Barcodes in level and sub-level persistence and Morse-Novikov theory

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Abstract

In this note we recall the relations between the barcodes in level and sub-level persistence and make precise their relation with the Morse-Novikov complex of a Morse real- or angle-valued map. The results in this paper are implicit in [4] and explicit in [1], but apparently not well known even to experts.

In this note we recall the relations between the barcodes in level and sub-level persistence and make precise their relation with the Morse-Novikov complex of a Morse real- or angle-valued map. The results in this Note are implicit in [4] and explicit in [1]. This note should be viewed as a preliminary section to [2].

A short summary of note is provided by the following two statements:

• From the perspective of classical (ELZ) persistence for a Morse real-valued map which is Lyapunov for a Morse-Smale vector field, the infinite barcodes determine the Betti numbers of the underlying manifold and the finite barcodes give information about the multitude of instantons (visible trajectories) between the rest points.

• From the perspective of level persistence for a real-valued or an angle-valued Morse map which is Lyapunov for a Morse-Smale vector field, the closed and the open barcodes determine the standard Betti numbers (for real-valued map) resp. the Novikov Betti numbers (for angle-valued map) while the closed-open barcodes give indication about the multitude of instantons.

More precisely the cardinality of the \( r \)-closed-open bar codes equals to \( N \) indicates that from at least \( N \) different rest points of Morse index \( r \) originate instantons and to at least \( N \) different rest points of Morse index \( r - 1 \) arrive instantons.

1 Barcodes in level and sub-level persistence

The case of real valued maps

Level persistence

For a nonnegative integer \( r \), a field \( \kappa \), a real-valued proper tame map \( f : X \rightarrow \mathbb{R} \) with \( X \) an ANR (and based on homology with coefficients in \( \kappa \)) the level persistence provides the following four type of barcodes c.f [3], [4] or [1] chapters 5 and 6:

1. level persistence closed \( r \)-barcodes, \([a, b], a \leq b, a, b \in \mathbb{R}\),

2. level persistence open \( r \)-barcodes, \((\alpha, \beta), \alpha < \beta, \alpha, \beta \in \mathbb{R}\),

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3. **level persistence closed-open** \( r \)-**barcodes**, \([m, n], m < n, m, n \in \mathbb{R}\),

4. **level persistence open-closed** \( r \)-**barcodes**, \((m', n'], m' < n', m', n' \in \mathbb{R}\).

Recall that \( f \) is **proper** if for any compact interval \( I \subset \mathbb{R}, f^{-1}(I) \) is compact, and **tame** if \( f^{-1}(t) \) is an ANR (cf. [1] Chapter 1 for definition) any \( t \in \mathbb{R} \). Recall that all simplicial complexes and in particular all manifolds, or more general stratified spaces, are ANR’s.

Each such barcode has a multiplicity \( \geq 1 \) and the collection of barcodes can be recored as the nonnegative integer-valued maps \( \delta^f_r : \mathbb{C} \rightarrow \mathbb{Z}_{\geq 0} \) and \( \gamma^f_r : \mathbb{C} \setminus \Delta \rightarrow \mathbb{Z}_{\geq 0} \)

defined as follows:

\[
\delta^f_r(z) := \begin{cases} 
\text{the multiplicity of the } r \text{-closed barcode } [a, b] \text{ for } z = a + ib \\
\text{the multiplicity of the } (r-1) \text{-open barcode } (\alpha, \beta) \text{ for } z = \beta + i\alpha \\
0 \text{ otherwise,}
\end{cases} \tag{1}
\]

\[
\gamma^f_r(z) := \begin{cases} 
\text{the multiplicity of the } r \text{-closed-open barcode } [m, n] \text{ for } z = m + in \\
\text{the multiplicity of the } r \text{-open-closed barcode } (m', n') \text{ for } z = n' + im' \\
0 \text{ otherwise}
\end{cases} \tag{2}
\]

cf [4] or [1], whose elements are the points in the support of these with multiplicity the value of these maps

If \( X \) is compact then the maps \( \delta^f_r \) and \( \gamma^f_r \) are configurations of points, i.e. maps with finite support.

