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Quantitative relations between short intervals and exceptional sets of cubic Waring-Goldbach problem

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Abstract: In this paper, we are able to prove that almost all integers \( n \) satisfying some necessary congruence conditions are the sum of \( j \) almost equal prime cubes with \( j = 7, 8 \), i.e., \( N = p_1^3 + \ldots + p_j^3 \) with \( \left| p_i - (N/j)^{1/3} \right| \leq N^{1/3 - \delta + \epsilon} \) (\( 1 \leq i \leq j \)), for some \( 0 < \delta \leq \frac{1}{9216} \). Furthermore, we give the quantitative relations between the length of short intervals and the size of exceptional sets.

Keywords: Circle method, Exponential sums over primes, Short intervals, Quantitative relations

MSC: 11P32, 11P05, 11N36, 11P55

1 Introduction

In the Waring-Goldbach problem, one studies the representation of positive integers by powers of \( j \) primes, where \( j \) is a positive integer. One of the most famous results of Hua [1] in 1938 states that each sufficiently large odd integer \( n \) can be written as the sum of nine cubes of primes.

When the number of variables \( j \) is becoming smaller, such as, \( 5 \leq j \leq 8 \), we could consider the exceptional sets of these problems. Denote by \( E_j(N) \) the set of integers \( n \in \mathcal{A}_j \), not exceeding \( N \) with \( j = 5, 6, 7, 8 \) such that

\[
n = p_1^3 + p_2^3 + \ldots + p_j^3,
\]

where

\[
\begin{align*}
\mathcal{A}_5 : &= \{ n \in \mathbb{N} : n \equiv 1 \pmod{2}, n \not\equiv 0, \pm 2 \pmod{9}, n \not\equiv 0 \pmod{7} \}, \\
\mathcal{A}_6 : &= \{ n \in \mathbb{N} : n \equiv 0 \pmod{2}, n \not\equiv \pm 1 \pmod{9} \}, \\
\mathcal{A}_7 : &= \{ n \in \mathbb{N} : n \equiv 1 \pmod{2}, n \not\equiv 0 \pmod{9} \}, \\
\mathcal{A}_8 : &= \{ n \in \mathbb{N} : n \equiv 0 \pmod{2} \}.
\end{align*}
\]

Hua [1] also proved that \( E_5(N) \ll N(\log N)^{-A} \), where \( A > 0 \) is arbitrary. In 2000, Ren [2] improved \( E_5(N) \ll N^{152/153+\epsilon} \) for \( j = 5 \). In 2005, Kumchev [3] proved the following theorem in this realm.

Theorem 1.1. Let \( j = 5, 6, 7, 8 \), let \( \mathcal{A}_j \) be defined as in (2), and define \( \theta_j \) by

\[
\theta_5 = 79/84, \quad \theta_6 = 31/35, \quad \theta_7 = 17/28, \quad \theta_8 = 23/84.
\]
Then we have

\[ E_j(N) \ll N^{\delta_j}. \]

In 2014, Zhao [4] improved the above to \( \theta_7 = 1/2 + \varepsilon, \theta_8 = 1/6 + \varepsilon \) for \( j = 7, 8 \). In this paper, if \( j = 7, 8 \), we investigate this problem with \( p_j \) taking values in short intervals, i.e.

\[ n = p_1^3 + p_2^3 + \ldots + p_j^3 \quad \text{with} \quad |p_i - (N/j)^{1/3}| \leq \gamma, \quad 1 \leq i \leq j, \tag{3} \]

where \( y = o(N^{1/3}) \) and \( p_j \) are primes. Let \( E_j(N, y) \) denote the number of integers that \( N \in A_j, N \leq n \leq N + N^{1/3}y \), which cannot be represented as in (3). In the case of \( j = 7, 8 \), our result is stated as follows.

**Theorem 1.2.** For \( y = N^{1/3-\delta} \) with \( 0 < \delta < 1/27 \), we have \( E_8(N, y) \ll N^{25/16} \), \( E_7(N, y) \ll N^{23/16} \), where \( 0 < \theta \leq 1, \delta \) and \( \theta \) satisfy \( 3\delta + \theta = 1 \).

Theorem 1.2 is proved by the circle method. When treating the major arcs, we apply the iterative method of Liu [5] and a mean value theorem of Choi and Kumchev [6] to establish the asymptotic formula. It is known that the estimation of exponential sums plays an essential role in this problem. The upper bound of the exponential sums leads to the final result directly. So the new estimate for exceptional sums over primes in short interval in Kumchev [7] plays an important role in treating the minor arcs.

For \( j = 9 \), Liu and Xu [8] proved that (3) holds unconditionally with \( \delta = 1/198 \), which is as strong as the result under the Generalized Riemann Hypothesis. In the direction of Waring-Goldbach problem in short intervals, there have been some developments in the last few years. Such as the recent work of Wei and Wooley [9], Huang [10] and Kumchev and Liu [11]. There are other similar problems (see [12, 13] and their references).

**Remarks:**

1. In this series of problems, we could not only focus our attention to the size of \( y \), but also concern with the cardinality of \( E_j(N, y) \) for \( j = 7, 8 \), such as Theorem 2 in Liu and Sun [14]. In this paper, we give the exact relation formula about the length of short intervals and the size of exceptional sets. Compared with Theorem 2 in [14], a wider range of the length of short intervals is given. At the same time, quantitative relation between size of exceptional sets and length of short intervals is obtained, i.e., \( 3\delta + \theta = 1 \).

