THERMODYNAMIC EXPANSION TO ARBITRARY MODULI

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Abstract. We extend the thermodynamic expansion results in [BGS11, MOW15] from square-free to arbitrary moduli by developing a novel decoupling technique and applying [BV12].

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1. Statements

In this short note, we use the “modular” expansion of [BV12], valid for arbitrary moduli, to extend the “archimedean”-thermodynamic expansion results in [BGS11, MOW15] from square-free to arbitrary moduli.

Theorem 1.1. Let $\Gamma$ be a finitely-generated, Zariski dense, Schottky (that is, free, convex-cocompact) subgroup of $\text{SL}_2(\mathbb{Z})$, and let $\delta \in (0, 1)$ be its critical exponent. For an integer $q$, let $\Gamma(q) := \{ \gamma \in \Gamma : \gamma \equiv I \pmod{q} \}$. Then there is an $\varepsilon > 0$ and $q_0 \geq 1$ such that, for all integers $q$ coprime to $q_0$, the resolvent of the Laplace operator

$$R_{\Gamma(q)} = (\Delta - s(1 - s))^{-1} : C^\infty_c(\Gamma(q) \backslash \mathbb{H}) \to C^\infty(\Gamma(q) \backslash \mathbb{H})$$

is holomorphic in the strip $\Re(s) > \delta - \varepsilon$, except for a simple pole at $s = \delta$.

This extends the statements of [BGS11, Theorems 1.4 and 1.5] and [OW14, Theorem 1.3] to arbitrary moduli $q$; see also the discussion below [MOW15, Theorem 1.1].

In a similar way, we deal with semigroups.

Date: July 30, 2015.
Bourgain is partially supported by NSF grant DMS-1301619.
Kontorovich is partially supported by an NSF CAREER grant DMS-1254788 and DMS-1455705, an NSF FRG grant DMS-1463940, an Alfred P. Sloan Research Fellowship, and a BSF grant.
Magee is partially supported by NSF grant DMS-1128155.
Theorem 1.2. The statement of [MOW15, Corollary 1.2] holds when specialized to the “Zaremba” (or continued fraction) setting of [MOW15, §6.1], without the restriction that the modulus $q$ be square-free.

In particular, this justifies Remark 8.8 in [BK14]. We expect analogous arguments will also prove the uniform exponential mixing result in [OW14, Theorem 1.1] for arbitrary moduli.

2. Proofs

The proofs are a relatively minor adaptation of the argument in [MOW15], which builds on the breakthrough in [OW14] (the latter is itself based on key ideas in [BGS11] combined with [Dol98, Nau05, Sto11]). The point of departure from the treatment in [MOW15] is in the analysis of the measure $\mu_{s,x,\alpha}$ in equation (135), culminating in Lemma 4.8, valid for arbitrary moduli $q$. We will follow this treatment, henceforth importing all the concepts and notation from that paper.

Thus we are lead to study the measure $\mu_{s,x,\alpha}$ on $G = SL_2(q)$ given by

$$\mu = \mu_{s,x,\alpha} = \sum_{\alpha^N > \alpha} \exp(\tau_N + ib\tau_N(\alpha_N^N x))\delta_{c_{\alpha^N x}},$$

(2.1)

as in [MOW15, (135)]. Here $x \in I$, $\alpha^M$ is a fixed branch of $T^{-M}$, and $\alpha^N = \alpha^M \alpha^R$. For ease of exposition, we first assume that we are treating the full shift as in Theorem 1.2, and that we can therefore view sums over branches $\alpha^N$ as sums over globally (on $I$) defined branches of $T^{-N}$. Moreover, assume for simplicity that $\Gamma(\text{mod } q) = SL_2(q)$. (Both of these assumptions are satisfied in the Zaremba setting of [BK14].)

Our goal in this paper is to prove the following

Theorem 2.2. For $|a - s_0| < a_0$ and $\varphi \in E_q$ (as defined in [MOW15, §4.1]), we have

$$\|\mu * \varphi\|_2 \leq C q^{-1/4} B \|\varphi\|_2,$$

(2.3)

where

$$\|\mu\|_1 < B.$$

This is the replacement of [MOW15, Lemma 4.5] (bypassing the property (MIX)), and the rest of the proof of [MOW15, Lemma 4.8] follows analogously.

