Regularization of a sharp shock by the defocusing nonlinear Schrödinger equation

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Abstract
The defocusing nonlinear Schrödinger (NLS) equation is studied for a family of step-like initial data with piecewise constant amplitude and phase velocity with a single jump discontinuity at the origin. Riemann–Hilbert and steepest descent techniques are used to study the long-time/zero-dispersion limit of the solutions to NLS associated to this family of initial data. We show that the initial discontinuity is regularized in the long time/zero-dispersion limit by the emergence of five distinct regions in the $(x,t)$ half-plane. These are left, right, and central plane waves separated by a rarefaction wave on the left and a slowly modulated elliptic wave on the right. Rigorous derivations of the leading order asymptotic behavior and error bounds are presented.

Keywords: integrable systems, nonlinear Schrödinger equations, dispersive shock waves, Riemann–Hilbert problems, dispersive regularization, long-time asymptotics, nonlinear optics
Mathematics Subject Classification: 35Q15, 35Q55, 37K15

(Some figures may appear in colour only in the online journal)

1. Introduction

In this paper we study the defocusing nonlinear Schrödinger equation (NLS), given here with the normalization

\[ i\epsilon \psi_t + \frac{\epsilon^2}{2} \psi_{xx} - |\psi|^2 \psi = 0, \quad (1.1) \]

for a fixed class of piecewise constant, steplike, initial data (see (1.5)). The NLS equation is a canonical model of dispersive wave dynamics, and has been shown to be an excellent model
for a wide variety of disparate physical systems, including water waves [34]; plasmas [39, 47]; nonlinear optics [1]; and Bose–Einstein condensates [26]. The case in which the dispersion parameter \( \epsilon \ll 1 \) is of particular interest as it is the natural scaling in both BECs and nonlinear optics [26, 32]. The NLS equation is also of intrinsic mathematical interest as one of the principal examples of a completely integrable nonlinear evolution equation.

The zero dispersion limit, i.e. \( \epsilon \to 0 \), of the NLS equation (1.1) is better understood by introducing the Madelung variables [37],

\[
\rho(x, t) = |\psi(x, t)|^2, \quad u(x, t) = \epsilon \text{ Im} \left[ \partial_t \log (\psi(x, t)) \right],
\]

which transform the NLS equation into the system of conservation laws

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) &= 0, \quad (1.3a) \\
\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x} \left( \rho u^2 + \frac{1}{2} \rho^2 \right) &= \frac{\epsilon^2}{4} \frac{\partial}{\partial x} \left( \rho \frac{\partial}{\partial x} \log \rho \right). \quad (1.3b)
\end{align*}
\]

When \( \epsilon = 0 \) these are the Euler equations for an ideal compressible fluid (gas) with local fluid density \( \rho \), velocity \( u \), and positive pressure \( P = \frac{1}{2} \rho^2 \). It is well known that the Euler system admits solutions which develop gradient catastrophes (infinite derivatives) in finite time. However, for \( \epsilon > 0 \), as the wave steepens the right hand side of the momentum conservation law (1.3b) cannot be treated as a perturbative term and shock formation is avoided by the emergence of expanding regions of (i) rarefaction waves and/or (ii) the onset of slowly modulating \( O(\epsilon) \) wavelength oscillations with \( O(1) \) amplitude known as dispersive (sometimes collisionless) shock waves (DSWs). Clearly, when DSWs emerge, a zero dispersion limit cannot exist in the classical sense. Nevertheless, a weak limit does exist for NLS as was shown in [30] following the work of [36, 43] on the Korteweg de Vries (KdV) equation. This weak limit can be understood in terms of the unique minimizer of a certain minimization problem with constraints. The minimizer itself is characterized by its support, which typically is a union of disjoint intervals. The endpoints of these intervals satisfy the system of quasilinear hyperbolic equations

\[
\frac{\partial \lambda_j}{\partial t} + v_j(\lambda) \frac{\partial \lambda_j}{\partial x} = 0, \quad j = 1, 2, \ldots, 2G + 2, \quad (1.4)
\]

where \( \lambda \in \mathbb{R}^{2G+2} \) and \( \lambda_j > \lambda_k \) for \( j < k \). This system is often called the Whitham equations after their first discoverer [44].

The dynamics of the DSWs themselves can be described as slowly modulating single or multiphase waves, whose modulations are also governed by the Whitham equations [23, 24]. The modulation theory was worked out for \( G = 1 \) in [45] and for \( G \geq 2 \) in [20] for KdV. The Whitham modulation theory for NLS was worked out in [21]. In the years following, Whitham theory has been used in the optics and fluid dynamic communities to investigate increasingly complicated structures: the initial data problem for piecewise constant data (the type considered in this paper) [3, 32]; the interaction of two DSWs [25]; and in [18] a classification of the types of solutions of the Whitham-NLS system for initial data with a discontinuity of the form (1.5) was given. This is but a few examples.

At the same time, the development of the inverse scattering technique for studying integrable nonlinear evolution equations has resulted in a huge amount of work on the NLS equation. In particular, the nonlinear steepest descent method of Deift and Zhou [13, 14] allows one to make completely rigorous arguments to obtain, in principle, full asymptotic expansions of the solutions of integrable systems in various asymptotic limits. The bulk of the work being done in the integrable systems community has focused on initial data \( \psi_0(x) \)
which decays to zero sufficiently fast as $|x| \to \infty$, [2, 9, 12, 16, 29, 31, 40]. Comparatively, much less time has been devoted to families of non-vanishing initial data. The family of finite density initial data $\psi_0(x)$ satisfying $\psi_0(x) \to g e^{i\varphi}$ as $x \to \pm \infty$ for constants $g > 0$ and $\varphi \in [0, 2\pi)$ is probably the best understood of these non-vanishing families. As was shown in [4, 10, 15, 19], the scattering theory for non-vanishing data must be constructed on multi-sheet Riemann surfaces, a complication which is not necessary for vanishing data. Results for long time asymptotics for (1.1) with finite density data were worked out first by Its et al in [27, 28] and recently Vartanian [41, 42] has found very detailed asymptotic formulae for the long time asymptotic behavior of finite density data with and without (dark) solitons. Another family of nonvanishing data are ‘step-like’ initial data which asymptotically approaches different plane wave states as $x$ approaches either infinity, both in the context of the NLS equation [4, 6, 7, 46] and other important integrable evolution equations [5, 17, 33].

In this paper it is our goal to make a completely rigorous study of the long-time/zero-dispersion behavior of the solutions of the NLS equation (1.1) for the family of sharp step initial data

$$\psi(x, t = 0) = \psi_0(x) := \begin{cases} 
1 & x < 0 \\
A \exp(-2i\mu x/\epsilon) & x > 0.
\end{cases} \quad (1.5)$$

for real constants $A > 0$ and $\mu$ using the machinery of inverse scattering and nonlinear steepest descent. We note that as the data (1.5) is scale invariant, the long-time and zero-dispersion limits are equivalent limits in this special case.

The hyperbolic nature of the Whitham modulation equations for the defocusing NLS equation suggests that for large $x$ the solution for initial data (1.5) should resemble a plane wave (zero phase oscillation) with Riemann invariants $\lambda_{\pm} = \mu \pm A$ (see section 2.1) whose values as $x \to -\infty$ approach $\pm 1$ and approach

$$\lambda_{\pm} = \mu \pm A \quad (1.6)$$

as $x \to +\infty$. In [18], using Whitham theory, the authors enumerate six possible long-time behaviors for the data (1.5) depending upon the relative ordering of these constants $\{-1, 1, \lambda_-, \lambda_+\}$. In each case the discontinuity is regularized by the emergence of two zones in which either DSWs or fan-like rarefactions connect three constant states, see figure 1.
Our results, which follow below, provide a completely rigorous proof that the leading order asymptotic behavior of the density ρ and velocity u are as predicted by the Whitham theory, and give bounds on the error. Moreover, our methods provide a superior description of the solution ψ(x, t) as we are able to compute the leading order phase of the solution ψ(x, t).

This includes terms which are lost in the Whitham averaging process, but nonetheless make O(1) contributions to the solution ψ of (1.1). Our paper provides all the tools necessary to easily deal with all six cases identified in [18]. However, for the sake of brevity, we will provide full details for only one case: −1 < λ− < λ+ < 1 (case i. in [18]), in which both DSWs and rarefaction waves emerge, see figure 1.

In order to compute the phase of the solution we need the reflection coefficient which is part of the scattering data computed in the inverse scattering procedure. For the initial data (1.5), and λ± defined by (1.6), the reflection coefficient generated from (1.5) is

\[ r(z) = \frac{1 - h(z)}{1 + h(z)}, \quad h(z) = \frac{\sqrt{z - 1}}{\sqrt{z - \lambda_-}} - \frac{\sqrt{z - \lambda_+}}{\sqrt{z + 1}}, \]

where each of the roots is principally branched. When (λ−, λ+) < (−1, 1), which is the setting or our result, it is easy to check that \( r(z) \) is cut on \((−1, 1)\) except at \( z = 0 \), with unit modulus on either side of the cut, i.e. \( |r(z + i0)| = 1 \) for \( z \in (−1, 1) \backslash (λ−, λ+) \), and \( r(z) \sim z^{-1} \) as \( z \to \infty \).

In theorem 1.1, \( r_+(z) \) refers to the boundary value of \( r \) along the cut from the upper half-plane: \( r_+(z) = \lim_{y \to 0^+} r(z + iy), \ z \in (−1, 1) \backslash (λ−, λ+) \).

**Theorem 1.1.** Given initial data (1.5), if the Riemann invariants \( \lambda_{\pm} = \mu \pm A \) satisfy −1 < λ− < λ+ < 1, then the long-time/small-dispersion asymptotic behavior of the solution \( ψ(x, t) \) of the NLS equation (1.1) is given by one of the five following formulae depending on the value of the similarity variable \( τ = x/t \) relative to the transition speeds \( τ_j \) identified as:

1. \( τ_1 = −1 \)
2. \( τ_2 = −\frac{1}{2} (−1 + 3λ+) \)
3. \( τ_3 = −\frac{1}{2} (−1 + 2λ+ + λ_) \)
4. \( τ_4 = −\frac{1}{2} (λ+ + λ_− − 2) + \frac{2(1+λ_−)(1+λ+)}{λ+ + λ_− + 2} \)

1. For \( τ < τ_1 \) the leading order behavior of the solution is given by a plane wave which, up to the phase \( e^{−iφ(x,t)} \), is the time-evolution of the left half of the initial data:

\[ ψ(x, t) = e^{−iφ/}\sqrt{t} + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \]

\[ φ(τ) = \frac{1}{π} \int_{−∞}^{1} \frac{ξ_+(τ)}{1 + \sqrt{z^2 − 1}} \frac{log(1 − |r(z)|^2)}{\sqrt{z^2 − 1}} dz + \frac{1}{π} \int_{−\infty}^{1} \frac{log(1 − |r(z)|^2)}{\sqrt{1 − z^2}} dz \]

\[ ξ_+(τ) = \frac{1}{4} \left[ \sqrt{τ^2 + 8} − τ \right] \]

2. For \( τ_1 < τ < τ_2 \), the solution is described by the rarefaction

\[ ψ(x, t) = \left(\frac{2τ − x}{3τ}\right) e^{−i\phi(x,t)} + \mathcal{O}\left(\frac{1}{τ}\right) \]

\[ φ(τ) = \frac{1}{π} \int_{−∞}^{-1} \frac{log(1 − |r(z)|^2)}{\sqrt{(z + 1)(z − λ_−(τ))}} dz + \frac{1}{π} \int_{−1}^{λ_−(τ)} \frac{log(1 − |r(z)|^2)}{\sqrt{(z + 1)(z − λ_−(τ))}} dz \]

\[ λ_−(τ) = \frac{(1 − 2τ)}{3} \]
3. For $\tau_2 < \tau < \tau_3$ the solution is asymptotically described by the (unmodulated) plane wave

$$\psi(x, t) = \sqrt{\rho} e^{i(\omega x - \epsilon t)} e^{-i\phi_0} + O\left(e^{-\epsilon t}\right)$$

$$\rho = \left(\frac{\lambda_s + 1}{2}\right)^2 \quad k = -\left(\lambda_s - 1\right) \quad \omega = \frac{1}{2} k^2 + \rho$$

$$\phi_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\log(1 - |z|^2)}{\sqrt{(z + 1)(z - \lambda_s)}} \, dz + \int_{-1}^{\lambda_s} \frac{\arg(r_\epsilon(z))}{\sqrt{V_x - \lambda_s - \lambda_s(z + 1)}} \, dz.$$  (1.9)

Note that, unlike the other four intervals, here the error bound is exponentially, not algebraically, small in $\epsilon/t$.

4. For $\tau_3 < \tau < \tau_4$ the asymptotic behavior of the solution is described by a slowly modulated one-phase (elliptic) wave, a dispersive shock wave,

$$\psi(x, t) = \sqrt{\rho(x, t)} e^{iS(x, t)} + O\left(\frac{\epsilon}{t}\right),$$

whose amplitude and phase are given by

$$\rho(x, t) = a^2_1 = (a^2_1 - a^2_2) \text{dn}^2\left(\sqrt{\frac{x - Vt}{\epsilon}} + \phi\left(\frac{x}{t}\right)\right) - K(m, m)$$

$$S(x, t) = \frac{V_x - (a^2_1 + a^2_2 + a^2_3 - V^2)t + (x - 2Vt)\eta}{\epsilon} + 2(V + \eta\phi)\left(\frac{x}{t}\right) + \arg\left\{\theta_1\left[\frac{\pi}{2K(m)}\text{dn}^2\left(\frac{x - Vt}{\epsilon} + \phi\left(\frac{x}{t}\right)\right) - \frac{1}{i\pi} F(\psi, 1 - m)\right]\right\}.$$  (1.10)

$$\eta = \lambda_1 - (\lambda_1 - \lambda_4)Z\left(-\frac{\lambda_3 - \lambda_4}{\lambda_1 - \lambda_4}, n\right),$$

$$n = -\left(\frac{\lambda_3 - \lambda_4}{\lambda_1 - \lambda_4}\right).$$  (1.11)

5. For $\tau > \tau_4$ the leading order behavior of the solution is given by a plane wave which, up to the phase $e^{-i\phi(x/t)}$, is the time-evolution of the right half of the initial data:

$$\psi(x, t) = A e^{-i2\mu x + (A^2 + 2\epsilon^2) t/4} e^{-i\phi(x/t)} + O\left(\frac{\epsilon}{t}\right)$$

$$\phi(\tau) = \frac{1}{\pi} \int_{-\infty}^{\tau} \frac{\log(1 - |z|^2)}{\sqrt{z - \lambda_+}} (z - \lambda_-) \, dz,$$

$$\xi(-\tau) = \frac{2\mu - \tau}{4} - \frac{\sqrt{(2\mu + \tau)^2 + 8A^2}}{4}.$$  (1.12)
Figure 2. The self-similar evolution of the Riemann invariants $\lambda_i(\tau)$ with respect to the similarity variable $\tau = x/t$. In the figure the constants are $A = 0.5$ and $\mu = 0.1$ (i.e. the Riemann invariants of the right side $\lambda_\pm = -u/2 \pm \sqrt{\rho}$ are $-0.4$ and $0.6$ respectively, which lie between the left invariants $\pm 1$).

Figure 3. Left: the leading order asymptotic behavior of the density $\rho = |\psi|^2$ and right: the leading order asymptotic behavior of the velocity $u = \text{Im} \frac{\psi_x}{\psi}$ related to the hydrodynamic interpretation (1.3) of the solution $\psi$ of NLS (1.1) in the small-dispersion/long-time limit for the initial data (1.5). The parameters used to generate the figures are the same used in figure 2 with $\epsilon = 0.001$. The initial discontinuity smooths itself by the emergence of a rarefaction zone on the left and a modulated elliptic wave front (a DSW) on the right connected by a constant central plateau.

Remark 1. The convergence of the solution $\psi(x, t)$ as $\epsilon \to 0$ to the given leading order formulae is uniform in any sector $(x, t) \in \mathbb{R} \times (T, \infty)$ with $\frac{x}{t} \notin [a, b]$ which avoids the transition speeds, i.e. $\tau_j \notin [a, b]$, $j = 1, 2, 3, 4$. Moreover, though perhaps not immediately obvious from the formulae, the leading order behavior is continuous across each of the four transitions as can be checked by hand, or as seen in figure 3.

Remark 2. The leading order hydrodynamic density $\rho = |\psi(x, t)|^2$ and velocity $u(x, t) = \epsilon \text{Im} \frac{\partial_t \log \psi(x, t)}{2}$ computed from the formulae in theorem 1.1 agree with the results predicted by Whitham theory techniques in [18]. The new contribution of this paper is the computation of the complex phase of $\psi(x, t)$ and the explicit bounds on the error. Though we do not pursue it here, one can use our results to compute the correction terms explicitly yielding a full asymptotic expansion of the solution in each of the five cases cases outlined above.

Remark 3. The slowly evolving phase term $\phi$ in each of the five formulae is new and does not appear in the Whitham theory as it constitutes a perturbative term in the computation of the velocity $u$ but nonetheless contributes an $O(1)$ correction to the complex phase of the solution $\psi(x, t)$. In the inverse analysis, it emerges from the quantity $D(\infty)$ appearing in (5.6)–(5.7) and describes the slow effect of the reflection coefficient on the phase of the solution.
Remark 4. Though we consider only the case $-1 < \lambda_- < \lambda_+ < 1$ the other five possible cases (i.e. orderings of $-1, 1, \lambda_-, \lambda_+$) regularize the initial discontinuity in a similar way, and we provide all the necessary tools to complete these computations. In each case, five sectors emerge in the $(x, t)$ half-plane as in figure 1: the far left and right fields exhibit plane wave (genus zero) oscillations which match the initial data for $t = 0$, while the three middle zones consist of either rarefaction and/or dispersive shock waves separated by a central plateau that is either a plane wave or, when $1 < \lambda_- < \lambda_+$, a standing (unmodulated) elliptic wave.

Remark 5. The choice to normalize the left half of the initial data (1.5) to have $\rho = 1$ and $\mu = 0$ is not a restriction, any sharp step of the form

$$\widetilde{\psi}(x, 0) = \widetilde{\psi}_0(x) := \begin{cases} A_L \exp \left(-2i\mu_L x/\epsilon\right) & x < 0 \\ A_R \exp \left(-2i\mu_R x/\epsilon\right) & x > 0. \end{cases}$$

with $A_L$ and $A_R$ not both zero can be reduce to our normalized data; in the case that $A_L \neq 0$, the change of variables

$$\widetilde{\psi}(x, t) = A_L \psi(AL(x - 2\mu_L t), A^2_L t)e^{-2i\mu_L(x + \mu t)/\epsilon}$$

results in a new unknown $\psi(x', t')$ solving (1.1) with initial data (1.5) in the new coordinate frame.

1.1. Organization of the rest of the paper

In section 2 we briefly review the NLS-Whitham equations for zero and one phase waves and discuss their self-similar solutions. In section 3 we discuss the integrable structure of the NLS equation, compute the scattering data for the step initial data (1.5), and state the Riemann–Hilbert problem satisfied by the solution of (1.1)–(1.5) in full detail. In section 4 we construct the so called $g$-functions that are needed in the inverse scattering analysis and show that their evolution is governed by the NLS-Whitham equations. Finally in section 5 we use the Deift–Zhou steepest descent procedure to derive the asymptotic behavior to the solution of Riemann–Hilbert problem 3.1 for every real value of $\tau = x/t$ not equal to one of the four transition speeds $\tau_j$, $j = 1, \ldots, 4$, which proves the results of theorem 1.1. The method presented here could be extended to derive asymptotic formulae valid in neighborhoods of the transition times, but we do not pursue this here.

Before proceeding we comment on notation. Throughout the paper we make use of the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. $$

In particular we use the matrix power notation $f^{\sigma_3} = \begin{pmatrix} f & 0 \\ 0 & f^{-1} \end{pmatrix}$ for any scalar $f$.

Regarding complex variable notation, $z^*$ denotes the complex conjugate of a complex number $z$; for a scalar function $f$, $f^*(z)$, or just $f^*$, denotes the Schwarz reflection through the real axis $f^*(z) = f(z^*)^*$. Given a piecewise smooth oriented contour $\gamma \subset \mathbb{C}$ and a function $f$ analytic in $\mathbb{C} \setminus \gamma$, for $z \in \gamma$, $f_{\pm}(z)$ is defined as the non-tangential limit of $f(w)$ as $w$ approaches $z$ from the left/right with respect to the orientation of $\gamma$. Finally, given a pair of real numbers $a, b$ or a vector $\lambda \in \mathbb{R}^{2G+2}$ with $G = 1, 2, \ldots$ we define

$$\mathcal{R}(z; a, b) = \sqrt{(z-a)(z-b)} \quad \mathcal{R}(z; \lambda) = \sqrt{\prod_{j=1}^{2G+2} (z - \lambda_j)} \quad (1.13)$$

to be finitely cut along the real axis such that $\mathcal{R}(z; a, b) \sim z$ and $\mathcal{R}(z; \lambda) \sim z^{G+1}$ as $z \to \infty$. 

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2. Hydrodynamic form and modulation theory

The Madelung change of variables (1.2) transforms the NLS equation into the system of conservation laws (1.3). If $\epsilon$ is formally set to zero in (1.3b) the resulting Euler system exhibits shock formation (infinite gradients) in finite time. For $\epsilon > 0$ the right hand side of (1.3b) ameliorates the formation of shocks by introducing growing regions of rapid oscillations into the solution. These rapid oscillations are well approximated by slowly modulating one-phase waves, whose modulations satisfy Whitham’s averaging equations [20, 21, 45]. For general initial data the number of phases needed to describe the wave may change as Riemann invariants are born or merge over the course of the evolution. For long times there is some evidence that the system may exhibit a simpler structure [22]. For the single shock initial data we consider here (1.5), we will see that only elliptic (one phase) oscillations develop. We summarize below the Whitham equations for zero and one phase oscillations only.

