A symmetric quantum calculus*

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Abstract We introduce the $\alpha, \beta$-symmetric difference derivative and the $\alpha, \beta$-symmetric Nörlund sum. The associated symmetric quantum calculus is developed, which can be seen as a generalization of the forward and backward $h$-calculus.

1 Introduction

Quantum derivatives and integrals play a leading role in the understanding of complex physical systems. The subject has been under strong development since the beginning of the 20th century [5–8,11]. Roughly speaking, two approaches to quantum calculus are available. The first considers the set of points of study to be the lattice $\mathbb{Q}$ or $\mathbb{h}\mathbb{Z}$ and is nowadays part of the more general time scale calculus [1,3,9]; the second uses the same formulas for the quantum derivatives but the set of study is the set $\mathbb{R}$ of real numbers [2,4,10]. Here we take the second perspective.

Given a function $f$ and a positive real number $h$, the $h$-derivative of $f$ is defined by the ratio $(f(x+h) - f(x))/h$. When $h \to 0$, one obtains the usual derivative of

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the function $f$. The symmetric $h$-derivative is defined by $(f(x+h) - f(x-h))/(2h)$, which coincides with the standard symmetric derivative \[1\] when we let $h \to 0$.

We introduce the $\alpha, \beta$-symmetric difference derivative and Nörlund sum, and then develop the associated calculus. Such an $\alpha, \beta$-symmetric calculus gives a generalization to (both forward and backward) quantum $h$-calculus.

The text is organized as follows. In Section 2 we recall the basic definitions of the quantum $h$-calculus, including the Nörlund sum, i.e., the inverse operation of the $h$-derivative. Our results are then given in Section 3: in §3.1 we define and prove the properties of the $\alpha, \beta$-symmetric derivative; in §3.2 we define the $\alpha, \beta$-symmetric Nörlund sum; and §3.3 is dedicated to mean value theorems for the $\alpha, \beta$-symmetric calculus: we prove $\alpha, \beta$-symmetric versions of Fermat’s theorem for stationary points, Rolle’s, Lagrange’s, and Cauchy’s mean value theorems.

2 Preliminaries

In what follows we denote by $|I|$ the measure of the interval $I$.

**Definition 1.** Let $\alpha$ and $\beta$ be two positive real numbers, $I \subseteq \mathbb{R}$ be an interval with $|I| > \alpha$, and $f : I \to \mathbb{R}$. The $\alpha$-forward difference operator $\Delta_\alpha$ is defined by

$$\Delta_\alpha[f](t) := \frac{f(t + \alpha) - f(t)}{\alpha}$$

for all $t \in I \setminus [\text{sup } I - \alpha, \text{sup } I]$, in case $\text{sup } I$ is finite, or, otherwise, for all $t \in I$. Similarly, for $|I| > \beta$ the $\beta$-backward difference operator $\nabla_\beta$ is defined by

$$\nabla_\beta[f](t) := \frac{f(t) - f(t - \beta)}{\beta}$$

for all $t \in I \setminus [\text{inf } I, \text{inf } I + \beta]$, in case $\text{inf } I$ is finite, or, otherwise, for all $t \in I$. We call to $\Delta_\alpha[f]$ the $\alpha$-forward difference derivative of $f$ and to $\nabla_\beta[f]$ the $\beta$-backward difference derivative of $f$.

**Definition 2.** Let $I \subseteq \mathbb{R}$ be such that $a, b \in I$ with $a < b$ and $\text{sup } I = +\infty$. For $f : I \to \mathbb{R}$ we define the Nörlund sum (the $\alpha$-forward integral) of $f$ from $a$ to $b$ by

$$\int_a^b f(t) \Delta_\alpha t = \int_a^{+\infty} f(t) \Delta_\alpha t - \int_b^{+\infty} f(t) \Delta_\alpha t,$$

where

$$\int_x^{+\infty} f(t) \Delta_\alpha t = \alpha \sum_{k=0}^{+\infty} f(x + k\alpha),$$

provided the series converges at $x = a$ and $x = b$. In that case, $f$ is said to be $\alpha$-forward integrable on $[a, b]$. We say that $f$ is $\alpha$-forward integrable over $I$ if it is $\alpha$-forward integrable for all $a, b \in I$. 

Remark 1. If \( f : I \to \mathbb{R} \) is a function such that \( \sup I < +\infty \), then we can easily extend \( f \) to \( \tilde{f} : \bar{I} \to \mathbb{R} \) with \( \sup \bar{I} = +\infty \) by letting \( \tilde{f}|_I = f \) and \( \tilde{f}|_{\bar{I}\setminus I} = 0 \).

