Zero-threshold expansion of two-loop self-energy diagrams

F.A. Berends\textsuperscript{a,}\textsuperscript{1}, A.I. Davydychev\textsuperscript{b,}\textsuperscript{2}, V.A. Smirnov\textsuperscript{c,}\textsuperscript{3} and J.B. Tausk\textsuperscript{d,}\textsuperscript{4}

\textsuperscript{a}Instituut-Lorentz, University of Leiden, P.O.B. 9506, 2300 RA Leiden, The Netherlands

\textsuperscript{b}Department of Physics, University of Bergen, Allégaten 55, N-5007 Bergen, Norway

\textsuperscript{c}Institute for Nuclear Physics, Moscow State University, 119899 Moscow, Russia

\textsuperscript{d}Department of Physics, University of Durham, Durham DH1 3LE, England

Abstract

An algorithm is constructed to derive a small momentum expansion for two-loop two-point diagrams in all cases where, due to the presence of physical thresholds, there are singularities at zero external momentum. The coefficients of this “zero-threshold” expansion are calculated analytically for arbitrary masses. Numerical examples, using diagrams occurring in the Standard Model, illustrate the convergence of the expansion below the first non-zero threshold.

\textsuperscript{1}This research is partly supported by the EU under contract numbers CHRX-CT92-0004 and INTAS-93-744.

\textsuperscript{2}E-mail address: berends@rulgm0.leidenuniv.nl

\textsuperscript{3}On leave from Institute for Nuclear Physics, Moscow State University, 119899 Moscow, Russia. E-mail address: davyd@vsfys1.fi.uib.no

\textsuperscript{4}E-mail address: smirnov@compnet.msu.su

\textsuperscript{5}Address after October 1st, 1994: Institut für Physik, Johannes Gutenberg Universität, Staudinger Weg 7, D-55099 Mainz, Germany
1. Introduction

The increasing precision of experiments [1, 2] testing the Standard Model (SM) of electroweak interactions will demand a matching precision from the theoretical predictions. The present LEP/SLC experiments measure the $Z$ mass, its total and partial widths, and various asymmetries with high accuracy. Theoretical predictions for these quantities depend on the unknown top quark and Higgs boson masses so that some freedom is left to fit theory to experiment. When knowledge of the top mass from direct detection becomes available the possibility to adjust $m_t$ in the theoretical predictions will be narrowed and the tests will become more stringent. At present there are hints of a value for the top mass of $174 \pm 10^{+13}_{-12} \text{GeV}$ [3].

A similar situation holds for the $W$ mass. At present its measured and predicted (from the Fermi constant $G_F$) values are in agreement when suitable $m_t$ and $M_H$ values are chosen. However the expected higher accuracy on $M_W$ from LEP2 measurements [4] will test the theory more severely.

When at some point in the future discrepancies arise between theoretical predictions and experiment it will be tempting to invoke extensions of the SM. This can only be done convincingly when SM predictions really fail. This brings us to the question how accurate the present theoretical calculations are.

The one-loop predictions of the SM have been studied by many people and are incorporated in various fitting programs in use at LEP1. Some special second order effects are often included as well. Although in principle this should be well under control, in practice there are subtle differences in implementing the theory, which may cause small deviations in theoretical predictions. An analysis of this situation is at the moment in progress [5].

For some quantities one-loop calculations are definitely not adequate and thus the dominant parts of some second order contributions should be added. The clearest examples are various QED radiative corrections to the $Z$-line shape [3] and to small angle Bhabha scattering [6, 7, 8]. The former are known to the desired accuracy, the latter not. For small angle Bhabha scattering one could for instance be interested in double box diagrams [10], which involve massless and massive particles. Examples of second order corrections in the non-QED part of the theory are contributions to the $\rho$ parameter [11, 12] and to $Z \rightarrow b \bar{b}$ decay, for a review see [13]. Here one encounters two-loop diagrams with many massive particles. The present calculations consider some limiting situations of heavy top mass or heavy Higgs mass, which allow the neglect of $M_Z$ or $M_W$ masses.

In the future these types of approximations will become less acceptable. Therefore it seems unavoidable that physics issues will dictate the study of exact two-loop corrections to the electroweak theory.

This is why during the last few years a number of studies have been made into the question of massive two-loop diagrams, the purely massless case being much better understood. It is a sensible strategy to study firstly the simplest two-loop diagrams, the self-energy diagrams. Not surprisingly, most papers deal with those diagrams. From a physical point of view this can also be justified, since at the one-loop level these diagrams are usually the most relevant ones. Since it has been shown that all two-loop self-energy diagrams can be reduced to scalar diagrams [14] (see also [15]), it is sufficient to consider only the latter ones.
The general case of massive two-loop diagrams occurs to be very complicated, even for two-point (self-energy) functions. Exact results in terms of known special functions are known for a restricted number of special cases only (some of them can be found in [16, 17, 18]). In all these examples at least one of the masses is equal to zero. A difficult problem, unknown at the one-loop level, arises when a diagram contains a three-particle cut where all particles are massive. There are some arguments that this result expanded around four dimensions is for general $k^2$ not expressible in terms of known functions like polylogarithms [19] (see also in [16] where a special case of such a diagram was investigated).

To be able to deal with diagrams that have not been evaluated exactly, a number of approaches have been developed, and some useful results have been obtained. One method is a reduction to a two-dimensional integral representation, both for convergent [20] and divergent [21] cases. A number of one-dimensional integral representations with more complicated integrands are available [23]. Some other methods [24, 25] can also be employed, although they involve representations with a larger number of parametric integrals. In all these cases, the final result is obtained by numerical integration.

When one is exclusively interested in numbers and less in the analytic structure the above method is sufficient, although there is always the question whether the numerical program works correctly in all cases. When one wants to understand the analytic structure and also wants to have another, independent method, series expansions in the variables of the problem, i.e. $k^2$ and the masses, provide a good alternative. Moreover, this approach can be very useful when some of the masses are unknown, because it gives analytic expressions for the coefficients of the expansion.

For a general massive scalar two-loop self-energy diagram the following expansions in $k^2$ have been studied. For small $k^2$ an algorithm was developed for the construction of the Taylor series coefficients [26] (the three-point case was considered in [27]). The expansion is valid below the smallest physical threshold. For large $k^2$ again an algorithm for the evaluation of the $1/k^2$ expansion coefficients was developed [28]. This is based on general results on asymptotic expansions of Feynman diagrams [24] where the coefficients are conveniently expressed in terms of more simple diagrams of a certain type. This expansion is valid above the largest physical threshold. Although general explicit expressions for coefficients of arbitrary order have not been obtained in such a way, for some special diagrams multiple series in $k^2$ and the masses, with closed expressions for the coefficients, are now available [23]. These multiple series expansions converge in certain regions and correspond to generalized hypergeometric functions.

From this discussion it is clear that there are $k^2$ regions, where no expansion is available. Namely, the behaviour above the smallest but below the largest physical threshold needs to be described. It is the purpose of the present paper to start to fill this gap. Clearly filling the most general gap would be the best. But we are going to start by filling a smaller gap, corresponding to the cases when the lowest physical threshold is zero, so besides massive particles some massless particles are present in the diagram. These are cases when an ordinary Taylor expansion [26] does not work because of infrared singular-

---

1 Analogous integral representations can be written also for three-point two-loop diagrams [22].

2 In ref. [27], conformal mapping and Padé approximations were also employed for numerical study of the behaviour above the threshold.
ities connected with the limit \( k^2 \to 0 \). By doing this, we cover all remaining holes in the small momentum expansion. Moreover, we expect that this specific case may be helpful for the most general problem of describing threshold behaviour.

In this paper, we are going to examine the region between the \( k^2 = 0 \) threshold and the first non-zero threshold. It will be shown that general results on asymptotic expansions of Feynman diagrams [29] can also be employed in this case.

