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This issue is dedicated to Gary Gruenhage

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1. Editor’s note

Those of you who missed the second SPM meeting in Lecce, may wish to have a look at Section 2.1. In fact, also those who attended may wish to do so.
It is encouraging that each issue comes with so many interesting announcements of new results in the field of SPM and in closely related field. This flourishing a good sign.

Contributions to the next issue are, as always, welcome.

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2. Research announcements

2.1. Lecce Workshop presentations available online. The slides of the talks given at the Second Workshop on Coverings, Selections, and Games in Topology (Lecce, December 2005) are now available online:

http://www.cs.biu.ac.il/~tsaban/SPMC05/slides.html

2.2. Borel cardinalities below \( c_0 \). The Borel cardinality of the quotient of the power set of the natural numbers by the ideal \( \mathcal{Z}_0 \) of asymptotically zero-density sets is shown to be the same as that of the equivalence relation induced by the classical Banach space \( c_0 \). We also show that a large collection of ideals introduced by Louveau and Velickovic, with pairwise incomparable Borel cardinality, are all Borel reducible to \( c_0 \). This refutes a conjecture of Hjorth and has facilitated further work by Farah.

http://www.ams.org/journal-getitem?pii=S0002-9939-06-08207-4

Michael Ray Oliver

2.3. Hereditarily non-topologizable groups. A group \( G \) is non-topologizable if the only Hausdorff group topology that \( G \) admits is the discrete one. Is there an infinite group \( G \) such that \( H/N \) is non-topologizable for every subgroup \( H \leq G \) and every normal subgroup \( N \trianglelefteq H \)? We show that a solution of this essentially group theoretic question provides a solution to the problem of \( c \)-compactness.

http://arxiv.org/abs/math.GR/0603513

Gábor Lukács

2.4. A hodgepodge of sets of reals. We prove a variety of results concerning singular sets of reals. Our results concern: Kysiak and Laver-null sets, Kocinac and \( \gamma_k \)-sets, Fleissner and square \( Q \)-sets, Alikhani-Koopaei and minimal \( Q \)-like-sets, Rubin and \( \sigma \)-sets, and Zapletal and the Souslin number. In particular we show that \( \sigma \)-sets are Laver-null, the union of \( \gamma_k \)-sets need not be \( \gamma_k \), the existence of \( Q \)-set implies an \( \aleph_1 \)-universal \( G_\delta \), minimal \( Q \)-like sets which are not \( Q \)-sets exist, thin sets need not exist, and \( sn^* \) is bounded by the cardinality of the smallest nonmeager set.

http://arxiv.org/abs/math.LO/0603691

Arnold W. Miller
2.5. Random gaps. It is proved that there exists an \((\aleph_1, \aleph_1)\) Souslin gap in the Boolean algebra \((L(\nu)/\text{Fin}, \subseteq^*_{ae})\) for every nonseparable measure \(\nu\). Thus a Souslin, also known as destructible, \((\aleph_1, \aleph_1)\) gap in \(P(\mathbb{N})/\text{Fin}\) can always be constructed from uncountably many random reals. We explain how to obtain the corresponding conclusion from the hypothesis that Lebesgue measure can be extended to all subsets of the real line (RVM), and why this runs counter to authoritative opinions on the nature of consequences of RVM.

http://arxiv.org/abs/math.LO/0604085
James Hirschorn

2.6. Covering a bounded set of functions by an increasing chain of slaloms. A slalom is a sequence of finite sets of length \(\omega\). Slaloms are ordered by coordinatewise inclusion with finitely many exceptions. Improving earlier results of Mildenberger, Shelah and Tsaban, we prove consistency results concerning existence and non-existence of an increasing sequence of a certain type of slaloms which covers a bounded set of functions in the Baire space.

http://arxiv.org/abs/math.LO/0604156
Masaru Kada

2.7. Baire-one mappings contained in a usco map. We investigate Baire-one functions whose graph is contained in a graph of usco mapping. We prove in particular that such a function defined on a metric space with values in \(\mathbb{R}^d\) is the pointwise limit of a sequence of continuous functions with graphs contained in the graph of a common usco map.

http://arxiv.org/abs/math.GN/0604238
Ondřej F.K. Kalenda

2.8. Applications of \(k\)-covers II. We continue the study of applications of \(k\)-covers to some topological constructions, mostly to function spaces and hyperspaces.

http://dx.doi.org/10.1016/j.topol.2005.07.015
A. Caserta, G. Di Maio, Lj. D.R. Kočinac, and E. Meccariello

2.9. Additivity numbers of covering properties. This is an invited chapter in the book “Selection Principles in Topology” (Ljubisa Kočinac, ed.), to appear in the book series Quaderni di Matematica.

