Effective-Range Corrections to the Proton-Proton Fusion Rate

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Abstract: Proton-proton fusion is considered in the effective field theory of Kaplan, Savage and Wise. Coulomb effects are included systematically in a non-perturbative way. Including the dimension-eight derivative coupling which determines the effective ranges of the scattering amplitudes, next-to-leading order corrections to the fusion rate are calculated. When the renormalization mass is well above the characteristic energies of the system, this contribution gives a rate which is eight percent below the standard value. The difference can be due to an unknown counterterm which comes in at this order.

During the last year much progress has been made in understanding the low-energy properties of few-nucleon systems from the effective field theory of Kaplan, Savage and Wise[1]. Both elastic and inelastic processes can be considered. Bound states like the deuteron can be treated via its interpolating field which replaces the need for explicit wavefunctions[2]. A more general review of the whole approach has recently been given by Kaplan[3].

Including the two lowest order interactions with coupling constants $C_0$ and $C_2$, the effective Lagrangian for the nucleon field $N^T = (p, n)$ with mass $M$ can be written as

$$\mathcal{L}_0 = N^\dagger \left( i\partial_t + \frac{\nabla^2}{2M} \right) N - C_0(N^T \Pi_N) \cdot (N^T \Pi_N)^\dagger$$
$$+ \frac{1}{2} C_2 \left\{ (N^T \nabla^2 \Pi_N) \cdot (N^T \Pi_N)^\dagger + h.c. \right\}$$

(1)

where the operator $\nabla = (\nabla - \nabla)/2$. The projection operators $\Pi_i$ enforce the correct spin and isospin quantum numbers in the channels under investigation. More specifically, for spin-singlet interactions $\Pi_i = \sigma_2 \tau_2 \gamma_i/\sqrt{8}$ while for spin-triplet interactions $\Pi_i = \sigma_2 \sigma_i \tau_2 /\sqrt{8}$. This theory is now valid below an upper momentum $\Lambda$ which will be the physical cutoff when the theory is regularized that way. For momenta much smaller than the pion mass, we can consider the pion field integrated out and all its effects soaked up in the two coupling constants $C_0$ and $C_2$. Then the value of the cutoff $\Lambda$ will be set by the pion mass $m_\pi$. In this momentum range all the main properties of few-nucleon systems should then in principle be given by the above Lagrangian. More accurate results will follow from higher order operators in this field-theoretic description[4].

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From calculations of proton-neutron elastic scattering using the PDS regularization scheme[1][2] or the equivalent OS scheme[5], the \textit{a priori} unknown coupling constants $C_0$ and $C_2$ can be determined in terms of experimental quantities. In the spin-triplet channel the deuteron will appear as a bound state and the corresponding lowest order renormalized coupling constant is

$$C_{d0}(\mu) = \frac{4\pi}{M} \frac{1}{\gamma - \mu}$$

where $\gamma = 45.7$ MeV is the momentum in the deuteron bound state[2][4]. The renormalization mass $\mu$ can be chosen freely in the interval $\gamma < \mu \leq m_\pi$ and physical results should be independent of its precise value. Similarly, for the dimension-eight coupling constant one finds

$$C_{d2}(\mu) = \frac{4\pi}{M} \left( \frac{1}{\gamma - \mu} \right)^2 \frac{\rho_d}{2}$$

where $\rho_d = 1.76$ fm is the spin-triplet $pn$ effective range scattering parameter evaluated at the deuteron pole. Together with the similar coupling constants in the spin-singlet channel, many properties of the proton-neutron system can be calculated. In particular, results for proton-neutron radiative capture $n + p \rightarrow d + \gamma$ have recently been obtained[4][6] which from the hadronic point of view is very similar to proton-proton fusion $p + p \rightarrow d + e^+ + \nu_e$.

The coupling constants in the $pp$ channel can also be matched to experimental scattering data at low energies. However, in this energy range the Coulomb effects become important and must be separated out. This has recently been done within the framework of the same effective theory[7] and also within standard quantum mechanics[8]. The leading order coupling constant in the spin-singlet $pp$ channel can now be written as

$$C_{p0}(\mu) = \frac{4\pi}{M} \frac{1}{1/a(\mu) - \mu}$$

where $a$ would be the strong scattering length when there were no Coulomb interactions. But these turn it into a $\mu$-dependent quantity which can be determined from the measured scattering length $a_p = -7.82$ fm using

$$\frac{1}{a(\mu)} = \frac{1}{a_p} + \alpha M \left[ \ln \left( \frac{\mu \sqrt{\pi}}{\alpha M} \right) + 1 - \frac{3}{2} C_E \right]$$

