THE DIRICHLET PROBLEM AT THE MARTIN BOUNDARY
OF A FINE DOMAIN

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Abstract. We develop the Perron-Wiener-Brelot method of solving the
Dirichlet problem at the Martin boundary of a fine domain in \( \mathbb{R}^n \) (\( n \geq 2 \)).

1. Introduction

The fine topology on an open set \( \Omega \subset \mathbb{R}^n \) was introduced by H. Cartan in
classical potential theory. It is defined as the smallest topology on \( \Omega \) in which
every superharmonic function on \( \Omega \) is continuous. Potential theory on a finely
open set, for example in \( \mathbb{R}^n \), was introduced and studied in the 1970’s by the
second named author \([8]\). The harmonic and superharmonic functions and the
potentials in this theory are termed finely [super]harmonic functions and fine
potentials. Generally one distinguishes by the prefix ‘fine(ly)’ notions in fine
potential theory from those in classical potential theory on a usual (Euclidean)
open set. Large parts of classical potential theory have been extended to fine
potential theory.

The integral representation of positive (= nonnegative) finely superharmonic
functions by using Choquet’s method of extreme points was studied by the
first named author in \([5]\), where it was shown that the cone of positive super-
harmonic functions equipped with the natural topology has a compact base.
This allowed the present authors in \([6]\) to define the Martin compactifica-
tion and the Martin boundary of a fine domain \( U \) in \( \mathbb{R}^n \). The Martin compactifica-
tion \( \overline{U} \) of \( U \) was defined by injection of \( U \) in a compact base of the cone \( \mathcal{S}(U) \)
of positive finely superharmonic functions on \( U \). While the Martin boundary
of a usual domain is closed and hence compact, all we can say in the present
setup is that the Martin boundary \( \Delta(U) \) of \( U \) is a \( G_\delta \) subset of the
compact Riesz-Martin space \( \overline{U} = U \cup \Delta(U) \) endowed with the natural topology.
Nevertheless we have defined in \([6]\) a suitably measurable Riesz-Martin kernel
\( K : U \times U \rightarrow [0, +\infty] \). Every function \( u \in \mathcal{S}(U) \) has an integral represen-
tation \( u(x) = \int_{\overline{U}} K(x, Y) d\mu(Y) \) in terms of a Radon measure \( \mu \) on \( \overline{U} \).
This representation is unique if it is required that \( \mu \) be carried by \( U \cup \Delta_1(U) \), where
\( \Delta_1(U) \) denotes the minimal Martin boundary of \( U \), which likewise is a \( G_\delta \) in
\( \overline{U} \). In this case of uniqueness we write \( \mu = \mu_u \). It was shown that \( u \) is a fine
potential, resp. an invariant function, if and only if \( \mu_u \) is carried by \( U \), resp.
by $\Delta(U)$. The invariant functions, likewise studied in [6], generalize the positive harmonic functions in the classical Riesz decomposition theorem. Finite valued invariant functions are the same as positive finely harmonic functions.

There is a notion of minimal thinness of a set $E \subset U$ at a point $Y \in \Delta_1(U)$, and an associated minimal-fine filter $\mathcal{F}(Y)$, which allowed the authors in [6] to obtain a generalization of the classical Fatou-Naïm-Doob theorem.

In a continuation [7] of [6] we studied sweeping on a subset of the Riesz-Martin space, both relative to the natural topology and to the minimal-fine topology on $\overline{U}$, and we showed that the two notions of sweeping are identical. In the present further continuation of [6] and [7] we investigate the Dirichlet problem at the Martin boundary of our given fine domain $U$ by adapting the Perron-Wiener-Brelot (PWB) method to the present setup. It is a complication that there is no Harnack convergence theorem for finely harmonic functions, and hence the infimum of a sequence of upper PWB-functions on $U$ may equal $-\infty$ precisely on some nonvoid proper finely closed subset of $U$. We define resolutivity of a numerical function on $\Delta(U)$ in a standard way and show that it is equivalent to a weaker, but technically supple concept called quasi-resolutivity, which possibly has not been considered before in the literature (for the classical case where $U$ is Euclidean open). Our main result implies the corresponding known result for the classical case, cf. [4, Theorem 1.VIII.8]. At the end of Section 3 we obtain analogous results for the case where the upper and lower PWB-classes are defined in terms of the minimal-fine topology on $\overline{U}$ instead of the natural topology. It follows that the two corresponding concepts of resolutivity are compatible. This result is possibly new even in the classical case. A further alternative, but actually equivalent, concept of resolutivity is discussed in the closing Section 4.

**Notations:** If $U$ is a fine domain in $\Omega$ we denote by $\mathcal{S}(U)$ the convex cone of positive finely superharmonic functions on $U$ in the sense of [8]. The convex cone of fine potentials on $U$ (that is, the functions in $\mathcal{S}(U)$ for which every finely subharmonic minorant is $\leq 0$) is denoted by $\mathcal{P}(U)$. The cone of invariant functions on $U$ is the orthogonal band to $\mathcal{P}(U)$ relative to $\mathcal{S}(U)$. By $G_U$ we denote the (fine) Green kernel for $U$, cf. [9], [10]. If $A \subset U$ and $f : A \to [0, +\infty]$ one denotes by $R_f^A$, resp. $\widehat{R}_f^A$, the reduced function, resp. the swept function, of $f$ on $A$ relative to $U$, cf. [8 Section 11]. If $u \in \mathcal{S}(U)$ and $A \subset U$ we may write $\widehat{R}_u^A$ for $\widehat{R}_f$ with $f := 1_{A}\cdot u$.

2. **The upper and lower PWB$^h$-classes of a function on $\Delta(U)$**

We shall study the Dirichlet problem at $\Delta(U)$ relative to a fixed finely harmonic function $h > 0$ on $U$. We denote by $\mu_h$ the measure on $\Delta(U)$ carried by $\Delta_1(U)$ and representing $h$, that is $h = \int K(\cdot, Y) d\mu_h(Y) = K\mu_h$. A function $u$ on $U$ (or on some finely open subset of $U$) is said to be finely
h-hyperharmonic, finely h-superharmonic, h-invariant, or a fine h-potential, respectively, if it has the form \( u = \nu / h \), where \( \nu \) is finely hyperharmonic, finely superharmonic, invariant, or a fine potential, respectively.

Let \( f \) be a function on \( \Delta(U) \) with values in \( \mathbb{R} \). A finely h-hyperharmonic function \( u = \nu / h \) on \( U \) is said to belong to the upper PWB\( ^h \)-class, denoted by \( \mathcal{U}_f^h \), if \( u \) is lower bounded and if

\[
\liminf_{x \to Y, x \in U} u(x) \geq f(Y) \quad \text{for every } Y \in \Delta(U).
\]

We define

\[
\hat{H}_f^h = \inf \mathcal{U}_f^h, \quad \overline{H}_f^h = \hat{H}_f^h = \inf \mathcal{U}_f^h (\leq \hat{H}_f^h).
\]

Both functions \( \overline{H}_f^h \) and \( \hat{H}_f^h \) are needed here, unlike the classical case where we have the Harnack convergence theorem and hence \( \overline{H}_f^h = \hat{H}_f^h \). In our setup, \( \hat{H}_f^h \) may be neither finely h-hyperharmonic nor identically \( -\infty \), but only nearly finely h-hyperharmonic on the finely open set \( \{ \overline{H}_f^h > -\infty \} \) which can be a nonvoid proper subset of \( U \), see Example 2.1 below, which also shows that \( \Delta(U) \) can be non-compact. According to the fundamental convergence theorem \([\S\text{ Theorem 11.8]}\) \( \overline{H}_f^h \) is finely h-hyperharmonic on \( \{ \overline{H}_f^h > -\infty \} \) and \( \overline{H}_f^h = \hat{H}_f^h \) quasi-everywhere (q.e.) there; furthermore, since \( \mathcal{U}_f^h \) is lower directed, there is a decreasing sequence \( (u_j) \subset \mathcal{U}_f^h \) such that \( \inf_j u_j = \hat{H}_f^h \). Clearly, \( \hat{H}_f^h \) is finely u.s.c. on all of \( U \), and \( \overline{H}_f^h \) is finely l.s.c. there.

The lower PWB\( ^h \) class \( \mathcal{U}_f^h \) is defined by \( \mathcal{U}_f^h = -\mathcal{U}_{-f}^h \), and we have \( \hat{H}_f^h = 0 \), hence also \( \overline{H}_f^h = \hat{H}_f^h = H_0^h = 0 \). It follows that if \( f \geq 0 \) then \( \hat{H}_f^h \geq \overline{H}_f^h \geq 0 \) and therefore only positive functions of class \( \mathcal{U}_f^h \) need to be considered in the definition of \( \hat{H}_f^h \) and hence of \( \overline{H}_f^h \). Moreover, \( \hat{H}_{\alpha f + \beta} = \alpha \hat{H}_f^h + \beta \) and hence \( \overline{H}_{\alpha f + \beta} = \alpha \overline{H}_f^h + \beta \) for real constants \( \alpha \geq 0 \) and \( \beta \) (when 0 times \( \pm \infty \) is defined to be 0).

