The onset of fluid-dynamical behavior in relativistic kinetic theory

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Based on G. Denicol, JN, arXiv:1608.07869 [nucl-th] + arXiv:170x.xxxxx

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The ubiquitousness of fluid dynamics

Based on conservation laws + large separation of length scales

Separation of scales → macroscopic: \( L \)  \( \quad \) microscopic: \( \ell \)

Knudsen number expansion:

\[ \frac{\ell}{L} \ll 1 \]

\[ \Rightarrow \text{FLUID} \]

Macroscopic: Gradient of velocity field

\[ \partial v \sim \frac{1}{L} \]

Example of microscopic scale:

\[ \ell \sim \frac{1}{(n\sigma)} \quad \text{Mean free path} \]

\[ L \sim 1 \text{ m} \]

\[ \ell \sim 10^{-7} \text{ m} \]
QGP initial condition

\( \tau = 0.2 \text{ fm} \)

**IP-Glasma**

Schenke, Tribedy, Venugopalan, PRL 2012

Knudsen number event-by-event

There is no reason to believe that Kn has to be small in this case.

Hydrodynamic behavior in small systems????

**CMS Preliminary**

\( \Delta \epsilon / \epsilon_0 \sim \Lambda_{QCD} \)

macro scale

microscopic scale ????

There is no reason to believe that Kn has to be small in this case.
**Hydrodynamics as a gradient expansion**

Assume deviations from local equilibrium

| Boltzmann equation | Chapman-Enskog series (1939) |
|--------------------|-------------------------------|
| \[ p^\mu \partial_\mu f = C[f, f] \] | \[ f = f_{eq} \left( 1 + \sum_{n=1}^{\infty} (K_N)^n f_n(x^\mu, p^\mu) \right) \] |

Convergence???

- At 0\(^{th}\) order in Kn → Ideal hydro, at 1\(^{st}\) order → **Navier-Stokes**

- Very hard to carry out procedure to higher orders

- Solution of Boltzmann completely defined in terms of local hydrodynamic fields:

\[ T, \mu, u^\mu \]
Hydrodynamics as a gradient expansion

- Gradient expansion “solution”, though systematic, is highly contrived
- Kn itself depends on the flow properties
- Problems with instabilities and acausality in the relativistic domain
- Procedure cannot describe all possible solutions of Boltzmann …

Ex: Homogeneous relaxation (set all spatial gradients to zero)

\[ \partial_t f = C[f, f] \rightarrow \text{Dynamics contains only non-hydro modes} \]

Non-hydro modes \( \rightarrow \) defined by nonzero eigenvalues of collision operator

\[ \lim_{k \to 0} \omega_{nh}(k) \neq 0 \]
Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Gradient expansion to all orders: Simplest “toy model” of QGP fluid

Bjorken expanding (conformal, transversely homogeneous) fluid:

\[ x^\mu = (\tau, x, y, \eta) \]

Spacetime rapidity \( \eta = \tanh^{-1}(z/t) \)

Milne propertime \( \tau = \sqrt{t^2 - z^2} \)

Flow velocity \( u^\mu = (1, 0, 0, 0) \)

Boltzmann equation

\[ \partial_\tau f = C[f, f] \]

By symmetry:

- \( f \rightarrow f(\tau, k_0, k_\eta) \)

- Any gradient \( \sim \frac{1}{\tau} \)
Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Relaxation time $\tau_R$ approximation (RTA)

(see Florkowski et al., Nucl.Phys. A916 (2013) 249-259)

$$\frac{\partial_{\tau} T}{T} + \frac{1}{3\tau} - \frac{\pi}{12\tau} = 0,$$

$$\partial_{\tau} f_k = -\frac{f_k - f_{eq}}{\tau_R},$$

Knudsen number

$$K_N = \frac{\tau_R}{\tau}$$

Shear stress tensor

$$\pi_{\eta} = \int \frac{d^3k}{(2\pi)^3 \tau} k_0 \left[ \frac{1}{3} - \left( \frac{k_\eta}{k_0\tau} \right)^2 \right] f_k$$

