The genus of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ not contained in quadrics

Vincenzo Di Gennaro

Abstract. A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree and the dimension of the space where they are embedded. Fixed integers $r, d, s$, Castelnuovo-Halphen's theory states a sharp upper bound for the genus of a non-degenerate, reduced and irreducible curve of degree $d$ in $\mathbb{P}^r$, under the condition of being not contained in a surface of degree $< s$. This theory can be generalized in several ways. For instance, fixed integers $r, d, k$, one may ask for the maximal genus of a curve of degree $d$ in $\mathbb{P}^r$, not contained in a hypersurface of a degree $< k$. In the present paper we examine the genus of curves $C$ of degree $d$ in $\mathbb{P}^r$ not contained in quadrics (i.e. $h^0(\mathbb{P}^r, I_C(2)) = 0$). When $r = 4$ and $r = 5$, and $d \gg 0$, we exhibit a sharp upper bound for the genus. For certain values of $r \geq 7$, we are able to determine a sharp bound except for a constant term, and the argument applies also to curves not contained in cubics.

Keywords: Projective curve. Castelnuovo-Halphen Theory. Quadric and cubic hypersurfaces. Veronese surface. Projection of a rational normal scroll surface. Maximal rank.

MSC2010: Primary 14N15; Secondary 14N25; 14M05; 14M06; 14M10; 14J26; 14J70

1. Introduction

A classical problem in the theory of projective curves is the classification of all their possible genera in terms of the degree and the dimension of the space where they are embedded [11], [9], [3]. Fixed integers $r, d, s$, Castelnuovo-Halphen’s theory states a sharp upper bound for the genus of a non-degenerate, reduced and irreducible curve of degree $d$ in $\mathbb{P}^r$, under the condition of being not contained in a surface of degree $< s$ [9], [10], [3]. This theory can be generalized imposing flag conditions [4]. For instance, fixed integers $r, d, k$, one may ask for the maximal genus of a curve of degree $d$ in $\mathbb{P}^r$, not contained in a hypersurface of a degree $< k$. As far as we know, in this case there are only rough estimates [13], [4] p. 726, (2.10)].

In the present paper we examine the case of curves in $\mathbb{P}^4$ and $\mathbb{P}^5$ not contained in quadrics. Our main results are the following Theorem 1.1 and Theorem 1.2

Theorem 1.1. Let $C \subset \mathbb{P}^4$ be a reduced and irreducible complex curve, of degree $d$ and arithmetic genus $p_a(C)$, not contained in quadrics (i.e. $h^0(\mathbb{P}^4, I_C(2)) = 0$).

• If $d > 16$, then

$$p_a(C) \leq \frac{1}{8}d(d - 6) + 1.$$
The bound is sharp, and \( p_a(C) = \frac{1}{5}d(d-6) + 1 \) if and only if \( d \) is even, and \( C \) is contained in the isomorphic projection in \( \mathbb{P}^4 \) of the Veronese surface.

- If \( d > 143 \), and either \( d \) is odd or \( C \) is arithmetically Cohen-Macaulay (a.C.M. for short), then

\[
p_a(C) \leq \frac{d^2}{10} - \frac{d}{2} - \frac{1}{10}(\epsilon - 4)(\epsilon + 1) + \left(\frac{\epsilon}{4}\right) + 1,
\]

where \( \epsilon \) is defined by dividing \( d - 1 = 5m + \epsilon \), \( 0 \leq \epsilon \leq 4 \). The bound is sharp. Every extremal curve is a.C.M., and it is contained in a flag like \( S \subset T \subset \mathbb{P}^4 \), where \( S \) is a unique surface of degree 5, and \( T \) is a cubic threefold. Moreover, the surface \( S \) has sectional genus \( \pi = 1 \).

When \( d \) is even, in contrast with the classical case, the extremal curves are not a.C.M.. Moreover, the asymptotic behaviour of the bound is different, depending on whether \( d \) is even or odd. As for the proof of Theorem 1.1, the first part, when \( d \gg 0 \), easily follows combining the geometry of the Veronese surface (see Lemma 3.1 below), with [9, p. 117, Theorem (3.22)]. In order to establish the bound under the hypothesis \( d > 16 \), we need to refine the analysis of the Hilbert function of the general hyperplane section of \( C \), similarly as in [8, p. 74-75]. It is a rather long numerical argument, relying on [3]. We relegate it to an appendix at the end of the paper (Section 7). We will not insist on this analysis in the other cases (i.e. in second part of Theorem 1.1 and in next Theorem 1.2 Proposition 1.3 and Proposition 1.4). We will be content with coarser hypotheses on \( d \), which can be obtained without too much effort, simply using [9, loc. cit.], [7, Corollary 3.11], and [3, Main Theorem]. However, we think our assumptions on \( d \), in the second part of Theorem 1.1 and in next Theorem 1.2 can be significantly improved. We hope to return on this question in a forthcoming paper. When \( d \) is odd, or \( C \) is a.C.M., the proof of Theorem 1.1 is more involved, and we need [4, Theorem 1] in order to extend certain results of [7] in the range \( r = 4 \) and \( s = 5 \).

**Theorem 1.2.** Let \( C \subset \mathbb{P}^5 \) be a reduced and irreducible complex curve, of degree \( d \) and arithmetic genus \( p_a(C) \), not contained in quadrics. Assume \( d > 215 \), and divide \( d - 1 = 6m + \epsilon \), \( 0 \leq \epsilon \leq 5 \). Then:

\[
p_a(C) \leq 6 \left(\frac{m}{2}\right) + m\epsilon.
\]

The bound is sharp. Every extremal curve is not a.C.M., and it is contained in a flag like \( S \subset T \subset \mathbb{P}^5 \), where \( S \) is a unique surface of degree 6, and \( T \) is a cubic hypersurface of \( \mathbb{P}^5 \). Moreover, \( S \) has sectional genus \( \pi = 0 \), and arithmetic genus \( p_a(S) = 0 \).

The analysis of the arithmetic genus of a surface of degree 6 in \( \mathbb{P}^5 \) appearing in [7, Corollary 3.6], combined with [9, loc. cit.], enables us to state the bound

---

1. The bound is the genus of a plane curve of degree \( d/2 \). It should be compared with the sharp lower bound \( K_S^2 \geq -d(d - 6) \) and \( \chi(O_S) \geq -\frac{1}{4}d(d - 6) \), for a smooth projective surface \( S \) of degree \( d \) [8, 9].
2. This is the Castelnuovo’s bound for curves of degree \( d \) in \( \mathbb{P}^7 \) (see Section 2, (v), below).
in Theorem 1.2. As for the sharpness, we simply project a general Castelnuovo’s curve in \( \mathbb{P}^7 \), which is not contained in quadrics because so is the general projection of a smooth rational normal scroll surface [1, Theorem 2, Lemma 3.1].

In the case of curves contained in \( \mathbb{P}^r \), for certain values of \( r \geq 7 \), we are able to compute the sharp bound except for a constant term. We obtain similar results also in the case of cubics. In fact, we prove the following Proposition 1.3 and Proposition 1.4. For the definition of the number \( d_1(r, s) \), appearing in the claims, see Section (2), (vi), and (14) below.

**Proposition 1.3.** Fix an integer \( r \geq 7 \), not divisible by 3, and divide

\[
\left( \frac{r + 2}{2} \right) = 3h + k, \quad 0 \leq k \leq 2.
\]

Set:

\[
s = \begin{cases} 
  h - 1 & \text{if } k = 0 \\
  h & \text{if } k = 2.
\end{cases}
\]

Let \( C \subset \mathbb{P}^r \) be a reduced and irreducible complex curve, of degree \( d \) and maximal arithmetic genus \( p_a(C) \) with respect to the conditions of being of degree \( d \) and not contained in a quadric hypersurface. Assume \( d > d_1(r, s) \), and divide \( d - 1 = ms + \epsilon \), \( 0 \leq \epsilon \leq s - 1 \). Then:

\[
p_a(C) = \frac{d^2}{2s} - \frac{d}{2s} (s + 2) + O(1), \quad \text{with } 0 < O(1) \leq \frac{s^3}{r - 2}.
\]

Moreover, \( C \) is not a.C.M., and it is contained in a flag like \( S \subset T \subset \mathbb{P}^r \), where \( S \) is a unique surface of degree \( s \), and \( T \) is a cubic hypersurface of \( \mathbb{P}^r \). The surface \( S \) has sectional genus \( \pi = 0 \).

