Zeta Functions for Function Fields

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Abstract. To (weighted) count semi-stable bundles on curves defined over finite fields, we introduce new genuine zeta functions. There are two types, i.e., the pure non-abelian zetas defined using semi-stable bundles, and the group zetas defined for pairs consisting of (reductive group, maximal parabolic subgroup). Basic properties such as rationality and functional equation are obtained. Moreover, conjectures on their zeros and uniformity are given.

The constructions and results were announced in our paper on ‘Counting Bundles’.

1 Pure Non-Abelian Zeta Functions

Non-Abelian zeta functions for function fields were introduced in [W1] about 10 years ago. However, due to the lack of the Riemann Hypothesis, we have faced some essential difficulties. Recently, with an old paper of Drinfeld ([D]) on counting rank two cuspidal $\mathbb{Q}_l$-representations for function fields, we realize that our old definition of zeta should be altered: instead of counting rational semi-stable bundles of all degrees as done in [W], only these of degree zero and more generally degrees of multiples of the rank should be counted. This then leads to the definition of pure high rank zeta functions of this paper. This purity proves to be very essential: We expect that the Riemann Hypothesis holds for pure zetas. Indeed, in this direction, we now have the work of Yoshida [Y] for the RH in rank two and of mine [W4] for elliptic curves.

1.1 Counting Semi-Stable Bundles

Let $X$ be an irreducible, reduced, regular projective curve of genus $g$ defined over $\mathbb{F}_q$. Denote by $\mathcal{M}_{X,r}(d)$ the moduli space of rank $r$ semi-stable bundles of degree $d$ consisting of the Seshadri Jordan-Hölder equivalence classes of $\mathbb{F}_q$-rational semi-stable bundles on $X$. For our own purpose, we consider $\mathcal{M}_{X,r}(d)$ in the sense of the fat moduli, meaning that ordinary moduli spaces equipped with an additional structure at Seshadri class $[\mathcal{E}]$ defined by the collection of semi-stable bundles in $[\mathcal{E}]$, namely, the set $\{\mathcal{E} : \mathcal{E} \in [\mathcal{E}]\}$, is added at the point $[\mathcal{E}]$. For our own convenience, $\mathcal{M}_{X,r}(d)$ equipped with such a structure is called a fat moduli space and denoted as $\mathbf{M}_{X,r}(d)$.

A natural question is to count these $\mathbb{F}_q$-rational semi-stable bundles $\mathcal{E}$ on $X$. For this purpose, two naive invariants, namely, the automorphism
group $\text{Aut}(\mathcal{E})$ and its global sections $h^0(X, \mathcal{E})$, will be used. This then leads to the refined Brill-Noether loci

$$W^{i}_{X, r}(d) := \left \{ [\mathcal{E}] \in M_{X, r}(d) : \min_{\mathcal{E} \in [\mathcal{E}]} \{ h^0(X, \mathcal{E}) \} \geq i \right \}$$

and

$$[\mathcal{E}]^j := \{ \mathcal{E} \in [\mathcal{E}] : \dim_{\mathbb{F}_q} \text{Aut} \mathcal{E} \geq j \}.$$  

Recall that there exist natural isomorphisms

$$M_{X, r}(d) \rightarrow M_{X, r}(d + rm), \quad \mathcal{E} \mapsto A^m \otimes \mathcal{E}$$

and

$$M_{X, r}(d) \rightarrow M_{X, r}(-d + r(2g - 2)), \quad \mathcal{E} \mapsto K_X \otimes \mathcal{E}^\vee,$$

where $A$ is an Artin line bundle of degree one on $X/\mathbb{F}_q$ and $K_X$ denotes the dualizing bundle of $X/\mathbb{F}_q$. So we only need to count $M_{X, r}(d_0)$ for $d_0 = 0, 1, \ldots, r(g - 1)$. Accordingly, we introduce

$$\alpha_{X, r}(d) := \sum_{\mathcal{E} \in M_{X, r}(d)} \frac{d h^0(X, \mathcal{E}) - 1}{\text{Aut}(\mathcal{E})}, \quad \beta_{X, r}(d) := \sum_{\mathcal{E} \in M_{X, r}(d)} \frac{1}{\text{Aut}(\mathcal{E})}$$

with $\beta$ a classical invariant ([HN]).

So to count bundles, the problem becomes how to control $\alpha_{X, r}(d_0)$’s with $d_0$ ranging as above, and $\beta_{X, r}(d)$ with $d = 0, 1, \ldots, r - 1$. For $\alpha$, two general principles can be used for counting semi-stable bundles, namely,

(i) the vanishing theorem claiming that, for semi-stable $\mathcal{E}$,

$$h^1(X, \mathcal{E}) = 0 \quad \text{if} \quad d(\mathcal{E}) \geq r(2g - 2) + 1;$$

(ii) the Clifford lemma claiming that, for semi-stable $\mathcal{E}$,

$$h^0(X, \mathcal{E}) \leq r + \frac{d}{2} \quad \text{if} \quad 0 \leq \mu(\mathcal{E}) \leq 2g - 2.$$

By contrasting, the invariant $\beta$ has already been understood, thanks to the high profile works of Harder-Narasimhan ([HN]), Desale-Ramanan ([DR]), Atiyah-Bott ([AB]), Witten ([Wi]) and Zagier ([Z]). To state it, let

$$\zeta_X(s) := \prod_{i=1}^{2g} \frac{(1 - \omega_i q^{-s})}{(1 - q^{-s})(1 - qq^{-s})}$$

be the Artin zeta function of $X/\mathbb{F}_q$,

$$v_n(q) := \prod_{i=1}^{2g} \frac{(1 - \omega_i q)}{q - 1} q^{(r^2 - 1)(g - 1)} \zeta_X(2) \cdots \zeta_X(r)$$

and for a partition $r = n_1 + \cdots + n_k$ of $r$, set

$$c_{r, d}(t) := \prod_{i=1}^{s-1} \frac{1}{1 - t^{n_i + n_{i+1}}}.$$
Theorem 1. ([HN], [DS], in particular, [Z, Thm 2]) For any pair \((r, d)\), we have

\[
\beta_{X,r}(d) = \sum_{n_1, \ldots, n_s > 0, \sum n_i = r} q^{(s-1)\sum_{i<j} n_in_j c_{r,d}(q)} \cdot \prod_{i=1}^{s} v_n(q).
\]

1.2 Pure Non-Abelian Zeta Functions

Practically, the difficulty of counting semi-stable bundles comes form the fact that direct summands of the associated Jordan-Hölder graded bundle, or equivalently, the Jordan-Hölder filtrations, of an \(\mathbb{F}_q\)-rational semi-stable bundle in general would not be defined over \(X/\mathbb{F}_q\), but rather its scalar extension \(X_n/\mathbb{F}_q^n\). Theoretically, this is the junction point where the abelian and non-abelian ingredients of curves interact. For examples, torsions of Jacobians, Weierstraß points and stable but not absolutely stable bundles are closely related and hence get into the picture naturally. Good examples may be found in [W1].

To uniformly study \(\alpha\) and \(\beta\)'s, we, in [W], introduce the non-abelian zeta functions with the hope that the Riemann Hypothesis would hold for them. Unfortunately, examples shows that there are zeros off the central line for these old zetas. (For details, see the examples below.) This, in practice, has prevented any further studies for such zetas. However, during my visit to IHES in September 2011, we got to know the work of Drinfeld ([D]). Learning from it, we now know where the problem lies for old zetas: We should count only the pure part, instead of counting all.

