TRANSVERSALITY THEOREMS ON GENERIC LINEARLY
PERTURBED MAPPINGS

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In memory of John Mather

Abstract. In his celebrated paper “Generic projections”, John Mather has
given a striking transversality theorem and its applications on generic pro-
jections. On the other hand, in this paper, two transversality theorems on
generic linearly perturbed $C^r$ mappings are shown ($r \geq 1$). Moreover, some
applications of the two theorems are also given.

1. Introduction

Throughout this paper, let $\ell$, $m$ and $n$ stand for positive integers. In this paper,
unless otherwise stated, all manifolds and mappings are assumed to be of class
$C^r$ ($r \geq 1$) and all manifolds are assumed to be without boundary and to have
countable bases.

Let $F : U \to \mathbb{R}^\ell$ be a $C^r$ mapping from an open subset $U$ of $\mathbb{R}^m$. Then, for any
linear mapping $\pi : \mathbb{R}^m \to \mathbb{R}^\ell$, set

$$F_\pi = F + \pi.$$  

Here, the mapping $\pi$ in $F_\pi = F + \pi$ is restricted to $U$.

Let $L(\mathbb{R}^m, \mathbb{R}^\ell)$ be the space consisting of all linear mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$.
Notice that we have the natural identification $L(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$. By $N$, we
denote a $C^r$ manifold of dimension $n$. For given $C^r$ mappings $f : N \to U$ and
$F : U \to \mathbb{R}^\ell$, a property of mappings $F_\pi \circ f : N \to \mathbb{R}^\ell$ (resp., $\pi \circ f : N \to \mathbb{R}^\ell$)
will be said to be true for a generic linearly perturbed mapping (resp., a generic
projection) if there exists a subset $\Sigma$ with Lebesgue measure zero of $L(\mathbb{R}^m, \mathbb{R}^\ell)$ such
that for any $\pi \in L(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ (resp., $\pi \circ f : N \to \mathbb{R}^\ell$)
has the property.

In his celebrated paper [5], for a given $C^\infty$ embedding $f : N \to \mathbb{R}^m$, John Mather
has given a striking transversality theorem on a generic projection $\pi \circ f : N \to \mathbb{R}^\ell$
($m > \ell$), where $N$ is a $C^\infty$ manifold (for details on this result, see [3] Theo-
rem 1 (p. 229))). Moreover, in [5], as an application of this result, he has also
shown that if $f : N \to \mathbb{R}^m$ is a $C^\infty$ embedding and $(n, \ell)$ is in the nice range of
dimensions (for the definition of nice range of dimensions, refer to [4]), then a generic
projection $\pi \circ f : N \to \mathbb{R}^\ell$ ($m > \ell$) is stable, where $N$ is a compact $C^\infty$ manifold.

In [3], an improvement of the transversality theorem of [5] is given by replacing
generic projections by generic linear perturbations. Namely, in [3], for a given $C^\infty$
embedding $f : N \to U$ and a given $C^\infty$ mapping $F : U \to \mathbb{R}^\ell$, a transversality

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theorem on a generic linearly perturbed mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is given, where $N$ is a $C^\infty$ manifold and $\ell$ is an arbitrary positive integer which may possibly satisfy $m \leq \ell$.

Moreover, in [2], for a given $C^\infty$ immersion or a given $C^\infty$ injection $f : N \to U$, transversality theorems on a generic linearly perturbed mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ are given, where $N$ is a $C^\infty$ manifold, $F : U \to \mathbb{R}^\ell$ is a $C^\infty$ mapping and $\ell$ is an arbitrary positive integer which may possibly satisfy $m \leq \ell$.

On the other hand, in this paper, as improvements of some results in [2], two main transversality theorems (Theorems 1 and 2 in Section 2) and their applications on generic linearly perturbed mapping are given in the case where manifolds and mappings are not necessarily of class $C^\infty$.

The first main theorem (Theorem 1) is as follows. Let $f : N \to U$ (resp., $F : U \to \mathbb{R}^\ell$) be a $C^r$ immersion (resp., a $C^r$ mapping), where $N$ is a $C^r$ manifold (for the value of $r$, see Theorem 1). Then, generally, the composition $F \circ f$ does not necessarily yield a mapping transverse to the subfiber-bundle of the jet bundle $J^1(N, \mathbb{R}^\ell)$ with a fiber $\Sigma^k$, where $k$ is a positive integer satisfying $1 \leq k \leq \min\{n, \ell\}$ and

$$\Sigma^k = \{j^1g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k\}.$$

Nevertheless, Theorem 1 asserts that a generic linearly perturbed mapping $F_\pi \circ f$ yields a mapping transverse to the subfiber-bundle of $J^1(N, \mathbb{R}^\ell)$ with $\Sigma^k$. The second main theorem (Theorem 2) is a specialized transversality theorem on crossings of a generic linearly perturbed mapping $F_\pi \circ f$, where $N$ is a $C^r$ manifold, $f : N \to U$ is a given $C^r$ injection and $F : U \to \mathbb{R}^\ell$ is a given $C^r$ mapping (for the value of $r$, see Theorem 2).

For a given $C^2$ immersion (resp., $C^1$ injection) $f : N \to U$ and a given $C^2$ mapping (resp., $C^1$ mapping) $F : U \to \mathbb{R}^\ell$, the following (1) and (2) (resp., (3)) are obtained as applications of Theorem 1 (resp., Theorem 2), where $N$ is a $C^2$ manifold (resp., a $C^1$ manifold).

1. If $(n, \ell) = (n, 1)$, then a generic linearly perturbed function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.
2. If $\ell \geq 2n$, then a generic linearly perturbed mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion.
3. If $\ell > 2n$, then a generic linearly perturbed mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an injection.

Furthermore, by combining the assertions (2) and (3), for a given $C^2$ embedding $f : N \to U$ and a given $C^2$ mapping $F : U \to \mathbb{R}^\ell$, we get the following assertion (4), where $N$ is a $C^2$ manifold.

4. If $\ell > 2n$ and $N$ is compact, then a generic linearly perturbed mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an embedding.

In Section 2 some definitions are prepared, and the two main transversality theorems (Theorems 1 and 2) are stated. Section 3 (resp., Section 4) is devoted to the proof of Theorem 1 (resp., Theorem 2). In Section 5, the above assertions (1)–(4) are shown. In Section 6, the important lemma for the proofs of Theorems 1 and 2 (Lemma 1 in Section 2) is shown as an appendix.
2. Preliminaries and the statements of Theorems 1 and 2

Firstly, the definition of transversality is given.

**Definition 1.** Let $N$ and $P$ be $C^r$ manifolds, and $Z$ be a $C^r$ submanifold of $P$ ($r \geq 1$). Let $g : N \to P$ be a $C^1$ mapping.

