Charm as a domain wall fermion in quenched lattice QCD

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We report a study describing the charm quark by a domain-wall fermion (DWF) in lattice quantum chromodynamics (QCD). Our study uses a quenched gauge ensemble with the DBW2 rectangle-improved gauge action at a lattice cutoff of $a^{-1} \sim 3$ GeV. We calculate masses of heavy-light (charmed) and heavy-heavy (charmonium) mesons with spin-parity $J^P = 0^\mp$ and $1^\mp$, leptonic decay constants of the charmed pseudoscalar mesons ($D$ and $D_s$), and the $D^0\bar{D}^0$ mixing parameter. The charm quark mass is found to be $m_{c^{\overline{MS}}}(m_c) = 1.24(1)(18)$ GeV. The mass splittings in charmed-meson parity partners $\Delta q,J=0$ and $\Delta q,J=1$ are degenerate within statistical errors, in accord with experiment, and they satisfy a relation $\Delta q=ud,J > \Delta q=s,J$, also consistent with experiment. A C-odd axial vector charmonium state, $h_c$, lies 22(11) MeV above the $\chi_{c1}$ meson, or $m_{hc} = 3533(11)_{\text{stat}}$ MeV using the experimental $\chi_{c1}$ mass. However, in this regard, we emphasize significant discrepancies in the calculation of hyperfine splittings on the lattice. The leptonic decay constants of $D$ and $D_s$ mesons are found to be $f_D = 232(7)_{\text{stat}}.(+6)^{-0}_{-0}\text{chiral}(11)_{\text{syst}}$. MeV and $f_{Ds}/f_D = 1.05(2)_{\text{stat}}.(+2)^{-0}_{-2}\text{chiral}(2)_{\text{syst}}$, where the first error is statistical, the second a systematic due to chiral extrapolation and the third error combination of other known systematics. The $D^0\bar{D}^0$ mixing bag parameter, which enters the $\Delta C = 2$ transition amplitude, is found to be $B_D(2 \text{ GeV}) = 0.845(24)_{\text{stat}}.(+24)^{-6}_{-6}\text{chiral}(105)_{\text{syst}}$.

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I. INTRODUCTION

Properties of hadrons containing one or more charm quarks have been intensively investigated in recent experiments [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. While the main purpose of these experiments is to acquire information useful or necessary to CKM phenomenology, it is also exciting that several new hadron states have been discovered and confirmed in these experiments. For the latest reviews on these new states, see, for example, Refs. [11, 12, 13]. Lattice QCD should be able to describe various features of these new states. However, in lattice calculations of heavy quark systems, the discretization error associated with masses as large as the lattice cutoff makes the interpretation of numerical results ambiguous. One possible way to avoid this systematic error is to rely on an effective theory such as HQET [14, 15, 16] or NRQCD [17, 18]. With such an approach, however, one must discard either the ability to study quarkonia or to take the continuum limit. An alternative is to rely on a relativistic method with brute numerical force. Thanks to rapid growth of computational resources, discretization errors in this approach for a relatively light heavy quark like charm are beginning to be brought under control. Several studies have already been made in this direction [19, 20, 21, 22, 23].

In this work, we explore the feasibility of applying the domain-wall fermion (DWF) [24, 25, 26] as a heavy quark within the quenched approximation. Domain-wall fermions have been successfully applied to calculations of the light hadron spectrum and weak matrix elements [27, 28, 29, 30, 31]. The primary advantage of working with this fermion action is its ability to retain continuum-like chiral symmetry even at a finite lattice spacing, by adding a fifth dimension to the lattice. The size of the symmetry breaking is represented by the residual mass, $am_{\text{res}}$, which is adjusted by changing the fifth-dimensional extent of the lattice and is of the order of $10^{-3}$ in lattice units in state-of-the-art calculations. It is known that the presence of exact chiral symmetry guarantees the absence of $O(a)$ discretization error, and in the domain-wall case, the violation of the order of $am_{\text{res}}$ implies that the leading error is significantly suppressed. Thus it seems to be a natural extension to apply the DWF formalism to heavier, $c$ and $b$, quarks. To make the next-to-leading-order discretization

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error ($\sim O((am_q)^2)$) as small as possible, our calculations are carried out on relatively fine lattices with $a^{-1} \sim 3$ GeV. Although this study is done in the quenched approximation, the experience should be useful in the future studies with dynamical fermions.

One way to test a given heavy quark formalism is to see how consistently the formalism describes both heavy-heavy and heavy-light systems with a single value of the mass parameter. While a single mass parameter does not have to describe both systems completely consistently in the quenched approximation and at a finite lattice spacing, making a quantitative test at some point will offer a reference for future work. Although we do not take a continuum limit here, such a study should become easier in the near future as available computational resources grow.

Once the charm quark mass is determined, it is interesting to see how well the lattice calculation reproduces the mass spectrum of recently discovered hadrons: the excited $D$ meson states with spin-parity $J^P = 0^+$ and $1^+$. The spectrum of these mesons had been studied before their discoveries by two groups based on chiral quark models [32, 33]. These turned out to describe some qualitative features well. After the experimental discoveries, these groups refined their analyses quantitatively [34, 35]. The studies suggest that chiral symmetry in the light quark sector plays an important role in the spectrum of these mesons. Since we apply the domain-wall formalism to both heavy and light quarks, we expect to get better descriptions of these mesons than earlier works with Wilson-type light quarks.

We would also like to make a lattice study of flavor physics, especially weak matrix elements of $D$ mesons. As an example, we present the $D$ meson leptonic decay constants and the bag parameter, $B_D$, relevant to $D^0-\bar{D}^0$ mixing. Experiments are currently searching for evidence of mixing in the neutral $D$-meson system [2]. Since the expected amplitude is very small in the Standard Model, once it has been discovered, it would provide an important probe of physics beyond the Standard Model (See, e.g., Ref. [36].) Unlike the neutral $B$ meson system, in $D^0-\bar{D}^0$ mixing it is not clear whether the short-distance contribution mediated by a local four-quark operator dominates over long-distance contributions. Even so, it is still sensible to have an idea about the size of the short-distance contribution. Knowing the size of $B_D$ will be also useful when evaluating $B_B$ by extrapolation in quark mass. In this paper we present the first DWF charm lattice study of this quantity. Note there were Wilson fermions studies earlier such as Refs. [37, 38].

The first exploratory study of massive DWF was done in Refs. [39, 40], which looked into...
the low-lying eigenmodes of the five-dimensional Hermitian domain-wall Dirac operator in
detail and their dependence on the bare quark mass, $a m_q$. All eigenmodes are classified into
one of two kinds of states. The first are the physical, or “decaying”, states which are bound
to the four-dimensional domain walls located at either end of the fifth dimension. Their
wavefunctions fall exponentially away from the wall. When $a m_q \ll 1$, these states dominate
the low-lying eigenmodes of the whole system and describe four-dimensional physics. The
other class contains unphysical, or “propagating”, states, which have non-zero momentum in
the fifth dimension. Their eigenvalues are large, $\sim O(1/a)$. From the viewpoint of the four-
dimensional effective theory, these states are unphysical, a source of non-locality that can
invalidate the effective theory. The gap between these two types of states is controlled by the
domain wall height “$M_5$” parameter. In the study of Refs. [39, 40], it turned out that as $a m_q$
increases the absolute values of the eigenvalues of the decaying states rapidly increase, and
the binding to the domain walls becomes less tight or even unbound. On the other hand, the
eigenvalues of the propagating states increase only slowly. Thus, if $a m_q$ further increases, at
some point the lowest eigenmode in the system will be one of the propagating states. Then
one might worry that something wrong happens to the four-dimensional effective theory.

In the past quenched studies [39, 40], this mass threshold can be as low as 0.2 (in lattice
units) for both $\beta = 6.0$ Wilson gauge action and $\beta = 0.87$ DBW2 gauge action, and 0.4 for
$\beta = 1.04$ DBW2 gauge action with $M_5 = 1.8$. In the present work, as will be described, a
higher lattice coupling is used for the DBW2 gauge action. This results in a threshold of
$a m_q \sim 0.5$ or higher in lattice units, while the bare charm quark mass is below 0.4.

The rest of the paper is organized as follows. In Sec. II we summarize our numerical
methods and their parameters. Then meson mass spectra and charm mass analyses are
presented in Sec. III. Section IV presents results for decay constants. We will also present
the first DWF calculation of the mixing parameter, $B_D$, in Section V. The $SU(3)$-breaking
ratio is discussed in Section VI. Extrapolations in $1/m_{\text{heavy}}$ to the static limit are reported
in Section VII. Some systematic errors are discussed in Sec. VIII. Section IX concludes
with a summary and future outlook for this program. Preliminary results on spectrum and
decay constants in this work are reported in Refs. [41, 42, 43].
II. NUMERICAL METHOD AND PARAMETERS

The numerical lattice QCD calculations reported in this paper were performed in the quenched approximation using the QCDSP computers at RIKEN-BNL Research Center and Columbia University. The gauge configurations were generated in a previous RBC work determining the neutral kaon mixing bag parameter, $B_K$ \cite{44}, with the DBW2 rectangle-improved gauge action \cite{45,46} with gauge coupling $\beta = 1.22$. With the 106 gauge configuration in total used in that study, the lattice cutoff measured from the $\rho$ meson mass is $a_{\rho}^{-1} = 2.914(54)$ GeV. This implies a physical spatial volume for the $24^3 \times 48$ lattices of about $(1.6 \text{ fm})^3$. We will use this cutoff estimate as a standard in this report. If we use the static quark potential instead, we obtain $a_{\rho}^{-1} = 3.07$ GeV \cite{44}. Notice that the difference between these two determination is about 5%, which is smaller than found on coarser lattices.

Of the 106 gauge configurations reported in the earlier study, in the present work we use only 103, due to the shutdown of the QCDSP computers in early 2006. This results in slight differences in the estimations of observables. We confirmed they are not significant.

