Quantiles as minimizers

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Abstract

A real random variable admits median(s) and quantiles. These values minimize convex functions on $\mathbb{R}$. We show by “Convex Analysis” arguments that the function to be minimized is very natural. The relationship with some notions about functions of bounded variation developed by J.J. Moreau is emphasized.

1 Introduction

Let $P$ a probability law on $\mathbb{R}$. Assume that $P$ is of order 1 (which writes $\int |x| dP(x) < +\infty$). A real number $\bar{m}$ is a median if and only if it minimizes the function of $m$, $\int_{\mathbb{R}} |x - m| dP(x)$. Without the order 1 hypothesis, the medians minimize

$$m \mapsto \Phi(m) := \int_{\mathbb{R}} (|x - m| - |x|) dP(x).$$

(This is not exactly, when $\tau = 1/2$ — the value of $\tau$ corresponding to medians —, the function defined in (6), because of a factor 2.) This extends to quantiles. This paper gives “Convex Analysis” proofs of these results. With our process the function $\Phi$ appears naturally. We emphasize the links with some notions about functions of bounded variation developed by J.J. Moreau. I began this paper being unaware of Koltchinskii [Kol]. This 1997 work is more devoted to the multivariate case. See some other comments in Section 4.

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2 Definitions and notations

Let $P$ be a probability law on $\mathbb{R}$. If necessary $X$ will denote a random variable obeying the law $P$. The law $P$ is of order $p$ ($p \in \{1, 2\}$) if $X$ is of order $p$, which writes $X \in L^p$. If no integrability condition on $X$ is assumed, specialists write $X \in L^0$ (order 0). When $P$ is of order 1 the mean $\mathbb{E}(X)$ does exist. When $P$ of order 0 there exists a non-empty compact interval of medians (the definition of a median is included in that of a quantile below). Let $F$ be the right-continuous distribution function:

$$F(x) := P([-\infty, x]) = P(X \leq x).$$

For $\tau \in ]0, 1[$ (usually in Statistics 0.95 or 0.99 etc.) a real number $\bar{q}$ is a $\tau$-quantile if

$$P(X \leq \bar{q}) \geq \tau \quad \text{and} \quad P(X \geq \bar{q}) \geq 1 - \tau$$

or maybe more plainly

$$P(X < \bar{q}) \leq \tau \leq P(X \leq \bar{q})$$

(note that $x \mapsto P(X < x)$ is the left-continuous version $F^-$ of $F$). When $\tau = 1/2$ one recovers median(s). A geometrical definition of $\tau$-quantiles is the following: their set is the projection on $\mathbb{R}$ of the intersection of $\mathbb{R} \times \{\tau\}$ with the filled-in graph\footnote{This graph is obtained from the graph of $F$ by adding vertical segments when there are gaps, thus obtaining an arcwise connected curve:}

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} ; y \in [F^-(x), F(x)]\}.$$

This comes from Monteiro Marques and Moreau works on the sweeping process: [MM1, p.147], [MM2, p.15], some ideas coming back to [Mor2]. In Section 4 the filled-in graph will be seen as a maximal monotone operator.
3 Medians and quantiles as minimizers

Well known\(^2\) is the result: If \(P\) is of order 1, \(m\) is a median if and only if it minimizes on \(\mathbb{R}\) the function

\[
m \mapsto \int_{\mathbb{R}} |x - m| \, dP(x).
\]

Surprising are: the assumption about order 1, and the fact that medians depend only on the structure of ordered space of \(\mathbb{R}\) and not on its metric (nor on its group structure, nor on Haar measure). An “answer”, at least relatively to the “order 1” assumption, is the following.

Well known too\(^3\) is the result: If \(P\) is any law, \(x \mapsto |x - m| - |x|\) is \(P\)-integrable (obviously \(|x - m| - |x| \leq |m|\)) and \(m\) is a median if and only if it minimizes on \(\mathbb{R}\) the function

\[
m \mapsto \int_{\mathbb{R}} (|x - m| - |x|) \, dP(x).
\]

The notion of median extends to \(\mathbb{R}^d\) and to Banach spaces: see [Kem, Kol, MD] and there exist conditional medians [V3]; Kemperman [Kem] and Milasevic & Ducharme [MD] gave in the multivariate case a sufficient condition implying uniqueness of the median (see already in 1948 Haldane [Ha]). These questions will not be considered here.

