On the Calabi flow

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1 Introduction

In [6], E. Calabi studied the variational problem of minimizing the so-called “Calabi energy” 1 in any fixed cohomology class of Kähler metrics. Any smooth critical point is called either constant scalar curvature (CscK) metric or extremal Kähler (CextK) metric depending on whether the Futaki Character vanishes or not in this class. CextK metrics include the more famous Kähler Einstein metric as a special case when the Kähler class is the canonical Kähler class. In recent years, the study of extremal Kähler (or CscK) metric has attracted intensive attention and many important works emerge. In particular, the uniqueness problem is completely settled, [11], [24], [38], and [18]. For existence, the situation is much more complicated. In 1976, S. T. Yau solved the famous Calabi

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1It is the $L^2$ norm of the scalar curvature function of the Kähler metric.
conjecture which implies that any Kähler manifold with vanishing first Chern class has a Calabi-Yau metric, that is, a Ricci-flat Kähler metric. Around the same time, T. Aubin and S. T. Yau independently proved existence of Kähler-Einstein metrics on compact Kähler manifolds with negative first Chern class. In [45], G. Tian proved that a complex surface with positive first Chern class admits a Kähler-Einstein metric if and only if its automorphism group is reductive. In 1997, a new algebraic invariant (K stability) was introduced by G. Tian [46] as an obstruction to the existence of Kähler Einstein metric. Later, S. K. Donaldson [24] gave a new definition of K-stability by using weights of Hilbert points... These invariants can also be extended to be the obstruction to the existence of Calabi's Extremal Kähler (CextK) metrics or CscK metrics, c.f. [39] and [42] for further references.

In 1984, Calabi constructed the first CextK metric, in \( \mathbb{C}P^n \) blown up at a point, by solving a 4th order ODE (the so-called “Calabi’s anstaz”). There are many beautiful works in constructing special CscK metric or CextK metric on many Kähler manifolds (cf. Lebrun-Simanca [35] [36] [4] and [43] etc and reference therein). More recently, J. Fine [30] has used adiabatic limit techniques to construct non-trivial CscK metrics on complex surfaces. Around the same time, using method of gluing and grafting Arezzo-Pacard [2][3] were able to construct new CscK metrics on the blow up of manifold if the original manifold has a CscK metric. In [27], Donaldson-Fine classify the toric anti-self-dual 4-manifold with some simple holomorphic data by using twistor correspondence.

However, there has not been much progress made on the existence of general Calabi's extremal Kähler metrics, via direct PDE method. Adopting the strategy of direct variational approach, the first author [10] studied the problem of minimizing the Calabi energy on Riemann surface, in an attempt to understand how compactness fails in general. Recent work of S. Donaldson [25] on toric surfaces, while still a special case, leads hope that the existence problem might be approachable on toric surfaces. Another ambitious program, not directly Kähler but very much related, is the recent work of Cheeger-Tian [9] and Tian-Viacosky [48]. However, for general Kähler manifold without boundary, the existence problem is still too difficult. Partially it is because the equation for CscK metric is a fully nonlinear 4th order partial differential equation. Envisioning this difficulty, in the same paper defining CscK metrics, E. Calabi proposed the so-called Calabi flow to attack the existence problem. Unlike the Kähler Ricci flow, (in Calabi flow) one deforms the Kähler potential in the direction of the scalar curvature. In [6], [7], Calabi shows that a) The Calabi energy is decreasing along the Calabi flow; b) The Calabi energy is weakly convex near a CscK or CextK metric. Therefore, at least conceptually, there is a good chance that the Calabi flow will converge in some fashion. These two properties play a crucial role in understanding the stability problem of the Calabi flow.

In general dimensional case, very few results are known or written down, even though several authors have comments on the importance of such a flow (cf. [26]). Even for the short time existence, which is known to the experts (because
the flow is parabolic) for smooth initial data, the precise/optimal result was never formulated in literature. In this paper, we first give a short time existence result:

**Theorem 1.1.** Let $(M, [\omega])$ be a polarized Kähler manifold without boundary. For any Kähler potential with $C^{3,\alpha}(M, \omega_g)$ norm bounded and such that the corresponding metric is uniform equivalent to $\omega_g$, then the Calabi flow exists for a short time and the Kähler metric becomes immediately smooth for $t > 0$.

In finite dimensions, once one establishes a contracting process to derive the short time existence, then it is immediately clear that, if one starts near a fixed point, it will stay in small neighborhood of the fixed point and eventually converge to the fixed point fast. However, in infinite dimensional case, the picture is much worse and peculiar phenomenon could occur. For certain parabolic flow with short time existence property, no matter how close you start from a fixed point, once the flow starts, it will diverge quickly away from fixed point. The hope for establishing a Stability theorem lies in the fact that the Hessian of the Calabi energy is strictly positive (unless it is in the direction of holomorphic isometry).

**Theorem 1.2.** (Stability theorem) Suppose $g$ is a CscK metric in $[\omega]$ on $M$. If the $C^{3,\alpha}(M)$ norm of the initial potential (w.r.t $g$) is small enough, then the flow will exist for all time and $g(t) = g_\varphi(t)$ will converge to a CscK metric $g_\infty$ in the same class $[\omega]$ in $C^\infty$ sense. Moreover, the Calabi energy will decay exponentially fast and consequently the convergence is exponentially fast along the flow.

**Remark 1.3.** The limit CscK metric $g_\infty$ could be different from the original CscK metric $g$ in holomorphic coordinates. By the recent result of [18], the CextK metric is unique in the fixed Kähler class up to automorphism. And so $g$ and $g_\infty$ differ by an automorphism. See the example in section 4 for more details.

To attack the long time existence of the Calabi flow, in the early 90s, the first author concentrated on the weak compactness of Kähler metrics with uniform bound on the Calabi energy [12], [13] and [14]. This is a crucial step in understanding the whole picture of the Calabi flow since the Calabi energy is decreasing along the flow. Using this weak compactness theorem, the first author [16] was able to prove the flow exists for all time in Riemann surface and the flow will converge to a CscK metric exponentially fast assuming the uniformization theorem. It gives a new proof to a theorem of P. Chruściel [20] on the Calabi flow in Riemann surface from a completely different perspective.

In this paper, we follow the path of [16] to tackle the long time existence under some suitable curvature assumption. In a fixed Kähler class, we obtain a compactness theorem under the uniform bound of Ricci curvature and potential.

**Theorem 1.4.** (Compactness theorem) All metrics $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$ in the space of Kähler metrics with both the potential $\varphi$ and the Ricci curvature $\text{Ric}_\varphi$
uniformly bounded are equivalent and compact in $C^{1,\alpha}$-topology for any $\alpha \in (0, 1)$ (It is equivalent to say $\varphi$ is uniformly bounded in $C^{3,\alpha}$ for any $\alpha \in (0, 1)$).

As a consequence, we prove that the Calabi flow will exist for all time if the evolving Ricci curvature is uniformly bounded.

**Theorem 1.5.** For the Calabi flow initiating from any smooth Kähler metric, the flow exists as long as the Ricci curvature stays uniformly bounded.

**Remark 1.6.** For Hamilton’s Ricci flow [33], it was proved by Hamilton that the flow will continue as long as the Riemannian curvature stays uniformly bounded. After Perelman’s work [40], [41], N. Sesum [44] was able to improve this important result saying that the Ricci flow will continue as long as the Ricci curvature is bounded. Readers are referred to [21] for a complete updated reference on this important subject.

In [16], the weak compactness of the space of Kähler metrics with uniform Calabi energy and area in Riemann surface is used critically in obtaining the long time existence of the Calabi flow. Motivated by [16], it is highly desirable to obtain a weak compactness theorem similar to the corresponding theorems in [12], [13], and [14] in general Kähler manifold. This will be discussed in a subsequent paper.

In [8], Calabi-Chen showed that the Calabi flow essentially decreases the geodesic distance in the space of Kähler metrics. This property must play an important role in the future study of the Calabi flow. In particular, this property suggests that the flow shall exist globally (cf. [15]) and the flow will converge to a CextK metric except a codimension 2 subvariety although the complex structure of the limiting manifold might be different from the original one. In [26], Donaldson describes precisely what the limit of the Calabi flow shall be under various situation, from perspective of sympletic moment map. Motivated by these discussions, it is very important to study the so-called “removing singularity” for CscK metric or CextK metric.

**Theorem 1.7.** Let $g$ be a weakly CscK metric in a punctured disc which is smooth in a small neighborhood of the boundary of the disc. If $g$ is uniformly bounded from above and below (with respect to the Euclidean metric) in the punctured disc and its tensors are weakly in $W^{1,2}_{\text{loc}}$, then it is a smooth CscK Kähler metric in the entire disc.

**Remark 1.8.** We simplify the proof by requiring the boundary data to be smooth in a small neighborhood of the boundary of the disc and provide a complete proof the last claim (Claim 6.6). Since the aim of this theorem is about removing possible singularities, the significance of the theorem is not affected. On the other hand, we add a remark that to prove the full version (we mean that the boundary data is smooth instead of smooth in a small neighborhood of the boundary), one need add one more step which is about to solve the Monge-Ampere equation

$$\log \det (\delta_{kl} + \varphi_{kl}) = f$$
in a strongly pseudo-convex domain with \( f \in C^\alpha \).

