Scattering with Baryon Number Violation
— The Case of Higgs Particle Production —

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ABSTRACT

A formalism based on path-integral expression of time-evolution operator during tunneling at a finite energy proposed by the authors is applied to $SU(2)$ gauge-Higgs system to produce Higgs particles with $\Delta B = 1$. Instead of starting from instanton tunneling at the zero energy, a classical bounce solution giving sphaleron (instanton) action at high (low) energies is used as the tunneling configuration. Fourier transform of the bounce configuration in coherent state expression at the entrance and exit of the tunneling plays an important role. Numerical results at various energies for $M_H/M_W = 1 \sim 2$ are given. Though the cross section with $\Delta B = 1$ results from a severe cancellation of several large quantities in the leading order as occured in the instanton calculus, it seems unlikely that the cross section grows as largely as to reach unitarity bound at energies $E \leq E_{sph}$. It is pointed out that the actual value $g^2 = 0.418$ of the $SU(2)$ gauge coupling constant may be too large to take the weak coupling limit.
1. Introduction

It is well-known that chiral anomaly in baryon number current of electroweak theory violates baryon number ($\Delta B \neq 0$) given by topologically nontrivial background gauge fields through quantum tunneling.\textsuperscript{1)} At low energies or low temperatures the probability of $\Delta B \neq 0$ processes through instanton tunneling is known to be unobservably small. Sphaleron\textsuperscript{2)} is a saddle-point solution to equations of motion in the electroweak theory, and its mass, $E_{sph} \sim M_W/\alpha_W$, gives the barrier height between the topologically inequivalent vacua. Though there remains problem of non-equilibrium process, the transition rate with $\Delta B \neq 0$ at high temperatures $\sim E_{sph}$ is commonly believed to be significant\textsuperscript{3)} and to be crucial for baryon asymmetry of the universe.

On the other hand, after works by Ringwald\textsuperscript{4)} and Espinosa,\textsuperscript{5)} which succeeded a pioneering one by Aoyama and Goldberg,\textsuperscript{6)} that the $\Delta B \neq 0$ process might be detected in future colliders, the scattering cross section at high energies has been evaluated by many authors. Almost all of these analyses are based on instanton calculus and the LSZ formalism, by which the cross section in the leading order consists of three factors; the instanton suppression factor, (residue)$^{n+2}$ and the $n$-body phase space volume with $n$ denoting the number of produced particles. The residue here is meant by that of the euclidean propagator in the instanton background of the particle. For $n$ large enough, the last two factors overcome the first one and raise the cross section even to the unitarity bound. It may be natural then that possible corrections to the leading order are large and tend to hinder the rise of the cross section. We are in a frustrated situation, as the corrections may be comparable with the leading order contribution so that no reliable answer has been obtained.\textsuperscript{7)}

In a previous paper,\textsuperscript{8)} the authors have proposed an alternative approach to treat the $\Delta B \neq 0$ high energy scattering, by noticing that the process is basically a tunneling process traversing not far from the barrier top so that starting from the instanton tunneling may not be appropriate. We have applied the formalism to a simple model accompanying both the instanton and the sphaleron, $O(3)$ nonlinear sigma model in 1+1 dimensions. In this paper we apply our formalism to more realistic model, 3+1 dimensional $SU(2)$ gauge-Higgs system, clarify structure of the amplitude, and numerically analyze the production cross section of $n$ Higgs particles from the two ones.

For tunneling configuration at non-zero energies, we use a classical bounce solution\textsuperscript{9)} instead of the instanton, because the tunneling exponent $W(E)$ at $E \neq 0$ is not given by the instanton action but rather by the Legendre transform of the bounce action as is
obvious in quantum mechanics of one degree of freedom. Here the tunneling exponent is given by
\[ W(E) \propto \int_{x_1(E)}^{x_2(E)} dx \sqrt{V(x) - E} \]
where \( x_1, x_2 \) are the turning points at energy \( E \). At \( E = 0 \), \( W(0) \) coincides with the instanton action while at \( E \neq 0 \), the transform leads to \( W(E) = S_b(T) + T(E)E \) where \( T(E) \) is the half period of the bounce motion and \( S_b \) is the half of the bounce action. Within the WKB approximation, the tunneling process in many-dimensional quantum mechanics and field theory may be forced to reduce to an essentially one-dimensional problem around some classical configurations.\(^{10,11}\)

Once we find a classical bounce solution employing some ansatz, the path-integral expression of time-evolution operator during tunneling at a given energy would be dominated by the configuration.

For the \( \Delta B = 1 \) process, the non-contractible loop parameter \( \mu \), that connects a neighboring pair of topologically inequivalent vacua,\(^2\) may be the most appropriate dynamical variable describing the tunneling through the sphaleron barrier. A variational parameter is introduced to compensate ignorance of other degrees of dynamical freedom in the tunneling (deformed sphaleron). Actually the classical bounce solution of this type of reduced model\(^{12}\) does bridge the gap between instanton and the sphaleron smoothly. Coherent state expression of the entrance and exit of the tunneling mediated by the bounce solution helps us to avoid singularities at the turning points,\(^{13}\) and plays an important role instead of the residue in the instanton background.

