Analysis of the stability of multi-mode systems with approximating nonlinear control laws

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Abstract. The paper analyzes the stability conditions for a multi-mode system with nonlinear links that approximate typical nonlinearities: relay ones with a dead zone and saturation. The problem of stability of systems associated with the implementation of nonlinear control laws based on approximating functions in the form of S-shaped sigma functions has not been considered until now. This problem is especially relevant for multi-mode systems, in which the requirements for the quality of transients in different modes are not the same. Using the example of a nonlinear system with an S-shaped control function obtained based on approximating the combination of functions «relay + relay with a dead band + saturation», averaging straight lines are obtained, the location of which confirms the sufficient conditions for the stability of the system with approximating control. The fulfillment of the obtained conditions has been verified by experimental research of systems with approximating control. Comparative analysis of transient processes in the original nonlinear system and in the system with approximating control made it possible to conclude that the existence of the averaging straight line and its location in the sector specified by the original relay system with a dead zone is a necessary and sufficient condition for the stability of the system with approximating control. This conclusion is based on the equivalence of the results obtained by line arising nonlinear characteristics using the idea of vibrolinearization and by approximating these characteristics by continuous nonlinear functions such as S-shaped sigma functions. Moreover, the performance of systems with approximating control in the transient mode is higher, and self-oscillations in the steady-state are absent in comparison with a system with relay control functions.

Keywords: stability, multi-mode system, nonlinear approximating control

1. Introduction

Ensuring the stability of particularly responsible control systems for marine underwater technological complexes, including the control of descent underwater vehicles in open sea conditions, transport marine objects, technical objects with electric drives and technological units, is a priority task, which is inherently associated with the synthesis of a controller that allows providing the specified quality indicators of its operation in transient and steady-state modes in automatic control systems (ACS). However, the problem of system stability associated with the implementation of nonlinear control laws based on approximating functions was not considered. It is especially relevant for multi-mode systems, in which the requirements for the quality of transients in different modes are not the same. In [1], approaches to solving the problem of approximation of nonlinear characteristics of real ACS elements
are considered, and in [2], the expediency and admissibility of using the method of approximating transformations in modeling and control problems are shown. The required properties in various modes can be achieved using systems with discontinuous controls in sliding modes [3, 4]. However, the presence of inertia, delays, backlashes and gaps and other dynamic imperfections in the actuators and the object leads to the breakdown of sliding modes, the appearance of self-oscillating modes and loss of stability of the system. The analysis of the works [5, 6] shows that nonlinear control laws are very effective for ensuring the high quality of the control process in each of the operating modes of the system. This is achieved by changing the structure and parameters of the control algorithm of the system [7]. These changes can be formed using the following dependence for the coefficients of the control algorithm in general:

\[ K_i = F_i(\epsilon, \dot{\epsilon}, \epsilon^2, \epsilon^3, \text{sign}(\epsilon + c \cdot \dot{\epsilon}) \cdot \epsilon), \text{sign} \epsilon \cdot \dot{\epsilon} \],

where \( \epsilon, \dot{\epsilon} \) is the error and its derivative; \( c \) is the adjustment coefficient; \( \text{sign} \) is the sign function.

When synthesizing a multi-mode system with a nonlinear control law, the required quality of processes in transient and steady-state modes, as can be seen from (1), can be provided with various combinations of information about the error and its derivative, by the sign of their product and a linear combination. At the same time, piecewise linear approximation should be excluded, in which the appearance of discontinuous and pulsed components corresponding to switching moments and violating the smoothness of the ongoing processes is possible. In addition, it can be noted that it is difficult to obtain high-quality derivatives under interference conditions. The use of approximating characteristics of nonlinear elements in the form of cubic, polynomial, power and exponential functions or other transcendental functions that allow significant control actions in amplitude and require special measures to limit them to eliminate the influence of saturation that occurs in the executive organs of the system is impractical in control algorithms, since the levels of these signals, as well as the ranges of active disturbances, are a priori unknown.