One convenes to denote by \( B^r_{\infty}(f), B^r_{\text{finite}}(f), B^{r,c}(f), B^{r,c,e}(f) \) the multi-sets (sets of elements with multiplicity) of level persistence closed \( r \)-barcodes, open \( r \)-barcodes, closed-open \( r \)-barcodes and \( r \)-open-closed barcodes.

**Sub-level persistence**

For a nonnegative integer \( r \), a field \( \kappa \), a real-valued tame proper map \( f : X \rightarrow \mathbb{R} \) bounded from below, with \( X \) an ANR, (and based on homology with coefficients in \( \kappa \)) the sub-level persistence provides the following two type of barcodes c.f [5]

1. **sub-level persistence infinite** \( r \)-**barcodes** \((a, \infty), a \in \mathbb{R}\),

2. **sub-level persistence finite** \( r \)-**barcodes** \((a, b), a < b, a, b \in \mathbb{R}\).

One convenes to denote by \( SB^r_{\infty}(f) \) and \( SB^r_{\text{finite}}(f) \) the multi-sets of the sub-level persistence infinite \( r \)-barcodes and finite \( r \)-barcodes respectively.

Level persistence barcodes refine the sub-level persistence barcodes as follows:

- A level persistence closed \( r \)-barcode \([a, b]\) corresponds to a sub-level persistence infinite \( r \)-barcode \((a, \infty)\) with the same multiplicity.
- A level persistence open \( r \)-barcode \((\alpha, \beta)\) corresponds to a sub-level persistence infinite \((r+1)\)-barcode \((\beta, \infty)\) with the same multiplicity.
- A level persistence closed-open \( r \)-barcode \([m, n]\) corresponds to a sub-level persistence finite \( r \)-barcode \((m, n)\) with the same multiplicity.

The level persistence open-closed \( r \)-barcode \((m', n]\) is not visible in sub-level persistence of \( f \) (but it does appear in the sub-level persistence of \(-f\)).

Note that the ends of any barcode for \( f \) are among the critical values of \( f \), the values \( t \) for which the homology of the level of \( f \) at \( t \) differs from the homology of levels at values in an arbitrary neighborhood of \( t \).
The case of angle-valued maps

If \( f : X \to S^1 \) is a tame angle-valued map let \( \tilde{f} : \tilde{X} \to \mathbb{R} \) be a lift of \( f \), i.e. given by a pullback diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\pi} & S^1 \\
\tilde{f} & \uparrow & \uparrow f \\
\tilde{X} & \longrightarrow & X
\end{array}
\]

with \( \pi \) the canonical infinite cyclic cover. Recall that tame means \( f^{-1}(\theta) \) is an ANR for any \( \theta \in S^1 \). One defines the multi-sets \( B_c^r(f), B_o^r(f), B_c^o(f), B_o^c(f) \) as the quotient sets

\[
B^r := B^r(\tilde{f})/2\pi \mathbb{Z}
\]

with respect to the obvious free action of \( \mathbb{Z} \) given by \( \mu(k, I) = I + 2\pi k \). The multiplicity of an element in the quotient set is the same as the multiplicity of any of its representatives.

Barcodes for exact or an integral (topological) closed one form

If one likes to consider tame real-valued maps \( f : X \to \mathbb{R} \) up to an additive constant, i.e. as an exact topological closed one form \( \omega = df \) or tame angle-valued maps \( f : X \to S^1 \) up to composition by a rotation of \( S^1 \), i.e. as a topological integral close one form \( \omega = f^*(dt) \) with \( dt \) the canonical differential one form on \( S^1 \) (the length) then the barcodes of \( \omega \) are actually real numbers given by the length of a barcode \( I \in B^r(\tilde{f}) \) or \( I \in B^c^r(\tilde{f}) \), precisely \( p(I) = b - a \), \( p : \mathbb{C} \equiv \mathbb{R}^2 \to \mathbb{R}, p(z = a + ib) = b - a \), with the multiplicity of the barcode \( x \in \mathbb{R} \) given the sum of the multiplicity of each barcode \( I \in p^{-1}(x) \).

This explains why the barcodes for an (arbitrary) closed one form which will be introduced in a future paper [2] are real numbers.