2.1. Zero-phase oscillations

Before wave breaking occurs the solution of (1.1) has bounded derivatives, and the limiting Euler equations for $\rho$ and $u$ approximate the solution well. That is, our solution is well described by the slowly modulating periodic wave

$$\psi_0(x, t) = \sqrt{\omega_0 - k_0^2} e^{i\theta_0}, \quad \partial_x \theta_0 = k_0/\epsilon, \quad \partial_t \theta_0 = -\omega_0/\epsilon,$$

(2.1)

whose density and velocity

$$\rho(x, t) = |\omega_0 - k_0^2|, \quad u(x, t) = k_0,$$

(2.2)

satisfy the Euler equations ((1.3) with $\epsilon = 0$). The Euler equations can be written in the Riemann invariant form (1.4), with $G = 0$ and

$$\lambda_1 = -\frac{u}{2} + \sqrt{\rho}, \quad \lambda_2 = -\frac{u}{2} - \sqrt{\rho},$$

(2.3)

$$v_1(\lambda) = -\frac{1}{2} (3\lambda_1 + \lambda_2), \quad v_2(\lambda) = -\frac{1}{2} (\lambda_1 + 3\lambda_2).$$

2.2. One-phase oscillations

If we suppose that the solution exhibits a single fast phase, then the density $\rho$ and velocity $u$ are instead described asymptotically in terms of modulating one-phase (elliptic) waves parameterized by four slowly varying Riemann invariants $\lambda = [\lambda_i]_{i=1}^4, \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$:

$$\rho(x, t; \lambda) = a_1^2 - (a_2^2 - a_3^2) \cdot \text{dn}^2 \left( \frac{\sqrt{a_1^2 - a_3^2}x - Vt}{\epsilon} \cdot m \right)$$

$$u(x, t; \lambda) = V - \frac{a_1a_2a_3}{\rho(x, t; \lambda)},$$

(2.4)

$$a_1 = -\frac{1}{2} (\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4),$$

$$a_2 = -\frac{1}{2} (\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4),$$

$$a_3 = -\frac{1}{2} (\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4),$$

(2.5)

$$V = -\frac{1}{2} (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4),$$

$$m = \frac{\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}.$$
The evolution of the Riemann invariants $\lambda_j$ is governed by (1.4) with $G = 1$ and:

$$v_j(\lambda) = V(\lambda) + \left(2 \frac{\partial}{\partial \lambda_j} \log L(\lambda)\right)^{-1},$$

$$L(\lambda) = \sqrt{2} \int_{\lambda_2}^{\lambda_1} \frac{dr}{\sqrt{-\prod_{j=1}^{4}(\tau - \lambda_j)}} = \frac{2\sqrt{2}K(m)}{\sqrt{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_4)}},$$

which can be obtained by averaging the first four conservation laws for NLS over a period of (2.4).

2.3. Self-similar evolution

If the Riemann invariants $\lambda_j$ depend on $(x, t)$ only through a similarity variable $\tau = x/t$, then the Whitham equations (1.4) are equivalent to

$$\left(v_j(\lambda) - \tau\right) \partial \lambda_j \tau = 0, \quad j = 1, 2, \ldots, 2G + 2. \quad (2.7)$$

So each $\lambda_j$ is either constant or its speed satisfies $v_j(\lambda) = \tau$. Moreover, since the NLS-Whitham system is strictly hyperbolic [3], i.e. $v_j(\lambda) < v_k(\lambda)$ for $j < k$ provided $\lambda_1 > \lambda_2 > \ldots > \lambda_{2G+2}$, it follows that at most one of the speeds can satisfy $v_j(\lambda) = \tau$, and therefore in a self-similar evolution at most one of the Riemann invariants is not constant.

3. Scattering of the shock initial data

It is well known that the NLS equation is completely integrable [48] in the sense that it is equivalent to the existence of simultaneous solution $\psi(x, t)$ of the Lax pair

$$\epsilon \Phi_x = -iz\sigma_3 \Phi + \Psi(x, t) \Phi, \quad (3.1a)$$

$$i\epsilon \Phi_t = z^2 \sigma_3 \Phi + iz \Psi(x, t) \Phi + \frac{1}{2} (\Psi(x, t)^2 + \epsilon \Psi_x(x, t)) \sigma_3 \Phi. \quad (3.1b)$$

Here $\Psi(x, t)$ is the matrix potential

$$\Psi(x, t) = \begin{pmatrix} 0 & \psi(x, t) \\ -\psi^*(x, t) & 0 \end{pmatrix}.$$

If we consider a plane wave solution of (1.1), $\psi^p(x, t) = A e^{i(kx - \omega t)/\epsilon}$, $\omega = \omega(k) = A^2 + k^2/2$, then an exact simultaneous solution of the Lax pair (3.1) is given by

$$\Phi^p(x, t) = e^{-\frac{i}{2} (kx - \omega t)/\epsilon} e(z; -k/2 - A, -k/2 + A) e^{\frac{i}{2} \Lambda(z; -k/2 - A, -k/2 + A)} e^{\frac{1}{2} \beta(z; A, B)}, \quad (3.2)$$

where

$$\Lambda(z; A, B) := \sqrt{z - A} \sqrt{z - B}, \quad \beta(z; A, B) := \left(\frac{z - B}{z - A}\right)^{1/4}, \quad (3.3)$$

and $\Lambda(z; A, B)$ and $\beta(z; A, B)$ are defined to be cut along $[A, B]$ and normalized such that as $z \to \infty$:

$$\Lambda(z; A, B) = z + O(1), \quad \beta(z; A, B) = 1 + O(z^{-1}).$$

For initial data $\psi_0(x)$ which is asymptotic to a plane wave for large $x$, i.e. $\psi_0(x) \sim \psi^p(x)$ as $x \to \pm \infty$, it is reasonable to define the Jost function solutions of (3.1a) to be those whose
asymptotic behavior is given by $\Phi^p$. For our particular family of initial data (1.5) this implies that our left and right normalized Jost functions satisfy
\[
\lim_{x \to -\infty} \Phi_L(x, t) e^{i\Lambda(z, -1) x \sigma_3} = E(z; 1, -1) \\
\lim_{x \to +\infty} e^{i\mu x \sigma_3} \Phi_R(x, t) e^{i\Lambda(z, \lambda+) x \sigma_3} = E(z; \lambda-, \lambda+)
\] (3.4)

For brevity we will use the shorthands
\[
\beta_L(z) := \beta(z, -1, 1), \quad \Lambda_L(z) := \Lambda(z, -1, 1), \quad E_L(z) := E(z, -1, 1),
\]
\[
\beta_R(z) := \beta(z, \lambda-, \lambda+), \quad \Lambda_R(z) := \Lambda(z, \lambda-, \lambda+), \quad E_R(z) := E(z, -\lambda-, \lambda+),
\]
and we denote the intervals where these functions are cut by:
\[
I_L = (-1, 1), \quad I_R = (\lambda-, \lambda+).
\]

Note that these branch points are exactly the Riemann invariants for the (constant) plane wave solutions (2.3) corresponding to each half of the initial data (1.5).

### 3.1. Forward scattering of our pure shock initial data

For general step-like initial data one can prove existence and analytic extension (in $z$) theorems for the Jost functions [4, 19]. However, for the initial data given by (1.5) the Jost functions are explicit:
\[
\Phi_L(x; z) = \begin{cases}
E_L(z) e^{-i\Lambda_L(z) x \sigma_3 / \epsilon} & x < 0 \\
E_R(z) e^{-i\Lambda_R(z) x \sigma_3 / \epsilon} & x > 0
\end{cases}
\]
\[
\Phi_R(x; z) = \begin{cases}
E_L(z) e^{-i\Lambda_L(z) x \sigma_3 / \epsilon} E_R(z) & x < 0 \\
E_R(z) e^{-i\Lambda_R(z) x \sigma_3 / \epsilon} E_L(z) & x > 0
\end{cases}
\] (3.5)

and we may proceed by exact calculation.

**Proposition 3.1.** For $k \in \{L, R\}$, let $\Phi_k(z; z)$ be defined by (3.5). The following properties are easily verified:

1. $\det \Phi_k = 1$.
2. $\Phi_k(x; z)$ is analytic for $z \in \mathbb{C} \setminus I_k$.
3. $e^{i\mu x \sigma_3 / \epsilon} \Phi_k(x; z) e^{i\Lambda_k(z) x \sigma_3 / \epsilon} = I + O(z^{-1})$ as $z \to \infty$.
4. For $z \in I_k$, $I_k$ oriented left-to-right, $\Phi_k(z)$ takes continuous boundary values satisfying
\[
\Phi_k(-z)^{-1} \Phi_k(z) = E_k(z)^{-1} E_k(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
z $\in I_k$.
5. $(z - p)^{1/4} \Phi_k(x; z)$ is bounded as $z \to p$ where $p$ is either endpoint of $I_k$.

Using the Jost functions, we define the scattering matrix
\[
S(z) := \Phi_R^{-1} \Phi_L(z) = E_R^{-1} (z) E_L(z) = \begin{pmatrix} a(z) & b^*(z) \\ b(z) & a^*(z) \end{pmatrix},
\] (3.6)

where the scattering functions and reflection coefficient are given by
\[
a(z) = \frac{\beta_L(z) \beta_R(z)^{-1} + \beta_L(z)^{-1} \beta_R(z)}{2},
\]
\[
b(z) = \frac{\beta_L(z) \beta_R(z)^{-1} - \beta_L(z)^{-1} \beta_R(z)}{2i}
\]
\[
r(z) = \frac{b(z)}{a(z)} = -i \frac{\beta_L(z)^2 - \beta_R(z)^2}{\beta_L(z)^2 + \beta_R(z)^2}.
\] (3.7)
By direct calculation, or as a consequence of (3.5), (3.6) and proposition 3.1, we see that the scattering functions are analytic in $C \setminus (I_L \triangle I_R)$ and satisfy the jump relations
\begin{align*}
a_+(z) &= b^+_+(z), \quad b_+(z) = a^+_+(z) \quad z \in I_L \setminus (I_L \cap I_R), \\
a_-(z) &= -b^-_+(z), \quad b_-(z) = -a^-_+(z) \quad z \in I_R \setminus (I_L \cap I_R).
\end{align*}
(3.8)
It follows that $r(z)$ is also analytic for $z \in C \setminus (I_L \triangle I_R)$ and
\begin{equation}
\frac{1}{r^+_+(z)} = z \in I_L \setminus I_R.
\end{equation}
(3.9)
Furthermore, from (3.7) it is easy to verify that
\begin{equation}
|\arg a(z)| < \pi.
\end{equation}
(3.7)

**Proposition 3.2.** The function $a(z)$ defined by (3.7) has no zeros in the complex plane.

**Proof.** The mapping $\beta_L(z) = \beta(z; -1, 1)$ is a conformal map of $C \setminus [-1, 1] \to \mathcal{U}$, where $\mathcal{U} = \{w \in C \setminus \{0\} : |\arg w| < \frac{\pi}{2}\}$, such that $\beta_L(C^\pm) = \mathcal{U} \cap C^\pm$. Since $\beta_R(z) = \beta_L(\frac{z-a}{b-a})$ (recall that $\lambda_\pm = \mu \pm A$), it is also a conformal mapping into $\mathcal{U}$. It follows that $\text{Re} \frac{\beta_L(z)}{\beta_R(z)} > 0$ and thus $\text{Re} a(z) > 0$ for all $z \in C$.

One consequence of (3.9)–(3.10) is that the transmission coefficient $1/a(z)$ does have zeros on the real axis. Indeed the squared transmission coefficient
\begin{equation}
\frac{1}{a(z)a^+(z)} = 1 - r(z)r^+(z) = \frac{4\beta_L(z)^2 \beta_R(z)^2}{(\beta_L(z)^2 + \beta_R(z)^2)^2}
\end{equation}
(3.11)
is analytic for $z \in C \setminus (I_L \triangle I_R)$ and vanishes as a square root at each of the four branch points. It has no other zeros or poles.

Using the time dependent Jost functions we construct the piecewise analytic function
\begin{align*}
m(z; x, t) &:= \begin{cases}
\phi_{R}^+(x, t; z) & \quad z \in C^+ \\
\phi_{L}^+(x, t; z) & \quad z \in C^-.
\end{cases}
\end{align*}
(3.12)
The function $m(z; x, t)$ satisfies the following Riemann-Hilbert problem:

**Riemann-Hilbert Problem 3.1.** for $m(z; x, t)$ Find a $2 \times 2$ function $m(z; x, t)$ with each of the following properties:

1. $m(z; x, t)$ is analytic in $C \setminus \mathbb{R}$, where $\mathbb{R}$ is oriented left-to-right.
2. $m(z; x, t) = I + O(1)$ as $z \to \infty$.
3. For $z \in \mathbb{R}$, $m$ satisfies the jump relation $m_+(z; x, t) = m_-(z; x, t)v(z; x, t)$ where
\begin{equation}
v(z, x, t) = \begin{cases}
1 - r e^{2ib/e} & \quad z \in \mathbb{R} \setminus (I_L \cup I_R) \\
r e^{2ib/e} & \quad z \in I_L \setminus (I_L \cap I_R) \\
(a+)^{-1} e^{2ib/e} & \quad z \in I_R \setminus (I_L \cap I_R) \\
e^{-2ib/e} & \quad z \in I_L \cap I_R
\end{cases}
\end{equation}
(3.13)

$A \triangle B$ denotes the symmetric difference of $A$ and $B$. 

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where

\[ r = r(z) \quad \text{and} \quad \theta = \theta(x, t, z) := xz + tz^2. \]

4. \( m(z; x, t) \) is bounded at each finite \( z \) except the points \( p, \ p \in [\lambda_-, \lambda_+] \) where it admits the singular behavior

\[
m(z; x, t) = \begin{cases} (z - p)^{1/4} & \text{if } z \in \mathbb{C}^+, \ p \in [\lambda_-, \lambda_+] \setminus J \setminus \{(p, 0)\}, \\ (z - p)^{-1/4} & \text{if } z \in \mathbb{C}^-, \ p \in [\lambda_-, \lambda_+]. \end{cases}
\]

\[ (3.14) \]

Let \( m_{12}(z; x, t) \) denote the \((1, 2)\)-entry of the matrix \( m(z; x, t) \). If a solution of the above Riemann Hilbert problem exists, then the solution of (1.1) is given by

\[ \psi(x, t) := -2i \lim_{z \to \infty} z m_{12}(z; x, t) \]

\[ (3.15) \]

4. Constructing the \( g \)-functions of self-similar wave motion

One of the essential tools in the steepest descent analysis of Riemann–Hilbert problems is the construction of what is known as a \( g \)-function, whose role is to renormalize oscillatory or exponentially large factors in the jump matrices. As in the KdV setting [43], this function can be characterized as the log transform of the minimizing measure of a certain minimization problem. For a large class of initial data this minimizer is supported on a finite union of disjoint intervals, and the deformation of the endpoints of these intervals as \((x, t)\) vary are governed by the Whitham equations for NLS. Here we construct the genus-0 and genus-1 \( g \)-functions admitted by self-similar motion following the method of [22]. The method clearly generalizes to higher genus.

Suppose that we are given a set of \( 2G + 2 \) real points \( \lambda_1, \lambda_2, \ldots, \lambda_{2G+2} \) ordered such that \( \lambda_1 > \lambda_2 > \ldots > \lambda_{2G+2} \). Label the intervals \( J_k = (\lambda_{2k+2}, \lambda_{2k+1}], \ k = 0, \ldots, G \) and \( J = \bigcup_{k=0}^{G-1} J_k \). We call the intervals \( J_k \) the ‘bands’. The intervals \( J_k = (\lambda_{2k+1}, \lambda_{2k}), \ k = 0, \ldots, G - 1 \) and \( \hat{J} = \bigcup_{k=1}^{G-1} \hat{J}_k \) we call the ‘gaps’. Finding the \( g \)-function, the log transform of the minimizer of the minimization problem, is equivalent to showing that there exist constants (possibly depending on \( \tau = x/t \)) \( \alpha_0, \ldots, \alpha_G \) and \( g_{\infty} \); and a scalar function \( g(z) \) with the following properties:

**Remark 6.** The growth condition at endpoints is often omitted in the literature, as it is generically understood to ‘3/2 vanishing’ at each endpoint. However, it is a necessary condition for uniqueness. In every case we consider \( \rho_k \in [3/2, 1/2] \) so that the problem for \( d\psi \) following (4.5) always has a unique solution.

Consider the Riemann surface of genus \( G \geq 0 \):

\[ S_G := \left\{ P = (z, \mathcal{R}), \ \mathcal{R}^2 = \prod_{j=1}^{2G+2} (z - \lambda_j) \right\}, \quad \lambda_1 > \lambda_2 > \ldots > \lambda_{2G+2} \]

The projection \( \pi(P) = z \), defines \( S_G \) as a two-sheet cover of \( \mathbb{C}P^1 \). By the upper (respectively lower) sheet of \( S_G \) we denote those points \((z, \mathcal{R}) \in S_G\) for which \( \mathcal{R} = \mathcal{R}(z, \lambda) \) (respectively \( \mathcal{R} = -\mathcal{R}(z, \lambda) \)) where \( \mathcal{R}(z, \lambda) \) is defined by (1.13). We take our basis \{\(a_j, b_j\}_{j=1}^{G}\) of the homology group \( H_1(S_G) \) so that \( a_j \) lies entirely on the upper sheet and encircles with positive (counterclockwise) orientation \( J_j = (\lambda_{2j+2}, \lambda_{2j+1}], \ j = 1, \ldots, G \), while \( b_j \) emerges from \( J_0 = (\lambda_2, \lambda_1) \) on the upper sheet passes counterclockwise to the lower sheet through
Figure 4. Our choice of basis \([a_1, \ldots, a_G, b_1, \ldots, b_G]\) for \(H_1(S_G)\), where \(S_G\) is the genus \(G\) hyperelliptic Riemann surface \(S_G\).

\(J_j = (\lambda_{2j+2}, \lambda_{2j+1})\) and returns to the initial point entirely on the lower sheet, see figure 4. We denote the pre-image of \(\infty\) on the upper sheet, where \(\mathcal{R}(z) \sim z^{G+1}\), by \((\infty, \infty)\) and the pre-image on the ‘bottom sheet’, where \(\mathcal{R}(z) \sim -z^{G+1}\), by \((\infty, -\infty)\).

Let \(v_j, j = 1, \ldots, G\) denote the canonical basis of holomorphic one-forms (Abelian differentials of the first kind) on \(\Sigma_G\):

\[v_j(z) = \frac{c_j,1z^{G-1} + c_j,2z^{G-2} + \ldots + c_j,G}{\mathcal{R}} dz\]

(4.1)

where the constants \(c_{j,i}\) are uniquely determined by the normalization conditions

\[\oint_{\nu_j} v_j = \delta_{kj}, \quad j, k = 1, \ldots, G.\]

Additionally let \(\omega^{(k)}, k = 0, 1\), denote the Abelian differentials of the second kind on \(S_G\) given by

\[\omega^{(k)} = \frac{P_k(z, \lambda)}{\mathcal{R}} dz\]

\[P_k(z; \lambda) = z^{k+G+1} + \Gamma_1 z^{k+G} + \ldots + \Gamma_{k+1} z^G + a_{k,1} z^{G-1} + \ldots + a_{k,G}\]

(4.2)

where \(\Gamma_j = \Gamma_j(\lambda)\) are the coefficients of the expansion

\[\mathcal{R}(z; \lambda) = \left(\prod_{k=1}^{2G+2} (z - \lambda_k)\right)^{1/2} = z^{G+1} \left(1 + \frac{\Gamma_1}{z} + \ldots + \frac{\Gamma_m}{z^m} + \ldots\right),\]

(4.3)

and the \(a_{k,j} = a_{k,j}(\lambda)\) are determined by the normalization condition

\[\oint_{\nu_j} \omega^{(k)} = 0, \quad j = 1, \ldots, G.\]

(4.4)

For large arguments \(\omega^{(k)}\) admits the expansion

\[\omega^{(k)} = \pm \left[z^k + \mathcal{O}\left(z^{-2}\right)\right] dz, \quad P \to (\infty, \pm\infty),\]

(4.5)

so it has poles of order \(k + 2\) at \((\infty, \pm\infty)\).
For any choice of moduli $\lambda$, $C$ is analytic in $\mathbb{C}\setminus\mathcal{J}$.

If the fourth condition is also satisfied, the function $\theta(z; x, t)$ is a soft edge if

\[ g(z) = \Theta((z - \lambda_k)^2) \quad \text{as} \quad z \to \lambda_k. \]

Now for given vanishing exponents $\rho_k \in (2N_0 + 1)/2$ we want to construct a differential $d\varphi$ which has the following properties:

1. $d\varphi$ is meromorphic on $S_G$ whose only poles are at $(\infty, \pm \infty)$.
2. $d\varphi \equiv d\theta$ is locally holomorphic as $P \to (\infty, \pm \infty)$.
3. $\int_{\lambda_1}^{\lambda_k} d\varphi = 0$ for $k = 1, \ldots, G$.
4. $d\varphi$ vanishes to order 2 at each branch point $\lambda_j$.

For any choice of moduli $\lambda$, the first three conditions define a meromorphic differential of the second kind, which given by

\[ d\varphi = 2t \omega^{(1)} + x \omega^{(0)}. \]  

(4.6)

If the fourth condition is also satisfied, the function

\[ g(z) = \theta(z) - \theta(\lambda_1) - \int_{\lambda_1}^{\lambda} d\varphi, \]

(4.7)

where the path of integration lies in $\mathbb{C}\setminus(\lambda_{2G+2}, \lambda_1)$, that is along the top sheet of $S_G$ on which $d\varphi \sim d\theta$, satisfies the conditions in table 1. Moreover, the function

\[ \psi(z) := \int_{\lambda_1}^{\lambda} d\varphi \]

(4.8)

is analytic in $\mathbb{C}\setminus\bigcup_{k=0}^{G}(\lambda_{2k+2}, \lambda_{2k+1})$ and satisfies the jump relations

\[ \psi_+(z) + \psi_-(z) = \left\{ \begin{array}{ll} 0 & z \in (\lambda_1, \lambda_2) \\ \int_{\lambda_k} \psi(z) & z \in (\lambda_{2k+1}, \lambda_{2k+2}) \end{array} \right., \quad k = 1, \ldots, G. \]

(4.9)

Table 1. Analytic properties of the $g$-function.

| Condition | Property |
|-----------|----------|
| 1. $g'(z)$ is analytic in $\mathbb{C}\setminus\mathcal{J}$.
| 2. $2g(z; x, t) - g_+(z) - g_-(z) = 2\alpha_k$, for $z \in J_k$, $k = 0, \ldots, G$.
| 3. $g(z) = g_\infty + \mathcal{O}(z^{-1})$ as $|z| \to \infty$.
| 4. $g(z) - \theta(z; x, t) = \mathcal{O}((z - \lambda_k)^2)$ as $z \to \lambda_k$.
| 5. $g(z) = g^+(z)$ for $z \in \mathbb{C}\setminus\mathcal{J}$.

