Chapter 1

Spin-Hall Effect in Chiral Electron Systems: from Semiconductor Heterostructures to Topological Insulators

P.G.Silvestrov\textsuperscript{1} and E.G.Mishchenko\textsuperscript{2}

\textsuperscript{1}Theoretische Physik III, Ruhr-Universität Bochum, 44780 Bochum, Germany
\textsuperscript{2}Department of Physics, University of Utah, Salt Lake City, Utah 84112, USA

The phenomenon of mesoscopic Spin-Hall effect reveals in a nonequilibrium spin accumulation (driven by electric current) at the edges of a ballistic conductor or, more generally, in the regions with varying electron density. In this paper we review our recent results on spin accumulation in ballistic two-dimensional semiconductor heterostructures with Rashba/Dresselhaus spin orbit interactions, and extend the method developed previously to predict the existence of spin-Hall effect on the surface of three-dimensional topological insulators. The major difference of the new Spin-Hall effect is its magnitude, which is predicted to be much stronger than in semiconductor heterostructures. This happens because in semiconductors the spin accumulation appears due to a small spin-orbit interaction, while the spin-orbit constitutes a leading term in the Hamiltonian of topological insulator.

1.1. Chiral electron systems

Chiral systems feature electron states whose quantum properties (e.g. spin) depend on the direction of propagation. One example of such chiral states arises from spin-orbit coupling that originates from relativistic corrections to the dispersion law of band electrons. In particular, for a typical two-dimensional electron gas (2DEG) the intrinsic asymmetry of a confining quantum well geometry is accompanied by a strong perpendicular (z) electric field that leads to spin-orbit coupling of the Rashba type,\textsuperscript{1}

\[ H_R = \lambda z \cdot (\sigma \times \mathbf{p}) = \lambda (\sigma_x p_y - \sigma_y p_z). \] (1.1)
As a result, the spin degeneracy is lifted via effective momentum-dependent Zeeman field, $\mathbf{h}_p = \lambda(-p_y, p_x)$. The electron spin eigenstate is thus determined by its momentum, $\mathbf{p}$.

Another important type of spin-orbit coupling occurs in 2DEG formed by semiconductors with broken inversion symmetry, e.g. GaAs, InAs. While only third order in momentum in the bulk GaAs, this coupling, known as Dresselhaus interaction, is “upgraded” in two dimensions to the linear order by virtue of transverse momentum quantization and becomes,

$$ H_D = \lambda_D (\sigma_x p_x - \sigma_y p_y). \quad (1.2) $$

Chiralities acquired from different spin-orbit couplings (1.1) and (1.2) are opposite, in a sense that the electron wave function acquires opposite Berry phases, $\pm \pi$, upon (adiabatic) completion of a loop in the momentum space (enclosing the degeneracy point $\mathbf{p} = 0$), depending on whether $\alpha > \beta$ or vice versa.

Dirac fermions in graphene represent another realization of the chiral system. Due to hexagonal symmetry of underlying two-dimensional honeycomb atom arrangement the low-energy electron excitations are combined into two Dirac cones (valleys), $K$ and $\bar{K}$, within the first Brillouin zone. The corresponding effective Hamiltonians are

$$ H_K = v (\tau_x p_x + \tau_y p_y), \quad H_{\bar{K}} = v (\tau_x p_x - \tau_y p_y). \quad (1.3) $$

Here $\tau$ stands for the pseudospin operator that acts in the sublattice space.

It is interesting to note the analogy between total graphene Hamiltonian and a semiconductor with equal Rashba and Dresselhaus couplings ($\alpha = \beta$) in the case of a conventional 2DEG. The latter case features spin eigenstates that are momentum-independent and, thus, somewhat trivial. However, since the two cones in graphene are separated in momentum space, the chiral physics can still be observed. (Yet some phenomena are trivially absent, e.g. pseudospin-Hall effect, cf. last section). It is important to emphasize that the chirality in graphene has nothing to do with spin-orbital coupling (which is rather weak in carbon allotropes) and is a consequence of the crystal geometry.

Another notable example of a chiral electron system is a topological insulator. The latter is different from the usual band insulator in that its valence and conduction bands originate not from different atomic orbitals but from the same spin-orbit-split orbital. In 2D HgTe quantum wells this gives rise to topologically protected edge states leading to a recently predicted and discovered quantum spin-Hall effect. Yet even more intriguing...
Spin-Hall Effect in Chiral Electron Systems

Twist has recently been added by a discovery of two-dimensional states on the surface of 3D topological insulators\textsuperscript{7,8} Bi\textsubscript{0.5}Sb\textsubscript{0.1}, Bi\textsubscript{2}Sb\textsubscript{3}, and Bi\textsubscript{2}Te\textsubscript{3}. The fundamental difference from graphene is that the number of pockets of the Fermi surface within the Brillouin zone is odd, with the complementary species residing on the opposite surface of a sample. Thus, these states are not simply separated in the momentum space (like in graphene), but also separated in real space. This removes the above mentioned “trivialization” that is present in graphene. In particular, spin-Hall effect can occur in topological insulators. As confirmed by the first-principle band structure calculations\textsuperscript{9} spin structure of these states is indeed chiral. In particular, the low-energy Hamiltonian of Bi\textsubscript{2}Te\textsubscript{3} can be deduced from symmetry considerations to be of the form,\textsuperscript{46}

\[ H_{TI} = v(\sigma_x p_y - \sigma_y p_x) + \frac{p^2}{2m^*} + \alpha \sigma_z p_x(p_x^2 + 3p_y^2). \] (1.4)

For an ungated and undoped Bi\textsubscript{2}Te\textsubscript{3} the last term is generally of the same order of magnitude as the first two. Still, as a starting point it is useful to neglect the effects of anisotropy.

1.2. Spin-Hall effect

Spin-Hall effect\textsuperscript{10} is the name given to a class of phenomena that exhibit boundary (surface, edge) spin polarization when electric current flows through a system with significant spin-orbit interaction. It has been observed in both 3D\textsuperscript{11–13} and 2D systems.\textsuperscript{14} It is customary to distinguish two mechanisms that could lead to this effect. The extrinsic mechanism is the dominant one in 3D semiconductors and originates from scattering off impurities.\textsuperscript{15–18} Presence of impurities is unavoidable in high carrier density 3D semiconductors that rely on doping. Quite contrary, intrinsic mechanism\textsuperscript{19,20} originates from spin-orbit-split band-structure, which induces spin precession when electric current is driven through the system. This mechanism can in principle be realized in coexistence with ballistic transport in 2D electron systems. Indeed, by placing dopant far enough from GaAs/AlGaAs interface one can reduce effects of disorder scattering.

In the present paper we concentrate on the intrinsic spin-Hall mechanism. It is known, however, in two-dimensional electron systems with spin-orbit coupling linear in momentum (typical for \textit{n}-doped heterostructures) any scattering that leads to a stationary electric current via deceleration of electrons by impurities, phonons, etc., will negate the precession due to
external electric field and result in the exact cancellation\textsuperscript{21–25} of the bulk spin-current in a dc case\textsuperscript{a}.

There are several ways to avoid such cancellation, in particular, to use ac currents with frequencies exceeding the inverse spin relaxation time\textsuperscript{21,26}. The second possibility and the one of interest to us here, is make a system sufficiently small and clean (ballistic) so that the electron mean free time exceeds the time of flight across the systems. The corresponding scenario is known as the \textit{mesoscopic spin-Hall effect}\textsuperscript{27}. While initial theories of spin-Hall effect in infinite systems had addressed the auxiliary quantity of spin current (for a review see Refs.\textsuperscript{28,29}), in a finite geometry it is both easier and more relevant to calculate spin polarization density, which is an experimentally measurable quantity.\textsuperscript{30} Such edge polarization was considered by numerical methods in several earlier publications\textsuperscript{27,31–34} as well as both analytically and numerically in our previous papers.\textsuperscript{35,36}

A crucial note is due. The edge spin polarization in ballistic systems appears not as a result of electric field-driven acceleration of electrons and associated with it precession of spins. Indeed, electric field in a ballistic system is absent as the electric potential drop occurs over a contact region with the leads rather than over a bulk of a ballistic conductor. Nevertheless, spin precession does occur. It originates from accelerated electron motion in the boundary potential that provides lateral confinement. Bias applied to the contacts ensures that populations of left- and right-moving states are different and the net spin precession appears. Naturally, it is proportional to the applied bias $V$. The net spin accumulation near an edge of a 2DEG is \textit{independent} of the shape of the boundary potential,

$$
\int_{-\infty}^{\infty} s_z(x) dx = \frac{\lambda^2 - \lambda^2_{D}}{12\pi^2 v_F^3} eV ,
$$

where $v_F$ is the bulk value of the Fermi velocity\textsuperscript{b}. Spin accumulation appears in the second order in spin-orbit interaction and vanishes for equal Rashba and Dresselhaus coupling strengths.

