Solving the QCD non-perturbative flow equation as a partial differential equation and its application to dynamical chiral symmetry breaking

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The non-perturbative renormalization group approach to dynamical chiral symmetry breaking is an effective method which can accommodate beyond the ladder (mean field) approximation. The usual method relying on field operator expansion suffers explosive behaviors of the 4-fermi coupling constant, which prevent us from evaluating the physical quantities in the broken phase. In order to overcome this difficulty, we solve the flow equation directly as a partial differential equation and calculate the dynamical mass and the chiral condensates. Also, we go beyond the ladder approximation to formulate an equation which gives almost gauge-independent results for the chiral condensates.

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1. Introduction

The Wilsonian renormalization group approaches to continuum quantum field theory have been formulated [1–3] and developed in past decades (see reviews [4–7]). In these approaches, the renormalization group (RG) flow equation is defined by a functional differential equation, the solution of which gives the partition function defined by the functional integral. Using this framework, we are able to obtain new approximation methods by extracting the non-perturbative information from the partition function. We therefore call this type of RG a “non-perturbative renormalization group” (NPRG).

In order to solve the flow equation approximately, we usually expand the equation in terms of the field operators and their derivatives. The derivative expansion has been applied to the evaluation of universal quantities such as critical exponents and anomalous dimensions. For example, in three-dimensional scalar field theories, expansion with respect to the field operator without the derivatives converges very well [8–11]. Although it is difficult to confirm the convergence with respect to the order of the derivatives, the result of the expansion up to the fourth derivatives agrees well with the Monte Carlo simulations [12].

NPRG has also been applied to the analysis of dynamical chiral symmetry breaking ($\chi$SB) in strong coupling gauge theories and its effective theory. As first noticed by Nambu and Jona-Lasinio [13], the scalar 4-fermi operators become the source of the $\chi$SB. From the viewpoint of the NPRG, when we lower the renormalization scale of the effective action, the strong gauge interactions induce the effective 4-fermi operators, which bring about the $\chi$SB at the low energy scale [14]. Unless there are explicit symmetry-breaking terms such as mass terms, the running 4-fermi coupling constant diverges at a low energy scale. This explosive behavior is nothing but a signal of the $\chi$SB [4,15].
On the other hand, the (improved) ladder Schwinger–Dyson (SD) equation has frequently been used for the analysis of $D\chi_{SB}$ in strong coupling gauge theories. This ladder approximation has a strong dependence on the gauge-fixing parameter. Moreover, it is difficult to improve the ladder approximation systematically. However, the framework of NPRG allows us to take account of the effects of the non-ladder diagrams in a systematic fashion [16,17].

Since the running 4-fermi coupling constants diverge at a critical scale, as mentioned above, the RG flow cannot go beyond the critical scale towards the infrared limit. In the simple framework of NPRG which maintains the chirally symmetric structure of the effective action, we cannot evaluate the physical quantities such as the dynamical mass of the fermion and the condensates of the fermion bilinear composite operator in the broken phase.

To overcome this problem, we introduce the bare mass term, which also works as a source term of the chiral condensates [18]. We may expect that the bare mass prevents the divergent behavior of the 4-fermi coupling constants, which might allow us to effectively evaluate the order parameter of the $D\chi_{SB}$ at the infrared limit scale. The field operator expansion of the NPRG equation, however, does not converge well, at least in the region of bare mass as small as the current masses of up and down quarks. Consequently, we will take another way by directly solving the NPRG flow equation as a partial differential equation (PDE) without relying on any field operator expansion.

By the way, another method like the Hubbard–Stratonovich transformation has been used in many works to avoid the divergent behavior [19–21]. In this method, the composite operators of fermions are partially transformed to scalar fields, one of which obtains the nonzero expectation value as a chiral order parameter. The introduction of these scalar fields has the merit that the meson physics can be argued simultaneously. However, it is more complicated to evaluate the convergence of the physical quantities with respect to the operator expansion including the scalar fields.

This paper is organized as follows. In Sect. 2 we briefly review the flow equation for the effective average action. We introduce the basic truncation which projects the complete operator space onto the subspace relevant to the $D\chi_{SB}$ so as to solve the flow equation approximately. In Sect. 3 we examine the truncation method in detail and obtain two types of truncation: one corresponds to the ladder approximation, and the other contains the main parts of the non-ladder corrections. In Sect. 4 we explain how to evaluate the chiral order parameters. In Sect. 5 we solve the flow equation by using the field operator expansion, and examine the convergence of chiral order parameters. In Sect. 6 we directly solve the partial differential equation and show the results for physical quantities. In Sect. 7 we summarize our methods and results, and discuss further issues along the lines of thought in this paper.

2. The NPRG flow equation and its application

2.1. Formulation

In order to evaluate the non-perturbative effects of quantum field theory, we introduce the so-called “effective average action” [3] that interpolates between the bare action and the full quantum effective action. For this purpose, we define the generating function with the infrared cutoff supplied by the cutoff term $\Delta S_\Lambda[\Omega]$ as follows:

$$e^{W_\Lambda[J]} = Z_\Lambda[J] := \int D\Omega \exp(-S_{\text{bare}}[\Omega] - \Delta S_\Lambda[\Omega] + \int J \cdot \Omega),$$

where $\Omega$ represents various fields generically and the functional integral is regularized by the ultraviolet (UV) momentum cutoff $\Lambda_0$. The cutoff term $\Delta S_\Lambda$ is defined as the following mass term depending
on the momentum:
\[
\Delta S_{\Lambda}[\Omega] = \int_p \frac{1}{2} \Omega^T (\mathbf{-} p) \cdot R_{\Lambda}(p) \cdot \Omega(p).
\] (2)

The regulator function \( R_{\Lambda}(p) \) suppresses quantum fluctuations with momentum lower than the infrared cutoff \( \Lambda \). Therefore, the regulator function satisfies
\[
\lim_{q^2/\Lambda^2 \to 0} R_{\Lambda}(p) > 0,
\] (3)

and it implements the “coarse graining” in the Wilsonian method.