Here, and in what follows, when $\varphi$ appears as a function of $z$ alone, this indicates the restriction of this function to the upper sheet of $S_G$, on which the variable $z$ is a suitable coordinate.

For our purposes we consider the following situation. The half-plane $(x, t \geq 0)$ is divided into distinct domains $D_m$ such that in each $D_m$ we have a fixed genus $G \geq 0$ and the moduli $\lambda_j$ are split into two types:

1. **Hard edges**: These $\lambda_j$ are known and constant for $(x, t) \in D_m$. There is no vanishing condition for $d\varphi$ at these $\lambda_j$.

2. **Soft edges**: These $\lambda_j$ are allowed to move for $(x, t) \in D_m$; their motion is described by the additional condition that $d\varphi$ vanish quadratically at $\lambda_j$ (when viewed as a branch point on $S_G$). Using (4.2) and (4.6) $d\varphi/\mathrm{d}z = (2tP_1 + xP_0)/R$, and the soft edge condition is equivalent to the condition that $2tP_1(z, \lambda) + xP_0(z, \lambda)$ have a simple zero at $z = \lambda_j$ or equivalently

\[ \lambda_j \text{ is a soft edge if } : x - v_j(\lambda)t = 0, \quad v_j(\lambda) = -2 \frac{P_1(\lambda_j, \lambda)}{P_0(\lambda_j, \lambda)}. \]

(4.10)

Equation (4.10) states that the motion of the branch points $\lambda_j$ are described by the self-similar solutions of the genus-$G$ Whitham equations (2.7). Also, as the Whitham equations for dNLS are strictly hyperbolic [32], any self-similar solution of the Whitham equations admits at most one soft edge.

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4.1. Self-similar genus zero \( g \)-functions

In the genus zero case \( \lambda = (\lambda_1, \lambda_2) \) and the first homology group is trivial as any closed loop is homotopic to a point. The polynomials associated with our second kind differentials (4.2) are given by

\[
P_0(z, \lambda) = z - \frac{1}{2} e_1(\lambda),
\]

\[
P_1(z, \lambda) = z^2 - \frac{1}{2} e_1(\lambda)z + \left( \frac{1}{2} e_2(\lambda) - \frac{1}{8} e_1(\lambda)^2 \right),
\]

where, for any \( G \geq 0 \), the functions

\[
e_1(\lambda) = \sum_{j=1}^{2G+2} \lambda_j, \quad e_2(\lambda) = \sum_{1 \leq j < k} \lambda_j \lambda_k
\]

are the first two elementary symmetric polynomials. The genus-0 speeds \( \lambda \) in (4.10) are given by

\[
\begin{align*}
V_j(\lambda) &= -\frac{1}{2} e_1(\lambda) - \lambda_j, & j &= 1, 2. \\
\end{align*}
\]

and

\[
d\varphi = \frac{2t P_1(z, \lambda) + x P_0(z, \lambda)}{R(z; \lambda)} \, dz = \frac{2t(z - \xi_+)(z - \xi_-)}{R(z; \lambda)} \, dz
\]

where \( R(z; \lambda) = \sqrt{(z - \lambda_1)(z - \lambda_2)} \) is cut on \((\lambda_2, \lambda_1)\) and \( R \sim z \) as \( z \to \infty \).

4.1.1. The one-cut, hard edged case (plane waves). If we suppose that \( \{\lambda_1, \lambda_2\} \) are known (constant) hard edges, then the stationary points, the zeros of \( d\varphi \), are given by

\[
\xi_{\pm} = \frac{\lambda_1 + \lambda_2 - \tau}{4} \pm \frac{1}{4} \sqrt{(\lambda_1 + \lambda_2 + \tau)^2 + 2(\lambda_1 - \lambda_2)^2}, \quad \tau = \frac{x}{t}.
\]

Each is a monotone decreasing function of \( \tau \) with the following special values:

\[
\begin{array}{c|cccc}
\xi_-(\tau) & -\infty & -\frac{1}{4} (3\lambda_1 + \lambda_2) & -\frac{1}{4} (\lambda_1 + 3\lambda_2) & \infty \\
\xi_+(\tau) & \infty & \frac{1}{4} (\lambda_1 + 3\lambda_2) & \frac{1}{4} (3\lambda_1 + \lambda_2) & \lambda_1
\end{array}
\]

With \( d\varphi \) defined by (4.13), the \( g \)-function, analytic for \( z \in \mathbb{C}\setminus(\lambda_2, \lambda_1) \), is given by:

\[
g(z) := \theta(z) - \theta(\lambda_1) = \int_{\lambda_1}^z d\varphi
\]

where the path of integration does not pass through the cut \((\lambda_2, \lambda_1)\). The integral term can be computed explicitly,

\[
\psi(z) := \int_{\lambda_1}^z d\varphi = tR(z, \lambda) \left( z + \frac{1}{2}(\lambda_1 + \lambda_2) + \tau \right).
\]

Clearly, \( g \) is bounded at infinity by virtue of the growth condition on \( dg \) and

\[
g_\infty = -\theta(\lambda_1) + x \left( \frac{\lambda_1 + \lambda_2}{2} \right) + t \left[ \left( \frac{\lambda_1 + \lambda_2}{2} \right)^2 + \frac{1}{2} \left( \frac{\lambda_1 - \lambda_2}{2} \right)^2 \right].
\]

For \( \psi \) defined by (4.17) the structure of the imaginary signature table depends on the position of the two real stationary points \( \xi_1(\tau) \) relative to the branch points \( \lambda_1 \) and \( \lambda_2 \). For \( \tau < \frac{1}{2}(\lambda_1 + 3\lambda_2) \), the level set \( \text{Im} \varphi = 0 \) consist of the real axis minus the cut and an asymptotically vertical
Figure 5. The topological structure of the sign table for Im $\varphi$ bifurcates as shown as the stationary phase points $\xi_{\pm}(\tau)$ pass through $\lambda_1$ and $\lambda_2$, the branch points of $\varphi$.

contour through $\xi_{-}(\tau) < \lambda_2$. For $\tau > \frac{1}{4}(3\lambda_1 + \lambda_2)$ the situation is reversed, and the vertical contour passes through $\xi_{+}(\tau) > \lambda_1$. For $\frac{1}{4}(\lambda_1 + 3\lambda_2) < \tau < \frac{1}{4}(3\lambda_1 + \lambda_2)$, both $\xi_{-}(\tau)$ and $\xi_{+}(\tau)$ lie on the cut. In this case the vertical component of Im $\varphi = 0$ passes through the point

$$\xi_0(\tau) = -\frac{1}{2}(\lambda_1 + \lambda_2 + 2\tau) \quad (4.19)$$

which lies between $\xi_{-}(\tau)$ and $\xi_{+}(\tau)$. See figure 5.

4.1.2. The one-cut, hard/soft edge case (rarefaction waves). If $dg$ is cut on a single interval $(\lambda_1, \lambda_2)$ where one branch point is a soft edge, $\lambda_s$, and the other is a known hard edge, $\lambda_h$, then conditions (4.10), (4.12) effectively ‘pin’ one zero of the numerator in (4.13) to $\lambda_s$, leaving one stationary point $\xi$. Solving these conditions gives the motion of the soft edge $\lambda_s$ and stationary point $\xi$ in terms of $x$, $t$, and $\lambda_h$:

$$\lambda_s = -\frac{1}{3}(2\tau + \lambda_h).$$

$$\xi = \frac{1}{4}(\lambda_s + 3\lambda_h) = \frac{1}{6}(4\lambda_h - \tau). \quad (4.20)$$

Note that $\xi$ always lies on the cut $(\lambda_2, \lambda_1)$.

In this notation $d\varphi$ has the explicit representation

$$d\varphi = 2t\left(\frac{z - \lambda_s}{z - \lambda_h}\right)^{1/2}(z - \xi)dz \quad (4.21)$$

As before we define

$$g(z) = \theta(z) - \theta(\lambda_1) - \varphi(z),$$

$$\varphi(z) = \int_{\lambda_1}^{z} d\varphi = 2t\int_{\lambda_1}^{z} \left(\frac{\lambda - \lambda_s}{\lambda - \lambda_h}\right)^{1/2}(\lambda - \xi)d\lambda = t(z - \lambda_s)^{3/2}(z - \lambda_h)^{1/2}. \quad (4.22)$$

The zero level set of Im $\varphi$ always consists of $\mathbb{R}\setminus(\lambda_1, \lambda_2)$ and two trajectories emerging from $\lambda_s$ into the upper and lower half-planes respectively. The resulting signature table for Im $\varphi$ is given in figure 6. Finally, we compute the limit

$$g_\infty = \frac{t}{8}(\lambda_h^2 - 6\lambda_h\lambda_s - 3\lambda_s^2) - \theta(\lambda_1). \quad (4.23)$$
4.2. Self-similar genus one g-functions

In the genus one case, there are four ordered branch points \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \), \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \). The polynomials associated with (4.2) are given by

\[
P_0(z, \lambda) = z^2 - \frac{1}{2} e_1(\lambda)z + a_{0, 1},
\]

\[
P_1(z, \lambda) = z^3 - \frac{1}{2} e_1(\lambda)z^2 + \left( \frac{1}{2} e_2(\lambda) - \frac{1}{8} e_1(\lambda)^2 \right) z + a_{1, 1},
\]

and the differential (4.6) is given by

\[
d\phi = \frac{2t P_1(z, \lambda) + x P_0(z, \lambda)}{R(z; \lambda)} dz,
\]

where \( R(z; \lambda) = \prod_{k=1}^4 \sqrt{z - \lambda_k} \) is cut on \( (\lambda_4, \lambda_3) \cup (\lambda_2, \lambda_1) \) and \( R \sim z^2 \) as \( z \to \infty \).

The coefficients \( a_{0, 1} \) and \( a_{1, 1} \) in (4.24) can be computed explicitly from (4.4) [8]:

\[
a_{0, 1} = \frac{1}{2} (\lambda_1 \lambda_2 + \lambda_3 \lambda_4) - \frac{1}{2} (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4) E(m) K(m)
\]

\[
a_{1, 1} = \frac{1}{8} (\lambda_1 \lambda_2 - \lambda_3 \lambda_4)(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) - \frac{1}{8} e_1(\lambda)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4) E(m) K(m).
\]

Here \( K(m) \) and \( E(m) \) are the complete elliptic integrals of the first and second kind respectively with modulus

\[
m = m(\lambda) = \frac{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}
\]

4.2.1. The two-cut, one soft edge case (modulated elliptic waves). If we suppose that one of the branch points, denoted \( \lambda_s \), is allowed to evolve as a soft edge while the other branch points are constant hard edges, then the cubic polynomial \( 2t P_1(z, \lambda) + x P_0(z, \lambda) \) has one zero in each band interval; this is a necessary consequence of the fact that \( d\phi \) has been normalized so that all of its \( a \)-cycles vanish. We label these zeros \( \xi_{\pm}(\tau) \in (\lambda_4, \lambda_3) \) and \( \xi_{\pm}(\tau) \in (\lambda_2, \lambda_1) \).

The remaining zero of the cubic polynomial lies at the soft edge, \( \lambda_s \):

\[
2P_1(\lambda_s, \lambda) + \tau P_0(\lambda_s, \lambda) = 0, \quad \tau = \frac{x}{t}, \quad \lambda \lambda_s \text{ constant}.
\]
This equation determines the motion of the soft edge and, as described by (2.7) and (2.6), the motion is exactly that of a self-similar solution of the Whitham equations for the genus-one Riemann invariants of defocusing NLS.

Writing

\[ 2tP_1(z, \lambda) + xP_0(z, \lambda) = 2t(z - \lambda_1(r))(z - \xi_-(r))(z - \xi_+(r)) \]

we find by comparing coefficients that

\[ \xi_+(r) + \xi_-(r) = \frac{1}{2}e_1(\lambda) - \frac{1}{2}\xi_+(r) + \frac{1}{2}\xi_-(r) = \frac{1}{2}\xi_+(r) + \frac{1}{2}\xi_-(r) \]

(4.27)

from which the motion of these station phase points are easily determined. We may write the differential

\[ d\varphi = \frac{2t(z - \lambda_1)(z - \xi_-(r))(z - \xi_+(r))}{\prod_{k=1}^4 \sqrt{z - \lambda_k}} \]

(4.28)

As before we define

\[ g(z) = \theta(z) - \theta(\lambda_1) - \varphi(z) \]

(4.29)

\[ \varphi(z) = 2t \int_{\lambda_1}^z \frac{(\lambda - \lambda_1)(\lambda - \xi_-(r))(\lambda - \xi_+(r))}{\prod_{k=1}^4 \sqrt{\lambda - \lambda_k}} d\lambda \]

so that \( \varphi(z) \) is analytic in \( \mathbb{C}(\lambda_4, \lambda_3) \cup (\lambda_2, \lambda_1) \) and satisfies the relations

\[ \varphi_+(z) + \varphi_-(z) = 0 \quad z \in (\lambda_2, \lambda_1), \]

(4.30)

\[ \varphi_+(z) + \varphi_-(z) = \oint_b \varphi \quad z \in (\lambda_2, \lambda_1). \]

Finally, we determine the structure of the signature table for \( \text{Im} \varphi \). The differential \( d\varphi \) is real valued on the real axis minus the bands, with vanishing a-cycles, and locally \( d\varphi = \mathcal{O}\left((z - \lambda_3)^{1/2}\right) \) at each hard edge and \( d\varphi = \mathcal{O}\left((z - \lambda_s)^{1/2}\right) \) at the soft edge. It follows that the zero level set of \( \text{Im} \varphi \) consists of the real axis minus the bands \( (\lambda_4, \lambda_3) \cup (\lambda_2, \lambda_1) \) and two trajectories emerging from the soft edge \( \lambda_s \) to infinity through the upper and lower half-planes respectively. The resulting signature table for \( \text{Im} \varphi \) is given in figure 7.

\[ \begin{array}{c|c|c|c|c}
\text{Im} \varphi < 0 & \text{Im} \varphi > 0 \\
\lambda_4 & \lambda_3 & \lambda_2 & \lambda_1 \\
\hline
\text{Im} \varphi > 0 & \text{Im} \varphi < 0 \\
\hline
\lambda_s & \lambda_3 \\
\end{array} \]

Figure 7. The topological structure of the sign table for \( \text{Im} \varphi \) corresponding to the genus one self-similar \( g \)-function (4.29) in the case where the soft edge \( \lambda_s \) is \( \lambda_3 \) and the other edges are fixed.
5. Steepest descent analysis

We are ready to begin studying solutions of RHP 3.1. Throughout the section we will refer to the constants $\lambda_{\pm} = \mu \pm A$ which represent the constant Riemann invariants corresponding to the right half of the initial data (1.5), and are the endpoints of the interval $I_R$ related to the branch cuts of the reflection coefficient (3.7). The course of the inverse analysis depends on the relative ordering of $\lambda_+, \lambda_-, -1$, and 1. Recall that $\pm 1$ are the Riemann invariants of the left half of (1.5). In theorem 1.1 we consider the case $-1 < \lambda_- < \lambda_+ < 1$ and so we only perform the inverse analysis in this case. It should be clear to the familiar reader how to adapt our calculations to the other five cases without much effort.

We begin the inverse analysis by cataloging a family of jump matrix transformations needed for the nonlinear steepest descent deformations. Next, we introduce the initial jump factorizations common to each of the five asymptotic zones identified in theorem 1.1 and briefly describe the generic sequence of transformations leading to a small norm error RHP which can be solved asymptotically via singular integrals. Finally, moving left-to-right, we go through the details of establishing the asymptotic behavior of the solution in each of the five zones. As we will see, in this case, when $-1 < \lambda_- < \lambda_+ < 1$, the initial shock is regularized by a region of rarefaction on the left and a shock wave on the right separated by a central planar plateau.

5.1. An almanac of matrix factorizations

Here we record several matrix factorizations that we will refer to when we deform contours onto steepest descent paths. The factorizations are grouped according to the intervals on which they will be used. The off-diagonal exponential factors are omitted but can be included by multiplying on the left and right by the appropriate diagonal factors.

For $z \in \mathbb{R} \setminus (I_L \cup I_R)$:

\[
\begin{pmatrix}
1 - rr^* & -r^* \\
r & 1
\end{pmatrix} = \begin{pmatrix} 1 & -r^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}
\] (5.1a)

\[
= \begin{pmatrix} 1 & 0 \\ \frac{1}{1-r^2} & 1 \end{pmatrix} (1 - rr^*)^{\theta_0} \begin{pmatrix} 1 & -r^* \\ 0 & 1 \end{pmatrix}
\] (5.1b)

For $z \in I_L \setminus (I_L \cap I_R)$, where $rr_+ = 1/r_+^*$:

\[
\begin{pmatrix} 0 & -r^* \\ r_+ & 1 \end{pmatrix} = \begin{pmatrix} 1 & -r^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r_+ & 1 \end{pmatrix}
\] (5.2a)

\[
= \begin{pmatrix} 1 & 0 \\ \frac{1}{1-r_+^2} & 1 \end{pmatrix} \begin{pmatrix} 0 & -r_+^* \\ r_+ & 0 \end{pmatrix} \begin{pmatrix} 1 & -r_+^* \\ 0 & 1 \end{pmatrix}
\] (5.2b)

For $z \in I_R \cap I_L$, where $r$ is analytic and $rr_+ = -r$:

\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -r^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}
\] (5.3a)

\[
= \begin{pmatrix} 1 & 0 \\ \frac{1}{1-r^2} & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -r^* \\ 0 & 1 \end{pmatrix}
\] (5.3b)

In our main theorem, theorem 1.1, we suppose that $I_R \subset I_L$, so the above factorizations are sufficient to perform the inverse analysis.
5.2. The sequence of matrix transformations

In the subsequent sections we describe the steepest descent analysis for RHP 3.1 in each of the six possible parameter regimes. In order to streamline this procedure, we record the sequence of transformations which lead from the initial RHP to one which is amenable to asymptotic expansion. In each case the transformation is the same up to redefinition of the \( g \)-functions, deformations of the various domains of definition, and the transition ‘times’. In what follows we will define the \( g \)-functions and domains for each instance and point out the critical behavior at each transition time appropriate to each case. It will then remain in each case to compute the leading order behavior of the solution of RHP 3.1.

The transformation to an asymptotically stable limit can be done in two steps. First, we introduce a \( g \)-function, as described in section 5, by introducing the transformation

\[
m(z) = e^{-ig_\infty \sigma_3/\epsilon} M(z) e^{ig(z)\sigma_3/\epsilon},
\]

which seeks to remove rapid oscillations from the problem. Second, we introduce steepest descent contours \( \Gamma_i \), \( i = 1, 2 \) in \( \mathbb{C}^+ \), oriented left-to-right, and their complex conjugate images \( \Gamma_i^* \) in \( \mathbb{C}^- \) in order to deform the jumps onto contours on which they are near identity. The exact shape of these contours is determined by the given \( g \)-function (they are essentially rays in the complex plane emanating from stationary phase points), but in each case \( \Gamma_1 \) lies to the right of \( \Gamma_2 \) and each returns to the real axis at exactly one point, which may or may not be distinct, see figures 8 and 9 for two examples. This divides \( \mathbb{C}^+ \) (and \( \mathbb{C}^- \)) into three regions which we label from right-to-left as \( \Omega_i, i = 1, 2, 3 \) (and \( \Omega_i^*, i = 1, 2, 3 \)). Using these regions we make the piecewise-analytic transformation \( M \mapsto N \) defined by

\[
M(z) = \begin{cases} 
N(z) \left( \begin{array}{cc} 1 & 0 \\
\frac{r(z)e^{2i(\varphi(z)+\theta(\lambda_1))/\epsilon}}{1-r(z)r^*(z)} & 1 \\
0 & 1 
\end{array} \right) & z \in \Omega_1 \\
N(z) \left( \begin{array}{cc} 1 & 0 \\
\frac{r^*(z)e^{-2i(\varphi(z)+\theta(\lambda_1))/\epsilon}}{1-r(z)r^*(z)} & 1 \\
0 & 1 
\end{array} \right) & z \in \Omega_1^* \\
N(z) \left( \begin{array}{cc} 1 & 0 \\
\frac{-r(z)e^{2i(\varphi(z)+\theta(\lambda_1))/\epsilon}}{1-r(z)r^*(z)} & 1 \\
0 & 1 
\end{array} \right) & z \in \Omega_2 \\
N(z) \left( \begin{array}{cc} 1 & 0 \\
\frac{-r^*(z)e^{-2i(\varphi(z)+\theta(\lambda_1))/\epsilon}}{1-r(z)r^*(z)} & 1 \\
0 & 1 
\end{array} \right) & z \in \Omega_2^* 
\end{cases}
\]

The new unknown \( N \) has jumps on the real axis and on each of the \( \Gamma_i \)'s. All of which have well defined asymptotic limits.

To analyze the asymptotic limit we introduce a scalar function \( D(z) \) to define another transformation \( Q(z) = D^\infty_N(z)D(z)^{-\infty} \) whose role is to reduce the limiting jumps of \( N(z) \) to new limits for \( Q(z) \) which are simple constants independent of the asymptotic parameters.

We then build an explicit, piecewise defined, global parametrix \( P(z) \) which uniformly approximates the solution of \( Q(z) \) and use it to define an error matrix \( E(z) = Q(z)P(z)^{-1} \).

5.2.1. Small-norm Riemann Hilbert problems. The error matrix \( E(z) \) described above will satisfy a RHP on some given contour \( \Gamma_E \) of the form:

- \( E(z) \) is analytic in \( \mathbb{C} \setminus \Gamma_E \) and \( E(z) = I + O(z^{-1}) \) as \( z \to \infty \);
- For \( z \in \Gamma_E \), the boundary values of \( E(z) \) satisfy \( E_+(z) = E_-(z)V_E(z) \), with \( \| V_E \|_{L^\infty(\Gamma_E)} = h \left( \frac{z}{\epsilon} \right) \)
where \( (z) = \sqrt{\frac{1}{1 + |z|^2}} \), \( p = 1, 2, \) or \( \infty \), and \( h(z) \) is some smooth function with \( h(0) = 0 \).

