Multi-parameter singular Radon transforms II: the $L^p$ theory

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Abstract

The purpose of this paper is to study the $L^p$ boundedness of operators of the form

$$f \mapsto \psi(x) \int f(\gamma_t(x)) K(t) \, dt,$$

where $\gamma_t(x)$ is a $C^\infty$ function defined on a neighborhood of the origin in $(t,x) \in \mathbb{R}^N \times \mathbb{R}^n$, satisfying $\gamma_0(x) \equiv x$, $\psi$ is a $C^\infty$ cutoff function supported on a small neighborhood of $0 \in \mathbb{R}^n$, and $K$ is a “multi-parameter singular kernel” supported on a small neighborhood of $0 \in \mathbb{R}^N$. We also study associated maximal operators. The goal is, given an appropriate class of kernels $K$, to give conditions on $\gamma$ such that every operator of the above form is bounded on $L^p$ ($1 < p < \infty$). The case when $K$ is a Calderón-Zygmund kernel was studied by Christ, Nagel, Stein, and Wainger; we generalize their work to the case when $K$ is (for instance) given by a “product kernel.” Even when $K$ is a Calderón-Zygmund kernel, our methods yield some new results. This is the second paper in a three part series. The first paper deals with the case $p = 2$, while the third paper deals with the special case when $\gamma$ is real analytic.

1 Introduction

The goal of this paper is to prove the $L^p$ boundedness of (a special case of) the multi-parameter singular Radon transforms introduced in [Str11b]. We consider operators of the form

$$T(f)(x) = \psi(x) \int f(\gamma_t(x)) K(t) \, dt,$$

where $\psi$ is a $C^\infty$ cut off function (supported near, say, $0 \in \mathbb{R}^n$), $\gamma_t(x) = \gamma(t,x)$ is a $C^\infty$ function defined on a neighborhood of the origin in $\mathbb{R}^N \times \mathbb{R}^n$ satisfying $\gamma_0(x) \equiv x$, and $K$ is a “multi-parameter” distribution kernel, supported near $0$ in $\mathbb{R}^N$. For instance, one could take $K$ to be a “product kernel” supported near $0$. To define this notion, suppose we have decomposed $\mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu}$, and write $t = (t_1, \ldots, t_\nu) \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu}$. A product kernel satisfies

$$|\partial_1^{\alpha_1} \cdots \partial_\nu^{\alpha_\nu} K(t)| \lesssim |t_1|^{-N_1-|\alpha_1|} \cdots |t_\nu|^{-N_\nu-|\alpha_\nu|},$$

along with certain “cancellation conditions.”\textsuperscript{2} The goal is to develop conditions on $\gamma$ such that $T$ is bounded on $L^p$ ($1 < p < \infty$). In addition, we will prove (under the same conditions on $\gamma$) $L^p$ boundedness ($1 < p \leq \infty$) for the corresponding maximal operator,

$$\mathcal{M}f(x) = \psi(x) \sup_{0 < \delta_1, \ldots, \delta_\nu \leq a} \frac{1}{\delta_1^{N_1} \delta_2^{N_2} \cdots \delta_\nu^{N_\nu}} \int |f(\gamma_t(x))| \, dt_1 \cdots dt_\nu,$$

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\textsuperscript{1}Our main theorem applies to classes of kernels other than product kernels.

\textsuperscript{2}The simplest example of a product kernel is given by $K(t_1, \ldots, t_\nu) = K_1(t_1) \otimes \cdots \otimes K_\nu(t_\nu)$, where $K_1, \ldots, K_\nu$ are standard Calderón-Zygmund kernels. That is, $K_j$ satisfies $|\partial_j^{\alpha} K_j(t_j)| \lesssim |t_j|^{-N_j-|\alpha_j|}$, again along with certain “cancellation conditions.” When $\nu = 1$, the class of product kernels is precisely the class of Calderón-Zygmund kernels. See Section 16 of [Str11b] for the statement of the cancellation conditions. We do not make them precise in this paper, since we will be working with more general kernels $K$.
1.1 Informal statement of the main results

In this section, we informally state the special case of our main results when \( K (t_1, \ldots, t_\nu) \) is a product kernel relative to the decomposition \( \mathbb{R}^N = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_\nu} \) (see the introduction for this notion). We suppose we are given a \( C^\infty \) function, \( \gamma (t, x) = \gamma_t (x) \in \mathbb{R}^n \) defined on a small neighborhood of the origin in \( (t, x) \in \mathbb{R}^N \times \mathbb{R}^n \), satisfying \( \gamma_0 (x) \equiv x \). For \( t \) sufficiently small, \( \gamma_t \) is a diffeomorphism onto its image. Thus, it makes sense to write \( \gamma_t^{-1} \), the inverse mapping. We define the vector field

\[
W (t, x) = \frac{d}{dt} \bigg|_{t=1} \gamma_t \circ \gamma_t^{-1} (x) \in T_x \mathbb{R}^n.
\]

For a collection of vector fields \( \mathcal{V} \), let \( D (\mathcal{V}) \) denote the involutive distribution generated by \( \mathcal{V} \). I.e., the smallest \( C^\infty \) module containing \( \mathcal{V} \) and such that if \( X, Y \in D (\mathcal{V}) \) then \([X, Y] \in D (\mathcal{V})\). For a multi-index \( \alpha \in \mathbb{N}^N \), write \( \alpha = (\alpha_1, \ldots, \alpha_\nu) \), with \( \alpha_\mu \in \mathbb{N}^{N_\mu} \).

Decompose \( W \) into a Taylor series in the \( t \) variable,

\[
W (t, x) \sim \sum_{\alpha} t^\alpha X_\alpha.
\]

We call \( \alpha = (\alpha_1, \ldots, \alpha_\nu) \in \mathbb{N}^N \) a pure power if \( \alpha_\mu \neq 0 \) for precisely one \( \mu \). Otherwise, we call it a non-pure power.

We assume that the following conditions hold “uniformly” for \( \delta = (\delta_1, \ldots, \delta_\nu) \in (0, 1]^\nu \), though we defer making this notion of uniform precise to Section 4.

- For every \( \delta \in (0, 1]^\nu \),

\[
\mathcal{D}_\delta := D \left( \left\{ \delta_1^{|\alpha_1|} \cdots \delta_\nu^{|\alpha_\nu|} X_{\alpha_1, \ldots, \alpha_\nu} : (\alpha_1, \ldots, \alpha_\nu) \text{ is a pure power} \right\} \right)
\]

is finitely generated as a \( C^\infty \) module, uniformly in \( \delta \).
- For every \( \delta \in (0, 1]^\nu \),

\[
W (\delta_1 t_1, \ldots, \delta_\nu t_\nu) \in \mathcal{D}_\delta,
\]

uniformly in \( \delta \).
Remark 1.2. If it were not for the “uniform” aspect of the above assumptions, they would be independent of \( \delta \). Thus it is the uniform part, which we have not made precise, that is the heart of the above assumptions.

Our main theorems are,

**Theorem 1.3.** Under the above assumptions (which are made precise in Section 4), the operator given by

\[ f \mapsto \psi (x) \int f (\gamma_\delta (x)) K (t) \, dt, \]

is bounded on \( L^p (1 < p < \infty) \), for every product kernel \( K (t_1, \ldots, t_\nu) \), with sufficiently small support, provided \( \psi \) has sufficiently small support.

**Theorem 1.4.** Under the same assumptions, the maximal operator given by

\[ f \mapsto \psi (x) \sup_{0 < \delta_1, \ldots, \delta_\nu < 1} \int_{|t| < 1} \left| f (\gamma_{\delta_1 t_1, \ldots, \delta_\nu t_\nu} (x)) \right| \, dt \]

is bounded on \( L^p (1 < p \leq \infty) \).

The precise statement of our main result (where more general kernels are considered) can be found in Theorems 5.1, 5.2 and 5.3.

2 Kernels

In this section, we will discuss the classes of kernels \( K (t) \) for which we will study operators of the form \( (1.1) \). The kernels which we study will be supported in \( B^N (a) = \{ x \in \mathbb{R}^N : |x| < a \} \), where \( a > 0 \) is some small number to be chosen later (depending on \( \gamma \)). Fix \( \nu \in \mathbb{N} \), we will be studying \( \nu \) parameter operators. Fix \( 1 \leq \mu_0 \leq \nu \), and define,

\[ \mathcal{A}_{\mu_0} = \{ \delta = (\delta_1, \ldots, \delta_\nu) \in [0, 1]^\nu : \delta_{\mu_0} \leq \delta_{\mu_0 + 1} \leq \cdots \leq \delta_\nu \}. \]

The class of kernels we define will depend on \( \mu_0 \) (by depending on \( \mathcal{A}_{\mu_0} \)). In \[11b\], the class of kernels depended on a subset \( \mathcal{A} \subseteq [0, 1]^\nu \). In that paper, one could use any subset \( \mathcal{A} \) such that if \( \delta_1, \delta_2 \in \mathcal{A} \), then \( \delta_1 \lor \delta_2 \in \mathcal{A} \) (where \( \delta_1 \lor \delta_2 \) is given by taking the coordinatewise maximum of \( \delta_1 \) and \( \delta_2 \)). In this paper, we restrict our attention to \( \mathcal{A} \) of the form \( \mathcal{A}_{\mu_0} \) for some \( \mu_0 \). This is the only difference between the setting in \[11b\] and the setting in this paper. Notice, \( \mathcal{A}_{\nu} = [0, 1]^\nu \) and \( \mathcal{A}_1 = \{ \delta_1 \leq \delta_2 \leq \cdots \leq \delta_\nu \} \); these make up the principle examples we are interested in.

We suppose we are given \( \nu \)-parameter dilations on \( \mathbb{R}^N \). That is, we are given \( e = (e_1, \ldots, e_N) \), with each \( 0 \neq e_j = (e^1_j, \ldots, e^\nu_j) \in [0, \infty) ^\nu \). For \( \delta \in [0, \infty)^\nu \) and \( t = (t_1, \ldots, t_N) \in \mathbb{R}^N \), we define \[1\]

\[ \delta t = (\delta^{e_1} t_1, \ldots, \delta^{e_N} t_N), \]

thereby obtaining \( \nu \)-parameter dilations on \( \mathbb{R}^N \). For each \( \mu, 1 \leq \mu \leq \nu \), let \( t_\mu \) denote those coordinates \( t_j \) of \( t = (t_1, \ldots, t_N) \in \mathbb{R}^N \) such that \( e^{\mu}_j \neq 0 \). For \( j = (j_1, \ldots, j_\nu) \in \mathbb{Z}^\nu \), define \( 2^j = (2^{j_1}, \ldots, 2^{j_\nu}) \).

The class of distributions we will define depends on \( N, a, e, \mu_0 \), and \( \nu \). Define,

\[ - \log_2 \mathcal{A}_{\mu_0} = \{ j \in \mathbb{N}^\nu : 2^{-j} \in \mathcal{A}_{\mu_0} \} = \{ j \in \mathbb{N}^\nu : j_{\mu_0} \geq j_{\mu_0 + 1} \geq \cdots \geq j_\nu \}. \]

Given a function \( \varsigma \) on \( \mathbb{R}^N \), and \( j \in \mathbb{N}^\nu \), define,

\[ \varsigma^{(2^j)} (t) = 2^{j_{e_1} + \cdots + j_{e_N}} \varsigma (2^j t). \]

Note that \( \varsigma^{(2^j)} \) is defined in such a way that,

\[ \int \varsigma^{(2^j)} (t) \, dt = \int \varsigma (t) \, dt. \]

---

\(^3\) Actually, this is not the most general form of \( \mathcal{A} \) that can be handled by our methods. See Section 4 for further details.

\(^4\) \( \delta^{e_j} \) is defined by standard multi-index notation: \( \delta^{e_j} = \prod_\mu \delta^{e_j}_\mu \).
Definition 2.1. We define $K = K(N,e,a,\mu_0,\nu)$ to be the set of all distributions, $K$, of the form

$$K = \sum_{j \in -\log_2 \mathcal{A}_{\mu_0}} \mathcal{C}^j(2^j),$$

(2.2)

where $\{\mathcal{C}^j\}_{j \in -\log_2 \mathcal{A}_{\mu_0}} \subset C_0^\infty \{B^N(a)\}$ is a bounded set, satisfying

$$\int \mathcal{C}^j(t) \, dt = 0,$$

unless $0 = j_\mu$ or $\mu > \mu_0$ and $j_\mu = j_{\mu-1}$. It was shown in [Str11b] that any sum of the form (2.2) converges in the sense of distributions.

See [Str11b] for a more in-depth discussion of the class $K$.

Remark 2.2. If each $e_j$ is non-zero in only one component, kernels in $K(N,e,a,\nu,\nu)$ are known as “product kernels,” and kernels in $K(N,e,a,1,\nu)$ are known as “flag kernels.” The classes we study here are more general: we do not insist that each $e_j$ are nonzero in only one component, and we allow for any $1 \leq \mu_0 \leq \nu$. See [Str11b] for a further discussion of these issues.

3 Multi-parameter Carnot-Carathéodory geometry

At the heart of the definition of the class of $\gamma$ which we will study lies multi-parameter Carnot-Carathéodory geometry. Thus, before we can even define the class of $\gamma$, it is necessary to review the relevant definitions of multi-parameter Carnot-Carathéodory balls. We defer the theorems we will use to deal with these balls to Section 7. Our main reference for Carnot-Carathéodory geometry is [Str11a], and we refer the reader there for a more detailed discussion.

Let $\Omega \subseteq \mathbb{R}^n$ be a fixed open set, and suppose $X_1, \ldots, X_q$ are $C^\infty$ vector fields on $\Omega$. We define the Carnot-Carathéodory ball of unit radius, centered at $x_0 \in \Omega$, with respect to the list $X$ by

$$B_X(x_0) := \left\{ y \in \Omega \mid \exists \gamma : [0,1] \to \Omega, \gamma(0) = x_0, \gamma(1) = y, \gamma'(t) = \sum_{j=1}^q a_j(t) X_j(\gamma(t)), a_j \in L^\infty([0,1]), \left\| \sum_{1 \leq j \leq q} |a_j|^2 \right\|_{L^\infty([0,1])} < 1 \right\}.$$

Now that we have the definition of balls with unit radius, we may define (multi-parameter) balls of any radius merely by scaling the vector fields. To do so, we assign to each vector field, $X_j$, a (multi-parameter) formal degree $0 \neq d_j = (d_{j,1}, \ldots, d_{j,q}) \in [0,\infty)^q$. For $\delta = (\delta_1, \ldots, \delta_q) \in [0,\infty)^q$, we define the list of vector fields $\delta X$ to be the list $(\delta_{d_1} X_1, \ldots, \delta_{d_q} X_q)$. Here, $\delta_{d_j}$ is defined by the standard multi-index notation: $\delta_{d_j} = \prod_{\mu=1}^q \delta_{\mu}^{d_{j,\mu}}$. We define the ball of radius $\delta$ centered at $x_0 \in \Omega$ by

$$B_{(X,\delta)}(x_0, \delta) := B_{\delta X}(x_0).$$

At times, it will be convenient to assume that the ball $B_{(X,\delta)}(x_0, \delta)$ lies “inside” of $\Omega$. To this end, we make the following definition.

Definition 3.1. Given $x_0 \in \Omega$ and $\Omega' \subseteq \Omega$, we say the list of vector fields $X$ satisfies $C(x_0, \Omega')$ if for every $a = (a_1, \ldots, a_q) \in \mathcal{A}_1$, with

$$\|a\|_{L^\infty([0,1])} = \left\| \sum_{j=1}^q |a_j|^2 \right\|_{L^\infty([0,1])} < 1,$$

we have $X_a : \Omega \to \Omega'$.
there exists a solution $\gamma : [0, 1] \to \Omega'$ to the ODE

$$
\gamma' (t) = \sum_{j=1}^{q} a_j (t) X_j (\gamma (t)), \quad \gamma (0) = x_0.
$$

Note, by Gronwall’s inequality, when this solution exists, it is unique. Similarly, we say $(X, d)$ satisfies $\mathcal{C} (x_0, \delta, \Omega')$ if $\delta X$ satisfies $\mathcal{C} (x_0, \delta, \Omega')$.

One of the main points of [Str11a] was to provide a detailed study of the balls $B_{(X, d)} (x_0, \delta)$, under appropriate conditions on the list $(X, d)$. To do this, we first need to pick a subset $A \subseteq [0, 1]^\nu$, and a compact set $K_0 \subseteq \Omega$. We will (essentially) be restricting our attention to those balls $B_{(X, d)} (x_0, \delta)$ such that $x_0 \in K_0$ and $\delta \in A$. One should think of $A = A_{\mu_0}$ for some $\mu_0$, as that case will be the primary one used in this paper.

**Definition 3.2.** We say $(X, d)$ satisfies $\mathcal{D} (K_0, A)$ if the following holds:

- Take $\Omega'$ with $K_0 \subseteq \Omega' \subseteq \Omega$ and $\xi > 0$ such that for every $\delta \in A$ and $x \in K_0$, $(X, d)$ satisfies $\mathcal{C} (x, \xi, \delta, \Omega')$.
- For every $\delta \in A$ and $x \in K_0$, we assume
  $$
  [\delta^d X_i, \delta^d X_j] = \sum_k c_{i,j}^{k,\delta, x} \delta^d X_k, \text{ on } B_{(X, d)} (x, \xi \delta).
  $$
- For every ordered multi-index $\alpha$ we assume\(^5\)
  $$
  \sup_{x \in K_0} \left\| (\delta X)^\alpha c_{i,j}^{k,\delta, x} \right\|_{C^0 (B_{(X, d)} (x, \xi \delta))} < \infty.
  $$

If we wish to be explicit about $\Omega'$ and $\xi$, we write $\mathcal{D} (K_0, A, \Omega', \xi)$.

It is under condition $\mathcal{D} (K_0, A)$ that the balls $B_{(X, d)} (x, \delta)$ were studied in [Str11a]. We refer the reader to Section 7 for an overview of the theorems from [Str11a] that we shall use.

In what follows, we will not be directly given a list of vector fields with formal degrees satisfying $\mathcal{D} (K_0, A)$,

$$
(X_1, d_1), \ldots, (X_q, d_q),
$$

but, rather, we will be given a list of $C^\infty$ vector fields with formal degrees which we will assume to “generate” such a list.

To understand this, let $(X_1, d_1), \ldots, (X_r, d_r)$ be $C^\infty$ vector fields with associated formal degrees $0 \neq d_j \in [0, \infty)^\nu$. For a list $L = (l_1, \ldots, l_m)$ where $1 \leq l_j \leq r$, we define,

$$
X_L = \text{ad} (X_{l_1}) \text{ad} (X_{l_2}) \cdots \text{ad} (X_{l_{m-1}}) X_{l_m},
$$

$$
d_L = d_{l_1} + d_{l_2} + \cdots + d_{l_m}.
$$

We define $\mathcal{S} = \{ (X_L, d_L) : L \text{ is any such list} \}$.

**Definition 3.3.** We say $\mathcal{S}$ is finitely generated or that $(X_1, d_1), \ldots, (X_r, d_r)$ generates a finite list if there exists finite subset, $\mathcal{F} \subseteq \mathcal{S}$, such that $\mathcal{F}$ satisfies $\mathcal{D} (K_0, A)$\(^6\) and

$$
(X_j, d_j) \in \mathcal{F}, \quad 1 \leq j \leq r.
$$

If we enumerate the vector fields in $\mathcal{F}$,

$$
\mathcal{F} = \{(X_1, d_1), \ldots, (X_q, d_q)\},
$$

\(^5\)We write $\|f\|_{C^0(U)} = \sup_{x \in U} |f (x)|$, and if we say the norm is finite, we mean (in addition) that $f$ is continuous on $U$.

\(^6\)Here, we are thinking of $K_0$ and $A$ fixed.
we say that \((X_1, d_1), \ldots, (X_r, d_r)\) generates the finite list \((X_1, d_1), \ldots, (X_q, d_q)\). Note that, if \(S\) is finitely generated, \((X_1, d_1), \ldots, (X_r, d_r)\) could generate many different finite lists. However, if we let \((X, d)\) and \((X', d')\) be two different such lists then either choice will work for our purposes. In fact, it is shown in \([Str11b]\) that \((X, d)\) and \((X', d')\) are equivalent in a sense that is made precise and discussed at length in that paper. It follows that, in every place we use these notions, it will not make a difference which finite list we use. Thus, we will unambiguously say “\((X_1, d_1), \ldots, (X_r, d_r)\) generates the finite list \((X_1, d_1), \ldots, (X_q, d_q)\)” to mean that \((X_1, d_1), \ldots, (X_r, d_r)\) generates a finite list and \((X_1, d_1), \ldots, (X_q, d_q)\) can be any such list.

4 Surfaces

In this section, we define the class of \(\gamma\) for which we will study operators of the form \([1.1]\). This is nothing but a reprise of the definitions in \([Str11b]\), and we refer the reader there for more details.

We assume we are given an open subset \(\Omega \subseteq \mathbb{R}^N\), a fixed \(\mu_0\), \(1 \leq \mu_0 \leq \nu\), and dilations \(e\) as in Section 2.

