A $\sqrt{n}$ ESTIMATE FOR MEASURES OF HYPERPLANE SECTIONS OF CONVEX BODIES

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ABSTRACT. The hyperplane (or slicing) problem asks whether there exists an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension. The problem is still open, with the best-to-date estimate $C \sim n^{1/4}$ established by Klartag, who slightly improved the previous estimate of Bourgain. It is much easier to get a weaker estimate with $C = \sqrt{n}$.

In this note we show that the $\sqrt{n}$ estimate holds for arbitrary measure in place of volume. Namely, if $L$ is an origin-symmetric convex body in $\mathbb{R}^n$ and $\mu$ is a measure with non-negative even continuous density on $L$, then

$$\mu(L) \leq \sqrt{n} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n},$$

where $c_n = |B_2^n|^{\frac{n-1}{n}} / |B_2^{n-1}| < 1$, and $B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$. We deduce this inequality from a stability result for intersection bodies.

1. INTRODUCTION

The hyperplane (or slicing) problem [Bo1, Bo2, Ba, MP] asks whether there exists an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|,$$  \hspace{1cm} (1)

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension. The problem is still open, with the best-to-date estimate $C \sim n^{1/4}$ established by Klartag [Kl], who slightly improved the previous estimate of Bourgain [Bo3]. We refer the reader to [BGVV] for the history and current state of the problem.

In the case where $K$ is an intersection body (see definition and properties below), the inequality (1) can be proved with the best possible
constant (\cite[p. 374]{G2}):
\[ |K|^\frac{n-1}{n} \leq \frac{|B_2^n|}{|B_2^{n-1}|} \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|, \]  
with equality when \( K = B_2^n \) is the unit Euclidean ball. Here \( |B_2^n| = \pi^{n/2}/\Gamma(1+n/2) \) is the volume of \( B_2^n \). Throughout the paper, we denote the constant in (2) by
\[ c_n = \frac{|B_2^n|^{\frac{n-1}{n}}}{|B_2^{n-1}|}. \]
Note that \( c_n < 1 \) for every \( n \in \mathbb{N} \); this is an easy consequence of the log-convexity of the \( \Gamma \)-function.

It was proved in \cite{K3} that inequality (1) holds for intersection bodies with arbitrary measure in place of volume. Let \( f \) be an even continuous non-negative function on \( \mathbb{R}^n \), and denote by \( \mu \) the measure on \( \mathbb{R}^n \) with density \( f \). For every closed bounded set \( B \subset \mathbb{R}^n \) define
\[ \mu(B) = \int_B f(x) \, dx. \]
Suppose that \( K \) is an intersection body in \( \mathbb{R}^n \). Then, as proved in \cite[Theorem 1]{K3} (see also a remark at the end of the paper \cite{K3}),
\[ \mu(K) \leq \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(K \cap \xi^\perp) |K|^{1/n}. \]  
The constant in the latter inequality is the best possible.

This note was motivated by a question of whether one can remove the assumption that \( K \) is an intersection body and prove the inequality (3) for all origin-symmetric convex bodies, perhaps at the expense of a greater constant in the right-hand side. One would like this extra constant to be independent of the body or measure. In this note we prove the following inequality.

**Theorem 1.** Let \( L \) be an origin-symmetric convex body in \( \mathbb{R}^n \), and let \( \mu \) be a measure with even continuous non-negative density on \( L \). Then
\[ \mu(L) \leq \sqrt{n} \frac{n}{n-1} c_n \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n}. \]  
In the case of volume, the estimate (1) with \( C = \sqrt{n} \) can be proved relatively easy (see \cite[p. 96]{MP} or \cite[Theorem 8.2.13]{G2}), and it is not optimal, as mentioned above. The author does not know whether the estimate (4) is optimal for arbitrary measures.
2. Proof of Theorem 1

We need several definitions and facts. A closed bounded set $K$ in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the *Minkowski functional* of $K$ defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body $K$ is defined by

$$\rho_K(x) = \|x\|^{-1}_K, \quad x \in \mathbb{R}^n.$$  

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of $K$ in the direction of $x$.

If $\mu$ is a measure on $K$ with even continuous density $f$, then

$$\mu(K) = \int_K f(x) \, dx = \int_{S^{n-1}} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r\theta) \, dr \right) d\theta. \quad (5)$$

Putting $f = 1$, one gets

$$|K| = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(\theta) \, d\theta = \frac{1}{n} \int_{S^{n-1}} \|\theta\|^n_K \, d\theta. \quad (6)$$

The *spherical Radon transform* $R : C(S^{n-1}) \mapsto C(S^{n-1})$ is a linear operator defined by

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) \, dx, \quad \xi \in S^{n-1}$$

for every function $f \in C(S^{n-1})$.