**Example 2.1.** In \( \Omega = \mathbb{R}^n \) with the Green kernel \( G(x, y) = |x - y|^{2-n} \), \( n \geq 4 \), let \( \omega \subset \Omega \) be a bounded Hölder domain such that \( \omega \) is irregular with a single irregular boundary point \( z \), cf. e.g. \([\S\text{ Remark 6.6.17]}\). Take \( U = \omega \cup \{ z \} \). According to \([\Pi \text{ Theorems 1 and 3.1]}\) the Euclidean boundary \( \partial \omega \) of \( \omega \) is topologically contained in the Martin boundary \( \Delta(\omega) \). In particular, \( z \) is non-isolated as a point of \( \Delta(\omega) \). But \( \Delta(U) = \Delta(\omega) \setminus \{ z \} \), where \( z \) is identified with \( P_z \) (see \([\S\text{ Section 3]}\)), and since \( \Delta(\omega) \) is compact we infer that \( \Delta(U) \) is noncompact. In \( \mathbb{R}^n \) choose a sequence \( (z_j) \) of points of \( \partial \omega \) such that \( |z_j - z| \leq 2^{-j} \). Then \( u := \sum_j 2^{-j} G(., z_j) \) is infinite at \( z \), but finite and harmonic on \( \omega \). Furthermore, \( u = \sup_k u_k \), where \( u_k := \sum_{j \leq k} 2^{-j} G(., z_j) \) is harmonic and bounded on \( \overline{\omega} \subset \mathbb{R}^n \). It follows that \( (u_k)_{|U} \) is of class \( \mathcal{U}_f^h \), where \( f := u|_{\Delta(U)} \).
In fact, 
\[ \lim_{x \to Y, x \in U} u_k(x) = u_k(Y) \leq u(Y) = f(Y) \]
for \( Y \in \Delta(U) \) (natural limit on \( U \cup \Delta(U) \), or equivalently Euclidean limit on \( \omega \cup ((\partial \omega) \setminus \{z\}) \)). Thus \( H^h_f \geq (u_k)_{U} \), and hence 
\[ H^h_f(z) \geq \sup_k u_k(z) = u(z) = +\infty. \]

To show that \( H^h_f < +\infty \) on \( U \setminus \{z\} \), let \( v \in U \setminus \{z\} \). Being upper bounded on the bounded open set \( \omega \), \( v \) is subharmonic on \( \omega \) by [8, Theorem 9.8], and so is therefore \( v - u \). For any \( Y \in \Delta(U) \) \((\equiv (\partial \omega) \setminus \{z\})\) we have 
\[ \limsup_{x \to Y, x \in \omega} v(x) \leq f(Y) < +\infty \]
(also with Euclidean limit), or equivalently 
\[ \limsup_{x \to Y, x \in \omega} (v(x) - u(x)) \leq 0. \]

Since \( \{z\} \) is polar and \( v - u \leq v \) is upper bounded, it follows by a boundary minimum principle that \( v - u \leq 0 \), that is, \( v \leq u \) on \( \omega \). By varying \( v \in U^h \) we conclude that \( H^h_f \leq u \) on \( \omega \equiv U \setminus \{z\} \) and hence by regularization that \( H^h_f \leq u < +\infty \) on \( U \setminus \{z\} \). Altogether, \( \overline{H}^h_f = -H^h_f \) equals \(-\infty \) at \( z \), but is finite on \( U \setminus \{z\} \).

Henceforth we fix the finely harmonic function \( h > 0 \) on \( U \), relative to which we shall study the Dirichlet problem at \( \Delta(U) \). Similarly to the classical case, cf. [4, p. 108], we pose the following definition, denoting by \( 1_A \) the indicator function of a set \( A \subset \Delta(U) \):

**Definition 2.2.** A subset \( A \) of \( \Delta(U) \) is said to be \( h \)-harmonic measure null if \( \overline{H}^h_{1_A} = 0 \).

It will be shown in Corollary 3.11 that \( A \) is \( h \)-harmonic measure null if and only if \( A \) is \( \mu_h \)-measurable with \( \mu_h(A) = 0 \).

**Proposition 2.3.** (a) Every countable union of \( h \)-harmonic measure null sets is \( h \)-harmonic measure null.

(b) A set \( A \subset \Delta(U) \) is \( h \)-harmonic measure null if and only if there is a finely \( h \)-superharmonic function \( u \) (positive if we like) on \( U \) such that \( \lim_{x \to Y, x \in U} u(x) = +\infty \) for every \( Y \in A \).

(c) If \( f : \Delta(U) \to [0, +\infty) \) has \( \overline{H}^h_f = 0 \) then \( \{f > 0\} \) is \( h \)-harmonic measure null.

(d) If \( f : \Delta(U) \to [0, +\infty] \) has \( \overline{H}^h_f < +\infty \) then \( \{f = +\infty\} \) is \( h \)-harmonic measure null.

(e) If \( f, g : \Delta(U) \to [0, +\infty] \) and if \( f \leq g \) off some \( h \)-harmonic measure null set then \( \overline{H}^h_f \leq \overline{H}^h_g \).
Proof. We adapt the proof in [4, p. 108, 111] for the classical case.

(a) Fix a point \( x_0 \) of the co-polar subset \( \bigcap_j \{ \hat{H}_1^{h_A} = 0 \} \) of \( U \). For given \( \varepsilon > 0 \) and integers \( j > 0 \) there are functions \( u_j \in \overline{U}_A \) with \( u_j(x_0) < 2^{-j} \varepsilon \). It follows that the function \( u := \sum_j u_j \) is of class \( \overline{U}_A \) because \( \sum_j 1_{A_j} \geq 1_A \) on \( \Delta(U) \). Consequently, \( \overline{H}_1^{h_A}(x_0) \leq \hat{H}_1^{h_A}(x_0) \leq u(x_0) < \varepsilon \), and the positive finely \( h \)-hyperharmonic function \( \overline{H}_1^{h_A} \) therefore equals 0 at \( x_0 \) and so indeed everywhere on \( U \).

(b) If \( \overline{H}_1^{h_A} = 0 \) then \( \hat{H}_1^{h_A} = 0 \) q.e., so we may choose \( x_0 \in U \) with \( \hat{H}_1^{h_A}(x_0) = 0 \). For integers \( j > 0 \) there exist positive finely \( h \)-superharmonic functions \( u_j \in \overline{U}_A \) on \( U \) such that \( u_j(x_0) < 2^{-j} \varepsilon \). The function \( u := \sum_j u_j \) is positive and finely \( h \)-superharmonic on \( U \) because \( u(x_0) < +\infty \). Furthermore, \( \liminf_{x \to Y, x \in U} u(x) = +\infty \) for every \( Y \in A \). Conversely, if there exists a function \( u \) as described in (b), we may arrange that \( u \geq 0 \) after adding a constant. Then \( \varepsilon u \in \overline{H}_1^{h_A} \) for every \( \varepsilon > 0 \). It follows that \( \hat{H}_1^{h_A} \leq \varepsilon u \) and by varying \( \varepsilon \) that \( \hat{H}_1^{h_A} = 0 \) off the polar set of infinities of \( u \), and hence q.e. on \( U \). It follows that indeed \( \overline{H}_1^{h_A} = 0 \).

(c) For integers \( j \geq 1 \) let \( f_j \) denote the indicator function on \( U \) for the set \( \{ f > 1/j \} \). Then \( 0 = \overline{H}_f \geq \overline{H}_f / j \), so the sets \( \{ f > 1/j \} \) are \( h \)-harmonic measure null, and so is by (a) the union \( \{ f > 0 \} \) of these sets.

(d) Choose \( x_0 \in U \) so that \( \hat{H}_f(x_0) = \overline{H}_f(x_0) \) \( < +\infty \) and \( u \in \overline{U}_f \). Then \( \lim_{x \to Y, x \in U} u(x) = +\infty \) for every \( Y \in A := \{ f = +\infty \} \). After adding a constant we arrange that the finely \( h \)-hyperharmonic function \( u \) is positive. According to (b) it follows that indeed \( \overline{H}_1^{h_A} = 0 \).

(e) Let \( v \in \overline{U}_f \) and let \( u \) be a positive \( h \)-superharmonic function on \( U \) with limit \( +\infty \) at every point of the \( h \)-harmonic measure null subset \( \{ f > g \} \) of \( \Delta(U) \). Then \( \hat{H}_f \leq v + \varepsilon u \in \overline{H}_f \) for every \( \varepsilon > 0 \). Hence \( \hat{H}_f \leq v \) q.e., and so \( \overline{H}_f \leq v \) everywhere on \( U \). By varying \( v \) it follows that \( \overline{H}_f \leq \hat{H}_g \) and so indeed by finely l.s.c. regularization \( \hat{H}_f \leq \overline{H}_g \).

Proposition 2.4. Let \( f \) be a function on \( \Delta(U) \) with values in \( \overline{H}_f \).

(a) \( \hat{H}_f \geq \overline{H}_f \) and hence \( \overline{H}_f \geq \hat{H}_f \) and \( \hat{H}_f \geq H_f \).

(b) \( \overline{H}_f \geq \hat{H}_f \) on \( \{ \overline{H}_f > -\infty \} \cup \{ \hat{H}_f < +\infty \} \).

(c) If \( f \) is lower bounded then \( \overline{H}_f(x) = \hat{H}_f(x) \) at any point \( x \in U \) at which \( \hat{H}_f(x) < +\infty \). If \( f \geq 0 \) on \( \Delta(U) \) then \( \overline{H}_f(0) \) is either identically \( +\infty \) or \( h \)-invariant on \( U \).

Proof. Clearly, \( \overline{U}_f \) is lower directed and \( \hat{U}_f \) is upper directed. The constant function \( +\infty \) belongs to \( \overline{U}_f \). If \( +\infty \) is the only function of class \( \overline{U}_f \) then obviously \( \hat{H}_f = +\infty \) and hence \( \overline{H}_f = +\infty \). In the remaining case it suffices to
consider finely $h$-superharmonic functions in the definition of $\hat{H}^h_f$ and hence of $\overline{H}^h_f$.

(a) Let $u \in \overline{U}^h_f$ and $v \in \underline{U}^h_f$. Then $u - v$ is well defined, finely $h$-hyperharmonic, and lower bounded on $U$, and

$$\liminf_{x \to y, x \in U} (u(x) - v(x)) \geq \liminf_{x \to y, x \in U} u(x) - \limsup_{x \to y, x \in U} v(x)$$

Hence

$$\geq f(Y) - f(Y) = 0$$

if $f(Y)$ is finite; otherwise $\liminf u(x) - \limsup v(x) = +\infty \geq 0$, for if for example $f(Y) = +\infty$ then $\liminf u = +\infty$ whereas $\limsup v < +\infty$ since $v$ is upper bounded. By the minimal-fine boundary minimum property given in [7, Corollary 3.13] together with [7, Proposition 3.5] applied to the finely superharmonic function $hu - hv$ (if $\neq +\infty$) it then follows that $u - v \geq 0$, and hence $u \geq v$. By varying $u$ and $v$ in either order we obtain $\hat{H}^h_f \geq \hat{H}^h_g$. Since $\hat{H}^h_f = \sup \underline{U}^h_f$ is finely l.s.c. it follows that $\overline{H}^h_f \geq \hat{H}^h_f$, and similarly $\overline{H}^h_g \geq \hat{H}^h_g$.

(b) Consider any point $x_0$ of the finely open set $V = \{\overline{H}^h_f > -\infty\}$. Since $\overline{H}^h_f$ is finely $h$-hyperharmonic and hence finely continuous on $V$ we obtain by

(a) $$\overline{H}^h_f(x_0) = \liminf_{x \to x_0, x \in V} \overline{H}^h_f(x) \geq \limsup_{x \to x_0, x \in V \setminus E} \overline{H}^h_f = \overline{H}^h_f(x_0).$$

The case $x_0 \in \{\overline{H}^h_f < +\infty\}$ is treated similarly, or by replacing $f$ with $-f$.

