DECOMPOSITION OF PAULI GROUPS
VIA WEAK CENTRAL PRODUCTS

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Abstract. For any \(n, m \geq 1\) and prime power \(p^m\), we show a result of decomposition for Pauli groups \(\mathcal{P}_{n,p^m}\) in terms of weak central products. This can be used to describe the underlying structure of Pauli groups on \(n\) qudits of dimension \(q = p^m\) and enables us to identify abelian subgroups of \(\mathcal{P}_{n,p^m}\). As a consequence of our main results, we show a similar factorisation for the so-called ‘lifted’ Pauli groups, recently introduced by Gottesman and Kuperberg in the context of error-correcting codes in quantum information theory.

1. Introduction

The algebraic properties of the Pauli group find several applications in quantum information and computation. Notable examples are quantum error correcting codes [10, 11], efficient classical simulation [1], and the theory of mutually unbiased basis [8]. Central to these applications are the abelian subgroups of the Pauli group.

Identifying abelian subgroups is in general a nontrivial problem. Here we present an algebraic decomposition of the Pauli group that enables us to identify its abelian subgroups. Our main observation is that nonabelian extraspecial \(p\)-groups admit decompositions via central products and this makes easier the identification of their abelian subgroups. Although the Pauli group does not belong to this class, it is close to be nonabelian extraspecial. In fact, it is a generalised nonabelian extraspecial \(p\)-group. These types of groups admit a decomposition result in terms of weak central products. Leveraging on this result we prove a decomposition for Pauli groups on prime power qudits that offers relevant information on the structure of its subgroups and quotients.

As a further application of our techniques we show a decomposition result for the ‘lifted’ Pauli groups introduced by Gottesman and Kuperberg in the context of quantum error correcting codes for qudits [13, 14].

The paper is structured as follows. We begin in Section 1 by recalling some facts on the Heisenberg and Pauli groups. In Section 3 we prove some lemmas of computational nature and additional facts which will be used for the proofs of the main results, which we present in Section 4. We conclude in Section 5 where we show a decomposition for ‘lifted’ Pauli groups and discuss the problem of finding abelian subgroups in Pauli groups.

Notation. For an arbitrary group \(G\), the set \(Z(G) = \{a \in G \mid ab = ba, \forall b \in G\}\) denotes the center of \(G\). \(Z(G)\) is a normal subgroup of \(G\). For \(a, b \in G\) let \([a, b] = a^{-1}b^{-1}ab\) be the commutator between \(a\) and \(b\). The derived subgroup of a group \(G\) is its smallest subgroup containing all \([a, b]\) and is denoted by \([G, G] = \langle [a, b] \mid a, b \in G \rangle\). As usual, a subgroup \(H\) of \(G\) is maximal, if \(H \neq G\) and for any subgroup \(K\) of \(G\) such that \(H \subseteq K \subseteq G\), then either

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$K = H$ or $K = G$. The Frattini subgroup of $G$ is the intersection of all maximal subgroups of $G$ and is denoted by $\Phi(G)$. $\Phi(G)$ is normal in $G$. Finally, given two subgroups $H$ and $K$ of $G$ we say that $G = H \rtimes K$ is the semidirect product of $H$ and $K$, if $H \cap K = 1$, $H$ is normal in $G$, and $HK = G$. We refer the reader to [17] for a more detailed presentation of these notions.

2. Preliminaries

We begin with a brief review of relevant notions on the Heisenberg and Pauli groups.

2.1. The Heisenberg group. Let $\mathbb{F}$ be a field of characteristic $\neq 2$ and triples $(p, q, t) \in \mathbb{F}^3$. Endow $\mathbb{F}^3$ with the binary operation

$$\Box : ((p_1, q_1, t_1), (p_2, q_2, t_2)) \in \mathbb{F}^3 \times \mathbb{F}^3 \mapsto (p_1, q_1, t_1) \Box (p_2, q_2, t_2) = (p_1 + p_2, q_1 + q_2, t_1 + t_2 + \frac{1}{2}(p_1q_2 + q_1p_2)) \in \mathbb{F}^3.$$  \hspace{1cm} (2.1)

One can see that $(\mathbb{F}^3, \Box)$ is a group and in fact it is the Heisenberg group $\mathbb{H}(\mathbb{F})$, which admits a more convenient matrix representation of the form

$$\mathbb{H}(\mathbb{F}) = \left\{ \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \mid p, q, t \in \mathbb{F} \right\} = \left\{ M(p, q; t) \mid p, q, t \in \mathbb{F} \right\} \hspace{1cm} (2.2)$$

with respect to the usual matrix product in the general linear group $\text{GL}_3(\mathbb{F})$. Note that the center of $\mathbb{H}(\mathbb{F})$ is the nontrivial proper normal subgroup

$$Z(\mathbb{H}(\mathbb{F})) = \{ M(0, 0; t) \mid t \in \mathbb{F} \} \hspace{1cm} (2.3)$$

so the group is nonabelian. It is easy to see that

$$[[\mathbb{H}(\mathbb{F}), \mathbb{H}(\mathbb{F})], \mathbb{H}(\mathbb{F})] = 1, \quad [\mathbb{H}(\mathbb{F}), \mathbb{H}(\mathbb{F})] = Z(\mathbb{H}(\mathbb{F})) \simeq \mathbb{F}, \quad \mathbb{H}(\mathbb{F})/Z(\mathbb{H}(\mathbb{F})) \simeq \mathbb{F}^2. \hspace{1cm} (2.4)$$

The condition $[[\mathbb{H}(\mathbb{F}), \mathbb{H}(\mathbb{F})], \mathbb{H}(\mathbb{F})] = 1$ means that $\mathbb{H}(\mathbb{F})$ is nilpotent of class two, following a classical terminology in group theory (see [23]). The other two conditions in (2.4) show that the derived subgroup coincides with the center and has the size of the ground field $\mathbb{F}$, but the whole group factorized through the center has a size which is double $\mathbb{F}^2 = \mathbb{F} \oplus \mathbb{F}$ than the ground field. We can deduce these structural properties of $\mathbb{H}(\mathbb{F})$ by the smallest numbers of generators and relations, that is, via group presentations for $\mathbb{H}(\mathbb{F})$ (see [16, 17, 23]). For instance, if $\mathbb{F} = \mathbb{Z}(p)$ is the finite field of order $p$ (with odd $p$), then

$$\mathbb{H}(\mathbb{Z}(p)) = \langle M(0, 1; 0), M(1, 0; 0), M(0, 0; 1) \mid [M(0, 1; 0), M(1, 0; 0)] = M(0, 0; 1), \hspace{1cm} (2.5) \rangle$$

$$M(0, 0; 1)^p = M(1, 0; 0)^p = M(0, 1; 0)^p = 1.$$  \hspace{1cm} (2.5)

