GEOMETRIC QUANTIZATION AND FAMILIES OF INNER PRODUCTS

PETER HOCHS AND VARGHESE MATHAI

Abstract. We formulate a quantization commutes with reduction principle in the setting where the Lie group \( G \), the symplectic manifold it acts on, and the orbit space of the action may all be noncompact. It is assumed that the action is proper, and the zero set of a deformation vector field, associated to the momentum map and an equivariant family of inner products on the Lie algebra \( \mathfrak{g} \) of \( G \), is \( G \)-cocompact. The central result establishes an asymptotic version of this quantization commutes with reduction principle. Using an equivariant family of inner products on \( \mathfrak{g} \) instead of a single one makes it possible to handle both noncompact groups and manifolds, by extending Tian and Zhang’s Witten deformation approach to the noncompact case.

Contents

1. Introduction 2
  1.1. Background 2
  1.2. Noncompact groups and manifolds 3
  1.3. Summary of results and strategy of proof 4
  1.4. Outline of this paper 6
  1.5. Acknowledgements 6
  1.6. Notation and conventions 7

2. Families of metrics on \( \mathfrak{g}^* \)
  2.1. Norms of momentum maps 7
  2.2. Two vector fields 8
  2.3. Induced vector fields on reduced spaces 10
  2.4. Critical points 10
  2.5. Rescaling the metric 12

3. Assumptions and results 12
  3.1. Assumptions 13
  3.2. Invariant quantization; the main results 13
  3.3. Special cases 15

4. A \( G \)-invariant index
  4.1. Sobolev spaces 17
  4.2. Properties of the spaces \( W_f^k(E)^G \) 18
  4.3. Free actions 19
  4.4. The Fredholm property 21
  4.5. The \( G \)-invariant index 22

2010 Mathematics Subject Classification. Primary 53D50, Secondary 81S10, 53C27, 53D20, 58J20.

Key words and phrases. Geometric quantization, locally compact groups, momentum map, Hamiltonian manifold, Hochs-Landsman conjecture, Guillemin-Sternberg conjecture, Dirac operator, Witten deformation, analytic localization, family of inner products.
1. Introduction

1.1. Background. Geometric quantization and the quantization commutes with reduction principle have been studied intensively for decades. Geometric quantization has its origins in physics, where it is a method to construct the quantum mechanical description of a physical system from its classical mechanical description. The quantization commutes with reduction principle states that geometric quantization is compatible with the ways symmetry works in classical and quantum mechanics.

The mathematical language of classical mechanics is symplectic geometry (or more generally, Poisson geometry). Symmetry in quantum mechanics leads to a unitary representation of the symmetry group involved. The quantization commutes with reduction principle has revealed deep connections between symplectic geometry and the theory of unitary representations, with various applications to representation theory and physics.

In their 1982 paper [11], Guillemin and Sternberg conjectured that quantization commutes with reduction, and proved this for compact Lie groups acting on compact Kähler manifolds, under a positivity assumption. A definition of geometric quantization that is valid for compact Lie groups acting on compact, possibly non-Kähler symplectic manifolds is attributed to Bott. Let \((M, \omega)\) be a compact symplectic manifold, on which a compact Lie group \(K\) acts, preserving \(\omega\). Let \(L \rightarrow M\) be a \(K\)-equivariant Hermitian line bundle whose first Chern class is \([\omega]\). Let \(D^L\) be the Dolbeault- or Spin\(^c\)-Dirac operator on \(M\), coupled to \(L\). If all structures involved in the definition of this operator are \(K\)-invariant, then \(D^L\) is \(K\)-equivariant. Bott’s definition of the geometric quantization \(Q_K(M, \omega)\) is

\[
Q_K(M, \omega) := K\text{-index}(D^L) \in R(K).
\]

Here \(R(K)\) is the representation ring of \(K\), and

\[
K\text{-index}(D^L) := \frac{1}{2} \left[ \ker D_+^L \right] - \left[ \ker D_-^L \right],
\]
with $D^L_{\pm}$ the even and odd parts of $D^L$. Since $M$ and $K$ are compact, this $K$-index is indeed a well-defined element of $R(K)$.

If the action of $K$ on $M$ is Hamiltonian, there is a momentum map

$$\mu : M \to \mathfrak{k}^*,$$

where $\mathfrak{k}^*$ is the dual of the Lie algebra $\mathfrak{k}$ of $K$. If $0 \in \mathfrak{k}^*$ is a regular value of $\mu$, then the space

$$M_0 := \mu^{-1}(0)/K$$

is an orbifold, since it can be shown that $K$ acts on $\mu^{-1}(0)$ with finite stabilizers. For simplicity, one can assume that these stabilizers are trivial, so that $M_0$ is a smooth manifold. The symplectic form $\omega$ naturally induces a symplectic form $\omega_0$ on $M_0$. The symplectic manifold $(M_0, \omega_0)$ is called the symplectic reduction or Marsden–Weinstein reduction $[19]$ of $(M, \omega)$. The geometric quantization of $(M_0, \omega_0)$ is defined as the index of the Dirac operator $D^L_0$ on $M_0$, coupled to the line bundle $L_0 \to M_0$ induced by $L$.

In terms of Bott’s definition of geometric quantization, Guillemin and Sternberg’s conjecture that quantization commutes with reduction states that the following diagram commutes:

$$\begin{array}{cccc}
K \otimes (M, \omega) & \xrightarrow{Q_K} & \text{K-index}(D^L) \\
\xrightarrow{\text{reduction}} R & & & \xrightarrow{\text{reduction}} R \\
(M_0, \omega_0) & \xrightarrow{Q} Q(M_0, \omega_0) & \xrightarrow{(K-\text{index}(D^L))^K}
\end{array}$$

That is, the quantization of $(M_0, \omega_0)$ is the $K$-invariant part of the quantization of $(M, \omega)$. The Guillemin–Sternberg conjecture in this generality was first proved 16 years after Guillemin and Sternberg’s paper, by Meinrenken $[22]$ (see also $[21]$), and by Meinrenken and Sjamaar $[23]$ for singular values of the momentum map. Other proofs were given by Tian and Zhang $[28, 29]$ and by Paradan $[25]$.

1.2. Noncompact groups and manifolds. After these results, the natural desire arose to generalize the quantization commutes with reduction principle to noncompact manifolds and groups. Such a generalization is very relevant to

- **physics**, since most classical mechanical phase spaces (such as cotangent bundles) are not compact; and to
- **representation theory**, since the representation theory for noncompact groups is much more intricate than for compact groups, and could benefit greatly from such a principle in the noncompact case.

However, even stating a quantization commutes with reduction principle in the noncompact setting proved highly challenging:

- if the manifold $M$ is noncompact, then the kernel of the Dirac operator $D^L$ is infinite-dimensional in general, so its index is not well-defined;
- if the compact group $K$ is replaced by a noncompact group $G$, the finite-dimensional representations of $G$, which make up $R(G)$, are not the interesting ones.

Ma and Zhang $[17, 18]$ solved an extended version of Vergne’s conjecture $[30]$ on quantization commutes with reduction for compact groups $G = K$ acting on noncompact manifolds, and later on Paradan $[26, 27]$ gave a new proof of it. They define quantization as an element of the *generalized representation ring* $R^{-\infty}(K)$, by taking localized indices of the
Dirac operator $D^L$ on expanding families of suitable relatively compact open subsets of $M$. Their approach applies to several interesting examples, such as cotangent bundles of a homogeneous spaces of compact Lie groups [20] and coadjoint orbits associated to holomorphic discrete series representations of reductive Lie groups [27].

Landsman [16] stated a conjecture \footnote{called the ‘Hochs–Landsman conjecture’ in [20]} for noncompact groups $G$ and manifolds $M$, assuming that the action is compact, i.e. the orbit space $M/G$ is compact. He used the analytic assembly map from the Baum–Connes conjecture to define geometric quantization. This assembly map generalizes the $K$-index, and is a map

$$\mu^G_M : K^*_G(M) \to K_*(C^*(G)).$$

Here $K^*_G(M)$ is the equivariant $K$-homology of $M$, and $K_*(C^*(G))$ is the $K$-theory of the $C^*$-algebra $C^*(G)$ of $G$. Landsman defined geometric quantization as

$$Q_G(M, \omega) := \mu^G_M[D^L] \in K_*(C^*(G)),$$

where $[D^L] \in K^*_G(M)$ is the $K$-homology class naturally defined by the Dirac operator $D^L$. There is a quantum reduction map $R_G : K_0(C^*(G)) \to \mathbb{Z}$, induced by averaging over the group $G$. Landsman’s quantization commutes with reduction conjecture asserts that

$$R_G(Q_G(M, \omega)) = Q(M_0, \omega_0) \in \mathbb{Z}.$$
zero set of this vector field is assumed to be cocompact, which implies that the symplectic reduction \( M_0 \) at zero of the action is compact.

The \( G \)-invariant part of the quantization of the action by \( G \) on \((M, \omega)\) may then be defined as the integer

\[
Q(M, \omega)^G := \dim \left( \ker_{L^2_t} ( (D^L_t)^+) \right)^G - \dim \left( \ker_{L^2_t} ( (D^L_t)^-) \right)^G,
\]

for \( t \) large enough. Here \( \ker_{L^2_t} \) denotes the space of sections in the kernel of an operator that are square-integrable transversally to orbits, in an appropriate sense. The first main result of this paper is that invariant quantization is well-defined in this way.

**Theorem 1.1.** For \( t \) large enough, the \( G \)-invariant part \( (\ker_{L^2_t} (D^L_t))^G \) of the vector space \( \ker_{L^2_t} (D^L_t) \) is finite-dimensional.

The proof of this fact involves some index theory on Sobolev spaces created from \( G \)-invariant sections, and a generalization of the Anghel–Gromov–Lawson criterion \([2, 9]\) for Fredholmness. For compact groups \( G \), Braverman used a similar approach in \([5]\). Using techniques from \([5]\) and the present paper, Braverman also developed an approach for non-compact groups \([6]\).

In terms of this definition of invariant quantization, one can state the quantization commutes with reduction principle as follows. (See Subsections 3.1 and 3.2 for details.)

**Conjecture 1.2** (Quantization commutes with reduction). Under the assumptions that the Hamiltonian \( G \)-action on the prequantizable symplectic manifold \((M, \omega)\) is proper, \( 0 \) is a regular value of the momentum map, and the zero set of the \( G \)-invariant vector field \( X^H_1 \) is \( G \)-cocompact, one has the equality

\[
Q(M, \omega)^G = Q(M_0, \omega_0),
\]

where \( Q(M_0, \omega_0) \) is the quantization of the symplectic reduction at zero.

If \( M/G \) is compact, this conjecture reduces to Landsman’s conjecture \([8]\) (see Corollary 3.9), which in turn reduces to commutativity of \([5]\) if both \( G \) and \( M \) are compact.

The second main result in this paper establishes an asymptotic version of this principle, under the simplifying assumption that \( G \) acts freely on \( \mu^{-1}(0) \) (rather than just locally freely).

**Theorem 1.3.** If \( G \) acts freely on \( \mu^{-1}(0) \), the equality \([9]\) holds for large enough multiples of \( \omega \):

\[
Q(M, p\omega)^G = Q(M_0, p\omega_0),
\]

for any integer \( p \) at least equal to a minimal value \( p_0 \).

As in \([29]\), the proofs of the main results start with a Bochner-type formula for the square of the deformed Dirac operator \( D^L_{tp} \) on \( G \)-invariant sections:

\[
(D^L_{tp})^2 = (D^L_t)^2 + tA + 4\pi pt^2H + \frac{t^2}{4} \|X^H_1\|^2,
\]

where the tensor term \( A \) is a generalization of the one in \([29]\). It can be decomposed as \( A = A_1 + A_2 + A_3 \) where \( A_1 \) is the Tian–Zhang tensor and \( A_2, A_3 \) are new tensors that vanish in the setting of \([29]\). In the non-cocompact case, it is possible for \( A \) to be unbounded, so...
it appears at first that the method in [29] does not work. However, we use the flexibility that one has in choosing an equivariant family of inner products on \( g^* \), and make a judicious choice of such a family to bound \( A \), which is a key ingredient of our approach.

In [29] and [20], adaptations of a result by Bismut and Lebeau [3] are used to localize the index of the deformed Dirac operator \( D_p^{L_p} \) to a suitable open set. This leads to proofs of the equality (11) in the special cases where \( M \) and \( G \), or \( M/G \), are compact. In [20], Mathai and Zhang used any inner product on \( g^* \), so that the Hamiltonian function given by the norm squared of the momentum map was no longer \( G \)-invariant. They used a weighted average of the Hamiltonian vector field, which is \( G \)-invariant. In the present paper, using a family of inner products on \( g^* \) rather than a single inner product allows us to apply the Bismut–Lebeau approach in a more general setting.

In Subsection 3.3, it is shown that the techniques developed here apply for example to physical systems where the configuration space is a Lie group \( G \), acted on by the group itself via left multiplication. Then \( M = T^*G \) is the cotangent bundle of \( G \), and the orbit space of the action is noncompact. The zero set of the vector field \( X_{H_1} \) is cocompact however, so Theorem 3.6 applies. This in particular applies to the case of a free particle in \( \mathbb{R}^n \) mentioned above, where \( G = \mathbb{R}^n \). Other examples discussed in Subsection 3.3 include the case when \( M/G \) is compact, and also the case when \( G \) itself is compact, which is relevant to the Vergne conjecture that was completely solved in [17, 18] using a very different index theorem. Finally, the cocompactness assumption in [12, 16, 20] precluded any form of the shifting trick for noncompact groups. In the present setting, a version of the shifting trick holds.

1.4. Outline of this paper. The key ingredient of our method is the use of \( G \)-invariant families of inner products on \( g^* \). These are introduced in Section 2. The vector fields defined via these families are also discussed.

The main results of this paper are Theorems 1.1 and 1.3, which are formulated in a precise way in Subsection 3.2. The index theory used to show that invariant quantization is well-defined, is developed in Section 4.

In Section 6, we will see that the kernels of the deformed Dirac operators we use localize in a suitable way. The argument for this localization is based on an explicit computation of the square of the deformed Dirac operator in Section 5. A relation between these deformed Dirac operators and certain Dirac operators on the symplectic reduction, proved in Section 7 then allows us to complete the proof that quantization commutes with reduction.

A version of elliptic regularity is proved in Appendix A. Appendices B and C contain some computations and estimates used in the main text, involving deformed Dirac operators.

1.5. Acknowledgements. The authors would like to thank Maxim Braverman for carefully reading a preliminary version of this paper, and giving useful advice. The authors also benefitted from discussions with Gert Heckman, Nigel Higson, Klaas Landsman, Paul–Émile Paradan, Maarten Solleveld and Weiping Zhang.

The first author was supported by the Alexander von Humboldt foundation and by the European Union, through a Marie Curie fellowship. A collaboration visit of the first author to the second was funded by the Australian Research Council, through Discovery Project DP110100072. The second author acknowledges funding by the Australian Research Council, through Discovery Project DP130103924.
1.6. **Notation and conventions.** For a smooth manifold \( M \), we denote the spaces of smooth functions, smooth \( k \)-forms and smooth vector fields on \( M \) by \( C^\infty(M) \), \( \Omega^k(M) \) and \( \mathfrak{X}(M) \), respectively. The value of a vector field \( v \) on \( M \) at a point \( m \) will be denoted by \( v_m \) or \( v(m) \), whichever seems clearer.

If \( E \to M \) is a vector bundle, the space of smooth sections of \( E \) is denoted by \( \Gamma^\infty(M; E) \) or \( \Gamma^\infty(E) \). We write \( \Omega^k(M; E) \) for the space of smooth sections of \( \bigwedge^k T^* M \otimes E \). For almost complex manifolds, \( \Omega^{0,k}(M) \) and \( \Omega^{1,k}(M; E) \) denote the analogous spaces of \((0,k)\)-forms.

A subscript \( c \) denotes compactly supported functions or sections. In the equivariant case, where a group \( G \) acts on the relevant structures, a superscript \( G \) denotes the space of \( G \)-invariant elements.

The Lie algebra of a Lie group \( G \) will be denoted by \( \mathfrak{g} \). We will denote the dimension of a manifold \( M \) by \( d_M \). An action with a compact orbit space will be called **cocompact**. A subset of a space acted on by a group is called **relatively cocompact** if its image under the quotient map is relatively compact.

We will write \( \mathbb{N} = \{1, 2, 3, \ldots\} \) for the set of natural numbers without the number 0.

2. **Families of metrics on \( \mathfrak{g}^* \)**

The methods used in this paper are based on deforming Dirac operators using a certain \( G \)-invariant vector field \( X^H \). This vector field is similar to the Hamiltonian vector field of the norm-squared function of a given momentum map, used in [29]. If a Lie group \( G \) is noncompact, an \( \text{Ad}^*(G) \)-invariant inner product on the dual \( \mathfrak{g}^* \) of the Lie algebra \( \mathfrak{g} \) may not exist. Then the Hamiltonian vector field of the norm-squared function of a momentum map is not \( G \)-invariant in general. To solve this problem, we work with **families** of inner products on \( \mathfrak{g}^* \), parametrized by a manifold on which \( G \) acts. Such a family of inner products allows one to define the vector field \( X^H \) mentioned above. The zero set of this vector field will later be assumed to be cocompact.

2.1. **Norms of momentum maps.** Let \( (M, \omega) \) be a symplectic manifold, and let \( G \) be a Lie group. Let a proper Hamiltonian action by \( G \) on \( M \) be given, and let

\[
\mu : M \to \mathfrak{g}^*
\]

be a momentum map. We will use the sign convention that for all \( X \in \mathfrak{g} \),

\[
d\mu_X = \omega(X^M, -),
\]

where \( \mu_X \) denotes the pairing of \( \mu \) and \( X \), and \( X^M \) is the vector field on \( M \) induced by \( X \).

As in [29], we will consider the norm-squared function of \( \mu \) and use this to deform a Dirac operator on \( M \). We would like this function to be \( G \)-invariant, so that the resulting deformation is again a \( G \)-equivariant operator. Since there is no \( \text{Ad}^*(G) \)-invariant inner product on the dual \( \mathfrak{g}^* \) of the Lie algebra \( \mathfrak{g} \) in general, we consider a smooth family of inner products \( \{(\cdot, \cdot)_m\}_{m \in M} \) on \( \mathfrak{g}^* \), which is \( G \)-equivariant, in the sense that for all \( m \in M \), \( g \in G \) and \( \xi, \xi' \in \mathfrak{g}^* \),

\[
(\text{Ad}^*(g)^\xi, \text{Ad}^*(g)^\xi')_{g \cdot m} = (\xi, \xi')_m.
\]

Put differently, the inner products \( (\cdot, \cdot)_m \) define a \( G \)-invariant smooth metric on the \( G \)-vector bundle \( M \times \mathfrak{g}^* \to M \), equipped with the \( G \)-action

\[
g \cdot (m, \xi) = (g \cdot m, \text{Ad}^*(g)^\xi),
\]
for \( g \in G \), \( m \in M \) and \( \xi \in g^* \). Such a metric was used by Kasparov in Section 6 of [15] in a different context, and always exists.

**Lemma 2.1.** There is a metric on the trivial bundle \( M \times g^* \to M \), which is invariant with respect to the \( G \)-action \((11)\).

**Proof.** By Palais’s theorem [24], the proper \( G \)-manifold \( M \times g^* \) has a \( G \)-invariant Riemannian metric. The vector bundle \( M \times g^* \) embeds into the restriction of \( T(M \times g^*) \) to \( M \times \{0\} \), via

\[
(m, \xi) \mapsto (0, \xi) \in T_m M \times g^* = T_{(m,0)}(M \times g^*),
\]

for \( m \in M \) and \( \xi \in g^* \). Restricting the Riemannian metric on \( M \times g^* \) to the subbundle \( M \times g^* \) of \( T(M \times g^*)|_{M \times \{0\}} \) in this way, one obtains the desired metric. □

Form now on, let \( \{(\cdot, m)\}_{m \in M} \) be a \( G \)-invariant metric on \( M \times g^* \to M \), and let \( \{\| \cdot \|_m\}_{m \in M} \) be the associated family of norms on \( g^* \). Consider the function \( \mathcal{H} \in C^\infty(M) \) defined by

\[
\mathcal{H}(m) = \|\mu(m)\|^2_m.
\]

It follows from equivariance of \( \mu \) and the property \((13)\) of the family of inner products on \( g^* \), that \( \mathcal{H} \) is a \( G \)-invariant function on \( M \).

