SPECIAL CYCLES ON UNITARY SHIMURA VARIETIES AT RAMIFIED PRIMES I

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ABSTRACT. In this paper, we study special cycles on unitary Shimura varieties at the ramified primes of an imaginary quadratic number field \(k\) and special cycles on the corresponding Rapoport-Zink spaces. We write down the support of these special cycles and relate them by a uniformization theorem. We then compute their dimensions.

1. Introduction

In [KR2], Kudla and Rapoport introduced moduli spaces of abelian varieties which are arithmetic models of Shimura varieties attached to unitary groups defined over an imaginary quadratic number field \(k\) of signature \((n-1,1)\). They then defined special cycles which are divisors in certain cases. To a positive definite Hermitian matrix \(T \in \text{Herm}_n(\mathcal{O}_k)\), one can associate a special cycle \(Z(T)\) which is a union of \(n\)-fold intersections of special divisors. It is showed in [KR2] that \(Z(T)\) is supported over finitely many inert or ramified primes \(p\) of \(k\). To be more specific, when the Hermitian \(k\)-vector space \(V_T\) determined by the inner product matrix \(T\) contains an almost self-dual lattice \(L\), which means that \(L^\vee/L = \mathbb{F}_p^2\) for an inert prime \(p\), \(Z(T)\) is supported on its special fiber over \(p\). In this case, [KR2] relate the intersection number of \(Z(T)\) to a Fourier coefficient of the derivative at \(s=0\) of a certain incoherent Eisenstein series for the group \(U(n,n)\) when the intersection is non-degenerate.

In this paper, we focus on the case when \(V_T\) contains a self-dual lattice. In this case results of [KR2] and this paper show that, \(Z(T)\) is supported on its special fibers over all the ramified primes of \(k\). We then compute the dimension of \(Z(T) \times_{\text{Spec} \mathcal{O}_k} \text{Spec} \mathbb{F}_p\) which only depends on \(p\)-adic expansion of \(T\) for a ramified prime \(p\). In order to do this, we first study the corresponding local intersection problem on the Rapoport-Zink space \(\mathcal{N}^0\) which uniformizes the super singular locus of the unitary Shimura variety over the ramified prime \(p\).

The goal of our project is to obtain an analogous result as that of [KR2] in the case when the intersection of special divisors is supported on its special fiber over the ramified primes. In other words, we want to relate the intersection number of \(Z(T)\) to a Fourier coefficient of the derivative at \(s=0\) of the incoherent Eisenstein series for the group \(U(n,n)\) defined in [KR2] when the intersection of special divisors is supported on its special fiber over the ramified primes. The current paper is the first step towards this goal.

We now describe our result in more detail. Let \(k = \mathbb{Q} \langle \sqrt{\Delta} \rangle\) be an imaginary quadratic extension of \(\mathbb{Q}\) and \(\mathcal{O}_k\) be its ring of integers. Let \(n \geq 1, 0 \leq r \leq n\) be integers. Define a groupoid \(\mathcal{M}(n-r, r)\) fibered over \((\text{Sch}/\text{Spec} \mathcal{O}_k)\) by associating...
to a locally noetherian \( \mathcal{O}_k \)-scheme \( S \) the groupoid of triples \((A, \iota, \lambda)\). Here \( A \) is an abelian scheme over \( S \), \( \iota: \mathcal{O}_S \to \text{End}_S(A) \) is an action of \( \mathcal{O}_k \) on \( A \) and \( \lambda: A \to A^\vee \) is a principal polarization such that

\[
\iota(a)^* = \iota(a^\sigma), \quad \forall a \in \mathcal{O}_k
\]

where \( * \) is the Rosati involution and \( a^\sigma \) is the Galois conjugate of \( a \). In addition we require that the action of \( \mathcal{O}_k \) on the Lie algebra of \( A \) satisfies the signature \((n - r, r)\) condition (see section [5]). \( \mathcal{M}(n - r, r) \) is a Deligne-Mumford stack over \( \text{Spec} \mathcal{O}_k \). We denote \( \mathcal{M}_0 = \mathcal{M}(1, 0) \). Suppose \((E, \iota_0, \lambda_0) \in \mathcal{M}_0(S) \) and \((A, \iota, \lambda) \in \mathcal{M}(n - r, r)(S) \). Assume \( S \) is connected. Consider \( S \) to be a ramified quadratic extension of \( H \).

Further, \( \mathcal{O}_{H_r} \) is a principal polarization such that \( \mathcal{O}_{H_r} \) is the Rosati involution and \( O \) is the Galois conjugate of \( O \) and by \( W \) its fractional field. Let \( A, \iota, \lambda, \iota_0, \lambda_0 \) be a ramified prime \( p \)-divisible group of dimension \( n \) and height 2n over \( \mathbb{F} \) with an action \( \mathcal{O}_H \to \text{End}(X) \). Let \( \lambda_\mathbb{F} \) be a principal polarization such that its Rosati involution induces on \( \mathcal{O}_H \) the non-trivial automorphism over \( \mathbb{Q}_p \).

Let \( \mathcal{N}_{(n-r,r)} \) be the set-valued functor on \( \text{Nil} \) which associates to \( S \in \text{Nil} \) the set of isomorphism classes of quadruples \((X, \iota, \lambda, g)\). Here \( X \) is a \( p \)-divisible group over \( S \) and \( \iota: \mathcal{O}_H \to \text{End}(X) \) satisfies the Kottwitz condition and the Pappas condition of signature \((n - r, r)\) (see section [2]). Further \( \lambda: X \to X^\vee \) is a principal quasi-polarization whose associated Rosati involution induces on \( \mathcal{O}_H \) the non-trivial automorphism over \( \mathbb{Q}_p \). Finally, \( g : X \times_S \tilde{S} \to X \times_{\text{Spec} \mathbb{F}} \tilde{S} \) is an \( \mathcal{O}_H \)-linear quasi-isogeny such that \( \lambda \) and \( g^* (\lambda_\mathbb{F}) \) differ locally on \( \tilde{S} \) by a factor in \( \mathbb{Q}_p^\times \). In this introduction, we mostly deal with signature \((n - 1, 1)\) so we simply use \( \tilde{N} \) to stand for \( \mathcal{N}_{(n-1,1)} \). We denote by \( \mathcal{N}^0 \) its open and closed hens where the quasi-isogeny has height 0, and by \( \mathcal{N}^0 \) the fiber of \( \mathcal{N}^0 \) over \( \mathbb{F} \). Let \( \mathcal{N}_\text{red}^0 \) be the underlying reduced scheme of \( \mathcal{N}^0 \). For the introduction it is enough to keep in mind that \( \mathcal{N}^0 \) uniformizes the formal completion of \( \mathcal{M}(n - 1, 1) \) along its super singular locus over a ramified prime \( p \) of \( k \).

To the triple \((X, \iota_\mathbb{F}, \lambda_\mathbb{F})\), there is a hermitian vector space \( C \) of dimension \( n \) over \( H \). A \( \mathcal{O}_H \) lattice \( \lambda \) in \( C \) is called a vertex lattice of type \( t = t(\Lambda) \) if the dual lattice \( \Lambda^t \) is contained in \( \Lambda \) and \( \Lambda/\Lambda^t \) is a \( \mathbb{F}_p \)-vector space of dimension \( t \). It turns out that \( t \) is an even integer and that all even integers \( 0 \leq t \leq n \) occur as types of vertex lattices, except when \( n \) is even and the Hermitian space \( C \) is non-split, in which case all even \( t \) with \( 0 \leq t \leq n - 2 \) occur but \( t = n \) does not.

In [RTW], the authors associated a locally closed irreducible subset \( \mathcal{N}_X^0 \) of \( \mathcal{N}_\text{red}^0 \) of dimension \( \frac{1}{2} t(\Lambda) \) such that the following properties hold (theorem 1.1 of [RTW]):
(1) There is a stratification of $\mathcal{N}_{red}^{0}$ by locally closed subschemes given by

$$\mathcal{N}_{red}^{0} = \bigcup_{\Lambda} \mathcal{N}_{\Lambda}^{0},$$

where $\Lambda$ runs over all vertex lattices.

(2) The closure $\mathcal{N}_{\Lambda}$ of $\mathcal{N}_{\Lambda}^{0}$ is the finite disjoint union

$$\mathcal{N}_{\Lambda} = \bigcup_{\Lambda' \subseteq \Lambda} \mathcal{N}_{\Lambda'}^{0},$$

where $\Lambda'$ runs over vertex lattice contained in $\Lambda$. Notice that $t(\Lambda') < t(\Lambda)$.

There is a unique closed point of $\mathcal{N}_{(0,1)}^{0}$ which we denote by $(\mathcal{Y}, v_{\mathcal{Y}}, \lambda_{\mathcal{Y}}, \varrho_{\mathcal{Y}})$. Let $V = \text{Hom}_{O_H}(\mathcal{Y}, X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

For $x, y \in V$, define the space of special homomorphisms to be

$$\hat{h}(x, y) = \lambda_{\mathcal{Y}} \circ \hat{y} \circ \lambda_{X} \circ x \in \text{End}_{O_H}(\mathcal{Y}) \otimes_{\mathbb{Q}} \mathbb{Q} \overset{\sim}{\rightarrow} H.$$
whose Hermitian inner product is defined by the matrix \(-\epsilon^{-1}\delta^2 T\). Fix a Jordan decomposition
\[
L = \bigoplus_{0 \leq \lambda \leq t} L_\lambda,
\]
such that \(sL_\lambda = \lambda\). Define
\[
t(L) = \begin{cases} 
\text{rank}_{\mathcal{O}_H}(L_{\geq 1}) - 1 & \text{if } \text{rank}_{\mathcal{O}_H}(L_{\geq 1}) \text{ is odd} \\
\text{rank}_{\mathcal{O}_H}(L_{\geq 1}) & \text{if } \text{rank}_{\mathcal{O}_H}(L_{\geq 1}) \text{ is even and } L_{\geq 1} \text{ is split} \\
\text{rank}_{\mathcal{O}_H}(L_{\geq 1}) - 2 & \text{if } \text{rank}_{\mathcal{O}_H}(L_{\geq 1}) \text{ is even and } L_{\geq 1} \text{ is non-split},
\end{cases}
\]
Then \(Z(x)\) is purely of dimension \(\frac{1}{2}t(L)\).
(iii) Define \(L\) and fix a Jordan decomposition of \(L\) as in (ii). Define
\[
n_{\text{odd}} = \sum_{sL_\lambda \geq 3, sL_\lambda \text{ is odd}} \text{rank}_{\mathcal{O}_H}(L_\lambda),
\]
and
\[
n_{\text{even}} = \sum_{sL_\lambda \geq 2, sL_\lambda \text{ is even}} \text{rank}_{\mathcal{O}_H}(L_\lambda).
\]
Then \(Z(x)\) is irreducible if and only if the following two conditions hold
(1) \(n_{\text{odd}} = 0\).
(2) \(n_{\text{even}} \leq 1\) or \(n_{\text{even}} = 2\) and \(L_{\geq 2}\) is non-split.
(iv) \(Z(x)\) is zero dimensional if and only if the following two conditions hold
(1) \(n_{\text{odd}} = \text{rank}_{\mathcal{O}_H}(L_1) = 0\).
(2) \(n_{\text{even}} \leq 1\) or \(n_{\text{even}} = 2\) and \(L_{\geq 2}\) is non-split.
If this is the case, then \(Z(x)\) is in fact a single point.

In Proposition \ref{prop:special_cycle} we relate the special cycle \(Z(T)\) on \(\mathcal{M}_0 \times_{\text{Spec} \mathcal{O}_k} \mathcal{M}(n-1,1)\) to the special cycles \(Z(x)\) defined on the Rapoport-Zink space \(N_{\mathcal{O}_H}^{n,1} \times_{\text{Spec} \mathcal{O}_H} N_{\mathcal{O}_H}^{1,n-1}\).

