Topological Magnon Insulator in NonCoplanar Kagomé Antiferromagnets:
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Topological magnon insulators in insulating Kagomé ferromagnets have been extensively studied in a series of papers. It has been established that Dzyaloshinskii-Moriya interaction (DMI) is the key ingredient to observe a nontrivial topological magnon with edge modes. However, insulating antiferromagnets on the Kagomé lattice are frustrated systems considered as a playground for studying quantum spin liquid physics. In these systems the DMI can induce a coplanar but noncollinear magnetic orders with a $\mathbf{q} = 0$ propagating wavevector. We show that topological magnon bands are absent in this coplanar spin texture in sharp contrast to collinear ferromagnets with DMI. Hence, geometrically frustrated Kagomé antiferromagnets can be deemed topologically trivial. The presence of an out-of-plane magnetic field in these frustrated magnets induces noncoplanar spin textures exhibiting a nonzero spin scalar chirality. We show that the field-induced spin chirality provides topological magnon bands in Kagomé antiferromagnets without the need of DMI and survives in the chiral spin liquid phase of frustrated magnets. Possible experimentally material includes iron jarosite $\text{KFe}_3(\text{OH})_6(\text{SO}_4)_2$.

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I. INTRODUCTION

In recent years, the concept of topological band theory has been extended to nonelectronic systems such as magnons [1,15] and phonons [10,21]. In the former, the DMI [22] is the primary source of topological magnon bands and magnon edge states [4,5], as well as thermal magnon Hall effect [1,2]. These systems are dubbed topological magnon insulators [4] in analogy to topological insulators in fermionic systems with spin-orbit coupling (SOC) [23,25]. However, as magnons are charge-neutral quasiparticles, the perfectly conducting edge states are believed to be useful for dissipationless transport applicable to magnon spintronics. Our conception of DMI being the primary source of topological effects in magnonic systems has been firmly established by recent experimental realization of topological magnon insulator in collinear Kagomé ferromagnet Cu(1-3, bdc) [6]. In these systems the DMI comes naturally because the Kagomé lattice lacks an inversion center.

In reality, however, there are more frustrated Kagomé magnets than collinear ferromagnets. The physics associated with the former has no analogy with the latter. The former are considered as a candidate for quantum spin liquid physics due to an extensive classical degeneracy that prevent magnetic ordering to lowest accessible temperatures. However, in recent experimental synthesis it has been shown that the effects of SOC or DMI are not negligible in most Kagomé antiferromagnets. The DMI appears as a perturbation to the Heisenberg spin exchange. One of the striking features of this perturbation in frustrated Kagomé systems is that it induces magnetic ordering with a $\mathbf{q} = 0$ propagation wavevector. Thus, it suppresses the spin liquid phase of Kagomé antiferromagnets up to a critical value [27,28]. Various experimentally accessible frustrated Kagomé antiferromagnets show evidence of coplanar/noncollinear $\mathbf{q} = 0$ magnetic ordering at specific temperatures. The famous one is iron jarosite $\text{KFe}_3(\text{OH})_6(\text{SO}_4)_2$ [29,30]. The unanswered question is how does topological effects arise in these systems? From the experimental perspective, it has been previously shown that iron jarosite possesses a finite spin scalar chirality induced by an out-of-plane magnetic field but no topological properties were measured [29]. In a recent Letter [26], we have provided evidence of magnon Hall effect and thermal Hall conductivity $\kappa_{xy}$ for this Kagomé material, which originates from non-coplanar spin texture and survives in the absence of DMI and magnetic ordering. Topological magnon insulator with magnon edge modes are another area of recent development [11,6].

In this report, we complete our analysis of topological magnon effects in geometrically frustrated Kagomé antiferromagnets by providing evidence of topological magnon edge modes. The main purpose of this report is to relate finite thermal Hall conductivity to topologically protected magnon edge states. We consider three different models, viz: (i) bilayer Kagomé ferromagnets coupled antiferromagnetically or layer antiferromagnets. (ii) The model Hamiltonian for iron jarosite $\text{KFe}_3(\text{OH})_6(\text{SO}_4)_2$ with DMI and second nearest neighbour interaction [30]. (iii) The XXZ model for Kagomé antiferromagnet without the DMI. The arrangement of this paper is as follows. In Sec. II we introduce the three models, analyze the magnon tight binding models, and present the distinctive magnon bands. Sec. III introduces the concept of topologically protected magnon edge modes and fictitious Chern numbers. We relate these concepts to experimentally accessible thermal magnon Hall effect. In Sec. IV we conclude and discuss potential methods to real-
ize topological magnon bands in geometrically frustrated Kagomé antiferromagnets with/without DMI.

