Radiation transfer in the cavity and shell of a planetary nebula

M. D. Gray, 1⋆ M. Matsuura2 and A. A. Zijlstra1

1Jodrell Bank Centre for Astrophysics, Alan Turing Building, University of Manchester, Manchester M13 9PL
2Department of Physics and Astronomy, UCL, Gower Street, London WC1E 6BT

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ABSTRACT
We develop an approximate analytical solution for the transfer of line-averaged radiation in the hydrogen recombination lines for the ionized cavity and molecular shell of a spherically symmetric planetary nebula. The scattering problem is treated as a perturbation, using a mean intensity derived from a scattering-free solution. The analytical function was fitted to Hα and Hβ data from the planetary nebula NGC 6537. The position of the maximum in the intensity profile produced consistent values for the radius of the cavity as a fraction of the radius of the dusty nebula: 0.21 for Hα and 0.20 for Hβ. Recovered optical depths were broadly consistent with observed optical extinction in the nebula, but the range of fit parameters in this case is evidence for a clumpy distribution of dust.

Key words: radiative transfer – stars: AGB and post-AGB – stars: atmospheres – circumstellar matter – stars: mass-loss – dust, extinction.

1 INTRODUCTION
Shells with internal cavities are found in almost all planetary nebulae (PNe) (Balick & Frank 2002). Such shells and cavities are related to the shaping of PNe by stellar winds. Low- and intermediate-mass stars lose their mass at a very high rate, forming a circumstellar envelope. Fast stellar winds from the central stars of PNe overtake the slower, denser, asymptotic giant branch (AGB) ejecta, resulting in the interaction of the fast and slow stellar wind components (Kwok, Purton & Fitzgerald 1978). The fast wind sweeps the slow wind, and leaves a cavity inside. Cavities are also found in supernova remnants (Dwek et al. 1987; Lagage et al. 1996), but these structures are more complicated (Bilikova et al. 2007). Shell-cavity dimorphism is also found in luminous blue variable (LBV) stars and in active galactic nuclei (AGN).

In one of the well-studied bipolar PNe, NGC 6537, the core consists of bright arcs tracing a shell surrounding an elongated cavity (Matsuura et al. 2005). Arcs are bright in Hα and other recombination lines, and extinction maps derived from Hα and Hβ suggest the presence of dust grains in the arcs. Matsuura et al. (2005) suggested that little dust exists in the cavity. Deriving accurate dust density from Hα and Hβ line maps is not straightforward, as the light from the central star is scattered within the arcs. In these arcs, both dust grains and gas are mixed together. Therefore, we developed a radiative transfer code to resolve cavity and shells for PNe, which includes scattered light in a shell.

A self-consistent radiative transfer code for this mixture has been developed by Ercolano, Barlow & Storey (2005). They have used a Monte Carlo method. Here, we use an analytical solution, with some approximations, and concentrate on the configuration of cavities and shells around them. The aim of this paper is therefore to obtain detailed physical insight through a simplified analytical model, which can complement the more complex information from numerical solutions.

2 THE MODEL OF NGC 6537
In this section, we describe the physical and radiative transfer models that are used in our analysis of NGC 6537. Although certain parameters are definitely specific to this object, taken from Matsuura et al. (2005), much of the model is generally applicable to any PNe of similar geometry, and in a similar state of evolution.

2.1 Physical model of the nebula
We assume that the nebula NGC 6537 is accurately spherically symmetric. The layout of the various radial zones of the object is summarized in Fig. 1. The central star is a white dwarf of negligible solid angle, both from an observer’s point of view and from the modeller’s point of view: no rays in the radiative transfer model are considered to start or end on the stellar surface.

The central star emits sufficient vacuum ultraviolet radiation to ionize a surrounding cavity. Ions in this cavity undergo frequent recombination and photoionization cycles, and it is the main source of Hα and Hβ line radiation. For NGC 6537 specifically, the observed flux ratio $F_{\text{Hß}}/F_{\text{Hα}} = 2.79$, where these fluxes are averages over the appropriate spectral-line bandwidth. From this ratio, we adopt radially constant values of the electron temperature, $T_e = 1.5 \times 10^5$ K, and electron number density, $n_e = 10^4$ cm$^{-3}$. 

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The angle-averaged intensity, $I_\nu$, is
\[ I_\nu = \frac{h}{4\pi} A_{e2}n_e(r) \phi_{e2}, \tag{1} \]
where $A_{e2}$ is the 'laboratory' line-centre frequency of the Hα line and $A_{e2}$ is the Einstein coefficient for spontaneous emission. The subscripts refer to the principal quantum numbers of atomic hydrogen, so $n_e(r)$ is the number density of H-atoms in the upper state of the transition. With an isothermal approximation, the lineshape function, $\phi_{e2}$, is not a function of radius, but may be a function of direction if there are significant radial motions in the cavity or shell. Given these definitions, a similar expression may be written down for the emission coefficient of Hβ:
\[ j_e(H\beta) = \frac{h}{4\pi} A_{e2}n_e(r) \phi_{e2}, \tag{2} \]

We assume complete velocity redistribution, so that absorption lineshape functions are identical to those specified above for emission.

The radiative transfer equation is
\[ \hat{\mathbf{h}} \cdot \nabla I_\nu = -[\kappa^e(r) + \sigma^e(r) + \kappa^s(r)]I_\nu + \kappa^s(r)B_c(T(r)) + j_e(r) + \sigma^e(r)J_e(r), \tag{3} \]

where $\hat{\mathbf{h}}$ is a unit vector along the ray of specific intensity $I_\nu$, opacity is provided by radiation-dependent absorption coefficients, $\kappa^e, \kappa^s$, for the line and continuum, respectively, and the continuum scattering coefficient, $\sigma^e$. Kirchoff's law allows the continuum emission from dust to be written as $\kappa^s(r)B_c(T(r))$, whilst the line emission coefficient is $j_e(r)$, and will be one of the specific forms in equations (1) or (2) for the appropriate transition. The final term on the right-hand side of equation (3), equal to $\sigma^e(r)J_e$, is the scattering integral assuming isotropic, elastic scattering, and $J_e$ is the angle-averaged intensity.

We expand the left-hand side of equation (3) in spherical polar coordinates (e.g. Peraiah 2002). This introduces $\mu = \cos \theta$ where $\theta$ is the angle between the direction of ray propagation and the radial direction. On the right, we combine the continuum absorption and scattering into an extinction coefficient, $\chi(r)$, and assume that the line absorption is negligible when compared to that in the continuum. The radiative transfer equation with these modifications is
\[ \frac{\partial I_\nu}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I_\nu}{\partial \mu} = -\chi(r)I_\nu + \kappa^s(r)B_c(T(r)) + \sigma^e(r)J_e + j_e(r). \tag{4} \]

We now integrate equation (4) over the spectral-line bandwidth appropriate to either the Hα or the Hβ line. We assume that this bandwidth is adequate to cover all the line radiation in the two hydrogen lines studied, regardless of all Doppler shifts within the source. In this context, we note that the velocity extent of the line due to the bulk motion of expansion is of the order of $20\,\text{km}\,\text{s}^{-1}$, whilst the thermal Doppler width is $21.4\,T_e^{1/2}\,\text{km}\,\text{s}^{-1}$, with $T_e$ equal to the kinetic temperature in the ionized cavity in units of $10^4\,\text{K}$. An unknown microturbulent width must be added in quadrature to the latter figure. It is therefore reasonable to suppose that even the extreme red- and blueshifted portions of the line are significantly blended by the thermal and microturbulent lineshape. In other words, the combination of a low terminal expansion velocity and a large thermal plus microturbulent line width allows us to neglect velocity field induced Doppler shifts that would otherwise complicate the analysis considerably. Let the spectral-line bandwidth be $\Delta \nu$, and the line-integrated intensity is given by
\[ I = \int_{-\Delta \nu/2}^{\Delta \nu/2} I_\nu \, dv. \tag{5} \]

A similar equation to equation (5) relates the angle-averaged intensities $I$ and $J_e$. If we assume also that the functions $\chi(r), \kappa^s, B_c(T)$ and $\sigma^e$ vary only very slightly over $\Delta \nu$, these functions can be removed from the filter integral when it is applied to equation (4). The result is
\[ \mu \frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} = -\chi(r)I + \Delta \nu \kappa^e(r)B_c(T(r)) + \sigma^e(r)J_e + j_e(r). \tag{6} \]

where $j_e(r) = \int_{-\Delta \nu/2}^{\Delta \nu/2} j_e \, dv$. The only frequency-dependent part of $j_e$ is the appropriate lineshape function (see equations 1 and 2). We assume that the filter width is sufficient for the normalization condition of the lineshape to hold, so that the integral $j_e(r)$ in equation (6) is either
\[ j_e(H\alpha) = \frac{h}{4\pi} A_{e2}n_e(r), \tag{7} \]
or the equivalent expression for Hβ.