This group has order $p^3$. Of course, there is a meaning for each relation: the first one is a nontrivial commuting relation and it is easy to check that all the other commuting relations between two generators of $\mathbb{H}(\mathbb{Z}(p))$ are trivial; the other three relations in (2.5) remind us that we are in a finite group. Of course, one can write $\mathbb{H}(\mathbb{F})$ when $\mathbb{F}$ is general field. In this case, we must change the last three equation in (2.5) up to the finiteness of $\mathbb{F}$. More details can be found in [16].

Having a presentation for $\mathbb{H}(\mathbb{F})$ and knowing the meaning of generators and relations helps to understand the behaviour of a dynamical systems whose group of symmetries is described by $\mathbb{H}(\mathbb{F})$. However it is possible to get similar information via the notion of semidirect product.
If we look at $\mathbb{H}(F)$, one can show that for any choice of $F$ of characteristic $\neq 2$, there are only two abelian maximal subgroups

$$A = \langle M(0, 0; 0) \rangle \oplus \langle M(1, 0; 0) \rangle \cong \mathbb{F}^2 \quad \text{and} \quad B = \langle M(0, 0; 1) \rangle \oplus \langle M(0, 1; 0) \rangle \cong \mathbb{F}^2$$

such that $A \cap B = Z(\mathbb{H}(F))$, $A \cap \langle M(0, 1; 0) \rangle = 1$, $B \cap \langle M(1, 0; 0) \rangle = 1$, and

$$\mathbb{H}(F) = A \rtimes \langle M(0, 1; 0) \rangle = B \rtimes \langle M(1, 0; 0) \rangle \cong \mathbb{F}^2 \rtimes \mathbb{F}. \quad (2.6)$$

More details can be found in [16, 17]. In particular, the following diagram is well known and describes the placement of the aforementioned subgroups in the lattice of subgroups of $\mathbb{H}(F)$.

![Hasse Diagram](image)

**Figure 1**: Portion of an Hasse diagram of $\mathbb{H}(F)$.

Note that all the subgroups involved in Fig. 1 are abelian, except $\mathbb{H}(F)$. At the first level we find the trivial subgroup. At the second level there are three subgroups isomorphic to the additive group of the ground field $F$. At the third level there are just two subgroups isomorphic to the additive group $\mathbb{F}^2$. At the fourth level we find the whole group. Finally, note that the Hasse diagram in Fig. 1 presents only the subgroups that can be directly deduced from (2.6) and not all the subgroups of $\mathbb{H}(F)$.

### 2.2. The Pauli group.

A character $\chi$ of a finite abelian group $G$ is a homomorphism from $G$ into the torus group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, that is, the additive group of the reals modulo the integers. Character theory of abelian groups (finite and infinite) is well known and the reader can refer to [17]. The set $\text{Hom}(G, \mathbb{T})$ of all characters of $G$ is again a group with respect to the composition of functions and is usually denoted with $\widehat{G}$, even known as dual group of $G$ (or also Pontryagin dual of $G$). Finite abelian groups have the pleasant property of being reflexive, that is, isomorphic to their respective dual groups. However, in general there is no canonical way of identifying $G$ with $\widehat{G}$.

Now consider the finite dimensional Hilbert space $\mathbb{C}^q$, any finite field $F$ of prime power order $q = p^m$ of characteristic $p \neq 2$, and $\chi \in \widehat{F}$. Label the standard basis of $\mathbb{C}^q$ by $\{ |\phi_1\rangle, \ldots, |\phi_q\rangle \}$ and let $p, q \in F$. Define the *shift* and *clock* operators as

$$X(q) : |\phi_k\rangle \mapsto |\phi_{k+q}\rangle, \quad Z(p) : |\phi_k\rangle \mapsto \chi(pk)|\phi_k\rangle \quad (2.7)$$

for all $k \in \{1, \ldots, q\}$.

The *Weyl representation* is the expression

$$w(p, q, t) := \chi \left( t - \frac{1}{2}pq \right) Z(p) X(q), \quad (2.8)$$
where the left side \( w(p, q, t) \) is called \textit{Weyl operator}. Note that the Heisenberg group \( \mathbb{H}(F) \) with \( F \) of characteristic \( \neq 2 \), that is, the group \( (2.2) \) possesses a Weyl representation as in \( (2.8) \). More generally, \( (2.1) \) may be extended replacing the role of \( F^2 \) with \( F^{2n} \) and \( n \geq 1 \), that is, we may consider

\[
\mathbb{H}(F^n) = \langle p_1, q_1, p_2, q_2, \ldots, p_n, q_n, t \mid [p_i, q_i] = t, p_i^q = q_i^q = 1, \forall i = 1, 2, \ldots, n \rangle, \tag{2.9}
\]

where the missing commuting relations \([p_i, q_j] = 1\) for all \( i, j \in \{1, 2, \ldots, n\} \) with \( i \neq j \). Of course, \( (2.9) \) gives \( (2.5) \) when \( n = 1 \). See details in [16, 15]. Now one can find that \( (2.9) \) possesses the Weyl representation

\[
w(p, q, t) = \chi(t)w(p_1, q_1) \otimes w(p_2, q_2) \otimes \cdots \otimes w(p_n, q_n), \tag{2.10}
\]

where \( \{p_i\}, \{q_i\} \) are the components of \( p \) and \( q \) with respect to the natural basis in \( F^n \).