The actual outline of the paper is as follows. In section 2 the algorithm for the expansion is constructed from the general theorem. All possible zero-threshold cases are considered. The actual calculation of several coefficients is presented in section 3. Also a comparison with some known analytical results is made. Section 4 contains numerical comparisons for some complicated, analytically unknown cases. Conclusions are given in section 5. In the appendices we present some formulae for two-loop vacuum integrals and discuss the evaluation of massless integrals with numerators.

2. Constructing the expansion

As in the previous papers [26, 28], we shall consider here two-loop Feynman integrals corresponding to the self-energy diagram of Fig. 1a, denoting them as

\[
J (\{\nu_i\}; \{m_i\}; k) = \int \int \frac{d^n p \, d^n q}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4} D_5^{\nu_5}},
\]

where \( n = 4 - 2\varepsilon \) is the space-time dimension (in the framework of dimensional regularization [30]), and \( \nu_i \) are the powers of the denominators \( D_i \equiv p_i^2 - m_i^2 + i0, p_i \) being the momentum of the corresponding line (\( p_i \) are constructed from the loop momenta \( p \) and \( q \) and the external momentum \( k \)). We are mainly interested in all \( \nu \)'s being integer. The cases when some of the \( \nu \)'s are zero correspond to reducing lines in Fig. 1a to points. In such a way, self-energy diagrams with four or three internal lines can also be described. Moreover, by trivial decomposition (partial fractioning) of the first and the fourth denominators (provided that \( \nu_1 \) and \( \nu_4 \) are integer) we can reduce the integral corresponding to Fig. 1b to the integrals [11] with \( \nu_1 \) or \( \nu_4 \) equal to zero (such a decomposition is required only if \( m_1 \neq m_4 \)). So, in the general case of self-energy diagrams (with integer \( \nu \)'s) it is sufficient to consider only the integrals [11].

In general, the diagram in Fig. 1a has four physical thresholds. Two of them correspond to two-particle cuts,

\[
k^2 = (m_1 + m_4)^2 \quad \text{and} \quad k^2 = (m_2 + m_5)^2,
\]

Figure 1: Two-loop self-energy diagrams.
while the other two are connected with three-particle cuts,

\[ k^2 = (m_1 + m_3 + m_5)^2 \quad \text{and} \quad k^2 = (m_2 + m_3 + m_4)^2. \]  

(3)

In special cases, some of these threshold values can be equal.

If all the thresholds (2)–(3) are not equal to zero, the integral (1) is an analytic function in a vicinity of \( k^2 = 0 \), so that the asymptotic expansion is given by its Taylor series in \( k^2 \). It is convergent when \( k^2 \) is less than the value of the first physical threshold. The general case of such an expansion of two-point functions was considered in [26], and some results for three-point functions can be found in [27]. The situation is different, however, when some of the physical thresholds (2)–(3) become equal to zero. In this case, there can be non-polynomial terms in \( k^2 \) as \( k^2 \to 0 \). Below we shall see that they appear as \( \ln(-k^2) \) terms as \( n \to 4 \). These are the “zero-threshold” cases that will be considered in the present paper.

For the diagram in Fig. 1a, four independent zero-threshold configurations exist, namely [3]:

- **Case 1**: one zero-2PT (e.g. \( m_2 = m_5 = 0 \));
- **Case 2**: two zero-2PT’s (\( m_1 = m_2 = m_4 = m_5 = 0 \));
- **Case 3**: one zero-3PT (e.g. \( m_2 = m_3 = m_4 = 0 \));
- **Case 4**: one zero-2PT and one zero-3PT (e.g. \( m_2 = m_3 = m_4 = m_5 = 0 \)).

![Figure 2: Different zero-threshold configurations for the diagram in Fig. 1a.](image)

These cases are shown in Fig. 2 where bold lines are massive, narrow lines are massless, and the cuts corresponding to zero-thresholds are denoted by dashed lines. Note that, in case 1, having one more vanishing mass (e.g. \( m_3 \) or \( m_4 \)) does not produce a new zero-threshold configuration, and the corresponding cases can be considered together with case 1, e.g.:

- **Case 1a** = Case 1 with \( m_3 = 0 \);
- **Case 1b** = Case 1 with one more mass (not \( m_3 \)) equal to zero.

The cases 1,2,3,4 are independent, and we shall discuss them separately.

As in the paper [28], we shall use general results on asymptotic expansions of Feynman diagrams. This subject was developed in several papers [29] (see also [31] for a review).

Before presenting these results in our case, we would like to discuss some issues connected with this approach. Let us first introduce the Taylor expansion operator \( \mathcal{T}_k \). Its\(^3\)}
action on the denominator can be represented as

$$T_k \left\{ \frac{1}{[(k-p)^2 - m^2]^{\nu}} \right\} = \sum_{j=0}^{\infty} \frac{(\nu)_j}{j!} \left\{ \frac{(2(kp) - k^2)^j}{[p^2 - m^2]^{\nu+j}} \right\}, \tag{4}$$

where

$$\left( \nu \right)_j \equiv \frac{\Gamma(\nu + j)}{\Gamma(\nu)} \tag{5}$$

is the Pochhammer symbol. It is understood that this operator $T$ acts on the integrands before the loop integrations are performed, and also that all the terms with equal powers of $k$ on the r.h.s. should be collected together (e.g. the terms with $k^2$ and $(kp)^2$ should be considered together). Below we shall also need a generalization of formula (4) to the expansion in two or more momenta ($k$ and one or two loop momenta). This generalization is straightforward: we can expand in all these momenta at the same time. We only have to remember that in this case the expansion is in the total power of these momenta, because inserting loop momenta in the numerator of the resulting massless integral effectively produces higher powers of $k$.

Since the limit $k^2 \to 0$ is not regular for the cases being considered, the true expansion cannot be obtained simply by applying eq. (4) to all denominators where the external momentum $k$ occurs. If we did this, we would obtain infrared singularities in the coefficients of the expansion, even if the initial diagram were finite. This is due to the fact that in all zero-threshold cases we always need to let the momentum $k$ go through at least one of the massless lines. Nevertheless, it is possible to correct this procedure by adding some extra terms. The general theorem on asymptotic expansions tells us how to construct this true expansion. Namely, the result is given by the naïve Taylor expansion in small parameters (in our case, $T_k$) plus terms given by Taylor expansions in a specific class of subgraphs (see below) which may also contain expansions in some of the loop momenta. So, this naïve Taylor series produces infrared singularities while the extra series also involve ultraviolet ones, due to the appearance of loop momenta in the numerator.

If the original diagram is finite, all the singularities should cancel\footnote{If it contains singularities of a specific type, those will survive.}. Moreover, infrared and ultraviolet singularities should cancel independently. It is possible to write general formulae of asymptotic expansions in such an explicitly finite form, but such a procedure is more complicated. However, if we use dimensional regularization for both types of singularities, some contributions correspond to massless tadpoles and vanish. In this case, the formulae are much simpler, and we are going to use this approach below. It might seem in the resulting formulae that now infrared poles in $\varepsilon$ are cancelled by ultraviolet ones. The explanation is that omitting massless tadpoles effectively leads to a “re-distribution” of both types of singularities. So, from a practical point of view it is more convenient to apply the formulae where, formally, poles in $\varepsilon$ corresponding to infrared and ultraviolet singularities of the integrand are mutually cancelled. Moreover, cancellation of these poles for convergent diagrams will be a good check of our calculational procedure.

