Generation of density waves in dipolar quantum gases by time-periodic modulation of atomic interactions

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We study the emergence of density waves in dipolar Bose-Einstein condensates (BEC) when the strength of dipole-dipole atomic interactions is periodically varied in time. The proposed theoretical model, based on the evolution of small perturbations of the background density, allows to compute the growth rate of instability (gain factor) for arbitrary set of input parameters, thus to identify the regions of instability against density waves. We find that among other modes of the system the roton mode is most effectively excited due to the contribution of sub-harmonics of the excitation frequency. The frequency of temporal oscillations of emerging density waves coincides with the half of the driving frequency, this being the hallmark of the parametric resonance, is characteristic to Faraday waves. The possibility to create density waves in dipolar BECs, which can persist after the emergence, has been demonstrated. The existence of a stationary spatially periodic solution of the nonlocal Gross-Pitaevskii equation has been discussed. The effect of three-body atomic interactions, which is relevant to condensates with increased density, upon the properties of emerging waves has been analyzed too. Significant modification of the condensate’s excitation spectrum owing to three-body effects is shown.

I. INTRODUCTION

During the last few years significant new achievements in the field of dipolar Bose-Einstein condensates (BEC) have been reported, such as self-bound quantum droplets [1, 2], experimental observation of the quasi-particle called “roton” [3, 4] and supersolid phase of matter [5–9]. These exotic phenomena can show up in dipolar quantum gases owing to a long-range and anisotropic nature of the atomic interactions. Besides, the dipolar interactions is responsible for a significant modification of well known phenomena that were observed and studied in ordinary quantum gases with short-range atomic interactions. For instance, the spectrum of elementary excitations, features of modulational instability, interactions between solitons, emergence and evolution of Faraday waves, and some other phenomena are substantially different in quantum gases with long-range (dipolar) atomic interactions as compared to ones with short-range (contact) interactions. Another notable feature of dipolar BECs is that they support stable bound states of solitons, so called soliton molecules [10–12].

The Faraday instability belongs to extensively studied subjects in physics. Here we briefly recall that emergence of characteristic wave patterns on the surface of a vertically vibrating water vessel, upon reaching certain frequency and amplitude, was observed for the first time by Michael Faraday [13]. The Faraday instability is the classical example of parametric instability, which occurs when the characteristic coefficients of the nonlinear system are periodically varied in time. As outcome of such instability a spatially periodic wave emerges in the medium with initially uniform properties, which is referred to as a Faraday wave. To date the Faraday waves are investigated in many areas of science, including the hydrodynamics, nonlinear optics, chemical reactions and biological systems, and still represents an active field of research (see e.g. review articles [14, 15]). Faraday waves are extensively studied also in BECs, both theoretically [16–24] and experimentally [15, 25]. An interesting situation, when the effective atomic interaction in a two-component BEC periodically changes due to the Rabi oscillations, leading to Faraday patterns through parametric resonance, has been reported in a recent paper [26]. This case is special because Faraday patterns are generated without any modulation of external parameters of the system. Here the recurrent population imbalance between two hyperfine states of a binary BEC leads to periodic change of the strength of effective interaction, which in its turn, triggers the Faraday instability.

Dipolar BECs represent an excellent system for investigation of nonlocal effects in quantum gases. The interplay between nonlinearity and nonlocality clearly shows up in the dynamics of dipolar BEC by significantly altering its excitation spectrum and regions of Faraday instability [27]. The existence of roton quasi-particles in dipolar BECs, i.e. rotonization of the excitation spectrum [8], can be revealed from features of emerging Faraday patterns. Specifically, the presence of even shallow roton minimum in the dispersion curve, leads to abrupt changes in the dependence of pattern size as a function of the driving frequency [18], as opposed to monotonic dependence in roton-free condensates [23]. Due to high sensitivity of density patterns to the presence of rotons, dipolar BECs can represent an excellent system to explore the physics of rotons.

In certain conditions the effect of higher order nonlinearities may become essential for consistent interpretation of experimental results. For instance, important role of three-body atomic interactions in the formation of quantum
droplet patterns in a $^{164}$Dy condensate, was demonstrated in numerical simulations of the nonlocal Gross-Pitaevskii equation with cubic-quintic nonlinearity [28].

