Generalized couplings and convergence of transition probabilities

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Abstract

We provide sufficient conditions for the uniqueness of an invariant measure of a Markov process as well as for the weak convergence of transition probabilities to the invariant measure. Our conditions are formulated in terms of generalized couplings. We apply our results to several SPDEs for which unique ergodicity has been proven in a recent paper by Glatt-Holtz, Mattingly, and Richards and show that under essentially the same assumptions the weak convergence of transition probabilities actually holds true.

1 Introduction

In this article, we provide sufficient conditions in terms of (generalized) couplings for the uniqueness of an invariant measure and weak convergence to the invariant measure for a Markov chain taking values in a Polish space $E$. Such criteria have already been established for the uniqueness of an invariant measure in [1] and [11] but – to the best of our knowledge – not for the weak convergence (or asymptotic stability) of transition probabilities. In [13] and [2], uniqueness and asymptotic stability were shown for so-called e-processes (which we explain below) and a similar approach was used in [11] to prove asymptotic stability. Our aim is to present a unified approach to both uniqueness and asymptotic stability in terms of generalized (asymptotic) couplings. Here, a probability measure $\xi$ on a product space is called a generalized coupling of $\mu$ and $\nu$ if the marginals of $\xi$ are not necessarily equal to $\mu$ and $\nu$ but only absolutely continuous. We point out that we do not assume the e-property to hold (which indeed does not hold in all cases of interest – see e.g. Example 5.4 – and even if it does it is often cumbersome to verify). Our uniqueness statement, Theorem 2.1, is a slight generalization of [11, Theorem 1.1]. We will show in Example 5.6 that our conditions are indeed strictly weaker than those in [11]. At the same time the proof is quite short and elementary.

We then proceed to formulate sufficient conditions for the convergence of transition probabilities assuming existence of an invariant measure $\mu$. Our main results are Theorems 2.3 and 2.7, the former one providing a sufficient condition for weak convergence of transition probabilities for $\mu$—almost all initial conditions and the latter one for weak convergence of the transition probabilities starting from a given point $x \in E$. In both

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theorems, the conditions are formulated in terms of generalized couplings. In Theorem 2.3 convergence is in fact in probability, i.e. the measure \( \mu \) of the set of initial conditions for which the distance of the transition probability to the invariant measure \( \mu \) after \( n \) steps is larger than \( \varepsilon \) converges to 0 for every \( \varepsilon > 0 \). It seems to be an open question if convergence even holds true in an almost sure sense. We point out that our proof of Theorem 2.3 requires the chain to be Feller, while Theorems 2.1 and 2.7 do not rely on this property. However, the Feller property is required to hold true with respect to some metric \( d \) on \( E \) which in general may differ from the original one, hence this assumption is quite flexible and non-restrictive.

The main results are proved in Sections 3 and 4. In Section 5 several examples are given which illustrate the conditions imposed in the main results and clarify the relations of these results with some other available in the field.

To illustrate the usefulness of our results, in Section 6 we give two groups of their applications. First, in Section 6.1, we consider the same example as in [11], namely a stochastic delay equation which has the space of continuous functions on \([-1,0]\) as its natural state space. The solution is a Feller process and it is not hard to find a generalized coupling which satisfies the conditions of Theorem 2.1. This coupling is actually a simplified version of the one used in the proof of [11, Theorem 3.1]; namely, our Theorem 2.1, unlike [11, Theorem 1.1], does not require the equivalence of the marginal distributions, thus the “localization in time” part of the construction of the generalized coupling can be omitted. Remarkably, such a simplified construction appears well applicable in Theorems 2.3 and 2.7 as well, so that without any additional work we get asymptotic stability for free (unlike in [11]).

This method to improve a result from unique ergodicity to asymptotic stability looks quite generic, and in our second group of applications essentially the same method leads to several new statements. We reconsider the results from the recent paper [8], where the generalized coupling technique (or asymptotic coupling, in their terminology) is used to prove uniqueness of an ergodic measure for several types of non-linear SPDEs. Each of the five SPDE models considered therein is analytically quite involved, and [8] perfectly illustrates the flexibility of the generalized coupling approach, which appears to be well applicable in complicated models. In Section 6.2 we show that just minor modifications in the construction of the generalized couplings from [8] make them applicable in our Theorems 2.3 and 2.7, as well, providing asymptotic stability (almost) for free and thus illustrating the power of our approach.

2 Main results

2.1 Basic definitions and notation

Let \((E, \rho)\) be a Polish (i.e. separable, complete metric) space with Borel \(\sigma\)-algebra \(\mathcal{E}\), and let \(X = \{X_n, n \in \mathbb{Z}_+\}\) be a Markov chain with state space \((E, \mathcal{E})\), where \(\mathbb{Z}_+ := \{0, 1, \cdots\}\). Transition probabilities and \(n\)-step transition probabilities for \(X\) are denoted respectively by \(P(x, dy)\) and \(P_n(x, dy)\). Let \(E^\infty := E^{\mathbb{Z}_+}\).

The law of the sequence \(\{X_n\}\) in \((E^\infty, \mathcal{E}^{\otimes \infty})\) with initial distribution \(\text{Law}(X_0) = \mu\) is denoted by \(\mathbb{P}_\mu\), the respective expectation is denoted by \(\mathbb{E}_\mu\); in case \(\mu = \delta_x\) we write simply \(\mathbb{P}_x, \mathbb{E}_x\).

Recall that an invariant probability measure for \(X\) is a probability measure \(\mu\) on \((E, \mathcal{E})\) such that

\[
\mu(dy) = \int_E P(x, dy) \mu(dx).
\]  

Equivalently, a probability measure \(\mu\) is invariant if the sequence \(\{X_n, n \in \mathbb{Z}_+\}\) is strictly stationary under \(\mathbb{P}_\mu\). An invariant probability measure \(\mu\) for \(X\) is ergodic, if the left shift on the space \((E^\infty, \mathcal{E}^{\otimes \infty})\) is ergodic.
with respect to $\mathbb{P}_\mu$. Recall that a strictly stationary sequence $\zeta_n, n \in \mathbb{Z}_+$ is called mixing if for any bounded measurable functions $f, g : E \to \mathbb{R}$
\[ Ef(\zeta_0)g(\zeta_n) \to Ef(\zeta_0)Eg(\zeta_0), \quad n \to \infty. \] (2)
For a measurable space $(S, \mathcal{S})$, we denote the set of all probability measures on $(S, \mathcal{S})$ by $\mathcal{P}(S)$. For given $\mu, \nu \in \mathcal{P}(S)$, define
\[ C(\mu, \nu) = \left\{ \xi \in \mathcal{P}(S \times S) : \pi_1(\xi) = \mu, \pi_2(\xi) = \nu \right\}, \]
where $\pi_i(\xi)$ denotes the $i$-th marginal distribution of $\xi, i = 1, 2$. Any $\xi \in C(\mu, \nu)$ is called a coupling for $\mu, \nu$. We also introduce the following two extensions of the notion of a coupling. Recall that $\mu \ll \nu$ means that $\mu$ is absolutely continuous with respect to $\nu$ and $\mu \sim \nu$ means that $\mu$ and $\nu$ are equivalent, i.e. mutually absolutely continuous. Define
\[ \tilde{C}(\mu, \nu) = \left\{ \xi \in \mathcal{P}(S \times S) : \pi_1(\xi) \sim \mu, \pi_2(\xi) \sim \nu \right\}, \]
and call any probability measure from one of the classes $\tilde{C}(\mu, \nu), \hat{C}(\mu, \nu)$ a generalized coupling for $\mu, \nu$. In what follows, we use this notation mainly in the following two frameworks: (a) $S = E$, (b) $S = E^\infty$; that is, the probability measures are considered either on the initial state space or on the trajectories space. To distinguish the notation, we denote probability measures by $\mu, \nu, \ldots$ and $\mathbb{P}, \mathbb{Q}, \ldots$ respectively in the first and the second cases.
For given $p > 1$ and $R \geq 1$, denote by $\hat{C}_p(\mathbb{P}, \mathbb{Q})$ the set of generalized couplings $\xi \in \hat{C}(\mathbb{P}, \mathbb{Q})$ such that
\[ \left( \int_{E^\infty} \left( \frac{d\pi_1(\xi)}{d\mathbb{P}} \right)^p d\mathbb{P} \right)^{1/p} \leq R, \quad \left( \int_{E^\infty} \left( \frac{d\pi_2(\xi)}{d\mathbb{Q}} \right)^p d\mathbb{Q} \right)^{1/p} \leq R. \]
Let $(S, \rho)$ be a metric space and $h : S \times S \to [0, 1]$ be a distance-like function; that is, $h$ is symmetric, lower semicontinuous, and $h(x, y) = 0 \Leftrightarrow x = y$. The associated minimal (or coupling) distance on $\mathcal{P}(S)$ (denoted by the same letter $h$) is defined by
\[ h(\mu, \nu) = \inf_{\eta \in C(\mu, \nu)} \int_{S \times S} h(x, y) \eta(dx, dy), \quad \mu, \nu \in \mathcal{P}(S). \]
When $\rho \leq 1$ and $h(x, y) = \rho(x, y)$, the above definition coincides with the definition of the Kantorovich-Rubinshtein metric (also commonly called 1-Wasserstein metric) on $\mathcal{P}(S)$, and it is well known that $\mathcal{P}(S)$ with this metric is a Polish space; cf. [6], Chapter 11.
Without loss of generality we assume furthermore that the metric $\rho$ on $E$ satisfies $\rho \leq 1$ (otherwise we introduce an equivalent metric $\rho \land 1$). We define the metric $\rho^{(\infty)}$ on $E^\infty$ by
\[ \rho^{(\infty)}(x, y) = \sum_{n=0}^{\infty} \frac{1}{2^n+1} d(x_n, y_n), \quad x = (x_n)_{n \geq 0}, y = (y_n)_{n \geq 0} \in E^\infty, \]
and the metric $\rho^{(\infty, \infty)}$ on $E^\infty \times E^\infty$ by
\[ \rho^{(\infty, \infty)}((x, x'), (y, y')) = \rho^{(\infty)}(x, y) + \rho^{(\infty)}(x', y'). \]
We consider $\mathcal{P}(E)$, $\mathcal{P}(E^\infty)$, and $\mathcal{P}(E^\infty \times E^\infty)$ as Polish spaces w.r.t. the corresponding Kantorovich-Rubinshtein metrics $\rho, \rho^{(\infty)}$ and $\rho^{(\infty, \infty)}$. The metric $\rho$ on $\mathcal{P}(E)$ induces weak convergence which we will denote by $\Rightarrow$.

Recall the following facts about the structure of the set of invariant probability measures, e.g. [5], Section 3.2 or [9, Theorem 5.7]:

- The set $\mathcal{I}_X$ of the invariant probability measures for $X$ is a convex compact set in $\mathcal{P}(E)$.
- Each two different ergodic invariant probability measures are mutually singular.
- Every extreme point of the set $\mathcal{I}_X$ is an ergodic invariant probability measure, and each invariant probability measure $\mu$ has a representation of the form
  \[ \mu = \int_{\mathcal{P}(E)} \nu \kappa(d\nu), \]  
  (3)
  where $\kappa$ is a probability measure on the space $\mathcal{P}(E)$ which is concentrated on the extreme points of the set $\mathcal{I}_X$.

Together with the initial metric $\rho$ on $E$, we will consider another metric $d$, and assume that it is bounded and continuous with respect to the metric $\rho$. The metric $d$ is not assumed to be complete. All measurability and continuity statements refer to $\rho$ rather than $d$ unless we explicitly say something different. Considering two metrics, one to deal with measurability issues and one to prove convergence, is motivated by applications to SPDE models, see Section 6.2 below. In many cases of interest however one can avoid such complications and choose $d$ and $\rho$ the same.

When $d \neq \rho$, we denote by $\overline{E}^d$ the completion of $E$ with respect to $d$, and regard $E$ as a subset in $\overline{E}^d$. Note that $(\overline{E}^d, d)$ is a Polish space (where we denote the extended metric again by $d$). We also assume that for any $y \in E$ there exist a sequence of $d$-continuous functions $\rho_n^y : \overline{E}^d \to [0, \infty)$ such that for $x \in \overline{E}^d$
\[ \rho_n^y(x) \to \begin{cases} \rho(x, y), & x \in E \\ \infty, & \text{otherwise} \end{cases}, \quad n \to \infty; \quad (4) \]
cf. [8], Appendix A. This ensures that the image in $\overline{E}^d$ of any open ball in $E$ is a $d$-Borel subset. Because $(E, d)$ is separable, this implies that $E$ is a $d$-Borel subset in $\overline{E}^d$ and guarantees that the trace $\sigma$-algebra
\[ \{A \cap E, A \in \mathcal{B}(\overline{E}^d)\} \]
on $E$ coincides with $\mathcal{E}$, hence allowing us to identify $\mathcal{P}(E)$ with the set of measures from $\mathcal{P}(\overline{E}^d)$ which provide a full measure for $E$.

We will use a separate notation $\Rightarrow_d$ for the weak convergence in $\mathcal{P}(E)$ with respect to $d$. We will call the Markov chain $d$-Feller if for each bounded and $d$-continuous function $f : E \to \mathbb{R}$, the map $x \mapsto \int f(y)P(x, dy)$ is $d$-continuous. Finally, recall that $X$ is called an e-chain with respect to the metric $d$, if its transition probability function is $d$-equicontinuous: for any $x \in E, \varepsilon > 0$ there exists $\delta > 0$ such that
\[ d(P_n(x, \cdot), P_n(y, \cdot)) \leq \varepsilon, \quad n \geq 0, \quad d(x, y) < \delta. \]
We note that our definition is equivalent to that in [13, Definition 2.1] by the Kantorovich-Rubinshtein duality theorem, see [6], Chapter 11.
2.2 Main theorems

Our first main result is aimed at uniqueness of an invariant probability measure. It is a slight generalization of [11, Theorem 1.1].

**Theorem 2.1.** Let $\mu_1$ and $\mu_2$ be ergodic invariant probability measures. Assume that for some set $M \in \mathcal{E} \otimes \mathcal{E}$ with $\mu_1 \otimes \mu_2(M) > 0$ for each $(x, y) \in M$ there exists $\alpha_{x,y} > 0$ such that for each $\varepsilon > 0$ there exists $\xi_{x,y}^\varepsilon \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)$ which satisfies

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_{x,y}^\varepsilon(d(X_i, Y_i) \leq \varepsilon) \geq \alpha_{x,y}. \quad (5)$$

Then $\mu_1 = \mu_2$.

Theorem 2.1 combined with the representation (3) immediately implies the following statement.

**Corollary 2.2.** Let $M \in \mathcal{E}$ be such that $\mu(M) > 0$ for every invariant probability measure $\mu$ and for each $x, y \in M$ there exists some $\alpha_{x,y} > 0$ such that for each $\varepsilon > 0$ there exists $\xi_{x,y}^\varepsilon \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)$ which satisfies

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi_{x,y}^\varepsilon(d(X_i, Y_i) \leq \varepsilon) \geq \alpha_{x,y}. \quad (6)$$

Then there exists at most one invariant probability measure, and if there exists one this measure is ergodic.

The second main result provides the weak convergence of transition probabilities to an invariant probability measure $\mu$ in a somewhat unusual form of the “weak convergence in probability”.