Let us introduce the following notation: $\Gamma$ is the original graph corresponding in our case to Fig. 1a, subgraphs of $\Gamma$ are denoted as $\gamma$, and the corresponding “reduced graph” $\Gamma/\gamma$ is obtained from $\Gamma$ by shrinking the subgraph $\gamma$ to a point. Furthermore, $J_{\gamma}$ is...
the dimensionally-regularized Feynman integral with the denominators corresponding to a graph $\gamma$. For example, $J_\Gamma$ corresponds to the integral (1). For our case, the general theorem yields

$$J_\Gamma \sim \sum_{k^2 \to 0} J_{\Gamma/\gamma} \circ T_{k, q_i} J_{\gamma},$$

where the sum goes over all the subgraphs $\gamma$ which (i) contain all the lines with large masses, and (ii) are one-particle irreducible with respect to the light (in our case, massless) lines. $T_{k, q_i}$ is an operator (4) expanding the integrand in $k$ and the loop momenta $q_i$ which are “external” for a given $\gamma$. The symbol “$\circ$” means that the polynomial in $q_i$, which appears as a result of applying $T$ to $J_{\gamma}$, should be inserted in the numerator of the integrand of $J_{\Gamma/\gamma}$. Note that although eq. (6) looks similar to the corresponding result for the large momentum expansion, eq. (2) of [28], there is an essential difference in the choice of subgraphs $\gamma$ contributing to the expansion.

Case 1
Case 1a
Case 1b
Case 2
Case 3
Case 4

Figure 3: The subgraphs $\gamma$ to be included in the sum (6).

Now, let us consider the contributions to the sum (6) for the different threshold configurations. We shall use the numbering given in Fig. 1a to indicate which lines are included in the subgraph: for example, $\{134\}$ denotes the subgraph containing only the “left” triangle (lines 1,3,4) of Fig. 1a, $\{12345\} \equiv \Gamma$, etc. These subgraphs are illustrated by Fig. 3, where bold lines correspond to massive propagators, narrow lines to massless ones, and dotted lines to the lines omitted in the subgraph (as compared with the “full” graph $\Gamma$). These dotted lines correspond to the reduced graphs $\Gamma/\gamma$. We do not list the vanishing contributions containing massless tadpoles.

Case 1. One 2PT is zero: $m_2 = m_5 = 0$. Two subgraphs contribute to the asymptotic expansion:
(i) $\gamma = \Gamma \Rightarrow$
\[
\int \frac{\frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k}{[(k-p)^2 - m^2]^{\nu_1} [(k-q)^2]^{\nu_2}}}{\int \frac{1}{\nu_4[q^2]^{\nu_4} T_k}} \, dq \, dp
\] (7)

(ii) $\gamma = \{134\} \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2]^{\nu_4} T_k} \, dq \, dp
\] (8)

In cases 1a and 1b we have the same contributing subgraphs (since the threshold configuration is the same), and the only thing to change is to set $m_3 = 0$ (case 1a) or $m_4 = 0$ (case 1b) in the formulae (7) - (8).

Case 2. Two 2PT’s are zero: $m_1 = m_2 = m_4 = m_5 = 0, m_3 \equiv m$. In this case, four subgraphs contribute:

(i) $\gamma = \Gamma \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2]^{\nu_4} T_k} \, dq \, dp
\] (9)

(ii) $\gamma = \{134\} \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2]^{\nu_4} T_k} \, dq \, dp
\] (10)

(iii) $\gamma = \{235\}$ contribution can be obtained from the previous one by the permutation $1 \leftrightarrow 2, 4 \leftrightarrow 5$;

(iv) $\gamma = \{3\} \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2]^{\nu_4} T_k} \, dq \, dp
\] (11)

Case 3. One 3PT is zero: $m_2 = m_3 = m_4 = 0$. Two subgraphs contribute:

(i) $\gamma = \Gamma \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2 - m_2^2]^{\nu_4} T_k} \, dq \, dp
\] (12)

(ii) $\gamma = \{15\} \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2 - m_2^2]^{\nu_4} T_k} \, dq \, dp
\] (13)

Case 4. One 2PT and one 3PT are zero: $m_2 = m_3 = m_4 = m_5 = 0, m_1 \equiv m$. For this case, three subgraphs contribute:

(i) $\gamma = \Gamma \Rightarrow$
\[
\int \frac{1}{\nu_3[p^2 - m_2^2]^{\nu_3} T_k} \, dq \, dp \frac{1}{\nu_4[q^2 - m_2^2]^{\nu_4} T_k} \, dq \, dp
\] (14)
(ii) $\gamma = \{134\} \Rightarrow$

$$\int \int d^n p \ d^n q \frac{1}{((k - q)^2)^{\nu_2} [q^2]^{\nu_3} T_{k, q}} \frac{1}{((k - p)^2 - m^2)^{\nu_1} [(p - q)^2]^{\nu_5} [p^2]^{\nu_4}}; \quad (15)$$

(iii) $\gamma = \{1\} \Rightarrow$

$$\int \int d^n p \ d^n q \frac{1}{((k - q)^2)^{\nu_2} [(p - q)^2]^{\nu_3} [p^2]^{\nu_4} [q^2]^{\nu_5} T_{k, p}} \frac{1}{((k - p)^2 - m^2)^{\nu_1}}. \quad (16)$$

Examination of eqs. (7)–(16) shows that, after partial fractioning is performed wherever necessary, the following types of contributions can occur in the expressions for the coefficients of the zero-threshold expansion:

(a) two-loop vacuum diagrams with two (or one) massive and one (or two) massless lines;

(b) products of a one-loop massless diagram with external momentum $k$ and a one-loop massive tadpole;

(c) two-loop massless diagrams with one (or two) powers of the denominators being non-positive.

The algebra of contributions of type (a) is discussed in Appendix A, which is based on a slightly modified algorithm of [26]. The problem of numerators of massive integrals occurring in types (a) and (b) has been studied in [28] (see Appendix B of that paper), and we use the same formulae here. For the numerators occurring in the integrals of type (c), we have used a procedure based on a formula in [32], and we explain the details in Appendix B.

All the formulae (7)–(16), along with eq. (4), can be used for any values of the space-time dimension $n$ and the powers of propagators $\nu_i$. In the next section, we shall consider the results produced by the expansion (3) for some special cases.

3. Analytic results

To get analytic results for the coefficients of the zero-threshold expansion, we have written a computer program using the REDUCE system for analytical calculations [33]. Like the algorithm itself, it works for any (integer) powers of the propagators, and can be applied for both convergent and divergent diagrams.

Many interesting physical applications require the calculation of the diagram in Fig. 1a with unit powers of propagators, the so-called “master” diagram. If all five $\nu_i$ are equal to one (and $k^2 \neq 0$) the corresponding diagram is finite as $n \to 4$, and we shall calculate the corresponding results at $n = 4$. The algorithm also makes it possible to consider higher terms of the expansion in $\varepsilon = (4 - n)/2$, or even results for arbitrary $n$. The individual contributions to the expansion (7)–(14) contain poles in $\varepsilon$, which, for finite diagrams such as this “master” diagram, cancel in the sum (3). This is one of the non-trivial checks on the algorithm.

Let us write the corresponding “master” integral as

$$J(m_1, m_2, m_3, m_4, m_5; k) \equiv J(1, 1, 1, 1; m_1, m_2, m_3, m_4, m_5; k) = -\pi^4 \sum_{j=0}^{\infty} C_j (k^2)^j, \quad (17)$$
Table 1: Coefficients of zero-threshold expansion for the cases with only one mass.

where, for zero-threshold configurations, the coefficients $C_j$ can depend on logarithms of $k^2$,

$$C_j = C_j^{(2)} \ln^2(-k^2) + C_j^{(1)} \ln(-k^2) + C_j^{(0)}.$$  (18)

Powers of $\ln(-k^2)$ higher than two cannot occur in the cases considered, because the separate terms contributing to the expansion can have at most double poles in $\varepsilon$.

