WEAK VERSION OF RESTRICTION ESTIMATES FOR SPHERES AND PARABOLOIDS IN FINITE FIELDS

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Abstract. We study $L^p - L^r$ restriction estimates for algebraic varieties in $d$-dimensional vector spaces over finite fields. Unlike the Euclidean case, if the dimension $d$ is even, then it is conjectured that the $L^{(2d+2)/(d+3)} - L^2$ Stein-Tomas restriction result can be improved to the $L^{(2d+4)/(d+4)} - L^2$ estimate for both spheres and paraboloids in finite fields. In this paper we show that the conjectured $L^p - L^2$ restriction estimate holds in the specific case when test functions under consideration are restricted to $d$-coordinate functions or homogeneous functions of degree zero. To deduce our result, we use the connection between the restriction phenomena for our varieties in $d$ dimensions and those for homogeneous varieties in $(d + 1)$ dimensions.

1. Introduction

Let $V$ be a subset of $\mathbb{R}^d$, $d \geq 2$, and $d\sigma$ a positive measure supported on $V$. The classical restriction problem asks us to determine $1 \leq p, r \leq \infty$ such that the following restriction estimate holds:

$$\|\hat{f}\|_{L^r(V,d\sigma)} \leq C_{p,r,d} \|f\|_{L^p(\mathbb{R}^d)}$$

for every Schwarz function $f : \mathbb{R}^d \to \mathbb{C}$. By duality, the restriction estimate (1.1) is same as the following extension estimate:

$$\|(gd\sigma)^\vee\|_{L^{p'}(\mathbb{R}^d)} \leq C_{p,r,d} \|g\|_{L^{r'}(V,d\sigma)},$$

where $p' = p/(p - 1)$ and $r' = r/(r - 1)$. This problem was addressed and studied by E.M. Stein ([18]). Much attention has been given to this problem, in part because it is closely related to other important problems such as the Falconer distance problem, the Kakeya problem, and the Bochner-Riesz problem (for example, see [5, 2, 3, 19]). The complete answer to the restriction problem is known only for certain lower dimensional hypersurfaces. For instance, Zygmund ([25]) established the restriction conjecture for the circle and the parabola in the plane. Barcelo ([1]) and Wolff ([24]) also solved it for the cone of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively. However, the restriction conjecture remains open in other higher dimensions. The best known result for the cone of $\mathbb{R}^d$, $d \geq 5$, is due to Wolff ([24]) who utilized the bilinear restriction method. Terence Tao ([20]) also used the method to derive the best known restriction results on the sphere and paraboloid of $\mathbb{R}^d$, $d \geq 3$. However, it has been believed that classically used analytical approaches are not enough to settle down
the restriction problem. We refer reader to Tao’s survey paper \[21\] and references therein for currently known skills to deduce restriction results in the Euclidean case.

In recent years, problems in the Euclidean space have been studied in the finite field setting. Motivation on the study of Euclidean problems in finite fields is to understand the original problems in simple finite field structure. In 1999, Tom Wolf (\[23\]) formulated the Kakeya problem in finite fields and new results on the problem were addressed in the subsequent papers (see \[17\] \[16\] \[22\]). Surprisingly, Dvir (\[4\]) proved the finite field Kakeya conjecture by beautifully simple, new argument based on the polynomial method. His work has inspired researchers to further efforts for seeking solutions to other analysis problems in finite fields. In \[16\], Mockenhaupt and Tao first investigated the Fourier restriction problem for various algebraic varieties in the finite field setting and they addressed interesting results on this problem. Further efforts to understand the finite field restriction problem have been made by other researchers (see, for example, \[7\] \[8\] \[10\] \[11\] \[12\] \[13\] \[14\]). In particular, the finite field restriction problem for cones, paraboloids, and spheres have been mainly studied, but known results are far from the conjectured results in higher dimensions.

When we study analogue of Euclidean problems in finite fields, we often find an unprecedented phenomenon which never occurs in the Euclidean case. It is well known that if \( V \subset \mathbb{R}^d \) is the sphere or a compact subset of the paraboloid, then \( p_0 = (2d + 2)/(d + 3) \) gives the sharp \( p \) exponent for \( L^p - L^2 \) restriction estimates for \( V \). The number \( p_0 \) is called the Stein-Tomas exponent for the \( L^p - L^2 \) restriction inequality. On the contrary to the Euclidean case, it is possible to improve the Stein-Tomas exponent \( p_0 \) if \( V \) is the paraboloid in even dimensional vector spaces over finite fields. For example, Mockenhaupt and Tao (\[16\]) proved the \( L^{4/3} - L^2 \) restriction estimate for the parabola lying in two dimensional vector spaces over finite fields. For even dimensions \( d \geq 4 \), A. Lewko and M. Lewko (\[14\]) obtained the \( L^{2d}/(d^2 + 2d - 2) - L^2 \) restriction result for the paraboloid in the finite field setting. These results are clearly better than the Stein-Tomas inequality. Here, we point out that if the dimension \( d \geq 3 \) is odd and \(-1\) is a square number, then it is impossible to improve the Stein-Tomas restriction estimate for spheres or paraboloids in finite field case. For this reason, we shall just focus on studying the \( L^p - L^2 \) restriction estimates for spheres or paraboloids in even dimensions.

