Esseen–Rozovskii type estimates for the rate of convergence in the Lindeberg theorem*

Ruslan Gabdullin\textsuperscript{1}, Vladimir Makarenko\textsuperscript{1}, and Irina Shevtsova\textsuperscript{1,2}

Abstract
We present structural improvements of Esseen’s (1969) and Rozovskii’s (1974) estimates for the rate of convergence in the Lindeberg theorem and also compute the appearing absolute constants. We introduce the asymptotically exact constants in the constructed inequalities and obtain upper bounds for them. We analyze the values of Esseen’s, Rozovskii’s, and Lyapunov’s fractions, compare them pairwise and provide some extremal distributions. As an auxiliary statement, we prove a sharp inequality for the quadratic tails of an arbitrary distribution (with finite second order moment) and its convolutional symmetrization.

1 Introduction
Let \( X_1, X_2, \ldots, X_n \) be independent random variables (r.v.’s) on a certain probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with distribution functions (d.f.’s) \( F_k(x) := \mathbb{P}(X_k < x), \) \( k = 1, \ldots, n, \) and such that
\[
EX_k = 0, \quad \sigma_k^2 := \mathbb{E}X_k^2 < \infty, \quad B_n^2 := \sum_{k=1}^{n} \sigma_k^2 > 0. \tag{1}
\]
Denote
\[
S_n = \sum_{k=1}^{n} X_k, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt,
\]
\[
\Delta_n = \Delta_n(F_1, \ldots, F_n) := \sup_{x \in \mathbb{R}} |\Phi(x) - \Phi(x)|, \quad n \in \mathbb{N},
\]
\[
\sigma_k^2(z) = \mathbb{E}X_k^2 1(|X_k| \geq z), \quad L_n(z) := \frac{1}{B_n^2} \sum_{k=1}^{n} \sigma_k^2(zB_n) = \frac{1}{B_n^2} \sum_{k=1}^{n} \mathbb{E}X_k^2 1(|X_k| \geq zB_n), \quad z \geq 0,
\]
so that \( \sup_{z \geq 0} \sigma_k^2(z) = \sigma_k^2(0) = \sigma_k^2 \) for all \( k = 1, \ldots, n \) and \( L_n(0) = 1. \)
In 1922 Lindeberg \cite{20} proved that \( \Delta_n \to 0, \) if the Lindeberg fraction \( L_n \) satisfies the condition
\[
\lim_{n \to \infty} L_n(z) = 0 \quad \text{for every } z > 0. \tag{L}
\]
In 1935 Feller \cite{3} completed Lindeberg’s theorem by proving the necessity of condition (L) for the CLT to hold, if the random summands are asymptotically negligible in the sense that
\[
\lim_{n \to \infty} B_n^{-2} \max_{1 \leq k \leq n} \sigma_k^2 = 0. \tag{F}
\]
Condition (F) is called the Feller condition.

\*Research supported by the Russian Foundation for Basic Research (projects 15-07-02984-a and 16-31-60110-mol_a_dk) and by the grant of the President of Russia No. MD-2116.2017.1.
\textsuperscript{1}Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, Moscow, Russia
\textsuperscript{2}Institute of Informatics Problems of Federal Research Center “Computer Science and Control,” Russian Academy of Sciences, Moscow, Russia; e-mail: ishevtsova@cs.msu.ru
Lindeberg’s theorem yields the celebrated Lyapunov theorem \([19]\), according to which \(\Delta_n \to 0\), if the Lyapunov fraction defined below tends to zero:

\[
L_{2+\delta,n} := \frac{1}{B^2_{n+\delta}} \sum_{k=1}^{n} E|X_k|^{2+\delta} \to 0, \quad n \to \infty, \quad \text{for certain } \delta > 0.
\]

In the case of independent and identically distributed (i.i.d.) random summands the Lyapunov fraction

\[
L_{2+\delta,n} = \frac{E|X_1|^{2+\delta}}{\sigma_1^{2+\delta} n^{\delta/2}}
\]

is of order \(O(n^{-\delta/2})\) as \(n \to \infty\) for every fixed distribution \(F_1\) of the random summands.

The numerical demonstration to the Lyapunov theorem is given by the inequality

\[
\Delta_n \leq C_{BE}(\delta) \cdot L_{2+\delta,n}, \quad 0 < \delta \leq 1,
\]

\(C_{BE}(\delta)\) being some absolute constants for every \(\delta \in (0,1]\), which was proved by Lyapunov himself in the same paper \([19]\) for \(0 < \delta < 1\), and, 40 years later, independently by Berry \([3]\) in the i.i.d. case and Esseen \([6]\) in the general situation for \(\delta = 1\). Inequality \([2]\) in the i.i.d. case with \(\delta = 1\) takes the form

\[
\Delta_n \leq C_{BE}(1) \frac{E|X_1|^{3}}{\sigma_1^3} \quad \text{for every } \varepsilon > 0,
\]

where

\[
\Lambda_n(\varepsilon) := \frac{1}{B_{n}} \sum_{k=1}^{n} E|X_k|^3 1(|X_k| < \varepsilon B_n), \quad \varepsilon > 0,
\]

and \(C\) is an absolute constant. The quantity \(\Lambda_n(\varepsilon) + L_n(\varepsilon)\) is called the Osipov fraction with the parameter \(\varepsilon > 0\). Since \(\Lambda_n(\varepsilon) \leq \varepsilon B_n^{-2} \sum_{k=1}^{n} E|X_k|^3 1(|X_k| \leq \varepsilon B_n) \leq \varepsilon\), inequality \([2]\) yields the bound

\[
\Delta_n \leq C(\varepsilon + L_n(\varepsilon)) \quad \text{for every } \varepsilon > 0,
\]

and hence, the Lindeberg theorem. Thus, in the case of asymptotically negligible random summands satisfying condition \((F)\), by the Feller theorem, the left-hand and right-hand sides of \((4)\) either tend or do not tend to zero simultaneously. In other words, inequality \([4]\) relates the criteria of convergence with the rate of convergence in the classical CLT and hence, according to Zolotarev’s terminology \([30]\), it can be called a natural convergence rate estimate.

It is easy to see that inequality \((4)\) with some \(\varepsilon > 0\) yields the Berry–Esseen inequality \((2)\) for every \(\delta \in (0,1]\), since

\[
\Lambda_n(\varepsilon) + L_n(\varepsilon) = \sum_{k=1}^{n} \left( \frac{E|X_k|^3 1(|X_k| < \varepsilon B_n)}{B_n^3} + \frac{EX_k^2 1(|X_k| \geq \varepsilon B_n)}{B_n^2} \right) \leq \sum_{k=1}^{n} \frac{E|X_k|^{2+\delta}}{B_n^{2+\delta}} = L_{2+\delta,n}.
\]

However, inequality \((4)\) with \(\varepsilon = 1\)

\[
\Delta_n \leq C(\Lambda_n(1) + L_n(1)) = \frac{C}{B_{n}^3} \sum_{k=1}^{n} \left( E|X_k|^3 1(|X_k| < B_n) + B_n EX_k^2 1(|X_k| \geq B_n) \right) = \frac{C}{B_{n}^2} \sum_{k=1}^{n} EX_k^2 \min\left\{ 1, \frac{|X_k|}{B_n} \right\} = C \int_0^1 L_n(z) dz, \quad (5)
\]
is of great interest, because \( \varepsilon = 1 \) minimizes the right-hand side of (4). Indeed, to find the minimizer one can observe, following the outline of the reasoning in [25], that for every \( \varepsilon > 0 \)
\[
\Lambda_n(\varepsilon) + L_n(\varepsilon) = \sum_{k=1}^{n} \left[ \mathbb{E} \left( \frac{X_k}{B_n} \right)^3 1 \left( \frac{X_k}{B_n} < \varepsilon \right) + \mathbb{E} \left( \frac{X_k}{B_n} \right)^2 1 \left( \frac{X_k}{B_n} \geq \varepsilon \right) \right]
\]
and for every r.v. \( X := X_k/B_n, k = 1 \ldots, n, \) with \( A \) being arbitrary Borel subset of \( \mathbb{R} \), we have
\[
\mathbb{E}|X|^3 1(|X| < 1) + \mathbb{E}X^2 1(|X| \geq 1) = \mathbb{E}X^2 \text{min}\{1,|X|\} (1(X \in A) + 1(X \notin A)) \leq \mathbb{E}|X|^3 1(X \in A) + \mathbb{E}X^2 1(X \notin A)
\]
(here the optimality of the set \( A = [-1, 1] \) was first noted, probably, by Loh [21] whose hardly available thesis we cite by [2]; however, in [25], this fact was proved independently). Now the claim follows by taking \( A := [-\varepsilon, \varepsilon] \).

Inequality (4) has an interesting history. First of all we note that it is a trivial corollary to earlier and formally more general results of Katz [12] and Petrov [26], but this connection remained unnoticed for a long time, even by Petrov himself (for example, see [27, Ch. 5, § 3, Theorems 5 and 8]). Furthermore, inequality (5) was re-proved formally in a more general, but in fact, in an equivalent form, by Feller [10] with \( C = 6 \) and by Paditz [23,24] in the i.i.d. case with \( C = 4.77 \). Two years later Paditz [25] announced a sharper bound \( C \leq 3.51 \) without a complete proof. The works of Feller and Paditz did not refer to the above mentioned work of Osipov. Inequality (4) was also re-proved by Barbour and Hall [2] who cited Feller's result, and applied the new Tikhomirov–Stein method and obtained a rougher bound \( C \leq 18 \). Chen and Shao [5] who cited only Feller's work re-proved (4) with \( C = 4.1 \). Finally, Korolev and his disciples successively improved the upper bounds for (4) to 2.011 in [16,17] and to 1.87 in [14].

Thus, inequalities (4) and (5) attracted much attention. However, as far back as in 1969 Esseen [8] proved the bound
\[
\Delta_n \leq \frac{C_1}{B_n^3} \sum_{k=1}^{n} \sup_{z>0} \left\{ \mathbb{E}X_k^3 1(|X_k| < z) \right\} + \varepsilon \mathbb{E}X_k^2 1(|X_k| \geq z) = C_1 \cdot \hat{L}_{e,n}^3(\infty),
\]
where
\[
\hat{L}_{e,n}^3(\varepsilon) := \frac{1}{B_n^3} \sum_{k=1}^{n} \sup_{0 < z < \varepsilon B_n} \{ \mu_k(z) + z\sigma_k^2(z) \}, \quad \mu_k(z) := \mathbb{E}X_k^3 1(|X_k| < z), \quad \varepsilon > 0, \quad z \geq 0,
\]
and, as its corollary, by the use of the traditional truncation techniques, deduced that
\[
\Delta_n \leq C_2 \cdot \hat{L}_{e,n}^3(1),
\]
with \( C_1 \) and \( C_2 \) being some absolute constants. Obviously, \( \hat{L}_{e,n}^3(1) \leq \hat{L}_{e,n}^3(\infty) \). Moreover, due to the left-continuity of the functions \( \mu_k(z), \sigma_k^2(z), \) \( k = 1, \ldots, n, \) for \( z > 0 \), the least upper bound in the definition of \( \hat{L}_{e,n}^3(\varepsilon) \) can be replaced by the one over the set \( z \in (0, \varepsilon B_n] \).

Comparing inequalities (6) and (7) with (4) we observe, first, that the fractions \( \hat{L}_{e,n}^3(\infty) \) and \( \hat{L}_{e,n}^3(1) \) here depend not on the absolute, but on the algebraic third-order truncated moments, which may vanish, for example, in the case of symmetrically distributed random summands. Second, Esseen’s fraction \( \hat{L}_{e,n}^3(1) \) never exceeds the Osipov fraction \( \Lambda_n(\varepsilon) + L_n(\varepsilon) \) with \( \varepsilon = 1 \) and hence with arbitrary \( \varepsilon > 0 \). Indeed, for every random summand \( X := X_k, k = 1, \ldots, n, \) we have
\[
\mathbb{E}X^3 1(|X| < z) + \varepsilon \mathbb{E}X^2 1(|X| \geq z) \leq \mathbb{E}|X|^3 1(|X| < z) + \varepsilon \mathbb{E}X^2 1(|X| \geq z) =: h(z),
\]
with \( h(z) \) being monotonically increasing with respect to \( z \geq 0, \) due to
\[
h(u) - h(v) = \mathbb{E} \left( |X|^3 - vX^2 \right) 1(v \leq |X| < u) + (u - v) \mathbb{E}X^2 1(|X| \geq u) \geq 0, \quad u \geq v \geq 0,
\]
and, hence, writing \( h_k(z) \) for the analogous function of the distribution of \( X_k, k = 1, \ldots, n, \) we get
\[
\hat{L}_{e,n}^3(1) \leq \frac{1}{B_n^3} \sum_{k=1}^{n} \sup_{0 < z < B_n} h_k(z) = \frac{1}{B_n^3} \sum_{k=1}^{n} h_k(B_n) = \Lambda_n(1) + L_n(1) = \inf_{\varepsilon > 0} \{ \Lambda_n(\varepsilon) + L_n(\varepsilon) \}.
\]
This fact implies, in turn, that \( C \leq C_2, \) in particular, inequality (7) is also a natural convergence rate estimate in the Lindeberg theorem.
Third, Esseen’s inequality \( \mu_1(z) = O(1), \quad z \sigma_1^2(z) = O(1), \quad z \to \infty. \) It is easy to see that Ibragimov’s condition is weaker than the condition \( E[X]^3 < \infty, \) and Esseen’s inequality \( (7) \) in the i.i.d. case

\[
\Delta_n \leq \frac{C_2}{\sigma_1^2 \sqrt{n}} \sup_{0 < z \leq \sigma_1 \sqrt{n}} \left\{ \left| \mu_1(z) \right| + \sigma_2^2(z) \right\}
\]

trivially yields the “if” part of Ibragimov’s criteria.

In 1974 Rozovskii \([30]\) generalized Ibragimov’s theorem and proved another estimate involving algebraic truncated third-order moments

\[
M_n(z) := \frac{1}{B^3_n} \sum_{k=1}^{n} \mu_k(zB_n) = \frac{1}{B^3_n} \sum_{k=1}^{n} \mathbb{E}X_k^3 1(|X_k| < zB_n), \quad z > 0,
\]

in the form

\[
\Delta_n \leq C_3 \cdot L^3_{n, n},
\]

where \( C_3 \) is an absolute constant and

\[
L^3_{n, n} := |M_n(1)| + \sup_{0 < z \leq 1} zL_n(z) = \frac{1}{B^3_n} \left( \left| \sum_{k=1}^{n} \mu_k(B_n) \right| + \sup_{0 < z \leq B_n} z \sum_{k=1}^{n} \sigma_k^2(z) \right).
\]

At first glance, Rozovskii’s inequality \( (8) \) is more favorable than Esseen’s inequalities \( (6) \) and \( (7) \). Indeed, the right-hand side of \( (8) \) is always finite, while the right-hand side of \( (7) \) may be infinite. In \( (8) \) the first term may vanish not only in the symmetric, but also in the non-symmetric case, for example, for even \( n \) if the appearing truncated third-order moments \( \mu_k(B_n), \) \( k = 1, \ldots, n, \) have the same absolute value, but alternating signs. One more advantage of inequality \( (8) \) over \( (6) \) and \( (7) \) is that the values of \( \mu_k(z) \) are used in \( (8) \) only in one and the same point \( z = B_n \) for all \( k = 1, \ldots, n, \) while the right-hand sides of \( (6) \) and \( (7) \) require the information on \( \mu_k(z) \) in every point of the interval \( z \in (0, B_n) \). However, a deeper analysis (see Theorem \( (8) \) below) shows that Rozovskii’s fraction \( L^3_{e, n} \) may take greater values than each of Esseen’s fractions \( \hat{L}^3_{e, n}(1), \hat{L}^3_{e, n}(\infty) \) and even greater than Lyapunov’s fraction \( L^3_{e, n} \), while Esseen’s fractions always satisfy

\[
\hat{L}^3_{e, n}(\infty) \leq L^3_{e, n}, \quad \hat{L}^3_{e, n}(1) \leq \Lambda_n(1) + L_n(1) \leq L^3_{2+\delta, n}, \quad \delta \in (0, 1].
\]

Thus the choice between inequalities \( (7) \) and \( (8) \) depends not only on the concrete values of the fractions \( \hat{L}^3_{e, n} \) and \( L^3_{e, n} \), but also on the values of the appearing absolute constants \( C_2 \) and \( C_3 \). However, the values of these constants, as well as of \( C_1 \), remain unknown.

In the present paper, we compute upper bounds for the constants \( C_1, C_2, \) and \( C_3 \). Moreover, we prove new natural convergence rate estimates in the Lindeberg-Feller theorem generalizing Esseen’s \( (6) \), \( (7) \) and Rozovskii’s \( (8) \) inequalities, and provide an explicit analytical representation with an algorithm of evaluation of the appearing constants. Namely, we introduce a truncation parameter \( \varepsilon > 0 \) and a balancing parameter \( \gamma > 0 \) and denote

\[
L^3_{e, n}(\varepsilon, \gamma) := \sup_{0 < z \leq \varepsilon} \left\{ \gamma |M_n(z)| + zL_n(z) \right\} = \frac{1}{B^3_n} \sup_{0 < z \leq \varepsilon B_n} \left\{ \gamma \left| \sum_{k=1}^{n} \mu_k(z) \right| + z \sum_{k=1}^{n} \sigma_k^2(z) \right\},
\]

\[
L^3_{e, n}(\varepsilon) := \sup_{0 < z \leq \varepsilon} zL_n(z) = \frac{1}{B^3_n} \left( \gamma \left| \sum_{k=1}^{n} \mu_k(z) \right| + z \sum_{k=1}^{n} \sigma_k^2(z) \right),
\]

where the least upper bounds with respect to \( 0 < z \leq \ldots \) can be replaced by those over the open sets \( 0 < z < \ldots \), due to the left-continuity of \( \mu_k(\cdot) \) and \( \sigma_k^2(\cdot) \). It is easy to see that

\[
L^3_{e, n}(1) = \sup_{0 < z \leq \varepsilon} \left\{ |M_n(z)| + zL_n(z) \right\} \leq \sup_{0 < z \leq \varepsilon} \left\{ \Lambda_n(z) + zL_n(z) \right\} = \Lambda_n(z) + \varepsilon L_n(z),
\]

**...**
sup_{\gamma \in (0,1]} L^3_{n,n}(\varepsilon, \gamma) \leq \frac{1}{B_3} \sum_{k=1}^{n} \sup_{0 < z \leq B_n} \left[ E|X_k|^3 I(|X_k| < z) + z E X_k^3 I(|X_k| \geq z) \right] \\
\leq \frac{1}{B_3} \sum_{k=1}^{n} \sup_{0 < z \leq B_n} z^{1-\delta} E|X_k|^{2+\delta} = \varepsilon^{1-\delta} L_{2+\delta,n} \quad \text{for every } \varepsilon > 0, \delta \in (0,1], \quad (10)
\]

moreover, $L^3_{n,n}(1,1) = L^3_{n,n}$, $L^3_{n,n}(1,1) \leq L^3_{n,n}(1)$ with equality sign, for example, in the i.i.d. case, and $L^3_{n,n}(\varepsilon, \gamma) = L^3_{n,n}(\varepsilon, \gamma)$ in the symmetric case.

**Theorem 1.** Under the above assumptions, for every $\gamma > 0$

\[
\Delta_n \leq C_\varepsilon(\varepsilon, \gamma) \cdot L^3_{n,n}(\varepsilon, \gamma), \quad \varepsilon \in (0, \infty], \quad (11)
\]

\[
\Delta_n \leq C_\varepsilon(\varepsilon, \gamma) \cdot L^3_{n,n}(\varepsilon, \gamma), \quad \varepsilon \in (0, \infty), \quad (12)
\]

where $C_\varepsilon(\varepsilon, \gamma)$ and $C_\varepsilon(\varepsilon, \gamma)$ depend only on the arguments in the brackets, take finite values for every $\gamma > 0$ and $\varepsilon$ specified above and can be computed for every $\varepsilon, \gamma$ under consideration by an algorithm provided below in the proof.

Both $C_\varepsilon(\varepsilon, \gamma)$ and $C_\varepsilon(\varepsilon, \gamma)$ are monotonically decreasing with respect to $\gamma > 0$, $C_\varepsilon(\varepsilon, \gamma)$ is also monotonically decreasing with respect to $\varepsilon > 0$. In particular,

\[
C_\varepsilon(\infty, 1) \leq \max \{ C_\varepsilon(\infty, 0.97), C_\varepsilon(4.35, 1) \} \leq 2.66, \quad C_\varepsilon(1, 1) \leq C_\varepsilon(1, 0.72) \leq 2.73,
\]

\[
C_\varepsilon(\infty, \infty) \leq \max \{ C_\varepsilon(\infty, 1.43), C_\varepsilon(4, 1.62), C_\varepsilon(2.74, 3), C_\varepsilon(2.56, \infty) \} \leq 2.65,
\]

\[
C_\varepsilon(1, 1) \leq \sup_{\gamma \geq \gamma_*} \inf_{\varepsilon > 0, \gamma > 0} C_\varepsilon(\varepsilon, \gamma) \leq \inf_{\gamma > 0, \gamma \geq \gamma_*} \sup_{\varepsilon > 0} C_\varepsilon(\varepsilon, \gamma) \leq C_\varepsilon(2.12, \gamma_*), \quad 2.66,
\]

\[
\sup_{\gamma \geq 0.4} \max \{ C_\varepsilon(2.63, \gamma), C_\varepsilon(1.76, \gamma) \} \leq 2.70, \quad \sup_{\gamma \geq 0.2} \max \{ C_\varepsilon(5.39, \gamma), C_\varepsilon(1.21, \gamma) \} \leq 2.87,
\]

where

\[
\gamma_* := \frac{1}{\sqrt{6\pi}} = 0.5599\ldots, \quad \varkappa := x^2 \sqrt{(\cos x - 1 + x^2/2)^2 + (\sin x - x)^2} \bigg|_{x=x_0} = 0.531551\ldots,
\]

and $x_0 = 5.487414\ldots$ is the unique root of the equation $8(\cos x - 1) + 8\sin x - 4x^2 \cos x - x^3 \sin x = 0$ on the interval $(\pi, 2\pi)$.

