Traces of weighted Sobolev spaces. The case \( p = 1 \)✩

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Abstract

A complete description of traces on \( \mathbb{R}^n \) of functions from the weighted Sobolev space \( W^l_1(\mathbb{R}^{n+1}, \gamma), l \in \mathbb{N} \), with weight \( \gamma \in A_{1}^{\text{loc}}(\mathbb{R}^{n+1}) \) is obtained. In the case \( l = 1 \) the proof of the trace theorems is based on a special nonlinear algorithm for constructing a system of tilings of the space \( \mathbb{R}^n \). As the trace of the space \( W^1_1(\mathbb{R}^{n+1}, \gamma) \) we have the new function space \( Z(\{\gamma_{k,m}\}) \).

Keywords: Besov spaces of variables smoothness, weighted Sobolev spaces, traces, Muckenhoupt weights

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1. Introduction

The problem of exact description of the trace space (on the boundary of a domain) of a weighted Sobolev space has a long history. A short survey of the available results in this direction is given in [1]. It is worth pointing out that, since the appearance of the pioneering work of Gagliardo [2] a long time ago, it was only in [1], [3] that a complete description of the trace space on \( \mathbb{R}^n \) of the weighted Sobolev space \( W^l_p(\mathbb{R}^{n+1}, \gamma), p \in (1, \infty) \), with weight \( \gamma \in A_p^{\text{loc}}(\mathbb{R}^{n+1}) \) was obtained. The solution of this problem, with such a high degree of generality, calls for the introduction of new modifications of Besov-type spaces of variable smoothness and new machinery for studying thereof.

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Thus, in the case $p \in (1, \infty)$ we have the result in the most general context possible at present.

The thing gets much worse in the case $p = 1$. Indeed, starting from 1957, as far as the author is aware, the bibliography on this subject lists only the papers [4], [5], which put forward an exact description of the trace on $\mathbb{R}^n$ by the weighted Sobolev space $W^1_1(\mathbb{R}^{n+1}, \gamma)$. However, a weight $\gamma$ in these papers was assumed to be a model function depending only on some highlighted group of variables. For example, in [4] it was assumed that a weight $\gamma$ depends only on the coordinate $x_{n+1}$ (note that in [4] a more general multiweight case was considered, when different derivatives in the Sobolev norm are integrated with different weights, but all weights depend on the same coordinate $x_{n+1}$), and in [3] it is assumed that $\gamma \in A_1(\mathbb{R}^n)$.

Of course, such a lack of knowledge in the case $p = 1$ is due to the great difficulty of the problem. Attempts to find the trace of the space $W^1_1(\mathbb{R}^{n+1}, \gamma)$ with fairly general $\gamma$ involve considerable difficulties.

The machinery of [1], [3] may not in principle be applied in this setting, because this approach depends on the Muckenhoupt theorem on the boundedness of the Hardy–Littlewood maximal operator in weighted Lebesgue spaces (this result fails for $p = 1$, see Ch. 5 of [6] for the details).

However, in the ‘simple’ nonlimiting case $l > 1$ one eventually succeeds in modifying the methods of [1], [3] (without having recourse to the Hardy–Littlewood maximal function!) to give a complete description of the trace space of the weighted Sobolev space $W^l_1(\mathbb{R}^{n+1}, \gamma)$ with weight $\gamma \in A^\text{loc}_1(\mathbb{R}^{n+1})$ in terms of the Besov spaces of variable smoothness that were introduced by the author [3]. We shall enlarge on these results in §3.

The case $l = 1$ presents the greatest challenge and calls for the development of a refreshingly different method. The situation is aggravated by the fact that even in the case $\gamma \equiv 1$ the extension operator from the trace space turns out to be nonlinear [2], [7]. At the same time in [4] it was shown that the extension operator from the corresponding trace space is linear if $\lim_{x_{n+1} \to 0} \gamma(x_{n+1}) = +\infty$ for a continuous weight $\gamma = \gamma(x_{n+1})$. 2
2. Basic notations and definitions

Throughout we shall fix \( n \in \mathbb{N} \), which will only be used to denote the dimension of the Euclidean space \( \mathbb{R}^n \). A point of the space \( \mathbb{R}^n \) will be written as \( x = (x_1, \ldots, x_n) \), and a point of the space \( \mathbb{R}^{n+1} \), as the pair \((x,t) (x \in \mathbb{R}^n, t \in \mathbb{R})\). The space \( \mathbb{R}^n \) will be identified with the hyperplane \( \mathbb{R}^n \times \{0\} \) of the space \( \mathbb{R}^{n+1} \).

The symbol \( C \) will be used to denote (different) insignificant constants in various estimates. Sometimes, if it is required for purposes of exposition, we shall indicate the parameters on which some or other constant depends.

As usual, \( \mathbb{Z}_+ \) will denote the set of nonnegative integers. Also, \( \mathbb{Z}^n \) is the linear space of vectors in \( \mathbb{R}^n \) with integer components.

The derivatives will be written in the multi-index notation: \( D^{\alpha} := \frac{\partial^{\alpha_1} \cdots \partial^{\alpha_n+1}}{\partial x_1^{\alpha_1} \cdots \partial x_{n+1}^{\alpha_{n+1}}} \), where \( \alpha \) is a vector from \( \mathbb{Z}^{n+1} \) with nonnegative components \((\alpha_1, \ldots, \alpha_{n+1})\), \( |\alpha| := |\alpha_1| + \cdots + |\alpha_{n+1}| \).

Given a measurable subset \( E \) of the space \( \mathbb{R}^d \), \( d = n, n+1 \), we denote by \( |E| \) the Lebesgue measure of \( E \), and by \( \chi_E \), the characteristic function of \( E \).

By an open cube \( Q \) in the space \( \mathbb{R}^d \), \( d = n, n+1 \) (or simply a cube, if the dimension of the ambient space is clear from the context) we shall mean a cube with sides parallel to coordinate axes. By \( \overline{Q} \) we denote the closure of a cube \( Q \) in the space \( \mathbb{R}^d \), \( d = n, n+1 \) which will be called a closed cube. By \( r(Q) \) we denote the side length of \( Q \).

Given \( k \in \mathbb{Z}_+ \), \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \), we let \( Q_{k,m} := \prod_{i=1}^{d} (\frac{m_i}{2^k}, \frac{m_i+1}{2^k}) \) denote an open dyadic cube of rank \( k \) in the space \( \mathbb{R}^d \), \( d = n, n+1 \).

Let \( I := \prod_{i=1}^{n} (-1,1) \) be the unit cube in the space \( \mathbb{R}^n \).

By a weight we shall imply a function \( \gamma \in L_1^w(\mathbb{R}^{n+1}) \) which is positive almost everywhere.

Next, for a measurable set \( E \subset \mathbb{R}^{n+1} \) of positive measure and a weight \( \gamma \), we define
\[
\gamma_E := \frac{1}{|E|} \int_{E} \gamma(x,t) \, dx \, dt.
\]
It what follows we shall be concerned only with weights that locally satisfy the Muckenhoupt condition. Following [8] we introduce

**Definition 2.1.** We say that a weight \( \gamma \in A^{1}_{\text{loc}}(\mathbb{R}^{n+1}) \) if

\[
\gamma Q \leq C_\gamma \operatorname{ess inf}_{(x,t) \in Q} \gamma(x,t) \quad (2.1)
\]

for any cube \( Q \) in \( \mathbb{R}^{n+1} \) of side length \( r(Q) \leq 1 \).

**Remark 2.1.** The next elementary observation will be used in the sequel. Let \( Q \) be a cube in \( \mathbb{R}^{n+1} \) with side \( r(Q) \leq 1 \), and let \( Q_1 \subset Q \) be a cube of halved side length. Using (2.1), we clearly have

\[
\operatorname{ess inf}_{(x,t) \in Q_1} \gamma(x,t) \leq \gamma Q_1 \leq |Q| \quad (2.2)
\]

From (2.2) one easily obtains that, for \( k \in \mathbb{Z}_+, m, m' \in \mathbb{Z}^{n+1}, |m - m'| \leq 1 \),

\[
\operatorname{ess inf}_{(x,t) \in Q_{k,m}} \gamma(x,t) \leq C(C_\gamma, n) \operatorname{ess inf}_{(x,t) \in Q_{k,m'}} \gamma(x,t),
\]

\[
\int \int_{Q_{k,m}} \gamma(x,t) \, dx \, dt \leq C(C_\gamma, n) \int \int_{Q_{k,m'}} \gamma(x,t) \, dx \, dt. \quad (2.3)
\]

Hence, by (2.3) it is seen that, for \( c > 1, \gamma \in A^{1}_{\text{loc}}(\mathbb{R}^{n+1}) \), and any cube \( Q \) in the space \( \mathbb{R}^{n+1} \) with side \( r(Q) \leq c \),

\[
\gamma Q \leq C(C_\gamma, n, c) \operatorname{ess inf}_{(x,t) \in Q} \gamma(x,t), \quad (2.4)
\]

and besides, for any cube \( Q \) in the space \( \mathbb{R}^{n+1} \) with side \( r(Q) \leq 1 \),

\[
\int \int_{Q} \gamma(x,t) \, dx \, dt \leq C(C_\gamma, n, c) \int \int_{Q} \gamma(x,t) \, dx \, dt. \quad (2.5)
\]

**Definition 2.2.** Assume that a weight \( \gamma \in A^{1}_{\text{loc}}(\mathbb{R}^{n+1}) \), \( l \in \mathbb{N} \), and \( \Omega \) is a domain in \( \mathbb{R}^{n+1} \). By \( W^l_1(\Omega, \gamma) \) we will denote the linear space of functions which are locally integrable on \( \Omega \) and have finite norm

\[
\|f|W^l_1(\Omega, \gamma)\| := \sum_{|\alpha| \leq l} \|\gamma D^\alpha f|L_1(\Omega)\|. \quad (2.6)
\]
For $\gamma \equiv 1$ we shall write $W^1_1(\Omega)$ instead of $W^1_1(\Omega, 1)$.

By $D^\alpha f$ in (2.6) we denote the (Sobolev) generalized derivatives of a function $f$ of order $\alpha$ (see Ch. 1 of [10] or Ch. 2 of [11] for equivalent definitions and basic properties).

Remark 2.2. Using (2.4) and Hölder’s inequality, we see that if the weight $\gamma \in A^0_{(n+1)}(\mathbb{R}^n)$, $l \in \mathbb{N}$ and $\Omega$ is a bounded domain (in the space $\mathbb{R}^{n+1}$), then the space $W^l_1(\Omega, \gamma)$ is continuously embedded in the space $W^l_1(\Omega)$ (with the embedding constant depending on $\text{diam} \Omega$ and the constant $C_\gamma$).

The following fact will be frequently useful. For completeness, we give the proof.