Using the above theorem we can derive our global result:

**Theorem 1.2.** For \(T \in \text{Herm}_n(\mathcal{O}_k),\) let \(V_T\) be the Hermitian \(k\)-vector space defined by the inner product matrix \(T\). Assume that \(V_T\) contains a self dual lattice. Define
\[
Z(T)[\frac{1}{2}] = Z(T) \times_{\text{Spec} \mathcal{O}_k} \text{Spec} \mathcal{O}_k[\frac{1}{2}].
\]
(i) \(Z(T)[\frac{1}{2}]\) is supported on its special fibers over the ramified primes of \(k\). Furthermore for any prime \(p\) such that \(p | \Delta\) and \(p \neq 2\), the special fiber \(Z(T)[\frac{1}{2}] \times_{\text{Spec} \mathcal{O}_k} \text{Spec} \mathcal{O}_p\) is nonempty.
(ii) Assume \(p | \Delta\) and \(p \neq 2\). Suppose \(\Delta = cp\). Let \(\eta \in W(F_{p}^2)\) such that \(\eta \cdot \eta^\sigma = \epsilon^{-1}\).
Choose any \(\delta \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^\times\) with \(\delta^2 \in \mathbb{Z}_p^\times\). Define
\[
Z_p(T) = Z(T) \times_{\mathcal{O}_k} (W(\bar{\mathbb{F}}_p) \otimes \mathcal{O}_k),
\]
where \(W(\bar{\mathbb{F}}_p)\) is the set of Witt vectors of \(\mathbb{F}_p\). Then \(Z_p(T)\) is equidimensional of dimension \(\frac{1}{2}t(L_p)\) when it is nonempty. Here \(L_p\) is the rank \(n\) free \(\mathcal{O}_H\)-module whose Hermitian inner product is defined by the matrix \(-\epsilon^{-1}\delta^2 T\) and \(t(L_p)\) is defined in Theorem \ref{thm:finite}. In particular, this dimension only depends on \(p\) and \(T\).

The paper is organized as follows. In section \ref{section:definition} we recall the definition and properties of the Rapoport-Zink space \(N\) as studied in \cite{RTW}. In section \ref{section:local} we define our local version of special cycles on Rapoport-Zink spaces and state their
properties. In section 4 we prove these properties. In section 5 we recall some results from [KR2]. To be more precise, we recall the definition of the arithmetic model of unitary Shimura varieties and the definition of special cycles on it. We then decompose the special cycles according to some naturally constructed Hermitian form as in [KR2]. Finally in section 6 we state a uniformization theorem for the (global) special cycles and prove Theorem 1.2.

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2. Rapoport-Zink spaces

In this sections we recall some background information on the Rapoport-Zink spaces defined in [RTW]. We assume $p \neq 2$ is a rational prime. We denote by $H$ a ramified quadratic extension of $\mathbb{Q}_p$. We fix a uniformizer $\pi$ of $H$ such that $\pi_0 = \pi^2 \in \mathbb{Q}_p$ is a uniformizer. Denote by $\mathbb{F}$ an algebraic closure of $\mathbb{F}_p$, and by $W = W(\mathbb{F})$ its ring of Witt vectors and by $W_\mathbb{Q}$ its fraction field. Denote by $\sigma$ the Frobenius automorphism on $\mathbb{F}$, $W$ and $W_\mathbb{Q}$.

Let $\bar{H} = W_\mathbb{Q} \otimes_{\mathbb{Q}_p} E$ and $\mathcal{O}_{\bar{H}} = W \otimes_{\mathbb{Z}_p} \mathcal{O}_H$ be its ring of integers. Let $\sigma = \sigma \otimes id$ on $\bar{H}$. We denote by $\psi_0: H \to \bar{H}$ the natural embedding, and by $\psi_1 = \psi_0 \circ ^c\psi$ its conjugate.

Let Nilp be the category of $\mathcal{O}_{\bar{H}}$-schemes $S$ such that $\pi \cdot \mathcal{O}_S$ is a locally nilpotent ideal sheaf. For $S \in \text{Nilp}$, set $\bar{S} = S \times_{\text{Spec} \mathcal{O}_{\bar{H}}} \text{Spec} \mathbb{F}$.

Fix $(X, \iota_X)$ a supersingular $p$-divisible group of dimension $n$ and height $2n$ over $\mathbb{F}$ with an action $\iota_X: \mathcal{O}_H \to \text{End}(X)$. Let $\lambda_X$ be a principal quasi-polarization such that its Rosati involution induces on $\mathcal{O}_H$ the non-trivial automorphism over $\mathbb{Q}_p$. When $n$ is odd, such an object is unique up to isogeny. In particular, we denote by $\mathcal{Y}$ the unique one dimensional $p$-divisible group satisfying above. When $n$ is even, we have two such objects up to isogeny. This is related to the classification of Hermitian spaces over $H$, see remark 4.2 of [RTW].

For a $p$-divisible group $X$ we denote by $M(X)$ its Dieudonné module and $X^t$ its dual. Denote by $E = W_\mathbb{Q}[F, V]$ the rational Cartier ring.

Let $N$ be the rational Dieudonné module of $X$. Then $N$ has an action of $H$ and a skew-symmetric $W_\mathbb{Q}$-bilinear form $\langle , \rangle$ induced by $\lambda_X$ satisfying

\[ \langle Fx, y \rangle = \langle x, Vy \rangle^\sigma, \]
\[ \langle \iota(a)x, y \rangle = \langle x, \iota(\overline{a})y \rangle, a \in E. \]

We abuse notation and denote by $\pi$ the action of $\pi \in H$ on $N$.

Let $\mathcal{N}_{(n-r,r)}$ be the set-valued functor on Nilp which associates to $S \in \text{Nilp}$ the set of isomorphism classes of quadruples $(X, \iota, \lambda, \varrho)$. Here $X$ is a $p$-divisible group over $S$ and $\iota: \mathcal{O}_H \to \text{End}(X)$ satisfies the Kottwitz condition and the Pappas condition of signature $(n-r, r)$

\[ \text{char}(\iota(\alpha)|\text{Lie}X) = (T - \psi_0(\alpha))^{n-r} \cdot (T - \psi_1(\alpha))^r, \]
\[ \bigwedge_{r+1} (\iota(\pi) - \pi|\text{Lie}X) = 0, \bigwedge_{n-r+1} (\iota(\pi) + \pi|\text{Lie}X) = 0, \text{if } n \geq 3. \]

Further $\lambda : X \to X^{\vee}$ is a principal quasi-polarization whose associated Rosati involution induces on $\mathcal{O}_H$ the non-trivial automorphism over $\mathbb{Q}_p$. Finally, $\varrho : X \times_S$
\( S \to X \times_{\text{Spec} F} \tilde{S} \) is a \( \mathcal{O}_H \)-linear quasi-isogeny such that \( \lambda \) and \( \varrho^*(\lambda_X) \) differ locally on \( \tilde{S} \) by a factor in \( \mathbb{Q}_p^\times \). The following is Proposition 2.1 of [RTW].

**Proposition 2.1.** The functor \( \mathcal{N}_{(1,n-1)} \) is representable by a separated formal scheme \( \mathcal{N}_{(1,n-1)} \), locally of finite type over \( \text{Spf} \mathcal{O}_H \). Furthermore, \( \mathcal{N}_{(1,n-1)} \) is flat over \( \mathcal{O}_H \). It is formally smooth over \( \mathcal{O}_H \) in all points of the special fiber except those corresponding to a points \((X,\tau,\lambda,\varrho) \in \mathcal{N}_{(1,n-1)}(\mathbb{F})\), where \( \text{Lie}(\tau) = 0 \) (these form an isolated set of points).

In the first half of the paper we mainly deal with the formal scheme \( \mathcal{N}_{(1,n-1)} \). When there is no confusion of signature, we simply denote \( \mathcal{N}_{(1,n-1)} \) by \( \mathcal{N} \). Also denote by \( \mathcal{N}_0 \) the formal scheme \( \mathcal{N}_{(0,1)} \). Denote by \( \mathcal{N}_0 \) the closed and open formal subscheme of \( \mathcal{N} \) where the height of \( \varrho \) is zero. Let \( \mathcal{N}_0 \) be the reduction modulo \( \pi \) of \( \mathcal{N}_0 \).

Suppose \( \pi_0 = cp \) and let \( \eta \in W \) such that \( \eta \cdot \eta^\sigma = \epsilon^{-1} \). Define a \( \sigma \)-linear operator \( \tau = \eta \pi V^{-1} \) on \( N \). Set \( C = N^\pi \), we obtain a \( \mathbb{Q}_p \)-vector space with an isomorphism

\[
C \otimes_{\mathbb{Q}_p} \mathcal{W}_q \simeq N.
\]

Then for \( x, y \in C \)

\[
\langle x, y \rangle = \langle \tau x, \tau y \rangle = \langle \eta \pi V^{-1} x, \eta \pi V^{-1} y \rangle
\]

\[
= \eta^2 (-pe) \langle V^{-1} x, V^{-1} y \rangle
\]

\[
= \eta^2 (-pe)p^{-1} \langle x, y \rangle^\sigma
\]

\[
= -\frac{\eta}{\eta^2} \langle x, y \rangle^\sigma.
\]

Choose \( \delta \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^\times \) such that \( \delta^2 \in \mathbb{Z}_p^\times \). Then the restriction of the form \( \langle , \rangle^\delta = \eta^\delta \langle , \rangle \) to \( C \) defines a skew symmetric form with values in \( \mathbb{Q}_p \), and

\[
(x, y) = \langle \pi x, y \rangle^\delta + \langle x, y \rangle^\pi
\]

defines a Hermitian form on \( C \) with

\[
\langle x, y \rangle^\pi = \frac{1}{2} \text{Tr}_{E/\mathbb{Q}_p}(\pi^{-1}(x, y)).
\]

**Remark 2.1.** We want to describe the Dieudonné lattice of \( Y \) explicitly. As a \( \mathbb{Z}_p \) lattice, it is of rank 2. We can choose a basis \( \{e_1, e_2\} \) such that \( Fe_1 = e_2, Fe_2 = pe_1, Ve_1 = e_2, Ve_2 = pe_1 \) and \( \langle e_1, e_2 \rangle = \delta \). With respect to this basis, \( \text{End}_\mathbb{Z}(N) \) is of the form

\[
\begin{pmatrix}
  a & bp \\
  b^\sigma & a^\sigma
\end{pmatrix}, \quad a, b \in \mathbb{Q}_p^2.
\]

which is the quaternion algebra \( \mathbb{H}_p \) over \( \mathbb{Q}_p \). By changing basis using elements in \( \mathbb{H}_p \cap \text{SL}_2(\mathbb{Q}_p^2) \) we can assume \( F, V \) are of the same matrix form as before and

\[
\pi = \begin{pmatrix}
  0 & p^\sigma \\
  p^{-\sigma} & 0
\end{pmatrix}
\]

Thus \( \tau \) is the diagonal matrix diag\( \{ p^{-\sigma}, 1 \} \). As \( \tau \) is \( \sigma \)-linear, it fixes the \( \mathbb{Q}_p \)-vector space span\( _{\mathbb{Q}_p} \{pe_1, e_2\} \). Let \( l_0 = pe_1, l_1 = e_2 \). Then \( (1_0, 1_0) = -\epsilon^{-1} \delta^2 \). As \( \mathcal{O}_H \) is a DVR and \( N \) is a one dimensional \( E \)-space, there is a unique self-dual \( \mathcal{O}_H \)-lattice w.r.t. \( (, \) spanned by \( \{l_0\} \). The \( \pi \) action shown above and the principal polarization corresponds to the form \( \langle , \rangle \) give an object \( (\mathbb{V}_\pi, \iota_\mathbb{V}, \lambda_\mathbb{V}) \). This is the unique closed point of both \( \mathcal{N}_0^{(0,1)} \) and \( \mathcal{N}_0^{(1,0)} \). In fact, one can check directly that
(Y, ν, λ) satisfies the signature conditions for signature (1, 0) and (0, 1) at the same time. By Theorem 6.1 of [RTW], both $\mathcal{N}^0_{(0,1)}$ and $\mathcal{N}^0_{(1,0)}$ have only one closed point.

