Forward-Secure Group Signatures from Lattices

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Abstract. Group signature is a fundamental cryptographic primitive, aiming to protect anonymity and ensure accountability of users. It allows group members to anonymously sign messages on behalf of the whole group, while incorporating a tracing mechanism to identify the signer of any suspected signature. Most of the existing group signature schemes, however, do not guarantee security once users’ secret keys are exposed. To reduce potential damages caused by key exposure attacks, Song (ACMCCS 2001) put forward the concept of forward-secure group signatures (FSGS). Nakanishi, Hira, and Funabiki (Pairing 2009) then defined a rigorous security model of FSGS for static groups, and demonstrated a pairing-based instantiation satisfying the model. Subsequently, Libert and Yung (ASIACCS 2010) extended Nakanishi et al.’s work to handle dynamically growing groups. For the time being, all known secure FSGS schemes are based on number-theoretic assumptions, and are vulnerable against quantum computers.

In this work, we construct the first lattice-based FSGS scheme. In Nakanishi et al.’s model, our scheme achieves forward-secure traceability under the Short Integer Solution (SIS) assumption, and offers full anonymity under the Learning With Errors (LWE) assumption. At the heart of our construction is a scalable lattice-based key-evolving mechanism, allowing users to periodically update their secret keys and to efficiently prove in zero-knowledge that the key-evolution process is done correctly. To realize this essential building block, we first employ the Bonsai-tree structure by Cash et al. (EUROCRYPT 2010) to handle the key evolution process, and then develop Langlois et al.’s construction (PKC 2014) to design its supporting zero-knowledge protocol. In comparison to all known lattice-based group signatures (that are not forward-secure), our scheme only incurs a very reasonable overhead: the bit-sizes of keys and signatures are at most $O(\log N)$, where $N$ is the number of group users; and at most $O(\log^2 T)$, where $T$ is the number of time periods.

Keywords. Group signatures, Key exposure, Forward-security, Lattice-based cryptography, Zero-knowledge proofs

1 Introduction

Group signatures. Initially introduced by Chaum and van Heyst [21], group signature allows members of a group controlled by a manager to sign messages
anonymously in the name of the group (anonymity). Nevertheless, there is a tracing mechanism to identify the signer of any signature in case of disputes (traceability). These two appealing features allow group signatures to find applications in various real-life scenarios, such as digital right management, anonymous online communications, e-commerce systems, and much more. On the theoretical front, designing secure and efficient group signature schemes is interesting and challenging, since those advanced constructions usually require a sophisticated combination of carefully chosen cryptographic ingredients: digital signatures, encryption schemes, and zero-knowledge protocols. In the last quarter-century, an abundance of group signature schemes with different security models, different levels of efficiency and functionality have been proposed (e.g., [5, 6, 11, 15, 30, 43]). Unfortunately, the exposure of group signing keys renders almost all the existing schemes unsatisfactory in practice, in the sense that the security of the scheme is no longer guaranteed when the key exposure happens. So now let us look at key exposure problem and the countermeasures to this problem.

Exposure of Group Signing Keys and Forward-Secure Group Signatures. Exposure of users’ secret keys is one of the greatest dangers to many cryptographic protocols in practice [59]. Forward-secure mechanisms, aiming to minimize the damages caused by secret key exposures, were first introduced by Anderson [4]. Since then, many forward-secure cryptosystems were constructed, such as forward-secure signatures [4, 7, 1, 31], forward-secure public key encryption systems [23, 18, 10], and forward-secure signatures with un-trusted update [14, 44, 45]. At the heart of these schemes is a key evolving technique that operates as follows. It divides the lifetime of the scheme into discrete $T$ time periods. Upon entering a new time period, a subsequent secret key is computed from the current secret key via a one-way key-evolution algorithm and the current one is deleted promptly. By the property of the one-way algorithm, the security of previous keys is preserved even when current secret key is exposed to the adversary. Therefore, by carefully choosing a secure scheme that operates well with a key evolving mechanism, forward-security of the scheme can be guaranteed.

As investigated by Song [59], secret key exposure in group signatures is much more damaging than in ordinary digital signatures. In group signatures, if one group member’s signing key is exposed to the attacker, then the latter can sign arbitrary messages. Under this circumstance, if the underlying group signature scheme is not secure against exposure of group signing keys, then the whole system has to be re-initialized, which is obviously unsuitable in practice. Furthermore, it renders all previously signed group signatures invalid since we do not have a mechanism to distinguish whether a signature is generated by a legitimate group member or by the attacker. This seriously violates the traceability property that group signatures are supposed to offer. Note that this kind of violation cannot be even solved by simply re-initializing the whole system. In fact, one of the easiest way for a misbehaving member Eve to repudiate her illegally signed signatures (i.e., to violate the traceability) is to reveal her group signing key secretly in the Internet and then claim to be a victim of the key exposure problem [31]. Now the users who had accepted signatures before Eve’s group
signing key is exposed are now at the mercy of all the group members, some of whom (e.g., Eve) would not reissue the signatures with the newly changed key. Therefore, traceability of group signatures is no longer guaranteed.

The aforementioned problems induced by the exposure of group signing keys motivated Song [59] to consider the concept of forward-secure group signatures (FSGS), where group members are able to update their group signing keys at each time period via a one-way key-evolution algorithm. Therefore, when some group member’s signing key is compromised, all the signatures generated during past periods remain valid. Furthermore, FSGS prevent dishonest group members from repudiating signatures generated during past periods by simply exposing keys. Later, Nakanishi, Hira, Funabiki [53] defined a rigorous security model of FSGS for static groups, where users are fixed throughout the scheme, and first achieved logarithmic complexity in total number of time periods $T$ for all algorithms and for sizes of all parameters. Subsequently, Libert and Yung [46] extended Nakanishi et al.’s work to capture the setting of the dynamically growing groups. Their scheme achieves better efficiency than all previous schemes. More precisely, signature size, signing and verification costs do not depend on number of time periods $T$ and other metrics are at most log-squared complexity in $T$. However, all these schemes are constructions based on number-theoretic assumptions and are fragile in the presence of quantum adversaries. In order not to put all eggs in one basket, it is imperative to consider instantiations based on alternative, post-quantum foundations, e.g., lattice assumptions. In view of this, let us now look at the topic of lattice-based group signatures.

**Lattice-based group signatures.** Lattice-based cryptography has been an exciting research area since the seminal works of Regev [57] and Gentry et al. [26]. Lattices not only allow to build powerful primitives (e.g., [25,27]) that have no feasible instantiations in conventional number-theoretic cryptography, but they also provide several advantages over the latter, such as conjectured resistance against quantum adversaries and faster arithmetic operations. Along with other primitives, lattice-based group signature has received noticeable attention in recent years. The first scheme was introduced by Gordon, Katz and Vaikuntanathan [28] whose solution produced signature size linear in the number of group users $N$. Camenisch et al. [17] then extended [28] to achieve anonymity in the strongest sense. Later, Laguillaumie et al. [36] put forward the first scheme with the signature size logarithmic in $N$, at the cost of relatively large parameters. Simpler and more efficient solutions with $O(\log N)$ signature size were subsequently given by Nguyen et al. [54] and Ling et al. [48]. Libert et al. [41] obtained substantial efficiency improvements via a construction based on Merkle trees which eliminates the need for GPV trapdoors [26]. More recently, a scheme supporting message-dependent opening (MDO) feature [58] was proposed in [42]. All the schemes mentioned above are designed for static groups.

Three lattice-based group signatures that have certain dynamic features were proposed by Langlois et al. [37], Libert et al. [38], and Ling et al. [49]. The first one is a scheme with verifier-local revocation (VLR) [12], which means that only the verifiers need to download the up-to-date group information. The second
one addresses the orthogonal problem of dynamic user enrollments, which was formalized by Kiayias and Yung [35] and by Bellare et al. [8]. The third one is a fully dynamic scheme that supports both features, following Bootle et al.’s model [13].

For the time being, there is no forward-secure group signatures from lattice assumptions. Considering the great threat of key exposure to group signatures and the vulnerability of group signatures from number-theoretic assumptions in front of quantum adversaries, it would be desirable to investigate the design of (non-trivial) lattice-based forward-secure group signatures. Furthermore, it would be inspiring to achieve it with reasonable overhead, e.g., with complexity at most poly-logarithmic in $T$.

Our Contributions. We introduce the first forward-secure group signature scheme from lattices. The scheme works in Nakanishi et al.’s model [53]. Under the Short Integer Solution (SIS) assumption and the Learning With Errors (LWE) assumption, our scheme achieves full anonymity and a stronger notion of traceability named forward-secure traceability, which captures the traceability in the setting of key exposure problems and was formally defined in [53]. Like all previous lattice-based group signatures, our scheme is proven to be secure in the random oracle model.

For a security parameter $\lambda$, maximum number of group members $N$, and total time periods $T$, our scheme features group public key size $\tilde{O}(\lambda^2 (\log N + \log T))$, signature size $\tilde{O}(\lambda (\log N + \log T))$ and secret key size $\tilde{O}(\lambda^2 (\log N + \log T)^2 \log T)$. In Table 1, we give a detailed comparison of our scheme with known lattice-based group signatures that are not forward-secure, in terms of efficiency and functionality.

Overview of Our Techniques. As we discussed earlier, designing secure group signatures typically requires a combination of digital signatures, encryption schemes and zero-knowledge protocols. To realize forward-secure group signatures, we further need a mechanism to update the group signing key periodically and a zero-knowledge protocol to prove that the key updating procedure is done honestly.

To start with, we utilize the Bonsai tree signature scheme [20] as one of three essential ingredients in constructing our forward-secure group signature scheme. Consider a group of $N = 2^\ell$ users, where each user is identified by a string $id \in \{0, 1\}^\ell$ that is the binary representation of this user’s index in the group. Let $n, m, \beta$ and $q \geq 2$ be positive integers (to be determined later). The main structure of our scheme is a Bonsai tree, specified by a matrix $A' = [A_0 | A_0|A_1| \cdots | A_0|A_{\ell}] \in \mathbb{Z}_q^{(2^\ell+1)m}$ and a vector $u$. Initially, each user with identity $id$ is issued a Bonsai signature on his identity as his group signing key, which is a small vector $v_{id} \in \mathbb{Z}^{(\ell+1)m}$ such that $A_{id} \cdot v_{id} = u \mod q$ and $\|v_{id}\|_\infty \leq \beta$ with $A_{id} = [A_0|A_{id}[1]| \cdots |A_{id}[\ell]]$. The scheme then follows the usual sign-then-encrypt-then-prove approach for constructing group signatures. When signing messages, the user $id$ first encrypts his identity $id$ to a ciphertext $c$ via a CCA-secure encryption scheme obtained by utilizing the CHK technique [19] to the identity-based encryption (IBE) scheme by Gentry et al. [26]. Then he
Thanks to the basis delegation algorithm from [20], users are able to derive keys.

The zero-knowledge protocol for fact (i) is developed from Langlois et al.'s technique [47] for proving possession of a valid message-signature pair for the Bonsai signature (which was also employed by Cheng et al. [22] in the context of policy-based signatures). Meanwhile, to prove fact (ii) in zero-knowledge, we rely on the decomposition-extension-permutation technique by Ling et al. [17], which is known to be well-suited for handling LWE-based relations [18, 33, 40].

The modular construction we discussed above actually only yields a group signature that is not forward-secure. To upgrade it into a forward-secure one, we exploit the hierarchical structure of the Bonsai tree to enable periodic key updating. Let $T$ be the total number of time periods. Now, each user id is additionally associated with a subtree of depth $d$, specified by the matrices $A_{t+1}^{0}, A_{t+1}^{1}, \ldots, A_{t+d}^{0}, A_{t+d}^{1}$. Let $z$ be a binary string of length $d_z \leq d$, define $A_{id||z} = [A_{id||z}^{0}||z] \cdots [A_{id||z}^{d_z}||z]$, and let $S_{id||z}$ be a trapdoor matrix such that $A_{id||z} \cdot S_{id||z} = 0 \mod q$. Group signing key of user id at time $t$ consists of $\{S_{id||z}, z \in \text{Nodes}_{(t \rightarrow T-1)}\}$ while group signing key at time $t' > t$ consists of $\{S_{id||z'}, z' \in \text{Nodes}_{(t' \rightarrow T-1)}\}$ such that for any $z' \in \text{Nodes}_{(t' \rightarrow T-1)}$, there exists an ancestor $z \in \text{Nodes}_{(t \rightarrow T-1)}$. Thanks to the basis delegation algorithm from [20], users are able to derive keys.

