Enhanced specialization and microlocalization

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Abstract
Enhanced ind-sheaves provide a suitable framework for the irregular Riemann–Hilbert correspondence. In this paper, we show how Sato’s specialization and microlocalization functors have a natural enhancement, and discuss some of their properties.

Keywords Sato’s specialization and microlocalization · Fourier-Sato transform · Irregular Riemann–Hilbert correspondence · Enhanced perverse sheaves

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Contents
1 Introduction ............................................. 2
2 Bordered normal deformation .................................... 3
3 Review on enhanced ind-sheaves .................................. 9
4 Specialization ............................................ 11
5 Fourier-Sato transform and microlocalization ............................ 21
6 Specialization at \(\infty\) on vector bundles ................................ 27
References ................................................ 32

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1 Introduction

1.1

Let $M$ be a real analytic manifold, and $N \subset M$ a closed submanifold. The normal deformation (or deformation to the normal cone) of $M$ along $N$ is a real analytic manifold $M^\text{nd}_N$ endowed with a map $(p, s) : M^\text{nd}_N \to M \times \mathbb{R}$, such that $s^{-1}(\mathbb{R}_{\neq 0}) \sim M \times \mathbb{R}_{\neq 0}$ and $s^{-1}(0)$ is identified with the normal bundle $T_N M$.

Sato’s specialization functor $\nu_N$, defined through $p : M^\text{nd}_N \to M$, associates to a sheaf $F \in \text{Db}(k_M)$ a conic sheaf on $T_N M$, describing the asymptotic behaviour of $F$ along $N$. Let $\cdot T N M$ be the complement of the zero section, identified with $N$. One has $\nu_N(F)|_N \cong F|_N$, and $\nu_N(F)|_{\cdot T N M}$ only depends on $F|_{M \setminus N}$.

Sato’s microlocalization functor $\mu_N$ is obtained from $\nu_N$ by Fourier-Sato transform, and provides a tool for the microlocal analysis of $F$ on the conormal bundle $T^*_N M$.

1.2

In this paper, we will define the enhanced version of the specialization and microlocalization functors. With notations as in §1.1, this proceeds as follows.

We start by showing that there exists a (unique) real analytic bordered space $(M^\text{nd}_N)_\infty$ such that the map $p : (M^\text{nd}_N)_\infty \to M$ is semiproper. (We call this a bordered compactification of $p$.)

Mimicking the classical construction, with $M^\text{nd}_N$ replaced by $(M^\text{nd}_N)_\infty$, we get an enhancement of Sato’s specialization. This associates to an enhanced ind-sheaf $K \in \text{E}^b(Ik_M)$ a conic enhanced ind-sheaf $E\nu_N(K)$ on the bordered compactification $(T_N M)_\infty$ of $T_N M \to N$. Consider the bordered space $M \setminus N := (M \setminus N, M)$. One has $E\nu_N(K)|_N \cong K|_N$, and $E\nu_N(K)|_{T_N M}$ only depends on $K|_{(M \setminus N)_\infty}$, not only on $K|_{M \setminus N}$.

Then, using the enhanced Fourier-Sato transform $L(\cdot)$, we get the enhanced microlocalization functor $E\mu_N := LE\nu_N$, with values in conic enhanced ind-sheaves on $(T^*_N M)_\infty$.

We establish some functorial properties of the functors $E\nu_N$ and $E\mu_N$. These are for the most part analogous to properties of the classical functors $\nu_N$ and $\mu_N$, but often require more geometrical proofs.

1.3

In view of future applications to the Fourier-Laplace transform of holonomic $\mathcal{D}$-modules, we also discuss the following situation.

Let $\tau : V \to N$ be a vector bundle, and $V_\infty$ its bordered compactification. In this setting, we consider the natural enhancement of the smash functor from [2, §6.1], described as follows. Let $\mathcal{S}V := ((\mathbb{R} \times V) \setminus (\{0\} \times N))/\mathbb{R}_{>0}$ be the fiberwise sphere compactification of $\tau$, and identify $V$ with the hemisphere $(\mathbb{R}_{>0} \times V)/\mathbb{R}_{>0}$. Consider

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1 We abusively call sheaf an object of the bounded derived category $\text{D}^b(k_M)$ of sheaves of $k$-vector spaces on $M$, for a fixed base field $k$. 

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the hypersurface $H := \partial V \subset S V$, and identify $\hat{V} := V \setminus N$ with the half of $T_H(S V)$ consisting of directions pointing to $V$. Then, the restriction of $Ev_H$ to $(\hat{V})_\infty$ can be thought of as a “specialization at infinity” on $V$. The enhanced smash functor $E\sigma_V$ (see §6.1) provides an extension of $Ev_H$ from $(\hat{V})_\infty$ to $V_\infty$.

If $K$ is an enhanced ind-sheaf on $V_\infty$, with the natural identification $V_\infty^* \simeq (T^*_N V)_\infty$ one has (see Proposition 6.6)

$$E\sigma_V^*(L K) \simeq E\mu_N(K).$$

### 1.4

The contents of this paper are as follows.

We introduce in Sect. 2 the notion of bordered compactification. This is a relative analogue of the classical one-point compactification. Then, we show that the normal deformation $p : M_N^{nd} \to M$ has a bordered compactification $(M_N^{nd})_\infty$ in the category of subanalytic bordered spaces.

Using the bordered normal deformation, and after recalling some notations in Sect. 3, the enhanced specialization is introduced and studied in Sect. 4. We also consider an analogous construction, attached to the real oriented blow-up of $M$ with center $N$. Moreover, we discuss the notion of conic enhanced ind-sheaf.

Section 5 establishes some complementary results on the enhanced Fourier-Sato transform, and uses it to enhance the microlocalization functor. Finally, in Sect. 6, we link the microlocalization along the zero section of a vector bundle with the so-called smash functor.

### 2 Bordered normal deformation

Here, after recalling the notion of bordered space from [3, §3], we introduce the notion of bordered compactification. We then show that the deformation to the normal cone, for which we refer to [7, §4.1], admits a canonical bordered compactification.

In this paper, a good space is a topological space which is Hausdorff, locally compact, countable at infinity, and with finite soft dimension.

#### 2.1 Bordered spaces

Denote by $\text{Top}$ the category of good spaces and continuous maps.

Denote by $b\text{-Top}$ the category of bordered spaces, whose objects are pairs $M = (M, C)$ with $M$ an open subset of a good space $C$. Set $\hat{M} := M$ and $\check{M} := C$. A morphism $f : M \to N$ in $b\text{-Top}$ is a morphism $\hat{f} : \hat{M} \to \hat{N}$ in $\text{Top}$ such that the projection $\Gamma_f \to \hat{M}$ is proper. Here, $\Gamma_f$ denotes the closure in $\hat{M} \times \hat{N}$ of the graph $\gamma_f$ of $\hat{f}$. 
The functor \( M \mapsto \hat{M} \) is right adjoint to the embedding \( \text{Top} \to \text{b-Top}, M \mapsto (M, M) \). We will write for short \( M \mapsto \hat{M} \) is not a functor.

We say that \( f : M \to N \) is semiproper if \( \overline{f} \) is proper. We say that \( M \) is semiproper if so is the natural morphism \( M \to \{ \text{pt} \} \). We say that \( f : M \to N \) is proper if it is semiproper and \( \hat{f} : \hat{M} \to \hat{N} \) is proper.

For any bordered space \( M \) there are canonical morphisms

\[
\hat{M} \xrightarrow{i_M} M \xrightarrow{j_M} \hat{M}.
\]

Note that \( j_M \) is semiproper.

By definition, a subset \( Z \) of \( M \) is a subset of \( \hat{M} \). We say that \( Z \) is open (resp. locally closed) if it is so in \( \hat{M} \). For a locally closed subset \( Z \) of \( M \), we set \( Z_\infty = (Z, \overline{Z}) \) where \( \overline{Z} \) is the closure of \( Z \) in \( M \). Note that, for an open subset \( U \subset M \), we have \( U_\infty \cong (U, \hat{M}) \).

We say that \( Z \) is relatively compact in \( M \) if it is contained in a compact subset of \( \hat{M} \). Then, for a morphism of bordered spaces \( f : M \to N \), the image \( f(\hat{Z}) \) is relatively compact in \( N \). In particular, the condition of \( Z \) being relatively compact in \( M \) does not depend on the choice of \( \hat{M} \).

### 2.2 Bordered compactification

Let \( S \) be a bordered space. Denote by \( \text{b-Top}_S \) the category of bordered spaces over \( S \), and by \( \text{Top}_S \) the category of good spaces over \( S \).

**Lemma 2.1** Let \( M \) and \( N \) be bordered spaces over \( S \). If \( N \) is semiproper over \( S \), then the natural morphism

\[
\text{Hom}_{\text{b-Top}_S}(M, N) \to \text{Hom}_{\text{Top}_S}(\hat{M}, \hat{N})
\]

is an isomorphism.

**Proof** Denote by \( p : M \to S \) and \( q : N \to S \) the given morphisms. In order to prove the statement, it is enough to show that any continuous map \( \hat{f} : \hat{M} \to \hat{N} \) which enters the commutative diagram

\[
\begin{array}{ccc}
\hat{M} & \xrightarrow{\hat{f}} & \hat{N} \\
\downarrow{\hat{p}} & & \downarrow{\hat{q}} \\
\hat{S} & \xrightarrow{\hat{f}} & \hat{N}
\end{array}
\]

induces a morphism \( f : M \to N \). That is, we have to prove that the map \( \overline{f} \) is proper.
It is not restrictive to assume that \( \hat{\rho} \) extends to a map \( \hat{\mathcal{M}} \to \hat{\mathcal{S}} \). Since \( \Gamma_f \subset \hat{\mathcal{M}} \times \hat{\mathcal{N}} \) is included in \( \hat{\mathcal{M}} \times \mathfrak{S} \), its closure \( \Gamma_f \) in \( \hat{\mathcal{M}} \times \mathfrak{N} \) is included in \( \hat{\mathcal{M}} \times \mathfrak{N} \). Since \( q \) is semiproper, the map \( \Gamma_q \to \mathcal{S} \) is proper. Hence so is the map \( \Gamma_f \to \hat{\mathcal{M}} \). \( \square \)

In particular, with notations as above, if \( \hat{\mathcal{M}} = \mathfrak{N} \) and both \( p \) and \( q \) are semiproper, then \( \mathcal{M} \simeq \mathfrak{N} \).

**Proposition 2.2** Let \( \mathcal{M} \) be a good space, and \( p: \mathcal{M} \to \mathcal{S} \) a morphism of bordered spaces. Then there exists a bordered space \( \mathcal{M}_\infty \), with \((\mathcal{M}_\infty)_{\partial} = \mathcal{M} \), such that \( \hat{\rho} \) induces a semiproper morphism \( p_\infty: \mathcal{M}_\infty \to \mathcal{S} \). Such an \( \mathcal{M}_\infty \) is unique up to a unique isomorphism.

**Definition 2.3** With notations as above, \( \mathcal{M}_\infty \) is called the **bordered compactification** of \( \mathcal{M} \) over \( \mathcal{S} \).