(c) The former assertion reduces easily to the case $f \geq 0$, whereby $\overline{U}^h_f$ consists of positive functions. We may assume that $\overline{H}^h_f \neq +\infty$, and hence $h\overline{U}^h_f \subset \mathcal{S}(U)$. Consider the cover of $U$ by the finely open sets $V_k$ from [7, Lemma 2.1 (c)]. Then $h\overline{U}^h_f$ is a Perron family in the sense of [7, Definition 2.2]. It therefore follows by [7, Theorem 2.3] that indeed $\overline{H}^h_f = \inf \overline{U}^h_f$ is $h$-invariant, and that $\overline{H}^h_f(x) = \hat{H}^h_f(x)$ at any point $x \in U$ at which $\hat{H}^h_f(x) < +\infty$. \qed

Proposition 2.5. Let $f, g$ be two functions on $\Delta(U)$ with values in $\mathbb{R}$.

1. If $f + g$ is well defined everywhere on $\Delta(U)$ then the inequality $\hat{H}^h_{f+g} \leq \hat{H}^h_f + \hat{H}^h_g$ holds at each point of $U$ where $\hat{H}^h_f + \hat{H}^h_g$ is well defined.

2. If $(f + g)(Y)$ is defined arbitrarily at points $Y$ of undetermination then the inequality $\overline{H}^h_{f+g} \leq \overline{H}^h_f + \overline{H}^h_g$ holds everywhere on $\{\overline{H}^h_f, \overline{H}^h_g > -\infty\}$.

3. For any point $x \in U$ we have $\hat{H}^h_f(x) < +\infty$ if and only if $\hat{H}^h_{f\vee 0}(x) < +\infty$.

4. Let $(f_j)$ be an increasing sequence of lower bounded functions $\Delta(U) \rightarrow [-\infty, +\infty]$. Writing $f = \sup_j f_j$ we have $\overline{H}^h_f = \sup_j \overline{H}^h_{f_j}$ and $\overline{H}^h_f = \sup_j \overline{H}^h_{f_j}$.

Proof. For 1., 2., and 4. we proceed much as in [7, 1.VIII.7, Proof of (c), (b), and (e)]. For Assertion 1., consider any two functions $u \in \overline{U}^h_f$ and $v \in \underline{U}^h_g$. Then $u + v \in \overline{U}^h_{f+g}$ and hence $\hat{H}^h_{f+g} \leq u + v$. By varying $v$ it follows that
\[ \hat{H}^h_{f+g} \leq u + \hat{H}^h_g \text{ on } \{ \hat{H}^h_g > -\infty \}. \]

By varying \( u \) this leads to \( \hat{H}^h_{f+g} \leq \hat{H}^h_f + \hat{H}^h_g \)

wherever the sum is well defined on \( \{ \hat{H}^h_g > -\infty \} \). By interchanging \( u \) and \( v \) we infer that \( \hat{H}^h_{f+g} \leq \hat{H}^h_f + \hat{H}^h_g \) altogether holds wherever the sum is well defined on \( \{ \hat{H}^h_f, \hat{H}^h_g > -\infty \} \). On the residual set \( \{ \hat{H}^h_f = -\infty \} \cup \{ \hat{H}^h_g = -\infty \} \) it is easily seen that \( \hat{H}^h_{f+g} = -\infty = \hat{H}^h_f + \hat{H}^h_g \) wherever the sum is well defined.

For Assertion 2., suppose first that \( f, g < +\infty \) (and so \( f + g \) is well defined).

In the proof of Assertion 1, we had \( \hat{H}^h_{f+g} \leq u + \hat{H}^h_g \) for \( \hat{H}^h_g > -\infty \), which is satisfied on \( \{ \hat{H}^h_g > -\infty \} \). It follows that \( \overline{H}^h_{f+g} \leq u + \overline{H}^h_g \) there, and hence that \( \overline{H}^h_{f+g} \leq \overline{H}^h_f + \overline{H}^h_g \) there (wherever well defined). In the general case define functions \( f_0 < +\infty \), resp. \( g_0 < +\infty \), which equal \( f \), resp. \( g \), except on the set \( \{ f = +\infty \} \), resp. \( \{ g = +\infty \} \). We may assume that these exceptional sets are \( h \)-harmonic measure null, for if e.g. \( \{ f = +\infty \} \) is not \( h \)-harmonic measure null then \( \overline{H}^h_f \equiv +\infty \) by Proposition 2.3 (d), in which case 1. becomes trivial.

It therefore follows in view of Proposition 2.3 (a), (e) that \( f + g = f_0 + g_0 \) off the \( h \)-harmonic measure null set \( \{ f = +\infty \} \cup \{ g = +\infty \} \) and hence by Proposition 2.3 (e) that

\[
\overline{H}^h_{f+g} = \overline{H}^h_{f_0+g_0} \leq \overline{H}^h_{f_0} + \overline{H}^h_{g_0} \leq \overline{H}^h_f + \overline{H}^h_g
\]

on the finely open set \( \{ \overline{H}^h_f > -\infty \} \cap \{ \overline{H}^h_g > -\infty \} \), the second inequality because \( f_0 \leq f \) and \( g_0 \leq g \).

For Assertion 3., let \( x \in U \) be given, and suppose that \( \hat{H}^h_f(x) < +\infty \). There is then \( u \in \overline{U}^h_f \) with \( u(x) < +\infty \), \( u \) being finely \( h \)-superharmonic \( \geq -c \) for some constant \( c \geq 0 \). It follows that \( u + c \in \overline{U}^h_{f+0} \) and hence \( \hat{H}^h_{f+0}(x) \leq u(x) + c < +\infty \). The converse follows from \( \hat{H}^h_f \leq \hat{H}^h_{f+0} \).

Assertion 4. reduces easily to the case of positive functions \( f_j \). Consider first the case of \( \overline{H} \). Then \( \overline{H}^h_f \) and each \( \overline{H}^h_{f_j} \) are positive and hence finely \( h \)-hyperharmonic by Proposition 2.4 (c). The inequality \( \overline{H}^h_f \geq \sup_j \overline{H}^h_{f_j} \) is obvious, and we may therefore assume that the positive finely \( h \)-hyperharmonic function \( \sup_j \overline{H}^h_{f_j} \) is not identically \( +\infty \), and therefore is \( h \)-invariant, again according to Proposition 2.4 (c). Denote \( E_j \) the polar subset \( \{ \overline{H}^h_{f_j} < \hat{H}^h_{f_j} \} \) of \( U \) and write \( E := \bigcup_j E_j \) (polar). For a fixed \( x \in U \setminus E \) and for given \( \varepsilon > 0 \) choose functions \( u_j \in \overline{U}^h_{f_j} \) so that

\[
(2.1) \quad u_j(x) < \hat{H}^h_{f_j}(x) + 2^{-j}\varepsilon = \overline{H}^h_{f_j}(x) + 2^{-j}\varepsilon.
\]

In particular, \( u_j \) is finely \( h \)-superharmonic. Define a finely \( h \)-hyperharmonic function \( u \) by

\[
(2.2) \quad u = \sup_j \overline{H}^h_{f_j} + \sum_j (u_j - \overline{H}^h_{f_j}) \geq \overline{H}^h_{f_k} + (u_k - \overline{H}^h_{f_k}) = u_k
\]
for any index \( k \). Then

\[
\liminf_{x \to Y, x \in U} u(x) \geq \liminf_{x \to Y, x \in U} u_k(x) \geq f_k(Y)
\]

for every \( Y \in \Delta(U) \) and every index \( k \). Thus \( u \in \overline{U}_f^h \) and \( \overline{H}_f^h \leq \hat{H}_f^h \leq u \). In particular, by the former equality (2.2) and by (2.1),

\[
(2.3) \quad \overline{H}_f^h(x) \leq u(x) \leq \sup_j \overline{H}_{f_j}^h(x) + \varepsilon,
\]

and hence the finely \( h \)-hyperharmonic function \( \overline{H}_f^h \) is finely \( h \)-superharmonic. Because \( \sup_j \overline{H}_f^h \) is \( h \)-invariant and majorized by \( \overline{H}_f^h \), the function \( \overline{H}_f^h - \sup_j \overline{H}_{f_j}^h \) is finely \( h \)-superharmonic \( \geq 0 \) on \( U \) by [6, Lemma 2.2], and \( \leq \varepsilon \) at \( x \). For \( \varepsilon \to 0 \) we obtain the remaining inequality \( \overline{H}_f^h \leq \sup_j \overline{H}_{f_j}^h \).

In the remaining case of \( \hat{H} \) we have \( \sup_j \hat{H}_{f_j}^h(x) \leq \hat{H}_f^h \). For any point \( x \in U \) at which \( \sup_j \hat{H}_{f_j}^h(x) < \hat{H}_f^h \) and for any \( \varepsilon > 0 \) choose functions \( u_j \in \overline{U}_{f_j}^h \) so that

\[
(2.4) \quad u_j(x) < \hat{H}_{f_j}^h(x) + 2^{-j}\varepsilon.
\]

Proceed as in the above case of \( \overline{H} \) by defining the finely \( h \)-hyperharmonic function by (2.2), replacing throughout \( \hat{H} \) by \( \hat{H} \). Corresponding to (2.3) we now end by

\[
\hat{H}_f^h(x) \leq u(x) \leq \sup_j \hat{H}_{f_j}^h(x) + \varepsilon,
\]

from which the remaining inequality \( \hat{H}_f^h(x) \leq \sup_j \hat{H}_{f_j}^h(x) \) follows for \( \varepsilon \to 0 \). \( \square \)

3. \( h \)-RESOLUTIVE AND \( h \)-QUASIRESolutive functions

**Definition 3.1.** A function \( f \) on \( \Delta(U) \) with values in \( \overline{\mathbb{R}} \) is said to be \( h \)-resolutive if \( \overline{H}_f^h = H_f^h \) on \( U \) and if this function, also denoted by \( H_f^h \), is neither identically \( +\infty \) nor identically \( -\infty \).

It follows that \( H_f^h \) is finely \( h \)-harmonic on the finely open set \( \{ \overline{H}_f^h > -\infty \} \cap \{ \overline{H}_f^h > +\infty \} = \{-\infty < H_f^h < +\infty \} \).

