Reflection principle for classical solutions of the homogeneous real Monge–Ampère equation

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Abstract: We consider reflection principle for classical solutions of the homogeneous real Monge–Ampère equation. We show that both the odd and the even reflected functions satisfy the Monge–Ampère equation if the second-order partial derivatives have continuous limits on the reflection boundary. In addition to sufficient conditions, we give some necessary conditions. Before stating the main results, we present elementary formulas for the reflected functions and study their differentiability properties across the reflection boundary. As an important special case, we finally consider extension of polynomials satisfying the homogeneous Monge–Ampère equation.

1. Introduction
Reflection is a method to extend functions and, in particular, solutions of homogeneous differential equations across a flat boundary. Classically, it is applied for some strong type equations but later on also for several weak type equations. The reflected function is usually equipped with negative sign in the reflected domain, but in our case it turns out to be profitable to study also a variant of the reflected function with positive sign.

Let \( G \) be a domain in the upper half complex plane \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Im} z < 0 \} \) and let \( G_0 \subset \partial \mathbb{C}^+ \) be open in \( \partial G \). If \( f = u + iv : G_0 \to \mathbb{C} \) is analytic and \( \lim_{z \to w} v(z) = 0 \) for all \( w \in G_0 \) then \( f \) has an analytic extension to \( G = G_0 \cup G_0 \cup G_- \) where \( G_- = PG_+ \) and \( P : \mathbb{C} \to \mathbb{C} \) is the reflection with respect to

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Mika Koskenoja is a university lecturer at the Department of Mathematics and Statistics in the University of Helsinki. After receiving his PhD in 2002, the author has continued to consider potential theoretic issues of several complex variables and spaces of variable exponents. Moreover, he has studied some specific questions of both real and complex Monge-Ampère equations. The results of this paper belong to the latter research topic.

PUBLIC INTEREST STATEMENT
Functions play a central role in all mathematical approaches. Sometimes it is important to be able to extend the domain of a function. Then the reflection may offer an admissible method for the extension. We consider the reflection principle for the Monge–Ampère equation which arises naturally in several areas of both mathematics and physics. We show that both the odd and the even reflected functions satisfy the Monge–Ampère equation if the second order partial derivatives have continuous limits on the reflection boundary.

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the real coordinate axis. The analytic extension from $G_+$ to $G_-$ is given by $f(\mathbb{Z}) = \overline{f(\mathbb{Z})}$. This classical reflection principle originates with H. A. Schwarz.

An analogous principle holds for harmonic functions given in the upper half space of $\mathbb{R}^n$ with $n \geq 2$, that is, $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, y < 0\}$. If $G_+ \subset \mathbb{R}^n_+$ is open and $f : G_+ \rightarrow \mathbb{R}$ is harmonic and extends continuously to zero on $G_0 \subset \partial \mathbb{R}^n_+$, then $f$ extends to a harmonic function in $G$ by reflection. This can be easily proved by using the mean value principle of harmonic functions, see Armitage and Gardiner (2001). Similar principles hold for biharmonic and, more generally, for polyharmonic functions, see Duffin (1955) and Huber (1955, 1957). Armitage (1978) showed that the classical reflection principle for $f$ harmonic in $G_+$ holds when one assumes (instead of $f$ tending to 0 at each point of $G_0 \subset \partial \mathbb{R}^n_+$) that $f$ converges locally in mean to 0 on $G_0$, that is, for all $(x, 0) \in G_0$ there exists $r > 0$ such that

$$
\lim_{r \to 0^+} \int_{|y-x|<r} f(y, t) \, dy = 0
$$

(1.1)

In higher real dimensions, Martio and Rickman (1972) introduced the reflection principle for quasiregular mappings as a generalization of the original result for plane analytic functions. Later on, Martio (1981) showed that the reflection principle holds for solutions of certain elliptic partial differential equations, and he also treated further the reflection principle for quasiregular mappings. Recently, Martio (2009) proved an equivalent principle for quasiminimizers in $\mathbb{R}^n$.

The real Monge–Ampère equation

$$
\det D^2 u = \det \left[ \frac{\partial^2 u}{\partial x_j \partial x_k} \right] = f
$$

(1.2)

is a second-order partial differential equation. It is fully non-linear which means that it is not elliptic, in general. If $\Omega \subset \mathbb{R}^n$ is open, then the Equation 1.2 is elliptic only for $u \in C^2(\Omega)$ that is uniformly convex at each point of $\Omega$ and for such a solution to exist, we must also have $f$ positive, see Gilbarg and Trudinger (1983). Roots of the real Monge–Ampère equation go back to the time of G. Monge (1784) in the end of the eighteenth century and Ampère (1820) in the beginning of the nineteenth century, but the mathematical theory including a few variants of weak solutions has mainly been developed during the latter part of the twentieth century, see e.g. Aleksandrov (1961), Bakelman (1957, 1983), Pogorelov (1971), Lions (1983), Caffarelli, Nirenberg, and Spruck (1984). Self-contained expositions of the real Monge–Ampère equation have been written by Pogorelov (1964) and later on by Gutiérrez (2001). Moreover, the second and later editions of the distinguished monograph in second-order partial differential equations by Gilbarg and Trudinger (1983) contain a part devoted to the Monge–Ampère equation.

We consider classical solutions of the homogeneous real Monge–Ampère Equation 1.2 where $f \equiv 0$. For example, all twice continuously differentiable real valued functions which are constant with respect to at least one variable satisfy the homogeneous Monge–Ampère equation, because then the Hesse matrix $D^2 u = \left[ \frac{\partial^2 u}{\partial x_j \partial x_k} \right]$ contains at least one zero row and one zero column. Note that $\det D^2 u$ is, in fact, the Jacobian determinant of the partial derivatives $\partial u/\partial x_1, \ldots, \partial u/\partial x_n$ of a function $u$.

In the one-dimensional case, the Monge–Ampère operator coincides with the Laplace operator since $\det D^2 u = \Delta u = u''$ for any twice differentiable function $u$. Therefore, reflection theory for the homogeneous Monge–Ampère equation provides nothing new in the case $n = 1$. However, we state our results and give proofs so that the one-dimensional case is included.

In higher dimensions $n \geq 2$, harmonic functions do not necessarily satisfy the homogeneous Monge–Ampère equation. The contrary is neither true: Twice continuously differentiable functions $u$ satisfying the homogeneous Monge–Ampère equation are not necessarily harmonic. However, there are functions which satisfy both the homogeneous Laplace equation and the homogeneous
2. Preliminaries including notation and terminology

We first set central notation connected to the reflection in \( \mathbb{R}^n \). Let \( G_+ \) be a domain in \( \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n < 0 \} \). Let \( P : \mathbb{R}^n \to \mathbb{R}^n \) be the reflection with respect to \( \partial \mathbb{R}^n_+ \), that is, \( P(x) = P(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, -x_n) \). Suppose that there is a non-empty set \( G_0 \subset \partial \mathbb{R}^n_+ \) open in \( \partial G_+ \). Set \( G = G_+ \cup G_0 \cup G_- \) where \( G_- = PG_+ \). Then \( G \) is a domain (open and connected set) in \( \mathbb{R}^n \).

