ON INDEXED ACTIONS

a mio padre

CLAUDIO PISANI

ABSTRACT.

We present some laws relating the Cat-indexed categories of left, right and bi-actions: by defining \((A \downarrow M)x = Mx^{Ax}\) one gets a biclosed monoidal action of Set
\(^{X^{op}}\) on (Set
\(^{X}\))\(^{op}\), while \(BX\) and Cat\(/
\(X\)\) act (partially) on their opposites by exponentials; both the inclusions \((BX, BX) \to (Set
\(^{X^{op}}\), Set
\(^{X}\)) \to (Cat/\(X\), Cat/\(X\))\) preserve the (cartesian) monoidal structures and the actions, and the same holds for substitutions along functors. These strong morphisms of strong indexed monoidal actions have in fact a wider range of applications; in particular, replacing Set with any (co)complete symmetric monoidal closed category \(V\), we consider the pair of biclosed indexed monoidal actions \((V
\(^{X^{op}}\), V
\(^{X}\))\)\(^{op}\) and its formal relationships with bi-actions and constant actions.

Some of the resulting laws also hold in a fragment of biclosed bicategory (with an object supporting a symmetric monoidal category) and are taken, in the second part, as the basis for developing some abstract category theory. Finally, we add Set
\(^{X^{op} \times X}\) to the picture and give a symmetrical version of the comprehension adjunction.

1. Introduction

We are concerned with actions on a twofold level: both as the main object of study and as the main tool to be used. On the one hand, we are interested in actions of categories \(X\) with respect to composition (presheaves or more generally functors valued in a monoidal closed \(V\)); the relevant morphisms of \(X\)-actions are the usual ones: natural transformations. On the other hand, the actions of monoidal categories with respect to the (tensor) product also play a major role in our technical development; now the relevant morphisms involve the acting category as-well. Consider the inclusion \(i_{\ell}^{X} : Set
\(^{X^{op}}\) \to Cat/\(X\)\) and \(i_{r}^{X} : Set
\(^{X}\) \to Cat/\(X\)\) via discrete (op)fibrations and let \(BX\) be their pullback, that is discrete bifibrations with projections \(j_{\ell}^{X} : BX \to Set
\(^{X^{op}}\)\) and \(j_{r}^{X} : BX \to Set
\(^{X}\)\) and \(k_{X} := i_{\ell}^{X}j_{\ell}^{X} = i_{r}^{X}j_{r}^{X}\). Exponentials (and products) in \(BX\) are computed as in Cat/\(X\): \(k_{X}(C^{B}) \cong (k_{X}C)^{k_{X}B}\). Otherwise stated, the pair \((k_{X}, k_{X})\) (to be precise, \((k_{X}, k_{X}^{op})\)) is an action morphism from the exponential action of \(BX\) on (Set
\(^{X^{op}}\))\(^{op}\) to the (partial) one of Cat/\(X\) on (Cat/\(X\))\(^{op}\). If we define the action \(\_\uparrow\) of Set
\(^{X^{op}}\) on (Set
\(^{X}\))\(^{op}\) pointwise by

\[(A \uparrow M)x := Mx^{Ax} ; \quad (A \uparrow M)f := Mf \circ \circ Af : Mx^{Ax} \to My^{Ay}\]
then \((k_X, k_X)\) factors through \((j_X^f, j_X^r)\) and \((i_X^f, i_X^r)\) which are also action morphisms:

\[
j_X^f(C^B) \cong j_X^fB \sqcap j_X^rC \quad ; \quad i_X^r(A \sqcup M) \cong (i_X^rM)^{i_X^fA}
\]

Furthermore, any functor \(f : X \to Y\) induces action morphisms at each of the three levels:

\[
f^*(C^B) \cong (f^*C)^{f^*B} \quad ; \quad f^r(A \sqcup M) \cong f^rA \sqcup f^rM \quad ; \quad f^{-1}(q^p) \cong (f^{-1}q)^{f^{-1}p}
\]

A similar but not strictly action-like situation is the following. Let \(X\) be a topological space and denote by \(\mathcal{O}X, \mathcal{C}X\) and \(\mathcal{B}X\) the posets of open, closed and clopen parts respectively, with inclusions \(j_X^f : \mathcal{B}X \to \mathcal{O}X, j_X^r : \mathcal{B}X \to \mathcal{C}X\), \(i_X^f : \mathcal{O}X \to \mathcal{P}X\) and \(i_X^r : \mathcal{C}X \to \mathcal{P}X\). The inclusion \(k_X : \mathcal{B}X \to \mathcal{P}X\) is a Boolean algebra morphism, preserving in particular the relative complement \(A \Rightarrow B = \mathcal{C}A \cup B\), which is the exponential action of \(\mathcal{B}X\) on \((\mathcal{B}X)^{\mathcal{op}}\) and of \(\mathcal{P}X\) on \((\mathcal{P}X)^{\mathcal{op}}\). The same formula gives an action \(\gamma : \mathcal{O}X \times (\mathcal{C}X)^{\mathcal{op}} \to (\mathcal{C}X)^{\mathcal{op}}\), and the action morphism \((k_X, k_X)\) factors through the action morphisms \((j_X^f, j_X^r)\) and \((i_X^f, i_X^r)\).

Again, any continuous map \(f : X \to Y\) also induce action morphisms at the three levels. The action \(\gamma : \mathcal{O}X \times (\mathcal{C}X)^{\mathcal{op}} \to (\mathcal{C}X)^{\mathcal{op}}\) is bicalled: the adjoint action \(\mathcal{O}X \times \mathcal{C}X \to \mathcal{C}X\) and the enrichment \((\mathcal{C}X)^{\mathcal{op}} \times \mathcal{C}X \to \mathcal{O}X\) are given respectively by \(\diamond_X^r(i_X^rA \cap i_X^rM)\) (the “closure” reflection in \(\mathcal{C}X\) of the product in \(\mathcal{P}X\)) and by \(\square_X^r(i_X^rM) \Rightarrow i_X^rN\) (the “interior” coreflexion in \(\mathcal{O}X\) of the exponential in \(\mathcal{P}X\)).

Since \(A \sqcup M\) is, in disguise, simply the intersection of \(A\) with the complement of \(M\), the associated right adjoints are simply exponentials in \(\mathcal{O}X\) in disguise. While this is not the case for the set valued context, the formulas

\[
\diamond_X^r(i_X^rA \times_X i_X^rM) \quad ; \quad \square_X^r(i_X^rN)^{i_X^rM}
\]

still give the associated right adjoints

\[
\circ : \text{Set}^{X^{\mathcal{op}}} \times \text{Set}^X \to \text{Set}^X \quad ; \quad \triangleright : (\text{Set}^X)^{\mathcal{op}} \times \text{Set}^X \to \text{Set}^{X^{\mathcal{op}}}
\]

(where \(i_X^r \Rightarrow \square_X^r : \text{Cat}/X \to \text{Set}^{X^{\mathcal{op}}}\) and \(\diamond_X^r \Rightarrow i_X^r \circ i_X^r : \text{Set}^X \to \text{Cat}/X\)).

In general we say that a bicalled monoidal action \(\gamma\) of \(\mathcal{V}\) on \(\mathcal{M}^{\mathcal{op}}\) is a “complemented category” and a morphism of complemented categories is a pair of functors \(f^\ell : \mathcal{V} \to \mathcal{V}'\) and \(f^r : \mathcal{M} \to \mathcal{M}'\) which preserve the monoidal structures and the actions and have right and left adjoint respectively. From \(f^*(A \sqcup M) \cong f^rA \sqcup f^rM\), we then get by adjunction \(\exists^r(f^\ell A \otimes M) \cong A \otimes f^rM\), which is a form of Frobenius law. For instance, for a topological space \(X\) the morphism \((i_X^r, i_X^r) : (\mathcal{O}X, \mathcal{C}X) \to (\mathcal{P}X, \mathcal{P}X)\) gives \(\diamond_X^r(i_X^rA \cap P) = A \otimes \diamond_X^rP = \diamond_X^r(i_X^rA \cap i_X^r \diamond_X^r P)\), which includes the fact that density is preserved on open parts. Similarly, for a category \(X\), the morphism \((i_X^r, i_X^r) : (\text{Set}^{X^{\mathcal{op}}}, \text{Set}^X) \to (\text{Cat}/X, \text{Cat}/X)\) gives \(\diamond_X^r(i_X^rA \times_X P) \cong A \otimes \diamond_X^rP \cong \diamond_X^r(i_X^rA \times_X i_X^r \diamond_X^r P)\), which is strictly related to the stability of final map with respect to discrete opfibrations (see [Pisani, 2008]).

As just sketched, left and right actions are united both by the (indexed) inclusion in \(\text{Cat}/X\) and by sharing the (indexed) subcategory \(\mathcal{B}X\). (They are also united by the Isbell adjunction, so that they share \(X\) as well, but this fact does not seem to be strictly related
to our present approach). While the first aspect can be useful for certain calculations, from an abstract point of view it has the drawback that the categories $\text{Cat}/X$ are not closed. Anyway, it is possible to capture the relevant formal laws of the second aspect by taking in account the fact that the functors $\odot$ and $\times$ on the one side and $\triangleright$ and $\triangleright$ on the other side collapse when one argument is restricted to $BX$. Thus we take the morphisms of indexed complemented categories $(j^L_X, j^R_X) : (BX, BX) \to (\text{Set}^{X^{op}}, \text{Set}^X)$ as the basis for an abstraction in which $B1$ has the role of “internal truth values category”: it turns out that all the categories and adjunctions involved in the definition are enriched in it. The abstraction includes a sort of $\mathcal{V}$-relative category theory, for a symmetric monoidal closed (co)complete category $\mathcal{V}$: the left and the right actions of a category $X$ on $\mathcal{V}_0$ have monoidal structures induced pointwise by that of $\mathcal{V}$, and the action of each of them on the opposite of the other one is induced pointwise by the internal hom of $\mathcal{V}$.

Further abstracting, we are naturally led to consider the concept of “indexed pair” $(\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})$ over a category $\mathcal{C}$ with a (not necessarily terminal) object $1 \in \mathcal{C}$. $\mathcal{L}1$ and $\mathcal{R}1$ are isomorphic and have a symmetric monoidal closed structure $\mathcal{V}$, $\mathcal{L}X$ and $\mathcal{R}X$ have quantifications and are enriched, powered and copowered over $\mathcal{V}$; furthermore, there are “mixed tensor” bifunctors $\ast_X : \mathcal{L}X \times \mathcal{R}X \to \mathcal{V}$ with enriched “absolute complement” adjoints $A \ast_X - \vdash A \triangleright_X - : \mathcal{V} \to \mathcal{R}X$ and $- \ast_X M \dashv M \triangleright_X - : \mathcal{V} \to \mathcal{L}X$, and substitution functors preserve powers, copowers and complements:

\[
\begin{array}{c c c c}
\{V \otimes^L_X A, B\}^L_X & \{V \otimes^L_X M, N\}^r_X & \{A \ast_X M, V\} \\
\{V, \{A, B\}\}^L_X & \{V, \{M, N\}\}^r_X & \{A, M \triangleright_X V\}^r_X \\
\{A, [V, B]\}^L_X & \{M, [V, N]\}^r_X & \{M, A \triangleright_X V\}^r_X \\
\end{array}
\]

\[
\begin{array}{c c c c}
f^L(V \otimes^L_X A) & f^L[V, A]^L_X & f^L(A \triangleright_X^L V) \\
V \otimes^L_X f^L A & [V, f^L A]^L_X & f^L(A \triangleright_X^L V) \\
\end{array}
\]

\[
\begin{array}{c c c c}
f^r(V \otimes^r_X M) & f^r[V, M]^r_X & f^r(M \triangleright_X^r V) \\
V \otimes^r_X f^r M & [V, f^r M]^r_X & f^r(M \triangleright_X^r V) \\
\end{array}
\]

\[
\begin{array}{c c c c}
\{f^L A, B\}^L_X & \{A, f^L B\}^L_X & \{f^r M, N\}^r_X & \{M, f^r N\}^r_X \\
\{A, \forall^L_X B\}^L_X & \{\exists^L_X A, B\}^L_X & \{M, \forall^r_X N\}^r_X & \{\exists^r_X M, N\}^r_X \\
\end{array}
\]

The same laws hold in a biclosed bicategory $\mathcal{M}$ (for instance, of $\mathcal{V}$-profunctors) with a selected object $1$ which supports a symmetric monoidal category, by taking as $\mathcal{C}$ the “maps” (right adjoint arrows) in $\mathcal{B}$ and posing $\mathcal{L}X := \mathcal{B}(X, 1)$ and $\mathcal{R}X := \mathcal{B}(1, X)$.

In Section 3, which can be red independently from the rest of the paper, we show that these axioms (along with some adequacy hypothesis) allow us to define weighted limits, (pointwise) Kan extensions, fully faithful, dense and absolutely dense maps and (if $1 \in \mathcal{C}$
is actually terminal) conical limits and final maps and to prove some of their familiar (and less familiar) properties with straightforward calculations.

