A MODIFICATION OF NAMBU’S MECHANICS

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Abstract. The Poisson, contact and Nambu brackets define algebraic structures on $C^\infty(M)$ satisfying the Jacobi identity or its generalization. The automorphism groups of these brackets are the symplectic, contact and volume preserving diffeomorphism groups. We introduce a modification of the Nambu bracket, which defines an evolution equation generating the whole diffeomorphism group. The relation between the modified and usual Nambu brackets is similar to the relation between the Poisson and contact structures. We briefly discuss the problem of quantization of the modified bracket.

1. Introduction

In 1973 Nambu constructed a generalization of Hamiltonian mechanics [14]. He defined a dynamical system on $\mathbb{R}^3$ by the trilinear Nambu bracket:

$$\frac{df}{dt} = \{H_1, H_2, f\} = \frac{\partial(H_1, H_2, f)}{\partial(x, y, z)} = L_{H_1, H_2}f,$$

(1.1)

where the third term is the Jacobian of $(H_1, H_2, f)$ with respect to $(x, y, z)$, and $L_{H_1, H_2}$ is a vector field on $\mathbb{R}^3$. The flow $\phi_t$ generated by $L_{H_1, H_2}$ is a canonical transformation in the sense that

$$\{H_1 \circ \phi_t, H_2 \circ \phi_t, f \circ \phi_t\} = \{H_1, H_2, f\} \circ \phi_t$$

(1.2)

is satisfied. Since the Nambu bracket can be defined in terms of the volume form of $\mathbb{R}^3$ [14] holds for any volume preserving transformation $\phi_t$, and indeed $\text{div} L_{H_1, H_2} = 0$. So Nambu dynamics is related to volume forms and volume preserving transformations just as Hamiltonian dynamics relates to symplectic two-forms $\omega$ and symplectic transformations leaving $\omega$ invariant.

Our main result is the introduction of a new bracket which generates transformation $\phi_t$ satisfying (1.2) although $\phi_t$ does not leave the volume form invariant. In some sense, the new bracket generates the whole diffeomorphism group. At the first sight the existence of such bracket seems to be unlikely, since no tensor is invariant under the action of the diffeomorphism group. However, the existence of the group of contact transformation and the contact bracket shows that strict invariance might be unnecessary, as contact transformations carry the contact one-form $\alpha$ into its scalar multiple $f\alpha$. The volume form $\nu$ behaves the same way under the action of a general diffeomorphism $\phi_t$: $\phi_t^*\nu = f\nu$. As the space of top dimensional differential forms at a point is one dimensional, the existence of $f$ is completely trivial!

We construct a trilinear bracket

$$[\cdot, \cdot, \cdot] : C^\infty(R^2) \otimes C^\infty(R^2) \otimes C^\infty(R^2) \longrightarrow C^\infty(R^2)$$

(1.3)

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(called the modified Nambu bracket) which fulfills (1.4):
\[ [H_1, H_2, f]_{mnN} = H_1\{H_2, f\} - H_2\{H_1, f\} + f\{H_1, H_2\}. \] (1.4)

({,}) is the standard Poisson bracket on \( \mathbb{R}^2 \). The vector fields \( L_{H_1,H_2} \) defined by
\[ L_{H_1,H_2}f = H_1\{H_2, f\} - H_2\{H_1, f\} \] (1.5)
generates the whole diffeomorphism group. Just as the Nambu bracket (1.1), this construction can be generalized to \( \mathbb{R}^n \).

To put our work into perspective, in Section 2 we briefly review the properties of the Poisson, contact and Nambu brackets. In Section 3 we study the modified Nambu bracket. Section 4 presents an embedding of the modified Nambu dynamics into Nambu’s original one. Section 5 contains some remarks on the problem of quantization of the new bracket.

2. Brackets on \( C^\infty (M) \).

In this section we review the properties of the Poisson, contact and Nambu brackets.