Main Definition 1. For an irreducible, reduced, regular projective curve \(X\) of genus \(g\) defined over finite field \(\mathbb{F}_q\), define its rank \(r\) pure non-abelian zeta function by

\[
\zeta_{X,r}(s) := \sum_{m=0}^{\infty} \sum_{V \in M_{X,r}(d), d=rm} q^{h^0(C,V) - 1} \cdot \frac{(q^s)^{r(V)}}{\# Aut(V)},
\]

\[
\hat{\zeta}_{X,r}(s) := \sum_{m=0}^{\infty} \sum_{V \in M_{X,r}(d), d=rm} q^{h^0(C,V) - 1} \cdot \frac{(q^s)^{\chi(X,V)}}{\# Aut(V)}.
\]

As usual, set

\[
Z_{X,r}(t) := \zeta_{X,r}(s) \quad \text{and} \quad \hat{Z}_{X,r}(t) := \hat{\zeta}_{X,r}(s) \quad \text{with} \quad t := q^{-s}.
\]

By [W1, Prop 1.2.1], when the rank is one,

\[
\zeta_{X,1}(s) = \zeta_X(s)
\]

is the Artin zeta function.
Moreover,
\[
\hat{Z}_{X,r}(t) = \sum_{m=0}^{2g-2} \sum_{V \in M_{X,r}(d), d = rm} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^{\chi(C,V)}
\]
\[
+ \sum_{m > 2g-2} \sum_{V \in M_{X,r}(d), d = rm, r[(2g-2)-m]} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^{\chi(C,V)}
\]
\[
= \left( \sum_{m=0}^{(g-1)-1} \sum_{V \in M_{X,r}(d), d = rm, r[(2g-2)-m]} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^{\chi(C,V)} \right)
\]
\[
+ \sum_{m > 2g-2} \sum_{V \in M_{X,r}(d), d = r(g-1)} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^{\chi(C,V)}
\]
\[
+ \sum_{m > 2g-2} \sum_{V \in M_{X,r}(0)} \frac{q^{r(m-(g-1))} - 1}{\#\text{Aut}(V)} \cdot t^{r(m-(g-1))}
\]
(by the Vanishing Thm and the Riemann–Roch Thm)
\[
= \left( \sum_{m=0}^{(g-1)-1} \left( \sum_{V \in M_{X,r}(rm)} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^{r(m-(g-1))} \right) \right)
\]
\[
+ \sum_{W \in M_{X,r}(rm)} \frac{q^{h^0(C,W)} - 1}{\#\text{Aut}(W)} \cdot (qt)^{r([g-1]-m)}
\]
\[
+ \sum_{V \in M_{X,r}(r(g-1))} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^0
\]
\[
+ \sum_{V \in M_{X,r}(0)} \frac{1}{\#\text{Aut}(V)} \cdot \left( \frac{(qt)^r t^g}{1 - (qt)^r} - \frac{t^g}{1 - t^r} \right)
\]
(by the duality)
\[
= \left[ \sum_{m=0}^{(g-1)-1} \left( \sum_{V \in M_{X,r}(rm)} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^{r(m-(g-1))} \right) \right]
\]
\[
+ \sum_{W \in M_{X,r}(rm)} \frac{q^{h^0(C,W)} - 1}{\#\text{Aut}(W)} \cdot (qt)^{r([g-1]-m)}
\]
\[
+ \sum_{W \in M_{X,r}(rm)} \frac{q^{-r[m-(g-1)]} - 1}{\#\text{Aut}(W)} \cdot t^{r([g-1]-m)}
\]
\[
+ \sum_{V \in M_{X,r}(r(g-1))} \frac{q^{h^0(C,V)} - 1}{\#\text{Aut}(V)} \cdot t^0
\]
\[
+ \sum_{V \in M_{X,r}(0)} \frac{1}{\#\text{Aut}(V)} \cdot \left( \frac{(qt)^r t^g}{1 - (qt)^r} - \frac{t^g}{1 - t^r} \right)
\]
\[
= \left[ \sum_{m=0}^{(g-1)-1} \alpha_{X,r}(rm) \cdot \left( t^{r(m-(g-1))} + \frac{1}{qt} r^{r(m-(g-1))} \right) + \alpha_{X,r}(r(g-1)) \right]
\]
\[
+ \beta_{X,r}(0) \cdot \left( \frac{(qt)^r}{1 - (qt)^r} - \frac{t^g}{1 - t^r} \right)
\]
Consequently,
\[ \hat{Z}_{X,r}(t) = \hat{Z}_{X,r}(t), \]
and, if we introduce \( T := t^r \) and \( Q := q^r \),
\[
Z_{X,r}(t) = \sum_{m=0}^{(g-1)-1} \alpha_{X,r}(mr) \cdot \left( T^m + Q^{(g-1)-m} \cdot T^{2(g-1)-m} \right) + \alpha_{X,r}(r(g-1)) \cdot T^{g-1} + (Q - 1)\beta_{X,r}(0) \cdot \frac{T^g}{(1 - T)(1 - QT)}. \tag{1}
\]
This then completes the proof of the following

**Theorem 2.** (Zeta Facts) (i) \( \zeta_{X,1}(s) = \zeta_X(s) \), the Artin zeta function for \( X/F_q \);

(ii) (Rationality) There exists a degree \( 2g \) polynomial \( P_{X,r}(T) \in \mathbb{Q}[T] \) of \( T \) such that
\[
Z_{X,r}(t) = \frac{P_{X,r}(T)}{(1 - T)(1 - QT)} \quad \text{with} \quad T = t^r, \quad Q = q^r;
\]

(iii) (Functional equation)
\[
\hat{Z}_{X,r}(t) = \hat{Z}_{X,r}(t).
\]

These non-abelian zetas give a systematical treatment of invariants \( \alpha \)'s and \( \beta \)'s in counting semi-stable bundles. With \( \beta_{X,r}(0) \) known, we expect the uniform control of
\[
\alpha_{X,r}(0), \alpha_{X,r}(0), \ldots, \alpha_{X,r}(r(g-1))
\]
through the following

**Riemann Hypothesis.** Let \( P_{X,r}(T) = P_{X,r}(0) \cdot \prod_{i=1}^{2g} (1 - \omega_{X,r}(i)T) \), then
\[
|\omega_{X,r}(i)| = Q^\frac{1}{2}, \quad \forall i, \ 1 \leq i \leq 2g.
\]

**Examples.** (i) ([W4]) Rank 2 Zeta for Elliptic Curves Let \( E \) be an elliptic curve defined over \( \mathbb{F}_q \) with \( N \) the number of \( \mathbb{F}_q \)-rational points. For rank two pure zeta, it suffices to calculate \( \alpha_{E,2}(0) \) and \( \beta_{E,2}(0) \). By Thm 1,
\[
\beta_{E,2}(0) = \frac{N}{q-1} \left( 1 + \frac{N}{q^2 - 1} \right).
\]
On the other hand, by the classification of Atiyah ([A]), over \( \overline{\mathbb{F}_q} \), the graded bundle associated to a Jordan-Hölder filtration of a semi-stable bundle \( V \otimes \mathbb{F}_q \overline{\mathbb{F}_q} \) is of the form \( \text{Gr}(V \otimes \mathbb{F}_q \overline{\mathbb{F}_q}) = L_1 \oplus L_2 \) with \( L_i \) degree zero line bundles,
which may not be defined over $\mathbb{F}_q$. Consequently, for $\mathbb{F}_q$-rational semi-stable bundles $V$ of rank two, $h^0(E, V) \neq 0$ if and only if $V = \mathcal{O}_E \oplus L$ or $V = I_2$ with $L$ a $\mathbb{F}_q$-rational line bundle of degree 0 and $I_2$ the only non-trivial extension of $\mathcal{O}_E$ by itself. Thus $\alpha_{E,2}(0)$ is given by

$$\sum_{L \in \text{Pic}^0(E), L \neq \mathcal{O}_E} \frac{q^{6h^0(E, \mathcal{O}_E \oplus L)} - 1}{\#\text{Aut}(\mathcal{O}_E \oplus L)}.$$