In the last section we came back to ordinary category theory, presenting a generalization of the comprehension adjunction between categories over $X$ and presheaves on $X$ [Lawvere, 1970]. The categories $X$ and $X^{op}$ are themselves united by their inclusions in the groupoidal reflection $\overline{X}$, which induce the inclusions $j_X^\ell$ and $j_X^r$ of biactions in presheaves. On the other hand, they can be also united by their product and the projections of $X^{op} \times X$ induce the (dummy) inclusion of $\text{Set}^{X^{op}}$ and $\text{Set}^X$ in $\text{Set}^{X^{op} \times X}$; the discrete (op)fibrations inclusions $i_X^\ell$ and $i_X^r$ factor through them and a “diagonal” comprehension functor giving the “extension” $i_X H = \{ x \in X | H(x, x) \}$ in $\text{Cat}/X$ of the “predicate” $H \in \text{Set}^{X^{op} \times X}$. In fact, the adjunctions $\otimes_X \dashv i_X^\ell : \text{Set}^{X^{op}} \to \text{Cat}/\overline{X}$ and $\otimes_X \dashv i_X^r : \text{Set}^X \to \text{Cat}/\overline{X}$ factor through $\otimes_X \dashv i_X : \text{Set}^{X^{op} \times X} \to \text{Cat}/\overline{X}$, where $\otimes_X p \cong \exists_{p^{op} \times p} \text{hom}_p$, for $p : P \to X$.

The present paper is a development of previous work by the author (see in particular [Pisani, 2010] and references therein) but can be read independently.

2. Complemented categories

Monoidal actions have been considered by several authors in different contexts. In this section we show how they can be usefully seen as monoidal categories with a “complement” functor and present various instances of indexed monoidal actions.

2.1. Definition. A **complemented category** $(\mathcal{V}, \mathcal{M}) = (\mathcal{V}, \otimes, I; \mathcal{M}, \rhd)$ is a symmetric monoidal closed category $(\mathcal{V}, \otimes, I)$ endowed with a complement in a category $\mathcal{M}$, that is a biclosed monoidal action $\rhd : \mathcal{V} \times \mathcal{M}^{op} \to \mathcal{M}^{op}$ of $\mathcal{V}$ on $\mathcal{M}^{op}$. A morphism $(f^\ell, f^r) : (\mathcal{V}, \mathcal{M}) \to (\mathcal{V}', \mathcal{M}')$ is a pair of functors which preserve the monoidal structure and the action up to isomorphisms and which have a right and a left adjoint respectively.

Thus, complemented categories and their morphisms are summarized (neglecting coherence and symmetry) by the following laws (natural isomorphisms):

\[
\begin{align*}
I \otimes A & \cong A; & (A \otimes B) \otimes C & \cong A \otimes (B \otimes C); & I \rhd M & \cong M; & (A \otimes B) \rhd M & \cong A \rhd (B \rhd M) \\
\mathcal{V}(A \otimes B, C) & \cong \mathcal{V}(A, [B, C]); & \mathcal{M}^{op}(A \rhd M, N) & \cong \mathcal{M}(N, A \rhd M); & \mathcal{V}(B, [A, C]) & \cong \mathcal{V}(A, N \triangleright M); & \mathcal{M}^{op}(A \otimes N, M) & \cong \mathcal{M}(A \otimes N, M) \\
\mathcal{V}(A, [B, C]) & \cong \mathcal{V}(A, N \triangleright M); & \mathcal{V}(A, N \triangleright M) & \cong \mathcal{V}(A, N \triangleright M)
\end{align*}
\]

\[
\begin{align*}
f^\ell I & \cong I; & f^\ell (A \otimes B) & \cong f^\ell A \otimes f^\ell B; & f^r (A \rhd M) & \cong f^r A \rhd f^r M
\end{align*}
\]
\[
\begin{array}{c}
\frac{\mathcal{V}(f^\ell A, B)}{\mathcal{V}(A, \forall^\ell_f B)} \quad \frac{\mathcal{M}'(M, f^r N)}{\mathcal{M}(\exists^r_f M, N)}
\end{array}
\tag{4}
\]

The laws (1) and (3) yield, by the adjunctions (2) and (4), the equivalent ones

\[
\begin{array}{c}
\frac{[I, A]}{A} \quad \frac{[A \otimes B, C]}{[A, [B, C]]}
\end{array}
\tag{5}
\]

\[
\begin{array}{c}
\frac{I \otimes M}{M} \quad \frac{(A \otimes B) \otimes M}{A \otimes (B \otimes M)} \quad \frac{(A \otimes M) \triangleright N}{[A, M \triangleright N]} \quad \frac{M \triangleright (A \triangleright N)}{}
\end{array}
\tag{6}
\]

\[
\begin{array}{c}
\frac{[A, \forall^\ell_f B]}{\forall^\ell_f [f^\ell A, B]} \quad \frac{A \otimes \exists^\ell_f M}{\exists^\ell_f (f^\ell A \circ M)} \quad \frac{\exists^r_f M \triangleright N}{\forall^r_f (M \triangleright f^r N)}
\end{array}
\tag{7}
\]

2.2. Remarks. Note in particular that

1. \(\otimes : \mathcal{V} \times \mathcal{M} \rightarrow \mathcal{M}\) is itself a biclosed monoidal action of \(\mathcal{V}\) on \(\mathcal{M}\).

2. \(\mathcal{M}\) is enriched (via \(\triangleright\)), powered and copowered over \(\mathcal{V}\) (so as \(\mathcal{V}\) itself); indeed, the adjunctions relating \(\triangleright\), \(\otimes\) and \(\triangleright\) are enriched over \(\mathcal{V}\) (so as those relating \(\otimes\) and \([-, -]\)).

3. Given a morphism \((f^\ell, f^r) : (\mathcal{V}, \mathcal{M}) \rightarrow (\mathcal{V}', \mathcal{M}')\), \(\mathcal{V}'\) and \(\mathcal{M}'\) are also enriched (via \(\forall^\ell_f [-, -]'\) and \(\forall^r_f (- \triangleright' -)\) respectively), powered and copowered over \(\mathcal{V}\).

4. The adjunctions \(f^\ell \dashv \forall^\ell_f\) and \(\exists^r_f \dashv f^r\) are also enriched over \(\mathcal{V}\).

2.3. Examples.

1. Any symmetric monoidal closed category \(\mathcal{V}\) gives rise to a complemented category \((\mathcal{V}, \mathcal{V})\), with \(A \triangleright B := [A, B]\). We say that \((\mathcal{V}, \mathcal{M})\) is \textbf{standard} if it is isomorphic to \((\mathcal{V}, \mathcal{V})\). Note that \((\mathcal{V}, \mathcal{M})\) is standard iff there is an isomorphism \(i : \mathcal{M} \rightarrow \mathcal{V}_0\) such that \(i(- \triangleright -) \cong [-, i-]\). A morphism of standard complemented categories is essentially a strong morphism of monoidal categories preserving also the closed structure.

In particular, any Heyting algebra gives rise to a complemented category where \(A \triangleright B\) is the exponential \(A \Rightarrow B\), that is the usual pseudocomplement of \(A\) relative to \(B\).
2. Any symmetric monoidal closed category \( \mathcal{V} \) gives rise to a complemented category \( (\mathcal{V}, \mathcal{V}^{\text{op}}) \), with \( A \dashv B := A \otimes B \). We say that \( (\mathcal{V}, \mathcal{M}) \) is topological if it is isomorphic to \( (\mathcal{V}, \mathcal{V}^{\text{op}}) \). Again, \( (\mathcal{V}, \mathcal{M}) \) is topological iff there is an isomorphism \( i : \mathcal{M} \rightarrow \mathcal{V}^{\text{op}} \) such that \( i(- \circ -) \cong - \otimes i- \). A morphism of topological complemented categories is essentially a strong morphism of monoidal categories.

In particular, any topological space \( X \) gives rise to a topological complemented category with \( \mathcal{V} := \mathcal{O}X \) (the Heyting algebra of open parts), \( \mathcal{M} := \mathcal{C}X \cong (\mathcal{O}X)^{\text{op}} \) (the poset of closed parts) and \( A \dashv M \) is given by the exponential \( A \Rightarrow M \) in \( \mathcal{P}X \), that is the relative complement \( \neg A \cup M \) in \( \mathcal{P}X \). Any continuous map gives rise to a morphism of complemented categories.

3. If \( \mathcal{V} \) is *-autonomous, then the standard \( (\mathcal{V}, \mathcal{V}) \) is isomorphic to the topological \( (\mathcal{V}, \mathcal{V}^{\text{op}}) \), via the isomorphism \((-)^* : \mathcal{V} \rightarrow \mathcal{V}^{\text{op}} \). (Conversely, if \( (\mathcal{V}, \mathcal{M}) \) is both standard and topological, then \( \mathcal{V} \) has a *-autonomous structure.)

Often complemented categories occur in a symmetrical fashion:

2.4. Definition. A complemented pair \( (\mathcal{L}, \mathcal{R}) = (\mathcal{L}, \otimes^\ell, \otimes^r, \mathcal{R}, \otimes^r, \mathcal{R}, \mathcal{L}, \mathcal{L}, \mathcal{R}, \mathcal{L}) \) consists of two symmetric monoidal closed categories, each one endowed with a complement in (the underlying category of) the other one; that is it consists of two complemented categories \( (\mathcal{L}, \mathcal{R}_0) \) and \( (\mathcal{R}, \mathcal{L}_0) \). A morphism \( (f^\ell, f^r) : (\mathcal{L}, \mathcal{R}) \rightarrow (\mathcal{L}', \mathcal{R}') \) is a pair of functors such that both \( (f^\ell, f^r) : (\mathcal{L}, \mathcal{R}_0) \rightarrow (\mathcal{L}', \mathcal{R}_0') \) and \( (f^r, f^\ell) : (\mathcal{R}, \mathcal{L}_0) \rightarrow (\mathcal{R}', \mathcal{L}_0') \) are morphisms of complemented categories.

Thus, half of the laws summarizing complemented pairs and their morphisms are

$$
\begin{align*}
\ell \otimes A & \quad ; \quad (A \otimes B) \otimes C & \quad ; \quad \ell \circ M & \quad ; \quad (A \otimes B) \circ M \\
A & \quad ; \quad A \otimes (B \otimes C) & \quad ; \quad M & \quad ; \quad A \circ (B \circ M)
\end{align*}
$$

$$
\begin{align*}
\mathcal{L}(A \otimes B, C) & \quad ; \quad \mathcal{R}^{\text{op}}(A \circ M, N) & \quad ; \quad \mathcal{R}(N, A \circ M) \\
\mathcal{L}(A, [B, C]^\ell) & \quad ; \quad \mathcal{R}^{\text{op}}(M, A \otimes N) & \quad \Rightarrow \quad \mathcal{R}(A \otimes N, M) \\
\mathcal{L}(B, [A, C]^r) & \quad ; \quad \mathcal{L}(A, N \triangleright^\ell M) & \quad \mathcal{L}(A, N \triangleright^r M)
\end{align*}
$$

$$
\begin{align*}
\ell \circ f^\ell & \quad ; \quad \ell \circ (A \otimes B) & \quad ; \quad \ell \circ (A \circ M) \\
\ell & \quad ; \quad \ell \circ A \otimes f^\ell B & \quad ; \quad \ell \circ A \circ f^\ell M
\end{align*}
$$

$$
\begin{align*}
\mathcal{L}'(f^\ell A, B) & \quad ; \quad \mathcal{R}'(M, f^r N) \\
\mathcal{L}(A, \triangleright^\ell f B) & \quad ; \quad \mathcal{R}(M, \triangleright^r f N) \\
\mathcal{L}(\exists^\ell f A, B) & \quad \mathcal{R}(\exists^r f M, N)
\end{align*}
$$

and the other half is obtained by exchanging \( \mathcal{L} \) and \( \mathcal{R} \) and the superscripts \( \ell \) and \( r \).
2.5. Definition. A $\mathcal{C}$-indexed complemented category (resp. pair) is a pseudofunctor from $\mathcal{C}^{\text{op}}$ to the category of complemented categories (resp. pairs):

$$(\mathcal{V}X, \mathcal{M}X; X \in \mathcal{C}) = (\mathcal{V}_X, \otimes_X, I_X; \mathcal{M}_X, \cap_X; X \in \mathcal{C})$$

$$(\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C}) = (\mathcal{L}_X, \otimes_X^l, I_X^l; \mathcal{R}_X, \otimes_X^r, I_X^r; \cap_X^l, \cap_X^r; X \in \mathcal{C})$$

A morphism of $\mathcal{C}$-indexed complemented categories (resp. pairs) is a family $(t_X^l, t_X^r; X \in \mathcal{C})$ of morphisms of complemented categories (resp. pairs) such that the obvious squares commute up to isomorphisms.

