Wavepackets on de Sitter spacetime

João C. A. Barata\textsuperscript{1} and Marcos Brum\textsuperscript{2}

Instituto de Física, Universidade de São Paulo, Rua do Matão 1371, São Paulo, Brasil. ZIP code: 05508-090.

E-mail: \texttt{jbarata@if.usp.br, mbrum@if.usp.br}

Abstract. We construct wavepackets on de Sitter spacetime, with masses consistently defined from the eigenvalues of an irreducible representation of a Casimir element in the universal enveloping algebra of the Lorentz algebra and analyze their asymptotic behaviour. Furthermore, we show that, in the limit as the de Sitter radius tends to infinity, the wavepackets tend to the wavepackets of Minkowski spacetime and the plane waves arising after contraction have support sharply located on the mass shell.

PACS numbers: 02.20.Qs, 02.30.Nw, 04.62.+v
1. Introduction

The particle concept on a curved spacetime is ambiguous. This is intrinsically related to the absence of a preferred Hilbert space of states and also presents itself in the absence of an S-matrix. Some consequences of this ambiguity are illustrated in the Hawking and Unruh effects \[22, 41\].

This problem can be analyzed in the particular case of the de Sitter spacetime (dS). It is a maximally symmetric spacetime, its group of isometries being the Lorentz group, a simply connected semisimple Lie group. Hence some results may be derived in a similar fashion to the usual approach to QFT on Minkowski spacetime \[10, 11, 35\] and the effects of the curvature of spacetime on physical quantities can be directly analyzed. Nevertheless, the particle concept on Minkowski spacetime stems from the Fourier decomposition of the solutions of the wave equation (the Klein-Gordon equation, in the present case). Such a decomposition is more involved in dS and has not yet been fully explored with this purpose. On the other hand, the classification of irreducible unitary representations of the Lorentz group, in the sense of Wigner \[45\], has already been performed \[2\]. Furthermore, the dS is a homogeneous space under the transitive action of the elements of the Lorentz group, therefore one can employ the Fourier analysis developed by Harish-Chandra \[20, 21, 44\]. These important results are on the basis of the approach we present now.

We intend to take the first step in the formulation of a particle state (and of multiparticle asymptotic states in the sense of the Haag-Ruelle theory \[18, 27, 26, 38\]), namely, we want to construct and analyze the asymptotic behaviour of wavepackets on dS. The first problem that we have met is the concept of mass, and it will be treated following \[2, 45\]. On Minkowski spacetime, the possible values of the mass of a field are the square roots of the eigenvalues of the irreducible representation of the Casimir element in the universal enveloping algebra of the Poincaré algebra which is physically interpreted as the square of the four-momentum, on the Hilbert space generated by smooth square integrable functions on Minkowski spacetime (with invariant measure – see \[4\]). There is also a Casimir element in the universal enveloping algebra of the Lorentz algebra which can be irreducibly represented on some Hilbert space, and the eigenvalue of this representation can be related to the mass. A “mass shell” is going to be defined in a different sense from the minkowskian definition. We will show how the massive solutions of the wave equation can be
interpreted as plane waves on dS and will construct the corresponding wavepackets. Their asymptotic behaviour, in the spirit of [27, 26, 38], will be analyzed.

Another desirable feature of any physical theory on a curved spacetime is that it has a sensible flat limit. More precisely, in the limit as the curvature of spacetime tends to zero, one must recover the corresponding physical theory on Minkowski spacetime. In the context of Lie group theory, such a limit can be obtained under the technique of group contraction [19, 29, 33]. It is well known that, in this limit, the Lorentz group contracts towards the Poincaré group (of the same dimension). We will explore this fact to prove that the wavepacket on dS tends to the usual wavepacket on Minkowski spacetime, thus clarifying some features which are not satisfactorily interpreted on dS.

Similar problems in the context of scalar quantum fields on de Sitter spacetime have been analyzed, for particular types of interactions, by the use of perturbation methods [5, 6, 7, 8, 9, 30, 32], particularly the Källen-Lehmann representation. However we intend here to formulate the equivalent of a Haag-Ruelle scattering theory [18, 27, 26, 38], identifying states that could be interpreted as asymptotic multiparticle states, without the necessity to specify the particular form of the interaction. Recent advances in the Haag-Ruelle scattering theory on Minkowski spacetime can be found in [13, 14] and references therein.

The plan of the paper is as follows: in section 2 we present the definitions which will be used throughout the text; in section 3 we define the wave equation invariant under the group of isometries and present the plane waves; in section 4 we present our first important result, that the Casimir operator of the Lorentz algebra, whose eigenvalue is related to the mass of the plane wave on dS, contracts towards the Casimir operator of the Poincaré algebra whose eigenvalue is related to the mass of the plane wave on Minkowski spacetime. This is a key result in the comparison between the wavepackets on dS and the ones on Minkowski spacetime. Finally, in section 5 we construct the wavepackets on dS, analyze their asymptotic behaviour and show that, in the flat limit, they converge to the usual wavepackets of the Minkowski spacetime. Moreover, the plane waves, after contraction, have support in momentum space sharply located on the mass shell. In section 6 we present our conclusions.
2. Definitions

The dS of dimension $n$ may be seen as a hyperboloid imbedded in Minkowski spacetime $\mathbb{M}_{n+1}$ of dimension $n+1$. Choosing a coordinate system which assigns to a point $p$ of $\mathbb{M}_{n+1}$ the point $(x_0(p), \ldots, x_n(p))$, the coordinates of any point in dS satisfy

$$x \cdot x := -x_0^2 + x_1^2 + \ldots + x_n^2 = R^2,$$

where $R$ is the curvature of dS (also called the de Sitter radius).

It is convenient to treat a particular point as the origin of the spacetime. Thus we define, without loss of generality, the point $\vartheta \in dS$ with coordinates $(0, 0, \ldots, 0, R)$, to be the origin of the de Sitter spacetime. The origin of $\mathbb{M}_{n+1}$ is the point $o \in \mathbb{M}_{n+1}$ with coordinates $(0, 0, \ldots, 0, 0)$. Moreover, the null cone $\mathcal{C}$ in $\mathbb{M}_{n+1}$ is the locus of points which are connected to $o \in \mathbb{M}_{n+1}$ through null curves. It is important to remark that $\mathcal{C}$ tangentiates the hyperboloid at infinity. The subset $\mathcal{C}^+ \subset \mathcal{C}$, called the future null cone, is the locus of points which are connected to $o \in \mathbb{M}_{n+1}$ through null curves whose tangent vectors are future directed.

The group of isometries of dS is the Lorentz group $L := SO_0(1, n)$, a simply connected semi-simple Lie group, and the corresponding Lie algebra $l = so(1, n)$ is the Lorentz algebra. Its elements act on dS as rotations and hyperbolic rotations of the points. It is implemented by the map

$$\kappa : L \times dS \ni (g, x) \mapsto xg \in dS.$$

The corresponding representation $\Pi$ of the elements of $L$ as operators on the Hilbert space of smooth complex-valued square integrable functions $f \in L^2(dS, d\Sigma)$, where $d\Sigma$ is the volume measure on dS (which is also invariant under the action of $L$), is given by

$$(\Pi(g)f)(x) = \mathcal{D}_l(g)f(xg),$$

where $\mathcal{D}_l$, in general, is a matrix, $l$ representing the mass and spin of the field. In the present case, $\mathcal{D}_l$ is a scalar and $l$ represents only the mass of $f$ (to be defined below). On the other hand, the group of isometries of $\mathbb{M}_{n+1}$ is the Poincaré group $P_{n+1} := SO_0(1, n) \ltimes \mathbb{R}^{n+1}$. Its elements act on the points of $\mathbb{M}_{n+1}$ as rotations, hyperbolic rotations (“boosts”), and translations.
Another space of functions that will appear below is the space $\mathcal{D}(X)$ of smooth compactly supported functions on some space $X$ (to be precisely specified in each case).