For cases when we have only one massive parameter $m$ (all non-zero masses are equal), a number of exact results are known [7]. [8], [9]: for some of them, closed expressions for the coefficients of the expansion are also available. They can be represented in terms of polylogarithms $\text{Li}_N$ with $N \leq 3$. While applying the algorithm described in Section 2 we have considered all possibilities to have only one mass parameter in zero-threshold configurations. They correspond to the diagrams in the first column of Fig. 3 with all non-zero masses equal. The results for the first few coefficients of the expansion are presented in Table 1, where the first line gives the mass arguments of $J$, and the dimensionless coefficients (and other notations) are defined by

$$J = -\frac{\pi^4}{m^2} \sum_{j=0}^{\infty} c_j \left( \frac{k^2}{m^2} \right)^j, \quad L \equiv \ln \left( -\frac{k^2}{m^2} \right), \quad \zeta_2 \equiv \zeta(2) = \frac{\pi^2}{6}.  \quad (19)$$

So, in Table 1 the following coefficients $c_j \, (j = 0, \ldots, 6)$ are presented,

$$c_j \equiv (m^2)^{j+1} C_j = c_j^{(2)} L^2 + c_j^{(1)} L + c_j^{(0)},$$  \quad (20)

and they agree with exact results wherever available (see the last line of the table). For cases 1b and 3 exact results were not available, but we checked the logarithmic terms by use of the dispersion relations technique.
In cases 2 and 4, we have only one massive line (one mass parameter), and the expansion (19) is sufficient. However, for other zero-threshold configurations we can have two (cases 1a, 1b, 3) or even three (general case 1) different masses.

Let us start by considering case 1 with three different masses. We have obtained the following results for the lowest coefficients of the expansion (17):

\[
C_0 = \frac{1}{4(m_1^2 - m_3^2)(m_4^2 - m_3^2)(m_1^2 - m_4^2)}
\times \left\{ 2 \left( \ln \left( -\frac{k^2}{m_1^2} \right) + \ln \left( -\frac{k^2}{m_4^2} \right) - 4 \right) \left( m_1^2(m_4^2 - m_3^2) \ln \frac{m_1^2}{m_4^2} - m_4^2(m_1^2 - m_3^2) \ln \frac{m_4^2}{m_3^2} \right) + (m_1^2 - m_3^2)(m_4^2 - m_3^2) \ln \frac{m_1^2}{m_4^2} \ln \frac{m_4^2}{m_3^2} \right. \\
+ (m_1^2 - m_3^2)(m_4^2 - m_3^2) \left( \ln \frac{m_1^2}{m_4^2} + \ln \frac{m_4^2}{m_3^2} \right) - 2m_3^2(m_1^2 - m_4^2) \ln \frac{m_1^2}{m_4^2} \ln \frac{m_4^2}{m_3^2} \\
\left. \right\} , \quad (21)
\]

\[
C_1 = \frac{1}{8(m_1^2 - m_3^2)^3(m_4^2 - m_3^2)^3(m_1^2 - m_4^2)^3}
\times \left\{ 2(m_1^2 - m_3^2)(m_4^2 - m_3^2) \left( \ln \left( -\frac{k^2}{m_1^2} \right) + \ln \left( -\frac{k^2}{m_4^2} \right) - 4 \right) \right. \\
\times \left\{ m_1^2(m_4^2 - m_3^2)^2(m_1^2 + m_4^2) - 2m_4^2m_3^2 \ln \frac{m_1^2}{m_4^2} \\
- m_3^2(m_1^2 - m_3^2)^2(m_4^2 + m_3^2) - 2m_1^2m_3^2 \ln \frac{m_1^2}{m_3^2} \\
+ (m_1^2 - m_3^2)(m_4^2 - m_3^2)(m_1^2 - m_4^2)(m_3^2(m_1^2 + m_4^2) - 2m_1^2m_4^2) \right\} \\
+ (m_1^2 - m_3^2)^2(m_4^2 - m_3^2)^2(m_1^2 + m_4^2) \left( \ln \frac{m_1^2}{m_4^2} + \ln \frac{m_4^2}{m_3^2} \right) \\
\times \left( (m_1^2 - m_3^2)(m_4^2 - m_3^2) \ln \frac{m_1^2}{m_4^2} + 2m_3^2(m_1^2 - m_4^2) \right) \\
+ 2m_3^4(m_1^2 - m_3^2)(m_4^2 - m_3^2)(m_1^2 - m_4^2)^3 \ln \frac{m_1^2}{m_4^2} \ln \frac{m_4^2}{m_3^2} \\
+ 4m_3^2(m_1^2 - m_4^2) \left( (m_1^2 - m_3^2)(m_4^2 - m_3^2) - (m_1^2 - m_4^2)^2 \right) \\
\times \left( m_1^2m_4^2(m_4^2 - m_3^2)^2 \ln \frac{m_1^2}{m_3^2} + m_4^2(m_1^2 - m_3^2)^2 \ln \frac{m_4^2}{m_3^2} \right) \\
+ 2(m_1^2 - m_3^2)(m_4^2 - m_3^2)^3(m_1^2m_3^2 + m_4^2m_3^2 + m_1^2m_4^2 - 3m_3^4) \\
+ 2(m_1^2 - m_3^2)(m_4^2 - m_3^2)^3 \left( (m_1^2 + m_4^2)(m_4^2 - m_3^2) + 2m_3^2(m_1^2 - m_4^2) \right) \mathcal{H}(m_1^2, m_3^2) \right. \\
\left. \right\} , \quad (21)
\]
\[-2(m_1^2 - m_2^2)^3(m_3^2 - m_3^2) \left(m_1^2 + m_2^2(m_3^4 - m_3^4) - 2m_2^2m_3^2(m_1^2 - m_1^2)\right) \mathcal{H}(m_2^2, m_3^2) \right\}. \quad (22)

In these formulae, \( \mathcal{H} \) is a dimensionless function which is defined as

\[ \mathcal{H}(m_1^2, m_2^2) = 2 \text{Li}_2 \left(1 - \frac{m_1^2}{m_2^2}\right) + \frac{1}{2} \ln^2 \left(\frac{m_1^2}{m_2^2}\right). \quad (23) \]

It is easy to see that the function \( \mathcal{H} \) is antisymmetric in its arguments and therefore it vanishes when they are equal. This function is connected with the finite part of the two-loop vacuum integral with one massless and two massive lines (for details of this definition, see Appendix A). We have also obtained higher coefficients for this case (up to \( C_3 \)), but do not present them here because they are more cumbersome.

If two of the masses are equal, \( m_1 = m_4 \equiv m \), eqs. (21)–(22) can be simplified to

\[
C_0 = \frac{1}{2(m^2 - m^2)} \left\{ 2 \left(m^2 - m^2 - m^2 \ln \frac{m^2}{m^2} \right) \ln \left(-\frac{k^2}{m^2}\right) 
- m^2 \ln^2 \frac{m^2}{m^2} - 4(m^2 - m^2) - 2m^2 \mathcal{H}(m^2, m^2) \right\}, \quad (24)
\]

\[
C_1 = \frac{1}{12m^2(m^2 - m^2)^2} \left\{ \left(6m^2m_3^4 \ln \frac{m^2}{m_3^2} + (m^2 - m^2)(m^4 - 5m^2m_3^2 - 2m_3^2)\right) \ln \left(-\frac{k^2}{m^2}\right) 
+ 3m^2m_3^4 \ln \frac{m^2}{m_3^2} - 2m_3^2(5m^4 - m_3^2) \ln \frac{m^2}{m_3^2} 
+ (m^2 - m_3^2)(m^4 + 17m^2m_3^2 + 2m_3^4) + 6m^2m_3^4 \mathcal{H}(m^2, m_3^2) \right\}. \quad (25)
\]

In this case, we have got the coefficients up to \( C_4 \).