Landau matching condition

$$\varepsilon = \int \frac{d^3k}{(2\pi)^3 \tau} k_0 f_k \equiv \int \frac{d^3k}{(2\pi)^3 \tau} k_0 f_{eq}$$

$$f_{eq} = \exp \left( -\frac{k_0}{T} \right)$$

Massless particles, constant relaxation time
Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Method of moments:

\[ \rho_{n,\ell} = \int \frac{d^3k}{(2\pi)^3} \left( k^0 \right)^n \left( \frac{k_\eta}{k^0} \right)^{2\ell} f_k \]

Moments eqs. \( \longleftrightarrow \) Boltzmann eq.

\[
\partial_\tau M_{n,\ell} + \frac{1}{\tau_R} M_{n,\ell} + \frac{6\ell - n}{3\tau} M_{n,\ell} - \frac{n + 3}{12\tau} M_{1,1} \left( 1 + M_{n,\ell} \right) \\
+ \frac{1}{\tau} \frac{(n - 2\ell)(1 + 2\ell)}{2\ell + 3} M_{n,\ell+1} = -\frac{1}{\tau} \frac{4\ell(n + 3)}{3(2\ell + 3)}
\]

Dimensionless moments

\[ M_{n,\ell} \equiv \frac{\rho_{n,\ell} - \rho_{n,\ell}^{eq}}{\rho_{n,\ell}^{eq}} \]

Nonlinearity

\[ M_{1,1} = -\pi/P \]
Divergence of the gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Solution of Boltzmann → reconstructed via $M_{n,\ell}(\tau)$

Gradient expansion series:

$$M_{n,\ell} = \sum_{p=0}^{\infty} \frac{\alpha_p^{(n,\ell)}}{\hat{T}^p}$$

Knudsen number

$$K_N = \frac{T_R}{\tau} = \frac{1}{\hat{T}}$$

Exact recursive relation

$$\alpha_{m+1}^{(n,\ell)} = -\frac{6\ell - n - 3m}{3} \alpha_m^{(n,\ell)} + \frac{n + 3}{12} \alpha_m^{(1,1)}$$

$$- \frac{(n - 2\ell)(1 + 2\ell)}{2\ell + 3} \alpha_m^{(n,\ell+1)} + \frac{n + 3}{12} \sum_{p=0}^{m} \alpha_p^{(1,1)} \alpha_{m-p}^{(n,\ell)}.$$
Gradient (Chapman-Enskog) series is clearly divergent !!!

$$\lim_{m \gg 1} \alpha^{(n, \ell)}_m \sim m!$$

First time this has been shown in relativistic kinetic theory

Hydrodynamics is not just a series in gradients ...
Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

Regularity of the system near initial condition: \( \tau \to \tau_0 \)

+ 

And the 1st order nature of the ODE's for the moments

Show that at early times
\[
M_{n,\ell}(\hat{\tau}) \sim e^{-(\hat{\tau} - \hat{\tau}_0)} \sim e^{-1/K_N}
\]

Dynamics contains highly non-perturbative terms !!!!

This should also hold for a QCD-like collision kernel

Essential singularity (given by non-hydro mode)
Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

- New terms not present in the usual gradient series
- They show that initial condition data is not easily “erased”
- They carry information about non-hydro mode dynamics

We propose a novel Generalized Chapman-Enskog (GCE) series

\[ M_{n,\ell} (\hat{\tau}) = \sum_{p=0}^{\infty} \frac{\beta_p^{(n,\ell)} (\hat{\tau})}{\hat{\tau}^p} \]

This introduces an expansion parameter in the moments method !!!
Generalized gradient expansion in kinetic theory

G. Denicol and JN, arXiv:1608.07869 [nucl-th]

CE series: \( \alpha_{m}^{(n,\ell)} \rightarrow \) obey algebraic relations

GCE series: \( \beta_{m}^{(n,\ell)}(\hat{\tau}) \rightarrow \) obey differential equations!!

\[
\partial_{\hat{\tau}} \beta_{m+1}^{(n,\ell)} + \beta_{m+1}^{(n,\ell)} = -\frac{4\ell (n + 3)}{3 (2\ell + 3)} \delta_{m,0} \\
- \frac{(n - 2\ell)(1 + 2\ell)}{2\ell + 3} \beta_{m}^{(n,\ell+1)} - \frac{6\ell - n - 3m}{3} \beta_{m}^{(n,\ell)} \\
+ \frac{n + 3}{12} \beta_{m}^{(1,1)} + \frac{n + 3}{12} \sum_{p=0}^{m} \beta_{m-p}^{(1,1)} \beta_{p}^{(n,\ell)}.
\]

New series describes the whole time evolution since initial condition !!!
Excellent agreement with full exact solution of Boltzmann already when truncated at 2\textsuperscript{nd} order !!!

