac{\partial}{\partial \Phi} , \quad \hat{\Phi} = \Phi , \quad [\hat{N}, \hat{\Phi}] = i \]  

(6.20)

Note that in the mentioned representation the states under the considerations are Φ—circular (in a similar way with the \( \varphi \)-circular states discussed above in connection with the \( L_{z} − \varphi \) case). The respective states are described by the wave functions

\[ \Psi_{N}(\Phi) = \frac{1}{\sqrt{2\pi}} e^{iN\Phi} \quad (N = 0, 1, 2, \ldots\ldots) \]  

(6.21)

For the N − Φ case the TIHR doctrine requires that through the Eq. (2.3) one should have the ordinary relation:

\[ \Delta_{\psi} N \cdot \Delta_{\psi} \Phi \geq \frac{1}{2} \]  

(6.22)

But it is easy to see that this relation is incorrect because with (6.20) and (6.21) one obtains \( \Delta_{\psi} N = 0 \) and \( \Delta_{\psi} \Phi = \pi/\sqrt{3} \).

The incorrectness of the Eq. (6.22) derives TIHR in another deadlock. With the aim of avoiding the respective deadlock TIHR partisans promoted the idea to replace the Eq. (6.22) with some adjusted relations, concordant with TIHR doctrine. So in literature (Fain and Khanin, 1965; Carruthers and Nieto, 1968; Davydov, 1973; Opatrny, 1995; Lindner et al., 1996) were promoted a few adjusted relations such as:

\[ \Delta_{\psi} N \cdot \Delta_{\psi} C \geq \frac{1}{2} \langle S \rangle \]  

(6.23)

\[ \Delta_{\psi} N \cdot \Delta_{\psi} S \geq \frac{1}{2} \langle C \rangle \]  

(6.24)
\[(\Delta_{\phi} \Phi)^2 \cdot [(\Delta_{\phi} N)^2 + \left(\langle \hat{N} \rangle_{\psi} + \frac{1}{2}\right) - \langle \hat{L}_z \rangle] \geq \frac{\hbar^2}{4} \left[1 - \frac{3 \pi^2 (\Delta_{\phi} \Phi)^2}{4}\right]\]  

(6.25)

The new quantities appearing in Eqs.(6.23)-(6.25) are defined through the relations:

\[\hat{C} = \frac{1}{2} (\hat{E}_- + \hat{E}_+) \quad , \quad \hat{S} = \frac{1}{2i} (\hat{E}_+ - \hat{E}_-)
\]

(6.26.a)

\[\hat{E}_- (N + 1)^{-1/2} \hat{a}, \quad \hat{E}_+ = \hat{a}^+ (N + 1)^{-1/2}
\]

(6.26.b)

\[\hat{N}' = \hat{I} \hat{L}_z - \frac{1}{2}, \quad \hat{L}_z = -i \hbar \frac{\partial}{\partial \Phi}\]

(6.27)

It is the place here to note the following observations. The replacement of the ordinary Eq. (6.22) with the adjusted Eqs. (6.23)-(6.25) is only a redundant mathematical operation without any true utility for physics. This happens because for the interests of physics those of real utility are the observables \(N\) and \(\Phi\) but not the above mentioned adjusted quantities \(N', C\) or \(S\). So if the interest of physics are connected on the particles (photons or phonons) radiated by quantum oscillators the real measuring instruments are counters respectively the polarizers. Such instruments measure directly \(N\) and \(\Phi\) but not \(N', C\) or \(S\). So the measuring uncertainties, appealed by TIHR as corner-stone pieces, must regard \(N\) and \(\Phi\) but not \(N', C\) or \(S\).

The above noted observations show that the relations like Eqs. (6.23)-(6.24) (or other adjusted formulae) do not solve the nonconformity of the pair \(N - \Phi\) with TIHR doctrine. The respective nonconformity remains an open question which can not be solved by means of inner elements of TIHR. This means that the \(N - \Phi\) case appears as an irrefutable defect of TIHR.

H. The energy - time case

The pair energy \(E - time t\) was and still is the subject of many debates in the literature (Aharonov and Bohm, 1961, 1964; Fock, 1962; Alcock, 1969; Bunge, 1970; Fujivara, 1970;
Surdin, 1973; Kijovski, 1974; Bauer and Mello, 1978; Voronstov, 1981; Kobe and Aquilera-Navaro, 1994). The respective debates originate in the following facts:

On the one hand $E$ and $t$ are considered as (canonically) conjugated variables whose ordinary operators $\hat{E} = i\hbar \frac{\partial}{\partial t}$ and $\hat{t} = t \cdot \cdot \cdot$ satisfy the commutation relation $[\hat{E}, \hat{t}] = i\hbar$. Then for these variables the theoretical HR expressed by Eq. (2.3) should take the ordinary form

$$\Delta_\psi E \cdot \Delta_\psi t \geq \frac{\hbar}{2}$$  \hspace{1cm} (6.28)

On the other hand as $t$ is not a random but a deterministic variable $\Delta_\psi t \equiv 0$ for any state (wave function). Moreover the energy is described by the Hamiltonian operator $\hat{H}$ and then $\Delta_\psi E = \Delta_\psi H$. Or $\Delta_\psi H$ is a null quantity (in the case of stationary states which are pure eigenstates of $\hat{H}$) or a non-null but finite quantity in the case of nonstationary states or of stationary ones which are mixtures of eigenstates of $\hat{H}$). Then one finds that in fact for the pair $E - t$ the theoretical HR given by Eqs. (2.3)/(6.28) degenerate into the absurd result $0 \geq (\hbar/2)$. With such a result the $E - t$ case radically deviates from the TIHR stipulations. For avoiding the respective deviation the TIHR partisans invented a lot of adjusted $E - t$ relations destined to replace Eq. (6.28) and to remain concordant with the alluded stipulations. More of the mentioned relations have the following generic form:

$$\Delta_v E \cdot \Delta_v t \geq \frac{\hbar}{2}$$  \hspace{1cm} (6.29)

where $\Delta_v E$ and $\Delta_v t$ have various significances such as: (i) $\Delta_1 E =$ the line breadth characterizing the decay of an excited quantum state and $\Delta_1 t =$ the duration life of the respective state (ii) $\Delta_2 E = h\Delta_2 \omega =$ the energetic width of a wave packet and $\Delta_2 t =$ temporal width of the packet (while Eq. (6.29) is introduced by means of Eq.(5.35) with $x = t$ and $k = \omega = E/\hbar$). (iii) $\Delta_3 E = \Delta_\psi H$ and $\Delta_3 t = \Delta_\psi A \cdot (d \langle A \rangle_\psi /dt)^{-1}$ with $A$ as an arbitrary variable.

Other substitutions of Eq. (6.28) were adjusted by means of some strange ideas such as (i) to take

$$\Delta t = [\langle t^2 \rangle - \langle t \rangle^2]^{1/2}$$  \hspace{1cm} (6.30)
with
\[
\langle t^n \rangle = \frac{\int_{-\infty}^{\infty} t^n |\Psi(q, t)|^2 dt}{\int_{-\infty}^{\infty} |\Psi(q, t)|^2 dt}
\] (6.31)

(ii) to fabricate a time (or tempus) operator \( \hat{T} \) capable of satisfying the commutation relation:
\[
[\hat{T}, \hat{E}] = i\hbar
\] (6.32)

In connection with the above mentioned substitutions of Eq. (6.28) the following major shortcomings must be notified: (i) The Eqs. (6.29)-(6.32) do not result from the standard mathematical procedures specific for the true theoretical HR (as presented in sections III, IV, and V), (ii) The variety of significances for \( \Delta_v E \) and \( \Delta_v t \) from Eq. (6.29) still generates persistent disputes among TIHR partisans, (iii) None of the substitution alternatives for Eq. (6.28) was ratified until now by natural and credible arguments.

The above notifications together with the presented observations about Eq. (6.28) clearly disclose an important and persistent defect of TIHR.

I. Bitemporal relations

The bitemporal relation given by Eq. (5.30) facilitates the detection of another defect of TIHR. If in the respective relation one takes \( |t_2 - t_1| \to \infty \) in the TIHR vision the quantities \( \Delta_{\Psi_1} A \) and \( \Delta_{\Psi_2} B \) refer to two variables \( A \) and \( B \) each of them considered as solitary. In such a case the TIHR assertion \( P - 2.10/A \) claims that both \( \Delta_{\Psi_1} A \) and \( \Delta_{\Psi_2} B \) should be boundlessly small quantities. But from Eq. (5.30) one can see that such a claim is refuted in the cases when \( (\delta_{\Psi_1, \hat{A}} \Psi_1, \delta_{\Psi_2, \hat{A}} \Psi_2) \neq 0 \). An example of such a case can be provided as follows.

For a QTP the time-dependent wave functions describing its states are
\[
\Psi_{nt} = \Psi_n(\xi, t) = \exp\left(-\frac{iE_n t}{\hbar}\right) \Psi_n(\xi)
\] (6.33)
where $\Psi_n(\xi)$ are given by Eq. (6.18) and $E_n = \hbar \omega \left( n + \frac{1}{2} \right)$. Then with $\hat{A} = \hat{L}_z$ and $\hat{B} = \hat{\varphi}$ the Eq. (5.30) becomes

$$\Delta_{\Psi_{nt_1}} L_z \cdot \Delta_{\Psi_{nt_2}} \varphi \geq \left| \left( \delta_{\Psi_{nt_1}} \hat{L}_z \Psi_{nt_1}, \delta_{\Psi_{nt_2}} \hat{\varphi} \Psi_{nt_2} \right) \right|$$  \hspace{1cm} (6.34)

with

$$\Delta_{\Psi_{nt_1}} L_z = \sqrt{\hbar J \omega \left( n + 1 \right)} \hspace{0.5cm}, \hspace{0.5cm} \Delta_{\Psi_{nt_2}} \varphi = \sqrt{\frac{\hbar}{J \omega}} \left( n + \frac{1}{2} \right)$$  \hspace{1cm} (6.35)

\[
\left( \delta_{\Psi_{nt_1}} \hat{L}_z \Psi_{nt_1}, \delta_{\Psi_{nt_2}} \hat{\varphi} \Psi_{nt_2} \right) = -\frac{i \hbar}{2} \exp \left\{ -i \omega \left( n + \frac{1}{2} \right) (t_2 - t_1) \right\} \hspace{1cm} (6.36)
\]

The above mentioned refutation of TIHR claimed by the bitemporal relations given by Eqs. (5.30)/(6.34) must be notified as a clear defect of TIHR. Various attempts (Levy-Leblond 1967; Ghanapragasam and Srinivas, 1979) of extrapolating TIHR vision onto the relations of Eq. (5.30) type seem to us to be without any real physical foundation.

**J. Multivariable relations**

In Sec. **V** we have shown that the multivariable relations from Eqs.(5.6) belong to the same family of theoretical formulae as the primary HR given by Eqs. (2.2)/(2.3). For an example of such a multivariable relation we can consider the QTP described by (6.18) and the following set of three variables: $A_1 = L_z, A_2 = \varphi$ and $A_3 = T = L_z^2/2J =$ the kinetic energy. Then from Eq. (5.6) one obtains:

\[
\begin{align*}
(\Delta_{\psi} L_z)^2 \cdot (\Delta_{\psi} \varphi)^2 \cdot (\Delta_{\psi} T)^2 & \geq (\Delta_{\psi} L_z)^2 \cdot \left| \left( \delta_{\psi} \hat{\varphi} \Psi, \delta_{\psi} \hat{T} \Psi \right) \right|^2 + \\
+ (\Delta_{\psi} \varphi)^2 \cdot \left| \left( \delta_{\psi} \hat{T} \Psi, \delta_{\psi} \hat{L}_z \Psi \right) \right|^2 + (\Delta_{\psi} T)^2 \cdot \left| \left( \delta_{\psi} \hat{L}_z \Psi, \delta_{\psi} \hat{\varphi} \Psi \right) \right|^2 & - \\
-2 \text{Re} \left\{ \left( \delta_{\psi} \hat{L}_z \Psi, \delta_{\psi} \hat{\varphi} \Psi \right) \cdot \left( \delta_{\psi} \hat{\varphi} \Psi, \delta_{\psi} \hat{T} \Psi \right) \cdot \left( \delta_{\psi} \hat{T} \Psi, \delta_{\psi} \hat{L}_z \right) \right\} \hspace{1cm} (6.37)
\end{align*}
\]

where $\text{Re} f = \text{real part of } f$, $\Delta_{\psi} L_z$ respectively $\Delta_{\psi} \varphi$ are given by Eq.(6.19) and
\[ \Delta_{\psi} T = \frac{\hbar \omega}{2} \sqrt{\frac{n^2 + n + 1}{2}}, \quad \left( \delta_{\psi} \hat{L}_z \Psi, \delta_{\psi} \hat{\varphi} \Psi \right) = -i \frac{\hbar}{2} \]

\[ \left( \delta_{\psi} \hat{\varphi}, \delta_{\psi} \hat{T} \Psi \right) = 0, \quad \left( \delta_{\psi} \hat{T} \Psi, \delta_{\psi} \hat{L}_z \Psi \right) = 0 \]  

(6.38)

The alluded relationship of the multivariable Eqs. (5.6), (6.37) with the HR from Eqs. (2.9)/(2.10) naturally requires an argued answer to the question: In what rapports must be the (physical) interpretation of the respective relations with TIHR? The question was approached (Synge, 1971) from a viewpoint of extrapolating TIHR vision. In the spirit of such a view there was promoted the idea that a 3-variable relation of Eq. (5.6) type (obtained by Synge on a different way) must be interpreted as describing a fundamental interconnection among the uncertainties characterizing the simultaneous measurements for the corresponding variables. Other scientists (Dodonov and Man’ko, 1987), who investigate from a mathematical viewpoint the generalizations of HR given by Eq. (2.2)/(2.3), have deliberately (and declaratively) omitted any approach of the above mentioned question.