**Proof of Proposition 2.2** Set \( \hat{\mathcal{M}} := \hat{\mathcal{M}} \sqcup \hat{\mathcal{S}} \), and endow it with the following topology. Let \( i: \hat{\mathcal{S}} \hookrightarrow \hat{\mathcal{M}} \) and \( j: \mathcal{M} \hookrightarrow \hat{\mathcal{M}} \) be the inclusions. Consider the map \( \hat{\rho}: \mathcal{M} \to \hat{\mathcal{S}} \). For \( s \in \hat{\mathcal{S}} \), a neighborhood of \( i(s) \) is a subset of \( \hat{\mathcal{M}} \) containing \( i(V) \cup j(\hat{\rho}^{-1}(V \cap \hat{\mathcal{S}}) \setminus K) \), where \( V \subset \hat{\mathcal{S}} \) is a neighborhood of \( s \), and \( K \subset \mathcal{M} \) is a compact subset. For \( x \in \mathcal{M} \), a neighborhood of \( j(x) \) is a subset of \( \hat{\mathcal{M}} \) containing \( j(U) \), where \( U \subset \mathcal{M} \) is a neighborhood of \( x \). It is easy to check that \( \hat{\mathcal{M}} \) is a good topological space containing \( \mathcal{M} \) as an open subset.

Define \( \hat{\rho}: \hat{\mathcal{M}} \to \hat{\mathcal{S}} \) by \( \hat{\rho}(j(x)) = \hat{\rho}(x) \) for \( x \in \mathcal{M} \), and \( \hat{\rho}(i(s)) = s \) for \( s \in \hat{\mathcal{S}} \). Then, \( \hat{\rho} \) is proper. It follows that, setting \( \mathcal{M}_\infty := (\mathcal{M}, \hat{\mathcal{M}}) \), the morphism \( p \) extends to a semiproper morphism \( \mathcal{M}_\infty \to \hat{\mathcal{S}} \). Such a morphism factors as \( \mathcal{M}_\infty \to \mathcal{S} \to \hat{\mathcal{S}} \), and hence also \( p_\infty \) is semiproper.

This proves the existence. Uniqueness follows from Lemma 2.1, as noticed before the statement of this proposition. \( \square \)

**2.3 Blow-ups and normal deformation**

Let \( \mathcal{M} \) be a real analytic manifold and \( N \subset \mathcal{M} \) a closed submanifold. Denote by \( \tau: T_N \mathcal{M} \to N \) the normal bundle, and by \( \bar{T}_N \mathcal{M} \subset T_N \mathcal{M} \) the complement of the zero-section. Recall that the multiplicative groups \( \mathbb{R}^\times := \mathbb{R}_{\neq 0} \) and \( \mathbb{R}_{>0}^\times := \mathbb{R}_{>0} \) act freely on \( \bar{T}_N \mathcal{M} \). Denote by \( S_N \mathcal{M} := \bar{T}_N \mathcal{M}/\mathbb{R}_{>0}^\times \) the sphere normal bundle, and by \( P_N \mathcal{M} := \bar{T}_N \mathcal{M}/\mathbb{R}^\times \) the projective normal bundle.

**Notation 2.4** (i) Denote by \( p_{tb}: M^\text{tb}_N \to \mathcal{M} \) the real oriented blow-up of \( \mathcal{M} \) with center \( N \). Recall that \( M^\text{tb}_N \) is a subanalytic space, that \( p_{tb} \) induces an isomorphism \( p_{tb}^{-1}(M \setminus N) \xrightarrow{\sim} M \setminus N \), and that \( p_{tb}^{-1}(N) = S_N \mathcal{M} \). In fact, \( M^\text{tb}_N \) is a real analytic
manifold with boundary $S_N M$. This is pictured in the commutative diagram

\[
\begin{array}{ccc}
S_N M & \xrightarrow{\cdot} & M^r_N \\
\downarrow & & \downarrow \\
N & \xrightarrow{p^r_N} & M
\end{array}
\]  \hspace{1cm} (2.1)

(ii) Denote by $p_{pb} : M^p_N \to M$ the real projective blow-up of $M$ with center $N$. Recall that $M^p_N$ is a real analytic manifold, that $p_{pb}$ induces an isomorphism $p_{pb}^{-1}(M \setminus N) \cong M \setminus N$, and that $p_{pb}^{-1}(N) = P_N M$. This is pictured in the commutative diagram

\[
\begin{array}{ccc}
P_N M & \xrightarrow{\cdot} & M^p_N \\
\downarrow & & \downarrow \\
N & \xrightarrow{p_{pb}} & M
\end{array}
\]  \hspace{1cm} (2.2)

We have a natural commutative diagram

\[
\begin{array}{ccc}
M^p_N & \xleftarrow{\cdot} & M^r_N \\
p_{pb} & \xleftarrow{\cdot} & p^r_N
\end{array}
\]

(iii) Denote by $(p_{nd}, s_{nd}) : M^{nd}_N \to M \times \mathbb{R}$ the normal deformation (or deformation to the normal cone) of $M$ along $N$ (see [7, §4.1]). Recall that $M^{nd}_N$ is a real analytic manifold, and that $(p_{nd}, s_{nd})$ induces isomorphisms $p_{nd}^{-1}(M \setminus N) \cong (M \setminus N) \times \mathbb{R}_{\neq 0}$ and $s^{-1}_{nd}(\mathbb{R}_{\neq 0}) \cong M \times \mathbb{R}_{\neq 0}$. One also has $s^{-1}_{nd}([0]) = T_N M$. This is pictured in the commutative diagram

\[
\begin{array}{ccc}
T_N M & \xleftarrow{i_{nd}} & M^{nd}_N \\
\downarrow & \xleftarrow{(p_{nd}, s_{nd})} & \\
M \times \{0\} & \xleftarrow{\cdot} & M \times \mathbb{R}
\end{array}
\]  \hspace{1cm} (2.3)

There is a natural action of $\mathbb{R}^\times$ on $M^{nd}_N$, extending that on $T_N M$. The map $s_{nd} : M^{nd}_N \to \mathbb{R}$ is smooth and equivariant with respect to the action of $\mathbb{R}^\times$ on $\mathbb{R}$ given by $c \cdot s = c^{-1} s$. For $\Omega := s^{-1}_{nd}(\mathbb{R}_{>0}) \cong M \times \mathbb{R}_{>0}$, consider the commutative diagram

\[
\begin{array}{ccc}
T_N M & \xleftarrow{i_{nd}} & M^{nd}_N \\
\downarrow & \xleftarrow{\cdot} & \\
N \times \{0\} & \xleftarrow{\cdot} & M \times \mathbb{R}
\end{array}
\]  \hspace{1cm} (2.4)
where we set \( p_\Omega := p_{nd}|_\Omega \). Note that \( \overline{\Omega} = s_{nd}^{-1}(\mathbb{R}_{\geq 0}) = \Omega \cup T_N M \). For \( S \subset M \), the normal cone to \( S \) along \( N \) is defined by
\[
C_N(S) := T_N M \cap j_{nd}(p_{\Omega}^{-1}(S)).
\]  
(2.5)

(iv) Denote by \( \tilde{\Omega} \) the complement of \( p_{nd}^{-1}(N) \setminus \hat{T}_N M \) in \( \overline{\Omega} \), i.e., \( \tilde{\Omega} = ((M \setminus N) \times \mathbb{R}_{> 0}) \cup \hat{T}_N M \). Thus, \( \tilde{\Omega} \) is an open subset of \( \overline{\Omega} \) which is invariant by the action of \( \mathbb{R}_x^\times \), and enters the commutative diagram
\[
\begin{array}{ccc}
T_N M^c & \xrightarrow{\Phi_{\rho}} & \overline{\Omega} \\
& \swarrow & \nearrow \rho_{nd} \mid \Omega \downarrow & \rho_{nd} \downarrow \\
\hat{T}_N M^c & \xrightarrow{\gamma} & M \\
& \swarrow & \nearrow \gamma \downarrow & \gamma \downarrow \\
S_N M^c & \xrightarrow{\Phi_{\rho}} & M^b_N
\end{array}
\]  
(2.6)

Note that \( \gamma : \tilde{\Omega} \to M^b_N \) is a principal \( \mathbb{R}_x^\times \)-bundle.

**Remark 2.5** Let us illustrate the above constructions in local coordinates. Consider a chart \( M \supset U \xrightarrow{\varphi} \mathbb{R}_x^m \times \mathbb{R}_y^n \) such that \( N \cap U = \phi^{-1}(x = 0) \).

(i) Let \( \mathbb{R}_x^\times \) act on \( \mathbb{R}_x^m \times \mathbb{R}_y^n \times \mathbb{R}_s \) by \( c \cdot (v, y, s) = (cv, y, c^{-1}s) \). Then \( p_{rb}^{-1}(U) \subset M^b_N \) has \( \mathbb{R}_x^\times \)-homogeneous coordinates \( [v, y, s] \) with \( v \neq 0, s \geq 0 \), and \( (sv, y) \in \varphi(U) \). One has \( p_{rb}([(v, y, s)]) = (sv, y) \).

(ii) Similarly, replacing the action of \( \mathbb{R}_x^\times \) by that of \( \mathbb{R}_x \), the open subset \( p_{pb}^{-1}(U) \subset M^b_N \) has \( \mathbb{R}_x \)-homogeneous coordinates \( [v, y, s] \) with \( v \neq 0 \) and \( (sv, y) \in \varphi(U) \). One has \( p_{pb}([(v, y, s)]) = (sv, y) \).

(iii) The open subset \( p_{nd}^{-1}(U) \subset M^d_N \) has coordinates \( (v, y, s) \in \mathbb{R}_x^{m+n+1} \), with \( (sv, y) \in \varphi(U) \). One has \( p_{nd}(v, y, s) = (sv, y) \) and \( s_{nd}(v, y, s) = s \). The action of \( \mathbb{R}_x^\times \) on \( M^d_N \) is given by \( c \cdot (v, y, s) = (cv, y, c^{-1}s) \). One has \( \tilde{\Omega} \cap p_{nd}^{-1}(U) = \{s > 0\} \) and \( \overline{\Omega} \cap p_{nd}^{-1}(U) = \{s \geq 0\} \).

(iv) One has \( \tilde{\Omega} \cap p_{nd}^{-1}(U) = \{(v, y, s) ; s \geq 0, \ v \neq 0\} \) and \( \gamma(v, y, s) = [v, y, s] \in M^b_N \).