To begin, we pick some $o \in I$, and define the measure $\nu$ by:

$$\nu \equiv \exp(\tau^M(a^Mo))\mu_1,$$

(2.4)

where $\mu_1$ is the measure given by

$$\mu_1 \equiv \sum_{\alpha^R} \exp(\tau^R(\alpha^Ro))\delta_{c_{\alpha^R o}},$$

(2.5)

Lemma 2.6. We have

$$|\mu| \leq C \nu.$$

(2.7)
Proof. Use the “contraction property” in [MOW15, (145-146)] and argue as in the proof of [MOW15, Lemma 4.4]. □

We will now manipulate $\mu_1$. We assume that $R$ can be decomposed further as

$$ R = R'L, \quad (2.8) $$

with $L$ to be chosen later (a sufficiently large constant independent of $R'$ and $q$).

Now split $\alpha^R$ as

$$ \alpha^R = \alpha^L \cdot \alpha^{L-1} \cdots \alpha^1, \quad (2.9) $$

where the $\alpha^k$ are branches of $T^{-L}$. This splitting (2.9) is uniquely determined by $\alpha^R$. For each $k \geq 2$, we also split

$$ \alpha_k^L = \alpha_{k-1}^L \cdot \alpha^1, $$

where $\alpha_k^L = g_{i_k}$ for some $i_k$.

Write out

$$ \tau_a^R(\alpha^R o) = \sum_{i=0}^{R-1} \tau_a(T^i \alpha^R o) $$

$$ = \sum_{i=0}^{R'-1} \sum_{\ell=0}^{L-1} \tau_a(T^{i\ell} \alpha^R o) $$

$$ = \sum_{i=0}^{R'-1} \sum_{\ell=0}^{L-1} \tau_a(T^{i\ell} \alpha^L_{R'-i} \alpha^L_{R'-i-1} \cdots \alpha^L_1 o) $$

$$ = \sum_{i=0}^{R'-1} \tau_a^L(\alpha^L_{R'-i} \alpha^L_{R'-i-1} \cdots \alpha^L_1 o) \quad (2.10) $$

We now perform decoupling term by term in the above. We will use the shorthand

$$ \alpha^{Lj} \equiv \alpha^L_j \cdot \alpha^{L-1} \cdots \alpha^1.$$

For $j \geq 2$, we compare each term in (2.10) of the form

$$ \tau_a^L(\alpha^{Lj}(o)) $$

to

$$ \tau_a^L(\alpha^L_j \alpha^{L-1}_{j-1} o). $$

This gives

$$ \tau_a^L(\alpha^{Lj}(o)) = \tau_a^L(\alpha^L_j \alpha^{L-1}_{j-1} o) + O\left(\sup |\tau_a^L \circ \alpha^L_j| d(\alpha^{L-1}_{j-1} o, \alpha^{L-1}_{j-1} \alpha^L_1 \cdots \alpha^L_1 o)\right) $$

$$ = \tau_a^L(\alpha^L_j \alpha^{L-1}_{j-1} o) + O(\gamma^{-(L-1)}), \quad (2.11) $$

where we used [MOW15, (68)], valid when $a$ is suitably close to $s_0$.

We will also use the formula

$$ \delta_{c}^{R}(\alpha^R o) = \delta_{c}^{L}(\alpha^L o) \cdot \delta_{c}^{L}(\alpha^{2L} o) \cdot \delta_{c}^{L}(\alpha^{3L} o) \cdot \cdots \cdot \delta_{c}^{L}(\alpha^{R'L} o). \quad (2.12) $$
Then combining (2.10) and (2.12), we write

$$\mu_1 = \sum_{\alpha_1^{L-1}, \ldots, \alpha_{R'}^{L-1}} \sum_{\alpha_1^1, \ldots, \alpha_{R'}^1} \exp(\tau_a^R(\alpha^R o)) \delta_{c_q^L(\alpha^R o)}$$

$$= \sum_{\alpha_1^{L-1}, \ldots, \alpha_{R'}^{L-1}} \sum_{\alpha_1^1, \ldots, \alpha_{R'}^1} \exp\left(\sum_{j=1}^{R'} \tau_{a_j}^L(\alpha^j L(o))\right) \times \delta_{c_q^L(\alpha^R o)} \delta_{c_q^L(\alpha^{2L} o)} \delta_{c_q^L(\alpha^{3L} o)} \ldots \delta_{c_q^L(\alpha^{R'L} o)},$$

(2.13)

We now decouple, replacing each term of the form

$$e^{\tau_{a_j}^L(\alpha^j L(o))} \mapsto e^{\tau_{a_j}^L(\alpha^j L-1 o)} \equiv \beta_j$$

with \(j \geq 2\), at a cost of a multiplicative factor of \(\exp(c_\gamma^{-L})\); here \(c\) is proportional to the implied constant of (2.11). When \(j = 1\), no replacement is performed, and we set \(\beta_1 \equiv e^{\tau_{a_1}^L(\alpha_1^L o)}\).