For two $O_H$-lattices $\Lambda$, $\Lambda'$, we use the notation $\Lambda \subset \Lambda'$ to stand for the situation when $p\Lambda' \subset \Lambda \subset \Lambda'$ and $\dim(\Lambda'/\Lambda) = 1$.

The following is Proposition 2.2 of [RTW].

**Corollary 2.1.** There is a bijection between $\tilde{\mathcal{N}}^0(\mathbb{F})$ and the set of $O_H$-lattices $\mathcal{V}(\mathbb{F}) = \{M \subseteq C \otimes_{E} \tilde{H} \mid M^2 = M, \pi \tau(M) \subseteq M \subseteq \pi^{-1} \tau(M), M \subseteq \leq_1 (M + \tau(M))\}$, where $M^2 = \{x \in N| (x, M) \subseteq O_H\} = M' = \{x \in N| x, M \geq \subseteq W\}$.

A vertex lattice in $C$ is an $O_H$-lattice $\Lambda$ such that $\pi \Lambda \subseteq \Lambda \subseteq \Lambda$. We denote the dimension of the $\mathbb{F}_p$-vector space $\Lambda/\Lambda^2$ by $t(\Lambda)$, and call it the type of $\Lambda$. It is an even integer (see Lemma 3.2 of [RTW]). The following is Proposition 4.1 of [RTW].

**Proposition 2.2.** Let $M \in \mathcal{V}(\mathbb{F})$ be an $O_H$-lattice in $C \otimes_{E} \tilde{H}$ which corresponds to a point of $\tilde{\mathcal{N}}^0(\mathbb{F})$ under the bijection of Proposition 2.1. There exist an integer $d \leq \frac{n}{2}$ such that there is a chain of inclusions of $O_H$-lattices

$$M \subseteq T_1(M) \subseteq \ldots \subseteq T_d(M) = T_{d+1}(M) = \ldots$$

The intersection $\Lambda = T_d(M) \cap C$ is a vertex lattice of type $2d$. It is the minimal lattice $\Lambda$ in $C$ such that $\Lambda \otimes_{O_H} O_H$ contains $M$. Dually, $\Lambda^d$ is the maximal lattice in $C$ such that $\Lambda \otimes_{O_H} O_H$ is contained in $M$.

Set

$$\mathcal{V}(\Lambda) = \{M \in \mathcal{V}(\mathbb{F}) \mid M \subseteq \Lambda \otimes_{O_H} O_H\}.$$ 

By Proposition 2.2,

$$\mathcal{V}(\mathbb{F}) = \bigcup_{\Lambda} \mathcal{V}(\Lambda).$$

We summarize the information we need from Theorem 6.10 of [RTW] as the following theorem.

**Theorem 2.2.** (i) For two vertex lattices $\Lambda_1$ and $\Lambda_2$

$$\mathcal{V}(\Lambda_1) \subseteq \mathcal{V}(\Lambda_2) \Leftrightarrow \Lambda_1 \subseteq \Lambda_2.$$ 

If $\Lambda_1 \cap \Lambda_2$ is a vertex lattice, then

$$\mathcal{V}(\Lambda_1 \cap \Lambda_2) = \mathcal{V}(\Lambda_1) \cap \mathcal{V}(\Lambda_2),$$

otherwise $\mathcal{V}(\Lambda_1) \cap \mathcal{V}(\Lambda_2) = \emptyset$.

(ii) For each vertex lattice $\Lambda$, $\mathcal{V}(\Lambda)$ is the set of $\mathbb{F}$ points of a variety $\mathcal{N}^0_{\Lambda}$. It is a Deligne Lusztig variety attached to a symplectic group of size $t(\Lambda)$ over $\mathbb{F}_p$ and a standard Coxeter element. The closure of any $\mathcal{N}^0_{\Lambda}$ in $\tilde{\mathcal{N}}^0$ is given by

$$\overline{\mathcal{N}^0_{\Lambda}} = \bigcup_{\Lambda' \subseteq \Lambda} \mathcal{N}^0_{\Lambda'} =: \mathcal{N}_{\Lambda},$$

which is a projective variety of dimension $t(\Lambda)/2$. The inclusion of points $\mathcal{V}(\Lambda_1) \subseteq \mathcal{V}(\Lambda_2)$ in (i) is induced by a locally closed embedding $\mathcal{N}^0_{\Lambda_1} \to \mathcal{N}^0_{\Lambda_2}$.
3. Special cycles on Rapoport-Zink spaces

In this section, we define special cycles on $N_0 \times N_0$. We then state our main results on the support of these cycles. First we need some background information on Hermitian lattices.

3.1. Hermitian lattices and Jordan splitting. We use notations from [J]. We use $\oplus$ to denote direct sum of mutually orthogonal spaces. For example, we use $(\alpha_1) \oplus \cdots \oplus (\alpha_n)$ to denote the $n$-dimensional $H$-space (or $\mathcal{O}_H$ lattice depending on the context) with a Hermitian form given by a diagonal matrix $\text{diag}\{\alpha_1, \ldots, \alpha_n\}$ with respect to an orthogonal basis. We also use $H(i)$ to denote the hyperbolic plane which is lattice of rank 2 with Hermitian form given by the matrix

$$
\begin{pmatrix}
0 & \pi^i \\
-\pi^i & 0
\end{pmatrix}
$$

with respect to a certain basis. We say a Hermitian lattice is split if it is an orthogonal sum of hyperbolic planes and non-split otherwise.

For a Hermitian lattice $L$ define $s_L$ to be $\text{ord}_{\pi}(\{x \cdot y \mid x, y \in L\})$. We say $x \in L$ is maximal if $x$ is not in $\pi L$. We say $L$ is $\pi^i$-modular if $x \cdot L = \pi^i \mathcal{O}_H$ for every maximal vector $x$ in $L$. We denote by $d_L$ the determinant of $L$ which is well defined up to $Nm(\mathcal{O}_H)$. Any Hermitian lattice $L$ has a Jordan splitting (3.1)

$$
L = \bigoplus_{1 \leq \lambda \leq t} L_{\lambda}
$$

where $L_{\lambda}$ is $\pi^{s_L\lambda}$-modular and $s_L t < \cdots < s_L 2 < s_L 1$. The splitting is canonical in the sense that, for any two Jordan splittings

$$
L = \bigoplus_{1 \leq \lambda \leq t} L_{\lambda}
$$

and

$$
L = \bigoplus_{1 \leq \lambda \leq T} K_{\lambda}
$$

of $L$, we have $t = T$, and for each $\lambda$, we have $s_L \lambda = s K_{\lambda}$, $\text{rank}(L_{\lambda}) = \text{rank}(K_{\lambda})$ and $d_L \lambda = d K_{\lambda}$ (see Theorem 8.2 of [J]). In particular, the following theorem (which is Proposition 8.1 of [J]) implies that $L_{\lambda}$ is isomorphic to $K_{\lambda}$ for each $\lambda$.

**Theorem 3.1.** Let $L$ be a $\pi^i$-modular lattice of rank $n$. Then

1. $L \simeq (\pi_0^i) \oplus (\pi_0^i) \oplus \cdots \oplus (\pi_0^{(n-1)i}dL)$ if $i$ is even.

2. $L \simeq H(i) \oplus H(i) \oplus \cdots \oplus H(i)$ if $i$ is odd.

In particular, no odd-rank lattice can be odd-modular.

We also need the following lemma.

**Lemma 3.2.** Assume $L'$ is a sub $\mathcal{O}_H$-module of $L$ such that $L'$ is $\pi^0$-modular and $sL = 0$. Then $L'$ is a direct summand of $L$. 
Proof. Since we assume $p \neq 2$, by Proposition 4.4 (c) of [J], $L'$ is diagonalizable with orthogonal basis $\{x_1, \ldots, x_m\}$ with $m \leq n$. Since $L'$ is $\pi^0$-modular, $(x_i, x_i) \in \mathbb{Z}_p^\vee$ for $1 \leq i \leq m$. For any $x \in L$,

$$x = \sum_{i=1}^m a^i x_i + x'$$

where $a^i = -\frac{(x-x_i)}{(x_i, x_i)}$. By the assumption that $sL = 0$, we have $(x, x_i) \in \mathcal{O}_H$. Then the fact $(x_i, x_i) \in \mathbb{Z}_p^\vee$ implies that $a^i \in \mathcal{O}_H$. So $\sum_{i=1}^m a^i x_i \in L'$ and $x' \in L \cap (L')^\perp$. This finishes the proof. 

3.2. Special cycles. Recall that we fix a supersingular $p$-divisible group $(X, \iota_X)$ which is of dimension $n$ and height $2n$ over $\mathbb{F}$ with an action $\iota_X : \mathcal{O}_H \to \text{End}(X)$. Let $\lambda_X$ be a principal quasi-polarization such that its Rosati involution induces on $\mathcal{O}_H$ the non-trivial automorphism over $\mathbb{Q}_p$. When $n$ is odd, such an object is unique up to isogeny. In particular, we denote by $\mathcal{Y}$ the unique one dimensional $p$-divisible group satisfying above. When $n$ is even, we have two such objects up to isogeny. Define the space of special homomorphisms to be the $H$-vector space

$$\mathbb{V} = \text{Hom}_{\mathcal{O}_H}(\mathcal{Y}, X) \otimes \mathbb{Q}$$

For $x, y \in \mathbb{V}$, let

$$h(x, y) = \lambda_X \circ \iota_X \circ \lambda_X \circ x \in \text{End}_{\mathcal{O}_H}(\mathcal{Y}) \otimes \mathbb{Q} \xrightarrow{c^{-1}} H.$$ 

This form is $\mathcal{O}_H$-valued on the lattice $\mathcal{L} = \text{Hom}_{\mathcal{O}_H}(\mathbb{V}, \mathbb{X})$.

For $x, y \in \mathbb{V}$ we abuse notation and denote the induced map between the corresponding Dieudonné modules still by $x, y$. Then

**Lemma 3.3.**

$$h(x, y)(1_0, 1_0)_{\mathbb{V}} = (x(1_0), y(1_0))_{\mathbb{X}}$$

where $(,)_X, (,)_V$ are defined as in equation (2.1) for the rational Dieudonné module of $\mathbb{X}$ and $\mathbb{Y}$ respectively.

**Proof.** We have to use the fact that $\lambda_{\mathbb{Y}}^{-1} \circ \iota_X \circ \lambda_X$ agrees with $y^*$ which is the adjoint operator of $y$ on $\text{Hom}_{\mathbb{Q}}(M(\mathbb{Y}), M(\mathbb{X}))$ w.r.t. $<,>_X$ and $<,>_\mathbb{Y}$. This is because we define $<,>_X$ by $e <, >, >$ where $e <, >$ is the pairing between $M(\mathbb{X})$ and $M(\mathbb{X})'$.

$$(x(1_0), y(1_0))_X = \langle \pi x(1_0), y(1_0) \rangle = \langle y^* x(1_0), 1_0 >_\mathbb{Y} + \langle y^* x(1_0), 1_0 >_\mathbb{Y} \pi$$

$$= \langle y^* x(1_0), 1_0 >_\mathbb{Y} + \langle y^* x(1_0), 1_0 >_\mathbb{Y} \pi$$

$$= \langle \pi y^* x(1_0), 1_0 >_\mathbb{Y} + \langle y^* x(1_0), 1_0 >_\mathbb{Y} \pi$$

$$= h(x, y)(1_0, 1_0)_{\mathbb{Y}}$$

$\square$

**Corollary 3.1.** Under the Hermitian form $h(,)$, the lattice $\mathcal{L} = \text{Hom}_{\mathcal{O}_H}(\mathcal{Y}, X)$ is self dual, i.e. $\mathcal{L} = \mathcal{L}^\perp$.