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| Scheme | Sig. size | Group PK size | Signer's SK size | Model |
|--------|-----------|--------------|------------------|-------|
| GKV [28] | $\tilde{O}(\lambda^2 \cdot N)$ | $\tilde{O}(\lambda^2)$ | $\tilde{O}(\lambda^2)$ | static |
| CNR [17] | $\tilde{O}(\lambda^2 \cdot N)$ | $\tilde{O}(\lambda^2)$ | $\tilde{O}(\lambda^2)$ | static |
| LLLS [36] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | static |
| LLNW [37] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | static |
| NZZ [54] | $\tilde{O}(\lambda + \ell^2)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda^2)$ | static |
| LNW [48] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda)$ | static |
| LLNW [48] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | static |
| LLNW [49] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda)$ | partially dynamic |
| LLM+ [50] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda)$ | MDO |
| LMN [51] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda)$ | fully dynamic |
| LNWX [52] | $\tilde{O}(\lambda \cdot t)$ | $\tilde{O}(\lambda^2 \cdot t)$ | $\tilde{O}(\lambda(t + d)^2 \cdot t)$ | forward secure |

Table 1. Comparison of known lattice-based group signatures, in terms of asymptotic efficiency and functionality. The comparison is done based on three governing parameters: security parameter $\lambda$, the maximum expected number of group users $N = 2^t$ and total time periods $T = 2^d$. 

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1. To be formally defined in Section 3.
for the descendant of node \( z \), and hence, to update their signing keys periodically. Furthermore, by the one-way property of the basis delegation algorithm, prior signing keys are guaranteed to be secure even when the current one is compromised. We remark that the above updating technique follows from previous works by Nakanishi, Hira and Funabiki \[53\] and Libert and Yung \[46\], which were inspired from \[18,10,14\].

Second, we modify the signing algorithm to capture the key-evolution procedure. Specifically, when signing a message at time \( t \), of which the binary representation is \( z \), user id now proves in zero-knowledge that: (i) he is a certified group member; (ii) he has done the key-evolution honestly; (iii) \( c \) is a correct encryption of \( \text{id} \). We note that fact (iii) can be handled in the same manner as in the basic scheme discussed earlier. So the question has reduced to how to handle facts (i) and (ii). To this end, we observe that it is sufficient for user id to prove knowledge of a small vector \( \mathbf{v}_{\text{id}} \parallel z \) such that \( A_{\text{id}} \parallel z \cdot \mathbf{v}_{\text{id}} \parallel z = u \mod q \). Note that when proving knowledge of such a short vector, \( \text{id} \) should be kept hidden while \( z \) is part of public input, therefore the problem reduces to proving knowledge of short vectors \( \mathbf{v}_{\text{id}} \) and \( \mathbf{w}_2 \), and a binary string \( \text{id} \) satisfying:

\[
A_{\text{id}} \cdot \mathbf{v}_{\text{id}} + A'' \cdot \mathbf{w}_2 = u \mod q.
\]

To handle this statement, we utilize Langlois et al.’s technique for the Bonsai signature to prove knowledge of \( \mathbf{v}_{\text{id}} \) and \( \text{id} \), and adapt Ling et al.’s decomposition-extension-permutation technique \[47\] to prove knowledge of \( \mathbf{w}_2 \). Finally, the protocol is transformed into a signature via the Fiat-Shamir heuristic \[24\].

To summarize, by incorporating a key updating mechanism into a secure group signature scheme, we constructed a forward-secure group signature scheme from lattices. Our scheme satisfies full anonymity thanks to the CCA-security of the underlying encryption scheme and the statistical zero-knowledge of the underlying zero-knowledge protocol, and achieves forward-secure traceability thanks to the security of the Bonsai tree structure. We believe that, our construction - while not being truly practical - would certainly help to enrich the field of lattice-based group signatures, and that our design approach may be of independent interest.

**Concurrent Work.** Recently, Kansal, Dutta and Mukhopadhyay \[33\] proposed a forward-secure group signature scheme from lattices in the model of Libert and Yung \[46\]. Unfortunately, it can be observed that their proposed scheme does not satisfy the correctness and security requirements. (For details, see Appendix \[B\]) Therefore, constructing forward-secure group signatures from lattices for dynamic groups remains an open problem.

**Organization.** In Section \[2\] we recall some background on forward-secure group signatures, lattice-based cryptography, the bonsai tree signature scheme, and Stern-like zero-knowledge protocols. In Section \[3\] our main scheme is constructed and analyzed. Section \[4\] describes the main zero-knowledge protocol used in our construction. Some details are deferred to the Appendix.
2 Preliminaries

2.1 Notations

In this paper, all vectors are column vectors. The column concatenation of matrices \( A \in \mathbb{R}^{n \times m} \) and \( B \in \mathbb{R}^{n \times k} \) is denoted by \( [A|B] \in \mathbb{R}^{n \times (m+k)} \) while the concatenation of two (column) vectors \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^k \) is denoted by \( (x|y) \in \mathbb{R}^{m+k} \). Let \( \| \cdot \| \) and \( \| \cdot \|_{\infty} \) be the Euclidean (\( \ell_2 \)) norm and infinity (\( \ell_{\infty} \)) norm of a vector, respectively. For \( a \in \mathbb{R} \), \( \log a \) denotes logarithm of \( a \) with base 2. For a vector \( v \in \mathbb{R}^m \), let \( v[i] \) be the \( i \)-th bit of \( v \). Denote \([m]\) as the set \( \{1,2,\cdots,m\} \).

2.2 Security Model of Forward-Secure Group Signature Scheme

We recall the definition and security notions of forward-secure group signatures (FSGS), as presented by Nakanishi et al. \[53\]. An FSGS scheme involves two authorities: a group manager (GM) who manages the group and a tracing manager (TM) who can identify the signer of any signature. The lifetime of an FSGS scheme is divided into \( T \) discrete periods \( 0,1,\cdots,T-1 \). At each time period \( t \), the secret key of group member \( i \) is denoted as \( \text{usk}_i[i] \). To enable forward-security, after each update, the old key is assumed to be discarded promptly. An FSGS scheme consists of the following polynomial-time algorithms.

**KeyGen:** This randomized key generation algorithm takes as input a tuple \((\lambda, T, N)\), where \( \lambda \) is the security parameter, \( T \) is the total number of time periods, and \( N \) is the maximum number of group members. It then returns \( \text{gpk} \), \( \text{msk} \), \( \text{mosk} \) and \( \text{usk}_0 \), where \( \text{gpk} \) is the group public key, \( \text{msk} \) is the secret key of GM, \( \text{mosk} \) is the secret key of TM, and \( \text{usk}_0 \) is an array of the initial \( N \) secret signing keys \( \{\text{usk}_0[0], \text{usk}_0[1], \cdots, \text{usk}_0[N-1]\} \), with \( \text{usk}_0[i] \) being the initial secret signing key of user \( i \).

**KeyUpdate:** This randomized algorithm, on inputs \( \text{gpk} \), \( \text{usk}_t[i] \), \( i \), and \( t+1 \), with \( \text{usk}_t[i] \) being the secret key of user \( i \) at time \( t \), outputs the secret key \( \text{usk}_{t+1}[i] \) of user \( i \) at time \( t+1 \).

**Sign:** On inputs \( \text{gpk} \), \( \text{usk}_t[i] \), user \( i \), time period \( t \) and message \( M \), this randomized algorithm returns a signature \( \Sigma \) on message \( M \).

**Verify:** This deterministic algorithm takes as inputs \( \text{gpk} \), time period \( t \), message \( M \) and signature \( \Sigma \), and outputs 1/0 indicating whether the signature is valid or not.

**Open:** On inputs \( \text{gpk} \), \( \text{mosk} \), \( t \), \( M \) and \( \Sigma \), this deterministic algorithm returns an index \( i \) or \( \perp \).

**Correctness.** The above scheme must satisfy the following correctness requirement: For all \( \lambda, T, N \), all \((\text{gpk}, \text{msk}, \text{mosk}, \text{usk}_0) \leftarrow \text{KeyGen}(\lambda, T, N)\), all \( i \in \{0,1,\cdots,N-1\} \), all \( M \in \{0,1\}^* \), all \( \text{usk}_t[i] \leftarrow \text{KeyUpdate}(\text{gpk}, \text{usk}_{t-1}[i], t) \) for all \( t \in \{0,1,\cdots,T-1\} \), the following equations hold:

\[
\text{Verify}(\text{gpk}, t, M, \text{Sign}(\text{gpk}, \text{usk}_t[i], t, M)) = 1,
\]
Open(gpk, mosk, t, M, Sign(gpk, usk_t[i], t, M)) = i.

**Forward-Secure Traceability.** This requirement demands that any PPT adversary, even if it can corrupt the tracing manager and some (or all) group members, is not able to create a valid signature (i) that is traced to one non-corrupted user or (ii) that is traced to one corrupted user, but the signature is signed at time period preceding the secret key query of this corrupted user. Note that (i) captures the standard traceability requirement as in [6] while (ii) deals with the new requirement in the context of forward-security. Details are modelled in the experiment in Fig. 1. In the experiment, the adversary can adaptively choose which user to corrupt, when to corrupt and when to halt, and when to output its forgery. Furthermore, it is allowed to query signature on any message of a member i through the signing oracle Sign(usk_t[i], ·) if i /∈ CU at time period t.

Define the advantage **Adv**_{FSGS, A}^{Trace}(λ, T, N) of adversary A against forward-secure traceability of an FSGS scheme as Pr[Exp_{FSGS, A}^{Trace}(λ, T, N) = 1]. An FSGS scheme is forward-secure traceable if the advantage of any PPT adversary A is negligible.

**Full Anonymity.** This requirement requires that it is infeasible for any PPT adversary to distinguish which of two signers of its choice signed the challenged message of its choice at time period t of its choice. Details of this requirement is modelled in the experiment in Fig. 2. In the experiment, the adversary is accessible to secret keys of all group members and GM, and can query the signer
of any signature except for the challenged one obtained from the opening oracle $\text{Open}(\text{msk}, \cdot)$.

Define $\text{Adv}^{\text{Anon}}_{\text{FSGS}, A}(\lambda, T, N)$ of adversary $A$ against full anonymity of an FSGS scheme as $\Pr[\text{Exp}^{\text{Anon}}_{\text{FSGS}, A}(\lambda, T, N) = 1]$. An FSGS scheme is fully anonymous if the advantage of any PPT adversary $A$ is negligible.

| Experiment $\text{Exp}^{\text{Anon}}_{\text{FSGS}, A}(\lambda, T, N)$ |
|---------------------------------------------------------------|
| $(\text{gpk}, \text{msk}, \text{msk}, \text{usk}) \leftarrow \text{KeyGen}(\lambda, T, N)$, |
| $(st, t, i_0, i_1, M) \leftarrow A^{\text{Open}}(\text{mask}, \cdot)$ (choose, gpk, msk, usk), |
| $b \leftarrow \{0, 1\}$, $\Sigma \leftarrow \text{Sign}(\text{gpk}, \text{usk}[i_b], t, M)$, |
| $b' \leftarrow A^{\text{Open}}(\text{mask}, \cdot)$ (guess, st, $\Sigma$), |
| If $b' = b$ then return 1, |
| Else return 0. |

**Fig. 2:** Experiment used to define full anonymity of an FSGS scheme.

### 2.3 Some Background on Lattices

Let $n \in \mathbb{Z}^+$ and $\Lambda$ be an $n$-dimensional lattice over $\mathbb{R}^n$. Let $S = \{s_1, \ldots, s_n\} \subset \mathbb{R}^n$ be a basis of $\Lambda$. For simplicity, we write $S = [s_1 | \cdots | s_n] \in \mathbb{R}^{n \times n}$. Denote $\|S\|$ as the $L_2$ length of the longest (column) vector in $S$, i.e., $\|S\| = \max_{i} \|s_i\|$. Let $\tilde{S} = [\tilde{s}_1 | \cdots | \tilde{s}_n]$ be the Gram-Schmidt orthogonalization of $S$. We refer to $\|S\|$ as the Gram-Schmidt norm of $S$. For any $c \in \mathbb{R}^n$ and $\sigma \in \mathbb{R}^+$, define the following:

\[
\rho_{\sigma, c}(x) = \exp(-\pi \frac{\|x-c\|^2}{\sigma^2}) \quad \text{and} \quad \rho_{\sigma, c}^{\Lambda}(x) = \sum_{x \in \Lambda} \rho_{\sigma, c}(x) \text{ for any } x \in \Lambda.
\]

Define the discrete Gaussian distribution over the lattice $\Lambda$ with parameter $\sigma$ and center $c$ to be $D_{\Lambda, \sigma, c}(x) = \rho_{\sigma, c}(x)/\rho_{\sigma, c}(\Lambda)$ for any $x \in \Lambda$. We often omit $c$ if it is 0.

Let $n, m, q$ be positive integers with $q \geq 2$. For $A \in \mathbb{Z}_q^{n \times m}$ and $u \in \mathbb{Z}_q^n$ in the subgroup of $\mathbb{Z}_q^n$ generated by the columns of $A$, define

\[
\Lambda^\perp(A) = \{e \in \mathbb{Z}^m : A e = 0 \mod q\},
\]

\[
\Lambda^u(A) = \{e \in \mathbb{Z}^m : A e = u \mod q\}.
\]

Note that we can define discrete Gaussian distribution over the set $\Lambda^u(A)$ in a similar way: $D_{\Lambda^u(A), \sigma, c}(x) = \rho_{\sigma, c}(x)/\rho_{\sigma, c}(\Lambda^u(A))$ for $x \in \Lambda^u(A)$.