Inserting this into (2.13) gives

$$\mu_1 \leq \sum_{\alpha_1^{L-1}, \ldots, \alpha_{R'}^{L-1}} \sum_{\alpha_1^1, \ldots, \alpha_{R'}^1} \beta_1 \delta_{c_q^L(\alpha^L o)} \exp(c_\gamma^{-L})^{R'-1} \left(\sum_{\alpha_2^1, \ldots, \alpha_{R'}^1} \prod_{j=2}^{R'} \beta_j \delta_{c_q^L(\alpha^{2L} o)} \delta_{c_q^L(\alpha^{3L} o)} \ldots \delta_{c_q^L(\alpha^{R'L} o)}\right).$$

(2.14)

Note that, although \(\beta_j\) depends on all of the indices in \(\alpha_j^L \alpha_{j-1}^{L-1}\), because \(\alpha_{j-1}^{L-1}\) are fixed in the outermost sum, we treat \(\beta_j\) as a function of \(\alpha_j^1\).

We claim that each term \(c_q^L(\alpha^j L o)\) also only depends on one \(\alpha_j^1\). This is because we have \(\alpha^{jL} = g_{k_1} \ldots g_{k_L} \alpha^{(j-1)L}\) for some choice of \(g_{km}\), and hence for whatever \(o\) is chosen, we have

$$c_q^L(\alpha^{jL} o) = c_q(g_{k_{L-1}} \alpha^{(j-1)L} o) c_q(g_{k_{L-2}} g_{k_{L-1}} \alpha^{(j-1)L} o) \ldots c_q(g_{k_1} \ldots g_{k_L} \alpha^{(j-1)L} o),$$

see [MOW15, (69)]. Since \(g_{km}\) maps \(I\) into \(I_{km}\), we have

$$c_q(g_{km} o') = g_{km} \mod q$$

for any \(o' \in I\). Thus

$$c_q^L(\alpha^{jL} o) = g_{k_L} \ldots g_{k_1} \mod q.$$

(2.15)

Here

$$g_{k_L} = \alpha_j^1.$$

(2.16)
This means we may distribute the convolution and product over the sum, writing (2.14) as

\[ \mu_1 \leq \exp(c\gamma^L)R'^{-1} \sum_{\alpha_{L-1}^1,\alpha_{L-1}^2,\ldots,\alpha_{R'}^1} \left( \sum_{\alpha_1^1} \beta_1 \delta_{c\gamma^L}^{\alpha_{L-1}} \right) \ast \left( \sum_{\alpha_2^1} \beta_2 \delta_{c\gamma^L}^{\alpha_{L-1}^2} \right) \ast \ldots \]

\[ \ldots \ast \left( \sum_{\alpha_{R'}^1} \beta_{R'} \delta_{c\gamma^L}^{\alpha_{R'}\alpha_{L-1}} \right). \]  

(2.17)

We give each convolved term in (2.17) a name, defining, for each \( j \geq 1 \), the measure

\[ \eta_j = \eta_j^{(\alpha_{L-1}^j,\alpha_{L-1}^{j-1})} = \sum_{\alpha_1^j} \beta_1 \delta_{c\gamma^L}^{\alpha_{L-1}^j}. \]  

(2.18)

We have thus proved the following

**Proposition 2.19.** We have

\[ \mu_1 \leq \exp(c\gamma^L)R'^{-1} \sum_{\alpha_{L-1}^1,\alpha_{L-1}^2,\ldots,\alpha_{R'}^1} \eta_1 \ast \eta_2 \ast \ldots \ast \eta_{R'}. \]  

(2.20)

Next we observe that each of the measures \( \eta_j \) is nearly flat, in that their coefficients in (2.18) differ by constants:

**Lemma 2.21.** For each \( j \geq 1 \) and any \( \alpha_1^j \) and \( \alpha_1'^j \), we have

\[ \frac{\beta_j'}{\beta_j} \leq \exp(c\gamma^{L+1}). \]  

(2.22)

**Proof.** The first \( L-1 \) terms of \( \beta_j \) and \( \beta_j' \) agree, so we again use the “contraction property” [MOW15, (145-146)]. \( \square \)

Since the measures \( \eta_j \) are nearly flat, we may now apply the expansion result in [BV12].

