Stability Analysis for the Helmholtz Equation with Many Frequencies

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Abstract.
This paper concerns the inverse source problem for the time- harmonic wave equation in a one dimensional domain. The goal is to determine the source function from the boundary measurements. The problem is challenging due to complexity of the Green’s function for the Helmholtz equation. Our main result is a logarithmic estimate consists of two parts: the data discrepancy and the high frequency tail.

Keywords: Inverse scattering source problem, time-harmonic wave equation, Green’s function

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1. Introduction and formulation of the problem

We formulate our problem as following:

(1.1) \[ u(x, \alpha)'' + k^2 u(x, \alpha) = f(x), \quad x \in (-1, 1), \]

where the direct solution \( u \) is required to satisfy the outgoing wave conditions:

(1.2) \[ u'(1, \alpha) +iku(1, \alpha) = 0, \quad u'(-1, \alpha) -iku(-1, \alpha) = 0 \]

Let \( f \in L^2(-1,1) \), it is well-known that the problem (1.1)-(1.2) has a unique solution:

(1.3) \[ u(x, \alpha) = \int_{-1}^{1} G(x-y)f(y) dy, \]

where \( G(x-y) \) is the Green function given as follows

(1.4) \[ G(x-y) = \frac{i e^{ik|x-y|}}{2k}. \]

This article concerns the inverse source problem when the source function \( f \) is compactly supported in the interval \((-1,1)\). This paper aims to recover the radiated source \( f \) using the boundary measurement \( u(1, \alpha) \) and \( u(-1, \alpha) \) with \( \alpha \in (0,k) \) where \( K > 1 \) is a positive real constant.

Motivated by these significant applications, inverse source problems have attracted many researchers from whole part of the world in many areas of science. It has vast applications in acoustical and biomedical/medical sciences, antenna synthesis,
geophysics, Elastography and material science ([1, 3]). It has been known that the
data of the inverse source problems for the time harmonic wave equation with a
single frequency can not guarantee the uniqueness ([14], Ch.4). Although, Many
studies showed that the uniqueness can be regained by taking many frequency
boundary measurement data in a non-empty interval \((0, K]\) noticing the analyticity
of wave-field on the frequency \([14, 18]\). Due to the significant applications, these
problems have been attracted considerable attention. These kinds of problems have
been extensively investigated by many researchers such as In the paper [2, 4, 7,
8, 10, 11, 12, 15, 16, 17, 19, 20] and [21]. It is worth mentioning that this
type of problems has also application in important system such classical elasticity
system and Maxwell system. For an example, in [13, 5], inverse source problems
was considered for classical elasticity system with constant coeffic ients.
In this paper, we assume that the domain is homogeneous in the whole space. In
this work, we try to establish an estimate to recover the source fu nctions for the
inverse source problem for the one-dimensional Helmholtz equation . In this work,
paper source function \(f \in H^2(\mathbb{R})\) is assumed to has a support in the domain
or \(\text{supp} f \subset (-1, 1)\). Our main result is as follows;

**Theorem 1.1.** Let’s \(u \in H^2((-1, 1))\) be the solution of the (1.1), Then there
exists a generic constant \(C\) depending on the domain \((-1, 1)\) such that

\[
\| f \|_{(0) (-1, 1)} \leq C \left( \epsilon^2 + \frac{M^2}{1 + KE^2} \right),
\]

where \(K > 1\). Here

\[
\epsilon^2 = \int_0^K \alpha^2 \left( |u(1, \alpha)|^2 + |u(-1, \alpha)|^2 \right) d\alpha,
\]

\(E = -\ln \epsilon\) and \(M = \max \{ \| f \|_{(1)} (-1, 1), 1 \}\) where \(\| \cdot \|_{(1)} (\Omega)\) is the standard
Sobolev norm in \(H^1(\alpha)\).

