Existence of a nontrivial solution for a strongly indefinite periodic Schrödinger-Poisson system

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Abstract: We consider the Schrödinger-Poisson system
\begin{align*}
-\Delta u + V(x)u + |u|^{p-2}u &= \lambda \phi u, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \text{in } \mathbb{R}^3.
\end{align*}
where \( \lambda > 0 \) is a parameter, \( 3 < p < 6 \), \( V \in C(\mathbb{R}^3) \) is 1-periodic in \( x \) for \( j = 1, 2, 3 \) and 0 is in a spectral gap of the operator \( -\Delta + V \). This system is strongly indefinite, i.e., the operator \( -\Delta + V \) has infinite-dimensional negative and positive spaces and it has a competitive interplay of the nonlinearities \( |u|^{p-2}u \) and \( \lambda \phi u \). Moreover, the functional corresponding to this system does not satisfy the Palais-Smale condition. Using a new infinite-dimensional linking theorem, we prove that, for sufficiently small \( \lambda > 0 \), this system has a nontrivial solution.

Key words: Schrödinger-Poisson system; infinite-dimensional linking; strongly indefinite.

2000 Mathematics Subject Classification: 35J20, 35J60

1 Introduction and statement of results

In this paper, we consider the Schrödinger-Poisson system
\begin{align}
-\Delta u + V(x)u + |u|^{p-2}u &= \lambda \phi u, \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi &= u^2, \quad \text{in } \mathbb{R}^3.
\end{align}
where \( 3 < p < 6 \) and \( \lambda > 0 \) is a parameter. The potential function \( V \) is a continuous function that is 1-periodic in \( x \) for \( j = 1, 2, 3 \). Under this assumption, \( \sigma(L) \), the spectrum of the operator
\begin{align}
L = -\Delta + V : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3),
\end{align}
is a purely continuous spectrum that is bounded below and consists of closed disjoint intervals (Theorem XIII.100). Thus, the complement \( \mathbb{R} \setminus \sigma(L) \) consists of open intervals called spectral gaps. We assume
\( (v) \). \( V \in C(\mathbb{R}^3) \) is 1-periodic in \( x \) for \( j = 1, 2, 3 \) and 0 is in a spectral gap \( (-\alpha, \beta) \) of \( -\Delta + V \), where \( 0 < \alpha, \beta < +\infty \).

A solution \((u, \phi)\) of (1.1) is called nontrivial if \( u \not\equiv 0 \) and \( \phi \not\equiv 0 \). Our main result is as follows:

**Theorem 1.1.** Suppose that \( 3 < p < 6 \) and \( (v) \) is satisfied. Then, there exists \( \lambda_0 > 0 \) such that, for any \( 0 < \lambda < \lambda_0 \), the problem (1.1) has a nontrivial solution.

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The Schrödinger-Poisson systems such as (1.1) arise in quantum mechanics, and are related to the study of the nonlinear Schrödinger equation for a particle in an electromagnetic field or the Hartree-Fock equation. Such systems have attracted much attention in recent years. For example, in [9] and [23], the system

$$- \Delta u + \omega u + u^2 = \phi u, \quad -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3$$

is used to describe a Hartree model for crystals, where $\omega$ is a positive constant. Problem (1.1) can also be seen as a nonlinear perturbation of the so-called Choquard system

$$- \Delta u + \omega u = \lambda \phi u, \quad -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3.$$ 

This system was introduced as an approximation to the Hartree-Fock theory for a one-component plasma ([1, 15], [24]-[32]). Problem (1.1) is also related to the so-called Schrödinger-Poisson-Slater system:

$$-\Delta u + V(x)u + \lambda \phi u = f(x, u), \quad -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3$$

where $\lambda > 0$. In recent years, there have been numerous studies of such systems under various assumptions on $V$ and $f$. One can, for example, see [2, 4, 5], [7, 14, 16, 20, 37] for the case $\inf_{\mathbb{R}^3} V > 0$ and [10, 11, 12] for the case where $V$ changes sign in $\mathbb{R}^3$. Finally, we should mention that, in [33], the author considers the system

$$-\Delta u + \omega u + W'_u(x, u) = \lambda u\phi, \quad -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3,$$

where $\omega$ and $\lambda$ are positive constants. The potential $W(x, u)$ is nonnegative, radially symmetric in $x$ and, even in $u$ and satisfies some growth conditions. The model nonlinearity is $\frac{1}{p}|u|^p$ with $2 < p < 6$. Using the symmetric mountain pass theorem (see [3]), the author obtained infinitely many radially symmetric solutions of this system.

From a mathematical point of view, problem (1.1) possesses some interesting properties. First of all, this problem has a variational structure. Let $D^{1,2}(\mathbb{R}^3)$ be the Hilbert space

$$D^{1,2}(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} |\nabla u|^2 < \infty \}$$

with inner product

$$(u, v)_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \nabla u \nabla v dx.$$ 

For $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, let

$$J_{\lambda}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx + \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} u^2 \phi dx.$$ 

Then, the critical points of $J_{\lambda}$ solve (1.1). Because $V$ is 1-periodic in every variable, this functional is invariant under a 1-periodic translation. As a consequence, the functional does not satisfy the Palais-Smale condition (see [38]). Second, because $0$ is in a spectral gap of $-\Delta + V$, the quadratic form $\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx$ has infinite-dimensional negative and positive spaces. This case is called strongly indefinite. Finally, since $J_{\lambda}$ is strongly indefinite, it is natural to use the infinite-dimensional linking theorem (see [21] Theorem 3.4) or [38] Theorem 6.10) to obtain critical points of $J_{\lambda}$. However, because of the competitive interplay of the nonlinearities $\frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$ and $\lambda \int_{\mathbb{R}^3} u^2 \phi dx$, $J_{\lambda}$ does not satisfy the global linking condition (see (6.4) in [38]) or the $\tau$-upper semi-continuous assumption (see (6.3) in [38]) in this theorem. To the best of our knowledge, variational problems possessing all these properties have never been studied before. Our study of this problem will shed some light on other variational problems possessing properties similar to those mentioned above.

In this paper, we first, modify the nonlinear terms of problem (1.1) such that the variational functional for the modified system of equations has a global infinite-dimensional linking structure. We then use a new infinite-dimensional linking theorem (see Theorem 5.2 in the appendix) to obtain a nontrivial solution for the modified system. This theorem replaces the $\tau$-upper semi-continuous assumption in the classical
infinite-dimensional linking theorem (see \cite{33} Theorem 6.10 or \cite{21} Theorem 3.4) with other assumption (see \cite{5.3} in Theorem 5.2). For reader’s convenience, we give the proof of this theorem in the appendix. Finally, we use a blow-up argument to show that if \( \lambda > 0 \) is sufficiently small, then the nontrivial solution of the modified system is in fact a nontrivial solution of (1.1).

**Notation.** \( B_r(a) \) denotes the open ball of radius \( r \) and center \( a \). For a Banach space \( X \), we denote the dual space of \( X \) by \( X' \), and denote strong and weak convergence in \( X \) by \( \to \) and \( \rightharpoonup \), respectively. For \( \varphi \in C^1(X; \mathbb{R}) \), we denote the Fréchet derivative of \( \varphi \) at \( u \) by \( \varphi'(u) \). The Gateaux derivative of \( \varphi \) is denoted by \( \langle \varphi'(u), v \rangle \), \( \forall u, v \in X \). \( L^q(\mathbb{R}^3) \) and \( L^q_{loc}(\mathbb{R}^3) \) denote the standard \( L^q \) space and the locally \( q \)-integrable function space, respectively (\( 1 \leq q \leq \infty \)), and \( H^1(\mathbb{R}^3) \) denotes the standard Sobolev space with norm \( \| u \|_{H^1} = (\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx)^{1/2} \). Let \( \Omega \) be a domain in \( \mathbb{R}^N \) (\( N \geq 1 \)). \( C_0^\infty(\Omega) \) is the space of infinitely differentiable functions with compact support in \( \Omega \).

