Localization Characteristics of Relativistic vs Nonrelativistic Fermions on a Lattice

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Relativistic fermions on a lattice are shown to correspond to the fluctuations in the localized nonrelativistic fermions. Therefore, in contrast to nonrelativistic case, the relativistic fermions are critical with universal exponents described by the strong coupling limit of the nonrelativistic problem. The fluctuations also describe anisotropic spin chain at the onset to long range magnetic order whose universality class is the Ising model. This generalizes the universality in spin models to include multifractal exponents. Finally, analogous to the nonrelativistic case, the relativistic fermions may exhibit ballistic character due to correlations.

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The problem of Dirac fermions with random mass has been the subject of various theoretical studies. \[1,2,3\] For weak disorder, the model describes one-dimensional (1D) spinless fermions with random hopping matrix elements \[2\], a toy model that exhibits localization transition in 1D. Both the spin-Peierls and the spin-ladder systems with non-magnetic impurities have been described by Dirac equation with random mass. \[3,4\] The zero-energy states which dominate the low temperature thermodynamic have been the subject of main focus as it provides an exactly solvable model whose multifractal spectrum can be computed analytically. \[3,5\] It turns out that in contrast to the usual 1D localization problem for nonrelativistic fermions, correlation functions for relativistic fermions exhibit power-law decay with universal exponents. \[3,5\]

The central result of this paper is that the differences between the relativistic and the nonrelativistic fermions in presence of disorder, can be attributed to simple relationships between these two problems. We show that relativistic fermions are described by the fluctuations in the exponential decaying wave functions of the nonrelativistic fermions. New insight into both the absence of localization and the universality of zero mode Dirac fermion is gained by their mapping to the anisotropic spin-1/2 chain at the onset to long range order (LRO). Zero mode Dirac fermions describe critical spin chains with \(O(1)\) symmetry while the corresponding non-relativistic fermions are related to \(O(2)\) spin chains. We argue that the massless excitations of the spin chain responsible for long range magnetic correlations provide mechanism for delocalization of relativistic fermions and the statement of universality is the well known statement that the anisotropic spin chains belong to the universality class of the Ising model. Although the setting we describe is quite general in the context of disordered systems, for concreteness we will consider quasiperiodic disorder which exhibits localization transition in nonrelativistic lattice problem. The reason for this particular choice is that the results from exact renormalization group (RG) \[6\] as well as rigorous mathematical analysis \[7\] can be used to compare and contrast the role of disorder in relativistic vs nonrelativistic problem. For quasiperiodic disorder, nonrelativistic problem known as Harper equation \[8\] has been studied extensively. Another interesting result of this paper is that it provides a new way to understand recently discussed strong coupling fixed point of Harper equation which describes the universal aspects of the localized phase. \[9\] The strong coupling universality is the statement that the universality class of anisotropic spin chain is determined by the Ising limit.

We consider a nearest-neighbor (nn) tight binding model (tbm) of spinless non-interacting fermions in a disordered potential \(V_n\),

\[
H = \sum [c_{n+1}^\dagger c_n + 2\lambda V_n c_{n}^\dagger c_n]
\]

Here the \(c\)'s are canonical Fermion operators and \(\lambda\) is the strength of the aperiodic potential \(V(x)\). The eigenstates of this lattice system are given by the following discrete Schroedinger equation,

\[
\psi_{n-1} + \psi_{n+1} + 2\lambda V_n \psi_n = E\psi_n.
\]

For disordered systems exhibiting exponential localization, the fermion wave functions \(\psi_n\) can be written as

\[
\psi_n = e^{-\gamma|x|} \eta_n
\]

where \(\gamma\) is the inverse localization length \(\gamma = \xi^{-1}\). The tbm describing the fluctuations \(\eta_n\) in the exponentially decaying envelope is given by the following pair of equations,

\[
e^{-\gamma} \eta_{n+1}^c + e^{\gamma} \eta_{n-1}^c + 2\lambda V_n \eta_n^c = E\eta_n^c
\]

\[
e^{\gamma} \eta_{n+1}^l + e^{-\gamma} \eta_{n-1}^l + 2\lambda V_n \eta_n^l = E\eta_n^l
\]

Here \(\eta^c\) and \(\eta^l\) respectively describe the fluctuations to the right and to the left of the localization center, which is chosen to be at \(n = 0\). As shown below, these equations describe relativistic fermions on a lattice for zero energy states as in the long wave length limit, the equations reduce to the Dirac equation. We replace \(n\) by \(x\) and write \(\eta_{n\pm 1} = e^{\pm ip\eta(x)}\), where \(p\) is the momentum canonically conjugate to \(x\). The equation (4) for the
fluctuations for $E = 0$ state can be described by the following non-Hermitian Hamiltonian $H_{flu}$ and its adjoint,

$$H_{flu} = e^{-\gamma} e^{i p} + e^{\gamma} e^{-i p} + 2\lambda V(x)$$

(5)

In the limit ($p \to 0$), the system for $E = 0$ reduces to the Dirac equation,

$$[\sigma_x p - i(2\lambda V(x) + 2\cosh(\gamma)) \sigma_y] \eta(x) = 0$$

(6)

where $\eta(x)$ is a two-dimensional spinor $\eta(x) = (\eta'(x), \eta''(x))$. Therefore, the two-component structure of Dirac spinor arises naturally when we consider fluctuations about exponentially localized wave functions. Here $g \equiv 2\sinh(\gamma)$ is the velocity of the Dirac fermions while the mass of the Dirac fermions $m(x) = 2((\lambda V(x) + \cosh(\gamma))$. The $\sigma_k$, $k = x, y, z$ are the Pauli matrices.