When \(-1\) is a square number in the underlying finite field, it is conjectured that the \( L^{(2d + 4)/(d + 4)} - L^2 \) restriction estimate is the best possible result on the \( L^p - L^2 \) estimate for the sphere or the paraboloid in even dimensional vector spaces over finite fields (see Conjecture \[2.2\]). The conjecture is open except for \( d = 2 \), and the aforementioned result due to A. Lewko and M. Lewko is far from the conjectured one. Furthermore, there is no known result for spheres in even dimensions \( d \geq 4 \) which improves on the Stein-Tomas exponent. The main purpose of this paper is to find a class of test functions for which the conjectured \( L^{(2d + 4)/(d + 4)} - L^2 \) restriction estimate holds for the sphere or the paraboloid in even dimensional vector spaces over finite fields. The main idea to derive our results is to use a connection between restriction estimates for homogeneous varieties in \((d + 1)\) dimensions and those for the sphere or the paraboloid in \(d\)-dimensional vector spaces over finite fields.
2. Weak version of restriction estimates

To precisely state our main results, we shall introduce the weak version of restriction problems in the finite field setting. Roughly speaking, we investigate the $L^p - L^2$ restriction estimates for algebraic varieties in the specific case when the test functions are restricted to specific classes of functions rather than all functions on vector spaces over finite fields. We begin by reviewing the restriction problem for algebraic varieties in finite fields.

2.1. Review of the restriction problem. Let $\mathbb{F}_q^d$, $d \geq 2$, be the $d$-dimensional vector spaces over finite fields $\mathbb{F}_q$ with $q$ elements. We assume that the characteristic of $\mathbb{F}_q$ is greater than two. The space $\mathbb{F}_q^d$ is equipped with a counting measure $dm$, by setting, for any function $g : (\mathbb{F}_q^d, dm) \to \mathbb{C}$,

$$
\int_{\mathbb{F}_q^d} g(m) \ dm = \sum_{m \in \mathbb{F}_q^d} g(m).
$$

Here and throughout the paper, we write the notation $(\mathbb{F}_q^d, dm)$ for the space $\mathbb{F}_q^d$ with the counting measure $dm$. On the contrary to the space $(\mathbb{F}_q^d, dm)$, we endow its dual space with a normalized counting measure $dx$. The dual space of $(\mathbb{F}_q^d, dm)$ is denoted by the notation $(\mathbb{F}_q^d, dx)$. Recall that if $g : (\mathbb{F}_q^d, dm) \to \mathbb{C}$, then its Fourier transform $\hat{g}$ is a function on the dual space $(\mathbb{F}_q^d, dx)$. Thus, for $x \in (\mathbb{F}_q^d, dx)$,

$$
\hat{g}(x) = \int_{\mathbb{F}_q^d} \chi(-m \cdot x)g(m) \ dm = \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x)g(m),
$$

where $\chi$ denotes a nontrivial additive character of $\mathbb{F}_q$. Also recall that if $f : (\mathbb{F}_q^d, dx) \to \mathbb{C}$, then its inverse Fourier transform $f^\vee$ can be defined by

$$
f^\vee(m) = \int_{\mathbb{F}_q^d} \chi(m \cdot x)f(x) \ dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x)f(x)
$$

where $m \in (\mathbb{F}_q^d, dm)$. Using the orthogonality relation of $\chi$, one can easily show that $(\hat{g})^\vee(m) = g(m)$ for $g : (\mathbb{F}_q^d, dm) \to \mathbb{C}$. This provides us of the Fourier inversion theorem:

$$
g(m) = \int_{\mathbb{F}_q^d} \chi(m \cdot x)\hat{g}(x) \ dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x)\hat{g}(x).
$$

Let $V$ be an algebraic variety in the dual space $(\mathbb{F}_q^d, dx)$. The variety $V$ is equipped with the normalized surface measure $d\sigma$, which is defined by the relation

$$
\int f(x) \ d\sigma(x) = \frac{1}{|V|} \sum_{x \in V} f(x),
$$

where $f : (\mathbb{F}_q^d, dx) \to \mathbb{C}$. Observe that we can write $d\sigma(x) = \frac{1}{|V|} V(x) \ dx$. Here, and throughout this paper, we write $A(x)$ for the characteristic function on a set $A \subset \mathbb{F}_q^d$ and $|A|$ denotes the cardinality of the set $A$.

The restriction problem for the variety $V$ is to determine $1 \leq p, r \leq \infty$ such that the following restriction estimate holds:

$$
\|\hat{g}\|_{L^r(V, d\sigma)} \leq C\|g\|_{L^p(\mathbb{F}_q^d, dm)} \quad \text{for all functions } g : \mathbb{F}_q^d \to \mathbb{C},
$$

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where the constant $C > 0$ is independent of functions $g$ and the size of the underlying finite field $\mathbb{F}_q$. The notation $R(p \to r) \lesssim 1$ is used to indicate that the restriction inequality (2.2) holds. In this case, we say that the $L^p - L^{r'}$ restriction estimate holds. By duality, inequality (2.2) is same as the following extension estimate:

$$\| (g d\sigma)^\vee \|_{L^{r'}(\mathbb{F}_q^d, dm)} \leq C \| g \|_{L^{r'}(V, d\sigma)}.$$  

When this extension inequality holds, we say that the $L^{r'} - L^{p'}$ extension estimate holds and we write $R^*(r' \to p') \lesssim 1$ for it. Thus, $R(p \to r) \lesssim 1$ if and only if $R^*(r' \to p') \lesssim 1$.