Consider the auxiliary function \( \tilde{\mathcal{H}} \in C^\infty(M \times M) \) defined by

\[
\tilde{\mathcal{H}}(m, m') = \|\mu(m)\|^2_{m'}.
\]

We write \( d_1 \mathcal{H} \) and \( d_2 \mathcal{H} \) for the derivatives of \( \tilde{\mathcal{H}} \) with respect to the first and second coordinates:

\[
(d_1 \mathcal{H})_m := d_m (m' \mapsto \tilde{\mathcal{H}}(m', m));
\]

\[
(d_2 \mathcal{H})_m := d_m (m' \mapsto \tilde{\mathcal{H}}(m, m')),
\]

for any \( m \in M \). In terms of these one-forms on \( M \), one has

\[
d \mathcal{H} = d_1 \mathcal{H} + d_2 \mathcal{H}.
\]

2.2. **Two vector fields.** Important roles will be played by the vector fields \( X^\mathcal{H}_j \) on \( M \) determined by

\[
d_j \mathcal{H} = \omega(X^\mathcal{H}_j, -) \in \Omega^1(M).
\]

The Hamiltonian vector field \( X^\mathcal{H} \) of \( \mathcal{H} \) decomposes as

\[
X^\mathcal{H} = X^\mathcal{H}_1 + X^\mathcal{H}_2.
\]

Note that \( X^\mathcal{H}_1 \) and \( X^\mathcal{H}_2 \) are not quite Hamiltonian vector fields. But they turn out to have similar useful properties.

One of these is \( G \)-invariance. This property will mean that one does not need to average them as in [20]. By \( G \)-invariance of \( \omega \), it is equivalent to \( G \)-invariance of \( d_1 \mathcal{H} \) and \( d_2 \mathcal{H} \). This follows from the following fact.

**Lemma 2.2.** Let \( M \) be a manifold on which a group \( G \) acts. Let \( F : M \times M \to \mathbb{R} \) be a smooth function which is invariant under the diagonal action by \( G \) on \( M \times M \). Then the one-forms \( d_1 F \) and \( d_2 F \) on \( M \) are \( G \)-invariant.

\(^2\)The notation \( d_1 F \) is not quite consistent with the notation \( d_j \mathcal{H} \), since \( \mathcal{H} \) is a function on \( M \). But the notation is hopefully self-explanatory.
Proof. We will prove the claim for $d_1 F$. Let $m \in M$, $g \in G$, and let $\gamma$ be a curve in $M$ with $\gamma(0) = m$. Then
\[
\langle (g^*(d_1 F))_m, \gamma'(0) \rangle = \langle (d_1 F)_{gm}, T_m g(\gamma'(0)) \rangle
\]
\[
= \left. \frac{d}{dt} \right|_{t=0} F(g \gamma(t), gm)
\]
\[
= \left. \frac{d}{dt} \right|_{t=0} F(\gamma(t), m)
\]
\[
= \langle (d_1 F)_m, \gamma'(0) \rangle.
\]
\[\Box\]

It will be useful to have explicit expressions for the vector field $X_H^1$. For any map $f : M \to g^*$, we will write $f^*$ for the map from $M$ to $g$ determined by
\[
(f(m), \xi)_m = \langle \xi, f^*(m) \rangle,
\]
for all $\xi \in g^*$. This dual map induces a vector field $V_f$ on $M$, by
\[
V_f(m) := f^*(m)^M_m = \left. \frac{d}{dt} \right|_{t=0} \exp(tf^*(m))m.
\]

**Lemma 2.3.** One has
\[X_H^1 = 2V_\mu.\]

**Proof.** For all $m \in M$ and $v \in T_m M$, we compute
\[
\omega_m(X_H^1(m), v) = \langle (d_1 H)_m, v \rangle
\]
\[
= 2(T_m \mu(v), \mu(m))_m
\]
\[
= 2\langle d_m \mu^*(m), v \rangle
\]
\[
= 2\omega_m(V_\mu(m), v).
\]
\[\Box\]

Let $h_1, \ldots, h_{d_G} : M \to g^*$ be an orthonormal frame for the vector bundle $M \times g^* \to M$, with respect to the given family of inner products. Write
\[
\mu = \sum_{j=1}^{d_G} \mu_j h_j,
\]
for functions $\mu_j \in C^\infty(M)$. For each $j$, write
\[
V_j := V_{h_j}
\]
for the vector field induced by $h_j$ as in (20). Then Lemma 2.3 implies that
\[
X_H^1 = 2 \sum_{j=1}^{d_G} \mu_j V_j.
\]
This is an analogue of (1.19) in [29].
2.3. **Induced vector fields on reduced spaces.** As noted in Subsection 2.2, the vector fields $X_1^H$ and $X_2^H$ are $G$-invariant, and hence descend to $M/G$ at points with trivial stabilizers. Because of Lemma 2.3, the vector field $X_1^H$ is tangent to $G$-orbits, so that it induces the zero vector field on the quotient.

The vector field $X_2^H$ is not necessarily tangent to orbits. It does have the weaker property that it is tangent to submanifolds of the form $\mu^{-1}(O)$, for any coadjoint orbit $O \subset g^*$ that consists of regular values of $\mu$. This follows from the following fact.

**Lemma 2.4.** Let $f \in C^\infty(M)$ be a $G$-invariant function, and let $X^f$ be its Hamiltonian vector field. Then at every point $m \in M$,

$$T_m \mu(X^f(m)) = 0 \in g^*.$$  

**Proof.** For every $X \in g$, one has

$$\langle T_m \mu(X^f(m)), X \rangle = \langle d_m \mu_X, X^f \rangle = \omega_m(X^M_m, X^f) = -\langle d_m f, X^M_m \rangle = 0,$$

since $f$ is $G$-invariant. □

**Corollary 2.5.** Let $m \in M$, and write $\xi := \mu(m)$ and $O := \text{Ad}^*(G)\xi$. One has

$$T_m \mu(X_2^H(m)) = -2\mu^*(m)_{\xi} \in T_\xi O \hookrightarrow g^*.$$  

So if $\xi$ is a regular value of $\mu$, then

$$X_2^H(m) \in T_m(\mu^{-1}(O)) = (T_m \mu)^{-1}(T_\xi O).$$

**Proof.** Because of $G$-invariance of $\mathcal{H}$, relation (18) and Lemmas 2.3 and 2.4 one has

$$T_m \mu(X_2^H(m)) = -2T_m \mu(V_\mu(m))$$

Equivariance of $\mu$ implies that the latter expression equals $-2\mu^*(m)_{\xi}$. □

Because $X_2^H$ is $G$-invariant and tangent to submanifolds of the form $\mu^{-1}(O)$ as above, it induces a vector field $(X_2^H)_O$ on every symplectic reduction $M_O = \mu^{-1}(O)/G$, if $O$ consists of regular values of $\mu$, and $G$ acts freely on $\mu^{-1}(O)$. As noted above, $X_1^H$ induces the zero vector field on the reduced spaces $M_O$.

We will mainly consider the case $O = \{0\}$.

**Lemma 2.6.** The vector field $(X_2^H)_0$ on $M_0$ induced by $X_2^H$ is the zero vector field.

**Proof.** Let $\iota: \mu^{-1}(0) \hookrightarrow M$ be the inclusion map. Then $\iota^*(d\mathcal{H}) = d(\iota^*\mathcal{H}) = 0$, so $X^H$ induces the zero vector field on $M_0$. Hence

$$(X_2^H)_0 = (X_1^H)_0 + (X_2^H)_0 = (X^H)_0 = 0.$$  

□

2.4. **Critical points.** For $j = 1, 2$, let $\text{Crit}_j(\mathcal{H})$ be the set of zeroes of $d_j \mathcal{H}$, which equals the set of zeroes of $X_j^H$. We will later assume that $\text{Crit}_1(\mathcal{H})$ is cocompact, and investigate that assumption in this subsection.

By Lemma 2.3, we have

$$\text{Crit}_1(\mathcal{H}) = \{ m \in M; \mu^*(m) \in g_m \}.  \quad (24)$$
We will assume that 0 is a regular value of \( \mu \), which by Smale’s lemma implies that \( g_m = 0 \) for all \( m \in \mu^{-1}(0) \). Since the minimal isotropy type occurs on an open dense subset \( U \subset M \) (see e.g. [1], Theorem 2.3), one has \( g_m = 0 \) for all \( m \in U \). Therefore,

\[
\text{Crit}_1(\mathcal{H}) \cap U = \mu^{-1}(0).
\]

If \( M_0 = \mu^{-1}(0)/G \) is compact, then (25) implies that any non-cocompact parts of \( \text{Crit}_1(\mathcal{H}) \) are contained in the positive codimension part \( M \setminus U \) of \( M \). We have therefore established the following sufficient condition for compactness of \( \text{Crit}_1(\mathcal{H})/G \)

**Lemma 2.7.** Suppose that 0 is a regular value of \( \mu \), and that \( M_0 \) is compact. Then \( \text{Crit}_1(\mathcal{H})/G \) is compact if \( (M \setminus U)/G \) is compact.

An example where \( \text{Crit}_1(\mathcal{H}) \) is cocompact is the action by a Lie group on its cotangent bundle.

**Example 2.8.** Let \( G \) be a Lie group, and consider the action by \( G \) on its cotangent bundle

\[
T^*G \cong G \times g^* \text{ induced by left multiplication.}
\]

A momentum map for this action is the projection \( \mu : G \times g^* \to g^* \), for which \( \mu^{-1}(0) = G \). Since the action is free, and \( \mu^{-1}(0)/G \) is a point, \( \text{Crit}_1(\mathcal{H}) \) is cocompact by Lemma 2.7. This holds for any family of inner products on \( g^* \).

Using a family of inner products on \( g^* \) rather than a fixed inner product provides a flexibility that can be used to give the vector field \( X_{\mathcal{H}} \) some desirable properties. This may allow one to make \( \text{Crit}_1(\mathcal{H}) \) more manageable, for example. Specifically, it follows from (24)

\[
\mu^{-1}(0) \cup M^G \subset \text{Crit}_1(\mathcal{H}),
\]

where \( M^G \) denotes the fixed point set of the action. In certain cases, the converse inclusion holds as well.

Indeed, let \( \text{Sym}^+(g^*) \) be the set of positive definite symmetric linear automorphisms of \( g^* \). Consider a smooth map

\[
b : M \to \text{Sym}^+(g^*)
\]

which has the equivariance property that for all \( m \in M \) and \( g \in G \),

\[
b(gm) = \text{Ad}^*(g)b(m) \text{Ad}^*(g)^{-1}.
\]

Then setting

\[
(-, -)^b_m := (-, b(m) - )_m,
\]

for \( m \in M \), defines a \( G \)-invariant metric \((- , - )^b \) on \( M \times g^* \to M \). Let \( \mathcal{H}^b \) be the resulting function defined as in (15).

For \( m \in M \), let \( b^*(m) \in \text{GL}(g) \) be the linear endomorphism dual to \( b(m) \). If the map \( b \) can be chosen such that \( b^*(m)\mu^*(m) \) points away from the stabilizer \( g_m \) where this is possible, then the converse inclusion to (26) holds as well.

**Lemma 2.9.** If for all \( m \in M \) outside \( \mu^{-1}(0) \cup M^G \), one has

\[
b^*(m)\mu^*(m) \not\in g_m
\]

then

\[
\text{Crit}_1(\mathcal{H}^b) = \mu^{-1}(0) \cup M^G.
\]
Proof. Let 

\[ \mu^*: M \to g \]

be the map dual to \( \mu \), with respect to the family of inner products \( \left\{ (-,-)^b_m \right\}_{m \in M} \) on \( g^* \), as defined in (19). One can check that for all \( m \in M \),

\[ \mu^*(m) = b^*(m) \mu^*(m). \]

Hence it follows from (24) that

\[ \text{Crit}_1(\mathcal{H}^b) = \{ m \in M; b^*(m) \mu^*(m) \in g^*_m \}. \]

The claim now follows from (27) and (26).

Note that if \( G \) is noncompact, properness of the action implies that \( M^G \) is empty. Hence, in the situation of Lemma 2.9, \( \text{Crit}_1(\mathcal{H}) \) is cocompact if and only if \( \mu^{-1}(0) \) is.

2.5. Rescaling the metric. Another important flexibility one can exploit is rescaling a family of inner products on \( g^* \) by a positive \( G \)-invariant function. As before, let \( \left\{ (-,-)_m \right\}_{m \in M} \) be a \( G \)-invariant metric on \( M \times g^* \to M \), and let \( \mathcal{H} \) be the associated norm squared function (15) of the momentum map. Let \( \psi \in C^\infty(M) \) be a positive, \( G \)-invariant function, and let \( \mathcal{H}_\psi \) be the analogous function associated to the family of inner products \( \{ \psi(m)(-,-)_m \}_{m \in M} \):

\[ \mathcal{H}_\psi(m) := \psi(m)\| \mu(m) \|^2. \]

We will write \( d_1\mathcal{H}_\psi \) and \( X_{\mathcal{H}_\psi} \) for the one-form and vector field constructed from the function \( \mathcal{H}_\psi \) in the same way as \( d_1\mathcal{H} \) and \( X_{\mathcal{H}}^1 \) were constructed from \( \mathcal{H} \).

Lemma 2.10. One has

\[ X_{\mathcal{H}_\psi}^1 = \psi X_{\mathcal{H}}^1. \]

Proof. Since computing the one-form \( d_1\mathcal{H}_\psi \) only involves differentiating with respect to the argument of \( \mu \), one has

\[ d_1\mathcal{H}_\psi = \psi d_1\mathcal{H}. \]

Remark 2.11. As mentioned in Subsection 2.4, the set \( \text{Crit}_1(\mathcal{H}) \) will be assumed to be cocompact. By Lemma 2.10, rescaling the metric by a positive, \( G \)-invariant function does not change the set \( \text{Crit}_1(\mathcal{H}) \), and hence does not influence this assumption.

3. Assumptions and results

In [29], Tian and Zhang give an analytic proof that quantization commutes with reduction in cases where the manifold \( M \) and the group \( G \) are compact. Their proof is based on a Witten-type deformation [32] of the Spin\(^c\)-Dirac operator on a symplectic manifold. In this deformation, they use the Hamiltonian vector field \( X_\mathcal{H} \) of the norm-squared function \( \mathcal{H} \) of the momentum map. Crucially, the norm on \( g^* \) used is invariant under the coadjoint action by \( G \), which is always possible for compact groups. In [20], Mathai and Zhang treat the cocompact case. Since an \( \text{Ad}^*(G) \)-invariant inner product on \( g^* \) is not always available then, they use a weighted average of the Hamiltonian vector field \( X_\mathcal{H} \), where \( \mathcal{H} \) is now defined with respect to a norm that is not necessarily \( \text{Ad}^*(G) \)-invariant. We will use a different deformation of Dirac operators, using the vector field \( X_{\mathcal{H}}^1 \) introduced in Section 2, instead of the vector field \( X_\mathcal{H} \).
3.1. Assumptions. We make the same assumptions as for example in [12, 20], with the important exception that the orbit space of the action considered need not be compact.

Let $\omega\in\Omega^2(M)$ be a symplectic manifold. Let $J$ be an almost complex structure on $M$ such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric on $TM$. Assume $M$ is complete with respect to this metric.

Assume there exists a Hermitian line bundle $L$ over $M$ carrying a Hermitian connection $\nabla^L$ such that $\sqrt{\frac{\omega}{2\pi}}(\nabla^L)^2 = \omega$. Then for any $p \in \mathbb{N}$, the $p$’th tensor power $L^p \to M$ is a prequantum line bundle for the symplectic manifold $(M, p\omega)$.

For such an integer $p$, let $D^{L^p}_\pm : \Omega^{0,\ast}(M; L^p) \to \Omega^{0,\ast}(M; L^p)$ be the Spin$^c$-Dirac operator on $M$ coupled to the line bundle $L^p$ via the given prequantum data (see Section 1 of [29], or [7, 8]). Let $D^{L^p}_+ \text{ and } D^{L^p}_-$ be the restrictions of $D^{L^p}$ to $\Omega^{0,\text{even}}(M; L^p)$ and $\Omega^{0,\text{odd}}(M; L^p)$, respectively.

Let $G$ be a unimodular Lie group, with Lie algebra $\mathfrak{g}$, acting properly and symplectically on $M$. We assume that the action of $G$ on $M$ lifts to $L$. Moreover, we assume the $G$-action preserves the above metrics and connections on $TM$ and $L$, as well as the almost complex structure $J$. Then the operators $D^{L^p}_\pm$ commute with the $G$-action.

The action of $G$ on $L$ naturally determines a momentum map $\mu : M \to \mathfrak{g}^*$ such that for any $X \in \mathfrak{g}$ and $s \in \Gamma^\infty(L)$, if $X^M$ denotes the induced Killing vector field on $M$, then the following Kostant formula for the Lie derivative $L_{X^M}$ holds:

\begin{equation}
L_{X^M}s = \nabla^L_{X^M}s - 2\pi \sqrt{-1}\mu_xs.
\end{equation}

For any integer $p$, and any section $s \in \Gamma^\infty(L^p)$, one then has

\begin{equation}
L_{X^M}s = \nabla^{L^p}_{X^M}s - 2\pi \sqrt{-1}p\mu_xs.
\end{equation}

We assume that $0 \in \mathfrak{g}^*$ is a regular value of $\mu$. It then follows from the definition of momentum maps that all stabilizers of the action by $G$ on $\mu^{-1}(0)$ are discrete, and hence finite by properness of the action. We will assume that these stabilizers are in fact trivial, i.e. that $G$ acts freely on $\mu^{-1}(0)$. Then the Marsden–Weinstein symplectic reduction [19] $(M_0, \omega_0)$ is a smooth symplectic manifold. Moreover, the prequantum line bundle $L$ descends to a line bundle $L_0$ on $M_0$. The connection $\nabla^L$ induces a connection $\nabla^{L_0}$ on $L_0$, such that the corresponding curvature condition $\sqrt{\frac{\omega_0}{2\pi}}(\nabla^{L_0})^2 = \omega_0$ holds. The $G$-invariant almost complex structure $J$ also descends to an almost complex structure $J_0$ on $M_0$, and the metrics on $L$ and $TM$ descend to metrics on $L_0$ and $TM_0$, respectively. Let $D^{L_0}$ denote the corresponding Spin$^c$-Dirac operator on $M_0$. These constructions generalize to yield the prequantum line bundle $L_0^p$ for the reduced space $(M_0, p\omega_0)$, and an associated Dirac operator $D^{L_0}_\pm$.

The manifold $M$ and the group $G$ are allowed to be noncompact independently, and there is no properness assumption on the momentum map $\mu$. The only compactness assumption made is that the set $\text{Crit}_1(\mathcal{H})/G$ is compact for a $G$-invariant metric on $M \times \mathfrak{g}^* \to M$ (see Subsection 2.3). By Lemma 2.3 this implies that the symplectic reduction $M_0$ is compact as well.

3.2. Invariant quantization; the main results. Let $X^\mathcal{H}_1$ be the vector field on $M$ introduced in Section 2 via a $G$-invariant metric on the trivial bundle $M \times \mathfrak{g}^* \to M$. 

13
**Definition 3.1.** For \( t \in \mathbb{R} \) and \( p \in \mathbb{N} \), the **deformed Dirac operator** on \( \Omega^{0,*}(M; L^p) \) is the operator

\[
D_t^{L^p} := D^{L^p} + \frac{\sqrt{-1} t}{2} c(X_1^m),
\]

where \( c \) denotes the Clifford action by \( TM \) on \( \Lambda^{0,*}T^*M \).

The Clifford action \( c \) by \( TM \) on \( \Lambda^{0,*}T^*M \) is explicitly defined as follows. Let \( m \in M \), \( v \in T_m M \), and let \( v_C = v^{1,0} + v^{0,1} \) be the decomposition of the complexification \( v_C \) of \( v \) according to \( T_m M \otimes \mathbb{C} = T_m^{1,0} M \oplus T_m^{0,1} M \). Let \( (v^{1,0})^* \in (T_m^{0,1} M)^* \) be the covector dual to \( v^{1,0} \) with respect to the metric. Then

\[
c(v) = \sqrt{2}( -i_{v^{0,1}} + (v^{1,0})^* \wedge - ) : \Lambda^{0,*}T_m^*M \to \Lambda^{0,*}T_m^*M.
\]

Here \( i_{v^{0,1}} \) denotes contraction by \( v^{0,1} \).

Invariant quantization will be defined in terms of the **transversally \( L^2 \)-kernel** of \( D_t^L \). The definition of this kernel involves the notion of a **cutoff function**.

Let \( dg \) be a left Haar measure on \( G \). In e.g. \cite{4}, Chapter VII, Section 2.4, Proposition 8, it is shown that a continuous, nonnegative function \( f \) on \( M \) exists, whose support intersects all \( G \)-orbits in compact sets, and satisfies

\[
\int_G f(g \cdot m)^2 dg = 1,
\]

for all \( m \in M \). Such functions are used in many applications in index theory (see e.g. \cite{15, 20}). If \( M/G \) is compact, one may take \( f \) to be compactly supported. We will call a function \( f \) with these properties a **cutoff function**.

**Definition 3.2.** Let \( E \to M \) be a \( G \)-vector bundle, equipped with a \( G \)-invariant Hermitian metric. The vector space of **transversally \( L^2 \)-sections** of \( E \) is the space \( L^2_T(E) \) of sections \( s \) of \( E \) (modulo equality almost everywhere) such that \( fs \in L^2(E) \) for every cutoff function \( f \). (Here integrals over \( M \) are defined with respect to the Liouville measure.)