The interest of medians comes from robustness, i.e., stability with respect to outliers values. Maybe the notions of means in metric spaces going back to Fréchet (see [Fr] and many other papers by the same author) should be revisited. See the papers by Armatte [A1, A2], the first one containing more than six pages of Fréchet’s references. Numerous authors studied random variables with values in a metric space: for instance [BH, RF].

Now we turn to quantiles. T.S. Ferguson [Fe1, Exercise 1.8.3 p.51, solution in [Fe2]] says that\(^4\): If \(\tau \in ]0, 1[\) and \(P\) is of order 1, \(\bar{q}\) is a \(\tau\)-quantile if and only if it minimizes on \(\mathbb{R}\) the function

\[
q \mapsto \int_{\mathbb{R}} \rho_\tau(x - q) \, dP(x).
\]

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\(^2\) In French textbooks: [DCD1, Exercice E.11 p.71] (solution page 28 of [DCD2]), and Théorème 10.1 page 93 in [FF]. See also [Fe1, ex.1.8.2 p.51] and its solution in [Fe2].

\(^3\) Exercises in French Universities and surely in many countries...

\(^4\) This result is appreciated by specialists: [Koe, first chapter] and already [KB, p.38].
where

\begin{equation}
\rho_\tau(x) = \frac{1}{2} |x| + \left( \tau - \frac{1}{2} \right) x = \begin{cases} 
(\tau - 1) x & \text{if } x \leq 0, \\
\tau x & \text{if } x \geq 0. 
\end{cases}
\end{equation}

My purpose here is to give a Convex Analysis proof of the extension to any probability law and to follow a natural way. Surely Koltchinskii [Kol] contains the statement but his framework is multivariate.

4 A Convex Analysis point of view

The filled-in graph \( G \) defined in Section 2 is a subset of \( \mathbb{R}^2 \) which is a \textit{maximal monotone operator} \([\text{RW, Chapter 12}]\) (and therefore a maximal cyclically monotone one because of dimension 1, cf. [\text{RW, 12.6 pp.547–548}]). Let \( F \) be the primitive (antiderivative) of \( F \) null at 0:

\[ F(x) = \begin{cases} 
\int_0^x F(u) \, du & \text{if } x \geq 0, \\
-\int_x^0 F(u) \, du & \text{if } x < 0. 
\end{cases} \]

Several textbooks treats convexity. One of the most fundamental is \([\text{Mor1}]\). Since \( F \) is nondecreasing \( F \) is convex. The \textit{sub-derivative} of \( F \) at \( x \) is

\[ \partial F(x) = \{ \ell \in \mathbb{R} ; \forall h \in \mathbb{R}, \ell h \leq F(x+h) \} . \]

Lemma 1 The graph of the multifunction \( x \mapsto \partial F(x) \) is nothing else but the filled-in graph \( G \) of the graph of \( F \):

\[ G = \{ (x, \ell) \in \mathbb{R}^2 ; \ell \in \partial F(x) \} . \]

Proof. Let \( F'(x; w) \) denote the directional derivative of \( F \) at \( x \) in the direction \( w \). There holds

\[ \partial F(x) = [-F'(x; -1), F'(x; 1)] . \]

Then, since \( F \) is right continuous,

\[ F'(x; 1) = \lim_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \downarrow 0} \frac{1}{h} \int_x^{x+h} F(u) \, du = F(x) . \]
and, since $F$ is as well the primitive of $F^{-}$,

$$
F'(x; -1) = \lim_{h \downarrow 0} \frac{F(x - h) - F(x)}{h} = -\lim_{h \downarrow 0} \frac{1}{h} \int_{x-h}^{x} F^{-}(u) \, du = -F^{-}(x).
$$

Therefore

$$
\partial F(x) = [F^{-}(x), F(x)]. \quad \square
$$

In such a situation a natural question is: what does give the minimisation of $F$? Obviously the infimum may be $-\infty$: this is why we cannot consider

$$
\int_{-\infty}^{x} F(u) \, du.
$$

But with a slope $\tau$ belonging to $]0,1[$ the function

$$
q \mapsto F(q) - \tau q
$$

does achieves minimum(s) and $\bar{q}$ is a minimum if and only if $0 \in -\tau + \partial F(\bar{q})$. This is equivalent to $(\bar{q}, \tau) \in G$, that is, $\bar{q}$ is a $\tau$-quantile.