The effective average action is defined by a slightly modified Legendre transformation:
\[
\Gamma_{\Lambda}[\Phi] \equiv \sup_J \left( \int \Phi \cdot J - W_{\Lambda}[J] \right) - \Delta S_{\Lambda}[\Phi].
\] (4)

It satisfies the following boundary conditions,
\[
\begin{aligned}
\Gamma_{\Lambda} &\to S_{\text{bare}}, \quad \Lambda \to \Lambda_0 \to \infty \\
\Gamma_{\Lambda} &\to \Gamma', \quad \Lambda \to 0,
\end{aligned}
\] (5)

provided that the regulator function \( R_{\Lambda}(p) \) has the following properties,
\[
\begin{aligned}
R_{\Lambda}(p) &\to \infty, \quad \Lambda \to \Lambda_0 \to \infty \\
R_{\Lambda}(p) &\to 0, \quad \Lambda^2/q^2 \to 0
\end{aligned}
\] (6)

The cutoff dependence of the effective average action is exactly given by the NPRG flow equation,
\[
\partial_t \Gamma_{\Lambda}[\Phi] = \frac{1}{2} \text{STr} \left[ \left( \Gamma^{(2)}_{\Lambda} + R_{\Lambda} \right)^{-1} \cdot \partial_t R_{\Lambda} \right],
\] (7)

where we define the dimensionless scale parameter \( t = \log \Lambda_0/\Lambda \). The right-hand side should be called the \( \beta \) functional, and consists of the second-order functional derivative of the effective average action,
\[
(\Gamma^{(2)}_{\Lambda})_{ij}(p, q) = \frac{\delta^2 \Gamma_{\Lambda}[\Phi]}{\delta \Phi_i(-p) \delta \Phi_j(q)}.
\] (8)

This is considered as a matrix with respect to the momentum and the species of fields. This flow equation was first derived by Wetterich [3]. Because of the boundary condition (5), the flow equation interpolates between the bare action \( S_{\text{bare}} \) and the full quantum effective action \( \Gamma' \).

2.2. Application to QCD

Hereafter, we consider \( N_f \)-flavor massless QCD with \( N_c \)-color. The bare action in Euclidean space is
\[
S_{\text{bare}} = \int_x \left\{ \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu}_a + \bar{\psi} \left( i \gamma_\mu \partial^\mu + i \bar{g}_s A^a T^a \right) \psi \right\},
\] (9)

where \( T^a \) is the generator of the fundamental representation of \( SU(N_c) \). This action has the chiral symmetry \( SU(N_f)_L \times SU(N_f)_R \), which is to be broken down to \( SU(N_f)_V \) dynamically by the strong gauge interactions.

To evaluate the dynamical chiral symmetry breaking (\( D\chi\text{SB} \)) by using the non-perturbative renormalization group (NPRG) flow equation, we truncate field operators which are not essential to the
$\Delta \chi_{SB}$ from the complete operator space of the effective average action. Here we define the following truncated effective average action:

$$\Gamma_\Lambda[\Phi] = \int_x \left\{ \frac{Z_F}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2\xi} (\partial_\mu A_\mu)^2 + \bar{\psi} (Z_\psi \partial + ig_s A) \psi - V(\psi, \bar{\psi}; \Lambda) \right\}, \quad (10)$$

where we use the covariant gauge with the gauge-fixing parameter $\xi$, and do not represent the ghost sector for simplicity. This truncated subspace of the complete effective action is spanned by the operators of the bare QCD action and the fermion self-interaction operator $V(\psi, \bar{\psi}; \Lambda)$, which we call the fermion potential. The discarded operators, including higher derivatives or gluon fields, may affect quantitative evaluation, but are not the essential sector to drive the $\Delta \chi_{SB}$. Actually, we will confirm that the operators of the bare QCD and the fermion potential bring about $\Delta \chi_{SB}$ using the NPRG flow equation [19].

Here it should be noted that the flow equation induces operators breaking the gauge symmetry, such as the mass operator of the gauge boson, because the cutoff function explicitly breaks the gauge symmetry. However, we do not discuss this issue because we truncate out such operators. Eventually, the flow equation of the gauge coupling constant agrees with that of the one-loop perturbation theory. We will concentrate on evaluating the flow equation for the fermion potential.

2.3. Techniques to derive the NPRG flow equation

In the rest of this section, we will explain some techniques for explicitly writing down the NPRG flow equation (7) for the fermion potential. We denote the degrees of freedom of the fields as a vector $\Phi^{(2)}(p, q) + R_\Lambda(p, q)$

$$= \left( \begin{array}{c} \delta A^{(2)}_\mu(-p) \\ \delta A^{(2)}_\mu(p) \\ \delta \psi(-p) \\ \delta \bar{\psi}(p) \\ \delta \bar{\psi}(q) \end{array} \right) (\Gamma_\Lambda[\Phi] + \Delta S_\Lambda[\Phi]) \left( \begin{array}{c} \frac{\delta}{\delta A^{(2)}_\mu(q)} \\ \frac{\delta}{\delta \psi(q)} \\ \frac{\delta}{\delta \bar{\psi}(q)} \end{array} \right). \quad (11)$$

The regulator function is defined by

$$R_\Lambda(p, q) = \left( \begin{array}{ccc} Z_A r(p) (D_0^{-1})_{ab}(p) & 0 & 0 \\ 0 & 0 & Z_\psi r_\psi(p) i \bar{\psi} \\ 0 & Z_\psi r_\psi(p) i \bar{\psi} & 0 \end{array} \right) \times \delta(p - q), \quad (12)$$

where $(D_0^{-1})_{ab}(p) \equiv p^2 \delta_{ab} (\delta_{\mu\nu} - q_{\mu}q_{\nu} (1 - 1/\xi))$ and $\delta(p - q) \equiv (2\pi)^4 \delta^{(4)}(p - q)$. The functions $r(p)$ and $r_\psi(p)$ are defined to satisfy properties (3) and (6).