**Definition 4.1.** Given a multi-index \(\alpha \in \mathbb{N}^N\), we define

\[
\deg(\alpha) = \sum_{j=1}^N \alpha_j e_j \in [0, \infty)^\nu.
\]

Let \(K_0 \subseteq \Omega' \subseteq \Omega'' \subseteq \Omega\) be subsets of \(\Omega\) with \(K_0\) compact and \(\Omega'\) and \(\Omega''\) open but relatively compact in \(\Omega\). Our goal in this section is to define a class of \(C^\infty\) functions

\[
\gamma(t, x) : B^N(\rho) \times \Omega'' \to \Omega
\]

such that \(\gamma(0, x) = x\). Here \(\rho > 0\) is a small number. This class of functions will depend on \(\mu_0\), \(N\), \(e\), and \(\Omega\) (nominally, the class will also depend on \(K_0\), \(\Omega'\), and \(\Omega''\), but this will not be an essential point). This class will be such that if \(\psi\) is a \(C^\infty_0\) function supported in the interior of \(K_0\), then there is an \(t > 0\) sufficiently small such that the operator given by \([1.1]\) is bounded on \(L^p\) (\(1 < p < \infty\)) for every \(K \in \mathcal{K}(N, e, a, \mu_0, \nu)\).

Note, by possibly shrinking \(\rho > 0\) we may assume that, for each \(t \in B^N(\rho)\), \(\gamma(t, \cdot)\big|_{\Omega'}\) is a diffeomorphism onto its image. From now on we will assume this. Also, as in the introduction, we will write \(\gamma_t(x) = \gamma(t, x)\).

Unlike the work in \([CNS99]\), we separate the condition on \(\gamma_t\) into two aspects. For the first, suppose we are given a \(C^\infty\) vector field on \(\Omega'\), \(X_1, \ldots, X_q\), with associated \(\nu\)-parameter formal degrees, \(d_1, \ldots, d_q\), satisfying \(D(K_0, A_{\mu_0}, \Omega', \xi)\) for some \(\xi > 0\) (we will see later where these vector fields will come from).

**Definition 4.2.** Suppose we are given a \(C^\infty\) vector field on \(\Omega'\), depending smoothly on \(t \in B^N(\rho)\), \(W(t, x) \in T_x \Omega'\). We say \((X, d)\) controls \(W(t, x)\) if there exists a \(\rho_1 \leq \rho\) and \(\tau_1 \leq \xi\) such that for every \(x_0 \in K_0\), \(\delta \in A_{\mu_0}\) there exist functions \(c_{\delta}^{x_0, \delta}\) on \(B^N(\rho_1) \times B_{(X, d)}(x_0, \tau_1\delta)\) satisfying

- \(W(\delta t, x) = \sum_{l=1}^q c_{\delta}^{x_0, \delta}(t, x) \delta^{d_l} X_l(x)\) on \(B^N(\rho_1) \times B_{(X, d)}(x_0, \tau_1\delta)\), where \(\delta t\) is defined as in \([2.1]\).
- \(\sup_{x_0 \in K_0} \sum_{|\alpha| + |\beta| \leq m} \left\| (\delta X)^\alpha \partial_\beta c_{\delta}^{x_0, \delta}\right\|_{C^0(B^N(\rho_1) \times B_{(X, d)}(x_0, \tau_1\delta))} < \infty\), for every \(m\).

**Definition 4.3.** We say \((X, d)\) controls \(\gamma_t(x)\) if \((X, d)\) controls \(W\) where

\[
W(t, x) = \frac{d}{de} \left. \gamma_{et} \circ \gamma_t^{-1}(x) \right|_{e=1}.
\]

Here, \(\epsilon(t_1, \ldots, t_N) = (\epsilon t_1, \ldots, \epsilon t_N)\), and so is unrelated to the dilations \(e\).
Part of our assumption will be that a particular family of vector fields \((X,d)\) controls \(\gamma_t\). Where these vector fields come from constitutes the other part of our assumption on \(\gamma\).

Let \(W\) be as in Definition 4.3. Let \(X_\alpha (x)\) be the Taylor coefficients of \(W\) when the Taylor series is taken in the \(t\) variable:

\[
W(t,x) \sim \sum_{|\alpha| > 0} t^\alpha X_\alpha (x),
\]

so that \(X_\alpha\) is a \(C^\infty\) vector field on \(\Omega'\).

Our assumption on \(\gamma_t\) is that if we take the set of vector fields with formal degrees:

\[
S = \{(X_\alpha, \deg(\alpha)) : \deg(\alpha) \text{ is non-zero in only one component}\},
\]

then there is a finite subset \(F \subseteq S\) such that \(F\) generates a finite list \((X,d) = (X_1,d_1), \ldots, (X_q,d_q)\) and this finite list controls \(\gamma_t\).

**Remark 4.4.** The list of vector fields \((X,d)\) depends on a few choices we have made in the above: it depends on the chosen finite subset \(F\) and it depends on the chosen list generated by \(F\). However, neither of these choices affects \((X,d)\) in an essential way. This is discussed in detail in [Str11b].

### 5 Statement of Results

Fix \(\Omega \subseteq \mathbb{R}^n\) open, and \(K_0 \in \Omega' \cap \Omega'' \subseteq \Omega\) with \(K_0\) compact (with nonempty interior) and \(\Omega'\) and \(\Omega''\) open but relatively compact in \(\Omega\). Let \(\gamma (t,x) : \mathbb{B}^N (\rho) \times \Omega'' \rightarrow \Omega\) be a \(C^\infty\) function such that \(\gamma (0,x) \equiv x\). Here, \(\rho > 0\) is a small number.

Fix \(\nu \in \mathbb{N}\) positive, and \(\mu_0, 1 \leq \mu_0 \leq \nu\). Furthermore, let \(e = (e_1, \ldots, e_N)\) be given, with \(0 \neq e_j \in [0, \infty)^\nu\). We suppose \(\gamma\) satisfies the assumptions of Section 4 with this \(K_0, \mu_0, \text{and} e\).

**Theorem 5.1.** For every \(\psi \in C_0^\infty (\mathbb{R}^n)\) supported in the interior of \(K_0\), there exists \(a > 0\) such that for every \(K \in \mathcal{K}(N,e,a,\mu_0,\nu)\) the operator

\[
f \mapsto \psi (x) \int f(\gamma_t (x)) K(t) \, dt
\]

extends to a bounded operator \(L^p (\mathbb{R}^n) \rightarrow L^p (\mathbb{R}^n)\), for every \(1 < p < \infty\).

Actually, as shown in [Str11b], Theorem 5.1 follows directly from the following, slightly more general, theorem.

**Theorem 5.2.** There exists \(a > 0\) such that for every \(\psi_1, \psi_2 \in C_0^\infty (\mathbb{R}^n)\) supported on the interior of \(K_0\), every \(K \in \mathcal{K}(N,e,a,\mu_0,\nu)\) and every \(C^\infty\) function \(\kappa (t,x) : \mathbb{B}^N (a) \times \Omega'' \rightarrow \mathbb{C}\) the operator

\[
Tf (x) = \psi_1 (x) \int f(\gamma_t (x)) \psi_2(\gamma_t (x)) \kappa (t,x) K(t) \, dt
\]

extends to a bounded operator \(L^p (\mathbb{R}^n) \rightarrow L^p (\mathbb{R}^n)\) for every \(1 < p < \infty\).

**Remark 5.3.** We focus on the more general Theorem 5.2. The importance of the form of the operator in Theorem 5.2 is that the class of operators is closed under taking \(L^2\) adjoints, which is not true of the class of operators in Theorem 5.1. See Section 12.3 of [Str11b] for the proof of this.

We also study maximal functions. Define, for \(\psi_1, \psi_2 \in C_0^\infty (\mathbb{R}^n)\), supported on the interior of \(K_0\) with \(\psi_1 \geq 0\),

\[
\mathcal{M}f(x) = \sup_{\delta \in \mathcal{A}_a} \psi_1(x) \int_{|\delta| < a} |f(\gamma_{\delta t}(x)) \psi_2(\gamma_{\delta t}(x))| \, dt,
\]

where \(a > 0\) is a fixed, small real number (this should be considered as the same \(a\) as in Theorem 5.2). Then, we have,
Theorem 5.4. Under the same assumptions as Theorem 5.2,

\[ \|Mf\|_{L^p} \lesssim \|f\|_{L^p}, \]

for every \( 1 < p \leq \infty \).

Most of the paper will be devoted to exhibiting the proofs of Theorems 5.2 and 5.4 in the case when \( \mu_0 = \nu \). That is, when \( A_{\mu_0} = [0,1]^n \). This case contains all the main ideas, but allows for simpler notation. We then describe the modifications needed to attack the more general situation in Section 14.

Remark 5.5. A major difficulty in the proofs of Theorems 5.2 and 5.4 is, there is no appropriate \textit{a priori} multi-parameter theory analogous to the standard theory of Calderón-Zygmund singular integrals priori to fall back on. Indeed, in the single parameter (\( \nu = 1 \)) case, one can “smooth out” the operators in question enough to apply the standard Calderón-Zygmund theory and obtain some \( L^p \) estimates (see Section 18 of [CNSW99]). To get around this issue, we will use square function techniques, that will allow us to introduce the Calderón-Zygmund theory in a more round about manner. Once Theorem 5.2 is proved, we will actually obtain, \textit{a posteriori}, a prototype for some aspects of a multi-parameter Calderón-Zygmund type theory. See Section 15.

Remark 5.6. While the maximal results in this paper are new, the class of \( \gamma \) we consider was more motivated by Theorem 5.2 than by Theorem 5.4. Indeed, we will see in Section 16 that there are choices of \( \gamma \) where the \( L^p \) boundedness of the singular integral fails, but the \( L^p \) boundedness of the maximal function holds. We have not attempted to state Theorem 5.4 in such a way to include these choices. This deficiency will be partially rectified in [SS11b]. Indeed, in [SS11b], we will obtain the following result. Let \( \gamma_t(x) : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^n \) be a real analytic function defined on a neighborhood of \((0,0) \in \mathbb{R}^N \times \mathbb{R}^m\), and satisfying \( \gamma_0(x) \equiv x \); assuming \textit{no additional definition}. Define a maximal operator by,

\[ \widetilde{M}f(x) = \sup_{\delta = (\delta_1, \ldots, \delta_N) \in [0,1]^N} \psi(x) \int_{|t| \leq a} |f(\gamma_{\delta_1 t, \ldots, \delta_N t N}(x))| \, dt. \]

Then, \( \widetilde{M} \) is bounded on \( L^p \) (\( 1 < p < \infty \)), provided \( \psi \) is supported on a sufficiently small neighborhood of 0, and \( a \) is sufficiently small. Note that this result is clearly not a special case of Theorem 5.4. See, also, [SS11a] for a further discussion of this point.

6 Basic Notation

Throughout the paper, for \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \), we write \( |v| \) for \( \left( \sum_j |v_j|^2 \right)^{\frac{1}{2}} \), and we write \( |v|_\infty \) for \( \sup_j |v_j| \). \( B^n(\eta) \) will denote the ball of radius \( \eta > 0 \) in the \( |\cdot| \) norm. For two numbers \( a, b \in \mathbb{R} \) we write \( a \vee b \) for the maximum of \( a \) and \( b \) and \( a \wedge b \) for the minimum. If instead, \( a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n \), we write \( a \vee b \) (respectively, \( a \wedge b \)) for \( (a_1 \vee b_1, \ldots, a_n \vee b_n) \) (respectively, \( (a_1 \wedge b_1, \ldots, a_n \wedge b_n) \)).

For a vectors \( \delta = (\delta_1, \ldots, \delta_n), d = (d_1, \ldots, d_n) \in \mathbb{R}^n \), we define \( \delta^d \) by the standard multi-index notation. I.e., \( \delta^d = \prod_{\mu=1}^n \delta_{\mu}^{d_{\mu}} \). Also we will write \( 2^d = (2^{d_1}, \ldots, 2^{d_n}) \).

Given a, possibly arbitrary, set \( U \subseteq \mathbb{R}^n \) and a continuous function \( f \) defined on a neighborhood of \( U \), we write

\[ \|f\|_{C^j(U)} = \sum_{|a| \leq j, x \in U} \sup_{x \in U} |\partial^a f(x)|, \]

and if we state that \( \|f\|_{C^j(U)} \) is finite, we mean that the partial derivatives up to order \( j \) of \( f \) exist on \( U \), are continuous, and the above norm is finite. If \( f \) is replaced by a vector field \( Y = \sum_k a_k(x) \partial_{x_k} \), then we write

\[ \|Y\|_{C^j(U)} = \sum_k \|a_k\|_{C^j(U)}. \]

Given a matrix \( A \), we write \( \|A\| \) for the usual operator norm. Given two integers \( 1 \leq m \leq n \), we let \( I(m,n) \) denote the set of all lists of integers \( (i_1, \ldots, i_m) \) such that

\[ 1 \leq i_1 < i_2 < \cdots < i_m \leq n. \]
Furthermore, suppose $A$ is an $n \times q$ matrix, and suppose $1 \leq n_0 \leq n \land q$. For $I \in \mathcal{I}(n_0, n)$, $J \in \mathcal{I}(n_0, q)$ we let $A_{I,J}$ denote the $n_0 \times n_0$ matrix given by taking the rows from $A$ which are listed in $I$ and the columns from $A$ which are listed in $J$. We define

$$\det_{n_0 \times n_0} A = (\det A_{I,J})_{I,J \in \mathcal{I}(n_0, n_0), J \in \mathcal{I}(n_0, q)}$$

so that, in particular, $\det_{n_0 \times n_0} A$ is a vector (it will not be important to us in which order the coordinates are arranged). $\det_{n_0 \times n_0} A$ comes up when one changes variables. Indeed, suppose $\Phi$ is a $C^1$ diffeomorphism from an open subset $U \subset \mathbb{R}^{n_0}$ mapping to an $n_0$ dimensional submanifold of $\mathbb{R}^n$, where this submanifold is given the induced Lebesgue measure $dx$. Then, we have

$$\int_{\Phi(U)} f(x) \, dx = \int_{U} f(\Phi(t)) \det_{n_0 \times n_0} d\Phi(t) \, dt.$$  

If $A = (A_1, \ldots, A_q)$ is a list of, possibly non-commuting, operators, we will use ordered multi-index notation to define $A^\alpha$, where $\alpha$ is a list of numbers $1, \ldots, q$. $|\alpha|$ will denote the length of the list. For instance, if $\alpha = (1, 4, 2, 1)$, then $|\alpha| = 5$ and $A^\alpha = A_1 A_4 A_2 A_1$. Thus, if $A_1, \ldots, A_q$ are vector fields, then $A^\alpha$ is an $|\alpha|$ order partial differential operator.

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a map, then we write,

$$df(x) \left( \frac{\partial}{\partial x_j} \right),$$

for the differential of $f$ at the point $x$ applied to the vector field $\frac{\partial}{\partial x_j}$. If $f$ is a function of two variables, $f(t,x): \mathbb{R}^N \times \mathbb{R}^n \to \mathbb{R}^m$, and we wish to view $df$ as a linear transformation acting on the vector space spanned by $\frac{\partial}{\partial x_j}$ ($1 \leq j \leq N$), then we instead write

$$\frac{\partial f}{\partial t}(t,x)$$

to denote this linear transformation. Hence, it makes sense to write,

$$\det_{n_0 \times n_0} \frac{\partial f}{\partial t}(t,x),$$

where $n_0 \leq m \land N$.

If $\psi_1, \psi_2 \in C^\infty_0(\mathbb{R}^n)$, we write $\psi_1 \prec \psi_2$ to denote that $\psi_2 \equiv 1$ on a neighborhood of the support of $\psi_1$.

We will have occasion to use vector valued functions. We denote by $L^p(\ell^q(\mathbb{N}^\nu))$ the set of sequences of measurable functions $\{f_j\}_{j \in \mathbb{N}^\nu}$ such that,

$$\left\| \left( \sum_{j \in \mathbb{N}^\nu} |f_j|^q \right)^{1/q} \right\|_{L^p} < \infty.$$

Finally, we will devote a good deal of notation to multi-parameter Carnot-Carathéodory geometry. See Sections 3 and 7.

### 7 Multi-parameter Carnot-Carathéodory geometry Revisited

In this section, we present the results that allow us to deal with Carnot-Carathéodory geometry. The results we outline here are contained in Section 4 of [Str11a]. The heart of this theory is the ability to “rescale” vector fields. This rescaling is obtained by conjugating by a particular diffeomorphism, which will be denoted by $\Phi$ in what follows.
Before we can enter into details, we must explain the connection between multi-parameter balls and single-parameter balls. We assume we are given \( C^\infty \) vector fields \( X_1, \ldots, X_q \) with associated formal degrees \( 0 \neq d_1, \ldots, d_q \in [0, \infty)^m \). Here, \( \nu \in \mathbb{N} \) is the number of parameters. Given the multi-parameter degrees, we obtain corresponding single parameter degrees, which we denote by \( \sum \delta \) and are defined by \( \delta_j := \sum_{\mu=1}^\nu \delta_j^\mu = |d_j|_1 \). Let \( \delta \in [0, \infty)^\nu \), and suppose we wish to study the ball
\[
B_{(X,d)}(x_0, \delta).
\]

Decompose \( \delta = \delta_0 \delta_1 \) where \( \delta_0 \in [0, \infty) \) and \( \delta_1 \in [0, \infty)^\nu \) (of course this decomposition is not unique). Then, directly from the definition, we obtain:
\[
B_{(X,d)}(x_0, \delta) = B_{(\delta_0 X, \sum \delta)}(x_0, \delta_0) = B_{(\delta X, \sum \delta)}(x_0, 1).
\]

Thus, studying a ball of radius \( \delta \) corresponding to \( (X,d) \) is the same as studying a ball of radius 1 corresponding to \( (\delta X, \sum \delta) \). For this reason, taking \( K_0 \) as in Section 3 and assuming \( (X,d) \) satisfies \( D(K_0, [0,1]^\nu) \), we will fix \( x_0 \in K_0 \) and \( \delta \in [0,1]^\nu \) and study balls of radius \( \approx 1 \) centered at \( x_0 \) corresponding to \( (\delta X, \sum \delta) \). In what follows, it will be important that all of the implicit constants are independent of \( x_0 \in K_0 \) and \( \delta \in [0,1]^\nu \).

We now turn to stating a theorem about a list of \( C^\infty \) vector fields \( Z_1, \ldots, Z_q \) defined on an open set \( \Omega \subseteq \mathbb{R}^n \), with associated single parameter formal degrees \( d_1, \ldots, d_q \in (0, \infty) \). The special case we are interested in is the case when \( Z, \hat{d} = (\delta X, \sum \delta) \); i.e., when \( Z_j = \delta^{\mu_j} X_j \) and \( \hat{d}_j = |d_j|_1 \).

Fix \( x_0 \in \Omega \) and \( 1 \geq \xi > 0 \). Let \( n_0 = \dim \text{span} \{ Z_1(x_0), \ldots, Z_q(x_0) \} \). For \( (j_1, \ldots, j_{n_0}) \in \mathcal{I}(n_0, q) \), let \( Z_j \) denote the list of vector fields \( Z_{j_1}, \ldots, Z_{j_{n_0}} \). Fix \( j_0 \in \mathcal{I}(n_0, q) \) such that
\[
\det_{n_0 \times n_0} Z_{j_0} (x_0) = \det_{n_0 \times n_0} Z(x_0) \bigg|_{\infty},
\]
where we have identified \( Z(x_0) \) with the \( n \times q \) matrix whose columns are given by \( Z_1(x_0), \ldots, Z_q(x_0) \) and similarly for \( Z_{j_0}(x_0) \). We assume \( \left( Z, \hat{d} \right) \) satisfies \( \mathcal{L}(x_0, \xi, \Omega) \). In addition we suppose that there are functions \( c_{i,j} \) on \( B_{(Z,\hat{d})}(x_0, \xi) \) such that
\[
[Z_i, Z_j] = \sum_{i,j} c_{i,j}^k Z_k, \quad \text{on } B_{(Z,\hat{d})}(x_0, \xi).
\]

We assume that:
\begin{itemize}
  \item \( \left\| Z_j \right\|_{C^m\left( B_{(Z,\hat{d})}(x_0, \xi) \right)} < \infty \) for every \( m \).
  \item \( \sum_{|\alpha| \leq m} \left\| Z_j^\alpha c_{i,j} \right\|_{C^0\left( B_{(Z,\hat{d})}(x_0, \xi) \right)} < \infty \), for every \( m \) and every \( i, j, k \).
\end{itemize}

We say that \( C \) is an \( m \)-admissible constant if \( C \) can be chosen to depend only on upper bounds for the above two quantities (for that particular choice of \( m \)), \( m \), upper and lower bounds for \( d_1, \ldots, d_q \), an upper bound for \( n \) and \( q \), and a lower bound for \( \xi \). Note that, in our primary example \( \left( Z, \hat{d} \right) = (\delta X, \sum \delta) \), \( m \)-admissible constants can be chosen independent of \( x_0 \in K_0 \) and \( \delta \in [0,1]^\nu \). We write \( A \lesssim B \) if \( A \leq CB \) where \( C \) is an \( m \)-admissible constant, and we write \( A \approx m \) if \( A \lesssim m B \) and \( B \lesssim m A \). Finally, we say \( \tau = \tau (\kappa) \) is an \( m \)-admissible constant if \( \tau \) can be chosen to depend on all the parameters an \( m \) admissible constant may depend on, and \( \tau \) may also depend on \( \kappa \).