II. MODEL HAMILTONIAN

A. Model I

In various frustrated Kagomé magnets, a strong out-of-plane magnetic field is sufficient to circumvent frustrated interactions and leads to magnetic ordering. A good example is the bilayer frustrated Kagomé magnet Ca_{10}Cr_{7}O_{28} [39], which shows evidence of ferromagnetic alignment at a magnetic field of magnitude \( h \sim 11 \) Tesla. The frustrated Kagomé volborthite Cu_{3}V_{2}O_{7}(OH)_{2} \cdot 2H_{2}O also shows evidence of magnetic ordering at several field values [40, 41]. In the ordered regimes, the magnetic excitations are definitely magnons. This is an example where both the out-of-plane DM vector (\( \mathbf{D}_{ij} = \mathbf{Dz} \)) and the in-plane DM (\( \mathbf{D}_{ij} = \mathbf{Dx} \)) contribute in the present system. This is a good example of field-induced canting of the system. It should be noted that this is not the case in previously studied Kagomé ferromagnets [11, 12]. To linear order in \( 1/S \) expansion we have

\[
\mathcal{H}_{DM,x} = \sigma D \sin \chi \sum_{(i,j) \tau} \mathbf{z} \cdot \mathbf{S}_{i}^{\tau} \times \mathbf{S}_{j}^{\tau} + O(1/S),
\]

where \( \sigma = \mp \) for top and bottom layers respectively. We see that the latter case has opposite signs on both layers.

Next, we study the excitations above the classical ground state by using the Holstein-Primakoff spin bosonic operators for the rotated prime coordinates:

\[
S_{i,\tau}^{x} = \sqrt{S/2}(b_{i,\tau}^{\dagger} + b_{i,\tau}), \quad S_{i,\tau}^{y} = i\sqrt{S/2}(b_{i,\tau}^{\dagger} - b_{i,\tau}), \quad S_{i,\tau}^{z} = S - b_{i,\tau}^{\dagger}b_{i,\tau}.
\]

The Hamiltonian maps to a magnon tight binding model

\[
\mathcal{H}_{SW} = v_{b} \sum_{i} n_{i,\tau} - v_{D} \sum_{(i,j) \tau} \left[ e^{-i\phi_{ij}}b_{i,\tau}^{\dagger}b_{j,\tau} + h.c. \right] + v_{t} \sum_{i \in \tau, j \in \tau'} \left[ (n_{i,\tau} + n_{j,\tau'}) \cos 2\chi \right.
\]

\[
\left. + \left( b_{i,\tau}^{\dagger} b_{j,\tau'} + h.c. \right) \cos^{2} \chi - \left( b_{i,\tau}^{\dagger} b_{j,\tau'} + h.c. \right) \sin^{2} \chi \right].
\]

where \( n_{i,\tau} = b_{i,\tau}^{\dagger}b_{i,\tau} \) is the occupation number, \( v_{0} = 4v_{J} + h \cos \chi \), \( v_{t} = J_{\tau}S \), \( v_{J} = |J|S \), and \( v_{D} = S\sqrt{J^{2} + D_{x,z}^{2}} \).
with $D_x = D \cos \chi$, $D_z = \sigma D \sin \chi$. The fictitious magnetic flux on each triangle of the Kagomé lattice is given by $\phi = \arctan(D_{x,z}/|J|)$. Using the vectors $\psi_k = (\psi_k^t, \psi_{-k})$, with $\psi_k^t = (b_{kA}, b_{kB}, b_{kC}, b_{kA'}^\dagger, b_{kB'}^\dagger, b_{kC'}^\dagger)$, the momentum space Hamiltonian is given by $H_{SW} = \frac{1}{2} \sum_k \textbf{S}_{AFM}(k) \cdot \textbf{S}_k$, where

$$H_{AFM}(k) = \begin{pmatrix} A(k, \phi) & B \\ B^\dagger & A^*(-k, \phi) \end{pmatrix},$$

$$A(k, \phi) = \begin{pmatrix} a(k, \phi) & b \\ b^\dagger & a(k, \phi)^\dagger \end{pmatrix}, \quad B = \begin{pmatrix} 0 & c \\ c^\dagger & 0 \end{pmatrix},$$