The following proposition gives a standard way to construct $\mathbf{Cat}$-indexed complemented categories (or pairs):

2.6. Proposition. If $(\mathcal{V}, \mathcal{M})$ is a complemented category with $\mathcal{V}_0$ complete and cocomplete then, for any $X \in \mathbf{Cat}$, $(\mathcal{V}, \mathcal{M})^X = (\mathcal{V}_0^X, \mathcal{M}_0^X)$ has also a complemented category structure and any functor $X \to Y$ gives rise to a morphism $(\mathcal{V}, \mathcal{M})^Y \to (\mathcal{V}, \mathcal{M})^X$. If $(\mathcal{V}, \mathcal{M})$ is topological, so is also $(\mathcal{V}, \mathcal{M})^X$. If $X$ is a groupoid and $(\mathcal{V}, \mathcal{M})$ is standard, so it is also $(\mathcal{V}, \mathcal{M})^X$.

Proof. The monoidal structure on $\mathcal{V}_0^X$ and the action on $(\mathcal{M}_0^X)^{\text{op}} \cong (\mathcal{M}^{\text{op}})^X$ are inherited “pointwise” by that of $(\mathcal{V}, \mathcal{M})$. Thus $A \cap M$ is the diagonal $X \to X \times X$ followed by $A \times M$ and by the internal hom $[-,-]$ of $\mathcal{V}$. The complement action is biclosed due to the (co)completeness of $\mathcal{V}$. Any functor $f : X \to Y$ gives rise, via substitution, to a morphism of complemented categories $(\mathcal{V}, \mathcal{M})^Y \to (\mathcal{V}, \mathcal{M})^X$, due to the pointwise nature of the structural operations. The rest can be seen by a routine check.

We henceforth tacitly assume that the symmetric monoidal closed categories $\mathcal{V}, \mathcal{L}$ and $\mathcal{R}$ underlying the complemented categories and the complemented pairs are complete and cocomplete.

2.7. Corollary. Any complemented category (resp. pair) gives rise to a $\mathbf{Cat}$-indexed complemented category (resp. pair).

2.8. Examples.

1. A locally cartesian closed category $\mathcal{C}$ gives rise to the (standard) indexed complemented pair $(\mathcal{C}/X, \mathcal{C}/X; X \in \mathcal{C})$. Substitution along $f : X \to Y$ in $\mathcal{C}$ is the morphism of (cartesian and standard) complemented pairs $(f^{-1}, f^{-1}) : (\mathcal{C}/Y, \mathcal{C}/Y) \to (\mathcal{C}/X, \mathcal{C}/X)$ given by pullback.

2. A topos $\mathcal{C}$ gives rise to the (standard) indexed complemented pair $(\mathcal{P}X, \mathcal{P}X; X \in \mathcal{C})$.

3. By applying Corollary 2.7 to the standard complemented pair $(\mathcal{V}, \mathcal{V})$, we get the indexed complemented pair $(\mathcal{V}_0^\mathcal{X}, \mathcal{V}_0^\mathcal{X}; X \in \mathbf{Cat})$, with $A \cap_M M$ defined as in the proof of Proposition 2.6 and $M \cap_M A$ symmetrically as the diagonal $X^{\text{op}} \to X^{\text{op}} \times X^{\text{op}}$ followed by $M^{\text{op}} \times A$ and by the internal hom $[-,-]$ of $\mathcal{V}$. For any functor $f : X \to Y$, the morphism of complemented pairs $(f^\mathcal{X}, f^\mathcal{X}) : (\mathcal{V}_0^\mathcal{X}, \mathcal{V}_0^\mathcal{X}) \to (\mathcal{V}_0^\mathcal{X}, \mathcal{V}_0^\mathcal{X})$ is obtained by substituting $f^{\text{op}}$ in $A : X^{\text{op}} \to \mathcal{V}$ and $f$ in $M : X \to \mathcal{V}$.
4. As an instance of example (3) above, for \( \mathcal{V} = \text{Set} \) we get the indexed complemented category \( (\text{Set}^{\mathcal{V}^{op}}, \text{Set}^{\mathcal{V}}; X \in \text{Cat}) \), where \( A \cap X : X \to \text{Set} \) is defined “pointwise” by \( (A \cap X)_x := \text{Set}(Ax, Mx) \). Any functor \( f : X \to Y \) gives rise, via substitution, to a morphism \( (\text{Set}^{\mathcal{V}^{op}}, \text{Set}^{\mathcal{V}}) \to (\text{Set}^{\mathcal{V}^{op}}, \text{Set}^{\mathcal{V}}) \). If \( X \) is a groupoid, then \( (\text{Set}^{\mathcal{V}^{op}}, \text{Set}^{\mathcal{V}}) \) is standard; indeed, the inverse functor \( (-)^{-1} : X^{op} \cong X \) induces \( s : \text{Set}^{\mathcal{V}} \cong \text{Set}^{\mathcal{V}^{op}} \), and \( A \cap M \) gives the exponential in \( \text{Set}^{\mathcal{V}^{op}} \) (modulo \( s \)):

\[
s(A \cap M) \cong A \Rightarrow sM
\]  

The inclusions (via discrete fibrations and opfibrations) \( i^l_X : \text{Set}^{\mathcal{V}^{op}} \to \text{Cat}/X \) and \( i^r_X : \text{Set}^X \to \text{Cat}/X \) form a morphism

\[
(\text{Set}^{\mathcal{V}^{op}}, \text{Set}^X; X \in \text{Cat}) \to (\text{Cat}/X, \text{Cat}/X; X \in \text{Cat})
\]

of indexed complemented pairs with a partial codomain (since \( \text{Cat}/X \) is not closed in general). Indeed, \( i^r_X(A \cap M) \) is the exponential \( i^r_X M i^l_X A \) in \( \text{Cat}/X \) and there are adjunctions \( \Diamond_X^l \Rightarrow i^l_X \Rightarrow \Box_X^r \) and \( \Diamond_X^l \Rightarrow i^r_X \Rightarrow \Box_X^r \).

The pullback (intersection) \( i^l_X \times_X i^r_X \) gives the category \( BX \) of discrete bifibrations with projections (inclusions) \( j^l_X : BX \to \text{Set}^{\mathcal{V}^{op}} \) and \( j^r_X : BX \to \text{Set}^X \) (as presheaves which act by bijections). The indexed inclusion \( (j^l_X, j^r_X; X \in \text{Cat}) \) gives, by (12), another instance of morphism of indexed complemented pairs:

\[
(BX, BX; X \in \text{Cat}) \to (\text{Set}^{\mathcal{V}^{op}}, \text{Set}^X; X \in \text{Cat})
\]  

Composing \( (j^l_X, j^r_X) \) with \( (i^l_X, i^r_X) \) one gets an inclusion \( (k^l_X, k^r_X) \)

\[
(BX, BX; X \in \text{Cat}) \to (\text{Cat}/X, \text{Cat}/X; X \in \text{Cat})
\]

of standard indexed complemented pairs (with a partial codomain).

5. A two-valued correspective of example (4) above is given by the morphisms of (topological) indexed complemented pairs

\[
(BX, BX; X \in \text{Pos}) \to (DX, UX; X \in \text{Pos}) \to (\mathcal{P}X, \mathcal{P}X; X \in \text{Pos})
\]

where \( DX \) (resp. \( UX \)) are the down-closed (resp. up-closed) subsets of \( X \), \( \mathcal{P}X \) are all subsets and \( BX \cong \mathcal{P}(\pi_0 X) \) are the up-down-closed subsets. The composite \( (BX, BX; X \in \text{Pos}) \to (\mathcal{P}X, \mathcal{P}X; X \in \text{Pos}) \) is of course a morphism of indexed boolean algebras.

6. Similarly, we have morphisms of (topological) indexed complemented pairs

\[
(BX, BX; X \in \text{Top}) \to (OX, CX; X \in \text{Top}) \to (\mathcal{P}X, \mathcal{P}X; X \in \text{Top})
\]

where \( BX \) are the clopen subsets of \( X \).
The morphism of indexed complemented pairs (13) in fact can be extended to the more general context of Example 2.8 (3), giving the inclusion

\[(j^L_X, j^R_X; X \in \text{Cat}) : (BX, BX; X \in \text{Cat}) \to (\nu^{X^\text{op}}_0, \nu^X_0; X \in \text{Cat})\]

of those actions which act by invertible maps in \(\nu_0\). Then it is easy to see that some of the operators on \((\nu^{X^\text{op}}_0, \nu^X_0; X \in \text{Cat})\) collapse when applied to biactions. We formalize this fact in the following

2.9. Definition. A morphism \((j^L_X, j^R_X; X \in \mathcal{C}) : (BX, BX; X \in \mathcal{C}) \to (\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) with a standard domain is said to endowe \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) with \textbf{biactions} if the following laws hold:

\[
\begin{align*}
\frac{j^L_X V \otimes^L_X A}{j^R_X V} & ; & \frac{M \triangleright^r_X j^L_X V}{M \triangleright_X^r j^L_X V} & ; & \frac{j^L_X V \otimes^L_X M}{j^R_X V \otimes^r_X M} & ; & \frac{A \triangleright^r_X j^R_X V}{A \triangleright_X^r j^R_X V}
\end{align*}
\] (14)

An object \(X \in \mathcal{C}\) is \textbf{groupoidal} if \(j^L_X\) and \(j^R_X\) are equivalences. For brevity, we refer to an indexed complemented pair \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) endowed with biactions and such that \(\mathcal{C}\) has a groupoidal terminal object as a \textbf{normal pair}.

Instances of normal pairs are thus \((\nu^{X^\text{op}}_0, \nu^X_0; X \in \text{Cat})\) with the biactions inclusion, \((\mathcal{D}X, \mathcal{U}X; X \in \text{Pos})\) with the inclusion of up-down-closed sets and \((\mathcal{O}X, \mathcal{C}X; X \in \text{Top})\) with the inclusion of clopen sets.

2.10. Remark. Given an indexed complemented pair \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) endowed with biactions and a groupoidal object \(G \in \mathcal{C}\), we get a normal pair \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C}/G)\).

2.11. Remark. If \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) is an indexed complemented pair and \(1 \in \mathcal{C}\) is a terminal object, by the remarks 2.2 it follows that:

1. All the categories \(\mathcal{L}X\) are enriched over \(\mathcal{L}1\) and over \(\mathcal{R}1\) by \(\forall^L_X \lbrack -, - \rbrack_k^L\) and \(\forall^r_X \lbrack -, \lbrack - \triangleright_X^r - \rbrack\) (where \(X : X \to 1\)). They also have copowers, \((X^\ell -) \otimes^L_X -\) and \((X^r -) \otimes^r_X -\), and powers \([X^\ell -, - \rbrack^L_X\) and \((X^r-) \triangleright^r_X -\). Symmetrically, the \(\mathcal{R}X\) are enriched, copowered and powered over \(\mathcal{R}1\) and over \(\mathcal{L}1\).

2. The adjunctions \(f^\ell \dashv \forall_f^\ell\) and \(\exists_f^\ell \dashv f^\ell\) are also enriched respectively over \(\mathcal{L}1\) and over \(\mathcal{R}1\) (and symmetrically for \(\exists_f^r \dashv f^r \dashv \forall_f^r\)).

3. The adjunctions relating \(\triangleright^r_X\), \(\otimes^L_X\) and \(\triangleright^r_X\) (so as those relating \(\otimes^L_X\) and \([- , - \rbrack^L_X\) are also enriched over \(\mathcal{L}1\) (and symmetrically).

If \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) is a normal pair, then Remark 2.11 and the laws (14) (or their adjoint ones) give that:
1. All the categories $\mathcal{L}X$ and $\mathcal{R}X$ are enriched over $B_1$ via

$$\{A, B\}_X^\ell := \frac{j^\ell \forall_X^\ell [A, B]_X^\ell}{j^\ell \forall_X^r (A \triangleright_X^r B)} ; \quad \{M, N\}_X^r := \frac{j^r \forall_X^r [M, N]_X^r}{j^r \forall_X^\ell (M \triangleright_X^\ell N)}$$

where $j^\ell$ and $j^r$ are adjoint to the equivalences $j^\ell = j_1^\ell$ and $j^r = j_1^r$. They are also copowered and powered over $B_1$:

$$V \otimes_X^\ell A := \frac{X^\ell j^\ell V \otimes_X^\ell A}{X^r j^r V \circ_X^r A} ; \quad V \otimes_X^r M := \frac{X^r j^r V \otimes_X^r M}{X^\ell j^\ell V \circ_X^\ell M}$$

$$[V, A]_X^\ell := \frac{[X^\ell j^\ell V, A]_X^\ell}{X^r j^r V \triangleright_X^r A} ; \quad [V, M]_X^r := \frac{[X^r j^r V, M]_X^r}{X^\ell j^\ell V \triangleright_X^\ell M}$$

2. There are absolute complement and mixed tensor adjoint functors $A \ast_X M \dashv_A A \triangleright_X^\ell V ; \quad A \ast_X M \dashv_M M \triangleright_X^r V$

given by

$$M \triangleright_X^r V := \frac{M \triangleright_X^r X^\ell j^\ell V}{M \triangleright_X^\ell X^r j^r V} ; \quad A \triangleright_X^\ell V := \frac{A \triangleright_X^\ell X^r j^r V}{A \triangleright_X^r X^\ell j^\ell V}$$

$$A \ast_X M := \frac{j^\ell \exists_X^\ell (M \otimes_X^\ell A)}{j^r \exists_X^r (A \otimes_X^r M)}$$

2.12. Nine laws. Now we summarize the basic laws which relate the (left or right) “actions” in a normal pair $(\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})$ and the “constant (bi)actions” in $B_1$ (which can also be seen as the category of “internal truth-values”), with respect to a given map $f : X \to Y$ in $\mathcal{C}$. We present them in three groups, such that the ones in the same group are each other equivalent by adjunction (we omit the other nine obtained by left-right symmetry).

