The order of the decay of the hole probability for Gaussian random $\text{SU}(m+1)$ polynomials.
by Scott Zrebiec

Abstract
We show that for Gaussian random $\text{SU}(m+1)$ polynomials of a large degree $N$ the probability that there are no zeros in the disk of radius $r$ is less than $e^{-c_1 r N^{m+1}}$, and is also greater than $e^{-c_2 r N^{m+1}}$. Enroute to this result, we also derive a more general result: probability estimates for the event where the volume of the zero set of a random polynomial of high degree deviates significantly from its mean.

1. Introduction and notation.
A hole refers to the event where a particular Gaussian random holomorphic function has no zeros in a given domain where many are expected. The order of the decay of the hole probability has been computed in several cases including for “flat” complex Gaussian random holomorphic functions on $\mathbb{C}^1$, [ST2], using a method which shall be used here. This work was subsequently refined to cover other large deviations in the distribution of the zeros sets, [Kri], and generalized to $\mathbb{C}^m$, [Zre]. Other results compute the hole probability for a class of complex Gaussian random holomorphic functions on the unit disk, [PV], and provide a weak general estimate for any one variable complex Gaussian random holomorphic functions, [Sod]. Additionally significant hole probability results have been discovered for real Gaussian random polynomials, ([DPSZ], [LS]).

Various properties of the zeros of random $\text{SU}(m+1)$ polynomials have been studied, in particular the zero point correlation functions have been computed. This is of particular interest in the physics literature as the zeros describe a random spin state for the Majorana representation (modulo phase), [Han]. Further this choice is intuitively pleasing as the zeros are uniformly distributed on $\mathbb{C}P^m$ (according to the Fubini-Study metric), or alternatively the average distribution of zeros is invariant under the $\text{SU}(m+1)$ action on $\mathbb{C}P^m$. These random $\text{SU}(m+1)$ polynomials can be written as:

$$
\psi_{\alpha,N}(z) = \sum_{|j|=0}^{N} \alpha_j \sqrt{\binom{N}{j}} z^j = \sum_{\sum j_k \leq N} \alpha_j \sqrt{\binom{N}{j_1, \ldots, j_m}} z_{j_1}^{j_1} z_{j_2}^{j_2} \ldots z_{j_m}^{j_m}, \quad (1)
$$
using standard multi index notation, and where $\forall j$, $\alpha_j$, are independent identically distributed standard complex Gaussian random variables (mean 0 and variance 1).

For these Gaussian random SU($m+1$) polynomials we will be computing the hole probability in a manner based on that used by Sodin and Tsirelson to solve the similar problem for flat random holomorphic functions on $\mathbb{C}$, \cite{ST2}. In particular, we shall be estimating the unintegrated counting function for a random SU($m+1$) variable polynomial, which is defined as

$$n_{\psi_{\alpha,N}}(r) = \frac{1}{r^{2m-2}} \cdot \text{Volume of the zero set of } \psi_{\alpha,N} \cap B(0,r),$$

where $B(0,r) = \{ z \in \mathbb{C}^m : |z| < r \}$.

Our first main result will be the following:

**Theorem 1.1.** Let $\psi_{\alpha,N}$ be a degree $N$ Gaussian random SU($m+1$) polynomial,

$$\psi_{\alpha,N}(z) = \sum_j \alpha_j \sqrt{\binom{N}{j}} z^j,$$

where $\alpha_j$ are independent identically distributed complex Gaussian random variables, and let $n_{\psi_{\alpha,N}}(r)$ be the unintegrated counting function.

For all $\Delta > 0$, and $r > 0$ there exists $A_{\Delta,r,m}$ and $N_{\Delta,r,m}$ such that for all $N > N_{\Delta,r,m}$

$$\left| n_{\psi_{\alpha,N}}(r) - \frac{N r^2}{1 + r^2} \right| \leq \Delta N$$

except for an event whose probability is less than $e^{-A_{\Delta,r,m}N^{m+1}}$.

Theorem 1.1 gives an upper bound on the rate of decay of the hole probability, and we will be able to prove a lower bound for the decay rate of the same order:

**Theorem 1.2.** Let $\psi_{\alpha,N}$ be as in theorem 1.1, and let

$$\text{Hole}_{N,r} = \{ \alpha : \forall z \in B(0,r), \ \psi_{\alpha,N}(z) \neq 0 \},$$

then there exists $c_{1,r,m}$, $c_{2,r,m} > 0$ and $N_r$ such that for all $N > N_{r,m}$

$$e^{-c_{2,r}N^{m+1}} \leq \text{Prob}(\text{Hole}_{N,r}) \leq e^{-c_{1,r}N^{m+1}}.$$

As an immediate consequence of this result, the order of the probability specified in the Theorem 1.1 is the correct order of decay.
Random polynomials of the form studied here are the simplest examples of a class of natural random holomorphic sections of large $N$ powers of a positive line bundle on a compact Kähler manifold. Most of the results stated in this paper may be restated in terms of Szegö kernels, which exhibit universal behavior in the large $N$ limit in an appropriately scaled neighborhood. Hopefully, this paper will provide insight into proving a similar decay rate for this more general setting. This has already been done for other properties of random holomorphic sections, e.g. correlation functions, [BSZ].

Acknowledgement: I would like to thank Bernard Shiffman and Misha Sodin for many useful discussions.