This is a first step in establishing a more ambitious regularity result (cf [15]) for weak CscK metrics on Kähler manifolds where the “suspected” singular locus is a union of subvarieties of codimension 2 or higher. In this regard, the assumption of uniform ellipticity is necessary in the following sense: if we blow up \( M \) at a point \( p \), we can denote the resulting Kähler manifold as \( \hat{M} \) and the exceptional divisor \( E \). Suppose \( M \) and \( \hat{M} \) is each equipped with a CscK metric. Then, one can view either of these CscK metrics as a weak CscK metric in the wrong manifold and singularities cannot be removed. One easily observes in both cases, the uniformly ellipticity is violated, but in different directions. In other words, one can perhaps weaken the assumption of uniform ellipticity in some way. However, some form of ellipticity must be presented.

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## 2 Preliminary

### 2.1 Notations in Kähler Geometry

Let \( M \) be a compact complex manifold of complex dimension \( n \). An Hermitian metric \( g \) on \( M \) in local coordinates is given by

\[
g = g_{i\bar{j}} dz^i \otimes dz^{\bar{j}},
\]

where \( \{g_{i\bar{j}}\} \) is a positive definite Hermitian matrix smooth function. And we use \( \{g^{i\bar{j}}\} \) to denote the inverse matrix of \( \{g_{i\bar{j}}\} \). The Kähler condition says that the corresponding Kähler form \( \omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \) is a closed \((1, 1)\) form. The Kähler class of \( \omega \) is its cohomology class \([\omega]\) in \( H^2(M, \mathbb{R}) \). By the Hodge theory, any other Kähler form in the same class is of the form

\[
\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0,
\]

for some real valued function \( \varphi \) on \( M \), where

\[
\partial \bar{\partial} \varphi = \sum_{i,j=1}^n \partial_i \bar{\partial}_j \varphi dz^i \wedge dz^{\bar{j}} = \varphi_{i\bar{j}} dz^i \wedge dz^{\bar{j}}.
\]
The corresponding Kähler metric is denoted by \( g_\varphi = (g_{ij} + \varphi, ij) \, dz^i \otimes \overline{dz}^j \), and we use \( \{ g_{ij}^\varphi \} \) to denote the inverse matrix of \( \{ g_{ij} + \varphi, ij \} \). For simplicity, we use both \( g \) and \( \omega \) to denote the Kähler metric. Define the space of Kähler potentials 
\[
H_\omega = \{ \varphi | \omega _\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0, \, \varphi \in C^\infty (M) \},
\]
which is called the space of Kähler metrics and is the main object we are interested in.

Given a Kähler metric \( \omega \), its volume form is
\[
\omega^n = (\sqrt{-1})^n \frac{1}{n!} \det (g_{ij}) dz^1 \wedge \overline{dz}^1 \wedge \cdots \wedge dz^n \wedge \overline{dz}^n.
\]
The Ricci curvature of \( \omega \) is locally given by
\[
R_{ij} = -\partial_i \partial_j \log \det (g_{kl}).
\]
Its Ricci form is of the form
\[
Ric_{\omega} = \sqrt{-1} R_{ij} dz^i \overline{dz}^j = -\sqrt{-1} \partial_i \partial_j \log \det (g_{kl}).
\]
It is a real, closed \((1, 1)\) form. The cohomology class of Ricci form is the famous first Chern class \( c_1 (M) \), independent of the metric.

2.2 The Calabi flow

Given a polarized compact Kähler manifold \((M, [\omega])\), for any \( \varphi \in H_\omega \), E. Calabi introduced the Calabi functional in [6], [7],
\[
Ca (\omega_\varphi) = \int_M R_{\varphi}^2 \omega_\varphi^n,
\]
where \( R_{\varphi} \) is the scalar curvature of \( \omega_\varphi \). Note that both the total volume
\[
V_\varphi = \int_M \omega_\varphi^n
\]
and the total scalar curvature
\[
S_\varphi = \int_M R_\varphi \omega_\varphi^n
\]
remain unchanged when \( \varphi \) varies in \( H_\omega \). As a consequence, the average of the scalar curvature
\[
\overline{R} = \frac{S_\varphi}{V_\varphi}
\]
is a constant depending only on the class \([\omega]\). Usually we use the following modified Calabi energy
\[
\tilde{Ca}(\omega_\varphi) = \int_M (R_\varphi - \overline{R})^2 \omega_\varphi^n
\]
to replace $Ca(\omega_\varphi)$ since they only differ by a topological constant. E. Calabi studied the variational problem to minimize $Ca(\omega_\varphi)$ in $H_\omega$. The critical point turns out to be either CscK or CextK metric depending on whether the Futaki character vanishes or not. The Futaki character $f = f_\varphi : h(M) \to \mathbb{C}$ is defined on the Lie algebra $h(M)$ of all holomorphic vector fields of $M$ as follows,

$$f_\varphi(X) = -\int_M X(F_\varphi)\omega_\varphi^n,$$

where $X \in h(M)$ and $F_\varphi$ is a real valued function defined by

$$F_\varphi = G_\varphi(R_\varphi).$$

$G_\varphi$ is the Hodge-Green integral operator, and $F_\varphi = G_\varphi(R_\varphi)$ is equivalent to $\triangle_\varphi F_\varphi = R_\varphi - \bar{R}$, where $\triangle_\varphi$ is the Laplace operator of the metric $\omega_\varphi$. In [7], E. Calabi showed that the Futaki character $f = f_\varphi$ is invariant when $\varphi$ varies in $H_\omega$, and the critical point of $Ca(\omega_\varphi)$ is CscK metric when $f$ vanishes, otherwise CextK metric when $f$ is not zero. Moreover, the second variation form (Hessian form) of $Ca(\omega_\varphi)$ is semi-positive definite at CscK metrics or CextK metrics with a finite dimensional kernel (corresponding to holomorphic vector fields).

The existence of CscK metric (or CextK metric) seems intractable at the first glance since the equation is a fully nonlinear 4th order equation (6th order for CextK metric). In [6], E. Calabi proposed the so-called Calabi flow to approach the existence problem. The Calabi flow is the gradient flow of the Calabi functional, defined as the following parabolic equation with respect to a real parameter $t \geq 0$,

$$\frac{\partial}{\partial t} g_{ij}(t) = \partial_i \partial_j R_{g(t)}.$$

On the potential level, the Calabi flow is of the form

$$\frac{\partial \varphi}{\partial t} = R_\varphi - \bar{R}.$$

Under the Calabi flow, we have

$$\frac{d}{dt} \int_M (R_\varphi - \bar{R})^2 \omega_\varphi^n = -2 \int_M (D_\varphi R_\varphi, R_\varphi) \omega_\varphi^n,$$

where $D_\varphi$ is Lichnérowicz operator with respect to $\omega_\varphi$. Lichnérowicz operator $D$ is defined by

$$Df = f_\alpha^\alpha \partial_\alpha,$$

where the covariant derivative is with respect to $\omega$. So the Calabi energy is strictly decreasing along the flow unless $\omega_\varphi$ is CextK or CscK metric.
3 Short time existence

A straightforward calculation shows that

\[ R_\varphi = g^{ij}_\varphi R_{ij}(\varphi) \]

\[ = -g^{ij}_\varphi \partial_i \partial_j \{ \log \det (g_{kl} + \partial_k \partial_l \varphi) \} \]

\[ = -g^{ij}_\varphi \partial_i \left\{ g^{kl}_\varphi \partial_j (g_{kl} + \partial_k \partial_l \varphi) \right\} \]

\[ = -g^{ij}_\varphi \left\{ g^{kl}_\varphi \partial_i \partial_j (g_{kl} + \partial_k \partial_l \varphi) + \partial_i g^{kl}_\varphi \partial_j (g_{kl} + \partial_k \partial_l \varphi) \right\} \]

\[ = -A(\nabla \varphi, \nabla^2 \varphi) \varphi + f(\nabla \varphi, \nabla^2 \varphi, \nabla^3 \varphi), \]

where

\[ A(\varphi)w = A(\nabla \varphi, \nabla^2 \varphi)w = g^{ij}_\varphi g^{kl}_\varphi \nabla_i \nabla_j \nabla_k \nabla_l w, \quad (3.1) \]

\( \nabla, \bar{\nabla} \) are both covariant derivatives with respect to \( \omega \) and \( f(\cdots) \) is some function of the derivatives of order \( \leq 3 \) of \( \varphi \). \( A(\varphi) \) is a strictly elliptic operator with coefficients depending on the derivatives of order \( \leq 2 \) of \( \varphi \), so the Calabi flow is a standard 4th order quasilinear parabolic equation. The quasilinear parabolic equation is well studied by many authors, and the existence of solution is established. In some sense the short time existence of the Calabi flow is standard and well known. We would like to carry out the details here for our purpose to tackle long time existence.

One approach for local existence and regularity theory of quasilinear parabolic equation is to use the maximal regularity theory of [23], which is based on the use of continuous interpolation spaces. In [1], [22], the maximal regularity was extended to functions which admit a prescribed singularity at \( t = 0 \). This extension allows us to take advantage of the quasilinear equation to establish the smoothing property. It turns out that this version of short time existence is very useful for the problem of the long time existence of the flow.