2 A modified system for (1.1)

Let \( \eta \in C_0^\infty(\mathbb{R}) \) be such that

\[
0 \leq \eta \leq 1; \quad \eta(t) = 0, \quad r \in [1, 1]; \quad \eta(0) = 0, \quad |t| \geq 2; \quad \eta(-t) = \eta(t), \quad \forall t \in \mathbb{R}.
\]

For \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \), let

\[
\eta_n(t) = \eta(t/n), \quad F_n(t) = t^2 \eta_n(t) \quad \text{and} \quad f_n(t) = F_n'(t)/2.
\]

Consider the following system

\[
\begin{align*}
\Delta u + V(x)u + |u|^{p-2}u &= \lambda f_n(u), & \text{in } \mathbb{R}^3, \\
\Delta \phi &= F_n(u), & \text{in } \mathbb{R}^3.
\end{align*}
\]

We can prove that the second equation has a unique solution \( \phi_n \). Substituting \( \phi_n \) into the first equation of (2.1), the problem can be transformed into a one variable equation. In fact, we have the following lemma:

**Lemma 2.1.** For any \( u \in H^1(\mathbb{R}^3) \), there exists a unique \( \phi_{n,u} \in D^{1,2}(\mathbb{R}^3) \) that is a solution of

\[
- \Delta \phi = F_n(u), \quad \text{in } \mathbb{R}^3.
\]

Moreover,

(i). \( \phi_{n,u}(x) = \int_{\mathbb{R}^3} \frac{F_n(u(y))}{4\pi|x-y|} \, dy, \quad x \in \mathbb{R}^3. \)

(ii). \( \int_{\mathbb{R}^3} |\nabla \phi_{n,u}|^2 \, dx = \int_{\mathbb{R}^3} F_n(u) \phi_{n,u} \, dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{F_n(u(x))F_n(u(y))}{4\pi|x-y|} \, dx \, dy. \)

(iii). Let \( C_* = \inf_{0 \neq v \in D^{1,2}(\mathbb{R}^3)} \frac{\int_{\mathbb{R}^3} |\nabla v|^2 \, dx}{(\int_{\mathbb{R}^3} |v|^p \, dx)^{1/3}} > 0 \)

be the Sobolev constant (see, for example, \cite{33} Theorem 1.8)). Then

\[
\int_{\mathbb{R}^3} |\nabla \phi_{n,u}|^2 \, dx \leq C_*^{-1} \left( \int_{\mathbb{R}^3} |u|^{2p} \, dx \right)^{1/2}, \quad \forall u \in H^1(\mathbb{R}^3).
\]

(iv). There exists a positive constant \( D \), which is independent of \( n \) and \( u \), such that \( 0 \leq \phi_{n,u} \leq Dn^2 \) in \( \mathbb{R}^3 \).
Proof. Because $0 \leq F_n(t) \leq t^2$ for all $n$ and $t$, the Hölder and the Sobolev inequalities implies that, for any $u \in H^1(\mathbb{R}^3)$ and $v \in D^{1,2}(\mathbb{R}^3)$,

$$
| \int_{\mathbb{R}^3} F_n(u) v dx | \leq \left( \int_{\mathbb{R}^3} |F_n(u)|^\frac{p}{2} dx \right)^\frac{2}{p} \left( \int_{\mathbb{R}^3} |v|^2 dx \right)^\frac{1}{2} 
$$

$$
\leq \left( \int_{\mathbb{R}^3} |u|^\frac{p}{2} dx \right)^\frac{2}{p} \left( \int_{\mathbb{R}^3} |v|^2 dx \right)^\frac{1}{2} \leq C_n^{-\frac{p}{2}} \left( \int_{\mathbb{R}^3} |u|^\frac{p}{2} dx \right)^\frac{2}{p} \left( \int_{\mathbb{R}^3} |v|^2 dx \right)^\frac{1}{2}. \quad (2.3)
$$

It follows that, for fixed $u \in H^1(\mathbb{R}^3)$, $v \in D^{1,2}(\mathbb{R}^3) \rightarrow \int_{\mathbb{R}^3} F_n(u) v dx$ is a bounded linear functional in $D^{1,2}(\mathbb{R}^3)$. Then, by the Riesz theorem, there exists a unique $\phi_{n,u} \in D^{1,2}(\mathbb{R}^3)$ such that

$$
\int_{\mathbb{R}^3} \nabla \phi_{n,u} \nabla v dx = \int_{\mathbb{R}^3} F_n(u) v dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3). \quad (2.4)
$$

It follows that $\phi_{n,u}$ is the unique solution of $(2.2)$. Then, by the theory of Poisson’s equation (see, for example, Theorem 2.2.1 of [18]), we obtain the expression for $\phi_{n,u}$ in $(i)$. Moreover, choosing $v = \phi_{n,u}$ in $(2.4)$, we obtain result $(ii)$ of this lemma.

Choosing $v = \phi_{n,u}$ in $(2.3)$ and using the first equality in $(ii)$, we obtain result $(iii)$.

Since $\phi_{n,u}$ is the solution of $(2.4)$, the regularity theory for elliptic equations (see, for example, [19, Theorem 8.17]) implies that there exists a positive constant $C_1$ that is independent of $n$, $u$ and $y \in \mathbb{R}^3$, such that, for any $y \in \mathbb{R}^3$,

$$
||\phi_{n,u}||_{L^\infty(B_1(y))} \leq C_1(\int_{B_2(y)} |F_n(u)|^2 dx)^\frac{1}{2}. \quad (2.5)
$$

From the definition of $F_n$, we have $0 \leq F_n(t) \leq (2n)^2$, $\forall t \in \mathbb{R}$. It follows that

$$
C_1(\int_{B_2(y)} |F_n(u)|^2 dx)^\frac{1}{2} \leq C_1(\int_{B_2(y)} (2n)^4 dx)^\frac{1}{2} = 4C_1(\int_{B_2(0)} dx)^{1/2} n^2. \quad (2.6)
$$

Choosing $D = 4C_1(\int_{B_2(0)} dx)^{1/2}$, by $(2.5)$ and $(2.6)$, we get that $\phi_{n,u} \leq Dn^2$ in $\mathbb{R}^3$. Finally, since $F_n(u) \geq 0$ in $\mathbb{R}^3$, the maximum principle (see, for example, [19]) implies that $\phi_{n,u} \geq 0$ in $\mathbb{R}^3$. \hfill \Box

For $u \in H^1(\mathbb{R}^3)$, let

$$
\Phi_{n,\lambda}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} F_n(u) \phi_{n,u} dx
$$

$$
= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx + \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx - \frac{\lambda}{4} \int_{\mathbb{R}^3} F_n(u(x)) \frac{F_n(u(y))}{4\pi|x-y|} dxdy. \quad (2.7)
$$

A direct computation shows that the derivative of $\Phi_{n,\lambda}$ is

$$
\langle \Phi_{n,\lambda}'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv) dx + \int_{\mathbb{R}^3} |u|^{p-2} uv dx
$$

$$
- \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_n(u(x))v(x)F_n(u(y))}{4\pi|x-y|} dxdy, \quad \forall u, v \in H^1(\mathbb{R}^3). \quad (2.8)
$$

From $(i)$ of Lemma $(2.4)$ and $(2.8)$, we have the following

**Lemma 2.2.** The following statements are equivalent:

$(i).$ $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of $(2.1)$.

$(ii).$ $u$ is a critical point of $\Phi_{n,\lambda}$ and $\phi = \phi_{n,u}$.
3 Existence of a nontrivial solution for (2.1)

Recall that $L$ is the operator defined by (1.2). We denote by $|L|^{1/2}$ the square root of the absolute value of $L$. The domain of $|L|^{1/2}$ is the space

$$X := H^1(\mathbb{R}^3).$$

On $X$, we choose the inner product $(u, v) = \int_{\mathbb{R}^3} |L|^{1/2} u \cdot |L|^{1/2} v dx$ and the corresponding norm $||u|| = \sqrt{(u, u)}$. Since $0$ lies in a gap of the essential spectrum of $L$, there exists an orthogonal decomposition $X = Y \oplus Z$ such that $Z$ and $Y$ are the positive and negative spaces corresponding to the spectral decomposition of $L$. Since $V$ is 1-periodic for all variables, they are invariant under the action of $\mathbb{Z}^3$, i.e., for any $u \in Y$ or $u \in Z$ and for any $k = (n_1, n_2, n_3) \in \mathbb{Z}^3$, $u(\cdot - k)$ is also in $Y$ or $Z$. Furthermore,

$$\forall u \in Y, \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) dx = (u, u) = ||u||^2, \quad (3.1)$$

$$\forall u \in Z, \int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) dx = -(u, u) = -||u||^2. \quad (3.2)$$

Let $Q : X \rightarrow Z$, $P : X \rightarrow Y$ be the orthogonal projections. By (3.1) and (3.2),

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2) dx = ||Pu||^2 - ||Qu||^2, \quad \forall u \in X. \quad (3.3)$$

From $X = Y \oplus Z$, we get that, for any $u \in X$,

$$u = Pu + Qu, \quad ||u||^2 = ||Pu||^2 + ||Qu||^2. \quad (3.4)$$

This is the standard variational setting for the quadratic form $\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx$. See section 6.4 of [38] for more details.