Position-resolved spin polarization $s_z(x)$ can be found analytically in two important situations. First is the case of a smooth confining potential $U(x)$ and the second is an infinite hard-wall boundary. We begin with

\textsuperscript{a}This exact cancellation does not occur in 3D- or 2D hole-systems that feature non-linear spin-orbit couplings.

\textsuperscript{b}Strictly speaking, the integration in Eq. (1.5) goes from a point far outside the 2DEG ($-\infty$) to some point deep inside 2DEG ($+\infty$), but still far away from the edges. Integration across the whole conductor would give zero, reflecting the fact that spin accumulation at the opposite edges has opposite signs.
analyzing semiclassical electron motion in smooth potentials in Sec. 1.3. In Sec. 1.4 we derive the result (1.5) for the net spin accumulation and illustrate it using the case of hard-wall boundary. In Sec. 1.5 we present a method of kinetic equation that allows to find local spin-polarization for smooth boundary potentials. In Sec. 1.6 singular dynamics near classical turning points is discussed. Finally, in Sec. 1.7 we use methods developed in the preceding chapters to describe spin-Hall effect in topological insulators.

Many of the results presented in this paper were published in the journal articles Refs. [35–37]. However, the derivation of the spin accumulation via the Kinetic equation in Sec. 1.5 and prediction of the nonequilibrium spin accumulation in topological insulators in Sec. 1.7 are presented here for the first time.

1.3. Semiclassical electron motion

Consider gated 2DEG with Rashba spin-orbit interaction described by the Hamiltonian

\[ H = \frac{p^2}{2m} + \lambda(p_y \sigma_x - p_x \sigma_y) + \frac{m \lambda^2}{2} + U(x,y). \]  

Potential \( U(x,y) \) is created by the external gates, or the edge potential ensuring the in-plane confinement of 2DEG. Possible effect of disorder on \( U(x,y) \) are going to be neglected.

Classical electron dynamics described by the short wave length limit of the Hamiltonian (1.6) reveals a number of very unusual features. As a first step, in this section we demonstrate how the trajectories-based approach allows to describe propagation of fully (in-plane) polarized electric currents through mesoscopic constrictions.

Construction of semiclassical solutions of the Schrödinger equation with the Hamiltonian (1.6) follows the reasoning of the conventional WKB approach, which is valid for a smooth potential, \( \hbar |\nabla U| \ll \min(p^3/m, p^2 \lambda) \).

Without the external potential \( U \), the electron spectrum consists of the two subbands, \( E_{\pm}(p_x, p_y) = (p \pm m \lambda)^2/2m \). The subbands meet at only one point, \( p = 0 \), and the spin in each subband is always aligned with one of the in-plane directions perpendicular to the momentum \( \vec{p} \). The semiclassical electron dynamics naturally captures the essential features of this translationally invariant limit. The classical motion in each subband
is determined by the equations of motion which follow from the effective Hamiltonian:

$$H_{\text{eff}} = \frac{(p \pm m\lambda)^2}{2m} + U(x, y). \quad (1.7)$$

Despite the fact that spin does not appear in this equation, one can easily construct semiclassical wave functions, which have spin pointed within the $xy$ plane perpendicular to the momentum:

$$\psi = ue^{iS/\hbar}, \quad u = \sqrt{\frac{\rho}{2p}} \left( \pm \frac{\sqrt{p_y + ip_x}}{\sqrt{p_y - ip_x}} \right). \quad (1.8)$$

Here the action $S$ is related to the momentum by $\vec{p} = \nabla S$, and $\rho = u^\dagger u$ is the classical density for a family of classical trajectories corresponding to a given energy $E$. The action $S$ obeys the classical Hamilton-Jacobi equation,

$$\frac{(|\nabla S| \pm m\lambda)^2}{2m} + U(x, y) = E. \quad (1.9)$$

During its motion, an electron described by Eq. (1.8) changes the momentum $p$ but always remains in the same spin-subband. To change the subband the electron trajectory should pass through the degeneracy point where both components of momentum vanish simultaneously, $\vec{p} = 0$, which is generically impossible. Moreover, with the proper use of potential barriers, one may realize a situation where electrons of only one subband are transmitted and the others are totally reflected. This leads to strong in-plane polarization of the transmitted electron flow.

To take into account the out-of-plane spin precession one has to go beyond the approximation Eq. (1.8), as was done in Ref. [37]. Instead of doing so we will describe in Sec. 1.4 a method allowing to find easily the expectation value of $\sigma_z$ for potential depending only on one coordinate $U = U(x)$ (boundary potential).

1.3.1. Sharvin conductance

To give an example of a spin-polarized current let us consider transmission through a barrier, $U(x)$, varying along the direction of a current propagation. We assume periodic boundary conditions in the perpendicular direction ($y + L \equiv y$), thus $p_y$ is an integral of motion\(^4\). For a smooth

---

\(^4\)Spin polarized currents on a cylinder with $x$-dependent spin-orbit interaction were considered recently in Ref. [42]
potential \( U(x) \) the conduction channels may either be perfectly transmitting or completely closed. The conserved transverse momentum takes the quantized values, \( p^0_n = 2\pi \hbar n/L \). Consider the functions

\[
E^\pm_n(p_x) = \frac{(p^0_n \pm m\lambda)^2}{2m}, \quad p^n = \sqrt{p_x^2 + p_y^2}
\]

(1.10)

For \( n \neq 0 \) the function \( E^\pm_n(p_x) \) splits into two distinct branches. At any point \( x \) the equation

\[
E^\pm_n(p_x) = E_F - U(x)
\]

(1.11)

yields solutions \( p^L_x \) and \( p^R_x \), corresponding to left- and right-moving electrons. Application of a small bias implies, e.g., the excess of right movers over left movers far to the left from the barrier. Particles are transmitted freely above the barrier if Eq. (1.11) has a solution, \( p^R_x \), for any \( x \). Let \( \mu = E_F - U_{\text{max}} \) be the difference between the Fermi energy and the maximum of the potential. The \( n \)th channel in the upper branch opens when

\[
\mu = (2\pi \hbar |n| + m\lambda L)^2 / 2mL^2.
\]

(1.12)

For the lower branch \( E^n_2(p_x) \) Eq. (1.11) has four solutions (two for right and two for left movers) for \( |n| < m\lambda L/2\pi \hbar \) and \( x \) close to the top of the barrier. However, far from the barrier (where the excess of right-movers is created) there are still only two crossings described by Eq. (1.11), one for right and
F. Author and S. Author

one for left movers. As a result, all the extra electrons injected at \( x = -\infty \) follow the evolution of a solution of Eq. (1.11) with the largest positive \( p_x \).

