New Models of $f(R)$ Theories of Gravity

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ABSTRACT: We introduce new models of $f(R)$ theories of gravity that are generalization of Hořava-Lifshitz gravity.

KEYWORDS: Hořava-Lifshitz gravity
1. Introduction and Summary

The acceleration of the universe was discovered 10 years ago and its explanation is one of the most secret enigmas of current theoretical physics. According to [1] and the other experiments that map the anisotropies of the cosmic microwave background, the approximately 76% of the energy content of the universe is not dark or luminous matter but it is instead a mysterious form of dark energy that is exotic, invisible and unclustered. In order to explain the origin of this form of matter three main classes of models for this acceleration exist:

1. A cosmological constant $\Lambda$
2. Dark Energy
3. Modified Gravity

The cosmological constant is the most obvious explanation of the acceleration of the universe [2, 3]. However $\Lambda$ suffers from well known cosmological problem that requires extreme fine-tuning.

The second class of models postulates the existence of a dark energy fluid with equation of state $P \approx -\rho$ where $\rho$ and $P$ are the energy density and pressure of the fluid [4].

The last class of models known as Extended Theories of Gravity (ETG) corresponds to the modification of the action of the gravitational fields. These theories are based on idea of extension of the Einstein Hilbert action by adding higher order curvature invariants such as $R^2, R^\mu\nu R_{\mu\nu}, \ldots$ and/or minimally or non-minimally coupled scalar fields to the dynamics $^1$. In this paper we consider the models corresponding to $f(R)$ theories that are general functions of the Riemann scalar $R$.

As we said above $f(R)$ theories should mainly explain the mysterious acceleration of the Universe and hence provide modification of the Einstein theory on large scales. On the other hand they do not solve the second fundamental problem considering theory of

$^1$For review and extensive list of references, see [6, 7, 8, 9, 10].
gravity which is the fact that the quantum gravity is non-renormalizable theory. In order
to improve this situation new formulation of the quantum gravity theory was proposed
recently in series of papers [11, 12, 4, 13]. The key idea of this proposal is to abandon
the fundamental role of the local Lorentz invariance and assume instead that this appears
only at low energies as an approximative symmetry. The breaking of Lorentz invariance is
achieved by introducing additional structure to the space-time that is a preferred foliation
by 3−dimensional space-like surfaces and then splitting of the coordinates into space and
time. This procedure allows to complete the Einstein’s general relativity with higher spatial
derivatives of the metric and then improve the UV behavior of the graviton propagator
and makes the theory renormalizable by power-counting.

Even if the Hořava-Lifshitz theory has many attractive properties investigations of
various aspects of Hořava gravity have started to reveal some potentially troublesome
features. The first one is considering the detailed balance condition that strongly restricts
the form of the potential terms in the gravity action. However the detailed investigation of
the Hořava-Lifshitz theory performed in past few months strongly suggests that this theory
has much better behavior when we abandon the condition of the detailed balance 3.

The second fundamental ingredient of the original Hořava-Lifshitz was the projectability
condition that says the lapse function in ADM formulation of gravity depends on time only.
Even if for the most important vacuum and cosmological solutions Einstein equations
can be put into the form of constant lapse function it seems that the consistency of the
Hořava-Lifshitz theory requires the projectability condition. In fact, the investigation of
the algebra of constraints of the Hořava-Lifshitz theory without projectability condition
performed in paper [13] implies some pathological behavior of this theory. On the other
hand it was already shown in [12] that the algebra of constraints of Hořava-Lifshitz theory
with projectability condition is closed. However it is important to stress that we should
be very careful considering the definite conclusion about the consistency/non-consistency
of Hořava-Lifshitz theory with/without projectability condition as was discussed recently
in [15, 22] in relation with another issue of Hořava-Lifshitz theory that is spin-0 scalar
graviton with addition to the standard spin-2 tensor graviton. The study of the properties
of this mode showed that it persists down to low energies and exhibits the pathological
behavior that should be disaster for the consistency of Hořava-Lifshitz theory. Explicitly,
it was shown in [32] that this mode is strongly coupled at energies above a very low energy
scale and it suffers from fast instabilities. It turns out that these properties appear both in
the non-projectable and projectable cases. On the other hand it was argued in very recent
paper [22] that it is possible to extend the original Hořava-Lifshitz action in such a way
that these dangerous properties are absent.