2\textsuperscript{nd} order truncation not the usual Israel-Stewart theory
Hydrodynamics cannot be just an expansion in gradients since any truncation of the theory expansion leads to acausal and unstable dynamics. Radius of convergence of expansion is zero.

- Israel-Stewart theory (used in 100% of hydro models):

\[ \tau_\pi D \pi^{\langle \mu \nu \rangle} + \pi^{\langle \mu \nu \rangle} = -\eta \sigma^{\mu \nu} + \ldots \rightarrow \text{non-hydro mode} \quad \tau_\pi \]

Just a possible resummation, not unique (or universal).
Conclusions

- The transition from $Kn << 1$ to $Kn \sim 1$ is a fundamental problem in fluid dynamics.

- This problem can be fully solved in a simple model for the QGP fluid within kinetic theory.

- In this case, the gradient / Knudsen series diverges. Same happens at strong coupling (see backup slides).

- Novel non-perturbative non-hydro modes are needed to make sense of fluid dynamics (at least in the relativistic regime).

Relativistic fluid dynamics is not just a series in gradients

- New effective theory of fluid dynamics, valid at any valid of the coupling, including both hydro + non-hydro modes must still be constructed.

- This should be relevant for understanding the fluid-like behavior of small systems in heavy ions.
EXTRA SLIDES
**Importance of non-hydro mode: Gubser flow**

Phys.Rev.Lett. 113 (2014) no.20, 202301

Not a simple Longitudinal Bjorken expansion

Gradient expansion solution extremely limited

- **Kinetic Exact**
- **Israel-Stewart-like (DNMR)** → contains non-hydro mode
- **1st-order Hydro**
- **Ideal Hydro**
- **Free Streaming**
Divergence of the gradient expansion at strong coupling

\[ K_N \sim \frac{\ell}{L} \ll 1 \]

Fluid/gravity correspondence (aka Chapman-Enskog at strong coupling) developed by Minwalla, Hubeny, Rangamani, and etc.

Gradient expansion at large orders at strong coupling done by Heller et al., PRL (2013), for N=4 SYM + Bjorken expansion

Energy density

\[ \epsilon = \frac{3}{8} \frac{N_c \pi^2}{\tau^{4/3}} \left( \epsilon_2 + \epsilon_3 \frac{1}{\tau^{2/3}} + \epsilon_4 \frac{1}{\tau^{4/3}} + \ldots \right) \]
Divergence of the gradient series at strong coupling

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Entropy production in N=2* theory (FLRW flow)

\[ \frac{d(a^3 s)}{dt} = \frac{N^2}{16\pi} a^{7-2\Delta} \mu^2 \delta^2_\Delta (4 - \Delta)^2 s_\Delta \times \Omega^2_\Delta, \]

\[ \Omega_\Delta \equiv \sum_{n=0}^{\infty} T_{\Delta,n+1}[a] \frac{F_{\Delta,n}(1)}{\mu^n}. \]

\[ T_{\Delta,n}[a] = \left( -\frac{1}{2} - \frac{3\omega}{2} \right)^n \Gamma\left( n + \frac{2(\Delta - 4)}{1 + 3\omega} \right) a^n H^n. \]

1st Analytical proof of divergence!!