The first group says that substitution commutes with (or preserves) copowers, that the universal quantification adjunction is enriched and that universal quantification commutes with powers:

$$f^\ell (V \otimes_X^\ell A) ; \quad \{A, \forall_f^r B\}_X^\ell ; \quad \{f^\ell A, B\}_X^\ell ; \quad \forall_f^r [V, B]_X^\ell$$

(15)
The second group says that existential quantification commutes with copowers, that the existential quantification adjunction is enriched and that substitution commutes with powers:

\[
\frac{V \otimes^\ell_Y (\exists^\ell_f A)}{\exists^\ell_f (V \otimes^\ell_X A)} \quad ; \quad \frac{\{\exists^\ell_f A, B\}^\ell_Y}{\{A, f^\ell B\}^\ell_X} \quad ; \quad \frac{f^\ell[V, B]^\ell_Y}{[V, f^\ell B]^\ell_X}
\]

(16)

The third group says that substitution can pass to the other argument inside a mixed tensor product becoming an existential quantification, that substitution commutes with absolute complement and that the absolute complement of an existentially quantified action is the same as the universal quantification of its absolute complement:

\[
\frac{\exists^\ell_f A *^Y M}{A *^X f^r M} \quad ; \quad \frac{f^\ell(M \dashv^r V)}{f^r M \dashv^r_X V} \quad ; \quad \frac{\exists^\ell_f A \dashv^r_X V}{\forall^r_f (A \dashv^r_X V)}
\]

(17)

2.13. Remarks.

1. Besides the copowers - powers adjunction, also the mixed tensor - absolute complement adjunction is enriched in \( B1 \). (In fact, since all the basic adjunctions defining a normal pair are enriched in \( B1 \), the same holds for the derived ones.) Explicitly, we have natural isomorphisms

\[
\frac{\{V \otimes^\ell_X A, B\}^\ell_X}{\{V, \{A, B\}^\ell_X\}} \quad ; \quad \frac{\{V \otimes^\ell_X M, N\}^\ell_X}{\{V, \{M, N\}^\ell_X\}} \quad ; \quad \frac{\{A *^X M, V\}}{\{A, M \dashv^r_X V\}}
\]

(15)

2. Most of the equations (15), (16) and (17) may be seen as expressing the fact that limits commute with limits, or that (co)limits can be defined in terms of limits (see the next section). On the other hand the first and the second groups may be seen as expressing the fact that being a left (resp. right) adjoint is equivalent to preserving some kinds of colimits (resp. limits).

3. The first group follows essentially from the fact that a morphism of complemented pairs preserves the monoidal structures, while the last two follow from the fact that it preserves the complement functors.

3. Some abstract category theory

In this section, which can be read independently from the rest of the paper, we develop some abstract category theory, resting on a few axioms which hold true in a normal pair as well as in (a fragment of) a biclosed bicategory (for instance, that of \( V \)-profunctors) with a suitable selected object (for instance the trivial \( V \)-category; see Remark 3.2); these axioms
allow us to define weighted limits, (pointwise) Kan extensions, fully faithful, dense and absolutely dense maps and (if \(1 \in \mathcal{C}\) is actually terminal) conical limits and final maps and to prove (using also some adequacy hypothesis) some of their familiar (and less familiar) properties with straightforward calculations.

### 3.1. Indexed Pairs

As we have seen in Section 2 (recall in particular Remark 2.13 (1)) any normal pair \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) gives rise to an **indexed pair** \((\mathcal{L}X, \mathcal{R}X; X \in \mathcal{C})\) consisting of the following data and axioms:

1. A category \(\mathcal{C}\) and an object (not necessarily terminal) \(1 \in \mathcal{C}\).

2. Two \(\mathcal{C}\)-indexed categories \(\mathcal{L}X\) and \(\mathcal{R}X\) with quantifications: \(\exists^\ell_f \dashv f^\ell \dashv \forall^\ell_f : \mathcal{L}X \to \mathcal{L}Y\), \(\exists^r_f \dashv f^r \dashv \forall^r_f : \mathcal{R}X \to \mathcal{R}Y\), for all \(f : X \to Y\).

3. \(\mathcal{L}1\) and \(\mathcal{R}1\) are isomorphic and have a symmetric monoidal closed structure \(\mathcal{V} = (\mathcal{V}_0 \cong \mathcal{L}1 \cong \mathcal{R}1, \otimes, I, [-,-])\).

4. All the categories \(\mathcal{L}X\) and \(\mathcal{R}X\) are enriched \((\{A, B\}^\ell_X\) and \(\{M, N\}^r_X\), powered \(([V, A]^r_X\) and \([V, M]^r_X\)) and copowered \((V \otimes^r_X A\) and \(V \otimes^r_X M\) over \(\mathcal{V}\).

5. For any \(X \in \mathcal{C}\) there is a “mixed tensor” bifunctor \(*_X : \mathcal{L}X \times \mathcal{R}X \to \mathcal{V}\) with enriched “absolute complement” adjoints \(A *_X - \dashv A \gamma^r_X - : \mathcal{V} \to \mathcal{R}X\) and \(- *_X M \dashv M \gamma^r_X - : \mathcal{V} \to \mathcal{L}X\).

6. All the operators collapse over \(1 \in \mathcal{C}\), becoming those of \(\mathcal{V}\): \(\{V, W\}^r_1 \cong \{V, W\}^\ell_1 \cong [V, W]_1^r \cong [V, W]_1^\ell \cong [V, W] \cong V \gamma^r_1 W \cong V \gamma^\ell_1 W\) (and similarly for tensors).

In the following, we will thus use \([-,-]\) in place of \([,-,-]\) for the internal hom of \(\mathcal{V}\).

7. Substitution functors preserve powers, copowers and complements.

The laws (natural isomorphisms) which summarize an indexed pair are thus (apart those concerning the associativity of \(\otimes\) and the quantification adjunctions):

\[
\begin{align*}
\{V \otimes^\ell_X A, B\}^\ell_X & \cong \{V \otimes X^r M, N\}^r_X \cong \{A * X M, V\} \\
\{V, \{A, B\}^\ell_X\} & \cong \{V, \{M, N\}^r_X\} \cong \{A, M \gamma^r_X V\}^\ell_X \\
\{A, [V, B]^\ell_X\} & \cong \{M, [V, N]^r_X\} \cong \{M, A \gamma^r_X V\}^r_X \\
\frac{f^\ell(V \otimes^\ell_X A)}{V \otimes^\ell_X f^\ell A} & \cong \frac{f^\ell[V, A]^\ell_X}{[V, f^\ell A]^\ell_X} \cong \frac{f^r(A \gamma^r_X V)}{f^\ell A \gamma^r_X V} \\
\frac{f^r(V \otimes^r_X M)}{V \otimes^r_X f^r M} & \cong \frac{f^r[V, M]^r_X}{[V, f^r M]^r_X} \cong \frac{f^\ell(M \gamma^r_X V)}{f^r M \gamma^r_X V}
\end{align*}
\]
From the above laws, by adjunction, we obtain:

\[
\begin{align*}
\{A, \forall f^\ell B\}_X \quad ; \quad \{f^\ell A, B\}_X \quad ; \quad \exists f^\ell A \ast_Y M \\
\{f^\ell A, B\}_X \quad ; \quad \{A, f^\ell B\}_X \quad ; \quad \exists f^\ell A \ast_X f^r M
\end{align*}
\]

(plus the symmetrical ones, obtained by exchanging \(\ell\) and \(r\)).

3.2. Remark.  Given a biclosed bicategory \(\mathcal{M}\) with a selected object \(1\) such that \(\mathcal{V} := \mathcal{M}(1, 1)\) is symmetric, we get an indexed pair by taking the maps (right adjoint arrows) in \(\mathcal{M}\) as \(\mathcal{C}\) and by posing \(\mathcal{L}X := \mathcal{M}(X, 1)\) and \(\mathcal{R}X := \mathcal{M}(1, X)\). Indeed, axioms (2) to (6) are a straightforward consequence of the closed structure of \(\mathcal{M}\), while axiom (7) is given by instances of associativity of composition in \(\mathcal{M}\) (or of their adjoints), when one of the arrows is a map. Likewise, “proarrow equipments” ([Wood, 1982]) give rise to indexed pairs.

Thus, what we do in this section can be also considered as an abstract approach to category theory which uses only a fragment of the (abstract) profunctor category, avoiding the full bicategorical machinery (of which \(\mathcal{L}X\) and \(\mathcal{R}X\) and the various bifunctors are obviously a trace).

3.3. External weighted limits.  Recall that \(A \in \mathcal{L}X\) (resp. \(M \in \mathcal{R}X\)) is to be thought of as a left (resp. right) action of \(X\) on \(\mathcal{V}\) (the trivial one, if \(X = 1\)), that is as a \((\mathcal{V})\)-functor \(X^{\text{op}} \to \mathcal{V}\) (resp. \(X \to \mathcal{V}\)). Thus \(\{A, B\}^\ell \in \mathcal{V}\) (resp. \(\{M, N\}^r \in \mathcal{V}\)) is the “internal truth value” of natural transformations \(A \to B\) (resp. \(M \to N\)). These \(\mathcal{V}\)-valued functors can be “composed” with the “functors” \(f : T \to X\) in \(\mathcal{C}\), giving \(f^\ell A \in \mathcal{L}T\) and \(f^r M \in \mathcal{R}T\); in particular, if \(T = 1\) we get the value \(x^\ell A \in \mathcal{V}\) (resp. \(x^r M \in \mathcal{V}\)) of \(A\) (resp. \(M\)) at the “point” \(x : 1 \to X\). (Of course, the value at \(x\) of \(f^\ell A\) is the value at \(fx\) of \(A\).)

We can also “compose” \(A\) (or \(M\)) with some functors \(\mathcal{V} \to \mathcal{V}\); namely, \([V, A]\) and \(A \rhd V\) can be seen as the substitution of \(A\) in the covariant and contravariant (enriched) functors represented by \(V\). Indeed, posing \(f = x : 1 \to X\) in (19)

\[
\begin{align*}
\frac{x^\ell [V, A]_X^\ell}{\{V, x^\ell A\}} \quad ; \quad \frac{x^r (A \rhd X^r V)}{\{x^\ell A, V\}}
\end{align*}
\]

we note that their value at \(x\) is an internal hom of \(\mathcal{V}\).

We now interpretate the indexed pair axioms in terms of “external” (that is \(\mathcal{V}\)-valued) weighted limits. (The “internal” ones are treated in Section 3.4.) We say (omitting the index \(X\) when superfluous) that \(\{A, B\}^\ell\) is the left limit of \(B\) weighted by \(A\) (and similarly \(\{M, N\}^r\) is a right limit) and that \(A \ast M\) is the colimit of \(M\) weighted by \(A\) (or conversely).
The indexed pair laws
\[
\begin{align*}
\{V, \{A, B\}\} & \quad ; \quad \{A \ast M, V\} \\
\{A, [V, B]\} & \quad ; \quad \{A, M \ract V\}
\end{align*}
\]
thus express limits and colimits in terms of limits or, more precisely, say that representables preserve limits and convert colimits into limits.

Similarly, the indexed pair laws
\[
\begin{align*}
\forall f [V, A] & \quad ; \quad \exists f A \ract V \\
\forall^f [V, A] & \quad ; \quad \forall^f (A \ract V)
\end{align*}
\]
say that representables preserve external right (Kan) extensions and convert the left ones into right ones.

Using the (21) for “points” \(x : 1 \to X\) we get the Yoneda and co-Yoneda isomorphisms:
\[
\begin{align*}
\{\exists^r I, A\}_X & \quad ; \quad \exists^r I \ast X M \\
\{I, x^r A\} & \quad ; \quad I \otimes x^r M
\end{align*}
\]
and symmetrically, \(\{\exists_x I, M\} \cong x^r M\) and \(A \ast \exists_x I \cong x^r A\).