Difficulties in the implementation in practice of systems with discontinuous controls in sliding modes, due to the presence of inertia, delay and final speed of movement of the regulating body, make it inexpedient to implement discontinuous controls in a nonlinear system. Therefore, it is effective to use continuous approximating characteristics of nonlinear elements based on their combinations in the form of S-shaped functions in control algorithms for multi-mode systems.

The use of approximating dependences in the control law makes it possible to exclude the appearance of discontinuous and impulse components that arise at the moments of control switching. Moreover, the determination of the switching points for the process of correcting the parameters of nonlinear laws in a multi-mode ACS with approximating control is not required [6]. Thus, using the approximation of nonlinearities, it is possible to overcome the contradiction known from the linear theory between the requirements of accuracy and speed in multi-mode ACS, and also to exclude, with large deviations, the frequent effects of maximum control actions on the actuators. Moreover, the analysis of the fulfillment of the stability conditions for a nonlinear system with approximating control remains a priority and urgent task.

2. Statement of the problem and method of its solution

The problem of analyzing the stability of nonlinear systems containing nonlinearities is considered in the following setting. It is assumed that the nonlinear control law is known and the stability of the system is ensured. It is assumed that the sector of stability of the system is determined by the coefficients of this control law, within which the approximating characteristic of the system is in this sector. For this, the choice of the functional dependence from (1) was carried out taking into account the requirements of the implemented modes in the system and the possibilities of obtaining reliable information about the derivative in the presence of interference. In this paper, the following approximating laws are analyzed:

- linear combination of error (\( \epsilon \)) and cubic nonlinearity (\( \epsilon^3 \));
- approximation of a relay characteristic with a dead zone and saturation.
It is easy to show that the ratio in the first version, for example, is typical for an asynchronous motor, when the angular velocity of the output shaft is low and represents the relationship between the torque on the shaft and the angular velocity [1]. In this case, the dependence in the form of the sum of the error and cubic nonlinearity can be averaged linear with the speed gain. In the second version, an averaged characteristic can also be obtained, which makes it possible to obtain a smoother transient process and smooth out the jumps that appear with an abrupt change in the setting and disturbing influences when changing modes. The presence of such averaged characteristics makes it possible to establish that the field of application of each of the selected variants of the approximating dependences from (1) does not lead to a violation of the stability of the state of the system. This provision is formulated as the following statement.

**Statement.** If the original system with a nonlinear control law is stable, then the system with a control law approximating the nonlinear control law of the original system is also stable.

This statement is based on the equivalence of the results obtained by linearization of nonlinear characteristics using the idea of vibrolinearization [9] and by approximation of these characteristics by continuous nonlinear functions of the sigmoid type [6]. Replacing the relay function with an approximating one, we obtain a model of the passage of external signals that are slow compared to self-oscillation through a relay system, which turns out to be close to linear, due to the so-called vibration smoothing effect of nonlinearities.

Suppose that for a system with a selected control law based on a function approximating a relay characteristic without a zone and with a dead zone with saturation, there is an unambiguous nonlinearity $F(\varepsilon)$, resulting from (1). There are two possible causes of the location of the characteristic $F(\varepsilon)$: the first – the nonlinear characteristic is located in the sector $[0, \lambda_m]$, as in Figure 1, the second – in the sector $[\lambda_0, \lambda_m]$, as in Figure 2. To solve the problem of synthesizing a system with a similar arrangement of the nonlinear characteristic, sufficient conditions for absolute stability are given by the V. M. Popov criterion [8]. The problem of ensuring such an arrangement of the approximating characteristic $F(\varepsilon)$ is reduced to checking the location of the averaging line in the first case, or the location of the tangent line to $F(\varepsilon)$ in the sector $[\lambda_0, \lambda_m]$ in the second case. The presence of such an arrangement of a straight line is a process of confirming the existence of stable transients in a system with an approximating characteristic. The boundaries of the region will be straight lines touching the branches of the border.