3.1 Function space and maximal regularity

This section follows [22] and we will state a main theorem in [22] which is used to prove the short time existence in next section. In the following we assume \( \theta \in (0, 1) \), and \( E \) is a Banach space, \( J = [0, T] \) for some \( T > 0 \). We consider functions defined on \( \tilde{J} = (0, T] \) with a prescribed singularity at 0. Set

\[ C_{1-\theta}(J, E) := \left\{ u \in C(\tilde{J}, E); [t \to t^{1-\theta} u] \in C(J, E), \lim_{t \to 0^+} t^{1-\theta} |u(t)|_E = 0 \right\}, \]

\[ |u|_{C_{1-\theta}(J, E)} := \sup_{t \in J} t^{1-\theta} |u|_E, \]

and

\[ C_{1}^{1-\theta}(J, E) := \left\{ u \in C^{1}(\tilde{J}, E); u, \dot{u} \in C_{1-\theta}(J, E) \right\}. \]
It is easy to verify $C_{1-\theta}(J, E)$, equipped with the norm $\| \cdot \|_{C_{1-\theta}(J, E)}$, is a Banach space and $C_{1-\theta}^1(J, E)$ is a subspace.

Let $E_1, E_0$ be two Banach space such that $E_1$ is continuously embedded into $E_0$. The set of bounded linear operators from $E_1$ to $E_0$ is denoted by $\mathcal{L}(E_1, E_0)$. Any operator $A \in \mathcal{L}(E_1, E_0)$ can be considered as an unbounded operator in $E_0$ with domain $E_1$. It can happen that an $A \in \mathcal{L}(E_1, E_0)$, seen as unbounded operator in $E_0$, is closed and generates a strongly continuous semigroup denoted by $\exp(-tA)(0 \leq t < \infty)$. And if the associated semigroup $\exp(-tA)$ is an analytic semigroup, we say $A \in \mathcal{H}(E_1, E_0)$. Set

$$E_0(J) := C_{1-\theta}(J, E_0),$$

and

$$E_1(J) := C_{1-\theta}^1(J, E_0) \cap C_{1-\theta}(J, E_1),$$

where

$$|u|_{E_1(J)} := \sup_{t \in J} t^{1-\theta} (|\dot{u}|_{E_0} + |u|_{E_1}).$$

By using $E_1(J), E_0(J)$, we can define the so-called “continuous interpolation spaces” of the couple $(E_1, E_0)$. This way of defining $E_\theta$ is described in [23]. For any $v \in E_1(J)$ we have $|\dot{v}|_{E_0} \leq Ct^{1-\theta}$ for some finite constant $C > 0$, so that $v$ extends to a continuous function from $[0, 1]$ to $E_0$. Indeed $v(0)$ is well defined by

$$v(0) = v(1) - \int_0^1 \dot{v}(t)dt.$$

We can define the Banach space

$$E_\theta = \{ u(0) : u \in E_1(J) \} , \quad |x|_{E_\theta} = \inf \{ |u|_{E_1(J)} : x = u(0), u \in E_1(J) \}.$$ 

Let $E_\theta, E_0(J), E_1(J)$ as above. Suppose that $A \in \mathcal{L}(E_1, E_0)$, we can consider the operator $\hat{A} : E_1(J) \rightarrow E_0(J) \oplus E_\theta$ defined by $\hat{A}u = (\dot{u} - Au, u(0))$. $\hat{A}$ is obviously bounded. We define $M_\theta(E_1, E_0)$ as follows: $M_\theta(E_1, E_0) = \{ A \in \mathcal{H} (E_1, E_0) : \hat{A} is an isomorphism between $E_1(J)$ and $E_0(J) \oplus E_\theta$\}. If $A \in M_\theta(E_1, E_0)$ then $(E_0(J), E_1(J))$ is called a pair of maximal regularity for $A$. In other words, $M_\theta(E_1, E_0)$ contains those generators of analytic semigroups $A$ for which

$$\dot{u} + Au(t) = f(t), \quad u(0) = u_0,$$

has a unique solution $u \in E_1(J)$ for any $f \in E_0(J)$, and $u_0 \in E_\theta$. In [1], S. Angenent has shown that many interesting operators belong to the class $M_\theta(E_1, E_0)$. In particular the operator $A(\varphi)$ in (3.1) belongs to the class $M_\theta(E_1, E_0)$ for appropriate Banach space $E_0, E_1$ (see next section).

Now we are in the position to state a main Theorem in [22].
Theorem 3.1 (Philippe Clément, Gieri Simonett [22]). Let \( \theta \in (0, 1) \) be fixed and let \( E_\theta = (E_0, E_1)_\theta \) be a continuous interpolation space. Assume \( V_\theta \subset E_\theta \) is open,

\[
(A, f) \in C^{1, -}(V_\theta, M_\theta(E_1, E_0) \times E_0).
\]

For every \( x_0 \in V_\theta \), there exists some positive constant \( \tau = \tau(x_0), \epsilon = \epsilon(x_0) \) and \( c = c(x_0) \) such that the following evolution equation

\[
\dot{u}(t) + A(u(t))u(t) = f(u(t)), \quad t \in \bar{J}, \quad u(0) = x,
\]

has a unique solution

\[
u(\cdot, x) \in C^{1, -}_1([0, \tau], E_0) \cap C_{1 - \theta}([0, \tau], E_1)
\]

in \([0, \tau]\) for any initial value \( x \in \bar{B}_{E_\theta}(x_0, \epsilon) \), where \( \bar{B}_{E_\theta}(x_0, \epsilon) \) denotes the ball in Banach space \( E_\theta \) with radius \( \epsilon \) centered at \( x_0 \). In particular, we have

\[
|u(\cdot, x_0)|_{C([0, \tau], E_\theta)} \leq c, \quad \text{and} \quad |u(\cdot, x_0)|_{E_1([0, \tau])} \leq c.
\]

Moreover, we have

\[
|u(\cdot, x) - u(\cdot, y)|_{E_1([0, \tau])} \leq c|x - y|_{E_\theta},
\]

and

\[
|u(\cdot, x) - u(\cdot, y)|_{C([0, \tau], E_\theta)} \leq c|x - y|_{E_\theta}, \quad x, y \in \bar{B}_{E_\theta}(x_0, \epsilon).
\]

3.2 Short time existence

We now exhibit appropriate Banach space \( E_0, E_1 \) to use Theorem 3.1 to prove the short time existence of the Calabi flow. The spaces we use are certain little Hölder space. Recall that if \( k \in \mathbb{N}, \alpha \in (0, 1) \), the Hölder space \( C^{k, \alpha} \) is the Banach space of all \( C^k \) functions \( f : \mathbb{R}^n \to \mathbb{R} \) which has finite Hölder norm. The subspace of \( C^\infty \) function in \( C^{k, \alpha} \) is not dense under the Hölder norm. One defines the little Hölder space \( c^{k, \alpha} \) to be the closure of smooth functions in the usual Hölder space \( C^{k, \alpha} \). And one can verify \( c^{k, \alpha} \) is a Banach space and that \( c^{l, \beta} \hookrightarrow c^{k, \alpha} \) is a continuous and dense imbedding for \( k \leq l \) and \( 0 < \alpha < \beta < 1 \). These definitions can be extended to functions on a smooth manifold \( M \) naturally [5]. For our purpose, the key fact [47] about the continuous interpolation spaces is that for \( k \leq l \) and \( 0 < \alpha < \beta < 1 \), and \( 0 < \theta < 1 \), there is a Banach space isomorphism

\[
(c^{k, \alpha}, c^{l, \beta})_\theta \cong c^{\theta l + (1 - \theta)k + \theta \beta + (1 - \theta)\alpha}, \quad (3.2)
\]

provided that the exponent \( \theta l + (1 - \theta)k + \theta \beta + (1 - \theta)\alpha \) is not an integer.

For any function \( \varphi \in \mathcal{H}_\omega \), the operator \( A(\varphi) \) is a strict elliptic operator. Under these conditions, \( A \) generates an analytic semigroup [37] in \( c^{\theta, \alpha}(M) \) with domain \( c^{4, \alpha}(M) \). By Theorem 3.1, the short time existence holds for any initial potential \( \varphi \in c^{3, \alpha}(M) \) satisfying \( \omega_\varphi > 0 \). We have
Theorem 3.2. If \( \omega_{\varphi_0} = \omega + \sqrt{-1} \bar{\partial} \partial \varphi_0 \) satisfies \( |\varphi_0|_{C^{3,\alpha}(M)} < K \), where \( K \) is some constant, and \( \lambda \omega < \omega_{\varphi_0} \leq \Lambda \omega \), where \( \lambda, \Lambda \) are two positive constants, then the Calabi flow initiating with \( \varphi_0 \) admits a unique solution
\[
\varphi(t) \in C([0,T], C^{3,\alpha}(M)) \cap C((0,T], C^{4,\alpha}(M))
\]
for some small \( T = T(\lambda, \Lambda, K, g) \). More specifically, for any \( t \in (0,T] \), there is a constant \( C = C(\lambda, \Lambda, K, g) \) such that
\[
t^{1/4} (|\varphi(t)|_{C^{3,\alpha}(M)} + |\varphi(t)|_{C^{4,\alpha}(M)}) \leq C, \quad |\varphi(t)|_{C^{3,\alpha}(M)} \leq C.
\]
In particular, when \( g \) is CscK, and \( |\varphi_0|_{C^{3,\alpha}(M)} \) is small enough, there is some uniform constant \( C_1 \) depending only on \( g \), such that
\[
|\varphi(t)|_{C^{3,\alpha}(M)} \leq C_1 |\varphi_0|_{C^{3,\alpha}(M)},
\]
and
\[
t^{1/4} (|\varphi(t)|_{C^{3,\alpha}(M)} + |\varphi(t)|_{C^{4,\alpha}(M)}) \leq C_1 |\varphi_0|_{C^{3,\alpha}(M)}.
\]