In this paper we study the density waves in dipolar BECs, emerging and evolving under periodically varying strength of the dipole-dipole atomic interactions. The contribution of three-body atomic interactions upon the dynamics of density waves has been investigated too. Related phenomena in the context of ordinary BECs with contact interactions were explored in Ref. [21]. However, as pointed out earlier, the long-range dipolar atomic interactions drastically modify the whole dynamics of the condensate, including the features of density waves. Therefore, we can expect to observe new facets of density waves in dipolar BECs when three-body atomic interactions are taken into account. The remaining part of the paper is organized as follows. In the next Sec. II we introduce the mathematical model and derive a Mathieu type equation to address the wave instability in dipolar BEC. In Sec. III we discuss the results of numerical simulations. Main results of this work is summarized in the last Sec. IV.

II. THE MODEL AND ANALYSIS OF WAVE INSTABILITY

Below we consider a dipolar BEC with two- and three- body atomic interactions. We assume that atomic dipoles are oriented along the $x$- axis of the trap. The dynamics of the condensate is described by the following 3D Gross-Pitaevskii equation (GPE):

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = \left[ \frac{-\hbar^2}{2m} \nabla^2 + V(\vec{r}) + g_1 |\Psi(\vec{r}, t)|^2 + g_2 |\Psi(\vec{r}, t)|^4 + \frac{C_d}{4\pi} \int_{-\infty}^{\infty} \frac{1 - \frac{3(x-x')^2}{|\vec{r}-\vec{r'}|^2}}{|\vec{r}-\vec{r'}|^3} |\Psi(\vec{r'}, t)|^2 d^3r' \right] \Psi(\vec{r}, t),$$

where $\Psi(\vec{r}, t)$ is the mean-field wave function of the condensate, normalized to the number of atoms $N = \int_{-\infty}^{\infty} |\Psi(\vec{r})|^2 d\vec{r}$, $m$ is the atomic mass. The BEC is trapped in external harmonic potential $V(\vec{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$. The strengths of two-body and three-body atomic interactions are represented by coefficients $g_1$ and $g_2$, respectively. The dipolar interactions are expressed through the parameter $C_d = \mu_0 \mu^2$ or $C_d = d^2/\varepsilon_0$, depending on whether the atoms possess a magnetic ($\mu$) or electric ($d$) dipole moment, where $\mu_0$, $\varepsilon_0$ are the permeability and the permittivity of vacuum, respectively. When the effective length of magnetic dipolar interactions $a_{dd} = \mu_0 \mu^2 m/12\pi \hbar^2$ is greater than the length of $s$-wave contact interactions $a_s$, the condensate is said to be in the dipolar dominated regime ($a_{dd} > a_s$).

To simplify the analysis, we consider a 1D geometry which assumes a strong confinement in the radial direction $\omega_z \gg \omega_x$. In this case the transverse (in $y - z$ plane) dynamics of the condensate is frozen to the ground state $\phi(y, z) = (1/\sqrt{\pi a_\perp}) \exp(-y^2 + z^2)/2a_\perp^2$ of the 2D harmonic potential with $a_\perp = \sqrt{\hbar/m\omega_z}$ being the harmonic oscillator length. Consequently, the wave function can be factorized as $\Psi(\vec{r}, t) = \phi(y, z) \psi(x, t)$. Then integration over transverse coordinates yields the following 1D GPE

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + \frac{m \omega_x^2 x^2}{2} \psi + \frac{g_1}{2\pi a_\perp^2} |\psi|^2 \psi + \frac{g_2}{3\pi a_\perp^4} |\psi|^4 \psi - \frac{C_d}{2\pi a_\perp^2} \psi \int_{-\infty}^{\infty} R \left( \frac{x-x'}{a_\perp} \right) |\psi(x', t)|^2 dx',$$

where $R(x) = \sqrt{\pi} \exp(x^2) (1 + 2x^2) \text{erfc}(|x|) - 2|x|$ is the response function characterizing the nonlocal feature of the dipolar BEC [29].