A careful examination of the facts shows that the above mentioned extrapolation of TIHR is unjustifiable at least because of the following reasons: (i) It cannot be sustained by real (nonfictitious) arguments regarding the true characteristics of the measurements, (ii) As it was pointed out in Sec. VI.A the theoretical quantities implied in Eqs. (5.6) and (6.37) (like \( \Delta_{\psi} A_j \) or \( (\delta_{\psi} \hat{A}_j \Psi, \delta_{\psi} A_i \Psi) \)) have not significance of measuring uncertainties. Then it results that, although mathematically the multivariable relations are closely related with HR, they cannot be interpreted in a manner consonant with TIHR. But such a result must be notified as another defect of TIHR regarded as a comprehensive doctrine.

**K. Thermal relations**

TIHR was promoted in connection with the quantum HR given by Eqs. (2.2)/(2.3). But we have shown that, mathematically, the respective HR are completely similar with the thermal relations from Eqs. (5.21)-(5.23). Firstly, we shall present here some concrete examples of such thermal relations.
So from the phenomenological (quasithermodynamic) theory of fluctuations we can quote (Dumitru, 1974 a, 1988) the following P-V (pressure-volume) formula

\[ \Delta_w V \cdot \Delta_w P \geq |\langle \delta_w V \delta_w P \rangle_w| \]  

(6.39)

where

\[ \Delta_w V = \sqrt{-kT \left( \frac{\partial V}{\partial p} \right)_T}, \quad \Delta_w P = \sqrt{-kT \left( \frac{\partial p}{\partial V} \right)_S} \]

\[ \langle \delta_w V \delta_w P \rangle_w = -kT \]  

(6.40)

with \( \overline{A} = \langle A \rangle_w \), \( k \) = Boltzmann’s constant and \( S \) = entropy.

Also we can quote a formula of Eq. (5.23) type from classical statistical physics. For this let us consider an ideal gas situated in a cylindrical recipient of height \( b \) on the Earth surface. The gas is supposed to be in a thermodynamical equilibrium state described by the canonical distribution \( w \sim \exp \{-H/kT\} \), with \( H \) = the Hamiltonian of the molecules. If the molecules are considered as identic point particles, we can take \( H \) of the form \( H = \sum_{i=1}^{3N} \left( \frac{p_i^2}{2m} \right) + \sum_{i=1}^{N} mgz_i \) \((N, p_i \text{ and } z_i \text{ represent respectively the number, linear momenta and altitude of the molecules; } m \text{ denotes the mass of a molecule and } g \text{ is the gravitational acceleration})\). As stochastic variables for the gas regarded as a statistical system we consider \( A = H = \text{Hamiltonian} \) and \( B = Z_c = \text{the altitude of the centre of mass} \). For such variables Eq. (5.23) transcribes as

\[ \Delta_w H \cdot \Delta_w Z_c \geq |\langle \delta_w H \delta_w Z_c \rangle_w| \]  

(6.41)

The terms from this relation are given by

\[ (\Delta_w H)^2 = k \nu RT^2 \left[ \frac{5}{2} - a^2 f(a) \right] \]  

(6.42a)

\[ (\Delta_w Z_c) = k \frac{b^2}{\nu R} \left[ \frac{1}{a^2} - f(a) \right] \]  

(6.42b)

\[ \langle \delta_w H \cdot \delta Z_c \rangle = kTb \left[ (e^a - 1)^{-1} - f(a) \right] \]  

(6.42c)
where \( \nu = \) number of moles, \( f(a) = e^a (e^a - 1)^{-2} \) and \( a = (\mu gb/RT) \) with \( \mu = \) molar mass of the gas, \( R = \) universal gas constant.

Another nontrivial exemplification of Eq. (5.22) is given by the early known Führth’s formula from the theory of Brownian motion (Führth, 1933). In the respective formula \( A = x \) and \( B = v \) i.e. the coordinate and velocity of a Brownian particle, whereas \( \langle \delta_w A \delta_w B \rangle_w = \langle \delta_w x \cdot \delta_w v \rangle_w = D \) diffusion coefficient of the particle. Then the Führth’s formula is:

\[
\Delta_w x \cdot \Delta_w v > D \tag{6.43}
\]

Now let us return to the mentioned mathematical similarity between thermal relations given by Eqs. (5.21)-(5.23), (6.39), (6.41), (6.43) and HR from Eqs. (2.2)/(2.3). The respective similarity suggests an investigation of the possible connections between the interpretation of the alluded thermal relations respectively the TIHR. Such an investigation was approached by some traditionalist-scientists (Frank-Kamenetsky, 1940; Bohm, 1957; Rosenfeld, 1961 and 1962; Terletsky, 1974). Mainly, they promoted the idea that the mentioned thermal relations must be interpreted in terms of a macroscopic complementarity. Consequently, by extrapolation of the TIHR statements \( P - 2.15 \) and \( P - 2.16 \), in Eq. (5.23) the variables \( A \) and \( B \) must be regarded as complementary (i.e. mutually exclusive) while \( \Delta_w A \) and \( \Delta_w B \) have to be interpreted as uncertainties in simultaneous measurements. The mentioned traditionalist idea was partially reviewed from various perspectives (Shaposhnikov, 1947; Bazarov, 1979; Uffink and van Lith, 1999). But in the alluded reviews there are not pointed out explicitly the facts that the quantities like \( \Delta_w A \) and \( \Delta_w B \) describe the thermal fluctuations respectively that such fluctuations are intrinsic properties of the thermodynamic systems.

Our opinion is that the alluded idea of macroscopic complementarity must be rejected due to the following reasons: (i) The quantity \( \Delta_w A \) is not a measuring uncertainty but a parameter (standard deviation) characterizing the thermal fluctuations. (ii) The value of \( \Delta_w A \) can be modified only by changing the inner state (i.e. the distribution function \( w \)) of the considered system but not by means of improvements of measuring precision. (iii)
Regarded as fluctuation parameter $\Delta_{w}A$ can be measured without restriction of principle. So in noise spectroscopy (Weissmann, 1981) it is possible to measure even the "constitutive" (i.e. spectral) components of $\Delta_{w}A$. (iv) In classical physics the variables characterizing a macroscopic system are not mutually exclusive (i.e. complementary), (v) The true conception about the macroscopic measurements does not include any reference to the mutual unavoidableness of simultan uncertainties for two (or more) variables.

The above mentioned reasons clearly guide us to the following conclusion. In spite of the mathematical similarity between HR and the discussed thermal relations, TIHR can not be extended (by similitude) to the interpretation of the respective relations. But as the mentioned mathematical similarity is of a conceptually fundamental nature the concluded inextensibility must be notified as a true defect of TIHR.

L. The so-called macroscopic operators

Controversies about TIHR also included several discussions regarding the macroscopic relation given by Eq. (5.29) (see Jancel, 1973 and references). The respective discussions were generated by the following conflicting findings: (i) On the one hand the respective relation appears within quantum theory and, mathematically, it is completely similar with HR expressed by Eqs. (2.2)-(2.3). Then by extrapolating TIHR the Eq. (5.29) should be interpreted as an interconnection of the macroscopic uncertainties $\Delta_{\rho}A$ and $\Delta_{\rho}B$ regarding the simultaneous measurements of the variables $A$ and $B$ afferent to a macroscopic system.

(ii) On the other hand, according to $\textbf{P - 2.4}$, TIHR operates with the hypothesis that a macroscopic variable can be measured without any uncertainty (i.e. with unbounded accuracy), irrespective of the fact that it is measured solitary or simultaneously with other variable. Then, of course, it is useless to speak of an interconnection between the uncertainties of two macroscopic variables-even if, in theoretical framework, they are described by quantum statistical operators.

To elude the mentioned conflict the TIHR partisans promoted the strange idea to abro-
gate the Eq. (5.29) and to replace it by an adjusted macroscopic formula concordant with TIHR vision. With this aim, the common operators \( \hat{A} \) and \( \hat{B} \) from Eq. (5.29) were substituted by the so-called "macroscopic operators" \( \hat{A}_M \) and \( \hat{B}_M \) which (in any representation?) can be pictured as quasi-diagonal matrices. Then one supposes that \( [\hat{A}_M, \hat{B}_M] = 0 \) and instead of Eq. (5.29) one obtains

\[
\Delta \rho_{A_M} \cdot \Delta \rho_{B_M} \geq 0 \tag{6.44}
\]

In this relation TIHR partisans view the fact that, simultaneously, the uncertainties \( \Delta \rho_{A} \) and \( \Delta \rho_{B} \) can be arbitrarily small. Such a view is concordant with the main concept of TIHR. Today many scientists believe that the adjusted Eq. (6.44) solves all the troubles of TIHR generated by the Eq. (5.29).

It is easy to remark that the mentioned belief proves to be unfounded if one takes into account the following observations:

(i) Equation (5.29) can not be abrogated unless the entire mathematical apparatus of quantum statistical physics is abrogated too. More exactly, the substitution of the common operators \( \hat{A}_M \) and \( \hat{B}_M \) with macroscopic operators \( \hat{A}_M \) and \( \hat{B}_M \) is a useless invention. This because in the practical domain of quantum statistical physics (see for example Tyablikov, 1975) the common operators but not the macroscopic ones are used.

(ii) The above mentioned substitution of operators does not metamorphose automatically Eq. (5.29) into Eq. (6.44). This because, if two operators are quasi-diagonal (in the sense required by the TIHR partisans) they can be non-commutable. As an example in this sense we refer to a macroscopic system formed by a large number \( N \) of independent \( \frac{1}{2} \)-spins (Dumitru, 1988, 1989). The hinted macroscopic variables are components \( M_j \) (\( j = x, y, z \)) of the magnetization \( \vec{M} \). The corresponding operators are

\[
\hat{A}_j = \hat{M}_j = \frac{\gamma \hbar}{2} \hat{\sigma}^{(1)}_j \oplus \frac{\gamma \hbar}{2} \hat{\sigma}^{(2)}_j \oplus \ldots \oplus \frac{\gamma \hbar}{2} \hat{\sigma}^{(N)}_j \tag{6.45}
\]

where \( \gamma = \) magneto-mechanical factor, \( \hat{\sigma}^{(n)}_j = \) Pauli matrices associated with the "\( n \)-th" spin (microparticle). One can see that the operators defined by Eqs. (6.45) are quasidiagonal
in the sense required for "macroscopic operators", but they are not commutable among
them, as we have for example \[ [\widehat{M}_x, \widehat{M}_y] = i\hbar \gamma \widehat{M}_z. \] Consequently one can say that by the
mentioned substitution of operators the Eq. (5.29) is transposed in fact not in (6.44) but
into the formula

\[ \Delta_\rho A_M \cdot \Delta_\rho B_M \geq \left| \left\langle \left[ \hat{A}_M, \hat{B}_M \right] \right\rangle_{\rho} \right| \]  

But such a formula is not helpful for TIHR if \[ \left\langle \left[ \hat{A}_M, \hat{B}_M \right] \right\rangle_{\rho} \neq 0, \] as in the case of operators
defined by Eqs. (6.45).

