TO SPECIFY SURFACES OF REVOLUTION WITH POINTWISE 1-TYPE GAUSS MAP IN 3-DIMENSIONAL MINKOWSKI SPACE

V. MILANI AND A. SHOJAEI-FARD

Abstract. In this paper, by the studying of the Gauss map, Laplacian operator, curvatures of surfaces in $\mathbb{R}^3_1$ and Bour’s theorem, we are going to identify surfaces of revolution with pointwise 1-type Gauss map property in 3--dimensional Minkowski space.

Introduction

The classification of submanifolds in Euclidean and Non-Euclidean spaces is one of the interesting topics in differential geometry and in this way one can find some attempts in terms of finite type submanifolds [1] [2] [3] [4] [5]. On the other hand Kobayashi in [8] classified space-like ruled minimal surfaces in $\mathbb{R}^3_1$ and its extension to the Lorentz version is done by de Woestijne in [11]. In continue, people encounter with the following problem:

Classify all surfaces in 3-dimensional Minkowski space satisfying the pointwise 1-type Gauss map condition $\Delta N = kN$ for the Gauss map $N$ and some function $k$.

In 2000, D.W.Yoon and Y.H.Kim in [9] classified minimal ruled surfaces in terms of pointwise 1-type Gauss map in $\mathbb{R}^3_1$.

On suitability oriented surface $M$ in $\mathbb{R}^3$ with positive Gaussian curvature $K$, one can induce a positive definite second fundamental form $II$ with component functions $e, f, g$. The second Gaussian curvature is defined by

\begin{equation}
K_{II} = \frac{1}{(eg - f^2)^2} \begin{vmatrix}
-\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t & 0 \\
\frac{1}{2}g_t & e & f & \frac{1}{2}g_s \\
\frac{1}{2}g_s & f & g & \frac{1}{2}g_f \\
\end{vmatrix}.
\end{equation}

We can extend it to the surfaces in $\mathbb{R}^3_1$.

In 2004, D.W.Yoon and Y.H.Kim in [10], classified ruled surfaces in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature in 3-dimensional Minkowski space. On the other hand, in 2001, Ikawa in [7] proved Bour’s theorem in $\mathbb{R}^3_1$. He showed that

A generalized helicoid is isometric to a surface of revolution in $\mathbb{R}^3_1$.

In this paper, the above problem is answered for the surfaces of revolution in 3-dimensional Minkowski space.

1991 Mathematics Subject Classification. (2000 MSC) 53A10; 53A35; 53B25; 53C50.
Key words and phrases. Minkowski Space, Surfaces of Revolution, Bour’s Theorem, Minimal Surfaces, Maximal Surfaces.
1. Classification

Let $\mathbb{R}^3_1$ be a 3-dimensional Minkowski space with the scalar product $\langle , \rangle$ of index 1 defined as

\[
\langle X, Y \rangle = X_1 Y_1 + X_2 Y_2 - X_3 Y_3
\]

for every vectors $X = (X_i)$ and $Y = (Y_i)$ in $\mathbb{R}^3_1$.

A vector $X$ of $\mathbb{R}^3_1$ is said to be space-like if $\langle X, X \rangle > 0$ or $X = 0$, time-like if $\langle X, X \rangle < 0$ and light-like or null if $\langle X, X \rangle = 0$ and $X \neq 0$. A time-like or light-like vector in $\mathbb{R}^3_1$ is said to be causal.

Lemma 1.1. There are no causal vectors in $\mathbb{R}^3_1$ orthogonal to a time-like vector.

A lorentz cross product $X \times Y$ is given by

\[
X \times Y = (X_2 Y_3 - X_3 Y_2, X_3 Y_1 - X_1 Y_3, X_2 Y_1 - X_1 Y_2)
\]