The most well-known bracket is the Poisson bracket on symplectic manifolds \((M^{2n}, \omega)\) where \( \omega \) is the symplectic two-form:
\[ \{f, g\}_P = \eta(df, dg). \] (2.1)

The two-vector \( \eta \) is the inverse of the map \( \omega : TM \to T^*M \), defined as \( \omega(X) = i_X\omega \).

The Poisson bracket satisfies
\[ \{f, g\}_P = -\{g, f\}_P, \] (2.2)
\[ \{f, \{g, h\}\}_P = \{\{f, g\}, h\}_P + \{g, \{f, h\}\}_P, \] (2.3)
\[ \{f, gh\}_P = \{f, g\}h + \{f, h\}g. \] (2.4)

Leibniz’s rule (2.4) implies that \( \{f, g\} = X_f g \ (X_f \in TM) \). As a consequence of the Jacobi identity (2.3) \( \phi_t^H = \exp tX_H \) satisfies
\[ \{f \circ \phi_t^H, g \circ \phi_t^H\} = \{f, g\} \circ \phi_t^H. \] (2.5)

(2.5) holds since \( \omega \) (and \( \eta \)) is invariant under the flow \( \phi_t^H \).

The next case is the contact bracket \[ \mathcal{L}_2 \]. It is defined on contact manifolds \((M^{2n+1}, \alpha)\) where \( \alpha \) is the contact one form \( \alpha \in \Lambda^1 M, \alpha \wedge (d\alpha)^n \neq 0 \). The contact one-form is defined only up to a scalar factor, so the contact structure is more invariantly defined by the field of \( 2n \) dimensional planes annihilating \( \alpha \). The infinite dimensional contact transformation group leaves the plane field invariant, and carries \( \alpha \) into \( f\alpha \) for some \( f \in C^\infty (M^{2n+1}) \). By a suitable coordinate transformation \( \alpha \) can be put into its normal form
\[ \alpha = dx^{2n+1} + \sum_{i=1}^n x^i dx^{i+n}. \] (2.6)
In this coordinate system the contact bracket is

\[ \{f, g\}_{\text{cont}} = \sum_{i=1}^{n} (\partial_i f \partial_{i+n} g - \partial_i g \partial_{i+n} f) + \left( \partial_{2n+1} f \sum_{i=1}^{2n} x_i \partial_i g - 2g \right) - \left( \partial_{2n+1} g \sum_{i=1}^{2n} x_i \partial_i f - 2f \right). \]  

(2.7)

(2.7) satisfies the Jacobi identity but not Leibniz’s rule since (2.7) has the structure

\[ \{f, g\}_{\text{cont}} = K(f)g + c(f)g = H(f)g, \]  

(2.9)

where \( K(f) \in TM \) and \( c(f) \in C^\infty(M) \), and the appearance of the multiplier term \( c(f) \) is incompatible with Leibniz’s rule.

The theory of Jacobi manifolds unifies the concepts of symplectic and contact manifolds \([43]\). The Jacobi bracket is given by

\[ \{f, g\}_J = \eta(df, dg) + fEg - gEf, \]  

(2.10)

where the \( \eta \in \Lambda^2 TM \) bivector and the \( E \in TM \) vector field satisfy compatibility conditions necessary for the fulfillment of the Jacobi identity

\[ [\eta, \eta] = 2E \wedge \eta, [E, \eta] = L_E \eta = 0, \]  

(2.11)

where \([,]\) is the Schouten-Nijenhuis bracket \([5]\). Jacobi manifolds are locally decomposable into an union of symplectic and contact leaves.