There are some works of interest by Shaposhnikov,\(^{14}\) Banks et al.\(^{15}\) and Khlebnikov. et al\(^{16}\), which would be close in spirit to ours in the meaning that they do not start from the zero energy instanton but treat the tunneling at a finite energy from the first. We will make some comments on them in due places.

This paper is organized as follows. In section 2 we briefly review our formalism. In section 3 we show calculations by the reduced model and clarify structure of the scattering amplitude. In section 4 numerical analyses of the cross section with \( \Delta B = 1 \) at various energies and for mass ratio \( M_H/M_W = 1 \sim 2 \) are presented. The high energy cross section results from a severe cancellation of several large quantities in the leading order as occurred in the instanton calculus. If we take the face values, it seems unlikely that the cross section grows as largely as to reach the unitarity bound at energies \( E \leq E_{sph} \). Section 5 is devoted to concluding remarks. The actual value \( g^2 = 0.418 \) of the \( SU(2) \) gauge coupling constant may be too large to make the weak coupling approximation. Gauge fixing conditions in \( R_\xi \) gauge of the reduced model are summarized in Appendix.
2. Scattering amplitude through bounce configuration

Characteristic points of our formalism are as follows.

2.1. \textit{S-matrix element}

The \textit{S}-operator is defined by

\[
\hat{S} = \lim_{t_F \to \infty} \lim_{t_I \to -\infty} e^{i\hat{H}_0 t_F/\hbar} e^{-i\hat{H}(t_F-t_I)/\hbar} e^{-i\hat{H}_0 t_I/\hbar},
\]

where \( \hat{H} \) is the hamiltonian and \( \hat{H}_0 \) is the free part of it. For simplicity we ignore initial- and final-state interactions. Then we have

\[
e^{-i\hat{H}(t_F-t_I)/\hbar} \simeq e^{-i\hat{H}_0(t_F-X^0-T/2)/\hbar} e^{-i\hat{H}T/\hbar} e^{-i\hat{H}_0(X^0-T/2-t_I)/\hbar},
\]

where \( T(E) \) is the half period of the bounce motion and \( X^0 \) is the time-like center of it.

By inserting the identity operator in terms of coherent state, the above \textit{S}-operator is written as

\[
\hat{S} \simeq e^{i\hat{H}_0(X^0+T/2)/\hbar} \int_{X^0-T/2}^{X^0+T/2} \frac{d\Phi(t)d\Pi(t)}{2\pi\hbar} [\Phi(X^0+T/2), \Pi(X^0+T/2)]
\]

\[
\times \exp\left\{\frac{i}{\hbar} \int_{X^0-T/2}^{X^0+T/2} d^d x [\Pi(x)\dot{\Phi}(x) - \mathcal{H}(\Phi, \Pi)]\right\}
\]

\[
\times \langle \Phi(X^0-T/2), \Pi(X^0-T/2) | e^{-i\hat{H}_0(X^0-T/2)/\hbar},
\]

where the state \( |\phi(x), \pi(x)\rangle \) is defined by

\[
|\phi(x), \pi(x)\rangle = \exp\left\{-\frac{i}{\hbar} \int d^{d-1} x [\phi(x)\dot{\Phi}(0, x) - \pi(x)\dot{\Pi}(0, x)]\right\} |0\rangle,
\]

with \( |0\rangle \) being vacuum annihilated by any annihilation operator \( \hat{a}_i(k) \) which composes \( \hat{\Phi}(0, x) \). Here \( \hat{\Phi} \) and \( \hat{\Pi} \) stand for the canonical variables in the theory. The reasons we use coherent state here are not only that the state is the one with minimal quantum uncertainty so that it will be suited to describe classical configurations,\textsuperscript{17} but also that the WKB approximation in terms of it is global and uniform in contrast to the usual one in coordinate representation which brings singularities into wave functions at the turning points.\textsuperscript{13} Since the residual states in (2.3) are just those on the turning points, the coherent state representation will be very suitable.
By extracting collective coordinates (such as $X^0$ and those for internal symmetry), we recast the expression (2.3) into an appropriate form suited for expansion about the classical solution. The $S$-matrix element is given by taking matrix element of the $S$-operator between the initial- and final-asymptotic states which are the Fock states with some definite baryon number $B$ such as $|\text{asym}\rangle_B = \hat{a}^\dagger(k_1)\hat{a}^\dagger(k_2)\cdots\hat{a}^\dagger(k_n)|0\rangle_B$ where the creation operators are the same as those constructing the coherent state within the approximation that the initial-final-state interactions are ignored. Then the $S$-matrix element in the leading order of the WKB approximation ($\hbar \sim 0$) reads

\[ S_{fi} \simeq (2\pi\hbar)^d \delta^d(P_f - P_i) e^{-W(E)/\hbar} \]

\[ \times \langle f|\phi_c(T/2, x), \pi_c(T/2, x)\rangle_{B=1} \langle \phi_c(-T/2, x), \pi_c(-T/2, x)|i\rangle_{B=0}. \] (2.5)

The tunneling exponent $W(E)$ is obtained by continuing the time variable to negative imaginary axis in path integral in (2.3);\footnotemark[18]

\[ \frac{i}{\hbar} \{ S[\phi_c, \pi_c] + ET(E) \} \to -\frac{1}{\hbar} W(E). \] (2.6)

