Subexponential instability implies infinite invariant measure

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We study subexponential instability to characterize a dynamical instability of weak chaos. We show that a dynamical system with subexponential instability has an infinite invariant measure, and then we present the generalized Lyapunov exponent to characterize subexponential instability.

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Chaotic dynamical systems with the zero Lyapunov exponent have attracted much interest in non-equilibrium statistical mechanics. Chaos in such dynamical systems is called weak chaos. We consider the Pomeau-Manneville map as an example of weak chaos, whose dynamical instability is not exponential one. Then, we define subexponential instability using the average of the logarithm of the separation of two nearby orbits. A dynamical instability of weak chaos is characterized as subexponential instability. We demonstrate the relation between subexponential instability and infinite invariant measure. In particular, we show that subexponential instability implies infinite invariant measure. The generalized Lyapunov exponent is studied to quantify a dynamical instability of weak chaos.

I. INTRODUCTION

In the foundation of statistical mechanics, chaos plays an important role in deriving a stochastic description from microscopic dynamics. The existence of an equilibrium state in dynamical systems requires ergodicity, i.e., time average being equal to space average. This is because macroscopic quantities, which result from time averages of microscopic observation functions, are almost equal to space average (constant).

Non-equilibrium phenomena can be classified into two different states: non-equilibrium stationary state and non-equilibrium non-stationary state. In non-equilibrium stationary state, the time average of an observation function is almost constant but it depends on subsystems. In other words, macroscopic observables restricted to subsystems are almost constant but depend on the position of subsystems. On the other hand, macroscopic observables are not constant in non-equilibrium non-stationary state. For example, an approach to an equilibrium state such as a diffusion is non-equilibrium non-stationary state because macroscopic observables depend on time. Other examples are random processes such as turbulence and chemical reaction, where macroscopic observables do not approach an equilibrium state but change randomly in time. In such phenomena, the time average of an observation function in underlying microscopic dynamics need to be intrinsically random. However, the foundation of non-equilibrium statistical mechanics on the basis of time average has not been studied at all.

Recently, random behaviors of physical quantities in non-equilibrium non-stationary phenomena such as anomalous diffusions and intermittency have been studied extensively in both experiments and theories. In particular, randomness of the generalized diffusion coefficient has been clearly observed in the diffusion of mRNA molecules inside the cell. Since then, this result has been elucidated theoretically through the continuous time random walk model of anomalous diffusion. In general, anomalous diffusion processes can be represented by deterministic dynamical systems. It has been known that dynamical systems describing subdiffusions, where mean square displacement increases as \( \langle x(t)^2 \rangle = \alpha(t) \), show aging and infinite invariant measure. Infinite measure systems, i.e., dynamical systems with infinite invariant measures, have attracted much interest to found the non-equilibrium statistical mechanics. The notable point in infinite measure systems is that the time average of an observation function is intrinsically random. More precisely, the time average of an observation function converges in distribution, and its distribution depends on the class of the observation function. This is reminiscent of the randomness of the time average in non-equilibrium non-stationary states.

In this paper, we develop the concept of ergodicity toward the foundation of non-equilibrium statistical mechanics based on time averages in dynamical systems. Our aim is to characterize the dynamical instability of infinite measure systems. In finite measure systems a dynamical instability can be characterized by the Lyapunov exponent. In a one-dimensional map \( T(x) \), the Lyapunov exponent defined by the time average of \( \ln |T'(x)| \) indicates the exponential rate of the separation between two nearby orbits. By Birkhoff’s individual ergodic theorem, the time average of \( \ln |T'(x)| \) converges to the average of \( \ln |T'(x)| \) with respect to the invariant measure for
almost all initial points. Here, we focus on the subexponential instability which is characterized by the subexponential growth of the separation of two nearby orbits. Accordingly, the Lyapunov exponent converges to zero in dynamical systems with subexponential instability. We show that dynamical systems with subexponential instability have infinite invariant measures. We then study the generalized Lyapunov exponent defined as the average growth rate of subexponential separation.

II. SUBEXPONENTIAL INSTABILITY

Gaspard and Wang suggested that the separation between nearby trajectories shows stretched exponential growth in Pomeau-Manneville maps with infinite invariant measures. However, the dependence of the growth rate on an initial point was not studied.

