Perturbative renormalization for overlap fermions

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Using lattice overlap fermions, we have computed the 1-loop renormalization factors of several operators that measure DIS structure functions and weak amplitudes. Computer codes written in the algebraic manipulation language FORM have been used. The improvement of the operators is also discussed.

1. INTRODUCTION

Overlap fermions\textsuperscript{[1,2]} exhibit an exact chiral symmetry on the lattice also for non-zero lattice spacing\textsuperscript{[3]}. Recent numerical works show the quite good accuracy with which chiral symmetry is attained in the overlap formulation\textsuperscript{[4,5]}.

The overlap-Dirac operator that we use is

\[ D_N = \frac{\rho}{a} \left[ 1 + \frac{X}{|X|} \right], \quad X = D_W - \frac{\rho}{a}, \quad 0 < \rho < 2, \]

where \( D_W \) is the Wilson-Dirac operator (\( r = 1 \)).

2. RENORMALIZATION

Monte Carlo computations of matrix elements of operators must be renormalized in order to obtain physical results. Lattice simulations need thus to be supported by the knowledge of the renormalization factors of the relevant operators.

Operator mixing for overlap fermions is simpler than for Wilson fermions. Chiral symmetry prohibits any mixings among operators of different chirality and in general reduces the number of operators which mix. Operators whose mixing coefficients are power-divergent like \( a^{-n} \) in the continuum limit do not mix in the overlap if they belong to multiplets with the wrong chirality.

The renormalization of the bilinears \( \bar{\psi} \Gamma \psi \) with overlap fermions was first computed by Alexandrou et al.\textsuperscript{[6]}. After that we have calculated the renormalization of several operators measuring the lowest moments of DIS structure functions\textsuperscript{[7]}. We have considered the unpolarized quark distribution \( q \), the helicity distribution \( \Delta q \), the transversity distribution \( \delta q \) and also the \( g_2 \) structure function. Their moments are proportional to the hadronic matrix elements of the towers of operators

\[ \langle x^n \rangle_q \sim \langle \bar{p}, \bar{s} | \bar{\psi} \gamma_\mu D_{\mu_1} \cdots D_{\mu_n} | \psi | \bar{p}, \bar{s} \rangle \]

In Table 1 we show the \( z \)'s of the multiplicatively renormalized operators for two values of the parameter \( \rho \) (and also for Wilson), for \( \beta = 6.0 \), in the \( \overline{\text{MS}} \) scheme. In some cases for a given moment we have computed two operators (\( a \) and \( b \)), which belong to two different representations of the discrete Euclidean Lorentz group. In the operators \( a \) all Lorentz indices are distinct.

In the Wilson case the operators measuring the moments of the \( g_2 \) structure function are not multiplicatively renormalized, and the additional mixings with wrong-chirality operators have power-divergent coefficients. Moreover, the \( Z \)'s of the corresponding moments of \( q \) and \( \Delta q \) are not constrained to be equal anymore.

It can be seen in Table 1 that the renormalization factors for \( \rho = 1.0 \) are large. Contrary to Wilson fermions, most of the renormalization comes from the quark self-energy, while the contributions of the remaining diagrams are small, close to the corresponding Wilson results and depend very little on \( \rho \). The quark self-energy instead decreases as \( \rho \) increases, and this suggests to consider higher \( \rho \)'s to get smaller \( Z \)'s. The \( Z \)'s for \( \rho = 1.9 \) are indeed smaller than for \( \rho = 1.0 \), and closer to the Wilson results (see Table 1).
Table 1
Renormalization constants of operators measuring moments of structure functions, for $\beta = 6.0$.

| moment | overlap $\rho = 1.0$ | overlap $\rho = 1.9$ | Wilson |
|--------|----------------------|----------------------|--------|
| $\langle x \rangle_q^{(a)}$ | 1.12123 | 1.21841 | 0.98920 |
| $\langle x \rangle_q^{(b)}$ | 1.40847 | 1.21309 | 0.97837 |
| $\langle x \rangle_q^{(b)}$ | 1.40847 | 1.21309 | 0.99859 |
| $\langle x \rangle_q^{(c)}$ | 1.51968 | 1.32436 | 1.09763 |
| $\langle x \rangle_q^{(d)}$ | 1.51968 | 1.32436 | 1.10231 |
| $\langle x \rangle_{g_2}$ | 1.61872 | 1.42279 | 1.19722 |
| $\langle x \rangle_{g_2}$ | 1.61872 | 1.42279 | 1.20040 |
| $\langle x \rangle_{g_2}$ | 1.63737 | 1.44159 | 1.21534 |
| $\langle x \rangle_{g_2}$ | 1.63737 | 1.44159 | 1.21944 |
| $\langle x \rangle_{g_2}$ | 1.34794 | 1.18456 | mixing |
| $\langle x \rangle_{g_2}$ | 1.47816 | 1.30997 | mixing |
| $\langle x \rangle_{g_2}$ | 1.58943 | 1.41900 | mixing |
| $\langle x \rangle_{g_2}$ | 1.27252 | 1.08648 | 0.85631 |
| $\langle x \rangle_{g_2}$ | 1.41153 | 1.21851 | 0.99559 |
| $\langle x \rangle_{g_2}$ | 1.51865 | 1.32355 | 1.10021 |

The renormalization of the four-fermion operators of the $\Delta F = 2$ and $\Delta S = 1$ effective weak Hamiltonians has been studied together with L. Giusti[3]. They describe physics like the $K^0$-$\bar{K}^0$ and $B^0$-$\bar{B}^0$ mixings, the $\Delta I = 1/2$ rule (octet enhancement) and the $CP$ violation parameter $\epsilon'/\epsilon$. The two latter cases are not easy to study with Wilson fermions, because of power-divergent operators that occur under renormalization and which have to be non-perturbatively subtracted. Neuberger’s fermions are appealing because the exact chiral symmetry forbids many mixings which occur in the Wilson case. The GIM mechanism, which as a consequence of chiral symmetry is quadratic instead of linear in the masses, is as powerful as in the continuum in eliminating unwelcome operators. Furthermore, corresponding parity-conserving and parity-violating operators are renormalized in the same way.