where $\alpha$ is the fine-structure constant and $C_E = 0.5772\ldots$ is Euler’s constant. Since the proton mass is so much heavier than $1/a_p$, the Coulomb correction is seen to be surprisingly large. This has been known for a long time and was previously expressed by the corresponding Jackson-Blatt relation of the same form and obtained from potential models[10]. The higher order coupling constant $C_{p2}$ in this channel will have the same form as $C_{d2}$ in (3) but with $\gamma$ replaced with $1/a(\mu)$ and $\rho_d$ with the proton-proton effective range $\rho_p = 2.79$ fm. We have shown that it is not affected by Coulomb corrections to this order in the effective theory. However, the $C_{p2}$ coupling gives an important contribution to the scattering length (5) which picks up an additional term $-\mu \rho_p/2$ in the parenthesis[7].
Including only the leading order couplings $C^{p,d}_0$, we have recently calculated the rate for proton-proton fusion $p + p \rightarrow d + e^+ + \nu_e$ as it takes place in the Sun[9]. Since the initial energy $E = p^2/M$ is then so low, it can be taken to zero. Thus also the momenta of the final state leptons can be ignored. Stating the result in terms of the standard reduced matrix element $\Lambda(E)$ which is dimensionless[11][12], we obtained the result

$$\Lambda_0(0) = e^\chi - 2\alpha Ma \, I(\chi)$$

(6)

where the parameter $\chi = \alpha M/\gamma$. Here we have introduced the function

$$I(\chi) = \frac{1}{\chi} - e^\chi E_1(\chi)$$

(7)

where $E_1(\chi)$ is the exponential integral function. This is in full agreement with the corresponding result obtained in the zero-range approximation of nuclear potential models[12]. With the above values for the different parameters, we have $\chi = 0.15$ and $I(0.15) = 4.96$. The matrix element then becomes $\Lambda_0(0) = 2.51$ so that in the rate we have $\Lambda_2^0(0) = 6.30$. This is 10% below the value $\Lambda_{2,g}^0(0) = 7.0 \pm 0.05$ obtained in the most complete calculations including higher order effects[13][14].

When we now include the effects of the next order coupling $C_2$ to the fusion process, it gives rise to the Feynman diagrams shown in Fig.1. In the chain of bubbles connected by $C_0$ interactions, the two protons interact via the Coulomb potential $V_C(r) = \alpha/r$. In a single bubble the particles propagate from zero separation and back to zero separation. The contribution of a single bubble in the chains is thus $J_0(p) = G_C(E; 0, 0)$ where $G_C(E; r, r')$ is the Coulomb propagator in coordinate space of the two protons with energy $E$. We have found it most convenient to calculate the value of the bubble quantity $J_0(p)$ in momentum space. With $\int_q \equiv \int d^3q/(2\pi)^3$ it is

$$J_0(p) = \int_q \int_{q'} G_C(E; q, q')$$

(8)

when expressed in terms of the Fourier-transformed Coulomb propagator $G_C(E; q, q')$. This integral can now be done using either dimensional regularization in the PDS scheme or a standard momentum cutoff to regularize the ultraviolet divergence it contains. The result has been used to obtain the Coulomb-corrected scattering length (5) and the leading order fusion result (6).

In some of the diagrams in Fig.1 the derivative operator $\hat{\nabla}^2$ acts on such Coulomb-dressed bubbles and gives rise to more divergent integrals. One of the simplest is

$$J_2(p) = \int_q \int_{q'} q^2 G_C(E; q, q')$$

(9)

Using a method based upon the the Fourier-transformed Coulomb wavefunctions, we have obtained a finite result for this integral[7]. However, the method lacks a firm foundation and cannot easily be applied to other integrals of a more general form. We will therefore
Figure 1: Feynman diagrams contributing to the fusion process to first order in the derivative coupling $C_2$ denoted by black square. The wiggly line indicates the weak current while the cross circle denotes the action of the deuteron interpolating field.

Here use a more direct and simpler method based upon the functional equation satisfied by the Coulomb Green’s function. Introducing the free propagator

$$G_0(E; q, q') = \frac{M}{p^2 - q^2 + i\epsilon} (2\pi)^3 \delta(q - q')$$

the functional equation is $G_C = G_0 + G_0 V_C G_C$. The integral can then be written as

$$J_2(p) = p^2 J_0(p) + \int_q \int_q' (q^2 - p^2) G_0(E; q, q')$$

$$+ \int_q \int_q' \int_k \int_{k'} (q^2 - p^2) G_0(E; q, k) V_C(k, k') G_C(E; k', q')$$

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This new method arose out of a discussion with Peter Lepage to whom we are much thankful.
where $V_C(k, k') = 4\pi\alpha/(k - k')^2$ is the Fourier transform of the Coulomb potential. The first integral is now zero by dimensional regularization. Integrating over $k$ in the second, we see that the first two factors just give $-M$. In the integration of $V_C(q, k')$ over $q$, we can now shift the integration variable $q \rightarrow q - k'$ and use

$$\int q \frac{4\pi\alpha}{q^2} = \alpha\mu$$

(11)

This integral is also zero with dimensional regularization, but it contains a PDS pole in $d = 2$ space dimensions which gives the non-zero contribution. The remaining integrations over $q'$ and $k'$ give then finally simply $J_0(p)$. Collecting all the factors, we thus obtain the simple result

$$J_2(p) = (p^2 - \alpha\mu M)J_0(p)$$

(12)

This agrees with the result using our previous method except for terms which are smaller by factors of the order of $\alpha M/\mu$. These can in practice be neglected since we always take the renormalization point $\mu \gg \alpha M$.

