On the plane-wave Riemann Problem in Fluid Dynamics *

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Abstract: This paper contains a stability analysis of the plane-wave Riemann problem for the two-dimensional hyperbolic conservation laws for an ideal compressible gas. It is proved that the contact discontinuity in the plane-wave Riemann problem is unstable under perturbations. The implications for Godunov’s method are discussed and it is shown that numerical post shock noise can set of a contact instability. A relation to carbuncle instabilities is established.

Key words: Riemann solver, Godunov-type methods, hyperbolic conservation laws, gas dynamic, carbuncle instability

AMS(MOS) subject classifications: 65M08, 65M12

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1 Introduction

Consider the compressible Navier-Stokes equations for a viscous, heat conducting gas. The governing equations can be found in many books. They are in two spatial dimensions

\[
\begin{align*}
\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) &= 0 \quad (1a) \\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u^2 + p) + \frac{\partial}{\partial y}(\rho uv) &= \frac{2}{3}\mu(\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v) + \mu(\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v) \quad (1b) \\
\frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + p) &= \frac{2}{3}\mu(\frac{\partial}{\partial y}v - \frac{\partial}{\partial x}u) \quad (1c) \\
\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}[u(E + p)] + \frac{\partial}{\partial y}[v(E + p)] &= \frac{2}{3}\mu(\frac{\partial}{\partial x}u - \frac{\partial}{\partial y}v) + \mu(\frac{\partial}{\partial y}u + \frac{\partial}{\partial x}v) \quad (1d)
\end{align*}
\]

The dependent variable are the density \( \rho \), the velocity field \( \vec{q} = (u, v) \) and the total Energy per unit volume \( E \). The total Energy per unit volume is given by

\[
E = \rho e + \frac{1}{2} \rho (u^2 + v^2) \quad (2)
\]

where \( e \) is the internal energy per unit mass. The thermodynamic variables \( \rho \) and \( e \) are related to the pressure \( p \) through the equation of state

\[
p = p(\rho, e) \quad \text{(equation of state)} \quad (3)
\]
If we assume that the fluid is a perfect gas, the equation of state state is

\[ p = (\gamma - 1)\rho e = (\gamma - 1)[E - \frac{1}{2}\rho(u^2 + v^2)] \quad \text{or} \quad T = \frac{\gamma - 1}{R}e \]  

(4)

where the constants \( \gamma \) and \( R \) are the ratio of specific heats and the gas constant, respectively. \( T \) is the temperature. For a perfect gas the specific heat at constant volume \( c_v \) and the specific heat at constant pressure \( c_p \) are related to \( \gamma \) and \( R \) by

\[ c_v = \frac{R}{\gamma - 1} \quad \text{and} \quad c_p = \frac{\gamma R}{\gamma - 1} \]  

(5)

We assume in the following that the coefficient of viscosity \( \mu \) is constant. We also assumed that the coefficient of bulk viscosity is negligible for the fluid, such that the second coefficient of viscosity \( \dot{\mu} \) is

\[ \dot{\mu} = -\frac{2}{3}\mu \]  

(6)

Furthermore we assume that Fourier’s law for heat transfer holds, with a constant coefficient of thermal conductivity \( k \). If the gas is polytropic resp. a perfect gas, the internal energy \( e \) is related to the temperature \( T \) by (4); i.e.

\[ e = c_v T \]  

(7)

For a vanishing viscosity \( \mu \) and thermal conductivity \( k \) the Navier-Stokes equations reduce to the hyperbolic conservation laws for an ideal compressible gas, also denoted as Euler equations. In vector form these equations are

\[ \frac{\partial}{\partial t} \mathbf{u}(x, y, t) + \frac{\partial}{\partial x} \mathbf{f}(\mathbf{u}(x, y, t)) + \frac{\partial}{\partial y} \mathbf{g}(\mathbf{u}(x, y, t)) = 0 \]  

(8)

where the conserved variable and flux functions are given by

\[ \mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E + p) \end{pmatrix}, \quad \mathbf{g}(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E + p) \end{pmatrix} \]  

(9)

The velocity field will be denoted by \( \mathbf{v} = (u, v) \). The density \( \rho \) and pressure \( p \) are related to the conserved quantities through the equation of state (4).

The Euler equations are a system of hyperbolic conservation laws; i.e. for any value \( \mathbf{u}_0 = (\rho_0, \rho_0 u_0, \rho_0 v_0, E_0)^T \) with positive density \( \rho_0 \) and positive internal energy \( e_0 \) the Jacobian matrix

\[ \dot{n}\mathbf{D}\mathbf{F}(\mathbf{u}_0) = n_x \mathbf{D}\mathbf{f}(\mathbf{u}_0) + n_y \mathbf{D}\mathbf{g}(\mathbf{u}_0) \]  

(10)
is diagonalizable with real eigenvalues for every unit vector $\mathbf{n} = (n_x, n_y)$. Therefore the structure of a plane-wave

$$u(x, y, t) = \varphi(\mathbf{n} \cdot (x, y) - \dot{s}t) \quad (11)$$

propagating at speed $\dot{s}$, is independent of the orientation in space; see for example [Lev2002]. The Euler equations are rotational symmetric and Galilean invariant. A special class of plane-waves is defined through the plane wave Riemann problem; an initial value problem for (8) with piecewise constant initial data, separated by a straight line:

$$u(\mathbf{x}) = \begin{cases} u_l & \text{for } \mathbf{n}_0 \cdot (\mathbf{x} - \mathbf{x}_0) < 0 \\ u_r & \text{for } 0 < \mathbf{n}_0 \cdot (\mathbf{x} - \mathbf{x}_0) \end{cases} \quad (12)$$

where $\mathbf{x}_0$ is a given point in (x,y)-plane and $\mathbf{n}_0$ a given unit-direction, $u_l$ and $u_r$ are initial states at time $t = t^n$.

The integral form of the conservation law (8) is

$$\frac{\partial}{\partial t} \int_D u(\xi, \eta, t) \, d\eta \, d\xi = - \int_{\partial D} \mathbf{n}(\xi) \cdot \mathbf{F}(u^R(\bar{x}(\xi), t^+; \mathbf{n}(\xi))) \, d\xi \quad (13)$$

where $\mathbf{n}(\xi) = (n_x(\xi), n_y(\xi))$ is the outward pointing unit-normal vector on $\partial D$ at a point $\bar{x}(\xi) = (x(\xi), y(\xi))$ on $\partial D$, where $\xi$ is the arclength parametrization of the boundary $\partial D$. $D$ is an arbitrary bounded convex set with a piecewise smooth boundary. $u^R(\bar{x}(\xi), t^+; \mathbf{n}(\xi))$ denotes the one sided limit in time of the solution to the plane-wave Riemann problem (12) at the cell boundary $\bar{x}(\xi)$ in the direction $\mathbf{n}(\xi)$; i.e.

$$u^R(\bar{x}(\xi), t^+; \mathbf{n}(\xi)) := \lim_{\epsilon \to 0} u^R(\bar{x}(\xi), t + \epsilon; \mathbf{n}(\xi)) \quad \text{on } \partial D \quad (14)$$

If the solution $u$ is smooth, then we can apply the divergence theorem and obtain the differential form (8) from the integral form (13). If a solution of the Euler equations contains discontinuities only the integral form is valid.

In this paper we consider the stability of the plane-wave Riemann problem (12) for the Euler equations and proof that the plane-wave Riemann problem is unstable under perturbations. A relation of this instability to the failing of Godunov’s and Roe’s method reported by Quirk [QUI1994], XU [XU1999], Elling [VE2006], Roe [ROE2007] is discussed.

The results of this paper can be extended to general systems of hyperbolic conservation laws, which have at least one genuinely nonlinear characteristic field and
two linear degenerate fields with a double eigenvalue. For clarity we restrict the description to the Euler equation for a $\gamma$-law gas \cite{8} which serves as a model equation for more general systems.

The outline of this paper is as follows. In section 2 we analyze the stability of the plane-wave Riemann problem and prove that the solution is unstable under perturbations. Viscosity and heat conduction are taken into account in section 3. In section 4 the relation to known nonphysical numerical results (carbuncles) in Godunov’s and Roe’s method is discussed and an explanation for carbuncle instabilities proposed, which is also consistent with Majda’s shock stability analysis \cite{MA1983}. The last section contains a short summary and some conclusions.