**Lemma 2.1.** Let $d \in \mathbb{N}$, $f \in L^1(\mathbb{R}^d)$, and let $N \in \mathbb{N}$, $\mathbb{R}^d = \bigcup_{j=1}^{\infty} R_j$, where measurable sets $R_j$ are such that any point $z \in \mathbb{R}^d$ lies in at most than $N$ sets from the family $\{R_j\}_{j=1}^{\infty}$. Then

$$\sum_{j=1}^{\infty} \int_{R_j} |f(z)| \, dz \leq N \| f \|_{L^1(\mathbb{R}^d)}.$$

**Proof.** From the hypotheses of the theorem we see at once that

$$\sum_{j=1}^{\infty} \chi_{R_j}(z) \leq N, \quad x \in \mathbb{R}^d.$$ 

Hence, we have the estimate

$$\sum_{j=1}^{\infty} \int_{R_j} |f(z)| \, dz = \sum_{j=1}^{\infty} \int_{\mathbb{R}^{n+1}} \chi_{R_j}(z)|f(z)| \, dz = \int_{\mathbb{R}^{n+1}} \sum_{j=1}^{\infty} \chi_{R_j}(z)|f(z)| \, dz \leq N \| f \|_{L^1(\mathbb{R}^{n+1})}.$$

3. The nonlimiting case

In this section we shall modify the methods of [4, 3] and give a complete description of the trace space of the space $W^l_1(\mathbb{R}^{n+1}, \gamma)$ on the plane $\mathbb{R}^n$ under
the condition that $\gamma \in A_1^{\text{loc}}(\mathbb{R}^{n+1}), \ l > 1$. Until the end of this section, $Q_{k,m}$ $(k \in \mathbb{Z}_+, \ m \in \mathbb{Z}^n)$ will denote dyadic cubes of rank $k$ in the space $\mathbb{R}^n$.

For the rest of this section we fix a parameter $l \in \mathbb{N}, \ l > 1$, and a weight $\gamma \in A_1^{\text{loc}}(\mathbb{R}^{n+1})$. Next, we set

$$
\gamma_{k,m} := 2^{kl} \int_{Q_{k,m}} \int_{(0,2^{-k})} \gamma(x,t) \, dx \, dt, \quad (k,m) \in \mathbb{Z}_+ \times \mathbb{Z}^n.
$$

**Remark 3.1.** From (2.3) it clearly follows that $\gamma_{k,m} \leq C \gamma_{k,m}$ for $k \in \mathbb{Z}_+, \ m,m' \in \mathbb{Z}^n, \ |m - m'| \leq 1$.

For further purposes we shall need the definition of the Besov space of variable smoothness. Actually, we give a particular case of Definition 2.5 of [1], because we shall not need the whole range of the parameters (and such general assumptions on the variable smoothness).

Given a measurable function $\varphi$ and $x,h \in \mathbb{R}^n$, we define $\Delta^l(h)\varphi(x) := \sum_{i=0}^l (-1)^{l-i} C_i \varphi(x + ih)$. Next, for a function $\varphi \in L_1^{\text{loc}}(\mathbb{R}^n)$ and a cube $Q$ in the space $\mathbb{R}^n$, we set

$$
\delta^l(Q)\varphi := \frac{1}{|Q|^2} \int_{r(Q)} \int_{Q} |\Delta^l(h)\varphi(x)| \, dx \, dh.
$$

By $E^l(Q)\varphi$ we shall denote the best $L_1(Q)$-approximation to a function $\varphi \in L_1^{\text{loc}}(\mathbb{R}^n)$ on a cube $Q$ by polynomials of degree $< l$.

**Definition 3.1.** By $\tilde{B}^l(\mathbb{R}^n, \{\gamma_{k,m}\})$ we shall denote the Besov space of variable smoothness $\{\gamma_{k,m}\}$ equipped with the norm

$$
\left\| \varphi |\tilde{B}^l(\mathbb{R}^n, \{\gamma_{k,m}\}) \right\| := \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \gamma_{k,m} \delta^l(Q_{k,m})\varphi + \sum_{m \in \mathbb{Z}^n} \gamma_{0,m} \varphi_{Q_{0,m}}. \quad (3.1)
$$

**Remark 3.2.** According to [1], for $c > 1$,

$$
\left\| \varphi |\tilde{B}^l(\mathbb{R}^n, \{\gamma_{k,m}\}) \right\| \sim \sum_{k=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \gamma_{k,m} E^l(cQ_{k,m})\varphi + \sum_{m \in \mathbb{Z}^n} \gamma_{0,m} \varphi_{Q_{0,m}}.
$$

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Definition 3.2. A function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be the trace of a function $f \in L^1_{\text{loc}}(\mathbb{R}^{n+1})$ on the hyperplane $\mathbb{R}^n$ (written $\text{tr} |_{t=0} f = \varphi$) if, for any open $Q$ (in the space $\mathbb{R}^n$),

$$\int_Q |\varphi(x) - f(x,t)| \, dx \to 0, \quad t \to 0. \quad (3.2)$$

Let $E \subset L^1_{\text{loc}}(\mathbb{R}^{n+1})$ be the linear space of functions $f$ that have the trace on the hyperplane $\mathbb{R}^n$. In what follows, by $\text{Tr}$ we shall denote the linear operator $\text{Tr} : E \to L^1_{\text{loc}}(\mathbb{R}^n)$ defined by $\text{Tr}[f] = \text{tr} |_{t=0} f = \varphi$.

Theorem 3.1. The linear operator $\text{Tr} : W^1_1(\mathbb{R}^{n+1}, \gamma) \to \tilde{B}^1(\mathbb{R}^n, \{\gamma_{k,m}\})$ is bounded. Moreover, there exists a bounded linear operator $\text{Ext} : \tilde{B}^1(\mathbb{R}^n, \{\gamma_{k,m}\}) \to W^1_1(\mathbb{R}^{n+1}, \gamma)$ such that $\text{Tr} \circ \text{Ext} = \text{Id}$ on the space $\tilde{B}^1(\mathbb{R}^n, \{\gamma_{k,m}\})$.

Proof. Step 1. Assume that a function $f \in W^1_1(\mathbb{R}^{n+1}, \gamma)$. Then Remark 2.2 and Theorem 2 of §5.2 of [10] show that the function $f$ has the trace (which we denote by $\varphi$) on $\mathbb{R}^n$.

Let us prove the estimate

$$\|\varphi|_{\tilde{B}^1(\mathbb{R}^n, \{\gamma_{k,m}\})}\| \leq C \|f|_{W^1_1(\mathbb{R}^{n+1}, \gamma)}\|, \quad (3.3)$$

where the constant $C > 0$ is independent of the function $f$.

Let $m \in \mathbb{Z}^n$ be fixed. Applying the fundamental theorem of calculus for almost all $x \in Q_{0,m}$ and using (2.1), (2.3), this gives

$$\gamma_{0,m} \int_{Q_{0,m}} \varphi(x) \leq \gamma_{0,m} \int_{Q_{0,m}} 2 \int_1^2 |f(x,\tau) - \varphi(x)| \, dx \, d\tau \leq$$

$$\leq C(n, C_\gamma) \int_{Q_{0,m}} 2 \int_0^2 \gamma(x,\tau) |D_1 f(x,\tau)| \, dx \, d\tau. \quad (3.4)$$

Summing estimate (3.4) over all $m \in \mathbb{Z}^n$, we see that

$$\sum_{m \in \mathbb{Z}^n} \gamma_{0,m} \int_{Q_{0,m}} \varphi(x) \leq C(n, C_\gamma) \|f|_{W^1_1(\mathbb{R}^{n+1}, \gamma)}\|. \quad (3.5)$$
Arguing as in the proof of Lemma 3.1 of [3], we obtain the estimate

\[ \delta^i(Q_{k,m}) \varphi \leq 2^{kn} \int_{C_i Q_{k,m}} \int_0^\gamma \sum_{|\alpha|=l} t^{l-1} |\partial^\alpha f(x,t)| \, dx \, dt, \quad (3.6) \]

in which the constants \( C_1, C_2 \) depend only on \( l, n \).

An application of (2.1) gives

\[ \sum_{j=1}^k \sum_{m' \in \mathbb{Z}^n} 2^{j(n)} \gamma_{j,m'} = \sum_{j=1}^k \sum_{m' \in \mathbb{Z}^n} 2^{j(n)} 2^{j(n+1)} \int_{Q_{j,m',x \in (0,2^{-j})}} \gamma(x,t) \, dx \, dt \leq \]

\[ C_{l,n} \sum_{j=1}^k \sum_{m' \in \mathbb{Z}^n} 2^{j(n)} \inf_{(x,t) \in Q_{j,m',x \in (0,2^{-j})}} \gamma(x,t) \leq C_{l,n} 2^{j(n+1)} \inf_{(x,t) \in Q_{j,m',x \in (0,2^{-j})}} \gamma(x,t). \]

(3.7)

For brevity, we put \( g(x,t) := \sum_{|\alpha|=l} t^{l-1} |\partial^\alpha f(x,t)| \) for \( (x,t) \in \mathbb{R}^{n+1} \). The sets \( C_1 Q_{k,m} \times (0, C_2 2^{-k}) \) have finite (depending only on \( n, l \)) overlapping multiplicity, and hence, using Remark 4.4 it follows from (3.6), (3.7) that

\[ \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} \gamma_{k,m} \delta^i(Q_{k,m}) \varphi \leq C \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} 2^{kn} \gamma_{k,m} \int_{C_i Q_{k,m}} \int_0^\gamma \sum_{|\alpha|=l} t^{l-1} |\partial^\alpha f(x,t)| \, dx \, dt \leq \]

\[ C \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} 2^{kn} \gamma_{k,m} \int_{Q_{k,m}} \int_0^\gamma g(x,t) \, dx \, dt = \]

\[ = C \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} \int_0^\gamma g(x,t) \, dx \, dt \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} 2^{j(n)} \gamma_{j,m'} \leq \]

\[ \leq C \sum_{k=1}^\infty \sum_{m \in \mathbb{Z}^n} 2^{k(n+1)} \inf_{(x,t) \in Q_{k,m,\times (0,2^{-k})}} \gamma(x,t) \int_{Q_{k,m,\times (2^{-k-1},2^{-k})}} g(x,t) \, dx \, dt \leq \]

\[ \leq C \int_{k=1}^\infty \int_{m \in \mathbb{Z}^n} \gamma(x,t) \sum_{|\alpha|=l} |\partial^\alpha f(x,t)| \, dx \, dt \leq C \| f \| W_1 L(\mathbb{R}^{n+1}, \gamma). \quad (3.8) \]

Now estimate (3.8) follows from (3.5), (3.8).

**Step 2.** The construction of the extension operator \( \text{Ext}: B^i(\mathbb{R}^n, \{ \gamma_{k,m} \}) \rightarrow \mathbb{R}^{n+1} \).
\(W^1_1(\mathbb{R}^{n+1}, \gamma)\) and the proof of its boundedness require minor modifications of the Step 2 in the proof of Theorem 3.1 from [1]. We omit the details.

4. The limiting case

In this section we shall be concerned with the problem of complete description of the trace space of the Sobolev space \(W^1_1(\mathbb{R}^{n+1}, \gamma)\) with \(l = 1\) and \(\gamma \in A^{loc}_1(\mathbb{R}^{n+1})\). We first note that this problem is equivalent to the problem of the description of the trace space of the Sobolev space \(W^1_1(\mathbb{R}^{n+1}, \gamma)\) on \(\mathbb{R}^n\). Indeed, this follows from the easily verified fact that the operator of even extension from \(W^1_1(\mathbb{R}^{n+1}, \gamma)\) into the space \(W^1_1(\mathbb{R}^{n+1}, \gamma)\) is continuous.