Thus, the “images of points” are to be thought of as functors (internally) represented by the point itself, and a “concrete representation” of the “abstract category” \(X \in \mathcal{C}\) is given by \(\overline{X}\), the full \(\mathcal{V}\)-enriched subcategory of \(\mathcal{L}X\) generated by \(\exists_x I\), for \(x : 1 \to X\). Since
\[
\begin{align*}
\{\exists^r I, \exists_x I\} & \quad ; \quad \exists^r I \ast \exists_x I \\
x^r \exists^r I & \quad ; \quad \exists^r I \ast \exists_x I
\end{align*}
\]
we see that \(\overline{X}\) and \(\overline{X}^r\) are dual. (In fact, one should check composition.)

3.4. Internal weighted limits. We define (internal) weighted limits by the internal correspective of the (24), that is using the substitutions \(f^\ell \exists_y I\) and \(f^r \exists^r y I\) in the internally representables functors. We say that \(\{M, f\} : 1 \to Y\) is a limit and that \(A \ast f : 1 \to Y\) is a colimit of \(f : X \to Y\), weighted by \(M \in \mathcal{R}X\) and \(A \in \mathcal{L}X\) respectively, if:
\[
\begin{align*}
\{M, f\} & \quad ; \quad \{A \ast f\} \\
\{M, f \ast \exists^r y I\} & \quad ; \quad \{A, f^r \exists^r y I\}
\end{align*}
\]
(note that the naturality conditions in $y$ refer to the categories $\overline{X}^\ell$ and $\overline{X}^r$). So, internal limits and colimits are defined in terms of external (left and right) limits.

3.5. Remark. Now, the limit-colimit duality is a perfect symmetry, while it is not the case externally. (One finds a similar situation in internal category theory.) Note also that since $\{ M, f \}^r \exists^r y I = \overline{\exists}^r (\exists^r_{(M, f)} I, \exists^r y I)$, the limit $\{ M, f \}$, as a functor of its weight $M$, is a (partially defined) adjoint of the restriction $\overline{\exists}^r \rightarrow RY \rightarrow RX$ of $f^r$; thus, it is also the (partially defined) reflection of $\exists^r_f M$ in $\overline{Y}$.

The internal corresspective of the laws

$$\frac{f^r(A \rightarrow^\ell V)}{f^\ell A \rightarrow^\ell V}; \quad \frac{f^\ell [V, A]^\ell}{[V, f^\ell A]^\ell}$$  \hspace{1cm} (29)

is simply the functoriality of substitution:

$$\frac{f^r(g^r \exists^r y I)}{(gf)^r \exists^r y I}; \quad \frac{f^\ell (g^\ell \exists^\ell y I)}{(gf)^\ell \exists^\ell y I}$$  \hspace{1cm} (30)

which (along with the adjunction $\exists^r_f \dashv f^\ell$) allows us to get

$$\frac{\{ M, gf \}}{\{ \exists^r_f M, g \}}; \quad \frac{A \ast gf}{\exists^r_f A \ast g}$$  \hspace{1cm} (31)

that are the internal version of

$$\frac{\{ M, f^r N \}^r}{\{ \exists^r_f M, N \}^r}; \quad \frac{A \ast f^r M}{\exists^r_f A \ast M}$$  \hspace{1cm} (32)

Thus (since $\{ I, x \} = x = I \ast x$) we also get the internal (co-)Yoneda isomorphisms:

$$\frac{\{ \exists^r_x I, f \}}{fx}; \quad \frac{\exists^r_x I \ast f}{fx}$$  \hspace{1cm} (33)

The laws which pose quantifications on the right side

$$\frac{\{ f^r M, N \}^r}{\{ M, \forall^r_f N \}^r}; \quad \frac{f^\ell A \ast M}{A \ast \exists^r_f M}$$  \hspace{1cm} (34)
become internally the definitions of the Kan extensions $\forall fg$ and $\exists fg$:

\[
\frac{\{f^r M, g\}}{\{M, \forall fg\}} ; \quad \frac{f^\ell A * g}{A * \exists fg}
\]

They can be equivalently defined by external Kan extensions, using the internal analogous of the (25):

\[
\frac{\forall f g \exists^\ell I}{\forall^\ell_f (g^\ell \exists^\ell I)} ; \quad \frac{(\exists g)^\ell \exists^\ell I}{\exists^\ell_f (g^\ell \exists^\ell I)}
\]

Indeed, there are natural isomorphisms

\[
\frac{\{M, (\forall fg)^\ell \exists^\ell I\}}{\{M, \forall fg\}^\ell \exists^\ell I} ; \quad \frac{\{M, g^\ell \exists^\ell I\}}{\{f^r M, g^\ell \exists^\ell I\}}
\]

3.6. The concrete representation of an indexed pair. The concrete representation of $X \in \mathcal{C}$ as the $\mathcal{V}$-category $\overline{X}^\ell$ (or $\overline{X}^r$) can be extended to “abstract functors” $f : X \to Y$; indeed, since $\exists^\ell_f \exists^\ell I = \exists^r_f I$, we get a $\mathcal{V}$-functor $\overline{f}^\ell : \overline{X}^\ell \to \overline{Y}^\ell$ (or $\overline{f}^r : \overline{X}^r \to \overline{Y}^r$), while for $A \in \mathcal{L}X$ we get a $\mathcal{V}$-functor $\overline{A}^\ell : (\overline{X})^{op} \to \mathcal{V}$ by restricting $\{-, A\}^\ell$ to the “representables” (and similarly for $M \in \mathcal{R}X$). By Yoneda, each of them can be seen as the restriction to $\overline{X}^\ell$ or $\overline{X}^r$ of (co)limits as functors of their weight, in two ways (which should be equivalent via the duality $\overline{X}^\ell \cong (\overline{X})^{op}$):

\[
\begin{align*}
\overline{A}^\ell & := \overline{X}^\ell \xrightarrow{\{\cdot, A\}^\ell} \mathcal{V} ; \\
\overline{M}^\ell & := \overline{X}^\ell \xrightarrow{\{-, M\}^\ell} \mathcal{V} ; \\
\overline{f}^\ell & := \overline{X}^\ell \xrightarrow{\{-, f\}^\ell} \overline{Y}^\ell
\end{align*}
\]

The extent to which this representation is “faithful” depends on the axioms we discuss in the following section.

3.7. Adequacy axioms. To develop abstract category theory in the frame of indexed pairs, we need some “reduction rules”, that is some adequacy (or density) axioms which (along with Yoneda reduction itself) allow us to eliminate or introduce variables.

In the following, we have in mind as a model the category of small categories, that is the indexed pair $(\text{Set}^{X^{op}}, \text{Set}^X, X \in \text{Cat})$. We do not consider here the question of the extent to which the axioms hold in $\mathcal{V}$-enriched contexts.
A (left or right) density condition for a functor $i$ with respect to a bifunctor $\cdot$ has the form of a reduction rule for fractions:

\[
\frac{ix \cdot y}{ix \cdot y'} \iff \frac{y}{y'} ; \quad \frac{y \cdot ix}{y' \cdot ix} \iff \frac{y}{y'}
\]

Of course, we are adopting the convention that “numerator” and “denominator” of a “fraction” are to be intended as functors of the variables which appear in both of them, while the fraction itself indicates the existence of an isomorphism between them. In fact, more generally, fractions are to be intended as morphisms (not necessarily isos) whose relative direction depends on the variance of the arguments.

We will use the following very general principle of functorial calculus:

**3.8. Proposition.** Suppose that the bifunctors $f$ and $g$ are “adjoint relatively to $h$ and $k$”:

\[
\frac{h(f(x, y), z)}{k(x, g(y, z))}
\]

and that $i$ is left dense for $k$ and $j$ is right dense for $h$; then $i$ is left dense for $f$ iff $j$ is right dense for $g$.

**Proof.**

\[
\frac{f(ix, y)}{f(ix, y')} \iff \frac{h(f(ix, y), jz)}{h(f(ix, y'), jz)} \iff \frac{k(ix, g(y, jz))}{k(ix, g(y', jz))} \iff \frac{g(y, jz)}{g(y', jz)}
\]

(We have supposed all functors covariant; otherwise, one has to do obvious changes.)

In the same way, when $h$ and $k$ are hom functors, $i$ and $j$ are identities and the intermediate category is $Y = 1 + 1$, one gets the usual “mates” correspondence for ordinary adjunctions $f \dashv g$ and $f' \dashv g'$.

The (left and right) density of the identity with respect to the enriched hom of $\mathcal{L}X$ or $\mathcal{R}X$ is expressed by “Yoneda reduction”:

\[
\begin{array}{ccc}
A & \iff & \{C, A\}^\ell \\
B & \iff & \{C, B\}^\ell \\
{\{A, C\}^\ell} & \iff & {\{B, C\}^\ell}
\end{array}
\] (37)

The first assumption says that the concrete representation of $A$ and $M$ as $\overrightarrow{A}$ and $\overrightarrow{M}$ are faithful, that is that the inclusions $\overrightarrow{X}^\ell \to \mathcal{L}X$ and $\overrightarrow{X}' \to \mathcal{R}X$ of “representables” in all “presheaves” are dense:
3.9. Axiom.

\[
\begin{align*}
A & \iff \{\exists_x I, A\}^\ell \quad ; \\
B & \iff \{\exists_x I, B\}^\ell \quad ; \\
M & \iff \{\exists_x I, M\}^r \\
N & \iff \{\exists_x I, N\}^r
\end{align*}
\]

(38)

or equivalently:

\[
\begin{align*}
A & \iff A \ast \exists_x I \\
B & \iff B \ast \exists_x I \quad ; \\
M & \iff \exists_x I \ast M \\
N & \iff \exists_x I \ast N
\end{align*}
\]

(39)

As in the proof of Proposition 3.8, Axiom 3.9 implies the “contraposition law” for absolute complement:

\[
\begin{align*}
A & \iff B \dashv V \\
B & \iff A \dashv V \quad ; \\
M & \iff N \dashv V \\
N & \iff M \dashv V
\end{align*}
\]

(40)

Explicitly, we have:

\[
\begin{align*}
N \dashv V & \iff \{\exists_x I, N \dashv V\} \\
M \dashv V & \iff \{\exists_x I, M \dashv V\} \iff \exists_x I \ast M \iff M \\
N & \iff \{\exists_x I \ast N, V\} \iff \exists_x I \ast N \iff N
\end{align*}
\]

(42)

Again by Proposition 3.8, (40) is equivalent to

\[
\begin{align*}
A & \iff B \ast M \\
B & \iff A \ast M \quad ; \\
M & \iff A \ast M \\
N & \iff A \ast N
\end{align*}
\]

(41)

We cannot assume a condition as Axiom 3.9 for maps \( f : X \to Y \) with respect to internal (co)limits, simply because \( \mathcal{C} \) is not (yet) a 2-category. Rather, we define such a structure on \( \mathcal{C} \) by

\[
\mathcal{C}(X, Y)(f, g) := \text{Cat}(X^f, Y^g)(\overline{f}^r, \overline{g}^r) \cong \text{Cat}(X^r, Y^r)(\overline{f}^r, \overline{g}^r)
\]

that is (following our convention on fraction notation)

\[
\begin{align*}
f \iff \exists_x I \ast f \\
g \iff \exists_y I \ast g \iff \exists_y I \ast f
\end{align*}
\]

(42)

Now, Axiom 3.9 and (42) imply

\[
\begin{align*}
f \iff g \ast \exists_y I \\
g \iff f \ast \exists_y I \iff f \ast \exists_y I
\end{align*}
\]

(43)
Indeed (considering for instance the left hand side):

$$
\frac{g^\ell \exists^\ell I}{f^\ell \exists^\ell y} \iff \left\{ \exists^\ell x_1, f^\ell \exists^\ell y_1 \right\} \iff \left( \exists^\ell x_1 \ast f \right)^\ell \exists^\ell y_1 \iff \exists^\ell I \ast f \iff f
$$

where the third equivalence is Yoneda reduction in $\mathbb{X}$, since $x^\ell \exists^\ell y \cong \mathbb{X} (\exists^\ell x_1, \exists^\ell y_1)$ by (26).

Again by Proposition 3.8, (43) is equivalent to

$$
\frac{f}{g} \iff \frac{A \ast f}{A \ast g} \iff \left\{ M, f \right\} \iff \left\{ M, g \right\}
$$

As our second and last density condition, we assume:

3.10. **Axiom.**

$$
\frac{A}{B} \iff A \ast_X f \iff \frac{M}{N} \iff \left\{ M, f \right\}_X
$$

(where naturality holds with respect to any category $\mathbb{C}(X, Y)$).