For all \( |n| < m\lambda L/2\pi\hbar \) such a solution does exist for any positive \( \mu \). Thus, at \( \mu = 0 \) as many as \( n_0 = m\lambda L/\pi\hbar \) channels open up simultaneously. The channels with higher values \( |n| > m\lambda L/2\pi\hbar \) in the lower subband \( E^- \) open when

\[
\mu = (2\pi\hbar|n| - m\lambda L)^2 / 2mL^2.
\]  

(1.13)

According to the Landauer formula, ballistic conductance is given by the total number of open channels multiplied by the conductance quantum \( G_0 = e^2/h \)

\[
G = G_0 \frac{L}{\pi\hbar} \left\{ \begin{array}{ll}
\sqrt{2m\mu} + m\lambda, & 0 < \mu < m\lambda^2/2 \\
2\sqrt{2m\mu}, & \mu > m\lambda^2/2
\end{array} \right.
\]  

(1.14)

This dependence \( G(\mu) \) is shown in Fig. 1.1. The striking evidence of the presence of spin-orbit interaction is the huge jump of the conductance at the pinch-off point, as opposed to the conventional square-root increase in the absence of spin-orbit coupling. This jump is a consequence of the “Mexican-hat” shape of the spectrum \( E^- (p_x, p_y) \). Accuracy of Eqs. (1.12) and (1.13) is sufficient to resolve the steps in the conductance due to the discrete values of \( |n| = 0, 1, 2, \ldots \), (conductance quantization), as shown in Fig. 1.1. The steps in \( G(\mu) \) are abrupt in the limit \( dU/dx \to 0 \).

Close to the pinch-off, at \( \mu \lesssim m\lambda^2 \), the conserved \( p_y \) component of the electronic momentum varies for different transmitted channels within the range \( |p_y| \lesssim m\lambda \). Therefore, far from the barrier, where the Fermi momentum is large \( p_F \gg m\lambda \), we have \( p_x \gg p_y \) and transmitted electrons propagate in a very narrow angle interval \( |\theta| < \sqrt{m\lambda^2/2E_F} \ll 1 \). Since the electron spin is perpendicular to its momentum, we conclude that the current due to electrons from each of the subbands is almost fully polarized. The total polarization of the transmitted current is given by the difference of two currents

\[
\langle \sigma_y \rangle = \frac{\langle \psi^\dagger \sigma_y v_x \psi \rangle}{\langle \psi^\dagger v_x \psi \rangle} = \min(1, \sqrt{m\lambda^2/2\mu}),
\]  

(1.15)

which is also depicted in Fig. 1.1. This current polarization may also be viewed as a creation of in-plain nonequilibrium spin density, maximal on the barrier.

Derivation of Eqs. (1.14) and (1.15) was greatly simplified because of the periodic boundaries. Our next example demonstrates semiclassical treatment of realistic boundary conditions.
1.3.2. Quantum Point Contact

Let us consider probably the most experimentally relevant example of a quantum point contact, described by the potential

$$U(x, y) = -\frac{m\Omega^2 x^2}{2} + \frac{m\omega^2 y^2}{2}. \tag{1.16}$$

Even in this simple model the electron flow in the presence of spin-orbit interaction acquires a number of interesting and peculiar features. This become clear already from the figure 1.2 where we show three types of trajectories in such potential. Each kind of trajectories is necessary for calculation of conductance.

Classical equations of motion follow in the usual manner from the effective Hamiltonian (1.7): 

$$\dot{\vec{r}} = \frac{\partial H_{\text{eff}}}{\partial \vec{p}}, \quad \dot{\vec{p}} = -\frac{\partial H_{\text{eff}}}{\partial \vec{r}}.$$ We consider quantum point contact (QPC) close to the opening with only the lower $E_-$ subband contributing to the conductance. A crucial property of the Hamiltonian $H_{\text{eff}}$, Eq. (1.7), is the existence of a circle of minima of the kinetic energy at $|p| = m\lambda$. Expanding around a point on this circle, $p_x = m\lambda \cos \alpha, p_y = m\lambda \sin \alpha$, one readily finds the equations of motion for $P = p_x \cos \alpha + p_y \sin \alpha - m\lambda \ll m\lambda$,

$$\ddot{P} + (-\Omega^2 \cos^2 \alpha + \omega^2 \sin^2 \alpha)P = 0, \quad \dot{\alpha} = 0. \tag{1.17}$$

The trajectory is found from the relations, $\dot{x} = P \cos \alpha/m, \quad \dot{y} = P \sin \alpha/m$. We observe from Eq. (1.17) that only the trajectories within the angle

$$\tan |\alpha| < \tan \alpha_0 = \Omega/\omega \tag{1.18}$$

are transmitted through QPC. Trajectories with larger angles are trapped (oscillate) within the point contact. Examples of both types of trajectories are presented in Fig. 1.2. Quantization of trapped trajectories would give rise to a set of (extremely) narrow resonances in the conductance, specific for spin-orbit interaction. Below we consider only the smoothed conductance.

To calculate the current $J$ through QPC one has to integrate over the phase space of the states which are transmitted from left to right,

$$J = \int dy \int e v_x \frac{d^2 p}{(2\pi\hbar)^2} = GV, \tag{1.19}$$

and have the energy within the interval $\mu - eV/2 < E_- < \mu + eV/2$, with $V$ standing for the applied voltage. In this section we define $\mu$ as the difference between the Fermi energy and the value of the potential at the saddle point $\mu = E_F - U(0, 0)$. The integral is most simply evaluated at $x = 0$ (with
Fig. 1.2. Three kinds of trajectories in the point contact. a, transmitted trajectory whose momentum is always collinear with the velocity. b, trajectory bouncing inside the QPC. This trajectory is periodic in the linearized approximation described in the text, while the exact calculation for finite amplitude shows its slow drift. c, transmitted trajectory whose momentum inside the contact is opposite to the velocity. Electrons flow from left to right. Arrows show momentum and spin orientations. Few equipotential lines are also shown.

The velocity given by $v_x = \mathcal{P} \cos \alpha / m$). The allowed absolute values of the momentum are

$$2\mu - eV - m\omega^2 y^2 < \mathcal{P}^2/m < 2\mu + eV - m\omega^2 y^2. \tag{1.20}$$

The angle interval of transmitting trajectories consists of two domains: $|\alpha| < \alpha_0$, $\mathcal{P} > 0$, and $|\alpha - \pi| < \alpha_0$, $\mathcal{P} < 0$. The appearance of the latter range of integration is highly non-trivial. A simple reasoning shows that the particles with the velocity antiparallel to the momentum ($v_x > 0$, $p_x < 0$) should not contribute to the conduction in the case of a transition through a one-dimensional barrier $U = U(x)$, see Eq. (1.14). Despite corresponding to the right-moving electrons, these states do not originate in the left lead. Indeed, they exist only in the vicinity of $x = 0$, but disappear as $x \to -\infty$ and, thus, cannot be populated by the excess electrons (except due to the tunneling transitions which are irrelevant in the semiclassical regime). Such trajectories, however, do exist in QPC, Eq. (1.16), as demonstrated in Fig. 1.2. After passing through QPC the trajectory bounces at the wall reversing its velocity. This kind of classical turning points, where both components of the velocity vanish simultaneously, are specific for the effective Hamiltonian (1.17). The existence of transmitting trajectories with $|\alpha - \pi| < \alpha_0$, $q < 0$ results in the doubling of the conductance. Simple
calculation yields

\[ G = G_0 \frac{4m\lambda \sin \alpha_0}{\pi \hbar} \sqrt{\frac{2\mu}{m}}. \] (1.21)

The presence of a threshold angle \( \alpha_0 \), as well as the square-root dependence of \( G(\mu) \), are in a sharp contrast to the well-known result \( G = G_0 \mu / \pi \hbar \omega \), in the absence of spin-orbit interaction.

Since Eq. (1.17) describes only the linearized electron dynamics, Eq. (1.21) is formally valid if \( \mu \ll m\lambda^2 \). Nevertheless, the current remains totally polarized for \( 0 < \mu < m\lambda^2/2 \) [similar to Eq. (1.15)]

\[ \langle \sigma_y \rangle = \frac{\langle \psi^\dagger \sigma_y v_x \psi \rangle}{\langle \psi^\dagger v_x \psi \rangle} = 1. \] (1.22)

With increasing the chemical potential, \( \mu > m\lambda^2/2 \), transmission via the upper subband \( E_+ \) kicks in and the degree of polarization gradually decreases, like it happened in Eq. (1.15).