and $a(k, \phi) = \tilde{v}_0 I_{3 \times 3} - 2 v_D A(k, \phi)$,

$$A(k, \phi) = \begin{pmatrix} 0 & \cos k_1 e^{-i \phi} & \cos k_3 e^{i \phi} \\ \cos k_1 e^{i \phi} & 0 & \cos k_2 e^{-i \phi} \\ \cos k_3 e^{-i \phi} & \cos k_2 e^{i \phi} & 0 \end{pmatrix},$$

$$b = -v_t \cos^2 \chi I_{3 \times 3}, \quad c = v_t \sin^2 \chi I_{3 \times 3},$$

where $I_{3 \times 3}$ is a $3 \times 3$ identity matrix, $\tilde{v}_0 = 4 v_D + h \cos \chi - v_t \cos 2 \chi = 4 v_D + v_t$, $k_i = k_i \cdot e_i$, $e_1 = -(1/2, \sqrt{3}/2)$, $e_2 = (1, 0)$, $e_3 = (-1/2, 3/2)$. At the saturation field $h = h_s$, $\chi = 0$, we obtain ferromagnetically coupled layers applicable to Cu(1-3, bdc) assuming strong interlayer coupling.

Plotted in Figs. 1 and 2 are the magnon bands along the Brillouin zone paths in Fig. 3(c), with the parameter values of Ca$_{10}$Cr$_7$O$_{28}$, $J = 0.16$ meV, $J_t/J = 0.11$ and the DM value $D/J = 0.2$. For $D_{ij} = D \hat{z}$, the DMI does not contribute at zero field because the spins are along the $x$-$y$ Kagomé plane. The resulting magnon bands are doubly degenerate between $S_z \rightarrow S_z = \pm S$. At finite magnetic field increases, each spin has a component along the $z$-axis, hence the degeneracy of the bands between $S_z = \pm S$ is lifted and the effects of the DMI results in a gap opening at $K$. At the saturation point $h = h_s$ (not shown) each layer is fully polarized along the field direction, but since the spins are along the $xy$ plane the DMI leads to gapped non-degenerate magnon bands. For the in-plane DMI $D_{ij} = D \hat{x}$ the situation is different. The degeneracy persists at zero field but since the spins are along the $xy$ plane the DMI leads to gap magnon bands. The degeneracy is always lifted at nonzero magnetic field, but in this case the there is a staggered flux emanating from both layers and the bands cross each other at $\pm K$. At the saturation point $h = h_s$ the spin are fully polarized along the $z$-axis and the in-plane DMI does not contribute and the non-degenerate magnon bands are gapless (not shown).

**B. Model II**

In the previous section, we studied one of the possible field-induced magnetically ordered phases in geometrically frustrated bilayer Kagomé magnets. In this section, we study another magnetically ordered phase which has been realized experimentally in various frustrated Kagomé magnets. Without loss of generality, we focus on the ideal Kagomé material KFe$_3$(OH)$_6$(SO$_4$)$_2$ [29] [30]. The Hamiltonian is given by

$$H = \sum_{i,j} J_{ij} S_i \cdot S_j + \sum_{<ij>} D_{ij} \cdot S_i \times S_j - h \hat{z} \cdot \sum_i S_i,$$

where $J_{ij} = J > 0$ and $J_2 > 0$ are the isotropic antiferromagnetic couplings for nearest-neighbour (NN) and
FIG. 3. Color online. (a) The zero field coplanar 120° Néel order on the Kagomé lattice corresponding to the $q = 0$ ground state of Kagomé antiferromagnets. Solid lines connect NN sites and dash lines connect NNN sites. (b) Field-induced non-coplanar out-of-plane spin canting with nonzero spin scalar chirality $\kappa$, where $\phi$ is the field-induced fictitious flux.