Therefore, on a lattice, the Dirac fermions with disordered mass are the fluctuations of the nonrelativistic localized fermions. This would imply the absence of exponential localization for relativistic fermions provided the equation (4) has a solution with $E = 0$.

Before we discuss the solution of the discrete Dirac equation, we show the equivalence between the disordered Dirac Hamiltonian and a spin chain with disordered magnetic field. The anisotropic XY spin-1/2 chain in a transverse magnetic field is,

$$H = -\sum [e^{-\sigma_n^x \sigma_{n+1}^x} + e^{\sigma_n^x \sigma_{n+1}^x} + 2\lambda V \sigma_n^z].$$

(7)

The $e^{\sigma_n^x}$ and $e^{\sigma_n^x}$ respectively describe the exchange interactions along the $x$ and the $y$ directions in spin space and therefore, the spin space anisotropy $g$ is given by $g = 2\sinh(\alpha)$. It is well known that the Jordan-Wigner transformation transforms the spin problem to spinless fermion problem where fermions are the quasiparticle excitations of the spin chain. The eigenstates of the excitations are described by the following coupled equations,

$$e^{-\sigma_n^x \eta_{n+1}^1} + e^{\sigma_n^x \eta_{n-1}^1} + 2\lambda V \eta_n^1 = E\eta_n^2$$

$$e^{\sigma_n^x \eta_{n+1}^2} + e^{-\sigma_n^x \eta_{n-1}^2} + 2\lambda V \eta_n^2 = E\eta_n^1$$

(8)

For $E = 0$, these equations reduce to equation (4) with the parameter $\alpha$ equals the inverse localization length $\gamma$. However, $E = 0$ is the solution of this equation only at the onset to LRO, and therefore, for anisotropic spin chain with disordered field $V_n$, this mapping between the spin and fermions provides an interesting way to describe the onset to LRO: the massless excitations of the critical anisotropic spin chain describes the fluctuations in the exponentially localized excitations of the isotropic chain provided the anisotropy parameter is the inverse localization length. Alternatively, the mapping provides a new method to determine the localization length of thms in the presence of disorder. For the case of quasiperiodic disorder, $V_n = \cos(2\pi(\sigma n + \phi))$, (where $\sigma$ is an irrational number and $\phi$ is a constant phase factor), the localization length is $\xi^{-1} = \log(\lambda)$. Therefore $\alpha = \log(\lambda)$ is the equation for the critical line, a result known from earlier numerical study for the spin chain.

The Ising limit ($\alpha \to \infty$) of the critical spin chain coincides with the $\lambda \to \infty$ limit of the Harper equation. This is the strong coupling limit of the Harper equation [7] that has been studied by exact RG as well as more rigorous analysis [8]. The existence of a strong coupling fixed point shows the universality of the fluctuations in the exponentially decaying solutions in the localized phase. The equivalence between the strong coupling limit of Harper equation and the critical Ising spin in a quasiperiodic transverse field provides a new way to understand this fixed point: the statement of universality of the strong coupling fixed point of Harper equation is equivalent to the statement that the anisotropic spin chain in a quasiperiodic field belongs to the universality class of the Ising model, a result well known for periodic spin systems in arbitrary dimension.

We now focus on the $E = 0$ solution of the Harper equation in the localized phase as the fluctuations about this describe the Dirac fermions. The universal features of this solution can be studied by an exact RG scheme. The earlier studies of this problem was confused to the band edges of the eigenspectrum. To show the universal features of the solution, with $\sigma$ equals inverse golden mean, all sites except those labeled by the Fibonacci numbers $F_m$ are decimated. At the $m^{th}$ decimation level, the tbm describing the fluctuations is expressed in the form

$$f_m(n)\eta_{n+F_m} = \eta_{n+F_m} + e_m(n)\eta_m.$$  

(9)

The additive property of the Fibonacci numbers provides an exact recursion relations for the decimation functions $e_m$ and $f_m$. In the strong coupling limit $f_m$ is found to approach zero simplifying the RG flow as,

$$e_{m+1}(n) = -e_{m-1}(n + F_m)e_m(n)$$

(10)

Numerical iteration of this equation provides an extremely accurate method to distinguish extended, localized and critical states. The extended and localized phase correspond to trivial asymptotic RG flow while the critical behavior is found to be characterized by a nontrivial asymptotic 6-cycle in the decimation function: $e_{m+6} = e_m$.

