A Generalized Two-Step Modulus-Based Matrix Splitting Iteration Method for Implicit Complementarity Problems of $H_+ $-Matrices

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Abstract. In this paper, a generalized two-step modulus-based matrix splitting iteration method for solving the implicit complementarity problems has been presented. The convergent analysis with the system matrix being $H_+ $-matrices are also discussed. Numerical experiments illustrate that our method is advantageous to the existing methods.

1. Introduction

The complementarity problem is a hot topic, which has several kinds of forms, such as linear complementarity problem [1, 24–41], nonlinear complementarity problem [2–5, 12], implicit complementarity problem [13–15], cone complementarity problem [16–18], etc.. The implicit complementarity problem is a special case in complementarity theory. It is frequently applied to stochastic optimal control problems[22, 23]. Therefore, it is significant to study this problem.

The implicit complementarity problem (ICP) is to find a pair $(u, w)$ of real vectors which satisfy

$$ u - m(u) \geq 0, \quad w := Au + q \geq 0, \quad (u - m(u))^T w = 0, $$

where $A$ and $q$ are a known matrix in $\mathbb{R}^{n \times n}$ and a known vector in $\mathbb{R}^n$, respectively. The mapping $m(\cdot): \mathbb{R}^n \to \mathbb{R}^n$ is known invertible, and $u - m(u)$ is supposed to be a invertible mapping.

When $m(u) = 0$, the implicit complementarity problem reduces to the linear complementarity problem [24, 25], abbreviated as LCP($A, q$). There are numerous existing methods to solve the LCP; see, for example, the projected iteration methods [6], the chaotic iteration methods [7], and the modulus-based multigrid methods [8]. In 2010, Bai first provided the modulus-based matrix splitting (MMS) iteration method [26] to solve this kind of problems. This method has attracted much attention, and most scholars modified the MMS iteration method by constructing different preconditioners [27–30] or generalized it by putting another parameterized diagonal matrix to the modulus equation of LCP [31–33] and so on [34–36]. And some researchers also give the synchronous multisplitting iteration methods [19, 20, 37–43]. For other iteration methods such as the matrix multisplitting iteration methods, parallel chaotic multisplitting
iteration methods, and the damped Newton methods for solving the LCP, we refer to [1, 9–11, 13, 21, 25, 44] and the references therein.

When \( m(u) \) is a general function, it is called as the implicit complementarity problem. In general, many studied ICP by variational or quasi-variational inequalities [45, 46]. Recently, the modulus-based matrix splitting has been widely used to derive numerical methods for the linear complementarity problems. For instance, Hong et al. raised the modulus-based matrix splitting iteration method [47] and Wang et al. consummated its details [48]. In order to improve the convergence rate, Cao et al. proposed the two-step modulus-based matrix splitting iteration method [49].

In this paper, in view of putting another parameterized diagonal matrix to the modulus equation and splitting the system matrix into two kinds of matrix splittings, we propose a general two-step modulus-based matrix splitting iteration method [49], to solve the ICP and its variants with different matrix splittings. We discuss its convergent theorems whose system matrix is an \( H_+ \)-matrix. Finally, numerical experiments are exhibited to indicate the efficiency of our presented method.

The paper is organized as follows. In Sec. 2, we give some symbolic representations and essential lemmas. The generalized two-step modulus-based matrix splitting method, which is the generalization of the two-step modulus matrix splitting iteration method [49], is given in Sec. 3 and Sec. 4, respectively. Numerical experiments illustrate that our proposed method is advantageous to some existing methods.

2. Preliminaries

In this section, we review some fundamental notations and indispensable lemmas.

For any two vectors \( u = (u_1, u_2, \cdots, u_n)^T \) and \( v = (v_1, v_2, \cdots, v_n)^T \), we denote \( u \geq v \) \((u > v)\) provided that the corresponding elements satisfy \( u_i \geq v_i \) \((u_i > v_i)\). \( \max(a, b) \) is the bigger one of \( a \) and \( b \). \( |a| \) means the absolute value of the vector \( u \), and \( u^T \) is its transpose. The notations of matrix is the same as the aforementioned. Denote \( A = (a_{ij}) \), then we review some special matrices as below:

- The matrix \( A \) is a \( Z \)-matrix iff \( a_{ij} \leq 0 \) for any \( i \neq j \).
- The \( Z \)-matrix \( A \) is an \( M \)-matrix iff \( A^{-1} \geq 0 \).
- The matrix \( A \) is an \( H \)-matrix iff its comparison matrix \( \langle A \rangle = (a_{ij}) \) is an \( M \)-matrix, where

\[
\langle a_{ij} \rangle = \begin{cases} |a_{ij}|, & \text{for } i = j, \\ -|a_{ij}|, & \text{for } i \neq j. \end{cases} i, j = 1, 2, \cdots , n.
\]

- The matrix \( A \) is an \( H_+ \)-matrix iff \( A \) is an \( H \)-matrix with the diagonal elements being positive; see [1].

Furthermore, \( A = E - F \) is a splitting of the matrix \( A \) iff \( E \) is nonsingular, and it is an \( H \)-compatible splitting iff it holds that \( \langle A \rangle = \langle E \rangle - |F| \). In the following sections, \( \alpha \) and \( \beta \) are parameters. \( D, -L \) and \( U \) are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix \( A \), respectively. Finally, we list six lemmas which are going to be used in subsequent convergent theorems.

Lemma 2.1. [50] Let \( A \) be an \( H \)-matrix, then \( |A^{-1}| \leq \langle A \rangle^{-1} \).

Lemma 2.2. [51] Let \( B \in \mathbb{R}^{n \times n} \) be a strictly diagonal dominant matrix. Then

\[
\|B^{-1}C\| \leq \max_{i \neq j} \frac{(|C|e)_i}{(|B|e)_i},
\]

holds for arbitrary matrix \( C \in \mathbb{R}^{n \times n} \), where \( e = (1, 1, \cdots , 1)^T \).

Lemma 2.3. [52] Let \( A \in \mathbb{R}^{n \times n} \), then \( \rho(A) < 1 \) iff \( \lim_{n \to \infty} A^n = 0 \).

Lemma 2.4. [53] Let \( A \in \mathbb{R}^{n \times n} \) be an \( M \)-matrix and \( B \in \mathbb{R}^{n \times n} \) be a \( Z \)-matrix. If \( A \leq B \), then \( B \) is an \( M \)-matrix.
Lemma 2.5. [54] Let $A$ be a $Z$-matrix, then the following statements are equivalent:
(i) $A$ is an $M$-matrix;
(ii) There exists a positive vector $x$, such that $Ax > 0$;
(iii) Let $A = M - N$ be a splitting of $A$ and $M^{-1} \geq 0$, $N \geq 0$, then $p(M^{-1}N) < 1$.