Let now $l = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $l$, where $\mathfrak{a}$ is the maximal abelian subalgebra of $l$, $\mathfrak{n}$ the nilpotent subalgebra normalized by $\mathfrak{a}$, $\mathfrak{k}$ the subalgebra on which the Cartan involution acts as the identity operator \[28\] and $\mathfrak{m} \subset \mathfrak{k}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ ($\mathfrak{m}$ also normalizes $\mathfrak{n}$). $\mathfrak{F}$ is the space of real functionals on $\mathfrak{a}$ and $\mathfrak{F}^+ \subset \mathfrak{F}$ the subset of positive real functionals. $\mathfrak{n}$ is identified as the disjoint union $\mathfrak{n} = \bigoplus_{\alpha \in \mathfrak{F}^+} l_\alpha$. Furthermore, define $\rho := (1/2) \sum_{\alpha \in \mathfrak{F}^+} m(\alpha) \alpha$, where $m(\alpha)$ is the dimension of the root space $l_\alpha$. Moreover, given two roots $\alpha$ and $\alpha'$, $(\alpha, \alpha') := \beta(h_\alpha, h_{\alpha'})$, where $h_\alpha \in l_\alpha$, $h_{\alpha'} \in l_{\alpha'}$ and $\beta(\cdot, \cdot)$ is the Cartan-Killing form of $l$ \[28\]. On $L$, this decomposition gives rise to the factorization $L = KAN = NA$, where $K := \langle \exp L \mathfrak{k} \rangle$ is the compact group generated by $\exp L \mathfrak{k}$, $A := \exp L \mathfrak{a}$ and $N := \exp L \mathfrak{n}$. $A$ and $N$ are simply connected closed subgroups of $L$ and $M := \langle \exp L \mathfrak{m} \rangle$.

The coset $L/K$ is a homogeneous space, $\dim(L/K) = \dim(L) - \dim(K) = n$, and the map

$$\kappa' : L/K \times dS \ni (gK, x) \mapsto xg \in dS$$

is an $L$-equivariant diffeomorphism \[28\].

A horosphere on $dS$ is an orbit of the subgroup $N$. Besides, the action of $N$ on $T_0^*M_{n+1}$ leaves invariant one generator of $\mathfrak{c}^+$, i.e., if $\xi \in T_0^*\mathfrak{c}^+$ is left invariant under the action of $N$, for every $\lambda > 0$, $\lambda \xi$ is also left invariant. Such a family of covectors is called the normal to the horosphere, or the absolute. The set of such covectors (not the whole of $T_0^*\mathfrak{c}^+$) will be designated by $\mathcal{A}$. We remark that all elements of $\mathcal{A}$ can be obtained one from the other by a rotation.

The concept of a horosphere just presented finds an analogue in the context of semi-simple Lie groups: in the homogeneous space $L/K$, a horosphere is an orbit of a subgroup of $L$ conjugate to $N$. Since $M$ normalizes $N$, this subgroup is $MN$. Denoting $\Xi$ the set of all horospheres in $L/K$, $\Xi \cong L/MN$. In addition, the horospheres are closed submanifolds of $L/K$. The origin of $L/K$ is defined to be the left coset $eK \equiv K$, where $e$ is the identity element of $L$. Let $o \in K$, $\xi_o = N.o$ is a horosphere passing through $o$. Helgason \[24\] showed that any horosphere in $L/K$ can be written as $kh.\xi_o =: \xi_{kh}$, where $km \in K/M$ and $h \in A$ are unique. His reasoning may be inverted to show that $\forall kM \in K/M$ and $\forall h \in A$, $kh.\xi_o$ is a horosphere in
Wavepackets on de Sitter spacetime

$L/K$, hence $L$ permutes the horospheres transitively. $kM$ is called the normal to the horosphere $\xi_{kh}$ and $h$ is the complex distance from the origin $o$ to $\xi_{kh}$ (see [24, 25] and the discussion at the end of section 9.2.1 in [44]).

Back to the Lorentz case, $a$ is actually one-dimensional, its generator is the generator of a hyperbolic rotation and $A$ leaves invariant one plane in $dS$. $n$ is an abelian subalgebra whose generators are the generators of the horospheric translations that leave invariant one of the null vectors in the tangent space of the plane left invariant by $A$ [17, 42]. Moreover, $\dim(n) = n - 1$ and $\forall n \in n$, $\text{ad} a(n) = n \Rightarrow \alpha(a) = 1$ and $m(\alpha) = (n - 1)/2$ [42], $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{m} \cong \mathfrak{so}(n - 1)$, and the action of $M$ leaves invariant each point of the plane left invariant by $A$.

3. Action on the spacetime

3.1. Group action

The action of $A$ on $dS$ is given by the matrix ($a \in A$)

$$a(\tau) := \begin{pmatrix} \cosh(\tau/R) & 0 & \ldots & 0 & \sinh(\tau/R) \\ 0 & 1_{n-1} & 0 \\ \sinh(\tau/R) & 0 & \ldots & 0 & \cosh(\tau/R) \end{pmatrix}, \ \tau \in \mathbb{R}.$$ 

$N$ is the group of horospheric translations leaving invariant the null vector in $T\mathbb{C}$ which, in the coordinate system (1), has components $(1, 0, \ldots, 0, 1)$. For $y \in \mathbb{R}^{n-1}$ and $n \in N$,

$$n(y) = \begin{pmatrix} 1 + \frac{|y|^2}{2R^2} & \frac{1}{R} y & \frac{1}{2} |y|^2 \\ \frac{1}{R} y^T & 1_{n-1} & \frac{1}{R} y^T \\ -\frac{1}{2} |y|^2 & -\frac{1}{R} y & 1 - \frac{1}{2} |y|^2 \end{pmatrix}.$$ 

Clearly, $a^{-1}(\tau) = a(-\tau)$ and $n^{-1}(y) = n(-y)$, and $N$ is an abelian group normalized by $A$. Moreover, $K$ is composed of matrices of the form

$$K := \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}(n) \end{pmatrix},$$
Wavepackets on de Sitter spacetime

and $M$ is composed of matrices of the form

$$M := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{SO}(n-1) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

In the following, we will always consider the indices $i$ and $j$ running through the set $\{1, \ldots, n-1\}$. We will also write $n_i := n(y_i)$, $m_{ij} \in M$ denoting rotation in the plane $x_i - x_j$ and $k_{in} \in K$ denoting rotation in the plane $x_i - x_n$. Moreover, almost every point of $dS$ can be reached from the origin $\vartheta$ by the composition of a hyperbolic rotation on the plane $x_0 - x_n$, a horospheric translation and a reflection:

$$x(\tau, y) = \vartheta \cdot a(\tau)n(y)\varepsilon_x$$

$$= R \left( \sinh(\tau/R) - \frac{1}{2} \frac{|y|^2}{R^2} e^{-\tau/R}, -\frac{1}{R} \bar{y} e^{-\tau/R}, \cosh(\tau/R) - \frac{1}{2} \frac{|y|^2}{R^2} e^{-\tau/R} \right) \varepsilon_x,$$

where $\varepsilon_x = \pm 1$. We remark that only points of the form $x_0 + x_n = 0$ are not covered by these charts, but these points form a set of measure zero in $dS$.

3.2. Representations of the Lorentz algebra

The action of $L$ on the points of $dS$ induces, for every point $p \in dS$, a homomorphism between $\mathfrak{l}$ and the Lie algebra of vectors $v \in T_p dS$ [28]. If we choose the coordinate system described in (1) and denote the generators of $\mathfrak{l}$ by $m_{i0}$ (the hyperbolic rotations in the planes 0-i) and $m_{ij}$ (the rotations in the planes i-j), with $i, j \in \{1, \ldots, n\}$, and consider a complex representation of the elements of $\mathfrak{l}$ as operators on the Hilbert space $L^2(dS, d\Sigma)$, these elements are represented as

$$m_{i0} = i \left( x_i \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_i} \right)$$

$$m_{ij} = i \left( x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right).$$

However, it must be taken into account that the coordinate $x_n$ is constrained by (1), hence $\partial/\partial x_n$ can be written in terms of the derivatives with respect to the other coordinates. We will give the explicit expressions in the coordinate system defined
in (3) in section 5.2. Besides, there exists a Noether charge associated to each \( m_{i0} \), \( m_{ij} \). The charges correspond, respectively, to the 0-th column and spatial part of the total angular momentum tensor [31].