### 2.4 Bordered normal deformation

Let \( M \) be a real analytic manifold and \( N \subset M \) a closed submanifold. Set \( X = M \times P \) and \( Y = N \times \{0\} \subset X \), where \( P := \mathbb{R} \cup \{\infty\} \) is the real projective line. There is a natural commutative diagram (see Fig. 1)
\[
\begin{array}{ccc}
M^d_N & \xrightarrow{(p_{nd}, s_{nd})} & X^b_Y \\
\downarrow & & \downarrow \rho_{pb} \\
M \times \mathbb{R}^c & \xrightarrow{\rho_{pb}} & M \times P = X
\end{array}
\]  
(2.7)
Fig. 1 The embedding $M^\text{nd} \hookrightarrow X^\text{pb}_Y$ pictured in the case $M = \mathbb{R}$ and $N = \{0\}$. The greyed out region does not belong to $X^\text{pb}_Y$. The red lines are fibers of the projections $M^\text{nd} \to M$ and $X^\text{pb}_Y \to X = M \times P \to M$, respectively. In the figure, we write $M_1 = s^{-1}(0)$ (Color figure online)

where the bottom arrow is induced by the inclusion of the affine chart $\mathbb{R} \subset P$, and the top arrow is the embedding described as follows. Recall that $M^\text{nd}_N = s^{-1}(\mathbb{R} \neq 0) \cup T_N M$. The natural identifications $(p_\text{nd}, s_\text{nd}) : s_\text{nd}^{-1}(\mathbb{R} \neq 0) \sim \to M \times \mathbb{R} \neq 0$ and $p_{\text{pb}} : p^\text{pb}_{\text{pb}}^{-1}(X \setminus Y) \sim \to (M \times P) \setminus (N \times \{0\})$, provide an open embedding $s_\text{nd}^{-1}(\mathbb{R} \neq 0) \subseteq X^\text{pb}_Y$. This extends to $M^\text{nd}_N$ by sending $v \in T_N M$ to $[v, 1] \in P_Y X = p^\text{pb}_{\text{pb}}^{-1}(Y)$. Note that one has

$$X^\text{pb}_Y \setminus M^\text{nd}_N = p^\text{pb}_{\text{pb}}^{-1}(M \times \{\infty\}) \cup p^\text{pb}_{\text{pb}}^{-1}((M \setminus N) \times \{0\})$$

$$= p^\text{pb}_{\text{pb}}^{-1}(M \times \{\infty\}) \cup p^\text{pb}_{\text{pb}}^{-1}((M \setminus N) \times \{0\})$$

$$\cup P_{\times \{0\}}(M \times \{0\}).$$

Remark 2.6 Let us describe the above constructions in the situation of Remark 2.5. Consider the action of $\mathbb{R}^\times$ on $\mathbb{R}^n_\text{nd} \times \mathbb{R}^n_\text{pb} \times \mathbb{R}_r \times \mathbb{R}_s$ given by $c \cdot (v, y, r, s) = (cv, y, cr, c^{-1}s)$. Let $\mathbb{R} = P \setminus \{\infty\}$ be the affine chart. Then $p^\text{pb}_{\text{pb}}^{-1}(U \times \mathbb{R}) \subseteq X^\text{pb}_Y$ has $\mathbb{R}^\times$-homogeneous coordinates $[v, y, r, s]$ with $(v, s) \neq (0, 0)$ and $(sv, y) \in \varphi(U)$. One has $p_{\text{pb}}([v, y, r, s]) = (sv, y, sr)$. The embedding $M^\text{nd}_N \hookrightarrow X^\text{pb}_Y$ is given by $(v, y, s) \mapsto [v, y, 1, s]$.

Recall from [3, §5.4] that a real analytic bordered space is a bordered space $M$ such that $\hat{M}$ is a real analytic manifold, and $\hat{N} \subset \hat{M}$ is a subanalytic open subset. A morphism $f : M \to N$ of real analytic bordered spaces is a morphism of bordered spaces such that $\hat{f}$ is a real analytic map, and $\hat{\Gamma}_f$ is a subanalytic subset of $\hat{M} \times \hat{N}$.

Lemma-Definition 2.7 The bordered compactification of $M^\text{nd}_N$ over $M$ has a realization in the category of real analytic bordered spaces by $(M^\text{nd}_N)_\infty := (M^\text{nd}_N, X^\text{pb}_Y)$, using the open embedding (2.7). Note that the projection $X^\text{pb}_Y \to M$ is proper.
Note that the closure $\overline{T_N M}$ of $T_N M$ in $P_Y X$ is the projective compactification of $T_N M$ along the fibers of $\tau: T_N M \to N$. Considering the bordered spaces $(T_N M)_\infty := (T_N M, \overline{T_N M})$ and $\Omega_\infty := (\Omega, X_Y^{pb})$, one has the commutative diagram with Cartesian squares of bordered spaces semiproper over $M$

$$
\begin{array}{ccc}
(T_N M)_\infty & \xrightarrow{\cong} & (M^\text{nd})_\infty \\
\downarrow & & \downarrow \Omega_\infty \\
N \times [0] & \xrightarrow{\cong} & M \times \mathbb{R}_\infty \\
\end{array}
$$

Note that the morphisms in the top row are $(\mathbb{R}^>_{\infty})_\infty$-equivariant. Here, $(\mathbb{R}^>_{\infty})_\infty := (\mathbb{R}_>, \mathbb{R})$ is a group object in $b\text{-}Top$, for $\mathbb{R} := \mathbb{R} \cup \{-\infty, +\infty\}$ the two-point compactification of $\mathbb{R}$.

## 3 Review on enhanced ind-sheaves

We recall here some notions and results, mainly to fix notations, referring to the literature for details. In particular, we refer to [7] for sheaves, to [12] (see also [4,5]) for enhanced sheaves, to [8] for ind-sheaves, and to [3] (see also [4,6,9,10]) for bordered spaces and enhanced ind-sheaves.

In this paper, $k$ denotes a base field.

### 3.1 Sheaves

Let $M$ be a good space.

Denote by $D^b(k_M)$ the bounded derived category of sheaves of $k$-vector spaces on $M$, and by $\otimes$, $f^{-1}$, $R f_!$ and $R \text{Hom}$, $R f_*$, $f^!$ the six operations. Here $f: M \to N$ is a morphism of good spaces.

For $S \subset M$ locally closed, we denote by $k_S$ the extension by zero to $M$ of the constant sheaf on $S$ with stalk $k$.

### 3.2 Ind-sheaves

Let $M$ be a bordered space.

We denote by $D^b(Ik_M)$ the bounded derived category of ind-sheaves of $k$-vector spaces on $M$, and by $\otimes$, $f^{-1}$, $R f_!$ and $R \text{Hom}$, $R f_*$, $f^!$ the six operations. Here $f: M \to N$ is a morphism of bordered spaces.

We denote by $\iota_M: D^b(k_M) \to D^b(Ik_M)$ the natural embedding, by $\alpha_M$ the left adjoint of $\iota_M$. One sets $R \text{Hom} := \alpha_M R \text{Hom}$.

For $F \in D^b(k_M)$, we often write simply $F$ instead of $\iota_M F$ in order to make notations less heavy.
3.3 Enhanced ind-sheaves

Denote by \( t \in \mathbb{R} \) the coordinate on the affine line, recall its two-point compactification \( \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\} \), and set \( \mathbb{R}_\infty := (\mathbb{R}, \overline{\mathbb{R}}) \). For \( M \) a bordered space, consider the projection

\[ \pi_M : M \times \mathbb{R}_\infty \to M. \]

Denote by \( E^b(I_kM) := D^b(I_kM \times \mathbb{R}_\infty) / \pi_M^{-1}D^b(I_kM) \) the bounded derived category of enhanced ind-sheaves of \( \mathbb{k} \)-vector spaces on \( M \). Denote by \( Q_M : D^b(I_kM \times \mathbb{R}_\infty) \to E^b(I_kM) \) the quotient functor.

For \( f : M \to N \) a morphism of bordered spaces, set

\[ f_{\mathbb{R}} := f \times \text{id}_{\mathbb{R}_\infty} : M \times \mathbb{R}_\infty \to N \times \mathbb{R}_\infty. \]

Denote by \( \otimes, Ef^{-1}, Ef_*, Ef^! \) and \( R\text{Hom}^+, Ef_*^+, Ef^! \) the six operations for enhanced ind-sheaves. Recall that \( \otimes \) is the additive convolution in the \( t \) variable, and that the external operations are induced via \( Q \) by the corresponding operations for ind-sheaves, with respect to the morphism \( f_{\mathbb{R}} \). Denote by \( D^E_M \) the Verdier dual.

There is a natural decomposition \( E^b(I_kM) \simeq E^b_+(I_kM) \oplus E^b_-(I_kM) \), there are embeddings

\[ \epsilon^\pm_M : D^b(I_kM) \to E^b_{\pm}(I_kM), \quad F \mapsto Q_M(\mathbb{k}_{\{t \geq 0\}} \otimes \pi_M^{-1}F), \]

and one sets \( \epsilon_M(F) := \epsilon^+_M(F) \oplus \epsilon^-_M(F) \in E^b(I_kM) \). Note that \( \epsilon_M(F) \simeq Q_M(\mathbb{k}_{\{t = 0\}} \otimes \pi_M^{-1}F) \).

3.4 Stable objects

Let \( M \) be a bordered space. Set

\[ \mathbb{k}_{[t \geq 0]} := \text{"lim"}_{a \to +\infty} \mathbb{k}_{[t \geq a]} \in D^b(I_kM \times \mathbb{R}_\infty), \]

\[ \mathbb{k}^E_M := Q_M\mathbb{k}_{[t \geq 0]} \in E^b_+(I_kM). \]

An object \( K \in E^b(I_kM) \) is called stable if \( \mathbb{k}^E_M \otimes K \xrightarrow{\sim} K \).

There is an embedding

\[ \epsilon_M : D^b(I_kM) \to E^b_+(I_kM), \quad F \mapsto \mathbb{k}^E_M \otimes \epsilon_M(F), \]

with values in stable objects.
4 Specialization

We discuss here the natural enhancement of the notions of conic object and Sato’s specialization. For the corresponding classical notions we refer to [7, §3.7] and [7, §4.2], respectively. We also link the specialization functor with the real oriented blow-up.

4.1 Conic objects

Recall that the bordered space \((\mathbb{R}^\times)_\infty := (\mathbb{R}_{>0}, \mathbb{R})\) is semiproper and has a structure of bordered group (i.e., is a group object in the category of bordered spaces). Let \(M\) be a bordered space endowed with an action of \((\mathbb{R}^\times)_\infty\), and consider the maps

\[ p, \mu : M \times (\mathbb{R}^\times)_\infty \to M, \]

where \(p\) is the projection and \(\mu\) is the action. Similarly to [7], one says that an object \(K \in \mathcal{E}^b(Ik_M)\) is \((\mathbb{R}^\times)_\infty\)-conic if there is an isomorphism

\[ E_p^{-1} K \simeq E \mu^{-1} K. \]

(Recall that if \(E_p^{-1} K\) and \(E \mu^{-1} K\) are isomorphic, then there exists a unique isomorphism which restricts to the identity on \(M \times \{1\}\).)

Denote by \(\mathcal{E}^b(\mathbb{R}^\times)_\infty (Ik_M)\) the full triangulated subcategory of conic objects.

We say that a morphism \(\gamma : M \to S\) is a principal \((\mathbb{R}^\times)_\infty\)-bundle if it is semiproper and if \(M\) is endowed with an action of \((\mathbb{R}^\times)_\infty\) such that the underlying map \(\tilde{\gamma} : \tilde{M} \to S\) is a principal \(\mathbb{R}^\times\)-bundle.

