Harmonic Bayesian prediction under $\alpha$-divergence

Yuzo Maruyama∗ and Toshio Ohnishi†
University of Tokyo∗ and Kyushu University†
e-mail: maruyama@csis.u-tokyo.ac.jp; ohnishi@econ.kyushu-u.ac.jp

Abstract: We investigate Bayesian shrinkage methods for constructing predictive distributions. We consider the multivariate normal model with a known covariance matrix and show that the Bayesian predictive density with respect to Stein’s harmonic prior dominates the best invariant Bayesian predictive density, when the dimension is greater than three. Alpha-divergence from the true distribution to a predictive distribution is adopted as a loss function.

AMS 2000 subject classifications: Primary 62C20; secondary 62J07.
Keywords and phrases: harmonic prior, minimaxity, Bayesian predictive density.

1. Introduction

Let $X \sim N_d(\mu, v_x I)$ and $Y \sim N_d(\mu, v_y I)$ be independent $d$-dimensional multivariate normal vectors with common unknown mean $\mu$. We assume that $d \geq 3$ and that $v_x$ and $v_y$ are known. Let $\phi(\cdot, \sigma^2)$ be the probability density of $N_d(0, \sigma^2 I)$. Then the probability density of $X$ and that of $Y$ are $\phi(x - \mu, v_x)$ and $\phi(y - \mu, v_y)$, respectively.

Based on only observing $X = x$, we consider the problem of obtaining a predictive density $\hat{p}(y | x)$ for $Y$ that is close to the true density $\phi(y - \mu, v_y)$. In most earlier papers on such prediction problems, a predictive density $\hat{p}(y | x)$ is often evaluated by

$$D_{KL}\{\phi(y - \mu, v_y) \mid \hat{p}(y | x)\} = \int_{\mathbb{R}^d} \phi(y - \mu, v_y) \log \frac{\phi(y - \mu, v_y)}{\hat{p}(y | x)} dy,$$

(1.1)

which is called the Kullback-Leibler divergence loss (KL-div loss) from $\phi(y - \mu, v_y)$ to $\hat{p}(y | x)$. The overall quality of the procedure $\hat{p}(y | x)$ for each $\mu$ is then summarized by the Kullback-Leibler divergence risk

$$R_{KL}\{\phi(y - \mu, v_y) \mid \hat{p}(y | x)\} = \int_{\mathbb{R}^d} D_{KL}\{\phi(y - \mu, v_y) \mid \hat{p}(y | x)\} \phi(x - \mu, v_x) dx.$$

(1.2)

Aitchison (1975) showed that the Bayesian solution with respect to a prior $\pi(\mu)$ under KL-div loss given by (1.1) is the Bayesian predictive density

$$\hat{p}_\pi(y | x) = \int_{\mathbb{R}^d} \phi(y - \mu, v_y) \pi(\mu | x) d\mu,$$

(1.3)
where $\pi(\mu|x) = \phi(x - \mu, v_x)\pi(\mu)/m_\pi(x, v_x)$ is the posterior density corresponding to $\pi(\mu)$ and
\[
m_\pi(x, v) = \int_{\mathbb{R}^d} \phi(x - \mu, v)\pi(\mu)\,d\mu
\] (1.4)
is the marginal density of $X \sim N_d(\mu, vI)$ under the prior $\pi(\mu)$.

For the prediction problems in general, many studies suggest the use of the Bayesian predictive density rather than plug-in densities of the form $\phi(y - \hat{\mu}(x), v_y)$, where $\hat{\mu}(x)$ is an estimated value of $\mu$. Liang and Barron (2004) showed that the Bayesian predictive density with respect to the uniform prior $\pi_U(\mu) = 1$,
\[
\hat{p}_U(y|x) = \int_{\mathbb{R}^d} \phi(y - \mu, v_y)\pi_U(\mu|x)\,d\mu = \phi(y - x, v_x + v_y) \quad (1.6)
\]
is best invariant and minimax. Although the best invariant Bayesian predictive density is generally a good default procedure, it has been shown to be inadmissible in some cases. Specifically, Komaki (2001) showed that the Bayesian predictive density with respect to Stein’s (1974) harmonic prior $\pi_H(\mu) = \|\mu\|^{-(d-2)}$
\[
\hat{p}_H(y|x) = \int_{\mathbb{R}^d} \phi(y - \mu, v_y)\pi_H(\mu|x)\,d\mu
\] (1.7)
dominates the best invariant Bayesian predictive density $\hat{p}_U(y|x)$. George, Liang and Xu (2006) extended Komaki’s (2001) result to general shrinkage priors including Strawderman’s (1971) prior.

From a more general viewpoint, the KL-div loss given by (1.1) is in the class of $\alpha$-divergence loss ($\alpha$-div loss) introduced by Csiszár (1967) and defined by
\[
D_{\alpha}\{\phi(y - \mu, v_y) \mid \hat{p}(y|x)\} = \int_{\mathbb{R}^d} f_{\alpha} \left(\frac{\hat{p}(y|x)}{\phi(y - \mu, v_y)}\right)\phi(y - \mu, v_y)\,dy, \quad (1.8)
\]
where
\[
f_{\alpha}(z) = \begin{cases} 4/(1 - \alpha^2), & |\alpha| < 1, \\ z \log z, & \alpha = 1, \\ -\log z, & \alpha = -1. \end{cases}
\]
When $\alpha = -1$, we have
\[
D_{-1}\{\phi(y - \mu, v_y) \mid \hat{p}(y|x)\} = D_{KL}\{\phi(y - \mu, v_y) \mid \hat{p}(y|x)\},
\]
where $D_{KL}$ is given by (1.1). When $\alpha = 0$, we have
\[
f_0(z) = 4(1 - z^{1/2}),
\]
where \( \sqrt{D_0 \{ \phi(y - \mu, v_y) \| \hat{p}(y | x) \}} / 2 \) is the Hellinger distance between \( \hat{p}(y | x) \) and \( \phi(y - \mu, v_y) \). As in the Kullback-Leibler divergence risk given by (1.2), the overall quality of the procedure \( \hat{p}(y | x) \) for each \( \mu \) is summarized by the \( \alpha \)-divergence risk

\[
R_\alpha \{ \phi(y - \mu, v_y) \| \hat{p}(y | x) \} = \int_{\mathbb{R}^d} D_\alpha \{ \phi(y - \mu, v_y) \| \hat{p}(y | x) \} \phi(x - \mu, v_x) dx.
\]

Corcuera and Giannolè (1999) showed that a Bayesian predictive density under \( \alpha \)-div loss is

\[
\hat{p}_\pi(y | x; \alpha) \propto \left\{ \int_{\mathbb{R}^d} \phi^{\frac{1-\alpha}{2\alpha}}(y - \mu, v_y) \phi(x - \mu, v_x) \pi(\mu | x)d\mu \right\}^{\frac{1}{2\alpha}}, \quad -1 \leq \alpha < 1,
\]

\[
\exp \left( \int_{\mathbb{R}^d} \{ \log \phi(y - \mu, v_y) \} \phi(x - \mu, v_x) \pi(\mu | x)d\mu \right), \quad \alpha = 1.
\]

By (1.9), in the prediction problem under \( \alpha \)-div loss with \( \alpha = 1 \) from the Bayesian point of view, the Bayesian solution is the normal density

\[
\hat{p}_\pi(y | x; 1) = \phi(y - \hat{\mu}_\pi(x), v_y),
\]

where \( \hat{\mu}_\pi(x) \) is the posterior mean given by

\[
\hat{\mu}_\pi(x) = \int_{\mathbb{R}^d} \mu \pi(\mu | x)d\mu = x + v_x \nabla_x \log m(x, v_x).
\]

In general, the Bayesian prediction problem under \( \alpha = 1 \) reduces to the estimation problem under the \( \alpha \)-div loss in the case of the exponential family density. This is because the exponential family density is closed under the calculation in (1.9) with \( \alpha = 1 \), as pointed out in Yanagimoto and Ohnishi (2009).

As demonstrated in Maruyama and Strawderman (2012), the \( \alpha \)-div loss in the case of \( \alpha = 1 \) is written as

\[
D_1 \{ \phi(y - \mu, v_y) \| \phi(y - \hat{\mu}_\pi(x), v_y) \} = \frac{\| \hat{\mu}_\pi(x) - \mu \|^2}{2v_y},
\]

and hence the prediction problem under \( \alpha = 1 \) reduces to the estimation problem of \( \mu \) under the quadratic loss. Stein (1981) showed that

\[
E_X \left[ \| \hat{\mu}_\pi(X) - \mu \|^2 \right] = dv_x + 4v_y^2 E_X \left[ \frac{\Delta_x m_\pi^{1/2}(X, v_x)}{m_\pi^{1/2}(X, v_x)} \right],
\]

which implies that the risk difference under \( \alpha = 1 \) is expressed as

\[
R_1 \{ \phi(y - \mu, v_y) \| \hat{p}_\pi(y | x; 1) \} - R_1 \{ \phi(y - \mu, v_y) \| \hat{p}(y | x; 1) \} = 2v_y^2 E_X \left[ \frac{\Delta_x m_\pi^{1/2}(X, v_x)}{m_\pi^{1/2}(X, v_x)} \right].
\]
Under the KL-div loss or α-div loss with \( \alpha = -1 \), George, Liang and Xu (2006) showed that the risk difference is given by

\[
R_1 \{ \phi(y - \mu, v_y) \mid \hat{p}_\alpha(y \mid x; -1) \} - R_1 \{ \phi(y - \mu, v_y) \mid \hat{p}_\pi(y \mid x; 1) \} = 2 \int_{v_*}^{v_*} E_Z \left[ \frac{\Delta_1 m_{\pi}^{1/2}(Z, v)}{m_{\pi}^{1/2}(Z, v)} \right] dv, \tag{1.14}
\]

where \( \hat{p}_\alpha(y \mid x; -1) \) is given by (1.6), \( Z \sim N_d(\mu, vI) \) and \( v_* = v_x v_y / (v_x + v_y) \). From this viewpoint, George, Liang and Xu (2006) and Brown, George and Xu (2008) considered the prediction problem under α-div loss with two extreme cases \( \alpha = \pm 1 \) and found a beautiful relationship of risk differences for two cases via \( \Delta_1 (m_\pi(z, v))^{1/2} \) for some \( v \). Under both risks \( R_1 \) and \( R_{-1} \), any shrinkage prior of the satisfier of the superharmonicity

\[
\Delta_1 m_{\pi}^{1/2}(z, v) \leq 0 \quad \text{for} \quad \begin{cases} v \in (v_x, v_x) & \text{for } \alpha = -1, \\ v = v_x & \text{for } \alpha = 1, \end{cases} \tag{1.15}
\]

implies the improvement over the best invariant Bayesian procedure. It is well-known that the superharmonicity of \( \pi(\mu) \), \( \Delta_1 \pi(\mu) \leq 0 \), implies the superharmonicity of \( m_\pi(z, v) \), \( \Delta_1 m_\pi(z, v) \leq 0 \). Further the superharmonicity of \( m_\pi(z, v) \) implies the superharmonicity of \( \{m_\pi(z, v)\}^{1/2} \). Hence the harmonic prior \( \pi_0(\mu) = ||\mu||^{-(d-2)} \) is one of the satisfiers of the superharmonicity of \( \{m_\pi(z, v)\}^{1/2} \).