**Remark 2.1.** $A \lesssim B$ for $A, B > 0$ means that there exists $C > 0$ independent of $q = |\mathbb{F}_q|$ such that $A \leq C B$. We also write $B \gtrsim A$ for $A \lesssim B$. In addition, $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$. We can define $R(p \to r)$ to be the best constant such that the restriction estimate (2.2) holds. $R(p \to r)$ may depend on $q$. The restriction problem is to determine $p, r$ such that $R(p \to r) \lesssim 1$.

When $V \subset \mathbb{F}_q^d$ is the sphere or the paraboloid, the necessary conditions for $R(p \to r) \lesssim 1$ are well known. In particular, necessary conditions for $R(p \to 2) \lesssim 1$ mainly depend on the biggest size of the affine subspaces lying in the variety $V$. For example, if $-1 \in \mathbb{F}_q$ is a square number and $V \subset \mathbb{F}_q^d$ is the sphere or the paraboloid, then one can construct an affine subspace $H \subset \mathbb{F}_q^d$ such that $|H| = q^{(d-1)/2}$ for $d \geq 3$ odd and $|H| = q^{d-2)/2}$ for $d \geq 2$ even (see [17] and [9]). Taking $g(x) = H(x)$ in (2.3), we can directly deduce that the necessary conditions for $R(p \to 2) \lesssim 1$ are given by

$$1 \leq p \leq \frac{2d + 2}{d + 3} \quad \text{for odd } d \geq 3$$  

and

$$1 \leq p \leq \frac{2d + 4}{d + 4} \quad \text{for even } d \geq 2.$$  

It was proved in [10] and [6] that the Stein-Tomas inequality holds for the sphere and the paraboloid, respectively. Therefore, if $d \geq 3$ is odd, then (2.4) is also the sufficient condition for $R(p \to 2) \lesssim 1$. However, when the dimension $d$ is even, it is not known that (2.5) is the sufficient condition for $R(p \to 2) \lesssim 1$ except for dimension two. For this reason, by the nesting property of norms, one may want to establish the following conjecture.

**Conjecture 2.2.** Let $V \subset \mathbb{F}_q^d$ be the sphere or the paraboloid. Assuming that $-1 \in \mathbb{F}_q$ is a square number and $d \geq 4$ is even, then

$$R\left(\frac{2d + 4}{d + 4} \to 2\right) \lesssim 1.$$  

2.2. $d$-coordinate lay functions and homogeneous functions of degree zero.

We introduce specific test functions on which the restriction operator for the sphere or the paraboloid acts. The following two definitions are closely related to a weak version of the restriction problem for the paraboloid.

**Definition 2.3.** A function $g : (\mathbb{F}_q^d, dm) \to \mathbb{C}$ is called a $d$-coordinate lay function if it satisfies that for each $(m', m_d) \in \mathbb{F}_q^{d-1} \times \mathbb{F}_q$, $g(m', m_d) = g(m', sm_d)$ for all $s \in \mathbb{F}_q \setminus \{0\}$. 

Theorem 2.6. Let \( d \)-coordinate lay functions \( g : (\mathbb{F}_q^d, dm) \to \mathbb{C} \).

The weak version of the restriction operator for the sphere shall be defined by taking homogeneous functions of degree zero as test functions. As usual, a function \( g : (\mathbb{F}_q^d, dm) \to \mathbb{C} \) is named a homogeneous function of degree zero if \( g(sm) = g(m) \) for \( m \in \mathbb{F}_q^d, s \in \mathbb{F}_q \setminus \{0\} \).

Definition 2.4. We write \( R_{d-lay}(p \to r) \lesssim 1 \) if the restriction estimate \( (2.2) \) holds for all \( d \)-coordinate lay functions \( g : (\mathbb{F}_q^d, dm) \to \mathbb{C} \).

Definition 2.5. We write \( R_{hom}(p \to r) \lesssim 1 \) if the restriction estimate \( (2.2) \) holds for all homogeneous functions of degree zero, \( g : (\mathbb{F}_q^d, dm) \to \mathbb{C} \).

2.3. Statement of main results. Our first result below is related to the parabolical restriction estimate for \( d-lay \) test functions.

Theorem 2.6. Let \( d \sigma \) be the normalized surface measure on the paraboloid \( P := \{ x \in \mathbb{F}_q^d : x_1^2 + \cdots + x_{d-1}^2 = x_d \} \). If \( d \geq 2 \) is even, then we have

\[
R_{d-lay} \left( \frac{2d+4}{d+4} \to 2 \right) \lesssim 1.
\]

When the test functions are homogeneous functions of degree zero, we obtain the strong result on the weak version of spherical restriction problems.