Black hole QNM's = non-hydro modes → singularities in the Borel plane (after resummation)
Divergence of the hydrodynamic series

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Hydrodynamic series

\[ \Omega_\Delta = \sum_{n=0}^{\infty} c_n g^n , \quad c_n \equiv \frac{\Gamma(n + 4 - \Delta) F_{\Delta,n}(1)}{(4\pi)^n} \text{ and } g \equiv \frac{H}{T} = \frac{4\pi}{\mu} a H. \]

Borel sum

\[ \Omega_\Delta^{(B)}(\xi) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \xi^n \]

Borel singularities = are the black hole quasinormal modes !!!
Divergence of the gradient expansion in kinetic theory

Simple argument to show that the series must diverge (a la Dyson)

If series converged around $K_N = \frac{\tau R}{\tau} \to 0$, there would be a nonzero radius of convergence $R$

But for RTA $\partial_\tau f = \frac{(f - f_{eq})}{\tau R}$

Since $\tau > 0$

$K_N < 0 \Rightarrow \tau R < 0$

INSTABILITY  !!!!!!!
Hydrodynamics as a series expansion

Scaled Boltzmann equation

\[ p^\mu \partial_\mu f = \frac{C[f, f]}{K_N} \]

Knudsen number

\[ K_N \sim \frac{\ell}{L} \]

Formal solution via a series

Hilbert Series

\[ f(x^\mu, p^\mu, K_N) = \sum_{n=0}^{\infty} (K_N)^n f_n(x^\mu, p^\mu) \]

\[ f_0 \sim e^{-p \cdot u / T} \rightarrow \text{derives (does not assume) ideal fluid dynamics} \]

No statement about existence of series is made
1\textsuperscript{st} analytical proof of the divergence of gradient expansion:

→ Knudsen gradient series has zero radius of convergence

→ Knudsen series leads to acausal and unstable dynamics

→ There must be a new way to define hydrodynamics beyond the gradient expansion

→ A recent way to understand that involves resurgence.
Friedmann-Robertson-Lemaitre-Walker (FRLW) spacetime

Maximally (spatially) symmetric spacetime

\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right] \]

\( K \sim 0 \) (spatially flat \( \rightarrow \) our universe)

\( K = 1, -1 \)

Einstein's equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \]

\( \dot{a} \)

\[ \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \varepsilon - \frac{K}{a^2} \]

\[ \ddot{a} = -\frac{4\pi G}{3} (\varepsilon + 3P) \]

\( \varepsilon \propto \begin{cases} a^{-3} & \text{matter} \\ a^{-4} & \text{radiation} \\ a^0 & \text{vacuum} \end{cases} \)
FLRW spacetime

Spatial isotropy + homogeneity

Isotropic and homogeneous expanding FLRW spacetime
(zero spatial curvature)

Ex: metric

$$ds^2 = dt^2 - a^2(t) (dx^2 + dy^2 + dz^2)$$

Determined from Einstein's equations
Friedmann-Lemaitre-Robertson-Walker spacetime

We consider an isotropic and homogeneous expanding FRW spacetime

\[ ds^2 = dt^2 - a^2(t) \left( dx^2 + dy^2 + dz^2 \right) \]

(zero spatial curvature)

Cosmological scale factor (e.g., radiation)
\[ a(t) \sim t^{1/2} \]

Hubble parameter \[ H = \frac{\dot{a}}{a} > 0 \]

Distances get stretched
\[ \Delta x_{phys}(t_0) = a(t) \Delta x \]

\[ \Delta x_{phys}(t_0) \quad \Delta x_{phys}(t_1) \]

\[ t_0 \quad t_1 \quad \text{time} \]
- Locally static system corresponding to an expanding, homogeneous Universe

- Provides a simple case to study the validity of gradient expansion at strong coupling

FLRW spacetime is conformal to Minkowski and, thus, one can perform gauge/gravity calculations in which the gauge theory lives in a curved 4-dimensional spacetime!!!!
Given that heavy ion data indicates that $T \sim$ QCD transition the QGP is a nearly perfect fluid …

There must have been nearly perfect fluidity in the early universe

Experimental consequences of that are not yet known (are there any??)