Next we explain how to calculate the “super trace” in the NPRG flow equation (7). We transform the flow equation as follows [5]:

$$\partial_t \Gamma_\Lambda[\Phi] = \tilde{\partial}_t \frac{1}{2} \text{Str} \log \left[ \Gamma_\Lambda^{(2)} + R_\Lambda \right]. \quad (13)$$
Here, the symbol $\tilde{\partial}_t$ is defined by

$$\tilde{\partial}_t = \int_p \left[ \frac{\partial_t(Z_A r(p))}{Z_A} \frac{\delta}{\delta r(p)} + \frac{\partial_t(Z_\psi r_\psi(p))}{Z_\psi} \frac{\delta}{\delta r_\psi(p)} \right],$$

(14)

where $\int_p \equiv \int \frac{d^4 p}{(2\pi)^4}$. Then we split the inverse propagator matrix (11) into submatrices as follows:

$$M \equiv \Gamma_k^{(2)} + R_k = \begin{pmatrix} M_{BB} & M_{BF} \\ M_{FB} & M_{FF} \end{pmatrix},$$

(15)

where $M_{BB}$ ($M_{FF}$) corresponds to the second derivative with respect to bosonic (fermionic) fields, while $M_{BF}$ and $M_{FB}$ correspond to bosonic and fermionic field derivatives. Physically, the submatrix $M_{BB}$ corresponds to the inverse free propagator of the gauge field, the submatrices $M_{BF}$ and $M_{FB}$ are the gauge interactions, and the submatrix $M_{FF}$ contains the inverse propagator of the fermion and the fermion self-interactions. Using this notation, we can rewrite the “super-trace log” in the flow equation (13) into the following formula [22]:

$$\text{STr} \log M = -\text{Tr} \log M_{FF} + \text{Tr} \log [M_{BB} - M_{BF}M_{FF}^{-1}M_{FB}].$$

(16)

Note that another expression,

$$\text{STr} \log M = \text{Tr} \log M_{BB} + \text{Tr} \log [M_{FF} - M_{FB}M_{BB}^{-1}M_{BF}],$$

(17)

has often been used in the NPRG analyses. Here, Eq. (16) is more appropriate for our purpose of improving the gauge parameter dependence.

To derive the flow equation for the fermion potential, the matrix $M$ in Eq. (15) should be evaluated by replacing the fields with their zero-momentum components. Replacing $\Phi(p) \to \Phi(p = 0)$, we have:

$$M(p, q) = \Gamma^{(2)}_\Lambda(\Phi)(p, q) + R_\Lambda(p, q) |_{\Phi(p) = \Phi(p = 0)} = \begin{pmatrix} Z_F(1 + r)(D_0^{-1})^ {ab}_{\mu\nu} & i\tilde{g}_s(\bar{\psi}\gamma_\mu T^a \psi) \\ -i\tilde{g}_s(\bar{\psi}\gamma_\mu T^b \psi)^T & -i\tilde{g}_s(\bar{\psi} \gamma^a T^b \psi) \end{pmatrix}$$

$$\times \delta(p - q).$$

(18)

### 3. Flow equation for the fermion potential

#### 3.1. Scalar 4-fermi operator

The gauge interactions induce all possible fermion operators respecting the symmetry of QCD, which are enhanced by themselves when we lower the cutoff scale. Even in the truncated subspace of the effective action (10), we cannot treat exactly all possible operators and interactions. Therefore, as an approximation of the flow equation for the fermion potential, we project its full operator space onto a specific subspace and restrict the interactions so that we evaluate the $D\chi$SB most effectively.

As in QED with one flavor [16], the central operator for $D\chi$SB is undoubtedly the following scalar 4-fermi operator:

$$\rho = \frac{1}{2} \sum_{l=0}^{N_f^2 - 1} [\langle \bar{\psi} \lambda^l \psi \rangle^2 + \langle \bar{\psi} \lambda^l i\gamma_5 \psi \rangle^2],$$

(19)

where $\lambda^I (I = 1, \ldots, N_f^2 - 1)$ are the generators of the fundamental representation of $SU(N_f)$, and $\lambda^0 = \frac{1}{\sqrt{2N_f}} \mathbf{1}_{\text{flavor}}$ is defined so that they satisfy the proper normalization, $\text{tr}[\lambda^I \lambda^J] = \delta^{IJ}/2$. The 4-fermi
operator $\rho$, often adopted in Nambu–Jona-Lasinio (NJL)-type models with $N_f = 3$, is invariant under the chiral transformation $SU(N_f)_L \times SU(N_f)_R$. It is the only chiral invariant 4-fermi operator which gives corrections to the mass operator, and it becomes the relevant operator in the region of the strong gauge coupling constant. Therefore, for a first-step approximation, we project the operator space of the fermion potential onto the subspace spanned by polynomials in the scalar operator $\rho$.

To project the flow equation onto the subspace defined above, we determine the coefficient from all possible operators included in the full fermion potential. This is equivalent to counting the coefficients of powers of $(\bar{\psi}\psi)^2$, even though this operator itself is not chirally invariant, because the $(\bar{\psi}\psi)^2$ operator does not appear in chiral invariant operators other than powers of $\rho$. Eventually, we may work with a potential function in the simplest scalar operator $\sigma = \bar{\psi}\psi$:

$$V(\psi, \bar{\psi}) \rightarrow V(\sigma).$$  \hspace{1cm} (20)

Here we note that the original chiral symmetry is not maintained in this subspace, but the discrete chiral symmetry still remains: the Lagrangian (the fermion potential) is invariant under the following discrete transformation:

$$\psi \rightarrow \gamma_5 \psi, \quad \bar{\psi} \rightarrow -\bar{\psi}\gamma_5, \quad \sigma \rightarrow -\sigma.$$  \hspace{1cm} (21)

The discrete chiral symmetry forbids the operators of the odd powers of $\sigma$, such as the fermion mass term.

It should be noted here that the fermion potential is a function of the Grassmannian variables which describes the total effective interactions at the scale. Therefore, the fermion potential should not be confused with more popular objects like the effective potential of the auxiliary field or the Legendre effective potential of the vacuum expectation value of the operator $\sigma$. The value of the fermion potential at the origin has a special physical meaning, the free energy at the scale, from which we can obtain the vacuum expectation value of the operator $\sigma$ at the scale by differentiating it with respect to the source field for $\sigma$. Also, the first derivative of the fermion potential, which will be introduced later, is called the mass function, since it is an effective mass operator of the shell mode propagator. The value of the mass function at the origin is the dynamical mass at the scale, the spontaneous generation of which is nothing but our target of the dynamical chiral symmetry breaking. Due to the even function property of the fermion potential, the spontaneous mass generation requires the emergence of singularity, non-differentiability, at the origin.