Because of the relationship given by (1.13), (1.14) and (1.15), it is of great interest to find the corresponding link via \( \Delta_1 (m_\pi(z, v))^{1/2} \) for α-div loss with general \( \alpha \in (-1, 1) \). To our knowledge, decision-theoretic properties seem to depend on the general structure of the problem (the general type of problem (location, scale), and the dimension of the parameter space) and on the prior in a Bayesian setup, but not on the loss function, as Brown (1979) pointed out in the estimation problem.

In this paper, we investigate the risk difference, \( \text{diff} R_{\alpha, u, \pi} \), in the case of α-div loss, defined by

\[
\text{diff} R_{\alpha, u, \pi} = R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_\pi(y \mid x; \alpha) \} - R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_u(y \mid x; \alpha) \}. \tag{1.16}
\]

In (1.16), \( \hat{p}_\pi(y \mid x; \alpha) \) is given by (1.9) and \( \hat{p}_u(y \mid x; \alpha) \) is the Bayesian predictive density under the uniform prior (1.5), the form of which will be derived in (2.5) of Section 2. As a generalization of Liang and Barron’s (2004) result, \( \hat{p}_u(y \mid x; \alpha) \) for general \( \alpha \in (-1, 1) \) is best invariant and minimax, as shown in Appendix A. Further, analyzing \( \text{diff} R_{\alpha, u, \pi} \), we provide some asymptotic results and a non-asymptotic decision-theoretic result.

**Asymptotic results** We show not only somewhat expected relationship

\[
\lim_{\alpha \to -1^-} \text{diff} R_{\alpha, u, \pi} = \text{diff} R_{1, u, \pi}, \quad \lim_{\alpha \to -1^+} \text{diff} R_{\alpha, u, \pi} = \text{diff} R_{1, u, \pi}, \tag{1.17}
\]
where \( \text{diff}R_{1,u,\pi} \) and \( \text{diff}R_{-1,u,\pi} \) are given in (1.13) and (1.14) respectively, but also the asymptotic relationship for general \( \alpha \in (-1,1) \),

\[
\lim_{v_x/v_y \to +0} \text{diff}R_{\alpha,u,\pi} = \text{diff}R_{1,u,\pi}.
\] (1.18)

Hence, the asymptotic situation \( v_x/v_y \to 0 \) corresponds to the case \( \alpha \to 1 \) and \( \Delta_z\{m_\pi(z,v)\}^{1/2} \) plays an important role for general \( \alpha \in (-1,1) \).

**Non-asymptotic result** We particularly investigate a decision-theoretic property of the Bayesian predictive density with respect to \( \pi_{H}(\mu) = ||\mu||^{-d/(d-2)} \) under \( \alpha \)-div loss with general \( \alpha \in (-1,1) \). We show that, the Bayesian predictive density with respect to \( \pi_{H}(\mu) = ||\mu||^{-d/(d-2)} \) dominates the best invariant Bayesian predictive density with respect to \( \pi_{U}(\mu) = 1 \) if

\[
\frac{v_x}{v_y} \leq \begin{cases} 
\frac{d+2}{d(1+\alpha)} & \text{if } \frac{2}{1-\alpha} \text{ is a positive integer,} \\
\left(\frac{2}{1-\alpha}\right)^2 \frac{d+2}{d} \left[ \frac{1}{\gamma} - \frac{2}{(1-\alpha)} \right] & \text{otherwise,}
\end{cases}
\]

where \( \kappa \) is the smallest integer larger than \( 2/(1-\alpha) \).

The organization of this paper is as follows. In Section 2, we derive the exact form of \( \hat{p}_\pi(y|x;\alpha) \), propose a general sufficient condition for \( \text{diff}R_{\alpha,u,\pi} \geq 0 \), where \( \text{diff}R_{\alpha,u,\pi} \) is given by (1.16), and demonstrate the asymptotic relationship described in (1.17) and (1.18). In Section 3, we propose the non-asymptotic result under the harmonic prior \( \pi_{H}(\mu) = ||\mu||^{-d/(d-2)} \) described in the above. Some technical proofs are given in Sections A and B of Appendix.

### 2. Bayesian predictive density under \( \alpha \)-divergence loss

As in (1.9), the Bayes predictive density under \( \alpha \)-div loss is

\[
\hat{p}_\pi(y|x;\alpha) \propto \left\{ \int_{\mathbb{R}^d} \phi(x-\mu,v_x) \phi^\beta(y-\mu,v_y)\pi(\mu)d\mu \right\}^{1/\beta},
\] (2.1)

where

\[
\beta = \frac{1-\alpha}{2}.
\] (2.2)

Clearly, it follows from \( \alpha \in (-1,1) \) that \( 0 < \beta < 1 \). Let

\[
\gamma = \frac{1}{1+\beta v_x/v_y}.
\] (2.3)

Since the relation of completing squares with respect to \( \mu \), for \( \phi(x-\mu,v_x) \phi^\beta(y-\mu,v_y) \), is given by

\[
\frac{1}{v_x}||x-\mu||^2 + \frac{\beta}{v_y}||y-\mu||^2
\]
\[ \begin{align*}
&= \frac{1}{v_x} \left( \|x - \mu\|^2 + \frac{1 - \gamma}{\gamma} \|y - \mu\|^2 \right) \\
&= \frac{1}{v_x} \left( \frac{1}{\gamma} \|\mu - \{\gamma x + (1 - \gamma) y\}\|^2 - \frac{\|\gamma x + (1 - \gamma) y\|^2}{\gamma} + \|x\|^2 + \frac{1 - \gamma}{\gamma} \|y\|^2 \right) \\
&= \frac{1}{v_x} \left( \frac{1}{\gamma} \|\mu - \{\gamma x + (1 - \gamma) y\}\|^2 + (1 - \gamma) \|y - x\|^2 \right) \\
&= \frac{1}{v_x\gamma} \|\mu - \{\gamma x + (1 - \gamma) y\}\|^2 + \beta \frac{\gamma}{v_y} \|y - x\|^2,
\end{align*} \]

we have the identity,
\[ \phi (x - \mu, v_x) \phi^\beta (y - \mu, v_y) = \gamma^{(1 - \beta)d/2} \phi (\gamma x + (1 - \gamma) y - \mu, v_x\gamma) \phi^\beta (y - x, v_y/\gamma). \] 

Under the uniform prior \( \pi_v(\mu) = 1 \), we have, from (2.4),
\[ \int_{\mathbb{R}^d} \phi (x - \mu, v_x) \phi^\beta (y - \mu, v_y) \pi_v(\mu) d\mu = \gamma^{(1 - \beta)d/2} \phi^\beta (y - x, v_y/\gamma) \]
in (2.1). Therefore the Bayesian predictive density under the uniform prior is
\[ \hat{p}_v(y|x; \alpha) = \phi (y - x, v_y/\gamma) = \phi (y - x, v_y + \beta v_x), \] 

which is the target predictive density so that the risk difference
\[ \text{diff} R_{u,v,\pi} = R_\alpha \{ \phi (y - \mu, v_y) \parallel \hat{p}_v(y|x; \alpha) \} - R_\alpha \{ \phi (y - \mu, v_y) \parallel \hat{p}_\pi(y|x; \alpha) \} \]
is going to be investigated in this paper. As shown in Appendix A, \( \hat{p}_v(y|x; \alpha) \) for general \( \alpha \in (-1, 1) \) is best invariant and minimax, which is regarded as a generalization of Liang and Barron’s (2004) minimaxity result. Hence \( \hat{p}_\pi(y|x; \alpha) \) with \( \text{diff} R_{\alpha,u,v,\pi} \geq 0 \) for all \( \mu \in \mathbb{R}^d \) is minimax.

The exact form of Bayes predictive density \( \hat{p}_\pi(y|x; \alpha) \) for (2.1) with normalizing constant, which is regarded as a generalization of Theorem 1 of Komaki (2001) as well as Lemma 2 of George, Liang and Xu (2006), is provided as follows.

**Theorem 2.1.** The Bayes predictive density under \( \pi(\mu) \) is
\[ \hat{p}_\pi(y|x; \alpha) = \frac{m_\pi^{1/\beta} (\gamma x + (1 - \gamma) y, v_x\gamma)}{E_{Z_1} \left[ m_\pi^{1/\beta} (x + \xi Z_1, v_x\gamma) \right]} \hat{p}_v(y|x; \alpha), \]
where \( Z_1 \sim N_d(0, I) \) and
\[ \xi = (1 - \gamma)(v_y/\gamma)^{1/2}. \]

**Proof.** By (2.1), (2.4) and (2.5), we have
\[ \hat{p}_\pi(y|x; \alpha) \propto \phi (y - x, v_y/\gamma) m_\pi^{1/\beta} (\gamma x + (1 - \gamma) y, v_x\gamma). \]
The normalizing constant of (2.8) is
\[
\int_{\mathbb{R}^d} \phi(y - x, v_y / \gamma) m_{\pi}^{1/\beta}(\gamma x + (1 - \gamma)y, v_x \gamma) dy
\]
\[= \int_{\mathbb{R}^d} \phi(z_1, 1) m_{\pi}^{1/\beta} \left( x + (1 - \gamma)(v_y / \gamma)^{1/2} z_1, v_x \gamma \right) dz_1
\]
\[= E_{Z_1} \left[ m_{\pi}^{1/\beta}(x + \xi Z_1, v_x \gamma) \right],
\]
where the first equality is from the transformation, \(z_1 = (\gamma/v_y)^{1/2}(y - x)\). □

In the following, as a generalization of the Bayes predictive density, we consider
\[
\hat{p}_f(y \mid x; \alpha) = \frac{f(\gamma x + (1 - \gamma)y)}{E_{Z_1} [f(x + \xi Z_1)]} \hat{p}_v(y \mid x; \alpha)
\]
(2.9)
where \(f : \mathbb{R}^d \to \mathbb{R}_+\) is general. As in the proof of Theorem 2.1, \(\int \hat{p}_f(y \mid x; \alpha) dy = 1\) follows. Also \(\hat{p}_f(y \mid x; \alpha)\) is nonnegative for any \(y \in \mathbb{R}^d\) and hence \(\hat{p}_f(y \mid x; \alpha)\) is regarded as a predictive density.

By the definition of the \(\alpha\)-div loss given by (1.8), the risk difference between \(\hat{p}_v\) and \(\hat{p}_f\) is written as
\[
\text{diff} R_{\alpha,v,f} = R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_v(y \mid x; \alpha) \} - R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_f(y \mid x; \alpha) \}
\]
\[= \frac{1}{\beta(1 - \beta)} \int_{\mathbb{R}^{2d}} \left\{ \left( \frac{\hat{p}_f(y \mid x; \alpha)}{\phi(y - \mu, v_y)} \right)^{1-\beta} - \left( \frac{\hat{p}_v(y \mid x; \alpha)}{\phi(y - \mu, v_y)} \right)^{1-\beta} \right\}
\times \phi(x - \mu, v_x) \phi(y - \mu, v_y) dx dy.
\]
(2.10)

Then we have a following result.