**Proof.** This is an easy consequence of remark 4.2 of [RTW]. When $n$ is odd, there is a unique object $(X, \iota, \lambda_X)$ that satisfies the description at the beginning of this subsection. We simply take $X = \mathcal{E}^n$ where $\mathcal{E}$ is the $p$-divisible group of a super singular elliptic curve over $\mathbb{F}$. $\mathcal{E}$ has an action of $\mathcal{O}_H$ and a natural principal
polarization which identify $\mathcal{E}$ with its dual. We take the action of $\mathcal{O}_H$ on $X$ to be the diagonal action and let $\lambda_X$ to be identity map (product of the natural principal polarization of $\mathcal{E}$).

When $n$ is even, we also let $X^+ = \mathcal{E}^n$ with its diagonal action by $\mathcal{O}_H$. We then define its polarization $\lambda_{X^+}$ to be given by the anti-diagonal matrix $M$ with 1's on the anti-diagonal. Let $X^- = \mathcal{E}^n$ be defined in the same way as $X^+$ except that we the polarization $\lambda_{X^-}$ is defined by the diagonal matrix diag(1, ..., 1, $u_1$, $u_2$) where $u_1, u_2 \in \mathbb{Z}^n_\mathbb{C}$ and $-u_1 u_2 \notin \text{Nm}_E/\mathbb{Q}_p(E^\times)$. $(X^+, \iota, \lambda_{X^+})$ and $(X^-, \iota, \lambda_{X^-})$ are the only two candidates that satisfies the description at the beginning of this subsection.

In all cases take $x_i$ to be the $i$-th diagonal embedding of $Y = \mathcal{E}$ into $X$. Then \{ $x_1, \ldots, x_n$ \} spans $\mathbb{L}$ since $X = \mathcal{E}^n$. By Lemma 3.3, the Hermitian form $h(\cdot, \cdot)$ is then represented by the matrix $I_n$, $M$ and diag($1, \ldots, 1, u_1, u_2$) respectively in the three cases. These matrices are all in $\text{GL}_n(\mathcal{O}_H)$, hence $\mathbb{L}$ is self dual. □

**Definition 3.4.** For a fixed $m$-tuple $x = [x_1, \ldots, x_m]$ of special homomorphisms $x_i \in V$, the associated special cycle $Z(x)$ is the subfunctor of collections $\xi = (Y, \iota, \lambda_Y, g_Y, X, \iota, \lambda_X, g_X)$ in $N^0(1, n-1)$ such that the quasi-homomorphism

\[
\varrho_X^{-1} \circ \mathfrak{g} \circ \varrho_Y : Y^m \times S_S \to X \times S_S
\]

extends to a homomorphism from $Y^m \to X$.

**Remark 3.5.** There is a natural isomorphism from $N(1, n-1)$ to $N(n-1, 1)$ by first changing the marking object $(X, \iota, \lambda_X)$ to $(X, \iota, \lambda_X)$ and then mapping $(X, \iota_X, \lambda, g) \in N(1, n-1)$ to $(\tilde{X}, \iota_X, \lambda, g) \in N(n-1, 1)$. The map

\[
N(1, n-1) \times_{\text{Spec} \mathcal{O}_R} N(1, n-1) \to N(1, n-1) \times_{\text{Spec} \mathcal{O}_R} N(n-1, 1)
\]

then induces natural identifications of special cycles on $N(1, n-1)$ to special cycles on $N(n-1, 1)$.

In this section we specialize to the case $x \in V^n$. Define

\[
L = \text{span}_{\mathcal{O}_m} [x_1(1_0), \ldots, x_n(1_0)]\]

(see remark 2 for definition of $1_0$). Define

\[
\mathcal{W}(L) = \{ M \in V | L \subseteq M \}.
\]

**Lemma 3.6.** The image of the projection of the special cycle $Z(x)$ on to $\mathcal{N}(\mathbb{F}) \simeq V(\mathbb{F})$ is $\mathcal{W}(L)$ and

\[
\mathcal{W}(L) = \bigcup_{\Lambda, L \subseteq \Lambda} \mathcal{V}(\Lambda)
\]

where $\Lambda$ is a vertex lattice.

**Proof.** The first claim follows from Dieudonné theory. Let $M \in V(\mathbb{F})$ and suppose $\Lambda$ is the smallest $\tau$-invariant $\mathcal{O}_H$ lattice containing $M$. Then $L \subseteq M \Rightarrow M \subseteq L^\tau$, as $M = M^2$

\[
\Leftrightarrow \Lambda \subseteq L^\tau, \ \text{as} \ L^\tau \text{is} \tau\text{-invariant and} \ \Lambda \text{is the smallest} \tau\text{-invariant lattice containing} \ M
\]

\[
\Leftrightarrow L \subseteq \Lambda^\tau.
\]

□

**Definition 3.7.** We use $\text{Vert}(L)$ to denote the set of vertex lattices $\Lambda$ such that $L \subseteq \Lambda^\tau$. 
Define
\[ T = h(x, x) = (h(x_i, x_j)) \in \text{Herm}_n(H) \]

**Lemma 3.8.** If \( \mathcal{Z}(\mathbb{F}) \) is non-empty, then \( T \in \text{Herm}(\mathcal{O}_H) \).

**Proof.** By the previous lemma, suppose \( M \in \mathcal{V}(\mathbb{F}) \) such that \( L \subseteq M \). By Lemma 3.3,
\[ h(x, x) = (h(x_i, x_j)) = h((1_0, 1_0)_Y) \in \text{Herm}(\mathcal{O}_H) \]

\( (x(1_0), x(1_0))_X \in (M, M)_Y \subseteq \mathcal{O}_H \) as \( M = M^l \), \( (1_0, 1_0)_Y \in \mathbb{Z}_p^\times \) by construction. \( \square \)

Take a Jordan splitting of \( L \) as in 3.1
\[ L = \bigoplus_{1 \leq \lambda \leq t} L_\lambda. \]

By the previous lemma and Lemma 3.3, we have \( 0 \leq sL_1 < sL_2 < \ldots < sL_t \). For simplicity we change the index of the above splitting to
\[ L = \bigoplus_{0 \leq \lambda \leq k} L_\lambda \]

where \( \lambda = sL_\lambda \). We define
\[ L_{\geq t} = \bigoplus_{t \leq \lambda \leq k} L_\lambda. \]

and
\[ m(L) = \text{rank}_{\mathcal{O}_H}(L_{\geq t}). \]

(3.4)

\[ n_{\text{odd}} = \sum_{\lambda \geq 3, \lambda \text{ is odd}} \text{rank}_{\mathcal{O}_H}(L_\lambda) \]

(3.5)

\[ n_{\text{even}} = \sum_{\lambda \geq 2, \lambda \text{ is even}} \text{rank}_{\mathcal{O}_H}(L_\lambda). \]

(3.6)

The following theorem is the analog of Theorem 4.2 of [KR2]

**Theorem 3.9.**

1. If \( T \notin \text{Herm}_n(\mathcal{O}_H) \), then \( \mathcal{W}(L) \) is empty.
2. If \( T \in \text{Herm}_n(\mathcal{O}_H) \). Then
\[ \mathcal{W}(L) = \bigcup_{\{\Lambda \in \text{Vert}(L)|t(\Lambda) = t(L)\}} \mathcal{V}(\Lambda) \]

where
\[ t(L) = \begin{cases} 
  m(L) - 1 & \text{if } m(L) \text{ is odd} \\
  m(L) & \text{if } m(L) \text{ is even and } L_{\geq 1} \text{ is split} \\
  m(L) - 2 & \text{if } m(L) \text{ is even and } L_{\geq 1} \text{ is non-split,} 
\end{cases} \]

and \( m(L) = \text{rank}_{\mathcal{O}_H}(L_{\geq 1}) \).

**Corollary 3.2.** If it is non-empty, \( \mathcal{W}(L) \) is the set of \( \mathbb{F} \)-points of a variety of pure dimension \( \frac{1}{2}t(L) \).

**Proof.** By Proposition 5.5 and 6.7 of [RTW], \( \mathcal{V}(\Lambda) \) is (the set of \( \mathbb{F} \)-points of) irreducible of dimension \( \frac{1}{2}t(\Lambda) \). The corollary now follows from Theorem 3.9. \( \square \)
The following theorem is the analog of Theorem 4.5 of [KR2]

**Theorem 3.10.** \( W(L) = \mathcal{V}(\Lambda) \) for a unique vertex lattice \( \Lambda \) if and only if the following two conditions are satisfied

1. \( n_{\text{odd}} = 0 \)
2. \( n_{\text{even}} \leq 1 \) or \( n_{\text{even}} = 2 \) and \( L \geq 2 \) is non-split.

**Corollary 3.3.** \( W(L) \) is the set of \( F \)-points of an irreducible variety if and only if the assumptions (1) and (2) in the previous theorem are satisfied.

**Proof.** By Proposition 5.5 and 6.7 of [RTW], \( \mathcal{V}(\Lambda) \) is (the set of \( F \)-points of) an irreducible variety of dimension \( \frac{1}{2} t(\Lambda) \). Conversely, \( \mathcal{V}(\Lambda), \mathcal{V}(\Lambda') \) intersect in \( \mathcal{V}(\Lambda \cap \Lambda') \). Hence we see that \( W(L) \) is irreducible if and only if \( W(L) = \mathcal{V}(\Lambda) \) for a unique vertex lattice \( \Lambda \). The corollary follows from Theorem 3.10. \( \square \)

**Corollary 3.4.** The underlying variety of \( W(L) \) is zero dimensional if and only if the following conditions are satisfied

1. \( n_{\text{odd}} = \text{rank}_{O_H}(L_1) = 0 \).
2. \( n_{\text{even}} \leq 1 \) or \( n_{\text{even}} = 2 \) and \( L \geq 2 \) is non-split.

If this is the case, then \( W(L) \) is a single point.

**Proof.** Notice that \( L_\lambda \) must be even dimensional for any odd \( \lambda \) by Theorem 3.1. The first part of the corollary then follows from Theorem 3.9. If this is the case, then \( W(L) \) is a single point by 3.10. \( \square \)

We will prove the above two theorems in Section 4.

### 4. Proof of Theorem 3.9 and Theorem 3.10

#### 4.1. Proof of Theorem 3.9

Notations are as in subsection 3.2. Fix a Jordan splitting of \( L = \text{span}_{O_H} [x_1(l_0), \ldots, x_n(l_0)] \) as in 3.3 and assume \( L \subseteq \Lambda^2 \) (see Lemma 3.6). Then

\[
L = L_0 \oplus L_{\geq 1}, \quad L^\sharp = L_0 \oplus (L_{\geq 1})^\sharp
\]

where \( L_{\geq 1}^\sharp \) is the integral dual of \( L_{\geq 1} \) in \( L_{\geq 1} \otimes \mathbb{Q} \). For any \( \Lambda \in \text{Vert}(L) \), we have

\[
L \subseteq \Lambda^\sharp \subseteq \Lambda^\sharp \subseteq L^\sharp.
\]

Assume \( L_0 \neq \{0\} \) then \( s\Lambda^\sharp = sL = 0 \). By lemma 3.2 we can assume

\[
\Lambda^\sharp = L_0 \oplus \Lambda'.
\]

Hence \( \Lambda = L_0 \oplus (\Lambda')^\sharp \) and we have the sequence

\[
L_{\geq 1} \subseteq \Lambda' \subseteq (\Lambda')^\sharp \subseteq (L_{\geq 1})^\sharp.
\]

As all the statements in Theorem 3.9 and Theorem 3.10 only involve \( L_{\geq 1} \), we can without loss of generality assume \( L_0 = 0 \). Let

\[
m = \text{rank}_{O_H}(L_{\geq 1}).
\]

The fact that \( t(L) \) can be no bigger than the bounds stated in Theorem 3.9 is a restatement of Lemma 3.3 of [RTW], our goal is to prove that it can achieve that number. To be more precise, we prove that if \( \Lambda \in \text{Vert}(L) \) and \( t(\Lambda) < t(L) \), then there is a \( \Lambda' \in \text{Vert}(L) \) such that \( \Lambda \subset \Lambda' \) (hence \( \mathcal{V}(\Lambda) \subset \mathcal{V}(\Lambda') \)) and \( t(\Lambda') = t(L) \).