2. SU($m + 1$) Invariance.

We begin by letting $Poly_N$ denote the set of polynomials in $m$ variables whose degree is less than or equal to $N$. $Poly_N$ becomes a Hilbert space with respect to the following SU($m + 1$) invariant norm, [BSZ]:

$$\|f\|_N^2 := \frac{N + m!}{N! \pi m} \int_{z \in \mathbb{C}^m} |f(z)|^2 \frac{dm(z)}{(1 + |z|^2)^{N + m + 1}};$$

where $dm$ is just the usual Lebesgue measure on $\mathbb{C}^m$. For this norm $\{\sqrt{\binom{N}{j}} z^j\}$ is an orthonormal basis, as is $\left\{ \sqrt{\binom{N}{j}} \prod_{k=1}^{m} (\sum_{l=1}^{m} a_{k,l} z_l + a_{k,0})^j \right\}$, where $A = (a_{k,l})$ and $A \cdot A^T = I$, and $j_0 = N - |j|$. Specifically, one alternate orthonormal basis is, for any $\zeta \in \mathbb{C}^1$,

$$\left\{ \sqrt{\binom{N}{j}} \left( \frac{z_1 - \zeta}{\sqrt{1 + |\zeta|^2}} \right)^j \left( \frac{1 + \bar{\zeta} z_1}{\sqrt{1 + |\zeta|^2}} \right)^{N-|j|} \prod_{k=2}^{m} \frac{z_k}{|z_k|} \right\}_{|j| \leq N}$$

Clearly, by line (1), a Gaussian random SU($m + 1$) polynomial is defined as, $\psi_{\alpha,N}(z) = \sum_{|j|=N}^{\infty} \alpha_j \psi_j(z)$, where $\alpha_j$ are i.i.d. standard complex Gaussian random variables, and $\{\psi_j\}$ is the first orthonormal basis. Any basis for $Poly_N$ could have been used and the Gaussian random SU($m + 1$) polynomials would be probabilistically identical, as for $\{\alpha_j\}$ a sequence of i.i.d. Gaussian random variables there exists another sequence of i.i.d. Gaussian
random variables, \( \{ \alpha_j' \} \), such that

\[
\sum \alpha_j \sqrt{\binom{N}{j}} z^j = \sum_{|j|=0}^{N} \alpha_j' \sqrt{\binom{N}{j}} \left( \frac{z_1 - \zeta}{\sqrt{1 + |\zeta|^2}} \right)^{j_1} \left( \frac{1 + \overline{\zeta} z_1}{\sqrt{1 + |\zeta|^2}} \right)^{N-|j|} \prod_{k=2}^{N} z_k^{j_k}.
\]

(2)

3. Large deviations of the maximum of a random SU\((m+1)\) polynomial.

In order to estimate \( \max_{B(0,r)} \log |\psi_{\alpha,N}| \), we will use following elementary estimates to compute upper and lower bounds for the probability of several events:

**Proposition 3.1.** Let \( \alpha \) be a standard complex Gaussian random variable, then

i) \( \text{Prob}(\{ |\alpha| \geq \lambda \}) = e^{-\lambda^2} \)

ii) \( \text{Prob}(\{ |\alpha| \leq \lambda \}) = 1 - e^{-\lambda^2} \in [\frac{\lambda^2}{2}, \lambda^2], \text{ if } \lambda \leq 1 \)

iii) if \( \lambda \geq 1 \) then \( \text{Prob}(\{ |\alpha| \leq \lambda \}) \geq \frac{1}{2} \)

This next lemma is key as it states that the maximum of the norm of a random SU\((m+1)\) polynomial on the ball of radius \( r \) tends to not be too far from its expected value.

**Lemma 3.2.** For all \( \delta \in (0,1) \), and for all \( r > 0 \) there exists \( a_{r,\delta,m} > 0 \) and \( N_{\delta,m} \) such that for all \( N > N_{\delta,m} \)

\[
\max_{B(0,r)} |\psi_{\alpha,N}(z)| \in \left[ (1 + r^2)^{\frac{N}{m}} (1 - \delta)^{\frac{N}{m}}, (1 + r^2)^{\frac{N}{m}} (1 + \delta)^{\frac{N}{m}} \right],
\]

except for an event whose probability is less than \( e^{-a_{r,\delta,m} N^{m+1}} \).

**Proof.** We will first prove a sharper decay estimate for the probability of the event where a random SU\((m+1)\) polynomial takes on large values in the ball of radius \( r \):

\[
\text{Prob} \left( \{ \max_{B(0,r)} |\psi_{\alpha,N}(z)| > (1 + r^2)^{\frac{N}{m}} (1 + \delta)^{\frac{N}{m}} \} \right) < e^{-c_m N^{2m}}.
\]

To do this we consider the event \( \Omega_N := \{ \forall j, \ |\alpha_j| \leq N^m \} \), the complement of which has probability \( \leq (N+1)^m e^{-N^{2m}} \), by Proposition 3.1. For \( \alpha \in \Omega_N \),
\[
\max_{z \in B(0,r)} |\psi_{\alpha,N}(z)| = \max_{z \in B(0,r)} \left| \sum \alpha_j \left( \frac{N}{j} \right)^{\frac{1}{2}} (z)^j \right|
\]

\[
\leq \max_{z \in B(0,r)} \sum |\alpha_j| \left( \frac{N}{j} \right)^{\frac{1}{2}} |z|^j
\]

\[
\leq \max_{z \in B(0,r)} N^m (N + 1)^{\frac{m}{2}} \left( 1 + \sum |z_i|^2 \right)^{\frac{N}{2}},
\]

by the Schwartz inequality.

\[
= N^m (N + 1)^{\frac{m}{2}} (1 + r^2)^{\frac{N}{2}}
\]

\[
\leq (1 + \delta)^{\frac{N}{2}} (1 + r^2)^{\frac{N}{2}}, \text{ if } N > N'_{\delta,m,1}
\]

In other words, if \(N > N'_{\delta,m,1}\) then

\[
\left\{ \max_{B(0,r)} |\psi_{\alpha,N}(z)| > (1 + r^2)^{\frac{N}{2}} (1 + \delta)^{\frac{N}{2}} \right\} \subset \Omega_N^c
\]

and thus, for all \(N > N'_{\delta,m}\), this first event has probability less than or equal to \((N + 1)^m e^{-N^{2m}} < e^{-\frac{1}{2}N^{2m}}\). This decay rate is independent of \(\delta\) and \(r\), and the estimate for the order of the decay of this probability could be improved upon.