Figure 1. A diagram of the PNe model. The spherically symmetric nebula is centred on the star, A, of negligible solid angle. This is surrounded by the cavity, B, and molecular shell, C. An infinitesimal volume, D, of the cavity is marked at radius, $r$. Optical radiation, $E$, escapes from the shell after suffering some degree of extinction. The particular values shown (in mag) apply to NGC 6537.

Surrounding the Hα cavity is a neutral shell, composed mainly of molecular material. This shell contains dust, which we assume, for the case of NGC 6537, to have a mass fraction of 0.01. Most of the shell material is, of course, in the form of molecular hydrogen. The shell produces approximately two magnitudes of visual extinction in NGC 6537, derived from the measured excess, $E(\text{Hα} - \text{Hβ})$. The shell is spatially resolved, and the extinction varies from 1.5 mag at the centre to 2.2 mag at 4 arcsec off centre.

At present, we leave the form of the density profile as an unknown function in both the cavity and shell zones of the nebula.

2.2 Radiative transfer model

The volume element $D$ in Fig. 1 has volume $\delta V$. If we let this be unit volume, then the radiated power per unit frequency and solid angle in the Hα line is
\[ j_e(H\alpha) = \frac{h}{4\pi} A_{e2}n_e(r) \phi_{e2}, \tag{1} \]

where $n_e(r)$ is the number density of H-atoms in the upper state of the transition. With an isothermal approximation, the lineshape function, $\phi_{e2}$, is not a function of radius, but may be a function of direction if there are significant radial motions in the cavity or shell. Given these definitions, a similar expression may be written down for the emission coefficient of Hβ:
\[ j_e(H\beta) = \frac{h}{4\pi} A_{e2}n_e(r) \phi_{e2}. \tag{2} \]
Finally, we re-write equation (6) in separate forms appropriate to
the cavity, and to the shell, respectively. In the cavity, we make the
approximation that there is no dust, and that other processes, such as
free–free and bound–free emission, make a negligible contribution to
the radiation flux within the filter bandwidths used (see e.g.
Matsuura et al. 2005). The cavity therefore acts effectively as a pure
source of line radiation, and the radiation transfer equation in this
zone is
\[ \frac{\mu}{r} \frac{dI}{dr} + \frac{(1 - \mu^2)}{r} \frac{dI}{d\mu} = j_0, \]
where \( j_0 \) is the constant emission coefficient. The left-hand side
of equation (10) is the spherical polar expansion of the dot product,
\( \hat{n} \cdot \nabla I \), but an alternative representation is as the derivative, \( dI/dr \).
4 MEAN INTENSITY IN THE CAVITY

To obtain the angle-averaged intensity, $I(r)$, we need to integrate over a number of different rays, which all meet at one point, where the radius, $r$, is a constant. At this point, $\theta$ is the final angle along its path, and can be used in the required solid angle integral. However, now $\omega$ is a variable, as we are integrating over many rays, and $\omega$ needs to be expressed in terms of $\theta$, or $\mu$. The situation is summarized in Fig. 3. Results from the cavity solution, which still apply to Fig. 3, are that $\sin \omega = (r/R) \sin \theta$ and $s = r \cos \theta + r \cos \omega$. We use the first of these relations to eliminate $\omega$ from the second. The expression for $s$ in turn be used to write the cavity solution, equation (16) in the form,

$$I(r, \mu) = I_0(\mu) + j_0(r \mu + [(R^2 - r^2) + r^2 \mu^2]^{1/2}),$$

(17)

noting that $\mu = \cos \theta$ is the only variable on the right-hand side of equation (17).

The solid angle integral for the mean intensity, with the integration over the azimuthal angle already carried out, is

$$J(r) = \frac{1}{2} \int_{-1}^{1} I(r, \mu) d\mu.$$  

(18)

After substitution of equation (17) into equation (18), and a partial evaluation of the resulting integral, we find

$$J(r) = J_0 + \frac{1}{2} j_0 r \int_{-1}^{1} (\alpha + \mu^2)^{1/2} d\mu,$$

(19)

where $\alpha = [(R/r)^2 - 1]$, and $J_0$ is the angle average of $I_0$. The result is a standard integral in terms of the arcsinh function, and after converting this to logarithmic form, the angle-averaged intensity is

$$J(r) = J_0 + \frac{j_0 R}{2} \left[ 1 + \frac{R}{r} \left( 1 - \frac{r^2}{R^2} \right) \ln \left[ \frac{r + R}{\sqrt{R^2 - r^2}} \right] \right].$$

(20)

We note that the apparent infinity in equation (20) at $r = 0$ disappears when this equation is replaced by a suitable expansion for the case $r \ll R$. The small radius form is

$$J(r) \approx J_0 + (1/2) j_0 R (2 - r^2/R^2).$$

(21)

We plot, in Fig. 4, the function $Q = (J(r) - J_0)/j_0 R$ from equation (20), as a function of the dimensionless radius $s = r/R$.

5 THE SHELL SOLUTION

We re-write the shell transfer equation as

$$\frac{\partial I}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial I}{\partial \mu} = -\chi'(r) I + f(r),$$

(22)

where $f(r)$ is an arbitrary function of the radius, which incorporates the continuum emission and line scattering. We change the left-hand side of equation (22), as in the cavity solution, to describe a solution along the ray characteristic. The equation can then be put in standard first-order linear form as

$$\frac{dI}{ds} + \chi'(r) I = f(r(s)).$$

(23)

Equation (23) may be integrated by the standard method of integrating factors. The boundary condition at $s = 0$ is that the specific intensity enters the shell from interstellar space with the intensity $I_{BG}$, assumed equal to a typical value for Galactic starlight. The extinction coefficient at this same position is zero, regardless of its variation within the shell. With this condition imposed, equation (23) has the formal solution,

$$I(s) = I_{BG} e^{-\int_0^s \chi'(r') dr'}$$

$$+ \int_0^s f(r(s')) \exp \left\{ - \int_0^{s'} \chi'(r(\sigma)) d\sigma \right\} ds'.$$

(24)

The ray distance, $s$, is related to the radius $r$ and direction cosine $\mu$ by a modified form of the relation which appears in equation (16). If the shell has an outer radius $R_2$, then

$$s = r \mu + R_2 \cos \omega,$$

(25)

where, for the moment, we ignore the presence of the cavity. With this same caveat, $r \sin \theta = R_2 \sin \omega$ is a constant along a given ray if $\omega$ is the angle with which the ray enters the shell from interstellar space. Defining $K_2 = R_2 \sin \omega$, we can express $\mu$ in terms of the radius as $\mu = (1 - K_2^2/r^2)^{1/2}$, which may be substituted into equation (25) to yield a relation between $r$ and $s$ with no other variables involved. This relation, with the radius as the subject, is

$$r = (s^2 - 2s R_2 \cos \omega + R_2^2)^{1/2}.$$  

(26)

The expression for the radius in equation (26) can be used to expand the radius in terms of $s$ in equation (24), provided that functional forms for $\chi'(r)$ and $f(r)$ are known.