The zero-field coplanar 120° Néel order on the Kagomé lattice corresponds to the $q = 0$ ground state of Kagomé antiferromagnets. Solid lines connect NN sites and dash lines connect NNN sites. (b) Field-induced non-coplanar out-of-plane spin canting with nonzero spin scalar chirality $\kappa$, where $\phi$ is the field-induced fictitious flux.

next-nearest-neighbour (NNN) sites respectively, $D_{ij} = (0, 0, \pm D_z)$, where $-/+\gamma$ denotes the directions of the out-of-plane DMI in the up/down triangles of the Kagomé lattice, and $h$ is the out-of-plane magnetic field in units of $g\mu_B$. For the iron jarosite $KFe_3(OH)_6(SO_4)_2$ the interlayer exchange interaction can be neglected for two reasons. First, it is very small compared to $J$ and $J_2$. Second, a single crystal of iron jarosite can be synthesized [30]. We have also neglected the in-plane DM vector because it neither stabilizes magnetic ordering nor induces topological effects. Therefore, it cannot change the results obtained here. The alternating out-of-plane DMI between the up and down triangles of the Kagomé lattice is fictitious in that only one ground state is selected for each sign with positive chirality ($D_z > 0$) or negative chirality ($D_z < 0$). At zero magnetic field, the out-of-plane DMI induces a coplanar 120° Néel order on the $x$-$y$ Kagomé plane with positive chirality [27, 29, 30] as shown in Fig. 3(a). For nonzero out-of-plane field, inelastic neutron scattering experiment on the ideal Kagomé material $KFe_3(OH)_6(SO_4)_2$ [27, 29, 30] has uncovered a non-coplanar spin texture with nonzero spin chirality $\kappa = \sum S_i \cdot (S_j \times S_k)$ [29] as shown in Fig. 3(b). However, the topological effects of this non-collinear system have not been studied both theoretically and experimentally. In a recent Letter [28], we showed for the first time that this material possesses a finite thermal Hall conductivity in this non-collinear regime. Thus, the iron jarosite $KFe_3(OH)_6(SO_4)_2$ is a possible candidate for investigating thermal Hall effect of magnon, which is accessible by using inelastic neutron scattering.

In the present study, we analyze the finite thermal Hall conductivity in terms of topological magnon edge states. We consider the $q = 0$ ground state with positive chirality $D_{ij} = (0, 0, \pm D_z)$ and $D_z > 0$. The basic procedure is similar to the bilayer system studied above. The rotation matrix is same as Eq. 3, however, with the oriented angles $\theta_A = 0$, $\theta_B = 2\pi/3$, $\theta_C = -2\pi/3$. The classical
energy is given by
\[ e_{cl} = \tilde{\mathcal{J}} (-1 + 3 \cos^2 \chi) - \sqrt{3} D_z \sin^2 \chi - \hat{h} \cos \chi, \]  
(14)
where \( e_{cl} = E_{cl}/3NS^2 \), \( \tilde{\mathcal{J}} = \mathcal{J} + \mathcal{J}_2 \), and the magnetic field is rescaled in units of \( S \). Minimizing \( e_{cl} \) yields the canting angle \( \cos \chi = \hat{h}/\hat{h}_s \), with \( \hat{h}_s = (6 \mathcal{J} + 2 \sqrt{3} D_z) \). We see that the DMI depends on the classical energy as it contributes to the stability of the \( q = 0 \) ground state. Besides, \( \mathcal{J}_2 > 0 \) can also stabilize the coplanar magnetic structure in the absence of the DMI. The topological excitations above the classical ground state is mediated by a field-induced scalar chirality \( \kappa_s = \cos \chi \sum S_i (S_i \times \mathbf{S}) \) defined as the solid angle subtended by three neighbouring spins. As mentioned previously, the scalar spin chirality originates from non-coplanar spin texture and does not need the presence of DMI or magnetic ordering. It is also the basis of chiral spin liquid states which suggests that topological effects may persist in the spin liquid regime of frustrated kagomé magnets. This model does not have an analogy to collinear ferromagnets [1–14] or bilayer collinear in Model I. It also differs significantly from field-induced topological magnons in bipartite frustrated honeycomb lattice because \( \kappa_s = 0 \) [15]. In fact, the bipartite honeycomb antiferromagnets fall into the class of Model I as they are doubly degenerate at zero magnetic field and require an explicit DMI. The magnon tight binding Hamiltonian for Model II is given by
\[ \mathcal{H}_{SW} = S \sum_{k,\alpha, \beta, 1,2} \left( \mathcal{M}_{\alpha, \beta, 1,2} b_{k\alpha}^\dagger b_{k\beta} + \mathcal{M}_{\alpha, \beta, 1,2}^\prime \left( b_{k\alpha}^\dagger b_{-k\beta}^\dagger + b_{k\alpha} b_{-k\beta} \right) \right), \]  
(15)
where \( \alpha, \beta = A, B, C \) and the coefficients are given by
\[ \mathcal{M}_0 = \zeta I_{3 \times 3}, \]  
and
\[ \mathcal{M}_1 = \Delta_{1,2} \begin{pmatrix} 0 & \gamma_{AB}^1 e^{-i\phi_1} & \gamma_{CA}^1 e^{i\phi_1} \\ \gamma_{AB}^2 e^{i\phi_1} & 0 & \gamma_{BC}^2 e^{-i\phi_1} \\ \gamma_{CA}^2 e^{-i\phi_1} & \gamma_{BC}^2 e^{i\phi_1} & 0 \end{pmatrix}, \]  
(16)
\[ \mathcal{M}_1' = \Delta_{1,2}' \begin{pmatrix} 0 & \gamma_{AB}^1 & \gamma_{BC}^1 \\ \gamma_{AB}^2 & 0 & \gamma_{BC}^2 \\ \gamma_{CA}^2 & \gamma_{BC}^2 & 0 \end{pmatrix}, \]  
(17)
and \( \Delta_{1,2} = \sqrt{(\Delta_{1,2}^R)^2 + (\Delta_{1,2}^M)^2} \), where
\[ \Delta_{1,2}^R = \mathcal{J} \left( -\frac{1}{2} + \frac{3}{4} \sin^2 \chi \right) - \sqrt{3} D_z \left( 1 - \frac{\sin^2 \chi}{2} \right), \]  
(18)
and
\[ \Delta_{1,2}^M = \frac{\cos \chi}{2} \left( -\sqrt{3} \mathcal{J} + D_z \right), \]  
(19)
Using the vector notation
\[ \Psi_k = (b_{kA}^\dagger, b_{kB}^\dagger, b_{kC}^\dagger, b_{-kA}, b_{-kB}, b_{-kC}), \]  
(23)
the Hamiltonian can be written as
\[ \mathcal{H}_{SW} = \mathcal{E}_0 + S \sum_k \Psi_k^\dagger \mathcal{H}(k) \Psi_k, \]  
(24)
where
\[ \mathcal{H}(k) = \mathbb{I}_{2 \times 2} \otimes (\mathcal{M}_0 + \mathcal{M}) + \sigma_z \otimes \mathcal{M}', \]  
(25)
and \( \mathcal{E}_0 \) is a constant. \( \mathbb{I}_2 \) is an identity \( 2 \times 2 \) matrix and \( \sigma_z \) is a Pauli matrix. \( \mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 \), the same for \( \mathcal{M}' \). The eigenvalues of this Hamiltonian cannot be obtained analytically as opposed to the zero field case, \( \chi = \pi/2 \), with coplanar \( 120^\circ \) Néel order on the \( x-y \) Kagomé plane. It is important to note that both fluxes do not vanish at zero DMI. This means that the DMI does not provide topological effects in stark contrast to ferromagnets [1–13]. As shown in Fig. 4 the magnon bands are not topological at zero magnetic field even in the presence of DMI.