(iii) The alluded substitution of operators does not solve the troubles of TIHR even if
the macroscopic operators are commutable. This because Eq. (5.29) is only a truncated
version of the more general Eq. (5.27). Then by the mentioned substitution, in fact, one
must consider the metamorphosis of Eq. (5.27) which gives

\[ \Delta_\rho A_M \cdot \Delta_\rho B_M \geq \left| \left\langle \delta_\rho \hat{A}_M \cdot \delta_\rho \hat{B}_M \right\rangle_{\rho} \right| \]  

In this formula, if \[ \left[ \hat{A}_M, \hat{B}_M \right] = 0, \] one obtains
\[ \left\langle \Delta_\rho \hat{A}_M \cdot \Delta_\rho \hat{B}_M \right\rangle_{\rho} = \frac{1}{2} \left\langle \left[ \delta_\rho \hat{A}_M, \delta_\rho \hat{B}_M \right] \right\rangle_{\rho} \]
i.e. a quantity which can have a non-null value. Then it results that the macroscopic product
\( \delta_\rho A_M \cdot \delta_\rho B_M \) can have a non-null lower bound. But such a result opposes to the agreements
of TIHR.

So we conclude that in fact the mentioned macroscopic operators cannot solve the TIHR
deficiencies connected with the Eq. (5.29). This means that the respective deficiencies
remain unsolved and they must be reported as another insurmountable defect of TIHR.

VII. INDUBITABLE FAILURE OF TIHR

A mindful examination of all the details of the facts discussed in the previous section
guide us to the following remarks:

\[ \mathbf{P - 7.1} \]: Taken together, in an ensemble, the above presented defects incriminate and
invalidate each of the main elements of TIHR. ▲
The mentioned defects are insurmountable for the TIHR doctrine, because they cannot be avoided or refuted by means of credible arguments from the framework of the respective doctrine.

The two remarks reveal directly the indubitable failure of TIHR which now appears as an unjustified doctrine. Then the sole reasonable attitude is to abandon TIHR and, as a first urgency, to search the genuine significance (interpretation) of the HR. As second urgency, probably, it is necessary a re-evaluation of those problems in which, by its implications, TIHR persists as a source of misconceptions and confusions.

VIII. THE GENUINE SIGNIFICANCE OF HR

A veritable search regarding the genuine significance of HR must be founded on the true meaning of the elements implied in the introduction of the respective relations. Then we have to take into account the following considerations.

Firstly, we opine that thought-experimental HR of the type given by Eq. (2.1) must be omitted from discussions. This because, as it was pointed out (see Sec. VI.B and comments about the Eq. (5.1)), such a type of relations has a circumstantial character dependent on the performances of the supposed measuring-experiment. Also in the respective relations the involved variables are not regarded as stochastic quantities such are the true quantum variables. So the equations of (2.1) - type have not a noticeable importance for the conceptual foundation of quantum mechanics. Moreover the usages of such relations in various pseudo-demonstration (Tarasov, 1980) have not a real scientific value. That is why we opine that the thought-experimental HR of the type given by Eq. (2.1) must be completely excluded from physics.

We resume the above opinions under the following point:

The thought-experimental HR like Eq. (2.1) must be disregarded being fictitious formulae without a true physical significance.

As regard the theoretical HR of the kind illustrated by Eqs. (2.2)/(2.3) the situation is
completely different. The respective HR are mathematically justified for precisely defined conditions within the theoretical framework of quantum mechanics. This means that the physical significance (interpretation) of the theoretical HR is a question of notifiable importance. Now note that the mentioned HR belong to the large family of correlation relations reviewed in Sec. V. This fact suggests that the genuine significance (interpretation) of the theoretical HR must be completely similar with that of the mentioned correlation relations.

We opine that the alluded suggestion must be taken into account with maximum consideration. Then, firstly, we remark that all of the mentioned correlation relations refer to the variables with stochastic characteristics. Such variables are specific both for quantum and non-quantum (i.e. classical) physical systems. Secondly, let us regard the quantities like $\Delta_w A$ or $\langle \delta_w A \delta_w B \rangle_w$ which appear in the corresponding correlation relations from classical statistical physics. In our days science the respective quantities are unanimously interpreted as fluctuation parameters of the considered variables $A$ and $B$. Also it is clearly accepted the fact that the respective parameters describe intrinsic properties of the viewed systems but not some characteristics (i.e. uncertainties) of the measurements on the respective properties. In this sense in some scientific domains, such as noise spectroscopy (Weissman, 1981), the evaluation of the quantities like $\Delta_w A$ is regarded as a tool for investigation of the intrinsic properties of the physical systems. In classical conception the description of the intrinsic properties of the physical systems is supposed to be not amalgamated with elements regarding the measuring uncertainties. The alluded description is made, in terms of corresponding physical variables, within the framework of known chapters of classical physics (i.e. mechanics, electrodynamics, optics, thermodynamics and statistical physics). For the mentioned variables, when it is the case, the description approaches also the fluctuations characteristics. Otherwise, in classical conception the measuring uncertainties /errors are studied within the framework of error analysis. Note that the respective analysis is a scientific branch which is independent and additional with respect to the mentioned chapters of physics (Worthing and Geffner, 1955).

The above mentioned aspects about the classical quantities $\Delta_w A$ and $\langle \delta_w A \delta_w B \rangle_w$ must
be taken into account for the specification of the genuine significance (interpretation) of the quantum quantities $\Delta_\Psi A$ and $\langle \delta_\Psi A \delta_\Psi B \rangle_\Psi$, as well as of the theoretical HR. We think that the respective specification can be structured through the following points:

**P – 8.2**: The quantum variables (operators) must be regarded as stochastic quantities which admit fluctuations.\▲

**P – 8.3**: According to the usual quantum mechanics, the time is not a stochastic variable but a deterministic quantity which does not admit fluctuations.\▲

**P – 8.4**: The theoretical quantities $\Delta_\Psi A$, $\Delta_\Psi B$, $\left( \delta_\Psi \hat{A} \Psi, \delta_\Psi \hat{B} \Psi \right)$ or $\left\langle \left[ \hat{A}, \hat{B} \right] \right\rangle_\Psi$ must be interpreted as parameters referring to the fluctuations regarded as intrinsic properties of quantum microparticles.\▲

**P – 8.5**: The theoretical HR of the kind illustrated by Eqs (2.2)/(2.3) must be considered through their accurate and complete forms presented in Sec.V. In such forms they have to be interpreted as fluctuation formulae regarding intrinsic characteristics of microparticles.\▲

**P – 8.6**: The quantities mentioned in **P – 8.4** as well as the theoretical HR have no connection with the description of measuring uncertainties.\▲

**P – 8.7**: The description of measurements, characteristics (e.g. of measuring uncertainties) for quantum microparticles must be made in the frame of a scientific branch which is independent and additional with respect to the usual quantum mechanics.\▲

A persuasive argumentation of these points results directly from the above presented considerations. For **P – 8.4** and **P – 8.6** the respective argumentation can be improved by the following observations: (i) the values of the quantities discussed in **P – 8.4** depend (through the wave function $\Psi$) only on the considered microparticle state, (ii) the respective values are independent of the measuring uncertainties which, for the same measured state, can be modified (by changing the accuracy of experimental instruments).

The above presented points **P – 8.1** — **P – 8.6** can be regarded as the main elements of a genuine reinterpretation of HR. It is easy to see that the respective reinterpretation is completely concordant with the working procedures of usual quantum mechanics (which, of
course, include the theoretical HR as particular formulae).

The mentioned reinterpretation assures a vision where all the defects of TIHR presented in Sec. VI. A — VI. L are eliminated as inadequate and unfounded statements. So the defect discussed in Sec. VI. A. loses any sense if one takes into account \( P - 8.1 \), \( P - 8.4 \) and \( P - 8.6 \). Also the defect revealed in Sec. VI. B has no value if we accept \( P - 8.1 \).

The facts presented in Sec. VI. C can be elucidated by considering \( P - 8.6 \). For the case of solitary variables approached in Sec. VI. D a pertinent answer is given by \( P - 8.4 \) and \( P - 8.6 \). The same points \( P - 8.4 \) and \( P - 8.6 \) offer a natural vision on the case of commutable variables approached in Sec. VI. E.

The cases of the pairs \( L_Z - \varphi \) and \( N - \Phi \) discussed in Secs. VI. F and VI. G can be brought in concordance with the proposed reinterpretation of HR as follows.

In the situations described by the wave functions given by Eqs.(6.6) and (6.21) the conditions expressed by Eqs. (5.8) are not satisfied. Then, according to \( P - 8.5 \), in such situations the discussions must refer to the complete and general correlation relations given by Eqs. (5.7) or (5.14) but not to the restrictive formulae like Eqs. (2.3)/(5.13)/(6.4).

Note that, from a general perspective, the situations described by the wave functions \( \Psi \) noted in Eqs. (6.6) or (6.21) refer to the cases which regard the uni-variable eigenstates. In such a state for two variables A and B we have \( \hat{A}\psi = a\psi \) and \( \hat{B}\psi \neq b\psi \) respectively \( \Delta_\psi A = 0 \) and \( 0 < \Delta_\psi B \neq \infty \). But if \( \left< \left[ \hat{A}, \hat{B} \right] \right> \neq 0 \) for the pair \( A - B \) the HR from Eq. (2.3) and the related TIHR assertions are not applicable. It must be reminded that an early modest notification (Davidson, 1965) of the TIHR shortcomings in respect with the uni-variable eigenstates seem to be ignored by the TIHR partisans.

Now we can see that the alluded inapplicability of Eq. (2.3) is generated by the fact that for the mentioned uni-variable eigenstates the Eqs. (5.8) are not satisfied. This because if for such states Eqs. (5.8) are satisfied we must admit the following row of relations.

\[
a < B >_\psi = a \left< \Psi, \hat{B}\Psi \right> = \left< \hat{A}\Psi, \hat{B}\Psi \right> = \left< \Psi, \hat{A}\hat{B}\Psi \right> =
\]
\[
\Psi, \hat{A} \hat{B} \Psi + (\Psi, \hat{B} \hat{A} \Psi) = \left\langle [\hat{A}, \hat{B}]_\psi \right\rangle + \langle B \rangle_\psi \cdot a
\] (8.1)

i.e. the absurd result \( a < B >_\psi = \left\langle [\hat{A}, \hat{B}]_\psi \right\rangle + a \langle B \rangle_\psi \) with \( \langle [A, B]_\psi \rangle \neq 0 \).

Add here the fact that for the discussed states instead of Eq. (2.3) the more general Eq. (5.7) remain valid (in the trivial form \( 0 = 0 \)).

It is quite evidently that the situations of uni-variable eigenstates come into the normality if \( \Delta_\psi A \) and \( \Delta_\psi B \) are regarded as parameters which describe the quantum fluctuations. Then in such situations A and B have respectively have not fluctuations (i.e. stochastic characteristics).

The situation described by the wave function given by Eqs. (6.15) must be discussed separately. Firstly it necessitates a confrontation with the conditions expressed by Eqs. (5.8). In this sense we note that for the respective situation one obtains the relation

\[
(\hat{L}_Z \Psi_t, \hat{\varphi} \Psi_t) - (\Psi_t, \hat{L}_Z \hat{\varphi} \Psi_t) =
\]

\[
= i\hbar \left\{ 1 + 2 \text{Im} \left[ \sum_m \sum_{m'} C^*_m C_{m'} \left( Y_{im}, \varphi Y_{im'} \right) \right] \right\}
\] (8.2)

Where \( \text{Im} \alpha \) denote the imaginary part of the complex quantity \( \alpha \). Then one observes that in the cases when the right hand term in Eq. (8.2) is null the variables \( L_Z \) and \( \varphi \) satisfy the Eqs.s (5.8). In such cases the Eqs. (2.3)/(5.13)/(6.4) are applicable. In other cases (when the mentioned term from Eq. (8.2) is non-null) the Eqs. (5.8) are infringed and for the pair \( L_Z - \varphi \) it must apply only the Eqs. (5.7) or (5.14).

Note that the \( L_Z - \varphi \) case in the situation described by wave functions from Eqs.(6.15) can be approached also by using the Fourier analysis procedures. So for the mentioned case and situations, similarly with Eq. (5.41), one obtains

\[
\Delta_\psi L_Z \cdot \Delta_\psi \varphi \geq \left| \text{Im} \left\{ \sum_m \sum_{m'} C^*_m C_{m'} \cdot m \left( Y_{im}, \varphi Y_{im'} \right) \right\} \right|
\] (8.3)

As regards the case of QTP described by the wave functions given by Eqs. (6.18) we note the following observations. In such a case the variables \( L_Z \) and \( \varphi \) satisfy the conditions
expressed by Eqs. (5.8). Consequently for the respective variables are applicable the Eqs. (2.3)/(5.13)(6.4). But the mentioned equations must be considered as resulting from more general Eqs. (5.7) or (5.14) which are referring to the quantum fluctuations.