**Theorem 2.3.** I. Assume that $X$ is $d$-Feller. Let $\mu$ be an invariant probability measure, and assume that for some $M \in \mathcal{E} \otimes \mathcal{E}$ with $(\mu \otimes \mu)(M) = 1$ the following condition holds true for each $(x, y) \in M$:

(i) $$\lim_{\varepsilon \downarrow 0+} \limsup_{n \to \infty} \sup_{\xi \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)} \xi(d(X_n, Y_n) \leq \varepsilon) > 0.$$ 

Then $$\mu \left( x : d\left( P_n(x, \cdot), \mu \right) > \varepsilon \right) \to 0, \quad n \to \infty, \quad \varepsilon > 0; \quad (7)$$

that is, $P_n(\cdot, \cdot)$, $n \geq 0$, considered as a sequence of $\mathcal{P}(\mathcal{E})$-valued random elements on $(\mathcal{E}, \mathcal{F}, \mu)$, $d$-converges in probability to the random element identically equal to $\mu$.

II. The following condition is equivalent to (i):

(ii) For some $p > 1$, $R \geq 1$,

$$\lim_{\varepsilon \downarrow 0+} \limsup_{n \to \infty} \sup_{\xi \in \tilde{C}_p^R(\mathbb{P}_x, \mathbb{P}_y)} \xi(d(X_n, Y_n) \leq \varepsilon) > 0.$$ 

Further, the following condition implies (ii) (and (i)):

(iii) $$\sup_{\xi \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)} \lim_{\varepsilon \downarrow 0+} \liminf_{n \to \infty} \xi(d(X_n, Y_n) \leq \varepsilon) > 0.$$
Combining the two statements of the theorem, we directly get the following corollary, formulated in terms similar to those used in Theorem 2.1.

**Corollary 2.4.** Let $X$ be $d$-Feller, $\mu$ be an ergodic invariant probability measure and $M \in \mathcal{E} \otimes \mathcal{E}$ be such that $\mu \otimes \mu(M) = 1$, and assume that for each $(x, y) \in M$ there exist $\xi_{x,y} \in \tilde{C}(P_x, P_y)$ and $\alpha_{x,y} > 0$ such that

$$\liminf_{n \to \infty} \xi_{x,y}(d(X_n, Y_n) \leq \varepsilon) \geq \alpha_{x,y}$$

for every $\varepsilon > 0$. Then (7) holds true.

In the following theorem a stronger (and more typical) type of convergence is obtained at the cost of making a stronger assumption: the e-chain property (which is essentially the uniform Feller property) instead of the usual Feller one.

**Theorem 2.5.** Let $X$ be an e-chain w.r.t. $d$, and one of the assumptions (i) – (iii) of Theorem 2.3 hold true. Then $P_n(x, \cdot) \xrightarrow{d} \mu$ for $\mu$-a.a. $x \in E$.

A proper benchmark for Theorem 2.5 is the modified Doob theorem, given in [14, Theorem 2]. Let, for a while, $d(x, y) = 1_{x \neq y}$ be the discrete metric (which is however is not included into our setting since it is not continuous). Then by the Coupling Lemma (e.g. [14, Lemma 1]) the corresponding probability distance equals $1/2$ of the total variation distance. In [14, Theorem 2] it is assumed that for $\mu \otimes \mu$-a.a. $(x, y)$ there exists $n = n_{x,y}$ such that $P_n(x, \cdot) \not\perp P_n(y, \cdot)$, which by the Coupling Lemma is equivalent to existence of a coupling $\xi_{x,y}$ and positive $\alpha_{x,y}$ such that

$$\xi_{x,y}(d(X_n, Y_n) = 0) \geq \alpha_{x,y}.$$

One can extend the coupling $\xi_{x,y}$ in such a way that $X_N = Y_N, N \geq n$, hence the above assumption actually coincides with the one from Corollary 2.4. That is, Theorem 2.5 is a direct analogue of the modified Doob theorem, which operates with weak convergence of the transition probabilities instead of total variation convergence.

Note that the discrete metric $d$ is non-expanding: since the discrete metric $d$ takes values $0, 1$ only,

$$d(P_n(x, \cdot), P_n(y, \cdot)) \leq d(x, y), \quad x, y \in E, \quad n \geq 1.$$

This property has the same meaning as the e-chain property, which in Theorem 2.5 is imposed as an additional assumption because general metric $d$ may fail to be non-expanding. The ergodicity under the e-chain (actually, the e-process) property was systematically studied in [2], [13], see also [11, Theorem 3.7], where the e-chain property was used essentially without naming it explicitly. We remark that the e-chain property, although being quite typical for ergodic processes, does not follow from the fact that the transition probabilities converge to the (unique) invariant probability distribution: see Example 5.4 below, which in particular shows that Proposition 6.4.2 in [16] is incorrect. In that concern, the clearly seen advantage of Theorem 2.3 is that there we avoid the quite non-elementary (and sometimes not easy to verify) e-chain assumption. We remark that both of the proofs of Theorem 2.3 and Theorem 2.5 exploit the typical “coupling” idea. Namely, we make one “coupling attempt” with the probability of success being close to the presumably maximal possible one, and then we show that if the latter probability is $< 1$, another “coupling attempt” will increase the overall probability of success significantly. In that strategy
of the proof, a kind of the “non-expansion” property is crucial in order to preserve the positive result of the first “coupling attempt”. We note that, in the proof of Theorem 2.3, only the basic $d$-Feller property is used to provide such “non-expansion”.

Theorem 2.3 provides the following important corollary.

**Corollary 2.6.** Under the conditions of Theorem 2.3, the (stationary) chain $X$ is mixing w.r.t. $\mathbb{P}_\mu$.

This corollary gives a good prerequisite for our third main result, which provides a sufficient condition for the transition probabilities $P_n(x,\cdot)$ of a given $x \in E$ to converge to the invariant measure $\mu$.

**Theorem 2.7.** Let $\mu$ be an invariant probability measure and $X$ be mixing w.r.t. $\mathbb{P}_\mu$. Fix $x \in E$ and assume that there exists a set $M \in \mathcal{E}$ such that $\mu(M) > 0$ and for every $y \in M$ there exists some $\xi_{x,y} \in \widehat{C}(\mathbb{P}_x, \mathbb{P}_y)$ such that $\pi_1(\xi_{x,y}) \sim \mathbb{P}_x$ and

$$\lim_{n \to \infty} \xi_{x,y}(d(X_n, Y_n) \leq \varepsilon) = 1$$

(8)

for every $\varepsilon > 0$. Then $P_n(x,\cdot) \xrightarrow{d} \mu$.

Combining Corollary 2.4, Corollary 2.6, and Theorem 2.7 we easily derive the following corollary, which provides weak convergence of $P_n(x,\cdot)$ for any starting point $x$ in terms of generalized couplings.

**Corollary 2.8.** Let $X$ be $d$-Feller, and assume that for any $(x, y) \in E \times E$ there exists some $\xi_{x,y} \in \widehat{C}(\mathbb{P}_x, \mathbb{P}_y)$ such that $\pi_1(\xi_{x,y}) \sim \mathbb{P}_x$ and (8) holds true for every $\varepsilon > 0$.

Then there exists at most one invariant probability measure, and if such a measure $\mu$ exists then $P_n(x,\cdot) \xrightarrow{d} \mu$ for any $x \in E$.

**Remark 2.9.** The assumption of Corollary 2.8 is well designed to be easily applied in various particular settings; we illustrate this in Section 6 below considering Markov chains generated by stochastic functional delay equations (SFDEs) and stochastic partial differential equations (SPDEs). On the other hand, this assumption is quite precise and can not be essentially weakened. For instance, the required statement may fail if one assumes only (8) for some $\xi_{x,y} \in \widehat{C}(\mathbb{P}_x, \mathbb{P}_y)$ without the additional condition $\pi_1(\xi_{x,y}) \sim \mathbb{P}_x$, see Example 5.5 below.

**Remark 2.10.** The existence of an invariant probability measure is a much easier topic, studied in great detail in the literature, and we do not address it here, referring e.g. to [5].

**Remark 2.11.** All the main statements, formulated above in the discrete-time case, have straightforward analogues in the continuous-time case. Namely, if it is assumed that the process $X_t, t \in [0, \infty)$ for any $x$ has a càdlàg modification with $X_0 = x$, we can repeat the arguments literally, with the space $E^\infty$ changed to $\mathbb{D}([0, \infty), E)$ and $\mathbb{P}_x, x \in E$ being the respective laws of $X$ in $\mathbb{D}([0, \infty), E)$. We remark that in some specific but important cases it may happen that the Markov process $X_t, t \in [0, \infty)$ is stochastically continuous, but fails to have a càdlàg modification. For such an example in the framework of Lévy driven SPDEs we refer to [4]. In this case the proofs of Theorem 2.3, Theorem 2.5 and Theorem 2.7 also can be adapted, and the statements of these theorems and respective Corollaries 2.4, 2.6, and 2.8 hold true. The technical difficulty which arise here is that now we do not have a “good” space of trajectories, hence the statements of Proposition 3.1 and Proposition 4.2 on the measurable choice can not be applied directly. These statements can be modified properly, but in order not to overburden the exposition we will not go into further details.
3  Proofs of Theorem 2.1 and Theorem 2.7

In this section we provide proofs of two of our main results, which are comparatively simple and are mainly based on the Ergodic theorem.

Proof of Theorem 2.1. If \( \mu_1 \neq \mu_2 \) then \( \mu_1 \perp \mu_2 \). The fact that \( E \) is Polish implies that for any probability measure \( \mu \) and each set \( A \in \mathcal{E} \), we have \( \mu(A) = \sup \mu(K) \), where the supremum is taken over all compact subsets of \( E \) which are contained in \( A \) (this is sometimes called *Ulam’s theorem* or *inner regularity* of \( \mu \); e.g. [3], Theorems 1.1 and 1.4). Therefore, for every \( m \geq 1 \) there exist compact sets \( K_{1,m}^n, K_{2,m}^n \) such that \( K_{1,m}^n \cap K_{2,m}^n = \emptyset \) and \( \mu_i(K_{i,m}^n) > 1 - 1/m \). Since \( d \) is continuous and \( K_{1,m}^n, K_{2,m}^n \) are compact and disjoint, the \( d \)-distance between \( K_{1,m}^n \) and \( K_{2,m}^n \) is positive. Then there exists a \( d \)-Lipschitz function \( f_m : E \to [0,1] \) such that \( f_m \mid_{K_{1,m}^n} \equiv 0, f_m \mid_{K_{2,m}^n} \equiv 1 \).

Choose \( (x, y) \in M \), and let \( \alpha_{x,y} \) be as in the statement of the theorem. We can and will assume in addition that \( x, y \) are chosen such that

\[
\frac{1}{n} \sum_{i=0}^{n-1} f_m(X_i) \to \int f_m \, d\mu_1 \quad \text{a.s. w.r.t. } \mathbb{P}_x, \quad \frac{1}{n} \sum_{i=0}^{n-1} f_m(Y_i) \to \int f_m \, d\mu_2 \quad \text{a.s. w.r.t. } \mathbb{P}_y \tag{9}
\]

for every \( m \geq 1 \). Take \( m_0 > 2/\alpha_{x,y} \) and fix \( \varepsilon > 0 \) such that

\[
\varepsilon < \frac{\alpha_{x,y} - 2/m_0}{\alpha_{x,y} \text{Lip}(f_{m_0})}. 
\]

Let \( \xi_{x,y}^\varepsilon \in \hat{C}(\mathbb{P}_x, \mathbb{P}_y) \) be as in the statement of the theorem, then

\[
\liminf_{n \to \infty} \mathbb{E}^{\xi_{x,y}^\varepsilon} \left( \frac{1}{n} \sum_{i=0}^{n-1} (f_{m_0}(Y_i) - f_{m_0}(X_i)) \right) \leq (1 - \alpha_{x,y}) + \varepsilon \text{Lip}(f_{m_0}) \alpha_{x,y} < 1 - 2/m_0.
\]

Because the distribution of \( \{X_i\} \) (resp. \( \{Y_i\} \)) w.r.t. \( \xi_{x,y}^\varepsilon \) is absolutely continuous w.r.t. \( \mathbb{P}_x \) (resp. \( \mathbb{P}_y \)) and \( f_{m_0} \) is bounded, it follows from (9) that

\[
\mathbb{E}^{\xi_{x,y}^\varepsilon} \left( \frac{1}{n} \sum_{i=1}^{n} (f_{m_0}(Y_i) - f_{m_0}(X_i)) \right) \to \int f_{m_0} \, d\mu_2 - \int f_{m_0} \, d\mu_1 \geq 1 - 2/m_0, \quad n \to \infty,
\]

which is a contradiction. Therefore \( \mu_1 = \mu_2 \) follows. \( \square \)

In the proof Theorem 2.7 we will use the following proposition, whose proof is postponed to Appendix B.

**Proposition 3.1.** Under the assumptions of Theorem 2.7, there exists a measurable mapping

\[
M \ni y \mapsto \xi_{x,y} \in \mathcal{P}(E^\infty \times E^\infty)
\]

such that for \( \mu \)-a.a. \( y \in M \) one has \( \xi_{x,y} \in \hat{C}(\mathbb{P}_x, \mathbb{P}_y) \), \( \pi_1(\xi_{x,y}) \sim \mathbb{P}_x \), and (8) holds true.
Proof of Theorem 2.7. Define the measure \( \xi \in \mathcal{P}(E^\infty \times E^\infty) \) as follows:

\[
\xi(A) = \frac{1}{\mu(M)} \int_M \xi_{x,y}(A) \mu(dy), \quad A \in \mathcal{E}^{\otimes \infty} \otimes \mathcal{E}^{\otimes \infty},
\]

where \( \{\xi_{x,y}, y \in M\} \) is defined in Proposition 3.1. Because \( \pi_1(\xi_{x,y}) \sim \mathbb{P}_x, \pi_2(\xi_{x,y}) \ll \mathbb{P}_\mu \) for \( \mu \)-a.a. \( y \in M \), we have that \( \xi \in \widehat{C}(\mathbb{P}_x, \mathbb{P}_\mu) \) and \( \pi_1(\xi) \sim \mathbb{P}_x \). In addition, we have

\[
\lim_{n \to \infty} \xi(d(X_n, Y_n) \leq \varepsilon) = 1 \tag{10}
\]

for every \( \varepsilon > 0 \).

Denote

\[
\mu_n(C) = \xi(X_n \in C), \quad \nu_n(C) = \xi(Y_n \in C), \quad C \in \mathcal{E}.
\]

Observe first that the family \( \{\nu_n, n \geq 1\} \subset \mathcal{P}(E) \) is tight: recall that \( \pi_2(\xi) \ll \mathbb{P}_\mu \) and thus for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( B \in \mathcal{E}^{\otimes \infty} \) with \( \mathbb{P}_\mu(B) \leq \delta \) one has \( \pi_2(\xi)(B) \leq \varepsilon \). Therefore

\[
\nu_n(K) = \xi(Y_n \in K) \leq \varepsilon, \quad n \geq 1
\]

if a compact set \( K \subset E \) is chosen such that \( \mu(K) \geq 1 - \delta \).

Since the embedding \( (E, \rho) \to (\overline{E}^d, d) \) is continuous, this yields that \( \{\nu_n, n \geq 1\} \subset \mathcal{P}(\overline{E}^d) \) is tight, as well. Then using using (10), we deduce that \( \{\nu_n, n \geq 1\} \subset \mathcal{P}(\overline{E}^d) \) is tight, as well. Because \( \mathbb{P}_x \ll \pi_1(\xi) \), this finally yields the tightness of \( \{P_n(x, .), n \geq 1\} \subset \mathcal{P}(\overline{E}^d) \).