Next, by introducing new variables $t \rightarrow \omega_z t$, $x \rightarrow x/a_\perp$ and $\psi \rightarrow \sqrt{2a_s/|\psi|} \psi$, we reduce the Eq. (2) into following dimensionless form

$$i\psi_t + \frac{1}{2} \psi_{xx} + q|\psi|^2 \psi + p|\psi|^4 \psi + g \psi(x, t) \int_{-\infty}^{\infty} R(|x-x'|) |\psi(x', t)|^2 dx' = 0,$$

where for the sake of clarity we denoted the dimensionless coefficient of attractive cubic term as $q = a_s/a_{bg}$, with $a_s$, $a_{bg}$ being the $s$-wave scattering length and its background value. The coefficients of quintic and dipolar terms have the form $p = g_2 m \omega_x^2/12\pi^2 \hbar^3 a_\perp^2$, $g = C_d/(4\pi \hbar^2 a_s/m)$. In addition, we have excluded the weak axial trap potential ($\omega_z \ll \omega_x$) and adopted periodic boundary conditions for the wave function. This equation is valid as long as the chemical potential of the condensate is much less than the energy of radial excitations $\mu/\hbar \omega_z = 2n_{1D} a_s |1 - \varepsilon_d| \ll 1$, where $\varepsilon_d = a_{dd}/a_s$ is the ratio between strengths of dipolar and contact interactions, $n_{1D}$ is the one-dimensional density of the condensate. Both the dipolar and contact interaction coefficients are tunable by external magnetic fields [30-32].
The density waves in BEC can be induced by periodic modulation of the parameters $q$, $p$, or $g$ in Eq. (3). Below we assume variable dipolar interactions with amplitude $\alpha$ and frequency $\omega$

$$g(t) = g_0[1 + 2\alpha \cos(2\omega t)].$$  \hspace{1cm} (4)

In the absence of modulations ($\alpha = 0$), the condensate remains in a uniform state with constant density. This state is described by the plane wave solution of Eq. (3) with amplitude $A$ and phase $\theta$

$$\psi = A e^{i\theta(t)}, \quad \theta(t) = A^2(q + g_0K + pA^2)t, \quad K = \int_{-\infty}^{\infty} R(|x|) \, dx.$$  \hspace{1cm} (5)

To study the emergence and dynamics of density waves, we introduce weak perturbation of the uniform state given by Eq. (5), and look for a solution in the form

$$\psi = (A + \delta \psi)e^{i\theta}, \quad \delta \psi \ll A.$$  \hspace{1cm} (6)

The equation for the perturbation is obtained by substituting Eq. (6) into Eq. (3), and keeping only the linear terms on $\delta \psi(x, t)$

$$i\delta \psi_t + \frac{1}{2}\delta \psi_{xx} + A^2(q + 2pA^2)(\delta \psi + \delta \psi^*) + gA^2 \int_{-\infty}^{\infty} R(|x - x'|)[\delta \psi(x', t) + \delta \psi^*(x', t)] \, dx' = 0.$$  \hspace{1cm} (7)

By representing $\delta \psi = u + iv$ and performing a Fourier transform, we obtain for the Fourier component $\bar{u}(k, t) = \int_{-\infty}^{\infty} u(x, t) e^{ikx} \, dx$ the Mathieu type equation

$$\bar{u}_{tt} + \left[\Omega^2(k) - b \cos(2\omega t)\right] \bar{u} = 0,$$  \hspace{1cm} (8)

where

$$\Omega^2(k) = \frac{k^2}{4}[k^2 - 4A^2(q + 2A^2p + g_0\bar{R}(k))],$$  \hspace{1cm} (9)

$$b = 2A^2\alpha g_0k^2\bar{R}(k),$$  \hspace{1cm} (10)

$$\bar{R}(k) = 2 \left[1 + \frac{k^2}{4} e^{k^2/4} \text{Ei}\left(-\frac{k^2}{4}\right)\right], \quad \text{with} \quad \text{Ei}(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t} \, dt.$$  \hspace{1cm} (11)

being the exponential integral function \[38\].

