Another multiplicity result for the periodic solutions of certain systems

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Abstract: In this paper, we deal with a problem of the type

\[
\begin{align*}
(\phi'(u'))' &= \nabla_x F(t, u) \quad \text{in } [0, T] \\
u(0) &= u(T), \quad u'(0) = u'(T),
\end{align*}
\]

where, in particular, \(\phi\) is a homeomorphism from an open ball of \(\mathbb{R}^n\) onto \(\mathbb{R}^n\). Using the theory developed by Brezis and Mawhin in [1] jointly with our minimax theorem proved in [3], we obtain a general multiplicity result, under assumptions of qualitative nature only. Three remarkable corollaries are also presented.

Key words: periodic solution; Lagrangian system of relativistic oscillators; minimax; multiplicity; global minimum.

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1. Introduction

In what follows, \(L, T\) are two fixed positive numbers. For each \(r > 0\), we set \(B_r = \{x \in \mathbb{R}^n : |x| < r\}\) (\(|\cdot|\) being the Euclidean norm on \(\mathbb{R}^n\)) and \(\overline{B}_r\) is the closure of \(B_r\).

We denote by \(\mathcal{A}\) the family of all homeomorphisms \(\phi\) from \(B_L\) onto \(\mathbb{R}^n\) such that \(\phi(0) = 0\) and \(\phi = \nabla \Phi\), where the function \(\Phi : \overline{B}_L \to ]-\infty, 0]\) is continuous and strictly convex in \(\overline{B}_L\), and of class \(C^1\) in \(B_L\). Notice that 0 is the unique global minimum of \(\Phi\) in \(\overline{B}_L\).

We denote by \(\mathcal{B}\) the family of all functions \(F : [0, T] \times \mathbb{R}^n \to \mathbb{R}\) which are measurable in \([0, T]\), of class \(C^1\) in \(\mathbb{R}^n\) and such that \(\nabla_x F\) is measurable in \([0, T]\) and, for each \(r > 0\), one has \(\sup_{x \in B_r} |\nabla_x F(\cdot, x)| \in L^1([0, T])\), with \(F(\cdot, 0) \in L^1([0, T])\).

Given \(\phi \in \mathcal{A}\) and \(F \in \mathcal{B}\), we consider the problem

\[
\begin{align*}
(\phi(u'))' &= \nabla_x F(t, u) \quad \text{in } [0, T] \\
u(0) &= u(T), \quad u'(0) = u'(T) .
\end{align*}
\]

A solution of this problem is any function \(u : [0, T] \to \mathbb{R}^n\) of class \(C^1\), with \(u([0, T]) \subset B_L\), \(u(0) = u(T), u'(0) = u'(T)\), such that the composite function \(\phi \circ u'\) is absolutely continuous in \([0, T]\) and one has \((\phi \circ u')(t) = \nabla_x F(t, u(t))\) for a.e. \(t \in [0, T]\).

Now, we set

\[
K = \{u \in \operatorname{Lip}([0, T], \mathbb{R}^n) : |u'(t)| \leq L \text{ for a.e. } t \in [0, T], u(0) = u(T)\} ,
\]

\(\operatorname{Lip}([0, T], \mathbb{R}^n)\) being the space of all Lipschitzian functions from \([0, T]\) into \(\mathbb{R}^n\).

Clearly, one has

\[
\sup_{[0, T]} |u| \leq LT + \inf_{[0, T]} |u| \tag{1.1}
\]

for all \(u \in K\).
Next, consider the functional $I : K \to \mathbb{R}$ defined by

$$I(u) = \int_0^T (\Phi(u'(t)) + F(t, u(t))) dt$$

for all $u \in K$.