A more recently introduced bracket operation is the generalized Nambu bracket on \( R^n \):

\[ \{1, \ldots, \} : C^\infty(TR^n)^n \rightarrow C^\infty(TM) \]

\[ f_1, f_2, \ldots, f_n}_N = \epsilon_{i_1 \ldots i_n} \partial_{i_1} f_1 \ldots \partial_{i_n} f_n = \eta^{(n)}(df_1, \ldots, df_n), \]  

(2.12)

where \( \epsilon \) is the alternating Levi-Civita symbol. The Nambu bracket satisfies

\[ \{f_1, f_2, \ldots, f_j, \ldots, f_n}_N = -\{f_1, \ldots, f_j, \ldots, f_i, \ldots, f_n}_N, \]  

(2.13)

\[ \{H_1, \ldots, H_{n-1}, gh\}_N = h}\{H_1, \ldots, H_{n-1}, g\}_N + g}\{H_1, \ldots, H_{n-1}, h\}_N, \]  

(2.14)

\[ \{H_1, \ldots, H_{n-1}, (g_1, \ldots, g_n)\}_N = \sum_{i=1}^{n} (g_1, \ldots, \{H_1, \ldots, H_{n-1}, g_i\}_N, \ldots, g_n)_N. \]  

(2.15)

At this point we remark that different generalizations of the Jacobi identity gained some popularity recently \([14, 15, 16, 17]\). Just as for the Poisson-bracket, (2.14) ensures that

\[ \{H_1, \ldots, H_{n-1}, g\}_N = X_{H_1, \ldots, H_{n-1}}, g, \]  

(2.16)

for some vector field \( X_{H_1, \ldots, H_{n-1}} \in TR^n \). Since the Fundamental Identity (FI) \( (2.15) \) holds \([16, 17] \), \( \phi_t = \exp tX_{H_1, \ldots, H_{n-1}} \) is a canonical transformation (an automorphism of the bracket):

\[ \{f_1, \ldots, f_n\}_N \circ \phi_t = \{f_1 \circ \phi_t, \ldots, f_n \circ \phi_t\}_N. \]  

(2.17)

Indeed, (2.17) is the derivative of (2.17). As

\[ \text{div } X_{H_1, \ldots, H_{n-1}} = \partial_{i_n} (\epsilon_{i_1 \ldots i_{n-1}} \partial_{i_1} H_1 \ldots \partial_{i_{n-1}} H_{n-1}) = 0, \]  

(2.18)
Nambu dynamics can be formulated in terms of the volume form $\nu$ instead of the n-vector $\eta$. For that purpose, let us define $\nu_\eta$ by the condition $\nu_\eta(\eta) = 1$. For a given set of Nambu Hamiltonians $H_1, \ldots, H_{n-1}$ the vector field $X_{H_1,\ldots,H_{n-1}}$ can be obtained by the following process: First find $n-1$ vectors $X_1, \ldots, X_{n-1}$ satisfying
\[ \langle X_i, dH_j \rangle = \delta_{ij}. \]
Then $X_{H_1,\ldots,H_{n-1}}$ is determined by the conditions
\[ X_{H_1,\ldots,H_{n-1}}H_k = 0, \quad \text{and} \quad \nu_\eta(X_1, \ldots, X_{n-1}, X_{H_1,\ldots,H_{n-1}}) = 1. \]  
(2.19)

Since the Nambu bracket is determined by the volume form, this implies (2.17) as divergenceless vector fields leave the volume form unchanged. As on a manifold two volume forms are equivalent if their total masses are equal, on $\mathbb{R}^n$ any nondegenerate n-vector has the form $\eta_0^{(n)}$ in a suitable coordinate system.

The vector fields $X_{H_1,\ldots,H_{n-1}}$ form only a subset of the Lie-algebra of volume preserving transformations. Since
\[ \frac{d}{dt}(H_1 \circ \phi_t) = 0, \]  
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3. **THE MODIFIED NAMBU BRACKET**

The General form of the Jacobi bracket (2.10) suggests the following generalization of the Nambu bracket:
\[ [f_1, \ldots, f_n]_{\eta_0} = \nu_\eta(\eta_0) \sum_{i=1}^{n} (-1)^{i+1} f_i \nu_\eta(\eta_0)(df_1, \ldots, \hat{f}_i, \ldots, f_n), \]  
(3.1)
where $\eta$ and $\nu_\eta$ are n and $n-1$ vectors. This bracket is antisymmetric, but does not satisfy the Leibniz identity. The Fundamental Identity imposes a set of complicated consistency conditions on $\eta$ and $\nu_\eta$. We prove that the following bracket on $C^\infty(\mathbb{R}^n)$ satisfies the FI:
\[ [\ldots, f_{n+1}]_{mN} = \sum_{i=1}^{n+1} (-1)^{i+1} f_i \nu_0^{(n)}(df_1, \ldots, \hat{f}_i, \ldots, f_{n+1}). \]  
(3.2)
($\nu_0^{(n)}$ is the standard Nambu tensor on $\mathbb{R}^n$.)