In the following, we review some facts about discrete Gaussian distribution.

**Lemma 1 ([26,56]).** Let $n$ and $q \geq 2$ be positive integers and let $m \geq 2n \log q$ and $\sigma \geq \omega(\sqrt{\log m})$.

- Then for all but a $2q^{-n}$ fraction of all $A \in \mathbb{Z}_q^{n \times m}$, the distribution of the syndrome $u = A \cdot e \mod q$ is statistically close to uniform over $\mathbb{Z}_q^n$ for $e \leftarrow D_{\mathbb{Z}_q^m, \sigma}$. Besides, the conditional distribution of $e \leftarrow D_{\mathbb{Z}_q^m, \sigma}$ given $A \cdot e = u \mod q$ is $D_{\Lambda^u(A), \sigma}$.
- Let $\beta = \lceil \sigma \cdot \log n \rceil$ and $x \leftarrow D_{\mathbb{Z}, \sigma}$. Then $\Pr[\|x\|_\infty > \beta]$ is negligible.
– The min-entropy of \( D_{Z^{m},\sigma} \) is at least \( m - 1 \).

We now recall two hard average-case problems: the Short Integer Solution (SIS) problem (in the \( \ell_{\infty} \) norm) and the Learning With Errors (LWE) problem.

**Definition 1 ([25][26], SIS\(_{n,m,q,\beta}^{\infty}\)).** Given a uniformly random matrix \( \mathbf{A} \in \mathbb{Z}_{q}^{n \times m} \), find a non-zero vector \( \mathbf{e} \in \mathbb{Z}^{m} \) such that \( \mathbf{A} \cdot \mathbf{e} = 0 \mod q \) and \( \|\mathbf{e}\|_{\infty} \leq \beta \).

If \( m, \beta = \text{poly}(n) \) and \( q > \beta \sqrt{n} \), then the average-case SIS\(_{n,m,q,\beta}^{\infty}\) problem (in the \( \ell_{\infty} \) norm) is at least as hard as the worst-case \( \text{SIVP}_{\gamma} \) problem for some \( \gamma = \beta \cdot \tilde{O}(\sqrt{n} \log m) \) (see [26][51]).

**Definition 2 ([57], LWE\(_{n,q,\chi}\)).** For \( \mathbf{s} \in \mathbb{Z}_{q}^{n} \), let \( \mathcal{A}_{\mathbf{s},\chi} \) be the distribution over \( \mathbb{Z}_{q}^{n} \times \mathbb{Z}_{q} \) obtained by sampling a vector \( \mathbf{a} \leftarrow \mathbb{Z}_{q}^{n} \) and an element \( e \leftarrow \chi \) and outputting the pair \( (\mathbf{a}, \mathbf{a}^\top \cdot \mathbf{s} + e) \). The LWE\(_{n,q,\chi}\) problem is to distinguish \( m = \text{poly}(n) \) samples from the distribution \( \mathcal{A}_{\mathbf{s},\chi} \) for some secret \( \mathbf{s} \in \mathbb{Z}_{q}^{n} \) and \( m \) samples from the uniform distribution over \( \mathbb{Z}_{q}^{n} \times \mathbb{Z}_{q} \).

Let \( q = \text{poly}(n) \) be a prime power, \( B \geq \sqrt{n} \cdot \omega(\log n) \), \( \gamma = \tilde{O}(nq/B) \) and \( \chi \) is a \( B \)-bounded distribution over \( \mathbb{Z} \), then the average-case LWE\(_{n,q,\chi}\) problem is at least as hard as the worst-case \( \text{SIVP}_{\gamma} \) problem (see e.g., [57][54]).

Now let us recall some algorithms from previous works that will be used extensively in this work. The TrapGen algorithm is used to generate a matrix \( \mathbf{A} \) that is statistically close to uniform together with a short basis of the lattice \( \Lambda^{\perp}(\mathbf{A}) \). The SampleD algorithm employs a short basis of \( \Lambda^{\perp}(\mathbf{A}) \) to output a short vector in \( \Lambda^{u}(\mathbf{A}) \) if \( \Lambda^{u}(\mathbf{A}) \) is not empty.

**Lemma 2 ([3]).** Let \( n \) and \( q \geq 2 \) be positive integers and \( m = \text{O}(n \log q) \). There is a PPT algorithm TrapGen\((n,m,q)\) that outputs a tuple \((\mathbf{A},\mathbf{S})\) such that

- \( \mathbf{A} \) is statistically close to uniform over \( \mathbb{Z}_{q}^{n \times m} \),
- \( \mathbf{S} \) is a basis for \( \Lambda^{\perp}(\mathbf{A}) \), i.e., \( \mathbf{A} \cdot \mathbf{S} = 0 \mod q \), and
- \( \|\mathbf{S}\| \leq L = \text{O}(\sqrt{n \log q}) \).

**Lemma 3 ([26]).** Let \( \mathbf{S} \in \mathbb{Z}^{m \times m} \) be a basis of \( \Lambda^{\perp}(\mathbf{A}) \) for some \( \mathbf{A} \in \mathbb{Z}_{q}^{n \times m} \) whose columns generate the entire group \( \mathbb{Z}_{q}^{n} \). Let \( \mathbf{u} \) be a vector over \( \mathbb{Z}_{q}^{n} \) and \( s \geq \omega(\sqrt{n}) \cdot \|\mathbf{S}\| \). There is a PPT algorithm SampleD\((\mathbf{A},\mathbf{S},\mathbf{u},s)\) that samples a vector \( \mathbf{v} \in \Lambda^{u}(\mathbf{A}) \) from a distribution that is statistically close to \( D_{\Lambda^{u}(\mathbf{A}),s} \).

We also need the following two algorithms to securely delegate basis.

**Lemma 4 ([20]).** Let \( \mathbf{S} \in \mathbb{Z}^{m \times m} \) be a basis of \( \Lambda^{\perp}(\mathbf{A}) \) for some \( \mathbf{A} \in \mathbb{Z}_{q}^{n \times m} \) whose columns generate the entire group \( \mathbb{Z}_{q}^{n} \). Let \( \mathbf{A}' \in \mathbb{Z}_{q}^{n \times m'} \) be any matrix containing \( \mathbf{A} \) as a submatrix. There is a deterministic polynomial-time algorithm ExtBasis\((\mathbf{S},\mathbf{A}')\) that outputs a basis \( \mathbf{S}' \in \mathbb{Z}^{m' \times m'} \) of \( \Lambda^{\perp}(\mathbf{A}') \) with \( \|\mathbf{S}'\| = \|\mathbf{S}\| \).
Lemma 5 (20). Let \( S \) be a basis of an \( m \)-dimensional integer lattice \( \Lambda \) and a parameter \( s \geq \omega(\sqrt{\log n} \cdot \|S\|) \). There exists a PPT algorithm \( \text{RandBasis}(S, s) \) that outputs a new basis \( S' \) of \( \Lambda \) with \( \|S'\| \leq s \cdot \sqrt{m} \). Moreover, for any two bases \( S_0, S_1 \) of \( \Lambda \) and any \( s \geq \max\{\|S_0\|, \|S_1\|\} \cdot \omega(\sqrt{\log n}) \), the outputs of \( \text{RandBasis}(S_0, s) \) and \( \text{RandBasis}(S_1, s) \) are statistically close.

Remark 1. Note that the \( \text{ExtBasis} \) algorithm is often combined with the algorithm \( \text{RandBasis} \) to securely delegate basis. Micciancio and Peikert (50) proposed more efficient mechanisms for this combined task and also for the \( \text{TrapGen} \) and \( \text{SampleD} \) algorithms, which is desired in practice, but for simplicity here, we present our scheme using the above ones.

2.4 The Bonsai Tree Signature Scheme

Our construction builds on the bonsai tree signature scheme proposed by Cash et al. (20). Now we describe it briefly. The scheme takes the following parameters: \( \lambda \) is the security parameter and \( n = \mathcal{O}(\lambda) \), \( \ell \) is the message length, integer \( q = \text{poly}(n) \) is sufficiently large, \( m = \mathcal{O}(n \log q) \), \( L = \mathcal{O}(\sqrt{n \log q}) \), \( s = \omega(\sqrt{\log n}) \cdot L \), and \( \beta = [s \cdot \log n] \). The verification key is the tuple \((A_0, A_0^0, A_1^0, \ldots, A_0^\ell, A_1^\ell, u)\) while the signing key is \( S_0 \), where \((A_0, S_0)\) is generated by the \( \text{TrapGen}(n, m, q) \) algorithm as described in Lemma 2 and matrices \( A_0^0, A_1^0, \ldots, A_0^\ell, A_1^\ell \) and vector \( u \) are all uniformly random and independent over \( \mathbb{Z}_q^n \times \mathbb{Z}_q^m \), respectively.

To sign a binary message \( \text{id} \in \{0, 1\}^\ell \), the signer first computes the matrix \( A_{\text{id}} := [A_0|A_0^0|A_1^0|\ldots|A_0^\ell|A_1^\ell] \in \mathbb{Z}_q^{n \times (\ell+1)m} \), and then outputs a vector \( v \in A^u(A_{\text{id}}) \) via \( \text{SampleD}(\text{ExtBasis}(S_0, A_{\text{id}}), u, s) \). To verify the validity of \( v \) on message \( \text{id} \), the verifier computes \( A_{\text{id}} \) as above and checks if \( A_{\text{id}} \cdot v = u \mod q \) and \( 0 < \|v\|_\infty \leq \beta \) hold. They proved that this signature scheme is existential unforgeable under static chosen message attacks based on the hardness of the SIS problem.

2.5 Zero-Knowledge Argument Systems and Stern-like Protocols

We will work with statistical zero-knowledge argument systems, namely, interactive protocols where the zero-knowledge property holds against any cheating verifier, while the soundness property only holds against computationally bounded cheating provers. More formally, let the set of statements-witnesses \( R = \{(y, w)\} \in \{0, 1\}^* \times \{0, 1\}^* \) be an \( \mathsf{NP} \) relation. A two-party game \( \langle \mathcal{P}, \mathcal{V} \rangle \) is called an interactive argument system for the relation \( R \) with soundness error \( e \) if the following conditions hold:

- Completeness. If \((y, w) \in R\) then \( \Pr[(\mathcal{P}(y, w), \mathcal{V}(y)) = 1] = 1 \).
- Soundness. If \((y, w) \notin R\), then \( \forall \text{ PPT } \mathcal{P}: \Pr[(\mathcal{P}(y, w), \mathcal{V}(y)) = 1] \leq e \).

An argument system is called statistical zero-knowledge if there exists a PPT simulator \( \hat{\mathcal{S}}(y) \) having oracle access to any \( \hat{\mathcal{V}}(y) \) and producing a simulated transcript that is statistically close to the one of the real interaction between
\( P(y, w) \) and \( \widehat{V}(y) \). A related notion is argument of knowledge, which requires the witness-extended emulation property. For protocols consisting of 3 moves (i.e., commitment-challenge-response), witness-extended emulation is implied by special soundness [29], where the latter assumes that there exists a PPT extractor which takes as input a set of valid transcripts with respect to all possible values of the “challenge” to the same “commitment”, and outputs \( w' \) such that \((y, w') \in R\).

**Stern-like protocols.** The statistical zero-knowledge arguments of knowledge presented in this work are Stern-like [60] protocols. In particular, they are \( \Sigma \)-protocols in the generalized sense defined in [32,9] (where 3 valid transcripts are needed for extraction, instead of just 2). The basic protocol consists of 3 moves: commitment, challenge, response. If a statistically hiding and computationally binding string commitment scheme, such as the KTX scheme [34], is employed in the first move, then one obtains a statistical zero-knowledge argument of knowledge (ZKAoK) with perfect completeness, constant soundness error \( 2/3 \).

In many applications, the protocol is repeated \( \kappa = \omega(\log n) \) times to make the soundness error negligibly small in \( n \).

**An abstraction of Stern’s protocol.** We recall an abstraction of Stern’s protocol, proposed in [38]. Let \( K, L, q \) be positive integers, where \( L \geq K \) and \( q \geq 2 \), and let \( \text{VALID} \) be a subset of \( \{-1, 0, 1\}^L \). Suppose that \( S \) is a finite set such that one can associate every \( \phi \in S \) with a permutation \( \Gamma_\phi \) of \( L \) elements, satisfying the following conditions:

\[
\begin{align*}
\{ \text{w} \in \text{VALID} & \iff \Gamma_\phi(\text{w}) \in \text{VALID}, \\
\text{If } \text{w} \in \text{VALID} \text{ and } \phi \text{ is uniform in } S, \text{ then } \Gamma_\phi(\text{w}) \text{ is uniform in } \text{VALID}. \}
\end{align*}
\quad (1)
\]

We aim to construct a statistical ZKAoK for the following abstract relation:

\[
R_{\text{abstract}} = \{(M, u), w \in \mathbb{Z}_q^K \times \mathbb{Z}_q^L \times \text{VALID} : M \cdot w = u \mod q. \}
\]

The conditions in (1) play a crucial role in proving in ZK that \( w \in \text{VALID} \): To do so, the prover samples \( \phi \overset{\$}{\leftarrow} S \) and let the verifier check that \( \Gamma_\phi(\text{w}) \in \text{VALID} \), while the latter cannot learn any additional information about \( w \) thanks to the randomness of \( \phi \). Furthermore, to prove in ZK that the linear equation holds, the prover samples a masking vector \( r_w \overset{\$}{\leftarrow} \mathbb{Z}_q^L \), and convinces the verifier instead that \( M \cdot (w + r_w) = M \cdot r_w + u \mod q \).