Based on the Darling and Kac theorem, Aaronson proved that the time average of the positive $L^1(m)$ function converges in distribution, where $m$ is an invariant measure. That is, the growth rate strongly depends on an initial point and is intrinsically random. When the logarithm of the derivative of a map is the positive $L^1(m)$ function, the normalized Lyapunov exponent converges in distribution:

$$\Pr \left( \frac{1}{a_n} \sum_{k=0}^{n-1} \ln |T'(x_k)| \leq t \right) \rightarrow M_\alpha(t) \quad \text{as } n \rightarrow \infty, \quad (1)$$

where $a_n$ is called the return sequence determined by the map and $M_\alpha(t)$ is the normalized Mittag-Leffler distribution. Generally, the return sequence is determined using the wandering rate defined by

$$w_n = m \left( \bigcup_{k=0}^{n} T^{-k} A \right), \quad (2)$$

where $0 < m(A) < \infty$. In particular, the return sequence is given by

$$a_n \sim \frac{n}{\Gamma(1 + \alpha) \Gamma(2 - \alpha) w_n}, \quad (3)$$

where $w_n$ is regularly varying at $\infty$ with index $\alpha$. For example, the sequence $a_n$ of the Pomeau-Manneville map,

$$x_{n+1} = T(x_n) = x_n + cx_n^{1+p} \mod 1, \quad (4)$$

where $c > 0, p \geq 1$, is given by

$$a_n \sim \begin{cases} n^{\frac{1}{p}} & (p > 1) \\ n \log n & (p = 1). \end{cases} \quad (5)$$

Because of $\ln |T'(x)| \in L^1(m)$, the distribution of the normalized Lyapunov exponent converges to a Mittag-Leffler distribution of order $\alpha$, where the order $\alpha$ is given by $1/p$. Figure 1 shows that the distribution of the normalized Lyapunov exponent, $\lambda = \frac{1}{a_n} \sum_{k=0}^{n-1} \ln |T'(x_k)|$, converges to a Mittag-Leffler distribution even when the time is finite ($n = 10^3$).

We consider the separation of two nearby orbits $(x_n$ and $x'_n)$ defined as

$$\Delta_n \equiv \lim_{\Delta x(0) \rightarrow 0} \frac{\Delta x(n)}{\Delta x(0)} = \prod_{k=0}^{n-1} T'(x_k), \quad (6)$$

where $\Delta x(n) = x_n - x'_n$. By $[\Pi]$, the distribution for the growth of the separation can be given by

$$\Pr \{ \Delta_n \leq t \} \rightarrow M_\alpha \left( \frac{\ln t}{a_n} \right) \quad \text{as } n \rightarrow \infty. \quad (7)$$

For large $n$, the average of the separation is given as

$$\langle \Delta_n \rangle \equiv \langle e^{a_n Y_\alpha} \rangle_{ML}, \quad (8)$$

where $\langle \cdot \rangle$ represents the average with respect to an initial ensemble, $\langle \cdot \rangle_{ML}$ represents the average with respect to a Mittag-Leffler distribution, and $Y_\alpha$ is a random variable with a Mittag-Leffler distribution of order $\alpha$. Accordingly, one can write

$$\langle e^{a_n Y_\alpha} \rangle_{ML} = \sum_{k=0}^{\infty} \frac{\Gamma(1 + \alpha)^k}{\Gamma(1 + k\alpha)} a_n^k. \quad (9)$$

We then have

$$\langle e^{a_n Y_\alpha} \rangle_{ML} > \sum_{k=0}^{\infty} \frac{\gamma a_n^k}{(|k\alpha| + 1)!} \exp[\gamma a_n^{|k\alpha|}], \quad (10)$$

where $\gamma = \Gamma(1 + \alpha)$. The upper bound is obtained as

$$\frac{1}{n} \ln \Delta_n = \frac{1}{n} \sum_{k=0}^{n-1} \ln |T'(x_k)| \leq M, \quad (11)$$

FIG. 1. Probability density function $P_d(t)$ of the normalized Lyapunov exponent of the Pomeau-Manneville map ($p = 1.5$). The sequence $a_n$ is given by $a_n = \langle \lambda \rangle^n$, where $\alpha = 2/3$ and $\langle \lambda \rangle = 2.622$.
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where $M = \ln[1 + c(1 + p)]$. For large $n$, we have the following inequality:

$$\exp(Bn^{|p|n}) < \langle \Delta_n \rangle < \exp(Mn),$$

where $B$ is constant. As shown in Fig. 2, the fluctuation of the separation $\Delta_n$ is extremely large and the average of $\Delta_n$ is almost determined by the largest value of $\ln \Delta_n$. This implies that subexponential instability should be characterized as the average of the logarithm of the separation, $\langle \ln \Delta_n \rangle$, rather than that of the separation, $\langle \Delta_n \rangle$. We call that a dynamical system has subexponential instability if there exists the sequence $a_n = o(n)$ such that

$$\left\langle \frac{1}{a_n} \sum_{k=0}^{n-1} \ln |T'(x_k)| \right\rangle \to 1 \quad \text{as} \quad n \to \infty. \quad (13)$$