**Theorem 2.2.** 1. The risk difference \(\text{diff} R_{\alpha,v,f}\) given by (2.10) is written by \(E[\rho(W, Z)]\) where \(W \sim N_d(\mu, v_x \gamma)\), \(Z \sim N_d(0, I)\) and
\[
\rho(w, z) = \frac{4 \gamma^{(1-\beta)d/2}}{\beta^2 f^{\beta-1}(w)} \int_0^\xi \frac{-\Delta_u \phi(w + tz; f)}{\phi^{2\beta-1}(w + tz; f)} dt
\]
(2.11)
where
\[
\phi(u; t; f) = \{E_{Z_1} [f(tZ_1 + u)]\}^{\beta/2}, \text{ for } Z_1 \sim N_d(0, I).
\]
(2.12)
2. A sufficient condition for \(\text{diff} R_{\alpha,v,f} \geq 0\) for \(\forall \mu \in \mathbb{R}^d\) is
\[
\Delta_u \phi(u; t; f) \leq 0 \quad \forall u \in \mathbb{R}^d, \quad 0 \leq \alpha \leq \xi.
\]
(2.13)

**Proof.** Part 2 easily follows from Part 1 and, in the following, we show Part 1. By (2.4), (2.5), and (2.9), the integrand of (2.10) is rewritten as
\[
\left\{ \left( \frac{\phi(y - \mu, v_y)}{\hat{p}_f(y \mid x; \alpha)} \right)^{\beta-1} - \left( \frac{\phi(y - \mu, v_y)}{\hat{p}_v(y \mid x; \alpha)} \right)^{\beta-1} \right\} \phi(y - \mu, v_y) \phi(x - \mu, v_x)
\]
\[= \gamma^{(1-\beta)d/2} \left\{ \left( \frac{E_Z [f(x + \xi Z_1)]}{f(\gamma x + (1-\gamma)y)} \right)^{\beta-1} - 1 \right\} \times \phi(\gamma x + (1-\gamma)y - \mu, v_x \gamma) \phi(y - x, v_y / \gamma). \]

By the change of variables, \( w = \gamma x + (1-\gamma)y \) and \( z = -(\gamma/v_y)^{1/2}(y - x) \), where Jacobian of the matrix below is \((\gamma/v_y)^{d/2}\),

\[
\begin{pmatrix}
  w \\
  z
\end{pmatrix} = \begin{pmatrix}
  \gamma I_d \\
  (\gamma/v_y)^{1/2} I_d \\
  (1-\gamma) I_d \\
  -(\gamma/v_y)^{1/2} I_d
\end{pmatrix} \begin{pmatrix}
  x \\
  y
\end{pmatrix},
\]

the risk difference is expressed as

\[
\frac{\gamma^{(1-\beta)d/2}}{\beta(1-\beta)} E_{W,Z} \left[ \left( \frac{f(W + \xi (Z_1 + Z))}{f(W)} \right)^{\beta-1} - 1 \right]
= \frac{\gamma^{(1-\beta)d/2}}{\beta(1-\beta)} E_W \left[ f(W)^{1-\beta} \{ g(\xi; W) - g(0; W) \} \right]
= \frac{\gamma^{(1-\beta)d/2}}{\beta(1-\beta)} E_W \left[ f(W)^{1-\beta} \int_0^\infty \frac{\partial}{\partial t} g(t; W) dt \right],
\]

where \( \xi = (1-\gamma)(v_y / \gamma)^{1/2} \) as in (2.7), \( W \sim N_d(\mu, v_x \gamma I), \ Z_1 \sim N_d(0, I) \), \( Z \sim N_d(0, I) \) and

\[
g(t; w) = E_Z \left[ E_{Z_1} [f(w + t \{ Z_1 + Z \})]^{\beta-1} \right].
\]

In the following, \( E_{Z_1} [f] = E_{Z_1} [f(w + t \{ Z_1 + z \})] \) for notational simplicity. Then we have

\[
\frac{\partial}{\partial t} g(t; w) = E_Z \left[ \frac{\partial}{\partial t} \{ E_{Z_1} [f] \}^{\beta-1} \right]
= (\beta - 1) E_Z \left[ \{ E_{Z_1} [f] \}^{\beta-2} E_{Z_1} [(Z_1 + Z)^t \nabla w f] \right]
= (\beta - 1) E_Z \left[ \{ E_{Z_1} [f] \}^{\beta-2} (E_{Z_1} [Z_1^t \nabla w f] + Z^t E_{Z_1} [\nabla w f]) \right].
\]

In (2.17), we have

\[
E_{Z_1} [Z_1^t \nabla w f] = E_{Z_1} \left[ Z_1^t \frac{1}{t} \nabla z_1 f \right] = \frac{1}{t} E_{Z_1} [\Delta z_1 f]
= t E_{Z_1} [\Delta_w f] = t \Delta_w E_{Z_1} [f]
\]

where the second equality follows from the Gauss divergence theorem. Similarly
we have

\[
(\beta - 1)E_Z \left[ \{E_{Z_1} [f]\}^{\beta - 2} Z^T E_{Z_1} [\nabla_v f] \right] \\
= (\beta - 1)E_Z \left[ \{E_{Z_1} [f]\}^{\beta - 2} Z^T \frac{1}{t} E_{Z_1} [\nabla_v f] \right] \\
= \frac{1}{t} (\beta - 1)E_Z \left[ \{E_{Z_1} [f]\}^{\beta - 2} Z^T \nabla_v E_{Z_1} [f] \right] \\
= \frac{1}{t} E_Z \left[ Z^T \nabla_v \{E_{Z_1} [f]\}^{\beta - 1} \right] \\
= \frac{1}{t} E_Z \left[ \Delta_w \{E_{Z_1} [f]\}^{\beta - 1} \right] \\
= tE_Z \left[ \Delta_w \{E_{Z_1} [f]\}^{\beta - 1} \right],
\]

where the fourth equality follows from the Gauss divergence theorem. By (2.17), (2.18) and (2.19), we have

\[
\frac{\partial}{\partial t} g(t; w) = tE_Z \left[ \Delta_w \{E_{Z_1} [f]\}^{\beta - 1} + (\beta - 1) \{E_{Z_1} [f]\}^{\beta - 2} \Delta_w E_{Z_1} [f] \right].
\]

Recall the formula of Laplacian for a function \( h(u) \),

\[
\Delta_u h^a(u) = ah^a(u) \left\{ \frac{\Delta_u h(u)}{h(u)} + (a - 1)\|\nabla_u \log h(u)\|^2 \right\},
\]

for \( a \neq 0 \). Then, in (2.20), we have

\[
\Delta_w \{E_{Z_1} [f]\}^{\beta - 1} + (\beta - 1) \{E_{Z_1} [f]\}^{\beta - 2} \Delta_w E_{Z_1} [f] \\
= \frac{(\beta - 1)}{\{E_{Z_1} [f]\}^{1 - \beta}} \left( \frac{2\Delta_w E_{Z_1} [f]}{E_{Z_1} [f]} + (\beta - 2)\|\nabla_w \log E_{Z_1} [f]\|^2 \right) \\
= \frac{2(\beta - 1)}{\{E_{Z_1} [f]\}^{1 - \beta}} \left( \frac{\Delta_w E_{Z_1} [f]}{E_{Z_1} [f]} + (\beta/2 - 1)\|\nabla_w \log E_{Z_1} [f]\|^2 \right) \\
= \frac{2(\beta - 1)}{\{E_{Z_1} [f]\}^{1 - \beta}} \left( \frac{\Delta_w E_{Z_1} [f]}{E_{Z_1} [f]} + (\beta/2 - 1)\|\nabla_w \log E_{Z_1} [f]\|^2 \right) \\
= \frac{4(\beta - 1)}{\beta} \frac{\Delta_w \{E_{Z_1} [f]\}^{\beta/2}}{\{E_{Z_1} [f]\}^{1 - \beta/2}}.
\]

By (2.15), (2.20) and (2.22), we completes the proof.

\]

\textit{Remark 2.1.} In the previous version of this article as well as George, Liang and Xu (2006), not only the Stein identity but also the heat equation

\[
\frac{\partial}{\partial v} \phi(u, v) = \frac{1}{2} \Delta_u \phi(u, v),
\]

was efficiently applicable for deriving a nice expression of the risk difference, like Part 1 of Theorem 2.2. It seemed to us that the heat equation was an additional
The superharmonicity of $f$ implies the superharmonicity of $E_{Z_1} [f(tZ_1 + u)]$. Furthermore, using the relationship (2.21), we see that the superharmonicity of $E_{Z_1} [f(tZ_1 + u)]$ implies the superharmonicity of

$$\varrho(u; t; f) = \left( E_{Z_1} [f(tZ_1 + u)] \right)^{\beta/2}$$

for $\beta \in (0, 1)$. Hence, for Part 2 of Theorem 2.2, we have a following corollary.

**Corollary 2.1.** Suppose $f : \mathbb{R}^d \to \mathbb{R}_+$ is superharmonic. Then the predictive density $\hat{p}_f(y|x; \alpha)$ given by (2.9) as

$$\hat{p}_f(y|x; \alpha) = \frac{f(\gamma x + (1-\gamma)y)}{E_{Z_1} [f(x + \xi Z_1)]} \hat{p}_\nu(y|x; \alpha),$$

dominates $\hat{p}_\nu(y|x; \alpha)$.

In Section 3, we will investigate the properties of the Bayesian predictive density $\hat{p}_\nu(y|x; \alpha)$ where

$$f(u) = \{m_\pi(u, v_x \gamma)\}^{1/\beta}$$

is assumed in Theorem 2.2 and Corollary 2.1. Actually in this case, Corollary 2.1 is not useful since the superharmonicity of $\{m_\pi(u, v_x \gamma)\}^{1/\beta}$ for $\beta \in (0, 1)$ is very restrictive. Recall the relationship given by (2.21). For example, the superharmonicity of $m_\pi(u, v_x \gamma)$ does not imply the superharmonicity of $\{m_\pi(u, v_x \gamma)\}^{1/\beta}$. Hence, in Section 3, we will seriously consider the superharmonicity of

$$\varrho(u; t; m_\pi^{1/\beta}) = \left( E_{Z_1} \left[ \{m_\pi(tZ_1 + u, v_x \gamma)\}^{1/\beta} \right] \right)^{\beta/2}.$$

Further, When $1/\beta = 2/(1-\alpha)$ is not an integer, $E_{Z_1} \left[ \{m_\pi(tZ_1 + u, v_x \gamma)\}^{1/\beta} \right]$ in Part 2 of Theorem 2.2 is not tractable for our current methodology in Section 3. Thus we propose a variant of Theorem 2.2 with $f(u) = \{m_\pi(u, v_x \gamma)\}^{1/\beta}$, for a non-integer $1/\beta$ as follows. Let $\kappa$ be the smallest integer among integers which is strictly greater than $1/\beta$,

$$\kappa = \min\{n \in \mathbb{Z} \mid n > 1/\beta\}. \quad (2.23)$$

Then $\kappa - 1 < 1/\beta < \kappa$. As in (2.15), the risk difference is expressed as

$$R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_\nu(y|x; \alpha) \} - R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_\nu(y|x; \alpha) \}$$

$$= \gamma^{(1-\beta)d/2} E_{W,Z} \left[ E_{Z_1} \left[ \left\{ \frac{m_\pi(W + \xi(Z_1 + Z), v_x \gamma)}{m_\pi(W, v_x \gamma)} \right\}^{1/\beta} \right]^\beta - 1 \right].$$
Y. Maruyama and T. Ohnishi / A Bayesian prediction under α-divergence

where $W \sim N_d(\mu, v_x \gamma I)$, $Z_1 \sim N_d(0, I)$ and $Z \sim N_d(0, I)$. From Jensen’s inequality, we have

\[
E_Z \left[ m_\pi^{1/\beta} (w + \xi (Z_1 + Z), v_x) \right] \\
= E_Z \left[ \{m_\pi^\kappa (w + \xi (Z_1 + Z), v_x)\}^{1/(\beta \kappa)} \right] \\
\leq \{E_Z \left[ m_\pi^\kappa (w + \xi (Z_1 + Z), v_x) \right]\}^{1/(\beta \kappa)},
\]

(2.24)

since $0 < 1/(\beta \kappa) < 1$ and hence

\[
R_\alpha \{\phi(y - \mu, v_y) \mid \hat{p}_\pi(y \mid x; \alpha)\} - R_\alpha \{\phi(y - \mu, v_y) \mid \hat{p}_\nu(y \mid x; \alpha)\} \\
\geq \frac{\gamma (1 - \beta)^{d/2}}{\beta (1 - \beta)} E_{W,Z} \left[ E_{Z_1} \left[ \frac{m_\pi^\kappa (W + \xi (Z_1 + Z), v_x)}{m_\pi^\kappa (W, v_x)} \right]^{(\beta - 1)/(\beta \kappa)} - 1 \right].
\]

Applying the same technique starting (2.15) through (2.22) to the lower bound above, we have a variant of Part 2 of Theorem 2.2.