2. This paper is focusing on the quantitative relation of short intervals and the exceptional sets. In the reference [15], improved results in shorter intervals are given. Compared with their fixed results for the number of prime variables, we actually obtain the wider range of short interval. Though the number of primes is different, the results are the same by the method in the paper.

3. As the number of variables is becoming smaller, the more difficult the question is. For example, when \( j = 4 \), it is a conjecture out of reach at present. Moreover, as the length of short intervals is becoming smaller, the size of exceptional sets is becoming larger. When \( j = 5, 6 \), this method does not work for this question, so we could not obtain similar results.

**Notation.** As usual, \( \varphi(n) \) and \( \Lambda(n) \) stand for the functions of Euler and von Mangoldt, respectively. The letter \( N \) is a large integer, and \( L = \log N \). The notation \( A \asymp B \) means that \( c_1A \leq B \leq c_2A \), \( r \sim R \) means \( R < r < 2R \). The letter \( \varepsilon \) denotes a positive constant, which is arbitrary small, but not the same at different occurrences.

## 2 Outline of the method

In this section, we give an outline of the proof of Theorem 1.2 (take \( j = 8 \) for example). In order to apply the circle method, for some \( \delta > 0 \), we set

\[ x = (N/8)^{1/3}, \quad y = N^{1/3-\delta+\varepsilon}, \tag{4} \]
and
\[
\begin{aligned}
P &= N^{1/21}, & Q &= N^{2/3}, \\
Q' &= N^{20/21}, & Q'' &= N^{31/36+2\varepsilon}.
\end{aligned}
\]

By Dirichlet’s lemma ([16], Lemma 2.1), each \( \alpha \in [1/Q', 1 + 1/Q'] \) may be written in the form
\[
\alpha = a/q + \lambda, \quad |\lambda| \leq 1/(qQ')
\]
for some integers \( a, q \) with \( 1 \leq a \leq q \leq Q' \) and \( (a, q) = 1 \). Denote by \( M(a, q) \) the set of \( \alpha \) satisfying (6), and define the major arcs \( \tilde{M} \) as follows:
\[
\tilde{M} := \bigcup_{1 \leq p < P} \bigcup_{\substack{q \leq Q' \leq 1 \leq \lambda < 1/(qQ')}} M(a, q).
\]

Again by Dirichlet’s lemma, each number \( \alpha \in [1/Q', 1 + 1/Q'] \setminus \tilde{M} \) can be written as
\[
\alpha = a/q + \lambda, \quad |\lambda| \leq 1/(qQ)
\]
with \( (a, q) = 1, 1 \leq a \leq q \leq Q \). Define the minor arcs \( C(\tilde{M}) \) to be the set of \( \alpha \in [1/Q', 1 + 1/Q'] \setminus \tilde{M} \) satisfying (8) with \( P \leq q \leq Q \).

Obviously, \( \tilde{M} \) and \( C(\tilde{M}) \) are disjoint. Let \( \bar{R} \) be the complement of \( \tilde{M} \) and \( C(\tilde{M}) \) in \([1/Q', 1 + 1/Q']\), so that
\[
[1/Q', 1 + 1/Q'] = \tilde{M} \cup C(\tilde{M}) \cup \bar{R}.
\]

Let \( N \) be a sufficiently large integer and \( n \in A_8 \) satisfying \( N \leq n \leq N + N^{2/3}y \). Denote by
\[
r(n) := \sum_{x - y \leq p \leq x + y} \log p_1 \ldots \log p_8 = \int_{1/Q'}^{1+1/Q'} S^8(a)e(-an) \, da,
\]
where \( e(t) = e^{2\pi it} \) and
\[
S(a) := \sum_{x - y \leq p \leq x + y} \log p e(ap^3).
\]

Then we can write
\[
r(n) = \int_{1/Q'}^{1+1/Q'} S^8(a)e(-an) \, da = \int_{\tilde{M}} + \int_{C(\tilde{M})} + \int_{\bar{R}}.
\]

Clearly, in order to prove Theorem 1.2, it is sufficient to show that \( r(n) > 0 \) for almost all integers \( n \in A_8 \cap [N, N + N^{2/3}y] \). The tools that we need are an estimate for exponential sums over primes in short intervals of Liu, Lü and Zhan [17], Kumchev [7] and a mean value theorem of Choi and Kumchev [6], which are stated as follows.

**Lemma 2.1.** For integer \( k \geq 1 \), let \( 2 < y \leq x \) and \( \alpha = a/q + \lambda \) be a real number with \( 1 \leq a \leq q \) and \( (a, q) = 1 \). Define
\[
\Xi := |\lambda|x^k + x^2y^2.
\]

Then for any fixed \( \varepsilon > 0 \), we have
\[
\sum_{x - y \leq p \leq x + y} \log p e(ap^k) \ll (qx)^\varepsilon \left\{ q^\varepsilon y \Xi^{1/2} + q^\varepsilon x^{1/2} \Xi + y^{1/4} x^{1/2} + x^{1/2} \Xi^{1/2} + \frac{x}{\Xi} + \frac{1}{q^{1/2} \Xi^{1/2}} \right\},
\]
where the implied constant depends on \( \varepsilon \) and \( k \) only.
Lemma 2.2. Let $\alpha$ be a positive integer, $R \geq 1$, $T \geq 1$, $X \geq 1$ and $\kappa = 1/ \log X$. Then there is an absolute positive constant $c$ such that

$$
\sum_{n \leq x} \sum_{r \leq T} \sum_{\nu \equiv r \mod T} \sum_{t \geq x/2X} \frac{\Lambda(n)\chi(n)}{n^{\nu + \tau}} \left| d\tau \ll (t^{-\frac{1}{3}}R^2TX^{11/20} + X)(\log RTX)^c,
\right.
$$

where the implied constant is absolute.