Figure 3. The angle-averaged intensity, $J$, is to be computed at point $P$, at radius $r$ from the centre. The average is over rays (examples marked with arrows) which make angles, $\theta$, with the radius vector at $P$, while $\omega$ may be in the range $0$ to $\pi$. A particular ray is shown entering the cavity at angle $\omega_1$, in direction $\theta$, and passing through a distance $s$ in order to reach $P$.

Figure 4. A dimensionless form of the angle-averaged intensity in the cavity, $[J(r) - I_0]/j_0 R$, see equation (20), plotted as a function of dimensionless radius, $r/R$. Equation (21) is used for the point at $r/R = 0$. 

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5.1 Shell solution with power-law density

To progress beyond equation (24) we require some analytical approximation for the behaviour of the radius-dependent functions. We start with the extinction, $\chi'(r)$. We assume that this depends on the radius only through the number density of the shell dust, so that the functional form is the same for both the absorption and scattering contributions. Further, we assume a density dependence which follows the inverse square behaviour of the singular isothermal sphere. Therefore, we suppose that the extinction in the shell behaves as

$$\chi'(r) = \chi'(R) (r/R)^{-2},$$

(27)

where $\chi'(R)$ is the maximum value of the extinction coefficient, found just outside the cavity boundary. With the help of equation (26), we can write the extinction as a function of $s$ rather than $r$. Integrals of the type which appear in equation (24) can now be written in the form

$$\int \chi'(r) ds = \frac{R^2 \chi'(R)}{s^2 - 2sR \cos \omega + R^2}. \quad (28)$$

Introducing the new variables $x = s - R \cos \omega$ and $a = R \sin \omega$, equation (28) may be written as the standard integral,

$$\int \chi'(r) ds = \frac{R^2 \chi'(R)}{a^2 + s^2}. \quad (29)$$

which has the solution (as an indefinite integral)

$$\int \chi'(r) ds = \frac{R^2 \chi'(R)}{r} \frac{\sin \omega}{\cos \omega} \arctan \left[ \frac{s - R \cos \omega}{R \sin \omega} \right]. \quad (30)$$

Equation (24) contains two definite forms of equation (30). The first of these has limits of 0 to $s$, and can be re-written, with the help of addition formulae for the arctangent from Gradshteyn & Ryzhik (1965), as

$$\int_0^s \chi'(r(s')) dr' = \frac{\beta}{\sin \omega} \left[ \arctan \left( \frac{\sin \omega}{2R} \right) - \frac{\sin \omega}{2R} \right], \quad (31)$$

where the $\tau$ is to be included only for ray distances such that $s > R \cos \omega$, and the group $\beta = R^2 \chi'(R)/R_2$. The second definite integral form from equation (24) has the limits $s'$ to $s$, and it is convenient to write it without combining the arctangents, since the part with $s$ can be removed from the integral over $s'$. Overall, the shell solution can now be written as

$$I(s) = I_{BG} \exp \left\{ -\frac{\beta}{\sin \omega} \left[ \arctan \left( \frac{\sin \omega}{2R} \right) - \frac{\sin \omega}{2R} \right] \right\} + e^{-\frac{\mu(s)}{\sin \omega}} \int_0^s f(s') \exp \left\{ \frac{\beta u(s')}{\sin \omega} \right\} ds', \quad (32)$$

where

$$u(s) = \arctan \left[ \frac{s - R \cos \omega}{R \sin \omega} \right] \quad (33)$$

and the functional form of $f$ remains to be determined.

5.2 Scattering as a perturbation

If we ignore continuum emission in the shell, the unknown function, $f$, in equation (32) reduces to line scattering alone, which depends on the angle-averaged mean intensity, $J(r)$. It is then possible, in principle, to average equation (32) over solid angle, leading to an integro-differential equation for $J(r)$. However, owing to the difficulty in attempting to solve such an equation analytically, we resort to the simpler procedure of treating the scattering term as a perturbation. For this approximation to be very good, the scattering contribution to the extinction should be small, so that $\sigma'(r)\chi'(r)$ is a small parameter for all radii. This is unlikely to be true for real dust. For example, silicate dust modelled by Ossenkopf, Henning & Mathis (1992) has an optical efficiency for scattering which is consistently approximately two to three times that for absorption over optical wavelengths. However, we still adopt the perturbative approximation as the only viable method of obtaining an approximate analytical solution. In this procedure, we first solve the radiative transfer equation in the shell, assuming that line extinction is the only contribution to the right-hand side of equation (22). This means that we can set $f = 0$ in equation (32), and take as the zero-order solution in the shell,

$$I(r, \mu) = I_{BG} \exp \left\{ \frac{\beta R_2}{(1 - \mu^2)^{1/2}} \left[ \arctan \left( \frac{1 - \mu^2}{(R_2^2 - r^2(1 - \mu^2)^2)^{1/2}} \right) \right] \right\}, \quad (34)$$

which is just the first term of equation (32), with $\theta$ and $\omega$ eliminated in favour of $\mu = \cos \theta$, and with the help of the relation $\sin \omega = (r/R_2) \sin \theta$ (see equation 14, which still holds in the shell). It is tedious, but straightforward, to show that equation (34) is indeed a solution of equation (22), for the case where $f(r) = 0$. Equation (34) cannot be used alone, as along any path through the nebula, a ray may encounter a sequence of shell, cavity and again shell conditions. We therefore need to determine some functional form for $I_{BG}$ for the case where a ray leaves the cavity and re-enters the shell, forming a combined solution along a ray.

As scattering is now to be treated as a perturbation, it can be computed from a mean intensity which is derived from the zeroth-order combined solution. An equation for a perturbation in the intensity, $\delta I$, can be constructed from equation (9) by expanding the specific intensity as $I = I_0 + \delta I$, and then subtracting off the equation in the zeroth-order estimate, $I_0$. The result is

$$\frac{\partial (\delta I)}{\partial r} + \frac{1 - \mu^2}{r} \frac{\partial (\delta I)}{\partial \mu} = -\chi'(r) \delta I + \sigma'(r) J(r), \quad (35)$$

where we have assumed that continuum emission makes a negligible contribution, so that $f(r) = \sigma'(r) J(r)$. So, mathematically, the equation in the perturbation may be treated in the same way as the shell equation (equation 22). The formal solution for $\delta I$ therefore looks like equation (24) with the scattering term replacing the unknown function $f$. The perturbation solution is therefore given by

$$\delta I(s) = \delta I(s_0) e^{-\delta(s)} + e^{-\delta(s)} \int_0^s \sigma'(r(s')) J(r(s')) e^{\delta(s')} ds', \quad (36)$$

where the function $\delta$ is given by equation (30). It is important to note that there are three possible versions of equation (36): a ray which avoids the cavity has $\delta(s_0) = 0$ and a lower limit of $s_0 = 0$ on the remaining integral. A ray which enters the cavity has two shell segments. In the first, $\delta(s_0) = 0$ and $s_0 = 0$, but the upper limit is the value of the path where the ray enters the cavity. The perturbation is then constant in the cavity, forming a finite value of $\delta I(s_0)$ for the second shell segment, where $s_0$ is now the path length where the ray leaves the cavity.

6 THE COMBINED SOLUTION ALONG A RAY

Given the approximations made in the previous section, it is now possible to combine the cavity and shell solutions for a single ray. The geometry of the combined solution is set out in Fig. 5. As we are ignoring, as this stage, rays which pass solely through the shell, each effective ray has three segments: the first is an absorptive
passage through the shell. However, if the input background at \( r = R_2 \) is very weak, we can ignore this segment, and treat the specific intensity of the ray at the cavity boundary, \( r = R \) as zero.