Let \( D \) be a linear operator on \( \Gamma^\infty(E) \). The **transversally \( L^2 \)-kernel** of \( D \) is the vector space

\[
\ker_{L^2_T(D)} := \{ s \in \Gamma^\infty(E) \cap L^2_T(E) ; Ds = 0 \}.
\]

**Remark 3.3.** The vector space \( L^2_T(E) \) can be given a locally convex topology via the seminorms

\[
\| s \|^f := \| fs \|_{L^2(E)},
\]

where \( s \in L^2_T(E) \), and \( f \) runs over the cutoff functions for the action by \( G \) on \( M \). (In fact, a set of cutoff functions whose supports cover \( M \) is enough.)

On the \( G \)-invariant part \( \overline{L^2_T(E)}^G \) of \( L^2_T(E) \), the expression

\[
(s, s')^f := (fs, fs')_{L^2(E)}
\]

defines an inner product, which is independent of the choice of \( f \) (as shown in the proof of Lemma 4.4). This turns \( L^2_T(E)^G \) into a Hilbert space.

---

\( ^3 \)This also holds for subspaces of \( L^2_T(E) \) with other kinds of transformation behavior under the action by \( G \), as long as \( \|(g \cdot s)(m)\| = \|s(m)\| \) for all \( g \in G, m \in M \) and \( s \) in such a subspace.
The first main result of this paper is that the $G$-invariant part of the transversally $L^2$-kernel of the deformed Dirac operator is finite-dimensional for large $t$, so that it can be used to define invariant quantization.

**Theorem 3.4.** For $t$ large enough, the $G$-invariant part $(\ker_{L^2_t}(D^L_t))^G$ of the vector space $\ker_{L^2_t}(D^L_t)$ is finite-dimensional.

This result will be proved in Sections 4–6. As a consequence, invariant quantization can be defined as follows.

**Definition 3.5.** The $G$-invariant geometric quantization of the action by $G$ on $(M,\omega)$ is the integer

$$Q(M,\omega)^G := \dim\left(\ker_{L^2_t}\left((D^L_t)_+\right)^G\right) - \dim\left(\ker_{L^2_t}\left((D^L_t)_-\right)^G\right),$$

for $t$ large enough.

The second main result of this paper is that Conjecture 1.2 is true for large enough powers of $L$.

**Theorem 3.6** (Quantization commutes with reduction for large $p$). There is a $p_0 \in \mathbb{N}$, such that for all integers $p \geq p_0$,

$$Q(M,p\omega)^G = Q(M_0,p\omega_0).$$

Theorem 3.6 will be proved in Sections 5–7.

**Remark 3.7.** Let $p \in \mathbb{N}$. If one includes the symplectic form $\omega$ in the notation for $L = L_\omega$ and $H = H_\omega$, then one has $L_{p\omega} = L^p_\omega$ and $H_{p\omega} = p^2 H_\omega$. Note that the invariant quantization $Q(M,p\omega)^G$ is defined via the operator

$$D^L_{p\omega} = D^L_{p\omega} + \frac{\sqrt{-1}}{2} c(X^H_{p\omega})$$

$$= D^L_{p\omega} + \frac{\sqrt{-1}}{2} p^2 t c(X^H_\omega)$$

$$= D^L_{p\omega}.$$

Since one takes $t$ large enough in Definition 3.5, one may therefore also use the deformed Dirac operator $D^L_{t\omega}$ of Definition 3.1 to define $Q(M,p\omega)^G$.

**Remark 3.8.** A particular consequence of Theorem 3.6 is that, for $t$ and $p$ large enough, the integer (30) is independent of the connection and Hermitian metric on $L$, the almost complex structure $J$ on $M$, the deformation parameter $t$, and the family of inner products on $\mathfrak{g}^*$ used (as long as they satisfy the assumptions listed). This is what one would expect based on the localization estimate in Proposition 6.1, as noted in Remark 6.2.

3.3. **Special cases.** Theorem 3.6 reduces to the main result in [20] in the cocompact case. In the setting of the Vergne conjecture [30] (a generalization of which was proved by Ma and Zhang [15]), it yields information about the reduction at zero. An example where neither $M/G$ nor $G$ needs to be compact is the case where $M = T^*G$ is the cotangent bundle of $G$. Also, because $M/G$ is not assumed to be compact, a version of the shifting trick applies.
Corollary 3.9. If $M/G$ is compact, Theorem 3.6 reduces to the case of Theorem 1.1 in [20] where no Ad$^*(G)$-invariant inner product on $\mathfrak{g}^*$ exists, and hence to the large $p$ case of Landsman’s conjecture [8].

Proof. If $M/G$ is compact, then $\text{Crit}_1(\mathcal{H})/G$ is always compact. Since all cutoff functions are compactly supported, all smooth sections are transversally $L^2$.

To see that one may take $t = 0$ in Definition 3.5 in the cocompact case, one can use the index theory from Section 4. If $M/G$ is compact, all $G$-invariant Sobolev norms of the same degree are equivalent. Hence the Sobolev spaces $W^t_p(M; L^p)^G$ defined in (16) are equal to $W^t_p(M; L^p)^G$ for all $t$. These spaces in turn are equal to the spaces $H^k_p(M, E)^G$ in the proof of Theorem 2.7 in [20]. These are isomorphic to the spaces $H^k_p(M, E)^G$ used there, via the map (2.22) in [20].

By taking $V = M$ in Proposition 4.7, one sees that the deformed Dirac operator $\tilde{D}_t^{L^p}$ is Fredholm for any $t$ and $p$. Since the vector field $X^\mathcal{H}_t$ has bounded norm, these deformed Dirac operators define a continuous family of Fredholm operators with respect to a single Sobolev norm, so that they have the same index. Hence the index of $\tilde{D}_t^{L^p}$ is independent of $t$, and equal to the index of $\tilde{D}^{L^p}$.

By (4.3) in [20], in the cocompact case the index of $\tilde{D}^{L^p}$ equals the index of the operator $P_{t}D^{L^p}$ used by Mathai and Zhang. Therefore, the invariant quantization $Q(M, p\omega)^G$ equals the left hand side of the equality in Theorem 1.1 in [20].

In the appendix to [20], Bunke shows that Theorem 1.1 in [20] implies the large $p$ case of Landsman’s conjecture. In fact, Conjecture 1.2 reduces to Landsman’s conjecture in the cocompact case. □

Remark 3.10. The case of Theorem 1.1 in [20] where $\mathfrak{g}^*$ admits an Ad$^*(G)$-invariant inner product and $p = 1$ is not a direct consequence of Theorem 3.6 but is closely related. Indeed, in that setting, the constant metric on $M \times \mathfrak{g}^* \to M$ defined by this inner product has the properties in Proposition 6.5 if one takes $V = M$. As noted in [20], the techniques from [29] generalize directly to that case.

Corollary 3.11. Consider the setting of the Vergne conjecture [30], where $G$ and $\text{Crit}(\mathcal{H})$ are compact\footnote{As noted on page 4 of [31], the set Crit(\mathcal{H}) is compact if \mu is proper and M is real-algebraic.} and $\mu$ is proper. If $G$ acts freely on $\mu^{-1}(0)$, then for $t$ and $p$ large enough, one has

$$\dim \left( \ker L^2((D_t^{L^p})_+) \right)^G - \dim \left( \ker L^2((D_t^{L^p})_-) \right)^G = Q(M_0, p\omega_0).$$

with $\ker L^2((D_t^{L^p})_{\pm})$ the spaces of $L^2$-sections in the kernels of the the even and odd parts $(D_t^{L^p})_{\pm}$ of $D_t^{L^p}$, respectively.

Proof. Since an Ad$^*(G)$-invariant inner product on $\mathfrak{g}^*$ exists if $G$ is compact, one can use a constant family of inner products on $\mathfrak{g}^*$. Then one has $\text{Crit}_1(\mathcal{H}) = \text{Crit}(\mathcal{H})$. Hence compactness of $\text{Crit}_1(\mathcal{H})$ is equivalent to (co)compactness of $\text{Crit}(\mathcal{H})$.

If $G$ is compact, then $f \equiv 1$ is a cutoff function. Hence $G$-invariant sections are transversally $L^2$ precisely if they are $L^2$. Therefore,

$$Q(M, p\omega)^G = \dim \left( \ker L^2((D_t^{L^p})_+) \right)^G - \dim \left( \ker L^2((D_t^{L^p})_-) \right)^G,$$

for $p$ and $t$ large enough. □
Let $T^*G$ be the cotangent bundle of $G$, equipped with the standard symplectic form $\omega$. Consider the action by $G$ on $T^*G$ induced by left multiplication (see also [7], p. 197).

**Corollary 3.12.** For $p$ large enough, one has

$$Q(T^*G, p\omega)^G = 1.$$ 

**Proof.** It was noted in Example 2.8 that $\text{Crit}_1(\mathcal{H}) \cong G$ is cocompact in this case. Furthermore, $G$ acts freely on $\mu^{-1}(0)$. Since $(T^*G)_0$ is a point, its quantization equals 1. Therefore, Theorem 3.6 implies that $Q(T^*G, p\omega)^G = 1$ for large enough $p$. □

**Remark 3.13.** Intuitively, one would expect the geometric quantization of $T^*G$ to be the space $L^2(G)$. This was made precise for compact $G$ by Paradan [26]. With Definition 3.5 of invariant quantization, one would expect the invariant quantization of $T^*G$ to be the space of $G$-invariant functions on $G$ that are square integrable after multiplication by a compactly supported function. This is the one-dimensional space of constant functions on $G$, in accordance with Corollary 3.12.

Because of the cocompactness assumption in [12, 16, 20], it was impossible to apply any version of the shifting trick. Indeed, even if $M/G$ is compact, the diagonal action by $G$ on $M \times O$ will not be, if $O$ is a noncompact coadjoint orbit. In the present setting, the shifting trick does apply to a certain extent.

4. A $G$-invariant index

This section contains some general index theory, which will be used to prove that invariant quantization is well-defined (Theorem 3.4). It will be shown that $G$-invariant quantization is the index of a Fredholm operator between Sobolev spaces defined in this section. Elliptic operators satisfying an invertibility condition outside a cocompact set will be shown to define such Fredholm operators. In Subsection 6.1, we will see that the deformed Dirac operator $D^p_t$ satisfies this condition, for $t$ large enough. This will complete the proof of Theorem 3.4.

4.1. Sobolev spaces. Let $M$ be a smooth manifold, and let $G$ be a unimodular Lie group acting properly on $M$. Let $dg$ be a Haar measure on $G$. Let $f$ be a smooth cutoff function for the action by $G$ on $M$ (see Subsection 3.2). Suppose that $M$ is equipped with a $G$-invariant Borel measure $dm$ (which will later be assumed to be given by the density associated to a $G$-invariant Riemannian metric). Let $E \to M$ be a $G$-vector bundle, equipped with a $G$-invariant Hermitian metric $(-,-)_E$. We denote the space of smooth sections of $E$ by $\Gamma^\infty(E)$, and the space of smooth, compactly supported sections of $E$ by $\Gamma^c(E)$. 

17
We will use Sobolev spaces constructed from spaces of transversally compactly supported sections of vector bundles.

**Definition 4.1.** The transversal support of a section $s$ of $E$ is the closure in $M/G$ of the set of orbits $O \in M/G$ such that there is a point $m \in O$ where $s(m) \neq 0$. If the transversal support of a section $s$ is compact, then $s$ is called transversally compactly supported.

The space of smooth, transversally compactly supported sections $s$ of $E$ will be denoted by $\Gamma_{\infty}^{tc}(E)$. The space of $G$-invariant sections in $\Gamma_{\infty}^{tc}(E)$ is denoted by $\Gamma_{\infty}^{tc}(E)^G$. For all $s \in \Gamma_{\infty}^{tc}(E)$, the product $fs$ is a smooth, compactly supported section of $E$.

Let $D : \Gamma_{\infty}(E) \to \Gamma_{\infty}(E)$ be a first order, $G$-equivariant, essentially self-adjoint, elliptic differential operator. We will write $\sigma_D$ for the principal symbol of $D$. Suppose that $E$ is $\mathbb{Z}/2$ graded, and write $E = E_+ \oplus E_-$ for the decomposition induced by this grading. Suppose that $D$ is odd with respect to the grading. Define the operator $	ilde{D}$:

$$
\tilde{D} : f\Gamma_{\infty}^{tc}(E)^G \to f\Gamma_{\infty}^{tc}(E)^G
$$

by

$$(31) \quad \tilde{D}(fs) = fDs$$

if $s \in \Gamma_{\infty}^{tc}(E)^G$.

Using the measure $dm$ on $M$ and the Hermitian metric $\langle -, - \rangle_E$ on $E$, one can define an $L^2$-inner product on compactly supported smooth sections of $E$. The operator $\tilde{D}$ is not symmetric with respect to this inner product in general. This fact will play a role in the proof of Proposition 6.6.

Consider the $k$'th Sobolev inner product on $f\Gamma_{\infty}^{tc}(E)^G$ defined by

$$(32) \quad (fs, fs')_{W^k_f(E)^G} := \sum_{j=0}^{k} (\tilde{D}^j(fs), \tilde{D}^j(fs'))_{L^2(E)},$$

for $s, s' \in \Gamma_{\infty}^{tc}(E)^G$.

**Definition 4.2.** The Sobolev space $W^{k}_f(E)^G$ is the completion of $f\Gamma_{\infty}^{tc}(E)^G$ in the inner product defined by (32).

**Definition 4.3.** The bounded operator

$$\tilde{D} : W^1_f(E)^G \to W^0_f(E)^G.$$

is the continuous extension of (31).

4.2. **Properties of the spaces $W^k_f(E)^G$.** The Sobolev spaces $W^k_f(E)^G$ are independent of $f$ if $G$ is unimodular. To prove this, we will use the measure $d\mathcal{O}$ on $M/G$ such that for all

$$
\int_{M/G} h_G(\mathcal{O}) d\mathcal{O} := \int_M (m)^2 h_G(Gm) dm.
$$

A version of this measure for non-unimodular groups $G$ also exists, but then the invariance condition on $dm$ is replaced by a condition involving the modular function.
\( h \in C_c(M), \)
\[
\int_M h(m) \, dm = \int_{M/G} \int_G h(g\tau(O)) \, dg \, dO,
\]
for any Borel section \( \tau : M/G \to M. \)

**Lemma 4.4.** Suppose \( G \) is unimodular. Let \( f_0 \) and \( f_1 \) be two cutoff functions for the action by \( G \) on \( M \). Then the map given by
\[
f_0 s \mapsto f_1 s
\]
for \( s \in \Gamma_{tc}^\infty(E)^G \) induces a unitary isomorphism \( W_k^f(E)^G = W_k^{f_1}(E)^G \).

**Proof.** The map (34) is a bijection from \( \Gamma_{tc}^\infty(E)^G \) to \( \Gamma_{tc}^\infty(E)^G \), because \( G \)-invariant sections are determined by their restrictions to \( \text{supp}(f_0) \) or \( \text{supp}(f_1) \).

For \( k = 0 \), one finds for all \( s, s' \in \Gamma_{tc}^\infty(E)^G \),
\[
(f_j s, f_j s')_{W_0^f(E)^G} = \int_M f_j(m)^2 \langle s(m), s'(m) \rangle_E \, dm
= \int_{M/G} \int_G f_j(g\tau(O))^2 \langle s(g\tau(O)), s'(g\tau(O)) \rangle_E \, dg \, dO.
\]
By the property (29) of cutoff functions, and \( G \)-invariance of \( s \) and \( s' \), the latter expression equals
\[
\int_{M/G} \langle s(\tau(O)), s'(\tau(O)) \rangle_E \, dO,
\]
which is independent of \( j \).

For general \( k \), one notes that for all \( s, s' \in \Gamma_{tc}^\infty(E)^G \),
\[
(\tilde{D}^k f_j s, \tilde{D}^k f_j s')_{L^2(E)} = (f_j D^k s, f_j D^k s')_{L^2(E)},
\]
which by the argument above is independent of \( j \). This implies that
\[
(f_0 s, f_0 s')_{W_k^f(E)^G} = (f_1 s, f_1 s')_{W_k^{f_1}(E)^G}
\]
for all \( k \), as required. \( \square \)

Since we assumed that \( G \) is unimodular, the Sobolev spaces \( W_k^f(E)^G \) are indeed independent of \( f \).

An analogue of the Rellich lemma holds for the restricted Sobolev spaces \( W_k^f(E|_V)^G \), if \( V \subset M \) is a \( G \)-invariant, relatively cocompact open subset of \( M \).

**Lemma 4.5.** Let \( V \subset M \) be \( G \)-invariant, relatively cocompact and open. Then for all \( k \geq 0 \), the inclusion map
\[
W_{k+1}^f(E|_V)^G \hookrightarrow W_k^f(E|_V)^G
\]
is compact.

**Proof.** Let \( Z \) be the intersection of \( V \) with the interior of \( \text{supp}(f) \). Restricting to \( Z \) is an injective map
\[
W_k^f(E|_V)^G \hookrightarrow W_k^f(E|_Z).
\]
This map is an isometry if \( W_k^f(E|_Z) \) is defined with respect to the Sobolev inner product coming from the elliptic operator \( D - \sigma_D(df) \). Since \( Z \) is relatively compact, all \( k \)'th Sobolev
norms on sections of $E|_V$ are equivalent, and one may as well use the one defined by $D - \sigma_D(df)$.

Consider the diagram

\[
\begin{array}{ccc}
W^{k+1}(E|_Z) & \xhookrightarrow{\subset} & W^k(E|_Z) \\
\uparrow & & \uparrow \\
W^{k+1}(E|_V) & \xhookrightarrow{\subset} & W^k(E|_V)^G.
\end{array}
\]

We have seen that the vertical inclusion maps are isometric, and the Rellich lemma implies that the top inclusion map is compact. Hence the bottom inclusion is compact as well, as required. □

4.3. Free actions. Suppose for now that the action by $G$ on $M$ is free, in addition to being proper. (The material in this subsection will be used in Section 7, where $M$ is replaced by a neighborhood of $\mu^{-1}(0)$.) For free actions, one has the induced vector bundle

\[E_G \to M/G,\]

such that $q^*E_G \cong E$, with $q : M \to M/G$ the quotient map. The Hermitian metric on $E$ induces one on $E_G$. The induced operator $D^G$ on sections of $E_G$,

\[D^G : \Gamma^\infty(E_G) \to \Gamma^\infty(F_G),\]

is again essentially self-adjoint and elliptic. Using this measure and this metric, and the operator $D^G$, one can define Sobolev spaces $W^k(E_G)$ of sections of $E_G$. These are the completions of $\Gamma^\infty_c(E_G)$ in the inner product given by

\[(s, s')_{W^k(E_G)} := \sum_{j=0}^k (\langle (D^G)^j s, (D^G)^j s' \rangle_{L^2(E_G)}),\]

for $s, s' \in \Gamma^\infty_c(E_G)$. Then the operator $D^G$ extends to a bounded operator

\[D^G : W^1(E_G) \to W^0(E_G).\]

**Lemma 4.6.** The composition

\[\Gamma^\infty_c(E_G) \xrightarrow{q^*} \Gamma^\infty_c(E)^G \xrightarrow{f} f\Gamma^\infty_c(E)^G\]

extends to a unitary isomorphism

\[\psi : W^k(E_G) \xrightarrow{\cong} W^k_f(E)^G.\]

**Proof.** We only need to check that multiplication by $f$ composed with $q^*$ is an isometry. The argument is analogous to the proof of Lemma 4.4. In the same way as there, one first computes that for all $s, s' \in \Gamma^\infty_c(E_G)$,

\[(fq^*s, fq^*s')_{W^0_f(E)^G} = (s, s')_{W^0(E_G)}.\]

For general $k$, one then notes that for all such $s, s'$,

\[\big(\tilde{D}^k fq^*s, \tilde{D}^k fq^*s'\big)_{L^2(E)} = \big(fq^*(D^G)^k s, fq^*(D^G)^k s'\big)_{L^2(E)} = \big((D^G)^k s, (D^G)^k s'\big)_{L^2(E_G)}.\]

This implies that

\[(fq^*s, fq^*s')_{W^k_f(E)^G} = (s, s')_{W^k(E_G)};\]
for all $k$, as required. \hfill \Box

The operators $D$, $\tilde{D}$ and $D^G$ are related by the commutative diagram

$$
\begin{array}{ccc}
\Gamma_c^\infty(E_G) & \xrightarrow{\varphi^*} & \Gamma_c^\infty(E)^G \\
\downarrow{D^G} & & \downarrow{D} \\
\Gamma_c^\infty(F_G) & \xrightarrow{\varphi^*} & \Gamma_c^\infty(F)^G
\end{array}
$$

In other words, the unitary isomorphism $\psi$ from Lemma 4.6 intertwines the operators $D^G$ and $\tilde{D}$.