Next, we pick up the interactions that are expected to be most important for the $D\chi^2$SB and for improvement of the gauge-fixing parameter $(\xi)$ dependence. In Eq. (18), we further select the large-$N_c$ leading interactions in the fermion self-interactions, which leads to the following simplification:

$$\frac{\partial^2 V}{\partial \psi \partial \bar{\psi}}, \frac{\partial^2 V}{\partial \bar{\psi} \partial \psi} \rightarrow 0,$$

$$\frac{\partial V}{\partial \bar{\psi} \partial \psi} - \frac{\partial V}{\partial \psi \partial \bar{\psi}} \rightarrow \partial_\psi V.$$  \hspace{1cm} (22)

Applying the above approximation and the usual field renormalization, $\psi \rightarrow \psi/Z_{\psi}^{1/2}$ and $A_\mu^a \rightarrow A_\mu^a/Z_A^{1/2}$, to Eqs. (13) and (16), we obtain the flow equation for the fermion potential:

$$\partial_t V(\sigma; t) = -\eta_{\bar{\psi}\psi} \partial_\sigma V + \int_p \text{tr} \tilde{\partial}_I \log S_{\bar{\psi}I}(p) - \frac{1}{2} \int_p \text{tr} \tilde{\partial}_I \log[1 + A(p) + B(p)].$$  \hspace{1cm} (23)
where $\eta_\psi$ is the anomalous dimension of the fermion field. Functions $A(p)$ and $B(p)$ are matrices in the space of the color adjoint representation and the Euclidean Lorentz vector space as follows:

\begin{align}
(A)^{ab}_{\mu\nu}(p) &= g_s^2 \bar{\psi} T^a \gamma_\mu S_\psi(p) \gamma_\rho D^b_{\rho\nu}(p) T^b \psi, \\
(B)^{ab}_{\mu\nu}(p) &= g_s^2 \bar{\psi} T^b D^b_{\nu\rho}(p) \gamma_\rho S_\psi(-p) \gamma_\mu T^a \psi.
\end{align}

Here we should pay attention to the difference between the two types of trace: $\text{tr}$ acts on the space of the fermion’s representation, the Dirac spinor, the color fundamental representation, and the flavor fundamental representation, and $\tilde{\text{tr}}$ acts on the space of the matrix $A$, $B$. The propagators of the fermions and the gauge bosons, including the cutoff function in Eqs. (24) and (25), are defined by

\begin{align}
S_\psi(p) &= \frac{-i(1+r_\psi) \hat{p} - \partial_\sigma V}{P_\psi(p) + (\partial_\sigma V)^2}, \\
(D)^{ab}_{\mu\nu}(p) &= \frac{\delta^{ab}}{P(p)} \left( \delta_{\mu\nu} - (1 - \xi) \frac{P_\mu P_\nu}{p^2} \right),
\end{align}

where $P_\psi(p) \equiv (1 + r_\psi)^2 p^2$, and $P(p) \equiv (1 + r) p^2$. According to the field renormalization, the anomalous dimension $\eta_\psi(\equiv \partial_t \log Z_\psi)$ appears in Eq. (23), and the gauge coupling constant is renormalized as $g_s = \frac{Z_1 g_s}{Z_\psi Z_A^{1/2}}$.

In the next two subsections, we will explain how to extract the scalar operators from the right-hand side of Eq. (23).

### 3.2. The ladder approximation

In Fig. 1, the flow equation (23) is diagrammatically expressed by using the “corrected vertex” defined in Fig. 2. We regard $A$ and $B$ as the ladder element and the crossed element of the diagrams, respectively.

The diagrams consisting only of the ladder element $A$ are the ladder diagrams. Moreover, we find that the diagrams consisting only of the crossed elements $B$ are also the ladder diagrams if...
we rotate all the fermion lines of the diagrams to untangle the crossed gluon lines. Therefore, the ladder approximation with only the ladder diagrams should be defined by:

\[
-\frac{1}{2} \mathrm{tr} \log[1 + A + B] = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{tr}(A + B)^n
\]

\[
\Rightarrow \text{ladder} \quad \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{tr}(A^n + B^n)
\]

\[
= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{tr} A^n. \tag{28}
\]

To project Eq. (28) onto the subspace of the scalar operators \(\sigma^n\), we adopt the following approximation rule of picking up \(\sigma\),

\[
\bar{\psi}_1 O_1 \psi_2 \bar{\psi}_2 O_2 \bar{\psi}_3 \cdots \bar{\psi}_n O_n \psi_1 \rightarrow (-1)^{n+1} F_n \prod_{i=1}^{n} \bar{\psi}_i \mathbf{1}_{\text{spinor}} \lambda^0 T^0 \psi_i, \tag{29}
\]

where we use \(T^0 \equiv \frac{1}{\sqrt{2N_c}} \mathbf{1}_{\text{color}}\). The right-hand side of the above formula is nothing but the scalar part of the general Fierz transformation obtained by using the completeness of the space of the spinor, the color, and the flavor. Thus, the coefficient \(F_n\) in the above formula is given by

\[
F_n = \mathrm{tr} \left[ \mathbf{1}_{\text{spinor}} \lambda^0 T^0 O_1 \cdot \mathbf{1}_{\text{spinor}} \lambda^0 T^0 O_2 \cdots \mathbf{1}_{\text{spinor}} \lambda^0 T^0 O_n \right]. \tag{30}
\]

According to this rule, the summation in Eq. (28) is calculated as follows:

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{tr} A^n \rightarrow \sum_{n=1}^{\infty} \mathrm{tr} \left[ \frac{(-1)^{n+1}}{n} \left( -C_2 g_s^2 3 + \xi \frac{3 + \xi}{4P(p)} S^\psi(p) \frac{\sigma}{N_lN_c} \right)^n \right]
\]

\[
= \mathrm{tr} \log \left( 1 - C_2 g_s^2 \frac{3 + \xi}{4P(p)} S^\psi(p) \frac{\sigma}{N_lN_c} \right), \tag{31}
\]

where \(C_2\) is the second Casimir invariant of the \(SU(N_c)\) representation, \(C_2 = \sum_{a=1}^{N_c^2-1} T^a T^a\).

Finally, we obtain the ladder flow equation:

\[
\partial_t V(\sigma; t) = -\eta_\psi \sigma \partial_\psi V + \int p \mathrm{tr} \tilde{\partial}_t \log \left( S^{-1}_\psi(p) - C_2 g_s^2 \frac{3 + \xi}{4P(p)} \sigma \right), \tag{32}
\]

where we rescaled \(V\) and \(\sigma\) by common factor \(N_cN_l\), \(V \rightarrow N_cN_l V\), and \(\sigma \rightarrow N_cN_l \sigma\).