Before proceeding with precise statements, we first give a brief 'heuristic' description of this problem in order to clarify, on the intuitive level, the principal impetuses for further constructions.

Unfortunately, the Besov space of variable smoothness \(\tilde{B}^1_1(\mathbb{R}^n, \{\gamma_{k,m}\})\) (considered in the previous section) are poor candidate for the role of trace space if a weight is only subject to the constraint \(\gamma \in A^{loc}_1(\mathbb{R}^{n+1})\).

It is not hard to see that for \(l = 1\) estimate (3.7) fails in general, and hence one may not expect an estimate like (3.8). In addition to this technical impediment there are much deeper reasons for the unfitness of the spaces \(\tilde{B}^1_1(\mathbb{R}^n, \{\gamma_{k,m}\})\).

Indeed, even in the case \(\gamma \equiv 1\) the classical Gagliardo's result shows that \(\text{Tr} \mid_{l=0} W^1_1(\mathbb{R}^{n+1}) = L_1(\mathbb{R}^n)\). In this case the space \(\tilde{B}^1_1(\mathbb{R}^n, \{\gamma_{k,m}\})\) coincides with the Besov space of smoothness zero \(B^{0,1}_{1,1}(\mathbb{R}^n)\) (see [13] for details). Next, Lemma 2 of [13] implies, in particular, that \(\tilde{B}^1_1(\mathbb{R}^n, \{\gamma_{k,m}\}) \neq L_1(\mathbb{R}^n)\) for \(\gamma \equiv 1\).

So, the trace space contains functions with inappropriate smoothness properties.

It is also worth pointing out that, according to Peetre [17], the extension operator \(\text{Ext} : L_2(\mathbb{R}^n) \to W^1_1(\mathbb{R}^{n+1})\) (which is the right inverse of the trace operator) may not be linear.

On the other hand, for the weight \(\gamma = \gamma(x,t) = |t|^\varepsilon, \varepsilon \in (0,1)\), the methods of the previous section also work! To this aim one needs to slightly modify
estimate (3.7). In spite of the fact that \( l - 1 = 0 \), we succeed in achieving a ‘geometric rate’ on account of the fact that \( \inf_{t \in (0, 2^{-k})} t^\varepsilon \geq 2^\varepsilon \inf_{t \in (0, 2^{-k+1})} t^\varepsilon \).

As a good candidate for the trace space in the general case one should consider a space whose elements are able to appreciably change their smoothness characteristics when transiting from a point to a point, because the ‘rate of decay’ of a weight may be substantially different at different points. As distinct from the case \( l > 1 \), in which, roughly speaking, the trace space is ‘quasi-homogeneous’, in the case \( l = 1 \) the trace space turns out to be ‘essentially nonhomogeneous’.

The thing is that in the case \( l > 1 \) a sufficiently rapidly growing geometric progression \( \{2^{kl}\} \) was instrumental, in a sense, in eliminating the strong nonhomogeneous of a weight. However, the limiting case \( p = l = 1 \) calls for a more subtle analysis of the local behaviour of the weight near each point of the hyperplane on which the trace is considered.

An important step in this analysis is the construction of a special system of tilings of the space \( \mathbb{R}^n \). This system of tilings will replace the standard system of tiling of the space \( \mathbb{R}^n \) composed of all dyadic cubes numbered by indexes \((k, m) \in \mathbb{Z}_+ \times \mathbb{Z}^n \). The cubes in our special system of tilings will be numbered by indexes \((k, m) \in A \subset \mathbb{Z}_+ \times \mathbb{Z}^n \). Here, the algorithm for construction of the index set \( A \) is based on combinatorial arguments and is nonlinear. Namely, the set \( A \) depends not only on the weight \( \gamma \), but also on the function \( f \).

In this section we shall denote by \( Q \) (respectively, \( \overline{Q} \)) an open (closed) cube in the space \( \mathbb{R}^n \).

**Definition 4.1.** Assume that we are given a set of dyadic closed cubes \( T = \{ \overline{Q}_\alpha \}_{\alpha \in A}, \ A \subset \mathbb{Z}_+ \times \mathbb{Z}^n \), in which different cubes have disjoint interiors and \( \mathbb{R}^n = \bigcup_{\alpha \in A} \overline{Q}_\alpha \). We shall call this family a *tiling of the space* \( \mathbb{R}^n \).

**Definition 4.2.** A tiling \( T' = \{ \overline{Q}_\alpha \}_{\alpha \in A'} \) will be said to succeed a tiling \( T = \{ \overline{Q}_\alpha \}_{\alpha \in A} \) (written \( T' \succ T \)) if each cube \( \overline{Q}_{\alpha'} \), \( \alpha' \in A' \), of the tiling \( T' \) is contained in some cube \( \overline{Q}_\alpha \), \( \alpha \in A \), of the tiling \( T \).
Definition 4.3. Assume that for any \( s \in \mathbb{Z}_+ \) we have a tiling \( T^s = \{ Q^s_\alpha \}_{\alpha \in A^s} \), \( A^s \subset \mathbb{Z}_+ \times \mathbb{Z}^n \) of the space \( \mathbb{R}^n \). Assume also that \( T^{s+1} \succ T^s \) for \( s \in \mathbb{Z}_+ \). Then the set \( T = \{ T^s \} = \{ T^s \}_{s=0}^\infty \) will be called a system of tilings of the space \( \mathbb{R}^n \).

The next lemma is an important combinatorial instrument required in the definition of the trace space.

Lemma 4.1. Let \( \{ \overline{Q}_\alpha \}_{\alpha \in A} \) be a tiling of the space \( \mathbb{R}^n \). Then, for each number \( \lambda = 2^{-k}, k \in \mathbb{N} \), there exists an index set \( \widetilde{A} \subset A \) such that

1) \( \mathbb{R}^n = \bigcup_{\alpha \in \widetilde{A}} \overline{Q}_\alpha, \overline{Q}_\alpha := (1 + \lambda)Q_\alpha \)

2) any point \( x \in \mathbb{R}^n \) lies in at most \( (n + 1)2^n \) cubes from the family \( \{ \overline{Q}_\alpha \}_{\alpha \in \widetilde{A}} \).

3) if \( \overline{Q}_\alpha \cap \overline{Q}_\alpha' \neq \emptyset \), then \( |\overline{Q}_\alpha \cap \overline{Q}_\alpha'| \geq C(n, \lambda) \min \{|Q_\alpha'|, |Q_\alpha|\} \).

Proof. At the first step we select from the family \( \{ \overline{Q}_\alpha \}_{\alpha \in A} \) the cubes with the largest side length; let \( S^1 \subset A \) be the index set corresponding to these cubes. At the second step we select from the cubes \( \{ \overline{Q}_\alpha \}_{\alpha \in S^1} \) the cubes with largest side length and such that the closure of any such a cube is not contained in \( \bigcup_{\alpha \in S^1} \overline{Q}_\alpha \). Let \( S^2 \subset A \) be the index set corresponding to these cubes. Given \( k \in \mathbb{N} \), assume that the set \( S^k \) is constructed. In order to build \( S^{k+1} \), we select from the cubes \( \{ \overline{Q}_\alpha \}_{\alpha \in S^k} \) the cubes with the largest side length and such that the closure of any such a cube is not contained in the set \( \bigcup_{\alpha \in S^k} \overline{Q}_\alpha \) (the union is taken over all indexes \( \alpha \in \bigcup_{i=1}^k S^i \)). As a result, either at some step \( l \) we get \( S^l = \emptyset \) or we obtain a countable family of sets \( \{ S^i \}_{i=1}^\infty \), for each of which \( \bigcup_{\alpha \in S^i} (1 + \lambda)Q_\alpha = \mathbb{R}^n \). We set \( \widetilde{A} := \bigcup_{i=1}^\infty S^i \). We claim that \( \widetilde{A} \) is the required index set.

1) If some point \( x \in \mathbb{R}^n \) is not covered, then the closed cube \( \overline{Q}_\alpha \ni x \) will not be completely covered, which contradicts the construction.

2) We claim that the overlapping multiplicity is at most \( (n + 1)2^n \). Indeed, we fix an arbitrary point \( x_0 \in \mathbb{R}^n \) and estimate the number of cubes from the family
\[ \{ \tilde{Q}_\alpha \}_{\alpha \in \tilde{A}} \] that contain this point. In doing so we shall modify one trick from Lemma 1.1 of \cite{12}. Namely, we draw through the point \( x_0 \) the planes that are parallel to the coordinate planes. This will give us \( 2^n \) quadrants (closed!) with vertex at \( x_0 \). We fix arbitrary quadrant and consider the cubes that contain \( x_0 \) and whose centers lie in this quadrant. Clearly, the lemma will be proved once we show that there are at most \((n + 1)\) such cubes.

Assume the contrary. Then, numbering these cubes in decreasing size, we see that the centre of the next cube (in the order of decreasing size) is contained in its direct predecessor. As a result, the centre of the cube with number \( n + 2 \) (which we denote by \( \tilde{Q}_{\alpha_0} \)) will be contained in at least \( n \) cubes from the family \( \{ \tilde{Q}_\alpha \}_{\alpha \in \tilde{A}} \). We claim that such a case is never realized (we shall obtain a contradiction with the algorithm for choosing the cubes).

Note that if the centres of the cubes \( \tilde{Q}_\alpha \ni x_0 \) and \( \tilde{Q}_{\alpha'} \ni x_0 \) lie in the same quadrant, then the center of one cube lies in the other cube. This implies, in particular, that either \( Q_\alpha \subset \tilde{Q}_{\alpha'} \) or \( Q_{\alpha'} \subset \tilde{Q}_\alpha \) (inasmuch as \( \lambda = 2^{-k}, k \in \mathbb{N} \)).

So, having a fixed quadrant and a cube \( \tilde{Q}_{\alpha_0} \) with center in this quadrant, we estimate the number of cubes \( \tilde{Q}_\alpha, \alpha \in \tilde{A} \), whose centers lie inside this quadrant, which contains the cube \( Q_{\alpha_0} \), but which do not contain the cube \( \tilde{Q}_{\alpha_0} \) (because otherwise the cube \( \tilde{Q}_{\alpha_0} \) will be excluded during the construction). There are at most \( n \) such cubes.