Remark 2. Definition 2 is valid for any two real points \( a, b \) and not only for points belonging to \( \alpha\mathbb{Z} \). This is in contrast with the theory of time scales [1, 2].

Similarly, one can introduce the \( \beta \)-backward integral.

Definition 3. Let \( I \) be an interval of \( \mathbb{R} \) such that \( a, b \in I \) with \( a < b \) and \( \inf I = -\infty \). For \( f : I \to \mathbb{R} \) we define the \( \beta \)-backward integral of \( f \) from \( a \) to \( b \) by

\[
\int_{a}^{b} f(t) \nabla \beta t = \int_{-\infty}^{b} f(t) \nabla \beta t - \int_{a}^{-\infty} f(t) \nabla \beta t,
\]

where

\[
\int_{-\infty}^{x} f(t) \nabla \beta t = \beta \sum_{k=0}^{+\infty} f(x - k\beta),
\]

provided the series converges at \( x = a \) and \( x = b \). In that case, \( f \) is said to be \( \beta \)-backward integrable on \( [a, b] \). We say that \( f \) is \( \beta \)-backward integrable over \( I \) if it is \( \beta \)-backward integrable for all \( a, b \in I \).

The \( \beta \)-backward Nörlund sum has similar results and properties as the \( \alpha \)-forward Nörlund sum.

3 Main Results

We begin by introducing in §3.1 the \( \alpha, \beta \)-symmetric derivative; in §3.2 we define the \( \alpha, \beta \)-symmetric Nörlund sum; while §3.3 is dedicated to mean value theorems for the new \( \alpha, \beta \)-symmetric calculus.

3.1 The \( \alpha, \beta \)-Symmetric Derivative

In what follows, \( \alpha, \beta \in \mathbb{R}^+_0 \) with at least one of them positive and \( I \) is an interval such that \( |I| > \max \{\alpha, \beta\} \). We denote by \( \mathcal{I}^\alpha_\beta \) the set

\[
\mathcal{I}^\alpha_\beta = \left\{ \begin{array}{ll}
I \setminus ([\inf I, \inf I + \beta] \cup [\sup I - \alpha, \sup I]) & \text{if } \inf I \neq -\infty \land \sup I \neq +\infty \\
I \setminus ([\inf I, \inf I + \beta]) & \text{if } \inf I \neq -\infty \land \sup I = +\infty \\
I \setminus ([\sup I - \alpha, \sup I]) & \text{if } \inf I = -\infty \land \sup I \neq +\infty \\
I & \text{if } \inf I = -\infty \land \sup I = +\infty.
\end{array} \right.
\]

Definition 4. The \( \alpha, \beta \)-symmetric difference derivative of \( f : I \to \mathbb{R} \) is given by

\[
D_{\alpha,\beta} [f] (t) = \frac{f(t + \alpha) - f(t - \beta)}{\alpha + \beta}.
\]
Remark 3. The $\alpha, \beta$-symmetric difference operator is a generalization of both the $\alpha$-forward and the $\beta$-backward difference operators. Indeed, the $\alpha$-forward difference is obtained for $\alpha > 0$ and $\beta = 0$; while for $\alpha = 0$ and $\beta > 0$ we obtain the $\beta$-backward difference operator.

Remark 4. The classical symmetric derivative [12] is obtained by choosing $\beta = \alpha$ and taking the limit $\alpha \to 0$. When $\alpha = \beta = h > 0$, the $\alpha, \beta$-symmetric difference operator is called the $h$-symmetric derivative.

Remark 5. If $\alpha, \beta \in \mathbb{R}^+$, then $D_{\alpha,\beta} [f] (t) = \frac{\alpha}{\alpha + \beta} \Delta_{\alpha} [f] (t) + \frac{\beta}{\alpha + \beta} \nabla_{\beta} [f] (t)$, where $\Delta_{\alpha}$ and $\nabla_{\beta}$ are, respectively, the $\alpha$-forward and the $\beta$-backward differences.