The situation changes drastically when \( F \) has characteristic \( 2 \). In this case \( (2.1) \) cannot be defined, because the symbol \( 1/2 \) in \( (2.8) \) cannot be defined (in a field of characteristic \( 2 \) we have that \( 2 = 0 \mod 2 \) which implies that \( 1/2 = 1/0 \)). However \( (2.7) \) is perfectly valid and \( (2.8) \) may be replaced in the term involving \( \chi(t - \frac{1}{2}pq) \) by

\[
w(p, q) := i^{-pq} Z(p) X(q). \tag{2.11}
\]

The group generated by \( \{w(p, q)\}_{p,q} \) is the \textit{Pauli group} on a qubit.

We may rewrite \( (2.10) \) considering a prime power \( q \) and the \((q \times q)\) Pauli matrices

\[
X = \begin{pmatrix}
0 & 1 & \cdots & \cdots & \cdots \\
1 & 0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
0 & \cdots & \cdots & \ddots & \cdots \\
1 & \cdots & \cdots & \cdots & 0
\end{pmatrix}, \quad Z = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots \\
0 & \omega & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots & \cdots \\
0 & \cdots & \cdots & \ddots & \cdots \\
0 & \cdots & \cdots & \cdots & \omega^{-1}
\end{pmatrix},
\]

where \( \omega = \exp(2\pi i/q) \). The Pauli group for \( n \) qudits of odd dimension \( q = p^n \) with \( p \) odd prime is then

\[
\mathcal{P}_{n,q} = \{\omega^c \bigotimes_{k=1}^n X^{\alpha_k} Z^{\beta_k} \mid c \in F_p, \alpha_k, \beta_k \in F_q\} \tag{2.12}
\]

but for \( q = 2 \) the above expression changes, becoming

\[
\mathcal{P}_{n,2} = \{i^d \bigotimes_{k=1}^n X^{\alpha_k} Z^{\beta_k} \mid d \in F_4, \alpha_k, \beta_k \in F_2\}, \tag{2.13}
\]

which is the well known Pauli group of order \( 4^{n+1} \) on \( n \) \textit{qubits}.

The structure of \( \mathcal{P}_{n,2} \) is significantly different from \( \mathbb{H}(F^n) \). In the next sections we introduce a purely algebraic characterisation of Pauli groups. Geometric aspects were studied in [9, 24, 27, 28, 29] using tools from projective geometries.

3. THE FORMALISM OF PAULI GROUPS IN COMPUTATIONAL GROUP THEORY

We present some arguments of finite group theory, adapted to the present context, in order to describe \( \mathcal{P}_{n,2} \). First of all we recall that the quotient group \( G/[G, G] \) of an arbitrary group \( G \) is always abelian, but not necessarily of the form

\[
G/[G, G] \simeq \bigoplus_{r \text{-times}} \mathbb{Z}(p),
\]
that is, $p$-elementary abelian of rank $r$. The presence of an elementary abelian quotient occurs for $P_{1,2}$ in the sense of the following result.

**Lemma 3.1.** The Pauli group $P_{1,2}$ can be presented both by

$$\langle X, Y, Z \mid X^2 = Y^2 = Z^2 = 1, (YZ)^4 = (ZX)^4 = (XY)^4 = 1 \rangle$$

and by

$$P_{1,2} = \langle u, a, b \mid u^4 = a^2 = 1, u^2 = b^2, a^{-1}ua = u^{-1}, ub = bu, ab = ba \rangle,$$

where

$$u = XY, \ a = Y, \ and \ b = XYZ.$$

Moreover, $P_{1,2}$ has no elements of order 8, $Z(P_{1,2}) \cong Z(4)$, $[P_{1,2}, P_{1,2}] \subseteq Z(P_{1,2})$, $[P_{1,2}, P_{1,2}] = \Phi(P_{1,2})$ and $P_{1,2}/Z(P_{1,2})$ is 2-elementary abelian of rank 2.

**Proof.** Let us consider the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ and \ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.1)$$

and check that $X^2 = Y^2 = Z^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so the first relation is satisfied. Now one can check that also the other relations $(YZ)^4 = (ZX)^4 = (XY)^4 = 1$ are satisfied, via the usual product of matrices. In addition $X^2 = 1$ implies $X = X^{-1}$ and similarly $Y = Y^{-1}$ and $Z = Z^{-1}$ so one can find the following calculus rules

$$(XYZ)^4 = 1, \ [XYZ, X] = 1, \ [XYZ, Y] = 1, \ [XYZ, Z] = 1,$$

which show that $Z(P_{1,2}) = \langle abc \rangle \cong Z(4)$. The absence of elements of order 8 can be checked either directly or looking at the relations in $P_{1,2}$, this means that the maximum order of an element in $P_{1,2}$ cannot exceed 4, or, equivalently, that the exponent of $P_{1,2}$ is 4. In order to do this, one can introduce

$$u = XY, \ a = Y, \ and \ b = XYZ,$$

and show that the original presentation is equivalent to the following

$$P_{1,2} = \langle u, a, b \mid u^4 = a^2 = 1, u^2 = b^2, a^{-1}ua = u^{-1}, ub = bu, ab = ba \rangle. \quad (3.2)$$

Here,

$$D_8 = \langle u, a \rangle = \langle u, a \mid u^4 = a^2 = 1, a^{-1}ua = u^{-1} \rangle \ and \ Z(4) = \langle b \mid b^4 = 1 \rangle \quad (3.3)$$

describe the dihedral group of order eight and the cyclic of order four, respectively. By motivations of order we get $|P_{1,2}/Z(P_{1,2})| = 4$, concluding that $[P_{1,2}, P_{1,2}] \subseteq Z(P_{1,2})$ and that no elements of order 8 are contained in $P_{1,2}$.

Further computations show that $P_{1,2}/Z(P_{1,2}) \cong Z(2) \times Z(2)$. In addition, one can check directly that $\Phi(P_{1,2}) = [P_{1,2}, P_{1,2}]$. $\square$

The previous results offers a fast and efficient way to detect abelian subgroups in $P_{1,2}$. Due to the decomposition that we have described, we can look at a portion of its Hasse diagram recognising $D_8 = \langle a, u \rangle$ and $Z(4) = \langle b \rangle$ and drawing the Hasse diagram of $D_8$ along with the information we have in the proof of Lemma 3.1.
Figure 2: Portion of an Hasse Diagram of $\mathcal{P}_{1,2}$.