We complete the proof by showing that:

\[
Prob \left( \left\{ \max_{B(0,r)} |\psi_{\alpha,N}(z)| < (1 + r^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}} \right\} \right) < e^{-a_{r,\delta} N^{m+1}}.
\]

This will be done when we prove the following claim concerning a polydisk, \(P(0, \frac{1}{\sqrt{m}} r) := \{ z \in \mathbb{C}^m : |z_1| < \frac{1}{\sqrt{m}} r, |z_2| < \frac{1}{\sqrt{m}} r, \ldots, z_m < \frac{1}{\sqrt{m}} r \} \):

\[
Prob \left( \left\{ \max_{P(0,r)} |\psi_{\alpha,N}(z)| < (1 + mr^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}} \right\} \right) < e^{-a_{r,\delta} N^{m+1}}.
\]

This second claim is stronger as \(\max_{P(0, \frac{1}{\sqrt{m}} r)} |\psi_{\alpha,N}(z)| \leq \max_{B(0,r)} |\psi_{\alpha,N}(z)|\).

Consider the event where

\[
M = \max_{P(0, \frac{1}{\sqrt{m}} r)} |\psi_{\alpha,N}(z)| < (1 + mr^2)^{\frac{N}{2}} (1 - \delta)^{\frac{N}{2}}.
\]

We will show that this event can only occur if certain Gaussian random variables, \(\alpha_j\), obey the inequality \(|\alpha_j| < e^{-c_m N}\), where \(c_m > 0\). Further we will show that this occurs whenever \(j\) is in a certain cube which will have sides of length \(c_{\delta,r} N\). This will give us the desired decay rate for the probability.
The Cauchy estimates for a holomorphic function state that:

\[ |\psi_{\alpha,N}^{(j)}(0)| \leq j! \frac{M}{r^{\|j\|}}. \]

By differentiating equation (1) we compute that

\[ \psi_{\alpha,N}^{(j)}(0) = \sqrt{\binom{N}{j}} j! \alpha_j. \]

Combining this with Stirling’s formula:

\[ \sqrt{2\pi} j_1^{j_1} e^{-j_1} < j! < \sqrt{2\pi} j_1^{j_1} e^{-\frac{j_1}{2}} \]

we get that:

\[ |\alpha_j| \leq \frac{(1 + m r^2)^{\frac{N}{2}}}{r^{\|j\|}} (1 - \delta)^{\frac{N}{2}} \]

\[ \leq e^{\frac{m}{2} + \frac{m}{2} \log(2\pi)} \left( \frac{(1 + m r^2)^{\frac{N}{2}}}{r^{\|j\|}} (1 - \delta)^{\frac{N}{2}} \right) \]

\[ \leq e^{\frac{m}{2} + \frac{m}{2} \log(2\pi)} N^{\frac{m}{2}} \left( \frac{(1 + m r^2)^{\frac{N}{2}}}{r^{\|j\|}} (1 - \delta)^{\frac{N}{2}} \right) \]

For the time being we focus on the term in parenthesis in the previous line which we call \( A \). Writing \( j \) as \( j = (j_k) = (x_k N) \), \( x_k \in (0,1) \), we now have:

\[ A = (1 - \delta)^{\frac{N}{2}} \left( \frac{(1 + m r^2)}{r^{2|x|}} (1 - |x|)^{(1-|x|)} \prod_{k=1}^{m} (x_k)^{x_k} \right) \]

If for all \( k \in \{1,2,\ldots,m\} \), \( x_k = \frac{x_k}{1+mr^2} \) then \( A = (1 - \delta)^{\frac{N}{2}} \), which inspires the following claim:

Claim: Let \( s_{r,m} = \frac{1}{2m(1+mr^2)} \min \{ r^2, 1 \} \).

If for each \( k \in \{1,2,\ldots,m\} \), \( x_k \in \left[\frac{r^2}{1+mr^2} - s_{r,m}, \frac{r^2}{1+mr^2}\right] \subset (0,1) \), and thus \( |x| < 1 \) then

\[ (1 + m r^2) \left( \frac{x}{r^2} \right)^x (1 - |x|)^{(1-|x|)} < (1 - \delta)^{-\frac{1}{2}}. \]

Proof: We begin by setting \( x_k = (1 - \Delta_k) \frac{r^2}{1+mr^2} \) and \( \Delta = \sum \Delta_k \). Therefore

\[ \Delta_k \in \left[0, \frac{1}{2m} \min \left\{ 1, \frac{1}{r^2} \right\} \delta \right] \quad \text{and} \quad \Delta \in \left[0, \frac{1}{2} \min \left\{ 1, \frac{1}{r^2} \right\} \delta \right]. \]
Thus, \(1 - |x| = \frac{1 + \Delta r^2}{1 + m r^2}\), and from this we compute that:

\[
\frac{(1 + m r^2)}{r^2|x|} (1 - |x|)^{1-x} (x_k)^{x_k} = \frac{1 + m r^2}{r^2|x|} \left( \frac{1 + \Delta r^2}{1 + m r^2} \right)^{1-|x|} \cdot \left( \frac{r^2}{1 + m r^2} \right)^{|x|} \prod (1 - \Delta_k)^{x_k} = (1 + \Delta r^2)^{1-|x|} \prod (1 - \Delta_k)^{x_k} \leq (1 + \Delta r^2) \prod (1 - \Delta_k)^{x_k} \leq 1 + \frac{1}{2} \delta \leq \left( \frac{1}{1 - \delta} \right)^{\frac{1}{2}}.
\]

Proving the claim.