1. We say that $g : N \to P$ is transverse to $Z$ at $q$ if $g(q) \notin Z$ or in the case of $g(q) \in Z$, the following holds:
   \[ dg_q(T_qN) + T_{g(q)}Z = T_{g(q)}P. \]
2. We say that $g : N \to P$ is transverse to $Z$ if for any $q \in N$, the mapping $g$ is transverse to $Z$ at $q$.

For the statement and the proof of Theorem 1 some definitions are prepared. Let $N$ be a $C^r$ manifold ($r \geq 2$) and $J^1(N, \mathbb{R}^\ell)$ be the space of 1-jets of mappings of $N$ into $\mathbb{R}^\ell$. Then, note that $J^1(N, \mathbb{R}^\ell)$ is a $C^{r-1}$ manifold. For a given $C^r$ mapping $g : N \to \mathbb{R}^\ell$ ($r \geq 2$), the mapping $j^1g : N \to J^1(N, \mathbb{R}^\ell)$ is defined by $q \mapsto j^1g(q)$. Then, notice that the mapping $j^1g : N \to J^1(N, \mathbb{R}^\ell)$ is of class $C^{r-1}$. For details on the space $J^1(N, \mathbb{R}^\ell)$ or the mapping $j^1g : N \to J^1(N, \mathbb{R}^\ell)$, see for example, [1].

Now, let $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ be a coordinate neighborhood system of $N$. Let $\Pi : J^1(N, \mathbb{R}^\ell) \to N \times \mathbb{R}^\ell$ be the natural projection defined by $\Pi(j^1g(q)) = (g(q), q)$. Let $\Phi_\lambda : \Pi^{-1}(U_\lambda \times \mathbb{R}^\ell) \to \varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times J^1(n, \ell)$ be the homeomorphism defined by

\[ \Phi_\lambda(j^1g(q)) = (\varphi_\lambda(q), g(q), j^1(\psi_\lambda \circ g \circ \varphi^{-1}_\lambda \circ \varphi_\lambda)(0)), \]

where $J^1(n, \ell) = \{j^1g(0) \mid g : (\mathbb{R}^n, 0) \to (\mathbb{R}^\ell, 0)\}$ and $\varphi_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ (resp., $\psi_\lambda : \mathbb{R}^m \to \mathbb{R}^m$) is the translation given by $\varphi_\lambda(0) = \varphi_\lambda(q)$ (resp., $\psi_\lambda(g)(q) = 0$). Then, $\{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda}$ is a coordinate neighborhood system of $J^1(N, \mathbb{R}^\ell)$. Set

\[ \Sigma^k = \{j^1g(0) \in J^1(n, \ell) \mid \text{corank } Jg(0) = k\}, \]

where $\text{corank } Jg(0) = \min\{n, \ell\} - \text{rank } Jg(0)$ and $k = 1, 2, \ldots, \min\{n, \ell\}$. Set

\[ \Sigma^k(N, \mathbb{R}^\ell) = \bigcup_{\lambda \in \Lambda} \Phi^{-1}_\lambda(\varphi_\lambda(U_\lambda) \times \mathbb{R}^\ell \times \Sigma^k). \]

Then, the set $\Sigma^k(N, \mathbb{R}^\ell)$ is a submanifold of $J^1(N, \mathbb{R}^\ell)$ satisfying

\[ \text{codim } \Sigma^k(N, \mathbb{R}^\ell) = \dim J^1(N, \mathbb{R}^\ell) - \dim \Sigma^k(N, \mathbb{R}^\ell) = (n - v + k)(\ell - v + k), \]

where $v = \min\{n, \ell\}$. (For details on $\Sigma^k$ and $\Sigma^k(N, \mathbb{R}^\ell)$, see for instance [1], pp. 60–61).

Then, the first main theorem in this paper is the following.

**Theorem 1.** Let $f$ be a $C^r$ immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$, where $N$ is a $C^r$ manifold of dimension $n$. Let $F : U \to \mathbb{R}^\ell$ be a $C^r$ mapping and $k$ be a positive integer satisfying $1 \leq k \leq \min\{n, \ell\}$. If

\[ r > \max\{\dim N - \text{codim } \Sigma^k(N, \mathbb{R}^\ell), 0\} + 1, \]

then there exists a subset $\Sigma$ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to the submanifold $\Sigma^k(N, \mathbb{R}^\ell)$. 

Moreover, if the mapping \( F \)

There is an advantage that the domain of the mapping \( F \)

Note that \( N^{(s)} \) is an open submanifold of \( N^s \). For any mapping \( g : N \rightarrow \mathbb{R}^\ell \), let \( g^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s \) be the mapping given by

\[
g^{(s)}(q_1, q_2, \ldots, q_s) = (g(q_1), g(q_2), \ldots, g(q_s)).
\]

Set \( \Delta_s = \{(y_1, \ldots, y) \in (\mathbb{R}^\ell)^s \mid y \in \mathbb{R}^\ell\} \). Then, \( \Delta_s \) is a submanifold of \( (\mathbb{R}^\ell)^s \) satisfying

\[
\text{codim} \Delta_s = \dim((\mathbb{R}^\ell)^s) - \dim \Delta_s = \ell(s - 1).
\]

**Definition 2.** Let \( g \) be a \( C^1 \) mapping of \( N \) into \( \mathbb{R}^\ell \), where \( N \) is a \( C^r \) manifold \((r \geq 1)\). Then, \( g \) is called a mapping with normal crossings if for any positive integer \( s \) \((s \geq 2)\), the mapping \( g^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s \) is transverse to \( \Delta_s \).

As in [2], for any injection \( f : N \rightarrow \mathbb{R}^m \), set

\[
s_f = \max \left\{ s \mid \forall(q_1, q_2, \ldots, q_s) \in N^{(s)}, \dim \sum_{i=2}^s \mathbb{R}f(q_1)f(q_i) = s - 1 \right\}.
\]

Since the mapping \( f \) is an injection, we have \( 2 \leq s_f \). Since \( f(q_1), f(q_2), \ldots, f(q_{s_f}) \) are points of \( \mathbb{R}^m \), it follows that \( s_f \leq m + 1 \). Hence, we get

\[
2 \leq s_f \leq m + 1.
\]

Moreover, in the following, for a set \( X \), we denote the number of its elements (or its cardinality) by \(|X|\). Then, the second main theorem in this paper is the following.