At the first level of Fig. 2, there is the trivial subgroup; at the second level some subgroups of order two, so abelian. Then some subgroups of order 4 again abelian at the third level. The level just below the whole group has two nonabelian subgroups.

We now investigate what happens to (2.12) in case $p$ is odd. A first important example can be illustrated, in order to understand the structure of Pauli groups on qudits.

Remark 3.2. Consider $p = 3$ in (2.12). Then $\mathcal{P}_{1,3}$ is a nonabelian 3-group of order 27. More explicitly, the elements of $\mathcal{P}_{1,3}$ can be visualized via the following $(3 \times 3)$ nonsingular matrices, endowed of the usual operation of row by column, and obtained by the set multiplication

$$\{1, \omega, \omega^2\} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{4\pi i / 3} & 0 \\ 0 & 0 & e^{2\pi i} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & e^{4\pi i / 3} & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{\pi i / 3} & 0 \\ 0 & 0 & e^{4\pi i} \end{bmatrix}, \begin{bmatrix} 0 & 0 & e^{2\pi i} \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & e^{4\pi i} \\ 0 & e^{2\pi i} & 0 \end{bmatrix}.$$

Now a direct computation (or looking at [26]) shows that there is only one nonabelian group of order 27 possessing nontrivial elements of order 9 and this is indeed our case, that is, it possible to select two matrices $x$ an $y$ from the previous 27 matrices such that $\mathcal{P}_{1,3} = \langle x, y | x^9 = y^3 = 1, y^{-1}xy = x^4 \rangle$.

The proof of Lemma 3.1 shows that $\mathcal{P}_{1,2}$ can be decomposed into the product of a subgroup, which is isomorphic to $D_8$, and of another which is cyclic of order four. Product decompositions have long been studied in group theory since they allow us to control the structural properties of groups. Some well known decompositions appear for various families of groups. For instance, abelian finitely generated groups may be decomposed in direct products of cyclic groups [23, 4.2.10]. Remak’s decomposition [23, 3.3.12] describes further conditions of splitting in direct products and the Schur–Zassenhaus Theorem introduces a decomposition more general than those obtained with direct products [23, 9.1.2].

Here we give a decomposition based on the notion of weak central product. The central product is an important concept in group theory since the structure of most nilpotent groups
can be described via central products (recall that the Heisenberg group is nilpotent). These kind of products have only recently been used to describe properties of physical systems [3]. The weak central product is defined as

**Definition 3.3** (Weak central product). A group $G$ is the weak central product of its normal subgroups $H$ and $K$, if simultaneously

(i) $G = HK$;

(ii) $[H, K] \subseteq Z(G)$.

When Definition 3.3 is satisfied with $[H, K] = C \subseteq Z(G)$, we write

$$G = H \bullet_C K$$

in order to specify the portion of the center which allows us to make the central product. Note that $[H, K] \subseteq H \cap K$ above, because $H$ and $K$ are both normal in $G$, but we do not know in general whether $[H, K] = 1$ or not. When $[H, K] = H \cap K = Z(G)$, we follow [23, Pages 145–146] calling $G$ central product of $H$ and $K$. In this case, we use the notation $G = H \Join K$, where we dropped the subscript for $C$.

**Remark 3.4.** The Heisenberg group $\mathbb{H}(\mathbb{Z}(p))$ can be decomposed in semidirect product as in (2.6), but also as central product of $A = \langle M(1, 0; 0), M(0, 0; 1) \rangle$ and $B = \langle M(0, 1; 0), M(0, 0; 1) \rangle$, noting that (i) and (ii) are satisfied in Definition 3.3. In fact

$$[A, B] = A \cap B = \langle M(0, 0; 1) \rangle = Z(\mathbb{H}(\mathbb{Z}(p)))$$

in this specific case.

We observe that it is possible to have (ii) of Definition 3.3 as a strict inclusion.

**Remark 3.5.** From Remark 3.4 note that $A \simeq B \simeq \mathbb{Z}(p) \oplus \mathbb{Z}(p)$ and so

$$\mathbb{H}(\mathbb{Z}(p)) = A \Join B \simeq (\mathbb{Z}(p) \oplus \mathbb{Z}(p)) \Join (\mathbb{Z}(p) \oplus \mathbb{Z}(p))$$

is an alternative description for the Heisenberg group $\mathbb{H}(\mathbb{Z}(p))$.

We recall the following definition from (23, Page 140):

**Definition 3.6** (Extraspecial $p$-group). A finite group $G$ of $|G| = p^n$ ($p$ prime and $n \geq 1$) is extraspecial if $Z(G) = [G, G]$ and $Z(G)$ has order $p$.

The structure of these groups can be described in terms of central products (Definition 3.3) and some relevant classes of $p$-groups. In order to accomplish this task we recall some further results of finite group theory.

**Lemma 3.7** (See [23]). If $p$ is odd, any nonabelian $p$-group of order $p^3$ must be isomorphic either to

$$E_1 = \langle x, y \mid x^p = y^p = 1, x^{-1}[x, y]x = y^{-1}[x, y]y = [x, y], \rangle,$$

which is called nonabelian $p$-groups of order $p^3$ and exponent $p$, or to

$$E_2 = \langle x, y \mid x^{p^2} = y^p = 1, y^{-1}xy = x^{1+p}, \rangle,$$

which is called nonabelian $p$-groups of order $p^3$ and exponent $p^2$.

