A DEGREE CONDITION FOR DIAMETER TWO ORIENTABILITY OF GRAPHS

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Abstract. For \( n \in \mathbb{N} \) let \( \delta_n \) be the smallest value such that every graph of order \( n \) and minimum degree at least \( \delta_n \) admits an orientation of diameter two. We show that \( \delta_n = \frac{n}{2} + \Theta(\ln n) \).

1. Introduction

Let \( G \) be a connected graph. An orientation of \( G \) is a digraph obtained from \( G \) by assigning a direction to every edge of \( G \). The diameter of strong digraph, i.e., of a digraph in which there is a directed path between any two vertices, is the maximum distance between any two vertices. The oriented diameter \( \overrightarrow{\text{diam}}(G) \) of \( G \) is the minimum diameter among all strong orientations of \( G \). A bridge is an edge of a connected graph whose removal renders the graph disconnected.

The well-known Robbin’s Theorem [18] states that every bridgeless connected graph has a strong orientation. For many applications it is desirable to have a strong orientation whose diameter is as small as possible. The natural question if bridgeless graphs of small diameter necessarily have orientations of small diameter was answered in the affirmative by Chvátal and Thomassen [3], who showed the existence of a (least) function \( f : \mathbb{N} \to \mathbb{N} \) such that every bridgeless graph of diameter \( d \) has an orientation of diameter at most \( f(d) \). Chvátal and Thomassen [3] showed that \( f(2) = 6 \), and that \( \frac{1}{4}d^2 + d \leq f(d) \leq 2d^2 + 2d \) for all \( d \in \mathbb{N} \). The value of \( f(3) \) is not known, but Egawa and Iida [5], and independently Kwok, Liu and West [17] gave the estimate \( 9 \leq f(3) \leq 11 \).

Upper bounds on \( \overrightarrow{\text{diam}}(G) \) are known in terms of other graph parameters, such as minimum degree [1, 19], maximum degree [4], and domination parameters [16, 7]. The oriented diameter has also been investigated for graphs from various special graph classes [6, 7, 8, 11].
For a survey of results on the diameter of orientations of graphs up to about 2000 see [15].

Of special interest is the question which graphs have an orientation of diameter two, the smallest possible diameter of an orientation of any graph on more than one vertex. It was shown by Chvátal and Thomassen [3] that the decision problem if a given input graph has an orientation of diameter two is NP-complete. Hence sufficient conditions for the existence of an orientation of diameter two are of interest. The smallest value \( m_n \), so that every graph of order \( n \) and size at least \( m_n \) admits an orientation of diameter two was conjectured in [15] to be \( \binom{n}{2} - n + 5 \) and this conjecture was proved in [2]. In this paper we present a sufficient condition for an orientation of diameter two in terms of minimum degree. For \( n \in \mathbb{N}, n \geq 3 \), define \( \delta_n \) to be the smallest value such that every graph of order \( n \) and minimum degree at least \( g(n) \) admits an orientation of diameter two. In view of the fact that for a graph \( G \) of order \( n \) we need the minimum degree to be at least \( \frac{n+1}{2} \) for \( G \) to guarantee that the diameter is not more than two, it is clear that \( \delta_n \geq \frac{n-1}{2} \). It is the aim of this paper to show that

\[
\delta_n = \frac{n}{2} + \Theta(\ln n).
\]

The notation we use in this paper is as follows. \( G \) always denotes a connected graph on \( n \) vertices. The neighborhood of a vertex \( v \) of \( G \), i.e., the set of all vertices adjacent to \( v \) is denoted by \( N_G(x) \), and the degree of \( v \) is \( |N_G(v)| \). In a digraph \( D \), \( N_D^+(v) \) is the set of vertices reached from \( v \) by an arc, and \( N_D^-(v) \) is the set of vertices that reach \( v \) by an arc. The minimum degree \( \delta(G) \) of \( G \) is the smallest of the degrees of the vertices of \( G \). The distance from a vertex \( u \) to a vertex \( v \), i.e., the minimum length of a \((u, v)\)-path in a graph or digraph is denoted by \( d(u, v) \). By a 2-path we mean a path of length two. In a given probability space we write \( \mathbb{P}[X] \) for the probability of an event \( X \), and \( \mu \) for the expected value of a random variable.

2. Results

Theorem 1. Let \( G \) be a graph on \( n \) vertices. If

\[
\delta(G) \geq \frac{n}{2} + \frac{\ln n}{\ln(4/3)} = \frac{1}{2}n + (3.476...) \ln n,
\]

then \( \overrightarrow{\text{diam}}(G) = 2 \).

Proof. Assume that the minimum degree of the graph \( G \) is at least \( n/2 + f(n) \), where \( f(n) = \frac{1}{\ln(4/3)} \ln n \). Fix two arbitrary vertices of \( G \), \( x \) and \( y \). By the inclusion-exclusion formula, \( |N_G(x) \cap N_G(y)| = |N_G(x)| + |N_G(y)| - |N_G(x) \cup N_G(y)| \geq n + 2f(n) - n = 2f(n) \).