**Proposition 1.4.** Fix an integer \( r \geq 9 \), and assume that the number

\[
s := \frac{1}{6} \left( \left\lfloor \frac{r + 3}{3} \right\rfloor - 4 \right)
\]

is an integer. Let \( C \subset \mathbb{P}^r \) be a reduced and irreducible complex curve, of degree \( d \) and maximal arithmetic genus \( p_a(C) \) with respect to the conditions of being of degree \( d \) and not contained in a cubic hypersurface. Assume \( d > d_1(r, s) \), and divide \( d - 1 = ms + \epsilon \), \( 0 \leq \epsilon \leq s - 1 \). Then:

\[
p_a(C) = \frac{d^2}{2s} - \frac{d}{2s} (s + 2) + O(1), \quad \text{with } 0 < O(1) \leq \frac{s^3}{r - 2}.
\]

Moreover, \( C \) is not a.C.M., and it is contained in a flag like \( S \subset T \subset \mathbb{P}^r \), where \( S \) is a unique surface of degree \( s \), and \( T \) is a quartic hypersurface of \( \mathbb{P}^r \). The surface \( S \) has sectional genus \( \pi = 0 \).

---

3 The number \( s \) is the minimal integer such that \( \left( \frac{r+2}{2} \right) - 3(s+1) \) is \( \leq 0 \) (e.g. \( (r, s) = (7, 11), (8, 14), (10, 21) \)). The number \( r \) is not divisible by 3 if and only if \( k \neq 1 \).

4 This is equivalent to say that the class \([r]\) of \( r \) modulo 36 is one of the following classes: [1], [2], [9], [10], [11], [18], [19], [27], [29] (e.g. \( (r, s) = (9, 36), (10, 47), (11, 60), (18, 221), (19, 256) \)).
The proof follows the same line of Theorem 2.2. However, this analysis, when $r \geq 7$, leads to examine surfaces whose degree is out of the range considered in [7, Theorem 2.2]. We partly overcome this difficulty, using an estimate appearing in [5, Lemma]. This explains why we are not able to determine the sharp bound (i.e. the constant $O(1)$).

2. Notations and preliminary remarks.

(i) For a projective subscheme $V \subseteq \mathbb{P}^r$ we will denote by $\mathcal{I}_V = \mathcal{I}_{V,\mathbb{P}^r}$ its ideal sheaf in $\mathbb{P}^r$, and by $M(V) := \oplus_{i \in \mathbb{Z}} H^1(V, \mathcal{I}_V(i))$ the Hartshorne-Rao module of $V$. We will denote by $h_V$ the Hilbert function of $V$, and by $\Delta h_V$ the first difference of $h_V$, i.e. $\Delta h_V(i) = h_V(i) - h_V(i - 1)$. Observe that
\begin{equation}
(4) \quad h_V(i) = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) - h^0(\mathbb{P}^r, \mathcal{I}_V(i)) \leq h^0(V, \mathcal{O}_V(i)).
\end{equation}

We denote by $p_a(V)$ the arithmetic genus of $V$. We say that $V$ is arithmetically Cohen-Macaulay (shortly a.C.M.) if all the restriction maps $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(i)) \to H^0(V, \mathcal{O}_V(i))$ $(i \in \mathbb{Z})$ are surjective, and $H^j(V, \mathcal{O}_V(i)) = 0$ for all $i \in \mathbb{Z}$ and $1 \leq j \leq \dim V - 1$. If $V$ is one-dimensional of degree $d$, and $V'$ denotes its general hyperplane section, then [9, p. 83-84]:
\begin{equation}
(5) \quad \Delta h_V(i) \geq h_{V'}(i) \text{ for every } i,
\end{equation}
and
\begin{equation}
(6) \quad p_a(V) \leq \sum_{i=1}^{+\infty} d - h_{V'}(i).
\end{equation}
Moreover, the equality occurs in (5) (resp. in (6)) if and only if $V$ is a.C.M.. If $V$ is integral (i.e. reduced and irreducible) and one-dimensional, we also have [9, p. 86-87, Corollary (3.5) and (3.6)]:
\begin{equation}
(7) \quad h_{V'}(i + j) \geq \min\{d, h_{V'}(i) + h_{V'}(j) - 1\} \text{ for every } i, j,
\end{equation}
and
\begin{equation}
(8) \quad h_{V'}(i) \geq \min\{d, i(r - 1) + 1\} \text{ for every } i.
\end{equation}
If $V$ is integral, and $\dim V \geq 2$, then $V$ is a.C.M. if and only if its general hyperplane section is.

(ii) Let $\Sigma \subset \mathbb{P}^{r-1}$ be an integral curve of degree $s$ in $\mathbb{P}^{r-1}$, $r \geq 4$, and $\Sigma' \subset \mathbb{P}^{r-2}$ its general hyperplane section. For every $i$ we have: $h^0(\Sigma, \mathcal{O}_\Sigma(i)) = 1 - \pi + is + h^1(\Sigma, \mathcal{O}_\Sigma(i))$, where $\pi$ denotes the arithmetic genus of $\Sigma$. From the exact sequence $0 \to \mathcal{O}_\Sigma(i - 1) \to \mathcal{O}_\Sigma(i) \to \mathcal{O}_{\Sigma'}(i) \to 0$ we get the exact sequence $0 \to H^0(\Sigma, \mathcal{O}_\Sigma(i - 1)) \to H^0(\Sigma, \mathcal{O}_\Sigma(i)) \to H^0(\Sigma', \mathcal{O}_{\Sigma'}(i)) \to H^1(\Sigma, \mathcal{O}_\Sigma(i - 1)) \to H^1(\Sigma, \mathcal{O}_\Sigma(i)) \to 0$. We deduce that $h^1(\Sigma, \mathcal{O}_\Sigma(i)) \leq h^1(\Sigma, \mathcal{O}_\Sigma(i - 1))$. In particular, when $i \geq 1$, we have $h^1(\Sigma, \mathcal{O}_\Sigma(i)) \leq \pi = h^1(\Sigma, \mathcal{O}_\Sigma)$. Observe that if, for some $i \geq 1$, one has $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = \pi$, then $\pi = 0$. In fact, in this case, we have $h^1(\Sigma, \mathcal{O}_\Sigma(1)) = \pi$, hence $h^0(\Sigma, \mathcal{O}_\Sigma(1)) = 1 + s$. This means that $\Sigma$ is contained in $\mathbb{P}^s$ as a non-degenerate curve of degree $s$, so $\pi = 0$. Similarly, if for some
\( i \geq 1, \) one has \( h^1(\Sigma, \mathcal{O}_{\Sigma}(i)) = \pi - 1, \) then \( \pi = 1. \) In fact, in this case, we have \( h^1(\Sigma, \mathcal{O}_{\Sigma}(1)) = \pi - 1, \) hence \( h^0(\Sigma, \mathcal{O}_{\Sigma}(1)) = s. \) This means that \( \Sigma \) is contained in \( \mathbb{P}^{s-1} \) as a non-degenerate curve of degree \( s. \) By Castelnuovo’s bound, it follows that \( \pi = 1. \) In conclusion, since \( h^0(\Sigma, \mathcal{O}_{\Sigma}(i)) = 1 - \pi + is + h^1(\Sigma, \mathcal{O}_{\Sigma}(i)), \) for every \( i \geq 1 \) we have:

\[
h^0(\Sigma, \mathcal{O}_{\Sigma}(i)) \leq \begin{cases} 1 + is & \text{if } \pi \geq 0, \\ is & \text{if } \pi \geq 1, \\ -1 + is & \text{if } \pi \geq 2. \\
\end{cases}
\]  

\( (9) \)