For any function \( f : \Delta(U) \to \overline{\mathbb{R}} \) we consider the following two subsets of \( U \):

\[
E_f^h = \{ \overline{H}_f^h = -\infty \} \cup \{ \overline{H}_f^h = +\infty \} \cup \{ \overline{H}_f^h \neq H_f^h \},
\]

\[
P_f^h = \{ \hat{H}_f^h > \overline{H}_f^h > -\infty \} \cup \{ H_f^h < \overline{H}_f^h < +\infty \},
\]

of which \( P_f^h \) is polar.
Definition 3.2. A function $f$ on $\Delta(U)$ with values in $\overline{\mathbb{R}}$ is said to be $h$-quasi-stable if $E_f^h$ is polar, or equivalently if the relations $\overline{H_f^h} > -\infty, H_f^h < +\infty$, and $\overline{H_f^h} = H_f^h$ hold quasi-everywhere on $U$.

When $f$ is $h$-quasi-resolutive on $U$ the functions $\overline{H_f^h}$ and $H_f^h$ are finely $h$-hyperharmonic and finely $h$-hypoharmonic, respectively, off the polar set $\{\overline{H_f^h} = -\infty\} \cup \{H_f^h = +\infty\}$, and they are actually equal and hence finely $h$-harmonic off the smaller polar set $E_f^h$. We then denote by $H_f^h$ the common restriction of $\overline{H_f^h}$ and $H_f^h$ to $U \setminus E_f^h$. Since $\overline{H_f^h}$ is finely $h$-hyperharmonic on $U \setminus E_f^h$, and $H_f^h$ is finely $h$-hypoharmonic there, it follows by Proposition 2.4 (a) that the equalities

$$H_f^h = H_f^h = \overline{H_f^h}$$

hold q.e. on $U \setminus E_f^h$ and hence q.e. on $U$.

Lemma 3.3. Every $h$-resolutive function $f$ is $h$-quasi-stable.

Proof. The sets $E_+ := \{H_f^h = +\infty\}$ and $E_- := \{H_f^h = -\infty\}$ are finely closed and disjoint. For any fine component $V$ of $U \setminus E_-$ such that $V \cap E_+$ is nonpolar we have $V \cap E_+ = V$, that is, $V \subseteq E_+$, because $\overline{H_f^h}$ is finely $h$-hyperharmonic on $V$. Denote by $W$ the union of these fine components $V$, and by $W'$ the (countable) union of the remaining fine components $V'$ of $U \setminus E_-$. Then $W \subseteq E_+$ whereas the set $P := W' \cap E_+$ is polar along with each $V' \cap E_+$. Since $E_+ \cap E_- = \emptyset$ we obtain

$$E_+ = (U \setminus E_-) \cap E_+ = (W \cup W') \cap E_+ = (W \cap E_+) \cup (W' \cap E_+) = W \cup P,$$

$\cup$ denoting disjoint union. Now, $(U \setminus P) \cap E_+ = E_+ \setminus P$ is finely closed relatively to the nonvoid fine domain $U \setminus P$ (cf. [8, Theorem 12.2]), but also finely open, being equal to $W$ as seen from the above display. Thus either $W = U \setminus P$ or $W = \emptyset$. But $W = U \setminus P$ would imply $E_+ = U$, contradicting $\overline{H_f^h} \neq +\infty$, and so actually $E_+ = P$ (polar). Similarly (or by replacing $f$ with $-f$) it is shown that $E_-$ is polar, and so $f$ is $h$-quasi-stable because $\overline{H_f^h} = H_f^h$ even holds everywhere on $U$.

In view of Lemma 3.3 an $h$-quasi-stable function $f$ is $h$-resolutive if and only if $E_f^h = \emptyset$ (any polar subset of $\Delta(U)$ being a proper subset). This implies that 1. and 2. in the following proposition remain valid with ‘$h$-quasi-stable’ replaced throughout by ‘$h$-resolutive’. It will be shown in Corollary 3.12 that $h$-resolutivity and $h$-quasi-resolutivity are actually identical concepts.

Proposition 3.4. Let $f, g : \Delta(U) \rightarrow \overline{\mathbb{R}}$ be $h$-quasi-stable. Then

1. For $\alpha \in \mathbb{R}$ we have $E_{\alpha f}^h \subseteq E_f^h$ and hence $\alpha f$ is $h$-quasi-stable. Furthermore, $H_{\alpha f}^h = \alpha H_f^h$ on $U \setminus E_f^h$. 


2. If \( f + g \) is defined arbitrarily at points of undetermination then \( E_{f+g}^h \subset E_f^h \cup E_g^h \) and hence \( f + g \) is \( h \)-quasiresolutive. Furthermore, \( H_{f+g}^h = H_f^h + H_g^h \) on \( U \setminus (E_f^h \cup E_g^h) \).

3. \( E_{f \vee g}^h, E_{f \wedge g}^h \subset E_f^h \cup E_g^h \cup P_f^h \) and hence \( f \vee g \) and \( f \wedge g \) are \( h \)-quasiresolutive. If for example \( H_{f \vee g}^h \geq 0 \) then \( H_{f \vee g}^h = (1/h) \hat{R}_{(H_f^h \vee H_g^h)}^h \) on \( U \setminus (E_f^h \cup E_g^h \cup P_f^h) \).

Proof. For Assertion 1., consider separately the cases \( \alpha > 0, \alpha < 0 \), and \( \alpha = 0 \). For 2. and 3. we proceed as in [4, 1.VIII.7 (d)]. For 2. we have

\[
H_f^h + H_g^h \geq H_{f+g}^h \geq H_f^h + H_g^h
\]

on \( U \setminus (E_f^h \cup E_g^h) \), the first inequality by 2. in Proposition 2.5, the third inequality by replacing \( f \) with \(-f \) in the first inequality, and the second inequality holds by Proposition 2.3 (b) on

\[
\{ \hat{H}_{f+g}^h > -\infty \} \supset \{ \hat{H}_f^h > -\infty \} \cap \{ \hat{H}_g^h > -\infty \} \supset U \setminus (E_f^h \cup E_g^h).
\]

Thus equality prevails on \( U \setminus (E_f^h \cup E_g^h) \) (and hence q.e. on \( U \)) in both of these inclusion relations. It follows that

\[
\{ \hat{H}_{f+g}^h = -\infty \} \subset \{ \hat{H}_f^h = -\infty \} \cup \{ \hat{H}_g^h = -\infty \} \subset E_f^h \cup E_g^h.
\]

and similarly \( \{ \hat{H}_{f+g}^h = -\infty \} \subset E_f^h \cup E_g^h \). Finally, by (3.1) with equality throughout,

\[
\{ \hat{H}_{f+g}^h \neq H_{f+g}^h \} \subset \{ \hat{H}_f^h \neq H_f^h \} \cup \{ \hat{H}_g^h \neq H_g^h \} \subset E_f^h \cup E_g^h.
\]

 Altogether, \( E_{f+g}^h \subset E_f^h \cup E_g^h \), and so \( f + g \) is indeed \( h \)-quasiresolutive along with \( f \) and \( g \).

For the notation in the stated equation in 3., see [3, Definition 11.4]. Since \( f \wedge g = -[(f) \vee (-g)] \) and \( f \vee g = [(f - g) + 0] + g \) it follows by 1. and 2. that 3. reduces to \( E_{f \vee 0}^h \subset E_f^h \cup P_f^h \), which implies the \( h \)-quasiresolutivity of \( f^+ = f \vee 0 \) and the stated expression for \( H_{f \vee 0}^h \) with \( g = 0 \). For given \( x \in U \setminus (E_f^h \cup P_f^h) \) and integers \( j > 0 \) choose \( u_j \in \hat{U}_f^h \) with \( u_j(x) = \hat{H}_f^h(x) + 2^{-j} = \hat{H}_f^h(x) + 2^{-j} \).

The series \( \sum_{j=0}^{\infty} (u_j - \hat{H}_f^h) \) of positive finely \( h \)-superharmonic functions on \( U \setminus (E_f^h \cup P_f^h) \) (\( u_j \) being likewise restricted to \( U \setminus (E_f^h \cup P_f^h) \)) has a positive finely \( h \)-superharmonic sum, finite at \( x \). Recall that \( H_f^h \) is defined and finely \( h \)-harmonic on \( U \setminus E_f^h \) and in particular on \( U \setminus (E_f^h \cup P_f^h) \). Consequently, \( H_f^h \) is finely \( h \)-subharmonic (and positive) on \( U \setminus (E_f^h \cup P_f^h) \) and majorized there by \( \hat{H}_f^h \), which is finite valued on \( U \setminus (E_f^h \cup P_f^h) \) by 3. in Proposition 2.5 because \( \hat{H}_f^h < +\infty \) on \( U \setminus (E_f^h \cup P_f^h) \) and because \( \hat{H}_f^h = H_f^h \) there. It follows by [3, Theorem 11.13], applied with \( f \) replaced by \( h \hat{H}_f^h \) or \( 0 \) on \( U \), which is finely subharmonic on \( U \setminus (E_f^h \cup P_f^h) \), that \( h \hat{R}_h^h \) (sweeping relative to \( U \)) is finely
The positive function
\[
\frac{1}{h} \widehat{R}_h h f_{\triangledown 0} + \sum_{j=k}^{\infty} (u_j - H^f_j)
\]
restricted to \( U \setminus (E^h_f \cup P^h_f) \) is therefore finely \( h \)-superharmonic. Moreover, this positive finely \( h \)-superharmonic function on \( U \setminus (E^h_f \cup P^h_f) \) majorizes \( u_k \in U^h_f \) there (being \( \geq \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} + (u_k - H^f_j) \geq u_k \) there), and this majorization remains in force after extension by fine continuity to \( U \), cf. [8, Theorem 9.14]. Thus the extended positive function (3.2) belongs to \( \overline{H}_{f_{\triangledown 0}}^h \). For \( k \to \infty \) it follows that \( \overline{H}_{f_{\triangledown 0}}^h \leq \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} \) on \( U \). On the other hand, \( H_{f_{\triangledown 0}}^h \) majorizes both \( H_{f_{\triangledown 0}}^h \) and 0, so \( H_{f_{\triangledown 0}}^h \geq \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} = \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} \) on \( U \), the equality because \( H_{f_{\triangledown 0}}^h = H_{f_{\triangledown 0}}^h \) on \( U \setminus (E^h_f \cup P^h_f) \) and hence q.e. on \( U \). It follows that
\[
\overline{H}_{f_{\triangledown 0}}^h \leq \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} = \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} \leq H_{f_{\triangledown 0}}^h \leq \overline{H}_{f_{\triangledown 0}}^h < +\infty
\]
because \( h \overline{H}_{f_{\triangledown 0}}^h \triangledown 0 = h H_{f_{\triangledown 0}}^h \triangledown 0 \) on \( U \setminus (E^h_f \cup P^h_f) \) and hence q.e. on \( U \). (The last inequality in the above display follows by Proposition 2.4 (b) because \( f \triangledown 0 > -\infty \).) Since \( \overline{H}_{f_{\triangledown 0}}^h \triangledown 0 \geq 0 \geq -\infty \) we conclude that \( f \triangledown 0 \) indeed is \( h \)-resolutive, resp. \( h \)-quasiresolutive, and that \( E^h_{f_{\triangledown 0}} \subset E^h_{f_{\triangledown 0}} \cup P^h_{f_{\triangledown 0}} \) and \( H_{f_{\triangledown 0}}^h = \frac{1}{h} \widehat{R}_h h f_{\triangledown 0} \) on \( E^h_{f_{\triangledown 0}} \cup P^h_{f_{\triangledown 0}} \). \( \square \)