Next we bound $S(\alpha)$ on $C(\mathbb{M}) \cup R$. We first estimate $S(\alpha)$ on $C(\mathbb{M})$, and this has been done in Kumchev [7] in which Theorem 1.2 states that

Lemma 2.3. Let $k \geq 3$ and $\theta$ be a real number with $(2k + 2)/(2k + 3) < \theta \leq 1$. Suppose that $0 < \rho < \rho_k(\theta)$, where

$$
\rho_k(\theta) = \begin{cases} 
\min(\frac{1}{T}, 2\theta + 1), & \text{if } k = 3, \\
\min(\frac{1}{T}, 3\theta + 1), & \text{if } k \geq 4,
\end{cases}
$$

with $\sigma_k$ defined by $\sigma_k^{-1} = \min(2^{k-1}, 2k(k-2))$. Then, for any fixed $\varepsilon > 0$,

$$
\sup_{x \geq \rho_k(\theta)} |\sum_{x \leq n \leq x^\theta} \Lambda(n)e(\alpha n^k)| \ll x^{\rho - \rho_k(\theta) + \varepsilon} + x^{\rho_k(\theta) - 1/2}.
$$

Therefore, by Lemma 2.3 for $\alpha \in C(\mathbb{M})$, we have

$$
S(\alpha) \ll yx^{\rho_1 + \varepsilon},
$$

in view of $y = x^\theta$, $8/9 < \theta \leq 1$ and our choice of $P$ in (5).

Now we estimate $S(\alpha)$ on $R$. To this end, also by Dirichlet's lemma on rational approximation, we further write $R = \mathbb{R}_1 \cup \mathbb{R}_2$, where

$$
\mathbb{R}_1 = \left\{ \alpha : 1 \leq q \leq P, \frac{1}{qQ} < |\alpha| \leq \frac{1}{qQ} \right\},
$$

and

$$
\mathbb{R}_2 = \left\{ \alpha : P < q \leq P, |\alpha| \leq \frac{1}{qQ} \right\}.
$$

For $\alpha \in \mathbb{R}_1$, we have $|\alpha| \geq \frac{1}{qQ} \geq \frac{1}{N^{2/3}y^2}$, and therefore,

$$
\Xi \asymp |\alpha|N + \frac{x^2}{y^2} \asymp |\alpha|N.
$$

Lemma 2.1 gives, for $\alpha \in \mathbb{R}_1$,

$$
S(\alpha) \ll N^\varepsilon \left\{ \frac{y\sqrt{q|\alpha|N}}{N^{\frac{1}{2}}} + N^{\frac{1}{2}}q^{\frac{1}{2}}(|\alpha|N)^{\frac{1}{2}} + y^\frac{1}{2}N^{\frac{1}{4}} + \frac{N^{\frac{1}{2}}}{\sqrt{q|\alpha|N}} \right\}
\ll yx^{\rho_1 + \varepsilon}.
$$

If $\alpha \in \mathbb{R}_2$, then

$$
P < q \leq P, |\alpha| \gg N^{2/3}y^{-2}, q\Xi \ll NQ^{-1} + PN^{2/3}y^{-2}.
$$

By Lemma 2.1,

$$
S(\alpha) \ll N^\varepsilon \left\{ \frac{y(q\Xi)^{\frac{1}{2}}}{N^{\frac{1}{2}}} + N^{\frac{1}{2}}q^{\frac{1}{2}}(q\Xi)^{\frac{1}{2}} + y^{\frac{1}{2}}N^{\frac{1}{4}} + \frac{N^{\frac{1}{2}}}{\Xi^{\frac{1}{2}}} + \frac{N^{\frac{1}{2}}}{(P\Xi)^{\frac{1}{2}}} \right\}
\ll yx^{\rho_1 + \varepsilon}.
$$

From (13)-(15), we get
Proposition 2.4. Let $C(M)$ and $\mathcal{R}$ be defined as the above, then we have
\[
\max_{a \in C(M) \cup \mathcal{R}} S(a) \ll y x^{-p+\varepsilon}.
\] (16)

For the major arcs, we have the following asymptotic formula, which will be proved in Section 4.

Proposition 2.5. Let $M$ be defined as in (7). Then for any sufficiently large $n \in A_8 \cap [N, N + N^{2/3}y]$, we have
\[
\int_M S^8(a) e(-an) \, da \sim C_8 E_8(n) N^{-\frac{1}{2}} y^7,
\] (17)
where $C_8$ is a positive constant, $\varphi(q)$ is the Euler function and
\[
E_8(n) := \sum_{q=1}^{\infty} \frac{\varphi(q)}{q} \sum_{a \in \mathbb{Z}, (a,q)=1} \left( \sum_{h=1}^{q} e \left( \frac{ah^3}{q} \right) \right)^{\frac{8}{5}} e \left( -\frac{an}{q} \right).
\]

In order to prove Theorem 1.2, we also need the following lemma, which can be viewed as a generalization of Hua’s lemma ([16], Lemma 2.5) in short intervals.