- \( (iii) \) Let \( S \subset \mathbb{P}^r \) (\( r \geq 4 \)) be an integral surface of degree \( s, \) and \( \Sigma \) its general hyperplane section. From the exact sequence \( 0 \to \mathcal{O}_S(i-1) \to \mathcal{O}_S(i) \to \mathcal{O}_{\Sigma}(i) \to 0 \) we get, for every \( i \geq 1, h^0(S, \mathcal{O}_S(i)) \leq \sum_{j=0}^{i} h^0(\Sigma, \mathcal{O}_{\Sigma}(j)). \) Hence, for every \( i \geq 1, \) we have (compare with \( (4) \)) \( h^0(\mathbb{P}^r, \mathcal{I}_S(i)) \geq \binom{i+1}{1} - \sum_{j=0}^{i} h^0(\Sigma, \mathcal{O}_{\Sigma}(j)). \) Combining with \( (9), \) it follows that:

\[
h^0(\mathbb{P}^r, \mathcal{I}_S(i)) \geq \begin{cases} \binom{i+1}{1} - \left[ i + 1 + \binom{i+1}{2} \right] s & \text{if } \pi \geq 0, \\ \binom{i+1}{1} - \left[ 1 + \binom{i+1}{2} \right] s & \text{if } \pi \geq 1, \\ \binom{i+1}{1} - \left[ 1 - i + \binom{i+1}{2} \right] s & \text{if } \pi \geq 2. \\
\end{cases}
\]  

\( (10) \)

- \( (iv) \) Let \( S \subset \mathbb{P}^{s+1} \) be a non-degenerate smooth rational normal scroll surface, of minimal degree \( s. \) Fix an integer \( k \geq 1. \) Since \( S \) is a.C.M., we have \( h^1(S, \mathcal{O}_S(k)) = 0. \) Moreover, one has \( K_S = -2\Sigma + (s - 2)W, \) where \( \Sigma \) is the general hyperplane section, and \( W \) the ruling. Therefore, one has \( h^2(S, \mathcal{O}_S(k)) = h^0(S, K_S - k\Sigma) = 0. \) By Riemann-Roch Theorem it follows that: \( h^0(S, \mathcal{O}_S(k)) = 1 + \frac{1}{2}k\Sigma \cdot (k\Sigma - K_S) = k + 1 + \binom{k+1}{2}s. \) In particular:

\[
h^0(S, \mathcal{O}_S(2)) = 3(1 + s), \quad \text{and} \quad h^0(S, \mathcal{O}_S(3)) = 4 + 6s.
\]  

- \( (v) \) Fix integers \( s \geq 2 \) and \( d \geq s + 1, \) and divide \( d - 1 = ms + \epsilon, 0 \leq \epsilon \leq s - 1. \) The number

\[
G(s + 1; d) := \left( \frac{m}{2} \right)s + me
\]

is the celebrated Castelnuovo’s bound for the genus of a non-degenerate integral curve of degree \( d \) in \( \mathbb{P}^{s+1} \) [9, p. 87, Theorem (3.7)]. Observe that

\[
G(s + 1; d) = \frac{d^2}{2s} + \frac{d}{2s}(-s - 2) + \frac{1 + \epsilon}{2s}(s + 1 - \epsilon) \leq \frac{d^2}{2s}.
\]  

\( (12) \)

- \( (vi) \) Fix integers \( r, d \) and \( s, \) with \( s \geq r - 1 \geq 2. \) Denote by \( G(r; d, s) \) the maximal arithmetic genus for an integral non-degenerate projective curve \( C \subset \mathbb{P}^r \) of degree \( d, \) not contained in any surface of degree \( < s \) [9]. Put

\[
d_0(r, s) := \frac{2s}{r - 2} \prod_{j=1}^{r-2}((r-1)!)^{1}. 
\]
By [3] Main Theorem and [5] Lemma, one knows that, for \( d > d_0(r, s) \), the number \( G(r; d, s) \) has the following form:
\[
G(r; d, s) = \frac{d^2}{2s} + \frac{d}{2s} \left[ 2G(r-1; s) - 2 - s \right] + R, \quad \text{with} \quad |R| \leq \frac{s^3}{r-2}.
\]
When \( r = 4 \) and \( s = 6 \), this holds true also for \( d > 143 \), and when \( r = 5 \) and \( s = 7 \), this holds true also for \( d > 179 \) [3] Theorem (3.22), p. 117. Moreover, taking into account (13), and that \( G(r-1; s+1) \leq \frac{(s+1)^2}{2(r-2)} \) (compare with (12)), an elementary computation, which we omit, shows that, for
\[
d > d_1(r, s) := \max \left\{ d_0(r, s), \frac{4s}{r-2}(s+1)^3 \right\},
\]
once has
\[
G(r; d, s+1) < G(s+1; d).
\]

- (vii) Let \( S \subset \mathbb{P}^r \) be an integral non-degenerate projective surface of degree \( s \geq r-1 \geq 2 \). Denote by \( \sigma \) the integer part of the number
\[
(s - r + 2) \left( \frac{s^2}{2(r-2)} + 1 \right) + 1.
\]
By [7] Lemma 3.8] we know that
\[
h^1(\mathbb{P}^r, \mathcal{I}_S(i)) = 0 \quad \text{for every} \quad i \geq \sigma.
\]
Now, suppose there exists an a.C.M. curve \( C \) on \( S \) of degree \( d \), with \( d-1 = ms + \epsilon \), \( 0 \leq \epsilon \leq s - 1 \), and \( m \geq \sigma \). Let \( \Sigma \) and \( \Gamma \) be the general hyperplane sections of \( S \) and \( C \). By Bezout’s theorem we have \( h_S(i) = h_C(i) \) and \( h_{\Sigma}(i) = h_{\Gamma}(i) \) for every \( i \leq m \). Since \( C \) is a.C.M., we have \( \Delta h_C(i) = h_{\Gamma}(i) \) for every \( i \) (compare with \( \ref{lem:3.1} \) and three lines below). It follows that \( \Delta h_S(i) = h_{\Sigma}(i) \) for every \( i \leq m \). Therefore, since
\[
(\mu_i := \Delta h_S(i) - h_{\Sigma}(i) = \dim_{\mathbb{C}} \ker (H^1(\mathbb{P}^r; \mathcal{I}_S(i-1)) \to H^1(\mathbb{P}^r; \mathcal{I}_S(i)))]
\]
(see [2] Lemma 3.4 and Remark 3.5), we get \( H^1(\mathbb{P}^r, \mathcal{I}_S(i-1)) \subseteq H^1(\mathbb{P}^r, \mathcal{I}_S(i)) \) for every \( i \leq m \). Combining with (16), it follows that \( M(S) = 0 \). Summing up: if a surface \( S \) contains an a.C.M. curve of degree \( d \) with \( m \geq \sigma \), then \( H^1(\mathbb{P}^r, \mathcal{I}_S(i)) = 0 \) for every \( i \in \mathbb{Z} \) (compare with [7] Remark 3.10, (i)). Notice that the condition \( m \geq \sigma \) is satisfied when \( r = 5 \) and \( s = 6 \) and \( d > 179 \), or when \( d > d_1(r, s) \).

### 3. The Proof of Theorem 1.1

We start with the proof of the first part of Theorem 1.1. We need the following Lemma 3.1, which is certainly well known. We prove it for lack of a suitable reference.

**Lemma 3.1.** Let \( V \subset \mathbb{P}^5 \) be the Veronese surface. Let \( \text{Sec}(V) \) be the secant variety of \( V \), \( x \in \mathbb{P}^5 \setminus \text{Sec}(V) \) be a point. Then the projection in \( \mathbb{P}^4 \) of \( V \) from \( x \) is a surface of degree 4, not contained in quadrics. Conversely, every integral surface of degree 4 in \( \mathbb{P}^4 \) not contained in quadrics is the projection of \( V \) from a point \( x \in \mathbb{P}^5 \setminus \text{Sec}(V) \).
Proof. Let $V'$ be the projection in $\mathbb{P}^4$ of $V$ from a point $x \in \mathbb{P}^5 \setminus \text{Sec}(V)$. Suppose there exists a quadric $Q$ in $\mathbb{P}^4$ containing $V'$. If $Q$ were smooth, then, by Severi’s theorem, $V$ would be a complete intersection of two quadrics. This implies that the sectional genus of $V$ is 1, in contrast with the fact that it is 0. If $Q$ were singular, then $Q$ would be a cone. Let $H \subset \mathbb{P}^4$ be a general hyperplane, passing from the vertex of the cone (if it is a point). Then $H \cap V$ is an integral curve of degree 4 of a quadric cone in $\mathbb{P}^3$. Therefore, the genus of $H \cap V$ would be 1, a contradiction, because it is 0.

