Abelian threshold models and forced weakening

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Mean field slider block models have provided an important entry point for understanding the behavior of discrete driven threshold systems. We present a method of constructing these models with an arbitrary frictional weakening function. This ‘forced weakening’ method unifies several existing approaches, and multiplies the range of possible weakening laws. Forced weakening also results in Abelian rupture propagation, so that an avalanche size depends only on the initial stress distribution. We demonstrate how this may be used to accurately predict the long-time event statistics of a simulation.

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Complex spatial or temporal patterns often emerge when systems are driven from equilibrium and experience instabilities. Coupled lattice maps have become a popular method of modelling such phenomena, seeking to capture the essential physics in a discrete form amenable to numerical simulation. In the study of earthquakes, simple slider-block models [1–4] have long been employed to investigate the origin of magnitude-frequency scaling. Similar models are used to describe such diverse phenomena as pinned charge density waves [5], flux lattices in type II superconductors [6], and creeping contact lines [7].

Recently, analytic results for the avalanche size distribution have been presented for mean-field slider-block models [8,9]. These models are characterized by homogeneous, infinite range elastic interactions, and exhibit complex event histories and regimes of behavior. Despite their simplicity, mean-field models remain sensitive to the choice of update rules and details of implementation. This is especially evident in models that impose different modes of behavior as the strength of the weakening is varied.

Here we describe a method of constructing a mean-field threshold model where one may simulate arbitrary weakening laws under identical rules of evolution. This unifies the analysis of previously incompatible models, and provides more freedom in numerical simulation. This technique also results in Abelian rupture propagation, where the size of a simulated earthquake is uniquely determined from initial conditions, leading to a rigorous and implementation-independent analysis.

The general slider-block model represents stick-slip motion along a fault plane with \( N \gg 1 \) discrete coordinates (or ‘sites’). Each site \( i \) is assigned a slip deficit \( u_i \), which measures the local distance from elastic equilibrium. The sites are pinned in place by frictional forces, and are subject to a restoring force (stress) proportional to their slip deficit. Internal disorder gives rise to an additional component of stress due to elastic interactions. The stress \( \bar{\sigma}_i \) at a site \( i \) is related to the slip deficits through a linear constitutive relation

\[
\bar{\sigma}_i = -K_L u_i - \sum_j K_{ij} (u_i - u_j) \quad (1)
\]

where \( K_L \) and \( K_{ij} \) are spring constants. If we impose uniform (mean-field) interactions between all the elements, \( K_{ij} = K_C/N \), the above relation becomes

\[
\bar{\sigma}_i = -K_L u_i - K_C (u_i - \langle u \rangle) \quad (2)
\]

where \( \langle u \rangle = N^{-1} \sum_i u_i \) will denote an average over all \( N \) sites in the model. We obtain a unitless expression by dividing by \( K_C a \), where \( a \) is a characteristic microscopic length. Defining the unitless slip deficit \( \phi = u/a \), stress \( \sigma = \bar{\sigma}/(K_C a) \), and spring constant ratio \( K_R = K_L/K_C \), Eq. (2) simplifies to

\[
\sigma_i = -(K_R + 1) \phi_i + \langle \phi \rangle. \quad (3)
\]

For finite \( N \) we will refer to this as the near mean-field (NMF) model. Note that it is easy to invert Eq. (3) for the slip deficits in terms of stresses,

\[
\phi_i = \frac{-\sigma_i}{K_R + 1} - \frac{\langle \sigma \rangle}{K_R (K_R + 1)} \quad (4)
\]

so the configuration is uniquely determined by the parameter \( K_R \) and either the slip deficits or stresses alone.

The model is slowly driven away from equilibrium by uniformly increasing the slip deficits. Eventually the stress at one site will surpass the maximum local frictional force and ‘fail’, sliding toward its equilibrium point. The motion of a failed site will change the mean slip deficit \( \langle \phi \rangle \), and produce a change in stress at other sites. If this change brings other sites to their threshold, they will also fail, producing an avalanche interpreted as a single event.
It is assumed that following the initiation of motion, the frictional force on a site will weaken, producing a transient dynamic instability. In discrete time we cannot model the dynamic slip or velocity of the site, but instead assign a residual stress $\sigma^R$ at which the motion arrests. This $\sigma^R$ is chosen from a probability distribution independently for each failed site. Since slips occur instantaneously, we lose the interplay between a continuously evolving stress field and frictional force at a site. The behavior of dynamical models is known to strongly depend on the form of frictional weakening [2], so the ability to include equivalent effects in discrete models would be advantageous.