C. Model III

The final model we shall consider is the XXZ Kagemé antiferromagnet without DMI subject to an out-of-plane magnetic field. The Hamiltonian is governed by
\[ \mathcal{H} = \mathcal{J} \sum_{(i,j)} (S_i \cdot S_j - \delta S_i^1 S_j^1) - \hat{h} \cdot \sum_i S_i, \]  
(26)
where \( \mathcal{J} > 0 \) and \( 0 \leq \delta \leq 1 \) is the easy-plane anisotropy, and \( \hat{h} \) is the magnetic field in units of \( g_0 \mu_B \). At zero field, the easy-plane anisotropy favours the positive chirality \( q = 0 \) ground state [27] [29] [31]. The canting angle is determined from the classical energy
\[ e_{cl} = \mathcal{J} \left( -1 + (3 - 2\delta) \cos^2 \chi \right) - \hat{h} \cos \chi, \]  
(27)
where \( \cos \chi = \hat{h}/\hat{h}_s \) and \( \hat{h}_s = 2\mathcal{J}(3 - 2\delta) \).

In this system, topological magnon bands originate from \( \kappa_s \) as in Model II. The fictitious magnetic flux encountered by propagating magnon is given by \( \tan \phi = \cdots \).
\[ \frac{\Delta_M}{\Delta_R} \text{ where } \Delta = \sqrt{\Delta_R^2 + \Delta_M^2}, \text{ and} \]
\[ \Delta_R = \left[ -\frac{1}{2} + \frac{3}{2} \left( 2 - \delta \right) \sin^2 \chi \right], \]
\[ \Delta_M = -\frac{\sqrt{3}}{2} \cos \chi, \quad \Delta' = \frac{\sin^2 \chi}{2} \left( 3 - \delta \right). \] (28) (29)

In Fig. 6 we show the magnon bands for \( \delta = 0.4 \). At zero field, the magnon bands are very similar to Model II. We see that the flat mode acquires a small dispersion at nonzero field and the magnon bands are gapped at all points in the Brillouin zone.