Conversely, let $S$ be a surface of degree 4 in $\mathbb{P}^4$ not contained in quadrics. Let $\pi$ be the sectional genus of $S$. By Castelnuovo’s bound, we have $\pi = 0$ either $\pi = 1$. If $\pi = 1$, then the general hyperplane section $H \cap S$ of $S$ is a Castelnuovo’s curve in $\mathbb{P}^3$. This curve is contained in a quadric, which lifts to a quadric containing $S$ because $H \cap S$ is a.C.M., hence also $S$ is. Therefore, $\pi = 0$. In this case, by [15] Lemma 7, p. 411, we know that $S$ is the projection of $V$ from some point $x \in \mathbb{P}^5 \setminus V$. Now we observe that $x \notin \text{Sec}(V) \setminus V$, otherwise $S$ is contained in a quadric [12] Remark 2.1, p. 60-61].

We are in position to prove Theorem 1.1 first part.

Let $C \subset \mathbb{P}^4$ be a curve of degree $d$ not contained in quadrics. Since a surface of degree 3 in $\mathbb{P}^4$ is contained in a quadric (Section 2, (10)), in view of previous Lemma 5.2 it suffices to prove that if $C$ is not contained in a surface of degree $< 5$, then $p_a(C) < \frac{1}{5}d(d-6) + 1$. To this purpose, set

$$G := \frac{1}{10}d^2 - \frac{3}{10}d + \frac{1}{5}v - \frac{1}{10}v^2 + w,$$

where $v$ is defined by dividing $d - 1 = 5n + v$, $0 \leq v \leq 4$, and $w := \max\{0, [\frac{d}{10}]\}$. By [9] Theorem (3.22), p. 117] (with the notation of [9] one has $G = \pi_2(d,4)$), we know that, if $C$ is not contained in a surface of degree $< 5$, and $d > 143$, then $p_a(C) \leq G$. An elementary computation shows that $G < \frac{1}{5}d(d-6) + 1$ for $d > 18$. This concludes the proof of the first part of Theorem 1.1 when $d > 143$. We may examine the remaining cases $16 < d \leq 143$ in a similar manner as in [8] p. 74-75]. For details, we refer to the Appendix (see Section 7 below).

Now we are going to prove Theorem 1.1 second part.

First, we notice that if $d$ is odd, or $C$ is a.C.M., and $C$ is not contained in quadrics, then $C$ is not contained in surfaces of degree $< 5$. In fact, by 110, every surface in $\mathbb{P}^4$ of degree 3 is contained in a quadric. This is true also for surfaces of degree 4, except for an isomorphic projection $S$ of the Veronese surface (Lemma 5.1). But on the Veronese surface every curve has degree even. And every curve on $S$ cannot be a.C.M. (otherwise, from the natural sequence: $0 \rightarrow \mathcal{I}_S \rightarrow \mathcal{I}_C \rightarrow \mathcal{O}_S(-C) \rightarrow 0$, we would get the exact sequence: $H^0(S, \mathcal{O}_S(H_S - C)) \rightarrow H^1(\mathbb{P}^4, \mathcal{I}_S(1)) \rightarrow H^1(\mathbb{P}^4, \mathcal{I}_C(1))$; this is impossible, because $H^0(S, \mathcal{O}_S(H_S - C)) = 0$ and $H^1(\mathbb{P}^4, \mathcal{I}_S(1)) \neq 0$).
On the other hand, if $C$ is not contained in a surface of degree $< 6$, and $d > 143$, then (compare with Section 2, (v)), and (13)

$$p_a(C) \leq G(4; d, 6) \leq \frac{d^2}{12} + 108.$$  

By an elementary comparison, it follows that, for $d > 143$, $p_a(C)$ is strictly less than the bound (11) appearing in our claim. Therefore, in order to prove the second part of Theorem 1.1, we may assume that $C$ is contained in an integral surface $S \subset \mathbb{P}^4$ of degree 5, not contained in quadrics. Observe that, by Bezout’s theorem, such a surface $S$ is unique, and is contained in a cubic hypersurface of degree 3 by (11).

Let $\Sigma \subset \mathbb{P}^3$ be the general hyperplane section of $S$. Since $\deg \Sigma = 5$, by Castelnuovo’s bound, the arithmetic genus $\pi$ of $\Sigma$ satisfies the condition $0 \leq \pi \leq 2$. The sectional genus $\pi$ cannot be 2, otherwise $\Sigma$ should be a Castelnuovo’s curve, hence a.C.M., and contained in a quadric. This would imply that also $S$ is a.C.M., and contained in a quadric (since $h^1(\mathbb{P}^4, \mathcal{I}_S(1)) = 0$, the restriction map $H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^3, \mathcal{I}_{\Sigma, \mathbb{P}^3}(2))$, induced by the exact sequence $0 \to \mathcal{I}_S(1) \to \mathcal{I}_S(2) \to \mathcal{I}_{\Sigma, \mathbb{P}^3}(2) \to 0$, is onto). Therefore, we have $0 \leq \pi \leq 1$. Since $\Sigma$ is non-degenerate, from the exact sequence $0 \to \mathcal{I}_{\Sigma, \mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_\Sigma(1) \to 0$ it follows that $h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma, \mathbb{P}^3}(1)) = 2 - \pi > 0$ (observe that $h^1(\Sigma, \mathcal{O}_\Sigma(i)) = 0$ for every $i \geq 1$, because $0 \leq \pi \leq 1$). Moreover, by [13, Theorem 1], we have

$$h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma, \mathbb{P}^3}(i - \pi)) = 0 \quad \text{for every } i \geq 3.$$  

Let $\Sigma' \subset \mathbb{P}^2$ be the general plane section of $\Sigma$. By [8] we have $h_{\Sigma'}(i) = 5$ for every $i \geq 2$. Hence, $h^1(\mathbb{P}^2, \mathcal{I}_{\Sigma', \mathbb{P}^2}(i)) = 0$ for every $i \geq 2$, and from the exact sequence $0 \to \mathcal{I}_{\Sigma', \mathbb{P}^2}(i - 1) \to \mathcal{I}_{\Sigma', \mathbb{P}^2}(i) \to \mathcal{I}_{\Sigma', \mathbb{P}^2}(i) \to 0$ it follows that $h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma', \mathbb{P}^2}(2)) \leq h^1(\mathbb{P}^3, \mathcal{I}_{\Sigma, \mathbb{P}^3}(1))$. Summing up, for every $i \geq 1$, we have:

$$h^1(\mathbb{P}^3, \mathcal{I}_C(i)) \leq \max\{0, 3 - \pi - i\} + \mu(i),$$

with $\mu(i) = 1$ if $\pi = 0$ and $i = 2$, and $\mu(i) = 0$ otherwise. Now, divide $d - 1 = 5m + \epsilon$, $0 \leq \epsilon \leq 4$, and set (recall that $0 \leq \pi \leq 1$):

$$h_{d, \pi}(i) = 1 - \pi + 5i - \max\{0, 3 - \pi - i\} - \mu(i)$$

for $1 \leq i \leq m$, and, for $i \geq m + 1$, $h_{d, \pi}(i) = d$, except for the case $\pi = 1$, $\epsilon = 4$, $i = m + 1$, in which we set $h_{d, \pi}(m + 1) = d - 1$. If we denote by $\Gamma$ the general hyperplane section of $C$, the same analysis appearing in the proof of [13, Lemma 3.3], shows that $h_\Gamma(i) \geq h_{d, \pi}(i)$ for every $i \geq 1$. By (19) it follows that

$$p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq \sum_{i=1}^{+\infty} d - h_{d, \pi}(i) =: G_{d, \pi}.$$  

Now, observe that

$$G_{d, 0} = 5 \left(\frac{m}{2}\right) + m\epsilon + 4, \quad \text{and} \quad G_{d, 1} = 5 \left(\frac{m}{2}\right) + m(\epsilon + 1) + 1 + \left(\frac{\epsilon}{4}\right).$$

Therefore, since $G_{d, 0} < G_{d, 1}$, and taking into account that $G_{d, 1}$ is exactly the bound appearing in (11), in order to complete the proof we only have to exhibit examples
with \( p_a(C) = G_{d,1} \) (in this case, by (19) and Section 2, (i), we know that \( C \) is a.C.M.).