Therefore if for each \(j, x_j \in \left[ \frac{r^2}{1 + m r^2} - s_{r,m\delta}, \frac{r^2}{1 + m r^2} \right]\), then \(A_k < (1 - \delta)^{\frac{N}{2}}\). This in turn guarantees that \(|\alpha_j| < e^{\frac{m}{12} + \frac{m}{2} \log(2\pi)} N^{\frac{m}{4}} (1 - \delta)^{\frac{N}{2}}\). The probability this occurs for a single \(\alpha_j\) is less than or equal to

\[
\left( e^{\frac{m}{12} + \frac{m}{2} \log(2\pi)} N^{\frac{m}{4}} (1 - \delta)^{\frac{N}{2}} \right)^2.
\]

Thus the probability this occurs for all \(\alpha_j, j_k \in \left[ \frac{r^2}{1 + m r^2} - s_{r,m\delta} \left( \frac{r^2}{1 + m r^2} \right) \right]\), is less than or equal to

\[
\left( e^{\frac{m}{12} + \frac{m}{2} \log(2\pi)} N^{\frac{m}{4}} (1 - \delta)^{\frac{N}{2}} \right)^{2\left(\left\lfloor N \frac{s_{r,m\delta}}{N} \right\rfloor \right)}.
\]

Hence, there exists \(a_{r,\delta,m} > 0\) and \(N''_{\delta,m}\) such that for all \(N > N''_{\delta,m}\),

\[
\left( e^{\frac{m}{12} + \frac{m}{2} \log(2\pi)} N^{\frac{m}{4}} (1 - \delta)^{\frac{N}{2}} \right)^{2\left(\left\lfloor N \frac{s_{r,m\delta}}{N} \right\rfloor \right)} < e^{-a_{r,\delta,m} N^{m+1}}.
\]

The result follows after setting \(N_{\delta,m} = \max\{N'_m, N''_{\delta,m}\}\)

\(\square\)

A nice application of this lemma, along with line (2) is the following:

**Lemma 3.3.** For all \(\Delta \in (0, 1)\) and \(a \in \mathbb{C}^n \setminus \{0\}\) there exists \(N_{\Delta,|a|,m}\) and \(c_{\Delta,|a|,m} > 0\), such that if \(N > N_{\Delta,|a|,m}\) then

\[
\max_{z \in B(0,\Delta)} |\psi_{\alpha,N}(z - a)| < (1 + |a|^2)^{\frac{N}{2}} (1 - \Delta)^{\frac{N}{2}},
\]

except for an event whose probability is less than \(e^{-c_{\Delta,|a|,m} N^{m+1}}\).
Proof. As Gaussian random SU($m + 1$) polynomials are rotationally invariant, as a random process, with out loss of generality we assume that $a$ is of the form: $a = (\zeta_1, 0, \ldots, 0)$.

Let $\delta = \frac{\Delta}{2 + 2|\zeta_1| + 2|\zeta_1^2|}$.

By Lemma 3.2 and line (2), there exists $c_{\Delta,m} > 0$ and $N_{\Delta,a,m}$ such that if $N > N_{\Delta,a,m}$ then, except for an event whose probability is less than $e^{-c_{\Delta,a}N^{m+1}}$,

$$(1 - \delta)\frac{N}{2} \leq \frac{\max_{B(0,\delta)}|\psi_{a,N}(z)|}{(1 + \delta)^{\frac{N}{2}}} \left| \sum \alpha' \sqrt{\binom{N}{j}} \frac{z_1 - \zeta_1}{\sqrt{1 + |\zeta_1|^2}} \right|^j \left( \frac{1 + \zeta_1}{\sqrt{1 + |\zeta_1|^2}} \right)^{N - j} \prod |z_k|^{j_k}.$$ 

In order to simplify this previous line, let

$$\phi(z) = \left( \frac{z_1 - \zeta_1}{1 + \zeta_1 z_1}, \frac{z_2}{1 + \zeta_1 z_1}, \ldots, \frac{z_m}{1 + \zeta_1 z_1} \right),$$

so that we may rewrite the previous equation as:

$$(1 - \delta)\frac{N}{2} \leq \left( \max_{B(0,\delta)} \frac{1 + \zeta_1 z_1}{1 + |\zeta_1|^2} \left( \frac{1 + |\zeta_1|^2}{1 + |\zeta_1|^2 + 2\zeta_1 \delta} \right)^{\frac{N}{2}} \right) \left( \max_{B(0,\delta)}|\psi_{a,N}(\phi(z))| \right) \left( \max_{B(0,\delta)} \frac{1 + \zeta_1 z_1}{1 + |\zeta_1|^2} \left( \frac{1 + |\zeta_1|^2}{1 + |\zeta_1|^2 + 2\zeta_1 \delta} \right)^{\frac{N}{2}} \right) \left( \max_{B(-\zeta_1, (2 + \delta^2)|\zeta_1|^2) \delta} |\psi_{\alpha',N}(z)| \right),$$

