HYBRID BOUNDS ON TWISTED L-FUNCTIONS ASSOCIATED TO MODULAR FORMS

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Abstract. For $f$ a primitive holomorphic cusp form of even weight $k \geq 4$, square-free level $N$, and $\chi$ a Dirichlet character mod $Q$ with $(Q, N) = 1$, we establish the following hybrid subconvex bound for $t \in \mathbb{R}$:

$$L\left(\frac{1}{2} + it, f \chi\right) \ll Q^{\frac{k}{2} + \varepsilon}(1 + |t|)^{\frac{1}{4} + \varepsilon}$$

where $\theta$ is the best bound toward the Ramanujan-Petersson conjecture at the infinite place. The implied constant only depends on $f$ and $\varepsilon$. This is done via amplification and taking advantage of a shifted convolution sum of two variables as defined and analyzed in [8].

Contents

1. Introduction 1
2. Amplifying both aspects 3
3. The diagonal portion $S_d$ 4
  3.1. The case of $l_1 = l_2$ 4
  3.2. The case of $l_1 \neq l_2$ 9
4. Off-diagonal portion $S_{o1}$, setting up the integrals 10
5. Off-diagonal, discrete spectrum 10
  5.1. The residues at $s = \frac{1}{2} \pm it_j - r$ 10
  5.2. The moved integral at Re $s = \frac{1}{2} - \frac{k}{2} - \theta - \varepsilon$ 11
  5.3. Proof of lemma (5.1) 16
6. Off-diagonal, continuous spectrum 24
  6.1. The residues at $s = \frac{1}{2} \pm z - r$ 24
  6.2. The moved integral at Re $s = \frac{1}{2} - \frac{k}{2} - \varepsilon$ 33
7. Proof of Theorem 1 38
Appendix A. Properties of $Z_Q(s, w)$ 39
Appendix B. Miscellaneous facts involving $\langle U, E_a(*, s) \rangle$ 41
Acknowledgements 42
References 42

1. Introduction

The growth of $L$-functions on the critical line Re $s = \frac{1}{2}$ has been one of the most studied problems in analytic number theory. This paper is concerned with $L$-functions of a holomorphic cusp form $f$, twisted by a character $\chi$ of conductor...
By using functional equation and Phragmén-Lindelöf principle, one can obtain the convexity bound
\[ L(\frac{1}{2} + it, f_{\chi}) \ll (Q(1 + |t|))^{\frac{1}{2} + \varepsilon}, \]
where we suppress the level and weight aspects here.

Throughout the years, there have been many attempts at lowering the exponents, most of which have focused on one chosen aspect. Since our result concerns \( Q \)- and \( t \)-aspects, we will state some known results in these directions.

In the \( t \)-aspect, Good showed in [5] that for \( f \) a holomorphic cusp form of the full modular group,
\[ L(\frac{1}{2} + it, f) \ll (1 + |t|)^{\frac{1}{2} + \varepsilon} \]
Meurman showed the same result for \( f \) Maass forms of full modular group in [10]. For number fields, subconvexity results were proved in Petridis and Sarnak [13] and Diaconu and Garrett [3].

In the \( Q \)-aspect, the first subconvexity result was obtained by Duke, Friedlander and Iwaniec [4] for holomorphic cusp forms of full level. Later, Bykovskii showed in [2] that for general level,
\[ L(\frac{1}{2} + it, f_{\chi}) \ll t^{\frac{3}{8} + \varepsilon}, \]
with a polynomial dependence in \((1 + |t|)\), provided that the nebentypus of \( f \) is trivial. This same bound without the nebentypus restriction is obtained in [8] and [11]. In [1], \( f \) can also be taken as a Maass form.

The first hybrid bound in \( Q \)- and \( t \)-aspects is given by Blomer and Harcos in [1]. Combining two methods which give subconvexity in each aspect separately, they were able to show that for \( f \) a holomorphic cusp form or Maass form,
\[ L(\frac{1}{2} + it, f_{\chi}) \ll (Q(1 + |t|))^{\frac{1}{2} - \frac{1}{18} + \varepsilon} \]
Michel and Venkatesh achieved subconvexity in all aspects in [11] with unspecified exponent. Later, using the circle method, Munshi gave an improved bound in [12]:
\[ L(\frac{1}{2} + it, f_{\chi}) \ll (Q(1 + |t|))^{\frac{1}{2} - \frac{1}{14} + \varepsilon} \]

The current best known bound that we are aware of is by Han Wu [14], which, following the method of Michel and Venkatesh, uses amplification. The bound obtained is:
\[ L(\frac{1}{2} + it, f_{\chi}) \ll (Q(1 + |t|))^{\frac{1}{2} + \frac{1}{20} + \varepsilon}, \]
where no complementary series with parameter \( \theta \) appears as a component of a cuspidal automorphic representation of \( GL_2(\mathbb{A}) \).

One thing to note is that these hybrid bounds do not reach the best known exponents in the \( Q \)-aspect. In this work, we partially resolve this situation by proving the following result.

**Theorem 1.** For \( f \) a primitive holomorphic cusp form of even weight \( k \geq 4 \), square-free level \( N \), and \( \chi \) a Dirichlet character mod \( Q \), where \((Q, N) = 1\), we have
\[ L(\frac{1}{2} + it, f_{\chi}) \ll (1 + |t|)^{\theta + \varepsilon}Q^{\frac{1}{2} + \varepsilon}, \]
where \( \theta \) is a bound toward the Ramanujan-Petersson conjecture at the infinite place.
Remark. A bound of $\theta$ for congruence subgroups of $SL_2(\mathbb{Z})$ is $7/64$ by Kim and Sarnak [9]. It should be noted that our theorem does not currently cover the case of Maass forms as the corresponding shifted convolution is not analyzed yet.

Our work also uses amplification. The major difference between this work and [14] is that we treat non-Archimedean and Archimedean places differently while Wu treated them uniformly. As such, we need more precise control on the $t$-aspect, which is achieved by relating the problem to the shifted convolution sum of two variables analyzed in [8].

The following is a brief summary of the plan of this paper.

Our goal is to bound $L(1/2, f, \chi)$ in the $Q$ and $t$-aspects. In section 2, using amplification methods, we reduce the problem to understanding the growth in $Q$ and $t$ of the following expression, where $G$ and $L$ are amplification parameters and $\alpha = \frac{1}{\log(Q(1+|t|))}$.

$$\varphi(Q) G \sum_{l_1, l_2 \sim L} \chi(l_1) \chi(l_2) \sum_{m_1, m_2 \geq 1, m_1 l_1 \equiv m_2 l_2 (Q)} \frac{A(m_1) A(m_2) (l_1 l_2)^{\alpha}}{m_1^{1/2 + it} m_2^{1/2 - it}} \times V(m_1 x) V(m_2 x) e^{-G |\log(l_1 m_1 l_2 m_2)|}$$

We then separate the analysis of this expression into the “diagonal” portion ($m_1 l_1 = m_2 l_2$) and two “off-diagonal” portions ($m_1 l_1 = m_2 l_2 + h_0 Q$ and $m_2 l_2 = m_1 l_1 + h_1 Q$, for $h_0, h_1 \geq 1$).

In section 3 we analyze the diagonal term with inverse Mellin transforms (propositions (3.1) and (3.2)).

For the off-diagonal portions, the first thing to note is they have the same contribution up to conjugation. Our analysis relies heavily on the shifted convolution sum of two variables $Z_Q(s, w)$ from [8]. By inverse Mellin transforms, we relate the off-diagonal term to a four-fold integral involving $Z_Q(s, w)$. This is done in section 4 (proposition 4.1).

The analysis of the off-diagonal then splits into the discrete part and the continuous part, due to the fact that such a splitting exists for $Z_Q(s, w)$. The analysis of each part is done by moving lines of integration, with the primary goal of reducing the $x$-exponent as much as possible, and a secondary goal of reducing contribution of the $t$-aspect where possible. The results can be found in propositions (5.1), (5.2), (6.1) and (6.2).

In the last section, we put the results of the previous sections together. Choosing $L$ and $G$ optimally yields the theorem.

2. Amplifying both aspects

Throughout this paper, fix a holomorphic cusp form $f$ of even weight $k \geq 4$, square-free level $N$: $f(z) = \sum_{n=1}^{\infty} A(n) n^{k-1/2} e^{2\pi i n z} = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$
Our aim here is to understand the growth of $L(\frac{1}{2} + it, f_\chi)$ in $t$- and $Q$-aspects. Since it is sufficient to prove the result on eigenforms, so we assume $f$ is an eigenform. We perform our investigation by averaging around $\frac{1}{2} + it$ for a small interval as well as applying the amplification technique.

For this, we choose a rapidly decreasing function $V: \mathbb{R} \to \mathbb{R}$ such that its Mellin transform $v(s)$ is meromorphic between $-5 < \text{Re } s < 5$. Moreover, $v(s)$ should only have a simple pole at $s = 0$ with residue 1 and exponential decay in $\text{Im } s$ as $\text{Im } s \to \infty$. An example of this is $v(s) = \frac{1}{5} \Gamma \left( \frac{s}{5} \right)$. Specifying $v(s)$ is enough, as:

$$V(x) = \frac{1}{2\pi i} \int_{(2)} v(s)x^{-s} \, ds$$

We start by writing the $L$-function as a rapidly converging series:

**Lemma 2.1.** As $x \to \infty$,

$$L \left( \frac{1}{2} + it, f_\chi \right) = \sum_{n \geq 1} A(n) \chi(n) \frac{n}{n^{\frac{1}{2} + it}} V \left( \frac{n}{x} \right) + O(x^{-\varepsilon}).$$

**Proof.** Consider the following inverse Mellin transform:

$$I_0 := \frac{1}{2\pi i} \int_{(2)} L \left( \frac{1}{2} + it + s, f_\chi \right) v(s)x^s \, ds$$

On the one hand, since the argument of $L$-function in $I_0$ is in the region of absolute convergence, we have:

$$I_0 = \frac{1}{2\pi i} \int_{(2)} \sum_{n \geq 1} \frac{A(n)\chi(n)}{n^{\frac{1}{2} + s + it}} v(s)x^s \, ds$$

$$= \sum_{n \geq 1} \frac{A(n)\chi(n)}{n^{\frac{1}{2} + it}} V \left( \frac{n}{x} \right)$$

On the other hand, we can move the line of integration to $-\frac{1}{2} - \varepsilon$, picking up the only simple pole at $s = 0$ and obtain:

$$I_0 = L \left( \frac{1}{2} + it, f_\chi \right) + O(x^{-\varepsilon})$$

Putting the two equivalent expressions of $I_0$ together proves the lemma. 

Our aim here is to get the bound on $L(\frac{1}{2} + it, f_\chi)$. To this end, we first amplify the character and obtain:

$$|L(\frac{1}{2} + it, f_\chi)|^2 \sum_{l \sim L} 1^2 \leq \sum_{\psi(\mathbb{Q})} |L(\frac{1}{2} + it, f_\psi)|^2 \sum_{l \sim L} \chi(l)\psi(l)^2$$

where the first summation runs over all Dirichlet characters $\psi \mod Q$ and the $l$-sums are running over primes that are relatively prime to $QN$. The parameter $L$ is to be chosen optimally later, subject to $L < Q$.

Next we perform amplification on the $t$-aspect, with modified ideas based upon [7]. The result is the following:
When put together with (2.0.6), we derive that via the positivity of the integrand. This leads to the desire to minimize Remark. In the proof, we will see that the introduction of \( G \) into the integral is via the positivity of the integrand. This leads to the desire to minimize \( G \) subject to the constraint above.

Proof. The proof relies on estimating \( L(\sigma + it, f_\psi) \) by averaging \( L(\sigma - \alpha + ir, f_\psi) \) over \( r \) in a small interval centered around \( t \). Each integral expression defined below is essentially illustrating this fact.

First we will show that \( L(\frac{1}{2} + it, f_\psi) \) is approximable by averaging the \( L \)-function over a small interval. To this end, consider the following integral:

\[
I_1 := \frac{1}{2\pi i} \int_{(2)} L(\frac{1}{2} + it, f_\psi) \frac{e^{s^2}}{s} \, ds
\]

On the one hand, \( I_1 \) is \( O(1) \) by bounding the \( L \)-function by a constant. On the other hand, if we move the line of integration down to \( \operatorname{Re} s = -\alpha \), then we have:

\[
I_1 = L(\frac{1}{2} + it, f_\psi) + \frac{1}{2\pi i} \int_{(-\alpha)} L(\frac{1}{2} + it, f_\psi) \frac{e^{s^2}}{s} \, ds.
\]

When put together with (2.0.6), we derive that

\[
L(\frac{1}{2} + it, f_\psi) = O(1) + \frac{1}{2\pi} \int_{-\infty}^{\infty} L(\frac{1}{2} - \alpha + it + ir, f_\psi) \frac{e^{(-\alpha + ir)^2}}{\alpha - ir} \, dr.
\]

After taking absolute values and squaring both sides, one gets:

\[
|L(\frac{1}{2} + it, f_\psi)|^2 \ll \left( \int_{-\infty}^{\infty} |L(\frac{1}{2} - \alpha + it + ir, f_\psi)| \frac{e^{\alpha^2 - r^2}}{\sqrt{\alpha^2 + r^2}} \, dr \right)^2 + O(1)
\]

To continue our investigation, we will split the integral into two parts, \(|r| \leq A\) and \(|r| > A\), where \( A = \sqrt{10 \log(Q(1 + |t|))} \).

We start by examining the part of the integral with \(|r| > A\), applying convexity for the \( L \)-function:

\[
\int_{|r| > A} |L(\frac{1}{2} - \alpha + it + ir, f_\psi)| \frac{e^{\alpha^2 - r^2}}{\sqrt{\alpha^2 + r^2}} \, dr \ll \int_{|r| > A} (Q|r + t|) \frac{\chi}{2 + \alpha \frac{e^{\alpha^2 - r^2}}{r}} \, dr
\]

\[
\ll (Q(1 + |t|)) \frac{\chi}{2 + \alpha} e^{\alpha^2 - A^2} \ll 1
\]
For the part $|r| \leq A$, we apply Cauchy’s inequality and functional equation:

$$
\left( \int_{|r| \leq A} |L \left( \frac{1}{2} - \alpha + it + ir, f_\psi \right)| \frac{e^{\alpha^2 - r^2}}{\sqrt{\alpha^2 + r^2}} \, dr \right)^2
\leq \int_{|r| \leq A} |L \left( \frac{1}{2} - \alpha + it + ir, f_\psi \right)|^2 \, dr \cdot \int_{|r| \leq A} \frac{e^{2\alpha^2 - 2r^2}}{\alpha^2 + r^2} \, dr
\ll \alpha^{-2}(Q(1 + |t|))^{4\alpha} \int_{|r| \leq A} |L \left( \frac{1}{2} + \alpha + ir, f_\psi \right)|^2 \, dr + O(1)
$$

Putting these into (2.0.7), recalling $A = \sqrt{10 \log(Q(1 + |t|))}$ and $\alpha = \frac{1}{\log(Q(1 + |t|))}$, we get

$$
|L \left( \frac{1}{2} + it, f_\psi \right)|^2 \ll \alpha^{-2}(Q(1 + |t|))^{4\alpha} \int_{|r| \leq A} |L \left( \frac{1}{2} + \alpha + it + ir, f_\psi \right)|^2 \, dr + O(1)
$$

We will multiply both sides by $| \sum_{l \sim L} \overline{\chi(l)} \psi(l) |^2$, obtaining:

$$
|L \left( \frac{1}{2} + it, f_\psi \right)|^2 | \sum_{l \sim L} \overline{\chi(l)} \psi(l) |^2 \ll \log^2(Q(1 + |t|)) \int |L \left( \frac{1}{2} + \alpha + it + ir, f_\psi \right)|^2 | \sum_{l \sim L} \overline{\chi(l)} \psi(l) |^2 \, dr + O(| \sum_{l \sim L} \overline{\chi(l)} \psi(l) |^2)
$$

(2.0.8)

Now, we approximate $L \left( \frac{1}{2} + \alpha + it, f_\psi \right) \sum_{l \sim L} \overline{\chi(l)} \psi(l)$ by an integral over a small interval on the critical line. To achieve this, we construct the following auxiliary integral:

$$
I_2 := \frac{1}{2\pi i} \int_{(2)} L \left( \frac{1}{2} + \alpha + it_2 + s, f_\psi \right) \left( \sum_{l \sim L} \overline{\chi(l)} \psi(l) l^{-s} \right) \frac{e^{2s^2}}{s} \, ds \quad (2.0.9)
$$

By the same reasoning as before, this is $O(1)$. Moving line of integration to $-\alpha$, we can also see this as:

$$
I_2 = L \left( \frac{1}{2} + \alpha + it_2, f_\psi \right) \left( \sum_{l \sim L} \overline{\chi(l)} \psi(l) \right)
+ \frac{1}{2\pi} \int_{-\infty}^{\infty} L \left( \frac{1}{2} + it_2 + ir', f_\psi \right) \left( \sum_{l \sim L} \overline{\chi(l)} \psi(l) l^{\alpha - ir'} \right) \frac{e^{(\alpha - ir')^2}}{-\alpha + ir'} \, dr'
$$

Splitting the integral into $|r'| \leq A'$ and $|r'| > A'$, where $A' = \sqrt{\log(Q|I_2|)}$, and doing the same analysis as before, we obtain:

$$
|L \left( \frac{1}{2} + \alpha + it_2, f_\psi \right)|^2 | \sum_{l \sim L} \overline{\chi(l)} \psi(l) |^2 \ll \log^2(Q|I_2|) \int \left| L \left( \frac{1}{2} + it_2 + ir', f_\psi \right) \right|^2 \left| \sum_{l \sim L} \overline{\chi(l)} \psi(l) l^{\alpha - ir'} \right|^2 \, dr' + O(1) \quad (2.0.10)
$$
Plugging (2.0.10) into (2.0.8), we obtain:

\[ |L(\frac{1}{2} + it, f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 \]
\[ \leq \log^4(Q(1 + |t|)) \int_{|r| \leq A} \int_{|r'| \leq A} |L(\frac{1}{2} + i(t + r) + ir', f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \, dr \, dr' + O(\log^2(Q(1 + |t|)) A) + O(\sum_{l \sim L} \overline{\chi(l)} |\psi(l)|^2) \quad (2.0.11) \]

where \( A_r = \sqrt{10 \log(Q|t + r|)} \). Note that all inequalities are independent of \( Q, L \) and \( t \), as long as \( L \ll Q \). Also note that we can make \( A_r \) uniform by enlarging the region to \(|r'| \leq \sqrt{40 \log(Q(1 + |t|)) = 2A} \), the inequality still holding due to positivity of the integrand.