Lemma 2.6. Let $k$ be a positive integer, $X \geq Y \geq 2$ and
\[
S_k^*(a) := \sum_{X-Y \leq n \leq X+Y} e(an^k).
\]
Then for any $\varepsilon > 0$ and $1 \leq s \leq k$, we have
\[
\int_0^1 |S_k^*(a)|^{2s} \, da \ll \varepsilon X^s Y^{2s-\varepsilon}.
\]

Proof. This is Lemma 4.1 in Li and Wu [18]. \qed

Proof of Theorem 1.2. To prove Theorem 1.2, we apply the method introduced by Wooley [19]. We only give detailed proof for $j = 8$. Denote by $E_8^*(N, y)$ the set of integers $n \in A_8 \cap [N, N + N^{2/3}y]$ such that
\[
n = p_1^3 + \ldots + p_8^3 \quad \text{with} \quad |p_i - (N/8)^{1/3}| \leq N^{1/3 - \delta + \varepsilon} (1 \leq i \leq 8).
\]
Introduce the function
\[
Z(a) := \sum_{n \in E_8^*(N, y)} e(-an).
\]
Clearly, we have
\[
\int_0^1 S^8(a) Z(a) \, da = 0,
\]
and write $\text{Card}(E_8^*(N, y)) = Z$
\[
\int_0^1 |Z(a)|^{2} \, da = |E_8^*(N, y)| = Z.
\]
By Proposition 2.5, we have
\[
\left| \int_{C(M) \cup \mathcal{R}} S^8(a) Z(a) \, da \right| = \left| \int_M S^8(a) Z(a) \, da \right| = \sum_{n \in E_8^*(N, y) \backslash M} \int S^8(a) e(-an) \, da
\]
\[ ZN^{-\frac{3}{4}}y^7. \]  

From (17) and (18), we deduce that
\[ |E^*_N(z)|N^{-\frac{3}{4}}y^7 \ll \int_{C(M) \cup \mathbb{R}} |S^8(a)Z(a)| \, da \ll \max_{a \in C(M) \cup \mathbb{R}} |S(a)|I_1^2I_2^2, \]
where \( I_1 = \int_0^1 |S(a)|^6|Z(a)|^2 \, da, I_2 = \int_0^1 |S(a)|^8 \, da. \) One could find that (see Lemma 6.2 in [19])
\[ I_1 \ll y^6(y^3Z^2 + y^4Z). \]  

And Lemma 2.6 implies
\[ \int_0^1 |S(a)|^8 \, da \ll \int_0^1 |S^3(a)|^8 \, da \ll N^3y^5. \]  

Collecting (16), (19), (20) and noting that \( x = (N/8)^{1/2} \), we obtain
\[ Zy^7x^{-2} \ll x^{-\rho+\varepsilon}y^5(Z + y^{1/2}Z^{1/2}) \]
If \( \rho \) is such that \( y^2 \gg x^{-\rho+\varepsilon} \), this leads to the bound
\[ Z \ll N^{(2\rho)/3+\varepsilon}x^{-3} \]
which implies \( Z \ll N^{2\rho/3+\varepsilon} \), by the choice of \( \rho \) in Lemma 2.3.

In the case of \( j = 7 \), we obtain the following asymptotic formula on major arcs by similar argument as described in Section 4,
\[ \int_{\mathcal{M}} S^7(a)e(-an) \, da \sim C_7\mathcal{E}_7(n)N^{-\frac{3}{4}}y^6. \]  

For \( j = 7 \), estimations on \( C(M) \cup \mathbb{R} \) are also similar to the case of \( j = 8 \). The other treatment is quite similar, so we omit the details. This completes the proof of Theorem 1.2. \( \square \)

### 3 Preliminaries for Proposition 2.5

For \( \chi \) mod \( q \), define
\[ C(\chi, a) := \sum_{h=1}^{q} \chi(h)e \left( \frac{ah^3}{q} \right), \quad C(q, a) := C(\chi^0, a). \]  

If \( \chi_1, \chi_2, \ldots, \chi_s \) are characters mod \( q \), then we write
\[ B_0(n, q; \chi_1, \ldots, \chi_s) = \sum_{a=0}^{q} e \left( -\frac{an}{q} \right) C(\chi_1, a)C(\chi_2, a) \ldots C(\chi_s, a), \]  
and
\[ B_0(n, q) = B_0(n, q; \chi^0, \ldots, \chi^0). \]  

The following lemma is important for proving Proposition 2.5.
Lemma 3.1. Let $\chi_i \mod r_i$ with $i = 1, \ldots, 8$ be primitive characters, $r_0 = [r_1, \ldots, r_8]$, and $\chi^0$ be the principal character $\mod q$. Then
\[
\sum_{q \leq z} \frac{1}{\varphi(q)} \left| B_0(n, q; \chi_1 \chi^0, \ldots, \chi_8 \chi^0) \right| \ll r_0^{-3} \log^c z.
\]

**Proof.** It is similar to that of Lemma 7 in [20], so we omit the details. \qed

Recall the definition of $x, y$ as in (4), and define
\[
S_0(\lambda) := \sum_{x-y \leq n \leq x+y} e(\lambda n^3),
\]
and
\[
W(\chi, \lambda) := \sum_{x-y \leq p \leq x+y} (\log p) \chi(p) e(\lambda p^3) - \delta_\chi S_0(\lambda),
\]
where $\delta_\chi = 1$ or 0 according as $\chi$ is principal or not. We also set
\[
W_\chi^d := \max_{|\lambda| \leq 1/\sqrt{d}} |W(\chi, \lambda)|, \quad \| W_\chi \|_2 := \left( \int_{-1/\sqrt{d}}^{1/\sqrt{d}} |W(\chi, \lambda)|^2 \, d\lambda \right)^{1/2}.
\]

Define further
\[
J(d) = \sum_{r \leq P'} [d, r]^{-3+\varepsilon} \sum_{\chi \mod r} W_\chi^d,
\]
\[
K(d) = \sum_{r \leq P'} [d, r]^{-3+\varepsilon} \sum_{\chi \mod r} \| W_\chi \|_2,
\]
where the sum $\sum_{\chi \mod r}$ denotes summation for all primitive characters modulo $r$. The proof of Proposition 2.5 depends on the following two lemmas, which will be proved in Section 5.