**Theorem 2.3.** Assume $1/\beta$ is not a positive integer. Let $\kappa$ be the smallest integer greater than $1/\beta$. A sufficient condition for $\text{diff} R_\alpha, U, \pi \geq 0$ is

\[
\Delta_u \left\{ E_{Z_1} \left[ m_\pi^\kappa (tZ_1 + u, v_x) \right] \right\}^{c(\beta)/\kappa} \leq 0, \quad \forall u \in \mathbb{R}^d, \quad 0 \leq \forall t \leq \xi
\]

(2.25)

where $Z_1 \sim N_d(0, I)$ and

\[
c(\beta) = \frac{\kappa - 1/\beta + 1}{2} \in (1/2, 1).
\]

(2.26)

**2.1. Asymptotics**

In this subsection, using Theorem 2.2 with $f = m_\pi^{1/\beta}$, we investigate asymptotics of the risk difference

\[
\text{diff} R_\alpha, U, \pi = R_\alpha \{\phi(y - \mu, v_y) \mid \hat{p}_\nu(y \mid x; \alpha)\} - R_\alpha \{\phi(y - \mu, v_y) \mid \hat{p}_\pi(y \mid x; \alpha)\}
\]

where $\hat{p}_\nu(y \mid x; \alpha)$ and $\hat{p}_\pi(y \mid x; \alpha)$ are given by (2.5) and (2.6), respectively.

**2.1.1. $\alpha \to -1$**

Let $v_* = v_x v_y/(v_x + v_y)$. When $\alpha \to -1$ or equivalently $\beta \to 1$, we have

\[
gamma \to \frac{1}{1 + v_x/v_y} = \frac{v_*}{v_x} \quad \text{and} \quad \xi^2 \to \frac{v_x^2}{v_x + v_y} = v_x - v_*
\]

and hence

\[
\frac{2 \gamma (1 - \beta)^{d/2}}{\beta^2} \{m_\pi(w, v_x \gamma)\}^{1/\beta - 1} \to 2,
\]

(2.27)
which are parts of \( \rho(w, z) \) given by (2.11). Further, in \( \varrho(t; u) \) given by (2.12), we have

\[
E_{Z} | m_{\pi}(tZ_{1} + u, v_{x}\gamma) | = m_{\pi}(u, v_{x}\gamma + t^{2}) \rightarrow m_{\pi}(u, v_{x} + t^{2}). \tag{2.28}
\]

By (2.27) and (2.28), we have

\[
\varrho(t; u) \rightarrow m_{\pi}^{1/2}(u, v_{x} + t^{2}),
\]

\[
E_{Z}[\rho(w, Z)] \rightarrow 4 \int_{0}^{v_{x}} \int_{R^{d}} \frac{t}{m_{\pi}^{1/2}(u, v_{x} + t)} \varphi(u-w, t) du dt \tag{2.29}
\]

\[
= 2 \int_{0}^{v_{x}} \left( \frac{t}{m_{\pi}^{1/2}(u, v_{x} + t)} \varphi(u-w, t) du \right) dt
\]

By (2.29), we have

\[
E_{W,Z}[\rho(W, Z)] \rightarrow 2 \int_{R^{d}} \left( \int_{0}^{v_{x}} \int_{R^{d}} \frac{t}{m_{\pi}^{1/2}(u, v_{x} + t)} \varphi(u-w, t) du dt \right)\]

\[
\times \varphi(w-\mu, v_{x}) dw
\]

\[
= 2 \int_{0}^{v_{x}} \left( \int_{R^{d}} \frac{t}{m_{\pi}^{1/2}(u, v_{x} + t)} \varphi(u-w, t) du \right) dt
\]

\[
= 2 \int_{v_{x}}^{v_{y}} E_{Z} \left[ \frac{t}{m_{\pi}^{1/2}(Z)} \frac{t}{m_{\pi}^{1/2}(Z,v)} \right] dv
\]

\[
= R_{-1}\{\varphi(y-\mu, v_{y}) \mid \hat{p}_{0}(y|\gamma; -1)\} - R_{-1}\{\varphi(y-\mu, v_{y}) \mid \hat{p}_{0}(y|\gamma; -1)\},
\]

where \( Z \sim N_{d}(\mu, vI) \) and \( v_{x} = v_{x}v_{y}/(v_{x} + v_{y}) \). The last equality follows from George, Liang and Xu’s (2006) result which was already explained in (1.14) of Section 1. Hence we have

\[
\lim_{\alpha \rightarrow -1+0} \text{diff}R_{\alpha,\mu,\pi} = \text{diff}R_{-1,\mu,\pi}.
\]

2.1.2. \( (1-\alpha)v_{x}/v_{y} \rightarrow 0 \)

Consider the asymptotic situation where

\[
(1-\alpha)v_{x}/v_{y} \rightarrow 0 \Leftrightarrow \beta(v_{x}/v_{y}) \rightarrow 0 \Leftrightarrow \gamma \rightarrow 1. \tag{2.30}
\]

Note that \( E_{Z}[\rho(w, Z)] \) is rewritten as the product \( \rho_{1}(w)\rho_{2}(w) \) where

\[
\rho_{1}(w) = \frac{2\gamma(1-\beta)t/2}{\beta^{2}} \{m_{\pi}(w, v_{x}\gamma)\}^{1/\beta-1} \xi^{2},
\]

\[
\rho_{2}(w) = \frac{2}{\xi^{2}} \int_{0}^{\xi} t \left\{ \int_{R^{d}} \frac{t}{\rho^{2/\beta-1}(t; u)} \varphi(u-w, t^{2}) du \right\} dt
\]
Since $\xi^2$ is rewritten as

\[ \xi^2 = \frac{(1 - \gamma)^2 v_y}{\gamma} = \left(1 - \frac{\gamma}{\gamma}\right)^2 v_y \gamma = \frac{v_x^2}{v_y} \beta^2 \gamma, \]  

we have

\[ \rho_1(w) = 2 \frac{v_x^2}{v_y} \gamma^{(1 - \beta)d/2 + 1} \{m_\pi(w, v_x \gamma)\}^{1/\beta - 1} \]  

and

\[ \lim_{\gamma \to 1} \rho_1(w) = 2 \frac{v_x^2}{v_y} \{m_\pi(w, v_x)\}^{1/\beta - 1}. \]  

When $\gamma \to 1$, we have $\xi^2 \to 0$ by (2.31) and hence

\[ \lim_{\gamma \to 1} \rho_2(w) = \lim_{t \to 0} \left\{ \int_{\mathbb{R}^d} \frac{-\Delta_u \phi(\sqrt{t}; u)}{\phi(\sqrt{t}; u)} \phi(u - w, t) du \right\} \]  

\[ = \int_{\mathbb{R}^d} \lim_{t \to 0} \left( \frac{-\Delta_u \phi(\sqrt{t}; u)}{\phi(\sqrt{t}; u)} \right) \delta(u - w) du, \]  

where $\delta(\cdot)$ is the Dirac delta function. By (2.33) and

\[ \lim_{t \to 0} \phi(\sqrt{t}; u) = \left\{ \int_{\mathbb{R}^d} m_\pi^{1/\beta}(u_1 + u, v_x \gamma) \delta(u_1) du_1 \right\}^{\beta/2} = m_\pi^{1/2}(u, v_x), \]  

we have

\[ \lim_{\gamma \to 1} \rho_2(w) = \left( -\Delta_u m_\pi^{1/2}(w, v_x) \right) m_\pi^{1/2 - 1/\beta}(w, v_x). \]  

By (2.32) and (2.34), we have

\[ \lim_{\gamma \to 1} E_Z [\rho(w, Z)] = \lim_{\gamma \to 1} \rho_1(w) \rho_2(w) = 2 \frac{v_x^2}{v_y} - \Delta_u m_\pi^{1/2}(w, v_x), \]  

which implies that

\[ \lim_{\alpha \to 1} \text{diff} R_{\alpha, u, \pi} = \text{diff} R_{1, u, \pi} = 2 \frac{v_x^2}{v_y} E \left[ \frac{-\Delta_u m_\pi^{1/2}(W, v_x)}{m_\pi^{1/2}(W, v_x)} \right], \]  

\[ \lim_{v_x/v_y \to 0} \frac{v_y}{v_x} \text{diff} R_{\alpha, u, \pi} = \frac{v_y}{v_x} \text{diff} R_{1, u, \pi} = 2v_x E \left[ \frac{-\Delta_u m_\pi^{1/2}(W, v_x)}{m_\pi^{1/2}(W, v_x)} \right]. \]  

Therefore the asymptotic situation $v_x/v_y \to 0$ corresponds to the case $\alpha \to 1$ and $\Delta_z \{m_\pi(z, v)\}^{1/2}$ plays an important role for general $\alpha \in (-1, 1)$. 
3. Improvement under the harmonic prior

Under the harmonic prior \( \pi_H(\mu) = \|\mu\|^{-(d-2)} \), let

\[
m_n(w, v) = \int_{\mathbb{R}^d} \phi(w - \mu, v) \pi_n(\mu) d\mu.
\] (3.1)

Let \( \nu \) be an integer larger than or equal to 2. The superharmonicity related to \( E_{Z_1}[m_n^\nu(tZ_1 + u, v)] \) with \( Z_1 \sim N_d(0, I) \) is as follows.

