ON THE GLOBALIZATION OF THE POISSON SIGMA MODEL IN THE BV-BFV FORMALISM

ALBERTO S. CATTANEO, NIMA MOSHAYEDI, AND KONSTANTIN WERNLI

Abstract. We construct a formal global quantization of the Poisson Sigma Model in the BV-BFV formalism using the perturbative quantization of AKSZ theories on manifolds with boundary and analyze the properties of the boundary BFV operator. Moreover, we consider mixed boundary conditions and show that they lead to quantum anomalies, i.e. to a failure of the (modified differential) Quantum Master Equation. We show that it can be restored by adding boundary terms to the action, at the price of introducing corner terms in the boundary operator. We also show that the quantum Grothendieck connection on the total space of states remains flat, which is necessary for a well-defined BV cohomology.

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1. INTRODUCTION

The goal of this paper is another step towards deformation quantization of the relational symplectic groupoid (RSG), ([9, 10]) through the Poisson Sigma Model (PSM), ([41, 40, 31], for the connection to the relational symplectic groupoid see [24, 16, 12]), using the BV-BFV formalism for the quantization of gauge theories on manifolds with boundary ([19, 20]). This possible application of the BV-BFV formalism was first discussed in [20]. In [22] we explained how the quantization of the RSG can be achieved in the case of constant Poisson structures. In [21], we generalize the methods of formal geometry used in [22] to describe the perturbative quantization of any polarized AKSZ theory ([1]), possibly on manifolds with boundary. In this paper we apply the results of [21] to the PSM, and extend them to the case of mixed boundary conditions. Let us briefly explain how we go about this task.

In Section 2 we give a very rough review of the classical and quantum BV-BFV formalism. For more details the reader is referred to the literature ([19, 20]). In particular we recall the Quantum Master Equation (QME) and its generalization to manifolds with boundary, called the modified Quantum Master Equation (mQME).

In Section 3 we recall the construction, and the results, of [21]. Most importantly, to apply the quantum BV-BFV formalism one needs to linearize the target around constant maps, which form a part of the moduli space of classical solutions of any polarized AKSZ theory ([1]). For non-linear targets, this can be done in a covariant way, as one varies the image of the constant map. In a natural way this leads to a family of quantizations parametrized by the target that satisfy a generalisation of the mQME, that we call the modified differential Quantum Master Equation (mdQME). This equation can be interpreted as the closedness of the state with respect to a flat “Quantum Grothendieck connection” $\nabla_G$. Moreover, under change of gauge choices the state changes by a

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1Here the letters BFV stand for Batalin, Fradkin and Vilkovisky, who introduced what is now known as BV [5, 4] and BFV [3, 2, 28, 27] formalisms for gauge fixing.
∇_G-exact term, so that there is a certain cohomology describing the physical states.

In Section 4 we apply the results recalled in Section 3 to the PSM, which is an example of an AKSZ theory. In particular, we describe the algebraic structure which is captured in the mdQME and the flatness of ∇_G.

In Section 6 we discuss what happens when one combines the globalization of the partition function over constant maps with the alternating or mixed boundary conditions that appear in the RSG. In particular, we describe an anomaly that arises form the curvature of the deformed Grothendieck connection D_G, and how it can be cancelled by a quantum counterterm in the action. We also describe how the mdQME gets spoilt by terms that come from the corners where the different boundary conditions meet.

In Section 7 we explain how one can restore the mdQME for the PSM with alternating boundary conditions. For this one has to extend both the space of operators and the space of states, and we sketch these extensions in Section 7.1 and 7.2. We also prove that the connection ∇_G remains flat.

Finally, in Section 8 we explain directions for further research. These are not restricted to the deformation quantization of the RSG. The methods developed in this paper could help understand both the globalization of other theories with more complicated moduli spaces of classical solutions, and the “extended” quantization (in the sense of extended TQFTs) of AKSZ theories on manifolds with corners (and possibly, defects of higher codimension).

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2. The BV-BFV formalism

The BV-BFV formalism is a gauge fixing formalism for gauge theories on manifolds with boundary, both at the classical ([19]) and quantum ([20, 21]) level. We briefly recall the most important ideas. We also refer to [21] for a more extensive introduction.

2.1. Lagrangian field theory. Let S_M = ∫_M L be a local action functional with some Lagrangian density L on a d-manifold M. S_M here is defined on some space of fields F_M, e.g. maps, sections, etc. We can associate to a (d − 1)-manifold Σ a space F_Σ of jets of fields at Σ × {0} on F_Σ × [0,ε], where ε > 0. By the variational calculus, we get a boundary 1-form term, called the “Noether form”, d̃α_Σ, for every Σ, with the property

δS = EL_M + π_M^* ̃α_Σ,

with ̃π_M: F_M → F_∂M the natural surjective submersion and EL_M the “EL 1-form” (EL denotes the part of the Euler-Lagrange equations). Moreover, we denote by δ the de Rham differential on the space of fields. One can then define d̃α_Σ := d̃α_Σ, and by techniques of symplectic geometry, such as symplectic reduction, one can construct a symplectic space of Cauchy data, which is given by a particular subset of the reduction space. Moreover, this construction is compatible with cutting and gluing ([19, 18]). This construction leads to a nice quantum formulation in the guise of path integrals choosing a suitable polarization ([20]).

2.2. Finite dimensional BV theory. Good references for gauge theories, different gauge fixing formalisms (including BV) and their perturbative quantization are [35, 36, 39]. Let M be a closed manifold and let F_M denote the space of fields associated to M. If we consider a regular Lagrangian
field theory $S_M: F_M \to \mathbb{R}$ the partition function in the path integral approach is

$$\psi_M = \int_{\Phi \in F_M} e^{iS_M(\Phi)} \mathcal{D}\Phi.$$ 

Usually, $F_M$ is infinite-dimensional, and one cannot define $\mathcal{D}\Phi$. The way out is usually to translate the formal asymptotics as $\hbar \to 0$ of finite-dimensional integrals to the infinite-dimensional case. The terms in the asymptotic expansion are conveniently labeled by Feynman diagrams [38]. If the critical points of the action functional $S_M$ are degenerate, one needs to gauge-fix the theory before one can use the formal asymptotics. To give a construction for gauge-fixing for a finite dimensional version, consider super coordinates $q^i, p_i$ and the odd symplectic form $\omega = \sum_i dq^i dp_i$. We can define the BV Laplacian

$$\Delta = \sum_i (-1)^{|q^i|} \frac{\partial^2}{\partial q^i \partial p_i}.$$ 

Then we get that $\Delta^2 = 0$ and, for two functions $f, g$, $\Delta(fg) = \Delta f g \pm f \Delta g \pm (f, g)$, where $( , )$ denotes the so-called BV bracket, which is the odd Poisson bracket induced by the symplectic form $\omega$. Moreover, for a function $f$ of $p$ and $q$ variables, $\psi$ a function only of the $q$ variables, we can define a BV integral

$$\int_{L_\psi} f := \int f(q^i, p_i = \partial_i \psi) dq^i \cdots dq^n$$

to be intended as the integral of $f$ on the Lagrangian submanifold $L_\psi = \text{graph } d\psi$ (here $dq^1 \cdots dq^n$ denotes Berezinian integration). If we assume that integrals converge, we get

- If $f = \Delta g$, then $\int_{L_\psi} f = 0$,
- If $\Delta f = 0$, then $\int_{L_\psi} f$ is invariant under deformations of $\psi$.

The latter allows us to exchange the integral over a Lagrangian, for which the integral is ill defined, by a well defined one. This procedure is called gauge-fixing. This construction can be extended to any (super)manifold. Moreover, considering $f = e^{i\hbar S}$, we need two other conditions, which are the Master Equations for the classical and quantum level:

- (Classical Master Equation (CME)) $(S, S) = 0,$
- (Quantum Master Equation (QME)) $(S, S) - 2i\hbar \Delta S = 0.$

The latter is equivalent to $\Delta e^{\frac{i}{\hbar} S} = 0$. The former is the classical limit of the latter.

2.3. BV theory. For $M$ a closed manifold, we can construct a BV manifold $(F_M, \omega_M, S_M)$, where $F_M$ is a supermanifold with $\mathbb{Z}$-grading, $\omega_M$ an odd symplectic form of degree $-1$ on $F_M$, and $S_M$ is an even function of degree zero on $F_M$, which extends the classical action and satisfies the CME.

2.4. The case with boundary. Consider now a manifold $M$ with boundary. Then the CME and the QME condition change ([20, 21]). The CME becomes the “modified” CME (mCME), which is given by

- (mCME) $\iota_{Q_M} \omega_M = \delta S_M + \pi_M^\ast \alpha_{\partial M},$

where $Q_M$ is a degree 1 cohomological vector field, i.e. $[Q_M, Q_M] = 0$. On closed manifolds it is defined as the Hamiltonian vector field of $S_M$. Here $\pi_M: F_M \to F_{\partial M}$ is the projection to the boundary fields, and $\alpha_{\partial M}$ a 1-form on $F_{\partial M}$ such that $\omega_{\partial M} = \delta \alpha_{\partial M}$. Again, we denote by $\delta$ the de Rham differential on the space of fields.

2.5. BV-BFV theory. The BV manifold construction can be extended to manifolds with boundary as was shown in [19, 20]. Let $M$ be a smooth manifold with boundary.

\(^2\)Only in special situations, i.e. $\dim M = 1$, and some examples discussed in [30].
2.5.1. Classical theory. We can define, according to subsection 2.3, a BFV manifold to be a triple 
\((\mathcal{F}_M, \omega_M, Q_M)\), where \(\mathcal{F}_M\) is a graded manifold, \(\omega\) an even symplectic form of degree 0, and \(Q_M\) a 1 degree 1 cohomological, symplectic vector field on \(\mathcal{F}_M\). If \(\omega_M\) is exact, there is a 1-form \(\alpha_{BM}\) with the same parity and degree, such that \(\omega_M = \delta \alpha_{BM}\); such A BFV manifold is called exact. A BV-BFV manifold over a given exact BFV manifold \((\mathcal{F}^0_{BM}, \omega^0_{BM} = \delta \alpha^0_{BM}, Q^0_{BM})\) is a quintuple \((\mathcal{F}_M, \omega_M, S_M, Q_M, \pi_M)\), where \(\mathcal{F}_M\) is a graded manifold, \(\omega_M\) is an even symplectic form of degree 0, \(S_M\) is an even function of degree 0, \(Q_M\) is a cohomological vector field, and \(\pi_M: \mathcal{F}_M \rightarrow \mathcal{F}^0_{BM}\) is a surjective submersion, such that \(S_M = \delta S^0 + \pi^* \alpha^0_{BM}\), and \(Q^0 = \delta \pi^* Q_M\), where \(\delta \pi_M\) denotes the differential of \(\pi_M\). If \(\mathcal{F}^0_{BM}\) is a point, we get that \((\mathcal{F}_M, \omega_M, S_M)\) is a BV manifold.

2.5.2. Quantization. The “modified” Quantum Master Equation (mQME) is given by ([20])

\[
(h^2 \Delta + \Omega) \psi_M = 0,
\]

where \(\Omega\) is a differential operator on \(\mathcal{H}_{BM}\), a certain space of functions on the leaf space \(B^P_{BM}\) of a compatible polarization \(P\). \(\Omega\) quantizes the boundary action \(S^0_{BM}\) and satisfies \(\Omega^2 = 0\). \(\Delta\) is the BV Laplacian on the space of residual fields \(V_M([20])\), and \(\psi_M\) is again constructed through Feynman diagrams, with sources in residual fields and the boundary. In this picture, it is a function (or half-density) on \(V_M \otimes B^P_{BM}\). If one changes the choices involved in the gauge-fixing, the state changes by a \((h^2 \Delta + \Omega)\)-exact term.