**Remark 1.** Within the method used: (i) $C_\varepsilon(1, \gamma)$ does not depend on $\gamma$ for $\gamma \geq \gamma_*$; (ii) further increase of $\gamma \geq 0.73$ does not reduce the constructed upper bound 2.73 for $C_\varepsilon(1, \gamma)$ by more than 0.01; (iii) the same concerns the presented upper bounds for $C_\varepsilon(\infty, 1)$ and $C_\varepsilon(\infty, \infty)$.

**Remark 2.** Within the method used, $C_\varepsilon(1, 1) = C_\varepsilon(1, 1)$, due to Remark [7].

The plot of the level curve $\gamma = \gamma(\varepsilon)$ delivering the constant value 2.65 to $C_\varepsilon(\varepsilon, \gamma)$ is given on Fig. [2] (right). The plot of the function $C_\varepsilon(\varepsilon, \gamma_*)$ constructed in the proof is given on Fig. [3] (right).

**Theorem 1** trivially yields

**Corollary 1.** The constants $C_1$, $C_2$, and $C_3$ in inequalities [3], [7], and [8] satisfy

\[
C_1 \leq C_\varepsilon(\infty, 1) \leq 2.66, \quad C_2 \leq C_\varepsilon(1, 1) \leq 2.73, \quad C_3 \leq C_\varepsilon(1, 1) \leq 2.73.
\]

The upper bounds for the constants $C_1, C_2, C_3$ presented in Corollary [1] are slightly greater than the best known upper bound 1.87, obtained in [14], for the absolute constant $C$ (which is no greater than $C_2$) in Osipov’s inequality [3]. In Theorem [8] (iv) of Section [5] we provide various examples of symmetric and non-symmetric distributions of the random summands $X_1, \ldots, X_n$ for which the right-hand sides of both inequalities [11] and [12] with $\varepsilon = \gamma = 1$, $C_\varepsilon(1, 1) = C_\varepsilon(1, 1) = 2.73$ are strictly less than the right-hand side of the Osipov inequality [3] with $C = 1.87$.

Similarly to Kolmogorov [13], where the classical Berry–Esseen inequality was discussed, we also introduce here the so-called asymptotically exact constants in [11], [12]

\[
C_{n,\varepsilon}^\gamma(\varepsilon, \gamma) := \limsup_{\ell \to 0} \sup_{n,F_1,\ldots,F_n} \left\{ \Delta_n(F_1, \ldots, F_n) : L^3_{n,n}(\varepsilon, \gamma) = \ell \right\}, \quad \varepsilon, \gamma > 0, \quad (13)
\]

\[
C_{n,\varepsilon}^\gamma(\varepsilon, \gamma) := \limsup_{\ell \to 0} \sup_{n,F_1,\ldots,F_n} \left\{ \Delta_n(F_1, \ldots, F_n) : L^3_{n,n}(\varepsilon, \gamma) = \ell \right\}, \quad \varepsilon, \gamma > 0, \quad (14)
\]

and present their upper bounds for every $\varepsilon > 0$ and $\gamma > 0$. 5
Theorem 2. For every $\varepsilon > 0$ and $\gamma > 0$ we have

$$C^*_B(\varepsilon, \gamma) \leq \frac{4}{\sqrt{2\pi}} + \frac{1}{\pi} \left[ \frac{\varepsilon}{\gamma} \gamma \left( 1, \frac{t_2^2}{2\varepsilon^2} \right) + \frac{\varepsilon}{12} \gamma \left( 2, \frac{t_2^2}{2\varepsilon^2} \right) + \frac{\sqrt{2(6\varepsilon^2 + 1)}}{6\gamma} \Gamma \left( \frac{3}{2}, \frac{t_2^2}{2\varepsilon^2} \right) \right] =: \hat{C}^*_B(\varepsilon, \gamma),$$

where $\Gamma(r, x) := \int_x^\infty t^{r-1}e^{-t}dt$, $\Upsilon(r, x) := \int_0^x t^{r-1}e^{-t}dt = \Gamma(r) - \Gamma(r, x)$, $r, x > 0$, are the upper and the lower incomplete gamma functions,

$t_\gamma := \frac{2}{\pi}(\sqrt{\gamma/\gamma_s}^2 + 1 - 1)$, \quad $t_{2, \gamma} = 2 \max \{ \gamma^{-1}, \gamma_s^{-1} \}$,

$t_{1, \gamma} := t_{2, \gamma} \cdot (1 - \sqrt{1 - \frac{(\gamma/\gamma_s)^2}{\gamma_s^2}})$,

$\gamma = 0.5315 \ldots$ and $\gamma_s = 0.5599 \ldots$ are defined in Theorem 1. Moreover, the function $\hat{C}^*_B(\varepsilon, \gamma)$ is monotonically decreasing with respect to $\varepsilon > 0$ and $\gamma > 0$, the function $\hat{C}^*_B(\varepsilon, \gamma)$ is monotonically decreasing with respect to $\gamma > 0$ being constant for $\gamma \geq \gamma_s$ for every fixed $\varepsilon > 0$. In particular,

$$C^*_B(\infty, \gamma) \leq \frac{4}{\sqrt{2\pi}} \left( 4 + \frac{1}{6} \sqrt{\gamma_s^2 + \gamma^{-2}} \right) = \begin{cases} \frac{1}{\sqrt{2\pi}} \left( 4 + \frac{\sqrt{2}}{\gamma} \gamma_s^{-1} \right) = 1.7636 \ldots, & \gamma = \gamma_s, \\ \frac{1}{\sqrt{2\pi}} \left( 4 + \frac{\sqrt{2}}{\gamma} \gamma_s^{-2} + 1 \right) = 1.7318 \ldots, & \gamma = 1, \\ \frac{1}{\sqrt{2\pi}} \left( 4 + \frac{1}{6\gamma} \right) = 1.7145 \ldots, & \gamma \to \infty, \end{cases},$$

$$\sup_{\gamma \geq \gamma_s} C^*_B(\varepsilon, \gamma_s) \leq \hat{C}^*_B(\varepsilon, \gamma_s) = \frac{4}{\sqrt{2\pi}} + \frac{1}{\pi} \left[ \frac{\varepsilon}{\gamma} \gamma \left( 1, \frac{2}{(\gamma_s)^2} \right) + \frac{\varepsilon}{12} \gamma \left( 1 + \Gamma \left( \frac{3}{2}, \frac{2}{(\gamma_s)^2} \right) \right) \right], \quad \varepsilon > 0,$$

$$\inf_{\varepsilon > 0} \inf_{\gamma > \gamma_s} C^*_B(\varepsilon, \gamma) \leq \inf_{\varepsilon > 0} \sup_{\gamma \geq \gamma_s} C^*_B(\varepsilon, \gamma) \leq \hat{C}^*_B(1.89, \gamma_s) \leq 1.75.$$

The values of the functions $\hat{C}^*_B(\varepsilon, \gamma)$ and $\hat{C}^*_B(\varepsilon, \gamma)$ for some $\varepsilon > 0$ and $\gamma > 0$ are given in the third columns of tables[2] and[2] respectively. The plot of the level curve $\gamma = \gamma(\varepsilon)$ with the constant value of $\hat{C}^*_B(\varepsilon, \gamma) = 1.72$, closest to the minimal one $\hat{C}^*_B(\infty, \infty)$ up to $10^{-2}$, is given on Fig. 2 (left). The plot of the function $\hat{C}^*_B(\varepsilon, \gamma_s)$ is given on Fig. 3 (left, solid line). The graphs of the functions $t_\gamma$, $t_{1, \gamma}$, $t_{2, \gamma}$ are given on Fig. 1.

Remark 3. Within the method used, the functions $C_B(\varepsilon, \gamma)$ and $C_B(\varepsilon, \gamma)$ constructed here in the proof of Theorem 7 are bounded below by the functions $\hat{C}^*_B(\varepsilon, \gamma)$ and $\hat{C}^*_B(\varepsilon, \gamma)$, respectively.

Remark 4. Since

$$\lim_{\varepsilon \to 0} L_{B,n}(\varepsilon, \gamma) = \lim_{\varepsilon \to 0} L_{B,n}(\varepsilon, \gamma) = 0 \text{ for every } \gamma > 0,$$

the functions $C_B(\varepsilon, \gamma)$, $C_B(\varepsilon, \gamma)$, $C^*_B(\varepsilon, \gamma)$, and $C^*_B(\varepsilon, \gamma)$ must be unbounded as $\varepsilon \to 0$.

Remark 5. Though the first terms $\mu_k(\cdot)$, $k = 1, 2, \ldots, n$, in the definitions of $L_{B,n}(\varepsilon, \gamma)$, $L_{B,n}(\varepsilon, \gamma)$ vanish for symmetric distributions of random summands, one cannot, in general, get rid of the third truncated moments $\mu_k(\cdot)$ in (11), (12), that is, the “constants” $C_B(\varepsilon, 0+)$, $C_B(\varepsilon, 0+)$, $C^*_B(\varepsilon, 0+)$, and $C^*_B(\varepsilon, 0+)$ are no more bounded. This fact follows from the observation that each of the above constants is bounded below by the so-called asymptotically best constant (we follow here the terminology introduced in [32]).

$$\sup_{F_1 = \ldots = F_n} \limsup_{n \to \infty} \frac{\Delta_n(F_1, \ldots, F_n)}{\sup_{0 < \varepsilon \leq \varepsilon} z L_n(z)}$$

for which, in Theorem 4 of Section 3, an infinite lower bound is constructed for every $\varepsilon > 0$. 

6
The proofs of Theorems 1 and 2 are based on the method of characteristic functions (ch.f.’s) realized in the Prawitz smoothing inequality [28] (see Lemma 8 below) and new estimates for ch.f.’s presented in Section 3. As an auxiliary result used in bounding the absolute value of a ch.f., we prove a sharp inequality
\[ E(X - X')^2 1(|X - X'| \geq z) \leq 4EX^2 1(|X| \geq z/2), \quad z \geq 0, \]
for arbitrary centered i.i.d. r.v.’s X and X’ with finite second moment, where the constant factor 4 on the R.H.S. cannot be made less (see Theorem 3 in Section 2). Similar inequality was proved and used in the preceding works with the constant factor 40 by Esseen [3], 12 and 8 by Rozovskii [30,31], respectively.

The proof of the main Theorem 1 is given in Section 4 and consists of two steps. First, we consider the case of small values of the fractions \( L_{n,n} \leq L \) and \( L_{n,n} \leq L \) with \( \bar{L} \) being small enough including the limit \( L \to 0 \), where the upper bounds for the corresponding “constants” (in fact, depending on \( L \)) are obtained in the analytical form (see Subsection 4.1) yielding, as a particular case, Theorem 2. Then we consider the remaining case, where the upper bounds for the appearing “constants” also depend on the concrete value of the fraction \( L := L_{n,n} \) or \( L := L_{n,n} \) and have a more complicated form assuming numerical evaluation by the use of a computer (see Subsection 4.2).

In Section 5 we compare the fractions \( L_{n,n}(\varepsilon, \gamma) \), \( L_{n,n}(\varepsilon, \gamma) \), \( L_{n,n} \) and also the right-hand sides of Osipov’s inequality [35] with our new inequalities [11, 12].

All the figures and tables are presented in the concluding Appendix.

2 An inequality for quadratic tails

Theorem 3. Let X and X’ be i.i.d. with the d.f. F and \( E X = 0 \). Then with
\[ \sigma^2(z) := E X^2 1(|X| \geq z) = \int_{|x| \geq z} x^2 dF(x), \]
\[ \sigma'^2(z) := E(X - X')^2 1(|X - X'| \geq z) = \int_{|x - y| \geq z} (x - y)^2 dF(x)dF(y), \quad z \geq 0, \]
we have
\[ \sigma'^2(z) \leq 2\sigma^2(\alpha z) + 2\sigma^2((1 - \alpha) z), \quad z \geq 0, \quad \alpha \in [0, 1], \]
In particular, with \( \alpha = 1/2 \) we have
\[ \sigma'^2(z) \leq 4\sigma^2(z/2), \quad z \geq 0, \quad \alpha = 1/2 \]
where the constant factor 4 on the right-hand side is the best possible in the sense of
\[ \sup_{X \colon E X = 0} \frac{\sigma'^2(z)}{\sigma^2(z/2)} = 4 \quad \text{for every} \quad z > 0, \quad (20) \]
where the least upper bound is taken over all distributions of the r.v. X with \( E X = 0 \) and delivered by the sequence of two-point distributions of the form \( P(X = qz) = p, P(X = -pz) = q := 1 - p \) with \( p \to 1/2 \) +.

Remark 6. In the original work of Esseen [3] inequality (19) was proved with the constant factor 40 on the right-hand side instead of 4, and in the works of Rozovskii [30,31] this factor was successively lowered to 12 and 8.

Proof. By \( F(x, y) = F(x) \cdot F(y) \) denote the d.f. of the random vector \((X, X')\). Then for every \( \alpha \in [0, 1] \) with \( \beta := 1 - \alpha \) we have
\[ \sigma'^2(z) = \int_{|x - y| \geq z} (x - y)^2 dF(x, y) \leq \int_{|x| \geq \alpha z} (x - y)^2 dF(x, y) + \int_{|y| \geq \beta z} (x - y)^2 dF(x, y) = \]
\[ = \int_{|x| \geq \alpha z} + \int_{|y| \geq \beta z} (x^2 - 2xy + y^2) dF(x, y) = \sigma^2(\alpha z) + \sigma^2(\beta z) + \sigma^2(\int_{|x| \geq \alpha z} dF(x) + \sigma^2(\int_{|y| \geq \beta z} dF(y), \]
since \( E X = EX' = 0 \). Observing that
\[ \sigma^2 \int_{|x| \geq \alpha z} dF(x) = \int_{|x| \geq \alpha z} dF(x) \int_{R} y^2 dF(y) \leq \int_{|y| < \alpha z \leq |x|} y^2 dF(x)dF(y) + \sigma^2(\alpha z) \int_{|x| \geq \alpha z} dF(x) \leq \]
\[ \int_{|y| < \alpha z \leq |x|} y^2 dF(x)dF(y) \]

...
we finally obtain
\[ \sigma^2(z) \leq 2\sigma^2(\alpha z) + 2\sigma^2(\beta z), \quad z \geq 0. \]

To prove (20), fix arbitrary \( z > 0 \) and consider a centred two-point distribution
\[ \begin{pmatrix} a, & \frac{b}{a+b} \\ -b, & \frac{a}{a+b} \end{pmatrix}, \]
where the choice of \( b > a > 0 \) will depend on \( z \). Then, with \( X' \) being an independent copy of \( X \), we have
\[ X - X' = \begin{pmatrix} a + b, & \frac{ab}{(a+b)^2}, \\ 0, & \frac{a^2 + b^2}{(a+b)^2}, \\ -a - b, & \frac{ab}{(a+b)^2} \end{pmatrix}, \]
\[ \sigma^2_s(z) = \begin{cases} 2ab, & 0 \leq z \leq a + b, \\ 0, & z > a + b, \end{cases} \quad \sigma^2_s \left( \frac{z}{2} \right) = \begin{cases} \frac{ab^2}{a + b}, & a < \frac{z}{2} \leq b, \\ 0, & z/2 > b. \end{cases} \]

Now let \( a \) and \( b \) satisfy \( a + b = z \). Then \( \sigma^2_s(z) = 2ab \) and \( \sigma^2(z/2) = ab^2/(a + b) = ab^2/z \), and hence for every \( z > 0 \)
\[ \sup_{0 < a < b: a + b = z} \frac{\sigma^2_s(z)}{\sigma^2(z/2)} = \sup_{z/2 < b < z} \frac{2z}{b} = \lim_{b \to z/2+} \frac{2z}{b} = 4. \]

3 Estimates for characteristic functions

In what follows we shall omit the arguments \( \varepsilon, \gamma \) of the fractions \( L_{\varepsilon, n}(\varepsilon, \gamma) \) and \( L_{\varepsilon, n}(\varepsilon, \gamma) \) assuming \( \gamma \in (0, \infty) \) and \( \varepsilon \in (0, \infty) \) for \( L_{\varepsilon, n}(\varepsilon, \gamma) \) being fixed and even simply using the notation \( L \) for each of the fractions \( L_{\varepsilon, n}, L_{\varepsilon, n} \). Furthermore, we shall also assume that the random summands \( X_1, \ldots, X_n \) are normalized so that
\[ B_n^2 := \sum_{k=1}^{n} \sigma^2_k = 1. \]

For \( k = 1, \ldots, n \) denote
\[ f_k(t) := \mathbb{E} e^{itX_k}, \quad \overline{f_n}(t) = \mathbb{E} \exp \left( \frac{itS_n}{\sqrt{DS_n}} \right) = \prod_{k=1}^{n} f_k(t), \quad t \in \mathbb{R}, \]
the ch.f.'s of \( X_k \) and of the (already normalized) sum \( S_n \). Recall that \( e^{-t^2/2}, \ t \in \mathbb{R}, \) is the ch.f. of the standard normal distribution on \( \mathbb{R} \).

3.1 Estimation of \( |\overline{f_n}(t)| \)

This section is aimed at bounding the absolute value \( |\overline{f_n}(t)| \) of the ch.f. of the normalized sum above. We provide two kinds of such estimates. The first estimate which is the most exact, but has a rather complicated form, is used below in the numerical method described in Subsection 4.2 for the case of \( L_{\varepsilon, n}, L_{\varepsilon, n} \) separated from the origin. The second estimate has a simpler form convenient for analytical work and is used below in Subsection 4.1 for the case of small \( L_{\varepsilon, n}, L_{\varepsilon, n} \).

We also note here that the form of the estimates for \( |\overline{f_n}(t)| \) presented below is the same for the both fractions \( L_{\varepsilon, n} \) and \( L_{\varepsilon, n} \) which is explained by the independence of the constructed bounds of the function \( M_n(\cdot) \).

Lemma 1 (see [29]). For every \( \theta \in [0, 2\pi] \) and \( x \in \mathbb{R} \) we have
\[ \cos x \leq 1 - a(\theta)x^2 + b(\theta)x^4, \]
with equality if and only if \( x \in \{ -\theta, 0, \theta \} \), where
\[ a(\theta) = 2 \frac{1 - \cos \theta}{\theta^2} - \frac{\sin \theta}{2\theta}, \quad b(\theta) = \frac{1 - \cos \theta}{\theta^4} - \frac{\sin \theta}{2\theta^3} \quad \text{for} \quad \theta \in (0, 2\pi] \]
and \( a(0) := 1/2, b(0) := 1/24 \) are defined by continuity.
Lemma 2. (i) The functions \(a(\theta)\) and \(b(\theta)\) on the open interval \(\theta \in (0,2\pi)\) are both strictly monotonically decreasing and strictly positive varying within the range

\[
\frac{1}{2} = a(0) > a(\theta) > a(2\pi) = 0, \quad \frac{1}{24} = b(0) > b(\theta) > b(2\pi) = 0.
\]

(ii) The function \(a(\theta)/b(\theta)\) is strictly monotonically increasing for \(\theta \in [0,2\pi)\), in particular,

\[
\sup_{0 < \theta < 2\pi} a(\theta) = \lim_{\theta \to 2\pi -} ^{\theta \to 0} b(\theta) = 4\pi^2.
\]

Proof. For \(\theta \in (0,2\pi)\) denote \(g(\theta) := a(\theta)/b(\theta)\). Then, omitting the argument \(\theta\) of the functions \(a(\theta)\) and \(b(\theta)\) for short, we have

\[
a' = -4 \frac{1 - \cos \theta}{\theta^3} + 5 \frac{\sin \theta}{2\theta} - \frac{\cos \theta}{\theta}, \quad b' = -4 \frac{1 - \cos \theta}{\theta^3} + 5 \frac{\sin \theta}{2\theta} - \frac{\cos \theta}{\theta^3},
\]

\[
\frac{g' }{\theta^2} = \frac{b' a - b'a}{b^2} \quad \text{and} \quad \frac{b' }{b^2} = \frac{b'(a - \theta^2 b)}{b^2} = \frac{b'}{b^2} - \frac{1 - \cos \theta}{\theta^2},
\]

hence, \(sgn g'(\theta) = -sgn b'(\theta)\) for all \(\theta \in (0,2\pi)\).