For the sake of clarity, we present a proof first for the $n = 2$ case. Let $\{f, g\}_P = \eta_0^{(2)}(df, dg)$ the standard Poisson-bracket on $\mathbb{R}^2$, and define the trilinear alternating modified Nambu bracket as
\[ [f, g, h]_{mN} = f\{g, h\}_P - g\{f, h\}_P + h\{f, g\}_P \]  
(3.3)
and
\[ = f\{g, h\}_P + g\{h, f\}_P + h\{f, g\}_P. \]  
(3.4)
Our goal is to prove
\[
[H_1, H_2, [f, g, h]]_{mN} = [[H_1, H_2, f]]_{mN} + [f, [H_1, H_2, g]]_{mN} + [g, h, [H_1, H_2, f]]_{mN} \tag{3.5}
\]
We introduce the shorthand notation
\[
[H_1, H_2, \phi]_{mN} = T\phi = L\phi + H\phi. \tag{3.6}
\]
In this expression \(L = X_{H_1, H_2} = H_1 X_{H_2} - H_2 X_{H_1}\) and \(H = \{H_1, H_2\}\), where \(X_{H_i}\) is the Hamiltonian vector field generated by \(H_i\) with respect to the Poisson-bracket.

The divergence of \(L\) is
\[
\text{div } L = -2H, \tag{3.7}
\]
so
\[
\mathcal{L}_L(dx \wedge dy) = -2H(dx \wedge dy), \text{and } \mathcal{L}_L \eta^2(0) = 2H\eta^2(0). \tag{3.8}
\]
We remark that the operator \(U_t = \exp tT_{H_1, H_2}\) preserve the 2-norm of \(L^2(\mathbb{R}^2)\), since
\[
dt \int h^2 dx \wedge dy = \int h^2 \mathcal{L}_L(dx \wedge dy) + (Hh)2h(dx \wedge dy) = 0. \tag{3.9}
\]
To prove the Fundamental Identity we evaluate the left-hand (LHS) and right-hand (RHS) sides of (3.5).

\[
(\mathcal{L} + H)(f\eta_0^{(2)}(dg, dh)) + (\text{cycl.perm.}) \quad (\text{LHS}_1)
\]
\[
[(\mathcal{L} + H)f]\eta_0^{(2)}(dg, dh) + f\eta_0^{(2)}(d(\mathcal{L} + H)g, dh) + f\eta_0^{(2)}(dg, d(\mathcal{L} + H)h) + (\text{cycl.perm.}) \quad (\text{RHS}_1)
\]
where \(\text{cycl.perm.}\) refers to the cyclical permutations of \(f, g\) and \(h\). The terms containing \(\mathcal{L}_L f, \ldots\), and \(Hf\eta_0^{(2)}(dg, dh), \ldots\) occurs the same way on both sides. After the deletion of these terms the following expressions remain:

\[
f(\mathcal{L}_L \eta_0^{(2)}) (dg, dh) + (\text{cycl.perm.}) \quad (\text{LHS}_2)
\]
\[
f\eta_0^{(2)}(d(Hg), dh) + f\eta_0^{(2)}(dg, d(Hh)) + (\text{cycl.perm.}) \quad (\text{RHS}_2)
\]
These expressions are equal to
\[
2Hf\eta_0^{(2)}(dg, dh) + (\text{cycl.perm.}) \quad (\text{LHS}_3)
\]
\[
2Hf\eta_0^{(2)}(dg, dh) + g f\eta_0^{(2)}(dH, dh) + h f\eta_0^{(2)}(dg, dH) + (\text{cycl.perm.}) \quad (\text{RHS}_3)
\]
However, the extra two terms of the RHS drops out. For example, \(hf\eta_0^{(2)}(dg, dh)\) is annihilated by \(fh\eta_0^{(2)}(dH, dg)\) which is generated by \(h\eta_0^{(2)}(d(Hf), dg)\). So \(LHS_3 = RHS_3\), consequently \(LHS = RHS\), i.e. the Fundamental Identity has been proven.

The same line of reasoning works for the bracket on \(\mathbb{R}^n\), too. To prove
\[
[H_1, \ldots, H_n, [f_1, \ldots, f_{n+1}]]_{mN} = \sum_{i=1}^n [f_1, \ldots, [H_1, \ldots, H_n, f_i]_{mN}, \ldots, f_{n+1}]_{mN} \tag{3.10}
\]
we introduce

\[ T = T_{H_1, \ldots, H_n} = \sum_{i=1}^{n} (-1)^{i+1} H_i X_{H_1, \ldots, H_i, \ldots, H_n} + \{H_1, \ldots, H_n\}_N = L + H, \]  
(3.11)

where \( X_{H_1, \ldots, H_n} \) is the vector field defined by the condition \( X_{H_1, \ldots, H_n} f = [H_1, \ldots, H_n, f]_N \).

The proof of (3.10) is very similar to the previous case. The difference between the left-hand side (LHS) and the right-hand side (RHS) of (3.10) evaluates to 0:

\[
\{\text{LHS}\} - \{\text{RHS}\} = \left\{ (\mathcal{L}_L + H) \left[ \sum_{i=1}^{n+1} (-1)^{i+1} f_i \eta_0^{(n)}(df_i, \ldots, \hat{d}f_i, \ldots, df_{n+1}) \right] \right\} \\
- \left\{ \sum_{i=1}^{n+1} (-1)^{i+1} \left( \mathcal{L}_L + H \right) f_i \eta_0^{(n)}(df_i, \ldots, \hat{d}f_i, \ldots, df_{n+1}) \right\} \\
+ \sum_{i=1}^{n+1} (-1)^{i+1} f_i \sum_{k=1, k \neq i}^{n+1} \eta_0^{(n)}(df_i, \ldots, d(\mathcal{L}_L + H) f_k, \ldots, \hat{d}f_i, \ldots, df_{n+1}) \\
= \left\{ \sum_{i=1}^{n+1} (-1)^{i+1} f_i (\mathcal{L}_L \eta_0^{(n)})(df_i, \ldots, \hat{d}f_i, \ldots, df_{n+1}) \right\} \\
- \left\{ \sum_{i=1}^{n+1} (-1)^{i+1} f_i \sum_{k=1, k \neq i}^{n+1} \eta_0^{(n)}(df_i, \ldots, d(H f_k), \ldots, \hat{d}f_i, \ldots, df_{n+1}) \right\} \\
= \left\{ \sum_{i=1}^{n+1} (-1)^{i+1} f_i (n H \eta_0^{(n)})(df_i, \ldots, \hat{d}f_i, \ldots, df_{n+1}) \right\} \\
- \left\{ \sum_{i=1}^{n+1} (-1)^{i+1} f_i \sum_{k=1, k \neq i}^{n+1} H \eta_0^{(n)}(df_i, \ldots, d f_k, \ldots, \hat{d}f_i, \ldots, df_{n+1}) \right\} \\
= 0.
\]

The FI means that \( T = L + H \) is a derivation (infinitesimal automorphism) of the bracket. The proof of the FI used the fact that \( T \) is generated by the Hamiltonians \( H_1, \ldots, H_n \) only for the computation of \( \text{div} L \). Consequently any \( T = L + H \) is a generator of an automorphism of the bracket if \( \text{div} L = -n H \) holds. So the assignment \( V \to T_V = V - 1/n \text{div} V \) maps any vector field to an infinitesimal automorphism of the modified Nambu bracket.