We use domain-wall fermions (DWF) \cite{24,25,26} to describe both heavy (charm) and light (up, down, strange) quarks. The number of sites in the fifth dimension, $L_s = 10$ and the domain-wall height, $M_5 = 1.65$, are taken to be the same as in the previous work \cite{44}. The simulation parameters and some numerical results obtained in Ref. \cite{44} which are relevant to this work are summarized in Table I. As seen from the Table, our choice of parameters results in a residual mass of $O(10^{-4})$ in lattice units, about 0.3 MeV. To study the quark mass dependence in a comprehensive way, we use five values each for heavy and light quark masses as listed in Table I. They respectively cover approximate ranges of $[\frac{1}{4}m_{\text{charm}}, \frac{5}{4}m_{\text{charm}}]$ and $[\frac{1}{4}m_{\text{strange}}, \frac{5}{4}m_{\text{strange}}]$. Note that our choice of the heavy mass appears reasonable: Figure I shows the eigenfunctions $|\Psi_2^s|_5$ in the fifth dimension for the lowest three eigenvalues with various masses: 0.03, 0.4, 0.5, 0.7. The largest mass in the present work for the heavy quark is 0.5, and as shown in the plot, its lowest three eigenstates drop exponentially away from the walls and so do not appear to be unphysical.

Table II lists the meson operators, $\bar{\psi} \Gamma \psi$, used in this work. We used source/sink combinations of wall-wall (denoted as $C_{\text{ww}}^{\Gamma_1 \Gamma_2}$) and wall-point (denoted as $C_{\text{wp}}^{\Gamma_1 \Gamma_2}$) types for two- and three-point functions. The source position is set at $t_{\text{src}} = 7$. The gauge fields are fixed to Coulomb gauge. The quark propagators are obtained by solving under periodic and
anti-periodic boundary conditions in time and averaging over them. This procedure gives us a twice larger period in time. In the calculations of two-point functions, we double the statistics by using additional set of correlators with \( t_{\text{src}} = 41 \).

As was mentioned in the introduction, we calculate the masses of heavy-light (\( D \) and \( D_s \)) and heavy-heavy (charmonium) mesons with spin-parity \( J^P = 0^\mp \) and \( 1^\mp \), decay constants of the pseudo-scalar mesons, and the \( D^0-\bar{D}^0 \) mixing parameter. We use the standard single-elimination jackknife method for statistical error analysis. Further details of numerical methods used in calculating respective quantities are summarized at the beginning of the sections reporting the results.

III. SPECTROSCOPY

The meson masses are extracted by fitting the wall-point two-point correlators to the hyperbolic form

\[
A \cosh(m(t - L_t)),
\]

after shifting the wall source time slice of the correlators to \( t = 0 \).

Using the 103 configurations, we first repeated the measurement of light meson spectroscopy as in Ref. [44], and confirmed that the differences from that reported in Ref. [44] using full 106 configurations are negligible. For example and for later use, we present the bare strange quark mass in lattice units here. We choose to use \( m_K \) and \( m_\rho \) as inputs to determine the strange quark mass. Following the same fit ranges and the same functional forms in chiral extrapolation as in Ref. [44], we obtain \( am_{\text{strange}} = 0.0298(13) \), which is consistent with \( am_{\text{strange}} = 0.0295(14) \) quoted in Ref. [44].

A. Charmed Mesons

The effective mass plots for the heavy-light mesons with the four different spin-parity states we are discussing are shown in Figure [2] for \( am_{\text{heavy}} = 0.4 \) and \( am_{\text{light}} = 0.032 \), which are the closest combination to the bare charm (obtained below) and strange (discussed above) mass, respectively. As seen from the figure, the plots for all four mesons show reasonable plateaux with this mass combination. Similar quality of plateaux are obtained for the other combinations of \( am_{\text{heavy}} = 0.2, 0.3, 0.4 \) and \( 0.5 \) and \( am_{\text{light}} = 0.024, 0.032 \) and 0.040 as well.
In addition, for the pseudoscalar \((0^-)\) and vector \((1^-)\) channels, we could also extract the masses for the mass combinations of \(am_{\text{heavy}} = 0.1\) and \(am_{\text{light}} = 0.008\) and \(0.016\). We take the fit ranges as \(15 \leq t \leq 23\) for \(0^-\) and \(1^-\) states, and \(10 \leq t \leq 16\) for \(0^+\) and \(1^+\). The numerical results for meson masses are summarized in Tables \(\text{III} \), \(\text{IV} \), \(\text{V} \) and \(\text{VI} \).

We use the pseudoscalar \(D_s\) meson mass \((1968.3(5)\) MeV), the bare strange quark mass estimate of \(am_{\text{strange}} = 0.0298(13)\) \([44]\), and \(\rho\)-meson mass \(m_{\rho} = 770\) MeV as inputs to determine the bare charm quark mass. The determination using quarkonium system will be discussed later. For this purpose, we first interpolate the heavy-light pseudoscalar masses in light quark mass to \(am_{\text{strange}}\), and then interpolate between the resulting data for \(am_{\text{heavy}} = 0.3\) and \(0.4\). Through this procedure we obtain an estimate of \(am_{\text{charm}}^{D_s/m_{\rho}} = 0.3583(22)\), where the superscript denotes the inputs used. Using \(am_{\text{charm}}^{D_s/m_{\rho}}\), we estimate the masses of the four different spin-parity states, resulting in Figure 3, which shows the comparison of our estimates (circles) with the experimental values (vertical lines), and in Table \(\text{VIII} \). Not surprisingly, the calculations are in reasonable agreement with the experiments, to within a few %.

A more stringent test is provided by the parity splittings, \(\Delta_{q,J} = m_{Dq(J^+)} - m_{Dq(J^-)}\), rather than the masses themselves. The numerical results in physical unit via \(a_{\rho}^{-1}\) are shown in Table \(\text{VII} \) together with their experimental values. First we extract some features from the experiments:

1. \(\Delta_{q,0} \approx \Delta_{q,1}\) for both \(q = ud\) and \(s\).
2. \(\Delta_{ud,J} > \Delta_{s,J}\) for both \(J = 0\) and \(1\).
3. The difference, \(\Delta_{ud,J} - \Delta_{s,J}\), is close to \(m_{Ds(J^-)} - m_{Dud(J^-)}\). This means that the positive parity state masses depend on the mass of the light spectator only weakly while the negative parity states change by \(m_{Ds(J^-)} - m_{Dud(J^-)} \approx m_s - m_{ud} \approx m_s\) between \(D_s(J^-)\) and \(D_{ud}(J^-)\).

There have been attempts to understand some of these features using model analyses. As we noted in the Introduction, Bardeen et al. \([34]\) described these charmed heavy-light mesons using a chiral quark model in a way that respected heavy quark symmetry, and predicted that \(\Delta_{q,0} \approx \Delta_{q,1}\) with an assumption that \(\Lambda_{\text{QCD}}/m_{\text{charm}}\) corrections are small. Nowak et al. \([35]\) made similar predictions with a slightly different model. Becirevic et al. \([47]\) noted
difficulties in understanding the experimental observation of $\Delta_{ud,J} > \Delta_{s,J}$ in terms of a version of chiral perturbation theory extended to include the four spin-parity states of the heavy-light mesons.

Importantly, many previous lattice calculations have systematically overestimated the parity splitting by about 50 to 200 MeV for $\Delta_{s,0}$. These works are summarized in Figure 4, which is the same plot given in Ref. [53] but the results from Dougall et al. [52] is modified by applying $r_0 = 0.50$ fm instead of their 0.55 fm.

Now let us turn to our results. Both $\Delta_{s,0}$ and $\Delta_{s,1}$ are overestimated compared with their experimental values by 1 – 1.5 standard deviations, which corresponds to a difference of 35 – 60 MeV. Although our results for $\Delta_{ud,J}$ are not as precise as $\Delta_{s,J}$ since the extrapolations of the positive parity states to the chiral limit of the light quark suffer from large statistical uncertainty (as described below), they show a consistency with the experimental values. The degeneracy between $\Delta_{q,0}$ and $\Delta_{q,1}$ are well reproduced for both $q = ud$ and $s$ within the statistical uncertainties. A comparison with the previous lattice calculations is made in Figure 4. Looking at the data around $m_c/m_Q = 1$, the present result obtained at our relatively fine lattice cutoff turns out to be consistent with that of Dougall et al. [52], which is obtained in the continuum limit.

The heavy quark mass dependence of $\Delta_{s,1}$ and $\Delta_{s,0}$ are shown in Figure 5, together with the available experimental data. The light quark mass is fixed to the strange quark mass. It is seen that $\Delta_{s,0}$ and $\Delta_{s,1}$ are statistically indistinguishable for $am_{\text{heavy}} > 0.3$, but that $\Delta_{s,0} > \Delta_{s,1}$ becomes clearer for smaller $am_{\text{heavy}}$ as $\Delta_{s,0}$ increases while $\Delta_{s,1}$ stays more or less constant. The latter behavior may be supported by the experimental values of the $K_1(1270)$ and $K^*(892)$ masses.

The light quark mass dependence of the splittings is shown in Figure 6, where $am_{\text{heavy}}$ is fixed to $am_{\text{charm}}$. Both $\Delta_{J=0}$ and $\Delta_{J=1}$ moderately increase toward lighter quark mass, which is consistent with the experimental observation.

The splitting between the pseudoscalar $D_s$ and $D$ mesons, $m_{D_s} - m_D$, is calculated to be 101(5) MeV. The central value is slightly above but completely consistent with the experimental value of 99 MeV.

As indicated in Figure 3, the estimates of the hyperfine splittings are significantly smaller than the experimental value. We find the 1S hyperfine splitting $m_{D_s^*} - m_D = 93(4)$ MeV and $m_{D_s^*} - m_{D_s} = 82(2)$ MeV, while the experimental results are 142 and 144 MeV re-
spectively. This has been a long-standing problem in lattice calculations of heavy quark systems. Take the $D_s$ system, for example: UKQCD [54] studied the splitting using a nonperturbatively $O(a)$-improved Wilson fermion action on quenched Wilson gauge action lattices at $\beta = 6.2$, giving $m_{D_s^*} - m_{D_s} = 95(6)$ MeV using $m_\rho$ to set the scale. In later studies, A. Dougall et al. [52], using the Sommer scale $r_0 = 0.55$ fm from the $K^*/K$ mass ratio, obtain $m_{D_s^*} - m_{D_s} = 97(6)$ MeV after taking the continuum limit. They also report a result of 96(2) MeV from two-flavor lattices at cutoff $\approx 1.7$ GeV. Di Pierro et al. [50] from 2+1 staggered dynamical fermion $20^3 \times 48$, $a \approx 0.13$ fm (from $r_0 = 0.5$ fm) lattices report the result $m_{D_s^*} - m_{D_s} \approx 112(20)$ MeV.