III. SUBEXPONENTIAL INSTABILITY IMPLIES INFINITE INVARINATE MEASURE

We show that invariant measure cannot be normalized when a dynamical system has subexponential instability, and then that the normalized Lyapunov exponent becomes intrinsically random. Here, we consider a one-dimensional conservative, ergodic, measure preserving map with an invariant measure absolutely continuous with respect to Lebesgue measure in the sense of infinite ergodic theory. In other words, we do not consider dynamical systems with the trivial zero Lyapunov exponents such as $T'(x) = 1$ for almost all $x$. Moreover, we do not consider dynamical systems where there exist attractive indifferent periodic points, $(T^m(x^*_k))' = 1$ for period $m$ points $x^*_1, \ldots, x^*_m$ ($k = 1, \ldots, m$) in the sense of Mihony. The typical example of such dynamical systems is the dynamical system at the period doubling bifurcation points. At the critical point $(m \to \infty)$, $\langle \Delta_n \rangle$ increases as power law. However, we do not consider this case because there is no sensitive dependence on initial conditions nor an invariant measure absolutely continuous with respect to Lebesgue measure. We note that there is at most one $\sigma$-finite invariant measure in a one-dimensional conservative, ergodic, measure preserving map.

First, we assume $X_k = \ln T'(x_k) \geq 0$. If $X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables, then by the laws of large numbers we have

$$\Pr \left\{ \frac{X_1 + \cdots + X_n}{n} - E(X_k) > \epsilon \right\} \to 0 \quad \text{as} \quad n \to \infty, \quad (14)$$

where $E(X_k) > 0$ is the expectation of $X_k$, which contradicts subexponential instability ($E(X_1 + \cdots + X_n) = o(n)$). This implies that $X_k$ is not an i.i.d. random variable. Moreover, by $E(X_1 + \cdots + X_n) = o(n)$ we have

$$E(X_n) \to 0 \quad \text{as} \quad n \to \infty. \quad (15)$$

It follows that the probability $\Pr\{X_k \leq \epsilon\}$ increases with time for all $\epsilon > 0$. That is, $X_k$ is not an i.i.d. random variable, but trapped near the indifferent fixed point ($T'(a) = 1$) for a very long time, where $a (\leq \infty)$ is an indifferent fixed point. In other words, the empirical distribution of an orbit $\{x_k\}$ converges to the delta distribution: For all $d > b > a$,

$$\int_b^d \rho_n(x) dx \to 0 \quad \text{as} \quad n \to \infty, \quad (16)$$

where

$$\rho_n(x) = \frac{\delta_{x_0}(x) + \delta_{x_1}(x) + \cdots + \delta_{x_n}(x)}{n}. \quad (17)$$

Therefore, the invariant measure is an infinite one. This is because there exists a unique invariant measure absolutely continuous with respect to Lebesgue measure and the empirical distribution of an orbit converges to the invariant measure if the map has a finite invariant measure (probability measure).

As will be seen later, an indifferent fixed point of Boolean transformation is at $\pm \infty$. In the case of $T'(\pm \infty) = 1$, a dynamical system is an open system, i.e. the space of a dynamical system is not compact. By the condition of subexponential instability, $E(X_1 + \cdots + X_n) = o(n)$, Eq. (16) is fulfilled for $-\infty < b < d < \infty$. Therefore, the invariant measure is an infinite one. This impossibility of the normalization of the invariant measure stems from the non-compactness of the space of a dynamical system. Performing a change of a variable, we can obtain a compact dynamical system with indifferent fixed point (see Appendix A). Intuitively speaking, the infinite invariant measure on the non-compact support is obtained by stretching the infinite invariant measure on the compact support near the indifferent fixed points.
Next, we consider the case of $X_k \in \mathbb{R}$, $E(|X_1| + \cdots + |X_n|) = O(n)$ and $E(X_1 + \cdots + X_n) = o(n)^{23}$. Let $X_1, X_2, \cdots$ be i.i.d. random variables with the mean 0. Then, there exists the sequence $a_n$ such that the limit distribution of $(X_1 + \cdots + X_n)/a_n$ converges to be a normal or a stable distribution. It follows that the probability of the negative Lyapunov exponent ($\lambda < 0$) remains positive and that the Lyapunov exponent converges to zero as $n$ goes to infinity. This means that the dynamical system has an attractive indifferent periodic points. As we mentioned above, we do not consider this dynamical system because there is no sensitivity to initial conditions. Therefore, $X_1, X_2, \cdots$ are not i.i.d. random variables with the mean 0.