In fact one can see that $E_1$ has no elements of order $p^2$, while $E_2$ has it. Note that $\mathcal{P}_{1,3}$ in Remark 3.2 is exactly $E_2$ when $p = 3$. In case $p = 2$, the situation is completely different.
Lemma 3.8 (See [23]). Any nonabelian 2-group of order 8 either is isomorphic to the dihedral group
\[ D_8 = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1} \rangle \]
or to the quaternion group
\[ Q_8 = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \rangle. \]

Note that any nonabelian p-group of order p^3 is extraspecial, both if p is even and if p is odd. For sure, the importance of these extraspecial groups appears the moment we observe the following fact

Remark 3.9. We have \( \mathbb{H}(\mathbb{Z}(p)) \simeq E_2 \) for p odd and \( \mathbb{H}(\mathbb{Z}(2)) \simeq D_8 \) for p = 2 and in both cases we have extraspecial groups. In a similar vein one can see that
\[ \mathbb{H}(\mathbb{Z}(p^n)) \simeq (\mathbb{Z}(p^n) \oplus \mathbb{Z}(p^n)) \rtimes \mathbb{Z}(p^n) \]
is always extraspecial (eventually with p = 2) of order p^{3n} and center of order p, but it is more interesting to describe
\[ \mathbb{H}(\mathbb{Z}(p^n)) \simeq (\mathbb{Z}(p^0) \oplus \cdots \oplus \mathbb{Z}(p^n)) \rtimes \mathbb{Z}(p) = \mathbb{Z}(p^n) \rtimes \mathbb{Z}(p), \]
which is again an extraspecial p-group of order p^{2n+1} and center always of order p. Note that the group in Remark 3.2 is isomorphic to \( \mathbb{H}(\mathbb{Z}(3)) \), and, more generally the Pauli group on one qudits \( \mathcal{P}_{1,p} \) for odd p is a nonabelian p-group of order p^3 and exponent p, that is, \( \mathcal{P}_{1,p} \simeq E_2 \simeq \mathbb{H}(\mathbb{Z}(p)) \).

Now we report the main classification of extraspecial p-groups.

Lemma 3.10 (See [23], Exercises 6 and 7). A nonabelian extraspecial group G of \( |G| = p^{2n+1} \)
has Z(G) of order p and p-elementary abelian quotient G/Z(G). Moreover, if p = 2, then G
is the central product of D_8’s or a central product of D_8’s and a single Q_8. If p > 2, then
either G has exponent p, or else it is a central product E_1’s and a single E_2.

The Pauli group \( \mathcal{P}_{1,2} \) cannot be described by Lemma 3.10 in fact \( |\mathcal{P}_{1,2}| = 16 \) and both
\( \mathcal{P}_{1,2} \neq D_8 \oplus \mathbb{Z}(2) D_8 \) and \( \mathcal{P}_{1,2} \neq D_8 \oplus \mathbb{Z}(2) Q_8 \). This means that \( \mathcal{P}_{1,2} \) is not an extraspecial 2-group. An alternative argument can be used if we note that \( |Z(\mathcal{P}_{1,2})| > 2 \) by Lemma 3.1.

Remark 3.11. If p > 2, then Lemma 3.10 says that G may be decomposed in the central
product of finitely many factors isomorphic either to E_1 or E_2. If p = 2, the same is true
but now the factors must be isomorphic either to D_8 or to Q_8. Because of these restrictive
conditions, it is usual to talk about the extraspecial p-group \( E_{p^{2n+1}} \) of order p^{2n+1}, up to
specify if p is even or odd.

In order to get a decomposition results for Pauli groups we need to introduce a more general
notion (see [23], Exercise 8).

Definition 3.12 (Generalised extraspecial p-group). A finite group G of order \( |G| = p^n \) (p
prime and n \geq 1) is generalized extraspecial (or generalized Heisenberg), if \( [G, G] \) is of order
p and Z(G) is cyclic.

One can find the following description for these groups.

Lemma 3.13. A generalized extraspecial p-group G satisfies the conditions:

(i) \( [G, G] \subseteq Z(G) \) and G/Z(G) is elementary abelian of even rank;
(ii) $G$ can be decomposed in weak central product;
(iii) $G = H / L$, where $H = E \times C$ with $E$ extraspecial, $C$ cyclic and $L$ is a subgroup of $H$ of exponent $p$;
(iv) $G$ is nonabelian but all of its proper quotients are abelian (i.e.: $G$ is just nonabelian).

Proof. (i) and (ii) follow from the definitions and are mentioned in [23, Exercises 8 and 9]. (iii) and (iv) can be deduced by [20, Theorem 11.2].

Note that Lemma 3.13 does not guarantee that a nonabelian generalized extraspecial group has center of prime order. In fact, the moment this is true, we get the case of extraspecial groups since the condition $[G, G] = Z(G)$ is automatically satisfied. Therefore Definition 3.12 is more general than Definition 3.3.

Remark 3.14. The Pauli group $P_{1,2}$ is the central product of $D_8$ by $Z(4)$ as noted in Lemma 3.1, that is, $P_{1,2} = D_8 \cdot Z(2) \cdot Z(4)$. In particular, $Z(P_{1,2}) \simeq Z(4)$ and so $P_{1,2}$ is not extraspecial by Lemma 3.10. On the other hand, $P_{1,2}$ satisfies the conditions in Definition 3.12 so it is generalized extraspecial. Note that here $D_8 \cap Z(4) = Z(2)$ is properly contained in $Z(P_{1,2})$ and Definition 3.3 is indeed satisfied.

4. Main results

The first main result is a direct consequence of what we noted in Lemmas 3.1 and 3.13. Recall that just nonabelian groups are groups that are nonabelian but whose all proper quotients are abelian (they have been extensively studied in [20, 25]). A dual case is when we have a group which is nonabelian but all of its proper subgroups are abelian. These are called minimal nonabelian groups and were classified by Redei and Schmidt (see [23]). Both definitions are important in the context of Pauli groups.

Theorem 4.1. The Pauli group $P_{1,2}$ is just nonabelian but is not minimal nonabelian. The Pauli group $P_{1,p}$ is both just nonabelian and minimal nonabelian when $p \neq 2$.

Proof. Consider $P_{1,2}$. The result follows from Lemmas 3.1 and 3.13. Now consider $P_{1,p}$ and note that $P_{1,p} \simeq \mathbb{F}(Z(p))$. Now Remarks 3.9 and 3.14 (or even the classifications known for these classes of groups in [23] and kos) show that the result is true.

Because of Remark 3.14 and the presentation in Lemma 3.1 one can see that there are just 6 nonabelian subgroups in $P_{1,2}$, three of these are isomorphic to $D_8$ and the remaining three are isomorphic to $Q_8$. All the other maximal subgroups of $P_{1,2}$ are abelian. This means that we have no hope to find that all subgroups of $P_{1,2}$ are abelian, but the situation is much better when we look at quotients because of Theorem 4.1. Thanks to the preliminary Lemmas 3.1, 3.10 and 3.13 we may prove another theorem of decomposition and this time for Pauli groups $P_{n,2}$ with $n \geq 1$ arbitrary.