The problems with the pair energy-time mentioned in Sec. VI. H become senseless if it is accepted $P - 8.3$. According to $P - 8.5$ the relations mentioned in sections VI. I and VI. J become simple generalizations of the theoretical HR without interpretational shortcoming. The relations discussed in Sec. VI. K and VI. L are nothing but macroscopic similars of the quantum theoretical HR. Also the respective relations do not imply any interpretational shortcoming. Moreover, the so-called macroscopic operators discussed in Sec. VI. L appear as pure inventions without any physical utility or significance.

♦ A reply addendum regarding the $L_z - \varphi$ pair

Our first opinions about the $L_z - \varphi$ pair in connection with TIHR were presented in earlier works (Dumitru 1977, 1980). Perhaps the respective presentations were more modest and less complete - e.g. we did not use at all the arguments resulting from the above mentioned examples of $L_z$ - degenerate states or of QTP. Nevertheless, we think that the alluded opinions were correct in their essence. However, in a review (Schroeck Jr., 1982) the respective opinions were judged as being erroneous. In this addendum, by using some of the above discussed facts, we wish to reply to the mentioned judgements.

The main error reproached by Prof. Schroeck to us is: ”most of the results stated concerning angular momentum and angle operators (including the supposed canonical commutation relations) are false, this being a consequence of not using Riemann-Stieljes integration theory which is necessitated since the angle function has a jump discontinuity”. In order to answer to this reproach we resort to the following specifications: (i) One can see that the respective reproach is founded, in fact, on the idea that the variable $\varphi$ has a jump (of magnitude $2\pi$ at $\varphi = 0$ or, equivalently at $\varphi = 2\pi$) and, consequently, on the commutation relation is $\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar + i\hbar 2\pi \delta$ (where $\delta = $ Dirac delta function at the boundary $\varphi = 0$ or $\varphi = 2\pi$). Note that the respective idea (confessed explicitly to us by Prof. Schroeck in two letters dated September 16, 1981 and April 2, 1982) can also be found in most TIHR
publications. (ii) The mentioned idea refers, in fact, only to the systems which are $\varphi$-circular and non-degenerate in respect with $L_z$ (defined in the sense precised above in Sec. VI. F). But, strangely, it is associated with the supposition that the range of $\varphi$ is the whole domain $(-\infty, \infty)$, but not the finite interval $[0, 2\pi]$. (iii) Here is the place to add also the cases presented in Sec. VI. F of QTP and of the $L_z$-degenerate states (the last ones for the situations with non-null term in the right hand side of Eq. (8.2)). For the respective cases we must consider, another commutation relation, namely $\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar$. (iv) Then in the spirit of the mentioned idea for the same pair of variables $L_z - \varphi$ one must tolerate two completely dissimilar commutation relations: $\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar + i\hbar 2\pi \delta$ and $\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar$. But such a toleration seems to be senseless and without any real (physical) substantion. (v) Our opinion about the $\hat{L}_z - \hat{\varphi}$ pair remains, as it was announced in previous works, and argued with more details in the present paper. It is founded on the necessity to approach in a unique manner all the alluded cases. In essence we think that, in all the respective cases, for the $L_z - \varphi$ pair we must have an unique commutation relation, namely $\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar$. The implementation of the respective relation in the mentioned cases for obtaining theoretical formulae of HR-type must be made by taking into account the natural (physical) range of $\varphi$ as well as the fulfillment of the Eqs. (5.8).

The ensemble of the above noted specifications proves as unfounded the reproaches of Professor F.E. Schroek Jr. regarding our opinions about the $L_z - \varphi$ pair.

***

The facts presented in this section show that all the problems directly connected with the interpretation of HR can be solved by the here-proposed genuine reinterpretation of the respective relations. But, as it is known, TIHR generated also disputes about the topics which are adjacent or additional with respect to the alluded problems. Several such topics will be briefly approached in next sections.
IX. A RECONSIDERATION REGARDING THE MEASUREMENTS

As it was mentioned in Sec. II the story of HR started with the primary questions regarding the measuring uncertainties. During the years the respective questions and more generally the description of the measurements generated a large number of studies (a good list of references, in this sense, can be obtained from the works: Yanase et al., 1978; Braginsky and Khalili 1992; Bush et. al., 1996; Hay and Peres, 1998; Sturzu, 1999 and surely from the bibliographical publications indicated in the end of Sec. I). It is surprising to see that many of the above alluded studies are contaminated one way or another by ideas pertaining to the TIHR doctrine. After the above exposed argumentation against TIHR, here we wish to present a few elements of a somewhat new reconsideration of the problems regarding the description of measurements (including of measuring uncertainties). We think that, even modestly limited, such a reconsideration can be of non-trivial interest for our days’ science. This because we agree with the opinion (Primas and Müller - Herold, 1978) that, in fact, ”there exists not yet a fundamental theory of actual measuring instruments”.

Firstly, we note that, in our opinion, the questions $P - 2.1 - P - 2.2$ are of real significance for the studies of the physical systems. This fact is due to the essential role of measurements (i.e. of quantitatively evaluated experiments) for the mentioned studies. Moreover, we think that, in principle, the alluded role appears both in quantum and non-quantum physics.

Then in our announced reconsideration we must try to search for natural answer to the questions.. $P - 2.1 - P - 2.2$ as well as to some other (more or less) directly connected problems. For such a purpose we shall note the remarks under the following points:

$P - 9.1$ : As a rule, all the measurements, both from macroscopic and microscopic physics, are confronted with measuring uncertainties.$\uparrow$

$P - 9.2$ : The respective uncertainties are generated by various factors. Among such factors the most important ones seem to be the intrinsic fluctuations within the experimental devices and the measuring perturbations (due to the interactions of the respective devices
P - 9.3: A quantitative description of the measuring uncertainties must be made in the framework of an authentic theory of measurements. The respective theory must be independent and additional with respect to the traditional chapters of physics (which describe the intrinsic properties of physical systems).

P - 9.4: The measurement of a stochastic variable should not be reduced to a sole detection. It must be regarded and managed as a *statistical sampling* (i.e. as a statistical ensemble of detections). Therefore, for such a variable, the finding of a single value from a sole detection does not mean the collapse of the corresponding stochastic characteristics (described by a wave function or by a probability density).

P - 9.5: In the spirit of the above remark the measuring uncertainties of stochastic variables must be described in terms of quantities connected with the afferent statistical sampling but not with solitary detections.

P - 9.6: As regards the above alluded theory of measurements we agree with the idea (Bunge, 1977b) that it must include some specific elements, but not only generic-universal aspects. This because every experimental apparatus, used in real measurements, has a well-defined level of performance and a restricted class of utilizations - i.e. it is not a generic-universal (for all-purpose) device.

P - 9.7: Together with the mentioned agreement we opine that the measurements theory can include also some elements/ideas with generic-universal characteristics. One such characteristic is connected with the fact that, in essence, every measurement can be regarded as an acquisition of some information about the measured system.

In the spirit of the latter remark we think that from a generic-universal viewpoint a measurement can be described as a process of information transmission, from the measured system to the receiver (recorder or observer). In such a view, the measuring apparatus can be represented as a channel for information transmission, whereas the measuring uncertainties can be pictured as an alteration of the processed information. Such informational approach is applicable for measurements on both macroscopic and microscopic (quantum)
systems. Also the mentioned approach does not contradict to the idea of specificity as regards the measurements theory. The respective specificity is implemented in theory by the concrete models/descriptions of the measured system (information source), of the measuring apparatus (transmission channel), and of the recorder/observer (information receiver).

For an illustration of the above ideas let us refer to the description of the measurement of a single stochastic variable $x$ having a continuous spectrum of values within the range $(-\infty, \infty)$. The respective variable can be of classical kind (such are the macroscopic quantities discussed in connection with Eqs. (5.19)-(5.23)) or of a quantum nature (e.g. a Cartesian coordinate of a microparticle).

The alluded measurement, being regarded as a statistical sampling, its usual task is to find certain global probabilistic parameters of $x$ such as: mean/expected value, standard deviation, or even higher order moments. But the respective parameters are evaluated by means of the elementary probability $dP = w(x) dx$ of finding the value of $x$ within the infinitesimal range $(x, x + dx)$. Here $w(x)$ denotes the corresponding probability density. Then the mentioned task can be connected directly to $w(x)$.

Related to the measured system’s own properties, $w(x)$ has an IN (input) expression $w_{IN}(x)$. So viewed $w_{IN}(x)$ is assimilable with: (a) a usual distribution from classical statistical physics - in the case of a macroscopic system, respectively (b) the quantity $|\Psi(x)|^2$; ($\Psi(x)$ = the corresponding wave function) - in the case of a quantum microparticle.

The fact that the measuring apparatus distorts (alters) the information about the values of $x$ means that the respective apparatus records an OUT (output) probability density $w_{OUT}(x)$ which generally differs from $w_{IN}(x)$. So, with respect to the measuring process, $w_{IN}(x)$ and $w_{OUT}(x)$ describe the input respectively the output information. Then it results that the measuring uncertainties (alterations of the processed information) must be described in terms of several quantities depending on both $w_{OUT}(x)$ and $w_{IN}(x)$.

A description of the mentioned kind can be obtained, for instance, if one uses the following mean values:
\[ \langle f \rangle_A = \int_{-\infty}^{\infty} f(x) \ w_A(x) \ dx, \quad (A = IN; \ OUT) \quad (9.1) \]

with \( f(x) \) = an arbitrary function of \( x \). Then a possible quantitative evaluation of the measuring disturbances can be made in terms of both the following parameters:

(i) the \textit{mean value uncertainty} given by

\[ \delta(\langle x \rangle) = \langle x \rangle_{OUT} - \langle x \rangle_{IN} \quad (9.2) \]

(ii) the \textit{standard deviation uncertainty} defined as

\[ \delta(\Delta x) = \Delta_{OUT}x - \Delta_{IN}x \quad (9.3) \]

where \( \Delta_Ax = \left[ \langle x^2 \rangle_A - \langle x_A \rangle^2 \right]^{1/2}, \quad (A = IN; OUT). \)

The mentioned evaluation can be enriched by also using the higher order probabilistic moments/correlations (in the sense discussed by Dumitru (1999)).

Another evaluation of the measuring uncertainties can be made by means of the \textit{informational entropy uncertainty}

\[ \delta H = H_{OUT} - H_{IN} \quad (9.4) \]

Here the informational entropies \( H_A \) \( (A = IN; OUT) \) are defined by

\[ H_A = - \int_{-\infty}^{\infty} w_A(x) \ln \left[ 1_x w_A(x) \right] dx \quad (9.5) \]

where \( 1_x = \) the unit of the physical variable \( x \).

Due to the fact that, in the present considerations, a measurement is regarded as a statistical sampling, the parameters defined by Eqs. (9.2)-(9.4) can be called \textit{statistical uncertainties}.

The uncertainty parameters introduced by Eqs. (9.2)-(9.4) can be detailed if one takes into account more elements regarding the characteristics of the measuring apparatus and/or of the measured system. So we can refer to an apparatus modeled as an (information) transmission channel with stationary and linear characteristics. Then we can write
\[ w_{\text{OUT}}(x) = \int_{-\infty}^{\infty} G(x, x') w_{\text{IN}}(x') \, dx' \]  

(9.6)

where the kernel \( G(x, x') \) must satisfy the normalization conditions

\[ \int_{-\infty}^{\infty} G(x, x') \, dx = \int_{-\infty}^{\infty} G(x, x) \, dx = 1 \]  

(9.7)

The measurement is ideal or real (non-ideal) in the cases when \( w_{\text{OUT}}(x) = w_{\text{IN}}(x) \) respectively \( w_{\text{OUT}}(x) \neq w_{\text{IN}}(x) \). In the above model such cases appear if we take \( G(x, x') = \delta(x - x') \) respectively \( G(x, x') \neq \delta(x - x') \), with \( \delta(x - x') \) as Dirac \( \delta \) function of \( x - x' \). Then by using Eqs. (9.4)-(9.6) and by taking into account the relation \( \ln y \leq y - 1 \), \( (y > 0) \), one obtains

\[
\delta H = - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' G(x, x') \, w_{\text{IN}}(x') \ln \left[ \frac{w_{\text{OUT}}(x)}{w_{\text{IN}}(x)} \right] \\
\geq - \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' G(x, x') \, w_{\text{IN}}(x') \left[ \frac{w_{\text{OUT}}(x)}{w_{\text{IN}}(x)} - 1 \right] = 0
\]  

(9.8)

So we find

\[ \delta H = H_{\text{OUT}} - H_{\text{IN}} \geq 0 \]  

(9.9)

This relation shows that for the processed information (measurement of \( x \)) the entropy \( H \) at the recorder is greater (or at least equal) than at the measured system. In terms of the entropy uncertainty we can speak of ideal respectively real (non-ideal) measurement as \( \delta H = 0 \) or \( \delta H \neq 0 \).