The second segment of the ray path, between points \( Q \) and \( Q' \) in Fig. 5, passes through the cavity. Here the specific intensity is assumed to increase through spontaneous emission in the line, according to the cavity solution, equation (16). In particular, we want an expression for the specific intensity at \( Q' \), where the cavity solution becomes the input value for the shell solution, which takes over as the third segment of the ray path, which continues until point \( Z \) is reached, and the ray passes into the vacuum.

At the point \( X \) in Fig. 5, which is at radius \( r \), the ray has travelled a distance \( s_{\text{shell}} \) through the shell, and it makes an angle \( \theta \) with the radial vector. At this point, the solution will be the shell solution, with an input specific intensity from the cavity.

The analysis of the triangle \( \text{MOQ} \) shows that the cavity segment of the ray path has length \( s' = 2r \cos \omega' \). The total distance from point \( P \) to point \( M \), from the triangle MOP, equal to \( s_M = R_2 \cos \omega \). Therefore, the ‘pre-cavity’ distance (from \( P \) to \( Q \)) is \( s_p = R_2 \cos \omega - r \cos \omega' \). The additional distance along the ray from \( M \) to \( X \) is \( s_M' = r \cos \theta = r \mu \). The total distance along the ray from \( P \) to \( X \) is therefore \( s = R_2 \cos \omega + r \mu \), just as in the shell solution, equation (25). From these results, we find that the distance \( s_{\text{shell}} \) in Fig. 5 is given by

\[
s_{\text{shell}} = r \mu - R \cos \omega',
\]

and by applying the sine rule to the triangle \( \text{OPQ} \), the angles \( \omega' \) and \( \omega \) are related by

\[
\sin \omega = (R_2/R) \sin \omega',
\]

which can be used to eliminate \( \omega' \) from equation (37), yielding

\[
s_{\text{shell}} = r \mu - [R^2 - R_2^2 \sin^2 \omega]^{1/2}.
\]

The limiting (maximum) value of \( \omega \), for a ray that just enters the cavity, occurs when \( \omega' = \pi/2 \), when \( \sin \omega_{\text{max}} = R/R_2 \).

### 6.1 Input solution to the shell

As we are assuming that the specific intensity at point \( Q \) in Fig. 5 is effectively zero, the input solution for the shell is just the cavity solution evaluated at \( Q' \). When this is done, the cavity solution (equation 16) applies, with \( \omega' \) replacing \( \omega \). We take this solution with \( I_0 = 0, r = R \) and \( \mu = \cos \omega' \), obtaining \( I(R, \omega') = 2j_0(R \cos \omega') \).

With the help of equation (38), this expression becomes

\[
I(R, \omega) = 2j_0[R^2 - R_2^2 \sin^2 \omega]^{1/2},
\]

which is the background for the solution in the shell beyond point \( Q' \).

### 6.2 Combined solution in the shell

As the limits of integration are modified for the combined solution, we generate the shell solution from the indefinite integral (equation 30). The upper limit is the distance along the ray path, \( s \), but the lower limit is now, from Fig. 5, \( s' + s_p = R_2 \cos \omega + [R^2 - R_2^2 \sin^2 \omega]^{1/2} \).

The result for the extinction integral is

\[
\int_{s'}^{s} x' \, dx' = \frac{R^2 \chi(R)}{R_2 \sin \omega} \left\{ \arctan \left[ \frac{s - R_2 \cos \omega}{R_2 \sin \omega} \right] - \arctan \left[ \frac{(R^2 - R_2^2 \sin^2 \omega)^{1/2}}{R_2 \sin \omega} \right] \right\}.
\]

(41)

When equation (25) has been applied, and equation (41) has been re-expressed entirely in terms of \( \mu \), the result is

\[
\int_{s'}^{s} x' \, dx' = \frac{\beta}{(1 - \mu^2)^{1/2}} \left\{ \arctan \left[ \frac{\mu}{(1 - \mu^2)^{1/2}} \right] - \arctan \left[ \frac{(1 - (r/R)^2(1 - \mu^2)^{1/2})}{(r/R)(1 - \mu^2)} \right] \right\}.
\]

(42)

The first arctangent can be converted to a simpler form, as an arcsine, via the relation

\[
\arctan \left[ \frac{x}{(1 - x^2)^{1/2}} \right] = \arcsin x,
\]

(43)

which is taken from Gradsteyn & Ryzhik (1965). The second arctangent in equation (42) can be written as the reciprocal of the form which appears in equation (43) by defining the new variable \( q = r(1 - \mu^2)/R \). An additional relation linking arctangents (Gradsteyn & Ryzhik 1965) for the case where \( x > 0 \), which is true of \( q \), is

\[
\arctan(1/x) = \pi/2 - \arctan x,
\]

(44)

and this relation allows us to use equation (43) directly on the second arctangent in equation (42). A third relation from Gradsteyn & Ryzhik, \( \arcsin x + \arccos x = \pi/2 \), allows us to re-write equation (42) as

\[
\int_{s'}^{s} x' \, dx' = \frac{\beta}{(1 - \mu^2)^{1/2}} \left[ \arcsin q - \theta \right].
\]

(45)

It is now a simple matter to write out the combined solution for a ray by inserting equation (45) into the shell solution, equation (24), with \( I_{BG} \) given by equation (40). With \( \beta \) and \( q \) fully expanded in terms of \( \theta \), the result is

\[
I(r, \theta) = 2j_0[R[1 - (R/R_2)^2 \sin^2 \omega]^{1/2} \times \exp\left\{ \frac{-R\chi(R)}{(r/R) \sin \theta} \left[ \arcsin \left( \frac{r \sin \theta}{R} \right) - \theta \right] \right\},
\]

(46)

which is valid for \( R \leq r \leq R_2 \). For plotting, we re-write equation (46) in the form

\[
I(x, a) = 2j_0[R(1 - a^2)^{1/2} \exp \left( -\frac{\pi}{a} \left[ \arcsin a - \arcsin \frac{a}{x} \right] \right],
\]

(47)
an equation in two parameters, \( a = (R_3/R) \sin \omega \), and the optical depth parameter, \( \tau = R_3'/(R) \). The latter parameter becomes equal to the radial optical depth of the shell in the limit where \( R_3 \gg R \). The independent variable is \( x = \tau R \). This last definition implicitly introduces a third parameter, the upper bound on \( x \), equal to \( R_3/R \). We note that equation (47) reduces to a sensible limiting form, that is

\[
I(x, a = 0) = 2j_0R \exp[-\tau(1 - R/R)],
\]

(48)

for the case where \( a = 0 \). In Fig. 6 we show the function \( U = I(x, a)/(2j_0R) \) for five values of \( a \) between its minimum value of zero, and maximum of 1. We set the optical depth parameter to be \( \tau = 1.38 \) (see Section 10) to agree approximately with the visual extinction of 1.5 mag near the centre of NGC 6537, and we plot the dimensionless radius out to \( x = 3 \).

## 7 MEAN INTENSITY IN THE SHELL

The approximation discussed in Section 6.1 – that rays do not cross the cavity contribute zero specific intensity – excludes all rays with negative values of \( \cos \theta \). Rays with non-zero specific intensity are limited to a subset of those with positive values of \( \cos \theta \), more precisely to those with \( \sin \theta < R/R \). With the additional restriction that any radius where \( J(r) \) is computed has \( R \leq r \leq R_3 \); the situation here is similar to that shown in Fig. 3. The function to be averaged is the combined solution specific intensity in the zeroth-order (no scattering) approximation, given by equation (46).