**Figure 1.** Nonlinear characteristic $F(\varepsilon)$ located in the sector $[0, \lambda_m]$.

**Figure 2.** Nonlinear characteristic $F(\varepsilon)$ located in the sector $[\lambda_0, \lambda_m]$.

Consider a continuous approximating characteristic in the form of cubic nonlinearity (Figure 3):

$$F(\varepsilon) = \alpha \cdot \varepsilon + \beta \cdot \varepsilon^3,$$

where $\alpha, \beta$ are coefficients ($\alpha, \beta$ – const).

Let us calculate the coefficient of harmonic linearization for cubic nonlinearity (1) with symmetric oscillations $\varepsilon = a \cdot \sin \psi$. Since the nonlinearity of $F(\varepsilon)$ is unique, according to [8], we obtain

$$F(\varepsilon) = q(a) \cdot \varepsilon,$$

where $q(a)$ is a linearization coefficient.
and with $\varepsilon = a \cdot \sin \varphi$ and $F(a \cdot \sin \varphi) = \alpha \cdot a \cdot \sin \varphi + \beta \cdot a^3 \cdot \sin^3 \varphi$ we obtain an expression for the harmonic linearization coefficient:

$$q(a) = \frac{2}{\pi \cdot a} \int_0^\pi (\alpha \cdot a \cdot \sin^2 \varphi + \beta \cdot a^3 \cdot \sin^4 \varphi) d\varphi = \frac{2}{\pi \cdot a} \cdot \alpha \cdot a \int_0^{\pi/2} \frac{1 - \cos 2\varphi}{2} d\varphi +$$

$$+ \frac{2}{\pi \cdot a} \cdot \beta \cdot a^3 \int_0^\pi \sin^2 \varphi \cdot \sin^2 \varphi d\varphi = \frac{2}{\pi} \cdot \alpha \cdot \frac{1}{2} \left[ \frac{\varphi}{\pi} - \frac{\alpha}{2} \cdot \sin 2\varphi \right]_0^\pi +$$

$$+ \frac{2}{\pi} \cdot \beta \cdot \frac{1}{4} \int_0^\pi (1 - \cos 2\varphi)^2 d\varphi = \frac{\alpha}{\pi} \cdot (\pi - 0) - \frac{1}{2 \cdot \pi} \cdot 0 +$$

$$+ \frac{\beta \cdot a^2}{4} \left[ \frac{1}{2} \cdot \cos 2\varphi + \cos^2 2\varphi \right] d\varphi = \alpha + \frac{\beta \cdot a^2}{2 \cdot \pi} \left( \varphi - \sin 2\varphi + \frac{1}{2} \cdot \sin 2\varphi + \frac{3}{4} \cdot \sin 4\varphi \right)_{0}^{\pi} =$$

$$= \alpha + \frac{\beta \cdot a^2}{2 \cdot \pi} \cdot \frac{3}{4} \cdot \pi = \alpha + \frac{3}{2} \cdot \beta \cdot a^2.$$

The characteristic $F(\varepsilon)$ and sector $[\lambda_1, \lambda_2]$ of a stable state of a nonlinear system with this characteristic are shown in Figure 3. Figure 3 shows that for a given amplitude $a$, the straight line $q(a) \varepsilon$ averages the nonlinear dependence $F(\varepsilon)$ on a given section and is in the sector $[\lambda_1, \lambda_2]$ of a stable system. It should be noted that the continuous approximation of form $F(\varepsilon) = k_1 \varepsilon + k_2 \varepsilon^2$ ($\varepsilon$ is the deviation of the controlled value from the set value) has the disadvantage that it allows a control action of unlimited amplitude and requires the use of special measures to limit the maximum values to eliminate the influence of saturation arising in the executive bodies of the system.