**Remark 1.1:** Our main estimate in (1.5) consists of two parts: the data discrep-
ancy and the high frequency part. The first part is the Lipschitz part comes form
the boundary measurement data type. The second part is of logarithmic type. Our
results shows when the wave number \(K\) grows, the problem is more st able. The
estimate (1.5) also implies the uniqueness of the inverse source problem as the norm
of data \(\epsilon \to 0\).

2. Increasing Stability for the Inverse source Problem

Let ’s define

\[
I(k) = I_1(k) + I_2(k)
\]

where

\[
I_1(k) = \int_0^K \alpha^2 |u(-1, \alpha)|^2 d\alpha, \quad I_2(k) = \int_0^K \alpha^2 |u(1, \alpha)|^2 d\alpha,
\]

using (1.3) and a simple calculation shows that

\[
\alpha u(1, \alpha) = \int_0^1 i \frac{e^{i \alpha (1-y)}}{2} f_1(y) dy, \quad \alpha u(-1, \alpha) = \int_{-1}^0 i \frac{e^{i \alpha (-1-y)}}{2} f_2(y) dy.
\]
where \( y \in (-1, 1) \). Functions \( I_1 \) and \( I_2 \) are both analytic with respect to the wave number \( k \in \mathbb{C} \) and play important roles in relating the inverse source problems of the Helmholtz equation and the Cauchy problems for the wave equations.

**Lemma 2.1.** Let \( \text{supp} \, f \subset (-1, 1) \) and \( f \in H^1(-1, 1) \). Then
\[
|I_1(k)| \leq C\left(|k| \| f \|_{H^1(0, 1)}^2\right) e^{2|k_2|}, \tag{2.3}
\]
\[
|I_2(k)| \leq C\left(|k| \| f \|_{H^1(0, 1)}^2\right) e^{2|k_2|}. \tag{2.4}
\]

**Proof.** Since we have \( k = k_1 + k_2 i \) is complex analytic on the set \( S \setminus [0, k] \), where \( S \) is the sector \( S = \{ k \in \mathbb{C} : |\arg k| < \frac{2\pi}{3} \} \) with \( k = k_1 + ik_2 \). Since the integrands in (2.1) are analytic functions of \( k \) in \( S \), their integrals with respect to \( \alpha \) can be taken over any path in \( S \) joining points 0 and \( k \) in the complex plane. Using the change of variable \( \alpha = k s \), \( s \in (0, 1) \) in the line integral (1.3), the fact that \( y \in (-1, 1) \),
\[
I_1(k) = \int_0^1 k s \int_0^1 \frac{1}{2} e^{i(k s)(1-y)} f_1(y) dy ds,
\]
and
\[
I_2(k) = \int_0^1 k s \int_0^1 \frac{1}{2} e^{i(k s)(-1-y)} f_2(y) dy ds.
\]
Noting
\[
|e^{ik s(-1-y)}| \leq e^{2|k_2|}, \quad |e^{ik s(1-y)}| \leq e^{2|k_2|},
\]
by the Cauchy-Schwartz inequality and integrating with respect to \( s \) and using the bound for \( |k| \) in \( S \), the proof of (2.3) is complete. Using the similar technique, the prove the (2.4) is straightforward.

Note that the functions \( I_1(k), I_2(k) \) are analytic functions of wave number \( k = k_1 + ik_2 \subset \mathbb{S} \) and \( |k_2| \leq k_1 \). The following steps are important to connect the unknown \( I_1(k) \) and \( I_2(k) \) for \( k \in [K, \infty) \) to the known value \( \epsilon \) in (1.1). Clearly
\[
|I_1(k)e^{-2k}| \leq C\left(|k_1| \| f \|_{H^1(0, 1)}^2\right) e^{-2k_1} \leq CM^2, \tag{2.7}
\]
where \( M = \max \{ \| f \|_{H^1(0, 1)}^2 \} \). With the similar argument bound (2.7) is true for \( I_2(k) \).