Assign one of the two possible orientations to every edge of \( G \) randomly and independently with probability 1/2. Define the random
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variable $X_{xy}$ to be 1 if we have no directed 2-path from $x$ to $y$, and 0 if there is such a path. Clearly, the expected value of $X_{xy}$ is $\mathbb{P}[X_{xy} = 1] = (\frac{3}{4})^{\left|N(x)\cap N(y)\right|} \leq (\frac{3}{4})^{2f(n)}$. Therefore, for the expected number $\mu$ of ordered pairs $x, y$ with no directed 2-paths from $x$ to $y$, we have

\begin{equation}
\mu = \sum_{x,y \in V(G), x \neq y} n(n-1) \left(\frac{3}{4}\right)^{2f(n)} \leq n^2 \left(\frac{3}{4}\right)^{2f(n)}.
\end{equation}

Let $X = \sum_{x,y \in V(G), x \neq y} X_{xy}$ be the number of distinct ordered pairs of vertices that are not joined by a directed 2-path. Then $X = 0$ means that a random orientation has diameter two. If $f(n)$ is selected so large that

$$n^2 \left(\frac{3}{4}\right)^{2f(n)} \leq 1,$$

then if follows from (1) that $\mu < 1$, and so at least one orientation satisfies $X = 0$, implying that $\text{diam}(G) = 2$. Easy calculations shows that this is indeed the case if $f(n) \geq \frac{1}{\ln(4/3)} \ln n$. \hfill \Box

**Theorem 2.** There exists an infinite family of graphs $G$ of order $n$ and

$$\delta(G) \geq \frac{n}{2} + \frac{\ln n}{2 \ln(27/4)} = \frac{n}{2} + (0.2618...) \ln n.$$  

which do not have an orientation of diameter 2.

**Proof.** For $k \in \mathbb{N}$ we construct the graph $G_k$ as follows. Let $N = (\binom{3k}{k})$. Let $H_1$, $H_2$ and $H_3$ be disjoint copies of the complete graph $K_N$, $K_{3k}$ and $K_N$, respectively. Let $G_k$ be obtained from the disjoint union of $H_1$, $H_2$ and $H_3$ by associating each 2k-subset $S$ of $V(H_2)$ with exactly one vertex $v_S$ of $H_1$ and exactly one vertex $w_S$ of $H_3$, and adding edges joining $v_S$ and $w_S$ to every vertex in $S$.

We now prove that $\text{diam}(G_k) \geq 3$. Let $D_k$ be an arbitrary orientation of $G_k$. Fix a vertex $v \in V(H_1)$. Then $|N^+ (v) \cap V(H_2)| + |N^- (v) \cap V(H_2)| = 2k$, and so $|N^+ (v) \cap V(H_2)| \leq k$ or $|N^- (v) \cap V(H_2)| \leq k$. Without loss of generality we assume $|N^+ (v) \cap V(H_2)| \leq k$. Then $|V(H_2) - (N^+ (v) \cap V(H_2))| \geq 2k$, and so there exists a set $S \subseteq V(H_2) - (N^+ (v) \cap V(H_2))$ with $|S| = 2k$. Consider vertex $w_S$. We claim that

\begin{equation}
d_{D_k}(v, w_S) \geq 3.
\end{equation}

Indeed, $w_S$ and $v$ are non-adjacent in $G_k$ and no vertex in $N_{G_k}(v) \cap N_{G_k}(w_S)$ is an out-neighbour of $v$ in $D_k$. Hence there exists no $(v, w_S)$-path of length two in $D_k$, and (2) follows.

Now (2) implies that $\text{diam}(D_k) \geq 3$, and since $D_k$ was arbitrary, we have $\text{diam}(G_k) \geq 3$.

We now determine the minimum degree of $G_k$. Let $n_k = |V(G_k)|$. Each vertex in $H_1$ or $H_3$ has degree $\binom{3k}{k} + 2k - 1 = \frac{1}{2}n_k + \frac{k}{2} - 1$. Every vertex in $H_2$ is adjacent to all vertices in $H_2$ except itself, and to two
thirds of the vertices in \( H_1 \) and \( H_3 \), and so has degree \( \frac{4}{3} \binom{3k}{k} + 3k - 1 = \frac{2}{3}n_k + k - 1 \). It follows that

\[
\delta(G_k) = \min\left(\frac{1}{2}n_k + \frac{k}{2} - 1, \frac{2}{3}n_k + k - 1\right) = \frac{1}{2}n_k + \frac{k}{2} - 1.
\]

Using Stirling’s Formula, we obtain for sufficiently large \( k \) that

\[
n_k = 2 \binom{3k}{k} + 3k \leq 2.5 \frac{(3k)!}{k!(2k)!} < 3 \frac{(\frac{3k}{e})^{3k} \sqrt{6k\pi}}{\left(\frac{k}{e}\right)^k \sqrt{2k\pi} (\frac{2k}{e})^{2k} \sqrt{4k\pi}} = \frac{3^{3/2}}{2} \left(\frac{3}{2}\right)^k \frac{1}{\sqrt{k\pi}}.
\]

Taking logarithms of both sides of (4), we get

\[
\ln n_k < \ln\left(\frac{3^{3/2}}{2\sqrt{\pi}}\right) + k \ln(27/4) - (1/2) \ln k.
\]

Substituting this into (3) yields

\[
\delta(G_k) - \frac{1}{2}n_k = \frac{k}{2} - 1 > \frac{\ln n_k - \ln\left(\frac{3^{3/2}}{2\sqrt{\pi}}\right)}{2 \ln(27/4)} - 1 + \frac{1}{4} \ln k > \frac{\ln n_k}{2 \ln(27/4)}
\]

since for large \( k \) the last term dominates the negative constant. \( \square \)

It follows from Theorems [1] and [2] that the least minimum degree \( \delta_n \) that guarantees the existence of an orientation of diameter two in a graph of order \( n \) and minimum degree not less than \( \delta_n \) satisfies \( \delta_n = \frac{1}{2}n + \Theta(\ln n) \).

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