2 An Instability in the plane-wave Riemann Problem

In this section we analyze the stability of the plane-wave Riemann problem. Since the Euler and Navier-Stokes equations are rotational symmetric, we can assume without loss of generality that the plane wave moves in the $x$-direction and that the initial states are separated by the line $x(y) = 0$. Assume the solution of the plane-wave Riemann is given by a single discontinuity, then the initial states $u_l$ and $u_r$ of the Riemann problem satisfy the Rankine-Hugoniot jump condition \cite{CF1948}.

$$\dot{s}[u_r - u_l] = f(u_r) - f(u_l)$$

where $\dot{s}$ is the shock speed. The normal velocity of the discontinuity can be computed from (15). Let us assume that the states $u_l$ and $u_r$ are connected by a near stationary 1-shock wave with normal shock speed $\dot{s}^1 = \dot{s}^1(u_l, u_r) < 0$. Since the shock is near stationary, we have $-\varepsilon^a < \dot{s}^1$, for a small positive number $\varepsilon^a$. Let $\tilde{u}_r$ be a small perturbation of the right state. The perturbed plane wave $\tilde{u}$ has the initial data

$$\tilde{u}(x, y, 0) = \begin{cases} u_l & \text{for } x < 0 \\ \tilde{u}_r & \text{for } 0 < x \end{cases}$$

and consists of a 1-shock, a contact discontinuity and a 3-wave, which is a weak rarefaction or weak shock wave.

In the following we neglect, without loss of generality, the 3-wave such that $\tilde{u}_{mr} = \tilde{u}_r$. A smooth function $F$ exists with

$$\tilde{u}_r = u_l + F(\varepsilon_1, \varepsilon_2, \varepsilon_3; u_l)$$


where $\varepsilon_2$ represents a parameter for the strength of the contact discontinuity and $\varepsilon_3$ is a parameter for the strength of a 3-wave; see [SM1983; Chapter 17]. $\varepsilon_1$ represents a parameter for the strength of a 1-shock. We choose $\varepsilon_1$ such that

$$u_r = u_l + F(\varepsilon_1, 0, 0; u_l)$$

(18)

For a perturbation of $\varepsilon_2$ with $\varepsilon_3 = 0$ the pressure $p$ and the x-component of the velocity are constant behind the 1-shock. We obtain from the Lax shock conditions

$$s_{i+1/2,j}^1 < \lambda_2(u_r)$$

where $\lambda_2(u_r)$ is the second eigenvalue of the Jacobian $A(u_r)$. For a near stationary strong shock wave we can assume that $0 < \lambda_2(u_r)$. Since the parameter $\varepsilon_2$ represents a contact discontinuity, the discontinuity speed $s^2$ is equal to the characteristic speed $\lambda_2(u_r)$, which is constant across a contact discontinuity; i.e. $s^2 = \lambda_2(u_r) = \lambda_2(\bar{u}_r)$. Where $\bar{u}_r$ is given by

$$\bar{u}_r = u_l + F(\varepsilon_1, \varepsilon_2, 0; u_l)$$

(19)

The y-component of the velocity $v_l$ and $v_r$ enters the solution of the plane-wave Riemann problem essentially as a parameter. We can first solve the one dimensional Riemann problem ignoring the momentum equation for $v$ and then introduce a jump in $v$ at the contact discontinuity to obtain the full plane-wave solution. However, $v_l$ and $v_r$ enter the one dimensional Riemann problem through the pressure as a function of the total energy, density and velocity. We assume that $v_r = O(\varepsilon)$ and $v_l = O(\varepsilon)$. For a small perturbation $O(\varepsilon)$ the change in (4) is of order $\varepsilon^2$.

Therefore we can assume that the wave structure for the perturbed plane-wave Riemann problem

$$\tilde{u}(x, y, 0) = \begin{cases} u_l^f(0-, y) & \text{for } x < 0 \\ \tilde{u}_r^e(0+, y) & \text{for } 0 < x \end{cases}$$

(20)

persists up to second order in $\varepsilon$, where $u_l^f(x, y) = (\rho_l, \rho_l u_l, \rho_l v^f(x, y), E_l)^T$ and $u_r^e(x, y) = (\tilde{\rho}_r, \tilde{\rho}_r u_r, \tilde{\rho}_r v^e(x, y), \tilde{E}_r)^T$.

If $v^f(x, y)$ is discontinuous at $x = 0$ the perturbation introduces a jump in the y-component of the velocity at the contact discontinuity. If $v_r \neq v_l$ the perturbed plane-wave Riemann problem contains a tangential or shear instability. In [LL1959;§81] it is proved that such a tangential instability in an incompressible non viscous flow is absolutely unstable and may lead to a turbulent flow, and it is further mentioned that these instabilities also exists in compressible flows. However, we assume in the following that the perturbation $v^f(x, y)$ is a smooth function.

We can consider the parameter $\varepsilon_1, \varepsilon_2$ in (19) as functions of $y$. We assume that
the variation of $\varepsilon_1$ in $y$ is small enough such that $s_{i+1/2,j}^{1} < u_r$ still holds. This variation can be defined independently of the $y$-component of the velocity and we can assume that $v$ is not affected through this perturbation. The flow for this perturbed plane-wave Riemann problem is defined through the two-dimensional Euler equation (8). A change of the parameter $\varepsilon_2$ in (19) affects only the contact discontinuity. Since the pressure and the $x$-component of the velocity are constant across a contact discontinuity, we have behind the 1-shock $p(x, y, t) = p_r = \tilde{p}_r$, $u(x, y, t) = u_r = \tilde{u}_r$. Thus the Euler equations reduce behind the 1-shock to:

\begin{align}
\frac{\partial}{\partial t} \rho + u_r \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho + \rho \frac{\partial}{\partial y} v &= 0 \\
\frac{\partial}{\partial t} v + u_r \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v &= 0 \\
\frac{\partial}{\partial t} E + u_r \frac{\partial}{\partial x} E + v \frac{\partial}{\partial y} E + E \frac{\partial}{\partial y} v &= 0
\end{align}

Using (2) to rewrite the total energy in (8) as a function of the density, pressure and velocities and using the other conservative equations in (8), the energy equation may be rewritten as a pressure equation

\begin{align}
\frac{\partial}{\partial t} p + u_r \frac{\partial}{\partial x} p + v \frac{\partial}{\partial y} p + \gamma p \left( \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v \right) &= 0
\end{align}

which for a constant pressure simply reduce to a divergence condition

\begin{align}
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v &= 0
\end{align}

Since the $x$-component of the velocity is constant behind the 1-Shock (21) reduces to:

\begin{align}
\frac{\partial}{\partial t} \rho + u_r \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho &= 0 \\
\frac{\partial}{\partial t} v + u_r \frac{\partial}{\partial x} v &= 0 \\
\frac{\partial}{\partial t} E + u_r \frac{\partial}{\partial x} E + v \frac{\partial}{\partial y} E &= 0
\end{align}

where the last equation follows from the equation of state (4) and from the first two. Therefore the perturbed solution satisfies behind the 1-shock the equations
with the initial data

\[ u(x, y, 0+) = \begin{cases} u^t(x, y) & \text{for } x < 0 \\ \tilde{u}^t(x, y) & \text{for } 0 < x \end{cases} \tag{25a} \]

with

\[ u^t(x, y) = (\rho_0(x, y), \rho_0(x, y)u_r, \rho_0(x, y)v_0^\varepsilon(x, y), E_0(x, y))^T \tag{25b} \]

and

\[ \tilde{u}^t(x, y) = (\tilde{\rho}_0(x, y), \tilde{\rho}_0(x, y)u_r, \tilde{\rho}_0(x, y)v_0^\varepsilon(x, y), E_0(x, y))^T \tag{25c} \]

and

\[ E_0(x, y) = \begin{cases} \frac{p_r}{\gamma - 1} + \frac{1}{2} \rho_0(x, y)(u_r^2 + v_0^\varepsilon(x, y)^2) & \text{for } x < 0 \\ \frac{p_r}{\gamma - 1} + \frac{1}{2} \tilde{\rho}_0(x, y)(u_r^2 + v_0^\varepsilon(x, y)^2) & \text{for } 0 < x \end{cases} \tag{25d} \]

where \( v_0^\varepsilon(x, y), \rho_0(x, y) \) and \( \tilde{\rho}_0(x, y) \) are smooth perturbed initial data.