### D. Finite thermal Hall conductivity at zero DMI

As mentioned in the text, the DMI does not provide any topological effects for the noncollinear \( q = 0 \) spin configuration on the kagomé lattice. This means that topological effects persist at zero DMI and nonzero out-of-plane magnetic field via an induced spin scalar chirality \( \mathcal{H}_\chi \sim \cos \chi \sum_i \mathbf{S}_i \cdot (\mathbf{S}_i \times \mathbf{S}_k) \), where \( \cos \chi = h/h_s \) with \( h_s = 6(J + f_2) \). However, DMI is usually present on the kagomé lattice due to lack of inversion center. In this section, we show that anomalous magnon Hall effect is present at zero DMI in contrast to collinear ferromagnets. Figures 6 and 7 show the magnon bands and the corresponding \( \kappa_{xy} \) respectively for zero DMI.

### III. TOPOLOGICAL MAGNON EDGE MODES

The Hamiltonians for insulating antiferromagnets are diagonalized by the generalized Bogoliubov transformation \( \Psi_k = \mathcal{P}_k \mathcal{Q}_k \), where \( \mathcal{P}_k \) is a \( 2N \times 2N \) paraunitary matrix and \( \mathcal{Q}_k = (\mathcal{Q}_k^\dagger, \mathcal{Q}_k) \) with \( \mathcal{Q}_k = (\beta_{kA}^\dagger \beta_{kB}^\dagger \beta_{kC}^\dagger) \) being the quasiparticle operators. The matrix \( \mathcal{P}_k \) satisfies the relations,
\[ \mathcal{P}_k^\dagger \mathcal{H}(k) \mathcal{P}_k = \mathcal{E}_k \] (30)
\[ \mathcal{P}_k^\dagger \tau_3 \mathcal{P}_k = \tau_3. \] (31)

where \( \mathcal{E}_k = \text{diag}(\epsilon_{kA}, \epsilon_{kC}), \tau_3 = \text{diag}(I_{N \times N}, -I_{N \times N}) \), and \( \epsilon_{kA} \) are the energy eigenvalues. From Eq. (31) we get \( \mathcal{P}_k^\dagger = \tau_3 \mathcal{P}_k^{-1} \tau_3 \), and Eq. (30) is equivalent to saying that we need to diagonalize the Hamiltonian \( \mathcal{H}'(k) = \tau_3 \mathcal{H}(k) \), whose eigenvalues are given by \( \tau_3 \mathcal{E}_k \) and the columns of \( \mathcal{P}_k \) are the corresponding eigenvectors. The paraunitary operator defines a Berry curvature given by
\[ \Omega_{ij,\alpha}(k) = -2 \text{Im}[\tau_3(\partial_{k_i} \mathcal{P}_k^\dagger) \tau_3(\partial_{k_j} \mathcal{P}_k)], \] (32)
with \( i, j = \{x, y\} \) and \( \mathcal{P}_k \) are the columns of \( \mathcal{P}_k \). This form of the Berry curvature simply extracts the diagonal components which are the most important. Suppose the explicit analytical form of \( \mathcal{P}_k \) is known as in honeycomb-lattice hardcore bosons [11], the Berry curvature can be computed directly from Eq. (32). Unfortunately, this is not the case in the present models. From Eq. (30) the Berry curvature can be written alternatively as
\[ \Omega_{ij,\alpha}(k) = -2 \sum_{\alpha' \neq \alpha} \text{Im}[\langle \mathcal{P}_k \mathcal{P}_k^\dagger \rangle \langle \mathcal{P}_k \mathcal{P}_k^\dagger \langle \mathcal{P}_k \mathcal{P}_k^\dagger \rangle \mathcal{P}_k \mathcal{P}_k^\dagger \rangle], \] (33)
where \( v = \partial \mathcal{H}'(k)/\partial k \) defines the velocity operators. The present form can be computed once the eigenvalues and eigenvectors of the Hamiltonian are obtained numerically. Similar to fermionic systems, the Chern number can still be defined for bosonic systems as the integration of the Berry curvature over the first Brillouin zone,
\[ C_\alpha = \frac{1}{2\pi} \int_{BZ} dk_i dk_j \Omega_{ij,\alpha}(k). \] (34)

Indeed, topologically protected magnon edge states are characterized by nonzero Chern numbers. However, the Chern numbers are fictitious because the notion of completely filled bands and Fermi energy do not apply to bosonic (magnonic) systems.

