Theory of multi-point probability densities for incompressible Navier-Stokes fluids

C. Asci\textsuperscript{1} and M. Tessarotto\textsuperscript{1,2}

\textsuperscript{1}Department of Mathematics and Informatics, University of Trieste, Trieste, Italy

\textsuperscript{2}Consortium for Magnetofluid Dynamics, Trieste, Italy

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Abstract

An open problem arising in the statistical description of turbulence is related to the theoretical prediction based on first principles of the so-called multi-point velocity probability density functions (PDFs) characterizing a Navier-Stokes fluid.

In this paper it will be shown that - based on a suitable axiomatic approach - a solution to this problem can actually be achieved based on the so-called inverse kinetic theory (IKT), recently developed for incompressible fluids. More precisely, we intend to show, based on the requirement that the Boltzmann-Shannon entropy for the s-point velocity PDF ($f_s$) is independent of the order $s$ and is also maximal at all times, that all multi-point PDFs are necessarily factorized in terms of the corresponding 1-point velocity PDF ($f_1$). As a consequence the multi-point PDFs usually considered for the phenomenological description of turbulence can be theoretically predicted based on the knowledge of $f_1$ achieved by means of IKT.

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I. INTRODUCTION

In the context of the statistical description of fluids, the problem of the determination of multi-point PDFs arises (at least) in two circumstances:

- the first one occurs in the phenomenological description of turbulence (see for example Monin and Yaglom [1], 1975 and Pope, 2000 [2]). In such a context, in fact, the statistical behavior of fluids is often described in terms of statistical frequencies defined for multi-point velocity spatial increments (however, similar frequencies can be established also for other fluid fields, such as vorticity, scalar pressure, temperature, etc.).

- the second one is the so-called Monin-Lundgren hierarchy [3, 4], based on the construction of an infinite set of equations for suitable ensemble-averaged multi-point PDFs (ML approach). Such a theory should provide, in principle, also a theoretical model for the phenomenological description of turbulence and as a consequence be able to predict also the precise form of the velocity-difference PDF observed experimentally in HIST (homogenous, isotropic and stationary turbulence). The goal the ML approach is actually to predict the time evolution of the ensemble average of the 1-point PDF, to be defined in terms of a suitable (and yet to be defined) ensemble-averaging operator.

Several open issues are related to the ML approach. These concern, in particular, the search of possible exact particular solutions of the ML hierarchy represented by a finite set of multi-point PDFs. It is well known that the construction of "closure conditions" of this type for the ML hierarchy (closure problem) remains one of the major unsolved theoretical problems in fluid dynamics. In practice, however, the program of constructing (exact) theories of this type or (in some sense) approximate, and holding for arbitrary fluid fields, is still open due to the difficulty of preserving the full consistency with the fluid equations. In fact, it is well known that many of the customary statistical models adopted in turbulence theory - which are based on closure conditions of various type - typically reproduce at most only in some approximate (i.e., asymptotic) sense the fluid equations.

This leaves fundamentally unsolved the problem of the construction of a consistent theoretical model for the multi-point PDFs arising in the phenomenological description of turbulence.
The goal of this paper is to prove (see THM.1 below in Section 2) that under suitable assumptions all multi-point velocity PDFs characterizing a turbulent NS fluid are factorizable in terms of the corresponding 1-point velocity PDF.

As a result (see Sec.3) the treatment of multi-point PDFs can be reached in the context of IKT (inverse kinetic theory [5–9]) based on the 1-point velocity statistics. It follows the fundamental consequence that the multi-point PDFs usually considered for the phenomenological description of turbulence can actually be theoretically predicted in this way! In particular, in the case of local Gaussian 1-point PDF [10] this permits to achieve explicit analytic representations of the multipoint velocity PDFs usually considered in the phenomenological description of turbulence.