Thus,

$$Z_{E,2}(t) = \alpha_{E,2}(0) \cdot \frac{1 + (N - 2)T + QT^2}{(1 - T)(1 - QT)},$$

the Riemann Hypothesis holds since

$$\Delta = (N - 2)^2 - 4Q = (N - 2 - 2q)(N - 2 + 2q) < 0$$

using Hasse’s theorem for the Riemann Hypothesis of elliptic curves, namely

$$N \leq 2\sqrt{q}.$$  

(ii) Rank Two Bundles on Genus Two Curves Let $X$ be a genus 2 curve. For rank two zeta,

$$Z_{X,2}(t) = \alpha_{X,2}(0) \cdot \left(1 + Q \cdot T^2\right) + \alpha_{X,r}(2) \cdot T^n + (Q - 1)\beta_{X,2}(0) \cdot \frac{T^2}{(1 - T)(1 - QT)}.$$

Thus

$$P_{X,2}(t) = \alpha_{C,2}(0) \left(1 + Q^2T^4\right) + \left(\alpha_{X,2}(2) - \alpha_{X,2}(0)(Q + 1)\right)(T + QT^3) + (2Q\alpha_{X,2}(0) - (Q + 1)\alpha_{X,2}(2) + \beta_{X,2}(0)(Q - 1))T^2.$$

So we need to consider the spaces $\mathcal{M}_{X,2}(0)$, $\mathcal{M}_{X,2}(2)$ (and $\mathcal{M}_{X,2}(4)$). By the Clifford lemma,

$$h^0(X, V) = \begin{cases} 0, 1, 2, & \text{if } V \in \mathcal{M}_{X,2}(0); \\ 0, 1, 2, & \text{if } V \in \mathcal{M}_{X,2}(2); \\ 2, 3, 4, & \text{if } V \in \mathcal{M}_{X,2}(4). \end{cases}$$

Consequently,

$$\alpha_{X,2}(0) = \sum_{V \in \mathcal{W}_{X,2}(0)} \frac{q - 1}{\#\text{Aut}(V)} + \frac{q^2 - 1}{(q^2 - 1)(q^2 - q)},$$

$$\alpha_{X,2}(2) = \sum_{V \in \mathcal{W}_{X,2}(0)} \frac{q - 1}{\#\text{Aut}(V)} + \sum_{V \in \mathcal{W}_{X,2}(0)} \frac{q^2 - 1}{\#\text{Aut}(V)}.$$
note that \( h^0(X, V) = 2 \) and \( d(V) = 0 \) iff \( V = \mathcal{O}_X \oplus \mathcal{O}_X \). Moreover, the Riemann Hypothesis now is equivalent to the conditions that

\[
A^2 < 4Q, \quad B^2 < 4Q
\]

with real constants \( A, B \) defined by

\[
P_{X,2}(T) = \alpha_{X,2}(0) \cdot (1 - AT + QT^2)(1 - BT + QT^2).
\]

That is to say,

\[
A + B = (Q + 1) - \alpha'_{X,2}(2),
\]

\[
AB = (Q - 1)\beta'_{X,2}(0) - (Q + 1)\alpha'_{X,2}(2),
\]

where

\[
\alpha'_{X,r}(d) := \frac{\alpha_{X,r}(d)}{\alpha_{X,r}(0)}, \quad \beta'_{X,r}(0) := \frac{\beta_{X,r}(0)}{\alpha_{X,r}(0)}.
\]

While the above does give a good control of \( \alpha' \)'s and \( \beta' \), it looks a bit clumsy. A much better way is to set

\[
Z_{X,r}(t) = \alpha_{X,r}(0) \cdot \exp \left( \sum_{m=1}^{\infty} N_{X,r}(m) \frac{T^m}{m} \right).
\]

Then

\[
N_{X,r}(m) = 1 + Q^m - \sum_{i=1}^{2g} \omega_{X,r}(i)^m
\]

and the Riemann Hypothesis gives a much elegant control of \( N_{X,r}(m) \)'s. We expect that \( N_{X,r}(m) \)'s measure rank \( r \) stable bundles over \( X/\mathbb{F}_q \). This is certainly the case in rank one through Weil’s counting zeta, and in rank two for elliptic curves, as indicated in the above example.

1.3 Why Purity

Next, we explain why purity is introduced for our study of zeta functions. Simply put, this is due to the Riemann Hypothesis.

For this purpose, let \( E \) be an irreducible, reduced regular elliptic curve defined over \( \mathbb{F}_q \). We will concentrate on ranks two and three. Due to isomorphisms

\[
M_{E,r}(d) \rightarrow M_{E,r}(d + rm), \quad \mathcal{E} \mapsto A^m \otimes \mathcal{E}
\]

and

\[
M_{E,r}(d) \rightarrow M_{E,r}(-d), \quad \mathcal{E} \mapsto K_E \otimes \mathcal{E}^\vee,
\]

among all invariants \( \alpha \) and \( \beta \)'s, for rank two and three, it suffices to understand \( \alpha_{E,r}(d) \) and \( \beta_{E,r}(d) \) for \( d = 0, 1 \).
1.3.1 Rank Two

From Ex(i) in §1.2, 

\[ \beta_{E,2}(0) = \frac{N}{q-1} \left( 1 + \frac{N}{q^2 - 1} \right) \quad \text{and} \quad \alpha_{E,2}(0) = \frac{N}{q-1}. \]

On the other hand, \( \alpha_{E,2}(1) \) and \( \beta_{E,2}(1) \) are easy to calculate. Indeed, all semi-stable bundles of rank 2 and degree 1 are stable. Thus, by the classification of Atiyah ([A]), \( M_{E,2}(1) \) via the determinant line bundle map is isomorphic to \( \text{Pic}^1(E) \). So

\[ \beta_{E,2}(1) = \frac{N}{q-1}. \]

Moreover, by the vanishing theorem, \( l^0(E, \mathcal{E}) = 1 \). Thus

\[ \alpha_{E,2}(1) = N. \]

As such, from [W1, §1.2.2], we know that the original zeta function \( \zeta \) of counting all degree semi-stable bundles defined as

\[
\sum_{V \in M_{E,2}(d)} \frac{q^{h^0(E,V)} - 1}{\# \text{Aut } V} =: \zeta_{E,2}(s) + \zeta^1_{E,2}(s)
\]

is given by

\[
\left( \alpha_{E,2}(0) + \beta_{E,2}(0) \right) \cdot \left( \frac{(q^2 - 1)t^2}{(1-t^2)(1-q^2t^2)} \right) + \beta_{E,2}(1) \left( \frac{qt}{1-q^2t^2} - \frac{t}{1-t^2} \right) = \frac{N}{q-1} \cdot \frac{1 + (q-1)t + (N-1)t^2 + (q-1)qt^3 + q^2t^4}{(1-t^2)(1-q^2t^2)}.
\]

Here

\[ \zeta^1_{E,2}(s) := \sum_{V \in M_{E,2}(2m+1), \ m \geq 0} \frac{q^{h^0(E,V)} - 1}{\# \text{Aut } V} . \]

Note that for the polynomial appeared in the numerator

\[ P_2(t) := 1 + (q-1)t + (N-1)t^2 + (q-1)qt^3 + q^2t^4, \]

by the functional equation ([W1]), we have the factorization

\[ P_2(t) = (qt^2 + A_+ t + 1)(qt^2 + A_- t + 1) \]

in \( \mathbb{R}[t] \). Assume, as we may, that \( |A_+| > |A_-| \). By Hasse’s theorem for Artin zeta functions, the coefficients of \( t^2 \) in \( P_2(t) \) is \( N - 1 \), which is of the same order as \( q - 1 \), the coefficient of \( t \). Consequently,

\[ A_+^2 - 4q > 0, \quad \text{while} \quad A_-^2 - 4q < 0. \]

So there is no RH for the zeta defined by counting semi-stable bundles of all degrees.
1.3.2 Rank Three

We already saw that for pure rank two zeta functions, the RH holds. In fact, one can show that this pattern persists ([W4]). This then leads to the problem of whether partial zeta functions defined by counting semi-stable bundles of other types of degrees satisfy the RH. Here, we use an example in rank 3 to indicate that another seemingly natural choice does not work either.