Next, we note that the facets of any \( Q \) can be canonically labeled by natural numbers from 1 to \( 2n \). Assume that a cube \( \tilde{Q}_{\alpha'} \supset Q_{\alpha_0} \) does not contain the cube \( \tilde{Q}_{\alpha_0} \) and \( r(Q_{\alpha'}) > r(Q_{\alpha_0}) \). Then, for some \( i \in \{1, \ldots, 2n\} \), the \( i \)th facet of the cube \( \tilde{Q}_{\alpha_0} \), lies in the \( i \)th facet of the cube \( \tilde{Q}_{\alpha'} \), for otherwise we would get the inclusion \( \tilde{Q}_{\alpha_0} \subset \tilde{Q}_{\alpha'} \) (because \( Q_{\alpha_0} \subset \tilde{Q}_{\alpha'} \)), which contradicts the construction. Next, if the \( i \)th facet of the cube \( \tilde{Q}_{\alpha'} \) contains the \( i \)th facet of the cube \( \tilde{Q}_{\alpha_0} \), then there is no other cube \( \tilde{Q}_{\alpha''} \) containing the cube \( Q_{\alpha_0} \) (whose center lies in the quadrant under consideration!) which has such a property. Indeed, if a cube \( \tilde{Q}_{\alpha''} \) (whose \( i \)th facet contains the \( i \)th facet of the cube \( \tilde{Q}_{\alpha_0} \)) has the same diameter as the cube \( \tilde{Q}_{\alpha'} \), then \( \tilde{Q}_{\alpha'} = \tilde{Q}_{\alpha''} \), because their centers lie in the same quadrant and \( \lambda < 1 \) (and hence, we have either \( Q_{\alpha'} \subset \tilde{Q}_{\alpha''} \) or \( Q_{\alpha''} \subset \tilde{Q}_{\alpha'} \)). If,
next, the size of the cube $\tilde{Q}_{\alpha''}$ is smaller than that of the cube $\tilde{Q}_{\alpha'}$, then the $i$th facet of the cube $\tilde{Q}_{\alpha''}$ may not contain the $i$th facet of the cube $\tilde{Q}_{\alpha'}$. Indeed, the cube $Q_{\alpha_0}$ is dyadic, and hence it is either wholly contained in the cube $Q_{\alpha''}$ or is disjoint from it. The first case cannot be realized by the construction. In the second case we either have $Q_{\alpha_0} \subset Q_{\alpha'}$ or $Q_{\alpha_0} \cap Q_{\alpha'} = \emptyset$. The former case is impossible by the construction, and in the latter case the cube $Q_{\alpha_0}$ is not contained in the cube $\tilde{Q}_{\alpha''}$ (because $\lambda < 1$ and since the $i$th facet of the cube $Q_{\alpha_0}$ is contained in the $i$th face of the cube $\tilde{Q}_{\alpha'}$).

Let $j(i)$ be the index corresponding to the facet which is parallel to the $i$th facet (recall that the facet are labeled in the canonical way and that the labeling is the same for each cube). It now remains to note that if the $i$th facet of the cube $\tilde{Q}_{\alpha''}$ lies in the $i$th facet of the cube $\tilde{Q}_{\alpha'}$, the $j$th facet of the cube $\tilde{Q}_{\alpha}$ lies in the $j$th facet of some $\tilde{Q}_{\alpha''} \supset Q_{\alpha_0}$, and besides, $r(Q_{\alpha'}) > r(Q_{\alpha_0})$, then the centers of the cubes $\tilde{Q}_{\alpha''}$ and $\tilde{Q}_{\alpha'}$ may not lie in the same quadrant.

Assertion 3) is clear from the construction of the cubes $\tilde{Q}_{\alpha}$. Indeed, for $\alpha, \alpha' \in \tilde{A}$ the intersection $\tilde{Q}_{\alpha} \cap \tilde{Q}_{\alpha'}$ contains a dyadic cube of side length $2^{-k} \min\{r(Q_{\alpha}), r(Q_{\alpha'})\}$.

**Notations.** We shall frequently use the following notation. Given $k \in \mathbb{Z}_+$, $m \in \mathbb{Z}^n$, we set

$$
\Pi_{k,m} := Q_{k,m} \times (0, r(Q_{k,m})), \quad \tilde{\Pi}_{k,m} := \tilde{Q}_{k,m} \times (0, r(Q_{k,m})),
$$

$$
\hat{\gamma}_{k,m} := \gamma_{\Pi_{k,m}}, \quad \gamma_{k,m} := (r(Q_{k,m}))^{n+1} \hat{\gamma}_{k,m}.
$$

Given a fixed parameter $\lambda = 2^{-k}$, $k \in \mathbb{N}$, and a cube $Q$ in $\mathbb{R}^n$, we set $\tilde{Q} := (1 + \lambda)Q$. If $T = \{Q_{\alpha}\}_{\alpha \in \tilde{A}}$ is a tiling of $\mathbb{R}^n$, then by $\tilde{A}$ we shall denote the index set which was constructed in Lemma 4.1

Assume we are given a system of tilings $T = \{T^s\}$ of the space $\mathbb{R}^n$. Applying Lemma 2.1 for each $s \in \mathbb{Z}_+$ to the tiling $T^s$, we obtain the covering $\Xi^s$ of the space $\mathbb{R}^n$ by cubes $\{\tilde{Q}_{\alpha}^s\}_{\alpha \in \tilde{A}^s}$.

In the cases when we know that $\alpha = (k, m) \in A^s \subset \mathbb{Z}_+ \times \mathbb{Z}^n$, then instead of $\hat{\gamma}_{k,m}, \gamma_{k,m}, \tilde{\Pi}_{k,m}, \Pi_{k,m}$, we shall write, respectively, $\hat{\gamma}_{k,m}^s, \gamma_{k,m}^s, \tilde{\Pi}_{k,m}^s, \Pi_{k,m}^s$.  

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For a function $\varphi \in L^1_{\text{loc}}(\mathbb{R}^n)$ we define

$$\varphi^s_\alpha = \frac{1}{|Q^s_\alpha|} \int_{Q^s_\alpha} \varphi(x) \, dx, \quad s \in \mathbb{Z}_+, \quad \alpha \in \tilde{A}^s.$$ 

By $\tilde{q}$ we shall denote the smallest of $C \geq 1$ for which

$$\frac{1}{8|\Pi_{k,m}|} \int_{8Q_{k,m}} \int_0^{r(Q_{k,m})} \tilde{\gamma}(x, t) \, dt \, dx \leq C \gamma_{k,m'},$$

where $k \in \mathbb{Z}_+, m \in \mathbb{Z}^n$, and $Q_{k,m'} \subset 8Q_{k,m}$.

Let $q := 8\max\{\tilde{q}, C_\gamma\}$. From the above estimate we have

$$\gamma_{k,m} \leq q\gamma_{k,m'}, \quad k \in \mathbb{Z}_+, |m_i - m'_i| \leq 1, i \in \{1, \ldots, n\}.$$ 

**Definition 4.4.** Let $c_1, c_2 \geq 1$. A system of tilings $T = \{T^s\} = \{T^s\}_{s=0}^\infty(c_1, c_2)$ of the space $\mathbb{R}^n$ ($T^s = \{Q^s_\alpha\}_{\alpha \in A^s}$, where $s \in \mathbb{Z}_+$), is called admissible for a weight $\gamma$ if, for each $s \in \mathbb{Z}_+$, the following conditions are satisfied:

1) if $\tilde{Q}^s_\alpha \cap \tilde{Q}^s_{\alpha'} \neq \varnothing$, then $\tilde{\gamma}^s_\alpha \leq c_1 \tilde{\gamma}^s_{\alpha'}$ and $\tilde{\gamma}^s_{\alpha'} \leq c_1 \tilde{\gamma}^s_\alpha$ for $\alpha, \alpha' \in A^s$;

2) if $Q_{\alpha' + 1}^s \subset Q^s_\alpha$, then $\tilde{\gamma}^s_\alpha \leq c_2 \tilde{\gamma}^{s+1}_{\alpha'}$ and $\tilde{\gamma}^{s+1}_{\alpha'} \leq c_2 \tilde{\gamma}^s_\alpha$ for $\alpha, \alpha' \in A^s, \alpha' \in A^{s+1}$;

3) $\max \{r(Q_{\alpha' + 1}^s) : x \in \tilde{Q}^{s+1}_\alpha\} \leq \frac{1}{2} \min \{r(Q^s_\alpha) : x \in \tilde{Q}^s_\alpha\}$ for any $x \in \mathbb{R}^n$;

4) $r(Q^s_\alpha) \geq 2^{-l_0}, \alpha \in A^s$, for some strictly increasing sequence of nonnegative integer numbers $\{l_j\}_{j=0}^\infty$ with $l_0 = 0$.

**Theorem 4.1.** Let a weight $\gamma \in A^1_{\text{loc}}(\mathbb{R}^{n+1})$, $\lambda = 1 + 2^{-k}$ ($k \in \mathbb{N}$). Then there exist constants $c_1(n, C_\gamma), c_2(n, C_\gamma) \geq 1$ such that, for any function $f \in W^1_1(\mathbb{R}^{n+1} + \gamma)$, there exists a system of tilings $T = \{T^s\}(c_1, c_2)$ of the space $\mathbb{R}^n$ that is admissible for the weight $\gamma$ and is such that

$$\sum_{m \in \mathbb{Z}^n} \tilde{\gamma}_{0,m} \varphi_{0,m} + \sum_{s=1}^\infty \sum_{\alpha \in A^s} \tilde{\gamma}^s_\alpha \int_{\tilde{Q}^s_\alpha} |\varphi^s_\alpha - \varphi(x)| \, dx \leq C(n, \gamma, c_1, c_2) ||f||_{W^1_1(\mathbb{R}^{n+1} + \gamma)}.$$

(4.1)

**Proof.** Step 1. For each function $f \in W^1_1(\mathbb{R}^{n+1} + \gamma)$ we construct a required system of tilings of the space $\mathbb{R}^n$ that is admissible for the weight $\gamma$. 

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First we construct an auxiliary system of tilings \( \{ \tilde{T}^s \} \) that will satisfy only properties 1), 2) and 4) of Definition 4.4. Next, for \( r \in \mathbb{N} \), \( r > 1 \) we choose a required subsystem \( \{ T^s \} := \{ \tilde{T}^s \} \) of the system \( \{ T^s \} \).

Let \( \{ l_j \}_{j=1}^{\infty} \) be a strictly increasing sequence of nonnegative integer numbers such that \( l_0 = 0 \) and

\[
\| f \|_{W^1_1(\mathbb{R}^n \times (0, 2^{-l_j+1})))} \leq \frac{1}{2} \| f \|_{W^1_1(\mathbb{R}^n \times (0, 2^{-l_j})))}, \quad j \in \mathbb{Z}_+. \tag{4.2}
\]

We construct the required system of tilings \( \{ T^k \} \) by induction.

**Induction basis.** We first build a tiling \( \tilde{T}^0 \). To do so we put \( \tilde{T}^0 := \{ Q^0_{0,m} \}_{m \in \mathbb{Z}^n} \), \( \tilde{A}^0 := \{ 0 \} \times \mathbb{Z}^n \), and for each \( m \in \mathbb{Z}^n \) we paint the cube \( Q^0_{0,m} \) yellow.