A version of Proposition 3.4 for \( h \)-resolutive functions instead of \( h \)-quasiresolutive functions will of course follow when the identity of \( h \)-resolutivity and \( h \)-quasiresolutivity has been established in Corollary 3.12. Before that, we do however need the following step in that direction, based on Proposition 2.5.

**Lemma 3.5.** Let \( f \) be an \( h \)-quasiresolutive function on \( \Delta(U) \). If \( f^+ \) and \( f^- \) are \( h \)-resolutive then so is \( f \), and the function \( H^f_j = H^f_j = H^f_{j_0} \) on \( U \) is finite valued.

**Proof.** According to 3. in Proposition 3.4, \( f^+ \) and \( f^- \) are \( h \)-quasiresolutive (besides being \( h \)-resolutive), and the functions \( H^f_{j_0} := H^f_{j_0} = H^f_{j_0} \) (defined on \( \{H^f_{j_0} > -\infty\} = U \) since \( f^+ \geq 0 \)) and similarly \( H^f_{j_0} := H^f_{j_0} = H^f_{j_0} \) are therefore finite valued. Since \( -f^- \leq f \leq f^+ \) it follows that \( -\infty < -H^f_{j_0} \leq H^f_{j_0} \leq H_{j_0} < +\infty \). Applying 2. in Proposition 2.5 to the sums \( f = f^+ + (-f)^+ = f^+ - f^- \) and \( -f = f^- - f^+ \), which are well defined on \( \Delta(U) \), we obtain
\[
\overline{H}_{j_0}^h \leq H^f_{j_0} - H^f_{j_0} \leq H^f_{j_0}
\]
on all of $U$, and hence $\overline{H}_f^h = H_f^h$ there because $\overline{H}_f^h \geq H_f^h$ on all of $U$, again by Proposition 2.4 (b) since we have seen that for example $\overline{H}_f^h > -\infty$. □

**Corollary 3.6.** Let $(f_j)$ be an increasing sequence of lower bounded $h$-resolutive, resp. $h$-quasiresolutive functions $\Delta(U) \rightarrow ]-\infty, +\infty]$, and let $f = \sup_j f_j$. If $\overline{H}_f^h \neq +\infty$ then $f$ is $h$-resolutive, resp. $h$-quasiresolutive.

**Proof.** By adding a constant to $f$ we reduce the claim to the case $f_j \geq 0$. For every $j$ we have $H_f^h \geq H_{f_j}^h = \overline{H}_{f_j}^h$, by Proposition 2.4 (b) because $\overline{H}_{f_j}^h > -\infty$. Hence $H_f^h \geq \sup_j \overline{H}_{f_j}^h = \overline{H}_f^h$ according to 4. in Proposition 2.4. By definition of $H_f^h$ we have at any point $x_0 \in U$

$$H_f^h(x_0) = \text{fine lim sup}_{x \to x_0, x \in U} H_f^h(x) \leq \text{fine lim sup}_{x \to x_0, x \in U} \overline{H}_f^h(x) = \overline{H}_f^h(x_0)$$

according to Proposition 2.4 (a), $\overline{H}_f^h$ being finely $h$-hyperharmonic by Proposition 2.4 (c). By Proposition 2.4 (b) we have $H_f^h \leq \overline{H}_f^h$ on $\{\overline{H}_f^h > -\infty\} = U$, and we conclude that $\overline{H}_f^h = H_f^h$. By hypothesis this finely hyperharmonic function on $U$ is finite q.e. on $\tilde{U}$, and in particular this positive function is not identically $+\infty$. Consequently, $f$ is indeed $h$-resolutive, resp. $h$-quasiresolutive. □

Recall that $\mu_h$ denotes the unique measure on $\overline{U}$ carried by $\Delta_1(U)$ and representing $h$, that is, $h = K \mu_h = \int K(.,Y) d\mu_h(Y)$.

**Proposition 3.7.** For any $\mu_h$-measurable subset $A$ of $\Delta(U)$ the indicator function $1_A$ is $h$-resolutive, and

$$H_{1_A}^h = \frac{1}{h} \int_A K(.,Y) d\mu_h(Y) = \frac{1}{h} \tilde{R}_A^h$$

on $U$. In particular, the constant function 1 on $\Delta(U)$ is $h$-resolutive and $H_1^h = 1$.

**Proof.** Because $h = K \mu_h$ and because $\mu_h$ is carried by $\Delta_1(U)$ we have by [7] Theorem 3.10 and Proposition 3.9

$$\tilde{R}_A^h = R_{K \mu_h}^A = \int_{\Delta_1(U)} \tilde{R}_{K(.,Y)}^A d\mu_h(Y) = \int K(.,Y) 1_A(Y) d\mu_h(Y).$$

Consider any finely $h$-hyperharmonic function $u = v/h \geq 0$ on $U$ such that $u \geq 1$ on some open set $W \subset \overline{U}$ with $W \supset A$. Then $u \in \overline{U}_{1_A}$ and hence $u \geq H_{1_A}^h \geq \overline{U}_{1_A}$. By varying $W$ it follows by [7] Definition 2.4 that $\frac{1}{h} \tilde{R}_A^h \geq H_{1_A}^h$. We have

$$\frac{1}{h} \int K(.,Y) 1_A(Y) d\mu_h(Y) = \frac{1}{h} \tilde{R}_A^h \geq \overline{U}_{1_A}.$$
Applying this inequality to the $\mu_h$-measurable set $\Delta(U) \setminus A$ in place of $A$ we obtain
\[
(3.5) \quad \frac{1}{h} \int K(.,Y)1_{\Delta(U) \setminus A}(Y)d\mu_h(Y) \geq \overline{\mathcal{T}}^h_{1_{\Delta(U) \setminus A}}.
\]
By adding the left hand, resp. right hand, members of (3.4) and (3.5) this leads by 2. in Proposition 2.5 to
\[
(3.6) \quad 1 = \frac{1}{h} \int K(.,Y)d\mu_h(Y) \geq \overline{\mathcal{T}}^h_{1_A} + \overline{\mathcal{T}}^h_{1_{\Delta(U) \setminus A}} \geq \overline{\mathcal{T}}^h = 1.
\]
Thus equalities prevail throughout in (3.4), (3.5), and (3.6). It follows altogether that
\[
\mathcal{H}^h_{1_A} = -\overline{\mathcal{T}}^h_{-1_A} = 1 - \overline{\mathcal{T}}^h_{1_{-1_A}} = 1 - \overline{\mathcal{T}}^h_{1_{\Delta(U) \setminus A}}
\]
\[
= \overline{\mathcal{T}}^h_{1_A} = \frac{1}{h} \tilde{\mathcal{R}}^h_A = \frac{1}{h} \int_A K(.,Y)d\mu_h(Y),
\]
so that indeed $1_A$ is $h$-resolutive and (3.3) holds.$\square$

For any function $f : \Delta(U) \rightarrow \mathbb{R}$ we define $f(Y)K(x,Y) = 0$ at points $(x,Y)$ where $f(Y) = 0$ and $K(x,Y) = +\infty$. If $f$ is $\mu_h$-measurable then so is $Y \mapsto f(Y)K(x,Y)$ for each $x \in U$ because $K(x,Y) > 0$ is $\mu_h$-measurable (even l.s.c.) as a function of $Y \in \Delta(U)$ according to [6, Proposition 2.2 (i)].

**Proposition 3.8.** Let $f$ be a $\mu_h$-measurable lower bounded function on $\Delta(U)$. Then
\[
\overline{\mathcal{T}}^h_{f_j} = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) > -\infty,
\]
and $\overline{\mathcal{T}}^h_{f_j}$ is either identically $+\infty$ or the sum of an $h$-invariant function and a constant $\leq 0$. Furthermore $f$ is $h$-quasiresolutive if and only if $f$ is $h$-resolutive, and that holds if and only if $\frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) < +\infty$ q.e. on $U$, or equivalently: everywhere on $U$. In particular, every bounded $\mu_h$-measurable function $f : \Delta(U) \rightarrow \mathbb{R}$ is $h$-resolutive.

**Proof.** Consider first the case of a positive $\mu_h$-measurable function $f$. Then $f$ is the pointwise supremum of an increasing sequence of positive $\mu_h$-measurable step functions $f_j$ (that is, finite valued functions $f_j$ taking only finitely many values, each finite and each on some $\mu_h$-measurable set; in other words: affine combinations of indicator functions of $\mu_h$-measurable sets). For any index $j$ it follows by Proposition 3.7 and by 1. and 2. in Proposition 3.4 (the latter extended to finite sums and with ‘$h$-resolutive’ throughout in place of ‘$h$-quasi-resolutive’, cf. the paragraph preceding Proposition 3.4) that each $f_j$ is $h$-resolutive and that
\[
H^h_{f_j} = \frac{1}{h} \int f_j(Y)K(.,Y)d\mu_h(Y)
\]
on \( U \), whence
\[
0 \leq \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) = \frac{1}{h} \sup \int f_j(Y)K(.,Y)d\mu_h(Y) = \sup H^h_{f_j} = \overline{H}^h_f
\]
by Proposition 2.5. For a general lower bounded \( \mu_h \)-measurable function \( f \) on \( \Delta(U) \) there is a constant \( c \geq 0 \) such that \( g := f + c \geq 0 \) and hence \( H^h_g = H^h_f + c \geq 0 \). It follows that
\[
\overline{H}^h_f = \frac{1}{h} \int g(Y)K(.,Y)d\mu_h(Y) - c = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) > -\infty
\]
and hence by Proposition 2.4 (c) applied to \( g \) that \( \overline{H}^h_f = \overline{H}^h_g - c \) has the asserted form.