Continuing to use the positivity of the integrand, for \( G \geq 2A \), the integrand is bounded by:

\[ \int_{|r| \leq A} \int_{|r'| \leq 2A} |L(\frac{1}{2} + i(t + r) + ir', f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \, dr \, dr' \]
\[ \ll \int_{|r| \leq A} \int_{|r'| \leq G} |L(\frac{1}{2} + i(t + r) + ir', f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \, dr \, dr' \]
\[ \ll \int_{|r| \leq A} \int_{r' \geq 0} |L(\frac{1}{2} + i(t + r) + ir', f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \, dr' \, dr \\
\quad \ll \int_{-\infty}^{|r| \leq A} \int_{-\infty}^{\infty} |L(\frac{1}{2} + i(t + r) + ir', f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \, dr' \, dr \
\quad \ll \log^4(Q(1 + |t|)) \sum_{\psi(Q)} \int_{|r| \sim \psi(Q)} \left( \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 \right) \left( \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \right) \, dr' \, dr \\
\quad \times \frac{\log^2(Q(1 + |t|))}{\pi(1 + \left(\frac{r'}{\psi(Q)}\right)^2)} + O(Q \log^2(Q(1 + |t|))) + O(QL) \quad (2.0.12) \]

Putting this into (2.0.11) gives the proposition. □

Putting this proposition together with (2.0.4), we derive the following:

**Proposition 2.1.** With the same values of \( A, \alpha \) and the same constraint on \( G \) as in lemma 2.2,

\[ |L(\frac{1}{2} + it, f_\chi)|^2 \sum_{l \sim L} 1^2 \]
\[ \ll \log^4(Q(1 + |t|)) \int_{|r| \leq A} \int_{-\infty}^{\infty} \sum_{\psi(Q)} |L(\frac{1}{2} + ir + r', f_\psi)|^2 \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 r'^2 \, dr' \, dr \\
\times \frac{\log^2(Q(1 + |t|))}{\pi(1 + \left(\frac{r'}{\psi(Q)}\right)^2)} + O(Q \log^2(Q(1 + |t|))) + O(QL) \quad (2.0.13) \]

**Proof.** The only part that requires a proof is the last error term. In particular, we should show that

\[ \sum_{\psi(Q)} \left( \sum_{l \sim L} \overline{\chi(l)} \psi(l)^2 \right) \ll QL \]
Starting with the left-hand side, we have:

\[
\sum_{\psi(Q)} \sum_{l \sim L} |\chi(l)\psi(l)|^2 = \sum_{\psi(Q)} \sum_{l_1, l_2 \sim L} \chi(l_1)\psi(l_1)\chi(l_2)\psi(l_2)
\]

\[
= \varphi(Q) \sum_l 1 \llQL
\]

The second equality is obtained by summing over the characters, which implies that \(l_1 = l_2\) since \(l_1 \equiv l_2(Q)\) and \(L < Q\).

Our next immediate goal is to execute the character sum and the \(r'\)-integral in (2.0.13). To proceed, fix \(r\) and set \(\tilde{t} = t + r\). Note that \(\tilde{t} \ll (1 + |t|)Q^r\), since \(|r| \leq A\). Replacing the \(L\)-series with (2.0.2), up to \(O(x^{-\varepsilon})\), one obtains:

\[
S := \int_{-\infty}^{\infty} \sum_{\psi(Q) \neq 0} \left| \sum_{m \geq 1} \sum_{l \text{ prime}} \frac{A(m)\chi(l)\alpha}{m^{s+it+ir'lir'}} V \left( \frac{m}{x} \right) \right|^2 \frac{dr'}{\pi(1 + \left( \frac{r'}{Q} \right)^2)}
\]

(2.0.14)

We apply Parseval here to obtain:

\[
S = \int_{-\infty}^{\infty} \varphi(Q) \left| \sum_{m \geq 1} \sum_{l \text{ prime}} \frac{A(m)\chi(l)\alpha}{m^{s+it+ir'lir'}} V \left( \frac{m}{x} \right) \right|^2 \frac{dr'}{\pi(1 + \left( \frac{r'}{Q} \right)^2)}
\]

\[
\leq \int_{-\infty}^{\infty} \varphi(Q) \left| \sum_{m \geq 1} \sum_{l \text{ prime}} \frac{A(m)\chi(l)\alpha}{m^{s+it+ir'lir'}} V \left( \frac{m}{x} \right) \right|^2 \frac{dr'}{\pi(1 + \left( \frac{r'}{Q} \right)^2)}
\]

\[
= \varphi(Q) \sum_{m_1, m_2 \geq 1} \sum_{l_1, l_2 \text{ prime}} \frac{A(m_1)A(m_2)\chi(l_1)\chi(l_2)(l_1l_2)^\alpha}{m_1^{s+it+ir'lir'} m_2^{s-it-lir'lir'}}
\]

\[
\times V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) \frac{dr'}{\pi(1 + \left( \frac{r'}{Q} \right)^2)}
\]

\[
= \varphi(Q) \sum_{l_1, l_2} \chi(l_1)\chi(l_2) \int_{-\infty}^{\infty} \sum_{m_1, m_2 \geq 1} \frac{A(m_1)A(m_2)(l_1l_2)^\alpha}{m_1^{s+it+ir'lir'} m_2^{s-it-lir'lir'}}
\]

\[
\times V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) \frac{dr'}{\pi(1 + \left( \frac{r'}{Q} \right)^2)}
\]

\[
= \varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2) \sum_{m_1, m_2 \geq 1} \frac{A(m_1)A(m_2)(l_1l_2)^\alpha}{m_1^{s+it} m_2^{s-it}}
\]

\[
\times V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) e^{-G \left| \log \left( \frac{l_1m_1}{l_2m_2} \right) \right|}
\]
where the diagonal portion \( S_d \) and the off-diagonal portions, \( S_{o_1} \) and \( S_{o_2} \), are defined as follows:

\[
S_d := \varphi(Q)G \sum_{l_1, l_2} \overline{\chi(l_1)} \chi(l_2) \sum_{m_1, m_2 \geq 1} \frac{A(m_1)A(m_2)(l_1 l_2)^\alpha}{m_1^{\frac{1}{2} + it} m_2^{\frac{1}{2} - it}} V\left(\frac{m_1}{x}\right) V\left(\frac{m_2}{x}\right)
\]

\[
S_{o_1} := \varphi(Q)G \sum_{l_1, l_2} \overline{\chi(l_1)} \chi(l_2) \sum_{h, m_2 \geq 1} \frac{A(m_1)A(m_2)(l_1 l_2)^\alpha}{m_1^{\frac{1}{2} + it} m_2^{\frac{1}{2} - it}} V\left(\frac{m_1}{x}\right) V\left(\frac{m_2}{x}\right) e^{-G|\log\left(\frac{1}{\Delta m_1 m_2}\right)|}
\]

\[
S_{o_2} := \varphi(Q)G \sum_{l_1, l_2} \overline{\chi(l_1)} \chi(l_2) \sum_{h, m_2 \geq 1} \frac{A(m_1)A(m_2)(l_1 l_2)^\alpha}{m_1^{\frac{1}{2} + it} m_2^{\frac{1}{2} - it}} V\left(\frac{m_1}{x}\right) V\left(\frac{m_2}{x}\right) e^{-G|\log\left(\frac{1}{\Delta m_1 m_2}\right)|}
\]

Remark. \( S_{o_2} = \overline{S_{o_1}} \).

At this point, we have converted the problem into studying \( S_d \) and \( S_{o_1} \).

3. The diagonal portion \( S_d \)

In this section, we focus on analyzing \( S_d \). The analysis breaks \( S_d \) into two sums, \( S_{d_1} \) corresponding to \( l_1 = l_2 \) and \( S_{d_2} \) corresponding to \( l_1 \neq l_2 \):

\[
S_d = S_{d_1} + S_{d_2},
\]

where

\[
S_{d_1} := \varphi(Q)G \sum_l l^{2\alpha} \sum_{m \geq 1} \frac{|A(m)|^2}{m} V\left(\frac{m}{x}\right) V\left(\frac{m}{x}\right)
\]

\[
S_{d_2} := \varphi(Q)G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) \sum_{m_1, m_2 \geq 1} \frac{A(m_1)A(m_2)(l_1 l_2)^\alpha}{m_1^{\frac{1}{2} + it} m_2^{\frac{1}{2} - it}} V\left(\frac{m_1}{x}\right) V\left(\frac{m_2}{x}\right)
\]

3.1. The case of \( l_1 = l_2 \). For \( S_{d_1} \), note that the \( m \)-sum does not depend on \( l \). The contribution of \( S_{d_1} \) is as follows:

Proposition 3.1. As \( x \to \infty \),

\[
S_{d_1} = \varphi(Q)G \sum_l l^{2\alpha} \left(\frac{4\pi}{T(k)}\right) |\langle f, f \rangle| \log x + O(QGL^{1+2\alpha}) + O(x^{-\varepsilon})
\]
Proof. Applying (2.0.1) twice to (3.0.1), we obtain:

$$S_{d_1} = \varphi(Q)G \sum_l t^{2\alpha} \left( \frac{1}{2\pi i} \right)^2 \int_2 \int_2 \sum_{m \geq 1} \frac{|A(m)|^2}{m^{1+s+w}} x^{s+w} v(s)v(w) \, ds \, dw$$

$$= \varphi(Q)G \sum_l t^{2\alpha} \left( \frac{1}{2\pi i} \right)^2 \int_2 \int_2 \sum_{m \geq 1} \frac{|A(m)|^2}{m^{1+s}} x^{s} v(s-w)v(w) \, ds \, dw \quad (3.1.2)$$

Moving the line of integration of $s$ down to $\Re s = -\frac{1}{3} - \varepsilon$, we pick up simple poles at $s = 0, s = w$, obtaining:

$$S_{d_1} = S_{d_1, \text{Res}_{s=0}} + S_{d_1, \text{Res}_{s=0}} + S_{d_1, \text{Res}_{s=-\frac{1}{3} - \varepsilon}} \quad (3.1.3)$$

For the residue at $s = w$:

$$S_{d_1, \text{Res}_{s=0}} = \varphi(Q)G \sum_l t^{2\alpha} \left( \frac{1}{2\pi i} \right) \int_2 \sum_{m \geq 1} \frac{|A(m)|^2}{m^{1+w}} x^{w} v(w) \, dw$$

$$= \varphi(Q)G \sum_l t^{2\alpha} \left( \frac{4\pi)^k}{\Gamma(k)} \right) (f, f) \log x + O(1) + O(x^{-\frac{1}{3} - \varepsilon}) \quad (3.1.4)$$

where the second equality is obtained by moving the line of integration $\Re w = 2$ down to $-\frac{1}{3} - \varepsilon$, picking up a double pole at $w = 0$.

We continue with the residue at $s = 0$:

$$S_{d_1, \text{Res}_{s=0}} = \varphi(Q)G \sum_l t^{2\alpha} \frac{4\pi)^k}{\Gamma(k)} (f, f) \left( \frac{1}{2\pi i} \int_2 v(-w)v(w) \, dw \right) \quad (3.1.5)$$

This is just $O(QGL^{1+2\alpha})$, upon noting the $l$-sum is $O(L^{1+2\alpha})$ and the $w$-integral above is a constant.

Finally, we deal with $\Re s = -\frac{1}{3} - \varepsilon$:

$$S_{d_1, \text{Res}_{s=-\frac{1}{3} - \varepsilon}} = \varphi(Q)G \sum_l t^{2\alpha} O(x^{-\frac{1}{3} - \varepsilon}) \quad (3.1.6)$$

Now plugging (3.1.4), (3.1.5) and (3.1.6) into (3.1.3), we have the proposition once we note that the $l$-sum is $O(L^{1+2\alpha})$. \hfill \Box

3.2. The case of $l_1 \neq l_2$. In $S_{d_2}$, we have the condition $m_1l_1 = m_2l_2$. When $l_1 \neq l_2$, it implies $m_1 = l_2m_2, m_2 = l_1m$ for some positive integer $m$. Hence, we have:

$$S_{d_2} = \varphi(Q)G \sum_{l_1, l_2 \neq l_1 \neq l_2} \frac{\chi(l_1)\chi(l_2)}{l_1^{2+it}l_2^{2-it}} (l_1l_2)^{\alpha} \sum_{m \geq 1} \frac{A(l_2m)A(l_1m)}{m} V \left( \frac{l_2m}{x} \right) V \left( \frac{l_1m}{x} \right) \quad (3.2.1)$$

Using methods similar to the proof of proposition 3.1, we have the following:

**Proposition 3.2.** As $x \to \infty$,

$$S_{d_2} = \varphi(Q)G \sum_{l_1, l_2 \neq l_1 \neq l_2} \frac{\chi(l_1)\chi(l_2)}{l_1^{2+it}l_2^{2-it}} (l_1l_2)^{\alpha} \frac{(4\pi)^k}{\Gamma(k)} (f, f) E_{l_1, l_2}(1) \log(x/l_2) + O(QGL^{1+2\alpha+\varepsilon}) + O(x^{-\varepsilon}) \quad (3.2.2)$$
where \( E_{l_1,l_2}(s) \) is defined as follows:

\[
E_{l_1,l_2}(s) := \left( \sum_{m \geq 1} \frac{A(l_2m)A(l_1m)}{m^s} \right) \left( \sum_{m \geq 1} \frac{|A(m)|^2}{m^s} \right)^{-1},
\]

which is essentially a product of ratios of Euler factors at the primes \( l_1 \) and \( l_2 \).

**Proof.** The claim about \( E_{l_1,l_2}(s) \) is analytic for \( \Re s > 0 \) and is bounded independent of \( l_1, l_2 \) in the region.

As in the proof of proposition \((3.1)\), we apply \((2.0.1)\) twice to \((3.2.1)\) and obtain:

\[
S_{d_2} = \varphi(Q)G \sum_{l_1,l_2 \neq \ell} \frac{\chi(l_1)\chi(l_2)}{l_2^{1/2} + it_1^{1/2} + it(l_1l_2)} \left( \frac{1}{2\pi i} \right)^2 \int (2) \int (4) \sum_{m \geq 1} \frac{A(l_2m)A(l_1m)}{m^{1+s}} x^s v(s-w)v(w) \frac{ds}{l_2^{1/2}} \frac{dw}{l_1^{1/2}}
\]

\[
\text{(3.2.3)}
\]

Again, we will move \( \Re s \) down to \(-\frac{1}{3} - \varepsilon\), hitting \( s = w \) and \( s = 0 \) as simple poles in the process:

\[
S_{d_2} = S_{d_2, \Re s = w} + S_{d_2, \Re s = 0} + S_{d_2, \Re s = -\frac{1}{3} - \varepsilon}
\]

\[
\text{(3.2.4)}
\]

Now we investigate the residue at \( s = w \):

\[
S_{d_2, \Re s = w} = \varphi(Q)G \sum_{l_1,l_2 \neq \ell} \frac{\chi(l_1)\chi(l_2)}{l_2^{1/2} + it_1^{1/2} + it(l_1l_2)} \left( \frac{1}{2\pi i} \right)^2 \int (2) \int (4) \sum_{m \geq 1} \frac{A(l_2m)A(l_1m)}{m^{1+w}} x^w v(s-w)v(w) dw
\]

\[
= \varphi(Q)G \sum_{l_1,l_2 \neq \ell} \frac{\chi(l_1)\chi(l_2)}{l_2^{1/2} + it_1^{1/2} + it(l_1l_2)} \left( \frac{4\pi}{\Gamma(k)} \right)^k (f, f) E_{l_1,l_2}(1) \log \left( \frac{x}{l_2} \right) + O(\log L) + O(x^{-\frac{1}{3} - \varepsilon})
\]

\[
\text{(3.2.5)}
\]

Next, we proceed to check out the residue at \( s = 0 \):

\[
S_{d_2, \Re s = 0} = \varphi(Q)G \left( \frac{4\pi}{\Gamma(k)} \right)^k (f, f) \sum_{l_1,l_2 \neq \ell} \frac{\chi(l_1)\chi(l_2)}{l_2^{1/2} + it_1^{1/2} + it(l_1l_2)} x^w E_{l_1,l_2}(1) \left( \frac{1}{2\pi i} \right)^2 \int (2) \frac{v(-w)v(w)}{l_2^{1/2}} dw
\]

\[
\text{(3.2.6)}
\]

The \( w \)-integral here is \( O(L^2) \), and hence this residue is \( O(QGL^{1+2\alpha+\varepsilon}) \). Finally, we obtain the following for the case \( \Re s = -\frac{1}{3} - \varepsilon \):

\[
S_{d_2, \Re s = -\frac{1}{3} - \varepsilon} \ll x^{-\varepsilon}
\]

\[
\text{(3.2.7)}
\]

Putting \((3.2.5)\), \((3.2.6)\) and \((3.2.7)\) together, we prove the proposition. \(\square\)
We have now obtained a complete understanding of the diagonal portion, and hence our next focus is to understand the off-diagonal sums $S_{o1}$ and $S_{o2}$. By the remark after (2.0.17), it is sufficient for us to understand $S_{o1}$.

4. Off-diagonal portion $S_{o1}$, setting up the integrals

Recall from (2.0.16),

$$S_{o1} = \varphi(Q)G \sum_{l_1, l_2} \frac{A(m_1)A(m_2)}{m_1^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} \sum_{m_1 l_1 = m_2 l_2 + hQ} \chi(l_1) \chi(l_2) \times V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) e^{-G \log \left( \frac{m_1 m_2}{l_2^2} \right)} (4.0.1)$$

We will show that this object can be converted into studying a four-fold integral involving the $Z_Q(s, w)$ function, which is defined as follows:

$$Z_Q(s, w) := \sum_{m_1 l_1 = m_2 l_2 + hQ} \frac{A(m_1)A(m_2)}{(l_2 m_2)^s (hQ)^{w+\frac{1}{2}}} (4.0.2)$$

This is one of the key objects extensively studied in [8]. The relevant details for the purpose of this work is quoted in the appendix.

We now resume the goal of showing $S_{o1}$ can be studied via a four-fold integral involving $Z_Q(s, w)$.

**Proposition 4.1.** As $G \to \infty$, we have

$$S_{o1} = \varphi(Q)G \sum_{l_1, l_2} \frac{\chi(l_1) \chi(l_2)}{l_1 l_2} \left( \frac{1}{2\pi i} \right)^4 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} Z_Q(s, s' - s - \frac{k}{2} + 1) \times \Gamma(s' - s + \frac{1}{2} - \beta) \Gamma(w + s + \beta + \frac{k}{2} - 1 - s' + it) \frac{1}{l_1^{w+\frac{1}{2}} l_2^{w'-\frac{1}{2}}} \times \Gamma(w + \frac{k}{2} + it) \times x'^{-\frac{1}{2}} v(s' - w) v(w - \frac{1}{2}) G^{-\beta} \Gamma(\beta) d\beta ds' ds dw + O(QGL^{3+2\alpha} e^{-G^{1/4}}) (4.0.3)$$

where $\text{Re } w = \gamma_1 = 1 + 2\theta + 2\varepsilon$, $\text{Re } s = \gamma_2 = \frac{3}{4}$, $\text{Re } s' = \gamma_3 = \frac{5}{4}$ and $\text{Re } \beta = \gamma_4 = \frac{3}{4}$.

**Remark.** As long as $G$ is chosen such that $G = (1 + |t|)^a \log^b(Q)$ with $a > 0$ and $b > 4$, then the last error term is $o(1)$ in both $Q$ and $t$. Note that any such $G$ satisfies $G \geq 2A = \sqrt{40 \log(Q(1 + |t|))}$ for large $Q$ and $t$.