The solution of this near identity problem can be expressed in terms of Cauchy singular integrals. Let \( C_{\omega} \) denote the minus Cauchy projection operator on \( \Gamma_1 \): 

\[
C_{\omega} f(z) = \lim_{z \to \Gamma_1} \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\xi)}{\xi - z} d\xi,
\]

where the limit is understood as \( z \to \Gamma_1 \) from the right side of the contour with respect to its orientation. The essential fact is that \( C_{\omega} \) is a bounded operator on \( L^2(\Gamma_1) \) for any piecewise smooth rectifiable contour \( \Gamma_1 \). Then in terms of the operator \( C_E \) defined by \( C_E[f](\xi) = C_{\omega}[f(VE - I)](\xi) \), the solution of the error Riemann–Hilbert problem is given by 

\[
E(z) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{E(\xi)}{(\xi - z)^2} d\xi \] 

where \( \mu \) is the unique solution of \( (1 - C_E)^{-1} \). The bounds on \( E - I \) also imply that \( |E - I| = O(\|C_E\|) = O(h(e/\iota)) \), the extra condition that \( s(VE - I) \) also have small norm is sufficient to expand \( E(z) \) for \( z \) large: 

\[
E(z) = I + z^{-1} E^{(1)} + O(z^{-2})
\]

By explicitly computing the Neumann iterates determining \( \mu \), one can replace the error bound on \( E^{(1)} \) with a full asymptotic expansion in \( \iota/\iota \).

Inverting the series of transformations from \( m(z) \mapsto E(z) \) we arrive at an explicit formula for the solution \( m(z) \) of the original NLS problem, RHP 3.1, in terms of the solution \( E(z) \) of the error matrix. For \( z \in \Omega_2 \):

\[
m(z) = e^{-\frac{1}{2}z\sigma_3} D^{-\sigma_3} E(z) P(z) D(z)^{\sigma_3} e^{\frac{1}{2}z\sigma_1(m_3)}.
\]

Then, using (3.15) we recover an asymptotic expansion of the solution of (1.1), (1.5) in the form

\[
q(x, t) = e^{-2z\sigma_3/c} \left[ \lim_{z \to \infty} \frac{P(z)}{D(z)^2} + O(h(e/\iota)) \right]
\]

5.3. The far left field: \( \tau < -1 \)

We expect that for large negative \( \tau \), that is \( x \ll -t \), the solution should resemble the plane wave specified by the left half of the initial data (1.5). At the level of the RHP this means that we expect that the \( g \)-function should be cut on \( I_L = (-1, 1) \) with two hard edges. Using the results of section 4.1 we define the \( g \)-function analytic for \( z \in \Omega \setminus I_L \) by

\[
g(z) = \int_1^1 d\theta - 2\iota \frac{(\lambda - \xi_+)(\lambda - \xi_-)}{R(\lambda; -1, 1)} d\lambda
\]

where

\[
\xi_\pm = \xi_\pm(\tau) = -\frac{\tau}{4} \pm \frac{1}{4} \sqrt{\tau^2 + 8}, \quad g_\infty = -\theta(1) + t/2.
\]

For \( \tau \leq -1 \), the stationary points satisfy \( \xi_+ \in [1, \infty) \) and \( \xi_- \in (-1, 0) \) with \( \xi_+ = 1 \) only when \( \tau = -1 \). As such the imaginary sign table for the function

\[
\psi(z) = 2\iota \int_1^1 \frac{(\lambda - \xi_+)(\lambda - \xi_-)}{R(\lambda; -1, 1)} d\lambda
\]

resembles figure 5(a). We open lenses along the steepest descent paths through \( \xi_\iota(\tau) \) as depicted in figure 8 and define the mapping from \( m \mapsto N \) using (5.4)–(5.5). This results is
Figure 8. The contours $\Gamma_i$ and regions $\Omega_i$ used to define the map $M \mapsto N$ (see (5.5)) for $x/t = \tau$ in the left planar zone (defined above). As $\tau$ increases the stationary phase points $\xi_\pm(\tau)$ decrease, at $\tau = -1$, the boundary of the zone, $\xi_+$ collides with 1; $\xi_-$ lies within $(-1,1)$ for all $\tau$ in the zone. Blue regions correspond to $\text{Im } \varphi > 0$ and white regions to $\text{Im } \varphi < 0$.

the following problem for the new unknown $N(z)$:

Riemann–Hilbert Problem 5.1. Find a $2 \times 2$ matrix $N$ with the following properties:

1. $N(z)$ is analytic in $\mathbb{C} \setminus \Gamma_N$, $\Gamma_N = (-\infty, \xi_+(\tau)) \cup \mathbb{R}^+$. 
2. $N(z) = I + O(z^{-1})$ as $z \to \infty$.
3. $N(z)$ takes continuous boundary values on $\Gamma_N$ away from points of self intersection and branch points which satisfy the jump relation $N_+(z) = N_-(z) V_N(z)$ where

$$V_N(z) = \begin{pmatrix} (1 - r(z)r^+(z))^{\frac{1}{2}} & 0 \\ 0 & -r^+(z)e^{-2i\theta_1(z)/\epsilon} \\ r_+(z)e^{2i\theta_1(z)/\epsilon} & 0 \\ 0 & e^{-2i\theta_1(z)/\epsilon} \end{pmatrix} \quad z \in (-\infty, \xi_+(\tau)) \setminus I_L, z \in I_L \setminus I_R, z \in I_R \cap I_L, z \in \Gamma_1$$

$$\begin{pmatrix} 1 & 0 \\ \frac{1}{1-r(z)r^+(z)}e^{-2i\theta_1(z)/\epsilon} & 1 \end{pmatrix} \quad z \in \Gamma_2$$

(5.11)

4. $N(z)$ is bounded except at the points $\{1, -1, \lambda_+, \lambda_-\}$ where

$$N(z) = O \left( \begin{pmatrix} 1 & (z-p)^{-1/4} \\ 0 & (z-p)^{-1/4} \end{pmatrix}, \quad z \in \Omega_3, \quad p \in \{-1, 1\} \right.$$ 

$$N(z) = O \left( \begin{pmatrix} 1 & (z-p)^{-1/4} \\ 0 & (z-p)^{-1/4} \end{pmatrix}, \quad z \in \Omega_3, \quad p \in \{-1, 1\} \right.$$ 

(5.12)

Remark 7. Throughout this section we give the jumps and growth bounds of the various Riemann–Hilbert problems only on the real axis and in the upper half-plane. The contour deformations we use all respect the original symmetry $m(z; x, t) = \sigma_1 m(z^*; x, t)^* \sigma_1$ of RHP 3.1. It follows that the jump along a contours $\Gamma_k^* \in \mathbb{C}^-$ is given by $\sigma_3 v(z^*; x, t)^* \sigma_3$ where $v(z; x, t)$ is the jump defined along $\Gamma_k \in \mathbb{C}^+$ and $A^\dagger$ is the hermitian conjugate of $A$. 

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5.3.1. Constructing a parametrix for \( r < -1 \). The jumps of \( N(z) \) along \( \Gamma_i, \ i = 1, 2 \) and their complex conjugates are all near identity at any positive distance from the real axis because the contours lie in regions in which the off diagonal entries are exponentially decaying. As a result, to leading order the solution \( N(z) \) should be given by the model problem produced by neglecting the jumps off the real axis in (5.11).

Define

\[
D(z) = \exp \left[ \frac{i \theta(1)}{\epsilon} + \frac{R(z; -1, 1)}{2 \pi i} \left\{ \left( \int_{-\infty}^{-1} + \int_{1}^{c_1} \right) \frac{\log(1 - r(\lambda)r^*(\lambda))}{R(\lambda; -1, 1)} \, d\lambda \right. \right.
\]

\[
+ \left. \left. \left( \int_{1}^{c_2} + \int_{c_1}^{-1} \right) \frac{\log(r_3(\lambda))}{R_3(\lambda; -1, 1)} \, d\lambda \right\} \right] \tag{5.13}
\]

As the following proposition describes, this function is constructed to remove the jumps along the real axis, or reduce to constants where they cannot be removed. Simultaneously, the growth behavior at the branch points is simplified.

**Proposition 5.1.** The function \( D : \mathbb{C}\setminus(-\infty, \xi_+) \rightarrow \mathbb{C} \) defined by (5.13) has the following properties:

1. \( D \) is analytic in \( \mathbb{C}\setminus(-\infty, \xi_+) \), and takes continuous boundary values on \( (-\infty, \xi_+) \) except at the endpoints of integration in (5.13).
2. As \( z \rightarrow \infty \), \( D(z) \rightarrow D_\infty + O\left( \frac{1}{z} \right) \) where

\[
D_\infty = e^{\frac{2 \theta(1)}{\epsilon}} e^{\frac{i}{\epsilon} \int_{-\infty}^{-1} \log(1 - r(z)r^*(z)) \, dz + \int_{-1}^{1} \log(r_3(z)) \, dz} \tag{5.14}
\]

3. For \( z \in (-\infty, \xi_+ \tau) \), \( D(z) \) satisfies the jump relations

\[
\begin{align*}
D_+(z)/D_-(z) &= 1 - r(z)r^*(z) & z \in (-\infty, \xi_+(\tau)) \setminus I_L \\
D_+(z)D_-(z) &= r_3(z)e^{2\theta(1)/\epsilon} & z \in I_L \setminus I_R \\
D_+(z)/D_-(z) &= e^{\frac{2 \theta(1)}{\epsilon}} & z \in I_L \cap I_R
\end{align*}
\]

4. \( D(z) \) exhibits the following singular behavior at each endpoint of integration:

\[
\begin{align*}
D(z) &= (z - p)^{\frac{i}{2} \text{sgn} \, \text{Im} \, D_0(z)} & z \rightarrow p \\
D(z) &= (z - \xi_+)^{\frac{i}{2} \text{sgn} \, \text{Im} \, D_0(z)} & z \rightarrow \xi_+
\end{align*}
\tag{5.15}
\]

where \( p \in [-1, 1, \lambda_-, \lambda_+] \) is any of the four branch points, \( \kappa(z) = -\frac{1}{\epsilon z} \log(1 - r(z)r^*(z)) \), and \( D_0(z) \) is a bounded function taking a definite limit as \( z \) approaches each singular point non-tangentially.

**Proof.** Each of these properties follows immediately from the general properties of Cauchy-type integrals and the local behavior of \( r \) and \( 1 - rr^* \) at the endpoints of integration which can be read off from (3.7) and (3.11). For the behavior at the endpoints of integration the standard reference is [38]. \( \square \)

Using the function \( D(z) \) the change of variables

\[
Q(z) = D_\infty^\xi N(z) D(z)^{-\xi_+} \tag{5.16}
\]

results in the following RHP for \( Q \).

**Riemann-Hilbert Problem 5.2.** Find a \( 2 \times 2 \) matrix \( Q \) with the following properties

1. \( Q(z) \) is analytic in \( \mathbb{C}\setminus \Gamma_Q, \ \Gamma_Q = (-1, 1) \cup \bigcup_{j \in \mathbb{Z}} (\Gamma_j \cup \Gamma_j^*) \).
2. \( Q(z) = I + O\left( \frac{1}{z} \right) \) as \( z \rightarrow \infty \).

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3. \( Q(z) \) takes continuous boundary values on \( \Gamma_N \) away from endpoints and points of self intersection satisfying the jump relation \( N_+(z) = N_-(z)V_Q(z) \) where

\[
V_Q(z) = \begin{cases} 
0 & -1 \ 1 & 0 \end{cases} \quad z \in (-1, 1) \\
\begin{pmatrix} 1 & 0 \\
(r(z)D^{-2}(z)e^{2i\phi(z)+\theta(1)/\epsilon} & 0 \\
1 & 1 \\
(1-r(z)d^2(z)e^{-2i\phi(z)+\theta(1)/\epsilon}) \\
0 & 1 \end{pmatrix} \quad z \in \Gamma_1 \\
(5.17)
\]

\[
\begin{pmatrix} 1 \ -r(z)d(z)e^{-2i\phi(z)+\theta(1)/\epsilon} \\
0 & 1 \end{pmatrix} \quad z \in \Gamma_2
\]

4. \( Q(z) \) is bounded except at the points \{1, -1\} where it admits 1/4-root singularities in each entry.

The jumps of \( Q(z) \) off the real axis converge pointwise to the identity, and the limiting problem on the real axis has a simple solution.

At any fixed distance from \( z = \xi_+ \), \( Q(z) \) is uniformly approximated by the solution of the limiting problem on the real axis given by

\[
P_\infty(z) = E(z; -1, 1),
\]

where \( E \), defined by (3.3), is related to the exact plane wave solution of the ZS scattering problem for the initial data produced by extending the left side of (1.5) to the entire real line.

Near \( \xi_+ \), the jumps \( V_Q \) are not uniformly near identity because of the quadratic vanishing of the phase \( \phi(z) \). To arrive at a global uniform approximation, we construct a local model \( P_{\xi_+} \) defined on a neighborhood \( U_{\xi_+} \) of \( \xi_+ \) which exactly matches the jumps \( V_Q \) inside \( U_{\xi_+} \). The construction is standard and we refer the reader to appendix and references within for more details. Briefly, we fix a neighborhood \( U_{\xi_+} \) such that the map \( \zeta = \sqrt{t} f(z) \) defined by

\[
\frac{1}{2} f(z)^2 = \frac{2}{t} (\phi(z) - \phi(\xi_+)) = 4 \int_{\xi_+}^{z} \frac{(\lambda - \xi_+)(\lambda - \xi_-)}{R(\lambda; -1, 1)} d\lambda, \quad (5.18)
\]

where the branch is chosen such that \( f'(\xi_+) > 0 \), is an invertible conformal map on \( U_{\xi_+} \). We define the local model \( P_{\xi_+}(z) \) by

\[
P_{\xi_+}(z) = E(z; -1, 1)h(z)^{-\kappa_{\xi_+}}M_{PC}(\sqrt{t} f(z), r(z) h(z)^{\kappa_{\xi_+}}, \quad (5.19)
\]

where \( h(z) \) is the analytic conjugating factor

\[
h(z) = e^{i(\phi(z)+\theta(1))}D(z)e^{-i\kappa(z)\log f(z)}, \quad \kappa(z) = -\frac{1}{2\pi} \log(1 + r(z)r^*(z)),
\]

\[
(5.20)
\]

where \( \log(z) = \log \left( \sqrt{t} f(z) \right) \) is principal branched. It is a straightforward calculation to show that \( P_{\xi_+} \) exactly satisfies the jumps (5.17) inside \( U_{\xi_+} \) using the results of appendix and the observation that \( E(\lambda; -1, 1), f(z) \) and \( D(z)e^{-i\kappa(z)\log f(z)} \) (also principally branched) are each analytic in \( U_{\xi_+} \), where the last follows from (5.18) and proposition 5.1.

The inverse analysis is completed by considering the error matrix \( E(z) \) defined by the relation

\[
Q(z) = \begin{cases} 
E(z)P_{\xi_+}(z) & z \in U_{\xi_+} \\
E(z)P_{\infty}(z) & \text{elsewhere.} \end{cases} \quad (5.21)
\]
The new unknown \( E(z) \) is analytic in \( \mathbb{C} \setminus \Gamma_E, \Gamma_E = (\Gamma_N \cup \partial \mathcal{U}_\epsilon) \cup \partial \mathcal{U}_\epsilon \), with uniformly near-identity jump matrices. As such it can be computed according to the small norm theory for RHPs. The essential step comes in computing the jump of \( E(z) \), where the dominant terms come from the mismatch along the positively oriented boundary \( \partial \mathcal{U}_\epsilon \). Using (A.8) we have

\[
E^{-1}(z)V_EE(z) - I = h(z)^{-\phi} \left[ \sqrt{\frac{\epsilon}{t}} \frac{1}{f(z)} \begin{pmatrix} 0 & i\beta_{12}(z) \\ -i\beta_{21}(z) & 0 \end{pmatrix} + O\left( \frac{z}{t} \right) \right] h(z)^{\phi}. \tag{5.22}
\]

Now, both the \( \beta_{jk}(z) \)'s and \( f(z) \) are independent of the asymptotic parameter, but \( h(z) \) is not. In fact \( |h(z)|^2 = O \left( (t/\epsilon)^{\max(1,|z|)} \right) \). Fix \( \eta > 0 \) arbitrarily small, but independent of the asymptotic parameters, by taking \( \mathcal{U}_\epsilon \) small enough (but still fixed with respect to \( t/\epsilon \)) we can make \( |h(z)|^2 \sim O \left( (t/\epsilon)^\eta \right) \). It easily follows that for \( p = 1, 2, \infty \) and \( |z| = \sqrt{1 + |z|^2} \) that

\[
\| (\cdot) (V_E - I) \|_{L^p(\Sigma)} = \begin{cases} O\left( \epsilon^{-\eta/\epsilon} \right) & \Sigma = \Gamma_N \setminus \mathcal{U}_\epsilon, \\ O\left( 1/\epsilon^{1/\eta} \right) & \Sigma = \partial \mathcal{U}_\epsilon, \end{cases} \tag{5.23}
\]

for fixed \( \epsilon > 0 \). This uniformly small bound on the jumps allows one to compute \( E(z) \) iteratively. By keeping more terms in (5.22) we can, in principle, compute a full asymptotic expansion for \( E(z) \) in \( t/\epsilon \). For our purposes it is enough that \( E(z) = I + E(1) + O \left( t/\epsilon \right) \) where \( E(1) = O \left( \sqrt{\frac{\epsilon}{t}} \right) \). The series of transformations from \( m(z) \) to \( E(z) \) can then be inverted to produce the asymptotic expansion of the original problem \( m(z) \). Taking \( |z| \gg 1 \) along the positive imaginary axis we have

\[
m(z) = \left[ e^{i\theta z/\epsilon} D_\infty \right]^{-\phi} E(z) P_\infty(z) \left[ D(z)e^{i\theta z/\epsilon} \right]^{\phi} \tag{5.24}
\]

Then using (3.15) we have

\[
\psi(x, t) = -2i \left[ \lim_{\zeta \to \infty} \zeta E_{12}(\zeta; 1, -1) + E_{12}^{(1)}(\zeta) \right] D_\infty^{-2} e^{-2i\theta z/\epsilon} \tag{5.25}
\]

From this formula using (5.9), (5.14), and (3.3) we find that the leading order behavior of the solution of (1.1)–(1.5) for \( \tau < -1 \) is given by

\[
\psi(x, t) = e^{-i\theta/\epsilon} e^{-\phi(x/\epsilon)} + O\left( \sqrt{\frac{\epsilon}{t}} \right), \tag{5.26}
\]

\[
\phi(\tau) = \exp \left[ \frac{1}{\pi} \left( \int_{-\infty}^{1} + \int_{1}^{\infty} \right) \log(1 - r(z)\rho(z)) \frac{d\lambda}{\sqrt{\lambda^2 - 1}} + \frac{1}{\pi} \int_{\mu_{\lambda}} \arg(r(z)\rho(z)) \frac{d\lambda}{\sqrt{1 - \lambda^2}} \right].
\]

5.4. Rarefaction zone: \(-1 < \tau < -\frac{1}{2} (1 + 3\lambda_+)\)

As \( \tau \) increases beyond \(-1 \) the stationary phase point \( \xi_+(\tau) \) of the far left field phase function (5.10) moves inside \( \mathcal{I}_K \) at \( z = 1 \). When this happens, the previous factorization (5.11) creates an exponentially large jump on the interval \((\xi_+, 1)\). So, for \( \tau > -1 \) we introduce a new \( g \)-function with a single cut \((-1, \lambda_+)\) whose soft edge \( \lambda_+ \) satisfies \( \lambda_+(\tau = -1) = 1 \). Using the

\^2 The error estimate of \( (\epsilon/\tau)^{1/2-\epsilon} \) in (5.23) is a consequence of the maximal value of \( |h(z)| \) on \( \partial \mathcal{U}_\epsilon \). However, the error jump (5.22) admits a meromorphic extension to \( \mathcal{U}_\epsilon \) and the leading term in \( E(1) \) comes from a residue computation at \( \xi_+ \). The result being that \( E(1) = O \left( \sqrt{\frac{\epsilon}{t}} \right) \) and not \( O(\sqrt{\frac{\epsilon}{t}} \log(\epsilon/t)) \) as is the case for non-analytic data. Details of this type of calculation can be found in [29].
The regions $\Omega_i$ and contours $\Gamma_i$ used to define the transformation $M \mapsto N$ for $x/t = \tau$ in the rarefaction zone (defined above). As $x/t = \tau$ increases across the zone, $\xi(\tau)$ and $\lambda_+(\tau)$ move to the right. The limits of the rarefaction zone are characterized by the soft edge $\lambda_+$ colliding with 1 and $\lambda_+$. Blue regions correspond to $\text{Im } \varphi > 0$ and white regions $\text{Im } \varphi < 0$.

results of section 4.1.2, define

$$g(z) = \int_{\lambda_s}^{z} d\theta - 2t \sqrt{\frac{\lambda - \lambda_s}{\lambda + 1}} (\lambda - \xi) d\lambda$$

(5.27)

analytic for $z \in \mathbb{C} \setminus (-1, \lambda_s)$ where

$$\lambda_s(\tau) = -\frac{1}{3} (2\tau - 1), \quad \xi(\tau) = -\frac{1}{6} (4 + \tau),$$

(5.28)

$$g_{\infty} = -\theta(\lambda_s) + \frac{t}{6} (2 - 2\tau - \tau^2)$$

Over the interval $-1 \leq \tau \leq -\frac{1}{3} (3\lambda_+ - 1)$, the soft edge $\lambda_s(\tau)$ decreases linearly from 1 to $\lambda_+$ and for each $\tau$ in this interval $-1 < \xi(\tau) < \lambda_s(\tau) < 1$.

The modified phase function

$$\varphi(z) = 2t \int_{\lambda_s}^{z} \sqrt{\frac{\lambda - \lambda_s}{\lambda + 1}} (\lambda - \xi) d\lambda = t(z - \lambda_s)^{3/2}(z + 1)^{1/2}$$

(5.29)

has an imaginary sign table of the form given in figure 6(b). We open lenses along the steepest descent paths through $\lambda_s$ and $\xi$ which define the contours $\Gamma_i$ and regions $\Omega_i$, see figure 9. The resulting problem for $N(z)$ defined by (5.4)–(5.5) is as follows.