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Since $O_H$ is a P.I.D, every torsion free module $M$ of $O_H$ is isomorphic to $O_H^r$. In particular
\[ \dim_{F_p}(M/\pi M) = \text{rank}_{O_H} M \]
Now suppose $t(\Lambda/\Lambda^t) = \dim_{F_p}(\Lambda/\Lambda^t) = t$, then $\pi \Lambda \subset \Lambda^t \subset \Lambda$. Also suppose
\[ r := \dim_{F_p}(\frac{1}{\pi} \Lambda^t \cap L^t/\Lambda), \]
Taking dual of the left hand side of the above equation we see that
\[ \dim_{F_p}(\Lambda^t/(\frac{1}{\pi} \Lambda^t \cap L^t)) = \dim_{F_p}(\Lambda^t/(\pi \Lambda + L)) \]. We have the following chain
\[ \pi \Lambda + L \subset \Lambda^t \subset \Lambda \subset \frac{1}{\pi} \Lambda^t \cap L^t. \]
Our assumption $L = L_{\geq 1}$ implies that $L \subset \pi L^t$. This together with the fact that $L \subset \Lambda^t$ implies that
\[ (4.1) \quad \pi \Lambda + L \subseteq \pi(\frac{1}{\pi} \Lambda^t \cap L^t) = \Lambda^t \cap \pi L^t. \]
This shows that
\[ \dim_{F_p}(\frac{1}{\pi} \Lambda^t \cap L^t/(\pi \Lambda + L)) \geq \dim_{F_p}(\frac{1}{\pi} \Lambda^t \cap L^t/\pi(\frac{1}{\pi} \Lambda^t \cap L^t)) = m \]
(Notice that the first quotient in the above inequality is a $F_p$ vector space). We have proved that
\[ (4.2) \quad 2r + t \geq m. \]
Define a symmetric form $S(,) : \mathbb{F}_p$ valued on the $\mathbb{F}_p$-vector space $\frac{1}{\pi} \Lambda^t/\Lambda$. The following is the analogue of Lemma 4.12 of [KR2].

**Lemma 4.1.** Suppose $\Lambda$ is a vertex lattice in $\text{Vert}(L)$ such that $\dim_{F_p}(\frac{1}{\pi} \Lambda^t \cap L^t/\Lambda) \geq 3$, then there exists a lattice $\Lambda' \in \text{Vert}(L)$ with $\Lambda \subset \Lambda'$ and $t(\Lambda') > t(\Lambda)$.

**Proof.** Recall that every quadratic form on a $\mathbb{F}_p$-vector space with dimension bigger or equal to three has an isotropic line (see for example 62:1 b of [O], Chapter VI ). Take an isotropic line $\ell$ in $\frac{1}{\pi} \Lambda^t \cap L^t/\Lambda$. Let $\Lambda' =: \text{pr}^{-1}(\ell)$ where pr is the natural projection $\frac{1}{\pi} \Lambda^t \to \frac{1}{\pi} \Lambda^t/\Lambda$. Then $L \subseteq \Lambda \subseteq \Lambda'$ and
\[ < \pi \Lambda', \Lambda' > = < \pi \ell + \pi \Lambda, \ell + \Lambda > = < \pi \Lambda, \ell > + < \pi \Lambda, \Lambda > + < \pi \ell, \ell > + < \pi \ell, \Lambda > = < \Lambda, \pi \ell > + < \pi \Lambda, \Lambda > + < \pi \ell, \ell > + < \pi \ell, \Lambda > . \]
We claim that every term in the last line is in $\mathbb{Z}_p$. Since $\ell \in \frac{1}{\pi} \Lambda^t$, $\Lambda', \pi \ell \subseteq \mathbb{Z}_p$. Since $\pi \Lambda \subseteq \Lambda^t$, $\pi \Lambda, \Lambda \subseteq \mathbb{Z}_p$. Since $\ell$ is isotropic, $p < \pi \ell, \ell > \subseteq p\mathbb{Z}_p$. Hence $< \pi \ell, \ell > \subseteq \mathbb{Z}_p$.
So $\pi \Lambda' \subseteq (\Lambda')^t \subseteq \Lambda'$, i.e. $\Lambda'$ is a vertex lattice. By definition, $\Lambda' \in \text{Vert}(L)$ and $t(\Lambda') = t(\Lambda) + 2$. 

\[ \square \]
By induction using the above lemma and the fact that $\mathcal{V}(\Lambda) \subset \mathcal{V}(\Lambda')$ if $\Lambda \subset \Lambda'$ (Proposition 4.3 of [RTW]), we reduce to the case when $r = \dim_{\mathbb{C}}(\frac{1}{\pi} \Lambda^2 \cap L^2) \leq 2$. Also keep in mind that equation (4.2) holds. There are (at most) four cases when $t(\Lambda)$ satisfies (4.2) but $t(\Lambda)$ is smaller than the claimed $t(L)$ in Theorem 3.9:

1. $m$ is even, $t(\Lambda) = m - 2, r = 2$.
2. $m$ is even, $t(\Lambda) = m - 2, r = 1$.
3. $m$ is even, $t(\Lambda) = m - 4, r = 2$.
4. $m$ is odd, $t(\Lambda) = m - 3, r = 2$.

We deal with them one by one.

**case (1):**

We have

$$\Lambda^2 \subset \Lambda \subset \frac{1}{\pi} \Lambda^2 \cap L^2.$$

But $\Lambda^2 \subset \frac{1}{\pi} \Lambda^2$, so actually $\frac{1}{\pi} \Lambda^2 \subseteq L^2$. Choose a Jordan splitting of $\Lambda$

$$\Lambda = \Lambda_0 \oplus \Lambda_{-1}.$$

Then we know $\text{rank}(\Lambda_0) = 2$, and $\Lambda_0^2 = \Lambda_0$ (w.r.t to the sub-space inner product), $\text{rank}(\Lambda_{-1}) = m - 2, \pi \Lambda_{-1} = \Lambda_{-1}^2$. By Theorem 3.1, $\Lambda_{-1}$ is split. If $\Lambda_0$ is split, then there exist $e_1, e_2 \in \Lambda_0$ such that $(e_1, e_1) = (e_2, e_2) = 0, (e_1, e_2) = 1$. Define

$$\Lambda' = \Lambda_{-1} \oplus [\pi e_1, \pi^{-1} e_2].$$

By the fact that $\frac{1}{\pi} \Lambda^2 \subseteq L^2$, we know that $\Lambda' \subseteq L^2$. Also

$$(\Lambda')^2 = \Lambda_{-1}^2 \oplus [\pi e_1, \pi^{-1} e_2].$$

So $t(\Lambda') = m$. So $t(\Lambda') = t(L)$ as stated in Theorem 3.9. If $\Lambda_0$ is nonsplit, then $t(\Lambda) = m - 2$ already obtains the number $t(L)$ as stated in Theorem 3.9.

**case (2):**

We have

$$\pi \Lambda + L \subset \frac{1}{\pi} \Lambda^2 \subset \Lambda \subset \frac{1}{\pi} \Lambda^2 \cap L^2.$$

We have already seen in equation (4.2) that

$$\pi \Lambda + L \subseteq \pi(\frac{1}{\pi} \Lambda^2 \cap L^2) = \Lambda^2 \cap \pi L^2$$

and $\pi(\frac{1}{\pi} \Lambda^2 \cap L^2) \subset \frac{1}{\pi} \Lambda^2 \cap L^2$. These together imply that in fact $\pi \Lambda + L = \Lambda^2 \cap \pi L^2$.

But

$$\pi \Lambda + L = (\frac{1}{\pi} \Lambda^2 \cap L^2)^2.$$ So define $\Lambda' = \frac{1}{\pi} \Lambda^2 \cap L^2$, we have $t(\Lambda') = m$. So $t(\Lambda') = t(L)$ as stated in Theorem 3.9.

**case (3):** Similar to case (2).

**case (4):**

We have

$$\Lambda^2 \subset \Lambda \subset \frac{1}{\pi} \Lambda^2 \cap L^2 \subset \frac{1}{\pi} \Lambda^2.$$

Choose a Jordan splitting of $\Lambda$

$$\Lambda = \Lambda_0 \oplus \Lambda_{-1}.$$
Then we know rank$(\Lambda_0) = 3$, $\Lambda_0^\perp = \Lambda_0$, rank$(\Lambda_{-1}) = m - 3$, $\pi \Lambda_{-1} = \Lambda^2_{-1}$. By assumption there is a basis $\{e_1, e_2, e_3\}$ of $\Lambda_0$ such that

$$\frac{1}{\pi} e_1, \frac{1}{\pi} e_2 \in \frac{1}{\pi} \Lambda^\perp \cap L^\perp, \frac{1}{\pi} e_3 \notin L^\perp.$$ 

By changing $\{e_1, e_2\}$ by an $O_H$ linear combination of them, we can assume $(e_1, e_i) = u_i (i = 1, 2)$ for $u_i \in \mathbb{Z}_p^\times$ and $(e_1, e_2) = 0$. As both $e_1$ and $e_2$ are maximal modular in $\Lambda_0$, by modifying $e_3$ using linear combinations of $e_1, e_2$ we can in fact assume that under the basis $\{e_1, e_2, e_3\}$, the form $(,)_|_{\Lambda_0}$ is represented by the diagonal matrix diag$\{u_1, u_2, u_3\}$ with $u_1, u_2, u_3 \in \mathbb{Z}_p^\times$. But this means that $e_3 \in L^\perp \cap L$ but $\frac{1}{\pi} e_3 \notin L^\perp$, which contradicts our assumption that $\frac{1}{\pi} L \subseteq L^\perp$. In conclusion, case (4) is not possible under our assumptions.

This finishes the proof of Theorem 3.9.

4.2. Proof of Theorem 3.10. Again we use notations from section 3.2. Recall that in Theorem 3.9 we define a number $t(L)$ depending on $L$ such that $t(\Lambda) \leq t(L)$ for all $\Lambda \in \text{Vert}(L)$ and

$$W(L) = \bigcup_{\{\Lambda \in \text{Vert}(L) | t(\Lambda) = t(L)\}} V(\Lambda).$$

Hence $W(L) = V(\Lambda)$ is true if and only if $\Lambda$ is the unique lattice in $\text{Vert}(L)$ with $t(\Lambda) = t(L)$.

Lemma 4.2. Assume that one of the following conditions holds,

1. $n_{\text{even}} \geq 3$ or $n_{\text{even}} = 2$ with $L_{\geq 2}$ split
2. $n_{\text{odd}} \geq 2$.

Then there is more than one $\Lambda$ in $\text{Vert}(L)$ such that $t(\Lambda) = t(L)$.

Proof. Fix a Jordan splitting of $L$ as in equation (3.8):

$$L = L = \bigoplus_{0 \leq \lambda \leq k} L_{\lambda}$$

and assume each $L_{\lambda}$ has the form as in Theorem 3.1.