The above considerations allow us to write a formal integral for the mean intensity, recalling that \( r \sin \theta = R_3 \sin \omega \):

\[
J(r) = \int_{0}^{\arcsin(R/r)} \left[ 1 - (r/R)^2 \sin^2 \theta \right]^{1/2} \times \exp \left\{ \frac{R_3' (R)}{r/R} \sin \theta \right\} \sin \theta d\theta,
\]

(49)

We simplify equation (49) via a series of substitutions. The first of these is to let the impact parameter be \( p = (r/R) \sin \theta \); we also define the new constant parameters, \( \gamma = R_3' (R) \), and \( \rho = R/R \). These definitions transform equation (49) to

\[
J(r) = j_0R \rho^2 \int_{0}^{1} p \left( \frac{1 - p^2}{1 - \rho^2 p^2} \right)^{1/2} \times \exp \left\{ \frac{\gamma}{p} \left( \arcsin p - \arcsin (\rho p) \right) \right\} dp.
\]

(50)

Figure 6. The function \( U = I(x, a)/(2j_0R) \) (see equation 47) plotted as a function of dimensionless radius \( x = \tau R \) for values of \( a = 0.0 \) (solid line), 0.4 (dotted line), 0.6 (dashed line), 0.8 (simple chain), 0.99 (solid grey line).

noting that \( p \) and \( \rho \) are always \( 1 \). Given this condition, we can expand the inverse sines in equation (50) in terms of Gauss hypergeometric functions (e.g. Abramowitz & Stegun 1965). For complex argument \( z \),

\[
\arcsin z = zF(1/2, 1/2; 3/2, z^2),
\]

(51)

where the power series forming the Gauss hypergeometric function, \( F \), is absolutely convergent within, and on, the unit circle for the arguments in equation (51) (Gradshteyn & Ryzhik 1965). The substitution of equation (51) into equation (50) has the beneficial consequence of cancelling \( p \) within the exponential, and leaving \( \rho^2 \) everywhere except for the product \( pdp \). This suggests the substitution \( x = \rho^2 \), yielding

\[
J(\rho) = \frac{j_0R \rho^2}{2} \int_{0}^{1} \left( \frac{1 - x}{1 - \rho^2 x} \right)^{1/2} \times \exp \left\{ -\gamma \left[ F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, x \right) - \rho F \left( \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \rho^2 x \right) \right] \right\} dx.
\]

(52)

At this point, we expand the hypergeometric series which appear in equation (52). For the case here, where the first two arguments are the same, the power series (Abramowitz & Stegun 1965) is

\[
F(a, a; g, z) = 1 + a^2 \frac{z}{g!} + a^2(a + 1) \frac{z^2}{(g + 1)!} + a^2(a + 1)(a + 2) \frac{z^3}{(g + 2)!} + \cdots,
\]

(53)

in the case of general \( a, g \), and for the specific case of \( a = 1/2 \) and \( g = 3/2 \), we find

\[
F(1/2, 1/2; 3/2, z) = 1 + \frac{z}{6} \left( 3 + \frac{5z}{\sqrt{2}} \right) + \ldots,
\]

(54)

Substitution of equation (54) in equation (52) leads to the expression

\[
J(\rho) = \frac{j_0R \rho^2}{2} \int_{0}^{1} \left( \frac{1 - x}{1 - \rho^2 x} \right)^{1/2} \times \exp \left\{ -\gamma \left[ (1 - \rho) + (1 - \rho^2) x + \frac{5(1 - \rho^2)^x}{40} + \frac{5(1 - \rho^2)^x}{112} + \cdots \right] \right\} dx.
\]

(55)

which is as far as it is possible to proceed without making further approximations.

### 7.1 Approximate forms

Although the series in equation (55) is convergent for all relevant values of \( x = [(r/R) \sin \theta]^2 \), convergence is rather slow when \( x \) is close to 1, the condition for a ray in glancing contact with the cavity. However, the contribution of large \( x \) to the integral is limited by the term in front of the exponential which tends to zero as \( x \to 1 \) (for the particular case of \( \rho = 1 \), see below). The physical reason for this is that rays at large angle have relatively short paths through the cavity, and correspondingly small specific intensities on entry to the shell.

Although values of \( \rho \) which are close to 1 (positions close to the shell/cavity boundary) can force the leading term in equation (55) to 1, this is not a problem because, in this situation, the argument of the exponential approaches zero, the whole exponential to 1 and integration of the leading term alone is sufficient for moderate accuracy. We evaluate equation (55) to two levels of accuracy in \( x \).

In the zero-order approximation, we abandon all but the first term.
in the series, but this does not depend on \( x \), and can therefore be removed from the integral, leaving

\[
J(\rho) \simeq \frac{j_0 R \rho^2}{2} e^{-\gamma(1-\rho)} \int_0^1 \left( \frac{1 - x}{1 - \rho^2 x} \right)^{1/2} dx. \tag{56}
\]

On the evaluation of the integral in equation (56), and reverting to the original radius variable, \( r = R \rho \), we obtain the zero-order approximation to the mean intensity:

\[
J(r) = \frac{j_0 R}{2} e^{-\gamma(1-R/r)} \left[ 1 - \frac{r^2 - R^2}{2rR} \ln \left( \frac{r + R}{r - R} \right) \right]. \tag{57}
\]

We note that at the cavity/shell boundary, where \( r = R \), the reduction of equation (57) agrees with the cavity solution (equation 20) evaluated at the same radius. Both equations yield \( J(R) = j_0 R/2 \).

It is also possible to obtain an analytic integral for a first-order approximation, keeping the term in \( x \) in the series in equation (55). The integral in the first-order case is a standard form in Gradshteyn & Ryzhik (1965). With the parameters specific to the current problem, the integral is

\[
\int_0^1 \left( \frac{1 - x}{1 - \rho^2 x} \right)^{1/2} \exp \left\{ -\frac{\gamma (1 - \rho^3)}{6} \right\} \Phi_1 \left( 1, \frac{5 \rho^2 + \rho^2 z}{2} \right). \tag{58}
\]

where \( z = -\gamma(1 - \rho^3)/6 \). The first function, \( B \), on the right-hand side of equation (58) is an Euler beta-function, whilst the second is a confluent hypergeometric series in two variables, \( \rho^2 \) and \( z \). Further standard formulae allow the beta-function \( B(1, 3/2) \) to be expressed as a ratio of factorials that reduces to \( 2/3 \), so that the mean intensity to the first order in \( x \) is

\[
J(\rho) = \frac{j_0 R \rho^2}{3} e^{-\gamma(1-\rho)} \Phi_1 \left( 1, \frac{5 \rho^2 + \rho^2 z}{2} \right). \tag{59}
\]

![Figure 7. The function \( Q = J(\rho)R/2 \) plotted as a function of dimensionless radius \( x = 1/\rho = r/R \) using the zeroth-order approximation (equation 57: solid line), the first-order approximation (equation 59: dashed line) and exact integrals from equation (50) (diamond symbols).](image)

8 PERTURBATION SOLUTION IN THE SHELL

We now proceed to solve equation (36) with the mean intensity in the scattering term given by equation (57): it is too complicated to use the more accurate equation (59). There are various forms of equation (36) which should be used for the appropriate zone of the source. For a ray which penetrates the cavity, we initially consider the case where this ray has entered the shell, but has not yet reached the cavity. Assuming a negligible input intensity from the vacuum, we have the boundary condition that \( \rho_0 = 0 \) and \( \delta I(0) = 0 \), and therefore the problem for this ray, and zone, reduces to solving the integral,

\[
\delta I(x) = e^{-h(x)} \int_0^x \sigma'(r(z')) J(r(z')) e^{h(z')} dz'. \tag{60}
\]

The most useful form for the function \( h(x) \), which appears in the integrating factor, for this zone is

\[
h(z') = R^2 z' \left( R \right) (\pi - \omega - \theta(z')), \tag{61}
\]

which is derived from equation (31), with a lower limit of 0 and an upper limit of \( z' \), followed by transformations similar to those used in working from equation (43) to equation (46). Note in particular that for a ray in this case, \( \omega \) is an acute angle, but \( \theta \) is obtuse, and the respective limits of these angles for a radial ray are zero and \( \pi \).