By (3.3) and (2.7), we have

$$- \Phi_{n,\lambda}(u) = \frac{1}{2}||Qu||^2 - \frac{1}{2}||Pu||^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx + \frac{\lambda}{4} \int_{\mathbb{R}^3} F_n(u)\phi_{n,u} dx. \quad (3.5)$$

Moreover, by (2.8), for any $u, v \in X$,

$$\langle -\Phi'_{n,\lambda}(u), v \rangle = (Qu, v) - (Pu, v) - \int_{\mathbb{R}^3} |u|^{p-2}uv dx + \lambda \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{f_n(u(x))v(x)F_n(u(y))}{4\pi|x - y|} dxdy$$

$$= (Qu, v) - (Pu, v) - \int_{\mathbb{R}^3} |u|^{p-2}uv dx + \lambda \int_{\mathbb{R}^3} \phi_{n,u} f_n(u) v dx. \quad (3.6)$$

We will prove that if $\lambda > 0$ is sufficiently small, then $-\Phi_{n,\lambda}$ satisfies the global linking condition (see Lemma 3.1). However, because of the nonlinearity, $\frac{1}{4} \int_{\mathbb{R}^3} F_n(u)\phi_{n,u} dx$, $-\Phi_{n,\lambda}$ does not satisfies the $\tau$-upper continuous assumption (see (6.3) in [38]). Therefore, to obtain critical points of $-\Phi_{n,\lambda}$, we have to use the new infinite-dimensional linking theorem (Theorem 5.2 in appendix).

Let $\{e_k\}$ be a total orthonormal sequence in $Y$ and

$$|||u||| = \max \left\{|||Qu|||, \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \|(Pu, e_k)\|\right\}. \quad (3.7)$$

For $R > r > 0$ and $u_0 \in Z$ with $||u_0|| = 1$, set

$$N = \{u \in Z \mid ||u|| = r\}, \quad M = \{u + tu_0 \mid u \in Y, \ t \geq 0, \ ||u + tu_0|| \leq R\}$$

and

$$\partial M = \{u \in Y \mid ||u|| \leq R\} \cup \{u + tu_0 \mid u \in Y, \ t > 0, \ ||u + tu_0|| = R\}.$$
**Lemma 3.1.** The functional $-\Phi_{n,\lambda}$ satisfies the following

(a) $-\Phi_{n,\lambda}'$ is weakly sequentially continuous, where the weakly sequential continuity is defined in Theorem 3.2 in the appendix.

(b) there exist $\delta > 0$, $R > r > 0$, $u_0 \in Z$ with $||u_0|| = 1$ and $\lambda'_n > 0$ for any $n \in \mathbb{N}$ such that if $0 < \lambda < \lambda'_n$, then

$$
\inf_N (-\Phi_{n,\lambda}) > \max \left\{ 0, \sup_{\partial M} (-\Phi_{n,\lambda}), \sup_{||u|| \leq \delta} (-\Phi_{n,\lambda}(u)) \right\} \tag{3.8}
$$

and

$$
\sup_M (-\Phi_{n,\lambda}) < +\infty. \tag{3.9}
$$

**Proof.** (a) Let $u \in X$ and $\{u_k\} \subset X$ be such that $u_k \rightharpoonup u$ as $k \to \infty$. It follows that

$$(Q u_k, v) \to (Q u, v), \quad (P u_k, v) \to (P u, v), \quad k \to \infty, \forall v \in X, \tag{3.10}$$

and $u_k \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for any $1 \leq s < 6$. As consequences, for any $v \in C^\infty_0(\mathbb{R}^3)$, as $k \to \infty$,

$$
\int_{\mathbb{R}^3} |u_k|^{p-2} u_k v dx \to \int_{\mathbb{R}^3} |u|^{p-2} u v dx, \tag{3.11}
$$

and

$$
\int_{\mathbb{R}^3} F_n(u_k) v dx \to \int_{\mathbb{R}^3} F_n(u) v dx. \tag{3.12}
$$

From (2.4) and (3.12), we have that, for any $v \in C^\infty_0(\mathbb{R}^3)$, as $k \to \infty$,

$$
\int_{\mathbb{R}^3} \nabla \phi_{n,u_k} \nabla v dx \to \int_{\mathbb{R}^3} \nabla \phi_{n,u} \nabla v dx.
$$

This implies $\phi_{n,u_k} \to \phi_{n,u}$ in $D^{1,2}(\mathbb{R}^3)$. Consequently, $\phi_{n,u_k} \to \phi_{n,u}$ in $L^s_{loc}(\mathbb{R}^3)$ for any $1 \leq s < 6$. Together with $u_k \to u$ in $L^s_{loc}(\mathbb{R}^3)$ for any $1 \leq s < 6$, this yields that, for any $v \in C^\infty_0(\mathbb{R}^3)$, as $k \to \infty$,

$$
\int_{\mathbb{R}^3} \phi_{n,u_k} f_n(u_k) v dx \to \int_{\mathbb{R}^3} \phi_{n,u} f_n(u) v dx. \tag{3.13}
$$

From (3.6), (3.10), (3.11) and (3.13), we get that, as $k \to \infty$,

$$
\langle -\Phi_{n,\lambda}'(u_k), v \rangle \to \langle -\Phi_{n,\lambda}'(u), v \rangle, \quad \forall v \in C^\infty_0(\mathbb{R}^3).
$$

Therefore, $-\Phi_{n,\lambda}'$ is weakly sequentially continuous. Moreover, $-\Phi_{n,\lambda}$ maps bounded sets into bounded sets, hence $\sup_{M'} (-\Phi_{n,\lambda}) < +\infty$.

(b) If $u \in Z$, then $P u = 0$ and $Q u = u$. As $\phi_{n,u} \geq 0$ (see Lemma 2.1) and $F_n \geq 0$, we have $\int_{\mathbb{R}^3} F_n(u) \phi_{n,u} dx \geq 0$ for all $u \in X$. Then, using the Sobolev inequality

$$
||u||_{L^p(\mathbb{R}^3)} \leq C' ||u||, \tag{3.14}
$$

we get for the definition of $-\Phi_{n,\lambda}$ that, for any $u \in Z$,

$$
-\Phi_{n,\lambda}(u) \geq \frac{1}{2} ||u||^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx \geq \frac{1}{2} ||u||^2 - \frac{C'p}{p} ||u||^p.
$$

Let $r = C'^{-p/(p-2)}$. Then, for $N = \{u \in Z \mid ||u|| = r\}$,

$$
\inf_N (-\Phi_{n,\lambda}) \geq (\frac{1}{2} - \frac{1}{p}) C'^{-2p/(p-2)} > 0. \tag{3.15}
$$
Let $C > 0$ be such that
\[
||u||_{L^2(\mathbb{R}^n)} \leq C||u||, \quad \forall u \in X.
\] (3.16)

Let
\[
\lambda'_n = (C^2 Dn^2)^{-1},
\]
where $D$ is the constant appearing in Lemma 2.1(ii). Then, for any $0 < \lambda \leq \lambda'_n$ and $u \in X$, $F_n(t) \leq t^2$ and $\phi_{n,u} \leq Dn^2$ (see Lemma 2.1(iv)) yield
\[
\Phi_n(\lambda, u) = \frac{1}{2} ||Qu||^2 - \frac{1}{2} ||Pu||^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p dx + \frac{\lambda}{4} \int_{\mathbb{R}^n} F_n(u)\phi_{n,u} dx
\]
\[
\leq \frac{1}{2} ||Qu||^2 - \frac{1}{2} ||Pu||^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p dx + \frac{\lambda}{4} \int_{\mathbb{R}^n} u^2 \cdot (Dn^2) dx
\]
\[
\leq \frac{1}{2} ||Qu||^2 - \frac{1}{2} ||Pu||^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p dx + \frac{1}{4} ||u||^2
\]
\[
= \frac{3}{4} ||Qu||^2 - \frac{1}{4} ||Pu||^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p dx.
\] (3.17)