**Theorem 7.1.** There exist 2-admissible constants \( \eta_1, \xi_1 > 0 \) such that if the map \( \Phi : B^{n_0}(\eta_1) \rightarrow B_{(Z,\hat{d})}(x_0, \xi) \) is defined by
\[
\Phi (u) = e^{u Z_{j_0} x_0},
\]
we have
\[9\]
\[10\]
Finally, 

- \( \Phi : B^{n_0} (\eta_1) \to B_{(z,d)} (x_0, \xi) \) is injective.
- \( B_{(z,d)} (x_0, \xi_1) \subseteq \Phi (B^{n_0} (\eta_1)) \).
- For all \( u \in B^{n_0} (\eta_1) \), \( |\det_{n_0 \times n_0} d\Phi (u)| \approx_2 |\det_{n_0 \times n_0} Z (x_0)| \).
- \( \text{Vol} (B_{(z,d)} (x_0, \xi_1)) \approx_2 |\det_{n_0 \times n_0} Z (x_0)| \).

Furthermore, if we let \( Y_j \) be the pullback of \( Z_j \) under the map \( \Phi \), then we have, for \( m \geq 0 \),

\[
\|Y_j\|_{C^m(B^{n_0}(\eta_1))} \lesssim_{m \vee 2} 1, \tag{7.1}
\]

\[
\|f\|_{C^m(B^{n_0}(\eta_1))} \approx (m-1) \vee 2 \sum_{|\alpha| \leq m} \|Y^\alpha f\|_{C^0(B^{n_0}(\eta_1))}. \tag{7.2}
\]

Finally,

\[
\left| \det_{n_0 \times n_0} Y_{j_0} (u) \right| \approx_2 1, \quad \forall u \in B^{n_0} (\eta_1). \tag{7.3}
\]

Note that, in light of (7.1) and (7.3), pulling back by the map \( \Phi \) allows us to rescale the vector fields \( Z \) in such a way that the rescaled vector fields, \( Y \), are smooth and span the tangent space (uniformly in any relevant parameters). We will also need the following technical result.

**Proposition 7.2.** Suppose \( \xi_2, \eta_2 > 0 \) are given. Then there exist 2-admissible constants \( \eta' = \eta' (\xi_2) > 0 \), \( \xi' = \xi' (\eta_2) > 0 \) such that,

\[
\Phi (B^{n_0} (\eta')) \subseteq B_{(z,d)} (x_0, \xi_2),
\]

\[
B_{(z,d)} (x_0, \xi') \subseteq \Phi (B^{n_0} (\eta_2)).
\]

**Proof.** The existence of \( \eta' \) can be seen by applying Theorem 7.1 with \( \xi \) replaced by \( \xi \wedge \xi_2 \). The existence of \( \xi' \) can be shown by combining the proof of Proposition 3.21 of [Str11a] with the proof of Proposition 4.16 of [Str11a]. \( \square \)

**Remark 7.3.** With a slight abuse of notation, when we say \( m \)-admissible constant, where \( m < 2 \), we will take that to mean a 2-admissible constant. Using this new notation, the \( \vee \) in (7.1) and (7.2) may be removed.

**Remark 7.4.** It is not hard to see that the single-parameter formal degrees \( \tilde{d} \) do not play an essential role in the above (see Remark 3.3 of [Str11a]). In fact, one could state Theorem 7.1 taking all the formal degrees \( \tilde{d}_j = 1 \) and that would be sufficient for our purposes. Moreover, in every place we use the single-parameter formal degrees \( \tilde{d} \), they are inessential. We have stated the result as above, though, to allow us to transfer seamlessly between the vector fields \( (X, d) \) and \( (Z, \tilde{d}) \), without any hand-waving about the formal degrees.

### 8 Some special single-parameter operators

In this section, we describe a certain single-parameter (i.e., \( \nu = 1 \)) special case of our main theorem. This special case will be easy to obtain using the theory described in Section 7 along with the classical Calderón-Zygmund theory of singular integrals. We will then, in Section 11, use the operators developed in this section to create an appropriate Littlewood-Paley theory adapted to the more general operators of this paper.

We suppose we are given \( K_0 \subseteq \Omega' \subseteq \Omega'' \subseteq \Omega \) as in Section 3 and a list of \( C^\infty \) vector fields \( X_1, \ldots, X_q \) on \( \Omega' \) with single-parameter formal degrees \( d_1, \ldots, d_q \in (0, \infty) \). We assume that there exists an \( \xi > 0 \) such that \( (X, d) \) satisfies \( D (K_0, [0, 1], \Omega', \xi) \).
Remark 8.1. In the case above, for $\delta \leq \frac{\xi}{8}$ we have,

$$\text{Vol} \left( B_{(x,d)}(x_0,2\delta) \right) \approx \left| \det_{n_0 \times n_0} (2\delta) X(x_0) \right|$$

$$\approx \left| \det_{n_0 \times n_0} \delta X(x_0) \right|$$

$$\approx \text{Vol} \left( B_{(x,d)}(x_0,\delta) \right),$$

where we have used our usual notation that $\delta X$ denotes the matrix whose columns are given by $\delta^{d_1} X_1, \ldots, \delta^{d_q} X_q$. (8.1) is the fundamental estimate involved in showing that the balls $B_{(x,d)}(\cdot, \cdot)$ form a space of homogeneous type. Thus, if $X_1, \ldots, X_q$ spanned the tangent space, then the above Carnot-Carathéodory balls would be open subsets of $\mathbb{R}^{n}$ and would endow $K_0$ with the structure of a space of homogeneous type. However, we are interested in the case when $X_1, \ldots, X_q$ do not, necessarily, span the tangent space. In this case, the classical Frobenius theorem applies to show that $X_1, \ldots, X_q$ foliate the $K_0$ into leaves, and each leaf is a space of homogeneous type. Using the coordinate charts $(\Phi)$ developed in Section 7, we will be able to exploit this fact in what follows. This idea was also used in Section 6.2 of [Str11a].

We consider the function,

$$\gamma(t,x) : B^{q}(p) \times \Omega'' \to \Omega,$$

given by,

$$\gamma(t_1,\ldots,t_q)(x) = e^{t_1 X_1 + \cdots + t_q X_q} x.$$

We define dilations on $\mathbb{R}^{q}$ by, for $\delta > 0$,

$$\delta(t_1,\ldots,t_q) = (\delta^{d_1} t_1, \ldots, \delta^{d_q} t_q),$$

and we define for $\zeta : \mathbb{R}^{q} \to \mathbb{C}$, and $j \geq 0$,

$$\zeta(2^j)(t) = 2^{j(d_1 + \cdots + d_q)} \zeta(2^j t).$$

Fix $a > 0$ a small number. Let $\{z_j\}_{j \in \mathbb{N}} \subset C^{\infty}_0(B^{q}(a))$ be a bounded subset satisfying,

$$\int z_j(t) \, dt = 0, \text{ if } j > 0,$$

where we are including $0 \in \mathbb{N}$. Let $K$ be the distribution defined by,

$$K(t) = \sum_{j \in \mathbb{N}} z_j(2^j(t),$$

where the sum is taken in the sense of distributions. I.e., $K \in K(q,e,a,1,1)$, where we are taking $e = (d_1, \ldots, d_q) \in (0,\infty)^q$.

Let $\kappa : B^{q}(a) \times \Omega'' \to \mathbb{C}$ be a $C^{\infty}$ function and let $\psi_1, \psi_2 \in C^{\infty}_0(\mathbb{R}^{n})$ be supported on the interior of $K_0$. Define the operator,

$$Tf(x) = \psi_1(x) \int f(\gamma_t(x)) \psi_2(\gamma_t(x)) \kappa(t,x) K(t) \, dt.$$

Theorem 8.2. There is an $a > 0$ such that $T$ (as defined above) is bounded on $L^p$, $1 < p < \infty$.

In addition we will need a maximal theorem. Take $\psi_1, \psi_2 \in C^{\infty}_0(\mathbb{R}^{n})$ supported on the interior of $K_0$ with $\psi_1 \geq 0$, and define

$$\mathcal{M}f(x) = \sup_{\delta \in (0,1]} \psi_1(x) \int_{|t| < a} |f(\gamma_{\delta t}(x)) \psi_2(\gamma_{\delta t}(x))| \, dt.$$

Then, we have,

---

8Here, $\xi_t$ is as in Theorem 4.1.

9The involutive distribution generated by $X_1, \ldots, X_q$ is finitely generated as a $C^{\infty}$ module in light of condition $D$. 

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Theorem 8.3. For \( a > 0 \) sufficiently small, \( \|Mf\|_{L^p} \lesssim \|f\|_{L^p} \), for \( 1 < p \leq \infty \).

Note that Theorems 8.2 and 8.3 are special cases of Theorems 5.2 and 5.4 respectively (this is proved in Section 17.1 of [Str11b]). However, we will see that Theorems 8.2 and 8.3 can be proven by reduction to the classical Calderón-Zygmund theory. We will then use these theorems to develop an appropriate Littlewood-Paley theory, with which to prove Theorems 5.2 and 5.4.

We separate the proofs of Theorem 8.2 and 8.3 into two cases. In the first case, we prove the results under the assumption that \( X_1, \ldots, X_q \) span the tangent space at each point of \( K_0 \); this is covered in Section 8.1. Then, we use the results in Section 8.1 to prove the more general case when \( X_1, \ldots, X_q \) do not necessarily span the tangent space at each point; this is covered in Section 8.2.

8.1 When the vector fields span

In this section, we prove Theorems 8.2 and 8.3 under the additional assumption that \( \inf_{x \in K_0} |\det_{n \times n} X(x)| \gtrsim 1 \). We will then be able to apply this special case to each leaf, in order to obtain the more general statement of Theorems 8.2 and 8.3.

Remark 8.4. In this particular case, Theorems 8.2 and 8.3 (and the methods used in this section) are already well understood. See [NRSW89, Koe02] for similar methods. We do not know of a reference that has these results in the exact form we need them, however, and so include the short proof here.

We focus now on the proof of Theorem 8.2 and will explain the proof of Theorem 8.3 at the end of the section. Thus, let \( T \) be as in Theorem 8.2. We already know from the theory in [Str11b] that \( \|T\|_{L^2 \to L^2} \lesssim 1 \).

Our goal is to show that \( T \) is a Calderón-Zygmund singular integral operator and it will follow that \( T \) is bounded on \( L^p \) for \( 1 < p \leq 2 \). Since the class of operators discussed in Theorem 8.2 is self-adjoint it will follow that \( T \) is bounded on \( L^p \) for \( 1 < p < \infty \).

Remark 8.5. Actually, it is not hard to see that, instead of using the \( L^2 \) theory in [Str11b], we could apply the \( T(b) \) theorem to obtain the \( L^p \) boundedness of \( T \). We leave this approach to the interested reader.

Let \( \rho(x,y) \) be the Carnot-Carathéodory metric corresponding to the vector fields with formal degrees \( (X_1, d_1), \ldots, (X_q, d_q) \). That is,

\[
\rho(x,y) = \inf \{ \delta > 0 : y \in B(x, \delta) \}.
\]

Let \( \tilde{K}(x,y) \) denote the Schwartz kernel of \( T \). We wish to show that

\[
\left| \int_{B(x,\delta)} \tilde{K}(x,y_1) - \tilde{K}(x,y_2) \, dx \right| \lesssim 1, \text{ if } y_2 \in B(x, \delta) \setminus B(x, \delta/2), \quad (8.2)
\]

and the \( L^p \) boundedness \((1 < p \leq 2)\) of \( T \) will follow from the classical theory of Calderón-Zygmund singular integrals (see, e.g., Theorem 3 on page 19 of [Ste93]). This uses the fact that the balls \( B(x, \delta) \) form a space of homogeneous type, as discussed in Remark 8.1.

We now turn to proving 8.2. As is well known, it suffices to prove the inequality,

\[
\left| X_\alpha^\alpha X_\beta^\beta \tilde{K}(x,y) \right| \lesssim \frac{\rho(x,y)^{-\deg(\alpha) - \deg(\beta)}}{\text{Vol}(B(x, \delta)(x,\rho(x,y)))}, \quad (8.3)
\]

where \( X_\alpha \) denotes the list of vector fields \( (X_1, \ldots, X_q) \) thought of a partial differential operators in the \( x \) variable, \( \alpha \) denotes an ordered multi-index, and

\[
\deg(\alpha) = \sum_{j=1}^q k_j d_j,
\]
where \( k_j \) is the number of times \( j \) appears in the ordered multi-index \( \alpha \). Similarly for \( X_y \) and \( \beta \).

Actually, it would suffice to prove (8.3) in the special case \( |\alpha| = 0, |\beta| = 1 \), but this is no simpler to prove.

For \( j \in \mathbb{N} \), let \( T_j \) be the operator given by

\[
T_j f(x) = \psi_1(x) \int f(\gamma_t(x)) \psi_2(\gamma_t(x)) \kappa(t, x) \zeta^{(2^j)}(t) \, dt.
\]

Let \( \tilde{K}_j(x, y) \) be the Schwartz kernel of \( T_j \). Thus, \( \tilde{K} = \sum_{j \in \mathbb{N}} \tilde{K}_j \).

To prove (8.3), it suffices to show that there is an \( a > 0 \) (independent of \( j \)) such that, when \( \tilde{K}_j \) is defined as above, we have,

- \( \tilde{K}_j(x, y) \) is supported on \( \{ (x, y) : \rho(x, y) \leq \xi_1 2^{-j} \} \), where \( \xi_1 \) is a constant, independent of \( j \), and
- \( \left| (2^{-j} X_y)^\alpha (2^{-j} X_y)^\beta \tilde{K}_j(x, y) \right| \lesssim \frac{1}{\text{Vol}(B_{1, \delta}(x, 2^{-j}))} \), where, as usual, \( \delta X \) denotes the list of vector fields \( \delta x_1, \ldots, \delta x_q \).

To prove the above we apply Theorem 7.1 to the list of vector fields \( (2^{-j} X, d) \) with \( x_0 \in K_0 \). Note that all of the assumptions in that section hold uniformly for \( x_0 \in K_0 \) and \( j \in \mathbb{N} \). Thus we obtain, \( \eta_1, \xi_1 > 0 \) and for each \( x_0 \in K_0 \) and \( j \in \mathbb{N} \) a map,

\[
\Phi_{j, x_0} : B^n(\eta_1) \to B_{(X, d)}(x_0, \xi_2^{-j}),
\]

as in Theorem 7.1. To prove the claim about the support of \( \tilde{K}_j \) it suffices to show that for \( x_0 \in K_0 \) and \( |t| \leq a \), we have

\[
e^{t_1 2^{-j_{d1}} X_1 + \cdots + t_q 2^{-j_{d_q}} X_q} x_0 \in B_{(X, d)}(x_0, \xi_1 2^{-j}).
\]

(8.4)

Let \( Y_1, \ldots, Y_q \) denote the pullbacks of \( X_1, \ldots, X_q \) via \( \Phi_{j, x_0} \). Pulling (8.4) back via \( \Phi_{j, x_0} \) it suffices to show,

\[
e^{t_1 Y_1 + \cdots + t_q Y_q} 0 \in B_{(Y, d)}(0, \xi_1).
\]

Take \( \eta' > 0 \) so small that \( B^n(\eta') \subseteq B_{(Y, d)}(0, \xi_1) \). It is easy to see that this is possible, since \( \xi_1 \geq 1 \) and \( \inf_{u \in B^n(\eta)} |\det_{n \times n} Y(u)| \geq 1 \). Since \( Y_1, \ldots, Y_q \in C^\infty \) uniformly in \( j \) and \( x_0 \) (see Theorem 7.1), it is follows that for \( |t| \leq a \), with \( a > 0 \) sufficiently small, we have,

\[
e^{t_1 Y_1 + \cdots + t_q Y_q} 0 \in B^n(\eta') \subseteq B_{(Y, d)}(0, \xi_1);
\]

which completes the proof of the support of \( \tilde{K}_j \).

Since \( |\det d\Phi_{j, x_0}(u)| \approx \text{Vol}(B_{(X, d)}(x_0, 2^{-j})) \) for \( u \in B^n(\eta_1) \) (and in light of the support of \( \tilde{K}_j \)), to prove the differential inequalities on \( \tilde{K}_j \), it suffices to show,

\[
\left| (\det d\Phi_{j, x_0}(u)) Y_a Y_v \tilde{K}_j(\Phi_{j, x_0}(u), \Phi_{j, x_0}(v)) \right| \lesssim 1,
\]

where \( u, v \in B^n(\eta_1) \). Using that \( Y_1, \ldots, Y_q \) and \( \Phi_{j, x_0} \) are \( C^\infty \) uniformly in any relevant parameters, it suffices to show that for all multi-indices \( \alpha \) and \( \beta \) (no longer ordered),

\[
\left| \partial^\alpha u \partial^\beta v \left( \tilde{K}_j(\Phi_{j, x_0}(u), \Phi_{j, x_0}(v)) \det d\Phi_{j, x_0}(v) \right) \right| \lesssim 1;
\]

that is, that \( \tilde{K}_j(\Phi_{j, x_0}(u), \Phi_{j, x_0}(v)) \det d\Phi_{j, x_0}(v) \) is \( C^\infty \) uniformly in any relevant parameters.

Let \( \Phi^\# \) denote the map \( \Phi^\# g = g \circ \Phi \). Then, \( \tilde{K}_j(\Phi_{j, x_0}(u), \Phi_{j, x_0}(v)) \det d\Phi_{j, x_0}(v) \) is the Schwartz kernel of the map

\[
\tilde{T}_j = \Phi_{j, x_0}^\# T_d \left( \Phi_{j, x_0}^\# \right)^{-1}.
\]

It is easy to see that,

\[
\tilde{T}_j g(u) = \psi_1(\Phi_{j, x_0}(u)) \int g(\gamma_t(u)) \psi_2(\gamma_t(u)) \kappa(2^{-j} t, \Phi_{j, x_0}(u)) \zeta_j(t) \, dt,
\]

which completes the proof of (8.3).
where \( \tilde{\gamma}_t (u) = e^{t_1 Y_1 + \cdots + t_n Y_n} u \). From Theorem 7.1, we have that,

\[
|\det Y_{J_1} (0)| \gtrsim 1,
\]

for some \( J_1 \in I (n, q) \). Without loss of generality, by reordering the coordinates, we may assume \( J_1 = (1, \ldots, n) \). Recall that \( \zeta_j \) is supported in \( B^q (a) \). For each \( u \) and \( t_{n+1}, \ldots, t_q \) fixed, define the map,

\[
\Psi_{u, t_{n+1}, \ldots, t_q} (t_1, \ldots, t_n) = e^{t_1 Y_1 + \cdots + t_q Y_q} u.
\]

Using the \( C^\infty \) bounds for \( Y_1, \ldots, Y_q \), we have that for \( |t| \leq a \) (with \( a > 0 \) sufficiently small),

\[
|\det d\Psi_{u, t_{n+1}, \ldots, t_q} (t_1, \ldots, t_n)| \approx 1.
\]

Applying the change of variables \( v = \Psi_{u, t_{n+1}, \ldots, t_q} (t_1, \ldots, t_n) \), it is immediate to see that the Schwartz kernel of \( \tilde{T}_j \) is \( C^\infty \) uniformly in any relevant parameters. This completes the proof of Theorem 8.2 in the case when \( X_1, \ldots, X_q \) span the tangent space.

The proof of Theorem 8.3 in this case, is merely a simpler reprise of the above. Indeed, the standard Calderón-Zygmund theory shows that the maximal function,

\[
\widetilde{M} f (x) = \sup_{\delta \in (0, 1]} \psi_1 (x) \frac{1}{\Vol (B_{(X, d)} (x, \delta))} \int_{y \in B_{(X, d)} (x, \delta)} |f (y) \psi_2 (y)| \, dy,
\]

is bounded on \( L^p \) (\( 1 < p \leq \infty \)). Hence we need only show that pointwise bound,

\[
\mathcal{M} f (x) \lesssim \widetilde{M} f (x).
\]

Let

\[
A_\delta f (x) = \psi_1 (x) \int_{|t| \leq a} f (\gamma_\delta (x)) |\psi_2 (\gamma_\delta (x))| \, dt,
\]

so that \( \mathcal{M} f (x) = \sup_{\delta \in (0, 1]} A_\delta |f| \). To show (8.5), it suffices to show, for \( a > 0 \) sufficiently small, independent of \( \delta \),

- If \( \tilde{K}_\delta (x, y) \) is the Schwartz kernel of \( A_\delta \), then \( \tilde{K}_\delta (x, y) \) is supported on \( (x, y) \) such that \( y \in B_{(X, d)} (x, \delta) \).
- \( |\tilde{K}_\delta (x, y)| \lesssim \frac{1}{\Vol (B_{(X, d)} (x, \delta))} \).

This follows just as above.