Lemma 3.2. Let $P$, $Q'$ be as in (2.2). We have
\[
K(d) \ll d^{-3+\varepsilon} y^{1/2} N^{-1/3} L^c.
\]

Lemma 3.3. Let $P$, $Q'$ be as in (2.2). We have
\[
J(d) \ll d^{-3+\varepsilon} y L^c.
\]

Further if $d = 1$, the estimate can be improved to
\[
J(1) \ll y L^{-A},
\]
where $A > 0$ is arbitrary.

## 4 Proof of Proposition 2.5

With Lemmas 3.2 and 3.3 known, we can use the iterative idea in Liu [5] to prove Proposition 2.5.

**Proof of Proposition 2.5.** Since $q \leq P$, we have $(p, q) = 1$ for $p \in (x-y, x+y]$. Using the orthogonality relation, we can write
\[
S \left( \frac{a}{q} + \lambda \right) = \frac{C(q, a)}{\varphi(q)} S_0(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \mod q} C(\chi, a) W(\chi, \lambda),
\]
where $\lambda = \frac{a}{q} + \lambda$. \qed
where $S_0(\lambda)$ and $W(\chi, \lambda)$ are as in (25). By (32), we can write
\[
\int_{\mathbb{H}} S^8(a) e(-aN) \, da = \sum_{0 \leq k \leq 8} C^k_8 I_k, \tag{33}
\]
where
\[
I_k := \sum_{1 \leq q \leq P} \frac{1}{q^8(q)} \sum_{a=1}^{q} C_0^{k}(q, a) e\left(\frac{-aN}{q}\right) \int_{-1/rQ'}^{1/rQ'} \left( \sum_{\chi \text{ mod } q} C(\chi, a) W(\chi, \lambda) \right)^k \, d\lambda.
\]
We will prove that $I_0$ produces the main term, and the other $I_k (1 \leq k \leq 8)$ contribute to error term.

The computation of $I_0$ is standard, and we can prove
\[
I_0 = C_8 \mathcal{E}_8(n) N^{-\varepsilon} y^7 \{1 + o(1)\}, \tag{34}
\]
where $C_8$ and $\mathcal{E}_8(n)$ are defined in Proposition 2.5.

It remains to estimate $I_k (1 \leq k \leq 8)$. We shall only treat $I_8$, the most complicated one. The treatment for $I_k (1 \leq k \leq 7)$ are similar.

Let
\[
I_8 = \sum_{1 \leq q \leq P} \frac{1}{q^8(q)} \sum_{a=1}^{q} \frac{e\left(\frac{-an}{q}\right)}{q^{1/rQ'}} \left( \sum_{\chi \text{ mod } q} C(\chi, a) W(\chi, \lambda) \right)^8 e(-\lambda \lambda) \, d\lambda.
\]

Suppose that $\chi_k (\text{mod } q)$ with $r_k | q$ being the primitive character inducing $\chi_k$. Thus we may write $\chi_k = \chi_k^0 \chi^0$, where $\chi^0$ is the principal character modulo $q$, $r_0 = [r_1, \ldots, r_8]$. It is easy to see that $W(\chi_k, \lambda) = W(\chi_k^0, \lambda)$. Since $r_0 = [r_1, \ldots, r_8] = [r_1, \ldots, r_7]$, by Lemma 3.1 and Cauchy’s inequality, we have
\[
|I_8| \ll L^4 \sum_{r_1 \leq P} \sum_{1 \leq x_1 \leq \frac{N}{r_1}} W_{x_1}^2 \sum_{r_2 \leq P} \sum_{1 \leq x_2 \leq \frac{N}{r_2}} W_{x_2}^2 \ldots \sum_{r_6 \leq P} \sum_{1 \leq x_6 \leq \frac{N}{r_6}} W_{x_6}^2 \|
\]
\[
\times \sum_{r_7 \leq P} \sum_{1 \leq x_7 \leq \frac{N}{r_7}} \| W_{x_7} \|_2 \sum_{r_8 \leq P} \sum_{1 \leq x_8 \leq \frac{N}{r_8}} \sum_{1 \leq x_8 \leq \frac{N}{r_8}} \| W_{x_8} \|_2.
\]

Now we introduce an iterative procedure to bound the above sums over $r_9, \ldots, r_1$ consecutively. Since $r_0 = [r_1, \ldots, r_8] = [r_1, \ldots, r_7, r_8]$, we use (29) two times, (30) five times and (31) once to get
\[
|I_8| \ll L^4 \gamma^{1/2} N^{-1/3} \sum_{r_1 \leq P} \sum_{1 \leq x_1 \leq \frac{N}{r_1}} W_{x_1}^2 \ldots \sum_{r_6 \leq P} \sum_{1 \leq x_6 \leq \frac{N}{r_6}} W_{x_6}^2 \]
\[
\times \sum_{r_7 \leq P} \sum_{1 \leq x_7 \leq \frac{N}{r_7}} \| W_{x_7} \|_2 \sum_{r_8 \leq P} \sum_{1 \leq x_8 \leq \frac{N}{r_8}} \sum_{1 \leq x_8 \leq \frac{N}{r_8}} \| W_{x_8} \|_2
\]
\[
\ll L^4 N^{-2/3} \gamma^{y} \sum_{r_1 \leq P} \sum_{1 \leq x_1 \leq \frac{N}{r_1}} W_{x_1}^2 \]
\[
\ll L^4 N^{-2/3} \gamma^{y}, \tag{35}
\]
for any fixed $A > 0$.