Let $u_0 \in Z$ be such that $||u_0|| = 1$. And let $u = v + tu_0 \in Y \oplus \mathbb{R} u_0$. By (3.17), we have
\[
- \Phi_n(\lambda, u) \leq \frac{3}{4} t^2 - \frac{1}{4} ||v||^2 - \frac{1}{p} \int_{\mathbb{R}^n} |v + tu_0|^p dx.
\] (3.18)

From (3.18) and the proof of Lemma 6.14 in [38], we obtain
\[
- \Phi_n(\lambda, u) \to -\infty, \quad \text{as} \quad ||u|| \to \infty, \quad u \in Y \oplus \mathbb{R} u_0.
\] (3.19)

Moreover, for $0 < \lambda \leq \lambda'_n$ and $u \in Y$, (3.17) implies
\[
- \Phi_n(\lambda, u) \leq -\frac{1}{4} ||u||^2 - \frac{1}{p} \int_{\mathbb{R}^n} |u|^p dx \leq 0.
\]

Together with (3.19), this yields
\[
\sup_{\partial M} (-\Phi_n(\lambda)) \leq \inf_N (-\Phi_n(\lambda)).
\] (3.20)

From (3.17) and the definition of $||| \cdot |||$ (see (3.7)), we have, for $0 < \lambda \leq \lambda'_n$,
\[
- \Phi_n(\lambda, u) \leq \frac{3}{4} ||Qu||^2 \leq \frac{3}{4} ||u||^2.
\] (3.21)

Choosing $\delta = \sqrt[4]{(\frac{3}{4})} \cdot 2 \cdot (p-2)^{1/2}$, (3.21) and (3.15) give that
\[
\sup_{||u|| \leq \delta} (-\Phi_n(\lambda))(u) < \inf_N (-\Phi_n(\lambda)).
\]

Together with (3.20), this yields (3.8).

\[\square\]

**Lemma 3.2.** Let $V_\infty = \max \{-V, 0\}$ and $M = \|V_\infty\|_{L^\infty(\mathbb{R}^3)} + 2$. For any $n \in \mathbb{N}$, there exists $\lambda''_n > 0$ such that if $0 < \lambda \leq \lambda''_n$ and $\{u_k\}$ is a $(C)_c$ sequence for $-\Phi_n(\lambda)$, i.e.
\[
\sup_n (-\Phi_n(\lambda)(u_k)) \leq c, \quad (1 + ||u_k||) - \Phi_n(\lambda)(u_k)|_{\chi'} \to 0, \quad \text{as} \quad k \to \infty,
\] (3.22)
then
\[
\int_{\omega_k} |u_k|^p dx \to 0, \quad k \to \infty,
\] (3.23)

where
\[
\omega_k = \{ x \in \mathbb{R}^3 \mid |u_k(x)| \geq M \}.
\]
Lemma 3.3. For any $n \in \mathbb{N}$, there exists $\lambda''_n > 0$ such that, if $0 < \lambda < \lambda''_n$ and $\{u_k\}$ is a $(\overline{C})_c$ sequence for $-\Phi_{n,\lambda}$, then $\{u_k\}$ is bounded in $X$.

Proof. From $\langle (1 + ||u_k||)|| - \Phi'_{n,\lambda}(u_k)||_{L'} \rightarrow 0$, we have

\[ \langle -\Phi'_{n,\lambda}(u_k), Q u_k \rangle = o(1) \quad \text{and} \quad \langle -\Phi'_{n,\lambda}(u_k), P u_k \rangle = o(1), \]

where $Q$ and $P$ are the orthogonal projections onto $L^1$ and $L^\infty$, respectively. Then, by the Sobolev inequality, we obtain

\[ \int_{\mathbb{R}^3} |\nabla u_k|^2 dx \leq C \int_{\mathbb{R}^3} |u_k|^2 dx \]

where $C$ is a constant depending only on $n$. Therefore, there exists $C'' > 0$ such that $\|u_k\| \leq C'' ||u_k||$, $\forall k \in \mathbb{N}$. Then, by (3.22), we get $\langle \Phi'_{n,\lambda}(u_k), u_k \rangle = o(1)$, here $o(1)$ denotes the infinitesimal depending only on $k$, i.e., $o(1) \rightarrow 0$ as $k \rightarrow \infty$. Together with (2.8), this yields

\[ o(1) = \int_{\mathbb{R}^3} \nabla u_k \nabla v_k dx + \int_{\mathbb{R}^3} V(x) u_k v_k dx + \int_{\mathbb{R}^3} |u_k|^{p-2} u_k v_k dx - \lambda \int_{\mathbb{R}^3} f_n(u_k) \phi_{n,u_k} v_k dx \]

where $V = V^+ + V^-$ and

\[ \overline{w}_k^+ = \{ x \in \mathbb{R}^3 | u_k(x) \geq M - 1 \}. \]

From $0 \leq \phi_{n,u_k} \leq Dn^2$ and $|t^{-1} f_n(t)| \leq 3$, we deduce that if

\[ 0 < \lambda \leq \frac{1}{6Dn^2} := \lambda''_n, \]

then, for any $x \in \overline{w}_k$, $|u_k|^{p-2} - V - \lambda u_k^{-1} f_n(u_k) \phi_{n,u_k} > 0$. Together with (3.24) and the fact that $V^+, u_k$ and $v_k$ are nonnegative on $\overline{w}_k$, this implies

\[ \int_{\overline{w}_k^+} |\nabla v_k|^2 dx = o(1). \]

Then, by the Sobolev inequality, we obtain

\[ \int_{\overline{w}_k^+} |v_k|^6 dx = o(1). \]

Let

\[ \overline{w}_k^+ = \{ x \in \mathbb{R}^3 | u_k(x) \geq M \}. \]

Then $\overline{w}_k^+ \subset \overline{w}_k$. And on $\overline{w}_k^-$, $v_k^6 \geq u_k^6 / M^6 \geq u_k^6 / M^p$. It follows that

\[ \int_{\overline{w}_k^-} |u_k|^p dx \leq M^p \int_{\overline{w}_k^+} |v_k|^6 dx = o(1). \]

Similarly, we have

\[ \int_{\overline{w}_k^-} |u_k|^p dx = o(1). \]