Despite of these open problems it is clear that Hořava-Lifshitz theory is very interesting
and stimulating proposal that should be extended in many directions. Our goal is to
generalize this theory in the similar way as the ordinary Einstein gravity is extended to

\footnote{For detailed study of this proposal, see [11, 12, 4, 13].}

\footnote{For nice discussion, see [42].}
$f(R)$ theories. We started this program in our previous paper [28] where we formulated the Hamiltonian form of $f(R)$ Hořava-Lifshitz theory based on the condition of the detailed balance. We considered one explicit example of $f(R)$ Hořava-Lifshitz theory with the function $f(x) = \sqrt{x}$ and we found the Lagrangian version of this theory. In this paper we generalize this analysis to the case of arbitrary $f(R)$ Hamiltonian form of Hořava-Lifshitz theory. We show that when we extend the original Hamiltonian of $f(R)$ Hořava-Lifshitz theory by introducing two non-propagating degrees of freedom we can find the Hamiltonian in such a form that allows us to perform the Legendre transformation from the Hamiltonian to the Lagrangian description. Then solving the equations of motion for these auxiliary fields we finally find a new form of $f(R)$ theories that are invariant under foliation of preserving diffeomorphism [13].

It is also possible to introduce the Hořava-Lifshitz theories of gravity in an alternative way when we replace 4-dimensional Ricci scalar $R$ with combination $R \rightarrow K_{ij}G^{ijkl}K_{kl} + \ldots$ known from the linear Hořava-Lifshitz theory of gravity. The resulting theory is invariant under foliation preserving diffeomorphism.

We hope that this short note could be the starting point in the study of the new form of $f(R)$ theories of gravity. It will be certainly very interesting to study the equations of motion that follow from these actions and find their possible cosmological solutions. It would be also interesting to perform the analysis of the fluctuation modes around classical background and study their physical properties. We hope to return to these problems in future.

2. The First Version of $f(R)$ Hořava-Lifshitz Theory of Gravity

In this section we consider $f(R)$ Hořava-Lifshitz theory of gravity that was introduced in [28]. These theories are defined by Hamiltonian

$$H = \int d^Dx \left( N(t)\mathcal{H}_0(t,x) + N_i(t,x)\mathcal{H}_i(t,x) \right),$$

where $\mathcal{H}_i$ is the generator of the spatial diffeomorphism

$$\mathcal{H}_i(t,x) = -2\nabla_j \pi^j_i(x)$$

(2.2)

that takes the same form as in standard Einstein theory of gravity. On the other hand the Hamiltonian constraint $\mathcal{H}_0$ of $f(R)$ Hořava-Lifshitz $\mathcal{H}_0$ is defined as

$$\mathcal{H}_0(t,x) = \kappa^2 \sqrt{f} \left( Q^{ij} \frac{1}{g} G^{ijkl} Q^{kl} \right),$$

(2.3)

where $\kappa$ is the coupling constant of the theory, $g = \det g_{ij}$ and where $f$ is an arbitrary function. Further, the functions $Q^{ij}$ and $Q^{ij}$ are defined as

$$Q^{ij} = i\pi^{ij} + \sqrt{g} E^{ij}, \quad Q^{ij} = -i\pi^{ij} + \sqrt{g} E^{ij},$$

(2.4)

where $g_{ij}, \ i, j = 1, \ldots, D$ are components of metric and $\pi^{ij}$ are conjugate momenta. These canonical variables have non-zero Poisson brackets

$${\{ g_{ij}(x), \pi^{kl}(y) \} = \frac{1}{2} (\delta_k^i \delta_l^j + \delta_l^i \delta_k^j) \delta(x - y).}$$

