NONCOMMUTATIVE GEOMETRY AND CONFORMAL GEOMETRY. I.
LOCAL INDEX FORMULA AND CONFORMAL INVARIANTS

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Abstract. This paper is the first of a series of papers on noncommutative geometry and conformal geometry. In this paper, elaborating on ideas of Connes and Moscovici, we establish a local index formula in conformal-diffeomorphism invariant geometry. The existence of such a formula was pointed out by Moscovici [Mo2]. Another main result is the construction of a huge class of global conformal invariants taking into account the action of the group of conformal diffeomorphisms (i.e., the conformal gauge group). These invariants are not of the same type as the conformal invariants considered by Spyros Alexakis in his solution of the Deser-Schwimmer conjecture. The arguments in this paper rely on various tools from noncommutative geometry, although ultimately the main results are stated in a differential-geometric fashion. In particular, a crucial use is made of the conformal invariance of the Connes-Chern character of conformal Dirac spectral triple of Connes-Moscovici [CM3].

1. Introduction

This paper is part of a series of papers whose aim is to use noncommutative geometry to study conformal geometry and noncommutative versions of conformal geometry. Conformal geometry is the geometry up to angle-preserving transformations. It has interactions with various areas of mathematical sciences, including geometric nonlinear PDEs, geometric scattering theory, parabolic geometry, asymptotically hyperbolic geometry, conformal field theory, or even conformal gravity. In particular, it plays a fundamental role in the AdS/CFT correspondence on the conjectured equivalence between string theory of anti-de Sitter spaces and some conformal field theory on their conformal boundaries. An important focus of interest in conformal geometry is the study of local and global conformal invariants, especially in the context of Fefferman’s program in parabolic geometry (see, e.g., [Al, BEG, BØ1, BØ2, FG1, FG2, PR, Po2]). Alternatively, given a conformal structure $\mathcal{C}$ on a manifold $M$ we are interested in the action of $M$ of the group of diffeomorphisms preserving this conformal structure (i.e., the conformal gauge group).

One main result of this paper is the reformulation of the local index formula in conformal-diffeomorphism invariant geometry (Theorem 8.3). Another main result is the construction and computation of a huge family of conformal invariants taking into account the action of the group of conformal-diffeomorphisms (Theorem 10.2). These conformal invariants are not of the same type as the conformal invariants considered by Alexakis [Al] in his solution of the conjecture of Deser-Schwimmer [DS]. However, there are closely related to conformal invariants exhibited by Branson-Ørsted [BØ2] (see Remark 10.3).

The main aim of noncommutative geometry is to translate the classical tools of differential geometry into the operator theoretic language of quantum mechanics, so as to be able to deal with geometric situations whose noncommutative nature prevents us from using the classical tools of differential geometry [Co4]. An example of such a noncommutative situation is provided by a group $G$ acting by diffeomorphisms on a manifold $M$. In general the quotient $M/G$ need not be Hausdorff, but in the framework of noncommutative geometry its algebra of smooth functions alway makes sense when realized as the (possibly noncommutative) crossed-product algebra $C^\infty(M) \rtimes G$ (see Section 7 for a definition of this algebra). The trade of spaces for algebras is the main impetus for noncommutative geometry.

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In the framework of noncommutative geometry the role of manifolds is played by spectral triples. A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) is given by an algebra \(\mathcal{A}\) represented on a Hilbert space \(\mathcal{H}\) and an unbounded selfadjoint operator \(D\) satisfying suitable conditions (see Section 2 for the precise definition). An example is provided by the spectral triple associated to the Dirac operator on a compact spin Riemannian manifold. In the setup of a diffeomorphism group \(G\) acting on a manifold \(M\), we thus seek for a spectral triple over the crossed-product algebra \(C^\infty(M) \rtimes G\). It is well known that the only differential structure on a manifold that is invariant under the full group of diffeomorphisms is the manifold structure itself. This prevents us from getting a natural representation of \(G\). A solution to this problem is to pass to the total space of metric bundle of \(P \to M\), which carries a wealth of diffeomorphism-invariant structures (see [Co2, CM2]). This passage is a geometric version of the reduction of type III factors to type II factors by taking crossed-products \([Co1]\).

Although there are Thom isomorphisms between the respective \(K\)-theory and cyclic cohomology of the crossed product algebras \(C^\infty(M) \rtimes G\) and \(C^\infty(P) \rtimes G\), it still is desirable to work directly with the former. As observed by Connes-Moscovici \([CM3]\), when the group \(G\) preserves a given conformal structure this can be done at the expense of twisting the definition of a spectral triple. More precisely, a twisted spectral \((\mathcal{A}, \mathcal{H}, D)_\sigma\) is like an ordinary spectral triple at the exception that the boundedness of commutators \([D, a], a \in \mathcal{A}\), is replaced by that of twisted commutators \([D, a]_\sigma = Da - \sigma(a)D\), where \(\sigma\) is a given automorphism of the algebra \(\mathcal{A}\). A natural example is given by conformal deformations of ordinary spectral triples,

\[
(\mathcal{A}, \mathcal{H}, D) \longrightarrow (\mathcal{A}, \mathcal{H}, kDk)_\sigma, \quad \sigma(a) = k^2ak^{-2},
\]

where \(k\) ranges over positive invertible elements of \(\mathcal{A}\) (see \([CM3]\)). A more refined example is the conformal Dirac spectral triple \((C^\infty(M) \rtimes G, L_0^2(M, \mathcal{S}), \mathcal{D}_\sigma)_\sigma\), associated to the Dirac operator \(\mathcal{D}\) on a compact Riemannian spin manifold \((M, g)\) and a group \(G\) of diffeomorphisms preserving a given conformal structure \(\mathcal{E}\) (see \([CM3]\); a review of this example is given in Section 7). The conformal invariance of the Dirac operator plays a crucial role in this construction. There are various other examples of twisted spectral triples (see \([CM3, GM, CT, IM, Mo2, PW3]\)).

An important motivation and application of Connes’ noncommutative geometry program is the reformulation and extension of the index formula of Atiyah-Singer \([AS1, AS2]\) to various new geometric settings. Given a Dirac operator \(\mathcal{D}\) on a closed Riemannian spin manifold \(M^n\) and a Hermitian vector bundle \(E\) we may twist \(\mathcal{D}\) with any Hermitian connection \(\nabla^E\) on \(E\) to form a Dirac operator \(\mathcal{D}_{\nabla^E}\) with coefficients in sections of \(E\). The main geometric lst order differential operators are locally of this form. The operators \(\mathcal{D}_{\nabla^E}\) are Fredholm and their Fredholm indices are computed by the Atiyah-Singer index formula,

\[
\text{ind}\mathcal{D}_{\nabla^E} = (2i\pi)^{-\frac{n}{2}} \int_M \hat{A}(M) \wedge \text{Ch}(F^E),
\]

where \(\hat{A}(M)\) is the \(\hat{A}\)-form of the Riemann curvature and \(\text{Ch}(F^E)\) is the Chern form of the curvature \(F^E\) of the connection \(\nabla^E\).

Likewise, given a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) and a Hermitian finitely generated projective module \(\mathcal{E}\) over the algebra \(\mathcal{A}\), we can twist the operator \(D\) with any Hermitian connection \(\nabla^\mathcal{E}\) on \(\mathcal{E}\) to get a Fredholm operator \(D_{\nabla^\mathcal{E}}\) with coefficients in \(\mathcal{E}\). The analogues of de Rham homology and cohomology in noncommutative geometry are provided by the cyclic cohomology and cyclic homology \([Co3, Ts]\). Furthermore, Connes \([Co3]\) associated to any (\(p\)-summable) spectral triple \((\mathcal{A}, \mathcal{H}, D)\) a cyclic cohomology class \(\text{Ch}(D)\), called Connes-Chern character, which computes the Fredholm indices \(\text{ind}\mathcal{D}_{\nabla^\mathcal{E}}\). Namely,

\[
(1.1) \quad \text{ind}\mathcal{D}_{\nabla^\mathcal{E}} = \langle \text{Ch}(D), \text{Ch}(\mathcal{E}) \rangle,
\]

where \(\text{Ch}(\mathcal{E})\) is the Chern character of \(\mathcal{E}\) in cyclic homology. Under further assumptions, the Connes-Chern character is represented by the CM cocycle \([CM2]\). The components of the CM cocycle are given by formulas that are local in the sense they involve a version for spectral triples of the noncommutative residue trace of Guillemin \([Gu]\) and Wodzicki \([Wo]\). Together with (1.1)
this provides us with the local index formula in noncommutative geometry. In the case of a Dirac
spectral triple this enables us to recover the Atiyah-Singer index formula (see [CM2, Po1]).

As for ordinary spectral triples, the datum of a twisted spectral triple \((\mathcal{A}, \mathcal{H}, D)\) gives rise
to an index problem by twisting the operator \(D\) with \(\sigma\)-connections (see [PW1]). The resulting
Fredholm indices are computed by a Connes-Chern character \(\text{Ch}(D)\) defined as a class in the
cyclic cohomology of \(\mathcal{A}\) (see [CM3, PW1]). However, although Moscovici [Mo2] produced an
Ansatz for a version of the CM cocycle for twisted spectral triple, this Ansatz has been verified
a priori only in a special class of examples (see [Mo2]). In particular, to date we still don’t know whether
Moscovici’s Ansatz holds for conformal deformations of ordinary spectral triples.

The first main result of this paper states that the Connes-Chern character of the conformal
Dirac spectral triple is a conformal invariant (Theorem 7.8). In fact, for our purpose it is crucial
to define the Connes-Chern character in the cyclic cohomology of \(\mathcal{A}\), which is
smaller than the ordinary cyclic cohomology of general cochains. In fact, we show that for a natural
class of twisted spectral triples \((\mathcal{A}, \mathcal{H}, D)\) over locally convex algebras (which we call smooth
twisted spectral triples) the Connes-Chern character descends to a class \(\text{Ch}(D) \in \text{HP}^0(\mathcal{A})\),
where \(\text{HP}^0(\mathcal{A})\) is the periodic cyclic cohomology of continuous cochains (see Proposition 5.9).

We then show that the invariance of this class under conformal perturbations of twisted spectral
triples (Proposition 6.8).

The construction of the conformal Dirac spectral triple \((\mathcal{C}^\infty(M) \rtimes G, L^2_{\sigma}(M, \mathcal{H}), D_\sigma)\) associated
to a conformal class \(\mathcal{C}\) on a compact spin manifold \(M\) alluded to above \(a\) \(\text{a priori}\) depends on
a choice of a metric \(g \in \mathcal{C}\). As it turns out, up to equivalence of twisted spectral triples, changing
a metric within \(\mathcal{C}\) only amounts to performing a conformal deformation in the realm pointed out
by Moscovici [Mo2]. Combining this with the invariance under conformal deformations of the
Connes-Chern character mentioned above shows that the Connes-Chern character \(\text{Ch}(D_\sigma) \in \text{HP}^0(\mathcal{C}^\infty(M) \rtimes G)\) is an invariant of the conformal class \(\mathcal{C}\).

The index formula in conformal-diffeomorphism invariant geometry is derived from the computation
of the Connes-Chern character \(\text{Ch}(D_\sigma)\). As we don’t know whether the conformal Dirac
spectral triple satisfies Moscovici’s Ansatz alluded to above, we \(a\) \(\text{a priori}\) cannot make use of a CM
cocycle representative to compute the Connes-Chern character \(\text{Ch}(D_\sigma)\). However, its conformal
invariance means that we can choose any metric in the conformal class \(\mathcal{C}\) to compute it. In particular,
we may take a \(G\)-invariant metric. It follows from Obata-Ferrand theorem that such a metric
always exists when the conformal structure is not flat (i.e., it is not equivalent to the conformal
structure of the round sphere). In the case the metric \(g\) is \(G\)-invariant the conformal spectral triple
becomes an ordinary spectral triple. We postpone to the sequel [PW2] (referred to as Part II) the
explicit computation of the Connes-Chern character of a general equivariant Dirac spectral triple
\((\mathcal{C}^\infty(M) \rtimes G, L^2_{\sigma}(M, \mathcal{H}), D_\sigma)\), where \(G\) is a group of smooth isometries. Note that for our purpose
it crucial to compute the Connes-Chern character defined as a class in the \(\text{continuous}\) cochain
cyclic cohomology \(\text{HP}^0(\mathcal{C}^\infty(M) \rtimes G)\). Using the results of Part II and given any \(G\)-invariant
metric \(g \in \mathcal{C}\), we then can express the Connes-Chern character \(\text{Ch}(D_\sigma)\) in terms of explicit
universal polynomials in curvatures and normal curvatures of fixed-point manifolds of the various
diffeomorphisms in \(G\) (see Theorem 8.3 for the explicit formulas). These terms are reminiscent
of the local equivariant index theorem for Dirac operators of Atiyah-Segal [AS]. Together with
the index formula (1.1) this provides us with a local index formula in conformal-diffeomorphism
invariant geometry (for non-flat conformal structures).

As cyclic cohomology is dual to cyclic homology, the conformal invariance of the Connes-Chern
cocycle \(\text{Ch}(D_\sigma)\) implies that any pairing with cyclic cycles produces a numerical conformal
invariant. In order to understand these invariants we construct geometric cyclic cycles spanning
the cyclic homology of \(\mathcal{C}^\infty(M) \rtimes G\). At this stage it is really important to have a representative
of the Connes-Chern character in \(\text{HP}^0(\mathcal{C}^\infty(M) \rtimes G)\), since the geometric cycles live in the dual
cyclic homology \(\text{HP}^0(\mathcal{C}^\infty(M) \rtimes G)\). This cyclic homology was computed by Brylinski-Nistor [BN]
(see also [Cr]). Namely,

\[(1.2) \quad \text{HP}_i(C^\infty(M) \rtimes G) \simeq \bigoplus_{\langle \phi \rangle \in \langle G \rangle} \bigoplus_{0 \leq a \leq n} \bigoplus_{q \geq 0} H^{2q+i}(M^a)^G, \quad i = 0, 1,\]

where \(\langle G \rangle\) is set of conjugacy classes of \(G\) and \(H^\bullet(M^a)\) is the \(G_\phi\)-invariant cohomology of the fixed point submanifold \(M^a\), \(\dim M^a = a\), where \(G_\phi\) is the stabilizer in \(G\) of a representative \(\phi\) in a given conjugacy class \(\langle \phi \rangle\). The results of [BN] and [Cr] are stated in a much more general case of the convolution algebra of an étale groupoid. Furthermore, although it can be exhibited by explicit geometric Hochschild cycles (see, e.g., [Da] in case \(G\) is finite), it is difficult to obtain explicit cyclic cycles due to some incompatibility properties between the cyclic operator and the action of the group \(G\).

The above issue is resolved by observing that the cyclic mixed cochain-complex of \(C^\infty(M) \rtimes G\) is quasi-isomorphic to a sub-complex consisting of what we call \(G\)-invariant cochains that are further invariant under the transformations,

\[(a^0, \ldots, a^m) \mapsto (a^0, \ldots, a^j u_\psi, u_\psi^{-1} a^{j+1}, \ldots, a^m), \quad a^i \in C^\infty(M) \rtimes G,\]

where \(\psi\) ranges over \(G\) and \(\psi \mapsto u_\psi\) is the representation of elements of \(G\) in the crossed product algebra \(C^\infty(M) \rtimes G\). It is fairly natural to consider \(G\)-invariant cyclic chains since, for instance, the transverse fundamental class cocycle of Connes [Co2] is \(G\)-invariant. Furthermore, the formulas for the Connes-Chern character \(\text{Ch}(\mathcal{P}_g)_{\sigma_g}\) mentioned above produce \(G\)-invariant representatives.

At the level of chains, the \(G\)-normalization condition essentially amounts to replacing the projector tensor product \(\hat{\otimes}\) by a coarser topological tensor product \(\hat{T}_G\) which takes into account the action of \(G\) and the \(G\)-bimodule structure of the crossed-product algebra \(C^\infty(M) \rtimes G\). The results of [BN, Cr] imply that the projection of the cyclic mixed complex onto the \(G\)-normalized cyclic complex is a quasi-isomorphism (see Proposition 9.12). As the Connes-Chern character \(\text{Ch}(\mathcal{P}_g)_{\sigma_g}\) is explicitly represented by \(G\)-invariant cochains, it follows that we only need to exhibit geometric \(G\)-normalized cyclic chains. This is done along the line of arguments of [BN, Cr]. Given any \(\phi \in G\) and a fixed-point submanifold \(M^a\), we then have a natural mixed complex morphism from the \(G_\phi\)-invariant de Rham complex of \(M^a\) to the \(G\)-normalized cyclic mixed complex of \(C^\infty(M) \rtimes G\) arising from the map,

\[\omega = f^0 df^0 \wedge \cdots \wedge df^m \mapsto \eta_\omega := \sum_{\sigma \in \mathfrak{S}_m} f^0 \hat{T}_G \hat{f}^{(1)} \hat{T}_G \cdots \hat{T}_G \hat{f}^{(m-1)} \hat{T}_G \hat{f}^{(m)} u_\phi,\]

where \(\mathfrak{S}_m\) is the \(m\)th symmetric group and \(\hat{f}^j\) is a suitable extension of \(f^j \in C^\infty(M^a)\) into a smooth function on \(M\). As it turns out, gathering all these maps yields the isomorphism (1.2). This also shows that, any \(G_\phi\)-invariant closed even form \(\omega\) on \(M^a\) gives rise to a \(G\)-normalized cyclic cycle \(\eta_\omega\) on \(C^\infty(M) \rtimes G\). Pulling back its homology class to a class \(\pi^*[\eta_\omega]\) in \(\text{HP}_0(C^\infty(M) \rtimes G)\) and pairing it with the Connes-Connes character \(\text{Ch}(\mathcal{P}_g)_{\sigma_g}\) we then obtain a conformal invariant,

\[(1.3) \quad I_\sigma(\omega) := \left( \text{Ch}(\mathcal{P}_g)_{\sigma_g}, \pi^*[\eta_\omega] \right).\]

Furthermore, using the geometric expression of the Connes-Chern character \(\text{Ch}(\mathcal{P}_g)_{\sigma_g}\), that we obtained in case of a \(G\)-invariant metric, we see that, for any \(G\)-invariant metric in the conformal class at stake,

\[(1.4) \quad I_\sigma(\omega) = (-i)^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}} \int_{M^a} \hat{A} \left( R^T M^a \right) \wedge \nu_\phi \left( R^{N^a} \right) \wedge \omega,\]

where \(\nu_\phi \left( R^{N^a} \right)\) is an explicit polynomial in the normal curvature \(R^{N^a}\).

The above conformal invariants are not of the same type of the conformal invariants considered by Alexakis [Al] in his solution of the conjecture of Deser-Schwimmer [DS] on the characterization of global conformal invariants. In [Al, DS] the conformal invariants appear as integrals over the whole manifold \(M\) of local Riemannian invariants, whereas the invariants (1.3)–(1.4) involve integrals over some fixed-point submanifolds. As explained in Remark 10.3, our invariants are related to conformal invariants exhibited by Branson-Ørsted [BØ2]. It would be interesting to
find, even conjecturally, a characterization of all conformal invariants encompassing the invariants of \([Al, DS]\) and the invariants (1.3)–(1.4) and those in \([BØ2]\).

In this paper, we focus on conformal structures of even dimension that are no equivalent to the conformal structure of the round sphere. We would like to stress out that our results can be extended to the odd dimension case. Although, the index theory in the odd-dimensional case is somewhat different, as it involves spectral flows, there is a Connes-Chern character which is almost identical to that in the even case. This enables us to obtain a local index formula and exhibit conformal invariants as in the even case. We defer the details to a sequel of this paper. In the case of round spheres, the conformal group is essential, and so the computation of the Connes-Chern character requires a different type of analysis than those carried in Part II (see also \([Mo2]\) for a discussion on this issue). We hope to deal with round spheres in a future article.

The paper is organized as follows. In Section 2, we review the main definitions and examples regarding twisted spectral triples and the construction of their index maps. In Section 3, we review the main facts about cyclic cohomology, cyclic homology and the Chern character in cyclic homology. In Section 4, we review the construction of the Connes-Chern character of a twisted spectral triple. In Section 5, we show that for smooth twisted spectral triples the Connes-Chern character descends to the cyclic cohomology of continuous cochains. In Section 6, we establish the invariance of the Connes-Chern character under conformal deformations. In Section 7, after reviewing the construction of the conformal Dirac spectral triple, we prove that its Connes-Chern Character is a conformal invariant. In Section 8, we compute this Connes-Chern character by using the results of Part II. This provides us with a local index formula in conformal-diffeomorphism invariant geometry. In Section 9, we construct geometric cyclic cycles spanning the cyclic homology of the crossed-product algebra \(C^\infty(M) \rtimes G\). In Section 10, we combine the results of the previous sections to construct our conformal invariants.

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2. Index Theory on Twisted Spectral Triples.

In this section, we recall how the datum of a twisted spectral triple naturally gives rise to an index problem ([CM3, PW1]). The exposition closely follows that of [PW1] (see also [Mo1] for the case of ordinary spectral triples).

In the setting of noncommutative geometry, the role of manifolds is played by spectral triples.

**Definition 2.1.** A spectral triple \((A, H, D)\) is given by

1. A \(\mathbb{Z}_2\)-graded Hilbert space \(H = H^+ \oplus H^-\).
2. A unital \(*\)-algebra \(A\) represented by bounded operators on \(H\) preserving its \(\mathbb{Z}_2\)-grading.
3. A selfadjoint unbounded operator \(D\) on \(H\) such that
   a. \(D\) maps \(\text{dom}(D) \cap H^\pm\) to \(H^\mp\).
   b. The resolvent \((D + i)^{-1}\) is a compact operator.
   c. \(a \text{dom}(D) \subset \text{dom}(D)\) and \([D, a]\) is bounded for all \(a \in A\).

**Remark 2.2.** The condition (3)(a) implies that with respect to the splitting \(H = H^+ \oplus H^-\) the operator \(D\) takes the form,

\[
D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \quad D^\pm : \text{dom}(D) \cap H^\pm \to H^\mp.
\]
The paradigm of a spectral triple is given by a Dirac spectral triple,
\[(C^\infty(M), L^2_g(M,\mathcal{S}),\mathcal{D}_g),\]
where \((M^n, g)\) is a compact spin Riemannian manifold of even dimension \(n\) and \(\mathcal{D}_g\) is its Dirac operator acting on the spinor bundle \(\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-\).

The definition of a twisted spectral triple is similar to that of an ordinary spectral triple, except for some “twist” given by the conditions (3) and (4) below.

**Definition 2.4** ([CM3]). A twisted spectral triple \((\mathcal{A}, \mathcal{H}, D, \sigma)\) is given by

1. A \(\mathbb{Z}_2\)-graded Hilbert space \(\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-\).
2. A unital \(*\)-algebra \(\mathcal{A}\) represented by even bounded operators on \(\mathcal{H}\).
3. An automorphism \(\sigma : \mathcal{A} \to \mathcal{A}\) such that \(\sigma(a)^* = \sigma^{-1}(a^*)\) for all \(a \in \mathcal{A}\).
4. An odd selfadjoint unbounded operator \(D\) on \(\mathcal{H}\) such that
   a. The resolvent \((D + i)^{-1}\) is compact.
   b. \(a(\text{dom } D) \subset \text{dom } D\) and \([D, a]_\sigma := Da - \sigma(a)D\) is bounded for all \(a \in \mathcal{A}\).

The relevance of the notion of twisted spectral triples is set in the setting of conformal geometry stems from the following observation. Let \((C^\infty(M), L^2_g(M,\mathcal{S}),\mathcal{D}_g)\) be a Dirac spectral triple as in Example 2.3, and consider a conformal change of metric,
\[\hat{g} = k^{-2} g, \quad k \in C^\infty(M), \quad k > 0.\]
We then can form a Dirac spectral triple \((C^\infty(M), L^2_{\hat{g}}(M,\mathcal{S}),\mathcal{D}_{\hat{g}})\) associated to the new metric \(\hat{g}\). As it turns out (see [PW1]) this spectral triple is equivalent to the following spectral triple,
\[(C^\infty(M), L^2_g(M,\mathcal{S}), \sqrt{k} \mathcal{D}_g \sqrt{k}).\]

We note that the above spectral triple continues to make sense if we only assume \(k\) to be a positive Lipschitz function on \(M\).

More generally, let \((\mathcal{A}, \mathcal{H}, D)\) be an ordinary spectral triple and \(k\) a positive element of \(\mathcal{A}\). If we replace \(D\) by its \textit{conformal deformation} \(kDk\) then, when \(\mathcal{A}\) is noncommutative, the triple \((\mathcal{A}, \mathcal{H}, kDk)\) need not be an ordinary spectral triple. However, as the following result shows, it always gives rise to a twisted spectral triple.

**Proposition 2.5** ([CM3]). Let \(\sigma : \mathcal{A} \to \mathcal{A}\) be the automorphism defined by
\[(2.1) \quad \sigma(a) := k^2 ak^{-2} \quad \forall a \in \mathcal{A}.\]
Then \((\mathcal{A}, \mathcal{H}, kDk, \sigma)\) is a twisted spectral triple.

**Remark 2.6.** A more elaborate version of the above example, and the main focus of this paper, is the conformal Dirac spectral triple of Connes-Moscovici [CM3]. This is a twisted spectral triple taking into account of the action of the group of diffeomorphisms preserving a given conformal structure. We refer to Section 7.1 for a review of this example.

**Remark 2.7.** We refer to [CM3, GMT, CT, IM, Mo2, PW3] for the constructions of various other examples of twisted spectral triples.

From now on, we let \((\mathcal{A}, \mathcal{H}, D, \sigma)\) be a twisted spectral triple. In addition, we let \(\mathcal{E}\) be a finitely generated projective right module over \(\mathcal{A}\), i.e., \(\mathcal{E}\) is a direct summand of a free module \(\mathcal{E}_0 \simeq \mathcal{A}^N\).