### 8.2 When the vector fields do not span

In this section, we complete the proof of Theorems 8.2 and 8.3 by proving the case when \( X_1, \ldots, X_q \) do not span the tangent space. The idea, as outlined in Remark 8.1, is to use the fact that the involutive distribution generated by \( X_1, \ldots, X_q \) is finitely generated as a \( C^\infty \) module. In fact, in light of \( D (K_0, [0, 1], \Omega, \xi), X_1, \ldots, X_q \) are generators of this distribution (as a \( C^\infty \) module). The goal is to apply the theory of Section 8.1 to each leaf. We will be able to do this by utilizing the coordinate charts on each leaf given to us by Theorem 7.1.

Let \( n_0 (x) = \dim \text{span} \{ X_1 (x), \ldots, X_q (x) \} \). Then, there exist \( \eta_1, \zeta_1 > 0 \) such that for each \( x \in K_0 \), we obtain a map

\[
\Phi_x : B^{n_0 (x)} (\eta_1) \rightarrow B_{(X, d)} (x, \xi),
\]

as in Theorem 7.1 by applying Theorem 7.1 to the vector fields \( (Z, d) = (X, d) \).

Let \( K_2 \subseteq K_0 \) be the support of \( \psi_1 \), and let \( K_1 \) be such that \( \tilde{K}_2 \subseteq K_1 \subseteq K_0 \). Here \( A \subseteq B \) denotes that \( A \) is a relatively compact subset of the interior of \( B \). For a function \( f \) defined on \( \Omega, \delta \leq \xi, \) and \( x \in K_0 \), let

\[
A_{B_{(X, d)} (x, \delta)} f (x) = \frac{1}{\Vol (B_{(X, d)} (x, \delta))} \int_{B_{(X, d)} (x, \delta)} f (y) \, dy,
\]

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where \( \text{Vol}(B(x,d)(x,\delta)) \) denotes the induced Lebesgue measure of \( B(x,d)(x,\delta) \) on the leaf in which \( x \) lies.

We restate Proposition 6.17 of \cite{Str11a}.

**Proposition 8.6 (Proposition 6.17 of \cite{Str11a}).** There exists a constant \( \xi_0 > 0 \), \( \xi_0 < \xi \), such that for every \( \xi' \leq \xi_0 \) and every measurable function \( f \) with \( f \geq 0 \), we have

\[
\int_{K_2} f(x) \, dx \lesssim \int_{K_1} A_{B(x,d)(\cdot,\xi')} f(x) \, dx \lesssim \int_{K_0} f(x) \, dx,
\]

where the implicit constants may depend on a lower bound for \( \xi' \).

We will prove,

**Proposition 8.7.** Let \( \xi' = \frac{\xi}{4} \wedge \frac{\xi_0}{2} \), then we have the pointwise bound for \( 1 < p < \infty \), \( x \in K_0 \),

\[
A_{B(x,d)(\cdot,\xi')} |Tf|^p(x) \lesssim A_{B(x,d)(\cdot,2\xi')} |f|^p(x),
\]

where the implicit constant may depend on \( p \), and we have taken \( a > 0 \) sufficiently small, in the definition of \( T \).

Before we prove Proposition 8.7, let us first see how it yields Theorem 8.2.

**Proof of Theorem 8.2 given Propositions 8.6 and 8.7.** Letting \( \xi' \) be as in Proposition 8.7, we have,

\[
\|Tf\|^p_{L^p} = \int_{K_2} |Tf(x)|^p \, dx
\]

\[
\lesssim \int_{K_1} A_{B(x,d)(\cdot,\xi')} |Tf|^p(x) \, dx
\]

\[
\lesssim \int_{K_1} A_{B(x,d)(\cdot,2\xi')} |f|^p(x) \, dx
\]

\[
\lesssim \int_{K_0} |f(x)|^p \, dx
\]

\[
\lesssim \|f\|^p_{L^p},
\]

completing the proof.

\[
\square
\]

We now turn to the proof of Proposition 8.7. It suffices to show that for each \( x_0 \in K_0 \), we have,

\[
A_{B(x,d)(\cdot,\xi')} |Tf|^p(\Phi_{x_0}(0)) \lesssim A_{B(x,d)(\cdot,2\xi')} |f|^p(\Phi_{x_0}(0)),
\]

since \( \Phi_{x_0}(0) = x_0 \).

Fix \( x_0 \) and let \( Y_1, \ldots, Y_q \) be the pullbacks of \( X_1, \ldots, X_q \) via the map \( \Phi_{x_0} \) to \( B_{\alpha_0(x_0)}(\eta) \). We have,

**Lemma 8.8.** For \( f \geq 0 \) a measurable function defined on \( \Omega \),

\[
A_{B(x,d)(\cdot,\xi')} f(x_0) \approx \int_{B(y,0)(\cdot,\xi')} f \circ \Phi_{x_0}(u) \, du.
\]

**Proof.** We apply a change of variables \( x = \Phi_{x_0}(u) \), using that \( \Phi_{x_0}(B(y,0)(0,\xi')) = B(x,d)(x_0,\xi') \) and that \( |\det_{n_0(x)}(x) \times n_0(x) \, d\Phi_{x_0}(u)| \approx \text{Vol}(B(x,d)(x_0,\xi')) \). See (B.2) of \cite{Str11a} for details on this sort of change of variables. It follows that,

\[
\int_{B(y,0)(0,\xi')} f(\Phi_{x_0}(u)) \, du \approx \frac{1}{\text{Vol}(B(x,d)(x_0,\xi'))} \int_{B(x,d)(x_0,\xi')} f(x) \, dx
\]

\[
= A_{B(x,d)(\cdot,\xi')} f(x_0),
\]

completing the proof.

\[
\square
\]
Completion of the proof of Proposition 8.7. In light of Lemma 8.8, it suffices to show
\[ \int_{B(Y,d)(0,\xi')} \left| \Phi_{x_0}^\# T (\Phi_{x_0}^\#)^{-1} \Phi_{x_0}^\# f(u) \right|^p du \lesssim \int_{B(Y,d)(0,2\xi')} \left| \Phi_{x_0}^\# f(u) \right|^p du. \]
This will follow from,
\[ \left\| \Phi_{x_0}^\# T (\Phi_{x_0}^\#)^{-1} \right\|_{L^p(B(Y,d)(0,\xi')) \to L^p(B(Y,d)(0,2\xi'))} \lesssim 1, \tag{8.6} \]
for \( 1 < p < \infty \), with the implicit constant independent of \( x_0 \).

To prove (8.6), we apply the theory in Section 8.1 to the operator \( \Phi_{x_0}^\# T (\Phi_{x_0}^\#)^{-1} \). Note that,
\[ \Phi_{x_0}^\# T (\Phi_{x_0}^\#)^{-1} g(u) = \psi_1(\Phi_{x_0}(u)) \int g(\tilde{\gamma}_t(u)) \psi_2(\tilde{\gamma}_t(u)) \kappa(t,\Phi(u)) K(t) \, dt, \]
where \( \tilde{\gamma}_t(u) = e^{t_1 Y_1 + \cdots + t_q Y_q} u \). We have, from Theorem 7.1 that \( |\det_{n_0(x) \times n_0(x)} Y(u)| \approx 1 \), for \( u \in B^{n_0(x)}(\eta_1) \). That is, that \( Y_1,\ldots,Y_q \) span the tangent space (uniformly in \( x_0 \)). It is easy to see that the methods in Section 8.1 apply to the operator \( \Phi_{x_0}^\# T (\Phi_{x_0}^\#)^{-1} \) uniformly in \( x_0 \), establishing (8.6) and completing the proof of Proposition 8.7.

The proof of Theorem 8.3 follows by a simpler reprise of the above. See also Section 6.2 of [Str11a].

9 Auxiliary operators

In this section, we introduce a number of operators, which will be useful in the proof of Theorems 5.2 and 5.3. Before we begin, we pick four \( C^\infty \) cut-off functions \( \psi_0, \psi_{-1}, \psi_{-2}, \psi_{-3} \geq 0 \), supported on the interior of \( K_0 \) with
\[ \psi_1, \psi_2 < \psi_0 < \psi_{-1} < \psi_{-2} < \psi_{-3}. \]

In the statement of Theorem 5.2, we took \( K \in K(N,e,a,\nu) \) (recall, we are first presenting the proof in the case \( \mu_0 = \nu \), and in Section 13 will present the necessary modifications to treat general \( \mu_0 \)). Thus,
\[ K(t) = \sum_{j \in \mathbb{N}^\nu} \zeta_j^{(2j)}(t), \]
where \( \{\zeta_j\} \subseteq C^\infty_0(B^N(a)) \) is a bounded set and the \( \zeta_j \) satisfy certain cancellation conditions (see Section 2 for details). Hence, there is a corresponding decomposition of \( T \). We define, for \( j \in \mathbb{N}^\nu \),
\[ T_j f(x) = \psi_1(x) \int f(\gamma_t(x)) \psi_2(\gamma_t(x)) \kappa(t,x) \zeta_j^{(2j)}(t) \, dt. \]
We have,
\[ \sum_{j \in \mathbb{N}^\nu} T_j = T. \]

We now turn to the operators which we will use to construct our Littlewood-Paley theory. For each \( \mu, 1 \leq \mu \leq \nu \), we obtain a list of vector fields with single-parameter formal degrees \( (X^\mu, d^\mu) \), by letting \( X_1^\mu,\ldots,X_{q_\mu}^\mu \) be those vector fields \( X_j \) such that \( d_j \) is non-zero in only the \( \mu \)-th component. We then assign the formal degree to be \( d^\mu \) (i.e., the value of the non-zero component). Using this definition,
\[ (\delta_\mu X^\mu, d^\mu) = (\hat{\delta} X, \sum d), \]
where \( \hat{\delta} \) is \( \delta_\mu \) in the \( \mu \)-th component and 0 in all other components, and we have suppressed the vector fields that are equal to 0. As a consequence, \( (X^\mu, d^\mu) \) satisfies \( \mathcal{D}(K_0,[0,1],\Omega',\xi) \), since \( (X,d) \) satisfies \( \mathcal{D}(K_0,[0,1]^\nu,\Omega',\xi) \).
We define (single-parameter) dilations on $\mathbb{R}^{q_{\nu}}$ by,

$$\delta (t_1, \ldots, t_{q_{\nu}}) = \left( \delta^{d_1 t_1}, \ldots, \delta^{d_{q_{\nu}} t_{q_{\nu}}} \right).$$  \hspace{1cm} (9.1)

Let $\phi_{\mu} \in C_0^\infty (B^{q_{\nu}} (a))$ be such that $\int \phi_{\mu} = 1$, and assume $\phi_{\mu} \geq 0$. Define,

$$\phi_{\mu,j} = \begin{cases} 
\phi_{\mu} & \text{if } j = 0, \\
\phi_{\mu} - \phi_{\mu} & \text{if } j > 0.
\end{cases}$$

Here, as usual, $\phi_{\mu}^{(2^j)} (t) = 2^j \left( \delta^{d_1 + \cdots + d_{q_{\nu}}} \right) \phi_{\mu} (2^j t)$. Define,

$$\tilde{\gamma}_{(t_1, \ldots, t_{q_{\nu}})}^{\mu} (x) = e^{t_1 x_1^{\nu} + \cdots + t_{q_{\nu}} x_{q_{\nu}}}. $$

For $j \in \mathbb{N}$, define,

$$D^\mu_j f (x) = \psi_{-3} (x) \int f \left( \tilde{\gamma}_{(t_1, \ldots, t_{q_{\nu}})}^{\mu} (x) \right) \psi_{-3} \left( \tilde{\gamma}_{(t_1, \ldots, t_{q_{\nu}})}^{\mu} (x) \right) \phi_{\mu,j} (t) \, dt;$$

so that $\sum_{j \in \mathbb{N}} D^\mu_j = \psi_{-3}^2$. For $j = (j_1, \ldots, j_{q_{\nu}}) \in \mathbb{N}^{q_{\nu}}$, define,

$$D_j = D^1_{j_1} D^2_{j_2} \cdots D^\nu_{j_{q_{\nu}}},$$

so that,

$$\sum_{j \in \mathbb{N}^{q_{\nu}}} D_j = \psi_{-3}^2.$$  \hspace{1cm} (9.2)

In Section 11 we will use the operators $D_j$ to create an appropriate Littlewood-Paley square function.

Now we turn to the operators which will be at the basis of the study of the maximal function. The study of the maximal function will proceed by induction on the number of parameters ($\nu$), with the base case being the trivial case $\nu = 0$ (we will explain this more in what follows). In what follows, we introduce operators that will facilitate this induction.

Let $N_{\infty} = \mathbb{N} \cup \{ \infty \}$. For a subset $E \subseteq \{1, \ldots, \nu\}$ and $j = (j_1, \ldots, j_{q_{\nu}}) \in \mathbb{N}^{q_{\nu}}$, define $j_E \in N_{\infty}$ to be equal to $j_\mu$ in those components $\mu \in E$, and equal to $\infty$ in the rest of the components. For $t \in \mathbb{R}^N$, we dilate $2^{-j_E} t$ in the usual way, where we identify $2^{-\infty} = 0$; thus, $2^{-j_E} t$ is zero in every coordinate $j$ such that $e_\mu^{(j)} \neq 0$ for some $\mu \in E^c$. We may think of these dilations as $|E|$-parameter dilations acting on a lower dimensional space consisting of those coordinates which are not mapped to 0 under this dilation. Notice that $j_{(1, \ldots, q_{\nu})} = j$ and $j_{_0} = (\infty, \infty, \ldots, \infty)$.

Let $\sigma \in C_0^\infty (B^{N} (a))$ satisfy $\sigma \geq 0$ and $\sigma \geq 1$ on a neighborhood of 0. We assume, further, that $\sigma$ is of the form,

$$\sigma (t_1, \ldots, t_N) = \sigma_0 (t_1) \cdots \sigma_0 (t_N),$$

where $\sigma_0 \in C_0^\infty (\mathbb{R})$, is supported near 0, is $\geq 0$, and is $\geq 1$ on a neighborhood of 0. We define for $j \in N_{\infty}^{q_{\nu}},$

$$M_j f (x) = \psi_0 (x) \int f \left( \gamma_{2^{-j} t} (x) \right) \psi_0 \left( \gamma_{2^{-j} t} (x) \right) \sigma (t) \, dt.$$

Notice,

$$M_{j_E} f (x) = \psi_0^2 (x) \left[ \int \sigma (t) \, dt \right] f (x).$$  \hspace{1cm} (9.3)

It is immediate to see,

$$M f (x) \leq \sup_{j \in \mathbb{N}_0} M_j |f| (x) + \psi_0 (x) \int_{|t| \leq \alpha} |f (\gamma_t (x))| \psi_0 (\gamma_t (x)) \, dt.$$  \hspace{1cm} (9.4)

The second term on the left hand side of (9.4) is easy to control, and so to prove Theorem 5.3 it suffices to prove the following proposition.
Proposition 9.1.

\[ \left\| \sup_{j \in \mathbb{N}_0} |M_j f| \right\|_{L^p} \lesssim \|f\|_{L^p}, \]

for \( 1 < p < \infty \).

Indeed, to deduce Theorem 5.4 merely apply Proposition 9.1 to \(|f|\) and use (9.4). The difficulty in Proposition 9.1 is that, unlike the operators \(T_j\), the operators \(M_j\) do not have any cancellation to take advantage of. We now turn to reducing Proposition 9.1 to an equivalent result where there will be cancellation to take advantage of.

We begin by explaining our induction. Given \( E \subseteq \{1, \ldots, \nu\} \), separate \( t \in \mathbb{R}^N \) into two variables \( t = (t_1^E, t_2^E) \): \( t_2^E \) will be those coordinates that are mapped to 0 under \( 2^{-jE} t \), and \( t_1^E \) will be the rest of the coordinates. I.e., \( t_2^E \) are those coordinates \( t_j \) such that \( c_j^\mu \neq 0 \) for some \( \mu \in E^c \). With an abuse of notation, we write \( 2^{-jE} t_1^E \) as the \( t_1^E \) coordinate of \( 2^{-jE} t \), and so \( 2^{-jE} t_1^E \) defines \(|E|\)-parameter dilations on \( t_1^E \). Furthermore, with another abuse of notation, we write \( \sigma(t) = \sigma(t_1^E \sigma(t_2^E) \), where \( \sigma(t_2^E) \) is a product of \( \sigma_0(t_j) \) such that \( t_j \) is a coordinate of \( t_2^E \), and similarly for \( t_2^E \). We may rewrite \( M_{jE} \) as follows,

\[
M_{jE} f(x) = \left[ \psi_0(x) \int f(\gamma_{2^{-jE} t_1^E}(x)) \psi_0(\gamma_{2^{-jE} t_1^E}(x)) \sigma(t_2^E) \, dt_2^E \right]\left[ \int \sigma(t_2^E) \, dt_2^E \right].
\]

The term \( \int \sigma(t_2^E) \, dt_2^E \) is a constant. It is easy to see from our assumptions that \( \gamma_{2^{-jE} t_1^E} \) is of the same form as \( \gamma_{2^{-j}} \) with \( \nu \) replaced by \(|E|\). I.e., \( \gamma_{t_1^E} \) satisfies the hypotheses of Theorem 5.4 with \( \nu \) replaced by \(|E|\). As a consequence, \( M_{jE} \) is a constant times an operator of the same form as \( M_j \), with \( \nu \) replaced by \(|E|\).

We will prove Proposition 9.1 by induction on \( \nu \). Due to the above discussion, our inductive hypothesis implies,

\[
\left\| \sup_{j \in \mathbb{N}_0} |M_{jE} f| \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad (9.5)
\]

for \( E \subseteq \{1, \ldots, \nu\} \) and \( 1 < p < \infty \). The base case of our induction will correspond to \( E = \emptyset \). In light of (9.3), the base case is trivial.

For each \( \mu \), \( 1 \leq \mu \leq \nu \), and each \( j \in \mathbb{N}_\infty \), define the operator,

\[
A_j^\mu f(x) = \psi_{-1}(x) \int_{\mathbb{R}^{\nu_\mu}} f(\tilde{\gamma}_j^{\mu_\mu}(x)) \psi_{-1}(\tilde{\gamma}_j^{\mu_\mu}(x)) \sigma(t) \, dt,
\]

where we have used the dilations on \( \mathbb{R}^{\nu_\mu} \) defined in (9.1) and we have identified \( 2^{-\infty} = 0 \); so that \( A^\infty_\nu = \left[ \int \sigma(t) \, dt \right] \psi_{-1}^2 \). Here we have abused notation and viewed \( \sigma \) as a function on \( \mathbb{R}^{\nu_\mu} \). By this we mean, \( \sigma(t_1, \ldots, t_{\nu_\mu}) = \prod_{j=1}^{\nu_\mu} \sigma_0(t_j) \). Define the maximal operator,

\[
M^\mu f(x) = \sup_{\delta \in [0,1]} \psi_{-\delta}(x) \int_{|t| \leq \delta} \left| f(\tilde{\gamma}_\delta^{\mu_\mu}(x)) \psi_{-\delta}(\tilde{\gamma}_\delta^{\mu_\mu}(x)) \right| \, dt.
\]

Note that Theorem 5.3 shows that,

\[
\|M^\mu f\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 < p \leq \infty.
\]

Also it is elementary to verify the pointwise inequality

\[
\sup_{j \in \mathbb{N}_\infty} \left| A_j^\mu f(x) \right| \lesssim M^\mu f(x), \quad (9.6)
\]

and so we have,

\[
\left\| \sup_{j \in \mathbb{N}_\infty} \left| A_j^\mu f(x) \right| \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 < p \leq \infty.
\]

For \( \mu = (j_1, \ldots, j_\nu) \in \mathbb{N}_\infty^\nu \) define

\[
A_j = A_{j_1}^1 A_{j_2}^2 \cdots A_{j_\nu}^\nu.
\]

(9.7)
Notice that
\[ A_{(\infty,\infty,\ldots,\infty)} = \left[ \int \sigma(t) \, dt \right]^\nu \psi_{-1} \nu. \]
And since \( \psi_{-1} M_j = M_j = M_{j(1,\ldots,\nu)} \), we see to prove Proposition 9.1 it suffices to prove,
\[ \left\| \sup_{j \in \mathbb{N}} |A_{jk} M_{j(1,\ldots,\nu)} f| \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 \leq p \leq \infty. \quad (9.8) \]
For \( E \subseteq \{1,\ldots,\nu\} \), combining (9.5) and (9.6), we see,
\[ \left\| \sup_{j \in \mathbb{N}} |A_{jE} M_{jE} f| \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 < p \leq \infty. \quad (9.9) \]
For \( j \in \mathbb{N}^\nu \), define the operator,
\[ B_j = \sum_{E \subseteq \{1,\ldots,\nu\}} (-1)^{|E|} A_{jE} M_{jE}. \]
From (9.9) we see that to prove (9.8) (and hence to prove Proposition 9.1 and Theorem 5.4) it suffices to prove,

**Proposition 9.2.**
\[ \left\| \sup_{j \in \mathbb{N}^\nu} |B_j f| \right\|_{L^p} \lesssim \|f\|_{L^p}, \quad 1 < p \leq \infty. \]

## 10 Preliminary \( L^2 \) results

In this section we describe some \( L^2 \) results concerning the operators defined in Section 9. These results, along with the results in Section 8, make up the main technical results on which our theory is based. All of the results in this section will follow from the results in [Str11b] (after some reductions).