Introduce then the function

\[ \zeta^{12}_{E,3}(s) := \sum_{V \in M_{E,2}(d)} \frac{q^{h^0(E,V)} - 1}{#\text{Aut} V}. \]

(The reason for taking both 1 and 2, not just a single one, 1 or 2, in the congruence classes is that otherwise the functional equation does not hold.) By the vanishing theorem and the fact that all rank three semi-stable bundles of degree 1 or 2 are stable, one checks that

\[ \zeta^{12}_{E,3}(s) = N \cdot \left( \frac{q^t + q^2 t^2 - t + t^2}{1 - q^3 t^3} \right) = \frac{P_{E,3}(t)}{(1 - t^3)(1 - q^3 t^3)}, \]

where

\[ P_{E,3}(t) = (q - 1)t \left[ q^2 t^4 + q(q - 1)t^3 + (q + 1)t + 1 \right]. \]

The polynomial

\[ q^2 t^4 + q(q - 1)t^3 + (q + 1)t + 1 \]

does not satisfy the Riemann Hypothesis.

2 Group Zeta Functions

2.1 Number Fields versus Function Fields

For number fields, we have yet another type of zeta functions defined for pairs consisting of (reductive group, maximal parabolic subgroup)’s ([W2,3]). We will introduce such zeta functions for function fields next. For this purpose, we first examine analogue between function fields and number fields in our setting. To be more precise, we will analysis Zagier’s formula for counting semi-stable bundles over curves on finite fields and our own volume formula for semi-stable lattices over number fields.

Set then

\[ \tilde{\zeta}_F(1) := \begin{cases} \text{Res}_{s=1} \zeta_F(s), & F \text{ number field;} \\ \text{Res}_{s=1} \zeta_F(s) \cdot \log q, & F \text{ function field.} \end{cases} \]

And denote by \( \mathcal{M}_{Q,r}[1] \) the moduli space of rank \( r \) semi-stable lattices of volume 1.
Theorem 3. (i) (Reformulation of [Z, Thm 2]) For an irreducible, reduced, regular projective curve $X/\mathbb{F}_q$ of genus $g$,

$$\frac{\beta_{X,r}(0)}{q^{-(g-1)}-2} = \sum_{n_1,\ldots,n_s>0, n_1+\cdots+n_k=r} (-1)^{k-1} \prod_{j=1}^{k-1} (q^{n_j+n_j+1} - 1) \prod_{i=1}^{k} \zeta_X(i);$$

(ii) ([W2, §4.8]) For a number field $F$,

$$\frac{1}{r} \cdot \text{Vol}(\mathcal{M}_{Q,r}[1]) = \sum_{n_1,\ldots,n_s>0, n_1+\cdots+n_k=r} (-1)^{k-1} \prod_{j=1}^{k-1} (n_j+n_j+1) \prod_{i=1}^{k} \zeta(i).$$

Put Zagier’s result in our form as above, the hidden parallel structures in these two worlds becomes crystal clear. That is to say, for the mass of moduli space of semi-stable objects, when shift from number fields to function fields, the integers $n_j+n_j+1$ should be replaces by $q^{(n_j+n_j+1)}-1$. This then would suggest that, more generally, when defining group zeta functions associated to $(G, P)$ with $G$ reductive and $P$ maximal parabolic for function fields, based on these for number fields investigated in [W2,3], we should replace the rational factor $\langle w\lambda - \rho, \alpha^\vee \rangle$ by $q^{\langle w\lambda - \rho, \alpha^\vee \rangle} - 1$. In reality, even this is the direction we would go, this is not exactly the path we really pave. As a matter of fact, when shifting from number fields to function fields, the rational factor $\langle w\lambda - \rho, \alpha^\vee \rangle$ should be replaced by $1 - q^{-\langle w\lambda - \rho, \alpha^\vee \rangle}$, instead of $q^{\langle w\lambda - \rho, \alpha^\vee \rangle} - 1$.

2.2 Definitions

Let $X$ be an irreducible, reduced, regular projective curve of genus $g$ defined on $\mathbb{F}_q$. Denote by $F$ its function field. Let $G$ be a split connected reductive group with $B$ a fixed Borel over $F$. Denote by $\Sigma(G) := \Sigma := \left(V, \langle \cdot, \cdot \rangle, \Delta = \{\alpha_1, \ldots, \alpha_n\}, \Lambda := \{\lambda_1, \ldots, \lambda_n\}, \Phi = \Phi^+ \cup \Phi^-, W \right)$ the associated root system with the Weyl vector $\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. For $w \in W$, set $\Phi_w := \Phi^+ \cap w^{-1}\Phi^-$, and for $\alpha \in \Phi$, denote its coroot by $\alpha^\vee := \frac{2}{(\alpha,\alpha)} \cdot \alpha$.

From Lie theory, (see e.g., [H]), there is a well-known one-to-one correspondence between standard parabolic subgroups of $G$ and subsets of $\Delta$. Consequently, for a maximal standard parabolic subgroup $P$, there exists a unique $p = p(P)$ such that the subset of $\Delta$ above for $P$ is given by

$$\Delta_p =: \Delta \setminus \{\alpha_p\} := \{\beta_{P,1}, \ldots, \beta_{P,n-1}\}.$$
For such $p$, let $\Phi_p$ be the corresponding root system. Then $\Phi_p$ is normal to the fundamental weight $\lambda_p$, since $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$. This similarly leads to positive $\Phi_p^+ := \Phi^+ \cap \Phi_p$, $\rho_p$, and $W_p$. Set
\[ c_p := 2\langle \lambda_p - \rho_p, \alpha_p \rangle. \]
Moreover, for $\lambda \in V$, introduce a specific coordinate system via
\[ \lambda = \sum_{j=1}^n (1 + s_j) \lambda_j = \rho + \sum_{j=1}^n s_j \lambda_j. \]

Main Definition 2. (i) The period of $G$ for $X$ is defined by
\[ \omega_X^G(\lambda) := \sum_{w \in W} \frac{1}{\prod_{\alpha \in \Delta} (1 - q^{-\langle w\lambda - \rho, \alpha \rangle})} \prod_{\alpha \in \Phi_w} \zeta_X^\alpha(\langle \lambda, \alpha \rangle) \prod_{\alpha \in \Phi_{\lambda}} \zeta_X^\alpha(\langle \lambda, \alpha \rangle + 1) \]
where $\zeta_X^\alpha$ denotes the complete Artin zeta function of $X$;
(ii) The period of $(G, P)$ for $X$ is defined by
\[ \omega_X^{(G, P)}(s) := \text{Res}_{\lambda = 0} \zeta_X^{\alpha}(\langle \lambda - \rho, \beta \rangle) \cdot \omega_X^G(\lambda) \]
with $s = s_p$.

It is clear that there exists a minimal number $I(G, P)$ and factors
\[ \zeta_X(a_1^{(G, P)} s + b_1^{(G, P)}), \zeta_X(a_2^{(G, P)} s + b_2^{(G, P)}), \ldots, \zeta_X(a_I^{(G, P)} s + b_I^{(G, P)}), \]
such that there are no zeta factors appeared in the denominators of all terms of the product $\prod_{i=1}^{I(G, P)} \zeta_X(a_i^{(G, P)} s + b_i^{(G, P)}) \cdot \omega_X^{(G, P)}(s)$.