3. Quantization of AKSZ Sigma Models

In [21] it was shown that one can construct a perturbative quantization for any possibly nonlinear AKSZ Sigma Model ([1]) on manifolds with boundary by considering techniques of formal geometry as in [29, 7] and the BV-BFV formalism as in [20]. Here one linearizes the space of fields around constant maps. Varying the image of the constant map one obtains a family of quantizations labeled by elements of the target. By choosing a Grothendieck connection on the target([7, 29, 6, 34, 25], see also the discussion in the appendix of [21]), we can add a new term to the action - corresponding to a new vertex in the Feynman diagrams - such that the new state \(\tilde{\psi}\) satisfies

\[
\tag{1}
d_x \tilde{\psi} = \left(\frac{i h}{\hbar} \Delta - \frac{i}{\hbar^2} \Omega\right) \tilde{\psi},
\]

i.e. if we vary the point of expansion, the state changes by a gauge equivalence. The new vertex is labeled with the coefficients of the connection 1-form of the Grothendieck connection and emits a single arrow, see [6, 21]. Equation (1) is an extension of the mQME, which is called the modified “differential” Quantum Master Equation (mdQME). The mdQME was first introduced in [22], where it has been shown that one can construct a globalized version of Kontsevich’s star product ([33, 11]) using the Poisson Sigma Model with constant Poisson structure on worldsheet manifolds with boundary, and the BV-BFV formalism, especially its compatibility with gluing. The mdQME can be formulated for a full covariant quantum state \(\tilde{\psi}\) ([21]) to be a flat section with respect to the quantum Grothendieck connection \(\nabla_G\). The mdQME is given by

\[
\tag{mdQME}
\nabla_G \tilde{\psi} = \left(d_x - \frac{i h}{\hbar^2} \Delta + \frac{i}{\hbar} \Omega\right) \tilde{\psi} = 0,
\]

where \(d_x\) represents the de Rham differential on the moduli space of classical solutions for the unperturbed theory \(\mathcal{M}_0 = \{(x, 0) | x \text{ constant map to the target}\}\).

Moreover, it was shown that \(\nabla_G\) is a flat connection, and that varying the choices involved in defining \(\tilde{\psi}\) - including the Grothendieck connection - the state changes by a \(\nabla_G\)-exact term.
4. Review of the Poisson Sigma Model

The Poisson Sigma Model (PSM) ([31, 41, 40]) is a 2-dimensional topological field theory, with important relation to deformation quantization ([33, 11, 15], see also Appendix A.5), and in particular a special case of an AKSZ Sigma model. In this section we will very briefly review some aspects of its classical version.

4.1. Classical Poisson Sigma Model. Fix a Poisson manifold \((\mathcal{P}, \Pi)\). The classical PSM associates to a smooth, oriented, compact and connected 2-manifold \(\Sigma\) (usually called the worldsheet) the space of fields \(F_\Sigma = \text{VBun}(T\Sigma, T^*\mathcal{P})\), which is the space of vector bundle maps from \(T\Sigma\) to \(T^*\mathcal{P}\). An element of the space of fields will be identified with a pair \((X, \eta)\) where \(X: \Sigma \rightarrow \mathcal{P}\) is the base map and \(\eta \in \Gamma(\Sigma, T^*\Sigma \otimes X^*T^*\mathcal{P})\) is a 1-form on \(\Sigma\) with values in \(X^*T^*\mathcal{P}\). The action functional is

\[
S_\Sigma[(X, \eta)] = \int_\Sigma \left( \langle \eta, dX \rangle + \frac{1}{2} \langle \Pi(X), \eta \wedge \eta \rangle \right).
\]

Here \(\langle \ , \ \rangle\) denotes the pairing between vectors and covectors. In local coordinates \(x^i\) on \(\mathcal{P}\), we can write \(\eta = \eta_i dx^i\) and \(X = (X^1, \ldots, X^n)\). Then the action reads

\[
S_\Sigma[(X, \eta)] = \int_\Sigma \left( \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j \right),
\]

where we use the Einstein summation convention.

4.2. BV-BFV extension. Since the PSM is a gauge theory, one needs to apply a gauge fixing formalism. Because the gauge symmetries only close on shell, the BRST formalism fails, and one needs to revert to the BV formalism (see [11, 15] for introductions to the BV formalism and detailed discussion of the gauge symmetries; see also section 2) on closed surfaces and to the BV-BFV formalism on surfaces with boundary ([19, 20]).

4.2.1. BV extension. The BV extended action and space of fields for the PSM can be constructed from the AKSZ formalism as discussed in [13]. The BV space of fields is the space of maps of supermanifolds

\[ F_\Sigma = \text{Map}(T[1]\Sigma, T^*[1]\mathcal{P}) \ni (X, \eta) \]

where \(X: T[1]\Sigma \rightarrow \mathcal{P}\) is a map and \(\eta\) is a section of \(X^*T^*[1]\mathcal{P}\). The BV action is given by

\[
S_\Sigma[(X, \eta)] = \int_{T[1]\Sigma} \langle \eta, DX \rangle + \frac{1}{2} \langle \Pi(X), \eta \wedge \eta \rangle,
\]

where \(D = \theta^\mu \frac{\partial}{\partial \theta^\mu}\) is the differential on \(T[1]\Sigma\). In local coordinates on \(\mathcal{P}\) we can write

\[
S_\Sigma[(X, \eta)] = \int_\Sigma \left( \eta_i \wedge dX^i + \frac{1}{2} \Pi^{ij}(X) \eta_i \wedge \eta_j \right).
\]

On closed surfaces, this action satisfies the Classical Master Equation (CME)

\[
(S_{\Sigma'}, S_{\Sigma}) = 0.
\]

Here \(\langle \ , \ \rangle\) is the odd Poisson bracket (BV bracket) associated to the odd symplectic form

\[
\omega_\Sigma = \int_\Sigma \delta X \wedge \delta \eta
\]

(where \(\delta\) denotes the de Rham differential on the space of fields). One can reformulate the CME as \(Q_\Sigma (S_\Sigma) = 0\) where \(Q_\Sigma = (S_{\Sigma'}, \ )\) in local coordinates on \(\mathcal{P}\) is given by

\[
Q_\Sigma = \int_\Sigma \left( dX^i + \Pi^{ij}(X) \eta_j \right) \wedge \delta + \left( d\eta_i + \frac{1}{2} \frac{\partial}{\partial x^i} \Pi^{jk}(X) \eta_j \wedge \eta_k \right) \wedge \frac{\delta}{\delta \eta_i}.
\]

The association \(\Sigma \mapsto (F_\Sigma, \omega_\Sigma, Q_\Sigma, S_\Sigma)\) is an example of a BV theory (see also subsection 2.3).
4.2.2. BV-BFV extension. In the BV-BFV formalism the boundary conditions are left unspecified and hence the CME no longer makes sense. However, one can still define the symplectic form \( \omega \) by (6), the action by (4) and the vector field \( Q \) by (7). One now also introduces the boundary data

\[
\mathcal{F}_{\partial \Sigma}^0 = \text{Map}(T[1] \partial \Sigma, T^*[1] \mathcal{P}),
\]

\[
\alpha_{\partial \Sigma}^0 = \int_{\partial \Sigma} \eta \wedge \delta X,
\]

\[
Q_{\partial \Sigma}^0 = \int_{\partial \Sigma} (dX^i + \Pi^{ij}(X) \wedge \eta_j) \wedge \frac{\delta}{\delta X^i} + \left( d\eta_i + \frac{1}{2} \frac{\partial}{\partial x^i} \Pi^{jk} \eta_j \wedge \eta_k \right) \wedge \frac{\delta}{\delta \eta_i},
\]

\[
S_{\partial \Sigma}^0 = \int_{\partial \Sigma} \langle \eta, dX \rangle + \frac{1}{2} \langle \Pi(X), \eta \wedge \eta \rangle,
\]

and the map \( \pi_{\Sigma}: \mathcal{F}_{\Sigma} \rightarrow \mathcal{F}_{\partial \Sigma}^0 \) given by restriction of maps. These then satisfy the axioms of a BV-BFV theory\(^3\):

\[
Q_{\Sigma}^2 = 0, \tag{8}
\]

\[
\delta \pi(Q_{\Sigma}) = Q_{\partial \Sigma}^0, \tag{9}
\]

\[
\iota_{Q_{\Sigma}} \omega_{\Sigma} = \delta S_{\Sigma}^0 + \pi_{\Sigma}^* \alpha_{\partial \Sigma}^0. \tag{10}
\]

Moreover, \( \iota_{Q_{\Sigma}}^0 \omega_{\partial \Sigma} = \delta S_{\partial \Sigma}^0 \), where \( \omega_{\partial \Sigma} = \delta \alpha_{\partial \Sigma}^0 \). Again, equation (10) is the mCME as already defined in subsection 2.4.

4.3. Perturbative quantization. We consider the PSM action as a perturbation of the quadratic part of the action,

\[
S_{0,\Sigma} = \int_{\Sigma} \langle \eta, dX \rangle.
\]

Since we expand around critical points of \( S_{0,\Sigma} \), this implies in particular that \( X \) is closed. Hence the ghost number 0 component of \( X \) is a constant map, which we denote by its image \( x \in \mathcal{P} \). As discussed in [14, 6, 22] and Appendix F of [20], it makes sense to perform perturbative quantization around points in the moduli space of classical solutions. Since the EL equations for the PSM are given by \( dX + \Pi(X) \eta = 0 \), we will perturb around the classical solution \( X = x = \text{const.} \) and \( \eta = 0 \) and gauge equivalent solutions. Hence for the PSM the appropriate moduli space is given by

\[
\mathcal{M}_0 = \{(x, 0)|x \text{ const map to } \mathcal{P}\} \cong \mathcal{P}.
\]

In this special case we have \( \mathcal{M}_0 \subset \mathcal{F}_{\Sigma} \). Instead of fixing a single classical solution \( x \in \mathcal{M}_0 \) and expanding around it, we want to vary \( x \) itself. We can do this using the methods of Appendix A.2. Considering the fields \( \hat{X} \) and \( \hat{\eta} \) given by \( X = \varphi_x \hat{X} \) and \( \eta = (d\varphi_x)^{-1} \hat{\eta} \), we get a formally globalized action for the PSM by

\[
\hat{S}_{\Sigma, x}[(\hat{X}, \hat{\eta})] = \int_{\Sigma} \hat{\eta}_i \wedge d_{\Sigma} \hat{X}^i + \frac{1}{2} \int_{\Sigma} (T \varphi_x^* \Pi)^{ij} \hat{X}_i \wedge \hat{\eta}_j + \int_{\Sigma} Y^j_i (x; \hat{X}) \hat{\eta}_j \wedge d_{\mathcal{M}_0} x^i,
\]

where we denote by \( d_{\mathcal{M}_0} \) and \( d_{\Sigma} \) the de Rham differentials on \( \mathcal{M}_0 \) and \( \Sigma \) respectively (we only write it once and leave out the indication every time it is clear).

5. Simply polarized boundaries

We want to describe the mdQME for the PSM in the case of a worldsheet where we have a single boundary component endowed with a certain polarization (see figure 1).

\(^3\)This is automatic for theories which admit an AKSZ formulation, see [19].
5.1. The boundary BFV operator. In this subsection we want to see how $\Omega$ is constructed without any globalization term, i.e., for $S^\Sigma$. We can formulate the boundary operator $\Omega$ for the PSM by the usual construction of the collapsing of subgraphs $\Gamma'$ of a graph $\Gamma$ in $\partial C\Gamma$ for the non-globalized theory. We briefly review the results of [20, Section 4.8], where the boundary operator of the non-globalized theory was computed. We consider two different boundary representation depending on the polarization, either we have an $E$-representation or an $X$-representation.