In [1], Brezis and Mawhin proved the following result:

**THEOREM 1.A** ([1], Theorem 5.2). - *Any global minimum of $I$ in $K$ is a solution of problem $(P_{\phi,F})$.*

In [4], using Theorem 1.A jointly with the theory developed in [2], we obtained the following multiplicity theorem:

**THEOREM 1.B** ([4], Theorem 3.1). - *Let $\phi \in A$, $F \in B$ and $G \in C^1(\mathbb{R}^n)$. Moreover, let $\gamma : [0, +\infty[ \to \mathbb{R}$ be a convex strictly increasing function such that $\lim_{s \to +\infty} \frac{\gamma(s)}{s} = +\infty$. Assume that the following assumptions are satisfied:

$(i_1)$ for a.e. $t \in [0, T]$ and for every $x \in \mathbb{R}^n$, one has

$$\gamma(|x|) \leq F(t, x) ;$$

$(i_2)$ $\lim \inf_{|x| \to +\infty} \frac{G(x)}{|x|} > -\infty$ ;

$(i_3)$ the function $G$ has no global minima in $\mathbb{R}^n$ ;

$(i_4)$ there exist two distinct points $x_1, x_2 \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{R}^n} \int_0^T F(t, x) dt < \max \left\{ \int_0^T F(t, x_1) dt, \int_0^T F(t, x_2) dt \right\}$$

and

$$G(x_1) = G(x_2) = \inf_{B_c} G$$

where

$$c = LT + \gamma^{-1} \left( \frac{1}{T} \max \left\{ \int_0^T F(t, x_1) dt, \int_0^T F(t, x_2) dt \right\} \right) .$$

Then, for every $\psi \in L^1([0, T]) \setminus \{0\}$, with $\psi \geq 0$, there exists $\tilde{\lambda} > 0$ such that the problem

$$\begin{cases}
(\phi(u'))' = \nabla_x (F(t, u) + \tilde{\lambda} \psi(t) G(u)) & \text{in } [0, T] \\
u(0) = u(T) , \ u'(0) = u'(T)
\end{cases}$$

has at least two solutions which are global minima in $K$ of the functional

$$u \to \int_0^T (\Phi(u'(t)) + F(t, u(t)) + \tilde{\lambda} \psi(t) G(u(t))) dt .$$

Clearly, condition $(i_4)$ is a little involved and has a typical quantitative nature, due to the presence of the constant $c$. This kind of drawback, however, is largely compensated by the great generality of the conclusion, due to its validity for any $\psi \in L^1([0, T]) \setminus \{0\}$, with $\psi \geq 0$.

The aim of the present short paper is to give a further contribution to the subject, adopting assumptions of qualitative nature only, in the spirit of Theorem 1.2 of [5].

2. Results

The two main tools we will use to prove our main theorem are as follows:
THEOREM 2.1 ([3], Theorem 1.2). - Let $X$ be a topological space, $E$ a real vector space, $Y \subseteq E$ a non-empty convex set and $J : X \times Y \to \mathbb{R}$ a function which is lower semicontinuous and inf-compact in $X$, and concave in $Y$. Moreover, assume that

$$\sup_y \inf_x J < \inf_x \sup_y J .$$

Then, there exists $\hat{y} \in Y$ such that the function $J(\cdot, \hat{y})$ has at least two global minima.

PROPOSITION 2.1 ([5], Proposition 2.2). - Let $X,Y$ be two non-empty sets and $f : X \to \mathbb{R}$, $g : X \times Y \to \mathbb{R}$ two given functions. Assume that there are two sets $A,B \subseteq X$ such that:

(a) $\sup_{A} f < \inf_{B} \sup_{Y} f$;
(b) $\sup_{y \in Y} \inf_{x \in A} g(x,y) < 0$;
(c) $\inf_{x \in B} \sup_{y \in Y} g(x,y) < 0$;
(d) $\inf_{x \in X \setminus B} \sup_{y \in Y} g(x,y) = +\infty$.

Then, one has

$$\sup_{y \in Y} \inf_{x \in X} (f(x) + g(x,y)) < \inf_{x \in X} \sup_{y \in Y} (f(x) + g(x,y)) .$$

A set $Y \subseteq L^1([0,T])$ is said to have property $P$ if

$$\sup_{\psi \in Y} \int_0^T \psi(t) h(t) dt = +\infty$$

for all $h \in C^0([0,T]) \setminus \{0\}.$

Our main result is as follows:

THEOREM 2.1. - Let $\phi \in \mathcal{A}$, $F \in \mathcal{B}$ and $G \in C^1(\mathbb{R}^n)$. Assume that:

(a1) there exists $q > 0$ such that

$$\lim_{|x| \to +\infty} \frac{\inf_{t \in [0,T]} F(t,x)}{|x|^q} = +\infty$$

and

$$\limsup_{|x| \to +\infty} \frac{|G(x)|}{|x|^q} < +\infty ;$$

(a2) there exists $r \in [\inf_{\mathbb{R}^n} G, \sup_{\mathbb{R}^n} G]$ such that

$$\max \left\{ \inf_{x \in G^{-1}([-\infty,r])} \int_0^T F(t,x) dt, \inf_{x \in G^{-1}([r,\infty])} \int_0^T F(t,x) dt \right\} < \int_0^T \inf_{x \in G^{-1}(r)} F(t,x) dt .$$

Then, for every non-empty convex set $Y \subseteq L^\infty([0,T])$ with property $P$, there exists $\psi \in Y$ such that the problem

$$\begin{cases}
(\phi(u'))' = \nabla_x (F(t,u) + \psi(t)G(u)) & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}$$

has at least two solutions which are global minima in $K$ of the functional

$$u \to \int_0^T (\Phi(u')(t) + F(t,u(t)) + \psi(t)G(u(t))) dt .$$

PROOF. Fix a non-empty convex set $Y \subseteq L^\infty([0,T])$ with property $(P)$. Let $C^0([0,T],\mathbb{R}^n)$ be the space of all continuous functions from $[0,T]$ into $\mathbb{R}^n$, with the norm $\sup_{[0,T]} |u|$. To achieve the conclusion,
we are going to apply Theorem 2.A taking $X = K$, regarded as a subset of $C^0([0,T], \mathbb{R}^n)$ with the relative topology, and $J : K \times Y \to \mathbb{R}$ defined by

$$J(u, \psi) = \int_0^T \left( \Phi(u'(t)) + F(t, u(t)) \right) dt + \int_0^T \psi(t) (G(u(t)) - r) dt$$

for all $(u, \psi) \in K \times Y$. Clearly, $J(u, \cdot)$ is concave in $Y$. Fix $\psi \in Y$. By Lemma 4.1 of [1], $J(\cdot, \psi)$ is lower semicontinuous in $K$. Let us show that $J(\cdot, \psi)$ is inf-compact in $K$. By $(a_1)$, there exist $k, \delta, \nu > 0$, with

$$\nu > k \|\psi\|_{L^\infty},$$

such that

$$|G(x)| \leq k(|x|^q + 1)$$

for all $x \in \mathbb{R}^n$ and

$$F(t, x) \geq \nu|x|^q$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^n \setminus B_\delta$. Since $F \in \mathcal{B}$, there exists $M \in L^1([0,T])$ such that

$$|\nabla_x F(t, x)| \leq M(t)$$

for all $t \in [0,T]$ and $x \in B_\delta$. By the mean value theorem, we have

$$F(t, x) - F(t, 0) = \langle \nabla_x F(t, \xi), x \rangle$$

for some $\xi$ in the segment joining $0$ and $x$. Consequently, for all $t \in [0,T]$ and $x \in B_\delta$, we have

$$|F(t, x)| - |F(t, 0)| \leq |F(t, x) - F(t, 0)| \leq \delta M(t)$$

and so, if we put

$$\beta(t) = \nu \delta^q + M(t) \delta + |F(t, 0)|,$$

we have

$$F(t, x) \geq \nu |x|^q - \beta(t)$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^n$. Now, set

$$\eta = -\int_0^T \beta(t) dt + \Phi(0) T - r \int_0^T |\psi(t)| dt,$$

and

$$\eta_1 = \eta - k \int_0^T |\psi(t)| dt.$$

For each $u \in K$, with $\sup_{[0,T]} |u| \geq LT$, taking (1.1), (2.1), (2.2), (2.3) into account, we have

$$J(u, \psi) \geq \int_0^T F(t, u(t)) dt - \int_0^T |\psi(t) G(u(t))| dt + \Phi(0) T - r \int_0^T |\psi(t)| dt$$

$$\geq \nu \int_0^T |u(t)|^q dt - \int_0^T |\psi(t) G(u(t))| dt + \eta \geq \nu \int_0^T |u(t)|^q dt - k \|\psi\|_{L^\infty} \int_0^T |u(t)|^q dt + \eta_1$$

$$\geq (\nu - k \|\psi\|_{L^\infty}) T \left( \inf_{[0,T]} |u| \right)^q + \eta_1 \geq (\nu - k \|\psi\|_{L^\infty}) T \left( \sup_{[0,T]} |u| - LT \right)^q + \eta_1$$