Main Definition 3. The zeta function of $X$ associated to $(G, P)$ is defined by
\[ \zeta_X^{(G, P)}(s) := \prod_{i=1}^{I(G, P)} \zeta_X(a_i^{(G, P)} s + b_i^{(G, P)}) \cdot \omega_X^{(G, P)}(s). \]

Theorem 4. (Functional Equation) We have
\[ \zeta_X^{(G, P)}(-c_p - s) = \zeta_X^{(G, P)}(s). \]

2.3 Proof of the Functional Equation

Using the Lie structures exposed, next, we give a proof of the functional equation for the group zetas of function fields, following [Ko], in which the group zetas for the field $\mathbb{Q}$ of rational numbers is treated.
2.3.1 Lie Structures

For \( w \in W \), denote by \( l(w) := |\Phi_w| \) the length of \( w \). Write the longest element of \( W \) as \( w_0 \). Then,

\[
w_0^2 = id, \quad w_0 \Delta = \Delta \quad \text{and} \quad w_0 \Phi^+ = \Phi^-.
\]

Similarly, for a fixed \( p \), denote by \( w_p \) the longest element of \( W_p \). Now, for \( w \in W \), introduce the subset \( W_p \) of \( W \) by

\[
W_p := \{ w \in W : w \Delta_p \subset \Delta \cup \Phi^- \}.
\]

One checks that \( id, w_0, w_p \in W_p \).

For each \( \alpha \in \Phi \), define its height by \( ht \alpha := \langle \rho, \alpha^\lor \rangle \). For \( w \in W_p \) and \((k, h) \in \mathbb{Z}^2\), set

\[
N_{p,w}(k, h) := \#\{ \alpha \in w^{-1} \Phi^- : \langle \lambda_p, \alpha^\lor \rangle = k, ht \alpha^\lor = h \},
\]

\[
N_p(k, h) := \#\{ \alpha \in \Phi : \langle \lambda_p, \alpha^\lor \rangle = k, ht \alpha^\lor = h \},
\]

\[
M_p(k, h) := \max_{w \in W_p} \{ N_{p,w}(k, h - 1) - N_{p,w}(k, h) \}, \quad \text{and}
\]

\[
\overline{M}_p(k, h) := \max_{w \in W_p} \{ \delta(N_{p,w}(k, h - 1) - N_{p,w}(k, h)) \},
\]

where \( \delta(a) = a \) if \( a > 0 \) and 0 otherwise.

**Lemma 5.** The following relations hold.

(i) If \( h \geq 1 \), \( M_p(k, h) = \overline{M}_p(k, h) \);

(ii) \( N_p(k, kc_p - h) - M_p(k, kc_p - h + 1) = N_p(k, h - 1) - M_p(k, h) \);

(iii) \( c_p \lambda_p - w_p \rho = \rho \).

They are various lemmas of [Ko]. More precisely, (i), (ii) and (iii) correspond to Lem. 5.4 (1), (2) and Lem 4.1, respectively.

2.3.2 A local decomposition

Write by

\[
\omega^G_X(\lambda) =: \sum_{w \in W} \omega^G_w(\lambda)
\]

where

\[
\omega^G_w(\lambda) := \left( \prod_{\alpha \in \Delta} \frac{1}{1 - q^{-\langle w \lambda - \rho, \alpha \rangle}} \right) \left( \prod_{\alpha \in \Phi_w} \zeta_X(\langle \lambda, \alpha^\lor \rangle + 1) \right).
\]

Since

\[
\langle w \lambda, \lambda' \rangle = \langle \lambda, w^{-1} \lambda' \rangle, \quad wa^\lor = (w \alpha)^\lor,
\]

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we have, for each $w \in W$, locally,

$$
\omega^G_w(\lambda) = \left( \prod_{\alpha \in \Delta} \frac{1}{1 - q^{-(w\alpha)}} \right) \left[ \prod_{\alpha \in \Phi_w \cap \Delta_p} \frac{1}{1 - q^{-(\lambda - \rho, \alpha)}} \right] 
\times \left[ \prod_{\alpha \in \Phi_w \cap \Delta_p} \left( 1 - q^{-(\lambda - \rho, \alpha)} \right) \cdot \frac{\widehat{\zeta}_X(\langle \lambda, \alpha \hat{\gamma} \rangle)}{\zeta_X(\langle \lambda, \alpha \hat{\gamma} \rangle + 1)} \right] 
\times \left[ \prod_{\alpha \in \Phi_w \cap \Delta_p} \left( 1 - q^{-(\lambda - \rho, \alpha)} \right) \cdot \frac{\widehat{\zeta}_X(\langle \lambda, \alpha \hat{\gamma} \rangle)}{\zeta_X(\langle \lambda, \alpha \hat{\gamma} \rangle + 1)} \right].
$$

2.3.3 Taking residues

Next, for $\omega^G_w(\lambda)$, we take the residues at $s_k = 0$ for $k \neq p$ and put $s_p = s$.

Recall that

$$
[\alpha \in \Delta \iff \langle \rho, \alpha \hat{\gamma} \rangle = 1] \implies \langle \lambda - \rho, \alpha \hat{\gamma} \rangle = \sum_{k=1}^{n} a_k s_k.
$$

Consequently, for each of four products appeared in the latest expression for $\omega^G_w(\lambda)$, (after taking the residue), we have

(i) For the first term,

$$
\prod_{\alpha \in (w^{-1}\Delta \cup \Phi_w) \cap \Delta_p} \frac{1}{1 - q^{-(\lambda - \rho, \alpha)}} = \prod_{\alpha \in (w^{-1}\Delta \cup \Phi_w) \cap \Delta_p} \frac{1}{1 - q^{-(\lambda - \rho, \alpha)}}.
$$

(ii) For the second term,

$$
\prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta_p} \frac{1}{1 - q^{-(\lambda - \rho, \alpha)}}|_{s_k=0, k \neq p; s_p=s} = \prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta_p} \frac{1}{1 - q^{-(\lambda_p, \alpha \hat{\gamma})s - \text{ht} \alpha \hat{\gamma} + 1}}.
$$

Since $\alpha \in (w^{-1}\Delta) \setminus \Delta_p$, $\text{ht} \alpha \hat{\gamma} \neq 1$ or $\langle \lambda_p, \alpha \hat{\gamma} \rangle \neq 0$. Thus the denominator do not vanish identically.

(iii) In the third term, for $\alpha_k \in \Phi_w \cap \Delta_p$, we have

$$
\left( 1 - q^{-(\lambda - \rho, \alpha_k \hat{\gamma})} \right) \cdot \frac{\widehat{\zeta}_X(\langle \lambda, \alpha_k \hat{\gamma} \rangle)}{\zeta_X(\langle \lambda, \alpha_k \hat{\gamma} \rangle + 1)} = \left( 1 - q^{-s_k} \right) \cdot \frac{\widehat{\zeta}_X(s_k + 1)}{\zeta_X(s_k + 2)} = \frac{\widehat{\zeta}_X(1)}{\zeta_X(2)} + o(s_k)
$$

as $s_k \to 0$, where $\widehat{\zeta}_X(1) := \text{Res}_{s=1} \widehat{\zeta}_X(s)$.