(2.5)
Finally we explain the meaning of the objects $E_{ij}$. In the original version of Hořava-Lifshitz theory based on the condition of the detailed balance they are given by variation of the $D$-dimensional functional $W = \int d^Dx \sqrt{g} w(g, \partial g)$

$$\sqrt{g} E_{ij}(x) = \frac{1}{2} \frac{\delta W}{\delta g_{ij}(x)}.$$ \hspace{1cm} (2.6)

However we can clearly abandon this requirement and consider the generalized Hořava-Lifshitz theories where the Hamiltonian constraint $H_0$ takes the form

$$H_0 = \kappa^2 \sqrt{g} f \left( \frac{1}{g} \pi^{ij} G_{ijkl} \pi^{kl} + E_{ij} G_{ijkl} E^{kl} + \ldots \right),$$ \hspace{1cm} (2.7)

where $\ldots$ denote the contribution from additional terms that violate the detailed balance condition and that are crucial for reproducing General relativity at low energy regime. However for simplicity of the notation we consider the first form of the Hamiltonian (2.3) keeping in mind that the analysis below is valid for the general case as well. Finally the "metric on the space of metric", $G^{ijkl}$ is defined as

$$G^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} - \lambda g^{ij} g^{kl} \right)$$ \hspace{1cm} (2.8)

with $\lambda$ an arbitrary real constant. The inverse $G_{ijkl}$ takes the form

$$G_{ijkl} = \frac{1}{2} \left( g_{ik} g_{jl} + g_{il} g_{jk} \right) - \tilde{\lambda} g_{ij} g_{kl},$$ \hspace{1cm} (2.9)

where $\tilde{\lambda} = \frac{\lambda}{D \lambda - 1}$. Note that (2.8) together with (2.9) obey the relation \footnote{We use the terminology introduced in [12] and that we review there. In case of relativistic theory, the full diffeomorphism invariance fixes the value of $\lambda$ uniquely to equal $\lambda = 1$. In this case the object $G_{ijkl}$ is known as the "De Witt metric". We use this terminology to more general case when $\lambda$ is not necessarily equal to one.}

$$G_{ijmn} G^{mnkl} = \frac{1}{2} \left( \delta^k_i \delta^l_j + \delta^l_i \delta^k_j \right).$$ \hspace{1cm} (2.10)

We should stress one important point. The Hamiltonian (2.3) contains the lapse function that depends on $t$ only. This property is known as projectability condition. It is still an open problem whether this condition should be imposed or whether we should consider the lapse function with the full space and time dependence. In this section we consider the theories that obey projectability condition. When this condition is imposed the functions $N(t)$ and $N_i(t, x)$ have conjugate momenta $\pi_N, \pi^i(x)$ that form the primary constraints of the theory

$$\pi_N \approx 0, \quad \pi^i(x) \approx 0.$$ \hspace{1cm} (2.11)

Since these constraints have to be preserved during the time evolution of the system we find

$$\partial_t \pi_N = \{\pi_N, H\} = \int d^Dx H_0(x) \equiv T_T \approx 0.$$ \hspace{1cm} (2.12)
Note that the global constraint $T_T \approx 0$ is fundamentally different from the local constraint $H_0(x) \approx 0$ that is imposed in case when the theory does not respect the projectability condition. Further, the consistency of the constraints $\pi^i(x) \approx 0$ imply the secondary constraints

$$\partial_t \pi^i(x) = \{\pi^i(x), H\} = -\mathcal{H}'(x) \approx 0.$$  \hfill (2.13)

It is natural to introduce the smeared form of the spatial diffeomorphism constraint

$$T_S \equiv \int d^Dx \xi_i(x) \mathcal{H}'(x).$$  \hfill (2.14)

The next goal is to calculate the Poisson bracket of constraints (2.12) and (2.14). These brackets have been determined in [28] with following result

$$\{T_T, T_T\} = 0,$$

$$\{T_S(\zeta), T_T\} = 0,$$

$$\{T_S(\zeta), T_S(\xi)\} = T_S(\zeta^i \partial_i \xi^k - \xi^i \partial_i \zeta^k).$$  \hfill (2.15)

In other words the constraint algebra of the gravity action that is invariant under foliation preserving diffeomorphism is closed. This is a crucial simplification with respect to theories that do not respect the projectability condition. More precisely, the algebra of constraints of the Hořava-Lifshitz theory without projectability condition was calculated in [43] with the result that suggests inconsistency of this theory.