5.1.1. $E$-representation. We look at first at the $E$-representation. Denote by

$$\text{Conf}_{n,k} = \{(u_1, ..., u_n, q_1, ..., q_m) \mid u_i \in \mathbb{H}, q_j \in \mathbb{R}, u_i \neq u_{i_2} \text{ for } i \neq i_2, q_{j_1} \neq q_{j_2} \text{ for } j \neq j_2\}$$

the configuration space of $n$ points in the bulk and $k$ points on the boundary, and by $C$ the quotient of $\text{Conf}_{n,k}$ by translation and scaling. Then we have $\text{dim } C = 2n + k - 2$, which has to be the same as the form degree of the weight of the appearing graph such that integrals do not vanish. Thus we need to have $2n + k - 2 = 2n$, since $n$ is the amount of points in the bulk which represent the Poisson tensor, i.e., emitting two arrows that have to remain inside the collapsing subgraph (otherwise the contribution vanishes by the boundary condition on the propagator). Hence we get $k = 2$. We label one boundary vertex by $u_0$ and the other one by $u_1$. Let $I, J, K, R, S$ be multi-indices. Consider a graph $\Gamma$ with incoming arrows at $u_0$ represented by the index $I$, which come from a subgraph $\Gamma'$ of $\Gamma$, and other incoming arrows at $u_0$, represented by the index $R$, not coming from $\Gamma'$. Moreover, consider incoming arrows, represented by the index $J$, which come from the same subgraph $\Gamma'$, and other incoming arrows at $u_1$, represented by the index $S$, not coming from $\Gamma'$. Moreover, we also consider arrows, represented by the index $K$, going into $\Gamma'$. Then $\Gamma'$ can collapse to the boundary $\partial C\Gamma$ (see figure 2). Summing over all subgraphs $\Gamma'$ of graphs $\Gamma$ that appear, we obtain the boundary operator

$$(12) \quad \Omega_{E} = \Omega_{E}^{\partial} + \int_{\partial \Sigma} \sum_{I,J,K,R,S} \frac{(-i\hbar)|K|-|I|-|J|+1}{(|K| + |R| + |S|)!} \partial_{K}B^{IJ}(x, \Pi)[E_{I}E_{R}][E_{J}E_{S}] \frac{\delta^{(|K|+|R|+|S|)}}{\delta E_{K} \delta E_{R} \delta E_{S}}$$

**Figure 1.** Example of two higher genus worldsheets with one connected boundary component and different polarization.
applied to the state, where $[E_I E_R]$ and $[E_J E_S]$ represent the so-called composite fields as in [20]. Here

$$\Omega_0^E = \int_{\partial \Sigma} dE_i \frac{\delta}{\delta E_i}.$$ 

**Figure 2.** An example of a subgraph collapsing as in the description. Here we have three incoming arrows to the boundary for the collapsing graph $\Gamma'$ on the right side corresponding to the index $S$, three incoming arrows to the boundary on the left side corresponding to the index $R$, three incoming arrows to $\Gamma'$ corresponding to the index $K$, two incoming arrows to $u_0$ from $\Gamma'$ corresponding to the index $I$ and one incoming arrow to $u_1$ from $\Gamma'$ corresponding to the index $J$.

**Remark 5.1.** Recall that $x$ represents the constant solution as a map $\Sigma \to \mathbb{R}^n$. We can describe the weights $B^{IJ}(x, \Pi)$, which depend on the Poisson tensor, as the coefficients in the star product on $C^\infty(\mathbb{R}^n)[[i\hbar]]$ by

$$f \star g = fg + \sum_{I,J} B^{IJ} \frac{\partial |I|}{\partial x^I} f \frac{\partial |J|}{\partial x^J} g = fg - \frac{i\hbar}{2} \sum_{ij} \Pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} + O(\hbar^2),$$

where $I, J$ are multi-indices and $i$ and $j$ are indices and $B^{IJ} = 0$ if $|I| = 0$ or $|J| = 0$.

**5.1.2. $\mathbb{X}$-representation.** In the $\mathbb{X}$-representation there are arbitrarily many points on the boundary. Denote the number of points on the boundary by $k$. Then we have a similar construction as for the $E$-representation, only with the difference that for each boundary vertex we can have arbitrarily many outgoing arrows either out of the collapsing graph $\Gamma'$ (left or right) and arbitrarily many outgoing arrows going into $\Gamma'$. Label the points on the boundary by $1, \ldots, k$ and let $L, I_1, \ldots, I_k, R_1, \ldots, R_k$ be multi-indices. Then for the $i$th vertex we have outgoing arrows, represented by the index $R_i$, going out of $\Gamma'$ and outgoing arrows, represented by $I_i$ going into $\Gamma'$. Moreover, we have outgoing arrows, represented by the index $L$, coming from $\Gamma'$. Then $\Gamma' \subset \Gamma$ can collapse to the boundary in $\partial C_\Gamma$ (see figure 3). This gives us the boundary operator

$$\Omega_0^\mathbb{X} = \Omega_0^\mathbb{X} + \sum_{k=0} \frac{1}{k!} \int_{\partial \Sigma} \sum_{L, I_1, \ldots, I_k, R_1, \ldots, R_k} \frac{(-i\hbar)^{|L|-(|I_1|+\cdots+|I_k|)+1}}{(|L|+|R_1|+\cdots+|R_k|)!} \cdot \partial^L a_{I_1,\ldots,I_k}(x, \Pi) \prod_{j=1}^k [\mathbb{X}_I^j \mathbb{X}_R^j] \frac{\delta^{[L]+|R_1|+\cdots+|R_k|}}{\delta \mathbb{X}_I^1 \delta \mathbb{X}_R^1 \cdots \delta \mathbb{X}_R^k},$$

where $a_{I_1,\ldots,I_k}$ are given by the sum of the weights over all Feynman diagrams with $k$ boundary points and $I_j$ outgoing arrows for $1 \leq j \leq k$. Here

$$\Omega_0^\mathbb{X} = \int_{\partial \Sigma} d\mathbb{X} \frac{\delta}{\delta \mathbb{X}}.$$
Figure 3. An example of a subgraph collapsing as in the description. We consider a term for $k = 2$ as before and we label them by $u_1$ and $u_2$. Here we have three outgoing arrows for the collapsing graph $\Gamma'$ on the right side corresponding to the index $R_1$, three outgoing arrows on the left side corresponding to the index $R_1$, three outgoing arrows to $\Gamma'$ corresponding to the index $L$ and one incoming to $\Gamma'$ out of each of the two boundary points corresponding to the indices $I_1$ and $I_2$.

5.2. The globalized BFV operator. We now give a formulation for $\Omega$ where we also consider the globalization term $S_{\Sigma,x,R}$. Recall that graphically this amounts to introducing new vertices emanating only a single arrow, representing the vector field $R$ as explained in the Feynman rules of [21]. This means that $\Omega$ now becomes an inhomogeneous form on $\mathcal{P}$, since $R$ is a 1-form on $\mathcal{P}$. As before, we distinguish between the $E$- and the $X$-representation.

5.2.1. $E$-representation. We have seen that degree counting implies that there are exactly two boundary vertices in a collapsing graph. Now we have to take the $R$ vertices into account. Consider a collapsing graph with $n$ bulk and $k$ boundary vertices. Then the dimension of the corresponding configuration space is $2n + k - 2$. On the other hand, there are now two types of bulk vertices: Suppose there are $m$ vertices labeled by the Poisson bivector field (emitting two arrows) and $r$ vertices labeled by the vector field $R$ (emitting one arrow). Since arrows cannot leave the collapsing graph, the total form degree is $2m + r$, which has to equal $2n + k - 2$. Since $n = m + r$, this implies that $r + k = 2$. This means there can be either zero, one or two vertices labeled by $R$ with two, one or zero boundary vertices respectively, as shown in figure 4.

Figure 4. Possible graphs in the $E$-representation.
The first contribution $r = 0, k = 2$ is exactly the operator $\Omega_\varepsilon$ given in (12) from the non-globalized case. In the case $r = 1, k = 1$ we get the graphs arising in the definition of the connection 1-form $A$ as in (39). Hence this contribution is given by

$$\Omega_{\varepsilon,r=1} = \sum_{I,J,L} \int_{\partial\Sigma} \left( -i\hbar \right)^{|L|-|J|} \frac{\partial L A^I(R, \Pi_\varphi)[E]E}{\delta \varepsilon_L \delta E_J}$$

where $A^I$ is the sum of weights of graphs where the incoming arrows at the boundary vertex are labeled by $I$. The contribution from graphs with $r = 2, k = 0$ can be expressed by the curvature term $F$ defined in (40):

$$\Omega_{\varepsilon,r=2} = \sum_L \int_{\partial\Sigma} \left( -i\hbar \right)^{|L|+1} \frac{\partial L F(R, R, \Pi_\varphi)}{\delta \varepsilon_L}$$

5.2.2. $\mathcal{X}$-representation. In the $\mathcal{X}$-representation, arrows can leave the collapsing graph, so we cannot do a degree count like in the $\mathcal{E}$-representation; in particular, the number of $R$ vertices in a collapsing graph is only bounded by the dimension of $\mathcal{P}$. So, we have $\Omega_{\mathcal{X},R} = \sum_{i=0}^{\dim \mathcal{P}} \Omega_{\mathcal{X},r=i}$, where $\Omega_{\mathcal{X},r=i}$ is the sum of all graphs with $i$ vertices labeled by $r$. In particular, $\Omega_{\mathcal{X},r=0} = \Omega_{\mathcal{X}}$.

5.3. Algebraic structure of the boundary operator. We know from [21] that $(\nabla_\varepsilon)^2 = 0$, and this is equivalent to $d_\varepsilon^2 + \frac{1}{2} [\Omega, \Omega] = 0$. For the PSM it is interesting to see how this condition can be derived by looking at the explicit structure of $\Omega$ as discussed in 5.2. We again consider the two different representations separately.

5.3.1. $\mathcal{E}$-representation. We collect the parts of $\Omega_\varepsilon$ in the flatness equation with the same form degree. Let $\Omega_\varepsilon = \Omega_0 + \Omega_{\varepsilon,r=0} + \Omega_{\varepsilon,r=1} + \Omega_{\varepsilon,r=2}$ as in 5.2.1. We write $\Omega_{(i)} := \Omega_{\varepsilon,r=i}$ for $i = 1, 2$ and $\Omega_{(0)} := \Omega_0 + \Omega_{\varepsilon,r=0}$. This gives us a partition of $\Omega_\varepsilon$ into the different form degrees.

**Proposition 5.1.** We have the following equations:

\begin{align}
[\Omega_{(0)}, \Omega_{(0)}] &= 0, \\
\partial_\varepsilon \Omega_{(0)} + [\Omega_{(0)}, \Omega_{(1)}] &= 0, \\
\partial_\varepsilon \Omega_{(1)} + [\Omega_{(0)}, \Omega_{(2)}] + \frac{1}{2} [\Omega_{(1)}, \Omega_{(1)}] &= 0, \\
\partial_\varepsilon \Omega_{(2)} + [\Omega_{(1)}, \Omega_{(2)}] &= 0, \\
[\Omega_{(2)}, \Omega_{(2)}] &= 0,
\end{align}

where $\Omega_{(i)}$ represents the part of $\Omega_\varepsilon$ with form degree $i$.

**Sketch of the proof.** We will only prove (17) in detail (but we are sloppy with signs), and sketch the idea of the proof of the other equations.