Observe that

\[
b_1(\theta) := 2\theta^5 b'(\theta) = -8(1 - \cos \theta) + 5\theta \sin \theta - \theta^2 \cos \theta, \quad b_1(0+) = 0, \quad b_1(2\pi) = -4\pi^2,
\]

\[
b_1'(\theta) = (\theta^2 - 3) \sin \theta + 3\theta \cos \theta, \quad b_1'(0+) = 0, \quad b_1'(2\pi) = 6\pi,
\]

\[
b_2(\theta) := b_1''(\theta) = \cos \theta - \frac{\sin \theta}{\theta}, \quad b_2(0+) = 0, \quad b_2(\pi) = -1, \quad b_2(2\pi) = 1.
\]

It is easy to see that there exists a unique \(\theta_0 \in (\pi,2\pi)\) such that

\[
b_2(\theta) < 0, \quad 0 < \theta < \theta_0,
\]

\[
b_2(\theta) > 0, \quad \theta_0 < \theta < 2\pi,
\]

With the account of \(sgn b_2 = sgn b_1''(\theta)\) and \(b_1'(0+) > 0, \quad b_1'(2\pi) > 0\), we conclude that \(b_1'\) has a unique sign change in a point \(\theta_1 \in (\theta_0,2\pi) \subset (\pi,2\pi)\) and

\[
b_1'(\theta) > 0, \quad 0 < \theta < \theta_0,
\]

\[
b_1'(\theta) < 0, \quad \theta_0 < \theta < 2\pi.
\]

Hence, \(b_1\) has the unique point of minimum \(\theta_1\) on \((0,2\pi)\) and for all \(\theta \in (0,2\pi)\)

\[
b_1(\theta) < \max\{b_1(0+), b_1(2\pi)\} = 0,
\]

hence, \(b_1'\) is a strictly decreasing function on \((0,2\pi)\) and for all \(\theta \in (0,2\pi)\)

\[
b_1(\theta) > b_1(2\pi) = -4\pi^2.
\]

Theorem 4. For \(\varepsilon > 0, \tau \geq 0, \, u \geq 0, \quad \text{and} \quad \theta \in (0,2\pi)\) let

\[
k(\tau, u, \theta) := a(\theta) - 4\tau u^{-1} (a(\theta) + b(\theta) u^2) \quad \text{for} \, \quad u > 0, \quad k(0,0,\cdot) := a(\theta), \quad k(\tau, 0, \cdot) := -\infty \quad \text{for} \, \tau > 0,
\]

\[
k(\tau, u) := \sup_{0 < \theta < 2\pi} \left\{ k(\tau, u, \theta) : 0 \leq u \leq u \sqrt{a(\theta) / b(\theta)} \right\}, \quad k(\tau, 0) := -\infty
\]

\[
k(\tau) = \sup_{\theta \in (0,2\pi], u > 0} k(\tau, u, \theta) = \sup_{0 < \theta < 2\pi} \left\{ k(\tau, \sqrt{a(\theta) / b(\theta)}, \theta) \right\},
\]

where the functions \(a(\theta), b(\theta)\) are defined in the formulation of Lemma \([\Box]\) Then for each of the fractions \(L \in \{L_{k,n}, L_{i,n}\}\) we have
(i) for every \( t \in \mathbb{R} \)

\[
|T_n(t)| \leq \exp \left\{ -k \left( L^3 |t|, 2t^2 \right) \right\};
\]

(ii) for every \( L_0 \geq L, \tau \geq T_1(L_0, \varepsilon) := \pi L_0^3/\varepsilon \) and \(|t| \leq \tau / L^3\)

\[
|T_n(t)| \leq \exp \left\{ -k(\tau)^2 \right\},
\]

with \( k(\tau) > 0 \) if and only if \( \tau < T_1 := \frac{\pi}{4} = 0.7853 \ldots \) and the interval \([T_1(L_0, \varepsilon), T_1]\) being nonempty if and only if \( L_0 < (\varepsilon/4)^{1/3} \).

(iii) Statements (i) and (ii) remain true with \( L^3 = \sup_{0 < z < \varepsilon} z L_n(z) \).

**Remark 7.** The right-hand side of (20) is monotonically increasing with respect to \( L > 0 \), monotonically decreasing with respect to \( \varepsilon > 0 \) and does not depend on \( \gamma \) with \( L \) being fixed. The right-hand side of (27) does not depend neither on \( L \), nor on \( \varepsilon \) and \( \gamma \).

**Proof.** Due to the symmetry, one can assume that \( t \geq 0 \). Moreover, inequalities (20) and (27) hold trivially true with \( t = 0 \), so that, in what follows, we shall assume that \( t > 0 \).

Consider the absolute value of the ch.f. of every single random summand. Namely, let \( X \) and \( X' \) be i.i.d. centred r.v.’s with the d.f. \( F \), the ch.f. \( f \), and \( \sigma^2 := \mathbb{E}X^2 < \infty \). Denote \( F^s(x) := \mathbb{P}(X - X' < x), x \in \mathbb{R} \), the symmetrization of \( F \),

\[
\sigma^2(z) := \int_{|x| \geq z} x^2 dF(x), \quad \text{and} \quad \sigma^2_s(z) := \int_{|x| \geq z} x^2 dF^s(x), \quad z \geq 0.
\]

Then for every \( t \) under consideration and \( \theta \in (0, 2\pi] \) we have

\[
|f(t)|^2 = \mathbb{E} \cos t(X - X') = 1 - 2a(\theta)\sigma^2 t^2 + J,
\]

with \( a(\theta) \) defined in Lemma 1 and

\[
J = J(t, \theta) := \int_{\mathbb{R}} (\cos tx - 1 + a(\theta)t^2 x^2) dF^s(x).
\]

Separating the range of integration in \( J \) into two domains: \(|x| \geq z \) and \(|x| < z \) with arbitrary \( z \geq 0 \), we use the inequality \( \cos x - 1 \leq 0, x \in \mathbb{R} \), to bound the integrand in the first integral and Lemma 1 to bound the second and obtain

\[
J \leq at^2 \int_{|x| \geq z} x^2 dF^s(x) + bt^4 \int_{|x| < z} x^4 dF^s(x) = (a - bt^2 z^2) t^2 \sigma^2_s(z) + 2bt^4 \int_0^z x \sigma^2_s(x) dx,
\]

where, for short, we omitted the argument \( \theta \) of the functions \( a(\theta) \) and \( b(\theta) \) defined in Lemma 1 and used the representation

\[
\int_{|x| < z} x^4 dF(x) = - \int_0^z x^2 dx = -z^2 \sigma^2_s(z) + 2 \int_0^z x \sigma^2_s(x) dx, \quad z \geq 0,
\]

with \( F := F^s \), which is valid for arbitrary d.f. \( F \) with finite second moment. Now assuming that \( a \geq bt^2 z^2 \) we may apply Theorem 3 to bound \( \sigma^2_s(\cdot) \) in the majorant for \( J \) above and obtain

\[
J \leq 4t^2 \left( a - bt^2 z^2 \right) \sigma^2_s(z/2) + 8bt^4 \int_0^z x \sigma^2_s \left( \frac{x}{2} \right) dx, \quad z \geq 0.
\]

Now repeating the same procedure for every random summand \( X := X_k, k = 1, \ldots, n, \) we construct an estimate for the absolute value of the corresponding ch.f. \( |f_k(t)| \) in the form

\[
|f_k(t)|^2 = 1 - 2a_k \sigma^2_k t^2 + J_k,
\]

where

\[
J_k = J_k(t, \theta) \leq 4t^2 \left( a - bt^2 z^2 \right) \sigma^2_k(z/2) + 8bt^4 \int_0^z x \sigma^2_k \left( \frac{x}{2} \right) dx, \quad z \geq 0,
\]
with
\[ \sum_{k=1}^{n} J_k \leq 4t^2 \left( a - bt^2 z^2 \right) L_n \left( \frac{z}{2} \right) + 16bt^4 \int_0^{z} \frac{x}{2} L_n \left( \frac{x}{2} \right) dx \leq \left( 8t^2 z^{-1} \left( a - bt^2 z^2 \right) + 16bt^4 z \right) \sup_{0 < x \leq z/2} x L_n \left( x \right) \sup_{0 < x \leq \varepsilon} x L_n \left( x \right) \]
for all \( \varepsilon \geq z/2 \geq 0 \), provided that \( a \geq bt^2 z^2 \). Thus, using the inequality \( 1 + x \leq e^x \), \( x \in \mathbb{R} \), and observing that
\[ \sup_{0 < x \leq \varepsilon} x L_n \left( x \right) \leq \min \{ L_n^3, L_n^3 \} \leq L^3 \]
(\text{compare with part (iii) of the theorem}), we obtain
\[ \left| \mathcal{F}_n \left( t \right) \right| = \prod_{k=1}^{n} \left| f_k \left( t \right) \right| \leq \exp \left\{ -at^2 + \frac{1}{2} \sum_{k=1}^{n} J_k \right\} \leq \exp \left\{ -at^2 + 4L^3 t^2 z^{-1} \left( a + bt^2 z^2 \right) \right\} = \exp \left\{ -t^2 \left( a + 4L^3 z^{-1} \left( a + bt^2 z^2 \right) \right) \right\} =: \exp \left\{ -t^2 \cdot k \left( L^3 t, tz, \theta \right) \right\} \]
for all \( t \in (0, \infty) \), \( 0 < z \leq \frac{1}{7} \sqrt{\frac{a(\theta)}{b(\theta)}} \wedge 2 \varepsilon =: z_* \left( t, \theta, \varepsilon \right) \), and \( \theta \in (0, 2\pi) \). Note that the function
\[ k \left( \tau, u, \theta \right) = a - 4\tau u^{-1} \left( a + bu^2 \right) \]
attains its global maximum with respect to \( u \in (0, \infty) \) for every fixed \( \tau > 0 \) and \( \theta \in (0, 2\pi) \) at the point \( u = \sqrt{\frac{a(\theta)}{b(\theta)}} =: u_* \left( \theta \right) \),
being monotonically increasing for \( 0 < u < u_* \left( \theta \right) \) and monotonically decreasing for \( u > u_* \left( \theta \right) \) (in fact, \( k \left( \tau, u, \theta \right) \) is concave with respect \( u > 0 \) for every fixed \( \tau > 0 \) and \( \theta \in (0, 2\pi) \)). Now choosing \( z = z_* \), to minimize the right-hand side of \([29]\) with respect to \( z \in (0, z_*) \) (which is equivalent to the choice of \( u = u_* \left( \theta \right) \wedge 2\varepsilon t \)) and optimizing then with respect to \( \theta \in (0, 2\pi) \) we arrive at \([29]\) with the right-hand side being a monotonically increasing function of \( L > 0 \) as the least upper bound to a family of monotonically decreasing functions.

Let us prove that \([29]\) (or \([29]\)) yields \([27]\) under the assumptions stated in the formulation of part (ii) of the theorem. For this purpose, observe that for every \( \theta \in (0, 2\pi) \)
\[ -k \left( L^3 t, tz_*, \theta \right) + a = \begin{cases} 2aL^3 \varepsilon^{-1} + 8bL^3 \varepsilon t^2 \leq 2aL^3 \varepsilon^{-1} + 4L^3 t \sqrt{ab}, & 2\varepsilon t \leq \sqrt{a/b}, \\ 8L^3 t \sqrt{ab}, & 2\varepsilon t > \sqrt{a/b}, \end{cases} \]
so that for all \( t \in \mathbb{R} \) we have
\[ -k \left( L^3 t, tz_*, \theta \right) + a \leq \max \left\{ 2aL^3 \varepsilon^{-1} + 4L^3 t \sqrt{ab}, 8L^3 t \sqrt{ab} \right\}. \]
Now let \( \tau \) be an arbitrary positive number. Then for \( 0 < t \leq \tau / L^3 \) we trivially obtain
\[ -k \left( L^3 t, tz_*, \theta \right) + a \leq \max \left\{ 2aL^3 \varepsilon^{-1} + 4\sqrt{ab}, 8\sqrt{ab} \right\} = 8\sqrt{ab}, \quad \text{if} \quad \tau \geq \frac{L^3}{2\varepsilon} \sqrt{\frac{a}{b}}, \]
and if for \( L_0 \geq L \) the uniform condition
\[ \tau \geq \frac{L^3}{2\varepsilon} \sup_{0 < \theta \leq 2\pi} \sqrt{\frac{a(\theta)}{b(\theta)}} = \frac{L^3}{2\varepsilon} \lim_{\theta \to 2\pi} \sqrt{\frac{a(\theta)}{b(\theta)}} = \frac{\pi L^3}{2\varepsilon} =: L_0, \varepsilon \]
holds, where the equality sign is due to Lemma \([4]\) then also
\[ -k \left( L^3 t, 2\varepsilon t \right) := - \sup_{0 < \theta \leq 2\pi} k \left( L^3 t, tz_*(t, \theta, \varepsilon), \theta \right) \leq - \sup_{0 < \theta \leq 2\pi} \left\{ a(\theta) - 8\sqrt{a(\theta)b(\theta)} \right\} =: -k(\tau). \]
for every $t \leq \tau/L^3$ and $L \geq L_0$. Note that for fixed $\varepsilon$ the quantity $\mathcal{L}_i(L_0, \varepsilon)$ can be made arbitrarily small by the choice of $L_0 = L_0(\varepsilon) > 0$ small enough.

Finally, observe that the function $h(\tau, \theta) := a(\theta) - 8\tau \sqrt{a(\theta)b(\theta)}$, $\tau > 0$, $\theta \in (0, 2\pi)$ is strictly positive if and only if $\tau < \frac{1}{8} \frac{a(\theta)}{b(\theta)} =: \tau_1(\theta)$, and hence, $k(\tau) := \sup_{0 < \theta < 2\pi} h(\tau, \theta) > 0$ if and only if there exists $\theta \in (0, 2\pi)$ such that $\tau < \tau_1(\theta)$. The latest condition is equivalent to $\tau < \sup_{0 < \theta < 2\pi} \tau_1(\theta) = \pi/4 := \tau_1$. The proof is completed by the remark that the set of admissible values of $\tau \in [\mathcal{L}_i(L_0, \varepsilon), \tau_1]$ is not empty if and only if

$$\mathcal{L}_i(L_0, \varepsilon) := \pi L_0^3/\varepsilon < \pi/4 := \tau_1,$$

i. e. $L_0 < (\varepsilon/4)^{1/3}$.

### 3.2 Estimation of $|\mathcal{F}_n(t) - e^{-t^2/2}|$.

The present section is aimed at bounding the absolute value of the difference $|\mathcal{F}_n(t) - e^{-t^2/2}|$ between the c.d.f. of the normalized sum and that of the standard normal law. Similarly to the preceding section, we construct two estimates, the first being the most exact and used in Subsection 4.2 and the second being the most convenient for analytical work in Subsection 3.3.

However, unlike in the previous section, here significant distinctions arise in the dependence of which of the fractions $L_{3,n}^3$ or $L_{3,n}$ is used. We pay special attention to the appearing distinctions and explain their reasons.

Before passing to the main theorem of the present subsection we provide four auxiliary statements.

**Lemma 3.** For every $x \in \mathbb{R}$ we have

$$|e^{ix} - 1 - ix - (ix)^2/2| \leq \kappa \cdot x^2,$$

where $\kappa := x^{-2} \sqrt{(\cos x - 1 + x^2/2)^2 + (\sin x - x)^2}|_{x = x_0} = 0.531551 \ldots$ and $x_0 = 5.487414 \ldots$ is the unique root of the equation

$$8(\cos x - 1) + 8x \sin x - 4x^2 \cos x - x^3 \sin x = 0$$

on the interval $(0, 2\pi]$ lying in $(\pi, 2\pi)$.

**Proof.** Due to the symmetry and triviality of the stated inequality for $x = 0$ we may assume that $x > 0$. Observe that the inequality of interest is equivalent to the relation

$$\max_{x > 0} g(x) = g(x_0) = 0.2825 \ldots = \kappa^2, \quad \text{where} \quad g(x) := \frac{(\cos x - 1 + x^2/2)^2 + (\sin x - x)^2}{x^4}, \quad x > 0.$$

We have

$$h(x) := x^5 g'(x) = 8(\cos x - 1) + 8x \sin x - 4x^2 \cos x - x^3 \sin x, \quad h'(x) = x^2 (\sin x - x \cos x).$$

Note that the function $h'(x)$ has a unique root $x_1 \in (\pi, 3\pi/2)$ on $(0, 2\pi]$, which is the point of maximum of $h(x)$. Since, $h(0) = 0$ and $h$ is strictly increasing on $(0, x_1)$, we conclude that $h(x_1) > 0$. Moreover, $h(2\pi) = -16\pi^2 < 0$, hence, $h$ has a unique sign change on $(0, 2\pi]$ and this sign change occurs in the points $x_1 \in (x_1, 2\pi) \subset (\pi, 2\pi)$ from $+$ to $-$. Thus, the function $g$ has a unique stationary point $x_0 \in (\pi, 2\pi)$ on $(0, 2\pi]$, which is the point of maximum. So, it remains to prove that $g(x) \leq g(x_0)$ for $x > 2\pi$.

If $x = 2\pi + y \in (2\pi, 9\pi/4)$, then, with the account of the monotone increase of $\sin y$ and monotone decrease of $\cos y$ for $y \in (0, \pi/4]$, we have

$$\frac{h'(x)}{x^2} = \sin y - (y + 2\pi) \cos y \leq \sin \frac{\pi}{4} - 2\pi \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}(1 - 2\pi) < 0.$$

Hence, $h(x)$ is strictly monotonically decreasing for $x \in (2\pi, 9\pi/4]$ with

$$\sup_{x \in (2\pi, 9\pi/4]} h(x) = h(2\pi +) = -16\pi^2 < 0,$$
so that $g'(x) := x^{-5}h(x) < 0$ for $x \in (2\pi, 9\pi/4]$ and
\[
\sup_{x \in (2\pi, 9\pi/4]} g(x) = g(2\pi+) = \frac{1}{4}(1 + \pi^{-2}) = 0.2753 \ldots < 0.2825 \ldots = \kappa^2.
\]
Finally, for $x > 9\pi/4$ we use the trivial bound
\[
g(x) = \frac{2(1 - \cos x) + x^4/4 + x^2 \cos x - 2x \sin x}{x^4} < \left(\frac{1}{4x^2} + \frac{1}{x^2} + \frac{2}{x^3}\right)_{x = 9\pi/4} < 0.2757 < \kappa^2,
\]
which completes the proof of the lemma.

\[\square\]

**Lemma 4.** For $\varepsilon > 0$ denote
\[
\alpha(\varepsilon) := \inf_{0 < x < \varepsilon} (x - x^3)^{-2/3} = \begin{cases} (\varepsilon - \varepsilon^3)^{-2/3}, & \text{if } \varepsilon \leq 3^{-1/2}, \\ 3 \cdot 2^{-2/3} = 1.8898 \ldots, & \text{otherwise.}
\end{cases}
\]
Then for every $\varepsilon > 0$ we have
\[
\max_{1 \leq k \leq n} \sigma_k^2 \leq \alpha(\varepsilon) \sup_{0 < x \leq \varepsilon} (xL_n(x))^{2/3}.
\]

**Proof.** From the definition of $L_n(z) := \sum_{k=1}^n \sigma_k^2(z)$ it follows that
\[
\sup_{0 < x \leq \varepsilon} xL_n(x) \geq z\sigma_k^2(z)
\]
for every $z \in (0, \varepsilon]$ and $k = 1, \ldots, n$. On the other hand, by the definition of $\sigma_k^2 := E X_k^2$, we have
\[
\sigma_k^2 = EX_k^21(|X_k| < z) + EX_k^21(|X_k| \geq z) \leq z^2 + \sigma_k^2(z), \quad z > 0, \quad k = 1, \ldots, n,
\]
whence it follows that
\[
\sup_{0 < x \leq \varepsilon} xL_n(x) \geq z \left(\sigma_k^2 - z^2\right) \quad \text{for} \quad 0 < z \leq \varepsilon.
\]
Now the choice of $z := \sigma_k/\sqrt{3} \wedge \varepsilon \sigma_k \leq \varepsilon$ as a maximizer to the right-hand side of the latest inequality completes the proof of the lemma.

\[\square\]

**Lemma 5.** For all $t \in \mathbb{R}$ and $0 < z \leq \varepsilon \leq \infty$ we have
\[
\sum_{k=1}^n |f_k(t)|^2 \leq t^4 \left[\frac{z}{2} \sup_{0 < x \leq \varepsilon} xL_n(x) + \frac{1}{4} \sum_{k=1}^n \sigma_k^2(z)\right] \leq t^4 \left[\frac{z}{2} \sup_{0 < x \leq \varepsilon} xL_n(x) + \frac{\alpha(\varepsilon)}{4z} \sup_{0 < x \leq \varepsilon} (xL_n(x))^{5/3}\right].
\]

**Proof.** For every random summand $X$ with the d.f. $F$, the ch.f. $f$ and $EX = 0$, $\sigma^2(z) := EX^21(|X| \geq z)$, $z \geq 0$, $\sigma^2 := \sigma^2(0) = EX^2$ we have
\[
|f(t)|^2 \leq \frac{(t^2\sigma^2)^2}{2} = \frac{t^4}{4} \left[\int_{|x| < z} x^2dF(x) + \sigma^2(z)\right]^2 = \frac{t^4}{4} \left[\left(\int_{|x| < z} x^2dF(x)\right)^2 + 2\sigma^2(z) \int_{|x| < z} x^2dF(x) + (\sigma^2(z))^2\right], \quad z \geq 0.
\]
Using the Jensen inequality for the first term, both the representation $\int_{|x| < z} x^2dF(x) = \sigma^2 - \sigma^2(z)$ and the bound $\int_{|x| < z} x^2dF(x) \leq z^2$ for the second term in square brackets in the first step below, and then equality (28) we obtain
\[
|f(t)|^2 \leq \frac{t^4}{4} \left[\int_{|x| < z} x^2dF(x) + z^2\sigma^2(z) + \sigma^2(z)^2\right] = \frac{t^4}{4} \left[2 \int_0^z x\sigma^2(x)dx + \sigma^2\sigma^2(z)\right].
\]
Summing up the constructed bounds for every random summand $X := X_k$ over all $k = 1, \ldots, n$, we get
\[
\sum_{k=1}^n |f_k(t)|^2 \leq \frac{t^4}{4} \left[2 \int_0^z xL_n(x)dx + \sum_{k=1}^n \sigma_k^2\sigma_k^2(z)\right] \leq t^4 \left[\frac{z}{2} \sup_{0 < x \leq \varepsilon} xL_n(x) + \frac{1}{4} \sum_{k=1}^n \sigma_k^2\sigma_k^2(z)\right].
\]
Now the observation that the least upper bound with respect to \( x \in (0, z] \) is a non-decreasing function of \( z \) yields the first claim of the lemma. The second claim follows from Lemma 4 yielding the chain of inequalities
\[
\sum_{k=1}^{\infty} \sigma_k^2 \sigma_k^2(z) \leq \alpha(\varepsilon) \sup_{0 < x \leq z} \left( x L_n(x) \right)^{2/3} \leq \alpha(\varepsilon) \sup_{0 < x \leq z} \left( x L_n(x) \right)^{5/3}, \quad z \in (0, \varepsilon].
\]

\[\square\]