**Scholium** Let $F$ denote a primitive of $F$. For $\tau \in ]0,1[,$

(2) $\bar{q}$ is a $\tau$-quantile $\iff \bar{q}$ minimizes $q \mapsto F(q) - \tau q$.

Next Lemma is an integration by parts result. We follow a proof by Schilling but we could deduce the statement from results by Rockafellar [Ro, Prop.1 pp.161–162] or Moreau [Mor3, Section 11]. This will be detailed elsewhere.

**Lemma 2** Let $a < b$ in $\mathbb{R}$. Then (note the half-open interval $]a,b]$)

(3)

$$
\int_{]a,b[} x \, dP(x) = b F(b) - a F(a) - \int_{a}^{b} F(x) \, dx.
$$

**Proof** (from Schilling [S1, Exercise 13.13 p.133] and its solution on the Net [S2, pp.10–12]). For the product of the Lebesgue measure (we denote it by $dx$) and of $P$

(4)

$$
(dx \otimes dP)(]a,b[) = (b - a) [F(b) - F(a)].
$$
And moreover

\[(dx \otimes d\mathcal{P})([a, b]^2) = \]
\[= \iint 1_{[a, b]}(x) 1_{[x, b]}(y) \, dx \, d\mathcal{P}(y) + \iint 1_{[a, b]}(x) 1_{[a, x]}(y) \, dx \, d\mathcal{P}(y) \]
\[= \iint 1_{[a, b]}(x) 1_{[x, b]}(y) \, dx \, d\mathcal{P}(y) + \iint 1_{[a, b]}(y) 1_{[y, b]}(x) \, dx \, d\mathcal{P}(y) \]
\[= \iint [F(b) - F(x)] \, dx + \int_{[a, b]} (b - y) \, d\mathcal{P}(y) \]
\[(5) \quad = (b - a) F(b) - \int_a^b F(x) \, dx + b [F(b) - F(a)] - \int_{[a, b]} y \, d\mathcal{P}(y). \]

Comparing (4) and (5) one gets (3). □

Now let 5 (\rho_\tau \text{ is defined in (1) above})

\[\Phi(q) := \int_{\mathbb{R}} (\rho_\tau(x - q) - \rho_\tau(x)) \, d\mathcal{P}(x) \]

(note that the function \(x \mapsto \rho_\tau(x - q) - \rho_\tau(x)\) is bounded on \(\mathbb{R}\)).

**Theorem 1** Let \(\tau \in ]0, 1[\) and \(\mathcal{P}\) be a law on \(\mathbb{R}\). There holds

\[\Phi(q) = \begin{cases} 
- \tau q + \int_{[0, q]} F(x) \, dx & \text{if } q \geq 0, \\
- \tau q - \int_{[q, 0]} F(x) \, dx & \text{if } q < 0,
\end{cases} \]

that is, \(\Phi(q) = F(q) - \tau q\). The function \(\Phi\) is convex and inf-compact. The value \(\bar{q}\) is a \(\tau\)-quantile if and only if it minimizes the function \(\Phi\).

**Comments.** Part 3) below is a bit technical. Surely careless calculus would be quicker: the formal calculus

\[\Phi'(q) = \int_{\mathbb{R}} \frac{d}{dq} (\rho_\tau(x - q) - \rho_\tau(x)) \, d\mathcal{P}(x) \]
\[= \int_{-\infty}^q (1 - \tau) \, d\mathcal{P} + \int_{q}^{+\infty} (-\tau) \, d\mathcal{P} \]
\[= (1 - \tau) F(q) - \tau (1 - F(q)) \]
\[= F(q) - \tau \]

5 Koltchinskii [Kol] considers the same functional. See specially two lines after his formula (1.1) page 436 where he sets \(f_{P, t}(s) := f_P(s) - s t\): there \(s\) is my variable \(q\) and \(t\) my threshold \(\tau\). Comparisons are not easy. He also uses (on pp.439–440) a multivalued integration result [IT, Th.8.3.4]. I got similar results earlier in [V1, V2] but Ioffe and Tihomirov published also some papers on these questions before their book.
leads to
\[ \Phi'(q) = 0 \iff F(q) = \tau ! \]

**Proof.** 1) Firstly \( x \mapsto \rho_\tau(x - q) - \rho_\tau(x) \) is less than \( \tau |q| \) if \( \tau \geq 1/2 \) or than \((1 - \tau) |q| \) if \( \tau \leq 1/2 \). Or more roughly \( \forall x, |\rho_\tau(x - q) - \rho_\tau(x)| \leq |q| \). So the integral is well defined.