(iv) In the forth term, for $\alpha \in \Phi_w \setminus \Delta_p$, we have

$$
\frac{\widehat{\zeta}_X(\langle \lambda, \alpha \hat{\gamma} \rangle)}{\zeta_X(\langle \lambda, \alpha \hat{\gamma} \rangle + 1)}|_{s_k=0, k \neq p; s_p=s} = \frac{\widehat{\zeta}_X(\langle \lambda_p, \alpha \hat{\gamma} \rangle s + \text{ht} \alpha \hat{\gamma})}{\zeta_X(\langle \lambda_p, \alpha \hat{\gamma} \rangle s + \text{ht} \alpha \hat{\gamma} + 1)}.
$$
Consequently, when taking the residues, all terms $\omega_w(G)(\lambda)$ vanish except for the $w$'s satisfying $\Delta_p \subset w^{-1} \Delta \cup \Phi_w$, i.e., $w \in \mathbb{W}_p$. Moreover, for $w \in \mathbb{W}_p,$

$$\text{Res}_{s_k=0,k \neq p} \omega_w^G(\lambda) = \left( \prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta_p} \frac{1}{1 - q^{-\langle \lambda_p, \alpha^\vee \rangle s - \text{ht} \alpha^\vee + 1}} \right) \times \left( \prod_{\alpha \in \Phi_w \cap \Delta_p} \frac{\zeta_X(1)}{\zeta_X(2)} \prod_{\alpha \in \Phi_w \setminus \Delta_p} \frac{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee + 1)}{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee)} \right) .$$

Therefore,

$$\omega_X^{(G,P)}(s) = \sum_{w \in \mathbb{W}_p} \left( \prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta_p} \frac{1}{1 - q^{-\langle \lambda_p, \alpha^\vee \rangle s - \text{ht} \alpha^\vee + 1}} \right) \times \left( \prod_{\alpha \in \Phi_w \cap \Delta_p} \frac{\zeta_X(1)}{\zeta_X(2)} \prod_{\alpha \in \Phi_w \setminus \Delta_p} \frac{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee + 1)}{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee)} \right) \cdot \hat{\zeta}_{p,w}(s)$$

where, for $w \in \mathbb{W}_p$, we let

$$\hat{\zeta}_{p,w}(s) := \left( \prod_{\alpha \in \Phi_w \cap \Delta_p} \frac{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee + 1)}{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee)} \right) \times \left( \prod_{\alpha \in \Phi_w \setminus \Delta_p} \frac{1}{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee + 1)} \right) .$$

### 2.3.4 Minimal number of factors

With the above decomposition, we are ready to find out the minimal number of factors used in the normalization process appeared in Main Definition 3. With the expression for $\omega_X^{(G,P)}(s)$ in (2), we concentrate the zeta factors in $\hat{\zeta}_{p,w}(s)$ for $w \in \mathbb{W}_p.$
By definition,
\[
\prod_{\alpha \in \Phi_w \setminus \Delta_p} \hat{\zeta}_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee) \\
= \hat{\zeta}_X(s + 1)^{N_{p,w}(1,1)} \prod_{\alpha \in \Phi_w \setminus \Delta} \hat{\zeta}_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee) \\
= \hat{\zeta}_X(s + 1)^{N_{p,w}(1,1)} \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(k s + h)^{N_{p,w}(k,h)},
\]
and
\[
\prod_{\alpha \in \Phi_w} 1 \\
= \prod_{k=0}^{\infty} \prod_{h=1}^{\infty} \hat{\zeta}_X(k s + h + 1)^{-N_{p,w}(k,h)} \\
= \prod_{k=0}^{\infty} \prod_{h=1}^{\infty} \hat{\zeta}_X(k s + h + 1)^{-N_{p,w}(k,h-1)}.
\]

Hence
\[
\hat{\zeta}_{p,w}(s) = \hat{\zeta}_X(s + 1)^{N_{p,w}(1,1)} \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(k s + h)^{N_{p,w}(k,h) - N_{p,w}(k,h-1)}.
\]

Therefore,
\[
\hat{\zeta}_X(k s + h) \text{ appears in the denominator of } \omega_{X}^{(G,P)}(s)
\]
\[
\hat{\zeta}_X(k s + h) \text{ appears in the denominator of } \omega_{X}^{(G,P)}(s)
\]
\[
N_{p,w}(k,h) - N_{p,w}(k,h - 1) < 0.
\]

Consequently,
\[
\prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(k s + h)^{M_p(k,h)} = \prod_{i=1}^{I(G,P)} \hat{\zeta}_X\left(a_i^{(G,P)} s + b_i^{(G,P)}\right)
\]

is exactly the minimal zeta factors appeared in the normalization process in defining \(\hat{\zeta}_X^{(G,P)}(s)\). Thus, by Lem. 5(i), we have proved the following

Theorem 6. The zeta function for \(X\) associated to \((G, P)\) is given by
\[
\hat{\zeta}_X^{(G,P)}(s) = \omega_X^{(G,P)}(s) \cdot \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(k s + h)^{M_p(k,h)}.
\]

2.3.5 A global decomposition

The factor
\[
\prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(k s + h)^{M_p(k,h)}
\]

is exactly the minimal zeta factors appeared in the normalization process in defining \(\hat{\zeta}_X^{(G,P)}(s)\). Thus, by Lem. 5(i), we have proved the following

Theorem 6. The zeta function for \(X\) associated to \((G, P)\) is given by
\[
\hat{\zeta}_X^{(G,P)}(s) = \omega_X^{(G,P)}(s) \cdot \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(k s + h)^{M_p(k,h)}.
\]
appeared in (3) proves to be a bit hard. To overcome this, we go back to the expression of \( \omega^{(G,P)}(s) \) in (2). Introduce the ‘overdone’ maximal factor

\[
M^{(G,P)}_X(s) := M_p(s) := \prod_{\alpha \in \Phi^+} \hat{\zeta}_X((\lambda_p, \alpha^\vee)s + \text{ht} \alpha^\vee + 1) = \prod_{\alpha \in \Phi^-} \hat{\zeta}_X((\lambda_p, \alpha^\vee)s + \text{ht} \alpha^\vee)
\]

Obviously, being maximal, \( M_p(s) \) does clear up all the zeta factors in the denominators of terms of (2). Moreover, by definition,

\[
M_p(s) = \prod_{k=0}^{\infty} \prod_{h=1}^{\infty} \hat{\zeta}_X(ks + h + 1)^{N_p(k,h)} = \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(ks + h)^{N_p(k,h-1)}.
\]

Here, in the last step, we have used the functional equation for Artin zetas. This then leads to the global decomposition

\[
\hat{\zeta}_X^{(G,P)}(s) = \frac{\Omega^{(G,P)}_X(s)}{D^{(G,P)}_X(s)}
\]

where we have set

\[
\Omega^{(G,P)}_X(s) := M^{(G,P)}_X(s) \cdot \omega^{(G,P)}_X(s), \quad \text{and}
\]

\[
D^{(G,P)}_X(s) := \prod_{k=0}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(ks + h)^{-M_p(k,h)+N_p(k,h-1)}.
\]

As such, then the functional equation of our zeta functions is equivalent to the following

**Proposition 7.**

\[
D^{(G,P)}_X(-c_p - s) = D^{(G,P)}_X(s), \quad \text{and} \quad \Omega^{(G,P)}_X(-c_p - s) = \Omega^{(G,P)}_X(s).
\]

**2.3.6 Functional Equation for \( D^{(G,P)}_X(s) \)**

This is rather easy. Decompose \( D \) according to whether it consists of special values of zetas or not to get

\[
D^{(G,P)}_X(s) := D^0_p \cdot D^1_p(s)
\]

where

\[
D^0_p := \prod_{h=2}^{\infty} \hat{\zeta}_X(h)^{N_p(0,h-1)-M_p(0,h)},
\]

\[
D^1_p(s) := \prod_{k=1}^{\infty} \prod_{h=2}^{\infty} \hat{\zeta}_X(ks + h)^{N_p(k,h-1)-M_p(k,h)}.
\]
It suffices to show that

\[ D_p^1(-c_p - s) = D_p^1(s). \]

Since \( N_{p,w}(k, h - 1) = 0 \) and \( M_p(k, h) = 0 \) for \( k \geq 1 \) and \( h \leq 1 \), we have

\[ D_p^1(s) = \prod_{k=1}^{\infty} \prod_{h=-\infty}^{\infty} \hat{\zeta}_X(ks + h)^{N_p(k, h - 1) - M_p(k, h)}. \]

Consequently,

\[ D_p^1(-c_p - s) = \prod_{k=1}^{\infty} \prod_{h=-\infty}^{\infty} \hat{\zeta}_X(-kc_p - ks + h)^{N_p(k, h - 1) - M_p(k, h)} \]