Proof. Set \( E_0 = c^{0,\alpha}(M), E_1 = c^{4,\alpha}(M) \) for any fixed \( \alpha \in (0,1) \). Choose \( \theta = 3/4 \) so that \( E_\theta = (E_0, E_1)_\theta = c^{3,\alpha}(M) \). For any \( \varphi_0 \in V_\theta \), where
\[
V_\theta := \{ u | u \in c^{3,\alpha}(M), \lambda g < g(u) < \Lambda g, |u|_{c^{3,\alpha}(M)} < K \}
\]
is open in \( c^{3,\alpha}(M) \), \( A(\varphi_0) = A \) generates a strongly continuous analytic-semigroup \( \exp(-tA) \) in \( c^{0,\alpha}(M) \) with domain \( c^{4,\alpha}(M) \) and \( A \in M_\theta(E_1, E_0) \). This fact follows from the construction in [1]. Because for any \( 0 < \alpha < 1, A \in \mathcal{H}(E_1, E_0) \). We can take \( F_0 = c^{3,\gamma}(M), F_1 = c^{4,\gamma}(M) \) for some \( 0 < \gamma < \alpha < 1 \), and \( A \in \mathcal{H}(F_1, F_0) \). Then define \( F_2 \) to be the domain of \( A^2 \), i.e.
\[
F_2 = \{ x \in F_1 : Ax \in F_1 \},
\]
and let \( F_2 \) have the graph topology, \( |x|_{F_2} = |x|_{F_1} + |Ax|_{F_1} \). In our case \( F_2 = c^{3,\gamma}(M) \). Pick up \( \delta = (\alpha - \gamma)/4 \) such that
\[
E_0 = F_\delta, \quad E_1 = F_{1+\delta} = (F_2, F_1)_\delta.
\]
Then by Theorem 2.14 in [1] we know actually \( A \in M_\delta(E_1, E_0) \).

To use Theorem 3.1, we need to show \( (A,f) \in C^{1,-}(E_0, M_\theta(E_1, E_0) \times E_0) \) for \( \theta = 3/4 \), and \( E_0 = c^{3,\alpha}(M), E_1 = c^{4,\alpha}(M) \). It suffices to show that for \( u, v \in V_\theta, w \in E_1 = c^{4,\alpha}(M) \),
\[
|f(u) - f(v)|_{C^{3,\alpha}(M)} \leq C_1 |u - v|_{C^{3,\alpha}(M)},
\]
\[
|A(u)w - A(v)w|_{E_0} \leq C_2 |u - v|_{C^{3,\alpha}(M)} |w|_{E_1},
\]
where \( C_1, C_2 \) are two constants. The first inequality is obvious since \( f \) is actually \( C^\infty \) in its arguments. The second one follows from
\[
|A(u)w - A(v)w|_{E_0} = 
\]
\[
\leq C |u - v|_{C^{3,\alpha}(M)} |w|_{E_1}
\]
\[
\leq C |u - v|_{C^{3,\alpha}(M)} |w|_{E_1}.
\]
Theorem 3.2 follows from Theorem 3.1 directly. \( \square \)
Theorem 3.3. *(Smoothing property)* Following Theorem 3.2, actually \( \varphi(t) \in C([0, T], C^3(M)) \cap C((0, T], C^\infty(M)) \).

*Proof.* Consider the following parabolic equation

\[
\frac{\partial \varphi}{\partial t} = -A(\varphi)\varphi + f(\varphi)
\]

\( \varphi(0) = \varphi_0. \)

By the maximal regularity, we know indeed \( \varphi(t) \in C^3(M) \) when \( t > 0. \) We can consider

\[
\frac{\partial \varphi}{\partial t} = -A(\varphi)\varphi + f(\varphi),
\]

with initial value \( \varphi(\delta) \in C^4(M) \) for any \( \delta > 0 \) small. Taking \( E_0 = c^1,\alpha(M), E_1 = c^5,\alpha(M) \), by (3.2) we have \( E_\theta = (E_0, E_1)_\theta = c^4,\alpha(M) \) where \( \theta = 3/4. \) Theorem 3.1 applies for \( (E_0, E_1) \), a pair of maximal regularity for \( A(\varphi(\delta)) \). So we can get \( \varphi(t) \in c^k,\alpha(M) \) when \( t \in (\delta, T] \). Similarly we can show \( \varphi(t) \in c^k,\alpha(M) \) for any \( k \in \mathbb{N} \) when \( t > 0. \)

Remark 3.4. One can show \( \varphi(t) \in C^\infty((0, T], C^\infty(M)) \). To show \( \varphi(t) \) is smooth in time argument, we need to consider some Banach space with time norm [37]. The idea is similar to Theorem 3.3, but we shall not use this fact.

4 Stability theorem

When \( g \) is a CscK metric, it is a fixed point under the Calabi flow. Since \( Ca(\omega_\varphi) \) is weakly convex at CscK metric \( g \), intuitively it is a stable fixed point under the Calabi flow. We will give an affirmative answer in this section. More precisely,

**Theorem 4.1.** *(Stability theorem)* Suppose \( g \) is a CscK metric in \([\omega]\) on \( M \). Consider the Calabi flow

\[
\frac{\partial \varphi}{\partial t} = R_\varphi - R, \quad \varphi(0) = \varphi_0.
\]

(4.1)

When \( |\varphi_0|_{C^3(M)} < \epsilon \), where \( \epsilon > 0 \) is small enough, then the flow exists for all time and \( g(t) = g_{\varphi(t)} \) converges to a CscK \( g_\infty \) metric in the same class \([\omega]\) in \( C^\infty \) sense. Moreover, the Calabi energy decays exponentially fast and the convergence is exponentially fast.

*Proof.* Suppose \( g \) is a CscK metric, we derive a priori estimates for the scalar curvature equation. Consider the following elliptic equation:

\[
R_\varphi - R = f,
\]

where \( \omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi \) is “\( C^\alpha \) equivalent” to \( \omega \) for some \( \alpha \in (0, 1). \) By “\( C^\alpha \) equivalent” we mean in a holomorphic local coordinates, \( \lambda \omega \leq \omega_\varphi = \omega + \)
$$\sqrt{-1} \partial \bar{\partial} \varphi \leq \Lambda \omega$$ for some constants $0 < \lambda \leq \Lambda$, and $|\varphi|_{C^{2,\alpha}(M, g)}$ is uniformly bounded. Rewrite the equation as

$$-g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \det (g_{kl} + \partial_k \partial_{\bar{l}} \varphi) + g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \det (g_{ki}) = f.$$ 

It follows that

$$-g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \frac{\det (g_{kl} + \partial_k \partial_{\bar{l}} \varphi)}{\det (g_{ki})} = f + \left( g^{i\bar{j}} - g^{\bar{i}j} \right) \partial_i \partial_{\bar{j}} \log \det (g_{ki}). \quad (4.2)$$

Denote

$$u = \log \frac{\det (g_{kl} + \partial_k \partial_{\bar{l}} \varphi)}{\det (g_{ki})},$$

$$h = f + \left( g^{i\bar{j}} - g^{\bar{i}j} \right) \partial_i \partial_{\bar{j}} \log \det (g_{ki}).$$

One can rewrite (4.2) as

$$-\Delta \varphi u = h. \quad (4.3)$$

By the standard $L^p$ theory ($p$ big enough), one can get that

$$|u - u|_{W^{2,p}} \leq C(|h|_{L^p} + |u - u|_{L^p}), \quad (4.4)$$

where $u$ is the average of $u$, the constant $C$ depends on the $C^\alpha$ Hölder norm of $\omega \varphi$ (actually $C^0$ norm is sufficient for $L^p$ theory). Through Moser’s iteration the estimate

$$|u - u|_{L^\infty} \leq C(|h|_{L^p} + |u - u|_{L^2})$$

holds for (4.3) when $g_\varphi$ is equivalent to $g$. And by (4.3), one can get

$$-(u - u) \Delta \varphi (u - u) = h(u - u).$$

It implies that

$$\int_M |\nabla u|^2 dg_\varphi = \int_M h(u - u) dg_\varphi \leq \left( \int_M h^2 dg_\varphi \right)^{1/2} \left( \int_M (u - u)^2 dg_\varphi \right)^{1/2}.$$ 

The Poincaré inequality gives that

$$\int_M |u - u|^2 dg_\varphi \leq C \int_M |\nabla u|^2 dg_\varphi.$$ 

And so

$$|u - u|_{L^2} \leq C|h|_{L^2}.$$ 

It implies that

$$|u - u|_{L^\infty} \leq C(|h|_{L^p} + |h|_{L^2}) \leq C|h|_{L^p}.$$ 

We need only the weaker version

$$|u - u|_{L^p} \leq C|h|_{L^p}. \quad (4.5)$$
It follows from (4.4) and (4.5) that

\[ |u - \underline{u}|_{W^{2,p}} \leq C|h|_{L^p}. \]  

(4.6)

We can rewrite (4.6) as

\[ |u - \underline{u}|_{W^{2,p}} \leq C \left( |f|_{L^p} + |\varphi|_{W^{2,p}} \right). \]  

(4.7)

Consider the Monge-Ampère equation,

\[ \log \frac{\det (g_{ij} + \partial_i \partial_j \varphi)}{\det (g_{ij})} = u. \]

We can rewrite it as

\[ \frac{\det (g_{ij} + \partial_i \partial_j \varphi)}{\det (g_{ij})} = \exp (u) \exp (u - \underline{u}). \]

Since \( g_{\varphi} \) is “\( C^\alpha \) equivalent” to \( g \), we know \( \exp (\underline{h}) \) is a uniformly bounded constant and so when \( p \) is big (\( p > 2n \))

\[ |\varphi|_{C^{3,\alpha}} \leq C |u - \underline{u}|_{C^{1,\alpha}} \leq C |u - \underline{u}|_{W^{2,p}} \]  

(4.8)

for some \( \alpha \in (0,1) \). Combine (4.7) and (4.8), we have

\[ |\varphi|_{C^{3,\alpha}} \leq C \left( |f|_{L^p} + |\varphi|_{W^{2,p}} \right). \]

By the interpolation inequality, actually we have

\[ |\varphi|_{C^{3,\alpha}} \leq C \left( |R_{\tau} - R|_{L^p} + |\varphi|_{L^1} \right). \]  

(4.9)

Since the metrics are equivalent, it does not matter which volume form we use as measure. Also \( C \) denotes a generic constant which could differ line by line.