In InAs-based heterostructures, typical value of spin-orbit coupling is \( \lambda \hbar = 2 \times 10^{-11} eV m \). Characteristic spin-orbit length \( l_R = \hbar/m^*\lambda = 100 \text{ nm} \) and energy \( m^*\lambda^2/2 = 0.1 \text{ meV} \). In order to have strongly spin-polarizing QPC, the latter should support many transmitting channels at chemical potential \( \mu \sim m^*\lambda^2/2 \gg \hbar \omega \). This condition can, equivalently, be written in terms of the width of the point contact \( \Delta y \), see Eq. (1.16), as \( \Delta y \gg l_R \).

### 1.4. Edge spin accumulation. Hard wall

Consider a semi-infinite ballistic 2DEG described by the Hamiltonian Eq. (1.6), where potential depending only on one coordinate, \( U \equiv U(x) \), ensures boundary confinement (see Fig. 1.3). The system is attached to two ideal reflectionless leads injecting equilibrium electrons into 2DEG. The chemical potentials of the leads are shifted by the applied voltage, \( eV \). The current flow along the \( y \)-direction in case of spin-orbit interaction Eq. (1.6) results in an edge spin accumulation \( s_z(x) \). In this section we first derive an exact formula for the total amount of spin accumulated at the edge, \( \int s_z dx \). Then we consider in more details the case of hard wall potential, \( U(x < 0) = \infty, U(x > 0) = 0 \), where exact results for \( s_z(x) \) are available\(^4\).

\(^4\)Still the number of open channels should be large for semiclassics.

\(^5\)Nonequilibrium edge spin accumulation at the hard wall boundary of disordered conductor was considered in Ref. [44]
1.4.1. **Net spin accumulation**

For the potential independent on \( y \) \((U = U(x))\) the corresponding momentum component, \( k_y \), is an integral of motion. It is convenient to use the Fourier representation along the \( y \)-axis for the electron operators, \( \hat{\psi}(r) = \sum_{k_y} \hat{c}_{k_y}(x)e^{ik_y y} \). We employ here the second quantization formalism. One can derive the equation of motion for the expectation value of the electron spin operator, \( s(k_y, x) = \frac{1}{2} \langle \hat{c}^\dagger_{k_y}(x)\hat{\sigma}_y\hat{c}_{k_y}(x) \rangle \), which can be readily written in the form,

\[
\partial_t s_y(k_y, x) = -\partial_x J^y_x(k_y, x) - 2\lambda k_y s_z(k_y, x). \tag{1.23}
\]

Here \( J^y_x \) stands for the conventional operator of spin-current, i.e.,

\[
J^y_x(k_y, x) = \frac{i}{4m}(\nabla_x \hat{c}^\dagger_{k_y} \hat{\sigma}_y \hat{c}_{k_y} - \hat{c}^\dagger_{k_y} \hat{\sigma}_y \nabla_x \hat{c}_{k_y}) - \frac{\lambda}{2} \langle \hat{c}^\dagger_{k_y} \hat{\sigma}_y \hat{c}_{k_y} \rangle.
\]

In a steady state the lhs of Eq. \( (1.23) \) vanishes. Integrating Eq. \( (1.23) \) over the \( x \)-direction, we obtain for the net spin polarization,

\[
\int_{-\infty}^{\infty} s_z(x) dx = -\frac{1}{2\lambda} \sum_{k_y} \frac{1}{k_y} J^y_x(k_y, \infty). \tag{1.24}
\]
It is straightforward to calculate the value of the \((k_y\text{-resolved})\) spin current \(J^y_x(k_y, \infty)\) inside the bulk of a 2D system:

\[
J^y_x(k_y, \infty) = -\frac{1}{2} \sum_{\beta = \pm 1} k_x \left( \lambda + \frac{\beta k_x^2}{mk} \right) n_\beta(k_x, k_y),
\]

where \(n_\beta(k_x, k_y)\) stands for the population of different momentum states in the subband \(\beta\). Only "uncompensated" states contribute to the non-equilibrium spin polarization given by Eqs. (1.24-1.25); these states describe electrons that originate in the left lead \((k_y > 0)\) and belong to the energy interval near the Fermi energy, \(E_F < \frac{(k+\beta\lambda)^2}{2m} < E_F + eV\). The integral (1.24) diverges logarithmically at \(k_y \to 0\). Assuming the same infrared cutoff in both subbands, \(\tilde{k}\), we observe that the diverging \(\ln \tilde{k}\)-contributions in the two subbands cancel each other, yielding in the linear \((in \ V)\) response,

\[
\int_{-\infty}^{\infty} s_z dx = \frac{eV}{2\lambda(2\pi)^2} \left( \frac{2\lambda}{v_F} - \ln \frac{v_F + \lambda}{v_F - \lambda} \right)
\]

(1.26)

where \(v_F = \sqrt{2E_F/m}\) is the Fermi velocity. Expanding this general result to the lowest non-vanishing order in \(\lambda/v_F\) we recover the net boundary polarization, Eq. (1.5).

1.4.2. Evanescent modes

We are now going to consider the edge spin density in the case of sharp (hard wall) edge potential. Since the sharp edge does not impose any finite length scale, the question arise, what would be the width of the edge spin distribution? Obvious candidate for that comes from the evanescent modes,\(^{33}\) whose wave function do have an explicitly decaying component \((\sim \exp(-m\sqrt{v_F}Ax))\).

The reflection at the sharp boundary mixes the two bulk subbands. Evanescent contributions in the upper subband appear when the reflecting states from the lower subband belong to the domain, \(m(v_F - \lambda) < k_y < m(v_F + \lambda)\). Repeating the calculations leading to Eq. (1.26), but now for the evanescent domain only, we obtain,

\[
\int_{-\infty}^{\infty} s^e_z dx = \frac{eV}{2\lambda(2\pi)^2} \left( \frac{\lambda}{v_F} - \ln \frac{1 + \sqrt{\lambda/v_F}}{1 - \sqrt{\lambda/v_F}} \right).
\]

(1.27)

Remarkably, the net evanescent contribution turns out to be much larger than the full result Eq. (1.26). This means that this contribution is largely cancelled by the contribution from the normal domain \(k_y < m(v_F - \lambda)\).
Similar cancellation of all smooth long wavelength contributions takes place for the local spin density, as we discuss in the next section.

Fig. 1.4. Dependence of the local spin polarization, in units of $eV m / 8 \pi^2$, on the distance to the boundary for different values of spin-orbit coupling constant. Solid (red) line: $\lambda/v_F = 0.1$, dotted (blue) line: $\lambda/v_F = 0.2$, solid (black) line utilizes the approximate formula (1.30) for $\lambda/v_F = 0.2$.

1.4.3. Hard wall, results

Exact lengthy explicit expression for $s_z(x)$ in the case of sharp boundary potential, Fig. 1.4, was given in Ref. [35]. Here we show only the spectral density

$$ s_z(q) = 2 \int_0^\infty dx s_z(x) \sin qx, \quad (1.28) $$

which is given by a simple piecewise continuous algebraic function defined in four domains. The surprising feature of the spectral distribution is its vanishing, $s_z(q) = 0$, in the whole long-wavelength domain, $0 < q < 2k^+$. In particular, this shows the exact cancelation between normal and evanescent modes. For larger values of $q$ we obtain to the leading order in $\lambda$,

$$ s_z(q) = \frac{eV q}{16 \pi m v_F^2} \left\{ \begin{array}{ll}
0, & q < 2m(v_F - \lambda), \\
-1, & 2m(v_F - \lambda) < q < 2mv_F, \\
1, & 2mv_F < q < 2m(v_F + \lambda), \\
-2/(q\xi)^4, & 2m(v_F + \lambda) < q.
\end{array} \right. \quad (1.29) $$

The plot of the spectral distribution is illustrated in Fig. 1.5. Remarkably, the net spin polarization (given by $\pi^{-1} \int dqs_z(q)/q$) comes from the large-$q$ tail ($\propto q^{-3}$) in the spectral density $s_z(q)$.
The approximate spin density may be written in a simple form ($\hbar = 1$),

$$s_z(x) \approx \frac{eV}{2\pi^2 v_F x} \cos(2mv_F x) \sin^2(m\lambda x).$$

(1.30)

It is remarkable that the spin-orbit coupling constant enters via the period of beating only.