There are many possible reasons for this underestimate in our calculation. Besides the quenched approximation, ambiguity in the lattice spacing, and the light quark mass being extrapolated from rather heavy values to the physical point, one possible reason is the absence of the “clover term”. If one requires accuracy through $O(a^2 m_{\text{charm}} \Lambda_{\text{QCD}})$ for on-shell quantities, it turns out that one needs to incorporate the clover term into the DWF with proper coefficients [55]. Although naively it is expected that the $O(a^2 m_{\text{charm}} \Lambda_{\text{QCD}})$ error is fairly small in the present calculation, this needs to be confirmed in future work.

From the above observations, it is interesting to look at the ratio,

$$\frac{\Delta_{q,J}}{\Delta_{\text{hyp}}},$$

(2)

to see how well a given heavy quark formalism describes the whole heavy-light system. Lattice calculations have overestimated the numerator while underestimating the denominator, thus obtaining the ratio closer to the experimental value seems to be extremely difficult for any formalism of lattice heavy quark. In our case, we obtain about 5 for $am_{\text{heavy}} = 0.4$, 4 for 0.3, and 3 for 0.2, and presumably any value of the heavy quark mass would not reproduce the experimental value of 2.4 for the charm-strange system. Although we need to take into account the mixing with the other $0^+$ or $1^+$ states before a reliable conclusion can be drawn, as $am_{\text{heavy}}$ becomes smaller, it is worthwhile to keep watching this ratio in future calculations.
B. Charmonium

The effective mass plots for the four spin-parity channels of the heavy-heavy meson system with $am_{\text{heavy}} = 0.4$ are shown in Figure 7. Similarly good plateaux are observed for other heavy quark masses of 0.1, 0.2 and 0.3. We summarize the meson mass estimates obtained from them in Table X.

If we interpolate to $am_{\text{charm}} = 0.3583(22)$ from subsection III A, we obtain the charmonium mass estimates shown in Table XI and Figure 8, where the experimental values are shown together. Notice that with these inputs the mass of the $\eta_c (J^P = 0^-)$ state is consistent with the experimental value. Alternatively we can get an estimate of $m_{\text{charm}} = 0.3561(11)$ for the bare charm quark mass from the $\eta_c (J^P = 0^-)$ and $\rho$. This of course is consistent with the $am_{\text{charm}}$ estimate.

On the other hand the calculated hyperfine splitting of 43(1) MeV is significantly smaller than the experimental value of 116 MeV, just like in the charmed meson cases. One of the reasons is what we described in the heavy-light case that the lack of “clover” term might cause the smallness of the hyperfine splitting. Secondly, heavy quarkonia hyperfine splittings are notoriously difficult to reproduce by lattice calculations. In the quenched approximation, a detailed study of the relation of hyperfine splitting and the lattice scale was carried out by QCD-TARO collaboration [56]. Their result is 77(2)(6) MeV after continuum extrapolation. In the dynamical three-flavor staggered case, S. Gottlieb et. al. [57] reported their hyperfine splitting to be 107(3) MeV from the “fine” MILC lattice configurations ($a \approx 0.086$ fm), with lattice scale obtained from bottomonium system. In the charmonium system, the problem appears independent of the heavy quark action adopted and is not solved even using dynamical configurations. For more details, see the review article in Ref. [58].

The masses of the $P$-wave states, in contrast to the charmed meson cases where they are overestimated, are several standard deviations underestimated from the experimental values.

Also we note a result of the mass of a yet-to-be-established $C$-odd axial vector state, $h_c (J^{PC} = 1^{+-})$: experimentally it has been a long-standing puzzle [58]. Recently two positive results with a mass of $m_{h_c} = 3524.4(6)(4)$ MeV [59, 60] and 3526.2(0.15)(0.2) MeV [61] were reported. A third experiment [62], though, did not confirm this. In the present lattice calculation the mass difference between $h_c$ and $\chi_{c1}$ is estimated as 22(11) MeV (see
Table XI. This is obtained from the ratio of the $h_c$ and $\chi_{c1}$ correlators in the same fitting range used for extracting the $\chi_{c0}$ and $\chi_{c1}$ masses, so is likely more reliable than the absolute values of the excited-state meson mass themselves. Note these masses are significantly underestimated in the present work. If we add this difference to the experimentally known $\chi_{c1}$ mass of $3510.59(10)$ MeV, we get a value of $3533(11)_{\text{stat}}$ MeV. We quote only the purely statistical error here. The $h_c$ correlator by itself yields $3361(21)_{\text{stat}}$ MeV, also with only purely statistical error.

Finally we discuss the charm mass and lattice cutoff estimations solely by charmonium. First we can determine the bare charm mass by looking at some charm onium mass ratio. For example, if we take a traditional choice spin-weighted mass ratio $s$, 
$$\frac{(m_{\chi_{c0}} + 3m_{\chi_{c1}}) - (m_{\eta_c} + 3m_{J/\psi})}{(m_{\eta_c} + 3m_{J/\psi})} = 0.13664(6),$$
we obtain an estimation of $m_{\text{charm}}a = 0.235(2)$. Then by matching the experimental mass of $(m_{\eta_c} + 3m_{J/\psi})/4 = 3.0676(12)$ GeV and the corresponding interpolation of the calculated mass, $0.804(8) a^{-1}$, we obtain a cutoff estimate of $a^{-1} = 3.82(4)$ GeV. If we use spin-0 mesons alone we obtain $m_{\text{charm}}a = 0.225(2)$ and $a^{-1} = 3.90(3)$ GeV and from spin-1 mesons $m_{\text{charm}}a = 0.239(2)$ and $a^{-1} = 3.79(4)$ GeV. This estimate from charmonium is about 30% larger than the one from $\rho$ meson mass \[44\]. The most likely cause of this is of course the quenched approximation. The rectangular improvement of the DBW2 action may also be playing some role here.

C. Charm Quark Mass

Presently available estimates of the charm quark mass are given in the PDG review \[63\]; the charm quark mass is estimated to be in the range
\begin{equation}
1.15 \text{ GeV} \leq m_{\text{MS}}(m_c) \leq 1.35 \text{ GeV}.
\end{equation}
The renormalized mass can be obtained from
\begin{equation}
m_{\text{ren}} = Z_{m}^{\text{MS}} (m_f + m_{\text{res}})
\end{equation}
where $Z_{m}^{\text{MS}} = Z_{m}^{\text{match}} Z_{m}^{\text{lat}}$ is the mass renormalization factor between lattice and continuum. $Z_{m}^{\text{lat}}$ can be obtained with lattice perturbation theory or RI/MOM-scheme non-perturbative renormalization either from matching the lattice quark propagator to the continuum one or from the scalar bilinear operator renormalization factor as $1/Z_S$ \[64, 65, 66\]. $Z_{m}^{\text{match}}$
matches between $\overline{\text{MS}}$ in the continuum and whatever scheme was used on the lattice; the matching between $\overline{\text{MS}}$ and RI/MOM scheme can be obtained from Ref. \[67\]. Aoki et al.\[68\] calculate one-loop $Z^{\overline{\text{MS}}}_m$ with DWF fermions in the DBW2 gauge action, giving $Z^{\overline{\text{MS}}}_m = 0.989441$; at the scale $\mu = 2 \text{ GeV}$ it gives $m_c = 1.0330(64) \text{ GeV}$ from the $D_s$ estimate of $am_{c,\text{charm}} = 0.3583(22)$. Chetyrkin calculates the anomalous dimension of the running of the quark mass to $O(\alpha_s^4)$ \[69, 70\] in $\overline{\text{MS}}$ scheme:

$$\mu^2 \frac{d}{d\mu^2}m^{(n_f)}(\mu) = m^{(n_f)}(\mu)\gamma_{m^{(n_f)}}\left(\alpha_s^{(n_f)}\right).$$  \hspace{1cm} (5)

The renormalization group invariant mass \[71\]

$$m_{\text{RI}} = \lim_{\mu \to \infty} m(\mu)\left[\frac{33 - 2N_f}{6\pi}\alpha_s\right]^{-\frac{\ln 12}{33 - 2N_f}},$$  \hspace{1cm} (6)

The running charm mass is then given by \[67, 72\]

$$m_c(\mu^2) = m_{c,\text{RI}}(\alpha_s)^{4/11} \left[1 + 0.68733 (\alpha_s) + 1.51211 (\alpha_s)^2 + 4.05787 (\alpha_s)^3\right];$$  \hspace{1cm} (7)

using $\alpha_s$ from Eq. \[87\] we find at the scale of $m_c$, our $m^{\overline{\text{MS}}}_c(m_c) = 1.239(8) \text{ GeV}$. The perturbative renormalization contains a systematic error of $O(\alpha_s^2) \approx 4\%$, giving us $m^{\overline{\text{MS}}}_c(m_c) = 1.24(1)(4) \text{ GeV}$. If we start from the charmonium estimate of $m_{c,a} = 0.235(2)$, we obtain $m^{\overline{\text{MS}}}_c(m_c) = 1.07(1)(4) \text{ GeV}$. We find that the difference between these two charm quark mass estimates dominates our systematic error; after combining this with the other known systematics, we find the charm quark mass to be $1.24(1)(18)$.

For the convenience of the reader, some past estimations from quenched lattice QCD with continuum extrapolation are listed in Table \[IX\]. Note, UKQCD recently \[73\] used a two-flavor clover sea to obtain $m^{\overline{\text{MS}}}_c(m_c) = 1.29(7)(13) \text{ GeV}$ after continuum extrapolation.

## IV. LEPTONIC DECAY CONSTANTS

The leptonic decay constants are obtained in a standard procedure from two-point correlation functions with heavy-light current $A_4 = \bar{\Psi}_l \gamma_\mu \gamma_5 \Psi_c$:

$$\langle 0 | A_4 | \text{PS at rest} \rangle = if_{\text{PS}} \cdot m_{\text{PS}}.$$  \hspace{1cm} (8)

We calculated both $C^{A_4}_P$ and $C^{PP}_P$ correlators for pseudoscalar mesons. Since these two correlators give the same pseudoscalar meson mass ($m_{\text{PS}}$), we can extract $m_{\text{PS}}$ along with
amplitudes from fitting these two correlators simultaneously. That is, we minimize the \( \chi^2 \) given by

\[
\chi^2 = \sum_{t=t_{\text{min}}}^{t_{\text{max}}} \left\{ \frac{C^{A_4P}(t,0) - A^{A_4P} \sinh(m_{PS}(t - L_t))}{\sigma^{A_4P}(t)} \right\}^2 + \left\{ \frac{C^{PP}(t,0) - A^{PP} \cosh(m_{PS}(t - L_t))}{\sigma^{PP}(t)} \right\}^2,
\]

(9)

where \( \sigma(t) \) is the jackknife error of the correlator at \( t \). The simultaneous fit gives us more stable amplitudes for two-point correlators than individual fits for the single correlators.