**Theorem 2.23.** Assume \( L \) is sufficiently large (depending only on \( \Gamma \)). Then for \( \varphi \in L^0_0(G) \), we have

\[ \| \eta_j \ast \varphi \|_2 \leq (1 - C_1) \| \eta_j \|_1 \| \varphi \|_2, \]  

(2.24)

Here \( C_1 > 0 \) depends on \( \Gamma \) but not on \( q \).

To prove this theorem, we need the following simple

**Lemma 2.25.** Let \( \pi \) be a unitary \( G \)-representation on a Hilbert space \( \mathcal{H} \), and assume that the operator \( A \) acts on \( \mathcal{H} \) via

\[ A \varphi = \sum_{j \in J} \pi(h_j) \varphi, \]
for some $h_j \in G$ and indexing set $J$. Assume that $A$ has the “spectral gap” property: there is some $C_0 > 0$ so that
\[
\langle A\varphi, \varphi \rangle \leq (1 - C_0) |J| \| \varphi \|^2. \tag{2.26}
\]
For some positive coefficients $\kappa_j > 0$, let $\tilde{A}$ act on $\mathcal{H}$ as
\[
\tilde{A}\varphi = \sum_{j \in J} \kappa_j \pi(h_j)\varphi,
\]
and assume that the $L^\infty$ norm of the coefficients is controlled by the $L^1$ norm, in the sense that for some $K \geq 1$,
\[
\max \kappa_j \leq K \bar{\kappa}, \tag{2.27}
\]
where
\[
\bar{\kappa} := \frac{1}{|J|} \sum_j \kappa_j
\]
is the coefficient average. Then $\tilde{A}$ has the following “spectral gap”:
\[
\langle \tilde{A}\varphi, \varphi \rangle \leq \bar{\kappa} (1 - C_0 + \sqrt{K - 1}) |J| \| \varphi \|^2. \tag{2.28}
\]

Proof. This is an exercise in Cauchy-Schwarz. \qed

With this lemma, it is a simple matter to give a

Proof of Theorem 2.23.

We will apply Lemma 2.25 with $\mathcal{H} = L^2_0(G)$ and $\pi$ the right-regular representation. Recalling (2.18), we can write
\[
\| \eta_j * \varphi \|_2^2 = \langle \tilde{A}\varphi, \varphi \rangle,
\]
where $\tilde{A}$ acts by convolution with the measure
\[
\sum_{\alpha_j, \alpha_j'} \beta_j \beta_j' \delta_{c_q(\alpha_j \cdot \alpha_j')^{-1}}.
\]
Using the notation of (2.15) and (2.16), note that
\[
c_q^{L}(\alpha^{L} \cdot \alpha'^{L})^{-1} = \alpha_j^1 \cdot g_{k_{L-1}} \cdots g_{k_1} (\alpha_j'^{L} \cdot g_{k_{L-1}} \cdots g_{k_1})^{-1} = \alpha_j^1 (\alpha_j'^{L})^{-1}.
\]
The indexing set $J$ of Lemma 2.25 then runs over pairs $\alpha_j, \alpha_j'$, the coefficients $\kappa_j$ are the products $\beta_j \beta_j'$, and the elements $h_j$ are $\alpha_j^1 (\alpha_j'^{L})^{-1}$.

That the operator $A$ (without coefficients) has a spectral gap (2.26) is precisely the statement proved in [BV12], with $C_0$ independent of $q$.\footnote{Here we need the products $\alpha_j^1 (\alpha_j'^{L})^{-1}$ to generate group with Zariski closure $SL_2$. In the Zaremba case, it is important that each $\alpha_j^1$ is a product of two generators $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Otherwise, e.g., the products $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ could all be lower-triangular.} The bound (2.27) follows
from (2.22) with

\[ K = \exp(2c\gamma^{-L+1}). \]

Note also that

\[ |J| \bar{\kappa} = \left( \sum_{\alpha_j} \beta_j \right)^2 = \| \eta_j \|_1^2. \]

Choosing \( L \) sufficiently large (depending only on \( \Gamma \)), one can make \( K \) sufficiently close to 1 so that (2.28) gives (2.24), as claimed. \( \square \)