For a bounce solution, $\phi(\pm T/2)$ is the edge of it and $\pi(\pm T/2) = 0$ is sitting at the turning points. The energy-momentum conserving $\delta$ function in the above expression comes from integrating out the collective coordinates of translation. In $SU(2)$ gauge-Higgs system in the $R_\xi$ gauge, the internal symmetry comes from global $SU(2)$ transformation; $A^a_\mu$ and BRS\footnotemark[19] quartet $(\chi^a, C^a, \bar{C}^a, B^a)$ transforming as $SU(2)$ triplets. If we set up the initial and final states to be eigenstates of conserved charges which generate the corresponding symmetries, integration of the collective coordinates yields $\delta$ functions of conservation of the symmetries. The inner products between the Fock states and the coherent states are given by the following formula;

\[ \langle 0|\hat{a}(k_1)\hat{a}(k_2)\cdots\hat{a}(k_n)|\phi(x), \pi(x)\rangle = e^{-\frac{i}{2} \int dk|\alpha(k)|^2 \alpha(k_1)\alpha(k_2)\cdots\alpha(k_n)}, \] (2.7)

where $\alpha(k)$ is defined by

\[ \alpha(k) = \int \frac{d^{d-1}x}{\sqrt{(2\pi)^{d-1}2\hbar\omega_k}} \left[ \omega_k\phi(x) + i\pi(x) \right] e^{-ik\cdot x} \] (2.8)

with $\omega_k$ being energy of asymptotic particle with momentum $k$. Hence one can evaluate the $S$-matrix element once one has the Fourier transform of the classical configuration.

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\* For more detailed, see I.
2.2. Reduced model

In general, a classical solution is given in some gauge and we perform expansion around it. So one must care about gauge-independence of $S$-matrix. The BRS formalism\textsuperscript{19}) is one of systematic methods which automatically gives gauge-independent $S$-matrix elements. We choose the $R_\xi$ gauge, which is suited to a problem with a classical background.

In this gauge the lagrangian is given by

$$\mathcal{L} = -\frac{1}{4} F^{a\mu}_\nu F^{a\mu\nu} + |D_\mu \Phi|^2 - \frac{\lambda}{2} (|\Phi|^2 - \frac{v^2}{2})^2 + \mathcal{L}_{R_\xi}. \quad (2.9)$$

Here the field strength and covariant derivative are defined respectively as

$$F^{a\mu}_\nu \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu,$$

$$D_\mu \Phi \equiv \partial_\mu \Phi - ig\frac{\tau^a}{2} A^a_\mu \Phi. \quad (2.10)$$

$v$ is the vacuum expectation value of the Higgs field; $\langle 0 | \Phi | 0 \rangle = v/\sqrt{2}$, by which the gauge boson mass $M_W = M_Z$ is $gv/2$, the Higgs boson mass $M_H$ is $\sqrt{\lambda}v$ and we parametrize the Higgs field as

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} v + \phi + i\tau^a \chi^a \\ 0 \end{pmatrix}. \quad (2.11)$$

The gauge fixing term $\mathcal{L}_{R_\xi}$ is given by

$$\mathcal{L}_{R_\xi} = (-i)\delta_B \left( - \partial_\mu \bar{C}^a A^{a\mu} + \frac{\alpha_g}{2} \bar{C}^a B^a + \alpha_g M_W \bar{C}^a \chi^a \right), \quad (2.12)$$

where $\delta_B$ means the BRS transformation and $\alpha_g$ is the gauge parameter. The path-integral expression of the $S$-operator is now given in terms of the following canonical variables;

$$(\Phi^a, \Pi^a_\Phi) \equiv \left\{ (A^a_0, B^a), (A^a_i, \Pi^a_\Phi), (B^a, \Pi_B^a), (\phi, \Pi_\phi), (\chi^a, \Pi^a_\chi), (C^a, \Pi^a_C), (\bar{C}^a, \Pi^a_{\bar{C}}) \right\}. \quad (2.13)$$

Under the spherically symmetric ansatz leading to the sphaleron as Ratra and Yaffe\textsuperscript{20)} introduced, we obtain the 1+1 dimensional action of gauge ann Higgs fields given by them together with $R_\xi$ gauge fixing term. These can be expressed, following the prescription by Aoyama, Goldberg and Ryzak,\textsuperscript{12)} in terms of the non-contractible loop parameter...
\( \mu \in [0, \pi] \) and three trial functions \( f, h \) and \( K^\star \). Here we have chosen that \( \mu \) depends only on time \( t \) and that \( f, h \) and \( K \) are functions of spatial variable \( r \). The neutral Higgs field \( \phi_H \) we are concerning is expressed as follows:

\[
\phi_H = \frac{2M_W}{g} \left[ \text{Re}[e^{-i\mu K(r)}(\cos \mu(t) + ih(r) \sin \mu(t))] \right] - 1. \tag{2.14}
\]