A curve in $\mathbb{R}^3_1$ is called space-like, time-like or light-like if the tangent vector at any point is space-like, time-like or light-like, respectively. A plane in $\mathbb{R}^3_1$ is space-like, time-like or light-like if its Euclidean unit normal is time-like, space-like or light-like, respectively. A surface in $\mathbb{R}^3_1$ is space-like, time-like or light-like if the tangent plane at any point is space-like, time-like or light-like, respectively. Let $M$ be a surface in $\mathbb{R}^3_1$. The Gauss map $N : M \to Q^2(\epsilon) \subset \mathbb{R}^3_1$ which sends each point of $M$ to the unit normal vector to $M$ at that point is called the Gauss map of surface $M$. Here $\epsilon (= \pm 1)$ denotes the sign of the vector field $N$ and $Q^2(\epsilon)$ is a 2-dimensional space form as follows:

\[
Q^2(\epsilon) = \begin{cases} S^2(1) & \text{in } \mathbb{R}^3_1(\epsilon = 1) \\ H^2(-1) & \text{in } \mathbb{R}^3_1(\epsilon = -1) \end{cases}
\]

It is well known that in terms of local coordinates $(x_i)$ of $M$ the Laplacian can be written by

\[
\Delta = -\frac{1}{\sqrt{|G|}} \sum_{i,j} \frac{\partial}{\partial x^i}(\sqrt{|G|} g^{ij} \frac{\partial}{\partial x^j})
\]

where $G = \det(g_{ij})$, $(g^{ij}) = (g_{ij})^{-1}$ and $(g_{ij})$ are the components of the metric of $M$ with respect to $(x_i)$.

Now, we define a ruled surface $M$ in a three-dimensional Minkowski space $\mathbb{R}^3_1$. Let $I$ be an open interval in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve in $\mathbb{R}^3_1$ defined on $I$ and $\beta = \beta(s)$ a transversal vector field along $\alpha$. For an open interval $J$ in $\mathbb{R}$, let $M$ be a ruled surface parameterized by:

\[
x = x(s, t) = \alpha(s) + t\beta(s) \quad s \in I, \ t \in J.
\]

According to the character of the base curve $\alpha$ and the director curve $\beta$ the ruled surfaces are classified into the following six groups.

If the base curve $\alpha$ is space-like or time-like, then the ruled surface $M$ is said to be of type $M_+$ or type $M_-$, respectively. Also the ruled surface of type $M_+$ can
be divided into three types. When \( \beta \) is space-like, it is said to be of type \( M^1_+ \) or \( M^2_+ \) if \( \beta' \) is non-null or light-like, respectively. By \( \square \) when \( \beta \) is time-like, \( \beta' \) must be space-like. In this case, \( M \) said to be of type \( M^1_+ \). On the other hand, for the ruled surface of type \( M_- \), it is also said to be of type \( M^1_- \) or \( M^2_- \) if \( \beta' \) is non-null or light-like, respectively. Note that in the case of type \( M_- \) the director curve \( \beta \) is always space-like. The ruled surface of type \( M^1_+ \) or \( M^2_+ \) (resp. \( M^1_- \), \( M^2_- \)) is clearly space-like (resp. time-like). If the base curve \( \alpha \) and the director curve \( \beta \) are light-like, then the ruled surface is called null scroll and it is a time-like surface.

Now we modify the definition of a surface of revolution and generalized helicoid in \( \mathbb{R}^3_1 \). For an open interval \( I \subset \mathbb{R} \), let \( \gamma : I \rightarrow \Pi \) be a curve in a plane \( \Pi \) in \( \mathbb{R}^3_1 \) (profile curve) and let \( l \) be a straight line in \( \Pi \) which does not intersect the curve \( \gamma \) (axis). A surface of revolution in \( \mathbb{R}^3_1 \) is defined by the Lorentzian rotation \( \gamma \) around \( l \).

Suppose the case when a profile curve \( \gamma \) rotates around the axis \( l \), it simultaneously displaces parallel to \( l \) so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called generalized helicoid.

In the following at first we give some examples of surfaces of revolution in \( \mathbb{R}^3_1 \) and next will use them.

**Example 1.2.** For the constants \( a, b \), let \( M \) be a surface in \( \mathbb{R}^3_1 \) with the parametric representation

\[
R(s, t) = (\sqrt{(t + a)^2 - b^2} \cos s, \sqrt{(t + a)^2 - b^2} \sin s, \int \sqrt{\frac{b^2}{(t + a)^2 - b^2}} dt)
\]

where \( b^2 < (t + a)^2 \). It is called surface of revolution of the 1st kind as space-like.