Using the same method to evaluate other similar integrals arising from the Feynman diagrams in Fig. 1, we can then simplify and gather all the contributions into a modified reduced matrix element. A part of this new contribution goes into the next-to-leading order scattering length $a_\rho$ in the lowest order result (6). The remaining terms simplify to

$$\Lambda_2(0) = C_2^d M\gamma^2(\mu - \gamma)\Lambda_0(0) - a_\rho\gamma^2(\mu - \gamma)\frac{C_0^p + C_2^d}{2C_0^p}$$

(13)

However, there is also a wavefunction renormalization constant $\sqrt{\Sigma'}$ which to this order will modify the lowest order result (6). It will enter in all calculations involving the bound state deuteron and has been calculated by Kaplan, Savage and Wise[2]. In this version of the effective theory without pions it is given by

$$\Sigma' = 1 - C_2^d M\gamma^2\frac{1}{2\pi}(\mu - \gamma)(\mu - 2\gamma)$$

(14)

The full result for the reduced matrix element in next-to-leading order is thus

$$\Lambda_{NLO} = \Lambda_0 + \frac{\Lambda_2}{\sqrt{\Sigma'}}$$

(15)

Expanding this now to first order in $C_2^d$ with the renormalized value (3), we get the final result

$$\Lambda_{NLO}(0) = \Lambda_0(0) \left(1 + \frac{1}{2}\gamma\rho_d\right) - a_\rho\gamma^2(\mu - \gamma)\frac{C_0^p + C_2^d}{2C_0^p}$$

(16)

The last term is seen to be dependent in general on the renormalization mass $\mu$. From a physical point of view, the result should be independent of this arbitrary parameter. What
makes this possible here, is the presence of a new, local interaction which comes in as a
counterterm at this order of perturbation theory. It will be $\mu$-dependent in such a way as
to make the overall result independent of $\mu$. A very similar situation arises in the process
$n + p \rightarrow d + \gamma$ where such a counterterm also is present. The a priori magnitudes
of these counterterms are determined by physics on scales shorter than included in the
effective theory. An absolute prediction of the proton-proton fusion rate is thus not possible
at this next-to-leading order as long as this counterterm is not determined by other means.

Here we will instead compare our result with the corresponding result from po-
tential models. When we take $\mu \gg \gamma$, the dependence of the result on this arbitrary
renormalization mass becomes negligible and we find

$$\Lambda_{NLO}(0)_{\mu} = \Lambda_0(0) \left(1 + \frac{1}{2} \gamma \rho_d\right) + \frac{1}{4} \alpha \rho_d^2 (\rho_p + \rho_d) \tag{17}$$

With the previous values of the different parameters, we obtain for the reduced matrix
element the value $\Lambda(0) = 2.54$ which is just a 1.4% addition to the leading order result.
This is surprisingly small, but results from an almost total cancellation between the two
effective-range corrections in (17). In the full rate, it corresponds to a value which is 8% below
the accepted value from nuclear potential models. However, the structure
of our result is very similar when compared to what one obtains in the corresponding
effective range approximation. The last term in (17) is then exactly the same, while in
the first term the factor $\left(1 + \frac{1}{2} \gamma \rho_d\right)$ is replaced by $\left(1 - \gamma \rho_d\right)^{-1/2}$ which in nuclear models
is the normalization factor of the deuteron wavefunction. To lowest order in the deuteron
effective range parameter $\rho_d$ this is then just the same. But $\gamma \rho_d = 0.41$ is really not a small
expansion parameter and higher order terms in the expansion of the square root give a
sizeable contribution. These can only be reproduced in the effective theory when including
even higher order interactions. At the same time this will then also bring in new and
unknown counterterms. It thus seems difficult for this effective theory to compete with
the more accurate results obtained from potential models where the deuteron effective
range corrections are included to all orders.

A more detailed presentation of the calculation and discussion of our results including
the presence of the counterterm, will be presented elsewhere.

We want to thank the organizers of the workshop on “Nuclear Physics with Effective
Field Theory” at the Institute of Nuclear Theory during which this work came to conclu-
sion. In addition, we are grateful to the Department of Physics and the INT for generous
support and hospitality. Xinwei Kong is supported by the Research Council of Norway.

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