4.4. The Fredholm property. In [2], Anghel gives a criterion for an elliptic, self-adjoint differential operator on a noncompact manifold to be Fredholm: an $L^2$-norm estimate outside a compact subset of the manifold. This generalizes Gromov and Lawson’s results for Dirac operators in Section 3 of [9]. In our setting, where one considers operators between the Sobolev spaces $W^1_j(E)^G$ and $W^0_j(E)^G$, the analogous estimate outside a cocompact subset is sufficient.

**Proposition 4.7.** Suppose $M$ carries a $G$-invariant Riemannian metric. Let the measure $dm$ be given by the Riemannian density, and suppose $M$ is complete. Let $K \subset M$ be a cocompact subset, and suppose there is a $C > 0$ such that for all $s \in \Gamma_c^\infty(E)^G$ with support disjoint from $K$,

$$
\|\tilde{D}fs\|_{L^2(E)} \geq C\|fs\|_{L^2(E)}.
$$

Then the operator $\tilde{D} : W^1_j(E)^G \to W^0_j(E)^G$ is Fredholm.

**Proof.** Let $(s_j)_{j=1}^\infty$ be a sequence in $\Gamma_c^\infty(E)^G$, such that

- the sequence $(fs_j)_{j=1}^\infty$ is bounded in $W^0_j(E)^G$;
- the sequence $(\tilde{D}fs_j)_{j=1}^\infty$ converges in $W^0_j(E)^G$.

It is enough to prove that there is a subsequence $(s_{jk})_{k=1}^\infty$ such that $(fs_{jk})_{k=1}^\infty$ converges in $W^0_j(E)^G$, i.e. in $L^2(E)$. This implies that $\tilde{D}$ has finite-dimensional kernel and closed range; see for example the last two paragraphs of the proof of Theorem 2.1 in [2].

Let $U$ be a relatively cocompact open neighborhood of $K$. Let $Z$ be the intersection of $U$ with the interior of the support of $f$. Then $Z$ is relatively compact. Hence all Sobolev norms of the same degree on $\Gamma_c^\infty(E)$ have equivalent restrictions to $\Gamma_c^\infty(E|_Z)$. The sequence $(fs_j)_{j=1}^\infty$ is bounded in $W^1_j(E)^G$, so its restriction to $Z$ is bounded with respect to any first Sobolev norm. Therefore, the Rellich lemma yields a subsequence $(s_{jk})_{k=1}^\infty$ such that $(fs_{jk}|_Z)$ converges in $L^2(E|_Z)$.

Because $f|_U$ is zero outside $Z$, the sequence $(fs_{jk}|_U)$ converges in $L^2(E|_U)$, and hence in $W^0_j(E|_U)^G$. Let $\chi \in C^\infty(M)^G$ be a smooth, $G$-invariant function, with values in $[0, 1]$, such that

- $\chi \equiv 1$ on $K$;
- $\chi \equiv 0$ on $M \setminus U$.

Then the sequence $(\chi fs_{jk})_{k=1}^\infty$ converges in $W^0_j(E)^G$. 


To show that the sequence \( (1 - \chi) f s_{j_k} \) converges in \( W^0_\infty(E)^G \), one first notes that the commutator \([D, \chi]\) equals \( \sigma_D(d\chi) \), with \( \sigma_D \) the principal symbol of \( D \). Hence for all \( s \in \Gamma_{tc}^\infty(E)^G \),
\[
[D, \chi] f s = f[D, \chi] s = \sigma_D(d\chi) f s.
\]
Since \( \sigma_D(d\chi) \) is \( G \)-equivariant, and zero outside the relatively cocompact set \( U \setminus K \), the operator
\[
\sigma_D(d\chi) : W^0_\infty(E)^G \to W^0_\infty(E)^G
\]
is bounded. Because \( \chi \) is supported in \( U \), convergence of \( (f s_{j_k} |_U) \) and boundedness of \( \sigma_D(d\chi) \) imply convergence of \( ([\tilde{D}, \chi] f s_{j_k}) _{k=1}^\infty \).

Because \( 1 - \chi \) is supported outside \( K \), the assumption (35) implies that for all \( k, l \),
\[
C \| (1 - \chi) f (s_{j_k} - s_{j_l}) \|_{W^0_\infty(E)^G} \leq \| \tilde{D} ((1 - \chi) f (s_{j_k} - s_{j_l})) \|_{W^0_\infty(E)^G} = \| (1 - \chi) \tilde{D} (f (s_{j_k} - s_{j_l})) - [\tilde{D}, \chi] f (s_{j_k} - s_{j_l}) \|_{W^0_\infty(E)^G} \leq \| \tilde{D} (f (s_{j_k} - s_{j_l})) \|_{W^0_\infty(E)^G} + \| [\tilde{D}, \chi] f (s_{j_k} - s_{j_l}) \|_{W^0_\infty(E)^G}.
\]
Both terms in the latter expression become arbitrarily small, since \( (\tilde{D} f s_{j_k}) _{j=1}^\infty \) and \( ([\tilde{D}, \chi] f s_{j_k}) _{k=1}^\infty \) converge. Hence the sequence \( (1 - \chi) f s_{j_k} \) converges as well, we conclude that \( (f s_{j_k}) _{k=1}^\infty \) converges in \( W^0_\infty(E)^G \), as required.

4.5. The \( G \)-invariant index. Suppose the conditions of Proposition 4.7 are satisfied. Then the Fredholm operator \( \tilde{D} \) has a well-defined index. The proof of Theorem 3.4 is based on the following explicit description of this index, in terms of the transversally \( L^2 \)-kernel of \( D \) (see Definition 3.2).

**Proposition 4.8.** In the situation of Proposition 4.7, the \( G \)-invariant part \( (\ker_{L^2_t} (D))^G \) of the transversally \( L^2 \)-kernel of \( D \) is finite-dimensional, and one has
\[
(36) \quad \text{index}(\tilde{D}) = \dim (\ker_{L^2_t} (D_+))^G - \dim (\ker_{L^2_t} (D_-))^G.
\]
A proof of this fact is given in Appendix A.

**Definition 4.9.** In the setting of Proposition 4.7, the \( G \)-invariant index of \( D \) is the number (36). It is denoted by \( \text{index}_G(D) \).

In Subsection 6.1, it is shown that the deformed Dirac operator \( D_t^{L^p} \) satisfies the hypotheses of Proposition 4.7 for \( t \) large enough. Hence invariant quantization is well-defined as its \( G \)-invariant index, and Theorem 3.4 follows. Note that this is different from the cocompact situation considered in [20], where the undeformed Dirac operators were already Fredholm on suitable Sobolev spaces.

5. The square of the deformed Dirac operator

As in [20, 29], the square of the deformed Dirac operator \( D_t^{L^p} \) plays an important role. An expression for this square is given in Theorem 5.1, which generalizes Theorem 1.6 in [29]. As in Corollary 1.7 in [29], a term involving Lie derivatives vanishes on \( G \)-invariant sections,
and one is left with the square of $D^{L_p}$ plus order zero terms. This is recorded in Corollary 5.2.

In the compact and cocompact cases, the zero order terms in the square of the deformed Dirac operator are automatically bounded. This is not true for non-cocompact manifolds. In Subsection 5.2 it will be shown that, for a well-chosen family of inner products on $g^*$, these zero order terms satisfy an estimate that can be used to localize the invariant index of $D_t^{L_p}$ to neighborhoods of Crit$(\mathcal{H})$ and $\mu^{-1}(0)$.

5.1. A Bochner formula. We first introduce some operators and vector fields that will be used in the expression for the square of $D_t^{L_p}$. Recall the notation of Subsection 2.2. In particular, we chose an orthonormal frame $\{h_1, \ldots, h_{d_G}\}$ for the bundle $M \times g^* \to M$, with respect to the given $G$-invariant metric on this bundle. Let $\{h_1^*, \ldots, h_{d_G}^*\}$ be the dual frame of $M \times g \to M$ as in (19). For each $j$, consider the operator $L_{h_j^*}$ on $\Omega^{0,*}(M; L^p)$ given by

$$(L_{h_j^*} s)(m) = (L_{h_j^*(m)} s)(m),$$

for all $s \in \Omega^{0,*}(M; L^p)$ and $m \in M$. We will use the fact that $L_{h_j^*}$ annihilates $G$-invariant sections.

In addition, for any vector field $v \in \mathfrak{X}(M)$, consider the commutator vector field $[v, (h_j^*)^M]$, given by

$$[v, (h_j^*)^M](m) = [v, h_j^*(m)^M](m).$$

Here $h_j^*(m)^M$ is the vector field induced by $h_j^*(m) \in g$, and $[-,-]$ is the Lie bracket of vector fields. Importantly, for fixed $m$, the vector fields $V_j$ in (22) and $h_j^*(m)^M$ are equal at the point $m$, but not necessarily at other points.

Finally, the one-form $\langle \mu, Th_j^* \rangle \in \Omega^1(M)$ is defined by

$$\langle \langle \mu, Th_j^* \rangle_m, v \rangle = \langle \mu(m), T_m h_j^*(v) \rangle$$

for all $m \in M$ and $v \in T_m M$. The dual vector field associated to $\langle \mu, Th_j^* \rangle$ via the Riemannian metric will be denoted by $\langle \mu, Th_j^* \rangle^*$. 

Theorem 5.1. Let $p \in \mathbb{N}$ and $t \in \mathbb{R}$ be given. The square of the deformed Dirac operator $D_t^{L_p}$ equals

$$\left( D_t^{L_p} \right)^2 = \left( D^{L_p} \right)^2 + t A + 4\pi p t \mathcal{H} + \frac{t^2}{4} \|X_1^H\|^2 - 2\sqrt{-1} t \sum_{j=1}^{d_G} \mu_j L_{h_j^*},$$

where $A$ is a vector bundle endomorphism of $\bigwedge^{0,*} T^* M \otimes L^p$, equal to $A = A_1 + A_2 + A_3$, with the endomorphisms $A_n$ defined as follows. Let $e_1, \ldots, e_{d_M}$ be a local orthonormal frame for $TM$, and write $e_k^{1,0}$ for the component of the complexification of $e_k$ in $T^{1,0} M$. Let $\nabla$ be the Levi–Civita connection on $TM$, and $\nabla^{T^{1,0}}$ the induced connection on $T^{1,0} M$. The endomorphism $A_1$ is the Tian–Zhang tensor that appears in Theorem 1.6 in [29].

$$A_1 = \frac{\sqrt{-1}}{4} \sum_{k=1}^{d_M} c(e_k) c(\nabla_{e_k} X_1^H) - \frac{\sqrt{-1}}{2} \text{tr} \left( \nabla^{T^{1,0}} X_1^H \big|_{T^{0,1} M} \right)$$

$$+ \frac{1}{2} \sum_{j=1}^{d_G} (\sqrt{-1} c(JV_j) c(V_j) + \|V_j\|^2).$$

23
The tensors $A_2$ and $A_3$ appear because the orthonormal frame $\{h_1, \ldots, d_{d_G}\}$ of $M \times g^* \rightarrow M$ may not be constant:

\begin{equation}
A_2 = \frac{\sqrt{-1}}{2} \sum_{j=1}^{d_G} c(\langle \mu, Th_j^* \rangle^*) c(V_j) + \sqrt{-1} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \langle \langle \mu, Th_j^* \rangle_m, e_k^0 \rangle^*(V_j, e_k^0) ;
\end{equation}

and

\begin{equation}
A_3 = -\frac{\sqrt{-1}}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j c(e_k) c([e_k, (h_j^*)^M - V_j]) \\
- \sqrt{-1} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j ([e_k^0, (h_j^*)^M - V_j], e_k^0).
\end{equation}

Since the operators $L_{h_j^*}$ in (37) map $G$-invariant sections to zero, one obtains the following analogue of Corollary 1.7 in [29].

**Corollary 5.2.** For $p \in \mathbb{N}$ and $t \in \mathbb{R}$, the restriction of $(D^{LP})^2$ to $\Omega^{0,*}(M; L)^G$ equals

\begin{equation}
(D^{LP})^2 + tA + 4\pi ptH + \frac{t^2}{4} \|X_H\|^2.
\end{equation}

**Remark 5.3.** If there is an Ad$^*(G)$-invariant inner product on $g^*$, then Theorem 5.1 and Corollary 5.2 imply Theorem 1.6 and Corollary 1.7 in [29], respectively. Indeed, in that case one may use a constant family of inner products on $g^*$, and a constant orthonormal frame for $M \times g^* \rightarrow M$. Then $X_H^1 = X_H^0$. Since $h_j^*$ is constant, one has $T_m h_j^* = 0$, so $\langle \mu, Th_j^* \rangle = 0$. Also, one has $h_j^*(m)^M = V_j$ for all $m \in M$, so that $[v, (h_j^*)^M - V_j] = 0$ for any vector field $v$. Therefore, $A_2 = A_3 = 0$, and the remaining part of (37) becomes the equality in Theorem 1.6 in [29].

Note that, even if $g^*$ admits an Ad$^*(G)$-invariant inner product, one may wish to use a non-constant family of inner products, for example to apply Lemma 2.9 or Proposition 6.5. Then one should use Theorem 5.1 and Corollary 5.2 rather than Theorem 1.6 and Corollary 1.7 in [29].

5.2. Proof of the Bochner formula. We give an outline of a proof of Theorem 5.1 here, and provide more details in Appendix 13.

**Proof of Theorem 5.1.** As in the equality (1.26) in [29], one finds that

\begin{equation}
(D_{LP}^t)^2 = (D^{LP})^2 + \frac{\sqrt{-1}}{2} t \sum_{k=1}^{d_M} c(e_k) c(\nabla e_k X_1^H) - \sqrt{-1} t \nabla X_1^H + \frac{t^2}{4} \|X_1^H\|^2.
\end{equation}

The bulk of the proof of Theorem 5.1 consists of computing an expression for the first order term $\nabla X_1^H$. Because of (39), one has

\[ \nabla X_1^H = 2 \sum_{j=1}^{d_G} \mu_j \nabla V_j. \]

For all $m \in M$, one has $V_j(m) = h_j^*(m)^M_m$. Since covariant derivatives $\nabla \nu$ depend locally on the vector field $\nu$, Lemma 1.5 in [29] therefore implies that for all $s \in \Omega^{0,*}(M; L)$ and all
\( m \in M, \)

\[
\begin{align*}
(43) \quad (\nabla V_j s)(m) &= (\nabla h_j^*(m) s)(m) = (L h_j^*(m) s)(m) + 2\pi \sqrt{-1} p \mu_j(m) s(m) \\
&+ \frac{1}{4} \sum_{k=1}^{d_M} (c(e_k) c(\nabla e_k h_j^*(m) M)) (m) s(m) \\
&+ \frac{1}{2} \text{tr} \left( \nabla^{T^{1,0} M} h_j^*(m) M_{T^{1,0} M} \right) (m) s(m).
\end{align*}
\]

Multiplying (43) by \( 2\mu_j \) and summing over \( j \) yields

\[
(44) \quad (\nabla X^h_{k,t} s)(m) &= 2 \sum_{j=1}^{d_G} \mu_j(m) (L h_j^*(m) s)(m) + 4\pi \sqrt{-1} p \mathcal{H}(m) s(m) \\
&+ \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(m) (c(e_k) c(\nabla e_k h_j^*(m) M)) (m) s(m) \\
&+ \sum_{j=1}^{d_G} \mu_j(m) \text{tr} \left( \nabla^{T^{1,0} M} h_j^*(m) M_{T^{1,0} M} \right) (m) s(m).
\]

In Proposition [3.1] in Appendix [3] it is deduced from (44) that

\[
(45) \quad (\nabla X^h_{k,t} s)(m) &= 2 \sum_{j=1}^{d_G} \mu_j(m) (L h_j^*(m) s)(m) + 4\pi \sqrt{-1} p \mathcal{H}(m) s(m) \\
&+ \left( \frac{1}{4} \sum_{k=1}^{d_M} c(e_k) c(\nabla e_k X^h_{1,t} \mathcal{H}) + \frac{1}{2} \sum_{j=1}^{d_G} (-c(JV_j) c(V_j) + \sqrt{-1} \|V_j\|^2) + \frac{1}{2} \text{tr} \left( \nabla^{T^{1,0} M} X^h_{1,t} M_{T^{1,0} M} \right) \right) (m) s(m) \\
&+ \sqrt{-1} (A_2 + A_3) (m) s(m),
\]

with \( A_2 \) and \( A_3 \) as in (39) and (40).

The equality (37) follows by inserting (43) into (42), and using the definition (38) of the operator \( A_1. \)

6. Localizing the deformed Dirac operator

The key steps in the proofs of the main results in this paper, Theorems 3.4 and 3.6, is showing that the kernel of \( D^L_{1,p} \) localizes to neighborhoods of \( \text{Crit}_1(\mathcal{H}) \) and \( \mu^{-1}(0) \), respectively. The localization estimates used to do this are Propositions 6.1 and 6.3, which will be proved in this section. Recall that the set \( \text{Crit}_1(\mathcal{H})/G \), and hence the set \( \mu^{-1}(0)/G \), was assumed to be compact.

6.1. Localization estimates. Let \( \Omega^0_{tc}(M; L^p) \) be the space of smooth sections of the vector bundle \( \bigwedge^0_* T^* M \otimes L^p \to M \) with compact transversal supports. Consider the Sobolev inner product (32) on \( \Omega^0_{tc}(M; L^p) \), defined with respect to the operator \( D = D^L_{1,p} \) on \( E = \bigwedge^0_* T^* M \otimes L^p \). We write \( (-, -)_{k,t} \) for the resulting \( k \)’th Sobolev inner product, and \( \| \cdot \|_{k,t} \) for the induced norm. In addition, we write \( (-, -)_{0,t} := (-, -)_{k,0} \) and \( \| \cdot \|_{k} := \| \cdot \|_{k,0} \). Note that \( (-, -)_{0,t} = (-, -)_{0} \) is the \( L^2 \) inner product for all \( t \).
Let
\[
W^k_J(M; L^p)^G := W^k_J(T^*M \otimes L^p)^G,
\]
be the completion of \(f\Omega^0(M; L^p)\) in the inner product \((-,-)_{\kappa,t}\). Then the deformed Dirac operator \(\tilde{D}^L_t\) induces a bounded operator
\[
\tilde{D}^L_t : W^1_J(M; L^p)^G \rightarrow W^0_J(M; L^p)^G.
\]

The first localization estimate will be used to prove Theorem \(3.4\) which states that invariant quantization is well defined.

**Proposition 6.1.** There is a family of inner products on \(g^*\), a relatively cocompact open neighborhood \(V\) of \(\text{Crit}_1(H)\), and there are constants \(C > 0, b > 0\) and \(t_0 > 0\), such that for every \(t \geq t_0\) and all \(s \in \Omega^0_{tc}(M; L)^G\) with support disjoint from \(V\),
\[
\|\tilde{D}^L_t(fs)\|^2_0 \geq C\left(\|fs\|^2_1 + (t - b)\|fs\|^2_0\right).
\]

**Remark 6.2.** Based on Proposition 6.1, one expects Definition 3.5 to be independent of the connection and metric on \(L\), the almost complex structure on \(M\), the deformation parameter \(t\), and the family of inner products on \(g^*\). Indeed, it means that the index of \(\tilde{D}^L_t\) is determined on the relatively cocompact set \(V\). On \(V\), the norm of \(X_t^H\) is bounded, and all \(G\)-invariant Sobolev norms are equivalent. Hence the family of operators \((\tilde{D}^L_t|_V)_{t \in \mathbb{R}}\) depends continuously on \(t\), with respect to a fixed Sobolev norm.

Furthermore, the deformed Dirac operators defined by two families of metrics on \(g^*\) have bounded difference on \(V\), with respect to the \(L^2\)-norm. Hence, by Lemma 4.5, this difference is a compact operator between the first and zero’th Sobolev spaces in question.

The second localization estimate is a further localization to neighborhoods of \(\mu^{-1}(0) \subset \text{Crit}_1(H)\), where one uses tensor powers of the line bundle \(L\).

**Proposition 6.3.** There is an equivariant family of inner products on \(g^*\) such that for any \(G\)-invariant\(^6\) open neighborhood \(U\) of \(\mu^{-1}(0)\), there are \(p_0 \in \mathbb{N}, t_0 > 0\) and \(C > 0, b > 0\), such that for all integers \(p \geq p_0\), real numbers \(t \geq t_0\) and all \(s \in \Omega^0_{tc}(M; L)^G\) with support disjoint from \(U\),
\[
\|\tilde{D}^L_t(fs)\|^2_0 \geq C\left(\|fs\|^2_1 + (t - b)\|fs\|^2_0\right).
\]

Importantly, one may use the same family of inner products on \(g^*\) in Propositions 6.1 and 6.3, namely the family constructed in Proposition 6.5.

**Remark 6.4.** It was assumed in Subsection 3.1 that 0 is a regular value of the momentum map \(\mu\). In fact, Proposition 6.1 also implies that quantization commutes with reduction if 0 is not a value of \(\mu\) at all. Indeed, one can then take \(U = \emptyset\) in Proposition 6.3 and conclude that \(\tilde{D}^L_t\) has trivial kernel for \(t > b\) and large \(p\). Therefore, by Proposition 4.8
\[
Q(M, p\omega)^G = 0
\]
in such cases. From now on, we suppose that \(0 \in \mu(M)\).