Lemma 2.6. [47] Let $A = E - F$ be a splitting of the matrix $A \in \mathbb{R}^{n \times n}$, $\gamma$ be a positive constant, $\Omega$ be a positive diagonal matrix and $\rho(u) = u - m(u)$ be an invertible mapping. For the ICP (1), the following statements hold true:
(i) if $(u, w)$ is a solution of the ICP(1), then $x = \frac{1}{\gamma}(u - \Omega^{-1}w - m(u))$ satisfies the implicit fixed-point equation

$$
(\Omega + E)x = Fx + (\Omega - A)|x| - \gamma Am[g^{-1}(\frac{1}{\gamma}(|x| + x))] - \gamma q. \tag{2}
$$

(ii) if $x$ satisfies the implicit fixed-pointed equation (2), then

$$
\begin{align*}
& u = \frac{1}{\gamma}(|x| + x) + m(u) \quad \text{and} \quad w = \frac{1}{\gamma}\Omega(|x| - x) \\
& \text{is a solution of the ICP(1).}
\end{align*}
$$

3. The general two-step modulus-based matrix splitting

Based on (2) and (3), Hong and Li provided the MMS iteration method [47]. However, they did not point out the inner-outer iteration, and some parameters of their method were indefinite. Then, in [48], they got rid of these advantages and presented the more complete version as below.

Method 3.1. [48] (The MMS iteration method for the ICP(1))

**Step 1.** Given $\varepsilon > 0$, $u_0 \in \mathbb{R}$, set $k := 0$;

**Step 2.** Find the solution $u^{(k+1)}$:

1. Compute the initial vector

$$
x^{(0)}(k) = \frac{u^{(k)} + q_j}{2}(u^{(k)} - m(u^{(k)}) - \Omega^{-1}w^{(k)}).
$$

Set $j := 0$.

2. Iteratively calculate $x^{(k+1)}$ by solving the equations

$$
(\Omega + E)x^{(k+1)} = Fx^{(k)} + (\Omega - A)|x^{(k)}| + \gamma Am(u^{(k)}) - \gamma q. \tag{4}
$$

3. Compute

$$
u^{(k+1)} = \frac{1}{\gamma}(\|x^{(k+1)}\| + x^{(k+1)}) + m(u^{(k)}). \tag{5}
$$

**Step 3.** If $RES = |(A\nu^{(k+1)} + q)^T(u^{(k+1)} - m(u^{(k+1}))| < \varepsilon$, then terminate. Otherwise, set $k := k + 1$ and return to Step 2.

Remark 3.2. We can utilize some special iterative schemes by choosing diverse matrix splittings to this method. For instance, set

$$
E = \frac{1}{\alpha}(D - \beta L), \quad F = \frac{1}{\alpha}((1 - \alpha)D + (\alpha - \beta)L + \alpha U).
$$

Based on the aforementioned matrix splitting, we can obtain the modulus-based accelerated over-relaxation (MAOR) iteration method. With different values of the parameters, we can get the other methods such as the modulus-based successive over-relaxation (MSOR) iteration method ($\alpha = \beta$), the modulus-based Gauss-Seidel (MGS) iteration method ($\alpha = \beta = 1$) and the modulus-based Jacobi (MJ) iteration method ($\alpha = 1, \beta = 0$) [48].
In [49], Cao and Wang raised the TMMS iteration method. Define a set:

$$Z = \{z | Az + q \geq 0, z - m(z) \geq 0\}.$$  

They need two splittings of the matrix $A$, i.e., $A = E_1 - F_1 = E_2 - F_2$. We show this method as the following is more standard:

**Method 3.3.** [49] (The TMMS iteration method for the ICP(1))

**Step 1.** Given $\epsilon > 0$, $u_0 \in Z$, set $k := 0$;

**Step 2.** Find the solution $u^{(k+1)}$:

1. Compute the initial vector

   $$u^{(0)} = \frac{q}{2}(u^{(0)} - m(u^{(0)}) - \Omega^{-1}u^{(0)}).$$

   Set $j := 0$.

2. Iteratively calculate $x^{(k+1)} \in \mathbb{R}^n$ by solving the equations

   $$\begin{align*}
   (\Omega + E_1)x^{(k+1)} &= F_1x^{(k)} + (\Omega - A)x^{(k)} + \gamma Am(u^{(k)}) - \gamma q, \\
   (\Omega + E_2)x^{(k+1)} &= F_2x^{(k+1)} + (\Omega - A)x^{(k)} + \gamma Am(u^{(k)}) - \gamma q.
   \end{align*}$$

   (6)

3. Compute

   $$u^{(k+1)} = \frac{1}{\gamma}(x^{(k+1)} + x^{(k+1)}) + m(u^{(k)}).$$

   (7)

**Step 3.** If $RES = |(Au^{(k+1)} + q)^T(u^{(k+1)} - m(u^{(k+1))))| < \epsilon$, then terminate. Otherwise, set $k := k + 1$ and return to Step 2.

**Remark 3.4.** We can utilize some special iterative schemes by choosing diverse matrix splittings to this method. For instance, set

$$E_1 = \frac{1}{\alpha}(D - \beta L),\; F_1 = \frac{1}{\alpha}(1 - \alpha)D + (\alpha - \beta)L + \alpha L),$$
$$E_2 = \frac{1}{\alpha}(D - \beta L),\; F_2 = \frac{1}{\alpha}(1 - \alpha)D + (\alpha - \beta)L + \alpha L).$$

Based on the aforementioned matrix splitting, we can obtain the two-step modulus-based accelerated over-relaxation (TMAOR) iteration method. With different values of the parameters, we can get the other methods such as the two-step modulus-based successive over-relaxation (TMSOR) iteration method ($\alpha = \beta$), the two-step modulus-based Gauss-Seidel (TMGS) iteration method ($\alpha = \beta = 1$) and the two-step modulus-based Jacobi (TMJ) iteration method ($\alpha = 1, \beta = 0$) [49].

In order to improve computing efficiency, on account of the general modulus-based matrix splitting iteration method [31], we get the the general two-step modulus-based matrix splitting (GTMMS) iteration method for solving the ICP(1) as below. There is a slight difference. We need two matrix splittings of the matrix $A\Omega_1$, instead of the matrix $A$. Hence, let $A\Omega_1 = E_{\Omega_1} - F_{\Omega_1} = E'_{\Omega_1} - F'_{\Omega_1}$.

**Method 3.5.** (The GTMMS iteration method for the ICP(1))

**Step 1.** Given $\epsilon > 0$, $u_0 \in Z$, set $k := 0$;

**Step 2.** Find the solution $u^{(k+1)}$:

1. Compute the initial vector

   $$u^{(0)} = \frac{q}{2}(u^{(0)} - m(u^{(0)}) - \Omega^{-1}u^{(0)}).$$

   Set $j := 0$.

2. Iteratively calculate $x^{(k+1)} \in \mathbb{R}^n$ by solving the equations

   $$\begin{align*}
   (E_1 + \Omega_1)x^{(k+1)} &= F_1x^{(k)} + (E_1 + \Omega_1)x^{(k+1)} + \gamma Am(u^{(k)}) - \gamma q, \\
   (E_2 + \Omega_1)x^{(k+1)} &= F_2x^{(k+1)} + (E_2 + \Omega_1)x^{(k+1)} + \gamma Am(u^{(k)}) - \gamma q.
   \end{align*}$$

   (6)

3. Compute

   $$u^{(k+1)} = \frac{1}{\gamma}(x^{(k+1)} + x^{(k+1)}) + m(u^{(k)}).$$

   (7)
Step 3. If \( \text{RES} = \| (A u^{(k+1)} + q) - m(u^{(k+1)}) \| < \varepsilon \), then terminate. Otherwise, set \( k := k + 1 \) and return to Step 2.