We can detail even more the above ideas by the following example. Let us consider \( x \) as coordinate of a rectilinear quantum oscillator situated in its lowest energy state. Such a state is described by the Gaussian wave function

\[ \Psi(x) = \left( \frac{\sqrt{2\pi}\sigma_{\text{IN}}}{\sqrt{2\pi}} \right)^{-1/2} \exp \left\{ -\frac{x^2}{4\sigma_{\text{IN}}^2} \right\} \]  

(9.10)

where \( \sigma_{\text{IN}} = \hbar/2m\omega \) with \( m \) and \( \omega \) denoting the mass respectively the angular frequency of the oscillator. Then \( w_{\text{IN}}(x) \) is given by
\[ w_{IN}(x) = |\Psi(x)|^2 = \left(\sqrt{2\pi\sigma_{IN}}\right)^{-1}\exp\left\{-\frac{x^2}{2\sigma_{IN}^2}\right\} \] (9.11)

The \( IN \) - values (calculated with \( w_{IN}(x) \)) of mean and standard deviation of \( x \) are

\[
\langle x \rangle_{IN} = 0, \quad \Delta_{IN}x = \sigma_{IN} = \sqrt{\frac{\hbar}{2m\omega}} \] (9.12)

If the errors induced by the measuring device are supposed to be small, the kernel \( G(x,x') \) can be taken of the Gaussian form:

\[
G(x,x') = \frac{1}{\sigma_D\sqrt{2\pi}}\exp\left\{-\frac{(x - \varepsilon - x')^2}{2\sigma_D^2}\right\} \] (9.13)

Here \( \varepsilon \) and \( \sigma_D \) denote the \textit{precision indices} of the device. So an ideal, respectively a real measurement correspond to \( \varepsilon \rightarrow 0 \) and \( \sigma_D \rightarrow 0 \) (when \( G(x,x') \rightarrow \delta(x-x') \)) respectively to \( \varepsilon \neq 0 \) and \( \sigma_D \neq 0 \). By introducing Eqs. (9.13) in (9.6) one finds

\[
w_{OUT}(x) = \frac{1}{\sigma_{OUT}\sqrt{2\pi}}\exp\left\{-\frac{(x - \varepsilon)^2}{2\sigma_{OUT}^2}\right\} \] (9.14)

with

\[
\sigma_{OUT}^2 = \sigma_{IN}^2 + \sigma_D^2 \] (9.15)

For the \( OUT \)-expressions (calculated with \( w_{OUT}(x) \)) of the mean value respectively of standard deviation of \( x \) one obtains

\[
\langle x \rangle_{OUT} = \varepsilon \] (9.16)

\[
\Delta_{OUT}x = \sigma_{OUT} = \sqrt{(\Delta_{IN}x)^2 + \sigma_D^2} \] (9.17)

Then the uncertainties defined by Eqs. (9.2) and (9.3) become

\[
\delta (\langle x \rangle) = \varepsilon \] (9.18)

\[
\delta (\Delta x) = \Delta_{IN}x \left\{ \sqrt{1 + \left(\frac{\sigma_D}{\Delta_{IN}x}\right)^2} - 1 \right\} \] (9.19)
In the same circumstances for the uncertainty given by Eq. (9.4) of the informational entropy regarding $x$ one finds

$$\delta H_x = \ln \left( \frac{\sigma_{OUT}}{\sigma_{IN}} \right) = \ln \sqrt{1 + \left( \frac{\sigma_D}{\Delta_{IN}x} \right)^2} \quad (9.20)$$

Now we can extend our discussion for the measurement of the momentum $p$ in the case of the same quantum oscillator described by the wave function from Eq. (9.10). As it is known the respective wave function is given in $x$-representation. But it can be transcribed in $p$-representation in the form:

$$\Psi(p) = \left( \sqrt{\frac{2\pi}{2\mu_{IN}}} \right)^{-1/2} \exp \left\{ -\frac{p^2}{2\mu_{IN}} \right\} \quad (9.21)$$

with $\mu_{IN} = (\hbar m\omega/2)$. So all the above considerations can be transcribed from a $x$-variable version in a $p$-variable form.

Then the $IN$-expressions for the probability distribution respectively for mean value and standard deviation of $p$ are

$$w_{IN}(p) = |\Psi(p)|^2 = \left( \sqrt{\frac{2\pi}{2\mu_{IN}}} \right)^{-1} \exp \left\{ -\frac{p^2}{2\mu_{IN}^2} \right\} \quad (9.22)$$

$$\langle p \rangle_{IN} = 0 \quad (9.23)$$

$$\Delta_{INP} = \mu_{IN} = \sqrt{\frac{\hbar m\omega}{2}} \quad (9.24)$$

If the $p$-measuring device is also characterized by small errors it can be described by the kernel

$$G(p,p') = \left( \sqrt{2\pi\mu_D} \right)^{-1} \exp \left\{ -\frac{(p - \eta - p')^2}{2\mu_D^2} \right\} \quad (9.25)$$

with $\eta$ and $\mu_D$ as precision indices of the device. Similarly with the $x$-variable case for the momentum $p$ one finds that the distribution $w_{OUT}(p)$ characterizing the output of the measuring process is given by

$$w_{OUT}(p) = \left( \sqrt{2\pi \cdot \mu_{OUT}} \right)^{-1} \exp \left\{ -\frac{(p - \eta)^2}{2\mu_{OUT}^2} \right\} \quad (9.26)$$
where

\[ \mu^2_{OUT} = \mu^2_{IN} + \mu^2_D \]  

(9.28)

Then the OUT-expression of the mean value and standard deviation for \( p \) are

\[ \langle p \rangle_{OUT} = \eta \]  

(9.29)

\[ \Delta_{OUT}p = \mu_{OUT} = \sqrt{(\Delta_{IN}p)^2 + \mu^2_D} \]  

(9.30)

Correspondingly, the uncertainties of the mean value and standard deviation of \( p \) are

\[ \delta (\langle p \rangle) = \eta \]  

(9.31)

\[ \delta (\Delta p) = \Delta_{IN}p \left\{ \sqrt{1 + \left( \frac{\mu_D}{\Delta_{IN}p} \right)^2} - 1 \right\} \]  

(9.32)

Also for the uncertainty regarding the informational entropy for the \( p \) variable one obtains

\[ \delta H_p = \ln \left( \frac{\mu_{OUT}}{\mu_{IN}} \right) = \ln \sqrt{1 + \left( \frac{\mu_D}{\Delta_{IN}p} \right)^2} \]  

(9.33)

Now, the above presented statistical uncertainties for \( x \) and \( p \) can be regarded together for a possible comparison with some supposed ideas from the TIHR doctrine. For such a purpose we consider the following products:

\[ \delta (\langle x \rangle) \cdot \delta (\langle p \rangle) = \varepsilon \cdot \eta \]  

(9.34)

\[ \delta (\Delta x) \cdot \delta (\Delta p) = \Delta_{IN}x \cdot \Delta_{IN}p \cdot \]  

\[ \cdot \left\{ \sqrt{1 + \left( \frac{\sigma_D}{\Delta_{IN}x} \right)^2} - 1 \right\} \cdot \left\{ \sqrt{1 + \left( \frac{\mu_D}{\Delta_{IN}p} \right)^2} - 1 \right\} \]  

(9.35)

\[ \delta H_x \cdot \delta H_p = \ln \sqrt{1 + \left( \frac{\sigma_D}{\Delta_{IN}x} \right)^2} \cdot \ln \sqrt{1 + \left( \frac{\mu_D}{\Delta_{IN}p} \right)^2} \]  

(9.36)
As we have discussed in sections II and VI, TIHR supposes that the product of the uncertainties for $x$ and $p$ has a non-null inferior limit. The respective limit is expressed only in terms of the fundamental constant $\hbar$, and it is completely independent of certain characteristics regarding the measuring device. Comparatively, from Eqs. (9.34)-(9.36) it results that the products of the statistical uncertainties for $x$ and $p$ are directly dependent on the precision parameters $\varepsilon, \eta, \sigma_D$ and $\mu_D$ of the measuring devices. If all the respective parameters are null (case of ideal measurements) the mentioned products are also null. This means that the products of the mentioned statistical uncertainties for $x$ and $p$ have not a non-null inferior limit.

Now note that, in the here-proposed model for describing the measurements, the $x$- and $p$- devices are considered as completely independent. In principle, the respective devices can be coupled in a more complex $x$-$p$- instrument. A theoretical model for the description of measurements with such an instrument can be obtained only by using a set of adequately justified hypotheses. In such a model probably that the above presented kernels $G(x,x')$ and $G(p,p')$ must be regarded as resulting from a more complex quantity dependent on both pairs $x - x'$ and $p - p'$. But a $x - p$ couplage of the mentioned kind still requires further investigations. Then, probably, the problem of the product of adequate $x$- and $p$-uncertainties will be also discussed.

X. THE SIMILAR STOCHASTIC SIGNIFICANCES OF PLANCK’S AND BOLTZMANN’S CONSTANTS

In sections V, VI, and VII we have argued that the theoretical HR from quantum mechanics given by Eqs. (2.2)-(2.3) have authentic nonquantum analogs. But the mentioned HR are commonly associated with Planck’s constant $\hbar$. Then there naturally arises the question whether $\hbar$ also has an authentic analog in nonquantum physics. Now, in this section we shall present a lot of elements which reveal that the answer to the above question is affirmative. The alluded analog of $\hbar$ is shown to be the Boltzmann’s constant $k$. The
viewed analogy is given by the fact that $\hbar$ and $k$ have similar roles of generic indicators of the onefold stochasticity (randomness) for well-specified classes of physical systems (i.e. for individual quantum microparticles and macroscopic nonquantum systems, respectively).

A physical system is considered to have stochastic respectively nonstochastic characteristics depending on the probabilistic nature of its specific variables. For a system, the level (degree) of stochasticity depends on the frame (approach) in which it is studied. So, for a macroscopic system consisting of a large ensemble of molecules, the stochasticity is significant in statistical physics approach but it is completely unimportant in the frame of continuous mechanics or of thermodynamics.. Also, in the case of a microparticle of atomic size, the stochastic characteristics are essential from a quantum mechanics view, but they are negligible from a classical mechanics perspective. Of course, the level of stochasticity can be described by means of certain fluctuation quantities such as the ones defined/implied in Eqs. (4.3), (5.4), (5.6), (5.15), (5.20) and (5.24). But the respective quantities take particular expressions (and values) for diverse variables or various systems. Therefore, they cannot be considered as generic indicators of stochasticity, i.e. as parameters indicating generically the stochasticity level for a set or variables or for a whole class of systems. Below we shall show that the roles of such generic indicators of stochasticity are played, in similar ways, by the constants $k$ and $\hbar$ for the macroscopic nonquantum systems and individual quantum microparticles, respectively.

Firstly, let us discuss the alluded role for $k$ with respect to the macroscopic systems. If such a system is studied in the framework of phenomenological (quasithermodynamic) theory of fluctuations (Munster, 1960, 1969; Dumitru, 1974a; Landau and Lifschitz, 1984), its microscopic-molecular structure is completely neglected. Also, its specific variables are global macroscopic quantities regarded as real stochastic variables with continuous spectra of values. For such a system, in the mentioned approach, the fluctuations of the variables as pressure $P$ and volume $V$ are described by the quantities $\Delta wV, \Delta wP$ and $\langle \delta _wV\delta _wP\rangle _w$ given in Eq. (6.40). More generally, for the same system we can consider two arbitrary variables $A = A (X_j)$ and $B = B (X_j)$ regarded as functions of certain independent variables
Then the correlation \( \langle \delta_w A \delta_w B \rangle \) describing the fluctuations of \( A \) and \( B \) is given by

\[
\langle \delta_w A \delta_w B \rangle = k \sum_{j=1}^{n} \sum_{l=1}^{n} \frac{\partial \overline{A}}{\partial X_j} \frac{\partial \overline{B}}{\partial X_l} \left[ \frac{\partial^2 S}{\partial X_j \partial X_l} \right]^{-1}
\]

(10.1)

where \( \overline{A} = \langle A \rangle \), \( [a_{jl}]^{-1} \) denotes the inverse of the matrix \( a_{jl} \) and \( S = S(X_j) \) is the entropy of the system.