Suppose that a function \( u : G_0 \to \mathbb{R} \) satisfies the following boundary condition on \( G_0 \):

\[
\lim_{x \to x_0} u(x) = 0 \quad \text{for all } x_0 \in G_0. \tag{2.1}
\]

Note that the boundary condition 2.1 is stronger than the boundary condition 1.1 used by Armitage (1978). In fact, we need to assume a much stronger boundary condition than 2.1 to ensure sufficiently nice behaviour of partial derivatives across the reflection boundary \( G_0 \), see 4.8 and 4.9.

We define the odd reflected function \( \tilde{u} : G \to \mathbb{R} \):

\[
\tilde{u}(x) = \begin{cases} 
  u(x), & x \in G_+, \\
  0, & x \in G_0, \\
  -u(P(x)), & x \in G_-.
\end{cases} \tag{2.2}
\]

Formula 2.2 is mainly used for a reflected function since the minus sign in the reflected domain \( G_- \) guarantees many useful properties towards the reflection boundary \( G_0 \).

All rather simple but substantial examples of this paper are given in the plane. The upper half plane is denoted by \( \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : y < 0 \} \) and the upper half unit disk by \( B_+ = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1 \text{ and } y < 0 \} \). Write \( B_0 = \{(x, y) \in \mathbb{R}^2 : -1 < x < 1 \text{ and } y = 0 \} \). Then \( B = B_+ \cup B_0 \cup B_- \) is the open unit disk where \( B_- = PB_+ \).

Our first example emphasizes that the second-order differentiability of a function may get broken on the reflection boundary.

**Example 2.3** Let \( u : B_+ \to \mathbb{R} \) be the function \( u(x, y) = y^2 \) where \( (x, y) \in B_+ \). Obviously, \( u \in C^2(B_-) \) and it extends continuously to zero in \( B_0 \). Now for every \( (x, y) \in B_- \) we have \( \tilde{u}(x, y) = -u(P(x, y)) = -u(x, -y) = -(y)^2 = -y^2 \), and hence

\[
\frac{\partial^2 \tilde{u}}{\partial y^2}(x, y) = \begin{cases} 
  2, & (x, y) \in B_+, \\
  -2, & (x, y) \in B_-,
\end{cases}
\]

but \( \frac{\partial \tilde{u}}{\partial y}(x_0, 0) \) does not exist for any \( (x_0, 0) \in B_0 \). In addition, the limit

\[
\lim_{(x,y) \to (0,0)} \frac{\partial^2 \tilde{u}}{\partial y^2}(x,y)
\]

does not exist for any \( (x_0, 0) \in B_0 \).

Consequently, if a given function \( u \) is smooth (at least twice differentiable) and extends continuously to zero in the reflection boundary, but we have no assumptions giving extra regularity towards the boundary, then the odd reflected function \( \tilde{u} \) is not necessarily smooth though it is continuous by the boundary condition 2.1 and definition 2.2.

**Example 2.4** The function \( u \) given in Example 2.3 and therefore also \( \tilde{u} \) satisfy the homogeneous Monge–Ampère equation in \( B_+ \). Moreover, \( \tilde{u} \) satisfies the homogeneous Monge–Ampère equation in \( B_- \), and hence the limit
In particular, the limit exists for every \((x_0, 0) \in \mathcal{B}_0\). However, the second-order partial derivative \(\frac{\partial^2 u}{\partial y^2}\) does not exist in \(\mathcal{B}_0\) and hence \(\det D^2 \hat{u}(x_0, 0)\) does not exist for any \((x_0, 0) \in \mathcal{B}_0\).

Therefore, we observe that nice limiting behaviour of \(\det D^2 \hat{u}\) does not guarantee the homogeneous Monge–Ampère equation to hold in the reflection boundary although it holds everywhere outside.

**Example 2.5**  Still, let \(u\) be given like in Example 2.3. If we define a function \(\hat{u}: \mathcal{B} \to \mathbb{R}\) by

\[
\hat{u}(x, y) = \begin{cases} 
  y^2, & (x, y) \in \mathcal{B}_+,
  
  0, & (x, y) \in \mathcal{B}_0,
  
  y^2, & (x, y) \in \mathcal{B}_-,
\end{cases}
\]

which means that \(\hat{u}(x, y) = y^2\) for all \((x, y) \in \mathcal{B}\), then \(\hat{u} \in C^2(\mathcal{B})\) and it satisfies the homogeneous Monge–Ampère equation in \(\mathcal{B}\).

Motivated by the previous Example 2.5 we introduce the following variant of the reflected function. Suppose that a function \(u: \mathcal{G}_+ \to \mathbb{R}\) satisfies the boundary condition 2.1. Then the even reflected function \(\check{u}: \mathcal{G} \to \mathbb{R}\) is given by

\[
\check{u}(x) = \begin{cases} 
  u(x), & x \in \mathcal{G}_+,
  
  0, & x \in \mathcal{G}_0,
  
  u(P(x)), & x \in \mathcal{G}_-.
\end{cases}
\] (2.6)

Both reflected functions \(\hat{u}\) and \(\check{u}\) adopt continuity of a function \(u\) which satisfies the boundary condition 2.1. Reflected functions provide optional methods to extend functions and solutions of equations across a flat boundary. It may happen that just one of the reflected functions works or both of them work for the purpose the extension is needed. However, there are situations where the extension over a flat boundary is available by using neither of the reflected functions, see Section 6.

**Remark 2.7**  Our setting to study reflection is valid for the one-dimensional case, indeed. Then \(\mathcal{G}_+ = (0, a)\) where \(a > 0\), \(\mathcal{G}_0 = \{0\}\), \(\mathcal{G}_- = (-a, 0)\) and \(\mathcal{G} = (-a, a)\). The boundary condition 2.1 for a function \(u: (0, a) \to \mathbb{R}\) means that \(\lim_{x \to 0^+} u(x) = 0\). The reflected functions \(\hat{u}, \check{u}: (-a, a) \to \mathbb{R}\) are given by

\[
\hat{u}(x) = \begin{cases} 
  u(x), & x \in (0, a),
  
  0, & x = 0,
  
  -u(-x), & x \in (-a, 0),
\end{cases}
\]

and

\[
\check{u}(x) = \begin{cases} 
  u(x), & x \in (0, a),
  
  0, & x = 0,
  
  u(-x), & x \in (-a, 0).
\end{cases}
\]

### 3. Formulas for the reflected functions

In this section, we give formulas for gradients and Hesse matrices of the reflected functions at points \(x \in \mathcal{G}_-\) with respect to the reflected points \(P(x) \in \mathcal{G}_+\). Most of the formulas are probably well known but cannot be found in the literature, so the elementary calculations needed for the proofs are presented here. These formulas are key tools to study the reflection principle for the homogeneous Monge–Ampère equation.