**Lemma 6.** For all \( t \in \mathbb{R}, \varepsilon > 0, \) and \( \gamma > 0 \) we have
\[
\left| \sum_{k=1}^{n} \left( f_k(t) - 1 + \sigma_k^2 t^2 / 2 \right) \right| \leq p_e(t, \varepsilon, \gamma) \cdot L_{k,n}(t, \varepsilon, \gamma), \quad (30)
\]
\[
\left| \sum_{k=1}^{n} \left( f_k(t) - 1 + \sigma_k^2 t^2 / 2 \right) \right| \leq p_n(t, \varepsilon, \gamma) \cdot L_{n,n}(t, \varepsilon, \gamma), \quad (31)
\]
where
\[
p_e(t, \varepsilon, \gamma) := t^2 \min_{0 < x \leq \varepsilon} \left\{ \frac{\varepsilon t^2}{24} + \max \left\{ \frac{|t|}{6\gamma}, \frac{\varepsilon t^2}{24} \right\} \right\} = \begin{cases} \frac{\varepsilon t^2}{24}, & \varepsilon |t| \leq \gamma, \\ \frac{\varepsilon t^2}{24}, & \varepsilon |t| > \gamma, \\ \end{cases}
\]
\[
p_n(t, \varepsilon, \gamma) := t^2 \max \left\{ \frac{|t|}{6\gamma}, \frac{\varepsilon t^2}{24}, \frac{\varepsilon t^2}{12} \right\} = \begin{cases} \frac{\varepsilon t^2}{12}, & \varepsilon |t| \leq \gamma, \\ \frac{\varepsilon t^2}{24}, & \varepsilon |t| > \gamma, \\ \end{cases}
\]
\[
t_\gamma := \frac{2}{\gamma} \left( \sqrt{(\gamma / \gamma_*)^2 + 1} - 1 \right), \quad t_\infty := \lim_{\gamma \to \infty} t_\gamma = 2 / \gamma_* = 3.5717 \ldots,
\]
\[
t_2, \gamma := 2 \max \left\{ \gamma^{-1}, \gamma_*^{-1} \right\}, \quad t_1, \gamma := t_2, \gamma (1 - \sqrt{(1 - (\gamma / \gamma_*)^2) / \gamma_*)},
\]
\( \gamma_* = 1 / \sqrt{6\pi} = 0.5599 \ldots \) and \( \gamma \) is defined in Lemma 3. Moreover, the functions \( t_\gamma \) and \( t_1, \gamma \) are monotonically increasing with respect to \( \gamma \), and \( t_\gamma \leq t_1, \gamma \leq t_\infty \) for all \( \gamma > 0 \). The functions \( p_n(t, \varepsilon, \gamma) \), \( p_n(t, \varepsilon, \gamma) \) are monotonically decreasing with respect to \( \gamma > 0 \) with \( p_n(t, \varepsilon, \gamma) \) do not depending on \( \gamma \) for \( \gamma \geq \gamma_* \), and \( p_n(t, \varepsilon, \gamma) \) also monotonically decreasing with respect to \( \varepsilon \).

The values of the function \( t_\gamma = 2\gamma^{-1} \left( \sqrt{(\gamma / \gamma_*)^2 + 1} - 1 \right) \) for some \( \gamma \) are given in Table 1. The plots of the functions \( t_\gamma \) and \( t_1, \gamma \) are given on Fig. 1 in the Appendix.

| \( \gamma \) | \( 0^+ \) | \( 0.25 \) | \( 0.41 \) | \( \gamma_* \) | \( 0.73 \) | \( 1 \) | \( 1.25 \) | \( 1.5 \) | \( \infty \) |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \( t_\gamma \) | 0.7611 | 1.1678 | 1.4794 | 1.7617 | 2.0935 | 2.3137 | 2.4791 | 3.5717 |

Table 1. Values of the function \( t_\gamma = 2\gamma^{-1} \left( \sqrt{(\gamma / \gamma_*)^2 + 1} - 1 \right) \) for some \( \gamma \) (rounded down).

**Remark 8.** For \( \gamma \geq \gamma_* \) we have \( t_1, \gamma = t_2, \gamma = 2 / \gamma_* = t_\infty \).

**Remark 9.** The functions \( p_n(t, \varepsilon, \gamma) \) and \( p_n(t, \varepsilon, \gamma) \) coincide on the interval \( |t| \leq t_\gamma / \varepsilon \) for every \( \gamma > 0 \).
Proof. Since \( \mathbb{E} X_k = 0 \) for all \( k = 1, \ldots, n \), for every \( t \in \mathbb{R} \) and \( z > 0 \) we have

\[
I := \left| \sum_{k=1}^{n} \left( f_k(t) - 1 + \sigma_k^2 t^2 / 2 \right) \right| \leq \sum_{k=1}^{n} \int_{\mathbb{R}} \left( e^{ixt} - 1 - itx - \frac{(ix)^2}{2} \right) dF_k(x) \leq \sum_{k=1}^{n} \int_{|x|<z} \left| e^{ixt} - 1 - itx - \frac{(ix)^2}{2} \right| dF_k(x) + \sum_{k=1}^{n} \int_{|x|\geq z} \left| e^{ixt} - 1 - itx - \frac{(ix)^2}{2} \right| dF_k(x).
\]

Using the inequalities \( |e^{iy} - 1 - iy - \frac{1}{2}(iy)^2 - \frac{1}{6}(iy)^3| \leq \frac{1}{24} y^4 \) and \( |e^{iy} - 1 - iy - \frac{1}{2}(iy)^2| \leq \kappa y^2 \) valid for all \( y \in \mathbb{R} \) and with the account of (28) we obtain

\[
I \leq \frac{t^4}{24} \sum_{k=1}^{n} \int_{|x|<z} x^4 dF_k(x) + \frac{|t|^3}{6} |M_n(z)| + \sigma^2 \sum_{k=1}^{n} \frac{\sigma_k^2(z)}{z} = \frac{t^4}{12} \int_0^z xL_n(x)dx - \frac{z^2 t^4}{24} L_n(z) + \frac{|t|^3}{6} |M_n(z)| + \sigma^2 L_n(z) \leq \frac{z t^4}{12} \sup_{0<x\leq z} xL_n(x) + \frac{\sigma^2}{z} \left( \frac{|t|^3}{6} |M_n(z)| \right)
\]

for all \( t \in \mathbb{R} \) and \( z > 0 \). Now choosing \( z = \frac{2}{t|t|} \left( \sqrt{6 \kappa^2 + 1} - 1 \right) \) and \( \varepsilon \) to minimize the R.H.S. of the expression in the large brackets with respect to \( z \in (0, \varepsilon] \) and observing that

\[
\sup_{0<x\leq z} \left\{ \frac{\sigma^2}{z} \left( \frac{|t|^3}{6} |M_n(z)| \right) \right\} = L_{\varepsilon,n}(z,\gamma) \leq L_{\varepsilon,n}(\varepsilon,\gamma)
\]

for every \( 0 < z \leq \varepsilon \) and \( \gamma > 0 \), we arrive at (30).

To prove (31), observe that (34) yields

\[
I \leq \frac{z t^4}{12} \sup_{0<x\leq z} xL_n(x) + \frac{\sigma^2}{z} \left( \frac{|t|^3}{6} |M_n(z)| \right) \leq \frac{z t^4}{12} \sup_{0<x\leq z} xL_n(x) + \frac{|t|^3}{6} |M_n(z)| \leq \frac{z t^4}{12} \sup_{0<x\leq z} xL_n(x) + \frac{|t|^3}{6} |M_n(z)| + \frac{|t|^3}{6} |M_n(z)| = p_n(t, z, \gamma) L_{\varepsilon,n}(z, \gamma)
\]

for every \( z > 0 \) and \( t \in \mathbb{R} \). The choice of \( z = \varepsilon \) completes the proof of inequality (31). The stated properties of the functions \( t\gamma, \bar{t}\gamma, p_\varepsilon \), and \( p_n \), as well as representations for \( p_\varepsilon \) and \( p_n \) in the right-hand sides of (32) and (33) are proved by use of elementary analysis.

Now we are ready to prove the main result of the present subsection.

Theorem 5. For \( \varepsilon > 0 \), \( \gamma > 0 \), \( L > 0 \), \( \tau > 0 \) denote

\[
\alpha(\varepsilon) := \inf_{0<x<\varepsilon} \left( x - x^3 \right)^{-2/3} = \begin{cases} (\varepsilon - \varepsilon^3)^{-2/3} & \text{if } \varepsilon \leq 1/\sqrt{\tau} = 0.5773 \ldots, \\ 3 \cdot 2^{-2/3} = 1.8898 \ldots & \text{otherwise}, \end{cases}
\]

\[\begin{align*}
\alpha(\varepsilon) := \inf_{0<x<\varepsilon} \left( x - x^3 \right)^{-2/3} &= \begin{cases} (\varepsilon - \varepsilon^3)^{-2/3} & \text{if } \varepsilon \leq 1/\sqrt{\tau} = 0.5773 \ldots, \\ 3 \cdot 2^{-2/3} = 1.8898 \ldots & \text{otherwise}, \end{cases} \\
\end{align*} \]

15
\[ \tau_0(\varepsilon) := \sqrt{\frac{2}{\alpha(\varepsilon)}}, \quad \tau_0(\varepsilon) := \varepsilon \sqrt{\frac{2}{\alpha(\varepsilon)}} = \frac{\varepsilon}{\tau_0(\varepsilon)}, \]

\[ \alpha_1(\varepsilon, L) = \min_{0 < u \leq L^{-1}} \left( \frac{u}{2} + \alpha(\varepsilon) \right) = \begin{cases} \sqrt{\frac{\alpha(\varepsilon)}{2}}, & L \leq \tau_0(\varepsilon), \\ \frac{\alpha(\varepsilon)}{2}, & L > \tau_0(\varepsilon), \end{cases} \]

\[ \kappa = \sup_{x > 0} x^{-2} \sqrt{(\cos x - 1 + x^2/2)^2 + (\sin x - x)^2} = 0.5315 \ldots, \quad \gamma_\ast = 1/\sqrt{6\kappa} = 0.5599 \ldots. \]

For \( \varepsilon > 0, \gamma > 0, L > 0, 0 < \tau < \tau_0(\varepsilon) \) also introduce the functions

\[ B(\tau, \varepsilon, L) = -\frac{4\alpha_1(\varepsilon, L)}{\alpha^2(\varepsilon)\tau^4} \left[ \ln \left( 1 - \frac{1}{2} \alpha(\varepsilon) \tau^2 \right) + \frac{1}{2} \alpha(\varepsilon) \tau^2 \right], \quad B(\tau, \varepsilon) = -\frac{\sqrt{\alpha(\varepsilon)}}{2} \frac{\sqrt{\tau^2}}{\alpha(\varepsilon)\. \tau^2}, \]

\[ A(\tau, \varepsilon) = \exp \left\{ \tau^4 B(\tau, \varepsilon) \right\}, \quad A_n(\tau, \varepsilon, \gamma, L) = A(\tau, \varepsilon) \exp \left\{ \frac{\tau^3}{6\gamma} + \sqrt{\frac{\kappa}{3}} \tau^3 + \frac{\kappa}{6\gamma} L \tau^2 \right\}, \]

\[ A_n(\tau, \varepsilon, \gamma, L) = A(\tau, \varepsilon) \exp \left\{ \frac{\tau^3}{6\gamma} \right\} \left( 1 < \gamma < \gamma_\ast \right) + \frac{\kappa}{6\gamma} L \tau^2, \]

(i) For every \( L \in \{ L_{n,\varepsilon}(\varepsilon, \gamma), L_{n,\varepsilon}(\varepsilon, \gamma) \}, \varepsilon > 0, \gamma > 0, \) and \( L|t| < \tau \leq \tau_0(\varepsilon) \) we have

\[ r_n(t) := |f_n(t)| - e^{-t^2/2} \leq \left( \exp \left\{ L^3 p(t, \varepsilon, \gamma) + L^4 t^4 B(L|t|, \varepsilon, L) \right\} - 1 \right) e^{-t^2/2}, \quad (35) \]

where \( p = p_n \) for \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \) and \( p = p_n \) for \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \) with \( p_n, p \) defined in Lemma 4. In particular, for \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \) with \( \varepsilon = \infty \) we have

\[ r_n(t) \leq \left( \exp \left\{ \frac{6\kappa \gamma^2 + 1}{6\gamma} \right\} L^3 - 4\sqrt{2} \cdot 3^{-3/2} \left[ \ln \left( 1 - 3 \cdot 2^{2-5/3} \tau^2 \right)irst}\right. + 3 \cdot 2^{-5/3} \tau^2 \right\} \tau^{-4} t^4 L^4 \right\} - 1 \right) e^{-t^2/2} \]

for \( L|t| < 2^{5/6}/\sqrt{3} = 1.0287 \ldots. \) Moreover, the right-hand side of (35) is monotonically increasing with respect to \( L > 0 \) and monotonically decreasing with respect to \( \gamma > 0 \) and, for \( L = L_{n,\varepsilon}, \) also with respect to \( \varepsilon > 0. \) The right-hand side of (35) with \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \) does not depend on \( \gamma \) for \( \gamma \geq \gamma_\ast. \)

(ii) For \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \leq \tau_0(\varepsilon) \) and \( L|t| < \tau \leq \tau_0(\varepsilon) \), we have

\[ r_n(t) \leq A_n(\tau, \varepsilon, \gamma, L) \left( p_n(t, \varepsilon, \gamma) + B(\tau, \varepsilon)Lt^4 \right) \cdot L^3 e^{-t^2/2}, \quad (36) \]

in particular, with \( \varepsilon = \infty, \)

\[ r_n(t) \leq \exp \left\{ \frac{\tau^3}{6\gamma} + \sqrt{\frac{\kappa}{3}} \tau^3 + \tau^4 B(\tau) \right\} \left( \frac{\sqrt{6\kappa \gamma^2 + 1}}{6\gamma} + B(\tau)L|t| \right) \cdot L^3 t^3 e^{-t^2/2}, \quad t \in \mathbb{R}, \]

where \( B(\tau) := B(\tau, \infty) = (\sqrt{3} \cdot 2^{-1/6}) \right) \left( 2^{5/3} - 3\gamma^2 \right). \) Moreover, the right-hand side of (36) is monotonically increasing with respect to \( L > 0 \) and monotonically decreasing with respect to \( \varepsilon > 0 \) and \( \gamma > 0. \)

For \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \leq \tau_0(\varepsilon) \) and \( L|t| < \tau \leq \tau_0(\varepsilon) \) with \( B := B(\tau, \varepsilon) \), we have

\[ r_n(t) \leq A_n(\tau, \varepsilon, \gamma, L) \left( p_n(t, \varepsilon, \gamma) + B(\tau, \varepsilon)Lt^4 \right) \cdot L^3 e^{-h(\tau, \varepsilon, L)t^2/2}, \quad (37) \]

where \( h(\tau, \varepsilon, L) := 1 - \varepsilon \tau^2 L/6 > 0 \) for all \( \tau \in (0, \tau_0(\varepsilon)) \) if and only if \( L \leq 6/(\varepsilon \tau_0(\varepsilon)) \). Moreover, the right-hand side of (37) is monotonically increasing with respect to \( L > 0 \) and monotonically decreasing with respect to \( \gamma > 0. \)

**Remark 10.** The right-hand sides of (35) with \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \) and (37) are unbounded as \( \varepsilon \to \infty. \)

**Proof.** Fix arbitrary \( \varepsilon, \gamma > 0 \) and let \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \) or \( L = L_{n,\varepsilon}(\varepsilon, \gamma) \). With \( u_k := f_k(t) - 1 \) Taylor’s formula and Lemma 4 yield

\[ |u_k| = |f_k(t) - 1| \leq \frac{t^2 \sigma_k^2}{2} \leq \frac{\alpha(\varepsilon)}{2} L^2 t^2 \]

(38)
for every \( k = 1, \ldots, n \) and \( t \in \mathbb{R} \), whence it follows that \( |u_k| < 1 \) for \( L|t| < \sqrt{2/\alpha(e)} =: \tau_0(e) \) and that \( \mathcal{T}_n(t) = \prod_{k=1}^{k} (1+u_k) \) does not vanish for \( L|t| < \tau_0(e) \). Hence, the logarithm \( \ln \mathcal{T}_n(t) \) is well-defined for \( L|t| < \tau_0(e) \). Considering the main branch of the logarithm and using the inequality \( |e^z - 1| \leq e^{|z|} - 1, \ z \in \mathbb{C} \), we get

\[
\begin{align*}
\delta_n(t) := & \left| \frac{t^2}{2} + \ln \mathcal{T}_n(t) \right| = \left| \sum_{k=1}^{n} \left( \ln (1 + u_k) + \frac{u_k^2}{2} \right) \right| \\
& \leq \left| \sum_{k=1}^{n} \ln (1 + u_k) - u_k \right| + \left| \sum_{k=1}^{n} u_k - \frac{\sigma_k^2 t^2}{2} \right| \\
& \leq \sum_{k=1}^{n} \sum_{j=2}^{\infty} \frac{|u_k|^j}{j} + \sum_{k=1}^{n} \left( f_k(t) - 1 + \frac{\sigma_k^2 t^2}{2} \right) =: I_2 + I_1.
\end{align*}
\]

By Lemma 6 we have

\[
I_1 \leq p(t, \varepsilon, \gamma) \cdot L^3 \quad \text{for} \quad t \in \mathbb{R}, \ \varepsilon > 0, \ \gamma > 0,
\]

where \( p = p_n \) if \( L = L_{\text{inf}}(\varepsilon, \gamma) \) and \( p = p_n \) if \( L = L_{\text{inf}}(\varepsilon, \gamma) \).

To bound \( I_2 \) above, observe that inequality (38) yields

\[
I_2 \leq \sum_{k=1}^{n} |u_k|^2 \sum_{j=2}^{\infty} \frac{1}{j} \left( \frac{\alpha(e)}{2} L^2 t^2 \right)^{j-2} = -\frac{4}{\alpha^2(e) L^4 t^4} \left[ \ln \left( 1 - \frac{\alpha(e)}{2} L^2 t^2 \right) + \frac{\alpha(e)}{2} L^2 t^2 \right] \sum_{k=1}^{n} |u_k|^2
\]

for \( L|t| < \tau_0(e) \), while Lemma 5 with the account of the inequality \( \sup_{0 < \varepsilon \leq L} x L_n(x) \leq L^3 \), implies that

\[
\sum_{k=1}^{n} |u_k|^2 = \sum_{k=1}^{n} |f_k(t) - 1|^2 \leq t^4 \left( \frac{z}{2} L^3 + \frac{\alpha(e)}{4} L^5 \right) =: t^4 L^4 \cdot \tilde{\alpha}_1 (\varepsilon, z L^{-1}) \quad t \in \mathbb{R}, \ z \in [0, \varepsilon] ,
\]

where

\[
\tilde{\alpha}_1 (\varepsilon, u) = \frac{u}{2} + \frac{\alpha(e)}{4u}, \quad \varepsilon > 0, \ u > 0.
\]

Choosing \( u = z L^{-1} = \sqrt{\alpha(e)/2} \wedge z L^{-1} := u_s(L, \varepsilon) \) to minimize \( \tilde{\alpha}_1 (\varepsilon, u) \) with respect to \( u \in (0, \varepsilon L^{-1}] \) we obtain

\[
\sum_{k=1}^{n} |u_k|^2 \leq \alpha_1 (\varepsilon, L) \cdot L^4 t^4
\]

with \( \alpha_1 (\varepsilon, L) := \min_{0 < \alpha \leq L^{-1}} \tilde{\alpha}_1 (\varepsilon, u) = \alpha_1 (\varepsilon, u_s(L, \varepsilon)) \) as in the formulation of the lemma. Thus, for \( L|t| < \tau_0(e) \) we have

\[
I_2 \leq -\frac{4 \alpha_1 (\varepsilon, L)}{\alpha^2(e)} \left[ \ln \left( 1 - \frac{1}{2} \alpha(e) L^2 t^2 \right) + \frac{1}{2} \alpha(e) L^2 t^2 \right] := B(L|t|, \varepsilon, L) \cdot L^4 t^4,
\]

and

\[
\delta_n(t) \leq I_1 + I_2 \leq p(t, \varepsilon, \gamma) \cdot L^3 + B(L|t|, \varepsilon, L) \cdot L^4 t^4,
\]

whence, with the account of the penultimate inequality in (39), we obtain (35). The observation that \( p(t, \varepsilon, \gamma) \) is monotonically decreasing with respect to \( \gamma \), the functions \( p_n(t, \varepsilon, \gamma), \alpha(e), \alpha_1 (\varepsilon, L) \) are monotonically decreasing with respect to \( \varepsilon \) and

\[
B(L|t|, \varepsilon, L) = \alpha_1 (\varepsilon, L) \sum_{j=0}^{\infty} \frac{\alpha(e) L^2 t^2/j^2}{j + 2}
\]

is monotonically increasing with respect to \( L > 0 \) and monotonically decreasing with respect to \( \varepsilon > 0 \) yields the stated properties of the right-hand side of (35).

