Light-front description of infinite spin fields in six-dimensional Minkowski space

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Abstract We present a new 6D infinite spin field theory in the light-front formulation. The Lorentz-covariant counterparts of these fields depend on 6-vector coordinates and additional spinor variables. Casimir operators in this realization are found. We obtain infinite-spin fields in the light-cone frame which depend on two sets of the SU(2)-harmonic variables. The generators of the 6D Poincaré group and the infinite spin field action in the light-front formulation are presented.

1 Introduction

The study of various aspects of classical and quantum field theory in higher dimensions attracts attention basically due to connections with the low-energy limit of superstring theory and miraculous cancelations of some divergences in supersymmetric field models. One of such aspects is a description of the massless representations of the Poincaré group in multi-dimensional spaces (see e.g. the recent works \cite{1–5}).\textsuperscript{1}

In this paper, we continue our study of field irreducible massless representations of the six-dimensional Poincaré group \cite{3–5} focusing on the infinite spin representations and their Lagrangian formulation.

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The study of the infinite spin representations of the Poincaré group \cite{8–10}, their field realizations and dynamical description aroused considerable interest, which led to the formation of a certain research branch mainly in the context of the theory of higher spin fields (see e.g. the review \cite{11} and earlier references therein, and recent papers \cite{12–28}) where aspects of interactions and supersymmetry of infinite spin fields have been examined. Since field realizations of the Poincaré group representations in each concrete dimension have specific features, infinite spin fields in higher dimensions deserve a separate study.

To construct infinite spin fields in six-dimensional Minkowski space, we should describe a possible spectrum of states corresponding to these fields and, first of all, to clarify the spin structure. It can be achieved by considering massless representations of the 6D Poincaré group in state space in terms of the canonically conjugate position and momentum operators, as well as the canonically conjugate pair of spinor operators. Irreducible representations are formulated in terms of second-, fourth- and sixth-order Casimir operators, respectively. The corresponding eigenvalues for these Casimir operators in the irreducible infinite spin representation are 0, $-\mu^2$, $-\mu^2 s(s+1)$ respectively, where $\mu$ is a nonzero real parameter and $s$ is a non-negative integer or half-integer number (see the details in \cite{3,4}).

To describe the spin structure of the 6D infinite spin fields, it is natural to refer to the light-cone frame for massless fields, where the eigenvalues of the energy-momentum operator are $\rho^0 = p^5 = k$, $\rho^5 = 0$, $\rho^a = 1, 2, 3, 4$ with some nonzero real parameter $k$. Here a remarkable result was unexpectedly discovered that any infinite spin field in this frame is necessarily a function on bi-harmonic space with the harmonics $u^\pm, v^\pm$ which were earlier essentially used to construct the
unconstrained superfield formulation of $4D, \mathcal{N} = 2$ supersymmetric field theories [29,30]. Taking into account this result, it is natural then to go to the light-front coordinate system $x^\pm, x^a, a = 1, 2, 3, 4$, which inherits the properties of the light-cone frame [31]. Thus, we arrive at the function of both $x^\pm, x^a$ and harmonics $u^\pm, v^\pm$ which is considered as the infinite spin field in the light-front coordinate system. The field dynamics in the light-front coordinates can be constructed following the generic scheme [31] (see also [32–37]).

The paper is organized as follows. In Sect.2, we discuss the description of irreducible infinite spin representations of the $6D$ Poincaré group in state space formulated in terms of the position and momentum operators and spin operators. These operators are $6D$ vectors and a pair of SU(2) Majorana–Weyl spinors. We find expressions of the fourth- and sixth-orders Casimir operators for the system considered and discuss the conditions leading to fixing the eigenvalues of these operators on physical states. In Sect.3, we derive infinite spin fields in the light-cone frame. Here we show that these fields are the function of bi-harmonic space with two sets of SU(2) harmonics $v^\pm_a$ and $v^\pm_2$. The harmonics obtained here describe the coset space $[SU(2) \otimes SU(2)]/U(1)$. Such a harmonic field possesses a harmonic charge which is determined by the eigenvalue of the sixth-order Casimir operator. We describe the general structure of the harmonic field in the light-cone frame and show that it is given by an infinite expansion in the harmonics. Using these results, in Sect.4, we develop the light-front dynamical formulation of an infinite spin field. We find the generators of the $6D$ Poincaré group for the fields under consideration and propose the corresponding action. An important point in this approach is the use of harmonics as additional coordinates, which greatly simplifies the field analysis. In Sect.5, we summarize the results obtained. Appendix A is devoted to the calculation of the sixth-order Casimir operator for the system considered. In Appendix B, we find the spinor part of the $6D$ Lorentz algebra generators.

2 Irreducible massless representation of the $D6$ Poincaré group

In this section, we discuss the construction of a massless irreducible representation of the six-dimensional Poincaré group emphasizing the specific use of spinor operators.

We consider the representations in the space of states described by vectors $|\Psi \rangle$. The basic operators acting in this space are

$$x^a, \ p_a; \ \xi^I_a, \ \rho^{aI}. \quad (2.1)$$

Here the Hermitian coordinate $x^a = (x^a)^\dagger$ and momentum $p_a = (p_a)^\dagger$ operators are components of the six-vectors, $a = 0, 1, \ldots, 5$ and they obey the standard commutation relations

$$[x^a, \ p_b] = i \delta^a_b. \quad (2.2)$$

The operators $\xi^I_a, \rho^{aI}$ are the SU(2) Majorana-Weyl spinors, where $\alpha = 1, 2, 3, 4$ and $I = 1, 2$ are, respectively, the spinorial SU$(4)$ and internal SU(2) indices. The Hermitian conjugation for these operators is defined as follows:

$$(\xi^I_a)^\dagger = \epsilon_{IJ} B^\alpha_\beta \xi^J_\beta, \quad (\rho^{aI})^\dagger = \epsilon_{IJ} \rho^{a\beta} (B^{-1})^\beta_\alpha \xi^I_\alpha, \quad (2.3)$$

where $B^\alpha_\beta$ is the matrix related to complex conjugation, and the antisymmetric tensors $\epsilon_{IJ}, \epsilon^{IJ}$ have the components $\epsilon_{12} = \epsilon^{21} = 1$ (see [5] for details). Nonzero commutation relations for the operators $\xi^I_a, \rho^{aI}$ have the form

$$[\xi^I_a, \rho^{aI}] = i \delta^a_b \xi^I_b. \quad (2.4)$$

The operators $\xi^I_a, \rho^{aI}$ are to describe the spin degrees of freedom. The state space will be specified in the next section.