Lemma 4.1 Let \(\gamma : M \to S\) be a principal \((\mathbb{R}^\times)_\infty\)-bundle. Then, for \(K \in \mathcal{E}^b(\mathbb{R}^\times)_\infty (Ik_M)\)

(i) One has

\[ K \simeq E\gamma^{-1} E\gamma_* K \simeq E\gamma^{-1} E\gamma!! K. \]

In particular, \(K \simeq E\gamma^{-1} H\) for some \(H \in \mathcal{E}^b(Ik_S)\).

(ii) One has \(E\gamma!! K \simeq E\gamma_* K[-1]\).

Proof (i) Since the proofs are similar, let us only discuss the first isomorphism. Consider the cartesian diagram

\[
\begin{array}{ccc}
M \times (\mathbb{R}^\times)_\infty & \xrightarrow{p} & M \\
\mu \downarrow & & \downarrow \\
M & \xrightarrow{\gamma} & S.
\end{array}
\]
Recalling that \( K \) is \((\mathbb{R}_{>0})^\infty\)-conic, one has

\[
E_{\gamma}^{-1}E_{\gamma}^* K \simeq E_{\gamma} E_{\gamma}^* K[-1] \simeq E_{p_*} E_{\mu}^1 K[-1] \simeq E_{p_*} E_{p}^1 K[-1].
\]

Then, Sublemma 4.2 implies

\[
E_{\gamma}^{-1}E_{\gamma}^* K \simeq \mathcal{R}I\hom^+ (\epsilon_M(R_{p_!} k_{\mathcal{M} \times \mathbb{R}}), K)[-1]
\]

\[
\simeq \mathcal{R}I\hom^+ (\epsilon_M(k_{\mathcal{M}}[-1]), K)[-1] \simeq K.
\]

(ii) One has

\[
E_{\gamma}! K \simeq E_{\gamma}!!E_{\gamma}^{-1}E_{\gamma}^* K \simeq E_{\gamma}^* K \oplus \epsilon_S(R_{\gamma}!(k_{\mathcal{M}})) \simeq E_{\gamma} K[-1],
\]

where the first isomorphism follows from (i), and the second isomorphism follows from Sublemma 4.2.

\[\square\]

**Sublemma 4.2** Let \( f : M \to N \) be a semiproper morphism of bordered spaces. Then, for any \( K \in \mathcal{E}^b(1k_M) \) one has

\[
Ef_!!Ef^{-1} K \simeq K \oplus \epsilon_N(R_{f_!} k_{\mathcal{M}}),
\]

\[
Ef_* Ef^1 K \simeq \mathcal{R}I\hom^+ (\epsilon_N(R_{f_!} k_{\mathcal{M}}), K).
\]

**Proof** The first isomorphism follows from

\[
Ef_!!Ef^{-1} K \simeq Ef_!!(Ef^{-1} K \oplus \epsilon_M(k_{\mathcal{M}}))
\]

\[
\simeq K \oplus Ef_!!(\epsilon_M(k_{\mathcal{M}}))
\]

\[
\simeq K \oplus \epsilon_N(R_{f_!} k_{\mathcal{M}})
\]

where the last isomorphism is due to the fact that \( f \) is semiproper.

Similarly, the second isomorphism follows from

\[
Ef_* Ef^1 K \simeq Ef_* \mathcal{R}I\hom^+ (\epsilon_M(k_{\mathcal{M}}), Ef^1 K)
\]

\[
\simeq \mathcal{R}I\hom^+ (Ef_!!(\epsilon_M(k_{\mathcal{M}})), K)
\]

\[
\simeq \mathcal{R}I\hom^+ (\epsilon_N(R_{f_!} k_{\mathcal{M}}), K).
\]

\[\square\]

### 4.2 Conic objects on vector bundles

Let \( \tau : V \to N \) be a real vector bundle over a good space \( N \), and let \( \hat{V} = V \setminus N \) be the complement of the zero-section. Let \( S_N V \to N \) be the associated sphere bundle
defined by \( S_N V := \hat{V}/\mathbb{R}_+^0 \). Consider the vector bundle \( W := \mathbb{R} \times V \to N \), and let \( \hat{W} := W \setminus \{0\} \times N \) be the complement of the zero section. The fiberwise sphere compactification \( \mathbb{S}V \to N \) of \( V \to N \) is the quotient \( \mathbb{S}V := \hat{W}/\mathbb{R}_+^0 \), where the action is given by \( c \cdot (u, x) = (cu, cx) \). The bordered compactification of \( V \to N \) is given by \( V_\infty = (V, \mathbb{S}V) \), with \( V \) embedded in \( \mathbb{S}V \) by \( v \mapsto [1, v] \). Note that \( V_\infty \) is endowed with a natural \( (\mathbb{R}_+^0)_\infty \)-action. Consider the morphisms

\[
\begin{array}{ccc}
N^C & \xrightarrow{o} & V_\infty \\
\downarrow{\tau} & & \downarrow{\tau} \\
\hat{V}_\infty & \xrightarrow{j} & S_N V,
\end{array}
\]

where \( o \) is the embedding of the zero section, \( j \) is the open embedding, and \( \gamma \) the quotient by the action of \( \mathbb{R}_+^0 \).

**Notation 4.3** For \( K \in E_b(\mathbb{R}_+^0)_\infty (I_k V_\infty) \), set

\[
K^{sph} := E\gamma K \in E_b(I_k S_N V).
\]

**Lemma 4.4** For \( K \in E_b(\mathbb{R}_+^0)_\infty (I_k V_\infty) \), one has the isomorphisms

(i) \( Ej^{-1}K \simeq E\gamma^{-1}K^{sph} \),
(ii) \( E\tau K \simeq Eo^{-1}K \),
(iii) \( E\tau!!K \simeq Eo^{1}K \),

and a distinguished triangle

(iv) \( E\tau!!E\gamma^{-1}K^{sph} \to Eo^{1}K \to Eo^{-1}K \xrightarrow{+1} \).

**Proof** (i) follows from Lemma 4.1.

(ii) We will adapt some arguments in the proof of [11, Lemma 2.1.12]. One has

\[
E\tau_*Eo_*Eo^{-1}K \simeq Eo^{-1}K,
\]

\[
E\tau_*Ej!!Ej^{-1}K \simeq E\tau_*Ej!!E\gamma^{-1}E\gamma K \simeq E\gamma K \simeq E\gamma K \xrightarrow{+1} .
\]

where the last isomorphism follows from Lemma 4.1. Applying \( E\tau_* \) to the distinguished triangle

\[
Ej!!Ej^{-1}K \to K \to E\gamma Eo^{-1}K \xrightarrow{+1} ,
\]

we are thus left to prove

\[
E\tau_*Ej!!E\gamma^{-1}H \simeq 0,
\]

for \( H = E\gamma K \). Let us prove it for an arbitrary \( H \in E_b(I_k S_N V) \).
Denoting by \((V_{rb}^N)_{\infty} := (V_{rb}^N, S V_{rb}^N)\) the bordered compactification of the real oriented blow-up \(p_{rb}: V_{rb}^N \to V_{\infty}\), consider the commutative diagram

\[
\begin{array}{ccc}
\hat{V}_{\infty} & \xrightarrow{\gamma} & (V_{rb}^N)_{\infty} \\
\downarrow j & \quad & \downarrow \gamma \\
V_{\infty} & \xrightarrow{p_{rb}} & S_N V
\end{array}
\]

Note that \(\hat{\gamma}: V_{rb}^N \to S_N V, [x, r] \mapsto [x]\), is an \(\mathbb{R}_{\geq 0}\)-fiber bundle. Hence one has (where we neglect for short the indices on \(\epsilon\))

\[
\begin{aligned}
E \hat{\gamma}^{-1} H & \simeq \epsilon(\hat{\gamma}^* k_{S_N V}) \otimes E \hat{\gamma}^{-1} H \\
& \simeq \epsilon(k_{\gamma(V)}) \otimes E \hat{\gamma}^{-1} H[1], \\
\end{aligned}
\]

(4.2)

\[
R \hat{\gamma}^* k_{V_{rb}^N} \simeq 0.
\]

(4.3)

Back to (4.1), one has

\[
\begin{aligned}
E \tau^! E j!! E \gamma^{-1} H & \simeq E \tau^* E p_{rb}^! E j!! E \gamma^{-1} H \\
& \simeq E \tau^* E p_{rb}^* E j!! E \gamma^{-1} H \\
& \simeq E q_* E \hat{\gamma}^* E j!! E \gamma^{-1} E \gamma^{-1} H \\
& \simeq E q_* E \gamma^* E \gamma^{-1} H[1] \\
& \simeq E q_* R I_{hom}^+(\epsilon(\hat{\gamma}^* k_{V_{rb}^N}), H)[-1] \\
& \simeq 0,
\end{aligned}
\]

where (1) is due to the fact that \(p_{rb}\) is proper, (2) follows from (4.2), (3) follows from Sublemma 4.2 since \(\hat{\gamma}\) is semiproper, and (4) follows from (4.3).

(iii) has a proof similar to (ii).

(iv) Let us show that the distinguished triangle

\[
E \tau^!! E j!! E \gamma^{-1} K \to E \tau^!! K \to E \tau^!! E o!! E o^{-1} K \xrightarrow{+1}
\]

is isomorphic to the distinguished triangle in the statement.

(iv-a) One has \(E \tau^!! E j!! E j^{-1} K \simeq E \tau^!! E j^{-1} K \simeq E \tau^!! E \gamma^{-1} K_{sph}\), where the last isomorphism follows from (i).

(iv-b) By (iii), one has \(E \tau^!! K \simeq E o^! K\).

(iv-c) One has \(E \tau^!! E o!! E o^{-1} K \simeq E o^{-1} K\), since \(\tau \circ o \simeq id_N\).
Lemma 4.5 For \( K \in \mathbb{E}^b_{(\mathbb{R}^\times_{>0})\infty}(\mathcal{I} k_{\mathcal{V}_\infty}) \), one has

\[
\operatorname{DE}(K^{sp}) \simeq (\operatorname{DE} K)^{sp}[-1].
\]

Proof One has

\[
\begin{align*}
\operatorname{DE}(E\gamma^*E j^{-1} K) & \simeq (\operatorname{DE} (E\gamma^!E j^{-1} K)[1]) \\
& \simeq E\gamma^*E j^!\operatorname{DE}(K)[-1] \\
& \simeq E\gamma^*E j^{-1}\operatorname{DE}(K)[-1],
\end{align*}
\]

where (*) follows from Lemma 4.1 (ii). \( \square \)

4.3 Enhanced specialization

Let \( M \) be a real analytic manifold, and \( N \subset M \) a closed submanifold. We will introduce here an enhancement of Sato’s specialization functor. We refer to [7, Chapter 4] for the classical construction.

Recall that the action of \( \mathbb{R}^\times_{>0} \) on (2.3) naturally extends to an action of \( (\mathbb{R}^\times_{>0})\infty \) on its bordered compactification. Consider the morphisms

\[
(T_N M)^\infty \xleftarrow{i_{nd}} (M_N^{nd})^\infty \xrightarrow{j_{nd}} \Omega^\infty \xrightarrow{p_{nd}\mid\Omega} M.
\]

In the following, when there is no risk of confusion, we will write for short \( i = i_{nd}, \)
\( j = j_{nd} \) and \( p = p_{nd}\mid\Omega \).