\(^6\)In \cite{29}, the analogous estimate is proved for any open neighborhood of \(\mu^{-1}(0)\), not necessarily \(G\)-invariant. If \(\mu^{-1}(0)\) is compact, any neighborhood contains a \(G\)-invariant one. In the noncompact case, one has to assume \(G\)-invariance. This is no restrictive assumption, however.
Before proving Propositions 6.1 and 6.3, we show how Propositions 6.1 and 4.7 imply Theorem 3.4.

Proof of Theorem 3.4. Let \( C, b \) and \( t_0 \), as well as a family of inner products on \( g^* \) and a set \( V \) as in Proposition 6.1 be given. For \( t \) larger than both \( t_0 \) and \( b + 1 \) and for \( s \in \Omega_{t_0^*}^0(M; L)^G \) with support disjoint from \( V \), one then has
\[
\| \tilde{D}_t^L(f s) \|_2^2 \geq C(\| f s \|_1^2 + (t - b)\| f s \|_0^2) \geq C\| f s \|_0^2.
\]
Proposition 4.7 therefore implies that \( \tilde{D}_t^L \) is Fredholm. Because of Proposition 4.8, the vector space \( (\ker L_2 T_2(D_t^L))^G \) is finite-dimensional, and the integer
\[
\dim \left( \ker L_2^g ((D_t^L)_+) \right)^G - \dim \left( \ker L_2^g ((D_t^L)_-) \right)^G
\]
is the Fredholm index of \( \tilde{D}_t^L \). \qed

6.2. Choosing the family of inner products on \( g^* \). One of the main difficulties in generalizing Tian and Zhang’s localization theorem to the non-cocompact setting, is that the operator \( A \) in (41) may not be bounded below. We overcome this difficulty by rescaling the family of inner products on \( g^* \) by a \( G \)-invariant positive function \( \psi \) on \( M \), as discussed in Subsection 2.5. Because of Lemma 2.10, rescaling the inner products in this way results in replacing the vector field \( X^H_{1^t} \) by \( \psi X^H_{1^t} \). A version of this technique, of deforming a Dirac operator by Clifford multiplication by a vector field, and then rescaling this vector field by a function, was used by Braverman [5] in the case where \( G \) is compact.

Fix a relatively cocompact open neighborhood \( V \) of \( \text{Crit}_1(\mathcal{H}) \). This is where cocompactness of \( \text{Crit}_1(\mathcal{H}) \) is used. We will also use the \( G \)-invariant, positive smooth function \( \eta \) on \( M \) defined by
\[
\eta(m) = \int_G f(gm)\|df\|(gm) \, dg,
\]
for \( m \in M \).

**Proposition 6.5.** The \( G \)-invariant metric on the bundle \( M \times g^* \to M \) can be chosen in such a way that for all \( m \in M \setminus V \),
\[
\mathcal{H}(m) \geq 1;
\]
\[
\|X^H_{1^t}(m)\| \geq 1 + \eta(m),
\]
and there is a positive constant \( C \), such that for all \( m \in M \), the operator \( A_m \) on \( \bigwedge^{0,*} T_m^* M \otimes L^p_m \) is bounded below by
\[
A_m \geq -C(\|X^H_{1^t}(m)\|^2 + 1).
\]

**Proof.** We outline a proof here, and refer to Appendix C for certain details.

By Lemma C.1 in Appendix C, there is an open cover \( \{\tilde{U}_i\}_i \) of \( M \) such that
- every open set \( \tilde{U}_i \) admits a local orthonormal frame for \( TM \);
- every compact subset of \( M \) intersects finitely many of the sets \( \tilde{U}_i \) nontrivially;
- there is a relatively compact subset \( U_i \subset \tilde{U}_i \) for all \( i \), such that \( \tilde{U}_i \subset \tilde{U}_i \), and \( \bigcup_i U_i = M \).
Fix a local orthonormal frame \( \{ e_1^l, \ldots, e_M^l \} \) for \( TM \) on every set \( \tilde{U}_l \). For all \( k \) and \( l \), let \((e_k^l)^{1,0}\) be the component of the complexification of \( e_k^l \) in \( T^{1,0} M \). Let \( W \subset M \) be a subset whose intersection with every nonempty cocompact subset of \( M \) is nonempty and compact.

Consider any \( G \)-invariant metric on \( M \times g^* \to M \), let \( \mathcal{H} \) be the associated function \( (15) \), and let \( X_1^{\mathcal{H}} \) be the vector field defined by \( (17) \). Fix an orthonormal frame \( \{ h_1, \ldots, h_d \} \) of \( M \times g^* \to M \), and let \( \{ h_1^*, \ldots, h_d^* \} \) be the dual frame of \( M \times g \to M \) as in \( (19) \).

By Lemma C.2 there are positive, \( G \)-invariant, continuous functions \( F_1, F_2, F_3 \in C(M)^G \) such that for all \( m \in W \), and for all \( l \) such that \( m \in U_l \), and all \( j = 1, \ldots, d \) and \( k = 1, \ldots, d_M \),

\[
\| \nabla e_k^l X_1^{\mathcal{H}} \| (m) \leq F_1(2m); \\
\| \nabla (e_k^l)^{1,0} X_1^{\mathcal{H}} \| (m) \leq F_1(2m); \\
\| \langle \mu, T h_j^* \rangle \| (m) \leq F_2(2m); \\
\| (e_k^l)^{1,0} (h_j^*)^M - V_j \| (m) \leq F_3(m); \\
\| e_k^l \| (m) \leq F_3(2m).
\]

Let \( N := \| X_1^{\mathcal{H}} \|^2 \) denote the norm-squared function of \( X_1^{\mathcal{H}} \). Define the continuous, nonnegative, \( G \)-invariant functions \( \varphi_0 \) and \( \varphi_1 \) on \( M \) by

\[
\varphi_0 = \min \left( \mathcal{H}, \frac{N^{1/2}}{F_1}, \min_j \frac{N}{\| V_j \| F_2}, \min_j \frac{N}{| \mu_j | F_3} \right); \\
\varphi_1 = \min \left( \frac{N^{1/2}}{\| \mu_j \| | V_j |} \right).
\]

Note that the functions \( \varphi_0 \) and \( \varphi_1 \) are strictly positive outside \( \text{Crit}_1(\mathcal{H}) \). Therefore, by Lemma C.3 there is a \( G \)-invariant, positive, smooth function \( \psi \) on \( M \), such that on \( M \setminus M \),

\[
\psi^{-1} \leq \varphi_0; \\
\| \delta(\psi^{-1}) \| \leq \varphi_1.
\]

As in Subsection 2.5 consider the family of inner products \( \{ \psi(m)(-, -)_m \}_{m \in M} \) on \( g^* \). This is again a \( G \)-invariant smooth metric on \( M \times g^* \to M \). We will show that this metric has the desired properties. As in Lemma 2.10 let \( \mathcal{H}_\psi \) be the corresponding norm-squared function of the momentum map \( \mu \), so that \( X_1^{\mathcal{H}_\psi} = \psi X_1^\mathcal{H} \). Write \( N_\psi := \| X_1^{\mathcal{H}_\psi} \|^2 = \psi^{-2} N \).

First of all, note that, outside \( V \), one has

\[
\mathcal{H}_\psi = \psi \mathcal{H} \geq \varphi_0^{-1} \mathcal{H} \geq 1; \\
\| X_1^{\mathcal{H}_\psi} \| = \psi \| X_1^\mathcal{H} \| \geq \varphi_0^{-1} \| X_1^\mathcal{H} \| \geq 1 + \eta,
\]

so \( (38) \) and \( (39) \) follow.

We now turn to a proof of the lower bound \( (40) \) for the operator \( A = A_1 + A_2 + A_3 \). We will find a bound for each of the operators \( A_n \) separately. Write \( A_n^\psi \) for the operators in \( (38) - (40) \), with \( \mathcal{H} \) replaced by \( \mathcal{H}_\psi \).

\[\text{If } f_1 \text{ and } f_2 \text{ are functions, and } f_2 \text{ is nonnegative, then by min}(f_1, 1/f_2) \text{ we mean the function equal to min}(f_1, 1/f_2) \text{ where } f_2 \text{ is nonzero, and to } f_1 \text{ where } f_2 \text{ is zero.}\]
We will use the orthonormal frame of $M \times g^* \to M$ made up of the functions
\[ h_j^\psi := \frac{1}{\psi^{1/2}} h_j. \]
The dual frame of $M \times g \to M$ consists of the functions
\[ (h_j^\psi)^* = \psi^{1/2} h_j^*. \]
Let $\mu_j$ be defined like the functions $\mu_j$ in (21), with $h_j$ replaced by $h_j^\psi$. Analogously, let $V_j$ be the vector field defined like $V_j$ in (22), with the same replacement. Then
\[ \mu_j^\psi = \psi^{1/2} \mu_j; \quad V_j^\psi = \psi^{1/2} V_j. \]

First, consider the operator $A_1^\psi$. On each of the sets $U_1$, we use the local orthonormal frame \{e_k := e_{k,l}^l dM_k = 1\} for $TM$, and set
\[
\tilde{A}_1^\psi := \frac{-1}{4} \sum_{k=1}^{d_M} c(e_k)c \left( \nabla_{e_k} X_1^\psi \right) - \frac{-1}{2} \text{tr} \left( \nabla^{T^{1,0}M} X_1^\psi |_{T^{0,1}M} \right) \\
= A_1^\psi - \frac{1}{2} \sum_{j=1}^{d_G} \left( \sqrt{-1} c(JV_j^\psi)c(V_j^\psi) + \|V_j^\psi\|^2 \right).
\]

By Lemma C.4 one has
\[ \|\nabla_v X_1^\psi\| \leq 2N_{\psi}, \]
on $(M \setminus V) \cap W$, if $v$ is one of the vector fields $e_k$ or $e_{k,0}$. By Lemma C.5, this implies that, on that set, the operator $\tilde{A}_1^\psi$ satisfies the pointwise estimate
\[ \|\tilde{A}_1^\psi\| \leq \frac{3}{2} d_M N_{\psi}. \]
Therefore,
\[ \text{Re}(\tilde{A}_1^\psi) \geq -\frac{3}{2} d_M N_{\psi}, \]
on $(M \setminus V) \cap W$. In addition, one has
\[ \sqrt{-1} c(JV_j^\psi)c(V_j^\psi) + \|V_j^\psi\|^2 \geq 0 \]
for all $j$ (see e.g. (2.13) in [29]). Therefore, on $(M \setminus V) \cap W$, we obtain
\[
\text{Re}(A_1^\psi) = \text{Re}(\tilde{A}_1^\psi) + \frac{1}{2} \sum_{j=1}^{d_G} \left( \sqrt{-1} c(JV_j^\psi)c(V_j^\psi) + \|V_j^\psi\|^2 \right) \geq -\frac{3}{2} d_M N_{\psi}.
\]

To estimate the norm of the operator $A_2^\psi$, we use the equality in Lemma C.6
\[ \langle \mu, T(h_j^\psi)^* \rangle = \mu_j d(\psi^{1/2}) + \psi^{1/2} \langle \mu, Th_j^* \rangle. \]
By Lemma C.7 this implies that, for all $j$, one has
\[ \|\langle \mu, T(h_j^\psi)^* \rangle \| \cdot \|V_j^\psi\| \leq 2N_{\psi} \]
on \((M \setminus V) \cap W\). This allows one to find a bound for \(\|A^\psi_2\|\). Indeed, one has

\[
\|A^\psi_2\| \leq \frac{1}{2} \sum_{j=1}^{d_G} \|\langle \mu, T(h^*_j)^\psi \rangle \| \cdot \|V^\psi_j\| + \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \|\langle \mu, T(h^*_j)^\psi \rangle \| \cdot \|V^\psi_j\|
\]

\[
\leq (d_G + 2d_Gd_M) N_\psi.
\]

To estimate the norm of the operator \(A^\psi_3\), we use Lemma C.8 which states that for all vector fields \(v \in X(M)\) and all \(j\),

\[
[v, ((h^*_j)^\psi)^M - V^\psi_j] = \psi^{1/2} [v, (h^*_j)^M - V_j] - v(\psi^{1/2})V_j.
\]

Let \(v\) be one of the vector fields \(e_k\) or \(e^{1,0}_k\). Then by Lemma C.9 one has for all \(j\), on \((M \setminus V) \cap W\),

\[
|\mu^\psi_j||[v, ((h^*_j)^\psi)^M - V^\psi_j]| \leq 2N_\psi.
\]

It then follows from the definition of the operator \(A^\psi_3\) that, on \((M \setminus V) \cap W\),

\[
\|A^\psi_3\| \leq \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} |\mu^\psi_j||[e^{1,0}_k, ((h^*_j)^\psi)^M - V^\psi_j]| + \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} |\mu^\psi_j||[e_k, ((h^*_j)^\psi)^M - V^\psi_j]|
\]

\[
\leq 3d_Gd_M N_\psi.
\]

Because of (54), (55) and (56), there is a constant \(C' > 0\) such that, on \((M \setminus V) \cap W\),

\[
A^\psi \geq -C'N_\psi.
\]

Since both sides of this inequality are \(G\)-invariant, it holds on all of \(M \setminus V\). Finally, because \(\nabla/G\) is compact, \(A^\psi\) is bounded below on \(V\). Therefore, the estimate (50) follows on all of \(M\).

From now on, we suppose the family of inner products on \(g^*\) has the properties in Proposition 6.5 and we omit the function \(\psi\) from the notation.

6.3. Estimates for adjoint operators. As noted in Subsection 4.1, the operator \(\widetilde{D}_t^{L^p}\) is not symmetric with respect to the \(L^2\)-inner product in general. Let \((\widetilde{D}_t^{L^p})^*\) be the operator on \(f\Omega_{tc}^{0,\ast}(M; L^p)\) such that for all \(s, s' \in \Omega_{tc}^{0,\ast}(M; L^p)\)

\[
((\widetilde{D}_t^{L^p})^* f s, f s')_0 = (fs, \widetilde{D}_t^{L^p} fs')_0.
\]

In particular, for \(t = 0\), one has the operator \((\widetilde{D}_0^{L^p})^* = (\widetilde{D}_0^{L^p})^\ast\). Theorem 5.1 and the estimates in Proposition 6.5 turn out to imply the following property of the operator \((\widetilde{D}_t^{L^p})^* \widetilde{D}_t^{L^p}\).

**Proposition 6.6.** One has

\[
(\widetilde{D}_t^{L^p})^* \widetilde{D}_t^{L^p} = (\widetilde{D}_t^{L^p})^* \widetilde{D}_t^{L^p} + tB + 4\pi ptH + \frac{t^2}{4} \|X_1^H\|^2,
\]

for an operator \(B\) for which there is a constant \(C' > 0\) such that for all \(m \in M\),

\[
B_m \geq -C'(\|X_1^H(m)\|^2 + 1).
\]

To prove Proposition 6.6, we use the following expression for \((\widetilde{D}_t^{L^p})^*\).
Lemma 6.7. For all \( s \in \Omega^0_{ic}(M; L^p) \), one has
\[
(\widetilde{D}_t^{L_p})^* f s = \widetilde{D}_t^{L_p} f s + 2c(df)s.
\]

Proof. For \( s, s' \in \Omega^0_{ic}(M; L^p) \), we compute
\[
(\widetilde{D}_t^{L_p})^* f s, f s')_0 = (f, f \tilde{D}_t^{L_p} s')_0 = (f, D_t^{L_p} f s')_0 - (f, c(df)s')_0.
\]
By symmetry of the operator \( D_t^{L_p} \), the first term on the right hand side of (57) equals
\[
(D_t^{L_p} f s, f s')_0 = (\tilde{D}_t^{L_p} f s + c(df)s, f s')_0.
\]
By antisymmetry of \( c(df) \), the second term on the right hand side of (57) equals
\[
-(f, c(df)s')_0 = (c(df)f s, s')_0,
\]
and the claim follows. \( \square \)

Lemma 6.8. For all \( s \in \Omega^0_{ic}(M; L^p) \), one has
\[
(\widetilde{D}_t^{L_p})^* \widetilde{D}_t^{L_p} f s = (\widetilde{D}_t^{L_p})^2 f s + 2c(df)D_t^{L_p} s.
\]

Proof. Because of Lemma 6.7, one has for all \( s \in \Omega^0_{ic}(M; L^p) \),
\[
(\widetilde{D}_t^{L_p})^* \widetilde{D}_t^{L_p} f s = (\widetilde{D}_t^{L_p})^2 f s + 2c(df)D_t^{L_p} s.
\]
Subtracting this equality for \( t = 0 \) from the equality for general \( t \), one gets
\[
((\widetilde{D}_t^{L_p})^* D_t^{L_p} - (\widetilde{D}_t^{L_p})^2) f s = ((\widetilde{D}_t^{L_p})^2 - (D_t^{L_p})^2) f s + itc(df)c(X_H^1)s.
\]
The desired equality therefore follows from Theorem 5.4. \( \square \)

A priori, it is not clear if the operator \( fs \mapsto \sqrt{-tc(df)c(X_H^1)s} \) that appears in (58) can be bounded in a suitable way. One has the following estimate, however.

Lemma 6.9. There is a constant \( C'' > 0 \) such that for all \( s \in \Omega^0_{ic}(M; L^p) \),
\[
\text{Re}(\sqrt{-tc(df)c(X_H^1)s}, fs)_0 \geq -C''(\|X_H^1\|^2 + 1) fs, fs)_0.
\]

Proof. Let \( s \in \Omega^0_{ic}(M; L^p) \). Then, using the metric \( d\mathcal{O} \) from (54), we find that
\[
|(\sqrt{-tc(df)c(X_H^1)s}, fs)_0| \leq \int_M (f\|df\| \cdot \|X_H^1\| \cdot \|s\|^2)(m) \, dm
\]
\[
\leq \int_{M/G} \left( \int_G f(g\tau(\mathcal{O})) \|df\| \|g\tau(\mathcal{O})\| \, dg \right) \|X_H^1(\tau(\mathcal{O}))\| \cdot \|s(\tau(\mathcal{O}))\|^2 \, d\mathcal{O}
\]
\[
= \int_{M/G} \eta(\tau(\mathcal{O})) \|X_H^1(\tau(\mathcal{O}))\| \cdot \|s(\tau(\mathcal{O}))\|^2 \, d\mathcal{O}.
\]
Here we have used \( G \)-invariance of the functions \( \|X_H^1\| \) and \( \|s\| \), and \( \eta \) is the function defined by (17).

Since we use the metric on \( M \times \mathfrak{g}^* \to M \) of Proposition 6.5, the function \( \eta \) satisfies \( \eta \leq \|X_H^1\| \) outside the set \( V \). And since \( \eta \) is \( G \)-invariant, it is bounded on the set \( V \). Hence there is a \( C'' > 0 \) such that
\[
\eta \leq C'' \left( \|X_H^1\| + 1 \right)
\]
on all of $M$. Let $C'' > 0$ be such that
\[ C''(\|X^H_1\|^2 + 1) \leq C''(\|X^H_1\|^2 + 1). \]
Then (59) is at most equal to
\[ C'' \int_{M/G} (\|X^H_1(\tau(O))\|^2 + 1) \|s(\tau(O))\|^2 d\tau(O). \]
It was shown in the proof of Lemma 4.4 that this expression equals
\[ C'' \int_M (\|X^H_1(m)\|^2 + 1) \|fs(m)\|^2 dm = C''((\|X^H_1\|^2 + 1)fs, fs)_0, \]
and the claim follows.

Proof of Proposition 6.6. Because of Lemma 6.9, the (real part of the) operator $fs \mapsto \sqrt{-1}e(df)c(X^H_1)s$ is bounded below by the multiplication operator $-C''(\|X^H_1\|^2 + 1)$ on $f_{\Gamma^\infty}(E)^G$. In addition, we had
\[ A \geq -C(\|X^H_1\|^2 + 1). \]
Lemma 6.8 therefore implies that
\[ tB := (\tilde{D}^L_t)^*\tilde{D}^L_t - \left( (\tilde{D}^L_t)^*\tilde{D}^L_t + 4\pi pt\mathcal{H} + \frac{t^2}{4}\|X^H_1\|^2 \right) \geq -t(C + C'')(\|X^H_1\|^2 + 1). \]
(Note that the operator $B$ is symmetric with respect to the $L^2$-inner product.)