Experimentally accessible magnon Hall effect has been previously analyzed in Ref. [26]. Now, we complete this
FIG. 8. Color online. Magnon edge states of bilayer Kagomé antiferromagnets (Model I) for a strip geometry with $\mathcal{D}_{ij} = \mathcal{D}^z$. The parameters are $\mathcal{D}/\mathcal{J} = 0.2$, $\mathcal{J}_t/\mathcal{J} = 0.11$.

FIG. 9. Color online. Magnon edge states of bilayer Kagomé antiferromagnets (Model I) for a strip geometry with $\mathcal{D}_{ij} = \mathcal{D}^x$. The parameters are $\mathcal{D}/\mathcal{J} = 0.2$, $\mathcal{J}_t/\mathcal{J} = 0.11$.

FIG. 10. Color online. Magnon edge states of Kagomé antiferromagnets for Model II with the parameter values of jarosite KFe$_3$(OH)$_6$(SO$_4$)$_2$ $\mathcal{D}_z/\mathcal{J} = 0.06$, $\mathcal{J}_z/\mathcal{J} = 0.03$.

FIG. 11. Color online. Magnon edge states of Kagomé antiferromagnets for Model III with $\delta = 0.4$.

Study by providing evidence of topological magnon insulator with Chern number protected magnon edge modes. We have solved for the magnon edge modes using a strip geometry with open boundary conditions along the $y$ direction and infinite along $x$ direction [23]. First, let us consider Model I with $\mathcal{D}_{ij} = \mathcal{D}^z$ shown in Fig. 8. In this case the magnon bulk bands are degenerate between $S_z \rightarrow S_z = \pm S$ sectors at $h = 0$ ($\chi = \pi/2$) and the DMI vanishes in the noninteracting limit as the spins are along the $x$-$y$ Kagomé plane at zero field. This results in gapless magnon bulk bands at $\pm \mathbf{K}$ and $\Gamma$ with a single edge mode connecting these points and $\kappa_{xy}$ vanishes as expected. For $h < h_s$, the spins are non-collinear and the degeneracy between $S_z \rightarrow S_z = \pm S$ is lifted because each spin has a component along the $z$-axis. The DMI opens a small gap between the bands. We see that pairs of gapless magnon edge states appear in the vicinity of bulk gap signifying the strong topology of the system, yielding nonzero $\kappa_{xy}$ [24]. At the saturation field $h = h_s$, the spins are collinear along the $z$-axis corresponding to bilayer ferromagnet coupled ferromagnetically. The DMI again leads to gap magnon bulk bands with counter-propagating gapless edge states and again with nonzero $\kappa_{xy}$.

For the in-plane DMI $\mathcal{D}_{ij} = \mathcal{D}^x$ shown in Fig. 9 the situation is different. There is no magnon Hall effect and $\kappa_{xy}$ vanishes in all regimes [24], but there is topological magnon insulator as we now explain. Indeed, we have degenerate magnon bulk bands between $S_z \rightarrow S_z = \pm S$ sectors at zero field, but the DMI has a profound effect since the spins are along the $x$-$y$ Kagomé plane. As shown in Fig. 9(a) there is a pair of edge state modes for each spin sector, and they are related by time-reversal symmetry. This is an evidence of topological magnon insulator. However, $\kappa_{xy}$ vanishes as a consequence of time-reversal symmetry between the degenerate spin sectors. In fact, this system is a magnonic counterpart to
fermionic topological insulator with imaginary second-nearest-neighbour SOC between electron spin up and down [23]. For \( h < h_s \), the bands cross at \( \pm K \) as shown above due to staggered flux configurations and the edge modes are not topologically protected as we have confirmed by computing the Berry curvatures and the Chern numbers. For this reason \( \kappa_{xy} \) again vanishes. At the saturation field \( h = h_s \), the in-plane DMI disappears in the noninteracting limit and the system is topologically trivial with vanishing \( \kappa_{xy} \). The key observation in the layer antiferromagnetic system is that although topologically protected edge states are present at zero magnetic field, \( \kappa_{xy} \) is suppressed by antiferromagnetic coupling.