To derive the long time existence, it is sufficient to control the \( C^{3,\alpha} \) norm of the potential by the short time existence derived in Section 3. By (4.9), essentially we just need to control the decay of the Calabi energy. Indeed, the Calabi energy decays exponentially fast around a CscK metric. Let us recall

\[ \frac{d}{dt} \int_M (R(t) - \underline{R})^2 \, dg(t) = -2 \int_M (D_t R(t), R(t)) \, dg(t). \]

By the definition of eigenvalues, the first eigenvalue of \( D_t \) is given by

\[ \lambda(t) = \inf_{f \in C_0^\infty(M)} \frac{\int_M f_{\alpha\beta} f^{\alpha\beta} \, dg(t)}{\int_M f^2 \, dg(t)}. \]

The definition involves only the metric and its first derivative. If \( g(t) \) is close to \( g \) in \( C^{3,\alpha} \) topology (by that we mean \( \varphi(t) \) is small in \( C^{3,\alpha} \), \( |\varphi(t)|_{C^{3,\alpha}} < \epsilon \), then
the eigenvalues of \( D_t \) is close to the eigenvalues of \( D \), and the corresponding eigenspace is close to the eigenspace of \( g \). The eigenvalues of \( D \) are nonnegative.

**Case 1.** If the first eigenvalue of \( D \) is strictly bigger than zero, then for any \( D_t \), the first eigenvalue is bounded away from zero. It implies
\[
\int_M (D_t R(t), R(t)) \, dg(t) \geq \delta \int_M (R(t) - \bar{R})^2 \, dg(t),
\]
for some constant \( \delta > 0 \).

**Case 2.** If the first eigenvalue of \( D \) is zero, by using the Futaki character, we can still prove the above estimate \([16]\). Notice that when the Kähler class admits a CscK metric, the Futaki character vanishes. It implies
\[
\int_M X F(t) \, dg(t) = 0,
\]
for any \( X \in \mathfrak{h}(M) \). Rewrite the Futaki character as the following,
\[
\int_M \theta_X (R(t) - \bar{R}) \, dg(t) = 0,
\]
where \( \theta_X \) is the potential of the vector field \( X \) with respect to the metric \( g(t) \).

The potential function \( \theta_X \) of a holomorphic vector field \( X \) is defined (up to addition of some constant) as the following:
\[
\int_M X(f) \, dg = - \int_M \theta_X \Delta f \, dg.
\]

The eigenvalue of \( D_t \) converges to the eigenvalue of \( D \), and the eigenspace \( \Upsilon_t \) of the first eigenvalue of \( D_t \) converges to the first eigenspace \( \Upsilon \) of \( D \) when \( \epsilon \) goes to zero. And if the first eigenvalue of \( D \) is zero, then the first eigenspace \( \Upsilon \) consists of functions which are the potential of some holomorphic vector fields (if \( f \in \Upsilon \), \( \nabla f \) is the real part of a holomorphic vector field). If we decompose \( \bar{R} - R \) into two components, then
\[
\bar{R} - R = \rho + \rho^\perp,
\]
where \( \rho \in \Upsilon_t \) and \( \rho^\perp \perp \Upsilon_t \). The vanishing of the Futaki character implies that
\[
\| \rho \|_{L^2(g(t))} \leq c(\epsilon) \| R(t) - \bar{R} \|_{L^2(g(t))},
\]
where \( c(\epsilon) \to 0 \) as \( \epsilon \to 0 \). If \( \epsilon \) is small enough, we can get
\[
\int_M (D_t R(t), R(t)) \, dg(t) \geq \lambda_1(t) \| \rho \|_{L^2(g(t))}^2 + \lambda_2(t) \| \rho^\perp \|_{L^2(g(t))}^2 \geq \frac{\lambda_2}{2} \| R(t) - \bar{R} \|_{L^2(g(t))}^2,
\]
15
where \( \lambda_i(t) \) is the i-th eigenvalue of \( D(t) \), and \( \lambda_i \) is the i-th eigenvalue of \( D \). By assumption we know \( \lambda_1 = 0, \lambda_2 > 0 \).

In both cases the Calabi energy exponentially decays:

\[
\int_M (R(t) - \bar{R})^2 \, dg(t) \leq \exp(-\delta t) \int_M (R(0) - \bar{R})^2 \, dg(0),
\]

where \( \delta \) is a positive constant bounded away from zero.

Now we are in the position to derive the long time existence and the convergence. By the smoothing property of the Calabi flow (Theorem 3.3), we can actually assume \( \varphi_0 \in C^\infty \) and \( |\varphi_0|_{C^{3,\alpha}} < \epsilon \). Define

\[
\mathcal{G} = \{ \varphi : \varphi \in C^\infty, \lambda g < \varphi < \Lambda g, |\varphi|_{C^{3,\alpha}} < K \}.
\]

If \( \epsilon > 0 \) is small, \( \varphi_0 \in \mathcal{G} \), by Theorem 3.2, we know the flow exists at least in \([0, T]\), where \( T \) depends on \( \mathcal{G} \). Moreover, \( |\varphi(t)|_{C^{3,\alpha}} \leq C|\varphi_0|_{C^{3,\alpha}} \) for any \( t \in [0, T] \), where \( C \) is a constant depending only on \( \mathcal{G} \). It implies in \([0, T]\), the Calabi energy decays exponentially fast because \( g(t) \) is close to \( g \) in \( C^{4,\alpha} \). And we know

\[
\varphi(t) = \varphi(0) + \int_0^t (R(s) - \bar{R}) \, ds,
\]

so we can get

\[
\int_M |\varphi(t)| \omega^n \leq \int_M |\varphi(0)| \omega^n + \int_M \left| \int_0^t (R(s) - \bar{R}) \, ds \right| \omega^n.
\]

Note that in \([0, T]\), all the metrics are close to CscK metric \( g \) in \( C^{4,\alpha} \), in particular all the metrics are “\( C^n \) equivalent” to \( g \). By Schwartz’s inequality, we can get

\[
\int_M \left| \int_0^t (R(s) - \bar{R}) \, ds \right| \omega^n \leq \int_0^t \int_M |R(s) - \bar{R}| \omega^n \, ds \leq C \int_0^t \int_M |R(s) - \bar{R}| \omega^n(s) \, ds \leq C \left( \int_0^t \left( \int_M (R(s) - \bar{R})^2 \omega^n(s) \right)^{1/2} \left( \int_M \omega^n(s) \right)^{1/2} \, ds \right)^{1/2} \leq C |R(0) - \bar{R}|_{L^2}.
\]

Since \( g(t) \) is close to \( g \) in \( C^{4,\alpha} \), we can assume \( |R(t) - \bar{R}| \leq 1 \), then

\[
|R_\varphi - \bar{R}|_{L^p} \leq C |R_\varphi - \bar{R}|_{L^2}^{2/p} \leq C |R(0) - \bar{R}|_{L^2}^{2/p}.
\]
By (4.9), we know \( \forall t \in [0, T] \)

\[
|\varphi(t)|_{C^{3,\alpha}} \leq C \left( |R_\varphi - R|_{L^p} + |\varphi(t)|_{L^1} \right)
\leq C \left( |R(0) - R|_{L^p}^{2/p} + |\varphi(0)|_{L^1} + |R(0) - R|_{L^2} \right),
\]

where \( C \) is a uniform constant independent of time \( t \). When \( \epsilon > 0 \) is small enough,

\[
|\varphi(T)|_{C^{3,\alpha}} \leq Cc(\epsilon),
\]

where \( c(\epsilon) \to 0 \) when \( \epsilon \to 0 \). Then we can extend the flow to \([T, 2T]\) starting from \( \varphi(T) \in \mathcal{G} \). And in \([0, 2T]\), all the metrics are close to CscK metric \( g \). By (4.9), we still have \( \forall t \in [0, 2T] \),

\[
|\varphi(t)|_{C^{3,\alpha}} \leq C \left( |R_\varphi - R|_{L^p} + |\varphi(t)|_{L^1} \right)
\leq C \left( |\varphi(0)|_{L^1} + |R(0) - R|_{L^p}^{2/p} + |R(0) - R|_{L^2} \right)
\leq Cc(\epsilon).
\]

Similarly we can extend the flow to \([2T, 3T]\) starting from \( \varphi(2T) \) with the uniform bounded \( C^{3,\alpha} \) norm, \( \forall t \in [0, 3T] \)

\[
|\varphi(t)|_{C^{3,\alpha}} \leq C \left( |\varphi(0)|_{L^1} + |R(0) - R|_{L^p}^{2/p} + |R(0) - R|_{L^2} \right)
\leq Cc(\epsilon).
\]