The metric space \( (\overline{E}^d, d) \) is complete by the construction and is separable since \( (E, \rho) \) is separable. Hence if we assume that that \( P_n(x, .), n \geq 1 \) does not weakly converge to \( \mu \) w.r.t. \( d \), then there exists some probability measure \( \nu \neq \mu \) on \( \overline{E}^d \) and a subsequence \( P_{n_k}(x, .) \xrightarrow{d} \nu \). Fix a bounded \( d \)-Lipschitz continuous function \( f : \overline{E}^d \to \mathbb{R} \) such that \( \bar{f} := \int_{\overline{E}^d} f \, d\mu \neq \int_{\overline{E}^d} f \, d\nu \) and put

\[
U_n = \frac{1}{n} \sum_{k=1}^n f(X_{m_k}).
\]

Recall that the chain \( X \) is stationary and mixing w.r.t. \( \mathbb{P}_\mu \), hence

\[
\text{Cov}_\mu(f(X_m), f(X_n)) := \mathbb{E}_\mu(f(X_m)f(X_n)) - (\bar{f})^2 \to 0, \quad |n-m| \to \infty.
\]

Then

\[
\mathbb{E}_\mu(U_n - \bar{f})^2 = \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{Cov}_\mu(f(X_{m_i}), f(X_{m_j})) \to 0, \quad n \to \infty,
\]

and furthermore there exists a sequence \( \{n_r\} \) such that \( U_{n_r} \to \bar{f}, r \to \infty \) a.s. w.r.t. \( \mathbb{P}_\mu \). Because \( \pi_2(\xi) \ll \mathbb{P}_\mu \), we have finally that \( U_{n_r} \to \bar{f}, r \to \infty \) a.s. with respect to \( \pi_2(\xi) \).

Recall that \( f \) is Lipschitz continuous, then by (10) the sequence

\[
\Delta_m := |f(X_m) - f(Y_m)| \leq \text{Lip}(f)d(X_m, Y_m), \quad m \geq 1
\]
converges to 0 in $\xi$-probability. Because $f$ is bounded, this convergence holds true also in the mean sense, which then implies
\[
\frac{1}{n} \sum_{k=1}^{n} f(X_{m_k}) - \frac{1}{n} \sum_{k=1}^{n} f(Y_{m_k}) \to 0, \quad n \to \infty
\]
in the mean sense w.r.t. $\xi$. Since $\{Y_n\}$ has the law $\pi_2(\xi)$ w.r.t. $\xi$ and $U_{n_r} \to \bar{f}, r \to \infty$ a.s. w.r.t. $\pi_2(\xi)$, we have then
\[
\left(\frac{1}{n_r} \sum_{k=1}^{n_r} f(X_{m_k}), \frac{1}{n_r} \sum_{k=1}^{n_r} f(Y_{m_k})\right) \to (\bar{f}, \bar{f}), \quad r \to \infty
\]
in $\xi$-probability, hence $U_{n_r} \to \bar{f}, r \to \infty$ in probability with respect to $\pi_1(\xi)$. Because $P_x \ll \pi_1(\xi)$, we deduce finally that $U_{n_r} \to \bar{f}, r \to \infty$ in probability with respect to $P_x$. Since $f$ is bounded, the sequence $U_{n_r}, r \geq 1$ is bounded by the same constant, and then we have
\[
\mathbb{E}_x U_{n_r} \to \bar{f}, \quad r \to \infty.
\]

On the other hand, by the assumption $P_{m_k}(x,.) \overset{d}{\Rightarrow} \nu$, we have $\int_{E^d} f(z) P_{m_k}(x, dz) \to \int_{E^d} f(d\nu)$ and therefore
\[
\mathbb{E}_x \frac{1}{n} \sum_{k=1}^{n} f(X_{m_k}) = \frac{1}{n} \sum_{k=1}^{n} \int_{E} f(z) P_{m_k}(x, dz) = \frac{1}{n} \sum_{k=1}^{n} \int_{E^d} f(z) P_{m_k}(x, dz) \to \int_{E} f d\nu \neq \bar{f}, \quad n \to \infty,
\]
which is a contradiction finishing the proof of the theorem. \qed

## 4 Proofs of Theorem 2.3, Theorem 2.5, and Corollary 2.6

In this section we prove Theorem 2.3, which is the most complicated of our main results. We also show how (parts of) the proof can be modified in order to obtain Theorem 2.5, and prove Corollary 2.6. Before proceeding with these proofs, we formulate several auxiliary results.

**Proposition 4.1.** I. Let $p > 1$ and $R > 0$ be fixed. Then for each $\alpha > 0$ there exists some $\alpha' > 0$ such that the following holds true: for every $P, Q \in \mathcal{P}(E^\infty)$ and every $\xi \in \mathcal{C}_p^R(P, Q)$ there exists some $\zeta \in C(P, Q)$ such that for each $A \in \mathcal{E}_{\infty} \otimes \mathcal{E}_{\infty}$ satisfying $\xi(A) \geq \alpha$ we have $\zeta(A) \geq \alpha'$.

II. For each $\alpha > 0$ there exists some $\alpha' > 0$ and $R \geq 1$ such that the following holds true: for every $p \geq 1$, every $P, Q \in \mathcal{P}(E^\infty)$ and every $\xi \in \mathcal{C}(P, Q)$ there exists some $\zeta \in \mathcal{C}_p^R(P, Q)$ such that for each $A \in \mathcal{E}_{\infty} \otimes \mathcal{E}_{\infty}$ satisfying $\xi(A) \geq \alpha$ we have $\zeta(A) \geq \alpha'$.

The proof of this proposition is given in Appendix A.

**Proposition 4.2.** Let $S_1, S_2$ be Polish spaces and $Q : S_1 \to \mathcal{P}(S_2)$ be a continuous mapping. Let $h : S_2 \times S_2 \to [0, 1]$ be a distance-like function. Then there exists a measurable mapping $\eta : S_1 \times S_1 \ni (x, y) \mapsto \eta_{x,y} \in \mathcal{P}(S_2 \times S_2)$ such that
\[
\eta_{x,y} \in C(Q(x), Q(y)), \quad \int_{S_2 \times S_2} h(x', y') \eta_{x,y}(dx', dy') = h(Q(x), Q(y)), \quad x, y \in S_1.
\]

The proof of this proposition is given in Appendix C.
Corollary 4.3. For a given $d$-Feller chain $X$ and given $n \in \mathbb{N}, \varepsilon > 0$, denote

$$
\gamma_{n,\varepsilon} := \sup_{\xi \in C(P_x, P_y)} \xi(d(X_n, Y_n) \leq \varepsilon), \quad (x, y) \in E \times E.
$$

The following statements hold.
I. The function $\gamma_{n,\varepsilon} : E \times E \to [0, 1]$ is $\mathcal{E} \otimes \mathcal{E} - \mathcal{B}([0, 1])$ measurable.
II. There exists a measurable function

$$
\xi_{n,\varepsilon} : E \times E \ni (x, y) \mapsto \xi_{n,\varepsilon}^{x,y} \in \mathcal{P}(E_\infty \times E_\infty)
$$

such that for every $(x, y) \in E \times E$ the following properties hold:

(i) $\xi_{n,\varepsilon}^{x,y} \in C(P_x, P_y)$;

(ii) $\xi_{n,\varepsilon}^{x,y}(d(X_n, Y_n) \leq \varepsilon) = \gamma_{n,\varepsilon}^{x,y}$.

Proof. Consider the Polish metric space $(E^d, d)$. Consider also the space $(E^d)^\infty$ with the metric $d(\infty)$ introduced in the same way with the metric $\rho(\infty)$, and the space $\mathcal{P}((E^d)^\infty)$ with the Kantorovich-Rubinshtein distance $d(\infty)$; see Section 2.1. By (4) we have that the image of $E^\infty$ under the natural embedding is a measurable subset in $(E^d)^\infty$, and $\mathcal{P}(E^\infty)$ can be identified as the (measurable) set of those measures from $\mathcal{P}((E^d)^\infty)$ which provide a full measure for $E^\infty$. The same remarks are valid for the spaces $(E^d)^\infty \times (E^d)^\infty$ and $\mathcal{P}((E^d)^\infty \times (E^d)^\infty)$ which are defined analogously.

Observe that, because the chain $X$ is $d$-Feller, the mapping

$$
E \ni x \mapsto P_x \in \mathcal{P}((E^d)^\infty)
$$

is continuous. Hence we can apply Proposition 4.2 with $S_1 = E, S_2 = (E^d)^\infty, Q(x) = P_x, x \in E$, and

$$
h(x, y) = 1_{d(x_n, y_n) > \varepsilon} = 1 - 1_{d(x_n, y_n) \leq \varepsilon}, \quad x = (x_k)_{k \geq 1}, \quad y = (y_k)_{k \geq 1} \in (E^d)^\infty.
$$

Then there exists a measurable function

$$
\xi_{n,\varepsilon} : E \times E \ni (x, y) \mapsto \xi_{n,\varepsilon}^{x,y} \in \mathcal{P}((E^d)^\infty \times (E^d)^\infty)
$$

which satisfies properties (i), (ii) in statement II of the corollary. In addition, each $P_x, x \in E$ assigns full measure to $E^\infty$, hence by the property (i) each measure $\xi_{n,\varepsilon}^{x,y}(x, y) \in E \times E$ assigns full measure to $E^\infty \times E^\infty$. Therefore $\xi_{n,\varepsilon}$ can be considered as a measurable mapping taking values in $\mathcal{P}(E^\infty \times E^\infty)$, which completes the proof of statement II.

Statement I follows immediately, because the mapping

$$
(x, y) \mapsto \xi_{n,\varepsilon}^{x,y}(d(X_n, Y_n) \leq \varepsilon)
$$

is measurable. \qed
Remark 4.4. Proposition 4.2 and Corollary 4.3 give a natural extension of the “Coupling Lemma for transition probabilities” (Lemma 1 in [14]). This lemma provides a probability kernel which in a pointwise sense minimizes the particular distance-like function \( h(x, y) = 1_{x \neq y} \), while Proposition 4.2 provides such a kernel for an arbitrary distance-like function. The proof of Lemma 1 in [14] exploits an explicit construction of a maximal coupling based on the splitting representation of a probability law, and it cannot be extended to our current setting. We use instead the general measurable selection theorem which dates back to Kuratovskii and Ryll-Nardzewski theorem combined with some measurability criteria for set-valued maps explained in [17], Chapter 12.1. We mention that our proof is similar to that of Lemma 4.13 in [11], which also provides a probability kernel which is maximal w.r.t. \( h \), but in our setting we avoid using an additional assumption on \( h \) to be continuous.

Remark 4.5. We mention for future reference that the kernel \( \xi^{n, \varepsilon} \) can be modified such that it possesses the following additional property, which is a direct analogue of property (ii) of the maximal coupling kernel constructed in [14, Lemma 1]:

(iii) the measure \( \xi^{n, \varepsilon}_{x,y} \) conditioned by \( \{d(X_n, Y_n) > \varepsilon\} \) is absolutely continuous w.r.t. \( \mathbb{P}_x \otimes \mathbb{P}_y \) with the respective Radon-Nikodym density being bounded from above by \( (1 - \gamma^{n, \varepsilon}_{x,y})^{-1} \).

Namely, denote by \( n^{n, \varepsilon}_{x,y} \) and \( c^{n, \varepsilon}_{x,y} \) the initial measure \( \xi^{n, \varepsilon}_{x,y} \) conditioned by the event \( \{d(X_n, Y_n) \leq \varepsilon\} \) and by its complement, respectively. Then it is easy to see that the modified function

\[
\gamma^{n, \varepsilon}_{x,y} n^{n, \varepsilon}_{x,y} + (1 - \gamma^{n, \varepsilon}_{x,y}) \pi_1(c^{n, \varepsilon}_{x,y}) \otimes \pi_1(c^{n, \varepsilon}_{x,y})
\]

satisfies (i) – (iii); see the proof of Theorem 2.3 below for a more detailed discussion of this construction in a slightly different setting.

Proposition 4.6. If \( \nu_1 \) and \( \nu_2 \) are singular probability measures on a Polish space \((E, \rho)\), then

\[
\lim_{\varepsilon \downarrow 0} \sup_{\zeta \in C(\nu_1, \nu_2)} \zeta \left( d(X, Y) \leq \varepsilon \right) = 0.
\]

The proof of this proposition is given in Appendix A.

Corollary 4.7. Under the conditions of Theorem 2.3, the measure \( \mu \) is ergodic.

Proof. Consider the ergodic decomposition (3) of \( \mu \). Then

\[
1 = (\mu \otimes \mu)(M) = \int_{E \times E} (\nu_1 \otimes \nu_2)(M) \kappa(d\nu_1) \kappa(d\nu_2).
\]

If \( \mu \) is not ergodic then \( \kappa \) is non-degenerate and there exist two mutually singular invariant probability measures \( \nu_1, \nu_2 \) such that

\[
(\nu_1 \otimes \nu_2)(M) = 1.
\]

(11)

Define for a given \( n \geq 1, \varepsilon > 0 \) the measure \( \eta^{n, \varepsilon} \in \mathcal{P}(E^\infty \times E^\infty) \) by

\[
\eta^{n, \varepsilon} = \int_{E \times E} \xi^{n, \varepsilon}_{x,y} \nu_1(dx) \nu_2(dy),
\]

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where $\xi_{x,y}^{n,\varepsilon}$ is defined as in Corollary 4.3, and denote by $\zeta_{n,\varepsilon}$ the law of $(X_n, Y_n)$ under $\eta_{n,\varepsilon}$. Because $\pi_1(\eta_{n,\varepsilon}) = \mathbb{P}_{\nu_1}$, $\pi_2(\eta_{n,\varepsilon}) = \mathbb{P}_{\nu_2}$, and $\nu_1, \nu_2$ are invariant, we have
\[\zeta_{n,\varepsilon} \in C(\nu_1, \nu_2)\]
for any $n \geq 1, \varepsilon > 0$.

On the other hand,
\[\zeta_{n,\varepsilon}(d(X, Y) \leq \varepsilon) = \int_{E \times E} \gamma_{x,y}^{n,\varepsilon} \nu_1(dx) \nu_2(dy)\]
and therefore
\[\sup_{\zeta \in C(\nu_1, \nu_2)} \zeta(d(X, Y) \leq \varepsilon) \geq \int_{E \times E} \gamma_{x,y}^{n,\varepsilon} \nu_1(dx) \nu_2(dy).\]

Denote
\[\gamma_{x,y}^{\varepsilon} = \liminf_{n \to \infty} \gamma_{x,y}^{n,\varepsilon}, \quad \gamma_{x,y} = \lim_{\varepsilon \to 0^+} \gamma_{x,y}^{\varepsilon},\]
then by the Fatou lemma and the monotone convergence theorem
\[\lim_{\varepsilon \downarrow 0^+} \sup_{\zeta \in C(\nu_1, \nu_2)} \zeta(d(X, Y) \leq \varepsilon) \geq \int_{E \times E} \gamma_{x,y} \nu_1(dx) \nu_2(dy).\]

By condition (i) of Theorem 2.3, we have $\gamma_{x,y} > 0$ for any $(x, y) \in M$, hence the above inequality combined with (11) contradicts Proposition 4.6.