Now, let us consider Model II which corresponds to the Kagomé material \( \text{KFe}_3(\text{OH})_6(\text{SO}_4)_2 \) [27, 29, 30]. As shown above this model differs significantly from Model I due to the presence of spin scalar chirality which survives in the absence of DMI and magnetic ordering. The associated magnon edge modes are depicted in Fig. 10. At zero field \( h = 0 \) it is evident that there are no protected chiral edge modes. This shows that the system is topologically trivial and \( \kappa_{xy} \) vanishes for any values of DMI. In the presence of a magnetic field perpendicular to the Kagomé plane there is an induced noncoplanar spin texture which provides spin scalar chirality [29]. Figures 10(b) and (c) show that the system is topologically nontrivial in this regime with protected gapless edge states which yield nonzero Chern number and finite \( \kappa_{xy} \) even without the presence of DMI [26]. Indeed, the presence of spin scalar chirality is the basis of chiral spin liquid physics, therefore it will not be surprising that the nontrivial topology of this system persists in the spin liquid phase of the frustrated Kagomé magnets. Model III explicitly ignores the DMI and the easy-plane anisotropy stabilizes the \( q = 0 \) magnetic ordering. This system is analogous to Model II as shown in Fig. 11. This is because the presence of the DMI does not have any topological effects in frustrated Kagomé lattice unlike insulating Kagomé ferromagnets. The layer antiferromagnetic system and the frustrated system have similarities and differences. In both systems \( \kappa_{xy} \) vanishes at zero field which can be attributed to zero net magnetic moment. In other words, the degeneracy at zero field between \( S_z \to S_x = \pm S \) sectors in layer antiferromagnetic system yields a zero net magnetic moment and for the coplanar/non-collinear system at zero field we have \( \sum \Delta S_\Delta = 0 \) on each triangular plaquette which also yields a zero net magnetic moment. However, the origin of finite \( \kappa_{xy} \) is different in both systems. For the layer antiferromagnets, topological magnon bands is induced by the DMI, whereas in the frustrated system with coplanar/non-collinear ordering the concept of topological magnon bands originates from field-induced spin scalar chirality which is nonzero even in the absence of DMI and magnetic ordering \( \langle S_j \rangle = 0 \).

IV. DISCUSSION AND CONCLUSION

It is natural to ask the importance of this investigation and whether such nontrivial topological effects can be experimentally realized in insulating antiferromagnets. Recently, topological magnon insulator has been realized in the Kagomé ferromagnet \( \text{Cu}(1-3, \text{bdc}) \) [3]. This material is also the first Kagomé ferromagnet that shows magnon Hall effect with finite \( \kappa_{xy} \) [2]. A recent experiment has reported a finite \( \kappa_{xy} \) in frustrated Kagomé volborthite \( \text{Cu}_3V_2O_7(\text{OH})_2 \cdot 2\text{H}_2\text{O} \) in the presence of an out-of-plane magnetic field \( h = 15 \) Tesla [41]. This result is attributed to spin excitations in the spin liquid regime. As mentioned previously, the Kagomé volborthite is known to exhibit different magnetic-field-induced ordered phases for \( h < 15 \) Tesla [40, 41], and the frustrated Kagomé compound \( \text{Ca}_{10}\text{Cr}_7\text{O}_{28} \) [39] also exhibits ferromagnetic ordered states for \( h \sim 11 \) Tesla. This suggests that the observed low temperature dependence of \( \kappa_{xy} \) in Kagomé volborthite might not be due to spin excitations in the spin liquid regime, but magnon excitations in the field-induced ordered phases. The iron jarosite \( \text{KFe}_3(\text{OH})_6(\text{SO}_4)_2 \) [29, 30] is an ideal Kagomé antiferromagnet with \( q = 0 \) ground state and nonzero field-induced spin scalar chirality [29]. This is a perfect material to search for topologically nontrivial excitations with finite \( \kappa_{xy} \) as described in the present study and Ref. [26]. At the moment, inelastic neutron scattering experiment has not figured out how to measure magnon edge state modes, which are consequences of the magnon bulk topology that gives rise to finite \( \kappa_{xy} \) because it is a bulk sensitive method. The magnon edge modes can be probed by edge sensitive methods such as light [35] or electronic [40] scattering method. This is not an impossible task in principle, and we believe it will be measured in the near future.

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