**Induction step.** Assume that for \( s \in \mathbb{Z}_+ \) the tiling \( \tilde{T}^s = \{ Q^s_{\alpha} \}_{\alpha \in \tilde{A}^s} \) is constructed. Let us construct the tiling \( \tilde{T}^{s+1} \). We fix a cube \( \tilde{Q}^s_{\alpha} \) for \( \alpha \in \tilde{A}^s \). Suppose that \( \int \int_{U^s_{\alpha}} \gamma(x,t) \, dx \, dt \in [q^j, q^{j+1}) \) for some \( j \in \mathbb{Z} \). We decompose the cube \( \tilde{Q}^s_{\alpha} \) into dyadic cubes \( \{ \tilde{Q}^s_{k,m} \} \), select those for which \( \tilde{\gamma}_{k,m} > q^{j+1} \) and paint these cubes blue. We decompose the remaining cubes into the cubes \( \tilde{Q}^s_{k,m'} \), select those for which \( \tilde{\gamma}_{k+1,m'} > q^{j+1} \) and paint these cubes blue. This process is repeated until the side length of a cube will be \( 2^{-l_{s+1}} \). In this case we either have a tiling of the cube \( \tilde{Q}^s_{\alpha} \) consisting of only blue cubes or there will be cubes \( \tilde{Q}^s_{l_{s+1},m''} \subset \tilde{Q}^s_{\alpha} \) for which \( \tilde{\gamma}_{l_{s+1},m''} \leq q^{j+1} \). In the latter case, we paint these cubes \( \tilde{Q}^s_{l_{s+1},m''} \) yellow. The resulting tiling of the cube \( \tilde{Q}^s_{\alpha} \) will be composed of the so-chosen blue cubes and the remaining yellow cubes. Combining the corresponding tilings of the cubes \( \tilde{Q}^s_{\alpha} \) over all \( \alpha \in \tilde{A}^s \), we obtain the tiling \( \tilde{T}^{s+1} \) of the space \( \mathbb{R}^n \). By \( \tilde{A}^{s+1} \) we shall denote the set of pairs of indices \( (k, m) \in \mathbb{Z}_+ \times \mathbb{Z}^n \) for which \( \tilde{Q}_{k,m} \in \tilde{T}^{s+1} \).

Clearly, for each \( s \in \mathbb{Z}_+ \), the tiling \( \tilde{T}^s \) is composed of at most countable set of dyadic cubes.

If we apply Lemma 2.1 for each \( s \in \mathbb{N} \) to the tiling \( \tilde{T}^s \), we obtain a covering \( \hat{\Xi}^s \) of the space \( \mathbb{R}^n \) by cubes \( \{ \tilde{Q}^s_{\alpha} \}_{\alpha \in \tilde{A}^s} \).

We next check that the system of tilings \( \{ \tilde{T}^s \} \) satisfies conditions 1), 2) and 4) of Definition 4.4 (in which the index sets \( A^s \) should be replaced by \( \tilde{A}^s \)).
Condition 4) is easily seen to hold.

We claim that Condition 1) of Definition 4.4 is satisfied with constant $c_1 = q^3$. Let $Q^s_\alpha \cap \overline{Q}^s_{\alpha'} \neq \emptyset$ for $\alpha, \alpha' \in \hat{A}^s$. Assume that $\hat{\gamma}_s^{\alpha'} > q^3 \gamma_\alpha^{s+1}$. For any cube $\overline{Q}^s_\alpha$, we let $b(Q^s_{\alpha})$ denote the number of blue cubes $Q^j_{\alpha'} \supset Q^s_\alpha$ (for $j \leq s$ and $\alpha' \in \hat{A}^j$). From our assumption it follows that there exists a natural $k_0 > 1$ such that the number of blue cubes containing the cube $\overline{Q}^s_\alpha$ is greater by $k_0$ than the number of blue cubes containing the cube $\overline{Q}^s_{\alpha'}$. But then there exist a blue cube $\overline{Q}^{s_0}_{\alpha_0} \supset Q^s_{\alpha'}$ and a yellow cube $\overline{Q}^{s_0}_{\alpha_0} \supset Q^s_\alpha$ such that $\hat{\gamma}_{s_0}^{\alpha_0} \geq q^{k_0-1}\gamma_{s_0}^{\alpha_0}$. By the construction, $r(Q^{s_0}_{\alpha_0}) \leq r(Q^{s_0}_{\alpha_0})$. Besides, $\overline{Q}^{s_0}_{\alpha_0} \cap \overline{Q}^{s_0}_{\alpha_0} \neq \emptyset$ by $Q^s_\alpha \cap \overline{Q}^s_{\alpha'} \neq \emptyset$. It follows that $\overline{Q}^{s_0}_{\alpha_0} \subset S\overline{Q}^{s_0}_{\alpha_0}$, and hence, $\hat{\gamma}_{s_0}^{\alpha_0} \geq \frac{3}{q} \overline{\gamma}_{s_0}^{\alpha_0}$. A contradiction is reached.

We now check condition 2). Let $Q^s_\alpha$ be the parent of the cube $Q^{s+1}_{\alpha'}$. By the construction of the system of tilings, we have $\hat{\gamma}_s^{\alpha'} \leq \hat{\gamma}_{s'}^{\alpha' \prime}$ and $\hat{\gamma}_{s+1}^{\alpha'} \leq q^3 \gamma_\alpha^{s+1}$. Let $r \in \mathbb{N}$, $r \geq 5$. Consider the system of tilings $\{T^s\} := \{T^r\}$ and define $A^s := \hat{A}^r$. Clearly, the system of tilings $\{T^s\}$ satisfies conditions 1), 2) (with the constants $c_1 = q^3$, $c_2 = q^r$) and 4) of Definition 4.4.

Let us check condition 3) of Definition 4.4. To this aim we fix a point $x \in \mathbb{R}^n$. Let $\overline{Q}^{r(s+1)}_\alpha \ni x$ be a cube with largest side length among the set of cubes $\{\overline{Q}^{r(s+1)}_\alpha\}_{\alpha \in A^{s+1}}$ that contain the point $x$ (this cube may not be unique). Let $\overline{Q}^{r(s+1)}_\alpha \ni x$ be a cube of smallest side length among all cubes from the family $\{\overline{Q}^{r(s+1)}_\alpha\}_{\alpha \in A^{s+1}}$, of which each contains the point $x$ (the cube $\overline{Q}^{r(s+1)}_\alpha \ni x$ may also be not unique). Consider the following chain of nested dyadic cubes $\overline{Q}^{r(s+1)}_\alpha \supset \ldots \supset \overline{Q}^{r(s+1)}_{\alpha''}$ (in this chain each succeeding dyadic cube is a unique parent of its predecessor). If this chain contains at least one yellow cube, then we have the result required. Suppose now that all cubes in this chain are blue. Assume that $\overline{Q}^{r(s+1)}_\alpha \cap \overline{Q}^{r(s+1)}_{\alpha'} \neq \emptyset$ and $r(Q^{r(s+1)}_\alpha) \geq \frac{1}{2}r(Q^{r(s+1)}_{\alpha'})$ for $\alpha' \in A^s$, $\alpha \in A^{s+1}$. Then $Q^{r(s+1)}_{\alpha'} \subset S\overline{Q}^{r(s+1)}_{\alpha'}$, and hence, $\hat{\gamma}_{r(s+1)}^{\alpha'} \geq \frac{2}{q} \gamma_{r(s+1)}^{\alpha'}$. On the other hand, by condition 1) of Definition 4.4 (as was pointed out above, this condition is satisfied with $c_1 = q^3$) and since all the cubes in the chain $\overline{Q}^{r(s+1)}_\alpha \supset \ldots \supset \overline{Q}^{r(s+1)}_{\alpha''}$ are blue and $r \geq 5$, we have the estimate $\hat{\gamma}_{r(s+1)}^{\alpha'} \geq q^3 \gamma_\alpha^{s+1}$. This contradiction completes the verification of condition 3).

Step 2. We claim that estimate (4.1) holds. Arguing as in the proof of
Lemma 3.1 of [1], we see that
\[
\int_{Q^s_\alpha} |\varphi^s_\alpha - \varphi(x)| \, dx \leq \frac{1}{|Q^s_\alpha|} \int_{Q^s_\alpha} \int |\varphi(x) - \varphi(y)| \, dx \, dy \leq \iint_{\tilde{\Pi}_\alpha^s} |\nabla f(x, t)| \, dtdx, \quad s \in \mathbb{Z}_+, \ \alpha \in \tilde{A}^s.
\]

From (4.3) we see at once that
\[
\sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}^s_\alpha \int \frac{|\varphi^s_\alpha - \varphi(x)|}{Q^s_\alpha} \, dx \leq \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}^s_\alpha \iint_{\tilde{\Pi}_\alpha^s} |\nabla f(x, t)| \, dtdx \leq \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}^s_\alpha \int \int |\nabla f(x, t)| \, dtdx + \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}^s_\alpha \iint_{\tilde{\Pi}_\alpha^s} |\nabla f(x, t)| \, dtdx = S_1 + S_2.
\]

(4.4)

The sum $S_1$ is easily estimated by (4.2). Using the finite (independent of $j$ and $m$) overlapping multiplicity of the sets $\tilde{\Pi}_{j,m}$ (when index $j$ is fixed and $m$ is variable) and Lemma 2.1, we arrive at the estimate
\[
S_1 \leq C \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \min_{(x,t) \in \tilde{\Pi}_\alpha^s} \gamma(x,t) \iint_{\tilde{\Pi}_\alpha^s} |\nabla f(x, t)| \, dtdx \leq C \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}_+} \int \int |\gamma(x,t)\nabla f(x,t)| \, dtdx \leq \sum_{j=0}^{\infty} \|f\|_{W_1^1(\mathbb{R}^{n+1}, \gamma)} \leq C \|f\|_{W_1^1(\mathbb{R}^{n+1}, \gamma)}.
\]

(4.5)

We note that the constant $C > 0$ on the right of (4.5) depends only on $\lambda, n, C_\gamma$.

Given $s \in \mathbb{Z}_+, \alpha \in \tilde{A}^s$, we set $G^s_\alpha := \tilde{\Pi}_\alpha^s \setminus \bigcup_{\alpha' \in \tilde{A}^{s+1}} \tilde{\Pi}_{\alpha'}^s$.

The sum $S_2$ is estimated from above as follows
\[
S_2 \leq \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \left( \sum_{j=0}^{s} \sum_{\alpha' \in \tilde{A}^j} \tilde{\gamma}^j_{\alpha'} \|f\|_{W_1^1(G^s_\alpha \cap \tilde{\Pi}_{\alpha'}^j)} \right). \tag{4.6}
\]
where the summation on the right of \((4.6)\) is carried only over those indexes \(\alpha' \in \tilde{A}^j\) for which the cube \(\overline{Q}_{\alpha'}^j\) is blue.

To continue estimating \((4.6)\) we need the following important observation. We fix indexes \(s \in \mathbb{Z}_+\) and \(\alpha \in \tilde{A}^s\). Given \(j \in \{0, \ldots, s\}\), we let \(\overline{Q}_{\beta_j(\alpha)}^j\) denote the unique dyadic cube from the tiling \(T^j\) that contains the cube \(Q_{\alpha}^s\).