**Theorem 2.** Let \( f \) be a \( C^r \) injection of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \), where \( N \) is a \( C^r \) manifold of dimension \( n \). Let \( F : U \rightarrow \mathbb{R}^\ell \) be a \( C^r \) mapping. If

\[
r > \max\{s_0, 0\},
\]

then there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), and for any \( s \) \((2 \leq s \leq s_f)\), the \( C^r \) mapping \((F_\pi \circ f)^{(s)} : N^{(s)} \rightarrow (\mathbb{R}^\ell)^s\) is transverse to the submanifold \( \Delta_s \), where

\[
s_0 = \max\{s(n - \ell) + \ell \mid 2 \leq s \leq s_f\}.
\]

Moreover, if the mapping \( F_\pi \) satisfies that \(|F_\pi^{-1}(y)| \leq s_f \) for any \( y \in \mathbb{R}^\ell \), then \( F_\pi \circ f : N \rightarrow \mathbb{R}^\ell \) is a \( C^r \) mapping with normal crossings.

**Remark 1.**  
(1) There is an advantage that the domain of the mapping \( F \) is an arbitrary open set. Suppose that \( U = \mathbb{R} \). Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be the function defined by \( x \mapsto |x| \). Since \( F \) is not differentiable at \( x = 0 \), we cannot apply Theorems 1 and 2 to \( F : \mathbb{R} \rightarrow \mathbb{R} \).

On the other hand, if \( U = \mathbb{R} - \{0\} \), then Theorems 1 and 2 can be applied to the restriction \( F|_U \).

(2) As in (1), there is a case of \( s_f = 3 \) as follows. If \( n + 1 \leq m \), \( N = S^n \) and \( f : S^n \rightarrow \mathbb{R}^m \) is the inclusion \( f(x) = (x, 0, \ldots, 0) \), then we get \( s_f = 3 \).

Indeed, suppose that there exists a point \((q_1, q_2, q_3) \in (S^n)^3\) satisfying \( \dim \sum_{i=2}^3 \mathbb{R}f(q_1)f(q_i) = 1 \). Then, since the number of the intersections of \( f(S^n) \) and a straight line of \( \mathbb{R}^m \) is at most two, this contradicts the
assumption. Thus, we have \( s_f \geq 3 \). From \( S^1 \times \{0\} \subset f(S^n) \), we get \( s_f < 4 \), where \( 0 = (0, \ldots, 0) \). Therefore, it follows that \( s_f = 3 \).

(3) The essential idea for the proofs of Theorems 1 and 2 is to apply Lemma 1 and it is similar to the idea of the proofs of Theorems 1 and 2. Note that in the special case \( r = \infty \), from some results in [2], the results in this paper (Theorems 1 and 2 in this section and Corollaries 1 to 7 in Section 5) can be obtained.

The following well known result is important for the proofs of Theorems 1 and 2. In [1], the proof of Lemma 1 in the case \( r = \infty \) is given. Hence, for the sake of readers’ convenience, the proof of Lemma 1 is given in Section 6 as an appendix.

**Lemma 1 ([1]).** Let \( N, A, P \) be \( C^r \) manifolds, \( Z \) be a \( C^r \) submanifold of \( P \) and \( \Gamma : N \times A \to P \) be a \( C^r \) mapping. If

\[
r > \max\{\dim N - \dim Z, 0\},
\]

and \( \Gamma \) is transverse to \( Z \), then there exists a subset \( \Sigma \) of \( A \) with Lebesgue measure zero such that for any \( q \in A - \Sigma \), the \( C^r \) mapping \( \Gamma_a : N \to P \) is transverse to \( Z \), where \( \dim Z = \dim P - \dim Z \) and \( \Gamma_a(q) = \Gamma(q, a) \).

### 3. Proof of Theorem 1

In this proof, for a positive integer \( \tilde{n} \), we denote the \( \tilde{n} \times \tilde{n} \) unit matrix by \( E_{\tilde{n}} \). Let \( (\alpha_{ij})_{1 \leq i \leq l, 1 \leq j \leq m} \) be a representing matrix of a linear mapping \( \pi : \mathbb{R}^m \to \mathbb{R}^l \). Set \( F_\alpha = F_\pi \). Then, we have

\[
F_\alpha(x) = \left( F_1(x) + \sum_{j=1}^{m} \alpha_{1j} x_j, F_2(x) + \sum_{j=1}^{m} \alpha_{2j} x_j, \ldots, F_l(x) + \sum_{j=1}^{m} \alpha_{lj} x_j \right),
\]

where \( F = (F_1, F_2, \ldots, F_l), \alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1m}, \ldots, \alpha_{l1}, \alpha_{l2}, \ldots, \alpha_{lm}) \in (\mathbb{R}^m)^l \) and \( x = (x_1, x_2, \ldots, x_m) \). For a given \( C^r \) immersion \( f : N \to U \), the \( C^r \) mapping \( F_\alpha \circ f : N \to \mathbb{R}^l \) is given as follows:

\[
F_\alpha \circ f = \left( F_1 \circ f + \sum_{j=1}^{m} \alpha_{1j} f_j, F_2 \circ f + \sum_{j=1}^{m} \alpha_{2j} f_j, \ldots, F_l \circ f + \sum_{j=1}^{m} \alpha_{lj} f_j \right),
\]

where \( f = (f_1, f_2, \ldots, f_m) \). Since we have the natural identification \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^l) = (\mathbb{R}^m)^l \), for the proof, it is sufficient to show that there exists a subset \( \Sigma \) with Lebesgue measure zero of \( (\mathbb{R}^m)^l \) such that for any \( \alpha \in (\mathbb{R}^m)^l - \Sigma \), the mapping \( j^1(F_\alpha \circ f) : N \to J^1(N, \mathbb{R}^l) \) is transverse to \( \Sigma^k(N, \mathbb{R}^l) \).

Now, let \( \Gamma : N \times (\mathbb{R}^m)^l \to J^1(N, \mathbb{R}^l) \) be the \( C^{r-1} \) mapping defined by

\[
\Gamma(q, \alpha) = j^1(F_\alpha \circ f)(q).
\]

Note that \( r - 1 > \max\{\dim N - \dim \Sigma^k(N, \mathbb{R}^l), 0\} \). Thus, if \( \Gamma \) is transverse to \( \Sigma^k(N, \mathbb{R}^l) \), then from Lemma 1 there exists a subset \( \Sigma \) of \( (\mathbb{R}^m)^l \) with Lebesgue measure zero such that for any \( \alpha \in (\mathbb{R}^m)^l - \Sigma \), the \( C^{r-1} \) mapping \( \Gamma_\alpha : N \to J^1(N, \mathbb{R}^l) \) is transverse to \( \Sigma^k(N, \mathbb{R}^l) \). Therefore, for the proof, it is sufficient to show that if \( \Gamma(q, \tilde{\alpha}) \in \Sigma^k(N, \mathbb{R}^l) \), then the following holds:

\[
d\Gamma_{(\tilde{q}, \tilde{\alpha})}(T_{(\tilde{q}, \tilde{\alpha})}(N \times (\mathbb{R}^m)^l)) + T_{\Gamma(q, \alpha)} \Sigma^k(N, \mathbb{R}^l) = T_{\Gamma(q, \alpha)} J^1(N, \mathbb{R}^l). \tag{3.3}
\]
As in Section 2, let \( \{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda} \) (resp., \( \{(\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda)\}_{\lambda \in \Lambda} \)) be a coordinate neighborhood system of \( N \) (resp., \( J^1(N, \mathbb{R}^\ell) \)). There exists a coordinate neighborhood \( (U_\lambda \times (\mathbb{R}^m)^\ell, \varphi_\lambda \times id) \) containing the point \((\tilde{q}, \tilde{\alpha})\) of \( N \times (\mathbb{R}^m)^\ell \), where \( id \) is the identity mapping of \((\mathbb{R}^m)^\ell \) into \((\mathbb{R}^m)^\ell \), and the mapping \( \varphi_\lambda \times id : U_\lambda \times (\mathbb{R}^m)^\ell \to \varphi_\lambda(U_\lambda) \times (\mathbb{R}^m)^\ell \) is given by \((\varphi_\lambda \times id)(q, \alpha) = (\varphi_\lambda(q), id(\alpha))\). There exists a coordinate neighborhood \( (\Pi^{-1}(U_\lambda \times \mathbb{R}^\ell), \Phi_\lambda) \) containing the point \( \Gamma(\tilde{q}, \tilde{\alpha}) \) of \( J^1(N, \mathbb{R}^\ell) \). Let \( t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n \) be a local coordinate on \( \varphi_\lambda(U_\lambda) \) containing \( \varphi_\lambda(\tilde{q}) \). Then, the mapping \( \Gamma \) is locally given by the following:

\[
(\Phi_\lambda \circ \Gamma \circ (\varphi_\lambda \times id)^{-1})(t, \alpha) = (\Phi_\lambda \circ j^1(F_\alpha \circ f \circ \varphi_\lambda^{-1})(t, \alpha)
\]

where \( j^1(F_\alpha \circ f \circ \varphi_\lambda^{-1}) \) is a subfiber-bundle of \( \tilde{\jmath} \) with the fiber \( J \) containing the point \( \Gamma(\tilde{q}, \tilde{\alpha}) \).

\[
\begin{align*}
\frac{\partial(F_{\alpha,1} \circ f \circ \varphi_\lambda^{-1})}{\partial t_1}(t), & \quad \frac{\partial(F_{\alpha,1} \circ f \circ \varphi_\lambda^{-1})}{\partial t_2}(t), \ldots, \frac{\partial(F_{\alpha,1} \circ f \circ \varphi_\lambda^{-1})}{\partial t_n}(t), \\
\frac{\partial(F_{\alpha,2} \circ f \circ \varphi_\lambda^{-1})}{\partial t_1}(t), & \quad \frac{\partial(F_{\alpha,2} \circ f \circ \varphi_\lambda^{-1})}{\partial t_2}(t), \ldots, \frac{\partial(F_{\alpha,2} \circ f \circ \varphi_\lambda^{-1})}{\partial t_n}(t), \\
& \quad \cdots, \\
\frac{\partial(F_{\alpha,\ell} \circ f \circ \varphi_\lambda^{-1})}{\partial t_1}(t), & \quad \frac{\partial(F_{\alpha,\ell} \circ f \circ \varphi_\lambda^{-1})}{\partial t_2}(t), \ldots, \frac{\partial(F_{\alpha,\ell} \circ f \circ \varphi_\lambda^{-1})}{\partial t_n}(t),
\end{align*}
\]

\[
= \left( t, (F_\alpha \circ f \circ \varphi_\lambda^{-1})(t),
\frac{\partial F_1 \circ \tilde{f}}{\partial t_1}(t) + \sum_{j=1}^{m} \alpha_{j1} \frac{\partial \tilde{f}_j}{\partial t_1}(t),
\frac{\partial F_2 \circ \tilde{f}}{\partial t_1}(t) + \sum_{j=1}^{m} \alpha_{j2} \frac{\partial \tilde{f}_j}{\partial t_1}(t),
\frac{\partial F_\ell \circ \tilde{f}}{\partial t_1}(t) + \sum_{j=1}^{m} \alpha_{\ell j} \frac{\partial \tilde{f}_j}{\partial t_1}(t),
\right)
\]

where \( F_\alpha = (F_{\alpha,1}, F_{\alpha,2}, \ldots, F_{\alpha,\ell}) \) and \( \tilde{f} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m) = (f_1 \circ \varphi_\lambda^{-1}, f_2 \circ \varphi_\lambda^{-1}, \ldots, f_m \circ \varphi_\lambda^{-1}) = f \circ \varphi_\lambda^{-1} \). The Jacobian matrix of \( \Gamma \) at \((\tilde{q}, \tilde{\alpha})\) is the following:

\[
J\Gamma(\tilde{q}, \tilde{\alpha}) = \begin{pmatrix}
E_n & 0 & \cdots & 0 \\
* & \cdots & \cdots & * \\
\text{t} J\tilde{f}_q & 0 & \cdots & \text{t} J\tilde{f}_q \\
0 & \cdots & \cdots & \text{t} J\tilde{f}_q
\end{pmatrix}_{(t, \alpha) = (\varphi_\lambda(q), \tilde{\alpha})}
\]

where \( J\tilde{f}_q \) is the Jacobian matrix of \( f \) at \( \tilde{q} \). Notice that \( \text{t} J\tilde{f}_q \) is the transpose of \( J\tilde{f}_q \) and that there are \( \ell \) copies of \( \text{t} J\tilde{f}_q \) in the above description of \( J\Gamma(\tilde{q}, \tilde{\alpha}) \). Since \( \Sigma^k(N, \mathbb{R}^\ell) \) is a subfiber-bundle of \( J^1(N, \mathbb{R}^\ell) \) with the fiber \( \Sigma^k \), in order to show
(3.3), it is sufficient to prove that the matrix $M_1$ given below has rank $n + \ell + n\ell$:

$$M_1 = \begin{pmatrix}
E_{n+\ell} & * & \cdots & * \\
0 & (J_fq) & 0 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & (J_fq)
\end{pmatrix}.$$

Notice that there are $\ell$ copies of $(J_fq)$ in the above description of $M_1$. Note that for any $i (1 \leq i \leq n\ell)$, the $(n + \ell + i)$-th column vector of $M_1$ coincides with the $(n + i)$-th column vector of $J(r_\alpha)$. Since $f$ is an immersion ($n \leq m$), the rank of $M_1$ is equal to $n + \ell + n\ell$. Therefore, we get (3.3). 

\[\square \]

4. Proof of Theorem \[\[\text{2}\]\]

As in the proof of Theorem \[\[\text{1}\]\] set $F_\alpha = F_\pi$, where $F_\alpha$ is given by (3.1) in Section \[\[\text{B}\]\]. For a given $C^r$ injection $f : N \to U$, the $C^r$ mapping $F_\alpha \circ f : N \to \mathbb{R}^\ell$ is given by the same expression as (3.2). Since we have the natural identification $L(\mathbb{R}^m, \mathbb{R}^\ell) = (\mathbb{R}^m)^\ell$, in order to prove that there exists a subset $\Sigma$ of $L(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in L(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any $s (2 \leq s \leq s_f)$, the $C^r$ mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to $\Delta_s$, it is sufficient to prove that there exists a subset $\Sigma$ of $L(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any $s (2 \leq s \leq s_f)$, the $C^r$ mapping $(F_\alpha \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to the submanifold $\Delta_s$.