There are two cases: (i) The expectation of $X_n$ is positive and $E(X_n) \to 0$ as $n \to \infty$. In a similar way we can show that the invariant measure is infinite. (ii) Due to canceling out of $X_1 + \cdots + X_n$, $E(X_1 + \cdots + X_n)$ does not grow linearly. We introduce some notations (see Fig. 3): the partial sum of $X_n$, $S_n = X_1 + \cdots + X_n$, the maximum of $S_n$, $m_n = \max\{0, S_1, \cdots, S_n\}$, the number of renewals of maximum, $N_n = \min\{k : m_k > m_{k+1}, m_k > N_{n-1}\}$, and the nth maximum, $M_n = m_{N_n}$. The condition $E(X_1 + \cdots + X_n) = o(n)$ leads to $E(N_n - N_{n-1}) = \infty$ or $E(M_n - M_{n-1}) = 0$ as $n \to \infty$, where the interevent times, $N_n - N_{n-1}$, are i.i.d. random variables.

In the case of $E(N_n - N_{n-1}) = \infty$ and $E(M_n - M_{n-1}) < \infty$, an orbit is occasionally trapped in the region near the fixed point becomes larger as the reinjection is closer to the fixed point. Since the probability density function of the residence time on $[a-\epsilon, a+\epsilon]$ for the tail part is invariant for all $\epsilon$, the expectation of the residence time on $[a-\epsilon, a+\epsilon]$ denoted by $T_A$, is infinite, where $a$ is the fixed point. By the law of large numbers, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} 1_{[a-\epsilon, a+\epsilon]}(x_k) \to \frac{T_A}{T_A + T_A^c} = 1,$$

for all $\epsilon$ as $n \to \infty$, where $T_A$ is the expectation of the residence time on the complement of the set $[a-\epsilon, a+\epsilon]$. This means that the invariant measure is an infinite one. It is noteworthy that the fixed point does not need to be the indifferent fixed point (see Fig. 4). As shown in

![Figure 3](image1.png)

**FIG. 3.** Schematic illustration of $N_n$ and $M_n$.

![Figure 4](image2.png)

**FIG. 4.** (Color online). The transformation with a flat critical point. The flat critical point is at $x = 0.5 + 0$.

Appendix B, the transformation with a flat critical point such as

$$T(x) = \begin{cases} 2x & x \in [0, 1/2] \\ \exp(-\frac{1}{x-1/2})/2 & x \in (1/2, 1] \end{cases}$$

(19)

has subexponential instability with $\ln |T'(x)| \in \mathbb{R}$ and an infinite invariant measure \cite{24,25}, which is compatible with the above result.

In the case of $E(M_n - M_{n-1}) \to 0$ as $n \to \infty$ and $E(N_n - N_{n-1}) < \infty$, this leads to $(T(N_n - N_{n-1})/x_{N_{n-1}})' \to 1$ as $n \to \infty$. It follows that an orbit is trapped near the indifferent periodic points. Hence, the invariant measure is an infinite one. Modifying the logistic map at the tangent bifurcation point to change the one-sided stable indifferent periodic points to the repulsive indifferent periodic points, we have a dynamical system with an infinite invariant measure. In particular, the modified logistic map is given by

$$T(x) = \begin{cases} -2(x - e) + (1 + 2\sqrt{2})e(1 - e) & x \in [0, e] \\ (1 + 2\sqrt{2})e(1 - x) & x \in (e, 1] \end{cases}$$

(20)

where $T^3(e) = e$ ($e < 0.2$). The three iterated map $T^3(x)$ has three indifferent fixed points (Fig. 5). In particular, $T^3(x) = x + C(x - e)^2 + o((x - e)^2)$ as $x \to e + 0$, where $C \geq 0$ is constant. Therefore, the invariant measure is infinite one\cite{23,24}.