Consequently

$$\sup_{[0,T]} |u| \leq \left( \frac{J(u, \psi) - \eta_1}{(\nu - k \|\psi\|_{L^\infty}) T} \right)^{\frac{1}{q}} + LT.$$
Fix $\rho \in \mathbb{R}$. By (2.4), the set $C_\rho := \{ u \in K : J(u, \psi) \leq \rho \}$ turns out to be bounded. Moreover, the functions belonging to $C_\rho$ are equi-continuous since they lie in $K$. As a consequence, by the Ascoli-Arzelà theorem, $C_\rho$ is relatively compact in $C^0([0, T], \mathbb{R}^n)$. By lower semicontinuity, $C_\rho$ is closed in $K$. But $K$ is closed in $C^0([0, T], \mathbb{R}^n)$ and hence $C_\rho$ is compact. The inf-compactness of $J(\cdot, \psi)$ is so shown. Now, to obtain the strict minimax inequality required by Theorem 2.1, we use Proposition 2.A. By $(a_2)$, there are $x_1, x_2 \in \mathbb{R}^n$ such that

$$G(x_1) < r < G(x_2)$$

and

$$\max \left\{ \int_0^T F(t, x_1) dt, \int_0^T F(t, x_2) dt \right\} < \inf_{x \in G^{-1}(r)} F(t, x).$$

Now, put

$$A = \{ x_1, x_2 \},$$

$$B = \{ u \in K : u([0, T]) \subseteq G^{-1}(r) \}$$

and define $f : K \to \mathbb{R}$, $g : K \times Y \to \mathbb{R}$ by

$$f(u) = \int_0^T (\Phi(u'(t)) + F(t, u(t))) dt,$$

$$g(u, \psi) = \int_0^T \psi(t)(G(u(t)) - r) dt$$

for all $u \in K$, $\psi \in Y$. Since the constant functions (from $[0, T]$ into $\mathbb{R}^n$) belong to $K$, we think of $A$ as a subset of $K$. With these choices, in connection with Proposition 2.A, $(a)$ is a simple consequence of (2.6); $(b)$ follows immediately from (2.5); $(c)$ is obvious since $g(u, \psi) = 0$ for all $u \in B$. Finally, concerning $(d)$, observe that, if $u \in K \setminus B$, then the continuous function $G \circ u - r$ is not zero and hence $\sup_{\psi \in Y} g(u, \psi) = +\infty$ since $Y$ has property $(P)$. Therefore, Proposition 2.A ensures that

$$\sup_{\psi \in Y} \inf_{K} J < \inf_{K} \sup_{Y} J.$$

Now, our conclusion follows directly from Theorem 2.1 and Theorem 1.A. \( \triangle \)

We now point out three remarkable corollaries of Theorem 2.1.

**COROLLARY 2.1.** - Let $\phi \in \mathcal{A}$, $F \in \mathcal{B}$ and $G \in C^1(\mathbb{R}^n)$. Besides condition $(a_1)$, assume that $F(t, \cdot)$ is even for all $t \in [0, T]$, that $G$ is odd and that

$$\inf_{x \in \mathbb{R}^n} \int_0^T F(t, x) dt < \int_0^T \inf_{x \in G^{-1}(0)} F(t, x) dt.$$

Then, the conclusion of Theorem 2.1 holds.

**PROOF.** By assumption, there is $\hat{x} \in \mathbb{R}^n$ such that

$$\int_0^T F(t, \hat{x}) dt < \int_0^T \inf_{x \in G^{-1}(0)} F(t, x) dt.$$

So, $G(\hat{x}) \neq 0$. Assume, for instance, that $G(\hat{x}) > 0$. Then, since $G$ is odd, $G(-\hat{x}) < 0$. But, since $F(t, \cdot)$ is even, we have

$$\int_0^T F(t, \hat{x}) dt = \int_0^T F(t, -\hat{x}) dt.$$

Therefore, condition $(a_2)$ is satisfied with $r = 0$, and the conclusion of Theorem 2.1 follows. \( \triangle \)
COROLLARY 2.2. - Let $\phi \in \mathcal{A}$ and let $F, G \in C^1(\mathbb{R}^n)$, with $\lim_{|x|\to\infty} G(x) = +\infty$. Besides condition (a$_1$), assume that there exists a point $x_0 \in \mathbb{R}^n$ which is, at the same time, the unique global minimum of $G$ and a strict local, not global, minimum of $F$.