Theorem 4.2. For all $n \geq 1$ there are normal subgroups $H_1, H_2, \ldots, H_n$ in $P_{n,2}$ such that

$$P_{n,2} = \cdots ((H_1 \ast L_1 H_2) \ast L_2 H_3) \ast L_3 H_4 \ldots$$

with $L_j$ abelian normal subgroup of order $2^{j+1}$ and $H_j \simeq P_{1,2}$ for all $j = 1, 2, \ldots, n$.

Proof. We proceed by a recursive argument, so we offer a constructive method at the same time. The case $n = 1$ is trivial (compare with Lemma 3.1 and Remark 3.14). Consider
$n = 2$ and the Pauli group $\mathcal{P}_{2,2}$ with 64 elements. Using the matrix representation of $\mathcal{P}_{1,2}$ and looking at Lemma 3.1 we get

$$Z(\mathcal{P}_{1,2}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \right\} \cong \mathbb{Z}(4).$$

Having in mind the proof of Lemma 3.1 and denoting by $X, Y, Z$ the Pauli matrices in (3.1), we have seen that $Z(\mathcal{P}_{1,2}) = \langle XYZ \rangle$. The generators of $\mathcal{P}_{2,2}$ are $\{X \otimes Y, X \otimes Z, X \otimes X, Z \otimes Z, Z \otimes X \}$ and have the following matrix representations

$$A = X \otimes X = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B = X \otimes Y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$C = Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = X \otimes Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$E = Z \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Making computations and using Lemma 3.1 we get

$$H_1 = \langle A, B, C \mid A^2 = B^2 = C^2 = 1, (BC)^4 = (CA)^4 = (AB)^4 = 1 \rangle \cong \mathcal{P}_{1,2};$$

$$H_1 = \langle u, x, y \mid u^4 = x^2 = 1, u^2 = y^2, x^{-1}ux = u^{-1}, uy = yu, xy = yx \rangle \cong \mathcal{P}_{1,2};$$

where $u = AB$, $x = B$, $y = ABC$;

$$H_2 = \langle A, D, E \mid A^2 = D^2 = E^2 = 1, (DE)^4 = (EA)^4 = (AD)^4 = 1 \rangle \cong \mathcal{P}_{1,2};$$

$$H_2 = \langle u, z, t \mid u^4 = z^2 = 1, u^2 = t^2, z^{-1}uz = u^{-1}, ut = tu, zt = tz \rangle \cong \mathcal{P}_{1,2};$$

where $u = AD$, $z = D$, $t = ADE$. Note that $Z(H_1) \cong Z(H_2) \cong \mathbb{Z}(4)$, in fact

$$Z(H_1) = \langle ABC \rangle = \left\{ \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \right\}, \quad Z(H_2) = \langle ADE \rangle = \left\{ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\},$$

but we are not yet ready to conclude that

$$Z(\mathcal{P}_{2,2}) \cong \mathbb{Z}(4) \oplus \mathbb{Z}(2) \quad \text{and} \quad [\mathcal{P}_{2,2}, \mathcal{P}_{2,2}] \cong \mathbb{Z}(4).$$

First of all, we define the set of generators $\text{gen}(\mathcal{P}_{1,2}) = \{u, x, y\} = \{U, X_1, Z_1\}$ with $u = U$, $X_1 = x$ and $Z_1 = y$, then

$$\text{gen}(\mathcal{P}_{n,2}) = \{U, X_1, X_2, \ldots, X_n, Z_1, Z_2, \ldots, Z_n\} \quad (4.1)$$

$$\text{gen}(\mathcal{P}_{n,2}) = \{U, X_1, Z_1\} \cup \{U, X_2, Z_2\} \cup \ldots \cup \{U, X_n, Z_n\} \quad (4.2)$$

and $\langle U, X_1, Z_1 \rangle \cong \langle U, X_2, Z_2 \rangle \cong \ldots \cong \langle U, X_n, Z_n \rangle \cong \mathcal{P}_{1,2}$, or equivalently $\langle U, X_1, Z_1 \rangle \cong H_1$, $\langle U, X_2, Z_2 \rangle \cong H_2$ and so on with all $H_j \cong \mathcal{P}_{1,2}$. Then we denote by $R(D_8)$ the relations in a presentation like (3.3) for $D_8$, with $R(\mathbb{Z}(4))$ the relation on the generator $Z_j$ such that its
This means that \( R(U, X_1, Z_1) \) may be written as
\[
R(U, X_1, Z_1) = R(D_8) \cup R(\mathbb{Z}(4)) \cup \{Z_1^2 = U^2\} \cup \{[X_1, Z_1] = 1\} \cup \{[U, X_1] = 1\},
\]
where \( Z_1^2 = U^2 \) shows that we are identifying the center of the factor \( \simeq D_8 \) with a cyclic subgroup of order two in the other factor \( \simeq \mathbb{Z}(4) \) when we form the weak central product (see Example 3.14). The remaining relations show that \( X_1 \) commutes both with \( Z_1 \) and \( U \). More generally, we have the same behaviour for all \( j = 1, 2, \ldots, n \)
\[
R(U, X_j, Z_j) = R(D_8) \cup R(\mathbb{Z}(4)) \cup \{Z_j^2 = U^2\} \cup \{[X_j, Z_j] = 1\} \cup \{[U, X_j] = 1\}.
\]
Therefore
\[
P_{n,2} = \langle \text{gen } (P_{n,2}) \mid R(U, X_j, Z_j) \rangle \quad \text{for all } j = 1, 2, \ldots, n \quad (4.3)
\]
Note that all the other relations can be derived from (4.3) in a general presentation for a Pauli group of type \( P_{n,2} \). In particular, we have for \( n = 2 \) that \([X_1, X_2] = [Z_1, Z_2] = U^2 \) and so \( Z(\langle U, X_1, Z_1, \rangle) = \langle Z_1 \rangle = Z(H_1), \langle (U, X_2, Z_2, \rangle) = \langle Z_2 \rangle = Z(H_2) \simeq \mathbb{Z}(4), \langle U \rangle = [H_1, H_2] \)

suitable for decomposition of \( P_{2,2} \) in terms of Definition 3.3 with two factors \( H_1 \simeq P_{1,2} \) and \( H_2 \simeq P_{1,2} \) identified via their common subgroup \( L_1 \). The thesis follows for \( n = 2 \).