As an example from classical statistical physics we can consider the system referred in connection with the Eqs. (6.41)-(6.42). The corresponding stochastic variables are \( H \) and \( Z_c \). Their fluctuations are described by the quantities \( \Delta_w H \), \( \Delta_w Z_c \) and \( \langle \delta_w H \delta_w Z_c \rangle \) whose expressions are given by Eqs. (6.42).

Now we can proceed to a direct examination of the expressions from Eqs. (6.40), (10.1) and (6.42) of the quantities \( \Delta_w A \) and \( \langle \delta_w A \delta_w B \rangle \) which describe the thermal fluctuations in macroscopic systems. One can observe that all the respective expressions are structured as products of \( k \) with terms which are independent from \( k \). The alluded independence is ensured by the fact that the mentioned terms are expressed only by means of macroscopic non-stochastic quantities. (Note that the mean values \( \overline{A} \) from the respective terms must coincide with deterministic (i.e. nonstochastic) quantities from usual thermodynamics). Due to the above observed structure, the examined fluctuation quantities are in a direct dependence of \( k \). So they are significant respectively negligible as we take \( k \neq 0 \) or \( k \to 0 \). Because \( k \) is a constant, the limit \( k \to 0 \) must be regarded in the sense that the quantities directly proportional with \( k \) are negligible comparatively with other terms of the same dimensionality but not containing \( k \). However, the fluctuations reveal the stochastic characteristics of the physical systems. Then we can conclude that the thermal stochasticity, for the system studied in nonquantum statistical physics, is an important respectively insignificant property as we consider \( k \neq 0 \) or \( k \to 0 \).

The mentioned features vis-a-vis the values of \( k \) are specific for all the macroscopic systems (e.g. gases, liquids and solids of various inner compositions) and for all their specific global variables. But such a remark reveals the fact that \( k \) has the qualities of an authentic
generic indicator for thermal stochasticity (i.e. for the stochasticity evidenced through the thermal fluctuations).

Now let us approach questions connected with the quantum stochasticity which is specific for the individual, nonrelativistic microparticles of atomic size. Such a kind of stochasticity is revealed by the specific quantum fluctuations of the corresponding variables (of orbital and spin nature). The respective fluctuations described by means of quantities like the standard deviations and correlations defined in Eqs. (5.2) and (5.15). Some expressions, e.g. those given by Eqs. (6.19) and (6.38), for the mentioned fluctuation quantities show the direct dependence of the respective quantities on the Planck’s constant $\hbar$. Then, there results that $\hbar$ can play the role of generic indicator for quantum stochasticity. Correspondingly, as $\hbar \neq 0$ or $\hbar \to 0$ the mentioned stochasticity appears as a significant respectively negligible property.

The above mentioned connection between the quantum stochasticity and $\hbar$ must be complemented with certain deeper considerations. Such considerations regard (Dumitru and Veriest, 1995) different behaviour patterns of various physical variables in the limit $\hbar \to 0$, usually called Quantum $\to$ Classical Limit (QCL).

Firstly, let us refer to the spin variables. We consider an electron whose spin state is described by the function (spinor) $\chi$ given by

\[
\chi = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad \alpha \in [0, \frac{\pi}{2}] 
\]

For a specific variable we take the $z$-component of the spin angular moment $\hat{S}_z = (\hbar/2) \hat{\sigma}_z$ ($\hat{\sigma}_z$ being the corresponding Pauli matrix). For the respective variable in the mentioned state we find

\[
\Delta_{\chi}S_z = \frac{\hbar}{2} \sin 2\alpha 
\]

The quantity $\Delta_{\chi}S_z$ describes the quantum fluctuations of spin kind i.e. the spin quantum stochasticity. The presence of $\hbar$ in Eq. (10.3) show that the respective stochasticity is
significant or not as $h \neq 0$ or $h \to 0$. This means that $h$ plays the role of generic indicator for the respective stochasticity. But in the state described by Eq. (10.2) one finds also $\langle S_z \rangle_x = (h/2) \cos \alpha$. This additional results shows that, in fact, for $h \to 0$ the variable $S_z$ disappears completely. Then we can note that for spin variables the behaviour pattern in quantum $\to$ classical limit consists of an annulment of both stochastic characteristics and mean values (i.e. in a complete disappearance).

In the case of orbital quantum variables the quantum $\to$ classical limit implies not only the condition $h \to 0$ but also the requirement that certain quantum numbers grow unboundedly. The mentioned requirement is due to the fact that certain significant variables connected with the orbital motion (e.g. the energy) pass from their quantum values to adequate classical values. So, with respect to the mentioned limit the orbital variables have two kinds of behaviour patterns.

As an example of the first kind we refer to the coordinate $x$ of a harmonic rectilinear oscillator considered in its n-th energy level. Then similarly with $\Delta \psi \varphi$ from Eq. (6.19) we have:

$$\Delta \psi x = \left[ \frac{\hbar}{m\omega} \left( n + \frac{1}{2} \right) \right]^{1/2}$$  \hspace{1cm} (10.4)

where $m, \omega$ and $n$ denote the mass, the angular frequency respectively the oscillation quantum number. For the mentioned example the quantum $\to$ classical limit means not only $h \to 0$ but also $n \to \infty$. This because the energy must pass from the quantum expression $E = \hbar \omega \left( n + \frac{1}{2} \right)$ to the corresponding classical expression $E_{cl} = \frac{1}{2} m\omega^2 x_0^2$, where $x_0$ is the coordinate amplitude. Then the standard deviation of $x$ passes from the quantum value given by Eq. (10.4) to the classical value

$$\Delta_{cl}x = \frac{x_0}{\sqrt{2}}$$  \hspace{1cm} (10.5)

But $\Delta \psi x$ and $\Delta_{cl}x$ are fluctuation parameters which describe the stochastic characteristics of $x$ in quantum respectively classical contexts. Then one can say that in quantum $\to$ classical limit the above considered coordinate $x$ preserves both its role of significant variable and its stochasticity.
As an example of the second kind of orbital variable we consider the distance $r$ between the electron and nucleus in a hydrogen atom. We refer to an electron in a state described by the wave function $\Psi_{nlm}$ with $l = n - 1$ (where $n$, $l$ and $m$ are respectively the principal, orbital and magnetic quantum numbers). Then for $\Delta_r = \Delta r$ we can use the expression given by Schwabl (1995), rewritten in the form

$$\Delta r = \frac{2\pi \varepsilon_0 \hbar^2 n(n + 1)^{1/2}}{m_0 e}$$  \hspace{1cm} (10.6)$$

with $m_0$ and $e$ denoting the mass respectively the charge of the electron. The energy of the electron is

$$E_n = -\frac{m_0 e^4}{32\pi^2 \varepsilon_0^2 \hbar^2 n^2}$$  \hspace{1cm} (10.7)$$

The quantum→classical limit requires that $E_n \to E_{cl}$ with $E_{cl}$ denoting the classical value of the energy. Then from Eqs. (10.6) and (10.7) it results that in the respective limit we have

$$\Delta r \to \left( \frac{\hbar e^4}{16\pi \varepsilon_0} \right)^{1/2} (-2m_0 E_{cl})^{-1/4}$$  \hspace{1cm} (10.8)$$

In the same circumstances we obtain

$$\langle r \rangle_\Psi \to r_{cl} = -\frac{e^2}{8\pi \varepsilon_0 E_{cl}}$$  \hspace{1cm} (10.9)$$

So it results that in the quantum→classical limit (when $\hbar \to 0$ and $E_n \to E_{cl}$) we have $\Delta r \to 0$ and $\langle r \rangle_\Psi \to r_{cl} \neq 0$. This means that $r$ preserves its role of significant variable but loses its stochasticity.

The above considerations can be concluded with the following remark:

**P – 10.1**: In the quantum→classical limit the physical variables display the following different behaviour patterns:

(i) The complete disappearance of both stochastic characteristics and mean values, as in the case of spin variables.

(ii) The preservation of both the role of significant variable and of stochastic characteristics, as in the case of oscillator coordinate...
(iii) The preservation of the role of significant variable but the loss of stochastic characteristics as in the case of electron-nucleus distance.▲

It is clear that the above remark corrects the traditional belief of a unique behaviour pattern compulsorily associated with the disappearance of the "uncertainties" (i.e. of the standard deviations $\Delta_{\Psi A}$).

Now let us return to the quantum stochasticity, specific for the variables of individual microparticles. We think that, in spite of the peculiarities mentioned in $\mathbf{P} - 10.1$, the Planck constant $\hbar$ can be considered as a generic indicator of such a stochasticity. Moreover, we consider that the respective role of $\hbar$ is completely similar with that the Boltzmann constant $k$ with respect to the macroscopic thermal stochasticity (see above).

Regarding the mentioned roles of $\hbar$ and $k$ another observation must be added. In the discussed cases, $\hbar$ and $k$ appear independently and singly. That is why one can say that the stochasticity of the corresponding systems (microparticles and classical macroscopic systems) has a onefold character. But there are also physical systems endowed with a twofold stochasticity characterized by a simultaneous connection with both $\hbar$ and $k$. Such systems are those studied in quantum statistical physics, i.e., the bodies of macroscopic size considered as statistical ensembles of quantum microparticles. The stochasticity of the respective systems is revealed by corresponding fluctuations described by the quantities given by Eqs. (5.24) which depend simultaneously on both $\hbar$ and $k$. The respective dependence is revealed by the so-called fluctuation-dissipation theorem. According to the respective theorem (Kubo 1957; Zubarev 1971; Balescu 1975) one can write

$$\langle \delta_\rho \hat{A} \delta_\rho \hat{B} \rangle_\rho + \langle \delta_\rho \hat{B} \delta_\rho \hat{A} \rangle_\rho =$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} \hbar \coth \left( \frac{\hbar \omega}{2kT} \right) \left[ \chi_{AB}^* (\omega) - \chi_{BA} (\omega) \right] d\omega$$

(10.11)

with $\chi_{AB}^*(\omega)$ as complex conjugate of $\chi_{AB}(\omega)$.

In Eq. (10.11) $\chi_{AB}(\omega)$ represent the generalized susceptibilities which appears also in the deterministic framework of nonequilibrium thermodynamics (De Groot and Mazur, 1962).
But it is a known fact that in the respective framework all the stochastic characteristics of physical variables are neglected and no microscopic (i.e. atomic or molecular) structure of the systems is taken into account. Another fact is that $\chi_{AB}(\omega)$ are directly connected (Landau and Lifschitz, 1984) with the macroscopic nonstochastic expression of the energy dissipated inside the thermodynamic systems actioned by external deterministic and macroscopic perturbations. The mentioned facts show that the susceptibilities $\chi_{AB}(\omega)$ do not depend on the constants $\hbar$ and $k$.

The above mentioned property of $\chi_{AB}(\omega)$ combined with Eq. (10.11) shows that the sole significant dependence of fluctuation quantities given by Eq. (5.24) on the constants $k$ and $\hbar$ is given by the factor $\hbar \coth (\hbar \omega / k T)$. For the respective factor one can write

$$\lim_{k\to0} \left\{ \lim_{\hbar\to0} \left[ \hbar \coth \left( \frac{\hbar \omega}{2kT} \right) \right] \right\} = 0 \quad (10.12)$$

This means that when both $\hbar$ and $k$ tend to zero the fluctuation quantities defined by Eq. (5.24) become null. So it results that in the mentioned limit the fluctuations in quantum statistical systems cease to manifest themselves. Consequently, for such a limit the respective systems lose their stochastic characteristics. Then in the spirit of the above presented opinions one can state that quantum statistical systems can be considered as endowed with a twofold stochasticity of the thermal and quantum type, revealed respectively by $k$ and $\hbar$ as generic indicators.