By iteration the flow exists for all time and actually, \( \varphi(t) \) is uniformly small in \( C^{3,\alpha} \). By smoothing property of the Calabi flow, indeed \( g(t) \) is uniformly close to \( g \) in \( C^\infty \) and the Calabi energy decays exponentially fast. In particular, \( \varphi(t) \) is uniformly bounded in \( C^\infty \) norm. And also we know that for any \( t, \bar{t} \in (0, \infty) \)

\[
|\varphi(t) - \varphi(\bar{t})|_{L^1} = \int_M |\varphi(t) - \varphi(\bar{t})| dg \\
= \int_M \int_{\bar{t}}^t \frac{\partial}{\partial s} \varphi(s) ds dg \\
\leq C \int_{\bar{t}}^t \int_M |R(\varphi(s)) - R| dg ds \\
\leq C \int_{\bar{t}}^t (|R(\varphi(s)) - R|_{L^2}) ds \\
\leq C|R(0) - R|_{L^2} \int_{\bar{t}}^t \exp(-\delta s/2) ds. \quad (4.10)
\]

So there exists a limit potential \( \varphi(\infty) \in C^\infty \), such that \( \varphi(t) \) converges to \( \varphi(\infty) \) by sequence in \( C^\infty \) and in \( L^1 \) in the flow sense. In particular, \( \varphi(\infty) \) defines a CscK metric \( g_\infty \) in the same class \( [\omega] \). Also we can get \textit{a priori} estimates for

\[
R(\varphi(t)) - R(\varphi(\infty)) = f.
\]
By the exact same argument, one can get

$$|\varphi(t) - \varphi(\infty)|_{C^{3,\alpha}(M)} \leq C (|f|_{L^p} + |\varphi(t) - \varphi(\infty)|_{L^1}).$$

(4.11)

By (4.10) and (4.11), the flow converges not only by sequence but exponentially fast along the flow.

**Remark 4.2.** The limit CscK metric $g_{\infty}$ could be different with $g$ in holomorphic coordinates. It means in general $\varphi(\infty) \neq 0$. To see this, we can pick up $\varphi(0) \neq 0$ such that $g_0$ is a CscK metric and is sufficient close to $g$ in holomorphic coordinates (but $g_0 \neq g$). But along the flow, $\varphi(t) \equiv \varphi(0) \neq 0$. For example, on Riemann sphere $S^2$, $(S^2, \lambda^2|dz|^2/(1 + \lambda^2|z|^2)^2)$ is a smooth continuous family of constant Gauss curvature metrics for $\lambda \in (0, \infty)$. From technical point of view, it explains also that the $L^1$ norm of the potential in (4.9) is necessary.

## 5 Long time existence

Essentially, to understand the whole picture of the Calabi flow, we need to understand some compactness behavior of $\omega_{\varphi} \in H_{\omega}$ with bounded Calabi energy since the Calabi flow is the gradient flow which decreases the Calabi energy. In this section, we derive a compactness theorem in $H_{\omega}$. As a consequence, the long time existence holds under the assumption of uniformly bounded Ricci curvature.

**Theorem 5.1.** (Compactness theorem) All metrics $\omega_{\varphi} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$ in the space of Kähler metrics $H_{\omega}$ with both the potential $\varphi$ and the Ricci curvature $Ric_{\omega}$ uniformly bounded are equivalent and compact in $C^{1,\alpha}$-topology for any $\alpha \in (0, 1)$ (It is equivalent to say $\varphi$ is uniformly bounded in $C^{3,\alpha}$ for any $\alpha \in (0, 1)$).

**Proof.** Denote

$$F = \log \left( \frac{\omega_{\varphi}}{\omega^n} \right).$$

We know

$$\triangle F = g^{i\bar{j}} \partial_i \partial_{\bar{j}} \log \left( \frac{\omega_{\varphi}}{\omega^n} \right) = -g^{i\bar{j}} R_{i\bar{j}}(\varphi) + R,$n

where $\triangle$ and scalar curvature $R$ are both with respect to the background metric $\omega$, and $R_{i\bar{j}}(\varphi)$ is the Ricci curvature of $\omega_{\varphi}$. Since $|Ric_{\omega}|$ is bounded, then

$$|\triangle F - R| \leq C(n + \triangle \varphi).$$

In other words,

$$\triangle (F - C \varphi) \leq C_1,$$

(5.1)
and
\[ \triangle (F + C\varphi) \geq -C_2 \] (5.2)
for some positive constant \( C_1, C_2 \). By (5.1) (5.2), we can deduce the \( C^0 \) bound of \( F \). For simplicity, set the volume of \( M \) to be 1,
\[ \int_M \omega^n = 1. \]
Let us recall the Green’s formula on a compact manifold,
\[(F + C\varphi)(p) = -\int_M G(p, q) \{\triangle (F + C\varphi)(q)\} \omega^n(q) + \int_M (F + C\varphi) \omega^n,\]
where \( G(p, q) \geq 0 \) is Green’s function of \( \omega \). The first term of right hand side is bounded from above since \( \triangle (F + C\varphi) \) is bounded from below. To bound the second term, we have
\[ \exp\left(\int_M F \omega^n\right) \leq \int_M \exp(F) \omega^n = 1 \]
provided that
\[ \int_M \omega^n = 1. \]
Because \( \varphi \) is uniformly bounded, we get that \( F \) is bounded from above. Before we derive the lower bound of \( F \), we try to show that \( \triangle \varphi \) is uniformly bounded following Yau’s celebrated work on Calabi conjecture. Consider the following Monge-Ampere equation
\[ \frac{\det(g_{ij} + \varphi_{ij})}{\det(g_{ij})} = \exp(F). \]
Just following Yau’s work [49], for some big constant \( C_3 \), we get at some point \( p \) (the maximum of \( \exp(-C_3\varphi)(n + \triangle \varphi)\)),
\[ \triangle \varphi \{\exp(-C_3\varphi)(n + \triangle \varphi)\}(p) \leq 0. \]
Follow Yau’s calculation, at the point \( p \),
\[ 0 \geq \triangle F - n^2 \inf_{i \neq l} R_{iil} - C_3 n(n + \triangle \varphi) + \left(C_3 + \inf_{i \neq l} R_{iil}\right) \exp\left\{\frac{-F}{n - 1}\right\} (n + \triangle \varphi)^{n/(n-1)}. \] (5.3)
By (5.2) (5.3), at the point \( p \) we have,
\[ 0 \geq -C \triangle \varphi - C_2 - n^2 \inf_{i \neq l} R_{iil} - C_3 n(n + \triangle \varphi) + \left(C_3 + \inf_{i \neq l} R_{iil}\right) \exp\left\{\frac{-F}{n - 1}\right\} (n + \triangle \varphi)^{n/(n-1)}. \] (5.4)
(5.4) implies that \((n + \triangle \varphi)(p)\) has an upper bound \(C_0\) depending only on \(\text{sup}_M F\) and \(M\). It follows that
\[
\exp(-C_3 \varphi)(n + \triangle \varphi) \leq \exp(-C_3 \varphi)(n + \triangle \varphi)(p) \leq C_0 \exp(-C_3 \varphi(p)).
\]
Because \(\varphi\) is uniformly bounded, we obtain
\[
0 < n + \triangle \varphi \leq C(C_3, \text{sup}_M F, M) . \tag{5.5}
\]
So \(\triangle \varphi\) is uniformly bounded. It follows from (5.1) (5.2) (5.5) that \(\triangle F\) is uniformly bounded.

The lower bound of \(F\) follows from the following result, which is essentially the proposition 3.14 in [17]:

**Proposition 5.2.** [17] If Ricci curvature is bounded from below and the Kähler potential is uniformly bounded. Then, there is a uniform constant \(C\) such that:
\[
\inf_M F \geq -4C \exp\left(2 + 2 \int_M \log \frac{\omega^n}{\omega^n} \right). \tag{5.6}
\]

**Proof.** Choose a constant \(c\) such that
\[
\int_M \log \frac{\omega^n}{\omega^n} \leq c.
\]
And remember we set
\[
\int_M \omega^n = 1.
\]
Put \(\epsilon\) to be \(\exp(-2 - 2c)\). Observe that
\[
\log \frac{\omega^n}{\omega^n} \geq -e^{-1} \omega^n,
\]
We have
\[
c \geq \left( \int_{\epsilon \omega^n > \omega^n} + \int_{\epsilon \omega^n \leq \omega^n} \right) \left( \log \frac{\omega^n}{\omega^n} \right) \geq \int_{\epsilon \omega^n > \omega^n} \left( \log \frac{1}{\epsilon} \right) \omega^n + \int_{\epsilon \omega^n \leq \omega^n} (-e^{-1} \omega^n) \geq 2(1 + c) \int_{\epsilon \omega^n > \omega^n} \omega^n - 1.
\]
It follows that
\[
\int_{\epsilon \omega^n > \omega^n} \omega^n < \frac{1}{2}.
\]
And so
\[
\int_{\omega_{\varphi} \leq \omega} \omega_{\varphi} > \frac{1}{2}.
\]  
(5.6)

Also we have
\[
\int_{\omega \leq 4 \omega_{\varphi}} \omega_{\varphi} = 1 - \int_{\omega > 4 \omega_{\varphi}} \omega_{\varphi} > 1 - \int_{\omega > 4 \omega_{\varphi}} \frac{\omega}{4} \geq 1 - \frac{1}{4} \int_M \omega = \frac{3}{4}.
\]  
(5.7)