**Lemma 3.1**  Let a point \(x \in \mathcal{G}_-\) be such that \(u\) is differentiable at \(P(x) \in \mathcal{G}_+\). Then

\[
D\check{u}(x) = \left( \frac{\partial u}{\partial x_1}(P(x)), \ldots, -\frac{\partial u}{\partial x_n}(P(x)), \frac{\partial u}{\partial x_{n+1}}(P(x)) \right) \] (3.2)
and

\[ D\tilde{u}(x) = \left( \frac{\partial \tilde{u}}{\partial x_1}(P(x)), \ldots, \frac{\partial \tilde{u}}{\partial x_{n-1}}(P(x)), \frac{\partial \tilde{u}}{\partial x_n}(P(x)) \right). \tag{3.3} \]

Moreover,

\[ D\tilde{u}(x) = -D\hat{u}(x). \tag{3.4} \]

In particular, if \( u \in C^1(G_\lambda) \), then \( \tilde{u}, \hat{u} \in C^1(G_\lambda) \).

**Proof**  Since \( P = (P_1, \ldots, P_n): \mathbb{R}^n \rightarrow \mathbb{R}^n \) is given coordinately by

\[ P_k(x) = \begin{cases} x_k, & k = 1, \ldots, n-1, \\ -x_k, & k = n, \end{cases} \]

we have

\[ \frac{\partial P_n}{\partial x_j}(x) = \begin{cases} 1, & j = k = 1, \ldots, n-1, \\ -1, & j = k = n, \\ 0, & \text{otherwise}. \end{cases} \]

Now by the chain rule of partial derivatives

\[ \frac{\partial \tilde{u}}{\partial x_j}(x) = \frac{\partial(-(u \circ P))}{\partial x_j}(x) = -\sum_{k=1}^n \frac{\partial u}{\partial x_k}(P(x)) \frac{\partial P_k}{\partial x_j}(x) \]

\[ = \begin{cases} -\frac{\partial u}{\partial x_k}(P(x)), & j = 1, \ldots, n-1, \\ \frac{\partial u}{\partial x_k}(P(x)), & j = n. \end{cases} \tag{3.5} \]

Therefore, formula 3.2 is valid. Formula 3.4 follows directly from the identity \( \tilde{u} = -\hat{u} \) in \( G_\lambda \) and then 3.3 follows from 3.2 and 3.4.

**Lemma 3.6**  Let a point \( x \in G_\lambda \) be such that \( u \) is twice differentiable at \( P(x) \in G_\gamma \). Then

\[ D^2 \tilde{u}(x) = \begin{bmatrix} -\frac{\partial^2 u}{\partial x_1^2}(P(x)) & \cdots & -\frac{\partial^2 u}{\partial x_1 \partial x_{n-1}}(P(x)) & \frac{\partial^2 u}{\partial x_1 \partial x_n}(P(x)) \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{\partial^2 u}{\partial x_{n-1} \partial x_1}(P(x)) & \cdots & -\frac{\partial^2 u}{\partial x_{n-1} \partial x_{n-1}}(P(x)) & \frac{\partial^2 u}{\partial x_{n-1} \partial x_n}(P(x)) \\ \frac{\partial^2 u}{\partial x_n \partial x_1}(P(x)) & \cdots & \frac{\partial^2 u}{\partial x_n \partial x_{n-1}}(P(x)) & \frac{\partial^2 u}{\partial x_n \partial x_n}(P(x)) \end{bmatrix} \tag{3.7} \]

and

\[ D^2 \hat{u}(x) = \begin{bmatrix} \frac{\partial^2 u}{\partial x_1^2}(P(x)) & \cdots & \frac{\partial^2 u}{\partial x_1 \partial x_{n-1}}(P(x)) & -\frac{\partial^2 u}{\partial x_1 \partial x_n}(P(x)) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial^2 u}{\partial x_{n-1} \partial x_1}(P(x)) & \cdots & \frac{\partial^2 u}{\partial x_{n-1} \partial x_{n-1}}(P(x)) & -\frac{\partial^2 u}{\partial x_{n-1} \partial x_n}(P(x)) \\ -\frac{\partial^2 u}{\partial x_n \partial x_1}(P(x)) & \cdots & -\frac{\partial^2 u}{\partial x_n \partial x_{n-1}}(P(x)) & \frac{\partial^2 u}{\partial x_n \partial x_n}(P(x)) \end{bmatrix}. \tag{3.8} \]

Moreover,

\[ D^2 \hat{u}(x) = -D^2 \tilde{u}(x). \tag{3.9} \]

In particular, if \( u \in C^2(G_\lambda) \), then \( \tilde{u}, \hat{u} \in C^2(G_\lambda) \).
Proof. The chain rule for partial derivatives implies now together with formula 3.5 that

\[
\frac{\partial^2 \hat{u}}{\partial x_i \partial x_j}(x) = \begin{cases} 
\frac{d}{dx_i} \left( - \frac{\partial}{\partial x_j} (P(x)) \right) = - \frac{d}{dx_i} \left( \frac{\partial}{\partial x_j} (\alpha P) (x) \right), & j = 1, \ldots, n - 1, \\
\frac{d}{dx_i} \left( \frac{\partial}{\partial x_j} (P(x)) \right) = \frac{d}{dx_i} \left( \frac{\partial}{\partial x_j} (\alpha P) (x) \right), & j = n, \\
\end{cases}
\]

\[
= \begin{cases} 
- \sum_{k=1}^{n} x_k \hat{u}_{x_k, x_i}(P(x)) \hat{u}_{x_j}(x), & j = 1, \ldots, n - 1, \\
\sum_{k=1}^{n} x_k \hat{u}_{x_k, x_i}(P(x)) \hat{u}_{x_j}(x), & j = n, \\
- \hat{u}_{x_i}(P(x)), & i, j = 1, \ldots, n - 1 \text{ or } i, j = n, \\
\frac{\partial}{\partial x_i} \hat{u}(P(x)), & \text{otherwise.}
\end{cases}
\]

(3.10)

Therefore, formula 3.7 is valid. Formula 3.9 follows directly from the identity \( \hat{u} = -\hat{\nu} \) in \( G_r \) and then 3.8 follows from 3.7 and 3.9.

Since the Laplacian is the trace of the Hesse matrix, we observe from the formulas 3.7 and 3.8 that

\[
\Delta \hat{u}(x) = -\Delta \hat{\nu}(x) = \Delta u(P(x))
\]

(3.11)

for all \( x \in G_r \) such that \( u \) is twice differentiable at \( P(x) \in G_r \). It appears that we obtain corresponding equations for the determinants of the Hesse matrices of the reflected functions \( \hat{u} \) and \( \hat{\nu} \), which is favourable since we study reflection for the homogeneous Monge–Ampère equation. These equations are stated and proved next.