![Figure 3. Graph of approximating cubic nonlinearity: $q(a) \varepsilon$ – equation of averaging straight line; $[\lambda_1, \lambda_2]$ – sector of the steady-state of the system.](image)

Let us consider the approximation of the discontinuous characteristic by a continuously differentiable function, the analytical expression of which for the combination of the approximated functions «relay + relay with a dead zone + saturation» has the following form:

$$U(\varepsilon) = \left[ \frac{M_1}{1 + \exp(-\lambda \cdot \varepsilon)} - \frac{M_1}{1 + \exp(\lambda \cdot \varepsilon)} \right] + \left[ \frac{M_2}{1 + \exp(-\lambda \cdot (\varepsilon - a))} - \frac{M_2}{1 + \exp(\lambda \cdot (\varepsilon + a))} \right].$$

(5)

where $M_1, M_2$ – the value of the control action in the dead zone (DZ) and beyond, respectively; $\varepsilon$ – control error; $\lambda$ – tuning parameter; $2a$ – magnitude of DZ.

Following [8], it can be argued that the approximating function for any nonlinearities, including relay ones with a dead zone, is a smooth curve. Therefore, this function can be linearized in the usual way, by determining the slope of the linear section by the formula:

$$K_n = M_2/b,$$

(6)

where $K_n$ is the nonlinearity gain in the control process; $b = M_2/\tan a$. 


As a result, it is sufficient to take the partial derivative with respect to \( \varepsilon \) of the expression \( U(\varepsilon) \) (5) and equate it to (6). Let us obtain an expression for determining the coefficient \( \lambda \), which determines the steepness of the linear section for the nonlinear approximating characteristic (5). With the value \( M_1 = 5 \), \( M_2 = 10 \) and \( 45^\circ \leq \alpha \leq 65^\circ \), we obtain \( k_1 = 1 \) u \( k_2 = 2.14 \), which is quite suitable for checking the fulfillment of the conditions for the location of the approximating characteristic in a certain angle \( (k_1, k_2) \). Taking into account the foregoing, preliminary values of the parameters of the controller with the approximating characteristic (5). At \( M_1 = 7 \), \( M_2 = 10 \), \( 2a = 0.2 \), \( b = 5.43 \) with the use (6) is got \( \lambda = 0.25 \). The got value \( \lambda \) provides the stable state of the system with a nonlinear approximating description, as this description will be in a sector \([0, \lambda]\), so it will not seem outside the sector of the initial steady system.

For this confirmation, we prospect non-linearity as approximating description (5). Static descriptions of nonlinear links (NL) are presented in Figure 4.

![Figure 4](image_url)

**Figure 4.** Static descriptions of nonlinear link with non-linearity of type «zone of insensitivity with a saturation»: a), c) initial; b), d) approximated.

Figure 4 (a, b) shows that most of the linear zones of characteristics and levels of saturation of signals at the output of the NL of the initial and approximated characteristics coincide with asymptotic convergence of signals to zero. This helps to maintain the stability of the system. Let us assume that the measured signal at the input of the NL has the form \( \varepsilon^* (t) = (\varepsilon(t) + \delta(t))\) (\( \varepsilon(t) \) is the value of the mismatch between the given \( g(t) \) and the current value of the variable \( x(t) \), \( \delta(t) \) is the noise signal). We will define a betweenness by amplification of linear area factor and zone of insensitivity in case of approximation of zone of insensitivity by a continuous nonlinear function. For this, we calculate the average component of the mismatch signal \( \varepsilon_i(t) \) at the NL output. The NL parameters (relay amplitudes, gains of linear zones, etc.), by analogy with work [9], as well as the variables \( g(t) \) and \( x(t) \) are assumed to be constant or slowly varying in time. When calculating \( \varepsilon_i(t) \), they can also be considered constant over the period of the interference oscillation. Let the acting noise \( \delta(t) \) on one oscillation period have the form of a triangular signal [10]:

\[
\delta(t) = \begin{cases} 
\frac{2h_1\omega}{\pi}t, & 0 \leq t \leq \frac{\pi}{2\omega}, \\
2h - \frac{2h_1\omega}{\pi}t, & \frac{\pi}{2\omega} \leq t \leq \frac{3\pi}{2\omega}, \\
4h + \frac{2h_1\omega}{\pi}t, & \frac{3\pi}{2\omega} \leq t \leq \frac{2\pi}{\omega},
\end{cases} \quad i = 1, p.
\]

where \( \omega \) is the frequency of vibrations of signal; \( h = const > 2\Delta \) \( (\Delta = \max(\Delta_1, \Delta_2)) \), \( \Delta_1, \Delta_2 \) are amplitudes of signals on the entrance of NL, at that the output signal of \( U \) goes out on the borders of \( M_1, M_2 \).