Noting
\[
|I_1(k)e^{-2k}| \leq \epsilon^2, \quad |I_2(k)e^{-2k}| \leq \epsilon^2 \text{ on } [0, K].
\]

Defining \( \mu(k) \) be the harmonic measure of the interval \([0, K]\) in \( \mathbb{S} \setminus [0, K] \), then as known (for example see [14], p.67), from two previous estimates and analyticity of the functions \( I_1(k)e^{-2k} \) and \( I_2(k)e^{-2k} \), we have that
\[
|I_1(k)e^{-2k}| \leq C\epsilon^{2\mu(k)} M^2, \tag{2.8}
\]
when \( K < k < +\infty \). Similarly it is easy to see that
\[
|I_2(k)e^{-2k}| \leq C\epsilon^{2\mu(k)} M^2, \tag{2.9}
\]
consequently
\[
|I(k)e^{-2k}| \leq C\epsilon^{2\mu(k)} M^2. \tag{2.10}
\]
To obtain the lower bound for the harmonic measure \( \mu(k) \), we use the following technical lemma. The proof is in [8].
\[
\alpha^2 |u(-1, \alpha)|^2 \leq C \left| \int_{-1}^0 e^{2\alpha y} f_2(y) dy \right|^2
\]

\[
\alpha^2 |u(1, \alpha)|^2 \leq C \left| \int_{0}^1 e^{2\alpha y} f_1(y) dy \right|^2
\]

Proof. It follows from (2.2) and \( y \in (-1, 1) \).

\[\square\]

3. Increasing stability for inverse source problem for the higher frequency

To proceed the estimate for reminders in (2.5) and (2.6) for \((k, \infty)\), we need the following lemma.

Lemma 3.1. Let \( u \) be a solution to the forward problem (1.1) with \( f_1 \in H^1(\alpha) \) with \( supp f \subset (-1, 1) \), then

\[
\int_{-\infty}^{\infty} \alpha^2 |u(-1, \alpha)|^2 d\alpha + \int_{-\infty}^{\infty} \alpha^2 |u(1, \alpha)|^2 d\alpha \leq C k^{-1} \left( \| f \|_{(1)}^2 (-1, 1) \right)
\]

Proof. Using (2.2), we obtain

\[
\int_{-\infty}^{\infty} \alpha^2 |u(-1, \alpha)|^2 d\alpha + \int_{-\infty}^{\infty} \alpha^2 |u(1, \alpha)|^2 d\alpha \leq C \left( \int_{-1}^{1} \left| \int_{0}^{1} e^{i\alpha y} f_1(y) dy \right|^2 d\alpha + \int_{-1}^{1} \left| \int_{-1}^{0} e^{i\alpha y} f_2(y) dy \right|^2 d\alpha \right)
\]

By integration by parts and our assumption \( supp f_1 \subset (0, 1) \) and \( supp f_2 \subset (0, 1) \), we derive

\[
\int_{0}^{1} e^{-i\alpha y} f_1(y) dy = \frac{1}{i\alpha} \int_{0}^{1} e^{-i\alpha y} (\partial_y f_1(y)) dy,
\]

and

\[
\int_{-1}^{0} e^{-i\alpha y} f_2(y) dy = \frac{1}{i\alpha} \int_{-1}^{0} e^{-i\alpha y} (\partial_y f_2(y)) dy,
\]

consequently for the first and second terms in (3.3), we have

\[
\left| \int_{0}^{1} e^{i\alpha y} f_1(y) dy \right|^2 \leq \frac{C}{\alpha^2} \| f_1 \|_{(1)}^2 (0, 1) \leq \frac{C}{\alpha^2} \| f_1 \|_{(1)}^2 (-1, 1)
\]

\[
\leq \frac{C}{\alpha^2} \| f \|_{(1)}^2 (-1, 1),
\]
utilizing the similar technique for the second term in (3.3) and integrating with respect to $\alpha$ the proof is complete. \[\square\]

Finally, we are ready for the proof of the main Theorem 1.1.