Next, consider a bounded \( \mu_h \)-measurable function \( f \) on \( \Delta(U) \). As just shown, we have
\[
\overline{H}^h_f = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y)
\]
and the same with \( f \) replaced by \(-f\), whence \( \overline{H}^h_f = \overline{H}^h_{-f} \), finite valued because \( f \) is bounded. Thus \( f \) is \( h \)-resolutive. Let \( c \geq 0 \) be a constant such that \( |f| \leq c \). Then \( \overline{H}^h_f = c - \overline{H}^h_{-f} \) which is finely \( h \)-harmonic because \( c - f \geq 0 \) and so \( H^h_{c - f} \) is \( h \)-invariant by Proposition 2.4 (c) and hence finely \( h \)-harmonic, being finely valued.

Returning to a general lower bounded \( \mu_h \)-measurable function \( f \), suppose first that \( f \) is \( h \)-quasiresolutive. Then, as shown in the first paragraph of the proof, \( \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) = \overline{H}^h_f \) is finite q.e. on \( U \). Conversely, if \( \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) < +\infty \) q.e., that is \( \overline{H}^h_f \neq +\infty \), then Corollary 3.6 applies to the increasing sequence of bounded \( \mu_h \)-measurable and hence \( h \)-resolutive functions \( f \land j \) converging to \( f \), and we conclude that \( f \) is \( h \)-resolutive (in particular \( h \)-quasiresolutive) and hence that \( \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) = \overline{H}^h_f \) is finite everywhere on \( U \).

**Corollary 3.9.** Let \( f : \Delta(U) \rightarrow \mathbb{R} \) be \( \mu_h \)-measurable. Then \( f \) is \( h \)-resolutive if and only if \( |f| \) is \( h \)-resolutive.

**Proof.** If \( f \) is \( h \)-resolutive, and therefore \( h \)-quasiresolutive by Lemma 3.3 then \( |f| = f \lor (-f) \) is \( h \)-quasiresolutive according to 3. and 1. in Proposition 3.3. Since \( |f| \) is lower bounded (and \( \mu_h \)-measurable) then by Proposition 3.8 \( |f| \) is even \( h \)-resolutive and \( |f|K(x,.) \) is \( \mu_h \)-integrable for every \( x \in U \). So are therefore \( f^+K(x,.) \) and \( f^-K(x,.) \), and it follows, again by Proposition 3.8 that \( f^+ \) and \( f^- \) are \( h \)-resolutive. So is therefore \( f = f^+ - f^- \) by Lemma 3.5. \( \square \)
Proposition 3.10. Every $h$-quasiresolutive function $f : \Delta(U) \rightarrow \mathbb{R}$ is $\mu_h$-measurable.

Proof. We begin by proving this for $f = 1_A$, the indicator function of a subset $A$ of $\Delta(U)$, cf. [4, p. 113]. Clearly, $H_f^h$ and $H_f^j(x)$ have their values in $[0, 1]$, and hence $\overline{H}_f^h = H_f^j$ and $\overline{H}_f^j = H_f^j$ according to Proposition 2.4 (c). Since $\overline{U}_f^j$ is lower directed there is a decreasing sequence of functions $u_j \in \overline{U}_f^j$ such that $\overline{U}_f^j(x_0) = \inf_j u_j(x_0)$. Replacing $u_j$ by $u_j \wedge 1 \in \overline{U}_f^j$ we arrange that $u_j \leq 1$. Denote by $g_j$ the function defined on $\overline{U}$ by

$$g_j(Y) = \lim\inf_{z \rightarrow Y, z \in U} u_j(z)$$

for any $Y \in \overline{U}$. Clearly, $g_j$ is l.s.c. on $\overline{U}$ and $1_A \leq g_j \leq 1$ on $\Delta(U)$. Write $f_2 = \inf_j g_j$ (restricted to $\Delta(U)$). Then $f_2$ is Borel measurable and $1 \geq f_2 \geq f = 1_A$, whence $\overline{H}_f^{f_2} \geq \overline{H}_f^j$. For the opposite inequality note that $u_j \in \overline{U}_f^{f_2}$ because $g_j \geq f_2$. Hence $\overline{H}_f^j = \inf_j u_j \geq \overline{H}_f^{f_2}$ with equality at $x_0$. Furthermore, $\overline{H}_f^j$ is invariant according to Proposition 2.4 (c), and hence $\overline{H}_f^{f_2} - \overline{H}_f^j$ is positive and finely $h$-superharmonic on $U$. Being 0 at $x_0$ it is identically 0, and so $\overline{H}_f^{f_2} = \overline{H}_f^j$. Clearly, $0 \leq f_1 \leq f_2 \leq 1$. Since $f$ is $h$-quasiresolutive we obtain from Proposition 2.4 (a) q.e. on $U$

$$H_f^j = H_f^{f_1} \leq \overline{H}_f^{f_1} \leq H_f^j \leq H_f^{f_2} \leq \overline{H}_f^{f_2} = H_f^j,$$

thus with equality q.e. all through. Hence $f_1$ and $f_2$ are $h$-quasiresolutive, and so is therefore $f_2 - f_1$ by 1. and 2. in Proposition 3.4, which also shows that $H_f^{f_2 - f_1} = H_{f_2} - H_{f_1} = 0$ q.e. Because $f_2 - f_1$ is positive and Borel measurable on $\Delta(U)$ it follows by Proposition 3.5 that $\frac{1}{h} \int (f_2(Y) - f_1(Y)) K(., Y) d\mu_h(Y) = 0$, and hence $f_1 = f_2$ is $\mu_h$-a.e. It follows that $f = 1_A$ is $\mu_h$-measurable, and so is therefore $A$.

Next we treat the case of a finite valued $h$-quasiresolutive function $f$ on $\Delta(U)$. Adapting the proof given in [4, p. 115] in the classical setting we consider the space $C(\overline{\mathbb{R}}, \mathbb{R})$ of continuous (hence bounded) functions $\mathbb{R} \rightarrow \mathbb{R}$, and denote by $\Phi$ the space of functions $\varphi \in C(\overline{\mathbb{R}}, \mathbb{R})$ such that $\varphi \circ f$ is $h$-quasiresolutive. In view of Proposition 3.4 $\Phi$ is a vector lattice, closed under uniform convergence because $|\varphi_j - \varphi| < \varepsilon$ implies $|H_{\varphi_j \circ f}^h - \overline{H}_{\varphi \circ f}^h| \leq \varepsilon$ and $|H_{\varphi \circ f}^h - \overline{H}_{\varphi \circ f}^h| \leq \varepsilon$ on $U \setminus E_f^h$, and so $|\overline{H}_{\varphi \circ f}^h - \overline{H}_{\varphi \circ f}^h| \leq 2\varepsilon$ on $U \setminus E_f^h$. We infer that $\overline{H}_{\varphi \circ f}^h = H_{\varphi \circ f}^h$ (finite values) q.e. on $U$, and so $\varphi \circ f$ is indeed resolutive. Furthermore, $\Phi$ includes the functions $\varphi_n : t \mapsto (1 - |t - n|) \vee 0$ on $\mathbb{R}$ for integers $n \geq 1$, again by Proposition 3.4. These functions separate points of $\mathbb{R}$. In fact, for distinct $s, t \in \mathbb{R}$, say $s < t$, take $n = [s]$ (that is, $n \leq s < n + 1$). If also $n \leq t < n + 1$ then clearly $\varphi_n(t) < \varphi_n(s) \leq 1$, and in the remaining case
\[ t \geq n + 1 \text{ we have } \varphi_n(t) = 0 < \varphi_n(s). \] It therefore follows by the lattice version of the Stone-Weierstrass theorem that \( \Phi = \mathcal{C}(\mathbb{R}, \mathbb{R}) \). Next, the class \( \Psi \) of (not necessarily continuous) functions \( \psi : \mathbb{R} \to \mathbb{R} \) for which \( \psi \circ f \) is \( h \)-quasi-resolutive is closed under bounded monotone convergence, by Corollary 3.6 (adapted to a bounded monotone convergent sequence of functions \( f_j \)). Along with the continuous functions \( \mathbb{R} \to \mathbb{R} \), \( \Psi \) therefore includes every bounded Borel measurable function \( \mathbb{R} \to \mathbb{R} \). In particular, the indicator function \( 1_J \) of an interval \( J \subset \mathbb{R} \) belongs to \( \Psi \), and hence \( 1_J \circ f \) is \( h \)-quasi-resolutive. We conclude by the first part of the proof that the \( h \)-quasi-resolutive indicator function \( 1_J \circ f = 1_{f^{-1}(J)} \) is \( \mu_h \)-measurable.