**Proposition 1:** a) If the initial tangential velocity \( v_0^\varepsilon \) does not depend on \( y \) then the constant pressure solution of the initial value problem (24), (25) is given by:

\[ \rho(x, y) = \begin{cases} \rho_0(x - u_r t, y - v^\varepsilon t) & \text{for } x < u_r t \\ \tilde{\rho}_0(x - u_r t, y - v^\varepsilon t) & \text{for } 0 < u_r t \end{cases} \tag{26a} \]

and

\[ v^\varepsilon = v_0^\varepsilon(x - u_r t) \tag{26b} \]

and

\[ E = \frac{p_r}{\gamma - 1} + \frac{1}{2} \rho(x, y)(u_r^2 + (v^\varepsilon)^2) \tag{26c} \]

b) If the initial tangential velocity \( v_0^\varepsilon \) depends on \( y \) a solution does not exist and the pressure is not constant.

Proof: The solution of the plane-wave Riemann problem has behind the 1-shock, i.e. for \( x > s_{i+1/2}^j t \), a constant pressure \( p_r \) and a constant x-component of the velocity \( u_r \). Therefore the Euler equations reduces to (24) and the solution is completely defined through the density \( \rho \), the tangential velocity \( v \), the constant
pressure and the constant x-component of the velocity. We have for \( x < u_r t \)
\[
\frac{\partial}{\partial t} \rho + u_r \frac{\partial}{\partial x} \rho + v^\varepsilon \frac{\partial}{\partial y} \rho \\
= \left[ -u_r + u_r \right] \frac{\partial}{\partial x} \rho_0 + \left[ -(v^\varepsilon + t \frac{\partial}{\partial t} v^\varepsilon) - u_r t \frac{\partial}{\partial x} v^\varepsilon + v^\varepsilon (1 - t \frac{\partial}{\partial y} v^\varepsilon) \right] \frac{\partial}{\partial y} \rho_0 \\
= -\left[ \frac{\partial}{\partial t} v^\varepsilon + u_r \frac{\partial}{\partial x} v^\varepsilon + \frac{\partial}{\partial y} v^\varepsilon \right] t \frac{\partial}{\partial y} \rho_0 
\]
(27)
and similar for \( x > u_r t \)
\[
\frac{\partial}{\partial t} \tilde{\rho} + u_r \frac{\partial}{\partial x} \tilde{\rho} + v^\varepsilon \frac{\partial}{\partial y} \tilde{\rho} \\
= \left[ -u_r + u_r \right] \frac{\partial}{\partial x} \tilde{\rho}_0 + \left[ -(v^\varepsilon + t \frac{\partial}{\partial t} v^\varepsilon) - u_r t \frac{\partial}{\partial x} v^\varepsilon + v^\varepsilon (1 - v^\varepsilon t \frac{\partial}{\partial y} v^\varepsilon) \right] \frac{\partial}{\partial y} \tilde{\rho}_0 \\
= -\left[ \frac{\partial}{\partial t} v^\varepsilon + u_r \frac{\partial}{\partial x} v^\varepsilon + v^\varepsilon \frac{\partial}{\partial y} v^\varepsilon \right] t \frac{\partial}{\partial y} \tilde{\rho}_0 
\]
(28)
Furthermore
\[
\frac{\partial}{\partial t} v^\varepsilon + u_r \frac{\partial}{\partial x} v^\varepsilon = \left[ -u_r + u_r \right] \frac{\partial}{\partial x} v^\varepsilon_0 = 0 
\]
(29)
If follows form (27), (28) for an initial tangential velocity \( v^\varepsilon_0 = v^\varepsilon_0(x) \) that (26a) and (26b) are solutions of the initial value problem (24), (25).

We obtain furthermore for the total energy \( E \) for \( x \neq u_r t \):
\[
\frac{\partial}{\partial t} E + u_r \frac{\partial}{\partial x} E + v^\varepsilon \frac{\partial}{\partial y} E \\
= \frac{1}{2} \left( u_r^2 + (v^\varepsilon)^2 \right) \frac{\partial}{\partial t} \rho + u_r \frac{\partial}{\partial x} \rho + v^\varepsilon \frac{\partial}{\partial y} \rho \\
+ \rho v^\varepsilon \left[ \frac{\partial}{\partial t} v^\varepsilon + u_r \frac{\partial}{\partial x} v^\varepsilon \right] = 0
\]

Therefore (26) defines a solution of the Euler equation (8) in smooth parts of the flow.

The jump condition for a plane-wave moving in the x-direction reduces for a constant pressure and constant x-component of the velocity to
\[
\dot{s}(t) \left[ \rho(x+, y, t) - \rho(x-, y, t) \right] = u_r \left[ \rho(x+, y, t) - \rho(x-, y, t) \right] \\
\dot{s}(t) u_r \left[ \rho(x+, y, t) - \rho(x-, y, t) \right] = (u_r)^2 \left[ \rho(x+, y, t) - \rho(x-, y, t) \right] \\
\dot{s}(t) \rho(x+, y, t) v^\varepsilon(x+, t) - \rho(x-, y, t) v^\varepsilon(x-, t) = u_r \left[ \rho(x+, y, t) v^\varepsilon(x+, t) - \rho(x-, y, t) v^\varepsilon(x-, t) \right] \\
\dot{s}(t) \left[ E(x+, y, t) - E(x-, y, t) \right] = u_r \left[ E(x+, y, t) - E(x-, y, t) \right]
\]
where \( \dot{s} \) is the speed of the discontinuity in the x-direction. Therefore we see that the jump conditions are satisfied along the curve \( x = s(t) = u_r t \), and (26) is a weak solution of (8).

Since the solution of initial value problem (24), (25) is unique for smooth velocities \( u_r \) and \( v^\varepsilon \), part a) of the proposition follows.

If the initial data \( v_0^\varepsilon \) depend on \( y \) the unique solution of (24b) is for every \( y \) given by (26b). In this case we obtain for the total energy \( E \) for \( x \neq u_r t \) and a constant pressure:

\[
\frac{\partial}{\partial t} E + u_r \frac{\partial}{\partial x} E + v^\varepsilon \frac{\partial}{\partial y} E = \frac{1}{2} \left( u_r^2 + (v^\varepsilon)^2 \right) \left[ \frac{\partial}{\partial t} \rho + u_r \frac{\partial}{\partial x} \rho + v^\varepsilon \frac{\partial}{\partial y} \rho \right] + \rho v^\varepsilon \left[ \frac{\partial}{\partial t} v^\varepsilon + u_r \frac{\partial}{\partial x} v^\varepsilon + v^\varepsilon \frac{\partial}{\partial y} v^\varepsilon \right] = \rho v^\varepsilon v^\varepsilon \frac{\partial}{\partial y} v^\varepsilon \tag{30}
\]

Thus the energy equation can only be satisfied if \( v^\varepsilon \) does not depend on \( y \). This must also hold for \( t = 0 \). If \( v \) depends on \( y \) we obtain from (22)

\[
\frac{\partial}{\partial t} p + u_r \frac{\partial}{\partial x} p + v \frac{\partial}{\partial y} p = -\gamma p \frac{\partial}{\partial y} v \tag{31}
\]

which cannot be satisfied for a constant pressure solution. This proves part b) and completes the proof. □

In a constant pressure flow the equations for the conservation of energy follows from the momentum and the continuity equations. The Euler equations (8) therefore simplify to

\[
\begin{align*}
\frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho &= 0 \tag{32a} \\
\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u &= 0 \tag{32b} \\
\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v &= 0 \tag{32c}
\end{align*}
\]

If the velocity field is given by \( u = u_r \) and \( v = v(x,t) \) these equations reduce to the equations (24) for a perturbed flow behind a shock. This system of partial differential equations is unstable under perturbations. Let \( \bar{\rho}, \bar{u} \) and \( \bar{v} \) be a constant
pressure solution of (32) and \( \bar{\rho}, \bar{u}, \bar{v} \) infinitesimal perturbations. It is assumed that

\[
\rho = \bar{\rho} + \tilde{\rho} \quad u = \bar{u} + \tilde{u} \quad v = \bar{v} + \tilde{v}
\]

is also a smooth constant pressure solution of (32). We obtain by neglecting quadratic perturbation terms