The commutation relations satisfied by the elements \( m_{i0} \) and \( m_{ij} \) are: let \( \eta_{ab} = \text{diag}[-1, 1, \ldots, 1] \) be the metric on \( \mathbb{M}_{n+1} \),

\[
[m_{ab}, m_{uv}] = i (\eta_{av} m_{bu} + \eta_{bu} m_{av} - \eta_{au} m_{bv} - \eta_{bv} m_{au}) = C_{abuv}^{rs} m_{rs}, \tag{6}
\]

where \( C_{abuv}^{rs} \) are the structure constants\(^\dagger\) of \( l \).

Moreover, let \( a \in \mathfrak{a} \) and \( n_i \in \mathfrak{n} \) correspond to \( n_i \in N \). These elements are written, in terms of the generators of \( l \) presented above, as

\[
a = m_{n0} \quad \text{and} \quad n_i = m_{i0} + m_{in}. \tag{7}
\]

One Casimir element in the universal enveloping algebra \( \mathcal{U} \) of \( l \) is

\[
C^2 := j^2 - m^2, \tag{8}
\]

where

\[
m^2 := \sum_{i=1}^{n} m_{i0}^2 \quad \text{and} \quad j^2 := \sum_{i<j} m_{ij}^2.
\]

Moreover,

\[
[m^2, j^2] = 0. \tag{9}
\]

The representation described above is irreducible, hence the Casimir element \( C^2 \) is represented as a multiple of the identity operator on \( \mathcal{L}^2(\text{d}S, \text{d}\Sigma) \). The multiplication constant (the eigenvalue of the operator) will be denoted by \(-\mu^2 R^2\).

The D’Alembert operator on \( \mathcal{L}^2(\text{d}S, \text{d}\Sigma) \) is related to the Casimir element \( C^2 \) by [12]

\[
\Box_{dS} = -\frac{1}{R^2} C^2 = \frac{1}{R^2} \left( m^2 - j^2 \right), \tag{10}
\]

where \( C^2 \) is the representation of \( C^2 \). Therefore a massive solution of the wave equation will be an eigenfunction of \( \Box_{dS} \) with eigenvalue \( \mu^2 \). The wave equation assumes the form

\[
\left( \Box_{dS} - \mu^2 \right) \psi = 0. \tag{11}
\]

\(^\dagger\) The structure constants are displayed with twice the usual number of indices because each element of the Lie algebra is represented by a pair of indices.
The eigenfunctions of $\Box_{dS}$ are actually eigenfunctions of $C^2$.

Harish-Chandra [20, 21] found a general solution of such an equation if the manifold is a symmetric space acted upon by a semi-simple Lie group, such as the homogeneous space $L/K$ above. Afterwards his result was given a more geometric interpretation (see [4, 43, 44]) based on concepts from hyperbolic geometry [17].

3.3. Solutions of the wave equation

Before we go in more detail into the structure of the Lorentz group, let us present the solution of the wave equation and some of its features.

One can verify, by direct inspection, that [10, 34, 36]

$$\psi_{\xi, \sigma}(x) = \left(\frac{x \cdot \xi}{\mu R}\right)^{\sigma} = \exp\left[\sigma \log\left(\frac{x \cdot \xi}{\mu R}\right)\right],$$  \hspace{1cm} (12)

is a solution of (11), where $\xi$ is a null covector. Furthermore, we will require that $\xi_0 > 0$, i.e., $\xi \in \mathcal{A}$. $\sigma$ is a complex number, and

$$\mu^2 R^2 = -\sigma(n - 1 + \sigma).$$  \hspace{1cm} (13)

However, $x \cdot \xi$ can be negative and we must carefully locate the branch cut of the logarithm. We choose, $\forall z \in \mathbb{C} \setminus ]-\infty, 0]$, $-\pi < \text{Arg}(z) < \pi$. Hence the logarithm is a holomorphic function on $\mathbb{C} \setminus ]-\infty, 0]$ and is real on $]0, \infty[$.

Consider now the sets $T^\pm := \{x + iy \mid x \in dS, y \in V^\pm \} \subset \mathbb{C}^{n+1}$, called *tuboids* in the literature [10, 11, 8], where

$$V^\pm := \{y \in \mathbb{R}^{n+1} \mid y^2 < 0, y_0 \geq 0\}. \hspace{1cm} (14)$$

Hence, for $z \in T^\pm$, $\text{Im}(z \cdot \xi) \geq 0$. If we consider $x_\pm$ to be the limit of $z \in T^\pm$ as $y$ tends to zero, we can define the logarithm of $x \cdot \xi$ in the case $x \cdot \xi < 0$ as the limit of $\log(z \cdot \xi)$, from the relation

$$(x_\pm \cdot \xi)^\sigma := \Theta(x \cdot \xi)|x \cdot \xi|^\sigma + e^{\pm i\sigma\Theta(-x \cdot \xi)}|x \cdot \xi|^\sigma,$$  \hspace{1cm} (15)

where $\Theta$ is the Heaviside step function ($\Theta(y) = 1$ if $y \geq 0$, otherwise $\Theta(y) = 0$). We remark that $\psi_{\xi, \sigma}$ is singular at $x \cdot \xi = 0$, but this is only possible in a set of (volume) measure zero, since $x \in dS$ and $\xi \in \mathcal{A}$. We remark that $(x_\pm \cdot \xi)^\sigma$ is well defined.
on the whole de Sitter spacetime (up to a set of measure zero). However, as we will prove in sections 4 and 5 only half of dS survives after the limit $R \to \infty$ is taken.

Furthermore, $\psi_{\xi,\sigma}(x)$ is the exponential of the distance from the origin to the horosphere with normal $\xi$ passing through $x$. Since a horosphere in dS is the analogue of a plane in $\mathbb{M}$, $\psi_{\xi,\sigma}$ is a plane wave of de Sitter spacetime.

We will search for real and complex solutions of (13). The complex solution is

$$\text{Re} \, \sigma = -\frac{n-1}{2},$$

$$\text{Im} \, \sigma = \pm \sqrt{\mu^2 R^2 - (n-1)^2/4} = \pm \mu'. \quad (16)$$

In this case, the mass $\mu$ assumes a minimum value $\mu_{\text{min}} = (n-1)/(2R)$ and $\pm \mu'$ can assume any real value. This solution corresponds to the so-called principal series of representations [2] and describes a massive field on dS.

The real solution of (13) is

$$\sigma = -\frac{n-1}{2} + \mu'',$$

$$\mu'' = \pm \sqrt{(n-1)^2/4 - \mu^2 R^2}.$$ 

Now, the mass is bounded from above and the corresponding Compton wavelength is of the order of the curvature radius. This corresponds to the so-called complementary series. These solutions present problems when one tries to interpret them as massive solutions, but they describe massless solutions of (10) [1, 3].

Henceforth we will concentrate on the principal series because, already on de Sitter spacetime, it provides a clearer interpretation of the mass. In this case, the plane waves indeed oscillate. However, $\psi_{\xi,\sigma}$ is neither an eigenfunction of $m^2$ nor of $\hat{j}^2$, although these operators commute. In addition, the determination of the mass $\mu$ does not impose any constraint on the covector $\xi$, but rather on the exponent $\sigma$. Therefore we will call the mass shell the point $\sigma = -\frac{n-1}{2} + i \mu'$ in the space of values that $\sigma$ can assume (the complex plane).

### 4. Contraction of the Lorentz algebra

After we construct the wavepackets on dS we will compare them with the wavepackets constructed on $\mathbb{M}_n$ [20] (note that $\mathbb{M}_n$ has the same number of dimensions

\[\uparrow\]

Afterwards, when we contract the Lorentz group into de Poincaré group, the mass will actually constrain $\xi$. 

\[\downarrow\]
as dS, differently from $\mathbb{M}_{n+1}$, in which dS is imbedded). Although the spacetimes are different, at any point of a curved spacetime (of dimension $n$) the components of the metric tensor can be made equal to the components of the metric tensor of $\mathbb{M}_n$, hence a (possibly infinitesimal) neighbourhood of the curved spacetime resembles (i.e., the effects of the curvature can be treated as perturbations) the Minkowski spacetime. Besides, on the level of Lie algebras, the contraction of algebras [29] allows us to transform the Lorentz algebra $\mathfrak{so}(1, n)$ into the Poincaré algebra $\mathfrak{p}_n := \mathfrak{so}(1, n - 1) \oplus \mathbb{R}^n$.