**Proof.** We will focus on the innermost sum of $S_{o1}$. For convenience, we define:

$$T_{o1} := \sum_{h_0, m_2 \geq 1, m_1 l_1 = m_2 l_2 + hQ} \frac{A(m_1)A(m_2)}{m_1^{\frac{1}{2}+it} m_2^{\frac{1}{2}-it}} V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) e^{-G \log \left( \frac{m_1 m_2}{l_2^2} \right)}$$
Starting with the definition of $T_{o_1}$, we will substitute in $m_1 l_1 = m_2 l_2 + h_0 Q$ in several places:

$$T_{o_1} = l_1^{\frac{1}{2} + it} l_2^{\frac{1}{2} - it} \sum_{m_2, h_0} A(m_1) \overline{A(m_2)} V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) \frac{1}{(m_2 l_2 + h_0 Q) \frac{1}{2} + it (m_2 l_2) \frac{1}{2} - it} e^{-G \log \left( 1 + \frac{h_0 Q}{m_2 l_2} \right)}$$

Now first observe that if $h_0 Q \geq m_2 l_2$, we can bound the exponential by $e^{-G \log^2}$.

Thus, we have the following:

$$\ll l_1^{\frac{1}{2} + it} l_2^{\frac{1}{2} - it} \sum_{m_2, h_0 \geq 1 \atop m_1 l_1 = m_2 l_2 + h_0 Q \atop h_0 Q \geq m_2 l_2} A(m_1) \overline{A(m_2)} V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) \frac{1}{(m_2 l_2 + h_0 Q)^{\frac{1}{2}} (m_2 l_2)^{\frac{1}{2}}} e^{-G \log^2}$$

$$\ll Le^{-G \log^2}$$

We conclude that the terms with large shift $h_0$ are negligible. We can actually do better. Say $\frac{2m_2 l_2}{G^4}$, $\leq h_0 Q \leq m_2 l_2$. By using the fact that $\log(1 + x) \geq \frac{x}{2}$ for $0 \leq x \leq 1$, we have:

$$e^{-G \log \left( 1 + \frac{h_0 Q}{m_2 l_2} \right)} \leq e^{-G(h_0 Q)/(2m_2 l_2)} \leq e^{-G^3/4}$$

Then using the same argument as above, we see that the sum of these terms are also negligible as they are $O(e^{-G^3/4})$.

Now we construct a double sum $\bar{T}$ from $T_{o_1}$ by replacing $\log \left( 1 + \frac{h_0 Q}{l_2 m_2} \right)$ with $\frac{h_0 Q}{l_2 m_2}$:

$$\bar{T} = l_1^{\frac{1}{2} + it} l_2^{\frac{1}{2} - it} \sum_{m_2, h_0} A(m_1) \overline{A(m_2)} V \left( \frac{m_1}{x} \right) V \left( \frac{m_2}{x} \right) \frac{1}{(m_2 l_2 + h_0 Q)^{\frac{1}{2} + it} (m_2 l_2)^{\frac{1}{2} - it}} e^{-G(h_0 Q)/(m_2 l_2)}$$

(4.0.4)

Using the same arguments as above, the only terms that matter in this sum are those with $h_0 Q \leq \frac{2m_2 l_2}{G^4}$.

In the following, we will show that $\bar{T}$ is a good estimate of $T_{o_1}$. In particular, we investigate their difference. Since the only relevant values of $h_0$ are those such that $h_0 Q \leq \frac{2m_2 l_2}{G^3}$, we will assume $h_0$ is in this range and start by bounding the difference of the exponentials:

$$e^{-G(h_0 Q)/(m_2 l_2)} - e^{-G \log \left( 1 + \frac{h_0 Q}{m_2 l_2} \right)}$$

$$= e^{-G(h_0 Q)/(m_2 l_2)} \left( 1 - e^{-G \left( \frac{h_0 Q}{m_2 l_2} - \log \left( 1 + \frac{h_0 Q}{m_2 l_2} \right) \right)} \right)$$

$$= e^{-G(h_0 Q)/(m_2 l_2)} O \left( G \left( \frac{h_0 Q}{m_2 l_2} \right)^2 \right)$$

$$= e^{-G(h_0 Q)/(m_2 l_2)} O \left( G^{-1/2} \right)$$

$$= e^{-G(h_0 Q)/(m_2 l_2)} O \left( G^{-1/2} \right)$$
In the above, we have used the fact that $1 - e^y = O(y)$ when $y$ is close to 0. Using this bound, we can conclude that as $G \to \infty$,

$$T_{n_1} - \bar{T} = \bar{T} \cdot O(G^{-1/2}) + O(L e^{-G^{1/4}})$$

Now we just need to show that $\bar{T}$ equals the 4-fold integral shown in the proposition. To this end, we start with $\bar{T}$ as in (4.0.4), apply (2.0.1) twice and express the exponential in its inverse Mellin transform, resulting in:

$$\bar{T} = \frac{1}{2\pi i} \int \frac{1}{2\pi i} \int \sum_{m_2, h_0} \frac{A(m_1) A(m_2)}{(m_2 l_2)^{s+w+1}} \times v(s) v(w) x^{s+w} l_2^{s+w} \left( \frac{G h_0 Q}{m_2 l_2} \right)^{-\beta} \Gamma(\beta) \, ds \, dw \, d\beta$$

We quote this identity from [6]:

$$\frac{1}{2\pi i} \int \frac{\Gamma(u) \Gamma(\beta - u)}{\Gamma(\beta)} t^{-u} \, du = (1 + t)^{-\beta}, \quad (4.0.5)$$

where $0 < \gamma < \Re \beta$. Manipulating the expression from before and using (4.0.5), we end up introducing the $ZQ$ function defined above:

$$\bar{T} = \frac{1}{2\pi i} \int \frac{1}{2\pi i} \int \sum_{m_2, h_0} \frac{A(m_1) A(m_2)}{(m_2 l_2)^{s+w+1}} \times \left( 1 + \frac{h_0 Q}{m_2 l_2} \right)^{-(s+w+1)} v(s) v(w) x^{s+w} l_2^{s+w} \left( \frac{G h_0 Q}{m_2 l_2} \right)^{-\beta} \Gamma(\beta) \, ds \, dw \, d\beta$$

$$= \frac{1}{2\pi i} \int \frac{1}{2\pi i} \int \sum_{m_2, h_0} \frac{A(m_1) A(m_2)}{(m_2 l_2)^{s+w+1}} \times \left( \Gamma(u) \Gamma(w + k + i\beta) - \Gamma(u) \Gamma(w + k + i\beta) \right)^{-u} v(s) v(w) x^{s+w} l_2^{s+w} \left( \frac{G h_0 Q}{m_2 l_2} \right)^{-\beta} \Gamma(\beta) \, ds \, dw \, d\beta$$

From here, we will do a series of change of variables. First we change $u \to u - \beta$:

$$= \frac{1}{2\pi i} \int \frac{1}{2\pi i} \int \sum_{m_2, h_0} \frac{A(m_1) A(m_2)}{(m_2 l_2)^{s+w+1}} \times \left( \Gamma(u - \beta) \Gamma(w + k + i\beta - u + i\beta) \right)^{-u} v(s) v(w) x^{s+w} l_2^{s+w} G^{-\beta} \Gamma(\beta) \, du \, dw \, d\beta$$
Now we do $s \mapsto s + u - 1$:

$$= l_1^{1+i \theta} l_2^{-i \theta} \left( \frac{1}{2 \pi i} \right)^4 \int_{(\epsilon_1')} \int_{(\epsilon_2')} \int_{(\epsilon_3')} \int_{(\epsilon_4')} Z_Q(s, w) x^s w^{-\frac{1}{2}} \Gamma(u - \frac{k - 1}{2}) \Gamma(w - \frac{k - 1}{2} + \beta + i \theta) \Gamma(w + \frac{k - 1}{2})$$

Then we will change $s \mapsto s - w, u \mapsto u + \frac{k - 1}{2}$:

$$= l_1^{1+i \theta} l_2^{-i \theta} \left( \frac{1}{2 \pi i} \right)^4 \int_{(\epsilon_1')} \int_{(\epsilon_2')} \int_{(\epsilon_3')} \int_{(\epsilon_4')} Z_Q(s, u) x^{s-u} w^{-\frac{1}{2}} \Gamma(u - \frac{k - 1}{2}) \Gamma(w - \frac{k - 1}{2} + \beta + i \theta) \Gamma(w + \frac{k - 1}{2})$$

Finally we will get rid of $u$ and introduce $s' = s + u + \frac{k - 1}{2}$, while also doing $w \mapsto w - \frac{1}{2}$:

$$= l_1^{1+i \theta} l_2^{-i \theta} \left( \frac{1}{2 \pi i} \right)^4 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} Z_Q(s, s') x^{s'} w^{-\frac{1}{2}} \Gamma(s' - \frac{1}{2} - \beta) \Gamma(s' - \frac{1}{2} - 1 + \beta + i \theta) \Gamma(s' - \frac{1}{2} + i \theta)$$

This ends the change of variables. We can take the following values for the $\gamma_i$'s: $\text{Re } s = 2, \text{Re } s' = 2 + \frac{1}{2} + \varepsilon, \text{Re } w = 1 + 2 \theta + 2 \varepsilon$ and $\text{Re } \beta = 2$.

We can move $\text{Re } \beta$ down to $\frac{3}{4}$ without hitting poles. Next, we can move $\text{Re } s'$ to $\frac{5}{4} + \varepsilon$ without hitting poles. Now we can move $\text{Re } s$ down to $\frac{3}{4}$ without picking up poles. Finally, we will move $\text{Re } s$ to $\frac{5}{4}$, again without poles. This proves the proposition. \[\square\]

We will separate our analysis of $S_{01}$ into 2 parts, the first part corresponding to the discrete spectrum $S_{01}^d$ and the second to the continuous spectrum $S_{01}^c$. We will also replace $Z_Q(s, s' - s - \frac{k}{2} + 1)$ with $\lim_{\delta \to 0} Z_Q(s, s' - s - \frac{k}{2} + 1; \delta)$, taking the $\delta$ limit where it is convenient to do so. In our analysis, our top priority is to bring down the effects of $x$; the second priority is to bring down the $t$-contribution.

To facilitate this process, we set $G = (1 + |t|)^a \log^b Q$, where $0 < a \leq 1$ is to be chosen in the next section. This ensures that the error term in proposition (4.1) is $o(1)$ in both $Q$ and $t$. 
5. Off-diagonal, discrete spectrum

We will be looking at the growth of the discrete terms in this section:

\[ S^d_{\alpha_1} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) \left( \frac{1}{2} + \alpha \frac{r}{2} - \frac{i}{2} \right) (1 + O\left(G^{-1/2}\right)) \]

\[ \times \left( \frac{1}{2\pi i} \right)^4 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} \frac{(4\pi)^k 2^s \frac{1}{2} (l_1 l_2) \frac{k}{k-1}}{\Gamma(s + k - 2\sqrt{i})} \right. \]

\[ \times v(s' - w) v(w - \frac{1}{2}) \Gamma(s' - \frac{1}{2}) \Gamma(s + w + \beta - s' - \frac{1}{2} - \frac{k-1}{2} + it) \]

\[ \times \sum_{t_j} L_Q(s', \overline{u_j}) M(s, t_j, \delta)(\overline{U, u_j}) G^{-\beta} \Gamma(\beta) d\beta ds' dw \]  

(5.0.1)

We will drop the factor \(1 + O\left(G^{-1/2}\right)\) in the following.

Since we are going to take the limit as \(\delta\) goes to 0 eventually, we will require \(\text{Re } s < \frac{1}{2} - \frac{\varepsilon}{2}\) at some stage so that the \(j\)-sum converges absolutely. Thus, we move \(\text{Re } s\) down to \(\text{Re } s = \frac{1}{2} - \frac{k-1}{2} - \theta - \varepsilon\), during which we will encounter poles at \(s = \frac{1}{2} \pm it_j - r\), with \(0 \leq r \leq \frac{k}{2}\) an integer. Hence, our analysis will split into 2 pieces, the residues and the moved integral.

5.1. The residues at \(s = \frac{1}{2} \pm it_j - r\). In this section, we will show the following:

Proposition 5.1. For \(G \asymp (1 + |t|)^{-2\sqrt{i}} \log^5 Q\),

\[ \sum_{r, t_j} \text{Res}_{s = \frac{1}{2} + it_j - r} S^d_{\alpha_1} \ll G^{1+\varepsilon} Q^{1+\varepsilon} L^{3+2\alpha+\varepsilon} + x^{-\varepsilon}, \]

where the \(r\)-sum runs over the integers \(0 \leq r \leq \frac{k}{2}\), and the \(t_j\)-sum runs through the eigenvalues \(\frac{1}{2} + t_j^2\) of the Laplacian.

Remark. In the proof of this proposition, we also justify the reason of having this condition on \(G\).

Proof. Taking the residue at \(s = \frac{1}{2} + it_j - r\) for \(0 \leq r \leq \frac{k}{2}\) and letting \(\delta \to 0\), we have:

\[ \text{Res}_{s = \frac{1}{2} + it_j - r} S^d_{\alpha_1} = \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) \left( \frac{1}{2} + \alpha \frac{r}{2} - \frac{i}{2} \right) \]

\[ \times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} \frac{L_Q(s', \overline{u_j}) c_{r, j} v(s' - w) v(w - \frac{1}{2})}{\Gamma(w + \frac{k-1}{2} + it)} \]

\[ \times x^{s' - \frac{1}{2} t_j - r - \beta} \Gamma(w + \beta - s' + \frac{k-1}{2} - r + it_j + it) \]

\[ \times \Gamma(s' - it_j + r - \beta) \Gamma(w + \beta - s' + \frac{k-1}{2} - r + it_j + it) \]

\[ \times x^{s' - \frac{1}{2} t_j - r - \beta} \Gamma(s' - \frac{1}{2} - \varepsilon) \Gamma(\beta) d\beta ds' dw \]  

(5.1.1)

Now we are going to move \(\text{Re } s'\) down to \(\frac{k}{2} - \varepsilon\), in the process of which, we hit the simple poles at \(s' = w\) and \(s' = \beta + it\) when \(r = 0\). The shifted integral at \(\text{Re } s' = \frac{k}{2} - \varepsilon\) is \(O(x^{-\varepsilon})\).
5.1.1. Contribution from the pole at $s' = w$. At the pole $s' = w$, we have residue:

$$\text{Res}_{s' = w} \text{Res}_{s = \frac{1}{2} + it_j - r} S_{O_1}^d = \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{1}{2} + \alpha \frac{it_j}{l_1 l_2}} L_Q(w, \overline{u}) \frac{\Gamma(w - it_j + r - \beta) \Gamma(\beta + \frac{k - 1}{2} - r + it_j + it) \Gamma(w + \frac{k - 1}{2} + it)}{\Gamma(\beta)} \times v(w - \frac{1}{2}) x^{\frac{1}{2}} L_1^{-\frac{1}{2}} G^\beta \Gamma(\beta) d\beta dw$$

Moving $\text{Re } w$ down to $\frac{1}{2} - \varepsilon$, we have a shifted integral that is of growth $O(x^{-\varepsilon})$ and encounter the simple poles at $w = \frac{1}{2}$ and $w = \beta + it_j$ when $r = 0$. At $w = \frac{1}{2}$,

$$\text{Res}_{w = \frac{1}{2}} \text{Res}_{s = \frac{1}{2} + it_j - r} S_{O_1}^d = \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{1}{2} + \alpha \frac{it_j}{l_1 l_2}} L_Q(\frac{1}{2}, \overline{u}) \frac{\Gamma(\frac{1}{2} - it_j + r - \beta) \Gamma(\beta + \frac{k - 1}{2} - r + it_j + it)}{\Gamma(\frac{k - 1}{2} + it)} G^\beta \Gamma(\beta) d\beta$$

We have to investigate this integral closely. To this end, it is convenient to have the following lemma.

**Lemma 5.1.** For $\text{Re } \beta = a \geq \frac{1}{2} + r + \varepsilon$ and $Q^{\frac{2}{k}} \ll L \ll Q$,

$$G \sum_{l_j} L_Q(\frac{1}{2}, \overline{u}) \frac{\Gamma(\frac{1}{2} + r - \beta) \Gamma(\beta + \frac{k - 1}{2} - r + it)}{\Gamma(\frac{k - 1}{2} + it)} \times v^2 \int_{(\alpha)} \frac{G^{-\beta + it} \Gamma(\beta - it_j)}{G^{\beta + it} \Gamma(\beta - it_j)} |\tilde{T}|^{a - r - \frac{1}{2} + \varepsilon} \ll G^{1-a+\theta} (1 + |\tilde{T}|)^{a - r - \frac{1}{2} + \varepsilon} (1 + |\tilde{T}|)^{1 + r + \varepsilon} Q^{-\frac{1}{2}} L^{1-k+\varepsilon}$$

In fact, to minimize this, the optimal choices are $\text{Re } \beta = a = \frac{3}{2} + 2r + \varepsilon$ and $G \asymp (1 + |t|)^{-\frac{1}{2}} \log^5 Q$. With these choices, the bound is $O(G^{1+\varepsilon} Q^{-\frac{1}{2}} L^{1-k+\varepsilon})$.

This relates to (5.1.2) by moving lines of integration and picking up relevant poles. We will delay the proof of this lemma to the end of this section.

Going back to the integral in (5.1.2), we will deal with the case $r = 0$ first. We move the line of integration up to $\text{Re } \beta = \frac{3}{2} + \varepsilon$, picking up a residue at $\beta = \frac{1}{2}$. Then by lemma (5.1) with the optimal choice of $G$, the moved integral is
We will move Re\(\beta\) down to \(\Re \beta = \frac{3}{2} + 2r + \varepsilon\). The moved integral is \(O(G^{1+\varepsilon}Q^{\frac{1}{2}+\varepsilon}L^{3+2a+\varepsilon})\). This whole sum is \(o(GQ^{\frac{1}{2}+\varepsilon}L^{3+2\alpha+\varepsilon})\). Hence, the residue at \(w = \frac{1}{2}\) has growth \(O(G^{1+\varepsilon}Q^{\frac{1}{2}+\varepsilon}L^{3+2\alpha+\varepsilon})\).

At \(w = \beta + it_j\) when \(r = 0\), the residue is:
\[
\varphi(Q)G \sum_{l_1,l_2} \frac{l_1^{\alpha}l_2^{-\alpha}}{l_1l_2} c_{0,j} \frac{1}{2\pi i} \int_{(\gamma_1)} L_Q(\beta + it_j, w_j) v(\beta + it_j) v(w) d\beta
\]

We will move Re\(\beta\) down to \(\frac{1}{2} - \theta - \varepsilon\), hitting the pole at \(\beta = \frac{1}{2} - it_j\). The shifted integral is \(O(x^{-\varepsilon})\). At the pole \(\beta = \frac{1}{2} - it_j\), the residue is:
\[
\varphi(Q)G \sum_{l_1,l_2} \frac{l_1^{\alpha}l_2^{-\alpha}}{l_1l_2} c_{0,j} L_Q(\beta + it_j, w_j) G^{\frac{1}{2}+\varepsilon} \Gamma(\beta) d\beta
\]

Summing up the \(t_j\)'s, this is \(O(G^{\frac{1}{2}+\varepsilon}Q^{\frac{1}{2}+\varepsilon}L^{3+2\alpha+\varepsilon})\).