Now let us assume that \( L \) is small and prove slightly rougher bounds than (35) that are more convenient for analytical work. By virtue of the inequality \(-\ln (1 - x) + x = \sum_{j=0}^{\infty} x^j/j \leq \frac{1}{2} \cdot \frac{x^2}{1-x^2}, \ 0 \leq x < 1 \), applied to \( x := \alpha(e) L^2 t^2/2 \) we have

\[
B(L|t|, \varepsilon, L) \leq \frac{\alpha_1 (\varepsilon, L)}{2 - \alpha(e) L^2 t^2} \leq \frac{\alpha_1 (\varepsilon, L)}{2 - \alpha(e) \tau^2} \quad \text{for} \quad L|t| < \tau \leq \sqrt{2/\alpha(e)} =: \tau_0(e),
\]

17
and if, in addition, $L \leq \varepsilon \sqrt{2/\alpha(\varepsilon)} =: L_0(\varepsilon)$, then $\alpha_1(\varepsilon, L) = \sqrt{\alpha(\varepsilon)/2}$ and we also get

$$B(L|t|, \varepsilon, L) \leq \frac{\sqrt{\alpha(\varepsilon)/2}}{2 - \alpha(\varepsilon)\tau^2} =: B(\tau, \varepsilon),$$

whence, with the account of (39), we obtain the factors $(p(t, \varepsilon, \gamma) + B(\tau, \varepsilon)Lt^4) \cdot L^3e^{-t^2/2}$ in (38) and (39), which are monotonically decreasing with respect to $\gamma > 0$ and $\varepsilon > 0$ and monotonically increasing with respect to $L > 0$. Finally, to bound $e^{\delta_n(t)}$ in (39) we observe that for all $L|t| \leq \tau$

$$p_n(t, \varepsilon, \gamma) = t^2 \min_{0 < z \leq \varepsilon} \left[ \frac{zt^2}{24} + \max \left\{ \frac{|t|}{6\gamma} + \frac{zt^2}{24}, \frac{zt^2}{24} \right\} \right] \leq L^{-2}\tau^2 \min_{0 < z \leq \varepsilon} \left[ \frac{zt^2}{24} + \max \left\{ \frac{\tau}{6\gamma} + \frac{zt^2}{24}, \frac{\tau}{6\gamma} \right\} \right].$$

Majorizing now the maximum of two non-negative numbers by their sum we obtain

$$L^3p_n(t, \varepsilon, \gamma) \leq \frac{\tau^3}{6\gamma} + L\tau^2 \min_{0 < z \leq \varepsilon} \left\{ \frac{z}{3} + \frac{zt^2}{12} \right\} = \frac{\tau^3}{6\gamma} + L\tau^2 \times \left\{ \frac{z^2}{24} + \frac{z}{6\gamma} \leq \frac{\sqrt{\tau^2}}{12} |t| + \frac{z}{6\gamma}, \varepsilon |t| \leq \frac{12z}{\sqrt{\tau}}, \frac{z}{6\gamma} \right\} \leq \frac{\tau^3}{6\gamma} + L\tau^2 \left( \frac{\tau}{3} |t| + \frac{\tau}{\varepsilon} \right) \leq \frac{\tau^3}{6\gamma} + \frac{\tau^3}{3} + \frac{\tau^2}{\varepsilon} L\tau^2,$$

with the latest expression being monotonically decreasing with respect to $\gamma > 0$ and $\varepsilon > 0$ and monotonically increasing with respect to $L > 0$. Thus the desired estimate takes the form

$$e^{\delta_n(t)} \leq \exp \left\{ \frac{\tau^3}{6\gamma} + \frac{\tau^3}{3} + \frac{\tau^2}{\varepsilon} L\tau^2 + B(\tau, \varepsilon)\tau^4 \right\} =: A_1(\tau, \varepsilon, \gamma, L) \quad \text{for} \quad L|t| \leq \tau,$$

whence, with the account of (39) and the above constructed estimate for $\delta_n(t)$, we obtain (39). Similarly, for $L = L_{n,n}$, using the explicit representation for $p_n$ in the right-hand side of (38), we have

$$L^3p_n(t, \varepsilon, \gamma) \leq L^3 \left( \frac{|t|^3}{6\gamma} \chi_{\varepsilon < \gamma, \tau} + \frac{zt^4}{\varepsilon} + \frac{\varepsilon t^4}{12} \right) \leq \frac{\tau^3}{6\gamma} \chi_{\varepsilon < \gamma, \tau} + \frac{\varepsilon L\tau^2}{12} - \frac{\varepsilon L\tau^2}{12} t^2,$$

$$e^{\delta_n(t)} \leq \exp \left\{ \frac{\tau^3}{6\gamma} \chi_{\varepsilon < \gamma, \tau} + \frac{\varepsilon L\tau^2}{12} - \frac{\varepsilon L\tau^2}{12} t^2 + B(\tau, \varepsilon)\tau^4 \right\} =: A_2(\tau, \varepsilon, \gamma, L) e^{\varepsilon L\tau^2 t^2/12} \quad \text{for} \quad L|t| \leq \tau,$$

with the right-hand side monotonically increasing with respect to $L > 0$ and monotonically decreasing with respect to $\gamma > 0$. Hence, with the account of (39), we obtain (37). \bbox

4 Proofs of the main results

The proof of Theorem 1 will be given simultaneously for both fractions $L = L_{n,n}(\varepsilon, \gamma)$ and $L = L_{n,n}(\varepsilon, \gamma)$ with $\varepsilon > 0$ and $\gamma > 0$ being fixed. In what follows by $C = C(\varepsilon, \gamma)$ we mean $C_1(\varepsilon, \gamma)$ or $C_2(\varepsilon, \gamma)$, sometimes omitting the arguments $\varepsilon, \gamma$. Following the outline of Zolotarev’s reasoning employed in [35] we construct an upper bound for $C(\varepsilon, \gamma)$ in the form $sup_{L \geq 0} C(\varepsilon, \gamma, L)$, where $C(\varepsilon, \gamma, L)$ is such a function of $L$ that the inequality

$$\Delta_n \leq C(\varepsilon, \gamma, L) \cdot L^3,$$

holds for all $n \in \mathbb{N}$ and all distributions of independent centered r.v.’s $X_1, \ldots, X_n$ with fixed value of the fraction of interest $L_{n,n}(\varepsilon, \gamma) = L$ or $L_{n,n}(\varepsilon, \gamma) = L$ for every $L > 0$. Due to the boundedness of the Kolmogorov distance $\Delta_n \leq 1$ for arbitrary d.f.’s, one can exclude from consideration large values of $L$. Moreover, the region of values of $L$ to be considered can be restricted even more by use of the following sharpened upper bound for $\Delta_n$ for standardized distributions.

**Lemma 7** (see [13]). For arbitrary r.v. $X$ with $0 < DX < \infty$ we have

$$\sup_{x \in \mathbb{R}} \left| P \left( \frac{X - EX}{\sqrt{DX}} < x \right) - \Phi(x) \right| \leq \sup_{x > 0} \left| \frac{1}{1 + x^2} - \Phi(-x) \right| = 0.5409 \ldots.$$
Lemma 7 implies that, in order to prove the inequality $\Delta_n \leq C \cdot L^3$ with the constant $C \geq C_{\min} := 2$, say, it suffices to consider $L < (0.541/C_{\min})^{1/3} < 0.65 =: L_1$. As this is so, we separate then the interval $L \in (0, L_1]$ into the two regions:

(i) $L \in (0, L_0]$ with some $L_0 = L_0(\varepsilon)$ small enough, where mostly analytical techniques is used;

(ii) $L \in (L_0, L_1]$, where numerical computations with the help of a computer are needed;

and construct an upper bound for $C$ as the maximum of the two corresponding bounds and the absolute constant $C_{\min}$:

$$C(\varepsilon, \gamma) \leq \max \left\{ C_{\min}, \max_{0 < L \leq L_0} C(\varepsilon, \gamma, L), \max_{L_0 < L \leq L_1} C(\varepsilon, \gamma, L) \right\},$$

where, in fact, the third term turns to be extremal. However, upper bounds for the asymptotically exact constants $C_{\varepsilon}(\varepsilon, \gamma)$, $C_{\min}(\varepsilon, \gamma)$ are obtained as limiting values of $C(\varepsilon, \gamma, L)$ as $L \to 0$.

To bound $\Delta_n$ above on each of the intervals $(0, L_0]$ and $(L_0, L_1]$, we use the method of characteristic functions realized by the Prawitz smoothing inequality.

**Lemma 8** (see [28]). For arbitrary d.f. $F$ with the ch.f. $f$ and for all $0 < T_0 \leq T_1$ we have

$$\sup_{x \in \mathbb{R}} |F(x) - \Phi(x)| \leq \frac{2}{T_1} \int_0^{T_0} \left| K \left( \frac{t}{T_1} \right) \right| \cdot |f(t) - e^{-t^2/2}| dt + \frac{2}{T_1} \int_{T_0}^{T_1} \left| K \left( \frac{t}{T_1} \right) \right| \cdot |f(t)| dt +$$

$$+ \frac{2}{T_1} \int_0^{T_0} \left| K \left( \frac{t}{T_1} \right) - \frac{iT_1}{2\pi t} \right| e^{-t^2/2} dt + \frac{1}{\pi} \int_{T_0}^{\infty} e^{-t^2/4} dt,$$

where

$$K(t) = \frac{1}{2} \left( 1 - |t| \right) + \frac{i}{2} \left( (1 - |t|) \cot \pi t + \frac{\text{sgn} t}{\pi} \right), \quad t \in (-1, 1) \setminus \{0\},$$

and $K(\pm 1) := 0$ defined by continuity. Moreover, the function $K(t)$ for all $t \in [-1, 1] \setminus \{0\}$ satisfies

$$|K(t)| \leq \frac{1.0253}{2\pi |t|}, \quad \left| K(t) - \frac{i}{2\pi t} \right| \leq \frac{1}{2} \left( 1 - |t| + \frac{\pi^2 t^2}{18} \right) \leq \frac{1}{2}. \quad (42)$$

By Lemma 8 we have

$$\Delta_n \leq I_1 + I_2 + I_3 + I_4,$$  \hspace{1cm} (43)

where

$$I_1 = \frac{2}{T_1} \int_0^{T_0} \left| K \left( \frac{t}{T_1} \right) \right| \cdot |\mathcal{F}_n(t) - e^{-t^2/2}| dt, \quad I_2 = \frac{2}{T_1} \int_{T_0}^{T_1} \left| K \left( \frac{t}{T_1} \right) \right| \cdot |\mathcal{F}_n(t)| dt,$$

$$I_3 = \frac{2}{T_1} \int_0^{T_0} \left| K \left( \frac{t}{T_1} \right) - \frac{iT_1}{2\pi t} \right| e^{-t^2/2} dt, \quad I_4 = \frac{1}{\pi} \int_{T_0}^{\infty} e^{-t^2/4} dt.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} 0 < T_0 \leq T_1.

In what follows we use the notation

$$T_0(\tau_0, L) := \frac{\tau_0}{L}, \quad T_1(\tau_1, L) := \frac{\tau_1}{L^3}, \quad \text{where} \quad \tau_0 \in (0, \tau_0(\varepsilon)), \quad \tau_1 \in (0, \tau_1)$$

are free parameters to be chosen below with $\tau_1 := \pi/4, \tau_0(\varepsilon) := \sqrt{2/\alpha(\varepsilon)}$ defined in Theorems 4, 5 respectively.

### 4.1 The case (i) $L \in (0, L_0]$ and the proof of Theorem 2

The purpose of the present subsection is to bound $C(\varepsilon, \gamma, L)$ above for $L \in (0, L_0]$ by an increasing function of $L$ with the best possible (within the method used) limiting value as $L \to 0$.

Let $L_0 > 0$ satisfy the conditions of Theorem 3(ii) and Theorem 5(ii):

$$L_0 < (\varepsilon/4)^{1/3} \quad \text{and} \quad L_0 \leq \tau_0(\varepsilon) := \varepsilon \sqrt{2/\alpha(\varepsilon)} = \varepsilon \tau_0(\varepsilon),$$

so that for $\varepsilon \geq 0.0557$ arbitrary $L_0 \leq 0.03$ surely fits. For $L = L_{n,n}$ we additionally assume that $L_0 \leq 6/(\tau_0^2(\varepsilon)) = 3\alpha(\varepsilon)/\varepsilon$, which provides the positiveness of the exponent in the right-hand side of (37). Since $\alpha(\varepsilon) \geq 3 \cdot 2^{-2/3}$, this assumption is surely satisfied if

$$\varepsilon L_0 \leq 9 \cdot 2^{-2/3} = 5.6696 \ldots,$$
in particular, for \( \varepsilon \leq 188 \) we may consider arbitrary \( L_0 \leq 0.03 \).

Let us describe the process of estimation of each term in (43) assuming that \( T_0 := T_0(\tau_0, L), T_1 := T_1(\tau_1, L) \) with \( \tau_1 \) also satisfying the condition of Theorem 5(ii): \( \tau_1 \geq \pi L_0^3/\varepsilon =: \tau_1(L_0, \varepsilon) \).

Adding and subtracting \((iI_1)/(2\pi t)\) from \( K(\frac{t}{T_1}) \) under the modulus sign in the integrand in \( I_1 \) and applying then the inequality \( |K(\frac{t}{T_1}) - i\frac{I_1}{2\pi t}| \leq \frac{1}{2}, |t| \leq T_1 \), from Lemma 8 we obtain

\[
I_1 \leq \frac{1}{\pi} \int_0^{T_0} \frac{r_n(t)}{t} \, dt + \frac{2}{T_1} \int_0^{T_0} \left| K \left( \frac{t}{T_1} \right) - i\frac{I_1}{2\pi t} \right| r_n(t) \, dt \leq I_{11} + I_{12},
\]

where

\[
r_n(t) := \left| \tilde{f}_n(t) - e^{-t^2/2} \right|, \quad I_{11} := \frac{1}{\pi} \int_0^{T_0} \frac{r_n(t)}{t} \, dt, \quad I_{12} := \frac{1}{T_1} \int_0^{T_0} r_n(t) \, dt.
\]

Further we use inequalities (46) and (47) from Theorem 5 to estimate the integrands \( r_n(t) \) in \( I_{11} \) and \( I_{12} \) and enlarge then the region of integration from \((0, T_0)\) to \((0, \infty)\). With the definitions of the upper and the lower incomplete gamma-functions yielding

\[
\int_{-\infty}^{\infty} t^s e^{-\kappa t^2} \, dt = \frac{k^{s+1/2}}{2} \Gamma \left( \frac{s+1}{2}, kx^2 \right), \quad \int_{0}^{\infty} t^s e^{-\kappa t^2} \, dt = \frac{k^{s+1/2}}{2} \gamma \left( \frac{s+1}{2}, kx^2 \right), \quad s, k, x > 0,
\]

- for the Essent-type fraction \( L = L_{e,n}(\varepsilon, \gamma) \) with arbitrary \( \varepsilon, \gamma > 0 \) we have

\[
\frac{I_{11}}{L^3} \leq \frac{A_k}{\pi} \left[ \int_0^{t_{1,1}/\varepsilon} \left( \frac{A_1}{\varepsilon} + \frac{\varepsilon \Gamma^2}{24} \right) e^{-t^2/2} \, dt + \frac{\sqrt{6\pi\gamma^2 + 1}}{6\gamma} \int_{t_{1,1}/\varepsilon}^{\infty} t^2 e^{-t^2/2} \, dt + BL \int_0^{\infty} t^2 e^{-t^2/2} \, dt \right] = \frac{A_k}{\pi} \left[ \frac{A_1}{\varepsilon} \gamma \left( \frac{3}{2}, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\varepsilon}{12} \gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\sqrt{6\pi\gamma^2 + 1}}{3\gamma} \Gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + 2BL \right],
\]

\[
\frac{I_{12}}{\tau_1} \leq \frac{A_k}{\tau_1} \left[ \int_0^{t_{1,1}/\varepsilon} \left( \frac{A_1}{\varepsilon} + \frac{\varepsilon \Gamma^2}{24} \right) e^{-t^2/2} \, dt + \frac{\sqrt{6\pi\gamma^2 + 1}}{6\gamma} \int_{t_{1,1}/\varepsilon}^{\infty} t^2 e^{-t^2/2} \, dt + BL \int_0^{\infty} t^4 e^{-t^2/2} \, dt \right] = \frac{A_k}{\tau_1} \left[ \frac{\sqrt{7}}{6\gamma} \Gamma \left( \frac{5}{2}, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\sqrt{6\pi\gamma^2 + 1}}{3\gamma} \Gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{3}{2} \sqrt{2\pi} BL \right],
\]

where \( A_k = A_k(\tau_0, \varepsilon, \gamma, L), B = B(\tau_0, \varepsilon) \);

- in particular, for the Essent-type fraction \( L = L_{e,n}(\infty, \gamma) \) with \( \varepsilon = \infty \) we obtain

\[
\frac{I_{11}}{L^3} \leq \frac{A_k}{\pi} \left[ \sqrt{2\pi(6\pi\gamma^2 + 1)} + 2BL \right], \quad \frac{I_{12}}{L^6} \leq \frac{A_k}{\tau_1} \left[ \frac{\sqrt{6\pi\gamma^2 + 1}}{3\gamma} + \frac{3}{2} \sqrt{2\pi} BL \right];
\]

- for the Rozovski-type fraction \( L = L_{r,n}(\varepsilon, \gamma) \) with arbitrary \( \varepsilon, \gamma > 0 \) we have

\[
\frac{I_{11}}{L^3} \leq \frac{A_k}{\pi} \left[ \int_0^{t_{1,1}/\varepsilon} \left( \frac{A_1}{\varepsilon} + \frac{\varepsilon \Gamma^2}{24} \right) e^{-h^2t^2/2} \, dt + \frac{1}{6\gamma} \int_{t_{1,1}/\varepsilon}^{t_{2,1}/\varepsilon} t^2 e^{-h^2t^2/2} \, dt + \frac{\varepsilon}{12} \int_{t_{2,1}/\varepsilon}^{\infty} t^3 e^{-h^2t^2/2} \, dt + \frac{1}{h^2} \int_0^{\infty} t^3 e^{-h^2t^2/2} \, dt + BL \int_0^{\infty} t^3 e^{-h^2t^2/2} \, dt \right] = \frac{A_k}{\pi} \left[ \frac{A_1}{\varepsilon} \gamma \left( \frac{3}{2}, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\varepsilon}{12h^2} \gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\sqrt{7}}{6\gamma} \Gamma \left( \frac{5}{2}, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\sqrt{6\pi\gamma^2 + 1}}{3\gamma} \Gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{3}{2} \sqrt{2\pi} BL \right],
\]

\[
\frac{I_{12}}{L^6} \leq \frac{A_k}{\tau_1} \left[ \int_0^{t_{1,1}/\varepsilon} \left( \frac{A_1}{\varepsilon} + \frac{\varepsilon \Gamma^2}{24} \right) e^{-h^2t^2/2} \, dt + \frac{1}{6\gamma} \int_{t_{1,1}/\varepsilon}^{t_{2,1}/\varepsilon} t^3 e^{-h^2t^2/2} \, dt + \frac{\varepsilon}{12} \int_{t_{2,1}/\varepsilon}^{\infty} t^4 e^{-h^2t^2/2} \, dt + \frac{1}{h^2} \int_0^{\infty} t^4 e^{-h^2t^2/2} \, dt + \frac{1}{3\gamma h^2} \left( 1 - \gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) - \Gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) \right) + \frac{\sqrt{7}}{6\gamma} \Gamma \left( \frac{5}{2}, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{\sqrt{6\pi\gamma^2 + 1}}{3\gamma} \Gamma \left( 2, \frac{t_{1,1}^2}{2\varepsilon^2} \right) + \frac{3}{2} \sqrt{2\pi} BL \right],
\]
where \( A_r = A_r(\tau_0, \varepsilon, \gamma, L) \), \( B = B(\tau_0, \varepsilon) \), \( h = h(\tau_0, \varepsilon, L) \) provided that \( \varepsilon L_0 < 9 \cdot 2^{-2/3} = 5.6696 \ldots \);

• in particular, for the Rozovskii-type fraction \( L = L_{n,n}(\varepsilon, \gamma) \) with \( \gamma \geq \gamma_* \), taking into account that \( t_{1,\gamma} = t_{2,\gamma} = t_{\infty} = 2/\gamma_* \), we obtain

\[
\frac{I_{11}}{L^3} \leq \frac{A_r}{\pi} \left[ \frac{\varepsilon}{\varepsilon h} \gamma \left( 1, \frac{2h}{(\varepsilon \gamma^2)^2} \right) + \frac{\varepsilon}{12h^2} \left( 1 + \Gamma \left( 2, \frac{2h}{(\varepsilon \gamma^2)^2} \right) \right) + \frac{2BL}{h^2} \right],
\]

\[
\frac{I_{12}}{L^6} \leq \frac{A_r}{\pi} \left[ \frac{\sqrt{2\varepsilon}}{\varepsilon h^{3/2}} \gamma \left( \frac{3}{2}, \frac{2h}{(\varepsilon \gamma^2)^2} \right) + \frac{\sqrt{2\varepsilon}}{12h^{5/2}} \left( \frac{3}{4} \sqrt{\pi} + \Gamma \left( \frac{5}{2}, \frac{2h}{(\varepsilon \gamma^2)^2} \right) \right) + \frac{3\sqrt{2}\pi BL}{2h^{5/2}} \right].
\]

Note that the constructed upper bounds for \( I_{11}/L^3 \) and \( I_{12}/L^6 \) are monotonically increasing with respect to \( L \), monotonically decreasing with respect to \( \gamma > 0 \) and, for \( L = L_{n,n} \), also with respect to \( \varepsilon > 0 \) as integrals of the functions possessing the stated properties. Moreover, they do not depend on \( \gamma \) for \( L = L_{n,n}(\varepsilon, \gamma) \) with \( \gamma \geq \gamma_* \), so that the further increase of \( \gamma \) has no effect on the value of the resulting constant \( C(\varepsilon, \gamma, L) \), hence, the value \( \gamma = \gamma_* \) is an optimal one.