In another limit, \( m_3 \to 0 \) (this corresponds to the case 1a), we get for the lowest coefficients

\[
C_0 = \frac{1}{2(m_1^2 - m_2^2)} \ln \frac{m_1^2}{m_2^2} \left\{ \ln \left(-\frac{k^2}{m_1^2}\right) + \ln \left(-\frac{k^2}{m_2^2}\right) - 4 \right\}, \quad (26)
\]

\[
C_1 = \frac{1}{4m_1^2m_4^2(m_1^2 - m_2^2)^3} \left\{ m_1^2m_4^2(m_1^2 + m_2^2) \left( \ln \left(-\frac{k^2}{m_1^2}\right) + \ln \left(-\frac{k^2}{m_2^2}\right) - 4 \right) \ln \frac{m_1^2}{m_4^2} 
- 2m_1^2m_4^2(m_2^2 - m_2^2) \ln \left(-\frac{k^2}{m_1^2}\right) + \ln \left(-\frac{k^2}{m_2^2}\right) - 4 \right) + (m_1^2 - m_2^2)^2 \right\}, \quad (27)
\]

\[
C_2 = \frac{1}{36m_1^2m_4^2(m_1^2 - m_2^2)^5} \left\{ 6m_1^4m_4^4(m_1^4 + 4m_2^2m_4^2 + m_2^4) \left( \ln \left(-\frac{k^2}{m_1^2}\right) + \ln \left(-\frac{k^2}{m_2^2}\right) - 4 \right) \ln \frac{m_1^2}{m_4^2} 
- 18m_2^4m_4^4(m_1^4 - m_2^4) \left( \ln \left(-\frac{k^2}{m_1^2}\right) + \ln \left(-\frac{k^2}{m_2^2}\right) - 4 \right) 
- 2m_1^4m_4^4(m_1^2 - m_2^2)^2 \ln \frac{m_1^2}{m_4^2} + (m_1^2 - m_2^2)^3 \left(m_1^2 + m_2^2\right) \left(m_1^4 - m_2^4m_4^2 + m_4^4\right) \right\}. \quad (28)
\]
For this case (with two different masses), the exact result was also available in [18], and we have successfully compared our coefficients with this exact expression. Now, let us present the results for the coefficients (18) for case 3 \((m_2 = m_3 = m_4 = 0)\), when \(m_1\) and \(m_5\) are different:

\[
C_0 = \frac{1}{4m_1^2m_5^8} \left\{ -(m_1^2 + m_5^2) \left( \ln^2 \frac{m_1^2}{m_5^2} + \frac{2\pi^2}{3} \right) - 2(m_1^2 - m_5^2) \mathcal{H}(m_1^2, m_5^2) \right\},
\]

\[
C_1 = \frac{1}{8m_1^4m_5^4(m_1^2 - m_5^2)} \left\{ 2m_1^2m_5^2(m_1^2 - m_5^2) \left( \ln \left( -\frac{k^2}{m_1^2} \right) + \ln \left( -\frac{k^2}{m_5^2} \right) - 3 \right) 
- (m_1^2 - m_5^2)(m_1^4 + m_5^4) \left( \ln^2 \frac{m_1^2}{m_5^2} + \frac{2\pi^2}{3} \right) + 2m_1^2m_5^2(m_1^2 + m_5^2) \ln \frac{m_1^2}{m_5^2}
- 2(m_1^2 - m_5^2)^2 (m_1^2 + m_5^2) \mathcal{H}(m_1^2, m_5^2) \right\},
\]

\[
C_2 = \frac{1}{36m_1^6m_5^6(m_1^2 - m_5^2)^3}
\times \left\{ 3m_1^2m_5^2(m_1^2 - m_5^2)^3(m_1^2 + m_5^2) \left( \ln \left( -\frac{k^2}{m_1^2} \right) + \ln \left( -\frac{k^2}{m_5^2} \right) \right)
- 3(m_1^2 - m_5^2)^3(m_1^4 + m_5^4)(m_1^2 - m_5^2) \left( \ln^2 \frac{m_1^2}{m_5^2} + \frac{2\pi^2}{3} \right)
+ 3m_1^2m_5^2 \left( 3(m_1^2 - m_5^2)^4 + 4m_1^2m_5^2(m_1^2 - m_5^2)^2 + 2m_1^4m_5^4 \right) \ln \frac{m_1^2}{m_5^2}
+ m_1^2m_5^2(m_1^2 - m_5^2)(m_1^2 + m_5^2) \left( (m_1^2 - m_5^2)^2 - 3m_1^2m_5^2 \right)
- 6(m_1^2 - m_5^2)^4(m_1^4 + m_1^2m_5^2 + m_5^4) \mathcal{H}(m_1^2, m_5^2) \right\}.
\]

Again, we do not present higher coefficients because they are more cumbersome. For this case, we have calculated them up to \(C_5\). For all cases with different masses considered in this section (except eqs. (26)–(28)), the exact results were not available. Nevertheless, we have checked all the terms containing \(\ln(-k^2)\) by use of dispersion relations. These logarithmic terms completely define the imaginary part of the results, which is non-zero for \(k^2 > 0\).

To see that the algorithm also works correctly for cases when the original integral is divergent, we considered an example corresponding to the diagram in Fig. 1b, where the first and the fourth lines were taken to be massive (with equal masses) while the other three lines were massless. Effectively, this example corresponds to case 3 (one zero-3PT) with \(\nu_1 = 2, \nu_2 = \nu_3 = \nu_4 = 1, \nu_5 = 0\). The exact result for this case was presented in [18] (eq. (102)), and we have checked that our approach produces the correct coefficients for both the divergent and finite parts (we have calculated terms up to \(C_6\)).
Another possibility is to compare the obtained results with programs based on numerical integration. This will also illustrate the radius of convergence of the expansion. This comparison is considered in the next section.

4. Numerical results

In order to demonstrate the use of the zero-threshold expansion as a way to obtain approximate numerical results for self-energy diagrams, we shall discuss a number of mass configurations corresponding to diagrams that occur in the Standard Model. For this purpose, we take the masses of the $W$ and $Z$ bosons, and of the top and bottom quarks to be:

$$M_W = 80 \text{ GeV}, \quad M_Z = 91 \text{ GeV}, \quad m_b = 5 \text{ GeV}, \quad m_t = 174 \text{ GeV}. \quad (32)$$

The first example is a diagram containing a top-bottom loop to which two $W$ bosons are attached, that contributes to the self-energy of the photon and the $Z$ boson. The corresponding scalar integral is

$$J(M_W, m_b, m_t, M_W, m_b; k). \quad (33)$$

Let us now neglect $m_b$ (below, we shall see that this is reasonable when $k^2 \gg m_b^2$), and consider instead

$$J(M_W, 0, m_t, M_W, 0; k). \quad (34)$$

It has one zero two-particle threshold and therefore it belongs to case 1. In Fig. 4 the approximations defined by

$$J^{(N)} = -\pi^4 \sum_{j=0}^{N} C_j (k^2)^j \quad (35)$$

are shown as curves, and, for comparison, values of $J$ obtained by numerical integration are shown as crosses. The position of the lowest non-zero threshold, at $k^2 = 4 M_W^2$ in this example, is marked by a vertical line. In the real part, we clearly see the logarithmic singularity as $k^2 \to 0$. This is the region where, by construction, the asymptotic expansion works extremely well. For instance, at $k^2 = 5000 \text{ GeV}^2$, the $N = 2$ approximation is already just as good as the numerical integration, which is accurate to 4 digits, for both the real and imaginary parts. However, the expansion converges all the way up to the first non-zero threshold at $k^2 = 25600 \text{ GeV}^2$. At that point, the real parts of $J^{(4)}$ and of $J$ differ by about 15%, and the imaginary parts by 25%.

We also examined the situation for $k^2 < 0$, where there are no physical thresholds and the imaginary part is zero. At small $k^2$, the behaviour of the real part for negative $k^2$ is quite similar to that for positive $k^2$, because it is defined by the $\ln |k^2|$ term in both cases. Moreover, because the signs of the terms alternate for $k^2 < 0$, the convergence of the asymptotic expansion is even a little better than for $k^2 > 0$.