II. MULTI-POINT STATISTICAL MODEL

The description of fluids, and more generally of continua, is based on the introduction of a suitable set of fluid fields \( \{ Z \} \equiv \{ Z_i, i = 1, k \} \) satisfying a closed set of PDEs denoted as fluid equations. In the case of a fluid obeying of the incompressible Navier-Stokes equations (INSE, NS fluid), they are \( \{ Z \} \equiv \{ \rho_0, V, p_1, S_T \} \). In particular, here \( \rho_0 \) (the mass density) and \( S_T \) (the thermodynamic entropy) are both assumed constant in \( \Omega \times I \), where the latter requirement implies that for isentropic flows the equation \( \partial S_T(t)/\partial t = 0 \) must hold identically for all \( t \in I \). In addition \( V \) and \( p_1 \) denote respectively the fluid velocity and the kinetic pressure; in particular, \( p_1 \) is defined as the strictly positive function

\[
p_1(r, t) = p(r, t) + p_0(t) + \phi(r, t),
\]

where \( p(r, t), p_0(t) \) and \( \phi(r, t) \) represent respectively the fluid pressure, the (strictly-positive) pseudo-pressure and the (possible) potential associated to the conservative volume force density acting on the fluid [see the Appendix, Eq.(44)].

The statistical description usually adopted for turbulent flows consists, instead, in the introduction of appropriate axiomatic approaches denoted statistical models, i.e., sets \( \{ f, \Gamma \} \) formed by a suitable probability density function (PDF) and a phase-space \( \Gamma \) (subset of \( \mathbb{R}^n \)) on which \( f \) is defined. By definition, a statistical model \( \{ f, \Gamma \} \) realizes a statistical description of the fluid if it is possible to define a mapping

\[
\{ f, \Gamma \} \Rightarrow \{ Z \},
\]
which allows the representation in terms of $f$ either:

A) of the complete set or more generally only B) of a subset of the fluid fields \( \{ Z \} \equiv \{ Z_i, i = 1, n \} \) which define the fluid state.

In particular, the fluid fields \( Z_i(\mathbf{r}, t) \in \{ Z \} \) are assumed as functionals of $f$ represented by suitable "velocity" moments (of $f$). In both cases their construction involves, besides the specification of the phase space (\( \Gamma \)) and the probability density function (PDF) $f$, the identification of the functional class to which $f$ must belong, denoted as \( \{ f \} \). Statistical approaches fulfilling either property A or B will be denoted respectively complete and incomplete statistical models. For definiteness in the remainder we shall consider only complete statistical models.

'A priori' the PDF $f$ to be used in a statistical model of this type may be identified with an $N$-point PDF of the form

\[
  f_N(\mathbf{x}, t) \equiv f_N(\mathbf{x}_1, ..., \mathbf{x}_N, t),
\]
and required to satisfy the normalization condition:

\[
  \int_{U^N} \prod_{j=1,N} d^3\mathbf{v}_j f_N(\mathbf{x}, t) = 1,
\]
i.e., to be a velocity probability density (in the velocity space $U^N$); moreover, $N \geq 1$ and for all $i = 1, N$, $\mathbf{x}_i = (\mathbf{r}_i, \mathbf{v}_i)$, $\mathbf{r}_i$ and $\mathbf{v}_i$ denote $N$ position and velocity vectors, respectively belonging to the configuration space of the fluid $\Omega$ and a suitable velocity space $U$ to be identified with $\mathbb{R}^3$. In particular, consistent with the physical requirement of a NS fluid [i.e. the existence of a strong solution of INSE in the set $\Omega \times I$], the following assumptions are introduced for $f \equiv f_N$:

- **Axiom #1 (symmetry condition):** $f_N(\mathbf{x}_1, ..., \mathbf{x}_N, t)$ is symmetric w.r. to arbitrary permutation $(\mathbf{x}_1, ..., \mathbf{x}_N)$, i.e., satisfying the invariance condition

  \[
  f_N((\mathbf{x}_1, ..., \mathbf{x}_N), t) = f_N((\mathbf{x}_1, ..., \mathbf{x}_N)'), t);
  \]

- **Axiom #2 (reduced $s$-body PDFs):** $f_N(\mathbf{x}_1, ..., \mathbf{x}_N, t)$ defines for all $s = 1, N - 1$ the reduced $s$-body PDFs

  \[
  f_s(\mathbf{x}_1, ..., \mathbf{x}_s, t) = \frac{1}{\mu(\Omega)} \int_{\Omega} d^3\mathbf{r}_{s+1} \int_{U} d^3\mathbf{v}_{s+1} f_{s+1}(\mathbf{x}_1, ..., \mathbf{x}_{s+1}, t),
  \]
where $\mu(\Omega) = \int d^3r_1$ is assumed finite and $> 0$. Hence each $f_s$ satisfies, thanks to (4), the normalization
\[
\int_{U_s} \prod_{j=1,s} d^3v_j f_s(x_1, ..., x_s, t) = 1; \tag{7}
\]