The symmetric difference operator has the following properties:

**Theorem 1.** Let $f, g : I \to \mathbb{R}$ and $c, \lambda \in \mathbb{R}$. For all $t \in I^a_{\beta}$ one has:

1. $D_{\alpha,\beta} [c] (t) = 0$;
2. $D_{\alpha,\beta} [f + g] (t) = D_{\alpha,\beta} [f] (t) + D_{\alpha,\beta} [g] (t)$;
3. $D_{\alpha,\beta} [\lambda f] (t) = \lambda D_{\alpha,\beta} [f] (t)$;
4. $D_{\alpha,\beta} [fg] (t) = D_{\alpha,\beta} [f] (t) g (t + \alpha) + f (t - \beta) D_{\alpha,\beta} [g] (t)$;
5. $D_{\alpha,\beta} [fg] (t) = D_{\alpha,\beta} [f] (t) g (t - \beta) + f (t + \alpha) D_{\alpha,\beta} [g] (t)$;
6. $D_{\alpha,\beta} \left[ \frac{f}{g} \right] (t) = \frac{D_{\alpha,\beta} [f] (t) g (t - \beta) - f (t - \beta) D_{\alpha,\beta} [g] (t)}{g(t + \alpha)g(t - \beta)}$
   provided $g(t + \alpha)g(t - \beta) \neq 0$;
7. $D_{\alpha,\beta} \left[ \frac{f}{g} \right] (t) = \frac{D_{\alpha,\beta} [f] (t) g (t + \alpha) - f (t + \alpha) D_{\alpha,\beta} [g] (t)}{g(t + \alpha)g(t - \beta)}$
   provided $g(t + \alpha)g(t - \beta) \neq 0$.

**Proof.** Property 1 is a trivial consequence of Definition 4. Properties 2, 3 and 4 follow by direct computations:

$$D_{\alpha,\beta} [f + g] (t) = \frac{(f + g)(t + \alpha) - (f + g)(t - \beta)}{\alpha + \beta} = \frac{f(t + \alpha) - f(t - \beta)}{\alpha + \beta} + \frac{g(t + \alpha) - g(t - \beta)}{\alpha + \beta} = D_{\alpha,\beta} [f] (t) + D_{\alpha,\beta} [g] (t);$$

$$D_{\alpha,\beta} [\lambda f] (t) = \frac{(\lambda f)(t + \alpha) - (\lambda f)(t - \beta)}{\alpha + \beta} = \lambda \frac{f(t + \alpha) - f(t - \beta)}{\alpha + \beta} = \lambda D_{\alpha,\beta} [f] (t);$$
Equality 5 follows from simple calculations:

\[ D_{\alpha,\beta}[fg](t) = \frac{(fg)(t+\alpha)-(fg)(t-\beta)}{\alpha+\beta} \]
\[ = \frac{f(t+\alpha)g(t+\alpha)-f(t-\beta)g(t-\beta)}{\alpha+\beta} \]
\[ = \frac{f(t+\alpha)-f(t-\beta)}{\alpha+\beta}g(t+\alpha) + \frac{g(t+\alpha)-g(t-\beta)}{\alpha+\beta}f(t-\beta) \]
\[ = D_{\alpha,\beta}[f](t)g(t+\alpha) + f(t-\beta)D_{\alpha,\beta}[g](t). \]

Equality 5 is obtained from 4 interchanging the role of \( f \) and \( g \). To prove 6 we begin by noting that

\[ D_{\alpha,\beta}\left[\frac{1}{g}\right](t) = \frac{\frac{1}{g}(t+\alpha)-\frac{1}{g}(t-\beta)}{\alpha+\beta} = \frac{\frac{1}{g(t+\alpha)}-\frac{1}{g(t-\beta)}}{\alpha+\beta} \]
\[ = \frac{g(t-\beta)-g(t+\alpha)}{(\alpha+\beta)g(t+\alpha)g(t-\beta)} = -\frac{D_{\alpha,\beta}[g](t)}{g(t+\alpha)g(t-\beta)}. \]

Hence,

\[ D_{\alpha,\beta}\left[\frac{f}{g}\right](t) = D_{\alpha,\beta}\left[\frac{1}{g}\right]D_{\alpha,\beta}[f](t) - \frac{1}{g(t+\alpha)} + f(t-\beta)D_{\alpha,\beta}[g](t) \]
\[ = \frac{D_{\alpha,\beta}[f](t)}{g(t+\alpha)} - f(t-\beta)\frac{D_{\alpha,\beta}[g](t)}{g(t+\alpha)g(t-\beta)} \]
\[ = D_{\alpha,\beta}[f](t)\frac{g(t-\beta)-f(t-\beta)D_{\alpha,\beta}[g](t)}{g(t+\alpha)g(t-\beta)}. \]