The construction in [14], recalled in Appendix A, yields a bundle $\mathcal{E} = \tilde{ST}^* \mathcal{P}[[\varepsilon]]$ of $*$-algebras on $\mathcal{P}$ by applying Kontsevich’s deformation quantization in every tangent space. Picking a Grothendieck connection $D_\mathcal{G} = d_\varepsilon + R$ on $\mathcal{P}$, and applying the Kontsevich formality map to $R$, one obtains a connection $D_\mathcal{G} = d_\varepsilon + A$ on $\mathcal{E}$. In [14] it is shown that this connection is a derivation of $\Gamma(\mathcal{E})$, i.e. for $\sigma, \tau \in \Gamma(\mathcal{E})$, we have

$$D_\mathcal{G}(\sigma \ast \tau) = (D_\mathcal{G}\sigma) \ast \tau + \sigma \ast (D_\mathcal{G}\tau).$$

We claim that this equation implies (17). This can be done directly by writing out (17) and (21) in coefficients, but it is best seen through Feynman diagrams (after all, $A$ and the star product are defined through Feynman diagrams). First, rewrite (21) into

$$d_\varepsilon(\sigma \ast \tau) - d_\varepsilon\sigma \ast \tau - \sigma \ast d_\varepsilon\tau = -A(\sigma \ast \tau) + (A\sigma) \ast \tau + \sigma \ast (A\tau).$$
\[ \sigma d_x = \sigma R + \tau R + \tau R \]

Figure 5. Schematics of the diagrammatic content of (21). \( \sigma \) and \( \tau \) are arbitrary sections of \( \Gamma(\mathcal{E}) \). We sum over all possible graphs. By \( d_x \) we mean that we apply \( d_x \) to the result of the integration. An \( R \) means that there is precisely one vertex labeled by \( R \) in every graph.

The left hand side of this equation is given by applying \( d_x \) to the coefficients of the star product. Schematically, we represent the diagrammatic content as in Figure 5. On the other hand, we recall from [21] that the commutator \( [\Omega_0, \Omega_1] \) can be expressed by replacing the boundary vertices in the graphs defining \( \Omega_1 \) by the graphs appearing in \( \Omega_0 \) and vice versa. If we ignore possible arrows arriving at the boundary vertices from outside the graph, this yields precisely the graphs on the right hand side of figure 5: The first term are the graphs of \( \Omega_0 \) placed at the boundary vertex of graphs appearing in \( \Omega_1 \), and the second and the third term represent the graphs of \( \Omega_1 \) placed at one of the boundary vertices of \( \Omega_0 \). Arriving arrows from outside the graph corresponds to taking derivatives of \( \sigma \) and \( \tau \). On the other hand, the left hand side yields precisely \( d_x \Omega_0 \).

Equation (16) holds, since the non-globalized boundary operator squares to zero (which is in turn a consequence of the CME, see [20] and [21]). Equation (20) holds, since there are no \( \mathcal{E} \)-field contributions in \( \Omega_2 \). Equation (19) corresponds to the Bianchi identity, and equation (18) corresponds to the equation \( d_x A + \frac{1}{2} [A, A] = [F, \ ]_\star \), in both cases, the proof is similar to the proof of the degree 1 case (17).

5.3.2. \( \mathcal{X} \)-representation. In the \( \mathcal{X} \)-representation, one can similarly decompose the boundary operator into form degrees \( \Omega_\mathcal{X} = \sum_{i=0}^{\dim \mathcal{X}} \Omega_i \), and for every \( k = 0, \ldots, r \) one obtains equations \( d_x \Omega_{(k-1)} + \frac{1}{2} \sum_{i+j=k} [\Omega_{(i)}, \Omega_{(j)}] = 0 \). The form degree 0 part is again the fact that the non-globalized boundary operator squares to 0. It would be interesting to investigate whether there is an algebraic structure underlying the equations in the other form degrees, similar to the \( \mathcal{E} \)-representation.

5.4. Modification of the action. We modify the classical BV action by using results of [6, 14, 17] as we also discuss in Appendix A. Let \( \gamma \in \Omega^1(\mathcal{P}, \mathcal{E}) \) be a solution of equation (48) for some choice of \( \omega \in \Omega^2(\mathcal{P}, \mathcal{E}) \) as discussed in A.3 (here the formal parameter \( \varepsilon \) is given by \( (-i\hbar)/2 \)). If \( \omega = 0 \), the modified formally globalized action for the PSM is given by

\[ S_{\Sigma,x}[\hat{\mathcal{X}}, \hat{\eta}] = \tilde{S}_{\Sigma,x}[\hat{\mathcal{X}}, \hat{\eta}] + S_{\Sigma,\gamma}, \]

where

\[ S_{\Sigma,\gamma} = \int_{\partial \Sigma} \gamma \left( x; \hat{\mathcal{X}} \right) = \sum_{k \geq 1} (-1)^k \left( -\frac{i\hbar}{2} \right)^k \sum_I dx^i \int_{\partial \Sigma} \gamma^{(k)} \left( x \right) \hat{\mathcal{X}}^I. \]

Remark 5.2. Here we integrate the 1-form part of \( \hat{\mathcal{X}} \) along the boundary, which, since the \( \hat{\mathcal{X}} \) fluctuation vanishes on components of the boundary in \( \mathcal{X} \)-representation, implies that for a single boundary with \( \mathcal{X} \)-representation \( S_{\Sigma,\gamma} \) does not give any contribution to \( \Omega_\mathcal{X} \). Therefore we only need to look at the \( \mathcal{E} \)-representation. Moreover, note that \( \gamma = O(\hbar) \), i.e. it is already a type of quantum counterterm which is not present classically, so it does not violate the mCME.
Remark 5.3. If we want to consider the case \( \omega \neq 0 \), we need to add another term to the action, which is given by

\[
S_{\Sigma,\omega} = \int_\Sigma \omega(x; \hat{X}) = \sum_{k \geq 1} (-1)^k \left( \frac{i\hbar}{2} \right)^k \sum_J \int_\Sigma \omega^{(k)}_{ij}(x) \hat{X}^J.
\]

We denote the action \( S_{\Sigma,x} \) together with the latter term by \( \tilde{S}_{\Sigma,x} \).

Considering the degree counting as in 5.2.1, we get different cases of boundary vertex configurations. For the case \( r = 0, k = 2 \), we can either have two \( E \)-field boundary vertices, one \( E \)-field and one \( \gamma \) boundary vertex or two \( \gamma \) boundary vertices. For the case \( r = 1, k = 1 \), we can have either one \( E \)-field boundary vertex or one \( \gamma \) boundary vertex. For the case \( r = 2, k = 0 \) we have the same contribution as before. In the case \( \omega \neq 0 \), there is a configuration where \( r = k = 0 \), but there is a single \( \omega \) vertex. These different diagrams contribute to different terms for the new boundary operator (they all have to be understood as sections of \( E = \hat{S}T^* M[\varepsilon] \), where we replace the formal variable \( y \) by \( \frac{\delta}{\delta y} \)), which are:

- \( r = 0, k = 2 \) (\( E, E \) on the boundary): Summing over all these graphs, this corresponds to the term
  \( \Omega_E \).
- \( r = 0, k = 2 \) (\( \gamma, \gamma \) on the boundary): Summing over all these graphs, this corresponds to \( \gamma \ast \gamma \).
- \( r = 0, k = 2 \) (\( E, \gamma \) on the boundary): Summing over all these graphs, this corresponds to \([e^{i\xi E}, \gamma]_\ast\).
- \( r = 1, k = 1 \) (\( E \) on the boundary): Summing over all these graphs, this corresponds to the connection term
  \( A(R, \Pi_\varphi)(e^{i\xi E}) \).
- \( r = 1, k = 1 \) (\( \gamma \) on the boundary): Summing over all these graphs, this corresponds to the curvature term
  \( F(R, R, \Pi_\varphi) \).
- \( r = 2, k = 0 \) (nothing on the boundary): Summing over all these graphs this just yields \( \omega \).

By equation (48) and (44), we obtain that the terms with two \( \gamma \)s, one \( \gamma \), nothing on the boundary, and possibly \( \omega \), can be put together as

\[
A(R, \Pi_\varphi)(\gamma) - F(R, R, \Pi_\varphi) - \gamma \ast \gamma - \omega = d_\varphi \gamma.
\]

Hence they do not contribute to the boundary operator, which we denote by \( \Omega_{E,\gamma} \).

5.4.1. \textit{mdQME}. We want to see how this modification changes the mdQME. In particular, we want to look at the mdQME for the full covariant state \( \tilde{\psi} \), which is defined using \( S_{\Sigma,x} \), and \( \Omega_{E,\gamma} \). Note that the mdQME holds by construction of \( \Omega_{E,\gamma} \) as shown in [21].

Moreover, by equation (45) and the fact that \( d_\varphi e^{i\xi E} = 0 \), the surviving terms will correspond to

\[
\overline{D_G} \circ e^{i\xi E} = D_G \circ e^{i\xi E} + [e^{i\xi E}, \gamma]_\ast.
\]

Hence the boundary operator is given by

\[
\Omega_{E,\gamma} = \Omega_E + \overline{D_G} \circ e^{i\xi E}.
\]
5.4.2. Flatness. Here $ee^{\frac{1}{\hbar}}$ is a multiplication operator on the space of states. For the flatness of $\nabla_G$, we have to prove that

$$d_x \Omega_{\Sigma, \gamma} + \frac{1}{2} [\Omega_{\Sigma, \gamma}, \Omega_{\Sigma, \gamma}] = 0.$$  

Separating the equation by form degree in $\mathcal{P}$ this is equivalent to

$$\frac{1}{2} [\Omega_{\Sigma}, \Omega_{\Sigma}] = 0$$  

Equation (29) is just saying that the standard $\Omega$ squares to zero. Equation (31) is true because $\mathcal{D}_G$ is a flat connection. Equation (30) means that $\Omega_{\Sigma}$ is a $\mathcal{D}_G$-closed section. This comes from the fact that the coefficients of $\Omega_{\Sigma}$ are the same as in the star product.

6. Alternating boundary conditions and the mdQME

6.1. Consistent boundary conditions. In [11] it was shown that the perturbative expansion of the QFT given from of the PSM on the disk coincides with Kontsevich’s star product, where we expand around the gauge equivalent classical solutions of the given EL equations, which are $X = x = \text{const.}$, $\eta = 0$ (recall subsection 4.3 and see also Appendix A). The boundary conditions on the disk $D$ are exactly set such that $\eta_{\partial D} = 0$ in order to be consistent with these type of solutions.

6.2. Construction of boundary conditions. We want to extend the methods developed in the last sections to describe deformation quantization of an object called the relational symplectic groupoid (see [16, 9, 10]) (RSG) as in [22]. This requires that we perform the BV-BFV quantization in the presence of “alternating” boundary conditions, which we can formulate for any higher genus worldsheet: Let $\partial \Sigma = \bigsqcup_{\ell} \partial \Sigma^{(\ell)}$ and consider a partition of another two distinguished components for every connected component $\partial \Sigma^{(\ell)}$ of the boundary given by $\partial \Sigma^{(\ell)} = \partial_1 \Sigma^{(\ell)} \sqcup \partial_2 \Sigma^{(\ell)}$. Each $\partial \Sigma^{(\ell)}$ is given as a disjoint union of an even number of intervals $I_1^{(\ell)}, ..., I_n^{(\ell)}$, such that $\partial_1 \Sigma^{(\ell)} = \bigsqcup_{j \text{ even}} I_j^{(\ell)}$ and $\partial_2 \Sigma^{(\ell)} = \bigsqcup_{j \text{ odd}} I_j^{(\ell)}$. Now the alternating condition is that on components of $\partial_1 \Sigma^{(\ell)}$ we set $\tilde{\eta} = 0$, and on components of $\partial_2 \Sigma^{(\ell)}$ we choose some polarization $\mathcal{P}_j$ for each $I_j$, and consider the corresponding boundary fields. We think of the endpoints of the intervals as “corners”.