5.1.2. Contribution from the pole \(s' = \beta + it_j\) when \(r = 0\). The residue here is:
\[
\varphi(Q)G \sum_{l_1,l_2} \frac{l_1^{\alpha}l_2^{-\alpha}}{l_1l_2} c_{0,j} \frac{1}{2\pi i} \int_{(\gamma_1)} \int_{(\gamma_4)} L_Q(\beta + it_j, w_j) v(\beta + it_j - \frac{1}{2}) v(w - \frac{1}{2}) d\beta dw
\]

We will move Re\(\beta\) down to \(\frac{1}{2} - \theta - \varepsilon\). This doesn't hit any poles and the resulting moved integral is \(O(x^{-\varepsilon})\).

Putting the two parts together, we have proved our proposition. \(\square\)
5.2. The moved integral at $\text{Re } s = \frac{1}{2} - \frac{b}{2} - \theta - \varepsilon$. We will show the following:

**Proposition 5.2.** For $G \asymp (1 + |t|)^\frac{-3s}{2} \log^5 Q$, the moved integral has the following upper bound:

$$S_{\text{mov}, \text{Re } s = \frac{1}{2} - \frac{b}{2} - \theta - \varepsilon} \ll G^{1+\varepsilon} Q^{\frac{3}{2}+\varepsilon} L^{3+2\alpha+\varepsilon} + x^{-\varepsilon}.$$  

**Proof.** Recalling from (5.0.1), this is the moved integral:

$$S_{\text{mov}, \text{Re } s = \frac{1}{2} - \frac{b}{2} - \theta - \varepsilon} = \lim_{\delta \to 0} \varphi(Q) \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{k}{2} + \alpha} t_{l_1}^{\frac{\theta}{2}} t_{l_2}^{\frac{-\theta}{2}}$$

$$\times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma_1)} \int_{(\gamma_2')} \int_{(\gamma_3)} \int_{(\gamma_4')} \sum_j (4\pi)^k 2^{s - \frac{k}{2}} M(s, t_j, \delta) \frac{L_Q(s', w)}{\Gamma(s + k - 1) 2\sqrt{\pi}}$$

$$\times v(s' - w) v(w - \frac{1}{2}) \Gamma(s' - s + \frac{k}{2} + \beta) \Gamma(s + w + \beta - s' + \frac{k}{2} - 1 + \i t)$$

$$\times x^{s' - \frac{k}{2}} t_{l_2}^{s' - w} t_{l_1}^{w - \frac{k}{2}} G^{-\beta} \Gamma(\beta) d\beta d s' d s d w$$

(5.2.1)

except when $\gamma_2' = \frac{1}{2} - \frac{k}{2} - \theta - \varepsilon$.

Similar to before, we are moving $\text{Re } s'$ to $\frac{1}{2} - \frac{k}{2}$, picking up a simple pole at $s' = w$. The shifted integral is $O(x^{-\varepsilon})$. For the residue at $s' = w$, we have:

$$\lim_{\delta \to 0} \varphi(Q) \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{k}{2} + \alpha} t_{l_1}^{\frac{\theta}{2}} t_{l_2}^{\frac{-\theta}{2}}$$

$$\times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma_1)} \int_{(\gamma_2')} \int_{(\gamma_3)} \int_{(\gamma_4')} \sum_j (4\pi)^k 2^{s - \frac{k}{2}} M(s, t_j, \delta) \frac{L_Q(s, w)}{\Gamma(s + k - 1) 2\sqrt{\pi}}$$

$$\times \frac{\Gamma(w - s + \frac{k}{2} + \beta) \Gamma(s + w + \beta + \frac{k}{2} - 1 + \i t)}{\Gamma(w + \frac{k-1}{2} + \i t)} (x l_1)^{w - \frac{k}{2}} G^{-\beta} \Gamma(\beta) d\beta d s d w$$

Now we can move $\text{Re } w$ down to $\frac{1}{2} - \varepsilon$, picking up the only pole at $w = \frac{1}{2}$. The shifted integral is again $O(x^{-\varepsilon})$, and the residue at $w = \frac{1}{2}$ is:

$$\lim_{\delta \to 0} \varphi(Q) \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{k}{2} + \alpha} t_{l_1}^{\frac{\theta}{2}} t_{l_2}^{\frac{-\theta}{2}}$$

$$\times \left( \frac{1}{2\pi i} \right)^2 \int_{(\gamma_2')} \int_{(\gamma_3)} \int_{(\gamma_4')} \sum_j (4\pi)^k 2^{s - \frac{k}{2}} M(s, t_j, \delta) \frac{L_Q(\frac{1}{2}, w)}{\Gamma(s + k - 1) 2\sqrt{\pi}}$$

$$\times \frac{\Gamma(1 - s - \beta) \Gamma(s + \beta + \frac{k}{2} - 1 + \i t)}{\Gamma(\frac{1}{2} + \i t)} G^{-\beta} \Gamma(\beta) d\beta d s$$

(5.2.2)

First, for $\text{Re } s < \frac{1}{2} - \frac{k}{2}$, we note the following bound which is true regardless of relative sizes of $|t_j|$ and $|\text{Im } s|$:

$$\lim_{\delta \to 0} \frac{2^{s - \frac{k}{2}} M(s, t_j, \delta)}{\Gamma(s + k - 1) 2\sqrt{\pi}} \ll (1 + |t_j|)^2 |\text{Re } s|^2 (1 + |\text{Im } s|)^{3 - 3|\text{Re } s - k|}$$
Hence, at \( \text{Re } s = \frac{1}{2} - \frac{k}{2} - \theta - \varepsilon \), using the above bound and \( \triangleq \), we have:

\[
\lim_{\delta \to 0} \frac{(4\pi)^k 2s^{-\frac{k}{2}}}{\Gamma(s + k - 1)2\sqrt{\pi}} \sum_j L_Q\left(\frac{1}{2}, \overline{u_j}\right)M(s, t_j, \delta)(U, u_j) \\
\ll (1 + |\text{Im } s|)^{\frac{\theta}{2} + \frac{\varepsilon}{2} + 3\theta + 3\varepsilon} Q^{-\frac{1}{2}} L^{1-k+\varepsilon}
\]

Using the fact noted above and the methods of proving lemma (5.1), we have the following auxiliary lemma:

**Lemma 5.2**. For \( G \simeq (1 + |t|)^{3+\delta s} \log^5 Q \) and \( \text{Re } \beta = a = \frac{5}{2} + k + 5\theta + 2\varepsilon \),

\[
G \left(\frac{1}{2\pi i}\right)^2 \int_{(\gamma)} \int_{(a)} \lim_{\delta \to 0} \frac{(4\pi)^k 2s^{-\frac{k}{2}} M(s, t_j, \delta)}{\Gamma(s + k - 1)2\sqrt{\pi}} L_Q\left(\frac{1}{2}, \overline{u_j}\right)(U, u_j) \\
\times \frac{\Gamma(1-s-\beta)\Gamma(s+\frac{k}{2}+1+\beta + \delta \bar{u})}{\Gamma(k/2+\bar{u})} G^{-\beta} \Gamma(\beta) d\beta d\bar{s}
\ll G^{1+\varepsilon} Q^{-\frac{1}{2}} L^{1-k+\varepsilon}
\]

(5.2.3)

Again in (5.2.2), we move \( \text{Re } \beta \) to \( \frac{5}{2} + k + 5\theta + 2\varepsilon \), hitting poles at \( \beta = 1 - s + \ell \), where \( 0 \leq \ell \leq \frac{1}{2} + 2 \). Using the proposition above and \( G \simeq (1 + |t|)^{3+\delta s} \log^5 Q \), the moved integral is \( O(G^{1+\varepsilon} Q^{\frac{1}{2}+\varepsilon} L^{3+2\theta+2\varepsilon}) \). The residue at \( \beta = 1 - s + \ell \) is:

\[
\lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{1}{2}+\alpha} |\bar{t} - \bar{t}| \bar{t}
\times \frac{1}{2\pi i} \int_{(\gamma)} \int_{(a)} \frac{(4\pi)^k 2s^{-\frac{k}{2}} M(s, t_j, \delta)}{\Gamma(s + k - 1)2\sqrt{\pi}} L_Q\left(\frac{1}{2}, \overline{u_j}\right)(U, u_j) \\
\times (-1)^{\ell} \frac{\Gamma(k/2 + \ell + \delta \bar{u})}{\Gamma(k/2 + \bar{u})} G^{1+\ell+\varepsilon} \Gamma(1-s+\ell) d\bar{s}
\]

These residues are \( o(GQ^{\frac{1}{2}+\varepsilon} L^{3+2\theta+2\varepsilon}) \). \( \square \)

5.3. **Proof of lemma (5.1)**. For convenience, we write \( \beta = a + it_\beta \), where \( a, t_\beta \in \mathbb{R} \). We will split the object to be analyzed as follows:

\[
G \sum_{t_j} L_Q\left(\frac{1}{2}, \overline{u_j}\right)c_{r,j} \\
\times \frac{1}{2\pi i} \int_{(a)} \frac{\Gamma(\frac{1}{2} + r - \beta) \Gamma(\beta + \frac{k-1}{2} - r + \delta \bar{u})}{\Gamma(k/2 + \bar{u})} G^{-\beta + it_j} \Gamma(\beta - it_j) d\beta
\]

\[
= G \sum_{t_j} L_Q\left(\frac{1}{2}, \overline{u_j}\right)c_{r,j}(P_1 + P_2 + P_3),
\]
where
\[
P_1 = \frac{1}{2\pi i} \int_{\beta=\alpha+i\beta}^{\beta=\alpha+it_{\beta}} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{1}{2} + i\tilde{t}\right)} G^{-\beta+it} \Gamma(\beta - it_j) \, d\beta
\]
\[
P_2 = \frac{1}{2\pi i} \int_{\beta=\alpha+it_{\beta}}^{\beta=\alpha+i\beta} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{1}{2} + i\tilde{t}\right)} G^{-\beta+it} \Gamma(\beta - it_j) \, d\beta
\]
\[
P_3 = \frac{1}{2\pi i} \int_{\beta=\alpha+it_{\beta}}^{\beta=\alpha+i\beta} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{1}{2} + i\tilde{t}\right)} G^{-\beta+it} \Gamma(\beta - it_j) \, d\beta
\]

In each part, we seek to bound the integrand using Stirling’s formula.

5.3.1. The case where \(|t_{\beta}| \leq |\tilde{t}| - \log^4 |\tilde{t}|\). The ratio of gammas is bounded by:
\[
\frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma(\beta - it_j)} \lesssim (1 + |\tilde{t}|)^{a-r-\frac{1}{2}}(1 + |t_{\beta}|)^{r-a}(1 + |t_{\beta} - t_j|)^{a+\theta-\frac{1}{2}} e^{-\frac{\pi}{2} |t_{\beta} - t_j|}
\]

We further separate this case by the relative sizes of \(|t_j|\) and \(|\tilde{t}|\):

1. If \(|t_j| \geq |\tilde{t}|\), then we can further conclude that the integrand is bounded by:
\[
\lesssim (1 + |\tilde{t}|)^{a-r-\frac{1}{2}}(1 + |t_{\beta}|)^{r-a}(1 + |t_{\beta} - t_j|)^{a+\theta-\frac{1}{2}} e^{-\frac{\pi}{2} |t_{\beta} - t_j|} G^{-a+\theta}
\]

Executing the \(\beta\)-integral and then summing over such \(|t_j|\)’s using the bound in proposition \(A.3\) we have that:
\[
G \sum_{|t_j| \geq |\tilde{t}|} L_{Q, \Gamma(\frac{1}{2}, \Gamma(\theta))} P_1 \lesssim G^{1-a+\theta}(1 + |\tilde{t}|)^{a-r-\frac{1}{2}+z} Q^{-\frac{1}{2}} L^{1-k+z}
\]

2. If \(|t_j| \leq |\tilde{t}|\), then we separate the integral as follows:
\[
P_1 = P_{1,1} + P_{1,2} + P_{1,3},
\]
where
\[
P_{1,1} = \frac{1}{2\pi i} \int_{\beta=\alpha+i\beta}^{\beta=\alpha+it_{\beta}} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{1}{2} + i\tilde{t}\right)} G^{-\beta+it} \Gamma(\beta - it_j) \, d\beta
\]
\[
P_{1,2} = \frac{1}{2\pi i} \int_{\beta=\alpha+it_{\beta}}^{\beta=\alpha+i\beta} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{1}{2} + i\tilde{t}\right)} G^{-\beta+it} \Gamma(\beta - it_j) \, d\beta
\]
\[
P_{1,3} = \frac{1}{2\pi i} \int_{\beta=\alpha+it_{\beta}}^{\beta=\alpha+i\beta} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(\beta + \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{1}{2} + i\tilde{t}\right)} G^{-\beta+it} \Gamma(\beta - it_j) \, d\beta
\]
(a) In the subcase $|t_\beta| \leq |t_j| - \log^4 |t_j|$, this subcase has the same effect as case 1:

$$G \sum_{|t_j| \leq \tilde{|t|}} L_Q \left( \frac{1}{2}, \overline{u_j} \right) c_{r,j} P_{1,1} \ll G^{1-a+\theta} (1 + \tilde{|\tilde{t}|^{a-r-\frac{1}{2}+\epsilon}} Q^{-\frac{1}{2}} L^{1-k+\epsilon}$$

(b) In the subcase $|t_\beta| - |t_j| \leq \log^4 |t_j|$, the integrand is bounded by:

$$\ll (1 + \tilde{|\tilde{t}|^{a-r-\frac{1}{2}+\epsilon}} (1 + |t_\beta|)^{r-a} G^{-a+\theta}$$

Now executing the $\beta$-integral and then summing $|t_j|$, we obtain that

$$G \sum_{|t_j| \leq \tilde{|t|}} L_Q \left( \frac{1}{2}, \overline{u_j} \right) c_{r,j} P_{1,2} \ll G^{1-a+\theta} (1 + \tilde{|\tilde{t}|^{1+r+\epsilon}} Q^{-\frac{1}{2}} L^{1-k+\epsilon}$$

(c) In the last subcase, $|t_j| + \log^4 |t_j| \leq |t_\beta| \leq |\tilde{t}| - \log^4 |\tilde{t}|$, the integrand is bounded by:

$$\ll (1 + \tilde{|\tilde{t}|^{a-r-\frac{1}{2}} (1 + |t_\beta|)^{r+\theta-\frac{1}{2}} e^{-\frac{2}{5} \epsilon} (1 + \log^4 |t_j|) G^{-a+\theta}$$

Now executing the $\beta$-integral and then summing $|t_j|$, we obtain that

$$G \sum_{|t_j| \leq \tilde{|t|}} L_Q \left( \frac{1}{2}, \overline{u_j} \right) c_{r,j} P_{1,3} \ll G^{1-a+\theta} (1 + \tilde{|\tilde{t}|^{a-r-\frac{1}{2}+\epsilon}} Q^{-\frac{1}{2}} L^{1-k+\epsilon}$$

Hence, in total:

$$G \sum_{j} L_Q \left( \frac{1}{2}, \overline{u_j} \right) c_{r,j} P_1 \ll G^{1-a+\theta} \left( (1 + \tilde{|\tilde{t}|}^{1+r+\epsilon} + (1 + \tilde{|\tilde{t}|}^{a-r-\frac{1}{2}+\epsilon}) Q^{-\frac{1}{2}} L^{1-k+\epsilon} \right)$$

### 5.3.2. The case where $|t_\beta| - |\tilde{t}| \leq \log^4 |\tilde{t}|$.

The ratio of gammas is bounded by:

$$\frac{\Gamma \left( \frac{1}{2} + r - \beta \right) \Gamma \left( \beta + \frac{2k}{r} - r + i \tilde{t} \right)}{\Gamma \left( \frac{1}{2} + i \tilde{t} \right)} \ll (1 + \tilde{|\tilde{t}|}^{r-a-\frac{1}{2}+\theta} (1 + |t_\beta|)^{a+r-\frac{1}{2}} e^{-\frac{2}{5} \epsilon} G^{-a+\theta}$$

We further separate this case by the relative sizes of $|t_j|$ and $|\tilde{t}|$:

1. If $|t_j| \geq |\tilde{t}| + 2 \log^4 |\tilde{t}|$, then we can further conclude that the integrand is bounded by:

$$\ll (1 + \tilde{|\tilde{t}|}^{r-a-\frac{1}{2}+\theta} (1 + |t_j|)^{a+r-\frac{1}{2}} e^{-\frac{2}{5} \epsilon} G^{-a+\theta}$$

Executing the $\beta$-integral and then summing over $|t_j|$'s, we have that:

$$G \sum_{|t_j| \geq \tilde{|t|} + 2 \log^4 |\tilde{t}|} L_Q \left( \frac{1}{2}, \overline{u_j} \right) c_{r,j} P_{2} \ll G^{1-a+\theta} (1 + \tilde{|\tilde{t}|}^{a-r-\frac{1}{2}+\epsilon} Q^{-\frac{1}{2}} L^{1-k+\epsilon} e^{-\frac{2}{5} \epsilon} G^{-a+\theta}$$

This is essentially negligible, since the decay in $|\tilde{t}|$ is faster than any polynomial.
If \(|t_j| - |\tilde{t}|| \leq 2 \log^4 |\tilde{t}|\), then the integrand is bounded by:

\[
\ll (1 + |\tilde{t}|)^r e^{-t_{j1}} G^{-a+\theta}
\]

Executing the \(\beta\)-integral and then summing over \(|t_j|\)'s, we have:

\[
G \sum_{|t_j| - |\tilde{t}| \leq 2 \log^4 |\tilde{t}|} L_Q\left(\frac{1}{2}, \frac{1}{2}, |t_j|\right)c_{r,j} P_2 \ll G^{1-a+\theta}\frac{1}{2}(1 + |\tilde{t}|)^2 2^r |1 + |t_j||^2 \epsilon L^{1-k+\epsilon}
\]

Noting that the \(|\tilde{t}|\)-exponent is actually less than \(1 + r + \epsilon\), we conclude that this term is covered by the bound in \(P_1\).

(3) If \(|t_j| \leq |\tilde{t}| - 2 \log^4 |\tilde{t}|\), then the integrand is bounded by:

\[
\ll (1 + |\tilde{t}|)^{r - a \frac{k-1}{2}} (1 + |t_\beta|)^{a+\theta - \frac{1}{2}\frac{r}{2}} e^{-\frac{1}{2} |t_j||\tilde{t}|} G^{-a+\theta}
\]

Executing the \(\beta\)-integral and then summing over \(|t_j|\)'s, we have that:

\[
G \sum_{|t_j| \leq |\tilde{t}| - 2 \log^4 |\tilde{t}|} L_Q\left(\frac{1}{2}, \frac{1}{2}, |t_j|\right)c_{r,j} P_2 \ll G^{1-a+\theta}\frac{1}{2}(1 + |\tilde{t}|)^2 2^r |1 + |t_j||^2 \epsilon L^{1-k+\epsilon}
\]

where \(C\) is some constant. This is negligible, since the decay in \(|\tilde{t}|\) is faster than any polynomial. In total, this term is completely overshadowed by the term in \(P_1\).