4. Embedding into Nambu’s dynamics

The Nambu bracket is a special case of the modified bracket:

\[ \{f_1, f_2, \ldots, f_n\} = [1, f_1, f_2, \ldots, f_n]_{mN}. \]  
(4.1)

In this section we study the reverse relation and express the modified bracket on \( \mathbb{R}^n \) with the help of the Nambu bracket on \( \mathbb{R}^{n+1} \).

The evolution equation of the modified Nambu dynamics

\[ \frac{d}{dt} f = [H_1, \ldots, H_n, f]_{mN} = (L + H) f \]  
(4.2)
To prove the existence of the Nambu tensor, we re-derive the previous result using the function those points as there the value of the Nambu bracket is zero anyway. Define the coordinate system where \((x_1, \ldots, x_n)\) is introduced, where \((x_1, \ldots, x_n, f(x_1, \ldots, x_n))\) the evolution of \(\eta\) is any \(n\)-vector \(y\) satisfies (4.7):

\[
\eta(dx_1 \wedge \ldots \wedge dx_n) = \eta(dx_1 \wedge \ldots \wedge dx_n, f_1, \ldots, f_n).
\]

To prove the existence of \(\tilde{\eta}\), we note that

\[
\tilde{\eta}(dy, d\tilde{f}_1, \ldots, d\tilde{f}_n) = y^n \tilde{\eta}(dy, \pi^*(f_1), \ldots, \pi^*(f_n)),
\]

so the following tensor satisfies

\[
\tilde{\eta} = \frac{1}{y^{n-1}} \partial_{\text{vert}} \wedge \eta_{\text{ext}},
\]

where \(\eta_{\text{ext}}\) is any \(n\)-vector on \(L\) satisfying

\[
\eta_{\text{ext}}(dx^*(f_1), \ldots, dx^*(f_n)) = \eta(df_1, \ldots, df_n),
\]

and \(\partial_{\text{vert}}\) is a vertical vector such that \(\partial_{\text{vert}}y = 1\). The 'pull-back' \(\eta_{\text{ext}}\) of \(\eta\) is determined only up to terms containing \(\partial_{\text{vert}}\), but these terms drop out of
anyway. Now we define the modified bracket by

\[ [f_1, \ldots, f_{n+1}]_{mN}(x_i) = \tilde{\eta}(d\tilde{f}_1, \ldots, d\tilde{f}_{n+1})(x_i, \xi). \]  \hfill (4.11)

Since \( \tilde{\eta}(d\tilde{f}_1, \ldots, d\tilde{f}_{n+1}) \) is one-homogeneous, the modified brackets inherits the Fundamental Identity from the Nambu bracket of \( L \). (Leibniz’s rule is not inherited, since it contains the product of two functions ruining one-homogeneity).

These constructions provide a second proof of the FI since we mapped the modified bracket onto the Nambu bracket which satisfies the FI. It show the possibility of an action formulation for the modified Nambu dynamics \[10\].

5. On the quantization of the modified Nambu bracket

The quantization of the Nambu bracket is not a terribly well-defined task, let alone the modified bracket. In this section we make a few remarks on the connection between the quantization of the two brackets. Since the modified and the usual Nambu mechanics can be embedded into each other it might be possible to induce the quantization of the modified bracket from the usual one’s. Nevertheless, it might be worth to see if it is possible to modify the constructions used so far for the quantization of Nambu’s mechanics.