After some manipulations, we obtain the reduced euclidean action with \( \mu \) as the dynamical variable:

\[
S_E[\mu] = \frac{4\pi}{g^2} \int dt dr \left[ \frac{1}{2} M(t, r) \dot{\mu}^2 + V(t, r) \sin^2 \mu \right] \tag{2.15}
\]

with \( \dot{\mu} = d\mu/dt \). Here

\[
M(r, t) = 4 \left( r^2 K'^2 + 2(f - K)^2 + M_W^2 r^2 (h - K)^2 \right) \\
+ 4 \left( 8f(1 - f)K(1 - K) + M_W^2 r^2 (1 - h^2)(1 - K^2) \right) \sin^2 \mu,
\]

\[
V(t, r) = 2 \left( 2f'^2 + M_W^2 r^2 h'^2 + 2M_W^2 (f - h)^2 \right) \\
+ \left( 8f^2(1 - f)^2/r^2 + 4M_W^2 f(1 - f)(1 - h^2) - 4M_W f(1 - h)^2 \right) \\
+ (1/2)M_W^2 M_H^2 r^2 (1 - h^2)^2 \right) \sin^2 \mu
\]

with \( f' = df/dr \). Two gauge conditions,

\[
h(r)\tan(\mu(t)) = \tan(\mu(t)K(r)), \\
\mu K'' + (1/r) \left[ (1/r) \left( f \sin(2\mu(1 - K)) - (1 - f) \sin(2\mu K) \right) + 2\mu K' \right] = 0, \tag{2.17}
\]

are obtained from the \( R_\xi \) gauge fixing term. (See Appendix.)

\* \( \Omega(t, r) \) in Ref. 12) is given by \( 2\mu(t)K(r) \).
3. Model Calculation

Let $\rho$ be the sphaleron size. We choose the trial functions of the following form;

$$f(r) = \tanh^2(r/\rho), \quad h(r) = K(r) = \tanh(r/\rho), \quad (3.1)$$

which do reproduce the required behavior of the sphaleron at $r \simeq 0$ and of the exponential damping of $e^{-M_W r} \sim e^{-M_H r}$ as $r \to \infty$. In the both regions, fortunately, they also satisfy the gauge conditions at $\mu = \pi/2$ (the sphaleron configuration) and $\mu = 0, \pi$ (the vacua). These may be satisfactory in views of simplicity of the trial functions. In terms of dimensionless time $\tau = M_W t$ and variational parameter $a = M_W \rho$, the euclidean action is written as

$$S_E[\mu] = \frac{4\pi}{g^2} \int d\tau \left[ \frac{1}{2} M(\mu, a) \dot{\mu}^2 + V(\mu, a) \sin^2 \mu \right] \quad (3.2)$$

with $\dot{\mu} = d\mu/d\tau$. Here

$$M(\mu, a) = \left(4(\pi^2/18 + 4\ln2 - 3)a\right) + \sin^2 \mu \left((8/5)a + 4((\pi^2/18) - (1/3))a^3\right),$$

$$V(\mu, a) = \left(32/15\right)a + 2(\pi^2/18 + 4\ln2 - 3)a + \sin^2 \mu \left(8C/a + 8(11/5 - \ln2)a + \sqrt{2M_H^2/M_W^2}((\pi^2/18) - (1/3))a^3\right) \quad (3.3)$$

with $C = \int_0^{\infty} dx (\sinh^4 x/(x^2 \cosh^8 x)) = 0.0916769$, which are dimensionless mass and potential height respectively.

The bounce solution within a self-consistent approximation is obtained as done in Ref. 21. We first regard time dependence coming from $\sin^2 \mu$ of $M(\mu, a)$ and $V(\mu, a)$ as weak and replace them by $M_0(a)$ and $V_0(a)$ respectively for a moment. Then the first integral of the equation of motion gives

$$(1/2)M_0(a) \dot{\mu}^2 - V_0(a) \sin^2 \mu = -V_0(a)(1 - \kappa^2), \quad (3.4)$$

where $\kappa$ is the integral constant such that $0 \leq \kappa \leq 1$. The periodic solutions to (3.4) are

$$\mu_b(\tau) = \arccos[-\kappa \text{sn}(b(\kappa, a)\tau; \kappa))] \quad (3.5)$$

with $b(\kappa, a) = \sqrt{2V_0(a)/M_0(a)}$, where $\text{sn}(x; \kappa)$ is the elliptic function and the period is
given by
\[ b(\kappa, a)T(E(\kappa)) = 4nK(\kappa) \quad \text{with} \quad n = 1, 2, 3, \cdots, \quad (3.6) \]
where \( K(\kappa) = \int_0^{\pi/2} d\theta (1 - \kappa^2 \sin^2 \theta)^{-1/2} \). We take the \( n = 1 \) solution without nodes and evaluate the average of \( \sin^2 \mu(\tau) \) by it as
\[ \langle \sin^2 \mu \rangle = (1 + \kappa \sqrt{1 - \kappa^2 / \arcsin \kappa}) / 2. \quad (3.7) \]
By putting this back to \( M(\mu, a) \) and \( V(\mu, a) \), we replace them by \( M(\kappa, a) \) and \( V(\kappa, a) \) respectively. Taking half (one way) of this bounce motion at \( E = V(\kappa = 0, a) (1 - \kappa^2) \), we obtain the tunneling exponent \( W(E)/g^2 \) by (2.6) as
\[ W(E)/g^2 = 8\pi \sqrt{2M(\kappa, a)V(\kappa, a)} \left[ E(\kappa) - (1 - \kappa^2)K(\kappa) \right] / g^2, \quad (3.8) \]
where \( E(\kappa) = \int_0^{\pi/2} d\theta (1 - \kappa^2 \sin^2 \theta)^{1/2} \).