5.3.3. The case where \(|t_\beta| \geq |\tilde{t}| + \log^4 |\tilde{t}|\). The ratio of gammas is bounded by:

\[
\frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(1 - \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{k}{2} + i\tilde{t}\right)} \ll (1 + |\tilde{t}|)^{-\frac{k-1}{2}} (1 + |t_\beta|)^{\frac{k-1}{2}} e^{-\pi |t_j||\tilde{t}|} (1 + |t_\beta|)^{a+\theta - \frac{1}{2}\frac{r}{2}} e^{-\frac{1}{2} |t_j||\tilde{t}|}
\]

Executing the \(\beta\)-integral will show that the result will have decay in \(|\tilde{t}|\) that is faster than every polynomial. Hence, this case is also negligible.

Putting the cases together, we realize that:

\[
G \sum_{t_j} L_Q\left(\frac{1}{2}, \frac{1}{2}, |t_j|\right)c_{r,j}
\]

\[
\times \frac{1}{2\pi i} \int_{(a)} \frac{\Gamma\left(\frac{1}{2} + r - \beta\right)\Gamma\left(1 - \frac{k-1}{2} - r + i\tilde{t}\right)}{\Gamma\left(\frac{k}{2} + i\tilde{t}\right)} G^{-\beta + i\tilde{t}} \Gamma(\beta - it_j) d\beta
\]

\[
\ll G^{1-a+\theta} \left(1 + |\tilde{t}|\right)^{1+r+\epsilon} + (1 + |\tilde{t}|)^{a-r-\frac{1}{2}+\epsilon} Q^{-\frac{1}{2}} L^{1-k+\epsilon}
\]

Our goal is to minimize \(G\). In order to do this, we want to increase \(a\), looking at the term \((1 + |\tilde{t}|)^{1+r+\epsilon}\), which has a fixed exponent. However, the other term dominates if \(a\) is too large, with increasingly worse behavior as \(a\) increases. Hence, we set the exponents of the terms to equal each other. This gives \(a = \frac{3}{2} + 2r + \epsilon\) (the \(\epsilon\) is added in to avoid the pole on the line \(\text{Re} \beta = \frac{3}{2} + 2r\)).

Finally, we can proceed to get a lower bound of \(G\) such that the above is \(O(G^{1+\epsilon}Q^{-\frac{1}{2}} L^{1-k+\epsilon})\), such that the contribution in \((1 + |\tilde{t}|)^{-\text{aspect}}\) is at most as much as in \(S_d\). It turns out that \(G \approx (1 + |\tilde{t}|)^{1+r+\epsilon} + (1 + |\tilde{t}|)^{a-r-\frac{1}{2}+\epsilon} Q^{-\frac{1}{2}} L^{1-k+\epsilon}\). The \(G\) required for larger values of \(r\) has a smaller \((1 + |\tilde{t}|)^{-\text{aspect}}\)-exponent. This proves the proposition.
6. Off-diagonal, Continuous Spectrum

Quoting the continuous spectrum from (A.0.2), the object to analyze here is:

\[ S^c_{01} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2) \frac{\bar{z}^2}{\bar{z}^2 + \alpha i} \bar{l_1} \bar{l_2} \bar{t} \]

\[ \times \left( \frac{1}{2\pi i} \right)^4 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} \frac{(4\pi)^k 2s - \frac{1}{2}}{\Gamma(s + k - 1)z^{2\sqrt{s}}} \]

\[ \times V_{\pi}(s, z) \sum_{a} \frac{1}{2\pi i} \int_{(0)} \frac{\zeta(s' + z) \zeta(s' - z) \kappa_{\alpha, Q}(s', -z)}{\zeta^*(1 - 2z)} \]

\[ \times U, E_o \langle \varphi, \frac{1}{2} + z \rangle \frac{\Gamma(s' - s + \frac{1}{2} - \beta) \Gamma(s + w + \beta - s' + \frac{k-1}{2} - 1 + i\bar{t})}{\Gamma(w + k-1 + i\bar{t})} \]

\[ \times v(s' - w) v(w - \frac{1}{2} x^{\beta + \frac{1}{2}}) \frac{1}{l_1} \frac{w^{\beta - \frac{1}{2}}}{l_2} G^{-\beta} \Gamma(\beta) dz \, d\beta \, ds' \, ds \, dw \]  

(6.0.1)

We will move Re\,z slightly to the right to a curve C, which has the property that if z is any complex number between 0 and C, \( \zeta^*(1 - 2z) \neq 0 \).

For the most part, there will be a lot of similarities in how we analyze this 5-fold integral compared to how we analyze the discrete spectrum expression in the last section. In particular, we are going to move Re\,s down to \( s = \frac{1}{2} - \frac{3}{2} - \varepsilon \), hitting simple poles at \( s = \frac{1}{2} \pm z - r \). Thus, our analysis will again split into 2 pieces, the residues and the moved integral.

6.1. The residues at \( s = \frac{1}{2} \pm z - r \). In this section, we will show the following:

Proposition 6.1. For \( G \asymp (1 + |t|)^{\frac{s}{2}} \log^5 Q \),

\[ \sum_{0 \leq r \leq \frac{1}{2}} \left( \text{Res}_{s=\frac{1}{2}+z-r} + \text{Res}_{s=\frac{1}{2}-z-r} \right) S^c_{01} \]

\[ = -\frac{\varphi(Q) G}{2\Gamma(k)} \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2) \frac{\bar{z}^2}{\bar{z}^2 + \alpha i} \bar{l_1} \bar{l_2} \bar{t} (4\pi)^k \langle f, f \rangle \log x \cdot b_{l_1, l_2} + O(x^{-\varepsilon}) \]

\[ + \{6.1.9a\} + \{6.1.13b\} + O(G^{1+\varepsilon} Q^{1+\varepsilon} L^{1+2\alpha+\varepsilon}) + O(G^{1+\varepsilon} Q^{\frac{1}{2}+\varepsilon} L^{2+2\alpha+\varepsilon}) \]  

(6.1.1)

where

\[ b_{l_1, l_2} = \begin{cases} l_1^{-1} & \text{if } l_1 = l_2 \\ (l_1 l_2)^{-1} E_{l_1, l_2}(1) & \text{if } l_1 \neq l_2 \end{cases} \]  

(6.1.2)

Remark. Note the sum of the (log x)-terms above is exactly \(-\frac{1}{2}\) of the (log x)-terms in \( S_d \).
Proof. Taking the residues at \( s = \frac{1}{2} \pm z - r \) for \( 0 \leq r \leq \frac{1}{2} \), we have the following expression:

\[
\left( \text{Res}_{s=\frac{1}{2}+z-r} + \text{Res}_{s=\frac{1}{2}-z-r} \right) S_{\delta}^c = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{1}{2}+\alpha} l_1^{\frac{r}{2}} l_2^{\frac{r}{2}} (4\pi)^k \mathcal{Y}_{N[l_1, l_2]}
\times \left( \frac{1}{2\pi i} \right)^4 \int_{(\gamma_1)} \int_{(\gamma_3)} \int_{(\gamma_4)} \int_{(C)} \sum_a \frac{\zeta(s+z)\zeta(s'-z)\kappa_a Q(s',-z)}{2\sqrt{\pi} \zeta^*(1-2z)} \right.
\times \left( \frac{2^{s-z} c_r(z, \delta)}{\Gamma(k - \frac{1}{2} + z - r)} \frac{\Gamma(s'-z + r - \beta)\Gamma(w - s' + \beta + \frac{k-1}{2} + z - r + it)}{\Gamma(w + \frac{k-1}{2} + it)} \right.
\times \left( \frac{2^{z-r} c_r(z, \delta)}{\Gamma(k - \frac{1}{2} + z - r)} \frac{\Gamma(s'-z + r - \beta)\Gamma(w - s' + \beta + \frac{k-1}{2} + z - r + it)}{\Gamma(w + \frac{k-1}{2} + it)} \right)
\times (U, E_a(s, \frac{1}{2} + z)) v(s' - w) v(w - \frac{1}{2}) x^{s'-z} l_1^{\frac{r}{2}} l_2^{\frac{r}{2}} G^{-\frac{\kappa}{2}} \Gamma(\beta) d\beta d\beta' dw \right)
\]

(6.1.3)

Moving \( \text{Re} s' \) down to \( \frac{1}{2} - \varepsilon \), we pick up simple poles at \( s' = w, s' = 1 \pm z \) and \( s' = \beta \pm z \) when \( r = 0 \). The shifted integral is \( O(x^{-\varepsilon}) \). Hence we have:

\[
\text{(6.1.3)} = R_{s'=w} + R_{s'=1+z} + R_{s'=1-z} + R_{s' = \beta + z} + R_{s' = \beta - z} + O(x^{-\varepsilon}),
\]

(6.1.4)

where the notation \( R_{s'=b} \) denotes the residue at \( s' = b \) in this subsection. It will be shown that the residue at \( s' = w \) is the major contribution, and the remaining residues get completely overshadowed.

6.1.1. Contribution from the pole at \( s' = w \). We have this residue at \( s' = w \):

\[
R_{s'=w} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{1}{2}+\alpha} l_1^{\frac{r}{2}} l_2^{\frac{r}{2}} (4\pi)^k \mathcal{Y}_{N[l_1, l_2]}
\times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma_1)} \int_{(\gamma_3)} \int_{(C)} \sum_a \frac{\zeta(w+z)\zeta(w-z)\kappa_a Q(w,-z)}{2\sqrt{\pi} \zeta^*(1-2z)} \right.
\times \left( \frac{2^{w-r} c_r(w, \delta)}{\Gamma(k - \frac{1}{2} + w - r)} \frac{\Gamma(w + z + r - \beta)\Gamma(\beta + \frac{k-1}{2} + z - r + it)}{\Gamma(w + \frac{k-1}{2} + it)} \right.
\times \left( \frac{2^{z-r} c_r(z, \delta)}{\Gamma(k - \frac{1}{2} + z - r)} \frac{\Gamma(w + z + r - \beta)\Gamma(\beta + \frac{k-1}{2} + z - r + it)}{\Gamma(w + \frac{k-1}{2} + it)} \right)
\times (U, E_a(s, \frac{1}{2} + z)) v(w - \frac{1}{2}) x^{s'-z} l_1^{\frac{r}{2}} l_2^{\frac{r}{2}} G^{-\frac{\kappa}{2}} \Gamma(\beta) d\beta d\beta' dw
\]

(6.1.5)

Then now we are moving \( \text{Re} w \) down to \( \frac{1}{2} - \varepsilon \), during which we pass simple poles at \( w = 1 \pm z, w = \frac{1}{2} \) and \( w = \beta \pm z \) when \( r = 0 \). The shifted integral is \( O(x^{-\varepsilon}) \). We obtain:

\[
R_{s'=w} = R_{s'=w \atop w=1+z} + R_{s'=w \atop w=1-z} + R_{s'=w \atop w=\beta+z \atop r=0} + R_{s'=w \atop w=\beta-z \atop r=0} + O(x^{-\varepsilon})
\]

(6.1.6)
(1) At the pole \( w = 1 + z \), we have residue:

\[
R \left. \frac{s'}{w} \right|_{w=1+z} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1,l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{k+\alpha}{2} + \frac{i}{2} i} (4\pi)^k V_N[l_1,l_2] \\
\times \left( \frac{1}{2\pi i} \right)^2 \int_{(\gamma_l)} \int_{(C)} \sum_a \frac{\zeta(1+2z)\kappa_{a,Q}(1+z,-z)}{2\sqrt{\pi} \zeta^*(1-2z)} \left( U, E_{a}(\frac{1}{2} + z) \right) \\
\times \left( \frac{2^{z-r}c_r(z,\delta)}{\Gamma(k - \frac{1}{2} + z - r)} \frac{\Gamma(1+r-\beta)\Gamma(\beta + \frac{k-1}{2} + z - r + it)}{\Gamma(z + \frac{k-1}{2} + it)} \\
+ \frac{2^{-z-r}c_r(-z,\delta)}{\Gamma(k - \frac{1}{2} - z - r)} \frac{\Gamma(1+2z+r-\beta)\Gamma(\beta + \frac{k-1}{2} - z - r + it)}{\Gamma(z + \frac{k-1}{2} + it)} \right) \\
\times \nu(\frac{1}{2} + z) (xl_1)^{\frac{k+\beta}{2}} G^{-\beta} \Gamma(\beta) dz \, d\beta \quad (6.1.7)
\]

We shift the line of integration of Re \( z \) down to \( -\frac{1}{2} - \varepsilon \), encountering simple poles at \( z = -\frac{1}{2}, z = \frac{1}{2} - \beta - it \) when \( r = \frac{k}{2} \) and \( z = \frac{1}{2} - \frac{k}{2} \) when \( r = 0 \). The shifted integral is \( O(x^{-\varepsilon}) \). We can deduce that:

\[
R \left. \frac{s'}{w} \right|_{w=1+z} = R \left. \frac{s'}{w} \right|_{w=1+z} + R \left. \frac{s'}{w} \right|_{w=1+z} + R \left. \frac{s'}{w} \right|_{w=1+z} + O(x^{-\varepsilon}) \quad (6.1.8)
\]

(a) Now, we look at the residue at \( z = -\frac{1}{2} \), bringing \( \delta \) to 0. Taking the sum \( 0 \leq r \leq \frac{k}{2} \), we obtain the residue sum is:

\[
\sum_{0 \leq r \leq \frac{k}{2}} R \left. \frac{s'}{w} \right|_{w=1+z} = \varphi(Q) G \sum_{l_1,l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{k+\alpha}{2} + \frac{i}{2} i} (4\pi)^k \sum_a \frac{\zeta(0)\kappa_{a,Q}(\frac{1}{2},\frac{1}{2})}{\zeta^*(2)} \left( U, E_{a}(\frac{1}{2},0) \right) \\
\times V_N[l_1,l_2] \frac{1}{2\pi i} \int_{(\gamma_l)} \left( - \frac{\Gamma(1+\frac{k}{2}-\beta)\Gamma(\beta - 1 + it)}{4\Gamma(\frac{k}{2} + 1)\Gamma(\frac{k}{2} - 1)\Gamma(\frac{k}{2} + it)} \\
+ \frac{\Gamma(-\beta)\Gamma(\beta + \frac{k}{2} + it)}{2\Gamma(k)\Gamma(\frac{k}{2} + it)} \right) G^{-\beta} \Gamma(\beta) d\beta
\]
We are splitting this into the two separate terms:

\[
\sum_{0 \leq r \leq \frac{k}{2}} R_{s' = w, z = \frac{1}{2} - \beta - i\delta} = -\varphi(Q) G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1 l_2)^{\frac{k}{2} + \alpha + i l_1 - i l_2} \sum_a \zeta(0)\kappa_a(Q)^{(\frac{1}{2}, \frac{1}{2})} \langle U, E_a(*, 0) \rangle \times \frac{(4\pi)^k V_N[l_1, l_2]}{2\Gamma(k)} \frac{1}{2\pi i} \int_{(\chi)} \frac{\Gamma(1 + \frac{k}{2} - \beta)\Gamma(-1 + i\delta)}{\Gamma(\frac{k}{2} + i\delta)} G^{-\beta} \Gamma(\beta) d\beta (6.1.9a)
\]

\[
+ \varphi(Q) G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1 l_2)^{\frac{k}{2} + \alpha + i l_1 - i l_2} \sum_a \zeta(0)\kappa_a(Q)^{(\frac{1}{2}, \frac{1}{2})} \langle U, E_a(*, 0) \rangle \times \frac{(4\pi)^k V_N[l_1, l_2]}{2\Gamma(k)} \frac{1}{2\pi i} \int_{(\chi)} \frac{\Gamma(-\beta)\Gamma(\beta + \frac{k}{2} + i\delta)}{\Gamma(\frac{k}{2} + i\delta)} G^{-\beta} \Gamma(\beta) d\beta (6.1.9b)
\]

We will analyze the term (6.1.9b) by first moving Re \(\beta\) down to \(\varepsilon\) and then splitting the situation into 3 cases: \(|t_{\beta}| \ll |\tilde{\ell}|\), \(|t_{\beta}| \sim |\tilde{\ell}|\) and \(|t_{\beta}| \gg |\tilde{\ell}|\).

For the case \(|t_{\beta}| \ll |\tilde{\ell}|\), the integrand is bounded by

\[
\ll (1 + |t_{\beta}|)^{-1} e^{-\frac{1}{2} - \zeta(\frac{1}{2})} (1 + |\tilde{\ell}|)^{\varepsilon} G^{-\varepsilon}
\]

Integrating the region of \(|t_{\beta}| \ll |\tilde{\ell}|\) and summing \(l\)'s, we obtain a bound of:

\[
\ll G^{1 - \varepsilon} (1 + |\tilde{\ell}|)^{\varepsilon} QL^{1 + 2\alpha}
\]

For the case \(|t_{\beta}| \sim |\tilde{\ell}|\), the integrand is bounded by

\[
\ll (1 + |\tilde{\ell}|)^{-\frac{1}{2} - \frac{k}{2} - \frac{1}{2}} e^{-\frac{1}{2} |\tilde{\ell}|} G^{-\varepsilon}
\]

We integrate the region of \(|t_{\beta}| \sim |\tilde{\ell}|\) and observe there is exponential decay in \(|\tilde{\ell}|\).

For the final case \(|t_{\beta}| \gg |\tilde{\ell}|\), the integrand is bounded by

\[
\ll (1 + |\tilde{\ell}|)^{\frac{1}{2} - \frac{k}{2} - \varepsilon} e^{\pi |\tilde{\ell}|} (1 + |t_{\beta}|)^{\frac{k}{2} + \varepsilon} e^{-\frac{1}{2} |\tilde{\ell}|} G^{-\varepsilon}
\]

Integrating \(|t_{\beta}|\), we realize there is again exponential decay in \(|\tilde{\ell}|\). In total, (6.1.9b) is \(O(G^{1 + \varepsilon} Q^{1 + \varepsilon} L^{1 + 2\alpha + \varepsilon})\).

(b) Next, we look at the simple pole at \(z = \frac{1}{2} - \beta - i\delta\) when \(r = \frac{k}{2}\):

\[
R_{s' = w, z = \frac{1}{2} - \beta - i\delta} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1 l_2)^{\frac{k}{2} + \alpha + i l_1 - i l_2} (4\pi)^k V_N[l_1, l_2]
\]

\[
\times \frac{1}{2\pi i} \int_{(\chi)} \sum_a \zeta(2 - 2\beta - 2i\delta)\kappa_a(Q)^{(\frac{1}{2}, \frac{1}{2})} (2\beta + 2i\delta)^{-\beta} \Gamma(2\beta + 2i\delta) v(1 - (1 - \beta - i\delta)
\]

\[
\times \langle U, E_a(*, 1 - \beta - i\delta) \rangle \frac{2^{\frac{1}{2} - \beta - i\delta} e^{\frac{1}{2} (\frac{1}{2} - \beta - i\delta, \delta)}}{2\sqrt{\pi} (\frac{k}{2} - \beta - i\delta)} (xt_1)^{-\beta - i\delta} G^{-\beta} \Gamma(\beta) d\beta
\]
We move $\text{Re} \beta$ up to $1 + \varepsilon$, hitting a simple pole at $\beta = 1 - \tilde{t}$ and no growth on $G$. Hence, we have:

$$R = \lim_{z \to \beta - i \tilde{t}} \frac{\phi(z)}{z - \beta}$$

$$= \frac{1}{2\pi i} \int_{\gamma_4} \sum_a \frac{\zeta(\beta) \kappa_{a,q}(\frac{\beta}{2}, \frac{1}{2})}{\zeta^*(2)} \frac{\langle U, E_a(*, 0) \rangle}{4\Gamma(\frac{1}{2} - 1)(\frac{1}{2} + 1)!} G^{-1 + i\tilde{t}} \Gamma(1 - i\tilde{t}) + O(x^{-\varepsilon})$$

This is exponential decay on $\tilde{t}$ and no growth on $G$.