6.3. Curvature Anomaly. Unlike in the constant case discussed in [22], upon quantization the mdQME fails to be satisfied. This effect arises from the curvature of the Grothendieck connection; namely, if we try to prove the mdQME as in [21], when integrating over the boundary of the compactified configuration space there are strata where a bulk graph collapses at a point $u \in \partial_1 \Sigma$ (i.e. one of the boundary components where $\tilde{\eta} = 0$). Summing over all these graphs one obtains the curvature of the Grothendieck connection (for more details see Appendix A). However, since there are no boundary fields on $\partial_1 \Sigma$, these terms cannot be cancelled by a term in $\Omega$. This can be interpreted as a quantum anomaly, since this problem is not present at the classical level. To restore the mdQME, we can add additional terms to the action, reminiscent to the addition of counterterms. This will yield new boundary terms, but they can be cancelled by adding appropriate terms to $\Omega$ as we have already seen in subsection 5.4, if we allow for a slight extension of the space of states (see subsection 7.1).

4Note that we are not performing “extended” quantization of a manifold with corners in the sense of extended TQFTs, but simply apply BV-BFV quantization where we allow boundary conditions to change along connected components of the boundary.
6.4. The new state. To cancel this anomaly we add quantum counterterms to the action, specifically, the terms $S_{\Sigma, \gamma}$ and $S_{\Sigma, \omega}$ defined in (23) and (24) respectively. The new terms in the action give rise to additional vertices. Namely, we now have vertices of arbitrary valence on components of the boundary where $\hat{\eta} \neq 0$, i.e. on the $\hat{\eta} = 0$ boundary components and the components of $\partial_2 \Sigma$ in $\mathbb{E}$-representation. At such a vertex we place the corresponding derivative of $\gamma$ in the formal directions. Also, there are new bulk vertices labeled by $\omega$, which are similarly labeled by derivatives of $\omega$ in the formal directions.

6.5. New boundary contributions in the proof of the mdQME. If we try to proceed with the proof of the mdQME as in [21], we get terms where a part of a graph collapses on $\partial_1 \Sigma$, i.e. the part of the boundary where $\hat{\eta} = 0$. We will now analyse these terms more closely. Let $\Gamma' \subset \Gamma$ be a subgraph that collapses on a point of the boundary, and denote by $\Gamma / \Gamma'$ the resulting graph. Suppose $\Gamma'$ has $n$ bulk and $k$ boundary vertices on $\partial_1 \Sigma$. Then the dimension of the corresponding boundary stratum is $2n + k - 2$, since it is the quotient of the compactification of $\mathbb{C}$. The contribution of the graph is non-vanishing only if the form degree of $\omega_{\Gamma'}$ is also $2n + k - 2$. The bulk vertices correspond to either $\Pi$ or $R$, the former has two outgoing arrows, the latter only one. If one of
these arrows points out of \( \Gamma' \), then \( \omega_{\Gamma'/\Gamma} = 0 \), since it contains a propagator with the tail evaluated on the \( \hat{\eta} = 0 \) boundary component. Hence all these arrows must point to another vertex in \( \Gamma' \). Suppose there are \( m \) vertices with two outgoing arrows and \( r \) vertices with one outgoing arrow. Then we must have the following system of equations:

\[
\begin{align*}
2n + k - 2 &= 2m + r \\
n &= m + r,
\end{align*}
\]

which is equivalent to \( r = 2 - k \) (\( m \) is arbitrary, and \( n = m + r \)). Since \( r \geq 0 \), we conclude that \( k \) is either 0, 1, or 2. Let us analyse these possibilities in more detail.

6.5.1. Terms with \( k = 0 \). In these terms there are no boundary vertices. They are also present if we do not add \( S_{\Sigma, \gamma} \) to the action. We have \( r = 2 - k = 2 \), so these terms are given by graphs with \( R \) at two vertices. Summing over all these terms yields the curvature of the Grothendieck connection, \( F \) (again, see Appendix A for details). This is what spoils the mdQME, since we cannot cancel it with terms in \( \Omega \), which can only cancel the boundary contributions on boundary components with free boundary fields. We are thus forced to add other terms to the action to cancel the appearance.

6.5.2. Terms with \( k = 1 \). In these terms there is one boundary vertex labeled by \( \gamma \), and one bulk vertex labeled by the vector field \( R \). If we sum over all such graphs, we get

\[
A(R, \Pi, \varphi)(\gamma) = A(R, \Pi, \varphi)(\gamma_i) dx^i
\]

by the definition of \( A \) as in Appendix A.

6.5.3. Terms with \( k = 2 \). In these terms there are two boundary vertices labeled by \( \gamma \), and no vertices labeled by the vector field \( R \). If we sum over all such terms, we get precisely the star product \( \gamma \ast \gamma \).

![Figure 8. Different contributions at the boundary](image)

6.6. New contributions at the corners. Introducing alternating boundary conditions means that the compactification of the configuration space changes. Namely, there are new boundary strata corresponding to the collapse of vertices at one of the corners. Such a collapse can be modeled on a configuration of points on the upper right quadrant, with a choice of boundary conditions on both sides. Here there is no translation symmetry, so the dimension of the boundary stratum is different. Adding \( S_{\Sigma, \gamma} \) to the action cancels the anomaly that comes from allowing for alternating boundary conditions. However, it results in new boundary contributions that come from graphs collapsing at the corners, as we will show presently.

Let \( \mathcal{C} \) denote the set of all corner points of \( \Sigma \). There are two types of corners: Let \( \mathcal{C}_2 \subset \mathcal{C} \) denote the subset containing those corner points which connect a \( \frac{\delta}{\partial x} \)-polarized connected component (i.e. a
component in \(E\)-representation) of \(\partial_1 \Sigma\) with a connected component of \(\partial_2 \Sigma\) and let \(\mathcal{C}_1 \subset \mathcal{C}\) denote the subset containing those corner points which connect a \(\frac{\delta}{\delta \phi}\)-polarized connected component of \(\partial_1 \Sigma\) with a connected component of \(\partial_2 \Sigma\).

\[
\hat{\eta} = 0
\]

\[
C \quad (a) \quad \text{A corner in } \mathcal{C}_2 \quad \hat{\eta} = 0 \quad C \quad (b) \quad \text{A corner in } \mathcal{C}_1
\]

**Figure 9.** The two types of corners.

The propagator still vanishes when its tail is evaluated at one of the corners (this can be checked from the explicit formula for the propagator in Appendix B). For this reason, as above if some subgraph \(\Gamma'\) of a graph \(\Gamma\) collapses at a corner, the contribution is only non-vanishing if no arrows leave \(\Gamma'\). Let us start at a corner \(C\) in \(\mathcal{C}_2\). Then we cannot have propagators ending at the \(\frac{\delta}{\delta \phi}\)-polarized boundary, since otherwise we need to evaluate the \(E\)-field at the corner point, which is equal to zero because of its boundary condition. So, any subgraph collapsing at \(C\) can only have bulk vertices, say \(n = m + r\) of them, where \(m\) denotes the number of interaction and \(r\) the number of \(R\) vertices, and vertices and \(\partial_2 \Sigma\), say \(k\) of them. Counting the dimensions we arrive at the following system of equations:

\[
2n + k - 1 = 2m + r \quad (34)
\]

\[
n = m + r \quad (35)
\]

which has the solutions \(k = 0, r = 1\) and \(k = 1, r = 0\), with \(m\) arbitrary. However, at these corners, graphs with bulk vertices do not contribute, this is the statement of the following lemma.

**Lemma 6.1.** If \(\Gamma'\) is a subgraph of \(\Gamma\) containing bulk points, then the integral of \(\omega_T\) over the boundary face of \(C_T\) where \(\Gamma'\) collapses at a corner \(C \in \mathcal{C}_2\) vanishes.

**Proof.** The point is that at these corners the boundary conditions are the same on both sides, so we can map the configuration to a configuration of points on the upper half plane, where we use the usual Kontsevich propagator, but without taking the quotient with respect to translations along the real axis. Instead we fix the image of the corner point to be a given point, e.g. 0. See also Figure 10. Now, observe that configurations with one bulk point evaluate to 0: These are either \(k = 0, m = 0, r = 1\), but this case is ruled out because there are no tadpoles, or \(k = 1, m = 1, r = 0\), but this is 0 because graphs cannot double edges. For more than two bulk points, note that the Kontsevich propagator depends on the the real parts of the points in the configuration only through their differences. Hence the product of propagators that is to be integrated has no component in the real part of the center of mass of the configuration, so integrating along this direction yields 0. \(\square\)

This means the only possibly nonzero contributions are those with \(k = 1, n = 0\), i.e. subgraphs \(\Gamma'\) consisting of a single \(\gamma\) vertex - possibly with any number of inward leaves - approaching the corner. This vertex can either lie on the \(\partial_1 \Sigma\) or \(\partial_2 \Sigma\) component and the corresponding boundary faces have opposite orientation. Hence all terms cancel out: there are no extra contributions from corners in \(\mathcal{C}_2\).

Next let us turn to corners \(C \in \mathcal{C}_1\). Here the boundary conditions change, so the propagator does not have translation symmetry along the axis. However, by continuity, now it vanishes when either the head or the tail are evaluated at the point of collapse. This implies that a subgraph collapsing
Here $h$ represents the mapping of the corner with the interior to the upper half plane, where the corner point is mapped to zero (with the same boundary conditions). The dashed circle represents some graph in the bulk with vertices corresponding to the Poisson structure $\Pi$ and the globalization term $R$, with some outgoing arrows deriving $\gamma$ on the boundary. In particular, the map $h$ is given by $z \mapsto z^2$ on the upper half plane.

At $C$ can have neither inward nor outward leaves, i.e. only entire connected components of graphs can collapse at corner $C \in \mathcal{C}_1$. Counting dimensions as above, we see that there are again the two possibilities $r = 0, k = 1$ and $r = 1, k = 0$, with $m$ arbitrary; in addition now we can have an arbitrary number $b$ of vertices at the boundary with $X$-representation.

Since only connected components of a graph can collapse, the corresponding action on the state is a multiplication operator $\Omega_{\mathcal{C}_1}$ that multiplies states with a functional of the values of $X$ at corners in $\mathcal{C}_1$, given by summing over all possible boundary contributions. This is not a regular functional as in [20], as it contains evaluation of fields on the corners. This means that we have to extend the space of states accordingly.