Following a similar procedure to treat $I_8$, we can show that
\[
|I_k| \ll L^k N^{-2/3} \gamma^{y}, \quad (1 \leq k \leq 7). \tag{36}
\]
Now the required asymptotic formula follows from (33), (34), (35) and (36).

\[\square\]
5 Estimation of $K(d)$

The proofs of Lemmas 3.2 and 3.3 are rather similar to those of Proposition 2.2 in [18]. In order to use Choi and Kumchev’s mean value theorem effectively, we need a preliminary lemma in [18] as follows.

**Lemma 5.1.** Let $\chi$ be a Dirichlet character modulo $r$. Let $2 \leq X < Y \leq 2X$, $T_0 = (\log(Y/X))^{-1}$, $T = X^6$ and $\kappa = 1/\log X$. Define

$$F(s, \chi) := \sum_{X < n \leq 2X} \Lambda(n)\chi(n)n^{-s}.$$

Then we have

$$\sum_{X < n \leq Y} \Lambda(n)\chi(n) \ll \log \left(\frac{Y}{X}\right) \int_{|\tau| \leq T_0} |F(\kappa + i\tau, \chi)| \, d\tau + \int_{T_0 < |\tau| < T} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|} \, d\tau + 1. \quad (37)$$

The implied constant is absolute.

**Proof of Lemma 3.2.** Introduce

$$\widetilde{W}(\chi, \lambda) := \sum_{X < n \leq Y} \Lambda(n)\chi(n)e(\lambda n^3) - \delta_xS_0(\lambda).$$

Then we have

$$\widetilde{W}(\chi, \lambda) - W(\chi, \lambda) \ll N^{1/6},$$

which implies

$$\| W_x \|_2 \ll \| \widetilde{W}_x \|_2 + N^{1/6} \left(\frac{r}{Q^2}\right)^{1/2}.$$

The contribution of $O\left(N^{1/6}(r/Q^2)^{1/2}\right)$ to the left-hand side of (29) is

$$\ll N^{1/6} \sum_{r \in P_r} [d, r]^{-3+\epsilon} \frac{T^{1/2}}{(Q^2)^{1/2}}$$

$$\ll d^{-3+\epsilon} N^{1/6} \frac{T^{-1/2}}{Q^{1/2}} \sum_{r \in P_r} \left(\frac{r}{T}\right)^{-3+\epsilon} r^{1/2}$$

$$\ll d^{-3+\epsilon} N^{1/6} \frac{T^{-1/2}}{Q^{1/2}} \sum_{l \in P_r} \sum_{l \in P_r} \left(\frac{T}{r}\right)^{-5+2\epsilon}$$

$$\ll d^{-3+\epsilon} N^{1/6} \frac{T^{-1/2}}{Q^{1/2}} \sum_{l \in P_r} \sum_{l \in P_r} \left(\frac{T}{r}\right)^{-5+2\epsilon} L^c,$$

where we have used $[d, r](d, r) = dr$, $l = (d, r)$, (4) and (5). Thus in order to prove (29), it suffices to show that

$$\sum_{r \sim R} [d, r]^{-3+\epsilon} \sum_{\chi \mod r} \| \tilde{W}_x \|_2 \ll d^{-3+\epsilon} N^{1/6} L^c \quad (38)$$

for any $R  \sim P_r$.

By Gallagher’s lemma ([21], Lemma 1), we have

$$\| \tilde{W}_x \|_2 \ll \frac{1}{RQ} \left( \int_{\infty}^{+\infty} \left| \sum_{n \sim \sqrt{RQ^3}} (\Lambda(n)\chi(n) - \delta_x) \right|^2 \, dv \right)^{1/2}$$

$$\ll \frac{1}{RQ} \left( \int_{-\infty}^{+\infty} \left| \sum_{n \sim \sqrt{RQ^3}} (\Lambda(n)\chi(n) - \delta_x) \right|^2 \, dv \right)^{1/2}, \quad (39)$$

where $\chi \mod r$, $l = (d, r)$, (4) and (5). Thus in order to prove (29), it suffices to show that

$$\sum_{r \sim R} [d, r]^{-3+\epsilon} \sum_{\chi \mod r} \| \tilde{W}_x \|_2 \ll d^{-3+\epsilon} N^{1/6} L^c \quad (38)$$

for any $R  \sim P_r$.
where
\[ X := \max\{(v - R Q'/3)^{1/3}, x - y\}, \quad Y := \min\{(v + R Q'/3)^{1/3}, x + y\}. \]

If \( R = 1 \), we have
\[
\left| \sum_{X \leq n \leq Y} (A(n) \chi(n) - \delta) \right| = \left| \sum_{X \leq n \leq Y} (A(n) - 1) \right| \ll (Y - X) L \\
\ll \{(v + Q'/3)^{1/3} - (x - y)\} L \\
\ll Q' N^{-2/3} L.
\]

which implies, in view of \( Q' < x^2 y \),
\[
d^{-3+\varepsilon} \| \tilde{W}_R \|_2 \ll d^{-3+\varepsilon} Q'^{-1} \left( (Q' N^{-2/3} L)^2 (x^2 y + Q') \right)^{1/2} \\
\ll d^{-3+\varepsilon} y^{1/2} N^{-1/3} L^c.
\]