**Corollary 2.29.** Assume that \( L \) is sufficiently large (depending only on \( \Gamma \)). Then there is some \( C_2 > 0 \) also depending only on \( \Gamma \) so that, for any \( \varphi \in L^2_0(G) \), we have

\[ \| \mu_1 * \varphi \|_2 \leq (1 - C_2)^R \| \mu_1 \|_1 \| \varphi \|_2. \]  
\[ \text{(2.30)} \]

**Proof.** Beginning with (2.20), apply (2.24) \( R' \) times to get

\[ \| \mu_1 * \varphi \|_2 \leq \exp(c\gamma^{-L}) \sum_{\alpha_1^{L-1} \ldots \alpha_{R'}^{L-1}} (1 - C_1)^{R'} \prod_{j=1}^{R'} \| \eta_j \|_1 \| \varphi \|_2. \]

Applying contraction yet again gives

\[ \sum_{\alpha_1^{L-1} \ldots \alpha_{R'}^{L-1}} \prod_{j=1}^{R'} \| \eta_j \|_1 \leq \exp(c\gamma^{-L}) \prod_{j=1}^{R'} \| \eta_j \|_1 \| \varphi \|_2. \]

whence (2.30) follows on taking \( L \) large enough and recalling (2.8). \( \square \)

Returning to the measure \( \nu \) in (2.4), we have from (2.30) that

\[ \| \nu * \varphi \|_2 \leq (1 - C_2)^R \| \nu \|_1 \| \varphi \|_2. \]  
\[ \text{(2.31)} \]

To conclude Theorem 2.2, we need the following

**Lemma 2.32.** Let \( \mu \) be a complex distribution on \( G = SL_2(q) \) and assume that \( |\mu| \leq C\nu \). Let \( E_q \subset L^2_0(G) \) be the subspace defined in [MOW15, §4.1], and let \( A : E_q \to E_q \) be the operator acting by convolution with \( \mu \). Then

\[ \| A \| \leq C' \left[ \frac{|G| \| \tilde{\nu} * \nu \|_2}{q} \right]^{1/4}. \]  
\[ \text{(2.33)} \]

Here \( \tilde{\mu}(g) = \mu(g^{-1}) \).

**Proof.** Note that the operator \( A^* A \) is self-adjoint, positive, and acts by convolution with \( \tilde{\mu} * \mu \). Let \( \lambda \) be an eigenvalue of \( A^* A \). Since \( A \) acts on \( E_q \), Frobenius gives that \( \lambda \) has multiplicity \( \text{mult}(\lambda) \) at least \( Cq \). We then have that

\[ \lambda^2 \text{ mult}(\lambda) \leq \text{tr}[(A^* A)^2] = \sum_{g \in G} \langle (A^* A)^2 \delta_g, \delta_g \rangle = \sum_{g \in G} \| \tilde{\mu} * \mu * \delta_g \|_2^2 \]

\[ = |G| \| \tilde{\mu} * \mu \|_2^2 \leq C^4 \| \tilde{\nu} * \nu \|_2^2. \]
The claim follows, as \( \|A\| = \max_\lambda \lambda^{1/2} \).

We apply the lemma to \( \mu \) in (2.1) using (2.7), giving
\[
\|\mu \ast \varphi\|_2 \leq C q^{1/2} \|\widetilde{\nu} \ast \nu\|_2^{1/2}.
\]
(2.34)

It remains to estimate the \( \nu \) convolution.

**Proposition 2.35.** Choosing \( R \) to be of size \( C \log q \) for suitable \( C \), we have that
\[
\|\widetilde{\nu} \ast \nu\|_2 \leq 2 \frac{\|\nu\|_2^2}{|G|^{1/2}}.
\]
(2.36)

**Proof.** Let
\[
\psi \equiv \delta_e - \frac{1}{|G|} 1_G \in L_0^2(G),
\]
and note that \( \|\psi\|_2 < 1 \). Then
\[
\|\widetilde{\nu} \ast \nu\|_2 = \|\widetilde{\nu} \ast \nu \ast \delta_e\|_2 \leq \|\widetilde{\nu} \ast \nu \ast \left( \frac{1}{|G|} 1_G \right)\|_2 + \|\widetilde{\nu} \ast \nu \ast \psi\|_2
\]
\[
\leq \frac{\|\nu\|_2^2}{|G|^{1/2}} + \|\nu\|_1 \|\nu \ast \psi\|_2,
\]
where we used the triangle inequality and Cauchy-Schwarz. Since \( \psi \in L_0^2(G) \), we apply (2.31), giving
\[
\|\nu \ast \psi\|_2 < (1 - C_2)^R \|\nu\|_1 < \frac{\|\nu\|_1}{|G|^{1/2}}
\]
by a suitable choice of \( R = C \log q \). The claim follows immediately. □