\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{\rho} + \bar{u} \frac{\partial}{\partial x} \tilde{\rho} + \bar{v} \frac{\partial}{\partial y} \tilde{\rho} + \bar{u} \frac{\partial}{\partial x} \bar{\rho} + \bar{v} \frac{\partial}{\partial y} \bar{\rho} + \tilde{u} \frac{\partial}{\partial x} \bar{\rho} + \tilde{v} \frac{\partial}{\partial y} \bar{\rho} &= 0 \\
\frac{\partial}{\partial t} \tilde{u} + \frac{\partial}{\partial x} \bar{u} + \bar{v} \frac{\partial}{\partial y} \bar{u} &= -\tilde{u} \frac{\partial}{\partial x} \bar{u} - \tilde{v} \frac{\partial}{\partial y} \bar{u} \\
\frac{\partial}{\partial t} \tilde{v} + \frac{\partial}{\partial x} \bar{v} + \bar{v} \frac{\partial}{\partial y} \bar{v} &= -\tilde{u} \frac{\partial}{\partial x} \bar{v} - \tilde{v} \frac{\partial}{\partial y} \bar{v}
\end{align*}
\]

which can be rewritten as a inhomogeneous hyperbolic system

\[
\begin{align*}
\frac{\partial}{\partial t} \tilde{\rho} + \bar{u} \frac{\partial}{\partial x} \tilde{\rho} + \bar{v} \frac{\partial}{\partial y} \tilde{\rho} &= -\tilde{u} \frac{\partial}{\partial x} \bar{\rho} - \tilde{v} \frac{\partial}{\partial y} \bar{\rho} \\
\frac{\partial}{\partial t} \tilde{u} + \frac{\partial}{\partial x} \bar{u} + \bar{v} \frac{\partial}{\partial y} \bar{u} &= -\bar{u} \frac{\partial}{\partial x} \bar{u} - \bar{v} \frac{\partial}{\partial y} \bar{u} \\
\frac{\partial}{\partial t} \tilde{v} + \frac{\partial}{\partial x} \bar{v} + \bar{v} \frac{\partial}{\partial y} \bar{v} &= -\bar{u} \frac{\partial}{\partial x} \bar{v} - \bar{v} \frac{\partial}{\partial y} \bar{v}
\end{align*}
\]

The solution of the homogeneous system (33) is

\[
\begin{align*}
\rho^h &= \rho^0(x - \bar{u}t, y - \bar{v}t) \\
u^h &= \bar{u}^0(x - \bar{u}t, y - \bar{v}t) \\
v^h &= \bar{v}^0(x - \bar{u}t, y - \bar{v}t)
\end{align*}
\]

where \( \rho^0, \bar{u}^0 \) and \( \bar{v}^0 \) are some arbitrary initial divergence free perturbations. Let

\[
\begin{align*}
S^\rho(x, y, t) &= -\tilde{u} \frac{\partial}{\partial x} \bar{\rho} - \tilde{v} \frac{\partial}{\partial y} \bar{\rho} \\
S^x(x, y, t) &= -\tilde{u} \frac{\partial}{\partial x} \bar{u} - \tilde{v} \frac{\partial}{\partial y} \bar{u} \\
S^y(x, y, t) &= -\tilde{u} \frac{\partial}{\partial x} \bar{v} - \tilde{v} \frac{\partial}{\partial y} \bar{v}
\end{align*}
\]

By formally applying Duhamel`s principle we obtain for the inhomogeneous equa-

\footnote{The results remain valid if pressure perturbations \( p = \bar{p} + \tilde{p} \) are included}
tion (33) with the source term (35)

\[ \tilde{\rho} = \tilde{\rho}^h + \int_0^t S^\rho(x - \tilde{u}(t - \tau), y - \tilde{v}(t - \tau), \tau) d\tau \]
\[ = \tilde{\rho}^h - t(\tilde{u}^h \frac{\partial}{\partial x} \tilde{\rho} + \tilde{v}^h \frac{\partial}{\partial y} \tilde{\rho}) \quad (36a) \]

\[ \tilde{u} = \tilde{u}^h + \int_0^t S^x(x - \tilde{u}(t - \tau), y - \tilde{v}(t - \tau), \tau) d\tau \]
\[ = \tilde{u}^h - t(\tilde{u}^h \frac{\partial}{\partial x} \tilde{u} + \tilde{v}^h \frac{\partial}{\partial y} \tilde{u}) \quad (36b) \]

\[ \tilde{v} = \tilde{v}^h + \int_0^t S^y(x - \tilde{u}(t - \tau), y - \tilde{v}(t - \tau), \tau) d\tau \]
\[ = \tilde{v}^h - t(\tilde{u}^h \frac{\partial}{\partial x} \tilde{v} + \tilde{v}^h \frac{\partial}{\partial y} \tilde{v}) \quad (36c) \]

where \( \bar{\rho} = \bar{\rho}(x, y, t) \), \( \bar{u} = \bar{u}(x, y, t) \) and \( \bar{v} = \bar{v}(x, y, t) \). Inserting \( \tilde{\rho}, \tilde{u} \) and \( \tilde{v} \) into (32) we can verify that (36) defines a solution of the inhomogeneous hyperbolic system (33). We obtained, that for a non constant background flow \( \bar{\rho}, \bar{u} \) and \( \bar{v} \), perturbation \( \tilde{\rho}, \tilde{u} \) and \( \tilde{v} \) grow linear in time.

The inhomogeneous hyperbolic system (33) can be rewritten in vector form as

\[ \frac{\partial}{\partial t} \tilde{w}(x, y, t) + \tilde{A}(x, y, t) \frac{\partial}{\partial x} \tilde{w}(x, y, t) + \tilde{B}(x, y, t) \frac{\partial}{\partial y} \tilde{w}(x, y, t) + \tilde{C}(x, y, t) \tilde{w}(x, y, t) = 0 \quad (37) \]

where the vector of primitive variables is \( \tilde{w} = (\tilde{\rho}, \tilde{u}, \tilde{v})^T \) and the 3 x 3 matrices \( \tilde{A}(x, y, t), \tilde{B}(x, y, t) \) and \( \tilde{C}(x, y, t) \) are defined by

\[ \tilde{A} = \begin{pmatrix} \tilde{u} & 0 & 0 \\ 0 & \tilde{u} & 0 \\ 0 & 0 & \tilde{u} \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} \tilde{v} & 0 & 0 \\ 0 & \tilde{v} & 0 \\ 0 & 0 & \tilde{v} \end{pmatrix} \quad \tilde{C} = \begin{pmatrix} 0 & \frac{\partial}{\partial x} \tilde{\rho} & \frac{\partial}{\partial y} \tilde{\rho} \\ 0 & \frac{\partial}{\partial x} \tilde{u} & \frac{\partial}{\partial y} \tilde{u} \\ 0 & \frac{\partial}{\partial x} \tilde{v} & \frac{\partial}{\partial y} \tilde{v} \end{pmatrix} \]

For a smooth background flow (37) is a symmetric hyperbolic system, with matrices \( \tilde{A}(x, y, t), \tilde{B}(x, y, t) \) and \( \tilde{C}(x, y, t) \) depending smoothly on \( x, y \) and \( t \). For any perturbation \( \tilde{w} \) with compact support in \((x, y)\) a energy inequality of the form

\[ \| \tilde{w}(t) \| \leq \exp(M \cdot |t|) \| \tilde{w}(0) \| \quad (38) \]

holds, where the constant \( M \) depends only on the magnitude of the symmetric part of \( \tilde{C} \) and of the first derivative of \( \tilde{A} \) with respect to \( x \) resp. \( \tilde{B} \) with respect to \( y \); see [LAX2006; Section 4.3]. This result can also be derived directly from (36). Now consider the solution (36) for a case where the initial background density \( \tilde{\rho} \)
is not differentiable at \( x = 0 \). For a constant x-component of the velocity \( \bar{u} = u_r \) and a initial tangential velocity \( \bar{v}_0 = \bar{v}_0(x) \), which is enforced through a constant pressure, the solution is given by \( (26) \). In this case the definition of the perturbed solution \( (36) \) remains valid, if we interpret the partial derivative of \( \bar{\rho} \) with respect to \( x \) at the contact discontinuity as a delta function; i.e.

\[
\frac{\partial}{\partial x} \bar{\rho}(u_r,t,y,t) = (\bar{\rho}_r - \bar{\rho}_l)\delta(x - u_r t) \tag{39}
\]

where \( \bar{\rho}_l \) and \( \bar{\rho}_r \) denote the left and right limits at the contact discontinuity. For an initial oscillatory velocity perturbation \( \tilde{u}^h \), large density perturbations are enforced. In this case the solution grows with \( t \) in a manner unlike what we would expect from a hyperbolic equation and no energy inequality of the form \( (38) \) can be derived. We denote such a flow in the following as unstable under perturbations. Consider for example a vanishing background tangential velocity \( \bar{v} \), a vanishing perturbed tangential velocity \( \tilde{v} \) and a piecewise constant background density \( \bar{\rho}(x,t) \) which is discontinuous at \( x = \bar{u}t \), then we obtain from \( (33a) \) for \( \bar{u} = u_r \),

\[
\frac{\partial}{\partial t} \tilde{\rho} + u_r \frac{\partial}{\partial x} \tilde{\rho} = -\tilde{u}(\bar{\rho}_r - \bar{\rho}_l)\delta(x - u_r t) \tag{40}
\]

and from \( (36a) \) for the solution

\[
\tilde{\rho} = \tilde{\rho}^h - t\bar{u}^h(\bar{\rho}_r - \bar{\rho}_l)\delta(x - u_r t) \tag{41}
\]