**Theorem 3.12** Let a point \( x \in G_r \) be such that \( u \) is twice differentiable at \( P(x) \in G_r \). Then

\[
\det D^2 \hat{u}(x) = (-1)^n \det D^2 u(P(x))
\]

(3.13)

and

\[
\det D^2 \hat{\nu}(x) = \det D^2 u(P(x)).
\]

(3.14)

In particular,

\[
\det D^2 \hat{u}(x) = (-1)^n \det D^2 \hat{\nu}(x).
\]

(3.15)

**Proof.** We apply elementary properties of determinants to matrices 3.7 and 3.8 getting

\[
\det D^2 \hat{u}(x) = \det(-D^2 u(P(x))) = (-1)^n \det D^2 u(P(x))
\]

and

\[
\det D^2 \hat{\nu}(x) = \det D^2 u(P(x)).
\]

Formula 3.15 follows now directly from formulas 3.13 and 3.14, or alternatively from formula 3.9.

### 4. Differentiability of the reflected functions

In this section, we present some examples and results clarifying the second-order differentiability of functions under reflection. We find necessary conditions for the existence and continuity of second-order partial derivatives in \( G \). Evidently, our study of classical solutions of the Monge–Ampère equation requires that all second-order partial derivatives exist which justifies our goal.

It is clear that the reflected functions \( \hat{u} \) and \( \hat{\nu} \) are continuous in \( G \) whenever the original function \( u \) is continuous in \( G \) and satisfies the boundary condition 2.1. In Section 3, we have confirmed that if \( u \) is once or twice continuously differentiable in \( G \), then \( \hat{u} \) and \( \hat{\nu} \) are once or twice continuously differentiable in \( G \).
differentiable in $G_0$, respectively. Therefore, differentiability requires extra care only in the reflection boundary $G_0$. Indeed, if $u \in C^2(G_0)$ satisfies the boundary condition 2.1, the reflected functions $\hat{u}$ and $\check{u}$ may behave badly in $G_0$. It helps none if $u$ satisfies, in addition, the homogeneous Monge–Ampère equation in $G_0$. This can be seen by the following examples.

**Example 4.1** The function $w: B_+ \rightarrow \mathbb{R}, w(x,y) = \sqrt{y}$, is $C^2$ in $B_+$ and satisfies the boundary condition 2.1. However,

$$\lim_{(x,y) \to (0,0)} \frac{\partial w}{\partial y}(x,y) = \lim_{y \to 0^+} \frac{1}{2\sqrt{y}} = \infty,$$

and the first order partial derivatives $\frac{\partial w}{\partial y}(x,0)$ and $\frac{\partial w}{\partial y}(x,0)$ do not exist for any $(x,0) \in B_0$. In particular, $w, \check{w} \not\in C^1(B)$.

**Example 4.2** The function $v: B_+ \rightarrow \mathbb{R}, v(x,y) = y^2 \sin\left( \frac{1}{y} \right)$, is $C^2$ in $B_+$ and satisfies the boundary condition 2.1. Hence $\check{v}$ and $\hat{v}$ are continuous in $B$. The first order partial derivatives of $\check{v}$ and $\hat{v}$ with respect to both $x$ and $y$ exist in $B$, indeed, we have $\frac{\partial \check{v}}{\partial x}(x,y) = \frac{\partial \hat{v}}{\partial x}(x,y) = 0$ for every $(x,y) \in B$,

$$\frac{\partial \check{v}}{\partial y}(x,y) = \begin{cases} -\cos \left( \frac{1}{y} \right) + 2y \sin \left( \frac{1}{y} \right), & (x,y) \in B_+, \\ 0, & (x,y) \in B_0, \\ \cos \left( -\frac{1}{y} \right) - 2y \sin \left( -\frac{1}{y} \right), & (x,y) \in B_. \end{cases}$$

and

$$\frac{\partial \hat{v}}{\partial y}(x,y) = \begin{cases} -\cos \left( \frac{1}{y} \right) + 2y \sin \left( \frac{1}{y} \right), & (x,y) \in B_+, \\ 0, & (x,y) \in B_0, \\ -\cos \left( -\frac{1}{y} \right) + 2y \sin \left( -\frac{1}{y} \right), & (x,y) \in B_. \end{cases}$$

and

$$\lim_{(x,y) \to (0,0)} \frac{\partial \check{v}}{\partial y}(x,y) \text{ and } \lim_{(x,y) \to (0,0)} \frac{\partial \hat{v}}{\partial y}(x,y)$$

do not exist for any $(x,0) \in G_0$ because sine and cosine of $\frac{1}{y}$ and $-\frac{1}{y}$ oscillate as $y$ tends to 0. Consequently, $\frac{\partial \check{v}}{\partial y}$ and $\frac{\partial \hat{v}}{\partial y}$ are not continuous at any $(x,0) \in B_0$. In particular, $\check{u}, \hat{u} \not\in C^1(B)$, even though all first-order partial derivatives of $\check{u}$ and $\hat{u}$ exist at every point of $B$.

Observe that in the previous examples both $w$ and $v$ satisfy the homogeneous Monge–Ampère equation in $B_+$. These counterexamples are important giving us two essential observations. If a function $u$ is (twice) differentiable in $G_0$ and satisfies the boundary condition 2.1, then the reflected functions $\check{u}$ and $\hat{u}$ are not always differentiable in $G$. On the other hand, if $u$ is (twice) continuously differentiable, then the reflected functions $\check{u}$ and $\hat{u}$ may be differentiable but not continuously differentiable in $G$.

Our primary requirement is that the studied functions are twice continuously differentiable, even though we need, in principle, the existence of all second-order partial derivatives only. Of course, we present only such conditions which are not true for all $u \in C^2(G_0)$ satisfying the boundary condition 2.1. In the first theorem, we present a necessary condition for the odd reflected function $\hat{u}$ such that the second-order partial derivatives exist and are continuous in $G_0$. 


Theorem 4.3 Let \( u \in C^2(G_u) \) satisfy the boundary condition 2.1. If \( \tilde{u} \in C^2(G) \) then
\[
\frac{\partial^2 \tilde{u}}{\partial x_n^2}(x_0) = 0 \quad \text{(4.4)}
\]
for every \( x_0 \in G_0 \).

Proof Suppose that \( \tilde{u} \in C^2(G) \) and \( x_0 \in G_0 \) Now 3.10 yields
\[
\lim_{x \to x_0} \frac{\partial^2 \tilde{u}}{\partial x_n^2}(x) = -\lim_{x \to x_0} \frac{\partial^2 \tilde{u}}{\partial x_n^2}(P(x)) = -\lim_{x \to x_0} \frac{\partial^2 \tilde{u}}{\partial x_n^2}(x),
\]
and hence the limit \( \lim_{x \to x_0} \frac{\partial^2 \tilde{u}}{\partial x_n^2}(x) \) exists if and only if
\[
\lim_{x \to x_0} \frac{\partial^2 \tilde{u}}{\partial x_n^2}(x) = \lim_{x \to x_0} \frac{\partial^2 \tilde{u}}{\partial x_n^2}(x) = 0.
\]
The continuity of \( \frac{\partial^2 \tilde{u}}{\partial x_n^2} \) at \( x_0 \) implies that the Equation 4.4 holds.