**Theorem 10.1.** For \( j_1, \ldots, j_r \in \mathbb{N}^\nu \), define,
\[ \text{diam} \{j_1, \ldots, j_r\} = \max_{1 \leq l, m \leq r} |j_l - j_m|. \]

If we take \( \alpha > 0 \) sufficiently small\(^{10}\) then there exists \( \varepsilon_2 > 0 \) such that,
\[ \begin{align*}
&\|B_{j_1} D_{j_2}\|_{L^2 \to L^2} \lesssim 2^{-\varepsilon_2 \text{diam}\{j_1, j_2\}}, \\
&\|D_{j_1} T_{j_2} D_{j_1}\|_{L^2 \to L^2} \lesssim 2^{-\varepsilon_2 \text{diam}\{j_1, j_2, j_3\}}, \\
&\|D_{j_1}^* D_{j_2}^* D_{j_3} D_{j_4}\|_{L^2 \to L^2} \lesssim 2^{-\varepsilon_2 \text{diam}\{j_1, j_2, j_3, j_4\}}, \\
&\|D_{j_1} D_{j_2} D_{j_3}^* D_{j_4}^*\|_{L^2 \to L^2} \lesssim 2^{-\varepsilon_2 \text{diam}\{j_1, j_2, j_3, j_4\}}.
\end{align*} \]

Here, \( j_1, j_2, j_3, \text{ and } j_4 \) are arbitrary elements of \( \mathbb{N}^\nu \).

The rest of this section is devoted to the proof of Theorem 10.1. We will see that each part of Theorem 10.1 follows from an application of the same general result. This result is proved in [Str11b], and we review the statement of the result in Section 10.1. In Section 10.2 we show how to reduce each part of Theorem 10.1 to the result in Section 10.1.

**Remark 10.2.** Using methods similar to the ones in this section, one can prove \( \|T_{j_1}^* T_{j_2}\|_{L^2 \to L^2} \cdot \|T_{j_1} T_{j_2}^*\|_{L^2 \to L^2} \lesssim 2^{-\varepsilon \text{diam}\{j_1, j_2\}} \). This shows, via the Cotlar-Stein lemma, that \( T \) is bounded on \( L^2 \). This is the proof used in [Str11b].

\(^{10}\)Recall, all of the operators in Section 9 were defined in terms of some small \( \alpha > 0 \).
10.1 A general $L^2$ result

In this section, we review the main technical result from [Str11b]; this result will imply Theorem 10.1. The setting is as follows. We are given operators $S_1, \ldots, S_L$, and $R_1, R_2$, and a real number $\zeta \in [0, 1]$. We will present conditions on these operators such that there exists $\epsilon > 0$ with,

$$\|S_1 \cdots S_L (R_1 - R_2)\|_{L^2 \to L^2} \lesssim \zeta^\epsilon. \quad (10.1)$$

In Section 10.2 we will show that the assumptions of this section hold uniformly in $j_1, j_2, j_3, j_4$ for the operators in Theorem 10.1 (with an appropriate choice of $\zeta$), and Theorem 10.1 will follow. The term $R_1 - R_2$ is how we make use of the cancellation implicit in the operators in Theorem 10.1.

To describe the operators above, suppose we are given $C^\infty$ operators in Theorem 10.1 (with an appropriate choice of $\zeta$). We assume that $C_{\nu} (\rho) \times \Omega'$ to $\Omega'$ be a uniform control for some fixed $0 < \rho$. In this section, we review the main technical result from [Str11b]; this result will imply Theorem 10.1. We assume that $\gamma$ is controlled by $\hat{\gamma}$ at the unit scale.

We assume, for each $j$, $r + 1 \leq j \leq q$, $Z_j$ may be written in the form,

$$Z_j = \text{ad} (Z_{l_1}) \text{ad} (Z_{l_2}) \cdots \text{ad}(Z_{l_m}) Z_{l_{m+1}}, \quad 1 \leq l_k \leq j, \quad 0 \leq m \leq M - 1.$$

**Definition 10.3.** Let $\hat{\gamma}: B^N (\rho) \times \Omega' \to \Omega$ be a $C^\infty$ function, satisfying $\hat{\gamma}_0 (x) \equiv x$. We say that $\hat{\gamma}$ is controlled by $\hat{\gamma}$ at the unit scale if the following holds. Define the vector field $\hat{W} (t, x)$ by,

$$\hat{W} (t, x) = \frac{d}{dt} \bigg|_{t=1} \hat{\gamma}_{t} \circ \hat{\gamma}_{t}^{-1} (x).$$

We suppose, there exist $\rho_1, \tau_1 > 0$ such that for every $x_0 \in K_0$,

- $\hat{W} (t, x) = \sum_{l=1}^q c_l (t, x) Z_l (x)$, on $B(z, d) (x_0, \tau_1),$

  $$\sum_{|\alpha| + |\beta| \leq m} \left\| Z^\alpha \partial_t^\beta \hat{W} \right\|_{C^m (B^N (B^N (\rho_1) \times B(z, d) (x_0, \tau_1)))} < \infty, \quad \text{for every } m.$$  

**Remark 10.4.** Note that the assumption that $X, d)$ controls $\gamma$ can be restated as $\delta X, \sum d) controls $\gamma$ at the unit scale for every $\delta \in [0, 1]$ uniformly in $\delta$.

We now turn to defining the operators $S_j$. We assume, for each $j$, we are given a $C^\infty$ function $\hat{\gamma}_j: B^{N_j} (\rho) \times \Omega' \to \Omega$ with $\hat{\gamma}_j (0, x) \equiv x$, and that this function is controlled by $\hat{\gamma}$ at the unit scale. As usual, we restrict our attention to $\rho > 0$ small, so that $\hat{\gamma}_j^{-1}$ makes sense wherever we use it. We suppose we are given $\psi_{j, 1}, \psi_{j, 2} \in C^\infty_0 (\mathbb{R}^n)$ supported on the interior of $K_0$ and $\kappa_j \in C^\infty_0 (B^{N_j} (a) \times \Omega')$. Finally, we suppose we are given $c_j \in C^\infty_0 (B^{N_j} (a))$. We define,

$$S_j f (x) = \psi_{j, 1} (x) \int f (\hat{\gamma}_{j, t} (x)) \psi_{j, 2} (\hat{\gamma}_{j, t} (x)) \kappa_j (t, x) c_j (t) \, dt.$$  

**Definition 10.5.** If $S_j$ is of the above form, we say $S_j$ is controlled by $\hat{\gamma}$ at the unit scale.

**Remark 10.6.** If $S_j$ is controlled by $\hat{\gamma}$ at the unit scale, then so is $S_j^*$. This is shown in [Str11b]. Furthermore, a simple change of variables shows that if $S_j$ is controlled at the unit scale by $\hat{\gamma}$, then $\|S_j\|_{L^2 \to L^2} \lesssim 1$.

---

\footnote{The implicit constants in (10.1) may depend on $M_1$ and $\xi$.}
We assume further, that for each \( l, 1 \leq l \leq r \), there is a \( j, 1 \leq j \leq L \), and a multi index \( \alpha \) (with \( |\alpha| \leq M_2 \) for some \( M_2 \)), such that,

\[
Z_l(x) = \frac{1}{a!} \frac{\partial^\alpha}{\partial t^\alpha} \left| \frac{d}{dt} \right|_{t=0} \frac{d}{de} \left| e=1 \right| \tilde{\gamma}_{t,s} \circ \psi_{t}^{-1}(x).
\]

This concludes our assumptions on \( S_1, \ldots, S_L \).

We now turn to the operators \( R_1 \) and \( R_2 \). It is here where \( \zeta \) plays a role. We assume we are given a \( C^\infty \) function \( \tilde{\gamma}_{t,s} \) which is controlled by \( (Z, d) \) at the unit scale:

\[
\tilde{\gamma}_{t,s}(x) : B^\infty N(\rho) \times [-1,1] \times \Omega' \to \Omega, \quad \tilde{\gamma}_{0,0}(x) \equiv x.
\]

**Remark 10.7.** Here we are thinking of \((t, s)\) as playing the role of the \( t \) variable in the definition of control.

We suppose we are given \( \tilde{\kappa}(t, s, x) \in C^\infty (B^N N(\rho) \times [-1,1] \times \Omega'), \xi \in L^1 (B^N N(\rho)) \), and \( \tilde{\psi}_1, \tilde{\psi}_2 \in C^\infty (\mathbb{R}^n) \) supported on the interior of \( K_0 \). We define, for \( \xi \in [-1,1] \),

\[
R^\xi f(x) = \tilde{\psi}_1(x) \int f(\gamma_{t,\xi}(x)) \tilde{\psi}_2(\gamma_{t,\xi}(x)) \tilde{\kappa}(t, \xi, x) \tilde{\zeta}(t) \ dt.
\]

We set \( R_1 = R^\xi \) and \( R_2 = R^0 \).

**Theorem 10.8** (Theorem 14.5 of [Str11b]). In the above setup, if \( a > 0 \) is sufficiently small, we have,

\[
\|S_1 \cdots S_L (R_1 - R_2)\|_{L^2} \lesssim \zeta^c,
\]

for some \( \epsilon > 0 \).

**Remark 10.9.** It is important in our applications of Theorem 10.8 that the various constants can be chosen independent of any relevant parameters. I.e., that if all of the hypotheses of this section hold "uniformly" then so does Theorem 10.8. Indeed, this is the case, and is discussed further and made precise in [Str11b]. In this paper, we merely say that in our proof of Theorem 10.1, all of our applications of Theorem 10.8 will satisfy the hypotheses of this section uniformly in the appropriate sense, and we leave the straight-forward verification of this fact to the reader.

### 10.2 Reduction to the general \( L^2 \) result

This section is devoted to proving Theorem 10.1 by applying Theorem 10.8. We will be implicitly choosing \( a > 0 \) by choosing it small enough that Theorem 10.8 applies. Since the assumptions of Theorem 10.8 will hold uniformly in \( j_1, j_2, j_3, j_4 \), we will have that \( a > 0, \epsilon > 0 \), and the implicit constant in Theorem 10.8 can all be chosen independent of \( j_1, j_2, j_3, j_4 \in \mathbb{N}^r \). See Remark 10.9 and [Str11b] for more details on this.

Recall the list of vector fields \((X, d) = (X_1, d_1), \ldots, (X_q, d_q)\) satisfying \( D(K_0, [0,1]^n) \) defined in Section 3.

Our assumptions on \( \gamma \) can be restated (by possibly reordering \((X_1, d_1), \ldots, (X_q, d_q)\)) as there exists \( r \leq q \) such that,

1. \((X, d)\) controls \( \gamma \).
2. For \( 1 \leq l \leq r \), \( d_l \) is nonzero in only one component.
3. Every \((X_j, d_j)\), with \( r < j \leq q \), can be written as,

\[
X_j = \text{ad} (X_{i_1}) \text{ad} (X_{i_2}) \cdots \text{ad} (X_{i_m}) X_{l_{m+1}},
\]

\[
d_j = d_{i_1} + d_{i_2} + \cdots + d_{l_{m+1}},
\]

with \( 1 \leq l_k \leq r \).

\[12\text{The implicit constants in (10.1) are allowed to depend on } M_2.\]
4. Every \((X_l, d_l), 1 \leq l \leq r\), is of the form,

\[
X_l = \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \bigg|_{t=0} \frac{d}{dx_{\ell=1}^\ell} \gamma_{\ell t} \circ \gamma_{t}^{-1} (x),
\]

\[
d_l = \deg (\alpha).
\] (10.2)

We now describe how the above assumptions come into play in what follows. Take \(\delta \in [0, 1]\). Define the list of vector fields with single-parameter formal degree \((Z, d) = (\delta X, \sum d)\). Note that \(Z_1, \ldots, Z_q\), in the sense of that every \(Z_j (r + 1 \leq j \leq q)\) can be written in the form,

\[
Z_j = \text{ad} (Z_{i_1}) \text{ad} (Z_{i_2}) \cdots \text{ad} (Z_{i_m}) Z_{i_{m+1}},
\]

with \(1 \leq l_k \leq r\) for every \(k\).

Fix \(\delta_1 = (\delta_1^1, \ldots, \delta_1^r), \delta_2 = (\delta_2^1, \ldots, \delta_2^r) \in [0, 1]^r\) and assume \(\delta_1^\mu \leq \delta_2^\mu\) for every \(\mu\). Then, using the fact that \((X, d)\) controls \(\gamma\), we have \((\delta_2 X, \sum d)\) controls \(\gamma_{\delta_2 t}\) at the unit scale, uniformly in \(\delta_1, \delta_2\). Furthermore, suppose that \(\delta_1^\mu = \delta_2^\mu\) for some fixed \(\mu\). Suppose further that for \(j_0\) fixed \((j_0 \leq r)\), \(d_{j_0}\) is nonzero in only the \(\mu\)th coordinate. Define \(\gamma_{t} = \gamma_{\delta_1 t}\). We then have,

\[
\delta_{l_{j_0}}^d X_{j_0} = \delta_{l_{j_0}}^d X_{j_0} = \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} \bigg|_{t=0} \frac{d}{dx_{\ell=1}^\ell} \gamma_{\ell t} \circ \gamma_{t}^{-1} (x),
\]

for some \(\alpha\).

We now turn to the proof of Theorem 10.1. We describe, in detail, the proof for \(B_j D_k\) as it is the most complicated (here we have replaced \(j_1\) with \(j\) and \(j_2\) with \(k\) for notational convenience). We then indicate the modifications necessary to study all of the other operators. Let \(\ell = j \wedge k\). Define \(\tilde{E} = (2^{-\ell} X, \sum d)\), and \(t_\infty = |j - k|\). Our goal is to show that there exists \(\epsilon > 0\) such that,

\[
\|B_j D_k\|_{L^2 \to L^2} \lesssim 2^{-\epsilon t_\infty}.
\] (10.4)

Note that this is trivial if \(t_\infty = 0\) and so we assume \(t_\infty > 0\) in what follows.

We separate the proof into two cases. The first case is when there is a \(\mu_1 (1 \leq \mu_1 \leq \nu)\) such that \(t_\infty = \mu_1 - j\). In this case, we need only use the cancellation in the operator \(D_k\), and so it suffices to prove,

\[
\|A_{j E} M_{j E} D_k\|_{L^2 \to L^2} \lesssim 2^{-\epsilon t_\infty},
\] (10.5)

for every \(E \subseteq \{1, \ldots, \nu\}\).

To prove (10.5), it suffices to prove,

\[
\left\| \left[ D_k^* M_{j E}^* A_{j E}^* A_{j E} M_{j E} D_k \right] \right\|_{L^2 \to L^2} \lesssim 2^{-\epsilon t_\infty},
\] (10.6)

where we have changed \(\epsilon\).

Using that \(\left\| D_k^* \right\|_{L^2 \to L^2} \cdot \left\| M_{j E}^* \right\|_{L^2 \to L^2} \cdot \left\| A_{j E}^* \right\|_{L^2 \to L^2} \lesssim 1\), to prove (10.6) it suffices to show,

\[
\left\| A_{j E} M_{j E} D_k D_k^* M_{j E}^* A_{j E} A_{j E} M_{j E} D_k \right\|_{L^2 \to L^2} \lesssim 2^{-\epsilon t_\infty}.
\] (10.7)

We now expand the last \(D_k\) in (10.7) into \(D_k = D_k^1 D_k^2 \cdots D_k^{\nu}\). Using that \(\left\| D_k^{\mu_1} \right\|_{L^2 \to L^2} \lesssim 1\) for every \(\mu\), to prove (10.7) it suffices to show,

\[
\left\| A_{j E} M_{j E} D_k D_k^* M_{j E}^* A_{j E} A_{j E} M_{j E} D_k \right\|_{L^2 \to L^2} \lesssim 2^{-\epsilon t_\infty}.
\] (10.8)

We will prove (10.8) by applying Theorem 10.8 with,

\[
S_1 \cdots S_L = A_{j E} M_{j E} D_k D_k^* M_{j E}^* A_{j E} A_{j E} M_{j E} D_k \cdots D_k^{\nu-1},
\]

\[
R_1 = D_k^{\nu_1}, \quad R_2 = 0.
\]

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In the above, we are thinking of $D_j$ and $A_{jE^c}$ as a product of terms (see \[9.2\] and \[9.7\]), and assigning an $S_l$ to each term in the product, similarly for the adjoints.

First we verify that each $S_l$ is controlled at the unit scale by \( (Z, \hat{d}) \). We will begin by showing that $A_{j\mu}^\mu$, $A_{j\infty}^\mu$, and $D_{j\mu}^\mu$ are controlled at the unit scale. We will also show that $M_{jE}^E$ is controlled at the unit scale. It will then follow that $D_{k^*}^\mu$, $M_{jE}^E$, and $A_{jE^c}^\mu$ are all products of operators which are controlled at the unit scale, since if $S_l$ is controlled at the unit scale, so is $S_l^*$ (Remark 10.6).

Consider,
\[
A_{j\mu}^\mu f(x) = \psi_\mu(x) \int_{t \in R^m} f\left(\tilde{\gamma}_{\mu j\mu}^\mu (x) \right) \psi_\mu(\tilde{\gamma}_{\mu j\mu}^\mu (x)) \sigma(t) \, dt,
\]
and so to show that $A_{j\mu}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) \), it suffices to show that $\tilde{\gamma}_{\mu j\mu}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) \). However,
\[
\tilde{\gamma}_{\mu j\mu}^\mu (x) = \exp \left( 2^{-j_\mu} t_1 X_1^\mu + \cdots + 2^{-j_\mu} t_{\mu_1} X_{\mu_1}^\mu \right) x.
\]
By definition, the list of vector fields \( (2^{-j_\mu} X, d^\mu) \) is a sublist of the list of vector fields \( (2^{-j} X, \sum d) \). Since $j \geq \ell$ coordinatewise, it follows immediately from Lemma 12.18 of [Str11b] that $\tilde{\gamma}_{\mu j\mu}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) = (2^{-\ell} X, \sum d) \). It is trivial that $A_{j\infty}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) \) since $\tilde{\gamma}_t(x) \equiv x$ is controlled at the unit scale by \( (Z, \hat{d}) \), trivially. The proof that $D_{k^*_\mu}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) \) follows just as the proof for $A_{j\mu}^\mu$.

We now turn to $M_{jE}^E$. Since,
\[
M_{jE}^E f(x) = \psi_0 (x) \int f(\gamma_{\mu j\mu}^\mu (x)) \psi_0 (\gamma_{\mu j\mu}^\mu (x)) \sigma(t) \, dt,
\]
we need only show that $\gamma_{\mu j\mu}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) \). As discussed at the beginning of this section, $\gamma_{\mu j\mu}^\mu$ is controlled at the unit scale by \( (Z, \hat{d}) \). Since $\gamma_{\mu j\mu}^\mu$ is the same as $\gamma_{\mu j\mu}$ except with some of the coordinates of $t$ set to 0, the result follows.

This completes the proof that each $S_p$ is controlled at the unit scale by \( (Z, \hat{d}) \). To complete our discussion of the $S_p$, we need to show that for each $l$, \( 1 \leq l \leq r \), $Z_l$ is of the form,
\[
Z_l(x) = \frac{1}{\alpha!} \left. \frac{\partial^{\alpha} \sigma}{\partial t^{\alpha}} \right|_{t=0} \frac{d}{de} \left. \tilde{\gamma}_t \circ \tilde{\gamma}_t^{-1} (x) \right|_{e=1}, \quad (10.9)
\]
for some $\alpha$, where $\tilde{\gamma}$ is one of the functions defining the maps $S_1, \ldots, S_L$.

Fix $l$, \( 1 \leq l \leq r \). Recall, $Z_l = 2^{-\ell} d_l X_l$, and $d_l$ is nonzero in precisely one component. Let us suppose that $d_l$ is nonzero in only the $\mu_2$ component. Thus, $Z_l = 2^{-\ell} d_l X_l$. There are two possibilities. Either $\ell_{\mu_2} = j_{\mu_2}$ or $\ell_{\mu_2} = k_{\mu_2}$. We deal with the second case first.

Suppose $\ell_{\mu_2} = k_{\mu_2}$. Pick $p$ such that $S_p = D_{k_{\mu_2}}^{\mu_2}$. We have,
\[
D_{k_{\mu_2}}^{\mu_2} f(x) = \psi_{-3}(x) \int f(\tilde{\gamma}_l (x)) \psi_{-3} (\tilde{\gamma}_l (x)) \phi_{\mu_2} (k_{\mu_2}) (t) \, dt,
\]
where,
\[
\tilde{\gamma}_l (x) = \tilde{\gamma}_{k_{\mu_2}}^{\mu_2} (x) = \exp \left( 2^{-k_{\mu_2}} d_{l_{\mu_2}} t_1 X_1^{\mu_2} + \cdots + 2^{-k_{\mu_2}} d_{l_{\mu_2}} t_{\mu_2} X_{\mu_2}^{\mu_2} \right) x.
\]
By the definition of $(X^{\mu_2}, d^{\mu_2})$, $(X_l, d^{\mu_2})$ appears in the list $(X^{\mu_2}, d^{\mu_2})$ (this uses the fact that $d_l$ is nonzero in only the $\mu_2$ coordinate). Hence, $Z_l$ is of the form $2^{-\ell} d_l^{\mu_2} X_l^{\mu_2} = 2^{-k_{\mu_2}} d_{l_{\mu_2}} X_{\mu_2}^{\mu_2}$ for some $m$. It follows that,
\[
Z_l(x) = \left. \frac{\partial}{\partial t_{\mu_2}} \right|_{t=0} \left. \frac{d}{de} \right|_{e=1} \tilde{\gamma}_t \circ \tilde{\gamma}_t^{-1} (x), \quad (10.9)
\]
which completes the proof of (10.9) in this case.