**Theorem 3.1.** Let \( c \in (0, 1) \) and \( Z_1 \sim N_d(0, I) \). Let \( \nu \) be an integer larger than or equal to 2. Then, we have

\[
\Delta_u \{ E_{Z_1}[m_n^\nu(tZ_1 + u, v)] \}^{c/\nu} \leq 0, \quad \forall u \in \mathbb{R}^d,
\] when

\[
0 \leq t \leq \left( \frac{(d + 2)(1 - c)v}{d\nu(\nu - 1)} \right)^{1/2}.
\] (3.2)

**Proof.** Section B of Appendix.

When \( 1/\beta \) is an integer larger than or equal to 2, namely,

\[
\alpha = 0, 1/3, 1/2, 3/5, 2/3, \ldots,
\]
\[
\beta = 1/2, 1/3, 1/4, 1/5, 1/6, \ldots,
\] (3.3)

let \( \nu = 1/\beta, v = v_x\gamma \) and \( c = 1/2 \) in Theorem 3.1 and compare (3.2) in Theorem 3.1 with \( 0 \leq t^2 \leq \xi^2 = \beta^2 v_x^2 \gamma^2 / v_y \) in Theorem 2.2. If

\[
\frac{\beta^2 v_x^2 \gamma}{v_y} \leq \left( \frac{(d + 2)(1 - c)}{d\nu(\nu - 1)} \right)^{1/2} v_x\gamma
\]

or equivalently

\[
\frac{v_x}{v_y} \leq \frac{d + 2}{d(1 + \alpha)} = \frac{d + 2}{2d(1 - \beta)},
\]

\( m_n(w, v_x\gamma) \) satisfies the sufficient condition of Theorem 2.2 and we have a following result of the Bayesian predictive density with respect to Stein’s harmonic prior \( \pi_H(\mu) = \|\mu\|^{-(d-2)} \), which is given by

\[
\hat{p}_n(y|x; \alpha) = \frac{m_n^{1/\beta}(\gamma x + (1 - \gamma)y, v_x\gamma)}{E_{Z_1}\left[ m_n^{1/\beta}(x + \xi Z_1, v_x\gamma) \right]} \hat{p}_C(y|x; \alpha).
\] (3.4)

**Theorem 3.2.** Suppose \( 2/(1 - \alpha) \) is an positive integer for \( \alpha \in (-1, 1) \). Suppose

\[
\frac{v_x}{v_y} \leq \frac{d + 2}{d(1 + \alpha)}.
\] (3.5)

Then, under \( \alpha \)-div loss, the Bayesian predictive density \( \hat{p}_n(y|x; \alpha) \) with respect to the harmonic prior \( \pi_n(\mu) = \|\mu\|^{-(d-2)} \) dominates the best invariant Bayesian predictive density \( \hat{p}_C(y|x; \alpha) = \phi(y - x, v_y/\gamma) \).
Remark 3.1. For any \( d \geq 3 \) and \( \alpha \in (-1,1) \), we have

\[
\frac{d + 2}{d(1 + \alpha)} > \frac{1}{2}.
\]

Note that, in most typical situations,

\[
\frac{v_x}{v_y} \leq \frac{1}{2},
\]

is easily assumed as follows. Suppose that we have a set of observations \( x_1, \ldots, x_n \) from \( N_d(\mu, \sigma^2 I) \). An unobserved set \( x_{n+1}, \ldots, x_{n+m} \) from the same distribution is predicted by using a predictive density as a function of \( x_1, \ldots, x_n \). From sufficiency,

\[
x = n^{-1} \sum_{i=1}^{n} x_i \sim N_d(\mu, \sigma^2 I/n) \quad \text{and} \quad y = m^{-1} \sum_{i=1}^{m} x_{n+i} \sim N_d(\mu, \sigma^2 I/m)
\]

and clearly \( v_x/v_y = m/n \) in this case. Since, \( m \) is typically 1 or 2 whereas \( n \) is relatively large, the condition (3.5) is satisfied.

When \( \beta = 2/(1 - \alpha) \) is not an integer, Theorem 2.3 can be applied. Let \( \kappa \) be the smallest integer greater than 1/\( \beta \). Suppose

\[
\beta^2 \frac{v_x}{v_y} v_x \gamma \leq \frac{(d + 2)(1 - c(\beta))v_x \gamma}{d \kappa(\kappa - 1)},
\]

where \( c(\beta) \) is given by (2.26) as \( c(\beta) = c(1 - \alpha)/2 = \{\kappa - 2/(1 - \alpha) + 1\}/2 \), the left-hand side is the upper bound of \( t \) of Theorem 2.3 and the right-hand side is the upper bound of \( t \) of Theorem 3.1. When

\[
\frac{v_x}{v_y} \leq \left( \frac{2}{1 - \alpha} \right)^2 \frac{d + 2(1 - \{\kappa - 2/(1 - \alpha)\})}{d \kappa(\kappa - 1)},
\]

which is equivalent to (3.6), \( m(w, v_x \gamma) \) satisfies the sufficient condition of Theorem 2.3 and we have a following result.

**Theorem 3.3.** Suppose \( 2/(1 - \alpha) \) is not a positive integer for \( \alpha \in (-1,1) \). Let \( \kappa \) be the smallest integer greater than \( 2/(1 - \alpha) \). Suppose

\[
\frac{v_x}{v_y} \leq \left( \frac{2}{1 - \alpha} \right)^2 \frac{d + 2(1 - \{\kappa - 2/(1 - \alpha)\})}{d \kappa(\kappa - 1)},
\]

Then the Bayesian predictive density \( \hat{p}_u(y | x; \alpha) \) with respect to the harmonic prior \( \pi_u(\mu) = ||\mu||^{-(d-2)} \) dominates the best invariant Bayesian predictive density \( \hat{p}_u(y | x; \alpha) = \phi(y - x, v_y \gamma) \).

By the definition of \( \kappa \),

\[
\kappa - 1 < \frac{2}{1 - \alpha} < \kappa.
\]
As $2/(1 - \alpha) \uparrow \kappa$, the upper bound given by (3.7) approaches $(d + 2)/\{d(1 + \alpha)\}$ which is exactly the upper bound given by (3.5) of Theorem 3.2. On the other hand, as $2/(1 - \alpha) \downarrow \kappa - 1$, the upper bound given by (3.7) approaches 0. Figure 1 gives a graph of behavior of the upper bound of $v_x/v_y$ for improvement. This undesirable discontinuity with respect to the upper bound of Theorem 3.3 is due to Jensen’s inequality (2.24) which was not used in the proof of Theorem 2.2. However, we would like to emphasize that, for any $\alpha \in (-1, 1)$, there exists a positive upper bound of $v_x/v_y$ for improvement. We can naturally make a conjecture that the lower bound of $v_y/v_x$ for improvement, $d(1 + \alpha)/(d + 2)$, of Theorem 3.2 is still valid even if $2/(1 - \alpha)$ is not an integer. For that purpose, the methodology for appropriately treating $E_{Z_1} \left[ \{m_{u}(tZ_1 + u, v_x \gamma)\}^{2/(1-\alpha)} \right]$ or more generally $E_{Z_1} \left[ \{m_{v}(tZ_1 + u, v_x \gamma)\}^{2/(1-\alpha)} \right]$ for non-integer $2/(1 - \alpha)$ is needed and it remains an open problem.

Appendix A: Minimaxity of $\hat{p}_U(y \mid x; \alpha)$

In this section, we show that

$$\hat{p}_U(y \mid x; \alpha) = \phi(y - x, v_y/\gamma) = \phi(y - x, v_y + \beta v_x)$$

is minimax, by following Sections II and III of Liang and Barron (2004). We start with the definition of invariance under location shift.
Definition A.1. A predictive density \( \hat{p}(y \mid x) \) is invariant under location shift, if for all \( a \in \mathbb{R}^d \) and all \( x, y \), \( \hat{p}(y + a \mid x + a) = \hat{p}(y \mid x) \).

Hence any invariant predictive density should be of the form
\[
\hat{p}(y \mid x) = q(y - x)
\]
which satisfies
\[
\int_{\mathbb{R}^d} q(y) dy = 1.
\]
Clearly \( \hat{p}_\alpha(y \mid x; \alpha) \) is invariant under location shift. Note that invariant procedures have constant risk since the risk of the invariant predictive density \( q(y - x) \) is
\[
R_{\alpha} \{ \phi(y - \mu, v_y) \mid q(y - x) \}
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f_\alpha \left( \frac{q(y - x)}{\phi(y - \mu, v_y)} \right) \phi(y - \mu, v_y) dy \right) \phi(x - \mu, v_x) dx \tag{A.2}
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} f_\alpha \left( \frac{q(z_y - z_x)}{\phi(z_y, v_y)} \right) \phi(z_y, v_y) dz_y \right) \phi(z_x, v_x) dz_x
\]
where \( z_x = x - \mu \) and \( z_y = y - \mu \), which does not depend on \( \mu \). More specifically, the risk of the invariant predictive density \( q(y - x) \) is as follows.

Lemma A.1. The risk of an invariant predictive density \( q(y - x) \) is
\[
R_{\alpha} \{ \phi(y - \mu, v_y) \mid q(y - x) \}
= \frac{1 - \gamma(1 - \beta)d/2}{\beta(1 - \beta)} + \gamma(1 - \beta)d/2 D_\alpha \{ \phi(z, v_y / \gamma) \mid q(z) \}. \tag{A.3}
\]

Proof. By (A.2) and the definition of \( \alpha \)-div loss,
\[
R_{\alpha} \{ \phi(y - \mu, v_y) \mid q(y - x) \}
= \frac{1}{\beta(1 - \beta)} \left\{ 1 - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q^{1 - \beta}(y - x) \phi^\beta(y, v_y) \phi(x, v_x) dx dy \right\}.
\]
By the identity (2.4) with \( \mu = 0 \), we have
\[
\phi(x, v_x) \phi^\beta(y, v_y) = \gamma(1 - \beta)d/2 \phi(\gamma x + (1 - \gamma)y, v_x \gamma) \phi^\beta(y - x, v_y/\gamma),
\]
and hence
\[
R_{\alpha} \{ \phi(y - \mu, v_y) \mid q(y - x) \}
= \frac{1}{\beta(1 - \beta)} \left\{ 1 - \gamma(1 - \beta)d/2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q^{1 - \beta}(y - x) \right. \\
\left. \times \phi^\beta(y - x, v_y / \gamma) \phi(\gamma x + (1 - \gamma)y, v_x \gamma) dx dy \right\}.
\]
By the change of variables,
\[
\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \gamma I_d & (1 - \gamma)I_d \\ -I_d & I_d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{A.4}
\]
where Jacobian of the matrix is 1, we have

\[
R_\alpha \{ \phi(y - \mu, v_y) \mid q(y - x) \} = \frac{1}{\beta(1 - \beta)} \left\{ 1 - \gamma(1 - \beta)d/2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q^{1 - \beta}(z) \phi(\beta(z, v_y/\gamma)) \phi(w, v_x \gamma) dz dw \right\}
\]

\[
= \frac{1}{\beta(1 - \beta)} \left\{ 1 - \gamma(1 - \beta)d/2 \int_{\mathbb{R}^d} q^{1 - \beta}(z) \phi(z, v_y/\gamma) dz \right\}
\]

\[
= \frac{1 - \gamma(1 - \beta)d/2}{\beta(1 - \beta)} + \gamma(1 - \beta)d/2 D_\alpha \{ \phi(z, v_y/\gamma) \mid q(z) \}.
\]

In (A.3) of Lemma A.1, \( D_\alpha \{ \phi(z, v_y/\gamma) \mid q(z) \} \) is non-negative and takes zero if and only if \( q(z) = \phi(z, v_y/\gamma) \). Hence the best invariant procedure is \( \hat{p}_\pi(y \mid x; \alpha) = \phi(y - x, v_y/\gamma) \), where the constant risk is

\[
1 - \frac{(1 - \beta)d/2}{\beta(1 - \beta)}.
\]

Since the risk is constant for invariant predictive density, the best invariant \( \hat{p}_\pi(y \mid x; \alpha) \) is the minimax procedure among all invariant procedures. If a constant risk procedure is shown to have an extended Bayes property defined in the below, then it is, in fact, minimax over all procedures. See Theorem 5.18 of Berger (1985) and Theorem 5.1.12 of Lehmann and Casella (1998) for the detail.