We next use the fact that the system of tilings \(T = \{T^s\}\) is admissible for the weight \(\gamma\) (assertion 1)), apply assertion 2) of Lemma \ref{lemma.4.1} and finally employ Lemma \ref{lemma.2.1} (For each fixed \(j\), the overlapping multiplicity of the sets \(\Pi_{\alpha'}^j\) is finite and independent of \(j\) and \(\alpha'\)). We have

\[
\sum_{j=0}^{s} \sum_{\alpha' \in \tilde{A}^j, \overline{Q}_{\alpha'}^j \cap \overline{Q}_{\alpha'}^j \neq \emptyset} \tilde{\gamma}_{j,\alpha'}^j \|f|W_1^1(G_{1,\alpha}^s \cap \Pi_{\alpha'}^j)\| \leq c_1 \sum_{j=0}^{s} \tilde{\gamma}_{j,\beta_j(\alpha)}^j \sum_{\alpha' \in \tilde{A}^j, \overline{Q}_{\alpha'}^j \cap \overline{Q}_{\alpha'}^j \neq \emptyset} \|f|W_1^1(G_{\alpha}^s \cap \Pi_{\alpha'}^j)\| \leq C(c_1, n) \sum_{j=0}^{s} \tilde{\gamma}_{j,\beta_j(\alpha)}^j \|f|W_1^1(G_{\alpha}^s)\|. \tag{4.7}
\]

We next partition the index set \(\{0, \ldots, s\}\) into two disjoint sets: \(\{0, \ldots, s\} = 1\Gamma_{\alpha}^s \cup 2\Gamma_{\alpha}^s\), where

\(1\Gamma_{\alpha}^s := \{j = 0, \ldots, s \mid \overline{Q}_{\beta_j(\alpha)}^j\) is blue\}, \(2\Gamma_{\alpha}^s := \{j = 0, \ldots, s \mid \overline{Q}_{\beta_j(\alpha)}^j\) is yellow\}.

We continue with estimate \((4.6)\). Using \((4.7)\), we have

\[
S_2 \leq \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s, \overline{Q}_{\alpha}^s \text{ is blue}} \left( \sum_{j \in 1\Gamma_{\alpha}^s} \tilde{\gamma}_{j,\beta_j(\alpha)}^j \|f|W_1^1(G_{\alpha}^s)\| \right) + \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s, \overline{Q}_{\alpha}^s \text{ is blue}} \left( \sum_{j \in 2\Gamma_{\alpha}^s} \tilde{\gamma}_{j,\beta_j(\alpha)}^j \|f|W_1^1(G_{\alpha}^s)\| \right) =: S_{2,1} + S_{2,2}. \tag{4.8}
\]

The following estimate is clear from the construction of the blue cubes:

\[
\sum_{j \in 1\Gamma_{\alpha}^s} \tilde{\gamma}_{j,\beta_j(\alpha)}^j \leq C(c_2) \tilde{\gamma}_{\alpha}^s. \tag{4.9}
\]

From \((4.9)\), using Lemma \ref{lemma.2.1} (here we use the finite overlapping multiplicity
of the sets $G^s_\alpha$, which is independent of $s$ and $\alpha$) and (2.4), (2.5), we get

$$S_{2,1} \leq C \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}} \sum_{Q_s^\alpha \text{ is blue}} \gamma^\alpha_s \|f|W^1_1(G^s_\alpha)\| \leq C \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}} \sum_{Q_s^\alpha \text{ is blue}} \|f|W^1_1(G^s_\alpha, \gamma)\| \leq C \|f|\mathbb{R}^{n+1}, \gamma\|. \quad (4.10)$$

Given $i \in \mathbb{Z}_+, m \in \mathbb{Z}^n$, we set $E_{l_i,m} := \Pi_{l_i,m} \setminus \bigcup_{m' \in \mathbb{Z}^n} \Pi_{l_i+1,m'}$.

To estimate $S_{2,2}$ we shall require the following key observation. We fix indexes $s \in \mathbb{Z}_+$ and $\alpha \in \tilde{A}$. Let $E_{l_i,m} \cap G^s_\alpha \neq \emptyset$ for some $i \in \mathbb{Z}_+, m \in \mathbb{Z}^n$. By elementary geometric considerations we see that $r(Q_{l_i,m}) \leq r(Q_{j_0}(\alpha))$, where $j_0 = \max\{j|j \in \mathbb{Z} \Gamma^s_\alpha\}$. Moreover, if the cube $Q^s_\alpha \subset Q^j_{j_0}(\alpha)$ with $j \in \mathbb{Z} \Gamma^s_\alpha$, then the cube $Q_{l_i,m} \subset Q^j_{j_0}(\alpha)$. Here, the dyadic cube $Q^j_{j_0}(\alpha)$ has common boundary points with the cube $Q^j_{j_0}(\alpha)$, and besides $r(Q^j_{j_0}(\alpha)) = r(Q^j_{j_0}(\alpha))$. But this in combination with Remark 3.1 implies that

$$\sum_{j \in \mathbb{Z} \Gamma^s_\alpha} \tilde{\gamma}^j_{j_0}(\alpha) \leq C(C_{\gamma}, n) \sum_{j=0}^{i} \sum_{m' \in \mathbb{Z}^n} \sum_{Q_{l_i,m} \subset Q_{l_j,m'}} \tilde{\gamma}_{l_j,m'} =: Ct_{l_i,m}. \quad (4.11)$$

Using (4.11) and Lemma 2.1 (here we use the finite overlapping multiplicity of the sets $G^s_\alpha$, which is independent of $s$ and $\alpha$), we change the order of summation, take into account the equality $\Pi_{l_j,m'} = \bigcup_{(i,m)} E_{l_i,m} \subset Q_{l_i,m} \subset Q_{l_j,m'}$, estimate (2.1) and estimate (4.2). As a result, we have

$$S_{2,2} \leq C \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}} \sum_{i=0}^{\infty} \sum_{m \in \mathbb{Z}^n} t_{l_i,m} \|f|W^1_1(G^s_\alpha \cap E_{l_i,m})\| \leq C \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}} \sum_{i=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \sum_{Q_{l_i,m} \subset Q_{l_j,m'}} \tilde{\gamma}_{l_j,m'} \|f|W^1_1(E_{l_i,m})\| \leq C$$
\[
\leq \sum_{j=0}^{\infty} \sum_{m' \in \mathbb{Z}^n} \tilde{\gamma}_{j, m'} \| f \|_{W^1_1(\Omega_{j, m'})} \leq \\
\leq C \sum_{j=0}^{\infty} \sum_{m' \in \mathbb{Z}^n} \| f \|_{W^1_1(\Omega_{j, m'}, \gamma)} \| \leq C \sum_{j=0}^{\infty} \| f \|_{W^1_1(\mathbb{R}^n \times (0, 2^{-j}), \gamma)} \| \leq \\
\leq C \| f \|_{W^1_1(\mathbb{R}^{n+1}, \gamma)}.
\]

Combining estimates (4.8), (4.10), (4.12), we find that

\[
S_2 \leq C \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}^s_{\alpha} \| f \|_{W^1_1(G^s_{\alpha})} \leq C \| f \|_{W^1_1(\mathbb{R}^{n+1}, \gamma)}.
\]  

Now estimate (4.1) follows from (3.2), (4.4), (4.5), (4.13). This completes the proof of the theorem.

For further purposes we shall require a special partition of unity on \( \mathbb{R}^n \times (0, 1) \). Let \( T = (T^*)^\infty_{s=0}(c_1, c_2) \) be a system of tilings of the space \( \mathbb{R}^n \) that is admissible for the weight \( \gamma \). Given \( s \in \mathbb{Z}_+, \alpha \in \tilde{A}^s \), assume that a function \( \theta^s_{\alpha} \in C^\infty(\mathbb{R}^n) \) is such that \( \theta^s_{\alpha}(x) \in (0, 1] \) for \( x \in Q^s_{\alpha} \), \( \theta^s_{\alpha}(x) = 0 \) for \( x \in \mathbb{R}^n \setminus Q^s_{\alpha} \), and \( |\nabla \theta^s_{\alpha}(x)| \leq \frac{C_\gamma}{r(Q^s_{\alpha})} \) for \( x \in \mathbb{R}^n \) with constant \( C_\gamma > 0 \) independent both of \( s \) and \( \alpha \). We also assume that \( \sum_{\alpha \in \tilde{A}^s} \theta^s_{\alpha} \equiv 1 \) on \( \mathbb{R}^n \).

Given \( s \in \mathbb{Z}_+, \alpha \in \tilde{A}^s \), assume that a function \( \psi^s_{\alpha} \in C^\infty((0, \infty)) \) is such that \( \psi^s_{\alpha}(t) = 1 \) for \( t \in (0, r(Q^s_{\alpha})) \), \( \psi^s_{\alpha}(x) \in (0, 1] \) for \( t \in (r(Q^s_{\alpha}), \frac{3}{2}r(Q^s_{\alpha})) \), \( \psi^s_{\alpha}(t) = 0 \) for \( t > \frac{3}{2}r(Q^s_{\alpha}) \), and \( \left| \frac{d\psi^s_{\alpha}}{dt}(t) \right| \leq \frac{C_\gamma}{r(Q^s_{\alpha})} \) for \( t > 0 \) with constant \( C_\gamma > 0 \) independent both of \( s \) and \( \alpha \).

Next, for \( s \in \mathbb{Z}_+, \alpha \in \tilde{A}^s \), we set

\[
g^s(x, t) := \sum_{\alpha \in \tilde{A}^s} \theta^s_{\alpha}(x) \psi^s_{\alpha}(t) - \sum_{\alpha' \in \tilde{A}^{s+1}} \theta^{s+1}_{\alpha'}(x) \psi^{s+1}_{\alpha'}(t), \quad (x, t) \in \mathbb{R}^{n+1}
\]

\[
g^s_{\alpha}(x, t) := \theta^s_{\alpha}(x) \psi^s_{\alpha}(t)(g^s(x, t) - g^{s+1}(x, t)), \quad (x, t) \in \mathbb{R}^{n+1}.
\]

Let us establish some elementary properties of the functions \( g^s_{\alpha}, s \in \mathbb{Z}_+, \alpha \in \tilde{A}^s \).

**Lemma 4.2.** Assume that \( g^s_{\alpha} \) is defined by (4.14). Then

1) \( \text{supp } g^s_{\alpha} \subset \overline{Q^s_{\alpha}} \times (0, \frac{3}{2}r(Q^s_{\alpha})) \).
2) \( \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} g^s(x, t) = 1 \) for \((x, t) \in \mathbb{R}^n \times (0, 1)\),

3) if \( s_1, s_2 \in \mathbb{Z}_+ \), \( |s_1 - s_2| \leq 1 \), \( \alpha_1 \in \tilde{A}^{s_1} \), \( \alpha_2 \in \tilde{A}^{s_2} \), then \( |\text{supp} \ g^{s_1}_\alpha \cap \text{supp} \ g^{s_2}_\alpha| \approx (\min\{r(Q^{s_1}_\alpha), r(Q^{s_2}_\alpha)\})^{n+1} \), provided that \( \tilde{Q}^{s_1}_\alpha \cap \tilde{Q}^{s_2}_\alpha \neq \emptyset \).

4) for any point \((x, t) \in \mathbb{R}^{n+1}_+\) the number of indexes \((s, \alpha)\) for which \( g^s(x, t) > 0 \) is at most \((n + 1)2^{n+1}\).