Since the NMF model is not dynamical we are only interested in large-scale features of its behavior that are independent of microscopic dynamics. Thus we are free to choose the simplest update rules that are consistent with the phenomenon of interest. In practice, we assume that a single site $j$ reaches its stress threshold first. Since the physics will depend only on changes in stress, we may impose a uniform failure threshold $\sigma^D$ by absorbing any variations into the residual stress distribution. The slip displacement $\Delta_j = \phi^{(j)}(n) - \phi^{(j)}(n-1)$ is related to the change in stress $\Delta \sigma_j = \sigma^R_j - \sigma^F$ by $\Delta_j = -\Delta \sigma_j/(K_R + 1 - N^{-1})$. The motion of the site will change the mean slip deficit $\langle \phi \rangle$ by $\Delta_j/N$. We may view this as a transfer of stress from failing sites to all others.

The above describes a series type dislocation where sites fail in sequence and the stress transfer occurs to other sites all at once. This makes it likely that any failing site (other than the single initiator) will have a stress slightly above the threshold, which subtly provides an order-of-failure dependence to the stress transfer. As a consequence, the exact stress transfer in simulation will depend on obscure factors like the order of iteration over sites. To eliminate this we must examine the stress transfer in more detail.

Suppose that in the course of a (possibly ongoing) event there have been $k$ block failures. Let $\{k\}$ represent the set of indices of failed sites. Call $k = k/N$ the fraction of failed sites. Then from Eq. (3) the change in stress for any stable site $i$ is

$$\Delta \sigma_{i \notin \{k\}} = \frac{1}{N} \sum_{j \in \{k\}} \Delta_j = \frac{\delta}{N} \sum_{j \in \{k\}} (\sigma^F_j - \sigma^R_j)$$

$$= \delta \kappa \langle \sigma^F \rangle_k - \langle \sigma^R \rangle_k$$ (5)

where $\delta = (K_R + 1 - 1/N)^{-1}$, and $\langle \rangle_k$ is an average applied over failed sites. We call this the external stress transfer to signify that it applies to sites that are not part of the rupture. The quantity $\sigma^F_j$ is the stress of site $j$ at failure, which may be greater than $\sigma^D$. Note that the slip displacement $\Delta_j$ is only dependent on the stress drop at failure because pinning occurs immediately, and subsequent stress changes will not affect the slip. Since $\sigma^F$ is typically very near $\sigma^D$ and the $\sigma^R_j$ are identically distributed random variables, the term in parenthesis is on average independent of $k$. Thus the external transfer grows linearly (on average) with the fraction of failed sites.

In ‘dynamic weakening’ models [1] the increased propensity for a failed site to slip further (due to a weakened pinning force) is simulated by imposing a lower threshold stress $\sigma^D < \sigma^F$ for the duration of a single event. After a site fails it will receive stress transfer from subsequent failures, and thus may reach this lower threshold and fail again. Re-failing sites will contribute more to the external transfer and enhance the likelihood of continued rupture growth.

This form of weakening is strange in that it first requires some failed sites to have their stress brought back up to the dynamical threshold, which will occur at some minimum rupture size. Following the onset of dynamical weakening the additional stress transfer typically results in a runaway event which fails every block in the system. This feature can be exploited to produce characteristic events which always occur once the minimum size is reached.

One way to visualize the effects of weakening is to examine the average stress of sites that have failed as a function of rupture size (Figure 1). Without weakening, the average stress of failed sites (the average internal stress) will itself grow linearly like the external transfer (with half the slope). However, with dynamic weakening, all failed sites with stress $\geq \sigma^D$ will fail again, putting a ceiling on the average internal stress.

We claim that a more realistic approach would be to have failed sites shed a certain fraction of the stress they receive after failure. Implementing this ‘fractional weakening’ would involve re-computing the slip of all failed sites with each new failure. There is an easier way to accomplish this if we note that the desired effect is to lower the slope of the average internal stress (AIS) function. There is a way to invert this relationship, such that the average internal stress function is given and the requisite slips and stress transfer computed as a result. In essence, in place of solving dynamical equations involving slip or velocity dependent friction, we can impose the effects of the weakening as they appear in a discrete time context. We call this approach ‘forced weakening’.