Suppose assumption (1) of the lemma holds. If $n_{\text{even}} \geq 3$, Proposition 63:19 of [O] shows that every quadratic space over a local field with dimension greater or equal to 5 is isotropic. We apply this to the trace form of $(,)|_{L_{\geq 2}}$ and conclude (by a proper scaling) that there is a maximal element in $L_{\geq 2}$ that has length zero. Hence there is a $H(i), i \geq 2$ which is a direct summand of $L_{\geq 2}$. Apparently we can conclude the same if $L_{\geq 2}$ is split. We scale the even modular part of $L$ that is perpendicular to the given $H(i)$ to be $\pi^0$-modular, and scale the odd modular part to be $\pi^1$-modular. We also scale the given $H(i)$ to $H(2)$. Denote the resulting lattice $L'$ so we have $L \subset L'$ and

$$L' \simeq L'_{0} \oplus L'_{1} \oplus H(2).$$

Suppose assumption (2) of the lemma holds. Theorem 3.1 implies that there is a $H(i), i \geq 3$ which is a direct summand of $L_{\geq 3}$. We scale the even modular part of $L$ to be $\pi^0$-modular, and scale the odd modular part that is perpendicular to the given $H(i)$ to be $\pi^1$-modular. We also scale the given $H(i)$ to $H(3)$. Denote the resulting lattice $L'$ so we have $L \subset L'$ (hence $\text{Vert}(L') \subset \text{Vert}(L)$). Then

$$L' \simeq L'_{0} \oplus L'_{1} \oplus H(3).$$
In any case, we can assume there is a lattice $L'$ such that $L \subseteq L'$ and

$$L' \simeq L_0' \oplus L_1' \oplus H(a)$$

where $a = 2$ or $3$.

Apparently, $t(L') = t(L_0' + L_1' + H(a)) = t(L_0' + a)$, $2$. Choose $\Lambda \subseteq L_0' + a$ such that $t(\Lambda) = t(L_0' + a)$ (this choice is actually unique as we will see soon). Let $\{e_1, e_2\}$ be a basis of $H(a)$ such that $(e_1, e_1) = (e_2, e_2) = 0$ and $(e_1, e_2) = \pi^a$. Define

$$\Lambda_1 = \Lambda + [\pi^{-a} e_1, \pi^{-1} e_2]$$

$$\Lambda_2 = \Lambda + [\pi^{-a} e_2, \pi^{-1} e_1].$$

Then

$$\Lambda_1^\sharp = \Lambda^\sharp + [\pi^{-a+1} e_1, e_2]$$

$$\Lambda_2^\sharp = \Lambda^\sharp + [\pi^{-a+1} e_2, e_1]$$

This shows that $t(\Lambda_1) = t(\Lambda_2) = t(L)$ but $\Lambda_1 \neq \Lambda_2$.

This proves the "only if" part of Theorem 3.10. To prove the converse, we start with a lemma.

**Lemma 4.3.** Suppose $L = L_0 \oplus L_1 \oplus L_2 \geq 1$ (Jordan splitting). If $\Lambda$ is a vertex lattice in $\Vert(L)$ such that $t(\Lambda) = t(L)$, then $L_0 \oplus L_1 \not\subseteq \Lambda$.

**Proof.** Suppose $L_0 \oplus L_1 \not\subseteq \Lambda$. Let $\Lambda' = \Lambda + L_0 \oplus L_1$. Since $L_1$ is $\pi^1$-modular, $\pi(\Lambda') \subseteq \Lambda'$. Also $\Lambda' \subseteq L_2$, so $\Lambda' \in \Vert(L)$. But now

$$(\Lambda')^\sharp < \Lambda^\sharp \subseteq \Lambda \subseteq \Lambda'.$$

Hence $t(\Lambda') > t(\Lambda)$ contradicting the maximality of $t(\Lambda)$.

Now we assume conditions (1) and (2) of 3.10 hold. By Theorem 3.11 we have the following three cases

1. $L \simeq L_0 \oplus H(1)^n$
2. $L \simeq L_0 \oplus H(1)^n \oplus (\pi_0^a)$ with $a \geq 1$.
3. $L \simeq L_0 \oplus H(1)^n \oplus ((-\pi_0)^a) \oplus (u(-\pi_0)^b)$, where $u \in \mathbb{Z}_p^\times$, $-u \notin Nm(E/\mathbb{Q}_p)$ and $a, b$ are integers greater or equal to 1.

Cases (1) and (2) are proved by the previous lemma ($L_1 = H(1)^n$) and the fact that $t(\Lambda)$ has to be even. Again by the previous lemma, in order to prove case (3), it suffices to prove the statement for $L = ((-\pi_0)^a) \oplus (u(-\pi_0)^b)$.

Let $L = [e_1, e_2]$ and $T = \text{diag}((-\pi_0)^a, u(-\pi_0)^b)$ is the Hermitian form with respect to $\{e_1, e_2\}$. Suppose $\Lambda = [e_1, e_2]S$ where $\Lambda \in \text{GL}_2(H) / \text{GL}_2(O_H)$. Then $\Lambda^\sharp = [e_1, e_2]^T T^{-1} \pi \Lambda^{-1}$. And

$$\Lambda^\sharp = \Lambda \iff S^{-1} T^{-1} \pi \Lambda^{-1} \in \text{GL}_2(O_H) \iff \pi \Lambda S \in \text{GL}_2(O_H)$$

$$L \subseteq \Lambda^\sharp \iff \pi \Lambda S \in M_2(O_H).$$

By Theorem 3.11 by multiplying $S$ on the right by an element in $\text{GL}_2(O_H)$, we can assume

$$\pi \Lambda S = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} =: T_1.$$ 

Assume

$$S = \begin{pmatrix} \pi^a & 0 \\ 0 & \pi^b \end{pmatrix} S_0,$$
then \( T_0S_0 = T_1 \). Assume \( S_0 = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \), then the previous equation implies that
\[
xz + uzx = 1 \\
yz + uwz = 0 \\
yw + uwv = u.
\]
If \( z = 0 \), then \( y = 0 \) and \( x, w \in \mathcal{O}_H \). Now \( uzx = -xz + 1 \), as \( u \notin Nm(H/\mathbb{Q}_p) \), \( x \neq 0 \).
Suppose \( x = x_0 \pi^e \) where \( e < 0 \), \( x_0 \in \mathcal{O}_H^\times \), then
\[
xz - 1 = (-\pi_0)^c(x_0z_0 - (-\pi_0)^{-c}).
\]
Since \( H \) is ramified over \( \mathbb{Q}_p \), \( Nm(\mathcal{O}_H^\times/\mathbb{Z}_p^\times) = (\mathbb{Z}_p^\times)^2 \) by class field theory. As \( x_0z_0 \in Nm(\mathcal{O}_H^\times/\mathbb{Z}_p^\times) = (\mathbb{Z}_p^\times)^2 \), by Hensel’s lemma, \( x_0z_0 - (-\pi_0)^{-c} \in Nm(\mathcal{O}_H^\times/\mathbb{Z}_p^\times) \). Then
\[
-eu = \frac{\varepsilon(xz)}{\varepsilon(z)} = \frac{(-\pi_0)^c(x_0z_0 - (-\pi_0)^{-c})}{\varepsilon(z)} \in Nm(\mathcal{O}_H^\times/\mathbb{Z}_p^\times),
\]
contradicts our assumption on \( -u \). This show that \( e \geq 0 \) and \( x \in \mathcal{O}_H \), but then \( z \in \mathcal{O}_H \) too.
Similarly \( \frac{1}{2}y\varepsilon + uw = 1 \) implies \( y, w \in \mathcal{O}_H \). So in fact \( S_0 \in GL_2(\mathcal{O}_H) \) since \( \det(S_0)\det(S_0) = 1 \). In other words
\[
\Lambda = [\pi^a e_1, \pi^h e_2].
\]
This proves the uniqueness of \( \Lambda \) and we finish the proof of Theorem 3.10.

5. Unitary Shimura varieties and special cycles

In this section we recall the definition of (the Pappas model) of unitary Shimura varieties and the definition of special cycles. All the information comes from [KR2] and [RTW].

5.1. Let \( k = \mathbb{Q}[\sqrt{\Delta}] \) be an imaginary quadratic extension of \( \mathbb{Q} \) and \( \mathcal{O}_k \) be its ring of integers. Let \( n \geq 1, 1 \leq r \leq n \) be integers. Define a groupoid (the Pappas model of unitary Shimura variety)
\[
\mathcal{M}(n-r, r) = \mathcal{M}(k, n-r, r)
\]
fibered over (Sch/Spec \( \mathcal{O}_k \)) by associating to a locally noetherian \( \mathcal{O}_k \)-scheme \( S \) the groupoid of triples \((A, \iota, \lambda)\). Here \( A \) is an abelian scheme over \( S \), \( \iota : \mathcal{O}_k \rightarrow \text{End}_S(A) \) is an action of \( \mathcal{O}_k \) on \( A \) and \( \lambda : A \rightarrow A^\vee \) is a principal polarization such that \( \iota(a)^* = \iota(a^\sigma), \quad \forall a \in \mathcal{O}_k \)
where \( \ast \) is the Rosati involution and \( a^\sigma \) is the Galois conjugate of \( a \). In addition the following conditions are satisfied
\begin{enumerate}
\item Kottwitz condition: \( \text{char}(T, \iota(a) | \text{Lie}(A)) = (T - \phi(a))^{n-r}(T - \phi(a^\sigma))^r \) where \( \phi : \mathcal{O}_k \rightarrow \mathcal{O}_S \) is the structure homomorphism.
\item Pappas condition: \( \wedge^{r+1}(\iota(a) - a) = 0, \wedge^{n-r+1}(\iota(a) - a^\sigma) = 0 \).
\end{enumerate}
For \( n \leq 2 \), the second condition follows from the first. \( \mathcal{M}(n-r, r) \) is a Deligne-Mumford stack over Spec \( \mathcal{O}_k \). It is flat of relative dimension \( (n-r)r \) over Spec \( \mathcal{O}_k \) and smooth over \( \mathcal{O}_k[\Delta^{-1}] \) (Proposition 2.1 and Theorem 2.5 of [KR2]).
5.2. As in [KR2], we denote by $\mathcal{R}(n-r,r)(k)$ the set of relevant Hermitian spaces, i.e., the set of isomorphism classes of Hermitian space $V$ over $k$ of dimension $n$ and signature $(n-r,r)$ which contains a self-dual $O_k$-lattice.

Let $G^V$ be the group of unitary similitudes of $V$. That is for any $\mathbb{Q}$-algebra $R$

$$G^V(R) = \{ g \in \text{End}_k(V) \otimes \mathbb{Q} \mid gg^* = \nu(g) \in R^\times \}.$$ 

Let $G^V_1 = U(V)$ be the isometry group of a $V \in \mathcal{R}(n-r,r)(k)$. Then $G^V_1(A_f)$ acts on the set of self dual lattices in $V$ by $g : L \mapsto V \cap (gL \otimes \hat{\mathbb{Z}})$. The orbit of $L$ is the $G^V_1$-genus of $L$, we denote it by $[L]$. Let $\mathcal{R}(n-r,r)(k)^1$ be the set of isomorphism classes of pairs $V^2 := (V,[L])$ where $V \in \mathcal{R}(n-r,r)(k)$ and $[L]$ is the $G^V_1$-genus of self dual Hermitian lattices in $V$.

The following is Proposition 2.14 of [KR2].

**Proposition 5.1.** Suppose that $V_p$ is a non-degenerate Hermitian space of dimension $n$ over $k_p/\mathbb{Q}_p$ and that $V_p$ contains a self-dual lattice. Then the unitary group $U(V_p)$ acts transitively on the set of self dual lattices in $V_p$ except in the following cases:

1. $p = 2$, $k_p/\mathbb{Q}_p$ is ramified, $n = \dim V_p$ is even and $V_p$ is a split space.
2. $p = 2$, $k_p = \mathbb{Q}_p(\sqrt{\Delta})$ where $ord_2(\Delta) = 3$, $n = \dim V_p$ is even and $V_p$ is the sum of a 2-dimensional anisotropic space and a split space of dimension $n-2$.

Then there are two $U(V_p)$-orbits of self dual lattices in $V_p$.

**Corollary 5.1.** For a relevant Hermitian space $V$ in $\mathcal{R}(n-r,r)(k)$, the number of $G^V_1$-genera of self dual lattices in $V$ is 2 or 1 depending on whether or not one of the exceptional cases in the above proposition occurs at $p = 2$.