For an initial oscillatory velocity perturbation \( \tilde{u}^h \) large density perturbations are enforced. For a constant normal velocity \( \bar{u} = u_r \) and a piecewise constant background tangential velocity \( \bar{v} \) which depends only on \( x \) and is discontinuous at \( x = u_r t \), equation \( (33c) \) becomes

\[
\frac{\partial}{\partial t} \tilde{v} + u_r \frac{\partial}{\partial x} \tilde{v} = -\tilde{u}(\bar{v}_r - \bar{v}_l)\delta(x - u_r t) \tag{42}
\]

with a weak solution given by \( (36c) \); i.e.

\[
\tilde{v} = \tilde{v}^h - t\bar{u}^h(\bar{v}_r - \bar{v}_l)\delta(x - u_r t) \tag{43}
\]

For an initial oscillatory velocity perturbation \( \tilde{u}^h \), large tangential velocity perturbations are enforced. Linear advection equation of the form \( (40), (42) \) with a singular source term are discussed by LeVeque in [LEV2002; Section 16.3.1].

We showed that the flow behind a perturbed plane stationary shock line is a constant pressure flow governed by \( (32) \) with \( u = u_r \) and \( v = v(x,t) \). Generally,
the flow between the nonlinear 1-wave and 3-wave in the Riemann problem is a constant pressure flow region. Since a discontinuity in a constant pressure plane wave flow region is a contact discontinuity, we obtained:

*If the solution of the plane-wave Riemann problem contains a contact discontinuity, then the solution is unstable under perturbations.*

Remark: The same perturbation analysis can be applied to an homentrop (constant entropy), nearly constant density flow, with a not necessary constant pressure and divergence. In this case we obtain for the perturbation equations

\[
\frac{\partial}{\partial t} \tilde{\rho} + \bar{u} \frac{\partial}{\partial x} \tilde{\rho} + \bar{v} \frac{\partial}{\partial y} \tilde{\rho} + \tilde{\rho}_0 (\frac{\partial}{\partial x} \tilde{u} + \frac{\partial}{\partial y} \tilde{v}) = 0 \tag{44a}
\]

\[
\frac{\partial}{\partial t} \tilde{u} + \bar{u} \frac{\partial}{\partial x} \tilde{u} + \bar{v} \frac{\partial}{\partial y} \tilde{u} = -\tilde{u} \frac{\partial}{\partial x} \bar{u} - \tilde{v} \frac{\partial}{\partial y} \bar{u} - \frac{1}{\tilde{\rho}_0} \frac{\partial}{\partial x} \tilde{p} \tag{44b}
\]

\[
\frac{\partial}{\partial t} \tilde{v} + \bar{u} \frac{\partial}{\partial x} \tilde{v} + \bar{v} \frac{\partial}{\partial y} \tilde{v} = -\tilde{u} \frac{\partial}{\partial x} \bar{v} - \tilde{v} \frac{\partial}{\partial y} \bar{v} - \frac{1}{\tilde{\rho}_0} \frac{\partial}{\partial y} \tilde{p} \tag{44c}
\]

\[
\frac{\partial}{\partial t} \tilde{p} + \bar{u} \frac{\partial}{\partial x} \tilde{p} + \bar{v} \frac{\partial}{\partial y} \tilde{p} + \gamma \tilde{p} (\frac{\partial}{\partial x} \tilde{u} + \frac{\partial}{\partial y} \tilde{v}) = 0 \tag{44d}
\]

where \( \tilde{\rho}_0 \) is the constant background density and \((\bar{u}, \bar{v})\) a divergence free background velocity field. For a constant background flow \( \tilde{\rho}_0, \bar{u}_0, \bar{v}_0, \tilde{p}_0 \), homentrop density perturbations are defined through

\[
\tilde{\rho} = \tilde{\rho}_A(x - \bar{u}_0 t, y - \bar{v}_0 t, t) \tag{45}
\]

where \( \tilde{\rho}_A \) satisfies a wave equation

\[
\frac{\partial^2}{\partial \tau^2} \tilde{\rho}_A - \bar{c}_0^2 (\frac{\partial^2}{\partial x^2} \tilde{\rho}_A + \frac{\partial^2}{\partial y^2} \tilde{\rho}_A) = 0 \tag{46}
\]

where \( \bar{c}_0 \) denotes the sound speed of the background flow and \( \tau \) refers to the differentiation with respect to the third argument in \( \tilde{\rho}_A \). Since the solutions of the wave equation are stable, density perturbations are stable.

A initial (acoustic) perturbation \( \tilde{\rho}_A \) propagates with sound speed relative to the background velocity into the flow. If generated at a shock, these acoustic perturbation propagate with sound speed relative to adjacent (constant) velocities, into the flow on both sides of the discontinuity. In contrast to acoustic perturbations, density perturbations in a constant pressure region can only exists on one side of a shock line. Since in an ideal gas, any density perturbation in a constant pressure region is equivalent to a entropy perturbation, the latter disturbances are also denoted as entropy disturbances.
In the analysis of the perturbed Riemann problem, the weak 3-wave was neglected. Since the change of the entropy across a shock wave is of third order in \( \varepsilon^3 \) in (17), the weak 3-wave can be regarded as a (discontinuous) homentrop perturbation of a constant state. The density of these acoustic perturbations is defined through (45), (46) and do not cause an unstable flow.

Note: To derive the Rankine-Hugoniot jump condition (15) from the integral form (13) of the conservation law, it is assumed that the integral

\[
\int_{-\Delta x/2}^{\Delta x/2} \frac{\partial}{\partial t} u(\xi, y, t^n) \, d\xi
\]

approaches zero for \( \lim \Delta x \rightarrow 0 \). This assumption fails for the perturbed two-dimensional plane-wave Riemann problem. Consider a discontinuity line \( S \). Without restriction of generality we may assume that for a sufficiently small portion, the discontinuity line \( S \) is perpendicular to the x-axis and we may assume that the flow is smooth in the y-direction. Furthermore for a sufficiently small time interval the shock speed \( \dot{s} \) may be considered as constant. Let \( x = s(t, y) = \dot{s}(y)t \) be the spatial-time discontinuity surface across which \( u \) has a jump. We obtain from the conservation law (8) for \( \Delta x > 0 \)

\[
\frac{\partial}{\partial t} \int_{-\Delta x/2}^{\Delta x/2} u(x, y, t) \, dx = - [f(u(\Delta x/2, y, t)) - f(u(-\Delta x/2, y, t))] - \int_{-\Delta x/2}^{\Delta x/2} \frac{\partial}{\partial y} g(u) \, dx
\]

The first integral can be rewritten as

\[
\frac{\partial}{\partial t} \int_{-\Delta x/2}^{\Delta x/2} u(x, y, t) \, dx = -\dot{s}[u(s+, y, t) - u(s-, y, t)] + \int_{-\Delta x/2}^{+\Delta x/2} \frac{\partial}{\partial t} u(x, y, t^n) \, dx
\]

where \( s+ \) and \( s- \) denote one-sided limits at the discontinuity line. Thus if (17) vanishes for \( \Delta x \rightarrow 0 \) we obtain the jump conditions (15) from the last two equations. The integral does not vanish, if density perturbations can grow infinitely fast in time.

Viscosity and heat conduction in the Navier-Stokes equations strongly affects steep gradients and perturbations. In reality a contact surface cannot be maintained for an appreciable length of time; (viscosity and) heat conduction between the permanently adjacent particles on either side of the discontinuity would soon make the idealized assumption unrealistic. While gas particles crossing a shock front are exposed to heat conduction for only a very short time, those that remain adjacent
on either side of a contact surface are exposed to heat conduction all the time. Hence a contact layer will gradually fade out; see [CF1948]. Therefore some odd behavior of the solution must be expected for a mathematical idealized contact discontinuity.