where $\overline{w}_k^- = \{ x \in \mathbb{R}^3 | - u_k(x) \geq M \}$. (3.23) follows from (3.25) and (3.26) immediately. \qed
where $o(1)$ denotes the infinitesimal depending only on $k$, i.e., $o(1) \to 0$ as $k \to \infty$. Together with (3.6), this yields
\[
||Qu_k||^2 = \int_{\mathbb{R}^3} |u_k|^{p-2}u_k \cdot Qu_k dx - \lambda \int_{\mathbb{R}^3} f_n(u_k)\phi_{n,u_k} \cdot Qu_k dx + o(1)
\] (3.27)
and
\[
||P_{u_k}||^2 = -\int_{\mathbb{R}^3} |u_k|^{p-2}u_k \cdot P_{u_k} dx + \lambda \int_{\mathbb{R}^3} f_n(u_k)\phi_{n,u_k} \cdot P_{u_k} dx + o(1).
\] (3.28)
For $\epsilon > 0$, let
\[
A_{\epsilon,k} = \{ x \in \mathbb{R}^3 \mid |u_k(x)| < \epsilon \}.
\]
And recall that $\varpi_k$ is the set defined in Lemma 5.2. Using $0 \leq \phi_{n,u_k} \leq Dn^2$, $|f_n(t)| \leq 5t$ and $|u_k| \leq M$ on $\mathbb{R}^3 \setminus \varpi_k$, we get from (3.27) that
\[
||Qu_k||^2
\]
\[= \int_{\mathbb{R}^3} |u_k|^{p-2}u_k \cdot Qu_k dx - \lambda \int_{\mathbb{R}^3} f_n(u_k)\phi_{n,u_k} \cdot Qu_k dx + o(1)
\]
\[\leq \bigg( \int_{A_{\epsilon,k}} + \int_{\mathbb{R}^3 \setminus (\varpi_k \cup A_{\epsilon,k})} \bigg) |u_k|^{p-1} \cdot |Qu_k| dx + 5\lambda Dn^2 \int_{\mathbb{R}^3} |u_k| \cdot |Qu_k| dx
\]
\[\leq C^2 \epsilon^{p-2}||u_k||^2 + \frac{C M \epsilon^{p-2}}{p} \int_{\mathbb{R}^3 \setminus (\varpi_k \cup A_{\epsilon,k})} |u_k|^2 dx \frac{1}{p} \|Qu_k\| + C'\int_{\varpi_k} |u_k|^p dx \frac{1}{p} \|Qu_k\|
\]
\[+ 5\lambda CDn^2||u_k||^2 + o(1),
\] (3.29)
where $C'$ and $C$ come from (3.14) and (3.16), respectively. Similarly, we have
\[
||P_{u_k}||^2 \leq C^2 \epsilon^{p-2}||u_k||^2 + \frac{C M \epsilon^{p-2}}{p} \int_{\mathbb{R}^3 \setminus (\varpi_k \cup A_{\epsilon,k})} |u_k|^2 dx \frac{1}{p} \|P_{u_k}\| + C'\int_{\varpi_k} |u_k|^p dx \frac{1}{p} \|P_{u_k}\|
\]
\[+ 5\lambda CDn^2||u_k||^2 + o(1).
\] (3.30)
Since $||u_k||^2 = ||P_{u_k}||^2 + ||Qu_k||^2$ (see (3.4)), these two inequalities (3.29) and (3.30) imply that
\[
||u_k||^2 \leq 2C^2 \epsilon^{p-2}||u_k||^2 + \frac{2CM \epsilon^{p-2}}{p} \int_{\mathbb{R}^3 \setminus (\varpi_k \cup A_{\epsilon,k})} |u_k|^2 dx \frac{1}{p} \|u_k\| + 2C'\int_{\varpi_k} |u_k|^p dx \frac{1}{p} \|u_k\|
\]
\[+ 10\lambda CDn^2||u_k||^2 + o(1)
\] (3.31)
From $\sup_n(-\Phi_{n,\lambda}(u_k)) \leq c$ and $(1 + ||u_k||) - \Phi_{n,\lambda}'(u_k)||x'| \to 0$, we obtain
\[
o(1) + c \geq -\Phi_{n,\lambda}(u_k) + \frac{1}{2} \Phi_{n,\lambda}'(u_k, u_k)
\]
\[= \left( \frac{1}{2} - \frac{1}{p} \right) \int_{\mathbb{R}^3} |u_k|^p dx - \frac{\lambda}{2} \int_{\mathbb{R}^3} (u_k f_n(u_k) - \frac{1}{2} F_n(u_k)) \phi_{n,u_k} dx.
\] (3.32)
From the definitions of $f_n$ and $F_n$, we have that
\[
|t^{-2}(tf_n(t) - \frac{1}{2} F_n(t))| \leq 5/2, \forall t \in \mathbb{R}.
\] (3.33)
From (3.32), (3.33), and \( \phi_{n,u_k} \leq Dn^2 \), we obtain

\[
e^{p-2} \int_{\mathbb{R}^3 \setminus (\pi_k \cup A_{k,k})} |u_k|^2 \, dx \\
\leq \int_{\mathbb{R}^3 \setminus (\pi_k \cup A_{k,k})} |u_k|^p \, dx \\
\leq c \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} + \lambda_1 \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} \int_{\mathbb{R}^3} (u_k f_n(u_k) - \frac{1}{2} f_n(u_k)) \phi_{n,u_k} \, dx - \int_{\pi_k \cup A_{k,k}} |u_k|^p \, dx + o(1) \\
\leq c \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} + \frac{5\lambda_1}{4} \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} Dn^2 \int_{\mathbb{R}^3} u_k^2 \, dx + o(1) \\
\leq c \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} + \left( \frac{5\lambda_1}{4} \right) \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} Dn^2 \cdot |\lambda||u_k||^2 + o(1). \tag{3.34}
\]

It follows that

\[
\left( \int_{\mathbb{R}^3 \setminus (\pi_k \cup A_{k,k})} |u_k|^2 \, dx \right)^{\frac{1}{2}} \\
\leq \epsilon^{1-p/2} c^{1/2} \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} + \epsilon^{1-p/2} \left( \frac{5\lambda_1}{4} \right) \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} Dn^2 \lambda^{1/2} |u_k| + o(1). \tag{3.35}
\]

If \( \epsilon \) and \( \lambda_n^{''''} \) are such that

\[
2C^2 \epsilon^{p-2} = \frac{1}{8}, \quad 2CM^p \epsilon^{1-p/2} \left( \frac{5\lambda_1}{4} \right) \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} Dn^2 \lambda^{1/2} \lambda^{''''} \leq \frac{1}{8}, \quad 10CD\lambda^{''''} \leq \frac{1}{8}, \quad \lambda^{'''} \leq \lambda_n^{''''},
\]

then, from (3.35), (3.31) and the fact that \( \int_{\pi_k} |u_k|^p \, dx = o(1) \) (Lemma 3.2), we deduce that, for \( 0 < \lambda \leq \lambda_n^{''''} \), \( \{ ||u_k|| \} \) is bounded. This completes the proof.

Let \( \lambda_n = \min \{ \lambda_n', \lambda_n'', \lambda_n^{''''} \} \), where \( \lambda_n', \lambda_n'' \) and \( \lambda_n^{''''} \) are the constants in Lemma 3.1, Lemma 3.2 and Lemma 3.3 respectively.

**Lemma 3.4.** For any \( n \in \mathbb{N} \) and \( 0 < \lambda < \lambda_n \), the system (2.1) has a nontrivial solution.

**Proof.** By Lemma 3.1, Lemma 3.3 and Theorem 5.2 in the appendix, we deduce that, for any \( n \in \mathbb{N} \) and \( 0 < \lambda < \lambda_n \), there exists a bounded \( (C)_e \) sequence \( \{ u_k \} \) for \( \Phi_{n,\lambda} \) with \( \inf_k ||u_k|| > 0 \). Up to a subsequence, either

(a) \( \lim_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_k|^2 \, dx = 0 \), or

(b) \( \exists \varrho > 0 \) and \( a_k \in \mathbb{Z}^3 \) such that \( \int_{B_1(a_k)} |u_k|^2 \, dx \geq \varrho \).

If (a) occurs, the Lions lemma (see, for example, [38, Lemma 1.21]) implies \( u_k \) satisfies \( u_k \to 0 \) in \( L^\infty(\mathbb{R}^3) \) for any \( 2 < s < 6 \). It follows that

\[
\int_{\mathbb{R}^3} |u_k|^p - u_k \cdot Qu_k \, dx \to 0, \quad \int_{\mathbb{R}^3} |u_k|^p - u_k \cdot Pu_k \, dx \to 0, \tag{3.36}
\]

and by \( |t^{-1} f(t)| \leq 3, \forall t \in \mathbb{R} \) and Lemma 2.1 (iii), we have

\[
\int_{\mathbb{R}^3} f_n(u_k) \phi_{n,u_k} \cdot Qu_k \, dx = \int_{\mathbb{R}^3} u_k^{-1} f_n(u_k) \phi_{n,u_k} \cdot u_k \cdot Qu_k \, dx \\
\leq 3 \int_{\mathbb{R}^3} u_k \phi_{n,u_k} \cdot Qu_k \, dx \\
\leq 3 \int_{\mathbb{R}^3} |\phi_{n,u_k}|^p \, dx \cdot \left( \int_{\mathbb{R}^3} |u_k|^{\frac{4}{4-p}} \, dx \right)^{\frac{p}{4}} \left( \int_{\mathbb{R}^3} |Qu_k|^{\frac{4}{4-p}} \, dx \right)^{\frac{p}{4}} \\
\leq 3C_n^{-1} \left( \int_{\mathbb{R}^3} |u_k|^{\frac{4}{4-p}} \, dx \right)^{\frac{p}{4}} \left( \int_{\mathbb{R}^3} |Qu_k|^{\frac{4}{4-p}} \, dx \right)^{\frac{p}{4}} \to 0. \tag{3.37}
\]
Similarly, we have \( \int_{\mathbb{R}^3} f_n(u_k) \phi_{n,u_k} \cdot Pu_k \, dx \to 0 \). Then from (3.36), (3.37), (3.27) and (3.28), we obtain \( \|u_k\| \to 0 \). This contradicts \( \inf_k \|u_k\| > 0 \). Therefore, case (a) cannot occur. As case (b) therefore occurs, \( w_k = u_k(\cdot + a_k) \) satisfies \( w_k \to u_0 \neq 0 \). From \( (1 + \|u_k\|) - \Phi_{n,\lambda}^\prime(w_k) \|x\| = (1 + \|u_k\|) - \Phi_{n,\lambda}^\prime(u_k) \|x\| \to 0 \) and the weakly sequential continuity of \( -\Phi_{n,\lambda}^\prime \) (see Lemma 5.1), we have that \( -\Phi_{n,\lambda}(u_0) = 0 \). Therefore, \( (u_0, \phi_{n,u_0}) \) is a nontrivial solution of (2.1). This completes the proof.