**Example 1.3.** For the constants \( a, b \), let \( M \) be a surface in \( \mathbb{R}^3_1 \) with the parametric representation

\[
R(s, t) = (\sqrt{b^2 - (t + a)^2} \sinh s, -b \sin^{-1}(\frac{\sqrt{b^2 - (t + a)^2}}{-b}), \sqrt{b^2 - (t + a)^2} \cosh s)
\]

where \( (t + a)^2 < b^2 \). It is called surface of revolution of the 2nd kind as space-like.

**Example 1.4.** For constants \( a, b \), let \( M \) be a surface in \( \mathbb{R}^3_1 \) with the parametric representation

\[
R(s, t) = (\sqrt{(t + a)^2 - b^2} \cosh s, -b \cosh^{-1}(\frac{\sqrt{(t + a)^2 - b^2}}{-b}), \sqrt{(t + a)^2 - b^2} \sinh s)
\]

where \( b^2 < (t + a)^2 \). It is called surface of revolution of the 2nd kind as time-like.

**Example 1.5.** For constants \( a, b \), let \( M \) be a surface in \( \mathbb{R}^3_1 \) with the parametric representation

\[
R(s, t) = (\int \sqrt{\frac{-b^2}{b^2 + (t + a)^2}} dt, \sqrt{b^2 + (t + a)^2} \sinh s, \sqrt{b^2 + (t + a)^2} \cosh s).
\]

It is called surface of revolution of the 3rd kind as Lorentzian.

**Proposition 1.6.** Let \( M \) be a helicoid of the 1st kind as space-like

\[
x(s, t) = ((t + a) \cos s, (t + a) \sin s, -bs)
\]
where $|a| > |b| > 0$, $t < \min(-a - b, -a + b)$ or $t > \max(-a - b, -a + b)$. This surface is isometric to a minimal surface of revolution with pointwise 1-type Gauss map property.

Proof. According to the Bour’s theorem in Minkowski 3-space [7], for each helicoidal surface there exists an isometric surface of revolution to it. Therefore one can see that helicoid of the 1st kind as space-like is isometric to the surface of revolution of the 1st kind as space-like. By the parametrization of this surface of revolution, its Gauss map is given by

$$N = \frac{R_s \times R_t}{\| R_s \times R_t \|} = -\frac{1}{\sqrt{(t + a)^2 - b^2}} (b \cos s, b \sin s, t + a).$$

The components $(g_{ij})$ of the metric with respect to the first fundamental forms of this surface are

$$E = g_{11} = \langle R_s, R_s \rangle = (t + a)^2 - b^2,$$
$$F = g_{12} = \langle R_s, R_t \rangle = 0,$$
$$F = g_{21} = \langle R_t, R_s \rangle = 0,$$
$$G = g_{22} = \langle R_t, R_t \rangle = 1.$$

By (1.4),

$$\Delta N = 2b^2 ((t + a)^2 - b^2)^{-\frac{3}{2}} (b \cos s, b \sin s, t + a).$$

Then for some function $k$, $\Delta N = kN$ such that $k = -2b^2 ((t + a)^2 - b^2)^{-2}$. In other words, this surface of revolution has pointwise 1-type Gauss map property. On the other hand, the second fundamental forms of surface of revolution of the 1st kind as space-like are

$$e = \langle R_{ss}, N \rangle = b,$$
$$f = \langle R_{st}, N \rangle = \langle R_{ts}, N \rangle,$$
$$g = \langle R_{tt}, N \rangle = -(t + a)^2 - b^2)^{-1}.$$

The mean curvature $H$ is given by

$$H = \frac{Eg - 2Ff + Ge}{2|EG - F^2|} = 0.$$

Therefore surface of revolution of the 1st kind as space-like is a maximal surface and its Gauss map is of pointwise 1-type.