We assume that the operators $p_a$ generate space-time translations. In this case, the generators $\{P_a, M_{ab}\}$ of the algebra $iso(1,5)$ of the Poincaré group are realized as

$$P_a = p_a, \quad (2.5)$$

$$M_{ab} = p_a x_b - p_b x_a + S_{ab}, \quad (2.6)$$

where the spin part of the Lorentz group generators looks like

$$S_{ab} = \xi^I_a (\tilde{\sigma}_{ab})_\beta \rho^{aI} - \rho^{aI} (\tilde{\sigma}_{ab})_\alpha \xi^I_a. \quad (2.7)$$

We consider the massless representations where the quadratic Casimir operator $C_2 = P^2 = P^a P_a$ of the algebra $iso(1,5)$ has zero eigenvalues

$$\rho^a p_a (\Psi) = 0. \quad (2.8)$$

In this case, the projection of the fourth-order Casimir operator in subspace (2.8) has the form [3,4]

$$C_4 = \Pi^a \Pi_a, \quad (2.9)$$

where

$$\Pi_a = P^b M_{ba}. \quad (2.10)$$

As a result, we can see that in the representation (2.5), (2.6) and under the condition (2.8) the operator (2.9) takes the following form:

$$C_4 = - \tilde{\ell} \ell, \quad (2.11)$$

where the scalar operators $\ell, \tilde{\ell}$ are defined by the relations

$$\ell := \frac{1}{2} \rho^a_\beta (p_a \sigma^\alpha)_{a\beta} \rho^{\alpha I}, \quad \tilde{\ell} := \frac{1}{2} \xi^I_a (p_a \tilde{\sigma}^\alpha a \beta) \xi^{a I}. \quad (2.12)$$

$\tilde{\sigma}$ We use the spinor conventions of the works [3–5].
When deriving expression (2.11), we used the relation (A.3) for the 6D $\sigma$-matrices. The algebra of operators (2.12) is written in the form

$$[\tilde{\ell}, \ell] = N p^a p_a$$

where the operator $N$ is defined by the anticommutator

$$N := \frac{i}{2} \{\xi_a^j, \rho^a_\ell\}$$

Besides, the operators (2.12) are the Poincaré group invariants and hence they commute with the generators (2.5), (2.6)

$$[P_a, \ell] = [\tilde{P}_a, \tilde{\ell}] = 0 \quad [M_{ab}, \ell] = [M_{ab}, \tilde{\ell}] = 0$$

The infinite spin representation is characterized by the condition that the fourth-order Casimir operator has nonzero negative eigenvalue

$$C_4 |\Psi\rangle = -\mu^2 |\Psi\rangle$$

where $\mu \neq 0$ is the dimensional real parameter which can be taken positive $\mu \in \mathbb{R}_{>0}$ without loss of generality. Using relations (2.16) and (2.11), we can see that it is sufficient to define infinite spin states by the constraints

$$\ell |\Psi\rangle = \mu |\Psi\rangle, \quad \tilde{\ell} |\Psi\rangle = \mu |\Psi\rangle$$

For massless representations (2.8), the sixth-order Casimir operator has the form [3,4]

$$C_6 = -\Pi^b M_{ba} \Pi_c M^{ca} + \frac{1}{2} \left(M^{ab} M_{ab} - 8\right) \Pi^a \Pi_a$$

where the operator $\Pi_a$ is defined in (2.10). In the representation (2.5), (2.6) and under the conditions (2.8), (2.16), we obtain

$$C_6 |\Psi\rangle = -\mu^2 J_i J_i |\Psi\rangle$$

where the operators $J_i$ ($i = 1, 2, 3$) are defined as follows:

$$J_i := \frac{i}{2} \xi_a^j (\sigma_i)_{j}^{\rho} \rho^a_\ell$$

Here $\sigma_i$ are the Pauli matrices. The operators $J_i$ form the $su(2)$ algebra

$$[J_i, J_j] = i \epsilon_{ijk} J_k$$

Expression (2.19) for the operator $C_6$ is the same as in [3,4] but the realization of the generators $J_i \in su(2)$ in [3,4] is different.

As it was shown in [3,4], the space $V$ of irreducible infinite spin representation is induced from the space of finite dimensional representation of (2.20) and the operator $C_6$ acts as follows:

$$C_6 |\Psi\rangle = -\mu^2 s(s + 1) |\Psi\rangle$$

where $s$ is a nonzero integer or half-integer number, $s \in \mathbb{Z}_{\geq 0}/2$. Therefore, the states corresponding to the infinite spin irreducible representation obey the constraints

$$J_i J_i |\Psi\rangle = s(s + 1) |\Psi\rangle$$

where the operators $J_i$ are defined in (2.20).

Note that the $su(2)$ algebra generators (2.20) commute with the SU(2) scalar operators (2.12):

$$[J_i, \ell] = [J_i, \tilde{\ell}] = 0$$

Besides, the operators (2.20) commute with generators of six-dimensional translations (2.5) and with the Lorentz algebra $so(1, 5)$ generators (2.6), (2.7):

$$[P_a, J_i] = 0, \quad [M_{ab}, J_i] = [S_{ab}, J_i] = 0$$

It is worth noting that the algebra $so(1, 5) = su^*(4)$ generated by (2.7) is dual to the algebra $su(2)$ with the generators (2.20) in the sense of Howe duality [38,39].

It was shown earlier [6,7,12] that in the vector approach an infinite spin representation of $so(1, 5)$ requires the use of the 6-dimensional Heisenberg algebra (2.2) generated by the operators of position $x^a$ and momentum $p_a$ and two additional 6-dimensional Heisenberg algebras with the coordinate operators $\gamma^a_1$, $\gamma^a_2$ and their momentum operators $p^a_1$, $p^a_2$. On the other hand, we proved in [5] that, in the twistor formulation, the $iso(1, 5)$ representations of infinite spin are necessarily described in the bi-twistor space, which is defined by two pairs of canonically-conjugated SU(2) Majorana–Weyl spinors of type (2.4) (but having nonzero mass dimensions). Here we have shown that 6D infinite spin representations can be described in the space defined by only one 6-dimensional Heisenberg algebra (2.2) and one pair of canonically conjugated SU(2) Majorana–Weyl spinors (2.4), as it was indicated in (2.1).

In the next section, we will describe the infinite spin vectors $|\Psi\rangle$ in terms of appropriate fields.

### 3 Infinite spin fields in the light-cone frame

We consider a structure of D6 infinite spin fields in the light-cone frame. This frame is defined by the following conditions on eigenvalues of the energy-momentum operator:

$$p^0 = p^5 = k, \quad p^\hat{a} = 0, \quad \hat{a} = 1, 2, 3, 4, \quad (3.1)$$

where $k$ is a nonzero real parameter of mass dimension. Using the light-cone coordinates $p^\pm = (p^0 \pm p^5)/\sqrt{2}$, one gets for the same frame

$$p^+ = \sqrt{2}k, \quad p^- = p^\hat{a} = 0. \quad (3.2)$$

Consider the SU$^*(4)$ spinors $\xi_a^j$, $\rho^a_\ell$ with a four-component spinor index $\alpha$ and present them as objects with
two-component indices as follows:
\[
\xi^I_\alpha = (\xi^I_\ell, \xi^J_\rho), \quad \rho^I_\alpha = (\rho^I_\ell, \rho^J_\rho),
\] (3.3)
where the two-component indices take the values \(i = 1, 2\) and \(j = 1, 2,\) i.e. \(i = \alpha\) for \(\alpha = 1, 2,\) and \(j = \alpha - 2\) for \(\alpha = 3, 4.\)