And we know
\[
\int_{\omega \leq 4 \omega_{\varphi}} \omega_{\varphi} \geq \int_{\epsilon \omega \leq \epsilon \omega_{\varphi} \leq \omega_{\varphi}} \omega_{\varphi} \geq \epsilon \int_{\epsilon \omega \leq \epsilon \omega_{\varphi} \leq \omega_{\varphi}} \omega_{\varphi}.
\]  
(5.8)

Set \( A = \{x \in M : \epsilon \omega_{\varphi}(x) \leq \omega(x)\} \) and \( B = \{x \in M : \omega(x) \leq 4 \omega_{\varphi}(x)\} \). By (5.6), (5.7),
\[
\int_{\epsilon \omega \leq \epsilon \omega_{\varphi} \leq \omega_{\varphi}} \omega_{\varphi} = \int_{A \cap B} \omega_{\varphi} = \int_{A} \omega_{\varphi} + \int_{B} \omega_{\varphi} - \int_{A \cup B} \omega_{\varphi} > \frac{1}{2} \cdot \frac{3}{4} - 1 = \frac{1}{4}.
\]  
(5.9)

Combine (5.8) and (5.9), we have
\[
\int_{\omega \leq 4 \omega_{\varphi}} \omega_{\varphi} > \epsilon/4.
\]  
(5.10)

By Green's formula, we have
\[
F(p) = - \int_M G(p, q) \Delta F(q) \omega^n(q) + \int_M F \omega^n,
\]  
(5.11)
where \( G(p, q) \geq 0 \) is a Green function of \( \omega \). By (5.10), (5.11) and the uniform bound of \( \Delta F \),
\[
\inf_M F \geq \int_M F \omega^n - C
\]
\[
\geq \inf_M F \int_{\omega \geq 4 \omega_{\varphi}} \omega^n + \int_{\omega < 4 \omega_{\varphi}} F \omega^n - C
\]
\[
\geq \inf_M F \int_{\omega \geq 4 \omega_{\varphi}} \omega^n - \log 4 \int_{\omega < 4 \omega_{\varphi}} \omega^n - C
\]
\[
\geq \left(1 - \frac{\epsilon}{4}\right) \inf_M F - C,
\]
where we can assume \( \inf_M F < 0 \). Therefore, we have
\[
\inf_M F \geq -4C \exp(2 + 2c).
\]

By the way we choose the constant \( c \) in the beginning of the proof, the proposition is proved. \( \square \)
So we get a uniform bound for $F$, $\triangle F$ and $\triangle \varphi$. It implies that all the metrics $g_\varphi$ are equivalent. For the Monge-Ampere equation

$$\frac{\det \left( g_{ij} + \varphi_{ij} \right)}{\det \left( g_{ij} \right)} = \exp(F).$$

The $C^2$ bound of $F$ implies that $\varphi \in W^{4,p}(M)$ is uniformly bounded for any $p > 1$. By the Sobolev’s embedding theorem, for any $\alpha \in (0, 1)$, $\varphi \in C^{3,\alpha}$ is uniformly bounded.

**Corollary 5.3.** For the Calabi flow initiating from any smooth Kähler metric, the flow exists as long as the Ricci curvature stays uniformly bounded.

**Proof.** By the short time existence of the Calabi flow, we can assume the maximal existence interval of the flow is $[0, T)$. If $T < \infty$, because $\text{Ric}_{\varphi(t)}(t)$ is uniformly bounded, so is the scalar curvature. Then by

$$\frac{\partial \varphi}{\partial t} = R_\varphi - \overline{R},$$

we know $\varphi$ is uniformly bounded. By Theorem 5.1, we know $\varphi(t)$ is uniformly bounded in $C^{3,\alpha}$ for any $\alpha \in (0, 1)$ and all metrics along the flow are equivalent. By Theorem 3.2, the flow can be extended. Contradiction.

**Corollary 5.4.** The maximal existence interval of the Calabi flow is $[0, T)$, starting with any initial smooth potential $\varphi(0)$. If the Ricci curvature and the potential are both uniformly bounded in $[0, T)$, then $T = \infty$ and the flow converges to some extremal metric in the same Kähler class. Moreover, when the limit metric is a CscK metric, the convergence is exponentially fast.

**Proof.** Obviously $T = \infty$. To get convergence, first we show the flow converges by sequence. Because Ricci curvature $R_{ij}(t)$ and the potential $\varphi(t)$ are both uniformly bounded, all metrics $g(t)$ are equivalent and $\varphi(t)$ is uniformly bounded in $C^{3,\alpha}(M)$ for any $t \in [0, \infty)$ and any $0 < \alpha < 1$. Then by the smoothing property of the Calabi flow (Theorem 3.3), we can get actually $\varphi(t)$ is uniformly bounded in $C^{k,\alpha}(M)$ for any $k \in \mathbb{N}$. It follows that the flow converges by sequence in $C^\infty$. To get the limit metric is extremal metric, recall that the Calabi energy is decreasing along the flow,

$$\frac{d}{dt} \int_M \left( R(t) - \overline{R} \right)^2 dg(t) = -2 \int_M R(t)_{\alpha\beta} R(t)^{-\alpha\beta} dg(t).$$

Denote

$$E_0 = \inf_{[0, \infty)} \int_M \left( R(t) - \overline{R} \right)^2 dg(t).$$

For any sequence $t_n \to \infty$, we have a subsequence $t_{n_k}$, such that $\varphi(t_{n_k})$ converges to $\varphi_\infty$ in $C^\infty$ topology. Then we have

$$E_0 = \int_M \left( R_\infty - \overline{R} \right)^2 dg_\infty.$$

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Moreover,

\[
E_{t_n} - E_0 = \lim_{k \to \infty} \left( \int_M (R(t_n) - R)^2 dg(t_n) - \int_M (R(t_{n_k}) - R)^2 dg(t_{n_k}) \right)
\]

\[
= \lim_{k \to \infty} \int_{t_n}^{t_{n_k}} \frac{d}{dt} \int_M (R(t) - R)^2 dg(t) dt
\]

\[
= -2 \lim_{k \to \infty} \int_{t_n}^{t_{n_k}} \int_M R(t)_{,\alpha\beta} R(t)^{,\alpha\beta} dg(t) dt
\]

\[
= -2 \int_{t_n}^{\infty} \int_M R(t)_{,\alpha\beta} R(t)^{,\alpha\beta} dg(t) dt.
\]

When \(t_n \to \infty\), \(E_{t_n} \to E_0\), it implies \(\int_M R(t)_{,\alpha\beta} R(t)^{,\alpha\beta} dg(t) \to 0\) when \(t \to 0\).

In particular,

\[
\int_M R_{\infty,\alpha\beta} R_{\infty}^{,\alpha\beta} dg_{\infty} = 0.
\]

So \(R_{\infty,\alpha\beta} = 0\), \(g_{\infty}\) is extremal metric. When \(R_{\infty} = R\), by similar argument as in Theorem 4.1, the convergence is exponentially fast.

6 Removing singularity

In this section we consider a Kähler metric which is defined locally in the punctured disc \(D \setminus \{0\}\), where \(D = \{z \in \mathbb{C}^n : |z| \leq 1\}\). If the metric admits constant scalar curvature in some weak sense, we show the metric can be extended to the whole disc smoothly.

**Definition 6.1.** A Kähler metric \(g_\varphi = (\delta_{ij} + \partial_i \bar{\partial}_j \varphi) dz^i \otimes d\bar{z}^j\) is defined in the punctured disc \(D \setminus \{0\}\), if \(\varphi \in C^2(D \setminus \{0\})\) is a real valued function such that \(\{\delta_{ij} + \partial_i \bar{\partial}_j \varphi\}\) is a positive Hermitian matrix function. The scalar curvature \(R_\varphi\) of \(g_\varphi\) is constant in weak sense in the punctured disc if \(\varphi \in C^2(D \setminus \{0\}) \cap W^{3,2}_{\text{loc}}(D \setminus \{0\})\) satisfies

\[-g_\varphi^{ij} \partial_i \partial_j \log \det(\delta_{kl} + \partial_k \bar{\partial}_l \varphi) = R, \quad \text{in} \quad D \setminus \{0\}\]

in weak sense.

**Theorem 6.2.** If \(g_\varphi\) is a Kähler metric defined in the punctured disc with constant scalar curvature in weak sense in the punctured disc, and \(\partial_i \partial_j \varphi \in L^\infty(D)\), \(L = g_\varphi^{ij} \partial_i \partial_j\) is uniformly elliptic on \(D\), moreover, to avoid regularity issues on the boundary, we assume \(\varphi\) is smooth in the neighborhood of the boundary \(\partial D\), then \(g_\varphi\) can be extended to a smooth metric with constant scalar curvature in the whole disc \(D\).

We need to prove a theorem first:
Theorem 6.3. Let \((a^i_j)\) be a uniformly elliptic matrix in \(D\). The following Dirichlet problem can be solved in \(W^{1,2}(D)\):

\[
-\partial_i \left(a^i_j \det(a_{kl}) \partial_j u\right) = R \det(a_{ij}) \quad \text{in } D,
\]

\[u|_{\partial D} = h.\]

Here \((a^i_j) \cdot (a_{kl}) = I_{n \times n}\).