To this purpose, fix an elliptic curve \( \Sigma \subset \mathbb{P}^3 \) of degree 5 (compare with [7, p. 101-103]). By (18) we know that \( h^1(\mathbb{P}^3, I_\Sigma(2)) = 0 \). It follows that \( \Sigma \) is not contained in quadrics, because \( H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \) and \( H^0(\Sigma, \mathcal{O}_\Sigma(2)) \) have the same dimension, 10. Let \( S = C(\Sigma) \subset \mathbb{P}^4 \) be the cone over \( \Sigma \). Let \( D \subset \mathbb{P}^4 \) be a subset formed by \( 4 - \epsilon \) points. Let \( C(D) \subset C(\Sigma) \) be the cone over \( D \). Let \( \mu \geq 3 \) be an integer. Let \( F \subset \mathbb{P}^4 \) be a hypersurface of degree \( \mu + 1 \) containing \( C(D) \), consisting of \( \mu + 1 \) sufficiently general hyperplanes. Let \( R \) be the residual curve to \( C(D) \) in the complete intersection of \( F \) with \( S \). Equipped with the reduced structure, \( R \) is a cone over \( k \) distinct points of \( \Sigma \), with \( k - 1 = 5\mu + \epsilon \). In particular, \( R \) is a (reducible) a.C.M. curve of degree \( k \) on \( S \), and, if we denote by \( R' \) the hyperplane section of \( R \) with the hyperplane \( \mathbb{P}^3 \subset \mathbb{P}^4 \) containing \( \Sigma \), we have \( p_a(R) = \sum_{i=1}^{\infty} k - h_{R'}(i) \) (compare with (19) and the first line below). On the other hand, since \( R' \subset \Sigma \), it is clear that \( h_{R'}(i) = h_{k,1}(i) \) for every \( i \geq 1 \). Hence, we have:

\[
p_a(R) = G_{k,1} = 5 \left( \frac{\mu}{2} \right) + \mu(1 + \epsilon) + 1 + \left( \frac{\epsilon}{4} \right).
\]

Now, let \( m \gg 0 \), and let \( G \subset \mathbb{P}^4 \) be a hypersurface of degree \( m + 1 \) containing \( C(D) \) such that the residual curve \( C \) in the complete intersection of \( G \) with \( S \), equipped with the reduced structure, is an integral curve of degree \( d = 5m + \epsilon + 1 \), with a singular point of multiplicity \( k \) at the vertex \( p \) of \( S \), and tangent cone at \( p \) equal to \( R \). We are going to prove that \( C \) is the curve we are looking for, i.e.

\[
p_a(C) = G_{d,1}.
\]

To this aim, let \( \tilde{S} \) be the blowing-up of \( S \) at the vertex. By [11, p. 374], we know that \( \tilde{S} \) is the ruled surface \( \mathbb{P}(\mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-1)) \to \Sigma \). Denote by \( E \) the exceptional divisor, by \( f \) the line of the ruling, and by \( L \) the pull-back of the hyperplane section. We have \( L^2 = 5 \), \( L \cdot f = 1 \), \( f^2 = 0 \), \( L \equiv E + 5f \) and \( K_{\tilde{S}} \equiv -2L + 5f \). Let \( \tilde{C} \subset \tilde{S} \) be the blowing-up of \( C \) at \( p \), which is nothing but the normalization of \( C \). Since \( C \) has degree \( d \), \( \tilde{C} \) belongs to the numerical class of \( (m + 1 + a)L + (1 + \epsilon - 5(a + 1))f \) for some integer \( a \). Moreover \( E \cdot \tilde{C} = 1 + \epsilon - 5(a + 1) = k \), so

\[
a = -\frac{k + 4 - \epsilon}{5} = -\mu - 1.
\]

By the adjunction formula we get

\[
p_a(\tilde{C}) = 5 \left( \frac{m}{2} \right) + m(\epsilon + 1) + 1 - \frac{5}{2}a^2 + a \left( \epsilon - \frac{3}{2} \right).
\]

On the other hand, we have \( p_a(C) = p_a(\tilde{C}) + \delta_p \), where \( \delta_p \) is the delta invariant of the singularity \( (C, p) \). Since the tangent cone of \( C \) at \( p \) is \( R \), the delta invariant is equal to the difference between the arithmetic genus of \( R \) and the arithmetic genus of \( k \) disjoint lines in the projective space, i.e.

\[
\delta_p = p_a(R) - (1 - k) = 5 \left( \frac{\mu}{2} \right) + \mu(1 + \epsilon) + 1 + \left( \frac{\epsilon}{4} \right) - (1 - k).
\]
It follows that
\[ p_a(C) = 5 \left( \frac{m}{2} \right) + m(\epsilon + 1) + 1 - \frac{5}{2} \mu + a \left( \epsilon - \frac{3}{2} \right) + 5 \left( \frac{\mu}{2} \right) + \mu(1 + \epsilon) + 1 + \left( \epsilon \frac{3}{4} \right) - (1 - k). \]

Taking into account that \( a = -\mu - 1 \), a direct computation proves that this number is exactly \( G_{d,1} \). This concludes the proof of Theorem 1.1.

4. The proof of Theorem 1.2

By (10), every surface of degree \( \leq 5 \) in \( \mathbb{P}^5 \) is contained in a quadric. Moreover, if \( C \) is not contained in a surface of degree \( < 7 \), and \( d > 179 \), then (see Section 2, (vi), and (13)):
\[ p_a(C) \leq G(5; d, 7) \leq \frac{d^2}{14} - \frac{3}{14} d + 115. \]

An elementary comparison, relying on (12), shows that this number is strictly less than \( G(7; d) = 6 \left( \frac{m}{2} \right) + m \epsilon \) for \( d > 179 \). Therefore, in order to prove Theorem 1.2 we may assume that \( C \) is contained in a surface \( S \) of degree 6. Such a surface is unique by Bezout’s theorem, and is contained in a hypersurface of degree 3 by (10). If \( \pi \) denotes the sectional genus of \( S \), by Castelnuovo’s bound we have \( 0 \leq \pi \leq 2 \), and \( \pi \) cannot be equal to 1 or 2, otherwise, by (10), \( S \) is contained in a quadric. It follows that \( \pi = 0 \).

Let \( S \subset \mathbb{P}^5 \) be a surface of degree 6 and sectional genus \( \pi = 0 \). By [7, Corollary 3.6] we know that \(-3 \leq p_a(S) \leq 0 \). Now we are going to prove that if \(-3 \leq p_a(S) \leq -1 \), then \( S \) is contained in a quadric.

- If \( p_a(S) = -3 \), by [2, Remark 3.7] we get \( h^1(\mathbb{P}^5, \mathcal{I}_S(1)) = 0 \). Therefore, the map \( H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \) is onto. This implies that \( S \) is contained in a quadric, because it is so for every curve of degree 6 in \( \mathbb{P}^4 \) by (3) and (4).

- Assume \( p_a(S) = -2 \). Since \( \Sigma \) has degree 6 and arithmetic genus 0, by [7, Proposition 3.1] we know that \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(1)) = 2 \), \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) \leq 1 \), and \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(i)) = 0 \) for \( i \geq 3 \).

In the case \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0 \), by [7, Lemma 3.4] we get (compare with (17)):
\[ -2 = p_a(S) = - \dim M(\Sigma) + \sum_{i=1}^{i=\infty} \mu_i = -2 + \sum_{i=1}^{i=\infty} \mu_i. \]
Therefore, \( \sum_{i=1}^{i=\infty} \mu_i = 0 \), and so \( h^1(\mathbb{P}^5, \mathcal{I}_S(1)) = 0 \). As before, we deduce that \( S \) lies on a quadric.

In the case \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 1 \), we have \( h^0(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 3 \). In view of the exact sequence \( H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \to H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \to H^1(\mathbb{P}^5, \mathcal{I}_S(1)) \), in order to prove that \( S \) lies on a quadric, it suffices to prove that
\[ h^1(\mathbb{P}^5, \mathcal{I}_S(1)) \leq 2. \]

This follows from the exact sequence \( 0 \to \mathcal{I}_S \to \mathcal{O}_{\mathbb{P}^5} \to \mathcal{O}_S \to 0 \), taking into account that \( h^0(\mathcal{O}_S(1)) \leq 8 \), in view of the exact sequence \( 0 \to \mathcal{O}_S \to \mathcal{O}_S(1) \to \mathcal{O}_S(1) \to 0 \).