We substitute for the scattering coefficient, \( \sigma'(r(z')) \), by assuming that it has the same \( 1/r^2 \) functional dependence as the absorption (see Section 5.1). The mean intensity is given by equation (57). It is perfectly possible to express all these functions in terms of the distance, \( z' \), along the ray, but the integral in equation (60) appears easier when working in terms of radius. Letting the integral be \( \Psi \), we have

\[
\Psi = \frac{\sigma'(R)R^2 j_0 \exp[\epsilon(\pi - \omega - a)]}{2} \times \int_0^\rho \frac{d\rho \exp[-\epsilon \arcsin(\rho a - a)]}{(1 - a^2 \rho^2)^{1/2}} \times \left[ 1 - \frac{1 - (1 - \rho^2)}{2\rho} \ln \left( \frac{1 + \rho}{1 - \rho} \right) \right]. \tag{62}
\]

where \( \rho = R/r \) as before, \( a = (Rz/R) \sin \omega, \epsilon = \gamma/\alpha \), and the limits are given by

\[
P_0 = R/R_z \tag{63}
\]

and

\[
P = R/[s^2 - 2s \cos \omega + R_z^2]^{1/2}, \tag{64}
\]

which reduces to 1 for a ray entering the shell from the vacuum, and reaching the edge of the cavity, where \( r = R \). If we divide equation (62) by the top line, which is independent of \( \rho \), we can carry out an integration by parts. The integrated part is

\[
V = \int \frac{d\rho}{(1 - a^2 \rho^2)^{1/2}} = -\frac{\epsilon a}{\epsilon a} \left( \int_0^P \frac{\exp[-\epsilon \arcsin(\rho a - a)]}{(1 - a^2 \rho^2)^{1/2}} d\rho \right), \tag{65}
\]

which removes the problematic square-root term. The result of this integration is

\[
\Psi_j = -e^{\epsilon(\pi - \omega)} \frac{1}{\rho} \left( 1 - \frac{1 - (1 - \rho^2)/2\rho}{\ln \left( \frac{1 + \rho}{1 - \rho} \right) \rho} \right) \left( R/R_z \right), \tag{66}
\]

where the function \( g(\rho) \), now entirely composed of logarithms and rational functions of \( \rho \), is defined by

\[
g(\rho) = 2a\epsilon - 2 + \frac{1}{\rho} \ln \left( \frac{1 + \rho}{1 - \rho} \right) \tag{67}
\]

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and $\Psi_j$ is given by

$$\Psi_j = \frac{4\epsilon a \Psi}{\sigma^*(R)R^2 j_0 e^{\epsilon \omega}}. \quad (68)$$

Note that the argument of the exponential in equation (68) is different from that in equation (62) because the original integral, that is the lower two lines of equation (62), was multiplied by a factor of $2e a e^{-\epsilon \omega}$ in order to obtain $\Psi_i$ in equation (66).

To calculate an approximation to the integral in equation (66), we expand the argument of the exponential, noting that $\epsilon \omega \leq 1$. The first term in the power-series expansion of the arcsine cancels with $a \rho$, and the next term is in $(a \rho)^3$ which we ignore, such that

$$e^{-\epsilon(a+\arcsin(a\rho)-a\rho)} \simeq e^{-\epsilon a (1+(a\rho)^3)/(6a)} \simeq e^{-\epsilon a}, \quad (69)$$

which is independent of $\rho$, and can be moved outside the integral, which has now been reduced to the integration of the function $g(\rho)$ from equation (67). The integration of $g(\rho)$ breaks down into six integrals, five of which can be solved in terms of elementary functions and the sixth in terms of the Riemann $\Phi(z, s, v)$ function (Gradshteyn & Ryzhik 1965). A full form for the solution, $\Psi_i$, appears in Appendix A.

8.1 Rays which avoid the cavity

A ray which does not enter the cavity obeys the same propagation equation, equation (60), but has a different upper limit on the distance, $s$. Instead of integrating to the cavity boundary, we integrate to the mid-point, where the radial vector is perpendicular to the direction of the ray. At this point, the value of $s = R_2 \cos \omega$, and the corresponding radius is $R_2 \sin \omega$. We can therefore still use equations (62), (63) and (64), but noting that the expression in equation (64) now reduces to $P = R/R_2 \sin \omega$, rather than 1. The integrations and approximations which follow still apply, providing we use the new value of $P$.

8.2 Outward bound rays

Provided that we are interested only in emergent rays, we can use the symmetry of the nebula to simplify matters greatly here. The integral along the ray, $\Psi_j$, is identical, whether we are integrating inwards to the mid-point (or cavity), or outwards again towards the edge of the nebula. These integrals become formally the same because when we convert from $s'$ to $r$ as the integration variable, we must use different roots of the expression

$$s' = R_2 \cos \omega \pm (r^2 - R_2^2 \sin^2 \omega)^{1/2}, \quad (70)$$

with the negative root required for the inwards segment, and the positive root holding for the outwards ray. The result is that the integrals over radius are identical. The form of equation (60) which applies to outwards rays is

$$\delta I(s) = \delta I(s_0) + e^{-\epsilon \omega} \int_{s_0}^s \sigma'(r(s')) I(r(s')) e^{\epsilon \omega(s')} ds', \quad (71)$$

where $s_0$ is now either the mid-point, for a ray which avoids the cavity, or the outwards cavity boundary. We assume in equation (71) that there is negligible scattering within the cavity. The input, $\delta I(s_0)$, is just the inbound solution, evaluated at the mid-point or the in-bound cavity boundary. If we define

$$\zeta = \sigma'(R)R^2 j_0/(4\epsilon a), \quad (72)$$

then we can re-write equation (71) as

$$\delta I(s) = \delta I(s_0) \left\{ e^{\epsilon \arcsin(R_2/(r(s)\sin \omega))) \sin \omega} + e^{\epsilon \arcsin(R_2/(r(s)\sin \omega))} \right\}. \quad (73)$$

9 SOLUTION FOR EMERGENT RAYS

To use the symmetry property of the integral $\Psi_j$, we have already evaluated the perturbation, $\delta I$, as an emergent quantity above. In

\[ \text{Figure 8.} \text{ The function } U = \delta I(s)/(2j_0 R) \text{ for complete path lengths } s = a(R_2) \text{ plotted as a function of entry angle into the nebula, } \omega. \text{ The optical depth parameter is } \chi'(R) = 1.38 \text{ as before, and the ratio of the shell to cavity radii is } R_2/R = 3.0. \text{ The ratio of the scattering coefficient to the extinction coefficient is } \sigma'(R)/\chi'(R) = 0.25. \]
general, the complete solution, for a given emergent ray, is given by
\[ I_{\text{em}}(s) = I(s) + \delta I(s), \]  
(76)
where \( s = 2R_z \cos \omega. \) For a ray which crosses the cavity, \( I(s) \) comes from a form of equation (46) where \( r = R_z \) and \( \theta = \omega, \) whilst \( \delta I(s) \) is given by equation (75). Overall,
\[ I_{\text{em}} = 2/jhR \left( 1 - a^2 \right)^{1/2} e^{-s/(\arcsin a - \omega)} + \zeta \Psi \left( e^{s \arcsin a} + e^{s\omega} \right). \]  
(77)
A ray which does not cross the cavity has only the \( \delta I \) contribution, and is given by equation (74) without modification.