7. The mdQME for the Globalized PSM with Alternating Boundary Conditions

Since the PSM is an example of an AKSZ theory, we can use the results of [21]. That means that the mdQME holds for $\tilde{S}_{\Sigma,x}[(\hat{X}, \hat{\eta})]$, if we do not impose alternating boundary conditions. Otherwise the proof of the mdQME fails, as we have shown in subsection 6.5. To restore it, we have to extend the allowed spaces of operators and states for $S_{\Sigma,x}$, and change $\Omega$ to account for the new corner contributions depending on $\gamma$, such that the quantum Grothendieck connection still squares to zero.
7.1. Extension of states. In the mdQME the boundary operator $\Omega$ acts on the state, and thus we need make sure that the state space is closed under this action. This is indeed not trivially satisfied, since the corner terms involve evaluating the boundary field at the corners points. There are two different corner terms in $\Omega_{C}$, namely the one where we have a single $\gamma$ on the boundary approaching the corner and no vector field $R$, or no boundary vertex on the $\vec{\eta} = 0$ component and one single vector field $R$ included in the graph from the bulk (see also figure 11). Note that since $\gamma \in \Omega^1(\mathcal{P}, \mathcal{E})$ and $R \in \Omega^1(\mathcal{P}, T\mathcal{P} \otimes \mathcal{E})$, we get that the collapsing graphs in the corner are given either by an operator $B_\gamma$, for the case that we have a $\gamma$ on the boundary, or $B_R$ for the case where we do not have any $\gamma$ on the boundary. These operators both depend on the boundary field evaluated at the corner, i.e. $\tilde{X}(C)$, where $C \in \mathcal{C}_1$. They multiply the state with the functional that evaluates the boundary fields at the corner, i.e. the operators are given by maps $B_F: \tilde{\psi} \mapsto F \cdot \tilde{\psi}$, where $F$ is a functional depending on the boundary fields evaluated at the corner. We construct the total bundle $\mathcal{H}^\mathcal{C}_t = \mathcal{H}_t \otimes \mathcal{H}_\mathcal{C}$, where $\mathcal{H}_\mathcal{C}$ is the space of functionals depending on the boundary fields evaluated at the corner. We have

$$\mathcal{H}_\mathcal{C} := \{ F: B_{\mathcal{C}_1} \to \mathbb{C}[[\hbar]] \mid F(\chi) \text{ is of the form } (\chi) B_J(C), \text{ where } B_J \in \Omega^\text{tot}(\mathcal{P}, \mathcal{C}_1) \}.$$ 

Now we can define a state to be given as a nonhomogeneous differential form on $\mathcal{P}$ with values in $\mathcal{H}_\mathcal{C}$, i.e. an element of $\Omega^\text{tot}(\mathcal{P}, \mathcal{H}_\mathcal{C})$.

7.2. Extension of operators. Recall from [20] that the algebra of the operators is generated by $\Omega_0$, which is the standard quantization of $S_0\Sigma$, and simple operators, which are of the form

$$\int_{\partial \Sigma} L^J_{I_1 \ldots I_r} [\chi^{I_1}] \ldots [\chi^{I_r}] \frac{\delta^{[J]} \chi^{[J]}}{\delta \chi^{[J]}},$$

where $L^J_{I_1 \ldots I_r}$ are some coefficients. Note that we can also have a similar expression for $E$. We want to extend this space by the multiplication operators coming from the corners as described above. The space of operators is extended by the multiplication operators that appear in the case of corners. The algebra of boundary operators acts on the algebra of corner operators by commutators. E.g. $\partial_k \Pi^{ij} \chi^k \frac{\delta}{\delta \chi^{ij}}$ is a boundary operator and $[\chi^{i} \chi^{j}](C) \partial_i \gamma \partial_j \gamma$ is a corner operator, with $C \in \mathcal{C}_1$. Then the commutator is given by

$$[\partial_k \Pi^{ij} \chi^k \frac{\delta}{\delta \chi^{ij}}] \chi^{i} \chi^{j}] (C) \partial_i \gamma \partial_j \gamma = \partial_k \Pi^{ij} \chi^k (C) \partial_i \gamma \partial_j \gamma.$$ 

The extended space now consists of operators taking a state in $\mathcal{H}_t$ and multiply it with an element in $\mathcal{H}_\mathcal{C}$.

7.3. mdQME and Flatness.

7.3.1. mdQME. The proof of the mdQME

$$\left( d_x - i\hbar \Delta + \frac{i}{\hbar} \Omega \right) \tilde{\psi} = 0$$

proceeds as in [21]. We observe that after using Stokes’ theorem for integration along the fiber we have to show that boundary contributions vanish. The new boundary contributions from the Grothendieck connection are 6.5.1. We can think of it as a new boundary vertex $(d_x - i\hbar \Delta + \frac{i}{\hbar} \Omega) \tilde{\psi}$. Similarly, there will be new boundary vertices corresponding to $A(R, \Pi_\psi)(\gamma)$ and $\gamma \star \gamma$ and $d_x \gamma$ in $(d_x - i\hbar \Delta + \frac{i}{\hbar} \Omega) \tilde{\psi}$. Since $F = D_G \gamma + \gamma \star \gamma$ and $D_G \gamma = d_x \gamma + A(R, \Pi_\psi)(\gamma)$, these vertices cancel out after summing over all graphs. Note that the term $d_x \gamma$ now appears in $d_x \tilde{\psi}$. This happens on the parts of the boundary where $S_{\Sigma, \gamma}$ is nonzero, i.e. everywhere but in the parts of $\partial_1 \Sigma$ in $\chi$-representation. Here we get new contributions to $\Omega_{\text{pert}}$ coming from the collapsing of graphs with $R$ vertices, and from graphs collapsing on the corners we get $\Omega_{\mathcal{C}_1}$. 


7.3.2. Flatness. The flatness condition $({\nabla_G})^2 = 0$ reduces to the proof that $\Omega = \Omega_\partial + \Omega_{\partial_1}$, where $\Omega_\partial$ is the part of the operator without corners, is a Maurer-Cartan element of the dgla of differential forms with values in $\text{End}(H_{\text{tot}})$.

**Proposition 7.1.** $d_x \Omega + \frac{1}{2} [\Omega, \Omega] = 0$.

*Sketch of the proof.* First of all note that $d_x \Omega_\partial + \frac{1}{2} [\Omega_\partial, \Omega_\partial] = 0$. This means we only need to prove

$$d_x \Omega_{\partial_1} + \frac{1}{2} [\Omega_{\partial_1}, \Omega_{\partial_1}] + [\Omega_{\partial_1}, \Omega_\partial] = 0.$$

We can show this similarly to [20, 21]. Namely, since $\Omega_\partial$ and $\Omega_{\partial_1}$ are given as sum of integrals over the boundary of the configuration space of collapsing graphs, we can use Stokes’ theorem:

$$d_x \Omega_{\partial_1} = \sum_{\Gamma \leq \Gamma'} \int_{C^{\partial}_{\Gamma'}} \sigma_{\Gamma'} = \sum_{\Gamma \leq \Gamma'} \int_{C^{\partial}_{\Gamma'}} d\sigma_{\Gamma'} - \int_{\partial C^{\partial}_{\Gamma'}} \sigma_{\Gamma'}$$

Here $C^{\partial}_{\Gamma'}$ is the configuration space describing the relative position of the vertices of the subgraph $\Gamma$ collapsing to the corner. In the first, the differential can act on the propagators, the boundary fields, or the vertex tensors $\Pi_{\phi, \gamma, R}$. The restriction of the propagators to this boundary face is closed, see Appendix B. If the differential acts on the boundary fields, this yields $[\Omega_\partial, \Omega_{\partial_1}]$. The differential acting on vertex tensors will be cancelled by boundary terms. Notice that on the boundary faces the dimension counting is different and we can have either two vertices labeled by $R$, one $R$ vertex and one $\gamma$ vertex on the boundary or two $\gamma$ vertices on the boundary. A boundary face of $C^{\partial}_{\Gamma'}$ corresponds to a collapse of a subgraph $\Gamma'' \leq \Gamma'$ to a single point. There are four distinct possibilities for that point:

- The point can be in the bulk. If $\Gamma''$ contains more than two vertices then the contribution is zero by a Kontsevich vanishing lemma. If it contains exactly two vertices, there is a cancellation similar to the proof of the mdQME using the classical master equation, the fact that vertex tensors are $d_x + R$ closed, and that $[R, R] = 0$.
- The point can be the corner. These terms yield $[\Omega_{\partial_1}, \Omega_{\partial_1}]$.
- The point can be at the boundary with the $\hat{\eta} \equiv 0$ boundary condition. In that case there is a cancellation similar to one in the proof of the mdQME in section 6.5 using the equation $d_x \gamma + A(R, \Pi_\phi)\gamma + \gamma * \gamma + F(R, R, \Pi_\phi) = 0$.
- The point can be on the upper boundary, this corresponds to $[\Omega_\partial, \Omega_{\partial_1}]$, the action of the algebra of boundary operators on the algebra of corner operators.

**Remark 7.1.** The failure of the (m)dQME and its resolution is somehow similar to what happens in the Landau–Ginzburg model. Namely, the classical boundary conditions turn out not to be compatible with quantization. The resolution consists in coupling the bulk theory with a boundary theory with action $S_{\Sigma, \gamma}$.

8. **Outlook**

8.1. **Relational Symplectic Groupoid.**

8.1.1. **Short description.** Symplectic groupoids are an important concept in Poisson and symplectic geometry ([42]). A groupoid is a small category whose morphisms are invertible. We denote a groupoid by $G \rightrightarrows M$, where $M$ is the set of objects and $G$ the set of morphisms. A Lie groupoid is, roughly speaking, a groupoid where $M$ and $G$ are smooth manifolds and all structure maps are smooth. Finally, a symplectic groupoid is a Lie groupoid with a symplectic form $\omega \in \Omega^2(G)$ such that the graph of the multiplication is a Lagrangian submanifold of $(G, \omega) \times (G, \omega) \times (G, -\omega)$. The manifold of objects $M$ has an induced Poisson structure uniquely determined by requiring that the
source map \( G \to M \) is Poisson. A Poisson manifold \( M \) that arises this way is called integrable. Not every Poisson manifold is integrable.

The reduced phase space of the PSM on a boundary interval with target an integrable Poisson manifold \( P \) is the source simply connected symplectic groupoid of \( P \) ([16]). In general, the reduced phase space is a topological groupoid arising by singular symplectic reduction. In ([24, 9, 10]) it was however shown that the space of classical boundary fields always has an interesting structure called relational symplectic groupoid (RSG). An RSG is, roughly speaking, a groupoid in the “extended category” of symplectic manifolds where morphisms are canonical relations. Recall that a canonical relation from \((M_1, \omega_1)\) to \((M_2, \omega_2)\) is an immersed Lagrangian submanifold of \((M_1, \omega_1) \times (M_2, -\omega_2)\). The main structure of an RSG \((G, \omega)\) is then given by an immersed Lagrangian submanifold \(L_1\) of \((G, \omega)\), which plays the role of unity, and by an immersed Lagrangian submanifold \(L_3\) of \((G, \omega) \times (G, -\omega)\), which plays the role of associative multiplication. (In addition, there is also an antisymplectomorphism \(I\) of \(G\) that plays the role of the inversion map.) In case \(M\) is integrable, it was also shown that the RSG \(G\) is equivalent, as an RSG, to the the symplectic groupoid \(G\).

8.1.2. Kontsevich’s star product. One can construct the Moyal product [37] (deformation quantization) as the gluing of canonical relations as it was shown in [22]. It still remains to show that one can also use the gluing of the RSG to construct a globalized version of Kontsevich’s star product using the gluing formulas of the BV-BFV formalism. One can thus use the results of this paper to deal with the \(L_3\) worldsheet structure, which is given as in figure 12 with mixed boundary conditions.

```
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure12.png}
\caption{The canonical relation \(L_3\) with its boundary structure. Here we have two \(\hat{\eta} = 0\)-polarized boundaries (the lower) and one \(\hat{\eta} = 0\)-polarized boundary (the upper), which would correspond to \(\partial_2 L_3\) and the \(\hat{\eta} = 0\) boundaries which are components of \(\partial_1 L_3\).}
\end{figure}
```

8.1.3. RSG with handles. Another interesting aspect would be to consider the RSG with handles. That is, one considers canonical relations \(L_3\) with non vanishing genus. Since our theory is topological, we are able to move the handle in arbitrary directions, which means that one has to understand what happens when a hole will approach an observable for the gluing of the disk in [21]. Moreover, one has to check what kind of structures appear for associativity.