In the above considerations the stochasticity appears as a property of exclusively intrinsic type. This means that it is connected only with the internal (inner) characteristics of the considered systems and does not depend on external (outside) factors. Moreover the mentioned stochasticity is directly and strongly associated with $\hbar$ and $k$ as generic indicators. But the stochasticity can be also of extrinsic type. In such cases it is essentially connected with factors from the outside (surroundings) of the considered physical systems. Also the extrinsic stochasticity is not (necessarily) associated with $\hbar$ and $k$. As examples of system with stochasticity of exclusively extrinsic type can be considered an empty bottle floating on a stormy sea or a die in a game.
In practice one finds also systems endowed with stochasticity of both the intrinsic and extrinsic types. Such are for example, the electric and electronic circuits. For a circuit the intrinsic stochasticity is caused by the thermal agitation of the charge carries and/or of elementary (electric or magnetic) dipoles inside of its constitutive elements (i.e. inside of resistors, inductances, condensers, transistors, integrated circuits, etc.). The respective agitation is responsible for fluctuations of macroscopic voltages and currents. Such fluctuations are known (Robinson, 1974) as thermal (or Nyquist) noises. Note that in the case of circuits the intrinsic stochasticity is characterized by generic indicators. Such indicators are \( k \) alone, if the circuit is considered as a classical (nonquantum) statistical system, and \( k \) together with \( \hbar \) when the circuit is viewed as a quantum statistical system. Otherwise, the stochasticity of a circuit can also be of extrinsic type when it is under the influence of a large variety of factors. Such factors can be: thermal fluctuations in the surrounding medium, accidental outside discharges and inductions, atmospheric (or even cosmic) electrical phenomena. The mentioned extrinsic stochasticity is also responsible for the noises in macroscopic currents and voltages in the circuits. But it must be noted that even for circuits, the extrinsic stochasticity is not connected in principle with generic indicators dependent on fundamental physical constants (such as \( \hbar \) and \( k \)). As an interesting case which implies stochasticity of both intrinsic and extrinsic type can be considered a measuring process viewed as in Sec. IX. In such a case the intrinsic stochasticity regards the inner properties of the measured system while the extrinsic one is due to the measuring apparatus. The corresponding intrinsic stochasticity is connected with \( \hbar \) and \( k \) as generic indicators in the above discussed sense. However, the extrinsic stochasticity, due to the apparatus, seems to be not connected with certain generic indicators. This because of the large diversity of apparata as regards their own structure and accuracy.
XI. A FEW REMARKS ON SEVERAL ADJACENT QUESTIONS

With respect to problematics of HR proper, in literature (see the bibliographical publications mentioned in the end of Sec. I) one knows of a large variety of adjacent questions which, one way or another, are allied with the subjects discussed in the previous sections of this paper. Now, we wish to note a few remarks on several such questions.

♦ Firstly, let us refer to the consequences of the here-proposed reconsideration of HR for both lucrative procedures and interpretational frame of quantum mechanics. Note that our reconsideration does not prejudice in any way the authentic version of the mentioned lucrative procedures, which, in fact, have met with unquestionable successes in both basic and aplicative researches. As regards the alluded interpretational frame, our reconsideration, mainly by the abandonment of TIHR, generates major and important changes. But such changes must be regarded as benefic, since they can offer a genuine elucidation to the controversial questions introduced in the frame of science by TIHR.

♦ By reinterpreting HR in the sense presented in Sec.VIII the respective relations lose their quality of crucial physical formulae. So, one can find a consonance with the prediction (Dirac, 1963): ”I think one can make a safe guess that uncertainty relations in their present form will not survive in the physics of future”. Note that the above prediction was founded not on some considerations about the essence of HR but on a supposition about the future role of $\hbar$ in physics. So it was supposed that $\hbar$ will be a derived quantity while $c$ and $e$ (speed of light and elementary charge) will remain as fundamental constants. That is why we wish to add here that our view about HR does not affect the actual position of $\hbar$ as a physical constant. More precisely, our findings cannot answer the question whether $\hbar$ will be a fundamental constant or a derived quantity (e.g. expressed in terms of $c$ and $e$).

♦ As it was pointed in Sec.X the Planck’s constant $\hbar$ has also the significance of estimator for the spin of microparticles (like the electron). So the spin appears to be a notable respectively absent property as $\hbar \neq 0$ or $\hbar \to 0$. On the other hand with the reference to the spin there are also some intriguing questions related to its relativistic justification. Usually
(Dirac, 1958; Blochintsev, 1981) for electrons the spin is regarded to be essentially explicable as consequence of relativistic theory. But, as it is known, the relativistic characteristics of a particle are evidenced by the relative value $v/c$ of its velocity $v$ comparatively with the light velocity $c$. Particularly, the respective characteristics must be insignificant when $v/c \ll 1$ or $c \to \infty$. Then the absence of the factor $v/c$ (or of some other equivalent factors) in the description of the electron spin variable appears at least as intriguing fact. Is such a fact a sufficient reason to consider $\hbar$ as a derived quantity in the sense guessed by Dirac (1963). In such a sense $\hbar = \frac{e^2}{4\pi\varepsilon_0 c \alpha} \ (\varepsilon_0 =$ the permittivity of vacuum and $\alpha = \frac{1}{137} =$ the fine structure constant) and the situations with $\hbar \to 0$ appear when $c \to \infty$. So the significance of $\hbar$ as spin estimator can be apparently related with some aspects of relativity. But here it must be noted the surprising fact that even in the nonrelativistic limit (i.e. when $v/c \ll 1$ or $c \to \infty$) the spin remains a significant variable of the electron. It is known (Ivanov, 1989) that the electron spin plays a decisive role (as a fourth quantum variable/number) in the electronic configuration of many-electronic atoms, in spite of the fact that for atomic electrons $v/c \ll 1$.

Due to the here mentioned features we think that the relativistic justification of the spin appears as a intriguing question which requires further investigations.

Our findings facilitate also a remark in connection with another supposition about $\hbar$. The respective supposition regards the possible existence of multiple Planck constants associated with various kinds of microparticles (e.g. with electrons, protons, neutrons). Currently, (Whichman, 1971; Fischbach et. al. 1991), the tendency is to contest such a possibility and to promote the idea of a unique Planck constant. For this one appeals either to experimental data or to some connection with the fundamental conservation laws. We think that our view about $\hbar$ pleads somewhat for the alluded idea of uniqueness. So, regarding $\hbar$ as generic indicator of quantum stochasticity, this one must have the same value for various kinds of microparticles. This because, similarly, the Boltzmann constant $k$ in its role of generic indicator for thermal stochasticity has a unique value for various kinds of macroscopic systems (e.g. hydrogen gas, liquid water or crystallin germanium).

The revealed stochastic similarity among quantum microparticles and macroscopic
systems facilitates another remark. In the macroscopic case the stochastic characteristics for an individual system is incorporated in the probability distribution \( w \) (see sections V and VI. K). As we have shown, the quantum similar of \( w(x) \) is the wave function \( \Psi \) (or the square \( |\Psi|^2 \) of its module). Such a \( w - \Psi \) similarity motivates us to agree the idea (Van Kampen 1978) that \( \Psi \) refer to a single system (microparticle). Simultaneously, we incline to a circumspect regard about the opinions that \( \Psi \) belongs to an ”ensemble of equally prepared systems” (Tschudi, 1987) or to an ”abstract physical object” (Mayants, 1984). Moreover, our agreement and opinion are also motivated by the observation that in practical applications both \( \Psi \) and \( w \) are calculated for individual systems (e.g. for an electron in a hydrogen atom or, respectively for an ideal gas).

A distinct group of remarks regards the reduction of stochasticity to subjacent elements of deterministic nature, for both cases of thermodynamic systems and quantum microparticles. In the first case the stochasticity refers to the macroscopic variables which characterize each system as a whole. But according to the classical statistical mechanics the respective variables are expressible in terms of subjacent molecular quantities (coordinates and momenta) which are considered as deterministic elements. In the case of quantum microparticles a similar problem was taken into account. So it was promoted the idea that the stochastic quantum variables (characterizing each microparticle as a whole) would be expressible in terms of some subjacent elements of deterministic nature, called ”hidden variables”. Viewing comparatively the two mentione d cases we think that is of nontrivial interest ti note the following observations:

(i) In the case of thermodynamic systems the subjacent molecular quantities can be justified in essence only by adequate experimental facts.

(ii) The mentioned molecular quantities are deterministic (i.e. depresionfree) only from a microscopic perspective, connected with the characteristics of the molecules. From a macroscopic perspective, coonected with a thermodynamic systems as a whole, they are stochastic variables. That is why, for example, in respect with a thermodynamic system like an ideal gas one speaks about the mean value and non-null dispersion of the molecular
velocity.

(iii) Even by taking into account the existence of sujacent molecular quantities the macroscopic variables, characterizing a thermodynamic system as a whole, keep their stochastic characteristics. Particularly the mentioned existence does not influence the verity or the significance of the macroscopic relations from the family of Eqs. (5.21) - (5.23).

(iv) The above observations (i) - (iii) reveal as unfounded the idea (Uffink and Van Lith 1999) that the sole examination of some theoretical formulas, from the mentioned family, can give a light on the problem of reduction of thermodynamic stochasticity to subjacent deterministic elements.

(v) By analogy with the fact noted in (i), in the case of quantum microparticles, the existence of the ”hidden variables” must be proved firstly by indubitale experimental facts. But, as far as we know, until now such an experimental proof was not ratified by scientific research.

(vi) The existence of the mentioned ”hidden variables” cannot be asserted only by means of considerations on some theoretical formulas regarding the global stochasticity of quantum microparticles, such are the HR.

(vii) The global description of a quantum microparticle remain equally probabilistic in both cases, with or without ”hidden variables”. More exactly in both cases for a variable refering to a quantum microparticle as a whole the theoretical predictions must be done in probabilistic terms while the experimental informations can be obtained only from measurements consisting in statistical samplings.

The discussions from Sec. X about the stochasticity suggest a remark connected with the Boltzmann’s constant $k$. As we have shown $k$ plays a major role in the evaluation of the level of the thermal stochasticity. But the respective stochasticity must be regarded as an important property of the macroscopic systems. So one finds as unfounded the idea, promoted in some publications (Wichman, 1971; Landau and Lifschitz, 1984; Storm, 1986) that, in physics $k$ has only a minor role of conversion factor between temperature scales (from energetic units into Kelvin degrees).
XII. CONCLUSIONS

We started the paper reminding the fact that even in our days TIHR persists as a source of unelucidated controversies about it defects. Motivated by the respective fact we proposed an investigation in the very core of the alluded controversies and defects. For such a purpose firstly we identified the main elements (assertions and arguments) of TIHR. Then, in reference with the mentioned elements, we localized the most known and critical defects of TIHR.

In such a reference frame we analyzed the reality of the respective defects. We found that all of them are veridical. Moreover, for TIHR, they are insurmountable and incriminate each of its main elements. So we can conclude that the sole reasonable attitude is to abandon TIHR as an unjustified doctrine.

The mentioned abandonment must be accompanied with a search for a new and genuine reinterpretation of HR. On this direction we opine that HR of troughs-experimental nature must be disregarded because they are fictitious formulae without a true physical significance. On the other hand we think that the theoretical HR are authentic physical formulae regarding the quantum fluctuations. So regarded the theoretical HR belong to a large class of formulae specific for systems, of both quantum and non-quantum nature, endowed with stochastic characteristics. By adopting the mentioned regards about HR all the controversies connected with the TIHR are elucidated on a natural way.

In the mentioned regard HR lose their traditional role of crucial physical formulae connected with the description of measurement characteristics (uncertainties). In the here promoted view the respective description must be done in terms (and formulae) which do not belong to the traditional chapter of physics (including the quantum mechanics). We suggested that a promising version for the description of measurements can be done in terms of information theory. So a measurement can be considered as an information transmission, from the measured system (information source) through the measuring device (transmission channel) to the device recorder (information receiver). Then the measuring uncertainties
appear as alternations of processed information. In the Sec. IX we illustrated the alluded informational model with some concrete considerations.

In our opinion the theoretical HR and their classical (non-quantum) similars are connected with the stochasticity regarded as an important property of physical systems. We showed that the respective property is characterized by generic indicators which are:

(i) the Planck’s constant \( \hbar \) (for quantum microparticles),

(ii) the Boltzmann’s constant \( k \) (for classical thermodynamical system), respectively

(iii) both \( \hbar \) and \( k \) (for quantum statistical systems).

In the end, in Sec. XI, we presented remarks on some questions which are adjacent with the subjects discussed in the other parts of the paper.

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| Acronym | Description                                           |
|---------|-------------------------------------------------------|
| CR      | correlation relation(s)                              |
| HR      | Heisenberg’s relation(s)                             |
| P       | point                                                 |
| P.../A  | assertion point                                      |
| P.../M  | motivation point                                      |
| QCL     | quantum—classical limit                              |
| QTP     | quantum torsion pendulum                              |
| SRTE    | super-resolution thought experiment(s/al)            |
| TE      | thought experiment(s/al)                              |
| TIHR    | traditional interpretation of Heisenberg’s relations |
REFERENCES

Aharonov Y. and Bohm D., 1961, Phys. Rev. 122, 1649.

Aharonov Y. and Bohm D., 1964, Phys. Rev. 134B, 1417.

Albertson J., 1963, Phys. Rev. 129, 940.

Alcock G.R. 1969 a, Ann. Phys. (N.Y.) 53, 253.

Alcock G.R. 1969 b, Ann. Phys. (N.Y.) 53, 286.

Alcock G.R. 1969 c, Ann. Phys. (N.Y.) 53, 311.