**Corollary 4.8.** Under the conditions of Theorem 2.3, the measure $\mu \otimes \mu$ is ergodic for the product chain.

**Proof.** Denote by the same symbol $d$ the metric on the product space $E \times E$
\[d((x,u),(y,v)) = d(x,y) \wedge d(u,v),\]
and by $\mathbb{P}_{(x,u)}$ the distribution of the product chain with the initial value $(x, u)$. For any $x, y, u, v \in E$ and $n \geq 1, \varepsilon > 0$ consider the probability measure on $E^\infty \times E^\infty \times E^\infty \times E^\infty$
\[\xi_{x,y,u,v}^{n,\varepsilon} = \xi_{x,y}^{n,\varepsilon} \otimes \xi_{u,v}^{n,\varepsilon},\]
where $\xi_{x,y}^{n,\varepsilon}$ is defined in Corollary 4.3. Then the projections $\pi_{1,3}$ and $\pi_{2,4}$ of this measure on the coordinates 1,2 and 2,4, respectively equal $\mathbb{P}_{(x,u)}$ and $\mathbb{P}_{(y,v)}$. On the other hand,
\[\lim_{\varepsilon \downarrow 0^+} \liminf_{n \to \infty} \xi_{x,y,u,v}^{n,\varepsilon} \left( d((X_n, U_n), (Y_n, V_n)) \leq \varepsilon \right) = \lim_{\varepsilon \downarrow 0^+} \liminf_{n \to \infty} \gamma_{x,y}^{n,\varepsilon} \gamma_{u,v}^{n,\varepsilon} \geq \gamma_{x,y} \gamma_{u,v} > 0\]
for any
\[(x, y, u, v) \in M' := M \times M.\]

Thus the above inequality yields the following analogue of condition (i) of Theorem 2.3 for the product chain: for any $(x, y, u, v) \in M'$,
\[\lim_{\varepsilon \downarrow 0^+} \liminf_{n \to \infty} \sup_{\pi_{1,3}(\xi) = \mathbb{P}_{(x,u)}, \pi_{2,4}(\xi) = \mathbb{P}_{(y,v)}} \xi \left( d((X_n, U_n), (Y_n, V_n)) \leq \varepsilon \right) > 0.\]

Clearly, $(\mu \otimes \mu \otimes \mu \otimes \mu)(M') = 1$, hence the required statement follows by the previous corollary.
Now we proceed with the proof of Theorem 2.3. The second statement of the theorem follows easily: since
\[ C(\mathbb{P}, \mathbb{Q}) \subset \mathcal{C}_R^p(\mathbb{P}, \mathbb{Q}) \]
for each \( p > 1 \) and \( R \geq 1 \), condition (i) immediately implies (ii). The inverse implication follows from the first statement in Proposition 4.1 while the second statement in this proposition shows that (iii) implies (ii).

**Proof of Theorem 2.3, statement I.** Our aim is to prove that for every \( \varepsilon > 0 \)
\[ \Gamma^{n, \varepsilon} := \int_{E \times E} \gamma_{x,y}^{n, \varepsilon} \mu(dx) \mu(dy) \rightarrow 1, \quad n \rightarrow \infty. \] (13)

This yields the required statement. Indeed, for given \( \varepsilon > 0 \) and \( n \geq 1 \), consider a random element \( \eta \) with law \( \mu \) and a sequence \( Z_k = (X_k, Y_k) \), \( k \geq 0 \) with \( Z_0 = (x, \eta) \) and the conditional law of \( Z \) under \( \sigma(Z_0) \) equal \( \xi_{x, \eta}^{n, \varepsilon} \); the measurable mapping \( \xi_{x, \eta}^{n, \varepsilon} \) is introduced in Corollary 4.3. By property (i) of this mapping, the law of \( Z \) belongs to \( C(\mathbb{P}_x, \mathbb{P}_\mu) \). We have assumed \( d \leq 1 \), hence
\[ d\left(P_n(x, \cdot), \mu \right) \leq \varepsilon + \xi_{x, \eta}^{n, \varepsilon}\left(d(X_n, Y_n) > \varepsilon \right). \]

By property (ii) of the mapping \( \xi_{x, \eta}^{n, \varepsilon} \), we have
\[ \xi_{x, \eta}^{n, \varepsilon}\left(d(X_n, Y_n) \leq \varepsilon \right) = \int_E \gamma_{x,y}^{n, \varepsilon} \mu(dy). \]
Consequently, it follows from (13) that
\[ \liminf_{n \rightarrow \infty} \int_E d\left(P_n(x, \cdot), \mu \right) \mu(dx) \leq \liminf_{n \rightarrow \infty} (\varepsilon + (1 - \Gamma^{n, \varepsilon})) = \varepsilon \]
for any \( \varepsilon > 0 \), which then yields (7).

Now we proceed with the proof of (13). Take an independent coupling \( Z_k = (X_k, Y_k) \), \( k \in \mathbb{Z}_+ \) with Law \( (X_0) = \text{Law} (Y_0) = \mu \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and observe that for every fixed \( \varepsilon, n, k \)
\[ \gamma_{Z_k}^{n+1, \varepsilon} \geq \mathbb{E}[^n_{Z_k} \gamma_{Z_k+1} | \mathcal{F}_k^{Z}]. \] (14)
Indeed, the expression on the left hand side means that one fixes the position of \( Z \) at the time instant \( k \) and optimizes the probability for the coordinates of \( Z \) to stay \( \varepsilon \)-close at the time instant \( n + k + 1 \), while the expression at the right hand side means that one makes an independent step first, and then optimizes the same the probability; the optimal probability in the second case is smaller because due to the more restricted set of possible couplings.

Using Fatou’s lemma and inequality (14) we obtain:
\[ \mathbb{E}[\gamma_{Z_k+1}^{n} | \mathcal{F}_k^{Z}] = \mathbb{E}[\liminf_n \gamma_{Z_k+1}^{n, \varepsilon} | \mathcal{F}_k^{Z}] \leq \liminf_n \mathbb{E}[\gamma_{Z_k+1}^{n, \varepsilon} | \mathcal{F}_k^{Z}] \leq \liminf_n \gamma_{Z_k}^{n+1, \varepsilon} = \gamma_{Z_k}^{\varepsilon}, \]
where \( \gamma, \gamma \) are defined as in the proof of Corollary 4.7. Hence \( \gamma_{Z_k}^\varepsilon, n \geq 1 \) is a non-negative super-martingale for every \( \varepsilon > 0 \), and so is \( \gamma_{Z_n}, n \geq 1 \). Therefore, the \( \mathbb{P}^Z \)-a.s. limits
\[ \gamma_{Z_n} \rightarrow \gamma, \quad \gamma_{Z_n}^\varepsilon \rightarrow \gamma^\varepsilon, \quad n \rightarrow \infty \]
exist. On the other hand, since $Z$ is stationary, the sequences $\gamma_{Z_n}^\varepsilon, n \geq 1$ and $\gamma_{Z_n}, n \geq 1$ are stationary as well, and thus each $\gamma_{Z_n}$ (resp. $\gamma_{Z_n}^\varepsilon$) has the same law as $\gamma$ (resp. $\gamma^\varepsilon$). By Corollary 4.8, the process $Z$ is ergodic and therefore Birkhoff’s ergodic theorem implies

$$\gamma = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \gamma_{Z_k}, \quad \gamma^\varepsilon = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \gamma_{Z_k}^\varepsilon$$

are almost surely constant. We can therefore assume that $\gamma$ and $\gamma^\varepsilon$ are deterministic. It follows that

$$\gamma_{x,y}^\varepsilon = \gamma^\varepsilon, \quad \gamma_{x,y} = \gamma$$

for $\mu \otimes \mu$-a.a. $(x, y) \in E \times E$. Observe that $\gamma^\varepsilon \geq \gamma$, and by assumption (i) of the theorem we have $\gamma > 0$. The same reasoning as the one we have used to prove (14) shows that

$$\Gamma^{n+1,\varepsilon} \geq \Gamma^{n,\varepsilon},$$

and clearly $\varepsilon \mapsto \Gamma^{n,\varepsilon}$ is non-decreasing. Hence there exist the limits

$$\Gamma^\varepsilon = \lim_{n \to \infty} \Gamma^{n,\varepsilon}, \quad \Gamma = \lim_{\varepsilon \to 0} \Gamma^\varepsilon.$$  

To show (13), we just need to show that $\Gamma = 1$. Observe that $\Gamma^{n,\varepsilon}$ equals the maximal probability of the event $\{d(X_n, Y_n) \leq \varepsilon\}$ over all couplings $Z = (X, Y)$ such that $\text{Law}(Z_0) = \mu \otimes \mu$ and the conditional distributions of $X, Y$ conditioned by $\sigma(Z_0)$ equal $\mathbb{P}_X, \mathbb{P}_Y$, respectively. Assuming $\Gamma < 1$, we will construct for any fixed $\varepsilon > 0$ and any $n$ large enough a coupling $Z = (X, Y)$ having the same properties as above such that

$$\mathbb{P}(d(X_n, Y_n) \leq \varepsilon) \geq \Gamma + \frac{(1 - \Gamma)\gamma}{2}. \quad (15)$$

This yields the contradictory inequality

$$\Gamma = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \Gamma^{n,\varepsilon} \geq \Gamma + \frac{(1 - \Gamma)\gamma}{2} > \Gamma,$$

and hence $\Gamma < 1$ is impossible. Note however that our proof does not show that $\gamma = 1$.

Fix $\gamma' \in (\gamma/2, \gamma)$ and $\Gamma' \in (0, \Gamma)$ close enough to $\Gamma$, so that

$$\Gamma' + (1 - \Gamma')\gamma' > \Gamma + \frac{(1 - \Gamma)\gamma}{2}.$$  

Then choose $\delta > 0$ small enough, so that

$$(1 - \delta)(\Gamma' - 2\delta) + \gamma'(1 - \Gamma') - \delta \geq \Gamma + \frac{(1 - \Gamma)\gamma}{2}. \quad (16)$$

Let us proceed with a preliminary analysis which will give us several auxiliary objects we will use in the construction below. First, since $\gamma_{x,y}^\varepsilon = \gamma^\varepsilon$ for $\mu \otimes \mu$-a.a. $(x, y) \in E \times E$, by the definition of $\gamma_{x,y}^\varepsilon$ we have that the time moment

$$T_{x,y}^\varepsilon = \min\{T : \gamma_{x,y}^{n,\varepsilon} > \gamma', \quad n \geq T\}$$
is finite for $\mu \otimes \mu$-a.a. $(x, y) \in E \times E$. Fix some $N$ such that

$$(\mu \otimes \mu)((x, y) : T_{x,y}^\varepsilon > N) < \delta,$$

and denote $O_N^\varepsilon = \{(x, y) : T_{x,y}^\varepsilon \leq N\}$.

Next, recall that $X$ is assumed to be $d$-Feller, and $d$ is continuous. Then for given $\varepsilon > 0$, $N \geq 1$ chosen above, and any compact set $K \subset E$

$$\gamma_{x,y}^N \to 1$$

when $d(x, y) \to 0, (x, y) \in K \times K$. Fix a compact set $K \subset E$ such that $\mu(K) > 1 - \delta$, and choose $\varepsilon_1 > 0$ such that

$$\gamma_{x,y}^N > 1 - \delta, \quad d(x, y) \leq \varepsilon_1, \quad (x, y) \in K \times K.$$

Finally, we observe that by the definition of $\Gamma^{\varepsilon_1} \geq \Gamma > \Gamma'$, there exists $N_0 \in \mathbb{N}$ such that $\Gamma^{\varepsilon_1}, N_0 \geq \Gamma'$. Hence for arbitrary $n \geq N_0$ there exists a coupling $Z^{n,\varepsilon_1} = (X^{n,\varepsilon_1}, Y^{n,\varepsilon_1})$ such that $\text{Law}(Z^{n,\varepsilon_1}_0) = \mu \otimes \mu$, the conditional distributions of $X^{n,\varepsilon_1}, Y^{n,\varepsilon_1}$ conditioned by $\sigma(Z^{n,\varepsilon_1})$ equal $\mathbb{P}_{X^{n,\varepsilon_1}_0}, \mathbb{P}_{Y^{n,\varepsilon_1}_0}$ respectively, and

$$\mathbb{P}\left(d(X^{n,\varepsilon_1}_n, Y^{n,\varepsilon_1}_n) \leq \varepsilon_1\right) \geq \Gamma'.$$

(17)

We modify this coupling by the same construction we have mentioned in Remark 4.5. Namely, denote the law of $Z$ by $\mathbb{P}^Z$ and consider the set

$$C = \left\{d(X^{n,\varepsilon_1}_n, Y^{n,\varepsilon_1}_n) \leq \varepsilon_1\right\}.$$

Then

$$\mathbb{P}^Z = \Gamma'\mathbb{P}^Z(\cdot|C) + (1 - \Gamma')\mathbb{Q}^{Z,\Gamma',C},$$

where

$$\mathbb{Q}^{Z,\Gamma',C} = (1 - \Gamma')^{-1}(\mathbb{P}^Z - \Gamma'\mathbb{P}^Z(\cdot|C))$$

is a probability measure on $E^\infty \times E^\infty$. Recall that the projections $\pi_1, \pi_2$ of $\mathbb{P}^Z$ equal $\mathbb{P}_\mu$, hence the projections of $\mathbb{Q}^{Z,\Gamma',C}$ are absolutely continuous w.r.t. $\mathbb{P}_\mu$ with their Radon-Nikodym derivatives $\leq (1 - \Gamma')^{-1}$. Taking instead of $\mathbb{P}^Z$ the measure

$$\Gamma'\mathbb{P}^Z(\cdot|C) + (1 - \Gamma')\pi_1(\mathbb{Q}^{Z,\Gamma',C}) \otimes \pi_2(\mathbb{Q}^{Z,\Gamma',C}),$$

we obtain a new coupling such that (17) for this coupling still holds true, but in addition the distribution conditioned by the complement to the set $C$ is absolutely continuous w.r.t. $\mathbb{P}_\mu \otimes \mathbb{P}_\mu$ with the Radon-Nikodym density bounded by $(1 - \Gamma')^{-1}$. With a slight abuse of notation which however does not cause misunderstanding, we denote this modified coupling by the same symbol $Z^{n,\varepsilon_1}$.

Now for an arbitrary $n \geq N_0 + N$ we construct the required coupling $Z$ such that (15) holds true. We define $Z$ as follows:

- the law of $Z_k, k \leq n - N$ is the same as the law of $Z^{n-N,\varepsilon_1}_k, k \leq n - N$ (recall that $n - N \geq N_0$ hence $Z^{n-N,\varepsilon_1}$ is well defined);

- the conditional law of $Z_{l+n-N}, l \geq 0$ w.r.t. $\sigma(Z_k, k \leq n - N)$ equals $\xi^{N,\varepsilon}_{X_{n-N}, Y_{n-N}}$, where $\xi^{n,\varepsilon}$ is the function constructed in Lemma 4.3.
To estimate the probability of the event $A = \{d(X_n, Y_n) \leq \varepsilon\}$, denote

$$B = \{d(X_{n-N}, Y_{n-N}) \leq \varepsilon_1\}, \quad C = \{Z_{n-N} \in K \times K\}, \quad D = \{Z_{n-N} \in O_N^\varepsilon\}.$$  

Observe that, when conditioned by $B \cap C$, the event $A$ has probability $\geq 1 - \delta$ because the components $X, Y$ start $\varepsilon_1$-close from the compact set $K$ and hence by the choice of $N$ they stay $\varepsilon$-close after the time $N$ with probability $\geq 1 - \delta$.