As for the regulator function, we adopt the following sharp regulator function:

\[
r_{\text{sharp}}(p) = Z_A \cdot r(p) = Z_\psi \cdot r_\psi(p) = \frac{1}{\theta(p^2 - 1)} - 1. \tag{33}
\]

Performing the momentum integration in Eq. (32), we obtain the ladder flow equation as a partial differential equation (PDE):

\[
\partial_t V(\sigma; t) = -\eta_\psi \sigma \partial_\psi V + \frac{\Lambda^4}{4\pi^2} \ln \left[ 1 + \Lambda^{-2} \left( \partial_\sigma V + (3 + \xi) \frac{C_2 g_s^2 \sigma}{4\Lambda^2} \right)^2 \right]. \tag{34}
\]

Apart from the anomalous dimension term, this flow equation agrees with the local potential approximated Wegner–Houghton equation, and it was proved that the flow equation gives results equivalent
to the improved ladder Schwinger–Dyson equation [19]. Actually, this is the reason why we call this approximated flow equation “the ladder”.

Using the momentum scale expansion [23,24], the anomalous dimension of the fermion field $\eta_\psi = \partial_t \log Z_\psi$ is given by

$$\eta_\psi = \frac{C_2 g_s^2}{8\pi^2} \left( \xi \frac{\Lambda^2}{\Lambda^2 + m^2(t)} + \frac{3 - \xi}{4} \frac{\Lambda^2 m^2(t)}{(\Lambda^2 + m^2(t))^2} \right).$$

where we define the running mass as

$$m(t) = \partial_\sigma V(\sigma; t)|_{\sigma=0}.\quad (36)$$

As will be seen in the numerical results in Sect. 6, the chiral order parameters given by the ladder flow equation (34) strongly depend on the gauge-fixing parameter $\xi$.

Here we explain the actual treatment of the above renormalization group equation for the fermion potential. At the beginning, we introduced the fermion potential as a local function of $\psi$ and $\bar{\psi}$ (without field derivative) in Eq. (10). Due to the Grassmannian property of $\psi$ and $\bar{\psi}$, higher powers of these fields vanish exactly. This situation is unchanged even after we limit the form of the potential $V$ to be a function of the $\sigma$ field. Instead, our subspace for the effective action should be considered to consist of a potential function $V$ of $\bar{\sigma} \equiv \int \sigma(x)dx$ [14,19]. Under our approximation rules of evaluating the beta function, the renormalization group equation for this $V(\bar{\sigma})$ has exactly the same form as above. Though the integrated field $\bar{\sigma}$ is still the Grassmannian variable, its higher powers do not vanish. Therefore, we may analyze the renormalization of function $V$ itself as if it were a usual function of a real variable, and also we can power expand it or plot it with respect to an argument of real values. Note that according to the above reformulation, the fermion potential $V$ does no longer consist of local terms in $\bar{\psi}(x)\psi(x)$. This situation is quite different from the local potential approximation for scalar theories.

3.3. Beyond the ladder approximation

In order to improve the gauge dependence, we have to add the crossed element as well. We evaluate the flow equation (23) using the full corrected vertex in Fig. 2, which is calculated as follows:

$$\begin{align*}
(A + B)_{\mu\nu}^{ab}(p) &= -\frac{2g_s^2}{P_\psi + (\partial_\sigma V)^2}\bar{\psi}T^a T^b \left( i\frac{(1 + r_\psi)P_{\alpha}}{P}(\epsilon_{\mu\nu\alpha\beta\gamma}\gamma_\beta) + D_{\mu\nu}\partial_\sigma V \right) \psi \\
&+ (\text{term including } [T^a, T^b]).
\end{align*}$$

Here we will ignore the commutator term for simplicity. Note that this is consistent with our approximation that we do not include diagrams with gluon self couplings, and from the following discussion we expect that it does not induce strong gauge dependence in the truncated subspace.

We associate the corrected vertex with the gauge-independent set of diagrams for the S-matrix in the case of Abelian gauge theory. In non-Abelian gauge theory, the diagram exchanging one gluon is also needed for the gauge independence of the S-matrix. On the other hand, such corrections cannot be added directly due to the one-loop nature of the NPRG $\beta$ function, and therefore the gauge dependence which would be canceled in the S-matrix appears in the commutator term of Eq. (37).

In the NPRG, the correction by the diagrams exchanging one gluon is treated through the effective operators such as $\partial_\mu F_{\mu\nu}\bar{\psi}\gamma_\nu\psi$. However, such operators are not included in the truncated subspace. Thus we may omit the commutator term without suffering strong gauge dependence.
Then, according to the general Fierz transformation (29), we can pick up the scalar operators as follows:

\[
\frac{(-1)^n}{2n} \text{tr}(A + B)^n \rightarrow 2N_cN_f (-1)^{n+1} \left[ \frac{g_s^2}{2P(\partial_\sigma V)^2 N_cN_f} \sigma \right]^n \times \left[ (\xi \partial_\sigma V)^n + \sum_{k=0}^{n} (-1)^k \binom{n}{2k} (2 + 4^k)(\partial_\sigma V)^n-2k P_\psi^k \right].
\] (38)

Finally, adopting the sharp regulator function, we obtain the flow equation beyond the ladder approximation as the following partial differential equation:

\[
\partial_t V(\sigma; t) = -\eta_\psi \sigma \partial_\sigma V + \frac{\Lambda^4}{4\pi^2} \log \left[ 1 + \frac{B^2}{\Lambda^2} \right] + \frac{\Lambda^4}{8\pi^2} \log \left[ \frac{\Lambda^2 + B^2}{\Lambda^2 + (\partial_\sigma V)^2} + \frac{3\Lambda^2 G^2}{(\Lambda^2 + (\partial_\sigma V)^2)^2} \right] \\
+ \frac{\Lambda^4}{4\pi^2} \log \left[ 1 + \xi \frac{\partial_\sigma V G}{\Lambda^2 + (\partial_\sigma V)^2} \right],
\] (39)

where \( B = \partial_\sigma V + C_2 \frac{g_s^2\sigma}{2\Lambda^2} \) and \( G = C_2 \frac{g_s^2\sigma}{2\Lambda^2} \).

4. **Chiral order parameters**

Now we explain how to evaluate the two chiral order parameters, the dynamical mass of quarks and the chiral condensates \( \langle \bar{\psi} \psi \rangle \), which are generated by the \( D_\chi \)SB.

In the framework of the NPRG, calculating the non-zero chiral order parameter is nontrivial because the NPRG flow equation maintains the chiral invariant structure of the effective action, which forbids the appearance of the dynamical mass operator.\(^1\)

On the other hand, the \( D_\chi \)SB shows itself as a divergent behavior of the 4-fermi coupling constant, which is the source of the \( D_\chi \)SB in the NJL model. We can define the \( \beta \) function for each operator by expanding the flow equation in powers of \( \sigma \). The \( \beta \) function for the 4-fermi coupling constant consists of itself and the gauge coupling constant, but does not include the higher-dimensional operators due to the chiral invariance. Solving the RG equation, we obtain the flow of the 4-fermi coupling constant as follows: Lowering the cutoff scale \( \Lambda(t) \), the gauge interactions generate the 4-fermi operator, which enhances itself, and consequently the 4-fermi coupling constant diverges at a finite infrared scale \( \Lambda_c \).