Equality 7 follows from simple calculations:

\[ D_{\alpha,\beta}\left[\frac{f}{g}\right](t) = D_{\alpha,\beta}\left[\frac{f}{g}\right]D_{\alpha,\beta}[f](t) - \frac{1}{g(t+\alpha)} + f(t-\beta)D_{\alpha,\beta}[g](t) \]
\[ = \frac{D_{\alpha,\beta}[f](t)}{g(t-\beta)} - f(t+\alpha)\frac{D_{\alpha,\beta}[g](t)}{g(t+\alpha)g(t-\beta)} \]
\[ = D_{\alpha,\beta}[f](t)\frac{g(t-\beta)-f(t+\alpha)D_{\alpha,\beta}[g](t)}{g(t+\alpha)g(t-\beta)}. \]

### 3.2 The \( \alpha,\beta \)-Symmetric Nörlund Sum

Having in mind Remark 5 we define the \( \alpha,\beta \)-symmetric integral as a linear combination of the \( \alpha \)-forward and the \( \beta \)-backward integrals.

**Definition 5.** Let \( f : \mathbb{R} \to \mathbb{R} \) and \( a, b \in \mathbb{R}, a < b \). If \( f \) is \( \alpha \)-forward and \( \beta \)-backward integrable on \([a, b] \), then we define the \( \alpha,\beta \)-symmetric integral of \( f \) from \( a \) to \( b \) by
\begin{align*}
\int_{a}^{b} f(t) d_{\alpha, \beta} t &= \frac{\alpha}{\alpha + \beta} \int_{a}^{b} f(t) \Delta t + \frac{\beta}{\alpha + \beta} \int_{a}^{b} f(t) \nabla \beta t.
\end{align*}

The function $f$ is $\alpha, \beta$-symmetric integrable if it is $\alpha, \beta$-symmetric integrable for all $a, b \in \mathbb{R}$.

**Remark 6.** Note that if $\alpha \in \mathbb{R}^+$ and $\beta = 0$, then $\int_{a}^{b} f(t) d_{\alpha, \beta} t = \int_{a}^{b} f(t) \Delta t$; if $\alpha = 0$ and $\beta \in \mathbb{R}^+$, then $\int_{a}^{b} f(t) d_{\alpha, \beta} t = \int_{a}^{b} f(t) \nabla \beta t$.

The properties of the $\alpha, \beta$-symmetric integral follow from the corresponding $\alpha$-forward and $\beta$-backward integral properties. It should be noted, however, that the equality $D_{\alpha, \beta} \left[ s \mapsto \int_{a}^{s} f(\tau) d_{\alpha, \beta} \tau \right] (t) = f(t)$ is not always true in the $\alpha, \beta$-symmetric calculus, despite both forward and backward integrals satisfy the corresponding fundamental theorem of calculus. Indeed, let $f(t) = \begin{cases} \frac{t}{2} & \text{if } t \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

Then, for a fixed $t \in \mathbb{N}$,

\[
\int_{0}^{t} \frac{1}{2^t} d_{1,1} \tau = \int_{0}^{t} \frac{1}{2^t} \Delta \tau + \int_{0}^{t} \frac{1}{2^t} \nabla \tau
= \int_{0}^{t} \left( \sum_{k=0}^{\infty} f(0+k) - \sum_{k=0}^{\infty} f(t+k) \right) + \frac{1}{2} \left( \sum_{k=0}^{\infty} f(t-k) - \sum_{k=0}^{\infty} f(0-k) \right)
= \frac{1}{2} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{t-1}} \right) + \frac{1}{2} \left( \frac{1}{2^t} + \frac{1}{2^{t-1}} + \cdots + \frac{1}{2} \right)
= \frac{1}{2} \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{t-1}} \right) = \frac{3}{2} \left( 1 - \frac{1}{2^t} \right)
\]

and $D_{1,1} \left[ s \mapsto \int_{0}^{s} \frac{1}{2^t} d_{1,1} \tau \right] (t) = \frac{3}{2} D_{1,1} \left[ s \mapsto 1 - \frac{1}{2^t} \right] (t) = -\frac{3}{2} \frac{1}{2^{t+1}} - \frac{1}{2^t} - \frac{1}{2} \left( \frac{1}{2} - \frac{1}{2^t} \right) = \frac{9}{2^{t+3}}$.