To estimate \( I_2 \), we use the first inequality in (22) from Lemma 3 and bound (27) from Theorem 4 to get

\[
\frac{I_2}{L^3} \leq \frac{1.0253}{\pi L^3} \int_{T_0}^{T_1} \left[ \frac{1}{t} \right] \frac{dL}{L^2} \leq \frac{1.0253}{\pi L^3 T_0} \int_{T_0}^{T_1} t^2 e^{-k(t)} dt = \frac{1.0253}{2\pi \tau_0^2 (k(\tau_0))/\gamma} \left( \frac{3}{2}, \frac{\tau_0^2}{2L^2} \right),
\]

with \( k(\tau_1) \) defined in the formulation of Theorem 4 (recall that \( k(\tau_1) > 0 \) if \( 0 < \tau_1 < \tau_1 \)). We also observe that the constructed upper bound for \( I_2/L^3 \) holds true for both fractions \( L = L_{n,n}(\varepsilon, \gamma) \) and \( L = L_{n,n}(\varepsilon, \gamma) \), independent of \( \varepsilon \) and \( \gamma \), and is monotonically increasing with respect to \( L \), moreover, \( I_2 = O(L^n) \) as \( L \to 0 \) for every \( \nu > 0 \).

Using the inequality \( |K(\frac{1}{\tau_1} - \frac{\tau_1}{\pi^2})| \leq \frac{\pi^2}{\tau_1} \) from Lemma 3 once again to bound \( I_3 \) and the condition \( 1 \leq t^3/T_0^3 \)

\[
\text{defining the region of integration in } I_4, \text{ we estimate the sum of the integrals } I_3 \text{ and } I_4 \text{ as}
\]

\[
\frac{I_3 + I_4}{L^3} \leq \frac{1}{\tau_1} \int_0^\infty e^{-t^2/2} dt + \frac{1}{\pi L^3 T_0} \int_0^\infty t^2 e^{-t^2/2} dt = \frac{1}{\tau_1} \sqrt{2} + \frac{1}{\pi T_0} \Gamma \left( \frac{3}{2}, \frac{\tau_0^2}{2L^2} \right),
\]

with the latest expression being monotonically increasing with respect to \( L \) and independent of \( \varepsilon \) and \( \gamma \).

Summing up the obtained bounds for the integrals \( I_{11}, I_{12}, I_3, I_4 \), we obtain an upper bound for \( \Delta_n \) in the form

\[
\Delta_n \leq L^3 \cdot C_0(\varepsilon, \gamma, L, \tau_0, \tau_1)
\]

with the function \( C_0(\varepsilon, \gamma, L, \tau_0, \tau_1) \) being monotonically increasing with respect to \( L \), monotonically decreasing with respect to \( \gamma > 0 \) and, for \( L = L_{n,n}(\varepsilon, \gamma) \), also with respect to \( \varepsilon > 0 \) and satisfying

\[
C_0(\varepsilon, \gamma, \tau_0, \tau_1) := \lim_{L \to 0} C_0(\varepsilon, \gamma, L, \tau_0, \tau_1) = \frac{1}{\tau_1} \sqrt{\frac{\pi}{2}} \left[ \frac{\alpha}{\tau_0^2} \left( \frac{1}{\tau_0^2} + \frac{1}{\gamma^2} \right) + \frac{\alpha}{\gamma^2} \left( 1, \frac{\gamma^2}{\tau_0^2} \right) - \frac{\alpha}{\gamma^2} \left( 2, \frac{\gamma^2}{2\tau_0^2} \right) + \frac{\gamma^2}{\tau_0^2} \Gamma \left( \frac{1}{2}, \frac{\gamma^2}{2\tau_0^2} \right) \right], 
\]

\[
\frac{\alpha}{\gamma^2} \left( 2, \frac{\gamma^2}{2\tau_0^2} \right) + \frac{\gamma^2}{\tau_0^2} \Gamma \left( \frac{1}{2}, \frac{\gamma^2}{2\tau_0^2} \right) \right],
\]

where \( h = h(\tau_0, \varepsilon, L) := 1 - \frac{2}{\varepsilon h_0^2}L^6 \).

Inequality (15) yields the following upper bound for the function \( C(\varepsilon, \gamma, L) \) from (11):

\[
C(\varepsilon, \gamma, L) \leq C_0(\varepsilon, \gamma, L) := \inf \left\{ C_0(\varepsilon, \gamma, L, \tau_0, \tau_1) : \tau_0 \in (0, \tau_0(\varepsilon)), \tau_1 \in (\tau_1(L_0, \varepsilon), \tau_1), \tau_1 \geq L^2 \tau_0 \right\}
\]

for every \( \varepsilon > 0, \gamma > 0 \), and \( L \leq L_0 \) with

\[
L_0 \leq T_0(\varepsilon) \wedge (\varepsilon/4)^{1/3}, \quad \text{also } L_0 \leq 9 \cdot 2^{-2/3}\varepsilon^{-1} \text{ for } L = L_{n,n}.
\]

and \( C_0(\varepsilon, \gamma, L) \) being a monotonically increasing function of \( L \) as the greatest lower bound to the increasing function \( \max_{0 < L \leq L_0} C(\varepsilon, \gamma, L, \tau_0, \tau_1), \) where the greatest lower bound is taken over a decreasing system of sets. Hence,

\[
\max_{0 < L \leq L_0} C(\varepsilon, \gamma, L) \leq \max_{0 < L \leq L_0} C_0(\varepsilon, \gamma, L) = C_0(\varepsilon, \gamma, L_0).
\]
Moreover, $C_0(\varepsilon, \gamma, L)$ is monotonically decreasing with respect to $\gamma > 0$ and, for $L = L_{e,n}(\varepsilon, \gamma)$, also with respect to $\varepsilon > 0$.

Inequality \((15)\) yields upper bounds for the asymptotically exact constants $C^* \in \{C^*_L, C^*_\kappa\}$ defined in \((13), (14)\). Namely, with the observation that conditions \((47)\) are trivially satisfied for every $\varepsilon > 0$ if $L \to 0$, we have

$$C^*(\varepsilon, \gamma, L) \leq \lim_{L \to 0} C(\varepsilon, \gamma, L) \leq C_0(\varepsilon, \gamma, 0+) \leq \inf \left\{ C_0(\varepsilon, \gamma, \tau_0, \tau_1) : \tau_0 \in (0, \pi_0(\varepsilon)), \tau_1 \in (0, \pi) \right\},$$

for every $\varepsilon, \gamma > 0$.

Hence, recalling that $\pi_1 = \pi/4$, $A_e(0+, \varepsilon, \gamma, 0+) = A_n(0+, \varepsilon, \gamma, 0+) = A(0+, \varepsilon) = 1$, $b(\tau_0, \varepsilon, 0+) = 1$, for every $\varepsilon > 0$ and $\gamma > 0$ we obtain:

- in the Esseen-type inequality

$$C^*_L(\varepsilon, \gamma) \leq C_0(\varepsilon, \gamma, 0+, \pi_1) = \frac{4}{\sqrt{2\pi}} + \frac{1}{\pi} \left[ \frac{\sqrt{\varepsilon}}{2} \Gamma \left( 1, \frac{t^2 \gamma}{2\varepsilon} \right) + \frac{\varepsilon}{12} \Gamma \left( 2, \frac{t^2 \gamma}{2\varepsilon} \right) + \frac{\varepsilon}{6} \Gamma \left( 2, \frac{t^2 \gamma}{2\varepsilon} \right) \right] \leq C^*_n(\varepsilon, \gamma),$$

- in the Rozovskii-type inequality

$$C^*_n(\varepsilon, \gamma) \leq C_0(\varepsilon, \gamma, 0+, \pi_1) = \frac{4}{\sqrt{2\pi}} + \frac{1}{\pi} \left[ \frac{\sqrt{\varepsilon}}{2} \Gamma \left( 1, \frac{t^2 \gamma}{2\varepsilon} \right) + \frac{\varepsilon}{12} \Gamma \left( 2, \frac{t^2 \gamma}{2\varepsilon} \right) + \frac{\varepsilon}{6} \Gamma \left( 2, \frac{t^2 \gamma}{2\varepsilon} \right) \right] \leq C^*_n(\varepsilon, \gamma),$$

with $C^*_L(\varepsilon, \gamma), C^*_n(\varepsilon, \gamma)$ defined in the formulation of Theorem 2 (see \((13), (19)\)). Moreover, the function $\tilde{C}^*_L(\varepsilon, \gamma), \tilde{C}^*_n(\varepsilon, \gamma)$ decreases with respect to $\varepsilon > 0$ and $\gamma > 0$ as an integral and then a limit of a function with the similar properties and is unbounded as $\varepsilon \to 0+$ or $\gamma \to 0+$. Similarly, the function $\tilde{C}^*_n(\varepsilon, \gamma)$ decreases with respect to $\gamma > 0$ being constant for $\gamma \geq \gamma_1$, for every fixed $\varepsilon > 0$ and infinitely grows as $\varepsilon \to 0+$, $\varepsilon \to \infty$, or $\gamma \to 0+$.

### 4.2 The case (ii) $L \in [L_0, L_1]$

Let $0 < T_0 \leq T_0(\pi_0(\varepsilon), L)$, $T_0 \leq T_1 \leq T_1(\pi_1, L)$, and $0 < L_0 \leq L \leq L_1 < \infty$.

Though the function $K(t)$ has a singularity of order $O(|t|^{-1})$ as $t \to 0$, the integrands in $I_1$ and $I_3$ have no singularities due to the presence of the factor $r_n(t) = O(t^2)$, $t \to 0$, in the integrand in $I_1$ and to the boundedness of the function $|K(t) - \frac{2\gamma}{\pi t^\gamma}|$ by \((12)\) in $I_3$.

Using estimates \((35)\) and \((20)\) from Theorems \(4\) and \(4\) to bound integrands in $I_1$ and $I_2$, we obtain, by \((33)\), an upper bound for $\Delta_n$ in the form

$$\Delta_n \leq D(\varepsilon, \gamma, L, T_0, T_1),$$

which is uniform in the class of all distributions of random summands with fixed value of the fraction under consideration $L \in [L_0, L_1]$, where $T_0 \leq T_1$ are free parameters. Moreover, the function $D(\varepsilon, \gamma, L, T_0, T_1)$ here is monotonically increasing with respect to $L$, monotonically decreasing with respect to $\gamma > 0$ and, for $L = L_{e,n}$, also with respect to $\varepsilon > 0$. Hence, we may construct an upper bound for $C(\varepsilon, \gamma, L)$ on the interval $L_0 \leq L \leq L_1$ in the form

$$C(\varepsilon, \gamma, L) \leq C_1(\varepsilon, \gamma, L) := \inf \left\{ \frac{D(\varepsilon, \gamma, L, T_0, T_1)}{L^3} : 0 < T_0 < \frac{\pi_0(\varepsilon)}{L}, T_0 < T_1 < \frac{\pi_1}{L^3} \right\},$$

where $\pi_1 := \pi/4$ and $\pi_0(\varepsilon) := \sqrt{2/\alpha(\varepsilon)}$ are defined in Theorems \(1\) and \(5\) respectively. The maximum value $\max_{L_0 \leq L \leq L_1} C_1(\varepsilon, \gamma, L)$ can be estimated similarly to \((35)\) by computation of $C_1(\varepsilon, \gamma, L)$ in a finite number of points using the inequality

$$\max_{L_0 \leq L \leq L_0'} \frac{C_1(\varepsilon, \gamma, L)}{L''} \leq C_1(\varepsilon, \gamma, L''') \cdot \left( \frac{L''}{L'} \right)^3,$$

which is valid due to the monotone growth of the function $C_1(\varepsilon, \gamma, L) \cdot L^3$ as the greatest lower bound to the monotonically increasing function $D(\varepsilon, \gamma, L, T_0, T_1)$ over a decreasing system of sets. Furthermore, since the function $D(\varepsilon, \gamma, L, T_0, T_1)$ is monotonically decreasing with respect to $\gamma$ and, for $L = L_{e,n}$, with respect to $\varepsilon$, so is $\max_{L_0 \leq L \leq L_1} C_1(\varepsilon, \gamma, L)$.
4.3 Numerical results

Summarizing what was said above, as the constants $C(\varepsilon, \gamma) \in \{C_\varepsilon(\varepsilon, \gamma), C_\gamma(\varepsilon, \gamma)\}$ in inequalities (11) and (12) we can take

$$C(\varepsilon, \gamma) := \max \left\{ C_{\min}, \sup_{0 < L \leq L_1} C(\varepsilon, \gamma, L) \right\} \text{ with } C(\varepsilon, \gamma, L) := \begin{cases} C_0(\varepsilon, \gamma, L), & 0 < L \leq L_0(\varepsilon), \\ C_1(\varepsilon, \gamma, L), & L_0(\varepsilon) < L \leq L_1, \end{cases}$$

$C_{\min} := 2$, $L_1 := 0.65 > (0.541/C_{\min})^{1/3}$, $L_0(\varepsilon) := (\varepsilon/4)^{1/3} \wedge \varepsilon r_0(\varepsilon)$ if $L = L_{e,n}$ and $L_0(\varepsilon) := (\varepsilon/4)^{1/3} \wedge 6/(\varepsilon r_0(\varepsilon))$ if $L = L_{e,n}$, for every $\varepsilon, \gamma > 0$. Since $C_0(\varepsilon, \gamma, L)$ and $C_1(\varepsilon, \gamma, L)$ are both monotonically decreasing with respect to $\gamma > 0$ and, for $L = L_{e,n}$, also with respect to $\varepsilon$, so is $C(\varepsilon, \gamma)$.

The concrete numerical values of $C_0(\varepsilon, \gamma, L)$ and $C_1(\varepsilon, \gamma, L)$ are processed with the help of a computer. Our computations were carried out in Python 3.6 using the library Scipy 1.0.0. The values of $\max_{0 < L \leq L_0} C_0(\varepsilon, \gamma, L) = C_0(\varepsilon, \gamma, L_0)$ for some $\varepsilon$ and $\gamma$ with $L_0 = 0.001$ and $L_0 = 0.03$ are given for the Esseen-type fraction $L = L_{e,n}(\varepsilon, \gamma)$ in table 2 in the fourth and seventh columns, respectively, accompanied by the optimal values of the parameters $\tau_0$ and $\tau_1$ in (16). The values of $\max_{0.03 \leq L \leq L_1} C_1(\varepsilon, \gamma, L) = C_1(\varepsilon, \gamma, L_*)$ for some $\varepsilon$ and $\gamma$ are given in table 4 for the Esseen fraction $L = L_{e,n}(\varepsilon, \gamma)$ accompanied by the optimal values of the parameters $T_0$ and $T_1$ in (15) specified in the form $\tau_0 := T_0L$ and $\tau_1 := T_1L^3$, in the extremal point $L = L_*$ which is also given in the fourth column of the same table. Columns 7–10 contain the normalized contributions of the integrals $I_k/L^3$, $k = 1, 2, 3, 4$, into the extreme value $C_1(\varepsilon, \gamma, L_*)$, so that $C_1(\varepsilon, \gamma, L_*) = (I_1 + I_2 + I_3 + I_4)/L^3$. Tables 3 and 4 contain similar results for the Rozovskii-type fraction $L = L_{e,n}(\varepsilon, \gamma)$.

For example, for the minimum value $\sup_{0 < L \leq L_0} C(\varepsilon, \gamma, L)$, we have

$$\sup_{0 < L \leq L_0} C(\varepsilon, \gamma, L) \leq 2.28, \quad \sup_{0.03 < L \leq L_1} C(\varepsilon, \gamma, L) \leq 2.65,$$

(the plot of the function $C_1(\varepsilon, \gamma, L)$ for $L := L_{e,n}(\varepsilon, \gamma, L) \in [0.03, L_1]$ is given on Fig. 2 (left)), hence,

$$\inf_{0 < \varepsilon, \gamma > 0} C(\varepsilon, \gamma) = C_\varepsilon(\varepsilon, \gamma, L_0) \leq \max \{ C_{\min}, 2.28, 2.65 \} = 2.65.$$

However, the same upper bound $C_\varepsilon(\varepsilon, \gamma) \leq 2.65$ can be reached, due to the rounding gap, already for finite values of $\varepsilon$ and $\gamma = \gamma(\varepsilon)$ plotted on Fig. 2 (right) and also at the points with one infinite component, for example,

$$(\varepsilon, \gamma) \in \{(2.56, \infty), (2.74, 3), (4.162), (\infty, 1.43)\},$$

for which, due to the monotonicity and according to tables 2, 4 we have

$$\max \{ C(2.56, \infty, L), C(2.74, 3, L), C(4.162, L), C(\infty, 1.43, L) \} =$$

$$= \begin{cases} C_0(0.6, 0.3, 0.03) \leq 2.64, & L \leq 0.03 \text{ (see table 2),} \\ \max \{ C_1(2.56, \infty, 0.4833 \ldots), \ldots \} \leq 2.65, & 0.03 < L \leq L_1 \text{ (see table 4).} \end{cases}$$

Similar level curve $\{(\varepsilon, \gamma): \widehat{C}_\varepsilon(\varepsilon, \gamma) = 1.72\}$ of the upper bound $\widehat{C}_\varepsilon(\varepsilon, \gamma) := C_0(\varepsilon, \gamma, 0+)$ for the asymptotically exact constant $C_\varepsilon(\varepsilon, \gamma)$ is plotted on Fig. 2 (left).

Since the constructed upper bound for $C_\varepsilon(\varepsilon, \gamma)$ is monotonically decreasing with respect to $\gamma$ and is constant for $\gamma \geq \gamma_*$, its global minimum with respect to $\gamma$ is attained at $\gamma = \gamma_* = 0.5599 \ldots$. Plot of the function $\max_{0 < L \leq L_1} C_1(\varepsilon, \gamma_*, L)$ with respect to $\varepsilon$ is given on Fig. 3 (right, solid line), and numerical computations show that its minimum is attained around the point $\varepsilon = 2.12$, for which, according to tables 5 and 6 we have

$$C_0(2.12, \gamma_*, 0.03) \leq 2.29, \quad \max_{0.03 < L \leq L_1} C_1(2.12, \gamma_*, L) = C_1(2.12, \gamma_*, 0.4827 \ldots) \leq 2.66,$$

(the plot of the function $C_1(2.12, \gamma_*, L)$ for $L = L_{e,n}(2.12, \gamma_*) \in [0.03, L_1]$ is given on Fig. 4 (right)). Hence,

$$\inf_{0 < \varepsilon, \gamma > 0} C(\varepsilon, \gamma) \leq C(2.12, \gamma_*) \leq \max \{ C_{\min}, 2.29, 2.66 \} = 2.66.$$

We also provide the least value of $\varepsilon$ (up to the second decimal digit), for which, within the numerical method used, the same upper bound 2.66 holds,

$$\inf \{ \varepsilon > 0: C_\varepsilon(\varepsilon, \gamma_*) \leq 2.66 \} \approx 1.99.$$
The interest to exactly the least value of $\varepsilon$ is stipulated by that the second term ($\sup_{0<\varepsilon\leq\varepsilon\{\ldots\}}$) in the definition of $L_{n,n}(\varepsilon, \gamma)$ is monotonically increasing with respect $\varepsilon > 0$.