Because of this divergence, the RG flow cannot go beyond the critical scale \( \Lambda_c \) toward the infrared limit in the chiral invariant operator space. Therefore, the divergence seems to imply that the chiral invariant RG flow cannot exist at a cutoff scale lower than \( \Lambda_c \), where the true RG flow might be in the chiral variant operator space including the mass operator. The relation between the divergence and \( D_\chi \)SB has been discussed in Refs. [4,16,25].

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\(^1\) In the theories including the scalar fields whose symmetries spontaneously break, we can evaluate the nonzero expectation values of the scalar fields as the order parameters by searching the minimum point of its effective potential. In the theory we consider, this method cannot be used directly because the chiral order parameters are not such expectation values of the fields. Therefore, the scalar fields corresponding to the \( \bar{\psi} \psi \) are introduced by methods like the Hubbard–Stratonovich transformation in many works. In this paper, however, we adopt another method as will be seen in Sect. 6.
Here, in order to go beyond the critical scale \( \Lambda_c \) and to effectively evaluate the chiral order parameters, we introduce the bare mass term, which explicitly breaks the chiral symmetry, in addition to the chiral invariant bare action \[18\]:

\[
S_{\text{bare}} = S_{\text{bare (invariant)}} - \int_x m_0 \bar{\psi} \psi. \tag{40}
\]

When the cutoff scale \( \Lambda(t) \) lowers, the running mass \( m(m_0; t) \), being \( m_0 \) at the initial scale \( t = 0 \), is rapidly enhanced around \( \Lambda_c \) by the 4-fermi interaction, and consequently the enhanced mass suppresses the \( \beta \) functions due to the decoupling effect. Therefore, the 4-fermi coupling constant is expected to stay finite at the infrared scale. Taking the zero bare mass limit after solving the flow equation, the infrared limit mass becomes a chiral order parameter, the so-called dynamical mass of quarks:

\[
m_{\text{dyn}} \equiv \lim_{m_0 \to 0} \lim_{t \to \infty} m(m_0; t). \tag{41}
\]

The bare mass \( m_0 \) also works as an external source for the composite operator \( \bar{\psi} \psi \), and its expectation value is evaluated as follows:

\[
\langle \bar{\psi} \psi \rangle \equiv \lim_{m_0 \to 0} \lim_{t \to \infty} \frac{\partial \tilde{G}_0(m_0; t)}{\partial m_0}, \tag{42}
\]

where \( \tilde{G}_0(m_0; t) \) denotes \( V(\psi, \bar{\psi}; t)|_{\psi = \bar{\psi} = 0} \). Here the infrared limit value \( \tilde{G}_0(m_0; \infty) \) corresponds to the Helmholtz free energy.

From the viewpoint of the Helmholtz free energy, we may describe the relation between the \( D \chi \) \( \text{SB} \) and the divergent behavior of the 4-fermi coupling constant. Due to the chiral invariance of the theory, \( \tilde{G}_0(m_0; t) \) is an even function of \( m_0 \). Hence, the derivative of \( \tilde{G}_0(m_0; t) \) at \( m_0 = 0 \), namely the chiral condensates, vanishes if it is an analytic function. On the other hand, the derivative of \( \tilde{G}_0(m_0; t) \) cannot be continuous at \( m_0 = 0 \), if the chiral condensates have non-vanishing values. When the cutoff scale \( \Lambda(t) \) decreases, the initially analytic function \( \tilde{G}_0(m_0; t) \) changes into a non-analytic one at the critical scale \( \Lambda_c \). Therefore its second derivative diverges at the scale \( \Lambda_c \). Actually, in the large-\( N \) approximated NJL model, the 4-fermi coupling constant is almost equal to the second derivative, and therefore its divergent behavior shows the emergence of \( D \chi \) \( \text{SB} \) at the scale \( \Lambda_c \).

5. Field operator expansion and its convergence

Here, we attempt to solve the flow equation using the field operator expansion so that we evaluate the chiral order parameters introduced in the previous section. As seen at the end of this section, however, the field operator expansion does not work well in this model.

In the subspace spanned by polynomials in \( \sigma \), the \( \beta \) function for the 4-fermi coupling constant includes the 6-fermi coupling constant owing to the chiral symmetry breaking effect of the bare mass \( m_0 \). In general, the \( \beta \) function for the \( 2n \)-fermi coupling constant includes the \( 2(n + 1) \)-fermi coupling constant. Therefore, we have to take account of the infinite number of \( 2n \)-fermi operators, and encounter an infinite tower of coupled RG equations.

Since such infinitely coupled equations cannot be evaluated numerically, we stop the expansion at some maximum power \( N \):

\[
V(\sigma; t) = \sum_{n=0}^{N} \frac{1}{n!} \tilde{G}_n(t) \sigma^n, \tag{43}
\]

where we call \( N \) the order of “truncation”. Then, expanding the flow equation in powers of \( \sigma \), we set up \( (N + 1) \)-coupled RG equations where the \( \beta \) function for the \( 2N \)-fermi operator does not include
the 2\((N + 1)\)-fermi coupling constant. The larger the truncation order \(N\) is, the better the solution of the truncated coupled RG equations approximates the solution of the original flow equation. Actually, this type of truncation approximation has worked well in many theories.

Let us see the truncation dependence of the dynamical mass \(m_{\text{dyn}}\) calculated for the ladder flow equation (34). In Appendix A we explain the input parameters and the running gauge coupling constant which obeys the one-loop perturbative \(\beta\) function. In Fig. 3, we plot the running mass \(m(m_0; t)\) at the infrared limit \(t \to \infty\) for each bare mass, and show its dependence on the truncation order from \(N = 2\) to \(N = 20\). For a bare mass larger than about 0.02 GeV, the truncated solutions converge well with respect to the truncation order \(N\). It also shows that the mass is dynamically generated by the \(D\chi\)SB since the infrared running mass is much larger than the corresponding bare mass. In the small bare mass region, however, the truncated solutions do not converge but diverge more for larger truncation order. Therefore, we cannot take the zero bare mass limit straightforwardly in this method with the field operator expansion. It is also difficult to make a reliable extrapolation. The non-ladder flow equation (39) shows similar behaviors to the ladder flow equation.