The decay constants can be obtained from

\[
f_{PS}^{\text{lat}} = \frac{A^{A_4P}_{pw}}{\sqrt{\frac{m_{PS}}{2} V^{PP}_{ww}}},
\]

(10)

where \( V \) denotes the spatial volume of the lattice.

Booth \[74\] and Sharpe and Zhang \[75\] calculate the chiral behavior of the heavy-light decay constants in the quenched approximation to be:

\[
\sqrt{m_{QQ}} f_{Qq} = F_1 + F_2 m_q + (F_3 m_q + F_4) \ln m_q,
\]

(11)

after replacement of \( m_{qq}^2 \propto m_q \). McNeile and Michael \[76\] reported that the effect of the chiral log on \( f_B \) with static-light is small. This indicates that we may be able to drop the most divergent term \( \ln m_q \). To be concrete, we enumerate below various fitting functions that we will use for the purpose of extrapolating/interpolating the light quark mass dependence:

\[
\sqrt{m_{QQ}} f_{Qq} = F_1 + F_2 m_q + (F_3 m_q + F_4) \ln m_q,
\]

(12)

\[
\sqrt{m_{QQ}} f_{Qq} = F_1 + F_2 m_q + F_3 m_q^2,
\]

(13)

\[
\sqrt{m_{QQ}} f_{Qq} = F_1 + F_2 m_q,
\]

(14)

at fixed \( am_{\text{heavy}} \). Then we examine the \( 1/am_{\text{heavy}} \) behavior and interpolate to \( am_{\text{charm}} \). The upper half of Figure 9 shows the light quark mass dependence at fixed \( am_{\text{heavy}} \) of the quantity \( \sqrt{m_{QQ}} f_{Qq} \) extrapolated to \( -m_{\text{res}} \) according to Eq. 12 and Figure 10 shows the light quark mass dependence of the quantity \( \sqrt{m_{QQ}} f_{Qq} \) interpolated to \( m_{\text{strange}} \). Then we further
linearly interpolate (in $1/M$) the heavy quark mass dependence to $a m_{\text{heavy}} = a m_{\text{charm}}$, and summarize the results in Table XII. The results from various expressions used for light quark extrapolation/interpolation are consistent among the fits. Here we take the fit from the quadratic expression, since this falls roughly in the middle of the range from the other expressions, and incorporate the difference from the other fits into the systematic error. Thus, we obtain

$$f_{D}^{\text{lat}} = 225(7)(^{+6}_{-0}) \text{MeV},$$

$$f_{D_s}^{\text{lat}} = 243(4)(^{+0}_{-3}) \text{MeV},$$

before renormalization.

In order to compare our lattice calculations with continuum results, we need to properly renormalize our lattice decay constants by a factor of:

$$Z_{h}^{hl} = Z_{l}^{ll} \times \sqrt{\frac{Z_{q,DWF}(am_{\text{heavy}})}{Z_{q,DWF}(am_{\text{light}})}},$$

The nonperturbative light-light current renormalization in the chiral limit, $m_f = -m_{\text{res}}$, is calculated in Ref. [44], as $0.88813(19)$. Here we employ a quark mass-dependent renormalization factor [55]:

$$Z_{q,DWF}^{(0)}(am_f, \omega) = \frac{am_f (1 + (am_f)^2) \cosh(m_p^{(0)}) - 2(am_f)^2 \omega \sinh(m_p^{(0)})}{(1 - (am_f)^2) \sinh(m_p^{(0)})},$$

in terms of the tree-level on-shell pole mass

$$m_p^{(0)} = \ln \left| \frac{-am_f \omega^2 + \sqrt{(1 + am_f^2)^2 + am_f^2 \omega^2 (\omega^2 - 4)}}{1 + am_f^2 - 2am_f \omega} \right|,$$

with $\omega = 2 - M_5$, and $Z_{q,DWF}^{\text{lat}}$ defined as

$$q_R = \sqrt{Z_{q,DWF}^{\text{lat}} q_{\text{lat}}}.$$

Therefore, the heavy-light current renormalization is 1.0492(22) with $m_{\text{charm}} = 0.3583(22)$; thus $f_D = 232(7)(^{+6}_{-0})$ and $f_{D_s} = 254(4)(^{+0}_{-3})$ MeV. Recent experimental measurements for these values are $f_{D^+} = 222.6 \pm 16.7 \pm 2.8$ MeV [74] and $f_{D_s} = 267 \pm 33$ MeV [63], and the recent lattice three-flavor dynamical results of MILC and Fermilab [78] are: $f_{D^+} = 201 \pm 3 \pm 17$ MeV, and $f_{D_s} = 249 \pm 3 \pm 16$ MeV.
To calculate the ratio of the decay constants, we first use the chiral form to obtain

\[
f_{D_s}/f_D = 1.11(2)\left(\frac{+0}{-2}\right);
\]

Figure 11 shows an example of our heavy quark mass interpolations of this ratio, with chiral interpolation/extrapolation on the light quark mass according to Eq. 12. We may then retrieve the ratio of decay constants for the \(D/D_s\) systems by multiplying in appropriate factors of mass:

\[
f_{D_s}/f_D = 1.05(2)\left(\frac{+0}{-2}\right).
\]

The experimental value suggesting the above ratio is 1.19(25) and the MILC number is 1.24(22). Our result has a smaller central value but agrees with the experimental and dynamical result within one \(\sigma\).

V. \(D-\bar{D}\) MIXING

In the Standard Model, \(D^0-\bar{D}^0\) mixing is strongly suppressed by CKM and GIM factors; if such a mixing is observed, it might be evidence for physics beyond the Standard Model. The \(D\)-mixing is the only probe for the dynamics for \(c\)-quark. The effective Hamiltonian of the QCD contribution to \(\Delta C = 2\) in the Standard Model comes from the box diagram

\[
\mathcal{H}_{\text{eff}}^{\Delta C = 2} = \frac{G_F^2}{4\pi^2}|V_{cs}V_{cd}|^2 \frac{(m_s^2 - m_d^2)^2}{m_c^2}\left(\mathcal{O} + 2\mathcal{O}'\right)
\]

where \(\mathcal{O} = \bar{c}\gamma^\mu(1 - \gamma_5)u\bar{c}\gamma_\mu(1 - \gamma_5)u\) and \(\mathcal{O}' = \bar{c}(1 + \gamma_5)u\bar{c}(1 + \gamma_5)u\) results from non-negligible external momentum. The \(b\)-quark contribution is proportional to \(V_{ub}m_b\) highly suppressed by the very small CKM matrix factor \(V_{ub}\), which is ignored here. The BaBar \(B\)-factory studies \(D - \bar{D}^0\) mixing from the semileptonic decay modes of \(D^{*+} \rightarrow \pi^+ D^0\) and \(D^0 \rightarrow K\nu\) \[79\]. In this work, we focus on the bag parameter of \(D\) meson, often defined for historical reasons as

\[
B_D = \frac{\langle D^0|\mathcal{O}|D^0\rangle}{\frac{8}{3}m_D^2f_D^2},
\]

where \(\mathcal{O} = \bar{c}\gamma^\mu(1 - \gamma_5)u\bar{c}\gamma_\mu(1 - \gamma_5)u\) and \(B_D\) is 1 in the vacuum-insertion approximation.

The parity-even operator for \(B_D\) in the continuum limit is

\[
\mathcal{O}_{VV+AA} = (\bar{c}_\gamma\mu u)(\bar{c}_\gamma\mu u) + (\bar{c}_\gamma\gamma_5\mu u)(\bar{c}_\gamma\gamma_5\mu u).
\]
However, since DWF does not have exact chiral symmetry, $O_{VV+AA}$ mixes with four other operators

$$O_{VV-AA} = (\bar{c}\gamma_\mu u)(\bar{c}\gamma_\mu u) - (\bar{c}\gamma_5\gamma_\mu u)(\bar{c}\gamma_5\gamma_\mu u),$$  \(26\)

$$O_{SS\pm PP} = (\bar{c}u)(\bar{c}u) \pm (\bar{c}\gamma_5 u)(\bar{c}\gamma_5 u),$$  \(27\)

$$O_{TT} = (\bar{c}\sigma_{\mu\nu} u)(\bar{c}\sigma_{\mu\nu} u).$$  \(28\)

The mixing coefficients for these operators do not go to zero as $m_f$ approaches the chiral limit, but are highly suppressed by $O((am_{res})^2)$, at least in the case for $B_K$ with DWF. This has been theoretically and numerically demonstrated in Ref. [44], and one can simply ignore the contribution from the above four operators. However, in the finite quark mass region, the correct way of solving this problem is to measure the matrix elements of all the operators in Eqns. 26, 27 and 28 directly on the lattice. These operators are relevant to beyond the Standard Model sources of mixing in $\Delta F = 2$ processes [44]. Since we did not do this measurement while the data were taken, we quote the maximal uncertainty from the mixing coefficients, which are expected to be $O((am_f)^2)$, due to the soft chiral symmetry breaking from the non-small $m_f$ term.

On the lattice we can rewrite Eq. 24 directly in terms of the correlation functions obtained on the lattice:

$$B_{PS}^{(lat)} = \frac{\langle 0 | \chi^\dagger(t_{snk})O(t)\chi(t_{src}) | 0 \rangle}{8CA_p \left( t_{snk}C_{pw}A_p(t,t_{src}) \right)_{t_{src} < t < t_{snk}}},$$  \(29\)

where $t_{src}$ and $t_{snk}$ are the source and the sink location of the quarks, and the $\chi(t)$ are quark bilinear interpolating meson fields. The results can be found in Figure with various $m_{heavy}$ and $m_{light}$. The plateau looks pretty good for extracting the value of $B_{PS}^{(lat)}$, and details are listed in Table XIII.