With overlap fermions the renormalization of the operators relevant for the $\Delta I = 1/2$ rule can be done without any power-divergent subtractions. The calculation of $\epsilon'/\epsilon$ is also greatly simplified compared to the Wilson case, although one power-divergent mixing still remains.

The flavor structure forbids mixings between $O_+$ and $O_-$ and chiral symmetry forbids mixings with other dimension-six operators. The complete renormalization is given by

$$\hat{O}_\pm(\mu) = Z_\pm(\mu \alpha, q_0^2)\hat{O}_\pm(\mu) + O(a^2)$$

$$\hat{O}_\pm(a) = O_\pm(a) + (m^2 - m^2)C_m^\pm(\mu^2)Q_m(a),$$

where $Q_m = (m_s + m_d)\bar{s}d + (m_s - m_d)\bar{s}\gamma_5d$. Thanks to the quadratic GIM factor $m^2 - m^2_u$, the mixing coefficients $C_m^\pm$ are finite. In principle the operators $O_\pm$ would mix also with $Q_m = g_0[(m_s + m_d)\bar{s}\gamma_5F_{\mu\nu}d + (m_s - m_d)\bar{s}\gamma_5F_{\mu\nu}d]$, however thanks to the quadratic GIM mechanism and to the $m_s \pm m_d$ factors coming from chiral symmetry these mixing coefficients are of $O(a^2)$.

In a regularization that breaks chiral symmetry, like Wilson, the GIM mechanism is only linear in the masses. Furthermore, the parity-conserving and parity-violating components of $Q_m$ and $Q_s$ behave differently; in particular, the $m_s \pm m_d$ factors are absent. The parity-conserving part of $C_m^\pm$ is then quadratically divergent $1^\text{st}$.

The $Z_\pm$ factors in $[3]$ are simple linear combinations of $Z_S$, $Z_V$ and $Z_\phi$. The $C_m^\pm$ coefficients are not needed for the physical $K \to \pi\pi$ matrix elements; if $K \to \pi\pi$ amplitudes are used, they can be determined by a 2-loop calculation or non-perturbatively using $K \to 0$ matrix elements.

All calculations have been done using codes written in the algebraic manipulation language FORM. The gauge invariance of the $Z$’s and the implementation of dimensional regularization (NDR and ’t Hooft-Veltman) and a mass regularization allow strong checks of the calculations,$^1$ Here $\gamma_{L,R}^\mu = \gamma^\mu(1 \mp \gamma_5)$ and $\alpha, \beta$ are color indices.

$^2$ The $m_s - m_d$ factors are present also for Wilson fermions thanks to CPS symmetry ($S$ is the interchange ($s \leftrightarrow d$)).
which we also did in some cases by hand. We also checked the Wilson results when known, and when not we computed them for the first time [7].

3. IMPROVEMENT

Overlap fermions present many advantages compared to Wilson fermions also when the issue is the Symanzik improvement of operators. Although Neuberger’s action (and therefore the spectrum of the theory) is already $O(a)$ improved, matrix elements of operators have in general $O(a)$ corrections. Operators of the form $O = \bar{\psi}O\psi$ are improved by considering [9,10]

$$O^{\text{imp}} = \bar{\psi} \left( 1 - \frac{1}{2} a D_N \right) \tilde{O} \left( 1 - \frac{1}{2} a D_N \right) \psi,$$

which works to all orders of perturbation theory.

Compared with Wilson fermions, the $O(a)$ improvement for Neuberger’s fermions presents a few very convenient simplifications:

1. the action is already improved, and there is no need of new interactions like for Wilson fermions (Sheikholeslami-Wohlert);
2. the renormalization constants for improved and unimproved operators are the same [1], while with Wilson fermions additional cumbersome calculations are needed to get the $Z$’s for the improved operators;
3. the improved operators are always given by Eq. (6), while for Wilson fermions the construction of the improved operator is different in each case;
4. full $O(a)$ improvement is achieved without tuning any coefficients, and it is valid to all orders of perturbation theory.

The last point is really a big calculational advantage. In fact, improving an operator with Wilson fermions means first finding out a complete basis of operator counterterms, and then determining for all counterterms the exact values of their coefficients that effectively improve the original operator, order by order in perturbation theory. This appears even at lowest order to be a highly demanding task, as it can be seen for the first moment of unpolarized structure functions, $\bar{\psi}\gamma_{\mu}(D_{\nu})\psi$. In this case two counterterms are needed,

$$O^{(1)}_{\mu\nu} = -\frac{1}{4} ac_1(\gamma^2_a) \sum_{\lambda} \bar{\psi} \sigma_{\lambda}[\mu \left( D_{\nu}, D_{\lambda} \right) \psi$$

$$O^{(2)}_{\mu\nu} = -\frac{1}{4} ac_2(\gamma^2_a) \bar{\psi} \left( D_{\mu}, D_{\nu} \right) \psi,$$

and up to now it has been possible to determine at 1 loop only one of the corresponding coefficients, while the other one remains unknown [10].

4. CONCLUSIONS

The overlap renormalization factors of most operators necessary in the study of DIS and weak processes are now known. The exact chiral symmetry of overlap fermions makes the study of long-standing problems like the $\Delta I = 1/2$ amplitudes and $\epsilon'/\epsilon$ much more accessible than before.

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