The polar formulas (5) and (6), applied to a hyperplane section of $K$, express volume of such a section in terms of the spherical Radon transform:

$$\mu(K \cap \xi^\perp) = \int_{K \cap \xi^\perp} f = \int_{S^{n-1} \cap \xi^\perp} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r\theta) \, dr \right) d\theta$$

$$= R \left( \int_0^{\|\theta\|_K^{-1}} r^{n-2} f(r \cdot) \, dr \right) (\xi). \quad (7)$$

and

$$|K \cap \xi^\perp| = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_K^{-n+1} d\theta = \frac{1}{n-1} R(\| \cdot \|_K^{-n+1})(\xi). \quad (8)$$
The spherical Radon transform is self-dual (see [Gr, Lemma 1.3.3]), namely, for any functions \( f, g \in C(S^{n-1}) \)
\[
\int_{S^{n-1}} R_f(\xi) \, g(\xi) \, d\xi = \int_{S^{n-1}} f(\xi) \, R_g(\xi) \, d\xi.
\] (9)

Using self-duality, one can extend the spherical Radon transform to measures. Let \( \mu \) be a finite Borel measure on \( S^{n-1} \). We define the spherical Radon transform of \( \mu \) as a functional \( R\mu \) on the space \( C(S^{n-1}) \) acting by
\[
(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) \, d\mu(x).
\]

By Riesz’s characterization of continuous linear functionals on the space \( C(S^{n-1}) \), \( R\mu \) is also a finite Borel measure on \( S^{n-1} \). If \( \mu \) has continuous density \( g \), then by (9) the Radon transform of \( \mu \) has density \( Rg \).

The class of intersection bodies was introduced by Lutwak [L]. Let \( K, L \) be origin-symmetric star bodies in \( \mathbb{R}^n \). We say that \( K \) is the intersection body of \( L \) if the radius of \( K \) in every direction is equal to the \((n-1)\)-dimensional volume of the section of \( L \) by the central hyperplane orthogonal to this direction, i.e. for every \( \xi \in S^{n-1} \),
\[
\rho_K(\xi) = \|\xi\|_{K}^{-1} = |L \cap \xi^\perp|.
\] (10)

All bodies \( K \) that appear as intersection bodies of different star bodies form the class of intersection bodies of star bodies.

Note that the right-hand side of (10) can be written in terms of the spherical Radon transform using (8):
\[
\|\xi\|_{K}^{-1} = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_{L}^{-n+1} d\theta = \frac{1}{n-1} R(\|\cdot\|_{L}^{-n+1})(\xi).
\]

It means that a star body \( K \) is the intersection body of a star body if and only if the function \( \|\cdot\|_{K}^{-1} \) is the spherical Radon transform of a continuous positive function on \( S^{n-1} \). This allows to introduce a more general class of bodies. A star body \( K \) in \( \mathbb{R}^n \) is called an intersection body if there exists a finite Borel measure \( \mu \) on the sphere \( S^{n-1} \) so that \( \|\cdot\|_{K}^{-1} = R\mu \) as functionals on \( C(S^{n-1}) \), i.e. for every continuous function \( f \) on \( S^{n-1} \),
\[
\int_{S^{n-1}} \|x\|_{K}^{-1} f(x) \, dx = \int_{S^{n-1}} Rf(x) \, d\mu(x).
\] (11)

We refer the reader to the books [G2, K2] for more information about intersection bodies and their applications. Let us just say that intersection bodies played a crucial role in the solution of the Busemann-Petty problem. The class of intersection bodies is rather rich. For
example, every origin-symmetric convex body in $\mathbb{R}^3$ and $\mathbb{R}^4$ is an intersection body [G1, Z]. The unit ball of any finite dimensional subspace of $L_p$, $0 < p \leq 2$ is an intersection body, in particular every polar projection body is an intersection body [K1].

We deduce Theorem 1 from the following stability result for intersection bodies.

**Theorem 2.** Let $K$ be an intersection body in $\mathbb{R}^n$, let $f$ be an even continuous function on $K$, $f \geq 1$ everywhere on $K$, and let $\varepsilon > 0$.

Suppose that

$$\int_{K \cap \xi^\perp} f \leq |K \cap \xi^\perp| + \varepsilon, \quad \forall \xi \in S^{n-1}, \quad (12)$$

then

$$\int_K f \leq |K| + \frac{n}{n-1} c_n |K|^{1/n} \varepsilon. \quad (13)$$

**Proof:** First, we use the polar formulas (7) and (8) to write the condition (12) in terms of the spherical Radon transform:

$$R\left( \int_0^{\|\cdot\|_K} r^{n-2} f(r \cdot) \, dr \right)(\xi) \leq \frac{1}{n-1} R(\|\cdot\|_K^{n+1})(\xi) + \varepsilon.$$