Booth [74] and Sharpe & Zhang [75] calculate the chiral behavior of the heavy-light meson bag parameter in the quenched approximation to be:

$$B_{Qq} = B_1 + B_2m_q + B_3m_q \ln m_q + B_4 \ln m_q.$$  \(30\)

The coefficient of the most divergent term, $\ln(m_q)$, is accompanied by a factor of $1 - 3g^2$, where $g$ is the coupling of $D - D^* - \pi$ that can be obtained from $D^* \to D\pi$ decay [80], which yields a value of $0.2(7)$. Note that this $g$ for $B - B^* - \pi$ is about the same due to heavy-quark
symmetry. However, it is still hard to judge the effect of this logarithmic dependence in our range of light quark mass. We will ignore it for the rest of paper and incorporate it into the systematic error by comparing with the unquenched calculation, where the \( \ln(m_q) \) term is absent. In order to estimate the systematic error due to our choice of fitting form, we adopt different expressions to extrapolate/interpolate the light quark dependence:

\[
B_{Qq} = B_1 + B_2 m_q + B_3 m_q \ln m_q; \quad (31)
\]

\[
B_{Qq} = B_1 + B_2 m_q + B_3 m_q^2; \quad (32)
\]

\[
B_{Qq} = B_1 + B_2 m_q. \quad (33)
\]

Here we show the extrapolations using the Eq. 31 in the upper graphs in Figure 13 to the chiral limit, \( m_f = -m_{\text{res}} \) and in Figure 14 to the strange quark mass. The results from various fits are summarized in Table XIV and different analyses agree with each other within statistical error bars. Therefore, we take the quadratic fit as the central value and absorb the other fits into the systematic error; thus we have

\[
B^{\text{lat}}_D = 0.859(24)(^{+24}_{-6}) \quad (34)
\]

\[
B^{\text{lat}}_{D_s} = 0.848(7)(^{+2}_{-0}) \quad (35)
\]

before proper renormalization. Note that since \( D_s \) is a charged meson, there cannot be oscillations between \( D_s \) and \( \overline{D_s} \). Here we calculate its bag parameter merely for the interest of studying the \( SU(3) \) breaking effect in charmed systems. Also, the extrapolation to the bag parameters from \( B \) and \( B_s \) mesons are simply side products of this study. It is interesting to check the goodness of \( 1/M \) extrapolation with the static quark lattice studies.

We take advantage of the bag parameter nonperturbative renormalization (NPR) done in Ref. [44] with RI/MOM scheme and further convert into \( \overline{\text{MS}} \). Ref. [81, 82] calculated the conversion factors in NLO perturbation theory. First, we convert the renormalization factor into an RI value

\[
Z^{\text{RI}}_B = [\alpha_s(\mu^\text{lat})]^{-2/11} \left[ 1 + \frac{\alpha_s(\mu^\text{lat})}{4\pi} J_{\text{RI/MOM}} \right] Z^{\text{lat,RI/MOM}}_B(\mu^\text{lat}) \quad (36)
\]
where

\[
\alpha_s(\mu) = \frac{12\pi}{(33 - 3N_f) \ln(\mu^2/\Lambda_{QCD}^2)} \left( 1 - \frac{918 - 90N_f - 24N_f^2 \ln(\mu^2/\Lambda_{QCD}^2)}{(33 - 2N_f)^2 \ln(\mu^2/\Lambda_{QCD}^2)} \right),
\]

(37)

with \(N_f = 0\), and \(\Lambda_{QCD}^0 = 238\) \(\text{MeV}\) and \(J_{RI/MOM}(N_f = 0) = 2.883\); that gives \(Z_B^{RI} = 1.409(5)\). Then we calculate the factor for \(N_f = 4\), as

\[
Z_B^{MS} = \alpha_s(\mu)^{25/3} \left[ 1 + \frac{\alpha_s(\mu^{\text{lat}})}{4\pi} J_{MS} \right]^{-1} \tilde{Z}_B.
\]

(38)

The \(\Lambda_{QCD}\) with \(N_f = 5\) is 217 \(\text{MeV}\) [63], suggesting that the \(\Lambda_{QCD}\) is 276 \(\text{MeV}\) for \(N_f = 4\). Then \(\alpha_s(\mu = 2\ \text{GeV})\) is 0.283. Therefore, \(Z_B^{MS}\), is 1.000(4) and our bag parameters are:

\[
B_D = 0.845(24)(^{+24}_{-0}),
\]

(39)

\[
B_{Ds} = 0.835(7)(^{+2}_{-0}),
\]

(40)

\[
B_{Ds}/B_D = 0.987(22)(^{+0}_{-27}).
\]

(41)

VI. SU(3) BREAKING RATIOS

In the long run, lattice QCD should provide a high-precision determination of \(SU(3)\) flavor-breaking of the \(\Delta F = 2\) heavy-light matrix element

\[
\frac{\mathcal{M}_{Qs}}{\mathcal{M}_{Ql}} = \frac{\langle \bar{P}_{Qs}|\bar{Q}\gamma_\mu(1 - \gamma_5)s|P_{Qs}\rangle}{\langle \bar{P}_{Ql}|\bar{Q}\gamma_\mu(1 - \gamma_5)l|P_{Ql}\rangle},
\]

(42)

where \(Q\) stands for heavy quark, \(l\) stands for light quark \((u,d)\), and \(P_{Ql}\) represents a pseudoscalar meson composed of \(Q\) and \(l\). There are two ways of obtaining this ratio: first, by calculating the matrix element directly [83]; secondly, by calculating the decay constants and bag parameters separately and combining them with Eq. 24. In this work, we will focus on the second method (often referred to as the “indirect” approach) to get the ratios by the combination of ratios of bag parameters and decay constants:

\[
r_Q = \frac{\mathcal{M}_{Qs}}{\mathcal{M}_{Ql}} = \frac{B_{Qs}(m_{Qs} f_{Qs})^2}{B_{Ql}(m_{Ql} f_{Ql})^2}.
\]

(43)
Therefore, it is important to obtain the ratio of
\[
\frac{f_{Qs}}{f_{ql}} \sqrt{\frac{B_{ql}}{B_{ql}}} = \xi_Q, \tag{44}
\]

Table XV summarizes $\xi_Q$ and the $SU(3)$ breaking for different chiral extrapolation formulae.

From the charm quark sector, we obtain
\[
\xi_c = 1.071(23)(^{+0}_{-31}), \tag{45}
\]
and
\[
r_c = 1.273(65)(^{+0}_{-67}). \tag{46}
\]

### VII. EXTRAPOLATION TO STATIC B MESONS

Although our main goal in this work is to apply the DWF to the charm quark physics directly, it is of some interest to extrapolate the heavy quark mass to the static limit. In the past, as is pointed out in Ref. \[84, 85\], some static limit $B$ parameters obtained from extrapolation from charm region with simple linear function in $1/m_{\text{heavy}}$ were found disagreeing with direct static calculations: after all the charm mass may not be sufficiently heavy to justify such a simple extrapolation to the static limit. However, as a mere by-product of our work on charm, the extrapolation may still be instructive.

#### A. Decay constants

Since we do not know how to renormalize the heavy-light decay constant for a static quark with DWF as the light quark action, we will only focus on ratios of decay constants. Since the bottom quark is so heavy compared to the scale of our physics, we may extrapolate to the static quark limit to get the result pertaining to $B$ mesons. The ratio is
\[
(f_{Bs}/\sqrt{m_{Bs}}/f_B\sqrt{m_B})^{\text{static}} = 1.08(4)(^{+1}_{-2}) \tag{47}
\]
Taking $m_{Bs}/m_B$ as 1.017 \[63\], we have
\[
(f_{Bs}/f_B)^{\text{static}} = 1.06(4)(^{+1}_{-2}). \tag{48}
\]
A previous quenched study using the static approximation for the heavy fermion action and the step-scaling technique \[86\] gives 1.11(16) when extrapolated to the continuum limit,
which is consistent with our extrapolation result here. A recent study using partially quenched two-flavor DWF lattices with the static approximation \cite{87} (with lattice cutoff \(\approx 1.7 \text{ GeV}\)) yields 1.29(4)(4)(2) for the same quantity, giving us an idea of the systematic error due to the quenched approximation. HPQCD use a 2+1 staggered fermion sea with NRQCD heavy quark action and lattice cutoffs \(\approx 1.6 \text{ GeV}\) and \(\approx 2.6 \text{ GeV}\) to obtain the ratio 1.20(2)(1) \cite{88}.

\textbf{B. Bag parameters}

Following the analyses in Sec. V, we extrapolate lattice bag parameter to the static limit

\begin{equation}
(B_{B}^{\text{lat}})^{\text{static}} = 0.940(33)(^{+30}_{-6})
\end{equation}

\begin{equation}
(B_{B_{s}}^{\text{lat}})^{\text{static}} = 0.919(9)(^{+3}_{-0}).
\end{equation}

We set the scale \(\mu\) at the mass of \(b\)-quark, 4.5 GeV, and the renormalization factor \(Z_{B}^{\text{MS}}(m_{b})(\text{with } N_{f} = 5)\) is 0.920(3), which suggests

\begin{equation}
(B_{B})^{\text{static}} = 0.865(33)(^{+30}_{-6})
\end{equation}

\begin{equation}
(B_{B_{s}})^{\text{static}} = 0.845(9)(^{+3}_{-0}).
\end{equation}

Ref. \cite{89} uses static heavy and Wilson light quark actions to get \(B_{B}(m_{b}) = 0.98(4)(^{+3}_{-18})\) at \(\approx 1.8 \text{ GeV}\) lattice cutoff. Another quenched study with static heavy and overlap light quarks gives \cite{90} (again, with the finest lattice cutoff \(\approx 1.8 \text{ GeV}\))

\begin{equation}
(B_{B_{s}}(m_{b}))^{\text{static}} = 0.940(16)(22).
\end{equation}