3 Viscosity and Heat Conduction

It is well known that viscosity and heat conduction can attenuate small scale oscillations. In this section we study the effect of heat conduction and viscosity on the solutions (26) derived in the previous section. Including viscosity in our considerations we obtain from the Navier-Stokes equation for a divergence free flow

\[
\frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho = 0
\]  

(48a)

\[
\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = - \frac{1}{\rho} \frac{\partial}{\partial x} p + \frac{\mu}{\rho} \left( \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u \right)
\]  

(48b)

\[
\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v = - \frac{1}{\rho} \frac{\partial}{\partial y} p + \frac{\mu}{\rho} \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right)
\]  

(48c)

\[
c_v \rho \frac{\partial}{\partial t} T + u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = k \left( \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T \right) + 2\mu \left( \frac{\partial}{\partial x} v \right)^2 + 2\mu \left( \frac{\partial}{\partial y} u \right)^2 + \mu \left( \frac{\partial}{\partial x} v + \frac{\partial}{\partial y} u \right)^2
\]  

(48d)

\[
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0
\]  

(48e)

see e.g. [ATP1984:Section 5]. \( T \) is the temperature given by (11). For a constant x-component of the velocity \( u = u_0 \) we obtain from the divergence condition (48e), that the tangential component \( v \) does not depend on \( y \) and equations (48) reduce to

\[
\frac{\partial}{\partial t} \rho + u_0 \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho = 0
\]  

(49a)

\[
\frac{\partial}{\partial t} v + u_0 \frac{\partial}{\partial x} v = - \frac{1}{\rho} \frac{\partial}{\partial y} p + \frac{\mu}{\rho} \frac{\partial^2}{\partial x^2} v
\]  

(49b)

\[
\frac{\partial}{\partial t} p + v \frac{\partial}{\partial y} p = (\gamma - 1) \mu \left( \frac{\partial}{\partial x} v \right)^2 + (\gamma - 1) k \left( \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T \right)
\]  

(49c)

\[
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0
\]  

(49d)
where we used (4) and (5) and assumed that the viscosity \( \mu \) and the coefficient of thermal conductivity \( k \) are positive constants. We obtain for a density of the form (26a) with \( u_r = u_0 \) and \( v^e = v \) for \( x < u_0 t \) with (19b)

\[
\frac{\partial}{\partial t} \rho + u_0 \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho = -\left[ \frac{\partial}{\partial t} v + u_0 \frac{\partial}{\partial x} v + \frac{\partial}{\partial y} v \right] t \frac{\partial}{\partial y} \rho_0
\]

\[
= \left[ \frac{1}{\rho} \frac{\partial}{\partial y} p - \frac{\mu}{\rho \frac{\partial}{\partial x} v} \right] t \frac{\partial}{\partial y} \rho_0
\]

For \( t > 0 \) (19a) requires

\[
\frac{\partial}{\partial y} p - \mu \frac{\partial^2}{\partial x^2} v = 0 \quad \text{or} \quad \frac{\partial \rho_0}{\partial \eta} = 0 \quad (50)
\]

Since the \( x \)-component of the velocity is constant, the pressure cannot depend on \( x \). Since \( v \) only depends on the spatial coordinate \( x \), this condition can only be satisfied if

\[
\frac{\partial}{\partial y} p = \mu \frac{\partial^2}{\partial x^2} v = f(t) \quad \text{or} \quad \frac{\partial \rho_0}{\partial \eta} = 0 \quad (51)
\]

for a function \( f(t) \). Therefore either the density does not depend on \( y \) or \( v \) is a quadratic function in \( x \); i.e., for a spatial oscillatory tangential velocity \( v \), the density \( \rho \) cannot vary in the \( y \)-direction. If viscosity is taken into account, perturbed solutions of the form (26) with an oscillatory tangential velocity \( v^0 \) are excluded. In a constant pressure flow any density perturbation is equivalent to an entropy perturbation. Since (26) is for \( x \neq u_r t \) a smooth solutions of the Euler equations, these solutions cannot be excluded through the established entropy conditions for weak solutions. Thus we may regard (51) as an additional "plane wave condition" for the Euler equations. A density perturbation of the form (26) which violates (51) will be denoted as an "inviscid entropy perturbation".
For a constant pressure $p = p_0$ equations (48) reduce to

$$\frac{\partial}{\partial t} \rho + u \frac{\partial}{\partial x} \rho + v \frac{\partial}{\partial y} \rho = 0$$  \hfill (52a)

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = \frac{\mu}{\rho} \left( \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u \right)$$  \hfill (52b)

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v = \frac{\mu}{\rho} \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right)$$  \hfill (52c)

$$\frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T = -2\frac{\mu}{k} \left( \frac{\partial}{\partial x} u \right)^2 - 2\frac{\mu}{k} \left( \frac{\partial}{\partial y} v \right)^2 - \frac{\mu}{k} \left( \frac{\partial}{\partial x} v + \frac{\partial}{\partial y} u \right)^2$$  \hfill (52d)

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0$$  \hfill (52e)

where we used (4) to rewrite the temperature as a function of the pressure and the density and used (52a) to obtain (52d). For a given velocity field equation (52d) is a Poisson equation for the temperature; where the ratio of the viscosity and thermal conductivity in (52d) can be computed from the Prandtl number $Pr$ and the specific heat at constant pressure $c_p$ by

$$\frac{\mu}{k} = \frac{Pr}{c_p}$$  \hfill (53)

The ratio $Pr/c_p$ is approximately constant for most gases. For air at standard conditions $Pr/c_p \simeq 0.00072$.

The temperature and constant pressure defines the density via the equation of state; i.e.

$$\rho = \frac{p}{RT}$$  \hfill (54)

For a non vanishing coefficient of thermal conductivity $k$ (52d) imposes a regularity condition on the density, which is not present in an flow governed by the Euler equations (8); e.g. using the Weyl Lemma [WA1994;Section 9] we obtain from (52d) for a velocity field which is constant outside a bounded set, that the density is as smooth as the velocity field, whereas the Euler equations admits contact discontinuities for such a velocity field; see Proposition 1.

A similair regularity condition holds for the pressure in the incompressible Navier-Stokes equations. For a constant density $\rho = \rho_0$ the equations (48) reduce to
\[
\begin{align*}
\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u &= -\frac{1}{\rho_0} \frac{\partial}{\partial x} p + \frac{\mu}{\rho_0} (\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u) 
\quad \text{(55a)} \\
\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v &= -\frac{1}{\rho_0} \frac{\partial}{\partial y} p + \frac{\mu}{\rho_0} (\frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v) 
\quad \text{(55b)} \\
\frac{\partial}{\partial t} p + u \frac{\partial}{\partial x} p + v \frac{\partial}{\partial y} p &= \frac{k}{\rho_0 c_v}(\frac{\partial^2}{\partial x^2} p + \frac{\partial^2}{\partial y^2} p) + 2(\gamma - 1)\mu (\frac{\partial}{\partial x} u)^2 + 2(\gamma - 1)\mu (\frac{\partial}{\partial y} v)^2 + (\gamma - 1)\mu (\frac{\partial}{\partial x} v + \frac{\partial}{\partial y} u)^2 \\
\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v &= 0 
\quad \text{(55d)}
\end{align*}
\]

where we used \((54)\) and \((5)\) to express the temperature as a function of the density and pressure. Equation \((55a), (55b)\) and \((55d)\) constitute the incompressible Navier-Stokes equations. The pressure in these equations can be computed from the velocity field through a Poisson equation

\[
\frac{\partial^2}{\partial x^2} p + \frac{\partial^2}{\partial y^2} p = -2\rho_0(\frac{\partial}{\partial x} v \frac{\partial}{\partial y} u - \frac{\partial}{\partial x} u \frac{\partial}{\partial y} v) 
\quad \text{(56)}
\]

see e.g. [KL1989; Section 9.1.3]. Again using the Weyl Lemma we obtain from \((56)\) for a velocity field which is constant outside a bounded set, that the pressure is as smooth as the velocity field. Inserting \((56)\) into \((55c)\) we obtain for a given smooth velocity field a inhomogeneous advection equation for the pressure. This equation defines the time evolution of the pressure and is generally neglected in an incompressible flow. Consider a constant pressure, constant density flow with a constant x-component of the velocity \(u = u_r\). We obtain from \((56)\) for a constant pressure, that the functional determinate of \((u, v)\) vanishes. Therefore \(v = \Phi(u)\) for some smooth function \(\Phi\) and we obtain that the tangential velocity is constant as well. From proposition 1 follows that \((u_r, v)\) with \(v = \Phi(x - u_r t)\) is a velocity field for the Euler equations with a constant pressure and constant density. These solutions are excluded for a finite viscosity \(\mu\) and coefficient of thermal conductivity \(k\) in the divergence free Navier-Stokes equations \((48)\).