Now, let $s$ be a positive integer satisfying $2 \leq s \leq s_f$. Let $\Gamma : N^{(s)} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^\ell)^s$ be the $C^r$ mapping given by

$$\Gamma (q_1, q_2, \ldots, q_s, \alpha) = ((F_\alpha \circ f)(q_1), (F_\alpha \circ f)(q_2), \ldots, (F_\alpha \circ f)(q_s)).$$

Note that from $r > \max\{s_0, 0\}$, we have

$$r > \max\{s(n - \ell) + \ell, 0\} = \max\{\dim N^{(s)} - \text{codim} \Delta_s, 0\}$$

for any positive integer $s (2 \leq s \leq s_f)$. Thus, if for any positive integer $s (2 \leq s \leq s_f)$, the mapping $\Gamma$ is transverse to $\Delta_s$, then from Lemma \[\[\text{I}\]\] for any positive integer $s (2 \leq s \leq s_f)$, there exists a subset $\Sigma_s$ of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero such that for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma_s$, the mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ ($\Gamma_\alpha = (F_\alpha \circ f)^{(s)}$) is transverse to $\Delta_s$. Then, $\Sigma = \bigcup_{s=2}^{s_f} \Sigma_s$ is a subset of $(\mathbb{R}^m)^\ell$ with Lebesgue measure zero. Thus, for any $\alpha \in (\mathbb{R}^m)^\ell - \Sigma$, and for any $s (2 \leq s \leq s_f)$, the $C^r$ mapping $\Gamma_\alpha : N^{(s)} \to (\mathbb{R}^\ell)^s$ ($\Gamma_\alpha = (F_\alpha \circ f)^{(s)}$) is transverse to $\Delta_s$.

Therefore, for this proof, it is sufficient to prove that for any positive integer $s (2 \leq s \leq s_f)$, if $\Gamma(q_\alpha) \in \Delta_s$ ($\bar{q} = (\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_s)$), then the following holds:

$$d\Gamma(q_\alpha)(T(q_\alpha)N^{(s)} \times (\mathbb{R}^m)^\ell) + T\Gamma(q_\alpha)\Delta_s = T\Gamma(q_\alpha)(\mathbb{R}^\ell)^s.$$

(4.1)

Let \{$(U_\lambda, \phi_\lambda)$\}$_{\lambda \in \Lambda}$ be a coordinate neighborhood system of $N$. There exists a coordinate neighborhood $(U_{\lambda_1} \times U_{\lambda_2} \times \cdots \times U_{\lambda_s} \times (\mathbb{R}^m)^\ell, \phi_{\lambda_1} \times \phi_{\lambda_2} \times \cdots \times \phi_{\lambda_s} \times id)$ containing $(\bar{q_\alpha})$ of $N^{(s)} \times (\mathbb{R}^m)^\ell$, where $id : (\mathbb{R}^m)^\ell \to (\mathbb{R}^m)^\ell$ is the identity mapping, and $\phi_{\lambda_1} \times \phi_{\lambda_2} \times \cdots \times \phi_{\lambda_s} \times id : U_{\lambda_1} \times U_{\lambda_2} \times \cdots \times U_{\lambda_s} \times (\mathbb{R}^m)^\ell \to (\mathbb{R}^n)^s \times (\mathbb{R}^m)^\ell$ is defined by $(\phi_{\lambda_1} \times \phi_{\lambda_2} \times \cdots \times \phi_{\lambda_s} \times id)(q_1, q_2, \ldots, q_s, \alpha) = (\phi_{\lambda_1}(q_1), \phi_{\lambda_2}(q_2), \ldots, \phi_{\lambda_s}(q_s), id(\alpha))$. 

\[\square \]
Let \( t_i = (t_{i1}, t_{i2}, \ldots, t_{in}) \) be a local coordinate around \( \varphi_{\lambda_i}(\bar{q}_i) \) (\( 1 \leq i \leq s \)). Then, \( \Gamma \) is locally given by the following:

\[
\Gamma \circ (\varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \cdots \times \varphi_{\lambda_s} \times \text{id})^{-1}(t_1, t_2, \ldots, t_s, \alpha)
= \left( \begin{array}{c}
(F_{\alpha} \circ f \circ \varphi_{\lambda_1}^{-1})(t_1), (F_{\alpha} \circ f \circ \varphi_{\lambda_2}^{-1})(t_2), \ldots, (F_{\alpha} \circ f \circ \varphi_{\lambda_s}^{-1})(t_s)
\end{array} \right)
= \left( \begin{array}{c}
F_1 \circ \bar{f}(t_1) + \sum_{j=1}^{m} \alpha_{1j} \tilde{f}_j(t_1), F_2 \circ \bar{f}(t_1) + \sum_{j=1}^{m} \alpha_{2j} \tilde{f}_j(t_1), \ldots, F_{\ell} \circ \bar{f}(t_1) + \sum_{j=1}^{m} \alpha_{\ell j} \tilde{f}_j(t_1),
\end{array} \right)
= \left( \begin{array}{c}
F_1 \circ \bar{f}(t_2) + \sum_{j=1}^{m} \alpha_{1j} \tilde{f}_j(t_2), F_2 \circ \bar{f}(t_2) + \sum_{j=1}^{m} \alpha_{2j} \tilde{f}_j(t_2), \ldots, F_{\ell} \circ \bar{f}(t_2) + \sum_{j=1}^{m} \alpha_{\ell j} \tilde{f}_j(t_2),
\end{array} \right)
= \ldots \ldots,
= \left( \begin{array}{c}
F_1 \circ \bar{f}(t_3) + \sum_{j=1}^{m} \alpha_{1j} \tilde{f}_j(t_3), F_2 \circ \bar{f}(t_3) + \sum_{j=1}^{m} \alpha_{2j} \tilde{f}_j(t_3), \ldots, F_{\ell} \circ \bar{f}(t_3) + \sum_{j=1}^{m} \alpha_{\ell j} \tilde{f}_j(t_3)
\end{array} \right),
\]

where \( \bar{f}(t_i) = (\bar{f}_1(t_i), \bar{f}_2(t_i), \ldots, \bar{f}_m(t_i)) = (f_1 \circ \varphi_{\lambda_1}^{-1}(t_i), f_2 \circ \varphi_{\lambda_2}^{-1}(t_i), \ldots, f_m \circ \varphi_{\lambda_s}^{-1}(t_i)) \) (\( 1 \leq i \leq s \)). For simplicity, set \( t = (t_1, t_2, \ldots, t_s) \) and \( z = (\varphi_{\lambda_1} \times \varphi_{\lambda_2} \times \cdots \times \varphi_{\lambda_s})(\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_s) \).