**Proof.** Assertions 1) and 2) are direct consequences of (4.14).

Let us prove 3). We first note that, for \( s \in \mathbb{Z}_+ \), \( \alpha' \in \tilde{A}^{s+1} \), we have \( g^{s+1}_\alpha(x, t) \in (0, 1) \) for \((x, t) \in \tilde{Q}^{s+1}_\alpha \times (r(Q^{s+1}_\alpha), \frac{3}{2} r(Q^{s+1}_\alpha))\). Hence, for \( \alpha' \in \tilde{A}^{s+1} \), it follows by the properties of the functions \( \theta^{s+1}_\alpha \) that

\[
\sum_{\alpha'' \in \tilde{A}^{s+1}} g^{s+1}_{\alpha''}(x, t) < 1, \quad (x, t) \in \tilde{Q}^{s+1}_{\alpha'} \times (r(Q^{s+1}_{\alpha'}), \frac{3}{2} r(Q^{s+1}_{\alpha'})).
\]

Hence, for \( s \in \mathbb{Z}_+ \), \( \alpha \in \tilde{A}^s \), \( \alpha' \in \tilde{A}^{s+1} \),

\[
\text{supp} \ g^s_\alpha \cap \text{supp} \ g^{s+1}_{\alpha'} \supset (\tilde{Q}^{s+1}_\alpha \cap \tilde{Q}^s_{\alpha'}) \times (r(Q^{s+1}_{\alpha'}), \frac{3}{2} r(Q^{s+1}_{\alpha'})).
\]

A similar analysis shows that

\[
\text{supp} \ g^s_\alpha \cap \text{supp} \ g^{s}_{\alpha'} \supset (\tilde{Q}^{s+1}_\alpha \cap \tilde{Q}^s_{\alpha'}) \times \left( \min\{r(Q^{s+1}_{\alpha'}), r(Q^s_\alpha)\}, \frac{3}{2} \min\{r(Q^{s+1}_{\alpha'}), r(Q^s_\alpha)\} \right)
\]

From property 3) of Definition 4.4 it clearly follows that the interior of each support of \( g^s_\alpha \) is nonempty, and moreover, \( \bigcup_{\alpha \in \tilde{A}^{s+1}} \text{supp} \ \theta^{s+1}_\alpha \psi^{s+1}_\alpha \subset \bigcup_{\alpha \in \tilde{A}^s} \text{supp} \ \theta^s_\alpha \psi^s_\alpha \) (this inclusion is strict!).

Let us prove assertion 4). From property 3) of Definition 4.4 it follows that \( \text{supp} \ g^s_\alpha \cap \text{supp} \ g^{s+2}_{\alpha'} = \emptyset \) for \( s \in \mathbb{Z}_+ \), \( \alpha \in \tilde{A}^s \), \( \alpha' \in \tilde{A}^{s+2} \). Now the required fact follows from assertion 2) of Lemma 4.4.

**Theorem 4.2.** Let a weight \( \gamma \in A^{1,\text{loc}}_1(\mathbb{R}^{n+1}) \), \( c_1, c_2 \geq 1 \). Assume that for a function \( \varphi \in L^{1,\text{loc}}_1(\mathbb{R}^n) \) there exists a system of tilings \( T = \{T^s\}(c_1, c_2) \) admissible for the weight \( \gamma \) such that

\[
\sum_{m \in \mathbb{Z}^n} \tilde{\gamma}_{0,m} \varphi_{0,m} + \sum_{s=1}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}^s_\alpha \int_{\tilde{Q}^s_{\alpha}} |\varphi^s_\alpha - \varphi(x)| \, dx < \infty
\]
Then there exists a function $f \in W^1_1(\mathbb{R}^{n+1}_+, \gamma)$ such that $\varphi = \text{tr}|_{t=0} f$, and moreover,

$$C\|f|W^1_1(\mathbb{R}^{n+1}_+, \gamma)\| \leq \sum_{m \in \mathbb{Z}^n} \tilde{\gamma}_{0,m}\|\varphi|L_1(Q_{0,m})\| + \sum_{s=1}^{\infty} \sum_{\alpha \in \tilde{A}^s} \tilde{\gamma}_\alpha \int_{\tilde{Q}_\alpha} |\varphi_{\alpha}^s - \varphi(x)| \, dx. \tag{4.15}$$

The constant $C > 0$ on the left of (4.15) depends only on $n, C_\gamma, c_1, c_2, C_\theta, C_\psi$.

**Proof.** Step 1. Given $(x, t) \in \mathbb{R}^{n+1}_+$, we set

$$f(x, t) = \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} g_{s,\alpha}^{\ast}(x, t)\varphi_{\alpha}^s.$$ \tag{4.16}

Note that the function $f \in C^\infty(\mathbb{R}^{n+1}_+)$. We claim that (4.15) holds.

To this aim we first estimate the integral

$$J := \int_0^1 \int_{\mathbb{R}^{n+1}_+} \gamma(x, t)|\nabla f(x, t)| \, dx \, dt = \int_0^1 \int_{\mathbb{R}^n} \gamma(x, t)|\nabla f(x, t)| \, dx \, dt + \int_0^1 \int_{\mathbb{R}^n} \gamma(x, t)|\nabla f(x, t)| \, dx \, dt =: J_1 + J_2.$$

From condition 3) of Definition 4.4 we have

$$|\nabla f(x, t)| \leq C \sum_{\alpha \in \tilde{A}^0} \varphi_{\alpha}^0 \chi_{\tilde{Q}_\alpha}(x), \quad x \in \mathbb{R}^n, \quad t \geq 1.$$

The cubes $\tilde{Q}_{0,m}$ have finite (independent of $m$) overlapping multiplicity, and hence by (2.3) we have

$$J_2 \leq C \sum_{m \in \mathbb{Z}^n} \tilde{\gamma}_{0,m} \left( \sum_{m' \in \mathbb{Z}^n} \varphi_{0,m'} \right) \leq C \sum_{m \in \mathbb{Z}^n} \varphi_{0,m} \tilde{\gamma}_{0,m}. \tag{4.17}$$

Clearly, the constant $C$ in (4.17) depends only on $n, C_\gamma, C_\theta, C_\psi$.

Now let us estimate the more difficult integral $J_1$. Using assertion 2) of Lemma 4.2 we clearly have

$$J_1 \leq \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \int_0^1 \int_{\mathbb{R}^n \times (0,1)} \gamma(x, t)|\nabla f(x, t)| \, dx \, dt \tag{4.18}.$$
Given a fixed index \( s_0 \in \mathbb{Z}_+ \) and \( \alpha_0 \in \tilde{A}^{s_0} \), we use Lemma 4.2 (assertions 1, 2, 4)) and recall that the system of tilings \( T \) is admissible (condition 3) of Definition 4.4. We have (if \( s_0 = 0 \) we set formally \( s_0 - 1 = 0 \))

\[
\int_{\text{supp } g_{s_0}^{\alpha_0} \cap \mathbb{R}^n \times (0,1)} \gamma(x,t) |\nabla f(x,t)| \, dx \, dt = \int_{\text{supp } g_{s_0}^{\alpha_0}} \gamma(x,t) \left| \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \nabla g_{s}^\alpha(x,t) \varphi_{\alpha}^s \right| \, dx \, dt =
\]

\[
\int_{\text{supp } g_{s_0}^{\alpha_0}} \gamma(x,t) \left( \sum_{s=0}^{\infty} \sum_{\alpha \in \tilde{A}^s} \nabla g_{s}^\alpha(x,y)(\varphi_{\alpha}^s - \varphi_{\alpha_0}^s) \right) \, dx \, dt \leq
\]

\[
\sum_{s=s_0-1}^{s_0+1} \sum_{\alpha \in \tilde{A}^s} \left( \int_{\text{supp } g_{s_0}^{\alpha_0} \cap \text{supp } g_{s}^\alpha} |\nabla g_{s}^\alpha(x,t)| \gamma(x,t) \, dx \, dt \right) |\varphi_{\alpha}^s - \varphi_{\alpha_0}^s|.
\]

(4.19)

The main crux now is to estimate \( |\nabla g_{s}^\alpha(x,t)| \) on the set \( \text{supp } g_{s_0}^{\alpha_0} \cap \text{supp } g_{s_0}^{s_0} \).

Consider the cubes \( Q_{s_0}^{s_0} \) with \( s' \in \{s_0 - 1, s_0, s_0 + 1\} \), \( \alpha' \in \tilde{A}^{s'} \), for which \( \tilde{Q}_{s_0}^{s_0} \cap \tilde{Q}_s^{s_0} \cap \tilde{Q}_{s'}^{s_0} \neq \emptyset \). For indexes \((s', \alpha')\) corresponding to these cubes, we consider the sets

\[
F_{s_0}^{s'} := \text{supp } g_{s_0}^{\alpha_0} \cap \text{supp } g_{s_0}^{s_0} \cap \left( \tilde{Q}_{s_0}^{s_0} \times (0, \frac{3}{2} r(Q_{s_0}^{s_0})) \right) \cup_{r(Q_{s_0}^{s_0})} \left( \tilde{Q}_{s_0}^{s_0} \times (0, \frac{3}{2} r(Q_{s_0}^{s_0})) \right).
\]

It is clear that

\[
\bigcup F_{s_0}^{s'} = \text{supp } g_{s_0}^{\alpha_0} \cap \text{supp } g_{s_0}^{s_0},
\]

(4.20)

where the union on the left of (4.20) is taken only over those \((s', \alpha')\) for which \( F_{s_0}^{s'} \neq \emptyset \).

We now made two key observations.

The first observation is that \( r(Q_{s_0}^{s_0}) \leq \min\{r(Q_{s_0}^{s_0}), r(Q_{s_0}^{s_0})\} \), provided that \( F_{s_0}^{s'} \neq \emptyset \). Hence, for the indexes \((s', \alpha')\) corresponding to nonempty sets \( F_{s_0}^{s'} \), we have

\[
|\tilde{Q}_{s_0}^{s_0} \cap \tilde{Q}_{s_0}^{s_0} \cap \tilde{Q}_{s_0}^{s_0}| \geq C(n, \lambda)|Q_{s_0}^{s_0}|,
\]

(4.21)

The second key observation is as follows:

\[
|\nabla g_{s_0}^{\alpha}(x,t)| \leq \frac{C}{r(Q_{s_0}^{s_0})}, \quad (x,t) \in F_{s_0}^{s'},
\]

(4.22)
where the constant $C > 0$ depends only on $n$, $C_\psi$, $C_\theta$.