The additivity number of a topological property (relative to a given space) is the minimal number of subspaces with this property whose union does not have the property. The most well-known case is where this number is greater than \(\aleph_0\), i.e. the property is \(\sigma\)-additive. We give a rather complete survey of the known results about the additivity numbers of a variety of topological covering properties, including those appearing in the Scheepers diagram (which contains, among others, the classical properties of Menger, Hurewicz, Rothberger, and Gerlits-Nagy). Some of the results proved here were not published beforehand, and many open problems are posed.

http://arxiv.org/abs/math.GN/0604451
2.10. **Combinatorial images of sets of reals and semifilter trichotomy.** Using a dictionary translating a variety of classical and modern covering properties into combinatorial properties of continuous images, we get a simple way to understand the interrelations between these properties in ZFC and in the realm of the trichotomy axiom for upward closed families of sets of natural numbers. While it is now known that the answer to the Hurewicz 1927 problem is positive, it is shown here that semifilter trichotomy implies a negative answer to a slightly weaker form of this problem.

http://arxiv.org/abs/math.LO/0604536

**Boaz Tsaban and Lyubomyr Zdomskyy**

2.11. **Another algebraic equivalent of the Continuum Hypothesis.** Enrico Zoli A renowned theorem due to Erdős and Kakutani states that the Continuum Hypothesis holds if and only if the set of all nonnull reals is a union of countably many Hamel bases. Adapting Erdős and Kakutani’s argument, here I show that the Continuum Hypothesis holds if and only if the set of all transcendental reals is a union of countably many transcendence bases.

2.12. **A connection between decomposable ultrafilters and possible cofinalities II.** We use Shelah’s theory of possible cofinalities in order to solve a problem about ultrafilters.

THEOREM. Suppose that $\lambda$ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter $D$ is $\kappa$-decomposable for all regular cardinals $\kappa$ with $\lambda' < \kappa < \lambda$. Then $D$ is either $\lambda$-decomposable, or $\lambda^+$-decomposable.

We give applications to topological spaces and to abstract logics.

http://arxiv.org/abs/math.LO/0605022

**Paolo Lipparini**

2.13. **Game Approach to Universally Kuratowski-Ulam Spaces.** We consider a version of the open-open game, indicating its connections with universally Kuratowski-Ulam spaces. We show that: Every I-favorable space is universally Kuratowski-Ulam; If a compact space $Y$ is I-favorable, then the hyperspace $\exp(Y)$ with the Vietoris topology is I-favorable, and hence universally Kuratowski-Ulam. Notions of uK-U and uK-U* spaces are compared.

http://arxiv.org/abs/math.GN/0605469

**A. Kucharski and Sz. Plewik**

2.14. **On the density of Banach spaces $C(K)$ with the Grothendieck property.** Using the method of forcing we prove that consistently there is a Banach space of continuous functions on a compact Hausdorff space $C(K)$ with the Grothendieck property and with density less than the continuum. It follows that the classical result stating that “no nontrivial complemented subspace of a Grothendieck space is separable” cannot be strengthened by replacing “is separable” by “has density less than that
of \( \ell_\infty \)”, without using an additional set-theoretic assumption. Such a strengthening was proved by Haydon, Levy and Odell, assuming Martin’s axiom and the negation of the continuum hypothesis. Moreover, our example shows that certain separation properties of Boolean algebras are quite far from the Grothendieck property.

http://www.ams.org/journal-getitem?pii=S0002-9939-06-08401-2

Christina Brech

2.15. **Antichains in partially ordered sets of singular cofinality.** In their paper from 1981, Milner and Sauer conjectured that for any poset \( \langle P, \leq \rangle \), if \( \text{cf}(P, \leq) = \lambda > \text{cf}(\lambda) = \kappa \), then \( P \) must contain an antichain of size \( \kappa \).

We prove that for \( \lambda > \text{cf}(\lambda) = \kappa \), if there exists a cardinal \( \mu < \lambda \) such that \( \text{cov}(\lambda, \mu, \kappa, 2) = \lambda \), then any poset of cofinality \( \lambda \) contains \( \lambda^\kappa \) antichains of size \( \kappa \).

The hypothesis of our theorem is very weak and is a consequence of many well-known axioms such as GCH, SSH and PFA. The consistency of the negation of this hypothesis is unknown.

http://arxiv.org/abs/math.LO/0606021

Assaf Rinot

2.16. **Bolzano-Weierstrass principle of choice extended towards ordinals.** The Bolzano-Weierstrass principle of choice is the oldest method of the set theory, traditionally used in mathematical analysis. We are extending it towards transfinite sequences of steps indexed by ordinals. We are introducing the notions: hiker’s tracks, hiker’s maps and statements \( P_n(X, Y, m) \); which are used similarly in finite, countable and uncountable cases. New proofs of Ramsey’s theorem and Erdős-Rado theorem are presented as some applications.