As observed in [19], almost every element $g \in L$ can be written as the product $g = ban$, with $b \in B$, where $B$ is the normalizer of a subgroup of $L$ isomorphic to $\text{SO}_0(1, n - 1)$. $B$ is composed of elements of the form

$$B := \begin{pmatrix} \text{SO}_0(1, n - 1) & 0 \\ 0 & \pm 1 \end{pmatrix}.$$ 

Let now $b \in \mathfrak{b}$ (the Lie algebra associated to $B$), $a \in \mathfrak{a}$ and $n_i \in \mathfrak{n}$ and define

$$b' := b, \quad a' := \frac{1}{R}a \quad \text{and} \quad n'_i := \frac{1}{R}n_i.$$ 

(17)

In the limit $R \to \infty$, one verifies that the commutation relations satisfied by $b'$, $a'$ and $n'$ are those satisfied by the generators of the Poincaré algebra $\mathfrak{p}_n$, where the elements $b'$ generate the subalgebra $\mathfrak{so}(1, n - 1)$ (as $b$ did), $a'$ is the generator of time translations and $n'$ are the generators of spatial translations. $a'$ and $n'$ comprise the abelian subalgebra which is normalized by $\mathfrak{so}(1, n - 1)$.

The two pictures above are comparable. On the de Sitter spacetime, one may consider a neighbourhood $O_p$ of a point $p \in \text{dS}$ such that $\forall x \in O_p$, the components of the metric tensor in the region $O_p$ are equal to the components of the metric tensor of the Minkowski spacetime $\mathbb{M}_n$. Hence, after the limit $R \to \infty$ is taken, the spacetime becomes the Minkowski spacetime $\mathbb{M}_n$ and its group of isometries is the Poincaré group $P_n$.

4.1. Representation of the Poincaré algebra

The representation of the Lorentz group defined in [2] can be induced from a representation of the closed subgroup

$$Q = MAN \subset L.$$ 

(18)
Moreover, the induction procedure shows that the functions $f \in L^2(L, dL)$ are completely determined if their values on the quotient space $B \cap N / M = B / M$ are known [15], where $dL$ is the unique left invariant Haar measure on $L$. However, $B / M$ is diffeomorphic to the hyperboloids [19, 33]

$$-x_0^2 + x_1^2 + \ldots + x_{n-1}^2 = -R^2,$$

which are the intersections between the hyperplane $x_n^2 = 2R^2$ and the de Sitter spacetime:

$$dS^\pm_0 := \left\{ p \in dS \mid -x_0(p)^2 + x_1(p)^2 + \ldots + x_{n-1}(p)^2 = -R^2, x_0(p) \geq 0 \right\}. \quad (19)$$

The quotient space $B / M$ remains unaltered by the contraction procedure. Therefore, after the contraction, the representation of the Poincaré group $P_n$ is determined by its representations on the two disjoint hyperboloids $dS^+_0$ and $dS^-_0$, i.e., it is the direct sum of these subrepresentations. The Hilbert spaces on which these subrepresentations act are $L^2(dS^+_0, d\Sigma)$ and $L^2(dS^-_0, d\Sigma)$, respectively, which are subspaces of $L^2(dS, d\Sigma)$. Therefore the identity operator $\mathbb{1}$ on $L^2(dS, d\Sigma)$ is also an identity operator on both $L^2(dS^+_0, d\Sigma)$ and $L^2(dS^-_0, d\Sigma)$.

On the other hand, the irreducible representation on $L^2(dS^+_0, d\Sigma)$ differs from the irreducible representation on $L^2(dS^-_0, d\Sigma)$ only by the change $\mu' \rightarrow -\mu'$ [33], where $\mu'$ is related to the mass of the plane wave of $dS$ by (16). Furthermore, the mass $\mu$ is related to the eigenvalue of the Casimir operator $C^2$ of the Lorentz algebra $l_n$. As we will see now, the eigenvalues of this operator can be easily related to the eigenvalues of a Casimir operator $P^2$ of the Poincaré algebra $p_n$.

Let us focus on the irreducible representation of the Lorentz algebra presented in (4) and (5), restricted to, say, $L^2(dS^+_0, d\Sigma)$. Both this representation and the one arising after contraction are irreducible. The Casimir operator is $(1 \leq i, j \leq n-1)$

$$\sum_{i<j} m_{ij} m_{ij} + \sum_i m_{ii} m_{ii} - \sum_i m_{i0} m_{i0} - a^2 = -\mu^2 R^2 \mathbb{1},$$

where $a$ is the representation of $a = m_{n0}$. Denoting the representations of $n_i$ by $n_i$,

$$\sum_{i<j} m_{ij} m_{ij} + \sum_i (n_i - m_{i0})(n_i - m_{i0}) - \sum_i m_{i0} m_{i0} - a^2 = -\mu^2 R^2 \mathbb{1} : .$$

$$\frac{1}{R^2} \sum_{i<j} m_{ij} m_{ij} + \sum_i \left( n'_i n'_i - n'_i \frac{1}{R} m_{i0} - \frac{1}{R} m_{i0} n'_i \right) - (a')^2 = -\mu^2 \mathbb{1}. \quad (20)$$
Hence, after we take the limit \( R \to \infty \), we find the Casimir operator \( \mathcal{P}^2 \) of the Poincaré algebra \( \mathfrak{p}_n \), together with its eigenvalue:

\[
\sum_i (n'_i)^2 - (a'_i)^2 = -\mu^2 \mathbb{1} =: \mathcal{P}^2 .
\]

(21)

If we had restricted the irreducible representation of the Lorentz group to the Hilbert space \( \mathcal{L}^2(d\Sigma^-_\mathbb{S}, d\Sigma) \), we would have found that the eigenvalue of the Casimir operator would be found from \( \mu' \mapsto -\mu' ; \; \mu^2 \mapsto \mu^2 \). Therefore the eigenvalue of the Casimir operator \( \mathcal{P}^2 \) of the Poincaré algebra \( \mathfrak{p}_n \) is uniquely determined by the eigenvalue of the Casimir operator \( \mathcal{C}^2 \) of the Lorentz algebra \( \mathfrak{l} \). This result is the first step in the comparison of the wavepackets on \( d\mathbb{S} \), to be constructed in the following section, and the usual wavepackets on \( M_n \). We record this result in the following

**Lemma 4.1.** The irreducible representation of the Casimir operator \( \mathcal{C}^2 \) of the Lorentz algebra \( \mathfrak{l} \), with eigenvalue \(-\mu^2 \mathbb{R}^2 \), denoted by \( \mu \mathcal{C}^2 \), under the contraction of the Lorentz algebra \( \mathfrak{l} \) into the Poincaré algebra \( \mathfrak{p}_n \) presented in the former section, is contracted towards the direct sum of two irreducible representations of the Casimir operator \( \mathcal{P}^2 \) of the Poincaré algebra \( \mathfrak{p}_n \), both of them with eigenvalue \(-\mu^2 \), denoted by \( \pm \mu \mathcal{P}^2 \),

\[\mu \mathcal{C}^2 \xrightarrow{R \to \infty} +\mu \mathcal{P}^2 \oplus -\mu \mathcal{P}^2 .\]

5. Wavepackets

Harish-Chandra’s analysis \([20, 21]\) is directly applicable to the homogeneous space \( L/K \). He consistently defined a Fourier transform on integrable functions (with a certain Haar measure) on this space and the inverse transform. The analogue Plancherel and Paley-Wiener theorems have been proved later (see \([23, 43, 44]\) for a complete analysis and references). We will not follow his approach here, but rather use his results to present some definitions and to interpret our own.