2.1 Lagrangian Formulation

In this section we formulate the Lagrangian formalism for the $f(R)$ theories defined by the Hamiltonian (2.1). In order to this we argue that the Hamiltonian constraint $\mathcal{H}_0$ defined in (2.1) can be written as

$$\mathcal{H}_0 = \kappa^2 \sqrt{g} \left[ B \left( Q^{i j} \frac{1}{g} g_{i j k l} Q^{k l} - A \right) + f(A) \right] + v_A P_A + v_B P_B,$$  \hfill (2.16)

where we introduced two new fields $A(x), B(x)$ with conjugate momenta $P_A(x), P_B(x)$ and corresponding Poisson brackets

$$\{A(x), P_A(y)\} = \delta(x - y), \quad \{B(x), P_B(y)\} = \delta(x - y).$$  \hfill (2.17)

Looking on the form of the Hamiltonian density (2.16) we see that it is natural to interpret $P_A(x), P_B(x)$ as the primary constraints of the theory

$$P_A(x) \approx 0, \quad P_B(x) \approx 0.$$  \hfill (2.18)

Then the requirement that the constraints (2.18) are preserved during the time evolution implies an existence of the secondary constraints $G_A(x), G_B(x)$ since

$$\partial_t P_A = \{P_A, H\} = \kappa^2 \sqrt{g} (B - f'(A)) \approx \kappa^2 \sqrt{g} G_A \approx 0,$$

$$\partial_t P_B = \{P_B, H\} = -\kappa^2 \sqrt{g} (Q^{i j} \frac{1}{g} g_{i j k l} Q^{k l} - A) = -\kappa^2 \sqrt{g} G_B \approx 0.$$  \hfill (2.19)
where $H$ is given in (2.1) with $\mathcal{H}_0$ defined in (2.16). Let us now calculate the Poisson brackets between $P_A, P_B$ and $G_A, G_B$. It is a straightforward exercise to calculate them and we find

\[
\begin{align*}
\{P_A(x), G_B(y)\} &= \delta(x - y), \\
\{P_A(x), G_A(y)\} &= f''(A)\delta(x - y), \\
\{P_B(x), G_A(y)\} &= -\delta(x - y), \\
\{P_B(x), G_B(y)\} &= 0.
\end{align*}
\] (2.20)

These results imply that $P_A, P_B, G_A, G_B$ form the second class constraints. As a consequence the requirement of the preservation of the constraints $G_A, G_B$ under the time evolution fixes $v_A, v_B$ uniquely. Since these constraints are second class they strongly vanish and can be explicitly solved with the result

\[
B = f'(A), \quad A = Q^{ij} g_{ijkl} Q_{kl}.
\] (2.21)

Then if we insert these results into the Hamiltonian density given in (2.16) we find that it takes the form

\[
\mathcal{H}_0 = \kappa^2 \sqrt{f} \left( Q^{ij} g_{ijkl} Q_{kl} \right)
\] (2.22)

that coincides with (2.1). As the final point note that for the system with the second class constraints we have to introduce the Dirac brackets instead of Poisson brackets

\[
\left\{ g_{ij}(x), \pi^{kl}(y) \right\}_D = \left\{ g_{ij}(x), \pi^{kl}(y) \right\} - \int dx' dy' \left\{ g_{ij}(x), \Phi_I(x') \right\} \Omega^{IJ}(x', y') \left\{ \Phi_J(y'), \pi^{kl}(y) \right\},
\] (2.23)