2) The function \( \rho_\tau \) defined by (1) being convex, the convexity of \( q \mapsto \rho_\tau(x - q) - \rho_\tau(x) \) is trivial, hence the convexity of \( \Phi \).

3) We now prove \( \Phi(q) \to +\infty \) as \( |q| \to +\infty \).

For \( q \geq 0 \),
\[
\rho_\tau(x - q) - \rho_\tau(x) = \begin{cases} 
(1 - \tau) q & \text{if } x \in \left[ -\infty, 0 \right], \\
(1 - \tau) q - x & \text{if } x \in \left] 0, q \right], \\
-\tau q & \text{if } x \in \left] q, +\infty \right[.
\end{cases}
\]

and, for \( q < 0 \),
\[
\rho_\tau(x - q) - \rho_\tau(x) = \begin{cases} 
(1 - \tau) q & \text{if } x \in \left] -\infty, q \right[, \\
x - \tau q & \text{if } x \in \left] q, 0 \right[, \\
-\tau q & \text{if } x \in \left] 0, +\infty \right[.
\end{cases}
\]

Hence, if \( q \geq 0 \) (we will apply formula (3) between lines 2 and 3),
\[
\Phi(q) = (1 - \tau) q \mathbb{P}(-\infty,0] + \int_{[0,q]} ((1 - \tau) q - x) d\mathbb{P}(x) - \tau q \mathbb{P}([q, +\infty])
\]
\[
= (1 - \tau) q \mathbb{P}(-\infty,q] - \int_{[0,q]} x d\mathbb{P}(x) - \tau q \mathbb{P}([q, +\infty])
\]
\[
= (1 - \tau) q \mathbb{P}(-\infty,q] - [x F(x)]_0^q + \int_{[0,q]} x F(x) dx - \tau q \mathbb{P}([q, +\infty])
\]
\[
= (1 - \tau) q F(q) - q F(q) + \int_{[0,q]} x F(x) dx - \tau q (1 - F(q))
\]
\[
(7) \quad = -\tau q + \int_{[0,q]} F(x) dx \cdot
\]

When \( q \to +\infty \) the mean value of \( F \) over \( [0,q] \) ultimately exceeds any value in \( \tau, 1 \), so \( \Phi(q) \to +\infty \). (More plainly for \( q > \tau \), \( \Phi(q) = \int_{[0,q]} (F(x) - \tau) dx + \int_{[\tau,q]} (F(x) - \tau) dx \\ and \( F(x) - \tau \to 1 - \tau > 0 \) as \( x \to \infty \).)
On the other hand, if \( q < 0 \), (we again apply (3) between lines 2 and 3),

\[
\Phi(q) = (1 - \tau) q P(-\infty, q) + \int_{[q,0]} \left( x - \tau q \right) dP(x) - \tau q P([0, +\infty])
\]

\[
= (1 - \tau) q P(-\infty, q) + \int_{[q,0]} x dP(x) - \tau q P(q, +\infty]
\]

\[
= (1 - \tau) q P(-\infty, q) + \left[ x F(x)q - \int_{[q,0]} x F(x) dx - \tau q P([q, +\infty]) \right]
\]

\[
= (1 - \tau) q F(q) - q F(q) - \int_{[q,0]} x F(x) dx - \tau q (1 - F(q))
\]

\[
= -\tau q - \int_{[q,0]} F(x) dx.
\]

When \( q \to -\infty \) the mean value of \( F \) over \([q,0]\) ultimately passes under any value in \([0, \tau]\), so \( \Phi(q) \to +\infty \).

Thus the finite valued convex function \( \Phi \) is inf-compact, so it achieves its infimum over a non-empty compact interval. By (7) and (8)

\[
\Phi(q) = F(q) - \tau q.
\]

From (2) the minimizers are the \( \tau \)-quantiles. □

**Corollary 1** Let \( \tau \in ]0, 1[ \) and \( P \) be a first order law on \( \mathbb{R} \) then

\( \bar{q} \) is a \( \tau \)-quantile \iff \( \bar{q} \) minimizes \( q \mapsto \int_{\mathbb{R}} \rho_{\tau}(x - q) dP(x) \).