Theorem 2.7. Let \( d \sigma \) be the normalized surface measure on the sphere with nonzero radius \( S_j := \{ x \in \mathbb{F}_q^d : x_1^2 + \cdots + x_d^2 = j \neq 0 \} \). Then if \( d \geq 2 \) is even, we have

\[
R_{hom} \left( \frac{2d+4}{d+4} \to 2 \right) \lesssim 1.
\]

Conjecture 2.2 claims that if \( d \geq 4 \) is even, then \( (2d+4)/(d+4) \) is the optimal \( p \) value for the \( L^p - L^2 \) restriction estimate for spheres and paraboloids in finite fields. According to Theorem 2.6 and 2.7, it seems that the conjecture is true. In dimension two, this conjecture was actually proved by Mockenhaupt and Tao (16) for the parabola and Iosevich and Koh (6) for the circle. Indeed, they obtained the \( L^2 - L^4 \) extension estimate which exactly implies that \( L^{2/3} - L^2 \) restriction estimate holds. However, it is still open in higher even dimensions \( d \geq 4 \) and the currently best known result for the paraboloid is \( R(2d^2/(d^2 + 2d - 2) \to 2) \lesssim 1 \) due to A. Lewko and M. Lewko (11). In fact, they proved the extension estimate, \( R^*(2 \to 2d^2/(d^2 + 2d + 2)) \lesssim 1 \) for even \( d \geq 4 \). Notice that this result is much better than the Stein-Tomas inequality, that is \( R((2d+2)/(d+3) \to 2) \lesssim 1 \). For the sphere in even dimensions \( d \geq 4 \), the Stein-Tomas inequality was only obtained by Iosevich and Koh (11) and it has not been improved.

2.4. Outline of the remain parts of the paper. The remain parts of this paper are constructed for providing proofs of Theorems 2.6 and 2.7. In Section 3 we deduce the \( L^p - L^2 \) restriction estimate for homogeneous varieties in \( d+1 \) dimensional vector spaces over finite fields \( \mathbb{F}_q \). Since homogeneous varieties are a collection of lines, it sounds plausible to expect that the Fourier decay of them is not so good. However, it is not always true. Indeed, we observe that if \( (d+1) \) is odd, then the Fourier decay of homogeneous varieties in \( (d+1) \) dimensions is enough to derive a good \( L^p - L^2 \) restriction result from the Stein-Tomas argument. In Section 4 we complete the proofs of Theorems 2.6 and 2.7 by deducing the connection between a weak version of restriction estimates for spheres or paraboloids in \( d \) dimensions and the restriction estimates for homogeneous varieties in \( d+1 \) dimensions.
3. Restriction phenomenon for homogeneous varieties

Let $d \geq 2$ be an integer. In this section, we derive the $L^p - L^2$ estimate for homogeneous varieties lying in $(\mathbb{F}_q^{d+1}, d\sigma)$ where $d\sigma$ denotes the normalized counting measure on $\mathbb{F}_q^{d+1}$. Define a variety $C \subset (\mathbb{F}_q^{d+1}, d\sigma)$ as

$$C = \{(x, x_{d+1}) \in \mathbb{F}_q^d \times \mathbb{F}_q : x_1^2 + \cdots + x_{d-1}^2 = x_{d+1}\}.$$ 

For each $j \in \mathbb{F}_q^*$, define a variety $H_j \subset (\mathbb{F}_q^{d+1}, d\sigma)$ by

$$H_j = \{(x, x_{d+1}) \in \mathbb{F}_q^d \times \mathbb{F}_q : x_1^2 + \cdots + x_d^2 = jx_{d+1}^2\}.$$ 

Throughout this paper, we denote by $d\sigma_c$ and $d\sigma_j$ the normalized surface measures on $C$ and $H_j$, respectively. In addition, $(\mathbb{F}_q^{d+1}, d\overline{\sigma})$ denotes the dual space of $(\mathbb{F}_q^{d+1}, d\sigma)$ where $d\overline{\sigma}$ is the counting measure on $\mathbb{F}_q^{d+1}$. Recall that if $\overline{\sigma} \in (\mathbb{F}_q^{d+1}, d\overline{\sigma})$, then

$$(d\sigma_j)^\vee(\overline{\sigma}) = \int_{H_j} \chi(\overline{\sigma} \cdot \overline{x}) \, d\sigma_c(\overline{x}) = \frac{1}{|H_j|} \sum_{\overline{x} \in H_j} \chi(\overline{\sigma} \cdot \overline{x})$$

and

$$(d\sigma_j)^\vee(\overline{\sigma}) = \int_{H_j} \chi(\overline{\sigma} \cdot \overline{x}) \, d\sigma_c(\overline{x}) = \frac{1}{|H_j|} \sum_{\overline{x} \in H_j} \chi(\overline{\sigma} \cdot \overline{x}).$$

With the above notation, we have the following result.