The interaction between prover \( P \) and verifier \( V \) is described in Fig. 3. The protocol employs a statistically hiding and computationally binding string commitment scheme \( \text{COM} \) from [34].

**Theorem 1 ([38]).** Assume that \( \text{COM} \) is a statistically hiding and computationally binding string commitment scheme. Then, the protocol in Fig. 3 is a statistical ZKAoK with perfect completeness, soundness error \( 2/3 \), and communication cost \( \mathcal{O}(L \log q) \). In particular:

- There exists a polynomial-time simulator that, on input \((M, u)\), outputs an accepted transcript statistically close to that produced by the real prover.
In the description below, for a binary tree of depth $d$, there exists a polynomial-time knowledge extractor that, on input a commitment CMT and 3 valid responses (RSP$_1$, RSP$_2$, RSP$_3$) to all 3 possible values of the challenge $Ch$, outputs $w' \in \text{VALID}$ such that $M \cdot w' = u \mod q$.

The proof of Theorem 1 appeared in [38], employs standard simulation and extraction techniques for Stern-like protocols [34,47]. For completeness, we recall the proof in Appendix A.

### 3 Our Lattice-Based Forward-Secure Group Signature

In the description below, for a binary tree of depth $k$, we identify each node at depth $j$ with a binary vector $z$ of length $j$ such that $z[1]$ to $z[j]$ are ordered from the top to the bottom and a 0 and a 1 indicate the left and right branch respectively in the order of traversal. Let $B$ be a positive integer. For a non-negative integer $b$ smaller than $B$, denote $\text{Bin}(b)$ as the binary representation of $b$ with length $\lceil \log B \rceil$.

In our forward-secure group signature scheme, lifetime of the scheme is divided into $T$ discrete periods $0, 1, \cdots, T-1$. For simplicity, let $T = 2^d$ for some positive integer $d$. Following previous works [11,16], time is arranged as the leaves of a binary tree of depth $d$ in chronological order. More precisely, each time period $t$ is associated with leave $\text{Bin}(t)$.
Following [11], for \( j = 1, \ldots, d + 1, \ t \in \{0, 1, \ldots, T - 1\} \), we define a time period’s “right sibling at depth \( j \)” as

\[
\text{sibling}(j, t) = \begin{cases} 
(Bin(t)[1], \ldots, Bin(t)[j - 1], 1)^T & \text{if } j \leq d \text{ and } Bin(t)[j] = 0, \\
\bot & \text{if } j \leq d \text{ and } Bin(t)[j] = 1, \\
Bin(t) & \text{if } j = d + 1.
\end{cases}
\]

Define node set \( \text{Nodes}_{(t \to T - 1)} \) to be \( \{\text{sibling}(1, t), \ldots, \text{sibling}(d + 1, t)\} \). For any \( t' > t \), one can check that for any non-\( \bot \) \( z' \in \text{Nodes}_{(t' \to T - 1)} \), there exists a \( z \in \text{Nodes}_{(t \to T - 1)} \) such that \( z \) is an ancestor of \( z' \).

For example, let \( T = 8 \), then time 0 corresponds to leave \((0, 0, 0)^T\) and time 4 corresponds to leave \((1, 0, 0)^T\). \( \text{Nodes}_{(0 \to 7)} = \{(1)^T, (0, 1)^T, (0, 0, 1)^T, (0, 0, 0)^T\} \) and \( \text{Nodes}_{(4 \to 7)} = \{\bot, (1, 1)^T, (1, 0, 1)^T, (1, 0, 0)^T\} \). For any non-\( \bot \) \( z' \in \text{Nodes}_{(4 \to 7)} \), \((1)^T\) is an ancestor of \( z' \).

### 3.1 Description of the Scheme

Our scheme works in the Nakashishi et al.’s (static) model [53]. Let the total number of time periods be \( T = 2^d \) and the maximum expected number of group members be \( N = 2^\ell \). The group public key consists of (i) a Bonsai tree of depth \( \ell + d \) specified by a matrix \( A = [A_0^1|A_0^2|\cdots|A_0^{d+1}] \in \mathbb{Z}_q^{m \times (2\ell+2d+1)^m} \) and a vector \( u \in \mathbb{Z}_q^n \), which is used to generate group signing keys; (ii) a public matrix \( B = \mathbb{Z}_q^{n \times m} \) of the IBE scheme by Gentry et al. [26], which is used to encrypt user identities when signing messages. The secret key of the group manager is the corresponding trapdoor matrix of the Bonsai tree while the secret key of the tracing manager is the corresponding trapdoor matrix of the IBE scheme.

Each user id \( \in \{0, 1 \}^\ell \) is assigned a node id. To enable periodical key updating, each user id is associated with a subtree of depth \( d \). In our scheme, all users are assumed to be valid group members from time 0 to \( T - 1 \). Let \( z \) be a binary string of length \( d \leq d \). Define \( A_{id \parallel z} = [A_0|A_0^1|\cdots|A_0^{d+1}] = \mathbb{Z}_q^{n \times (\ell + d)^m} \). Specifically, the initial group signing key of user id at time \( t \) is \( S_{id \parallel z} \in \text{Nodes}_{(t \to T - 1)} \), which satisfies \( A_{id \parallel z} \cdot S_{id \parallel z} = 0 \mod q \). Thanks to the basis delegation technique [24], users are able to compute the trapdoor matrices for all the descendents of nodes in the set \( \text{Nodes}_{(t \to T - 1)} \) and hence are able to derive all the subsequent signing keys. We remark that since we do not need to delegate basis for leave nodes, we generate short vectors \( v_{id \parallel Bin(t)} \) instead of \( S_{id \parallel Bin(t)} \) in the group signing key for all \( t \).

Once received the group signing key, each user can issue signatures on behalf of the group. When signing a message at time \( t \), user id first generates a one-time signature key pair \((ovk, osk)\), and then encrypts his identity id to ciphertext \( c \) using the IBE scheme with respect to “identity” \( ovk \). Next, he proves in zero-knowledge that: (i) he is a certified group member; (ii) he has done key-evolution honestly; (iii) \( c \) is a correct encryption of id. To prove that facts (i) and (ii) hold, it is sufficient to prove knowledge of a short vector \( v_{id \parallel Bin(t)} \) such that \( A_{id \parallel Bin(t)} \cdot v_{id \parallel Bin(t)} = u \mod q \). The protocol is developed from Langlois et
al.’s technique [37] and Ling et al.’s technique [47], and is repeated \( \kappa = \omega(\log n) \) times to achieve negligible soundness error, and is made non-interactive via Fiat-Shamir transform [24] as a triple \( \Pi \). Finally, the user generates a one-time signature sig on the pair \((c, \Pi)\), and outputs the group signature consisting of \((c, \Pi, \text{sig})\).

To verify a group signature, one simply checks the validity of \( \text{sig} \) and \( \Pi \). In case of dispute, the tracing manager can decrypt the ciphertext with respect to the “identity” \( \text{ovk} \) using his secret key. Details of the scheme are described below.

**KeyGen(\( \lambda, T, N \)):** On inputs security parameter \( \lambda \), total number of time periods \( T = 2^d \) for some \( d \in \mathbb{Z}_+ \) and maximum number of group members \( N = 2^\ell \) for some \( \ell \in \mathbb{Z}_+ \), this algorithm does the following:

1. Choose \( n = \mathcal{O}(\lambda), \) \( q = \widetilde{\mathcal{O}}(n^{2.5}), \) \( m = \mathcal{O}(n \log q) \). Let \( k = \ell + d \) and \( \kappa = \omega(\log n) \).
2. Run \( \text{TrapGen}(n, m, q) \) as described in Lemma 2 to obtain \( A_0 \in \mathbb{Z}_{q}^{n \times m} \) and \( S_0 \in \mathbb{Z}^{m \times m} \).
3. Sample \( u \leftarrow \mathcal{D}_n \), and \( A_i \leftarrow \mathcal{D}_{n \times m} \) for all \( i \in [k] \) and \( b \in \{0, 1\} \).
4. Choose a one-time signature scheme \( OTS = \{OGen, OSign, OVer\} \), and a statistically hiding and computationally binding commitment scheme \( COM \) from [34] that will be used in our zero-knowledge argument system.

5. Let \( \mathcal{H}_0 : \{0, 1\}^* \rightarrow \mathbb{Z}_{q}^{n \times \ell} \) and \( \mathcal{H}_1 : \{0, 1\}^* \rightarrow \{1, 2, 3\}^\kappa \) be collision-resistant hash functions, which will be modelled as random oracles in the security analysis.
6. Choose Gaussian parameter \( s_i \) to be \( \mathcal{O}(\sqrt{n k \log q})^{i-\ell+1} \cdot \omega(\log n)^{i-\ell+1} \) that is used to generate short bases or sample short vectors at level \( i \) for \( i \in \{\ell, \ell + 1, \ldots, k\} \).
7. Choose integer integer bounds \( \beta = \left\lfloor s_k \cdot \log n \right\rfloor \), \( B = \widetilde{\mathcal{O}}(\sqrt{n}) \), and let \( \chi \) be a \( B \)-bounded distribution over \( \mathbb{Z} \).
8. Generate a master key pair \((B, S) \in \mathbb{Z}_{q}^{n \times m} \times \mathbb{Z}^{m \times m} \) for the IBE scheme by Gentry et al. [20] via the \( \text{TrapGen}(n, m, q) \) algorithm.
9. For user \( i \in \{0, 1, \ldots, N - 1\} \), let \( \text{id} = \text{Bin}(i) \in \{0, 1\}^\ell \). Let node id be the identifier of user \( i \). Determine the node set \( \text{Nodes}_{0 \rightarrow T-1} \), if \( z = 1 \), set \( \text{usk}_0[i][z] = 1 \). Otherwise denote \( d_z \) as the length of \( z \) with \( d_z \leq d \), compute the matrix
   \[ A_{\text{id}[z]} = [A_0|A_{1}[\text{id}[z]] | \cdots | A_{\ell}[\text{id}[z]] | A_{\ell+1}[\text{id}[z]] | \cdots | A_{\ell+d_z}[\text{id}[z]]] \in \mathbb{Z}_{q}^{n \times (\ell + d_z + 1)m} \]
   and proceed as follows:
   - If \( z \) is of length \( d \), i.e., \( d_z = d \), it computes a vector \( v_{\text{id}[z]} \in A^n(A_{\text{id}[z]}) \) via
     \[ v_{\text{id}[z]} \leftarrow \text{SampleD(ExtBasis}(S_0, A_{\text{id}[z]}), u, s_k) \].
   - Set \( \text{usk}_0[i][z] = v_{\text{id}[z]} \).
If \( z \) is of length less than \( d \), i.e., \( 1 \leq d_z < d \), it computes a matrix \( S_{id\|z} \in \mathbb{Z}^{(\ell_d + d_z + 1)m \times (\ell_d + d_z + 1)m} \) via

\[
S_{id\|z} \leftarrow \text{RandBasis}(\text{ExtBasis}(S_0, A_{id\|z}), s_{\ell_d + d_z}).
\]

Let \( \text{usk}_0[i][z] = \{ \text{usk}_0[i][z], z \in \text{Nodes}_{(0 \rightarrow T-1)} \} \) be the initial secret key of user \( i \).

Let public parameter be \( \text{pp} \), group public key be \( \text{gpk} \), secret key of \( \text{GM} \) be \( \text{msk} \), secret key of \( \text{TM} \) be \( \text{mosk} \) and initial secret key be \( \text{usk}_0 \), which are defined as follows:

\[
\text{pp} = \{ n, q, m, \ell, d, k, \kappa, \mathcal{OTS}, \mathcal{COM}, \mathcal{H}_0, \mathcal{H}_1, s_\ell, \ldots, s_k, \beta, B \},
\]

\[
\text{gpk} = \{ \text{pp}, A_0, A_1, \ldots, A_k, k, u, B \},
\]

\[
\text{msk} = S_0, \quad \text{mosk} = S,
\]

\[
\text{usk}_0 = \{ \text{usk}_0[0], \text{usk}_0[1], \ldots, \text{usk}_0[N-1] \}.
\]

KeyUpdate(\( \text{gpk}, \text{usk}_t[i], i, t+1 \)): Compute the identifier of user \( i \) as \( \text{id} = \text{Bin}(i) \), parse \( \text{usk}_t[i] = \{ \text{usk}_t[i][z], z \in \text{Nodes}_{(t \rightarrow T-1)} \} \), and determine the node set \( \text{Nodes}_{(t+1 \rightarrow T-1)} \).