Remark 3.6. Similar to the TMMS iteration method, we can also acquire analogous methods by appropriate choices of the parameter and the matrix splitting. Set

\[
\begin{align*}
\Omega_1 &= E_{\Omega_1} - F_{\Omega_1} = E_{\Omega_1} - F_{\Omega_1}, \\
\Omega_2 &= \Omega_1 + \Omega_2 = \Omega_1 + \Omega_2, \\
E_{\Omega_1} &= \frac{1}{a}(L - \beta U), \\
F_{\Omega_1} &= \frac{1}{a}(1 - \alpha) \hat{D} + (\alpha - \beta) \hat{L} + a \hat{U},
\end{align*}
\]

where \( \hat{D} \), \( \hat{L} \) and \( \hat{U} \) are the diagonal, the strictly lower-triangular and the strictly upper-triangular matrices of the matrix \( \Lambda \Omega_1 \), respectively. Based on the aforementioned matrix splitting, we can obtain the general two-step modulus-based accelerated over-relaxation (GTMAOR) iteration method. With different values of the parameters, we can get the other methods such as the general two-step modulus-based successive over-relaxation (GTMSOR) iteration method (\( \alpha = \beta \)), the general two-step modulus-based Gauss-Seidel (GTMGS) iteration method (\( \alpha = \beta = 1 \)) and the general two-step modulus-based Jacobi (GTMJ) iteration method (\( \alpha = 1, \beta = 0 \)).

4. Convergence theorems

In this section, we are going to discuss convergence properties of Method 3.5 when the system matrix \( A \) of the ICP(1) is an \( H^* \)-matrix.

Theorem 4.1. Let \( A \) be an \( H^* \)-matrix in \( \mathbb{R}^{m \times m} \), and \( \Omega_1 \) and \( \Omega_2 \) be known positive diagonal matrices. \( \Lambda \Omega_1 = E_{\Omega_1} - F_{\Omega_1} \) are \( H \)-compatible splittings. \( m(\cdot) \) is a Lipschitz continuous function, i.e., it holds that

\[
\| m(a) - m(b) \| \leq |a - b| \quad \forall a, b \in \mathbb{R}^n,
\]

wherein \( l \) is the Lipschitz constant. Set \( \zeta_1 = (\Omega_2 + (E_{\Omega_1})^{-1}(\Omega_2 - \Lambda \Omega_1) + |F_{\Omega_1}|) \), \( \zeta_2 = |(\Omega_2 + E_{\Omega_1})^{-1}A| \), \( \zeta_3 = |(\Omega_2 + (E_{\Omega_1})^{-1}(\Omega_2 - \Lambda \Omega_1) + |F_{\Omega_1}|)| \) and \( \zeta_4 = |(\Omega_2 + E_{\Omega_1})^{-1}A| \). If \( l(\frac{2\|\zeta_1\|+\zeta_2\|+\zeta_3\}}{\|\zeta_1\|+\zeta_2\|+\zeta_3\|}) < 1 \) and the parameter matrices \( \Omega_1 \) and \( \Omega_2 \) satisfy

\[
\Omega_2 e > \max(DD_1 e - T^{-1}((E_{\Omega_1}) - |F_{\Omega_1}|)T e, DD_1 e - T^{-1}((E_{\Omega_1}) - |F_{\Omega_1}|)T e),
\]

where \( D \) is the diagonal matrix of \( A \) and \( T \) is a positive diagonal matrix such that \( ((E_{\Omega_1}) - |F_{\Omega_1}|)T \) and \( ((E_{\Omega_1})-|F_{\Omega_1}|)T \) are s.d.d. matrices. Then for any initial vector \( u^{(0)} \in Z \), the iteration sequence \( \{ u^{(k)} \} \) \( k=0 \) resulted from Method 3.5 converges to the unique solution \( u^* \) of the ICP(1).

Proof. Assume that \( u^* \in Z \) is the solution of the ICP(1), then \( w^* = Au^* + q \). According to Method 3.5, it holds that

\[
x^* = \frac{1}{2}(\Omega_1^{-1}w^* - \Omega_2^{-1}w^* - \Omega_1^{-1}m(u^*)) \quad |x^*| = \frac{1}{2}(\Omega_1^{-1}u^* + \Omega_2^{-1}u^* - \Omega_1^{-1}m(u^*))
\]
are the solutions of the implicit fixed-point modulus equations

\[
\begin{cases}
(\Omega_2 + E_{\Omega_1}^* x^* = F_{\Omega_2} x^* + (\Omega_2 - A \Omega_1) |x^*| + \lambda(u^*) - q, \\
(\Omega_2 + E_{\Omega_1}^* x^* = F_{\Omega_2} x^* + (\Omega_2 - A \Omega_1) |x^*| + \lambda(u^*) - q.
\end{cases}
\]

(10)

By Lemma 2.6, we have

\[
u^* = (\Omega_2(|x^*| + x^*) + m(u^*).
\]

(11)

Subtracting (9) from (11) and taking the absolute values on both sides, it is easy to get

\[
|u^{(k+1)} - u^*| = |\Omega_2(|x^{(k,j+1)}| + x^{(k,j+1)}) + m(u^{(k)}) - \Omega_1(|x^*| + x^*) - m(u^*)|
\leq |m(u^{(k)}) - m(u^*)| + \Omega_1(|x^{(k,j+1)}| - |x^*|) + |x^{(k,j+1)} - x^*)|
\leq ||u^{(k)} - u^*|| + 2\Omega_1|u^{(k,j+1)} - x^*|.
\]

(12)

According to \(A \Omega_2 = E_{\Omega_2}^* - F_{\Omega_2}^* = F_{\Omega_2}^* - F_{\Omega_2}^*\) being H-compatible splittings, i.e., \(A \Omega_2 = (E_{\Omega_2}^* - |F_{\Omega_2}^*| = (E_{\Omega_2}^* - |F_{\Omega_2}^*|, it holds that

\[
(A \Omega_2) \leq (E_{\Omega_2}^*) \leq \text{diag}(E_{\Omega_2}^*) \quad \text{and} \quad (A \Omega_2) \leq (E_{\Omega_2}^*) \leq \text{diag}(E_{\Omega_2}^*).
\]

In the light of Lemma 2.4, it is obvious that \(E_{\Omega_2}^*\) and \(E_{\Omega_2}^*\) are \(H_+\)-matrices. Hence, based on Lemma 2.1, we have

\[
||Ω_2 + Ω_{Ω_2}^*||^{-1} \leq (Ω_2 + (E_{Ω_2}^*))^{-1}, \quad ||Ω_2 + Ω_{Ω_2}^*||^{-1} \leq (Ω_2 + (E_{Ω_2}^*))^{-1}.
\]

By subtracting (10) from (8), we can acquire the following equations

\[
\begin{cases}
|x^{(k,j+1)} - x^*| = (Ω_2 + E_{Ω_2}^*)^{-1}[(F_{Ω_2}^* (x^{(k,j)}) - x^*) + (Ω_2 - A Ω_1)(|x^{(k,j)}| - |x^*|) + A(m(u^{(k)}) - m(u^*))], \\
x^{(k,j+1)} - x^* = (Ω_2 + E_{Ω_2}^* x^{(k,j+1)} - x^*) + (Ω_2 - A Ω_1)(|x^{(k,j+1)}| - |x^*|) + A(m(u^{(k)}) - m(u^*)].
\end{cases}
\]