For \( R \geq 2 \) and \( r \sim R \), we have \( \delta = 0 \). Thus, we can apply (37) to write
\[
\| \tilde{W}_R \|_2 \ll \left( \frac{y}{x^2} \right)^{1/2} \int_{|\tau| < T_0} |F(x + it \tau, \chi)| \, d\tau \\
+ \frac{(x^2 y)^{1/2}}{R Q'} \int_{T_0 < |\tau| < T} \frac{|F(x + it \tau, \chi)|}{|\tau|} \, d\tau + \frac{(x^2 y)^{1/2}}{R Q'}. \tag{41}
\]

Since
\[ T_0^{-1} = \log(Y/X) = R Q' v^{-1} \ll R Q' x^{-3}, \]
and
\[ (x + y)^3 + R Q'/3 - (x - y)^3 + R Q'/3 \ll x^2 y, \]

Therefore, the contribution of the first term of (41) to the left-hand side of (38) is
\[
\ll d^{-3+\varepsilon} (x^{-4} y)^{1/2} \sum_{\mathbb{Z} : l \leq R} \left( \frac{R}{T} \right)^{-3+\varepsilon} (l^{-1} R^2 T_0 x^{11/20} + x) \\
\ll d^{-3+\varepsilon} N^{-1/3} y^{1/2} (N^{17/20} Q^{-1} + 1) L^c \\
\ll d^{-3+\varepsilon} N^{-1/3} y^{1/2} L^c. \tag{42}
\]

Introducing
\[ M(l, R, T', x) := \sum_{\mathbb{Z} : l \leq R} \sum_{\mathbb{Z} : x \mod r} \int_{T} \frac{d\tau}{|\tau|} |F(x + it \tau, \chi)| \, d\tau. \]

The contribution of the second term of (41) to the left-hand side of (38) is
\[
\ll d^{-3+\varepsilon} (x^2 y)^{1/2} (R Q')^{-1} \sum_{\mathbb{Z} : l \leq R} \left( \frac{R}{T} \right)^{-3+\varepsilon} \max_{T_0 < T \leq T'} \left( T'^{-1} M(l, R, T', x) \right) \\
\ll d^{-3+\varepsilon} y^{1/2} (R Q')^{-1} \sum_{\mathbb{Z} : l \leq R} \left( \frac{R}{T} \right)^{-3+\varepsilon} (l^{-1} R^2 x^{11/20} + T_0^{-1} x) L^c \\
\ll d^{-3+\varepsilon} N^{-1/3} y^{1/2} (N^{17/20} Q^{-1} + 1) L^c \\
\ll d^{-3+\varepsilon} N^{-1/3} y^{1/2} L^c. \tag{43}
\]

Finally, the contribution of the last term of (41) to the left-hand side of (38) is
\[
\ll d^{-3+\varepsilon} (x^2 y)^{1/2} (R Q')^{-1} \sum_{\mathbb{Z} : l \leq R} \sum_{\mathbb{Z} : x \mod r} \left( \frac{R}{T} \right)^{-3+\varepsilon} \]


\[ \ll d^{-3+\varepsilon} N^{-1/3} y^{1/2} Q^{-1} N^{2/3} \]
\[ \ll d^{-3+\varepsilon} N^{-1/3} y^{1/2} L^c. \]

Now the inequality (38) follows from (40), (42), (43) and (44). This completes the proof of Lemma 3.2. \(\square\)

### 6 Estimation of \(J(d)\)

In this section, we establish Lemma 3.3. The idea of the proof is similar to that of Lemma 3.2, but there are several differences.

**Estimation of \(J(d)\).** Replacing \(W(\chi, \lambda)\) by \(\tilde{W}(\chi, \lambda)\) as in §5, we get that the resulting error is

\[ \ll \sum_{r \in P_r} [d, r]^{-3+\varepsilon} r N^{1/6} \ll d^{-3+\varepsilon} N^{1/6} \sum_{l \in P_r} e^{-3+\varepsilon} \sum_{i \in P} \ll d^{-3+\varepsilon} N^{1/6} p_r^3 \ll d^{-3+\varepsilon} y L^c. \]

Hence, Lemma 3.3 is a consequence of the estimate

\[ \sum_{r \in P_r} [d, r]^{-3+\varepsilon} \sum_{\chi \mod r} \max_{|\lambda| \leq 1/(16Q)} \tilde{W}(\chi, \lambda) \ll d^{-3+\varepsilon} y L^c, \quad (45) \]

where \(R \leq P_r\) and \(c > 0\) is some constant.