**Definition A.2.** A predictive procedure \( \hat{p}_\pi(y \mid x) \) is called extended Bayes, if there exists a sequence of Bayes procedures \( \hat{p}_{\pi_c}(y \mid x; \alpha) \) with proper prior densities \( \pi_c(\mu) \) for \( c = 1, \ldots, \) such that their Bayes risk differences go to zero, that is,

\[
\lim_{c \to \infty} \left( \int_{\mathbb{R}^d} R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_{\pi_c}(y \mid x) \} \pi_c(\mu) d\mu \right.
\]

\[
- \left. \int_{\mathbb{R}^d} R_\alpha \{ \phi(y - \mu, v_y) \mid \hat{p}_{\pi_c}(y \mid x; \alpha) \} \pi_c(\mu) d\mu \right) = 0.
\]

Recall that

\[
\hat{p}_{\pi_c}(y \mid x; \alpha) \propto \left\{ \int_{\mathbb{R}^d} \phi^\beta(y - \mu, v_y) \phi(\mu, v_x) \pi(\mu) d\mu \right\}^{1/\beta}
\]

\[
(A.5)
\]

for \( \beta = (1 - \alpha)/2 \) and \( \alpha \in (-1, 1) \). Under the prior \( \mu \sim N_d(0, \{cv_x \gamma\} I) \) with the density \( \pi_c(\mu) = \phi(\mu, cv_x \gamma) \), the Bayesian solution is

\[
\hat{p}_{\pi_c}(y \mid x; \alpha) = \phi \left( y - \frac{c \gamma}{1 + c \gamma} x, v_y \frac{1 + c}{1 + c \gamma} \right)
\]
by the identity
\[ \phi^\beta(y - \mu, v_y) \phi(x - \mu, v_x) \phi(\mu, cv_x \gamma) \]
\[ = \left( \frac{1 + c\gamma}{1 + c} \right)^{d(1 - \beta)/2} \phi\left( \mu - \frac{c\gamma x + (1 - \gamma)y}{1 + c}, \frac{cv_x \gamma}{1 + c} \right) \]
\[ \times \phi^\beta\left( y - \frac{c\gamma x}{1 + c\gamma}, v_y \frac{1 + c}{1 + c\gamma} \right) \phi(x, v_x (1 + c\gamma)) . \]

Furthermore, by the identity (A.6), the Bayes risk of \( \hat{p}_{\pi_c}(y \mid x; \alpha) \) is given by
\[ \frac{1}{\beta(1 - \beta)} \left( 1 - \int_{R^d} \int_{R^d} \int_{R^d} \{ \hat{p}_{\pi_c}(y \mid x; \alpha) \}^{1 - \beta} \phi(x - \mu, v_x) \phi(y - \mu, v_y) \phi(\mu, cv_x \gamma) \right) \]
\[ \times \phi^\beta\left( y - \frac{c\gamma x}{1 + c\gamma}, v_y \frac{1 + c}{1 + c\gamma} \right) \phi(x, v_x (1 + c\gamma)) d\mu dy d\mu \]
\[ = \frac{1}{\beta(1 - \beta)} \left( 1 - \left( \frac{1 + c\gamma}{1 + c} \right)^{d(1 - \beta)/2} \int_{R^d} \int_{R^d} \int_{R^d} \phi\left( \mu - \frac{c\gamma x + (1 - \gamma)y}{1 + c}, \frac{cv_x \gamma}{1 + c} \right) \]
\[ \times \phi\left( y - \frac{c\gamma x}{1 + c\gamma}, v_y \frac{1 + c}{1 + c\gamma} \right) \phi(x, v_x (1 + c\gamma)) d\mu dy d\mu \right) \]
\[ = \frac{1}{\beta(1 - \beta)} \left( 1 - \left( \frac{1 + c\gamma}{1 + c} \right)^{d(1 - \beta)/2} \right) \]

which approaches \( (1 - \gamma(1 - \beta)d/2)/\beta(1 - \beta) \) as \( c \) goes to infinity, the constant risk of \( \hat{p}_{\pi_c}(y \mid x; \alpha) \). Hence \( \hat{p}_{\pi_c}(y \mid x; \alpha) \) is extended Bayes and hence minimax.

**Appendix B: Proof of Theorem 3.1**

Recall the identity
\[ \|\mu\|^{-(d-2)} = b \int_0^\infty g^{d/2-2} \exp\left(-g\frac{\|\mu\|^2}{2v}\right) dg \] (B.1)

for any \( v > 0 \), where \( b = 1/\{\Gamma(d/2 - 1)2^{d/2-1}v^{d/2-1}\} \). Then we have
\[ m_\mu(w, v) = \int_{R^d} \phi(w - \mu, v)\|\mu\|^{-(d-2)} d\mu \]
\[ = b \int_0^\infty g^{d/2-2} dg \int_{R^d} \frac{1}{(2\pi)^{d/2}v^{d/2}} \exp\left(-\frac{\|w - \mu\|^2}{2v} - g\frac{\|\mu\|^2}{2v}\right) d\mu \]
\[ = b \int_0^\infty g^{d/2-2} \exp\left(-\frac{g\|w\|^2}{2(1 + g)v}\right) dg \]
\[ = b \int_0^1 \lambda^{d/2-2} \exp\left(-\frac{\lambda\|w\|^2}{2v}\right) d\lambda ,\]
where the third equality is from the relation of completing squares with respect to \( \mu \)

\[
\|w - \mu\|^2 + g\|\mu\|^2 = (g + 1)\|\mu - w/(g + 1)\|^2 + \{g/(g + 1)\}\|w\|^2
\]

and the fourth equality is from the transformation \( \lambda = g/(g + 1) \).

Note that \( m_{\nu}^*(w, v) \) for a positive integer \( \nu \) is expressed as

\[
m_{\nu}^*(w, v) = b^\nu \int_{\mathcal{D}_\nu} \prod_{\nu=1}^\nu \lambda_{i}^{d/2 - 2} \exp \left( -\sum_{i=1}^\nu \lambda_{i} \|w\|^2 \right) \prod \lambda_{i} d\lambda,
\]

where \( \mathcal{D}_\nu \) is \( \nu \)-dimensional unit hyper-cube. In the following, \( d\lambda \) denotes \( \prod_{\nu=1}^\nu d\lambda_{i} \) for notational simplicity. Furthermore the subscript and superscript of \( \prod \) and \( \sum \) is omitted for simplicity if they are \( i = 1 \) and \( i = \nu \) respectively. Hence \( m_{\nu}^*(w, v) \) in the above is written as

\[
m_{\nu}^*(w, v) = b^\nu \int_{\mathcal{D}_\nu} \prod_{\nu=1}^\nu \lambda_{i}^{d/2 - 2} \exp \left( -\sum_{i=1}^\nu \lambda_{i} \|w\|^2 \right) d\lambda.
\]

For the calculation of

\[
E_{Z_1}[m_{\nu}^*(tZ_1 + u, v)] = \int_{\mathbb{R}^d} m_{\nu}^*(x + u, v)\phi(x, t^2)dx \quad (B.2)
\]

under \( Z_1 \sim N_d(0, I) \), note the relation of completing squares with respect to \( x \),

\[
\frac{(\sum \lambda_i)\|x + u\|^2}{\nu} + \|x\|^2 = \frac{1}{\nu} \left\{ \sum \lambda_i\|x + u\|^2 + s\|x\|^2 \right\}
= \frac{1}{\nu} \left\{ \left( \sum \lambda_i + s \right) \|x + \sum \lambda_i/s u\|^2 + s \sum \lambda_i \sum \lambda_i/s \|u\|^2 \right\}, \quad (B.3)
\]

where \( s = v/t^2 \). Then, by (B.3), we have

\[
E_{Z_1}[m_{\nu}^*(tZ_1 + u, v)] = \frac{b^\nu v^{d/2}}{t^d} \int_{\mathcal{D}_\nu} \frac{\prod \lambda_{i}^{d/2 - 2}}{(\sum \lambda_{i} + s)^{d/2}} \exp \left( -\frac{s \sum \lambda_{i} \|u\|^2}{v(\sum \lambda_{i} + s)} - \frac{1}{2} \right) d\lambda.
\]

Re-define \( u := \{s/v\}^{1/2} u \) and let

\[
\psi(u; \nu, s) = \int_{\mathcal{D}_\nu} \frac{\prod \lambda_{i}^{d/2 - 2}}{(\sum \lambda_{i} + s)^{d/2}} \exp \left( -\frac{\sum \lambda_{i} \|u\|^2}{2} \right) d\lambda. \quad (B.4)
\]

By (2.21), the super-harmonicity of \( \{E_{Z_1}[m_{\nu}^*(tZ_1 + u, v)]\}^{c/\nu} \) with respect to \( u \in \mathbb{R}^d \) is equivalent to

\[
\left( \frac{c}{\nu} - 1 \right) \|\nabla_u \psi\|^2 + \psi \Delta_u \psi \leq 0, \quad \forall u \in \mathbb{R}^d. \quad (B.5)
\]
The integrand of \( \psi \) given by (B.4) is denoted by

\[
\zeta(\lambda) = \zeta(\lambda_1, \ldots, \lambda_\nu) = \frac{\prod \lambda_i^{d/2-2}}{(\sum \lambda_i + s)^{d/2}} \exp \left( -\sum \frac{\lambda_i}{\lambda_i + s} z \right)
\]

where \( z = \|u\|^2/2 \). Then we have

\[
\frac{\partial}{\partial u_j} \psi = -u_j \int \zeta(\lambda) \frac{\sum \lambda_i}{\sum \lambda_i + s} d\lambda,
\]

for \( j = 1, \ldots, d \) and

\[
\frac{\partial^2}{\partial u_j^2} \psi = \int \zeta(\lambda) \left\{ -\frac{\sum \lambda_i}{\sum \lambda_i + s} + u_j^2 \left( \frac{\sum \lambda_i}{\sum \lambda_i + s} \right)^2 \right\} d\lambda.
\]

Noting \( z = \|u\|^2/2 \), we have

\[
\|\nabla u \psi\|^2 = 2z \left( \int \zeta(\lambda) \frac{\sum \lambda_i}{\sum \lambda_i + s} d\lambda \right)^2 = 2\nu^2z \left( \int \zeta(\lambda) \frac{\lambda_1}{\sum \lambda_i + s} d\lambda \right)^2 \tag{B.6}
\]

and

\[
\Delta_a \psi = -d \int \zeta(\lambda) \frac{\sum \lambda_i}{\sum \lambda_i + s} d\lambda + 2z \int \zeta(\lambda) \left( \frac{\sum \lambda_i}{\sum \lambda_i + s} \right)^2 d\lambda
\]

\[
= -d\nu \int \zeta(\lambda) \frac{\lambda_1}{\sum \lambda_i + s} d\lambda + 2\nu z \int \zeta(\lambda) \frac{\lambda_1^2}{(\sum \lambda_i + s)^2} d\lambda + 2\nu(\nu - 1)z \int \zeta(\lambda) \frac{\lambda_1\lambda_2}{(\sum \lambda_i + s)^2} d\lambda \tag{B.7}
\]

In (B.6) and (B.7), the second equalities are from symmetry with respect to \( \lambda_i \)'s.