On the other hand, when conditioned by $\overline{B} \cap D$, event $A$ has probability $\geq \gamma'$ by the definitions of the set $O_N^\varepsilon$ and the event $D$, and according to our construction of the coupling $Z$. Therefore

$$\mathbb{P}(A) \geq (1 - \delta)\mathbb{P}(B \cap C) + \gamma'\mathbb{P}(\overline{B} \cap D).$$

Recall that each of the components $X, Y$ has law $\mathbb{P}_\mu$, hence

$$\mathbb{P}(B \cap C) \geq \mathbb{P}(B) - \mathbb{P}(X_{n-N} \not\in K) - \mathbb{P}(Y_{n-N} \not\in K) \geq \mathbb{P}(B) - 2\delta.$$  

Next, the law of $Z_{n-N}$ conditioned by $\overline{B}$ is absolutely continuous w.r.t. $\mu \otimes \mu$ with Radon-Nikodym density $\leq (1 - \Gamma')^{-1}$. Hence

$$\mathbb{P}(\overline{D} \mid \overline{B}) \leq (1 - (1 - \Gamma')^{-1}(\mu \otimes \mu)((E \times E) \setminus O_N^\varepsilon)) \leq (1 - \Gamma')^{-1}\delta$$  

and therefore

$$\mathbb{P}(\overline{B} \cap D) = \mathbb{P}(D \mid \overline{B})(1 - \mathbb{P}(B)) \geq \left(1 - (1 - \Gamma')^{-1}\delta\right)(1 - \mathbb{P}(B)).$$

Recall that $\mathbb{P}(B) = \Gamma'$, so we finally obtain

$$\mathbb{P}(A) \geq (1 - \delta)(\Gamma' - 2\delta) + \gamma'(1 - \Gamma') - \delta.$$  

By (16), this yields (15) and completes the proof. \hfill \square

**Proof of Theorem 2.5.** Like in the previous proof, it is sufficient to show that for any $\varepsilon > 0$ the constant $\gamma^\varepsilon$ constructed above equals 1. Fix $x_0$ in the (topological) support of $\mu$ and observe that by the e-chain property and the triangle inequality for the metric $d$ on $\mathcal{P}(E)$, for any $\varepsilon > 0$ there exists $r > 0$ such that

$$d(P_n(x, \cdot), P_n(y, \cdot)) \leq \kappa, \quad n \geq 0, \quad x, y \in B(x_0, r),$$  

where $B_d(x_0, r)$ is the open ball in $E$ w.r.t. $d$ with center $x_0$ and radius $r$. Note that for any $\varepsilon > 0$ and any coupling $\xi \in C(\mathbb{P}_x, \mathbb{P}_y)$,

$$\mathbb{E}^\xi d(X_n, Y_n) \geq \varepsilon \xi(d(X_n, Y_n) > \varepsilon) \geq \varepsilon(1 - \gamma_{x,y}^n),$$  

hence

$$\gamma_{x,y}^n \geq 1 - \frac{1}{\varepsilon} \mathbb{E}^\xi d(X_n, Y_n)$$  

Since

$$d\left(P_n(x, \cdot), P_n(y, \cdot)\right) = \min_{\xi \in C(\mathbb{P}_x, \mathbb{P}_y)} \mathbb{E}^\xi d(X_n, Y_n),$$  

combining (18) and (19) we get

$$\gamma_{x,y}^n \geq 1 - \frac{\kappa}{\varepsilon}, \quad n \geq 0, \quad x, y \in B(x_0, r).$$
Because \( B(x_0, r) \times B(x_0, r) \) has positive measure \( \mu \otimes \mu \) for any \( r > 0 \), and
\[
\liminf_{n \to \infty} \gamma_{x,y}^{n,\varepsilon} = \gamma^\varepsilon
\]
for \( \mu \otimes \mu \)-a.a. \((x, y)\), the above inequality yields
\[
\gamma^\varepsilon \geq 1 - \frac{\kappa}{\varepsilon}
\]
for any \( \kappa > 0 \); that is, \( \gamma^\varepsilon = 1 \).

Proof of Corollary 2.6. Let \( g : E \to \mathbb{R} \) be Lipschitz continuous w.r.t. \( d \). Then
\[
\left| \mathbb{E}_\mu [g(X_n) | X_j, j \leq 0] - \mathbb{E}_\mu g(X_0) \right| \leq \text{Lip}(g) d(P_n(X_0, \cdot), \mu).
\]
Since \( \text{Law}(X_0) = \mu \), (2) follows from (7) by the dominated convergence theorem (recall that we assume \( d \leq 1 \)).

For an arbitrary bounded \( g \), the usual approximation arguments can be applied since the time shift is an isometry on \( L^2(E^\infty, \mathbb{P}_\mu) \) and the class of \( d \)-Lipschitz continuous functions is dense in \( L^2(E, \mu) \).

5 Examples

In this section we give several examples which illustrate the conditions imposed in our main results and clarify the relations of these results with some other available in the field.

Example 5.1. This example shows that assumption (5) in Theorem 2.1 cannot be replaced by the assumption
\[
\limsup_{n \to \infty} \xi_{x,y} \left( d(X_n, Y_n) \leq \varepsilon \right) = 1,
\]
even if the chain is Feller and generalized couplings are replaced by couplings. Consider the torus \( E = [0, 1) \) equipped with the Euclidean metric \( d(x, y) = |y - x| \wedge (1 - |y - x|) \) and consider the deterministic map \( x \mapsto 2x \mod 1 \), \( \mu_1 = \delta_0 \), \( \mu_2 = \lambda \), where \( \lambda \) is the Lebesgue measure on \( E \). Both \( \delta_0 \) and \( \lambda \) are invariant and ergodic and for \( \lambda \)-almost all \( y \in E \) there exists a (deterministic) sequence along which the transition probabilities starting from \( y \) converge to \( \delta_0 \) weakly.

Example 5.2. This example shows that the assumptions of Theorem 2.1 or Corollary 2.2 do not guarantee weak convergence of transition probabilities. Take \( E = \{0, 1\} \) with transition probabilities \( p_{0,1} = p_{1,0} = 1 \). The assumptions hold with \( M = \{(0, 0)\} \) respectively \( M = \{0\} \) and \( \alpha = 1 \) and there exists a unique invariant measure \( \mu \) but the transition probabilities do not converge to \( \mu \). Note however that under the assumptions of Theorem 2.1 or Corollary 2.2, for any ergodic invariant measure \( \mu \) the \((\text{time-})\text{averaged}\) transition probabilities converge to \( \mu \) for \( \mu \)-almost all initial conditions \( y \in E \) by the ergodic theorem.

Example 5.3. This example shows that Theorem 2.7 fails if (8) is replaced by a corresponding averaged limit. Consider a deterministic dynamics on an unbounded countable subset \( E = \{0, a_1, a_2, \ldots\} \) of \([0, \infty)\) which maps 0 to 0 and \( a_1 \to a_2 \to \ldots \). Choosing the sequence such that it has only 0 as an accumulation point we can ensure that the chain is Feller. Clearly, \( \delta_0 \) is an invariant measure. On the other hand, if the average of the first \( n \) members of the sequence converges to 0 then for every \( x = a_j \in E \) and \( y = 0 \) the (deterministic) coupling \( X_n = a_{j+n}, Y_n = 0, n \geq 0 \) satisfies the averaged analogue of (8). However, if \( a_n \not\to 0 \), we have \( P_n(x, \cdot) \not\Rightarrow \delta_0 \) for each \( x \neq 0 \).
Example 5.4. This example shows that an ergodic Feller chain is not necessarily an e-chain. Consider a deterministic dynamics on the unbounded countable set $E = \{0\} \cup \{2^{-k}, k \geq 0\}$ which maps 0 to 0, 1 to 0, and $2^{-k}$ to $2^{-k+1}$, $k \geq 1$. Clearly, the chain is Feller and for every $x \in E$, $P_n(x, \cdot)$ converge as $n \to \infty$ to the unique invariant measure $\mu = \delta_0$ – even in the total variation distance. However, for any two points $x, y \in E \setminus \{0, 1\}$ with, say, $x \succ y$ there exists $n \geq 1$ such that $2^n x = 1, 2^ny \in (0, 1/2]$, and therefore

$$
\sup_n d(P_n(x, \cdot), P_n(y, \cdot)) \geq \frac{1}{2}.
$$

On the other hand, for any $\delta > 0$ there exist $k, m$ large enough so that $d(2^{-k}, 2^{-m}) < \delta$. That is, this chain is not an e-chain. Note that this example is also not asymptotically strong Feller (see [10] for the definition of this concept). The example does however satisfy the assumptions of all theorems in Section 2.

Example 5.5. This example shows that Theorem 2.7 may fail if the assumption $\pi_1(\xi_{x,y}) \sim \mathbb{P}_x$ is omitted, and (8) holds true just for $\xi_{x,y} \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)$. Consider $E = \{0, 1, 2, \ldots\}$ with transition probabilities $p_{0,0} = 1$ and $p_{i,i-1} = 1/3$ and $p_{i,i+1} = 2/3$ for $i = 1, 2, \ldots$. Clearly, $\mu = \delta_0$ is the unique invariant measure, transition probabilities $P_n(x, \cdot)$ do not converge to $\mu$ for $x \neq 0$ and for each $x \in \mathbb{N}$ there exists some $\xi \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_0)$ such that $X_n \to 0$ almost surely under $\xi$.

Example 5.6. This final example clarifies the relation between condition (8) and the condition

$$
\xi_{x,y}(\lim_{n \to \infty} d(X_n, Y_n) = 0) > 0,
$$

which was used in [11, Theorem 3.1]. Namely, we show that the “convergence in probability” type assumption (8) is strictly weaker than the “convergence with positive probability” one (21). Since this difference may not be too crucial, in order not to overburden the exposition we just outline the construction and omit detailed proofs.

Let $E = \{0, 1\} \times \{-1, 1\}$ be equipped with the metric $d((u, i), (v, j)) = \bar{d}(u, v) + |j - i|$, where $\bar{d}$ denotes the Euclidean metric on the torus $T = [0, 1)$ and let $r \in (0, 1) \setminus \mathbb{Q}$. Define a Markov operator $P$ on $E$ as follows: for any $x = (u, i) \in E$,

$$
P((u, i), \{(u + r \mod 1, i)\}) = 1/2$$

and

$$
P((u, i), \{(u, -i)\}) = 1/2.
$$

It is clear that $P$ is Feller, and there exists at least one invariant probability measure, namely $\mu = \lambda \otimes \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right)$, where $\lambda$ denotes Lebesgue measure on $T$.

In the following we distinguish between components and coordinates, the former referring to the first or second element of a pair $(X, Y) \in E^\infty \times E^\infty$, and the latter referring to the first or second element of a point $x = (u, i) \in E$. For any generalized coupling $\xi_{x,y} \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)$ the distance of the two components remains constant as long as their second coordinates are the same, and if the second coordinates differ the distance is at least two. Therefore the only way the distance of the two components can converge to zero is that they coincide eventually. Hence for any $x = (u, i), y = (v, j)$ such that $u - v$ is not an integer multiple of $r$ (mod 1), it is clear that there is no $\xi \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)$ for which the distance between the two components converges to zero with positive probability. That is, there are no sets $M_1, M_2 \in \mathcal{E}$ of positive $\mu$-measure such that for each $(x, y) \in M_1 \times M_2$ there exists some $\xi_{x,y} \in \tilde{C}(\mathbb{P}_x, \mathbb{P}_y)$ satisfying (21).
On the other hand, for any \( x, y \in E \) we can find a coupling \( \xi_{x,y} \in C(\mathbb{P}_x, \mathbb{P}_y) \) such that (8) holds (and hence \( P_n(x, \cdot) \to \mu \) for any \( x \)). Fix \( x, y \in E \) and define \( \xi_{x,y} \in C(\mathbb{P}_x, \mathbb{P}_y) \) as a Markov chain \( \{(X_n, Y_n), n \geq 0\} \) defined as follows with the function \( p(z), z \in [0, 1) \) yet to be determined:

- if the second coordinates of \( X_n, Y_n \) differ, then with probability 1/2, \( X_{n+1} \) changes the second coordinate and \( Y_{n+1} \) doesn’t and the same holds for \( X \) and \( Y \) interchanged, so in both cases the second coordinates of \( X_{n+1}, Y_{n+1} \) coincide;

- if the second coordinates of \( X_n, Y_n \) coincide, and the difference (mod 1) between the first coordinates of \( X_n, Y_n \) is \( z \in [0, 1) \), then the second coordinates of \( X_{n+1} \) and \( Y_{n+1} \) either change or stay the same simultaneously, with the probability of each of these two possibilities 1/2(1 – \( p(z) \)), and the probabilities that the second coordinate of \( X_{n+1} \) (resp. \( Y_{n+1} \)) changes while \( Y_{n+1} \) (resp. \( Y_{n+1} \)) doesn’t, are equal 1/2\( p(z) \).

By construction, if at some moment the second coordinates differ, they become equal immediately afterwards. Consider the sequence \( \{Z_n\} \) of differences of the first coordinates of \( \{(X_n, Y_n)\} \). If \( Z_n = z \) and the second coordinates of \( X_n \) and \( Y_n \) coincide, then \( Z \) will keep taking the value \( z \) for a geometric number of steps with expected value \( 1/p(z) \), then the second coordinates of \( X, Y \) will be different for one time unit after which they become the same again and \( Z \) takes the values \( z, z + 2r \), and \( z - 2r \) with probabilities \( 1/2, 1/4, \) and \( 1/4 \) respectively. It is not hard to see (and easy to believe) that if the continuous function \( z \mapsto p(z) \) is chosen such that \( p(0) = 0, p(z) > 0 \) for \( z \neq 0 \) and \( p(z) \) approaches 0 as \( z \to 0 \) sufficiently fast, then both \( Z_n \) and the indicator \( 1_{X_n \neq Y_n} \) will converge to 0 in probability since \( Z_n \) is very likely to take a value close to 0 when \( n \) is large.

6 Applications: SFDEs and SPDEs

In this section we illustrate our main results applying them to stochastic functional differential equations (SFDEs) and stochastic partial differential equations (SPDEs).

6.1 Stochastic delay equations

Denote \( C := C([-1, 0], \mathbb{R}^m) \), and for a function or a process \( X \) defined on \([-1, t] \) write \( X_t(s) := X(t+s), s \in [-1, 0] \). Consider the SFDE

\[
\begin{align*}
\mathrm{d}X(t) &= F(X_t) \, \mathrm{d}t + G(X_t) \, \mathrm{d}W(t), \\
X_0 &= f \in C,
\end{align*}
\]

where \( F : C \to \mathbb{R}^m \) and \( G : C \to \mathbb{R}^{m \times m} \) satisfy a global Lipschitz condition with respect to the supremum norm and \( W \) is a standard Wiener process in \( \mathbb{R}^m \). Assume the non-degeneracy condition

\[
\sup_{f \in C} |G^{-1}(f)| < \infty,
\]

where \( G^{-1}(f) \) denotes the generalized (or Moore-Penrose) inverse matrix of \( G(f), f \in C \).

This model was well studied in [11], where it was proved that the \( C \)-valued solution process \( X_t, t \geq 0 \) is uniquely defined, is a Feller process, and has at most one invariant probability measure \( \mu \) in which case
all transition probabilities converge to $\mu$ weakly (for ease of exposition we have imposed slightly stronger assumptions on $F$ and $G$ compared to [11]).

Here we use this model to benchmark our results. Namely, we will show that these results can be applied yielding the same conclusions, but in a considerably easier and more straightforward way.