We now turn to the case when \( \ell \mu_2 = j \mu_2 \). We separate this case into two cases: when \( \mu_2 \in E \) and when \( \mu_2 \in E^c \). First we deal with the case when \( \mu_2 \in E^c \). In that case, we use \( p \) such that \( S_p = A^{\mu_2}_{\nu_2} \), and the proof of (10.3) proceeds just as in the case for \( D^{\mu_2}_{\nu_2} \) above.

We turn to the case when \( \mu_2 \in E \). In this case, we use \( p \) such that \( S_p = M_{jE} \). We have,

\[
M_{jE} f (x) = \psi_0 (x) \int f (\gamma_{2-jE} (x)) \psi_0 (\gamma_{2-jE} (x)) \sigma (t) \, dt.
\]

Hence, we take \( \gamma_{2-jE} = \gamma_{2-jE} \). Note, if \( \gamma_{2-jE} \) were instead taken to be equal to \( \gamma_{2-jE} \), then (10.9) would follow immediately from (10.2) and (10.3). \( \gamma_{2-jE} \) is just \( \gamma_{2-jE} \) with some of the coordinates set to 0. Thus, to prove (10.9), we need only show that \( \partial_t^p \) in (10.2) only involves those coordinates of \( 2^{-jE} \) which are not identically 0. Let \( \alpha \) be the multi-index from (10.2). We know that \( \deg (\alpha) = d_1 \) and therefore is nonzero in only the \( \mu_2 \) coordinate. Thus if \( t_m \) is a coordinate appearing in \( \partial_t^p \), then \( e_m \) must be nonzero in only the \( \mu_2 \) coordinate. Since \( jE \) is only in those coordinates \( \mu \in E^c \), it follows that \( 2^{-jE} t \) is not identically 0 in any of the coordinates appearing in \( \partial_t^p \). (10.9) follows.

In conclusion, the operators \( S_1, \ldots, S_L \) satisfy all the hypotheses of Theorem 10.8. We now turn to \( R_1, R_2 \). Recall, we are presently considering the case \( \ell \alpha = k_{\mu_1} - j_{\mu_1} \). We have,

\[
D_{\mu}^p f (x) = \psi_3 (x) \int f (\tilde{\gamma}_{2-k_{\mu_1} t} (x)) \psi_3 (\tilde{\gamma}_{2-k_{\mu_1} t} (x)) \phi_{\mu_1,k_{\mu_1}} (t) \, dt
\]

where we have used that \( \int \phi_{\mu_1,k_{\mu_1}} = 0 \) (since \( k_{\mu_1} > 0 \), which follows from our assumption that \( \ell \alpha > 0 \) and therefore \( R_2 = 0 \).

Let \( c_0 = \min_{1 \leq m \leq q_0} d_m^\mu > 0 \). Define \( \zeta = 2^{-c_0} \). Let,

\[
\tilde{\gamma}_{t,s} (x) = \exp \left( 2^{-\ell_{\mu_1} d_1^\mu} 2^{-k_{\mu_1} t} (d_{\mu_1}^\mu - c_0) st_1 + \ldots + 2^{-\ell_{\mu_1} d_1^\mu} 2^{-k_{\mu_1} t} (d_{\mu_1}^\mu - c_0) st_{q_0} \right) x.
\]

It is simple to verify that \( \left( Z, \tilde{d} \right) \) controls \( \gamma_{t,s} \) at the unit scale. In particular, this follows from the fact that \( \left( Z, \tilde{d} \right) \) controls \( \gamma_{2-\ell_{\mu_1} t} \) at the unit scale (see Lemma 12.18 of [Str11b]), and \( \tilde{\gamma}_{t,s} \) is just \( \gamma_{2-\ell_{\mu_1} t} \) with \( t_m \) replaced by \( 2^{-k_{\mu_1} t} (d_{\mu_1}^\mu - c_0) st_m \).

Let \( \zeta = 2^{-c_0} (k_{\mu_1} - t_{\mu_1}) = 2^{-c_0} \). Note that,

\[
\tilde{\gamma}_{2-k_{\mu_1} t} = \tilde{\gamma}_{t,\zeta}.
\]

We therefore have,

\[
(R_1 - R_2) f (x) = \psi_{-3} (x) \int f (\gamma_{t,\zeta} (x)) \psi_{-3} (\gamma_{t,\zeta} (x)) \phi_{\mu_1,k_{\mu_1}} (t) \, dt
\]

This completes the proof that \( R_1 - R_2 \) has the desired form.

Theorem 10.8 applies to show that there exists \( \epsilon > 0 \) such that,

\[
\| S_1 \cdot S_L (R_1 - R_2) \|_{L^2 \to L^2} \lesssim \epsilon = 2^{-\epsilon' \ell \alpha},
\]

for some \( \epsilon' > 0 \). This completes the proof of (10.8) and therefore shows,

\[
\| B_j D_k \|_{L^2 \to L^2} \lesssim 2^{-\epsilon' (j-k)}.
\]
in this case.

We return to deal with the second case: there is a \( \mu_1 \) such that \( \ell_\infty = j_{\mu_1} - k_{\mu_1} \). We wish to show,

\[
\|B_j D_k\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty}.
\]

Applying the triangle inequality, it suffices to show for every \( E \subseteq \{1, \ldots, \nu\} \setminus \{\mu_1\} \),

\[
\left\| \left[ A_{j(E \cup \{\mu_1\})^c} M_{E \cup \{\mu_1\}} - A_{jE^c} M_{jE} \right] D_k \right\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty}.
\]

Let \( O_{j,k,E} = \left[ A_{j(E \cup \{\mu_1\})^c} M_{E \cup \{\mu_1\}} - A_{jE^c} M_{jE} \right] D_k \). Thus, we wish to show,

\[
\left\| O_{j,k,E}^* O_{j,k,E} \right\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty}.
\]

Applying the triangle inequality to the term \( O_{j,k,E}^* \), we see that it suffices to show for every \( F \subseteq \{1, \ldots, \nu\} \),

\[
\left\| D_k^* M_{jF} A_{jF^c} O_{j,k,E} \right\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty}.
\]

Using that \( \|O_{j,k,E}\|_{L^2 \to L^2} \lesssim 1 \), we see,

\[
\left\| D_k^* M_{jF} A_{jF^c} O_{j,k,E} \right\|_{L^2 \to L^2}^2 = \|O_{j,k,E}^* A_{jF^c} M_{jF} D_k D_k^* M_{jF}^* A_{jF^c} O_{j,k,E}\|_{L^2 \to L^2} \lesssim \|A_{jF^c} M_{jF} D_k D_k^* M_{jF}^* A_{jF^c} O_{j,k,E}\|_{L^2 \to L^2}.
\]

Thus, it suffices to show,

\[
\|P_{j,k,F} O_{j,k,E}\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty},
\]

where \( P_{j,k,F} = A_{jF^c} M_{jF} D_k D_k^* M_{jF}^* A_{jF^c}^* \).

We now use that \( \|D_k\|_{L^2 \to L^2}, \|M_{jF}\|_{L^2 \to L^2}, \|M_{E \cup \{\mu_1\}}\|_{L^2 \to L^2} \lesssim 1 \), to see,

\[
\|P_{j,k,F} O_{j,k,E}\|_{L^2 \to L^2} \lesssim \|P_{j,k,F} A_{j(E \cup \{\mu_1\})^c} - A_{jE^c}\|_{L^2 \to L^2} + \sum_{G \in \{E^c, (E \cup \{\mu_1\})^c\}} \|P_{j,k,F} A_{jG} \left[ M_{j(E \cup \{\mu_1\})} - M_{jE}\right]\|_{L^2 \to L^2}.
\]

Thus it suffices to show, for every \( F, G \subseteq \{1, \ldots, \nu\} \) and every \( E \subseteq \{1, \ldots, \nu\} \setminus \{\mu_1\} \),

\[
\|P_{j,k,F} A_{jG} \left[ M_{j(E \cup \{\mu_1\})} - M_{jE}\right]\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty},
\]

\[
\|P_{j,k,F} A_{jE} \left[ M_{j(E \cup \{\mu_1\})} - M_{jE}\right]\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty},
\]

where we have reversed the roles of \( E \) and \( E^c \) in \( 10.12 \).

We begin with \( 10.12 \). Write \( j_E = (j_{E_1}, \ldots, j_{E_\nu}) \in \mathbb{N}_{\infty}^\nu \). Note that,

\[
A_{j_{E \cup \{\mu_1\}}} - A_{j_E} = A_{j_{E_1}}^1 A_{j_{E_2}}^2 \cdots A_{j_{E_{\mu_1}-1}}^{\mu_1-1} \left[ A_{j_{E_{\mu_1}}}^{\mu_1} - A_{j_{E_{\mu_1}+1}}^{\mu_1+1} \cdots A_{j_{E_\nu}}^\nu \right].
\]

Using the fact that \( \|A_{j_E}^\mu\|_{L^2 \to L^2} \lesssim 1 \) for every \( \mu \), to prove \( 10.12 \) it suffices to show,

\[
\|P_{j,k,F} A_{j_E}^1 \cdots A_{j_{E_{\mu_1}-1}}^{\mu_1-1} \left[ A_{j_{E_{\mu_1}}}^{\mu_1} - A_{j_{E_{\mu_1}+1}}^{\mu_1+1} \cdots A_{j_{E_\nu}}^\nu \right]\|_{L^2 \to L^2} \lesssim 2^{-\ell_\infty}.
\]

To prove \( 10.13 \) we will apply Theorem \( 10.8 \) with,

\[
S_1 \cdots S_L = P_{j,k,F} A_{j_E}^1 \cdots A_{j_{E_{\mu_1}-1}}^{\mu_1-1} = A_{j_{E \cup \{\mu_1\}}} M_{jF} D_k D_k^* M_{jF}^* A_{j_{E \cup \{\mu_1\}}}^* A_{j_{E \cup \{\mu_1\}}}^{\mu_1-1} \cdots A_{j_{E_{\mu_1}-1}}^{\mu_1-1},
\]

\[
R_1 = A_{j_{E_{\mu_1}}}^\mu, \quad R_2 = A_{j_{E_{\mu_1}+1}}^{\mu_1+1}.
\]
As before, we take \((Z, \hat{d}) = (2^{-\ell} X, \sum d)\). The proof that the term \(S_1 \cdots S_L\) is of the proper form follows just as before. We, therefore, concern ourselves only with showing that \(R_1\) and \(R_2\) have the proper form.

We have,

\[
R_1 f (x) = \psi_1 (x) \int f \left( \gamma_{2^{-j\mu_1}} (x) \right) \psi_{-1} (\gamma_{2^{-j\mu_1}} (x)) \sigma (t) \, dt,
\]

\[
R_2 f (x) = \psi_2 (x) \left[ \int f (x) = \psi_{-1} (x) \int f (\gamma_{2^j} (x)) \psi_{-1} (\gamma_{0} (x)) \sigma (t) \, dt.\right.
\]

Setting \(c_0 = \min_{1 \leq m \leq \eta_{\mu_1}} d^{\mu_1}_m\) and \(\zeta = 2^{-c_0}\), the proof that \(R_1 - R_2\) is of the proper form follows just as in the previous case. Theorem 10.8 applies to show (10.13), thereby establishing (10.12).

We turn, finally, to showing (10.11). We will apply Theorem 10.8 with

\[
S_1 \cdots S_L = P_{j,k,F} A_{jG} = A_{jF} D_k D_k^* M_{jF} A_{jF}^* A_{jG},
\]

\[
R_1 = M_{j \in \nu_1}, \quad R_2 = M_{j E}.
\]

As before, we take \((Z, \hat{d}) = (2^{-\ell} X, \sum d)\). That \(S_1 \cdots S_L\) has the proper form to apply Theorem 10.8 follows just as before. We therefore only concern ourselves with \(R_1\) and \(R_2\).

Consider,

\[
M_{jE} f (x) = \psi_0 (x) \int f (\gamma_{2^{-jE}} (x)) \psi_0 (\gamma_{2^{-jE}} (x)) \sigma (t) \, dt,
\]

with a similar formula for \(M_{j \in \nu_1}\). For \(t \in \mathbb{R}^N\), separate \(t\) into two variables: \(t = (t_1, t_2)\). \(t_1\) will consist of those coordinates \(t_i\) such that \(e^{\mu_i}_1 \neq 0\), and \(t_2\) will denote the rest of the coordinates. Write \(t_1 = (t_1, \ldots, t_{N_1})\). Let \(i \in \mathbb{N}^N\) be given by,

\[
i_{\mu} = \begin{cases} j_{\mu}^E & \text{if } \mu \neq \mu_1, \\
j_{\mu_1} & \text{if } \mu = \mu_1. \end{cases}
\]

Notice,

\[
2^{-j_{\in \nu_1}} (t_1, t_2) = 2^{-i} \left(2^{(k_{\mu_1} - j_{\mu_1})} e^{\mu_1}_1 t_1, \ldots, 2^{(k_{\mu_1} - j_{\mu_1})} e^{\mu_1}_1 t_{N_1}, t_2\right),
\]

\[
2^{-j_E} (t_1, t_2) = 2^{-i} (0, t_2).
\]

Define \(c_0 = \min \left\{e^{\mu_1}_{i_1}, \ldots, e^{\mu_1}_{i_{N_1}}\right\} > 0\), and \(\zeta = 2^{-c_0} (j_{\mu_1} - k_{\mu_1}) = 2^{-c_0} \zeta\). Thus,

\[
2^{-j_{\in \nu_1}} (t_1, t_2) = 2^{-i} \left(2^{(k_{\mu_1} - j_{\mu_1})} (e^{\mu_1}_{i_1} - c_0) \zeta t_1, \ldots, 2^{(k_{\mu_1} - j_{\mu_1})} (e^{\mu_1}_{i_{N_1}} - c_0) \zeta t_{N_1}, t_2\right).
\]

Define,

\[
\tilde{\gamma} ((t_1, t_2, s) x) = \gamma \left(2^{-i} \left(2^{(k_{\mu_1} - j_{\mu_1})} (e^{\mu_1}_{i_1} - c_0) s t_1, \ldots, 2^{(k_{\mu_1} - j_{\mu_1})} (e^{\mu_1}_{i_{N_1}} - c_0) s t_{N_1}, t_2\right), x\right).
\]

We claim that \((Z, \hat{d})\) controls \(\tilde{\gamma}\) at the unit scale. Indeed, we know \((Z, \hat{d})\) controls \(\gamma_{2^{-i} s}\) at the unit scale\(^\text{13}\) and since \(i \geq \ell\) coordinatewise, we have that \((Z, \hat{d})\) controls \(\gamma_{2^{-i} s}\) at the unit scale. Now the result follows easily from the definition of control, since \((k_{\mu_1} - j_{\mu_1}) (e^{\mu_1}_{i_m} - c_0) < 0\) for every \(m\).

Note that,

\[
R_1 f (x) = \psi_0 (x) \int f (\tilde{\gamma}_{i,\zeta} (x)) \psi_0 (\tilde{\gamma}_{i,\zeta} (x)) \sigma (t) \, dt,
\]

\(^{13}\)As discussed before, this follows directly from our assumptions on \(\gamma\).
Thus, $R_1 - R_2$ has the proper form for Theorem 10.8. Theorem 10.8 applies to show,

$$\|S_1 \cdots S_L (R_1 - R_2)\|_{L^2 \rightarrow L^2} \lesssim \zeta^c = 2^{-\epsilon \gamma_{\infty}},$$

which establishes (10.11) and completes the proof of (10.4).

We now make comments on the modifications of the above necessary to deal with the other parts of Theorem 10.1. When considering $D_{j_1} T_{j_2} D_{j_3}$, one takes $\ell = j_1 \wedge j_2 \wedge j_3$ and $\gamma_{\infty} = \max_{1 \leq k, l \leq 3} |j_k - j_l|_{\infty} \approx \text{diam } \{j_1, j_2, j_3\}$. Also, we set $\left(\tilde{Z}, \tilde{d}\right) = (2^{-\ell} X, \sum d)$. One proceeds in essentially the same manner as $B_{j_1} D_{j_2}$. The $D_j$ terms behave just as before. When $T_{j_2}$ appears as a $S_p$ for some $p$, it can be treated just as $M_{j(1,\ldots,\nu)}$ was treated above. When $\gamma_{\infty} = j_2 \wedge j_3$ or $\gamma_{\infty} = j_1 \wedge j_3$, for some $\mu_1$, $T_{j_2}$ must also be used as $R_1 - R_2$ in the above argument. In that case, one uses that $\int \zeta_{j_2} (t) dt\mu_1 = 0$ and setting $R_2 = 0$, one can write $T_{j_2} = R_1 = R_1 - R_2$ in a form that works just as $M_{j(1,\ldots,\nu)} - M_{j(1,\ldots,\nu)\setminus\mu_1}$ did above. See also [Str11] for details on this.

When considering, instead, $D_{j_1} D_{j_2} D_{j_3}^* D_{j_4}^*$ or $D_{j_1}^* D_{j_2}^* D_{j_3} D_{j_4}$, one takes $\ell = j_1 \wedge j_2 \wedge j_3 \wedge j_4$ and $\gamma_{\infty} = \max_{1 \leq k, l \leq 4} |j_k - j_l|_{\infty} \approx \text{diam } \{j_1, j_2, j_3, j_4\}$. Also, one takes $\left(Z, \tilde{d}\right) = (2^{-\ell} X, \sum d)$. With these choices everything proceeds as above with simple modifications. We leave the details to the interested reader.

11 Square functions and the reproducing formula

Using the operators $D_j$ defined in Section 9 we develop, in this section, a Littlewood-Paley square function and a Calderón-type “reproducing formula” which will be essential to our proof of Theorems 5.2 and 5.4.

Recall,

$$\sum_{j \in \mathbb{N}^n} D_j = \psi_{\leq 3}^2.$$

For notational convenience, we define $D_0 = 0$ for $j \in \mathbb{Z}^n \setminus \mathbb{N}^n$. For $M \in \mathbb{N}$, define,

$$U_M = \sum_{j \in \mathbb{N}^n, |j| \leq M} D_j D_{j+1},$$

$$R_M = \sum_{j \in \mathbb{N}^n, |j| > M} D_j D_{j+1};$$

so that $U_M + R_M = \psi_{\leq 3}^2$. The main results of this section are:

**Theorem 11.1** (A Calderón-type “reproducing formula”). Fix $p_0$, $1 < p_0 < \infty$. There exists $M = M (p_0)$, and a bounded map $V_M : L^{p_0} \rightarrow L^{p_0}$ such that,

$$\psi_{-2} U_M V_M = \psi_{-2} = V_M U_M \psi_{-2}.$$

**Remark** 11.2. Strictly speaking, Theorem 11.1 does not give a reproducing formula (one would need $V_M = I$ for it to be a reproducing formula). However, we will use it in the same way that one often uses the Calderón reproducing formula, which is why we have labeled it such.

**Theorem 11.3** (The Littlewood-Paley square function). For every $p$, $1 < p < \infty$,

$$\|\psi_{-2} f\|_{L^p} \lesssim \left\| \left( \sum_{j \in \mathbb{N}^n} |D_j \psi_{-2} f|^2 \right)^{\frac{1}{2}} \right\|_{L^p},$$

(11.1)
The rest of this section is devoted to the proofs of Theorems 11.1 and 11.3. We begin with the proof of (11.2):

**Lemma 11.4.** For every \(1 < p < \infty\), we have:

\[
\left\| \left( \sum_{j \in \mathbb{N}} |D_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \le C_p \|f\|_{L^p}.
\]

(11.3)

The same result holds with \(D_j\) replaced with \(D_j^*\).

**Proof.** As is well known, to prove (11.3), it suffices to prove for every set of \(\nu\) sequences \(\{\epsilon_j^1\}_{j \in \mathbb{N}}, \ldots, \{\epsilon_j^\nu\}_{j \in \mathbb{N}}\), taking values of \(\pm 1\), the operator

\[
\sum_{j_1, \ldots, j_\nu \in \mathbb{N}} \epsilon_{j_1}^1 \epsilon_{j_2}^2 \cdots \epsilon_{j_\nu}^\nu D_{(j_1, \ldots, j_\nu)}
\]

(11.4)

is bounded on \(L^p\), with bound independent of the choice of the sequence. To see why (11.4) is enough, see for example, page 267 of [Ste93] and Chapter 4, Section 5 of [Ste70]. We have,

\[
\sum_{j_1, \ldots, j_\nu \in \mathbb{N}} \epsilon_{j_1}^1 \epsilon_{j_2}^2 \cdots \epsilon_{j_\nu}^\nu D_{(j_1, \ldots, j_\nu)} = \left( \sum_{j_1 \in \mathbb{N}} \epsilon_{j_1}^1 D_{j_1}^1 \right) \cdots \left( \sum_{j_\nu \in \mathbb{N}} \epsilon_{j_\nu}^\nu D_{j_\nu}^\nu \right).
\]

For each \(\mu\),

\[
\sum_{j_\mu \in \mathbb{N}} \epsilon_{j_\mu}^\mu D_{j_\mu}^\mu
\]

is of the form covered by Theorem 8.2 and hence bounded on \(L^p\) (\(1 < p < \infty\)). It is easy to see that the Theorem 8.2 holds uniformly in the choice of the sequence \(\{\epsilon_{j_\mu}^\mu\}\) taking values of \(\pm 1\). The result now follows.