Let $\mu$ be the measure on $S^{n-1}$ corresponding to $K$ by the definition of an intersection body (11). Integrating both sides of the latter inequality over $S^{n-1}$ with the measure $\mu$ and using (11), we get

$$\int_{S^{n-1}} \|\theta\|^{-1}_K \left( \int_0^{\|\theta\|^{-1}_K} r^{n-2} f(r \theta) \, dr \right) d\theta$$

$$\leq \frac{1}{n-1} \int_{S^{n-1}} \|\theta\|^{-n}_K \, d\theta + \varepsilon \int_{S^{n-1}} d\mu(\xi). \quad (14)$$

Recall (5), (6) and the assumption that $f \geq 1$. We write the integral in the left-hand side of (14) as

$$\int_{S^{n-1}} \|\theta\|^{-1}_K \left( \int_0^{\|\theta\|^{-1}_K} r^{n-2} f(r \theta) \, dr \right) d\theta$$

$$= \int_{S^{n-1}} \left( \int_0^{\|\theta\|^{-1}_K} r^{n-1} f(r \theta) \, dr \right) d\theta$$

$$+ \int_{S^{n-1}} \left( \int_0^{\|\theta\|^{-1}_K} (\|\theta\|^{-1}_K - r) r^{n-2} f(r \theta) \, dr \right) d\theta$$
\[ \geq \int_K f + \int_{S^{n-1}} \left( \int_0^{\|\theta\|_{K}} (\|\theta\|_{K}^{-1} - r) r^{n-2} \, dr \right) d\theta \]

\[ = \int_K f + \frac{1}{(n-1)n} \int_{S^{n-1}} \|\theta\|_{K}^{-n} \, d\theta = \int_K f + \frac{1}{n-1} |K|. \] (15)

Let us estimate the second term in the right-hand side of (14) by adding the Radon transform of the unit constant function under the integral \((R1(\xi) = |S^{n-2}| \text{ for every } \xi \in S^{n-1})\), using again the fact that \(\| \cdot \|_{K}^{-1} = R\mu\) and then applying Hölder’s inequality:

\[ \epsilon \int_{S^{n-1}} R\mu(\xi) = \epsilon |S^{n-2}| \int_{S^{n-1}} R1(\xi) \, d\mu(\xi) \]

\[ = \epsilon |S^{n-2}| \int_{S^{n-1}} \|\theta\|_{K}^{-1} \, d\theta \]

\[ \leq \epsilon \frac{|S^{n-2}|}{|S^{n-2}|} |S^{n-1}|^{\frac{n}{n-1}} \left( \int_{S^{n-1}} \|\theta\|_{K}^{-n} \, d\theta \right)^{\frac{1}{n}} \]

\[ = \epsilon \frac{|S^{n-2}|}{|S^{n-2}|} |S^{n-1}|^{\frac{n}{n-1}} n^{1/n} |K|^{1/n} = \frac{n}{n-1} c_n |K|^{1/n} \epsilon. \] (16)

In the last step we used \(|S^{n-1}| = n|B_2^n|, |S^{n-2}| = (n-1)|B_2^{n-1}|.\) Combining (14), (15), (16) we get

\[ \int_K f + \frac{1}{n-1} |K| \leq \frac{n}{n-1} |K| + \frac{n}{n-1} c_n |K|^{1/n} \epsilon. \] \(\Box\)

Now we prove our main result.

**Proof of Theorem 1:** Let \(g\) be the density of the measure \(\mu\), so \(g\) is an even non-negative continuous function on \(L\). By John’s theorem [J], there exists an origin-symmetric ellipsoid \(K\) such that

\[ \frac{1}{\sqrt{n}} K \subset L \subset K. \]

The ellipsoid \(K\) is an intersection body (see for example [G2, Corollary 8.1.7]). Let \(f = \chi_K + g\chi_L\), where \(\chi_K, \chi_L\) are the indicator functions of \(K\) and \(L\). Clearly, \(f \geq 1\) everywhere on \(K\). Put

\[ \epsilon = \max_{\xi \in S^{n-1}} \left( \int_{K \cap \xi \perp} f - |K \cap \xi \perp| \right) = \max_{\xi \in S^{n-1}} \int_{L \cap \xi \perp} g \]
and apply Theorem 2 to $f, K, \varepsilon$ (the function $f$ is not necessarily continuous on $K$, but the result holds by a simple approximation argument). We get

$$\mu(L) = \int_L g = \int_K f - |K|$$

$$\leq \frac{n}{n-1} c_n |K|^{1/n} \max_{\xi \in S^{n-1}} \int_{L \cap \xi^\perp} g$$

$$\leq \sqrt{n} \frac{n}{n-1} c_n |L|^{1/n} \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp),$$

because $|K|^{1/n} \leq \sqrt{n} |L|^{1/n}$.

\[\square\]

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