- **Axiom #3 (fluid moments):** $f_N(x_1, ..., x_N, t)$ determines uniquely the local fluid fields. Thus, introducing suitable weight functions $G_i(r_k, v_k, t)$, for all $k = 1, n$ the local fluid fields $Z_i(r_k, t)$ to be identified with $V, p_1$ [both evaluated at the local position $r_k$ and time $t$ belonging to $\Omega \times I$] are taken of the form:
\[
\frac{1}{\mu(\Omega)^N} \int_{\Omega_{N-1}} \prod_{h=1,N:h \neq k} d^3r_h \int_{U^N} \prod_{j=1,N} d^3v_j G_i(r_k, v_k, t)f_N(x, t) = \int_{U} d^3v_k G_i(r_k, v_k, t)f_1(r_k, v_k, t) = Z_i(r_k, t). \tag{8}
\]
As suggested by classical statistical mechanics (CSM) [11, 12], $G_i(r, v, t)$ are identified respectively with
\[
G_i(r, v, t) = v, \rho_0u^2/3 \tag{9}
\]
[with $u \equiv v - V(r, t)$ the relative velocity] for $V(r, t)$ and $p_1(r, t)$;

- **Axiom #4 (entropy moments):** $f_N(t) \equiv f_N(x_1, ..., x_N, t)$ determines uniquely the global fluid field $S_T(t)$. Again based on CSM, the thermodynamic entropy $S_T(t)$ can be identified with the Boltzmann-Shannon (BS) statistical entropy. For this reason, consistent with Ref. [7] we require that for all $t \in I$:
\[
S_T(t) = S(f_1(t)), \tag{10}
\]
where $f_1(t) \equiv f_1(x_1, t)$ and $f_1(x_1, t)$ is defined in terms of $f_N(t)$ by means of Eq. (10).
Furthermore we impose also that for arbitrary $N \in \mathbb{N}_1$ and $t \in I$
\[
K^2_N S(f_N) = S(f_1) \tag{11}
\]
(entropy constraint). Here, denoting by $\Gamma^N$ the product phase-space $\Gamma^N \equiv \prod_{i=1,N} \Gamma_1$, with $\Gamma_1 = \Omega \times U$, the BS entropy for the $N$-point PDF $f_N$ is defined as
\[
S(f_N) = -\int_{\Gamma^N} dx f_N \ln f_N, \tag{12}
\]
where $dx = \prod_{k=1,N} dr_k dv_k$ and $K^2_N$ are suitable constants independent $f_N$ to be determined;
• **Axiom #5 (entropic principle):** for all \( N \in \mathbb{N}_1 \), \( f_N(x_1, ..., x_N, t) \) satisfies the principle of entropy maximization requiring

\[
\delta S(f_1) = 0 \quad (13)
\]

(PEM variational principle [13]). The variational principle (13) is imposed either solely subject to **Axiom #5a (local entropic principle) at some initial time** \( t = t_0 \) or to **Axiom #5b (global entropic principle) for all** \( t \in I \).

Let us analyze the physical interpretation of the previous assumptions.

First we notice that #1, #2, #3 and #4 follow from the requirement that the state of the fluid is **solely** prescribed by the set of **local** and **global** fluid fields \( \{Z\} \). In particular the locality of the fluid fields, together with the assumption that they are defined everywhere in \( \Omega \), implies manifestly the symmetry requirement (5) (see #1). In fact, the positions \( r_1, ..., r_N \) can be manifestly interchanged arbitrarily among them (and similarly the velocity vectors \( v_1, ..., v_N \)) without affecting the determination of the fluid fields. This justifies the definitions given above in terms of the 1-point PDF both for the local and global fluid fields [see Eqs. (8), (10) and (11)].