**Lemma 3.1.** Let $d \geq 2$ be even. Then

$$|C| = |H_j| = q^d \quad \text{for } j \in \mathbb{F}_q^*.$$ 

Moreover, if $\overline{\sigma} \in \mathbb{F}_q^{d+1} \setminus \{(0, \ldots, 0)\}$, then

$$|(d\sigma_j)^\vee(\overline{\sigma})| \leq q^{-d/2}$$

and

$$|(d\sigma_j)^\vee(\overline{\sigma})| \leq q^{-d/2} \quad \text{for all } j \in \mathbb{F}_q^*.$$ 

**Proof.** Before we proceed with the proof, we recall preliminary knowledge for exponential sums. Let $\eta$ be a quadratic character of $\mathbb{F}_q$. For each $a \in \mathbb{F}_q$, the absolute value of the Gauss sum $G_a$ is given by

$$|G_a| := \left| \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(as) \right| = \left| \sum_{s \in \mathbb{F}_q^*} \eta(s)\chi(a/s) \right| = \begin{cases} q^{\frac{d}{2}} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0. \end{cases}$$

It is not hard to see that

$$\sum_{s \in \mathbb{F}_q^*} \chi(as^2) = G_1 \eta(a) \quad \text{for any } a \neq 0.$$ 

It follows from the orthogonality relations of $\chi$ and $\eta$ that

$$\sum_{s \in \mathbb{F}_q^*} \chi(as) = \begin{cases} 0 & \text{if } a \in \mathbb{F}_q^* \\ q & \text{if } a = 0, \end{cases}$$

and

$$\sum_{s \in \mathbb{F}_q^*} \eta(as) = \begin{cases} 0 & \text{if } a \in \mathbb{F}_q^* \\ q - 1 & \text{if } a = 0. \end{cases}$$
For (3.1), (3.2), and (3.3), see Chapter 5 in [15]. Completing the square and using a change of variables, (3.2) can be generalized by the formula:

\[
\delta_j \left( \sum_{x \in \mathbb{F}_q} \chi(a x^2 + b x) \right) = G_1 \eta(a) \chi(b^2 / (-4a)) \quad \text{for } a \in \mathbb{F}_q^*, b \in \mathbb{F}_q.
\]

Next, we can directly deduce by the previous argument that if \( j \) then

\[
(3.1) \quad \sum_{x \in \mathbb{F}_q} \chi(2s^2 + bs) = G_1 \eta(a) \chi((b^2 / (-4a)))
\]

From the definition of \((\delta_j)\) the \( \delta_j \) where

\[
\left| \sum_{x \in \mathbb{F}_q} \chi(a x^2 + b x) \right| = q^{d/2} \quad \text{for } a \in \mathbb{F}_q^*, b \in \mathbb{F}_q.
\]

Now we are ready to prove the lemma. First, we estimate \((d\sigma_c)^\vee\). For \( \overline{m} = (m_1, \cdots, m_{d+1}) \in \mathbb{F}_q^{d+1} \), it follows from the orthogonality relation of \( \chi \) that

\[
(d\sigma_c)^\vee(\overline{m}) = \frac{1}{\|C\|} \sum_{x \in \mathbb{F}_q} \chi(\overline{m} \cdot \overline{x})
\]

\[
= \frac{1}{q^{d/2}} \sum_{\overline{x} \in \mathbb{F}_q^{d+1}} \sum_{t \in \mathbb{F}_q} \chi(\overline{m} \cdot \overline{x}) \chi(t(2x_1^2 + \cdots + x_{d-1}^2 - x_dx_{d+1}))
\]

\[
= \frac{q^d}{\|C\|} \delta_0(\overline{m}) + \frac{1}{q^{d/2}} \sum_{\overline{x} \in \mathbb{F}_q^{d+1}} \chi(\overline{m} \cdot \overline{x}) \chi(t(2x_1^2 + \cdots + x_{d-1}^2 - x_dx_{d+1}))
\]

\[
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\]

where \( \delta_0(\overline{m}) = 1 \) for \( \overline{m} = (0, \ldots, 0) \), and 0 otherwise. Applying (3.2), we see that

\[
\left| \sum_{\overline{x} \in \mathbb{F}_q^{d+1}} \chi(\overline{m} \cdot \overline{x}) \right| \leq q^{d/2} \quad \text{for } \overline{m} \neq (0, \ldots, 0).
\]

Then we obtain that for \( \overline{m} \in \mathbb{F}_q^{d+1} \),

\[
(3.5) \quad (d\sigma_c)^\vee(\overline{m}) = \frac{q^d}{\|C\|} \delta_0(\overline{m}) + \frac{G_1^{d-1}}{q^{d/2}} \sum_{t \neq 0} \eta(t) \chi((\|m'\|^2 - 4m.dm_{d+1}) / (-4t)).
\]

From the definition of \((d\sigma_c)^\vee\) and the orthogonality relation of \( \eta \), we see that

\[
1 = (d\sigma_c)^\vee(0, \ldots, 0) = \frac{q^d}{\|C\|}.
\]

Thus, it completes the proof of \(|C| = q^d\) and it follows immediately from (3.5) and (3.1) that \(|(d\sigma_c)^\vee(\overline{m})| \leq q^{-d/2} \) for \( \overline{m} \neq (0, \ldots, 0) \).