We say two Hermitian forms $(,)$ and $(,,)'$ on $V$ are strictly similar if $(,)=c(,,)'$ for some $c \in \mathbb{Q}_k^\times$. When $c \in \text{Nm}(k^\times)$, $(,)$ and $(,,)'$ are isomorphic to each other. The fact that $V$, $(,)$ contains a self dual lattice is equivalent to the condition that the determinant of $(,)$ lies in $\mathbb{Z}_p^\times \text{Nm}(k^\times)$ for all $p$. For split and ramified primes, this condition is automatic. For inert primes, it is equivalent to $\text{inv}_p(V) = (\text{det}V,\Delta)_p = 1$ where $(,)_p$ is the quadratic Hilbert symbol for $\mathbb{Q}_p$. Hence if both $(,)$ and $(,,)'$ are in $\mathcal{R}(n-r,r)(k)$ and $(,)=c(,,)'$, $c$ is of the form

$$c = a \cdot a^7 \prod_{\ell \mid \Delta} \ell^{\epsilon_\ell}, \quad \epsilon_\ell = 0 \text{ or } 1, \quad a \in k,$$

such that $(,)=c(,,)'$. In this case, multiplying by $c$ takes a self dual lattice $L$ of $(,)$ to the self dual lattice $L' = cL$ of $(,,)'$. In this sense, we can talk about strictly similarity in $\mathcal{R}(n-r,r)(k)$.

5.3. Suppose $F$ is an algebraically closed field of characteristic $p$. Assume that $(A,\iota,\lambda) \in \mathcal{M}(n-r,r)(F)$. Let $TP(A)$ be its Tate module and $TP(A)^0 = TP(A) \otimes \mathbb{Q}$ be its rational Tate module. We fix once and for all a trivialization of prime-to-$p$ root of unity over $F$

$$\hat{\mathbb{Z}}^p(1) \cong \hat{\mathbb{Z}}^p.$$

Using the Weil pairing

$$e_A : TP(A)^0 \times TP(A^\vee)^0 \to \mathbb{A}_f^p$$
and the polarization $\lambda$, we can define a Hermitian form $h_\lambda$ on $T^p(A)^0$ by
\begin{equation}
2h_\lambda(x, y) = e_A(\delta x, \lambda(y)) + \delta e_A(x, \lambda(y)),
\end{equation}
where $\delta = \sqrt{\lambda}$. The following comes from Proposition 2.12 and Proposition 2.19 of [KR2].

**Proposition 5.2.** (i) There is a natural disjoint decomposition of algebraic stacks:
\[ \mathcal{M}(n - r, r) = \bigcup_{V \in \mathcal{R}(n - r, r)(k)/\text{str.sim.}} \mathcal{M}(n - r, r)^{V}. \]
Here $(A, t, \lambda) \in \mathcal{M}(n - r, r)^{V}$ if and only if $(T^p(A)^0, h_\lambda)$ is isomorphic to $V \otimes \mathbb{A}_f^p$ as Hermitian spaces.

(ii) There is a natural disjoint decomposition of algebraic stacks:
\[ \mathcal{M}(n - r, r)\left[\frac{1}{2}\right] = \bigcup_{V^s \in \mathcal{R}(n - r, r)(k)^s/\text{str.sim.}} \mathcal{M}(n - r, r)^{\left[\frac{1}{2}\right]}V^s. \]
where $\mathcal{M}(n - r, r)\left[\frac{1}{2}\right] = \mathcal{M}(n - r, r) \times_{\text{Spec} \mathcal{O}_k \text{Spec} \mathcal{O}_k} \mathcal{M}(n - r, r)$ and $V^s = (V, [L])$. $(A, t, \lambda) \in \mathcal{M}(n - r, r)^{V^s}$ if and only if $(T^p(A)^0, h_\lambda)$ is isomorphic to $V \otimes \mathbb{A}_f^p$ as Hermitian spaces, and $[T^p(A)] = [L \otimes \mathbb{Z}^p]$.

5.4. **Special cycles.** We denote $\mathcal{M}_0 = \mathcal{M}(1, 0)$ and define
\[ \mathcal{M} = \mathcal{M}(n - r, r) \times_{\text{Spec} \mathcal{O}_k} \mathcal{M}_0. \]
Suppose $(E, t_0, \lambda_0) \in \mathcal{M}_0(S)$ and $(A, t, \lambda) \in \mathcal{M}(n - r, r)(S)$. Assume $S$ is connected. Consider the torsion free $\mathcal{O}_k$ module of finite rank
\[ \tilde{V}^r(A, E) = \text{Hom}_{\mathcal{O}_k}(E, A). \]

On this module there is a $\mathcal{O}_k$-valued Hermitian form given by
\begin{equation}
\tilde{h}'(x, y) = t_0^{-1}(\lambda_0^{-1} \circ y^\vee \circ \lambda \circ x) \in \mathcal{O}_k,
\end{equation}
where $y^\vee$ is the dual of $y$. By Lemma 2.7 of [KR2], $\tilde{h}'$ is positive definite.

**Definition 5.1.** Let $T \in \text{Herm}_m(\mathcal{O}_k)$ be an $m \times m$ Hermitian matrix with coefficients in $\mathcal{O}_k$. The special cycle $\mathcal{Z}(T)$ attached to $T$ is the stack of collections $(A, t, \lambda; E, t_0, \lambda_0; x)$ where $(A, t, \lambda) \in \mathcal{M}(n - r, r)(S)$, $(E, t_0, \lambda_0) \in \mathcal{M}_0(S)$ and $x \in \text{Hom}_{\mathcal{O}_k}(E, A)^m$ is an $m$-tuple of homomorphisms such that
\begin{equation}
h'(x, x) = T.
\end{equation}
$\mathcal{Z}(T)$ is representable by a Deligne-Mumford stack and the natural morphism $\mathcal{Z}(T) \to \mathcal{M}$ is finite and unramified (Proposition 2.9 of [KR2]). Note that $\mathcal{Z}(T)$ is empty if $T$ is not positive definite.

Suppose $\mathbb{F}$ is an algebraically closed field with characteristic $p$. Let $(E, t_0, \lambda_0) \in \mathcal{M}_0^0(\mathbb{F})$ and $(A, t, \lambda) \in \mathcal{M}(n - r, r)^V(\mathbb{F})$. Let $h_{\lambda_0}(\cdot)$ and $h_{\lambda}(\cdot)$ be the Hermitian forms on $T^p(E)^0$ and $T^p(A)^0$ respectively as defined in equation (5.1). Together they determine Hermitian form $h_{\lambda}(\cdot)$ on $\text{Hom}_{k \otimes \mathbb{A}_f^p}(T^p(E)^0, T^p(A)^0)$ by equation
\begin{equation}
h_{\lambda}(x(a), y(b)) = h(x, y)h_{\lambda_0}(a, b), \quad x, y \in \text{Hom}_{k \otimes \mathbb{A}_f^p}(T^p(E)^0, T^p(A)^0).
\end{equation}

Proposition 5.2 implies that there exist $V_0 \in \mathcal{R}(1, 0)(k)$ and $V \in \mathcal{R}(n - r, r)(k)$ such that
\[ T^p(E)^0 \cong V_0 \otimes \mathbb{A}_f^p, \quad T^p(A)^0 \cong V \otimes \mathbb{A}_f^p. \]
as Hermitian spaces. Hence
\[ \text{Hom}_k(A_p)(T^p(E)^0, T^p(A)^0) \cong \tilde{V} \otimes A_p, \]
as Hermitian spaces where \( \tilde{V} := \text{Hom}_k(V_0, V) \in \mathcal{R}(n - r, r)(k) \). In other words, we get

**Corollary 5.2.** (i) There are natural disjoint decompositions of algebraic stacks:
\[ \mathcal{M} = \bigsqcup_{\tilde{V} \in \mathcal{R}(n-r,r)(k) / \text{str.sim.}} \mathcal{M}^{\tilde{V}}, \]
and
\[ \mathcal{M}^{\tilde{V}} = \bigsqcup_{V_0 \in \mathcal{R}(1,0)(k) / \text{str.sim.}} \mathcal{M}^{(V,V_0)}, \]
where \( \mathcal{M}^{(V,V_0)} = \mathcal{M}(1,0)^{V_0} \times_{\mathcal{O}_k} \mathcal{M}(n-r,r)^{V} \). Similar decompositions hold if we replace \( \mathcal{M} \) by \( \mathcal{M}[\frac{1}{2}] \) and \( \tilde{V} \) by \( \tilde{V}^\natural \) as in Proposition 5.2.

Let \((A, \lambda; E, \iota_0, \lambda_0) \in \mathcal{M}(\mathcal{F})\). The Hermitian form \( h(\cdot, \cdot) \) defined in equation (5.3) induces a Hermitian form on \( \tilde{V}_p := \text{Hom}_k(A_p)(T^p(E)^0, T^p(A)^0) \) by the same formula
\[ h'(x,y) = \iota_0^{-1}(\lambda_0^{-1} \circ y^V \circ \lambda \circ x) \in k \otimes A_p, \quad x, y \in \tilde{V}_p. \]
The natural embedding \( V'(A, E) \to \tilde{V}_p \) is isometric.

The following lemma is Lemma 2.10 of [KR2].

**Lemma 5.2.** The two Hermitian forms \( h(\cdot, \cdot) \) and \( h'(\cdot, \cdot) \) are identical on \( \text{Hom}_k(A_p)(T^p(E)^0, T^p(A)^0) \).

5.5. **Decomposition of special cycles.** From now on we assume \( r = 1 \). We define
\[ (5.6) \quad Z(T)^{\tilde{V}} = Z(T) \times_{\mathcal{M}} \mathcal{M}^{\tilde{V}}, \quad Z(T)^{(V,V_0)} = Z(T) \times_{\mathcal{M}} \mathcal{M}^{(V,V_0)}. \]

We use the subscript \( p \) to denote the base change to the field \( \mathbb{F}_p \) of a \( \mathcal{O}_k \) scheme. Let \( \mathcal{M}_p(1, n - 1)^{V,ss} \) be the super singular locus of \( \mathcal{M}_p(1, n - 1)^{V} \). Let \( \mathcal{M}_p^{V,ss} \) be the locus of \( \mathcal{M}_p^{\tilde{V}} \) where the natural projection to \( \mathcal{M}_p(1, n - 1) \) lies in the super singular locus.

The following is Lemma 2.21 of [KR2].

**Lemma 5.3.** Let \( 0 < r < n \). When \( T \in \text{Herm}_n(\mathcal{O}_k)_{>0} \), the support of \( Z(T) \) is contained in the union over finitely many inert or ramified \( p \) of the super singular locus of \( \mathcal{M}_p \).

We now restate Proposition 2.22 of [KR2].

**Proposition 5.3.** Let \( T \in \text{Herm}_n(\mathcal{O}_k)_{>0} \) and \( V_T \) be the Hermitian space determined by the Hermitian form \( T \). Let \( \text{Diff}_0(T) \) be the set of primes \( p \) that are inert in \( k \) for which \( \text{ord}_p(\det(T)) \) is odd.
(i) If \(|\text{Diff}_0(T)| > 1\), then \(\mathcal{Z}(T)\) is empty.

(ii) If \(\text{Diff}_0(T) = \{p\}\), then

\[
\mathcal{Z}(T) = \bigsqcup_{V_0 \in \mathcal{R}(1,0)(k)/\text{str.sim.}} \mathcal{Z}(T)^{(V,V_0)},
\]

where \(\tilde{V}\) is the Hermitian form in \(\mathcal{R}(1,n-1)(k)\) with \(\text{inv}_\ell(V) = \text{inv}_\ell(\tilde{V})\) for all primes \(\ell \neq p\). Moreover,

\[
\text{supp}(\mathcal{Z}(T)^{(V,V_0)}) \subset \mathcal{M}_p^{(V,V_0)}.
\]

(iii) If \(\text{Diff}_0(T)\) is empty, then for each \(p|\Delta\), there is a unique relevant Hermitian space \(V^{(p)} \in \mathcal{R}(1,n-1)(k)\) for which \(\text{inv}_\ell(V^{(p)}) = \text{inv}_\ell(\tilde{V})\) for all primes \(\ell \neq p\). Then we have

\[
\mathcal{Z}(T) = \bigsqcup_{p|\Delta} \mathcal{Z}(T)^{(p)},
\]

\[
\mathcal{Z}(T)^{(p)} = \bigsqcup_{V_0 \in \mathcal{R}(1,0)(k)/\text{str.sim.}} \mathcal{Z}(T)^{(V,V_0)}
\]

for \(p|\Delta\). We also have

\[
\text{supp}(\mathcal{Z}(T)^{(V,V_0)}) \subset \mathcal{M}_p^{(V,V_0)}.
\]

When \(p \neq 2\), we can replace \(\mathcal{R}(1,n-1)(k)\) by \(\mathcal{R}(1,n-1)(k)^\ell\) and \(V\) by \(V^\ell\) in the decompositions of (2) and (3).