Let

\[
\rho(j_1, j_2, l) = \int_{D_{\nu}} \lambda_1^{j_1} \lambda_2^{j_2} (\sum \lambda_i + s)^l \zeta(\lambda) d\lambda,
\]

\[
\eta(j_2, l) = \int_{D_{\nu-1}} \lambda_2^{j_2} \left( 1 + \sum_{i=2}^\nu \lambda_i + s \right)^l \zeta(1, \lambda_2, \ldots, \lambda_\nu) \prod_{i=2}^{\nu} d\lambda_i,
\]

where \( j_1 \) and \( j_2 \) are nonnegative integers. Then \( \|\nabla u \psi\|^2 \) and \( \Delta_a \psi \) given by (B.6) and (B.7) is rewritten as

\[
\|\nabla u \psi\|^2 = 2\nu^2z \rho(1, 0, -1)^2,
\]

\[
\Delta_a \psi = -d\nu\rho(1, 0, -1) + 2\nu z \rho(2, 0, -2) + 2\nu(\nu - 1)z \rho(1, 1, -2). \tag{B.8}
\]

Here are some useful relationships and inequalities.
Lemma B.1.

\[ s \rho(j_1, j_2, l) = - \eta(j_2, l + 2) + (j_1 + d/2 - 2) \rho(j_1 - 1, j_2, l + 2) + (l - d/2 + 2) \rho(j_1, j_2, l + 1), \quad \text{for } j_1 \geq 1, \]
\[ \rho(0, 0, l) = \nu \rho(1, 0, l - 1) + s \rho(0, 0, l - 1), \]
\[ \rho(1, 0, l) = \rho(2, 0, l - 1) + (\nu - 1) \rho(1, 1, l - 1) + s \rho(1, 0, l - 1), \]
\[ \eta(0, 1) = \eta(0, 0) + (\nu - 1) \eta(1, 0) + s \eta(0, 0), \]
\[ \frac{\rho(1, 0, -1)}{\rho(1, 0, 0)} \geq \frac{1}{\nu d/(d + 2) + s}. \]

**Proof.** See Sub-section B.1. ∎

Applying the identity (B.9) to \( \| \nabla_u \psi \|^2 \) and \( \Delta_u \psi \) given in (B.8), we have

\[ s \| \nabla_u \psi \|^2 = 2 \nu^2 \{ s \rho(1, 0, -1) \} \rho(1, 0, -1) \]
\[ = \nu^2 \{ -2 \eta(0, 1) + (d - 2) \rho(0, 0, 1) - (d - 2) \rho(1, 0, 0) \} \rho(1, 0, -1), \]
\[ s \Delta_u \psi = -d s \rho(1, 0, -1) + \nu \{ -2 \eta(0, 0) + d \rho(1, 0, 0) - d \rho(2, 0, -1) \}
\[ + \nu (\nu - 1) \{ -2 \eta(1, 0) + (d - 2) \rho(1, 0, 0) - d \rho(1, 1, -1) \} \]
\[ = -2 \nu \{ \eta(0, 0) + (\nu - 1) \eta(1, 0) \} + \nu (\nu - 1) (d - 2) \rho(1, 0, 0), \]

where the second equality of \( s \Delta_u \psi \) follows from (B.11). Then we have

\[ \frac{s}{\nu} \left( \frac{c - \nu}{\nu} \| \nabla_u \psi \|^2 + \psi \Delta_u \psi \right) \]
\[ = (\nu - c) \{ 2 \eta(0, 1) - (d - 2) \{ \rho(0, 0, 1) - \rho(1, 0, 0) \} \} \rho(1, 0, -1) \]
\[ - 2 \{ \eta(0, 0) + (\nu - 1) \eta(1, 0) \} \rho(0, 0, 0) + (\nu - 1) (d - 2) \rho(1, 0, 0) \rho(0, 0, 0). \]

By applying (B.10), (B.12) and (B.13), the terms of (B.15) including \( \eta(\cdot, \cdot, \cdot) \), divided by 2, is

\[ (\nu - c) \eta(0, 1) \rho(1, 0, -1) - \eta(0, 0) + (\nu - 1) \eta(1, 0) \rho(0, 0, 0) \]
\[ = (\nu - c) \eta(0, 1) \rho(1, 0, -1) - \eta(0, 1) - s \eta(0, 0) \rho(0, 0, 0) \]
\[ = (\nu - c) \eta(0, 1) \rho(1, 0, -1) - \eta(0, 1) \{ \nu \rho(1, 0, -1) + s \rho(0, 0, -1) \}
\[ + s \eta(0, 0) \rho(0, 0, 0) \]
\[ = -s \eta(0, 1) \rho(1, 0, -1) - s \eta(0, 1) \rho(0, 0, -1) - \eta(0, 0) \rho(0, 0, 0) \]
\[ \leq 0, \]

where the first equality follows from (B.12), the second equality follows from (B.10) and the inequality follows from (B.13).
The terms of (B.15) not including \(\eta(\cdot, \cdot)\), divided by \((d - 2)\), are rewritten as

\[
\begin{align*}
(\nu - c) \{ - \rho(0, 0, 1) + \rho(1, 0, 0) \} & \rho(1, 0, -1) + (\nu - 1)\rho(1, 0, 0)\rho(0, 0, 0) \\
= - (\nu - c)(\nu - 1)\rho(1, 0, 0)\rho(1, 0, -1) & - (\nu - c)\rho(0, 0, 0)\rho(1, 0, -1) \\
+ (\nu - 1)\rho(1, 0, 0)\rho(0, 0, 0) & \\
\leq - \left\{ \frac{(\nu - c)s}{\nu d/(d + 2) + s} - (\nu - 1) \right\} \rho(1, 0, 0)\rho(0, 0, 0) \\
= - \frac{(1 - c)s - \nu(\nu - 1)d/(d + 2)}{\nu d/(d + 2) + s} \rho(1, 0, 0)\rho(0, 0, 0),
\end{align*}
\]

(B.17)

which is nonpositive for \(s \geq \nu(\nu - 1)d/(1 - c)(d + 2)\), where the first equality follows from (B.10) and the inequality follows from (B.14).

By (B.16) and (B.17), we have

\[
\left( \frac{c}{\nu} - 1 \right) \| \nabla_u \psi \|^2 + \psi \Delta_u \psi \leq 0, \quad \forall u \in \mathbb{R}^d
\]

or equivalently

\[
\Delta_u \left\{ \mathcal{E}_{Z_1} [m_Z(tZ_1 + u, v)] \right\}^{c/\nu} \leq 0, \quad \forall u \in \mathbb{R}^d,
\]

when \(t \leq \{(d + 2)(1 - c\nu)/(\nu(\nu - 1))\}^{1/2}\).

**B.1. Proof of Lemma B.1**

[Part of (B.9)]

Note

\[
\frac{\partial}{\partial \lambda_i} \exp \left( - \frac{z}{\sum \lambda_i} \right) = - \frac{s z}{(\sum \lambda_i + s)^2} \exp \left( - \frac{z}{\sum \lambda_i + s} \right).
\]

Then, by an integration by parts, we have

\[
sz \int_0^1 \lambda_i^{d/2 - 2} \frac{\lambda_i}{\lambda_i + s} \frac{d}{d\lambda_1} \left\{ \prod_{j = 3}^{\nu - 1} \lambda_j^{d/2 - 2} \right\} (\sum \lambda_i + s)^{\nu - d/2 + 2} \frac{d}{d\lambda_1} \exp \left( - \frac{z}{\sum \lambda_i} \right) d\lambda_1
\]

\[
= - \lambda_i^{d/2 - 2} \prod_{j = 3}^{\nu - 1} \lambda_j^{d/2 - 2} \int_0^1 \lambda_1^{d/2 - 2} \left\{ \lambda_1^{d/2 - 2} \frac{d}{d\lambda_1} \left( \sum \lambda_i + s \right)^{\nu - d/2 + 2} \right\} \frac{d}{d\lambda_1} \exp \left( - \frac{z}{\sum \lambda_i} \right) d\lambda_1
\]

\[
= - \lambda_i^{d/2 - 2} \prod_{j = 3}^{\nu - 1} \lambda_j^{d/2 - 2} \left\{ \int_0^1 \lambda_1^{d/2 - 2} \frac{d}{d\lambda_1} \left( \sum \lambda_i + s \right)^{\nu - d/2 + 2} \exp \left( - \frac{z}{\sum \lambda_i} \right) d\lambda_1
\]

\[
= -(d/2 - 2 + j) \int_0^1 \lambda_1^{d/2 - 3} \frac{d}{d\lambda_1} \left( \sum \lambda_i + s \right)^{\nu - d/2 + 2} \exp \left( - \frac{z}{\sum \lambda_i} \right) d\lambda_1
\]

\[
= -(l - d/2 + 2) \int_0^1 \lambda_1^{d/2 - 2} \frac{d}{d\lambda_1} \left( \sum \lambda_i + s \right)^{\nu - d/2 + 1} \exp \left( - \frac{z}{\sum \lambda_i} \right) d\lambda_1.
\]
(B.9) follows from integration with respect to \(\lambda_2, \ldots, \lambda_{\nu}\) in the both hand side of the above equality.

[Parts of (B.10), (B.11) and (B.12)] The equalities (B.10), (B.11) and (B.12) easily follows from symmetry with respect to \(\lambda_i\)’s.

[Part of (B.13)] Note that (B.13) is equivalent to

\[
\eta(0,0)\rho(0,0,0) - \eta(0,1)\rho(0,0,-1)
\]

\[
= \{\rho(0,0,0) - \rho(0,0,-1)\} \eta(0,1) - \{\eta(0,1) - \eta(0,0)\}\rho(0,0,0)
\]

\[
= \int_{D_{\nu-1}} f_1(\lambda_2, \ldots, \lambda_{\nu}) \prod_{i=2}^{\nu} d\lambda_i \int_{D_{\nu-1}} f_2(\lambda_2, \ldots, \lambda_{\nu}) \prod_{i=2}^{\nu} d\lambda_i
\]

\[- \int_{D_{\nu-1}} f_3(\lambda_2, \ldots, \lambda_{\nu}) \prod_{i=2}^{\nu} d\lambda_i \int_{D_{\nu-1}} f_4(\lambda_2, \ldots, \lambda_{\nu}) \prod_{i=2}^{\nu} d\lambda_i
\]

\[\leq 0,
\]

where

\[
f_1(\lambda_2, \ldots, \lambda_{\nu}) = \int_0^1 \left(1 - \frac{1}{\sum \lambda_i + s}\right) \zeta(\lambda_1, \ldots, \lambda_{\nu}) d\lambda_1
\]

\[
f_2(\lambda_2, \ldots, \lambda_{\nu}) = (1 + \sum_{i=2}^{\nu} \lambda_i + s)\zeta(1, \lambda_2, \ldots, \lambda_{\nu})
\]

\[
f_3(\lambda_2, \ldots, \lambda_{\nu}) = \left(\sum_{i=2}^{\nu} \lambda_i + s\right)\zeta(1, \lambda_2, \ldots, \lambda_{\nu})
\]

\[
f_4(\lambda_2, \ldots, \lambda_{\nu}) = \int_0^1 \zeta(\lambda_1, \ldots, \lambda_{\nu}) d\lambda_1.
\]

Since both \(1 - 1/ (\sum \lambda_i + s)\) and \(\sum \lambda_i + s\) are increasing in each of its arguments, we have

\[
\left\{1 - 1/ (\sum \lambda_i + s)\right\} \left(1 + \sum_{i=2}^{\nu} \xi_i + s\right)
\]

\[
\leq \left\{1 - \frac{1}{\lambda_1 \vee 1 + \sum_{i=2}^{\nu} (\lambda_i \vee \xi_i) + s}\right\} \left(\lambda_1 \vee 1 + \sum_{i=2}^{\nu} (\lambda_i \vee \xi_i) + s\right)
\]

\[
= \sum_{i=2}^{\nu} (\lambda_i \vee \xi_i) + s,
\]