In the light-cone frame (3.1) the operators (2.12) take the form
\[
\tilde{\theta} = k e^{ij} \xi^I_\ell \xi^J_\rho, \quad \ell = k e^{ij} \xi^I_\ell \rho^I_\ell \rho^I_\rho.
\] (3.4)
where the matrices \(\sigma^-\) (B.5) and \(\sigma^+\) (B.4) were used. Then in this frame the constraints
\[
\tilde{\theta} = \mu, \quad \ell = \mu
\] (3.5)
from (2.17) are written in the form
\[
\xi_{IJ} u^I_i v^J_j = \epsilon_{ij}, \quad \xi^I_\ell u^I_i = \epsilon^I_\ell.
\] (3.6)
where we have used the spinor variables
\[
u_i^I := \sqrt{2k/\mu} \xi_i^I, \quad \nu^I_\ell := \sqrt{2k/\mu} \rho^I_\ell.
\] (3.7)
\[
\rho^I_\alpha = \xi^I_\alpha \rho^I_\alpha.
\] (3.8)
The conditions (2.3) in terms of the SU(2) spinors (3.7) look like
\[
(u^I_i)^* = -\epsilon_{IJ} e^{ij} u^J_i, \quad (v^I_i)^* = -\epsilon_{IJ} e^{ij} v^J_i.
\] (3.9)
Conditions (3.6), (3.8) are nothing but the ones of unimodularity, det \(u = 1,\) det \(v = 1,\) and unitarity, \(u^T u = 1, v^T v = 1,\) of the 2x2 matrices
\[
u := \|\nu_i^I\|, \quad \nu := \|\nu^I_\ell\|.
\] (3.10)
As a result, in the light-cone frame the variables \(u^I_i\) and \(v^I_i\) (3.7) are the elements of the SU(2) groups and parameterize the compact space. Further, analogously to [29,30], we will use the following notation:
\[
\nu_i^I = u_i^I, \quad \nu^I_\ell = u^I_\ell, \quad v^I_\ell = v_i^I, \quad v^I_\ell = v^I_\ell.
\] (3.11)
In this notation, relations (3.6) are rewritten in the form
\[
u^I_\ell v^I_i = \epsilon_{ij}, \quad \nu^I_i v^I_\ell = \epsilon_{ij}
\] (3.12)
or, in the equivalent form
\[
u^I_\ell v^I_i = 1, \quad \nu^I_i v^I_\ell = 1.
\] (3.13)
where \(u^I_\pm = e^{ij} u^I_\pm, u^I_\pm = e^{ij} u^I_\pm.\) Note that both \(u^I_\pm\) and \(v^I_\pm\) have the same indices \(\pm,\) since they are obtained from the common SU(2)-index \(I\) for the SU(2) Majorana–Weyl spinors (3.3).

We emphasize that the variables \(u^\pm\) and \(v^\pm\) introduced in (3.7) completely determine the operators \(\ell, \tilde{\ell}\) in (3.4) and represent half of the different canonical pairs in the algebra (2.4). We treat the second half of the operators in (2.4) as differential operators. Thus, one considers a representation where the operators \(\rho^I_\ell\) and \(\xi^I_\ell\) in the algebra (2.4) are realized as differential operators
\[
\rho^I_\ell = -i \frac{\partial}{\partial \xi^I_\ell} = -i \sqrt{2k/\mu} \frac{\partial}{\partial \xi^I_\ell},
\] (3.14)
\[
\xi^I_\ell = i \frac{\partial}{\partial \rho^I_\ell} = i \sqrt{2k/\mu} \epsilon_{ij} \epsilon^{IJ} \frac{\partial}{\partial v^J_\ell}.
\] (3.15)
In the representation chosen, the operators \(\ell\) and \(\tilde{\ell}\) are realized by operators of multiplication by the functions of \(u^\pm, v^\pm.\)

In such a representation, the \(\text{su}(2)\)-generators \(J_{\pm} := J_1 \pm i J_2\) and \(J_3,\) given by (2.20), are written as follows
\[
J_\pm = D_u^{\pm \pm} + D_v^{\pm \pm}, \quad J_3 = \frac{1}{2} \left( D_u^0 + D_v^0 \right),
\] (3.16)
where
\[
D_u^{\pm \pm} := u_i^\pm \frac{\partial}{\partial u_i^\pm}, \quad D_u^0 := u_i^+ \frac{\partial}{\partial u_i^-} - v_i^+ \frac{\partial}{\partial v_i^-},
\] (3.17)
\[
D_v^{\pm \pm} := v_i^\pm \frac{\partial}{\partial v_i^\pm}, \quad D_v^0 := v_i^+ \frac{\partial}{\partial v_i^-} - v_i^- \frac{\partial}{\partial v_i^+}.
\] (3.18)
occur in the derivatives in the notation [29, 30].

As a solution to the irreducibility condition (2.19), (2.22) for 6D representations we take the highest weight vector \(|\Psi^{(2s)}\rangle\) which is defined by the equations
\[
J_+ |\Psi^{(2s)}\rangle = 0, \quad (J_3 - s) |\Psi^{(2s)}\rangle = 0,
\] (3.19)
where the operators \(J_{\pm}\) and \(J_3\) are expressed via harmonic derivatives (3.15), (3.16) in (3.14). Recall that the vector \(|\Psi^{(2s)}\rangle\) also obeys the conditions (2.17):
\[
\ell |\Psi^{(2s)}\rangle = \mu |\Psi^{(2s)}\rangle, \quad \tilde{\ell} |\Psi^{(2s)}\rangle = \mu |\Psi^{(2s)}\rangle.
\] (3.20)

Now, we show that the vectors of the states \(|\Psi^{(2s)}\rangle\) are realized as fields. In the representation (3.13) the corresponding fields in the light-cone frame are the functions \(\psi^{(2s)}(u^\pm, v^\pm)\) of four SU(2) spinors \(u^\pm, v^\pm.\) Then, it is natural to present the solution of equations (3.19) by using \(\delta\)-functions
\[
\psi^{(2s)}(u^\pm, v^\pm) = \delta(\ell - \mu) \delta(\tilde{\ell} - \mu) \Phi^{(2s)}(u^\pm, v^\pm),
\] (3.21)
where the arguments of the field \( \Phi^{(2)}(u^\pm, v^\pm) \) satisfy (3.11), (3.12), and it means that the field \( \Psi^{(2)}(u^\pm, v^\pm) \) is a function on the SU(2) \( \otimes \) SU(2) group.