\[ \Box \]

4 Proof of Theorem 1.1

**Proposition 4.1.** Suppose that \( 3 < p < 6 \). Then for any \( \Lambda > 0 \), there exists \( N_\Lambda > 0 \) such that, if \( (u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) is a solution of (2.1) with \( n \geq N_\Lambda \) and \( 0 < \lambda \leq \Lambda \), then

\[ |u(x)| \leq n, \ \forall x \in \mathbb{R}^3. \]

**Proof.** We apply an indirect argument, and assume by contradiction that there exist \( \Lambda_0 > 0 \), a real number sequence \( \{\mu_n\} \) and a sequence \( \{(u_n, \phi_n)\} \) in \( H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3) \) such that \( 0 < \mu_n \leq \Lambda_0 \), \( (u_n, \phi_n) \) is a solution of (2.1) with \( \lambda = \mu_n \) and

\[ \|u_n\|_{L^\infty(\mathbb{R}^3)} > n. \tag{4.1} \]

Since \( \phi_{n,u} \) is a bounded function in \( \mathbb{R}^3 \) (see Lemma 2.1(iv)) and \( u_n \in H^1(\mathbb{R}^3) \) is a solution of

\[ -\Delta u + V(x)u + |u|^{p-2}u = \mu_n f_n(u) \phi_{n,u} \quad \text{in} \ \mathbb{R}^3, \tag{4.2} \]

the bootstrap argument of elliptic equations (see [19]), implies that \( u_n \in C^1(\mathbb{R}^3) \). Let \( x_n \in \mathbb{R}^3 \) be such that

\[ |u_n(x_n)| = \max_{x \in \mathbb{R}^3} |u_n(x)| = \|u_n\|_{L^\infty(\mathbb{R}^3)}. \tag{4.3} \]

We shall use the blow-up argument of [6] to induce a contradiction. Let \( M_n = \max_{x \in \mathbb{R}^3} |u_n(x)| \) and

\[ \tilde{u}_n(x) = M_n^{-\alpha} u_n(x + M_n^{-\alpha}), \ x \in \mathbb{R}^3 \]

with \( \alpha = (p - 2)/2 \). Then,

\[ |\tilde{u}_n| \leq 1 \ \text{in} \ \mathbb{R}^3, \ |\tilde{u}_n(0)| = 1, \ \forall n \in \mathbb{N}, \tag{4.4} \]

and

\[ u_n(x) = M_n \tilde{u}_n(M_n^\alpha(x - x_n)), \ x \in \mathbb{R}^3. \]

Substituting this expression into (4.2), we obtain

\[ \begin{align*}
-\Delta \tilde{u}_n + M_n^{1+2\alpha}(\Delta \tilde{u}_n)(M_n^\alpha(x - x_n)) + M_n V(x) \tilde{u}_n(M_n^\alpha(x - x_n)) \\
+ M_n^{p-1}(|\tilde{u}_n|^{p-2} \tilde{u}_n)(M_n^\alpha(x - x_n)) = \mu_n f_n(M_n \tilde{u}_n(M_n^\alpha(x - x_n))) \\
&= \mu_n f_n(M_n \tilde{u}_n(M_n^\alpha(x - x_n))) \int_{\mathbb{R}^3} \frac{F_n(M_n \tilde{u}_n(M_n^\alpha(y - x_n)))}{4\pi |x - y|} \, dy. \tag{4.5}
\end{align*} \]

By a direct computation, we have

\[ \int_{\mathbb{R}^3} \frac{F_n(M_n \tilde{u}_n(M_n^\alpha(y - x_n)))}{4\pi |x - y|} \, dy = M_n^{-2\alpha} \int_{\mathbb{R}^3} \frac{F_n(M_n \tilde{u}_n(y))}{4\pi |M_n^\alpha(x - x_n) - y|} \, dy, \]

Together with (4.5), this yields

\[ \begin{align*}
-\Delta \tilde{u}_n + M_n^{2-p} V(x_n + x M_n^{-\alpha}) \tilde{u}_n + |\tilde{u}_n|^{p-2} \tilde{u}_n \\
= M_n^{-2(p-3)} \cdot \mu_n f_n(M_n \tilde{u}_n) \int_{\mathbb{R}^3} \frac{F_n(M_n \tilde{u}_n(y))}{4\pi |x - y|} \, dy. \tag{4.6}
\end{align*} \]
From the definition of $f_n$, we have $|f_n(t)| \leq 6n, \forall t \in \mathbb{R}$. Then, Lemma 2.1 (iv) gives

$$0 \leq \phi_{n,M_n\tilde{u}_n} = \int_{\mathbb{R}^3} \frac{F_n(M_n\tilde{u}_n(y))}{4\pi|x-y|} \, dy \leq Dn^2.$$  

Then by $p > 3$, $\mu_n \leq \Lambda_0$ and $M_n > n$, we obtain

$$M_n^{-2(2p-3)} \cdot \mu_n |f_n(M_n\tilde{u}_n)| \int_{\mathbb{R}^3} \frac{F_n(M_n\tilde{u}_n(y))}{4\pi|x-y|} \, dy \leq 6D\Lambda_0n^3 M_n^{-2(p-3)} \to 0, \ n \to \infty. \quad (4.7)$$

Moreover, because $V$ is a bounded function in $\mathbb{R}^3$, we have that

$$M_n^{2-p}V(x_n + x M_n^{-\alpha}) \to 0, \ n \to \infty \quad (4.8)$$

holds uniformly for $x \in \mathbb{R}^3$. Then, by the standard elliptic estimates (see Section 9.2 of [19]), we deduce from $|\tilde{u}_n| \leq 1$ in $\mathbb{R}^3$, (4.6), (4.7) and (4.8) that, for any $2 \leq q < \infty$ and $0 < R < \infty$, $\tilde{u}_n$ is bounded in $W^{2,q}(B_R(0))$.

Without loss of generality, we may assume that $\tilde{u}_n$ converges weakly in $W^{2,q}(B_R(0)) (\forall R < +\infty, \forall p < +\infty)$ and thus in particular in $C^1(B_R(0))$ to some $u_0$ satisfying $|u_0(0)| = 1$. From (4.6), (4.7) and (4.8), we can see that $u_0$ satisfies

$$- \Delta u + |u|^{p-2}u = 0, \ u \in C^1(\mathbb{R}^3). \quad (4.9)$$

However, by Theorem 1 of [8], the only solution to this equation is $u = 0$, which contradicts $|u_0(0)| = 1$. This completes the proof.

**Proof of Theorem 1.1.** We choose $\Lambda = 1$ in Proposition 4.1 and choose $n_* \in \mathbb{N}$ satisfying $n_* \geq N_\Lambda$. Then, by Lemma 3.4 and Proposition 4.1 for any $0 < \lambda < \lambda_0 := \min\{\lambda_{n_*}, 1\}$, problem (2.1) with $n = n_*$ has a nontrivial solution $(u, \phi)$ satisfying

$$||u||_{L^\infty(\mathbb{R}^3)} \leq n_*.$$  

It follows that $f_{n_*}(u) = u$ and $F_{n_*}(u) = u^2$. Hence, $(u, \phi)$ is a nontrivial solution of (1.1). \hfill \square

## 5 Appendix: A variant infinite-dimensional linking theorem

In this section, we give a new infinite-dimensional linking theorem. This theorem replaces the $\tau$-upper semi-continuous assumption (see (6.3) in [38]) in the Kryszewski and Szulkin’s infinite-dimensional linking theorem (see [38] Theorem 6.10) or [21] Theorem 3.4]) with other assumptions (see (5.5) in Theorem 5.3). Our theorem is a generalization of the classical finite-dimensional linking theorem (see [34], Theorem 5.3])