To perform this inversion we first observe the change in stress of a site $j$ as it depends on $k$, the number of failed sites. Let $\Delta_k^i$ denote the slip displacement of site $j$ after $k$ failures. If it is nonzero it includes all block motion, including initial failure, additional failures from weakening, or continuous sliding. Consider the system of equations for the stress changes of the failed sites $j \in \{k\}$

$$\Delta \sigma_{j \in \{k\}} = \sigma^F_j - \sigma^D_j = -\delta \Delta_k^j + \frac{1}{N} \sum_{i \notin \{k\}} \Delta_k^i$$

$$= -(K_R + 1) \Delta_k^j + \frac{1}{N} \sum_{i \in \{k\}} \Delta_k^i.$$

(6)

The last line demonstrates the simple linear form of the
relationship between stress drops and slip displacements. This may be obtained via a matrix with diagonal elements \( N^{-1} - (K_R + 1) \) and off-diagonal elements \( N^{-1} \). This matrix may easily be inverted to obtain the slips \( \Delta^k_j \) in terms of the current stress drops \( \Delta^k_j \) as in Eq. 5. Summing over them yields a new expression to calculate the external transfer, we need not specify the forced weakening method has two main advantages. Practically, it allows the simulation of models with arbitrary weakening characteristics, some of which would not be obtainable with modified CA rules. Formally, the model is Abelian, so that the event size is a unique function of the initial stress configuration. Using this fact we can seek an expression which will determine the event size given adequate information of the initial stresses.

To accomplish this we first approximate the stress configuration with a continuous distribution \( p_\sigma(\chi) \), where \( p_\sigma(\chi) \) is the probability that a randomly selected site will have stress \( \sigma \) between \( \chi \) and \( \chi + d\chi \). There are two ways of obtaining a continuous probability distribution from a discrete set of stresses: allow the number of sites \( N \) to become infinite, or to consider averages over a course grained time. In the latter case the continuous driving of the slip deficits will quickly fill in the spectrum of possible stresses.

When considering a course grained time, we realize that the distribution is only an average, and that the actual instantaneous distribution of stresses will fluctuate. Similarly, the solution we obtain will be an average with statistical properties we would like to calculate. Under ordinary conditions, the distribution of stresses is statistically stationary [11], so the expected distribution of event sizes at any time will predict the long-time average behavior.

To write down the next-event-size expression, it is convenient to define the stress deficit \( \Sigma = \sigma^F - \sigma \), and the cumulative distribution

\[
P_\Sigma(\chi) = \int_0^\chi p_\Sigma(\chi')d\chi'
\]

where \( p_\Sigma(\chi) = p_\sigma(\sigma^F - \chi) \). Given this distribution the event size \( \kappa \) is the solution to

\[
\frac{1}{K_R + 1 - \kappa} \left[ \kappa \sigma^F - \kappa f(\kappa) - \sigma^F \int_0^\kappa P_\Sigma(\chi)d\chi \right] = -\sigma^F \Sigma^{-1}(\kappa + d\kappa) = 0
\]

where \( f(\kappa) \) is again the AIS function. A full derivation and analysis of the results of this equation will be published elsewhere. For an example solution, suppose each stress in the system is chosen independently from a uniform distribution between zero and \( \sigma^F \). Then for large \( N \), Eq. [11] determines that \( \kappa \) is the first crossing of a 1-d random walk constrained to return to zero at \( \kappa = 1 \). Without that constraint, this produces a power law distribution of event sizes with exponent \(-3/2\). With the constraint in place, there is an upturn at large event sizes, which is observed in simulations (so long as \( K_R \) is low enough to allow such large events). A comparison of this analytical solution with a simulation is shown in Figure 2.

In summary, the forced weakening method extends the basic slider block model to include arbitrary weakening functions in an efficient discrete time simulation. Additionally, forced weakening results in Abelian avalanches which leads to a rigorous and implementation-independent analysis. We have presented a brief example
of this showing excellent agreement between theory and simulation. This approach should be equally applicable to related discrete threshold models. While the resulting formalism is dependent on the mean-field character of the model, the numerical techniques may find wider use whenever an inversion like Eq. (1) is available.

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FIG. 1. Average internal stress (AIS) functions for several weakening laws. The AIS function measures the average stress of failed sites as a function of rupture size. ‘Dynamic weakening’ as implemented in the literature places an upper bound on the AIS starting at the dynamic threshold \( \sigma^D \). Fractional weakening, proposed here, sheds a fixed fraction of the internal transfer to unfailed sites. In this figure we have taken the average residual stress to be 0.05.

FIG. 2. Comparison of a solution of Eq. (11) for uniformly distributed stresses (a) with simulation results (b). The power laws have identical mean-field exponents of -3/2 and the predicted upturn at large event sizes is evident in the simulated data. The parameters of the simulation are \( K_R = 0.01 \) and an average residual stress of 0.05.