The case \(R < 1\) contributes to \(d^{-3+\varepsilon} y L\) which is obviously acceptable. For \(R \geq 1\), we have \(\delta_\chi = 0\). Thus,

\[ \tilde{W}(\chi, \lambda) = \sum_{x-y \in \mathbb{N} \times y} \Lambda(m) \chi(m) e(m^3 \lambda). \]

Define

\[ H(s, \chi) = \sum_{x-y \in \mathbb{N} \times y} \Lambda(m) \chi(m) m^{-s}, \quad V(s, \lambda) = \int_{x-y} w^{s-1} e(\lambda w^2) dw. \]

By partial summation and Perron’s summation formula, we get

\[ \tilde{W}(\chi, \lambda) = \frac{1}{2\pi i} \int_{b-iT} H(s, \chi) V(s, \lambda) ds + O(1), \quad (46) \]

where \(0 < b < L^{-1}\) and \(T = (1 + |\lambda| N) y L^2\). Using Lemmas 4.3, 4.5 of [22] and a trivial estimate, we have

\[ V(s, \lambda) \ll N^{b/3} \min \left\{ \frac{y}{N^{1/3}}, \frac{1}{\sqrt{|t| + 1}}, \max_{x-y \in \mathbb{N} \times y} \frac{1}{|t + 6\pi \lambda w^2|} \right\}. \]

Take

\[ T = \frac{N^{2/3}}{y^2}, \quad T_* = \frac{12\pi N}{RQ}. \]

Then for \(b \to 0\), \(\tilde{W}(\chi, \lambda)\) is bounded by

\[ \tilde{W}(\chi, \lambda) \ll \frac{y}{N^{1/3}} \int_{|t| < T} |H(it, \chi)| dt + \int_{\mathbb{R} \times |t| < T} |H(it, \chi)| \frac{dt}{\sqrt{|t| + 1}}. \]
Thus, it suffices to show that the estimates
\[
\sum_{r \sim R} \sum_{\chi \mod r} [d, r]^{-3+\varepsilon} \sum_{r \sim R} \chi \mod r \int_{T_1}^{2T_1} |H(it, \chi)| dt \ll d^{-3+\varepsilon} N^{1/3} L^c
\]
holds for \( R \leq P \) and \( 0 < T_1 \leq \hat{T} \);
\[
\sum_{r \sim R} [d, r]^{-3+\varepsilon} \sum_{\chi \mod r} \int_{T_2}^{2T_2} |H(it, \chi)| dt \ll \varepsilon d^{-3+\varepsilon} U(T_2 + 1)^{1/2} L^c
\]
holds for \( R \leq P \) and \( \hat{T} < T_2 \leq T_* \); and
\[
\sum_{r \sim R} \sum_{\chi \mod r} (d, r)^{-3+\varepsilon} \sum_{\chi \mod r} \int_{T_3}^{2T_3} |H(it, \chi)| dt \ll d^{-3+\varepsilon} N(RQ^*)^{-1} UL^c
\]
holds for \( R \leq P \) and \( T_* < T_3 \leq T \).

The estimates (49), (50) and (51) follow from Lemma 2.2 via an argument similar to that leading to (43), so we omitted the details. The first part of Lemma 3.3 is proved.

**Estimation of \( J(1) \).** The result is the same as that of \( J(d) \) except for the saving of \( L^{-A} \) on its right hand side.

To order to get this saving, we have to distinguish two cases \( L^c < R \leq P \) and \( R \leq L^c \), where \( C \) is a constant depending on \( A \). The proof of the first case is the same as that of \( J(d) \), so we omit the details.

Now we prove the second case \( R \leq L^c \). We use the well-known explicit formula
\[
\sum_{m \leq u} \Lambda(m) \chi(m) = \delta_u u - \sum_{|y| \leq T} \frac{U^\rho}{\rho} + O \left( \left\{ \frac{U}{T} + 1 \right\} \log^2 (ruT) \right),
\]
where \( \rho = \beta + iy \) is a non-trivial zero of the function \( L(s, \chi) \), and \( 2 \leq u \leq T \) is a parameter. Then by inserting (44) into \( \hat{W}(\chi, A) \), and applying partial summation formula, we get
\[
\hat{W}(\chi, \lambda) = \sum_{x \sim y} e(u^3 \lambda) \left\{ \sum_{x < y < u} (\Lambda(m) \chi(m) - \delta_x) \right\}
\]
\[
= \sum_{x \sim y} e(u^3 \lambda) \sum_{|y| \leq T} u^{\rho - 1} du + O \left( (1 + |\lambda|) y T^{-1} L^2 \right)
\]
\[
\ll y \sum_{|y| \leq T} N^{\beta_1} + O(NyQ^{*-1} T^{-1} L^2).
\]

Now let \( \eta(T) = c_2 \log^{-4/5} T \). By Prachar [23], \( \prod_{|y| \leq T} L(s, \chi) \) is zero-free in the region \( \sigma \geq 1 - \eta(T) \), \( |t| \leq T \) except for the possible Siegel zero. But by Siegel’s theorem (see [24], section 21), the Siegel zero does not exist in the present situation, since \( r \sim R \leq L^c \). Thus by the large-sieve type zero-density estimates for Dirichlet \( L \)-functions (see [25]), we have
\[
\sum_{r \sim R, X \mod r} \sum_{|y| \leq T} N^{\beta_1} \ll L^c \int_0^{1 - \eta(T)} T \frac{\log T}{2} N^{\alpha_1} d\alpha
\]
\[
\ll L^c \int_0^{1 - \eta(T)} N^{\frac{11\alpha_1 - \alpha}{5}} d\alpha \ll L^c N^{-\frac{11\alpha_1 - \alpha}{5}}
\]
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\[ \ll \exp(-c_2 \epsilon L^{1/5}) \]

provided that \( T = N^{1/10} \). Consequently,

\[ \sum_{\substack{d, r \neq P, \chi \mod r \ast \max_{|\lambda|<1/6Q'} \tilde{W}(\chi, \lambda) \ll d^{-3\epsilon_y} y L^{-A} \] provided that \( Q' = N^{1/136+2\epsilon} \). Then the lemma follows.

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