Harish-Chandra defined the Fourier transform, which later became known as the *Fourier-Helgason transform*, of a smooth compactly supported function \( f \in \mathcal{D}(L/K) \), as a function in \( \mathcal{F} \times (K/M) \), according to the following definition:

**Definition 5.1.** Let \( f \in \mathcal{D}(L/K) \). The Fourier-Helgason transform of \( f \) is the function \( \hat{f} \) on \( \mathcal{F} \times (K/M) \) given by

\[
\hat{f}(\nu, \hat{k}) := \int_L f(x)e^{-(i\nu + \rho)(H^{-1}x)}dL .
\]

(22)
Here, $H(x^{-1}k) \in a$ denotes the log of the distance from the origin of $L/K$ to the (unique) horosphere passing through $xK$ with normal $k = kM$. $dL$, as above, is the unique left-invariant Haar measure on $L$ (unimodular, because $L$ is semi-simple) and $\nu \in \mathcal{F}$.

The inverse transform is given by

$$f(x) = [W]^{-1} \int_{\mathcal{F} \times (K/M)} \hat{f}(\nu, \dot{k}) e^{-(\nu + \rho)(H(x^{-1}k))} |c(\nu)|^{-2} d\nu d(K/M),$$

where $[W]$ is the order of the Weyl group, $d\nu$ is the measure on $\mathcal{F}$, $d(K/M)$ is the left-invariant Haar measure on $K/M$ and $c(\nu)$ is the Harish-Chandra function, $c(\nu) = I(i\nu)/I(\rho)$, where

$$I(\nu) = \prod_{\alpha \in \mathcal{P}_+} B \left( \frac{m(\alpha)}{2}, \frac{m(\alpha/2)}{4} + \frac{(\nu, \alpha)}{(\alpha, \alpha)} \right),$$

and $B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x + y)$ is the Beta function. We note that $c^{-1}$ is analytic on $\mathcal{F}$. Moreover, $\forall \nu \in \mathcal{F}, \tau(\nu) = c(-\nu)$, $|c|^{-2}$ is also analytic on $\mathcal{F}$.

**Remarks 5.2.** There are some important results following the above definitions (for details and further references, see [4, 24, 44]):

(i) Let $\hat{\nu} \in L^2(K/M, d(K/M))$, $E_\nu$ be the vector space consisting of those functions on $L$ which can be written as

$$v_\nu(x) = \int_{K/M} e^{-(\nu + \rho)(H(x^{-1}k))}\hat{\nu}(\dot{k})d(K/M).$$

$E_\nu$ can be completed to a Hilbert space with norm

$$||v_\nu|| := \left\{ \int_{K/M} |\hat{\nu}(\dot{k})|^2 d(K/M) \right\}^{1/2},$$

hence we have a direct integral decomposition

$$L^2(L/K, dL) = \bigoplus E_\nu |c(\nu)|^{-2} d\nu.$$

Therefore the Fourier-Helgason transform extends to an isometric isomorphism:

$$L^2(L/K, dL) \simeq L^2 \left( \mathcal{F} \times K/M, |c(\nu)|^{-2} d\nu d(K/M) \right).$$
(ii) The functions $e_{b,\nu} : L \ni x \mapsto e^{-(i\nu + \rho)(H(x^{-1}k))}$, $b = kM$, are eigenfunctions of the Casimir element $C^2 \in \mathcal{U}$, represented as a differential operator on $\mathcal{L}^2(L/K, dL)$. Under the mapping induced by the diffeomorphism $\kappa'$ between $L/K$ and dS defined in \ref{eq:2}, $e_{b,\nu} \mapsto \psi_{\xi,\sigma}$. The functions $e_{b,\nu}$ are interpreted as the “plane waves” on the homogeneous space $L/K$. This reinforces the interpretation of $\psi_{\xi,\sigma}$ as the plane waves of de Sitter spacetime.

(iii) The measure $dL$ can be taken to be $dL(x) = e^{2\rho(H)}dA(h)dN(n)$, where $e^H = h \in A$, $dK$, $dA$ and $dN$ are the invariant Haar measures on $K$, $A$ and $N$ respectively. All these measures are unimodular \cite{15}.

Summarizing, $\mathcal{L}^2(L/K, dL)$ can be decomposed into a direct integral (sum) of eigenspaces of $C^2$. Therefore $\mathcal{L}^2(dS, d\Sigma)$ is decomposed into a direct integral (sum) of eigenspaces of $\square_{dS}$.

Recalling the definition of mass shell given in section 5.3, the determination of the mass $\mu$ constrains the value that the functional $\nu$ can assume, when operating on $a \in \mathfrak{a}$. $\nu(a) = \mu'd$, where $d$ is the distance from the origin $\vartheta$ of dS to the horosphere through $x$ with normal $kM$, in the notation of 5.2. The plane wave restricted to the mass shell will be designated by $\psi_{\mu}$, as a function on $dS \times \mathfrak{a}$. Hence, we may define a wavepacket as follows:

**Definition 5.3.** A wavepacket is the smearing of a compactly supported smooth function on the absolute with the plane wave restricted to the mass shell, i.e., let $\hat{f} \in \mathcal{D}(\mathfrak{a})$, the wavepacket is the function on dS given by

$$f(x) = N \int \hat{f}(\xi) \psi_{\mu}(x, \xi) [i_U \omega](\xi) ,$$

where

$$\psi_{\mu}(x, \xi) = \Theta(x \cdot \xi) \left| \frac{x \cdot \xi}{\mu R} \right|^{-(n-1)/2+\mu'} e^{\pi i(n-1)/2+\mu'} \Theta(-x \cdot \xi) \left| \frac{x \cdot \xi}{\mu R} \right|^{-(n-1)/2+\mu'},$$

$[i_U \omega]$ is the measure on the absolute $\mathfrak{a}$, the contraction of the invariant volume form $\omega$ on $\mathfrak{C}^+$ with the vector $U$ tangent to a curve that intercepts each (or almost every) generator of $\mathfrak{a}$ exactly once (in reality, any two homotopic curves would give the same measure \cite{10}) and $N$ is a normalization factor (containing the Harish-Chandra function and the order of the Weyl group).
In components,
\[
\omega = \frac{1}{2|\xi_0|} d\xi_1 \wedge \ldots \wedge d\xi_n \quad \text{and} \quad U = (\xi_0, \ldots, \xi_n).
\]

Therefore,
\[
[i_U \omega] = \frac{1}{2|\xi_0|} \sum_{j=1}^n (-1)^{j+1} \xi_j d\xi_1 \wedge \ldots \wedge \widehat{d\xi_j} \wedge \ldots \wedge d\xi_n,
\]
(27)

where \(\widehat{d\xi_j}\) means that the differential \(d\xi_j\) is not present in the product. The measure on \(A\) will be necessary when we analyze the flat limit of the wavepacket.

\(\hat{f} \psi \mu\) is a bounded and infinitely differentiable function in the region where \(\psi \mu\) is holomorphic. Then, from the dominated convergence theorem, the limit and integration operations commute, hence the derivatives of the wavepacket are the smearing of \(\hat{f}\) with the derivatives of the plane wave. Therefore the wavepacket is also a solution of the wave equation (11).

5.1. Asymptotic behaviour

The estimate of the asymptotic behaviour of the wavepacket is crucial for the construction of a state which, asymptotically, can be interpreted as a free particle. The wavepacket on dS is defined in (25) above. Since \(\xi \in A\), \(\xi \cdot \xi = 0\) and \(\xi_0 = \sqrt{\sum_{i=1}^n (\xi_i)^2}\).