□

Proposition 1.7. Let $M$ be a helicoid of the 2nd kind as space-like

$$x(s, t) = ((t + a) \cosh s, -bs, (t + a) \sinh s)$$

where $|b| > |a|$, $\min(-a - b, -a + b) < t < \max(-a - b, -a + b)$. This surface is isometric to a minimal surface of revolution with pointwise 1-type Gauss map property.
Proof. According to the Bour’s theorem in Minkowski 3-space [7], for every helicoidal surface one can find its isometric surface of revolution. Therefore the helicoid of the 2nd kind as space-like is isometric to the surface of revolution of the 2nd kind as space-like. By the parametrization of this surface of revolution, its Gauss map is given by

\[ N = \frac{R_s \times R_t}{\| R_s \times R_t \|} = \frac{1}{\sqrt{b^2 - (t + a)^2}}(-b \sinh s, t + a, -b \cosh s). \]

The components \((g_{ij})\) of the metric with respect to the first fundamental forms of this surface are

\[ E = g_{11} = \langle R_s, R_s \rangle = b^2 - (t + a)^2, \]
\[ F = g_{12} = \langle R_s, R_t \rangle = 0, \]
\[ F = g_{21} = \langle R_t, R_s \rangle = 0, \]
\[ G = g_{22} = \langle R_t, R_t \rangle = 1. \]

By (1.4),

\[ \Delta N = -2b^2(b^2 - (t + a)^2)^{-2}(-b \sinh s, t + a, -b \cosh s). \]

Then for some function \(k\), \(\Delta N = kN\) such that \(k = -2b^2(b^2 - (t + a)^2)^{-2}\). In other words, this surface of revolution has pointwise 1-type Gauss map property. And also, the second fundamental forms of surface of revolution of the 2nd kind as space-like are

\[ e = \langle R_{ss}, N \rangle = b, \]
\[ f = \langle R_{st}, N \rangle = \langle R_{ts}, N \rangle = 0, \]
\[ g = \langle R_{tt}, N \rangle = -b(b^2 - (t + a)^2)^{-1}. \]

The mean curvature \(H\) is given by

\[ H = \frac{Eg - 2Ff + Ge}{2|EG - F^2|} = 0. \]

Therefore surface of revolution of the 2nd kind as space-like is a maximal surface and its Gauss map has pointwise 1-type property.

\[ \square \]

Proposition 1.8. Let \(M\) be a helicoid of the 2nd kind as time-like

\[ x(s, t) = ((t + a) \cosh s, -bs, (t + a) \sinh s) \]

where \(|a| > |b| > 0\), \(t < \min(-a - b, -a + b)\) or \(t > \max(-a - b, -a + b)\). This surface is isometric to a minimal surface of revolution with pointwise 1-type Gauss map property.

Proof. By Bour’s theorem in Minkowski 3-space, the helicoid of the 2nd kind as time-like is isometric to the surface of revolution of the 2nd kind as time-like. The Gauss map of this surface of revolution is

\[ N = \frac{R_s \times R_t}{\| R_s \times R_t \|} = \frac{1}{\sqrt{(t + a)^2 - b^2}}(-b \sinh s, t + a, -b \cosh s). \]

The components \((g_{ij})\) of the metric with respect to the first fundamental forms of this surface are

\[ E = g_{11} = \langle R_s, R_s \rangle = -((t + a)^2 - b^2), \]
\[ F = g_{12} = \langle R_s, R_t \rangle = 0, \]
\[ F = g_{21} = \langle R_t, R_s \rangle = 0, \]
\[ G = g_{22} = \langle R_t, R_t \rangle = 1. \]
Then for some function $k$ as space-like) \[7\], one can see that the mean curvature $H$ of the surface of revolution of the 2nd kind as time-like (similar to the surface of revolution of the 2nd kind as space-like) \[7\], one can see that the mean curvature $H$ is given by

$$H = \frac{Eg - 2Ff + Gc}{2|EG - F^2|} = 0.$$ 

Therefore surface of revolution of the 2nd kind as time-like is a minimal surface and its Gauss map is of pointwise 1-type.

**Proposition 1.9.** Let $M$ be a helicoid of the 3rd kind as Lorentzian

$$x(s, t) = (bs, (t + a) \sinh s, (t + a) \cosh s)$$

where $|a| < |b|$, $\min(-a - b, -a + b) < t < \max(-a - b, -a + b)$. This surface is isometric to a minimal surface of revolution with pointwise 1-type Gauss map property.