For \( z' \in \text{Nodes}_{(t+1 \rightarrow T-1)} \), if \( z' = s_0 \), set \( \text{usk}_{t+1}[i][z'] = s_0 \). Otherwise, there exists exactly one \( z \in \text{Nodes}_{(t \rightarrow T-1)} \) as its prefix, i.e., \( z' = z\|y \) for some suffix \( y \). Consider the following two cases.

1. If \( z' = z \), let \( \text{usk}_{t+1}[i][z'] = \text{usk}_t[i][z] \).
2. If \( z' = z\|y \) for some non-empty \( y \), then \( \text{usk}_t[i][z] = S_{id\|z} \). Consider the following two subcases.

- If \( z' \) is of length \( d \), run

\[
\text{v}_{id\|z'} \leftarrow \text{SampleD}(\text{ExtBasis}(S_{id\|z}, A_{id\|z'}), u, s_k).
\]

and set \( \text{usk}_{t+1}[i][z'] = \text{v}_{id\|z'} \).

- If \( z' \) is of length less than \( d \), run

\[
S_{id\|z'} \leftarrow \text{RandBasis}(\text{ExtBasis}(S_{id\|z}, A_{id\|z'}), s_{\ell_d + d_z}).
\]

and set \( \text{usk}_{t+1}[i][z'] = S_{id\|z'} \).

Output updated key as \( \text{usk}_{t+1}[i] = \{ \text{usk}_{t+1}[i][z'], z' \in \text{Nodes}_{(t+1 \rightarrow T-1)} \} \).

Sign(\( \text{gpk}, \text{usk}_t[i], i, t, M \)): Compute the identifier \( \text{id} = \text{Bin}(i) \). By the structure of the node set \( \text{Nodes}_{(t \rightarrow T-1)} \), there exists some \( z \in \text{Nodes}_{(t \rightarrow T-1)} \) such that \( z = \text{Bin}(t) \) is of length \( d \) and \( \text{usk}_t[i][z] = \text{v}_{id\|z} \).

To sign a message \( M \in \{0, 1\}^* \), the signer then performs the following steps.
1. First, generate a one-time signature key pair \((ovk, osk) \leftarrow \text{OGen}(n)\), and then encrypt id with respect to “identity” \(ovk\) as follows. Let \(G = H_0(ovk) \in \mathbb{Z}_q^{n \times \ell}\). Sample \(s \leftarrow \chi^n\), \(e_1 \leftarrow \chi^m\), \(e_2 \leftarrow \chi^\ell\), and compute ciphertext \((c_1, c_2) \in \mathbb{Z}_q^m \times \mathbb{Z}_q^\ell\) as

\[
(c_1 = B^T \cdot s + e_1, \quad c_2 = G^T \cdot s + e_2 + \lfloor \frac{q}{2} \rfloor \cdot \text{id}).
\]

2. Second, compute the matrix \(A_{id\parallel z}\) and generate a \textit{NIZK}AuK \(\Pi\) to demonstrate the possession of a valid tuple

\[
\xi = (\text{id}, s, e_1, e_2, v_{id\parallel z})
\]

such that

(a) \(A_{id\parallel z} \cdot v_{id\parallel z} = u \mod q\), and \(\|v_{id\parallel z}\|_\infty \leq \beta\).

(b) Equations in [2] hold with \(\|s\|_\infty \leq B\), \(\|e_1\|_\infty \leq B\) and \(\|e_2\|_\infty \leq B\).

This is done by running the argument system described in Section 4.2 with public input

\[
\gamma = (A_0, A_1, \ldots, A_k, u, B, G, c_1, c_2, t)
\]

and witness tuple \(\xi\) as above. The protocol is repeated \(\kappa = \omega(\log n)\) times to obtain negligible soundness error and made non-interactive via the Fiat-Shamir heuristic [24] as a triple \(\Pi = ((\text{CMT}_i)_{i=1}^\kappa, \text{CH}, (\text{RSP}_i)_{i=1}^\kappa)\) with \(\text{CH} = H_1(M, (\text{CMT}_i)_{i=1}^\kappa, c_1, c_2, t)\).

3. Third, compute a one-time signature \(\text{sig} = \text{OSign}(osk; c_1, c_2, \Pi)\) and output the signature as \(\Sigma = (ovk, c_1, c_2, \Pi, \text{sig})\).

\textbf{Verify}(gpk, t, M, \Sigma):\) This algorithm proceeds as follows:

1. Parse \(\Sigma\) as \(\Sigma = (ovk, c_1, c_2, \Pi, \text{sig})\). If \(\text{OVer}(ovk; \text{sig}; c_1, c_2, \Pi) = 0\), then return 0.

2. Parse \(\Pi\) as \(\Pi = ((\text{CMT}_i)_{i=1}^\kappa, (\text{CH}_i)_{i=1}^\kappa, (\text{RSP}_i)_{i=1}^\kappa)\).

If \((\text{CH}_1, \ldots, \text{CH}_\kappa) \neq H_1(M, (\text{CMT}_i)_{i=1}^\kappa, c_1, c_2, t)\), then return 0.

3. For \(i \in [\kappa]\), run the verification step of the protocol from Section 4.2 to check the validity of \(\text{RSP}_i\) with respect to \(\text{CMT}_i\) and \(\text{CH}_i\). If any of the conditions does not hold, then return 0.

4. Return 1.

\textbf{Open}(gpk, mosk, t, M, \Sigma):\) If \(\text{Verify}(gpk, t, M, \Sigma) = 0\), abort. Otherwise, let \(\text{mosk} \in \mathbb{Z}^{m \times m}\) and parse \(\Sigma\) as \(\Sigma = (ovk, c_1, c_2, \Pi, \text{sig})\). Then it decrypts \((c_1, c_2)\) as follows:

1. Compute \(G = H_0(ovk) = [g_1 | \cdots | g_\ell] \in \mathbb{Z}_q^{n \times \ell}\). Then use \(S\) to compute a small norm matrix \(F_{ovk} \in \mathbb{Z}_q^{m \times \ell}\) such that \(B \cdot F_{ovk} = G \mod q\). This is done by computing \(f_i \leftarrow \text{SampleD}(B, S, g_i, s_i)\) for all \(i \in [\ell]\) and let \(F_{ovk} = [f_1 | \cdots | f_\ell]\).

2. Use \(F_{ovk}\) to decrypt \((c_1, c_2)\) by computing

\[
id' = \lfloor \frac{c_2 - F_{ovk}^T \cdot c_1}{q/2} \rfloor \in \{0, 1\}^\ell.
\]
3. Return id’ ∈ \{0, 1\}^ℓ.

Remark 2. Note that Song [59] and Libert and Yung [46] consider a mechanism that deals with time-limited group membership, where each user i is only a valid group member from time \(t_{i,1}\) to \(t_{i,2}\). In our construction, for simplicity, we only consider fixed group membership from time 0 to \(T−1\) as in the work by Nakanishi et al. [53]. However, our construction can directly cater to time-limited group membership as in [59,46] without any extra overhead. This can be done by generating secret key to be \(\{usk_{(t_{i,1}→t_{i,2})}[i][z], z \in \text{Nodes}_{(t_{i,1}→t_{i,2})}\}\) for user i with group membership from \(t_{i,1}\) to \(t_{i,2}\) in the KeyGen algorithm.

3.2 Analysis of the Scheme

Efficiency. We first analyze the complexity of the scheme described in Section 3.1, with respect to security parameter \(\lambda\) and parameters \(\ell = \log N\) and \(d = \log T\). Recall \(k = \ell + d\).

- The group public key \(gpk\) has bit-size \(\tilde{O}(\lambda^2 \cdot k)\).
- The user secret key \(usk_i[i]\) has at most \(d + 1\) trapdoor matrices, and has bit-size \(\tilde{O}(\lambda^2 \cdot k^2d)\).
- The size of signature \(\Sigma\) is dominated by that of the Stern-like NIZK AoK \(\Pi\), which is \(\tilde{O}(|\xi| \cdot \log q) \cdot \omega(\log \lambda)\), where \(|\xi|\) denotes the bit-size of the witness-tuple \(\xi\). Overall, \(\Sigma\) has bit-size \(\tilde{O}(\lambda \cdot k)\).

Correctness. The correctness of the above scheme follows from the following facts: (i) the underlying argument system is perfectly complete; (ii) the underlying encryption scheme obtained by transforming the IBE scheme in [26] via CHK transformation [19] is correct.

Specifically, for an honest user, when he signs a message at time period \(t\), he is able to demonstrate the possession of a valid tuple \(\xi\) of the form (3). Therefore, with probability 1, the resulting signature \(\Pi\) will be accepted by the Verify algorithm, implied by the perfect completeness of the underlying argument system. As for the correctness of the Open algorithm, note that

\[
c_2 - F_{ovk}^T \cdot c_1 = G^T \cdot s + e_2 + \left[\frac{q}{2}\right] \cdot id - F_{ovk}^T \cdot (B^T \cdot s + e_1)
= \left[\frac{q}{2}\right] \cdot id + e_2 - F_{ovk}^T \cdot e_1
\]

where \(\|e_1\| \leq B\), \(\|e_2\|_\infty \leq B\), and \(\|f_i\|_\infty \leq [s_{\ell} \cdot \log m] = \tilde{O}(\sqrt{n \cdot k})\), which is implied by Lemma 1. Recall that \(q = \tilde{O}(n^{2.5})\), \(m = \mathcal{O}(n \log q)\) and \(B = \tilde{O}(\sqrt{n})\). Hence

\[
\|e_2 - F_{ovk}^T \cdot e_1\|_\infty \leq B + m \cdot B \cdot \tilde{O}(\sqrt{n \cdot k}) = \tilde{O}(n^2) \leq \left[\frac{q}{5}\right] = \tilde{O}(n^{2.5}).
\]

With probability 1, the Open algorithm will recover id and correctness of the Open algorithm holds.

Security. In Theorem 2, we prove that our scheme satisfies the security requirements defined in Section 2.2.
Theorem 2. In the random oracle model, the forward-secure group signature described in Section 3 satisfies full anonymity and forward-secure traceability requirements under the LWE and SIS assumptions.

The proof of Theorem 2 is established by Lemma 6 and Lemma 7.

Lemma 6. Suppose that one-time signature scheme OTS is strongly unforgeable. In the random oracle model, the forward-secure group signature scheme described in Section 3 is fully anonymous under the hardness of the LWE_{n,A,X} problem.

Proof. Denote C as the challenger and A as the adversary. Following [48], we prove this lemma using a series of computationally indistinguishable games. The first game Game 0 is the real experiment Exp_{FSGS,Anon}(\lambda,T,N) while the last game is such that the advantage of the adversary is 0.

Game 0: In this game, C runs the experiment Exp_{FSGS,Anon}(\lambda,T,N) faithfully. In the challenge phase, A outputs a message M∗ together with two users 0 ≤ i0, i1 ≤ N - 1 for the targeted time t∗. C responds by sending back a signature Σ∗ = (ovk∗, c1∗, c2∗, Π∗, sig∗) ← Sign(gpk, usk, i0, i1, t∗, M∗) for a random bit b ∈ {0,1}. Then the adversary outputs a bit b′ ∈ {0,1} and this game returns 1 if b′ = b and 0 otherwise. In this experiment, C replies with all random strings for oracles queries of H0, H1.

Game 1: In this game, we modify Game 0 in two aspects: (i) We generate the pair (ovk∗, osk∗) in the very beginning of the experiment; (ii) For the signature opening queries, if A asks for a valid signature of the form Σ = (ovk, c1, c2, Π, sig) such that ovk = ovk∗, then C outputs a random bit and aborts the experiment. Now we argue that the probability that C aborts is negligible and hence Game 0 and Game 1 are computationally indistinguishable. Actually, before the challenged signature is given to A, ovk∗ is independent of A’s view, hence it is negligible that A queries a signature containing ovk∗. Furthermore, after the challenged signature is sent to A, if A queries a new valid signature of the form (ovk∗, c1, c2, Π, sig), then ((c1, c2, Π), sig) is a successful forgery of the OTS scheme, which breaks the strong unforgeability of the OTS scheme. This proves that C aborts with negligible probability. From now on, we assume that A will not query valid signature containing ovk∗.