(by the functional equation \( \hat{\zeta}_X(1 - s) = \hat{\zeta}_X(s) \))

\[ = \prod_{k=1}^{\infty} \prod_{h=-\infty}^{\infty} \hat{\zeta}_X(ks + h)^{N_p(k, kcp - h) - M_p(k, kcp - h + 1)} \]

(by Lem 5(ii))

\[ = D_p^1(s). \]

### 2.3.7 Involution Structure on \( \mathfrak{W}_p \)

We are left with the proof of the functional equation for \( \Omega_X^{(G,P)}(s) \). For this, we use an involution structure on \( \mathfrak{W}_p \) given by \( w \mapsto w_0 w w_p \). Set then

\[ f_{p,w}(s) := \prod_{\alpha \in (w^{-1} \Delta \setminus \Delta_p)} \frac{1}{1 - q^{-(\lambda_p, \alpha^\vee) s - \text{ht} \alpha^\vee + 1}}, \]

\[ g_{p,w}(s) := \prod_{\alpha \in (w^{-1} \Phi^- \setminus \Delta_p)} \hat{\zeta}_X((\lambda_p, \alpha^\vee) s + \text{ht} \alpha^\vee). \]

Similarly as in [Ko], we have the following

**Proposition 8.** (i) **Involution Structure**

\[ f_{p,w}(-c_p - s) = f_{p,w_0 w w_p}(s), \quad g_{p,w}(-c_p - s) = g_{p,w_0 w w_p}(s); \]

(ii)

\[ \Omega_X^{(G,P)}(s) = \sum_{w \in \mathfrak{W}_p} f_{p,w}(s) \cdot g_{p,w}(s). \]
Proof. (i) For a fixed subset $A \subset \Phi, w \in W$, set

$$S_{p,A}(s;w) := \{ (\lambda_p, \alpha^\vee) s + \text{ht} \alpha^\vee : \alpha \in (w^{-1}A) \setminus \Delta_p \}.$$ 

Then, note that, for $A = \Delta$ or $\Phi^-$, $w_0A = -A$ and

$$-w_p(w^{-1}A \setminus \Delta_p) = (w_p w^{-1}(A)) \setminus (w_p(-\Delta_p)) = (w_p w^{-1}w_0A) \setminus \Delta_p.$$ 

So, we have

$$f_{p,w}(s) = \prod_{as+b \in S_{p,\Delta}(s:w)} \frac{1}{1 - q^{-as - b + 1}},$$

$$g_{p,w}(s) = \prod_{as+b \in S_{p,\Phi^-}(s:w)} \frac{1}{\zeta_X(as + b)},$$

Moreover,

$$S_{p,A}(-c_p - s;w) = \{(\lambda_p, \alpha^\vee)(-c_p - s) + \text{ht} \alpha^\vee : \alpha \in (w^{-1}A) \setminus \Delta_p \}$$

$$= \{(\lambda_p, -w_p \alpha^\vee)s + (c_p \lambda_p - w_p \rho, -w_p \alpha^\vee) : \alpha \in (w^{-1}A) \setminus \Delta_p \}$$

$$= \{(\lambda_p, \beta^\vee) + s + (\rho, \beta^\vee) : \beta \in (w_p w^{-1}w_0A) \setminus \Delta_p \}$$

(by Lem 5(iii))

$$= S_{p,A}(s;w_0w w_p).$$

(ii) In [Ko], the following Lie structures are exposed.

(a) $\Phi^- \setminus (-\Phi_w) = \Phi^- \setminus (\Phi^- \cap w^{-1} \Phi^+) = \Phi^- \setminus w^{-1} \Phi^+ = \Phi^- \cap w^{-1}(\Phi^-)$,

(b) $(\Phi_w \setminus \Delta_p) \cup (\Phi^- \cap w^{-1}\Phi^-) = ((\Phi^+ \cap w^{-1}\Phi^-) \setminus \Delta_p) \cup (\Phi^- \cap w^{-1}\Phi^-)$

$$= ((\Phi^+ \cap w^{-1}\Phi^-) \cup (\Phi^- \cap w^{-1}\Phi^-)) \setminus \Delta_p = w^{-1} \Phi^- \setminus \Delta_p.$$

Consequently,

$$\Omega^{(G,P)}_X(s) \overset{(a)}{=} \sum_{w \in W, \Delta_p \subset w^{-1}(\Delta \cup \Phi^-)} \left( \prod_{\alpha \in (w^{-1}\Delta) \setminus \Delta_p} \frac{1}{1 - q^{-\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee + 1}} \right) \times \left( \prod_{\alpha \in (\Phi_w \setminus \Delta_p) \cup (\Phi^- \cap w^{-1}\Phi^-)} \hat{\zeta}_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee) \right)$$

$$\overset{(b)}{=} \sum_{w \in W_\Phi} \left( \prod_{\alpha \in (w^{-1}\Delta \setminus \Delta_p)} \frac{1}{1 - q^{-\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee + 1}} \right) \times \left( \prod_{\alpha \in (w^{-1}\Phi^- \setminus \Delta_p)} \hat{\zeta}_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \alpha^\vee) \right).$$

Proof for Thm 4. With Prop. 8, the functional equation

$$\Omega^{(G,P)}_X(c_P - s) = \Omega^{(G,P)}_X(s)$$

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is a direct consequence of the fact that \( w_0, w_p \in \mathfrak{M}_p \) so \( w \mapsto w_0 w w_p \) induces an involution structure of \( \mathfrak{M}_p \). This then completes the proof of Thm 4 as well.

Remark. The functional equation for \( \hat{\zeta}_X^{(G,P)}(s) \) can be more directly proved using the involution structure \( w \mapsto w_0 w \) on \( \mathfrak{M}_p \) directly, based on the relation ([KKS, Cor 8.7])

\[
M_p(k, h) = N_{p,w_0}(k, h - 1) - N_{p, w_0}(k, h).
\]

3 Counting Bundles

3.1 Uniformity and the Riemann Hypothesis

Recall that, by Thm 2, or better, the equality (1), we have

\[
\zeta_{X,r}(s) = \sum_{m=0}^{(g-1)-1} \alpha_{X,r}(mr) \cdot \left( (q^{-rs})^m + (q^r)^{(g-1)-m} \cdot (q^{-rs})^{2(g-1)-m} \right)
\]

\[
+ \alpha_{X,r}(r(g - 1)) \cdot (q^{-rs})^{g-1} + (q^r - 1)\beta_{X,r}(0) \cdot \frac{(q^{-rs})^g}{1 - q^{-rs}(1 - q^r q^{-rs})}.
\]

Thus, the following conjecture, motivated by our works on zetas for number fields ([W2,3]), counts semi-stable bundles decisively.

**Conjecture 9.** (1) (Uniformity) There are universal constants \( a_{F,r}, b_{F,r} \) and rational functions \( c_{F,r}(q) \) depending on \( F \) and \( r \) such that

\[
\hat{\zeta}_{F,r}(s) = c_{F,r}(q) \cdot \zeta_{X}^{(SL^r, P_{r-1,1})}(a_{F,r} \cdot s + b_{F,r}).
\]

(2) (The Riemann Hypothesis)

\[
\hat{\zeta}_{F,r}(s) = 0 \quad \Rightarrow \quad \text{Re}(s) = \frac{1}{2}.
\]

That is to say, all weighted counts on semi-stables via the invariants \( \alpha \)'s and \( \beta \)'s can be read from Artin’s zeta functions defined using only line bundles, while the Riemann Hypothesis gives an effective control of the invariants \( \alpha \)'s and \( \beta \)'s.

We have the following supportive evidences.