**Riemann-Hilbert Problem 5.3.** Find a $2 \times 2$ matrix $N$ with the following properties

1. $N(z)$ is analytic in $\mathbb{C} \setminus \Gamma_N$, $\Gamma_N = (-\infty, \lambda_s) \cup \bigcup_{i=1}^{2} (\Gamma_i \cup \Gamma_i^*)$.
2. $N(z) = I + O(z^{-1})$ as $z \to \infty$.
3. $N(z)$ takes continuous boundary values on $\Gamma_N$ away from points of self intersection and branch points which satisfy the jump relation $N_+(z) = N_-(z)V_N(z)$ where

$$V_N(z) = \begin{pmatrix}
(1 - r(z)r^*(z))e^{\varphi(z)/\epsilon} & 0 \\
\begin{pmatrix}
1 & 0 \\
\frac{r(z)e^{2\varphi(z)/\epsilon}}{1 - r(z)r^*(z)} & 1
\end{pmatrix} & 1
\end{pmatrix}$$

(5.30)

$z \in \Gamma_4$
5.4.1. Rarefaction parametrix. The jump matrices of the RHP for \( N(z) \) take well defined asymptotic limits whose values are independent of the ordering of \( \xi = \xi(\tau) \) relative to \( \lambda_{\pm} \): 

\[
N(z) = O \left( \frac{(z + 1)^{-1/2}}{(z - 1)^{-1/2}} \right), \quad z \in \mathbb{C}^+
\]

\[
N(z) = O \left( \frac{(z - p)^{1/4}}{(z - p)^{-1/4}} \right), \quad z \in \mathbb{C}^+ \quad p \in [\lambda_-, \lambda_+]
\]

The precise form of the jump \( T(z) \) in (5.30) depends on the position of \( \xi = \xi(\tau) \) relative to \( \lambda_{\pm} \):

\[
T(z) = \begin{cases}
0 & z \in ((-1, \lambda_-) \cup (\lambda_+, \lambda_+)) \cap \{z < \xi\}
-r^*_+(z)e^{-2i\theta(\lambda_+)/\epsilon} & z \in ((-1, \lambda_-) \cup (\lambda_+, \lambda_+)) \cap \{z > \xi\}
0 & z \in (\lambda_-, \lambda_+)
\end{cases}
\]

(5.32)

5.4.1. Rarefaction parametrix. The jump matrices of the RHP for \( N(z) \) take well defined asymptotic limits whose values are independent of the ordering of \( \xi = \xi(\tau) \) and \( \lambda_- \). The jumps off the real axis approach identity pointwise, and along the real axis the jumps take well defined limits, up to imaginary phase constants depending on \( \epsilon \). As before, we first introduce a scalar function \( D(z) \) which simplifies the limiting problem by reducing the limiting problem to one with constant jumps. Define

\[
D(z) = \exp \left[ \frac{i\theta(\lambda_+)}{\epsilon} + \frac{R(z; -1, \lambda_+)}{2\pi i} \left( \int_{(-\infty, -1)} \frac{\log(1 - r_+(\lambda))}{R(\lambda; -1, \lambda_+)} \frac{d\lambda}{\lambda - z} \right) \right]
\]

(5.33)

Proposition 5.2. The function \( D : \mathbb{C} \setminus (-\infty, \lambda_+) \to \mathbb{C} \) defined by (5.33) has the following properties:

1. \( D \) is analytic in \( \mathbb{C} \setminus (-\infty, \lambda_+) \), and takes continuous boundary values on \( (-\infty, \lambda_+) \) except at the endpoints of integration in (5.13).
2. As \( z \to \infty \), \( D(z) \to D_\infty + O \left( z^{-1} \right) \) where

\[
D_\infty = e^{i\theta(\lambda_+)/\epsilon} \left[ \frac{1}{\pi} \int_{(-\infty, -1)} \frac{\log(1 - r_+(\lambda))}{R(\lambda; -1, \lambda_+)} \frac{d\lambda}{\lambda - z} \right]
\]

(5.34)

3. For \( z \in (-\infty, \lambda_+) \), \( D(z) \) satisfies the jump relations

\[
\begin{align*}
D_+(z)/D_-(z) &= 1 - r_+(z) & z \in (-\infty, -1) \\
D_+(z)D_-(z) &= r_+(z)e^{2i\theta(\lambda_+)/\epsilon} & z \in (-1, \lambda_-) \cup (\lambda_+, \lambda_+) \\
D_+(z)D_-(z) &= e^{2i\theta(\lambda_+)/\epsilon} & z \in (\lambda_-, \lambda_+)
\end{align*}
\]

4. \( D(z) \) exhibits the following singular behavior at each endpoint of integration:

\[
D(z) = (z - p)^{1/2}\text{sgn } \arg z D_0(z) \quad z \to p
\]

(5.35)

where \( p \in [-1, \lambda_-, \lambda_+] \) and \( D_0(z) \) is a bounded function taking a definite limit as \( z \) approaches each point non-tangentially.

Using \( D(z) \), the change of variables

\[
Q(z) = D^{\sigma_+}(z)N(z)D(z)^{-\sigma_-}
\]

(5.36)
results in the following RHP for $Q$:

**Riemann-Hilbert Problem 5.4.** Find a $2 \times 2$ matrix $Q$ with the following properties:

1. $Q(z)$ is analytic in $\mathbb{C} \setminus \Gamma_Q$, $\Gamma_Q = (-1, \lambda_s) \cup \bigcup_{i=1}^{2} (\Gamma_i \cup \Gamma_i^*)$.
2. $Q(z) = I + O \left( z^{-1} \right)$ as $z \to \infty$.
3. $Q(z)$ takes continuous boundary values on $\Gamma_N$ away from endpoints and points of self intersection satisfying the jump relation $N_+ (z) = N_- (z) V_Q(z)$ where

\[
V_Q(z) = \begin{cases}
    \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & z \in (\lambda_-, \lambda_+), \\
    \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & z \in ((-1, \lambda_-) \cup (\lambda_+, \lambda_s)) \cap \{ z < \xi \}, \\
    \begin{pmatrix} 1 & -1 & -2 \phi(z) \epsilon \\ \frac{D_2(z)}{D(z)} e^{-2 \phi(z) \epsilon} \\ 1 \\
    \end{pmatrix} & z \in \Gamma_1 \\
    \begin{pmatrix} 1 & -r \phi(z) \epsilon \\ \frac{D_1(z)}{D(z)} e^{-2 \phi(z) \epsilon} \\ 0 \\
    \end{pmatrix} & z \in \Gamma_2.
\end{cases}
\]

4. $Q(z)$ is bounded except at the point $z = -1$ where the entries are $O(z + 1)^{-1/4}$.

The jumps of $Q(z)$ off the real axis converge pointwise to identity, and on the interval $(-1, \lambda_s)$ the jump of $Q(z)$ is either constant, or exponentially close to the same constant. The convergence of the jumps to constants is uniform away from a neighborhood $U_{\lambda_s}$ of the point $\lambda_s$ where the exponential phase $\phi = O(z - \lambda_s)^{3/2}$. As is usual we construct a global parametrix $P$ consisting of an outer model $P_\infty$ and a local model $P_{\lambda_s}$ such that the error matrix $E(z)$ defined by

\[
E(z) = \begin{cases}
    E(z) P_{\lambda_s}(z) & z \in U_{\lambda_s}, \\
    E(z) P_\infty(z) & \text{elsewhere}.
\end{cases}
\]

has jumps uniformly and asymptotically close to identity.

The outer model $P_\infty(z)$ is the solution of the limiting problem with jump $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $(-1, \lambda_s)$ given by

\[
P_\infty(z) = E(z; -1, \lambda_s),
\]

where $E$, defined by (3.7), is related to the Jost functions for the plane wave initial data whose (scaled) Riemann invariants are $-1$ and $\lambda_s = \lambda_s(\tau)$ given by (5.28).

Inside $U_{\lambda_s}$, the jump matrices cannot be uniformly approximated as $\phi = O(z - \lambda_s)^{3/2}$. We construct a local model $P_{\lambda_s}$ which exactly matches the jumps of $Q$ inside $U_{\lambda_s}$. Using (5.29), the function $f(z)$ defined by the relation

\[
\frac{2}{3} (-f(z))^{3/2} = \frac{1}{i} \text{sgn}(\text{Im } z) \psi(z) = (\lambda_s - z)^{3/2} (z + 1)^{1/2}
\]

the roots are taken such that $f'(\lambda_s) > 0$) is a conformal change of coordinates inside $U_{\lambda_s}$. In terms of $\xi = (i/\epsilon)^{2/3} f(z)$ the jumps of $Q(z)$ near $\lambda_s$ are, up to reflection and an explicit conjugation, those of the Airy model problem, RHP A.2 described in appendix. To describe the conjugation let

\[
h_+ (z) = r(z)^{-1/2} D(z) e^{i \phi(\lambda_s) / \epsilon} \quad h_- (z) = r^* (z)^{-1/2} D(z)^{-1} e^{i \phi(\lambda_s) / \epsilon}
\]
The local model \(P_{\lambda_s}\) is given by

\[
P_{\lambda_s}(z) = \begin{cases} 
K(z)\Psi_{\lambda_1} \left( -\left(\frac{1}{2}\right)^{2/3} f(z) \right) \sigma_1 h_1(z)^{\sigma_1} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) & z \in \mathcal{U}_{\lambda_s} \cap \mathbb{C}^+ \\
K(z)\Psi_{\lambda_1} \left( -\left(\frac{1}{2}\right)^{2/3} f(z) \right) \sigma_1 h_{-1}(z)^{\sigma_1} & z \in \mathcal{U}_{\lambda_s} \cap \mathbb{C}^- \end{cases}
\]

(5.42)

where \(K(z)\) is the matching factor, analytic in \(\mathcal{U}_{\lambda_s}\), given by

\[
K(z) = \begin{cases} 
e^{-\frac{3}{2}\sqrt{T}} e^{\frac{2}{3} \mathcal{E}(z)} h_1(z)^{-\sigma_1} \sigma_1 e^{\frac{T}{\lambda_s} \sigma_1} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \left( \frac{-t^{1/3} f(z)}{e^{\sigma_1}} \right)^{2/3} & z \in \mathcal{U}_{\lambda_s} \cap \mathbb{C}^+ \\
e^{-\frac{3}{2}\sqrt{T}} e^{\frac{2}{3} \mathcal{E}(z)} h_{-1}(z)^{-\sigma_1} \sigma_1 e^{\frac{T}{\lambda_s} \sigma_1} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) \left( \frac{-t^{1/3} f(z)}{e^{\sigma_1}} \right)^{2/3} & z \in \mathcal{U}_{\lambda_s} \cap \mathbb{C}^- \end{cases}
\]

(5.43)

That \(K\) is analytic in \(\mathcal{U}_{\lambda_s}\) follows from the fact that \(K(z) = \mathcal{O}\left((z - \lambda_s)^{-1/2}\right)\) and, by direct computation, using proposition 3.1, proposition 5.2, (3.9) and (5.41) \(K\) has no jump across the real axis. The definition of \(K\) is chosen to match the outer model on the boundary \(\partial \mathcal{U}_{\lambda_s}\); from (5.39), (5.42) and (A.14) it follows that

\[
P_{\lambda_s}(z)P_{\lambda_s}(z)^{-1} = I + \mathcal{O} \left( \epsilon t^{-1} \right) \quad z \in \partial \mathcal{U}_{\lambda_s}.
\]

(5.44)

The error \(E(z)\) defined by (5.38) is analytic in \(\mathbb{C}\setminus \Gamma_E\), \(\Gamma_E = \partial \mathcal{U}_{\lambda_s} \cup (\Gamma_Q \setminus \mathcal{U}_{\lambda_s})\); there are no jumps on \(\Gamma_Q \cap \mathcal{U}_{\lambda_s}\) since the local model \(P_{\lambda_s}\) has exactly the same jump matrices as \(Q\) inside \(\mathcal{U}_{\lambda_s}\). Letting \(\langle z \rangle := \sqrt{1 + |z|^2}\), for some \(c > 0\) its jumps satisfy

\[
\| \langle z \rangle (\mathcal{V}_E - I) \|_{L^2(\mathbb{C})} = \begin{cases} \mathcal{O}(\langle z \rangle^{-c}) & \Sigma = \Gamma_Q \setminus \mathcal{U}_{\lambda_s} \\
\mathcal{O}(\langle z \rangle^{-1}) & \Sigma = \partial \mathcal{U}_{\lambda_s}\end{cases}
\]

(5.45)

It follows from the small norm theory for Riemann–Hilbert problems that \(E(z)\) exists and, by expanding (5.44) explicitly, one can compute its full asymptotic expansion in powers of \(\epsilon t^{-1}\).

For our purposes it is enough that \(E(z) = I + \mathcal{O} (\epsilon t^{-1})\) where \(E^{(1)} = \mathcal{O} (\epsilon t^{-1})\). The series of transformations from \(m(z)\) to \(E(z)\) can now be inverted to produce the asymptotic expansion of the original problem \(m(z)\). Using (5.7), (5.28), (5.34), and (5.39) the leading order behavior of the solution of (1.1)–(1.5) for \(-1 < \tau < \frac{1}{2}(1 - 3\lambda_s)\) is given by

\[
\psi(x, t) = \left( \frac{2 - \tau}{3} \right) e^{-\frac{2}{3}(3 - 2\tau - \tau^2)} e^{-\frac{2}{3}(x - t)} + \mathcal{O} \left( \frac{\tau}{t} \right)
\]

(5.46)

\[
\phi(\tau) = \frac{1}{\pi} \left( \int_{-\infty}^{-1} \frac{\log(1 - r(\lambda)^{\tau}(\lambda))}{\sqrt{(\lambda + 1)(\lambda - \lambda_s)}} d\lambda \right) + \int_{(1 - \lambda_s)(\lambda_s, \lambda_s)}^{\arg(r(\lambda)^{\tau}(\lambda))} \frac{\lambda_s - \lambda_s}{\sqrt{(\lambda + 1)(\lambda - \lambda_s)}} d\lambda
\]

\[
5.5. \text{The central plateau: } -\frac{1}{2} (-1 + 3\lambda_s) < \tau < -\frac{1}{2} (-1 + \lambda_s + 2\lambda_s)
\]

For \(\tau = -\frac{1}{2} (-1 + 3\lambda_s)\) the soft edge \(\lambda_s\) (see (5.28)) of the rarefaction \(g\)-function (5.27) collides with \(\lambda_s\), the upper boundary of \(I_B\). If \(\lambda_s < \lambda_s\) then the factorization (5.5) leaves a non-vanishing component in the \((1, 1)\)-entry of \(V_N\) on \(\lambda_s, \lambda_s\) which is exponentially large. The \(g\)-function must be modified to account for this. For \(\tau > -\frac{1}{2} (-1 + 3\lambda_s)\) we use the results of section 4.1.1 to define a \(g\)-function, with a single fixed cut \((-1, \lambda_s)\):

\[
g(z) = \int_{\lambda_s}^{\lambda_s} d\theta - 2\tau \left( \frac{\lambda_s - \xi_s}{\mathcal{R}(\lambda_s, \lambda_s)} \right) = \theta \big|_{\lambda_s}^{\lambda_s} - \tau \mathcal{R}(\xi, -1, \lambda_s)(z - \xi_s),
\]

(5.47)
\[ -\frac{1}{2} (-1 + 3\lambda_+) < \tau < -\frac{1}{2} (-1 + 2\lambda_- + \lambda_+) \]

Figure 10. The contours \( \Gamma_i \) and regions \( \Omega_i \) used to define the map \( M \rightarrow N \) (see (5.5)) for \( s/t = \tau \) in the central plateau (defined above). As \( \tau \) varies across the region, \( -1 < \xi_-(\tau) < \xi_0(\tau) < \xi_+(\tau) < \lambda_+ \) are each decreasing. The lower bound on \( \tau \) in this region is characterized by the collision \( \xi_-(\tau) = \lambda_- \) and the upper bound by \( \xi_0(\tau) = \lambda_+ \). The lesser stationary phase point \( \xi_-(\tau) \) may lie on either side of \( \lambda_- \) for allowed values of \( \tau \). Blue regions correspond to \( \Im \varphi > 0 \) and white regions \( \Im \varphi < 0 \).

where
\[
\xi_0 = -\frac{1}{2} (-1 + \lambda_+) - \tau \quad (5.48)
\]
\[
\xi_{\pm} = \xi_{\pm}(\tau) = \frac{1}{4} \left( -1 + \lambda_+ - \tau \pm \sqrt{(-1 + \lambda_+ + \tau)^2 + 2(\lambda_+ + 1)^2} \right).
\]

are ordered such that \( -1 < \xi_- < \xi_0 < \xi_+ < \lambda_+ \) for \( \frac{-1 + \lambda_+}{2} < \tau < \frac{-1 + \lambda_+ + 2\lambda_+}{2} \). As such, both of the stationary points of the modified phase function
\[
\varphi(z) = 2t \int_{\xi_-}^{\xi_+} \frac{\lambda - \xi_-(\lambda - \xi_+) R(\lambda; -1, \lambda_+)}{R(\lambda; -1, \lambda_+)} d\lambda = -t R(z; -1, \lambda_+)(z - \xi_0)
\]
lie on its branch cut and the transition point for the signature of \( \Im \varphi \) occurs at \( \xi_0 \) which lies between them, see figure 10. The lens contours \( \Gamma_i \) used to define (5.5) are taken as the steepest descent contours through \( \xi_{\pm} \). The contours \( \Gamma_i \) and corresponding regions \( \Omega_i \) are depicted as in figure 10.

The result of (5.4)–(5.5) using (5.47) is the following RHP for \( N(z) \):

**Riemann-Hilbert Problem 5.5.** Find a \( 2 \times 2 \) matrix-valued function \( N \) with the following properties:

1. \( N(z) \) is analytic in \( \mathbb{C} \setminus \Gamma_N \), \( \Gamma_N = (-\infty, \lambda_+) \bigcup_{i=1}^{2}(\Gamma_i \cup \Gamma_i^*) \).
2. \( N(z) = I + \mathcal{O}(z^{-1}) \) as \( z \rightarrow \infty \).
3. \( N(z) \) takes continuous boundary values on \( \Gamma_N \) away from points of self intersection and branch points which satisfy the jump relation \( N_i(z) = N_-(z)V_N(z) \) where

\[
V_N(z) = \begin{cases} 
(1 - r(z)r^*(z))^0 & z \in (-\infty, -1) \\
T(z) & z \in (-1, \xi_-) \\
\begin{pmatrix} 0 & -e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & 0 \end{pmatrix} & z \in (\xi_-, \lambda_-) \\
\begin{pmatrix} 1 & 0 \\
r(z)e^{2i(\varphi(z)+\theta(\lambda_+)/\epsilon)} & 1 \end{pmatrix} & z \in \Gamma_1 \\
\begin{pmatrix} 1 & -r^*(z) \frac{e^{-2i(\varphi(z)+\theta(\lambda_+)/\epsilon)} - 1}{1-r(z)r^*(z)} \\
0 & 1 \end{pmatrix} & z \in \Gamma_2
\end{cases}
\]
4. \( N(z) \) is bounded except at the points \( z = \{-1, \lambda_-, \lambda_+\} \) where the local growth bound at each point are given for \( z \in \mathbb{C}^* \) by

\[
N(z) = O\left( \begin{array}{cc}
(z+1)^{-1/2} & 0 \\
(z+1)^{-1/2} & 0
\end{array} \right)
\]

\[
N(z) = O\left( \begin{array}{cc}
(z-\lambda_-)^{1/4} & (z-\lambda_-)^{-1/4} \\
(z-\lambda_-)^{1/4} & (z-\lambda_-)^{-1/4}
\end{array} \right)
\]

\[
N(z) = O\left( \begin{array}{cc}
(z-\lambda_+)^{1/4} & (z-\lambda_+)^{-1/4} \\
(z-\lambda_+)^{1/4} & (z-\lambda_+)^{-1/4}
\end{array} \right)
\]

In (5.50) \( T(z) \) is one of the following sets of twist matrices, depending on the ordering of \( \xi_- \) and \( \lambda_- \): 

If \( \xi_- > \lambda_- \) then

\[
T(z) = \begin{cases} 
0 & -r_\lambda^*(z)e^{-2i\theta(\lambda_+)/\epsilon} \\
r_\lambda(z)e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
e^{2i\theta(\lambda_+)/\epsilon} & -e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
0 & e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
0 & e^{-2i\theta(\lambda_+)/\epsilon} \\
0 & e^{-2i\theta(\lambda_+)/\epsilon}
\end{cases} 
\]

or if \( \xi_- < \lambda_- \), then

\[
T(z) = \begin{cases} 
0 & -r_\lambda^*(z)e^{-2i\theta(\lambda_+)/\epsilon} \\
r_\lambda(z)e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
e^{2i\theta(\lambda_+)/\epsilon} & -e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
0 & e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
0 & e^{-2i\theta(\lambda_+)/\epsilon} \\
0 & e^{-2i\theta(\lambda_+)/\epsilon}
\end{cases} 
\]

Examining \( T(z) \), \( N(z) \) has near identity jump matrices only if \( \xi_0 > \lambda_- \); if \( \xi_0 < \lambda_- \), then on the segment \( (\xi_0, \lambda_-) \subset (\xi_-, \lambda_-) \) the jump (5.52b) is exponentially large in the \((2,2)\)-entry. This defines the upper boundary of the central plateau region. The upper boundary is the unique \( \tau \) such that \( \xi_0 = \lambda_- \):

\[
\tau = -\frac{1}{2}(-1 + \lambda_+ + 2\lambda_-) \iff \xi_0(\tau) = \lambda_- .
\]

Provided that \( \xi_0 > \lambda_- \), i.e. \(-\frac{1}{2}(-1 + 3\lambda_+) < \tau < -\frac{1}{2}(-1 + \lambda_+ + 2\lambda_-) \), the limiting value of the jump matrices of \( N \) are the same in all cases

\[
V_N(z) \sim \begin{cases} 
(1 - r(z) r^*(z))^\phi \\
r_\lambda(z)e^{2i\theta(\lambda_+)/\epsilon} & -r_\lambda^*(z)e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & -e^{-2i\theta(\lambda_+)/\epsilon} \\
e^{2i\theta(\lambda_+)/\epsilon} & 0 \\
0 & e^{-2i\theta(\lambda_+)/\epsilon}
\end{cases} 
\]

\[
z \in (-\infty, -1) \\
z \in (-1, \lambda_-) \\
z \in (\lambda_-, \lambda_+).
\]

5.5.1. Constructing the parametrix in the central plateau. The RHP for \( N(z) \) has a well defined asymptotic limit—independent of the ordering of \( \xi_0(\tau) \) and \( \lambda_- \). The jumps off the real axis approach identity pointwise, and the jumps along the real axis take well defined limits, up to imaginary phase constants depending on \( \epsilon \). Again we introduce a scalar function \( D(z) \).
which reduces the limiting problem to one with constant jumps. Define

\[
D(z) = \exp \left[ \frac{\text{i} \theta(\lambda_+)}{\epsilon} + \frac{R(z; -1, \lambda_+)}{2\pi i} \left( \int_{[1, -1]} \frac{\log(1 - r(\lambda) r^*(\lambda))}{R(\lambda; -1, \lambda_+)} \frac{d\lambda}{\lambda - z} \right) \right] + \int_{(-1, \lambda_-)} \frac{\log(r_+ (\lambda))}{R_+(\lambda; -1, \lambda_+)} \frac{d\lambda}{\lambda - z}.
\]

(5.53)

**Proposition 5.3.** The function \( D : \mathbb{C} \setminus (-\infty, \lambda_+) \rightarrow \mathbb{C} \) defined by (5.53) has the following properties:

1. \( D \) is analytic in \( \mathbb{C} \setminus (-\infty, \lambda_+) \), and takes continuous boundary values on \((-\infty, \lambda_+)\) except at the endpoints of integration in (5.13).
2. As \( z \rightarrow \infty \), \( D(z) \rightarrow D_\infty + O \left( \frac{1}{z} \right) \) where

\[
D_\infty = e^{\theta(\lambda_+)/\epsilon} \exp \left[ \frac{1}{\epsilon} \left( \int_{[1, -1]} \log(1 - r(\lambda) r^*(\lambda)) \frac{d\lambda}{\lambda - z} \right) \right]
\]

(5.54)

3. For \( z \in (-\infty, \lambda_+) \), \( D(z) \) satisfies the jump relations

\[
\begin{align*}
D_+(z)/D_-(z) &= 1 - r(z) r^*(z) \quad z \in (-\infty, -1) \\
D_+(z)D_-(z) &= r_+(z) e^{2i\theta(\lambda_+)/\epsilon} \quad z \in (-1, \lambda_-) \\
D_+(z)D_-(z) &= e^{2i\theta(\lambda_+)/\epsilon} \quad z \in (\lambda_-, \lambda_+).
\end{align*}
\]

4. \( D(z) \) exhibits the following singular behavior at each endpoint of integration:

\[
D(z) = \left( z - p \right)^{\frac{1}{2} \text{sgn} \Im z} D_0(z) \quad z \rightarrow p
\]

(5.55)

where \( p \in (-1, \lambda_-) \) and \( D_0(z) \) is a bounded function taking a definite limit as \( z \) approaches each point non-tangentially.