where $\Phi_I, I = 1, 2, 3, 4$ is a collection of the second class constraints $\Phi_I = (P_A, G_A, P_B, G_B)$ and where $\Omega^{IJ}$ is inverse of the matrix of Poisson brackets (2.20). Explicitly

\[
\Omega_{IJ}(x, y) \equiv \left\{ \Phi_I(x), \Phi_J(y) \right\} = \begin{pmatrix}
0 & f'' & 0 & 1 \\
-f'' & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix} \delta(x - y)
\] (2.24)

so that

\[
\Omega^{IJ}(x, y) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & -f'' \\
1 & 0 & f'' & 0
\end{pmatrix} \delta(x - y)
\] (2.25)

However due to the structure of the matrix $\Omega^{IJ}$ and the fact that $g_{ij}$ and $\pi^{ij}$ have vanishing Poisson brackets with $P_A, P_B$ we find that the Dirac brackets coincide with Poisson brackets.
In summary, we claim that the Hamiltonian density that is useful for the developing of the Lagrangian formalism takes the form

\[ H = -2N_i \nabla_j \pi^{ij} + N \kappa^2 \sqrt{g} \left[ B \left( Q^{ij} \frac{1}{g} g_{ijkl} Q^{kl} - A \right) + f(A) \right] + v_A P_A + v_B P_B \ . \tag{2.26} \]

The main advantage of the Hamiltonian (2.26) density with respect to (2.1) is that it depends on the momenta \( \pi^{ij} \) quadratically and hence the Legendre transformation from the Hamiltonian to the Lagrangian description can be easily performed. Explicitly, the canonical equations of motion for \( A, B \) and \( g_{ij} \) take the form

\[
\begin{align*}
\partial_t A &= \{ A, H \} = v_A , \\
\partial_t B &= \{ B, H \} = v_B , \\
\partial_t g_{ij} &= \{ g_{ij}, H \} = \frac{2BN\kappa^2}{\sqrt{g}} G_{ijkl} \pi^{kl} + \nabla_i N_j + \nabla_j N_i . \tag{2.27}
\end{align*}
\]

It is natural to define

\[ K_{ij} = \frac{1}{2N}(\partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i) \tag{2.28} \]

so that the equation on the second line in (2.27) implies

\[ K_{ij} = \frac{B\kappa^2}{\sqrt{g}} G_{ijkl} \pi^{kl} . \tag{2.29} \]

Then it is easy to determine the corresponding Lagrangian

\[
\begin{align*}
L &= \int d^Dx (\partial_t g^{ij} \pi_{ij} + \partial_t A P_A + \partial_t B P_B - H) = \\
&= \int d^Dx N \sqrt{g} \left( \frac{1}{B\kappa^2} K_{ij} G_{ijkl} K_{kl} - \kappa^2 B(E^{ij} G_{ijkl} E^{kl} - A) - \kappa^2 f(A) \right) . \tag{2.30}
\end{align*}
\]

Finally we eliminate non-dynamical fields \( A \) and \( B \) from (2.30). Varying (2.30) with respect to \( B \)

\[ - \frac{1}{B^2\kappa^2} K_{ij} G^{ijkl} K_{kl} - (E^{ij} G_{ijkl} E^{kl} - A) = 0 , \tag{2.31} \]

and varying (2.30) with respect to \( A \) we find

\[ B - f'(A) = 0 . \tag{2.32} \]

Inserting (2.32) into (2.31) we obtain the equation

\[ - \frac{1}{f'^2(A)\kappa^2} K_{ij} G^{ijkl} K_{kl} - \kappa^2 (E^{ij} G_{ijkl} E^{kl} - A) = 0 \]
that, for known function \( f \), allows us to express \( A \) as a function of \( K_{ij}, g_{ij} \)
\[
A = \Psi(K_{ij}, g_{ij}) .
\]

Inserting these results into the Lagrangian (2.31) we find the new class of \( F(R) \) theories of gravity in the form
\[
L = \int d^Dx \mathcal{L} ,
\]
\[
\mathcal{L} = N \sqrt{g} \left[ \frac{1}{2} K_{ij} g^{ijkl} K_{kl} - \kappa^2 f(\Psi(K_{ij}, g_{ij})) \right] .
\]