Game 2: In this game, we change Game 1 in the following ways. First, instead of generating B using the TrapGen algorithm, we generate a uniformly random matrix B∗ over Zq^{n×m}. This change is indistinguishable to A since the matrix B is statistically close to uniform by Lemma 2. Second, we program the random oracle H0 as follows. For query of ovk∗, it generates a uniformly random matrix G∗ ∈ Zq^{n×ℓ}, let H0(ovk∗) = G∗, and return G∗ to A. This change also makes no difference to A since the output of H0 is uniformly random. For query of ovk ≠ ovk∗, we first sample F_{ovk} ← Dq_{n×ℓ}, let H0(ovk) = G = B∗ · F_{ovk} mod q, and return G to A. We then keep a record of (ovk, F_{ovk}, G). G generated in this way is statistically close to uniform by
Lemma 1. Hence, this change does not affect \( A \)'s view non-negligibly. When \( A \) queries the opening oracle a signature \((ovk, c_1, c_2, \Pi, \text{sig})\) with \( ovk \neq ovk^* \), \( C \) can use the recorded \( F_{ovk} \) to decrypt the ciphertext \((c, c_2)\). Hence \( C \) can answer all signature opening queries. It then follows that Game 2 and Game 1 are statistically indistinguishable.

**Game 3:** In this game, we modify Game 2 as follows. Instead of generating a real proof \( \Pi^* \) for the challenged signature, we generate a simulated one without using the witness by programming the random oracle \( H_1 \). Since our argument system is statistically zero-knowledge, the view of adversary \( A \) is statistically indistinguishable between Game 3 and Game 2.

**Game 4:** In this game, we change Game 3 as follows. Instead of computing \((c^*_1, c^*_2) = (B^* \cdot s + e_1, G^* \cdot s + e_2 + \lfloor \frac{q}{2} \rfloor \cdot \text{Bin}(i_b)) \in \mathbb{Z}_q^m \times \mathbb{Z}_q^\ell \) in the challenged phase, where \( s \in \chi^n, e_1 \in \chi^m, e_2 \in \chi^\ell \), we let
\[
(c^*_1, c^*_2) = (z^*_1, z^*_2 + \lfloor \frac{q}{2} \rfloor \cdot \text{Bin}(i_b)),
\]
where \( z^*_1, z^*_2 \) are uniformly random vectors over \( \mathbb{Z}_q^m \) and \( \mathbb{Z}_q^\ell \). We claim that this modification is computationally indistinguishable to the view of the adversary \( A \) assuming the hardness of \( \text{LWE}_{n,q,\chi} \). Indeed, if we let \( D = [B^* | G^*] \in \mathbb{Z}_q^{n \times (m+\ell)}, e = (e_1 | e_2) \in \chi^{m+\ell}, \) and \( z = (z^*_1 | z^*_2) \in \mathbb{Z}_q^{m+\ell} \), then to distinguish Game 3 and Game 4 is to distinguish \((D, D^T \cdot s + e)\) and \((D, z)\). Recall that \( B^* \) and \( G^* \) are uniformly random matrices, so is \( D \).

**Game 5:** In this game, we slightly change Game 4 by substituting \((c^*_1, c^*_2)\) with a new independent and uniform tuple \((z^*_1, z^*_2)\). It is straightforward that Game 5 and Game 4 are statistically indistinguishable. Furthermore, the challenged signature in this game does not depend on the challenged bit \( b \) any more, and hence the advantage of \( A \) in this game is 0.

It then follows that \( \text{Adv}^{\text{Anon}}_{\text{FSGS},A}(\lambda, T, N) \) is negligible in \( \lambda \) because of the indistinguishability of the above games. This concludes the proof. \( \square \)

**Lemma 7.** In the random oracle model, the forward-secure group signature scheme described in Section 3.1 is forward-secure traceable under the hardness of the \( \text{SIS}_{n,\overline{m},q,2\beta} \) problem, where \( \overline{m} = (k+1)m \).

**Proof.** Assume there is a PPT adversary \( A \) attacking the forward-secure traceability of the forward-secure group signature scheme with non-negligible probability, then we construct a new PPT adversary \( B \) attacking the \( \text{SIS}_{n,\overline{m},q,2\beta} \) problem with non-negligible probability.

Given a \( \text{SIS} \) instance \( C \in \mathbb{Z}_q^{n \times \overline{m}} \), the goal of \( B \) is to find a non-zero vector \( v \in \mathbb{Z}_q^{\overline{m}} \) such that \( C \cdot v = 0 \mod q \) and \( \|v\|_{\infty} \leq 2\beta \). Initially, \( B \) lets \( t = 0 \) and the set \( CU \) be empty and then proceeds as follows.
– Parse $C$ as $C = [C_0| C_1| \ldots | C_k]$ for $C_j \in \mathbb{Z}_q^{n \times m}$, $j \in \{0, 1, \ldots, k\}$. It then generates the remaining public parameters as in Section 3.1.

– Sample $z = (z_0||z_1|| \ldots ||z_k) \in \mathbb{Z}^m$, where each $z_i$ is sampled from $D_{\mathbb{Z}^m, s_k}$. If $\|z\|_\infty > \beta$, then repeat the sampling. Compute $u = C \cdot z \mod q$.

– Guess the targeted user $i^* \in \{0, 1, \ldots, N - 1\}$ and targeted forgery time $t^* \in \{0, 1, \ldots, T - 1\}$ uniformly.

– Let $id^* = Bin(i^*)$ and $z^* = Bin(t^*)$. Define $A^\text{id^*}_j$ to be $C_j$ for $j \in [\ell]$ and $A^\text{z^*}_j$ to be $C_{\ell + j}$ for $j \in [d]$.

– Generate $A^\text{1-id^*}_j$ for $j \in [\ell]$ via $(A^\text{1-id^*}_j, S_j) \leftarrow \text{TrapGen}(n, m, q)$ and $A^\text{1-z^*}_j$ for $j \in [d]$ via $(A^\text{1-z^*}_j, S_{\ell + j}) \leftarrow \text{TrapGen}(n, m, q)$.

– Generate a master key pair $(B, S)$ via TrapGen$(n, m, q)$.

$B$ sets $\text{gpk, mosk}$ honestly and sends them to $A$. At the start of each time period $t \in \{0, 1, \ldots, T - 1\}$, $B$ announces the beginning of $t$ to $A$. At current time period $t$, $B$ responds to $A$’s queries as follows.

– When $A$ queries the random oracles $H_0, H_1$, $B$ replies with uniformly random strings and keeps a record of the queries.

– When $A$ queries the secret key of member $i^*$, if $i \in \text{CU} \land t \leq t^*$, $B$ then aborts. Otherwise, for each node $z \in \text{Nodes}_{(t \to T - 1)}$, it first computes the smallest index $d_{z,t}$ such that $1 \leq d_{z,t} < d$ and $z^*[d_{z,t}] \neq z[d_{z,t}]$. Then $B$ computes $\text{usk}_i[i][z]$ via SampleD$(\text{ExtBasis}(S_{\ell + d_{z,t}}, A^\text{id^*}_i), s_k)$ if $z$ is of length $d$ or via RandBasis$(\text{ExtBasis}(S_{\ell + d_{z,t}}, A^\text{id^*}_i), s_{\ell + d_{z,t}})$ if $z$ is of length $d < d_{z,t}$. Finally, $B$ sets $\text{usk}_i[i]$ as in our construction and sends it to $A$. Add $i^*$ to the set CU.

– When $A$ queries the secret key of member $i \neq i^*$, if $i \in \text{CU}$, $B$ aborts. Otherwise, let $id = Bin(i)$ and $\ell_i$ be the smallest index such that $1 \leq \ell_i \leq \ell$ and id[$\ell_i] \neq \text{id^*}[\ell_i]$. Compute $\text{usk}_i[i][z]$ via SampleD$(\text{ExtBasis}(S_{\ell_i}, A^\text{id}_i), u, s_k)$ if $z$ is of length $d$ or via RandBasis$(\text{ExtBasis}(S_{\ell_i}, A^\text{id}_i), s_{\ell + d})$ if $z$ is of length $d < d_{z,t}$ for $z \in \text{Nodes}_{(t \to T - 1)}$. Set $\text{usk}_i[i]$ as in our construction and send it to $A$. Finally, add $i$ to the set CU.

– When $A$ queries a signature on a message for user $i$ with $i \neq i^*$ and $i \notin \text{CU}$ at current time $t$, $B$ responds as in our algorithm $\text{Sign}$ using the corresponding witness. For user $i$ with $i = i^*$ and $i \notin \text{CU}$ at current time $t$, $B$ performs the same as in our algorithm $\text{Sign}$ except that it generates a simulated proof $\Pi'$ by programming the hash oracle $H_1$ and returns $\Sigma = (\text{ovk, c_1, c_2, \Pi', sig})$ to $A$.

We claimed that $A$ cannot distinguish whether it interacts with the real challenger or with $B$. First, group public key gpk given to $A$ is indistinguishable from the real one. This is because the output matrix $A$ of the TrapGen algorithm is statistically close to uniform by Lemma 2 and $u$ is statistically close to a uniform vector over $\mathbb{Z}_q^n$ by Lemma 4. Second, the secret signing key given to $A$ is indistinguishable from the real one due to the fact that the outputs of RandBasis
using two different bases are within statistical distance by Lemma 5. Third, the signature queries make no difference to the view of $A$. This can be implied by the statistical zero-knowledge property of the underlying argument system.

When $A$ halts and outputs a message $M^*$ and a signature $\Sigma^*$ at the targeted time period $t'$ such that $\text{Verify}(\text{gpk}, t', M^*, \Sigma^*) = 1$ and $I^*$ is not obtained by making a signing query at $M^*$, check $t' = t^*$ holds or not. If not, then $B$ aborts. Otherwise, $A$ outputs $(t^*, M^*, \Sigma^*)$. Parse $\Sigma^*$ as

$$(ovk^*, c_1^*, c_2^*, (\text{CMT}^*)_{i=1}^c, (\text{Ch}^*_i)_{i=1}^c, (\text{RSP}^*_i)_{i=1}^c, \text{sig}^*).$$

Run $id' \leftarrow \text{Open}(\text{gpk}, \text{mosk}, t^*, M^*, \Sigma^*)$. If $id' \neq id^*$, indicating that the guess of $i^*$ fails, then $B$ aborts. Otherwise, $B$ makes use of the forgery to solve the SIS problem as follows.

First, $A$ must have queried $H_1$ for the tuple $(M^*, (\text{CMT}^*)_{i=1}^c, c_1^*, c_2^*, t^*)$, since the probability of guessing this value is at most $3^{-\kappa}$, which is negligible by our choice of $\kappa$. Let $(M^*, (\text{CMT}^*)_{i=1}^c, c_1^*, c_2^*, t^*)$ be the $h$-th oracle query and $Q_{H_1}$ be the total oracle queries $A$ has made to $H_1$. Next, $B$ lets $h$ be the targeted forking point and replays $A$ polynomial-number times. For each new run, $B$ starts with the same random tape and random input as in the original run. Further, for the first $h - 1$ queries of $H_1$, $B$ replies with the same random value as in the original run, but from $h$-th query on, $B$ replies with fresh and independent value. Besides, for queries of $H_0$, $B$ always replies as in the original run.

Constructed in this way, $(M^*, (\text{CMT}^*)_{i=1}^c, c_1^*, c_2^*, t^*)$ is always the $h$-th oracle query $A$ made to $H_1$. The improved forking lemma [10] implies that with probability $\geq 1/2$, $B$ obtains 3-fork involving the same tuple $(M^*, (\text{CMT}^*)_{i=1}^c, c_1^*, c_2^*, t^*)$ with pairwise distinct hash values $\text{CH}^{(1)}_h$, $\text{CH}^{(2)}_h$, $\text{CH}^{(3)}_h$ and corresponding valid responses $\text{RSP}^{(1)}_h$, $\text{RSP}^{(2)}_h$, $\text{RSP}^{(3)}_h$. A simple calculation shows that with probability $1 - (\frac{7}{9})^\kappa$, we have $\{\text{CH}^{(1)}_{h,j}, \text{CH}^{(2)}_{h,j}, \text{CH}^{(3)}_{h,j}\} = \{1, 2, 3\}$ for some $j \in [\kappa]$. Therefore, $\text{RSP}^{(1)}_{h,j}$, $\text{RSP}^{(2)}_{h,j}$, $\text{RSP}^{(3)}_{h,j}$ are 3 valid responses for all the challenges $1, 2, 3$ w.r.t. the same commitment $\text{CMT}^*_j$. Since COM is computationally binding, $B$ is able to extract the witness tuple

$$\xi^* = (id, s, e_1, e_2, v_{id\|z})$$

such that $\|v_{id\|z}\|_\infty \leq \beta$, $\|s\|_\infty \leq B$, $\|e_1\|_\infty \leq B$, $\|e_2\|_\infty \leq B$ and

$$A_{id\|z} \cdot v_{id\|z} = u \mod q,$$

$$(c_1^* = B^T \cdot s + e_1, c_2^* = (G^*)^T \cdot s + e_2 + \left\lfloor \frac{z}{q} \right\rfloor \cdot \text{id}) \in \mathbb{Z}_q^m \times \mathbb{Z}_q^t,$$

where $G^* = H_0(ovk^*)$. Conditioned on guessing correctly $i^*$, $t^*$, we have $id = id^*$ and $z = z^*$. Therefore, $A_{id\|z} = C$. Now we have $C \cdot v_{id\|z} = u = C \cdot z \mod q$. We claim that $v_{id\|z} \neq z$ with overwhelming probability. This is because $A$ either queried the secret key of user id at time after $t^*$ or never queried the secret key at all (by successfully attacking forward-secure traceability), then $z$ is not known to $A$. Further, from the view of the adversary, $z$ is from the distribution $D_{A^{\infty}(C), sk}$ and hence has large min-entropy, which are implied by Lemma 11.
Therefore, $\mathbf{v}_{id} \parallel z \neq z$ with overwhelming probability. Hence $x = \mathbf{v}_{id} \parallel z \neq 0$ and $|x| \parallel \leq 2\beta$. This implies that we solve the $\text{SIS}_{n,\pi,q,2\beta}$ problem with non-negligible probability and hence our scheme is forward-secure traceable.