as the image of $\phi \mid_{B(0,\delta)} \subset B(-\zeta_1, (2 + 2|\zeta_1|^2)\delta)$, since:

$$\max_{z \in \partial B(0,\delta)} \left| \frac{z_1 - \zeta_1}{1 + \zeta_1 z_1} + \zeta \right|^2 \leq \max_{z \in \partial B(0,\delta)} \left| \frac{z_1 - \zeta_1 + \zeta_1 z_1}{1 + \zeta_1 z_1} \right|^2 \leq \delta^2 \max_{z \in \partial B(0,\delta)} \left| \frac{z_1 - \zeta_1 + \zeta_1 z_1}{1 + \zeta_1 z_1} \right|^2 \leq 4\delta^2.$$
Rearranging the previous sets of equations we get the result:

\[
\max_{B(0, \Delta)} |\psi_{\alpha', N}(z - \zeta_1)| \geq \frac{(1 + |\zeta_1|^2)^{\frac{\Delta}{2}}(1 + 2^\delta)^{\frac{\Delta}{2}}}{(1 + |\zeta_1|\delta)^N} \cdot (1 - \delta)^{\frac{\Delta}{2}} \\
\geq (1 + |\zeta_1|^2)^{\frac{\Delta}{2}}(1 - (2 + 2|\zeta_1|\delta)^{\frac{\Delta}{2}}) \\
\geq (1 - \Delta)^{\frac{\Delta}{2}}(1 + |\zeta_1|^2)^{\frac{\Delta}{2}} 
\]

\[\Box\]

4. Second key lemma.

The goal of this section will be to estimate \(\int_{S_r} \log |\psi_{\alpha, N}(z)| d\mu_r(z)\), where \(d\mu_r(z)\) is the rotationally invariant probability measure of the sphere of radius \(r\), \(S_r = \partial B(0, r)\), which will be accomplished when we prove lemma 4.2 using the same techniques as in [ST2]. As \(\log(x)\) becomes unbounded near 0, we will first prove a deviation result for the event where the \(L^1\) norm of \(\log |\psi_{\alpha, N}|\) is significantly larger than its max on the same region.

Lemma 4.1. For all \(r > 0\) there exists \(c_{m, r}\), and \(N_m\) such that for all \(N > N_m\),

\[
\int_{S_r} |\log(|\psi_{\alpha, N}(z)||) d\mu_r(z) \leq \left(\frac{32m}{2} + \frac{1}{2}\right) N \log ((2)(1 + r^2))
\]

except for an event whose probability is < \(e^{-c_{m, r} N^{m+1}}\).

Proof. By Lemma 3.2 there exists \(c_{m, r}\), and \(N_{m, r}\) such that if \(N > N_{m, r}\) then, with the exception of an event whose probability is less than \(e^{-c_{m, r} N^{m+1}}\), there exists \(\zeta_0 \in \partial B(0, \frac{r}{2})\) such that \(\log(|\psi_{\alpha, N}(\zeta_0)||) > 0\). This also implies that:

\[
\int_{z \in S_r} P_r(\zeta_0, z) \log(|\psi_{\alpha, N}(z)||) d\mu_r(z) \geq \log(|\psi(\zeta_0)||) \geq 0,
\]

Where \(P_r\) is the Poisson kernel for the sphere of radius \(r\): \(P_r(\zeta, z) = \frac{r^{2m - 2} r^2 - |\zeta|^2}{|z - \zeta|^{2m}}\). Hence,

\[
\int_{z \in S_r} P_r(\zeta_0, z) \log^-(|\psi_{\alpha, N}(z)||) d\mu_r(z) \leq \int_{z \in S_r} P(\zeta_0, z) \log^+(|\psi_{\alpha, N}(z)||) d\mu_r(z)
\]

Now given the event where

\[
\log \max_{B(0, r)} |\psi_{\alpha}(z)| < \frac{N}{2} \log ((2)(1 + r^2)),
\]
(whose complement for \( N > N_{m,r} \) has probability less than \( e^{-c_m N^{m+1}} \)), we may estimate that

\[
\int_{z \in S_r} \log^+ (|\psi_{\alpha,N}(z)|) d\mu_r(z) \leq \frac{N}{2} \log ((2)(1 + r^2)).
\]

Since \( \zeta_0 \in \partial B(0, \frac{1}{2}r) \) and \( z = re^{i\theta} \), we have: \( \frac{r^2}{4} \leq |z - \zeta_0|^2 \leq \frac{9}{4}r^2 \). Hence, by using the formula for the Poisson Kernel,

\[
\frac{2^{2m-2}}{3^{2m-1}} \leq P(\zeta_0, z) \leq 3 \cdot 2^{2m-2}.
\]

Putting the pieces together proves the result:

\[
\int_{z \in S_r} P_r(\zeta_0, z) \log^+ (|\psi_{\alpha,N}(z)|) d\mu_r(z) \leq 3 \cdot 2^{2m-3} N \log (2(1 + r^2))
\]

\[
\int_{z \in S_r} \log^-(|\psi_{\alpha,N}(z)|) d\mu_r(z) \leq \frac{1}{\min P(\zeta_0, z)} \int_{z \in S_r} P_r(\zeta_0, z) \log^+ (|\psi_{\alpha,N}|) d\mu_r(z)
\]

\[
\leq \frac{3}{2} 2^{2m} N \log (2(1 + r^2))
\]

\[\square\]

We now arrive at the main result of this section:

**Lemma 4.2.** For all \( r > 0 \) and for all \( \Delta \in (0, 1) \) there exists \( c_{\Delta, r, m} > 0 \) and \( N_{\Delta, r, m} \) such that for all \( N > N_{\Delta, r, m} \),

\[
\int_{z \in S_r} \log (|\psi_{\alpha,N}(z)|) d\mu_r(z) > \frac{N}{2} \log ((1 + r^2)(1 - \Delta))
\]

except for an event whose probability is less than \( e^{-c_{\Delta, r, m} N^{m+1}} \).