Our number does not agree too well with previous static results. This might be due to the coarse lattices used in their simulations. However, our number agrees better with JLQCD’s quenched calculation with NRQCD \cite{91}, where the finest lattice spacing is 2.3 GeV:

\begin{equation}
B_{B_{d}}(m_{b}) = 0.84(3)(5),
\end{equation}

\begin{equation}
B_{B_{s}}/B_{B_{d}} = 1.020(21)(^{+15}_{-16})(^{+5}_{-0}),
\end{equation}
\[ B_{S_s}(m_h) = 0.85(1)(5)(^{+1}_{-0}), \] (56)
after taking the continuum limit. Our results for the \( B \) case are consistent with the two-flavor DWF static-light calculation in Ref. [87]:
\[ (B_B)^{\text{static}} = 0.812(48)(67)(^{+0}_{-300}), \] (57)
\[ (B_{B_s})^{\text{static}} = 0.864(28)(71)(^{+0}_{-320}), \] (58)
\[ (B_{B_s}/B_B)^{\text{static}} = 1.06(6)(3)(1), \] (59)
and the two-flavored \( O(a) \)-improved Wilson fermion sea with NRQCD heavy quark study done by JLQCD
\[ (B_B) = 0.836(27)(^{+56}_{-62})(B_{B_s}/B_B) = 1.017(16)(^{+56}_{-17}). \] (60)

C. \( SU(3) \) breaking ratio

Extending the discussions in Sec. VI, we have the results for the bottom sector: \( \xi_b = 1.019(37)(^{+16}_{-34}) \) and the \( SU(3) \) breaking ratio \( r_b = 1.274(94)(^{+32}_{-180}) \). It is useful to recall that in the first calculation in Ref. [83], using a Wilson fermion action, found \( \xi_b = 1.30(4)(^{+21}_{-15}) \) and \( 1.17(2)(^{+12}_{-6}) \) after linear continuum extrapolations from direct and indirect methods, respectively. A previous static quark study on a two-flavor DWF sea gives: \( \xi_b = 1.33(8)(8) \). Using JLQCD ratio of bag parameters and HPQCD 2+1 result on decay constants, the Lattice '05 review gives \( \xi_b = 1.210(47) \) and the previous world average gives \( \xi_b = 1.23(6) \). Our central value is smaller than these previous results but quite consistent with other studies given our relatively large errors. We first take the light quark mass limit according to the heavy-light meson bag parameter chiral formula and then take the heavy quark mass to \( m_{\text{charm}}^{\text{lat}} \). Figures 13 and 14 show that the chiral behavior of the light quark part makes the bag parameters for mesons with up and down quarks somewhat larger than those with a strange quark. However, the other studies get their bag parameters by fitting the bag parameter as a function of heavy-light pseudoscalar mass, instead of the heavy-light chiral forms; see the dependence for our data in Figure 15. The bag parameter increases
with the pseudoscalar mass. Therefore, if one fits the bag parameter function as input of heavy-light pseudoscalar mass, from a relation $m_{\{B,D\}}/m_{\{B,D\}} > 1$, which always holds, it follows that $B_{\{B,D\}}/B_{\{B,D\}} > 1$. Therefore, $\xi_Q$ in these studies must come out larger than our values.

VIII. SYSTEMATIC ERRORS

In the previous sections, we present our calculation with statistical errors which are computed using the jackknife procedure, along with systematic errors mostly caused by chiral extrapolation. There are other potential sources of systematic errors caused by the quenched approximation, finite-volume effects, operator mixing and matching calculations. Here is a detailed estimate of the various systematic errors.

**Finite volume:** The proper way to estimate this error is to perform the calculation on at least two different volumes and extrapolate to the infinite-volume limit. However, in this work, we only perform our calculation in one lattice volume, which is about (1.6 fm)$^3$. Since the scale of $D$ system is in between $K$ and $B$, we will estimate our finite-volume error by quoting whichever system ($K$ or $B$) has larger finite-volume error in previously published studies. Ref. [44], which uses the same lattice box we do, quoted a 2% error in $B_K$ by comparing with a 2.4 fm box. We also compare our $B_B$ found by extrapolation to the static limit with a quenched, larger-volume ($\approx 2.4$ fm), calculation [91], which gives $B_{B_d}(m_b) = 0.84(3)(5)$ with NRQCD fermions; this gives us an estimation of 0.4% error. Taking the larger of these two estimates, we conclude that the finite-volume effect on the $B_D$ should be smaller than 2%.

**Continuum extrapolation:** In this paper, we only perform our calculation at one lattice spacing, and therefore we cannot extrapolate our result to the continuum limit. This should be checked carefully in future works. The next-best thing we can do is to estimate the error from a continuum extrapolation study from the same gauge configuration. Such a study has been performed on the decay constants and matrix elements of the $K$ system in Ref. [44] with an additional lattice cutoff at 1.982(30) GeV. The result shows mild dependence from $a^{-1} \approx 3$ GeV to the continuum. Therefore, we add 0.2% to our systematic errors due to continuum extrapolation.
RI formulation: We need the $\alpha_s$ to first convert the scale-dependence RI/MOM NPR factor, $Z_B^{RI/MOM}$, to an RI value and further convert it to an $\overline{\text{MS}}$ one. In the expression, we use the NLO order formulation, which leaves the remaining leading error up to $O(\alpha_s^2) \approx 4\%$.

NPR renormalization factor in bag parameters: We adopt the RI/MOM NPR renormalization factor from Ref. [44], and we quote the estimation within that paper as 1%.

Operators mixing: We have discussed mixing with wrong chirality operators, which contributes around 10% uncertainties to our final $B_D$ calculation.

Quenched approximation: The quenched approximation ignores sea quark loop contributions which in general are considered to be a major contribution to systematic errors. Ref. [44], which uses the same quenched lattice ensemble as in the present work, quotes a 6% error on the $B_K$ factor due to this approximation, after comparing with the number calculated on two-flavor DWF lattices. Similarly, we can compare our static quark limit value with the one calculated on two-flavor DWF lattices [87], which gives us 6%. Therefore, it seems reasonable to use a 6% systematic quenching error for $B_D$ as well.

IX. CONCLUSION

In this work, we use the domain-wall fermions (DWF) formulation for charm quark as well as the three lighter flavors, up, down and strange, on the lattice with a relatively high cutoff (around 3 GeV). We use the mass ratio $m_K/m_\rho$ to set the bare strange quark mass $m_{\text{strange}} a = 0.0298(13)$ in lattice units. Then combining this and another hadronic mass ratio $m_{D_s}/m_\rho$ we obtain the bare charm mass, $m_{\text{charm}} a = 0.3583(22)$.

Using these bare quark mass values we found, we conclude the following for charmed and charmed-strange meson states:

- the masses of the $J^P = 0^\pm$ and $1^\pm$ $D$, $D^*$, $D_s$ and $D_{sJ}$ states are well reproduced to within a few percent.

- Their parity splitting, $\Delta_J$, are better reproduced than previous works, with only 10-20% over estimations.
• The experimental observation of $\Delta_{ud} > \Delta_s$ is reproduced.

• The hyperfine splittings are only 60-65% reproduced.

Regarding the dependence on heavy quark mass,

• $\Delta_{J=0}$ and $\Delta_{J=1}$ are degenerate for $m_{\text{heavy}}a > 0.2 - 0.3$.

• $\Delta_{J=0}$ increases as $m_{\text{heavy}}$ decreases further, while

• $\Delta_{J=1}$ does not.

Also with the bare charm quark mass we worked out the renormalization factor, $Z_m$, from the existing one-loop domain-wall perturbation calculation. After RG scaling, the $m_c^{\overline{\text{MS}}} (m_c) = 1.239(8)$ GeV. The perturbative renormalization contains a systematic error of $O(\alpha_s^2) \approx 4\%$, giving us $m_c^{\overline{\text{MS}}} (m_c) = 1.24(1)(18)$ GeV.

In the charmonium system we find the $\eta_c$ mass agrees well with the experimental value. The $J/\psi$ mass (hence the hyperfine splitting) is smaller than the experimental one, confirming the long puzzle in lattice QCD. We also note a prediction for the mass of yet-to-be-discovered C-odd $h_c$ state: the mass difference between $h_c$ and $\chi_{c1}$ is estimated as $22(11)$ MeV. This would translate to the mass of $3533(11)_{\text{stat}}$ MeV for this meson.

The leptonic decay constants are also calculated. We tried to estimate the systematic uncertainty from the associated quenched logarithm by adopting three different fitting formulations. Our result after considering other systematic uncertainties due to the quenched approximation and continuum extrapolation give us

$$f_D = 232(7)(^{+6}_{-0})(11) \text{ MeV}$$

(61)

$$f_{D_s}/f_D = 1.05(2)(^{+0}_{-2})(2).$$

(62)

We also discussed extrapolation to the static limit in terms of $1/m_{\text{heavy}}$ which gives results consistent with previous static calculations.

The bag parameters in the $D$ and $D_s$ (purely theoretical but interesting in regard of $SU(3)$ breaking effect) are studied. The use of domain-wall fermions on the current fine lattice with only softly broken chiral symmetry gives us some advantages such as absence of complicated mixing and the availability of the RI/MOM nonperturbative renormalization techniques.
We include a detailed estimation of systematic uncertainties. The biggest systematic errors come from the quenched approximation and the mixing of wrong-chirality operators. Our result is:

$$B_D(2 \text{ GeV}) = 0.845(24)(^{+24}_{-6})(105),$$  \tag{63}

$$B_{D_s}/B_D = 0.987(22)(^{+0}_{-27})(23),$$  \tag{64}

where the first error is statistical, the second systematic from fitting, and the third combining all other known systematics. Thus a minor $SU(3)$-breaking is seen in our calculation. We also discussed extrapolation to the static limit in terms of $1/m_{\text{heavy}}$ which suggests the $D$ and $D_s$ meson results are reliable.

In conclusion, using DWF to simulate the charm quark on a quenched ensemble at a moderately high cutoff of about 3 GeV obtained with the DBW2 action results in reasonable descriptions for most of the calculated meson observables: the masses and their splittings, leptonic decay constant, and $\Delta C = 2$ mixing. It seems the charm quark propagation is successfully described with the current set up. This obviously is an attractive direction to proceed, especially with dynamical QCD ensembles that are becoming available.