**Theorem 10.** (i) (Uniformity, [W4]) For elliptic curves, the uniformity holds when \( r = 1, 2, 3, 4, 5 \).

(ii) (Riemann Hypothesis) The Riemann Hypothesis holds for

(a) (Weil) \( \hat{\zeta}_X(s) \);

(b) ([Y]) \( \hat{\zeta}_{X}^{(SL^2, P_{1,1})}(s) \);

(c) ([W4]) \( \hat{\zeta}_{E,r}(s) \) for \( r = 2, 3, 4, 5 \) with \( E \) an elliptic curve.
3.2 Parabolic Reduction, Stability and the Mass

To end this paper, we explain the reasons why $\zeta_{X,r}(s)$ are non-abelian zeta functions of $X$, despite the uniformity claiming that, up to certain rational function factors, $\zeta_{X,r}(s)$ can be read from abelian Artin zetas.

The central reason is certainly that $\zeta_{X,r}(s)$’s are defined using moduli spaces of semi-stable bundles, highly non-commutative objects associated to $X$. Furthermore, even assuming the uniformity, from the equation (2), we can still detect where the non-abelian structure lies on. More precisely, in each term $\omega^G_w(s)$, $w \in \mathcal{W}_p$, while, for the zeta factor part

$$\hat{\zeta}_{p,w}(s) := \prod_{\alpha \in \Phi_w} \frac{1}{\zeta_X(\langle \lambda_p, \alpha^\vee \rangle s + \text{ht} \, \alpha^\vee + 1)},$$

there are involved only Artin zetas which are abelian, the non-abelian structure, through the group structure, is naturally reflected via the rational function factors

$$\prod_{\alpha \in (w^{-1} \Delta) \setminus \Delta_p} \frac{1}{1 - q^{-\langle \lambda_p, \alpha^\vee \rangle s - \text{ht} \, \alpha^\vee + 1}}.$$

To properly understand this, let us examine the so-called parabolic reduction structure appeared in the mass formula for function fields, and similarly, the volume formula for number fields, respectively.

As usual, let $D_{Q,r}$ be the volume of fundamental domain of $SL_r(\mathbb{Z})$, and $\mathcal{M}_{Q,r}[1]$ the moduli space of rank $r$ semi-stable lattices of volume 1. (For background materials, please refer to [W2].) Then we have

**Theorem 11.** (i) (Siegel)

$$\text{Vol}(D_{Q,r}) = r \cdot \prod_{i=1}^{r} \hat{\zeta}(i);$$

(ii) (Reformulation of [KS, §16])

$$\text{Vol}(D_{Q,r}) = \sum_{n_1, \ldots, n_k \geq 1, \atop n_1 + \cdots + n_k = r} \frac{\prod_{j=1}^{k} \text{Vol}(\mathcal{M}_{Q,n_j}[1])}{n_1(n_1+n_2) \cdots (n_1+\cdots+n_k) \cdots (n_k-1+n_k)n_k};$$

(iii) (Weng [W2, §4.8])

$$\frac{1}{r} \cdot \text{Vol}(\mathcal{M}_{Q,r}[1]) = \sum_{n_1, \ldots, n_s \geq 0, \atop n_1 + \cdots + n_k = r} (-1)^{k-1} \prod_{j=1}^{k} (n_j + n_{j+1}) \prod_{j=1}^{k} \text{Vol}(D_{Q,n_j}).$$
Remarks. (1) Siegel’s formula claims that the volume of non-abelian fundamental domain can be measured using special values of the abelian zeta;
(2) Even the roots for [KS, §16] and [W2] are very much different: the former uses arithmetic truncation of Harder-Narasimhan filtration, and the later uses analytic truncation and Eisenstein series, they share a common origin, as we observed, namely, the parabolic reduction structure;
(3) The part of non-abelian group structure and the part of the abelian zeta are well-organized so that they fit into a uniform theory naturally. For example, roughly, we see that fundamental domains consists of an essential part coming from stable lattices and boundary parts coming from tubular neighborhoods of cusps associated to proper parabolic subgroups.

Motivated by this, more generally, for a split reductive group $G$ defined over a number field $F$, $B$ a fixed Borel ... denote by $G(\mathbb{A})^{ss}$ the adelic elements of $G$ corresponding to semi-stable principle $G$-lattices ([G]). Write $\mathbb{K}_G$ for the associated maximal compact subgroup. Also for a standard parabolic subgroup $P$, write its Levi decomposition as $P = U M$ with $U$ the unipotent radical and $M$ its Levi factor. Denote the corresponding simple decomposition of $M$ as $\prod_i M_i$ with $M_i$’s the simple factors of $M$. Introduce invariants

$$\nu_P := \prod_i \text{Vol}\left( \mathbb{K}_{M_i} \mathbb{Z}_{M_i(\mathbb{A})} \setminus M_i(\mathbb{A}) / M_i(F) \right)$$

and

$$\mu_P := \prod_i \text{Vol}\left( \mathbb{K}_{M_i} \mathbb{Z}_{M_i(\mathbb{A})} \setminus M_i(\mathbb{A})^{ss} / M_i(F) \right).$$

In parallel, we have similar constructions for function fields $F = F_q(X)$. Based on all this, then we have the following

Conjecture 12. (Parabolic Reduction) Let $G/F$ be a split reductive group with $B/F$ a fixed Borel. Then, for each standard parabolic subgroup $P$ of $G$, there exist constants $c_P \in \mathbb{Q}$, $e_P \in \mathbb{Q}_{>0}$ such that

$$\nu_G = \sum_P c_P \cdot \nu_P, \quad \mu_G = \sum_P \text{sgn}(P) e_P \cdot \nu_P,$$

where $P$ runs over all standard parabolic subgroups of $G$, and $\text{sgn}(P)$ denotes the sign of $P$.

The exact values of $e_P$’s can be written out in terms of the associated root system. Indeed, if

$$W_0 := \left\{ w \in W : \{ \alpha \in \Delta : w\alpha \in \Delta \cup \Phi^- \} = \Delta \right\},$$

then there is a natural one-to-one correspondence between $W_0$ and the set of subsets of $\Delta$, and hence to the set of standard parabolic subgroups of $G$. Thus we will write

$$W_0 := \left\{ w_P : P \text{ standard parabolic subgroup} \right\},$$
and, for $w = w_P \in W_0$, write $J_P \subset \Delta$ the corresponding subset.

**Conjecture 13. (Parabolic Reduction, Stability & the Mass, [W5])**

Let $G$ be a split type reductive group with $P$ its maximal parabolic subgroup.

1. **Over a number field $F$,**
   (i) The volume of moduli space of semi-stable principal lattices is given by
   $$\nu_G = \text{Res}_{s=-c_P} \zeta^{(G,P)}_F(s) = \text{Res}_{\lambda=\rho} \omega^{G}_F(\lambda);$$
   (ii) We have the following formula
   $$\mu_G = \sum_P (-1)^{\text{rank}(P)} \prod_{\alpha \in \Delta \setminus w_J P \{1 - \langle w_J \rho, \alpha^\vee \rangle\}} \nu_P;$$

2. **Over an irreducible reduced regular projective curve $X$,**
   (i) The mass of moduli space of semi-stable principal bundles is given by
   $$\log q \cdot \nu_G = \text{Res}_{s=-c_P} \zeta^{(G,P)}_X(s) = \text{Res}_{\lambda=\rho} \omega^{G}_X(\lambda);$$
   (ii) We have the following formula
   $$\mu_G = \sum_P (-1)^{\text{rank}(P)} \prod_{\alpha \in \Delta \setminus w_J P \{1 - q^{\langle w_J \rho, \alpha^\vee \rangle} - 1\}} \nu_P;$$

**Remarks.**

1. We expect that $c_P > 0$ for and only for number fields.
2. Calculations in [Ad] for lower ranks groups indicates that, for number fields, $\frac{1}{c_P} \in \mathbb{Z}_{>0}$. It would be very interesting to find a close formula for them.

All this indicates that non-commutative group structures are naturally embedded into our pure high rank zeta functions.

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