The Jacobian matrix of \( \Gamma \) at \( (\bar{q}, \bar{\alpha}) \) is the following:

\[
J\Gamma_{(\bar{q}, \bar{\alpha})} = \begin{pmatrix}
\ast & B(t_1) \\
\ast & B(t_2) \\
\vdots & \vdots \\
\ast & B(t_s) \\
\end{pmatrix}_{(t, \alpha) = (z, \bar{\alpha})},
\]

where

\[
B(t_i) = \left[ \begin{array}{cccc}
b(t_{i1}) & b(t_{i2}) & 0 & \cdots \\
0 & b(t_{i2}) & \cdots & b(t_{is}) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & b(t_{is})
\end{array} \right]_{\ell \text{ rows}}
\]

and \( b(t_i) = (\bar{f}_1(t_i), \bar{f}_2(t_i), \ldots, \bar{f}_m(t_i)) \). By the construction of \( T_{\Gamma(\bar{q}, \bar{\alpha})} \Delta_s \), in order to prove (4.1), it is sufficient to prove that the rank of the following matrix \( M_2 \) is equal to \( \ell s \):

\[
M_2 = \left[ \begin{array}{ccc}
E_\ell & B(t_1) \\
E_\ell & B(t_2) \\
\vdots & \vdots \\
E_\ell & B(t_s)
\end{array} \right]_{t = z}.\]
There exists an $\ell s \times \ell s$ regular matrix $Q_1$ satisfying
\[
Q_1 M_2 = \begin{pmatrix}
E_\ell & B(t_1) \\
0 & B(t_2) - B(t_1) \\
\vdots & \vdots \\
0 & B(t_s) - B(t_1)
\end{pmatrix}_{t=z},
\]

There exists an $(\ell + m\ell) \times (\ell + m\ell)$ regular matrix $Q_2$ satisfying
\[
Q_1 M_2 Q_2 = \begin{pmatrix}
E_\ell & 0 \\
0 & B(t_2) - B(t_1) \\
\vdots & \vdots \\
0 & B(t_s) - B(t_1)
\end{pmatrix}_{t=z}
\]

where $\bar{f}(t_1)\bar{f}(t_i) = (\bar{f}_1(t_i) - \bar{f}_1(t_1), \bar{f}_2(t_i) - \bar{f}_2(t_1), \ldots, \bar{f}_m(t_i) - \bar{f}_m(t_1))$ (2 \( i \leq s \)) and \( t = z \). From \( s - 1 \leq s_f - 1 \) and the definition of \( s_f \), we have
\[
\dim \sum_{i=2}^{s} \mathbb{R} \bar{f}(t_1)\bar{f}(t_i) = s - 1,
\]

where $t = z$. Hence, by the construction of $Q_1 M_2 Q_2$ and $s - 1 \leq m$, the rank of $Q_1 M_2 Q_2$ is equal to $\ell s$. Therefore, the rank of $M_2$ must be equal to $\ell s$. Hence, we get (4.1). Therefore, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, and for any $s$ (2 \( s \leq s_f \)), the $C^r$ mapping $(F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s$ is transverse to $\Delta_s$.

Moreover, suppose that the $C^r$ mapping $F_\pi$ satisfies that $|F_\pi^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$. Since $f : N \to \mathbb{R}^m$ is injective, it follows that $|(F_\pi \circ f)^{-1}(y)| \leq s_f$ for any $y \in \mathbb{R}^\ell$. Thus, for any positive integer $s$ with $s \geq s_f + 1$, we have $(F_\pi \circ f)^{(s)}(N^{(s)}) \cap \Delta_s = \emptyset$. Namely, for any positive integer $s$ with $s \geq s_f + 1$, the $C^r$ mapping $(F_\pi \circ f)^{(s)}$ is transverse to $\Delta_s$. Hence, $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a $C^r$ mapping with normal crossings. \( \square \)
5. Applications of Theorems 1 and 2

In Section 5.1 (resp., Section 5.2), applications of Theorem 1 (resp., Theorem 2) are stated and proved. In Section 5.2, applications obtained by combining Theorems 1 and 2 are also given.

5.1. Applications of Theorem 1. A $C^2$ function $g : N \to \mathbb{R}$ is called a Morse function if all of the critical points of $g$ are nondegenerate, where $N$ is a $C^2$ manifold of dimension $n$ (for details on Morse functions, see for instance, [1] p. 63). In the case of $(n, \ell) = (n, 1)$, we have the following.

Corollary 1. Let $f$ be a $C^2$ immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$, where $N$ is a $C^2$ manifold of dimension $n$. Let $F : U \to \mathbb{R}$ be a $C^2$ function. Then, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the $C^2$ function $F_\pi \circ f : N \to \mathbb{R}$ is a Morse function.

Proof. We have $\dim N - \operatorname{codim} \Sigma^1(N, \mathbb{R}) = 0$. Therefore, from Theorem 1 there exists a subset $\Sigma$ with Lebesgue measure zero of $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$ such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R})$ is transverse to $\Sigma^1(N, \mathbb{R})$. Therefore, if $q \in N$ is a critical point of the function $F_\pi \circ f$, then the point $q$ is nondegenerate. □

In the case of $\ell \geq 2n$, we have the following.

Corollary 2. Let $f$ be a $C^2$ immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$, where $N$ is a $C^2$ manifold of dimension $n$. Let $F : U \to \mathbb{R}^\ell$ be a $C^2$ mapping ($\ell \geq 2n$). Then, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ is a $C^2$ immersion.