8.1.4. Generalization of Kontsevich’s star product. Kontsevich’s star product arises from the computation of expectation values of observables in the Poisson Sigma Model for a genus zero worldsheet surface. As in string theory, one expects that we should sum over all genera. Since a particular gluing of the RSG gives rise to Kontsevich’s star product, one can relate this structure to the RSG construction with handles.

We will return on these questions in a forthcoming paper.
8.2. Manifolds with corners. The methods developed in this paper can be useful to give a description for the the quantization of manifolds with corners. Here the corners arose from the structure of mixed boundary conditions, but in principle the methods that we develop might be adapted to the general case. Another paper in this direction is [32].

8.3. Globalization of other theories. AKSZ theories have a particularly nice subset of classical solutions, the space of constant maps. This subset admits for a natural globalization, as was shown in [21]. It would be interesting to see whether the methods we used carry over to more complicated moduli spaces of classical solutions. E.g. in Chern-Simons theory, this subset is just the trivial connection, since the body of the target in that case is just a point, but one would like to take non-trivial connections into account as well.

Appendix A. Deformation quantization and the Poisson Sigma Model

In this section we recollect the connection between globalization of Kontsevich’s star product and the Poisson Sigma Model that was discussed in [17, 15, 14, 8, 23, 33, 25].

A.1. Kontsevich’s formality map on \( \mathbb{R}^d \). Kontsevich’s formality map is an \( L_\infty \) (quasi-)iso)morphism from multivector fields \( T_{poly}\mathbb{R}^d := \Gamma ( \bigwedge^\bullet \mathbb{R}^d ) \) to multidifferential operators \( D_{poly}^\bullet \mathbb{R}^d \) on \( \mathbb{R}^d \). As such it consists of a family of maps

\[
\mathcal{U}_n : \Gamma \left( \bigwedge^{k_1} T\mathbb{R}^d \right) \oplus \cdots \oplus \Gamma \left( \bigwedge^{k_n} T\mathbb{R}^d \right) \to D_{poly}^\bullet \mathbb{R}^d \\
(\xi_1, \ldots, \xi_n) \mapsto \mathcal{U}_n(\xi_1, \ldots, \xi_n) := \sum_{\Gamma \in \mathcal{G}_{n,\ell}} w_{\Gamma, \xi_1, \ldots, \xi_n},
\]

where \( \mathcal{G}_{n,\ell} \) is the set of \( n + \ell \) numbered vertices, with \( \ell := 2 - 2n + \sum_{i=1}^{n} k_i \), such that the \( j \)th vertex for \( 1 \leq j \leq n \) emanates exactly \( k_j \) arrows (without short loops). Here \( k_i \) represents the degree of the multivector field \( \xi_i \). Note that \( \mathcal{U}_n(\xi_1, \ldots, \xi_n) \) acts on \( \ell \) functions. Here \( B_{\Gamma, \xi_1, \ldots, \xi_n} \) are multidifferential operators, depending a graph \( \Gamma \) and also on the vector fields \( \xi_1, \ldots, \xi_n \), and the \( w_{\Gamma, \xi_1, \ldots, \xi_n} \) are weights corresponding to a graph \( \Gamma \) as in [33]. For a vector field \( \xi \) (i.e. \( \xi \) is of degree 1) and a bivector field \( \Pi \) (i.e. \( \Pi \) is of degree 2) we can define

\[
P(\Pi) := \sum_{j=0}^{\infty} \varepsilon^j_j \mathcal{U}_j(\Pi, \ldots, \Pi),
\]
\[
A(\xi, \Pi) := \sum_{j=0}^{\infty} \varepsilon^j_j \mathcal{U}_{j+1}(\xi, \Pi, \ldots, \Pi),
\]
\[
F(\xi_1, \xi_2, \Pi) := \sum_{j=0}^{\infty} \varepsilon^j_j \mathcal{U}_{j+2}(\xi_1, \xi_2, \Pi, \ldots, \Pi).
\]

We have chosen the letters in this way, because later we will think of \( P \) to be Kontsevich’s star product for \( \Pi \) a given Poisson tensor, \( A \) as a connection 1-form and \( F \) as its curvature. Let us take a look at some of the graphs appearing for some chosen multivector fields. For example, for a bivector field \( \Pi \), we get that the term \( U_1(\Pi) \) corresponds to the first graph of figure 13, whereas for a multivector field \( V \) of degree \( r \) we get for \( U_1(V) \) the second graph of figure 13. Let now \( \xi \) be a vector field. Note that the number \( \ell \) for \( \mathcal{U}_n(\xi, \ldots, \Pi) \) will always be 1 for every \( n \), which implies that \( A(\xi, \Pi) \) takes a smooth map \( f \) as an argument.

We want to look at graphs appearing for higher terms in \( A \). We can, e.g., consider the \( n = 3 \) term, i.e. \( \mathcal{U}_3(\xi, \Pi, \Pi) \). Some example of graphs in \( \mathcal{G}_{3,1} \), which are taken in account for the sum, are given in figure 14.
We can also explicitely say what the differential operator given by a graph will be. E.g. for the graph as in 14 (b) we get
\[ \partial_{i_1} \partial_{i_3} \xi^{i_5} \partial_{i_2} \Pi^{i_3 i_4} \partial_{i_5} \Pi^{i_1 i_2} \partial_{i_4} (f). \]

By definition of $F$, there are only derivatives of the vector fields in the bulk, i.e. for every $n$ we get that $\ell = 0$, i.e. the image of $U_n$ will be a differential operator of degree zero, which is a smooth function. Some examples for graphs in $G_{3,0}$ are given in figure 15.

**Figure 13.** The graphs $U_1(\Pi)$ and $U_1(\mathcal{V})$.

**Figure 14.** Example of graphs in $G_{3,1}$.

**Figure 15.** Example of graphs in $G_{3,0}$.

**A.2. Notions of formal geometry.** We want to give the most important notions of formal geometry as in [29] following the presentation as in [14] and [6]. For a smooth manifold $P$ we can consider a formal exponential map $\varphi \in \Gamma(T P)$, such that for $x \in P$ we have $\varphi_x : T_x P \to P$, and we define a vector field $R \in \Gamma(T^* P \otimes T P \otimes S^* P)$, which is a 1-form with values in derivations...
of $\tilde{ST}^*\mathcal{P}$. Here $\tilde{S}$ denotes the completed symmetric algebra. In local coordinates we have $R_i dx^i$ with

$$
R_i(x; y) = \left( \left( \frac{\partial \varphi}{\partial y} \right)^{-1} \right)^k \left( \frac{\partial \varphi^j}{\partial x^i} \frac{\partial}{\partial y^k} \right) =: Y^k_i(x; y) \frac{\partial}{\partial y^k}.
$$

Then we can define the classical Grothendieck connection $D_G := d + R$, which is flat. For a vector field $\xi = \xi^i \frac{\partial}{\partial x^i}$ we have $D_G^\xi = \xi + \hat{\xi}$, where

$$
\hat{\xi}(x; y) = \iota_{\xi} R(x; y) = \xi^i Y^k_i(x; y) \frac{\partial}{\partial y^k}.
$$

### A.3. Globalization.

Now let us describe how to generalize the above procedure to an arbitrary Poisson manifold $(\mathcal{P}, \Pi)$. Namely, let $x \in \mathcal{P}$, and $\varphi$ a formal exponential map on $\mathcal{P}$. Then $\Pi_{\varphi, x}$, the Taylor expansion of $\Pi$ around $x$ defined using $\varphi$, is a Poisson tensor on $\tilde{ST}^*_x \mathcal{P}$. Any choice of coordinates on $T_x \mathcal{P}$ now allows us to identify $\tilde{ST}^*_x \mathcal{P} \equiv \mathbb{R}[[y_1, \ldots, y_d]]$ and define Kontsevich’s star product $P(\Pi_{\varphi, x})$. See [17] for a discussion of the equivariance of this construction in the choice of coordinates. In this way we get a new bundle $\mathcal{E} := \tilde{ST}^*_x \mathcal{P}[[\varepsilon]]$ of $\ast$-algebras. One can use the Grothendieck connection defined in A.2 to give a description of a subalgebra $\mathcal{A} \subset \Gamma(\mathcal{E})$ which is a deformation quantization of $C^\infty(\mathcal{P})$ seen as a subalgebra of $\Gamma(\mathcal{E})$. Formally we have

$$
\Gamma(\mathcal{E}) \supset C^\infty(\mathcal{P}) \xrightarrow{\text{Deformation Quantization}} \mathcal{A} \subset \Gamma(\mathcal{E}).
$$

The algebra $\mathcal{A}$ is given by closed sections under a deformation of the Grothendieck connection, which is defined in two steps: For a tangent vector $\xi \in T_x \mathcal{P}$, we let

$$
D_G^\xi := \xi + \Pi_{\varphi} \left( \hat{\xi}, \Pi_{\varphi} \right) = D_G^\xi + O(\varepsilon),
$$

where again we denote by $\Pi_{\varphi}$ the Poisson tensor $\Pi$ lifted to a formal neighborhood and $\hat{\xi}$ is defined as in (42). One can write

$$
D_G = d + A(R, \Pi_{\varphi})
$$

interpreting $A(R, \Pi_{\varphi})$ as a one-form valued in differential operators on $\mathcal{E}$. At some point $x \in \mathcal{P}$, in coordinates $x^i$ around $x$, it is given by

$$
A(R, \Pi_{\varphi}) = dx^i A(R_i(x; ), \Pi_{\varphi, x}) = dx^i A \left( Y^k_i(x; ) \frac{\partial}{\partial y^k}, \Pi_{\varphi, x} \right).
$$

One can then show (see [17]) that $D_G$ is a globally defined connection on $\Gamma(\mathcal{E})$, a derivation, and that $(D_G)^2$ is an inner derivation, i.e.

$$
(D_G)^2 \sigma = [F^{\mathcal{P}}, \sigma], := F^{\mathcal{P}} \ast \sigma - \sigma \ast F^{\mathcal{P}},
$$

for any $\sigma \in \Gamma(\mathcal{E})$, where $F^{\mathcal{P}}$ is the Weyl curvature tensor of $D_G$ given by $F^{\mathcal{P}}(\xi_1, \xi_2) := F(\hat{\xi}_1, \hat{\xi}_2, \Pi_{\varphi})$, where $\xi_1, \xi_2 \in T_x \mathcal{P}$ are two tangent vectors on $\mathcal{P}$. More precisely, $F^{\mathcal{P}}$ is a 2-form valued in sections of $\mathcal{E}$ which in local coordinates can be expressed as

$$
F^{\mathcal{P}}_x = dx^i \wedge dx^j F(R_i(x; ), R_j(x; ), \Pi_{\varphi, x}).
$$

For the Weyl tensor we get $D_G F^{\mathcal{P}} = 0$. The task is to modify the globalized connection $D_G$ slightly more, such that it becomes flat but still remaining a derivation. One can set $^5$

$$
\bar{D}_G := D_G + [\gamma, ]_*,
$$

$^5$For any two $\mathcal{E}$-valued 1-forms $\gamma = \gamma_1 dx^i, \sigma = \sigma_j dx^j \in \Omega^1(\mathcal{P}, \mathcal{E})$ one defines their star product by $\gamma \ast \sigma := (\gamma_1 \ast \sigma_j) dx^i \wedge dx^j$.
and observe that for any 1-form $\gamma \in \Omega^1(\mathcal{P}, E)$ this connection is a derivation. Moreover, its Weyl curvature tensor is then given by
\begin{equation}
\mathcal{F}^\mathcal{P} = F^\mathcal{P} + \mathcal{D}_G \gamma + \gamma \star \gamma.
\end{equation}
We call (43) the deformed Grothendieck connection and (45) the modified deformed Grothendieck connection. One then needs to find $\gamma \in \Omega^1(\mathcal{P}, E)$ such that $\mathcal{F}^\mathcal{P} = 0$, which implies that $(\mathcal{D}_G)^2 = 0$, then $\mathcal{D}_G$-closed sections will form the algebra $\mathcal{A}$ as a deformation quantization of $C^\infty(\mathcal{P})$. If we compute $(\mathcal{D}_G)^2$ explicitly, by using (45) we get
\begin{equation}
(\mathcal{D}_G)^2 = (\mathcal{D}_G)^2 + \mathcal{D}_G[\gamma, ] + [\gamma, [\gamma, ]]*.
\end{equation}
More precisely, $\gamma$ has to satisfy
\begin{equation}
F^\mathcal{P} + \mathcal{D}_G \gamma + \gamma \star \gamma = 0.
\end{equation}
The existence of such a $\gamma$ was shown in [14, 17] by homological perturbation theory. Now we want to focus on some special cases. We want to look at two important examples of Poisson structures.