Equation (8) for the Fourier component of the perturbation is equivalent to that of the parametrically excited oscillator with an internal frequency $\Omega(k)$. Similar equation in the context of Faraday waves in dipolar BEC was derived in \[27\]. In our setting Eq. (8) extends the results of that work by taking into regard the effect of three-body atomic interactions.

The spectrum of elementary excitations defines the basic properties of the condensed matter. For dipolar BEC it is represented by dispersion relation Eq. (9), which is illustrated in Fig. 1. A remarkable property of dipolar BEC is that, it supports the roton quasi-particles. A peculiar shape of the dispersion curve with a local minimum at $\frac{\pi}{k_{ro}}$ is the hallmark of the roton mode (red curve in Fig. 1a). Originally, the existence of rotons in quantum fluids was postulated by L. Landau in 1941 to explain the properties of superfluid $^4\text{He}$ \[34\], and later they were observed in neutron scattering experiments \[32\]. The presence of rotons in dipolar quantum gases was theoretically predicted in Ref. \[35\] and they had been observed in recent experiments \[3 \& 4\]. Below we shall refer to condensates, whose excitation spectra possess (do not possess) a local minimum as rotonized (roton-free) condensates.

Rotons are responsible for the tendency of a system to establish a local order, with typical length scale $\sim 1/k_{ro}$, which is equal approximately to the inter-particle distance. While in strongly interacting systems, such as superfluid $^4\text{He}$, this indeed holds true, in dilute quantum gases the atomic correlations are due to the long-range dipolar interactions with a length scale, greatly exceeding the inter-particle distances. As a consequence of the roton mode, a long-lived super-solid phase in dipolar BEC can emerge \[5\]. The quintic nonlinearity, designated by a coefficient $p$ in Eq. (3), can significantly modify the dispersion curve of a dipolar BEC. Even relatively small quintic nonlinearity may lead to emergence of a roton minimum in a roton free condensate, as shown in Fig. 1b.

Time-periodic modulation of the coefficient of nonlinearity may lead to generation of different kinds of elementary excitations (quasi-particles) in the system, such as phonons, rotons, resonance waves, Faraday waves, etc. Which kinds of these waves are predominantly generated depends primarily on the properties of the condensate, and external parameters, such as the amplitude and frequency of the perturbation. In some regions of the parameter space, several types of waves may be simultaneously generated. All above mentioned types of excitations reveal themselves as density modulations in the condensate, thus below we refer to them as density waves.
Since the coefficient which indicates the presence of roton quasi-particles in the condensate (red solid line). If the strength of contact interactions be represented as a system of two first order equations

$$t_c \approx \frac{\sqrt{t_c^2 - 4 d_c}}{2}$$

where $$t_c = u_1(T) + v_2(T)$$ and $$\lambda_{1,2} = t_c \pm \sqrt{t_c^2 - 4 d_c}$$ are the trace and determinant of the matrix $$C$$, respectively. It is easy to show using Eqs. (12) and (13) that the determinant of the fundamental solution matrix is equal to unity at any time $$d_c(t) = d_c(0) = d_c(T) = 1$$, therefore $$\lambda_{1,2} = (t_c \pm \sqrt{t_c^2 - 4})/2$$. Since the product of the two eigenvalues is

FIG. 1: (Color online) The excitation spectrum of a dipolar BEC according to Eq. (9) for different sets of input parameters. (a) The dispersion curve in absence of the quintic term ($p = 0$) features a local minimum at finite momentum $$k_{rot} \approx 2.35$$, which indicates the presence of roton quasi-particles in the condensate (red solid line). If the strength of contact interactions is slightly reduced ($q = 0.8$) or increased ($q = 1.0$), the roton minimum disappears (blue dashed line) or deepens (purple dashed line), respectively. (b) A small attractive quintic nonlinearity may induce a roton minimum (purple dashed line) in a roton-free condensate (red solid line). Parameter values: $$A = 2$$, $$q = 1$$, $$g_0 = -1$$ (a), $$g_0 = -3.5$$ (b).