Definition 4.6 For \( K \in \mathbb{E}^b(\mathcal{I} k_M) \), we set

\[
\begin{align*}
E\nu_N(K) & := Ei^{-1}E j_\Omega E p^{-1}_\Omega K \in \mathbb{E}^b_{(\mathbb{R}^\times_{>0})\infty}(\mathcal{I} k_{(T_N M)^\infty}), \\
E\nu_N^{sp}(K) & := (E\nu_N(K))^{sp} \in \mathbb{E}^b(\mathcal{I} k_{SN M}).
\end{align*}
\]

The functor \( E\nu_N \) is called enhanced specialization along \( N \).

With a proof similar to that of Lemma 4.12 (or of [7, Lemma 4.2.1]), one has

Lemma 4.7 For \( K \in \mathbb{E}^b(\mathcal{I} k_M) \), one has

\[
E\nu_N(K) \simeq E i^!E j^{!!}E p^{!!}_\Omega K.
\]

Note that there is an isomorphism

\[
e \circ \nu_N \simeq E\nu_N \circ e,
\]

and similarly for \( e \) replaced by \( \epsilon, \epsilon^+ \) or \( \epsilon^- \).
Consider the morphisms

$$M \xleftarrow{i_N} N \xrightarrow{o} (T_N M)_\infty \xrightarrow{\tau} (\hat{T}_N M)_\infty \xrightarrow{\gamma} S_N M,$$

where $o$ is the zero-section.

**Lemma 4.8** For $K \in E^b(Ik_M)$, one has the isomorphisms

(i) $E \tau_\ast E \nu_N(K) \simeq E o^{-1} E \nu_N(K) \simeq E i_N^{-1} K$,

(ii) $E \tau!! E \nu_N(K) \simeq E o! E \nu_N(K) \simeq E i!_N K$,

and distinguished triangles

(iii) $E \tau!! E \nu^{-1} E \nu_N K \to E i^{-1}_N K \to E i^{-1}_N K + 1$,

(iv) $E \nu^{-1} E \nu_N^{sph} K \to E \nu_N K \to E o\ast E i_N^{-1} K \to E i^{-1}_N K$,

(v) $E o\ast E i_N^{-1} K \to E \nu_N K \to E \nu\ast E i^{-1}_N K + 1$.

**Proof** (i-a) The isomorphism $E \tau_\ast E \nu_N(K) \simeq E o^{-1} E \nu_N(K)$ follows from Lemma 4.4 (ii).

(i-b) Let us show that the composition

$$E \tau_\ast E \nu_N(K) \to E \tau_\ast E \nu_N(E i_N \ast E i_N^{-1} K) \to E i^{-1}_N K$$

is an isomorphism. Since the problem is local on $N$, we may work in coordinates as in Remark 2.5 (iii).

Recall that $\overline{\Omega} = s_{nd}^{-1}(\mathbb{R}_{\geq 0}) = \Omega \sqcup T_N M$ is the closure of $\Omega$ in $M^0_N$, and consider the map $r: \overline{\Omega} \to M \times \mathbb{R}$ given by $r(v, y, s) = (sv, y, s - |v|)$. Then $r$ is a proper map since, in the commutative diagram

$$\begin{array}{ccc}
\overline{\Omega} & \xrightarrow{r} & M \times \mathbb{R} \\
\downarrow f & & \downarrow g \\
N \times \mathbb{R}_{\geq 0} \times \mathbb{R} & \xrightarrow{h} & N \times \mathbb{R}_{\geq 0} \times \mathbb{R},
\end{array}$$

$f$ and $h$ are proper. Here, $f(v, y, s) = (y, |v|, s)$, $g(x, y, s) = (y, |x|, s)$, and $h(y, u, s) = (y, su, s - u)$.

Setting $Z = N \times \mathbb{R}_{\leq 0} \subset M \times \mathbb{R}$ and $U = (M \times \mathbb{R}) \setminus Z$, the continuous map $r$ induces a homeomorphism $\Omega \xrightarrow{\sim} U$ (see Fig. 2). Consider the commutative diagram of bordered spaces semiproper over $M$, whose two top squares are cartesian,
Fig. 2 The map $r : \Omega \to M \times \mathbb{R}$ pictured in the case $M = \mathbb{R}$ and $N = \{0\}$. The red lines are fibers of the projections $\Omega \xrightarrow{p} M$ and their images by $r$, respectively (Color figure online).

Here, $q$, $qU$ and $qZ$ denote the first projections, and $\overline{p} = p_{\text{nd}}|_{\Omega}$. One has

\[
\begin{align*}
\text{Er}_*\text{Ev}_N(K) &\simeq \text{Er}_*\text{E}i^{-1}\text{E}j_*\text{E}j^{-1}\overline{p}^{-1}K \\
&\simeq \text{Eq}_*\text{E}r_i\text{E}i^{-1}\text{E}j_*\text{E}j^{-1}\text{E}r^{-1}\text{E}q^{-1}K \\
&\simeq \text{Eq}_*\text{E}r_i\text{E}i^{-1}\text{E}j_*\text{E}r_j^{-1}\text{E}j^{-1}\text{E}q^{-1}K \\
&\simeq \text{Eq}_*\text{E}i^{-1}\text{E}r_i\text{E}j_*\text{E}r_j^{-1}\text{E}j^{-1}\text{E}q^{-1}K \\
&\simeq \text{Eq}_*\text{E}i^{-1}\text{E}j_*\text{E}j^{-1}\text{E}q^{-1}K,
\end{align*}
\]

where $(*)$ follows from the properness of $r$. Consider the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
U_\infty & \xrightarrow{j} & M \times \mathbb{R}_\infty \\
qU & \downarrow & q \\
M & \xleftarrow{i} & Z_\infty
\end{array}
\end{array}
\]
\[
\begin{align*}
E \tau_* E^N(K) &\simeq E \tilde{o}^{-1} E \tilde{i}^{-1} E j_* E j^{-1} E q^{-1} K \\
&\simeq E i_N^{-1} E \tilde{i}_0^{-1} E j_* E q_U^{-1} K \\
&\simeq E i_N^{-1} E q_* E j_* E q_U^{-1} K \\
&\simeq E i_N^{-1} E q_U_* E q_U^{-1} K,
\end{align*}
\]

where \((*)\) holds since \(E \tilde{i}_0^{-1} E j_* E q_U^{-1} K\) is \((\mathbb{R}^\times_0)_\infty\)-conic, and \((**)\) holds since \(E j_* E q_U^{-1} K\) is \((\mathbb{R}^\times_0)_\infty\)-conic. Then, one has

\[
E \tau_* E^N(K) \simeq E i_N^{-1} E q_U_* E q_U^{-1} K[-1] \\
\simeq E i_N^{-1} R hom^+(\epsilon(R \tilde{q}_U, K), K[-1]) \\
\simeq E i_N^{-1} R hom^+(\epsilon(k, K), K[-1]) \\
\simeq E i_N^{-1} K,
\]

where \((*)\) holds by Sublemma 4.2, and \((***)\) is due to the fact that the fibers of \(q_U\) are homeomorphic to \(\mathbb{R}\). Thus (4.5) is an isomorphism.

(ii) has a similar proof to (i).

(iii) follows from (i) and (ii), using Lemma 4.4 (iv).

(iv) Consider the distinguished triangle

\[
E u!! E u^{-1} E^N(K) \rightarrow E^N(K) \rightarrow E o!! E o^{-1} E^N(K) \rightarrow E^N(K+1) 
\]

Then the statement follows from (i) and Lemma 4.4 (i).

(v) has a proof similar to that of (iv), using the distinguished triangle

\[
E o!! E o^{-1} E^N(K) \rightarrow E^N(K) \rightarrow E u!! E u^{-1} E^N(K) \rightarrow E^N(K+1) 
\]

\[\square\]

Recall the definition of the normal cone in (2.5). Here is an analogue of [7, Exercise IV.2]

**Lemma 4.9** Let \(S \subset M\) be a closed subset. Then \(E^N\) induces a functor \(E^N|_{M \setminus S}\) (see (4.6) below) entering the quasi-commutative diagram

\[
\begin{array}{ccc}
E^b(Ik_M) & \buildrel E^N \over \longrightarrow & E^b(Ik_{(TN M)_\infty}) \\
\downarrow E j_S^{-1} \quad & & \downarrow E j_C^{-1} \\
E^b(Ik_{(M \setminus S)_\infty}) & \buildrel E^N|_{M \setminus S} \over \longrightarrow & E^b(Ik_{(TN M \setminus C_N S)_\infty}),
\end{array}
\]

where \(j_S: (M \setminus S)_\infty \rightarrow M\) and \(j_C: (TN M \setminus C_N S)_\infty \rightarrow (TN M)_\infty\) are the open embeddings.
Proof Consider the commutative diagram with cartesian squares

\[
\begin{array}{ccc}
M & \xleftarrow{p_\Omega} & \Omega_\infty \\
\downarrow j_S & & \downarrow j \\
(M\backslash S)_\infty & \xleftarrow{p_\Omega} & (\Omega \backslash p_\Omega^{-1}S)_\infty \\
\end{array}
\]

where all the vertical arrows are open embeddings. Then

\[
Ej_C^{-1}Ev_N(K) = Ej_C^{-1}Ei^{-1}Ej_sEp_\Omega^{-1}K
\]
\[
\simeq Ei'^{-1}Ej'_sEp_\Omega'^{-1}Ej_S^{-1}K.
\]

For \( K' \in E^b(I_k(M\backslash S)_\infty) \), set

\[
Ev_{N|M\backslash S}(K') : = Ei'^{-1}Ej'_sEp_\Omega'^{-1}Ej_S^{-1}K'.
\] (4.6)

Then the statement is clear.

Here is an analogue of [7, Exercise IV.5]

Lemma 4.10 Let \( \tau : V \to N \) be a vector bundle, and denote by \( o : N \to V \) the embedding of the zero-section. For \( K \in E^b_{(R^<_0)}(I_kV_\infty) \), one has

\[
Ev_N(K) \simeq K,
\]

where we use the identifications \( N \simeq o(N) \subset V \) and \( TNV \simeq V \).

Proof One has \( (V^{nd}_N)_\infty \cong V \times R_\infty \), with \( s_{nd}(x, s) = s \) and \( p_{nd}(x, s) = sx \). Hence \( \Omega_\infty \cong V \times (R^>_0)_\infty \), and \( p_\Omega = \mu \), where \( \mu \) is the \( (R^>_0)_\infty \)-action. Then,

\[
Ep_\Omega^{-1}K = E\mu^{-1}K \cong K \boxplus e(k_{R^>_0}),
\]

where the last isomorphism is due to the fact that \( K \) is \( (R^>^\infty)_{\infty} \)-conic. Recalling Definition 4.6, the statement easily follows.