Next, we give a necessary condition for the even reflected function \( \tilde{u} \) such that, firstly, the first-order partial derivatives exist and are continuous in \( G_0 \), and secondly, the second-order partial derivatives exist in \( G_0 \).

Theorem 4.5 Let \( u \in C^2(G_u) \) satisfy the boundary condition 2.1. If \( \tilde{u} \in C^1(G) \) then
\[
\frac{\partial \tilde{u}}{\partial x_n}(x_0) = 0 \quad \text{(4.6)}
\]
for every \( x_0 \in G_0 \).

Proof Suppose that \( \tilde{u} \in C^1(G) \) and \( x_0 \in G_0 \). It follows from 3.3 that
\[
\lim_{x \to x_0} \frac{\partial \tilde{u}}{\partial x_n}(x) = -\lim_{x \to x_0} \frac{\partial \tilde{u}}{\partial x_n}(P(x)) = -\lim_{x \to x_0} \frac{\partial \tilde{u}}{\partial x_n}(x),
\]
and hence the limit \( \lim_{x \to x_0} \frac{\partial \tilde{u}}{\partial x_n}(x) \) exists if and only if
\[
\lim_{x \to x_0} \frac{\partial \tilde{u}}{\partial x_n}(x) = \lim_{x \to x_0} \frac{\partial \tilde{u}}{\partial x_n}(x) = 0.
\]
Theorem 4.5 yields now that the Equation 4.6 holds.

Corollary 4.7 Let \( u \in C^1(G_u) \) satisfy the boundary condition 2.1. If all second-order partial derivatives of \( \tilde{u} \) exist in \( G_0 \), then 4.6 holds for every \( x_0 \in G_0 \).

Proof The existence of the second-order partial derivatives of \( \tilde{u} \) in \( G_0 \) implies continuity of the first order partial derivatives of \( \tilde{u} \) in \( G_0 \), therefore \( \tilde{u} \in C^1(G) \). Theorem 4.5 yields now that the Equation 4.6 holds for every \( x_0 \in G_0 \).

Note that the necessary conditions 4.4 and 4.6 concern only the \( n \)th first- and second-order partial derivatives of the reflected functions \( \tilde{u} \) and \( \hat{u} \). In our rather restrictive setting other partial derivatives of the reflected functions are not so crucial.

Finally, to ensure that the determinant of the Hesse matrix (that is, the Monge–Ampère operator) of a reflected function is defined in \( G_0 \), the second-order partial derivatives need to behave nicely around \( G_0 \). Therefore, whenever \( u : G_0 \to \mathbb{R} \) is \( C^2 \), we set the following boundary conditions on \( G_0 \).
\[
\lim_{x \to x_0} \frac{\partial^2 \hat{u}}{\partial x_j \partial x_k}(x) = \frac{\partial^2 \tilde{u}}{\partial x_j \partial x_k}(x_0) \text{ for all } x_0 \in G_0 \text{ and for every } j, k = 1, \ldots, n,
\]  

(4.8)

\[
\lim_{x \to x_0} \frac{\partial^2 \hat{u}}{\partial x_j \partial x_k}(x) = -\frac{\partial^2 \tilde{u}}{\partial x_j \partial x_k}(x_0) \text{ for all } x_0 \in G_0 \text{ and for every } j, k = 1, \ldots, n.
\]  

(4.9)

We will see (and have partly seen already) that if \( u \in C^2(G_\pm) \), it may happen that none, only one, or both of the conditions 4.8 and 4.9 hold. It is clear that if \( u \in C^2(G_\pm) \) and the boundary condition 4.8 holds, then \( \tilde{u} \in C^2(G) \). Correspondingly, if \( u \in C^2(G_\pm) \) and the boundary condition 4.9 holds, then \( \hat{u} \in C^2(G) \). The boundary conditions 4.8 and 4.9 mean that all second order partial derivatives have continuous limits on the reflection boundary.

5. Reflection principles for the homogeneous real Monge–Ampère equation

We are ready to state our first reflection principle for the homogeneous Monge–Ampère equation.

Theorem 5.1 Let \( u \in C^2(G_\pm) \) satisfy the boundary conditions 2.1 and 4.8. If \( u \) satisfies the equation \( \det D^2 u = 0 \) in \( G_+ \), then the odd reflected function \( \hat{u} \) satisfies \( \det D^2 \hat{u} = 0 \) in \( G \).

Proof Let \( x \in G_- \). Then by 3.13,

\[ \det D^2 \hat{u}(x) = (-1)^n \det D^2 u(P(x)) = 0, \]

because \( P(x) \in G_- \). Hence \( \hat{u} \) satisfies the homogeneous Monge–Ampère equation in \( G_- \) and further, in the union \( G_+ \cup G_- \) since it is clear that \( \det D^2 \hat{u} = 0 \) in \( G_- \) where \( \hat{u} \equiv u \).

We need to show that \( \hat{u} \) satisfies the homogeneous Monge–Ampère equation in \( G_- \). If \( n = 1 \), then 0 is the only point in \( G_- \) and the continuity of \( D^2 \hat{u} = \hat{u}'' \) at 0 yields

\[ \det D^2 \hat{u}(0) = \hat{u}''(0) = \lim_{x \to 0} \hat{u}''(x) = 0 \]

because \( \hat{u}'' = \det D^2 \hat{u} = 0 \) in \( G_- \cup G_- \).

Suppose then that \( n \geq 2 \) and let \( x_0 \in G_- \). Since \( \hat{u} \equiv 0 \) in \( G_+ \), \( \hat{u} \) is constant in \( G_- \) with respect to the variables \( x_1, \ldots, x_{n-1} \). Hence we have \( \frac{\partial^2 \hat{u}}{\partial x_j \partial x_k}(x_0) = 0 \) for each \( j, k = 1, \ldots, n - 1 \). If \( n = 2 \), continuity of the second-order partial derivatives yields

\[
\det D^2 \hat{u}(x_0) = \det \begin{bmatrix} 0 & \frac{\partial^2 \hat{u}}{\partial x_1 \partial x_2}(x_0) \\ \frac{\partial^2 \hat{u}}{\partial x_2 \partial x_1}(x_0) & 0 \end{bmatrix} = -\left( \frac{\partial^2 \hat{u}}{\partial x_1 \partial x_2}(x_0) \right)^2
\]

\[ = \lim_{x \to x_0} \left( \frac{\partial^2 \hat{u}}{\partial x_1 \partial x_2}(x) \frac{\partial^2 \hat{u}}{\partial x_2 \partial x_1}(x) - \left( \frac{\partial^2 \hat{u}}{\partial x_1 \partial x_2}(x) \right)^2 \right) = \lim_{x \to x_0} \det D^2 \hat{u}(x) = 0,
\]