Proof. It is clearly seen that $F_\pi \circ f : N \to \mathbb{R}^\ell$ is an immersion if and only if $j^1(F_\pi \circ f)(N) \cap \bigcup_{k=1}^n \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$. From $\ell \geq 2n$, for any positive integer $k (1 \leq k \leq n)$, we have

$$\dim N - \operatorname{codim} \Sigma^k(N, \mathbb{R}^\ell) = n - k(\ell - n + k) \leq 0.$$ 

Thus, for any positive integer $k (1 \leq k \leq n)$, from Theorem 1 there exists a subset $\Sigma_k$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_k$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$. Set $\Sigma = \bigcup_{k=1}^n \Sigma_k$. Note that $\Sigma$ has Lebesgue measure zero. Let $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$ be an arbitrary element. Then, suppose that there exists a point $q \in N$ and a positive integer $k (1 \leq k \leq n)$ such that $j^1(F_\pi \circ f)(q) \in \Sigma^k(N, \mathbb{R}^\ell)$. Since $j^1(F_\pi \circ f)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$, we have the following:

$$d(j^1(F_\pi \circ f))_q(T_q N) + T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell) = T_{j^1(F_\pi \circ f)(q)} J^1(N, \mathbb{R}^\ell).$$

Hence, we have

$$\dim d(j^1(F_\pi \circ f))_q(T_q N) \geq \dim T_{j^1(F_\pi \circ f)(q)} J^1(N, \mathbb{R}^\ell) - \dim T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell)$$

$$= \dim T_{j^1(F_\pi \circ f)(q)} \Sigma^k(N, \mathbb{R}^\ell).$$

Thus, we get $n \geq k(\ell - n + k)$. This contradicts the assumption $\ell \geq 2n$. Therefore, we get $j^1(F_\pi \circ f)(N) \cap \bigcup_{k=1}^n \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$.

A $C^1$ mapping $g : N \to \mathbb{R}^\ell$ has singular points of corank at most $k$ if

$$\sup \{\operatorname{corank} dg_q \mid q \in N\} \leq k,$$
where corank $dg_q = \min\{n, \ell\} - \operatorname{rank} dg_q$.

**Corollary 3.** Let $f$ be a $C^r$ immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$, where $N$ is a $C^r$ manifold of dimension $n$. Let $F : U \to \mathbb{R}^\ell$ be a $C^r$ mapping. Let $k_0$ be the maximum integer satisfying $(n - v + k_0)(\ell - v + k_0) \leq n$ \((v = \min\{n, \ell\})\). If

$$r > \max\{\dim N - \operatorname{codim} \Sigma^1(N, \mathbb{R}^\ell), 0\} + 1,$$

then there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the $C^r$ mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has singular points of corank at most $k_0$.

**Proof.** For any positive integer $k$ \((1 \leq k \leq v)\), we have

$$r > \max\{\dim N - \operatorname{codim} \Sigma^k(N, \mathbb{R}^\ell), 0\} + 1 \geq \max\{\dim N - \operatorname{codim} \Sigma^k(N, \mathbb{R}^\ell), 0\} + 1.$$

From Theorem 1 for any positive integer $k$ satisfying $1 \leq k \leq v$, there exists a subset $\Sigma_k$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma_k$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$. Then, $\Sigma = \bigcup_{k=1}^v \Sigma_k$ has Lebesgue measure zero. Hence, there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ for any positive integer $k$ satisfying $1 \leq k \leq v$.

In the case of $\ell = 1$, we have $k_0 = 1$. Thus, in this case, the assertion clearly holds.

Now, we will consider the case of $\ell \geq 2$. In this case, note that $k_0 + 1 \leq v$. Indeed, suppose that $v \leq k_0$. Then, by $(n - v + k_0)(\ell - v + k_0) \leq n$, we get $n\ell \leq n$. This contradicts the assumption $\ell \geq 2$. For the proof of Corollary 3 it is sufficient to show that the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ satisfies that $j^1(F_\pi \circ f)(N) \cap \Sigma^k(N, \mathbb{R}^\ell) = \emptyset$ for any positive integer $k$ satisfying $k_0 + 1 \leq k \leq v$.

Suppose that there exist a positive integer $k$ \((k_0 + 1 \leq k \leq v)\) and a point $q \in N$ such that $j^1(F_\pi \circ f)(q) \in \Sigma^k(N, \mathbb{R}^\ell)$. Since the mapping $j^1(F_\pi \circ f) : N \to J^1(N, \mathbb{R}^\ell)$ is transverse to $\Sigma^k(N, \mathbb{R}^\ell)$ at the point $q$, the following holds:

$$d(j^1(F_\pi \circ f))_q(T_qN) + T_{j^1(F_\pi \circ f)(q)}\Sigma^k(N, \mathbb{R}^\ell) = T_{j^1(F_\pi \circ f)(q)}J^1(N, \mathbb{R}^\ell).$$

Hence, we have

$$\dim d(j^1(F_\pi \circ f))_q(T_qN) \geq \dim T_{j^1(F_\pi \circ f)(q)}J^1(N, \mathbb{R}^\ell) - \dim T_{j^1(F_\pi \circ f)(q)}\Sigma^k(N, \mathbb{R}^\ell) = \operatorname{codim} T_{j^1(F_\pi \circ f)(q)}\Sigma^k(N, \mathbb{R}^\ell).$$

Thus, we get $n \geq (n - v + k_0)(\ell - v + k_0)$. Since the given integer $k_0$ is the maximum integer satisfying $n \geq (n - v + k_0)(\ell - v + k_0)$, it follows that $k \leq k_0$. This contradicts the assumption $k_0 + 1 \leq k$. \hfill \Box

5.2. Applications of Theorem 2

**Corollary 4.** Let $f$ be a $C^r$ injection of $N$ into an open subset $U$ of $\mathbb{R}^m$, where $N$ is a $C^r$ manifold of dimension $n$. Let $F : U \to \mathbb{R}^\ell$ be a $C^r$ mapping. If

$$(s_f - 1)\ell > ns_f \text{ and } r > \max\{2n - \ell, 0\},$$

then there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the $C^r$ mapping $F_\pi \circ f : N \to \mathbb{R}^\ell$ has singular points of corank at most $k_0$. \hfill \Box
then there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), \( F_\pi \circ f : N \to \mathbb{R}^\ell \) is a \( C^r \) mapping with normal crossings satisfying \( (F_\pi \circ f)^{(s)}(N^{(s)}) \cap \Delta_{sf} = \emptyset \).

**Proof.** From \((sf - 1)\ell > nsf\), we have \( n - \ell < 0 \). Thus, we get
\[
s_0 = \max \{ s(n - \ell) + \ell \mid 2 \leq s \leq sf \} = 2n - \ell.
\]

Hence, note that \( r > \max\{s_0, 0\} \). From Theorem [3] there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), and for any \( s \) \((2 \leq s \leq sf)\), the mapping \((F_\pi \circ f)^{(s)} : N^{(s)} \to (\mathbb{R}^\ell)^s\) is transverse to \( \Delta_s \).

Therefore, for this proof, it is sufficient to prove that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \((F_\pi \circ f)^{(sf)}\) satisfies that \((F_\pi \circ f)^{(sf)}(N^{(sf)}) \cap \Delta_{sf} = \emptyset \).