As is evident from Eq. (1.29) one should speak about the spin accumulation at the hard wall with certain caution. The spin density in this case comes from quickly oscillating functions and the notion of spin accumulation should be understood in the same mathematical sense as the finite value of the integral

$$\int_0^\infty dx \sin x = \lim_{\eta \to 0} \int_0^\infty dx \ e^{-\eta x} \sin x = 1.$$  

(1.31)

Fig. 1.5. Spectral distribution (1.29) of spin density in units of $eV/4\pi v_F$ for different values of spin-orbit coupling constant, $\lambda/v_F = 0.1$, and $\lambda/v_F = 0.2$.

1.5. Smooth edge: Kinetic equation

The second ballistic spin-Hall problem that allows analytical solution involves a smooth boundary potential, the exact condition to be presented below. Away from classical turning points (see the next section) spin accumulation can be obtained with the help of the matrix kinetic equation. In addition this method allows to find a complete electron distribution along the way. In the semiclassical approximation the $2 \times 2$ matrix $\tilde{f}_p(x)$ generalizes the usual distribution function. The equation for the matrix
distribution function in the presence of a smooth confining potential \( U(x) \) and in the absence of disorder takes the form:

\[
\frac{p_x}{m} \partial_x \hat{f}_p - \lambda \left\{ \hat{s}_y, \partial_x \hat{f}_p \right\} + 2i\lambda \left[ p_y \hat{s}_x - p_x \hat{s}_y, \hat{f}_p \right] = \frac{\partial \hat{f}_p}{\partial p_x} \partial_x U(x). \tag{1.32}
\]

To the zeroth order in the potential gradient \( \partial_x U \) the solution is trivial and reduces to the equilibrium Fermi-Dirac distributions for the two spin-split subbands \((\beta = \pm 1)\),

\[
f_\beta(x) = n_F \left( \frac{1}{2m} \left( p + \beta \lambda \right)^2 + U(x) - \mu \right),
\]

with the local value of the potential \( U(x) \) determining the elevation of the bottom of the subbands. To this order the electron spin remains adiabatically within the plane of 2DEG during the course of electron motion in the potential \( U(x) \).

The out-of-plane spin component arises from the non-adiabatic corrections that are first order in \( \partial_x U \). The solution of kinetic equation (1.32) is rather straightforward and yields,

\[
\hat{f}_p = \frac{1}{2} \sum_\beta \left[ 1 + 2\beta(n_y \hat{s}_x - n_x \hat{s}_y) \right] f_\beta + \frac{\lambda n_y \hat{s}_z}{2p^2} \partial_x U \sum_\beta \frac{\partial (f_\beta / \lambda)}{\partial \lambda}, \tag{1.33}
\]

where \( \vec{n} \) is the direction of the electron momentum. The local value of spin-Hall density \( s_z(x) = \text{Tr} \hat{s}_z \sum_p \hat{f}_p \) is obtained by integrating Eq. (1.33) over excess electron states that originate in the left lead, namely over those with \( v_y > 0 \) and the energy within the interval \([E_F, E_F + eV]\),

\[
s_z(x) = \frac{dU}{dx} \frac{m\lambda}{2(2\pi)^2} \frac{\partial}{\partial \lambda} \int \frac{dE}{\sqrt{2m[E - U(x)]}} \left( \frac{1}{p^+} - \frac{1}{p^-} \right) \left[ n_F(E - eV) - n_F(E) \right]. \tag{1.34}
\]

Here \( p^\pm \) are the momenta corresponding to a given energy: \( p^\pm(x) = \sqrt{2m[E - U(x)]} \mp m\lambda \) when both subbands are occupied, \( \sqrt{2m[E - U(x)]} < m\lambda \); and \( p^\pm = m\lambda \mp \sqrt{2m[E - U(x)]} \) when the upper subband is empty, \( \sqrt{2m[E - U(x)]} > m\lambda \). Calculation of this integral in the linear order in \( eV \) yields

\[
s_z(x) = -\frac{\lambda^2 eV}{(2\pi)^2 m(v_F^2 - \lambda^2)v_F^2} \frac{dU}{dx}. \tag{1.35}
\]

The expression (1.35) has a number of interesting features. The local spin polarization is proportional to the force exerted on electrons by the boundary. As long as the Fermi energy is well above the bottom of the bands, \( v_F \gg \lambda \), spin accumulation is small and only second order in the spin-orbit coupling constant. When the bottom of the band is elevated high enough, \( v_F < \lambda \), the local spin polarization increases dramatically. The vicinities of the two singularities at \( v_F(x) = 0 \) (classical turning point)
and \(v_F(x) = \lambda\) (degeneracy point) have to be studied by means beyond
kinetic equation (1.32).

In addition to a smooth classical spin distribution (1.35) quantum wiggles in \(s_z(x)\) are present whose magnitude is not necessarily small compared to \(\bar{s}_z(x)\). A convenient quantity (especially for numerical calculations) that averages out these wiggles is the "integrated spin density"

\[
\bar{S}(x) = \sum_j \int_{-\infty}^{x} s_z(x') dx'.
\]  (1.36)

Integrating Eq. (1.35) over \(x\) yields smooth part of the integrated spin density

\[
\bar{S}(x) = \frac{eV}{2\lambda(2\pi)^2} \left( \frac{2\lambda}{v_F(x)} - \ln \frac{v_F(x) + \lambda}{|v_F(x) - \lambda|} \right). \]  (1.37)

This formula presents a generalization of Eq. (1.26) for the case of a smooth variation of the confining potential.

Note that the net spin polarization across the edge is again independent of the shape of the boundary potential, \(S(\infty) = -\lambda^2 eV/12\pi^2 v_F^3\), and is expressed via the bulk value of the Fermi velocity \(v_F(\infty)\). Here by \(x = \infty\) we assume a point deep inside the 2DEG but yet far from its opposite edge. The latter has spin accumulation of the same absolute value and opposite sign.

1.6. Smooth edge: singular spin dynamics near turning points

Semiclassical Eqs. (1.35,1.37) do not offer the important information about the edge spin. First, these equations predict a singular spin density at \(v_F(x) = 0\) and \(v_F(x) = \lambda\). In addition, Eqs. (1.35,1.37) ignore any interference effects, which may be important for realistic boundary potentials. Both these problems may be addressed analytically, as it was done in Ref. [36]. In order to have more pedagogical discussion here we concentrate mostly on numerical results. Let us approximate the boundary by a linear potential

\[
U(x) = -Fx,
\]  (1.38)

with the constant force \(F\).

Figure 1.6 shows the "raw" numerical data for \(s_z(x)\) and \(\bar{S}(x)\). The smoothness of the boundary implies that \(F \ll m^2\lambda^3/\hbar\). We see from
Fig. 1.6. a). Spin density for a force strength \( F = 0.01m^2\lambda^3/\hbar \) \((U = -Fx)\). [The curve offset both vertically and horizontally.] All coordinates are measured in units of \( \hbar/m\lambda \). A smooth component of the density is hardly visible because of oscillating contributions. b). The smooth component is recovered in the integrated spin \( \mathcal{S}(x) = \int_{-\infty}^{x} s_z dx' \). Blue, green and red curves show the integrated spin for the force strength \( F = \alpha m^2\lambda^3/\hbar \) with \( \alpha = 0.25, 0.05, 0.01 \), respectively. In all these cases we see three regions with different spin behavior. First, the spin density is the largest in narrow outer \((x \approx 0)\) strip along the edge. This spin is compensated (and overcompensated) by the wide strip of negative smooth spin density. Finally, in the third strip (at \( x > 50 \) for the red curve \( \alpha = 0.01 \)) the smooth component of the density changes sign to positive again. The width of all three strips and the amount of accumulated spin, which in each strip is much larger than in Eq. (1.5), increase [formally unlimited] with decreasing slope of the boundary potential. To obtain the values of the spin density and accumulated spin one need to multiply the numbers in the figure by \( eVm/8\pi^2 \) and \( eV/8\lambda\pi^2 \) respectively.

Fig. 1.6 a, how the rapid quantum oscillations make it hard to observe the mean value \( \bar{s}_z(x) \) Eq. (1.35) even for \( F\hbar/m^2\lambda^3 = 0.01 \). The smooth component is recovered in the integrated spin on Fig. 1.6 b even for relatively steep boundary \( F\hbar/m^2\lambda^3 = 0.25 \).