Balescu R., 1975 Equilibrium and Nonequilibrium Statistical Mechanics (Wiley, New York).

Ballentine L. E., 1987, Am. J. Phys. 55, 785.

Bauer M. and P.A. Mello, 1978, Ann. Phys. (New York) 111, 38.

Bazarov I. P., 1979, Methodological Problems of Statistical Physics and Thermodynamics (Moscow University Press, Moscow) (in Russian).

Bell J. S., 1985, Private letter to the author, dated January 29.

Blochinstsev D., 1981, Principles de mecanique quantique (Mir, Moscov).

Bohm D., 1957, Causality and Chance in Modern Physics (Routlege Kegan Paul, London).

Bouten M., N. Maene, P. Van Leuven, 1965, Nuovo Cimento 37, 119.

Braginsky V. B. and Khalili, 1992, Quantum Measurement (Cambridge University Press, Cambridge).

Bransden B. H., C.J. Joachain, 1994, Introduction to Quantum Mechanics (Longman, Essex).

Bunge M., 1970, Can. J. Phys. 8, 183.

Bunge M., 1977 a, in: Denken und Umdenken (zu Werk und Werkung von Werner Heisen-
berg), edited by H. Pfeffer (Piper R., München).

Bunge M., 1977b, Int. J. Quantum Chem. (Suppl 1), 12, 1.

Bush P., P.J. Lathi and P. Mittelstaedt, 1996, The Quantum Theory of Measurement, (Cambridge University Press, Cambridge).

Carruthers P. and M. Nietto, 1968, Rev. Mod Phys. 40, 411.

Cramer J.C., 1986, Rev. Mod Phys. 58, 647.

Croca J.R., A. Rica da Silva and J.S. Ramos, 1996, "Experimental Violation of Heisenberg’s Uncertainty Relations by the Scanning Near-Field Optical Microscope" preprint, University of Lisboa, Portugal. (This work discusses the potential implications of performances attained in optical experiments such as those reported by Pohl et al. 1984, and by Heiselmann and Pohl 1984).

Davidson R.E., 1968, J. Chem. Phys. 42, 1491.

Davydov A.S., 1973, Quantum Mechanics (Nauka, Moscow), (in Russian)

De Bruijn N.G., 1996, in; Inequalities, edited by O. Sisha (Academic, New York).

De Groot S.R. and P. Mazur, 1962, Nonequilibrium Thermodynamics (North Holland, Amsterdam).

De Witt B.S., Graham R.N., 1971, Am. J. Phys., 39, 724.

Dirac P.A.M., 1958, The Principles of Quantum Mechanics (Clarendon Press, Oxford).

Dirac P.A.M., 1963, Sci. Am. 208(5) 45

Dodonov V.V. and V.I. Man’ko, 1987, Proceedings of the Lebedev Physics Institute 183, 5 (in Russian; English version Nova Science Pub., Commack, New York, 1989).

Dumitru S., 1974 a, Physica Scripta 10, 101.
Dumitru S., 1974 b, Phys. Lett  A 48, 109.

Dumitru S., 1977, Epistemological Letters 15,1.

Dumitru S., 1980, What are in fact the Uncertainty Relations? (An Analysis of the Insufficiencies of the Conventional Interpretations of Uncertainty Relations) Preprint 120p (University of Brasov).

Dumitru S., 1984 Microphysics - Solved Problems and a Critical Examination of the Question of Uncertainty Relations (Dacia, Cluj-Napoca), (in Romanian).

Dumitru S., 1987, in: Recent Advances in Statistical Physics (Proceedings of International Bose Symposium on Statistical Physics, Calcutta, India, 28-31 dec. 1984) edited by B. Datta and M. Dutta (World Scientific, Singapore)

Dumitru S., 1989, Rev. Roum. Phys. 34, 329.

Dumitru S., 1991, in: Quantum Field Theory, Quantum Mechanics and Quantum Optics. Part I. Symmetries and Algebraic Structures. (Proceedings 18th International Colloquium on Group Theoretical Methods in Physics, Moscow - June 4-9, 1990) edited by V.V. Dodonov and V.I. Man’ko (Nova Science, New York).

Dumitru S., 1993, Physics Essays 6, 5.

Dumitru S., 1996, Romanian Reports in Physics 48, 891.

Dumitru S., 1999, Optik (Stuttgart) 110, 110.

Dumitru S. and E.I. Verriest, 1995, Int. J. Theor. Phys. 34, 1785.

Evett A. A. and H.M. Mahmoud, 1965, Nuovo Cimento, 38, 295.

Fain V.M. and A.I. Khaimnin, 1965, Quantum Radiophysics (Sov. Radio, Moscow) (in Russian)

Fischbuch E., G.J.Green and R.J. Hughes, 1991, Phys. Rev. Lett. 66, 256.
Fock V., 1962, Zh. Eksp. Thepr. Phys. 42, 1135.

Frank - Kamenetsky D.A. 1940, Zh. Eksp. Theor. Phys. 10, 700.

Fujiwara I., 1970, Progr. Theor. Phys. 44, 1701.

Fürth R., 1933, Z. Phys. 81, 143.

Galiniski V., B. Karnakov and V. Kogan 1985, *Problèmes de Mécanique Quantique* (Mir, Moscou)

Gellert W., et al (Ed), 1975, *Kleine Enzyklopädie der Mathematik* (VEB Bibliogr., Leipzig).

Ghanapragasam B. and M. D. Srinivas, 1979, Pranama, 12, 699.

Gudder S. P., 1979, *Stochastic Methods in Quantum Mechanics* (North Holland, Amsterdam)

Harris R.A., and H. Strauss, 1978, J. Chem. Education 55, 374.

Hasse R.W., 1980, J. Phys. A, 13, 307.

Hay O., and A. Peres 1998, Phys. Rev. A, 58, 116.

Heiselman H. and D. W. Pohl, 1994, Appl. Phys. A, 58, 89.

Heisenberg W., 1927, Z. Phys. 43, 172.

Heisenberg W., 1930, *The Physical Principles of Quantum Theory* (First German Edition, Leipzig 1930; English version, Dover Pub., New York 1949)

Heisenberg W., 1977 in: *The Uncertainty Principle and Foundation of Quantum Mechanics*, edited by W. C. Price and S.S. Chissick (Wiley, New York)

Holevo A.S., 1981, *Probabilistic and Statistical Aspects of Quantum Theory*, (Nauka, Moscow) (in Russian)

Ivanov B.N., 1989, *Fundamentals of Physics* (Mir, Moscou)

Jammer M., 1966, *The Conceptual Development of Quantum Mechanics* (Mc Graw Hiil,
Jammer M., 1974, *The Philosophy of Quantum Mechanics* (Wiley, New York).

Jancel R., 1973, *Foundations of Classical and Quantum Statistical Mechanics* (Pergamon, New York).

Judge D., 1963, Phys. Lett. **5**, 189.

Judge D., 1964, Nuovo Cimento **31**, 322.

Judge D., and J.T. Levis, 1963, Phys. Lett. **5**, 190.

Kijowskii J., 1974, Rep. Math. Phys. **6**, 361.

Kobe D.H. and V.C. Aquilera-Navaro, 1994, Phys. Rev. A. **50**, 933.

Kompaneyets A.S., 1966, *Basic Concepts in Quantum Mechanics* (Reinhold, New York).

Korn G.A. and T.M. Korn, 1968, *Mathematical Handbook (For Scientists and Engineers)* (Mc Graw Hill, New York).

Krauss K., 1965, Z. Phys. **188**, 374.

Krauss K., 1968, Z. Phys. **201**, 134.

Kubo R. 1957, J. Phys. Soc. Japan **12**, 570.

Landau L. and E. Lifchitz, 1984, *Physique Statistique* (Mir, Moscou).

Levy-Leblond J.-M., 1972, Am. J.Phys. **40**, 899.

Levy-Leblond J.-M., 1976, Ann. Phys. **101**, 319.

Linder A., Reiβ., Wassiliadis G. and Freese H., 1996, Phys. Lett. A. **218**, 1

Martens H., 1991, *Uncertainty Principle*, Ph. D. Thesis (Tehnical University, Eidhoven).

Mayants L.S., 1984, *The Enigma of Probability and Physics* (D. Reidel, Dordrecht).
Mehra J., and H. Rechenberg 1982, *The Historical Development of Quantum Theory* vol 1-9 (Springer, Berlin) (1982 for vol. 1-4 and to be published for vol. 5-9).

Münster A., 1960, in *Thermodinamica dei processi irreversibili*, Redinconti della Scuola Internazionale de fisica “E. Fermi” corso X (Soc. Italiana de Fisica, Bologna).

Münster A., 1969, *Statistical Thermodinamics*, vol. I, (Springer, Berlin).

Nilson D.R., 1976, in: *Logic and Probability in Quantum Mechanics*, edited by P. suppes (D. Reidel, Dordrecht).

Omnes R., 1992, Rev. Mod. Phys. **64**, 339.

Omnes R., 1994, *The Interpretation of Quantum Mechanics* (*Princeton University Press, Princeton*).

Opatrny T., 1995, J. Phys. A **28**, 6961.

Piron C., 1982, Lect. Notes. Phys. (Springer) **153**, 179.

Pohl D.W., W. Denk and M. Lanz, 1984, Appl. Phys. Lett. **44**, 651.

Primas H., 1981, *Chemistry, Quantum Mechanics and Reductionsm*, Lecture Notes in Chemistry, vol. 24 (Springer, Berlin).

Primas H. and U. Müller-Herold, 1978, Adv.Chem.Phys. **38**, 1. pg.1-107.

Robinson F.N.H., 1974, *Noises and Fluctuations in Electronic Devices and Circuits* (Clarendon, Oxford).

Ror C.L. and A.B. Samigrahi, 1979, Am. J. Phys. **47**, 965.

Rosenfeld L., 1961, Nature, (London) **190**, 384.

Rosenfeld L., 1962, in: *Questions of Irreversibility and Ergodicity in Ergodic Theories* edited by P. Caldirola (Zanichelli, Bologna).
Roychoudhuri C., 1978, Found. Phys. 8, 845.

Ruppeiner G., 1995, Rev. Mod. Phys. 67, 605.

Schaposhnikov I.G., 1947, Zh. Eksp. Theor. Phys. 17, 485.

Scheer J., K. Götsch, T. Koch, G. Lüning, M. Schmidt and H. Ziggel, 1989, Found. Phys. Lett. 2, 71.

Schilling H., 1972 Statistische Physik in Beispielen (Veb Fechbuchverlang, Leipzig).

Schroeck F.E. Jr., 1982, Mathematical Review 82d : 81007.

Schwabl F., 1995, Quantum Mechanics, 2nd. rev., ed. (Springer, Berlin).

Storm L., 1986 in: Noise in Physical Systems and 1/f Noise-1985 (North Holland, Amsterdam).

Sturzu I., 1999, Revista de Filosofie, 410, 3-4 (in press).

Surdin M., 1973, Int. J. Theor. Phys. 8, 183.

Synge J.L., 1971, Proc. R. Soc. London A325, 151.

Tarascov L., 1980, Physique Quantique et Operateurs Lineaires (Mir, Moscou).

Terletsky Ya. P., 1974, Porc. Univ. “ P. Lumumba” - Theor. Phys. 70/8, 3.

Tschudi H.R., 1987, Helv. Phys. Acta 60, 363.

Tyablikov S.V., 1975, Methods of Quantum Theory of Magnetism (Nauka, Moscow)(in Russian).

Uffink J. and van Lith J., 1999, Found. Phys. 29, 655.

Van Kampen N.G., 1988, Physica A (Utrecht) 53, 97.

Vorontsov Yn. I., 1981, Uspekhi Fiz. Nauk 133, 351.
Weissman M., 1981, Ann. Rev. Phys. Chem. 32, 205.

Wichman E.H., 1971, Quantum Physics - Berkley Physics Course (Mc Graw Hill, New York).

Worthing A.G. and J. Geffner, 1955, Treatment of Experimental Data (Wiley, New York).

Yamada K., 1982, Phys. Rev.D 25, 3256.

Yanase M.M., M. Namiki and S. Makida (editors) 1978 Selected Papers on the Theory of Measurement in Quantum Mechanics (Phys. Soc. Japan, Tokyo).

Zubarev D.N., 1971, Nonequilibrium Statistical Thermodynamics (in Russian: Nauka, Moscow; English version: Consultants Bureau, New York, 1974).
FIGURES

FIG. 1. Private paper from J.S. Bell to the author (dated January 29, 1985)
This figure "Fig.1.gif" is available in "gif" format from:

http://arxiv.org/ps/quant-ph/0004013v1