In a similar way, since by assumption the fluid fields cannot depend on the level adopted for the statistical description of the fluid, all moments which define the fluid field must be independent of the choice of the \( N \)-point PDF. Indeed, for arbitrary \( N \in \mathbb{N}_1 \), it must be possible to represent the thermodynamic entropy in terms of Boltzmann-Shannon entropy associated to the \( N \)-point PDF \( f_N \), as well to \( f_1 \). This implies, that besides the position (8) invoked for the local fluid fields also the additional constraint (11) must be placed on all the BS entropies associated to multi-point PDFs. This constraint manifestly should hold identically (for all \( t \in I \)).

Finally, the hypothesis that the Boltzmann-Shannon entropy is maximal (see #5) implies the validity of the entropic principle (13). We stress that, in principle, PEM can be assumed to hold either at the initial time \( t_0 \) or, more generally, for arbitrary \( t \in I \). The second requirement is consistent with the assumption of isentropic flow. In fact, the positions (10) and (11) imply that also the BS entropy must be constant (i.e., independent of time). Hence, the requirement that it is maximal at some initial time \( t_0 \) may not be at variance with the requirement placed by the global entropic principle #5b.
Basic issues are related to the, possibly non-unique, determination of the appropriate statistical model \( \{f, \Gamma\} \). These concern in particular:

1. (PROBLEM #1) the search of the (possible) minimum level \( (N) \) of the statistical description to be adopted for \( \{f, \Gamma\} \);

2. (PROBLEM #2) the determination of the time-evolution of the multi-point PDFs \( f_N \);

3. (PROBLEM #3) the determination of the initial and boundary conditions for \( f_N \).

Regarding the first problem the following remarkable result holds:

**THM.1 - Factorization theorem for \( f_N \).**

*Let us impose Axioms #1-#4 with #5b. Then it follows necessarily that:*

1) the variational constraint

\[ \delta \left\{ K_N^2 S(f_N) - S(f_1) \right\} = 0 \]  \hspace{1cm} (14)

must hold for all \( t \in I \);

2) for all \( N \in \mathbb{N}_1 \), the \( N \)-point PDF \( f_N(x_1, ..., x_N, t) \) is of the form:

\[ f_N(x_1, ..., x_N, t) = \prod_{i=1,N} f_1(x_1, t), \]  \hspace{1cm} (15)

with \( f_1(x_1, t) \) denoting the corresponding 1-point PDF defined by Eq.(6). Hence, it follows also that for all \( s = 1, N-1 \):

\[ f_s(x_1, ..., x_s, t) = \int_U d^3v_{s+1} f_{s+1}(x_1, ..., x_{s+1}, t). \]  \hspace{1cm} (16)

3) the constant \( K_N^2 \) in Eq.(11) reads

\[ K_N^2 = N\mu(\Omega)^{N-1}. \]  \hspace{1cm} (17)

**PROOF** First we notice that the entropy constraint (11) together the global entropic principle #5b [i.e., the requirement that Eq.(13) holds for all \( t \in I \)] imply that, for all \( N \) and for all \( t \in I \), also the variational constraint (14) must be fulfilled. To prove that the
factorization property of the $N$-point PDF must hold for all $t \in I$, let us consider for illustration (and without loss of generality) the case $N = 2$. Denoting $f_2(x_1, x_2, t) \equiv f_2(1, 2)$ and $f_1(x_1, t) \equiv f_1(1)$, Eq. (14) delivers for arbitrary variations $\delta f_1(3)$:

$$
\int_{\Gamma^2} d\mathbf{x} \delta f_1(3) \{ f_2(1, 2) \ln f_2(1, 2) - f_1(1)f_1(2) [\ln f_1(1) + \ln f_1(2)] \} = 0.
$$

This implies necessarily that the factorization condition $f_2(1, 2) = f_1(1)f_1(2)$ must hold identically in $\Gamma^2 \times I$. The proof can easily be extended to arbitrary $N > 2$, yielding Eq. (15). In turn, thanks to Eq. (15), equations (16) and (17) immediately follow, respectively from Eqs. (6) and (11). Q.E.D.