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| Quantities     | numbers                                      |
|---------------|----------------------------------------------|
| # of conf.    | 103                                          |
| $a^{-1}$      | 2.914(54) GeV                               |
| $L_s$         | 10                                           |
| $M_5$         | 1.65                                         |
| $am_{res}$    | $0.9722(27) \times 10^{-4}$                 |
| $am_{light}$  | 0.008, 0.016, 0.024, 0.032, 0.040            |
| $am_{heavy}$  | 0.1, 0.2, 0.3, 0.4, 0.5                      |

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TABLE II: Meson states in this study are created by local operators of the form $\bar{\psi} \Gamma \psi$ in this table. Their spin and parity are also listed, as well as charge conjugation for quarkonium cases.

| $\Gamma$ | $2S+1L_J$ | $J^{PC}$ |
|----------|-----------|----------|
| $\gamma_5$ | $^{1}S_{0}$ | 0$^{--}$ |
| $\gamma_i$ | $^{3}S_{1}$ | 1$^{--}$ |
| 1         | $^{3}P_{0}$ | 0$^{++}$ |
| $\gamma_5\gamma_i$ | $^{3}P_{1}$ | 1$^{++}$ |
| $\gamma_i\gamma_j$ |     | 1$^{+-}$ |
TABLE III: Heavy-light $J^P = 0^-$ meson mass obtained from fitting to the hyperbolic cosine form in the range of $15 \leq t \leq 23$. 

| $m_{\text{heavy}}$ | $m_{\text{light}}$ | $m_{\text{PS}}$ | $\chi^2$ per d.o.f. |
|---------------------|---------------------|------------------|---------------------|
| 0.008               | 0.3254(15)          | 0.22             |                     |
| 0.016               | 0.3373(13)          | 0.15             |                     |
| 0.1                 | 0.024               | 0.3495(12)       | 0.09               |
| 0.032               | 0.3617(11)          | 0.08             |                     |
| 0.040               | 0.3739(11)          | 0.07             |                     |
| 0.008               | 0.4715(17)          | 0.36             |                     |
| 0.016               | 0.4812(14)          | 0.30             |                     |
| 0.2                 | 0.024               | 0.4913(13)       | 0.23               |
| 0.032               | 0.5016(12)          | 0.19             |                     |
| 0.040               | 0.5120(11)          | 0.18             |                     |
| 0.008               | 0.5916(19)          | 0.51             |                     |
| 0.016               | 0.6003(15)          | 0.46             |                     |
| 0.3                 | 0.024               | 0.6095(13)       | 0.37               |
| 0.032               | 0.6189(12)          | 0.32             |                     |
| 0.040               | 0.6285(11)          | 0.31             |                     |
| 0.008               | 0.6917(21)          | 0.71             |                     |
| 0.016               | 0.7000(16)          | 0.70             |                     |
| 0.4                 | 0.024               | 0.7086(14)       | 0.59               |
| 0.032               | 0.7175(13)          | 0.52             |                     |
| 0.040               | 0.7266(12)          | 0.49             |                     |
| 0.008               | 0.7734(22)          | 0.84             |                     |
| 0.016               | 0.7815(17)          | 0.92             |                     |
| 0.5                 | 0.024               | 0.7898(15)       | 0.84               |
| 0.032               | 0.7984(14)          | 0.78             |                     |
| 0.040               | 0.8072(13)          | 0.75             |                     |
TABLE IV: Heavy-light $J^P = 1^-$ meson mass obtained from fitting to the hyperbolic cosine form in the range of $15 \leq t \leq 23$.

| $m_{\text{heavy}}$ | $m_{\text{light}}$ | $m_{\text{PS}}$ | $\chi^2$ per d.o.f. |
|---------------------|--------------------|----------------|---------------------|
| 0.008               | 0.4083(45)         | 0.05           |
| 0.016               | 0.4153(35)         | 0.40           |
| 0.024               | 0.4223(30)         | 0.27           |
| 0.032               | 0.4310(26)         | 0.20           |
| 0.040               | 0.4401(24)         | 0.17           |
| 0.008               | 0.5259(37)         | 0.65           |
| 0.016               | 0.5324(28)         | 0.52           |
| 0.024               | 0.5400(24)         | 0.36           |
| 0.032               | 0.5484(21)         | 0.27           |
| 0.040               | 0.5572(19)         | 0.23           |
| 0.008               | 0.6304(34)         | 0.63           |
| 0.016               | 0.6373(25)         | 0.56           |
| 0.024               | 0.6450(21)         | 0.41           |
| 0.032               | 0.6532(19)         | 0.33           |
| 0.040               | 0.6618(17)         | 0.29           |
| 0.008               | 0.7203(32)         | 0.59           |
| 0.016               | 0.7275(24)         | 0.57           |
| 0.024               | 0.7351(20)         | 0.44           |
| 0.032               | 0.7433(18)         | 0.36           |
| 0.040               | 0.7518(16)         | 0.33           |
| 0.008               | 0.7943(31)         | 0.64           |
| 0.016               | 0.8017(23)         | 0.68           |
| 0.024               | 0.8095(20)         | 0.57           |
| 0.032               | 0.8176(18)         | 0.50           |
| 0.040               | 0.8260(16)         | 0.47           |
TABLE V: Heavy-light $J^P = 0^+$ meson mass obtained from fitting to the hyperbolic cosine form in the range of $10 \leq t \leq 16$.

| $m_{\text{heavy}}$ | $m_{\text{light}}$ | $m_{\text{PS}}$ | $\chi^2$ per d.o.f. |
|-----------------|-----------------|----------------|-----------------|
| 0.024           | 0.656(16)       | 0.01           |
| 0.2             | 0.032           | 0.654(17)      | 0.01           |
|                 | 0.040           | 0.658(10)      | 0.01           |
|                 | 0.024           | 0.762(13)      | 0.02           |
| 0.3             | 0.032           | 0.760(10)      | 0.02           |
|                 | 0.040           | 0.764(8)       | 0.02           |
|                 | 0.024           | 0.854(14)      | 0.01           |
| 0.4             | 0.032           | 0.853(11)      | 0.01           |
|                 | 0.040           | 0.857(9)       | 0.01           |
|                 | 0.024           | 0.934(14)      | 0.01           |
| 0.5             | 0.032           | 0.932(10)      | 0.01           |
|                 | 0.040           | 0.936(8)       | 0.01           |
TABLE VI: Heavy-light $J^P = 1^+$ meson mass obtained from fitting to the hyperbolic cosine form in the range of $10 \leq t \leq 16$.

| $m_{\text{heavy}}$ | $m_{\text{light}}$ | $m_{\text{PS}}$ | $\chi^2$ per d.o.f. |
|-------------------|--------------------|----------------|-----------------------|
| 0.024             | 0.689(17)          | 0.004          |                       |
| 0.2               | 0.032              | 0.688(12)      | 0.002                 |
|                   | 0.040              | 0.692(10)      | 0.002                 |
|                   | 0.024              | 0.792(16)      | 0.011                 |
| 0.3               | 0.032              | 0.790(12)      | 0.006                 |
|                   | 0.040              | 0.793(10)      | 0.005                 |
|                   | 0.024              | 0.881(16)      | 0.017                 |
| 0.4               | 0.032              | 0.879(12)      | 0.010                 |
|                   | 0.040              | 0.881(9)       | 0.008                 |
|                   | 0.024              | 0.955(16)      | 0.02                  |
| 0.5               | 0.032              | 0.952(12)      | 0.01                  |
|                   | 0.040              | 0.955(9)       | 0.01                  |

TABLE VII: Summary of mass splitting results (in MeV).

| $\Delta_0$ | 444(36) | 533(90) |
|------------|--------|--------|
| $\Delta_1$ | 420(36) | 452(78) |
| $1^- - 0^-$ | 140.64(10) | 93(4) |
| $\Delta_{S0}$ | 348.4(9) | 411(40) |
| $\Delta_{S1}$ | 345.9(1.2) | 380(37) |
| $1^- - 0^-$ | 143.8(4) | 82(2) |
TABLE VIII: Summary of the heavy-light spectrum (in MeV).

| Meson ($J^P$) | Experiment | This work |
|---------------|-------------|-----------|
| $D^\pm (0^-)$ | 1869.4(5)   | 1867(4)   |
| $D^{*\pm} (1^-)$ | 2010.0(5)  | 1961(4)   |
| $D_0^+(0^+)$   | 2352(50)    | 2401(89)  |
| $D_1^+(1^+)$   | 2427(26)(25)| 2413(76)  |
| $D_5^\pm (0^-)$ | 1968.3(5)  | –         |
| $D_7^{*\pm} (1^-)$ | 2112.1(7)  | 2051(2)   |
| $D_9^{*\pm} (0^+)$ | 2317.4(9)  | 2379(40)  |
| $D_{11}^{*\pm} (1^+)$ | 2459.3(1.3)| 2431(37)  |

TABLE IX: A list of the past charm quark mass estimations from quenched lattice QCD with continuum extrapolation.

| Group                  | $m_c^{\overline{MS}}(m_c)$ GeV | action |
|------------------------|---------------------------------|--------|
| Kronfeld [95]          | 1.33(8)                         | Clover |
| Hornbostel et al. [96] | 1.20(4)(11)(2)                  | NRQCD  |
| Becirevic et al. [97]  | 1.26(4)(12)                     | Clover |
| Juge [98]              | 1.27(5)                         | Clover |
| Rolf and Sint [99]     | 1.301(34)                       | Clover |
| de Divitiis et al. [19]| 1.319(28)                       | NRQCD  |
| Nobes et al. [100]     | 1.22(9)                         | Fermilab |
| This work              | 1.24(1)(18)                     | DWF    |
TABLE X: Summary of quarkonium spectrum in lattice unit.

| $am_{\text{heavy}}$ | $0^{-+}$ | $1^{--}$ | $0^{++}$ | $1^{++}$ | $1^{-+}$ |
|---------------------|---------|---------|---------|---------|---------|
| 0.1                 | 0.4614(12) | 0.5113(24) | 0.604(11) | 0.632(9) | 0.656(17) |
| 0.2                 | 0.7118(10) | 0.7392(14) | 0.824(7) | 0.848(6) | 0.858(8) |
| 0.3                 | 0.9226(8) | 0.9406(11) | 1.026(6) | 1.046(5) | 1.053(7) |
| 0.4                 | 1.1008(8) | 1.1129(9) | 1.200(6) | 1.218(6) | 1.227(6) |
| 0.5                 | 1.2467(7) | 1.2541(8) | 1.350(6) | 1.364(7) | 1.377(5) |

TABLE XI: Summary of the charmonium spectrum (in MeV). The last row presents a separate estimate on the mass difference between $h_c$ and $\chi_{c1}$ obtained directly from themeson's propagator ratio: it is likely more reliable than the calculated excited mesons mass themselves which are underestimated.