A.3.1. Constant Poisson structure. The situation of a constant Poisson structure is a first example to think about. Let $(\mathcal{P}, \Pi)$ be a Poisson manifold with constant Poisson structure $\Pi$ and $\xi \in T_x \mathcal{P}$ for $x \in \mathcal{P}$ be a fixed tangent vector. By the definition of $\mathcal{A}$, and the fact that each vertex has only one outgoing and no incoming arrow, we get $A(\xi, \Pi_{\phi}) = \xi$, which leads to the fact that
\begin{equation}
\mathcal{D}_G^\xi = (\xi + \hat{\xi}) = D_G^\xi.
\end{equation}
Therefore we get $(\mathcal{D}_G)^2 = 0$ and thus $\mathcal{F}^\mathcal{P} = 0$.

A.3.2. Linear Poisson structure. Let now $(\mathcal{P} = \mathfrak{g}^*, \Pi)$ be a Poisson manifold with linear Poisson structure $\Pi(x) = \Pi_{ij}^k x^k \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$, where $\Pi_{ij}^k$ represent the structure constants of $\mathfrak{g}$, and $\xi \in T_x \mathcal{P}$ for $x \in \mathcal{P}$ be a fixed tangent vector. As in the constant case, we observe that $A(\xi, \Pi_{\phi}) = \xi$, which is the case since the integral of a bulk vertex with one incoming and one outgoing arrow is zero, and since there is at most one incoming arrow for each vertex.

A.4. Generalized curvature. One can actually construct $\gamma$ to be a solution of the more general equation given by
\begin{equation}
\mathcal{F}^\mathcal{P}_\omega = F^\mathcal{P} + \varepsilon \omega + \mathcal{D}_G \gamma + \gamma \star \gamma = 0,
\end{equation}
where $\omega \in \Omega^2(\mathcal{P}, E)$ such that $\mathcal{D}_G \omega = 0$ and $[\omega, ]* = 0$ ([17]).

A.5. Connection to the Poisson Sigma Model. In [11] and [15] it was shown that Kontsevich’s formality map on $\mathbb{R}^d$ can be interpreted as the perturbative computation of expectation values of observables of the PSM on the upper half plane (or respectively the disk) with values in $\mathbb{R}^d$. The graphs which appear in the construction of Kontsevich’s star product on Poisson manifolds ([33]) are given on the upper half plane, where they can collapse, according to the boundary of the configuration space, on the boundary of the upper half plane. This means that the graphs that appear in the PSM are exactly the graphs that appear for Kontsevich’s star product. More precisely, if one considers the disk $D$ in $\mathbb{R}^2$ and the classical action of the PSM on $D$ given by $S_D[\eta] = \int_D \left( \langle \eta, d\eta \rangle + \frac{1}{2}(\Pi(\eta, \eta) \wedge \eta) \right)$, we can asymptotically write Kontsevich’s star product for two smooth maps $f$ and $g$ as a perturbative expansion of the following path integral:
\begin{equation}
f \star g(x) = \int_{X(\infty) = x} f(X(0))g(X(1))e^{\frac{1}{\hbar}S_D[\eta]},
\end{equation}
where $0, 1, \infty$ represent some marked points on the boundary of $D$. Note that $x \in \text{Map}(D, \mathbb{R}^2)$ is a constant map, i.e. the we get a local representation of the star product. If one considers a general
Poisson manifold \((\mathcal{P}, \Pi)\), one can consider the constant map \(x \in \text{Map}(D, \mathcal{P})\) as a point sitting in \(\mathcal{P}\) giving a local product on each fiber. As already described in \(\text{A.3}\), one can then algebraically construct the star product on all of \(\mathcal{P}\).

**Appendix B. On the Propagator**

We have an explicit propagator for the PSM structure, i.e. using the superfields of it, on a disk with alternating boundary conditions, which was computed in [12], in [23] and, in full generality, in [26].

**B.1. Construction of the branes.** Consider an \(n\)-sided polygon \(P_n = u(\mathbb{H}^+)\) with \(u : \mathbb{H}^+ \to P_n\) is a suitable homeomorphism between the compactified complex upper half plane \(\mathbb{H}^+\) and \(P_n\), depending on the number of the branes considered. Let \(G_{S_i}\), be the relevant superpropagators for the PSM with \(n\) branes defined by constraints \(C_j = \{x^{\mu_j} = 0 \mid \mu_j \in I_j\}\) (also called branes) and index sets \(S_1 = I_1^C \cap I_2 \cap I_3 \cap \ldots \cap I_n\), \(S_2 = I_1 \cap I_2^C \cap I_3 \cap \ldots \cap I_n^C\) for \(n\) even, and \(S_1 = I_1^C \cap I_2 \cap I_3^C \cap \ldots \cap I_n^C\), \(S_2 = I_1 \cap I_2^C \cap \ldots \cap I_n\) for \(n\) odd, which are called relevant. It turns out that the \(C_i \subset \mathcal{P}\) are coisotropic submanifolds of \(\mathcal{P}\) ([12]).

**B.2. Constructing integral kernels.** The integral kernels \(\theta(Q, P)_{S_i} := -\frac{1}{\pi} \langle \hat{X}^i(Q) \hat{\eta}_i(P) \rangle\) for the two brane case are given by:

\[
\theta(Q, P)_{S_1} = \frac{1}{2\pi} \arg \frac{(u - v)(\bar{u} - v)}{(\bar{u} + v)(u + v)},
\]

\[
\theta(Q, P)_{S_2} = \frac{1}{2\pi} \arg \frac{(u - v)(\bar{u} + v)}{(\bar{u} - v)(u + v)},
\]

where \(P_2 := u(\mathbb{H}^+)\) with \(u(z) = \sqrt{z}, v := u(w), d = d_u + d_v.\) We identify \((P, Q)\) with the couple \((u, v)\). Consider e.g. \(P_2\) to be the worldsheet disk \(\Sigma\) with boundary \(\partial \Sigma = \bigcup_{1 \leq j \leq 6} J_j\) (we denote the intervals here by \(J\) instead of \(I\) such that there is no confusion with the index sets) and the branes \(C_1 = \{x^{\mu_1} = 0 \mid \mu_1 \in I_1 = \{1, \ldots, n\}\}\) and \(C_2 = \{x^{\mu_2} = 0 \mid \mu_2 \in I_2 = 2\}\), which correspond to the boundary conditions of \(\partial_1 \Sigma\) and \(\partial_2^{\text{tot}} \Sigma\) respectively. The components \(\partial_1 \Sigma\) and \(\partial_2^{\text{tot}} \Sigma\) are such that \(\partial_1 \Sigma = \partial_3 \Sigma \cup \partial_2^{\text{tot}} \Sigma\), where \(\partial_3 \Sigma\) is chosen to be some \(J_1\) endowed with the \(\frac{d}{dx}\) polarization and \(\partial_2^{\text{tot}} \Sigma = \bigcup_{2 \leq j \leq 6} J_j\) such that \(J_j\) is endowed with the \(\frac{d}{dz}\) polarization and with the boundary condition \(\hat{\eta}_j \equiv 0\) for \(j\) odd and even respectively. Now we get \(S_1 = I_1^C \cap I_2 = \emptyset\) and \(S_2 = I_1 \cap I_2^C = \{1, \ldots, n\}\). Now \(P_2\) is defined by \(P_2 = u(\mathbb{H}^+)\), where \(u\) is the map \(z \mapsto \sqrt{z}\). Points \((P, Q) \in P_2 \times P_2\) are represented respectively by a pair of complex numbers \((u, v)\) in the first quadrant, with \(u = u(z), v = u(w)\) for all \((z, w) \in \mathbb{H}^+ \times \mathbb{H}^+\). The boundary \(\partial_1 P_2\) (corresponding to \(\partial_1 \Sigma\)) is given by the positive imaginary axis, while \(\partial_2 P_2\) (corresponding to \(\partial_2^{\text{tot}} \Sigma\)) is given by the positive real axis.

**B.3. Construction of superpropagators.** The boundary conditions imposed by the index sets \(S_i\) are \(\theta(v, u \in \partial_1 P_2)_{S_1} = \theta(u \in \partial_2 P_2, u)_{S_1} = 0, \theta(v, u \in \partial_2 P_2)_{S_2} = \theta(v \in \partial_1 P_2, u)_{S_2} = 0.\) Let

\[
\psi(u, v)_{S_1} = \arg \frac{(u - v)(\bar{u} - v)}{(\bar{u} + v)(u + v)},
\]

\[
\psi(u, v)_{S_2} = \arg \frac{(u - v)(\bar{u} + v)}{(\bar{u} - v)(u + v)},
\]

which satisfy the same boundary conditions as \(\theta(v, u)_{S_i}\). Now for vanishing cohomology, we get the following theorem.
Theorem B.1. The integral kernels for the superpropagators $G_{S_i}$ in presence of two branes are given by

\[ \theta(v, u)_{S_i} = \frac{1}{2\pi} d\psi(u, v)_{S_i}, \]

with mirror maps (52) and (53). The integral kernels satisfy the additional boundary conditions $\theta(v, u)_{S_1} = \theta(v, \bar{u}) = \theta(-\bar{v}, u)_{S_1}$, $\theta(v, u)_{S_2} = \theta(v, -\bar{u})_{S_2} = \theta(\bar{v}, u)_{S_2}$, i.e. every boundary component of $P_2$ is labeled by a boundary condition for both the variables $(u, v)$. By construction $\theta(v, u)_{S_1} = \theta(u, v)_{S_2}$, $\theta(v, u)_{S_2} = \theta(u, v)_{S_1}$.

B.4. Relation to Kontsevich’s propagator. Let $\phi$ be Kontsevich’s angle 1-form. Then, one can show that

\[ \theta(v, u)_{A_1} = \frac{1}{2\pi} d\arg \frac{(u-v)(u+v)}{(u+v)(u-v)} = \frac{1}{2\pi} d\arg \frac{(z-w)}{(z-w)} = \frac{1}{2\pi} d\phi(z, w), \]
\[ \theta(v, u)_{A_2} = \frac{1}{2\pi} d\arg \frac{(u-v)(u+v)}{(u-v)(u+v)} = \frac{1}{2\pi} d\arg \frac{(z-w)}{(z-w)} = \frac{1}{2\pi} d\phi(w, z), \]

where $A_1 = I_2 \cap I_2$ and $A_2 = I_2^C \cap I_2^C$.

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190 CH-8057 Zürich
E-mail address, A. S. Cattaneo: cattaneo@math.uzh.ch

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190 CH-8057 Zürich
E-mail address, N. Moshayedi: nima.moshayedi@math.uzh.ch

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190 CH-8057 Zürich
E-mail address, K. Wernli: konstantin.vernli@math.uzh.ch