**Definition 2.8** ([PW1]). A \(\sigma\)-translate for \(\mathcal{E}\) is a finitely generated projective right module \(\mathcal{E}^\sigma\) equipped with the following data:

1. A linear isomorphism \(\sigma^\mathcal{E} : \mathcal{E} \to \mathcal{E}^\sigma\).
2. An idempotent \(e \in M_N(\mathcal{A}), \ N \in \mathbb{N}\).
3. Right module isomorphisms \(\phi : \mathcal{E} \to e \mathcal{A}^N\) and \(\phi^\sigma : \mathcal{E}^\sigma \to \sigma(e)\mathcal{A}^N\) such that
   \[(2.2) \quad \phi^\sigma \circ \sigma^\mathcal{E} = \sigma \circ \phi.\]

**Remark 2.9.** The condition (2.2) implies that
\[\sigma^\mathcal{E}(\xi a) = \sigma^\mathcal{E}(\xi) \sigma(a) \quad \text{for all } \xi \in \mathcal{E} \text{ and } a \in \mathcal{A}.\]
Remark 2.10. When $\mathcal{E} = e\mathcal{A}^N$ with $e = e^2 \in M_N(\mathcal{A})$ we always may take $\mathcal{E}^\sigma = \sigma(e)\mathcal{A}^N$ as $\sigma$-translate of $e\mathcal{A}^N$. In this case $\sigma^\mathcal{E}$ agrees on $e\mathcal{A}^N$ with the lift of $\sigma$ to $\mathcal{A}^N$. In particular, $\mathcal{E}^\sigma = \mathcal{E}$ when $\sigma = \text{id}$.

Throughout the rest of the section we let $\mathcal{E}^\sigma$ be a $\sigma$-translate of $\mathcal{E}$. In addition, we consider the $(\mathcal{A}, \mathcal{A})$-bimodule of twisted 1-forms,

$$\Omega^1_{D, \sigma}(\mathcal{A}) = \{ \Sigma a[D, b^i]_\sigma : a^i, b^i \in \mathcal{A} \}.$$ 

The “twisted” differential $d_\sigma : \mathcal{A} \to \Omega^1_{D, \sigma}(\mathcal{A})$ is given by

(2.3) $$d_\sigma a := [D, a]_\sigma \quad \forall a \in \mathcal{A}. $$

This is a $\sigma$-derivation, in the sense that

(2.4) $$d_\sigma(ab) = (d_\sigma a)b + \sigma(a)d_\sigma b \quad \forall a, b \in \mathcal{A}. $$

Definition 2.11. A $\sigma$-connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E}^\sigma \otimes_\mathcal{A} \Omega^1_{D, \sigma}(\mathcal{A})$ such that

(2.5) $$\nabla(\varepsilon) = (\nabla \varepsilon) a + \sigma^2(\varepsilon) \otimes d_\sigma a \quad \forall \varepsilon \in \mathcal{E} \forall a \in \mathcal{A}. $$

Example 2.12. Suppose that $\mathcal{E} = e\mathcal{A}^N$ with $e = e^2 \in M_N(\mathcal{A})$. Then a natural $\sigma$-connection on $\mathcal{E}$ is the Grassmannian $\sigma$-connection $\nabla^\mathcal{E}$ defined by

(2.6) $$\nabla^\mathcal{E}_\xi = \sigma(a)(d_\sigma \xi_j) \quad \text{for all } \xi = (\xi_j) \in \mathcal{E}. $$

Definition 2.13. A Hermitian metric on $\mathcal{E}$ is a map $(\cdot, \cdot) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that

(1) $(\cdot, \cdot)$ is $\mathcal{A}$-sesquilinear, i.e., it is $\mathcal{A}$-antilinear with respect to the first variable and $\mathcal{A}$-linear with respect to the second variable.

(2) $(\cdot, \cdot)$ is positive, i.e., $(\xi, \xi) \geq 0$ for all $\xi \in \mathcal{E}$.

(3) $(\cdot, \cdot)$ is nondegenerate, i.e., $\xi \mapsto (\xi, \cdot)$ is an $\mathcal{A}$-antilinear isomorphism from $\mathcal{E}$ onto its $\mathcal{A}$-dual $\text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A})$.

Example 2.14. The canonical Hermitian structure on the free module $\mathcal{A}^N$ is given by

(2.7) $$(\xi, \eta)_0 = \xi_1^* \eta_1 + \cdots + \xi_q^* \eta_q \quad \text{for all } \xi = (\xi_j) \text{ and } \eta = (\eta_j) \text{ in } \mathcal{A}^N. $$

It gives a Hermitian metric on any direct summand $\mathcal{E} = e\mathcal{A}^N$, $e = e^2 \in M_N(\mathcal{A})$ (see, e.g., [PW1]).

From now on we assume that $\mathcal{E}$ and its $\sigma$-translate carry a Hermitian metric. We denote by $\mathcal{H}(\mathcal{E})$ the pre-Hilbert space consisting of $\mathcal{E} \otimes_\mathcal{A} \mathcal{H}$ equipped with the Hermitian inner product,

(2.8) $$\langle \xi_1 \otimes \xi_2 \otimes \xi_3 \rangle := \langle \xi_1, (\xi_1, \xi_2) \xi_3 \rangle, \quad \xi_1, \xi_2, \xi_3 \in \mathcal{E},$$

where $(\cdot, \cdot)$ is the Hermitian metric of $\mathcal{E}$. It can be shown $\mathcal{H}(\mathcal{E})$ actually is a Hilbert space and its topology is independent of the choice of the Hermitian inner product of $\mathcal{E}$ (see, e.g., [PW1]). We also note there is a natural $\mathbb{Z}_2$-grading on $\mathcal{H}(\mathcal{E})$ given by

(2.9) $$\mathcal{H}(\mathcal{E}) = \mathcal{H}^+ \mathcal{E} \otimes \mathcal{H}^- \mathcal{E}, \quad \mathcal{H}^\pm(\mathcal{E}) := \mathcal{E} \otimes_\mathcal{A} \mathcal{H}^\pm. $$

We denote by $\mathcal{H}(\mathcal{E}^\sigma)$ the similar $\mathbb{Z}_2$-graded Hilbert space associated to $\mathcal{E}^\sigma$ and its Hermitian metric.

Let $\nabla^\mathcal{E}$ be a $\sigma$-connection on $\mathcal{E}$. Regarding $\Omega^1_{D, \sigma}(\mathcal{A})$ as a subalgebra of $L(\mathcal{H})$ we have a natural left-action $c : \Omega^1_{D, \sigma}(\mathcal{A}) \otimes_\mathcal{A} \mathcal{H} \to \mathcal{H}$ given by

$$c(\omega \otimes \zeta) = \omega(\zeta) \quad \text{for all } \omega \in \Omega^1_{D, \sigma}(\mathcal{A}) \text{ and } \zeta \in \mathcal{H}. $$

We then denote by $c(\nabla^\mathcal{E})$ the composition $(1_{\mathcal{E}^\sigma} \otimes c) \circ (\nabla^\mathcal{E} \otimes 1_\mathcal{H}) : \mathcal{E} \otimes \mathcal{H} \to \mathcal{E}^\sigma \otimes \mathcal{H}$. Thus, for $\xi \in \mathcal{E}$ and $\zeta \in \mathcal{H}$, and upon writing $\nabla^\mathcal{E} \xi = \sum \xi_\alpha \otimes \omega_\alpha$ with $\xi_\alpha \in \mathcal{E}^\sigma$ and $\omega_\alpha \in \Omega^1_{D, \sigma}(\mathcal{A})$, we have

(2.10) $$c(\nabla^\mathcal{E})(\xi \otimes \zeta) = \sum \xi_\alpha \otimes \omega_\alpha(\zeta). $$

In what follows we regard the domain of $D$ as a left $\mathcal{A}$-module, which is possible since the action of $\mathcal{A}$ on $\mathcal{H}$ preserves $\text{dom} \, D$. 

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Definition 2.15. The operator $D_{\nabla^\xi}: \mathcal{E} \otimes_{\mathcal{A}} \text{dom}(D) \to \mathcal{H}(\mathcal{E}^\sigma)$ is defined by
\begin{equation}
D_{\nabla^\xi}(\xi \otimes \zeta) := \sigma^\xi(\xi) \otimes D\zeta + c(\nabla^\xi)(\xi \otimes \zeta) \quad \text{for all } \xi \in \mathcal{E} \text{ and } \zeta \in \text{dom } D.
\end{equation}

Remark 2.16. Although the operators $\sigma^D$, $D$ and $\nabla^\xi$ are not module maps, the operator is well defined as a linear map with domain $\mathcal{E} \otimes_{\mathcal{A}} \text{dom}(D)$ (see [PW1]).

Remark 2.17. With respect to the $\mathbb{Z}_2$-gradings (2.9) for $\mathcal{H}(\mathcal{E})$ and $\mathcal{H}(\mathcal{E}^\sigma)$ the operator $D_{\nabla^\xi}$ takes the form,
\begin{equation}
D_{\nabla^\xi} = \begin{pmatrix} 0 & D_{\nabla^\xi}^- \\ D_{\nabla^\xi}^+ & 0 \end{pmatrix}, \quad D_{\nabla^\xi}^\pm : \mathcal{E} \otimes_{\mathcal{A}} \text{dom } D^{\pm} \to \mathcal{H}^\pm(\mathcal{E}^\sigma).
\end{equation}

That is, $D_{\nabla^\xi}$ is an odd operator.

Example 2.18 (See [PW1]). Suppose that $\mathcal{E} = eA^N$ with $e = e^2 \in M_N(\mathcal{A})$ and let $\nabla^\xi_0$ be the Grassmanian $\sigma$-connection of $\mathcal{E}$. Then up to the canonical unitary identifications $\mathcal{H}(\mathcal{E}) \simeq e\mathcal{H}^N$ and $\mathcal{H}(\mathcal{E}^\sigma) \simeq \sigma(e)\mathcal{H}^N$ the operators $D_{\nabla^\xi_0}^\pm$ agrees with
\[ \sigma(e)(D \otimes 1_N) : e(\text{dom } D)^N \to \sigma(e)\mathcal{H}^N. \]

Example 2.19. In the case of a Dirac spectral triple $(C^\infty(M), L^2_0(M, \mathcal{S}), \mathcal{D}_g)$, we may take $\mathcal{E}$ to the module $C^\infty(M, E)$ of smooth sections of a vector bundle $E$ over $M$. Any Hermitian metric and connection on $E$ give rise to a Hermitian metric and a connection $\nabla^\xi$ on $\mathcal{E}$. Furthermore, if we set $\mathcal{H} = L^2_0(M, E)$, then, under the natural identification $\mathcal{H}(\mathcal{E}) \simeq L^2(M, \mathcal{S} \otimes E)$, the operator $(\mathcal{D}_g)_{\nabla^\xi}$ agrees with the usual twisted Dirac operator $\mathcal{D}_{\nabla^\xi}$ as defined, e.g., in [BGV].

Proposition 2.20 ([PW1]). The operator $D_{\nabla^\xi}$ is closed and Fredholm.

Note that the above result implies that the operators $D^\pm_{\nabla^\xi}$ in (2.12) are Fredholm. This leads us to the following definition.

Definition 2.21. The index of the operator $D_{\nabla^\xi}$ is
\[ \text{ind } D_{\nabla^\xi} = \frac{1}{2}(\text{ind } D^+_{\nabla^\xi} - \text{ind } D^-_{\nabla^\xi}), \]
where $\text{ind } D^\pm_{\nabla^\xi}$ is the usual Fredholm index of $D^\pm_{\nabla^\xi}$ (i.e., $\text{ind } D^\pm_{\nabla^\xi} = \dim \ker D^\pm_{\nabla^\xi} - \dim \text{coker } D^\pm_{\nabla^\xi}$).

Remark 2.22. In general the indices $\pm \text{ind } D^\pm_{\nabla^\xi}$ do not agree, so that it is natural take their mean to define the index of $D_{\nabla^\xi}$. However, as shown in [PW1], the index $\text{ind } D_{\sigma}$ is an integer when the automorphism $\sigma$ is ribbon, in the sense there is another automorphism $\tau : \mathcal{A} \to \mathcal{A}$ such that $\sigma = \tau \circ \sigma$ and $\tau(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$. The ribbon condition is satisfied in all main examples of twisted spectral (see [PW1]). When this condition holds, it is shown in [PW3] that we always can endow $\mathcal{E}$ with a “$\sigma$-Hermitian structure” (see [PW3] for the precise definition). In that case, for any connection $\nabla^\xi$ compatible with the $\sigma$-Hermitian structure,
\[ \text{ind } D_{\nabla^\xi} = \dim \ker D^+_{\nabla^\xi} - \dim \ker D^-_{\nabla^\xi} \in \mathbb{Z}. \]

The above formula is the generalization of the usual formula for the index of a Dirac operator twisted by a Hermitian connection on a Hermitian vector bundle.

As it turns out the index $\text{ind } D_{\nabla^\xi}$ only depends on the $K$-theory class of $\mathcal{E}$ (see [PW1]). More precisely, we have the following result.

Proposition 2.23 ([CM3, PW1]). There is a unique additive map $\text{ind } D_{\sigma} : K_0(\mathcal{A}) \to \frac{1}{2}\mathbb{Z}$ such that, for any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and any $\sigma$-connection on $\mathcal{E}$, we have
\[ \text{ind } D_{\sigma}[\mathcal{E}] = \text{ind } D_{\nabla^\xi}. \]

3. Cyclic Cohomology and the Chern Character

Cyclic cohomology, and its dual version cyclic homology, were discovered by Connes [Co3] and Tsygan [Ts] independently. In this section, we review the main facts about cyclic cohomology and cyclic homology, and how this enables us to define a Chern character in cyclic homology. We refer to [Co3, Co4, Lo] for more complete accounts on these topics. Throughout this section we let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$. 


3.1. Cyclic cohomology. The Hochschild cochain-complex of $\mathcal{A}$ is defined as follows. The space of $m$-cochains $C^m(\mathcal{A})$, $m \in \mathbb{N}_0$, consists of $(m+1)$-linear maps $\varphi : \mathcal{A}^{m+1} \to \mathbb{C}$. The Hochschild coboundary $b : C^m(\mathcal{A}) \to C^{m+1}(\mathcal{A})$, $b^2 = 0$, is given by

\begin{equation}
    b\varphi(a^0, \ldots, a^{m+1}) = \sum_{j=0}^{m} (-1)^j \varphi(a^0, \ldots, a^ja^{j+1}, \ldots, a^{m+1})
\end{equation}

\begin{equation}
    \quad + (-1)^{m+1} \varphi(a^{m+1}, a^0, \ldots, a^m), \quad a^j \in \mathcal{A}.
\end{equation}

A cochain $\varphi \in C^m(\mathcal{A})$ is called cyclic when $T\varphi = \varphi$, where the operator $T : C^m(\mathcal{A}) \to C^m(\mathcal{A})$ is defined by

\begin{equation}
    T\varphi(a^0, \ldots, a^{m}) = (-1)^m \varphi(a^m, a^0, \ldots, a^{m-1}), \quad a^j \in \mathcal{A}.
\end{equation}

We denote by $C^m_\omega(\mathcal{A})$ the space of cyclic $m$-cochains. As $b(C^*_{\omega}(\mathcal{A})) \subset C^{*+1}_{\omega}(\mathcal{A})$, we obtain a sub-complex $(C^*_{\omega}(\mathcal{A}), b)$, the cohomology of which is denoted $HC^*_{\omega}(\mathcal{A})$ and called the cyclic cohomology of $\mathcal{A}$.

The operator $B : C^m(\mathcal{A}) \to C^{m-1}(\mathcal{A})$ is given by

\begin{equation}
    B = AB_0(1-T), \quad \text{where} \quad A = 1 + T + \cdots + T^m,
\end{equation}

and the operator $B_0 : C^m(\mathcal{A}) \to C^{m-1}(\mathcal{A})$ is defined by

\begin{equation}
    B_0\varphi(a^0, \ldots, a^{m-1}) = \varphi(1, a^0, \ldots, a^{m-1}), \quad a^j \in \mathcal{A}.
\end{equation}

Note that $B$ is annihilated by cyclic cochains. Moreover, it can be checked that $B^2 = 0$ and $bB + Bb = 0$. Therefore, in the terminology of [Lo, §2.5.13], we obtain a mixed cochain-complex $(C^*_{\omega}(\mathcal{A}), b, B)$, which is called the cyclic mixed cochain-complex of $\mathcal{A}$. Associated to this mixed complex is the periodic cyclic complex $(C^*[\omega](\mathcal{A}), b + B)$, where

\begin{equation}
    C^{[i]}_{\omega}(\mathcal{A}) = \bigoplus_{q=0}^{\infty} C^{2q+i}_{\omega}(\mathcal{A}), \quad i = 0, 1,
\end{equation}

and we regard $b$ and $B$ as operators between $C^0(\mathcal{A})$ and $C^1(\mathcal{A})$. The corresponding cohomology is called the periodic cyclic cohomology of $\mathcal{A}$ and is denoted by $HC^*_{\omega_{\mathcal{A}}}(\mathcal{A})$. Note that a periodic cyclic cocycle is a finitely supported sequence $\varphi = (\varphi_{2q+i})$ with $\varphi_{2q+i} \in C^{2q+i}_{\omega_{\mathcal{A}}}(\mathcal{A})$, $q \geq 0$, such that

\begin{equation}
    b\varphi_{2q+i} + B\varphi_{2q+2+i} = 0 \quad \text{for all} \quad q \geq 0.
\end{equation}

As the operator $B$ is annihilated by cyclic cochains, any cyclic $m$-cocycle $\varphi$ is naturally identified with the periodic cyclic cocycle $(0, \ldots, 0, \varphi, 0, \ldots) \in C^{[i]}_{\omega}(\mathcal{A})$, where $i$ is the parity of $m$. This gives rise to natural morphisms,

\begin{equation}
    HC_{2q+*}^{*}(\mathcal{A}) \to HC^*_{\omega_{\mathcal{A}}}(\mathcal{A}), \quad q \geq 0.
\end{equation}

Connes’ periodicity operator $S : C^m_{\lambda}(\mathcal{A}) \to C^{m+2}_{\lambda}(\mathcal{A})$ is obtained from the cup product with the unique cyclic 2-cocycle on $\mathbb{C}$ taking the value 1 at $(1, 1, 1)$ (see [Co3, Co4]). Equivalently,

\begin{equation}
    S = \frac{1}{(m+1)(m+2)} \sum_{j=1}^{m+1} (-1)^j S_j,
\end{equation}

where the operator $S_j : C^m_{\lambda}(\mathcal{A}) \to C^{m+2}_{\lambda}(\mathcal{A})$ is given by

\begin{equation}
    S_j \varphi(a^0, \ldots, a^{m+2}) = \sum_{0 \leq i \leq j-2} (-1)^i \varphi(a^0, \ldots, a^ia^{i+1}, \ldots, a^j, a^{j+1}, \ldots, a^{m+2})
\end{equation}

\begin{equation}
    + (-1)^{j+1} \varphi(a^0, \ldots, a^{j-1}, a^j, a^{j+1}, \ldots, a^{m+2}).
\end{equation}

Here (cf. [Co4]) the operator $S$ is normalized so that the induced map on $HC^*(\mathcal{A})$ satisfies

\begin{equation}
    S = -bB^{-1} \quad \text{on} \quad HC^*(\mathcal{A}).
\end{equation}

In particular, if $\varphi$ is any cyclic cocycle, then $S\varphi$ is a cyclic cocycle whose class in $HC^*(\mathcal{A})$ agrees with that of $\varphi$. Furthermore, Connes [Co3, Theorem II.40] proved that

\begin{equation}
    \lim (HC_{2q+*}^{*}(\mathcal{A}), S) = HC^*(\mathcal{A}),
\end{equation}

where the operator $S$ is normalized so that the induced map on $HC^*(\mathcal{A})$ satisfies

\begin{equation}
    S = -bB^{-1} \quad \text{on} \quad HC^*(\mathcal{A}).
\end{equation}
where the left-hand side is the inductive limit of the direct system $(\HC^{2q+\bullet}(A), S)$.

It is sometimes convenient to “normalize” the cyclic mixed complex. More precisely, we say that a cochain \( \varphi \in C^m(A) \) is normalized when

\[
\varphi(a^0, \ldots, a^m) = 0 \quad \text{whenever} \quad a^j = 1 \text{ for some } j \geq 1.
\]

We denote by \( C^m_0(A) \) the space of normalized \( m \)-cochains. As the operators \( b \) and \( B \) preserve the space \( C^m_0(A) \), we obtain a subcomplex \( (C^m_0(A), b, B) \) of the cyclic mixed complex. Note that \( B = B_0(1 - T) \) on \( C^m_0(A) \). We denote by \( H^*_{\HC}(A) \) the cohomology of the normalized periodic complex \( (C^m_0(A), b + B), \) where \( C^m_0(A) = \bigoplus_{q \geq 0} C^{2q+\bullet}(A) \). Furthermore (see [Lo, Corollary 2.1.10]), the inclusion of \( C^m_0(A) \) in \( C^m(A) \) gives rise to an isomorphism,

\[
H^*_\HC(A) \simeq H^*(A).
\]

**Example 3.1.** Let \( A = C^\infty(M) \), where \( M \) is a closed manifold. For \( m = 0, 1, \ldots, n \), let \( \Omega_m(M) \) be space of \( m \)-dimensional currents. Any current \( C \in \Omega_m(M) \) defines a cochain \( \varphi_C \in C^m(A) \) by

\[
\varphi_C(f^0, \ldots, f^m) = \frac{1}{m!} \langle C, f^0 df^1 \wedge \cdots \wedge df^m \rangle, \quad f_j \in C^\infty(M).
\]

Note that \( \varphi_C \) is a normalized cochain. Moreover it can be checked that \( b\varphi_C = 0 \) and \( B\varphi_C = \varphi_d\varphi_C \), where \( d \) is the de Rham boundary for currents. Therefore, we obtain a morphism from the mixed complex \( (\Omega_\bullet(M), 0, d^\bullet) \) to the cyclic mixed complex of \( A = C^\infty(M) \). In particular, we have a natural linear map,

\[
\alpha^M : H^\bullet_\HC(M, \mathbb{C}) \longrightarrow H^\bullet_\HC(C^\infty(M)), \quad H^i_\HC(M, \mathbb{C}) := \bigoplus_{q \geq 0} H_{2q+i}(M, \mathbb{C}), \quad i = 0, 1,
\]

where \( H_{2q+i}(M, \mathbb{C}) \) is the de Rham homology of \( M \) of degree \( 2q + i \).

**3.2. Cyclic homology.** The cyclic mixed cochain-complex defined above is the dual of a mixed chain-complex defined as follows. The space of \( m \)-chains, \( m \geq 0 \), is \( C_m(A) = A^\otimes(m+1) \). The Hochschild boundary \( b : C_m(A) \to C_{m-1}(A) \), \( b^2 = 0 \), is given by

\[
b(a^0 \otimes \cdots \otimes a^m) = \sum_{j=0}^{m-1} (-1)^j a^0 \otimes \cdots \otimes a^j a^{j+1} \otimes \cdots \otimes a^m
\]

\[
+ (-1)^m a^m a^0 \otimes \cdots \otimes a^{m-1}, \quad a^j \in A.
\]

A chain \( \eta \in C_m(A) \) is cyclic when \( T\eta = \eta \), where the cyclic operator \( T : C_m(A) \to C_m(A) \) is given by

\[
T(a^0 \otimes \cdots \otimes a^m) = (-1)^m a^m \otimes a^0 \otimes \cdots \otimes a^{m-1}, \quad a^j \in A.
\]

We denote by \( C^\Lambda_m(A) \) the space of cyclic \( m \)-chains. This provides us with a sub-complex \( (C^\Lambda_m(A), b) \) of the Hochschild chain-complex. Its homology is called the cyclic homology of \( A \) and is denoted by \( \HC^\Lambda(A) \).

The operator \( B : C_m(A) \to C_{m+1}(A) \) is given by

\[
B := (1 - T)B_0 A, \quad \text{where} \quad A := 1 + T + \cdots + T^m,
\]

and the operator \( B_0 : C_m(A) \to C_{m+1}(A) \) is defined by

\[
B_0(a^0 \otimes \cdots \otimes a^m) = 1 \otimes a^0 \otimes \cdots \otimes a^m, \quad a^j \in A.
\]

It can be checked that \( B^2 = 0 \) and \( bb + Bb = 0 \), so that we obtain a mixed chain-complex \( (C^\bullet(A), b, B) \), called the cyclic mixed chain-complex of \( A \). The associated periodic chain-complex is \( (C^\bullet_\Lambda(A), b + B) \), where

\[
C^i_\Lambda(A) = \prod_{q=0}^\infty C_{2q+i}(A), \quad i = 0, 1.
\]

The homology of this complex is the periodic cyclic homology of \( A \) and is denoted by \( \HC^\Lambda(A) \).
The periodicity operator \( S : C^0_m(\mathcal{A}) \to C^0_{m-2}(\mathcal{A}) \) is given by

\[
S = \frac{1}{m(m-1)} \sum_{j=1}^{m-1} (-1)^j S_j,
\]

where the operator \( S_j : C^0_m(\mathcal{A}) \to C^0_{m}(\mathcal{A}) \) is defined by

\[
S_j(a^0 \otimes \cdots \otimes a^m) = \sum_{0 \leq l \leq j-2} (-1)^l a^0 \otimes \cdots \otimes a^l a^{j+1} \otimes \cdots \otimes a^m + (-1)^{j+1} a^0 \otimes \cdots \otimes a^{j-1} a^j a^{j+1} \otimes \cdots \otimes a^{m+2}.
\]

The operator \( S \) commutes with the Hochschild boundary \( b \) and any cyclic cycle \( \eta \) is cohomologous to \( S\eta \) in \( \text{HP}_*(\mathcal{A}) \). Furthermore, the dual version of (3.8) holds, namely,

\[
\lim \text{ (HC}_{2q+\bullet}(\mathcal{A}), S) = \text{HP}_*(\mathcal{A}),
\]

where the left-hand side is the projective limit of the system \( \text{HC}_{2q+\bullet}(\mathcal{A}), S \).