2 Chain complexes of finite dimensional vector spaces over a field \( \kappa \)

Some elementary linear algebra

It is a straightforward consequence of elementary linear algebra that a chain complex, cf [1] chapter 8,

\[
(C_*, \partial_*) := \cdots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_0 \xrightarrow{\partial_0} 0
\]

is determined up to a non canonical isomorphism by two of the three systems of integers:

1. \( c_r := \dim C_r \)
2. \( \beta_r := \dim H_r(C_*, \partial_*) \)
3. \( \rho_r := \text{rank } \partial_r \)

related by

\[
c_r = \beta_r + \rho_r + \rho_{r-1}
\]

Actually the chain complex is isomorphic to the chain complex \( (hC_*, h\partial_*) \) with \( hC_n \simeq \kappa^{\beta_r} \oplus \kappa^{\rho_r} \oplus \kappa^{\rho_{r-1}} \)

\[
h\partial_r = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \text{id}
\end{pmatrix}.
\]

This chain complex \( (hC_*, h\partial_*) \) is referred to as the Hodge form of \( (C_*, \partial_*) \) and is isomorphic although non canonically to \( (C_*, \partial_*) \).

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1. the concept of topological closed one form will be defined in [2] and is not used in this note
Suppose that the chain complex \((C'_r, \partial'_r)\) is a sub-complex of \((C_r, \partial_r)\) i.e. \(C'_r \subseteq C_r\) and \(\partial'_r |_{C'_r}\) and therefore \(c'_r \leq c_r\) and \(\rho'_r \leq \rho_r\).

It is straightforward to check that any isomorphism from \(i'_*: (C'_r, \partial'_r) \to (hC'_r, h\partial'_r)\) can be extended to an isomorphism \(i_*: (C_r, \partial_r) \to (hC_r, h\partial_r)\).

**Morse complex**

Let \(M^n\) be a closed manifold, \(X\) a smooth vector field which is Morse-Smale and \(f: M^n \to \mathbb{R}\) a Morse function which is Lyapunov for \(X\). Each rest point \(x \in X := \{y \in M^n \mid X(y) = 0\}\), equivalently critical point of \(f\), has a Morse index \(r \in \{0, 1, \cdots n\}\). Let \(\kappa\) be a field.

To these data one associate the \(\kappa\)-vector spaces \(C_r := \kappa[X_r]\) generated by the set \(X_r\) of the rest points of Morse index \(r\) and the linear maps \(\partial_r : C_r \to C_{r-1}\) given by the matrix \(\partial_r\) with entries \(\partial_r(x, y)\) which counts the algebraic cardinality of the set of trajectories of \(X\) from the rest point \(x \in X_r\) to \(y \in X_{r-1}\) (instantons) viewed as an element in \(\kappa\).

For \(t \in \mathbb{R}\) one denotes by \(X_r(t) := \{x \in X_r, f(x) \leq t\}\) and one observes that the subspace \(C_r(t) := \kappa[X_r(t)]\) is invariant to \(\partial_r\), hence \(\partial_r(C_r(t)) \subseteq C_{r-1}(t)\). Let \(\partial_r(t)\) be the restriction of \(\partial_r\) to \(C_r(t)\). This equips \((C_r, \partial_r)\) with the filtration indexed by \(t \in \mathbb{R}\),

\[
0 \subseteq (C_r(t'), \partial_r(t')) \subseteq (C_r(t), \partial_r(t)) \cdots \subseteq (C_r, \partial_r)
\]

for \(t' < t\).

One of the main results of Morse theory is the following theorem.

**Theorem 2.1 (Morse theory)**

1. \(\partial_{r-1} \circ \partial_r = 0\), hence \((C_r, \partial_r)\) defines a chain complex of finite dimensional vector spaces referred below as the Morse complex of the pair \((M, X)\) and \((C_r(t), \partial_r(t))\) provides a filtration of the chain complex \((C_r, \partial_r)\).

2. \(H_r(C_r, \partial_r) \cong H_r(M; \kappa)\) and \(H_r(C_r(t), \partial_r(t)) = H_r(f^{-1}((-\infty, t]), \kappa)\),

3. Up to a non canonical isomorphism the Morse complex \((C_r, \partial_r)\) and its sub-complexes \((C_r(t), \partial_r(t))\) depend only on the Morse function \(f\).