(13)

By taking the absolute values on both sides, we acquire

\[
|x^{(k,j+1)} - x^*| = ||Ω_2 + E_{Ω_2}^*||^{-1}[(F_{Ω_2}^* (x^{(k,j)}) - x^*) + (Ω_2 - A Ω_1)(|x^{(k,j)}| - |x*|) + A(m(u^{(k)}) - m(u^*))]
\leq ||(Ω_2 + E_{Ω_2}^*||^{-1}(Ω_2 - A Ω_1) + ||(Ω_2 + E_{Ω_2}^*||^{-1}A)||m(u^{(k)}) - m(u^*)|
\leq (Ω_2 + (E_{Ω_2}^*))^{-1}[(Ω_2 - A Ω_1) + |F_{Ω_2}^*||x^{(k,j)} - x^*| + ||Ω_2 + E_{Ω_2}^*||^{-1}A||u^{(k)} - u^*|
= ζ_3|x^{(k,j)} - x^*| + l(ζ_4|u^{(k)} - u^*|
\]

(14)

where \(ζ_3 = (Ω_2 + (E_{Ω_2}^*))^{-1}[(Ω_2 - A Ω_1) + |F_{Ω_2}^*||x^{(k,j+1)}| - x^*| + ||Ω_2 + E_{Ω_2}^*||^{-1}A||u^{(k)} - u^*|
\)

(15)

where \(ζ_3 = (Ω_2 + (E_{Ω_2}^*))^{-1}[(Ω_2 - A Ω_1) + |F_{Ω_2}^*||x^{(k,j+1)}| - x^*| + ||Ω_2 + E_{Ω_2}^*||^{-1}A||u^{(k)} - u^*|
\)

Combining (14) and (15), it is appreciable that
\[ |x^{(k+1)} - x^*| \leq \zeta_3 \zeta_1 |x^{(k)} - x^*| + I(\zeta_3 \zeta_2 + \zeta_4)|u^{(k)} - u^*|. \]

It is obvious that
\[ |x^{(k,0)} - x^*| \leq \frac{1}{2}(\Omega_1^{-1} + |\Omega_2^{-1}A|)|u^{(0)} - u^*|. \]

According to (12) and the above discussion, we obtain
\[
|u^{(k+1)} - u^*| \leq l|u^{(k)} - u^*| + 2\Omega_1 [\zeta_3 \zeta_1 |x^{(k,0)} - x^*| + I(\zeta_3 \zeta_2 + \zeta_4)]|u^{(k)} - u^*| \\
\leq 2\Omega_1 (\zeta_3 \zeta_1)\zeta_1 |x^{(k,0)} - x^*| + I(2\Omega_1 (\zeta_3 \zeta_2 + \zeta_4) + I)|u^{(k)} - u^*| \\
\leq [\Omega_1 (\zeta_3 \zeta_1)^{j+1}(\Omega_1^{-1} + I\Omega_1^{-1} + |\Omega_2^{-1}A|) + \zeta_5]|u^{(k)} - u^*| \\
= \Omega_1(\zeta_3 \zeta_1)^{j+1}(\Omega_1^{-1} + I\Omega_1^{-1} + |\Omega_2^{-1}A|) + \zeta_5 \\
where \( \bar{\Gamma} = \Omega_1(\zeta_3 \zeta_1)^{j+1}(\Omega_1^{-1} + I\Omega_1^{-1} + |\Omega_2^{-1}A|) + \zeta_5 \) and \( \zeta_5 = 2\Omega_1 \sum_{m=0}^{j} (\zeta_3 \zeta_1)^m(\zeta_3 \zeta_2 + \zeta_4) + I. \)

Then, we are going to research the conditions that ensure the convergence of Method 3.5, i.e.,
\[ \rho(\bar{\Gamma}) < 1. \] (16)

On the basis of the definition of \( \bar{\Gamma} \), it is appreciable that
\[
\rho(\bar{\Gamma}) = \rho(\Omega_1(\zeta_3 \zeta_1)^{j+1}(\Omega_1^{-1} + I\Omega_1^{-1} + |\Omega_2^{-1}A|) + \zeta_5) \\
\leq \|\Omega_1(\zeta_3 \zeta_1)^{j+1}(\Omega_1^{-1} + I\Omega_1^{-1} + |\Omega_2^{-1}A|) + \zeta_5\| \\
\leq \|\Omega_1\| \cdot \|[(\zeta_3 \zeta_1)^{j+1}]\| \cdot \|\Omega_1^{-1} + I\Omega_1^{-1} + |\Omega_2^{-1}A|| + l\|\Omega_1\| (2 \sum_{m=0}^{j} (\zeta_3 \zeta_1)^m(\zeta_3 \zeta_2 + \zeta_4) + 1).
\]

Analogous to analysis of Theorem 5 [31], it holds that \( \rho(\zeta_3 \zeta_1)^{j+1} < 1 \) if the parameter matrices \( \Omega_1 \) and \( \Omega_2 \) satisfy \( \Omega \geq \max(D\Omega_1 - T^{-1}(E_\Omega' - E_\Omega)Tr, D\Omega_2 - T^{-1}(E_\Omega' - E_\Omega)Tr) \) for any positive diagonal matrix \( T \) such that \((E_\Omega')^T(E_\Omega)\) and \((E_\Omega')^T(E_\Omega)\) are s.d.d. matrix. Based on Lemma 2.3, we have \( (\zeta_3 \zeta_1)^{j+1} = 0, j \rightarrow \infty. \)

Thus, for \( \forall j \geq N \), it holds that
\[
\rho(\bar{\Gamma}) \leq \rho(\Omega_1)\|\|2 \sum_{m=0}^{j} (\zeta_3 \zeta_1)^m(\zeta_3 \zeta_2 + \zeta_4)\| + 1) \\
\leq \rho(\Omega_1)\|\|2\|\|\zeta_3 \zeta_2 + \zeta_4\| \cdot \|\Omega_1\| + 1 \leq \rho(\Omega_1)\|\|\zeta_3 \zeta_2 + \zeta_4\| \cdot \|\Omega_1\| + 1 \\
\leq \rho(\Omega_1)\|\|2\|\|\zeta_3 \zeta_2 + \zeta_4\| \cdot \|\Omega_1\| + 1 \leq \rho(\Omega_1)\|\|\zeta_3 \zeta_2 + \zeta_4\| \cdot \|\Omega_1\| + 1.
\]

Owing to \( l \) satisfying the condition \( \|\|\|\Omega_1\||\|2\|\|\zeta_3 \zeta_2 + \zeta_4\| \cdot \|\Omega_1\| < 1 \), we obtain \( \rho(\bar{\Gamma}) < 1 \). Therefore, \( \lim_{k \rightarrow \infty} u^{(k)} = u^* \).

**Remark 4.2.** In the above discussions, we extend the convergent theoretics of Method 3.3 to the general case. When \( \Omega_1 = I \) and \( \Omega_2 = \Omega \), Theorem 4.1 reduces to Theorem 3.3 in [49] with \( \Omega \geq \max(diag(E_1), diag(E_2)) \).