The mixed chain-complex \( (C^\bullet (\mathcal{A}), b, B) \) is normalized as follows. For \( m \in \mathbb{N} \) and \( l = 1, \ldots, m \), the operator \( N_{m,l} : C_{m-1}(\mathcal{A}) \to C_m(\mathcal{A}) \) is defined by

\[
N_{m,l}(a^0 \otimes \cdots \otimes a^{m-1}) = a^0 \otimes \cdots \otimes a^l \otimes 1 \otimes a^{l+1} \otimes \cdots \otimes a^{m-1}, \quad a^j \in \mathcal{A}.
\]

We then set \( N_m(\mathcal{A}) := \sum_{l=1}^{m} N_{m,l}(C_m(\mathcal{A})) \). The spaces of normalized chains \( C^0_m(\mathcal{A}), m \geq 0 \), are then defined by

\[
C^0_m(\mathcal{A}) = C_0(\mathcal{A}) \quad \text{and} \quad C^0_m(\mathcal{A}) = C_m(\mathcal{A})/N_m(\mathcal{A}) \quad \text{for} \ m \geq 1.
\]

Note that \( C^0_m(\mathcal{A}) \cong \mathcal{A} \otimes (\mathcal{A}/C) \otimes^m \) for \( m \geq 1 \). The operators \( b, B, \) and \( S \) descend to operators on normalized chains. In particular, we obtain another mixed chain-complex \( (C^\bullet(\mathcal{A}), b, B) \). The associated periodic complex is \( (C^\bullet(\mathcal{A}), b + B) \), where \( C^0_{[i]}(\mathcal{A}) = \prod_{q=0}^\infty C^0_{2q+i}(\mathcal{A}) \). Its homology is denoted \( \text{HP}^\bullet_*(\mathcal{A}) \). Moreover, the canonical projection \( C^\bullet_*(\mathcal{A}) \to C^0_*(\mathcal{A}) \) gives rise to an isomorphism,

\[
\text{HP}_*(\mathcal{A}) \cong \text{HP}^0_*(\mathcal{A}).
\]

The duality pairing between \( C^\bullet(\mathcal{A}) \) and \( C_\bullet(\mathcal{A}) \) is given by

\[
\langle \varphi, a^0 \otimes \cdots \otimes a^m \rangle = \varphi(a^0, \ldots, a^m), \quad \varphi \in C^m(\mathcal{A}), \ a^j \in \mathcal{A}.
\]

This induces a duality pairing between cyclic cochains and cyclic chains. This also extends to a duality pairing between \( C_0^\bullet(\mathcal{A}) \) (resp., \( C^\bullet(\mathcal{A}) \)) and \( C^\bullet_0(\mathcal{A}) \) (resp., \( C^\bullet(\mathcal{A}) \)). For instance, for \( \varphi = (\varphi_{2q+i}) \in C^0_{[i]}(\mathcal{A}) \) and \( \eta = (\eta_{2q+i}) \in C_{[i]}(\mathcal{A}) \), we have

\[
\langle \varphi, \eta \rangle = \sum_{q \geq 0} \langle \varphi_{2q+i}, \eta_{2q+i} \rangle.
\]

Moreover, the operators \( b, T, A, B_0, B, S \) on cochains are the transposes of the corresponding operators on chains. In particular, the aforementioned pairings descend to duality pairings between the cohomology space \( \text{HC}^\bullet(\mathcal{A}) \) (resp., \( \text{HP}^\bullet(\mathcal{A}), \text{HP}^0_*(\mathcal{A}) \)) and the homology space \( \text{HC}_\bullet(\mathcal{A}) \) (resp., \( \text{HP}_*(\mathcal{A}), \text{HP}^0_*(\mathcal{A}) \)).

**Example 3.2.** Let \( \mathcal{A} = C^\infty(M) \), where \( M \) is a closed manifold. Let \( \Omega^m(M) \) be the space of differential forms on \( M \) of degree \( m \). There is a natural linear map \( \alpha_M : C_m(\mathcal{A}) \to \Omega^m(M) \) given by

\[
\alpha_M(f^0 \otimes \cdots \otimes f^m) = \frac{1}{m!} f^0 df^1 \wedge \cdots \wedge df^m, \quad f^i \in C^\infty(M).
\]

It is a mixed-complex morphism from the cyclic mixed chain-complex of \( \mathcal{A} \) to \( (\Omega^\bullet(M), 0, d) \), where \( d \) is de Rham’s differential. This thus gives rise to a morphism of chain-complexes from the
periodic complex \((\Omega^\bullet(C^\infty(M)), b + B)\) to the de Rham complex \((\Omega^\bullet(M), d)\), where \(\Omega^i(M) = \bigoplus_{q=0}^{\infty} \Omega^{2q+i}(M)\), \(i = 0, 1\). In particular, we get a linear map,

\[
\alpha_M : HP^\bullet(C^\infty(M)) \to H^\bullet(M, \mathbb{C}), \quad H^{[i]}(M, \mathbb{C}) := \bigoplus_{q \geq 0} H^{2q+i}(M, \mathbb{C}), \quad i = 0, 1,
\]

where \(H^{2q+i}(M, \mathbb{C})\) is the de Rham cohomology of \(M\) in degree \(2q + i\). This map is the transpose of the map \(\alpha^M\) defined by (3.11). Thus, for any de Rham homology class \(\omega \in H^{[i]}(M, \mathbb{C})\) and any periodic cyclic homology class \(\eta \in HP^i(C^\infty(M))\), we have

\[
(3.21) \quad \langle \alpha^M(\omega), \eta \rangle = \langle \omega, \alpha_M(\eta) \rangle,
\]

where the pairing on the r.h.s. is the pairing between de Rham homology and de Rham cohomology.

3.3. Chern character in cyclic homology. Given a positive integer \(N\), the trace \(\text{tr}\) on the algebra \(M_N(A) = A \otimes M_N(\mathbb{C})\) gives rise to a linear map \(\text{tr} : C_m(M_N(A)) \to C_m(A)\) defined by

\[
\text{tr} \left( (a^0 \otimes \mu^0) \otimes \cdots \otimes (a^m \otimes \mu^m) \right) = \text{tr} \left[ \mu^0 \cdots \mu^m \right] a^0 \otimes \cdots \otimes a^m, \quad a^j \in A, \ \mu^j \in M_N(\mathbb{C}).
\]

This map is compatible with the operators \(b, T, \) and \(B\) and yields isomorphisms at the level of cyclic homology and periodic cyclic homology (see [Co3, Lo]).

The Chern character in cyclic homology [Co3, GS, Lo] is defined as follows. Let \(e\) be an idempotent in \(M_N(A)\) and define the even normalized chain \(\text{Ch}(e) = (\text{Ch}_2(e))_{q \geq 0} \in C^0_{[0]}(A)\) by

\[
(3.22) \quad \text{Ch}_0(e) = \text{tr}[e], \quad \text{Ch}_2q(e) = (-1)^q \frac{(2q)!}{q!} \text{tr} \left[ e \otimes e \otimes \cdots \otimes e \right], \quad q \geq 1.
\]

It can be checked that \(\text{Ch}(e) = 0\) in a cocycle in \(C^0_{[0]}(A)\) whose class in \(\text{HP}^0_0(A) \simeq \text{HP}^0(A)\) depends only on the \(K\)-theory class of \(e\). In fact, this gives rise to an additive map Chern character map,

\[
(3.23) \quad \text{Ch} : K_0(A) \to \text{HP}^0(A).
\]

Incidentally, given any finite generated projective module \(E\) over \(A\), we define its Chern character by

\[
(3.24) \quad \text{Ch}(E) = \text{class of } \text{Ch}(e) \text{ in } \text{HP}^0(A),
\]

where \(e\) is any idempotent in some algebra \(M_N(A)\), \(N \geq 1\), such that \(E \simeq eA^N\).

Composing the Chern character map \(\text{Ch} : K_0(A) \to \text{HP}^0(A)\) with the duality pairing between \(\text{HP}^0(A)\) and \(\text{HP}^0(A)\) provides us with a pairing between \(\text{HP}^0(A)\) and \(K_0(A)\). Furthermore, given any cyclic 2\(q\)-cocycle \(\varphi\) it can be shown (see, e.g., [PW1, Remark 6.4]) that

\[
(3.25) \quad \langle \varphi, \text{Ch}(e) \rangle = (-1)^q \frac{(2q)!}{q!} \text{tr} \left[ e^\otimes(2q+1) \right] \quad \forall e \in M_N(A), \ e^2 = e.
\]

Therefore, the pairing between \(\text{HP}^0(A)\) and \(K_0(A)\) given by the above Chern character map agrees with the original pairing defined by Connes [Co3, Co4].

Example 3.3. Suppose now that \(A = C^\infty(M)\), where \(M\) is closed manifold, and let \(e \in M_m(C^\infty(M))\), \(e^2 = e\). Consider the vector bundle \(E = \text{ran } e\), which we regard as a subbundle of the trivial vector bundle \(E_0 = M \times \mathbb{C}^m\). By Serre-Swan theorem any vector bundle over \(M\) is isomorphic to a vector bundle of this form. We then equip \(E\) with the Grassmannian connection \(\nabla^E\) defined by \(e\), i.e.,

\[
\nabla^E_X \xi(x) = e(x) \cdot ((X \xi_j)(x)) \quad \text{for all } X \in C^\infty(M, TM) \text{ and } \xi = (\xi_j) \in C^\infty(M, E).
\]

As the curvature of \(\nabla^E\) is \(F^E = e(de)^2 = e(de)^2 e\), it can be checked that

\[
(3.26) \quad \alpha_M(\text{Ch}(E)) = \alpha_M(\text{Ch}(e)) = \text{Ch}(F^E) = \text{Ch}(E) \quad \text{in } H^{[0]}(M).
\]
3.4. Locally convex algebras. In various geometric situations we often work with algebras that carry natural locally convex algebra topologies (e.g., \( A = C^\infty(M) \), where \( M \) is a smooth manifold). In such cases it is often more convenient to work with continuous versions of cyclic cohomology (see [Co3]). We shall now briefly recall how the corresponding cyclic cohomologies and homologies are defined in such a context.

We shall now assume that \( A \) is a (unital) locally convex algebra, i.e., it carries a locally convex space topology with respect to which its product is continuous. Given \( m \in \mathbb{N}_0 \), we denote by \( \mathcal{C}^m(A) \) the space of *continuous* cochains on \( A \), i.e., continuous \((m+1)\)-linear forms on \( A^{m+1} \). All the operators \( b, T, A, B_0, B, S \) introduced earlier preserve \( \mathcal{C}^\bullet(A) \). Therefore, all the above-mentioned results for arbitrary cochains hold *verbatim* for continuous cochains. In particular, we get cyclic mixed complexes \( (\mathcal{C}^\bullet(A), b, B) \) and \( (\mathcal{C}^\bullet_0(A), b, B) \), where \( \mathcal{C}^\bullet_0(A) \) is the space of normalized continuous cochains. This then gives rise to the following cochain-complexes:

- The cyclic complex \( (\mathcal{C}^\bullet_0(A), b), \) where \( \mathcal{C}^\bullet_0(A) \) is the space of cyclic continuous cochains.
- The periodic cyclic complex \( (\mathcal{C}^\bullet[0](A), b + B), \) where \( \mathcal{C}^\bullet[0](A) = \bigoplus_{q \geq 0} \mathcal{C}^{2q+i}(A), \ i = 0, 1. \)
- The normalized periodic cyclic complex \( (\mathcal{C}^\bullet_0[0](A), b + B), \) where \( \mathcal{C}^\bullet_0[0](A) = \bigoplus_{q \geq 0} \mathcal{C}^{2q+i}_0(A). \)

The respective cohomologies of these complexes are denoted by \( \mathcal{H}C^\bullet(A), \mathcal{H}P^\bullet(A), \) and \( \mathcal{H}P^\bullet_0(A). \) Furthermore, the analogues of (3.8) and (3.10) hold. Namely,

\[
\lim_{m \to \infty} (\mathcal{H}C^{2q+i}(A), S) = \mathcal{H}P^\bullet(A) \quad \text{and} \quad \mathcal{H}P^\bullet_0(A) \simeq \mathcal{H}P^\bullet(A).
\]

The corresponding chains are defined by replacing the algebraic tensor product \( \otimes \) (over \( \mathbb{C} \)) by the projective topological tensor product \( \widehat{\otimes} \) (see [Gr, Tr]). Thus, the space of \( m \)-chains is

\[
\mathcal{C}_m(A) = A^{\widehat{\otimes}(m+1)} = \bigwedge^{m+1} A.
\]

All the operators \( b, T, A, B_0, B, S \) on algebraic chains extends continuously to operators on topological chains. Therefore, all the results on algebraic chains mentioned continue to hold for topological chains. In particular, we obtain a mixed chain-complex \( (\mathcal{C}^\bullet(A), b, B) \). This gives rise to a cyclic complex \( (\mathcal{C}^\bullet_0(A), b) \) and a periodic cyclic complex \( (\mathcal{C}^\bullet[0](A), b + B) \). Here \( \mathcal{C}^\bullet_0(A) \) is the space of cyclic topological chains and \( \mathcal{C}^\bullet[0](A) = \bigoplus_{q \geq 0} \mathcal{C}^{2q+i}(A), \ i = 0, 1. \) The respective homologies of these chain-complexes are denoted by \( \mathcal{H}C^\bullet(A) \) and \( \mathcal{H}P^\bullet(A). \)

The mixed complex \( (\mathcal{C}^\bullet(A), b, B) \) is normalized as follows. The space of normalized \( m \)-chains is given by the topological quotient,

\[
\mathcal{C}^\bullet_0(A) = \mathcal{C}^\bullet(A)/\mathcal{N}^\bullet_0(A).
\]

This provides us with a normalized mixed complex \( (\mathcal{C}^\bullet_0(A), b, B) \) and a normalized periodic cyclic chain-complex \( (\mathcal{C}^\bullet_0[0](A), b + B), \) where \( \mathcal{C}^\bullet_0[0](A) = \bigoplus_{q \geq 0} \mathcal{C}^{2q+i}_0(A), \ i = 0, 1. \) The homology of the normalized periodic complex is denoted by \( \mathcal{H}P^\bullet_0(A). \)

Finally, the inclusion of \( \mathcal{C}^\bullet_0(A) \) into \( \mathcal{C}^\bullet(A) \) gives rise to an embedding of \( \mathcal{H}P^\bullet(A) \) into \( \mathcal{H}P^\bullet_0(A). \)

The mixed complex \( (\mathcal{C}^\bullet(A), b, B) \) is normalized as follows. The space of normalized \( m \)-chains is given by the topological quotient,

\[
\mathcal{C}^\bullet_0(A) = \mathcal{C}^\bullet(A)/\mathcal{N}^\bullet_0(A).
\]

This provides us with a normalized mixed complex \( (\mathcal{C}^\bullet_0(A), b, B) \) and a normalized periodic cyclic chain-complex \( (\mathcal{C}^\bullet_0[0](A), b + B), \) where \( \mathcal{C}^\bullet_0[0](A) = \bigoplus_{q \geq 0} \mathcal{C}^{2q+i}_0(A), \ i = 0, 1. \) The homology of the normalized periodic complex is denoted by \( \mathcal{H}P^\bullet_0(A). \)

Example 3.4. Suppose that \( A = C^\infty(M) \), where \( M \) is a closed manifold. Then \( A \) carries a natural Fréchet algebra topology. The map \( C \to \varphi_C \) in Example 3.1 maps even/odd currents to *continuous* periodic (normalized) cochains. As proved by Connes [Co3], it descends to an isomorphism,

\[
\mathcal{H}P^\bullet(C^\infty(M)) \simeq H^\bullet(M, \mathbb{C}),
\]

where \( H^\bullet(M, \mathbb{C}) \) is the even/odd de Rham homology. Likewise, the map \( \alpha_M \) given by (3.20) is continuous and extends to \( \mathcal{C}^\bullet_0(C^\infty(M)) \) and gives rise to an isomorphism,

\[
\mathcal{H}P^\bullet(C^\infty(M)) \simeq H^\bullet(M, \mathbb{C}),
\]

where \( H^\bullet(M, \mathbb{C}) \) is the even/odd de Rham cohomology. Furthermore, under the above isomorphisms the pairing (3.19) is the usual duality pairing between de Rham homology and de Rham cohomology.
4. The Connes-Chern Character of a Twisted Spectral Triple

In this section, we shall recall the construction the Connes-Chern character of a twisted spectral triple and how this enables us to compute the associated index map \([CM3, PW1]\). This extends to twisted spectral triples the construction of the Connes-Chern character of an ordinary spectral triple by Connes \([Co3]\). The exposition follows closely that of \([PW1]\).

Let \((\mathcal{A}, \mathcal{H}, D)\) be a twisted spectral triple. We assume that \((\mathcal{A}, \mathcal{H}, D)\) is \(p\)-summable for some \(p \geq 1\), that is,

\[
\text{Tr} \left| D \right|^{-p} < \infty.
\]

In what follows, letting \(\mathcal{L}^1(\mathcal{H})\) be the ideal of trace-class operators on \(\mathcal{H}\), we denote by \(\text{Str} \) its supertrace, i.e., \(\text{Str}[T] = \text{Tr}[\gamma T]\), where \(\gamma := \text{id}_{\mathcal{H}^+} - \text{id}_{\mathcal{H}^-}\) is the \(\mathbb{Z}_2\)-grading of \(\mathcal{H}\).

**Definition 4.1.** Assume \(D\) is invertible and let \(q\) be an integer \(\geq \frac{1}{2}(p - 1)\). Then \(\tau_{2q}^{D,\sigma}\) is the \(2q\)-cochain on \(\mathcal{A}\) defined by

\[
\tau_{2q}^{D,\sigma}(a^0, \ldots, a^{2q}) = c_q \text{Str} \left( D^{-1}[D, a^0] \cdots D^{-1}[D, a^{2q}] \right) \quad \forall a^j \in \mathcal{A},
\]

where we have set \(c_q = \frac{1}{2}(-1)^q q!\).

**Remark 4.2.** The right-hand side of (4.2) is well defined since the \(p\)-summability condition (4.1) and the fact that \(q \geq \frac{1}{2}(p - 1)\) imply that

\[
D^{-1}[D, a^0] \cdots D^{-1}[D, a^{2q}] \in \mathcal{L}^1(\mathcal{H}) \quad \forall a^j \in \mathcal{A}.
\]

**Proposition 4.3** ([CM3, PW1]). Assume \(D\) is invertible and let \(q\) be an integer \(\geq \frac{1}{2}(p - 1)\). Then

1. The cochain \(\tau_{2q}^{D,\sigma}\) is a normalized cyclic cocycle whose class in \(\text{HP}^0(\mathcal{A})\) is independent of the value of \(q\).

2. For any finitely generated projective module \(\mathcal{E}\) over \(\mathcal{A}\) and \(\sigma\)-connection on \(\mathcal{E}\),

\[
\text{ind} D_{\mathcal{E}} = \left\langle \tau_{2q}^{D,\sigma}, \text{Ch}(\mathcal{E}) \right\rangle,
\]

where \(\text{Ch}(\mathcal{E})\) is the Chern character of \(\mathcal{E}\) in cyclic homology.

When \(D\) is not invertible, we can reduce to the invertible case by passing to the unital invertible double of \((\mathcal{A}, \mathcal{H}, D)\) as follows.

Consider the Hilbert space \(\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}\), which we equip with the \(\mathbb{Z}_2\)-grading given by

\[
\tilde{\gamma} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix},
\]

where \(\gamma\) is the grading operator of \(\mathcal{H}\). On \(\tilde{\mathcal{H}}\) consider the selfadjoint operator,

\[
\tilde{D} = \begin{pmatrix} D & 1 \\ 1 & -D \end{pmatrix}, \quad \text{dom}(\tilde{D}) := \text{dom}(D) \oplus \text{dom}(D).
\]

Noting that

\[
\tilde{D}^2 = \begin{pmatrix} D^2 + 1 & 0 \\ 0 & D^2 + 1 \end{pmatrix},
\]

we see that \(\tilde{D}\) is invertible and \(\left| \tilde{D} \right|^{-p}\) is trace-class. Let \(\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}\) be the unitalization of \(\mathcal{A}\) whose product and involution are given by

\[
(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda \mu), \quad (a, \lambda)^* = (a^*, \bar{\lambda}), \quad a, b \in \mathcal{A}, \quad \lambda, \mu \in \mathbb{C}.
\]

The unit of \(\tilde{\mathcal{A}}\) is \(1_{\tilde{\mathcal{A}}} = (0, 1)\). Thus, identifying any element \(a \in \mathcal{A}\) with \((a, 0)\), any element \(\tilde{a} = (a, \lambda) \in \tilde{\mathcal{A}}\) can be uniquely written as \((a, \lambda) = a + \lambda 1_{\tilde{\mathcal{A}}}\). We represent \(\tilde{\mathcal{A}}\) in \(\mathcal{H}\) using the representation \(\pi : \tilde{\mathcal{A}} \to \mathcal{L}(\mathcal{H})\) given by

\[
\pi(1_{\tilde{\mathcal{A}}}) = 1, \quad \pi(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \forall a \in \mathcal{A}.
\]

In addition, we extend the automorphism \(\sigma\) into the automorphism \(\tilde{\sigma} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}\) given by

\[
\tilde{\sigma}(a + \lambda 1_{\tilde{\mathcal{A}}}) = \sigma(a) + \lambda 1_{\tilde{\mathcal{A}}} \quad \forall (a, \lambda) \in \mathcal{A} \times \mathbb{C}.
\]
It can be verified that any twisted commutator $[\tilde{D}, \pi(\tilde{a})]_\sigma$, $\tilde{a} \in \tilde{A}$, is bounded. We then deduce that $(\tilde{A}, \mathcal{H}, \tilde{D})_\sigma$ is a $p$-summable twisted spectral triple. Moreover, as $\tilde{D}$ is invertible we may define the normalized cyclic cocycles $\tau^{D,\sigma}_{2q}$, $q \geq \frac{1}{2}(p-1)$.

**Definition 4.4.** Let $q \geq \frac{1}{2}(p-1)$. Then $\tau^{D,\sigma}_{2q}$ is the $2q$-cochain on $\mathcal{A}$ defined by

$$\tau^{D,\sigma}_{2q}(a^0, \ldots, a^{2q}) = c_q \text{Str} \left( \tilde{D}^{-1}[\tilde{D}, \pi(a^0)]_\sigma \cdots \tilde{D}^{-1}[\tilde{D}, \pi(a^{2q})]_\sigma \right) \quad \forall a^j \in \mathcal{A}.$$ 

**Remark 4.5.** The cochain $\tau^{D,\sigma}_{2q}$ is the restriction to $\mathcal{A}^{2q+1}$ of the cochain $\tau^{\tilde{D},\sigma}_{2q}$. Note that, as the restriction to $\mathcal{A}$ of the representation $\pi$ is not unital, unlike in the invertible case, we don’t obtain a normalized cochain.

**Proposition 4.6** ([PW1]). Let $q$ be an integer $\geq \frac{1}{2}(p-1)$.

1. The cochain $\tau^{D,\sigma}_{2q}$ is a cyclic cocycle whose class in $\text{HP}^0(\mathcal{A})$ is independent of $q$.
2. If $D$ is invertible, then the cocycles $\tau^{D,\sigma}_{2q}$ and $\tau^{D,\sigma}_{2q}$ are cohomologous in $\text{HP}^0(\mathcal{A})$.
3. For any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and $\sigma$-connection on $\mathcal{E}$,

$$\text{ind } D_{\mathcal{E}} = \left\langle \tau^{D,\sigma}_{2q}, \text{Ch}(\mathcal{E}) \right\rangle.$$ 

All this leads us to the following definition.

**Definition 4.7.** The Connes-Chern character of $(\mathcal{A}, \mathcal{H}, D)_\sigma$, denoted by $\text{Ch}(D)_\sigma$, is defined as follows:

- If $D$ is invertible, then $\text{Ch}(D)_\sigma$ is the common class in $\text{HP}^0(\mathcal{A})$ of the cyclic cocycles $\tau^{D,\sigma}_{2q}$ and $\tau^{D,\sigma}_{2q}$, with $q \geq \frac{1}{2}(p-1)$.
- If $D$ is not invertible, then $\text{Ch}(D)_\sigma$ is the common class in $\text{HP}^0(\mathcal{A})$ of the cyclic cocycles $\tau^{D,\sigma}_{2q}$, $q \geq \frac{1}{2}(p-1)$.

**Remark 4.8.** When $\sigma = \text{id}$ the Connes-Chern character is simply denoted $\text{Ch}(D)$; this is the usual Connes-Chern Character of an ordinary spectral triple constructed by Connes [Co3].

With this definition in hand, the index formulas (4.3) and (4.4) can be merged onto the following result.

**Proposition 4.9** ([CM3, PW1]). For any Hermitian finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ and any $\sigma$-connection on $\mathcal{E}$,

$$\text{ind } D_{\mathcal{E}} = \left\langle \text{Ch}(D)_\sigma, \text{Ch}(\mathcal{E}) \right\rangle,$$

where $\text{Ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ in cyclic homology.

**Example 4.10** ([Co3, BF, Po1]). Let $(M^n, g)$ be a compact Riemannian manifold. The Connes-Chern character $\text{Ch}(\mathcal{D}_g)$ of the Dirac spectral triple $(C^\infty(M), L^2(M, \mathbb{S}), \mathcal{D}_g)$ is cohomologous to the even periodic cocycle $\varphi = (\varphi_{2q})_{q \geq 0}$ given by

$$\varphi_{2q}(f^0, \ldots, f^{2q}) = \frac{(2\pi)^{-q}}{2^q(2q)!} \int_M \tilde{A}(R^M) \wedge f^0 df^1 \wedge \ldots \wedge df^{2q}, \quad f^i \in C^\infty(M).$$

Together with (3.20), (3.22) and (3.26) this enables us to recover the index theorem of Atiyah-Singer [AS1, AS2] for Dirac operators.