We remark that in principle THM.1 can be generalized by requiring that PEM holds only at the initial time $t_o \in I$ (Axiom #5a). Nevertheless, in this case the constraint (11) only warrants that the factorization condition (16) holds at the initial time $t_o$, unless the form of the statistical (Liouville) equations holding for the $s$-point velocity PDFs is explicitly prescribed as done in Ref. [9].

Invoking, however, the validity of Axiom #5b and consequently of THM.1, the statistical model $\{ f, \Gamma \}$ can be identified with the IKT statistical model for the 1-point PDF [5–8].

III. IKT FOR MULTI-POINT PDFS

The construction of multi-point PDFs is a problem of "practical" interest in experimental/numerical research in fluid dynamics, usually adopted for the statistical analysis of turbulent fluids. In fact, they can be experimentally measured in terms of velocity differences between different fluid elements.

Let us assume, for definiteness, that $f_1(x_i, t)$ is the 1-point PDF which is particular solution of the Liouville equation [or inverse kinetic equation (IKE)] provided by IKT [5]. Then, denoting $f_1(i) \equiv f_1(x_i, t)$ (for $i = 1, s$) the same PDF evaluated at the states $x_i \equiv (r_i, v_i)$ (for $i = 1, s$), the $s$-point PDF is the probability density

$$
f_s(1, 2, .. s) \equiv \prod_{i=1,s} f_1(i),
$$

defined in the product phase-space $\Gamma^s \equiv \prod_{i=1,s} \Gamma$. The statistical equation advancing in time $f_s$ follows trivially from the Liouville equation for the 1-point PDF see [5]).
for \( i = 1, s \) by \( \mathbf{F}(i) \equiv \mathbf{F}(\mathbf{x}_i, t; f_1) \) the 1-point mean-field force per unit mass acting on the \( i \)-th particle (with state \( \mathbf{x}_i \)) [defined in Refs. 5 and 6] and introducing the \( s \)-point Liouville operator

\[
L_s(1, \ldots, s) \equiv \frac{\partial}{\partial t} + \sum_{i=1,s} \left[ \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{r}_i} + \frac{\partial}{\partial \mathbf{r}_i} \cdot \{ \mathbf{F}_i(i) \} \right],
\]

(20)

it follows that \( f_s(1,2,..s) \) satisfies identically the \( s \)-point Liouville equation

\[
L_s(1, \ldots, s)f_s(1, 2, \ldots s) = 0.
\]

(21)

A. Explicit evaluation of 2-point velocity PDFs

In terms of the 2-point PDF, \( f_2(1,2) \), a number of reduced probability densities can be defined in suitable subspaces of \( \Gamma^2 \). To introduce them explicitly let us first introduce the transformation to the center of mass coordinates of the two point-particles with states \((\mathbf{r}_i, \mathbf{v}_i)\) (for \( i = 1, 2 \))

\[
\{\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2\} \rightarrow \{\mathbf{r}, \mathbf{R}, \mathbf{v}, \mathbf{V}\}
\]

(22)

[here \( \mathbf{r} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{2}, \mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \); furthermore, \( \mathbf{v}, \mathbf{V} \) can be identified with \( \mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2 \) and \( \mathbf{V} = \mathbf{v}_1 + \mathbf{v}_2 \)]. Then, these are respectively:

1) the local (in configuration space) velocity-difference 2-point PDF \( g_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}, t) \) defined in the phase-space \( \Omega^2 \times U \) and obtained integrating the 2-point velocity PDF w.r. to the mean velocity \( \mathbf{V} \)

\[
g_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{v}, t) = \int_U d^3 \mathbf{V} f_2(1,2) \equiv \\
\equiv \int d^3 \mathbf{V} f_1(\mathbf{r}_1, \mathbf{v} + \mathbf{V}, t)f_1(\mathbf{r}_2, \mathbf{V} - \mathbf{v}, t);
\]

(23)

2) the velocity-difference 2-point PDF \( \hat{f}_2(\mathbf{r}, \mathbf{v}, t) \) defined in \( \Gamma_1 = \Omega \times U \) and obtained integrating also on the center-of-mass position vector \( \mathbf{R} \). Thus denoting by

\[
\langle \cdot \rangle_{\mathbf{R}, \Omega} = \frac{1}{\mu(\Omega)} \int_{\Omega} d^3 \mathbf{R}.
\]