Because of the constructive method, we can go ahead with \( n = 3 \) and find an \( H_3 = \langle U, X_3, Z_3, R(U, X_3, Z_3) \rangle \simeq P_{1,2} \) such that \( P_{3,2} = (H_1 \bullet L_1 H_2) \bullet L_2 H_3 \) with \( L_2 = [(H_1 \bullet L_1 H_2), H_3] \simeq [P_{2,2}, P_{1,2}] \) which turns out to be an abelian group of order 8 contained in \( Z(P_{3,2}) \).

We may modify the argument of Theorem 4.2 for \( P_{n,p} \) when \( p \) is an odd prime.

**Corollary 4.3.** Given an odd prime \( p \) and \( m \geq 1 \), the Pauli group \( P_{n,p} \) is isomorphic to a Heisenberg group \( \mathbb{H}(\mathbb{Z}(p)^m) \) of order \( p^{2m+1} \).

**Proof.** We can use an argument as in Theorem 4.2. First we consider \( P_{n,p} \) and we observe that \( P_{n,p} \simeq \mathbb{H}(\mathbb{Z}(p)) \). Then we consider \( P_{n,p^2} \) and find that
\[
P_{n,p^2} = P_{n,p} \cdot P_{n,p} \simeq \mathbb{H}(\mathbb{Z}(p)) \cdot \mathbb{H}(\mathbb{Z}(p)) \simeq \mathbb{H}(\mathbb{Z}(p)^2)
\]
This is a \( p \)-group with center of order \( p \), exponent \( > p \) and order \( p^5 \). Iterating this observation and noting that
\[
\mathbb{H}(\mathbb{Z}(p)^2) \cdot \mathbb{H}(\mathbb{Z}(p)) \simeq \mathbb{H}(\mathbb{Z}(p)^3)
\]
the result follows.

One of the consequences of the above results is expressed below.
Corollary 4.4. The Pauli groups $P_{n,2}$ are just nonabelian if and only if $n = 1$. On the other hand, $P_{n,p^m}$ with $p$ odd is just nonabelian for all $m, n \geq 1$.

Proof. The sufficient condition is clear from Theorem 4.1. The necessary condition can be extrapolated from the argument in the proof of Theorem 4.2. In fact any time $n \geq 2$, $P_{n,2}$ possesses always a nontrivial normal subgroup $N$ which is isomorphic to $P_{1,2}$ and $G/N \cong G_{n-1}$ is manifestly nonabelian. In case of odd primes, it is sufficient to apply Corollary 4.3 and Lemma 3.13.

5. Applications of the main results

In this section we give two applications of our main results. First we show that our decomposition can be applied to the 'lifted' Pauli groups of Gottesman and Kuperberg [13, 14]. Second, we show how to detect families of abelian subgroups directly from the decomposition in Theorem 4.2 and Corollary 4.3.

5.1. Decomposition result for 'lifted' Pauli groups. Stabiliser codes for quantum error correction were introduced in [11] and constitute an active field of research in quantum information theory. A stabiliser is an abelian subgroup of the Pauli group and the error correcting properties of stabiliser codes are firmly rooted in the group structure of Pauli groups. The reader may refer to [10] for a formal introduction to quantum error correction and stabiliser codes.

Quantum error-correcting codes on the qudits enable a greater variety of codes [6, 12, 19, 22] and it is interesting to consider stabiliser codes for qudits. It is possible to extend the notion of stabiliser codes to qudits of any dimension [19] but part of the structure of the qubit stabiliser is lost. An interesting case where it is possible to extend the theory of qubit stabilisers without loss—by exploiting the fact that there is a unique finite field $\mathbb{F}_q$ for every prime power $q = p^m$—is for $q$-dimensional qudits where $q$ is a prime power [2, 18]. Gottesman and Kuperberg recently discussed how the standard way of extending stabiliser codes for prime power qudits does not exploit the full field structure and propose a new definition of stabiliser codes, based on the notion of 'lifted' Pauli groups, over $q$-dimensional registers [13, 14].

We present the definition of 'lifted' Pauli groups given in [13, 14].

Definition 5.1. Let $q = p^m$ where $p$ is an odd prime, and let $n \geq 1$. The lifted Pauli group is the group of unitriangular matrices

$$\hat{P}_{n,q} = \left\{ \begin{pmatrix} \begin{array}{ccc} 1 & \tilde{\alpha} & \eta \\ 0 & I & \tilde{\beta}^T \\ 0 & 0 & 1 \end{array} \end{pmatrix} \right\}$$

where $\tilde{\alpha}, \tilde{\beta} \in \mathbb{F}_q^n$, $[,]^T$ indicates the transpose of a vector, and $\eta \in \mathbb{F}_q$.

Elements of the lifted Pauli group are also denoted by

$$P = \begin{pmatrix} 1 & \tilde{\alpha} & \eta \\ 0 & I & \tilde{\beta}^T \\ 0 & 0 & 1 \end{pmatrix} = \omega^n X^{\tilde{\alpha}} Z^{\tilde{\beta}}$$

for an element $P \in \hat{P}_{n,q}$. Note that in the last expression $X$ and $Z$ are just symbols and not Pauli operators; similarly, $\omega$ is just a symbol and not a root of unity. Using a multiplication
rule similar to the one used for the standard Pauli group that is is possible to show that there exist a surjective homomorphism

$$\Pi : \omega^a X^{\vec{a}} Z^{\vec{b}} \in \hat{\mathcal{P}}_{n,q} \mapsto \Pi \left( \omega^a X^{\vec{a}} Z^{\vec{b}} \right) = \omega^a X^{\vec{a}} Z^{\vec{b}} \in \mathcal{P}_{n,q}. \quad (5.1)$$

Therefore we have the following relevant result.

**Corollary 5.2.** Assume $\hat{\mathcal{P}}_{n,q}$ is a lifted Pauli group of $\mathcal{P}_{n,q}$, where $q = p^m$ and $p$ odd prime. Then $\hat{\mathcal{P}}_{n,q}$ is isomorphic to $\mathbb{H}(Z(p)^m)$ up to a quotient.