The succeeding part is the convergent theorem of GTMAOR iteration method.
Theorem 4.3. Let $A$ be an $H_+$-matrix in $\mathbb{R}^{n \times n}$, and $\Omega_1$ and $\Omega_2$ be known positive diagonal matrices such that $A\Omega_1$ is an $H_+$-matrix. $A\Omega_1 = D - L - U := D - B$ satisfies

$$\langle \hat{D} - 2|B| \rangle y > 0 \text{ and } \Omega_2 \geq \max(\hat{D}, \frac{1}{\alpha}\hat{D}),$$

wherein $\hat{D}$, $\hat{B}$, $\hat{L}$ and $\hat{U}$ are the diagonal, the non-diagonal, the strictly lower triangular and the strictly upper triangular parts of the matrix $A\Omega_1$, respectively. $m(\cdot)$ is a Lipschitz continuous function, i.e., it holds that

$$|m(a) - m(b)| \leq \|a - b\|, \quad \forall a, b \in \mathbb{R}^n,$$

wherein $l$ is the Lipschitz constant. Set

$$\Psi_1 = \|\Omega_2 + (E_{\Omega_1}')(\hat{A}_2 - A\Omega_2)\|_1 + \|E_{\Omega_1}'\|_1, \quad \Psi_2 = \|(\Omega_2 + E_{\Omega_1}')^{-1}A\|,$$

$$\Psi_3 = \|(\Omega_2 + (E_{\Omega_1}')(\hat{A}_2 - A\Omega_2)\|_1 + \|E_{\Omega_1}'\|_1 \text{ and } \Psi_4 = \|(\Omega_2 + E_{\Omega_1}')^{-1}A\|. \text{ If } l \text{ satisfies } l < \frac{1-\Psi_1\Psi_2}{\|\Omega_2\|_1(\Omega_2 + (E_{\Omega_1}')(\hat{A}_2 - A\Omega_2))}, \text{ then, for arbitrary initial vector, the GMAOR iteration method is convergent for } 0 < \beta \leq \alpha \leq 2 \text{ and } \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$

Proof. Based on Theorem 4.1, we only need to justify the condition (16). Set

$$E_{\Omega_1}' = \frac{1}{\alpha}(\hat{D} - \beta L), \quad E_{\Omega_1}'' = \frac{1}{\alpha}(1 - \alpha)\hat{D} + (\alpha - \beta)L + a\hat{U},$$

and

$$E_{\Omega_1}' = \frac{1}{\alpha}(\hat{D} - \beta \hat{U}), \quad E_{\Omega_1}'' = \frac{1}{\alpha}(1 - \alpha)\hat{D} + (\alpha - \beta)\hat{U} + a\hat{L}.$$  

Denote $A\Omega_1 := G = (g_{ij}) \in \mathbb{R}^{n \times n}$, and construct an irreducible matrix $G$ as

$$g_{ij} = \begin{cases} \epsilon, & g_{ii} = 0, \quad i, j = 1, 2, \ldots, n. \\ g_{ij}, & g_{ij} \neq 0. \end{cases}$$

Since the diagonal elements of the matrix $A$ are positive and the matrix $\Omega_1$ is positive diagonal, the diagonal elements of the matrix $G$ is not zero, which means they are the same as the diagonal elements of the matrix $G$. Let $G = D - L - U = D - B$ with $B$, $L$ and $U$ being the non-diagonal, the strictly lower triangular and the strictly upper triangular matrices of the matrix $G$, respectively. $G$ is an $H_+$-matrix, then $\rho(D^{-1}|B|) < 1$. For sufficiently small $\epsilon > 0$, based on the continuity of the spectral radius, it holds that $\rho(D^{-1}|B|) < 1$, which means that $\langle G \rangle$ is an $M$-matrix, i.e., $G$ is an $H_+$-matrix. Furthermore, $G$ is irreducible means that $D^{-1}|B|$ is nonnegative irreducible. According to Perron-Frobenius theorem, there is a vector $y = (y_1, y_2, \ldots, y_n)^T > 0$ such that

$$D^{-1}|B|y = \rho(D^{-1}|B|)y,$$

i.e.,

$$|B|y = \rho(D^{-1}|B|)\hat{D}y,$$

which means $\rho(D^{-1}|B|) > 0$. Based on the above discussion, it is appreciate that $0 < \rho(D^{-1}|B|) < 1$. Similarly, set

$$E_{\Omega_1}' = \frac{1}{\alpha}(\hat{D} - \beta L), \quad E_{\Omega_1}'' = \frac{1}{\alpha}(1 - \alpha)\hat{D} + (\alpha - \beta)L + a\hat{U},$$

and

$$E_{\Omega_1}' = \frac{1}{\alpha}(\hat{D} - \beta \hat{U}), \quad E_{\Omega_1}'' = \frac{1}{\alpha}(1 - \alpha)\hat{D} + (\alpha - \beta)\hat{U} + a\hat{L}.$$  

Then, we will prove that $\Omega_2 + E_{\Omega_1}'$ is an $H_+$-matrix. We only need to prove $\langle \Omega_2 + E_{\Omega_1}' \rangle$ is an $M$-matrix. Via the direct calculation and the conditions $0 < \beta \leq \alpha < 2$, we have

$$\langle \Omega_2 + E_{\Omega_1}' \rangle = \Omega_2 + \frac{1}{\alpha}\hat{D} - \frac{1}{\alpha}|L| \geq \frac{1}{\alpha}\hat{D} - |B|.$$  

From $\alpha < \frac{1}{\rho(D^{-1}|B|)}$, we can obtain that $\frac{1}{\alpha}\hat{D} - |B|$ is an $M$-matrix, which means $\langle \Omega_2 + E_{\Omega_1}' \rangle$ is an $M$-matrix. Based on Lemma 2.1, we have
\[ |(\Omega_2 + E'_{\Omega_2})^{-1}| \leq (\Omega_2 + E'_{\Omega_2})^{-1} = (\Omega_2 + (E'_{\Omega_2}))^{-1}. \]

Denote \( Y = \text{diag}(y_1, y_2, \ldots, y_n) \). Then
\[
Y^{-1} L(E_{\Omega_2}, F'_{\Omega_2}, \Omega_2) Y = (\Omega_2 + E'_{\Omega_2}) Y^{-1} (|F'_{\Omega_2}| + |\Omega_2 - \tilde{G}|) Y.
\]

Let \( \Omega_1 = \text{diag}(\omega_1', \omega_2', \ldots, \omega_n') \) and \( \Omega_2 = \text{diag}(\omega_1'', \omega_2'', \ldots, \omega_n'') \). For sufficiently small \( \varepsilon > 0 \), when \( 0 < \beta \leq \alpha \leq 2 \) and \( \Omega_2 \geq \max (\tilde{D}, \frac{1}{\alpha} \tilde{D}) \), \( (\Omega_2 + E'_{\Omega_2}) Y e \) satisfies
\[
(\Omega_2 + E'_{\Omega_2}) Y e = \omega_1'' y_1 + \frac{1}{\alpha} g_a y_i - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j \\
\geq \frac{2}{\alpha} g_a y_i - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j \\
= \frac{2}{\alpha} g_a y_i - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j + \frac{\beta}{\alpha} \sum_{j=i+1}^{n} |g_{ij}| y_j - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j \\
= \frac{2 - \beta \cdot \rho (\tilde{D}^{-1}|\beta|)}{\alpha} g_a y_i + \frac{\beta}{\alpha} \sum_{j=i+1}^{n} |g_{ij}| y_j \\
\geq \frac{2 - \beta \cdot \rho (\tilde{D}^{-1}|\beta|)}{\alpha} g_a y_i > 0.
\]