Using \( D(z) \), the change of variables

\[
Q(z) = D_{\infty}^\ast N(z) D(z)^{-\ast}
\]

(5.56)

results in the following RHP for \( Q \):

**Riemann-Hilbert Problem 5.6.** Find a \( 2 \times 2 \) matrix \( Q \) with the following properties:

1. \( Q(z) \) is analytic in \( \mathbb{C} \setminus \Gamma_Q \), \( \Gamma_Q = (-1, \lambda_+) \bigcup_{\ell=1}^{\ell+1} (\Gamma_\ell \cup \Gamma_\ell^+) \).
2. \( Q(z) = 1 + O \left( \frac{1}{z} \right) \) as \( z \rightarrow \infty \).
3. \( Q(z) \) takes continuous boundary values on \( \Gamma_N \) away from endpoints and points of self intersection satisfying the jump relation \( N_+(z) = N_-(z) V_Q(z) \) where

\[
V_Q(z) = \begin{pmatrix}
0 & -1 \\
1 & \frac{D_+(z; -1, \lambda_+)}{\epsilon} e^{2i\psi(z)/\epsilon} \\
\frac{1}{r(z) D(z; -1, \lambda_+)} e^{2i\phi(z) + 2i\theta(1)/\epsilon} & 0 \\
\frac{1 - r(z) D(z; -1, \lambda_+)}{r(z) D(z; -1, \lambda_+)} & 1
\end{pmatrix}
\]

(5.57)

Here, \( 1_{a<z<b} \) is the indicator function of the set \( (a, b) \). If \( b < a \) than this is the indicator of the empty set and the function is identically zero.

4. \( Q(z) \) is bounded except at the points \( z = -1, \lambda_+ \) where it admits \( 1/4 \)-root singularities in each entry.
Unlike the other four cases, the convergence of $Q(z)$ to its limiting problem is uniform in the plane and we can construct a parametrix globally without the need to introduce a local model. The reason for this happy coincidence is that, in the plateau region, the stationary phase points $\xi_{\pm}$, where the steepest descent contours $\Gamma_{\tau}$ return to the real axis, now both lie on the cut $(-1, \lambda_+)$ of the phase $\psi$ where $\text{Im} \, \psi_{\pm}(z) \neq 0$. As a consequence, there exist a constant $c > 0$ such that the off-diagonal entries of $V_Q$ in (5.57) on $\Gamma_{\tau}$, $\tau = 1, 2$, and the $(2, 2)$ entry for $z \in (-1, \lambda_+)$ are $O(e^{-ct/\epsilon})$. So, what has been an outer model is, here, a uniformly accurate approximation in the whole plane and no local model is needed. We have

$$Q(z) = E(z)E(z; -1, \lambda_+),$$  \hspace{1cm} (5.58)

where the error $E(z)$ is the solution of a Riemann–Hilbert problem with jumps $V_E$ on $\Gamma_E := \chi_{[\lambda_-, \lambda_+]} \bigcup_{\tau = 1}^{\tau = 2} (\Gamma_{\tau} \cup \Gamma_{\tau}^+)$ which are exponentially near identity: $\| (\cdot) (V_E - I) \|_{L^p(\Gamma_E)} = O(e^{-ct/\epsilon})$. Using the small norm theory for Riemann–Hilbert problem the error matrix $E(z)$ is itself an exponential perturbation of identity $E(z) = I + E^{(1)}(z) + O(z^{-2})$, $E^{(1)} = O(e^{-ct/\epsilon})$ for each $\tau \in \left(-\frac{1 + \lambda_+, 2\lambda_+, 2\lambda_+ + 1}{2}, \frac{-1 + 3\lambda_+ - \lambda_+}{2}\right)$.

The resulting behavior of the solution of (1.1)–(1.5) using (5.7), (5.47), (5.54), and (5.58), is

$$\psi(x,t) = \left(\frac{\lambda_+ + 1}{2}\right) e^{-i(2\chi - \omega t)/\epsilon} + O(e^{-ct/\epsilon})$$

$$k = \lambda_+ - 1 \quad \omega = -\frac{1}{2} (\lambda_+ - 1)^2 - \frac{1}{4} \left(\lambda_+ + 1\right)^2$$

$$\phi_0 = \frac{1}{\pi} \left(\int_{-\infty}^{-1} \log(1 - r(\lambda)r^*(\lambda)) d\lambda + \int_{-1}^{\lambda_+} \frac{\arg(r(x, \lambda))}{\sqrt{(\lambda + 1)(\lambda - \lambda_+)} d\lambda}\right)$$

5.6. The modulation zone: $-\frac{1}{2} (-1 + \lambda_+ + 2\lambda_-) < \tau < -\frac{1}{2} (\lambda_+ + \lambda_- - 2) + \frac{2(1 + \lambda_+ - 1 + \lambda_-)}{\lambda_+ + \lambda_- - 2}$

When $\tau$ increases beyond $-\frac{1}{2} (-1 + 2\lambda_- + \lambda_+)$ the point $\xi_0(\tau)$ (defined by (5.48)) lies to the right of $\lambda_-$. This makes the $(2, 2)$ entry of the jump $V_N$ defined by (5.50) exponentially large on the interval $(\xi_0, \lambda_-)$. To arrive at a stable limit problem we modify the $g$-function to include a gap interval below $\lambda_-$, with a soft upper edge. Following section 4.2 define the $g$-function, analytic for $z \in \mathbb{C} \setminus \{(-1, \lambda_+) \cup (\lambda_-, \lambda_+)\}$:

$$g(z) = \int_{\lambda_-}^{z} d\theta - 2I \frac{(\lambda_+ - \lambda_+)(\lambda - \xi_0)(\lambda - \xi_0)}{R(\lambda_+ - 1, \lambda, \lambda_-, \lambda_+)} d\lambda.$$  \hspace{1cm} (5.60)

The motion of the soft edge $\lambda_s = \lambda_s(x/t)$ is given by the self-similar solution of the Whitham equations:

$$\frac{x}{t} = V_3(\lambda_+, \lambda_-, \lambda_s, -1) = -\frac{1}{2} (-1 + \lambda_+ + \lambda_+ + \lambda_-) - \frac{\lambda_+ + 1}{1 - \frac{\lambda_+ + 1}{\lambda_+ - \lambda_+}} \frac{E^{(m)}}{K^{(m)}}$$

$$m = \frac{(\lambda_+ - \lambda_-)(\lambda_s + 1)}{(\lambda_- - \lambda_+)(\lambda_s + 1)}$$

where $K^{(m)}$ and $E^{(m)}$ are the complete elliptic integrals of the first and second kind respectively. The above equation is solvable for each $\lambda_s \in (-1, \lambda_-)$. Using (5.61) it’s easy to
verify the two-band solution degenerates when:

\[
\begin{align*}
\lambda_s &\to \lambda_-, & \tau &\to -\frac{1}{2} (\lambda_s + 2\lambda_+ - 1), \\
\lambda_s &\to -1, & \tau &\to -\frac{1}{2} (\lambda_+ + \lambda_+ - 2) + \frac{2(\lambda_+ + 1)(\lambda_+ - 1)}{\lambda_s + \lambda_+ + 2},
\end{align*}
\]

(5.62)

which define the transition from the modulation zone to the plane wave zones which it separates. The two stationary phase points \(\xi_-(\tau)\), and \(\xi_+(\tau)\) which lie one in each band can be computed from (4.27).

The phase function

\[
\varphi(z) = 2t \int_{\lambda_s}^{z} \frac{(\lambda - \lambda_+)(\lambda - \xi_-)(\lambda - \xi_+)}{R(\lambda; -1, \lambda_s, \lambda_-, \lambda_+)} d\lambda
\]

(5.63)

is analytic in \(\mathbb{C}\setminus(-1, \lambda_s) \cup (\lambda_-, \lambda_+)\) and satisfies the jump relation

\[
\varphi_+(z) + \varphi_-(z) = \begin{cases} 
0 & z \in (\lambda_-, \lambda_+) \\
\gamma = 2(\varphi_+(\lambda_s) - \varphi(\lambda_+)) & z \in (-1, \lambda_s)
\end{cases}
\]

(5.64)

The structure of the zero level set of \(\text{Im } \varphi\) resembles that in figure 7. In order to define the mapping from \(m \mapsto N\) given by (5.4)–(5.5) in the two band case we open lenses from \(\xi_-(\tau)\) (opening to the left) and from \(\lambda_s(\tau)\) (opening to the right) as shown in figure 11. The result of (5.4)–(5.5) with \(g\) given by (5.60) is the following RHP for \(N(z)\):

**Riemann-Hilbert Problem 5.7.** Find a \(2 \times 2\) matrix-valued function \(N\) with the following properties:

1. \(N(z)\) is analytic in \(\mathbb{C}\setminus\Gamma_N, \Gamma_N = (-\infty, \lambda_s(\tau)) \cup (\lambda_-, \lambda_+) \bigcup_{i=1}^{2}(\Gamma_i \cup \Gamma_i^*)\).
2. \(N(z) = I + O(z^{-1})\) as \(z \to \infty\).
3. $N(z)$ takes continuous boundary values on $\Gamma_N$ away from points of self intersection and branch points which satisfy the jump relation $N_+(z) = N_-(z)V_N(z)$ where

$$
V_N(z) = \begin{cases}
(1 - r(z)r^+(z))^{\theta s} & z \in (-\infty, -1) \\
\begin{pmatrix}
0 & -r^+(z)e^{-i\theta/\epsilon}e^{-2i\theta(\lambda_s)/\epsilon} \\
r(z)e^{i\theta/\epsilon}e^{2i\theta(\lambda_s)/\epsilon} & 0
\end{pmatrix} & z \in (-1, \xi_-(\tau)) \\
\begin{pmatrix}
0 & -r^+(z)e^{-i\theta/\epsilon}e^{-2i\theta(\lambda_s)/\epsilon} \\
r(z)e^{i\theta/\epsilon}e^{2i\theta(\lambda_s)/\epsilon} & e^{i\theta/\epsilon}e^{2i\theta(\lambda_s)/\epsilon}
\end{pmatrix} & z \in (\xi_-(\tau), \lambda_s(\tau)) \\
\begin{pmatrix}
1 & r(z)e^{2i\theta(\lambda_s)/\epsilon}e^{2(\lambda_c(\tau))/\epsilon} \\
e^{-2i\theta(\lambda_s)/\epsilon} & 1
\end{pmatrix} & z \in \Gamma_1 \\
\begin{pmatrix}
0 & e^{-2i\theta(\lambda_s)/\epsilon} \\
e^{2i\theta(\lambda_s)/\epsilon} & 1
\end{pmatrix} & z \in \Gamma_2
\end{cases}
$$

4. $N(z)$ is bounded except at the point $z = -1, \lambda_-, \lambda_+$ where

$$
N(z) = O\left(\begin{pmatrix}
(z + 1)^{-1/2} \\
(z + 1)^{-1/2}
\end{pmatrix}, \quad z \in \Omega_3
\right)
$$

$$
N(z) = O\left(\begin{pmatrix}
(z - \lambda_\pm)^{-1/4} \\
(z - \lambda_\pm)^{-1/4}
\end{pmatrix}, \quad z \in \Omega_3
\right)
$$

5.6.1. Constructing the parametrix in the modulation zone. In the long-time/small dispersions limit, the jumps of $N(z)$ along the real axis have well defined limits up to $\epsilon$-dependent constants, while the jumps on the non-real contours approach identity uniformly at any distance from $\lambda_s$ (the convergence at $\xi_-$ is uniform provided $\lambda_s$ and $-1$ are well separated):

$$
V_N(z) \sim_{\epsilon \to 0} \begin{cases}
(1 - r(z)r^+(z))^{\theta s} & z \in (-\infty, -1) \\
\begin{pmatrix}
0 & -r^+(z)e^{-i\theta/\epsilon}e^{-2i\theta(\lambda_s)/\epsilon} \\
r(z)e^{i\theta/\epsilon}e^{2i\theta(\lambda_s)/\epsilon} & 0
\end{pmatrix} & z \in (-1, \lambda_s(\tau)) \\
\begin{pmatrix}
0 & -e^{-2i\theta(\lambda_s)/\epsilon} \\
e^{2i\theta(\lambda_s)/\epsilon} & 1
\end{pmatrix} & z \in (\lambda_-, \lambda_+).
\end{cases}
$$

In order to build a uniformly accurate parametrix, we introduce a scalar function $D(z)$ which reduces this limiting problem to one with constant jumps. Define

$$
D(z) = D_0(z)D_1(z)
$$

$$
D_0(z) = \exp\left[\frac{\pi i}{4} + \frac{\mathcal{R}(z, \lambda)}{2\pi} \int_{-\infty}^{-1} \log(1 - r(s)r^+(s)) \frac{ds}{s - z} + \int_{-1}^{\lambda_s} \log r_+(s) \frac{ds}{s - z} \right],
$$

$$
D_1(z) = \exp\left[\frac{i\theta(\lambda_\pm)}{\epsilon} + \frac{\mathcal{R}(z, \lambda)}{2\pi} \int_{-1}^{\lambda_s} \frac{i\epsilon}{s - z} \frac{ds}{s - z} \right].
$$

Here $\lambda = [-1, \lambda_-, \lambda_-, \lambda_+]$ are the branch points of $\mathcal{R}(z, \lambda)$ and $g(z)$.

**Proposition 5.4.** The function $D : \mathbb{C}\setminus(-\infty, \lambda_s) \to \mathbb{C}$ defined by (5.68) has the following properties:

1. $D$ is analytic in $\mathbb{C}\setminus(-\infty, \lambda_s)$, and takes continuous boundary values on $(-\infty, \lambda_s)$ except at the endpoints of integration in (5.13).
2. As \( z \to \infty \), \( D(z) \to D_{\infty}(z) \left[ 1 + O\left(z^{-1}\right) \right] \) where

\[
D_{\infty}(z) = e^{-iz^4/4} \phi_{\lambda}(z) e^{i(\phi(z) + e^{-1}\phi(z))} \tag{5.69}
\]

and \( \phi_k(z) \), \( k = 0, 1 \) are the linear functions

\[
\phi_0(z) = \sum_{j=0}^{\lambda} \frac{z^j}{2\pi} \int_{-\infty}^{\infty} \left( w + V \right)^j \left( \log(1 - |r(w)|^2) \Pi_{-\infty,-1} + \log r_e(w) \Pi_{-\infty,-1} \right) \frac{1}{R_0(w, \lambda)} \text{d}w,
\]

\[
\phi_1(z) = \sum_{j=0}^{\lambda} \frac{z^j}{2\pi} \int_{-1}^{\lambda} \frac{y_j (w + V)^j}{R_0(w, \lambda)} \text{d}w.
\]

where \( V = -\frac{\sigma_1}{2} = -\frac{1}{2} \sum_{j=1}^{\lambda} \lambda_j \).

Note, that as \( |r_e(s)| = 1 \) and \( \text{Re} \left( R_\lambda(s, \lambda) \right) = 0 \) \( \forall s \in (-1, \lambda_\lambda) \), each \( \phi_k(z) \) is a real (linear) polynomial.

3. For \( z \in (-\infty, \lambda_\lambda) \), \( D(z) \) satisfies the jump relations

\[
D_+(z)/D_-(z) = 1 - r(z)\sigma_+ (z) \quad z \in (-\infty, -1)
\]

\[
D_+(z)/D_- (z) = 1 - r(e^\epsilon) e^{i\theta(\lambda_\lambda)/\epsilon} \quad z \in (-1, \lambda_\lambda)
\]

\[
D_+(z)/D_- (z) = 1 - e^{i\theta(\lambda_\lambda)/\epsilon} \quad z \in (\lambda_\lambda, \lambda_+),
\]

4. \( D(z) \) exhibits the following singular behavior at each endpoint of integration:

\[
\begin{align*}
D(z) &= (z + 1)^{\frac{1}{2} \text{sgn} \Im z} \hat{D}(z) \quad z \to -1 \\
D(z) &= \hat{D}(z) \quad z \to \lambda_\lambda
\end{align*}
\]

where in each case \( \hat{D}(z) \) is a (different) bounded function taking a definite limit as \( z \) approaches each point non-tangentially.

We also introduce

\[
\sigma(z) = \left( \frac{z - \lambda_\lambda}{z - \lambda_\lambda} \right)^{1/4} \left( \frac{z - \lambda_+}{z + 1} \right)^{1/4}
\]

cut along \((-1, \lambda_\lambda)\) and \((\lambda_-, \lambda_\lambda)\) and normalized such that \( \sigma(z) \sim 1 \) as \( z \to \infty \) to define the transformation

\[
Q(z) = \sigma(z)^{-1} N(z) D(z)^{-\sigma_+} \tag{5.74}
\]

then \( Q \) must satisfy the following constant jump RHP:

**Riemann-Hilbert Problem 5.8.** Find a \( 2 \times 2 \) matrix valued function \( Q(z) \) such that

1. \( Q(z) \) is analytic in \( \mathbb{C} \setminus (-1, \lambda_\lambda) \cup (\lambda_-, \lambda_\lambda) \cup \bigcup_{\tau=1}^{\lambda_\lambda} (\Gamma_\tau \cup \Gamma_\lambda^*) \).

2. \( Q(z)D_{\infty}(z)^{\sigma_+} = I + O\left(z^{-1}\right) \) as \( z \to \infty \).

3. \( Q(z) \) takes continuous boundary values on \( \Gamma_Q \) away from the points of self intersection and endpoints, which satisfy the jump relation \( Q_+ (z) = Q_-(z) V_Q (z) \) where

\[
V_Q (z) = \begin{cases} \sigma_1, & z \in (-1, \xi_-(\tau)) \cup (\lambda_-, \lambda_+) \\ 0 & z \in (\xi_-(\tau), \lambda_\lambda) \\ 1 & z \in \Gamma_1 \\ 1 & z \in \Gamma_2. \end{cases}
\]

4. \( Q(z) \) admits \( 1/4 \)-root singularities at \( \lambda_\lambda \) and \( 1/2 \)-root singularities at \( \lambda_+ \).
The jump matrix for $Q(z)$ converges pointwise to identity away from the real axis, and to constants on the real axis. The convergence is uniform away from the soft edge $\lambda_s$, where the lens contours return to the real axis. We take $U_{\lambda_s}$ a local neighborhood of $\lambda_s$ and build local and outer parametrices $P_{\lambda_s}$ and $P_\infty$ respectively so that the relation

$$
Q(z) = \begin{cases} 
E(z)P_{\lambda_s}(z) & z \in U_{\lambda_s} \\
E(z)P_\infty(z) & \text{elsewhere}
\end{cases}
$$

(5.76)

results in a residual problem for $E(z)$ which can be proven to exist and asymptotically expanded using the small-norm theory for RHPs.

To build the outer solution, we replace the jump condition (5.75) in RHP 5.8 with

$$
P_\infty(z) = P_\infty(z) - \sigma_1 \quad \text{for } z \in (-1, \lambda_1) \cap (\lambda_-, \lambda_+),
$$

(5.77)

and admit at most $1/2$-root singularities at $\lambda_+$ and $\lambda_s$. The solution of such a multi-cut problem is constructed from theta functions on the hyperelliptic Riemann surface associated with $R(z; \lambda)$.

The construction is standard, so we will provide only the necessary formula to define the solution.