Let \( f : M_1 \to M_2 \) be a morphism of real analytic manifolds, let \( N_i \subset M_i \) \((i = 1, 2)\) be closed submanifolds, and assume \( f(N_1) \subset N_2 \). Consider the associated morphism, given by the composition

\[
T_{N_1}f : (T_{N_1}M_1)_\infty \xrightarrow{f'} N_1 \times N_2 (T_{N_2}M_2)_\infty \to (T_{N_2}M_2)_\infty.
\]

The enhanced specialization functor satisfies the analogous functorial properties as those in Propositions 4.2.4, 4.2.5 and 4.2.6 of [7]. The proofs in loc. cit. immediately extend to the enhanced framework.
For example, if \( f \) and \( f|_{N_1}: N_1 \to N_2 \) are smooth, one has
\[
E(T_{N_1}f)^{-1} \circ \text{Ev}_{N_2} \simeq \text{Ev}_{N_1} \circ E f^{-1}.
\]

### 4.4 Blow-up transform

Let \( M \) be a real analytic manifold, and \( N \subset M \) a closed submanifold. With notations as in (2.1), consider the real oriented blowup \( M_{rb}^N \) and the commutative diagram of bordered spaces
\[
\begin{array}{ccc}
S_N M & \xrightarrow{i_{rb}} & M_{rb}^N \\
\sigma \downarrow & & \downarrow j_{rb} \\
N & \xrightarrow{i_N} & M
\end{array}
\]

Note that \((M\setminus N) \simeq (M_{rb}^N \setminus S_N M)\). In the following, when there is no risk of confusion, we will write for short \( i = i_{rb}, j = j_{rb} \) and \( p = p_{rb} \).

**Definition 4.11** For \( K \in \text{E}^b(I k_M) \), consider the object
\[
\text{Ev}_{rb}^N(K) := E i^{-1} E j_* E j_N^{-1} K \in \text{E}^b(I k_{SN M}).
\]

We denote by
\[
\nu_{rb}^N: \text{D}^b(k_M) \to \text{D}^b(k_{SN M})
\]
the analogous functor for sheaves.

Note that, by definition, \( \text{Ev}_{rb}^N \) factors through a functor, that we denote by the same name,
\[
\text{Ev}_{rb}^N: E^b(I k_{(M\setminus N)\infty}) \to E^b(I k_{SN M}).
\]

Note also that one has
\[
e \circ \nu_{rb}^N \simeq \text{Ev}_{rb}^N \circ e,
\]
and similarly for \( e \) replaced by \( e \circ \iota, \epsilon, \epsilon^+ \) or \( \epsilon^- \).

**Lemma 4.12** For \( K \in E^b(I k_M) \), one has
\[
\text{Ev}_{rb}^N(K) \simeq E i^1 E j_{!} E j_N^{-1} K[1].
\]

**Proof** For \( L \in E^b(I k_{M_{rb}^N}) \), there is a distinguished triangle
\[
E i_* E i^1 L \to L \to E j_* E j^1 L \to E j_* E j_N^{-1} K[1].
\]

When \( L = E j_{!} E j_N^{-1} K \), the above distinguished triangle reads
\[
E i_* E i^1 E j_{!} E j_N^{-1} K \to E j_{!} E j_N^{-1} K \to E j_* E j_N^{-1} K \to E j_* E j_N^{-1} K[1].
\]
The statement follows by applying $E_i^{-1}$, and noticing that $E_i^{-1}E_j!! \simeq 0$. □

**Lemma 4.13** For $K \in E^b(1k_M)$, one has

$$E_N^{\text{sph}}(K) \simeq E_N^{\text{rb}}(K).$$

**Proof** Considering the morphisms

$$(T_N M)_{\infty} \xleftarrow{\gamma}(\hat{T}_N M)_{\infty} \xrightarrow{\gamma} S_N M,$$

it is equivalent to prove

$$Eu^{-1}E_N(K) \simeq E\gamma^{-1}E_N^{\text{rb}}(K).$$

Consider the commutative diagram, extending (2.6), whose squares are Cartesian with smooth vertical arrows

We have to prove

$$Eu^{-1}E_i^{-1}E_{j_{\text{nd}}*}EP_{\Omega}^{-1}K \simeq E\gamma^{-1}E_i^{-1}E_{j_{\text{rb}}*}E_j^{-1}K. \quad (4.10)$$

This is obtained by chasing the above diagram. □

5 Fourier-Sato transform and microlocalization

We recall here the natural enhancement of the Fourier-Sato transform from [12, §3] (see also [1] and [9]), referring to [7, §3.7] for the classical case. We then define the natural enhancement of Sato’s microlocalization, referring to [7, §4.3] for the classical case.
5.1 Kernels

Let \( X \) and \( Y \) be bordered spaces. A kernel from \( X \) to \( Y \) is a triple \((p, q, C)\), where \( p \) and \( q \) are morphisms of bordered spaces

\[
\begin{array}{c}
X \xleftarrow{p} S \xrightarrow{q} Y,
\end{array}
\]

and \( C \in \text{E}^b(\text{Ik}_S) \). To such a kernel one associates the functors

\[
\Phi_{(p,q,C)} : \text{E}^b(\text{Ik}_X) \to \text{E}^b(\text{Ik}_Y),
\]

\[
\Psi_{(p,q,C)} : \text{E}^b(\text{Ik}_X) \to \text{E}^b(\text{Ik}_Y),
\]

defined by

\[
\Phi_{(p,q,C)}(K) := E^q!(C \otimes E^p K),
\]

\[
\Psi_{(p,q,C)}(K) := E^q R \mathcal{I} \text{hom}^+(C, E^p K).
\]

Given a commutative diagram

\[
\begin{array}{ccc}
X & & Y \\
p & S & q \\
p' & & q'
\end{array}
\]

one has

\[
\Phi_{(p,q,C)} \simeq \Phi_{(p',q',\text{Rr} \circ C)}, \quad \Psi_{(p,q,C)} \simeq \Psi_{(p',q',\text{Rr} \circ C)}. \tag{5.1}
\]

If there is no fear of confusion, we will write for short \( C = (p, q, C), \ C^{\text{prod}} = (q_1, q_2, \text{R}(p, q), C), \ C' = (q, p, C) \),

where \( q_1 \) and \( q_2 \) are the projections from \( X \times Y \) to \( X \) and \( Y \), respectively, and \((p, q) : S \to X \times Y \) is the morphism induced by \( p \) and \( q \). Then, (5.1) implies

\[
\Phi_C \simeq \Phi_{C^{\text{prod}}}, \quad \Psi_C \simeq \Psi_{C^{\text{prod}}},
\]

and the kernel \( C' \) from \( Y \) to \( X \) gives functors

\[
\Phi_{C'}, \Psi_{C'} : \text{E}^b(\text{Ik}_Y) \to \text{E}^b(\text{Ik}_X).
\]

Note that \( \Phi_C \) is left adjoint to \( \Psi_{C'} \) (and \( \Psi_C \) is right adjoint to \( \Phi_{C'} \)). Note also that for \( K \in \text{E}^b(\text{Ik}_X) \) one has

\[
D_Y^E \Phi_C(K) \simeq \Psi_C(D_X^E K). \tag{5.2}
\]
Note that, if \( C \in E_b^\pm(I_k\Sigma) \), then \( \Phi_C \) and \( \Psi_C \) take value in \( E_b^\pm(I_k\mathcal{Y}) \). In this case, we set

\[
\Phi_C^\pm := \Phi_C|_{E_b^\pm(I_k\chi)}, \quad \Psi_C^\pm := \Psi_C|_{E_b^\pm(I_k\chi)},
\]

so that we have functors

\[
\Phi_C^\pm, \Psi_C^\pm : E_b^\pm(I_k\mathcal{Y}) \to E_b^\pm(I_k\chi).
\]

For \( * \in \{\emptyset, +, -\} \), consider the kernel \( 1^*_{\chi} := (q_1, q_2, \epsilon^*(k_{\Delta X})) \), where \( X = \check{X} \) and \( \Delta X \subset X \times X \) is the diagonal. Note that one has \( 1^*_{\chi} \simeq (\text{id}_X, \text{id}_X, \epsilon^*(k_X))^{\text{prod}} \), so that in particular

\[
\Psi_{1^*_{\chi}} \simeq \text{id}_{E_b^\pm(I_k\chi)} \simeq \Phi_{1^*_{\chi}}.
\]

Given another bordered space \( Z \), and a kernel \( D = (T \to Y, T \to Z, D) \) from \( Y \) to \( Z \), consider the diagram with cartesian square

\[
\begin{array}{ccc}
S \times Y & \xrightarrow{r} & T \\
\downarrow{q} & & \downarrow{s} \\
X & \xrightarrow{r'} & Z.
\end{array}
\]

Setting

\[
C \circ D := E_{r'}^{-1}C \otimes E_{q'}^{-1}D \in E_b(I_k\Sigma \times Y T),
\]

one gets a kernel \( C \circ D = (p \circ r', s \circ q', C \circ D) \) from \( X \) to \( Z \) such that

\[
\Phi_D \circ \Phi_C \simeq \Phi_{C \circ D}, \quad \Psi_D \circ \Psi_C \simeq \Psi_{C \circ D}.
\]

Let \( * \in \{\emptyset, +, -\} \), \( C \in E_b^*(I_k\Sigma) \) and \( D \in E_b^*(I_k\mathcal{T}) \). Assume that \( Z = X \) and that

\[
(C \circ D)^{\text{prod}} \simeq 1^*_X, \quad (D \circ C)^{\text{prod}} \simeq 1^*_Y.
\]

Then, the functors \( \Phi_C^* \) and \( \Phi_D^* \) (resp. \( \Psi_C^* \) and \( \Psi_D^* \)) are equivalences of categories quasi-inverse to each other. Moreover, by uniqueness of the adjoint, one has

\[
\Phi_C^* \simeq \Psi_{D^r}^*, \quad \Psi_C^* \simeq \Phi_{D^r}^*.
\]
**Lemma 5.1** Consider a commutative diagram of bordered spaces with cartesian squares

\[
\begin{array}{ccc}
X' & \xrightarrow{p'} & S' & \xrightarrow{q'} & Y' \\
\downarrow f & & \downarrow h' \quad & & \downarrow g \\
X & \xrightarrow{p} & S & \xrightarrow{q} & Y.
\end{array}
\]

Let \( C \in \text{E}^b(\Gamma S) \), and set

\[
C' := E h^{-1} C \in \text{E}^b(\Gamma S').
\]

Consider \( C = (p, q, C) \) and \( C' = (p', q', C') \) as kernels. Then, one has

\[
\begin{align*}
\Phi_C \circ Ef!! & \simeq Eg!! \circ \Phi_{C'}, \\
\Phi_{C'} \circ Ef^{-1} & \simeq Eg^{-1} \circ \Phi_C, \\
\Psi_C \circ Ef_* & \simeq Eg_* \circ \Psi_{C'}, \\
\Psi_{C'} \circ Ef' & \simeq Eg' \circ \Psi_C.
\end{align*}
\]

**5.2 Enhanced Fourier-Sato transform**

Let \( M \) be a bordered space, and \( U \subset M \) an open subset. For \( \varphi : U \rightarrow \mathbb{R} \) a continuous function, consider the object of \( \text{E}^b_+ (\Gamma M) \)

\[
E^\varphi_U := \mathcal{Q} \otimes \{ \{ \{ t + \varphi(x) \geq 0 \} \} \}.
\]

where we write for short

\[
\{ t + \varphi(x) \geq 0 \} = \{ (x, t) \in U \times \mathbb{R}; \ t + \varphi(x) \geq 0 \}.
\]

Let \( \tau : V \rightarrow N \) be a real vector bundle and \( \sigma : V^* \rightarrow N \) its dual bundle. Denote by \( V_\infty \) and \( V^*_\infty \) their bordered compactifications, consider the projections

\[
V_\infty \leftarrow^p V_\infty \times_N V^*_\infty \rightarrow^q V^*_\infty,
\]

and let \( \langle \cdot, \cdot \rangle : V \times_N V^* \rightarrow \mathbb{R} \) denote the map induced by the pairing.