6.4. Proofs of localization estimates. Let us prove Propositions 6.1 and 6.3.

Proof of Proposition 6.1. Let a family of inner products on $g^*$ and a set $V \subset M$ as in Proposition 6.5 be given. Let the operator $B$ and the constant $C' > 0$ be as in Proposition 6.6. Choose any number $\varepsilon > 0$ and set $t_0 := 8C' + 4\varepsilon$. Then for all $t \geq t_0$ and $m \in M \setminus V$, the fact that $\|X^H_1(m)\| \geq 1$ implies that
\[ tB_m + 4\pi t\mathcal{H}(m) + \frac{t^2}{4}\|X^H_1(m)\|^2 \geq t \left( \left( \frac{t}{4} - C' \right) \|X^H_1(m)\|^2 - C' + 4\pi \mathcal{H}(m) \right) \]
\[ \geq t \left( \frac{t}{4} - 2C' \right) \]
\[ \geq \varepsilon t. \]

So for all such $t$, and all $s \in \Omega^{0,*}_{tc}(M; L)^G$ with supp$(s) \subset M \setminus V$,
\[ \|\tilde{D}^L_t(fs)\|^2_0 = \left( (\tilde{D}^L_t)^*\tilde{D}^L_t(fs), fs \right)_0 \]
\[ \geq \left( \tilde{D}^L_t \right)^*\tilde{D}^L_t(fs, fs)_0 + \varepsilon t\|fs\|^2_0 \]
\[ \geq \|fs\|^2_1 + (\varepsilon t - 1)\|fs\|^2_0. \]

If one sets $\tilde{C} := \min(\varepsilon, 1)$ and $b := 1$, the latter expression is at least equal to
\[ \tilde{C}(\|fs\|^2_1 + (t - b)\|fs\|^2_0). \]
Proof of Proposition 6.3. Consider a family of inner products on $g^*$ as in Proposition 6.5. Fix a $G$-invariant open neighborhood $U$ of $\mu^{-1}(0)$. Since $H \geq 1$ outside the set $V$, and $H$ is positive and $G$-invariant on the cocompact set $\overline{V} \setminus U$, there is a $\zeta > 0$ such that $H \geq \zeta$ outside $U$.

Let the operator $B$ and the constant $C' > 0$ be as in Proposition 6.6. Let $p_0 \in \mathbb{N}$ be such that
\[ \varepsilon := 4\pi \zeta p_0 - C' > 0. \]
Then for all $p \geq p_0$ and $t \geq t_0 := 4C'$, one has for all $m \in M \setminus U$,
\[ tB_m + 4\pi ptH(m) + \frac{t^2}{4}\|X_1^H(m)\|^2 \geq t \left( \left( \frac{t}{4} - C' \right) \|X_1^H(m)\|^2 - C' + 4\pi p \right) \geq \varepsilon t. \]
So for all such $p$ and $t$, and all $s \in \Omega^\alpha_{tc}(M; L^p)^G$ with $\text{supp}(s) \subset M \setminus U$, analogously to the proof of Proposition 6.1, we find that
\[ \|\tilde{D}_t^Lp(fs)\|_0^2 \geq \left( \left( \tilde{D}_t^Lp \right)^* \tilde{D}_t^Lp(fs), fs \right)_0 + \varepsilon t\|fs\|_0^2 \]
\[ = \|fs\|_1^2 + (\varepsilon t - 1)\|fs\|_0^2. \]
As in the proof of Proposition 6.1 the latter expression is at least equal to
\[ \tilde{C}(\|fs\|_1^2 + (t - b)\|fs\|_0^2), \]
for $\tilde{C} := \min(\varepsilon, 1)$ and $b := 1$. \hfill \square

Remark 6.10. In the proof of Proposition 6.3 it was necessary to take both $p$ and $t$ large enough, so that the term $\frac{t^2}{4}\|X_1^H(m)\|^2$ compensates for the term $tB_m \geq -tC'(\|X_1^H(m)\|^2 + 1)$. This is different from the arguments in [29] and [20] for large powers of $L$, since the deformation vector field and the operator $A$ are bounded in the compact and cocompact situations considered there. Then it is enough that just $p$ is large.

6.5. Discrete localized spectrum. In [29], the fact that the restriction of $(D_t^Lp)^2$ to $G$-invariant sections has discrete spectrum is used. This fact generalizes to the current setting as follows.

Lemma 6.11. Let $\lambda > 0$. Then for $t$ and $p$ large enough, the intersection of the interval $[\lambda - \infty, \lambda]$ with the spectrum of the restriction of $(D_t^Lp)^2$ to $G$-invariant sections is discrete, and the corresponding eigenspaces are finite-dimensional.

Proof. Let $U$ be a $G$-invariant relatively cocompact open neighborhood of $\mu^{-1}(0)$, on which $G$ acts freely. By Proposition 6.3 there are $C > 0$, $b > 0$, $t_0 > 0$ and $p_0 > 0$, such that for all $t > t_0$ and $p \geq p_0$, and all $s \in \Omega^\alpha_{tc}(M; L)^G$ with support disjoint from $U$,
\[ \|\tilde{D}_t^Lp(fs)\|_0^2 \geq C(\|fs\|_1^2 + (t - b)\|fs\|_0^2). \]
Let $\chi_{[\lambda - \infty, \lambda]}$ be the characteristic function of the interval $[\lambda - \infty, \lambda]$. Set
\[ E_t(\lambda) := \text{image} \left( \chi_{[\lambda - \infty, \lambda]} \left( \tilde{D}_t^Lp \right)^2 \right). \]
Then for all sections $\sigma \in E_t(\lambda)$,
\[ \|\tilde{D}_t^Lp(\sigma)\|_0 \leq \lambda^{1/2}\|\sigma\|_0. \]
Hence by (60), one has
\[ \lambda \| \sigma \|_0^2 \geq C(t - b)\| \sigma \|_0^2, \]
if supp(\sigma) \subset M \setminus U. For
\[ t > t_0(\lambda) := \max \left( t_0, \frac{\lambda + b}{C} \right), \]
this implies that \( \sigma = 0 \). That is, for \( t \geq t_0(\lambda) \), sections in \( E_t(\lambda) \) localize to \( U \). Using Rellich’s lemma on the relatively compact set \( U/G \), we see that the space of all \( G \)-invariant sections in \( E_t(\lambda) \) is spanned by eigensections, and the claim follows. \( \square \)

7. Dirac operators on \( M \) and \( M_0 \)

In [29], Tian and Zhang prove a relation between the deformed Dirac operator \( D^L_{t, p} \) on \( M \), and a Dirac-type operator \( D^L_{p, G} \) on \( M_0 \). This relation allows them to use apply the techniques in [3] to the operator \( D^L_{t, p} \). We will generalize this relation to the noncompact case.

A version of this relation in the cocompact case was (implicitly) used in [20]. In the present setting, one basically arrives at the cocompact situation after localizing to a relatively cocompact set \( U \). The authors though it worthwhile to include an explicit discussion for the operator \( \tilde{D}^L_{t, p} \), however.

7.1. Vector bundles on orbit spaces. We begin by briefly recalling some facts and notation concerning vector bundles on orbit spaces of free actions, induced by equivariant vector bundles on the space acted on.

Let \( U \) be a manifold on which a Lie group \( G \) acts properly and freely. (We will apply what follows to an open neighborhood \( U \) of \( \mu^{-1}(0) \) in \( M \).) Let \( q : U \to U/G \) be the quotient map. Let \( E \to U \) be a \( G \)-vector bundle. As in Subsection 4.3, let
\[ E_G \to U/G \]
be the induced vector bundle, such that \( E \cong q^*E_G \) as \( G \)-vector bundles over \( U \). Consider the Sobolev spaces \( W^k_f(E)^G \) as in Definition 4.2. Let \( R = \psi^{-1} : W^k_f(E)^G \xrightarrow{\cong} W^k(E_G) \)
be the inverse of the unitary isomorphism of Lemma 4.6. (We will use the same notation for the restriction of \( R \) to the dense subspace \( f\Gamma^\infty(E)^G \).)

For any \( G \)-invariant submanifold \( N \subset U \), consider the inclusion map \( i : N \hookrightarrow U \), and the induced inclusion map
\[ i^*_G : \Gamma^\infty(N/G) \hookrightarrow U/G. \]
This induces the restriction map
\[ \tilde{i}^*_G : \Gamma^\infty(U/G, E_G) \to \Gamma^\infty(N/G, i^*_G E_G). \]

In the setting of Subsection 3.1, let now \( U \) be a \( G \)-invariant open neighborhood of \( \mu^{-1}(0) \), on which \( G \) acts freely. Let \( E = \bigwedge^{0, *} T^*\mu U \otimes (L^p|_U) \), and \( N = \mu^{-1}(0) \). Then, as in Subsection 3e of [29], one has the projection map
\[ \pi : i^*_G \left( \bigwedge^{0, *+} T^* M_0 \otimes L^p \right) \to \bigwedge^{0, *+} T^* M_0 \otimes L^p, \]
defined as follows. Let $N_G \to M_0$ be the normal bundle to $M_0$ in $U/G$. Consider the almost complex structure $J_G$ on $(TU)_{U/G}|_{M_0}$ induced by the almost complex structure $J$ on $M$. Then

$$i^*_G(TU)_G = N_G \oplus J_G N_G \oplus TM_0,$$

so

$$i^*_G(\bigwedge^{0,*}T^*U)_G \cong \bigwedge^{0,*}(N_G^* \oplus J_G N_G^*) \otimes \bigwedge^{0,*}T^*M_0.$$ 

The map $\pi$ is defined via this identification, as projection onto the term

$$\bigwedge^{0,0}(N_G^* \oplus J_G N_G^*) \otimes \bigwedge^{0,*}T^*M_0 \otimes i^*_G(L^p|_U)_G \cong \bigwedge^{0,*}T^*M_0 \otimes L^p_0.$$ 

Let

$$\iota : \bigwedge^{0,*}T^*M_0 \otimes L^p_0 \hookrightarrow i^*_G(\bigwedge^{0,*}T^*U \otimes (L^p|_U))_G$$

be the embedding induced by the same identification, so that $\pi \circ \iota$ is the identity on $\bigwedge^{0,*}T^*M_0 \otimes L^p_0$.

In the next subsections, we will see how the deformed Dirac operator $D^p_l$ on $M$ is related to a Dirac-type operator $D^L_Q$ on $M_0$ by the maps

\[
\begin{align*}
    f \Omega_{lc}^0(U; L^p|_U)^G & \xrightarrow{R} \Gamma_c^\infty \left(U/G, \left(\bigwedge^{0,*}T^*U \otimes (L^p|_U)\right)_G\right) \\
    & \xrightarrow{\iota_G^*} \Gamma^\infty \left(M_0, i^*_G(\bigwedge^{0,*}T^*U \otimes (L^p|_U))_G\right) \\
    & \xleftarrow{\pi} \Omega^{0,*}(M_0; L^p_0).
\end{align*}
\]

(Since $M_0$ is compact, all sections of vector bundles over $M_0$ are compactly supported. Therefore, the subscript $c$ is dropped in the notation for spaces of sections of such bundles.)

7.2. **Intermediate operators.** There are several operators on the vector bundles considered in the previous subsection that are relevant to our purposes. These operators are related to each other by the maps $R$, $i^*_G$, and $\pi$ in (61). Let us define these operators.

Choose a $G$-invariant local orthonormal frame $\{e_1, \ldots, e_{d_M}\}$ for $TU$, such that $\{e_{d_M+1}, \ldots, e_{d_M}\}$ is a frame for the vertical tangent bundle $\ker(Tq)$. For a $G$-invariant vector field $v$ on $U$, consider the vector field $q_*v$ on $U/G$, given by

$$(q_*v)_{q(m)} := T_m q(v_m).$$

For $j = 1, \ldots, d_{M/G}$, write $f_j := q_*e_j$. Then $\{f_1, \ldots, f_{d_{M/G}}\}$ is an orthonormal frame for $T(U/G)$. Suppose that $i^*_G f_1, \ldots, i^*_G f_{d_{M_0}}$ is an orthonormal frame for $TM_0$.

Consider the operator $D_G$ on

$$\Gamma^\infty_c \left(U/G, \left(\bigwedge^{0,*}T^*U \otimes (L^p|_U)\right)_G\right)$$

and the operator $i^*_G D_G$ on

$$\Gamma^\infty \left(M_0, i^*_G(\bigwedge^{0,*}T^*U \otimes (L^p|_U))_G\right).$$
given by

\[ D_G := \sum_{j=1}^{d_{M/G}} c(f_j) \nabla_{f_j}^{(\wedge^0 T^*U \otimes (L^p|_U))_G}; \]

\[ i_G^* D_G := \sum_{j=1}^{d_{M/G}} c(i_G^* f_j) i_G^* \left( \nabla^{(\wedge^0 T^*U \otimes (L^p|_U))_G} \right) i_G^* f_j \]

\[ = \sum_{j=1}^{d_{M_0}} c(i_G^* f_j) \nabla^{(\wedge^0 T^*U \otimes (L^p|_U))_G} i_G^* f_j. \]

Here, for any \( G \)-vector bundle \( E \to U \), with a \( G \)-invariant connection \( \nabla^E \), the connection \( \nabla^{E_G} \) on \( E_G \) is defined by commutativity of

\[ \Gamma^\infty(E_G) \xrightarrow{\nabla_{\nabla^E_G}} \Gamma^\infty(E_G) \]

\[ \Gamma^\infty(E)^G \xrightarrow{\nabla^E} \Gamma^\infty(E)^G, \]

for all \( G \)-invariant vector fields \( v \) on \( U \).

Also, consider the operator

\[ B := \sum_{j=d_{M/G}+1}^{d_M} c(e_j) \nabla^{0,* T^*U \otimes (L^p|_U)}_{e_j} \]

on \( \Omega^{0,*}(U; L^p)^G \). Because the vector fields \( e_{d_{M/G}+1}, \ldots, e_d \) are tangent to \( G \)-orbits, \( B \) has the following property.

**Lemma 7.1.** The operator \( B \) is given by a vector bundle endomorphism of \( \wedge^0 T^*U \otimes (L^p|_U) \).

**Proof.** See Lemma 3.3 in [29]. \( \square \)

Let \( B_G \) be the operator on

\[ \Gamma^\infty\left(U/G, (\wedge^0 T^*U \otimes (L^p|_U))_G\right) \]

induced by the \( G \)-equivariant vector bundle endomorphism \( B \), and let the operator \( i_G^* B_G \) on

\[ \Gamma^\infty(M_0, i_G^* (\wedge^0 T^*U \otimes (L^p|_U))_G) \]

be the restriction of \( B_G \) to \( M_0 \).

### 7.3. An operator on \( M_0 \).

Using the operators \( D_G, i_G^* D_G, B_G \) and \( i_G^* B_G \) from the previous subsection, we define a Dirac-type operator \( D_{L^p_Q}^{L^p} \) on \( M_0 \), and show that the maps (61) relate this operator to \( \tilde{D}_{L^p} \). This is a version of Corollary 3.6 and Definition 3.12 in [29] for the noncompact setting.

The first step is a relation between the undeformed Dirac operator \( \tilde{D}_{L^p} \) and the operator \( D_{L^p_Q}^{L^p} \) on \( M_0 \) defined by

\[ D_{L^p_Q}^{L^p} = \pi \circ (i_G^* D_G + i_G^* B_G) \circ \iota : \Omega^0(M_0; L^p_0) \to \Omega^0(M_0; L^p_0). \]
Proposition 7.2. The following diagram commutes:

\[
\begin{array}{c}
\Gamma_c^\infty \left( U/G, \left( \bigwedge^{0,*} T^* U \otimes (L^p|_U) \right) G \right) \\
\downarrow \quad \left( i_G^* D_G + i^*_G B_G \right) \\
\Gamma_c^\infty \left( M_0, i_G^* \left( \bigwedge^{0,*} T^* U \otimes (L^p|_U) \right) G \right) \\
\downarrow \quad \pi \\
\Omega^{0,*}(M_0; L^p_0)
\end{array}
\]}

\[
\begin{array}{c}
\tilde{D}^{L^p}_Q \\
\downarrow \quad D_G^L \circ R^{-1}
\end{array}
\]}

\[
\begin{array}{c}
f \Omega^{0,*}(U; L^p|_U)^G \\
\downarrow \quad \left( \tilde{D}^{L^p}_Q \circ R\right)
\end{array}
\]}

\[
\begin{array}{c}
f \Omega^{0,*}(U; L^p|_U)^G \\
\downarrow \quad \left( \tilde{D}^{L^p}_Q \circ R\right)
\end{array}
\]}

Proof. The bottom part of Diagram (62) commutes by definition of $D_G^L$. The middle part of the diagram commutes by definition of the pulled-back connection $\nabla^G \left( \bigwedge^{0,*} T^* U \otimes (L^p|_U) \right) G$, and the facts that $i_G B_G$ is the restriction to $M_0$ of $B_G$, and $c(i_G f_j)$ is the restriction of $c(f_j)$.

To show that the top part commutes, let $s_G \in \Gamma_c^\infty \left( U/G, \left( \bigwedge^{0,*} T^* U \otimes (L^p|_U) \right) G \right)$ be given, and set $s := q^* s_G \in \Omega^{0,*}(U; L^p|_U)^G$. Then

\[
(\tilde{D}^{L^p} \circ R^{-1}) s_G = f D_G^L s
\]

\[
= f \sum_{j=1}^{d_M} c(e_j) \nabla_{e_j} \left( \bigwedge^{0,*} T^* U \otimes (L^p|_U) \right) G s + f B s.
\]

The first of the latter two terms equals

\[
f q^* (D_G s_G) = (R^{-1} \circ D_G) s_G.
\]

The second term equals $f q^* (B_G s_G)$, so that indeed $\tilde{D}^{L^p} \circ R^{-1} = R^{-1} \circ (D_G + B_G)$.

An important property of the operator $\tilde{D}_Q^{L^p}$ is that it has the same index as the Spin$^c$-Dirac operator $D_0^L$ on $M_0$ coupled to $L^p_0$.

Lemma 7.3. One has

\[
\text{index } D_Q^{L^p} = \text{index } D_0^L.
\]

Proof. By Lemma 7.1, the operator $i_G^* D_G + i^*_G B_G$ in the third horizontal arrow in (62) has the same principal symbol as the term $i_G^* D_G$ on its own. The principal symbol of that term induces the Clifford action on $\bigwedge^{0,*} T^* M_0 \otimes L^p_0$. Hence the principal symbol of $D_Q^{L^p}$ is given by the Clifford action, and is equal to the principal symbol of $D_0^L$. Because $M_0$ is compact, the Fredholm indices of $D_Q^{L^p}$ and $D_0^L$ are therefore equal.

The final step is to relate the deformed Dirac operator $\tilde{D}_i^{L^p}$ on $M$ to the operator $D_Q^{L^p}$ on $M_0$. To this end, note that

\[
R \circ c(X_1^H) = c(q^* X_1^H) \circ R,
\]

where $X_1^H$ is the horizontal component of $X_1$.
and that
\[ i_G \circ c(q_*X^H) = c(i_G^* q_* X^H) = 0. \]

For the last equality, we have used the fact that \( X^H_1 \equiv 0 \) on \( \mu^{-1}(0) \), by Lemma \[23\]. We therefore obtain the following result.

**Corollary 7.4.** The following diagram commutes:

\[
\begin{array}{c}
\Gamma_\infty \left( M_0, i_G^* \left( \bigwedge^{0, *}_U \otimes (L^p|_U) \right)_G \right) \xrightarrow{i^*_G \circ R} \Gamma_\infty \left( M_0, i_G^* \left( \bigwedge^{0, *}_U \otimes (L^p|_U) \right)_G \right) \\
\Omega^{0, *}(M_0; L_0^p) \xrightarrow{D^0_G} \Omega^{0, *}(M_0; L_0^p).
\end{array}
\]

**Remark 7.5.** The deformed Dirac operator used in the compact setting considered in \[29\] had the form
\[ D^L + \frac{\sqrt{-1}t}{2} c(X^H) = e^{-\frac{\sqrt{-1}t}{2} \mathcal{H}} D^L e^{\frac{\sqrt{-1}t}{2} \mathcal{H}}. \]

That is, the deformation could be described as conjugation by a nonzero function as in Witten’s paper \[32\].

In the present setting, the analogous expression is
\[ D_G^L p = D^L + \frac{\sqrt{-1}t}{2} c(X^H). \]

Now the deformed Dirac operator is a conjugate of the operator \( D^L \) by \( e^{-\frac{\sqrt{-1}t}{2} \mathcal{H}} \left( D^L - \frac{\sqrt{-1}t}{2} c(X^H) \right) e^{\frac{\sqrt{-1}t}{2} \mathcal{H}} \) in Diagram (63), the bottom arrow in that diagram initially becomes
\[
e^{-\frac{\sqrt{-1}t}{2} i^*_G \mathcal{H}_G} \left( D^L_G - \frac{\sqrt{-1}t}{2} c((X^H)_0) \right) e^{\frac{\sqrt{-1}t}{2} i^*_G \mathcal{H}_G},
\]

where \( \mathcal{H}_G \) is the function on \( U/G \) induced by \( \mathcal{H} \), so that \( i_G^* \mathcal{H}_G = 0 \). Since also \( (X^H)_0 = 0 \), as noted in Lemma \[26\] the operator (64) equals \( D^L_G \). This is in agreement with Corollary 7.4.