Let $\lambda = (\lambda_+, \lambda_-, \lambda_s, -1)$ denote the moduli of the genus-one Riemann surface

$$
S_1 := \left\{ P = (z, R), \ R^2 = \prod_{i=1}^{4}(z - \lambda_i) \right\}
$$

and fix the homology basis as in figure 4. Define the holomorphic differential

$$
\nu(z, R) = c_v \frac{dz}{R}, \quad c_v = \frac{i\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{4K(m)}
$$

(5.78)

normalized so that

$$
\oint_a \nu = 2c_v \int_{\lambda_1}^{\lambda_2} \frac{dz}{R(z, \lambda)} = 1.
$$

(5.79)

Then we also have the elliptic half-period ratio

$$
\tau := \oint_b \nu = 2c_v \int_{\lambda_3}^{\lambda_4} \frac{dz}{R(z, \lambda)} = \frac{K(1-m)}{K(m)}, \quad m = \frac{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}
$$

(5.80)

where $K(m)$ denotes the complete elliptic integral of the first kind with parameter $m$.

Using these quantities, define the Siegel theta function

$$
\Theta(z) = \Theta(z; \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i (nz + \frac{1}{2}n^2 \tau)} = \theta_3(\pi z, e^{i \tau})
$$

(5.81)

Here, $\theta_3(z, q)$ is the standard Jacobi theta function with nome $q$. Note that $\Theta(z)$ is a quasi-doubly periodic function satisfying:

$$
\Theta(z + 1) = \Theta(z), \quad \Theta(z + \tau) = \Theta(z)e^{-2\pi i \tau}e^{-i \tau^2/2},
$$

(5.82)

and vanishes at the lattice of half periods:

$$
\Theta(z) = 0, \quad \text{iff } z = \frac{1}{2} + \frac{\tau}{2} + \mathbb{Z} + \tau \mathbb{Z}
$$

(5.83)
The half-period ratio $\tau$ appearing only here, as it relates to the theta functions, should not be confused with the similarity variable $\tau = \lambda / t$ used throughout the paper.

Let $A(z)$ denote the restriction of the standard Abel map to the complex plane:

$$A(z) = \int_{z_0}^{z} \frac{e_v}{R(z; \lambda)} \, dz$$

where the path of integration lies in $\mathbb{C} \setminus \{\lambda_4, \lambda_3\} \cup \{\lambda_2, \lambda_1\}$.

We also need the following normalized differential of the second kind

$$\nu = \nu_0 + \epsilon^{-1} \nu_1$$

$$\nu_k = \phi_k \omega^{(0)}, \quad k = 0, 1$$

where $\phi_k$ is the coefficient of the linear term of $\phi_1(z)$ given by (5.70) and $\omega^{(0)}$ is the normalized differential of the second kind defined by (4.2). Let $\gamma$ be the $b$-period of this differential

$$\gamma = \gamma_0 + \epsilon^{-1} \gamma_1$$

The purpose of this differential is to cancel the behavior of $D(z)$ at infinity. Define $\chi$ by the relation

$$\chi = \chi_0 + \epsilon^{-1} \chi_1, \quad \chi_k = -i \log \left( \lim_{z \to \infty} D_k(z) e^{-i f_k(z) u_k} \right).$$

Clearly, both $\gamma$ and $\chi$ are real quantities.

The outer model $P_\infty(z)$ approaching the solution of RHP 5.8 away from $\lambda_s$ is given by

$$P_\infty(z) = \frac{\Theta(0)}{\Theta(\frac{\pi i}{2})} e^{-i \gamma_0} e^{-i \varepsilon / 2} \left( \frac{1 + i \varepsilon}{2} \frac{1 + i \varepsilon}{2} \frac{1 + i \varepsilon}{2} \frac{1 + i \varepsilon}{2} \right) e^{-i f(z) u_k} \right).$$

At first glance, it seems the outer model depends in a complicated way on the asymptotic parameter. However, it is a simple calculation to show that

$$d(\log D_i) = d \left( \frac{R(z; \lambda)}{2\pi i} \right) = \int_{-1}^{1} \frac{i \gamma}{R_n(w; \lambda)} \, dw = \nu_1$$

and such it follows that

$$\chi_1 = \theta(\lambda_s) \quad \gamma_1 = \gamma = 4\pi i c_r(x + \frac{1}{2} e_1(\lambda) t)$$

where the last equality comes from explicit computation, by identifying $\gamma = \int_{0}^{\infty} d\varphi$ and making use of the Riemann bilinear relations.

Putting the parts together, the ratio $P_\infty(z) D(z)^{-\gamma_0}$ for large $z$ is given by

$$P_\infty(z) D(z)^{\gamma_0} = \frac{\Theta(0)}{\Theta(\frac{\pi i}{2})} e^{-i \varepsilon / 2} e^{-i \varepsilon / 2} \left( \frac{1 + i \varepsilon}{2} \frac{1 + i \varepsilon}{2} \frac{1 + i \varepsilon}{2} \frac{1 + i \varepsilon}{2} \right) e^{-i f(z) u_k} \right).$$

where

$$T(z) = \left( \begin{array}{ccc} \frac{a(z) e^{-1} \theta(A(z) + A(\infty) + 2 \varepsilon)}{2} & \frac{a(z) e^{-1} \theta(A(z) + A(\infty) + 2 \varepsilon)}{2} \\ \frac{a(z) e^{-1} \theta(A(z) + A(\infty) + 2 \varepsilon)}{2} & \frac{a(z) e^{-1} \theta(A(z) + A(\infty) + 2 \varepsilon)}{2} \end{array} \right)$$

Note importantly that the coefficients $\gamma$, $\gamma_0$, and $\theta(\lambda_s)$ are all real so that the $\epsilon$-dependence of the solution corresponds only to rapid oscillations.

The outer model is uniformly accurate except in any fixed neighborhood $U_0$, of $\lambda_s$. A local model must be inserted inside $U_0$. At $\lambda_s$ we have the usual critical behavior of a soft
edge \( \varphi(z) - \varphi(\lambda_4) = O((z - \lambda_4)^{3/2}) \), and the appropriate local model is constructed from the Airy model described in appendix A. The construction is standard and we summarize it here for completeness. It follows from (5.63)–(5.64) that the relation
\[
\frac{2}{3} \left( -f(z) \right)^{2/3} = \frac{i}{t} \text{sgn}(\text{Im } z) \left( \varphi(z) - \frac{y}{2} \right) = -\frac{2}{3} \int_{\lambda_4}^{z} \frac{\lambda - \xi_- (\lambda - \xi_+)}{\sqrt{\lambda - \lambda_+ \sqrt{\lambda - \lambda_- \sqrt{\lambda + 1}}} d\lambda, \tag{5.91}
\]
where the root is taken such that \( f'(\lambda_4) > 0 \), is a locally analytic and invertible mapping of \( \mathcal{U}_{\lambda_4} \) onto a neighborhood of the origin. In terms of \( \varphi(z) \) and the nonlinearity \( 28 \)
\[Nonlinearity 28 (2015) 2131 R Jenkins\]

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\[Nonlinearity 28 (2015) 2131 R Jenkins\]

Then the local model \( P_{\lambda_4} \) is given by
\[
P_{\lambda_4}(z) = \begin{cases} 
\hat{K}(z) \alpha(z)^{-1} \mathcal{M}_\lambda \left( -\left( \frac{z}{\lambda} \right)^{2/3} f(z) \right) \hat{h}_s(z) \gamma, & z \in \mathcal{U}_{\lambda_4} \cap \mathbb{C}^+, \\
\hat{K}(z) \alpha(z)^{-1} \mathcal{M}_\lambda \left( -\left( \frac{z}{\lambda} \right)^{2/3} f(z) \right) \hat{h}_s(z) (-i \sigma_1), & z \in \mathcal{U}_{\lambda_4} \cap \mathbb{C}^-,
\end{cases}
\tag{5.93}
\]
where \( \hat{K}(z) \) is the matching factor, analytic in \( \mathcal{U}_{\lambda_4} \), given by
\[
\hat{K}(z) = \begin{cases} 
e^{-\frac{\pi i}{3} \sqrt{\pi} \alpha(z) P_{\infty}(z)} \hat{h}_s(z)^{-} \gamma_{\alpha} \left( \frac{1}{1} \right) \left( -\frac{e^{3/2} f(z)}{e^{3/2}} \right)^{\frac{2}{3} \xi} & z \in \mathcal{U}_{\lambda_4} \cap \mathbb{C}^+, \\
e^{-\frac{\pi i}{3} \sqrt{\pi} \alpha(z) P_{\infty}(z)} \hat{h}_s(z)^{-} \gamma_{\alpha} \left( \frac{1}{1} \right) \left( -\frac{e^{3/2} f(z)}{e^{3/2}} \right)^{\frac{2}{3} \xi} & z \in \mathcal{U}_{\lambda_4} \cap \mathbb{C}^-.
\end{cases}
\tag{5.94}
\]
The analyticity of \( \hat{K}(z) \) follows from the observation that \( \hat{K}(z) \) is analytic in \( \mathcal{U}_{\lambda_4} \setminus \mathbb{R} \) and \( \hat{K}(z) = O((\lambda - \lambda_4)^{-1/2}) \) and by direct calculation using (5.71) and (5.77) we find that \( K_{-}(z)^{-1} \hat{K}(z) = I \) for \( z \in \mathcal{U}_{\lambda_4} \cap \mathbb{R} \).

The definition of \( K(z) \) is such that the leading order term on the inner model matches the outer model on the boundary \( \partial \mathcal{U}_{\lambda_4} \); from (A.14) and the \( \epsilon \) and \( t \) scaling in (5.91), (5.93) we have
\[
P_{\infty}(z) P_{\lambda_4}(z)^{-1} = I + O\left( \frac{\epsilon^2}{t} \right), \quad z \in \partial \mathcal{U}_{\lambda_4}. \tag{5.95}
\]

The explicit formulas for the model solutions \( P_{\infty} \) and \( P_{\lambda_4} \) imply that the error matrix \( E(z) \) defined by (5.76) is analytic inside \( \mathcal{U}_{\lambda_4} \) as the local model \( P_{\lambda_4} \) exactly matches the local jump matrices of \( Q \). The error \( E(z) \) is analytic in \( \mathbb{C} \setminus \Gamma_E \), \( \Gamma_E = \partial \mathcal{U}_{\lambda_4} \cup (\Gamma_\sigma \setminus \mathcal{U}_{\lambda_4}) \) and, letting \( (z) := \sqrt{1 + |z|^2} \), for some \( c > 0 \) its jumps satisfy
\[
\| (z) (V_E - I) \|_{L^p(\mathbb{C})} = \begin{cases} 
O(\epsilon^{-c/t}), & z \in \Gamma_\sigma \setminus \mathcal{U}_{\lambda_4}, \\
O(\epsilon^{-1}), & z \in \partial \mathcal{U}_{\lambda_4}.
\end{cases} \tag{5.96}
\]

It follows from the small norm theory for Riemann–Hilbert problems that \( E(z) \) exists and has a full asymptotic expansion in powers of \( \epsilon/t \) which can be computed in principle using (A.14) to expand (5.95). For our purposes it is enough that \( E(z) = I + O\left( \epsilon^{1/4} \right) \). The series of transformations from \( m(z) \) to \( E(z) \) can now be inverted to produce the asymptotic expansion of the original problem \( m(z) \). It follows that the solution \( \psi(x, t) \) of (1.1)–(1.5) has the resulting expansion valid for each \( x, t \) in the modulation zone:
\[
\psi(x, t) = \frac{\theta_1 - \lambda_2 + \lambda_3 - \lambda_4}{2} \frac{\Theta(0)}{\Theta(\frac{\theta_1 - \lambda_2 + \lambda_3 - \lambda_4}{2})} \frac{\Theta(2A(\infty) + \frac{\gamma_{\theta_1 - \lambda_2 + \lambda_3 - \lambda_4}{2}}{2\pi})}{\Theta(2A(\infty))} e^{2i(\chi_0 + \gamma_{\theta_1 - \lambda_2 + \lambda_3 - \lambda_4})/\epsilon} + O\left( \frac{\epsilon}{t} \right). \tag{5.97}
\]

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5.6.2. Computing the leading order square modulus. Recognizing that $\gamma_0$ and $\gamma$ are real, while $A(\infty)$ is pure imaginary, write

$$\rho(x,t) = \frac{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4}{2} \frac{\theta_3(0)^2 \theta_3(w + iv) \theta_3(w - iv)}{\theta_1^2(w) \theta_1^2(iv)} \frac{\theta_3(0)^2 \theta_3(0) \theta_3(iv)}{\theta_1^2(0) \theta_1^2(iv)} s_d(w \theta_1^2, m)$$

Then in terms of these real variables we have

$$\rho(x,t) := |\psi(x,t)|^2 = \left( \frac{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4}{2} \frac{\theta_3(0)^2 \theta_3(w + iv) \theta_3(w - iv)}{\theta_1^2(w) \theta_1^2(iv)} \right)^2 \left( \frac{\theta_3(0)^2 \theta_3(0) \theta_3(iv)}{\theta_1^2(0) \theta_1^2(iv)} s_d(w \theta_1^2, m) \right)$$

To simplify the formula further we can evaluate the ratio of theta functions as follows. Write $iv := 2\pi f_{\infty} v = 2x$ and make use of duplication formulae [35] to write

$$\frac{\theta_1^2(iv)}{\theta_1^2(iv)} = \frac{4 \theta_1^2(0)}{\theta_1^2(0)} \left( \frac{\theta_1(x) \theta_3(x) \theta_3(x) \theta_3(x)}{\theta_1^2(x) \theta_1^2(x)} \right)^2 = 4 \sqrt{1 - m} \frac{T^2}{T^2 - 1},$$

where $T = \frac{\eta(x)/(\eta(x))}{\eta(3)(\eta(3)) \eta(3)}$. To compute $T$, consider the function $F(P)$ defined on the Riemann surface $S_1$ by

$$F(P) := \frac{\Theta(f_{\infty}, v + \frac{1}{2} + \frac{z}{2})}{\Theta(f_{\infty}, v + \frac{1}{2} + \frac{z}{2})} e^{-2\pi i f_{\infty} v}.$$

It follows from (5.82) that $F$ is single-valued on $S_1$ and by construction $F$ has simple zeros at $\lambda_3$ and $\lambda_4$ and simple poles at $\lambda_1$ and $\lambda_2$. That is, $F$ is meromorphic on $S_1$ and we have

$$F(P) = \frac{\theta_3(0)^2 \theta_3(0)}{\theta_3(0) \theta_3(0) \frac{dP}{d\lambda_1}} \frac{(z - \lambda_3)^{1/2}}{(z - \lambda_3)^{1/2}} \frac{(z - \lambda_4)^{1/2}}{(z - \lambda_4)^{1/2}}$$

where the normalization comes from matching the residues at $\lambda_1$, and $dv/dP(\lambda_1)$ is computed in the local coordinate on $S_1$ near $\lambda_1$. Computing $F(\infty, \nu)$ using both representations of $F$ gives:

$$F(\infty, \nu) = \frac{\Theta(f_{\infty}, v)}{\Theta(f_{\infty}, v + \frac{1}{2} + \frac{z}{2})} e^{-2\pi i f_{\infty} v} \frac{\theta_3(x) \theta_3(x)}{\theta_1^2(x) \theta_1^2(x)}$$

Comparing the two values we see that $T = \frac{\theta_3(x) \theta_3(x)}{\theta_1^2(x) \theta_1^2(x)} = \frac{(\lambda_1 - \lambda_2)^{1/2}}{(\lambda_3 - \lambda_4)^{1/2}}$. Inserting this into (5.99) and simplifying (5.98) gives the formula

$$\rho(x,t) = a_1^2 + (a_3^2 - a_2^2) \frac{2 + (\frac{x - Vt}{\epsilon} + \phi) - K(m, \nu)}{a_1^2 - a_3^2} n^2 \left( \frac{x - Vt}{\epsilon} + \phi - K(m, \nu) \right)$$

(5.102)
where \( a_1, a_2, a_3, \) and \( V \) are given by (2.5) and \( \phi \) is as given in (1.11).

### 5.6.3. Computing the leading order phase.

Using (5.97), the leading order phase contribution is given by

\[
\arg \psi(x, t) = -2(\chi_0 + \pi/4) - \frac{2}{\epsilon}(\theta(\lambda_1) + g_\infty) + \arg \theta_3(w + iv).
\]

These terms can be evaluated explicitly in terms of elliptic integrals [8]

\[
-2(\chi_0 + \pi/4) = -2\phi \left( \lambda_1 - \int_{\lambda_1}^{\infty} (\omega^{(0)} - dz) \right) = 2\phi [V + \eta],
\]

\[
-2(\theta(\lambda_1) + g_\infty) = \theta(\lambda_1) - 2i \int_{\lambda_1}^{\infty} (\omega^{(1)} - zdz) - x \int_{\lambda_1}^{\infty} (\omega^{(0)} - dz) = 2i \left[ \Gamma_2 - V^2 - V\eta \right] + x [V + \eta].
\]

Putting it all together we have

\[
\epsilon \arg \psi(x, t) = 2\epsilon \left( \Gamma_2 - V^2 - V\eta \right) + 2x (V + \eta)
\]

\[
+ \epsilon \arg \left\{ H \left( \frac{\pi}{2K(m)} \sqrt{\frac{a}{m}} \left( \frac{x - V}{\epsilon} + \phi \right) - i\pi \left( \frac{1}{K(m)} \right) \right) \right\} + 2\epsilon\phi(V + \eta),
\]

where \( \Gamma_2 = \sum_{j,k \geq 1} \lambda_j \lambda_k \) and \( \eta, n, \) and \( \phi \) are as defined in (1.11).

### 5.6.4. Computing the fluid velocity.

Using formula (5.97) for the leading order behavior of \( \psi \) in the modulation zone, the velocity \( u \) defined by the hydrodynamic change of variables for NLS (1.2) becomes

\[
u(x, t) = \epsilon \Im \left[ \frac{\partial \log(\psi(x, t))}{\partial x} \right] = \frac{\gamma_\delta}{2} \Im \left[ \frac{\theta_3'(w + iv)}{\theta_3(w + iv)} \right] + 2 \int_{\lambda_+}^{\infty} (\omega^{(0)} - d\lambda) - 2\lambda_+ + O(\epsilon/t)
\]

The first term can be simplified as follows

\[
\frac{\gamma_\delta}{2} \Im \left[ \frac{\theta_3'(w + iv)}{\theta_3(w + iv)} \right] = \frac{\gamma_\delta}{4i} \left[ \frac{\theta_3'(w + iv)}{\theta_3(w + iv)} - \frac{\theta_3'(w - iv)}{\theta_3(w - iv)} \right]
\]

taking the log derivative of 1.4.25 in [35] and using (5.98) this becomes

\[
\frac{\gamma_\delta}{2} \Im \left[ \frac{\theta_3'(w + iv)}{\theta_3(w + iv)} \right] = \frac{\gamma_\delta}{2i} \left[ \frac{\theta_3'(iv)}{\theta_3(iv)} - \frac{a_2}{\rho} \frac{d}{d\zeta} \log \left( \frac{\theta_3(\zeta)}{\theta_3(i\zeta)} \right) \right] \bigg|_{i\zeta = iv} = 2\pi c_v \frac{\theta_3'(iv)}{\theta_3(iv)} + \frac{a_1 a_2 a_3}{\rho}
\]

where in the last step the logarithmic derivative is evaluated using (5.99)–(5.101) and we use that fact that \( \gamma_\delta = \int_0^1 \omega(0) = 4\pi ic_v \). Inserting this into (5.104) we have

\[
u(x, t) = \frac{a_1 a_2 a_3}{\rho} + 2\pi c_v \frac{\theta_3'(iv)}{\theta_3(iv)} + 2 \int_{\lambda_+}^{\infty} (\omega^{(0)} - d\lambda) - 2\lambda_+ + O(\epsilon/t)
\]

### Proposition 5.5.

\[
2\pi c_v \frac{\theta_3'(iv)}{\theta_3(iv)} + 2 \int_{\lambda_+}^{\infty} (\omega^{(0)} - d\lambda) - 2\lambda_+ = -\frac{1}{2} \sum_{k=1}^{d} \lambda_k := V
\]

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Proof. The function
\[ G(P) = 2\pi c_0 \left\{ \frac{\theta_0}{\theta_1} \left( \frac{\pi J_{\lambda_+}^P v}{2\pi J_{\lambda_+}^P v} \right) + 2 \int_{\lambda_+}^P (\omega^{(0)} - d\lambda) \right\} = \sum_{k=1}^4 \frac{\theta_0}{\theta_k} \left( \frac{\pi J_{\lambda_+}^P v}{2\pi J_{\lambda_+}^P v} \right) + 2 \int_{\lambda_+}^P (\omega^{(0)} - d\lambda) \]
is single valued on the Riemann surface \( S_1 \) and by definition \( \lim_{P \to \infty} G(P) = K \). The single-valuedness follows from the relations \( d\log \theta_k(x + n\pi + m\pi\tau) = d\log \theta_k(x) - 2im \), \( \int_{\lambda_+}^P (\omega^{(0)} - d\lambda) = 0 \) and \( \int_{\lambda_+}^P (\omega^{(0)} - d\lambda) = -2\pi ic_\nu \). From the second representation for \( G(P) \) above it is clear that \( G \) is meromorphic over \( S_1 \) with five simple poles at \( \lambda_k, k = 1, \ldots, 4 \), and \( \infty \), with residues
\[ \text{Res}_{P=\lambda_k} G(P) = \frac{1}{2} \prod_{j \neq k} (\lambda_k - \lambda_j)^{1/2}, \quad \text{Res}_{P=\infty} G(P) = -4. \]

The function
\[ \tilde{G}(P) = \frac{dR}{d\lambda}(\lambda(P)) - 2(\lambda(P) - \lambda_+) \]
is also meromorphic on \( S_1 \) with the same poles and residues, so that the difference \( G(P) - \tilde{G}(P) \) is constant. However, expanding the difference at \( \lambda_+ \) we find that \( G(P) - \tilde{G}(P) \) is \( O((\lambda(P) - \lambda_+)^{1/2}) \) so \( G(P) = \tilde{G}(P) \). The result follows from observing that \( \lim_{P \to \infty} \tilde{G}(P) = V \).

It immediately follows from the proposition that
\[ u(x, t) = \frac{a_1 d_2 d_1}{\rho} + V, \quad (5.106) \]
which is in perfect agreement with the Whitham theory for the genus one self-similar solutions of NLS (2.4).