Using the relations (3.14), one rewrites the remaining conditions (3.17) and (3.18) in the form

\[
(D_u^+ + D_v^+) \Phi^{(2)}(u^\pm, v^\pm) = 0, \tag{3.21}
\]

\[
(D_u^0 + D_v^0 - 2s) \Phi^{(2)}(u^\pm, v^\pm) = 0. \tag{3.22}
\]

Equation (3.22) means the U(1) covariance of the field \( \Phi^{(2)}(u^\pm, v^\pm) \):

\[
\Phi^{(2)}(e^{\pm i\eta}u^\pm, e^{\pm i\alpha}v^\pm) = e^{2s(i\eta)} \Phi^{(2)}(u^\pm, v^\pm). \tag{3.23}
\]

The charge \( 2s \) in the notation of the field \( \Phi^{(2)} \) reflects the property (3.23) of this field. Transformations of the arguments of the right action on the matrices (3.9) by the diagonal unitary matrix \( h \):

\[
u_i J \rightarrow u_j K h_{K J}, \quad v_i J \rightarrow v_j K h_{K J},
\]

\[
h \equiv \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix},
\tag{3.24}
\]

where \( K, J = (+, -) \). In the standard stereographic parametrization of the SU(2) matrices

\[
\begin{pmatrix} u_1^+ & u_2^+ \\ u_1^- & u_2^- \end{pmatrix} = \frac{1}{\sqrt{1 + t_1 t_2}} \begin{pmatrix} 1 & -t_1 \\ t_1 & 1 \end{pmatrix},
\]

\[
\begin{pmatrix} v_1^+ & v_2^+ \\ v_1^- & v_2^- \end{pmatrix} = \frac{1}{\sqrt{1 + t_2 t_2}} \begin{pmatrix} 1 & -t_2 \\ t_2 & 1 \end{pmatrix},
\tag{3.25}
\]

the transformation (3.24) is represented by the phase shift \( \psi \rightarrow \psi + \alpha \). Moreover, due to the fact that the field \( \Phi^{(2)}(u^\pm, v^\pm) \) has a fixed U(1)-charge equal to \( 2s \), its dependence on the phase variable is factorized:

\[
\Phi^{(2)}(u^\pm, \pm v^\pm) = e^{2s(i\eta)} \Phi(t_1, t_2, t_1, t_2, \psi).
\tag{3.26}
\]

The field \( \Phi(t_1, t_2, t_1, t_2, \psi) \) on the right-hand side of equality (3.26) is the function on the coset \([SU(2) \otimes SU(2)]/U(1)\) where the variable \( \psi \) is the coordinate of the stability subgroup U(1). Thus, the field \( \Phi^{(2)}(u^\pm, v^\pm) \) having a fixed U(1)-charge is in a one-to-one correspondence with the function on the coset space \([SU(2) \otimes SU(2)]/U(1)\) [29, 30].

For this reason, we may refer to the variables \( u_1^\pm, v_1^\pm \) used here as the \([SU(2) \otimes SU(2)]/U(1)\) harmonics. Since the variables \( u_1^\pm, v_1^\pm \) consist of twice the number of harmonics used in [29, 30], the space parameterized by these four SU(2) spinors can be called the bi-harmonic space.

Note that a slightly different type of the bi-harmonic space was previously used in the study of various super-symmetric models. For example, two types of harmonics were employed in [41, 42] for constructing an off-shell superfield formulation of the 2D \((4, 4)\) sigma-model. In those papers, the harmonics were used to parameterize the coset space \( SU_L(2)/U_L(1) \otimes SU_R(2)/U_R(1) \), where the spaces \( SU_L(2)/U_L(1) \) and \( SU_R(2)/U_R(1) \) were associated with the harmonics \( u^\pm, v^\pm \), and \( u^0, v^0 \), respectively, having the charges of different U(1) groups. As a result, the fields on this harmonic coset have two U(1) charges, which are defined as eigenvalues of the operators \( D^0 \) and \( D^0 \). In the other hand, in the case considered here, the field is defined by Eq. (3.22), where the only U(1) charge \( 2s \) of the field \( \Phi^{(2)}(u^\pm, v^\pm) \) is given as the eigenvalue of the U(1) generator \( D^0 = D^0 + D^0 \). Besides, the U(1) charges \( s \) of the two pairs of harmonics \( u^\pm, v^\pm \) coincide unlike the harmonics in [41, 42]. As discussed above, this means that the field \( \Phi^{(2)}(u^\pm, v^\pm) \) of a special type is defined on bi-harmonic space where the coordinates \( u^\pm \) and \( v^\pm \) parameterize the coset \( [SU(2) \otimes SU(2)]/U(1) \), as shown in (3.23). Another type of bi-harmonics was used, e.g., in [43], to describe the effective actions \( \mathcal{N} = 4 \) SYM theory (see the details in [43] and the references therein).

Now we describe the general solution to equations (3.21) and (3.22).

First, we note that any function of the variables

\[
\gamma_{ij} := u_i^+ v_j^- - u_i^- v_j^+ \tag{3.27}
\]

satisfies Eqs. (3.21), (3.22) for \( s = 0 \).

Then, in general, the field \( \Phi^{(2)}(u^\pm, v^\pm) \), obeying equations (3.21), (3.22) for \( 2s \in \mathbb{Z} \geq 0 \), is written in the form

\[
\Phi^{(2)}(u^\pm, v^\pm) = \sum_{r=0}^{2s} \Phi_{k(r)}^{(2)}(u^+, v^+) y^{k(r)}_{-}(u^+, v^+), \tag{3.28}
\]

where

\[
\Phi_{k(r)}^{(2)}(u^+, v^+) = \sum_{p,q=0}^{2s} \delta_{k(r)}^{(p,q)} u_i^+(p) v_j^+(q). \tag{3.29}
\]

Expressions (3.28) and (3.29) use the following concise notation for the monomials:

\[
u_i^+(p) := u_i^+ \cdots u_i^+, \quad v_j^+(q) := v_j^+ \cdots v_j^+,
\]

\[
y^{k(r)}_{-}(u^+, v^+) := y^{k(r)}_{-}(u^+, v^+) \tag{3.30}
\]

and we use the standard convention \( y^{k}_{-} = e^{ik} \). For raising and lowering the SU(2) indices.

The field \( \Phi^{(2)}(u^\pm, v^\pm) \) in (3.28) that satisfies (3.21) and (3.22) is a linear combination with the constant coefficients \( \delta_{k(r)}^{(p,q)} \) of an infinite number of basis states \( \Phi_{k(r)}^{(2)}(u^+, v^+) y^{k(r)}_{-}(u^+, v^+) \). The corresponding combinations of these basis vectors allow us to define the space for the irreducible infinite spin iso(1, 5) representation in the light-cone frame.

As a solution of the irreducibility condition (2.19), we chose one of the \( 2s+1 \) possible vectors in the space of the
su(2) irreps with spin s, namely, we took the higher weight vector \(|\hat{\Psi}^{(2s)}\rangle\) defined by conditions (3.17), (3.18). This choice does not lead to loss of generality. The remaining 2s vectors are obtained from this vector \(|\hat{\Psi}^{(2s)}\rangle\) by acting of the operator \((J \cdots)^{\cdots} - 1\) at \(k = 1, \ldots, 2s\). In the representation (3.14), (3.21) and (3.22) the fields \(\Phi^{(2s - 2k)}\) are obtained by the action of the operator \((D \cdots)^{\cdots} - 1\) on \(\Phi^{(2s)}\). Note that the action of \((D \cdots)^{\cdots} - 1\) decreases the degree of the polynomial \(\Phi^{(2s)}_{k(r)l(r)}(u^+, v^+)\) (3.29) in the variables \((u^+, v^+)\) and increases it in the variables \((u^-, v^-)\). Choosing any other fields \(\Phi^{(2s - 2k)}\), \(k = 1, \ldots, 2s\) leads to the equivalent infinite spin representations of \(\text{iso}(1, 5)\).