The same proof works with \(D_j\) replaced by \(D_j^*\). \(\Box\)

To prove Theorem 11.1, we first state a preliminary lemma.

**Lemma 11.5.** For \(p_0\) fixed, \(1 < p_0 < \infty\),

\[
\lim_{M \to \infty} \|R_M\|_{L^{p_0} \to L^{p_0}} = 0.
\]

**Proof of Theorem 11.1 given Lemma 11.5** Note that,

\[
U_M = \psi_{j_3} - R_M.
\]

Take \(\psi \in C_0^\infty\) with \(\psi_{-2} \prec \psi \prec \psi_{-3}\). Take \(M = M(p_0)\) so large that \(\|R_M\psi\|_{L^{p_0} \to L^{p_0}} < 1\). Define,

\[
V_M = \sum_{m=0}^{\infty} \psi (R_M\psi)^m,
\]

with convergence in the uniform operator topology \(L^{p_0} \to L^{p_0}\). It is direct to verify that \(V_M\) satisfies the conclusions of Theorem 11.1. \(\Box\)

Lemma 11.5 follows by interpolating the next two lemmas,
Lemma 11.6. For $1 < p < \infty$, $M \geq 1$,
\[
\| R_M \|_{L^p \rightarrow L^p} \leq C_M^p.
\]

Lemma 11.7.
\[
\| R_M \|_{L^2 \rightarrow L^2} \lesssim 2^{-\epsilon M},
\]
for some $\epsilon > 0$.

Proof of Lemma 11.6. Since $R_M = \psi_{4\nu} - U_M$, it suffices to prove the result with $U_M$ in place of $R_M$. Let $q$ be dual to $p$. Fix $f \in L^p$ and $g \in L^q$ with $\|g\|_{L^q} = 1$. Consider,
\[
|\langle g, U_M f \rangle| \leq \sum_{|l| \leq M} \left| \sum_{j \in \mathbb{N}^\nu} \langle D_j^* g, D_{j+l} f \rangle \right|^{1/2} \leq \left( \sum_{j \in \mathbb{N}^\nu} |D_j^* g| \right)^{1/2} \left( \sum_{j \in \mathbb{N}^\nu} |D_j f| \right)^{1/2} \lesssim M^{\nu} \left( \sum_{j \in \mathbb{N}^\nu} |D_j g| \right)^{1/2} \left( \sum_{j \in \mathbb{N}^\nu} |D_j f| \right)^{1/2} \lesssim M^{\nu} \|g\|_{L^q} \|f\|_{L^p} = M^{\nu} \|f\|_{L_p},
\]
where, in the second to last line, we have applied Lemma 11.4 twice. Taking the supremum over all $g$ with $\|g\|_{L^q} = 1$ yields the result. \[\Box\]

Proof of Lemma 11.7. We wish to apply the Cotlar-Stein lemma to,
\[
\sum_{|l| \leq M} D_j D_{j+l}.
\]
Applying Theorem 10.1 we have,
\[
\| D_{j_1} D_{j_1+l_1} D_{j_2} D_{j_2+l_2} \|_{L^2 \rightarrow L^2} \lesssim 2^{-\epsilon \text{diam}\{j_1, j_1+l_1, j_2, j_2+l_2\}}.
\]
The Cotlar-Stein lemma states,
\[
\| R_M \|_{L^2 \rightarrow L^2} \lesssim \sup_{|l_1| > M} \sum_{j_2 \in \mathbb{N}} 2^{-\epsilon \text{diam}\{j_1, j_1+l_1, j_2, j_2+l_2\}/2} \lesssim 2^{-\epsilon M},
\]
completing the proof. \[\Box\]

We have now completed the proof of Theorem 11.1. We end this section with the completion of the proof of Theorem 11.3 by proving (11.3).
Proof of Lemma 12.1. Fix \( p_0, 1 < p_0 < \infty \), and let \( M = M (p_0) \) be as in Theorem 11.1. Let \( q_0 \) be dual to \( p_0 \), so that \( V_M^* : L^{q_0} \to L^{q_0} \). Let \( g \in L^{q_0} \) be such that \( \| g \|_{L^{q_0}} = 1 \). We have for \( f \in L^{q_0} \),

\[
|\langle g, \psi^{-2} f \rangle| = |\langle V_M^* g, U_M \psi^{-2} f \rangle| \\
\leq \sum_{|l| \leq M} \| \sum_{j \in \mathbb{N}^\nu} \langle D_j^* V_M^* g, D_j \psi^{-2} f \rangle \| \\
\leq \sum_{|l| \leq M} \left( \left\| \sum_{j \in \mathbb{N}^\nu} |D_j^* V_M^* g| \right\|^2 \right)^{1/2} L^{q_0} \left( \left\| \sum_{j \in \mathbb{N}^\nu} |D_j \psi^{-2} f|^2 \right\|^2 \right)^{1/2} L^{q_0} \\
\lesssim M^\nu \| V_M^* g \|_{L^{q_0}} \left( \sum_{j \in \mathbb{N}^\nu} |D_j \psi^{-2} f|^2 \right)^{1/2} ;
\]

where in the second to last line we applied Lemma 11.1 and in the last line we used that \( M \) is fixed (since \( p_0 \) is), \( V_M^* \) is bounded on \( L^{q_0} \), and \( \| g \|_{L^{q_0}} = 1 \). Taking the supremum over all \( g \) with \( \| g \|_{L^{q_0}} = 1 \) yields the result.

12 The maximal result (Theorem 5.4)

In this section, we prove Theorem 5.4. The proof proceeds by a bootstrapping argument. In fact, there are at least two, well-known, bootstrapping arguments that can be used to prove results like Theorem 5.4. One can be found in [NSW78], another in Section 4 of [GSW99]. Either of these arguments will suffice for our purposes. We proceed using the methods of [NSW78].

Let us review a few of the reductions covered in Section 9. First, for \( E \subseteq \{1, \ldots, \nu\} \), define,

\[
\mathcal{M}_E f (x) = \sup_{j \in \mathbb{N}^\nu} M_{jE} |f| (x).
\]

Then, to prove Theorem 5.4 it suffices to prove that \( \mathcal{M}_{\{1, \ldots, \nu\}} \) is bounded on \( L^p \) (1 < \( p \leq \infty \)). We proceed by induction on \( \nu \). As discussed in Section 9, the base case (\( \nu = 0 \)) is trivial, and we may assume by our inductive hypothesis that \( \mathcal{M}_E \) is bounded on \( L^p \) for \( E \subseteq \{1, \ldots, \nu\} \). Note that \( \mathcal{M}_{\{1, \ldots, \nu\}} \) is clearly bounded on \( L^\infty \), and so our goal is to show that it is bounded on \( L^p \), 1 < \( p \leq 2 \).

The following lemma was proved in Section 9 (under our inductive hypothesis),

Lemma 12.1. For each \( p, 1 < p < \infty \),

\[
\| \mathcal{M}_{\{1, \ldots, \nu\}} f \|_{L^p} \lesssim \| f \|_{L^p}, \quad \forall f \in L^p,
\]

if and only if,

\[
\left\| \sup_{j \in \mathbb{N}^\nu} |B_j f| \right\|_{L^p} \lesssim \| f \|_{L^p}, \quad \forall f \in L^p.
\]

Remark 12.2. Actually, only the if part of Lemma 12.1 was shown in Section 9. The only if part is immediate, and is also not used in what follows. We, therefore, leave it to the interested reader.

In what follows, \( D_j \) for \( j \in \mathbb{Z}^\nu \setminus \mathbb{N}^\nu \) is defined to be 0. For \( k \in \mathbb{Z}^\nu \) define a new operator acting on sequences of measurable functions \( \{ f_j (x) \}_{j \in \mathbb{N}^\nu} \) by,

\[
B_k \{ f_j \}_{j \in \mathbb{N}^\nu} = \{ B_j D_j + k f_j \}_{j \in \mathbb{N}^\nu}.
\]

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Proposition 12.3. Fix $p_0$, $1 < p_0 < \infty$. If there is $\epsilon > 0$ such that

$$\|B_k\|_{L^{p_0}(\ell^2(N^\nu)) \rightarrow L^{p_0}(\ell^2(N^\nu))} \lesssim 2^{-\epsilon|k|},$$

then $\mathcal{M}_{\{1,\ldots,p\}}$ is bounded on $L^{p_0}$.

Proof. In light of Lemma 12.1 and because $\|g\|_{L^{p_0}} \lesssim \|f\|_{L^{p_0}}$ and

$$\|B_j f\|_{L^{p_0}} \leq \left( \sum_{j \in \mathbb{N}^\nu} |B_j f|^2 \right)^{1/2},$$

it suffices to show

$$\left( \sum_{j \in \mathbb{N}^\nu} |B_j f|^2 \right)^{1/2} \lesssim \|f\|_{L^{p_0}}.$$

Fix $M = M(p_0)$ as in Theorem 11.1. Note, $B_j = B_j \psi_{-2} = B_j \psi_{-2} U_M V_M = B_j U_M V_M$. Let $g = V_M f$. Note that $\|g\|_{L^{p_0}} \lesssim \|f\|_{L^{p_0}}$. Thus, it suffices to show,

$$\left( \sum_{j \in \mathbb{N}^\nu} |B_j U_M g|^2 \right)^{1/2} \lesssim \|g\|_{L^{p_0}}.$$

Consider, using the triangle inequality,

$$\left( \sum_{j \in \mathbb{N}^\nu} |B_j U_M g|^2 \right)^{1/2} = \left( \sum_{j \in \mathbb{N}^\nu} \sum_{k \in \mathbb{Z}^\nu, ||k|| \leq M} |B_j D_{j+k} D_{j+k+i} g|^2 \right)^{1/2} \lesssim \sum_{k \in \mathbb{Z}^\nu, ||k|| \leq M} \left( \sum_{j \in \mathbb{N}^\nu} |B_j D_{j+k} D_{j+k+i} g|^2 \right)^{1/2} \lesssim \sum_{k \in \mathbb{Z}^\nu, ||k|| \leq M} \|B_k \|_{L^{p_0}(\ell^2(N^\nu))} \|D_j g\|_{L^{p_0}(\ell^2(N^\nu))} \lesssim \|g\|_{L^{p_0}}.$$

where, in the last line, we have applied Theorem 11.3. This completes the proof of the proposition. □

Define,

$$\mathcal{P} = \left\{ p \in (1,2) : \exists \epsilon > 0, \|B_k\|_{L^{p}(\ell^2(N^\nu)) \rightarrow L^{p}(\ell^2(N^\nu))} \lesssim 2^{-\epsilon|k|} \right\}.$$
In light of Proposition \ref{prop:M12-v}, the $L^p$ boundedness of $M_{\{1,\ldots,\nu\}}$ will follow directly from the following proposition.

**Proposition 12.4.** $\mathcal{P} = (1, 2]$.

Proposition \ref{prop:M12-v} in turn, follows directly from the next lemma.

**Lemma 12.5.**

- $2 \in \mathcal{P}$,

- If $q \in \mathcal{P}$, then $(\frac{2q}{q+1}, 2] \subseteq \mathcal{P}$.

It is easy to see that any subset of $(1, 2]$ satisfying the conclusions of Lemma \ref{lem:2} must equal $(1, 2]$. The rest of this section is devoted to the proof of Lemma \ref{lem:2} which then completes the proof of Theorem \ref{thm:main}.

**Proof of Lemma \ref{lem:2}** That $2 \in \mathcal{P}$ follows directly from Theorem \ref{thm:M12-v}. In fact, if $\epsilon_2 > 0$ is as in Theorem \ref{thm:M12-v} we have,

\[
\|B_k\|_{L^2(\ell^2(\nu)) \to L^2(\ell^2(\nu))} \lesssim 2^{-\epsilon_2|k|};
\]

merely by interchanging the norms. Moreover, using that,

\[
\|B_j D_{j+k}\|_{L^1 \to L^1} \lesssim 1,
\]

we have,

\[
\|B_k\|_{L^1(\ell^1(\nu)) \to L^1(\ell^1(\nu))} \lesssim 1; \tag{12.2}
\]

also by interchanging the norms. Interpolating (12.1) and (12.2) shows for $1 < p \leq 2$,

\[
\|B_k\|_{L^p(\ell^p(\nu)) \to L^p(\ell^p(\nu))} \lesssim 2^{-\epsilon_p|k|}; \tag{12.3}
\]

where $\epsilon_p = \left(2 - \frac{2}{p}\right) \epsilon_2 > 0$.

Now suppose $q \in \mathcal{P}$. By Proposition \ref{prop:M12-v} $M_{\{1,\ldots,\nu\}}$ is bounded on $L^q$. We claim,

\[
\|B_k\|_{L^q(\ell^\infty(\nu)) \to L^q(\ell^\infty(\nu))} \lesssim 1. \tag{12.4}
\]

Before we verify (12.4), recall the maximal functions $M^\mu$ defined in Section \ref{sec:9}. $M^\mu$ is bounded on $L^p$ ($1 < p \leq \infty$), and by (9.6) and the definition of $A_{jE}$, we have

\[|A_{jE} f(x)| \lesssim \left[\prod_{\mu \in E} M^\mu\right] f(x),\]

where the product is taken in order of increasing $\mu$. A similar result holds for $D_j$ with $E$ replaced by $\{1, \ldots, \nu\}$.

We now turn to verifying (12.4).

\[
\left\|\sup_{j \in \mathbb{N}^\nu} |B_j D_{j+k} f_j|\right\|_{L^q} \leq \sum_{E \subseteq \{1, \ldots, \nu\}} \left\|\sup_{j \in \mathbb{N}^\nu} |A_{jE} M_{jE} D_{j+k} f_j|\right\|_{L^q}
\]

\[\lesssim \sum_{E \subseteq \{1, \ldots, \nu\}} \left[\prod_{\mu \in E} M^\mu\right] M_E \left[\prod_{\mu = 1}^\nu M^\mu\right] \sup_{j \in \mathbb{N}^\nu} |f_j| L^q \lesssim \left\|\sup_{j \in \mathbb{N}^\nu} |f_j|\right\|_{L^q}.\]

In the last line, we have used our inductive hypothesis when $E \neq \{1, \ldots, \nu\}$ and we used that $M_{\{1,\ldots,\nu\}}$ is bounded on $L^q$ when $E = \{1, \ldots, \nu\}$. This completes the verification of (12.4).

Interpolating (12.4) with (12.3) as $p \to 1$ proves that $(\frac{2q}{q+1}, 2] \subseteq \mathcal{P}$. Here, we have implicitly used the fact that (by interpolation) if $r \in \mathcal{P}$, then $[r, 2] \subseteq \mathcal{P}$ (since $2 \in \mathcal{P}$).
13 Proof of Theorem 5.2

This section is devoted to proving Theorem 5.2 \( T : L^p \rightarrow L^p, 1 < p < \infty \). Since the class of operators covered in Theorem 5.2 is closed under adjoints (see Section 12.3 of \textit{Str11b}), it suffices to prove the result for \( 1 < p \leq 2 \).

We decompose \( T = \sum_{j \in \mathbb{Z}^\nu} T_j \) as in Section 9. In what follows, \( T_j \) and \( D_j \) for \( j \in \mathbb{Z}^\nu \setminus \mathbb{N}^\nu \) are defined to be 0. For \( k_1, k_2 \in \mathbb{Z}^\nu \), define a new operator, acting on sequences of measurable functions \( \{ f_j(x) \}_{j \in \mathbb{N}^\nu} \) by

\[
T_{k_1, k_2} \{ f_j \}_{j \in \mathbb{N}^\nu} = \{ D_j T_j + k_1 D_j + k_2 f_j \}_{j \in \mathbb{N}^\nu}.
\]

Theorem 5.2 follows immediately from a combination of the following two propositions.

**Proposition 13.1.** Fix \( p_0, 1 < p_0 < \infty \). If there exists \( \epsilon > 0 \) such that

\[
\| T_{k_1, k_2} \|_{L^{p_0}(\ell^2(\mathbb{N}^\nu)) \rightarrow L^{p_0}(\ell^2(\mathbb{N}^\nu))} \lesssim 2^{-\epsilon(|k_1| + |k_2|)},
\]

then \( T \) is bounded on \( L^{p_0} \).

**Proposition 13.2.** For each \( p, 1 < p \leq 2 \), there is an \( \epsilon = \epsilon(p) > 0 \) such that,

\[
\| T_{k_1, k_2} \|_{L^p(\ell^2(\mathbb{N}^\nu)) \rightarrow L^p(\ell^2(\mathbb{N}^\nu))} \lesssim 2^{-\epsilon(|k_1| + |k_2|)}.
\]

**Proof of Proposition 13.1.** Fix \( p_0, 1 < p_0 < \infty \). Take \( M = M(p_0) \) as in Theorem 11.1. We have, \( T = T_{\psi} - 2 = T_{\psi} - 2 U_M V_M = T U_M V_M \). Since \( V_M \) is bounded on \( L^{p_0} \), it suffices to show \( T U_M \) is bounded on \( L^{p_0} \).

Consider, using Theorem 11.3

\[
\| T U_M f \|_{L^{p_0}} = \| \psi_{-2} T U_M f \|_{L^{p_0}} \approx \left\| \left( \sum_{j \in \mathbb{N}^\nu} |D_j \psi_{-2} T U_M f|^2 \right)^{1/2} \right\|_{L^{p_0}} = \left\| \left( \sum_{j \in \mathbb{N}^\nu} |D_j T U_M f|^2 \right)^{1/2} \right\|_{L^{p_0}}.
\]

Thus, to complete the proof, it suffices to show,

\[
\left\| \left( \sum_{j \in \mathbb{N}^\nu} |D_j T U_M f|^2 \right)^{1/2} \right\|_{L^{p_0}} \lesssim \| f \|_{L^{p_0}}.
\]
where, in the last line, we have applied Theorem 11.3. This completes the proof.

\[ \left\| \sum_{j \in \mathbb{N}} |D_j T M f|^2 \right\|_{L^p_{0}}^{1/2} = \left\| \sum_{j \in \mathbb{N}} \left( \sum_{j_1, j_2 \in \mathbb{N}} D_j T_{j_1} D_{j_2} D_{j_1 + j_2 + 1} f \right)^2 \right\|_{L^p_{0}}^{1/2} \]

\[ = \left\| \sum_{j \in \mathbb{N}} \left( \sum_{j_1, j_2 \in \mathbb{N}} D_j T_{j + k_1} D_{j + k_2} D_{j + k_1 + k_2 + 1} f \right)^2 \right\|_{L^p_{0}}^{1/2} \]

\[ \leq \sum_{k_1, k_2 \in \mathbb{Z}^2} \left\| \sum_{j \in \mathbb{N}} |D_j T_{j + k_1} D_{j + k_2} D_{j + k_1 + k_2 + 1} f|^2 \right\|_{L^p_{0}}^{1/2} \]

\[ = \sum_{k_1, k_2 \in \mathbb{Z}^2} \left\| T_{k_1, k_2} \{ D_j f \}_{j \in \mathbb{N}} \right\|_{L^p_{0}(\ell^2(\mathbb{N}^2))} \]

\[ \lesssim \sum_{k_1, k_2 \in \mathbb{Z}^2} 2^{-\epsilon(|k_1| + |k_2|)} \left\| (D_j f)_{j \in \mathbb{N}} \right\|_{L^p_{0}(\ell^2(\mathbb{N}^2))} \]

\[ \lesssim \left\| \sum_{j \in \mathbb{N}} |D_j f|^2 \right\|_{L^p_{0}}^{1/2} \]

\[ \lesssim \|f\|_{L^p_{0}}, \]

Proof of Proposition 13.2. We first prove the result for \( p = 2 \). In this case, the result follows immediately from Theorem 10.1. Indeed if \( \epsilon_2 > 0 \) is as in Theorem 10.1, we have,

\[ \|T_{k_1, k_2}\|_{L^2(\ell^2(\mathbb{N}^2)) \to L^2(\ell^2(\mathbb{N}^2))} \lesssim 2^{-\frac{2}{3}(|k_1| + |k_2|)}; \]

merely by interchanging the norms. In addition, since,

\[ \|D_j T_{j + k_1} D_{j + k_2}\|_{L^1 \to L^1} \lesssim 1, \quad (13.1) \]

we have,

\[ \|T_{k_1, k_2}\|_{L^1(\ell^1(\mathbb{N}^2)) \to L^1(\ell^1(\mathbb{N}^2))} \lesssim 1; \quad (13.2) \]

also by interchanging the norms. Interpolating (13.1) and (13.2) shows that for every \( p, 1 < p \leq 2 \),

\[ \|T_{k_1, k_2}\|_{L^p(\ell^p(\mathbb{N}^2)) \to L^p(\ell^p(\mathbb{N}^2))} \lesssim 2^{-\epsilon_p(|k_1| + |k_2|)}; \quad (13.3) \]

where \( \epsilon_p = \left( 1 - \frac{1}{p} \right) \epsilon_2 > 0 \).