Figure 1 shows the self-similar fluctuations about exponentially localized solutions in Harper equation, which describes zero-mode Dirac problem with quasiperiodic mass. In the strong coupling limit, which also describes the massless mode of the Ising chain the wave function is given by the following algebraic equation,

$$\frac{\eta_n}{\eta_0} = \exp[-\Phi(n)] = \exp[-\sum_{j=1}^n \log(2|V_j|)]$$

(11)
For random disorder, the above summation can be done explicitly resulting in exact solution [5][3], that facilitated an exact calculation of multifractal spectrum.

In order to seek the physical meaning of universal 3-cycle, and see whether the statement that the critical exponents of spin chains are determined by the symmetry of the spin chain applies to the multifractal exponents, we compute the $f(\alpha)$ curve describing the multifractal spectrum associated with the self-similar wave function or the inverse participation ratios $P$,

$$P(q,N) = \frac{\sum |\eta_n|^{2q}}{\sum |\eta_n|^2} \sim N^{-\tau(q)}$$

$$\alpha = \frac{d\tau}{dq}$$

$$f(\alpha) = \alpha q - \tau(q)$$  (12)

The free energy function $\tau(q)$ and its Legendre transform $f(\alpha)$ were found to be $\lambda$ independent only for positive values of $q$ and hence only left half of the $f(\alpha)$ curve is universal. Therefore, for quasiperiodic spin chains at the onset to LRO, scaling exponents for only positive moments of the participation ratio are determined by the spin space symmetry.

We next show that the presence of correlated disorder can result in propagating solutions for the disordered Dirac equation, as is known to be the case for the non-relativistic fermions. [3] For quasiperiodic disorder, dimer-type correlations can be introduced by replacing $\theta_n = 2\pi \sigma n$ in $V(\theta_n)$ by the iterates of the supercritical standard map, describing Hamiltonian systems with two degrees of freedom. [14][15]

$$\theta_{n+1} + \theta_{n-1} - 2\theta_n = -\frac{K}{2\pi} \sin(2\pi \theta_n).$$  (13)

We use iterates that describe golden-mean cantorus, which has been shown to exhibit dimer-type correlations, and leads to Bloch-type states for the nonrelativistic fermions. [14] Here, we will confine to the Ising model, described by,

$$\eta_{n+1} + 2\lambda \cos(2\pi \theta_n)\eta_n^1 = E\eta_n^2$$

$$\eta_{n-1}^2 + 2\lambda \cos(2\pi \theta_n)\eta_n^1 = E\eta_n^1$$  (14)

We determine the critical $\lambda$ as a function of $K$, and analyze the massless mode of the Ising model using the RG methodology [3]. As shown in figure 3, nontrivial 6-cycle degenerates to trivial fixed points at certain special parameter values. The origin of these trivial fixed points, has been traced to a hidden dimer in the quasiperiodic iterates describing the golden-cantorus. [14] At these points, the relativistic massless mode of the Ising model is ballistic. Further details of the effects of correlated disorder on magnetic and spectral transitions will be discussed elsewhere.

We would like to point out that although the relationship between the fermion and the spin problem is well known, what is new here is the that disordered spin chains provide important key in understanding the role of disorder in transport properties of relativistic and non-relativistic fermions. The differences between these two fermions is traced to the differences between spin chains with $O(1)$ and $O(2)$ symmetry where the spin chains with $O(1)$ symmetry exhibits magnetic long range correlations. The correspondence with the spin chain also explains why $E = 0$ mode of Dirac equation is special: the root of criticality of $E = 0$ Dirac fermions can be attributed to the fact that it corresponds to infinite magnetic correlation length. We argue that these magnetic correlations account for the absence of exponential localizations in relativistic fermions. It is rather interesting that the relativistic fermions can be viewed as the fluctuations about localized nonrelativistic fermions and this picture also explains the absence of localization of Dirac fermions. Furthermore, the universal aspects of relativistic fermions are described by the strong coupling limit of the nonrelativistic fermions. This universality implying that the positive moments of the participation ratio have universal exponents, generalizes the well known statements of universality of periodic spin chains to quasiperiodic spins exhibiting multifractal exponents. Finally, we would like to mention that many of our conclusions are valid for random as well as pseudorandom disorder. [7]

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FIG. 1. (a) Absolute value of the fluctuations for Harper equation for $E = 0$ states with $\phi = .25$. At the Fibonacci sites, we see period-6 behavior(period-3 in absolute value): $|\eta_{F_m}| = |\eta_{F_{m+3}}|$ which is independent of $\lambda$.

FIG. 2. Numerically obtained (a) $f(\alpha)$ curves for $\lambda \rightarrow \infty$(solid curve) and $\lambda = 1.5$ (lines with crosses).

FIG. 3. (a) Critical $\lambda$ as a function of $K$ for the Ising model. (b) RG 6-cycle showing the variation in the renormalized coupling $e$ after Fibonacci decimation.