**Notation 5.2** Let \( K \in \text{E}^b_+ (\Gamma V_\infty) \).

(i) The Fourier-Sato transforms (see [7, §3.7] for the case of sheaves) are defined by

\[
K^\wedge := \Phi^+_F (K), \quad K^\vee := \Phi^+_\mathcal{Q} (K),
\]

for \( F, \mathcal{Q} \in \text{E}^b_+ (\Gamma V_\infty \times_N V^*_\infty) \) given by

\[
F := \epsilon^+ (k_{\{(x, y) \leq 0\}}), \quad \mathcal{Q} := \epsilon^+ (k_{\{(x, y) \geq 0\}} \otimes p^{-1}_* \omega_{V/N}).
\]
Note that the kernels $F$ and $F$ are $(\mathbb{R}_{>0})_\infty$-bi-conic for the actions $c \cdot (x, y) = (cx, y)$ and $d \cdot (x, y) = (x, dy)$. Hence, the Fourier-Sato transforms take values in $E(b)_{(\mathbb{R}_{>0})_\infty}(I_kV) \cap E(b)_{+}(I_kV)$. 

(ii) The enhanced Fourier-Sato transforms (see [12, §3.1.3] for the case of enhanced sheaves) are defined by

$$L_K := \Phi_L^+(K), \quad L_k := \Phi_k^+(K),$$

for $L, K \in E(b)_{+}(I_kV)$. Hence the enhanced Fourier-Sato transforms send $E(b)_{(\mathbb{R}_{>0})_\infty}(I_kV) \cap E(b)_{+}(I_kV)$ to $E(b)_{(\mathbb{R}_{>0})_\infty}(I_kV) \cap E(b)_{+}(I_kV)$.

Note that one has

$$F = Qk[t \geq (x, y)], \quad L = Qk[t - (x, y) \geq 0]. \quad (5.4)$$

It is shown in [12] (see also [9]) that one has

$$\left( L \circ L \right)^{prod} \simeq 1_{V_\infty}^+, \quad \left( L \circ L \right)^{prod} \simeq 1_{V_\infty}^+. \quad (5.5)$$

It follows that $L(\cdot)$ and $L(\cdot)$ are quasi-inverse to each other and that, by (5.3),

$$L_K \simeq \Psi_L^+(K), \quad L_k \simeq \Psi_k^+(K). \quad (5.6)$$

Note that, for $K$ an ind-sheaf, one has

$$e(K^\wedge) \simeq (e(K))^\wedge, \quad e(K^\vee) \simeq (e(K))^\vee, \quad (5.7)$$

and the same for $e$ replaced by $e^+$.

The following result was obtained in [1,9] for conic sheaves, and we generalize it to enhanced ind-sheaves.

**Proposition 5.3** For $K \in E(b)_{(\mathbb{R}_{>0})_\infty}(I_kV) \cap E(b)_{+}(I_kV)$, one has

$$L_K \simeq K^\wedge, \quad L_k \simeq K^\vee.$$

**Proof** We will adapt the proof of [9, Theorem 5.7]. Since the arguments are similar, we will only treat the first isomorphism.
Recall (5.4). The inclusion \( \{t - \langle x, y \rangle \geq 0\} \supset \{t \geq 0 \geq \langle x, y \rangle\} \) induces a distinguished triangle

\[
Q_k[t \geq \langle x, y \rangle \geq 0] \oplus Q_k[0 > t \geq \langle x, y \rangle] \to Q_k[t - \langle x, y \rangle \geq 0] \to Q_k[t \geq 0 \geq \langle x, y \rangle] \xrightarrow{+1}.
\]

We are thus left to prove

\[
\Phi_{Q_k[t \geq \langle x, y \rangle > 0]}(K) \simeq 0 \simeq \Phi_{Q_k[0 > t \geq \langle x, y \rangle]}(K).
\]

Since the arguments are similar, we will only treat the first isomorphism. Consider the morphisms

\[
V_\infty \xleftarrow{p} V_\infty \times V_\infty^* \xrightarrow{h} V_\infty^* \times \mathbb{R}_\infty \xrightarrow{j} V_\infty^* \times (\mathbb{R}_\infty > 0)_\infty
\]

where \( h(x, y) = (y, (x, y)) \), \( j \) is the embedding, and \( p, q, r, s \) are the projections. Then, denoting by \( \lambda \) the coordinate of \( \mathbb{R}_\infty \),

\[
\Phi_{Q_k[t \geq \langle x, y \rangle > 0]}(K) = Eq!!\left(Q_k[t \geq \langle x, y \rangle > 0] \oplus E p^{-1} K\right)
\]

\[
\simeq Er!!\left(E h!!(E h^{-1} Q_k[t \geq \lambda > 0] \oplus E p^{-1} K\right)
\]

\[
\simeq Er!!\left(Q_k[t \geq \lambda > 0] \oplus E h!! E p^{-1} K\right)
\]

\[
\simeq Er!!\left(\epsilon(\mathbf{k}_{\lambda > 0}) \oplus Q_k[t \geq \lambda] \oplus E h!! E p^{-1} K\right)
\]

\[
\simeq Er!!(E j!! E j^{-1}(Q_k[t \geq \lambda] \oplus E h!! E p^{-1} K)
\]

\[
\simeq Es!!\left(Q_k[t \geq \lambda] \oplus E j^{-1} E h!! E p^{-1} K\right).
\]

Since \( E j^{-1} E h!! E p^{-1} K\) is \( (\mathbb{R}_{\geq 0}^\times)^\infty \)-conic for the action \( c \cdot (y, \lambda) = (y, c\lambda) \), we have \( E j^{-1} E h!! E p^{-1} K \simeq E s^{-1} H\) for some \( H \in E^b(1_{kV_{\leq 0}})\). Hence

\[
\Phi_{Q_k[t \geq \langle x, y \rangle > 0]}(K) \simeq Es!!\left(Q_k[t \geq \lambda] \oplus E s^{-1} H\right)
\]

\[
\simeq Es!!\left(Q_k[t \geq \lambda]\right) \oplus H
\]

\[
\simeq Q\left(R s_{\mathbb{R} \times k[t \geq \lambda]}\right) \oplus H \simeq 0,
\]

where (\(\ast\)) follows since \( s \) is semiproper. (Recall that \( s_{\mathbb{R}} = s \times id_{\mathbb{R}}\).) \( \Box \)

By Lemma 5.1, we obtain the following analogue of [7, Proposition 3.7.13].
Lemma 5.4 Let $V \to N$ be a vector bundle and let $f: N' \to N$ be a morphism. Set $V' = V \times_N N'$ and $V'^* = V^* \times_N N'$, and let $g: V' \to V$ and $h: V'^* \to V^*$ be the induced morphism. Then

(i) For any $K \in \mathcal{E}^b_+(\mathcal{I}^k_{V_{\infty}})$, we have

$$Eh^{-1}(K) \simeq (Eg^{-1}K) \quad \text{and} \quad Eh'(K) \simeq (Eg'K).$$

(ii) For any $K' \in \mathcal{E}^b_+(\mathcal{I}^k_{(V')}_{\infty})$, we have

$$Eh(K') \simeq (Eg_*K') \quad \text{and} \quad Eh!!(K') \simeq (Eg!!K').$$

The enhanced Fourier functor satisfies also other functorial properties, as those in Propositions 3.7.14 and 3.7.15 of [7]. The first one was already pointed out in [9, §5.2], and the second one easily follows from

$$E^{-\langle x_1, y_1 \rangle}_{V_1 \times_N V_1^*} \Delta_{N}^{v} E^{-\langle x_2, y_2 \rangle}_{V_2 \times_N V_2^*} \simeq E^{-\langle (x_1, x_2), (y_1, y_2) \rangle}_{(V_1 \times_N V_2) \times_N (V_1 \times_N V_2)^*}.$$

5.3 Enhanced microlocalization

As in § 4.3, let $M$ be a real analytic manifold, and $N \subset M$ a closed submanifold.

Definition 5.5 For $K \in \mathcal{E}^b_+(\mathcal{I}_M)$, we set

$$E\mu_N(K) := L(E\nu_N(K)) \simeq (E\nu_N(K))^\wedge \in \mathcal{E}^b_+(\mathcal{I}^k_{T_N^* M}) \cap \mathcal{E}^b_+(\mathcal{I}^k_{T_N^* S_N^* M}),$$

$$E\mu^\text{sph}_N(K) := (E\mu_N(K))^\text{sph} \in \mathcal{E}^b_+(\mathcal{I}^k_{S_N^* M}),$$

where the isomorphism follows from Proposition 5.3, since $E\nu_N(K)$ is $(\mathbb{R}^>_0)_{\infty}$-conic. The functor $E\mu_N$ is called enhanced microlocalization along $N$.

Note that one has

$$e \circ \mu_N \simeq E\mu_N \circ e,$$

and similarly for $e$ replaced by $\epsilon^+$. The enhanced microlocalization functor satisfies the analogous functorial properties as those in Propositions 4.3.4, 4.3.5 and 4.3.6 of [7]. The proofs in loc. cit. immediately extend to the enhanced framework.

6 Specialization at $\infty$ on vector bundles

On a vector bundle $\tau: V \to N$, we construct an enhancement of the so-called smash functor from [2, §6.1], which is related to “specialization at $\infty$”, and we compute its enhanced Fourier-Sato transform.
6.1 Smash functor

Let \( \tau : V \to N \) be a vector bundle, and consider the morphisms of bordered vector bundles over \( N \)

\[
V_\infty \xleftarrow{p_{sm}} V_\infty \times (\mathbb{R}_{>0})_\infty \xrightarrow{j_{sm}} V_\infty \times \mathbb{R}_\infty \xrightarrow{i_{sm}} V_\infty,
\]

where \( p_{sm}(x, s) = s^{-1}x \), \( i_{sm}(x) = (x, 0) \), and \( j_{sm} \) is the open embedding. In the rest of this section we will write for short \( p, i \) and \( j \) instead of \( p_{sm}, i_{sm} \) and \( j_{sm} \), respectively, if there is no fear of confusion.

Note that \( p, i, j \) are \( (\mathbb{R} \times \mathbb{R}_{>0})_\infty \)-equivariant with respect to the ordinary actions of \( (\mathbb{R} \times \mathbb{R}_{>0})_\infty \) on \( V_\infty \) and \( \mathbb{R}_\infty \), except the trivial action on the leftmost \( V_\infty \).

**Definition 6.1** For \( K \in E^b(Ik_{V_\infty}) \), set

\[
E\sigma_V(K) := E_i^{-1}E_j^*E p^{-1}K \in E^b_{(\mathbb{R}_{>0})_\infty}(Ik_{V_\infty}).
\]

This is called the enhanced smash functor.