Using the inclusion $F_{s' \alpha'} \subset 4Q_{s' \alpha'} \times (0, 4r(Q_{s' \alpha'}))$ and \(2.5\), it is easily follows that
\[
\int F_{s' \alpha'} \gamma(x, t) \, dx \, dt \leq C \gamma_{s' \alpha'}.
\]
Hence, employing \(1.21\), \(1.22\), and since the tiling $T$ is admissible for the weight $\gamma$ (conditions 1), 2) of Definition 4.4, we have the estimate
\[
|\varphi^s_\alpha - \varphi^{s_0}_{\alpha_0}| \int_{F_{s' \alpha'}} |\nabla g^s_\alpha(x, t)| \gamma(x, t) \, dx \, dt \leq \frac{C|\varphi^s_\alpha - \varphi^{s_0}_{\alpha_0}|}{r(Q_{s' \alpha'})} \gamma_{s' \alpha'} \leq \]
\[
\leq C \gamma_{s' \alpha'} \int_{\tilde{Q}_{s' \alpha'}} \varphi^s_\alpha - \varphi^{s_0}_{\alpha_0} \, dx \leq \]
\[
\leq C \gamma_{s_0 \alpha_0} \int_{\tilde{Q}_{s_0 \alpha_0}} \varphi(x) - \varphi^{s_0}_{\alpha_0} \, dx + C \gamma_{s_0 \alpha_0} \int_{\tilde{Q}_{s_0 \alpha_0}} \varphi(x) - \varphi^{s_0}_{\alpha_0} \, dx. \quad (4.23)
\]

Summing estimate \(4.23\) for all indexes $(s', \alpha')$ for which $\tilde{Q}_{s_0 \alpha_0} \cap \tilde{Q}_{s' \alpha'} \cap \tilde{Q}_{s' \alpha'} \neq \emptyset$, taking into account the finite (depending only on $n$) overlapping multiplicity of the cubes $\tilde{Q}_{s' \alpha'}$, it follows by Lemma 2.1 that
\[
\left( \int_{\supp g_{s_0}^\alpha} \int_{\supp g_{s_0}^\alpha} |\nabla g^s_\alpha(x, t)| \gamma(x, t) \, dx \, dt \right) |\varphi^s_\alpha - \varphi^{s_0}_{\alpha_0}| \leq \]
\[
\leq C \gamma_{s_0 \alpha_0} \int_{\tilde{Q}_{s_0 \alpha_0}} |\varphi(x) - \varphi^{s_0}_{\alpha_0}| \, dx + C \gamma_{s_0 \alpha_0} \int_{\tilde{Q}_{s_0 \alpha_0}} |\varphi(x) - \varphi^{s_0}_{\alpha_0}| \, dx. \quad (4.24)
\]

Combining estimates \(4.19\), \(4.24\), we have, for $s_0 \in \mathbb{N}$, $\alpha \in \tilde{A}_{s_0}$,
\[
\int_{\supp g_{s_0}^\alpha} \gamma(x, t) |\nabla f(x, t)| \, dx \, dt \leq C \sum_{s=s_0-1}^{s_0+1} \sum_{\alpha \in \tilde{A}_{s_0}} \tilde{\gamma}_s^\alpha \int_{\tilde{Q}_{s_0}^\alpha \cap \tilde{Q}_{s_0}^\alpha} |\varphi(x) - \varphi^{s_0}_{\alpha_0}| \, dx + \]
\[
+ C \gamma_{s_0 \alpha_0} \int_{\tilde{Q}_{s_0}^\alpha \cap \tilde{Q}_{s_0}^\alpha} |\varphi(x) - \varphi^{s_0}_{\alpha_0}| \, dx. \quad (4.25)
\]

Summing estimate \(4.25\) over all indexes $s_0$, $\alpha_0$, taking into account assertion 4) of Lemma 4.2 and employing Lemma 2.1 (with $d = n$), we finally have
\[ J_1 \leq C(c_1, c_2, C_\theta, C_\psi, n) \left( \sum_{s=1}^{\infty} \sum_{\alpha \in A_s} \hat{\gamma}_s^\alpha \int_{\tilde{Q}_s^\alpha} |\varphi_s^\alpha - \varphi(x)| \, dx + \sum_{m \in \mathbb{Z}^n} \hat{\gamma}_{0,m} \varphi_{0,m,0} \right). \]  

(4.26)

Arguing as in the estimate 3.11 of [1] we have

\[ \iint_{\mathbb{R}^{n+1}} \gamma(x, t) |f(x, t)| \, dx \, dt \leq C \iint_{\mathbb{R}^{n+1}} \gamma(x, t) |\nabla f(x, t)| \, dx \, dt. \]  

(4.27)

Now (4.15) follows from (4.17), (4.26), (4.27).

Step 2. We now claim that \( \varphi = \text{tr}|_{t=0} f \). Note that the set \( \tilde{Q}_s^\alpha \) of cubes containing the point \( x \) forms a regular family in the sense of §1.8 of [14]. Combining the arguments of §1.8 of [14] with condition 3 from Definition 4.4 it is easily seen that

\[ \varphi(x) = \lim_{t \to +0} f(x, t) \quad \text{for almost all } x \in \mathbb{R}^n. \]  

(4.28)

By Remark 2.2 using the definition of the (Sobolev) generalized derivative of \( f \), it is found from (4.28) that

\[ f(x, t) - \varphi(x) = \int_0^t D_t f(x, \tau) \, d\tau \quad \text{for almost all } x \in \mathbb{R}^n. \]  

(4.29)

Next, by (4.29) and Remark 2.2 we have, for any cube \( Q \),

\[ \int_Q |f(x, t) - \varphi(x)| \, dx \leq \int_Q \left| \frac{\partial f}{\partial t}(x, \tau) \, d\tau \right| \leq C(C_\gamma, Q) \| f \|_{W_1^1(Q \times (0, t), \gamma)} \to 0, \quad t \to +0. \]

Definition 4.5. Assume that a weight \( \gamma \in A^{1,\infty}_{1,1}(\mathbb{R}^{n+1}) \) and \( c_1, c_2 \geq 1 \). By \( Z = Z(\{\gamma_{k,m}\}, c_1, c_2) \) we shall denote the linear space of all functions \( \varphi \in L^{1,\infty}_{1,1}(\mathbb{R}^n) \) with finite norm

\[ \| \varphi \|_Z := \inf_T \sum_{s=1}^{\infty} \sum_{\alpha \in A_s} \tilde{\gamma}_s^\alpha E(\tilde{Q}_s^\alpha) \varphi + \sum_{m \in \mathbb{Z}^n} \hat{\gamma}_{0,m} \varphi_{0,m,0}, \]  

(4.30)

where the infimum on the right of (4.30) is taken over tilings \( T = \{T^s\}_{s=0}^{\infty}(c_1, c_2) \) of the space \( \mathbb{R}^n \) that are admissible for the weight \( \gamma \).
The following main result of the present section is a direct corollary of Theorems 4.1 and 4.2 and the elementary estimate
\[ E(\tilde{Q}_\alpha^s) \leq \int_{\tilde{Q}_\alpha^s} |\varphi(x) - \varphi_{\alpha}^s| \, dx \leq 2E(\tilde{Q}_\alpha^s), \quad s \in \mathbb{Z}_+, \alpha \in \tilde{A}^s. \]

**Corollary 4.1.** Assume that a weight \( \gamma \in A_1^{\text{loc}}(\mathbb{R}^{n+1}) \). Then there exist numbers \( c_1 \geq q^3 \), \( c_2 \geq q^5 \) such that the operator \( \text{Tr} : W_1^1(\mathbb{R}^{n+1}, \gamma) \to Z(\{\gamma_{k,m}\}, c_1, c_2) \) is continuous and there exists a (nonlinear) continuous operator \( \text{Ext} : Z(\{\gamma_{k,m}\}, c_1, c_2) \to W_1^1(\mathbb{R}^{n+1}, \gamma) \), which is the right inverse of the operator \( \text{Tr} \).

**Remark 4.1.** From the proof of Theorems 4.1 and 4.2 it follows that for \( c_1 \geq q^3 \), \( c_2 \geq q^5 \) the space \( Z(\{\gamma_{k,m}\}, c_1, c_2) \) is independent of the choice of constants \( c_1, c_2 \), the corresponding norms being equivalent. Of course, the parameters \( q^3 \), \( q^5 \) may be fairly large. But for us it is important that they are determined only from the sequence \( \{\gamma_{k,m}\} \). Similarly, the space \( Z(\{\gamma_{k,m}\}, c_1, c_2) \) is independent of the choice of the parameter \( \lambda \) (which controls the expansion of the cubes \( Q_\alpha^s \)). Hence in what follows the space \( Z(\{\gamma_{k,m}\}, c_1, c_2) \) will be denoted by \( Z(\{\gamma_{k,m}\}) \).

The following fairly subtle question is still open: find the constants \( \sigma_1, \sigma_2 \) such that for \( c_1 > \sigma_1, \ c_2 > \sigma_2 \), the corresponding norms in the space \( Z(\{\gamma_{k,m}\}) \) are equivalent, but for \( c_1 \leq \sigma_1 \) or \( c_2 \leq \sigma_2 \) the resulting norm is not equivalent to the norm of the space \( Z(\{\gamma_{k,m}\}) \). However, by author’s opinion, this question plays no critical role for applications.

Let us establish some elementary properties of the space \( Z(\{\gamma_{k,m}\}) \).

**Lemma 4.3.** Assume that a weight \( \gamma \in A_1^{\text{loc}}(\mathbb{R}^{n+1}) \) and \( c_1 \geq q^3 \), \( c_2 \geq q^5 \). Then, for the space \( Z = Z(\{\gamma_{k,m}\}) \), we have the following continuous embeddings:
\[ \tilde{B}^1(\mathbb{R}^n, \{\gamma_{k,m}\}) \subset Z(\{\gamma_{k,m}\}) \subset L_1^{\text{loc}}(\mathbb{R}^n). \]

The proof of the continuity of the embedding \( \tilde{B}^1(\mathbb{R}^n, \{\gamma_{k,m}\}) \subset Z(\{\gamma_{k,m}\}) \) is clear. The second embedding follows from Corollary 4.1, Remark 2.2 and the simple estimate.
\[ \| \text{tr}_{t=0} f | L_1(Q) \| \leq \| f | W_1^1(Q \times (0,1)) \|, \]

where \( Q \) is a cube in the space \( \mathbb{R}^n \).

**Lemma 4.4.** Assume that a weight \( \gamma \in A_1^{\text{loc}}(\mathbb{R}^{n+1}) \) and \( c_1 \geq q^3, c_2 \geq q^5 \). Then the space \( Z(\{\gamma_{k,m}\}) = Z(\{\gamma_{k,m}\}, c_1, c_2) \) is complete.

**The proof** follows from Corollary 4.1 and the fact that the space \( W_1^1(\mathbb{R}^n, \gamma) \) is complete.

**Remark 4.2.** We claim that for \( \gamma \equiv 1 \) Gagliardo’s result follows from Corollary 4.1. The embedding \( L_1(\mathbb{R}^n) \supset Z(\{\gamma_{k,m}\}, c_1, c_2) \) with \( c_1 \geq q^3, c_2 \geq q^5 \) is clear. To prove the converse embedding we note that

\[ \| \varphi | Z(\{\gamma_{k,m}\}) \| \leq \inf_{\{l_j\}} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \varphi_{l_j,m} \leq C \| \varphi | L_1(\mathbb{R}^n) \|, \]

where the infimum is taken over all sequences \( \{l_j\} \) for which \( l_0 = 0 \) and the corresponding series is converging.

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