The wavepacket is a function of fast decrease, in the sense that, if \(s := |x_0| + |x| \to \infty\), where \(|x| := \sqrt{\sum_{i=1}^n (x_i)^2}\), the wavepacket decays faster than any inverse power of \(s\). This is stated and proved in the following

**Theorem 5.4.** A wavepacket on de Sitter spacetime is a function of fast decrease.

**Proof.** Since the derivatives of the wavepacket are the smearing of \(\hat{f}\) with the derivatives of the plane wave, we will focus only on \(\psi \mu\) (see equation [26]). We will apply the stationary phase method to estimate the asymptotic behaviour of \(f\) [40]. The oscillatory terms are

\[
\left|\frac{x \cdot \xi}{\mu R}\right|^{-(n-1)/2+i\mu'} = \exp \left\{ \left(-\frac{n-1}{2} + i\mu'\right) \log \left|\frac{-x_0 \sqrt{\sum_{i=1}^n (\xi_i)^2} + \sum_{i=1}^n x_i \xi_i}{\mu R}\right| \right\}
\]

\[
= \exp \left\{ (|x_0| + |x|) \left(-\frac{n-1}{2} + i\mu'\right) \Phi_\mu(\xi) \right\},
\]
where
\[ \Phi_x(\xi) := \frac{1}{|x_0| + |x|} \log \left| -x_0 \sqrt{\sum_{i=1}^{n} (\xi_i)^2} + \sum_{i=1}^{n} x_i \xi_i \right| \mu R. \] (28)

It must first be checked whether \( \Phi_x(\xi) \) has any fixed points.

\[ \text{grad } \Phi_x(\xi) = \frac{1}{(|x_0| + |x|)(\xi \cdot \xi)} \left( x_1 - x_0 \frac{\xi_1}{\xi_0}, \ldots, x_n - x_0 \frac{\xi_n}{\xi_0} \right) \] (29)

(this expression is valid for the two possible signs of \( x \cdot \xi \)). However, \( \forall i \in \{1, \ldots, n\}, -1 \leq \frac{\xi_i}{\xi_0} \leq 1 \). Besides, \( (x_0)^2 + R^2 = \sum_{i=1}^{n} (x_i)^2 \). Therefore, if \( \forall i \in \{1, \ldots, n\}, x_i \frac{\xi_i}{\xi_0} = x_i, \)

\[ (x_0)^2 + R^2 = \left( \frac{x_0}{\xi_0} \right)^2 \sum_{i=1}^{n} (\xi_i)^2 = (x_0)^2 \] \( \therefore R = 0 \),

which is an absurd. Therefore \( \text{grad } \Phi_x(\xi) \neq 0 \), hence the stationary phase method tells that, \( \forall m > 0, \exists c > 0 \) such that

\[ \lim_{s \to \infty} f(x) < c (1 + s)^{-m}. \]

\[ \square \]

The asymptotic behaviour of the wavepacket on de Sitter spacetime is different from the minkowskian case \[26, 38\]. There the phase of the wavepacket had critical points located along a trajectory in spacetime whose tangent vector was the phase velocity of the packet. Inside a neighbourhood of that trajectory the amplitude of the wavepacket decayed at a certain rate. Outside of that neighbourhood, the amplitude decreased fast. In the present case there is no stationary point. Geometrically, a null vector cannot be tangent to the de Sitter spacetime. This is characteristic of harmonic analysis on semi-simple Lie groups \[16, 21, 23, 44\]. We remark, in addition, that when the limit \( s \to \infty \) is taken, we may have \( |x| \to \infty \) only if \( |x_0| \to \infty \), because the dS is spatially compact.

5.2. Flat limit

At last we want to compare the behaviour of the wavepacket (25) in the limit \( R \to \infty \) with the usual construction performed on the Minkowski spacetime \( M^4 \). Besides showing consistency of our construction, this comparison will allow us to find an interpretation for \( \xi \).
The expression (15) exhibits the two possible solutions of the wave equation. The (complex) phase factor that multiplies the term which can be nonzero when \( x \cdot \xi < 0 \) reads

\[
\exp \left[ \mp \pi \left( \frac{i(n-1)}{2} + \mu' \right) \right],
\]

whether \( x = x_\pm \) in (15). Let us first choose \( x = x_+ \), i.e., \( x \) is the limit of a complex variable \( z \in T^+ \subset \mathbb{C}^{n+1} \) when the imaginary part of \( z \) tends to zero. Then, as \( R \to \infty \), the above phase factor decays exponentially and the only term in (15) which survives the flat limit is the one that is nonzero when \( x \cdot \xi > 0 \). If we choose \( x = x_- \), the phase factor blows up exponentially, and we regard this choice as nonphysical. Henceforth we assume that \( x = x_+ \) and \( x \cdot \xi > 0 \). We will show the consequences of this choice for the analysis of the limit \( R \to \infty \).

\( \hat{f} \) is a compactly supported function on \( \mathcal{A} \) and we consider its support to be contained in a small neighbourhood of a covector \( \xi' \in \mathcal{A} \). If we choose a coordinate system in which \( \xi' \) has components \((1, 0, \ldots, 0, 1)\), then

\[
x \cdot \xi' > 0 \Rightarrow x_n > x_0,
\]

which corresponds to a half of dS which will be designated as \( \mathbb{H} \). If, however, we pick a different element \( \xi'' \) in the support of \( \hat{f} \) and choose a new coordinate system such that, now, \( \xi'' \) has those same components, then

\[
x \cdot \xi'' > 0 \Rightarrow x'_n > x_0.
\]

Since the elements of \( \mathcal{A} \) can be obtained one from another by a rotation, the half of dS characterized by \( x'_n > x_0 \) is just a rotation of \( \mathbb{H} \). Therefore the region where \( x \cdot \xi > 0 \) for every \( \xi \in \text{supp} \hat{f} \), the intersection of all the regions just described, is contained in \( \mathbb{H} \). On the other hand, the region where \( x \cdot \xi > 0 \) for some \( \xi \in \text{supp} \hat{f} \) comprises a neighbourhood containing \( \mathbb{H} \). But since \( \text{supp} \hat{f} \) is contained in a small neighbourhood of \( \xi' \), the regions where \( x \cdot \xi > 0 \) for some \( \xi \in \text{supp} \hat{f} \) and \( x_n < x_0 \) are contained in small neighbourhoods of \( x_n = x_0 \).

We are first going to analyze the asymptotic behaviour of the plane wave in \( \mathbb{H} \) using the chart described in (3), with \( \varepsilon = +1 \).

The contraction of the Lorentz algebra presented in section 4 showed that the generators of the horospheric translations contract into the generators of spatial translations and a contracts into the generator of time translations. Moreover,
(3) gives the change of coordinates from \((x_0, \ldots, x_n)\) to \((\tau, y)\). Actually, this is a restriction to a submanifold, from \(\mathbb{M}_{n+1}\) to \(\mathbb{H}\). This fact will play an important role in the following. Analyzing (3) we find
\[
\frac{x_n - x_0}{R} = e^{-\tau/R} \therefore \tau = -R \log \left(\frac{x_n - x_0}{R}\right)
\]
\[
\therefore y_i = -\frac{Rx_i}{x_n - x_0}.
\]
These equations are necessary in order to write the partial derivatives in the new coordinate system. The generators \(a\) and \(n_i\) are represented as differential operators as (see (4), (5) and (7))
\[
a = iR \left(\frac{\partial}{\partial \tau} + \sum \frac{y_i}{R} \frac{\partial}{\partial y_i}\right),
\]
and
\[
n_i = iR \frac{\partial}{\partial y_i}.
\]
We remark here that the horospheric translation is represented simply as a partial differentiation. Moreover, all the other terms in the differential operator (20) have higher powers of \(1/R\).

Now, we are going to act with the operators \(n'_i n'_i, (1 \leq i \leq n - 1)\) and \(a'^2\) (defined in (17)) on the plane wave \(\psi_\mu\) and analyze the behaviour of the result in the limit \(R \to \infty\).