Hence, \( \Omega_2 Y + (E'_{\Omega_2}) Y \) is a s.d.d. matrix. In addition, since \( 0 < \beta \leq \alpha < 2 \), it holds that
\[
||Y^{-1} L(E_{\Omega_2}, F'_{\Omega_2}, \Omega_2) Y||_{\infty} = ||(\Omega_2 + E'_{\Omega_2}) Y^{-1} (|F'_{\Omega_2}| + |\Omega_2 - \tilde{G}|) Y||_{\infty} \\
= ||(\Omega_2 + \frac{1}{\alpha} \tilde{D} - \frac{\beta}{\alpha} |\tilde{L}|) Y^{-1} \cdot [\frac{1}{\alpha} (1 - \alpha) \tilde{D} + (\alpha - \beta) \tilde{L} + \alpha \tilde{U}] Y || \leq \max_{1 \leq i \leq n} \left( \frac{\omega_1'' y_1 + \frac{\alpha - \beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j + \alpha \sum_{j=i+1}^{n} |g_{ij}| y_j + \sum_{j=1}^{i-1} |g_{ij}| y_j}{\omega_1'' y_1 - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j} \right),
\]

which holds according to Lemma 2.2.

Since \( (\tilde{D} - 2|\beta|) Y e > 0 \), then
\[
(\omega_1'' y_1 + \frac{1}{\alpha} g_a y_i - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j) - \left( (\omega_1'' y_1 + \frac{|1 - \alpha| - \alpha}{\alpha} g_a) y_i \right) \\
+ \frac{\alpha - \beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j + \sum_{j=i+1}^{n} |g_{ij}| y_j + \sum_{j=1}^{i-1} |g_{ij}| y_j \\
> g_a y_i + \frac{1}{\alpha} g_a y_i - \frac{\beta}{\alpha} \sum_{j=1}^{i-1} |g_{ij}| y_j > 0.
\]
Hence, it is simple that \( \| Y^{-1} \mathcal{L}(E_{\Omega_1}'', F_{\Omega_1}'', \Omega_2) Y \|_\infty < 1 \).

Analogously, we can also get \( \| Y^{-1} \mathcal{L}(E_{\Omega_1}', F_{\Omega_1}', \Omega_2) Y \|_\infty < 1 \). It holds that

\[
\rho(\zeta_3\zeta_4) < \| \zeta_3 \zeta_4 \| = \lim_{k \to \infty} (\| Y^{-1} \mathcal{L}(E_{\Omega_1}'', F_{\Omega_1}'', \Omega_2) Y \|_\infty)^\frac{1}{k} < 1,
\]

which means \( (\zeta_3\zeta_4)^{j+1} \to 0, \ j \to \infty \). Since \( \| \Omega_1 \| \) and \( \| \Omega_1^{-1} + I \Omega_2^{-1} + |\Omega_2^{-1}\mathbf{A}| \) are constants.

Hence, for arbitrary positive number \( \epsilon \), there is \( N \in \mathbb{N}^+ \) such that

\[
\| \Omega_1 \| (\zeta_3\zeta_4)^{j+1} \| \Omega_1^{-1} + I \Omega_2^{-1} + |\Omega_2^{-1}\mathbf{A}| \leq \epsilon, \ \forall j \geq N.
\]

Let \( \| \zeta \| = \Psi_{ij}, \ i = 1, 2, 3, 4 \). Thus, for \( \forall j \geq N \), it holds that

\[
\rho(\Gamma) \leq \epsilon + l\| \Omega_1 \|(2 \sum_{m=0}^{j} (\Psi_1\Psi_3)^m(\Psi_2\Psi_3 + \Psi_4) + 1)
\leq \epsilon + l\| \Omega_1 \|\left(\frac{2\Psi_2\Psi_3 + \Psi_4}{1 - \Psi_1\Psi_3} + 1\right)
\leq \epsilon + l\| \Omega_1 \|\left(\frac{2\Psi_2\Psi_3 + \Psi_4 + 1}{1 - \Psi_1\Psi_3}\right).
\]

Because of \( l \) satisfying the condition \( l < \frac{1}{\rho(\Omega_1\Omega_2\psi_3\psi_4)} \), we obtain \( \rho(\Gamma) < 1 \). Therefore, \( \lim_{k \to \infty} u^{(k)} = u^* \), which prove the GTMAOR iteration method is convergent. \( \square \)

**Remark 4.4.** Set \( \Omega_1 = I \) and \( \Omega_2 = \Omega \), then Theorem 4.3 is the convergent results of the TMAOR method.

**Remark 4.5.** We can directly obtain that the GTMSOR iteration method is convergent for

\[
0 < \alpha < \min\left(2, \frac{1}{\rho(\Omega_3\Omega_4)}\right),
\]

and other conditions of Theorem 4.3 remain unchanged.

5. Numerical results

In this section, in order to demonstrate the efficiency of our proposed methods, we will do some numerical experiments, which include three aspects: the elapsed CPU time in seconds (CPU), the norm of absolute residual vectors (RES), and the iteration numbers (IT), respectively. All of these numerical results were performed in Matlab (R2017a) on an Intel(R) Core(TM) i5-4210U CPU @ 1.70GHz, 4.00GB RAM.

In the numerical computations, we choose the initial vector \( u^{(0)} \) to be zero vector. 'RES' is defined as

\[
\text{RES}(u^{(k)}) := \|(Au^{(k)} + q)(u^{(k)} - m(u^{(k)}))\|,
\]

where \( u^{(k)} \) is the \( k \)th approximate solution to the ICP, and all iterations are terminated either the maximum iteration numbers exceed 2000 or \( \text{RES}(u^{(k)}) \leq 10^{-5} \). The inner iteration steps \( j \) is set to be 3.

We choose three methods such as MSOR, TMSOR and GTMSOR in the following experiments. Furthermore, on the basis of the existing literature, we set the parameters \( \alpha = 1.2, \ t = 2.5 \) and \( \gamma = 2 \). We take \( \Omega = \gamma t D_A \) in TMMSOR methods, and \( \Omega_1 = \frac{t}{\alpha} I \) and \( \Omega_2 = t D_A \) in both GMMSOR and TGMMSOR methods, respectively.