**Proof.** It suffices to prove this result for small \( \Delta \). Set \( \delta = 3^{-4m} \Delta^{4m} \). Let \( s = \lceil \frac{1}{\delta} \rceil \), let \( Q = (2m)s^{2m-1} \), and let \( \kappa = 1 - \delta^{\frac{1}{4m}} \).

In [Zre] it was shown that by projecting a tiling of the \( 2m \) cube by \( 2m - 1 \) cubes onto the sphere of radius \( \kappa r \) one gets a partition consisting of \( Q \) measurable disjoint sets \( \{I_1^{kr}, I_2^{kr}, \ldots, I_Q^{kr}\} \) such that

\[
diam(I_j^{kr}) \leq \frac{\sqrt{2m - 1}}{s} \kappa r = \frac{c_m}{Q^{2m-1}} \kappa r.
\]

We choose such a partition and then we choose a \( \zeta_j \) within \( \delta r < 1 \) of \( I_j^{kr} \) such that

\[
\log (|\psi_{\alpha,N}(\zeta_j)|) > \frac{N}{2} \log ((1 + \kappa^2 r^2)(1 - \delta r))
\]
for which, by Lemma 3.3, there exists $c'_{\Delta,r}$ and $N'_{\Delta,r}$ such that if $N > N'_{\Delta,r}$ then the probability that this does not occur is less than $e^{-c'_{\Delta,r}N^{m+1}}$. Therefore there exists $c_{\Delta,r} > 0$ and $N_{\Delta,r}$ such that if $N > N_{\Delta}$ the union of these $m$ events has probability less than or equal to

$$\left(2m \left[ \frac{1}{\delta} \right]^{2m-1} \right) e^{-c_{\delta,r}N^{m+1}} < e^{-c_{\delta,r}N^{m+1}}. \tag{4}$$

Let $\mu_k = \mu_{k\rho}(I^k_{\rho})$. As $\{I^1_{\rho}, I^2_{\rho}, \ldots, I^Q_{\rho}\}$ form a partition of $S_{\rho\rho}$, $\sum_k \mu_k = 1$.

We now turn to investigating the average of $\log|\psi_{\alpha,N}(z)|$ on the sphere of radius $r$ by approximating said integral with a Riemann sum which makes use of line (3):

$$\frac{N}{2} \log((1 + \frac{1}{2} \kappa^2 r^2)(1 - \delta)) \leq \sum_{k=1}^{k=Q} \mu_k \log|\psi_{\alpha,N}(\zeta_k)|$$

$$\leq \int_{z \in S_r} \left( \sum_k \mu_k P_r(\zeta_k, z) \log(|\psi_{\alpha,N}(z)|) d\mu_r(z) \right)$$

$$= \int_{z \in S_r} \left( \sum_k \mu_k (P_r(\zeta_k, z) - 1) \right) \log(|\psi_{\alpha,N}(z)|) d\mu_r(z)$$

$$+ \int_{z \in S_r} \log(|\psi_{\alpha,N}(z)|) d\mu_r(z)$$

This will simplify to:

$$\int \log(|\psi_{\alpha,N}(z)|) d\mu_r(z) \geq \frac{N}{2} \log((1 + \kappa^2 r^2)(1 - \delta r))$$

$$-(\int \log(|\psi_{\alpha,N}(z)|) d\mu_r(z)) \max_{z \in S_r} \sum_k \mu_k (P_r(\zeta_k, z) - 1)$$

In [Zre], it was computed that in exactly this situation that:

$$\max_{|z|=r} \left| \sum_k \mu_k (P_r(\zeta_k, z) - 1) \right| \leq C_m \delta^{\frac{1}{2m-1}}$$

Hence by Lemma 4.1 and line (4), there exists $c_{\delta,r,m} > 0$ and $N_{\delta,r,m}$ such that if $N > N_{\delta,r,m}$, except for an event of probability $< e^{-c_{\delta,r,m}N^{m+1}}$: 
\[ \int \log(|\psi_{\alpha,N}|) d\mu(z) \geq \frac{N}{2} \log \left((1 + \kappa^2 r^2)(1 - \delta r)\right) - C_m N \log \left(2(1 + r^2)\right) \delta^{\frac{1}{2m-1}}, \]
\[ = \frac{N}{2} \log \left((1 + r^2 - 2\delta \frac{1}{2m} r^2 + O(r^2 \delta \frac{1}{2m} + \delta \frac{4m+1}{4m} r^3 + \delta r)\right) - C_m N \log \left(2(1 + r^2)\right) \delta^{\frac{1}{2m-1}}, \]
\[ \geq \frac{N}{2} \log((1 + r^2)(1 - 3\delta \frac{1}{2m})), \text{ for sufficiently small } \delta. \]

The proof is thus completed by choosing sufficiently small \( \Delta \) so that the previous line holds, (and \( \delta r < 1 \)).

5. Main Results.

We will now be able to estimate the value of the un-integrated counting function for a random \( \operatorname{SU}(m+1) \) polynomial, \( \psi_{\alpha,N} \).