**Remark 4.11.** The definitions of the cocycles $\tau^{D,\sigma}_{2q}$ and $\tau^{D,\sigma}_{2q}$ involve the usual (super)trace on trace-class operators, but this is not a local functional since it does vanish on finite rank operators. As a result this cocycle is difficult to compute in practice (see, e.g., [BF]). Therefore, it stands for reason to seek for a representative of the Connes-Chern character which is easier to compute. For ordinary spectral triples, and under further assumptions, such a representative is provided by the CM cocycle of Connes-Moscovici [CM2]. This cocycle is an even periodic cycle whose components are given by formulas which are **local** in the sense of noncommutative geometry. More precisely,
they involve a version for spectral triples of the noncommutative residue trace of Guillemin [Gu] and Wodzicki [Wo]. This provides us with the local index formula in noncommutative geometry. In Example 4.10 the cocycle \( \varphi = (\varphi_{2q}) \) given by (4.6) is precisely the CM cocycle of the Dirac spectral triple \((C^\infty(M), L^2(M, \mathcal{S}), \mathcal{D}_g)\) (see [CM2, Remark II.1] and [Po1]).

**Remark 4.12.** In the case of twisted spectral triples, Moscovici [Mo2] attempted to extend the local index formula to the setting of twisted spectral triples. He devised an Ansatz for such a local index formula and verified it in the special case of a ordinary spectral triples twisted by scaling automorphisms (see [Mo2] for the precise definition). Whether Moscovici's Ansatz holds for a larger class of twisted spectral triples still remains an open question to date. For instance, it is not known if Moscovici Ansatz holds for conformal deformations of ordinary spectral triples satisfying the local index formula in noncommutative geometry.

5. Twisted Spectral Triples over Locally Convex Algebras

In this section, we shall explain how to refine the construction of the Connes-Chern character for twisted spectral triples over locally convex algebras.

In what follows by *locally convex *-algebra we shall mean a *-algebra equipped with a locally convex topology with respect to which the product and involution are continuous.

**Definition 5.1.** A twisted spectral triple \((\mathcal{A}, \mathcal{H}, D)\) over a locally convex *-algebra \(\mathcal{A}\) is called smooth when the following conditions hold:

1. The representation of \(\mathcal{A}\) in \(\mathcal{L}(\mathcal{H})\) is continuous.
2. The map \(a \rightarrow [D, a]_\sigma\) is continuous from \(\mathcal{A}\) to \(\mathcal{L}(\mathcal{H})\).
3. The automorphism \(\sigma : \mathcal{A} \rightarrow \mathcal{A}\) is a homeomorphism.

**Example 5.2.** Let \((\mathcal{M}^n, g)\) be a closed spin Riemannian manifold of even dimension. Then the associated Dirac spectral triple \((C^\infty(M), L^2(M, \mathcal{S}), \mathcal{D}_g)\) is smooth.

**Example 5.3.** Any conformal deformation of a smooth ordinary spectral triple yields a smooth twisted spectral.

**Remark 5.4.** As we shall see in Section 7 the conformal Dirac spectral triple of [CM3] is smooth.

Throughout the rest of this subsection we let \((\mathcal{A}, \mathcal{H}, D)\) be a smooth twisted spectral triple which is \(p\)-summable for some \(p \geq 1\). We shall now show that the Connes-Chern character of \((\mathcal{A}, \mathcal{H}, D)\), which is originally defined as a class in \(\text{HP}^0(\mathcal{A})\), actually descends to a class in \(\text{HP}^0(\mathcal{A})\).

**Lemma 5.5.** Let \(q\) be any integer \(\geq \frac{1}{2}(p-1)\).

1. If \(D\) is invertible, then the cyclic cocycle \(\tau_{2q}^{D, \sigma}\) is continuous and its class in \(\text{HP}^0(\mathcal{A})\) is independent of \(q\).
2. The cyclic cocycle \(\tau_{2q}^{D, \sigma}\) is continuous and its class in \(\text{HP}^0(\mathcal{A})\) is independent of \(q\).

**Proof.** Assume that \(D\) is invertible and let \(q\) be an integer \(\geq \frac{1}{2}(p-1)\). By assumption the map \(a \rightarrow [D, a]_\sigma\) is continuous from \(\mathcal{A}\) to \(\mathcal{L}(\mathcal{H})\). Combining this with Hölder’s inequality for Schatten ideals we deduce that the map \((a^0, \ldots, a^{2q}) \rightarrow \gamma D^{-1}[D, a^0] \cdots D^{-1}[D, a^{2q}]\) is continuous from \(\mathcal{A}^{2q+1}\) to \(\mathcal{L}(\mathcal{H})\). As \(\tau_{2q}^{D, \sigma}\) is (up to a multiple constant) the composition of this map with the operator trace, we deduce that \(\tau_{2q}^{D, \sigma}\) is a continuous cochain.

Moreover, by Lemma 7.4 and Lemma 7.5 of [PW1, ] we may write

\[
\tau_{2q+2}^{D, \sigma} = b(\varphi_{2q+1} - \psi_{2q+1}), \quad \tau_{2q}^{D, \sigma} = -B(\varphi_{2q+1} - \psi_{2q+1}),
\]

where, up to normalization constants, the cochains \(\varphi_{2q+1}\) and \(\psi_{2q+1}\) are given by

\[
\varphi_{2q+1}(a^0, \ldots, a^{2q+1}) = \text{Str}(a^0 D^{-1}[D, a^1] \cdots D^{-1}[D, a^{2q+1}]_\sigma),
\]
\[
\psi_{2q+1}(a^0, \ldots, a^{2q+1}) = \text{Str}(\sigma(a^0)[D, a^1]_\sigma D^{-1} \cdots [D, a^{2q+1}]_\sigma D^{-1}), \quad a^j \in \mathcal{A}.
\]

Note that \(\varphi_{2q+1}\) and \(\psi_{2q+1}\) are normalized cochains. Moreover, in the same way as with the cocycle \(\tau_{2q}^{D, \sigma}\) we can show that these cochains are continuous. As (5.1) implies that

\[
\tau_{2q+2}^{D, \sigma} - \tau_{2q}^{D, \sigma} =
\]
$(B + b)(\varphi_{2q+1} - \psi_{2q+1})$, we then deduce that $\tau_{2q}^{D,\sigma}$ and $\tau_{2q+2}^{D,\sigma}$ define the same class in $\text{HP}^0(\mathcal{A})$. Alternatively, using (3.7) we get

$$
(5.2) \quad S\tau_{2q}^{D,\sigma} = -SB(\varphi_{2q+1} - \psi_{2q+1}) = b(\varphi_{2q+1} - \psi_{2q+1}) = \tau_{2q+2}^{D,\sigma} \quad \text{in} \quad \text{HC}^{2q+2}(\mathcal{A}).
$$

In case $D$ is not invertible, we observe that the unitalization $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ is a locally convex *-algebra with respect to the direct sum topology. It then can be checked that the invertible double $(\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{D})$ considered in Section 4 is a smooth twisted spectral triple. Therefore, by part (1) the cocycle $\tau_{2q}^{D,\sigma}$ is continuous. As the inclusion of $\mathcal{A}$ into $\tilde{\mathcal{A}}$ is continuous and $\tau_{2q}^{D,\sigma}$ is the restriction to $\mathcal{A}$ of $\tau_{2q}^{\tilde{D},\sigma}$, we then see that $\tau_{2q}^{\tilde{D},\sigma}$ is a continuous cocycle on $\tilde{\mathcal{A}}$. Note also that $S\tau_{2q}^{D,\sigma}$ is the restriction to $\mathcal{A}$ of $S\tau_{2q}^{\tilde{D},\sigma}$ (cf. Eq. (3.6)). Therefore using (5.2) we deduce that $S\tau_{2q}^{D,\sigma} = \tau_{2q+2}^{D,\sigma}$ in $\text{HC}^{2q+2}(\mathcal{A})$, which implies that $S\tau_{2q}^{D,\sigma}$. and $\tau_{2q+2}^{D,\sigma}$ define the same class in $\text{HP}^0(\mathcal{A})$. The proof is complete. \qed

In addition, we will also need the following version for smooth twisted spectral triples of [PW1, Proposition C.1] on the homotopy invariance of the Connes-Chern character of a twisted spectral triple.

**Lemma 5.6.** Assume $D$ is invertible and consider an operator homotopy of the form,

$$
D_t = D + V_t, \quad 0 \leq t \leq 1,
$$

where $(V_t)_{0 \leq t \leq 1}$ is a $C^1$-family of selfadjoint operators in $\mathcal{L}(\mathcal{H})$ such that $D_t$ is invertible for all $t \in [0, 1]$ and $(D_t^{-1})_{0 \leq t \leq 1}$ is a bounded family in $\mathcal{L}^p(\mathcal{H})$. Then

1. $(\mathcal{A}, \mathcal{H}, D_t)_{\sigma}$ is a smooth $p$-summable twisted spectral triple for all $t \in [0, 1]$,
2. For any $q \geq \frac{1}{2}(p + 1)$, the cocycles $\tau_{2q}^{D_t,\sigma}$ and $\tau_{2q}^{D_t,\sigma}$ are cohomologous in $\text{HC}^{2q}(\mathcal{A})$, and hence define the same class in $\text{HP}^0(\mathcal{A})$.

**Proof.** We know from [PW1, Proposition C.1] that $(\mathcal{A}, \mathcal{H}, D_t)_{\sigma}$ is a $p$-summable twisted spectral triple for all $t \in [0, 1]$ and, for any $q \geq \frac{1}{2}(p + 1)$, the cocycles $\tau_{2q}^{D,\sigma}$ and $\tau_{2q}^{D,\sigma}$ are cohomologous in $\text{HC}^{2q}(\mathcal{A})$. Therefore, we only need to show that the twisted spectral triples $(\mathcal{A}, \mathcal{H}, D_t)_{\sigma}$, $t \in [0, 1]$, are smooth and the cocycles $\tau_{2q}^{D,\sigma}$ and $\tau_{2q}^{D,\sigma}$ differ by the coboundary of a continuous cyclic cochain.

By assumption $(\mathcal{A}, \mathcal{H}, D)_{\sigma}$ is a smooth twisted spectral triple. In particular, the representation of $\mathcal{A}$ in $\mathcal{L}(\mathcal{H})$ is continuous and the automorphism $\sigma$ is a homeomorphism. Moreover, for all $t \in [0, 1]$ and $a \in \mathcal{A}$,

$$
[D_t, a]_{\sigma} = [D, a]_{\sigma} + V_t(a - \sigma(a)V_t).
$$

We then see that the map $(t, a) \rightarrow [D_t, a]_{\sigma}$ is continuous from $[0, 1] \times \mathcal{A}$ to $\mathcal{L}(\mathcal{H})$. It then follows that $(\mathcal{A}, \mathcal{H}, D_t)_{\sigma}$ is a smooth twisted spectral triple for all $t \in [0, 1]$.

Let $q$ be an integer $\geq \frac{1}{2}(p + 1)$. In [PW1] the explicit homotopy between $\tau_{2q}^{D,\sigma}$ and $\tau_{2q}^{D,\sigma}$ is realized as follows. For $t \in [0, 1]$ and $a \in \mathcal{A}$ set

$$
\delta_t(a) = D_t^{-1}[\tilde{V}_t D_t^{-1}, \sigma(a)] D_t.
$$

In addition, for $j = 0, \ldots, 2q + 1$ we set

$$
\alpha_j^t(a) = a \quad \text{if} \quad j \quad \text{even} \quad \text{and} \quad \alpha_j^t(a) = D_t^{-1}\sigma(a)D_t \quad \text{if} \quad j \quad \text{odd}.
$$

We note that

$$
\delta_t(a) = D_t^{-1}(\tilde{V}_t\alpha_j^t(a) - \sigma(a)\tilde{V}_t) \quad \text{and} \quad \alpha_j^t(a) = a - D_t^{-1}[D_t, a]_{\sigma}
$$

In particular we see that the maps $(t, a) \rightarrow \alpha_j^t(a)$ are continuous from $[0, 1] \times \mathcal{A}$ to $\mathcal{L}(\mathcal{H})$. Moreover, it is shown in [PW1, Appendix C] that $(D_t^{-1})_{0 \leq t \leq 1}$ is a $C^1$-family in $\mathcal{L}^p(\mathcal{H})$. Therefore, we also see that the map $(t, a) \rightarrow \delta_t(a)$ is continuous from $[0, 1] \times \mathcal{A}$ to $\mathcal{L}^p(\mathcal{H})$.

Bearing this mind it is shown in [PW1] that

$$
(5.3) \quad \tau_{2q}^{D_t,\sigma} - \tau_{2q}^{D_0,\sigma} = B_t,
$$

where $B_t$ is a boundary.
where $\eta$ is the Hochschild $(2q+1)$-cocycle given by
\[
\eta(a^0, \ldots, a^{2q+1}) = c_q \sum_{j=0}^{2q+1} \int_0^1 \text{Str} \left( \alpha_j(a^0) D_t^{-1} |D_t, a^1| \cdots \delta_t(a^q) \cdots D_t^{-1} [D_t, a^{2q+1}] \right) dt,
\]
where $c_q$ is some normalization constant. It follows from all the previous observations and the fact that $q \geq \frac{1}{2}(p-1)$ that all the maps
\[
(t, a^0, \ldots, a^{2q+1}) \to \alpha_j(a^0) D_t^{-1} [D_t, a^1] \cdots \delta_t(a^q) \cdots D_t^{-1} [D_t, a^{2q+1}]
\]
are continuous from $[0, 1] \times A^{2q+2}$ to $L^1(H)$. Therefore the above formula for $\eta$ defines a continuous Hochschild cocycle on $A^{2q+2}$. Moreover, using (5.2) and applying the operator $S$ to both sides of (5.3) gives
\[
S_{2q+2} - S_{2q+2} = SB\eta - b\eta = 0 \quad \text{in } HC^{2q+2}(A).
\]
This proves the 2nd part of the lemma and completes the proof. □

When $D$ is invertible we know that, for $q \geq \frac{1}{2}(p-1)$, the cyclic cocycles $\tau_{2q}^{D,\sigma}$ and $\tau_{2q}^{D,\sigma}$ define the same class in $HP^0(A)$. If we argue along the same lines as that of the proof in [PW1] and use Lemma 5.6, then we obtain the following result.

**Lemma 5.7.** Assume $D$ is invertible and let $q$ be a an integer $\geq \frac{1}{2}(p-1)$. Then the cyclic cocycles $\tau_{2q}^{D,\sigma}$ and $\tau_{2q}^{D,\sigma}$ define the same class in $HP^0(A)$.

Granted Lemma 5.5 and Lemma 5.7 the Connes-Chern character descends to a class in $HP^0(A)$ as follows.

**Definition 5.8.** $\text{Ch}(D)_{\sigma}$ is the common class in $HP^0(A)$ of the cocycles $\tau_{2q}^{D,\sigma}$, $q \geq \frac{1}{2}(p-1)$, and of the cocycles $\tau_{2q}^{D,\sigma}$, $q \geq \frac{1}{2}(p-1)$, when $D$ is invertible.

**Proposition 5.9.** Let $(A, H, D)_{\sigma}$ be a $p$-summable smooth twisted spectral triple. Then

1. The class $\text{Ch}(D)_{\sigma}$ agrees with the Connes-Chern character $\text{Ch}(D)_{\sigma}$ under the linear map $\text{HP}^0(A) \to \text{HP}^0(A)$ induced by the inclusion of $C^*(A)$ into $C^*(A)$.

2. For any Hermitian finitely generated projective module $E$ over $A$ and $\sigma$-connection on $E$,
\[
\text{ind } D_{eE} = \langle \text{Ch}(D)_{\sigma}, \text{Ch}(E) \rangle,
\]
where $\text{Ch}(E)$ is the Chern character of $E$ seen as a class in $\text{HP}_0(A)$.

**Proof.** The first part is immediate since by their very definitions of $\text{Ch}(D)_{\sigma}$ and $\text{Ch}(D)_{\sigma}$ are represented by the same cyclic cocycles. As for the 2nd part, consider an idempotent $e$ in some $M_N(A)$, $N \geq 1$ such that $E \simeq eA^N$. Then, for any $q \geq \frac{1}{2}(p-1),$
\[
\langle \text{Ch}(D)_{\sigma}, \text{Ch}(E) \rangle = \langle \tau_{2q}^{D,\sigma}, \text{Ch}(e) \rangle = \langle \text{Ch}(D)_{\sigma}, \text{Ch}(E) \rangle.
\]
Combining this with Proposition 4.9 gives the result. □

**Remark 5.10.** In what follows by Connes-Chern character of a smooth $(p$-summable) twisted spectral triple $(A, H, D)_{\sigma}$ we shall mean the cohomology class $\text{Ch}(D)_{\sigma} \in \text{HP}^0(A)$.

## 6. Invariance of the Connes-Chern Character

In preparation for the next section, we prove in this section the invariance of the Connes-Chern character under equivalences and conformal deformations of twisted spectral triples.
6.1. Equivalence of twisted spectral triples. The equivalence of two twisted spectral triples over the same algebra is defined as follows.

Definition 6.1. Let \((\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}\) and \((\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}\) be twisted spectral triples over the same algebra. For \(i = 1, 2\) let us denote by \(\pi_i\) the representation of \(\mathcal{A}\) into \(\mathcal{H}_i\). Then we say that \((\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}\) and \((\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}\) are equivalent when there is a unitary operator \(U : \mathcal{H}_1 \to \mathcal{H}_2\) such that

\[
\begin{align*}
D_1 &= U^* D_2 U, \\
\pi_1(a) &= U^* \pi_2(a) U \quad \text{and} \quad \pi_1(\sigma_1(a)) = U^* \pi_2(\sigma_2(a)) U \quad \text{for all } a \in \mathcal{A}.
\end{align*}
\]

Remark 6.2. We may also define the equivalence of a pair of spectral triples \((\mathcal{A}_1, \mathcal{H}_1, D_1)_{\sigma_1}\) and \((\mathcal{A}_2, \mathcal{H}_2, D_2)_{\sigma_2}\) with \(\mathcal{A}_1 \neq \mathcal{A}_2\) by requiring the existence of a \(*\)-algebra isomorphism \(\psi : \mathcal{A}_1 \to \mathcal{A}_2\) and replacing the condition (6.2) by

\[
\pi_1(a) = U^* \pi_2(\psi(a)) U \quad \text{and} \quad \pi_1(\sigma_1(a)) = U^* \pi_2(\psi(\sigma(a))) U \quad \text{for all } a \in \mathcal{A}_1.
\]

Proposition 6.3. Let \((\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}\) and \((\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}\) be equivalent twisted spectral triples that are \(p\)-summable for some \(p \geq 1\). In addition, let \(q\) be an integer \(\geq \frac{1}{2}(p - 1)\). Then

1. The cyclic cocycles \(\tau^{D_1, \sigma_1}_{2q}\) and \(\tau^{D_2, \sigma_2}_{2q}\) agree.
2. The same result holds for the cocycles \(\tau^{D_1, \sigma_1}_{2q}\) and \(\tau^{D_2, \sigma_2}_{2q}\) when \(D_1\) and \(D_2\) are invertible.
3. The twisted spectral triples \((\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}\) and \((\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}\) have same Connes-Chern character.

Proof. The last part is an immediate consequence of the first two parts, so we only need to prove these parts. In addition, we note that the invertible doubles of \((\mathcal{A}, \mathcal{H}_1, D_1)_{\sigma_1}\) and \((\mathcal{A}, \mathcal{H}_2, D_2)_{\sigma_2}\) are equivalent, where the equivalence is implemented by the unitary operator \(U \otimes U\) acting on \(\mathcal{H} = \mathcal{H} \oplus \mathcal{H}\). Therefore, it is enough to assume that \(D_1\) and \(D_2\) are invertible and prove the 2nd part.

Under the aforementioned assumption and using (6.1)–(6.2) we see that, for all \(a \in \mathcal{A}\),

\[
D_1^{-1}[D_1, a]_{\sigma_1} = U^* D_2^{-1}[D_2, a]_{\sigma_2} U.
\]

Therefore, for all \(a^i \in \mathcal{A}\), we have

\[
\tau^{D_1, \sigma_1}_{2q}(a^0, \ldots, a^{2q}) = \frac{1}{2} (-1)^q \frac{q!}{(2q)!} \text{Str} \left\{ U^* D_2^{-1}[D_2, a^0]_{\sigma_2} U \cdots U^* D_2^{-1}[D_2, a^{2q}]_{\sigma_2} U \right\} = \tau^{D_2, \sigma_2}_{2q}(a^0, \ldots, a^{2q}).
\]

This proves the result. \(\square\)

6.2. Conformal deformations of twisted spectral triples. For future purpose it will be useful to extend to the setting of twisted spectral triples the conformal deformations of ordinary spectral triples. Let \((\mathcal{A}, \mathcal{H}, D)_{\sigma}\) be a twisted spectral triple and \(k\) a positive element of \(\mathcal{A}\). We let \(\hat{\sigma} : \mathcal{A} \to \mathcal{A}\) be the automorphism of \(\mathcal{A}\) given by

\[
\hat{\sigma}(a) = k \sigma(k a k^{-1}) k^{-1} \quad \forall a \in \mathcal{A}.
\]

Proposition 6.4. \((\mathcal{A}, \mathcal{H}, kDk)_{\hat{\sigma}}\) is a twisted spectral triple.

Proof. We only need to check the boundedness of twisted commutators \([kDk, a]_{\hat{\sigma}}, a \in \mathcal{A}\). To see this we note that

\[
[kDk, a]_{\hat{\sigma}} = k Da \quad \text{and} \quad \hat{\sigma}(a) k D k = k D(k a k^{-1}) - (k^{-1} \hat{\sigma}(a) k) D k = k[D, a]_{\sigma} k.
\]

As the twisted commutator \([D, a]_{\sigma}\) is bounded, it follows that \([kDk, a]_{\hat{\sigma}}\) is bounded as well. The proof is complete. \(\square\)

The following shows that the Connes-Chern character is invariant under conformal deformations.
Proposition 6.5. Assume that \((A, H, D)_\sigma\) is \(p\)-summable for some \(p \geq 1\). Then, for any positive element \(k \in A\),

\[
\text{Ch}(kDk)_\sigma = \text{Ch}(D)_\sigma \in \text{HP}^0(A).
\]

Remark 6.6. The above result is proved in [CM3] in the special case \(\sigma = \text{id}\) and \(D\) is invertible.

Proof of Proposition 6.5. Set \(D_k = kDk\). We shall first prove the result when \(D\) is invertible, as the proof is simpler in that case. Given an integer \(q > \frac{1}{2}(p-1)\) let \(a^j \in A, j = 0, \ldots, 2q\). Using (4.2) and (6.3) we get

\[
\begin{align*}
\tau_{2q}^{D_k, \sigma}(a^0, \ldots, a^{2q}) &= c_\sigma \text{Str} \{ D_k^{-1}[D_k, a^0]_{\sigma} \cdots D_k^{-1}[D_k, a^{2q}]_{\sigma} \} \\
&= c_\sigma \text{Str} \{ (k^{-1}D^{-1}[D, ka^0])_{\sigma} \cdots (k^{-1}D^{-1}[D, ka^{2q}])_{\sigma} \} \\
&= c_\sigma \text{Str} \{ D^{-1}[D, ka^0]_{\sigma} \cdots D^{-1}[D, ka^{2q}]_{\sigma} \} \\
&= \tau_{2q}^{D, \sigma}(ka^0, \ldots, ka^{2q}).
\end{align*}
\]

As cyclic cohomology is invariant under the action of inner automorphisms (see [Co4, Prop. III.1.8] and [Lo, Prop. 4.1.3]), we deduce that the cyclic cocycles \(\tau_{2q}^{D_k, \sigma}\) and \(\tau_{2q}^{D, \sigma}\) are cohomologous in \(\text{HC}^{2q}(A)\). Therefore, they define the same class in \(\text{HP}^0(A)\), and so the twisted spectral triples \((A, H, D)_\sigma\) and \((A, H, D_k)_\sigma\) have same Connes-Chern character.

Let us now prove the result when \(D\) is not invertible. To this end consider the respective unital invertible doubles \((\tilde{A}, \tilde{H}, \tilde{D})_\sigma\) and \((\tilde{A}, \tilde{H}, \tilde{D}_k)_\sigma\) of \((A, H, D)_\sigma\) and its conformal deformation \((\tilde{A}, \tilde{H}, \tilde{D}_k)_\sigma\), where by a slight abuse of notation we have denoted by \(\sigma\) and \(\tilde{\sigma}\) the extensions to \(\tilde{A}\) of the automorphisms \(\sigma\) and \(\sigma\). As it turns out, \((\tilde{A}, \tilde{H}, \tilde{D}_k)_\sigma\) is not a conformal deformation of \((\tilde{A}, \tilde{H}, \tilde{D})_\sigma\) so that the proof in the invertible case does not extend to the invertible doubles. Nevertheless, as we shall see, \((\tilde{A}, \tilde{H}, \tilde{D}_k)_\sigma\) is a pseudo-inner twisting in the sense of [PW3] of a twisted spectral triple which is homotopy equivalent to \((\tilde{A}, \tilde{H}, \tilde{D})_\sigma\).