4 Field theory in the light-front coordinates

In the previous section, we have developed a description of the irreducible 6D infinite spin representations in the light-cone frame and shown that this description is formulated in terms of fields in bi-harmonic space. Now we extend this analysis to the light-front coordinate system and construct the corresponding field theory.

The formulation of the field theory on the light-front was proposed by Dirac [31], its further development and applications were considered by many authors (see e.g. [32, 33, 36, 37] and the references therein). The light-front is defined as the surface \(x^+ = \text{const}\) in the six-dimensional Minkowski space \(\mathbb{R}^{1, 5}\). It means that the coordinate \(x^+\) is interpreted as a “time” evolution parameter. Therefore, the role of the Hamiltonian in the case under consideration is played by the operator

\[ H = P^-. \] (4.1)

To define an infinite spin field in the light-front coordinates, we will use the results of the previous section, where the corresponding field is given by (3.28), (3.29). Note that the light-cone coordinate system is obtained from the light-front coordinate system by vanishing the coordinates \(x^\pm\) and fixing the coordinates \(x^\pm\). Therefore, it is natural to assume that the infinite spin field in the light-front coordinates should have the form (3.28), (3.29), where, however, the coefficients \(\hat{\Phi}^{i(p)j(q)}_{k(r)l(r)}\) are functions of \(x^\pm\) and \(x^\pm\). As a result, the irreducible infinite spin field depending on the light-front coordinates is defined as

\[ \Phi^{(2s)}(x^\pm, x^\pm, u^\pm, v^\pm) = \sum_{p, q = 0}^{2s} \sum_{r=0}^{\infty} \Phi^{i(p)j(q)}_{k(r)l(r)}(x^\pm, x^\pm) u^\pm_i v^\pm_j x^\pm_k x^\pm_l. \] (4.2)

Taking into account a general principle of the light-cone dynamics [31], one concludes that the equation of motion for the field (4.2) is the Schrödinger-type equation

\[ \left( -i \frac{\partial}{\partial x^-} - H \right) \Phi^{(2s)}(x^\pm, x^\pm, u^\pm, v^\pm) = 0, \] (4.3)

where the coordinate \(x^+\) plays the role of time.

As usual, the generators \(P_\mu\) and \(M_{\mu\nu}\) of \(\text{iso}(1, 5)\) in the light-front formulation are divided into kinematic and dynamic generators. One can show that these divisions in the field realization (4.2) have the form

- Kinematic generators

\[ P^+ = p^+, \quad P^- = p^-; \] (4.4)

\[ M^{\hat{a}b} = x^b \epsilon_\hat{a}^\mu \epsilon_\hat{b}^\nu \mathcal{P}^\mu \mathcal{P}^\nu; \]

\[ M^{\hat{a}b} = x^b H - x^+ p^+ + S^-; \] (4.5)

where

\[ p^\hat{a} = i \frac{\partial}{\partial x^\hat{a}}; \quad p^\pm = -i \frac{\partial}{\partial x^\pm}. \] (4.8)

and all spin parts of the Lorentz rotation generators \(S^{\hat{a}b} = (S^{\hat{a}b}, S^{\hat{a}b}, S^{\hat{a}b})\) depend on the spinors \(u^\pm\) and \(v^\pm\) in the same way as the operators (B.10), (B.14), (B.17), (B.18) depend on the spinors \(\xi_\hat{i}^\mu\) and \(\rho_\hat{i}^\nu\).

\[ S^{\hat{a}b} = \frac{1}{2} \eta^{ij}_{ab} \left[ \frac{\partial}{\partial u_i^\mu} (\tau_i)^j \hat{\xi}_j^\nu + \frac{\partial}{\partial u_i^\nu} (\tau_i)^j \hat{\xi}_j^\mu \right] \]

\[ + \frac{1}{2} \eta^{ij}_{ab} \left[ \frac{\partial}{\partial v_i^\mu} (\tau_i)^j \hat{\xi}_j^\nu + \frac{\partial}{\partial v_i^\nu} (\tau_i)^j \hat{\xi}_j^\mu \right], \] (4.9)

\[ S^{\hat{a}b} = \frac{i}{\sqrt{2}} \left[ \frac{\partial}{\partial u_i^\mu} (\tau_\hat{a})_j J^j + \frac{\partial}{\partial v_i^\mu} (\tau_\hat{a})_j J^j \right], \]

at \(\hat{a} = 1, 2, 3,\) (4.10)

\[ S^{\hat{a}b} = \frac{i}{\sqrt{2}} \left[ v_i^\mu \epsilon_i^k (\tau_\hat{a})_l J^l - v_i^\mu \epsilon_i^k (\tau_\hat{a})_l J^l \right], \]

at \(\hat{a} = 1, 2, 3,\) (4.11)

\[ S^{\hat{a}b} = \frac{1}{\sqrt{2}} \left[ \frac{\partial}{\partial u_i^\mu} \epsilon_i^k \frac{\partial}{\partial u_i^k} - \frac{\partial}{\partial u_i^\mu} \epsilon_i^k \frac{\partial}{\partial v_i^k} \right], \] (4.12)

\[ S^{\hat{a}b} = \frac{1}{\sqrt{2}} \left[ v_i^\mu \epsilon_i^k u_i^+ + v_i^\mu \epsilon_i^k u_i^- \right], \] (4.13)

\[ S^{\hat{a}b} = \frac{1}{\sqrt{2}} \left[ u_i^+ \frac{\partial}{\partial u_i^+} + u_i^- \frac{\partial}{\partial u_i^-} + v_i^+ \frac{\partial}{\partial v_i^+} + v_i^- \frac{\partial}{\partial v_i^-} + 4 \right]. \] (4.14)
Turn attention that all the generators $S^{ab} = (S^{āb}, S^{±a}, S^{+−})$ defined in (4.9)–(4.14) have zero U(1)-charge.

After acting by the operator $p^+$ on Eq. (4.3), this equation takes the form
\[
\Box \Phi^{(2s)}(x^+, x^a, u^\pm, v^\pm) = 0,
\]
where $\Box$ is the d’Alambertian operator in the six-dimensional Minkowski space in the light-front coordinates
\[
\Box := \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} - \frac{\partial}{\partial x^a} \frac{\partial}{\partial x^a}.
\]
Equation (4.15) is the equation of motion corresponding to the action
\[
S = \int d^6x \, du \, dv \, \tilde{\Phi}^{(-2s)} \Box \Phi^{(2s)},
\]
where $d^6x = dx^+dx^-d^4x$ is the 6D Minkowski space measure and $du \, dv$ is the bi-harmonic space measure. The function $\tilde{\Phi}^{(-2s)}$ is obtained by complex conjugation of the function $\Phi^{(2s)}$:
\[
\tilde{\Phi}^{(-2s)} = (\Phi^{(2s)})^*.
\]
Integration over harmonics is defined by simple rules (see [29, 30] for details). The integral is a linear operation and it does not vanish only for SU(2)-scalars with the following normalization condition:
\[
\int du = 1, \quad \int dv = 1.
\]
For all other harmonic monomials, the harmonic integral is equal to zero:
\[
\int du \, u^+_i \cdots u^+_m \, u^-_j \cdots u^-_n = 0,
\]
\[
\int dv \, v^+_l \cdots v^+_m \, v^-_j \cdots v^-_n = 0,
\]
at arbitrary integers $m$ and $n$ which are not equal to zero simultaneously.