(B.19)

where \(\vee\) is the maximum operator, i.e. \(\lambda_i \vee \xi_i = \max(\lambda_i, \xi_i)\). In the following, \(\wedge\) denotes the minimum operator, i.e. \(\lambda_i \wedge \xi_i = \min(\lambda_i, \xi_i)\). Note that a function \(h: \mathbb{R}^\nu \rightarrow \mathbb{R}\) is said to be multivariate totally positive of order two (MTP2) if it satisfies

\[
h(x_1, \ldots, x_{\nu})h(y_1, \ldots, y_{\nu}) \leq h(x_1 \vee y_1, \ldots, x_{\nu} \vee y_{\nu})h(x_1 \wedge y_1, \ldots, x_{\nu} \wedge y_{\nu})
\]

for any \(x, y \in \mathbb{R}^\nu\). By Lemma B.2 below, \(\zeta(\lambda_1, \ldots, \lambda_{\nu})\) is MTP2 as a function of \(\nu\)-variate function and hence the inequality

\[
\zeta(\lambda_1, \ldots, \lambda_{\nu}) \zeta(1, \xi_2, \ldots, \xi_{\nu})
\]

\[
\leq \zeta(\lambda_1 \vee 1, \lambda_2 \vee \xi_2, \ldots, \lambda_{\nu} \vee \xi_{\nu}) \zeta(\lambda_1 \wedge 1, \lambda_2 \wedge \xi_2, \ldots, \lambda_{\nu} \wedge \xi_{\nu})
\]

\[
= \zeta((1, \lambda_2 \vee \xi_2, \ldots, \lambda_{\nu} \vee \xi_{\nu}) \zeta(\lambda_1, \lambda_2 \wedge \xi_2, \ldots, \lambda_{\nu} \wedge \xi_{\nu})
\]

(B.20)
follows. By (B.19) and (B.20), we have
\[
\begin{align*}
f_1(\lambda_2, \ldots, \lambda_\nu)f_2(\xi_2, \ldots, \xi_\nu) \\
\leq \int_0^1 \left\{ \sum_{i=2}^\nu (\lambda_i \vee \xi_i) + s \right\} \zeta(1, \lambda_2 \vee \xi_2, \ldots, \lambda_\nu \vee \xi_\nu) \\
\times \zeta(\lambda_1, \lambda_2 \wedge \xi_2, \ldots, \lambda_\nu \wedge \xi_\nu) \right\} d\lambda_1 \\
= f_3(\lambda_2 \vee \xi_2, \ldots, \lambda_\nu \vee \xi_\nu)f_4(\lambda_2 \wedge \xi_2, \ldots, \lambda_\nu \wedge \xi_\nu).
\end{align*}
\]
(B.21)

From Theorem B.1 below, shown by Karlin and Rinott (1980), the theorem follows. 

[Part of (B.14)] By Jensen’s inequality, we have
\[
\frac{\rho(1,0,-1)}{\rho(1,0,0)} = \int \frac{1}{\lambda_1 + \sum_{i=2}^\nu \lambda_i + s} \frac{\lambda_1 \zeta(\lambda)}{\rho(1,0,0)} d\lambda \\
\geq \frac{\rho(2,0,0)}{\rho(1,0,0)} + (\nu - 1) \frac{\rho(1,1,0)}{\rho(1,0,0)} + s.
\]
(B.22)

Let \( f \) be a probability density given by
\[
f(\lambda_1, \ldots, \lambda_\nu) = \frac{d}{2} \left( \frac{d}{2} - 1 \right)^{\nu-1} \lambda_1^{d/2-1} \prod_{i=2}^\nu \lambda_i^{d/2-2},
\]
which is clearly MTP2. Also let
\[
g_1(\lambda_1, \ldots, \lambda_\nu) = \lambda_1, \quad g_2(\lambda_1, \ldots, \lambda_\nu) = -\frac{\exp \left( sz/(\sum \lambda_i + s) \right)}{(\sum \lambda_i + s)^{d/2}},
\]
which are both increasing increasing in each of its arguments. Hence, by so-called FKG inequality given in Theorem B.2 below,
\[
\int_{D_\nu} g_1(\lambda_1, \ldots, \lambda_\nu)g_2(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda \\
\geq \int_{D_\nu} g_1(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda \int_{D_\nu} g_2(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda
\]
or equivalently
\[
\int_{D_\nu} g_1(\lambda_1, \ldots, \lambda_\nu)g_2(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda \\
\leq \int_{D_\nu} g_1(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda,
\]
since \( g_2 < 0 \). Since \( \rho(2,0,0)/\rho(1,0,0) \) is expressed as
\[
\frac{\rho(2,0,0)}{\rho(1,0,0)} = \frac{\int_{D_\nu} g_1(\lambda_1, \ldots, \lambda_\nu)g_2(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda}{\int_{D_\nu} g_2(\lambda_1, \ldots, \lambda_\nu)f(\lambda_1, \ldots, \lambda_\nu)d\lambda},
\]
we have
\[
\frac{\rho(2, 0, 0)}{\rho(1, 0, 0)} \leq \frac{d}{d + 2}. \tag{B.23}
\]
Similarly we have
\[
\frac{\rho(1, 1, 0)}{\rho(1, 0, 0)} \leq \frac{d - 2}{d} \leq \frac{d}{d + 2}. \tag{B.24}
\]
Hence, by (B.22), (B.23) and (B.24), we have
\[
\frac{\rho(1, 0, -1)}{\rho(1, 0, 0)} \geq \frac{1}{\nu d / (d + 2) + s}.
\]

**Lemma B.2.** Let
\[
\zeta(\lambda_1, \ldots, \lambda_\nu) = \prod \frac{\lambda_i^{d/2 - 2}}{(\sum \lambda_i + s)^{d/2}} \exp \left( -\frac{\sum \lambda_i}{\sum \lambda_i + s} z \right).
\]
Then \(\zeta(\lambda_1, \ldots, \lambda_\nu)\) is MTP2.

**Proof.** Note
\[
\exp \left( -\frac{\sum \lambda_i}{\sum \lambda_i + s} z \right) = \exp(-z) \exp \left( \frac{s z}{\sum \lambda_i + s} \right).
\]
From the form of \(\zeta\), we have only to check
\[
(\sum \lambda_i + s)(\sum \xi_i + s) \geq (\sum \lambda_i \lor \xi_i + s)(\sum \lambda_i \land \xi_i + s)
\]
or equivalently
\[
(\sum \lambda_i)(\sum \xi_i) \geq (\sum \lambda_i \lor \xi_i)(\sum \lambda_i \land \xi_i).
\]
We have
\[
(\sum \lambda_i)(\sum \xi_i) - (\sum \lambda_i \lor \xi_i)(\sum \lambda_i \land \xi_i)
\]
\[
= \sum_{i \neq j} \{\lambda_i \xi_j + \lambda_j \xi_i - (\lambda_i \lor \xi_i)(\lambda_j \land \xi_j) - (\lambda_j \lor \xi_j)(\lambda_i \land \xi_i)\}.
\]
Without the loss of generality, assume \(\lambda_i \geq \xi_i\). Then we have
\[
\lambda_i \xi_j + \lambda_j \xi_i - (\lambda_i \lor \xi_i)(\lambda_j \land \xi_j) - (\lambda_j \lor \xi_j)(\lambda_i \land \xi_i)
\]
\[
= \lambda_i \xi_j + \lambda_j \xi_i - \lambda_i(\lambda_j \land \xi_j) - (\lambda_j \lor \xi_j)\xi_i
\]
\[
= \lambda_i\{\xi_j - (\lambda_j \land \xi_j)\} - \xi_i\{(\lambda_j \lor \xi_j) - \lambda_j\}
\]
\[
= (\lambda_i - \xi_i)\{\xi_j - (\lambda_j \land \xi_j)\}
\]
\[
\geq 0,
\]
which completes the proof. \(\square\)
Theorem B.1 (Theorem 2.1 of Karlin and Rinott (1980)). Let \( f_1, f_2, f_3 \) and \( f_4 \) be nonnegative functions satisfying for all \( x, y \in \mathbb{R}^\nu \)
\[
f_1(x)f_2(y) \leq f_3(x \vee y)f_4(x \wedge y).
\]
Then
\[
\int f_1(x)dx \int f_2(x)dx \leq \int f_3(x)dx \int f_4(x)dx.
\]

Theorem B.2 (FKG Inequality, e.g. Theorem 2.3 of Karlin and Rinott (1980)). Let \( f(x) \) for \( x \in \mathbb{R}^\nu \) be a probability density satisfying MTP2. Then for any pair of increasing functions \( g_1(x) \) and \( g_2(x) \), we have
\[
\int g_1(x)g_2(x)f(x)dx \geq \int g_1(x)f(x)dx \int g_2(x)f(x)dx.
\]

References

Aitchison, J. (1975). Goodness of prediction fit. *Biometrika* **62** 547–554. MR0391353

Berger, J. O. (1985). *Statistical decision theory and Bayesian analysis*, second ed. *Springer Series in Statistics*. Springer-Verlag, New York.

Brown, L. D. (1979). A heuristic method for determining admissibility of estimators—with applications. *Ann. Statist.* **7** 960–994. MR536501

Brown, L. D., George, E. I. and Xu, X. (2008). Admissible predictive density estimation. *Ann. Statist.* **36** 1156–1170. MR2418653

Corcuera, J. M. and Giummolè, F. (1999). A generalized Bayes rule for prediction. *Scand. J. Statist.* **26** 265–279. MR1707658

Csiszár, I. (1967). Information-type measures of difference of probability distributions and indirect observations. *Studia Sci. Math. Hungar.* **2** 299–318. MR0219345

George, E. I., Liang, F. and Xu, X. (2006). Improved minimax predictive densities under Kullback-Leibler loss. *Ann. Statist.* **34** 78–91. MR2275235

Karlin, S. and Rinott, Y. (1980). Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions. *J. Multivariate Anal.* **10** 467–498. MR599685

Komaki, F. (2001). A shrinkage predictive distribution for multivariate normal observables. *Biometrika* **88** 859–864. MR1859415

Lehmann, E. L. and Casella, G. (1998). *Theory of point estimation*, second ed. *Springer Texts in Statistics*. Springer-Verlag, New York. MR1639875

Liang, F. and Barron, A. (2004). Exact minimax strategies for predictive density estimation, data compression, and model selection. *IEEE Trans. Inform. Theory* **50** 2708–2726. MR2096988

Maruyama, Y. and Strawderman, W. E. (2012). Bayesian predictive densities for linear regression models under \( \alpha \)-divergence loss: some results and open problems. In *Contemporary developments in Bayesian analysis and
statistical decision theory: a Festschrift for William E. Strawderman. Inst. Math. Stat. (IMS) Collect. 8 42–56. Inst. Math. Statist., Beachwood, OH. MR3202501

Stein, C. (1974). Estimation of the mean of a multivariate normal distribution. In Proceedings of the Prague Symposium on Asymptotic Statistics (Charles Univ., Prague, 1973), Vol. II 345–381. Charles Univ., Prague. MR0381062

Stein, C. (1981). Estimation of the mean of a multivariate normal distribution. Ann. Statist. 9 1135–1151. MR630098

Strawderman, W. E. (1971). Proper Bayes minimax estimators of the multivariate normal mean. Ann. Math. Statist. 42 385–388. MR0397939

Yanagimoto, T. and Ohnishi, T. (2009). Bayesian prediction of a density function in terms of $e$-mixture. J. Statist. Plann. Inference 139 3064–3075. MR2535183