To wit consider the selfadjoint unbounded operator on \(\tilde{H} = H \oplus H\) given by

\[
\tilde{D}_1 = \begin{pmatrix} D & -k^{-2} \\ k^{-2} & -D \end{pmatrix}, \quad \text{dom}(\tilde{D}_1) = \text{dom}(D) \oplus \text{dom}(D).
\]

As \(\tilde{D}_1\) agrees with \(\tilde{D}\) up to a bounded operator, we see that \((\tilde{A}, \tilde{H}, \tilde{D}_1)_\sigma\) is a \(p\)-summable twisted spectral triple. Note also that

\[
\tilde{D}_k = \begin{pmatrix} Dk & 1 \\ 1 & -kDk \end{pmatrix} = \omega \tilde{D}_1\omega, \quad \text{where } \omega := \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}.
\]

In particular, we see that \(\tilde{D}_1\) is invertible. An explicit homotopy between \((\tilde{A}, \tilde{H}, \tilde{D})_\sigma\) and \((\tilde{A}, \tilde{H}, \tilde{D}_1)_\sigma\) is given by the family of twisted spectral triples \((\tilde{A}, \tilde{H}, \tilde{D}_t)_\sigma, t \in [0, 1]\), with

\[
\tilde{D}_t = \begin{pmatrix} D & k^{-2t} \\ k^{-2t} & -D \end{pmatrix} = \tilde{D} + V_t, \quad \text{where } V_t = \begin{pmatrix} 0 & k^{-2t} - 1 \\ k^{-2t} - 1 & 0 \end{pmatrix}.
\]

We observe that \((V_t)_{t \in [0, 1]}\) is a \(C^1\)-family in \(\mathcal{L}(\tilde{H})\) and, in the same way as for \(\tilde{D}_1\), it can be shown that the operator \(\tilde{D}_t\) is invertible for all \(t \in [0, 1]\). Therefore, we may use the homotopy invariance of the Connes-Chern character in the form of [PW1, Appendix C] to deduce that, for all \(q \geq \frac{1}{2}(p+1)\), the cyclic cocycles \(\tau_{2q}^{\tilde{D}, \sigma}\) and \(\tau_{2q}^{\tilde{D}_1, \sigma}\) are cohomologous in \(\text{HC}^{2q}(\tilde{A})\). Incidentally, if denote by \(\tilde{\tau}_{2q}^{\tilde{D}, \sigma}\) and \(\tilde{\tau}_{2q}^{\tilde{D}_1, \sigma}\) their respective restrictions to \(A\), then we see that \(\tilde{\tau}_{2q}^{\tilde{D}, \sigma}\) and \(\tilde{\tau}_{2q}^{\tilde{D}_1, \sigma}\) are cohomologous cyclic cocycles in \(\text{HC}^{2q}(A)\).

Bearing this in mind, let \(a \in A\). Using (6.8) and noting that \(\omega \tilde{\pi}(a)\omega^{-1} = \tilde{\pi}(ka^{-1})\), we see that

\[
\tilde{\pi}(a)\tilde{D}_k = \omega \tilde{\pi}(a)\omega^{-1} = \tilde{\pi}(ka^{-1})\omega.
\]

Thus,

\[
\tilde{D}_k^{-1}[\tilde{D}_k, \tilde{\pi}(a)]_\sigma = (\omega^{-1}\tilde{D}_1^{-1}\omega^{-1}) \left( \omega[\tilde{D}_1, \tilde{\pi}(ka^{-1})]_\sigma\omega \right) = \omega^{-1}\tilde{D}_1^{-1}[\tilde{D}_1, \tilde{\pi}(ka^{-1})]_\sigma\omega.
\]
Given an integer $q > \frac{1}{2}(p-1)$ let $a^j \in \mathcal{A}$, $j = 0, \ldots, 2q$. Using (4.2) and (6.10) and arguing as in (6.7) we obtain
\begin{align}
(6.11) \quad \tau_{2q}^{D_0, \sigma}(a^0, \ldots, a^{2q}) &= c_q \text{Str} \left\{ \tilde{D}_k [\tilde{D}_k, \tilde{\pi}(a^0)] \cdots \tilde{D}_k [\tilde{D}_k, \tilde{\pi}(a^{2q})] \right\} \\
(6.12) &= c_q \text{Str} \left\{ (\omega^{-1} D_1 [\tilde{D}_1, \tilde{\pi}(ka^0k^{-1})] \cdots (\omega^{-1} D_1 [\tilde{D}_1, \tilde{\pi}(ka^{2q}k^{-1})] \right\} \\
(6.13) &= c_q \text{Str} \left\{ \tilde{D}_1 [\tilde{D}_1, \tilde{\pi}(ka^0k^{-1})] \cdots \tilde{D}_1 [\tilde{D}_1, \tilde{\pi}(ka^{2q}k^{-1})] \right\} \\
(6.14) &= \tau_{2q}^{D_1, \sigma}(ka^0k^{-1}, \ldots, ka^{2q}k^{-1}).
\end{align}
Therefore, in the same way as in the invertible case, we deduce that the cocycles $\tau_{2q}^{D_0, \sigma}$ and $\tau_{2q}^{D_1, \sigma}$ are cohomologous in $HC^{2q}(\mathcal{A})$. It then follows that, for $q \geq \frac{1}{2}(p+1)$, the cocycles $\tau_{2q}^{D_0, \sigma}$ and $\tau_{2q}^{D_0, \sigma}$ are cohomologous in $HC^{2q}(\mathcal{A})$, and hence define the same class in $HP^0(\mathcal{A})$. This shows that the twisted spectral triples $(\mathcal{A}, \mathcal{H}, D)_\sigma$ and $(\mathcal{A}, \mathcal{H}, D)_\sigma$ have same Connes-Chern character. The proof is complete.

6.3. Smooth twisted spectral triples. We shall now explain how to extend to smooth twisted spectral triples the previous results of this section. First, we have the following result.

**Proposition 6.7.** Let $(\mathcal{A}, \mathcal{H}_1, D_1)_\sigma_1$ and $(\mathcal{A}, \mathcal{H}_2, D_2)_\sigma_2$ be equivalent smooth twisted spectral triples that are $p$-summable for some $p \geq 1$. Then $\text{Ch}(D_1)_\sigma_1 = \text{Ch}(D_2)_\sigma_2$ in $HP^0(\mathcal{A})$.

**Proof.** This is immediate consequence of the first two parts of Proposition 6.5, since they imply that $\text{Ch}(D_1)_\sigma_1$ and $\text{Ch}(D_2)_\sigma_2$ are represented by the same cocycles. $\square$

**Proposition 6.8.** Assume that $(\mathcal{A}, \mathcal{H}, D)_\sigma$ is $p$-summable for some $p \geq 1$. Then, for any positive element $k \in \mathcal{A}$,
\[ \text{Ch}(kDk)_\sigma = \text{Ch}(D)_\sigma \in HP^0(\mathcal{A}). \]

**Proof.** We shall continue using the notation of the proof of Proposition 6.5. This proof shows that, for any $q \geq \frac{1}{2}(p+1)$, the cocycles $\tau^{D_0, \sigma}$ and $\tau^{D_0, \sigma}$ define the same class in $HC^{2q}(\mathcal{A})$ by establishing they both are cohomologous to the cocycle $\tau^{D_1, \sigma}$. In order to completes the proof we only need to show that $\tau^{D, \sigma}$ and $\tau^{D_0, \sigma}$ define the same class in $HC^{2q}(\mathcal{A})$.

The equality between the classes of $\tau^{D, \sigma}$ and $\tau^{D_1, \sigma}$ is a consequence of the homotopy invariance of the Connes-Chern character and the fact that the operators $\pi$ and $\tilde{\pi}$ can be connected by an operator homotopy of the form (6.9). Therefore, using Lemma 5.6 we see that the cocycles $\tau^{D_0, \sigma}$ and $\tau^{D_1, \sigma}$ are cohomologous in $HC^{2q}(\mathcal{A})$, and so are their restrictions to $\mathcal{A}$, that is, the cocycles $\tau^{D_0, \sigma}$ and $\tau^{D_1, \sigma}$, define the same class in $HC^{2q}(\mathcal{A})$.

In addition, Eq. (6.14) shows that the cocycle $\tau^{D_k, \sigma}$ is obtain from $\tau^{D_1, \sigma}$ by composing with the inner automorphism defined by $k$. The proof of the invariance of cyclic cohomology by the action of inner automorphisms in [Lo] holds verbatim for the cyclic cohomology of continuous cochains. It then follows that the cocycles $\tau^{D_k, \sigma}$ and $\tau^{D_1, \sigma}$ define the same class $HC^{2q}(\mathcal{A})$, and so $\tau^{D, \sigma}$ and $\tau^{D_k, \sigma}$ are cohomologous in $HC^{2q}(\mathcal{A})$. The proof is complete. $\square$

7. The Conformal Connes-Chern Character

In this section, after recalling the construction of the conformal Dirac spectral triple of [CM3] associated to any given conformal structure, we show that its Connes-Chern character (defined in continuous cyclic cohomology) is actually a conformal invariant.

7.1. The conformal Dirac spectral triple. Throughout this section and the rest of the paper we let $M$ be a compact (closed) spin oriented manifold of even dimension $n$. We also let $\mathcal{C}$ be a conformal structure on $M$, i.e., a conformal class of Riemannian metrics on $M$. We then denote by $G$ the identity component of the group of (smooth) orientation-preserving diffeomorphisms of $M$ preserving the conformal structure $\mathcal{C}$ and the spin structure of $M$. Let $g$ be a representative metric in the conformal class $\mathcal{C}$, and consider the associated Dirac operator $D_g : C^\infty(M, \mathcal{S}) \rightarrow C^\infty(M, \mathcal{S})$.
on the sections of the spinor bundle $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. In addition, we denote by $L_2^G(M, \mathcal{S})$ the corresponding Hilbert space of $L^2$-spinors.

In the setup of noncommutative geometry, the role of the quotient space $M/G$ is played by the (discrete) crossed-product algebra $C^\infty(M) \rtimes G$. The underlying vector space of $C^\infty(M) \rtimes G$ is $C^\infty(M \times G)$, where $G$ is equipped with the discrete topology and smooth structure. The product and involution of $C^\infty(M) \rtimes G$ are given by

$$ F_1 \star F_2(x, \phi) = \sum_{\phi_1 \circ \phi_2 = \phi} F_1(x, \phi_1) F_2(\phi_2^{-1}(x), \phi_2), \quad F^*(x, \phi) = \overline{F(x, \phi^{-1})}. $$

Alternatively, if we denote by $u_\phi$ the characteristic function of $M \times \{\phi\}$, then $u_\phi \in C^\infty_c(M \times G)$ and any $F \in C^\infty_c(M \times G)$ is uniquely written as a finitely supported sum,

$$ F = \sum_{\phi \in G} f_\phi u_\phi, $$

where $f_\phi(x) := F(x, \phi) \in C^\infty(M)$. Moreover, we have the following relations hold:

$$ u_{\phi_1} u_{\phi_2} = u_{\phi_1 \circ \phi_2}, \quad u_{\phi_1}^* = u_{\phi_1^{-1}}, \quad \phi_j \in G, \quad (7.1) $$

$$ u_\phi f = (f \circ \phi^{-1}) u_\phi, \quad f \in C^\infty(M), \quad \phi \in G. \quad (7.2) $$

Let $\phi \in G$. As $\phi$ is a diffeomorphism preserving the conformal class $\mathcal{C}$, there is a unique function $k_\phi \in C^\infty(M)$, $k_\phi > 0$, such that

$$ \phi_* g = k_\phi^2 g. \quad (7.3) $$

Moreover, $\phi$ uniquely lifts to a unitary vector bundle isomorphism $\phi^\#: \mathcal{S} \to \phi_* \mathcal{S}$, i.e., a unitary section of $\text{Hom}(\mathcal{S}, \phi_* \mathcal{S})$ (see [BG]). We then let $V_\phi : L^2_\phi(M, \mathcal{S}) \to L^2_\phi(M, \mathcal{S})$ be the bounded operator given by

$$ V_\phi u(x) = \phi^\# (u \circ \phi^{-1}(x)) \quad \forall u \in L^2_\phi(M, \mathcal{S}) \forall x \in M. \quad (7.4) $$

The map $\phi \to V_\phi$ is a representation of $G$ in $L^2_\phi(M, \mathcal{S})$, but this is not a unitary representation. In order to get a unitary representation we need to take into account the Jacobian $|\phi'(x)| = k_\phi(x)^n$ of $\phi \in G$. This is achieved by using the unitary operator $U_\phi : L^2_\phi(M, \mathcal{S}) \to L^2_\phi(M, \mathcal{S})$ given by

$$ U_\phi = k_\phi^{\frac{n}{2}} V_\phi, \quad \phi \in G. \quad (7.5) $$

Then $\phi \to U_\phi$ is a unitary representation of $G$ in $L^2_\phi(M, \mathcal{S})$. Combining this with the action of $C^\infty(M)$ by multiplication operators provides with a $*$-representation of the crossed-product algebra $C^\infty(M) \rtimes G$. In addition, we let $\sigma_g$ be the automorphism of $C^\infty(M) \rtimes G$ given by

$$ \sigma_g (f u_\phi) := k_\phi^{\frac{n}{2}} f u_\phi, \quad \forall f \in C^\infty(M) \forall \phi \in G. \quad (7.6) $$

**Proposition 7.1** ([CM3]). The triple $(C^\infty(M) \rtimes G, L^2_\phi(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ is a twisted spectral triple.

**Remark 7.2.** The bulk of the proof is showing the boundedness of the twisted commutators $[\mathcal{D}_g, U_\phi]_{\sigma_g}$, $\phi \in G$. We remark that

$$ U_\phi \mathcal{D}_g U_\phi^* = k_\phi^{\frac{n}{2}} (\mathcal{D}_g V_\phi^{-1}) k_\phi^{-\frac{n}{2}} = k_\phi^{\frac{n}{2}} \mathcal{D}_{\phi_* g} k_\phi^{-\frac{n}{2}} = k_\phi^{\frac{n}{2}} \mathcal{D}_g k_\phi^{-\frac{n}{2}}. $$

Combining this with the conformal invariance of the Dirac operator (see, e.g., [Hi]) we obtain

$$ U_\phi \mathcal{D}_g U_\phi^* = k_\phi^{\frac{n}{2}} \left( k_{\phi_* g}^{\frac{n+1}{2}} \mathcal{D}_g k_{\phi_* g}^{\frac{n-1}{2}} \right) k_{\phi}^{-\frac{n}{2}} = k_{\phi_* g}^{\frac{n}{2}} \mathcal{D}_g k_{\phi_* g}^{-\frac{n}{2}}. $$

Using this we see that the twisted commutator $[\mathcal{D}_g, U_\phi]_{\sigma_g} = \mathcal{D}_g U_\phi - k_\phi U_\phi \mathcal{D}_g$ is equal to

$$ \left( \mathcal{D}_g k_{\phi}^{\frac{n}{2}} - k_{\phi} (U_\phi \mathcal{D}_g U_\phi)^* \right) k_{\phi}^{-\frac{n}{2}} U_\phi = \left( \mathcal{D}_g k_{\phi}^{\frac{n}{2}} - k_{\phi} \mathcal{D}_g \right) k_{\phi}^{\frac{n}{2}} U_\phi = \mathcal{D}_g k_{\phi}^{\frac{n}{2}} k_{\phi}^{\frac{n}{2}} U_\phi. $$

This shows that $[\mathcal{D}_g, U_\phi]_{\sigma_g}$ is bounded.

**Remark 7.3.** We shall refer to $(C^\infty(M) \rtimes G, L^2_\phi(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ as the conformal Dirac spectral triple associated to the representative metric $g$. 

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Remark 7.4. The automorphism $\sigma_\phi$ is ribbon in the sense mentioned in Remark 2.22. This is seen by using the automorphism $\tau_\phi$ of $C^\infty(M) \rtimes G$ defined by

$$\tau_\phi(fu_\phi) = \sqrt{k_\phi}fu_\phi \quad \forall f \in C^\infty(M) \forall \phi \in G,$$

where $k_\phi$ is the conformal factor in the sense (7.3).

In addition, recall that $C^\infty(M) \rtimes G$ has a natural locally convex $*$-algebra topology defined as follows. For any finite set $F \subseteq G$ set

$$C^\infty(M) \rtimes F := \left\{ \sum_{\phi \in F} f_\phi u_\phi; \ f_\phi \in C^\infty(M) \right\}.$$

There is a natural linear isomorphism from $C^\infty(M) \rtimes F$ onto $C^\infty(M)^F$, which enables us to pullback to $C^\infty(M) \rtimes F$ the locally convex space topology of $C^\infty(M)^F$. The locally convex space topology of $C^\infty(M) \rtimes G$ is then obtained as the inductive limit of the locally convex space topologies of the spaces $C^\infty(M) \rtimes F$. In particular, the following result holds.

**Lemma 7.5.** Given any topological vector space $X$, a linear map $\Phi : C^\infty(M) \rtimes G \to X$ is continuous if and only if, for all $\phi \in G$, the map $f \to \Phi(fu_\phi)$ is a continuous from $C^\infty(M)$ to $X$.

Using this we see that the product and involution of $C^\infty(M) \rtimes G$ are continuous. Furthermore, we have the following.

**Lemma 7.6.** For any metric $g \in \mathcal{C}$, the twisted spectral triple $(C^\infty(M) \rtimes G, L^2_g(M,\mathcal{S}),\mathcal{D}_g)_{\sigma_\phi}$ is smooth in the sense of Definition 5.1.

**Proof.** Let $\phi \in G$. The map $f \to fu_\phi$ is continuous from $C^\infty(M)$ to $L^2(L^2_g(M,\mathcal{S}))$. As $\sigma_\phi(fu_\phi) = k_\phi fu_\phi$ and $\sigma_\phi^{-1}(fu_\phi) = k_\phi^{-1}fu_\phi$, the maps $f \to \sigma_\phi(fu_\phi)$ are continuous from $C^\infty(M)$ to $C^\infty(M) \rtimes G$. In addition, note that, for any $f \in C^\infty(M)$,

$$[\mathcal{D}_g, fu_\phi]_{\sigma_\phi} = [\mathcal{D}_g, f]u_\phi + f[\mathcal{D}_g, u_\phi]_{\sigma_\phi} = ic(df)u_\phi + f[\mathcal{D}_g, u_\phi]_{\sigma_\phi},$$

where $c(df)$ is the Clifford representation of the differential $df$. Thus, the map $f \to [\mathcal{D}_g, fu_\phi]_{\sigma_\phi}$ is continuous from $C^\infty(M)$ to $L^2(L^2_g(M,\mathcal{S}))$. Combining all this with Lemma 7.5 shows that the conditions (1)-(3) of Definition 5.1 are satisfied, and so $(C^\infty(M) \rtimes G, L^2_g(M,\mathcal{S}),\mathcal{D}_g)_{\sigma_\phi}$ is a smooth twisted spectral triple. The lemma is thus proved.  

### 7.2. Conformal invariance of the Connes-Chern character.

The construction of the conformal Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_g(M,\mathcal{S}),\mathcal{D}_g)_{\sigma_\phi}$, depends on the choice of a representative metric $g$ in the conformal class $\mathcal{C}$. The following proposition describes the dependence on this choice.

**Proposition 7.7.** Let $\hat{g}$ be another metric in the conformal class $\mathcal{C}$, i.e., $\hat{g} = k^2g$ for some function $k \in C^\infty(M)$, $k > 0$. Let $\hat{\sigma}$ be the automorphism of $C^\infty(M) \rtimes G$ defined by

$$(7.7) \quad \hat{\sigma}(a) = k^{-\frac{1}{2}}\sigma_\phi(k^{-\frac{1}{2}}ak^{-\frac{1}{2}})k^{\frac{1}{2}} = k^{-1}\sigma_\phi(a)k, \quad a \in C^\infty(M) \rtimes G.$$

Then the conformal Dirac spectral $(C^\infty(M) \rtimes G, L^2_{\hat{g}}(M,\mathcal{S}),\mathcal{D}_{\hat{g}})_{\sigma_\phi}$ associated to $\hat{g}$ is equivalent to the conformal deformation $(C^\infty(M) \rtimes G, L^2_g(M,\mathcal{S}),k^{-\frac{1}{2}}\mathcal{D}_gk^{-\frac{1}{2}})_{\sigma_\phi}$.

**Proof.** Let $U : L^2_g(M,\mathcal{S}) \to L^2_{\hat{g}}(M,\mathcal{S})$ be the operator given by the multiplication by $k^{\frac{1}{2}}$. Let $u \in L^2_g(M,\mathcal{S})$. As $|dx|_{\hat{g}} = k(x)|dx|_g$, we have

$$\|Uu\|^2_{L^2_{\hat{g}}(M,\mathcal{S})} = \int_M (k(x))^{\frac{3}{2}}u(x)k(x)^{\frac{3}{2}}u(x))|dx|_g = \int_M (u(x), u(x))|dx|_{\hat{g}} = \|u\|^2_{L^2_g(M,\mathcal{S})}.$$  

This shows that $U$ is a unitary operator. Moreover, using the conformal invariance of the Dirac operator [Hit] we see that

$$(7.8) \quad \mathcal{D}_{\hat{g}} = k^{-\frac{1}{2}(n+1)}\mathcal{D}_gk^{\frac{1}{2}(n-1)} = U^*k^{-\frac{1}{2}}\mathcal{D}_gk^{-\frac{1}{2}}U.$$
Let $\phi \in G$. We have two representations of $\phi$. One is the unitary operator $V_\phi$ of $L^2(M, \mathcal{S})$ given by (7.4) using the representative metric $g$. We have another representation of $\phi$ as a unitary operator $\hat{V}_\phi$ of $L^2(M, \mathcal{S})$ given by the same formula using the metric $\hat{g}$. That is,

$$\hat{V}_\phi u = e^{nh\phi} \phi_* u, \quad u \in L^2(M, \mathcal{S}),$$

where $\hat{h}_\phi \in C^\infty(M, \mathbb{R})$ is so that $\phi_* \hat{g} = e^{2\hat{h}_\phi} \hat{g}$. Set $h = \log k$. Then

$$\phi_* \hat{g} = \phi_* (k^2 g) = (k \circ \phi^{-1})^2 \phi_* g = e^{2h} e^{2h\phi} g = e^{2(h\phi^{-1} + h_\phi - h)} \hat{g},$$

which shows that $\hat{h}_\phi = h \circ \phi^{-1} + h_\phi - h$. Let $u \in L^2(M, \mathcal{S})$. Then

$$\hat{V}_\phi u = e^{-nh} e^{nh\phi} e^{-nho\phi} \phi_* u = k^{-\frac{1}{2}} e^{nh\phi} (k^{\frac{1}{2}} u) = U^* V_{\phi} U u.$$

Likewise,

$$\sigma_\phi (\hat{V}_\phi) u = e^{-(n+1)h} \phi_* (k^{n+1} u) = U k^{-1} \sigma_g (V_{\phi}) U u = U^{-1} \sigma (V_{\phi}) U u.$$ 

We then deduce that, for all $f \in C^\infty(M)$ and $\phi \in G$,

$$f \hat{V}_\phi = U^*(f V_{\phi}) U \quad \text{and} \quad \sigma_\phi (f \hat{V}_\phi) = U^* \sigma (f V_{\phi}).$$

Combining this with (7.8) shows that the twisted spectral triples $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ and $(C^\infty(M) \rtimes G, L^2(M, \mathcal{S}), k^{-\frac{1}{2}} \mathcal{D}_g k^{-\frac{1}{2}})_{\sigma_\phi}$ are equivalent. The proof is complete. \qed

Given any metric $g \in \mathcal{C}$, the $j$-th eigenvalue of $|\mathcal{D}_g|$ grows like $j^{\frac{1}{2}}$ as $j \to \infty$. Therefore, the associated twisted spectral triple $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ is $p$-summable for all $p > n$. Combining this with Proposition 6.3 shows that the Connes-Chern character of $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ is well defined as a class in $\text{H}^\mathcal{P} (C^\infty(M) \rtimes G)$ for every metric $g$ in the conformal class $\mathcal{C}$.

We are now in a position to state the main result of this section.

**Theorem 7.8.** The Connes-Chern character $\text{Ch} (\mathcal{D}_g)_{\sigma_g} \in \text{H}^\mathcal{P} (C^\infty(M) \rtimes G)$ is independent of the choice of the metric $g \in \mathcal{C}$, i.e, it is an invariant of the conformal structure $\mathcal{C}$.

**Proof.** Let $\hat{g}$ be another metric in the conformal class $\mathcal{C}$, so that $\hat{g} = k^2 g$ with $k \in C^\infty(M), k > 0$. Combining Proposition 7.7 with Proposition 6.3 and Proposition 6.8 we get

$$\text{Ch} (\mathcal{D}_g)_{\sigma_g} = \text{Ch} (k^{-\frac{1}{2}} \mathcal{D}_g k^{-\frac{1}{2}})_{\hat{\sigma}} = \text{Ch} (\mathcal{D}_\hat{g})_{\sigma_{\hat{g}}} \quad \text{in } \text{H}^\mathcal{P} (C^\infty(M) \rtimes G),$$

where $\hat{\sigma}$ is defined as in (7.7). This proves the result. \qed

This leads us to the following definition.

**Definition 7.9.** The Connes-Chern character of the conformal class $\mathcal{C}$, denoted by $\text{Ch} (\mathcal{C})$, is the common class in $\text{H}^\mathcal{P} (C^\infty(M) \rtimes G)$ of the Connes-Chern characters of the conformal Dirac spectral triples $(C^\infty(M) \rtimes G, L^2_g(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ as the metric $g$ ranges over the conformal class $\mathcal{C}$.

Combining this with Proposition 5.9 we obtain the following index formula.

**Proposition 7.10.** Let $\mathcal{E}$ be a finitely generated projective over $C^\infty(M) \rtimes G$. Then, for any metric $g \in \mathcal{C}$ and any $\sigma_g$-connection on $\mathcal{E}$, it holds that

$$\text{ind } D_{\nabla g} = (\text{Ch} (\mathcal{C}), \text{Ch} (\mathcal{C})).$$
8. Local Index Formula in Conformal Geometry

In this section, we shall compute the conformal Connes-Chern character $\text{Ch}(\mathcal{C})$ when the conformal structure $\mathcal{C}$ is not flat. Together with Proposition 7.10 this will provide us with the local index formula in conformal-diffeomorphism invariant geometry. We recall that the conformal structure $\mathcal{C}$ is flat when it is equivalent to the conformal structure of the round sphere $\mathbb{S}^n$. In case $M$ is simply connected and has dimension $\geq 4$ this is equivalent to the vanishing of the Weyl curvature tensor of $M$ (see [Ku]).