\[
n'_i n'_i \left(\frac{x \cdot \xi}{\mu R}\right)^\sigma = \]
\[
= -\frac{\sigma}{\mu R} \left(\frac{x \cdot \xi}{\mu R}\right)^{\sigma - 2} \left\{ \frac{\sigma - 1}{\mu R} \left[ \frac{y_i}{R} (\xi_0 - \xi_n) - \xi_i \right]^2 e^{-2\tau/R} + \left(\frac{x \cdot \xi}{\mu R}\right) \frac{\xi_0 - \xi_n}{R} e^{-\tau/R} \right\},
\]
and
\[
a'^2 \left(\frac{x \cdot \xi}{\mu R}\right)^\sigma = \]
\[
= -\frac{\sigma}{\mu R} \left(\frac{x \cdot \xi}{\mu R}\right)^{\sigma - 2} \left\{ \frac{\sigma - 1}{\mu R} \left[ -\cosh \left(\frac{\tau}{R}\right) \xi_0 + \sinh \left(\frac{\tau}{R}\right) \xi_n + \frac{1}{2} \frac{|y|^2}{R^2} e^{-\tau/R} (\xi_0 - \xi_n) \right]^2 + \left(\frac{x \cdot \xi}{\mu R}\right) \frac{1}{R} \left[ -\sinh \left(\frac{\tau}{R}\right) \xi_0 + \cosh \left(\frac{\tau}{R}\right) \xi_n + \frac{1}{2} \frac{|y|^2}{R^2} e^{-\tau/R} (\xi_0 - \xi_n) \right] \right\},
\]
Yet,
\[
\frac{\sigma}{\mu R} = -\frac{n-1}{2\mu R} + i \sqrt{1 - \left(\frac{n-1}{2\mu R}\right)^2} = i + \mathcal{O}(1/R)
\]
and the same is valid for \(\frac{\sigma-1}{\mu R}\). Hence
\[
n'_i n'_i \left(\frac{x \cdot \xi}{\mu R}\right) ^ \sigma = \left(\frac{x \cdot \xi}{\mu R}\right) ^ {\sigma-2} [\xi_i]^2 + \mathcal{O}(1/R) \quad \text{and} \quad (35)
\]
\[
a'^2 \left(\frac{x \cdot \xi}{\mu R}\right) ^ \sigma = \left(\frac{x \cdot \xi}{\mu R}\right) ^ {\sigma-2} [\xi_0]^2 + \mathcal{O}(1/R) . \quad (36)
\]
Besides, \(x \cdot \xi = -\tau \xi_0 + \sum_i y_i (\xi_i) + R\xi_n + \mathcal{O}(1/R)\), hence
\[
\frac{x \cdot \xi}{\mu R} = -\tau \xi_0 + \sum_i y_i (\xi_i) + \frac{\xi_n}{\mu} + \mathcal{O}(1/R^2) . \quad (37)
\]
Since the coordinates of a point of \(\mathbb{H}\) are parametrized by the pair \((\tau, y)\), we define \(y = (\tau, y), \bar{\xi} = (\xi_0, -\xi_i)\) and \(y \cdot \bar{\xi} := -\tau \xi_0 + \sum_i y_i (-\xi_i)\). Therefore
\[
\left(\frac{x \cdot \xi}{\mu R}\right) ^ {\sigma-2} = \left(\frac{y \cdot \bar{\xi} + \xi_n}{\mu R}\right)^{\mu R(i + \mathcal{O}(1/R))}
\]
\[
= \left(\frac{y \cdot \bar{\xi}}{\xi_n R + 1}\right)^{\xi_n R(i + \mathcal{O}(1/R))} \left(\frac{\xi_n}{\mu}\right)^{\mu R(i + \mathcal{O}(1/R))} \quad (37)
\]
If \(\xi_n \neq \mu\), the last term will oscillate extremely rapidly in the limit and
\[
\lim_{R \to \infty} \left(\frac{x \cdot \xi}{\mu R}\right) ^ {\sigma-2} = 0 .
\]
However, if \(\xi_n = \mu\),
\[
\lim_{R \to \infty} \left(\frac{x \cdot \xi}{\mu R}\right) ^ {\sigma-2} = \lim_{R \to \infty} \left(\frac{y \cdot \bar{\xi}}{\mu R + 1}\right)^{\mu R(i + \mathcal{O}(1/R))} = e^{iy \bar{\xi}} \quad (38)
\]
and
\[
\bar{\xi} \cdot \bar{\xi} = -\mu^2 . \quad (39)
\]
The same result would be found if one had calculated \(\lim_{R \to \infty}(\ldots)^{\sigma}\) instead.
Therefore, collecting the results (38), (36) and (35) and considering the action of the operator (21) on the plane wave, one finds

\[
\lim_{R \to \infty} \left[ \sum_i (n'_i)^2 - (a'_i)^2 \right] \left( \frac{x \cdot \xi}{\mu R} \right)^\sigma = \left[ (\xi_1)^2 - (\xi_0)^2 \right] e^{i y \cdot \overline{\xi}} = -\mu^2 e^{i y \cdot \overline{\xi}}. \tag{40}
\]

\(y\) represents the coordinates of a point of the resulting Minkowski spacetime \(\mathbb{M}_n\) and \(\overline{\xi}\) is a timelike vector on the mass shell of \(\mathbb{M}_n\) (39). Their origin, however, are restrictions of the coordinates \(x\) of a point on dS and the null covector \(\xi\) on the absolute, respectively. All other terms of \(\Box_{dS}\) have higher powers of \(1/R\), and therefore their contribution would converge to zero.

We still have to analyze the asymptotic behaviour of \(\psi_{\mu}\) in the points where \(x \cdot \xi > 0\) for some \(\xi \in \text{supp} \hat{f}\), but \(x_n < x_0\). This region can be covered by the chart (31), with \(\varepsilon = -1\). Since those points are contained in a small neighbourhood of \(x_n = x_0\), the previous analysis shows that, after the limit \(R \to \infty\), these points will be at infinity. Thus they are irrelevant, because of the decay of the wavepacket. Therefore we have shown that the plane wave on the whole \(\mathbb{M}_n\) is the limit \(R \to \infty\) of the plane wave on half of dS. Collecting these results in a sentence, on \(\mathbb{H}\),

\[
\lim_{R \to \infty} \psi_{\mu}(x, \xi) = \begin{cases} 
 e^{i y \cdot \overline{\xi}}, & \text{if } \xi_n = \mu; \\
 0, & \text{if } \xi_n \neq \mu. 
\end{cases} \tag{41}
\]

This function is supported on a set of measure zero in the variable \(\xi_n\). Hence when we multiply this limit with \(\hat{f}\) and integrate in the measure (27), we obtain

\[
\lim_{R \to \infty} f(x) = N' \int \hat{f}(\overline{\xi}) e^{i y \cdot \overline{\xi}} \frac{1}{2|\xi_0|} d\xi_1 \wedge \ldots \wedge d\xi_{n-1}, \tag{42}
\]

a wavepacket on Minkowski spacetime.

Before collecting the results of this section, let us note that the plane wave and wavepacket on dS are of fast decay, but this is not the behaviour of a wavepacket on Minkowski spacetime \([26, 27]\). We can see the change in the asymptotic behaviour as \(R\) becomes larger. In (38) one notes that the leading term in the exponent of the plane wave (above (28)), for large \(R\), is

\[
\frac{1}{s} \mu R \log \left[ 1 + \frac{y \cdot \overline{\xi}}{\mu R} \right] =: \Gamma_y(\overline{\xi})
\]
(the term $i$ is not important now). Hence if we subtract and add this term to the phase,

$$\left[ \mu R \Phi_x(\xi) - \Gamma_y(\xi) \right] + \Gamma_y(\xi), \quad (43)$$

the term between brackets goes to zero in the limit $R \to \infty$, but $\text{grad} \Gamma_y(\xi)$ has fixed points. Hence as the de Sitter radius $R$ gets larger, the wavepacket decreases more slowly, until it reaches the rate $s^{-3/2}$ given by the stationary phase principle [40] and calculated in [27, 26].

All the results of this section are collected in the following

**Theorem 5.5.** The limit, as $R \to \infty$, of the wavepacket on the $n$-dimensional de Sitter spacetime is a wavepacket on Minkowski spacetime, analytic in the whole $\mathbb{M}_n$, with mass sharply constrained to the mass shell (now in the sense of (39)) and determined by the mass of its precedent wavepacket on $dS$.

### 6. Conclusions

We have shown that one can consistently construct wavepackets on the de Sitter spacetime whose mass is defined from one of the Casimir elements in the universal enveloping algebra of the Lorentz algebra. The mass shell, at this level, is a restriction on the space of functionals on the maximal abelian subalgebra of $\mathfrak{l}$. The wavepacket is a function of fast decrease, differently from the wavepacket defined on Minkowski spacetime in [26, 27]. As we emphasized before, this is a general feature of harmonic analysis on semi-simple Lie groups.

The physical interpretation of the wavepacket became clearer after the evaluation of its flat limit, with the wavepackets converging to the usual one defined on Minkowski spacetime, but with support sharply constrained on the mass shell (now, a subset of momentum space). This is an important difference with respect to the wavepackets defined, from the beginning, on the Minkowski spacetime. These have momenta located on a neighbourhood of the mass shell.