9.1 The observer’s view of the model
To an observer with perfect angular resolution, an emergent ray of given exit angle, \( \omega, \) and characteristic a circular strip of a spherical surface. This strip has an area \( 2\pi R_z^2 \sin u \, du, \) where \( u \) is the polar angle measured from the line of sight towards the limb of the nebula. The observer cannot see this surface as a whole, but only a 2D projection of it. The projected area of the strip is
\[ dA_{\perp} = 2\pi R_z^2 \sin u \cos \omega \, du. \]  
(78)
It is straightforward to see that for any given strip, the polar angle \( u \) is identical to the entry/exit angle of the ray, \( \omega, \) which has allowed values between 0 and \( \pi/2. \) The observer with perfect angular resolution will therefore be able to pick out a small piece of a given (projected) strip, and will measure the associated specific intensity, \( I_{\text{em}} \), as a brightness. For a nebula of radius \( R_z, \) at a distance \( d \) from the observer, the measured brightness at an angle \( \Theta \) from the centre of the nebula, as seen by the observer, corresponds to exit angle \( \omega = \arcsin(\Theta d/R_z). \) The brightness itself can be found by substituting this angle into either equation (77) or equation (74).

The other extreme observer’s view is that of a telescope with a beam that is much larger than the nebula. In this case, the specific intensity cannot be measured directly, and a flux, averaged over the whole object, is obtained instead. To a very good approximation, this flux is given by
\[ F = \frac{2\pi R_z^2}{d^2} \int_0^{\pi/2} I(\omega) \sin \omega \cos \omega \, d\omega, \]  
(79)
where, as previously, the functional form of \( I(\omega) \) is taken from equation (77) or equation (74), depending on whether or not the ray at angle \( \omega \) traverses the cavity.

10 FITS TO NGC 6537
The function summarized in equations (76) and (77) was fitted to observational data describing the brightness variation of the H\( \alpha \) and H\( \beta \) spectral lines as functions of angular position across the nebula. The fits were made with respect to four variable parameters in the theoretically derived function: \( x = R/R_z, \) the ratio of the cavity radius to the overall radius of the nebula; \( S = \sigma_x/\sigma_y, \) the ratio of the scattering to extinction coefficients in the continuum; \( \tau = R_x/(R), \) the optical depth parameter, and an overall intensity-axis scale factor, \( Y, \) allowing a fit to normalized observational data. The optimum fit for each observational data set was taken to be that with the minimum value of the \( \chi^2 \)-statistic,
\[ \chi^2 = \frac{1}{N-4} \sum_{i=1}^{N} \left( \frac{I_i - I_{\text{em}}(\sin \omega_i, x, S, \tau, Y)}{\sigma_i} \right)^2, \]  
(80)
for a data set with \( N \) entries of the form \( (\sin \omega_i, I_i), \) and the four free parameters introduced above. Details of the treatment of the observational data, and of the fitting function, are described below.

10.1 Data from NGC 6537
Data were extracted from HST H\( \alpha \) and H\( \beta \) images (Matsuura et al. 2005). Seven straight-line slices were taken through the nebula, each at a different angle on the sky. For each slice, data were recorded for both H\( \alpha \) and H\( \beta, \) and for each slice and spectral line, data were organized into pairs consisting of a coordinate position along the slice, and a corresponding specific intensity. The following operations were applied to convert these data into a form suitable for fitting: for each slice and line, two pixel positions were found for the emission peaks, corresponding to the intersections of the slice with the cavity boundary. The coordinate origin was then shifted to the mid-point of the peak positions, and the absolute value of the slice coordinate taken, transforming the data to a pair of radial brightness profiles measured from an origin at the approximate centre of the nebula. An approximate correction for the aspherical nature of the nebula was imposed by re-scaling the radial axis such that the origin to peak distance was the same for all radial profiles. The outer radius of the nebula was taken to be the smallest maximum value on the new scale, and the remaining profiles truncated to the same distance. Finally, the new radial coordinate was re-scaled once more to the range 0.0–1.0, with 1.0 corresponding to the new common outer radius. We note that the H\( \beta \) data covered a larger angular range on average than the H\( \alpha \) data, leading to a larger value of \( R_z \) in H\( \beta \) by a factor of 1.51, and a different radial scaling for the two lines.

For individual radial profiles, the specific intensities were scaled such that the peak brightness was set to 1.0. Mean profiles for each spectral line were also constructed by averaging over all the un-normalized individual brightness profiles (on the normalized radial scale for each line defined above), and then normalizing the averaged brightnesses to a peak height of 1.0.

In the averaged profiles, the standard errors resulting from the averaging process far exceeded the formal measurement errors in the observational data. The former were calculated at the peak position to be 0.098 for H\( \alpha \) and 0.054 for H\( \beta, \) as fractions of the mean peak height. The latter were only of the order of \( 1.7 \times 10^{-4} \) and \( 3.0 \times 10^{-4}, \) respectively. For the normalized data sets, with peak heights of 1.0 we assumed absolute uncertainties at all radial points to be equal to those at the respective peaks: 0.098 for H\( \alpha \) and 0.054 for H\( \beta. \) The same absolute errors were assumed for the fits to the individual radial profiles. Values of \( N = 4 \) were 75 for H\( \alpha \) and 120 for H\( \beta. \)

10.2 Parameter ranges
The parameter \( x \) has the formal range 0.0–1.0, but a much smaller realistic range can be selected for any radial profile by observing the position of the scattering peak (see also Fig. 8). Both the optical depth parameter, \( \tau, \) and the scale-factor, \( Y, \) have the possible range 0.0–\( \infty. \) The most problematic parameter is \( S, \) the ratio of the continuum scattering to extinction coefficients. The scattering has been treated as a perturbation in Section 8, so strictly speaking only the range 0.0–0.1 is available to \( S. \) However, from dust models, we expect the ratio of the scattering to absorption cross-sections to be of the order of 2–3 (corresponding to \( S = 0.666–0.75) \) in the optical region, as already discussed in Section 5.2. We have therefore fitted each profile twice: once for a true perturbation, with \( S \) limited to the range 0.0–0.1, and once with the range 0.0–0.666, with the upper limit dictated by computed dust parameters. For each spectral line, we have fitted the averaged profile, and one selected individual profile. The fitting parameters for H\( \alpha \) and H\( \beta \) have been taken to be entirely independent in this preliminary study.
10.3 Results of fitting

In Fig. 9, we show the results of the fits where the allowed range of \( S \) is 0.0–0.666. The upper graphs are fits to \( \text{H}\alpha \) data, and the lower graphs to \( \text{H}\beta \). For each line, the left-hand panels are fits to the averaged profile, whilst the right-hand panels show fits to a selected individual profile. Note that the individual \( \text{H}\beta \)-fit (bottom right) is the only case in which the best fit fell in the perturbation range. A general difficulty with fits to averaged data is the broadening of the scattering peak in the averaging process, a consequence of the asphericity of the nebula that has only been approximately mitigated by the rescaling operations described in Section 10.1. The fitting parameters and quality estimates are given in Table 1.

Fig. 10 shows the fits in which \( S \) was restricted to the perturbation range, 0.0–0.1. Only in the case of the fit to the selected \( \text{H}\beta \)-profile was the best overall fit found in the perturbation range, and this fit has already been shown in Fig. 9. We note that this particular fit has rather peculiar parameters (see Table 1), with a much larger optical depth parameter (and much smaller scattering parameter, \( S \)) than the others.

The easiest parameter to compare with observations is \( x \). Values of \( x = R/R_2 \) are markedly different for the two spectral lines, but this is simply a consequence of the greater angular extent of the data (larger \( R_2 \)) for \( \text{H}\beta \). When corrected to the value of \( R_2 \) for \( \text{H}\alpha \), by multiplying by 1.51, the values of \( x \) for \( \text{H}\beta \), in the order they appear in Table 1, are 0.1993, 0.2001 and 0.2032. These values are then consistent with those found for \( \text{H}\alpha \) for \( \text{H}\beta \). We now consider the results for the optical depth parameter.