Numerical methods for the Euler equations employ an artificial numerical viscosity model to resolve discontinuities. If this numerical viscosity model is not properly related to the physical viscosity and thermal conductivity, numerical artefact’s may be introduced into a approximate solution, which are not related to a real viscous, heat conducting flow.

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4 Instability of Godunov’s method

In a first order finite volume methods the assumption is generally made, that the solution is constant inside a cell at a time-level \( t = t^n \); e.g.

\[
\bar{u}(\vec{x}, t) := u_{i,j} \quad \text{for} \quad \vec{x} \in [x_{i-1/2}, x_{i+1/2}] \times [y_{i-1/2}, y_{i+1/2}]
\]  
(57)

and it is assumed that discontinuities are moved to the cell boundary. We assume for simplicity that \( D_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{i-1/2}, y_{i+1/2}] \) is a rectangle defined through a constant cartesian grid \( x_i = i \Delta x \) and \( y_j = j \Delta y \) and denote by

\[
\bar{u}_{i,j}(t) = \frac{1}{V(D_{i,j})} \int_{D_{i,j}} u(\xi, \eta, t) \, d\eta \, d\xi
\]  
(58)

the cell-average, where \( V(D_{i,j}) \) is the Volume of \( D_{i,j} \). Finite-Volume Godunov-type methods are derived from the integral form of the conservation law. The integral form \([13]\) can be rewritten as

\[
\frac{d}{dt} \bar{u}_{i,j}(t) = - \frac{1}{V(D_{i,j})} \int_{\partial D_{i,j}} \vec{n}(\xi) \cdot \vec{F}(u^R(\vec{x}(\xi), t^n; \vec{n}(\xi))) \, d\xi
\]  
(59)

This equations says that we can evolve the cell averages in time, by solving one-dimensional Riemann problems at a the cell boundary at time \( t = t^n \) and then solve the system of ordinary differential equation \([59]\) to obtain the cell average at time \( t = t^n + \tau, \tau > 0 \). Taking for granted that the solution of the Riemann problem at the cell interface can be locally advanced in time; i.e. we require that

\[
\frac{\partial}{\partial t} u^R(\vec{x}(\xi), t^n; \vec{n}(\xi))|_{t^n+} = 0
\]  
(60)

should exist at the cell-boundary \( \partial D_{i,j} \).

In Godunov’s method a plane-wave Riemann problem is solved at the cell boundary, for example at the cell-boundary \((x_{i+1/2}, y_j)\) at time \( t = t^n \) in the x-direction. Denote by \( u^R(x_{i+1/2}, y_j, t) \) the solution of the plane-wave Riemann problem in the x-direction at the cell boundary, then \( u^R_{t+1/2,j} := \lim_{t \to t^n+} u^R(x_{i+1/2}, y_j, t) \) is computed and used to evaluate the physical flux function.

In the last two sections we saw that a constant pressure region which a contact discontinuity is unstable under perturbations. In this section we analyze Godunov’s method for this critical region. Assume that the velocity \( u \) and the pressure \( p \) is constant, then the solution of the Riemann problem in the x-direction consists of contact discontinuity and we obtain

\[
u^R(x_{i+1/2}, y_j, t^n) = \begin{cases} 
    u_{i+1,j} & \text{for } u < 0 \\
    u_{i,j} & \text{for } 0 < u
\end{cases}
\]  
(61)
If $v$ depends only on the spatial coordinate $x$ and $p$ is constant, then the solution in the $y$-direction is also a contact discontinuity and we obtain

$$u^R(x_i, y_{j+1/2}, t^n) = \begin{cases} \bar{u}_{i,j+1} & \text{for } v_i < 0 \\ \bar{u}_{i,j} & \text{for } 0 < v_i \end{cases}$$  \hspace{1cm} (62)$$

Therefore for $u < 0$ and $v_i < 0$ we obtain

$$\bar{u}_{i,j}^{n+1} = \bar{u}_{i,j} - \frac{\tau}{\Delta x} [f(\bar{u}_{i,j}^n) - f(\bar{u}_{i,j}^n)] - \frac{\tau}{\Delta y}[g(\bar{u}_{i,j+1}^n) - g(\bar{u}_{i,j}^n)]$$  \hspace{1cm} (63)

which is equivalent to the four difference equations

$$\rho_{i,j}^{n+1} = \rho_{i,j} - \frac{\tau}{\Delta x} u(\bar{p}_{i+1,j}^n - \bar{p}_{i,j}^n) - \frac{\tau}{\Delta y} \bar{v}_i(\bar{p}_{i,j+1}^n - \bar{p}_{i,j}^n)$$

$$u\rho_{i,j}^{n+1} = u[\rho_{i,j} - \frac{\tau}{\Delta x} u(\bar{p}_{i+1,j}^n - \bar{p}_{i,j}^n)] - \frac{\tau}{\Delta y} \bar{v}_i(\bar{p}_{i,j+1}^n - \bar{p}_{i,j}^n)]$$

$$(\bar{p}v)_{i,j}^{n+1} = (\bar{p}v)_{i,j} - \frac{\tau}{\Delta x} u((\bar{p}v)_{i+1,j}^n - (\bar{p}v)_{i,j}^n) - \frac{\tau}{\Delta y} \bar{v}_i((\bar{p}v)_{i,j+1}^n - (\bar{p}v)_{i,j}^n)$$

$$\bar{E}_{i,j}^{n+1} = \bar{E}_{i,j} - \frac{\tau}{\Delta x} u(\bar{E}_{i+1,j}^n - \bar{E}_{i,j}^n) - \frac{\tau}{\Delta y} \bar{v}_i(\bar{E}_{i,j+1}^n - \bar{E}_{i,j}^n)$$

Since the second equation is just the first equation multiplied with the constant velocity $u$, this can be reduced to three difference equations

$$\rho_{i,j}^{n+1} = \rho_{i,j} - \frac{\tau}{\Delta x} u(\bar{p}_{i+1,j}^n - \bar{p}_{i,j}^n) - \frac{\tau}{\Delta y} \bar{v}_i(\bar{p}_{i,j+1}^n - \bar{p}_{i,j}^n)$$

$$(\bar{p}v)_{i,j}^{n+1} = (\bar{p}v)_{i,j} - \frac{\tau}{\Delta x} u((\bar{p}v)_{i+1,j}^n - (\bar{p}v)_{i,j}^n) - \frac{\tau}{\Delta y} \bar{v}_i((\bar{p}v)_{i,j+1}^n - (\bar{p}v)_{i,j}^n)$$

$$\bar{E}_{i,j}^{n+1} = \bar{E}_{i,j} - \frac{\tau}{\Delta x} u(\bar{E}_{i+1,j}^n - \bar{E}_{i,j}^n) - \frac{\tau}{\Delta y} \bar{v}_i(\bar{E}_{i,j+1}^n - \bar{E}_{i,j}^n)$$

This is a discrete approximation to

$$\frac{\partial}{\partial t} \bar{\rho} + u \frac{\partial}{\partial x} \bar{\rho} + \bar{v} \frac{\partial}{\partial y} \bar{\rho} = 0$$  \hspace{1cm} (64a)

$$\frac{\partial}{\partial t} (\bar{p}v) + u \frac{\partial}{\partial x} (\bar{p}v) + \bar{v} \frac{\partial}{\partial y} (\bar{p}v) = 0$$  \hspace{1cm} (64b)

$$\frac{\partial}{\partial t} \bar{E} + u \frac{\partial}{\partial x} \bar{E} + \bar{v} \frac{\partial}{\partial y} \bar{E} = 0$$  \hspace{1cm} (64c)

Rewriting the second equation as

$$\bar{p}[\frac{\partial}{\partial t} \bar{v} + u \frac{\partial}{\partial x} \bar{v}] + \bar{v}[\frac{\partial}{\partial t} \bar{\rho} + u \frac{\partial}{\partial x} \bar{\rho} + \bar{v} \frac{\partial}{\partial y} \bar{\rho}] = 0$$
we obtain that (64), is equivalent to (24) in this flow region. This holds also for $u < 0$, $v \geq 0$, $u \geq 0$, $v < 0$ and $u \geq 0$, $v \geq 0$. Therefore Godunov’s scheme is a discrete approximation to (24) in a regions with a constant pressure $p$, constant x-component of the velocity $u$ and a y-component of the velocity $v$, which does only depend on the spatial coordinate $x$.