Let us consider the specific example of the theory defined by the function \( f(A) = \sqrt{1 + A} - 1 \). For this function the equation (2.32) implies
\[
B = \frac{1}{2\sqrt{1 + A}} .
\]

Inserting this result into (2.31) we find
\[
A = \Psi(g_{ij}, K_{ij}) = \frac{E_{ij} G_{ijkl} E_{kl} + \frac{4}{\kappa^2} K_{ij} G_{ijkl} K_{kl}}{1 - \frac{1}{\kappa^2} K_{ij} G_{ijkl} K_{kl}} .
\]

and hence the Lagrangian density takes the form
\[
\mathcal{L} = -N \sqrt{g} (1 - 4K_{ij} G_{ijkl} K_{kl}) \sqrt{1 + \Psi} + \kappa^2 \sqrt{g} N =
\]
\[
= -\kappa^2 N \sqrt{g} \left( \sqrt{1 - \frac{4}{\kappa^4} K_{ij} G_{ijkl} K_{kl} \sqrt{1 + E_{ij} G_{ijkl} E_{kl}} - 1} \right) .
\]

We see that (2.38) coincides with the Lagrangian density studied in [28].

Finally we show that the action \( S = \int dt L \) where \( L \) is given in (2.38) is invariant under the foliation-preserving diffeomorphism that by definition consists a space-time dependent spatial diffeomorphisms as well as time-dependent time reparameterization. These symmetries are generated by infinitesimal transformations
\[
\delta x^i \equiv x'^i - x^i = \zeta^i(t, x) , \quad \delta t \equiv t' - t = f(t) .
\]

It was shown in [12] that the metric transform under (2.39) as
\[
g^{ij}(t', x') = g_{ij}(t, x) - g_{il}(t, x) \partial_j \zeta^l - \partial_i \zeta^k g_{kj}(t, x) ,
\]
\[
g^{ij}(t', x') = g^{ij}(t, x) + \partial_i \zeta^j g^{mj}(t, x) + g^{in}(t, x) \partial_n \zeta^j
\]

and the fields \( N_i(t, x), N(t) \) transform under (2.39) as
\[
N^i(t', x') = N_i(t, x) - N_i(t, x) \dot{f} - N_j(t, x) \partial_i \zeta^j - g_{ij}(t, x) \dot{\zeta}^i ,
\]
\[
N^t(t', x') = N^t(t, x) + N^j(t, x) \partial_j \zeta^i(t, x) - N_i(t, x) \dot{f} - \dot{\zeta}^i(t, x) ,
\]
\[
N^t(t') = -N(t) \dot{f} .
\]
Further, the transformation property of the metric components (2.40) imply following transformation prescription for $G_{ijkl}$

\[
G'_{ijkl}(t', x') = G_{ijkl}(t, x) + \partial_i \zeta^m(t, x) G_{mjk}(t, x) + \partial_j \zeta^m(t, x) G_{ikm}(t, x) + \partial_k \zeta^m(t, x) G_{ijm}(t, x) .
\]

(2.42)

It can be also shown that $K_{ij}$ transform covariantly under (2.39)

\[
K'_{ij}(t', x') = K_{ij}(t, x) - K_{ik}(t, x) \partial_j \zeta^k(t, x) - \partial_i \zeta^k(t, x) K_{kj}(t, x) .
\]

(2.43)

Using these formulas it is easy to see that the expression $K_{ij} G_{ijkl} K_{kl}$ is invariant under (2.39).

Finally we consider the expression $E_{ij} G_{ijkl} E_{kl}$. As was shown in [28] $E_{ij}$ transforms under (2.39) as

\[
E'_{ij}(t', x') = E_{ij}(t, x) + \partial_k \zeta^i(t, x) E_{kj}(t, x) + E_{ik}(t, x) \partial_k \zeta^j(t, x) .
\]

(2.44)

Then $E_{ij} G_{ijkl} E_{kl}$ is clearly invariant under (2.39). Finally if we use the fact that

\[
N'(t') \sqrt{g(x', t')} dt' d^D x' = N(t) \sqrt{g(x, t)} dt d^D x
\]

(2.45)

under (2.39) we find that the action (2.30) is invariant under foliation preserving diffeomorphism (2.39).