Before state the infinite-dimensional linking theorem, we give some notations and definitions. Let $X$ be a separable Hilbert space with inner product $(\cdot, \cdot)$ and norm $|| \cdot ||$, respectively. $Y$ and $Z$ are closed subspaces of $X$ and $X = Y \oplus Z$. Let $\{e_k\}$ be a total orthonormal sequence in $Y$. Let

$$Q : X \to Z, \ P : X \to Y \quad (5.1)$$

be the orthogonal projections. We define

$$|||u||| = \max \left\{ ||Qu|| \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} ||(Pu, e_k)|| \right\} \quad (5.2)$$

on $X$. Then, $||Qu|| \leq ||u|| \leq ||u||, \forall u \in X,$ and if $||u_n||$ is bounded and $||u_n - u|| \to 0$, then $\{u_n\}$ weakly converges to $u$ in $X$. The topology generated by $||| \cdot |||$ is denoted by $\tau$, and all topological notations related to it will include this symbol.
Let \( R > r > 0 \) and \( u_0 \in Z \) with \(|u_0| = 1\). Set
\[
N = \{ u \in Z \mid ||u|| = r \}, \quad M = \{ u + t u_0 \mid u \in Y, \ t \geq 0, \ ||u + t u_0|| \leq R \}. \tag{5.3}
\]
Then, \( M \) is a submanifold of \( Y \oplus \mathbb{R}^+ u_0 \) with boundary
\[
\partial M = \{ u \in Y \mid ||u|| \leq R \} \cup \{ u + t u_0 \mid u \in Y, \ t > 0, \ ||u + t u_0|| = R \}. \tag{5.4}
\]

**Definition 5.1.** Let \( J \in C^1(X, \mathbb{R}) \). A sequence \( \{ u_n \} \subset X \) is called a \((C_c)\) sequence for \( J \), if
\[
\sup_n J(u_n) \leq c \quad \text{and} \quad (1 + ||u_n||)||J'(u_n)||_{X'} \to 0, \ \text{as} \ n \to \infty.
\]

**Theorem 5.2.** If \( H \in C^1(X, \mathbb{R}) \) satisfies
(a) \( H' \) is weakly sequentially continuous, i.e., if \( u \in X \) and \( \{ u_n \} \subset X \) are such that \( u_n \rightharpoonup u \), then, for any \( \varphi \in X \), \( \langle H'(u_n), \varphi \rangle \to \langle H'(u), \varphi \rangle \).
(b) there exist \( \delta > 0 \), \( u_0 \in Z \) with \( ||u_0|| = 1 \), and \( R > r > 0 \) such that
\[
\inf_N H > \max \left\{ \sup_{\partial M} H, \sup_{||u|| \leq \delta} H(u) \right\} \tag{5.5}
\]
and
\[
\sup_M H < +\infty \tag{5.6}
\]
Then there exists a \((C_c)\) sequence \( \{ u_n \} \) for \( H \) with \( c = \sup_M H \) and \( \inf_n ||u_n|| \geq \delta/2 \).

**Proof.** Arguing indirectly, assume that the result does not hold. Then, there exists \( \epsilon > 0 \) such that
\[
(1 + ||u||)||H'(u)||_{X'} \geq \epsilon, \ \forall u \in E \tag{5.7}
\]
where
\[
E = \{ u \in X \mid H(u) \leq d + \epsilon \} \cap \{ u \in X \mid ||u|| \geq \delta/2 \}
\]
and
\[
d = \sup_M H.
\]
From \( (5.5) \), we can choose \( \epsilon \) such that
\[
0 < \epsilon < \inf_N H - \max \left\{ \sup_{\partial M} H, \sup_{||u|| \leq \delta} H(u) \right\} \tag{5.8}
\]

**Step1.** A vector field in a \( \tau \)-neighborhood of \( E \).
Let
\[
b = \inf_N H, \quad T = 2(d - b + 2\epsilon)/\epsilon, \quad R = (1 + \sup_{u \in M} ||u||) \epsilon^T \tag{5.9}
\]
and
\[
B_R = \{ u \in X \mid ||u|| \leq R \}. \tag{5.10}
\]
For every \( u \in E \cap B_R \), there exists \( \phi_u \in X \) with \( ||\phi_u|| = 1 \) such that \( \langle H'(u), \phi_u \rangle \geq \frac{\epsilon}{2} ||H'(u)||_{X'} \).
Then, \( (5.7) \) implies
\[
(1 + ||u||)\langle H'(u), \phi_u \rangle > \frac{1}{2} \epsilon. \tag{5.11}
\]
From the definition of $||| \cdot |||$, we deduce that if a sequence $\{u_n\} \subset E \cap B_R$ $\tau$-converges to $u \in X$, i.e., $|||u_n - u||| \to 0$, then $u_n \rightharpoonup u$ in $X$ (see Remark 6.1 of [38]). By the weakly sequential continuity of $H'$, we get that for any $\varphi \in X$, $\langle H'(u_n), \varphi \rangle \to \langle H'(u), \varphi \rangle$. This implies that $H'$ is $\tau$-sequentially continuous in $E \cap B_R$. By (5.11), the $\tau$-sequential continuity of $H'$ in $E \cap B_R$ and the weakly lower semi-continuity of the norm $||| \cdot |||$, we get that there exists a $\tau$-open neighborhood $V_u$ of $u$ such that
\[
\langle H'(v), (1 + ||u||)\phi_u \rangle > \frac{1}{2} \epsilon, \forall v \in V_u, \tag{5.12}
\]
and
\[
||| (1 + ||u||)\phi_u ||| = 1 + ||u|| \leq 2(1 + ||v||), \forall v \in V_u. \tag{5.13}
\]

Because $B_R$ is a bounded convex closed set in the Hilbert space $X$, $B_R$ is a $\tau$-closed set. Therefore, $X \setminus B_R$ is a $\tau$-open set.

The family
\[
\mathcal{N} = \{V_u \mid u \in E \cap B_R\} \cup \{X \setminus B_R\}
\]
is a $\tau$-open covering of $E$. Let
\[
\mathcal{V} = \left( \bigcup_{u \in E \cap B_R} V_u \right) \bigcup (X \setminus B_R).
\]
Then, $\mathcal{V}$ is a $\tau$-open neighborhood of $E$.

Since $\mathcal{V}$ is metric, hence paracompact, there exists a local finite $\tau$-open covering $\mathcal{M} = \{M_i \mid i \in I\}$ of $\mathcal{V}$ finer than $\mathcal{N}$. If $M_i \subset V_u$, for some $u_i \in E$, we choose $\varpi_i = (1 + ||u_i||)\phi_{u_i}$ and if $M_i \subset X \setminus B_R$, we choose $\varpi_i = 0$. Let $\{\lambda_i \mid i \in I\}$ be a $\tau$-Lipschitz continuous partition of unity subordinated to $\mathcal{M}$. And let
\[
\xi(u) := \sum_{i \in I} \lambda_i(u)\varpi_i, \quad u \in \mathcal{V}. \tag{5.14}
\]

Since the $\tau$-open covering $\mathcal{M}$ of $\mathcal{V}$ is local finite, each $u \in \mathcal{V}$ belongs to only finite many sets $M_i$. Therefore, for every $u \in \mathcal{V}$, the sum in (5.14) is only a finite sum. It follows that, for any $u \in \mathcal{V}$, there exist a $\tau$-open neighborhood $U_u \subset \mathcal{V}$ of $u$ and $L_u > 0$ such that $\xi(U_u)$ is contained in a finite-dimensional subspace of $X$ and
\[
||| \xi(v) - \xi(w) ||| \leq L_u||v - w||, \quad \forall v, w \in U_u. \tag{5.15}
\]
Moreover, by the definition of $\xi$, (5.12) and (5.13), we get that, for every $u \in \mathcal{V}$,
\[
||| \xi(u) ||| \leq 1 + ||u|| \quad \text{and} \quad \langle H'(u), \xi(u) \rangle \geq 0 \tag{5.16}
\]
and for every $u \in E \cap B_R$,
\[
\langle H'(u), \xi(u) \rangle > \frac{1}{2} \epsilon. \tag{5.17}
\]

**Step2.** Let $\theta$ be a smooth function satisfying $0 \leq \theta \leq 1$ in $\mathbb{R}$ and
\[
\theta(t) = \begin{cases} 
0, & t \leq \frac{2\delta}{3}, \\
1, & t \geq \delta.
\end{cases}
\]

Let
\[
\chi(u) = \begin{cases} 
-\theta(||u||)\xi(u), & u \in \mathcal{V}, ||u|| \leq \frac{2\delta}{3} , \\
0, & ||u|| > \frac{2\delta}{3}.
\end{cases}
\]

Then, $\chi$ is a vector field defined in
\[
\mathcal{W} = \mathcal{V} \cup \{u \in X \mid ||u|| < \delta\}.
\]