Given that around those temperatures QCD is not conformal, we would like to use a nonconformal gravity dual in a FLRW spacetime

This was done by A. Buchel, M. Heller, JN in arXiv:1603.05344 [hep-th] PRD (2016)
**Toy model for QCD: N=2* gauge theory**

Pilch, Warner, Buchel, Peet, Polchinski, 2000  
A. Buchel, S. Deakin, P. Kerner and J. T. Liu, NPB 784 (2007) 72

A relevant deformation of SYM:  
Breaking of SUSY

\[
\begin{align*}
N = 4 \text{ SYM theory} &+ \quad \delta \mathcal{L} = -2 \int d^4 x \left[ m_b^2 \mathcal{O}_b + m_f \mathcal{O}_f \right] \\
\mathcal{O}_b &= \frac{1}{3} \text{Tr} \left( |\phi_1|^2 + |\phi_2|^2 - 2 |\phi_3|^2 \right) , \\
\mathcal{O}_f &= -\text{Tr} \left( i \psi_1 \psi_2 - \sqrt{2} g_{YM} \phi_3 [\phi_1, \phi_1^\dagger] + \sqrt{2} g_{YM} \phi_3 [\phi_2^\dagger, \phi_2] \right) \\
&+ \text{h.c.} \right) + \frac{2}{3} m_f \text{Tr} \left( |\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 \right)
\end{align*}
\]

Bosonic mass  
Fermionic mass

C. Hoyos, S. Paik, and L. G. Yaffe, JHEP 10, 062 (2011)
To toy model for QCD: N=2* gauge theory

Pilch, Warner, Buchel, Peet, Polchinski, 2000

Classical gravity dual action:

\[ S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R - 12(\partial \alpha)^2 - 4(\partial \chi)^2 - V \right), \]

Scalar potential

\[ V = -e^{-4\alpha} - 2e^{2\alpha} \cosh 2\chi + \frac{1}{4} e^{8\alpha} \sinh^2 2\chi. \]

- Well defined stringy origin

- Non-conformal strongly interacting plasma: \( \varepsilon \neq 3p \)

- Used in tests of holography in non-conformal setting

Bulk viscosity

\[ \frac{\zeta}{\eta} \sim \mathcal{O}(1) \left( \frac{1}{3} - c_s^2 \right) \]
**N=2* gauge theory in a FLRW Universe**

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Characteristic formulation of gravitational dynamics in asymptotically AdS5 spacetimes

Assuming spatial isotropy and homogeneity $x = \{x, y, z\}$ leads to

$$ds_5^2 = 2dt \ (dr - A dt) + \Sigma^2 \ d\mathbf{x}^2,$$

$$\Sigma = a \frac{r}{r} + \mathcal{O}(r^{-1}), \quad A = \frac{r^2}{8} - \frac{\dot{a}r}{a} + \mathcal{O}(r^0)$$

$$\alpha = -\frac{8m_b^2 \ln r}{3r^2} + \mathcal{O}(r^{-2}), \quad \chi = \frac{2m_f}{r} + \mathcal{O}(r^{-2}).$$

Encode non-equilibrium dynamics in an expanding Universe !!!
N=2* gauge theory in a FLRW Universe

Buchel, Heller, JN, arXiv:1603.05344 [hep-th], PRD 94, 106011 (2016)

Conformal limit When \( m_b = m_f = 0 \),

Analytical solution for SYM in FLRW spacetime

\[
\alpha = \chi = 0, \quad \Sigma = \frac{ar}{2}, \quad A = \frac{r^2}{8} \left( 1 - \frac{\mu^4}{r^4 a^4} \right) - \frac{\dot{a}}{a} r,
\]

First studied by P. S. Apostolopoulos, G. Siopsis, and N. Tetradis, PRL, (2009)

Temperature

\[
T = \frac{\mu}{4\pi a}.
\]

Energy density

\[
\epsilon = \frac{3}{8} \pi^2 N^2 T^4 + \frac{3N^2(\dot{a})^4}{32\pi^2 a^4}
\]

Pressure

\[
P = \frac{1}{3} \epsilon - \frac{N^2(\dot{a})^2\ddot{a}}{8\pi^2 a^3}
\]

Conformal anomaly!!!!

\[
-\epsilon + 3P = \frac{N^2}{32\pi^2} \left( R_{\mu\nu}R^{\mu\nu} - \frac{1}{3} R^2 \right)
\]
In our FLRW case, the gradient expansion corresponds to

\[ T_{\mu\nu} = T_{\mu\nu}^{eq} + \Pi_{\mu\nu}(\dot{a}, \{\dot{a}^2, \ddot{a}\}, \cdots), \]