To generalize the methods of \[29\] to the present setting, the final ingredients one needs are versions of Remarks 3.7 and 3.8 in \[29\]. These remarks generalize to the current setting because the form \[23\] of the vector field \( X^H_1 \) is analogous to (1.19) in \[29\]. Indeed, Remark 3.7 in \[29\] generalizes because the vector field \( X^H_1 \) is still tangent to orbits. Hence, for any local frame \( \{ f_1, \ldots, f_{\dim G} \} \) of \( T(U/G) \), the operators \( c(q_* X^H) \) and \( c(f_j) \) anticommute. Remark 3.8 in \[29\] can be generalized because of the following estimate.
Lemma 7.6. There is a neighborhood $U$ of $\mu^{-1}(0)$ and a constant $C > 0$ such that
$$
\mathcal{H}|_U \leq C\|X^H_1\|_U^2.
$$

Proof. Since $G$ acts freely on a neighborhood of $\mu^{-1}(0)$, the vector fields $V_j$ are linearly independent there. Hence every point $m \in \mu^{-1}(0)$ has a neighborhood $U_m$ that admits a vector bundle automorphism $B$ of $TU_m$ such that $\{BV_1|_U, \ldots, BV_{d_G}|_U\}$ is orthonormal. Then, on $U_m$, (21) and (23) imply that
$$
H = \sum_{j=1}^{d_G} \mu_j^2 = \left\| \sum_{j=1}^{d_G} \mu_j BV_j \right\|^2 \leq \|B\|^2 \|X^H_1\|^2 \leq C_m\|X^H_1\|^2,
$$
if $U_m$ is chosen small enough so that there is a $C_m > 0$ such that $\|B\| \leq C_m$ on $U_m$.

Since $H$ and $\|X^H_1\|^2$ are $G$-invariant, we obtain the estimate $H \leq C_m\|X^H_1\|^2$ on $G \cdot U_m$, for any $m \in \mu^{-1}(0)$. Since $\mu^{-1}(0)$ is cocompact, one can cover it with finitely many such sets $G \cdot U_m$. □

Because of this lemma, one has
$$
c(q^*X^H_1)^2 \geq \frac{1}{C}\mathcal{H}_G
$$
on $U/G$, where $\mathcal{H}_G$ is the function on $U/G$ induced by $\mathcal{H}$. This generalizes Remark 3.8 in [29], which is a version of Proposition 8.14 in [3].

Because of Proposition 6.3, Lemma 6.11 and Corollary 7.4, and the generalizations of Remarks 3.7 and 3.8 in [29] mentioned above, the methods of [29] generalize to the noncompact setting considered here. The key result in [29] is Theorem 3.13, which is an analogue of (9.156) in [3] in the Spin$^c$ setting. A generalization of that result to the cocompact setting was used (implicitly) at the end of Section 3 in [20]. Since, in the present setting, $M_0$ is compact, Proposition 6.3 implies that the kernel of the Dirac operator $\bar{D}_t^L$ localizes to a relatively cocompact neighborhood of $\mu^{-1}(0)$. There, one applies the same generalization of Theorem 3.13 in [29] as the one used in [20].

More precisely, using the notation in Section 6.5, recall from Lemma 6.11 that given $\lambda > 0$, there is $t_0 = t_0(\lambda) > 0$ such that for all $t \geq t_0(\lambda)$, the $G$-invariant part of the space $E_t(\lambda)$ is spanned by eigensections of $(D^L_0)^2$. Then one has the following result.

Theorem 7.7. Let $\lambda > 0$ be such that there are no eigenvalues of $(D^L_0)^2$ in the interval $(0, \lambda]$. Then there is $t_0 = t_0(\lambda) > 0$ such that for all $t \geq t_0(\lambda)$, one has
$$
(65) \quad \dim E_t(\lambda) = \dim \ker(D^L_0).
$$

Since the positive and negative eigenvalues of $D^L_0$ of absolute value at most $\lambda$ are in bijection with each other, one has, for $t \geq t_0(\lambda)$,
$$
\text{index}_G(D^L_t) = \text{index}(D^L_0) \quad \text{by Theorem 7.7}
$$
$$
= \text{index}(D^L_0) \quad \text{by Lemma 7.3}
$$
Thus one obtains a proof of Theorem 3.6.
Appendix A. Elliptic regularity and transversally $L^2$-kernels

Consider the setting of Proposition 4.7. Recall that the transversally $L^2$-kernels of the operators $D_\pm$ were defined as

$$\ker_{L^2_t}(D_\pm) = \{ s \in \Gamma_\infty(E_\pm) \cap L^2_t(E); Ds = 0 \}.$$ 

The space $L^2_t(E)$ of transversally $L^2$-sections of $E$ was defined in Definition 3.2. One can give an explicit characterization of the $G$-invariant index of the operator $D$ in terms of its transversally $L^2$-kernel as follows.

**Proposition A.1.** In the situation of Proposition 4.7, one has

$$\text{index}_G(D) = \dim(\ker_{L^2_t}(D_+))^G - \dim(\ker_{L^2_t}(D_-))^G.$$ 

The proof of this proposition is based on a version of elliptic regularity. It will be convenient to use a slightly different realization of the Sobolev spaces $W^{k,f}(E_G)$ from the one used in Section 4. This realization is defined in terms of the Sobolev norms $\| \cdot \|_{W^{k,f}_G}$ on $\Gamma_\infty^\infty(E)$, equal to

$$\left(\|s\|_{W^{k,f}_G}^2\right)^2 := \sum_{j=0}^k \|fD^js\|_{L^2(E)}^2 = \|fs\|_{W^{k,f}_G(E)}^2,$$

for $s \in \Gamma_\infty^\infty(E)^G$. Let $\tilde{W}^k_f(E)^G$ be the completion of $\Gamma_\infty^\infty(E)^G$ in the norm $\| \cdot \|_{W^{k,f}_G}$. Multiplying by $f$ then extends to a unitary isomorphism $\tilde{W}^k_f(E)^G \cong W^k_f(E)^G$. The operator $D$ on $\Gamma_\infty^\infty(E)^G$ extends continuously to an operator

$$D_f : \tilde{W}^1_f(E)^G \to \tilde{W}^0_f(E)^G.$$ 

The isomorphism $\tilde{W}^k_f(E)^G \cong W^k_f(E)^G$ just mentioned intertwines this operator and the operator $\tilde{D}$.

Let $\varphi \in C^\infty_c(M)$ be a function whose compact support is contained in the interior of $\text{supp}(f)$. Then

$$\varepsilon := \min_{m \in \text{supp}(\varphi)} |f(m)| > 0.$$ 

**Lemma A.2.** Consider the multiplication operator by $\varphi$,

$$m_\varphi : \Gamma_\infty^\infty(E)^G \to \Gamma_\infty^\infty(E).$$

It is bounded with respect to the Sobolev norm $\| \cdot \|_{W^{k,f}_G}$ defined above on the domain, and the norm $\| \cdot \|_{W^{k,f}_G}$ on the codomain defined by

$$\|s\|_{W^{k,f}_G}^2 := \sum_{j=0}^k \|D^js\|_{L^2(E)}^2,$$

for $s \in \Gamma_\infty^\infty(E)$.

**Proof.** For $k = 0$, we note that for all $s \in \Gamma_\infty^\infty(E)^G$,

$$\|\varphi s\|_{L^2(E)} \leq \frac{1}{\varepsilon} \|\varphi f s\|_{L^2(E)} \leq \frac{\|\varphi\|_{\infty}}{\varepsilon} \|s\|_{W^{0,f}_G}.$$ 

40
For $k = 1$, one has

$$\|D\varphi s\|_{L^2(E)} = \|\varphi Ds + \sigma_D(d\varphi)s\|_{L^2(E)}$$

$$\leq \frac{1}{\varepsilon} \left( \|\varphi fDs\|_{L^2(E)} + \|\sigma_D(d\varphi)f s\|_{L^2(E)} \right)$$

$$\leq \frac{1}{\varepsilon} \left( \|\varphi\|_{\infty} \|s\|_f^1 + \|\sigma_D(d\varphi)\| \cdot \|s\|_f^1 \right).$$

Hence $m_\varphi$ is also bounded with respect to the first Sobolev norms.

For general $k$, one similarly notes that $D^k\varphi s$ equals a sum of the form

$$\sum_{j=0}^k R_j D^j s,$$

where $R_j$ is a repeated commutator of $D$ and $m_\varphi$. All these commutators are bounded, since $\varphi$ is compactly supported. Hence

$$\|D^k\varphi s\|_{L^2(E)} \leq \frac{1}{\varepsilon} \sum_{j=0}^k \|f R_j D^j s\| \leq \frac{1}{\varepsilon} \sum_{j=0}^k \|R_j\| \cdot \|s\|_f^j.$$

We conclude that $m_\varphi$ is bounded with respect to the $k$’th Sobolev norms used, for all $k$. □

Let $W^k(E)$ be the completion of $\Gamma^\infty_c(E)$ in the norm given by (66). Because of Lemma A.2, we obtain the continuous extension

$$m^{(k)}_\varphi : \tilde{W}^k_f(E)^G \rightarrow W^k(E).$$

The diagram

(67)

$$\begin{array}{ccc}
\tilde{W}^k_f(E)^G & \longrightarrow & \tilde{W}^{k-1}_f(E)^G \\
\downarrow m^{(k)}_\varphi & & \downarrow m^{(k-1)}_\varphi \\
W^k_f(E) & \longrightarrow & W^{k-1}(E)
\end{array}$$

commutes on the dense subspace $\Gamma^\infty_c(E)^G$ of $\tilde{W}^k_f(E)^G$. Because all maps in the diagram are bounded, it therefore commutes on all of $\tilde{W}^k_f(E)^G$.

**Lemma A.3.** The kernel of the operator $D_f$ consists of smooth, $G$-invariant sections of $E$.

**Proof.** Since $G$ acts trivially on the space $\Gamma^\infty_c(E)^G$, the continuous extension of the action to $\tilde{W}^k_f(E)^G$ is trivial as well. Hence the kernel of $D_f$ consists of $G$-invariant sections of $E$.

Let $s \in \ker(D_f)$ be given. Since $D^j_f s = 0$ for all $j$, the section $s$ is in all Sobolev spaces $\tilde{W}^k_f(E)^G$. Hence, by commutativity of (67), we find that

$$m^{(0)}_\varphi(s) = m^{(k)}_\varphi(s) \in W^k(E)$$

for all $k$. Therefore, the section $m^{(0)}_\varphi(s)$ is smooth. Since $m^{(0)}_\varphi s$ is the pointwise product of $\varphi$ and $s$, we conclude that $s$ is smooth on the interior of the support of $\varphi$. Since this argument holds for any function with the properties of $\varphi$, we find that $s$ is smooth on the interior of the support of $f$. By $G$-invariance of $s$, it is smooth everywhere. □
Proof of Proposition A.1. We have seen that
\[ \ker(\tilde{D}_\pm) \cong \ker((D_f)_\pm) \subset \Gamma^\infty(E)^G. \]
On smooth sections, the operator \( D_f \) is equal to \( D \).
Any section \( s \in \tilde{W}_f^1(E)^G \) satisfies
\[ \|fs\|_{L^2(E)} = \|s\|_{f}^2 \leq \|s\|_1^2 \]
(with equality if and only if \( s \in \ker(D_f) \)). Since by Lemma 4.4 for \( G \)-invariant \( s \), the norm \( \|fs\|_{L^2(E)} \) is independent of the cutoff function \( f \), one has
\[ \tilde{W}_f^1(E)^G \subset L^2(E). \]
We conclude that
\[ \ker((D_f)_\pm) \subset \Gamma^\infty(E)^G \cap L^2(E), \]
and the claim follows. □

Appendix B. Computing the square of \( D_L^{lp} \)

This appendix contains a proof of an explicit expression for the covariant derivative \( \nabla_{X^H} \).
This is the main computation in the proof of Theorem 5.1. We will use the notation of Section 5.

Proposition B.1. For all \( s \in \Omega^{0,*}(M; L^p) \) and all \( m \in M \), one has
\[ (\nabla_{X^H} s)(m) = 2 \sum_{j=1}^{d_G} \mu_j(m) (L_{h_j^*(m)})s(m) + 4\pi \sqrt{-1} p\mathcal{H}(m)s(m) \]
\[ + \left( \frac{1}{4} \sum_{k=1}^{d_M} c(e_k)c(\nabla_{e_k} X_1^H) + \frac{1}{2} \sum_{j=1}^{d_G} (c(JV_j)c(V_j) + \sqrt{-1} \|V_j\|^2) + \frac{1}{2} \operatorname{tr} \left( \nabla^{T^{1,0}_M X_1^H} \right) (m)s(m) \right) \]
\[ + \sqrt{-1}(A_2 + A_3)(m)s(m), \]
with \( A_2 \) and \( A_3 \) as in (39) and (40).

The proof of this proposition is based on a number of intermediate lemmas.

Lemma B.2. For all \( s \in \Omega^{0,*}(M; L^p) \) and \( m \in M \), one has
\[ \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(m) \left( c(e_k)c(\nabla_{e_k} h_j^*(m)^M) \right) (m)s(m) = \]
\[ \left( \frac{1}{4} \sum_{k=1}^{d_M} c(e_k)c(\nabla_{e_k} X_1^H) - \frac{1}{2} \sum_{j=1}^{d_G} c(\operatorname{grad} \mu_j) c(V_j) \right) \left( \frac{1}{2} \sum_{j=1}^{d_M} \mu_j c(e_k)c([e_k, (h_j^*)^M - V_j]) \right) (m)s(m). \]
Proof. Since the Levi-Civita connection $\nabla$ is torsion-free, we have, for all $j$ and $k$, and all $m \in M$,
\begin{align}
\left(\nabla_{e_k} (h^*_j(m))^M - \nabla_{e_k} V_j\right)(m) &= \left(\left(\nabla_{h^*_j(m)^M} - \nabla_{V_j}\right)e_k\right)(m) + [e_k, h^*_j(m)^M - V_j](m) \\
&= [e_k, (h^*_j)^M - V_j](m),
\end{align}

since $h^*_j(m)^M = V_j(m)$. So
\begin{align}
\frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(m) \left( c(e_k) c(\nabla_{e_k} h^*_j(m)^M) \right)(m)s(m) &= \\
\frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(m) \left( c(e_k) c(\nabla_{e_k} V_j) \right)(m)s(m) + \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(m) \left( c(e_k) c([e_k, (h^*_j)^M - V_j]) \right)(m)s(m).
\end{align}

By (23), we find that
\begin{align}
\sum_{j=1}^{d_G} \mu_j c(\nabla_{e_k} V_j) &= \frac{1}{2} c(\nabla_{e_k} X_1^H) - \sum_{j=1}^{d_G} e_k(\mu_j) c(V_j).
\end{align}

Therefore, the first term on the right hand side of (70) equals
\begin{align}
\frac{1}{4} \sum_{k=1}^{d_M} \left( c(e_k) c(\nabla_{e_k} X_1^H) \right)(m)s(m) - \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \left( e_k(\mu_j) c(e_k) c(V_j) \right)(m)s(m).
\end{align}

Because
\begin{align}
\sum_{k=1}^{d_M} e_k(\mu_j)e_k = \text{grad} \mu_j,
\end{align}

the desired equality follows. \qed

Lemma B.3. One has
\begin{align}
\text{grad} \mu_j = JV_j + \langle \mu, Th^*_j \rangle^*.
\end{align}

Proof. Note that, for all $m \in M$,
\begin{align}
\mu_j(m) &= \langle \mu(m), h^*_j(m) \rangle = \mu_{h^*_j(m)}(m).
\end{align}

Hence, by (12),
\begin{align}
d_m \mu_j = d_m (m' \mapsto \mu_{h^*_j(m)}(m')) + d_m (m' \mapsto \mu_{h^*_j(m')}(m)) \\
&= \omega_m (V_j(m), -) + \langle \mu, Th^*_j \rangle_m \\
&= \omega_m (V_j(m), -) + \omega_m (-, J(\mu, Th^*_j)^*) \\
&= \omega_m (V_j(m) - J(\mu, Th^*_j)_m, -).
\end{align}

Now also
\begin{align}
d \mu_j = \omega(-, J \text{grad} \mu_j) \\
&= - \omega(J \text{grad} \mu_j, -),
\end{align}

and the claim follows. \qed
Lemma B.4. One has, for all \( m \in M \),

\[
\sum_{j=1}^{d_G} \mu_j(m) \operatorname{tr} \left( \nabla^{T,0}_M h^*_j(m)|_{T^{1,0}_M} \right) (m) = \\
\left( \frac{1}{2} \operatorname{tr} \left( \nabla^{T,0}_M X^*_1|_{T^{1,0}_M} \right) - \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} e^{1,0}_k (\mu_j)(V_j, e^{1,0}_k) \right) + \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j \left( [e^{1,0}_k, (h^*_j)^M - V_j], e^{1,0}_k \right) (m).
\]

Proof. Note that, for all \( m \in M \),

\[
\operatorname{tr} \left( \nabla^{T,0}_M h^*_j(m)|_{T^{1,0}_M} \right) = \sum_{k=1}^{d_M} \left( \nabla^{T,0}_M h^*_j(m), e^{1,0}_k \right) (m).
\]

Analogously to (69), we find

\[
\left( \nabla^{T,0}_M (h^*_j(m))^M \right) (m) = \left( \nabla^{T,0}_M V_j \right) (m) + [e^{1,0}_k, h^*_j(m)^M - V_j] (m).
\]

Hence, at \( m \), (72) equals

\[
\operatorname{tr} \left( \nabla^{T,0}_M V_j|_{T^{1,0}_M} \right) (m) + \sum_{k=1}^{d_M} \left( [e^{1,0}_k, h^*_j(m)^M - V_j], e^{1,0}_k \right) (m).
\]

As in (71), one has

\[
\sum_{j=1}^{d_G} \mu_j \nabla^{T,0}_M V_j = \frac{1}{2} \nabla^{T,0}_M X^*_1 - \sum_{j=1}^{d_G} e^{1,0}_k (\mu_j) V_j.
\]

The claim follows. \( \square \)

Lemma B.5. One has

\[
\sum_{k=1}^{d_M} (JV_j, e^{1,0}_k)(V_j, e^{1,0}_k) = \frac{1}{2} \frac{1}{\sqrt{-1}} ||V_j||^2.
\]

Proof. Since \( \frac{1}{2} \left( 1 + \frac{J}{\sqrt{-1}} \right) \) is the projection \( TM \otimes \mathbb{C} \to T^{1,0}M \), we have

\[
\sum_{k=1}^{d_M} (JV_j, e^{1,0}_k)(V_j, e^{1,0}_k) = \sum_{k=1}^{d_M} \left( \frac{1}{2} \left( 1 + \frac{J}{\sqrt{-1}} \right) JV_j, e_k \right) \left( \frac{1}{2} \left( 1 + \frac{J}{\sqrt{-1}} \right) V_j, e_k \right) = \frac{1}{4} \left( \left( 1 + \frac{J}{\sqrt{-1}} \right) JV_j, \left( 1 + \frac{J}{\sqrt{-1}} \right) V_j \right) = \frac{1}{4} \left( (JV_j, V_j) + (JV_j, \frac{J}{\sqrt{-1}} V_j) + (-\frac{1}{\sqrt{-1}} V_j, V_j) + (-\frac{1}{\sqrt{-1}} V_j, \frac{J}{\sqrt{-1}} V_j) \right).
\]

Since \( JV_j \) is orthogonal to \( V_j \), the first and last terms in the latter expression vanish. Since the extension of the Riemannian metric to \( TM \otimes \mathbb{C} \) was (tacitly) taken to be complex-antilinear in the first coordinate, the remaining two terms add up to \( \frac{1}{2} ||V_j||^2 \). \( \square \)
Proof of Proposition B.1. Let $s \in \Omega^{0,*}(M; L^p)$ and all $m \in M$. We saw in (44) that

\[(\nabla X^H_{\mu} s)(m) = 2 \sum_{j=1}^{d_G} \mu_j(m)(L_{h_j^*}(m)s)(m) + 4\pi \sqrt{-1}p\mathcal{H}(m)s(m)\]

\[+ \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(m)\left(c(e_k)c(\nabla e_k h_j^*(m)^M)\right)(m)s(m)\]

\[+ \sum_{j=1}^{d_G} \mu_j(m) \text{tr}\left(\nabla^{T,0} h_j^*(m)^M_{|T,0} \right) \right)(m)s(m)\].

By Lemma B.2, the third term on the right hand side of (73) equals

\[(14) \left(\frac{1}{4} \sum_{k=1}^{d_M} c(e_k)c(\nabla e_k X^H_{\mu}) - \frac{1}{2} \sum_{j=1}^{d_G} c(\text{grad } \mu_j)c(V_j)\right)

\[+ \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(c(e_k)c(\left[e_k, (h_j^*)^M - V_j\right]) \right)(m)s(m)\].