(24)

the configuration-space average operator acting on the center of mass coordinates \( \mathbf{R} \), there it follows

\[
\hat{f}_2(\mathbf{r}, \mathbf{v}, t) = \langle g_2(\mathbf{r} + \mathbf{R}, \mathbf{R} - \mathbf{r}, \mathbf{v}, t) \rangle_{\mathbf{R}, \Omega}.
\]

(25)
In particular, in the case of a Gaussian PDF \[ g_2 \], Eq.(23) delivers again a Gaussian-type PDF

\[ g_2(r_1, r_2, v, t) = \frac{1}{\pi^{3/2} v_{th}^3} \exp \left\{ - \frac{\|v - \frac{1}{2}(V(1) - V(2))\|^2}{v_{th}^2} \right\} , \quad (26) \]

where \( V(i) \equiv V(r_i, t) \), \( v_{th,p}^2(i) = v_{th,p}^2(r_i, t) \) and \( v_{th}^2 \) denotes

\[ v_{th}^2 = \frac{v_{th,p}^2(1) + v_{th,p}^2(2)}{4} . \quad (27) \]

In a similar way it is possible to obtain explicit representations for the following additional 2-point PDFs:

1. **the velocity-difference 2-point PDF for parallel velocity increments.** Introducing the representations \( v = n v \) and \( r = n r \), \( n \) denoting a unit vector, \( \hat{f}_{2\parallel}(r, v, t) \) can be simply defined as the solid-angle average

\[ \hat{f}_{2\parallel}(r, v, t) = \int d\Omega(n) \hat{f}_2(r = n r, v = n v, t) ; \quad (28) \]

2. **the velocity-difference 2-point PDF for perpendicular velocity increments.** Introducing, instead, the representations \( v = n v \) and \( r = n \times b r \), \( n \) and \( b \) denoting two independent unit vectors, \( \hat{f}_{2\perp}(r, v, t) \) can be defined as the double-solid-angle average

\[ \hat{f}_{2\perp}(r, v, t) = \int d\Omega(n) \int d\Omega(b) \]

\[ \hat{f}_2(r = n \times b r, v = n v, t) . \quad (29) \]

An interesting property which emerges from these results is that in all cases indicated above [i.e., Eqs. (25), (28) and (29)] the definition of \( g_2 \) given above [Eq. (23)] implies that non-Gaussian features, respectively in \( \hat{f}_{2\parallel}, \hat{f}_{2\perp} \) and \( \hat{f}_{2\perp} \), may arise even if the 1–point PDF is Gaussian. This occurs due to velocity and pressure fluctuations occurring between different spatial positions \( r_1 \) and \( r_2 \). More generally, however, we can infer that, due to the constraint here imposed on the 1-point PDF

\[ \langle f_1(t) \rangle_{r, \Omega} = \hat{f}_1^{(freq)}(t) \quad (30) \]

[where \( \langle \cdot \rangle_{r, \Omega} \) it the averaging operator \( \langle \cdot \rangle_{r, \Omega} \equiv \frac{1}{\mu(\Omega)} \int d^3 r_o \) acting on of a function \( F(x, t) \)], it is obvious that, if the fluid velocity \( V(r, t) \) is bounded in the domain \( \Omega \), the same 1-point PDF, and hence the 2-point PDFs, cannot be Gaussian distributions.
B. Statistical evolution equation for the velocity-difference 2-point PDF

From the 2-point IKE (21) (obtained in the case \( s = 2 \)) it is immediate to obtain the corresponding evolution equation for the reduced PDFs indicated above. For example, the velocity-difference 2-point PDF \( \hat{f}_2 \) satisfies the equation

\[
\frac{\partial \hat{f}_2}{\partial t} + \mathbf{v} \cdot \frac{\partial \hat{f}_2}{\partial \mathbf{r}} = -\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D} \tag{31}
\]

where \( \mathbf{D} \) is the diffusion vector

\[
\mathbf{D} = \int d^3 \mathbf{V} \left\langle \frac{F_1(1) - F_2(2)}{2} f_2(1, 2) \right\rangle_{\mathbf{R}, \Omega} \tag{32}
\]