**Proof.** Bour’s theorem gives an isometric between the helicoid of the 3rd kind as Lorentzian and surface of revolution of the 3rd kind as Lorentzian. Its Gauss map is given by

$$N = \frac{R_s \times R_t}{\| R_s \times R_t \|} = -(t + a)^2 + b^2)^{-\frac{3}{2}}(t + a, ib \sinh s, ib \cosh s).$$

The components $(g_{ij})$ of the metric with respect to the first fundamental forms of this surface are

$$E = g_{11} = \langle R_s, R_s \rangle = (t + a)^2 + b^2,$$

$$F = g_{12} = \langle R_s, R_t \rangle = 0,$$

$$F = g_{21} = \langle R_t, R_s \rangle = 0,$$

$$G = g_{22} = \langle R_t, R_t \rangle = -1.$$ 

By \[13\],

$$\Delta N = 2b^2((t + a)^2 + b^2)^{-\frac{3}{2}}(t + a, ib \sinh s, ib \cosh s).$$

Then $\Delta N = kN$ for some function $k$ such that $k = -2b^2((t + a)^2 + b^2)^{-2}$. It means that, this surface of revolution has pointwise 1-type Gauss map property.

The second fundamental forms of the surface of revolution of the 3rd kind as Lorentzian are

$$e = \langle R_{ss}, N \rangle = ib,$$

$$f = \langle R_{st}, N \rangle = \langle R_{ts}, G \rangle,$$

$$g = \langle R_{tt}, N \rangle = ib((t + a)^2 + b^2)^{-1}.$$
The mean curvature $H$ is given by

$$H = \frac{Eg - 2Ff + Ge}{2|EG - F^2|} = 0.$$ 

Therefore surface of revolution of the 3rd kind as Lorentzian is a minimal surface and its Gauss map is of pointwise 1-type. □

**Proposition 1.10.** Let $M$ be a conjugate of Enneper’s surface of the 2nd kind as space-like or time-like

$$x(s, t) = (hs^2 + t, h(s^3 - s) + ts, h(s^3 + s) + ts).$$

This surface is isometric to a minimal or maximal surface of revolution with pointwise 1-type Gauss map property.

**Proof.** According to the Bour’s theorem in Minkowski 3-space \[7, 11\], we can see that the conjugate of Enneper’s surface of the 2nd kind as space-like or time-like is isometric to minimal or maximal surface of revolution Enneper of the 2nd, 3rd kind

$$x(s, t) = (at^3 + s^2t + b, -2st, at^3 - t^2s + b)$$

or

$$x(s, t) = (-at^3 + s^2t + b, -2st, -at^3 - t^2s + b)$$

respectively, where $a > 0, b \in \mathbb{R}$ and $t \neq 0$.

On the other hand, the conjugate of Enneper’s surface of the 2nd kind as space-like and the surface of revolution Enneper of the 2nd kind have the same Gauss map. Also, conjugate of Enneper’s surface of the 2nd kind as time-like and surface of revolution Enneper of the 3rd kind have the same Gauss map. Since the conjugate of Enneper’s surface of the 2nd kind as space-like or time-like has pointwise 1-type Gauss map property \[9\], the proof is completed. □

**Proposition 1.11.** Let $M$ be a non-developable ruled surface of type $M_1^+$ or $M_3^+$ in $\mathbb{R}^3_1$, such that has one of the following conditions:

$$aK_{II} + bH = \text{constant}, \quad a, b \in \mathbb{R} - \{0\}, \quad 2a - b \neq 0, \text{ along each ruling}$$

or

$$aH + bK = \text{constant}, \quad a \neq 0, \quad b \in \mathbb{R}, \text{ along each ruling}$$

or

$$aK_{II} + bK = \text{constant}, \quad a \neq 0, \quad b \in \mathbb{R}, \text{ along each ruling}.$$ 

Then $M$ is isometric to an open part of one of the following surfaces of revolution:

1. Surface of revolution of the 1st kind as space-like,
2. Surface of revolution of the 2nd kind as space-like,
3. Surface of revolution of the 3rd kind as Lorentzian.

**Proof.** According to theorems 4.1, 4.2 and 4.4 in \[10\], we know that every non-developable ruled surfaces of type $M_1^+$ or $M_3^+$ in $\mathbb{R}^3_1$ such that have one of the above conditions, are open parts of one of the following surfaces:

The helicoid of the 1st, 2nd kind as space-like,

The helicoid of the 3rd kind as Lorentzian.

On the other hand \[L6, L7\] and \[L9\] show that these surfaces are isometric to
Surface of revolution of the 1st, 2nd kind as space-like,
Surface of revolution of the 3rd kind as Lorentzian, respectively.