We will consider the calculation of the middle constituent of signal \( \varepsilon_i(t) \) on the exit of the link with an approximating function for non-linearity «zone of the insensitivity with a satiation», presented in Figure 4.a. For this type of non-linearity and signal of hindrance \( \delta(t) \) of three-cornered form on the period of vibrations, following work [9], we will get

\[
\frac{2h_1\omega}{\pi}t, \quad 0 \leq t \leq \frac{\pi}{2\omega};
\]

\[
2h - \frac{2h_1\omega}{\pi}t, \quad \frac{\pi}{2\omega} \leq t \leq \frac{3\pi}{2\omega};
\]

\[
4h + \frac{2h_1\omega}{\pi}t, \quad \frac{3\pi}{2\omega} \leq t \leq \frac{2\pi}{\omega}, \quad i = 1, p.
\]
\[ e_C(t) = -\frac{\alpha}{2\pi} \int_0^{\infty} \text{sign}(\bar{g}(t) - \bar{x}(t) + \delta(t))d\tau = -\frac{M_1 - M_2}{2} - \frac{k_1}{4h} (A_1^2 - A_1^2) - \frac{k_2}{4h} (A_2^2 - A_2^2) + \frac{M_1 A_1'}{2h} - \frac{M_2 A_2'}{2h} - \frac{M_1 + M_2}{2h} (\bar{g}(t) - \bar{x}(t)), \] (8)

where \( k_1 = \frac{M_1}{A_1' - A_1}, k_2 = \frac{M_2}{A_2' - A_2} \) are amplification of linear zones factors; \( M_1, M_2 = \text{const} > 0 \) are maximal values of signal on the exit of link; \( \bar{g}(t), \bar{x}(t) \) are slowly changing variables.

Figure 5 illustrates the quality picture of making the hindrances of three-cornered form. Size of middle constituent of signal \( e_c(t) \) for non-linearity of type «zone of the insensitivity with a satiation», represented on a Figure 4, b, and the signal described by expression (8), can be certain as

\[ e_C(t) = -\frac{M_1 - M_2}{2} - \frac{A_2'}{2h} M_2 + \frac{A_1}{2h} M_1 - k(\bar{g}(t) - \bar{x}(t)), \] (9)

where \( k = \frac{M_1 + M_2}{2h} \) is an amplification of linear zones factor.

As be obvious from expressions (8) and (9), constituent \( e_c(t) \) for a slowly changing signal in the nonlinear system with approximated DZ has a smooth kind in relation to primary saltatory one, due to the smoothed influence of approximating link.

We will eliminate the indignations conditioned by the asymmetry of the description of approximating NL, accepting the parameters of \( M_1 = M_2 = M \) and \( \Delta_1 = \Delta_2 \). In the case when the description of initial non-linearity is symmetric, mu-factors on a linear area correspond within a permanent size to the coefficients calculated in general a way linearizing, for example using tangent, i.e. \( k_c = M/h \). Taking this into account, a straight line equation was obtained for the averaged signal component \( e_c(t) \), which, in this case, according to (8) and (9), does not depend on the form of nonlinearity, i.e.

\[ e_C(t) = -k_c(\bar{g}(t) - \bar{x}(t)). \]

This equation averages the nonlinear dependence in a given area \(-a < e < a\). Naturally, the steepness of the slope of this averaging curve does not go beyond the sector of the nonlinear initial characteristic. According to [8], stability conditions are preserved in this case and, therefore, stability in a system with nonlinear links approximating relay links with a dead zone is preserved.
3. Results of experimental researches and their discussion