because \( \det D^2 \hat{u} = 0 \) in \( G_- \cup G_- \). Otherwise, if \( n \geq 3 \),

\[
\det D^2 \hat{u}(x_0) = \det \begin{bmatrix} 0 & \ldots & 0 & \frac{\partial^2 \hat{u}}{\partial x_1 \partial x_n}(x_0) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & \frac{\partial^2 \hat{u}}{\partial x_{n-1} \partial x_n}(x_0) \\ \frac{\partial^2 \hat{u}}{\partial x_1 \partial x_n}(x_0) & \ldots & \frac{\partial^2 \hat{u}}{\partial x_{n-1} \partial x_n}(x_0) & 0 \end{bmatrix} = 0,
\]

since in the cofactor expansion along the last row the last cofactor matrix is the zero matrix and other cofactor matrices have \( n - 2 \) zero columns, that is, at least one zero column. We conclude that
\( \hat{u} \) satisfies the homogeneous Monge–Ampère equation in \( G \).

Next, we state our second reflection principle for the homogeneous Monge–Ampère equation.

**Theorem 5.2** Let \( u \in C^2(G) \) satisfy the boundary conditions 2.1 and 4.9. If \( u \) satisfies the equation \( \det D^2u = 0 \) in \( G \), then the even reflected function \( \hat{u} \) satisfies \( \det D^2\hat{u} = 0 \) in \( G \).

**Proof** Let \( x \in G \). Then by 3.14,
\[
\det D^2\hat{u}(x) = \det D^2u(P(x)) = 0,
\]
because \( P(x) \in G \). Hence \( \hat{u} \) satisfies the homogeneous Monge–Ampère equation in \( G \), and further, in the union \( G_+ \cup G_- \). The rest of the proof is similar to the end of the proof of Theorem 5.1.

**Remark 5.3** In any case, even without having the boundary conditions 4.8 and 4.9, the even reflected functions \( \hat{u} \) and \( \hat{u} \) satisfy the equations \( \det D^2\hat{u} = 0 \) and \( \det D^2\hat{u} = 0 \) in the open components \( G_+ \) and \( G_- \) of \( G \). Note that the union \( G_+ \cup G_- \) is disconnected since \( G_0 \) separates it into two components.

### 6. Continuation of solutions of the homogeneous real Monge–Ampère equation

If a solution of the homogeneous Monge–Ampère equation satisfies either the boundary condition 4.8 or 4.9 in addition to the boundary condition 2.1, then by Theorems 5.1 and 5.2 an extension of the solution over a flat boundary can always be found by using the reflected functions. Therefore, we may ask if an extension is always available by our two variants of the reflection. And if not, is it nevertheless possible that an extension is available. The answer for the first question is negative but for the second question positive. This can be seen by the following example.

**Example 6.1** Let \( v: B \rightarrow \mathbb{R} \) be the function \( v(x, y) = -y + y^2 \) if \( (x, y) \in B \). Then \( v \) satisfies the homogeneous Monge–Ampère equation in \( B \) and the boundary condition 2.1 on \( B \). Firstly,
\[
\lim_{y \to 0} \frac{\partial v}{\partial y}(x, y) = \lim_{y \to 0}(-1 + 2y) = -1
\]
for every \( (x_0, 0) \in B \). Hence by Theorem 4.5, the even reflected function \( \hat{v} \) is not \( C^1 \) in \( B \). This can easily be seen by a straightforward calculation. Since \( \hat{v}(x, y) = y + y^2 \) if \( (x, y) \in B \), we have
\[
\lim_{y \to 0} \frac{\partial \hat{v}}{\partial y}(x, y) = \lim_{y \to 0}(1 + 2y) = 1 \neq -1 = \lim_{y \to 0}(-1 + 2y) = \lim_{y \to 0} \frac{\partial v}{\partial y}(x, y)
\]
for every \( (x_0, 0) \in B \). Hence the first-order partial derivative \( \frac{\partial \hat{v}}{\partial y} \) does not exist in \( B \). In particular, \( v \) does not satisfy the boundary condition 4.9 and it can not be a classical solution of the homogeneous Monge–Ampère equation for any \( (x_0, 0) \in B \).

Secondly, the second-order partial derivative with respect to the second variable \( y \) satisfies
\[
\lim_{y \to 0} \frac{\partial^2 v}{\partial y^2}(x, y) = 2
\]
for every \( (x_0, 0) \in B \). Hence by Theorem 4.3, the odd reflected function \( \tilde{v} \) is not \( C^2 \) in \( B \). Like above, this can be seen by a straightforward calculation. Since \( \tilde{v}(x, y) = -y - y^2 \) if \( (x, y) \in B \), we have
\[
\lim_{y \to 0} \frac{\partial^2 \tilde{v}}{\partial y^2}(x, y) = -2 \neq 2 = \lim_{y \to 0} \frac{\partial^2 v}{\partial y^2}(x, y)
\]
for every \((x_0, 0) \in B_\alpha\). Hence the second-order partial derivative \(\frac{\partial^2 v}{\partial x_1 \partial x_n}\) does not exist in \(B_\alpha\). In particular, \(v\) does not satisfy the boundary condition 4.8 and it cannot be a classical solution of the homogeneous Monge–Ampère equation for any \((x_0, 0) \in B_\alpha\).

However, the real analytic continuation of \(v\), that is, the function \(\tilde{v}(x, y) = -y + y^2, (x, y) \in B\), gives an extension of \(v\) over the boundary \(B_\alpha\). Therefore, an extension may exist if it is not available by using either the odd reflected function or the even reflected function.

The class of polynomials is undeniably one of the most important categories of functions. On the other hand, polynomials which are constant with respect to at least one of the variables \(x_1, \ldots, x_n\) and of which every term contains the variable \(x_\alpha\) satisfy both the homogeneous Monge–Ampère equation and the boundary condition 2.1. This means that a large family of polynomials is relevant to our study. Hence, consider finally if for polynomials satisfying the homogeneous Monge–Ampère equation an extension is always available. We already observed in Example 6.1 that an extension for polynomials cannot be always found by our two reflection methods.

Since the \(n\)th variable is in a special position in our considerations, we write a polynomial \(p : G_\alpha \to \mathbb{R}\) of degree \(k\) in the form

\[
p(x_1, \ldots, x_n) = q_0(x_1, \ldots, x_{n-1}) + q_1(x_1, \ldots, x_{n-1})x_n + \cdots + q_k(x_1, \ldots, x_{n-1})x_n^k,
\]

where each \(q_i : G_\alpha \to \mathbb{R}\) is a polynomial of the variables \(x_1, \ldots, x_{n-1}\) and of degree \(k\) at most, that is,

\[
q_i(x_1, \ldots, x_{n-1}) = \sum_{i_1 = 0}^{k} \cdots \sum_{i_{n-1} = 0}^{k} a_{i_1, \ldots, i_{n-1}} x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}, \quad i = 0, 1, 2, \ldots, k.
\]

Our first lemma gives an equivalent expression to the boundary condition 2.1 for polynomials.