(c) Now we should look at the pole at $z = \frac{\beta}{2} - \frac{1}{2}$ when $r = 0$. The residue equals:

$$R = \lim_{z \to \beta - i \tilde{t}} \frac{\phi(z)}{z - \beta}$$

$$= \frac{1}{2\pi i} \int_{\gamma_4} \sum_a \frac{\zeta(\beta) \kappa_{a,q}(\frac{\beta}{2}, \frac{1}{2})}{\zeta^*(2)} \frac{\langle U, E_a(*, 0) \rangle}{4\Gamma(\frac{1}{2} - 1)(\frac{1}{2} + 1)!} G^{-1 + i\tilde{t}} \Gamma(1 - i\tilde{t}) + O(x^{-\varepsilon})$$

We move $\text{Re} \beta$ down to $-\varepsilon$, picking up a double pole at $\beta = 0$. Since the moved integral is $O(x^{-\varepsilon})$, we compute that

$$R = \lim_{z \to \beta - i \tilde{t}} \frac{\phi(z)}{z - \beta}$$

$$= \frac{1}{2\pi i} \int_{\gamma_4} \sum_a \frac{\zeta(\beta) \kappa_{a,q}(\frac{\beta}{2}, \frac{1}{2})}{\zeta^*(2)} \frac{\langle U, E_a(*, 0) \rangle}{4\Gamma(\frac{1}{2} - 1)(\frac{1}{2} + 1)!} G^{-1 + i\tilde{t}} \Gamma(1 - i\tilde{t}) + O(x^{-\varepsilon})$$

where $c_1$ is as follows:

$$c_1 = \lim_{\delta \to 0} \text{Res}_{\beta=0} \left( \sum_a \frac{\zeta(\beta) \kappa_{a,q}(\frac{\beta}{2}, \frac{1}{2})}{\zeta^*(2)} \frac{\langle U, E_a(*, 0) \rangle}{4\Gamma(\frac{1}{2} - 1)(\frac{1}{2} + 1)!} \left( \log(xl_1) \frac{4\Gamma(k)}{2\Gamma(k)} - \log G \right) + c_1 \right) + O(x^{-\varepsilon})$$

An important property of $c_1$ is that it is $O(L^{1-k+\varepsilon}Q^\varepsilon)$.

Hence for the pole at $w = 1 + z$, putting the cases together into (6.1.8) and using lemma (B.2), we obtain:
Lemma 6.1. For $G > (1 + |t|)\frac{2}{3} \log^5 Q$,

\[
\sum_{0 \leq r < \frac{1}{2}} \text{Res}_{w=1+z} \left( \text{Res}_{s'=w} \left( \left( \text{Res}_{s_{1} = \frac{1}{2} - r} + \text{Res}_{s_{2} = \frac{1}{2} - r} \right) S'_{s_{1} s_{2}} \right) \right)
\]

\[
= -\varphi(Q) G \sum_{l_{1}, l_{2}} \frac{\chi(l_{1}) \chi(l_{2}) (l_{1} l_{2})^{\frac{1}{2} + \alpha_{1} l_{1}^{i} l_{2}^{i}}}{(4\pi)^{k} \Gamma(f, f)} \log \frac{x}{4 \Gamma(k) b_{1}, b_{2}}
\]

\[
+ O(G^{1+\varepsilon} Q^{1+2\alpha+\varepsilon}) + \text{(6.1.9a)} + O(x^{-\varepsilon}), \tag{6.1.11}
\]

where $b_{1}, b_{2}$ is defined as in \text{(6.1.2)}.

(2) Now we deal with the pole $w = 1 - z$. Taking the residue for $w = 1 - z$ from \text{(6.1.2)} and recalling proposition 4.5

\[
R'_{s' = w} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_{1}, l_{2}} \frac{\chi(l_{1}) \chi(l_{2}) (l_{1} l_{2})^{\frac{1}{2} + \alpha_{1} l_{1}^{i} l_{2}^{i}}}{(4\pi)^{k} \Gamma(f, f)} V_{N \{l_{1}, l_{2}\}} \times \left( \frac{1}{2\pi i} \right)^{2} \int_{(\sigma_{4})} \frac{\pi^{\frac{1}{2} - z} \zeta_{0, 0}(1 - z, -z)}{\Gamma(\frac{1}{2} - z)} \frac{(U, E_{0}(s, \frac{1}{2} + z))}{\Gamma(\frac{1}{2} + z)}
\]

\[
\times \left( \frac{2\pi r}{(2\pi)^{2}} \frac{\Gamma(1 - 2z + r - \beta)}{\Gamma(\beta + \frac{k}{2} - z + r + it)} \frac{\Gamma(1 + \tilde{r} - \beta)}{\Gamma(\beta + \frac{k}{2} - z + \tilde{r} + it)} \right)
\]

\[
\times \frac{\Gamma(k - \frac{1}{2} - z - r)}{\Gamma(k - \frac{1}{2} - z + \tilde{r})} \right)_{z = \frac{1}{2}}
\]

\[
\times \pi^{\frac{1}{2} - z} G^{\beta} \Gamma(\beta) dz \, d\beta
\]

Moving Re $z$ up to $\frac{1}{2} + \varepsilon$, we encounter simple poles at $z = \frac{1}{2}, z = \beta - \frac{1}{2} + \tilde{i} t$ when $r = \frac{k}{2}$ and $z = \frac{1}{2} - \beta - \tilde{i} t$ when $r = 0$. The shifted integral is $O(x^{-\varepsilon})$. So we have:

\[
R'_{s' = w} = R'_{s = w} + R'_{s = w} + R'_{s' = w} + O(x^{-\varepsilon}) \tag{6.1.12}
\]

(a) We start with the residue at $z = \frac{1}{2}$. Summing over residues for $0 \leq r \leq \frac{1}{2}$, we get:

\[
\sum_{0 \leq r < \frac{1}{2}} R'_{s' = w} \left( \frac{1}{2\pi i} \right)^{2} \int_{(\sigma_{4})} \frac{\pi^{\frac{1}{2} - z} \zeta_{0, 0}(1 - z, -z)}{\Gamma(\frac{1}{2} - z)} \frac{(U, E_{0}(s, \frac{1}{2} + z))}{\Gamma(\frac{1}{2} + z)}
\]

\[
\times \left( \frac{2\pi r}{(2\pi)^{2}} \frac{\Gamma(1 - 2z + r - \beta)}{\Gamma(\beta + \frac{k}{2} - z + r + it)} \frac{\Gamma(1 + \tilde{r} - \beta)}{\Gamma(\beta + \frac{k}{2} - z + \tilde{r} + it)} \right)
\]

\[
\times \frac{\Gamma(k - \frac{1}{2} - z - r)}{\Gamma(k - \frac{1}{2} - z + \tilde{r})} \right)_{z = \frac{1}{2}}
\]

\[
\times \pi^{\frac{1}{2} - z} G^{\beta} \Gamma(\beta) dz \, d\beta
\]

\[
- \varphi(Q) G \sum_{l_{1}, l_{2}} \frac{\chi(l_{1}) \chi(l_{2}) (l_{1} l_{2})^{\frac{1}{2} + \alpha_{1} l_{1}^{i} l_{2}^{i}}}{(4\pi)^{k} \Gamma(f, f)} \times \left( \frac{1}{2\pi i} \right)^{2} \int_{(\sigma_{4})} \frac{\pi^{\frac{1}{2} - z} \zeta_{0, 0}(1 - z, -z)}{\Gamma(\frac{1}{2} - z)} \frac{(U, E_{0}(s, \frac{1}{2} + z))}{\Gamma(\frac{1}{2} + z)}
\]

\[
\times \left( \frac{2\pi r}{(2\pi)^{2}} \frac{\Gamma(1 - 2z + r - \beta)}{\Gamma(\beta + \frac{k}{2} - z + r + it)} \frac{\Gamma(1 + \tilde{r} - \beta)}{\Gamma(\beta + \frac{k}{2} - z + \tilde{r} + it)} \right)
\]

\[
\times \frac{\Gamma(k - \frac{1}{2} - z - r)}{\Gamma(k - \frac{1}{2} - z + \tilde{r})} \right)_{z = \frac{1}{2}}
\]

\[
\times \pi^{\frac{1}{2} - z} G^{\beta} \Gamma(\beta) dz \, d\beta
\]

\[
- \varphi(Q) G \sum_{l_{1}, l_{2}} \frac{\chi(l_{1}) \chi(l_{2}) (l_{1} l_{2})^{\frac{1}{2} + \alpha_{1} l_{1}^{i} l_{2}^{i}}}{(4\pi)^{k} \Gamma(f, f)} \times \left( \frac{1}{2\pi i} \right)^{2} \int_{(\sigma_{4})} \frac{\pi^{\frac{1}{2} - z} \zeta_{0, 0}(1 - z, -z)}{\Gamma(\frac{1}{2} - z)} \frac{(U, E_{0}(s, \frac{1}{2} + z))}{\Gamma(\frac{1}{2} + z)}
\]

\[
\times \left( \frac{2\pi r}{(2\pi)^{2}} \frac{\Gamma(1 - 2z + r - \beta)}{\Gamma(\beta + \frac{k}{2} - z + r + it)} \frac{\Gamma(1 + \tilde{r} - \beta)}{\Gamma(\beta + \frac{k}{2} - z + \tilde{r} + it)} \right)
\]

\[
\times \frac{\Gamma(k - \frac{1}{2} - z - r)}{\Gamma(k - \frac{1}{2} - z + \tilde{r})} \right)_{z = \frac{1}{2}}
\]

\[
\times \pi^{\frac{1}{2} - z} G^{\beta} \Gamma(\beta) dz \, d\beta
\]

\[
(6.1.13a)
\]

\[
+ \varphi(Q) G \sum_{l_{1}, l_{2}} \frac{\chi(l_{1}) \chi(l_{2}) (l_{1} l_{2})^{\frac{1}{2} + \alpha_{1} l_{1}^{i} l_{2}^{i}}}{(4\pi)^{k} \Gamma(f, f)} \times \left( \frac{1}{2\pi i} \right)^{2} \int_{(\sigma_{4})} \frac{\pi^{\frac{1}{2} - z} \zeta_{0, 0}(1 - z, -z)}{\Gamma(\frac{1}{2} - z)} \frac{(U, E_{0}(s, \frac{1}{2} + z))}{\Gamma(\frac{1}{2} + z)}
\]

\[
\times \left( \frac{2\pi r}{(2\pi)^{2}} \frac{\Gamma(1 - 2z + r - \beta)}{\Gamma(\beta + \frac{k}{2} - z + r + it)} \frac{\Gamma(1 + \tilde{r} - \beta)}{\Gamma(\beta + \frac{k}{2} - z + \tilde{r} + it)} \right)
\]

\[
\times \frac{\Gamma(k - \frac{1}{2} - z - r)}{\Gamma(k - \frac{1}{2} - z + \tilde{r})} \right)_{z = \frac{1}{2}}
\]

\[
\times \pi^{\frac{1}{2} - z} G^{\beta} \Gamma(\beta) dz \, d\beta
\]

\[
(6.1.13b)
\]
The analysis of (6.1.13a) is similar to that of (6.1.9b) and actually yields the same bound $O(G^{1+\varepsilon}Q^{1+\varepsilon}L^{1+2\alpha+\varepsilon})$.

(b) The residue at $z = \beta - \frac{1}{2} + i\tilde{t}$ when $r = \frac{\varepsilon}{2}$ is similar to the analysis of the residue at $w = 1 + z$ and $z = \frac{1}{2} - \beta - i\tilde{t}$. It is also exponential decay in $\tilde{t}$ and constant growth on $G$.

(c) Now we examine the residue at $z = \frac{1}{2} - \frac{\alpha}{2}$ when $r = 0$:

$$R_{z'\equiv w, z=\frac{1}{2} - \frac{\alpha}{2}} = \lim_{\delta \to 0} \varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1l_2)^{\frac{1}{2} + \alpha} t_1^{-i\tilde{t}}(4\pi)^k V_{N[l_1, l_2]}$$

$$\times \frac{1}{2\pi i} \int_{(\gamma_1)} \pi^\delta \mathcal{K}_0, Q \left(\frac{1}{2} + \beta, \frac{1}{2} + \beta, \frac{1}{2}, \frac{1}{2}\right) \langle U, E_0(*, 1 - \beta) \rangle v \left(\frac{\beta}{2}\right)$$

$$\times \frac{2^{\frac{1}{2} - \frac{\varepsilon}{2}}}{4\sqrt{\pi} \Gamma(k - \frac{\varepsilon}{2})} c_0 \left(\frac{1}{2} - \frac{\beta}{2}, \delta \right) \langle x^2 \rangle G^{-\beta} \Gamma(\beta) d\beta$$

We move $\text{Re} \beta$ down to $-\varepsilon$, picking up a double pole at $\beta = 0$. Since the moved integral is $O(x^{-\varepsilon})$, we have:

$$R_{z'\equiv w, z=\frac{1}{2} - \frac{\alpha}{2}} = \varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1l_2)^{\frac{1}{2} + \alpha} t_1^{-i\tilde{t}}(4\pi)^k \langle f, f \rangle b_{l_1, l_2} \left(-\frac{\log(xl_1)}{4\Gamma(k)} + \frac{\log G}{2\Gamma(k)}\right)$$

$$+ \varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1l_2)^{\frac{1}{2} + \alpha} t_1^{-i\tilde{t}} c_2 + O(x^{-\varepsilon})$$

where $c_2$ is defined as follows:

$$c_2 = \lim_{\delta \to 0} \text{Res}_{\beta=0} \left(4\pi)^k V_{N[l_1, l_2]} \pi^\delta \mathcal{K}_0, Q \left(\frac{1}{2} + \beta, \frac{1}{2} + \beta, \frac{1}{2}, \frac{1}{2}\right) \langle U, E_0(*, 1 - \beta) \rangle v \left(\frac{\beta}{2}\right)$$

$$\times \frac{2^{\frac{1}{2} - \frac{\varepsilon}{2}}}{4\sqrt{\pi} \Gamma(k - \frac{\varepsilon}{2})} c_0 \left(\frac{1}{2} - \frac{\beta}{2}, \delta \right) \Gamma(\beta) \right) \right) \quad (6.1.14)$$

$c_2$ has the property that it is $O(L^{1-k+\varepsilon}Q^\varepsilon) \times \begin{cases} L^{-1} & \text{if } l_1 = l_2 \\ L^{-2} & \text{if } l_1 \neq l_2. \end{cases}$

To summarize:

**Lemma 6.2.** For $G \sim (1 + |t|)^{\frac{1}{2}}$, $\frac{1}{\sqrt{\pi}} \log^3 Q$,

$$\sum_{0 \leq \varepsilon \leq \frac{1}{2}} \text{Res}_{w=1-z} \left(\text{Res}_{s'=w} \left((\text{Res}_{s=\frac{1}{2}+z-r} + \text{Res}_{s=\frac{1}{2}-z-r}) S_{\alpha_1}^e\right)\right)$$

$$= -\varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1l_2)^{\frac{1}{2} + \alpha} t_1^{-i\tilde{t}}(4\pi)^k \langle f, f \rangle \frac{\log x}{4\Gamma(k)} b_{l_1, l_2}$$

$$+ (6.1.13b) + O(G^{1+\varepsilon}Q^{1+\varepsilon}L^{1+2\alpha+\varepsilon}) + O(x^{-\varepsilon}) \quad (6.1.15)$$
(3) Now we have to deal with the pole at \( w = \frac{1}{2} \). From the expression (5.1.5), we take residue at \( w = \frac{1}{2} \):

\[
R'_{\sigma' = w} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{1}{2} + \sigma} i^{l_1 - i l_2} (4\pi)^k \mathcal{V}_{N|l_1, l_2} \\
\times \left( \frac{1}{2\pi i} \right)^2 \int_{(\gamma_1)} \int_{(C)} \sum_a \zeta(\frac{1}{2} + z) \zeta(\frac{1}{2} - z) \kappa_{a,Q}(\frac{1}{2} - z) \zeta^*(1 - 2z) (U, E_a(*, \frac{1}{2} + z)) \\
\times \left( \frac{2^{z - r} c_r(z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} + z - r)} \Gamma(\frac{1}{2} - z + r + \beta) \Gamma(\beta + \frac{k - 1}{2} + z - r + i\delta) \right) \\
\times \left( \frac{2^{z - r} c_r(-z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} - z - r)} \Gamma(\frac{1}{2} + z + r - \beta) \Gamma(\beta + \frac{k - 1}{2} - z - r + i\delta) \right) \\
\times G^{-\beta} \Gamma(\beta) d\beta \\
(6.1.16)
\]

Similar to the previous section, we can prove the following proposition as in lemma [5.1] with slight modifications:

**Lemma 6.3.** For \( G \approx (1 + |t|)^{-2\pi + r} \log^5 Q \) and \( \Re \beta = a = \frac{3}{2} + 2r + \epsilon \),

\[
\lim_{\delta \to 0} G \left( \frac{1}{2\pi i} \right)^2 \int_{(a)} \int_{(C)} \sum_a \zeta(\frac{1}{2} + z) \zeta(\frac{1}{2} - z) \kappa_{a,Q}(\frac{1}{2} - z) \zeta^*(1 - 2z) (U, E_a(*, \frac{1}{2} + z)) \\
\times \left( \frac{2^{z - r} c_r(z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} + z - r)} \Gamma(\frac{1}{2} - z + r - \beta) \Gamma(\beta + \frac{k - 1}{2} + z - r + i\delta) \right) \\
\times \left( \frac{2^{z - r} c_r(-z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} - z - r)} \Gamma(\frac{1}{2} + z + r - \beta) \Gamma(\beta + \frac{k - 1}{2} - z - r + i\delta) \right) \\
\times G^{-\beta} \Gamma(\beta) \mathcal{V}_{N|l_1, l_2} d\beta \\
\ll G^{1 + \epsilon} L^{1 - k + \epsilon} Q^{-\frac{1}{2} + \epsilon} \times \begin{cases} 
L^{-\frac{3}{2}} & \text{if } l_1 = l_2 \\
L^{-1} & \text{if } l_1 \neq l_2
\end{cases} \\
(6.1.17)
\]

By the same arguments that we made after proving lemma [5.1], we can conclude that the double integral (6.1.16) is \( O(G^{1 + \epsilon} Q^{\frac{3}{2} + \epsilon} L^{2 + 2a + \epsilon}) \).