When $\ln |T'(x)|$ is the positive $L^1(m)$ function, by the Darling-Kac and Aaronson theorem, the normalized Lyapunov exponent converges in distribution to a Mittag-Leffler distribution. On the other hand, the normalized Lyapunov exponent converges in distribution to the generalized arcsine distribution\cite{23}. When $\ln |T'(x)|$ is $L^1_{\text{loc}}(m)$ with finite mean function, the Lyapunov exponent converges in distribution to a stable distribution\cite{23}. However, these cases do not show subexponential instability. Therefore, the subexponential instability makes the Lyapunov exponent intrinsically random, and the normalized Lyapunov ex-
IV. GENERALIZED LYAPUNOV EXPONENT

Indicators characterizing weak chaos have been studied by quantifying the complexity or the sensitivity to initial conditions. To characterize subexponential instability we use the generalized Lyapunov exponent proposed by Korabel and Barkai. The generalized Lyapunov exponent \( \lambda(a_n) \) is defined as

\[
\lambda(a_n) = \left\langle \frac{1}{a_n} \sum_{k=0}^{n-1} \ln |T'(x_k)| \right\rangle,
\]

where \( \langle \cdot \rangle \) represents the average with respect to initial points. It is remarkable that one can choose an arbitrary initial ensemble if it is continuous with respect to Lebesgue measure. Note that there exists the sequence \( a_n \) such that \( 0 < \lambda(a_n) < \infty \) as \( n \to \infty \) in dynamical systems with subexponential instability. The sequence \( a_n \) presents a global instability of a dynamical instability and \( \lim_{n \to \infty} \lambda(a_n) \) represents the rate of a subexponential growth. We call \((a_n, \lambda)\) the Lyapunov pair.

We review the Darling-Kac and Aaronson theorem. Let \( T \) be a transformation on \( X \) with subexponential instability, and let \( \ln |T'(x)| \) be the positive \( L^1(m) \) function. Then the normalized Lyapunov exponent converges in distribution to a Mittag-Leffler distribution: There exists \( a_n \) such that

\[
\Pr \left\{ \frac{1}{a_n} \sum_{k=0}^{n-1} \ln |T'(x_k)| \leq t \right\} \to M_\alpha(t) \int_X \ln |T'(x)| dm
\]
as \( n \to \infty \).

We demonstrate subexponential instability for Boole transformation. Boole transformation \( T : \mathbb{R} \to \mathbb{R} \) is given by

\[
x_{n+1} = T(x_n) = x_n - \frac{1}{x_n}.
\]

The invariant measure is uniform, i.e., of infinite measure. The return sequence \( a_n \) is given as

\[
a_n = \frac{\sqrt{2n}}{\pi}.
\]

By the Darling-Kac and Aaronson theorem, we can obtain the generalized Lyapunov exponent

\[
\lambda(\sqrt{2n}/\pi) = \int_{-\infty}^{\infty} \ln |1 + 1/x^2| dx = 2\pi.
\]

Thus, the subexponential instability is represented by

\[
\langle \ln \Delta_n \rangle \sim 2\sqrt{2n} \quad \text{as} \quad n \to \infty.
\]

V. CONCLUSIONS

In conclusion, we have shown that dynamical systems with subexponential instability have infinite invariant measures. It follows that a dynamical system with a finite invariant measure does not show subexponential instability. Considering Birkhoff’s ergodic theorem, we see that a dynamical system has the positive Lyapunov exponent if the invariant measure \( m \) is a probability measure and the function \( \ln |T'(x)| \) is an \( L^1(m) \) function. Furthermore, we have presented that one dimensional maps with the non-trivial zero Lyapunov exponent are described as infinite measure systems such as transformations with indifferent fixed or periodic points which are repellers and transformations with flat critical points.

While it has been known that random dynamical systems have aperiodic orbits with the negative Lyapunov exponent, the relation between an invariant measure and the Lyapunov exponent is still unknown in random dynamical systems. However, the invariant measure becomes infinite when the Lyapunov exponent is zero in the drift-diffusion model. Moreover, the transformation \( T x = D x \) with probability 1/2 and \( T x = x / D \) with probability 1/2 does not have attractors but have an infinite invariant measure absolutely continuous with respect to Lebesgue measure. We conjecture that the non-trivial zero Lyapunov exponent would imply an infinite invariant measure.