Definition 5.1. For \( f \in O(B(0, r)), f(0) \neq 0, B(0, r) \subset \mathbb{C}^m \), the un-integrated counting function,

\[ n_f(r) := \int_{B(0, t) \cap Z_f} \left(\frac{i}{2\pi} \partial \bar{\partial} \log |z|^2\right)^{m-1} = \int_{B(0, t)} \left(\frac{i}{2\pi} \partial \bar{\partial} \log |z|^2\right)^{m-1} \wedge \left(\frac{1}{i} \partial \bar{\partial} \log |f|\right)^{m-1}. \]

The equivalence of these two definitions follows by the Poincare-Lelong formula. The above form \( \left(\frac{i}{2\pi} \partial \bar{\partial} \log |z|^2\right)^{m-1} \) gives a projective volume, with which it is more convenient to measure the zero set of a random function. The Euclidean volume may be recovered as

\[ \int_{B(0, t) \cap Z_f} \left(\frac{i}{2\pi} \partial \bar{\partial} \log |z|^2\right)^{m-1} = \int_{B(0, t) \cap Z_f} \left(\frac{i}{2\pi} \partial \bar{\partial} |z|^2\right)^{m-1}. \]

Lemma 5.2. If \( u \in L^1(\overline{B_r}), \) and \( \partial \bar{\partial} u \) is a measure, then

\[ \int_{t=r=0}^{t=R} \int_{B_r} \frac{i}{2\pi} \partial \bar{\partial} u \wedge \left(\frac{i}{2\pi} \partial \bar{\partial} \log |z|^2\right)^{m-1} = \frac{1}{2} \int_{S_R} u \, d\mu_R - \frac{1}{2} \int_{S_r} u \, d\mu_r \]

A proof of this result is available on page 390-391 of Griffiths and Harris, \([\text{GH}]\). Using this we may now prove one of our two main theorems, Theorem 1.1.

Proof. (of theorem 1.1). It suffices to prove the result for small \( \Delta \). Let \( \delta = \frac{\Delta^2}{4} < 1 \). Let \( \kappa = 1 + \sqrt{\delta} = 1 + \frac{\Delta}{2} \). As \( n_{\psi_{\alpha,N}}(r) \) is increasing,

\[ n_{\psi_{\alpha,N}}(r) \log(\kappa) \leq \int_{t=r=r}^{t=\kappa r} n_{\psi_{\alpha,N}}(t) \frac{dt}{t} \leq n_{\psi_{\alpha,N}}(\kappa r) \log(\kappa) \]
There exists $c_{δ,r,m} > 0$ and $N_{δ,r,m}$ such that for all $N > N_{δ,r,m}$, except for an event of probability $\leq e^{-c_{δ,r,m} N^{m+1}}$, we get that:

$$n_{ψ,α,N}(r) \log(κ) \leq \int_{S_κ} \log |ψ_{α,N}(z)| dμ_κ(z) - \int_{S_r} \log |ψ_{α,N}(z)| dμ_r(z) \leq \frac{N}{2} \left( \log \left( (1 + κ^2 r^2)(1 + δ) \right) - \int_{S_r} \log |ψ_α(re^{iθ})| dμ_r \right), \text{ by Lemma 3.2}$$

$$\leq \frac{N}{2} \left( \log \left( (1 + κ^2 r^2)(1 + δ) \right) - \log \left( (1 + r^2)(1 - δ) \right) \right), \text{ by Lemma 4.2}$$

$$\leq \frac{N}{2} \left( \frac{2\sqrt{δr^2} + δr^2 + 2δ + 2δr^2}{(1 + r^2)} - \frac{2δr^4}{(1 + r^2)^2} + O(δ^3) \right).$$

Therefore,

$$n_{ψ,α,N}(r) \leq N \left( \frac{r^2 + \frac{1}{2} \sqrt{δr^2} + \sqrt{δ} + \sqrt{δr^2}}{1 + r^2} - \frac{\sqrt{δr^2}}{(1 + r^2)^2} + O(δ) \right) \cdot \left( 1 + \frac{δ}{2} + O(δ) \right) \leq \frac{Nr^2}{1 + r^2} + 3N\sqrt{δ} + O(δ).$$

This proves the probability estimate when the value of the unintegrated counting function $n_{ψ,α,N}(r)$ is significantly above its typical value. We now modify the above the argument to finish the proof. There exists $c_{δ,r,m}$ and $N_{δ,r,m}$ such that if $N > N_{δ,r,m}$ then, except for an event whose probability is less than $e^{-c_{δ,r,m} N^{m+1}}$, the following inequalities hold:

$$n_{ψ,α,N}(r) \log(κ) \geq \int_{S_κ} \log |ψ_{α,N}(z)| dμ_κ(z) - \int_{S_{κ^{-1}}} \log |ψ_{α,N}(z)| dμ_{κ^{-1}}(z) \geq \frac{N}{2} \left( \log \left( (1 + r^2)(1 - δ) \right) - \int_{S_r} \log |ψ_α(re^{iθ})| dμ_r \right), \text{ by Lemma 4.2}$$

$$\geq \frac{N}{2} \left( \log \left( (1 + r^2)(1 - δ) \right) - \log \left( (1 + κ^{-2} r^2)(1 + δ) \right) \right), \text{ by Lemma 3.2}$$

$$\geq \frac{N}{2} \left( \log(1 - δ) - \log \left( 1 - \frac{2\sqrt{δr^2}}{1 + r^2} + \frac{δr^2}{1 + r^2} + δ + O(δ^\frac{3}{2}) \right) \right).$$
Therefore,
\[ n_{\psi_{\alpha,N}}(r) \geq N \left( -\sqrt{\delta} + \frac{2r^2}{1 + r^2} - \sqrt{\delta} + \frac{2\sqrt{\delta}r^4}{(1 + r^2)^2} + O(\delta) \right) \]
\[ \cdot \left( 1 + \frac{\sqrt{\delta}}{2} + O(\delta) \right) \]
\[ \geq \frac{Nr^2}{1 + r^2} - 2N\sqrt{\delta} + O(\delta). \]

\[ \square \]

We have just implicitly proven an upper bound on the order of the decay of the hole probability. We will now compute the lower bound to finish the proof theorem 1.2.