As pointed out in Remark 4.12, the Ansatz of Moscovici [Mo2] is not known to hold for conformal deformations of ordinary spectral triples satisfying the local index formula in noncommutative geometry of [CM2]. As a result, unless the conformal structure $\mathcal{C}$ is flat and $G$ is a maximal parabolic subgroup of $\text{PO}(n+1,1)$, we cannot claim that for a general metric in $\mathcal{C}$ the corresponding conformal Dirac spectral triple $(C^\infty(M) \rtimes G, L_2^2(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ satisfies Moscovici’s Ansatz.

Having said this, the conformal invariance of the Connes-Chern character $\text{Ch}(\mathcal{C})$ as defined by the orientations of $\sigma_g$ gives rise to a section of $\Lambda^g \otimes \mathcal{S}$ which we shall refer to as Part II. In order to use the results of Part II a bit of notation need to be introduced.

**Proposition 8.1** (Ferrand-Obata [Ba, Fe, Sc]). *If the conformal structure $\mathcal{C}$ is not flat, then the group of smooth diffeomorphisms of $M$ preserving $\mathcal{C}$ is a compact Lie group and there is a metric in $\mathcal{C}$ that is invariant by this group.*

The relevance of using a $G$-invariant metric $g \in \mathcal{C}$ stems from the observation that in this case $\phi_* g = g$ for all $\phi \in G$, and so, for every $\phi \in G$, the conformal factor $k_\phi$ is always the constant function 1. This implies that, for any $\phi \in G$, the unitary operator $U_\phi$ agrees with the pushforward by $\phi$. This also implies that the automorphism $\sigma_g$ given by (7.6) is trivial, so that the conformal Dirac spectral triple $(C^\infty(M) \rtimes G, L_2^2(M, \mathcal{S}), \mathcal{D}_g)_{\sigma_g}$ is actually an ordinary spectral triple. Therefore, we arrive at the following statement.

**Proposition 8.2.** The conformal Connes-Chern character $\text{Ch}(\mathcal{C})$ agrees with the ordinary Connes-Chern character of the equivariant Dirac spectral triple $(C^\infty(M) \rtimes G, L_2^2(M, \mathcal{S}), \mathcal{D}_g)$ associated to any $G$-invariant metric $g \in \mathcal{C}$ in the conformal class $\mathcal{C}$.

We postpone the computation of the Connes-Chern character of an equivariant Dirac spectral triple $(C^\infty(M) \rtimes G, L_2^2(M, \mathcal{S}), \mathcal{D}_g)$ associated to any $G$-invariant metric $g \in \mathcal{C}$ to the sequel [PW2] (which we shall refer to as Part II). In order to use the results of Part II a bit of notation need to be introduced.

In what follows we let $g$ be a $G$-invariant metric in the conformal class $\mathcal{C}$. Let $\phi \in G$ and denote by $M^\phi$ its fixed-point set. As $\phi$ preserves the orientation and the metric $g$, the fixed-point set $M^\phi$ is a disconnected union $M^\phi_0$ of submanifolds of even dimension $a = 0, 2, \cdots, n$. Therefore, we can merely treat $M^\phi$ as if it were a manifold. Let $N^\phi = (TM^\phi)^\perp$ be the normal bundle of $M^\phi$, which we regard as a vector bundle over each of the submanifold components of $M^\phi$. We denote by $\phi^N$ the isometric vector bundle isomorphism induced on $N^\phi$ by $\phi$. We note that the eigenvalues of $\phi^N$ are either $-1$ (which has even multiplicity) or complex conjugates $e^{i\theta}, \theta \in (0, \pi)$, with same multiplicity. In addition, we shall orient $M^\phi_0$ like in [BGV, Prop. 6.14], so that the vector bundle isomorphism $\phi^N : \mathcal{S} \to \phi_* \mathcal{S}$ gives rise to a section of $\Lambda^b(N^\phi)^\perp$ which is positive with respect to the orientation of $N^\phi$ defined by the orientations of $M$ and $M^\phi$. Here $b$ is the dimension of the fibers of $N^\phi$.

Let $R^TM$ be the curvature of $(M, g)$, seen as a section of $\Lambda^2 T^* M \otimes \text{End}(TM)$. As the Levi-Civita connection $\nabla^TM$ is preserved by $\phi$, it preserves the splitting $TM|_{M^\phi} = TM^\phi \oplus N^\phi$ over $M^\phi$, and so it induces connections $\nabla^{TM^\phi}$ and $\nabla^{N^\phi}$ on $TM^\phi$ and $N^\phi$ respectively, in such a way that

$$\nabla^{TM^\phi}_{TM^\phi} = \nabla^{TM^\phi} \oplus \nabla^{N^\phi} \quad \text{on } M^\phi.$$
We note that $\nabla^{TM^\phi}$ is the Levi-Civita connection of $TM^\phi$. Let $R^{TM^\phi}$ and $R^{N^\phi}$ be the respective curvatures of $\nabla^{TM^\phi}$ and $\nabla^{N^\phi}$. We define

$$
(8.1) \quad \hat{A}(R^{TM^\phi}) := \det^{\frac{1}{2}} \left( \frac{R^{TM^\phi}}{\sinh(R^{TM^\phi}/2)} \right) \quad \text{and} \quad \nu_\phi(R^{N^\phi}) := \det^{-\frac{1}{2}} \left( 1 - \phi^N e^{-R^{N^\phi}} \right),
$$

where $\det^{-\frac{1}{2}} \left( 1 - \phi^N e^{-R^{N^\phi}} \right)$ is defined in the same way as in [BGV, Section 6.3]. In addition, given a differential form $\omega$ on $M$ we shall denote by $\int_{M_n^\phi} \omega$ the integral over $M_n^\phi$ of the top-degree component of $\iota^* \omega$, where $\iota$ is the inclusion of $M_n^\phi$ in $M$.

Combining the results of Part II with Proposition 8.2 we obtain the following index formula in conformal-diffeomorphism invariant geometry.

**Theorem 8.3.** Assume that the conformal structure $\mathcal{C}$ is non-flat.

1. For any $G$-invariant metric $g \in \mathcal{C}$, the conformal Connes-Chern character $\text{Ch}(\mathcal{C})$ is represented in $\text{HP}^0(C^\infty(M) \rtimes G)$ by the cocycle $\varphi = (\varphi_{2q})_{q \geq 0}$ given by

$$
(8.2) \quad \varphi_{2q}(f^0 u_{\phi_0}, \ldots, f^{2q} u_{\phi_{2q}}) = \frac{(-i)^{\frac{n}{2}}}{(2\pi)^{q}} \sum_{\alpha \leq q, \alpha \text{ even}} (2\pi)^{-\frac{n}{2}} \int_{M_n^\phi} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi(R^{N^\phi}) \wedge f^0 d\hat{j}_1 \wedge \cdots \wedge d\hat{j}_{2q},
$$

where we have set $\phi := \phi_0 \circ \cdots \circ \phi_{2q}$ and $\hat{j}_j := f^j \circ \phi_{j-1}^{-1} \circ \cdots \circ \phi_0^{-1}$, $j = 1, \ldots, 2q$.

2. Let $E$ be a finitely generated projective module over $C^\infty(M) \rtimes G$. Then, for any metric $g \in \mathcal{C}$ and any $\sigma_0$-connection on $E$, it holds that

$$
\text{ind} D_{\varphi E} = \langle \varphi, \text{Ch}(E) \rangle,
$$

where $\varphi$ is the cocycle (8.2) associated to any $G$-invariant metric in $\mathcal{C}$ and $\text{Ch}(E)$ is the Chern character of $E$.

**Remark 8.4** (See also [Mo2]). For $q = \frac{1}{2}n$ the right-hand side of (8.2) reduces to an integral over $M_n^\phi$ and this submanifold is empty unless $\phi = \text{id}$. Thus,

$$
(8.3) \quad \varphi_n(f^0 u_{\phi_0}, \ldots, f^n u_{\phi_n}) = \begin{cases} 
\frac{(2\pi)^{-\frac{n}{2}}}{n!} \int_M f^0 d\hat{j}_1 \wedge \cdots \wedge d\hat{j}_n & \text{if } \phi_0 \circ \cdots \circ \phi_n = \text{id}, \\
0 & \text{otherwise}.
\end{cases}
$$

That is, $\varphi_n$ agrees with the transverse fundamental class cocycle of Connes [Co2].

**Remark 8.5.** The computation in Part II of the Connes-Chern character in $\text{HP}^0(C^\infty(M) \rtimes G)$ of an equivariant Dirac spectral triple $(C^\infty(M) \rtimes G, L^2_g(M, S), \partial_g)$ has two main steps. The first step is showing that the Connes-Chern is represented in $\text{HP}^0(C^\infty(M) \rtimes G)$ (not just in $\text{HP}^0(C^\infty(M) \rtimes G)$) by the CM cocycle of [CM2]. This requires extending the local index formula to the framework of smooth spectral triples. The second step is the explicit computation of that CM cocycle. Related computations were carried out independently by Azmi [Az] in the case of a finite group and Chern-Hu [CH] in the equivariant setting by elaborating on the approach of Lafferty-Yu-Zhang [LYZ] to the proof of the equivariant local index theorem of Patodi [Pa], Donnelly-Patodi [DP] and Gilkey [Gi] (see also [Bi, BV, BGV, LM]). In Part II this computation is obtained as a simple consequence of a new proof of the equivariant local index theorem.

9. **The Cyclic Homology of $C^\infty(M) \rtimes G$**

In this section, we present a geometric construction of cycles in $\text{HP}^0(C^\infty(M) \rtimes G)$. This will be used in the next section to construct conformal invariants. In what follows we continue to assume that the conformal structure $\mathcal{C}$ is non-flat.
9.1. The Cyclic Homology of $C^\infty(M) \rtimes G$. The periodic cyclic homology $\text{HP}_\bullet(C^\infty(M) \rtimes G)$ is known [BN, Cr] and can be described as follows.

Let $(G)$ be the set of conjugacy classes of $G$. Given $\phi \in G$ we denote by $\langle \phi \rangle$ its conjugacy class and let $G_\phi = \{ \psi \in G; \psi \circ \phi = \phi \circ \psi \}$ be its stabilizer. Note that the action of $G_\phi$ preserves each fixed-point submanifold $M_\phi^a, a = 0, 2, \ldots, n$. For $i = 0, 1$ we then set

$$H[i](M_\phi^a)^{G_\phi} = \bigoplus_{2q+i \leq a} H^{2q+i}(M_\phi^a)^{G_\phi},$$

where $H^{2q+i}(M_\phi^a)^{G_\phi}$ is the $G_\phi$-invariant de Rham cohomology of degree $2q+i$ of $M_\phi^a$. We then have the following description of $\text{HP}_\bullet(C^\infty(M) \rtimes G)$.

**Proposition 9.1** (Brylinski-Nistor [BN]; see also [Cr]). It holds that

$$\text{HP}_\bullet(C^\infty(M) \rtimes G) \simeq \bigoplus_{\phi \in (G)} \bigoplus_{0 \leq a \leq n, a \text{ even}} H[i](M_\phi^a)^{G_\phi}.$$ 

**Remark 9.2.** Brylinski-Nistor [BN] actually computed the cyclic homology of the convolution algebras of Hausdorff étale groupoids. Crainic [Cr] extended their results to non-Hausdorff groupoids. In the case of the discrete transformation groupoid associated to the action of $G$ on $M$ we obtain the isomorphism (9.1). We refer to [BC, BGJ, Co4, Da, FT, GJ, KKL, Ni1, Ni2, NPPT] for various results related to the cyclic homology of crossed-product algebras.

In what follows, we will construct explicit maps from each cohomology space $H[i](M_\phi^a)^{G_\phi}$ to $\text{HP}_\bullet(C^\infty(M) \rtimes G)$. This construction is carried out in several steps. It essentially follows the approach of [BN, Cr] except for the specialization of the arguments to group actions and the use of the mixed complexes of $G$-normalized chains, which does not appear explicitly in [BN, Cr].

We refer to Definition 9.3 and Definition 9.9 below for the precise definitions of $G$-normalized cochains and $G$-normalized chains. It will be shown in the next section that, for any $G$-invariant metric $g \in \mathcal{C}$, the cocycle (8.2) representing the conformal Connes-Chern character $\text{Ch}(\xi)$ is $G$-normalized. Therefore it is a crucial step in our approach to obtain cycles arising from explicit $G$-normalized chains.

9.2. $G$-normalized cochains. In what follows we set $\mathcal{A} = C^\infty(M)$ and $\mathcal{A}_G = C^\infty(M) \rtimes G$. We also note that the group $G$ acts on $C^\infty(\mathcal{A}_G)$ as follows: given $\varphi \in C^\infty(\mathcal{A}_G)$ and $\psi \in G$, the action of $\psi$ on $\varphi$ is the cochain $\psi^* \varphi \in C^\infty(\mathcal{A}_G)$ defined by

$$\psi^* \varphi(a^0, \ldots, a^m) = \varphi(u_\psi a^0 u_\psi^{-1}, \ldots, u_\psi a^m u_\psi^{-1}) \quad \forall a^j \in \mathcal{A}_G.$$ 

**Definition 9.3.** A continuous cochain $\varphi \in C^\infty(\mathcal{A}_G)$ is $G$-normalized when, for all $a^0, \ldots, a^m$ in $\mathcal{A}_G$ and $\psi \in G$, the following holds

$$\varphi(a^0, \ldots, a^j u_\psi^{-1}, u_\psi a^{j+1}, \ldots, a^m) = \varphi(a^0, \ldots, a^m) \quad \text{for } j = 0, \ldots, m - 1,$$

$$\varphi(u_\psi a^0, a^1, \ldots, a^{m-1}, a^m u_\psi^{-1}) = \varphi(a^0, \ldots, a^m).$$

We denote by $C^\infty_{G}(\mathcal{A}_G)$ the space of $G$-normalized $m$-cochains on $\mathcal{A}_G$.

The following lemma is useful for checking the $G$-normalization conditions (9.3) and (9.4).

**Lemma 9.4.** Let $\varphi \in C^\infty(\mathcal{A}_G)$. Then the following are equivalent:

(i) The cochain $\varphi$ is $G$-normalized.

(ii) The cochain $\varphi$ is satisfy (9.3) and is $G$-invariant with respect to the action (9.2).

(iii) For all $f^0, \ldots, f^m$ in $C^\infty(M)$ and $\phi_0, \ldots, \phi_m, \phi, \psi$ in $G$, we have

$$\varphi(f^0 u_{\phi_0}, \ldots, f^m u_{\phi_m}) = \varphi(f^0, f^1 \circ \phi_0^{-1}, \ldots, f^m \circ \phi_{m-1}^{-1} \circ \cdots \circ \phi_0^{-1} u_{\phi_0 \circ \cdots \circ \phi_m}),$$

$$\varphi(f^0 \circ \psi^{-1}, \ldots, f^m \circ \psi^{-1}, f^m \circ \psi^{-1} u_{\psi \circ \cdots \circ \psi^{-1}}) = \varphi(f^0, \ldots, f^m - 1, f^m u_{\psi}).$$
Proof. If (9.3) holds, then, for all \(a^0, \ldots, a^m\) in \(A_G\) and \(\psi \in G\), we have
\[
\psi^* \varphi(a^0, \ldots, a^m) = \varphi(u_\psi a^0 u_\psi^{-1}, u_\psi a^1 u_\psi^{-1}, \ldots, u_\psi a^m u_\psi^{-1}) = \varphi(a^0, a^1, \ldots, a^m u_\psi^{-1}).
\]
Therefore, in this case (9.4) is equivalent to the \(G\)-invariance of \(\varphi\). This gives the equivalence of (i) and (ii).

Assume that (iii) holds. Let \(f^0, \ldots, f^m\) be in \(C^\infty(M)\) and let \(\phi_0, \ldots, \phi_m\) and \(\psi\) be in \(G\). Then (7.2) implies that, for \(j = 1, \ldots, m\),
\[
\varphi(f^0 u_{\phi_0}, \ldots, f^m u_{\phi_m}) = \varphi(f^0 u_{\phi_0}, \ldots, f^j u_{\phi_j} \circ \psi^{-1} u_{\phi_{j+1}}, \ldots, f^m u_{\phi_m})
\]
\[
= \varphi(f^0 u_{\phi_0}, \ldots, (f^j u_{\phi_j}) u_{\psi}^{-1} (f^{j+1} u_{\phi_{j+1}}), \ldots, f^m u_{\phi_m}).
\]
Note also that
\[
\varphi(u_{\psi}(f^0 u_{\phi_0}) u_{\psi}^{-1}, \ldots, u_{\psi}(f^m u_{\phi_m}) u_{\psi}^{-1}) = \varphi((f^0 \circ \psi^{-1}) u_{\psi_{\phi_0} \circ \psi^{-1}}, \ldots, (f^m \circ \psi^{-1}) u_{\psi_{\phi_m} \circ \psi^{-1}}).
\]
Therefore, using (9.5) and (9.6) we see that \(\varphi(u_\psi(f^0 u_{\phi_0}) u_\psi^{-1}, \ldots, u_\psi(f^m u_{\phi_m}) u_\psi^{-1})\) is equal to
\[
\varphi(f^0 \circ \psi^{-1}, f^1 \circ \phi_0^{-1} \circ \psi^{-1}, \ldots, (f^m \circ \phi_{m-1}^{-1} \circ \cdots \circ \phi_0^{-1}) \circ \psi^{-1} u_{\psi(\phi_{m-1} \circ \cdots \circ \phi_0) \circ \psi^{-1}})
\]
\[
= \varphi(f^0, f^1 \circ \phi_0^{-1}, \ldots, (f^m \circ \phi_{m-1}^{-1} \circ \cdots \circ \phi_0^{-1}) u_{\phi_0 \circ \cdots \circ \phi_m})
\]
Together with (9.8) this shows that \(\varphi\) satisfies (ii).

Conversely, assume that \(\varphi\) satisfies (ii) and let us show that (iii) then holds. As (9.6) is only a special case of (9.2) (cf. Eq. (7.2)), we only have to check (9.5). To see this let \(f^0, \ldots, f^m\) be in \(C^\infty(M)\) and \(\phi_0, \ldots, \phi_m\) in \(G\). We observe that the relation \(f = u_\psi^{-1}(f \circ \phi^{-1}) u_\psi\) for all \(f \in C^\infty(M)\) and \(\phi \in G\) implies that \(\varphi(f^0 u_{\phi_0}, \ldots, f^m u_{\phi_m})\) is equal to
\[
\varphi(f^0 u_{\phi_0}, u_{\phi_0}^{-1}(f \circ \phi_0^{-1}) u_{\phi_0 \circ \phi_1}, \ldots, u_{\phi_0 \circ \cdots \circ \phi_m}^{-1}(f^m \circ (\phi_0 \circ \cdots \circ \phi_{m-1})^{-1}) u_{\phi_0 \circ \cdots \circ \phi_m}).
\]
Combining this with (9.3) yields (9.5) and shows that \(\varphi\) satisfies (ii). The proof is complete. \(\square\)

Remark 9.5. As we shall see (cf. Proposition 10.1), for any given \(G\)-invariant metric \(g \in \mathcal{C}\), the cocycle given by (8.2) is \(G\)-normalized. In particular, this implies that Connes’ transverse fundamental class cocycle (8.3) is \(G\)-normalized.

Remark 9.6. When the metric \(g\) is \(G\)-invariant and the Dirac operator \(\mathcal{D}_g\) is invertible, it also can be shown that the cocycles \(\tau_{2q}^T\), \(q \geq \frac{n}{2}\), are \(G\)-normalized and differ from the cocycle \(\varphi\) in (8.2) by \(G\)-normalized coboundaries.

Lemma 9.7. The operators \(b\) and \(B\) preserve \(\mathbf{C}_G(A_G)\).

Proof. Let \(\varphi \in \mathbf{C}_G^m(A_G)\). The operator \(b\) is equivariant with respect to the action of \(G\). Therefore, in view of Lemma 9.4, in order to check that \(b \varphi\) is \(G\)-normalized we only have to check it satisfies the condition (9.3). Let \(a^0, \ldots, a^{m+1}\) be in \(A_G\) and \(\psi \in G\). Using (3.1) it is straightforward to check that \(b \varphi(a^0, \ldots, a^j u_\psi^{-1}, u_\psi a^{j+1}, \ldots, a^{m+1})\) agrees with \(b \varphi(a^0, \ldots, a^m)\) for \(j = 0, \ldots, m\). Likewise, it is immediate to check that \(b \varphi(a^0, \ldots, a^m u_\psi^{-1}, u_\psi a^{m+1}) - b \varphi(a^0, \ldots, a^m)\) is equal to
\[
\varphi(u_\psi a^{m+1} a^0, a^1, \ldots, a^m u_\psi^{-1}) - \varphi(a^{m+1} a^0, a^1, \ldots, a^m),
\]
which is seen to be zero by using (9.4). Therefore, the cocycle \(b \varphi\) satisfies (9.3)-(9.4), and hence is \(G\)-normalized. This shows that the operator \(b\) preserves the \(G\)-normalization condition.

As \(B = (1 - T)B_0 T\), in order to check that the \(G\)-normalization condition is preserved by the operator \(B\), it is sufficient to check this property for the operators \(T\) and \(B_0\). Let \(\varphi \in \mathbf{C}_G^m(A_G)\). As with the coboundary \(b \varphi\) above, in order to check that \(T \varphi\) and \(B_0 \varphi\) are \(G\)-normalized we only have to check they satisfy (9.3). Let \(\psi\) be in \(G\) and let \(a^0, \ldots, a^m\) be in \(A_G\). Using (3.3)
it is straightforward to check that \((T\varphi)(a^0, \ldots, a^j, u_\varphi^{-1}, u_\varphi a^{j+1}, \ldots, a^m) = (T\varphi)(a^0, \ldots, a^m)\) for \(j = 0, \ldots, m - 2\). In addition, \((T\varphi)(a^0, a^1, \ldots, a^{m-1} u_\varphi^{-1}, u_\varphi a^m) = (T\varphi)(a^0, \ldots, a^m)\) is equal to
\[(-1)^m \varphi(u_\varphi a^m, a^0, a^1, \ldots, a^{m-1} u_\varphi^{-1}) = (-1)^m \varphi(a^m, a^0, \ldots, a^{m-1}),\]
which like (9.9) is seen to be zero by using (9.4). This implies that \(T\varphi\) satisfies (9.3)-(9.4), and hence is \(G\)-normalized.

It is also immediate to check that \(B_0 \varphi(a^0, a^1, \ldots, a^{m-2}, a^m) = \varphi(u_\varphi^{-1} a^0, a^1, \ldots, a^{m-2}, a^{-1} u_\varphi^{-1}) = B_0 \varphi(a^0, \ldots, a^m)\). This implies that \(\psi\) given by
\[\psi(u_\varphi^{-1} a^0, a^1, \ldots, a^{m-2}, a^{-1} u_\varphi^{-1}) = \varphi(u_\varphi^{-1} a^0, a^1, \ldots, a^{m-2}, a^{-1} u_\varphi^{-1}) = B_0 \varphi(a^0, \ldots, a^m)\].
Thus, the cochain \(B_0 \varphi\) satisfies (9.3)-(9.4), and hence is \(G\)-normalized. All this shows that the \(G\)-normalization condition is preserved by the operators \(T\) and \(B_0\). The proof is complete. \(\square\)

It follows from Lemma 9.7 that we obtain a mixed sub-complex \((\mathcal{C}^*_G(A_G), b, B)\) of the cyclic mixed complex \((\mathcal{C}^*_G(A_G), b, B)\). In particular we obtain a periodic complex \((\mathcal{C}^*_G(A_G), b + B)\), where \(\mathcal{C}^*_G(A_G) = \bigoplus_{i \geq 0} \mathcal{C}^{2i}_G(A_G), i = 0, 1\). We denote by \(\text{HP}^*_G(A_G)\) the cohomology of this complex. We note that the inclusion of \(\mathcal{C}^*_G(A_G)\) into \(\mathcal{C}^*_G\) gives rise to an injective linear map,
\[(9.10) \quad \iota_* : \text{HP}^*_G(A_G) \rightarrow \text{HP}^*_G(A_G)\].

Remark 9.8. As we shall see (cf. Proposition 9.12), the results of [BN, Cr] imply that this map is actually an isomorphism. There are similar isomorphisms at level of Hochschild cohomology and (non-periodic) cyclic cohomology.

9.3. \(G\)-normalized chains. The \(G\)-normalization of chains is defined as follows. There is a natural action of \(G\) on \(\mathcal{C}^*_G(A_G)\) so that the action of \(\psi \in G\) on a chain \(a^0 \otimes \cdots \otimes a^m\), \(a^j \in A_G\), is given by
\[(9.11) \quad \psi(a^0 \otimes \cdots \otimes a^m) := (u_\psi^{-1} a^0 u_\psi^{-1}) \otimes \cdots \otimes (u_\psi^{-1} a^m u_\psi^{-1}).\]
For \(j = 0, \ldots, m\) we denote by \(\psi_{*, m, j}\) the endomorphism of \(\mathcal{C}^*_G(A_G)\) defined by
\[\psi_{*, m, j}(a^0 \otimes \cdots \otimes a^m) = \left\{ \begin{array}{ll}
\psi_{*, m, j}(a^0 \otimes \cdots \otimes a^m) = a^0 \otimes \cdots \otimes a^j, & j \neq 0, \ldots, m - 1, \\
& a^0 \otimes \cdots \otimes a^j u_\psi^{-1} \otimes u_\psi a^{j+1} \otimes \cdots \otimes a^m & \text{for } j = 0, \ldots, m - 1, \\
& u_\psi a^0 \otimes a^1 \otimes \cdots \otimes a^{m-1} u_\psi^{-1} & \text{for } j = m.
\end{array} \right.\]
In addition, we have an inhomogeneization linear map \(\theta : \mathcal{C}^*_G(A_G) \rightarrow \mathcal{C}^*_G(A_G)\) such that, for all \(f^0, \ldots, f^m\) in \(A\) and \(\phi_0, \ldots, \phi_m\) in \(G\),
\[(9.12) \quad \theta (f^0 u_{\phi_0} \otimes \cdots \otimes f^m u_{\phi_m}) = f^0 \otimes f^1 \circ \phi_0^{-1} \otimes \cdots \otimes (f^m \circ \phi_{m-1}^{-1} \circ \cdots \circ \phi_0^{-1}) u_{\phi_0 \cdots \phi_m}\].
We note that \(\theta\) is an idempotent, i.e., \(\theta^2 = \theta\). We also observe that
\[\theta (C_m(A_G)) = \text{Span} \left\{ f^0 \otimes \cdots \otimes f^m u_{\phi}; \ f^j \in A, \ \phi \in G \right\}.\]
For \(m \in \mathbb{N}_0\), set
\[N^G_m(A_G) = \sum_{j=0}^{m} \sum_{\psi \in G} (\psi_{*, m, j} - \text{id}) (C_m(A_G))\].