Finally, we characterized the separation of nearby orbit as the generalized Lyapunov exponent. There are different definitions of the generalized Lyapunov exponent. Kuzovkov et al. applied the generalized Lyapunov exponent to the Anderson localization problem. Compared to these definitions, the generalized Lyapunov exponent in this paper focuses on weak instabilities such as stretched exponential and algebraic instabilities. It would be interesting to make clear the
relation between the macroscopic observables in non-equilibrium state and the generalized Lyapunov exponent. We expect that the generalized Lyapunov exponent will be useful to understand the non-equilibrium non-stationary phenomena. In particular, we hope that random behaviors of transport coefficients in anomalous diffusions will be characterized by the generalized Lyapunov exponent.

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Appendix A: Change of a variable

In Boole transformation a change of a variable, \( \phi(x) = x - 2 + \sqrt{4 + x^2} \), yields a map with indifferent fixed points on \([0, 1]\). The transformed map \( S : [0, 1] \to [0, 1] \),

\[
S(x) = \phi \circ T \circ \phi^{-1}(x),
\]

is given by

\[
S(x) = \begin{cases} 
\frac{x(1-x)}{1-x-x^2} & x \in [0, 1/2] \\
1 - \frac{2x}{1-3x+x^2} & x \in (1/2, 1].
\end{cases}
\]

This map has two indifferent fixed points at \( x = 0 \) and \( 1 \), and the asymptotic behavior at \( x = 0 \) is given by

\[
S(x) \approx x + x^3 + o(x^3).
\]

Hence, we see that the invariant measure is an infinite one.

Appendix B: Subexponential instability of a map (19)

Due to the flat critical point, the invariant measure is an infinite invariant measure and the invariant density is not bounded at \( x = 0 \). The inequality, \( 2^{-N-1} < T(x) < 2^{-N} \), is fulfilled when the residence time in \([0, 1/2]\) is \( N \) and \( x \in (1/2, 1] \). This inequality is solved as

\[
\frac{1}{N \ln 2 + 2} + \frac{1}{2} < x < \frac{1}{(N - 1) \ln 2 + 2} + \frac{1}{2},
\]

Using

\[
\ln |T'(x)| = \begin{cases} 
\ln 2 & x \in [0, 1/2] \\
\frac{-1}{x - \frac{1}{2}} + 2 - \ln 2 + \ln(x - \frac{1}{2})^{-2} & x \in (1/2, 1],
\end{cases}
\]

we have the inequality for the partial sum \( S_{N+1} = X_1 + \cdots + X_{N+1} \):

\[
2 \ln \{(N - 1) \ln 2 + 2\} < S_{N+1} = \ln |T'(x)| + N \ln 2 < -\ln 2 + 2 \ln(N \ln 2 + 2).
\]

Let \( l_n \) denote the number of reinjections on \([0, 1/2]\) until \( n \) iterations. Since the distribution of the injection on \([1/2, 1]\) is uniform, the distribution of the residence time in the interval \([0, 1/2]\) is given as

\[
F(N) = \Pr\{N_k \leq N\} = \Pr\{T(x) \leq 2^{-N}\}
\]

\[
= \Pr\left\{ x \geq \frac{1}{N \ln 2 + 2} + \frac{1}{2} \right\}
\]

\[
= 1 - \frac{2}{2 + N \ln 2}.
\]

Therefore,

\[
E(N_k) = \int_0^\infty NF'(N)dN > \frac{1}{2 + 2N} \int_1^\infty \frac{1}{N}dN = \infty.
\]

By the renewal theory\(^2\), there exists a constant \( K > 0 \) such that

\[
l_n = K \frac{n}{\ln n} + o\left( \frac{n}{\ln n} \right) \quad \text{as } n \to \infty.
\]

Using the arithmetic and geometric means inequality, we have

\[
E(X_1 + \cdots + X_n) < E(S_{N_1+1} + \cdots + S_{N_n+1})
\]

\[
< 2E[\ln((N_1 + 1) \cdots (N_n + 1))] = 2E(l_n \ln(n/l_n))
\]

\[
\propto \frac{n \ln n}{\ln n} \quad \text{as } n \to \infty,
\]

where \( N_1 + 1 + \cdots + N_n + 1 = n \). This means subexponential instability, \( E(X_1 + \cdots + X_n) = o(n) \).

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15. The random variable $Y_\alpha$ on $\mathbb{R}$ has the normalized Mittag-Leffler distribution of order $\alpha$ if

$$E(e^{\lambda Y_\alpha}) = \sum_{k=0}^{\infty} \frac{\Gamma(1+\alpha)\lambda^k}{\Gamma(1+k\alpha)},$$

where $E(\cdot)$ is the expectation.
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