Then, for each $\gamma \in L^1([0,T])$, with $\inf_{[0,T]} \gamma > 0$, and for each non-empty convex set $Y \subseteq L^\infty([0,T])$ with property (P), there exists $\psi \in Y$ such that the problem

$$
\begin{cases}
(\phi(u'))' = \nabla_x (\gamma(t) F(u) + \psi(t) G(u)) & \text{in } [0,T] \\
u(0) = u(T), \ u'(0) = u'(T)
\end{cases}
$$

has at least two solutions which are global minima in $K$ of the functional

$$u \to \int_0^T (\Phi(u'(t)) + \gamma(t) F(u(t)) + \psi(t) G(u(t))) dt .$$

PROOF. By assumption, there are $x_1 \in \mathbb{R}^n$ and $\rho > 0$ such that

$$F(x_1) < F(x_0) < F(x)$$

for all $x \in B(x_0, \rho) \setminus \{x_0\}$. Now, observe that, since $G$ is inf-compact (being coercive) and $x_0$ is the unique global minimum of $G$, for each sequence $\{y_k\}$ in $\mathbb{R}^n$ such that $\lim_{k \to \infty} G(y_k) = G(x_0)$, we have $\lim_{k \to \infty} y_k = x_0$. As a consequence, we can fix $r > G(x_0)$ so that

$$G^{-1}(]-\infty, r[)) \subseteq B(x_0, \rho) .$$

From (2.7) and (2.8), it follows that

$$G(x_1) > r$$

as well as, by compactness,

$$F(x_0) < \inf_{x \in G^{-1}(r)} F(x) .$$

At this point, it is clear that, for each $\gamma \in L^1([0,T])$, with $\inf_{[0,T]} \gamma > 0$, the function $(t,x) \to \gamma(t) F(x)$ satisfies conditions (a$_1$) and (a$_2$) and the conclusion follows. \( \triangle \)

Recall that a real-valued function on a convex subset of a vector space is said to be quasi-convex if its sub-level sets are convex.

COROLLARY 2.3. - Let $n = 1$ and let $\phi \in \mathcal{A}$, $F \in \mathcal{B}$, $G \in C^1(\mathbb{R})$. Besides condition (a$_1$), assume that $G$ is strictly monotone and that $x \to \int_0^T F(t,x) dt$ is not quasi-convex.

Then, the conclusion of Theorem 2.1 holds.

PROOF. By assumption, there are $x_1, x_2, x_3 \in \mathbb{R}$, with $x_1 < x_3 < x_2$, such that

$$\max \left\{ \int_0^T F(t,x_1) dt, \int_0^T F(t,x_2) dt \right\} < \int_0^T F(t,x_3) dt .$$

Moreover, the numbers $G(x_1) - G(x_3)$ and $G(x_2) - G(x_3)$ have opposite signs and $G^{-1}(G(x_3)) = \{x_3\}$. Therefore, condition (a$_2$) is satisfied with $r = G(x_3)$, and the conclusion follows. \( \triangle \)

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References

[1] H. BREZIS and J. MAWHIN, *Periodic solutions of Lagrangian systems of relativistic oscillators*, Commun. Appl. Anal., 15 (2011), 235-250.

[2] B. RICCERI, *Well-posedness of constrained minimization problems via saddle-points*, J. Global Optim., 40 (2008), 389-397.

[3] B. RICCERI, *On a minimax theorem: an improvement, a new proof and an overview of its applications*, Minimax Theory Appl., 2 (2017), 99-152.

[4] B. RICCERI, *Multiple periodic solutions of Lagrangian systems of relativistic oscillators*, in “Current Research in Nonlinear Analysis - In Honor of Haim Brezis and Louis Nirenberg”, Th. M. Rassias ed., 249-258, Springer, 2018.

[5] B. RICCERI, *Miscellaneous applications of certain minimax theorems II*, Acta Math. Vietnam., to appear.

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