In Fig. 5, values of the integrals (33) (dashed lines) and (34) (solid lines) are plotted to show the effect of neglecting $m_b$. At low $k^2$, their behaviour is obviously very different, the latter having a singularity at $k^2 = 0$, whereas the former has one at $k^2 = 4m_b^2$, but as $k^2$ increases above $4m_b^2$, they soon come close together.
Figure 4: The real and imaginary parts of $J(M_W, 0, m_t, M_W, 0; k)$. 
Figure 5: The real and imaginary parts of \( J(M_W, m_b, m_t, M_W, m_b; k) \) and \( J(M_W, 0, m_t, M_W, 0; k) \).
For momenta close to the $Z$ mass shell, $k^2 \approx 8300 \text{ GeV}^2$, the error we make by neglecting $m_b$ is of the order of 1%. The truncation error of the asymptotic expansion is already less than that if we use $J^{(2)}$.

The second diagram we studied contains a top-bottom loop with a $W$ boson going across the inside. Neglecting $m_b$ as before, it corresponds to

$$J(m_t, 0, M_W, m_t, 0; k).$$

This mass configuration also belongs to case 1. Below the first non-zero threshold, the behaviour of (36) is qualitatively very similar to the previous example, but because this threshold is now a three-particle threshold, it is different above the threshold. In this case the truncation error of $J^{(4)}$ on the threshold is only about 2% in the real part and 0.4% in the imaginary part.

Finally, we study an example of case 3,

$$J(M_W, 0, 0, 0, M_Z; k),$$

which corresponds to diagrams that contribute to, e.g., the electron or muon self-energy, if $m_e$ or $m_{\mu}$ is neglected with respect to $k^2$. The results are shown in Fig. 6. The position of the first non-zero threshold, at $k^2 = M_W^2$, is again indicated by a vertical line. For this case, we also checked that the numerical values (shown as crosses) agree with the results of other numerical programs [22, 24].

Contrary to case 1, the first term of the expansion, $C_0$, given by eq. (29), does not depend on $\ln(-k^2)$, and therefore, the integral has a finite limit as $k^2 \to 0$. At the threshold, where the truncation errors are largest, the errors in the real and imaginary parts of $J^{(5)}$ are approximately 2% and 16%, respectively.

In all the examples, the zero-threshold expansion provides approximations that are at least as accurate as numerical integration in a large part of the region of convergence, and can be evaluated much faster.

5. Conclusions

In this paper we have studied all possibilities for a two-loop self-energy diagram to have its lowest physical threshold at zero external momentum squared. Due to specific infrared singularities at $k^2 = 0$, constructing the small momentum expansion is more complicated than when all the thresholds are non-vanishing. Namely, instead of a regular Taylor expansion, we get a cut along the positive $k^2$ axis, starting with a branchpoint at $k^2 = 0$. As a consequence, our results in four dimensions contain $\ln(-k^2)$ terms, or even $\ln^2(-k^2)$ terms if two physical thresholds vanish. So, technically this zero-threshold expansion is closer to the large momentum expansion [28] than to a regular small momentum expansion [26].

For special cases, we have compared our results with known analytic ones. For some more complicated cases corresponding to diagrams with different masses occurring in the Standard Model, we made a comparison with the results of a numerical integration program based on the algorithm [20] (for one case, we have also compared with [24]). It shows that the zero-threshold expansion converges up to the first non-zero threshold,
Figure 6: The real and imaginary parts of $J(M_W, 0, 0, 0, M_Z, k)$.
and that, unless \( k^2 \) is very close to the threshold, only a few terms are needed to obtain accurate results. This comparison can also be considered as a check of the numerical programs.

The algorithm constructed in the present paper covers the only “hole” remaining in the small momentum expansion of two-loop self-energy diagrams. Together with [26], it solves this problem completely. The large momentum expansion of these diagrams is described in [28], and there are no remaining “holes” there. It is interesting to note that all these algorithms can be applied to two-loop three-point functions as well.

The general problem of describing the threshold behaviour is still waiting for a solution. Here we have considered only a part of this problem, corresponding to the zero-threshold cases. In principle, the general theory of asymptotic expansions can be applied in some regions between the thresholds, whilst the description of the behaviour near the non-zero thresholds requires other approaches.

A. D. would like to thank the Instituut-Lorentz, University of Leiden, and J.B. T. the Department of Physics, University of Bergen, for hospitality during visits when parts of this work were done. We are grateful to A. Czarnecki and J. Fujimoto for their help in comparisons with results based on other numerical approaches. A. D.’s research was supported by the Research Council of Norway.

Appendix A. Two-loop massive vacuum diagrams

To evaluate two-loop vacuum massive diagrams occurring in the separate terms (10)–(16) contributing to the zero-threshold expansion, we mainly use the algorithms and formulae presented in [26, 28].

The integrals we are interested in are defined as

\[
I(\nu_1, \nu_2, \nu_3; m_1, m_2, m_3) \equiv \int \int \frac{d^n p \, d^n q}{[p^2 - m_1^2]^\nu_1 [q^2 - m_2^2]^\nu_2 [(p - q)^2 - m_3^2]^\nu_3}. \tag{38}
\]

The general case of this integral with arbitrary \( n \) and \( \nu \)'s was considered in [26]. By use of the technique [34], a result in terms of hypergeometric functions of two variables was obtained. For the case when all \( \nu \)'s are equal to one and \( n \) is arbitrary, this result was reduced to hypergeometric functions \( _2F_1 \) of one variable (see also in [19]). Expanding in \( \varepsilon \), we get standard results in terms of dilogarithms or Clausen’s function given by eqs. (4.9), (4.10) and (4.15) of [26] (see also in [11, 35]).

In this paper, however, we always have at least one of the masses in (38) equal to zero. Generally, this simplifies the calculations but requires some changes in the algorithms as compared with the general mass case [23].

If one of the masses (e.g. \( m_3 \)) is zero, the result for the integral (38) can be expressed in terms of a Gauss hypergeometric function,

\[
I(\nu_1, \nu_2, \nu_3; m_1, m_2, 0) = \pi^n t^{2-2n} (-m_2^2)^{n-\nu_1-\nu_2-\nu_3} \\
\times \frac{\Gamma \left( \frac{n}{2} - \nu_3 \right) \Gamma \left( \nu_1 + \nu_3 - \frac{n}{2} \right) \Gamma \left( \nu_2 + \nu_3 - \frac{n}{2} \right) \Gamma \left( \nu_1 + \nu_2 + \nu_3 - n \right)}{\Gamma (\nu_1) \Gamma (\nu_2) \Gamma \left( \frac{n}{2} \right) \Gamma (\nu_1 + \nu_2 + 2\nu_3 - n)}
\]
where it is understood that $i^{-2n}(-m^2)^n = (m^2)^n$. This result can easily be obtained using the formulae of \cite{34, 26}. The symmetry $(m_1, \nu_1) \leftrightarrow (m_2, \nu_2)$ can be seen by use of the well-known transformation
\begin{equation}
\begin{split}
2F_1\left( \begin{array}{c} a, b \\ c \\ \end{array} \bigg| z \right) = (1-z)^{-b} 2F_1\left( \begin{array}{c} c-a, b \\ c \\ \end{array} \bigg| \frac{z}{z-1} \right). 
\end{split}
\end{equation}

For the cases when $m_1 = m_2$ or $m_2 = 0$ the formula \cite{34} gives the correct answers:
\begin{equation}
I(\nu_1, \nu_2, \nu_3; m, m, 0) = \pi^{n/2-2n}(-m^2)^{n-\nu_1-\nu_2-\nu_3}
\begin{split}
\times \frac{\Gamma \left( \frac{n}{2} - \nu_3 \right) \Gamma \left( \nu_1 + \nu_2 - \frac{n}{2} \right) \Gamma \left( \nu_2 + \nu_3 - \frac{n}{2} \right) \Gamma \left( \nu_1 + \nu_2 + 2\nu_3 - n \right)}{\Gamma (\nu_1) \Gamma (\nu_2) \Gamma \left( \frac{n}{2} \right) \Gamma \left( \frac{n}{2} + 2\nu_3 - n \right)};
\end{split}
\end{equation}
\begin{equation}
I(\nu_1, \nu_2, \nu_3; m, 0, 0) = \pi^{n/2-2n}(-m^2)^{n-\nu_1-\nu_2-\nu_3}
\begin{split}
\times \frac{\Gamma \left( \frac{n}{2} - \nu_2 \right) \Gamma \left( \frac{n}{2} - \nu_3 \right) \Gamma \left( \nu_2 + \nu_3 - \frac{n}{2} \right) \Gamma \left( \nu_1 + \nu_2 + 2\nu_3 - n \right)}{\Gamma (\nu_1) \Gamma (\nu_2) \Gamma (\nu_3) \Gamma \left( \frac{n}{2} \right)}.
\end{split}
\end{equation}

These results are well-known, and we present them here for completeness only.