One can extend the above result to the case \(M\) is a compact manifold with boundary \(\partial M\) with \(X\) transversal to \(\partial M\) and pointing out towards the interior and \(f\) is constant on \(\partial M\). The statement remains the same.

A similar complex is obtained in case \(f: X \to S^1\) is an angle-valued map which is Lyapunov for \(X\). In this case the set of trajectories from \(x \in X_r\) to \(y \in X_{r-1}\) can have countable cardinality but their "counting" can still be done as shown by Novikov cf [6] or [7] with the result an element in the field \(\kappa[t^{-1}, t]\) of Laurent power series with coefficients in \(\kappa\).

Finally the complex referred below as the Novikov complex has the components \(C_k\) the \(\kappa[t^{-1}, t]\)-vector space generated by \(X_k\) and the linear maps \(\partial_k\) given by matrices with coefficients \(\partial_k(x, y), x \in X_r, y \in X_{r-1}\), elements in \(\kappa[t^{-1}, t]\).

There is no Morse type filtration in this case\(\footnote{but there are other interesting filtrations}\).

The above Morse complex and its Morse filtration as well as the Novikov complex can be recovered up to a non canonical isomorphism from the barcodes of the level persistence. More precisely one has.
Theorem 2.2
If \( f : M^n \to \mathbb{R} \) is a Morse real-valued function then \( f \) is tame and

1. \[
\beta_r = \sum_{z \in \text{supp} f_r} \delta_f^r(z) = \sharp(B^c_r \sqcup B^o_r) = \sharp S B^c_r
\]
2. \[
\rho_r = \sum_{z \in \text{supp} f_r, R z < \Omega z} \gamma_f^r(z) = \sharp S B^c_r \leq \sharp B^c_r
\]

and then \( c_r = \beta_r + \rho_r + \rho_{r-1} \).

In view of the observation that the end of each barcode is a critical value the knowledge of the \((C_*(t), \partial_*(t))\), for all critical values \( t \), determines the entire collection of bar codes \( B^c_r, B^o_r, \) and \( B^c_r \).

For \( I \) a barcode with ends \( a, b \) denote by \( l(I) = a \) and \( r(I) = b \). As shown in [1] Chapter 4 one has:

1. \[ H_r(t) := \begin{cases} 
\{ I \in B^c_r : l(I) \leq t \} \sqcup \\
\{ I \in B^o_r : l(I) \leq t \} \sqcup \\
\{ I \in B^o_r : l(I) \leq r(I) < t \} 
\end{cases} \text{ with } H_r(C_*(t), \partial_*(t)) \simeq \kappa[H_r(t)], \]
2. \[ C^+_r(t) := \{ I \in B^c_r : l(I) \leq t, r(I) \leq t \} \text{ with } C^+_r(t) \simeq \kappa[C^+_r(t)], \]
3. \[ C^-_r(t) := \{ I \in B^c_r : l(I) \leq r(I) < t \} \text{ with } C^-_r(t) \simeq \kappa[C^-_r(t)], \]
4. \[ C_r(t) := H_r(t) \sqcup C^+_r(t) \sqcup C^-_r(t) \subseteq B^c_r \sqcup B^o_r \sqcup B^c_r \sqcup \text{ with } C_r \simeq \kappa[C_r]. \]

For \( t \leq t' \) let \( i_r(t, t') : \kappa[C_r(t)] \to \kappa[C_r(t')] \) be the inclusion induced linear map and denote by \( \partial_r \) the linear extension of \( h \partial_r(I) := \begin{cases} 0 \text{ if } I \in H_r \cup C^+_r \\
I \text{ if } I \in C^-_{r-1} \end{cases} \).

Theorem 2.3
Suppose \( f : M^n \to \mathbb{R} \) is a Morse function defined on a closed manifold. The linear maps \( i_r(t, t') \) and \( i_{r-1}(t, t') \) intertwine \( h \partial_r(t) \) and \( h \partial_{r}(t') \) and then for \( t < t' \), the complexes \((h C_*(t), h \partial_*(t))\) provide a filtration of \((h C_*, h \partial_*)\) which makes the filtered chain complex \((h C_*, h \partial_*)\) non canonical isomorphic to the Morse complex equipped with a Morse filtration.
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