It is a $\tau$-open neighborhood of $H^{d+\epsilon} \cup (X \setminus B_R)$, where
\[
H^{d+\epsilon} := \{u \in X \mid H(u) \leq d + \epsilon\}.
\]

From (5.15), (5.17) and the definition of $\chi$, we deduce that the mapping $\chi$ satisfies that
(a). each \( u \in \mathcal{W} \) has a \( \tau \)-open set \( V_u \) such that \( \chi(V_u) \) is contained in a finite-dimensional subspace of \( X \),
(b). for any \( u \in \mathcal{W} \), there exist a \( \tau \)-open neighborhood \( U_u \) of \( u \) and \( L'_u > 0 \) such that
\[
||\chi(v) - \chi(w)|| \leq L'_u||v - w||, \forall v, w \in U_u.
\] (5.18)

This means that \( \chi \) is locally Lipschitz continuous and \( \tau \)-locally Lipschitz continuous,
(c).
\[
||\chi(u)|| \leq 1 + ||u||, \forall u \in \mathcal{W},
\] (5.19)
(d).
\[
\langle H'(u), \chi(u) \rangle \leq 0, \forall u \in \mathcal{W}.
\] (5.20)

and
\[
\langle H'(u), \chi(u) \rangle < -\frac{1}{2}t, \forall u \in \{u \in E \mid ||u|| \geq \delta\} \cap B_R.
\] (5.21)

**Step 3.** From (5.18) and the fact that \( ||v|| \leq ||v||, \forall v \in X \), we have
\[
||\chi(v) - \chi(w)|| \leq L'_u||v - w||, \forall v, w \in U_u.
\] This implies that \( \chi \) is a local Lipschitz mapping under the \( || \cdot || \) norm. Then by the standard theory of ordinary differential equation in Banach space, we deduce that the following initial value problem
\[
\left\{ \begin{array}{l}
d\eta/dt = \chi(\eta), \\
\eta(0, u) = u \in \mathcal{W}.
\end{array} \right.
\] (5.22)
has a unique solution in \( \mathcal{W} \), denoted by \( \eta(t, u) \), with right maximal interval of existence \( [0, T(u)) \). Furthermore, using (5.18) and the Gronwall inequality (see, for example, Lemma 6.9 of [38]), the similar argument as the proof of c) in [38, Lemma 6.8] yields that

(A). \( \eta \) is \( \tau \)-continuous, i.e., if \( u_n \in \mathcal{W}, u_0 \in \mathcal{W}, 0 \leq t_n < T(u_n) \) and \( 0 \leq t_0 < T(u_0) \) satisfy
\[
||u_n - u_0|| \rightarrow 0 \text{ and } t_n \rightarrow t_0, \text{ then } \|\eta(t_n, u_n) - \eta(t_0, u_0)\| \rightarrow 0.
\] From (5.20), we have
\[
\frac{d}{dt}H(\eta(t, u)) = \langle H'(\eta(t, u)), \eta(t, u) \rangle \leq 0.
\]
Therefore, \( H \) is non-increasing along the flow \( \eta \). It follows that \( \{\eta(t, u) \mid 0 \leq t \leq T(u)\} \subset H^{d+\epsilon} \) if \( u \in H^{d+\epsilon} \), i.e., \( H^{d+\epsilon} \) is an invariant set of the flow \( \eta \). Then, (5.19) and Theorem 5.6.1 of [22] (or Corollary 4.6 of [36]) implies that, for any \( u \in H^{d+\epsilon}, T(u) = +\infty \).

**Step 4.** We shall prove that
\[
\{\eta(t, u) \mid 0 \leq t \leq T, u \in M\} \subset B_R.
\] (5.23)

Let \( u \in H^{d+\epsilon} \). By the result in Step 3, we have \( T(u) = +\infty \) and
\[
\eta(t, u) = u + \int_0^t \chi(\eta(s, u))ds, \forall t \in [0, +\infty).
\]
Together with (5.19), this yields
\[
||\eta(t, u)|| \leq ||u|| + \int_0^t ||\chi(\eta(s, u))||ds \leq ||u|| + \int_0^t (1 + ||\eta(s, u)||)ds.
\]
Then, by the Gronwall inequality (see, for example, Lemma 6.9 of [38]), we get that
\[ ||\eta(t,u)|| \leq (1 + ||u||)^e - 1, \quad \forall t \in [0, +\infty). \] (5.24)
Since \( M \subset H^{d+\epsilon} \), by (5.24) and the definition of \( R \) (see (5.9), we get (5.23).

**Step 5.** From the choice of \( \epsilon \) (see (5.8)), we have
\[ \sup_{||u|| \leq \delta} H < b - \epsilon. \]
It follows that
\[ \{u \in X \mid ||u|| \leq \delta\} \subset H^{b-\epsilon} := \{u \in X \mid H(u) \leq b - \epsilon\}. \]
Together with (5.21), this yields
\[ \langle H'(u), \chi(u) \rangle < -\frac{1}{2} \epsilon, \quad \forall u \in H^{d+\epsilon}_b \cap B_R, \] (5.25)
where
\[ H^{d+\epsilon}_b := \{u \in X \mid b - \epsilon \leq H(u) \leq d + \epsilon\}. \]

We show that, for any \( u \in M \), \( H(\eta(T,u)) \leq b - \epsilon \). Arguing indirectly, assume that this were not true. Then, there exists \( u \in M \) such that \( H(\eta(T,u)) > b - \epsilon \). Since \( H \) is non-increasing along the flow \( \eta \), from (5.23), we deduce that \( \{\eta(t,u) \mid 0 \leq t \leq T\} \subset H^{d+\epsilon}_b \cap B_R \). Then, by (5.25),
\[ H(\eta(T,u)) = H(\eta(0,u)) + \int_0^T \langle H'(\eta(s,u)), \chi(\eta(s,u)) \rangle ds \]
\[ \leq H(\eta(0,u)) + \int_0^T (-\frac{1}{2} \epsilon) ds \]
\[ \leq d + \epsilon - \frac{1}{2} \epsilon T = b - \epsilon. \] (5.26)
This contradicts \( H(\eta(T,u)) > b - \epsilon \). Therefore, we have
(B). \( \eta(T,M) \subset H^{b-\epsilon} \).

Moreover, using the result (a) in Step 2 and and the fact that \( \eta \) is \( \tau \)-continuous (see (A)), the similar argument as the proof of the result (b) of [38] Lemma 6.8] yields that
(C). Each point \((t,u) \in [0,T] \times H^{d+\epsilon}\) has a \( \tau \)-neighborhood \( N_{(t,u)} \) such that
\[ \{v - \eta(s,v) \mid (s,v) \in N_{(t,u)} \cap ([0,T] \times H^{d+\epsilon})\} \]
is contained in a finite-dimensional subspace of \( X \).

**Step 6.** Let
\[ h : [0,T] \times M \rightarrow X, \quad h(t,u) = P\eta(t,u) + (||Q\eta(t,u)|| - r)u_0 \]
where \( P, Q, r \) and \( u_0 \) are defined in (5.1), (5.3) and (5.4). Then
\[ 0 \in h(t,M) \Leftrightarrow \eta(t,M) \cap N \neq \emptyset. \]
From \( \inf_N H > \sup_{\partial M} H \) (see (5.3)) and the fact that, for any \( u \in X \), the function \( H(\eta(\cdot,u)) \) is non-increasing, we deduce that \( \inf_N H > \sup_{u \in \partial M} H(\eta(t,u)), \forall t \in [0,T] \). Therefore,
\[ 0 \notin h(t,\partial M), \quad \forall t \in [0,T]. \] (5.27)
Since \( \eta \) has the properties (A) and (C) obtained in step 3 and step 5 respectively and \( h \) satisfies (5.27),
there is an appropriate degree theory for \( \deg(h(t,\cdot), M, 0) \) (see Proposition 6.4 and Theorem 6.6 of [38]). Then, the same argument as the proof of Theorem 6.10 of [38] yields that
\[ \deg(h(T,\cdot), M, 0) = \deg(h(0,\cdot), M, 0) \neq 0. \]
It follows that \( 0 \in h(T,M) \) and \( \eta(T,M) \cap N \neq \emptyset \). Therefore, there exists \( u \in M \) such that \( H(\eta(T,u)) \geq b \). It contradicts the property (B) obtained in step 5. This completes the proof of this theorem. □
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