3.11. **Kan extensions.** We have defined Kan extensions in (35); from the properties of the 2-category structure of $\mathbb{C}$ it easily follows that they are defined up to isomorphisms: if $f \cong f'$ and $g \cong g'$ then $\forall_f g \cong \forall_{f'} g'$ and $\exists_f g \cong \exists_{f'} g'$. Furthermore, they are really extensions in $\mathbb{C}$:

$$
\begin{array}{ll}
\forall_f g & \rightarrow t \\
\exists_f g & \rightarrow t \\
\end{array}
\begin{array}{ll}
\forall_f g \rightarrow t \\
\exists_f g \rightarrow \left( \forall_f g \right)^\ell \exists^\ell x_1 \\
\exists_f g \rightarrow \left( \exists_f g \right)^\ell \exists^\ell y_1 \\
\forall_f \left( g^\ell \exists^\ell y_1 \right) & \rightarrow \forall_f \left( g^\ell \exists^\ell y_1 \right) \\
f \ast f \rightarrow g^\ell \exists^\ell y_1 & \rightarrow g^\ell \exists^\ell y_1 \\
\left( tf \right)^\ell \exists^\ell x_1 & \rightarrow g^\ell \exists^\ell x_1 \\
t f & \rightarrow g \\
\end{array}
$$

(46)

3.12. **Remark.** Our Kan extensions are “pointwise”: indeed they are preserved by rappresentables (36), and are given by the (co)limit formulas:

$$
\begin{array}{ll}
\left( \forall_f g \right) x & \rightarrow \left( \exists_f g \right) x \\
\left\{ \exists^\ell x_1, \forall_f g \right\} & \rightarrow \left\{ \exists^\ell x_1, \exists_f g \right\} \\
\left\{ f^\ell \exists^\ell x_1, g \right\} & \rightarrow \left\{ f^\ell \exists^\ell x_1, \exists_f g \right\} \\
\end{array}
$$

(47)
3.13. Fully faithful maps. Given a map \( f : X \to Y \) in \( \mathcal{C} \), \( \overrightarrow{f} : \overrightarrow{X} \to \overrightarrow{Y} \) is fully faithful iff \( \overleftarrow{f} : \overleftarrow{X} \to \overleftarrow{Y} \) is such and iff anyone of the following properties holds:

\[
\begin{align*}
\exists_x I, \exists_y I & \quad ; \quad \exists_x I, \exists_y I & \quad ; \quad \exists_x I \ast \exists_y I & \quad \text{ (48)} \\
\exists_x f x, \exists_y f y I & \quad ; \quad \exists_x f x, \exists_y f y I & \quad ; \quad \exists_x f x I \ast \exists_y f y I \\
\{A, B\} & \quad ; \quad \{M, N\} & \quad ; \quad \{M, g\} & \quad \text{ (49)} \\
\exists f A, \exists F B & \quad ; \quad \exists f M, \exists F N & \quad ; \quad \exists f M, \exists f g \\
\{A, B\} & \quad ; \quad \{M, N\} & \quad ; \quad \{M, g\} & \quad \text{ (50)} \\
\forall f A, \forall f B & \quad ; \quad \forall f M, \forall f N & \quad ; \quad \forall f M, \forall f g \\
\end{align*}
\]

\[
\begin{align*}
A \ast M & \quad ; \quad \exists f A \ast \exists f M & \quad ; \quad \exists f A \ast \exists f g & \quad \text{ (51)} \\
\end{align*}
\]

\[
\begin{align*}
\frac{f \exists f A}{A} & \quad ; \quad \frac{f \exists F M}{M} & \quad ; \quad \frac{(\exists f g) f}{g} & \quad \text{ (52)} \\
\frac{f \forall f A}{A} & \quad ; \quad \frac{f \forall f M}{M} & \quad ; \quad \frac{(\forall f g) f}{g} & \quad \text{ (53)} \\
\frac{\exists f I \ast g}{\exists f x I \ast \exists f g} & \quad ; \quad \frac{f \exists f x I}{\exists f x I} & \quad ; \quad \frac{f \exists f x I}{\exists f x I} & \quad \text{ (54)} \\
\end{align*}
\]

For the proof one uses the density rules (introduction and elimination of variables) of Section 3.7 and the adjunction-like laws, that is the introduction and elimination of quantifications rules summarized below for the reader convenience:

\[
\begin{align*}
\{f^\ell A, B\} & \quad ; \quad \{f^r M, N\} & \quad ; \quad \{f^r M, g\} \quad \text{ (55)} \\
\{A, \forall f B\}^\ell & \quad ; \quad \{M, \forall f N\} & \quad ; \quad \{M, \forall f g\} \\
\{A, f^\ell B\} & \quad ; \quad \{M, f^r N\} & \quad ; \quad \{M, gf\} \quad \text{ (56)} \\
\{\exists f A, B\}^\ell & \quad ; \quad \{\exists f M, N\} & \quad ; \quad \{\exists f M, g\} \\
\end{align*}
\]

\[
\begin{align*}
\frac{f^\ell A \ast M}{A \ast \exists f M} & \quad ; \quad \frac{f^\ell A \ast g}{A \ast \exists f g} & \quad \frac{A \ast f^r M}{\exists f A \ast M} & \quad \frac{A \ast gf}{\exists f A \ast g} \quad \text{ (57)} \\
\end{align*}
\]
Proof. Let us prove some of the equivalences; the other ones can be proven with the same technique and we leave them to the reader.

\[
\begin{align*}
\{ \exists_x I, \exists_y I \} & \iff \{ \exists_x I, \exists_y I \} \\
\{ \exists_f \exists_x I, \exists_f \exists_y I \} & \iff \{ \exists_f \exists_x I, \exists_f \exists_y I \} \\
\{ \exists_x I, f^\ell \exists_y I \} & \iff \{ \exists_x I, f^\ell \exists_y I \} \\
\exists^\ell I & \iff \{ \exists^\ell I, A \} \\
f^{\ell} \exists^\ell I & \iff \{ f^{\ell} \exists^\ell I, A \} \\
\exists^\ell I & \iff \exists^\ell I \ast M \\
f^{\ell} \exists^\ell I \ast M & \iff \exists^\ell I \ast f^{\ell} \exists^\ell M \\
M & \iff A \ast M \\
f^{\ell} A \ast M & \iff f^{\ell} \exists^\ell A \ast M \\
A & \iff A \ast g \\
f^{\ell} A \ast g & \iff A \ast (\exists f g) f \\
\exists^\ell I & \iff \exists^\ell I \ast \exists x I \\
f^{\ell} \exists^\ell I \ast \exists x I & \iff f^{\ell} \exists^\ell I \ast \exists x I \\
A & \iff A \ast \exists y I \\
f^{\ell} A \ast \exists y I & \iff f^{\ell} A \ast \exists y I
\end{align*}
\]

\(\exists\) and so on. Note that in some cases one can choose different paths; for instance

\[
\begin{align*}
A & \iff \{ B, A \} \\
f^{\ell} A & \iff \{ f^{\ell} A, f^{\ell} B \} \\
B & \iff f^{\ell} \exists^\ell B \\
f^{\ell} B & \iff \{ f^{\ell} \exists^\ell B, f^{\ell} \exists^\ell A \} \\
f^{\ell} A & \iff f^{\ell} A \ast \exists y I
\end{align*}
\]

3.14. Absolutely dense maps. While fully faithful maps are those for which the unit of \(\exists_f^\ell \dashv f^\ell\) is an iso, absolutely dense maps are those for which the counit of the same adjunction is an iso. (In the context of a bicategory \(\mathcal{M}\) of proarrows, these would become adjunctions in \(\mathcal{M}\).) Absolutely dense, or “connected,” functors are treated (in an enriched context) in [El Bashir and Velebil, 2002] where one finds some of the characterizations below.

Given a map \(f : X \to Y\) in \(\mathcal{C}\), the following properties are equivalent:

\[
\begin{align*}
\{ A, B \} & \iff \{ f^{\ell} A, f^{\ell} B \} \\
\{ M, N \} & \iff \{ f^r M, f^r N \} \\
\{ M, g \} & \iff \{ f^r M, g f \} \\
\{ h, g \}_Y & \iff \{ h f, g f \}_Y \\
\exists^\ell I \ast \exists^\ell I & \iff A \ast M \\
f^{\ell} \exists^\ell I \ast \exists^\ell I & \iff f^{\ell} A \ast f^{\ell} M \\
A \ast g & \iff f^{\ell} A \ast g f
\end{align*}
\]
3.15. Remark. So as full faithfulness is a strong “injectivity” property, absolute density is a strong “surjectivity” property. We will treat presently other (weaker) surjectivity properties: left or right density and final or initial maps.

In fact, this is more than a vague analogy: in the indexed pair \((\mathcal{P}X, \mathcal{P}X; X \in \text{Set})\) we find again the usual concepts for mappings. Note, by the way, that in that case \(LX = RX, \mathcal{V} = 2 = \{\text{true, false}\}, \{P, Q\} \text{ and } P \star Q \) are the truth values of \(P \subseteq Q\) and of \(P \cap Q \neq \emptyset\) respectively, and \(P \neg \text{false}\) is the usual complementary set. The “representables” become the singletons and the weighted (co)limit \(\{P, f\} = P \star f\) exists iff \(f\) is constant on \(P\).

3.16. Dense maps. Given a map \(f : X \to Y\) in \(\mathcal{C}\), the following properties are equivalent:

\[
\frac{\exists f f^\ell A}{A} \quad ; \quad \frac{\exists f f^r M}{M} \quad ; \quad \frac{\exists f (gf)}{g} \quad (60)
\]

\[
\frac{\forall f f^\ell A}{A} \quad ; \quad \frac{\forall f f^r M}{M} \quad ; \quad \frac{\forall f (gf)}{g} \quad (61)
\]

\[
\frac{\exists f f^\ell \exists x I}{\exists x I} \quad ; \quad \frac{\exists f f^r \exists x I}{\exists x I} \quad (62)
\]

Proofs are very similar to those for fully faithful maps, and we leave them to the reader.

A map satisfying these properties is said to be left dense. Indeed, by the last of (63), \(f : X \to Y\) is left dense iff any “object” of \(Y\) is a colimit of \(f\) “canonically” weighted. A map is right dense if it satisfies the “dual” (that is “symmetrical”) properties:

\[
\frac{\{\exists x I, \exists y I\}}{\{f^r \exists x I, f^r \exists y I\}} \quad ; \quad \frac{\forall f f^r \exists x I}{\exists x I} \quad ; \quad \frac{\forall f f}{\text{id}_Y} \quad ; \quad \frac{\{f^r \exists x I, f\}}{x} \quad (64)
\]

In the indexed pair \((\mathcal{P}X, \mathcal{P}X; X \in \text{Set})\) one finds again surjectivity.
3.17. Limits preservation. Given \( f : X \to Y \), we say that \( g : Y \to Z \) preserves the limit \( \{ M, f \} \) if \( \{ M, gf \} \cong g \{ M, f \} \) (and the same for colimits). Similarly, if \( h \forall f g \cong \forall f (hg) \), then we say that \( h \) preserves the right Kan extension \( \forall f g \) (and the same for the left ones).

We can now motivate the term “absolute density” for the strong surjectivity notion of Section 3.14: the conditions

\[
\exists_f(gf) \quad ; \quad \forall_f(gf)
\]

imply (for \( g = \text{id} \)) left and right density and show that left and right Kan extensions

\[
\exists_f \quad ; \quad \forall_f
\]

are absolute, that is preserved by any map. Furthermore, they are equivalent to

\[
f^\ell \exists_x^f I \ast gf \quad ; \quad \{ f^r \exists_x^f I, gf \}
\]

that is to the fact that the density (co)limits

\[
f^\ell \exists_x^f I \ast f \quad ; \quad \{ f^r \exists_x^f I, f \}
\]

are absolute.

3.18. Conical limits. Suppose now that \( 1 \in C \) is in fact terminal. We can then define constant “functors” and “presheaves” as those that factor through \( X : X \to 1 \); thus

\[
yX : X \to Y \quad ; \quad X^\ell V \in \mathcal{L}X \quad ; \quad X^r V \in \mathcal{R}X
\]

are the constant functors and presheaves whose values at \( x : 1 \to X \) are \( yXx = y \), \( x^\ell X^\ell V = V \) and \( x^r X^r V = V \).

If we define \( \Gamma_X : = X^\ell I \in \Gamma_X = X^r I \), then

\[
\exists_X M \quad ; \quad \forall_X A
\]

\[
I \ast \exists_X^r M \quad ; \quad \{ I, \forall_X^\ell A \}
\]

\[
X^\ell I \ast M \quad ; \quad \{ X^\ell I, A \}
\]

\[
\Gamma_X \ast M \quad ; \quad \{ \Gamma_X^r, A \}
\]
that is, external (co)limits weighted by the constant “trivial actions” \( I^e_X \) give quantifications “on all \( X \).