3. The Second Version of $f(R)$ Hořava-Lifshitz Theory of Gravity

In this section we present the second way how to define $f(R)$ Hořava-Lifshitz theories of gravity. We begin with known $f(R)$ theory of gravity with the action

\[
S = -\frac{1}{\kappa^2} \int dt d^D x \sqrt{g} f(R^{(4)}) ,
\]

(3.1)

where $R^{(4)}$ is four dimensional covariant Ricci scalar. Then in order to find $f(R)$ Hořava-Lifshitz theory of gravity we simply replace $R^{(4)}$ in (3.1) with the combination $K_{ij} K^{ij} - \lambda^2 K - E_{ij} G_{ijkl} E^{kl}$ that appears in the Hořava-Lifshitz theory that obeys the detailed balance condition 5.

As in previous section we introduce the auxiliary fields $A$ and $B$ and write the action (3.1) in the equivalent form

\[
S = \kappa^2 \int dt d^D x \sqrt{g} N (B(K_{ij} K^{ij} - \lambda^2 K^2 - E_{ij} G_{ijkl} E^{kl} - A) + f(A)) .
\]

(3.2)

5We would like to stress that this procedure can be easily generalized to theories that break the detailed balance condition. Generally, we could replace $R^{(4)}$ with $K_{ij} K^{ij} - \lambda^2 K - V$ where the explicit form of $V(g)$ was given in [4].
Clearly the solution of the equation of motion for $B$ reproduces the original action while the equation of motion for $A$ implies $f'(A) = B$. Inserting this result into (3.2) we find
\[ S = \kappa^2 \int dt d^Dx \sqrt{g} N(f'(A)(K_{ij}K^{ij} - \lambda^2 K^2 - E^{ij}G_{ijkl}E^{kl} - A) + f(A)) . \]  
(3.3)

Our goal is to find the Hamiltonian corresponding to the action (3.2). Even if this action is invariant under foliation preserving diffeomorphism we can consider more general case and presume that the theory does not obey the projectability condition so that $N$ depends on $x$. Then we introduce the set of conjugate momenta $\pi^{ij}(x), \pi^{i}(x), \pi_{N}(x)$ and $P_{A}(x), P_{B}(x)$ where
\[ \pi^{ij} = \frac{\delta S}{\delta \partial_{t}g_{ij}} = \kappa^2 \sqrt{g} B G^{ijkl} K_{kl} , \]  
(3.4)

and where the remaining momenta form the primary constraints of the theory
\[ \pi_{N}(x) \approx 0 , \quad \pi^{i}(x) \approx 0 , \quad P_{A}(x) \approx 0 , \quad P_{B}(x) \approx 0 . \]  
(3.5)

Then it is straightforward to find corresponding Hamiltonian
\[
H = \int d^Dx \mathcal{H}(x) = \int d^Dx (N(x)\mathcal{H}_0(x) + N_i(x)\mathcal{H}_i(x) + \nonumber \\
+ v_i(x)\pi^i(x) + v_{N}(x)\pi_{N}(x) + v^{A}(x)P_{A}(x) + v^{B}(x)P_{B}(x) ) , \\
\mathcal{H}_0 = \left[ \frac{1}{\kappa^2 B \sqrt{g}} \pi^{ij}G_{ijkl}\pi^{kl} + \kappa^2 \sqrt{g} B (E^{ij}G_{ijkl}E^{kl} + A) - \kappa^2 \sqrt{g} f(A) \right] , \\
\mathcal{H}_i = -2\nabla_j \pi^{ij} .
\]  
(3.6)