We now show the results using another method of avoiding the explosive behavior of the 4-fermi coupling constant. In Eq. (43), the fermion potential is expanded around the vanishing value \((\sigma = 0)\). We expand it around the nonvanishing value \(\sigma_0\) \((\neq 0)\),

\[
V(\sigma; t) = \sum_{n=0}^{N} \frac{1}{n!} \tilde{H}_n(k)(\sigma - \sigma_0)^n.
\]  

(44)

In this expansion, we can go beyond the critical scale \(\Lambda_c\) without introducing the bare mass because the nonvanishing value \(\sigma_0\) plays a similar role to the bare mass. Note that it does not work as an external source for the chiral condensates. The dynamical mass corresponding to Eq. (41) is given by the following limit:

\[
m_{\text{dyn}} = \lim_{\sigma_0 \to +0} \lim_{t \to \infty} \tilde{H}_1(\sigma_0; t).
\]

(45)

As seen in Fig. 4, however, expansion around the nonvanishing value does not converge in the small value of the expansion point \(\sigma_0\).
6. Solving the flow equation as a PDE

In Sects. 4 and 5, so as to evaluate the chiral order parameters, we have introduced two methods, the introduction of the bare mass and expansion around the non-zero point, both of which are expected to avoid the divergent behavior of the 4-fermi coupling constant. Note that expansion around the non-zero point means that the fermion potential $V(\sigma; t)$ is evaluated at the non-zero point, $\sigma \neq 0$. In these methods, however, the field operator expansion does not converge in the small value region of the bare mass or the expansion point, and we cannot obtain a reliable chiral limit of the dynamical mass.

Obviously, the poor convergence of the field operator expansion originates from the divergent behavior of the 4-fermi coupling constant, which corresponds to the second derivative of the fermion potential $V(\sigma; t)$. On the other hand, if the flow equation is solved as a partial differential equation (PDE), the fermion potential is expected to behave analytically, at least away from the origin. Therefore, we will directly solve the flow equation as a PDE without the field operator expansion.

In the practical calculation we solve the flow equation in terms of the mass function, $M(\sigma; t) = \partial_{\sigma} V(\sigma; t)$, because the numerical solution of the PDE for the mass function is more stable than that for the fermion potential. Moreover, for numerical stability around $\sigma = 0$, $\sigma$ is transformed into the logarithmic variable $x = \log \sigma / \sigma_{\text{norm}}$, where the normalization scale $\sigma_{\text{norm}}$ is set to be 1 (GeV)$^3$. We do not directly treat the origin of $\sigma$ and consider the limit $x \to -\infty$ of the mass function $M(x; t)$ to calculate the dynamical mass as seen in Eq. (45).

Now we adopt the simple formulation of the grid method where derivatives with respect to $x$ are replaced with finite differences using n-point formulas [26]. It is worth mentioning that the matching method formulated in [27] is most frequently adopted in many works for the grid calculation of the flow equation. The n-point formulas adopted here are simpler than the matching method, but actually

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2 Strictly speaking, such a function without total analyticity cannot be a global solution of the PDE. Adopting the “weak” solution of the PDE, we can exactly solve the PDE and obtain physical quantities straightforwardly without any extrapolation. This method of the weak solution will be reported in a separate article: K.-I. Aoki, D. Sato, and S.-I. Kumamoto, in preparation (2013).
Fig. 5. RG evolution of the mass function $M(\sigma; t)$. The green dashed lines denote the mass function at dimensionless scales $t$ increasing by $\Delta t = 0.3$ steps, and the red line denotes the mass function in the infrared limit, $t \to \infty$. The mass function is dramatically increased from the vanishing value below a critical scale $t_c \simeq 5.3$. Give sufficiently good results in our case. The finite differences are defined by the 7-point formula where, for example, the finite difference corresponding to the first derivative $\partial_x M$ consists of values on 6 points. These points are the nearest neighbors to the point where the derivative is evaluated.

For the numerical calculation, we will set a finite region for $x$, $x_L \leq x \leq x_R$. In order to approximate the mass function at the end point $x_L$ to be the dynamical mass, we need to choose a small enough $x_L \ll -1$. Near the boundaries, where we cannot take the full 6 points, we use the 5-point formula, the 3-point formula, and eventually at the very end we take only the next point. Usually these constraints enhance the numerical error near boundaries. It will be found that actually these low-order approximated definitions of the finite differences near the boundaries do not induce instability. In the practical calculation, we choose the boundaries such that $x_L = -19$ and $x_R = 0$.

Through the procedure explained above, the coupled ordinary differential equations for the discretized mass function on the grid are obtained, and we numerically solve the equations with respect to the dimensionless scale $t$ using the fourth-order Runge–Kutta method. In Fig. 5 we present the RG evolution of the mass function given by the ladder flow equation (34). Here, the Landau gauge $\xi = 0$ is adopted, and the anomalous dimension is ignored, which is consistent with the local potential approximation (LPA). We can see that the mass function is dramatically increased from the vanishing value when lowering the cutoff scale $\Lambda(t)$. In particular, dynamical mass generation is observed below a critical scale, rather rapidly in a short range of scale $t$. The infrared limit is reliably evaluated, whose size reaches the $\Lambda_{QCD}$ scale, and thus the $D\chi$SB occurs. If the chiral symmetry is not spontaneously broken, the mass function becomes zero in the limit $x \to -\infty$.

The mass function converges well to a certain value when $x$ goes towards the end point $x_L$. Therefore we can conclude that the approximated definition of the finite differences near the boundaries does not induce the instability of the mass function at $x_L$, and the value can be identified with the dynamical mass $m_{\text{dyn}}$. To evaluate the chiral condensates $\langle \bar{\psi} \psi \rangle$ by using Eq. (42), we introduce the bare mass and calculate the free energy $\tilde{G}_0$ through the evolution of $V$ at the end point $x_L$. As mentioned in Sect. 3.2, the LPA ladder flow equation with the Landau gauge has been proved to give results equivalent to the improved ladder SD equation. Actually, the two chiral order parameters

\[ \text{Infrared limit} \]
obtained now agree well with the ones obtained by the SD approach in Ref. [29], which assures the total consistency of our method.