III. NUMERICAL SIMULATIONS

For numerical simulations of the nonlocal GPE we use the split-step fast Fourier transform method with 2048 Fourier modes and periodic boundary conditions. The integration domain was selected sufficiently large $$x \in [-40\pi, 40\pi]$$ to accommodate ~100 periods of emerging waves on average, and the time step was $$\Delta t = 0.0005$$. To evaluate the integral term of the equation the convolution theorem has been employed. To minimize disturbances, caused by a possible mismatch between the domain length and integer number of periods of emerging waves, we consider only the small amplitude limit. As initial perturbation of the background amplitude we use random numbers $$x \in \mathbb{R}$$ with 2048 Fourier modes and periodic boundary conditions. The integration domain was selected sufficiently large

Next, we produce the matrix $$C$$, which is the fundamental solution matrix, evaluated at time $$T$$

$$C = \begin{bmatrix} u_1(T) & u_2(T) \\ v_1(T) & v_2(T) \end{bmatrix}$$

According to Floquet’s theory, stability of solutions to Eq. (12) is determined by the eigenvalues of the matrix $$C$$

$$\lambda_{1,2} = \frac{t_c \pm \sqrt{t_c^2 - 4 d_c}}{2}$$

It is easy to show using Eqs. (12) and (13) that the determinant of the fundamental solution matrix is equal to unity at any time $$d_c(t) = d_c(0) = d_c(T) = 1$$, therefore $$\lambda_{1,2} = (t_c \pm \sqrt{t_c^2 - 4})/2$$. Since the product of the two eigenvalues is
\( \lambda_1 \cdot \lambda_2 = 1 \), the instability sets in if either of the eigenvalues has modulus larger than unity (because the other one is less than unity). Therefore, the instability develops in all cases when \( |t_c| > 2 \). If \( |t_c| < 2 \), then Eq. (15) has two complex conjugate roots, both lying on the unit circle, since their product is unity. This situation corresponds to stable quasi-periodic motion. In particular case \( t_c = 2 \) (\( t_c = -2 \)) the motion is stable with period \( T \) (2\( T \)).

An important quantity, relevant to the unstable solution of Eq. (12) is called a characteristic exponent of the system, or gain factor (G), and is defined as follows

\[
e^{G_T} = \lambda, \quad G = \frac{1}{T} \ln(\lambda),
\]

where the principal value of the logarithm is assumed. The amplitude of the unstable wave, once emerged, grows exponentially with time \( |\delta \psi| \sim e^{G t} \).

A. The domains of instability and features of density waves

The proposed theoretical approach, expressed through Eqs. (12)-(15), allows to identify the domains of instability in the parameter space against generation of density waves in dipolar BECs. The growth rate of instability, also known as gain factor, given by Eq. (16), can be considered as a function of system’s parameters \( G = F(A, q, g_0, p, \alpha, k, \omega) \), and its numerical value can be calculated for arbitrary set of these parameters using the Floquet’s theory.

In Fig. 2 we show the domains of instability (dark regions) in the space of two parameters: the wave vector of emerging density waves (\( k \)), and the strength of modulation of the dipolar interactions (\( \alpha \)), at fixed values of other parameters. The upper row in this figure corresponds to the case, when the roton mode in the dispersion curve is present (red line in Fig. 1a). Conspicuous island of instability appears around \( k_{\text{rot}} = 2.35 \) at parameter values \( \alpha = 0.055 \), \( \omega = 0.52 \), and expands in size as the modulation frequency increases. This island of instability disappears if the roton mode is eliminated from the spectrum of excitations (see Fig. 1) by reducing the strength of contact interactions from \( q = 1 \) to \( q = 0.8 \) (blue dashed line in Fig. 1a), which is shown in the lower panels of Fig. 2. The island of instability does not appear also in other settings without the roton mode, for instance, due to contribution of a quintic nonlinearity in Eq. (9). These observations suggest, that the revealed island of instability corresponds to excitation of the roton mode in dipolar BECs, subject to periodic variation of the strength of dipolar interactions. It should be noted, that similar island of instability was found in numerical simulations of the 2D dipolar BEC [18], however its relevance to the roton mode was not explicitly commented there. A qualitative difference between the instability domains of rotonized and roton-free dipolar condensates is clearly evident from the upper and lower panels of Fig. 2.