Like in [11], we fix a pair of initial conditions $f$ and $g$ in $C$, and consider the pair of equations

$$dX(t) = F(X_t)\, dt + G(X_t)\, dW(t), \quad X_0 = f,$$
$$dY(t) = F(Y_t)\, dt + \lambda (X(t) - Y(t))\, dt + G(Y_t)\, dW(t), \quad Y_0 = g.$$  

It is shown in [11], Section 3 that if $\lambda > 0$ is sufficiently large (when compared with the Lipschitz constants for $f, g$), then with probability 1

$$|X(t) - Y(t)| \to 0, \quad t \to \infty$$

exponentially fast, and thus

$$\int_0^\infty |X(t) - Y(t)|^2 \, dt < \infty$$  \hspace{1cm} (25)

(the proofs are not very long and are based on basic stochastic calculus arguments). Observe that the equation for $Y$ can be re-written to the form

$$dY(t) = F(Y_t)\, dt + \lambda (X(t) - Y(t))\, dt + G(Y_t)\, \widehat{dW}(t), \quad Y_0 = g$$  \hspace{1cm} (26)

with

$$\widehat{W}(t) = W(t) + \int_0^t \beta_s \, ds, \quad \beta_t := \lambda (X(t) - Y(t))G^{-1}(Y_t).$$

Combining (25) with (24), we see that

$$\int_0^\infty \beta_t^2 \, dt < \infty$$

with probability 1. Then by the Girsanov theorem the law of $\widehat{W}$ on $C([0, \infty), \mathbb{R}^m)$ is absolutely continuous w.r.t. the law of the Wiener process $W$; cf. [15], Theorem 7.4. Because $Y$ is the strong solution to (26), this yields immediately that the law of of $Y(t), t \in [-1, \infty)$ is absolutely continuous with respect to the law of the solution to (22) with initial condition $g$. On the other hand, $X$ is just the solution to (22) with initial condition $f$, hence the joint law $\xi$ of the pair $X, Y$ is a generalized coupling from the class $\overline{C}(\mathbb{P}_f, \mathbb{P}_g)$ which satisfies the additional condition $\pi_1(\xi) = \mathbb{P}_f$. Applying the continuous-time version of Corollary 2.8, we directly obtain weak convergence of all transition probabilities to the unique invariant probability measure (in the case it exists).

We note that the simple construction explained above can not be applied directly within the approach developed in [11]. Theorem 3.1 in [11], which provides uniqueness of the invariance measure, exploits a generalized coupling which belongs to the class $\overline{C}(\mathbb{P}_f, \mathbb{P}_g)$. It is difficult to guarantee the equivalence of the law of $Y$ to $\mathbb{P}_g$ using just the Girsanov theorem; this is the reason why in the proof of uniqueness in [11] a more sophisticated construction of the generalized coupling is used which involves localization in time. The proof of Theorem 3.7 in [11], which states the weak convergence of transition probabilities to the invariant measure, contains an extra analysis which actually shows that $X$ is an e-process. None of these additional considerations are required in our approach. This is a clearly seen advantage, which makes it possible to extend the uniqueness results to asymptotic stability (almost) for free. Below we show that such a possibility is quite generic and is available as well in SPDE setting.
6.2 SPDEs

In this section we show that in each of the five SPDE models studied in [8] only a minor modification of the constriction of a generalized coupling allows us to apply Corollary 2.8 and thus to obtain the asymptotic stability of the model rather than just unique ergodicity. Such a drastic improvement becomes possible thanks to Theorem 2.3 and Theorem 2.7, and illustrates the usefulness of these results. To simplify the cross-references, within this section we mainly adopt the notation from [8] even if it does not correspond to the notation introduced in Section 2.1. The methodology will be similar for all the five models, hence we explain most details for the first one and then just sketch the argument for the other four. Throughout this section we denote by $H$ the Sobolev classes $H^2(D)$ with a domain $D$ which varies from model to model. The $L^2$-norm and the $H^1$-norm are denoted $| \cdot |$ and $\| \cdot \|$ respectively, for all other norms are indicated explicitly. We also denote by $\lambda_n, n \geq 1$ the increasingly enumerated eigenvalues of an operator $A$, which will be specified in each model separately, and by $P_N$ the projector onto the span of the respective first $N$ eigenvectors.

6.2.1 2D Navier-Stokes on a domain

Consider the 2D stochastic Navier-Stokes equation on $D \subset \mathbb{R}^2$

$$\begin{align*}
\frac{du}{dt} + u \cdot \nabla u &= (\nu \Delta u + \nabla \pi + f)dt + \sum_{k=1}^{m} \sigma_k dW_k, \quad \nabla \cdot u = 0, \quad (27)
\end{align*}$$

with the unknown velocity field $u = (u_1, u_2)$ and the unknown pressure $\pi$. The bounded domain $D$ is assumed to have smooth $\partial D$, and the no-slip (Dirichlet) boundary condition on $u$ is imposed:

$$u|_{\partial D} = 0. \quad (28)$$

The deterministic vector fields $f, \sigma_1, \ldots, \sigma_m \in L^2(D)^2$ and independent standard Brownian motions $W_1, \ldots, W_m$ are fixed.

Denote by $V$ the subspace of $H^1(D)^2$, which contains $u$ such that $\nabla \cdot u = 0$ and $u \cdot n = 0$ (where $n$ denotes the outward normal for $\partial D$). Denote by $H$ the completion of $V$ w.r.t. the $L^2(D)^2$-norm, by $P_H$ the projector in $L^2(D)^2$ on $H$, and by $A = -P_H \Delta$ the Stokes operator.

It is known that for any $u_0 \in H$ the system (27), (28) with the initial data $u_0 \in H$ admits a unique (strong) solution with values in $H$, and this solution depends continuously on $u_0 \in H$. That is, (27), (28) defines a Feller Markov process valued in $H$; we refer for details to [8], Section 3.1.1.

Now we explain the generalized coupling construction for this system. Fix arbitrary $u_0, \tilde{u}_0 \in H$ and consider $u = u(\cdot, u_0)$ solving (27), (28) with initial data $u_0$, and $\tilde{u}$ solving

$$\begin{align*}
\frac{d\tilde{u}}{dt} + \tilde{u} \cdot \nabla \tilde{u} dt &= (\nu \Delta \tilde{u} + \lambda P_N(u - \tilde{u}) + \nabla \omega + f)dt + \sum_{k=1}^{m} \sigma_k dW_k, \quad \nabla \cdot \tilde{u} = 0,
\end{align*}$$

with initial data $\tilde{u}_0$; here $\lambda$ and the number $N$ are yet to be chosen; recall that $P_N$ is the projector which is defined in the terms of the operator $A$.

This construction is based on the “stochastic control” argument, similar to the one developed in [11], Section 3; see also [8], Section 2.4. The similar coupling construction in [8], Section 3.1.2 involves an
additional localization term \(1_{\tau_K > t}\), and the corresponding generalized coupling is defined as the conditional law of the pair \((u, \tilde{u})\) on the set \(\{\tau_K = \infty\}\). This gives a generalized coupling from the class \(\mathcal{C}(P_u, \mathbb{P}_u)\). Because of the conditioning, the law of the first component have no reason to be equivalent to \(P_u\). The latter condition is however crucial for our Theorem 2.7; see Remark 2.9 and Example 5.5. We resolve this difficulty in a similar way we did in Section 6.1. Namely, we remove the localization term and consider the law of the pair \((u, \tilde{u})\) as the required generalized coupling. This leads only to minor modifications in the respective calculus, as we explain below, but it allows to apply our main results in order to derive asymptotic stability.

The difference \(v := u - \tilde{u}\) satisfies

\[
dv - \nu \Delta v dt + 1_{\tau > t} \lambda P_N v dt = -\nabla \pi + \nabla \omega + \nabla u \cdot u + \tilde{u} \cdot \nabla \tilde{u}, \quad \nabla \cdot v = 0, \quad v_{|\partial D} = 0.
\]

(29)

Like in [8], Section 3.1.2, multiplying (29) by \(v\), integrating over \(D\), and using that \(u, \tilde{u}, v\) are all divergence-free and satisfy the Dirichlet boundary condition, one gets

\[
\frac{1}{2} d|v|^2 + \nu|v|^2 dt + \lambda |P_N v|^2 dt \leq \int_D v \cdot \nabla u \cdot v dx dt
\]

\[
\leq C_D |v||v||\nabla u||dt \leq \left(\frac{\nu}{2} |v|^2 + \frac{\nu C_D}{2\nu} |v|^2 |\nabla u|^2\right) dt
\]

with a universal constant \(C_D\) which involves the quantities from Sobolev embedding. By the Poincaré inequalities [8] (3.3), for the particular choice \(\lambda = \nu \lambda_N/2\) we get

\[
\lambda |P_N v|^2 + \frac{\nu}{2} |v|^2 \geq \lambda |v|^2; \quad \lambda \leq \nu \lambda_N/2.
\]

Taking \(\lambda = \nu \lambda_N/2\) we obtain

\[
d|v|^2 \leq \left(-\nu \lambda_N 1_{\tau > t} + \frac{C_D}{\nu} |\nabla u|^2\right) |v|^2 dt,
\]

and finally by Gronwall’s lemma

\[
|v(t)|^2 \leq |u_0 - \tilde{u}_0|^2 \exp\left(-\nu \lambda_N t + \frac{C_D}{\nu} \int_0^t |u(s)|^2 ds \right), \quad t \geq 0.
\]

(30)

One has with probability 1

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t |u(s)|^2 ds \leq \frac{|A^{-1/2} f|^2}{2\nu^2} + \frac{\sigma^2}{\nu^2},
\]

(31)

where \(\sigma^2 := \sum_{k=1}^m |\sigma_k|^2\); this follows from the energy estimate [8] (3.5). Hence \(N\) satisfies

\[
\lambda_N > C_D \left(\frac{|A^{-1/2} f|^2}{2\nu^4} + \frac{\sigma^2}{\nu^3}\right),
\]

(32)

with probability 1 the right hand side term in (30) tends to 0 exponentially fast.

On the other hand, consider \(\sigma\) as a linear operator \(\mathbb{R}^m \to H\) and assume that, for the given \(N\),

\[
H_N := P_N H \subset \text{Range}(\sigma) = \text{Span}(\sigma_k, k = 1, \ldots, m).
\]

(33)
Then the corresponding pseudo-inverse operator $\sigma^{-1} : H_N \to \mathbb{R}^m$ is well defined and bounded. Then the principal equation for $\tilde{u}$ can be written in the form

$$d\tilde{u} + \tilde{u} \cdot \nabla \tilde{u} \, dt = (\nu \Delta \tilde{u} + \nabla \varpi + f) \, dt + \sum_{k=1}^{m} \sigma_k d\tilde{W}_k$$

with

$$\tilde{W}(t) = W(t) + \int_0^t \beta_s \, ds, \quad \beta_t := \lambda \sigma^{-1} P_N v(t)$$

Since $\sigma^{-1}$ is bounded and $|v(t)|$ tends to 0 exponentially fast, we have

$$\mathbb{P} \left( \int_0^t \|\beta_s\|_{\mathbb{R}^m}^2 \, ds < \infty \right) = 1,$$

and the law of $\tilde{W}$ is absolutely continuous w.r.t. the law of $W$. Thus the law of $\tilde{u}$ is absolutely continuous w.r.t. the law of the solution to (27), (28) with initial data $\tilde{u}_0$. Note that the law of the first component w.r.t. this coupling just equals $\mathbb{P}_{u_0}$ and the distance between the components tend to 0 exponentially fast as $t \to \infty$. Hence the law of the pair $(u(\cdot), \tilde{u}(\cdot))$ can be used as the coupling required in Corollary 2.8 with $E = H$, $\rho = d = |\cdot| - |\cdot| \wedge 1$. We conclude that in the framework of Proposition 3.1 [8], which states unique ergodicity for (27), (28), the following stabilization property actually holds true:

"For any $u \in H$, the transition probabilities $P_t(u, \cdot) \in \mathcal{P}(H)$ weakly converge as $t \to \infty$ to the unique invariant measure."

### 6.2.2 2D Hydrostatic Navier-Stokes Equations

Next, following [8] Section 3.2, we consider a stochastic version of the 2D Hydrostatic Navier-Stokes equation

$$du + (u \partial_x u + w \partial_z u + \partial_x p - \nu \Delta u) \, dt = \sum_{k=1}^{m} \sigma_k dW_k,$$

$$\partial_x p = 0,$$

$$\partial_x u + \partial_z w = 0$$

for an unknown velocity field $(u, w)$ and pressure $p$ evolving on the domain $D = (0, L) \times (h, 0)$. The boundary $\partial D$ is decomposed into its vertical sides $\Gamma_v = [0, L] \times \{0, h\}$ and lateral sides $\Gamma_l = \{0, L\} \times [h, 0]$, where the boundary conditions are imposed:

$$u = 0 \quad \text{on} \ \Gamma_l, \quad \partial_x u = w = 0 \quad \text{on} \ \Gamma_v.$$  \hspace{1cm} (36)

Denote

$$H = \left\{ f \in L_2(D) : \int_{-h}^{0} u \, dz \equiv 0 \right\}, \quad V = \left\{ u \in H^1(D) : \int_{-h}^{0} u \, dz \equiv 0, \ u|_{\Gamma_l} = 0 \right\}.$$ 

Denote also by $P_H$ the projector in $L_2(D)$ on $H$, and put $A = -P_H \Delta$.

It is known (see [8], Section 3.2.1) that under a proper condition on the family $\{\sigma_k\}$ for a given $u_0 \in V$ the system (35), (36) has a unique strong solution, which in addition depends continuously on $u_0 \in V$. Thus the system (35), (36) defines a Feller Markov process in $E = V$.  

\[24\]
Now we explain the generalized coupling construction for (35), (36). For fixed \( u_0, \tilde{u}_0 \in V \), consider the solution \( u \) to (35), (36) with the initial data \( u_0 \) and the solution \( \tilde{u} \) to a similar system with the first equation changed to
\[
d\tilde{u} + (\tilde{u}\partial_x \tilde{u} + w\partial_z \tilde{u} + \partial_z \tilde{p} - \nu \Delta \tilde{u} + \lambda P_N (u - \tilde{u})) \, dt = \sum_{k=1}^m \sigma_k dW_k
\]
with \( \lambda = \nu \lambda_N / 2 \). One has
\[
|v(t)| \leq \exp \left( -2\lambda t + C \int_0^t \left( \|u(s)\|^2 + |\partial_z u(s)||\partial_z u(s)| \right) \, ds \right) |v(0)|, \quad t \geq 0
\] (37)
with a constant \( C \) depending only on \( \nu \) and \( D \); see (3.22), [8]. Next, there exists \( C_1 \) depending only on \( \nu, D \), and \( |\sigma|^2 + |\partial_z \sigma|^2 \) such that
\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \left( \|u(s)\|^2 + |\partial_z u(s)|^2 \right) \, ds \leq C_1
\] (38)
with probability 1; this follows from the energy estimates (3.16), (3.18) [8]. If \( N \) is large enough, so that \( 2\lambda = \nu \lambda_N > CC_1 \), the above inequalities yield that the \( H \)-norm \( |v(t)| \) tends to zero as \( t \to \infty \) exponentially fast. If in addition for such \( N \) (33) holds true, then one can interpret \( \tilde{u} \) as the solution to (35), (36) with the initial data \( \tilde{u}_0 \) and \( W \) changed to
\[
\tilde{W}(t) = W(t) + \int_0^t \beta_s \, ds, \quad \beta_t := \lambda \sigma^{-1} P_N v(t).
\]
Since the pseudo-inverse operator \( \sigma^{-1} : H_N \to \mathbb{R}^m \) is bounded and \( |v(t)| \) decays exponentially fast, we have (34). Hence the law of \( \tilde{W} \) is absolutely continuous w.r.t. the law of \( W \) and therefore the law of \( \tilde{u} \) in \( C((0, \infty), V) \) is absolutely continuous w.r.t. \( \mathbb{P}_{\tilde{u}_0} \). Recall that the Markov process which corresponds to (35), (36) is well defined and is Feller on \( V \). However, it is an easy observation that this process is \( H \)-Feller, as well. Namely, inequality (37) actually holds true for any \( \lambda \leq \nu \lambda_N / 2 \), and taking \( \lambda = 0 \) we easily deduce the \( H \)-continuity of the semigroup.