Similarly, the constructed upper bound $\hat{C}_n^*(\varepsilon, \gamma)$ (defined in (18)) for the asymptotically exact constant $C^*_n(\varepsilon, \gamma)$ (defined in (14)) attains its minimum value at the point $\gamma = \gamma_*, \varepsilon \approx 1.89$ with
\[
\inf_{\varepsilon > 0, \gamma > 0} \hat{C}_n^*(\varepsilon, \gamma) \leq \hat{C}_n^*(1.89, \gamma_*) \leq 1.75;
\]
the plot of $\hat{C}_n^*(\varepsilon, \gamma_*)$ is given on Fig. 3 (left, solid line), while the least value of $\varepsilon$, for which the inequality $\hat{C}_n^*(\varepsilon, \gamma_*) \leq 1.75$ still holds, is around 1.52. Furthermore, as Fig. 3 demonstrates, with the decrease of $\gamma$, the graphs of $\hat{C}_n^*(\varepsilon, \gamma)$ and $C^*_n(\varepsilon, \gamma)$ become more and more flat around the points of minimum, so that the intervals, where the values of the functions $\hat{C}_n^*(\cdot, \gamma)$ and $C^*_n(\cdot, \gamma)$ differ slightly from their minimal values, $\min_{\varepsilon > 0} \hat{C}_n^*(\varepsilon, \gamma)$ and $\min_{\varepsilon > 0} C^*_n(\varepsilon, \gamma)$, enlarges. For example, for $\gamma = 0.4$ (see Fig. 3 (left, dashdot line) and Table 3 the point of minimum of $\hat{C}_n^*(\cdot, \gamma)$ is located around $\varepsilon \approx 1.99$ with $\hat{C}_n^*(1.99, 0.4) \leq 1.78$, while the inequality $\hat{C}_n^*(\varepsilon, 0.4) \leq 1.78$ still holds for $\varepsilon = 1.41$; similarly, for $\gamma = 0.2$
\[
\inf_{\varepsilon > 0} \hat{C}_n^*(\varepsilon, 0.2) \leq \hat{C}_n^*(2.77, 0.2) \leq 1.93, \quad \inf\{\varepsilon > 0: \hat{C}_n^*(\varepsilon, 0.2) \leq 1.93\} \approx 1.89,
\]
see Fig. 3 (left, dotted line) and Table 3. For the “absolute” constant $C_n(\varepsilon, \gamma)$ with $\gamma \in \{0.2, 0.4\}$ we have
\[
\inf_{\varepsilon > 0, \gamma \geq 0.4} C_n(\varepsilon, \gamma) \leq C_n(2.63, 0.4) \leq \max\{C_{\min}, C_0(2.63, 0.4, 0.03), C_1(2.63, 0.4, 0.4822 \ldots\} \leq 2.70,
\]
\[
\inf\{\varepsilon > 0: C_n(\varepsilon, 0.4) \leq 2.70\} \approx 1.76, \quad C_n(1.76, 0.4) \leq \max\{2, 2.37, 2.70\} = 2.70;
\]
\[
\inf_{\varepsilon > 0, \gamma \geq 0.2} C_n(\varepsilon, \gamma) \leq C_n(5.39, 0.2) \leq \max\{2, 2.68, 2.87\} = 2.87,
\]
\[
\inf\{\varepsilon > 0: C_n(\varepsilon, 0.2) \leq 2.87\} \approx 1.21, \quad C_n(1.21, 0.2) \leq \max\{2, 2.64, 2.87\} = 2.87;
\]
see Fig. 3 (right, dashdot and dotted lines) and Tables 3 and 5.

Another particular case is concerned with the historical values $\gamma = 1$ and $\varepsilon \in \{1, \infty\}$, for which, according to tables 2, 3, 4 and 5 we have
\[
C_n(1, 1) \leq C_n(1, \gamma_*) \leq 2 \lor C_0(1, \gamma_*, 0.03) \lor C_1(1, \gamma_*, 0.4834 \ldots) \leq 2 \lor 2.35 \lor 2.73 = 2.73,
\]
\[
C_n(1, 1) \leq C_n(1, 0.72) \leq 2.73 \geq C_n(1, \infty),
\]
\[
C_n(\infty, 1) \leq 2.66, \quad \text{which also follows from} \quad C_n(\infty, 0.97) \leq 2.66 \quad \text{or} \quad C_n(4.35, 1) \leq 2.66.
\]

It is interesting to note that, as it follows from Tables 3 and 5 the largest contribution into extreme values of $C(\varepsilon, \gamma) = C_1(\varepsilon, \gamma, L_*)$ in both inequalities $\{11\}, \{12\}$ for all the presented values of $\varepsilon$ and $\gamma$ is provided by the integral $I_3$ which depends on the constructed estimates for characteristic functions through the maximal length of the interval, where the absolute value of a characteristic function can be estimated by a majorant strictly less than 1. Hence, to get further improvements of the constructed upper bounds for $C(\varepsilon, \gamma)$, one should improve, in first turn, upper bounds for absolute values of characteristic functions presented in Theorem 4.

5 The comparison of Osipov’s, Lyapunov’s, and modified Esseen's and Rozovskii’s fractions. Lower bound for $\gamma \to 0$.

In the present section we compare the fractions $L_{n,n}(\varepsilon, 1), L_{n,n}(\varepsilon, 1)$ with $L_{3,n} + \Lambda_1(1)$ and $L_{3,n} + \Lambda_1(1)$, and demonstrate that our new inequalities $\{11\}, \{12\}$ with $C_n(1, 1) = 2.73 = C_n(1, 1)$ may be sharper than Osipov’s inequality $[3]$ with the best known constant $C = 1.87$ $[14]$. In what follows we emphasize the dependence of the above fractions on the distributions of random summands $X_1, \ldots, X_n$ with the d.f.’s $F_1, \ldots, F_n$ by writing $L_{n,n}(\varepsilon, \gamma_1, \ldots, \gamma_n), L_{n,n}(\varepsilon, \gamma_1, \ldots, \gamma_n)$ and $L_{3,n}(F_1, \ldots, F_n)$ using the three-argument notation $L_{n,n}(\varepsilon, \gamma, F), L_{n,n}(\varepsilon, \gamma, F)$, $L_{n,n}(F)$, $n \in N$, in the i.i.d. case, that is, for $F_1 = \ldots = F_n = F$. Let $\mathcal{F}$ denote the set of all d.f.’s on $\mathbb{R}$ with zero mean and finite second-order moment and $F_p \in \mathcal{F}$ be the d.f. of the two-point distribution prescribing the masses $p \in [\frac{1}{2}, 1)$ and $q = 1 - p$ to the points $\sqrt{q/p}$ and $-\sqrt{q/p}$. It easy to see that $F_p$ has zero mean and unit variance.
Theorem 6. (i) For all \( n \in \mathbb{N}, \varepsilon > 0 \), and \( F_1, \ldots, F_n \in \mathcal{F} \) such that \( B_n > 0 \) we have

\[
L_{6,n}^3(\varepsilon, 1) \leq L_{3,n}^3
\]

where the equality takes place for every \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) such that \( n\varepsilon^2 \geq 1 \). As for the extremal, one can take a common d.f. \( F_1 = \ldots = F_n = F_p \) with arbitrary \( p \in [\frac{1}{2}, 1] \) satisfying \( p/(1-p) \leq n\varepsilon^2 \).

(ii) For all \( n \in \mathbb{N}, \varepsilon > 0 \), and \( p \in (\frac{1}{2}, \frac{\sqrt{p-1}}{2}) \) such that \( n\varepsilon^2 > (1-p)/p \) we have

\[
L_{6,n}^3(\varepsilon, 1, F_p) > L_{3,n}^3(\varepsilon, 1, F_p)
\]

in particular, \( L_{6,n}^3(\varepsilon, 1, F_p) > L_{6,n}^3(\varepsilon, 1, F_p), L_{6,n}^3(\varepsilon, 1, F_p) > L_{6,n}^3(\varepsilon, 1, F_p) \).

(iii) There exists a d.f. \( F \in \mathcal{F} \) such that for all \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) satisfying the condition \( n\varepsilon^2 \geq \frac{49}{27} \) we have

\[
L_{6,n}^3(\varepsilon, 1, F) < L_{6,n}^3(\varepsilon, 1, F)
\]

in particular, \( \sup_{\varepsilon > 0} L_{6,n}^3(\varepsilon, 1, F) < L_{3,n}^3(F) \), where one can consider \( F \) to be a discrete tree-point d.f.

(iv) There exists a d.f. \( F \in \mathcal{F} \) such that \( L_{6,n}^3(\varepsilon, 1, F) < L_{6,n}^3(\varepsilon, 1, F) \), with Osipov's inequality [13] with \( C = 1.87, \) namely,

\[
2.73 \cdot \max \{ L_{6,n}^3(1,1), L_{6,n}^3(1,1) \} < 1.87(\Lambda_n(1) + L_n(1)) \quad \text{for every} \quad n \geq 9.
\]

Proof. The inequality in (i) follows from [10] with \( \delta = 1 \). To prove that equality also occurs, consider the sequence of i.i.d. r.v.’s \( X_1, \ldots, X_n \) with the d.f. \( F_p \in \mathcal{F} \) for \( p \in [\frac{1}{2}, 1] \):

\[
X_1 = \begin{cases} \frac{\sqrt{p}}{\sqrt{q}}, & p, \\ -\frac{\sqrt{q}}{\sqrt{p}}, & q := 1 - p. \end{cases}
\]

We have \( B_n = \sqrt{n} \),

\[
\sqrt{n}L_{3,n}^3 = \int_{\mathbb{R}} |x|^3 dF_p(x) = \frac{p^2 + q^2}{\sqrt{pq}} \quad \text{for all} \quad n \in \mathbb{N},
\]

\[
\mu(x) := \int_{|x| < z} x^3dF_p(x) = \begin{cases} 0, & 0 < z \leq \frac{\sqrt{p}}{\sqrt{q}}, \\ \frac{q}{\sqrt{pq}}, & \frac{\sqrt{p}}{\sqrt{q}} < z \leq \frac{\sqrt{p}}{\sqrt{q}}, \\ \frac{p-q}{\sqrt{pq}}, & z > \frac{\sqrt{p}}{\sqrt{q}}. \end{cases}
\]

With the account of \( \mu(z) \geq 0 \) for all \( z > 0 \) and of the left-continuity of the functions \( \mu(z) \) and \( \sigma^2(z) \) for \( z > 0 \), for all \( n \in \mathbb{N} \) and \( p \in [\frac{1}{2}, 1] \) we obtain for every \( \varepsilon > 0 \)

\[
\sqrt{n}L_{6,n}^3(\varepsilon, 1, F_p) = \sup_{0 < \varepsilon \leq \varepsilon \sqrt{n}} \{ \mu(z) + z\sigma^2(z) \} =
\]

\[
\begin{cases} \varepsilon \sqrt{n}, & \varepsilon \sqrt{n} \leq \frac{4}{p}, \\ \max \{ \sqrt{\frac{\varepsilon}{p}} \sqrt{pq} \} + p\varepsilon \sqrt{n}, & \varepsilon \sqrt{n} \leq \frac{4}{p}, \end{cases}
\]

\[
\begin{cases} \frac{q}{\sqrt{pq}}, & \frac{q}{\sqrt{pq}} \leq \varepsilon \sqrt{n} \leq \frac{4}{p}, \\ \frac{p-q}{\sqrt{pq}}, & \frac{p-q}{\sqrt{pq}} \leq \varepsilon \sqrt{n} \leq \frac{4}{p}. \end{cases}
\]

\[
\sqrt{n}L_{6,n}^3(\varepsilon, 1, F_p) = \mu \left( \varepsilon \sqrt{n} \right) + \sup_{0 < \varepsilon \leq \varepsilon \sqrt{n}} z\sigma^2(z) =
\]

\[
\begin{cases} \varepsilon \sqrt{n}, & \varepsilon \sqrt{n} \leq \frac{4}{p}, \\ \frac{q^2}{\sqrt{pq}} + \max \{ \sqrt{\frac{\varepsilon}{p}} \sqrt{pq} \} + p\varepsilon \sqrt{n}, & \varepsilon \sqrt{n} \leq \frac{4}{p}, \end{cases}
\]

\[
\begin{cases} \frac{p-q}{\sqrt{pq}}, & \frac{p-q}{\sqrt{pq}} \leq \varepsilon \sqrt{n} \leq \frac{4}{p}, \end{cases}
\]
Now it is easy to see that for all \( p \in \left[ \frac{1}{2}, 1 \right) \) and \( n \varepsilon^2 \geq p/q \),
\[
\sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1, F_p) = \frac{q^2}{\sqrt{pq}} + p \sqrt{\frac{p}{q}} = \frac{q^2 + p^2}{\sqrt{pq}} = \sqrt{n} L^3_{3,n}(F_p),
\]
where, for \( n \varepsilon^2 = 1 \), the unique value \( p = \frac{1}{2} \) is admissible. Thus, (i) is proved.

Now let \( \frac{1}{2} < p < \frac{\sqrt{3} - 1}{2} \). If \( n \varepsilon^2 > p/q \), then
\[
\sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1, F_p) = \frac{p}{\sqrt{pq}} > \frac{p + q(q - p)}{\sqrt{pq}} = \frac{p^2 + q^2}{\sqrt{pq}} = \sqrt{n} L^3_{3,n}(F_p),
\]
while for \( q/p < n \varepsilon^2 \leq p/q \) we have \( q > p^2 \geq p \varepsilon \sqrt{pq} \) and, hence,
\[
\sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1, F_p) = \frac{q^2 + q}{\sqrt{pq}} > \frac{q^2 + p^2}{\sqrt{pq}} = \sqrt{n} L^3_{3,n}(F_p),
\]
so that \( L^3_{\epsilon,n}(\varepsilon, 1, F_p) > L^3_{3,n}(F_p) \geq \sup_{\varepsilon' > 0} L^3_{\epsilon,n}(\varepsilon', 1, F_p) \) for all \( n \in \mathbb{N} \), \( \varepsilon > 0 \), and \( p \in \left( \frac{1}{2}, \frac{\sqrt{3} - 1}{2} \right) \) such that \( n \varepsilon^2 > (1 - p)/p \), which proves (ii).

To prove (iii), consider the common three-point distribution of the random summands \( X_1, \ldots, X_n \) concentrated in the points
\[
x_1 = \frac{4}{3}, \quad x_2 = -1, \quad x_3 = \frac{7}{3},
\]
with masses \( p, q, r \geq 0 \) such that
\[
\begin{align*}
& p + q + r = 1, \\
& \mathbb{E} X_1 = px_1 + qx_2 + rx_3 = 0, \\
& \mathbb{E} X_1^2 = px_1^2 + qx_2^2 + rx_3^2 = 1,
\end{align*}
\]
that is,
\[
p = \frac{10}{27} = 0.3703 \ldots, \quad q = \frac{53}{108} = 0.4907 \ldots, \quad r = \frac{5}{36} = 0.1388 \ldots.
\]
Now for \( n \varepsilon^2 \geq (|x_1| \vee |x_2| \vee |x_3|)^2 = x_2^2 = \frac{49}{25} = 1.96 \) with \( B_n = \sqrt{n} \) we have
\[
\sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1) = \max \{|x_1|, |x_1^3 p| + (1 - x_1^2 p)|x_2|, |x_1^3 p + x_2^3 q| + x_2^3 r \cdot |x_3|, |x_1^3 p + x_2^3 q + x_3^3 r|\} =
\]
\[
= \frac{643}{675} = 0.9525 \ldots,
\]
\[
\sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1) = |x_1^3 p + x_2^3 q + x_3^3 r| + \max \{|x_1|, (1 - x_1^2 p)|x_2|, x_2^3 r \cdot |x_3|\} = \frac{22}{25} = 0.88 < \sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1),
\]
which proves (iii) for \( n \varepsilon^2 \geq \frac{49}{25} \).

To prove (iv), let us consider the common symmetric four-point distribution of the random summands \( X_1, \ldots, X_n \) of the form
\[
X_1 = \begin{cases} \pm x_1, & p/2 \geq 0, \\
\pm x_2, & q/2 \geq 0, \end{cases} \quad p + q = 1,
\]
and such that \( \mathbb{E} X_1 = 0, \mathbb{E} X_1^2 = 1 \). Setting \( x_1 = 0.9 \) and \( x_2 = 3 \), we find \( p = \frac{860}{819} = 0.9768 \ldots \) and \( q = \frac{49}{819} = 0.0231 \ldots \), so that for \( n \varepsilon^2 \geq x_1^2 \vee x_2^2 = x_2^2 = 9 \) with \( B_n = \sqrt{n} \) we have
\[
\sqrt{n} L^3_{\epsilon,n}(\varepsilon, 1) = \sqrt{n} L^3_{n,n}(\varepsilon, 1) = \max \{ x_1, x_2 (1 - px_1^2) \} = 0.9,
\]
while for \( n \geq x_1^2 \vee x_2^2 = 9 \) we obtain
\[
\sqrt{n}(A_n(1) + L_n(1)) = x_1^3 p + x_2^3 q = \frac{87}{65} = 1.3384 \ldots.
\]
In particular, for \( \varepsilon = 1 \) and \( n \geq 9 \)
\[
2.73 \cdot \sqrt{n} \max \{L^3_{\epsilon,n}(1, 1), L^3_{n,n}(1, 1)\} = 2.457 < 2.5029 \ldots = 1.87 \cdot \sqrt{n}(A_n(1) + L_n(1)),
\]
which proves (iv). We can also propose another “non-symmetric” example for even \(n\). Let \(X_1, \ldots, X_n\) be independent r.v.’s with the distributions \(X_k \overset{d}{=} X\) if \(k\) is odd and \(X_k \overset{d}{=} -X\) if \(k\) is even, where \(X\) is a three-point r.v. with
\[
X = \begin{cases} 
  x_1 = 1/2, & p = 4/9, \\
  x_2 = -1, & q = 4/9, \quad p + q + r = 1, \quad \text{so that } E X = 0, \quad E X^2 = 1.
\end{cases}
\]

Taking into account that \(\sum_{k=1}^{n} E X_k^3 I(\lvert X_k \rvert < z) = 0\) for even \(n\) and all \(z > 0\), for even \(n \geq \max\{x_1^2, x_2^2, x_3^2\} = 4\) we have
\[
\sqrt{n} L_{n,1}^3 (1, 1) = \sqrt{n} L_{n,1}^3 (1, 1) = \sup_{0 < x \leq \sqrt{n}} \left\{ z E X^2 I(\lvert X \rvert \geq z) \right\} = \max \left\{ x_1, (1 - px_1^2) \lvert x_2 \rvert, rx_3^2 \right\} = \frac{8}{9},
\]
\[
\sqrt{n} (\Lambda_n (1) + L_n (1)) = px_1^3 + q \lvert x_2 \rvert^3 + rx_3^3 = \frac{25}{18} = 1.3888 \ldots,
\]
\[
2.73 \cdot \sqrt{n} \max \left\{ L_{n,1}^3 (1, 1), L_{n,1}^3 (1, 1) \right\} = 2.4266 \ldots < 2.5972 \ldots = 1.87 \cdot \sqrt{n} (\Lambda_n (1) + L_n (1)).
\]

\[\square\]

**Theorem 7.** For the asymptotically best constant we have
\[
\sup_{F_1, \ldots, F_n \in \mathcal{F}: \text{\textit{B}_\text{\textit{B}}}_n > 0} \limsup_{n \to \infty} \frac{\Delta_n (F_1, \ldots, F_n)}{\sup_{0 < z \leq \varepsilon} z L_n (z)} = \infty \quad \text{for every } \varepsilon > 0.
\]

**Proof.** Using his asymptotic expansion, Esseen deduced \([7]\) that in the i.i.d. case with \(E X_1 = 0, E X_1^2 = \sigma^2 > 0, E |X_1|^3 < \infty\) the limit below exists and
\[
\lim_{n \to \infty} \sqrt{n} \Delta_n = \frac{|a_3| + 3 h \sigma^2}{6 \sqrt{2 \pi} \sigma^3},
\]
where \(a_3 := E X_1^3, h > 0\) is the span in case of a lattice distribution of \(X_1\), and \(h := 0\) otherwise. Now let us consider an absolute continuous distribution of \(X_1\) whose d.f. \(F_\theta\) is defined by the density
\[
p_\theta (x) = \begin{cases} 
  ax^{-1-\theta}, & x > 1 \\
  0, & \lvert x \rvert \leq 1, \quad \text{with } a = a(\theta) := \frac{4 (2 + \theta)(3 + \theta)}{17 + 7 \theta}, \quad b = b(\theta) := \frac{12 (3 + \theta)}{17 + 7 \theta}, \quad \theta \in (0, 1).
\end{cases}
\]

Then \(h = 0, E X_1 = 0, \)
\[
\sigma^2 = \sigma^2 (\theta) = \frac{a}{1 + \theta} + \frac{b}{2} = \frac{2 (3 + \theta)(7 + 5 \theta)}{(1 + \theta)(17 + 7 \theta)}, \quad a_3 = a_3 (\theta) = a - b = \frac{4 (3 + \theta)(2 - \theta - 3 \theta^2)}{\theta(17 + 7 \theta)},
\]
\[
E X_1^3 I(\lvert X_1 \rvert \geq z) = \begin{cases} 
  \sigma^2, & z \in (0, 1), \\
  \frac{a_3}{1 + \theta} z^{-1-\theta} + \frac{b}{2} z^{-2}, & z > 1,
\end{cases}
\]
\[
\lim_{n \to \infty} \sqrt{n} \sup_{0 < z \leq \varepsilon} z L_n (z) = \sigma^{-3} \lim_{n \to \infty} \sup_{0 < z \leq \varepsilon \sigma \sqrt{n}} z E X_1^2 I(\lvert X_1 \rvert \geq z) = \sigma^{-3} \sup_{z > 0} z E X_1^2 I(\lvert X_1 \rvert \geq z) = \sigma^{-1},
\]
so that
\[
\sup_{F_1, \ldots, F_n \in \mathcal{F}: \text{\textit{B}_\text{\textit{B}}}_n > 0} \limsup_{n \to \infty} \frac{\Delta_n (F_1, \ldots, F_n)}{\sup_{0 < z \leq \varepsilon} z L_n (z)} \geq \sup_{\theta \in (0, 1)} \lim_{n \to \infty} \sqrt{n} \sup_{0 < z \leq \varepsilon} z L_n (z) = \frac{1}{6 \sqrt{2 \pi} \theta^{-10}} \frac{|a_3(\theta)|}{\sigma^2 (\theta)} = \infty.
\]

\[\square\]
6 Appendix: Figures and Tables

Figure 1. Graphs of the functions $t_\gamma := \frac{2}{\gamma} \left( \sqrt{\frac{\gamma}{\gamma_\ast}} \right) + 1$ (solid line) and $t_{1,\gamma} := \frac{2}{\gamma} \left( 1 - \sqrt{1 - \left( \frac{\gamma}{\gamma_\ast} \right)^2} \right)$ (dashdot line) for $\gamma > 0$; dashed line represents the limiting value $t_\infty := \lim_{\gamma \to \infty} t_\gamma = t_{1,\gamma_\ast} = 2/\gamma_\ast = 3.5717\ldots$.