We present the numerical results of the two chiral order parameters obtained from the two approximated flow equations, the ladder one (34) and the non-ladder one (39), with the various values of the gauge-fixing parameter $\xi$. Table 1 (Table 2) shows the numerical values of the dynamical mass (the chiral condensates) with or without the anomalous dimension. Here, the chiral condensates are the renormalized ones at 1 GeV, $\langle \bar{\psi} \psi \rangle_{1\text{GeV}}$ [29].

We show these results graphically in Figs. 6 and 7. Now we find that the gauge dependence of the results with the anomalous dimension (A. D. in the figures) are suppressed much better than ones without. Additionally, as for the results with the anomalous dimension, the gauge dependence of the chiral condensates obtained from the non-ladder flow equation is almost vanishing. It looks perfect and this is nothing but what we expected to get by adding the crossed gluon diagrams to go beyond the ladder. On the other hand, such great improvement in the non-ladder calculation is not seen for the dynamical mass. This is due to the fact that the chiral condensates are the on-shell quantities while the dynamical mass is not. Consequently there is no reason that the dynamical mass does not depend on the gauge-fixing parameter. We confirmed that our non-ladder extended calculation respects the gauge independence almost perfectly.

Finally, it should be noted that in the Landau gauge ($\xi = 0$) the gauge-dependent ladder result of the chiral condensates coincides with the almost gauge-independent non-ladder extended one. This feature of the Landau gauge proves the folklore that the ladder approximation is particularly good in the Landau gauge.

7. Summary and discussion

In this paper we have derived a new approximated flow equation beyond the ladder approximation so as to improve dependence on the gauge-fixing parameter. We have developed various methods
to evaluate the two chiral order parameters, the dynamical mass of quarks, and the chiral condensates. These methods are expected to avoid the explosive behavior of the 4-fermi coupling constant in the course of solving the flow equation. Within these methods, however, the field operator expansion, which is usually applied to solve the flow equations, shows poor convergence induced by the infrared singularity. Then, we stopped the field operator expansion, and solved the flow equation as a partial differential equation by using the grid method. We have obtained the chiral order parameters successfully without any instability or extrapolation.

As for the chiral condensates, we have seen that the gauge dependence of the non-ladder extended result almost disappeared. Therefore our non-ladder extended approximation almost respects the gauge invariance.
A next step for further improvement of the approximation would be to take account of higher-order operators, including the first derivatives. This improvement is implemented by replacing the coefficient of the kinetic term \( Z_\psi(t) \) with a function of \( \sigma \), \( Z_\psi(\sigma; t) \). Consequently, we need to solve coupled partial differential equations in terms of the fermion potential \( V(\sigma; t) \) and the kinetic function \( Z_\psi(\sigma; t) \).

We have also seen that, particularly in the Landau gauge, the gauge-dependent ladder result of the chiral condensates coincides with the almost gauge-independent non-ladder extended one. This agreement might be related to the statement that only in the Landau gauge does the ladder approximation satisfy the Ward–Takahashi identity. However, at finite temperature and chemical potential, this relation is broken. Therefore we would encounter the new phase structures by applying this non-ladder extended approximation to the hot and dense QCD in the framework of the non-perturbative renormalization group.

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Appendix A. Running gauge coupling constant and input parameter

In our truncated subspace, the \( \beta \) function for the gauge coupling constant, \( \alpha_s \equiv g_s^2/4\pi \), agrees with the result of the 1-loop perturbation:

\[
\partial_t \alpha_s = \frac{\beta_0}{2\pi} \alpha_s^2,
\]

where \( \beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \). The solution of the RG equation is

\[
\alpha_s = \frac{2\pi}{\beta_0} \frac{1}{t_{\text{QCD}} - t},
\]

where the scale \( \Lambda_{\text{QCD}} \equiv \Lambda_0 e^{-t_{\text{QCD}}} \) gives the scale of the infrared quantities such as the chiral order parameters. Here, in order to go beyond the scale, lower than \( \Lambda_{\text{QCD}} \), we introduce the infrared cutoff effect [28] whereby the gauge coupling constant stops increasing at a proper infrared scale, which is naturally expected by the confinement. Adopting the cutoff scheme in Ref. [29], the running gauge coupling constant is given as follows:

\[
\alpha_s(t) = \begin{cases} 
\frac{2\pi}{\beta_0} \frac{1}{t_{\text{QCD}} - t}, & t < t_{ir} \\
\frac{2\pi}{\beta_0} \frac{1}{t_{\text{QCD}} - t_{ir}} + \frac{\pi}{\beta_0} \frac{(t - t_1)^2 - (t_{ir} - t_1)^2}{(t_{ir} - t_1)(t_{\text{QCD}} - t_{ir})^2}, & t_{ir} < t < t_1, \\
\frac{2\pi}{\beta_0} \frac{1}{t_{\text{QCD}} - t_{ir}} - \frac{\pi}{\beta_0} \frac{t_{ir} - t_1}{(t_{ir} - t_1)^2}, & t_1 < t 
\end{cases}
\]

where we set a fixed dimensionless scale for \( t_1 \) to be \( t_{\text{QCD}} + 1 \), and \( t_{ir} \) is left as an infrared cutoff scale parameter. Obeying Ref. [29], \( t_{ir} \) should be parametrized by \( \Delta_{\text{ir}} \) as \( t_{ir} = t_{\text{QCD}} - 0.5 \cdot (\Delta_{\text{ir}} + 1) \), and we take the following parameter:

\[
\Lambda_{\text{QCD}} = 484 \text{ MeV}, \quad \Delta_{\text{ir}} = -0.5.
\]

By the analysis using the ladder Schwinger–Dyson equation, it is confirmed that the physical quantities such as the chiral condensates are not sensitive to the choice of the infrared cutoff scale parameter \( t_{ir} \).
Finally, we discuss the initial condition of the fermion potential $V(\sigma; t)$. The initial cutoff scale $\Lambda_0$ has to be large enough so that the RG flow obtained approximates well the renormalized trajectory starting from the ultraviolet limit, $\Lambda_0 \rightarrow \infty$. At the ultraviolet limit, the fermion potential vanishes because the effective average action agrees with the bare QCD action, or if it exists there it would be strongly suppressed soon by its higher dimensionality. In practical calculations, we take the vanishing fermion potential as the initial condition, and set the initial cutoff scale $\Lambda_0$ to be large enough so that the infrared quantities do not depend on $\Lambda_0$ within a given numerical precision. Actually, we set $\Lambda_0$ to be the $Z$ boson mass scale, $M_Z = 91.2$ GeV.

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