For the case when all $\nu$'s are equal to one and $n \to 4$ ($\varepsilon \to 0$), the result for \cite{34} can be written as
\begin{equation}
I(1, 1, 1; m_1, m_2, 0) = \pi^{4-2\varepsilon} \frac{\Gamma^2(1 + \varepsilon)}{2(1-\varepsilon)(1-2\varepsilon)}
\begin{split}
\times \left\{ -\frac{1}{\varepsilon^2}(m_1^2 + m_2^2) + \frac{2}{\varepsilon}(m_1^2 \ln(m_1^2) + m_2^2 \ln(m_2^2)) \\
-2(m_1^2 \ln^2(m_1^2) + m_2^2 \ln^2(m_2^2)) + \frac{1}{2}(m_1^2 + m_2^2) \ln \frac{m_2^2}{m_1^2} + (m_1^2 - m_2^2)H(m_1^2, m_2^2) \right\},
\end{split}
\end{equation}
where $H$ is a dimensionless antisymmetric function defined as
\begin{equation}
H(m_1^2, m_2^2) = 2Li_2 \left( 1 - \frac{m_1^2}{m_2^2} \right) + \frac{1}{2} \ln \frac{m_1^2}{m_2^2}.
\end{equation}

It is connected with the function $F$ used in \cite{28} (see Appendix B, eqs. (B.3)–(B.7) of \cite{28}) via
\begin{equation}
H(m_1^2, m_2^2) = - \lim_{m_3 \to 0} \left\{ \frac{F(m_1^2, m_2^2, m_3^2)}{m_1^2 - m_2^2} - \frac{1}{2} \ln \frac{m_1^2}{m_2^2} \left( \ln \frac{m_1^2}{m_3^2} + \ln \frac{m_2^2}{m_3^2} \right) \right\}.
\end{equation}

We cannot use the original function $F$ here because it has a logarithmic singularity as $m_3 \to 0$. 

\begin{equation}
\times F_1\left( \begin{array}{c} \nu_1 + \nu_3 - \frac{n}{2}, \nu_1 + \nu_2 + \nu_3 - n \\ \nu_1 + \nu_2 + 2\nu_3 - n \\ \end{array} \bigg| 1 - \frac{m_1^2}{m_2^2} \right),
\end{equation}
We need to calculate integrals (38) with arbitrary (integer) powers of denominators. For the cases when we have one mass parameter only, it is convenient to use the simple explicit formulae (41), (42). If we have different masses, however, it is better to use recurrence relations obtained by using the integration-by-parts technique [36] (in a way analogous to [37]). For the case of the integrals (38), these relations were considered in [26]. The solution to these relations presented there (see eq. (5.3) of [26]) contains masses in the denominator. This can be avoided by employing the connection (5.4) of the same paper. Note that for \( m_3 = 0 \) the determinant of the system of equations corresponding to the recurrence relations becomes \( \Delta(m_1^2, m_2^2, 0) = -(m_1^2 - m_2^2)^2 \).

Of course, application of these recurrence relations is equivalent to using connections between contiguous hypergeometric functions \( _2F_1 \) (see, e.g., in [38]) corresponding to eq. (39). Furthermore, in cases when we have close values of the masses, it may be better to use the explicit result (39) to expand in the mass difference up to the order corresponding to the required accuracy.

Appendix B. Two-loop massless integrals with numerators

Here we shall consider massless integrals

\[
J^{(0)}(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \equiv J(\{\nu_i\}; 0, 0, 0, 0; k),
\]

where \( J \) corresponds to the original integral \( [1] \). The main properties of these integrals are well-known, and many of them can be obtained by use of the integration-by-parts technique [36] (we had collected some of them in Appendix A of [28]). In particular, when any of \( \nu \)'s is equal to zero, the corresponding integral can be calculated in terms of gamma functions, for any value of the space-time dimension \( n \). If all \( \nu \)'s are equal to one and \( n = 4 \), the result is well-known \((-6\zeta(3)\pi^4/k^2)\). For higher powers of propagators, integration by parts [36] can be used, which was implemented in programs like [39].

In this appendix, we would like to discuss a “non-standard” way of calculating the integrals in cases when some of the \( \nu \)'s are negative. So, in fact, we get the corresponding denominators in the numerator. This approach is based on the formula presented in ref. [32] and was used in this paper. The general case of this formula can be applied to any one-loop integral with tensor structure in the numerator. As a result, one gets a sum of tensor structures multiplied by scalar integrals in other dimensions. We shall need only a simple corollary of this general formula below.

In fact, we have two different cases to consider: (i) \( \nu_3 < 0 \) and (ii) any other \( \nu \) (for example, \( \nu_5 \)) is negative. Let us consider the first possibility, the second one can be treated analogously. If \( \nu_3 < 0 \), this corresponds to getting

\[
[(p - q)^2]^{\nu_3} \equiv \left[ p^2 + q^2 - 2(pq) \right]^{\nu_3}
\]

in the numerator. After expanding and cancelling \( p^2 \) and \( q^2 \) against the corresponding denominators, we get integrals of the form

\[
\int \int d^n p \ d^n q \ \frac{(pq)^M}{[p^2]^{\nu_1} [q^2]^{\nu_2} [(k - p)^2]^{\nu_4} [(k - q)^2]^{\nu_5}}.
\]
Now, let us write \((pq)^M\) as \(p^{\mu_1} \ldots p^{\mu_M} q_{\mu_1} \ldots q_{\mu_M}\) and consider one of the subloops (an analogy with [14] can be noticed). According to [32], it can be written as

\[
\int d^n q \frac{q_{\mu_1} \ldots q_{\mu_M}}{[q^2]^{\nu_2} [(k-q)^2]^{\nu_5}} = \pi^{n/2} \Gamma^{n/2-\nu_2-\nu_5} \Gamma(\nu_2) \Gamma(\nu_5) \Gamma(n-\nu_2-\nu_5+M)^{-1} \times \sum_{\lambda, \kappa} \left(\frac{k^2}{2}\right)^\lambda \left\{[g]^\lambda[k]^\kappa\right\}_{\mu_1 \ldots \mu_M} \Gamma\left(\frac{n}{2}-\nu_2+\lambda+\kappa\right) \Gamma\left(\frac{n}{2}-\nu_5+\lambda\right) \Gamma\left(\nu_2+\nu_5-\frac{n}{2}-\lambda\right),
\]

where \(\left\{[g]^\lambda[k]^\kappa\right\}_{\mu_1 \ldots \mu_M}\) denotes a symmetric tensor structure each term of which is constructed from \(\lambda\) metric tensors \(g_{\mu_i\mu_j}\) and \(\kappa\) vectors \(k_{\mu_i}\), the coefficient of each term being equal to one. It is easy to show that such a tensor structure contains

\[
\frac{(2\lambda + \kappa)!}{2^\lambda \lambda! \kappa!}
\]

terms. Moreover, when contracted with \(p^{\mu_1} \ldots p^{\mu_M}\), each term gives the same result, namely \((p^2)^\lambda (kp)^\kappa\).