We remark some physical features of importance. Obviously at \( \kappa = 0 \), \( W(E) = 0 \) while at \( \kappa = 1 \), \( W(E) \) is minimized at \( a = 0 \) giving \( W(E)/g^2 = 1.027S_{inst} \) where \( S_{inst} = 8\pi^2/g^2 \). Such a smooth connection between the almost free passing over the sphaleron barrier at high energies and the strong instanton suppression at low energies is one of natural implications of our formalism. Given the size \( a \), the sphaleron mass is
\[ E_{sph}(a)/M_W = (4\pi/g^2)V_0(\kappa = 0, a) \quad (3.9) \]
as shown in Fig.1, which is minimized at \( a = a_0 \) and is compared with Yaffe’s numerical result of \( E_{sph}^{Y, 22} \) in Table I. Such small deviations from the \( S_{inst} \) and \( E_{sph}^{Y} \) may prove that our trial functions are satisfactory. Note that the parameter \( a \) (or \( \rho \)) is not only of the variational character but discriminates the symmetry breaking of the electroweak theory. That is, \( a_0 \neq 0 \) to minimize the sphaleron realizes the broken phase while \( a = 0 \) of the instanton does the symmetric phase as understood from a dimensional consideration.\(^{21}\)

**Fig.1 Table I**

Now we evaluate the Fourier transform of the classical Higgs configuration. The edge of the bounce is obtained by substituting the solution (3.5) at the edge back into \( \phi_H \) in (2.14). It is given by
\[ \phi_c^{(in, out)}(r/\rho) = -\frac{2\sqrt{2\pi}M_W}{g} \sin^2 \mu_b^{(in, out)} \frac{1 - h^2(r/\rho)}{1 + \sqrt{\cos^2 \mu_b^{(in, out)} + h^2(r/\rho) \sin^2 \mu_b^{(in, out)}}}, \quad (3.10) \]
where \( \mu_b^{(in, out)} \equiv \mu_b(\mp T/2) = \arccos(\pm \kappa) \). Here we have used the first of the gauge conditions (2.17). In calculating the Fourier profile, we approximate the denominator in
(3.10) by a constant $\langle D \rangle$ between 1 and 2 noting that this is a slowly varying function within this interval. Then we have an analytic expression;

$$\alpha^{(\text{in,out})}(k)/g = \alpha^{*(\text{in,out})}(k)/g \approx -\frac{2\pi}{g} \sin^2 \mu_{b}^{(\text{in,out})} M_{W} \rho^2 (|k|^2 + M_{H}^2)^{\frac{1}{2}} \left( \frac{\pi \rho |k|/2}{\tanh(\pi \rho |k|/2)} - 1 \right). \quad (3.11)$$

Equipped with the tunneling exponent $W(E)/g^2$ and the Fourier profile $\alpha(k)/g$'s of the classical configuration, we are ready to calculate the $S$-matrix element of $2 \rightarrow n$ Higgs production according to (2.5) and (2.7). In covariant normalization, it reads

$$\langle k_1, \cdots, k_n | \hat{S} | p_1, p_2 \rangle = (2\pi)^4 \delta^{(4)} \left( \sum_{i=1,2} p_i - \sum_{j=1}^{n} k_j \right)$$

$$\times e^{-W(E)/g^2} e^{-A^2(E)/g^2} \frac{\sqrt{(2\pi)^3 2p_1^0 \alpha^{(\text{in})}(p_1)}}{g^2} \frac{\sqrt{(2\pi)^3 2p_2^0 \alpha^{(\text{in})}(p_2)}}{g^2} \prod_{j=1}^{n} \frac{\sqrt{(2\pi)^3 2k_j^0 \alpha^{(\text{out})}(k_j)}}{g}, \quad (3.12)$$

where the normalization exponent of the coherent state is given by

$$\frac{A^2(E)}{g^2} = \frac{1}{g^2} \int_{-\infty}^{\infty} d^3 k \alpha^{(\text{in})}(k)^2 = \frac{1}{g^2} \int_{-\infty}^{\infty} d^3 k \alpha^{(\text{out})}(k)^2.$$

The cross section obtained from the above $S$-matrix element in center of momentum system at the incident energy $E$ is of the following form.

$$\sigma_{2 \rightarrow n} = X_0 X_n, \quad (3.13)$$

where

$$X_0 = \frac{(2\pi)^{10} E^2}{4E |p|} e^{2W(E)/g^2} e^{-2A^2(E)/g^2} \left( \frac{\alpha^{(\text{in})}(p)}{g} \right)^4 M_{W}^8,$$

$$X_n = \frac{1}{n!} \int \prod_{j=1}^{n} \frac{d^3 k_j^0}{(2\pi)^3 2k_j^0} \left( \frac{\sqrt{(2\pi)^3 2k_j^0 \alpha^{(\text{out})}(k_j)}}{g} \right)^2 \delta \left( \sum_{j=1}^{n} k_j^0 - E \right) \delta^{(3)} \left( \sum_{j=1}^{n} k_j \right). \quad (3.14)$$

with $|p| = \sqrt{(E/2)^2 - M_{H}^2}$ and $k_j^0 = \sqrt{k_j^2 + M_{H}^2}$. 
4. Numerical analysis—suppression versus enhancement