Explicit comparison between numerics and analytical expression Eq. (1.37) is made on Fig. 1.7, where we plot the rescaled numerical results for different values of the slope of the boundary potential.

According to Eq. (1.37) we find two regions of different smooth spin behavior. First, within the strip where \( 0 < v_F(x) < \lambda \) spin density is negative (which is seen in a downward slope of the integrated density \( \mathcal{S}(x) \) in Fig. 1.7). Farther away, \( s_z(x) \) changes sign for \( v_F(x) > \lambda \), where both \( s_z(x) \) and \( \mathcal{S}(x) \) decrease gradually with increasing \( x \).
Fig. 1.7. Integrated spin density \( S(x) = \int_{-\infty}^{x} s_z dx \) for the potential \( U(x) = -\alpha m^2 \lambda^2 x/\hbar \) in units of \( eV/8\lambda^2 \). The curves for \( \alpha = 8, 4, 2, 1 \times 10^{-3} \) are shown in yellow, green, blue and red respectively. The horizontal coordinate is scaled differently for different curves, as \( x \) is measured in units of \( 10^3 \alpha \times \hbar / m \lambda \). Narrow black lines stand for the semiclassical prediction, Eq. (1.26). The logarithmic behavior, \( \sim \log \alpha \), of the dip at \( U(x) = -m\lambda^2/2 \) \( (x = 500) \) is clearly seen. Inset magnifies the region near the edge of 2DEG \( (x \approx 0) \).

The most interesting is the behavior of spin at the borders of these regions, \( v_F = 0 \) and \( v_F = \lambda \). At \( v_F(x) = \lambda \) the accumulated spin \( \mathfrak{S}(x) \) Eq. (1.37) diverges logarithmically. This singularity originates from the accumulation of classical turning points taking place when the conical crossing point in the spectrum of the Hamiltonian (1.6), see Fig. 1.3C, passes through the Fermi energy. This singularity is regularized as \( S \sim \log F \), according to Fig. 1.7.

Yet more peculiar is the behavior of both \( s_z(x) \) and \( S(x) \) at the edge of 2DEG, near the point where \( v_F(x) = 0 \). The smooth part of the accumulated spin, Eq. (1.37), has an infinite jump here (from Eq. (1.37) it follows that \( \mathfrak{S}(+0) = \infty \), while obviously \( \mathfrak{S}(-0) = 0 \)). Development of such jump with decreasing slope of the potential is seen in the inset in Fig. 1.7. The jump in \( S(x) \) corresponds to the formation of a narrow strip with extremely large values of spin \( s_z > 0 \) along the border. This behavior will now be analyzed in more detail.

Classical dynamics of electrons with Rashba spin-orbit interaction is
described by the effective Hamilton Eq. (1.7) with the boundary potential approximated by the linear function $U = -Fx$. The family of classical trajectories generated by this Hamiltonian, shown in Fig. 1.8, demonstrate a number of unusual features.

As seen from Fig. 1.8, those electrons from the lower subband that have $|p_y| < m\lambda$, pass three turning points in the course of their motion in the $x$ direction, corresponding to three solutions of the equation $\partial H_{\text{eff}}/\partial p_x = 0$. Two of these turning points (those with $p = m\lambda$) correspond to simultaneous vanishing of both velocity components, $\vec{v}(x) = 0$, the behavior generically impossible in a 2DEG with the parabolic spectrum, $H = p^2/2m$.

![Fig. 1.8](image)

**Fig. 1.8.** *Bottom:* Family of classical trajectories at $E = E_F$ for different values of $p_y$, and $m = \lambda = F$. Trajectories for both lower (red) and upper (green) spin-orbit split subbands are shown. An example of a trajectory contributing to the peak in spin density at $x \approx 0$, Eq. (1.30), is shown for $|p_x| \ll m\lambda$ (see the text). *Top:* The electron density $\rho = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2$ for given longitudinal momentum and energy. Here $\psi_{1,2}$ are two eigenfunctions with $p_y = 0.2m\lambda, E_F = 0, F = 0.003m^2\lambda^3/\hbar, \rho$ in arbitrary units, $x$ in units of $\hbar/m\lambda$. Three classical turning points can be seen. The interference of incoming and reflected waves in the upper subband causes smooth oscillations to the right of the inner turning point I ($x > m\lambda^2/2F$). At the other turning points, II and III, the two kinds of oscillations are seen. Slow oscillations are caused by the interference of the incoming wave and the wave reflected at the turning point. Fast oscillations (wavelength $\sim \hbar/m\lambda$) are due to the interference of distant (in time) segments of the same trajectory. *Inset:* Kinetic energy (arbitrary units) $T_\pm(p_x) = (p \pm m\lambda)^2/2m$ for fixed $p_y = 0.2m\lambda$. 
Spin-Hall Effect in Chiral Electron Systems

Analytical treatment\(^{36}\) of the expectation value of the \(z\)-component of electron spin in the vicinity of turning point \(v_F(x) = 0\) yields \((x \ll m\lambda^2/F)\)

\[
s_z = \frac{3meV}{4\pi\hbar} \frac{\partial}{\partial \bar{x}} \int_{0}^{1} dz \ \text{Ai}^2(-\bar{x}/z),
\]

where \(\bar{x} = x(2Fm/\hbar^2)^{1/3}\). In the asymptotic region \(x \gg (2Fm/\hbar^2)^{-1/3}\) one can average over the oscillations of the Airy function. This allows us to recover the singular behavior of the smooth spin density \((1.35)\):

\[
\langle s_z \rangle \sim x^{-3/2}.
\]

With the logarithmic accuracy the (properly regularized) height of the peak of spin density is\(^{36}\)

\[
s_z(0) = \frac{meV}{10\sqrt{3\pi}^2\hbar} \ln \left( \frac{m^2\lambda^3}{\hbar F} \right).
\]

Striking feature of this result is that this maximal value is virtually independent of the strength of spin-orbit coupling or the shape of the boundary potential (up to a weak logarithmic factor).

We thus conclude that the nonequilibrium spin-Hall spin accumulation near a smooth boundary of 2DEG ballistic conductor with spin-orbit interaction develops a narrow peak at the edge, with the width \(\sim (\hbar^2/mF)^{1/3}\) and height given by Eq. \((1.40)\). It is followed by a slow non-monotonic decay, as shown on Fig. 1.7. This smooth tail of spin density persists to much larger distances, \(\gtrsim m\lambda^2/F\). The amount of spin accumulated in the peak (found as a maximum of the function \(S(x) = \int_{x}^{\infty} s_z(dx)\)) equals \(S_{\text{max}} \approx 0.04eV(m^2/\hbar F)^{1/3} > 0.04eV/\lambda\), where in the last inequality we utilize the fact that \(F < m^2\lambda^3/\hbar\). We see that the spin accumulated at the edge described by a semiclassical boundary potential is inversely proportional to the strength of spin-orbit interaction and becomes progressively larger for smoother slopes. This prediction can be used for experimental observation of spin-Hall effect in realistic two-dimensional electron systems.

1.7. Spin-Hall effect on surfaces of topological insulators

The method developed in Sections 1.4 and 1.5 is fully applicable to other chiral systems mentioned in the Introduction, as long as the corresponding terms in the Hamiltonians are linear in electron momentum. This is a good approximation in graphene and an acceptable one in Bi\(_2\)Te\(_3\). In graphene, however, chiral structure arises from pseudospin (in sublattice
space). There is nothing straightforward about experimental detection of pseudospin in graphene. In addition, as explained in the Introduction, the case of graphene is mapped onto “Rashba-like” chirality for one cone within the first Brillouin zone and “Dresselhaus-like” chirality for the other one. As a result, the net pseudospin accumulation vanishes when contributions from the two cones are added together, cf. Eq. (1.5).

The situation changes dramatically for topological insulators. First, the chirality originates from true spin, accessible by standard experimental techniques. Second, there is an odd number of Fermi valleys per (geometric) surface. This is in a sharp contrast to graphene which has two Dirac cones. Note that despite the equal number of “species” with opposite chiralities residing on the opposite surface, one can probe them individually, by virtue of their spatial separation.