**Example 5.1.** [28] Let \( m \) be a prescribed positive integer and \( n = m^2 \). Consider the ICP(1), in which \( A \in \mathbb{R}_+^{n \times n} \) is a block tridiagonal matrix.
Table 1: Numerical results for Example 5.1

| Algorithm | m=10       | m=20       | m=30       | m=40       |
|-----------|-----------|-----------|-----------|-----------|
| MSOR      | CPU 0.0222| 3.5643    | 85.9189   | 772.4247  |
|           | RES 1.2815e-06 | 9.8599e-06 | 9.9055e-06 | 9.9520e-06 |
|           | IT 25     | 133       | 450       | 956       |
| TMSOR     | CPU 0.0452| 4.3827    | 91.7140   | 824.1270  |
|           | RES 1.6585e-06 | 4.6080e-06 | 9.6860e-06 | 9.7956e-06 |
|           | IT 20     | 75        | 250       | 529       |
| GTMSOR    | CPU 0.0115| 1.5031    | 29.6515   | 259.7250  |
|           | RES 9.4070e-07 | 3.4515e-06 | 8.7211e-06 | 9.5469e-06 |
|           | IT 7      | 23        | 79        | 168       |

\[
A = \text{tridiag}(-I, S, -I) = \begin{pmatrix}
S & -I & 0 & \cdots & 0 & 0 \\
-I & S & -I & \cdots & 0 & 0 \\
0 & -I & S & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & S & -I \\
0 & 0 & 0 & \cdots & -I & S
\end{pmatrix} \in \mathbb{R}^{n \times n}
\]

and the block diagonal matrix \( S \) is defined as a tridiagonal matrix

\[
S = \text{tridiag}(-1, 4, -1) = \begin{pmatrix}
4 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 4 & -1 & \cdots & 0 & 0 \\
0 & -1 & 4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 4 & -1 \\
0 & 0 & 0 & \cdots & -1 & 4
\end{pmatrix} \in \mathbb{R}^{m \times m}
\]

The vector \( q \in \mathbb{R}^n \) and the point-to-point mapping \( m(u) \) are

\[
q = (-1, 1, -1, 1, \cdots, (-1)^{n-1}, (-1)^{n})^T \in \mathbb{R}^n \text{ and } m(u) = (u_1^3, u_2^3, \cdots, u_n^3)^T \in \mathbb{R}^n,
\]

respectively.

In this example, we set four different sizes, i.e., \( n = 100, 400, 900, 1600 \). From Table 1, it is simple to find that the GTMSOR method outperforms the MSOR and TMSOR methods in both IT and CPU, which manifests the GTMSOR method has an advantage over the others. The convergent rate becomes faster as the system matrix \( A \) size \( n \) is increasing. Numerical experiment illustrates the convergence speed is accelerated by the general two-step modulus-based matrix splitting.

**Example 5.2.** [28] Let \( m \) be a prescribed positive integer and \( n = m^2 \). Consider the ICP(1), in which \( A \in \mathbb{R}^{m \times m} \) is a block tridiagonal matrix

\[
A = \text{tridiag}(-I, S, -I) = \begin{pmatrix}
S & -0.5I & 0 & \cdots & 0 & 0 \\
-1.5I & S & -0.5I & \cdots & 0 & 0 \\
0 & -1.5I & S & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & S & -0.5I \\
0 & 0 & 0 & \cdots & -1.5I & S
\end{pmatrix} \in \mathbb{R}^{m \times m}
\]

and the block diagonal matrix \( S \) is defined as a tridiagonal matrix
The vector $q \in \mathbb{R}^n$ and the point-to-point mapping $m(u)$ are

$$q = (-1, 1, -1, 1, \cdots, (-1)^{n-1}, (-1)^{n})^T \in \mathbb{R}^n \text{ and } m(u) = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n,$$

respectively.

### Example 5.3.

[47] Let $m$ be a prescribed positive integer and $n = m^2$. Consider the ICP(1), in which $A \in \mathbb{R}^{n \times n}$ is

$$A = \begin{bmatrix} S & -I & -I \\ -I & S & -I \\ & & \ddots & \ddots \\ & & & S & -I \end{bmatrix} \in \mathbb{R}^{n \times n}$$

and the block diagonal matrix $S$ is defined as a tridiagonal matrix

$$S = \text{tridiag}(-1, 8, -1) = \begin{bmatrix} 8 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 8 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 8 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 8 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 8 \end{bmatrix} \in \mathbb{R}^{m \times m}.$$

The vector $q \in \mathbb{R}^n$ and the point-to-point mapping $m(u)$ are

$$q = (-1, 1, -1, 1, \cdots, (-1)^{n-1}, (-1)^n)^T \in \mathbb{R}^n \text{ and } m(u) = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n,$$

respectively.

In Table 3, as for the three mentioned aspects, our suggested method is faster in CPU and IT and smaller in RES. Furthermore, we can also gain the conclusion that the GTMSOR method has an advantage over the MSOR and TMSOR methods. These results all prove our proposed method is a better method.
Table 3: Numerical results for Example 5.3

| Algorithm | $m=10$  | $m=20$  | $m=40$  | $m=60$  |
|-----------|---------|---------|---------|---------|
|           | CPU     | RES     | IT      | CPU     | RES     | IT      | CPU     | RES     | IT      |
| MSOR      | 0.0177  | 9.2304e-06 | 12     | 0.4805  | 8.3774e-06 | 14     | 13.0513 | 5.3743e-06 | 16     |
|           | 122.3386 | 4.5922e-06 | 17     |         |         |         |         |         |         |
| TMSOR     | 0.0108  | 7.8810e-07 | 7      | 0.4004  | 7.2184e-06 | 7      | 12.3242 | 1.3861e-06 | 8      |
|           | 117.8681 | 3.7938e-06 | 8      |         |         |         |         |         |         |
| GTMSOR    | 0.0254  | 1.6485e-06 | 2      | 0.1810  | 4.6253e-06 | 2      | 4.6610  | 1.4303e-10 | 3      |
|           | 40.5694  | 4.9490e-10 | 3      |         |         |         |         |         |         |