Reality conditions (3.8) are now written as follows:
\[
(u^{i±})^* = \pm u^{i±}, \quad (v^{i±})^* = \pm v^{i±}.
\]
Therefore, at complex conjugation the charge $2s$ of the harmonic field $\Phi^{(2s)}$ changes to $-2s$ in accordance with (4.18).

As a result, the integrand in (4.17) has a zero harmonic charge as it should be for the non-vanishing harmonic integral.

In expansion of the harmonic field (4.2) the indices $i$ and $j$ of the component fields $\Phi^{(i(m)/(n))\hat{v}(k)/(j)(n)}(x)$ are half of the 6D SU*(4)-indices $\alpha$. It means, for half-integer $s$, the harmonic field $\Phi^{(2s)}(x, u^\pm, v^\pm)$ is an odd order polynomial in $u^\pm, v^\pm$ and describes half-integer spin fields with an odd number of indices. Therefore, the fermionic fields should be endowed by the corresponding odd statistics. Besides, in the fermionic case, the natural Lagrangian is the one of the first order in space derivatives. This type of Lagrangian in the light-front formalism is obtained from the Lagrangian (4.17) by replacement
\[
\Psi^{(2s)} = \sqrt{p^+} \Phi^{(2s)}.
\]
Then, for the field $\Psi^{(2s)}$ expression (4.17) leads to the following Lagrangian:
\[
\tilde{\Psi}^{(-2s)} (p^- - H) \Psi^{(2s)}.
\]

Thus, the action (4.17) determines the field dynamics of infinite spin fields on the light front. A specific feature of the obtained theory is its formulation in terms of harmonic variables.

5 Summary

We have developed the 6D Minkowski space infinite spin free Lagrangian field theory in the light-cone formalism. First, we have studied this theory in the light-cone frame and unexpectedly found that the corresponding infinite spin field is a function on a special bi-harmonic space associated with the coset $[SU(2) \otimes SU(2)]/U(1)$. Second, the result obtained was generalized to the light-front coordinate system, where the infinite spin field is described by the function $\Phi^{(2s)}(x^+, x^a, u^\pm, v^\pm)$ (4.2) depending on the light-front coordinates and harmonics. Representations of all the 6D Poincaré group generators in this coordinate system are constructed. The field equation of motion in the light-front coordinate system has the form of Schrödinger-type Eq. (4.3) with the Hamiltonian (4.1). The corresponding action is given by (4.17).

The harmonic light-front approach formulated in this paper opens a possibility to construct an interacting theory for 6D infinite spin fields. One can expect that introducing an interaction will lead to a modification of the dynamic generators (4.6), (4.7) by the interaction terms (see the description of interactions in the light-front formalism, e.g., in [37] and the references therein). In particular, the Hamiltonian (4.6) should go to
\[
H \rightarrow H + H_{int}.
\]
The harmonic formalism allows one from the very beginning to make some simple predictions on the structure of the interacting Hamiltonian $H_{int}$. To preserve zero harmonic
charge of the action, this Hamiltonian should have zero harmonic charge as well. It immediately means that an arbitrary order self-interaction of the same harmonic fields $\Phi(2s)$ is possible only for $s = 0$ if other charged harmonic quantities in the action are absent. Self-interaction of charged fields $\Phi(2s)$, $s \neq 0$ can only be of an even order, such as $\sim \Phi(-2s) \Phi(-2s) \Phi(2s) \Phi(2s)$. Although for fields with different charges there is an additional choice in the structure of the interaction Lagrangian. For example, the following interacting terms $\sim \Phi_1^{(-2s)} (\Phi_2^{(0)} + \Phi_3^{(0)}) \Phi_1^{(2s)}$ or $\sim (\Phi_1^{(q_1)} \Phi_2^{(q_2)} \Phi_3^{(q_3)} + c.c.)$ at $q_1 + q_2 + q_3 = 0$ are allowed in the action. In general, the requirement of zero charge of interacting contributions to the action controls both charges of interacting fields and their number. We plan to construct interacting infinite spin 6$D$ theories in the forthcoming works.

We also think that the appearance of bi-harmonic space in the infinite spin representations of $iso(1, 5)$ can indicate the existence of manifest $N = (1, 0)$ supersymmetrization of the theory (4.17), (5.1).9

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Appendix A: Calculation of the Casimir operator $C_6$

The six-order Casimir operator in the $6D$ massless theory is given by (2.18). The derivation of this operator in [3,4] was based on the relations:

$$C_6 = \frac{1}{64} \gamma^a \gamma_a, \quad \gamma_a = \varepsilon_{abcdef} P^b M^{cd} M^{ef} \quad (A.1)$$

These expressions show that only the spin part $S_{ab}$ (2.7) of the Lorentz group generators $M_{ab}$ (2.6) contributes to the Casimir operator (A.1). However, if we substitute $S_{ab}$ for $M_{ab}$ into (2.18), an incorrect result is obtained, since when passing from (A.1) to (2.18), one has to rearrange the operators $P_0$ and $M_{ab}$ using commutators. At the replacement $M_{ab} \rightarrow S_{ab}$, a correct expression $C_6$ can be obtained only after preliminary “untangling” of the generators $P_0$ and $M_{ab}$ in expression (2.18).

We will act in the following way. First, we rearrange with the help of commutation relations all the operators $P_0$ to the right on all the operators $M_{ab}$ in expression (2.18). Second, after such an ordering is done, we replace the operator $M_{ab}$ by the operator $S_{ab}$ in the obtained expression.

Using the commutator $[M_{ab}, \Pi_i] = i (\eta_{ac} \Pi_b - \eta_{bc} \Pi_a)$, we rearrange the operators $\Pi_i$ to the right in the first term of expression (2.18). All the terms proportional to $2 \Pi_{[a} \Pi_{b]} = [\Pi_a, \Pi_b] = -i M_{ab} P^2$ can be omitted for a massless representation where $P^2 = 0$. Besides, since we consider irreducible infinite spin for which the condition (2.16) holds, we replace the operator $C_4$ by its eigenvalue $-\mu^2$ in the second term of (2.18). Now all operators $M_{ab}$ to the left of the operators $\Pi_i$ are replaced by the operators $S_{ab}$. As a result, one obtains

$$C_6 = - S^{(b \ a} S^{c) a} \Pi_b \Pi_c - \frac{1}{2} \mu^2 \left( S^{bc} S_{bc} - 8 \right). \quad (A.2)$$