For a stationary shock wave at $x_{1/2} = 0$ with a shock line perpendicular to the x-axis, the x-component of the flux function of Godunov’s scheme satisfies

$$f_{1/2,j}^G = f(\bar{u}_{1,j}^n) = f(\bar{u}_{0,j}^n)$$

i.e. one-dimensional stationary shocks are resolved exactly. Therefore Godunov’s method is a discrete approximation to (24) behind the shock with boundary conditions

$$\bar{\rho}(0, t) = \text{const.}$$
$$\bar{v}(0, t) = \text{const.}$$

where $u_r$ in (24) is the constant normal component of the velocity behind the shock. If the shock is near stationary these boundary conditions introduce perturbation. Thus on a sufficiently fine grid entropy perturbations of the form (26) and acoustic perturbations of the form (45) are resolved by Godunov’s method behind a near stationary shock. In the last section we saw that inviscid entropy perturbations do not relate to a real viscous flow. Flow structures not related to a real viscous, heat conducting flow may therefore appear in numerical solutions of Godunov’s method.

Numerical examples for a plane shock wave aligned with the grid, which is moving down a duct are given in [QUI 1994; Figure 5]. At the grid center line, a small perturbation is introduced in the computation. Downstream of the shock an unstable density profile develops, which over time leads to an unstable numerical shock front. If we associate the center of the duct with the x-axis, then the perturbation depend behind the shock front on $y$. This numerical example reflects the situation discussed analytically at the beginning of section 2.

Roe’s method [Roe1980] is more prone to generate these perturbations then Godunov’s method, due to the fact that a rarefaction wave is replaced by a rarefaction shock. Therefore noise from the rarefaction shock, can also lead to an unstable growth of the density, in addition to the noise from the shock.

An example from an aerodynamic simulation, which results in incorrect numerical results is given in [PEIM1988], for a bow shock over a blunt body placed in a high Mach number flow. Along the stagnation line the bow shock is approximately aligned with grid used for the calculation. A perturbation of the shock profile is
given through the curvature of the shock. At the stagnation point we have approximately a plane-wave near stationary shock wave, with a disturbed shock profile. This again is the situation discussed at the beginning of section 2.

Based on the numerical observation, that shock capturing methods which try to capture contact discontinuities exactly, generally suffer from failings, a link between the carbuncle phenomenon and the resolution of the contact discontinuities was suggested by Gressier and Moschetta [GRE1998].

The dissipation model for Godunov’s scheme was studied by Xu [XU1999] in a series of numerical experiments. He concluded that Godunov’s method gives accurate results in both unsteady shock structure and boundary layer calculations, but that the absence of dissipation in the gas evolution model in Godunov’s scheme amplifies post-shock oscillations. He found that the numerical dissipation model for Godunov’s method is in the multidimensional case mesh-oriented and not consistent with the Navier-Stokes equations. Xu also mentioned that it is well known that the inviscid Euler equations cannot give a correct representation of the fluid motion in the discontinuity flow region. Which is consistent with our analysis regarding a contact discontinuity line (vortex sheet).

Real weak shock fronts are transition layers of finite width and the representation of weak shock fronts through a discontinuity line in the Euler equations is a mathematical approximation. A justification for this approximation was given by Majda [MA1983; Proposition 3]. Based on a linear stability analysis Majda found that planar compressive shock fronts in an ideal gas are uniformly stable. In [LL1959; §87] it is furthermore noted that the front depth of a non weak shocks is so small, that a transition layer becomes meaningless. Also by no means obvious, it may be safely assumed, that the mathematical representation of a single smooth shock front in an ideal gas through a discontinuity line resp. surface is reasonable. Since Majda’s stability analysis does not apply to a contact discontinuity line/surface and we assumed in this paper that shock fronts only generate perturbations without assuming an unstable growth at the smooth shock front itself, this result is consistent with the analysis in this paper.

The source of small perturbations in Godunov’s methods is the displacement of the shock curves at the cell-boundary and the nonlinear interaction of the dependent variables in the numerical shock layer. A non stationary or stationary displaced shock wave is approximated through a smeared profile with at least one intermediate cell. Whenever the smeared shock profile changes, perturbations are generated from the characteristic fields. If the shock curve is not exactly a plane-wave in the x-direction, the shock-curvature will introduce an y dependence in the perturbations. In regions of low (numerical) viscosity, the instability under perturbation of the constant pressure region behind the shock will result in an unstable flow - if a contact discontinuity is present.
For the case reported by Perry and Inlay [PEIM1988] entropy perturbations are generated at the shock. Since according to the proposition in section 2, entropy perturbations can only exist if the tangential velocity does not depend on \( y \), these perturbations are initially restricted to the symmetry axis. If the flow is near stationary, cell-averaging of the form (58) in the projection state in Godunov’s method leads to density perturbation in these cells. At the boundary of these cells Godunov’s scheme then resolves a contact discontinuity, which results in an unstable flow according to the analysis in section 2 and 4. The unstable flow behind the shock interacts with the near stationary shock front resulting in carbuncle structures. Since the interaction destroys also the smoothness of the shock front itself, the stability analysis of Majda no longer applies.

For the case reported by Quirk [QUI 1994; Figure 5] perturbations are introduced externally at the grid center line. The grid center line corresponds to the symmetry line in results of [PEIM1988]. In [QUI 1994; Figure 5a] perturbations are first visible behind the plane shock, but no carbuncle is visible at this stage. Only after the shock has propagated further down the duct the carbuncle becomes visible in front of the shock.

If discontinuous density or velocity perturbation are artificially introduced in a background flow then according to the analysis in section 2, unstable structures can even be generated in numerical solutions of the isentropic Euler equations. This was demonstrated by Elling [VE2006] for Godunov’s method. In his case a steady plane shock line parallel to the \( y \)-axis is perturbed through a one-cell-high filament along the \( x \)-axis in front of the shock. In the filament the normal component of the velocity \( u = \bar{u} \) is set to zero. The flow in the filament is therefore a constant pressure flow governed by (32) with a constant normal velocity component \( v = \bar{v} \). From proposition 1 follows (with the tangential and normal velocity interchanged) that \( \bar{u} = \bar{u}(y, t) \). The linear stability analysis results again in an advection equation with a singular source term of the form

\[
\frac{\partial}{\partial t} \bar{u} + \bar{v} \frac{\partial}{\partial y} \bar{u} = -\bar{v}(\bar{u}_r - \bar{u}_l)\delta(y - \bar{v}t) \tag{66}
\]

This shows that the flow in the filament is unstable. Interactions of this unstable flow with the shock wave results in a carbuncle instability. Elling noted that the carbuncle instability can also be observed for the local Lax-Freidrich/Rusanov and Osher-Solomon schemes. His conjecture is that carbuncles can be related to a special class of non-physical entropy solutions for the continuum equations, which is supported by our analysis.

A discussion of the carbuncle instability for several numerical methods, can be found in [DMG;2004] and [Roe2007]. The focus in these papers is on the perturbations generated through a numerical shock profile. The cause for the carbuncle
instability is often related to the numerical shock profile itself and the instability also denoted as a shock instability. However, in this paper it is shown, that the instability may be related to the central constant pressure region in the plane wave Riemann problem. This is an intrinsic instability of the Euler equations and not related to a particular numerical method. The numerical shock profile manifests this instability in regions of low numerical dissipation through the generation of perturbations. This manifestation of the instability, is problem and method dependent.

5 Summary and Conclusion

In this paper it is proved that a constant pressure flow region, governed by the hyperbolic conservation laws for an ideal gas, is unstable under perturbations if a discontinuity is present. The instability is immanent to the linear degenerate fields in the multidimensional hyperbolic conservation laws, with a double eigenvalue. The mathematical idealized assumption of a zero thickness contact discontinuity in the plane wave Riemann problem is unrealistic for real flows. If infinitesimal viscosity and thermal conductivity are taken into account, inviscid entropy perturbations are excluded for an oscillatory tangential velocity in a perturbed plane wave Riemann problem. Since smooth entropy disturbances in a constant pressure flow are transported with the fluid, they cannot be excluded through the established entropy conditions for weak solutions of hyperbolic conservation laws. Additional conditions are required to guarantee that a solution of the plane wave Riemann problem is as a limit solution of the Navier-Stokes equations for a vanishing viscosity and thermal conductivity.

Immanent perturbations (acoustic and entropy) generated at a perturbed shock line and the numerical impossibility to differentiate exactly between admissible and non admissible entropy perturbations in a high Reynolds number (low viscous) flow, are challenges for discontinuity resolving methods. Godunov’s method closely relates to the physics of the Euler equations. Numerical artefacts observed in numerical solutions for Godunov’s method are a numerical manifestation of the immanent instability in the Euler equations. The HLLE scheme [EIN1988] on the other hand can be regarded as a vortex sheet averaged approximation to the Navier-Stokes equations for high Reynolds number flows.
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