Table 1. Values of the fitting parameters, \( x, S, \tau \) and \( Y \) (see Section 10), that gave the best value of \( \chi^2 \), as defined by equation (80), for the associated data set. Type ‘Avg’ refers to averaged data, and type ‘Ind’ to an individual profile. The symbol \( S' \) has the value \( 3.2 \times 10^{-4} \). Note that values of \( x \) for \( \text{H}\beta \) should be multiplied by 1.51 to place them on the same radial scale as those of \( \text{H}\alpha \).

| Line/type  | \( S_{\text{max}} \) | \( x \)   | \( S \) | \( \tau \) | \( Y \) | \( \chi^2 \) |
|------------|-----------------|---------|--------|--------|------|--------|
| \( \text{H}\alpha/\text{Avg} \) | 0.666 | 0.2326 | 0.666  | 1.665  | 0.372 | 0.491  |
| \( \text{H}\alpha/\text{Avg} \) | 0.100 | 0.2120 | 0.100  | 2.413  | 0.099 | 1.423  |
| \( \text{H}\alpha/\text{Ind} \) | 0.666 | 0.2060 | 0.585  | 2.850  | 0.160 | 0.384  |
| \( \text{H}\alpha/\text{Ind} \) | 0.100 | 0.2060 | 0.100  | 3.175  | 0.785 | 0.434  |
| \( \text{H}\beta/\text{Avg} \) | 0.666 | 0.1320 | 0.300  | 1.602  | 0.750 | 0.755  |
| \( \text{H}\beta/\text{Avg} \) | 0.100 | 0.1325 | 0.100  | 1.995  | 1.818 | 1.079  |
| \( \text{H}\beta/\text{Ind} \) | 0.666 | 0.1346 | \( S' \) | 6.193  | 26.10 | 0.146  |

We now consider the results for the optical depth parameter. In Section 2.1, the observed magnitude drops across the nebula, owing to extinction, ranged from 1.5 to 2.2 mag, corresponding to intensity ratios of 0.251 and 0.132 along a line of sight. The respective optical depths are 1.38 and 2.03. We note that the fitted values of \( \tau \) in Table 1 are of the optical depth parameter introduced at the end of Section 6.2, and that this is related to the optical depth of a radial ray passing through the full thickness of the nebula by \( \tau = \tau(1 - x) \) (see equation 48 with the limiting value of \( r \to R_2 \)). Fitted values of \( \tau \) from the averaged data of 1.67 for \( \text{H}\alpha \) and 1.60 for \( \text{H}\beta \) with the full range of \( S \) then yield respective optical depths of 1.32 and 1.39. If the perturbation restriction is enforced, the optical depths are 1.90 and 1.74, results that are reasonably consistent with the observational values. Note that considerably different optical depths are recovered from the selected individual profile, suggesting a locally clumpy medium.

11 CONCLUSION

An approximate analytical function has been derived that predicts the brightness of a spectral line, within some filter bandwidth, as a
function of angular distance from the centre of a cavity/shell PNe. The scattering part of this function is derived as a perturbation on the radiative transfer solution. This function has been fitted to observational data from NGC 6537 in the Hα and Hβ lines. The best-fitting optical depth parameter in the model gives reasonable agreement with the observed optical extinction in the nebula. However, best-fitting values for the scattering parameter are mostly too high to be consistent with a true perturbation. Consistent values were also found for the sizes of the cavity in both lines when the outer radius for Hα was applied to both spectral lines. The mean of the values of $x$ from Table 1 was 0.21, and the mean of the adjusted values for Hβ was 0.20.

The analytical function may also have applications to other sources with a hot, largely ionized, interior, surrounded by a shell rich in dust. Examples include post-AGB stars, supernova remnants, symbiotic stars, quasars and highly magnetized planets in addition to the PNe discussed here.

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**APPENDIX A: THE INTEGRAL OF G(P)**

The solution shown here is for any ray. For a ray which enters the cavity, the upper limit of these integrals should be set to $P = 1$. For a ray which avoids the cavity, the upper limit should be $R/R_2 \sin \omega$.

The integral of $g(p)$ in equation (67) breaks down into six integrals. The first two, trivially, are

$$I_1 = 2\epsilon a \int_{P/R_2}^{R/R_2} \frac{d\rho}{\rho} = 2\epsilon a (P - R/R_2) \quad \text{(A1)}$$

and

$$I_2 = \int_{R/R_2}^{P} \frac{d\rho}{\rho} = 2 \ln(P_2/R). \quad \text{(A2)}$$

The third can be integrated by parts, with the logarithm as the differentiated part. The result is

$$I_3 = \int_{R/R_2}^{P} \ln \left( \frac{1 + \rho}{1 - \rho} \right) \frac{d\rho}{\rho}$$

$$= (1 + P) \ln(1 + P) + (1 - P) \ln(1 - P)$$

$$- \frac{R}{R_2} \ln \left( \frac{R_2 + R}{R_2 - R} \right) - \ln \left( 1 - \frac{R_2^2}{R^2} \right). \quad \text{(A3)}$$

The fourth integral is the most problematic, as the result cannot be expressed in terms of elementary functions. If the integral is split,
so that
\[
\frac{I_4}{\epsilon a} = \int_{R/R_2}^P \rho \ln \left( \frac{1 + \rho}{1 - \rho} \right) d\rho
\]
\[
= \int_{R/R_2}^P \frac{\ln(1 + \rho)}{\rho} d\rho - \int_{R/R_2}^P \frac{\ln(1 - \rho)}{\rho} d\rho,
\]  \hspace{1cm} (A4)

then both the expressions on the bottom line of equation (A4) conform to a standard integral (Gradshteyn & Ryzhik 1965), which is their no. 2.728 (version 2). The solution is written in terms of the function \( \Phi(z, s, v) \), which is discussed in detail in section 9.55 of Gradshteyn & Ryzhik (1965). The final result is
\[
I_4 = \epsilon a \{ P [\Phi(-P, 2, 1) + \Phi(P, 2, 1)]
- \frac{R}{R_2} \left[ \Phi \left( -\frac{R}{R_2}, 2, 1 \right) + \Phi \left( \frac{R}{R_2}, 2, 1 \right) \right] \}.
\]  \hspace{1cm} (A5)

The fifth integral can be solved in a similar way to the simpler \( I_5 \), integrating by parts with the logarithm as the differentiated part. The integrated part of the result can be solved by an expansion in partial fractions, and the overall expression is
\[
I_5 = \int_{R/R_2}^P \frac{\rho}{\rho^2} \ln \left( \frac{1 + \rho}{1 - \rho} \right) d\rho
\]
\[
= \frac{R}{R_2} \ln \left( \frac{R_2 + R}{R_2 - R} \right) + \ln \left( \frac{R_2^2 - R^2}{R^2} \right)
+ 2 \ln P - \left( \frac{1}{P} + 1 \right) \ln(1 + P) + \left( \frac{1}{P} - 1 \right) \ln(1 - P).
\]  \hspace{1cm} (A6)

The final integral can be solved in a broadly similar manner to \( I_5 \), and the solution is
\[
\frac{I_6}{\epsilon a} = \int_{R/R_2}^P \rho \ln \left( \frac{1 + \rho}{1 - \rho} \right) d\rho
\]
\[
= \frac{1}{2} \left( 1 - \frac{R_2}{R} \right) \ln \left( \frac{R_2 + R}{R_2 - R} \right) + P - \frac{R}{R_2}
+ \frac{1}{2} (1 - P^2) \ln \left( \frac{1 - P}{1 + P} \right).
\]  \hspace{1cm} (A7)

The integral \( \Psi_j \), in its approximate analytical form, is now given by equation (66), with its lower line replaced by the expression
\[
e^{-\epsilon a} \sum_{k=1}^6 I_k,
\]  \hspace{1cm} (A8)

where \( I_k \) are the integrals in equations (A1) to (A7). The necessary approximation to the exponential is discussed in Section 8 (see equation 69).