4 The Underlying Zero-Knowledge Argument System

In Section 4.1, we recall the extension, decomposition, and permutation techniques from [17]. Then we describe in Section 4.2 our statistical ZK AoK protocol that will be used in generating group signatures.

4.1 Extension, Decomposition, and Permutation

EXTENSIONS. For $m \in \mathbb{Z}$, let $\mathcal{B}_{3m}$ be the set of all vectors in $\{-1, 0, 1\}^{3m}$ having exactly $m$ coordinates $-1$, $m$ coordinates 1, and $m$ coordinates 0 and $\mathcal{S}_m$ be the set of all permutations on $m$ elements. Let $\oplus$ be the addition operation modulo 2.

Define the following functions

\[ \begin{align*}
\text{ext}_3 &: \{-1, 0, 1\}^m \rightarrow \mathcal{B}_{3m} \text{ that transforms a vector } \mathbf{v} = (v_1, \ldots, v_m) \rightarrow \text{vector } (|\mathbf{v}||(-1)^{m-n_0}|1^{m-n_1}) \text{ for any } \mathbf{v} \in \{-1, 0, 1\},
\end{align*} \]

\[ \begin{align*}
\text{enc}_2 &: \{0,1\}^m \rightarrow \{0,1\}^{2m} \text{ that transforms a vector } \mathbf{v} = (v_1, \ldots, v_m) \rightarrow \text{vector } (v_1, 1-v_1, \ldots, v_m, 1-v_m). 
\end{align*} \]

DECOMPOSITIONS AND PERMUTATIONS. We now recall the integer decomposition technique. For any $B \in \mathbb{Z}^+$, define $p_B = [\log B] + 1$ and the sequence $B_1, \ldots, B_{p_B}$ as

\[ B_j = \left\lfloor \frac{B + 2^{j-1}}{2} \right\rfloor \text{ for each } j \in [p_B]. \]

As observed in [17], it satisfies $\sum_{j=1}^{p_B} B_j = B$ and any integer $v \in [B]$ can be decomposed to $\text{idec}_B(v) = (v^{(1)}, \ldots, v^{(p_B)})$ such that $\sum_{j=1}^{p_B} B_j \cdot v^{(j)} = v$. This decomposition procedure is described in a deterministic manner as follows:

1. $v':=v$
2. For $j=1$ to $p_B$ do:
   (i) If $v' \geq B_j$ then $v^{(j)} := 1$, else $v^{(j)} := 0$;
   (ii) $v':=v'-B_j \cdot v^{(j)}$;
3. Output $\text{idec}_B(v) = (v^{(1)}, \ldots, v^{(p_B)})$.

Next, for any positive integers $m, B$, we define the function $\text{vdec}_{m,B}$ that transforms a vector $\mathbf{w} = (w_1, \ldots, w_m) \rightarrow [\mathcal{B}, B]^m$ to a vector of the following form:

\[ \mathbf{w}' = (\sigma(w_1) \cdot \text{idec}_B(|w_1|)) \cdots (\sigma(w_m) \cdot \text{idec}_B(|w_m|)) \in \{-1, 0, 1\}^{mp_B}, \]

where $\forall j \in [m]: \sigma(w_j) = 0$ if $w_j = 0$; $\sigma(w_j) = -1$ if $w_j < 0$; $\sigma(w_j) = 1$ if $w_j > 0$.

Define the matrix $H_{m,B} = \begin{bmatrix} B_1, \ldots, B_{p_B} \\ \vdots \\ B_1, \ldots, B_{p_B} \end{bmatrix} \in \mathbb{Z}^{m \times p_B}$ and its extension $\hat{H}_{m,B} = [H_{m,B}]^{0 \times 2mmp_B} \in \mathbb{Z}^{m \times 3mp_B}$. Let $\hat{\mathbf{w}} = \text{ext}_3(\mathbf{w}') \in \mathcal{B}_{3mp_B}$. 

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then one can see that \( \widehat{H}_{m,B} \cdot \widehat{w} = w \) and for any \( \psi \in S_{3m,p,B} \), the following equivalence holds:

\[
\widehat{w} \in B_{3m,p,B} \Leftrightarrow \psi(\widehat{w}) \in B_{3m,p,B}.
\]

(4)

Define the following permutation.

– For any \( e = (e_1, \ldots, e_m)^\top \in \{0,1\}^m \), define \( \Pi_{e} : \mathbb{Z}^{2m} \to \mathbb{Z}^{2m} \) that maps a vector \( v = (v_1^0, v_1^1, \ldots, v_m^0, v_m^1)^\top \) to \( (v_1^{e_1}, v_1^{1-e_1}, \ldots, v_m^{e_m}, v_m^{1-e_m})^\top \).

One can see that, for any \( z, e \in \{0,1\}^m \), the following equivalence holds:

\[
v = \text{enc}_2(z) \Leftrightarrow \Pi_{e}(v) = \text{enc}_2(z \oplus e).
\]

(5)

4.2 The Underlying Zero-Knowledge Argument System

We now describe a statistical ZKAoK that will be invoked by the signer when generating group signatures. The protocol is developed from Stern-like techniques proposed by Ling et al. [37] and Langlois et al. [37].

**Public input** \( \gamma \): \( A_0 \in \mathbb{Z}_q^{n \times m} \), \( A_j^b \in \mathbb{Z}_q^{n \times m} \) for \( (b, j) \in \{0,1\} \times [k] \), \( u \in \mathbb{Z}_q^n \), \( B \in \mathbb{Z}_q^{n \times m} \), \( G \in \mathbb{Z}_q^{n \times \ell} \), \( (c_1, c_2) \in \mathbb{Z}_q^{m \times \ell} \), \( t \in \{0,1, \ldots, T - 1\} \).

**Secret input** \( \xi \): \text{id} \in \{0,1\}^T, \text{s} \in \chi^n, e_1 \in \chi^m, e_2 \in \chi^\ell, v_{\text{id}||z} \in \mathbb{Z}^{(t+\ell+1)m} \) with \( z = \text{Bin}(t) \).

**Prover’s goal:**

\[
\begin{cases}
A_{\text{id}||z} \cdot v_{\text{id}||z} = u \mod q, & ||v_{\text{id}||z}||_\infty \leq \beta; \\
c_1 = B^\top \cdot s + e_1 \mod q, & c_2 = G^\top \cdot s + e_2 + \left\lfloor \frac{q}{2} \right\rfloor \cdot \text{id} \mod q; \\
||s||_\infty \leq B, & ||e_1||_\infty \leq B, ||e_2||_\infty \leq B.
\end{cases}
\]

(6)

We first rearrange the above conditions. Let \( A' = [A_0 A_0^0 | A_1 | \cdots | A_0^\ell A_1^\ell] \in \mathbb{Z}_q^{(2\ell+1)m} \), \( A_{\text{id}} = [A_0 A_0^0 | A_1 | \cdots | A_0^\ell A_1^\ell] \in \mathbb{Z}_q^{(t+1)m} \) and \( A'' = A_{t+1}^2 | \cdots | A_{t+1}^\ell \in \mathbb{Z}_q^{dm} \). Then \( A_{\text{id}||z} = [A_{\text{id}} A''] \in \mathbb{Z}_q^{(t+\ell+1)m} \). Let \( v_{\text{id}} = (v_0 \parallel v_1 \parallel \cdots \parallel v_\ell) \), \( w_2 = (v_{\ell+1} \parallel \cdots \parallel v_{\ell+d}) \) with each \( v_i \in \mathbb{Z}^m \). Then \( v_{\text{id}||z} = (v_{\text{id}} \parallel w_2) \). Therefore \( A_{\text{id}||z} \cdot v_{\text{id}||z} = u \mod q \) is equivalent to

\[
A_{\text{id}} \cdot v_{\text{id}} + A'' \cdot w_2 = u \mod q.
\]

(7)

Since \text{id} is part of secret input, \( A_{\text{id}} \) should not be explicitly given. We note that Langlois et al. [37] already addressed this problem. The idea is as follows: they first added \( \ell \) suitable zero-blocks of size \( m \) to vector \( v_{\text{id}} \) and then obtained the extended vector \( w_1 = (v_0 \parallel v_1 \parallel \cdots \parallel v_\ell) \) with each \( v_i \in \mathbb{Z}^{(2\ell+1)m} \), where the added zero-blocks are \( v_{1-\text{id}||[\ell]} \), \( \ldots, v_{\ell-\text{id}||[\ell]} \) and \( v_{\ell||[\ell]} = v_i, \forall i \in [\ell] \). Now one can check that equation (7) is equivalent to

\[
A' \cdot w_1 + A'' \cdot w_2 = u \mod q.
\]

(8)
Let $B' = \begin{bmatrix} B^\top & \mathbf{I}_m & 0^{m \times \ell} \\ \mathbf{G}^\top & 0_{\times m} & \mathbf{I}_\ell \end{bmatrix}$, $B'' = \begin{bmatrix} 0^{m \times \ell} \\ \mathbf{g}/2 \mathbf{I}_3 \end{bmatrix}$, and $\mathbf{w}_3 = (s|\mathbf{e}_1|\mathbf{e}_2) \in \mathbb{Z}^{n+m+\ell}$. Then one can check that $c_1 = B^\top \cdot s + \mathbf{e}_1 \text{ mod } q$, $c_2 = \mathbf{G}^\top \cdot s + \mathbf{e}_2 + \left[ \frac{q}{2} \right] \cdot \mathbf{id} \text{ mod } q$ is equivalent to $B' \cdot \mathbf{w}_3 + B'' \cdot \mathbf{id} = (c_1|c_2) \text{ mod } q$. 

Using basic algebra, we can transform equations (8) and (9) into one equation of the following form:

$$M_0 \cdot \mathbf{w}_0 = \mathbf{u}_0 \mod q,$$

where $M_0, \mathbf{u}_0$ are built from $A', A'', B', B''$ and $\mathbf{u}, (c_1|c_2)$, respectively, and $\mathbf{w}_0 = (\mathbf{w}_1||\mathbf{w}_2||\mathbf{w}_3||\mathbf{id})$.

Now we can use the decomposition and extension techniques described in Section 4.1 to handle our secret vectors. Let $L_1 = 3(2\ell+1)m\beta$, $L_2 = 3dmp\beta$, $L_3 = 3(n+m+\ell)p\beta$, and $L = L_1 + L_2 + L_3 + 2\ell$. We transform our secret vector $\mathbf{w}_0$ to vector $\mathbf{w} = (\mathbf{w}_1||\mathbf{w}_2||\mathbf{w}_3||\mathbf{id}) \in \{-1,0,1\}^L$ of the following form:

- $\tilde{\mathbf{w}}_1 = (\mathbf{v}_0||\mathbf{v}_1||\mathbf{v}_2||\cdots||\mathbf{v}_\ell) \in \{-1,0,1\}^{L_1}$ with $\mathbf{v}_0 = \text{ext}_3(\text{vdec}_{m,\beta}(\mathbf{v}_0)) \in \mathcal{B}_{3mp\beta}$, $\mathbf{v}_i = \mathbf{0}^{3mp\beta}$ and $\mathbf{v}_i^\beta = \text{ext}_3(\text{vdec}_{m,\beta}(\mathbf{v}_i^\beta)) \in \mathcal{B}_{3mp\beta}$;
- $\tilde{\mathbf{w}}_2 = \text{ext}_3(\text{vdec}_{d,\beta}(\mathbf{w}_2)) \in \mathcal{B}_{3dmp\beta}$;
- $\tilde{\mathbf{w}}_3 = \text{ext}_3(\text{vdec}_{n+m+\ell,\beta}(\mathbf{w}_3)) \in \mathcal{B}_{3(n+m+\ell)p\beta}$;
- $\mathbf{id} = \text{enc}_2(\mathbf{id}) \in \{0,1\}^{2\ell}$.

Using basic algebra, we can form public matrix $M$ such that $M \cdot \mathbf{w} = M_0 \cdot \mathbf{w}_0 = \mathbf{u}_0 \mod q$.