Definition 9.9. For \(m \in \mathbb{N}_0\), the space of \(G\)-normalized \(m\)-chains is the topological quotient,
\[\mathcal{C}^G_m(A_G) = C_m(A_G) / N^G_m(A_G).\]

Remark 9.10. \(\mathcal{C}^G_m(A_G)\) is naturally identified with the topological dual of \(\mathcal{C}^G_m(A_G)\).

Lemma 9.11. Let \(m \in \mathbb{N}_0\). Then
\[(9.13) \quad N^G_m(A_G) = \sum_{j=0}^{m-1} \sum_{\psi \in G} (\psi_{*, m, j} - \text{id}) (C_m(A_G)) + \sum_{\psi \in G} (\psi - \text{id}) (C_m(A_G))\]
\[(9.14) \quad = \ker \theta \cap C_m(A_G) + \sum_{\psi \in G} (\psi - \text{id}) \circ \theta (C_m(A_G)).\]
\textbf{Proof.} A (pre-)dualization of the arguments of the proof of Lemma 9.4 together with (9.12) gives
\begin{align*}
N_{m}^{G}(A_{G}) &= \sum_{j=0}^{m-1} \sum_{\psi \in G} \phi_{m,j} - \text{id}) (C_{m}(A_{G})) + \sum_{\psi \in G} (\psi - \text{id}) (C_{m}(A_{G})) \\
&= (\theta - \text{id}) (C_{m}(A_{G})) + \sum_{\psi \in G} (\psi - \text{id}) \circ \theta (C_{m}(A_{G})).
\end{align*}

The proof is completed by noting that the idempotency of $\theta$ implies that $\text{ran}(\theta - \text{id}) = \ker \theta$. \hfill $\square$

In what follows, given $a^{0}, \ldots, a^{m}$ in $A_{G}$, we shall denote by $a^{0} \hat{\otimes} \cdots \hat{\otimes} a^{m}$ the class of the chain $a^{1} \otimes \cdots \otimes a^{m}$ modulo $N_{m}^{G}(A_{G})$. As an immediate consequence of the very definition of $N_{m}^{G}(A_{G})$ and Lemma 9.11 we obtain that, for all $a^{0}, \ldots, a^{m}$ in $A_{G}$ and $\psi \in G$, we have
\begin{align*}
(9.15) & \quad a^{0} \hat{\otimes} \cdots \hat{\otimes} a^{j} \hat{\otimes} \cdots \hat{\otimes} a^{m} = a^{0} \hat{\otimes} \cdots \hat{\otimes} a^{m} \quad \text{for } j = 1, \ldots, m - 1, \\
(9.16) & \quad u_{\psi} a^{0} \hat{\otimes} \cdots \hat{\otimes} a^{j} \hat{\otimes} \cdots \hat{\otimes} a^{m} u_{\psi}^{-1} = u_{\psi} a^{0} \hat{\otimes} \cdots \hat{\otimes} a^{m} u_{\psi}^{-1} = a^{0} \hat{\otimes} \cdots \hat{\otimes} a^{m}.
\end{align*}

It also follows from (9.14) that, for all $f^{0}, \ldots, f^{m}$ in $A$ and all $\phi_{0}, \ldots, \phi_{m}, \phi, \psi \in G$, we have the following identities:
\begin{align*}
(9.17) & \quad (f^{0} \circ \psi^{-1}) \hat{\otimes} \cdots \hat{\otimes} (f^{m} \circ \psi^{-1}) u_{\psi^{-1} \circ \psi}^{-1} = (f^{0} \hat{\otimes} \cdots \hat{\otimes} f^{m} \hat{\otimes} u_{\phi}, \\
(9.18) & \quad f^{0} u_{\phi_{0}} \hat{\otimes} \cdots \hat{\otimes} f^{m} u_{\phi_{m}} = f^{0} \hat{\otimes} f^{1} \hat{\otimes} \cdots \hat{\otimes} f^{m} \hat{\otimes} \phi_{m-1} \circ \cdots \circ \phi_{0}^{-1} u_{\phi_{0} \cdots \phi_{m}}.
\end{align*}

The operators $b$ and $B$ descend to continuous operators on $G$-normalized chains. Therefore, we obtain a mixed chain-complex $(C_{\bullet}^{G}(A_{G})), b, B, C_{G}(A_{G}))$ and a periodic complex $(C_{\bullet}^{G}(A_{G}), b + B)$, where $C_{G}^{G}(A_{G}) = \prod_{G \geq 0} C_{G}^{G}(A_{G}), i = 0, 1$. We shall denote by $H_{C}^{G}(A_{G})$ the homology of this periodic complex.

The canonical projection $\pi : C_{\bullet}(A_{G}) \rightarrow C_{\bullet}^{G}(A_{G})$ is a morphism of mixed complexes, and hence gives rise to linear map,
\[ \pi_{\star} : H_{C}^{G}(A_{G}) \rightarrow H_{C}^{G}(A_{G}). \]

In addition, the duality between $C_{\bullet}^{G}(A_{G})$ and $C_{\bullet}^{-G}(A_{G})$ yields a duality pairing,
\[ \langle \cdot, \cdot \rangle : H_{C}^{-G}(A_{G}) \times H_{C}^{G}(A_{G}) \rightarrow C. \]

In particular, the linear map $\iota_{\star}$ given by (9.10) is related to the linear map $\pi_{\star}$ by
\[ \iota_{\star} \varphi, \eta = \langle \varphi, \pi_{\star} \eta \rangle \quad \forall \varphi \in H_{C}^{-G}(A_{G}) \forall \eta \in H_{C}^{G}(A_{G}). \]

\textbf{Proposition 9.12.} The linear map $\pi_{\star}$ is an isomorphism.

\textbf{Proof.} As $\pi$ is onto, we only have to establish that $\pi_{\star}$ is one-to-one. That result is a consequence of Proposition 4.1 of [Cr]. To see this, it is convenient to identify each chain space $C_{m}^{G}(A_{G})$ with the function space $C_{m}^{\infty}(M^{m+1} \times G^{m+1})$ in such way that any $m$-cochain $f^{0} u_{\phi_{0}} \cdots \otimes f^{m} u_{\phi_{m}}$ with $f^{j} \in A$ and $\phi_{j} \in G$ is identified with the function,
\[ (x_{0}, \ldots, x_{m}, \psi_{0}, \ldots, \psi_{m}) \rightarrow f^{0}(x_{0}) \cdots \otimes f^{m}(x_{m}) \delta_{\phi_{0}}(\psi_{0}) \cdots \delta_{\phi_{m}}(\psi_{m}), \]
where $\delta_{\phi_{j}}(\psi)$ is the characteristic function of $\{ \phi_{j} \}$ on $G$. Under this identification an $m$ cochain appears as a finite sum,
\[ \eta = \sum_{\eta_{\phi_{0}, \ldots, \phi_{m}}} \delta_{\phi_{0}}(\psi_{0}) \cdots \delta_{\phi_{m}}(\psi_{m}), \quad \eta_{\phi_{0}, \ldots, \phi_{m}} \in C_{m}^{\infty}(M^{m+1}). \]

In addition, the Burghelea space $B^{(m)}$ is the subspace of $M^{m+1} \times G^{m+1}$ given by
\[ B^{(m)} := \{ (x_{0}, \ldots, x_{m}, \phi_{0}, \ldots, \phi_{m}) ; x_{j+1} = \phi_{j}^{-1}(x_{j}), j = 0, \ldots, m - 1, x_{0} = \phi_{m}^{-1}(x_{m}) \}. \]

In the terminology of [Cr] this the $(m + 1)$-th Burghelea space of the transformation groupoid associated to the right action $(x, \phi) \rightarrow \phi^{-1}(x)$ of $G$ on $M$. We also note that
\[ B^{(m)} = \bigcup_{\phi \in G} \{ (x, \phi_{m}^{-1}(x), \ldots, \phi_{m}^{-1} \circ \cdots \circ \phi_{0}^{-1}(x), \phi_{0}, \ldots, \phi_{m}) ; x \in M^{\phi}, \phi_{0} \circ \cdots \circ \phi_{m} = \phi \}. \]
Bearing this in mind, by [Cr, Proposition 4.1] a periodic cycle $\eta = (\eta^{2q+i})_{q \geq 0}$ in $C_{ij}^G(A_G)$, $i = 0, 1$, is a boundary if and only if it is cohomologous to a cycle $\tilde{\eta} = (\tilde{\eta}^{2q+i})_{q \geq 0}$ such that

$$ \left(\tilde{\eta}^{2q+i}\right)_{|B^{2q+i}} = 0 \quad \text{for all } q \geq 0. $$

Thus in order to prove that (9.21) we observe that $\eta$ is trivial, since cyclic homology is invariant under inner automorphisms (cf. [Co4, Lo]).

Combining this with (9.14) we deduce that in order to check that any periodic cycle contained in $\Pi N^G_{2q+i}(A_G)$ is cohomologous to a cycle of the form (9.21) in $\Pi N^G_{2q+i}(A_G)$ it is sufficient to show that any chain in $\ker \theta \cap C^G_m(A_G)$ vanishes on the Burghelea space $B^{(m)}$. To see this we observe that if $\eta$ is an $m$-chain of the form (9.20), then $(\theta \eta)(x_0, \ldots, x_m, \psi_0, \ldots, \psi_m)$ is equal to

$$ \sum_{\phi_0, \ldots, \phi_m} \eta_{\phi_0, \ldots, \phi_m} (x_1, \phi_0^{-1}(x_1), \ldots, (\phi_m^{-1} \circ \cdots \circ \phi_0^{-1})(x_m)) \delta_1(\psi_0) \cdots \delta_1(\psi_m) \delta_{\phi_0} \cdots \delta_{\phi_m}(\psi_m). $$

Therefore, we see that $\theta \eta = 0$ if and only if

$$ \sum_{\phi_0, \ldots, \phi_m} \eta_{\phi_0, \ldots, \phi_m} (x_0, \phi_0^{-1}(x_1), \ldots, (\phi_m^{-1} \circ \cdots \circ \phi_0^{-1})(x_m)) = 0 \quad \forall \phi \in G. $$

This condition clearly implies that $\eta$ vanishes on $B^{(m)}$. Therefore, we see that the kernel of $\theta$ consists of chains vanishing on Burghelea spaces. This completes the proof.

**Remark 9.13.** A dual version of Proposition 9.12 shows that the map $\iota_* : \text{HP}^*_G(A_G) \rightarrow \text{HP}^*_G(A_G)$ is an isomorphism as well.

### 9.4. Splitting along conjugacy classes.

The same way as in [BN, Co4, Cr, FT, Ni1, Ni2], the cyclic homologies $\text{HP}^*_G(A_G)$ and $\text{HP}^*_G(A_G)$ split along the set $\langle G \rangle$ of conjugacy classes of $G$ as follows. Let $\phi \in G$ and denote by $\langle \phi \rangle$ its conjugacy class. For $m \in \mathbb{N}_0$, define

$$ C^G_m(A_G)_\phi = C^G_m(A_G)_\phi \quad \text{and} \quad C^G_m(A_G)_\phi = C^G_m(A_G)_\phi $$

where

$$ C^G_m(A_G)_\phi = \text{Span} \left\{ f^0 u_{\phi_0} \otimes \cdots \otimes f^m u_{\phi_m} ; \quad f^j \in C^\infty(M), \quad \phi_j \in G, \quad \phi_0 \circ \cdots \circ \phi_m \in \langle \phi \rangle \right\}, $$

$$ C^G_m(A_G)_\phi = \text{Span} \left\{ f^0 u_{\phi_0} \otimes \cdots \otimes f^m u_{\phi_m} ; \quad f^j \in C^\infty(M), \quad \phi_j \in G, \quad \phi_0 \circ \cdots \circ \phi_m \in \langle \phi \rangle \right\}. $$

We also set

$$ N^G_m(A_G)_\phi = \sum_{j=0}^m \sum_{\psi \in G} (\psi_{\ast, m-j} - \text{id}) (C^G_m(A_G)_\phi), $$

$$ = \ker \theta \cap C^G_m(A_G)_\phi + \sum_{\psi \in G} (\psi_{\ast} - \text{id}) \circ \theta (C^G_m(A_G)_\phi). $$

We observe that

$$ \theta (C^G_m(A_G)_\phi) = \text{Span} \left\{ f^0 \otimes \cdots \otimes f^{m-1} \otimes f^m u_{\kappa^{-1} \circ \phi_m \kappa} ; \quad f^j \in A, \quad \kappa \in G \right\}. $$

Moreover, the canonical projection $C^G_m(A_G) \rightarrow C^G_m(A_G)$ induces a topological vector space isomorphism,

$$ C^G_m(A_G)_\phi \simeq C^G_m(A_G)_\phi / N^G_m(A_G)_\phi. $$

Bearing this in mind, we have the obvious splitting,

$$ C^G_m(A_G) = \bigoplus_{\langle \phi \rangle \in \langle G \rangle} C^G_m(A_G)_\phi, $$

with a similar decomposition for each chain space $C^G_m(A_G)$. We also observe that the condition $\phi_0 \circ \cdots \circ \phi_m \in \langle \phi \rangle$ is preserved by any cyclic permutation of the diffeomorphisms $\phi_0, \ldots, \phi_m$. It then follows that the space $C^G_m(A_G)_\phi$ is preserved by the action of the cyclic homology operators $b, T, B_0, B, S$. Therefore, we obtain a mixed sub-complex $(C^G_m(A_G)_\phi, b, B)$ of the mixed complex $(C^G_\bullet(A_G), b, B)$. This yields a periodic complex $(C^G_\bullet(A_G)_\phi, b + B)$, where $C^G_0(A_G)_\phi = \text{ker} \theta \cap C^G_m(A_G)_\phi + \sum_{\psi \in G} (\psi_{\ast} - \text{id}) \circ \theta (C^G_m(A_G)_\phi).$
\[ \prod_{q=0}^{\infty} C_{2q+i}^G(A_G)_\phi, \ i = 0, 1. \] The direct sum over conjugacy classes of these complexes yields a complex \((\bigoplus_{(\phi) \in (G)} C_{2q}^G(A_G)_\phi, b + B)\). The splitting (9.24) then yields an inclusion of complexes,

(9.25)
\[ \bigoplus_{(\phi) \in (G)} C_{2q}^G(A_G)_\phi \to C_{2q}^G(A_G). \]

9.5. **Twisted cyclic mixed complexes.** Let \( \phi \in G \). The twisted cyclic mixed complex \((C_\bullet(A), b_\phi, B_\phi)\) (cf. [FT, Ni1, Cr]) is defined as follows. The boundary \( b_\phi : C_m(A) \to C_{m-1}(A) \) is given by

\[ b_\phi(f^0 \otimes \cdots \otimes f^m) = \sum_{j=0}^{m-1} (-1)^j f^0 \otimes \cdots \otimes f^j f^{j+1} \otimes \cdots \otimes f^m + (-1)^m (f^m \circ \phi) f^0 \otimes \cdots \otimes f^{m-1}, \quad f^j \in A. \]

The operator \( B_\phi : C_m(A) \to C_{m+1}(A) \) is \((1 - T_\phi)B_0A_\phi, \) where the operator \( B_0 \) is defined as in (3.15) and \( A_\phi = 1 + T_\phi + \cdots T_\phi^{m-1}, \) where \( T_\phi : C_m(A) \to C_m(A) \) is given by

\[ T_\phi(f^0 \otimes \cdots \otimes f^m) = (-1)^m (f^m \circ \phi) \otimes f^0 \otimes \cdots \otimes f^{m-1}, \quad f^j \in A. \]

In addition, the group \( G \) acts on \( C_m(A) \) by

\[ \psi_*((f^0 \otimes \cdots \otimes f^m)) = f^0 \circ \psi^{-1} \otimes \cdots \otimes f^m \circ \psi^{-1}, \quad f^j \in A, \ \psi \in G. \]

Note this action agrees with the action (9.11) on chains of the form \( f^0 \otimes \cdots \otimes f^m, \ f^j \in A. \) Furthermore, this induces an action on \( C_m(A)_\phi \) by the stabilizer \( G_\phi := \{ \psi \in G; \phi \circ \psi = \psi \circ \phi \}. \)

We then observe that the operators \( b_\phi \) and \( B_\phi \) are equivariant with respect to the action of \( G_\phi. \) Therefore, we obtain a mixed complex \((C_\bullet(A)^{G_\phi}, b_\phi, B_\phi)\), where \( C_\bullet(A)^{G_\phi} \) is the space of \( G_\phi \)-invariant chains in \( C_\bullet(A). \) This yields a periodic complex \((C_\bullet(A)^{G_\phi}, b_\phi + B_\phi)\), where \( C_{i|}(A)^{G_\phi} = \bigoplus_{q \geq 0} C_{2q+i}(A)^{G_\phi}, \ i = 0, 1. \) The homology of this periodic complex is denoted by \( \text{HP}_{2q}^G(A)_\phi. \)

Bearing this in mind, the mixed complexes \((C_\bullet(A)^{G_\phi}, b_\phi, B_\phi)\) and \((C_\bullet(A)_\phi, b, B)\) are related as follows. Let \( \tilde{\chi}_\phi : C_\bullet(A)_\phi \to C_\bullet(A^{G_\phi})_\phi \) be the linear map defined by

(9.26)
\[ \tilde{\chi}_\phi(f^0 \otimes \cdots \otimes f^m) = f^0 \hat{\otimes} \cdots \hat{\otimes} f^{m-1} \hat{\otimes} f^m u_\phi, \quad \forall f^j \in A. \]

**Lemma 9.14.** The map \( \tilde{\chi}_\phi \) is a morphism of mixed complexes.

**Proof.** Let \( f^1, \ldots, f^m \) be in \( A \) and set \( \eta = f^0 \otimes \cdots \otimes f^m. \) Then

\[ b\tilde{\chi}_\phi(\eta) = \sum_{j=0}^{m-1} (-1)^j f^0 \hat{\otimes} \cdots \hat{\otimes} f^j f^{j+1} \hat{\otimes} f^j \cdots \hat{\otimes} f^m u_\phi \]
\[ + (-1)^m f^m \hat{\otimes} \cdots \hat{\otimes} f^m \hat{\otimes} f^{m-1} \hat{\otimes} f^m u_\phi + (-1)^m f^m u_\phi f^0 \hat{\otimes} \cdots \hat{\otimes} f^m f^{m-1}. \]

Using (9.16) we see that \( f^m u_\phi f^0 \hat{\otimes} \cdots \hat{\otimes} f^m f^{m-1} \) is equal to

(9.27)
\[ u_\phi(f^m \circ \phi) f^0 \hat{\otimes} \cdots \hat{\otimes} f^m f^{m-1} = (f^m \circ \phi) f^0 \hat{\otimes} \cdots \hat{\otimes} f^m f^{m-1} u_\phi. \]

It then follows that \( b\tilde{\chi}_\phi(\eta) = \tilde{\chi}_\phi(b_\phi(\eta)). \)

We also have \( B_0\tilde{\chi}_\phi(\eta) = 1 \hat{\otimes} f^0 \hat{\otimes} \cdots \hat{\otimes} f^m \hat{\otimes} f^{m-1} \hat{\otimes} f^m u_\phi = \tilde{\chi}_\phi(B_0\eta). \) Moreover, in the same way as in (9.27) we see that \( T\tilde{\chi}_\phi(\eta) \) is equal to

\[ (-1)^m f^m u_\phi \hat{\otimes} f^0 \hat{\otimes} \cdots \hat{\otimes} f^m f^{m-1} = (-1)^m (f^m \circ \phi) \hat{\otimes} f^0 \hat{\otimes} \cdots \hat{\otimes} f^m f^{m-1} u_\phi = \tilde{\chi}_\phi(T_\phi(\eta)). \]

We then deduce that \( B\tilde{\chi}_\phi(\eta) = \tilde{\chi}_\phi(B_\phi(\eta)). \) The proof is complete. \( \square \)

We shall denote by \( \chi_\phi \) the restriction of \( \tilde{\chi}_\phi \) to \( G_\phi \)-invariant chains. Thanks to Lemma 9.14 this provides us with a mixed complex morphism from \((C_\bullet(A)^{G_\phi}, b_\phi, B_\phi)\) to \((C_\bullet(A)_\phi, b, B)\).

We also observe that the linear maps \( \chi_\phi \) and \( \tilde{\chi}_\phi \) are related by means of the average map \( \Lambda_\phi : C_\bullet(A) \to C_\bullet(A)^{G_\phi} \) defined by

\[ \Lambda_\phi(\eta) = \int_{G_\phi} \psi_* \eta d\lambda(\psi), \quad \forall \eta \in C_\bullet(A). \]
where $\lambda$ is the Haar probability measure of $G_\phi$. More precisely, we have the following result.

**Lemma 9.15.** It holds that $\bar{\chi}_\phi = \chi_\phi \circ \Lambda_\phi$.

**Proof.** Let $f^0, \ldots, f^m$ be in $A$ and set $\eta = f^0 \otimes \cdots \otimes f^m$. As $\bar{\chi}_\phi$ is a continuous linear map,

$$
\chi_\phi \circ \Lambda_\phi(\eta) = \bar{\chi}_\phi \left( \int_{G_\phi} \psi_\eta d\lambda(\psi) \right) = \int_{G_\phi} \bar{\chi}_\phi(\psi, \eta) d\lambda(\psi).
$$

Let $\psi \in G_\phi$. Then

$$
\bar{\chi}_\phi \circ \psi_\eta(\eta) = \bar{\chi}_\phi \left( f^0 \circ \psi^{-1} \otimes \cdots \otimes f^m \circ \psi^{-1} \right) = f^0 \circ \psi^{-1} \hat{G} \cdots \hat{G} f^{m-1} \circ \psi^{-1} \hat{G} (f^m \circ \psi^{-1}) u_\phi.
$$

As $\phi = \psi \circ \phi \circ \psi^{-1}$ since $\psi \in G_\phi$, using (9.17) we see that $\bar{\chi}_\phi \circ \psi_\eta(\eta)$ is equal to

$$
f^0 \circ \psi^{-1} \hat{G} \cdots \hat{G} f^{m-1} \circ \psi^{-1} \hat{G} (f^m \circ \psi^{-1}) u_{\psi \circ \phi \circ \psi^{-1}} = f^0 \hat{G} \cdots \hat{G} f^{m-1} \hat{G} f^m u_\phi = \bar{\chi}_\phi(\eta).
$$

Combining this with (9.28) then gives

$$
\chi_\phi \circ \Lambda_\phi(\eta) = \int_{G_\phi} \bar{\chi}_\phi(\eta) d\lambda(\psi) = \bar{\chi}_\phi \left( \int_{G_\phi} \eta d\lambda(\psi) \right) = \bar{\chi}_\phi(\eta).
$$

This proves the lemma. \hfill \Box

We shall now construct an explicit inverse for $\bar{\chi}_\phi$. To this end let $f^0, \ldots, f^m$ be in $A$ and $\phi_0, \ldots, \phi_m$ in $G$ such that $\phi_0 \circ \cdots \circ \phi_m \in \langle \phi \rangle$, i.e., $\phi_0 \circ \cdots \circ \phi_m = \kappa^{-1} \circ \phi \circ \kappa$ for some $\kappa \in G$. Using (9.17)–(9.18) and Lemma 9.15 we observe that

$$
f^0 u_{\phi_0} \hat{G} \cdots \hat{G} f^m u_{\phi_m} = f^0 \hat{G} f^1 \circ \phi_0^{-1} \hat{G} \cdots \hat{G} (f^m \circ \phi_m^{-1} \cdots \phi_0^{-1}) u_{\kappa^{-1} \circ \phi \circ \kappa}
$$

$$
= f^0 \circ \kappa^{-1} \hat{G} f^1 \circ \phi_0^{-1} \circ \kappa^{-1} \hat{G} \cdots \hat{G} (f^m \circ \phi_m^{-1} \cdots \phi_0^{-1}) u_\phi
$$

$$
= \bar{\chi}_\phi \circ \kappa_\phi (f^0 \circ f^1 \circ \phi_0^{-1} \cdots \circ f^m \circ \phi_m^{-1} \cdots \circ \phi_0^{-1}).
$$

Combining this with Lemma 9.15 then gives

$$
f^0 u_{\phi_0} \hat{G} \cdots \hat{G} f^m u_{\phi_m} = \chi_\phi \circ \Lambda_\phi \circ \kappa_\phi (f^0 \circ f^1 \circ \phi_0^{-1} \cdots \circ f^m \circ \phi_m^{-1} \cdots \circ \phi_0^{-1}).
$$

This leads us to define

$$
\bar{\mu}_\phi(f^0 u_{\phi_0} \hat{G} \cdots \hat{G} f^m u_{\phi_m}) = \Lambda_\phi \circ \kappa_\phi (f^0 \circ f^1 \circ \phi_0^{-1} \cdots \circ f^m \circ \phi_m^{-1} \cdots \circ \phi_0^{-1})).
$$

We observe that $\kappa$ is unique up to the left-composition with a diffeomorphism in $G_\phi$. Moreover, as $\Lambda_\phi \circ \psi_\phi = \Lambda_\phi$ for all $\psi \in G_\phi$, we deduce that the r.h.s. of (9.30) is independent of the choice of $\kappa$. We thus obtain a continuous linear map $\bar{\mu}_\phi : C_\bullet(AG_\phi) \to C_\bullet(A)G^\phi$.