- Assume \( p_a(S) = -1 \). As before, we have two cases, \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 0 \), or \( h^1(\mathbb{P}^4, \mathcal{I}_\Sigma(2)) = 1 \).
If \( h^1(\mathbb{P}^4, \mathcal{I}_S(2)) = 0 \), then \(-1 = p_a(S) = -\dim M(\Sigma) + \sum_{i=1}^{+\infty} \mu_i = -2 + \sum_{i=1}^{+\infty} \mu_i \). Hence \( \sum_{i=1}^{+\infty} \mu_i = 1 \). Combining with the exact sequence \( H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \rightarrow H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \rightarrow H^1(\mathbb{P}^5, \mathcal{I}_S(1)) \rightarrow H^1(\mathbb{P}^4, \mathcal{I}_S(2)) \), we get \( H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \neq 0 \), otherwise the map on the right should have a kernel of dimension \( \geq 2 = h^0(\mathbb{P}^4, \mathcal{I}_S(2)) \), in contrast with the fact that \( \mu_2 \leq 1 \).

If \( h^1(\mathbb{P}^4, \mathcal{I}_S(2)) = 1 \), then \( h^0(\mathbb{P}^4, \mathcal{I}_S(2)) = 3 \). Hence, the map \( H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \rightarrow H^1(\mathbb{P}^5, \mathcal{I}_S(1)) \) has non trivial kernel, because \( h^1(\mathbb{P}^5, \mathcal{I}_S(1)) \leq 2 \) (see (20)). This implies \( H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \neq 0 \).

Summing up, in order to prove Theorem [22], we may assume that \( C \) is contained in a surface \( S \) of degree 6, sectional genus \( \pi = 0 \), and arithmetic genus \( p_a(S) = 0 \).

On such a surface, the genus of \( C \) satisfies the bound in view of [7, Corollary 3.11] (here we need \( d > 215 \)). Moreover, \( C \) cannot be a.C.M.. Otherwise, \( M(S) = 0 \) (see Section 2, (vii)). In particular, \( H^1(\mathbb{P}^5, \mathcal{I}_S(1)) = 0 \), and so the restriction map \( H^0(\mathbb{P}^5, \mathcal{I}_S(2)) \rightarrow H^0(\mathbb{P}^4, \mathcal{I}_S(2)) \) is onto. This is impossible, because \( \Sigma \) is contained in a quadric, and \( S \) not.

It remains to prove that the bound is sharp. To this purpose, let \( S' \subset \mathbb{P}^7 \) be a smooth rational normal scroll surface, of degree 6. Let \( C' \) be a Castelnuovo’s curve on \( S' \) of degree \( d \). Let \( S \) be the general projection of \( S' \) in \( \mathbb{P}^5 \). Then the image \( C \) of \( C' \) is an extremal curve. In fact, \( S \) is not contained in quadrics. To prove this, we use the fact that \( S \) is of maximal rank [11, Theorem 2, Lemma 3.1]. In particular, the map \( H^0(\mathbb{P}^5, \mathcal{O}_{S'}(2)) \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_S(2)) \) is injective or surjective. Since both spaces have the same dimension (see (11)), it follows that \( h^0(\mathbb{P}^5, \mathcal{I}_S(2)) = 0 \).

This concludes the proof of Theorem [12].

5. The proof of Proposition 1.3

By the definition of \( s \), we have \( -(r+2) - (3+3(s-1)) > 0 \). Hence, by (10), every surface of degree \( < s \) is contained in a quadric. In particular, the extremal curve \( C \) cannot be contained in a surface of degree \( < s \). Moreover, if \( C \) is not contained in a surface of degree \( < s + 1 \), then (compare with (14), (15), and (12))

\[
(21) \quad p_a(C) \leq G(r; d, s + 1) < G(s + 1; d) = \frac{d^2}{2s} - \frac{d}{2s}(s + 2) + \frac{1+\epsilon}{2s}(s + 1 - \epsilon).
\]

On the other hand, let \( S' \subset \mathbb{P}^{s+1} \) be a smooth rational normal scroll surface, of degree \( s \). Let \( D' \) be a Castelnuovo’s curve on \( S' \) of degree \( d \). Let \( S \equiv S' \) be the general projection of \( S' \) in \( \mathbb{P}^r \). By [11, loc. cit.], \( S \) is of maximal rank. Since

\[
h^0(\mathbb{P}^r, \mathcal{O}_{D'}(2)) = \left( \frac{r+2}{2} \right) \leq 3 + 3s = h^0(\mathbb{P}^4, \mathcal{O}_S(2))
\]

(compare with the definition of \( s \) and (11)), it follows that the restriction map \( H^0(\mathbb{P}^r, \mathcal{O}_{D'}(2)) \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_S(2)) \) is injective. Hence, \( S \), and so the image \( D \) of \( D' \), are not contained in quadrics. Moreover, since \( D \) is a Castelnuovo’s curve, we have (compare with (12))

\[
(22) \quad p_a(D) = p_a(D') = G(s + 1; d) > \frac{d^2}{2s} - \frac{d}{2s}(s + 2).
\]
In view of (21) and (22), in order to prove (2), we may assume that $C$ is contained in a surface $S$ of degree $s$. By (10), the sectional genus of $S$ should be $\pi = 0$, otherwise $S$ is contained in a quadric (here we have to assume $k \neq 1$, i.e. that $r$ is not divisible by 3). Now, if $C$ is contained in a surface $S$ of degree $s$ and sectional genus $\pi = 0$, then by [5, Lemma] we know that
\[
p_a(C) \leq \frac{d^2}{2s} - \frac{d}{2s}(s + 2) + O(1), \quad \text{with} \quad O(1) \leq \frac{s^3}{r^2}.
\]
Taking into account (22), we deduce (2).

Since $d > d_1(r, s)$, by Bezout’s theorem the surface $S$ containing $C$ is unique. By (10) and the definition of $s$, it follows that $S$ is contained in a cubic hypersurface. Moreover, $C$ cannot be a.C.M.. Otherwise, $M(S) = 0$ (see Section 2, (vii)). In particular, $H^1(\mathbb{P}^r, I_S(1)) = 0$, and so the restriction map $H^0(\mathbb{P}^r, I_S(2)) \to H^0(\mathbb{P}^r, I_{\mathbb{P}^r-1, S}(2))$ is onto ($\Sigma =$ the general hyperplane section of $S$). This is impossible, because, in view of (1) and (9), $\Sigma$ is contained in a quadric, and $S$ not. This concludes the proof of Proposition 1.3.

Remark 5.1. When 3 divides $r$, i.e. when $k = 1$, previous argument does not work, because it may happen that, for $\pi > 0$, a surface of degree $s (3s + 1 = \frac{r^2 + 1}{2})$ is not contained in quadrics (however, it is necessary that $\pi \leq 1$ by (10)). For instance, one may consider a general projection $S$ in $\mathbb{P}^6$ of a 3-ple Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^9$. In this case, repeating the same argument as before (compare with [1, Theorem 3] and [5, loc. cit.]), one may prove that the sharp bound (for curves in $\mathbb{P}^6$ not contained in quadrics, and of degree $d = 0$ mod. 3, $d > d_1(r, s)$) is
\[
p_a(C) = \frac{d^2}{2s} - \frac{d}{2} + O(1),\]
with $s = 9$, and $0 < O(1) \leq 182$. The formula (23) remains true (with $0 < O(1) \leq \frac{s^3}{r^2}$) if there exists a surface of degree $s$ in $\mathbb{P}^r$, with sectional genus $\pi = 1$, and not contained in quadrics (at least when $\epsilon = s - 1$, i.e. when $d$ is a multiple of $s$).

6. The proof of Proposition 1.4

The proof is quite similar to the proof of previous Proposition 1.3. Hence, we omit some details.