Up to this point, we have transformed the considered relations into equation of the desired form $M \cdot \mathbf{w} = \mathbf{u} \mod q$. We now specify the set VALID that contains the secret vector $\mathbf{w}$, the set $\mathcal{S}$ and permutations $\{\Gamma_\phi : \phi \in \mathcal{S}\}$ such that the conditions in (11) hold.

Define VALID to be the set of vectors of the form $\mathbf{z} = (\mathbf{z}_1||\mathbf{z}_2||\mathbf{z}_3||\mathbf{z}_4) \in \{-1,0,1\}^L$ such that there exists $x \in \{0,1\}^\ell$

- $\mathbf{z}_1 = (y_0||y_1^0||y_1^1||\cdots||y_1^\ell) \in \{-1,0,1\}^{3(2\ell+1)m\beta}$ with $y_0 \in \mathcal{B}_{3mp\beta}$ and for each $i \in [\ell]$, $y_i^{\ell-x[i]} = \mathbf{0}^{3mp\beta}$, $y_i^{x[i]} \in \mathcal{B}_{3mp\beta}$;
- $\mathbf{z}_2 \in \mathcal{B}_{3dmp\beta}$ and $\mathbf{z}_3 \in \mathcal{B}_{3(n+m+\ell)p\beta}$;
- $\mathbf{z}_4 = \text{enc}_2(\mathbf{x}) \in \{0,1\}^{2\ell}$.

Clearly, our vector $\mathbf{w}$ belongs to the tailored set VALID.

Now, let $\mathcal{S} = (\mathcal{S}_{3mp\beta})^{2\ell+1} \times \mathcal{S}_{3dmp\beta} \times \mathcal{S}_{3(n+m+\ell)p\beta} \times \{0,1\}^\ell$. For any

$$\phi = (\psi_0, \psi_1^0, \psi_1^1, \psi_2^0, \psi_2^1, \eta_2, \eta_3, \mathbf{e}) \in \mathcal{S}, \mathbf{e} = (e_1, \ldots, e_\ell)^\top,$$

define the permutation $\Gamma_\phi : \mathbb{Z}^L \rightarrow \mathbb{Z}^L$ as follows. When applied to a vector $\mathbf{z} = (y_0||y_1^0||y_1^1||\cdots||y_\ell^0||y_\ell^1||\mathbf{z}_2||\mathbf{z}_3||\mathbf{z}_4) \in \mathbb{Z}^L$. 

25
where the first $2\ell + 1$ blocks are of size $3m p_3$ and the last three blocks are of size $3d m p_3$, $3(n + m + \ell) p_B$ and $2\ell$, respectively; it transforms $z$ to vector $\Gamma_\phi(z)$ of the following form:

$$(\psi(y_0)\|\psi^e_1(y_1)\|\psi^{1-e_1}_{e_1}(y_1^{1-e_1})\|\cdots\|\psi^e_\ell(y_\ell)\|\psi^{1-e_\ell}_{e_\ell}(y_\ell^{1-e_\ell})\|\eta_2(z_2)\|\eta_3(z_3)\|\Pi_e(z_4)).$$

Based on the equivalences observed in (4) and (5), it can be checked that if $z \in \text{VALID}$ for some $x \in \{0, 1\}^\ell$, then $\Gamma_\phi(z) \in \text{VALID}$ for some $x \oplus e \in \{0, 1\}^\ell$. In other words, the conditions in (1) hold, and therefore, we can run the interactive protocol described in Fig. 3 and obtains a statistical ZKAoK protocol.

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A Proof of Theorem\textsuperscript{1}

We provide the proof of Theorem\textsuperscript{1} as appeared in \textsuperscript{RS}. We first restate the theorem.

**Theorem 3.** Assume that COM is a statistically hiding and computationally binding string commitment scheme. Then, the protocol in Fig. 3 is a statistical ZK\textsuperscript{AOK} with perfect completeness, soundness error $2/3$, and communication cost $O(L \log q)$. In particular:

- There exists a polynomial-time simulator that, on input $(M, u)$, outputs an accepted transcript statistically close to that produced by the real prover.
- There exists a polynomial-time knowledge extractor that, on input a commitment $C_{MT}$ and 3 valid responses $(R_{SP_1}, R_{SP_2}, R_{SP_3})$ to all 3 possible values of the challenge $Ch$, outputs $w' \in \text{VALID}$ such that $M \cdot w' = u \mod q$.

**Proof.** It can be checked that the protocol has perfect completeness: If an honest prover follows the protocol, then he always gets accepted by the verifier. It is also easy to see that the communication cost is bounded by $O(L \log q)$.

We now prove that the protocol is a statistical zero-knowledge argument of knowledge.

**Zero-Knowledge Property.** We construct a PPT simulator $SIM$ interacting with a (possibly dishonest) verifier $\hat{V}$, such that, given only the public input, $SIM$ outputs with probability negligibly close to $2/3$ a simulated transcript that is statistically close to the one produced by the honest prover in the real interaction.
The simulator first chooses a random $\overline{Ch} \in \{1, 2, 3\}$ as a prediction of the challenge value that $\hat{V}$ will not choose.

**Case $\overline{Ch} = 1$:** Using basic linear algebra over $\mathbb{Z}_q$, $\text{SIM}$ computes a vector $w' \in \mathbb{Z}_q^L$ such that $M \cdot w' = u \mod q$. Next, it samples $r_w \overset{\$}{\leftarrow} \mathbb{Z}_q^L$, $\phi \overset{\$}{\leftarrow} S$, and randomness $\rho_1, \rho_2, \rho_3$ for $\text{COM}$. Then, it sends the commitment $CMT = (C_1', C_2', C_3')$ to $\hat{V}$, where

$$
\begin{align*}
C_1' &= \text{COM}(\phi, M \cdot r_w; \rho_1), \\
C_2' &= \text{COM}(\Gamma_\phi(r_w); \rho_2), \\
C_3' &= \text{COM}(\Gamma_\phi(w' + r_w); \rho_3).
\end{align*}
$$

Receiving a challenge $Ch$ from $\hat{V}$, the simulator responds as follows:

- If $Ch = 1$: Output $\perp$ and abort.
- If $Ch = 2$: Send $\text{RSP} = (\phi, w' + r_w, \rho_1, \rho_3)$.
- If $Ch = 3$: Send $\text{RSP} = (\phi, r_w, \rho_1, \rho_2)$.

**Case $\overline{Ch} = 2$:** $\text{SIM}$ samples $w' \overset{\$}{\leftarrow} \text{VALID}$, $r_w \overset{\$}{\leftarrow} \mathbb{Z}_q^L$, $\phi \overset{\$}{\leftarrow} S$, and randomness $\rho_1, \rho_2, \rho_3$ for $\text{COM}$. Then it sends the commitment $CMT = (C_1', C_2', C_3')$ to $\hat{V}$, where

$$
\begin{align*}
C_1' &= \text{COM}(\phi, M \cdot r_w; \rho_1), \\
C_2' &= \text{COM}(\Gamma_\phi(r_w); \rho_2), \\
C_3' &= \text{COM}(\Gamma_\phi(w' + r_w); \rho_3).
\end{align*}
$$

Receiving a challenge $Ch$ from $\hat{V}$, the simulator responds as follows:

- If $Ch = 1$: Send $\text{RSP} = (\Gamma_\phi(w'), \Gamma_\phi(r_w), \rho_2, \rho_3)$.
- If $Ch = 2$: Output $\perp$ and abort.
- If $Ch = 3$: Send $\text{RSP} = (\phi, r_w, \rho_1, \rho_2)$.

**Case $\overline{Ch} = 3$:** $\text{SIM}$ samples $w' \overset{\$}{\leftarrow} \text{VALID}$, $r_w \overset{\$}{\leftarrow} \mathbb{Z}_q^L$, $\phi \overset{\$}{\leftarrow} S$, and randomness $\rho_1, \rho_2, \rho_3$ for $\text{COM}$. Then it sends the commitment $CMT = (C_1', C_2', C_3')$ to $\hat{V}$, where $C_2' = \text{COM}(\Gamma_\phi(r_w); \rho_2)$, $C_3' = \text{COM}(\Gamma_\phi(w' + r_w); \rho_3)$ as in the previous two cases, while

$$
C_1' = \text{COM}(\phi, M \cdot (w' + r_w) - u; \rho_1).
$$

Receiving a challenge $Ch$ from $\hat{V}$, it responds as follows:

- If $Ch = 1$: Send $\text{RSP}$ computed as in the case $(\overline{Ch} = 2, Ch = 1)$.
- If $Ch = 2$: Send $\text{RSP}$ computed as in the case $(\overline{Ch} = 1, Ch = 2)$.
- If $Ch = 3$: Output $\perp$ and abort.

We observe that, in every case we have considered above, since $\text{COM}$ is statistically hiding, the distribution of the commitment $CMT$ and the distribution of the challenge $Ch$ from $\hat{V}$ are statistically close to those in the real interaction.
Hence, the probability that the simulator outputs ⊥ is negligibly close to 1/3. Moreover, one can check that whenever the simulator does not halt, it will provide an accepted transcript, the distribution of which is statistically close to that of the prover in the real interaction. In other words, we have constructed a simulator that can successfully impersonate the honest prover with probability negligibly close to 2/3.

**Argument of Knowledge.** Suppose that $RSP_1 = (t_w, t_r, \rho_2, \rho_3), RSP_2 = (\phi_2, w_2, \rho_1, \rho_3), RSP_3 = (\phi_3, w_3, \rho_1, \rho_2)$ are 3 valid responses to the same commitment $CMT = (C_1, C_2, C_3)$, with respect to all 3 possible values of the challenge. The validity of these responses implies that:

$$\begin{align*}
\text{if } t_w \in \text{VALID}; \\
C_1 &= \text{COM}(\phi_2, M \cdot w_2 - u \mod q; \rho_1) = \text{COM}(\phi_3, M \cdot w_3; \rho_1); \\
C_2 &= \text{COM}(t_r; \rho_2) = \text{COM}(\Gamma_{\phi_3}(w_3); \rho_2); \\
C_3 &= \text{COM}(t_w + t_r \mod q; \rho_3) = \text{COM}(\Gamma_{\phi_2}(w_2); \rho_3).
\end{align*}$$

Since $\text{COM}$ is computationally binding, we can deduce that

$$\begin{align*}
\text{if } t_w \in \text{VALID}; & \quad \phi_2 = \phi_3; \\
& \quad t_r = \Gamma_{\phi_3}(w_3); \\
& \quad t_w + t_r = \Gamma_{\phi_2}(w_2) \mod q; \\
& \quad M \cdot w_2 - u = M \cdot w_3 \mod q.
\end{align*}$$

(10)

Since $t_w \in \text{VALID}$, if we let $w' = (\Gamma_{\phi_2})^{-1}(t_w)$, then $w' \in \text{VALID}$. Furthermore, we have

$$\Gamma_{\phi_2}(w') + \Gamma_{\phi_2}(w_3) = \Gamma_{\phi_2}(w_2) \mod q,$$

which implies that $w' + w_3 = w_2 \mod q$, and that $M \cdot w' + M \cdot w_3 = M \cdot w_2 \mod q$.

As a result, we have $M \cdot w' = u \mod q$. This concludes the proof. \qed

### B Some Remarks on [33] (ePrint 2017/1128)

In [33], Kansal, Dutta, and Mukhopadhyay proposed a forward-secure group signature scheme from lattices in the model of Libert and Yung [46]. Unfortunately, it can be observed that their proposed scheme does not satisfy the correctness and security requirements.

The version of the scheme posted on 27-Nov-2017 15:26:21 UTC contains the following shortcomings.

- The scheme does not satisfy the correctness requirement. The opening algorithm, on input a signature generated by user $i$, does not output $i$. In fact, the transcript for user $i$ stored by the group manager is transcript$_i = (V_i^{(0)}, i, u v_k[i], \sigma_i, [t_1, t_2])$. However, when signing messages at time $t_1 + j$ with $0 < j \leq t_2 - t_1$, the user $i$ encrypts $V_i^{(j)}$, which is never seen by the group manager and which is unrelated to $V_i^{(0)}$. Hence, the decryption procedure can only recovers $V_i^{(j)}$ and no user is being traced in this case. (Readers are referred to Page 22 (Join algorithm) and Page 25 (Sign algorithm) in [33] for more details.)
The scheme does not satisfy the anonymity requirement. The signature generated by user $i$ at time period $t_j$ contains matrix $C^{(j)}_i$, which is part of the updated certificate and which should be kept secret. Therefore, two signatures generated by the same user at the same period can easily be linked.

(Readers are referred to Page 24 (Update algorithm) and Page 27 (equation (13)) in [33] for more details.)

We note that in an updated version of the scheme, posted on 18-Jan-2018 17:33:37 UTC, the signature does not contain matrix $C^{(j)}_i$, but the validity of the signature now cannot be publicly verified. That is because, in order to verify the underlying zero-knowledge argument system of Section 5 (Page 27), one needs to be given matrix $C^{(j)}_{id}$ that encodes the secret identity of the signer and that is not publicly known. In other words, this updated scheme also does not work.