In terms of the energy density and pressure out-of-equilibrium

\[ \epsilon = \epsilon^{eq} + \mathcal{O}(\dot{a}^2, \ddot{a}), \quad P = P^{eq} - \zeta(\nabla \cdot u) + \mathcal{O}(\dot{a}^2, \ddot{a}), \]

Bulk viscosity
Universality and perfect fluidity

\[ \lambda \gg 1 \quad \text{in QFT} \rightarrow \text{string theory in weakly curved backgrounds} \]

\[ \text{d.o.f. / vol.} \rightarrow \infty \quad \text{in QFT} \rightarrow \text{vanishing string coupling} \]

\[ T, \mu \quad \text{in QFT} \rightarrow \text{spatially isotropic black brane} \]

For anisotropic models there is violation
see PRD 2014
arXiv:1406.6019

 Universality of shear viscosity

\[ \frac{\eta}{s} = \frac{1}{4\pi} \]

Kovtun, Son, Starinets, 2005

Universality of black hole horizons

HOLOGRAPHY

Universality of transport coefficient in QFT
Chapman-Enskog expansion: Non-relativistic regime

Santos, Brey, Dufty, 1571, vol 51 PRL (1986)

Newtonian fluid: \( P_{xy} = -\eta_0 \partial u_x / \partial y \)

Uniform shear flow
\[
\frac{\partial u_i}{\partial x_j} = \gamma \delta_{ix} \delta_{jy}
\]

Pressure tensor
\[
P_{xy} = - \sum_{k=0}^{\infty} \eta_k \left( \frac{\partial u_x}{\partial y} \right)^{2k+1}
\]

BGK Boltzmann
\[
(\partial_t + \mathbf{v} \cdot \nabla) f = -\nu (f - f_0)
\]

Series converges if \( \nu \sim \text{const} \)
(Maxwell molecules)

DIVERGES \( \nu \sim p^\alpha, \quad \alpha = (n - 4) / 2n \) \( r^{-n} \) potential
(e.g., hard spheres, n=2)
Hydrodynamics from the method of moments

Harold Grad, 1948

(See Israel-Stewart for Relativistic case)

\[ p^\mu \partial_\mu f = C[f, f] \]

Define infinite set of moments such as

\[ \varepsilon = \int_p (u \cdot p)^2 f \quad T^{\mu\nu} = \int_p p^\mu p^\nu f \]

- energy density
- Energy-momentum tensor

- Use Boltzmann equation to find exact equations for the moments

- Reconstruct solution of Boltzmann using a complete set of moments

- In the relativistic domain, 14 moments truncation → Israel-Stewart eqs.
PROS:

- Moments method played a major role in the derivation of the hydrodynamic equations for the QGP – stability and causality!!!
- Used in many approaches: Israel-Stewart, DNMR, (v)AHYDRO ...
- Describe interactions between hydro and non-hydro modes
- Can provide consistent (and convergent) solution of Boltzmann
- Used to derive the 1\textsuperscript{st} analytical solution of expanding Boltzmann gas by BDHMN in PRL 116, (2016)

CONS:

- Absence of a small expansion parameter
- Very hard to implement general equations in practice
Resurgence

Recent works by Dunne, Unsal, Basar, Cherman, Heller, Janik ...

\[ \langle \mathcal{O}(\lambda) \rangle = \sum_{n=0}^{\infty} p_{0,n} \lambda^n + \sum_{c} e^{-S_c/\lambda} \sum_{k=0}^{\infty} p_{c,n} \lambda^n \]

\[ \sum_{n=0,k=0}^{\infty} c_{n,k,q} g^{2n} \left[ e^{-S/g^2} \right]^k \left[ \log \left( \frac{1}{g^2} \right) \right]^q \]

Heller, Spalinski, PRL 2015

Hydro expansion via resurgence

\[ f(w) = \sum_{m=0}^{\infty} c^m \Omega(w)^m \sum_{n=0}^{\infty} a_{m,n} w^{-n} \]

\[ \Omega \equiv w^{-\gamma} \exp(-w \xi_0) \]