By the definition of $s$ and by (10), every surface of degree $< s$ is contained in a cubic (here we need that $6s + 4 \leq \frac{r^2 + 3}{3}$). In particular, the extremal curve $C$ cannot be contained in a surface of degree $< s$. Moreover, if $C$ is not contained in a surface of degree $< s + 1$, then (compare with (14), (15), and (12))
\[
p_a(C) \leq G(r; d, s + 1) < G(s + 1; d) = \frac{d^2}{2s} - \frac{d}{2s}(s + 2) + \frac{1 + \epsilon}{2s}(s + 1 - \epsilon).
\]
On the other hand, let $S' \subset \mathbb{P}^{s+1}$ be a smooth rational normal scroll surface, of degree $s$. Let $D'$ be a Castelnuovo’s curve on $S'$ of degree $d$. Let $S (\cong S')$ be the
THE GENUS OF CURVES IN \( P^4 \) AND \( P^5 \) NOT CONTAINED IN QUADRICS

### Appendix

In order to conclude the proof of Theorem 1.1, first, it remains to prove that if \( C \) is not contained in a surface of degree \( < 5 \), then \( p_a(C) < \frac{1}{8}d(d - 6) + 1 \), when \( 16 < d \leq 143 \). We are going to do an analysis similar to the one that appears in [8, pp. 74-75]. The same calculation in [8, loc. cit.] proves that \( p_a(C) < \frac{1}{8}d(d - 6) + 1 \) for \( d > 30 \). We need to refine the argument to deal with the case \( 16 < d \leq 30 \). Let \( \Gamma \subset P^3 \) be the general hyperplane section of \( C \). In the sequel, we will apply [3], [4], [7], and [8] (see Section 2, (i)).

- **Case I:** \( h^0(P^3, I_{\Gamma}(2)) \geq 2 \).

  This case can’t happen. Otherwise, since \( d > 4 \), by monodromy [3, Proposition 2.1], \( \Gamma \) would be contained in an integral curve of \( P^3 \) of degree \( \leq 4 \). Since \( d > 16 \), from [2, Theorem (0.2)] we would deduce that \( C \) is contained in a surface of degree \( \leq 4 \), against our hypothesis.

- **Case II:** \( h^0(P^3, I_{\Gamma}(2)) = 1 \) and \( h^0(P^3, I_{\Gamma}(3)) > 4 \).

  This is the most complicated case. Since \( d > 6 \), by monodromy [3, Proposition 2.1], \( \Gamma \) is contained in an integral curve \( X \) of \( P^3 \) of degree \( \leq 6 \). Based on what was said in the previous case, we may suppose the degree of \( X \) is 5 or 6.

  **First assume** \( \text{deg } X = 5 \).

  If \( d \geq 21 \), by Bezout’s theorem we have \( h_{\Gamma}(i) = h_{X}(i) \) for all \( i \leq 4 \). Let \( X' \) be the general hyperplane section of \( X \). By [5] we get \( h_{X}(i) \geq \sum_{j=0}^{i} h_{X'}(j) \). By [8] it

The general projection of \( S' \) in \( P^r \). By [1, loc. cit.], \( S \) is of maximal rank. Since

\[
h^0(P^r, \mathcal{O}_{P^r}(3)) = \binom{r + 3}{3} = 6s + 4 = h^0(S, \mathcal{O}_S(3))
\]

(compare with the definition of \( s \) and (11)), it follows that the restriction map \( H^0(P^r, \mathcal{O}_{P^r}(3)) \to H^0(S, \mathcal{O}_S(3)) \) is injective (actually, here we only need that \( 6s + 4 \geq \binom{r + 3}{3} \)). Hence, \( S \), and so the image \( D \) of \( D' \), are not contained in quadrics. Moreover, since \( D \) is a Castelnuovo’s curve, we have (compare with (12))

\[
(25) \quad p_a(D) = p_a(D') = G(s + 1; d) > \frac{d^2}{2s} - \frac{d}{2s}(s + 2).
\]

In view of (24) and (25), in order to prove (3), we may assume that \( C \) is contained in a surface \( S \) of degree \( s \). By (10), the sectional genus of \( S \) should be \( \pi = 0 \), otherwise \( S \) is contained in a cubic (again, here we need that \( 6s + 4 \leq \binom{r + 3}{3} \)). Now, if \( C \) is contained in a surface \( S \) of degree \( s \) and sectional genus \( \pi = 0 \), then by [5, Lemma] we know that

\[
p_a(C) \leq \frac{d^2}{2s} - \frac{d}{2s}(s + 2) + O(1), \quad \text{with } O(1) \leq \frac{s^3}{r - 2}.
\]

Taking into account (25), we deduce (3). One may prove the remaining properties in a similar way as in Proposition 1.3. This concludes the proof of Proposition 1.4.
follows that \( h_X(3) \geq 14 \) and \( h_X(4) \geq 19 \). Therefore, if \( d \geq 21 \), taking into account (7), we get:

\[
\begin{align*}
   h_\Gamma(1) &= 4, \quad h_\Gamma(2) = 9, \quad h_\Gamma(3) \geq 14, \quad h_\Gamma(4) \geq 19, \\
   h_\Gamma(5) &\geq \min\{d, 22\}, \quad h_\Gamma(6) \geq \min\{d, 27\}, \quad h_\Gamma(7) = d.
\end{align*}
\]

If \( 28 \leq d \leq 30 \), using (9) we deduce:

\[
p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 9) + (d - 14) + (d - 19) + 8 + 3 = 4d - 35,
\]

which is \( < \frac{1}{8} d(d - 6) + 1 \).

If \( 24 \leq d \leq 27 \), we have:

\[
p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 9) + (d - 14) + (d - 19) + 5 = 4d - 41,
\]

which is \( < \frac{1}{8} d(d - 6) + 1 \), except for \( d = 24 \) for which we have \( 4d - 41 = \frac{1}{8} d(d - 6) + 1 \). However, \( p_a(C) \) should be strictly less than such number, otherwise \( p_a(C) = \sum_{i=1}^{+\infty} d - h_\Gamma(i) \) and \( C \) would be a.C.M. (Section 2, (i)). This is impossible, because \( C \) is not contained in quadrics, while \( h^0(\mathbb{P}^3, \mathcal{I}_C(2)) = 1 \) (we will use this argument later as well).

If \( 21 \leq d \leq 23 \), we have:

\[
p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 9) + (d - 14) + (d - 19) + 1 = 4d - 45,
\]

which is \( < \frac{1}{8} d(d - 6) + 1 \).

Now assume \( 17 \leq d \leq 20 \). By Bezout’s theorem we have \( h_\Gamma(3) = h_X(3) \). Hence we get:

\[
h_\Gamma(1) = 4, \quad h_\Gamma(2) = 9, \quad h_\Gamma(3) \geq 14, \quad h_\Gamma(4) \geq 17, \quad h_\Gamma(7) = d.
\]

And we may conclude with a calculation similar to the previous one (when \( 18 \leq d \leq 20 \), we use again the fact that \( C \) cannot be a.C.M.).

\textit{Assume} \( \deg X = 6 \).

In this case \( X \) is a complete intersection of bi-degree \((2, 3)\).

If \( d \geq 25 \), by Bezout’s theorem we have \( h_\Gamma(i) = h_X(i) \) for every \( i \leq 4 \). Let \( X' \) be the general plane section of \( X \). Similarly as before, since \( \deg X = 6 \), we have \( h_X(3) \geq 15 \) e \( h_X(4) \geq 21 \). Therefore, if \( d \geq 25 \), we have:

\[
\begin{align*}
   h_\Gamma(1) &= 4, \quad h_\Gamma(2) = 9, \quad h_\Gamma(3) \geq 15, \quad h_\Gamma(4) \geq 21, \\
   h_\Gamma(5) &\geq \min\{d, 23\}, \quad h_\Gamma(6) \geq \min\{d, 29\}, \quad h_\Gamma(7) = d.
\end{align*}
\]

It follows that:

\[
p_a(C) \leq \sum_{i=1}^{+\infty} d - h_\Gamma(i) \leq (d - 4) + (d - 9) + (d - 15) + (d - 21) + 7 = 4d - 42,
\]

which is \( < \frac{1}{8} d(d - 6) + 1 \).