Using the relations for the $\sigma$-matrices from [5], the identity

$$\eta^b_{\ [a} \sigma_{\ \ c]} (\bar{\sigma} \sigma^c) \gamma^a \delta = \frac{1}{4} \eta^{bc} \left( \delta^a_{\ [\delta} \delta^c_{\ \gamma]} - 2 \delta^a_{\ [\gamma} \delta^c_{\ \delta]} \right) + \frac{1}{2} \delta^a_{\ [\delta} \delta^c_{\ \gamma]} - \frac{1}{2} \delta^a_{\ [\gamma} \delta^c_{\ \delta]}, \quad (A.3)$$

the commutation relations (2.4) and the realization (2.7) for the operators $S_{ab}$, one gets the equality

$$S^{(b \ a} S^{c) a} = \eta^{bc} \left[ \frac{1}{2} (\xi_1 \rho_K) (\xi_1 \rho^K) + i (\xi_1 \rho_1) \right] + \frac{1}{2} \xi_1 \rho_1^a \xi_1 \rho_1^b \delta (\bar{\sigma} (b) \sigma^c) \gamma^a \delta, \quad (A.4)$$

which leads to

$$S^{bc} S_{bc} = - \frac{1}{2} (\xi_1 \rho_1)^2 + 4i (\xi_1 \rho_1) + 2 (\xi_1 \rho_K) (\xi_1 \rho^K), \quad (A.5)$$

where the notation $(\xi_1 \rho_K) := \bar{\xi}_1 \rho_1^a \delta$ has been used. After substituting (A.4) and (A.5) into (A.2), one gets

$$C_6 = \mu^2 \left[ \frac{1}{4} (\xi_1 \rho_1)^2 - \frac{1}{2} (\xi_1 \rho_1) (\xi_1 \rho^1) - i (\xi_1 \rho_1) + 4 \right].$$
\[-\frac{1}{2} \xi^I_\alpha (\xi^I_\rho \rho^I_\delta) (\sigma^\beta \rho^\gamma) (\sigma^\epsilon)_{\beta \delta} \Pi_b \Pi_c.\]  
(A.6)

The last term in this expression is represented in the following form:

\[-\frac{1}{2} \xi^I_\alpha (\xi^I_\rho \rho^I_\delta) (\sigma^\beta \rho^\gamma) (\sigma^\epsilon)_{\beta \delta} \Pi_b \Pi_c = -\frac{i}{2} \mu^2 (\xi^I_\rho I)\]

\[+ \frac{1}{4} (\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\]  
(A.7)

where \((\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\) which is the same as (2.19).

The last step in deriving the expression for \(C_b\) is to move the operators \(P_m\) to the right in expression (A.7). Using the equality \(\Pi_a = M_{ba} P^b - 5i P_a\), we write the expression \(\Pi_b \Pi_c\) in the form:

\[\Pi_b \Pi_c = M_{eb} M_{fc} P^e P^f - 6i M_{eb} P^e P_c - 5i M_{ec} P^e P_b - 30 P_b P_c.\]  
(A.8)

Now that all operators \(M_{ab}\) on the right side of (A.8) are to the left of all operators \(P_c\), we replace the operators \(M_{ab}\) with their spin parts \(S_{ab}\).

After such a replacement \(M_{ab} \rightarrow S_{ab}\), where \(S_{ab}\) are defined in (2.7), the substitution (A.8) into (A.7) and (A.6) and using the equalities

\[\frac{1}{4} (\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\]  
\[= \mu^2 \left[ -\frac{1}{4} (\xi^I_\rho I)^2 + 2i(\xi^I_\rho I) - 6 \right],\]  
(A.9)

\[-\frac{3i}{2} (\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\]  
\[= \mu^2 \left[ -3i(\xi^I_\rho I) + 12 \right],\]  
(A.10)

\[-\frac{5i}{4} (\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\]  
\[= \mu^2 \left[ \frac{5i}{2} (\xi^I_\rho I) + 20 \right],\]  
(A.11)

\[-\frac{15}{2} (\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\]  
\[= -30 \mu^2,\]  
(A.12)

which are valid at \(P^2 = 0\) and \((\xi^I_\sigma (\sigma^\beta \rho^\epsilon) (\rho_I \sigma^\epsilon \rho^J) \Pi_b \Pi_c,\) one obtains

\[C_b = -\frac{1}{2} \mu^2 (\xi_\epsilon (\rho_I \rho^I) (\xi^I_\rho I).\]  
(A.13)

When using the operators (2.20), this final expression (A.13) is represented as

\[C_b = -\mu^2 J_b J_c.\]  
(A.14)

which is the same as (2.19).

**Appendix B: Spinor part of the \(so(1, 5)\)-generators**

We consider a representation where the \((4 \times 4)\) \(\sigma\)-matrices \([5, 45, 46]\) \(\sigma^a = \| (\sigma^a)_{ab} \|, \tilde{\sigma}^a = \| (\tilde{\sigma}^a)_{\alpha \beta} \|\) with the 6D vector index \(a = 0, 1, \ldots, 5\) and spinor indices \(\alpha, \hat{\alpha} = 1, \ldots, 4\) are realized in the form of the following matrices:

\[\sigma^a = (\sigma^0, \sigma^1, \sigma^2), \quad \tilde{\sigma}^a = (\sigma^0, -\sigma^1, -\sigma^2),\]  
\(\hat{\alpha} = 1, \ldots, 4,\]  
(B.1)

where

\[\sigma^0 = 14, \quad \tilde{\sigma}^0 = \tau_2 \otimes \tau_3 \text{ at } \hat{\alpha} = 1, 2, 3,\]  
\[\sigma^4 = -\tau_1 \otimes \tau_2, \quad \sigma^5 = \tau_3 \otimes \tau_2\]  
(B.2)

and \(\tau_{1,2,3}\) are the Pauli matrices.\(^\text{10}\) The antisymmetric \(\sigma\)-matrices with non-dotted spinor subscripts and superscripts are defined as follows:

\[(\sigma^a)_{ab} = (\sigma^a)_{ab} (B^{-1})^{\beta \gamma}, \quad (\tilde{\sigma}^a)_{\alpha \beta} = B_{\beta \alpha} (\tilde{\sigma}^a)^{\gamma \beta},\]  
(B.3)

where \(B = \| B_{\beta \alpha} \| = 1_2 \otimes i \tau_2 = \begin{pmatrix} i \tau_2 & 0 \\ 0 & i \tau_2 \end{pmatrix}\) is the matrix defining complex conjugation of the 6D Weyl spinors \([5, 45, 46]\). In particular, the matrices \(\sigma^{\pm} = (\sigma^0 \pm \sigma^5)/\sqrt{2}, \quad \tilde{\sigma}^{\pm} = (\tilde{\sigma}^0 \pm \tilde{\sigma}^5)/\sqrt{2}\) with undotted indices have the form:

\[(\sigma^+)_{ab} = \sqrt{2} \begin{pmatrix} i \tau_2 & 0 \\ 0 & 0 \end{pmatrix},\]  
\[(\sigma^-)_{ab} = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\]  
(B.4)

\[\tilde{\sigma}^+ a b = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & -i \tau_2 \end{pmatrix},\]  
\[\tilde{\sigma}^- a b = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 0 & i \tau_2 \end{pmatrix},\]  
(B.5)