References

[1] Z.-Z. Bai, On the convergence of the multisplitting methods for the linear complementarity problem, SIAM J. Matrix Anal. Appl., 21 (1999) 67-78.
[2] M. A. Noor, Fixed point approach for complementarity problems, J. Math. Anal. Appl., 133 (1988) 437-448.
[3] Z.-Z. Bai, The monotone convergence of a class of parallel nonlinear relaxation methods for nonlinear complementarity problems, Comput. Math. Appl., 31 (1996) 17-33.
[4] Z.-Z. Bai, New comparison theorem for the nonlinear multispli廷ing relaxation method for the nonlinear complementarity problems, Comput. Math. Appl., 32 (1996) 41-48.
[5] Z.-Z. Bai, D.-R. Wang, A class of parallel nonlinear multispli廷ing relaxation methods for the large sparse nonlinear complementarity problems, Comput. Math. Appl., 32 (1996) 79-95.
[6] Z.-Z. Bai, On the monotone convergence of the projected iteration methods for linear complementarity problem, Numer. Math., J. Chin. Univ. (English Series), 5 (1996) 228-233.
[7] Z.-Z. Bai, D. J. Evans, Chaotic iterative methods for the linear complementarity problems, J. Comput. Appl. Math., 96 (1998) 127-138.
[8] Z.-Z. Bai, L.-L. Zhang, Modulus-based multigrid methods for linear complementarity problems, Numer Linear Algebra Appl., 24 (2017) e2105: 1-15.
[9] Z.-Z. Bai, On the monotone convergence of matrix multispli廷ing relaxation methods for the linear complementarity problem, Numer Linear Algebra Appl., 24 (2017) 1-15.
[10] Z.-Z. Bai, Parallel chaotic multispli廷ing iterative methods for the large sparse linear complementarity problem, J. Comput. Math., 19 (2001) 281-292.
[11] Z.-Z. Bai, J.-L. Dong, A modified damped Newton method for linear complementarity problems, Numer. Algor., 42 (2006) 207-228.
[12] C.-L. Li, Z.-C. Xia, Modulus-based matrix splitting iteration methods for a class of nonlinear complementarity problem, Appl. Math. Comput., 271 (2015) 34-42.
[13] J.-S. Pang, On the convergence of a basic iterative method for the implicit complementarity problem, J. Optim. Theory Appl., 37 (1982) 149-162.
[14] Q. Yuan, H.-Y. Yin, Stationary points of the minimization reformulations of implicit complementarity problems, Numer. Math., 31 (1999) 11-18.
[15] C.-L. Li, J.-T. Hong, Modulus-based synchronous multispli廷ing iteration methods for an implicit complementarity problem, ESAIM M2AN, 7 (2017) 363-375.
[16] X. Wang, X. Li, L.-H. Zhang, R.-C. Li, An efficient numerical method for the symmetric positive definite second-order cone linear complementarity problem, J. Sci. Comput., 79 (2019) 1608-1629.
[17] L.-H. Zhang, W.-H. Yang, An efficient algorithm for second-order score linear complementarity problems, Math. Comput., 83 (2013) 1701-1726.
[18] L.-H. Zhang, W.-H. Yang, An efficient matrix splitting method for the second-order cone complementarity problem, Math. Comput., 24 (2014) 1178-1205.
[19] X. Wang, W.-W. Li and L.-Z. Mao, On positive-definite and skew-Hermitian splitting iteration methods for continuous Sylvester equation $AX + XB = C$. Comput. Math. Appl. 66 (2013), 2352-2361.
[20] X. Wang, Y. Li, L. Dai, On Hermitian and skew-Hermitian splitting iteration methods for the linear matrix equation $AXB = C$. Comput. Math. Appl. 65 (2013), 657-664.
[21] X. Wang, X.-Y. Xiao, Q.-Q. Zheng, A single-step iteration method for non-Hermitian positive definite linear systems. J Comput. Appl. Math., 346 (2019), 471-482.
[22] G. Isac, The implicit complementarity problem. In: Complementarity Problems, Lect. Notes Math. Springer, Berlin, Heidelberg (1992).
[23] M. C. Ferris, J.-S. Pang, Engineering and economic applications of complementarity problems, SIAM Rev., 39 (1997) 669-713.
[24] S. C. Billups, K. G. Murty, Complementarity problems, J. Comput. Appl. Math. 124 (2000) 303-328.
[25] R. W. Cottle, J.-S. Pang, R. E. Stone, The Linear Complementarity Problem, Academic Press, San Diego (1992).
[26] Z.-Z. Bai, Modulus-based matrix splitting iteration methods for linear complementarity problems, Numer. Linear Algebra Appl., 17 (2010) 917-933.

[27] W. Li, H. Zheng, A preconditioned modulus-based iteration method for solving linear complementarity problems of $H$-matrices, Linear and Multilinear Algebra, 64 (2016) 1390-1403.

[28] P.-F. Dai, J.-C. Li, J.-C. Bai, J.-M. Qiu, A preconditioned two-step modulus-based matrix splitting iteration method for linear complementarity problem, Appl. Math. Comput., 348 (2019) 542-551.

[29] X.-P. Wu, X.-F. Peng, W. Li, A preconditioned general modulus-based matrix splitting iteration method for linear complementarity problems of $H$-matrices, Numer. Algor., 79 (2018) 1131-1146.

[30] H. Ren, X. Wang, X.-B. Tang, A preconditioned general two-step modulus-based matrix splitting iteration method for linear complementarity problems of $H$-matrices, Numer. Algor., (2018) https://doi.org/10.1007/s11075-018-0367-5.

[31] W. Li, A general modulus-based matrix splitting iteration method for linear complementarity problems of $H$-matrices, Appl. Math. Lett., 26 (2013) 1159-1164.

[32] S.-M. Liu, H. Zheng, W. Li, A general accelerated modulus-based matrix splitting iteration method for solving linear complementarity problems, Calcolo, 53 (2016) 189-199.

[33] H. Ren, X. Wang, X.-B. Tang, T. Wang, The general two-sweep modulus-based matrix splitting iteration method for solving linear complementarity problems, Comput. Math. Appl., 77 (2019) 1071-1081.

[34] L.-L. Zhang, Two-step modulus-based matrix splitting iteration method for linear complementarity problems, Appl. Math. Comput., 348 (2019) 542-551.

[35] Z.-Z. Bai, D. Evans, Matrix multisplitting relaxation methods for linear complementarity problems, Int. J. Comput. Math., 63 (1997) 309-326.

[36] Z.-Z. Bai, D. Evans, Matrix multisplitting methods with applications to linear complementarity problems: parallel asynchronous methods, Int. J. Comput. Math., 79 (2002) 205-232.

[37] Z.-Z. Bai, L.-L. Zhang, Modulus-based synchronous multisplitting iteration methods for linear complementarity problems, Numer. Linear Algebra Appl., 20 (2013) 425-439.

[38] Z.-Z. Bai, L.-L. Zhang, Modulus-based synchronous two-stage multisplitting iteration methods for linear complementarity problems, Numer. Algor., 62 (2013) 59-77.

[39] L.-L. Zhang, Two-step modulus-based synchronous multisplitting iteration methods for linear complementarity problems, J. Comput. Math., 33 (2015) 100-112.

[40] R. Zhou, X. Wang, P. Zhou, A Modified HSS iteration method for solving the complex linear matrix equation $AXB = C$, J. Comput. Math., 34(2016), 437-450.

[41] R. Zhou, X. Wang, X.-B. Tang, Preconditioned positive-definite and skew-Hermitian splitting iteration methods for continuous Sylvester equations $AX + XB = C$, E. Asian J. Appl. Math., 7(2017) 55-69.

[42] R. Zhou, X. Wang, X.-B. Tang, A generalization of the Hermitian and skew-Hermitian splitting iteration method for solving Sylvester equation, Appl. Math. Comput., 271(2015) 609-617.

[43] D. Chan, J.-S. Pang, The generalized quasi-variational problem, Math. Oper. Res., 7 (1982) 211-222.

[44] R. W. Cottle, F. Giannessi, J. L. Lions, Variational inequalities and complementarity problems: Theory and applications, Wiley, Chichester (1980).

[45] J.-T. Hong, C.-L. Li, Modulus-based matrix splitting iteration methods for a class of implicit complementarity problems, Numer. Linear Algebra Appl., 23 (2016) 629-641.

[46] A. Wang, Y. Cao, Q. Shi, Convergence analysis of modulus-based matrix splitting iterative methods for implicit complementarity problems, J. Inequal. Appl., (2016) https://doi.org/10.1186/s13660-017-1593-7.

[47] Y. Cao, A. Wang, Two-step modulus-based matrix splitting iteration methods for implicit complementarity problems, Numer. Algor. (2019) https://doi.org/10.1007/s11075-019-00660-7.

[48] A. Frommer, G. Mayer, Covergence of relaxed parallel multisplitting methods, Linear Algebra Appl., 119 (1989) 141-152.

[49] J.-G. Hu, Estimates of $\|B^{-1}A\|_\infty$ and their applications, Math. Numer. Sin. 4 (1982) 272-282.

[50] R. S. Varga, Matrix Iterative Analysis (Second Edition), Springer, Berlin, Heidelberg, (2000).

[51] A. Frommer, D. B. Szyld, H-splittings and two-stage iterative methods, Numer. Math., 63 (1992) 345-356.

[52] A. Berman, R. J. Plemmons, Nonnegative matrices in the mathematical sciences, SIAM Publisher, Philadelphia (1994).