The $n$-independent part $X_0$ always suppresses $\sigma_{2\rightarrow n}$ as strongly as the instanton does. At low energies, the tunneling exponent $2W(E \simeq 0)/g^2 \simeq 2S_{\text{inst}}$ at $a = 0$ leads to this suppression as remarked before. At high energies with $|p| \simeq E/2$ and $E \simeq E_{\text{sph}}$, the tunneling suppression does not work while the incident profiles damp as

$$\left(\alpha^{(\text{in})}(p)\right)^4 \equiv \alpha^{(\text{4})}_{\text{(in)}}(p) \sim e^{-\pi p E_{\text{sph}}} = e^{-a(\pi E_{\text{sph}}/M_W)}.$$ (4.1)

Referring to Table I, we see that $(\pi E_{\text{sph}}/M_W)$ is comparable to or even larger than $2S_{\text{inst}}$. The physical reason for this high energy suppression is that the Fourier profile of the sphaleron as an extended object damps for large $|p|$ more rapidly than any power of $|p|$.

Hereafter we fix the gauge boson mass as $M_W = 80.6$ GeV and $SU(2)$ gauge coupling $g^2 = 0.418$. Then the sphaleron mass in Table I is $E_{\text{sph}} = 8.053$, $9.845$, $10.93$ and $11.95$ TeV for $M_H/M_W = 0$, $1.0$, $1.5$ and $2.0$ respectively. We also fixed $\langle D \rangle = 1.0$ in the denominator of (3.11), though this may somewhat over-estimate $\alpha^{(\text{in,out})}$.

The next task is to evaluate how the multiple of profiles and the phase space volume of the outgoing $n$ particles compete with the suppression factor. As $X_n$ has no axial symmetry, let us first imagine that the Higgs particles would be produced isotropically provided that $n$ is large enough, so that the angular part of the $k$ integration could be done. Then a single outgoing particle contributes as

$$\int d^3k \left(\frac{\alpha^{(\text{out})}(k)}{g}\right)^2 \rightarrow \int dk \frac{\alpha^{(\text{2})}_{\text{(out)}}(k)}{g^2} \quad \text{with} \quad \alpha^{(\text{2})}_{\text{(out)}}(k) = 4\pi k^2\left(\alpha^{(\text{out})}(k)\right)^2.$$ (4.2)

$\alpha^{(\text{2})}_{\text{(out)}}(k = 0) = 0$ independently of $\mu$ and $a$ while it damps rapidly as $e^{-\pi pk}$ for $k \gg M_W$, so that $\alpha^{(\text{2})}_{\text{(out)}}(k)$ always has a sharp peak at $kp \sim O(1)$ corresponding to the sphaleron size with the width of $O(1/M_W)$. An example is given in Fig.2. Let us denote this peak position as $k^*$ and the number of the corresponding produced particles as $n^* \simeq E/\sqrt{k^* + M_H^2}$. These together with the isotropic distribution would provide the saddle point of $X_n$ satisfying the energy-momentum conservation, so that we may make an approximation for $n \gg 1$;

$$X_n \rightarrow \bar{X}_n = \left(\frac{1}{n!}\right)(M_W \alpha^{(\text{2})}_{\text{(out)}}(k^*)/g^2)^n/M_W^4.$$ (4.3)

From this we obtain $\sigma_{2\rightarrow n}$ by (3.13) and $\sigma_{\text{tot}}$ by summing it to the maximum possible $n$. 


In order to have some notions how the enhancement by $X_n$ struggles against the suppression by $X_0$, we make a step further to obtain $\sigma_{\text{tot}}$ by exponentiation:

$$\sigma_{\text{tot}} = \sum_n \sigma_{2\to n} \sim \frac{1}{M_W^2} X_0 \times e^{M_W \alpha_{\text{out}}^{(2)} (k^*)/g^2}.$$  \hfill (4.4)

We should say that this would give an over estimate of the enhancement factor as $n$ may be rather limited around $n^*$.

**Fig.2**

Fig.3 shows the typical pattern of the sphaleron deformation. At the point $a = a_0$ that minimizes $E_{\text{sph}}$, $\sigma_{\text{tot}}(a_0)$ there does not give the maximum of $\sigma_{\text{tot}}$, but $\sigma_{\text{tot}}$ has two local maxima: one $\sigma_{\text{tot}}(a^{(+)}_0)$ at $a^{(+)}_0 > a_0$ and the other $\sigma_{\text{tot}}(a^{(-)}_0)$ at $a^{(-)}_0 < a_0$ respectively. The mechanism to give $\sigma_{\text{tot}}(a^{(\pm)})$ can be traced back to $\alpha^{(\text{in, out})} \propto a^2$ in (3.11). As $a$ increases from $a_0$, the barrier becomes higher and the sphaleron gets fatter indeed, but the $n$-multiple of $\alpha_{\text{out}}^{(2)} \propto a^4$ gives the larger effect to increase $\sigma_{\text{tot}}$. On the other hand, at $a^{(-)}_0 \ll 1$, all the effects coming from $\alpha^{(\text{in, out})}$ vanish while $2W(E)/g^2$ in this symmetric phase coincides with $2S_{\text{inst}}$, so that $\sigma_{\text{tot}}(a^{(-)}_0)$ is controlled as $(1/M_W^2) e^{-2S_{\text{inst}}}$ which is numerically larger than $\sigma_{\text{tot}}(a_0)$.