Finally, for an arbitrary \( h \)-quasi-resolutive function \( f : \Delta(U) \to \mathbb{R} \), write \( A_+ := \{ f = +\infty \} \) and \( A_- := \{ f = -\infty \} \). By 3. in Proposition 3.4, \( f \vee 0 \) is \( h \)-quasi-resolutive and
\[
\overline{\mathcal{H}}^h_{(+)1_{A_+}} = \overline{\mathcal{H}}^h_{f1_{A_+}} \leq \overline{\mathcal{H}}^h_{f \vee 0} = \hat{\mathcal{H}}^h_{f \vee 0} < +\infty
\]
on \( U \setminus (E_0^h \cup P^h) \), cf. the opening of the present section. It follows that \( \overline{\mathcal{H}}_{1_{A_+}} = 0 \) q.e. on \( U \) and hence \( \overline{\mathcal{H}}^h_{(+)1_{A_+}} = 0 \) q.e. according to 3. in Proposition 2.5. Since \( -f \) likewise is \( h \)-quasi-resolutive we have \( \overline{\mathcal{H}}^h_{(+)1_{A_-}} = 0 \) q.e. Furthermore, \( 0 \leq \overline{\mathcal{H}}_{(+)1_{A_-}} \leq \overline{\mathcal{H}}^h_{(+)1_{A_-}} = 0 \) q.e. by Proposition 2.4 (a), and similarly \( \overline{\mathcal{H}}_{(+)1_{A_-}} = 0 \) q.e. Writing \( A = A_+ \cup A_- = \{ |f| = +\infty \} \) we infer from 2. in Proposition 3.4 that \( 1_A = 1_{A_+} + 1_{A_-} \) is \( h \)-quasi-resolutive, and hence so is \( (+\infty)1_A \) in view of Corollary 3.6. As shown in the first paragraph of the proof it follows that \( A \) is \( \mu_h \)-measurable. Define \( g : \Delta(U) \to \mathbb{R} \) by \( g = f \) except that \( g = 0 \) on \( A = \{|f| = +\infty \} \). Then \( g \leq f + (+\infty)1_A \) and \( f \leq g + (+\infty)1_A \), and hence by Proposition 2.5 (recall that \( \overline{\mathcal{H}}_{1_{A_+}} = 0 \) q.e. on \( U \))
\[
\overline{\mathcal{H}}^h_g \leq H^h_f + \overline{\mathcal{H}}^h_{(+\infty)1_A} = H^h_f \leq \overline{\mathcal{H}}^h_g + H^h_{(+\infty)1_A} = H^h_g
\]
quasieverywhere, actually with equalities q.e., and hence \( g \) is \( h \)-quasi-resolutive along with \( f \). As established in the beginning of the proof, \( g \) is \( \mu_h \)-measurable and so is \( f \). In fact, \( \{ f \neq g \} = A \) is \( \mu_h \)-measurable with \( \mu_h(A) = 0 \) by the following corollary.

\[ \square \]

**Corollary 3.11.** (Cf. [4, p. 114].) A set \( A \subset \Delta(U) \) is \( h \)-harmonic null if and only if \( A \) is \( \mu_h \)-measurable with \( \mu_h(A) = 0 \).

**Proof.** If \( A \) is \( \mu_h \)-measurable with \( \mu_h(A) = 0 \) then \( \overline{\mathcal{H}}^h_{1_A} = \frac{1}{h} \int 1_A K(.,Y) d\mu_h(Y) = 0 \) according to Proposition 3.8. Conversely, if \( \overline{\mathcal{H}}^h_{1_A} = 0 \) then \( 0 \leq \overline{\mathcal{H}}^h_{1_A} \leq \overline{\mathcal{H}}^h_{1_A} = 0 \) and so \( 1_A \) is \( h \)-resolutive. It follows by Proposition 3.10 that \( 1_A \) is \( \mu_h \)-measurable, and again by Proposition 3.8 that \( \frac{1}{h} \int 1_A K(.,Y) d\mu_h(Y) = \overline{\mathcal{H}}^h_{1_A} = 0 \). Since \( K(x,.) > 0 \) is l.s.c. by [6, Proposition 3.2] we conclude that \( \mu_h(A) = 0 \). \[ \square \]
Corollary 3.12. A function \( f \) on \( \Delta(U) \) with values in \( \mathbb{R} \) is \( h \)-resolutive if and only if \( f \) is \( h \)-quasiresolutive.

Proof. The ‘only if’ part is contained in Lemma 3.3. For the ‘if’ part, suppose that \( f \) is \( h \)-quasiresolutive and hence \( \mu_h \)-measurable, by Proposition 3.10. If \( f \geq 0 \) then \( f \) is \( h \)-resolutive according to Proposition 3.8. For arbitrary \( f \):
\[
\Delta(U) \rightarrow \mathbb{R}
\]
this applies to \( f^+ \) and \( f^- \), which are \( h \)-quasiresolutive according to 3. in Proposition 3.4. Consequently, \( f = f^+ - f^- \) is likewise finely \( h \)-resolutive by 1. and 2. in Proposition 3.4. \( \Box \)

Theorem 3.13. A function \( f : \Delta(U) \rightarrow \mathbb{R} \) is \( h \)-resolutive if and only if the function \( Y \mapsto f(Y)K(x,Y) \) on \( \Delta(U) \) is \( \mu_h \)-integrable for quasievery \( x \in U \).

In the affirmative case \( Y \mapsto f(Y)K(x,Y) \) is \( \mu_h \)-integrable for every \( x \in U \), and we have everywhere on \( U \)
\[
H^h_f = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y).
\]

Proof. Suppose first that \( f \) is \( h \)-resolutive. By Proposition 3.10 \( f \) is then \( \mu_h \)-measurable. According to 3. in Proposition 3.4 the function \( |f| \) is also \( h \)-resolutive, and it follows by Proposition 3.8 that \( H^h_{|f|}(x) = \frac{1}{h} \int |f(Y)|K(x,Y)d\mu_h(Y) < +\infty \) for every \( x \in U \). Conversely, suppose that \( fK(.,.) \) is \( \mu_h \)-integrable for quasievery \( x \in U \setminus E \). For any \( x \in U \), \( K(x,.). > 0 \) is l.s.c. and hence \( \mu_h \)-measurable on \( \overline{U} \) by \ref{prop:measure} Proposition 3.2], and so \( f^+ \) and \( f^- \) must be \( \mu_h \)-measurable. By Proposition 3.8, \( f^+ \) and \( f^- \) are therefore \( h \)-quasiresolutive, that is \( h \)-resolutive by Corollary 3.12 and so is therefore \( f = f^+ - f^- \) by 1. and 2. in Proposition 3.4. \( \Box \)

Remark 3.14. In the case where \( U \) is Euclidean open it follows by the Harnack convergence theorem for harmonic functions (not extendable to finely harmonic functions) for any numerical function \( f \) on \( \Delta(U) \) that \( \overline{H^h_f} \) is \( h \)-hyperharmonic on \( U \) (in particular \( > -\infty \)) and hence equal to \( \hat{H}^h_f \) (except if \( \overline{H^h_f} \equiv -\infty \)). It follows by Proposition 2.4 (a) that \( \overline{H^h_f} = \hat{H}^h_f \geq H^h_f = H^h_f \). If \( f \) is resolutive then by Definition 3.1 equality prevails on all of \( U \). Summing up, Theorem 3.13 is a (proper) extension of the corresponding classical result, cf. e.g. \ref{prop:measure} Theorem 1.VIII.8).

We close this section with a brief discussion of an alternative, but compatible concept of \( h \)-resolutivity based on the minimal-fine (mf) topology, cf. \ref{prop:measure} Definition 3.4. The mf-closure of \( U \) is \( U \cup \Delta_1(U) \), and the relevant boundary functions \( f \) are therefore now defined only on the mf-boundary \( \Delta_1(U) \).

Given a function \( f \) on \( \Delta_1(U) \) with values in \( \mathbb{R} \), a finely \( h \)-hyperharmonic function \( u \) on \( U \) is now said to belong to the upper PWB\(^h \) class, denoted again
by $\mathcal{U}_f^h$, if $u$ is lower bounded and if
\[ \operatorname{mf-lim} \inf_{x \rightarrow Y, x \in U} u(x) \geq f(Y) \quad \text{for every } Y \in \Delta_1(U). \]
This leads to new, but similarly denoted concepts
\[ \hat{H}^h_f = \inf_{U} h^h_f, \quad H^h_f = \check{\hat{H}}^h_f = \inf_{\check{U}^h_f} (\leq \hat{H}^h_f), \]
and hence new concepts of $h$-quasiresolutivity and $h$-resolutivity.

When considering reduction $R^A_u$ and sweeping $\hat{R}^A_u$ of a finely $h$-hyperharmonic function $u$ on $U$ onto a set $A \subset \overline{U}$ we similarly use the alternative, though actually equivalent mf-versions [7, Definition 3.14], cf. [7, Theorem 3.16]. This occurs in the proof of Proposition 3.7 (after the first display), where now $W \subset \overline{U}$ is mf-open (and contains $A$).

The changes as compared with the case of $h$-resolutivity relative to the natural topology are chiefly as follows. A set $A \subset U \cup \Delta_1(U)$ is of course now said to be $h$-harmonic null if $H^h_{1_A} = 0$ (with the present mf-version of $H^h$). In Proposition 2.4 (b) we apply [7, Proposition 3.12] in place of its corollary. In the beginning of the proof of Proposition 3.10 the function $g_j$ shall now be defined at $Y \in \Delta_1(U)$ by
\[ g_j(Y) = \operatorname{mf-lim} \inf_{z \rightarrow Y, z \in U} u_j(z). \]
And $g_j$ is $\mu_h$-measurable on $\Delta_1(U)$ because $g_j$ equals $\mu_h$-a.e. the $\mu_h$-measurable function defined $\mu_h$-a.e. on $\Delta_1(U)$ by $Y \mapsto \operatorname{mf-lim}_{z \rightarrow Y, z \in U} u_j(z) = \frac{d\mu_{u_j}}{d\mu_h}(Y)$ according to the version of the Fatou-Naïm-Doob theorem established in [6, Theorem 4.5]. Here $\mu_{u_j}$ denotes the representing measure for $u_j$, that is $K\mu_{u_j} = u_j$, and $d\mu_{u_j}/d\mu_h$ denotes the Radon-Nikodým derivative of the $\mu_h$-continuous part of $\mu_{u_j}$ (carried by $\Delta_1(U)$) with respect to $\mu_h$. The $\mu_h$-measurability of $g_j$ on $\Delta_1(U)$ thus established is all that is needed for the proof of the mf-version of Proposition 3.10, replacing mostly $\Delta(U)$ with $\Delta_1(U)$.

The following result is established like Theorem 3.13.

**Theorem 3.15.** A function $f : \Delta_1(U) \rightarrow \mathbb{R}$ is $h$-resolutive relative to the mf-topology if and only if the function $Y \mapsto f(Y)K(x,Y)$ on $\Delta_1(U)$ is $\mu_h$-integrable for quasievery $x \in U$. In the affirmative case $Y \mapsto f(Y)K(x,Y)$ is $\mu_h$-integrable for every $x \in U$, and we have everywhere on $U$
\[ H^h_f = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y). \]

**Corollary 3.16.** For any $h$-resolutive function $f : \Delta(U) \rightarrow \mathbb{R}$ relative to the natural topology, the restriction of $f$ to $\Delta_1(U)$ is resolutive relative to the mf-topology. Conversely, for any $h$-resolutive function $f : \Delta_1(U) \rightarrow \mathbb{R}$ relative
to the mf-topology, the extension of $f$ by 0 on $\Delta(U) \setminus \Delta_1(U)$ is $h$-resolutive relative to the natural topology.