A **conical (co)limit** of \( f : X \to Y \) is a (co)limit weighted by \( I^e_X \):

\[
\lim f := \{ I^e_X, f \} \quad ; \quad \colim f := I^e_X \ast f
\]

Conical (co)limits can be obtained as Kan extensions along \( X \to 1 \):

\[
\begin{array}{c}
\exists_X f \\
\colim f
\end{array}
\quad ; \quad
\begin{array}{c}
\forall_X f \\
\lim f
\end{array}
\]

which is the internal correspective of (69). Indeed, considering for instance colimits and using (69) itself:

\[
(colim f)^e \exists^e_x I
\]

\[
\begin{array}{c}
\{ I^e_X, f^e \exists^e_x I \} \\
\forall^e_X (f^e \exists^e_x I)
\end{array}
\]

\[
(\exists_X f)^e \exists^e_x I
\]

Under the hypothesis that the canonical \( \bowtie^e_X : C/X \to \mathcal{L}X \) e \( \bowtie^r_X : C/X \to \mathcal{R}X \) have right adjoints \( \bowtie^e_X \dashv i^e_X \) and \( \bowtie^r_X \dashv i^r_X \) (the comprehension scheme of [Lawvere, 1970]) and that these are fully faithful, the weighted (co)limits can be canonically reduced to conical (co)limits. Indeed, in that case any weight \( A \in \mathcal{L}X \) is isomorphic to \( \exists_t I^e_T \) (for \( t = i^e_X A : T \to X \)) so that:

\[
\begin{array}{c}
A \ast f \\
\exists_t I^e_T \ast f
\end{array}
\]

\[
\begin{array}{c}
I^e_T \ast ft \\
\colim ft
\end{array}
\]

(Of course, if \( C = \textbf{Cat} \) then \( t = i^e_X A : T \to X \) is the discrete fibration associated to \( A \).)

**3.19. Final maps.** Again in the hypothesis that \( 1 \in C \) is terminal, we have a further notion of “surjectivity”; the following properties are equivalent for a map \( f : X \to Y \) in \( C \):

\[
\begin{array}{c}
\exists^e_I X \quad ; \\
\exists^r_X f^e M
\end{array}
\]

\[
\begin{array}{c}
\forall^e_X f^r A \quad ; \\
\forall^r_Y A
\end{array}
\]

\[
\begin{array}{c}
\exists^r_X f^e \exists^e_Y I \\
\colim_X (gf)
\end{array}
\]

\[
\begin{array}{c}
\exists^r_Y M \\
\colim_Y A
\end{array}
\]

\[
\begin{array}{c}
\exists^r_Y A
\end{array}
\]

\[
\begin{array}{c}
\colim_X \ast f^e \exists^e_Y I \\
\colim_Y g
\end{array}
\]
**Proof.** First note that $\exists^f_x I = \exists^f_{X,x} I = I$. (If $C = \text{Cat}$, this corresponds to the fact that (the total of) the discrete fibration associated to a representable, that is a slice, is connected.) Then, by (69) we have:

\[
\begin{align*}
\exists^f_{I_Y} &\iff \exists^f_{I_Y} \ast \exists^r_Y I \\
I_Y^r &\iff I_Y^r \ast \exists^r_Y I \\
I_Y^r &\iff \exists^r_Y I \\
I_Y^r &\iff \exists^r_Y I \\
I_Y^r &\iff \exists^r_Y I
\end{align*}
\]

These maps are called **final**, and symmetrically one defines **initial** maps.

In the indexed pair $(\mathcal{P}X, \mathcal{P}X; X \in \text{Set})$ one finds again surjectivity.

3.20. **Remark.** Once we chose that the “true” concrete representation of $X$ is, say, $\overline{X}^f$ (rather than $\overline{X}^r$), $A \in \mathcal{L}X$ is a “presheaf” on $X$ (that is a “contravariant functor” $X \to \mathcal{V}$) via the inclusion $\overline{X}^f \to \mathcal{L}X$, while $M \in \mathcal{R}X$ is a “covariant functor” $X \to \mathcal{V}$. So, reasoning “on the left side”, $f^\ell A$ is to be thought of as the substitution of $f^{op}$ (rather than $f$) in $A : X^{op} \to \mathcal{V}$. Thus the conditions

\[
\frac{\exists^f_X f^r M}{\exists^r_Y M} ; \quad \frac{\text{colim}_X(gf)}{\text{colim}_Y g}
\]

express the fact that precomposing with a final functor preserves colimits, while

\[
\frac{\forall^f_X f^\ell A}{\forall^r_Y A}
\]

expresses the fact that precomposing with $f^{op}$ (which is initial) preserves limits.

3.21. **Remark.** Since the absolute density of $f : X \to Y$ is equivalent to each one of the conditions:

\[
\frac{\exists^f f^\ell A}{A} ; \quad \frac{\exists^r f^r M}{M}
\]

and since $f^\ell I_Y^r \leq I_X^r$ and $f^r I_Y^r \leq I_X^r$, an absolutely dense map is both final and initial.
3.22. Adjunctible and adjoint maps. A simple characterization of left adjoint functors \( f : X \to Y \) in \( \textbf{Cat} \) is that, for all \( y \in Y \), \( f^\ell \exists_y^I \cong \exists_{f^{-1}y}^I \), for a suitable \( f^{-1}y \in X \) (a reflection of \( y \) along \( f \)). In the frame of indexed pairs, we call such a map left adjunctible, rather than “adjoint”, since it does not imply the existence of a \( g : Y \to X \) such that \( gy \cong f^{-1}y \). Symmetrically one defines right adjunctible maps.

3.23. Remarks.

1. In the indexed pair \((\mathcal{P}X, \mathcal{P}X; X \in \text{Set})\) one finds bijections (see Remark 3.15).

2. If \( 1 \in \mathcal{C} \) is terminal, a left (resp. right) adjunctible map is initial (resp. final). Indeed

\[
\exists^\ell_X f^\ell \exists^\ell_y I \cong \exists^\ell_X \exists^\ell_{f^{-1}y} I \cong I.
\]

Right (resp. left) adjunctible maps preserve limits (resp. colimits) and right (resp. left) Kan extensions:

\[
\begin{array}{cccc}
(f\{M, g\})^r \exists^r_y I & (f\{M, g\})^r \exists^r_y I \\
\{M, g\}^r f^r \exists^r_y I & (\forall_h f^r g)^r \exists^r_y I \\
\{M, g\}^r \exists^r_{f^{-1}y} I & (\forall_h f^r g)^r \exists^r_{f^{-1}y} I \\
\{M, g\}^r f^r \exists^r_{f^{-1}y} I & (\forall_h f^r g)^r \exists^r_{f^{-1}y} I \\
\{M, (fg)\}^r \exists^r_y I & (\forall_h (fg)^r \exists^r_y I) \\
\{M, (fg)\}^r \exists^r_y I & (\forall_h (fg)^r \exists^r_y I) \\
\{M, (fg)\}^r \exists^r_y I & (\forall_h (fg)^r \exists^r_y I)
\end{array}
\]

It seems natural to define (left and right) adjoint maps by their classical characterization in terms of Kan extensions: \( g \dashv f \) iff

\[
\begin{array}{cccc}
t g & \iff & \{M, tg\} & \iff & \exists^r_y M & \iff & f^r M
d \{M, tg\} & \iff & \{M, \forall_f t\} & \iff & \{f^r M, t\} & \iff & \exists^r_y M & \iff & f^r M
\end{array}
\]

which, since \( \exists^r_y \dashv g^r \), is equivalent to \( f^r \dashv g^r \). Symmetrically, \( \exists^r_y \dashv g^r \) is equivalent to \( g^\ell \dashv f^\ell \).

Of course, a left (resp. right) adjoint map is left (resp. right) adjunctible: \( f^r \exists^r_y I \cong \exists^r_y I \).

4. The symmetrical comprehension adjunction

In this section we consider the \( \textbf{Cat} \)-indexed category \( \text{Set}^{\text{Set}^{X^{\text{op}}} \times X} \), with substitution along \( f : X \to Y \) given by \( \tilde{f}H(x, x') := H(fx, fx') \), and its relationships with \( \text{Set}^{\text{Set}^{X^{\text{op}}} \times X} \) and \( \text{Cat}/X \). While we do not propose here any abstraction or generalization, some of the results presented may suggest steps in that direction.
4.1. The Indexed Category of Endoprofunctors. Note that $(\text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}; \mathbf{X} \in \mathbf{Cat})$ can be seen as $(\text{Set}^{\mathbf{X}}; \mathbf{X} \in \mathbf{Cat})$ restricted to the categories of the form $\mathbf{X}^{\text{op}} \times \mathbf{X}$ and to the functors of the form $f^{\text{op}} \times f$, so that $\bar{f}$ becomes $(f^{\text{op}} \times f)^r$ and has adjoints $\exists f \dashv \bar{f} \dashv \forall f$ given by $\exists f := \exists f^{\text{op}} f$ and $\forall f := \forall f^{\text{op}} f$. Since $\mathbf{X}^{\text{op}} \times \mathbf{X}$ is canonically self-dual, any one of its right actions $H$ correspond to a left action $H'$ given by $H'(x, y) := H(y, x)$. Furthermore, $(f \times f^{\text{op}})H' = ((f \times f)^r)H'$. The projections of $\mathbf{X}^{\text{op}} \times \mathbf{X}$ induce the “dummy inclusion” indexed functors $\delta^\ell : \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \to \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$, $\delta^r : \text{Set}^{\mathbf{X}} \to \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$.

Given $A : \mathbf{X}^{\text{op}} \to \text{Set}$ and $M : \mathbf{X} \to \text{Set}$, we define $A \mapsto M$ by $(A \mapsto M)(x, y) = Ax \times My$, that is as the composite

$$\begin{array}{c}
\text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \xrightarrow{\text{Set} \times \text{Set}} \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \xrightarrow{\text{Set} \times \text{Set}} \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}
\end{array}$$

This “operation” $\mapsto : \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \to \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$ is indexed, in the sense that $f^\ell A \mapsto f^r M \cong \bar{f}(A \mapsto M)$. (Note that $A \mapsto M$ is the product $\delta^\ell A \times \delta^r M$ in $\text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$.) Another operation $\Rightarrow^r : \text{Set}^{\mathbf{X}} \times \text{Set}^{\mathbf{X}} \to \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$ is obtained by posing $(M \Rightarrow^r N)(x, y) = [Mx, Ny]$:

$$\begin{array}{c}
\text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \xrightarrow{\text{Set} \times \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}} \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \xrightarrow{\text{Set} \times \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}} \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}
\end{array}$$

(and similarly one defines $\Rightarrow^\ell : \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \to \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$). Note that $I_\mathbf{X} \mapsto M \cong \delta^r M \cong I_\mathbf{X} \Rightarrow^r M$, $A \mapsto I_\mathbf{X} \cong \delta^\ell A \cong I_\mathbf{X} \Rightarrow^\ell A$, where $I_\mathbf{X}$ is the terminal presheaf on $\mathbf{X}$.

Now, given $H \in \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}}$, let us define a category $i_\mathbf{X} H$ over $\mathbf{X}$ as follows:

- the objects over $x \in \mathbf{X}$ are the elements of $H(x, x)$;
- given $\lambda : x \to y$ in $\mathbf{X}$, there is at most one arrow from $a \in H(x, x)$ to $b \in H(y, y)$ over $\lambda$, and this is the case iff $H(x, \lambda)a = H(\lambda, y)b \in H(x, y)$.

Then one easily verifies that (see also [Pisani, 2007] and the references therein):

1. This constructions is the object map of an indexed functor $i_\mathbf{X} : \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \to \mathbf{Cat}/\mathbf{X}$.
2. The “category of elements” functors $i^\ell_\mathbf{X} : \text{Set}^{\mathbf{X}^{\text{op}}} \to \mathbf{Cat}/\mathbf{X}$ and $i^r_\mathbf{X} : \text{Set}^{\mathbf{X}} \to \mathbf{Cat}/\mathbf{X}$ factor through it: $i^\ell_\mathbf{X} \cong i_\mathbf{X} \delta^\ell$, $i^r_\mathbf{X} \cong i_\mathbf{X} \delta^r$.
3. $i_\mathbf{X}(A \mapsto M)$ is the product $i^r_\mathbf{X} A \times i^\ell_\mathbf{X} M$ in $\mathbf{Cat}/\mathbf{X}$. In particular, for objects $x, y : 1 \to \mathbf{X}$, $i_\mathbf{X}(\exists^\ell y \exists^r 1) \cong x \setminus X \times Y/y$ is the “interval” category $[x, y]$ (over $\mathbf{X}$) with objects $x \to z \to y$.
4. $i_\mathbf{X}(M \Rightarrow^r N)$ is the exponential $(i^r_\mathbf{X} N)^{i^\ell_\mathbf{X} M}$ in $\mathbf{Cat}/\mathbf{X}$ (and similarly for $i_\mathbf{X}(A \Rightarrow^\ell B)$).
5. The inclusion $i_\mathbf{X} : \text{Set}^{\mathbf{X}^{\text{op}} \times \mathbf{X}} \to \mathbf{Cat}/\mathbf{X}$ is not full. Indeed,

$$\mathbf{Cat}/\mathbf{X}(i_\mathbf{X} H, i_\mathbf{X} K) \cong \text{Din}_\mathbf{X}(H, K)$$

that is one gets the strong dinatural transformations, also known as “Barr dinatural”. Recall also that

$$\text{Din}_\mathbf{X}(A \mapsto M, K) \cong \text{Din}_\mathbf{X}(A \mapsto M, K)$$
that is, the dinatural transformations with domain $A \times M$ are also strongly dinatural, for any $K$.