**Fig.3**

Fig.4 shows energy dependence of $\sigma_{\text{tot}}(a^{(\pm)})$ and $\sigma_{\text{tot}}(a_0)$. In the case of $M_H/M_W = 1.0$, $\sigma_{\text{tot}}(a^{(-)}_0)$ of the instanton suppression should turn into $\sigma_{\text{tot}}(a^{(+)})$ as $E$ increases. The reason why $\sigma_{\text{tot}}(a^{(+)})$ is smaller than $\sigma_{\text{tot}}(a^{(-)}_0)$ for $M_H/M_W > 1.2$ will be explained in the next section. Anyhow, $\sigma_{\text{tot}}$ are all lower than the unitarity bound $\sigma_{\text{unitarity}} = 16\pi/E^2$.

**Fig.4**

5. Concluding remarks

1. The structure of the $\Delta B \neq 0$ scattering is quite clear in our formalism based on path-integral expression of the time-evolution operator during tunneling at non-zero energies, which is dominated by the classical bounce solution giving the tunneling exponent that bridges from the instanton to the sphaleron. At low energies, the exponent gives the instanton suppression as usual. The Fourier profile $\alpha^{(\text{in, out})}(k)/g$ at the edges of the tunneling may correspond to the residue of Green’s function in the instanton background in the LSZ formalism, but the former damps rapidly for large momenta since the sphaleron is a classical lump of field with a finite size $\sim 1/M_W$. At high energies the overlap of
the incident wave with the entrance of tunneling leads to such a suppression comparable with that given by the instanton. Importance of overlap of the incident waves with the entrance of tunneling configuration at finite energies, which could keep $\sigma_{\text{tot}}$ to be very small, was also stressed by Banks et al.\textsuperscript{15)}

2. The multiple of the outgoing profiles together with the phase space volume indeed enhances $\sigma_{\text{tot}}$, but, in so far as the maximum number of produced particles is $n \sim 100$, the multiple may be too weak to overcome the suppression at high energies. We should mention that in the case of extremely large number of produced particles, the enhancement factors exceed the instanton-like suppression and do not prevent $\sigma_{\text{tot}}$ from a rapid increase. We show two examples.

(i) $\sigma_{\text{tot}}(a^{(+)})$ at $E = E_{\text{sph}}$ in Fig.4 sharply depends on the maximum number $n$ through $M_H/M_W$. If $M_H/M_W \sim 0.5$ fictitiously so that $n \sim 200$, an extrapolation of $\sigma_{\text{tot}}(a^{(+)})$ would surely reach $\sigma_{\text{unitarity}}$. For $M_H/M_W > 1.2$, in contrast, $n$ is too small to overcome the suppression factor.

(ii) In the $O(3)$ model in I, $\sigma_{\text{tot}} \sim 10^{-30}$ for $g^2 \sim 0.1 (n \sim 100)$ while $\sigma_{\text{tot}} > 1$ (unitarity bound in $1 + 1$ dimensions) for $g^2 < 0.022 (n > 362)$.

In other words, $g^2 = 0.418$ of the $SU(2)$ gauge coupling constant may be too large to take the weak coupling limit, or, realistic $n \sim 100$ may be dangerous to make an easy exponentiation etc.

One might claim that any $\sigma_{\text{tot}}$ should reproduce the one by the instanton calculus at low energies,\textsuperscript{16)} given by\textsuperscript{7)} $\sigma_{\text{tot}} \sim \exp \left[ \left( 16\pi^2/g^2 \right) \left[ -1 + \text{const.} \times (E/E_{\text{sph}})^2 \right] \right]$. We feel, however, that the weak coupling approximation used there would be inadequate as just mentioned.

3. The strong dependence of $\sigma_{\text{tot}}$ on $n$ may be related to how drastically a severe cancellation takes place to give $\sigma_{\text{tot}}$ at high energies as shown in Fig. 3. Because of the severe and delicate cancellation together with possible corrections in the next-to-leading order,\textsuperscript{*} we do not assert here that $\sigma_{\text{tot}}$ with $\Delta B \neq 0$ is actually extremely small as shown in Fig.4.\textsuperscript{†} It seems unlikely, however, that it grows as largely as to reach the unitarity bound at energies $E \leq E_{\text{sph}}$. An exponential suppression of $\sigma_{\text{tot}}$ at high energies was strongly sugested by Shaposhnikov.\textsuperscript{14)}

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\* We might expect that the corrections would not be significant in our cases, since the leading order result itself is much smaller than the unitarity bound.

\† We have also to take the gauge boson production into account, as the sphaleron would decay dominantly into them\textsuperscript{23)}.
4. We finally repeat a remark in I that unitarity is a subtle issue in the $\Delta B \neq 0$ high energy scattering. In any formalism the scattering is considered to be a transition between states each constructed on minima of classical energy functional space and these (quasi-stationary) states belong to the eigenstate of $B$. But the hamiltonian here does not commute with $B$ because of the chiral anomaly. Hence the asymptotic states, which belong to the eigenstate of hamiltonian in usual scattering problem, are not the eigenstate of the hamiltonian. So the $\Delta B \neq 0$ process corresponds to off-diagonal $S$-matrix elements. We have tacitly assumed that these elements are small compared with the diagonal ones, which would be consistent with our $\sigma_{\text{tot}}$ far below the unitarity bound.