Since the Fermi energy in topological insulators lies within the bandgap for 3D bulk electronic states, electron transport occurs only across the crystal surface. While not yet implemented in practice to our knowledge, it makes possible local control (via gate electrodes) of spatial distribution of electron density. We envisage the following experimental setup, see Fig. 1.9. Metallic contacts are attached to the surface of topological insulator and drive dc electric current along $y$-direction. Gates are positioned (without direct contact) some distance above the surface. Application of electric potentials to the gates modulates position of electronic bands. Here, similarly to the rest of the paper, we assume that the topological insulator is disorder-free (ballistic). At the present time it is unclear how well this assumption is satisfied in contemporary samples, but undoubtedly sample quality is only going to improve in the near future.

Effective low-energy Hamiltonian for Bi$_2$Te$_3$ (neglecting cubic terms)

$$H = -i v (\sigma_x \partial_y - \sigma_y \partial_x) + U(x),$$

(1.41)

has a single Dirac point whose elevation is determined by the gate potential $U(x)$. There are two cases of interest here:

i) Potential $U(x)$ is not very strong, so that the Dirac point always lies below the Fermi-energy (which is also normal situation for ungated material).

ii) Potential is strong enough to lift the Dirac point above the Fermi level in some region on the surface, thus forming a $p$-$n$ junction.

We now present our general method before analyzing these two cases separately. The method extends the approach of Sec. 1.4 for the Hamiltonian (1.41) and we present it here in detailed form for the reader’s convenience. Utilizing the fact that in a ballistic system $k_y$ is the integral of
motion, we write the electron operators in the mixed representation,
\[ \hat{\psi} (\vec{r}) = \sum_{k_y} \hat{c}_{k_y} (x) e^{i k_y y}. \] (1.42)

Similarly, spin density in the mixed representation reads
\[ \vec{s} (k_y, x) = \langle \hat{c}_{k_y}^\dagger (x) \vec{\sigma} \hat{c}_{k_y} (x) \rangle \] (1.43)

and satisfies the equation
\[ \partial_t s_y (k_y, x) = - \partial_x J^y_x (k_y, x) - 2v k_y s_z (k_y, x). \] (1.44)

Here \( J^y_x \) is spin current operator (in general \( J^i_j = \frac{1}{2}(\hat{\sigma}_i \hat{v}_j + \hat{v}_j \hat{\sigma}_i) \)),
\[ J^y_x (k_y, x) = -v \langle \hat{c}_{k_y}^\dagger (x) \hat{c}_{k_y} (x) \rangle. \] (1.45)

In a steady state the left-hand side of Eq. (1.43) vanishes and we arrive at
\[ s_z (k_y, x) = - \frac{1}{2v k_y} \partial_x J^y_x (k_y, x). \] (1.46)
This yields the net spin polarization across the profile of the gate potential $U(x)$
\[
\int_{-\infty}^{\infty} s_z(x)dx = -\frac{1}{2v} \sum_{k_y} \frac{1}{k_y} [J^y_x(k_y, \infty) - J^y_x(k_y, -\infty)], \tag{1.47}
\]
expressing it via the values of spin current far away from the region where $U(x)$ is varied,
\[
J^y_x(k_y, \pm \infty) = -v \sum_{k_x} n(k_x, k_y, \pm \infty), \tag{1.48}
\]
with $n(k_x, k_y, \infty)$ denoting the corresponding distributions of electrons.

The summation in Eqs. (1.47)-(1.48) has to be performed over “uncompensated” states that originate from the left lead and belong to the energy interval $E_F < E < E_F + eV$, where $V$ is the bias applied between electrical contacts. (The uncompensated states originate in the contact that has higher chemical potential and propagate towards the other contact.) The subsequent analysis will be performed separately for the two cases mentioned above.

i) $n-n$ junction. When gate potential $U(x)$ is weaker than needed to elevate Dirac point above the Fermi-energy the electric current is carried by electrons only. Still, a smooth step-like potential $U(x)$ create a “junction” between half planes with different values of the Fermi momenta, $k_L$ and $k_R$, respectively.

Since $dk_x = dE k_x / k_x v$ and $dE \to eV$ the summation over $k_x$ is performed according to
\[
\sum_{k_x} \to \frac{eV k_n}{\pi \sqrt{k^2_n - k^2_y}}, \tag{1.49}
\]
(note that both $k_x > 0$ and $k_x < 0$ contribute to this expression). From Eq. (1.47),
\[
\int_{-\infty}^{\infty} s_z(x)dx = \frac{eV}{2\pi v} \left\{ \int_0^{k_L} \frac{dk_y}{k_y} \frac{k_L}{\sqrt{k^2_L - k^2_y}} - \int_0^{k_R} \frac{dk_y}{k_y} \frac{k_R}{\sqrt{k^2_R - k^2_y}} \right\}. \tag{1.50}
\]
Regularizing these formally divergent integrals (by assuming the same lower cutoff which subsequently drops out from the difference of the two terms) we find the amount of net spin accumulated at the junction,
\[
\int_{-\infty}^{\infty} s_z(x)dx = \frac{eV}{2\pi v} \ln \left( \frac{k_L}{k_R} \right). \tag{1.51}
\]
In addition, if the gate potential changes smoothly on the scale of $1/k_F$ the local spin polarization can be written as,

$$s_z(x) = \frac{eV}{2\pi v^2 k_F(x)} \partial_x U(x). \quad (1.52)$$

ii) $p$-$n$ junction. In the real experiment the strength of the gate potential $U(x)$ can be made significant enough to lift Dirac point above the Fermi energy over some region of the surface. For this setup we find

$$\int_{-\infty}^{\infty} s_z(x) dx = \frac{eV}{2\pi v} \ln \left( \frac{4k_p k_n}{q_{\min}^2} \right), \quad q_{\min} = \frac{1}{\hbar v} \left| \frac{dU}{dx} \right|_{x=0}. \quad (1.53)$$

This result differs from Eq. (1.50), and from the spin accumulation around $v_F(x) = \lambda$ considered in section 1.5, in that here the excess electrons to the left and to the right of the potential step have similar velocity, $v_y$, but opposite momentum, $p_y$, in the direction of current. In Eq. (1.53) the excess electrons originate from the lead $E_F + eV$ and thus have to have $v_y > 0$. On the contrary, in the case of edge spin accumulation (Sec. 1.5) the electrons arrive to the strip $0 < v_F(x) < \lambda$ from the bulk 2DEG, as is shown in Fig. 1.8 and thus must carry the bulk momentum $p_y > 0$, but may have the "wrong" sign of velocity. As a result the spin density in Eq. (1.53) has the same sign on both sides of the $n-p$ junction and two divergent logarithms add. Introducing a proper quantum mechanical cutoff to these integrals yields

$$\int_{-\infty}^{\infty} s_z(x) dx = \frac{eV}{2\pi v} \ln \left( \frac{4k_p k_n}{q_{\min}^2} \right), \quad q_{\min} = \frac{1}{\hbar v} \left| \frac{dU}{dx} \right|_{x=0}. \quad (1.54)$$

Electrons with very small $k_y$ start to tunnel thorough the $n-p$ junction, which would change our semiclassical predictions. The rude estimate of the quantum mechanical tunnelling exponent gives $\kappa = \int_0^{k_p} k_y(x) dx = k_p^2 \hbar v / (2dU/dx)$. Requiring a small tunnelling probability, $e^{-\kappa} \ll 1$, gives the above cutoff $q_{\min}$.

We may now compare the strength of the Spin-Hall effect in semiconductor heterostructures Eq. (1.5) with the result for topological insulators Eqs. (1.51, 1.54). The main difference is that in conventional semiconductors the chiral electron states appear because of the weak spin-orbit interaction. Consequently the result Eq. (1.5) acquires a small factor $(\lambda^2 - \lambda_D^2)/v_F^2$. There is now such suppression for the surface states in topological insulators.\(^8\)

\(^8\) Spin accumulation Eq. (1.5) and Eqs. (1.51, 1.54) depends also on the value of the
1.8. Summary

In this paper we discussed several nonequilibrium spin-related phenomena occurring in 2-dimensional chiral electron systems.