**Proof.** Without loss of generality, we can assume that $\epsilon < 1$ and $3\pi E^{-\frac{1}{4}} < 1$, otherwise the bound (1.1) is obvious. Let's

\begin{equation}
\begin{cases}
K^\frac{1}{2}E^{\frac{1}{2}} & \text{if } 2^\frac{1}{2}K^\frac{1}{2} < E^{\frac{1}{2}} \\
K & \text{if } E^{\frac{1}{2}} \leq 2^\frac{1}{2}K^\frac{1}{2},
\end{cases}
\end{equation}

if $E^{\frac{1}{2}} \leq 2^\frac{1}{2}K^\frac{1}{2}$, then $k = K$, using the (2.8) and (2.10), we can conclude

\begin{equation}
|I(k)| \leq 2^\epsilon^2.
\end{equation}

If $2^\frac{1}{2}K^\frac{1}{2} < E^{\frac{1}{2}}$, we can assume that $E^{-\frac{1}{4}} < \frac{1}{15}$, otherwise $C < E$ and hence $K < C$ and the bound (1.5) is straightforward. From (3.4), Lemma 2.2, (2.8) and the equality $\epsilon = \frac{1}{eE}$ we obtain

\begin{equation}
|I(k)| \leq CM^2 e^{\frac{2K}{E^{\frac{1}{2}}}} \left(\left(\frac{1}{5}\right)^{\frac{1}{2}} - 1\right)^{\frac{-1}{2}}
\end{equation}

\begin{equation}
|I(k)| \leq CM^2 e^{-\frac{1}{2}K^\frac{1}{2}E^{\frac{1}{2}}(1 - \frac{2\pi}{3}E^{-\frac{1}{4}})}
\end{equation}

using the trivial inequality $e^{-t} \leq \frac{1}{t^2}$ for $t > 0$ and our assumption at the beginning of the proof, we obtain

\begin{equation}
|I(k)| \leq CM^2 \frac{1}{K^2E^{\frac{1}{2}}(1 - \frac{5\pi}{3}E^{-\frac{1}{4}})^3}.
\end{equation}

Due to the (2.5), (3.5), (3.6), and Lemma 2.5, we can conclude

\begin{equation}
\int_0^{+\infty} \alpha^2|u(-1, \alpha)|^2d\alpha + \int_0^{+\infty} \alpha^2|u(1, \alpha)|^2d\alpha
\end{equation}

\begin{equation}
\leq I(k) + \int_k^{\infty} \alpha^2|u(-1, \alpha)|^2d\alpha + \int_k^{\infty} \alpha^2|u(1, \alpha)|^2d\alpha
\end{equation}

\begin{equation}
\leq 2\epsilon^2 + \frac{CM^2}{K^2E^\frac{1}{2}} + \frac{\| f \|_{(2)}^2 (-1, 1)}{K^\frac{1}{2}E^\frac{1}{2}}
\end{equation}

Using the inequalities in (3.7) and Lemma 2.3, we finally obtain

\begin{equation}
\| f \|_{(0)}^2 (\alpha) \leq C\left(\epsilon^2 + \frac{M^2}{K^2E^\frac{1}{2}} + \frac{\| f \|_{(1)}^2 (-1, 1)}{K^\frac{1}{2}E^\frac{1}{2} + 1}\right)
\end{equation}

Due to the fact that $K^\frac{1}{2}E^\frac{1}{2} < K^2E^\frac{1}{2}$ for $1 < K, 1 < E$, the proof is complete. \[\square\]
4. Conclusion

In this work, we investigate the inverse source problem with many frequencies in a one dimensional domain with many frequencies. The result showed that if the wave number $k$ grows the estimate improves and the problems more stable. It also showed that if we have data exists for all wave number $k \in (0, \infty)$, the estimate will be a Lipschitz estimate. It will be very interesting if we can investigate these type of problems in time domain with different speed of propagation or in a domain with different number of layers and densities. It is also very amazing if one can study these type of problem in domain with corners. From computational point of view, these type of problems have a vast application in different area of science.

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