So, now we obtain a sum of the terms of the form

\[
\int d^n p \frac{(kp)^\kappa}{[p^2]^{\nu_1-\lambda} [(k-p)^2]^{\nu_4}},
\]

with known coefficients (51). Finally, the scalar products \((kp)\) in the numerator can be represented in terms of the denominators and \(k^2\), and the resulting integrals are trivial.

References

[1] J. Lefrançois, in Proceedings of the International EPS Conference on HEP, Marseille, July 22-28, 1993, J. Car, M. Perrottet, eds. (Editions Frontières, Gif-sur-Yvette, 1994), p.671.

[2] D. Schaile, in Proceedings of the 27th International Conference on High Energy Physics, Glasgow, 20-27 July 1994, to be published.

[3] F. Abe et al. (CDF collaboration), Phys. Rev. Lett. 73 (1994) 225.

[4] S. Katsanevas et al., Internal note DELPHI 92-166 PHYS. (1992), p.250.

[5] G. Passarino, talk at Zeuthen Workshop on Elementary Particle Theory: Physics at LEP200 and Beyond, Teupitz, Germany, 10-15 Apr 1994.

[6] D. Yu. Bardin et al., in “Z physics at LEP1”, G. Altarelli, R. Kleiss and C. Verzegnassi, eds., CERN 89-08, Vol.1, p.89.

\(^5\)There was a misprint in the corresponding formula (17) of [32]: the last gamma function on the first line of eq. (17) should be \(\Gamma(n-\nu_2-\nu_5+M)\). The general formula (11) is correct.
[7] W. Beenakker, F.A. Berends and S.C. van der Marck, *Nucl. Phys.* B 355 (1991) 281.
[8] S. Jadach, E. Richter-Was, B.F.L. Ward and Z. Was, *Phys. Lett.* B 260 (1991) 438.
[9] M. Caffo, H. Czyz and E. Remiddi, *Phys. Lett.* B 327 (1994) 369.
[10] K.S. Bjørkevoll, G. Fäl dt and P. Osland, *Nucl. Phys.* B 386 (1992) 280, 303; B 391 (1993) 591; G. Fäl dt, P. Osland, *Nucl. Phys.* B 413 (1994) 16, 64.
[11] J.J. van der Bij and M. Veltman, *Nucl. Phys.* B231 (1984) 205; J.J. van der Bij and F. Hoogeveen, *Nucl. Phys.* B283 (1987) 477.
[12] R. Barbieri, M. Beccaria, P. Ciafaloni, G. Curci and A. Viceré, *Phys. Lett.* B288 (1992) 95; *Nucl. Phys.* B409 (1993) 105; J. Fleischer, O.V. Tarasov and F. Jegerlehner, *Phys. Lett.* B319 (1993) 249; G. Degrassi, S. Fanchiotti and P. Gambino, Preprint CERN-TH-7180-94 (hep-ph 9403250).
[13] B. A. Kniehl, in *Proceedings of the International EPS Conference on HEP*, Marseille, July 22-28, 1993, J. Car, M. Perrottet, eds. (Editions Frontières, Gif-sur-Yvette, 1994), p. 639.
[14] G. Weiglein, R. Scharf and M. Böhm, *Nucl. Phys.* B 416 (1994) 606.
[15] D. Kreimer, *Mod. Phys. Lett.* A9 (1994) 1105.
[16] D.J. Broadhurst, *Z. Phys.* C 47 (1990) 115.
[17] D.J. Broadhurst, J. Fleischer and O.V. Tarasov, *Z. Phys.* C 60 (1993) 287.
[18] R. Scharf and J.B. Tausk, *Nucl. Phys.* B 412 (1994) 523.
[19] R. Scharf, Diploma Thesis, Würzburg, 1991.
[20] D. Kreimer, *Phys. Lett.* B 273 (1991) 277.
[21] F.A. Berends and J.B. Tausk, *Nucl. Phys.* B 421 (1994) 456.
[22] D. Kreimer, *Phys. Lett.* B292 (1992) 341; A. Czarnecki, U. Kilian and D. Kreimer, Mainz preprint MZ-TH/94-13 (hep-ph 9405423).
[23] F.A. Berends, M. Böhm, M. Buza, R. Scharf, *Z. Phys.* C 63 (1994) 227; S. Bauberger, F.A. Berends, M. Böhm and M. Buza, Leiden preprint INLO-PUB-9/94 (hep-ph 9409388); S. Bauberger, F.A. Berends, M. Böhm, M. Buza and G. Weiglein, Leiden preprint INLO-PUB-11/94 (hep-ph 9406404).
[24] J. Fujimoto, Y. Shimizu, K. Kato and Y. Oyanagi, KEK preprint 92-213.
[25] A. Ghinculov and J.J. van der Bij, Freiburg preprint THEP-94-05 [hep-ph 9405418].

[26] A. I. Davydychev and J. B. Tausk, *Nucl. Phys.* B 397 (1993) 123;  
*Yad Fiz.* 56, No.11 (1993) 137 [ *Phys. At. Nucl.* 56 (1993) 1531 ].

[27] J. Fleischer and O. V. Tarasov, *Z.Phys.* C 64 (1994) 413.

[28] A. I. Davydychev, V. A. Smirnov and J. B. Tausk, *Nucl. Phys.* B 410 (1993) 325.

[29] F.V. Tkachov, Preprint INR P-358 (Moscow, 1984);  
G.B. Pivovarov and F.V. Tkachov, Preprints INR P-0370, II-459 (Moscow, 1984);  
K.G. Chetyrkin and V.A. Smirnov, Preprint INR G-518 (Moscow, 1987);  
S.G. Gorishny and S.A. Larin, *Nucl.Phys.* B283 (1987) 452;  
K.G. Chetyrkin, *Teor.Mat.Fiz.* 75 (1988) 26; 76 (1988) 207;  
Preprint MPI-PAE/PTTh 13/91 (Munich, 1991);  
S.G. Gorishny, *Nucl.Phys.* B319 (1989) 633;  
V.A. Smirnov, *Commun.Math.Phys.* 134 (1990) 109.

[30] G.’t Hooft and M. Veltman, *Nucl.Phys.* B44 (1972) 189;  
C.G. Bollini and J.J. Giambiagi, *Nuovo Cim.* 12B (1972) 20.

[31] V.A. Smirnov, *Renormalization and asymptotic expansions* (Birkhäuser, Basel, 1991).

[32] A.I. Davydychev, *Phys. Lett.* B263 (1991) 107.

[33] A.C. Hearn, *REDUCE user’s manual*, RAND publication CP78 (Santa Monica, 1987);  
version 3.5 (Rev. 10/1993).

[34] E.E. Boos and A.I. Davydychev, *Teor. Mat. Fiz.* 89 (1991) 56 [ *Theor. Math. Phys.* 89 (1991) 1052 ];  
A.I. Davydychev, *J. Math. Phys.* 32 (1991) 1052; 33 (1992) 358.

[35] C. Ford, I. Jack and D.R.T. Jones, *Nucl.Phys.* B387 (1992) 373.

[36] F.V. Tkachov, *Phys. Lett.* B100 (1981) 65;  
K.G. Chetyrkin and F.V. Tkachov, *Nucl. Phys.* B192 (1981) 159;  
F.V. Tkachov, *Teor. Mat. Fiz.* 56 (1983) 350.

[37] A.I. Davydychev, *J. Phys.* A25 (1992) 5587.

[38] A. Erdelyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher transcendental functions* (Bateman manuscript project), Vol.1 (McGraw-Hill, New York, 1953).

[39] S.G. Gorishny, S.A. Larin, L.R. Surguladze and F.V. Tkachov, *Comp.Phys.Commun.* 55 (1989) 381.