Suppose that there exists an element \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \) such that there exists a point \( q \in N^{(sf)} \) satisfying \((F_\pi \circ f)^{(sf)}(q) \in \Delta_{sf} \). Since \((F_\pi \circ f)^{(sf)}\) is transverse to \( \Delta_{sf} \), we have the following:
\[
d((F_\pi \circ f)^{(sf)})_q(T_q N^{(sf)}) + T_{(F_\pi \circ f)^{(sf)}(q)} \Delta_{sf} = T_{(F_\pi \circ f)^{(sf)}(q)}(\mathbb{R}^\ell)^{sf}.
\]

Thus, we get
\[
\dim d((F_\pi \circ f)^{(sf)})_q(T_q N^{(sf)}) \\
\geq \dim T_{(F_\pi \circ f)^{(sf)}(q)}(\mathbb{R}^\ell)^{sf} - \dim T_{(F_\pi \circ f)^{(sf)}(q)} \Delta_{sf} \\
= \text{codim } T_{(F_\pi \circ f)^{(sf)}(q)} \Delta_{sf}.
\]

Hence, we have \( nsf \geq (sf - 1)\ell \). This contradicts the assumption \((sf - 1)\ell > nsf\).

\( \square \)

In the case of \( \ell > 2n \), we have the following.

**Corollary 5.** Let \( f \) be a \( C^1 \) injection of \( N \) into an open subset \( U \) of \( \mathbb{R}^m \), where \( N \) is a \( C^1 \) manifold of dimension \( n \). Let \( F : U \to \mathbb{R}^\ell \) be a \( C^1 \) mapping. If \( \ell > 2n \), then there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the \( C^1 \) mapping \( F_\pi \circ f : N \to \mathbb{R}^\ell \) is injective.

**Proof.** Since \( sf \geq 2 \) and \( \ell > 2n \), it is easily seen that the dimension pair \((n, \ell)\) satisfies the assumption \((sf - 1)\ell > nsf\) of Corollary [4]. Indeed, from \( \ell > 2n \), we get \((sf - 1)\ell > 2n(sf - 1)\). From \( sf \geq 2 \), it follows that \( 2n(sf - 1) \geq nsf \).

Since \( \max\{2n - \ell, 0\} = 0 \), from Corollary [4] there exists a subset \( \Sigma \) of \( \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma \), the mapping \((F_\pi \circ f)^{(2)} : N^{(2)} \to (\mathbb{R}^\ell)^2\) is transverse to \( \Delta_2 \). For this proof, it is sufficient to prove that the mapping \((F_\pi \circ f)^{(2)}\) satisfies that \((F_\pi \circ f)^{(2)}(N^{(2)}) \cap \Delta_2 = \emptyset \).

Suppose that there exists a point \( q \in N^{(2)} \) such that \((F_\pi \circ f)^{(2)}(q) \in \Delta_2 \). Then, we get the following:
\[
d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) + T_{(F_\pi \circ f)^{(2)}(q)} \Delta_2 = T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2.
\]

Thus, we have
\[
\dim d((F_\pi \circ f)^{(2)})_q(T_q N^{(2)}) \\
\geq \dim T_{(F_\pi \circ f)^{(2)}(q)}(\mathbb{R}^\ell)^2 - \dim T_{(F_\pi \circ f)^{(2)}(q)} \Delta_2 \\
= \text{codim } T_{(F_\pi \circ f)^{(2)}(q)} \Delta_2.
\]
Hence, we have $2n \geq \ell$. This contradicts the assumption $\ell > 2n$. \hfill \Box

By combining Corollaries 6 and 7, we have the following.

**Corollary 6.** Let $f$ be an injective immersion of $N$ into an open subset $U$ of $\mathbb{R}^m$, where $N$ is a $C^2$ manifold of dimension $n$ and $f$ is of class $C^2$. Let $F : U \rightarrow \mathbb{R}^\ell$ be a $C^2$ mapping. If $\ell > 2n$, then there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the $C^2$ mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is an injective immersion.

From Corollary 6, we get the following.

**Corollary 7.** Let $N$ be a compact $C^2$ manifold of dimension $n$. Let $f$ be a $C^2$ embedding of $N$ into an open subset $U$ of $\mathbb{R}^m$. Let $F : U \rightarrow \mathbb{R}^\ell$ be a $C^2$ mapping. If $\ell > 2n$, then there exists a subset $\Sigma$ of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ with Lebesgue measure zero such that for any $\pi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell) - \Sigma$, the $C^2$ mapping $F_\pi \circ f : N \rightarrow \mathbb{R}^\ell$ is an embedding.

### 6. Proof of Lemma 1

#### 6.1. Preliminaries for the proof of Lemma 1

Let $N$ and $P$ be $C^r$ manifolds, and let $g : N \rightarrow P$ be a $C^1$ mapping ($r \geq 1$). A point $x \in N$ is called a critical point of $g$ if it is not a regular point, i.e., the rank of $dg_x$ is less than the dimension of $P$. We say that a point $y \in P$ is a critical value if it is the image of a critical point. A point $y \in P$ is called a regular value if it is not a critical value. The following is Sard’s theorem.

**Theorem 3**. If $N$ and $P$ are $C^r$ manifolds, $g : N \rightarrow P$ is a $C^r$ mapping, and $r > \max\{\dim N - \dim P, 0\}$, then the set of critical values of $g$ has Lebesgue measure zero.

#### 6.2. Proof of Lemma 1

In this proof, by $\pi : N \times A \rightarrow A$, we denote the natural projection defined by $\pi(x, a) = a$.

Since $\Gamma$ is transverse to $Z$, the set $\Gamma^{-1}(Z)$ is a $C^r$ submanifold of $N \times A$ satisfying

$$
\dim N + \dim A - \dim \Gamma^{-1}(Z) = \dim P - \dim Z. 
$$

Firstly, suppose that $\dim \Gamma^{-1}(Z) = 0$. Then, since $\Gamma^{-1}(Z)$ is a countable set, $\pi(\Gamma^{-1}(Z))$ has Lebesgue measure zero in $A$. It is clearly seen that for any $a \in A - \pi(\Gamma^{-1}(Z))$, the mapping $\Gamma_a$ is transverse to $Z$.

Finally, we will consider the case $\dim \Gamma^{-1}(Z) > 0$. It is not hard to see that if $a \in A$ is a regular value of $\pi|_{\Gamma^{-1}(Z)}$, then $\Gamma_a$ is transverse to $Z$, where $\pi|_{\Gamma^{-1}(Z)}$ is the restriction of $\pi$ to $\Gamma^{-1}(Z)$. Let $\Sigma$ be the set of critical values of $\pi|_{\Gamma^{-1}(Z)}$. From $r > \max\{\dim N + \dim Z - \dim P, 0\}$ and (1), we have $r > \max\{\dim \Gamma^{-1}(Z) - \dim A, 0\}$. From Theorem 3, $\Sigma$ has Lebesgue measure zero in $A$. Therefore, if $a \in A - \Sigma$, then $\Gamma_a$ is transverse to $Z$. \hfill \Box

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