We were mostly interested in the effects which may be explained in terms of semiclassical electron motion in smooth potentials. In this case it is easy (at least theoretically) to produce the strongly in-plane spin polarized currents (Sec. 1.3).

The main part of the paper was devoted to calculation of the out of plane spin density $\langle \sigma_z \rangle$. It is the out of plane spin component, that is usually investigated in the experiment via the measuring of the optical Kerr rotation.\(^{11}\)

Finally we discussed the mesoscopic spin-Hall effect in novel materials such as graphene and topological insulators. Experimental investigation of pseudospin-Hall effect might prove not to be easy. On the contrary, observation of spin-Hall $\sigma_z$ accumulation at the surface of 3D topological insulator looks most appealing. While the net spin accumulation in semiconductor heterostructures is of the second order in small spin-orbit interaction, $s \sim \lambda^2/v_F^3$, there is no such restriction for topological insulator, $s \sim 1/v^2$. As a result, the spin accumulation in the latter case should exceed that possible in traditional 2-dimensional electron gas by orders of magnitude.

Acknowledgments

We gratefully acknowledge our collaborator V. Zyuzin, who participated in obtaining a number of previously published results reviewed in this paper; we also have benefited from discussions with A. Andreev, G.E.W. Bauer, C.W.J. Beenakker, B.I. Halperin, M. Raikh and O. Starykh. P.G.S. was supported by the SFB TR 12; E.G.M. was supported by the DOE, Office of Basic Energy Sciences, Award No. DEFG02-06ER46313.

References

1. Yu.A. Bychkov and E.I. Rashba, J. Phys. C 17 6039 (1984); F.T. Vas’ko, JETP Lett. 30, 540 (1979).
2. G. Dresselhaus, Phys. Rev. 100, 580 (1955).

Fermi velocity. The value of $v_F$ in Eq. (1.5) depends on the bulk density of 2DEG. In real experiment however, the Fermi velocity in our two examples will probably be of the same order of magnitude.
3. A.H. Castro Neto, F. Guinea, N.M. Peres, K.S. Novoselov, and A.K. Geim, Rev. Mod. Phys. 81, 109 (2009).
4. J. Schliemann, J.C. Egues, and D. Loss, Phys. Rev. Lett. 90, 146801 (2004).
5. B.A. Bernevig, T.L. Hughes, S.-C. Zhang, Science 314, 1757 (2006).
6. M. König, S. Wiedmann, Ch. Brune, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science 318, 766770 (2007).
7. D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M.Z. Hasan, Nature 452, 970-974 (2008).
8. Y. Xia, D. Qian, D. Hsieh, L. Wray, A. Pal, H. Lin, A. Bansil, D. Grauer, Y.S. Hor, R.J. Cava, M.Z. Hasan, Nature Physics 5, 398 (2009).
9. H.-J. Zhang, C.-X. Liu, X.-L. Qi, X.-Yu Deng, X. Dai, S.-C. Zhang, Z. Fang, Phys. Rev. B 80, 085307 (2009).
10. M.I. Dyakonov, V.I. Perel, Phys. Lett. A 35, 459 (1971).
11. Y.K. Kato, R.C. Myer, A.C. Gossard, and D.D. Awschalom, Science 306, 1910 (2004).
12. V. Sih, R. C. Myers, Y. K. Kato, W. H. Lau, A. C. Gossardand D. D. Awschalom, Nature Physics 1, 31 (2005).
13. S.O. Valenzuela and M. Tinkham, Nature 442, 176 (2006).
14. J. Wunderlich, B. Kaestner, J. Sinova, and T. Jungwirth, Phys. Rev. Lett. 99, 047204 (2005).
15. J.E. Hirsch, Phys. Rev. Lett. 83, 1834 (1999).
16. Shufeng Zhang, Phys. Rev. Lett. 85, 393 (2000).
17. H.A. Engel, B.I. Halperin, and E.I. Rashba, Phys. Rev. Lett. 95, 166605 (2005).
18. W.-K. Tse, S. Das Sarma, Phys. Rev. Lett. 96, 056601 (2006).
19. S. Murakami, N. Nagaosa, and S.-C. Zhang, Science 301, 1348 (2003); Phys. Rev. B 69, 235206 (2004).
20. J. Sinova, D. Culcer, Q. Niu, N.A. Sinitsyn, T. Jungwirth, and A.H. MacDonald, Phys. Rev. Lett. 92, 126603 (2004).
21. E. G. Mishchenko, A. V. Shytov, and B. I. Halperin, Phys. Rev. Lett. 93, 226602 (2004).
22. J.J. Inoue, G.E.W. Bauer, and L.W. Molenkamp, Phys. Rev. B 70, 41303(R) (2004).
23. A. Khaetskii, Phys. Rev. Lett. 96, 056602 (2006).
24. R. Raimondi and P. Schwab, Phys. Rev. B 71, 33311 (2005).
25. O.V. Dimitrova, Phys. Rev. B 71, 245327 (2005).
26. O. Entin-Wohlman, A. Aharony, Y. M. Galperin, V. I. Kozub, V. Vinokur, Phys. Rev. Lett. 95, 086603 (2005).
27. B.K. Nikolic, L.P. Zarbo, and S.Souma, Phys. Rev. B 72, 075361 (2005); B.K. Nikolic, S.Souma, L.P. Zarbo, and J. Sinova, Phys. Rev. Lett. 95, 046601 (2005).
28. S. Murakami, Adv. in Solid State Phys. 45, 197 (2005).
29. H.A. Engel, E.I. Rashba, and B.I. Halperin, Theory of Spin Hall Effects in Semiconductors, in Handbook of Magnetism and Advanced Magnetic Materials, H. Eds. Kronmüller and S. Parkin (John Wiley & Sons, 2007).
30. Non-local detection of spin Hall currents in multiterminal devices has re-
F. Author and S. Author

cently been reported in K.C. Weng, et al., [arXiv:0804.0096] C. Bruene, et al.,
[arXiv:0812.3768].
31. K. Nomura, J. Wunderlich, J. Sinova, B. Kaestner, A.H. MacDonald, and T.
Jungwirth, Phys. Rev. B 72, 245330 (2005).
32. E.M. Hankiewicz, L.W. Molenkamp, T. Jungwirth, and J. Sinova, Phys. Rev.
B 70, 241301(R) (2004).
33. G. Usaj and C.A. Balseiro, Europhys. Lett. 72, 621 (2005); A. Reynoso, G.
Usaj and C.A. Balseiro, Phys. Rev. B 73, 115342 (2006).
34. T.D. Stanescu and V. Galitski, Phys. Rev. B 74, 205331 (2006).
35. V.A. Zyuzin, P.G. Silvestrov, and E.G. Mishchenko, Phys. Rev. Lett. 99,
106601 (2007).
36. P.G. Silvestrov, V.A. Zyuzin, and E.G. Mishchenko, Phys. Rev. Lett. 102,
196802 (2009).
37. P.G. Silvestrov and E.G. Mishchenko, Phys. Rev. B 74, 165301 (2006).
38. R.G. Littlejohn and W.G. Flynn, Phys. Rev. A 44, 5239 (1991); 45, 7697
(1992).
39. J. Bolte and S. Keppeler, Phys. Rev. Lett. 81, 1987 (1998).
40. M. Pletyukhov and O. Zaitsev, J. Phys. A: Math. Gen. 36, 5181 (2003).
41. D. Culcer, J. Sinova, N. A. Sinitsyn, T. Jungwirth, A. H. MacDonald, and
Q. Niu, Phys. Rev. Lett. 93, 046602 (2004).
42. O. Entin-Wohlman, A. Aharony, Y. Tokura, Y. Avishai, [arXiv:0911.1347].
43. D. Grundler, Phys. Rev. Lett. 84, 6074 (2000).
44. E.B. Sonin, [arXiv:0909.3156].
45. E.G. Mishchenko and B.I. Halperin, Phys. Rev. B 68, 045317 (2003).
46. L. Fu, [arXiv:0908.1418].