Figure 3 illustrates the domains of instability and gain factors, corresponding to generation of density waves with \( k = k_{\text{rot}} \) in a rotonized, and roton-free condensates. As can be noted from comparing the upper and lower panels of
this figure, for the rotonized case the contribution of the main modulation frequency and its sub-harmonics towards generation of density waves greatly exceeds the similar contribution for the roton-free condensate. This implies, that if the excitation spectrum of a dipolar BEC features a local minimum at \( k = k_{\text{rot}} \) (see Fig. 1a, red line), then the density waves are excited much more effectively.

![Figure 3](image)

**FIG. 3:** (Color online) The domains of instability (a, c) and gain factors (b, d) concerning generation of a density wave with \( k = k_{\text{rot}} \) for the rotonized (a,b) and roton-free (c,d) condensates. Vertical dashed lines represent sub-harmonics of the main modulation frequency \( \omega/n \), with \( \omega = 0.82 \) (a), \( \omega = 2.25 \) (c), \( n = 1, 2, 3, 4 \). Parameter values: \( A = 2, g_0 = -1, q = 1 \) (a,b), \( q = 0.8 \) (c,d), \( k_{\text{rot}} = 2.35 \).

In Fig. 4 we show the gain factor as a function of two input parameters: the frequency of modulations of the dipolar interactions and wave vector of emerging density waves for the roton-free and rotonized condensates. In the same figures we show by symbols the result of numerical simulations. To obtain these numerical data we solve the GPE [3] with time-periodic modulation of the strength of dipolar interactions as per Eq. (4). When the amplitude of emerging waves reaches few percent of the background amplitude, the simulation is terminated and the spatial period of emerging density waves is measured, and therefore its wave vector \( k \) is determined. In the roton-free case (Fig. 4a) the dependence \( \omega(k) \) is monotonic, and the results of numerical simulations are in good agreement with

![Figure 4](image)

**FIG. 4:** (Color online) The growth rate of instability \( G(\omega, k) \) as a function of the modulation frequency and wave vector of emerging waves, according to Floquet’s theory [10], represented through a density plot. The darker regions correspond to higher gain factors \( G(\omega, k) \). Red symbols designate values obtained from numerical simulations of Eq. (3). The parameters correspond to absence (a) and presence (b) of the roton mode in the excitation spectrum. Parameter values: \( A = 2, g_0 = -1, p = 0, \alpha = 0.02, q = 0.8 \) (a), \( q = 1 \) (b).
the prediction of the Floquet theory. For the same reason, a good agreement is observed also in the rotonized case for the frequency values above the local maximum, corresponding to wave vectors \( k > k_{\text{rot}} \). In the rotonized case (Fig. 4a), for one frequency value \( \omega_0 \) from the interval between the local minimum and local maximum, there are three roots of the equation \( \omega(k) = \omega_0 \). Despite the possibility of three resonant wave vectors, the density wave with the greatest \( k \) (out of three) emerges. This can be understood by looking at Eq. (5), where the strength of modulation is increasing function of the wave vector \( b \sim k^2 \tilde{R}(k) \). For the values of modulation frequency, below the roton minimum, the contribution of sub-harmonics dominate (see Fig. 6a,b), resulting in generation of density waves mainly with \( k \approx k_{\text{rot}} \). Thus, in the condensate with a rotonized excitation spectrum, low frequency modulations of the dipolar interactions gives rise to emergence of density waves, corresponding to rotons.

B. Generation of persistent density waves in dipolar condensates

From the analysis of previous sections it follows that time periodic modulation of the strength of dipolar interactions gives rise to spatially periodic density waves in the condensate. In those regions of the parameter space, where the gain factor is a broad function of the wave vector \( k \in [k_{\text{min}}, k_{\text{max}}] \), as for example near the roton minimum, the emerging density wave represents a superposition of many wave components. Constructive and destructive interference of these components produce a picture, where a high frequency wave appears as modulated by a low frequency envelope (Fig. 5b). In the opposite case of a narrow, delta function-like gain factor \( G(k) \sim \delta(k) \), a wave with a single spatial period is generated (Fig. 5a).