Since the constraints $\pi^{i}(x) \approx 0 , \pi_{N}(x) \approx 0$ have to be preserved during the time evolution of the systems we find
\[ \partial_{t}\pi(x) = \{ \pi_{N}(x), H \} = -\mathcal{H}_0(x) \approx 0 , \nonumber \\
\partial_{t}\pi^{i}(x) = \{ \pi^{i}(x), H \} = -\mathcal{H}^{i}(x) \approx 0 . \]  
(3.7)

On the other hand the requirement of the preservation of the constraints $P_{A}(x) \approx 0 , P_{B}(x) \approx 0$ during the time evolution of the system imply
\[ \partial_{t}P_{A}(x) = \{ P_{A}(x), H \} = G_{A}(x) \approx 0 \]  
(3.8)

and
\[ \partial_{t}P_{B}(x) = \{ P_{B}(x), H \} = NG_{B}(x) \approx 0 , \]  
(3.9)
where
\[
G_A = -B + f'(A), \quad G_B = \frac{1}{\kappa^2 B^2 \sqrt{g}} \pi^{ij} G_{ijkl} \pi^{kl} - \kappa^2 \sqrt{g} (E^{ij} \tilde{G}_{ijkl} E^{kl} + A) .
\]

(3.10)

Now we calculate the Poisson brackets of these constraints when we again introduce the notation \(\Phi_I = (P_A, G_A, P_B, G_B)\), \(I = 1, 2, 3, 4\). Then we have
\[
\begin{align*}
\{\Phi_1(x), \Phi_2(y)\} &= -f''(A)(x) \delta(x - y), \\
\{\Phi_1(x), \Phi_3(y)\} &= 0, \\
\{\Phi_1(x), \Phi_4(y)\} &= \kappa^2 B(x) \sqrt{g}(x) \delta(x - y), \\
\{\Phi_2(x), \Phi_3(y)\} &= -\delta(x - y), \\
\{\Phi_2(x), \Phi_4(y)\} &= 0, \\
\{\Phi_3(x), \Phi_4(y)\} &= \kappa^2 \sqrt{g}(E^{ij} \tilde{G}_{ijkl} E^{kl} + A)(x) \delta(x - y).
\end{align*}
\]

(3.11)

We see that \(\Phi_I\) is collection of the second class constraints that can be set strongly to zero and explicitly solved \(^6\). First of all, \(\Phi_1 = \Phi_3 = 0\) imply that \(P_A = P_B = 0\). Further, from \(\Phi_2 = 0\) we find
\[
B = f'(A) .
\]

(3.12)

Finally the constraint \(\Phi_4 = 0\) together with (3.12) gives
\[
\frac{1}{\kappa^2 f'(A) \sqrt{g}} \pi^{ij} G_{ijkl} \pi^{kl} = \kappa^2 \sqrt{g} (E^{ij} \tilde{G}_{ijkl} E^{kl} + A) .
\]

(3.13)

Again, for known \(f\) the equation above allows us to find \(A\) as a function of \(\frac{1}{\sqrt{g}} \pi^{ij} G_{ijkl} \pi^{kl}\) and \(\sqrt{g} E^{ij} \tilde{G}_{ijkl} E^{kl}\) so that
\[
A = \Psi\left(\frac{1}{\sqrt{g}} \pi^{ij} G_{ijkl} \pi^{kl}, \sqrt{g} E^{ij} \tilde{G}_{ijkl} E^{kl}\right) .
\]

(3.14)

Using these results we finally find the Hamiltonian constraint \(\mathcal{H}_0\) in the form
\[
\mathcal{H}_0 = \frac{2}{\kappa^2 \sqrt{g} f'(\Psi)} \pi^{ij} G_{ijkl} \pi^{kl} - \kappa^2 \sqrt{g} f(\Psi) .
\]

(3.15)

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\(^6\) It the similar way as in previous section we can easily show that the Dirac bracket \(\{g_{ij}(x), \pi^{kl}(y)\}_D\) coincides with the Poisson bracket \(\{g_{ij}(x), \pi^{kl}(y)\}\).
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