We intend to use these wavepackets to try to formulate a Haag-Ruelle scattering theory on de Sitter spacetime. The next step, subject of a future work, is the construction of the Haag-Ruelle scattering states on $dS$ and the proof of existence of the S-matrix. The interpretation of these states as particle states might also be possible only after their flat limit is analyzed. This is also going to be inquired in future work.
MB acknowledges financial support from the São Paulo Research Foundation (FAPESP) under grant #2015/02975-4.

References

[1] E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer. Massless particles, conformal group, and de Sitter universe. *Phys. Rev. D*, 23:1278–1289, Mar 1981.
[2] V. Bargmann. Irreducible Unitary Representations of the Lorentz Group. *Annals of Mathematics*, 48(3):568–640, 1947.
[3] A. O. Barut and A. Böhm. Reduction of a Class of O(4, 2) Representations with Respect to SO(4, 1) and SO(3, 2). *Journal of Mathematical Physics*, 11(10):2938–2945, 1970.
[4] A. O. Barut and R. Rączka. *Theory of group representations and applications*. World Scientific Publishing, Singapore, 1986.
[5] J. Bros, H. Epstein, M. Gaudin, U. Moschella, and V. Pasquier. Triangular Invariants, Three-Point Functions and Particle Stability on the de Sitter Universe. *Communications in Mathematical Physics*, 295(1):261–288, 2010.
[6] J. Bros, H. Epstein, and U. Moschella. Analyticity Properties and Thermal Effects for General Quantum Field Theory on de Sitter Space-Time. *Communications in Mathematical Physics*, 196(3):535–570, 1998.
[7] J. Bros, H. Epstein, and U. Moschella. The lifetime of a massive particle in a de Sitter universe. *Journal of Cosmology and Astroparticle Physics*, 2008(02):003, 2008.
[8] J. Bros, H. Epstein, and U. Moschella. Particle Decays and Stability on the de Sitter Universe. *Annales Henri Poincaré*, 11(4):611–658, 2010.
[9] J. Bros, J.-P. Gazeau, and U. Moschella. Quantum field theory in the de Sitter universe. *Phys. Rev. Lett.*, 73:1746–1749, 1994.
[10] J. Bros and U. Moschella. Two-point functions and quantum fields in de Sitter universe. *Reviews in Mathematical Physics*, 08(03):327–391, 1996.
[11] J. Bros and U. Moschella. Fourier analysis and holomorphic decomposition on the one-sheeted hyperboloid. In *Géométrie complexe. II. Aspects contemporains*
REFERENCES

dans les mathématiques et la physique, pages 27–58. Hermann Éd. Sci. Arts, Paris, 2004.

[12] J. Dixmier. Représentations intégrables du groupe de de Sitter. Bulletin de la Societe mathematique de France, 89:9–41, 1961.

[13] W. Dybalski. From Faddeev-Kulish to LSZ. Towards a non-perturbative description of colliding electrons. Nuclear Physics B, 925:455 – 469, 2017.

[14] W. Dybalski and C. Gérard. A Criterion for Asymptotic Completeness in Local Relativistic QFT. Communications in Mathematical Physics, 332(3):1167–1202, Dec 2014.

[15] G. B. Folland. A Course in Abstract Harmonic Analysis. Textbooks in Mathematics. Chapman and Hall/CRC, 2nd edition, 2015.

[16] R. Gangolli. On the Plancherel Formula and the Paley-Wiener Theorem for Spherical Functions on Semisimple Lie Groups. Annals of Mathematics, 93(1):150–165, 1971.

[17] I. M. Gel’fand, M. I. Graev, and N. Ya. Vilenkin. Generalized functions. Vol. 5. Integral geometry and representation theory. AMS Chelsea Publishing, Providence, RI, 2016.

[18] R. Haag. Quantum Field Theories with Composite Particles and Asymptotic Conditions. Phys. Rev., 112:669–673, 1958.

[19] K. C. Hannabuss. The localizability of particles in de Sitter space. Mathematical Proceedings of the Cambridge Philosophical Society, 70(2):283–302, 1971.

[20] Harish-Chandra. Spherical Functions on a Semisimple Lie Group, I. American Journal of Mathematics, 80(2):241–310, 1958.

[21] Harish-Chandra. Spherical Functions on a Semisimple Lie Group II. American Journal of Mathematics, 80(3):553–613, 1958.

[22] S. W. Hawking. Particle creation by black holes. Communications in Mathematical Physics, 43:199–220, 1975.

[23] S. Helgason. An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces. Mathematische Annalen, 165(4):297–308, Dec 1966.

[24] S. Helgason. Lie Groups and Symmetric Spaces. In Cecile M. DeWitt and John A. Wheeler, editors, Battelle Rencontres – 1967 Lectures in Mathematics and Physics, pages 1 – 71. W. A. Benjamin, New York, 1968.
REFERENCES

[25] S. Helgason. *Groups and Geometric Analysis: Integral Geometry, Invariant Differential Operators, and Spherical Functions*. Mathematical surveys and monographs. American Mathematical Society, 1984.

[26] K. Hepp. On the connection between the LSZ and Wightman quantum field theory. *Communications in Mathematical Physics*, 1(2):95–111, 1965.

[27] K. Hepp. On the Connection Between Wightman and LSZ Quantum Field Theory. In *Lecture in Theoretical Physics: Axiomatic Field Theory*, Brandeis Summer Institute, New York, 1965. Gordon and Breach.

[28] J. Hilgert and K.-H. Neeb. *Structure and Geometry of Lie Groups*. Springer, New York, 2012.

[29] E. Inonu and E. P. Wigner. On the Contraction of Groups and Their Representations. *Proceedings of the National Academy of Sciences of the United States of America*, 39(6):510–524, 1953.

[30] D. P. Jatkar, L. Leblond, and A. Rajaraman. Decay of massive fields in de Sitter space. *Phys. Rev. D*, 85:024047, 2012.

[31] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*. Course of Theoretical Physics, volume 2. Butterworth-Heinemann, 1975.

[32] D. Marolf, I. A. Morrison, and M. Srednicki. Perturbative S -matrix for massive scalar fields in global de Sitter space. *Classical and Quantum Gravity*, 30(15):155023, 2013.

[33] J. Mickelsson and J. Niederle. Contractions of representations of de Sitter groups. *Communications in Mathematical Physics*, 27(3):167–180, 1972.

[34] V. F. Molchanov. Harmonic analysis on a hyperboloid of one sheet. *Soviet Math. Dokl.*, 7(6):1553, 1966.

[35] U. Moschella. Infrared surprises in the de Sitter universe. *International Journal of Modern Physics D*, 25(09):1641020, 2016.

[36] U. Moschella and R. Schaeffer. Quantum theory on lobatchevski spaces. *Classical and Quantum Gravity*, 24(14):3571, 2007.

[37] B. O’Neill. *Semi-Riemannian geometry*. Academic Press Inc., New York, 1983.

[38] D. Ruelle. On the asymptotic condition in quantum field theory. *Helvetica Physica Acta*, 35(3):147–163, 1962.

[39] E. Schrödinger. *Expanding Universes*. Cambridge University Press, 2011.
[40] E. M. Stein and R. Shakarchi. *Functional analysis*, volume 4 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2011. Introduction to further topics in analysis.

[41] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14:870–892, 1976.

[42] N. Ja. Vilenkin and A. U. Klimyk. *Representation of Lie groups and special functions. Vol. 2*, volume 74 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1993. Class I representations, special functions, and integral transforms, Translated from the Russian by V. A. Groza and A. A. Groza.

[43] G. Warner. *Harmonic Analysis on Semi-Simple Lie Groups. I*. Springer-Verlag, New York-Heidelberg, 1972.

[44] G. Warner. *Harmonic Analysis on Semi-Simple Lie Groups. II*. Springer-Verlag, New York-Heidelberg, 1972.

[45] E. P. Wigner. On Unitary Representations of the Inhomogeneous Lorentz Group. *Annals of Mathematics*, 40(1):149–204, 1939.