http://arxiv.org/abs/math.LO/0606028

W. Kulpa, Sz. Plewik and M. Turzański

2.17. **Nonequality of Dimensions for Metric Groups.** An embeddability criterion for zero-dimensional metrizable topological spaces in zero-dimensional metrizable topological groups is given. A space which can be embedded as a closed subspace in a zero-dimensional metrizable group but is not strongly zero-dimensional is constructed; thereby, an example of a metrizable group with noncoinciding dimensions \( \text{ind} \) and \( \text{dim} \) is obtained. It is proved that one of Kulesza’s zero-dimensional metrizable spaces cannot be embedded in a metrizable zero-dimensional group.

http://arxiv.org/abs/math.GN/0606146

Ol’ga V. Sipacheva

2.18. **Not all pure states on \( B(H) \) are diagonalizable.** Assuming the continuum hypothesis, we prove that \( B(H) \) has a pure state whose restriction to any masa is not pure. This resolves negatively an old conjecture of Anderson.

http://arxiv.org/abs/math.OA/0606168

Charles Akemann and Nik Weaver
2.19. A comment on $p < t$. Dealing with the cardinal invariants $p$ and $t$ of the continuum we prove that

$$m \geq p = \aleph_2 \Rightarrow t = \aleph_2.$$ 

In other words, if $\text{MA}_{\aleph_1}$ (or a weak version of this) holds, then (of course $\aleph_2 \leq p \leq t$ and) $p = \aleph_2 \Rightarrow p = t$. The proof is based on a criterion for $p < t$.

http://arxiv.org/abs/math.LO/0404220

Saharon Shelah

3. Problem of the Issue

A set of reals is totally imperfect if it has no perfect subsets. There are in the literature several examples of totally imperfect sets of reals satisfying $U_{\text{fin}}(\mathcal{O}, \Gamma)$, see [3, 1, 5]. However, all of them actually satisfy $S_1(\Gamma, \Gamma)$ [4] or at least $S_1(\Gamma, \mathcal{O})$ [2, 5]. Consequently, all of their continuous images satisfy $S_1(\Gamma, \mathcal{O})$ and are therefore totally imperfect [3].

Problem 3.1. Could there be a totally imperfect set of reals $X$ satisfying $U_{\text{fin}}(\mathcal{O}, \Gamma)$ that can be mapped continuously onto $\{0, 1\}^\mathbb{N}$?

Tomasz Weiss

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http://arxiv.org/abs/math.GN/0507043
4. PROBLEMS FROM EARLIER ISSUES

Issue 1. Is \( \binom{\Omega}{\Gamma} = \binom{\Omega}{\Gamma} \)?

Issue 2. Is \( \cup_{\text{fin}}(\Gamma,\Omega) = S_{\text{fin}}(\Gamma,\Omega) \)? And if not, does \( \cup_{\text{fin}}(\Gamma,\Gamma) \) imply \( S_{\text{fin}}(\Gamma,\Omega) \)?

Issue 4. Does \( S_1(\Omega,\Gamma) \) imply \( \cup_{\text{fin}}(\Gamma,\Gamma) \)?

Issue 5. Is \( p = p^* \)? (See the definition of \( p^* \) in that issue.)

Issue 6. Does there exist (in ZFC) an uncountable set satisfying \( S_1(\mathcal{B},\mathcal{B}) \)?

Issue 8. Does \( X \notin \text{NON}(\mathcal{M}) \) and \( Y \notin \mathcal{D} \) imply that \( X \cup Y \notin \text{COF}(\mathcal{M}) \)?

Issue 9 (CH). Is \( \text{Split}(\Lambda,\Lambda) \) preserved under finite unions?

Issue 10. Is \( \text{cov}(\mathcal{M}) = \mathfrak{d} \)? (See the definition of \( \mathfrak{d} \) in that issue.)

Issue 11. Does \( S_1(\Gamma,\Gamma) \) always contain an element of cardinality \( \mathfrak{b} \)?

Issue 12. Could there be a Baire metric space \( M \) of weight \( \aleph_1 \) and a partition \( \mathcal{U} \) of \( M \) into \( \aleph_1 \) meager sets where for each \( \mathcal{U}' \subset \mathcal{U} \), \( \bigcup \mathcal{U}' \) has the Baire property in \( M \)?

Issue 14. Does there exist (in ZFC) a set of reals \( X \) of cardinality \( \mathfrak{d} \) such that all finite powers of \( X \) have Menger’s property \( S_{\text{fin}}(\mathcal{O},\mathcal{O}) \)?

Issue 15. Can a Borel non-\( \sigma \)-compact group be generated by a Hurewicz subspace?

Issue 16 (MA). Is there an uncountable \( X \subseteq \mathbb{R} \) satisfying \( S_1(\mathcal{B},\mathcal{B}) \)?

Issue 17 (CH). Is there a totally imperfect \( X \) satisfying \( \cup_{\text{fin}}(\mathcal{O},\Gamma) \) that can be mapped continuously onto \( \{0,1\}^\mathbb{N} \)?