Proof. To solve the Dirichlet equation, define the following functional:

\[
I(u) = \int_D \left(a^i_j \partial_i u \partial_j u + R u\right) \det(a_{ij}) dx,
\]

where \(u \in A = \{u|u \in W^{1,2}(D), u|_{\partial D} = h\}\), and \(dx\) is the Euclidean measure. Here \(u|_{\partial D} = h\) is in trace sense [28].

Claim 6.4. \(I(u)\) is bounded from below.

Proof. \(\forall u \in A\)

\[
I(u) = \int_D \left(a^i_j \partial_i u \partial_j u + R u\right) \det(a_{ij}) dx
\]

\[
\geq \frac{1}{C} \int_D |Du|^2 dx - C \int_D |u| dx
\]

\[
\geq \frac{1}{C} \int_D |Du|^2 dx - C \int_D u^2 dx - \frac{C}{\epsilon},
\]

where \(D\) is the derivative under the Euclidean metric. Let \(\omega \in A\), we know

\[
\int_D |Du|^2 dx \geq \int_D |Du - Dw|^2 dx - \int_D |Dw|^2 dx,
\]

\[
\int_D u^2 dx \leq 2 \int_D (u - w)^2 dx + 2 \int_D w^2 dx.
\]

So we have

\[
I(u) \geq \frac{1}{C} \int_D \left(|D(u - w)|^2 - |Dw|^2\right) dx - 2C \int_D (u - w)^2 dx - 2C \int_D w^2 dx - \frac{C}{\epsilon}
\]

\[
\geq \frac{1}{C} \int_D |D(u - w)|^2 dx - 2C \int_D (u - w)^2 dx - C(\epsilon, w)
\]

\[
\geq \frac{1}{2C} \int_D |D(u - w)|^2 dx - C(\epsilon, w)
\]

\[
\geq -C(\epsilon, w).
\]

The above inequality follows from Poincaré inequality because \(u - w \in W_0^{1,2}(D)\) if \(\epsilon > 0\) small enough. Denote \(m = \inf_{u \in A} I(u)\). Taking a minimizing sequence \(\{u_k\}\) of \(I(u)\), we have

\[
\frac{1}{2C} \int_D |D(u_k - w)|^2 dx \leq I(u_k) + C(\epsilon, w),
\]

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for some fixed \( w \in A \). It is easy to show that \( u_k \) is uniformly bounded in \( W^{1,2}(D) \). So we can get a subsequence of \( \{u_k\} \) converge to \( u_0 \) weakly in \( W^{1,2}(D) \), and strongly in \( L^2(D) \).

**Claim 6.5.** \( u_0 \) is the minimizer of \( I(u) \) and \( u_k \) converge to \( u_0 \) strongly in \( W^{1,2}(D) \).

**Proof.** We know

\[
\int_D a^{ij} \partial_i (u_k - u_0) \partial_j (u_k - u_0) \det (a_{ij}) dx \geq 0.
\]

It implies

\[
\lim_{k} \int_D a^{ij} \partial_i (u_k - u_0) \partial_j (u_k - u_0) \det (a_{ij}) dx \geq \int_D a^{ij} \partial_i (u_0) \partial_j (u_0) \det (a_{ij}) dx,
\]

because \( u_k \) converges to \( u_0 \) weakly in \( W^{1,2}(D) \). Then we have

\[
m = \lim_{k} I(u_k) = \lim_{k} \int_D \left( a^{ij} \partial_i \partial_j u_k + Ru_k \right) \det (a_{ij}) dx \\
\geq \int_D \left( a^{ij} \partial_i \partial_j u_0 + Ru_0 \right) \det (a_{ij}) dx = I(u_0).
\]

\( m \) is the minimum of \( I(u) \), so \( u_0 \) is the minimizer. It easily follows that \( u_k \) converge to \( u_0 \) strongly.

The minimizer \( u_0 \) of \( I(u) \) satisfies the following equation in weak sense:

\[
-\partial_i \left( a^{ij} \partial_j u \right) = R \det (a_{ij}) \text{ in } D.
\]

Theorem 6.3 follows.

Now we are in the position to prove Theorem 6.2.

**Proof.** Taking \( a_{ij} = \delta_{ij} + \partial_i \partial_j \varphi \), \( h = \log \det (\delta_{ij} + \partial_i \partial_j \varphi)|_{\partial D} \), Theorem 6.3 says there exists a weak solution \( u_0 \) of the following equation,

\[
-\partial_i \left( g^{ij}_\varphi \det (\delta_{kl} + \partial_k \partial_l \varphi) \partial_j u_0 \right) = R \det (\delta_{ij} + \partial_i \partial_j \varphi) \text{ in } D,
\]

\[
u_0|_{\partial D} = h.
\]

Since \( \varphi \in C^2(D \setminus \{0\}) \cap W^{3,2}_{loc}(D \setminus \{0\}) \), by straightforward calculation, \( u_0 \) satisfies the following equation in weak sense:

\[
-g^{ij}_\varphi \partial_i \partial_j u_0 = R \text{ in } D.
\]

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Denote \( v = \log \det (\delta_{ij} + \partial_i \partial_j \varphi) - u_0 \), where \( v \) is the weak solution of
\[
-g^{ij} \partial_i \partial_j v = 0, \quad \text{in } D \setminus \{0\},
\]
\[
v|_{\partial D} = 0.
\]
If we can show \( v \) is identically zero, then \( \varphi \) will satisfy the following equation
\[
-g^{ij} \partial_i \partial_j \log \det (\delta_{kl} + \partial_k \partial_l \varphi) = R \tag{6.1}
\]
in the whole disc. It implies that we can extend the metric \( g_\varphi \) to the whole disc with constant scalar curvature in weak sense.

To show \( v \) is actually 0, we need to use maximum principle for elliptic equation with an isolated singularity [31]. In general, we need some further assumption for the elliptic operator around the singularity (cf. [31]). In this complex setting, it is nice that we don’t need any assumption other than the uniform ellipticity.

Let \( f = |z|^p \), then
\[
f_i = p |z|^i |z|^{p-1} = \frac{p}{2} \xi_i |z|^{p-2}.
\]
and
\[
f_{ij} = \frac{p}{2} |z|^{p-2} \delta_{ij} + \frac{p(p-2)}{4} \xi_i \xi_j |z|^{p-4}.
\]
Set
\[
\psi = vf, \quad \text{or } v = \psi |z|^{-p} = \psi |z|^q, \quad q = -p < 0.
\]
Then,
\[
v_i = \psi_i |z|^q + \psi \frac{q}{2} \xi_i |z|^{q-2}
\]
and
\[
v_{ij} = \psi_{ij} |z|^q + \psi_j \frac{q}{2} \xi_i |z|^{q-2} + \psi_i \frac{q}{2} \xi_j |z|^{q-2} + \psi \left( \frac{q}{2} |z|^{q-2} \delta_{ij} + \frac{q(q-2)}{4} \xi_i \xi_j |z|^{q-4} \right).
\]
Claim that \( \psi \leq 0 \). Otherwise, there is at least one interior point \( |z| \neq 0 \) such that
\[
(\psi_{ij}) \leq 0, \quad \text{and } \psi > 0.
\]
Note that \( g^{ij} v_{ij} = 0 \). Thus, we have (at maximum point)
\[
0 \leq g^{ij} \psi \left( \frac{q}{2} |z|^q \delta_{ij} + \frac{q(q-2)}{4} \xi_i \xi_j |z|^{q-4} \right).
\]
Since \( q < 0 \) and \( |z| \neq 0 \), this means that \( (q < 0) \)
\[
|z|^2 \sum_{i=1}^{n} g^{ii} + \frac{q(q-2)}{2} g^{ij} \xi_i \xi_j \leq 0.
\]
This is a contradiction when \( -q > 0 \) is very small. Similarly, we can show that \( \psi \geq 0 \). Consequently, \( \psi = 0 \).
Claim 6.6. $g_\varphi$ is a smooth metric with constant scalar curvature in the entire disc.

Proof. By (6.1), $\varphi$ is the weak solution of

$$-g_\varphi^{ij}\partial_i\partial_j \log \det(\delta_{kl} + \partial_k\partial_l \varphi) = R.$$ 

Since $L = g_\varphi^{ij}\partial_i\partial_j$ is uniformly elliptic, the De Giorgi-Nash-Moser’s estimate [32] gives that

$$u = \log \det(\delta_{\bar{i}\bar{j}} + \varphi_{\bar{i}\bar{j}})$$

is in $C^\alpha(D)$. Also by the Evans-Krylov estimate [29] for Monge-Ampere equation ($L = g_\varphi^{ij}\partial_i\partial_j$ is uniformly elliptic), we can deduce from (6.2) that $\varphi \in C^{2,\alpha}(D)$. Go back to (6.1),

$$-g_\varphi^{ij}\partial_i\partial_j \log \det(\delta_{kl} + \partial_k\partial_l \varphi) = R.$$ 

Now $g_\varphi \in C^\alpha(D)$, and so the elliptic theory gives that $u = \log \det(\delta_{\bar{i}\bar{j}} + \varphi_{\bar{i}\bar{j}}) \in C^{2,\alpha}(\bar{D})$. The standard Monge-Ampere theory implies that $\varphi \in C^{4,\alpha}$. Using the standard boot-strapping argument, we can get that $\varphi \in C^\infty(D)$. To avoid the issue of boundary regularity, we assume in addition the $\varphi$ is smooth in the neighborhood of the boundary $\partial D$. So $\varphi \in C^\infty(\bar{D})$. \qed

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