Figure 2. Level curves $\gamma = \gamma(\varepsilon)$ for the upper bounds to the asymptotically exact “constant” $C_\ast^k(\varepsilon, \gamma)$ defined in (13) and to the absolute $C_k(\varepsilon, \gamma)$ “constant” in the Esseen-type inequality (11). Left: $\{ (\varepsilon, \gamma) : \hat{C}_\ast^k(\varepsilon, \gamma) = 1.72 \}$ with $\hat{C}_\ast^k(\varepsilon, \gamma) := C_0(\varepsilon, \gamma, 0^+)$ defined in (14). Right: $\{ (\varepsilon, \gamma) : \max_{L_0 \leq L \leq L_1} C_1(\varepsilon, \gamma, L) = 2.65 \}$ with $C_1(\varepsilon, \gamma, L)$ defined in (48).
Figure 3. Left: Graphs of the function $\hat{C}_R^*(\varepsilon, \gamma)$ (see (18)) which bounds from above the asymptotically exact constant $C_0^*(\varepsilon, \gamma)$ (see (14)), with respect to $\varepsilon$, for $\gamma \geq \gamma_* = 0.5599\ldots$ (solid line), $\gamma = 0.4$ (dashdot line), $\gamma = 0.3$ (dashed line), and $\gamma = 0.2$ (dotted line).

Right: Graphs of the upper bounds $\max_{L_0 \leq L \leq L_1} C_1(\varepsilon, \gamma, L)$ (see (48)) for the “constants” $C_R(\varepsilon, \gamma)$ in the Rozovskii-type inequality, with respect to $\varepsilon$, for $\gamma \geq \gamma_* = 0.5599\ldots$ (solid line), $\gamma = 0.4$ (dashdot line), $\gamma = 0.3$ (dashed line), and $\gamma = 0.2$ (dotted line).

Figure 4. Demonstration to the evaluation of the absolute constants $C_1(\infty, \infty)$ and $C_0(2.12, \gamma_*)$: Plots of the functions $C_1(\infty, \infty, L)$, $L \in [L_0, L_1]$, defined in (18) with $L := L_{n, n}(\infty, \infty)$ (left) and $L := L_{n, n}(2.12, \gamma_*)$ (right).
| $\varepsilon$ | $\gamma$ | $C_0(\varepsilon, \gamma, 0+)$ | $L = 0.001$ | Optimal | $L = 0.03$ | Optimal |
|---|---|---|---|---|---|---|
| $0.6$ | $0.3$ | $1.92245$ | $1.98369$ | $0.33196$ | $0.76348$ | $2.63565$ | $0.61269$ | $0.58888$ |
| $1.21$ | $0.2$ | $1.95457$ | $2.02040$ | $0.30850$ | $0.76192$ | $2.70866$ | $0.58750$ | $0.58212$ |
| $2.06$ | $0.2$ | $1.94999$ | $2.01563$ | $0.30911$ | $0.76197$ | $2.70121$ | $0.58906$ | $0.58256$ |
| $\infty$ | $0.2$ | $1.94879$ | $2.01437$ | $0.30943$ | $0.76199$ | $2.69812$ | $0.58998$ | $0.58279$ |
| $1.48$ | $0.4$ | $1.80997$ | $1.86401$ | $0.37487$ | $0.76589$ | $2.45637$ | $0.65732$ | $0.59975$ |
| $\infty$ | $0.4$ | $1.80005$ | $1.85350$ | $0.37849$ | $0.76607$ | $2.44001$ | $0.66177$ | $0.60081$ |
| $1.89$ | $\gamma_*$ | $1.77136$ | $1.82167$ | $0.39960$ | $0.76705$ | $2.38689$ | $0.68022$ | $0.60488$ |
| $2.03$ | $\gamma_*$ | $1.76995$ | $1.82017$ | $0.40115$ | $0.76711$ | $2.38467$ | $0.68083$ | $0.60502$ |
| $\infty$ | $\gamma_*$ | $1.76370$ | $1.81351$ | $0.40416$ | $0.76725$ | $2.37413$ | $0.68386$ | $0.60563$ |
| $1$ | $\gamma_*$ | $1.80596$ | $1.85831$ | $0.38771$ | $0.76651$ | $2.43955$ | $0.66679$ | $0.60193$ |
| $1$ | $0.67$ | $1.79961$ | $1.85099$ | $0.39328$ | $0.76673$ | $2.42541$ | $0.67230$ | $0.60312$ |
| $1$ | $\infty$ | $1.79149$ | $1.83892$ | $0.42035$ | $0.76791$ | $2.38889$ | $0.69303$ | $0.60759$ |
| $2.24$ | $1$ | $1.73996$ | $1.78661$ | $0.43002$ | $0.76828$ | $2.32719$ | $0.70218$ | $0.60952$ |
| $\infty$ | $1$ | $1.73186$ | $1.77796$ | $0.44157$ | $0.76870$ | $2.31385$ | $0.70646$ | $0.61040$ |
| $3.07$ | $\infty$ | $1.71998$ | $1.76313$ | $0.45717$ | $0.76925$ | $2.28233$ | $0.72259$ | $0.61358$ |
| $3.2$ | $5$ | $1.71997$ | $1.76354$ | $0.45694$ | $0.76926$ | $2.28502$ | $0.72045$ | $0.61316$ |
| $3.28$ | $4$ | $1.71999$ | $1.76368$ | $0.45267$ | $0.76914$ | $2.28573$ | $0.71991$ | $0.61306$ |
| $4$ | $2.4$ | $1.71998$ | $1.76401$ | $0.45217$ | $0.76907$ | $2.28788$ | $0.71820$ | $0.61272$ |
| $5$ | $2.06$ | $1.71997$ | $1.76413$ | $0.45634$ | $0.76902$ | $2.28870$ | $0.71753$ | $0.61259$ |
| $5.37$ | $2$ | $1.72090$ | $1.76420$ | $0.45158$ | $0.76907$ | $2.28892$ | $0.71737$ | $0.61256$ |
| $\infty$ | $1.83$ | $1.71995$ | $1.76423$ | $0.45335$ | $0.76907$ | $2.28913$ | $0.71712$ | $0.61251$ |
| $\infty$ | $\infty$ | $1.71451$ | $1.75725$ | $0.46845$ | $0.76952$ | $2.27337$ | $0.72554$ | $0.61416$ |

Table 2. Demonstration to the evaluation of the upper bound $C_0(\varepsilon, \gamma, L_0)$ defined in (16) for the constant $C_0(\varepsilon, \gamma)$ in the Esseen-type inequality (14) for small values of $L := L_{x,n}(\varepsilon, \gamma) \leq L_0$ and some $\varepsilon, \gamma$: Values of $C_0(\varepsilon, \gamma, L_0)$, rounded up, for $L_0 = 0.001$ (the fourth column) and $L_0 = 0.03$ (the seventh column) accompanied by the corresponding optimal values of the parameters $\tau_0, \tau_1$ in (16). The third column provides values of the function $C_0^*(\varepsilon, \gamma) := C_0(\varepsilon, \gamma, 0+)$ defined in (15) which bounds from above the asymptotically exact constant $C_0^*(\varepsilon, \gamma)$ defined in (13). Recall that $\gamma_* = 0.5599$ . . .
Table 3. Demonstration to the evaluation of the upper bound $C_0(\varepsilon, \gamma, L_0)$ defined in (19) for the constant $C_0(\varepsilon, \gamma)$ in the Rozovskii-type inequality (12) for small values of $L = L_{0, \varepsilon}(\varepsilon, \gamma) \leq L_0$ and some $\varepsilon, \gamma$. Values of $C_0(\varepsilon, \gamma, L_0)$, rounded up, for $L_0 = 0.001$ (the fourth column) and $L_0 = 0.03$ (the seventh column) accompanied by the corresponding optimal values of the parameters $\tau_0, \tau_1$ in (19). The third column provides values of the function $\tilde{C}_0(\varepsilon, \gamma) := C_0(\varepsilon, \gamma, 0^+)$ defined in (18) which bounds from above the asymptotically exact constant $C_0(\varepsilon, \gamma)$ defined in (14). Recall that $\gamma_s = 0.5599 \ldots$ .

| $\varepsilon$ | $\gamma$ | $C_0(\varepsilon, \gamma, 0)$ | $L = 0.001$ ($C_0(\varepsilon, \gamma, L)$) | $L = 0.03$ ($C_0(\varepsilon, \gamma, L)$) | $\tau_0$ | $\tau_1$ | $\tau_0$ | $\tau_1$ |
|---|---|---|---|---|---|---|---|---|
| 1.21 | 0.2 | 1.93474 | 1.99463 | 0.33997 | 0.76401 | 2.63576 | 0.62018 | 0.59077 |
| 1.89 | 0.2 | 1.92998 | 1.98967 | 0.33862 | 0.76391 | 2.62929 | 0.62130 | 0.59105 |
| 2.77 | 0.2 | 1.92890 | 1.98855 | 0.33849 | 0.76389 | 2.62857 | 0.62119 | 0.59103 |
| 5.39 | 0.2 | 1.95832 | 2.01917 | 0.33339 | 0.76358 | 2.67331 | 0.61213 | 0.58874 |
| 1.41 | 0.4 | 1.77974 | 1.82650 | 0.42470 | 0.76809 | 2.37242 | 0.69715 | 0.60847 |
| 1.76 | 0.4 | 1.77249 | 1.81886 | 0.43083 | 0.76830 | 2.36208 | 0.69946 | 0.60894 |
| 1.99 | 0.4 | 1.77128 | 1.81759 | 0.43528 | 0.76848 | 2.36048 | 0.69982 | 0.60902 |
| 2.63 | 0.4 | 1.77841 | 1.82511 | 0.42595 | 0.76814 | 2.37148 | 0.69707 | 0.60846 |
| 0.5 | $\gamma_s$ | 1.94743 | 1.99139 | 0.43510 | 0.76847 | 2.54844 | 0.68154 | 0.60513 |
| 1 | $\gamma_s$ | 1.79154 | 1.83112 | 0.48527 | 0.77017 | 2.34604 | 0.72358 | 0.61378 |
| 1.52 | $\gamma_s$ | 1.74995 | 1.78796 | 0.49835 | 0.77051 | 2.29031 | 0.73728 | 0.61641 |
| 1.89 | $\gamma_s$ | 1.74383 | 1.78159 | 0.50808 | 0.77078 | 2.28222 | 0.73934 | 0.61679 |
| 1.99 | $\gamma_s$ | 1.74412 | 1.78189 | 0.50748 | 0.77083 | 2.28271 | 0.73918 | 0.61676 |
| 2.12 | $\gamma_s$ | 1.74542 | 1.78324 | 0.50216 | 0.77063 | 2.28462 | 0.73863 | 0.61666 |
| 3 | $\gamma_s$ | 1.77092 | 1.80977 | 0.49999 | 0.77029 | 2.32025 | 0.72921 | 0.61487 |
| 5 | $\gamma_s$ | 1.86500 | 1.90688 | 0.45666 | 0.76924 | 2.44632 | 0.70070 | 0.60920 |
Table 4. Demonstration to the evaluation of the upper bound $\max_{L_0 \leq L \leq L_1} C_1(\varepsilon, \gamma, L)$ (see (48)) for the constant $C_0(\varepsilon, \gamma)$ in the Esseen-type inequality (11) for moderate values of $L = L_{n,n}(\varepsilon, \gamma) \in [L_0, L_1] = [0.03, 0.65]$ and some $\varepsilon, \gamma$: Extreme values of $C_1(\varepsilon, \gamma, L) \leq C_1(\varepsilon, \gamma, L_*)$ on the interval $L \in [L_0, L_1]$ (column 3); maximizer $L_*$, rounded down (column 4); optimal values of the parameters $T_0$ and $T_1$ in (48) multiplied by $L$ and $L^4$ in the extremal point $L = L_*$, rounded down (columns 5, 6); values of the normalized integrals $I_k/L^3$, $k = 1, 2, 3, 4$, rounded down (columns 7–10), so that $C_1(\varepsilon, \gamma, L_*) = (I_1 + I_2 + I_3 + I_4)/L^4$.

Recall that $\gamma_* = 0.5599 \ldots$.
Table 5. Demonstration to the evaluation of the upper bound $\max_{L_0 \leq L \leq L_1} C_1(\varepsilon, \gamma, L)$ (see (48)) for the constant $C_1(\varepsilon, \gamma)$ in the Rozovskii-type inequality (12) for moderate values of $L := L_{n,n}(\varepsilon, \gamma) \in [L_0, L_1] = [0.03, 0.65]$ and some $\varepsilon, \gamma$: Extreme values of $C_1(\varepsilon, \gamma, L) \leq C_1(\varepsilon, \gamma, L_*)$ on the interval $L \in [L_0, L_1]$ (column 3); maximizer $L_*$, rounded down (column 4); optimal values of the parameters $T_0$ and $T_1$ in (48) multiplied by $L$ and $L^3$ in the extremal point $L = L_*$, rounded down (columns 5, 6); values of the normalized integrals $I_k/L^3_*$, $k = 1, 2, 3, 4$, rounded down (columns 7–10), so that $C_1(\varepsilon, \gamma, L_*) = (I_1 + I_2 + I_3 + I_4)/L^3_*$. Recall that $\gamma_* = 0.5599\ldots$.

| $\varepsilon$ | $\gamma$ | $C_1 \leq L_*$ | Optimal $T_0, T_1$ | Contributions of $I_k/L^3_*$ |
|---------------|-----------|----------------|-------------------|-----------------------------|
|               |           |                | $T_0L_*$          | $T_1L^3_*$                  | $k = 1$ | $k = 2$ | $k = 3$ | $k = 4$ |
| 1.21          | 0.2       | 2.86991        | 0.47901           | 0.88360                    | 0.69388 | 0.43734 | 0.80872 | 1.51596 | 0.10790 |
| 5.39          | 0.2       | 2.86343        | 0.47902           | 0.88371                    | 0.69384 | 0.43101 | 0.80851 | 1.51608 | 0.10784 |
| 1.76          | 0.4       | 2.69985        | 0.48249           | 0.93002                    | 0.69288 | 0.32875 | 0.75737 | 1.52923 | 0.08452 |
| 2.63          | 0.4       | 2.69323        | 0.48229           | 0.92948                    | 0.69296 | 0.32154 | 0.75779 | 1.52922 | 0.08470 |
| 0.5           | $\gamma_*$| 3.03953        | 0.50011           | 0.92200                    | 0.66074 | 0.52576 | 0.85947 | 1.55924 | 0.09507 |
| 1             | $\gamma_*$| 2.72857        | 0.48346           | 0.93595                    | 0.69239 | 0.36458 | 0.75136 | 1.53053 | 0.08211 |
| 1.99          | $\gamma_*$| 2.65991        | 0.48300           | 0.93912                    | 0.69274 | 0.30011 | 0.74764 | 1.53169 | 0.08048 |
| 2.12          | $\gamma_*$| 2.65925        | 0.48273           | 0.93728                    | 0.69293 | 0.29727 | 0.74960 | 1.53117 | 0.08122 |
| 3             | $\gamma_*$| 2.67687        | 0.48125           | 0.92365                    | 0.69352 | 0.29824 | 0.76363 | 1.52790 | 0.08710 |
| 5             | $\gamma_*$| 2.75611        | 0.47832           | 0.89074                    | 0.69456 | 0.33467 | 0.79931 | 1.51873 | 0.10341 |
References

[1] R. P. Agnew, “Estimates for global central limit theorems”, *Ann. Math. Stat.*, **28**, 26–42 (1957).

[2] A. D. Barbour and P. Hall, “Stein’s method and the Berry–Esseen theorem”, *Australian Journal of Statistics*, **26**, 8–15 (1984).

[3] A. C. Berry, “The accuracy of the Gaussian approximation to the sum of independent variates”, *Trans. Amer. Math. Soc.*, **49**, 122–136 (1941).

[4] R. N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expansions*, Wiley, New York (1976).

[5] L. H. Y. Chen and Q. M. Shao, “A non-uniform Berry–Esseen bound via Stein’s method”, *Probab. Theory Relat. Fields*, **120**, 236–254 (2001).

[6] C.-G. Esseen, “On the Liapounoff limit of error in the theory of probability”, *Ark. Mat. Astron. Fys.*, **A28**, No. 9, 1–19 (1942).

[7] C.-G. Esseen, “Fourier analysis of distribution functions. A mathematical study of the Laplace–Gaussian law”, *Acta Math.*, **77**, No. 1, 1–125 (1945).

[8] C.-G. Esseen, “On the remainder term in the central limit theorem”, *Arkiv för Matematik*, **8**, No. 1, 7–15 (1969).

[9] W. Feller, “Über den zentralen Genzwertsatz der Wahrscheinlichkeitsrechnung”, *Math. Z.*, **40**, 521–559 (1935).

[10] W. Feller, “On the Berry–Esseen theorem”, *Z. Wahrsch. Verw. Geb.*, **10**, 261–268 (1968).

[11] I. A. Ibragimov, “On the accuracy of the approximation of distribution functions of sums of independent variables by the normal distribution”, *Theory Probab. Appl.*, **11**, No. 4, 559–579 (1966).

[12] M. L. Katz, “Note on the Berry–Esseen theorem”, *Ann. Math. Statist.*, **34**, 1107–1108 (1963).

[13] A. N. Kolmogorov, “Some recent works in the field of limit theorems of probability theory”, *Bulletin of Moscow University* [in Russian], **10**, No. 7, 29–38 (1953).

[14] V. Korolev and A. Dorofeyeva, “Bounds of the accuracy of the normal approximation to the distributions of random sums under relaxed moment conditions”, *Lith. Math. J.*, **57**, No. 1, 38–58 (2017).

[15] V. Korolev and I. Shevtsova, “An improvement of the Berry–Esseen inequality with applications to Poisson and mixed Poisson random sums”, *Scand. Actuar. J.*, **2012**, No. 2, 81–105 (2012). Available online since 04 June 2010.

[16] V. Yu. Korolev and S. V. Popov, “An improvement of convergence rate estimates in the central limit theorem under absence of moments higher than the second”, *Theory Probab. Appl.*, **56**, No. 4, 682–691 (2012).

[17] V. Yu. Korolev and S. V. Popov, “Improvement of convergence rate estimates in the central limit theorem under weakened moment conditions”, *Dokl. Math.*, **86**, No. 1, 506–511 (2012).

[18] V. Yu. Korolev and I. G. Shevtsova, “On the upper bound for the absolute constant in the Berry–Esseen inequality”, *Theory Probab. Appl.*, **54**, No. 4, 638–658 (2010).

[19] A. Liapunoff, “Nouvelle forme du théorème sur la limite de probabilité”, *Mém. Acad. Sci. St-Pétersbourg*, **12**, No. 5, 1–24 (1901).

[20] J. W. Lindeberg, “Eine neue Herleitung des Exponentialgesetzes in der Wahrscheinlichkeitsrechnung”, *Mathematische Zeitschrift*, **15**, No. 1, 211–225 (1922).

[21] W. Y. Loh, *On the normal approximation for sums of mixing random variables*, Master thesis, Department of Mathematics, University of Singapore (1975).

[22] L. V. Osipov, “Refinement of Lindeberg’s theorem”, *Theory Probab. Appl.*, **10**, No. 2, 299–302 (1966).
[23] L. Paditz, “Bemerkungen zu einer Fehlerabschätzung im zentralen Grenzwertsatz”, Wiss. Z. Hochschule für Verkehrswesen “Friedrich List”. Dresden., 27, No. 4, 829–837 (1980).

[24] L. Paditz, “On error–estimates in the central limit theorem for generalized linear discounting”, Math. Operationsforsch. u. Statist., Ser. Statistics, 15, No. 4, 601–610 (1984).

[25] L. Paditz, “Über eine globale Fehlerabschätzung im zentralen Grenzwertsatz”, Wiss. Z. Hochschule für Verkehrswesen “Friedrich List”. Dresden., 33, No. 2, 399–404 (1986).

[26] V. V. Petrov, “An estimate of the deviation of the distribution function of a sum of independent random variables from the normal law”, Soviet Math. Dokl., 6, No. 5, 242–244 (1965).

[27] V. V. Petrov, Sums of independent random variables, Springer–Verlag, Berlin–Heidelberg (1975).

[28] H. Prawitz, “Limits for a distribution, if the characteristic function is given in a finite domain”, Skand. Aktuarietskr., 55, 138–154 (1972).

[29] H. Prawitz, “Noch einige Ungleichungen für charakteristische Funktionen”, Scand. Actuar. J., No. 1, 49–73 (1991).

[30] L. V. Rozovskii, “On the rate of convergence in the Lindeberg–Feller theorem”, Bulletin of Leningrad University [in Russian], No. 1, 70–75 (1974).

[31] L. V. Rozovskii, “An estimate of the speed of convergence in the multidimensional central limit theorem without moment hypotheses”, Math. Notes, 23, No. 4, 343–351 (1978).

[32] I. G. Shevtsova, “On the asymptotically exact constants in the Berry–Esseen–Katz inequality”, Theory Probab. Appl., 55, No. 2, 225–252 (2011).

[33] I. G. Shevtsova, “On the absolute constant in the Berry–Esseen inequality and its structural and non-uniform improvements”, Informatics and its Applications [in Russian], 7, No. 1, 124–125 (2013).

[34] I. G. Shevtsova, “On the absolute constants in the Berry–Esseen-type inequalities”, Dokl. Math., 89, No. 3, 378–381 (2014).

[35] V. M. Zolotarev, “An absolute estimate of the remainder term in the central limit theorem”, Theory Probab. Appl., 11, No. 1, 95–105 (1966).

[36] V. M. Zolotarev, Modern Theory of Summation of Random Variables, VSP, Utrecht, The Netherlands (1997).