Proposition 1.12. Let $M$ be a non-developable ruled surface of type $M_1^-$ and $M$ isn’t an open part of the helicoid of the 1st kind as time-like in $\mathbb{R}^3_1$ such that this ruled surface satisfies one of the following conditions:

$$aK_{II} + bH = \text{constant, } a, b \in \mathbb{R} - \{0\}, 2a - b \neq 0, \text{ along each ruling}$$
or$$aH + bK = \text{constant, } a \neq 0, b \in \mathbb{R}, \text{ along each ruling}$$
or$$aK_{II} + bK = \text{constant, } a \neq 0, b \in \mathbb{R}, \text{ along each ruling}.$$Then $M$ is isometric to an open part of the following surface:

Surface of revolution of the 2nd kind as time-like.

Proof. According to theorems 4.1, 4.2 and 4.4 in [10], we know that every non-developable ruled surfaces of type $M_1^-$ and not an open part of the helicoid of the 1st kind as time-like in $\mathbb{R}^3_1$ such that have one of the above conditions, are open parts of the helicoid of the 2nd kind as time-like. On the other hand [15] shows that this surface is isometric to the surface of revolution of the 2nd kind as time-like. This completes the proof. □

Proposition 1.13. Let $M$ be a non-developable ruled surface of type $M_2^+$ or $M_2^-$ in $\mathbb{R}^3_1$, such that has one of the following conditions:

$$aK_{II} + bH = \text{constant, } a, b \in \mathbb{R} - \{0\}, 2a - b \neq 0, \text{ along each ruling}$$
or$$aH + bK = \text{constant, } a \neq 0, b \in \mathbb{R}, \text{ along each ruling}.$$Then $M$ is isometric to an open part of the following surfaces of revolution:

surfaces of Enneper of the 2nd, 3rd kind.

Proof. By theorems 4.1 and 4.2 in [10], one can see that every non-developable ruled surfaces of type $M_2^+$ or $M_2^-$ in $\mathbb{R}^3_1$ such that have one of the above conditions, are open parts of the conjugate of Enneper’s surfaces of the 2nd kind as space-like or time-like. On the other hand [15] shows that these surfaces isometric to surfaces of Enneper of the 2nd, 3rd kind. □

Corollary 1.14. Let $R$ be a surface of revolution such that is isometric to an open part of the non-developable ruled surface $M$ of type $M_1^+$ or $M_3^+$ where $M$ satisfies one of the following conditions:

$$aK_{II} + bH = \text{constant, } a, b \in \mathbb{R} - \{0\}, 2a - b \neq 0, \text{ along each ruling}$$
or$$aH + bK = \text{constant, } a \neq 0, b \in \mathbb{R}, \text{ along each ruling}$$
or
\[ aK_{II} + bK = \text{constant}, \quad a \neq 0, \quad b \in \mathbb{R}, \quad \text{along each ruling}. \]

Then \( R \) has pointwise 1-type Gauss map property.

**Proof.** According to 1.11, \( R \) is an open part of one of the following surfaces of revolution:

- Surface of revolution of the 1st kind as space-like,
- Surface of revolution of the 2nd kind as space-like,
- Surface of revolution of the 3rd kind as Lorentzian.

On the other hand 1.6, 1.7 and 1.9 show that these surfaces have pointwise 1-type Gauss map property. \( \square \)

**Corollary 1.15.** Let \( R \) be a surface of revolution such that is isometric to an open part of the non-developable ruled surface \( M \) of type \( M^1_\sigma \) and \( M \) isn’t an open part of the helicoid of the 1st kind as time-like in \( \mathbb{R}^3_1 \) and also this ruled surface satisfies one of the following conditions:

\[ aK_{II} + bH = \text{constant}, \quad a, b \in \mathbb{R} \setminus \{0\}, \quad 2a - b \neq 0, \quad \text{along each ruling} \]
or
\[ aH + bK = \text{constant}, \quad a \neq 0, \quad b \in \mathbb{R}, \quad \text{along each ruling} \]
or
\[ aK_{II} + bK = \text{constant}, \quad a \neq 0, \quad b \in \mathbb{R}, \quad \text{along each ruling}. \]

Then \( R \) has pointwise 1-type Gauss map property.