With a proof similar to that of Lemma 4.12, one has the following enhancement of [2, Lemma B.1].

**Lemma 6.2** With the above notations, one has

\[
E\sigma_V(K) \simeq E^{i_!}E^{j_!}E p^{-1}K.
\]

One also has the following enhancement of [2, Lemma B.2].

**Lemma 6.3** Let \( o : N \to V \) be the zero section. Then for \( K \in E^b(Ik_{V_\infty}) \), one has

\[
Eo^{-1}E\sigma_V(K) \simeq E\tau_! K, \quad Eo^{i_!}E\sigma_V(K) \simeq E\tau_! K.
\]

**Proof** Since the proofs are similar, let us only discuss the first isomorphism. Let \( \tilde{\tau} : V_\infty \times \mathbb{R}_\infty \to N \) be the projection, and \( \tilde{o} : N \to V_\infty \times \mathbb{R}_\infty \) the zero section. one has

\[
Eo^{-1}E\sigma_V(K) \simeq Eo^{-1}Ei^{-1}E j_! E p^{-1}K
\]

\[
\simeq E\tilde{o}^{-1}E j_! E p^{-1}K
\]

\[
\simeq E\tau_* E j_* E p^{-1}K
\]

\[
\simeq E\tau_* E p_* E^{-1}K
\]

\[
\simeq E\tau_* K.
\]

Here \((*)\) follows from \( E j_* E p^{-1}K \in E^b_{(\mathbb{R}_{>0})_\infty}(Ik_{V_\infty \times \mathbb{R}_\infty})\). \( \Box \)
Consider the vector bundle $W := \mathbb{R} \times V \to N$, and let $\hat{W} := W \backslash \{0\} \times N$ be the complement of the zero section. Recall from §4.2 that the fiberwise sphere compactification\(^2\) of $\tau$ is $SV := \hat{W} / \mathbb{R}_{>0}$. We denote by $q: \hat{W} \to SV$ the quotient map, and set $[u, x] := q(u, x)$.

There is a natural decomposition
\[
SV = V^+ \sqcup H \sqcup V^-,
\]
(6.1)
corresponding to $u > 0$, $u = 0$ and $u < 0$, respectively. Note that the fibers of $H \to N$ are great spheres of codimension one in the fibers of $SV \to N$. Note also that there are natural identifications $i^\pm: V \stackrel{\sim}{\to} V^\pm$, $x \mapsto [\pm 1, x]$. Set $N^\pm = i^\pm(N) \subset V^\pm$, and $N_0^\pm = N^\pm \times \{0\} \subset SV \times \mathbb{R}$.

In order to compute the functor $E_{SV}$, let us describe the normal deformation $SV_H^{nd}$ and the bordered compactification $(SV_H^{nd})_\infty$ of $p_{nd}: SV_H^{nd} \to SV$.

**Lemma 6.4 With notations as above,**

(i) one has $SV_H^{nd} \simeq (SV \times \mathbb{R}) \backslash (N_0^+ \cup N_0^-)$, with $p_{nd}((\tilde{u}, \tilde{x}), s) = [s\tilde{u}, \tilde{x}]$ and $s_{nd}((\tilde{u}, \tilde{x}), s) = s$ (see Fig. 3). The $\mathbb{R}^\times$-action on $SV_H^{nd}$ is given by $c \cdot ((\tilde{u}, \tilde{x}), s) = ((c\tilde{u}, \tilde{x}), c^{-1}s)$.

(ii) One has $(SV_H^{nd})_\infty \simeq ((SV \times \mathbb{R}) \backslash (N_0^+ \cup N_0^-), SV \times \mathbb{R})$.

**Proof** (i) Set $Z = (u = 0) \subset W$. Then $W^Z_H = W \times \mathbb{R} = \mathbb{R} \times V \times \mathbb{R}$ with $p_{nd}^W((\tilde{u}, \tilde{x}), s) = (s\tilde{u}, \tilde{x})$ and $s_{nd}^W((\tilde{u}, \tilde{x}), s) = s$. The $\mathbb{R}^\times$-action on $W^Z_H$ is given by $c \cdot ((\tilde{u}, \tilde{x}), s) = (c\tilde{u}, \tilde{x}, c^{-1}s)$.

One has $W_H^{nd} = (p_{nd}^W)^{-1}(\hat{W}) = W^Z_H \backslash \left((\mathbb{R} \times N \times \{0\}) \cup (\{0\} \times N \times \mathbb{R})\right)$. Consider the $\mathbb{R}^\times_{>0}$-action on $W_H^{nd}$ induced by the $\mathbb{R}^\times_{>0}$-action on $\hat{W}$, which is given by $c \cdot (\tilde{u}, \tilde{x}, s) =$

---

\(^2\) Here we choose a different compactification from the one in [2, §B.2].
Its quotient is the map $q : \hat{W}_Z^{nd} \to (SV \times \mathbb{R}) \setminus (N_0^+ \cup N_0^-)$, given by $q(\tilde{u}, \tilde{x}, s) = ([\tilde{u}, \tilde{x}], s)$. Setting $\tilde{p}(\tilde{u}, \tilde{x}, s) = \tilde{s}(\tilde{u}, \tilde{x}, s) = s$, there is a commutative diagram with cartesian square

\[
\begin{array}{ccc}
\hat{W}_Z^{nd} & \xrightarrow{q} & (SV \times \mathbb{R}) \setminus (N_0^+ \cup N_0^-) \\
\downarrow p_{nd}^W & & \downarrow \tilde{p} \\
W & \xrightarrow{q} & SV.
\end{array}
\]

Since the quotient maps $q$ are principal $\mathbb{R}^\times \times \mathbb{R}^0$-bundles, it follows that $SV_H^{nd} = (SV \times \mathbb{R}) \setminus (N_0^+ \cup N_0^-)$, $p_{nd} = \tilde{p}$ and $s_{nd} = \tilde{s}$.

(ii) follows by uniqueness of bordered compactifications. \hfill \Box

Denote by $T_H^+ SV \subset \hat{T}_H SV$ the normal vectors pointing to $V^+$. Since

\[ T_H^+ SV = s_{nd}^{-1}(0) = (\dot{V}^+ \cup H \cup \dot{V}^-) \times \{0\}, \]

this gives a natural identification $T_H^+ SV = \dot{V}^+$. We will also use the identification $V = V^+$ given by $t^+$. Note that the $\mathbb{R}_{>0}^\times$-action on $\dot{V}^+ \subset T_H^+ SV$ induced by the one on $T_H^+ SV$ is given by $c \cdot t^+(x) = t^+(c^{-1}x)$.

By Lemma 4.9, with the above identifications, $E_{\nu H}$ induces a functor

\[ Ev_{H|\dot{V}} : E^b(Ik_{\dot{V}^\infty}) \to E^b(Ik_{\dot{V}^\infty}). \]

Similarly, $E_{\sigma V}$ induces a functor (see (6.2) below)

\[ E_{\sigma V|\dot{V}} : E^b(Ik_{\dot{V}^\infty}) \to E^b(Ik_{\dot{V}^\infty}). \]

**Lemma 6.5** With the above notations, one has

\[ Ev_{H|\dot{V}} \simeq E_{\sigma V|\dot{V}}. \]

**Proof** Consider the diagram with cartesian squares

\[
\begin{array}{ccc}
SV & \xrightarrow{p_{nd}} & \Omega_\infty \\
\downarrow j_{SV} & \downarrow j_{nd} & \downarrow i_{nd} \\
\hat{V}_\infty \times (\mathbb{R}_{>0})_\infty & \xrightarrow{i} & V_\infty = (T_H^+ SV)_\infty
\end{array}
\]

Here, $p_{sm}'$ and $p_{nd}'$ are induced by $p_{sm}$ and $p_{nd}$, respectively.
Since
\[
j_SV(p'_N(x,s)) = p_N([1,x],s) = [s,x] = [1,s^{-1}x],
\]
\[
j(jV(p'_S(x,s))) = j(p_S(x,s)) = j(s^{-1}x) = [1,s^{-1}x],
\]
one has in fact \( p'_N = p'_S \).

By (4.6), one has
\[
E_{\nu|V} \simeq Ei^{-1} \circ Ej_* \circ E(p'_N)^{-1}.
\]

Similar arguments give
\[
E_{\sigma|V} \simeq Ei^{-1} \circ Ej_* \circ E(p'_S)^{-1}. \tag{6.2}
\]

\[
\square
\]

6.2 Smash functor and microlocalization

As in the previous section, let \( \tau : V \to N \) be a vector bundle. Denote by \( o : N \to V \) the embedding of the zero section. Consider the natural identifications
\[
N = o(N) \subset V, \quad T^*_N V = V^*.
\]

There is the following relation between the smash functor and Sato’s microlocalization

**Proposition 6.6** For \( K \in \mathcal{E}^b(Ik_{V_{\infty}}) \) there is an isomorphism in \( \mathcal{E}^b(Ik_{V_{\infty}}) \)
\[
E\mu_N(K) \simeq E\sigma_{V^*}(\cdot K).
\]

In other words, one has \( \mathcal{L}(\cdot) \circ E\nu_N \simeq E\sigma_{V^*} \circ \mathcal{L}(\cdot) \).

**Proof** Consider the following diagram with cartesian squares.

\[
\begin{array}{cccccccc}
V_{\infty} & \xleftarrow{p} & V_{\infty} \times_N V^*_{\infty} & \xrightarrow{q} & V^*_{\infty} \\
p_N & & p_N \times p_S & & p_S \\
V_{\infty} \times (\mathbb{R}^*_0)_{\infty} & \xleftarrow{p} & (V_{\infty} \times_N V^*_{\infty}) \times (\mathbb{R}^*_0)_{\infty} & \xrightarrow{q} & V^*_{\infty} \times (\mathbb{R}^*_0)_{\infty} \\
j_V & & j & & j_V^* \\
V_{\infty} \times \mathbb{R}_\infty & \xleftarrow{p} & (V_{\infty} \times_N V^*_{\infty}) \times \mathbb{R}_\infty & \xrightarrow{q} & V^*_{\infty} \times \mathbb{R}_\infty \\
i_V & & k & & i_V^* \\
V_{\infty} & \xleftarrow{p} & V_{\infty} \times_N V^*_{\infty} & \xrightarrow{q} & V^*_{\infty} \\
\end{array}
\]
We have to prove
\[
L(E_i^{-1}E_jv^*E_{\text{nd}}^{-1}K) \simeq E_i^{-1}E_jv^*E_{\text{sm}}^{-1}(LK).
\] (6.3)

In the diagram above, consider the left and right columns as morphisms of vector bundles over \(N, N \times \mathbb{R}_>, N \times \mathbb{R}\) and \(N\), respectively. Then the left column is the dual of the right column, and the middle column is the vector bundle product of the left and the right columns. Moreover, since \((p_{\text{rb}} \times p_{\text{sm}})(x, y, s) = (sx, s^{-1}y)\) and \(\langle sx, s^{-1}y \rangle = \langle x, y \rangle\), the maps from the second row to the top row are compatible with the coupling of the left columns and the right columns.

Therefore (6.3) follows from Lemma 5.4. \(\square\)

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