We take \( E = V, \rho = \| \cdot \| \land 1, d = | \cdot | \land 1 \); note that condition (4) holds true with \( \rho_n'(x) = |P_n(y - x)|, n \geq 1, y \in E \). In this setting, we apply continuous time version of Corollary 2.8 with the generalized coupling \( \xi \) defined as the joint law of processes \( u(\cdot), \tilde{u}(\cdot) \) defined above. We conclude that in the framework of Proposition 3.2 [8], which states unique ergodicity for (35), (36), in addition the following (weak) \( L_2 \)-stabilization property holds:

for any \( u \in V \), the transition probabilities \( P_t(u, \cdot) \in \mathcal{P}(V) \) weakly converge in the \( L_2 \)-topology as \( t \to \infty \) to the unique invariant measure \( \mu \in \mathcal{P}(V) \).

### 6.2.3 The fractionally dissipative Euler model

Next, following [8] Section 3.3, we consider the fractionally dissipative Euler model, described by the system
\[
d\xi + (\Lambda^\gamma \xi + u \cdot \nabla \xi) \, dt = \sum_{k=1}^m \sigma_k dW_k, \quad u = \mathcal{K} * \xi
\] (39)
for an unknown vorticity field \( \xi \) (this is the notation borrowed from [8], which is not to be mixed with the notation for a coupling we used previously). Here \( \Lambda^\gamma = (-\Delta)^\gamma \) is the fractional Laplacian with \( \gamma \in (0,2] \), \( K \) is the Biot-Savart kernel, so that \( \nabla_\perp \cdot u = \xi \) and \( \nabla u = 0 \), and (39) is posed on the periodic box \( \mathbb{T}^2 = [\pi, \pi]^2 \). In the velocity formulation, (39) has the form

\[
du + \left( \Lambda^\gamma u + u \cdot \nabla u + \nabla \pi \right) dt = \sum_{k=1}^{m} \sigma_k dW_k, \quad \nabla \cdot u = 0,
\]

where the unknowns are the velocity field \( u \) and the pressure \( \pi \). It is known that for a fixed \( r > 2 \) for any given \( u_0 \in H^r \) there exists a unique strong solution to (40) taking values in \( H^r \) and this solution depends on the initial data \( u_0 \in H^r \) continuously. That is, (40) defines a Feller Markov process valued in \( H^r \); see [8], Section 3.3.1.

For fixed \( u_0, \bar{u}_0 \in H^r \), consider the function \( u(\cdot) \) solution to (40) with the initial data \( u_0 \) and the function \( \bar{u}(\cdot) \) solving

\[
d\bar{u} + \left( \Lambda^\gamma \bar{u} - \lambda P_N(u - \bar{u}) + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{\pi} \right) dt = \sum_{k=1}^{m} \sigma_k dW_k, \quad \nabla \cdot \bar{u} = 0
\]

with the initial data \( \bar{u}_0 \) and \( P_N \) which now denotes the projector which corresponds to the eigenfunctions of \( A = \Lambda^\gamma \). This is actually the generalized coupling construction from [8], Section 3.3.2, where in the additional control term we remove the localization term \( 1_{x_K > t} \). Denote \( \nu = u - \bar{u} \), then for \( \lambda \leq \lambda_N/2 \)

\[
|\nu(t)|^2 \leq \exp \left( -2\lambda t + C \int_0^t \| \xi(s) \|_{L_p}^2 \, ds \right) |\nu(0)|^2, \quad t \geq 0
\]

with properly chosen \( p > 1 \) and universal \( C \); see [8], (3.28). On the other hand, there exists a universal \( C_1 \) such that

\[
\limsup_{t \to \infty} \frac{1}{t} \int_0^t \| \xi(s) \|_{L_p}^2 \, ds \leq C_1 \| \sigma \|_{L_p}^2
\]

with probability 1, where

\[
\| \sigma \|_{L_p} = \left( \int_{\mathbb{T}^2} \left( \sum_{k=1}^{m} \sigma_k^2 \right)^{p/2} \, dx \right)^{1/p}.
\]

This follows from the energy estimate (3.31) [8].

Now we can repeat literally the argument from the previous subsection. Taking in the above calculation \( \lambda = 0 \), we see that the Markov process is \( H \)-Feller with \( H = L_2(\mathbb{T}^2) \). Taking \( \lambda = \lambda_N/2 \), we get that if, for some \( N, \lambda_N > CC_1 \) and (33) holds, then the law of the pair \( (u(\cdot), \bar{u}(\cdot)) \) can be used as the coupling required in Corollary 2.8 with \( E = H^r, \rho = \| \cdot \|_{H^r} \land 1, d = \| \cdot \|_{H^r} \land 1 \) (again, condition (4) is easy to verify). We conclude that in the framework of Proposition 3.3 [8], which states unique ergodicity for (40), in addition the following (weak) \( L_2 \)-stabilization property holds:

\textit{for any} \( u \in H^r \), transition probabilities \( P_t(u, \cdot) \in \mathcal{P}(H^r) \) weakly converge in the \( L_2 \)-topology as \( t \to \infty \) to the unique invariant measure \( \mu \in \mathcal{P}(H^r) \).
6.2.4 The damped stochastically forced Euler-Voigt model

Next, we consider an inviscid “Voigt-type” regularization of a damped stochastic Euler equation:

\[
du + \left( \gamma u + u_\alpha \cdot \nabla u_\alpha + \nabla p \right) dt = \sum_{k=1}^{m} \sigma_k dW_k, \quad \nabla \cdot u = 0,
\]

with for some \( \gamma > 0 \) and the unknown vector field \( u \), where the non-linear terms are subject to an \( \alpha \)-degree regularization

\[
(-\Delta)^{\alpha/2} u_\alpha = \Lambda^\alpha u_\alpha = u.
\]

The absence of a parabolic regularization mechanism brings specific difficulties to the analysis of the model, we refer to [8], Sections 3.4.1 – 3.4.3 for details. Surprisingly, the construction of the generalized coupling which leads to the stability of the model does not bring substantial novelties and can be provided within the same lines we discussed previously. To shorten the exposition, we consider only the case of a 2D model evolving on the periodic box \( \mathbb{T}^2 \), and assume \( \alpha > 2/3 \). In this case, Proposition 3.4, [8] shows that for any \( u_0 \in H^{1-\alpha/2} \) there exists unique strong solution to (42) with the initial data, and the corresponding semigroup is Feller w.r.t. \( H^{-\alpha/2} \) norm.

Next, for fixed \( u_0, \tilde{u}_0 \in H^{1-\alpha/2} \), consider the function \( u(\cdot) \) solution to (42) with the initial data \( u_0 \) and the function \( \tilde{u}(\cdot) \) solving

\[
d\tilde{u} + \left( \gamma u - \lambda P_N(u - \tilde{u}) + \tilde{u}_\alpha \cdot \nabla \tilde{u}_\alpha + \nabla \tilde{p} \right) dt = \sum_{k=1}^{m} \sigma_k dW_k, \quad \nabla \cdot u = 0
\]

with the initial data \( \tilde{u}_0 \); now \( P_N \) denotes the projector on the span of \( N \) first elements in the sinusoidal basis. Since \( \alpha > 2/3 \), there exists \( \delta > 0 \) such that \( H^{\alpha/2-\delta} \subset L^3 \). For such \( \delta \), inequalities (3.45), (3.46) [8] provide the following bound for \( v = u - \tilde{u} \):

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{H^{-\alpha/2}}^2 + \left( \gamma - C \left( \lambda^{-1} + N^{-\delta} \right) \left( 1 + \| \xi \|_{H^{-\alpha/2}}^2 \right) \right) \| v \|_{H^{-\alpha/2}}^2 dt \leq 0,
\]

where \( \xi = \text{curl } v \) and constant \( C \) depends only on \( \delta, \alpha \). On the other hand, with probability 1

\[
\limsup_{t \to \infty} \frac{\gamma}{t} \int_0^t \| \xi(s) \|_{H^{-\alpha/2}}^2 ds \leq \| \xi \|_{H^{-\alpha/2}}^2
\]

with \( \xi = \text{curl } \sigma \); see [8], Section 3.4.1. Hence, if \( \lambda, N \) are taken large enough, \( \| v(t) \|_{H^{-\alpha/2}}^2 \) tends to 0 as \( t \to \infty \) exponentially fast. Repeating literally the same arguments as before we obtain that, if \( H_N \subset \text{Range}(\sigma) \), the law of the pair \((u(\cdot), \tilde{u}(\cdot))\) can be used as the coupling required in Corollary 2.8 with \( E = H^{1-\alpha/2}, \rho = \| \cdot \|_{H^{1-\alpha/2}} \land 1, d = \| \cdot \|_{H^{1-\alpha/2}} \land 1 \) (again, condition (4) is easy to verify). We conclude that in the framework of Proposition 3.4 [8], which states unique ergodicity for (42), in addition the following (weak) \( H^{-\alpha/2} \)-stabilization property holds:

for any \( u \in H^{1-\alpha/2} \), the transition probabilities \( P_t(u, \cdot) \in \mathcal{P}(H^{1-\alpha/2}) \) weakly converge in the \( H^{-\alpha/2} \)-topology as \( t \to \infty \) to the unique invariant measure \( \mu \in \mathcal{P}(H^{1-\alpha/2}) \).
6.2.5 The damped nonlinear wave equation

Finally, we consider the damped Sine-Gordon equation which is written as the system of stochastic partial differential equations

\[ dv + \left( \alpha v - \Delta u + \beta \sin(u) \right) dt = \sum_{k=1}^{m} \sigma_k dW_k, \quad du = v dt, \]  

(43)

where the unknown \( u \) evolves on a bounded domain \( D \subset \mathbb{R}^n \) with smooth boundary, and satisfies the Dirichlet boundary condition \( u|_{\partial D} \equiv 0 \). The parameter \( \alpha \) is strictly positive and \( \beta \) is a real number.

It is known (see [8], Section 3.5.1) that for any initial data \( U_0 = (u_0, v_0) \in X := H^1_0(D) \times L^2(D) \) there exists a unique strong solution to (43), and moreover (43) defines a Feller Markov process in \( X \). In this final example the generalized coupling construction proposed in [8], Section 3.5.2 is already well adapted for our purposes. Within this construction, they put

\[ d\tilde{v} + \left( \alpha \tilde{v} - \Delta \tilde{u} + \beta \sin(\tilde{u}) - \beta_1 \tau_K > t P_N(\sin(u) - \sin(\tilde{u})) \right) dt = \sum_{k=1}^{m} \sigma_k dW_k, \quad du = v dt, \]

with the initial data \( (\tilde{u}_0, \tilde{v}_0) \) and \( \tau_K = \inf \left\{ t : \int_0^t \left| u(t) - \tilde{u}(s) \right|^2 ds \geq K \right\} \).

They prove that for \( N, K \) sufficiently large \( \tau_K = \infty \) a.s., and the difference \( w = u - \tilde{u} \) satisfies

\[ \|w(t)\|^2 + |\partial_t w(t)|^2 \to 0, \quad t \to \infty \]

exponentially fast. This means that, if \( H_N \subset \text{Range}(\sigma) \), the joint law of the solutions \( U = (u, v), \tilde{U} = (\tilde{u}, \tilde{v}) \) can be used as the coupling required in Corollary 2.8 with \( E = X, \rho = d = \| \cdot \|_X \wedge 1 \). We conclude that in the framework of Proposition 3.5 [8], which states unique ergodicity for (43), in addition the following stabilization property holds:

for any \( (u, v) \in X = H^1_0(D) \times L^2(D) \), transition probabilities \( P_t((u, v), \cdot) \in \mathcal{P}(X) \) weakly converge as \( t \to \infty \) to the unique invariant measure.

A Proofs of Propositions 4.1 and 4.6

Proof of Proposition 4.1. I. Take an arbitrary \( \xi \in \hat{C}_p^{\infty}({\mathbb{P}, \mathbb{Q}}) \), and consider the sets

\[ B^1_\gamma = \left\{ x : \frac{d\pi_1(\xi)}{d\mathbb{P}}(x) \leq \gamma^{-1} \right\}, \quad B^2_\gamma = \left\{ x : \frac{d\pi_2(\xi)}{d\mathbb{Q}}(x) \leq \gamma^{-1} \right\}, \quad C_\gamma = B^1_\gamma \times B^2_\gamma, \quad \gamma \in (0, 1). \]

Define the sub-probability measure \( \eta_\gamma \) on \( (E^\infty \times E^\infty, \mathcal{E}^{\otimes \infty} \otimes \mathcal{E}^{\otimes \infty}) \) by

\[ \eta_\gamma(A) = \gamma \xi(A \cap C_\gamma). \]

Then the “marginal distributions” \( \pi_i(\eta_\gamma), i = 1, 2 \) (which now are sub-probability measures, as well) satisfy

\[ \pi_1(\eta_\gamma) \leq \mathbb{P}, \quad \pi_2(\eta_\gamma) \leq \mathbb{Q}. \]
Denote
\[ \beta_\gamma = \eta_\gamma(E^\infty \times E^\infty) = \gamma \xi(C_\gamma) \leq \gamma < 1, \]
then each of the measures \( \mathbb{P} - \pi_1(\eta_\gamma), Q - \pi_2(\eta_\gamma) \) has total mass \( 1 - \beta_\gamma \). We put
\[ \zeta_\gamma = \eta_\gamma + (1 - \beta_\gamma)^{-1}(\mathbb{P} - \pi_1(\eta_\gamma)) \otimes (Q - \pi_2(\eta_\gamma)), \]
which by construction belongs to \( C(\mathbb{P}, Q) \). Let us show that \( \gamma \) can be chosen small enough, so that \( \zeta = \zeta_\gamma \) possesses the required property.

Let \( \alpha > 0 \). For \( A \in \mathcal{E} \otimes \mathcal{E} \) satisfying \( \xi(A) \geq \alpha \), we have
\[ \zeta_\gamma(A) \geq \eta_\gamma(A) \geq \gamma \left( \alpha - \xi((E^\infty \times E^\infty) \setminus C_\gamma) \right). \]

Next,
\[ \xi((E^\infty \times E^\infty) \setminus C_\gamma) \leq \xi((E^\infty \setminus B_\gamma^1) \times E^\infty) + \xi(E^\infty \times (E^\infty \setminus B_\gamma^2)) = \pi_1(\xi)(E^\infty \setminus B_\gamma^1) + \pi_2(\xi)(E^\infty \setminus B_\gamma^2) \]
and by the definition of the sets \( B_\gamma^i, i = 1, 2 \)
\[ \pi_1(\xi)(E^\infty \setminus B_\gamma^1) = \int_{E^\infty \setminus B_\gamma^1} \frac{d\pi_1(\xi)}{d\mathbb{P}} \, d\mathbb{P} \leq \gamma^{p-1} \int_{E^\infty} \left( \frac{d\pi_1(\xi)}{d\mathbb{P}} \right)^p \, d\mathbb{P} \leq \gamma^{p-1} R^p, \]
\[ \pi_2(\xi)(E^\infty \setminus B_\gamma^2) = \int_{E^\infty \setminus B_\gamma^2} \frac{d\pi_2(\xi)}{dQ} \, dQ \leq \gamma^{p-1} \int_{E^\infty} \left( \frac{d\pi_2(\xi)}{dQ} \right)^p \, dQ \leq \gamma^{p-1} R^p. \]
Hence, if \( \gamma \) is taken small enough for \( 4 \gamma^{p-1} R \leq \alpha \), for every \( A \) with \( \xi(A) \geq \alpha \) we have for \( \zeta = \zeta_\gamma \)
\[ \zeta(A) \geq \frac{\gamma \alpha}{2} =: \alpha', \]
which completes the proof of statement I.