**Proof.** The term

\[
\int_{\mathbb{R}} \rho_{\tau}(x) dP(x)
\]

is finite and does not depend on \( q \). Hence one can add it to the right-hand side of (6). □

## 5 A direct proof

Classically the convex function \( \Phi \) on \( \mathbb{R} \) achieves a minimum at \( \bar{q} \) if and only if the left and right derivatives of \( \Phi \) at \( \bar{q} \) are respectively \( \leq 0 \) and \( \geq 0 \).

**Theorem 2** Let \( \tau \in ]0, 1[ \) and \( P \) be a law on \( \mathbb{R} \). The right derivative of \( \Phi \) at \( \bar{q} \) is \( \geq 0 \) if and only if \( P([\bar{q}, +\infty)) \) is \( \geq \tau \). The derivative of \( \Phi \) at \( \bar{q} \) in direction \( -1 \) is \( \geq 0 \) if and only if \( P([-\infty, \bar{q}]) \) is \( \leq \tau \).
Proof. We have to reformulate the inequalities

\[
\Phi'(\bar{q}; 1) = \lim_{h \searrow 0} \frac{\Phi(\bar{q} + h) - \Phi(\bar{q})}{h} \geq 0
\]

and

\[
\Phi'(\bar{q}; -1) = \lim_{h \searrow 0} \frac{\Phi(\bar{q} - h) - \Phi(\bar{q})}{h} \geq 0
\]

the second one expressing the positiveness of the directional derivative at \( \bar{q} \) in the direction \(-1 \) (another way to say that the left derivative is \( \leq 0 \)).

a) Firstly let us consider the right derivative at \( \bar{q} \). One has

\[
\frac{\Phi(\bar{q} + h) - \Phi(\bar{q})}{h} = \frac{1}{h} \int_{\mathbb{R}} \left( \rho_r(x - \bar{q} - h) - \rho_r(x - \bar{q}) \right) dP(x).
\]

The integral splits into

\[
\int_{]-\infty, \bar{q}]} + \int_{[\bar{q}, \bar{q} + h[} + \int_{[\bar{q} + h, +\infty[}
\]

where the (non written) integrands are respectively \((1 - \tau)\), a function bounded above by \(h\) (on a vanishing domain\(^6\)) and \(-\tau h\) (on a domain which converges to \([\bar{q}, +\infty[\)). Hence

\[
\lim_{h \searrow 0} \frac{\Phi(\bar{q} + h) - \Phi(\bar{q})}{h} = (1 - \tau) \mathbf{P}(]-\infty, \bar{q}]) + (-\tau) \mathbf{P}([\bar{q}, +\infty[)
\]

\[
= \mathbf{P}(]-\infty, \bar{q}]) - \tau.
\]

Thus the right derivative is \( \geq 0 \) if and only if \( \mathbf{P}(]-\infty, \bar{q}]) \) is \( \geq \tau \).

b) We turn now\(^7\) to the directional derivative in the direction \(-1\). One has

\[
\frac{\Phi(\bar{q} - h) - \Phi(\bar{q})}{h} = \frac{1}{h} \int_{\mathbb{R}} \left( \rho_r(x - \bar{q} + h) - \rho_r(x - \bar{q}) \right) dP(x).
\]

The integral splits into

\[
\int_{]-\infty, \bar{q} - h[} + \int_{[\bar{q} - h, \bar{q}[} + \int_{[\bar{q}, +\infty[}
\]

\(^6\) The open intervals \([\bar{q}, \bar{q} + h[\) decrease when \(h \searrow 0\) and have empty intersection. In order to apply convergence theorems of Integration Theory one should consider — and this is sufficient — a sequence \((h_n)_{n \in \mathbb{N}}\) satisfying \(h_n \searrow 0\).

\(^7\) Minus signs are always more perilous.
where the integrands are respectively $-(1 - \tau) h$ (over a domain which converges to $]-\infty, \bar{q}[)$, a function bounded above by $h$ (the domain being vanishing) and $\tau h$. Whence
\[
\lim_{h \to 0^+} \frac{\Phi(\bar{q} - h) - \Phi(\bar{q})}{h} = -(1 - \tau) P[-\infty, \bar{q}[) + \tau P([\bar{q}, +\infty[)
\]
\[
= -P[-\infty, \bar{q}[) + \tau .
\]
Thus the directional derivative in the direction $-1$ is $\geq 0$ if and only if $P[-\infty, \bar{q}[)$ is $\leq \tau$. □

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