Finally, we give a

**Proof of Theorem 2.2.** Insert (2.36) into (2.34) and use (2.7) and \( |G| > C q^3 \). Clearly (2.3) holds with \( B = C \|\nu\|_1 \). □

### 2.1. Modifications for Subshifts.

We sketch here the modifications needed to handle the case \( \Gamma \) is a Schottky group as in Theorem 1.1. Then \( I = \bigcup_k I_k \), where to each \( I_k \) is assigned some \( g_k \in \text{SL}_2(\mathbb{Z}) \) such that \( T \mid I_k = g_k^{-1} \) and \( c_0 \mid I_k \equiv g_k \). The shift is restricted to exclude any letter \( g_k \) being followed by \( g_k^{-1} \). Note that while in [MOW15] it is stated that the values \( c_0(I) \) should freely generate a semigroup, the arguments also apply equally to the Schottky case.

In the decomposition (2.13), each sum on \( \alpha^1_j \) needs to be restricted to be admissible, once \( \alpha^L_{j-1} \) and \( \alpha^L_j \) are chosen (and each itself is an admissible sequence). The base points \( o \in I \) need to be chosen in the appropriate domains of branches of \( T^{-L} \), etc.; we only ever use the contraction principle, so these choices have no effect.

The following issue arises when \( \Gamma \) is generated by two elements, \( g \) and \( h \), say. Suppose \( \alpha^L_j \) ends in \( g \) while \( \alpha^L_{j-1} \) starts with \( g^{-1} \). Then in the \( \alpha^1_j \) sum, only \( h \) and \( h^{-1} \) are admissible, and this does not generate a Zariski dense group for the
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operator $A$ in the proof of Theorem 2.23. To fix this issue, one instead decomposes each block $\alpha_j^L$ as $\alpha_j^{L-2}\alpha_j^2$, that is, isolating two indices instead of one. With this adjustment, even if $\alpha_j^{L-2}$ ends in $g$ and $\alpha_j^{L-2}$ starts in $g^{-1}$, the admissible $\alpha_j^2$ sum runs over the elements $gh, gh^{-1}, hh, h^{-1}g^{-1}, h^{-1}h^{-1}$. It is then easy to see that the operator $A$ in the proof of Theorem 2.23 generates a Zariski dense group (if $\Gamma$ has more than two generators, this is clear). Now, this group and its generator set (and hence also its expansion constant $C_0$ in (2.26)) depend on $\alpha_j^{L-2}$ and $\alpha_j^{L-2}$ (or rather just their starting/ending letters). But as $\Gamma$ is finitely generated, only a finite number of groups/generators arise in this way, and we simply take $C_0$ to be the worst one. With these modifications, the proof goes through as before.

References

[BGS11] J. Bourgain, A. Gamburd, and P. Sarnak. Generalization of Selberg’s 3/16th theorem and affine sieve. Acta Math., 207:255–290, 2011. 1, 2

[BK14] J. Bourgain and A. Kontorovich. On Zaremba’s conjecture. Annals Math., 180(1):137–196, 2014. 2

[BV12] Jean Bourgain and Péter P. Varjú. Expansion in $SL_d(\mathbb{Z}/q\mathbb{Z})$, $q$ arbitrary. Invent. Math., 188(1):151–173, 2012. 1, 5, 6

[Dol98] Dmitry Dolgopyat. On decay of correlations in Anosov flows. Ann. of Math. (2), 147(2):357–390, 1998. 2

[MOW15] Michael Magee, Hee Oh, and Dale Winter. Expanding maps and continued fractions, 2015. Preprint, arXiv:1412.4284v2. 1, 2, 3, 4, 5, 7, 8

[Nau05] Frédéric Naud. Expanding maps on Cantor sets and analytic continuation of zeta functions. Ann. Sci. École Norm. Sup. (4), 38(1):116–153, 2005. 2

[OW14] Hee Oh and Dale Winter. Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of SL(2,\mathbb{Z}), 2014. Preprint, arXiv:1410.4401v2. 1, 2

[Sto11] Luchezar Stoyanov. Spectra of Ruelle transfer operators for axiom A flows. Nonlinearity, 24(4):1089–1120, 2011. 2

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