Next, we can directly deduce by the previous argument that if \( j \in \mathbb{F}_q^* \) and \( \overline{m} \in \mathbb{F}_q^{d+1} \), then

\[
(d\sigma_j)^\vee(\overline{m}) = \frac{q^d}{|H_j|} \delta_0(\overline{m}) + \frac{G_1^{d+1}}{q^d} \sum_{t \in \mathbb{F}_q^*} \eta(-j) \chi((m_{d+1}^2 + j\|m\|^2) / (4jt)),
\]

where \( \|m\|^2 = m_1^2 + \cdots + m_d^2 \). This implies that \(|H_j| = q^d\) for \( j \in \mathbb{F}_q^* \) and \(|(d\sigma_j)^\vee(\overline{m})| \leq q^{-d/2} \) for \( \overline{m} \neq (0, \ldots, 0) \). We leave the detail to readers.
Remark 3.2. If $d$ is odd, then the Fourier decays become much worse than those in conclusions of Lemma 3.1. To see this, notice that if $d$ is odd, then $\eta^{d-1} = 1$. Thus, the term $\eta$ disappears in the formula (3.5). Consequently, if $m = (m', m_d, m_{d+1}) \neq (0, \ldots, 0)$ and $\|m'\|^2 - 4m_dm_{d+1} = 0$, then $|(d\sigma_c)^\vee(m)| \sim q^{(-d+1)/2}$.

Applying the well known Stein-Tomas argument in finite fields, Lemma 3.1 enables us to deduce the $L^p - L^2$ restriction theorem for the homogeneous varieties $C$ and $H_j$ for $j \in \mathbb{F}_q^*$.

**Lemma 3.3.** Let $d \geq 2$ be an even integer. Then we have

\[(3.6) \quad \|\hat{G}\|_{L^2(C, d\sigma_c)} \lesssim \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \quad \text{for all functions} \quad G : \mathbb{F}_q^{d+1} \to \mathbb{C}.\]

We also have that if $j \in \mathbb{F}_q^*$, then

\[(3.7) \quad \|\hat{G}\|_{L^2(H_j, d\sigma_j)} \lesssim \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \quad \text{for all functions} \quad G : \mathbb{F}_q^{d+1} \to \mathbb{C}.\]

**Proof.** Since the proof of (3.6) is exactly the same as that of (3.7), we shall only introduce the proof of (3.6). By duality and Hölder’s inequality, we see

\[
\|\hat{G}\|_{L^2(C, d\sigma_c)}^2 = \sum_{m \in \mathbb{F}_q^{d+1}} |G(m) (G * (d\sigma_c)^\vee)(\overline{m})|^2 
\leq \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \|G * (d\sigma_c)^\vee\|_{L^{-\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})}^2.
\]

It is enough to prove that for every function $G$ on $(\mathbb{F}_q^{d+1}, d\overline{m})$,

\[
\|G * (d\sigma_c)^\vee\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \lesssim \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})}.
\]

Define $K(\overline{m}) = (d\sigma_c)^\vee(\overline{m}) - \delta_0(\overline{m})$. Since $(d\sigma)^\vee(0, \ldots, 0) = 1$, we see that $K(\overline{m}) = 0$ for $m = (0, \ldots, 0)$, and $K(\overline{m}) = (d\sigma_c)^\vee(\overline{m})$ for $\overline{m} \in \mathbb{F}_q^{d+1} \setminus \{(0, \ldots, 0)\}$. Since $G * (d\sigma_c)^\vee = G * \delta_0 + G * K$, it will be enough to prove the following two inequalities:

\[(3.8) \quad \|G * \delta_0\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \lesssim \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})}
\]

and

\[(3.9) \quad \|G * K\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \lesssim \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})}.
\]

Since $G * \delta_0(\overline{m}) = G(\overline{m})$ for $\overline{m} \in (\mathbb{F}_q^{d+1}, d\overline{m})$, (3.8) follows by observing that

\[
\|G * \delta_0\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} = \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})} \leq \|G\|_{L^{\frac{2d+4}{d+4}}(\mathbb{F}_q^{d+1}, d\overline{m})},
\]

where the last line follows from the facts that $d\overline{m}$ is the counting measure and $(2d+4)/(d+4) < (2d+4)/d$. In order to prove (3.9), we assume for a moment that

\[(3.10) \quad \|G * K\|_{L^2(\mathbb{F}_q^{d+1}, d\overline{m})} \lesssim q \|G\|_{L^2(\mathbb{F}_q^{d+1}, d\overline{m})}
\]

and

\[(3.11) \quad \|G * K\|_{L^\infty(\mathbb{F}_q^{d+1}, d\overline{m})} \lesssim q^{-\frac{d}{2}} \|G\|_{L^1(\mathbb{F}_q^{d+1}, d\overline{m})}.
\]
Then (3.9) follows immediately by interpolating (3.10) and (3.11). Thus, our final task is to show that both (3.10) and (3.11) hold. As a direct consequence from the Plancherel theorem, (3.10) can be proved. Indeed, we have

\[ \|G \ast K\|_{L^2(\mathbb{R}^{d+1},d\sigma)} = \|\hat{G}\hat{K}\|_{L^2(\mathbb{R}^{d+1},d\sigma)} \leq \|\hat{K}\|_{L^1(\mathbb{R}^{d+1},d\sigma)}\|\hat{G}\|_{L^1(\mathbb{R}^{d+1},d\sigma)} \leq q\|G\|_{L^2(\mathbb{R}^{d+1},d\sigma)}, \]

where the last line is obtained by observing that for each \( \mathcal{F} \in (\mathbb{R}^{d+1},d\mathcal{F}) \)

\[ |\hat{K}(\mathcal{F})| = |\sigma_d(\mathcal{F}) - \delta(\mathcal{F})| = |q^{d+1}|C(\mathcal{F}) - 1| < q. \]