Therefore, $D_{1,1} \left[ s \mapsto \int_{0}^{s} \frac{1}{2^t} d_{1,1} \tau \right] (t) \neq \frac{1}{2^t}$.

### 3.3 Mean Value Theorems

We begin by remarking that if $f$ assumes its local maximum at $t_0$, then there exist $\alpha, \beta \in \mathbb{R}_0^+$ with at least one of them positive, such that $f(t_0 + \alpha) \leq f(t_0)$ and $f(t_0) \geq f(t_0 - \beta)$. If $\alpha, \beta \in \mathbb{R}^+$, this means that $\Delta_{\alpha}[f](t) \leq 0$ and $\nabla_{\beta}[f](t) \geq 0$. Also, we have the corresponding result for a local minimum. If $f$ assumes its local minimum at $t_0$, then there exist $\alpha, \beta \in \mathbb{R}^+$ such that $\Delta_{\alpha}[f](t) \geq 0$ and $\nabla_{\beta}[f](t) \leq 0$.

**Theorem 2** (The $\alpha, \beta$-symmetric Fermat theorem for stationary points). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f$ assumes a local extremum at $t_0 \in ]a, b[$, then there exist two positive real numbers $\alpha$ and $\beta$ such that $D_{\alpha, \beta}[f](t_0) = 0$. 

Proof. We prove the case where \( f \) assumes a local maximum at \( t_0 \). Then there exist \( \alpha_1, \beta_1 \in \mathbb{R}^+ \) such that \( \Delta_{\alpha_1} \ g \) \((t_0) \leq 0 \) and \( \nabla_{\beta_1} \ g \) \((t_0) \geq 0 \). If \( f(t_0 + \alpha_1) = f(t_0 - \beta_1) \), then \( \Delta_{\alpha_1, \beta_1} \ g \) \((t_0) = 0 \). If \( f(t_0 + \alpha_1) \neq f(t_0 - \beta_1) \), then let us choose \( \gamma = \min \{\alpha_1, \beta_1\} \). Suppose (without loss of generality) that \( f(t_0 - \gamma) > f(t_0 + \gamma) \). Then, \( f(t_0) > f(t_0 - \gamma) > f(t_0 + \gamma) \) and, since \( f \) is continuous, by the intermediate value theorem there exists \( \rho \) such that \( 0 < \rho < \gamma \) and \( f(t_0 + \rho) = f(t_0 - \gamma) \). Therefore, \( D_{\rho, \gamma} \ g \) \((t_0) = 0 \).

**Theorem 3 (The \( \alpha, \beta \)-symmetric Rolle mean value theorem).** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function with \( f(a) = f(b) \). Then there exist \( \alpha, \beta \in \mathbb{R}^+ \) and \( c \in [a, b] \) such that \( D_{\alpha, \beta} \ g \) \((c) = 0 \).

**Proof.** If \( f = \text{const} \), then the result is obvious. If \( f \) is not a constant function, then there exists \( t \in [a, b] \) such that \( f(t) \neq f(a) \). Since \( f \) is continuous on the compact set \([a, b] \), \( f \) has an extremum \( M = f(c) \) with \( c \in [a, b] \). Since \( c \) is also a local extremizer, then, by Theorem 2, there exist \( \alpha, \beta \in \mathbb{R}^+ \) such that \( D_{\alpha, \beta} \ g \) \((c) = 0 \).

**Theorem 4 (The \( \alpha, \beta \)-symmetric Lagrange mean value theorem).** Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then there exist \( c \in [a, b] \) and \( \alpha, \beta \in \mathbb{R}^+ \) such that \( D_{\alpha, \beta} \ g \) \((c) = \frac{f(b) - f(a)}{b - a} \).

**Proof.** Let function \( g \) be defined on \([a, b] \) by \( g(t) = f(a) - f(t) + (t - a) \frac{f(b) - f(a)}{b - a} \). Clearly, \( g \) is continuous on \([a, b] \) and \( g(a) = g(b) = 0 \). Hence, by Theorem 3 there exist \( \alpha, \beta \in \mathbb{R}^+ \) and \( c \in [a, b] \) such that \( D_{\alpha, \beta} \ g \) \((c) = 0 \).

\[
D_{\alpha, \beta} \ g \ g(t) = \frac{g(t + \alpha) - g(t + \beta)}{\alpha + \beta} \\
= \frac{1}{\alpha + \beta} \left( f(a) - f(t + \alpha) + (t + \alpha - a) \frac{f(b) - f(a)}{b - a} \right) \\
- \frac{1}{\alpha + \beta} \left( f(a) - f(t - \beta) + (t - \beta - a) \frac{f(b) - f(a)}{b - a} \right) \\
= \frac{1}{\alpha + \beta} \left( f(t - \beta) - f(t + \alpha) + (\alpha + \beta) \frac{f(b) - f(a)}{b - a} \right) \\
= \frac{f(b) - f(a)}{b - a} - D_{\alpha, \beta} \ g \ g(t),
\]

we conclude that \( D_{\alpha, \beta} \ g \ g) \ g(c) = \frac{f(b) - f(a)}{b - a} \).