The gradient of $\mu_j$ was computed in Lemma B.3

\[(75) \text{grad } \mu_j = JV_j + \langle \mu, Th_j^* \rangle^*\]

Therefore, (74) equals

\[(76) \left(\frac{1}{4} \sum_{k=1}^{d_M} c(e_k)c(\nabla e_k X^H_{\mu}) - \frac{1}{2} \sum_{j=1}^{d_G} c(JV_j)c(V_j) - \frac{1}{2} \sum_{j=1}^{d_G} c(\langle \mu, Th_j^* \rangle\rangle)c(V_j)\right)

\[+ \frac{1}{2} \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j(c(e_k)c(\left[e_k, (h_j^*)^M - V_j\right]) \right)(m)s(m)\].

By Lemma B.4 the last term on the right hand side of (44) equals

\[(77) \left(\frac{1}{2} \text{tr}\left(\nabla^{T,0} X_{1,0}^H_{\mu} |_{T,0} \right) - \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} e_{k,1,0}^{1,0}(\mu_j)(V_j, e_{k,1,0}^{1,0})\right)

\[+ \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j\left([e_{k,1,0}^{1,0}, (h_j^*)^M - V_j], e_{k,1,0}^{1,0}\right) \right)(m)s(m)\].

Because of (75), one has

\[e_{k,1,0}^{1,0}(\mu_j) = (JV_j, e_{k,1,0}^{1,0}) + \langle \langle \mu, Th_j^* \rangle, e_{k,1,0}^{1,0}\rangle\]

It was computed in Lemma B.5 that

\[\sum_{k=1}^{d_M} (JV_j, e_{k,1,0}^{1,0})(V_j, e_{k,1,0}^{1,0}) = \frac{1}{2\sqrt{-1}}\|V_j\|^2.\]
Hence (77) equals
\[
(78) \quad \left( \frac{1}{2} \text{tr} \left( \nabla^{T_{1,0}^M} X^H_{T_{1,0}^M} \right) - \frac{1}{2\sqrt{-1}} \sum_{j=1}^{d_G} \|V_j\|^2 - \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \langle \mu, T h_j^* \rangle \langle V_j, e_k^{1,0} \rangle \right) + \sum_{j=1}^{d_G} \sum_{k=1}^{d_M} \mu_j \left( \langle e_k^{1,0}, (h_j^*)^M - V_j \rangle, e_k^{1,0} \right) (m)s(m).
\]

Using expression (76) for the third term in (73) and expression (78) for the last term yields the desired equality. \qed

**Appendix C. Estimates used to bound the operator A**

The facts in this appendix were used in the proof of Proposition 6.5.

**Lemma C.1.** Let $M$ be a Riemannian manifold. There is an open cover $\{\widetilde{U}_l\}$ of $M$ such that

- every open set $\widetilde{U}_l$ admits a local orthonormal frame for $TM$;
- every compact subset of $M$ intersects finitely many of the sets $\widetilde{U}_l$ nontrivially;
- there is a relatively compact subset $U_l \subset \widetilde{U}_l$ for all $l$, such that $\overline{U}_l \subset \widetilde{U}_l$, and $\bigcup_l U_l = M$.

**Proof.** Let $\{K_j\}_{j=1}^\infty$ be a sequence of compact subsets of $M$ such that

- for all $j$, the set $K_j$ is contained in the interior of $K_{j+1}$;
- the sets $K_j$ cover $M$.

Write $K_{-1} = K_0 = \emptyset$. For every $j \in \mathbb{N}$, the complement of the interior of $K_{j-1}$ in $K_j$ is compact, and hence admits a cover $\{\widetilde{U}_{j,1}, \ldots, \widetilde{U}_{j,n_j}\}$ by open subsets of $K_{j+1} \setminus K_{j-2}$ that admit local orthonormal frames of $TM$ and have relatively compact subsets $U_{j,l} \subset \widetilde{U}_{j,k}$ such that $\overline{U}_{j,k} \subset \widetilde{U}_{j,k}$, which cover $K_j \setminus K_{j-1}$. The cover
\[
\{\widetilde{U}_{j,k}; j \in \mathbb{N}, k \in \{1, \ldots, n_j\}\}
\]
has the desired properties. \qed

**Lemma C.2.** Let $X$ be a topological space, and let $\{\widetilde{U}_l\}$ be an open cover of $X$ such that

- every compact subset of $X$ intersects finitely many of the sets $\widetilde{U}_l$ nontrivially;
- there is a relatively compact subset $U_l \subset \widetilde{U}_l$ for all $l$, such that $\overline{U}_l \subset \widetilde{U}_l$, and $\bigcup_l U_l = X$.

For all $l$, let $h_l$ be a continuous function on $\widetilde{U}_l$.

Let $G$ be a group acting on $X$. Let $W \subset X$ be a subset whose intersection with every nonempty cocompact subset of $X$ is nonempty and compact. Then there is a positive, $G$-invariant, continuous function $F \in C(X)^G$ such that for all $x \in W$, and for all $l$ such that $x \in U_l$,
\[
h_l(x) \leq F(x).
\]

**Proof.** For every $l$, the restriction $h|_{\overline{U}_l}$ is bounded, so there is a constant $C_l > 0$ such that $h_l \leq C_l$ on $U_l$. For $x \in X$, set
\[
B(x) := \max\{C_l; (G \cdot x) \cap W \cap U_l \neq \emptyset\}
\]
Since the set \((G \cdot x) \cap W\) is compact for all \(x\), it intersects finitely many of the sets \(U_i\). Hence \(B(x)\) is finite for all \(x\). In fact, any cocompact set \(Z \subset X\) has compact intersection with \(W\), and hence intersects finitely many sets \(U_i\). Therefore, the function \(B\) is bounded on \(Z\). In addition, it is \(G\)-invariant, but not continuous in general.

Let \(B_G\) be the function on \(X/G\) induced by \(B\). Since it is bounded on compact subsets, there is a continuous function \(F_G \in C(X/G)\) such that \(B_G \leq F_G\). The pullback \(F\) of \(F_G\) along the quotient map \(X \to X/G\) has the desired property. (It is not necessary that \(F\) is positive in general, but one may choose \(F\) in this way.)

**Lemma C.3.** Let \(G\) be a Lie group, acting properly and isometrically on a Riemannian manifold \(M\). Let \(\varphi_0,\varphi_1 \in C(M)^G\) be continuous, positive, \(G\)-invariant functions on \(M\). Then there exists a positive, \(G\)-invariant, smooth function \(\psi\) on \(M\), such that

\[
\psi \leq \varphi_0; \\
\|d\psi\| \leq \varphi_1.
\]

**Proof.** Let \(\{U_j\}_{j=1}^{\infty}\) be a countable, locally finite open cover of \(M\), by \(G\)-invariant, relatively cocompact open subsets. For example, one can use

\[
U_j := \{m \in M; d(G \cdot m_0, m) \in [j - 2, j]\},
\]

for a fixed orbit \(G \cdot m_0 \in M/G\), where \(d\) denotes the Riemannian distance on \(M\). Since the action is proper, there is a \(G\)-invariant partition of unity \(\{\tilde{\chi}_j\}_{j=1}^{\infty}\) on \(M\), subordinate to the cover \(\{U_j\}_{j=1}^{\infty}\) (see e.g. [10], Corollary B.33). Define the functions \(\chi_j\) by

\[
\chi_j = \frac{\tilde{\chi}_j}{\max_{m \in U_j} \|d_m \tilde{\chi}_j\| + 1}.
\]

Then for all \(m \in M\) and \(j \in \mathbb{N}\),

\[
\chi_j(m) \leq 1; \\
\|d_m \chi_j\| \leq 1; \\
\sum_{k=1}^{\infty} \chi_k(m) > 0.
\]

For every \(j \in \mathbb{N}\), set

\[
J_j := \{k \in \mathbb{N}; U_k \cap U_j \neq \emptyset\}; \\
a_j := \min_{m \in U_j} \varphi_0(m); \\
b_j := \min_{m \in U_j} \varphi_1(m); \\
\alpha_j := \min_{k \in J_j} \frac{\min(a_k, b_k)}{|J_k|}.
\]

Because \(\varphi_0\) and \(\varphi_1\) are continuous, positive and \(G\)-invariant, and \(U_j/G\) is compact, the numbers \(a_j\) and \(b_j\), and hence \(\alpha_j\), are positive for all \(j\). Define the function \(\psi\) by

\[
\psi := \sum_{j=1}^{\infty} \alpha_j \chi_j.
\]
This function is $G$-invariant and smooth, and positive everywhere. Let us show that $\psi$ and its derivative satisfy the desired estimates.

Fix $j \in \mathbb{N}$, and let $m \in U_j$. Then

$$\psi(m) = \sum_{k \in J_j} \alpha_k \chi_k(m) \leq \sum_{k \in J_j} \min_{l \in J_k} \frac{a_l}{\#J_l}.$$ 

Since $k \in J_j$ if and only if $j \in J_k$, the summands in the latter sum are at most equal to $\frac{a_j}{\#J_j}$. Therefore,

$$\psi(m) \leq \sum_{k \in J_j} \frac{a_j}{\#J_j} = a_j \leq \varphi_0(m).$$

Similarly, one estimates

$$\|d_m \psi\| = \left\| \sum_{k \in J_j} \alpha_k d_m \chi_k \right\| \leq \sum_{k \in J_j} \alpha_k \leq \sum_{l \in J_k} \min_{l \in J_k} \frac{b_l}{\#J_l} \leq b_j \leq \varphi_1(m).$$

\[\square\]

**Lemma C.4.** Let $v$ be one of the vector fields $e_k$ or $e_k^{1,0}$ in the proof of Proposition 6.3. Then one has

$$\|\nabla_v X_1^H \psi\| \leq 2N_\psi,$$

on $(M \setminus V) \cap W$.

**Proof.** Note that, outside $\text{Crit}_1(\mathcal{H})$, one has for all vector fields $v$ and all $k$,

$$\frac{\|\nabla_v X_1^H \psi\|}{N_\psi} \leq \frac{|v(\psi)|}{\psi^2} \frac{1}{N^{1/2}} + \frac{1}{N} \frac{\|\nabla_v X_1^H \psi\|}{N}.$$

Now suppose $v$ is one of the vector fields $e_k$ or $e_k^{1,0}$. Then $v$ has norm at most 1, so that the first term on the right hand side of (79) can be estimated on $M \setminus V$ by

$$\frac{|v(\psi)|}{\psi^2} \frac{1}{N^{1/2}} = \frac{|v(\psi^{-1})|}{\psi^2} \frac{1}{N^{1/2}} \leq \frac{\|d(\psi^{-1})\|}{N^{1/2}} \leq 1,$$

by (52). On $(M \setminus V) \cap W$, the estimate (51) implies that the second term in (79) is at most equal to

$$\frac{1}{\psi} \frac{\|\nabla_v X_1^H \psi\|}{N} \leq \frac{1}{\psi} \frac{F_1}{N} \leq 1.$$ 

\[\square\]

**Lemma C.5.** In the setting of the proof of Proposition 6.3, the operator $\tilde{A}_1^\psi$ satisfies the pointwise estimate

$$\|\tilde{A}_1^\psi\| \leq \frac{3}{2} d_M N_\psi,$$

on $(M \setminus V) \cap W$.

**Proof.** By Lemma C.4 one has on $(M \setminus V) \cap W$, for all $k$,

$$\left\| c(e_k) c_1 \left( \nabla_{e_k} X_1^H \psi \right) \right\| \leq \|\nabla_{e_k} X_1^H \psi\| \leq 2N_\psi.$$
Similarly, one has on \((M \setminus V) \cap W\),

\[
\left| \text{tr} \left( \nabla^{T^{1,0}M} X_1^{H_v} \right) \right| = \left| \sum_{k=1}^{d_M} \left( \nabla^{T^{1,0}M} X_1^{H_v}, e_k \right) \right|
\leq \sum_{k=1}^{d_M} \left\| \nabla^{T^{1,0}M} X_1^{H_v} \right\|
\leq 2d_M N_{\psi}.
\]

The estimate now follows from the definition (53) of \(\tilde{A}_1^\psi\).

\[\square\]

**Lemma C.6.** For all \(j\), one has

\[
\langle \mu, T(h_j^\psi)^* \rangle = \mu_j d(\psi^{1/2}) + \psi^{1/2}\langle \mu, Th_j^* \rangle.
\]

**Proof.** Since \((h_j^\psi)^* = \psi^{1/2}h_j^*\), one has for all \(m \in M\),

\[
T_m(h_j^\psi)^* = h_j^*(m) d_m(\psi^{1/2}) + \psi^{1/2}T_m h_j^*.
\]

Furthermore, one has \(\langle \mu, h_j^* \rangle = \mu_j\).

\[\square\]

**Lemma C.7.** In the setting of the proof of Proposition 6.3, one has for all \(j\),

\[
\| \langle \mu, T(h_j^\psi)^* \rangle \| \cdot \| V_j^\psi \| \leq 2N_{\psi},
\]

on \((M \setminus V) \cap W\).

**Proof.** By Lemma C.6, we find that for all \(j\),

\[
\frac{\| \langle \mu, T(h_j^\psi)^* \rangle \| \cdot \| V_j^\psi \|}{N_{\psi}} \leq \frac{\| d(\psi^{1/2}) \| \| \mu_j \| \cdot \| V_j \|}{\psi^{3/2} N} + \frac{1}{\psi} \frac{\| \langle \mu, Th_j^* \rangle \| \cdot \| V_j \|}{N}.
\]

By (51), the second of these terms is at most equal to 1 on \((M \setminus V) \cap W\). The first term is equal to

\[
\frac{\| d(\psi^{-1}) \| \| \mu_j \| \cdot \| V_j \|}{2 N},
\]

which is less than or equal to 1 on \(M \setminus V\), because of (52).

\[\square\]

**Lemma C.8.** For all vector fields \(v \in \mathfrak{X}(M)\) and all \(j\), one has

\[
[v, ((h_j^*)^\psi)^M - V_j^\psi] = \psi^{1/2} [v, (h_j^*)^M - V_j] - v(\psi^{1/2})V_j.
\]

**Proof.** The Leibniz rule for the Lie derivative of vector fields implies that for all \(m, m' \in M\),

\[
[v, ((h_j^*)^\psi(m))^M - V_j^\psi(m')] = [v, \psi(m)^{1/2}h_j^*(m)^M - \psi^{1/2}V_j(m')] (m')
= \psi(m)^{1/2} [v, h_j^*(m)^M] (m') - (\psi(m')^{1/2} [v, V_j(m')] + v(\psi^{1/2})(m') V_j(m')) .
\]

Taking \(m = m'\) yields the desired equality.

\[\square\]

**Lemma C.9.** Let \(v\) be one of the vector fields \(e_k\) or \(e_k^{1,0}\) in the proof of Proposition 6.3. Then one has for all \(j\), on \((M \setminus V) \cap W\),

\[
| \| \mu_j^\psi \| | \| v, ((h_j^*)^\psi)^M - V_j^\psi \| \leq 2N_{\psi}.
\]
Proof. By Lemma C.8 one has for all $v$ vector fields $v \in \mathfrak{X}(M)$, and for all $j$, and $m \in (M \setminus V) \cap W$,

\[
\left( \frac{\mu_j^N}{N} \left[ v, ((h_j^*)^\psi(m))^M - V_j^\psi \right] \right)(m)
\leq \left( \frac{1}{\psi} \left[ v, h_j^\psi(m)^M - V_j \right] \right)(m) + \left( \frac{\psi^{1/2}}{\psi^{3/2}} \frac{|v|}{N} \left[ |\mu_j| V_j \right] \right)(m).
\]

Now suppose $v$ is one of the vector fields $e_k$ or $e_k^{1,0}$. Then, because of (51), the first term on the right hand side is at most equal to 1. The second term is equal to

\[
\left( \frac{\psi^{-1}}{2} \frac{|v|}{N} \left[ |\mu_j| V_j \right] \right)(m).
\]

Because of (52), this expression is also at most equal to 1. \qed

REFERENCES

[1] D. Alekseevsky, A. Kriegl, M. Losik, P.W. Michor, The Riemannian geometry of orbit spaces: the metric, geodesics, and integrable systems. Dedicated to Professor Lajos Tamássy on the occasion of his 80th birthday. Publ. Math. Debrecen 62 (2003), no. 3-4, 247-276. MR2008095

[2] N. Anghel, An abstract index theorem on non-compact Riemannian manifolds, Houston J. Math. 19, (1993) 223–237. MR1225459

[3] J.-M. Bismut and G. Lebeau, Complex immersions and Quillen metrics, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp. (1992). MR1188532

[4] N. Bourbaki, Intégration, Éléments de mathématique vol. VI, ch. 7–8 (Hermann, Paris, 1963). MR0179291

[5] M. Braverman, Index theorem for equivariant Dirac operators on noncompact manifolds, K-Theory 27 (2002), no. 1, 61–101. MR1936585

[6] M. Braverman, The index theory on non-compact manifolds with proper group action, arXiv:1403.7587

[7] J.J. Duistermaat, The heat kernel Lefschetz fixed point theorem for the Spin$^c$-Dirac operator, Progress in nonlinear differential equations and their applications 18 (Birkhäuser, Basel, 1996). MR1365745

[8] T. Friedrich, Dirac operators in Riemannian geometry, Graduate studies in Mathematics 26 (American Mathematical Society, Providence, RI, 2000). MR177732

[9] M. Gromov and H.B. Lawson, Positive scalar curvature and the Dirac operator on complete riemannian manifolds, Publ. Math. Inst. Hautes Études Sci. 58 (1983) 83–196. MR0720933

[10] V. Guillemin, V. Ginzburg and Y. Karshon, Moment map, cobordisms, and Hamiltonian group actions, Mathematical Surveys and Monographs 98, (Amer. Math. Soc., Providence, RI, 2002). MR1929136

[11] V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (1982), no. 3, 515–538. MR0664118

[12] P. Hochs and N.P. Landsman, The Guillemin-Sternberg conjecture for noncompact groups and spaces, J. K-theory 1 (2008) 473–533. MR2433278

[13] P. Hochs, Quantisation commutes with reduction at discrete series representations of semisimple groups, Adv. Math. 222 (2009), no. 3, 862–919. MR2553372

[14] ———, Quantisation of presymplectic manifolds, K-theory and group representations, Proc. Amer. Math. Soc. (to appear), 17 pages, arXiv:1211.0107

[15] G. G. Kasparov, K-theoretic index theorems for elliptic and transversally elliptic operators J. K-theory (to appear), preprint 2013, 89 pages.

[16] N.P. Landsman, Functorial quantization and the Guillemin-Sternberg conjecture, Twenty years of Bialowieza: a mathematical anthology (eds. S. Ali, G. Emch, A. Odzijewicz, M. Schlichenmaier, & S. Woronowicz, World scientific, Singapore, 2005) 23–45. MR2181545
[17] X. Ma and W. Zhang, *Geometric quantization for proper moment maps*, C. R. Acad. Sci. Paris, Ser. I 347 (2009) 389–394. MR2537236
[18] ———, *Geometric quantization for proper moment maps: the Vergne conjecture*, Acta Math. (to appear), 40 pages, arXiv:0812.3989
[19] J.E. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. Math. Phys. 5 (1974), no. 1, 121–130. MR0402819
[20] V. Mathai and W. Zhang, *Geometric quantization for proper actions* (with an appendix by U. Bunke), Adv. Math. 225 (2010), no. 3, 1224-1247. arXiv:0806.3138 MR2673729
[21] E. Meinrenken, *On Riemann-Roch formulas for multiplicities*. J. Amer. Math. Soc. 9 (1996), no. 2, 373-389. MR1325798
[22] ———, *Symplectic surgery and the Spin\(^c\)-Dirac operator*, Adv. Math. 134 (1998) 240–277. MR1617809
[23] E. Meinrenken and R. Sjamaar, *Singular reduction and quantization*, Topology 38 (1999) 699–762. MR1679797
[24] R. Palais, *On the existence of slices for actions of non-compact Lie groups*, Ann. Math. 73 (1961), no. 2, 295–323. MR0126506
[25] P.-E. Paradan, *Localisation of the Riemann-Roch character*, J. Funct. Anal. 187 (2001), no. 2, 442–509. MR1875155
[26] ———, *Formal geometric quantization II*, Pacific J. of Math. 253 (2011) 169–212. MR2869441
[27] ———, *Quantization commutes with reduction in the noncompact setting: the case of the holomorphic discrete series*, arXiv:1201.5451
[28] Y. Tian and W. Zhang, *Symplectic reduction and quantization*. C. R. Acad. Sci. Paris Sr. I Math. 324 (1997), no. 4, 433–438. MR1440962
[29] ———, *An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg*, Invent. Math. 132 (1998), no. 2, 229–259. MR1621428
[30] M. Vergne, *Applications of equivariant cohomology*, International Congress of Mathematicians, vol. I, Eur. Math. Soc., Zürich (2007) 635664. MR2334206
[31] M. Vergne, *Transversally elliptic operators and quantization*, Lusztig’s anniversary conference, MIT (2006).
[32] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982), no. 4, 661-692. MR0683171

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ADELAIDE, ADELAIDE 5005, AUSTRALIA
E-mail address: peter.hochs@adelaide.edu.au

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ADELAIDE, ADELAIDE 5005, AUSTRALIA
E-mail address: mathai.varghese@adelaide.edu.au