II. We fix \( \xi \in \tilde{C}(\mathbb{P}, Q) \) and modify slightly the construction from the previous part of the proof. Let \( B_\gamma^i, i = 1, 2, C_\gamma \) be as above, then we define
\[ \tilde{\eta}_\gamma(A) = \xi(A \cap C_\gamma). \]

We fix \( \gamma \in (0, 1) \) small enough, so that
\[ \xi((E^\infty \times E^\infty) \setminus C_\gamma) \leq \alpha/2, \]
where \( \alpha \) is as in the statement of the lemma.
We have \( \tilde{\eta}_\gamma = \gamma^{-1} \eta_\gamma \), and thus the total mass of the measure \( \tilde{\eta}_\gamma \) equals \( \gamma^{-1} \beta_\gamma = \xi(C_\gamma) \leq 1 \). In addition,
\[ \pi_1(\tilde{\eta}_\gamma) \leq \gamma^{-1} \mathbb{P}, \quad \pi_2(\tilde{\eta}_\gamma) \leq \gamma^{-1} Q, \]
and the total mass for each of the measures \( \gamma^{-1} \mathbb{P} - \pi_1(\tilde{\eta}_\gamma), \gamma^{-1} Q - \pi_2(\tilde{\eta}_\gamma) \) equals \( \gamma^{-1}(1 - \beta_\gamma) \geq \gamma^{-1} - 1 \). Then
\[ \zeta_\gamma := \tilde{\eta}_\gamma + (1 - \gamma^{-1} \beta_\gamma) \left( \gamma^{-1} (1 - \beta_\gamma) \right)^{-2} \left( \gamma^{-1} \mathbb{P} - \pi_1(\tilde{\eta}_\gamma) \right) \otimes \left( \gamma^{-1} Q - \pi_2(\tilde{\eta}_\gamma) \right) \]
is a probability measure with
\[ \tilde{\zeta}_\gamma(A) \geq \xi(A) - \xi((E^\infty \times E^\infty) \setminus C_\gamma) \geq \xi(A) - \frac{\alpha}{2}; \]
that is, \( \tilde{\zeta}_\gamma(A) \geq \alpha' := \alpha/2 \) as soon as \( \xi(A) \geq \alpha \). In addition, the marginal distributions of \( \tilde{\zeta}_\gamma \) equal
\[
(1 - \beta_\gamma)^{-1} \left( (1 - \gamma^{-1} \beta_\gamma) P + (1 - \gamma) \pi_1(\tilde{\eta}_\gamma) \right), \quad (1 - \beta_\gamma)^{-1} \left( (1 - \gamma^{-1} \beta_\gamma) Q + (1 - \gamma) \pi_2(\tilde{\eta}_\gamma) \right),
\]
and their Radon-Nikodym densities w.r.t. \( P, Q \) respectively are bounded by
\[ R := (1 - \beta_\gamma)^{-1} \left( (1 - \gamma^{-1} \beta_\gamma) + (1 - \gamma) \gamma^{-1} \right) = \gamma^{-1}, \]
hence \( \tilde{\zeta}_\gamma \in C^R_P(P, Q) \) for every \( p \geq 1 \).

**Proof of Proposition 4.6.** There exist two increasing sequences \( K_1^n, K_2^n, n \geq 1 \) of compact subsets of \( E \) such that \( K_1^n \cap K_2^n = \emptyset \), \( \nu_1(K_1^n) \geq 1 - 1/n \), and \( \nu_2(K_2^n) \geq 1 - 1/n \) for \( n \geq 1 \). Let
\[ \delta_n = d(K_1^n, K_2^n), \quad n \geq 1. \]
Clearly \( \delta_n, n \geq 1 \) is non-increasing and \( \delta_n > 0 \) for all \( n \geq 1 \) since \( d : E \times E \to [0, \infty) \) is continuous with respect to \( \rho \otimes \rho \). On the other hand, for any \( \xi \in C(\nu_1, \nu_2) \) we have
\[ \xi(d(X, Y) < \delta_n) \leq \frac{2}{n}, \]
proving the proposition. \( \square \)

### B Jankov’s lemma and the proof of Proposition 3.1

Recall that a measurable space \( (X, \mathcal{X}) \) is called *(standard) Borel* if it is measurably isomorphic to a Polish space equipped with its Borel \( \sigma \)-algebra. For any Borel space \( (X, \mathcal{X}) \) and any set \( A \in \mathcal{X} \), this set endowed with its trace \( \sigma \)-algebra
\[ \mathcal{X}_A := \{ A \cap B, B \in \mathcal{X} \} \]
is a Borel measurable space, see [12, Corollary 13.4].

Our proof of Proposition 3.1 is based on the following lemma.

**Lemma B.1.** (Jankov’s lemma, [7, Appendix 3 §1]). Let \( (X, \mathcal{X}), (Y, \mathcal{Y}) \) be Borel measurable spaces and let \( f : Y \to X \) be a measurable mapping with \( f(Y) = X \). Then for any probability measure \( \nu \) on \( (X, \mathcal{X}) \) there exists a measurable function \( \phi : X \to Y \) such that \( f(\phi(x)) = x \) for \( \nu \)-a.a. \( x \in X \).

In the framework of Proposition 3.1, we put \( X = M, \mathcal{X} = E_M \) (the trace \( \sigma \)-algebra), then \( (X, \mathcal{X}) \) is a Borel space. We define \( \nu \) as the measure \( \mu \) conditioned by \( M \).

Before proceeding with the construction, we mention several simple facts we will use. First, let \( S \) be a Polish space and \( \mathcal{P}(S) \) be endowed by the corresponding Kantorovich-Rubinshtein metric. Then the subset \( \Delta \subset \mathcal{P}(S) \) consisting of all \( \delta \)-measures (that is, measures concentrated in one point) is closed, and
$S$ and $\Delta$ are isomorphic. Second, the mapping $\theta$ from $\mathcal{P}(E^\infty \times E^\infty)$ to $\mathcal{P}(E \times E)$ which maps the law of $(X_n, Y_n), n \geq 0$ to the law of $(X_0, Y_0)$ is (Lipschitz) continuous. Hence, the subset

$$\Xi := \{\xi \in \mathcal{P}(E^\infty \times E^\infty) : \theta(\xi) \text{ is a } \delta\text{-measure}\}$$

is closed. In addition, the mapping $\varrho : \Xi \to E \times E$ which transforms $\xi \in \Xi$ to the (unique) point $(x, y) \in E$ such that $\theta(\xi) = \delta_{(x, y)}$, is continuous. Then $\Xi$ endowed with the trace $\sigma$-algebra is a Borel space and $\varrho$ is a measurable mapping on this space with $\varrho(\Xi) = E \times E$. Denote by $\varrho_{1,2}$ the (measurable) mappings $\Xi \to E$ such that $\varrho(\xi) = (\varrho_1(\xi), \varrho_2(\xi)), \xi \in \Xi$.

Now we can proceed with the construction which deduces Proposition 3.1 from Jankov’s lemma. We fix $x \in E$, put

$$Y = \{\xi \in \Xi : \varrho_1(\xi) = x, \varrho_2(\xi) \in M, \pi_1(\xi) \sim \mathbb{P}_x, \pi_2(\xi) \ll \mathbb{P}_{\varrho_2(\xi)}\},$$

and $f(\xi) = \varrho_2(\xi), \xi \in Y$. Clearly, $f(Y) = M$ and $f$ is a restriction on $Y$ of a measurable mapping $\Xi \to E$ (the projection of $\varrho$ on the second coordinate). Hence in order to be able to apply Jankov’s lemma we need only to show that $Y$ is a measurable subset of $\Xi$. Because $\varrho_{1,2}$ are measurable and $\{x\}, M \in \mathcal{E}$, the sets

$$\{\xi \in \Xi : \varrho_1(\xi) = x\}, \quad \{\xi \in \Xi : \varrho_2(\xi) \in M\}$$

are measurable.

Next, recall that for any two probability measures $\mathbb{P}, \mathbb{Q}$ on $(E^\infty, \mathcal{E}^{\otimes \infty})$ one has $\mathbb{P} \ll \mathbb{Q}$ if, an only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbb{P}(A) \leq \varepsilon \quad \text{for any } A \in \mathcal{E}^{\otimes \infty} \text{ such that } \mathbb{Q}(A) \leq \delta.$$

Because $E^\infty$ is a Polish space, there exists a countable algebra $\mathcal{A}$ which generates $\mathcal{E}^{\otimes \infty}$, and then for any $\gamma > 0, A \in \mathcal{E}^{\otimes \infty}$ there exists $A_\gamma \in \mathcal{A}$ such that

$$\mathbb{P}(A \Delta A_\gamma) < \gamma, \quad \mathbb{Q}(A \Delta A_\gamma) < \gamma.$$

Then in the above characterization of the absolute continuity the class $\mathcal{E}^{\otimes \infty}$ can be replaced by $\mathcal{A}$. Hence

$$\{\xi \in \Xi : \pi_2(\xi) \ll \mathbb{P}_{\varrho_2(\xi)}\} = \bigcap_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{A \in \mathcal{A}} B_{m,k}(A),$$

where

$$B_{m,k}(A) = \{\xi : \pi_2(\xi)(A) \leq m^{-1}, \mathbb{P}_{\varrho_2(\xi)}(A) \leq k^{-1}\} \cup \{\xi : \mathbb{P}_{\varrho_2(\xi)}(A) > k^{-1}\}.$$

Since the mappings $\varrho_2 : \Xi \to E, \pi_2 : \Xi \to \mathcal{P}(E^\infty), E \ni v \mapsto \mathbb{P}_v \in \mathcal{P}(E^\infty)$, and

$$\mathcal{P}(E^\infty) \ni \mathbb{P} \mapsto \mathbb{P}(A) \in \mathbb{R}, \quad A \in \mathcal{A}$$

are measurable, each of the sets $B_{m,k}(A)$ is measurable. Therefore the set

$$\{\xi \in \Xi : \pi_2(\xi) \ll \mathbb{P}_{\varrho_2(\xi)}\}$$

is measurable, as well. Finally, a similar and simpler argument shows that the set

$$\{\xi \in \Xi : \pi_1(\xi) \sim \mathbb{P}_x\}$$

is measurable (we omit the explicit expression for this set here).

Summarizing, we have that $Y$ is a measurable subset of $\Xi$ and therefore, being endowed with the trace $\sigma$-algebra, is a Borel space. We finish the proof of Proposition 3.1 by applying Jankov’s lemma to the Borel spaces $X, Y$, the mapping $f$, and the measure $\nu$ specified above.
C Kuratovskii and Ryll-Nardzewski’s theorem and the proof of Proposition 4.2

Our proof of Proposition 4.2 is based on measurability and measurable selection results discussed in [17], Chapter 12.1. Let us survey the required results briefly.

Let \( X \) be a Polish space with complete metric \( \rho \). Denote by \( \text{comp}(X) \) the space of all non-empty compact subsets of \( X \), endowed with the Hausdorff metric.

**Theorem C.1.** ([17, Theorem 12.1.10] Let \((E, \mathcal{E})\) be a measurable space and \( \Phi : E \to \text{comp}(X) \) be a measurable map. Then there exists a measurable map \( \phi : E \to X \) such that \( \phi(q) \in \Phi(q), q \in E \).

The above theorem is a weaker version of the Kuratovskii and Ryll-Nardzewski’s theorem on measurable selection for a set-valued mapping which takes values in the space of closed subsets of \( X \); e.g. [18].

In the set-up of Proposition 4.2, for \( \mu, \nu \in \mathcal{P}(S_2) \), we denote by \( \text{C}_{\text{opt}}(\mu, \nu) \) the subset of \( \text{C}(\mu, \nu) \) consisting of all couplings which minimize the distance-like function \( h \); that is,

\[
\eta \in \text{C}_{\text{opt}}(\mu, \nu) \iff \eta \in \text{C}(\mu, \nu), \quad \int_{S_2 \times S_2} h(u, v)\eta(du, dv) = h(\mu, \nu).
\]

We prove the following simple facts.

**Lemma C.2.** For any \( \mu, \nu \in \mathcal{P}(S_2) \):

(i) the set \( \text{C}_{\text{opt}}(\mu, \nu) \) is non-empty;

(ii) the sets \( \text{C}(\mu, \nu), \text{C}_{\text{opt}}(\mu, \nu) \) are compact.

**Proof.** Since \( \pi_1, \pi_2 : \mathcal{P}(S_2 \times S_2) \to \mathcal{P}(S_2) \) are continuous, any weak limit point of a sequence from \( \text{C}(\mu, \nu) \) belongs to \( \text{C}(\mu, \nu) \). Because the marginal distributions of all \( \eta \in \text{C}(\mu, \nu) \) are the same, the set \( \text{C}(\mu, \nu) \) is tight, which by the Prokhorov theorem completes the proof of compactness of \( \text{C}(\mu, \nu) \).

Next, the mapping

\[
\mathcal{P}(S_2 \times S_2) \ni \eta \mapsto I_h(\eta) := \int_{S_2 \times S_2} h(u, v)\eta(du, dv) \in [0, 1]
\]

is lower semicontinuous. To see that, consider a sequence \( \eta_n \Rightarrow \eta \) and use the Skorokhod “common probability space principle”: there exist random elements \( X_n, n \geq 1, X \) with \( \text{Law}(X_n) = \eta_n, \text{Law}(X) = \eta \) such that \( X_n \to X \) a.s. (see [6, Theorem 11.7.2]. Since \( h \) is bounded and lower semicontinuous, we have

\[
\mathbb{E}h(X) \leq \mathbb{E}\liminf_n h(X_n) \leq \liminf_n \mathbb{E}h(X_n),
\]

which proves the required semicontinuity of \( I_h \). By this semicontinuity (a) the function \( I_h \) attains its minimum on the compact set \( \text{C}(\mu, \nu) \), i.e. \( \text{C}_{\text{opt}}(\mu, \nu) \) is non-empty; (b) the set \( \text{C}_{\text{opt}}(\mu, \nu) \) is closed, and since it is a subset of the compact set \( \text{C}(\mu, \nu) \), it is compact.

To prove Proposition 4.2, we apply Theorem C.1 in the following setting: \( E = S_1 \times S_1, X = \mathcal{P}(S_2 \times S_2) \), and

\[
\Phi((x, y)) = \text{C}_{\text{opt}}(Q(x), Q(y)), \quad (x, y) \in E.
\]
We represent $\Phi$ as a composition of $\Psi$ and $\Upsilon$, where

$$
\Psi((x, y)) = C(Q(x), Q(y)), \quad (x, y) \in E
$$

and

$$
\Upsilon(K) = \left\{ \eta \in \mathbb{X} : I_h(\eta) = \min_{\zeta \in K} I_h(\zeta) \right\} \in \text{comp}(\mathbb{X}), \quad K \in \text{comp}(\mathbb{X}).
$$

Clearly, the minimization of $I_h$ is equivalent to maximization of $1 - I_h$, and $1 - I_h$ is upper semicontinuous. Hence the mapping $\Upsilon : \text{comp}(\mathbb{X}) \to \text{comp}(\mathbb{X})$ is measurable by [17], Lemma 12.1.7. On the other hand, for any sequence $(x_n, y_n) \rightarrow (x, y)$ and $\eta_n \in \Psi((x_n, y_n))$ we have that the marginal distributions of $\eta_n$ weakly converge to $Q(x), Q(y)$ respectively. Then by the Prokhorov theorem there exist a weakly convergent subsequence $\eta_{n_k}$, and in addition the weak limit has the marginal distributions $Q(x), Q(y)$, that is, belongs to $\Psi((x, y))$. Then the mapping $\Psi : E \rightarrow \text{comp}(\mathbb{X})$ is measurable by [17, Lemma 12.1.8]. Hence $\Phi$ is measurable, as well, and we obtain the statement of Proposition 4.2 as a straightforward corollary of Theorem C.1. □

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