![FIG. 5: (Color online) Typical density waves in a dipolar BEC with rotonized (a) and roton-free (b) excitation spectrum, emerging at particular time instance \( t_0 = 87 \) (a) and \( t_0 = 600 \) (b), in numerical simulations of the GPE (3). The period of density waves \( \nu = 2 \pi/k; \nu = 2.69 \) (a), \( \nu = 15.23 \) (b). Parameter values: \( A = 2, q = 1 \) (a), \( q = 0.8 \) (b), \( g_0 = -1, p = 0, \alpha = 0.02, \omega = 0.8 \).](https://example.com/figure5)

The Fig. 5 illustrates the correlation between the predictions of theoretical Floquet analysis for the gain factor, and the results of direct numerical GPE simulations with additional Fourier spectral analysis. A good qualitative agreement between the theoretical approach and GPE simulations, observed in this figure, confirms the adequacy of the developed model.

The density waves, considered in this work are the consequence of resonance phenomena triggered by time-periodic modulation of the system parameters. Hence, the amplitude of waves steadily grows with time, as depicted in Fig. 6, obtained by numerical solution of the governing Eq. (4). The frequency of oscillations, determined from this figure, is equal to half of the driving frequency in Eq. (4), which is the evidence of the parametric resonance phenomenon (42). The main characteristic feature of Faraday waves is that, they emerge at the modulation frequency, corresponding to parametric resonance (44–45).

An interesting possibility would be to generate a steady wave in the condensate, which can persist after the creation. To this end we performed numerical experiments in which the periodic modulation in Eq. (4) was kept until some time instance \( t_0 \), after that was set to zero, i.e. \( \alpha(t) = \alpha_0 \Theta(t_0 - t) \) with \( \Theta(t) \) being the Heaviside theta function. The result for the condensate with a roton-free excitation spectrum is shown in Fig. 7. As can be seen in this figure, the density waves persist afterwards the creation, provided that the modulation of the coefficient of dipolar interactions is stopped at some time instance \( t \approx 600 \). In fact this is expected outcome, since the original model GPE (3) is conservative.

In addition, the result of Fig. 8 suggests, that GPE (3) may have a stationary spatially periodic solution. To verify this conjecture, we pick the density wave at some particular time instance, namely at \( t = 600 \), and subject it to relaxation procedure. Iterative relaxation method of Nijhof (44) yields the result, shown in Fig. 9b. Next, by inserting
FIG. 6: (Color online) Theoretical gain factor $G(k)$ (red solid line, left axis) and Fourier transform $\mathcal{F}(k)$ (blue dashed line, right axis) of the density wave, shown in Fig. 5a.

FIG. 7: (Color online) The amplitude of the density wave at origin resonantly increases with time, according to numerical solution of Eq. (3). The temporal frequency, counted from this figure is $\omega_0 \approx 0.79$, which is the half of the driving frequency $2\omega = 1.6$ in Eq. (4), suggesting the presence of the parametric resonance phenomenon. Parameter values are similar to those in Fig. 5b.

C. The effect of three-body atomic interactions

The density waves, which are under consideration in this work are the consequence of nonlinear properties of the host medium. In the original experiment [25], the strength of radial confinement of BEC in a cigar-shaped trap was periodically varied in time, which resulted in generation of Faraday waves in the longitudinal direction. This would not be possible in a system, described by a linear Schrödinger equation, because without the nonlinearity the radial perturbation would not be transferred to the axial direction in the form of Faraday waves. Therefore, one can expect enhancement of density waves in condensates with higher order nonlinearities. A well known example is the quintic nonlinearity originating from three-body atomic interactions in the condensate, whose strength is designated by $\rho$ in Eq. (3). The explicit form of this coefficient in terms of material parameters was derived in Ref. [44].

In discussions about BECs with cubic and cubic-quintic nonlinearities, it is usually assumed that two- and three-
FIG. 8: (Color online) Creation of persistent density waves by time-periodic variation of the strength of dipolar interactions according to Eq. (4) with \(\alpha(t) = \alpha_0 \cdot \Theta(t_0 - t)\), where \(\Theta(t)\) is the Heaviside step function. The results are obtained by numerical solution of the nonlocal GPE and represented through a density plot (a) and 1D waveforms at particular time sections (b). Parameter values: \(A = 2, q = 0.8, g_0 = -1, p = 0, \alpha_0 = 0.02, \omega = 0.8, t_0 = 600\).