We claim, for every \( p, 1 < p < \infty \),

\[ \|T_{k_1, k_2}\|_{L^p(\ell^\infty(\mathbb{N}^2)) \to L^p(\ell^\infty(\mathbb{N}^2))} \lesssim 1. \quad (13.4) \]

We use the maximal operators \( M^\nu \) defined in Section 9 along with the maximal operator \( M \) from

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In the previous sections, we exhibited the proofs of Theorems 5.2 and 5.4 in the special case $\mu_0 = \nu$. That is, in the case when $A_{\mu_0} = [0, 1]^\nu$. In this section, we describe the modifications necessary to prove the result for general $\mu$. At the end of the section we make some remarks about even more general sets $A \subseteq [0, 1]^\nu$ for which our methods apply.

Fix $\mu_0, 1 \leq \mu_0 \leq \nu$. Recall,

$$A_{\mu_0} = \{ \delta = (\delta_1, \ldots, \delta_\nu) \in [0, 1]^\nu : \delta_0 \leq \delta_1 \leq \cdots \leq \delta_\nu \}.$$  

Our decomposition of $T$ now takes the form,

$$T = \sum_{j \in -\log_2 A} T_j,$$

where

$$-\log_2 A = \{ j \in \mathbb{N}^\nu : 2^{-j} \in A_{\mu_0} \} = \{ j \in \mathbb{N}^\nu : j_0 \geq j_{\mu_0+1} \geq \cdots \geq j_\nu \};$$

see Section 2 for more details.

The proof in this case remains almost exactly the same, provided we make a few different choices when defining the auxiliary operators in Section 9. In the case when $\mu_0 = \nu$, we defined the vector fields with single parameter formal degrees $(X^\mu, d^\mu)$ so that,

$$(\delta_\mu X^\mu, d^\mu) = \left( \hat{\delta} X, \sum d \right),$$  

(14.1)

where $\hat{\delta} \in [0, 1]^\nu$ was $\delta_0$ in the $\mu$ coordinate and 0 in every other coordinate. In the case when $\mu_0 < \mu$, this choice no longer works. Instead, we choose $(X^\mu, d^\mu)$ so that (14.1) holds where $\hat{\delta}$ is defined to be $\delta_\mu$ in the $\mu^\prime$ coordinate, for every $\mu^\prime \geq \mu$, and defined to be 0 in the rest of the coordinates. The operators $A_{\mu}^\mu$ and $D_{\mu}^\mu$ are defined with this choice of $(X^\mu, d^\mu)$.

For $E \subseteq \{1, \ldots, \nu\}, j_E \in \mathbb{N}_\infty^\nu$ must be defined differently, so that $2^{-j_E} \in A$. For $1 \leq \mu \leq \mu_0$, we define,

$$j_E^\mu = \begin{cases} j_\mu & \text{if } \mu \in E, \\ \infty & \text{otherwise}. \end{cases}$$

For $\mu > \mu_0$, we recursively define,

$$j_E^\mu = \begin{cases} j_\mu & \text{if } \mu \in E, \\ \min \left\{ \infty, j_E^{\mu-1} \right\} & \text{otherwise}. \end{cases}$$

One defines $A_{j_E}, M_{j_E}$ in the same manner as before, but with this choice of $j_E$.

Now the proof goes through just as before. Whenever one uses $B_j$ or $T_j$ one must restrict attention to $j \in -\log_2 A$. However, when one considers $D_j$, one allows $j$ to range over $j \in \mathbb{N}_\infty^\nu$. 

In the last line, we used the $L^p$ boundedness of the various maximal functions. (13.3) follows. Interpolating (13.3) and (13.4) yields the result. 

**Theorem 5.4**

We have,

$$\left\| \sup_j \left| T_{k_1, k_2} \{ f_j \}_{j \in \mathbb{N}^\nu} \right| \right\|_{L^p} = \left\| \sup_j \left| D_j T_{j+k_1} D_{j+k_2} f_j \right| \right\|_{L^p} \lesssim \left\| \prod_{\mu=1}^{\nu} \mathcal{M}^{\mu} \right\| \mathcal{M} \left[ \prod_{\mu=1}^{\nu} \mathcal{M}^{\mu} \right] \left\| \sup_j |f_j| \right\|_{L^p} \lesssim \left\| \sup_j |f_j| \right\|_{L^p}.$$
In fact, the above methods work for more general \( \mathcal{A} \subseteq [0,1]^\nu \). For instance, just by changing \( e \) and \( a \), studying the operators associated to

\[
\mathcal{A} = \left\{ \delta = (\delta_1, \ldots, \delta_\nu) \in [0,1]^\nu : \delta_{\mu_0}^b \leq C_{\mu_0+1} \delta_{\mu_0+1}^b \leq \cdots \leq C_\nu \delta_{\nu}^b \right\},
\]

where \( b_\nu \) and \( C_\nu \) are positive numbers, is equivalent to studying the operators associated to \( \mathcal{A}_{\mu_0} \). See [Str11b] for the definition of the class of kernels \( \mathcal{K} \) for more general sets \( \mathcal{A} \).

There were two main reasons that our methods applied to \( \mathcal{A}_{\mu_0} \).

1. There was a natural choice of \((X^\mu, d^\mu)\) for each \( \mu \).
2. Given \( \delta_\mathcal{E} \) as above (for \( \delta \in \mathcal{A} \)), there was a natural ("minimal") choice \( \hat{\delta} \in \mathcal{A} \) with \( \hat{\delta}_{\mu} = \delta_{\mu}, \forall \mu \in \mathcal{E} \).

There are, of course, many subsets \( \mathcal{A} \subseteq [0,1]^\nu \) where no such natural choices can be made. There are more examples (which do satisfy the above in an appropriate way) which can be covered by our methods. However, we know of no simple general condition unifying these examples. Moreover, for all the applications we have in mind, \( \mathcal{A}_{\mu_0} \) will suffice. We, therefore, say no more on this issue, here.

15 \textbf{Singular integrals not of Radon transform type}

The main point of Section 8 was that a single-parameter special case of the operators studied in this paper, fell under the general Calderón-Zygmund singular integral framework. The point of this section is to make a few remarks of the multi-parameter special case of our main theorems which is analogous to this single-parameter special case. The operators studied in this section can be considered as a prototype for a multi-parameter analog of parts of the Calderón-Zygmund theory.

Remark 15.1. Often, when one hears of \textit{multi-parameter singular integrals}, it is the product theory of singular integrals to which is being referred. See, e.g., [Fed81, NS04]. The operators discussed in this section are not necessarily of product type.

Suppose we are given \( \nu \) families of \( \mathcal{C}^\infty \) vector fields with single-parameter formal degrees, \((X^\mu, d^\mu) = (X^\mu_1, d^\mu_1), \ldots, (X^\mu_\nu, d^\mu_\nu)\). \( 1 \leq \mu \leq \nu \). We suppose that each \((X^\mu, d^\mu)\) satisfies \( \mathcal{D}(K_0, [0,1]) \).

Let \((X_1, d_1), \ldots, (X_\nu, d_\nu)\) be the list of vector fields consisting of \( (X_j^\mu, \hat{d}_j^\mu) \) for every \( \mu \) and \( j \), where \( \hat{d}_j^\mu \in [0, \infty)^\nu \) is \( d_j^\mu \) in the \( \mu \) coordinate and 0 in every other coordinate.

Taking \( \mu_0 = \nu \) (i.e., \( \mathcal{A} = [0,1]^\nu \)), we suppose \((X_1, d_1), \ldots, (X_\nu, d_\nu)\) generates a finite list \((X_1, d_1), \ldots, (X_\nu, d_\nu)\). We take \( N = q \) and define \( \nu \)-parameter dilations on \( \mathbb{R}^N \) by,

\[
\delta \left( t_1, \ldots, t_\nu \right) = \left( \delta^{d_1 t_1}, \ldots, \delta^{d_\nu t_\nu} \right),
\]

for \( \delta \in [0, \infty)^\nu \). Define,

\[
\gamma_{(t_1, \ldots, t_\nu)}(x) = e^{d_1 X_1 + \cdots + d_\nu X_\nu} x.
\]

We consider operators, \( T \), of the form covered in Theorem 5.2 where \( K \in \mathcal{K}(q, d, a, \nu, \nu) \) for some small \( a > 0 \). It follows from the remarks in Section 17.1 of [Str11b] that all of the assumptions of Theorem 5.2 are satisfied with the above choices. Hence, \( T \) is bounded on \( L^p \), \( 1 < p < \infty \).

For \( K \in \mathcal{K}(q, d, a, \nu, \nu) \) decompose \( K \),

\[
K = \sum_{j \in \mathbb{N}^\nu} \zeta_j^{(2j)},
\]

where \( \zeta_j \) is as in Definition 2.1. Corresponding to this decomposition of \( K \), one obtains a decomposition of \( T, T = \sum_{j \in \mathbb{N}^\nu} T_j \).

Let \( T_j(x, y) \) denote the Schwartz kernel of \( T_j \). It follows directly from Proposition 4.22 of [Str11a] that
• $T_j(x,y)$ is supported on $y \in B_{(X,d)}(x,2^{-j})$,

• $|T_j(x,y)| \lesssim \text{Vol} \left( B_{(X,d)}(x,2^{-j}) \right)^{-1}$.

In the one parameter situation, this is just the fact that a Calderón-Zygmund singular integral can be decomposed into dyadic scales in the usual way. Thus, the above is a prototype for a multi-parameter generalization of the usual dyadic decomposition of a single-parameter Calderón-Zygmund singular integral operator.

Remark 15.2. In light of the above, $T_j^* T_j$ is no “smoother” than $T_j$. The reader used to the single parameter theory might then suspect that a $T^* T$ type iteration argument will not be helpful in our studies. However, this is not the case. Indeed, a $T^* T$ type iteration argument was essential to our proof (in Section 11.1). The idea is that when $j_1, j_2 \in \mathbb{N}'$ and $j_1 \wedge j_2 \neq j_1, j_2$, $(T_{j_1}^* T_{j_2})^k T_{j_1} T_{j_2}$ is smoother than $T_{j_1}^* T_{j_2}$.

We now turn to maximal functions. With all the same choices as above, we define $\mathcal{M}$ as in Theorem 5.4. With $\psi_1, \psi_2$ as in Theorem 5.4, define a new maximal operator by,

$$\tilde{\mathcal{M}} f(x) = \sup_{|\delta| \leq a'} \psi_1(x) \frac{1}{\text{Vol} \left( B_{(X,d)}(x,\delta) \right)} \int_{B_{(X,d)}(x,\delta)} |f(y)\psi_2(y)| \ dy.$$  

It follows directly from Proposition 4.22 of [Str11a] that $\tilde{\mathcal{M}} f(x) \lesssim \mathcal{M} f(x)$, provided $a' > 0$ is sufficiently small. Hence, $\tilde{\mathcal{M}}$ is bounded on $L^p$, $1 < p \leq \infty$. This generalizes the maximal results of [Str11a].

Reduction to Theorem 5.4 is not the only way to prove the $L^p$ boundedness of $\tilde{\mathcal{M}}$. Indeed, for each $\mu$, define the maximal operator,

$$\tilde{\mathcal{M}}_\mu f(x) = \sup_{0 < \delta \leq a''} \psi_0(x) \frac{1}{\text{Vol} \left( B_{(X',d')}((x,\delta_\mu) \right)} \int_{B_{(X',d')}((x,\delta_\mu)} |f(y)\psi_0(y)| \ dy.$$  

It is shown in Section 6.2 of [Str11a] that $\tilde{\mathcal{M}}_\mu$ is bounded on $L^p$ ($1 < p \leq \infty$) for each $\mu$. This proceeds in a similar manner to the methods in Section 8.2 by reduction to the classical Calderón-Zygmund theory.

It can be shown that,

$$\tilde{\mathcal{M}} f(x) = \left( \tilde{\mathcal{M}}_1 \cdots \tilde{\mathcal{M}}_{\nu} \right)^M f(x),$$  

for some large $M$ (provided $a'$ is sufficiently smaller than $a''$); and the $L^p$ boundedness of $\tilde{\mathcal{M}}$ follows. The proof of [Str11a] is somewhat lengthy and technical, and does not seem to yield Theorem 5.4 in the general case. We, therefore, say no more about this here.

In this situation, we can develop a Littlewood-Paley square function of an appropriate type. While the operators $D_j$ from Section 9 were sufficient to create a Littlewood-Paley square function to prove the $L^p$ boundedness of $T$, they are not of the same type as $T_j$—and therefore take us out of the class of operators we are discussing.

Instead, one uses that the distribution $\delta_0 \in \mathcal{K}(q,d,a,\nu,\nu)$ (Proposition 16.3 of [Str11b]). Write,

$$\delta_0 = \sum_{j \in \mathbb{N}''} \zeta_j^{(2')},$$  

where $\zeta_j$ is as in Definition 2.1. Define,

$$\tilde{D}_j f(x) = \psi_{-2}(x) \int f(\gamma_t(x)) \psi_{-2}(\gamma_t(x)) \zeta_j^{(2')}(t) \ dt.$$  

Thus, $\psi_{-2}^2 = \sum_{j \in \mathbb{N}''} \tilde{D}_j$. One can recreate the theory in Section 11 with $D_j$ replaced by $\tilde{D}_j$, so long as one uses Theorem 5.2 instead of Theorem 8.2 throughout. We therefore obtain a Calderón-type “reproducing formula” and a Littlewood-Paley square function in terms of $\tilde{D}_j$.

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16 Some comments on maximal operators

There are a number of maximal operators in the literature which are related to the one discussed in Theorem 5.3. The ones most closely related are those discussed in [Chr92], where certain strong maximal functions on nilpotent Lie groups are discussed. Of course, our methods also apply to convolution operators on nilpotent groups. See Section 17.2 of [Str11b]. Our results can be used to study some of the maximal operators which were covered in [Chr92], and we discuss this below. At the end of this section, we discuss the fact that not all of the maximal operators from [Chr92] are covered by our results. Nevertheless, these maximal operators can be covered by the methods of this paper, and this will be taken up (and generalized) in [SS11b].

In what follows, we describe the connection between the results in this paper and the results in [Chr92] in the special case of the three dimensional Heisenberg group, \( \mathbb{H}^1 \). All of the comments that follow work more generally for, say, stratified nilpotent Lie groups, but we leave those details to the interested reader. As a manifold, \( \mathbb{H}^1 = \mathbb{C} \times \mathbb{R} \), and we give it coordinates \((z, t) = (x, y, t)\). A basis for the left invariant vector fields on \( \mathbb{H}^1 \) is \( X = \partial_x - 2y\partial_y, Y = \partial_y + 2x\partial_t, T = \partial_t \).

In [Chr92], the following strong maximal function is considered,

\[
\widetilde{M} f(\xi) = \sup_{\delta_1, \delta_2, \delta_3 > 0} \int_{(x, y, t) \leq 1} \left| f(e^{\delta_1 x + \delta_2 y + t \delta_3 T} \xi) \right| \, dx \, dy \, dt.
\]

It is shown that \( \widetilde{M} \) is bounded on \( L^p \), \( 1 < p \leq \infty \). We claim that this result follows from Theorem 5.4. To do this, we show,

\[
\widetilde{M}_N f(\xi) = \sup_{N \geq \delta_1, \delta_2, \delta_3 > 0} \int_{(x, y, t) \leq 1} \left| f(e^{\delta_1 x + \delta_2 y + t \delta_3 T} \xi) \right| \, dx \, dy \, dt,
\]

is bounded on \( L^p \) \((1 < p < \infty)\) with bound independent of \( N \).

Indeed, let \( \psi \geq 0 \) be a \( C^\infty_0 \) function which equals 1 on a neighborhood of 0. Define

\[
\mathcal{M} f(\xi) = \sup_{1 \geq \delta_1, \delta_2, \delta_3 > 0} \psi(\xi) \int_{(x, y, t) \leq a} \left| f(e^{\delta_1 x + \delta_2 y + t \delta_3 T} \xi) \right| \psi(e^{\delta_1 x + \delta_2 y + t \delta_3 T} \xi) \, dx \, dy \, dt,
\]

where \( a > 0 \) is some small number. It is easy to verify that Theorem 5.3 applies (see Section 17.2 of [Str11b]). Thus \( \mathcal{M} \) is bounded on \( L^p \). Note, for \( f \) with small support near 0, we trivially have,

\[
\widetilde{M}_a f(\xi) \lesssim \mathcal{M} f(\xi),
\]

for every \( \xi \).

Now consider the one-parameter dilations on \( \mathbb{H}^1 \) given by,

\[
r(x, y, t) = (rx, ry, r^2 t),
\]

(16.1)

for \( r \in (0, \infty) \). Define, for \( 1 < p < \infty \),

\[
f^{(r)}(\xi) = r^{4/p} f(r\xi),
\]

so that \( \left\| f^{(r)} \right\|_{L^p} = \left\| f \right\|_{L^p} \).

It is easy to see that,

\[
\left( \mathcal{M}_{N/r} f^{(r)} \right)^{(1/r)} \lesssim \mathcal{M} f.
\]

Fix \( f \in L^p \) with compact support. Taking \( r \) so large \( N/r \leq a \), and \( f^{(r)} \) has small support, we see,

\[
\left\| \mathcal{M}_N f \right\|_{L^p} = \left\| \mathcal{M}_{N/r} f^{(r)} \right\|_{L^p} \lesssim \left\| f^{(r)} \right\|_{L^p} \lesssim \left\| f \right\|_{L^p}.
\]

Thus, we have,

\[
\left\| \mathcal{M}_N f \right\|_{L^p} \lesssim \left\| f \right\|_{L^p}.
\]
for every $f$ with compact support. A limiting argument completes the proof.

There is another approach which can be used to prove the $L^p$ boundedness of $\tilde{M}$. Namely, one could simply recreate the entire proof in this paper, without using cutoff functions, and instead of restricting to $\delta \in [0, 1]^\nu$, one allows $\delta \in [0, \infty)^\nu$. It is easy to see that, in this special case, all of our methods go through. This is due to the fact that one has global one-parameter dilations, \([16.1]\), on $\mathbb{H}^1$ which respect each aspect of our proof.

The proof method for the $L^2$ boundedness of $\tilde{M}$ in [Chr92] is closely related to the proof in this paper. One main difference is that (for certain maximal operators more general than $\tilde{M}$), [Chr92] uses transference methods to lift the problem to a higher dimensional maximal function. This allows [Chr92] to deal with certain maximal functions on nilpotent groups which are not directly applicable by our methods.

It turns out that all of the maximal operators covered in [Chr92] can be covered by our methods, with some modifications. This will be discussed in [SS11b], where (among other things) the results of [Chr92] will be generalized.

To understand where Theorem 5.4 falls short, consider the function $\gamma : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by $\gamma_{s,t}(x) = x - st$. It is easy to see, using the methods of Section 17.5 of [Str11b] that there is a product kernel $K(s,t) \in K(2, ((1,0),(0,1)), a, 2, 2)$ supported on $B^2(a)$ (with $a$ as small as we like) such that the corresponding singular Radon transform (as in Theorem 5.1) is not bounded on $L^2$. This fact was first noted in [NW77].

However, if $\psi \in C_0^\infty$, $\psi \geq 0$, is supported sufficiently close to 0, the maximal function,

$$Mf(x) = \sup_{0<\delta_1, \delta_2 \leq a} \psi(x) \int |f(\gamma_{\delta_1 s, \delta_2 t}(x))| \, ds \, dt,$$

is bounded on $L^p(\mathbb{R})$ ($1 < p \leq \infty$), for $a > 0$ sufficiently small. Thus, in an ideal world, Theorem 5.4 would be generalized to apply to this choice of $\gamma$ (and other, more complicated like it). However, since we used the same class of $\gamma$ for Theorem 5.1 and Theorem 5.4, and Theorem 5.1 fails for this choice of $\gamma$, our methods need to be modified to attack this sort of example.

In fact, it is possible to modify our methods in a natural way to deal with cases the same type as this choice of $\gamma$. This will be discussed in detail in [SS11b]. The reason we have not done so here, is that in order to include $\gamma$ in the class of functions we study, we will need to strengthen other aspects of our assumptions in a few technical ways. Thus, the maximal theorem proven in [SS11b] will not be strictly stronger than the one in this paper. This is an issue, since the proof of Theorem 5.2 used Theorem 5.4. Thus, we would end up weakening Theorem 5.2 if we attempted to modify Theorem 5.4. See Remark 5.6 for further details on the sort of maximal results we will prove in [SS11b].

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\[\text{Because of the form of } \gamma, \text{ the operator } M \text{ is actually equal to a one-parameter maximal operator, which is covered by Theorem 5.4. There are other choices of } \gamma \text{ of the same general type where this is not the case, yet the maximal operator is still bounded.}\]
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