**Proof.** According to 1.12, \( R \) is an open part of the surface of revolution of the 2nd kind as time-like. On the other hand 1.8 shows that this surface has pointwise 1-type Gauss map property. \( \square \)

**Corollary 1.16.** Let \( R \) be a surface of revolution such that is isometric to an open part of the non-developable ruled surface \( M \) of type \( M^2_\sigma \) or \( M^2_\omega \) where \( M \) satisfies one of the following conditions:

\[ aK_{II} + bH = \text{constant}, \quad a, b \in \mathbb{R} \setminus \{0\}, \quad 2a - b \neq 0, \quad \text{along each ruling} \]
or
\[ aH + bK = \text{constant}, \quad a \neq 0, \quad b \in \mathbb{R}, \quad \text{along each ruling} \]

Then \( R \) has pointwise 1-type Gauss map property.

**Proof.** According to 1.13, \( R \) is an open part of the surfaces Enneper of the 2nd and 3rd kind. On the other hand 1.10 shows that these surfaces have pointwise 1-type Gauss map property. \( \square \)

**Corollary 1.17.** Let \( R \) be one of the following surfaces:

1. Surface of revolution of the 1st kind as space-like,
2. Surface of revolution of the 2nd kind as space-like or time-like,
3. Surface of revolution of the 3rd kind as Lorentzian,
4. Surfaces of Enneper of the 2nd, 3rd kind.

Then \( R \) is a part of one of the following surfaces:

1. A space-like or time-like plane,
2. The catenoids of the 1st, 2nd, 3rd, 4th, 5th kind,
Surfaces of Enneper of the 2nd, 3rd kind.

Proof. By 1.6, 1.7, 1.8, 1.9 and also 1.10 these surfaces of revolution are minimal or maximal. On the other hand by [11], we know that the only minimal or maximal surfaces of revolution in Minkowski 3-space are an open part of one of the following surfaces: A space-like or time-like plane, The catenoids of the 1st, 2nd, 3rd, 4th, 5th kind, Surfaces of Enneper of the 2nd, 3rd kind.

And finally one can reach to a nice characterization of the minimal and maximal surfaces of revolution by the pointwise 1-type property. We have

**Proposition 1.18.** Let \( R \) be a surface of revolution in \( \mathbb{R}^3_1 \). \( R \) has pointwise 1-type Gauss map property if and only if \( R \) be an open part of the minimal or maximal surfaces of revolution.

**References**

[1] C.Baikoussis, B.Y.Chen, L.Verstraelen, *Surfaces with Finite Type Gauss Map*, Geometry and Topology of submanifolds, vol.IV, World Scientific, Singapore, 1992, pp. 214-216.
[2] B.Y.Chen, *On submanifolds of finite type*, Soochow J. Math. 9 (1983), 65-81.
[3] B.Y.Chen, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, Singapore, 1984.
[4] B.Y.Chen, *A report on submanifolds of finite type*, Soochow J.Math. 22 (1996), 117-337.
[5] B.Y.Chen, P.Picciinni, *Submanifolds with finite type Gauss map*, Bull. Aust. Math. Soc. 35 (1987), 161-186.
[6] W.Greub, *Linear Algebra*, Springer, New York, 1963.
[7] T.Ikawa, *Bour’s Theorem in Minkowski Geometry*, Tokyo J. Math(24), No2 (2001), 377-394.
[8] O.Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space* \( L^3 \), Tokyo J. Math 6 (1983), 297-309.
[9] Y.H.Kim, D.W.Yoon, *Ruled surfaces with pointwise 1-type Gauss map*, J. Geom. Phys. 34(2000), 191-205.
[10] Y.H.Kim, D.W.Yoon, *Classification of ruled surfaces in Minkowski 3-spaces*, J. Geom. Phys. 49 (2004), 89-100.
[11] I.V.de Woestijne, *Minimal surfaces in the 3-dimensional Minkowski space*, in: Geometry and Topology of submanifolds: II, World Scientific, Singapore, 1990, P: 344-369.

Department of Mathematics, Shahid Beheshti University, 1983963113 Tehran, Iran
E-mail address: v-milani@cc.sbu.ac.ir
E-mail address: a_shojaei@sbu.ac.ir