Then, since the end and the (strong) coend of $H$ are representations of the functors $\text{Din}_X(\Delta X S, H)$, $\text{Din}_X^*(H, \Delta X S)$ and $\text{Din}_X(H, \Delta X S)$ respectively, we get:

4.2. Proposition. The set of sections of $i_X H$

$$\text{Cat}/X(\text{id}_X, i_X H) \cong \forall^\ell_X \Box^\ell_X (i_X H) \cong \forall^r_X \Box^r_X (i_X H)$$

gives the end of $H : X^\text{op} \times X \to \text{Set}$. The components of (the total of) $i_X H$

$$\bigotimes X (i_X H) \cong \exists^\ell_X \bigotimes^\ell_X (i_X H) \cong \exists^r_X \bigotimes^r_X (i_X H)$$

gives the strong coend of $H$. The coend of $A \times M$ coincides with its strong coend and with the mixed tensor product of Section 2:

$$A \ast M \cong \bigotimes X (i_X^l A \times i_X^r M) \cong \exists^\ell_X \bigotimes^r_X (i_X^l A \times i_X^r M) \cong \exists^r_X \bigotimes^r_X (i_X^r A \times i_X^r M)$$

4.3. Remarks.

1. It is natural to see $i_X H$ as a sort of “diagonal” extension of $H$ and to denote it by $\{ x \in X | H(x, x) \}$ (see [Lawvere, 1970] and Remark 4.6). Then the usual end notation $\int_{x \in X} H(x, x)$ can be replaced (for set-valued functors) by the end formula $\int_X \{ x \in X | H(x, x) \}$ of the above proposition, where $\int_X$ is the sections functor, right adjoint to $\text{Set} \to \text{Cat}/X$.

2. As a corollary, one gets the naturality formula for set-valued functors:

$$\begin{array}{c}
\text{end}(M \Rightarrow^r N) \\
\text{Cat}/X(\text{id}_X, i_X (M \Rightarrow^r N)) \\
\text{Cat}/X(\text{id}_X, (i_X^r N)(i_X^r M)) \\
\text{Cat}/X(i_X^r N, i_X^r M) \\
\text{Set}^X(M, N)
\end{array}$$

3. While strong dinaturality has the advantage over dinaturality of arising naturally as a full subcategory of $\text{Cat}/X$ (so that, for instance, strong dinatural transformations always compose) there are other facts that conversely seem to indicate a prevalent role for dinaturality. For instance $\text{Din}_X(H, K)$ (and not $\text{Din}_X^*(H, K)$) can be expressed as an end. Furthermore, the formula (74) below has a correspondent for coends: the mixed tensor product $H' \ast \text{hom}_X (\cong \text{hom}_{X^\text{op}} \ast H)$ gives the (not strong) coend of $H$. 
The following (rephrased) is referred to, in [MacLane, 1965], as “diagonal Yoneda”:

\[
\text{Cat}/X(id_X, i_X H) \cong \text{Set}^{\text{op} \times X}(\text{hom}_X, H)
\]  

and says that there are two ways to express the end of an endoprofunctor. In fact, it is also the key fact to prove the

4.4. Proposition. \[i_X : \text{Set}^{\text{op} \times X} \to \text{Cat}/X\] has a left adjoint \(\ddagger_p \hom_P\), which takes \(p : P \to X\) to \(\exists_p \hom_P\).

Proof.

\[
\begin{array}{c}
\text{Set}^{\text{op} \times X}(\ddagger_p \hom_P, H) \\
\text{Set}^{\text{op} \times P}(\hom_P, \ddagger H) \\
\text{Cat}/P(id_P, i_P(\ddagger H)) \\
\text{Cat}/P(id_P, p^{-1}(i_X H)) \\
\text{Cat}/X(p, i_X H)
\end{array}
\]

In particular, \(\text{hom}_X\) is the reflection of the terminal \(\text{id}_X\) of \(\text{Cat}/X\).

The value of \(\ddagger_X p\) at \((x, y)\) can be expressed in various ways:

\[
(\ddagger_X p)(x, y) \\
p^f \exists_y^I \star p^r \exists_y^I \\
\text{coend}_P(p^f \exists_y^I \tilde{\times} p^r \exists_y^I) \\
\text{coend}^*_p(p^f \exists_y^I \tilde{\times} p^r \exists_y^I) \\
\text{coend}^*_p \ddagger p(\exists_y^I \tilde{\times} \exists_y^I) \\
\ddagger \Sigma_P i_P \ddagger p(\exists_y^I \tilde{\times} \exists_y^I) \\
\ddagger \Sigma_P p^{-1} i_X(\exists_y^I \tilde{\times} \exists_y^I) \\
\ddagger \Sigma_P p^{-1}[x, y] \\
\ddagger \Sigma_X(p \times [x, y])
\]

Indeed, using the usual formula for left Kan extensions (see also Section 3)

\[
(\exists^f M)x \cong f^r \exists_y^I \star M
\]

and observing that, for \((x, y) : 1 \to X^{\text{op}} \times X\), \(\exists^f(x, y) I \cong \exists_y^I \tilde{\times} \exists_y^I\), we get

\[
(\exists^r \rho_{\rho \times \rho} \hom_P)(x, y) \cong (p^{\text{op}} \times p)^f \exists^f(x, y) I \star \hom_P \cong (p^r \exists_x^I \tilde{\times} p^r \exists_y^I) \star \hom_P
\]

which is the third row above (see Remark 4.3 (3)). The other equivalences are immediate.
4.5. Remark. Since \(i_X^* \cong i_X \delta_X^c\) and \(\delta_X^c \cong \pi_2^c\) for the projection \(\pi_2 : X^{op} \times X \to X\), we also have \(\chi_X \cong \exists_\pi^c \times X\), that is \(\exists_\pi^{op} \times X \cong \exists_X \chi^{op} \times X\hom_P \cong \exists_X \exists_\pi^{op} \times X\hom_P\), for any \(p \in \mathbf{Cat}/X\). Indeed, \(\exists_X \chi^{op} \times X\hom_P \cong 1_P\), as one can easily verify directly:

\[
(\exists_\pi^{op} \times X\hom_P)x \cong \exists_\pi^c \times X \ast \hom_P \cong \text{coend}(\delta^c \exists^c I) \cong \exists_X \chi \times_X i_X \chi \exists^c I \cong 1
\]

4.6. Remark. Classically, the third row of (4.3) is written \(\int^{x \in X} \mathcal{P}(x, p,a) \times \mathcal{P}(p,a,y)\). As argued elsewhere, the last row of (4.3) can be seen as the set-valued version of the predicate \(p\) “meets” \([x, y]\). In fact, in two valued contexts we have simplified forms of the adjunction \(\chi \dashv i_X\). For instance, let \(X\) be a poset, \(2^{X^{op} \times X}\) the poset of binary relations on \(X\) compatible with the order, and \(\mathcal{P}X\) the poset of all parts of \(X\). Then we have \(\chi \dashv i_X : 2^{X^{op} \times X} \to \mathcal{P}X\), where \(i_X H = \{x \in X|H(x, x)\}\) and \(x(\chi P)y \iff \exists a \in P(x \leq a \leq y) \iff P \cap [x, y] \neq \emptyset\), where \([x, y] = \{z \in X|x \leq z \leq y\}\); indeed, as before, \([x, y] = i_X H_{xy}\) where \(H_{xy}\) is the product of representables of opposite variance: \(zH_{xy}w \iff x \leq w \& z \leq y\).

4.7. Absolutely dense and fully faithful functor. We conclude by adding further characterizations of absolutely dense and of fully faithful functors (see Section 3).

4.8. Proposition. Each of the following is equivalent to the absolute density of the functor \(f : X \to Y\):

1. \(\exists f^{op} \times f\hom_X \cong \hom_Y\);
2. \(\text{end}_Y H \cong \text{end}_X \tilde{f}H\), naturally in \(H\);
3. \(\text{coend}_Y H \cong \text{coend}_X \tilde{f}H\), naturally in \(H\);
4. \(\chi Zg \cong \chi(gf)\), for any \(g : Y \to Z\);
5. \(\chi_Y f \cong \hom_Y\);
6. \(g\) and \(g(f^{op} \times f)\) have the same (co)end, for any \(g : Y^{op} \times Y \to Z\).

Proof. The first condition is equivalent to absolute density by (75) (see Section 3.14) and is equivalent to the second one by

\[
\begin{array}{ccc}
\text{end}_X \tilde{f}H & \in \text{Set}^{X^{op} \times X}((\hom_X, \tilde{f}H) \\
\text{Set}^{Y^{op} \times Y}((\exists_f \hom_X, H)) & ; & \text{Set}^{Y^{op} \times Y}((\exists_f \hom_X, H)) \\
\text{Set}^{Y^{op} \times Y}((\exists_f \hom_X, H)) & \in \text{Set}^{Y^{op} \times Y}((\hom_Y, H)) \\
\text{end}_Y H & \in \text{Set}^{Y^{op} \times Y}((\hom_Y, H))
\end{array}
\]
Similarly one proves the equivalence between the first one and the third one:

\[
\begin{array}{c}
\text{coend}_X \tilde{f}H \\
\hom_{X^{\text{op}}} * \tilde{f}H \\
\exists_{f^{\text{op}}} \hom_{X^{\text{op}}} * H \\
\hom_{Y^{\text{op}}} * H \\
\text{coend}_Y H
\end{array}
\]

\[
\begin{array}{c}
\exists_{f^{\text{op}}} \hom_{X^{\text{op}}} * H \\
\hom_{X^{\text{op}}} * \tilde{f}H \\
\text{coend}_X \tilde{f}H \\
\\end{array}
\]

The fourth one follows from the first one by Proposition 4.4:

\[
\begin{array}{c}
\Diamond_z (gf) \\
\exists_{gf} \hom_X \\
\exists_g \exists_f \hom_X \\
\exists_g \hom_Y \\
\Diamond_z g
\end{array}
\]

and implies the fifth one (for \( g = \text{id}_Y \)) which (again by Proposition 4.4) is equivalent to the first one. As for the last condition, since a (co)end is a (co)limit weighted by \( \hom \) the technique of Section 3 applies:

\[
\begin{array}{c}
\text{coend} g (f^{\text{op}} \times f) \\
\hom_{X^{\text{op}}} * g (f^{\text{op}} \times f) \\
\exists_{f^{\text{op}}} \hom_{X^{\text{op}}} * g \\
\hom_{Y^{\text{op}}} * g \\
\text{coend} g
\end{array}
\]

\[
\begin{array}{c}
\exists_{f^{\text{op}}} \hom_{X^{\text{op}}} * g \\
\hom_{X^{\text{op}}} * g (f^{\text{op}} \times f) \\
\text{coend} g (f^{\text{op}} \times f) \\
\\end{array}
\]

\[
\begin{array}{c}
\text{coend} g \\
\hom_{Y^{\text{op}}} * g
\end{array}
\]

4.9. Remark. From a two-valued point of view, the second, third and fourth conditions of Proposition 4.8 become respectively: “\( H(-, -) \) is reflexive on \( Y \) iff \( H(f-, f-) \) is so on \( X \)”, “\( H(-, -) \) has ‘fixed points’ on \( Y \) iff \( H(f-, f-) \) has them on \( X \)” and “\( g \) and \( gf \) have the same (symmetric) image”.

4.10. Corollary. If \( f \) and \( gf \) are absolutely dense, so is \( g \).
4.11. **Proposition.** Each of the following is equivalent to the full faithfulness of the functor $f : X \to Y$:

1. $\tilde{f} \text{hom}_Y \cong \text{hom}_X$;
2. $\text{end}_X H \cong \text{end}_Y \tilde{\nu}_f H$, naturally in $H$;
3. $\text{coend}_X H \cong \text{coend}_Y \tilde{\exists}_f H$, naturally in $H$.

**Proof.** The equivalence between the first two conditions is given by

\[
\begin{array}{c|c}
\text{end}_Y \tilde{\nu}_f H & \text{Set}^{X^{\text{op}} \times Y}(\tilde{f} \text{hom}_Y, H) \\
\text{Set}^{X^{\text{op}} \times Y}(\text{hom}_Y, \tilde{\nu}_f H) & ; \\
\text{Set}^{X^{\text{op}} \times X}(\tilde{f} \text{hom}_Y, H) & \text{end}_Y \tilde{\nu}_f H \\
\text{Set}^{X^{\text{op}} \times X}(\text{hom}_X, H) & \text{end}_X H \\
\text{end}_X H & \text{Set}^{X^{\text{op}} \times X}(\text{hom}_X, H)
\end{array}
\]

and similarly one gets the equivalence between the first and the third ones.

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*via Saluzzo 67,*

10125 Torino, Italy.

Email: pisclau@yahoo.it