**Theorem 5 (The \( \alpha, \beta \)-symmetric Cauchy mean value theorem).** Let \( f, g : [a, b] \to \mathbb{R} \) be continuous functions. Suppose that \( D_{\alpha, \beta} \ g \ g) \ g \) \((t) \neq 0 \) for all \( t \in ]a, b[ \) and all \( \alpha, \beta \in \mathbb{R}^+ \). Then there exists \( \bar{\alpha}, \bar{\beta} \in \mathbb{R}^+ \) and \( c \in ]a, b[ \) such that \( \frac{f(t) - f(a)}{g(b) - g(a)} = \frac{D_{\bar{\alpha}, \bar{\beta}} \ f\ g(c)}{D_{\bar{\alpha}, \bar{\beta}} \ g) \ g(c)} \).

**Proof.** From condition \( D_{\alpha, \beta} \ g \ g) \ g \) \((t) \neq 0 \) and the \( \alpha, \beta \)-symmetric Rolle mean value theorem (Theorem 3), it follows that \( g(b) \neq g(a) \). Let us consider function \( F \) defined on \([a, b] \) by \( F(t) = f(t) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} [g(t) - g(a)] \). Clearly, \( F \) is continuous on \([a, b] \) and \( F(a) = F(b) \). Applying the \( \alpha, \beta \)-symmetric Rolle mean value
theorem to the function $F$, we conclude that exist $\bar{\alpha}, \bar{\beta} \in \mathbb{R}^+$ and $c \in [a, b]$ such that

$$0 = D_{\bar{\alpha}, \bar{\beta}} [F](c) = D_{\bar{\alpha}, \bar{\beta}} [f](c) - \frac{f(b) - f(a)}{g(b) - g(a)} D_{\bar{\alpha}, \bar{\beta}} [g](c),$$

proving the intended result.

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References

1. R. Agarwal, M. Bohner, D. O’Regan and A. Peterson, Dynamic equations on time scales: a survey, J. Comput. Appl. Math. 141 (2002), no. 1-2, 1–26.
2. R. Almeida and D. F. M. Torres, Nondifferentiable variational principles in terms of a quantum operator, Math. Methods Appl. Sci. 34 (2011), no. 18, 2231–2241. arXiv:1106.3831
3. M. Bohner and A. Peterson, Dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2001.
4. A. M. C. Brito da Cruz, N. Martins and D. F. M. Torres, Higher-order Hahn’s quantum variational calculus, Nonlinear Anal. 75 (2012), no. 3, 1147–1157. arXiv:1101.3653
5. J. Cresson, G. S. F. Frederico and D. F. M. Torres, Constants of motion for non-differentiable quantum variational problems, Topol. Methods Nonlinear Anal. 33 (2009), no. 2, 217–231. arXiv:0805.0720
6. T. Ernst, The different tongues of $q$-calculus, Proc. Est. Acad. Sci. 57 (2008), no. 2, 81–99.
7. F. H. Jackson, $q$-Difference Equations, Amer. J. Math. 32 (1910), no. 4, 305–314.
8. V. Kac and P. Cheung, Quantum calculus, Universitext, Springer, New York, 2002.
9. A. B. Malinowska and D. F. M. Torres, On the diamond-alpha Riemann integral and mean value theorems on time scales, Dynam. Systems Appl. 18 (2009), no. 3-4, 469–481. arXiv:0804.4420
10. A. B. Malinowska and D. F. M. Torres, The Hahn quantum variational calculus, J. Optim. Theory Appl. 147 (2010), no. 3, 419–442. arXiv:1006.3765
11. L. M. Milne-Thomson, The calculus of finite differences, Macmillan and Co., Ltd., London, 1951.
12. B. S. Thomson, Symmetric properties of real functions, Monographs and Textbooks in Pure and Applied Mathematics, 183, Dekker, New York, 1994.