If $\sigma_{\text{tot}}$ with $\Delta B \neq 0$ were to reach the unitarity bound, one can not rely on any formalism available at present but has to develop a basically new formalism to treat the $\Delta B \neq 0$ scattering.

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Appendix: Gauge fixing conditions in $R_\xi$ gauge

After performing the BRS transformation, the original gauge fixing term (2.12) in the text is expressed as

$$L_{R_\xi} = -\partial_\mu B^a A^{a\mu} + \alpha g M_W B^a \chi^a + \frac{\alpha g}{2} B^a B^a - i\partial_\mu \bar{C}^a D^\mu C^a + i\bar{C}^a [\alpha g M_W^2 \delta^{ac} + \frac{\alpha g}{2} g M_W (\phi \delta^{ac} + \epsilon_{abc} \chi^b)] C^c. \quad \text{(A.1)}$$

Let us introduce the spherical symmetric ansatz following Ratra and Yaffe;\textsuperscript{20)}

$$A_0 \equiv A_0^a \tau^a \equiv \frac{1}{2g} a_0(t, r) \tau^a \hat{x}^a,$$

$$A^j \equiv A^j a^{\tau a} \tau^a \equiv \frac{1}{2g} \left[ \frac{\alpha(t, r)}{r} \tau^j(1) + \frac{1 + \beta(t, r)}{r} \tau^j(2) + a_1(t, r) \tau^j(3) \right], \quad \text{(A.2)}$$

$$\Phi = \frac{1}{g} \left( \sigma(t, r) + i\eta(t, r) \tau^a \hat{x}^a \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with

$$\tau^j(1) \equiv \tau^j - \tau^a \hat{x}^a \hat{x}^j, \quad \tau^j(2) \equiv i(\tau^a \hat{x}^a \tau^j - \hat{x}^j) \quad \text{and} \quad \tau^j(3) \equiv \tau^a \hat{x}^a \hat{x}^j.$$
After dealing with the subsidiary fields in the standard way, $\mathcal{L}_{R_\xi}$ is written as

$$\mathcal{L}_{R_\xi} = \frac{v}{\sqrt{2g}} \left( a_1 \eta' + 2 \frac{\alpha \eta}{r^2} \right) - \alpha g \frac{v^2}{4} \eta^2 - \frac{1}{\alpha g^2} Y^2; \quad (A.3)$$

where $Y$ is given by

$$Y \equiv \dot{a}_0 - a_1' + \frac{2}{r^2} \alpha - \frac{2}{r} a_1. \quad (A.4)$$

We put $\mathcal{L}_{R_\xi} = 0$, which leads to two gauge fixing conditions,

$$\eta(t, r) = 0 \text{ and } Y(t, r) = 0. \quad (A.5)$$

They are rewritten in terms of $\mu(t), f(r), h(r)$ and $K(r)$ as (2.17) in the text. Note that $\eta(t, r) = 0$ implies vanishing of the imaginary part of $\phi_H$ in (2.14).

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Table I. The minimized $E_{sph}(a_0)$ by (3.9). $r$ denotes the ratio of $E_{sph}(a_0)$ to $E_{sph}^Y$ obtained by minimizing the energy functional.

| $M_H/M_W$ | $E_{sph}/(M_W/g^2)$ | $a_0$ | $r$ |
|-----------|----------------------|------|-----|
| 0.0       | 41.76                | 1.742| 1.09|
| 1.0       | 51.06                | 1.150| 1.12|
| 1.5       | 56.71                | 0.938| 1.18|
| 2.0       | 62.00                | 0.874| 1.25|
Figure Captions

**Fig.1.** $E_{\text{sph}}(a)$ vs. $a$. The minimized point $a = a_0$ is shown by the arrow.

**Fig.2.** $\alpha^{(2)}_{(\text{out})}(k)$ vs. $k$ which is peaked at $k^*$ (denoted by the dotted line) for $M_H/M_W = 1.0$ and $a = a_0 = 1.150$, where $E_{\text{sph}} = 9.845$ TeV.

**Fig.3.** $\ln\sigma_{\text{tot}}$ (solid curve) by (4.4) vs. $a$ for $M_H/M_W = 1.0$ and at $E=9.5$ TeV. The relevant sphaleron mass is $E_{\text{sph}}(a_0) = 9.845$ TeV. Note that $\ln \sigma_{\text{tot}}$ is given by the difference between two large quantities, $\ln(M_W^2 X_0)$ and $M_W\alpha^{(2)}_{(\text{out})}(k^*)/g^2$ (the both denoted by broken curves). Three quantities contributing to the former (dotted curves) are shown. The corresponding $k^*$ and $n^*$ explained in the text are also shown.

**Fig.4.** $\sigma_{\text{tot}}(a)$ obtained by making use of (4.3). $\sigma_{\text{tot}}(a_0)$ (dotted curve) is $\sigma_{\text{tot}}$ in the case when the the sphaleron does not deform. $\sigma_{\text{tot}}(a^{(+)}_0)$ (solid curve) and $\sigma_{\text{tot}}(a^{(-)}_0)$ (broken curve), the latter being almost independent of $M_H/M_W$, are the local maxima. These quantities are plotted vs. $E$ for $0 \leq E \leq E_{\text{sph}}$, $E_{\text{sph}}$ being given in Table I.