It follows, in particular, that in the case of a Gaussian 1-point PDF this equation reduces to the Fokker-Planck equation

\[
\frac{\partial \hat{f}_2}{\partial t} + \mathbf{v} \cdot \frac{\partial \hat{f}_2}{\partial \mathbf{r}} = -\frac{\partial}{\partial \mathbf{v}} \cdot \hat{\mathbf{D}} \tag{33}
\]

where the Fokker-Planck diffusion vector \( \hat{\mathbf{D}} \) reads

\[
\hat{\mathbf{D}} = \langle \mathbf{F}^{(T)} g_2(\mathbf{r} + \mathbf{R}, \mathbf{R} - \mathbf{r}, \mathbf{v}, t) \rangle_{\mathbf{R}, \Omega} \tag{34}
\]

and the vector field \( \mathbf{F}_1^{(T)} \equiv \mathbf{F}_1^{(T)}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{V}, t; f_M) \) is reported in Ref. [5]. It follows that both equations are manifestly non-Markovian as a consequence of the non-local dependencies arising (in both cases) in the Fokker-Planck coefficients \( \mathbf{D} \) and \( \hat{\mathbf{D}} \).

An interesting issue is here provided by the comparison with the statistical formulation developed by Peinke and coworkers [14–18]. Their approach, based on the statistical analysis of experimental observations, indicates that in case of stationary and homogeneous turbulence both the 2-point PDFs for parallel and velocity increments obey stationary Fokker-Planck equations. In particular, according to experimental evidence [17, 18] a reasonable agreement with a Markovian approximation for Eq.(33) - at least in some limited subset of parameter space- is suggested. Our theory implies, however, that a breakdown of the Markovian property should be expected due to non-local contributions appearing in the previous statistical equations (31) and (33).

IV. CONCLUSIONS

In this paper we have shown that the multi-point PDFs used in customary phenomenological approaches to turbulence can be explicitly evaluated in terms of the 1-point velocity
PDF \((f_1)\) determined in the framework on the IKT-statistical model \([5–9]\).

The starting point is provided by THM.1, which shows that under suitable hypotheses the multi-point PDF \(f_N\) is necessarily factorized in terms of the 1-point PDF \(f_1\). The requirements here imposed include, in particular, the assumption that \(\{f_N, \Gamma^N\}\) is a complete statistical model, i.e., that in terms of the multi-point PDF the complete set of fluid fields (defining the fluid state) can be represented by means of suitable velocity and phase-space moments [see Axioms #1-#6]. Then, provided:

A) the entropy constraint (11) is invoked ((Axiom #4); 
B) the validity of PEM is imposed at all times \(t \in I\) (Axiom #5b);
the factorization condition (15) for \(f_N\) in terms of the 1-point PDF \(f_1\) necessarily follows.

As a result, in validity of the previous requirements, the statistical model for NS fluid can be identified with the IKT-statistical model \(\{f_1, \Gamma_1\}\) earlier developed \([5–8]\) and based on the 1-point PDF \(f_1\). The theory has important consequences:

1. arbitrary multi-point PDFs can be uniquely represented in terms of the 1-point PDF characterizing the IKT-statistical model \(\{f_1, \Gamma_1\}\);
2. the time evolution of the multi-point PDFs is uniquely determined by \(\{f_1, \Gamma_1\}\);
3. the theoretical prediction of multipoint PDFs is actually possible.
4. qualitative properties of the multi-point PDFs can be investigated. As a particular case, the example of a Gaussian 1-point PDF has been pointed out.