For confirmation of statement, an experimental way using the mathematical design is get transients in the nonlinear system of adjusting of concentration of cut-in oxygen in a culture medium in the reactor of synthesis of biologics. For definiteness study of the system is undertaken an at external saltatory entrance influence of \( f(t) = H_1(t) \) and zero initial conditions for the moment of time of \( t = -0 \). The prospected object of management is approximated by an interval model with a transfer function of the form

\[
W(p) = K_{ob} \exp(-\tau p)/(T_2^2 p^2 + T_1 p + 1),
\]

where values of transmitivity of \( K_{ob} \), permanent to time of \( T_1 \), \( T_2 \) and time of delay \( \tau \) are in intervals:

\[
0.029 \leq K_{ob} \leq 0.402;\quad 7.81 \leq T_1 \leq 14.8;\quad 4.61 \leq T_2 \leq 10.08;\quad 2.0 \leq \tau \leq 7.5.
\]

The parameters of the nominal (computational) model of the object were specified in the form of average interval values. Using these values, the relay controller and the controller with approximating control were tuned. Transient processes in a system with relay control of the form

\[
U(t) = M_1 \text{sign}[\varepsilon(t)] + M_2 \text{sign}[\varepsilon(t) - a],
\]

where \( M_1, M_2 > 0 - \text{const} \), control value; \( \text{sign}[\varepsilon(t)] \) – sign function (signum function); \( \varepsilon \) – deviation \((\varepsilon = g - x)\) between the controlled value \( x \) and the specified action \( g \); \( a \) – the magnitude of the control delay \( M_2 \), with the values \( M_1 = M_2 = M \) are shown in Figure 6. Reduction of value \( M_1 \) in 2 times results in the decline of the amplitude of self-excited oscillations in the mode of stabilizing, that evidently from the charts of the transients presented in Figure 7. For reduction of self-excited oscillations and smooth control, we will take advantage of a controller with the approximating control of kind (5).

![Figure 6. Transient in relay ACS at the values of \( M_1 = M_2 = M = 5.0 \).](image)

![Figure 7. Transient in relay ACS at the values of \( M_1 = 2.5; M_2 = 5.0 \).](image)

From the graph of the transient process in the system with approximating control at \( M_1 = M_2 = M = 5.0 \), shown in Figure 8, one can see the preservation of stability and speed in the transient mode (the rise time is 17.9 min), the absence of oscillations in the steady-state and the approach of the transient process to the aperiodic, as well as the absence of frequent maximum impacts on the actuator, which has a beneficial effect on its work.
Thus, it has been confirmed that with the stability of the original relay system, the stability of the system with approximating control is preserved. An increase in the value of $M$ by 2 times, as can be seen from Figure 9, provides an increase in speed (the rise time decreased by 1.88 times), a decrease in the static error by a factor of 1.39, and the system remains in a stable region, despite a slight increase in the oscillation of the controlled value and control action. At the same time, an increase in the speed of response is provided due to the maximum values of the control action of the controller only at the beginning of the transient process, which is important for the transient mode of the MCS.

4. Results
The statement considered in this paper allows us to conclude that the existence of the averaging straight line and its location in the sector given by the original nonlinear system is a necessary and sufficient condition for the stability of a multimode system with approximating control.

5. Conclusion
The fulfillment of stability conditions for a system with nonlinear links approximating relay links with a dead zone is confirmed by experimental studies of control systems and the resulting transient processes in the original nonlinear system and in the system with approximating control. Based on the fact that the approximating function for any nonlinearities, including relay ones with a dead zone, is a smooth curve that can be linearized in the usual order and determine the slope of the linear section, a formula for calculating the coefficient characterizing the curvature of the linear section of the approximating characteristic that does not deduce this characteristic is obtained from the region of existence of stable transient processes in a system with an approximating characteristic. It can be noted that since the two-mode control of many objects is the main one, the results obtained can be of both theoretical and practical significance for the control of this class of multi-mode objects.

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