FIG. 9: (Color online) (a) The density wave of Fig. 8b at \(t = 600\) (blue dashed line), subjected to relaxation procedure, transforms to a stationary periodic wave (red solid line). (b) When the relaxed wave function is inserted as initial condition to GPE and propagated in time, a standing wave pattern is observed.

body atomic interactions are relatively weak. However, in some situations, when the inter-atomic coupling is moderate or strong, models with only quintic nonlinearity (without the cubic term in Eq. (3), so \(q = 0\)) are most accurate [45, 46]. In Fig. 10 we show the excitation spectrum Eq. (9) and growth rate of instability according to Floquet theory Eq. (16) for the condensate with only quintic and dipolar atomic interactions. By comparing this figure with its counterpart possessing only cubic and dipolar interactions, shown in Fig. 1a, one notices a strong shift of the roton minimum to greater values of the wave vector. Besides, the contribution of sub-harmonics of the main frequency towards generation of density waves with \(k \approx k_{\text{rot}}\) appears to be much more pronounced in this case (Fig. 10b, Fig. 11).

Figure 11 illustrates the domains of instability as a function of the strength and frequency of modulation in Eq. (4), and gain factor for the condensate with only quintic and dipolar interactions. From the Floquet analysis and numerical simulations of this section one can conclude, that the higher order nonlinearity greatly enhances the production of density waves in dipolar BECs.
The strength of three-body atomic interactions $K_3 = K_r + i K_4$ in $^{164}$Dy condensate was reported to be $K_r = 5.87 \times 10^{-39} \text{h} \cdot \text{m}^6/\text{s}$ (real, conservative part) and $K_4 = 7.8 \times 10^{-42} \text{h} \cdot \text{m}^6/\text{s}$ (imaginary part, characterizing the three-body recombination rate) \cite{38}. Since the imaginary part of this parameter is smaller than its real part by three orders of magnitude, we have omitted it in the GPE (3), leaving only the conservative part $g_2 = K_r$. The coefficient of quintic nonlinearity, computed using this value and above defined parameters is found to be $p \approx 0.01$. Numerical simulations are performed using the integration domain of length $L = 80 \pi$, which corresponds in physical units to $\mathcal{L} = L a_1 \approx 162 \mu$m. The amplitude of the background wave in dimensionless units is defined as $A = \sqrt{2 a_s N/\mathcal{L}} \approx 2.14$, where $N$ stands for the number of atoms in the condensate.

The above estimates of dimensionless parameters $A$, $q$, $g_0$ and $p$ derived from experimentally feasible quantities are in the range of values, used in our numerical simulations. Some deviations can be adjusted by changing the tunable parameters of the system, such as the $s$-wave scattering length, frequency of radial confinement and the strength of dipolar interactions.
IV. CONCLUSIONS

We have shown, that time-periodic variation of the strength of atomic interactions gives rise to spatially periodic density waves in dipolar quantum gases. The features of emerging density waves are found to be quite different when the roton mode in the excitation spectrum is present, compared to a roton-free case. In the former case the emerging density waves show up as a superposition of waves with different spatial periods, which interfere with each-other and produce a beating-like patterns. In the latter case the density modulations are close to a monochromatic wave. In numerical experiments we observed persistent density waves, which can be created by keeping the time-periodic modulation of the dipolar interaction until the waves emerge, and afterwards setting the modulation part to zero. By subjecting the emergent density wave to a relaxation procedure, and subsequently evolving thus obtained wave function with GPE, we observed a standing wave pattern. The existence of a stationary spatially periodic solution of the nonlocal GPE is conjectured. Theoretical analysis, based on the instability of small perturbations of the background density allowed to evaluate the gain factor for arbitrary set of parameters in the governing nonlocal GPE. The quintic nonlinearity is found to greatly enhance the production of density waves in dipolar BECs. The domains of instability in the parameter space of the problem have been identified, and theoretical predictions are confirmed by direct numerical GPE simulations.

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