**Lemma 6.4** A polynomial \(p : G_\alpha \to \mathbb{R}\) satisfies the boundary condition 2.1 if and only if \(q_0\) is the zero polynomial.

**Proof** Since we suppose that the reflection boundary \(G_\beta\) is non-empty and open in \(\partial G_\beta\), there is a point \(\xi = (\xi_1, \ldots, \xi_{n-1}, 0) \in G_\alpha\) such that \(\xi_i \neq 0\) for every \(i = 1, \ldots, n-1\). Hence from 6.2 we see that

\[
\lim_{x \to \xi} p(x) = \lim_{x \to \xi} p(x_1, \ldots, x_n) = q_0(\xi_1, \ldots, \xi_{n-1}) = 0
\]

if and only if \(q_0 \equiv 0\).

If \(x = (x_1, \ldots, x_n) \in G_\beta\), then

\[
\hat{p}(x_1, \ldots, x_n) = -p(x_1, \ldots, x_{n-1}, -x_n) = \sum_{i=0}^{k} (-1)^{i+1} q_i(x_1, \ldots, x_{n-1})x_n^i
\]

and

\[
\hat{p}(x_1, \ldots, x_n) = p(x_1, \ldots, x_{n-1}, -x_n) = \sum_{i=0}^{k} (-1)^i q_i(x_1, \ldots, x_{n-1})x_n^i.
\]

In case of polynomials, the next lemma gives equivalent expressions to the conditions 4.4 and 4.6 being necessary for the boundary conditions 4.8 and 4.9.

**Lemma 6.7** Let a polynomial \(p : G_\alpha \to \mathbb{R}\) be such that \(q_0 \equiv 0\). Then
(i) the even reflected function \( \hat{p} \) satisfies 4.6 if and only if \( q_1 \equiv 0 \),
(ii) the odd reflected function \( \check{p} \) satisfies 4.4 if and only if \( q_2 \equiv 0 \).

**Proof** As in the proof of Lemma 6.4, we may suppose that there is a point \( \xi = (\xi_1, \ldots, \xi_{n-1}, 0) \in G_0 \) such that \( \xi_i \neq 0 \) for every \( i = 1, \ldots, n-1 \). Now

\[
\lim_{x \to \xi} \frac{\partial \hat{p}}{\partial x_n}(x) = \lim_{x \to \xi} \sum_{m=1}^{k} i q_m(x_1, \ldots, x_{n-1}, 0) x_n^{-1} = q_1(\xi_1, \ldots, \xi_{n-1}) = 0
\]

if and only if \( q_1 \equiv 0 \). Then by 6.6 we have

\[
\lim_{x \to \xi} \frac{\partial \hat{p}}{\partial x_n}(x) = \lim_{x \to \xi} \sum_{m=1}^{k} (-1)^m i q_m(x_1, \ldots, x_{n-1}) x_n^{-1} = 0
\]

Therefore, 4.6 holds for every \( x_0 \in G_0 \).

Similarly,

\[
\lim_{x \to \xi} \frac{\partial^2 \hat{p}}{\partial x_n^2}(x) = \lim_{x \to \xi} \sum_{m=1}^{k} i(i - 1) q_m(x_1, \ldots, x_{n-1}) x_n^{-2} = 2q_2(\xi_1, \ldots, \xi_{n-1}) = 0
\]

if and only if \( q_2 \equiv 0 \). Then by 6.5 we have

\[
\lim_{x \to \xi} \frac{\partial^2 \hat{p}}{\partial x_n^2}(x) = \lim_{x \to \xi} \sum_{m=1}^{k} (-1)^{i+1} i(i - 1) q_m(x_1, \ldots, x_{n-1}) x_n^{-2} = 0
\]

Therefore, 4.4 holds for every \( x_0 \in G_0 \).

**Lemma 6.8** Let a polynomial \( p : G_0 \to \mathbb{R} \) be such that \( q_0 \equiv 0 \). Then all partial derivatives \( \frac{\partial p}{\partial x_i} \) extend continuously from \( G_0 \) to \( G_0 \). Correspondingly, if a polynomial \( p \) is defined in \( G_0 \) and \( q_0 \equiv 0 \), then all partial derivatives \( \frac{\partial p}{\partial x_i} \) extend continuously from \( G_0 \) to \( G_0 \).

**Proof** Since \( p \) extends continuously to 0 in \( G_0 \), we have \( p(x) = \hat{p}(x) = \check{p}(x) \) for every \( x \in G_0 \cup G_0 \) meaning that \( \hat{p} \) and \( \check{p} \) have the same terms with the same coefficients in \( G_0 \) than \( p \) in \( G_0 \). Therefore, all partial derivatives \( \frac{\partial p}{\partial x_i} \) extend continuously from \( G_0 \) to \( G_0 \). Note that here \( \frac{\partial p}{\partial x_i} \) is considered \( G_0 \)-sided in \( G_0 \), because the \( G_0 \)-sided limit is not defined in \( G_0 \). The second part of the lemma follows similarly.

**Lemma 6.9** Let a polynomial \( p : G_0 \to \mathbb{R} \) be such that \( q_0 \equiv 0 \). Then

(i) \( \hat{p} \in C^2(G) \) if and only if \( q_1 \equiv 0 \),
(ii) \( \check{p} \in C^2(G) \) if and only if \( q_2 \equiv 0 \).

**Proof** By Lemma 6.7(i) and Corollary 4.7, \( q_1 \equiv 0 \) is a necessary condition to have \( \hat{p} \in C^2(G) \). We need show that \( q_0 \equiv 0 \) and \( q_1 \equiv 0 \) imply the boundary condition 4.9. By 6.6,

\[
\frac{\partial^2 \hat{p}}{\partial x_n^2}(x) = \begin{cases} 
\sum_{i=1}^{k} i(i - 1) q_i(x_1, \ldots, x_{n-1}) x_n^{-2}, & x \in G_0, \\
\sum_{i=1}^{k} (-1)^i i(i - 1) q_i(x_1, \ldots, x_{n-1}) x_n^{-2}, & x \in G_0.
\end{cases}
\]

which yields

\[
\lim_{x \to \xi} \frac{\partial^2 \hat{p}}{\partial x_n^2}(x) = 2q_2(x_1, \ldots, x_{n-1}) = \frac{\partial^2 \hat{p}}{\partial x_n^2}(\xi_0)
\]
for every $x_0 = (x_1, \ldots, x_{n-1}, 0) \in G_\circ$. Above, the second equation follows from Lemma 6.8 since partial derivatives of polynomials are polynomials. Otherwise, suppose that $j \neq n$ or $k \neq n$. Then

$$
\frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x) = \left\{ \begin{array}{ll}
\sum_{i=2}^{k} \frac{\partial^i}{\partial x_{j}^i} \left( q_i(x_1, \ldots, x_{n-1}) x_n^i \right), & x \in G_+, \\
\sum_{i=2}^{k} (-1)^i \frac{\partial^i}{\partial x_{k}^i} \left( q_i(x_1, \ldots, x_{n-1}) x_n^i \right), & x \in G_.
\end{array} \right.
$$

When we evaluate the second-order partial derivatives in the sum expressions above, we observe that every term achieved includes variable $x_n$ with power $i - 1 \geq 1$, that is, $i \geq 2$. This implies again by Lemma 6.8 that

$$
\lim_{x \to x_0} \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x) = 0 = \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x_0)
$$

for every $x_0 = (x_1, \ldots, x_{n-1}, 0) \in G_\circ$. We conclude that the boundary condition 4.9 holds.