Now, we prove (3.11). It follows from Young’s inequality that

\[ \|G \ast K\|_{L^\infty(\mathbb{R}^{d+1},d\sigma)} \leq \|K\|_{L^\infty(\mathbb{R}^{d+1},d\sigma)}\|G\|_{L^1(\mathbb{R}^{d+1},d\sigma)}. \]

From the definition of \( K \) and the Fourier decay estimate in Lemma 3.1, we conclude that (3.11) holds. Thus, the proof is complete.

\[ \Box \]

4. Proofs of Theorem 2.6 and 2.7

As a key ingredient of proving our main results, we use the relation between the restriction theorem for \( C \) and \( H_j \) in \( \mathbb{R}^{d+1} \) and the weak restriction theorem for paraboloids and spheres in \( \mathbb{F}_q \). Theorem 2.6 shall be deduced from (3.9) in Lemma 3.3. Similarly, we shall prove Theorem 2.7 by applying (3.7) in Lemma 3.3.

4.1. Proof of Theorem 2.6

We must prove that if \( d \geq 2 \) is even, then

\[ \|\hat{g}\|_{L^2(P,d\sigma)} \lesssim \|g\|_{L^2(\mathbb{F}_q^{(d+4)/(d+4)},d\mathcal{F},d\mu)} \quad \text{for all } d\text{-coordinate lay functions} \quad g: \mathbb{F}_q^d \to \mathbb{C}. \]

Given a \( d\)-coordinate lay function \( g: (\mathbb{F}_q^d,d\mathcal{F}) \to \mathbb{C} \), we define \( G_g: (\mathbb{F}_q^{d+1},d\sigma) \to \mathbb{C} \) by the relation

\[ G_g(x',x_d,s) = \begin{cases} \hat{g}(x',x_d)s & \text{if } s \neq 0 \\ 0 & \text{if } s = 0, \end{cases} \]

where \((x',x_d,s) \in (\mathbb{F}_q^{d+1},d\sigma)\) with \( x' \in \mathbb{F}_q^{d-1}, x_d, s \in \mathbb{F}_q \). We need the explicit form of \( G_g \).

**Proposition 4.1.** For \((m,l) \in \mathbb{F}_q^d \times \mathbb{F}_q \),

\[ G_g(m,l) = \frac{g(m)}{q} \sum_{s \in \mathbb{F}_q^d} \chi(ls). \]

**Proof.** By the Fourier inversion theorem (2.4) for \( d+1 \) dimensions, and the definition of \( \hat{G}_g \) in (4.1), we see that if \((m',m_d,l) \in \mathbb{F}_q^{d-1} \times \mathbb{F}_q \times \mathbb{F}_q = \mathbb{F}_q^{d+1} \), then

\[ G_g(m',m_d,l) = \frac{1}{q^{d+1}} \sum_{x' \in \mathbb{F}_q^{d-1}, x_d \in \mathbb{F}_q} \chi(m' \cdot x' + m_d x_d + ls) \hat{G}_g(x',x_d,s) = \frac{1}{q^{d+1}} \sum_{s \neq 0} \sum_{(x',x_d) \in \mathbb{F}_q^d} \chi(m' \cdot x' + m_d x_d + ls) \hat{g}(x',x_d,s). \]
By a change of variables, $x_d \to x_d/s$, and the Fourier inversion formula \(2.1\),
\[
G_g(m', m_d, l) = \frac{1}{q^{d+1}} \sum_{s \neq 0} \chi(ls) \sum_{x \in \mathbb{F}_q^d} \chi(x \cdot (m', m_d/s)) \hat{g}(x)
\]
\[
= \frac{1}{q^d} \sum_{s \neq 0} \chi(ls) g(m', m_d/s).
\]

Since $g$ is a $d$-coordinate lay function, $g(m', m_d/s) = g(m)$ for all $s \in \mathbb{F}_q^*$. Hence, the proof of Proposition 4.1 is complete. \(\square\)