(4) Now we have to look at the residue at \( w = \beta + z \) when \( r = 0 \).

\[
R_{\sigma' = w} \bigg|_{w = \beta + z} \\
= \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{1}{2} + \alpha} i^{l_1 - i l_2} (4\pi)^k \mathcal{V}_{N|l_1, l_2} \\
\times \left( \frac{1}{2\pi i} \right)^2 \int_{(\gamma_1)} \int_{(C)} \sum_a \zeta(\beta + 2z) \zeta(\beta + z, -z) \kappa_{a,Q}(\beta + z, -z) \zeta^*(1 - 2z) (U, E_a(*, \frac{1}{2} + z)) \\
\times \left( \frac{2^{z - r} c_r(z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} + z)} \right) v(\beta + z - \frac{1}{2})(xl_1)^{\beta + z - \frac{1}{2}} G^{-\beta} \Gamma(\beta) d\beta
\]
We move \(\text{Re} \beta\) down to \(\frac{1}{2} - \varepsilon\). The only pole we encounter this way is \(\beta = \frac{1}{2} - z\). As the moved integral is \(O(x^{-\varepsilon})\), we deduce that:

\[
R_{s' = w}\quad \lim_{\delta \to 0} \varphi(Q) G_{\delta} \sum_{l_1,l_2} \chi(l_1) \chi(l_2)(l_1l_2)^{\frac{k}{2} + \alpha} \tilde{l}_1 \tilde{l}_2 (4\pi)^k v_{N[l_1,l_2]} \\
\times \frac{1}{2\pi i} \int_{(C)} \sum_{a} \frac{\zeta(z)}{\zeta(1-2z)} \frac{\zeta(z)}{\zeta(1-2z)} \frac{\kappa_a(Q)}{(U,E_a(*, \frac{1}{2} + z))} \\
\times \frac{2z e_0(z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} + z)} G^{-\frac{1}{2} - z} \Gamma(\frac{1}{2} - z) \, dz + O(x^{-\varepsilon})
\]

This is \(O(Q^{\frac{1}{2} + \varepsilon} L^{2 + 2\alpha + \varepsilon})\).

(5) Last but not least, to finish the investigation of the residue at \(s' = w\), we need to check out the residue at \(w = \beta - z\) when \(r = 0\). This is completely similar to the residue at \(w = \beta + z\) when \(r = 0\) and results in the same bounds.

Putting every item together proves part of proposition (6.1.1). Now we just need to show the other parts are contained in the big-O terms.

### 6.1.2. Contribution from the poles at \(s' = 1 \pm z\)

From (6.1.3), we have the following residue at \(s' = 1 + z\):

\[
R_{s' = 1 + z} = \lim_{\delta \to 0} \varphi(Q) G_{\delta} \sum_{l_1,l_2} \chi(l_1) \chi(l_2)(l_1l_2)^{\frac{k}{2} + \alpha} \tilde{l}_1 \tilde{l}_2 (4\pi)^k v_{N[l_1,l_2]} \\
\times \left( \frac{1}{2\pi i} \int_{(C)} \right)^3 \left( \frac{1}{2\pi i} \sum_{\alpha} \frac{\zeta(1+2z) \kappa_a(Q)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} + z)} \right) \\
\times \left( \frac{2^{z-r} e_0(z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} + z)} \right) \\
\times \left( \frac{\Gamma(1+r-\beta) \Gamma(w + \beta + \frac{k-1}{2} - r - 1 + it)}{\Gamma(w + \frac{k-1}{2} + it)} \right) \\
\times \left( \frac{2^{z-r} e_0(-z, \delta)}{2\sqrt{\pi} \Gamma(k - \frac{1}{2} - z - r)} \right) \\
\times (1 + 2z - r) \Gamma(1+2z-w) u(w - \frac{1}{2}) G^{-\beta} \Gamma(\beta) \, dz \, dw
\]

We will move \(\text{Re} z\) down to \(-\frac{1}{2} - \varepsilon\). We are hitting a simple pole at \(z = \frac{\beta}{2} - \frac{1}{2}\) when \(r = 0\). As usual, the shifted integral is \(O(x^{-\varepsilon})\). We have:

\[
\sum_{0 \leq r \leq \frac{1}{2}} R_{s' = 1 + z} = R_{s' = 1 + z}^{z = \frac{\beta}{2} - \frac{1}{2}} + O(x^{-\varepsilon})
\]
When $r = 0$, the residue at $z = \frac{\beta}{2} - \frac{1}{2}$ is:

\[
R_{s' = 1 + z}^{\delta \rightarrow 0} = \lim_{\delta \rightarrow 0} \varphi(Q)G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{\beta}{2} - \frac{1}{2} + \frac{\alpha}{2}} \Gamma(\zeta)(4\pi)^k \mathcal{V}_{N[l_1, l_2]} \times \left( \frac{1}{2\pi i} \right)^2 \int_{(\gamma_1)} \int_{(\gamma_2)} \frac{\zeta(\beta) \kappa_a Q(\frac{\beta}{2} + \frac{1}{2}, -\frac{\beta}{2} + \frac{3}{2})}{\zeta^*(2 - \beta)} (U, E_a(*, \frac{1}{2}))(l_1 l_2)^{\beta - w} G^{-\delta} \Gamma(\beta) d\beta dw
\]

\[
\times \frac{2^{\frac{\beta}{2} + \frac{1}{2} - \delta} c_0(\frac{\beta}{2} + \frac{3}{2}, \delta)}{\Gamma(k - \frac{\beta}{2} + 4\sqrt{\pi})} x^{\frac{\beta}{2} + \frac{1}{2} - w} v(1 - w) v(w - \frac{1}{2}) + O(x^{-\varepsilon})
\]

Now we move Re $\beta$ down to $-\varepsilon$, picking up a residue at $\beta = 0$. The moved integral is $O(x^{-\varepsilon})$. The residue is:

\[
R_{s' = 1 + z}^{\delta \rightarrow 0} = \varphi(Q)G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{\beta}{2} - \frac{1}{2} + \frac{\alpha}{2}} \Gamma(\zeta)(4\pi)^k \mathcal{V}_{N[l_1, l_2]} \times \left( \frac{1}{2\pi i} \right) \int_{(\gamma_1)} \int_{(\gamma_2)} \frac{\zeta(0) \kappa_a Q(\frac{1}{2}, \frac{1}{2})}{\zeta^*(2)} (U, E_a(*, 0))(l_1 l_2)^{\beta - w} G^{-\delta} \Gamma(\beta) d\beta dw + O(x^{-\varepsilon})
\]

This is $O(GQL^{1 + 2\alpha + \varepsilon})$. For the pole at $s' = 1 - z$, similar calculations also give the same bound for that residue.

6.1.3. The contribution from the poles at $s' = \beta \pm z$. From (6.1.3), we have the following residue at $s' = \beta + z$ when $r = 0$:

\[
R_{s' = \beta + z}^{\delta \rightarrow 0} = \lim_{\delta \rightarrow 0} \varphi(Q)G \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{\beta}{2} + \frac{\alpha}{2}} \Gamma(\zeta)(4\pi)^k \mathcal{V}_{N[l_1, l_2]} \times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \frac{\zeta(\beta + 2z) \zeta(\beta + z, -z)}{2^{\frac{\beta}{2} - \frac{1}{2} + \alpha} \Gamma(\zeta)(1 - 2z)} (U, E_a(*, -\frac{1}{2} + z)) \times \frac{2^{\frac{\beta}{2} + \frac{1}{2} - \delta} c_0(\frac{\beta}{2} + \frac{3}{2}, \delta)}{\Gamma(k - \frac{\beta}{2} + 4\sqrt{\pi})} x^{\beta + z} v(1 - w) v(w - \frac{1}{2}) x^{\beta + z - \frac{1}{2} l_2^2 - w} G^{-\delta} \Gamma(\beta) dz d\beta dw
\]

We move Re $\beta$ down to $\frac{1}{2} - \varepsilon$. During this process, we do not encounter any poles, and hence we can estimate this as $O(x^{-\varepsilon})$. A similar calculation gives the same bound for the residue at the pole $s' = \beta - z$.

Putting the results together, we see $R_{s' = w}$ dominates and the proposition is proved.

6.2. The moved integral at $Re s = \frac{1}{2} - \frac{\beta}{2} - \varepsilon$. The way that we deal with the moved integral will look similar to previous sections. However, there will also be subtle differences when we move $z$ around, picking up poles from the $M$-function.
Proposition 6.2. For $G \asymp (1 + |t|)^{-\frac{3}{2}} \log^5 Q$, the moved integral has the following upper bound:

$$S_{\alpha, \text{Re} s = \frac{1}{2} - \frac{k}{2} - \varepsilon}^c = -\left(0.139a - 0.113b\right) + O(G^{1+\varepsilon}Q^{\frac{4}{2}+\varepsilon}L^{1+2\alpha+\varepsilon}) + O(x^{-\varepsilon}).$$

Proof. We will reiterate the moved integral here for convenience:

$$S_{\alpha, \text{Re} s = \frac{1}{2} - \frac{k}{2} - \varepsilon}^c = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1l_2)^{\frac{k}{2} + \alpha} \int\frac{dx}{l_1 l_2}$$

$$\times \left(\frac{1}{2\pi i}\right)^4 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} \left(\frac{4\pi i}{\Gamma(s + k - 1)}\right)^{2s - \frac{d}{2}} M(s, \frac{z}{\alpha}, \delta) d\gamma$$

$$\times \sum_{a} \left(\frac{\zeta(s') + \zeta(s' - z)}{\zeta(1 - 2z)}\right)(s' + w + \beta - s' + \frac{k}{2} - 1 + i\delta)$$

$$\times v(s' - w)v(w - \frac{1}{2}) \Gamma(s' - s + 1 - \beta) \Gamma(s + w + \beta - s' + \frac{k}{2} - 1 + i\delta)$$

$$\times x^{s' - \frac{1}{2}} y^{s' - w - \frac{1}{2}} G^{-\beta} \Gamma(\beta) d\beta ds' ds dw$$

where $\gamma_1 = \text{Re} w = 1 + 2\varepsilon$, $\gamma_2 = \text{Re} s = \frac{1}{2} - \frac{k}{2} - \varepsilon$, $\gamma_3 = \text{Re} s' = \frac{5}{4}$, $\gamma_4 = \text{Re} \beta = \frac{3}{4}$ and $C$ is the curve described before.

The start of our analysis involved moving $\text{Re} s'$ down to $\frac{1}{2} - \frac{k}{2}$, encountering simple poles at $s' = w$ and $s' = 1 + z$. The moved integral is $O(x^{-\varepsilon})$. We have:

$$S_{\alpha, \text{Re} s = \frac{1}{2} - \frac{k}{2} - \varepsilon}^c = U_{s' = w} + U_{s' = 1 + z} + U_{s' = 1 - z} + O(x^{-\varepsilon}), \quad (6.2.1)$$

where $U_{s' = b}$ denotes the residue at $s' = b$ for this subsection.

6.2.1. The contribution from the pole at $s' = w$. Taking residue at $s' = w$, we have the following expression:

$$U_{s' = w} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1l_2)^{\frac{k}{2} + \alpha} \int\frac{dx}{l_1 l_2}$$

$$\times \left(\frac{1}{2\pi i}\right)^4 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} \int_{(\gamma_4)} \left(\frac{4\pi i}{\Gamma(s + k - 1)}\right)^{2s - \frac{d}{2}} M(s, \frac{z}{\alpha}, \delta) d\gamma$$

$$\times \sum_{a} \left(\frac{\zeta(w + z)}{\zeta(1 - 2z)}\right)(w + \beta + \frac{k}{2} - 1 + i\delta)$$

$$\times \Gamma(w - s + \frac{1}{2} - \beta) \Gamma(s + w + \beta - s' + \frac{k}{2} - 1 + i\delta)$$

$$\times v(w - \frac{1}{2}) \Gamma(s' - s + 1 - \beta) \Gamma(s + w + \beta - s' + \frac{k}{2} - 1 + i\delta)$$

$$\times x^{s' - \frac{1}{2}} y^{s' - w - \frac{1}{2}} G^{-\beta} \Gamma(\beta) d\beta ds' ds dw$$

We will move $\text{Re} w$ down to $\frac{1}{2} - \varepsilon$, picking up simple poles at $w = 1 + z$ and $w = \frac{1}{2}$. The shifted integral is $O(x^{-\varepsilon})$. Then we have:

$$U_{s' = w} = U_{s' = w}^{s' = w} + U_{s' = w}^{s' = 1 + z} + U_{s' = w}^{s' = 1 - z} + O(x^{-\varepsilon}) \quad (6.2.2)$$
(1) At \( w = 1 + z \):

\[
U_{s' = w, \, w = 1 + z} = \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{s}{2} + \alpha} \Gamma(l_1 - i) \left( \frac{1}{2 \pi i} \right)^3 \int_{(\gamma_2)} \int_{(\gamma_4)} \frac{(4\pi)^{k} s^{-\frac{k}{2}}}{\Gamma(s + k - 1)}
\]

\[
\times \mathcal{V}_{N[l_1, l_2]} \sum_{a} \frac{\zeta(1 + 2z) \kappa_a, Q(1 + z, -z)}{2\sqrt{\pi} \zeta(1 - 2z)} M(s, \frac{s}{2}, \delta)(U, E_{a}(*, \frac{1}{2} + z)) v(\frac{1}{2} + z)
\]

\[
\times \frac{\Gamma(\frac{s}{2} + z - s - \beta) \Gamma(s + \beta + \frac{k}{2} - 1 + \iota t)}{\Gamma(z + \frac{k+1}{2} + \iota t)} (x l_1)^{\frac{s}{2} + \frac{k}{2} + \beta} \Gamma(\beta) dz \, d\beta \, ds
\]

The difference comes here when we move \( \text{Re} \, z \) down to \(-\frac{1}{2} - \varepsilon\). We only hit a simple pole at \( z = s - \frac{1}{2} + \frac{k}{2} \) from the \( M \)-function. There is no pole at \( z = -\frac{1}{2} \) since the \( M \)-function evaluated at \( z = -\frac{1}{2} \) is zero. The shifted integral is \( O(x^{-\varepsilon}) \). We can then conclude that:

\[
U_{s' = w, \, w = 1 + z} = U_{s' = w, \, w = 1 + z} + O(x^{-\varepsilon}) \quad (6.2.3)
\]

The residue at \( z = s - \frac{1}{2} + \frac{k}{2} \) is:

\[
U_{s' = w, \, w = 1 + z} = \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{s}{2} + \alpha} \Gamma(l_1 - i) \left( \frac{1}{2 \pi i} \right)^3 \int_{(\gamma_2)} \int_{(\gamma_4)} \frac{(4\pi)^{k} s^{-\frac{k}{2}}}{\Gamma(s + k - 1)}
\]

\[
\times \mathcal{V}_{N[l_1, l_2]} \sum_{a} \frac{\zeta(1 + 2z) \kappa_a, Q(1 + z, -z)}{2\sqrt{\pi} \zeta(1 - 2z)} M(s, \frac{s}{2}, \delta)(U, E_{a}(*, \frac{1}{2} + z)) v(\frac{1}{2} + z)
\]

\[
\times \frac{\Gamma(\frac{s}{2} + z - s - \beta) \Gamma(s + \beta + \frac{k}{2} - 1 + \iota t)}{\Gamma(z + \frac{k+1}{2} + \iota t)} (x l_1)^{\frac{s}{2} + \frac{k}{2} + \beta} \Gamma(\beta) dz \, d\beta \, ds
\]

Now we will move \( \text{Re} \, s \) further down to \(-\frac{k}{2} - \varepsilon\), picking up simple poles at \( s = -\frac{k}{2} \) and \( s = 1 - \frac{k}{2} - \beta - \iota t \). The shifted integral is \( O(x^{-\varepsilon}) \). We get that:

\[
U_{s' = w, \, w = 1 + z} = U_{s' = w, \, w = 1 + z} + U_{s' = w, \, w = 1 + z} + O(x^{-\varepsilon}) \quad (6.2.4)
\]
(a) We start with the residue at \( s = -\frac{b}{2} \):

\[
U_{s' = w, w = 1 + z}^{z = s = -\frac{b}{2}} = \varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1 l_2)^{\frac{1}{2} + \alpha l_1 l_2} \sum_{a} \frac{\zeta(0)\kappa_a Q}{\zeta^{*}(2)}(U, E_a(*, 0))
\]

\[
\times \frac{1}{2\pi i} \int_{(\gamma_4)} \frac{(4\pi)^k \nu_N[l_1, l_2]}{4\Gamma(\frac{\beta}{2} - \beta - it)} \frac{\Gamma(1 + \frac{\beta}{2} - \beta - 2it\tilde{t})\Gamma(\frac{\beta}{2} + \beta + it)}{\Gamma(\frac{\beta}{2} + it)} G^{-\beta} \Gamma(\beta) d\beta
\]

This is exactly \((6.1.1.3b)\).

(b) We now investigate the other pole at \( s = 1 - \frac{b}{2} - \beta - it \):

\[
U_{s' = w, w = 1 + z}^{z = s = 1 - \frac{b}{2} - \beta - it} = \varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1 l_2)^{\frac{1}{2} + \alpha l_1 l_2} \tilde{t}^{\alpha l_1 l_2}
\]

\[
\times \frac{1}{2\pi i} \int_{(\gamma_4)} \frac{(4\pi)^k \nu_N[l_1, l_2]}{4\Gamma(\frac{\beta}{2} - \beta - it)} \frac{\Gamma(1 + \frac{\beta}{2} - 2\beta - 2it\tilde{t})\Gamma(\frac{\beta}{2} + \beta + it)}{\Gamma(\frac{\beta}{2} + it)} \frac{\zeta(0)\kappa_a Q}{\zeta^{*}(2)}(U, E_a(*, 1 - \beta - it))
\]

\[
\times v(1 - \beta - it)(x|l_1) \Gamma(1 - \beta - it) G^{-\beta} \Gamma(\beta) d\beta
\]

We move \( \Re \beta \) up to \( 1 + \epsilon \), hitting the pole at \( \beta = 1 - \tilde{t} \). The moved integral is \( O(x^{-\epsilon}) \). The residue at \( \beta = 1 - \tilde{t} \) is:

\[
\varphi(Q)G \sum_{l_1, l_2} \chi(l_1)\chi(l_2)(l_1 l_2)^{\frac{1}{2} + \alpha l_1 l_2} \tilde{t}^{\alpha l_1 l_2} \frac{(4\pi)^k \nu_N[l_1, l_2]}{4\Gamma(\frac{\beta}{2} - 1)(\frac{\beta}{2} + 1)!}
\]

\[
\times \frac{\zeta(0)\kappa_a Q}{\zeta^{*}(2)}(U, E_a(*, 0)) G^{-1 + \tilde{t}^2} \Gamma(1 - \tilde{t})
\]

Note that this has exponential decay in \( |\tilde{t}| \) multiplied by \( Q^{1 + 2\epsilon} L^{1 + 2\alpha + \epsilon} \).