5.7. The far right field: \( \tau > -\frac{1}{2}(\lambda_+ + \lambda_- - 2) + \frac{2(1 + \lambda_+ - \lambda_-)}{\lambda_+ + \lambda_-} \)

When the moving branch point of the two cut \( g \)-function collides with \(-1 \), the left cut closes and what remains is a one-cut \( g \)-function on the interval \( (\lambda_-, \lambda_+) \), as one would expect for the far right field. The right field \( g \)-function is given by
\[ g(z) = \int_{\lambda_-}^z d\theta - 2(\frac{\lambda - \xi_-}{R(\lambda; \lambda_-, \lambda_+)} - t \frac{d\lambda}{R(\lambda; \lambda_-, \lambda_+)})(\lambda - \xi_0) \quad (5.107) \]
where
\[ \xi_0 = \frac{1}{2}(\lambda_+ + \lambda_- + 2\tau) \]
\[ \xi_{\pm} = \frac{1}{4} \left( \lambda_+ + \lambda_- - \tau \pm \sqrt{(\lambda_+ + \lambda_- + \tau)^2 + 2(\lambda_+ - \lambda_-)^2} \right) \quad (5.108) \]

In order for the two-cut \( g \)-function to degenerate continuously into this equation we need \( \xi_-(\tau) = -1 \). This is exactly the condition which bounds the far-right field:
\[ \xi_-(\tau) < -1 \iff \tau > -\frac{1}{2}(\lambda_+ + \lambda_- - 2) + \frac{2(1 + \lambda_-)(1 + \lambda_+)}{\lambda_+ + \lambda_-} \]

As such the modified phase function
\[ \varphi(z) = 2(t \frac{(\lambda - \xi_-)(\lambda - \xi_+)}{R(\lambda; \lambda_-, \lambda_+)} - \frac{\lambda_+ + \lambda_- + \tau}{2}) \quad (5.109) \]
has an imaginary sign table resembling figure 5(a). We open contours \( \Gamma_i \), \( i = 1, 2 \) from \( \xi_-(\tau) \) which divide \( \mathbb{C}^+ \) into three sectors \( \Omega_i \), \( i = 1, 2, 3 \) as shown in figure 12.
The contours $\Gamma_i$ and regions $\Omega_i$ used to define the map $M \mapsto N$ (see (5.5)) for $t = x/t$ in the right planar zone (defined above). The stationary point $\xi_-(\tau)$ is a decreasing function of $\tau$; the lower boundary of the zone is characterized by the collision $\xi_-(\tau) = -1$. Blue regions correspond to $\text{Im} \varphi > 0$ and white regions $\text{Im} \varphi < 0$.

The result of (5.4)–(5.5) using (5.107) is the following RHP for $N(z)$:

**Riemann-Hilbert Problem 5.9.** Find a $2 \times 2$ matrix-valued function $N$ with the following properties:

1. $N(z)$ is analytic in $\mathbb{C} \setminus \Gamma_N$, $\Gamma_N = (-\infty, \xi_-(\tau)) \cup (\lambda_-, \lambda_+) \cup \bigcup_{i=1}^{2} (\Gamma_i \cup \Gamma_i^*)$.
2. $N(z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.
3. $N(z)$ takes continuous boundary values on $\Gamma_N$ away from points of self intersection and branch points which satisfy the jump relation $V_N(z) = N_-(z)V_N(z)$ where

$$
V_N(z) = \begin{cases}
(1 - r(z)r^*(z))^{\gamma_1} & z \in (-\infty, \xi_-(\tau)) \\
0 & z \in (\lambda_-, \lambda_+) \\
-\frac{e^{-2i\theta(\lambda_+)/\epsilon}}{r(z)e^{2i\theta(\lambda_+)/\epsilon}} & z \in \Gamma_1 \\
1 & z \in \Gamma_2
\end{cases}
$$

(5.110)

4. $N(z)$ is bounded except at the points $\lambda_+$ and $\lambda_-$ where

$$
N(z) = \mathcal{O}(z-p)^{-1/4} \quad \text{as} \quad z \to \pm \lambda_i, \quad p \in \{\lambda_+, \lambda_-, \lambda_i\}.
$$

(5.111)

**5.7.1. Constructing a parametrix for the far right field.** Clearly, the jump matrices along $\Gamma_i$, $i = 1, 2$ are near identity at any fixed distance from $\xi_-$. The remaining jumps on the real axis can be dealt with as before. In fact, comparing RHP 5.9 to RHP 5.1 we see that the problems in the far right field is a simpler version of that for the left field, the twist jumps along $(-1, 1)$ are exchanged for a simpler twist along $(\lambda_-, \lambda_+)$, and the diagonal jump $(1 - |r(z)|^2)^{\phi}$ lies only on $(-\infty, \xi_-(\tau))$ which is separated from the twist. As such the parametrix is constructed in the same way, but with less effort needed to construct the scalar function $D(z)$. We give the solution in the right field without reiterating the details which can be found in section 5.3.1.

Define

$$
D(z) = \exp \left[ \frac{i\phi(\lambda_+)}{\epsilon} + \frac{R(z; \lambda_+, \lambda_-)}{2\pi i} \int_{-\infty}^{\xi_-(\tau)} \log \left( 1 - |r(\lambda)|^2 \right) \frac{d\lambda}{R(\lambda; \lambda_+, \lambda_-)} \right].
$$

(5.112)
so that
\[ D_\infty = \exp \left[ \frac{i\phi(\lambda_\pm)}{\epsilon} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log (1 - |r(\lambda)|^2)}{R(\lambda; \lambda_+, \lambda_-)} \, d\lambda \right] \]  
(5.113)

Then introducing a fixed neighborhood \( \mathcal{U}_\xi \) of the stationary point \( \xi(\tau) \) which remains bounded away from \(-1\), we can write the solution \( N(z) \) to RHP 5.9 in the form
\[ N(z) = \begin{cases} 
D_\infty^{-\sigma_1} E(z)E(z; \lambda_+, \lambda_-)D(z)^{\sigma_1} & z \in \mathbb{C} \setminus \mathcal{U}_\xi,
D_\infty^{-\sigma_1} E(z)P_\xi(z)D(z)^{\sigma_1} & z \in \mathcal{U}_\xi
\end{cases} 
\]  
(5.114)

where \( E(z; \lambda_+, \lambda_-) \), defined by (3.3), is related to the plane wave solution of the Lax-Pair (3.1) for a constant plane wave with Riemann invariants \( \lambda_\pm \), and \( P_\xi(z) \) is a local model which exactly matches the jumps of \( ND(z)^{-\sigma_1} \) for \( z \in \mathcal{U}_\xi \). The construction of this function from solution of the parabolic cylinder problem, RHP A.1, goes exactly along the same lines as in the far left field; we have
\[ P_\xi(z) = E(z; \lambda_+, \lambda_-)h(z)^{-\sigma_1}M_{PC} \left( \sqrt[\epsilon]{f(z), r(z)} \right) h(z)^{\sigma_1} \]  
(5.115)

where \( M_{PC} \) is the solution of RHP A.1,
\[ \frac{1}{4} f(z)^2 = \frac{1}{t} (\varphi(z) - \varphi(\xi_\pm)) = 2 \int_{\xi_\pm}^{z} \frac{(\lambda - \xi_\pm)(\lambda - \xi_\pm)}{R(\lambda; \lambda_+, \lambda_-)} \, d\lambda, \]
\[ h(z) = e^{i(\epsilon) \log((z/\epsilon)^{1/2}f(z))} D(z)^{-1} e^{i(\theta(\lambda_+ + \varphi(\xi_\pm))}, \]
(5.116)

and the logarithmic term in \( h(z) \) is principally branched.

The remainder \( E(z) \) in this construction satisfies a small norm Riemann–Hilbert problem with jumps on the contours \( (\Gamma_i \cup \Gamma^*_i) \setminus \mathcal{U}_\xi \), \( i = 1, 2 \) and on the matching boundary \( \partial \mathcal{U}_\xi \). The essential fact is that these jumps are uniformly small: fixing any constant \( 0 < \eta < 1/2 \) independent of \( \epsilon/t \) we can find a sufficient small neighborhood \( \mathcal{U}_\xi \) (fixed independent of \( \epsilon/t \)) such that for \( p = 1, 2, \infty \), there exist \( c > 0 \) such that
\[ \| \cdot \|_{L^p(\Sigma)} \leq \begin{cases} 
O \left( e^{-c(\tau/\epsilon)} \right) & \Sigma = \bigcup_{i=1}^{\infty} (\Gamma_i \cup \Gamma^*_i) \setminus \mathcal{U}_\xi, \\
O \left( t^{1/2-\eta} \right) & \Sigma = O \left( \partial \mathcal{U}_\xi \right),
\end{cases} \]
(5.117)

where \( \langle z \rangle = \sqrt{1 + |z|^2} \). Moreover, we have here a full expansion of the error jump on \( \partial \mathcal{U}_\xi \) akin to (5.22). The small norm theory allows us one to prove that \( E(z) \) exist and compute its full asymptotic expansion. For us it is enough to know that \( E(z) = I + \frac{E^{(1)}}{z} + O(z^{-2}) \) and \( E^{(1)} = O \left( \sqrt{z} \right) \). The series of explicit transformations from \( m(z) \) to \( E(z) \) can then be inverted to give an asymptotic expansion of the original problem for \( m(z) \). The behavior of the solution of (1.1)–(1.5) for \( \tau > -\frac{1}{2} \left( \lambda_+ + \lambda_- \right) - 2 + \frac{2(1 + \lambda_+ \lambda_-)}{\lambda_+ + \lambda_-} \) is given by
\[ \psi(x, t) = \sqrt{\rho} e^{i(kx - \omega t)/\epsilon} e^{-i\phi(\lambda_+ t)} + O \left( \frac{\epsilon}{\sqrt{t}} \right), \]
\[ \rho = \left( \frac{\lambda_+ - \lambda_-}{2} \right)^2, \quad k = -\lambda_+ + \lambda_-, \quad \omega = \frac{1}{2} k^2 + \rho \]
(5.118)
Figure 13. The contours $\Gamma_j$ and sectors $S_j$ in the $\zeta$-plane defining RHP $A.1$.

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Appendix. Local models: the airy and parabolic cylinders RHPs

In this section we provide, for completeness, the necessary details concerning the local model Riemann Hilbert problems which we find necessary to construct local parametrices in the steepest decent analysis of our problem. The models are completely standard in the literature, and we do not intend to prove or re-derive the results stated here; those details can be found in the references provided.

A.1. The parabolic cylinder model

For $j = 1, 2$, let $\Gamma_j$ denote the complex contour
\[
\Gamma_j = \{ \zeta \in \mathbb{C} | \arg \zeta = \frac{2j - 1}{4} \pi \}, \quad \Gamma_j^* = \{ \zeta \in \mathbb{C} | \arg \zeta = -\frac{2j - 1}{4} \pi \}
\] oriented left-to-right and let $S_j$, $j = 0, 1, 2$, denote the sector in $\mathbb{C}^+$
\[
S_j = \{ \zeta \in \mathbb{C}^+ | \max \left( 0, \frac{2j - 1}{4} \pi \right) < \arg \zeta < \min \left( \frac{2j + 1}{4} \pi, \pi \right) \}
\]
and let $S_j^*$ denote the conjugate sections in $\mathbb{C}^-$. See figure 13.

Fixed $p \in \mathbb{C}$ and let
\[
\kappa = \kappa(p) := -\frac{1}{2\pi} \log(1 + pp^*).
\]

Then consider the following Riemann–Hilbert problem

Riemann–Hilbert Problem A.1 for the Parabolic Cylinder Model Find a $2 \times 2$ matrix valued function $\mathcal{M}_{PC}(\zeta; p)$ such that for each fixed $p \in \mathbb{C}$:

1. $\mathcal{M}_{PC}$ is a holomorphic function of $\zeta$ for $\zeta \in \mathbb{C} \setminus \bigcup_{j=1}^{2}(\Gamma_j \cup \Gamma_j^*)$. 
2. $\mathcal{M}_{PC}(\zeta; p) = 1 + \mathcal{M}_{\text{RHP}}^{\text{A}1}(\zeta) + O(\zeta^{-2})$ uniformly as $\zeta \to \infty$.

3. $\mathcal{M}_{PC}(\zeta; p)$ takes continuous boundary values $\mathcal{M}_{PC+}$ and $\mathcal{M}_{PC-}$ for $\zeta \in \bigcup_{j=1}^{2}(\Gamma_j \cup \Gamma_j^*)$ satisfying the jump relation $\mathcal{M}_{PC+}(\zeta; p) = \mathcal{M}_{PC-}(\zeta; p)V(\zeta, p)$ where

$$
V_{PC}(\zeta; p) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} & \text{arg } \zeta = \pi/4 \\
\begin{pmatrix} 1 & \rho^* \\
0 & 1 \end{pmatrix} & \text{arg } \zeta = -\pi/4 \\
\begin{pmatrix} 1 & \rho_{\text{ppr}} \zeta^{2i}e^{-i\zeta^2/2} \\
0 & 1 \end{pmatrix} & \text{arg } \zeta = 3\pi/4 \\
\begin{pmatrix} 1 & \rho_{\text{ppr}} \zeta^{-2i}e^{i\zeta^2/2} \\
0 & 1 \end{pmatrix} & \text{arg } \zeta = -3\pi/4
\end{cases}
$$

(A.4)

For $p = 0$ the solution is trivial; we consider $p \neq 0$. In this case, RHP A.1 has an explicit piecewise analytic solution $\mathcal{M}_{PC}(\zeta; p)$ which can be expressed in terms of the solutions $D_a(\pm \zeta)$ of the parabolic cylinder equation, $(\zeta^2 + \left(\frac{1}{2} - \frac{i}{2} + a\right)) D_a(\zeta) = 0$, as follows:

$$
\mathcal{M}_{PC}(\zeta; p) = \Psi(\zeta; p)\mathcal{P}(\zeta, p)e^{i\zeta^2/2}A_{\text{e}^{-i\zeta^2/2}}
$$

(A.5)

where

$$
\mathcal{P}(\zeta, p) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
-p & 1 \end{pmatrix} & \zeta \in S_0 \\
\begin{pmatrix} 1 & \rho^* \\
0 & 1 \end{pmatrix} & \zeta \in S_0^* \\
\begin{pmatrix} 1 & \rho_{\text{ppr}} \zeta^{2i}e^{-i\zeta^2/2} \\
0 & 1 \end{pmatrix} & \zeta \in S_2 \\
\begin{pmatrix} 1 & \rho_{\text{ppr}} \zeta^{-2i}e^{i\zeta^2/2} \\
0 & 1 \end{pmatrix} & \zeta \in S_2^*
\end{cases}
$$

(A.6)

$$
\Psi(\zeta; p) = \begin{cases} 
\begin{pmatrix} e^{i\zeta} D_a \left(e^{\frac{i\pi}{2} \zeta}\right) & -i\beta_{12}e^{i\frac{\pi}{2}(\zeta-i)} D_{a-1} \left(e^{\frac{i\pi}{2} \zeta}\right) \\
i\beta_{21}e^{-i\frac{\pi}{2}(\zeta+i)} D_{a-1} \left(e^{-\frac{i\pi}{2} \zeta}\right) & e^{i\frac{\pi}{2} \zeta} D_{a} \left(e^{-\frac{i\pi}{2} \zeta}\right) \end{pmatrix} & \zeta \in \mathbb{C}^+ \\
\begin{pmatrix} e^{i\zeta} D_a \left(e^{\frac{i\pi}{2} \zeta}\right) & -i\beta_{12}e^{i\frac{\pi}{2}(\zeta-i)} D_{a-1} \left(e^{\frac{i\pi}{2} \zeta}\right) \\
i\beta_{21}e^{-i\frac{\pi}{2}(\zeta+i)} D_{a-1} \left(e^{-\frac{i\pi}{2} \zeta}\right) & e^{i\frac{\pi}{2} \zeta} D_{a} \left(e^{-\frac{i\pi}{2} \zeta}\right) \end{pmatrix} & \zeta \in \mathbb{C}^-
\end{cases}
$$

and $\beta_{12}$ and $\beta_{21}$ are the complex constants

$$
\beta_{12} = \frac{\sqrt{2\pi e^{i\pi/4}e^{-\pi\kappa/2}}}{\rho \Gamma(-i\kappa)}, \quad \beta_{21} = -\frac{\sqrt{2\pi e^{-i\pi/4}e^{-\pi\kappa/2}}}{\rho^* \Gamma(i\kappa)} = \frac{\kappa}{\beta_{12}}.
$$

(A.7)

A derivation of this result is given in [13, 14], a direct verification of the solution in given in the appendix of [29]. The essential fact for our needs is the asymptotic behavior of the solution
given in the above references, as is easily verified using the well known asymptotic behavior of \(D_a(z)\),

\[
\mathcal{M}_{\text{PC}}(\zeta; p) = I + \frac{1}{\zeta} \begin{pmatrix}
0 & -i\beta_{12} \\
i\beta_{21} & 0
\end{pmatrix} + \mathcal{O}(\zeta^{-2}).
\] (A.8)

**Remark 8.** Here we have here treated the parameter \(p\) as a constant, but as the functions \(D_a(z)\) are entire in \(a\), the solution \(\mathcal{M}_{\text{PC}}(\zeta; p)\) defined by (A.5) is an analytic function of \(p\) on any bounded set where \(\kappa(p)\) is single-valued and non-vanishing. As such, if \(p, \zeta\) are independent analytic functions of a third variable \(z\): \(\zeta(z) = \zeta(p(z))\) is analytic and satisfies the jump condition (A.4) in any regime such that the quantity \(1 + p(z)p^*(z)\) does not vanish. Further, the asymptotic expansions hold in any regime where \(\zeta(z) \gg 1\) and \(p(z)\) remains bounded.

### A.2. The Airy local model

Fix the four complex rays

\[
\Gamma_1 = \{ \zeta \in \mathbb{C} \mid \arg \zeta = 0 \}, \quad \Gamma_2 = \{ \zeta \in \mathbb{C} \mid \arg \zeta = 2\pi/3 \},
\]

\[
\Gamma_3 = \{ \zeta \in \mathbb{C} \mid \arg \zeta = \pi \}, \quad \Gamma_4 = \{ \zeta \in \mathbb{C} \mid \arg \zeta = -2\pi/3 \},
\] (A.9)

and their conjugate sectors \(S^*_j\), \(j = 1, 2\), in the lower half-plane, see figure 14. Consider the following problem

**Riemann–Hilbert Problem A.2 for the Airy Model Problem** Find a \(2 \times 2\) matrix valued function \(\mathcal{M}_{\text{AI}}\) such that

1. \(\mathcal{M}_{\text{AI}}\) is a holomorphic function of \(\zeta\) for \(\zeta \in \mathbb{C}\setminus \Gamma\), \(\Gamma := \bigcup_{j=1}^4 \Gamma_j\).
2. \(\mathcal{M}_{\text{AI}}(\zeta) = I + \mathcal{O}(z^{-1})\) as \(z \to \infty\).
3. \(\mathcal{M}_{\text{AI}}(\zeta)\) takes continuous boundary values \(\mathcal{M}_{\text{AI}+}\) and \(\mathcal{M}_{\text{AI}-}\) for \(\zeta \in \Gamma\) satisfying

\[
\mathcal{M}_{\text{AI}+(\zeta)} = \mathcal{M}_{\text{AI}-(\zeta)} V_{\text{AI}}(\zeta)
\] (A.11)

where

\[
V_{\text{AI}}(\zeta) =
\begin{cases}
\begin{pmatrix}
1 & e^{\frac{2}{3}i\zeta^{3/2}} \\
0 & 1
\end{pmatrix} & z \in \Gamma_1 \\
\begin{pmatrix}
1 & 0 \\
e^{\frac{2}{3}i\zeta^{3/2}} & 1
\end{pmatrix} & z \in \Gamma_2 \cup \Gamma_4 \\
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} & z \in \Gamma_3
\end{cases}
\]
The fact is [11] that the solution of RHP A.2 is given by

\[
\mathcal{M}_{\text{Ai}}(\zeta) = \begin{cases} 
\Psi(\zeta)e^{\frac{i\pi}{6}2^{\frac{3}{2}}\sigma_3} & \zeta \in S_1 \\
\Psi(\zeta) \begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix} e^{\frac{i\pi}{3}2^{\frac{3}{2}}\sigma_3} & \zeta \in S_2 \\
\Psi(\zeta) \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}2^{\frac{3}{2}}\sigma_3} & \zeta \in S_3 \\
\Psi(\zeta)e^{\frac{i\pi}{6}2^{\frac{3}{2}}\sigma_3} & \zeta \in S_4 
\end{cases} \tag{A.12}
\]

where

\[
\Psi(\zeta) = \begin{pmatrix} \text{Ai}(\zeta) & \text{Ai}(\omega^2\zeta) \\
\text{Ai}'(\zeta) & \omega^2\text{Ai}'(\omega^2\zeta) \end{pmatrix} e^{-\frac{i\pi}{2}\sigma_3} \quad \zeta \in \mathbb{C}^+ \\
\begin{pmatrix} \text{Ai}(\zeta) & -\omega^2\text{Ai}(\omega\zeta) \\
\text{Ai}'(\zeta) & -\omega^2\text{Ai}'(\omega\zeta) \end{pmatrix} e^{-\frac{i\pi}{2}\sigma_3} \quad \zeta \in \mathbb{C}^- \tag{A.13}
\]

A proof that \( \mathcal{M}_{\text{Ai}} \) solves RHP A.2 is given in [11]. In this reference, they also show that the solution \( \mathcal{M}_{\text{Ai}} \) admits a full asymptotic expansion as \( \zeta \to \infty \), valid in each of the four sectors, given by

\[
\mathcal{M}_{\text{Ai}}(\zeta) = \frac{e^{i\pi}}{2\sqrt{\pi}} \zeta^{\sigma_3/4} \sum_{k=0}^{\infty} \left( \frac{(-1)^k s_k}{(-1)^k t_k} \right) e^{-\frac{i}{2}\sigma_3} \left( \frac{2}{3} \right)^{-k} \tag{A.14}
\]

where

\[
s_0 = t_0 = 1, \quad s_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})}, \quad t_k = -\frac{6k + 1}{6k - 1} s_k, \quad \text{for } k \geq 1. \tag{A.15}
\]

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