**Lemma 9.16.** The linear map $\bar{\mu}_\phi$ is $G_\phi$-equivariant and annihilated by $NG(AG_\phi)$.

**Proof.** In view of (9.14) we only have to show that $\bar{\mu}_\phi$ vanishes on $\ker \theta \cap C_m(AG_\phi)$ and, for every $\psi \in G$, the map $\bar{\mu}_\phi \circ (\psi_* - \text{id})$ vanishes $\theta(C_m(AG_\phi))$. It is immediate from the definition of $\bar{\mu}_\phi$ that $\bar{\mu}_\phi \circ \theta = \bar{\mu}_\phi$. As $\ker \theta \cap C_m(AG_\phi) = (\theta - \text{id}) (C_m(AG_\phi))$, it follows that $\ker \theta \cap C_m(AG_\phi)$ is contained in the kernel of $\bar{\mu}_\phi$.

Let $f^0, \ldots, f^m$ be in $A$, and $\kappa$ and $\psi$ in $G$. Then

$$
\psi_* (f^0 \otimes \cdots \otimes f^{m-1} \otimes f^m u_{\kappa^{-1} \circ \phi \circ \kappa}) = (f^0 \circ \psi^{-1} \otimes \cdots \otimes f^{m-1} \circ \psi^{-1} \otimes f^m \circ \psi^{-1} u_{(\kappa \psi^{-1})^{-1} \circ \phi \circ (\kappa \psi^{-1})}.
$$

Therefore, by the very definition (9.30) of $\bar{\mu}_\phi$ we have

$$
\bar{\mu}_\phi \circ \psi_* (f^0 \otimes \cdots \otimes f^{m-1} \otimes f^m u_{\kappa^{-1} \circ \phi \circ \kappa}) = \Lambda_\phi \circ (\kappa \circ \psi^{-1})*(f^0 \otimes \cdots \otimes f^m \circ \psi^{-1})
$$

$$
= \Lambda_\phi \circ \kappa_\phi (f^0 \circ \cdots \circ f^m)
$$

$$
= \bar{\mu}_\phi (f^0 \otimes \cdots \otimes f^{m-1} \otimes f^m u_{\kappa^{-1} \circ \phi \circ \kappa}).
$$

Combining this with (9.31) shows that $\bar{\mu}_\phi \circ (\psi_* - \text{id})$ vanishes on $\theta(C_m(AG_\phi))$. The proof is thus complete. \hfill \Box
Combining Lemma 9.16 with (9.23) shows that $\tilde{\mu}_\phi$ descends to a continuous linear map,

$$
\mu_\phi : C(G(A_\phi)) \rightarrow C_\bullet(A)^G_\phi.
$$

**Proposition 9.17.** The linear maps $\chi_\phi$ and $\mu_\phi$ are inverses of each other.

**Proof.** It immediately follows from (9.29) that $\chi_\phi \circ \mu_\phi = \text{id}$. Moreover, we have

$$
\mu_\phi \circ \chi_\phi (f^0 \otimes \cdots \otimes f^m) = \tilde{\mu}_\phi (f^0 \otimes \cdots \otimes f^m) = f^0 \otimes \cdots \otimes f^m.
$$

Thus $\mu_\phi \circ \chi_\phi = \text{id}$. The proof is complete. \qed

9.6. Concentration on fixed-point sets. Let $\phi \in G$. For $a = 0, 2, \ldots, n$ set $A_\phi^a = C^\infty(M_\phi^a)$. As the action of $G_\phi$ on $M$ preserves each fixed-point component $M_\phi^a$, we see that the restriction map $f \rightarrow f_{|M_\phi^a}$ gives rise to a $G_\phi$-equivariant continuous linear map,

$$
\tilde{\rho}_\phi^a : C_\bullet(A) \rightarrow C_\bullet(A_\phi^a).
$$

As $\phi$ acts like identity on $M_\phi^a$ this actually provides with a morphism of mixed complexes from the twisted cyclic mixed complex $(C_\bullet(A), b_\phi, B_\phi)$ to the cyclic mixed complex $(C_\bullet(A_\phi^a), b, B)$. A right-inverse of $\tilde{\rho}_\phi^a$ is obtained as follows.

Let $g$ be a $G$-invariant metric in the conformal class $\mathcal{C}$. Given $r > 0$ we denote by $B_r(TM)$ the ball bundle in $TM$ of radius $r$ about the zero-section. For $r$ small enough the exponential map is a diffeomorphism from $B_r(TM)$ onto its image. Let $N_\phi^a$ be the restriction of the normal bundle $N_\phi$ to $M_\phi^a$. This is a smooth vector bundle over $M_\phi^a$ such that the fiber at $x \in M_\phi^a$ is $(T_x M_\phi^a)^\perp \subset T_x M$. Let $p_a : N_\phi^a \rightarrow M_\phi^a$ be the corresponding fibration map. In addition, we denote by $B_r(N_\phi^a)$ the ball bundle in $N_\phi^a$ of radius $r$ about the zero-section. Then the map $\Phi_a : X \mapsto \exp_{p_a(X)}(X)$ is a diffeomorphism from $B_r(N_\phi^a)$ onto a tubular neighborhood $V_{a,r}$ of $M_\phi^a$. Composing $p_a$ with $\Phi_a^{-1}$ we then obtain a smooth fibration $\tilde{p}_a : V_{a,r} \rightarrow M_\phi^a$ such that $\tilde{p}_a|_{M_\phi^a} = \text{id}_{M_\phi^a}$.

As $G$ is contained in the isometry group of the metric $g$, its action preserves the ball bundle $B_r(TM)$ and the exponential map is a $G$-equivariant map. If we restrict this action to $G_\phi$, then the action further preserves $M_\phi^a$ and the ball bundle $B_r(N_\phi^a)$ and the fibration $p_a$ is a $G_\phi$-equivariant map. Therefore, the diffeomorphism $\Phi_a$ is a $G_\phi$-equivariant map and the action of $G_\phi$ preserves the tubular neighborhood $V_{a,r}$. It then follows that $\tilde{p}_a$ is a $G_\phi$-equivariant fibration.

In addition, let $r_1$ and $r_2$ be real numbers such that $0 < r_1 < r_2 < r$. For $j = 1, 2$ set $V_{a,r_j} = \Phi_a(B_r(N_{\phi^a}^j))$, where $B_r(N_{\phi^a}^j)$ is the ball the ball bundle in $N_{\phi^a}^j$ of radius $r_j$ about the zero-section. As $V_{a,r}$ both $V_{a,r_1}$ and $V_{a,r_2}$ are preserved by the action of $G_\phi$. Therefore, we can find a $G_\phi$-invariant function $\psi_a \in C^\infty(V_{a,r})$ such that $\psi_a = 1$ on $V_{a,r_1}$ and $\psi_a = 0$ outside $V_{a,r_2}$. Extending $\psi_a$ to be zero outside $V_{a,r}$ we obtain a $G_\phi$-equivariant smooth function on $M$ which is equal to 1 near $M_\phi^a$.

For any given $f \in C^\infty(M_\phi^a)$ we denote by $\tilde{f}$ the smooth function on $M$ defined by

$$
\tilde{f}(x) = \psi_a(x)(f \circ \tilde{p}_a)(x) \quad \forall x \in M.
$$

This provides us with a continuous linear map from $A_\phi^a$ to $A$. We note that $\tilde{f} = f$ on $M_\phi^a$, so that we have obtain a right-inverse of the restriction map $f \rightarrow f_{|M_\phi^a}$. Furthermore, given $b \in \{0, 2, \ldots, n\} \setminus \{a\}$ the fact that $M_\phi^a$ and $N_{\phi^a}^b$ are disjoint implies that

$$
M_\phi^a \cap V_{a,r} = \Phi(M_\phi^a) \cap \Phi(B_r(M_\phi^a)) = \emptyset.
$$

Therefore $\tilde{f}$ vanishes on $M_\phi^a$.

The map $f \rightarrow \tilde{f}$ gives rise to a continuous linear map $\tilde{\rho}_\phi^a : C_\bullet(A_\phi^a) \rightarrow C_\bullet(A)$, so that

$$
\tilde{\rho}_\phi^a(f^0 \otimes \cdots \otimes f^m) = \tilde{f}^0 \otimes \cdots \otimes \tilde{f}^m \quad \forall f^j \in A_\phi^a.
$$

This is a right-inverse of $\tilde{\rho}_\phi^a$. Furthermore, the $G_\phi$-equivariance of $\tilde{p}_a$ and the $G_\phi$-invariance of $\psi_a$ imply that $\tilde{\rho}_\phi^a$ is a $G_\phi$-equivariant linear map. In addition, as $\phi$ is contained in $G_\phi$ and acts like the identity on $M_\phi^a$, we further see that $\tilde{\rho}_\phi^a$ is a mixed complex morphism from $(C_\bullet(A_\phi^a), b, B)$ to $(C_\bullet(A), b_\phi, B_\phi)$. 34
As pointed out in Example 3.3, the map $\alpha_{M_0}$ given by (3.20) is a quasi-isomorphism from the periodic cyclic complex $(C_\bullet(A_0^\phi), b + B)$ to the de Rham complex $(\Omega^\bullet(M_0^\phi), d)$. A quasi-inverse (cf. [Co3]) is provided by the linear map $\beta_{M_0} : \Omega^\bullet(M_0^\phi) \to C_\bullet(A_0^\phi)$ defined by

$$\beta_{M_0}(\sigma) = \sum_{\sigma \in \mathcal{E}_m} \epsilon(\sigma) f^0 \otimes f^{\sigma(1)} \otimes \cdots \otimes f^{\sigma(m)}, \quad f^j \in A_0^\phi,$$

where $\mathcal{E}_m$ is the $m$-th symmetric group. Note that $\beta_{M_0}$ is actually a right-inverse of $\alpha_{M_0}$. Moreover, both $\alpha_{M_0}$ and $\beta_{M_0}$ are $G_\phi$-equivariant maps.

Setting $\rho_a^\phi = \alpha_{M_0} \circ \rho_a^\phi$ and $\epsilon_a^\phi = \epsilon_a^\phi \circ \beta_{M_0}$ we obtain $G_\phi$-equivariant morphisms of complexes,

$$\rho_a^\phi : C_\bullet(A) \to \Omega^\bullet(M_0^\phi) \quad \text{and} \quad \epsilon_a^\phi : \Omega^\bullet(M_0^\phi) \to C_\bullet(A),$$

such that $\epsilon_a^\phi$ is a right-inverse of $\rho_a^\phi$. Note also that

$$(9.32) \quad \epsilon_a^\phi(f^0 \sigma_1 \otimes \cdots \otimes d f^m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{E}_m} \epsilon(\sigma) f^0 \otimes f^{\sigma(1)} \otimes \cdots \otimes f^{\sigma(m)}, \quad f^j \in A_0^\phi.$$

In addition both $\rho_a^\phi$ and $\epsilon_a^\phi$ are $G_\phi$-equivariant maps, and hence descends to morphisms of complexes,

$$\rho_a^\phi : C_\bullet(A)^{G_\phi} \to \Omega^\bullet(M_0^\phi)^{G_\phi} \quad \text{and} \quad \epsilon_a^\phi : \Omega^\bullet(M_0^\phi)^{G_\phi} \to C_\bullet(A)^{G_\phi},$$

such that the latter is a right-inverse of the former.

9.7. Geometric classes in $H_{\bullet}^*(A_G).$ We are now in a position to construct geometric cyclic homology classes in $H_{\bullet}^*(A_G).$ Let $\phi \in G$ and $a \in \{0, 2, \ldots, n\}$. For any $\omega \in \Omega^m(M_0^\phi)^{G_\phi}$ we set

$$\eta_\omega = \{1/x_{\phi} \circ \epsilon_a^\phi(\omega) \in C_m^\phi(A_G).$$

If we write $\omega$ in the form $\omega = \sum f^0_1 \sigma_1 \otimes \cdots \otimes d f^m_1$, then (9.32) and (9.28) give

$$(9.33) \quad \eta_\omega = \sum f^0_j \sigma_j \otimes \cdots \otimes d f^m_j \in A_0^\phi.$$ 

The map $\omega \to \eta_\omega$ is a morphism of mixed complexes from $(\Omega^\bullet(M_0^\phi)^{G_\phi}, 0, d)$ to $(C^\bullet(A_G), b, B)$. Therefore, if $\omega \in \Omega^m(M_0^\phi)^{G_\phi}$ is closed, then $\eta_\omega$ is a cycle in $H_{\bullet}^i(A_G)$, where $i$ is the parity of $m$. We shall then denote by $[\eta_\omega]$ its class in $H_{\bullet}^i(A_G)$.

By Proposition 9.12, the canonical projection $\pi : H_{\bullet}^*(A_G) \to H_{\bullet}^*(C_\bullet(A_G))$ is an isomorphism. Denoting by $\pi^*$ its inverse we arrive at the following result.

**Proposition 9.18.** Let $\phi \in G$ and $a \in \{0, 2, \ldots, n\}$. Then any closed form $\omega \in \Omega^\bullet(M_0^\phi)^{G_\phi}$ defines a cyclic homology class $\pi^*[\eta_\omega] \in H_{\bullet}^*(A_G)$ which depends only on the class of $\omega$ in $H^\bullet(M_0^\phi)^{G_\phi}$.

9.8. Proof of Proposition 9.1. Although this is not our main purpose, let us briefly disgress and explain how the previous constructions enable us to compute the cyclic homology of $A_G$.

Given $\phi \in G$ and using the splitting $\Omega^\bullet(M^\phi) = \bigoplus_a \Omega^\bullet(M_0^\phi)$ we may combine together all the maps $\rho_a^\phi$ (resp., $\epsilon_a^\phi$), $a = 0, 2, \ldots, n$, so as to get linear maps,

$$(9.34) \quad \rho : C_\bullet(A)^{G_\phi} \to \Omega^\bullet(M^\phi)^{G_\phi} \quad \text{and} \quad \epsilon : \Omega^\bullet(M^\phi)^{G_\phi} \to C_\bullet(A)^{G_\phi},$$

such that $\epsilon$ is a right-inverse of $\rho$. These maps are mixed complex morphisms between the de Rham mixed complex $(\Omega^\bullet(M^\phi)^{G_\phi}, 0, d)$ and the twisted cyclic mixed complex $(C_\bullet(A)^{G_\phi}, b_\phi, B_\phi)$. A result of Brylinski-Nistor [BN, Lemma 5.2] ensures us that $\rho_\phi$ gives rise to a quasi-isomorphism from the twisted Hochschild complex $(C_\bullet(A)^{G_\phi}, b_\phi)$ to $(\Omega^\bullet(M_0^\phi)^{G_\phi}, 0)$. As this is a morphism of mixed complexes, a routine exercise in cyclic homology shows that $\rho_\phi$ gives rise to a quasi-isomorphism from the periodic complex $(C_\bullet(A)^{G_\phi}, b_\phi + B_\phi)$ to the de Rham complex $(\Omega^\bullet(M^\phi)^{G_\phi}, d)$. Using the $G_\phi$-equivariance of $\rho_\phi$ and Proposition 9.17 we then obtain the isomorphisms,

$$H_{\bullet}^*(A_G) \simeq H_{\bullet}^*(C_\bullet(A)) \simeq H^\bullet(M^\phi) = \bigoplus_a H^\bullet(M_0^\phi)^{G_\phi}. $$

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This implies that the inclusion \( \omega \to \eta_\omega \) in Proposition 9.18 is actually a quasi-isomorphism. Combining all this with Proposition 9.12 gives the isomorphisms,

\[
\text{HP}_* (A_G) \cong \text{HP}_*^G (A_G) \cong \bigoplus_{(\phi) \in (G)} \text{HP}_* (A_G)_\phi \cong \bigoplus_{(\phi) \in (G)} H^*(M^\phi)_G.
\]

This proves Proposition 9.1. In particular, this shows that \( \text{HP}_* (A_G) \) is spanned by the cyclic homology classes provided by Proposition 9.18.

10. Conformal Invariants

In this section, we shall combine together the results from the previous sections to construct and compute a wealth of new conformal invariants. We shall continue using the notation and assumptions of the previous sections.

To a large extent the introduction of \( G \)-normalized cochains in the previous section was motivated by the following result.

**Proposition 10.1.** Let \( g \) be a \( G \)-invariant metric in \( \mathcal{C} \). Then the cocycle \( \varphi = (\varphi_{2q}) \) defined by (8.2) is \( G \)-normalized.

**Proof.** Let \( q \) be a nonnegative integer \( \leq \frac{1}{2} n \). For \( j = 0, \ldots, 2q \) let \( f^j \in C^\infty (M) \) and \( \phi_j \in G \). It immediately follows from the formula (8.2) that

\[
\varphi_{2q} (f^0 u_{\phi_0}, \ldots, f^{2q} u_{\phi_{2q}}) = \varphi_{2q} (f^0, f^1 \circ \phi_0^{-1}, \ldots, (f^{2q} \circ \phi_{2q-1} \circ \cdots \circ \phi_0^{-1}) u_{\phi_0 \circ \cdots \circ \phi_{2q}}).
\]

In addition, let \( \phi \) and \( \psi \) be in \( G \), and set \( \hat{\phi} = \psi \circ \phi \circ \psi^{-1} \). Then, for \( a = 0, 2, \ldots, n \), we have

\[
\int \int \int \hat{A} \left( R^{TM^\phi} \right) \wedge \nu_\phi \left( R^{N^\psi} \right) \wedge (f^0 \circ \psi^{-1}) d(f^1 \circ \psi^{-1}) \wedge \cdots \wedge d(f^{2q} \circ \psi^{-1})
\]

\[
= \int \int \int \hat{A} \left( R^{TM^\phi} \right) \wedge \nu_\phi \left( R^{N^\psi} \right) \wedge f^0 df^1 \wedge \cdots \wedge df^{2q}.
\]

Thus,

\[
\varphi_{2q} (f^0 \circ \psi^{-1}, \ldots, f^{2q} \circ \psi^{-1}, (f^{2q} \circ \psi^{-1}) u_{\hat{\phi}}) = \varphi_{2q} (f^0, \ldots, f^{2q} \circ \psi^{-1}, f^{2q} u_{\hat{\phi}}).
\]

Combining all this with Lemma 9.4 shows that \( \varphi_{2q} \) is \( G \)-normalized, proving the lemma. \( \square \)

We are now in a position to extract the geometric contents of Theorem 9.1. More precisely, we have the following result.

**Theorem 10.2.** Given \( \phi \in G \) and \( a \in \{0, 2, \ldots, n\} \), let \( \omega \) be a \( G_\phi \)-invariant closed even form on \( M^\phi \). For any metric in the conformal class \( g \in \mathcal{C} \) define

\[
I_g (\omega) = \left< \text{Ch}(D_g)_{\pi_\sigma}, \pi^* [\eta_\omega] \right>,
\]

where \( \text{Ch}(D_g)_{\pi_\sigma} \) is the Connes-Chern character of the conformal Dirac spectral triple associated to the metric \( g \) and the class \( \pi^* [\eta_\omega] \in \text{HP}_0 (A_G) \) is defined as in Proposition 9.18. Then

1. The number \( I_g (\omega) \) is an invariant of the conformal structure \( \mathcal{C} \) which depends only on the class of \( \omega \) in \( H^0 (M^\phi)_G \).
2. For any \( G \)-invariant metric \( g \in \mathcal{C} \), we have

\[
I_g (\omega) = (-i)^{\frac{n}{2}} (2\pi)^{-\frac{n}{2}} \int \hat{A} \left( R^{TM^\phi} \right) \wedge \nu_\phi \left( R^{N^\psi} \right) \wedge \omega.
\]

**Proof.** The first part is an immediate consequence of Theorem 7.8 and Proposition 9.18. To prove the second part, let us assume that \( g \) is a \( G \)-invariant metric in \( \mathcal{C} \). Then by Theorem 8.3 the Connes-Chern character \( \text{Ch}(D_g)_{\pi_\sigma} \) is represented by the cocycle \( \varphi = (\varphi_{2q}) \) given by (8.2). By Proposition 10.1 this cocycle is \( G \)-normalized, so that \( \text{Ch}(D_g)_{\pi_\sigma} = \iota_* [\varphi] \). Therefore, using (9.19) we get

\[
I_g (\omega) = \left< \iota_* [\varphi], \pi^* [\eta_\omega] \right> = \left< \varphi, \eta_\omega \right> = \sum_{q \geq 0} \left< \varphi_{2q}, \eta_{2q} \omega \right>,
\]

where \( \eta_{2q} \omega \) is defined as in Proposition 9.18.
where \( \omega_{2q} \) is the component of degree \( 2q \) of \( \omega \). Let us write \( \omega_{2q} = \sum_{q=1}^{n} f^q_l df^{0}_l \wedge \cdots \wedge df^{2q}_l \), with \( f^q_l \in A^q_{(2q)} \). Then (9.33) gives

\[
\langle \varphi_{2q}, \omega_{2q} \rangle = \sum_{q=1}^{n} \sum_{\sigma \in S_{2q}} \epsilon(\sigma) \left( \varphi_{2q} \cdot f^0_l \otimes G f^{(1)}_l \otimes G \cdots \otimes G \right) f^{(2q-1)}_l \otimes G f^{(2q)}_l u_{\phi}
\]

Recall that if \( f \in A^q_{(2q)} \), then \( \tilde{f} = f \) on \( M^0_a \) and \( f = 0 \) on \( M^0_b \) with \( b \neq 0 \). Therefore, the formula (10.5) for \( \varphi_{2q} \) gives

\[
\varphi_{2q} \left( f^{0}_l, f^{(1)}_l, \cdots, f^{(m)}_l, \phi \right) = \int_{M^0_a} \Upsilon^{(2q)}_a \wedge f^{(1)}_l df^{(1)}_l \cdots \wedge df^{(2q)}_l,
\]

where we have set \( \Upsilon^{(2q)}_a = (-i)^{2q} (2\pi)^{-\frac{d}{2}} A \left( R^{M^0_a} \right) v \). Using this together with (10.5) shows that the pairing \( \langle \varphi_{2q}, \omega_{2q} \rangle \) is equal to

\[
\frac{1}{(2q)!} \sum_{q=1}^{n} \sum_{\sigma \in S_{2q}} \epsilon(\sigma) \int_{M^0_a} \Upsilon^{(2q)}_a \wedge f^{(1)}_l df^{(1)}_l \cdots \wedge df^{(2q)}_l = \int_{M^0_a} \Upsilon^{(2q)}_a \wedge \omega_{2q}.
\]

Combining this with (10.3) yields the formula (10.2) for \( I_g(\omega) \). The proof is complete.

**Remark 10.3.** The conformal invariants \( I_g(\omega) \) are closely related to conformal invariants constructed by Branson-Ørsted [BO2, §1]. It can be shown (see, e.g., [Gi] or Part II) that, given a \( G \)-invariant metric \( g \in \mathcal{C} \) and \( \phi \in G \), we have

\[
\text{Str} \left[ e^{-t\Phi^2_g \phi} U_{\phi} \right] \sim \sum_{0 \leq a \leq n} \sum_{j \geq 0} t^{-\frac{d}{2} + j} \int_{M^0_a} I_{\phi,a}^{(j)}(g)(x) \sqrt{g(x)} d^a x \quad \text{as } t \to 0^+,
\]

where the \( I_{\phi,a}^{(j)}(g) \) are smooth functions on \( M^a_a \). Using variational formulas for equivariant heat kernel asymptotics Branson-Ørsted [BO2] proved that each constant coefficient,

\[
\int_{M^0_a} I_{\phi,a}^{(a/2)}(g)(x) \sqrt{g(x)} d^a x, \quad a = 0, 2, \ldots, n,
\]

remains constant when \( g \) ranges over all \( \phi \)-invariant metrics in \( \mathcal{C} \). Thus, they give rise to invariants of the conformal class \( \mathcal{C} \). The results of Branson-Ørsted actually hold for any even power of a conformally invariant elliptic self-adjoint differential operator. Bearing this in mind, given \( \phi \in G \) and any closed \( G_\phi \)-invariant form \( \omega \in \Omega^{2q}(M^0_a) \), the computation in Part II [PW2] of the Connes-Chern character \( \text{Ch}(D_g) \) for a \( G \)-invariant metric in \( \mathcal{C} \) shows that

\[
I_g(\omega) = \lim_{t \to 0^+} t^q \text{Str} \left[ c(\omega) e^{-t\Phi^2_g \phi} U_{\phi} \right],
\]

where \( c(\omega) \) is the Clifford action of \( \omega \) on spinors. In particular, the r.h.s. above agrees with (10.1) by taking \( \omega \) to be a scalar multiple of the volume form of \( M^0_a \). Therefore, Theorem 10.2 provides us with a cohomological interpretation and explicit computations of Branson-Ørsted’s conformal invariants in the case of Dirac operators.

**Remark 10.4.** In general, the invariants of the form (10.2) are not of the same type of the global conformal invariants considered by Alexakis [AI] in his solution of the conjecture of Deser-Schwimmer [DS] on the characterization of global conformal invariants. His invariants are of the form,

\[
I_g = \int_M P_g(R)(x) \sqrt{g(x)} d^a x,
\]

where \( P_g(R)(x) \) is a linear combination of complete metric contractions of tensor products of the Riemann curvature tensor and its covariant derivatives in such way that \( I_g \) remains constant on
each conformal class of metrics. Such a conformal invariant is of the form (10.2) when $\phi = \text{id}$ and $\omega$ is a constant function. However, in this case Theorem 10.2 reduces to the conformal invariance of the total $\hat{A}$-class of $M$, which is obvious since this is a topological invariant.

Remark 10.5. As mentioned in the Introduction, it would be interesting to find, at least conjecturally, a characterization of conformal invariants encompassing the invariants of [AI, DS] and the invariants from [BO2] and Theorem 10.2.

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