**Proof.** This follows from an application of Corollary 4.3 and from the First Isomorphism Theorem for groups, applied to the epimorphism (5.1). □

In analogy to (5.1), one can define an epimorphism as in [13] when we deal with $p = 2$,

$$\varepsilon : \hat{\mathcal{P}}_{n,2} \mapsto \mathcal{P}_{n,2},$$

and we have by Theorem 4.2 that:

**Corollary 5.3.** The lifted Pauli group $\hat{\mathcal{P}}_{n,2}$ is isomorphic the weak central product of $\mathcal{P}_{1,2}$ up to a quotient.

In particular, we can see that recognising abelian subgroups in lifted Pauli groups is reduced to recognise those in the usual Pauli groups up to a quotient. In the successive subsection we will see how to do this via Theorems 4.1 and 4.2.

5.2. **Identification of abelian subgroups.** In this section we show how our main results can be used to detect families of abelian subgroups. This may have applications in the theory of mutually unbiased basis [8].

**Corollary 5.4.** The Pauli group $\mathcal{P}_{n,p^m}$ is minimal nonabelian for all $p \neq 2$ and $m \geq 1$. Moreover, it contains always two distinct maximal abelian normal subgroups $A, B$ and a nonnormal abelian subgroup $H$ such that

$$\mathcal{P}_{n,p^m} = A \rtimes H = B \rtimes H,$$

where $[A, B] = A \cap B = Z(\mathcal{P}_{n,p^m})$ and $A \simeq B$.

**Proof.** Apply Corollary 4.3. Now the considerations in the previous Paragraph 2.2 along with Remark 3.9 conclude the proof. □

Note that $\mathcal{P}_{n,p^m}$ are also minimal nonabelian when $p \neq 2$, since so are Heisenberg groups. In fact they satisfy the classification results in [23] being extraspecial $p$-groups.

In order to study the case $p = 2$ we must recall some facts on dihedral groups from [23]. We first introduced the dihedral group $D_8$ in Lemma 3.1. Fig. 2 contains the lattice of subgroups of $D_8 = \langle u, a \rangle$ which is involved in the central product with $Z(4)$ for the formation of $\mathcal{P}_{1,2}$. The lattice of subgroups of $D_8$ is

$$\mathcal{L}(D_8) = \{1, \langle u \rangle, \langle u^2 \rangle, \langle a \rangle, \langle ua \rangle, \langle u^2 a \rangle, \langle u^3 a \rangle, \langle u^2, a \rangle, \langle u^2, ua \rangle, D_8 \}.$$

The normal subgroups in $D_8$ are of course $D_8$ and 1, but also $U = \langle u \rangle$, $Z(D_8) = \langle u^2 \rangle$, $M_1 = \langle u^2, a \rangle$ and $M_2 = \langle u^2, ua \rangle$. Then there are subgroups of order two below these, namely $V, W, H, K$. It is clear the analogy with the left side of Fig. 2.
Denoting by $\sigma(r)$ the sum of divisors of $r$ and by $\tau(r)$ the number of divisors of $r$, it is known that the number of subgroups of $D_8$ can be counted as

$$|\mathcal{L}(D_8)| = \sigma(4) + \tau(4) = 7 + 3 = 10 \quad (5.2)$$

and all of them are abelian up to $D_8$, so the number of nontrivial abelian subgroups of $D_8$ is $c_{ab}(D_8) = 8$.

**Corollary 5.5.** The number of nontrivial abelian subgroups $c_{ab}(\mathcal{P}_{n,2})$ of $\mathcal{P}_{n,2}$ is bounded by

$$2(c_{ab}(\mathcal{P}_{n-1,2}) + 1) \geq c_{ab}(\mathcal{P}_{n,2}) \geq 10n.$$  

**Proof.** We begin by proving the lower bound. Apply Theorem 4.2 noting that each factor $H_j$ in the decomposition in weak central product contains at least one dihedral group of order 8. We get $8n$ abelian subgroups in this way. Now consider Fig. 2 and note that in each $H_j$ we also find two abelian subgroups like $\langle b \rangle$ and $\langle ub \rangle$ of order two, so the result follows.

For the upper bound, we refer again to Fig. 2 and Lemma 3.1. We observe from [26] that all nontrivial abelian subgroups of $\mathcal{P}_{1,2}$ are the following: the unique normal subgroup $\langle u^2 \rangle$ of order two; the six nonnormal subgroups of order two $\langle a \rangle$, $\langle ua \rangle$, $\langle u^2a \rangle$, $\langle u^3a \rangle$, $\langle ub \rangle$; four cyclic subgroups of order four $\langle b \rangle$, $\langle u \rangle$, $\langle ab \rangle$ and $\langle uab \rangle$; three 2-elementary abelian 2-subgroups of rank two $\langle u^2, a \rangle$, $\langle u^2, ua \rangle$ and $\langle u^2, ub \rangle$; finally the subgroups of order eight $\langle u, b \rangle$, $\langle a, b \rangle$ and $\langle ua, b \rangle$ which are isomorphic to $\mathbb{Z}(4) \oplus \mathbb{Z}(2)$. This shows that $c_{ab}(\mathcal{P}_{1,2}) = 1 + 6 + 4 + 3 + 3 = 17 \leq 2 c_{ab}(D_8) + 2$,

because in the process of forming the central product we find at least two subgroups in $\mathcal{P}_{1,2}$ isomorphic to $D_8$. By Theorem 4.2 we have that

$$c_{ab}(\mathcal{P}_{2,2}) \leq 2 c_{ab}(\mathcal{P}_{1,2}) + 2$$

because in the process of forming the central product we find at least two subgroups in $\mathcal{P}_{1,2}$ and a subgroup of order 4 which gives the additional term equal to 2. Therefore we find that

$$c_{ab}(\mathcal{P}_{n,2}) \leq 2c_{ab}(\mathcal{P}_{n-1,2}) + 2$$

from which the result follows. \qed

Note that there exist more sophisticated counting formulas than the one we presented (e.g. [4]). Our purpose was to show a simple application of our main results for the problem of counting abelian subgroups of Pauli groups. This follows from the dihedral component of the groups in the factorisation given in Theorem 4.2.
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