**Proof.** (of theorem 1.2) The desired upper bound for the order of the decay of the hole probability is a consequence of the previous theorem.

We must still prove the lower bound for the order of the decay of the hole probability, and we start this by considering the event, \( \Omega \) which consists of \( \alpha_{\bar{0}} \) where:
\[ |\alpha_{\bar{0}}| \geq 1 \]
\[ |\alpha_j| < \left( \begin{array}{c} N \\ j \end{array} \right)^{\frac{1}{2}} N^{-m_r - |j|}. \]

If \( \alpha \in \Omega \), then \( |\alpha_0| > \sum_{|j| > 0} |\alpha_j| \left( \begin{array}{c} N \\ j \end{array} \right)^{\frac{1}{2}} r^j \). Hence for all \( z \in B(0, r) \), \( \psi_{\alpha,N}(z) \neq 0 \Rightarrow \Omega \subset Hole_{N,r} \). A lower bound for the probability of \( \Omega \) will thus give a lower bound for the probability of \( Hole_{N,r} \). First we restrict ourselves to considering the Gaussian random variables, \( \alpha_j \), for whose indices, \( j \),
\[
{N\choose j} N^{-2m_r -2|j|} \leq 1.
\]

\[
\text{Prob}\left( \left\{ |\alpha_j| < \left( {N\choose j} \right)^{\frac{1}{2}} N^{-m_r -|j|} \right\} \right) \geq \frac{1}{2} \frac{1}{N^{2m} {N\choose j} r^{-2|j|}}
\]

by Proposition 3.1

\[
= \frac{1}{2} (N - |j|)! |j|! r^{-2|j|}
\]

\[
\geq \frac{N^{2m} N!}{(2\pi)^{\frac{1}{2}}} r^{-2|j|} e^{\frac{1}{2}}
\]

\[
\geq e^{-(N+\frac{m}{2}) \log(m+1)+c_m - |j| \log(r) - 2m \log(N)}
\]

\[
\geq e^{-c_{m,r}(N)}
\]

Please note that the last inequality still holds even if \( r < \frac{1}{m+1} \) since

\[
\text{Prob}\left( \left\{ |\alpha_j| < \left( {N\choose j} \right)^{\frac{1}{2}} N^{-m_r -|j|} \right\} \right) \approx e^{c_{m,r} N} \geq e^{-c_{m,r} N}
\]

Whereas if for the index \( j \), \( {N\choose j} N^{-2m_r -2|j|} < 1 \) then

\[
\text{Prob}\left( \left\{ |\alpha_j| < \left( {N\choose j} \right)^{\frac{1}{2}} N^{-m_r -|j|} \right\} \right) \geq \text{Prob}(\{ |\alpha_j| < 1 \})
\]

\[
> \frac{1}{2} e^{-N \log(2)}
\]

Further, \( \text{Prob}(\{|\alpha_0| > N\}) = e^{-1} \). Hence, \( \text{Prob}(\Omega) \geq e^{-c_{r,m} N^{m+1}} \)

\[\square\]

**References**

[BSZ] P. Bleher, B. Shiffman, and S. Zelditch, Universality and scaling of correlations between zeros on complex manifolds, *Inventiones Mathematicae*, no. 2. (2000), 351-395.

[DPSZ] A. Dembo, B. Poonen, Q. M. Shao, and O. Zeitouni, Random polynomials having few or no zeros, *Journal of the American Mathematical Society*, 15, no. 4 (2002), 857-892.

[EK] A. Edelman and E. Kostlan, How many zeros of a random polynomial are real?, *Bull. Amer. Math. Soc.*, 32 (1995), 1-37.
[GH] P. Griffith and J. Harris, *Principles of Algebraic Geometry*, (New York, New York: John Wiley and Sons, Inc., 1994).

[Han] J. Hannay *Chaotic Analytic zero points: exact statistics for those of a random spin state* J. Phys. A, (29) (1996), no. 5, L101-L105.

[Kri] M. Krishnapur, Zeros of Random Analytic Functions, e-print Archive, ArXiv:math.PR/0607504 (2006).

[LS] W. Li, Q. Shao, *A normal comparison inequality and its applications*, Probab. Theory Related Fields, 122 (2002), no. 4, 494-508.

[PV] Y. Peres, B. Virag, Zeros of i.i.d. Gaussian powerseries: a conformally invariant determinental process, *Acta Math.*, 194, no. 1, (2005) 1-35.

[Sod] M. Sodin, *Zeros of Gaussian Analytic Functions* Mathematical Research Letters, 7 (La Jolla, CA: 2000).

[ST1] M. Sodin and B. Tsirelson, Random Complex Zeros I: Asymptotic normality, *Israel J. Math.*, 144, (2004), 125-149.

[ST2] M. Sodin and B. Tsirelson, Random Complex Zeros III, Decay of the hole probability, *Israel J. Math.*, 147 (2005), 371-379.

[Zre] S. Zrebiec, The zeros of Gaussian random holomorphic functions on $\mathbb{C}^n$, and hole probability, E-print Archive, ArXiv:math.CV/0603696 (2006).