4. Further equivalent concepts of $h$-resolutivity

We again consider functions $f : \Delta(U) \to \mathbb{R}$. We show that the equivalent concepts of $h$-resolutivity and $h$-quasiresolutivity do not alter when $H^h_f, H^h_f$ in Definitions 3.1, 3.2 are replaced by $\dot{H}^h_f$ and $H^h_f$, respectively. Recall from Proposition 2.4 (c) that $\dot{H}^h_f = H^h_f$ if $\dot{H}^h_f < +\infty$. This applies, in particular, to the indicator function $1_A$ of a set $A \subset \Delta(U)$. Therefore the “dot”-version of the concept of an $h$-harmonic null set $A$ is identical with the version considered in Definition 2.2.

**Definition 4.1.** A function $f$ on $\Delta(U)$ with values in $\mathbb{R}$ is said to be $h$-dot-resolutive if $\dot{H}^h_f = H^h_f$ on $U$ and if this function, also denoted by $H^h_f$, is neither identically $+\infty$ nor identically $-\infty$.

For any function $f : \Delta(U) \to \mathbb{R}$ we consider the following subset of $U$:

$$\dot{E}^h_f = \{\dot{H}^h_f = -\infty\} \cup \{H^h_f = +\infty\} \cup \{\dot{H}^h_f \neq H^h_f\}.$$  

**Lemma 4.2.** A function $f$ on $\Delta(U)$ with values in $\mathbb{R}$ is $h$-quasiresolutive if and only if $f$ is $h$-dot-quasiresolutive in the sense that $\dot{E}^h_f$ is polar, or equivalently that the relations $\dot{H}^h_f > -\infty$, $H^h_f < +\infty$, and $\dot{H}^h_f = H^h_f$ all hold quasieverywhere on $U$.

**Proof.** If these three relations hold q.e. on $U$ then analogously $\overline{H}^h_f = \dot{H}^h_f = H^h_f > -\infty$ q.e. and similarly $\overline{H}^h_f < +\infty$ q.e. But $\overline{H}^h_f = \dot{H}^h_f > -\infty$ q.e. on $\{\overline{H}^h_f > -\infty\}$, hence also q.e. on $U$. Similarly, $\overline{H}^h_f < +\infty$ q.e. on $U$, and altogether $\overline{H}^h_f = H^h_f$ q.e. on $U$. Thus $f$ is quasiresolutive. The converse is obvious. 

**Lemma 4.3.** Every $h$-dot-resolutive function $f$ is $h$-resolutive, and hence $h$-quasiresolutive (now also termed $h$-dot-quasiresolutive).

**Proof.** Suppose that $f$ is $h$-dot-resolutive then $f$. Then $f$ is $h$-resolutive, for $\dot{H}^h_f, \overline{H}^h_f, \underline{H}^h_f$, and $H^h_f$ are all equal because there is equality in the general inequalities $\dot{H}^h_f \geq \overline{H}^h_f \geq \underline{H}^h_f \geq H^h_f$, cf. Proposition 2.4 (a). The rest follows from Lemma 3.3.

In view of Lemma 4.3 an $h$-(dot-)quasiresolutive function is $h$-dot-resolutive if and only if $\dot{E}^h_f = \emptyset$. Assertions 1. and 2. of Proposition 3.4 therefore remain valid when $E$ is replaced throughout by $\dot{E}$. The proof of the dot-version of eq. (3.1) uses 1. of Proposition 2.5 in place of 2. there. The following lemma is analogous to Lemma 3.5.
Lemma 4.4. Let \( f \) be an \( h \)-quasiresolutive function on \( \Delta(U) \). If \( f^+ \) and \( f^- \) are \( h \)-dot-resolutive then so is \( f \), and the function \( H^h_f = \hat{H}^h_f = \hat{H}^h_f \) on \( U \) is finite valued.

Proof. Since \( f \) is \( h \)-quasiresolutive, so are \( f^+, f^- \) (besides being \( h \)-dot-resolutive) by 3. in Proposition 3.4. Hence the functions \( H^h_{f^\pm} = \hat{H}^h_{f^\pm} = \hat{H}^h_{f^\pm} \) are finite valued (co-polar subsets of \( U \) being non-void). From \( -f^- \leq f \leq f^+ \) it therefore follows by Proposition 2.4 (a) that

\[
-\infty < -\hat{H}^h_{f^-} \leq \hat{H}^h_f \leq \hat{H}^h_{f^+} < +\infty.
\]

Applying 1. in Proposition 2.5 to the sums \( f = f^+ + (-f)^+ = f^+ - f^- \) and \( -f = f^- - f^+ \), which are both well defined on \( \Delta(U) \), we obtain

\[
\hat{H}^h_f \leq H^h_{f^+} - H^h_{f^-} \leq \hat{H}^h_f
\]
on all of \( U \). It follows that \( \hat{H}^h_f = H^h_f = H^h_{f^+} - H^h_{f^-} \) holds there, again by Proposition 2.4 (a).

\[\square\]

Corollary 4.5. Let \( (f_j) \) be an increasing sequence of lower bounded \( h \)-dot-resolutive functions \( \Delta(U) \to ]-\infty, +\infty[ \), and let \( f = \sup_j f_j \). If \( \hat{H}^h_f \neq +\infty \) then \( f \) is \( h \)-dot-resolutive.

Proof. For every \( j \) we have \( H^h_f \geq H^h_{f_j} = \hat{H}^h_{f_j} \), and hence \( H^h_f \geq \sup_j \hat{H}^h_{f_j} = \hat{H}^h_f \) according to 4. in Proposition 2.5. Hence equality prevails on account of Proposition 2.4 (a). By hypothesis, \( \hat{H}^h_f \neq +\infty \), and clearly \( H^h_f > -\infty \), so we conclude that \( f \) indeed is \( h \)-dot-resolutive.

\[\square\]

Proposition 4.6. For any \( \mu_h \)-measurable subset \( A \) of \( \Delta(U) \) the indicator function \( 1_A \) is \( h \)-dot-resolutive and (3.3) holds. In particular, the constant function 1 on \( \Delta(U) \) is \( h \)-dot-resolutive and \( H^h_1 = 1 \).

Proof. Since \( \hat{H}^h_{1_A} \leq \hat{H}^h_1 = 1 < +\infty \) it follows from Proposition 2.4 (c) that \( \overline{H}^h_{1_A} = \hat{H}^h_{1_A} \), and the assertions reduce to the analogous Proposition 3.7.

\[\square\]

Proposition 4.7. Let \( f \) be a \( \mu_h \)-measurable lower bounded function on \( \Delta(U) \). Then

\[
\hat{H}^h_f = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) > -\infty,
\]

and \( \hat{H}^h_f \) is either identically \( +\infty \) or the sum of an \( h \)-invariant function and a constant \( \leq 0 \). Furthermore \( f \) is \( h \)-quasiresolutive if and only if \( f \) is \( h \)-dot-resolutive, and that holds if and only if \( \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y) < +\infty \) q.e. on \( U \), or equivalently: everywhere on \( U \). In particular, every bounded \( \mu_h \)-measurable function \( f : \Delta(U) \to \mathbb{R} \) is \( h \)-dot-resolutive.

Proof. In view of the case of \( \hat{H} \) in 4. of Proposition 2.5 the proof of the analogous Proposition 3.8 carries over mutatis mutandis.

\[\square\]
Corollary 4.8. Let \( f : \Delta(U) \to \mathbb{R} \) be \( \mu_h \)-measurable. Then \( f \) is \( \mu_h \)-dot-resolutive if and only if \( |f| \) is \( \mu_h \)-dot-resolutive.

Proof. If \( f \) is \( \mu_h \)-dot-resolutive, and therefore \( \mu_h \)-quasiresolutive by Lemma 4.3 then \( |f| = f \vee (-f) \) is \( \mu_h \)-quasiresolutive according to 1. and 3. in Proposition 3.4. Now, \( |f| \) is lower bounded (and \( \mu_h \)-measurable), and \( |f| \) is therefore even \( \mu_h \)-dot-resolutive, by Proposition 4.7. Consequently, \( |f|K(x,.) \) is \( \mu_h \)-integrable for every \( x \in U \). So are therefore \( f^+K(x,.) \) and \( f^-K(x,.) \). Again by Proposition 4.7 it follows that \( f^+ \) and \( f^- \) are \( \mu_h \)-dot-resolutive along with \( f^+ \) and \( f^- \) (positive) by Lemma 4.7. So is therefore \( f = f^+ - f^- \) by Lemma 4.4. \( \square \)

Corollary 4.9. A function \( f \) on \( \Delta(U) \) with values in \( \mathbb{R} \) is \( \mu_h \)-dot-resolutive if and only if \( f \) is \( \mu_h \)-quasiresolutive.

Proof. The ‘only if’ part is contained in Lemma 4.3. For the ‘if’ part, suppose that \( f \) is \( \mu_h \)-quasiresolutive and hence \( \mu_h \)-measurable according to Proposition 3.10. If \( f \geq 0 \) then \( f \) is \( \mu_h \)-dot-resolutive according to Proposition 4.7. For arbitrary \( f : \Delta(U) \to \mathbb{R} \) this applies to \( f^+ \) and \( f^- \), which are \( \mu_h \)-quasiresolutive according to 3. in Proposition 3.4. Consequently, \( f = f^+ - f^- \) is likewise finely \( \mu_h \)-dot-resolutive by 1. and 2. in Proposition 3.3. \( \square \)

Theorem 4.10. A function \( f : \Delta(U) \to \mathbb{R} \) is \( \mu_h \)-dot-resolutive if and only if the function \( Y \mapsto f(Y)K(x,Y) \) on \( \Delta(U) \) is \( \mu_h \)-integrable for every \( x \in U \), or equivalently for quasievery \( x \in U \). In the affirmative case the solution of the PWB-problem on \( U \) with boundary function \( f \) is

\[
H^h_f := \overline{H}^h_f = H^h_f = \hat{H}^h_f = \frac{1}{h} \int f(Y)K(.,Y)d\mu_h(Y).
\]

Proof. The proof of the analogous Theorem 3.15 carries over mutatis mutandis. \( \square \)

The alternative concept of \( h \)-resolutivity relative to the mf-topology discussed at the end of the preceding section likewise has a similarly established compatible version based on \( \hat{H}^h_f \) and \( H^h_f \) instead of \( \overline{H}^h_f \) and \( H^h_f \).

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