Correspondingly, by Lemma 6.7(ii) and Theorem 4.3, $q_2 \equiv 0$ is a necessary condition to have $\tilde{p} \in C^2(G)$. We need show that $q_3 \equiv 0$ and $q_4 \equiv 0$ imply the boundary condition 4.8. Then by 6.5

$$
\frac{\partial^3 \tilde{p}}{\partial x^2}(x) = \left\{ \begin{array}{ll}
\sum_{i=3}^{n} i(i-1) q_i(x_1, \ldots, x_{n-1}) x_n^{i-2}, & x \in G_+, \\
\sum_{i=3}^{n} i(i-1)(-1)^i q_i(x_1, \ldots, x_{n-1}) x_n^{i-2}, & x \in G_.
\end{array} \right.
$$

which yields by Lemma 6.8 that

$$
\lim_{x \to x_0} \frac{\partial^3 \tilde{p}}{\partial x^2}(x) = 0 = \frac{\partial^3 \tilde{p}}{\partial x^2}(x_0)
$$

for every $x_0 = (x_1, \ldots, x_{n-1}, 0) \in G_\circ$. Otherwise, suppose that $j \neq n$ or $k \neq n$. Then

$$
\frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x) = \left\{ \begin{array}{ll}
\frac{\partial^i}{\partial x_{j}^i} \left( q_i(x_1, \ldots, x_{n-1}) x_n^i \right) + \sum_{i=2}^{k} \frac{\partial^i}{\partial x_{k}^i} \left( q_i(x_1, \ldots, x_{n-1}) x_n^i \right), & x \in G_+, \\
\frac{\partial^i}{\partial x_{k}^i} \left( q_i(x_1, \ldots, x_{n-1}) x_n^i \right) + \sum_{i=2}^{k} (-1)^i \frac{\partial^i}{\partial x_{j}^i} \left( q_i(x_1, \ldots, x_{n-1}) x_n^i \right), & x \in G_-
\end{array} \right.
$$

If $j = n$ and $k \neq n$, then by Lemma 6.8

$$
\lim_{x \to x_0} \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x) = \frac{\partial q_1}{\partial x_k}(x_1, \ldots, x_{n-1}) = \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x_0)
$$

for $x_0 = (x_1, \ldots, x_{n-1}, 0) \in G_\circ$. Similarly, if $j \neq n$ and $k = n$, then by Lemma 6.8

$$
\lim_{x \to x_0} \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_n}(x) = \frac{\partial q_1}{\partial x_n}(x_1, \ldots, x_{n-1}) = \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_n}(x_0)
$$

for $x_0 = (x_1, \ldots, x_{n-1}, 0) \in G_\circ$. If $j \neq n$ and $k \neq n$, then Lemma 6.8 again yields

$$
\lim_{x \to x_0} \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x) = 0 = \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k}(x_0)
$$

for every $x_0 = (x_1, \ldots, x_{n-1}, 0) \in G_\circ$. We conclude that the boundary condition 4.8 holds.

**Theorem 6.10** Let a polynomial $p: G_+ \to \mathbb{R}$ be such that $q_3 \equiv 0$ and $\det D^2 p = 0$ in $G_+$. Then
(i) the even reflected function \( \tilde{p} \) satisfies \( \det D^2 \tilde{p} = 0 \) in \( G \) if and only if \( q_1 \equiv 0 \).

(ii) the odd reflected function \( p \) satisfies \( \det D^2 \tilde{p} = 0 \) in \( G \) if and only if \( q_2 \equiv 0 \).

Proof. By Lemma 6.9, we only need to verify that the homogeneous Monge–Ampère equation holds in \( G \). But for \( \tilde{p} \) this follows immediately from Theorem 5.1 and for \( \tilde{p} \) from Theorem 5.1.

If a polynomial \( p = q_0 + q_1 x_n + \ldots + q_n x_n^k : G \to \mathbb{R} \) is such that \( q_1 \neq 0 \) and \( q_2 \neq 0 \), then it follows from Theorem 6.10 that an extension to \( G \) cannot be found by using the reflected functions. However, an extension can always be found, which was tentatively observed in Example 6.1.

THEOREM 6.11 Let a polynomial \( p : G \to \mathbb{R} \) be such that \( q_0 \equiv 0 \) and \( \det D^2 p = 0 \) in \( G \). Then there is an extension \( \tilde{p} : G \to \mathbb{R} \) of \( p \) such that \( \tilde{p} \) satisfies \( \det D^2 \tilde{p} = 0 \) in \( G \).

Proof. Write \( p(x) = q_0 (x_1, \ldots, x_{n-1}) + q_1 (x_1, \ldots, x_{n-1}) x_n + \ldots + q_n (x_1, \ldots, x_{n-1}) x_n^k \) where \( x = (x_1, \ldots, x_n) \in G \). If \( q_1 \neq 0 \) or \( q_2 \neq 0 \), then by Theorem 6.10 an extension is found by choosing \( \tilde{p} = \tilde{p} \) or \( \tilde{p} = \tilde{p} \) in \( G \).

Otherwise, and also simultaneously, we may simply extend \( p \) to \( \mathbb{R} \) real analytically so that the polynomial \( \tilde{p} : G \to \mathbb{R} \) has the same terms with the same coefficients than \( p \) in \( G \). Note that \( \tilde{p} \) is then \( C^2 \) in \( G \). Entries of the Hesse matrix of \( \tilde{p} \) are polynomials, and hence the determinant \( \det D^2 \tilde{p} \) is a polynomial as a sum of products of polynomials. Since \( \det D^2 p(x) = 0 \) at every \( x \in G \) and \( G \) is open and non-empty, \( \det D^2 \tilde{p} \) has uncountably many zeroes. Hence \( \det D^2 \tilde{p} \equiv 0 \) and \( \det D^2 \tilde{p}(x) = 0 \) at every \( x \in G \).

In fact, a polynomial \( p \) can always be extended real analytically using the latest method. Even the boundary condition 2.1 is not necessary. In particular, our considerations show that an extension of a solution of the homogeneous Monge–Ampère equation is not necessarily unique.

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