| Charmonium ($J^{PC}$) | Experiment | This work |
|------------------------|------------|-----------|
| $\eta_c(0^{-+})$       | 2980(1)    | 2987(12)  |
| $J/\psi(1^{--})$       | 3096.916(11) | 3030(11) |
| $\chi_{c0}(0^{++})$    | 3415.19(34) | 3282(21) |
| $\chi_{c1}(1^{++})$    | 3510.59(12) | 3336(21) |
| $h_c(1^{-+})$          | not established | 3360(21) |
| $h_c - \chi_{c1}$      |            | 22(11)    |

TABLE XII: Fit expressions and various decay constant related values.

| fit | $a^{3/2} \sqrt{m_D f_D}$ | $a^{3/2} \sqrt{m_{D_s} f_{D_s}}$ | $\left( a^{3/2} \sqrt{m_B f_B} \right)_{\text{stat}}$ | $\left( a^{3/2} \sqrt{m_{B_s} f_{B_s}} \right)_{\text{stat}}$ | $\left( \sqrt{m_{B_s} f_{B_s}} \right)_{\text{stat}}$ |
|-----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| $a + bm_q$  | 0.0621(18)               | 0.0677(13)               | 1.090(19)               | 0.0718(31)               | 0.0773(25)               |
| $a + bm_q + cm_q^2$ | 0.0621(18)               | 0.0688(13)               | 1.107(19)               | 0.0718(31)               | 0.0783(21)               |
| $a + bm_q + cm_q \ln m_q$ | 0.0630(21)               | 0.0688(13)               | 1.092(26)               | 0.0725(39)               | 0.0783(21)               |
### TABLE XIII: List of the $B_{\text{PS}}^{\text{lat}}$ values.

| $am_{\text{heavy}}$ | $am_{\text{light}}$ | $B_{\text{PS}}^{\text{lat}}$ | $am_{\text{PS}}$ |
|---------------------|---------------------|-----------------------------|-----------------|
| 0.2                 | 0.016               | 0.792(10) 0.4839(14)        |                 |
|                     | 0.024               | 0.792(7)  0.4940(12)        |                 |
|                     | 0.032               | 0.795(6)  0.5042(11)        |                 |
|                     | 0.040               | 0.799(5)  0.5146(11)        |                 |
| 0.3                 | 0.016               | 0.830(11) 0.6033(15)        |                 |
|                     | 0.024               | 0.828(8)  0.6124(13)        |                 |
|                     | 0.032               | 0.830(6)  0.6218(11)        |                 |
|                     | 0.040               | 0.834(6)  0.6314(11)        |                 |
| 0.4                 | 0.016               | 0.855(12) 0.7027(15)        |                 |
|                     | 0.024               | 0.853(9)  0.7113(13)        |                 |
|                     | 0.032               | 0.855(7)  0.7202(12)        |                 |
|                     | 0.040               | 0.858(6)  0.7293(11)        |                 |
| 0.5                 | 0.016               | 0.874(13) 0.7865(23)        |                 |
|                     | 0.024               | 0.871(9)  0.7943(20)        |                 |
|                     | 0.032               | 0.873(7)  0.8024(18)        |                 |
|                     | 0.040               | 0.876(6)  0.8110(17)        |                 |

### TABLE XIV: Fit expressions and various bag parameter values.

| fit                  | $B_{D}$ | $B_{D_{s}}$ | $B_{D_{s}}/B_{D}$ | $(B_{B})^{\text{static}}$ | $(B_{B_{s}})^{\text{static}}$ | $(B_{B_{s}}/B_{B})^{\text{static}}$ |
|----------------------|---------|-------------|-------------------|---------------------------|-------------------------------|---------------------------------|
| $a + bm_{q}$         | 0.845(16) | 0.849(8)    | 1.001(12)         | 0.923(22)                 | 0.921(10)                     | 0.998(14)                      |
| $a + bm_{q} + cm_{q}^{2}$ | 0.859(24) | 0.848(7)    | 0.987(22)         | 0.940(33)                 | 0.919(9)                      | 0.977(28)                      |
| $a + bm_{q} + cm_{q} \ln m_{q}$ | 0.874(33) | 0.848(7)    | 0.970(32)         | 0.958(45)                 | 0.919(9)                      | 0.959(39)                      |

### TABLE XV: Fit expressions and various bag parameter values.

| fit                  | $f_{D_{s}}/f_{D}$ | $B_{D_{s}}/m_{D_{s}} f_{D}$ | $(B_{D_{s}}^{m_{D_{s}}} f_{D})^{2}$ | $B_{D}/B_{B}$ | $(B_{D_{s}}/B_{B})^{2}$ | $B_{B_{s}}/B_{B}$ | $(B_{B_{s}}^{m_{B_{s}}} f_{B})^{2}$ | $(B_{B_{s}}/B_{B})^{2}$ |
|----------------------|-------------------|------------------------------|--------------------------------------|---------------|----------------------------|-----------------|--------------------------------------|----------------------------|
| $a + bm_{q}$         | 1.064(29)         | 1.257(73)                    | 1.019(53)                            | 1.27(13)      |                             |                 |                                      |                            |
| $a + bm_{q} + cm_{q}^{2}$ | 1.071(23)         | 1.273(65)                    | 1.019(37)                            | 1.274(94)     |                             |                 |                                      |                            |
| $a + bm_{q} + cm_{q} \ln m_{q}$ | 1.048(31)         | 1.219(80)                    | 1.001(53)                            | 1.23(13)      |                             |                 |                                      |                            |
FIG. 1: Fifth-dimensional $s$-dependence of the three eigenvectors of $D_H$ with smallest eigenvalues (displayed in right- and left-projection pairs): $|\Psi^2_\alpha|$ (with $y$-axis in log scale) with $m_f \in \{0.03, 0.4, 0.5, 0.7\}$. The propagating states only arise for $am_f > 0.5$. 
FIG. 2: Effective mass plots with “close-to-quark-mass” simulation points: \( am_{\text{Heavy}} = 0.4 \) and \( am_{\text{light}} = 0.032 \).
FIG. 3: The spectrum of the $D_s$ (above) and $D$ (below) systems. The circles are our results with statistical errorbars and the horizontal lines correspond to experimental data with one $\sigma$ error.
FIG. 4: The summary plots of the previous lattice estimate for $\Delta_{sJ}$ including ours.
FIG. 5: The heavy quark mass dependence of parity splitting in lattice units, where $am_{\text{light}} = am_{\text{strange}}$. The stars denote $am_{\text{heav}} = am_{\text{charm}}$.

FIG. 6: The light quark mass dependence of parity splitting in lattice units where $am_{\text{heavy}} = am_{D_s/m_{\rho}}$. The stars denote $am_{\text{light}} = am_{\text{strange}}$. 
FIG. 7: Effective mass plots in the heavy-heavy sector at the simulation point $am_{\text{Heavy}} = 0.4$.

FIG. 8: The spectrum of the charmonium system. The circles are our results with statistical errorbars and the horizontal lines correspond to experimental values.
FIG. 9: The light (above) and heavy (below) quark mass dependence of $a^{3/2} \sqrt{m_D} f_D$ with the chiral expression: $\sqrt{m_{Qq}} f_{Qq} = F_1 + F_2 m_q + (F_3 m_q) \ln m_q$ and linear (in $1/m_{\text{Heavy}}$) fit respectively. The top figure displays fixed $am_{\text{Heavy}} \in \{0.2, 0.3, 0.4, 0.5\}$ from top to bottom. In the bottom figure, the triangle (blue) point represents the physical $D$ meson point, 0.0630(21), and the pentagon (red) point represents the static quark limit point of 0.0725(39).
FIG. 10: The light (above) and heavy (below) quark mass dependence of $a^{3/2} \sqrt{m_{Ds}} f_{Ds}$ with the chiral expression: $\sqrt{m_{Qq}} f_{Qq} = F_1 + F_2 m_q + (F_3 m_q) \ln m_q$ and linear (in $1/m_{\text{Heavy}}$) fit respectively. In the bottom figure, the triangle (blue) point represents the physical $D_s$ meson point, 0.0688(13), and the pentagon (red) point represents the static quark limit point of 0.0783(21).
FIG. 11: The heavy quark mass dependence of the $SU(3)$ breaking ratio $\frac{\sqrt{m_{D_s}}}{\sqrt{m_{D_{s1}}}}$ with chiral interpolation/extrapolation (Eq. 12) in light quark mass (black points). The triangle (blue) point represents the physical charm quark point, 1.092(26), and the pentagon (red) point represents the static quark limit point of 1.076(53).
FIG. 12: The time dependence of pseudoscalar meson bag parameters at fixed light quark mass: 0.016, 0.024, 0.032, 0.040 from top to bottom. In each subgraph, we display the heavy quark mass dependence of $B_{PS}$. 
FIG. 13: The light (top) and heavy (bottom) quark mass dependence of $B_{PS}$ with the chiral:

$$B_{Qq} = B_1 + B_2 m_q + B_3 m_q \ln m_q$$

and linear (in $1/M$) fit respectively. The top figure displays light quark mass interpolation to $-m_{res}$ at fixed $am_{heavy}$: 0.2, 0.3, 0.4, 0.5 from top to bottom. In the bottom figure, the triangle (blue) point represents the physical $D$ meson point and the pentagon (red) point represents the static quark limit point.
FIG. 14: The light (top) and heavy (bottom) quark mass dependence of $B_{PS}$ with the chiral:

$$B_{Qq} = B_1 + B_2 m_q + B_3 m_q \ln m_q$$

and linear (in $1/M$) fit respectively. The top figure displays light quark mass interpolation to $m_{strange}$ at fixed $am_{heavy}$: 0.2, 0.3, 0.4, 0.5 from top to bottom. In the bottom figure, the triangle (blue) point represents the physical $D_s$ meson point and the pentagon (red) point represents the static quark limit point.
FIG. 15: The $B_{PS}^{\text{lat}}$ as a function of the pseudoscalar meson. The circle points are the data points and the black line is the fit of the form: $B_{PS}(m_{PS}) = B_1 + B_2 m_{PS}^2 + B_3 m_{PS}^2 \ln m_{PS}^2$. The star (blue) point is the $B_{D}^{\text{lat}}$ point obtained from Fig.13, the triangle (red) point is the $B_{D}^{\text{lat}}$ from quadratic fit, and the pentagon (purple) point is the $B_{D}^{\text{lat}}$ from linear fit.