(2) A similar situation occurs in the residue at \( w = 1 - z \), the contribution being \((6.1.13b)\) and a term which has exponential decay in \( \tilde{t} \) multiplied by \( Q^{1 + \epsilon} L^{1 + 2\alpha + \epsilon} \).
(3) The remaining case here is \( w = \frac{1}{2} \):

\[
U_{s' = w} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{1}{2} + \alpha} l_1^i l_2^i \mathcal{V}_{N[l_1,l_2]} \\
\times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma'_2)} \int_{(\gamma_4)} \int_{(C)} \frac{(4\pi)^k 2^{s - \frac{1}{2}} \Gamma(1 - s - \beta) \Gamma(s + \beta + \frac{1}{2} - 1 + i\tilde{t})}{\Gamma(s + k - 1)} \Gamma(s + k - 1) 2\sqrt{\pi} \right.
\]
\[
\times \sum_a \frac{\zeta(\frac{1}{2} + z) \zeta(\frac{1}{2} - z) \kappa_a Q(\frac{1}{2} - z)}{2\sqrt{\pi} \zeta^* (1 - 2z)} M(s, \tilde{\gamma}, \delta)(U, E_a(*, \frac{1}{2} + z)) \\
\times G^{-\beta} \Gamma(\beta) \, dz \, d\beta \, ds
\]

\[ (6.2.5) \]

Similar to the previous section, we can prove the following proposition as in lemma \[5.1\] with slight modifications:

**Lemma 6.4.** For \( G \times (1 + |t|)^{\frac{2}{2} - 2\pi} \log^b Q \) and \( \text{Re} \beta = a = \frac{5}{2} + k + 2\varepsilon \),

\[
\lim_{\delta \to 0} G \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma'_2)} \int_{(\gamma_4)} \int_{(C)} \frac{(4\pi)^k V_{N[l_1,l_2]} 2^{s - \frac{1}{2}}}{\Gamma(s + k - 1) 2\sqrt{\pi}} \right.
\]
\[
\times \sum_a \frac{\zeta(\frac{1}{2} + z) \zeta(\frac{1}{2} - z) \kappa_a Q(\frac{1}{2} - z)}{2\sqrt{\pi} \zeta^* (1 - 2z)} M(s, \tilde{\gamma}, \delta)(U, E_a(*, \frac{1}{2} + z)) \\
\times \frac{\Gamma(1 - s - \beta) \Gamma(s + \beta + \frac{1}{2} - 1 + i\tilde{t})}{\Gamma(s + k - 1) 2\sqrt{\pi}} G^{-\beta} \Gamma(\beta) \, dz \, d\beta \, ds
\]

\[ \ll G^{1 + \varepsilon} L^{1 - k + \varepsilon} Q^{-\frac{1}{2} + \varepsilon} \times \begin{cases} L^{-\frac{1}{2}} & \text{if } l_1 = l_2 \\ L^{-1} & \text{if } l_1 \neq l_2 \end{cases} \]

\[ (6.2.6) \]

By the same arguments that we made after stating lemma \[5.2\], we can conclude that the triple integral \[(6.2.5)\] is \( O(G^{1 + \varepsilon} L^{\frac{1}{2} + \varepsilon} L^{2 + 2\alpha + \varepsilon}) \).

6.2.2. The contribution from the poles at \( s' = 1 \pm \varepsilon \). Again, we will focus on the residue at \( s' = 1 + \varepsilon \) first:

\[
U_{s' = 1 + \varepsilon} = \lim_{\delta \to 0} \varphi(Q) G \sum_{l_1, l_2} \chi(l_1) \chi(l_2)(l_1 l_2)^{\frac{1}{2} + \alpha} l_1^i l_2^i \\
\times \left( \frac{1}{2\pi i} \right)^4 \int_{(\gamma_1)} \int_{(\gamma_4)} \int_{(C)} \frac{(4\pi)^k V_{N[l_1,l_2]} 2^{s - \frac{1}{2}}}{\Gamma(s + k - 1) 2\sqrt{\pi}} v(1 + z - w) v(w - \frac{1}{2}) \\
\times \sum_a \frac{\zeta(1 + 2z) \kappa_a Q(1 + z, -z)}{\zeta^* (1 - 2z)} M(s, \tilde{\gamma}, \delta)(U, E_a(*, \frac{1}{2} + z)) l_1^{1 + z - w} w^{-\frac{1}{2}} G^{-\beta} \\
\times \frac{\Gamma(\frac{s}{2} + z - s - \beta)(s + w + \beta + \frac{1}{2} - 2 - z + i\tilde{t})}{\Gamma(w + k - 1 + i\tilde{t})} x^{\frac{1}{2} + z} \Gamma(\beta) \, dz \, d\beta \, ds \, dw
\]

We will now move \( \text{Re} \, z \) down to \(-\frac{1}{2} - \frac{k}{2}\), encountering only one simple pole at \( z = s - \frac{1}{2} + \frac{k}{2}\). The shifted integral is \( O(x^{-\varepsilon}) \) as usual. The residue at \( z = s - \frac{1}{2} + \frac{k}{2} \)
is:
\[
\varphi(Q) \sum_{l_1, l_2} \chi(l_1) \chi(l_2) (l_1 l_2)^{\frac{1}{2} + \alpha} l_1^{-\frac{r}{2}} l_2^{-\frac{r}{2}} \\
\times \left( \frac{1}{2\pi i} \right)^3 \int_{(\gamma_1)} \int_{(\gamma_2)} \int_{(\gamma_3)} (4\pi)^k \mathcal{V}_N[l_1, l_2] \frac{(-1)^k \Gamma(2s - 1 + \frac{3}{4}) \Gamma(1 - s - \frac{3}{4})}{2\Gamma(s + k - 1) \Gamma(s + \frac{3}{4}) \Gamma(1 - s - \frac{3}{4})} \zeta(2s + k) \kappa_a Q (s + \frac{k}{2} + \frac{k}{4} - s - \frac{3}{4}) \zeta^*(2 - 2s - k) \left( U, E_{\alpha}(s + \frac{3}{2}) \right) \nu(s + \frac{k+1}{2} - w)v(w - \frac{1}{2}) \Gamma(1 + \frac{k}{2} - \beta) \Gamma(w + \frac{k-1}{2} + i\beta) \\
\times \mathcal{O}(x^{\beta-\gamma}) \mathcal{O}(x^{-\gamma}) \mathcal{O}(x^{-\frac{3}{2} + \frac{1}{2} - \sum_{i} (2 - 2s - k)} G^{-\beta} \Gamma(\beta) d\beta d\alpha dw
\]

Now we move \( \text{Re} \ s \) further down to \(-\frac{k}{2} - \varepsilon\), which goes through without poles. Hence this term is \( O(x^{-\varepsilon}) \).

A similar situation happens for \( s' = 1 - z \), and again, the same growth bounds do apply. \( \square \)

## 7. Proof of Theorem 1

Putting the results of the sections together (in particular, propositions (3.1), (3.2), (5.2), (5.3) and (6.1)), we obtain that \( S \) as defined in (2.0.14) has the following bound:

\[
S = O(G^{1+\varepsilon} Q^{1+\varepsilon} L^{1+2\alpha+\varepsilon} + O(G^{1+\varepsilon} Q^{1+\varepsilon} L^{3+2\alpha+\varepsilon}) + O(x^{-\varepsilon}),
\]

where \( G \asymp (1 + |t|)^{-\varepsilon} \log^5 Q \) and \( \alpha = \frac{1}{\log(Q(1 + |t|))} \). Note that this is independent of \( r \) when \( t = t + r \) with \( |r| \leq A \), where \( A = \sqrt{10 \log(Q(1 + |t|))} \).

Plugging this into the right-hand side of proposition (2.1), we have:

\[
\sum_{t \leq L} |L(\frac{1}{2} + it, f_\chi)|^2 \ll \log^2(Q(1 + |t|)) \left( G^{1+\varepsilon} (Q^{1+\varepsilon} L^{1+2\alpha+\varepsilon} + Q^{\frac{1}{2}+\varepsilon} L^{3+2\alpha+\varepsilon}) + x^{-\varepsilon} \right) + QL + (1 + |t|)^{\varepsilon} Q^{1+\varepsilon}
\]

Taking \( x \to \infty \), we can drop the \( x \)-term above. Note that

\[
\sum_{t \leq L} \frac{1}{(t, QN) = 1} \ll \frac{L}{\log L}
\]

Hence, we can conclude that:

\[
|L(\frac{1}{2} + it, f_\chi)|^2 \ll (1 + |t|)^{\frac{3}{2}+\varepsilon} \left( Q^{1+\varepsilon} L^{-1+2\alpha+\varepsilon} + Q^{\frac{1}{2}+\varepsilon} L^{1+2\alpha+\varepsilon} \right) + QL^{-1+\varepsilon} + (1 + |t|)^{\varepsilon} QL^{-2+\varepsilon}
\]
In \[8\], it is shown that
\[
|L(\frac{1}{2} + it, f_\chi)|^2 \ll (1 + |t|)^{\frac{1}{2} + \varepsilon} Q^{\frac{3}{2} + \varepsilon}
\]
and that \(Z\) is convergent.

We actually require \(\Re s < \frac{1}{2} - \frac{\varepsilon}{2}\) for the sum and integral there to be absolutely convergent.

Taking square roots of the above, this is the theorem.

**Appendix A. Properties of \(Z_Q(s, w)\)**

Most of the following is quoted from \[8\], quoted here for convenience.

First, we reiterate the definition of \(Z_Q(s, w)\):

\[
Z_Q(s, w) := \sum_{h_0, m_2 \geq 1 \atop m_1l_1 = m_2l_2 + h_0Q} \frac{A(m_1)A(m_2)}{(l_2m_2)^s(h_0Q)^v + \frac{1}{2}}
\]

(A.0.1)

In \[8\], it is shown that
\[
\lim_{\delta \to 0} Z_Q(s, u; \delta) = Z_Q(s, u)
\]
and that \(Z_Q(s, u; \delta)\) has the following spectral expansion
\[
Z_Q(s, s' - s - \frac{k}{2} + 1; \delta)
= \frac{(4\pi)^k(l_1l_2)^{\frac{k}{2} + \frac{1}{2} + \frac{s}{2} - \frac{k}{4}}}{\Gamma(s + k - 1)2\sqrt{\pi}} \left( \sum_j L_Q(s', \overline{u_j})M(s, t_j, \delta)(U, u_j) \right)
\]
\[
+ \mathcal{V}(\\|l_1, l_2\\|) \sum_a \frac{1}{2\pi i} \int_{(0)} \frac{\zeta(s' + z)\zeta(s' - z)\kappa_{a,Q}(s', -z)M(s, \frac{z}{l_1}, \delta)(U, E_a(s, \frac{1}{2} + z))}{\zeta^*(1 - 2z)} dz
\]
(A.0.2)

where \(a\)-sum is over cusps of \(\Gamma_0(\\|l_1, l_2\\|)\) and
\[
L_Q(s', \overline{u_j}) := \sum_{h \geq 1} \frac{\rho_j(-hQ)}{(hQ)^{s'}}
\]
\[
\kappa_{a,Q}(s', z) := \frac{\zeta^*(1 + 2z)}{2\zeta(s' + z)\zeta(s' - z)} \sum_{h \geq 1} \frac{\rho_a(-hQ, z)}{(hQ)^{s'}}
\]
\[
U(z) := y^k f(l_1z)f(l_2z)
\]
\[
\mathcal{V}(N) := \pi [\text{SL}_2(\mathbb{Z}) : \Gamma_0(N)]
\]
(A.0.3)

\(\rho_j(n)\) being \(n\)-th Fourier coefficient of Maass form \(u_j\) and \(\rho_a(n, z)\) being \(n\)-th Fourier coefficient of Eisenstein series at cusp \(a\) with holomorphic argument at \(\frac{1}{2} + z\), \([l_1, l_2]\) denotes the least common multiple of \(l_1\) and \(l_2\).

In \[8\], it is also shown that in (A.0.2), if we are to take the limit as \(\delta\) goes to 0, we actually require \(\Re s < \frac{1}{2} - \frac{\varepsilon}{2}\) for the sum and integral there to be absolutely convergent.

The properties of the \(M\) and \(Z\) functions that are relevant for this work are quoted in the following two propositions:
Proposition A.1. Let \( z \in \mathbb{C} - \frac{1}{2}\mathbb{Z} \). Then \( M(s, \frac{z}{r}; \delta) \) has simple poles at \( s = \frac{1}{2} \pm z - r \), for \( r \) a nonnegative integer. We denote the following:

\[
\text{Res}_{s=\frac{1}{2} \pm z - r} M(s, \frac{z}{r}; \delta) = c_r(\pm z; \delta),
\]

where \( c_r(\pm z, \delta) \) has the following explicit expression as \( \delta \to 0 \):

\[
\lim_{\delta \to 0} c_r(\pm z, \delta) = \frac{(-1)^r \sqrt{\pi^2 \Gamma^2(\mp 2z - r)} \Gamma(\frac{1}{2} \mp z + r)}{r! \Gamma(\frac{1}{2} \pm z) \Gamma(\frac{1}{2} - z)}
\]

And we have the following values at \( z = \pm \frac{1}{2} \), as \( \delta \to 0 \):

\[
c_r(-\frac{1}{2}, \delta) \to -\frac{2^r \sqrt{\pi}}{(2r + 1)!} \text{ for } r \geq 0 \quad (A.0.4)
\]

\[
c_0(\frac{1}{2}; \delta) \to \sqrt{\pi} \quad (A.0.5)
\]

\[
c_r(\frac{1}{2}, \delta) \to \frac{2^{r-\frac{1}{2}} \sqrt{\pi}}{2r!} \text{ for } r \geq 1 \quad (A.0.6)
\]

Also for \( \text{Re}(s + z) \leq \frac{1}{2} + \max(0, |\text{Re} z|) \), with \( s \) and \( z \) at least a distance \( \varepsilon \) away from poles,

\[
\lim_{\delta \to 0} M(s, \frac{z}{r}; \delta) = \frac{\sqrt{\pi} 2^{s-z} \Gamma(s - \frac{1}{2} + z) \Gamma(s - \frac{1}{2} - z) \Gamma(1 - s)}{\Gamma(\frac{1}{2} + z) \Gamma(\frac{1}{2} - z)} \quad (A.0.7)
\]

Proposition A.2. \( Z(s, u; \delta) \) has simple poles at \( s = \frac{1}{2} \pm it_j - r \), where \( r \) is a nonnegative integer. Taking the residues at those points and \( \delta \to 0 \), we have the following:

\[
\lim_{\delta \to 0} \text{Res}_{s=\frac{1}{2} + it_j - r} Z(s, u; \delta) = (l_1 l_2)^{\frac{1}{2} + t_j} c_{r, j} L_Q(s'; \overline{\nu}_j),
\]

where \( c_{r, j} \) has growth, when \( T \gg 1 \),

\[
\sum_{|t_j| \sim T} |c_{r, j}|^2 e^{|t_j|} \ll \log(T)(l_1 l_2)^{-k} T^{2r+1} \quad (A.0.8)
\]

We will also require the following estimate concerning \( L \)-functions:

Proposition A.3. For \( \text{Re}s' \geq \frac{1}{2} \) and \( L \gg Q^{\frac{1}{2}} \log Q \),

\[
\sum_{|t_j| \sim T} |L_Q(s', \overline{\nu}_j)|^2 e^{|t_j|} \ll Q^{-2s'} L^{2+\varepsilon} (1 + |s'| + T)^{2+\varepsilon} \quad (A.0.9)
\]

Together with (A.0.8) and the following fact

\[
\sum_{|t_j| \sim T} |\langle U, u_j \rangle|^2 e^{|t_j|} \ll L^{-2k} T^{2k} \log T,
\]

we have

Proposition A.4.

\[
\sum_{|t_j| \sim T} L_Q(s', \overline{\nu}_j) \langle U, u_j \rangle \ll Q^{-s'} L^{1-k+\varepsilon} T^{1+k+\varepsilon} \quad (A.0.10)
\]

\[
\sum_{|t_j| \sim T} L_Q(s', \overline{\nu}_j) c_{r, j} \ll Q^{-s'} L^{1-k+\varepsilon} T^{\frac{3}{2}+r+\varepsilon} \quad (A.0.11)
\]
These equations can be proved by Cauchy’s inequality with facts quoted before the proposition.

Here is a proposition on some properties of $\kappa_{a,Q}(s',-z)$:

**Proposition A.5.** Take $a$ to be the cusp $\frac{1}{w}$, where $w|N$. Then:

$$
\kappa_{a,Q}(s',-z) = Q^{-(s'+z)} \left( \frac{1}{wN} \right)^{\frac{s'}{2}-z} \prod_{p|N} (1 - p^{-(1-2z)-1}) \times \prod_{p|w} (1 - p^{-(s'-z)}) \prod_{p|w} (-1 + p^{1-(s'+z)}) \\
\times \prod_{p^\gamma|Q} (1 - p^{2z})^{-1} \left( (1 - p^{-(s'-z)}) - p^{2(\gamma+1)z}(1 - p^{-(s'+z)}) \right)
$$

If we take $s' = 1 + z$, then we have the following facts:

1. **Evaluating at** $z = -\frac{1}{2}$:

   $$
   \kappa_{a,Q}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{N} \prod_{p|N} (1 + \frac{1}{p})^{-1} = \prod_{p|N} \frac{1}{p + 1}
   $$

2. **On the line** $\Re z = -\frac{1}{2}$:

   $$
   \kappa_{a,Q}(1 + z, -z) \ll \kappa_{a,Q}\left(\frac{1}{2}, \frac{1}{2}\right)
   $$

If we take $s' = 1 - z$, we only have nonzero function if $a$ is the cusp $0$.

1. **Evaluating at** $z = \frac{1}{2}$:

   $$
   \kappa_{0,Q}\left(\frac{1}{2}, \frac{1}{2}\right) = 1
   $$

2. **On the line** $\Re z = \frac{1}{2}$:

   $$
   \kappa_{0,Q}(1 - z, -z) \ll Q^{\varepsilon}
   $$

If we take $s' = \frac{1}{2}$, then we have the following bound when $\Re z = 0$:

$$
\kappa_{a,Q}\left(\frac{1}{2}, -z\right) \ll Q^{-\frac{1}{2} + \varepsilon}
$$

**Remark.** Note that since $N$ is square-free, for any $w|N$, we have $(w, \frac{N}{w}) = 1$

Hence, the $p|w$ and $p|\frac{N}{w}$ refers to disjoint set of primes.

**Appendix B. Miscellaneous Facts Involving $(U, E_a(*, s))$**

**Lemma B.1.** The inner product $(U, E_a(*, s))$ has the following properties:

1. **Res**$_{s=1}(U, E_a(*, s)) = \frac{(l_1l_2)^{-\frac{1}{2}}}{{l_1l_2}^{\frac{1}{2}}v_{l_1l_2}}(f, f)$ if $l_1 = l_2$.

2. **Res**$_{s=1}(U, E_a(*, s)) = \frac{(l_1l_2)^{-\frac{1}{2}}}{{l_1l_2}^{\frac{1}{2}}v_{l_1l_2}}(f, f)E_{l_1l_2}(1)$ if $l_1 \neq l_2$. 


Lemma B.2.  

1. If \( l_1 \neq l_2 \), 
\[
\mathcal{V}_{Nl_1l_2} \sum_a \frac{\zeta(0)\kappa_a Q_1 (\frac{1}{2}, \frac{1}{2})}{\zeta(a)(2)} (U, E_a(*, 0)) = -(l_1 l_2)^{-\frac{1}{2} - 1} \langle f, f \rangle E_{l_1l_2}(1)
\]

2. If \( l_1 = l_2 \), 
\[
\mathcal{V}_{Nl_1} \sum_a \frac{\zeta(0)\kappa_a Q_1 (\frac{1}{2}, \frac{1}{2})}{\zeta(a)(2)} (U, E_a(*, 0)) = -(l_1 l_2)^{-\frac{1}{2} - 1} \langle f, f \rangle
\]

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