Nambu proposed a generalization of the Heisenberg commutation relation:

\[ cI = [\hat{A}_1, \hat{A}_2, \hat{A}_3] = \sum_{\pi \in S_3} \epsilon(\pi)\hat{A}_{\pi(1)}\hat{A}_{\pi(2)}\hat{A}_{\pi(3)}, \]  \hfill (5.1)

Takhtajan \[16\] constructed a representation for this relation and its generalization. This alternating product might be suitable for Nambu’s mechanics. However, in the terms of the modified bracket \([f, g, h]\) on \( \mathbb{R}^2 \), \( f, g \) and \( h \) play a somewhat different role, so probably the following ternary product might be more appropriate:

\[ [\hat{A}_1, \hat{A}_2, \hat{A}_3] = \rho(\hat{A}_1)[\hat{A}_2, \hat{A}_3] + \rho(\hat{A}_2)[\hat{A}_3, \hat{A}_1] + \rho(\hat{A}_3)[\hat{A}_1, \hat{A}_2], \]  \hfill (5.2)

where \( \rho \) is some linear functional.

Deformation quantization \[3\] of an algebra satisfying certain identities is a somewhat better defined task. For Nambu’s mechanics a quite novel approach was developed in the papers \[8, 9\], while the straightforward generalization of Weyl’s quantization does not seems to work very well. In our case, the analog of Weyl’s quantization could be the following deformed triple product:

\[ (e, f, g)^{\pi}_h = \pi \circ \exp (hD)(e \otimes f \otimes g) \]

\[ D = \partial_x \otimes \partial_y \otimes I + I \otimes \partial_x \otimes \partial_y + (\partial_x \otimes I \otimes \partial_y - \partial_y \otimes I \otimes \partial_x), \]  \hfill (5.3)

where \( \pi \) is the restriction to the diagonal. However, it is unlikely that this deformed product satisfies the FI.

The n-norm preserving nature of the modified evolution equation deserve some attention. Since neither the 1-norm nor the 2-norm is invariant, the usual probabilistic an quantum mechanical interpretations might be inappropriate in this case. Although the generator \( T_{H_1,\ldots,H_n} \) of the evolution is a linear operator, in a theory similar to quantum mechanics one can not expect to use linear operators if a p-norms \( p \neq 2 \) need to be preserved. Indeed, the only linear operators leaving the norm \( ||x||_p^p = \sum |x_i|^p \) are basically the permutations of the basis vectors and sign changes.
The modified bracket has an interesting relation to cyclic cocycles [6]. On the algebra $C^\infty(M)$ of smooth functions of an $n$-dimensional manifold a cyclic cocycle is given by

$$\tau(f_0, \ldots, f_n) = \int_{M^n} f_0 df_1 \wedge \ldots \wedge df_n. \quad (5.4)$$

This expression is proportional to the integral of the modified Nambu bracket of $f_0, \ldots, f_n$ over $M^n$ for the Nambu $n$-vector $\eta$ satisfying $\text{vol}(\eta) = 1$.

$$\tau(f_0, \ldots, f_n) = \frac{1}{n!} \int_{M^n} [f_0, \ldots, f_n] \text{dvol}. \quad (5.5)$$

As our last remark, we show that the space of abelian gauge fields $A_i(x)$ over a three-manifold carries a fairly natural Nambu structure. We define a trilinear alternating bracket by

$$A_i(x), A_j(y), A_k(z) = c(x)\delta^3(x-y)\delta^3(y-z), \quad (5.6)$$

where $c(x)$ is some nonzero function. This bracket can be extended for polynomial functions of $A$ by Leibnitz’ rule. The FI is satisfied, since the three-vector generating this bracket is the integral of the three-vectors

$$\eta(x) = c(x) \frac{\delta}{\delta A_1(x)} \wedge \frac{\delta}{\delta A_2(x)} \wedge \frac{\delta}{\delta A_3(x)}, \quad (5.7)$$

and $\eta(x)$ has the same form as the standard Nambu three-vector $\partial_1 \wedge \partial_2 \wedge \partial_3$ on $\mathbb{R}^3$. Gauge invariant functions form a closed subalgebra with respect to this bracket. The variation $\delta_{\delta(x)} A_i(y) = \partial_x, \delta(x-y)$ of $A$ by a gauge transformation can be expressed as

$$\delta_{\delta(x)} A_i(y) = \partial_x, \delta(x-y) = \frac{1}{4c(x)} \epsilon_{lmn} \{ F_{lm}(x), A_m(x), A_i(y) \}. \quad (5.8)$$