In the IKT-statistical model the statistical equation advancing in time the 1-point PDF \(f_1\) coincides with the Liouville equation. As a consequence, its explicit evaluation is actually made possible \([9]\). In particular, as shown in Ref. \([10]\), in the presence of HIST the 1-point PDF necessarily coincides with a Gaussian distribution. Thanks to the factorization theorem (THM.1) this implies that also the multi-point velocity PDFs are uniquely determined. As result, as indicated in Section 3 (see subsection 3.1), two-point PDFs relevant for the phenomenological description of hydrodynamic turbulence can be explicitly determined.
V. APPENDIX: INSE PROBLEM

The fluid equations for a NS fluid are the so-called incompressible Navier-Stokes equations (INSE) for the fluid fields \( \{Z\} \equiv \{\rho_0, V(r,t), p_1(r,t), S_T(t)\} \):

\[
\begin{align*}
\rho &= \rho_0, \\
\nabla \cdot V &= 0, \\
NV &= 0, \\
\frac{\partial}{\partial t} S_T &= 0, \\
Z(r,t_0) &= Z_o(r), \\
Z(r,t)|_{\partial \Omega} &= Z_w(r,t)|_{\partial \Omega},
\end{align*}
\]

Eqs. (35)- (40) denote respectively the incompressibility, isochoricity, Navier-Stokes and constant thermodynamic entropy equations and the initial and Dirichlet boundary conditions for \( \{Z\} \), with \( \{Z_o(r)\} \) and \( \{Z_w(r,t)|_{\partial \Omega}\} \) suitably prescribed initial and boundary-value fluid fields, defined respectively at the initial time \( t = t_o \) and on the boundary \( \partial \Omega \), In particular, this means that they are are required to be at least continuous in all points of the closed set \( \overline{\Omega \times I} \), with \( \overline{\Omega} = \Omega \cup \partial \Omega \) closure of \( \Omega \). In the remainder we shall require that:

1. \( \Omega \) (configuration domain) is a bounded subset of the Euclidean space \( E^3 \) on \( \mathbb{R}^3 \);
2. \( I \) (time axis) is identified, when appropriate, either with a bounded interval, i.e., \( I = [t_0, t_1] \subseteq \mathbb{R} \), or with the real axis \( \mathbb{R} \);
3. in the open set \( \Omega \times I \) the functions \( \{Z\} \), are assumed to be solutions of Eqs. (36)-(38) subject, while in \( \overline{\Omega \times I} \) they satisfy the whole set of Eqs. (35)-(40). In particular: Eqs. (35)- (40) define the initial-boundary value INSE problem,
4. by assumption, the fluid fields are strong solutions of the fluid equations. Hence Eqs. (35)- (40) are required to define a well-posed problem with unique strong solution defined everywhere in \( \Omega \times I \).

Here the notation as follows. \( N \) is the NS nonlinear operator

\[
NV = \frac{D}{Dt} V - F_H,
\]
with \( \frac{D}{Dt} \mathbf{V} \) and \( \mathbf{F}_H \) denoting respectively the Lagrangian fluid acceleration and the total force per unit mass

\[
\frac{D}{Dt} \mathbf{V} = \frac{\partial}{\partial t} \mathbf{V}(\mathbf{r},t) + \mathbf{V}(\mathbf{r},t) \cdot \nabla \mathbf{V}(\mathbf{r},t),
\]

\[
\mathbf{F}_H \equiv -\frac{1}{\rho_0} \nabla p(\mathbf{r},t) + \frac{1}{\rho_0} \mathbf{f}(\mathbf{r},t) + \nu \nabla^2 \mathbf{V}(\mathbf{r},t),
\]

while \( \rho_0 > 0 \) and \( \nu > 0 \) are the constant mass density and the constant kinematic viscosity. In particular, \( \mathbf{f} \) is the volume force density acting on the fluid, namely which is assumed of the form

\[
\mathbf{f} = -\nabla \phi(\mathbf{r},t) + \mathbf{f}_R(\mathbf{r},t),
\]

\( \phi(\mathbf{r},t) \) being a suitable scalar potential, so that the first two force terms [in Eq.(43)] can be represented as

\[
-\nabla p(\mathbf{r},t) + \mathbf{f}(\mathbf{r},t) = -\nabla p_1(\mathbf{r},t) + \mathbf{f}_R(\mathbf{r},t),
\]

with \( p_1(\mathbf{r},t) \) defined by Eq.(1) denoting the kinetic pressure. As a consequence the fluid pressure necessarily satisfies the Poisson equation

\[
\nabla^2 p(\mathbf{r},t) = S(\mathbf{r},t),
\]

where the source term \( S \) reads

\[
S(\mathbf{r},t) = -\rho_0 \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) + \nabla \cdot \mathbf{f}.
\]

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