If \( 19 \leq d \leq 24 \) we have \( h_\Gamma(3) = h_X(3) \), and so:

\[
h_\Gamma(1) = 4, \quad h_\Gamma(2) = 9, \quad h_\Gamma(3) \geq 15, \quad h_\Gamma(4) \geq 18, \quad h_\Gamma(5) \geq \min\{d, 23\}, \quad h_\Gamma(6) = d.
\]
Hence:

\[ p_a(C) \leq \sum_{i=1}^{+\infty} d - h_{\Gamma}(i) \leq (d - 4) + (d - 9) + (d - 15) + (d - 18) + 1 = 4d - 45, \]

which is \( \leq \frac{1}{5}d(d - 6) + 1 \). It remains to examine the cases \( d = 18 \) or \( d = 19 \).

If \( d = 18 \) and \( h_{\Gamma}(3) = h_X(3) \), then \( h_{\Gamma}(3) \geq 15 \). Otherwise, \( \Gamma \) is a complete intersection of type \( (2, 3, 3) \), and from Koszul's complex we get \( h_{\Gamma}(3) \geq 14 \). Therefore, in every case, we have:

\[ h_{\Gamma}(1) = 4, h_{\Gamma}(2) = 9, h_{\Gamma}(3) \geq 14, h_{\Gamma}(4) \geq 17, h_{\Gamma}(5) = 18. \]

Hence \( p_a(C) < \sum_{i=1}^{+\infty} 18 - h_{\Gamma}(i) \leq 28 = \frac{1}{5}18(18 - 6) + 1. \)

When \( d = 17 \), previous argument does not work. We may argue as follows.

If \( h_{\Gamma}(3) = h_X(3) \) we may repeat previous computation (in this case \( h_X(3) \geq 15 \)). Otherwise, \( h^0(\mathbb{P}^3, \mathcal{I}_X(3)) \geq h^0(\mathbb{P}^3, \mathcal{I}_X(3)) \). On the other hand, since \( X \) is a complete intersection of bi-degree \( (2, 3) \), we have \( h^1(\mathbb{P}^3, \mathcal{I}_X(2 + i)) = h^2(\mathbb{P}^3, \mathcal{I}_X(2 + i)) = 0 \) for all \( i \geq 0 \). It follows that [3 Corollary 1.2]

\[ \Delta h_{\Gamma}(4) \leq \max \{0, \Delta h_{\Gamma}(3) - 2\}. \]

If \( \Delta h_{\Gamma}(4) \leq 0 \) then \( h_{\Gamma}(3) = 17 \). Otherwise, \( \Delta h_{\Gamma}(4) \leq \Delta h_{\Gamma}(3) - 2 \), hence \( h_{\Gamma}(3) \geq 14 \), and we may proceed as in previous computation.

- Case III: \( h^0(\mathbb{P}^3, \mathcal{I}_{\Gamma}(2)) = 1 \) or \( h^0(\mathbb{P}^3, \mathcal{I}_{\Gamma}(3)) = 4 \).

We have:

\[ h_{\Gamma}(1) = 4, h_{\Gamma}(2) = 9, h_{\Gamma}(3) = 16, h_{\Gamma}(4) \geq \min\{d, 19\}, \]

\[ h_{\Gamma}(5) \geq \min\{d, 24\}, h_{\Gamma}(6) = d. \]

If \( 26 \leq d \leq 30 \), we have:

\[ p_a(C) < \sum_{i=1}^{+\infty} d - h_{\Gamma}(i) \leq (d - 4) + (d - 9) + (d - 16) + 11 + 6 = 3d - 12, \]

which is \( \leq \frac{1}{5}d(d - 6) + 1. \)

If \( 21 \leq d \leq 25 \), we have:

\[ p_a(C) < \sum_{i=1}^{+\infty} d - h_{\Gamma}(i) \leq (d - 4) + (d - 9) + (d - 16) + 6 + 1 = 3d - 22, \]

which is \( \leq \frac{1}{5}d(d - 6) + 1 \), except the case \( d = 21 \). In this case \( 3d - 22 = 41 \), hence \( p_a(C) \leq 40 < \frac{1}{5}21(21 - 6) + 1 = 40 + \frac{4}{5}. \)

If \( 17 \leq d \leq 20 \), we have:

\[ p_a(C) < \sum_{i=1}^{+\infty} d - h_{\Gamma}(i) \leq (d - 4) + (d - 9) + (d - 16) + 1 = 3d - 28, \]

which is \( \leq \frac{1}{5}d(d - 6) + 1. \)

- Case IV: \( h^0(\mathbb{P}^3, \mathcal{I}_{\Gamma}(2)) = 0. \)

We have:

\[ h_{\Gamma}(1) = 4, h_{\Gamma}(2) = 10, h_{\Gamma}(3) \geq 13, h_{\Gamma}(4) \geq \min\{d, 19\}, \]
So, we may conclude with a similar computation as in the previous case.

Remark 7.1. From previous analysis it follows that if \( C \subset \mathbb{P}^4 \) is not contained in quadrics, and \( d > 16 \), then \( h^0(\mathcal{O}_S, \mathcal{I}_T(2)) \leq 1 \). In fact, otherwise \( C \) should be contained in an isomorphic projection of the Veronese surface. Therefore, \( \Gamma \) should be contained in a rational quartic \( X \) in \( \mathbb{P}^3 \). And \( h_{\Gamma}(2) = h_X(2) = 9 \) (in fact \( h^1(\mathbb{P}^3, \mathcal{I}_X(2)) = 0 \)).

References

[1] Ballico, E. - Ellia, Ph.: On Projections of Ruled and Veronese Surfaces, Journal of Algebra, 121, 477-487 (1989).
[2] Chiantini, L. - Ciliberto, C.: A few remarks on the lifting problem, Astérisque, Vol. 218**, (1993), pp. 95-109.
[3] Chiantini, L. - Ciliberto, C. - Di Gennaro, V.: The genus of projective curves, Duke Math. J., 70(2), 229-245 (1993).
[4] Chiantini, L. - Ciliberto, C. - Di Gennaro, V.: On the genus of projective curves verifying certain flag conditions, Bollettino U.M.I. (7) 10-B (1996), 701-732.
[5] Di Gennaro, V.: Hierarchical structure of the family of curves with maximal genus verifying flag conditions, Proceedings of the American Mathematical Society, Volume 136, Number 3, March 2008, Pages 791-799.
[6] Di Gennaro, V.: A lower bound for \( \chi(\mathcal{O}_S) \), Rendiconti del Circolo Matematico di Palermo, Series 2, https://doi.org/10.1007/s12215-021-00618-6, OnlineFirst, Published on line 24 May 2021, pp. 7.
[7] Di Gennaro, V. - Franco, D.: Refining Castelnuovo-Halphen bounds, Rendiconti del Circolo Matematico di Palermo, Volume 61, Number 1, 91-106 (2012).
[8] Di Gennaro, V. - Franco, D.: A lower bound for \( K_S^2 \), Rendiconti del Circolo Matematico di Palermo, II. Ser (2017) 66:69-81.
[9] Eisenbud, D. - Harris, J.: Curves in Projective Space, Sém. Math. Sup. 85 Les Presses de l’Université de Montréal, 1982.
[10] Gruson, L. - Peskine, C.: Genre des courbes dans l’espace projectif, Algebraic Geometry: Proceedings, Norway, 1977, Lecture Notes in Math., Springer-Verlag, New York, 687 (1978), 31-59.
[11] Hartshorne, R.: Algebraic Geometry, GTM, 52, Springer-Verlag, 1983.
[12] El Majnouni, A. - Laytimi, F. - Nagaraj, D.S.: Projections of Veronese surface and morphisms from projective plane to Grassmannian, Proc. Indian Acad. Sci. (Math. Sci.) Vol. 127, No. 1, February 2017, pp. 59-67.
[13] Nagel, U. - Vogel, W.: Bounds for Castelnuovo’s regularity and the genus of projective varieties, Topics in Algebra, Banach Center Publications, Volume 26, Part 2, PWN Polish Scientific Publishers, Warsaw 1990, 163-183.
[14] Nonn, A.: A bound on the Castelnuovo-Mumford regularity for curves, Math. Ann. 322, 69-74 (2002)
[15] Swinnerton-Dyer, H. P. F.: An Enumeration of All Varieties of Degree 4, American Journal of Mathematics, Vol. 95, No. 2 (Summer, 1973), pp. 403-418.

Università di Roma “Tor Vergata”, Dipartimento di Matematica, Via della Ricerca Scientifica, 00133 Roma, Italy.

Email address: digennar@mat.uniroma2.it