If $\Phi_i[A], i = 1, 2, 3$ are gauge invariant functionals, then so is $\Psi[A] = \{ \Phi_1, \Phi_2, \Phi_3 \}$, since the FI implies that

$$
\delta_{\delta(x)} \Psi = \frac{1}{4c} \epsilon_{lmn} \{ F_{lm}(x), A_n, \Phi_1 \}
= \frac{1}{4c} \epsilon_{mn} \left( \{ \{ F_{lm}(x), A_n, \Phi_1 \}, \Phi_2, \Phi_3 \} + \{ \Phi_1 \{ F_{lm}(x), A_n, \Phi_2 \}, \Phi_3 \} 
+ \{ \Phi_1, \Phi_2, \{ F_{lm}(x), A_n, \Phi_3 \} \right) = 0.
$$

Unfortunately, the natural generalization of this bracket for nonabelian gauge fields is not Nambu type:

$$\{ A^a_i(x), A^b_j(y), A^c_k(z) \} = c(x)\epsilon(x) f^{abc} \delta^3(x-y)\delta^3(y-z), \quad (5.10)$$

where $f^{abc}$ is the antisymmetric structure constant of a compact Lie-group, as this bracket is symmetric in its arguments.

REFERENCES

[1] V.Arnold,A.Givental’, Symplectic geometry, in: Dynamical Systems IV, Encyclopaedia of Mathematical Sciences, vol. 4. Springer-Verlag, 1990. pp. 77-82.
[2] J.A.de Azcarraga, A.M.Perelomov and J.C.Pérez Bueno, New generalized Poisson structures, J.Phys. A29 (1996) L151-157. hep-th/9703019.
[3] J.A.de Azcarraga, A.M.Perelomov and J.C.Pérez Bueno, The Schouten-Nijenhuis bracket, cohomology and generalized Poisson structures, J.Phys. A29 (1996) 7993-8009.
[4] J.A.de Azcárraga, J.C.Pérez Bueno, *Higher-order simple Lie-algebras*, Comm.Math.Phys, **184** (1996) 669-681.

[5] F.Bayen, M.Flato, C.Fronsdal and D.Steinheimer, *Deformation theory and quantization, I-II*. Ann. Phys. **110** (1978) 67-110, 111-151.

[6] A.Connes, *Non Commutative Geometry*, (1994) Paris.

[7] R.Chatterjee, L.Takhtajan, *Aspects of classical and quantum Nambu mechanics*, Lett.Math.Phys. **37** (1996) 475-482.

[8] G.Dito, M.Flato, *Generalized abelian deformations*, [hep-th/9609113](http://arxiv.org/abs/hep-th/9609113).

[9] G.Dito, M.Flato, D.Steinheimer and L.Takhtajan, *Deformation quantization and Nambu mechanics*, [hep-th/9602016](http://arxiv.org/abs/hep-th/9602016).

[10] P.Hanlon, M.Wachs, *On Lie k-Algebras*, Adv.Math. **113** (1997) 206-236.

[11] J.Hietarinta, *Nambu tensors and commuting vector fields*, [solv-int/9608010](http://arxiv.org/abs/solv-int/9608010).

[12] N.Hurt, *Geometric Quantization in Action*, D.Reidel, 1983.

[13] A.Kirillov, *Local Lie algebras*, Russian Math. Surveys **31** (1976), 55-75.

[14] Y.Nambu, *Generalized Hamiltonian mechanics*, Phys.Rev. **D7** (1973) 2405-2412.

[15] J.C.Pérez Bueno, *Generalized Jacobi structures*, [hep-th/9707032](http://arxiv.org/abs/hep-th/9707032).

[16] L.Takhtajan, *On the foundation of generalized Nambu mechanics*, Comm.Math.Phys. **160** (1994) 160.

[17] I.Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Birkhauser, 1994.

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