We continue to prove Theorem 2.6. It is enough to show that\[
\|\hat{g}\|^2_{L^2(P, d\sigma)} \lesssim \|g\|^2_{L^{(2d+4)/(d+4)}(\mathbb{F}_q^d, d\sigma)}.
\]
Since $|C| = q^d = q|P|$, it follows that\[
\|\hat{g}\|^2_{L^2(P, d\sigma)} = \frac{1}{|P|} \sum_{x \in P} |\hat{g}(x)|^2 \sim \frac{1}{|C|} \sum_{s \in \mathbb{F}_q^*} \sum_{x \in P} |\hat{g}(x)|^2
\]
\[
= \frac{1}{|C|} \sum_{s \in \mathbb{F}_q^*} \sum_{(x', x_d) \in \mathbb{F}_q^d} |\hat{g}(x', x_d s)|^2
\]
\[
= \|\hat{G}_g\|^2_{L^2(C, d\sigma)},
\]
where the last line follows from (4.1). By (3.9) in Lemma 3.3 to prove Theorem 2.6, it therefore suffices to show that\[
\|G_g\|^2_{L^{(2d+4)/(d+4)}(\mathbb{F}_q^d, d\sigma), \mathbb{F}_q^d, d\sigma)} \lesssim \|g\|^2_{L^{(2d+4)/(d+4)}(\mathbb{F}_q^d, d\sigma)}.
\]
Letting $\alpha = (2d + 4)/(d + 4) > 1$, it will be enough to prove that\[
\|G_g\|^\alpha_{L^\alpha(\mathbb{F}_q^d, d\sigma)} \lesssim \|g\|^\alpha_{L^\alpha(\mathbb{F}_q^d, d\sigma)}.
\]
From the explicit form of $G_g$ in Proposition 4.1, it follows that\[
\|G_g\|^\alpha_{L^\alpha(\mathbb{F}_q^d, d\sigma)} = \sum_{(m, l) \in \mathbb{F}_q^d \times \mathbb{F}_q^d} |G_g(m, l)|^\alpha
\]
\[
= \sum_{l \in \mathbb{F}_q^d} \sum_{m \in \mathbb{F}_q^d} |g(m)|^\alpha q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(ls) \left| q^{-\alpha} |g(m)|^\alpha \sum_{l \neq 0} m \in \mathbb{F}_q^d |g(m)|^\alpha
\]
\[
\leq \sum_{m \in \mathbb{F}_q^d} |g(m)|^\alpha + q^{-\alpha} (q-1) \sum_{m \in \mathbb{F}_q^d} |g(m)|^\alpha \leq 2 \|g\|_{L^\alpha(\mathbb{F}_q^d, d\sigma)}^\alpha.
\]

Thus, the proof of Theorem 2.6 is complete.
4.2. Proof of Theorem 2.7. We aim to prove that for every \( j \in \mathbb{F}_q^* \),

\[
\|\hat{g}\|_{L^2(S_j, d\sigma)} \lesssim \|g\|_{L^{(2d+4)/(d+4)}(\mathbb{F}_q^d, dm)}
\]

for all homogeneous functions of degree zero \( g : \mathbb{F}_q^d \to \mathbb{C} \). Let \( g : (\mathbb{F}_q^d, dm) \to \mathbb{C} \) be a homogeneous function of degree zero. By the definition of the homogeneous function of degree zero, we see that for every \( t \in \mathbb{F}_q \) and \( x \in (\mathbb{F}_q^d, dx) \),

\[
\hat{g}(x) = \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m/t).
\]

From this observation and a change of variables, \( m \to tm \), it follows that

\[
\|\hat{g}\|_{L^2(S_j, d\sigma)}^2 = \frac{1}{|S_j|(q - 1)} \sum_{t \in \mathbb{F}_q^*, x \in S_j} \left| \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m/t) \right|^2
\]

\[
= \frac{1}{|S_j|(q - 1)} \sum_{t \in \mathbb{F}_q^*, x \in S_j} \left| \sum_{m \in \mathbb{F}_q^d} \chi(-tm \cdot x) g(m) \right|^2
\]

\[
= \frac{1}{|S_j|(q - 1)} \sum_{t \in \mathbb{F}_q^*, x \in \mathbb{F}_q^d} \left| \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot tx) g(m) \right|^2.
\]

Applying a change of variables, \( x \to x/t \), we have

\[
\|\hat{g}\|_{L^2(S_j, d\sigma)}^2 = \frac{1}{|S_j|(q - 1)} \sum_{t \in \mathbb{F}_q^*, x \in \mathbb{F}_q^d} \left| \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m) \right|^2
\]

\[
\leq \frac{1}{|S_j|(q - 1)} \sum_{(x,t) \in H_j} \left| \sum_{m \in \mathbb{F}_q^d} \chi(-m \cdot x) g(m) \right|^2.
\]

Now, consider a function \( G_g : (\mathbb{F}_q^{d+1}, dm) \to \mathbb{C} \) defined by

\[
G_g(m, m_{d+1}) = \begin{cases} 
0 & \text{if } m_{d+1} \neq 0, \\
g(m) & \text{if } m_{d+1} = 0.
\end{cases}
\]

Then the last expression above can be written by

\[
\frac{1}{|S_j|(q - 1)} \sum_{(x,t) \in H_j} \left| \sum_{(m, m_{d+1}) \in \mathbb{F}_q^d \times \mathbb{F}_q} \chi(-m \cdot x) \chi(-m_{d+1} \cdot t) G_g(m, m_{d+1}) \right|^2.
\]

Since \( |S_j|(q - 1) \sim q^d = |H_j| \), we see that

\[
\|\hat{g}\|_{L^2(S_j, d\sigma)}^2 \lesssim \frac{1}{|H_j|} \sum_{(x,t) \in H_j} |\hat{G}_g(x,t)|^2 = \|\hat{G}_g\|_{L^2(H_j, d\sigma_j)}^2.
\]

Applying (3.7) in Lemma 3.3, we conclude from the definition of \( G_g \) that

\[
\|\hat{g}\|_{L^2(S_j, d\sigma)}^2 \lesssim \|G_g\|_{L^{(2d+4)/(d+4)}(\mathbb{F}_q^{d+1}, dm)}^2 = \|g\|_{L^{(2d+4)/(d+4)}(\mathbb{F}_q^d, dm)}^2.
\]
which completes the proof.

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