Also we use the standard representation for the ‘t Hooft symbols \(\eta_{ab}^i = -\eta_{ab}^{-i}, i = 1, 2, 3\) and \(\tilde{\eta}_{ab}^i = -\tilde{\eta}_{ab}^{-i}, i = 1, 2, 3\) (see e.g. \([3, 4, 47, 48]\))

\[\eta_{ab}^i = \begin{cases} \epsilon_{i \hat{\alpha} \hat{b}} \hat{\alpha}, \hat{\beta} = 1, 2, 3, & i = 1, 2, 3, \\
\delta_{i \hat{\alpha} \hat{b}} \hat{\alpha}, \hat{\beta} = 4, & -\delta_{i \hat{\alpha} \hat{b}} \hat{\alpha}, \hat{\beta} = 4.
\end{cases}\]  
(B.6)

First, we will consider the \(so(4)\)-part of the generators (2.7), i.e. the operators

\[S_{ab} = \xi^I_\rho (\sigma^\alpha a b \rho^\beta \rho^I ) , \quad \hat{\alpha} = 1, 2, 3, 4,\]  
(B.7)

\(\text{In} [5, 45]\) the representation \(\sigma^1 = \tau_1 \otimes 1_2, \sigma^2 = \tau_2 \otimes \tau_3 \otimes 1_2\) at \(\hat{\alpha} = 2, 3, 4\) and \(\sigma^0, \sigma^5\) was used as in (B.2). That is, distinction between the representations [5, 45] and (B.2) lies in the difference in the notation of the four space coordinates labeled by \(\hat{\alpha}\). However, the representation (B.2) is more convenient when using the standard realizations (B.6) for the ‘t Hooft symbols.
These six generators $S_{\alpha\beta}^+$ are written as the sum

$$S_{ab}^\pm = S_{ab}^{(\pm)} + \tilde{S}_{ab}^{(\pm)},$$

(B.8)

where the SO\((4)\)-(anti-)self-dual parts $S_{ab}^{(\pm)}$ are expressed in terms of the SO\((3)\)-vectors $S_i^{(+)}$, $S_i^{(-)}$:

$$S_{ab}^{(\pm)} = -\eta_{ab}^{\pm}S_i^{(+)}S_{i'}^{(\pm)},$$

(B.9)

if we use the 't Hooft symbols (B.6). Thus, the generator (B.7) has the expansion

$$S_{ab} = -\eta_{ab}S_i^{(+)} - \tilde{\eta}_{ab}^{\pm}S_i^{(-)},$$

(B.10)

where the operators $S_i^{(+)}$ and $S_i^{(-)}$ form two su\((2)\) algebras:

$$\{S_i^{(\pm)}, S_j^{(\pm)}\} = i\epsilon_{ijk}S_k^{(\pm)}, \quad \{S_i^{(\pm)}, S_j^{(\pm)}\} = i\epsilon_{ijk}S_k^{(\pm)},$$

$$\{S_i^{(\pm)}, S_j^{(\pm)}\} = 0.$$  

(B.11)

Using the equalities $\eta_{ab}^{\pm}\eta_{ab} = 4\delta^{ab}$, $\tilde{\eta}_{ab}^{\pm}\tilde{\eta}_{ab}^{\pm} = 4\delta^{ab}$ and $\eta_{ab}\tilde{\eta}_{ab} = 0$, one finds the inverse to (B.9) relations

$$S_i^{(\pm)} = -\frac{1}{4}\eta_{ab}S_{ab}, \quad S_i^{(\pm)} = -\frac{1}{4}\tilde{\eta}_{ab}S_{ab}.$$  

(B.12)

However, using (B.1), (B.2), (B.3) and (B.6) we obtain that the matrices present in the definition of the generators (B.12) have only one diagonal $2\times2$ matrix block:

$$-\frac{1}{4}\eta_{ab}^{\pm}(\bar{\sigma}_{ab})^{\alpha\beta} = \frac{i}{2}\begin{pmatrix} -\tau_i^T & 0 \\ 0 & 0 \end{pmatrix},$$

$$-\frac{1}{4}\tilde{\eta}_{ab}^{\pm}(\bar{\sigma}_{ab})^{\alpha\beta} = \frac{i}{2}\begin{pmatrix} 0 & 0 \\ 0 & -\tau_i^T \end{pmatrix}.$$  

(B.13)

Substituting (B.7) and (B.13) into (B.12), one finds

$$S_i^{(\pm)} = -\frac{i}{2}\rho_i^{\pm}(\tau_i^T)\tau_i^{\pm}, \quad S_i^{(-)} = -\frac{i}{2}\rho_i^T(\tau_i^T)\tau_i^{\pm}.$$  

(B.14)

Thus, the generators $S_i^{(\pm)}$ are built using the canonical pairs $(\xi_i^\pm, \rho_i^\pm)$ from (3.3), while the generators $S_i^{(-)}$ are built using the other canonical pairs $(\xi_i^\pm, \rho_i^T)$. Now using the matrix expressions

$$\begin{aligned}
(\sigma^{\pm\hat{a}})_{\hat{\alpha}\hat{\beta}} &= \frac{i}{\sqrt{2}}\begin{pmatrix}
0 & 0 \\
0 & -\tau_a^T
\end{pmatrix}, \quad (\sigma^{\pm\hat{A}})_{\hat{\alpha}\hat{\beta}} &= \frac{i}{\sqrt{2}}\begin{pmatrix}
0 & \tau_a^T \\
0 & 0
\end{pmatrix}, \\
(\sigma^{\hat{A}\hat{a}})_{\hat{\alpha}\hat{\beta}} &= -\frac{1}{\sqrt{2}}\begin{pmatrix}
0 & 0 \\
0 & 1_{2a}
\end{pmatrix}, \quad (\sigma^{\hat{A}\hat{A}})_{\hat{\alpha}\hat{\beta}} &= -\frac{1}{\sqrt{2}}\begin{pmatrix}
0 & 1_{2a} \\
0 & 0
\end{pmatrix}, \\
(\sigma^{\hat{A}\hat{A}})_{\hat{\alpha}\hat{\beta}} &= -\frac{1}{\sqrt{2}}\begin{pmatrix}
1_{2a} & 0 \\
0 & 0
\end{pmatrix}, \\
(\sigma^{\hat{A}\hat{A}})_{\hat{\alpha}\hat{\beta}} &= \frac{1}{2}\begin{pmatrix}
0 & 0 \\
0 & 1_{2a}
\end{pmatrix}
\end{aligned}$$  

(B.15)

obtained from (B.1), (B.2), (B.3), and expansion (3.3), we find the spin part of the remaining Lorentz group generators

$$(2.7):$$

$$S^{\pm\hat{a}} = -\frac{i}{\sqrt{2}}\rho_i^{\pm}(\tau_a^T)\tau_i^{\pm}, \quad S^{\pm\hat{A}} = \frac{i}{\sqrt{2}}\rho_i^T(\tau_a^T)\tau_i^{\pm},$$

at $\hat{a} = 1, 2, 3$.

(B.17)

$$S^{\hat{A}\hat{a}} = -\frac{1}{\sqrt{2}}\rho_i^T(\tau_a^T)\tau_i^{\pm}, \quad S^{\hat{A}\hat{A}} = -\frac{1}{\sqrt{2}}\rho_i^{\pm}(\tau_a^T)\tau_i^{\pm},$$

(B.18)

The found expressions (B.10), (B.14), (B.17), (B.18) are used in Sect. 4 to construct the spin part of the Lorentz algebra generators in the bi-harmonic space.

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