**Proof.** This is proved in [KR2]. We recall the proof of (iii).

Suppose \(\text{Diff}_0(T) = 0\). By the previous lemma, the support of \(\mathcal{Z}(T)\) is contained in the union over finitely many inert or ramified \(p\) of the super singular locus of \(\mathcal{M}_p\).

Fix a prime \(p\) such that \(\mathcal{Z}(T)({\overline{F}_p}) \neq \emptyset\). Let \((A,\iota,\lambda; E,\iota_0,\lambda_0; x) \in \mathcal{Z}(T)({\overline{F}_p})\). By Corollary 5.2 we can assume that \((A,\iota,\lambda; E,\iota_0,\lambda_0) \in \mathcal{M}^{(V,V_0)}\). Let \(\tilde{V} = \text{Hom}(V_0, V) \in \mathcal{R}(1,n-1)(k)\). We know that

\[
h'(x, x) = T,
\]

hence \(\text{Hom}_{\mathfrak{O}_k}(E, A) \otimes \mathbb{Q} \cong V_T\) by definition of \(V_T\). By Lemma 5.2 \(V_T\) agrees with \(\tilde{V}\) at all finite places except for \(p\). Moreover the Hasse invariant of \(V_T\) at \(\infty\) is 1 while that of \(\tilde{V}\) is \(-1\). Hence by product formula, the Hasse invariant of \(V_T\) and \(\tilde{V}\) differ at \(p\) as well. But both \(V_T\) and \(\tilde{V}\) contain self dual lattices which is equivalent to the condition on Hasse invariants that \(\text{inv}_p(V_T) = \text{inv}_p(\tilde{V}) = 1\) for all inert primes \(p\). So \(p\) must be a ramified prime.

Let \(V^{(p)}\) be defined as in the statement of (iii). Then \(V^{(p)}\) and \(\tilde{V}\) agree at all finite places except for \(p\) and have the same signature. Thus they agree by Hasse principle and product formula. This proves (iii). \(\square\)
6. Special cycles in the super singular locus

In this section we fix a prime \( p \) such that \( p|\Delta \) and \( p \neq 2 \). Let \( \mathbb{F} = \overline{\mathbb{F}}_p \) and \( W = W(\mathbb{F}) \) be the Witt vectors of \( \mathbb{F} \). Let \( W_\mathbb{Q} \) be the fraction field of \( W \). Let \( H \) be a ramified extension of \( \mathbb{Q}_p \). Let \( \tilde{H} = W \otimes_{\mathbb{Q}_p} H \) and \( \mathcal{O}_\tilde{H} \) be its ring of integers.

We will write down a uniformization theorem for special cycles in \( M_{p}^{\text{ss}} \). We then use the local information on special cycles on the Rapoport-Zink space \( N \) to compute the dimension of \( Z(T) \times_{\mathcal{O}_H} \overline{\mathbb{F}}_p \).

6.1. First we briefly recall the uniformization theorem of [RTW] for \( \tilde{M}_{p}^{V,ss}(n-1,1) \) in the style of section 5 of [KR2]. Here \( \tilde{M}_{p}^{V,ss}(n-1,1) \) is the formal completion of \( M^{V}(n-1,1) \times_{\text{Spec} \mathcal{O}_H} \text{Spec}(\mathcal{O}_\tilde{H}) \) along its super singular locus. Fix a \( V = (V, [L]) \in \mathcal{N}(n-1,1)(\mathbb{F}) \). Fix a base point \( \xi = (A^\circ, \iota^0, \lambda^0) \in \tilde{M}_{p}^{V,ss}(n-1,1)(\mathbb{F}) \).

By the same proof with that of Lemma 5.1 in [KR2], such a point always exist.

We fix a self dual lattice \( L \in V \) in the given \( G_1^\circ \)-genus and an isomorphism

\[
\eta^o : T^p(A^0) \cong V(\mathbb{A}_f^p)
\]

such that \( \eta^o \) is an isometry and \( \eta^o(T^p(A^0)) = L \otimes \hat{\mathbb{Z}}^p \). Let \( K^p \) be the stabilizer of \( L \) in \( G(\mathbb{A}_f^p) \). \( K^p \) is a subgroup of

\[
G^V(\mathbb{A}_f^p)^0 = \{ g \in G(\mathbb{A}_f^p) \mid \nu(g) \in (\hat{\mathbb{Z}}^p)^\times \}.
\]

Let \( (X, \iota, \lambda_X) \in \mathcal{N}^0(\mathbb{F}) \) be the underlying \( p \)-divisible group of \( (A^o, \iota^0, \lambda^0) \). Choose a lift \( \tilde{X} \) of \( X \) to \( \mathcal{O}_\tilde{H} \) and let \( \tilde{A}^o \) be the corresponding lift of \( A^o \). For \( S \in \text{Nilp}_{\mathcal{O}_H} \), let \( \tilde{A}_S^o := \tilde{A}^o \times_{\mathcal{O}_H} S \). There is a canonical isomorphism

\[
\tilde{A}_S^o \times_{\mathcal{O}_H} \mathbb{F} = A^o \times_S \tilde{S},
\]

where \( \tilde{S} = S \times_{\mathcal{O}_H} \mathbb{F} \). There is an \( \mathcal{O}_k \)-action

\[
\iota^o_S = \iota^o : \mathcal{O}_k \to \text{End}_S(A^o \times \tilde{S}) = \text{End}(A^o),
\]

and a polarization

\[
\lambda^o_S = \lambda^o \times 1_{\tilde{S}} : A^o \times_S \tilde{S} \to (A^o)^\vee \times_S \tilde{S}.
\]

There are unique extensions of these to \( \tilde{A}_S^o \). i.e., there is an \( \mathcal{O}_k \)-action by quasi-isogenies

\[
\tilde{\iota}^o_S : \mathcal{O}_k \to \text{End}_S(\tilde{A}_S^o) = \text{End}^0(A^o),
\]

and a quasi-polarization

\[
\tilde{\lambda}^o_S : \tilde{A}_S^o \to (\tilde{A}_S^o)^\vee.
\]

There is an isomorphism

\[
\tilde{\eta}^o_S : T^p(\tilde{A}_S^o) \to T^p(A^0)^0 \xrightarrow{\eta^o} V(\mathbb{A}_f^p).
\]

Note that all the above constructions are functorial in \( S \).

**Proposition 6.1.** For each object \( \xi = (X, \iota, \lambda_X, \rho_X) \) in \( \mathcal{N}(S) \) and coset \( gK^p \in G(\mathbb{A}_f^p)/K^p \), there is an object \( \Theta(\xi, gK^p) = (A, \iota, \lambda) \) of \( \mathcal{M}(n-1,1)(S) \) and a \( \mathcal{O}_k \)-linear quasi-isogeny \( \phi : A \to \tilde{A}_S^o \) uniquely characterized by the following properties:

(i) The polarization \( \lambda \) agrees with \( \phi^\circ \circ \tilde{\lambda}_S^o \circ \phi \).

(ii) Let

\[
\eta = \tilde{\eta}^o_S \circ \phi : T^p(A^0) \to V(\mathbb{A}_f^p).
\]
Then
\[ \eta(T^n(A)) = g \cdot (L \otimes \hat{\mathbb{Z}}^p). \]

(iii) Let \((X(A), \iota)\) be the \(p\)-divisible group of \((A, \iota)\) with \(\mathcal{O}_k\)-action. Then there is an isomorphism
\[ i: (X(A), \iota) \overset{\sim}{\longrightarrow} (X, \iota) \]
such that the quasi-isogeny \(\phi\) induces \(\rho_X\) over \(\tilde{S}\), i.e. the diagram
\[
\begin{array}{ccc}
(X(A), \iota) \times_W \mathbb{F} & \overset{\sim}{\longrightarrow} & (X, \iota) \times_W \mathbb{F} \\
\downarrow & & \downarrow \\
(X, \iota) & \phi_* & (X, \iota)
\end{array}
\]
commutes. The map \(\lambda_{X(A)}\) and \(i^*(\lambda_X)\) agree up to a factor in \(\mathbb{Z}_p^X\).

The proof of the above proposition can be found in section 6 of [RZ].

Let \(I'\) be the groups of quasi-isogenies in \(\text{End}_{\mathcal{O}_k}(A^p)\) that preserve the polarization \(\lambda^0\). Any \(\gamma \in I'\) induces a quasi-isogeny \(\alpha_p(\gamma)\) of height 0 of the \(p\)-divisible group \((X, \iota', \lambda_X)\) of \((A^p, \iota, \lambda^0)\) and hence acts on \(\mathcal{N}\) by sending \(\xi = (X, \iota, \lambda_X, \rho_X)\) to \(\alpha_p(\gamma)\xi = (X, \iota, \lambda_X, \alpha_p(\gamma) \circ \rho_X)\). The element \(\gamma\) also induces an automorphism \(\gamma_*\) of \(T^n(A^p)^0\) and hence defines an element
\[ \alpha_p(\gamma) = \eta^{p, o} \circ \gamma_* \circ (\eta^{p, o})^{-1} \in G(A^p_p). \]

The following theorem is described in section 7 of [RTW] which is a special case of the uniformization theorem of [RZ].

**Theorem 6.1.** Let \(\mathcal{M}^{V^s, ss}(n-1, 1)\) be the formal completion of \(\mathcal{M}^{V^s, ss}(n-1, 1) \times_{\text{Spec } \mathcal{O}_k} \text{Spec } \mathcal{O}_H\) along its super singular locus. The map \(\Theta\) induces an isomorphism
\[
\Theta: [I'(\mathbb{Q}) \times G^V(A^p_p)^0 / K_p)] \overset{\sim}{\longrightarrow} \mathcal{M}(n, 1-1 V^s, ss)
\]
of formal algebraic stacks over \(\mathcal{O}_H\).

6.2. Throughout this subsection we base change from \(\mathcal{O}_k\) to \(\mathcal{O}_H\). We write \(\mathcal{M}(n, 1, 1)\) for \(\mathcal{M}(n, 1, 1) \times_{\text{Spec } \mathcal{O}_k} \text{Spec } \mathcal{O}_H\), \(\mathcal{M}(n, 1, 1) V^s\) for \(\mathcal{M}(n, 1, 1) V^s \times_{\text{Spec } \mathcal{O}_k} \text{Spec } \mathcal{O}_H\), \(\mathcal{M}\) for \(\mathcal{M} \times_{\text{Spec } \mathcal{O}_k} \text{Spec } \mathcal{O}_H\) and so on. We also write \(\mathbb{Z}_p(T)\) for \(\mathbb{Z}(T) \times_{\text{Spec } \mathcal{O}_k} \text{Spec } \mathcal{O}_H\). We introduce the fiber product
\[
\mathbb{Z}_p(T)^{(V^s, V_0), ss} \overset{\sim}{\longrightarrow} \mathcal{M}^{(V^s, V_0), ss} \mathcal{M}.
\]

First we want to use Theorem 6.1 to get a uniformization of \(\mathbb{Z}_p(T)^{(V^s, V_0), ss}\).

Fix a base point \((A^p, \iota^0, \lambda^0; E^a, t_0^a, \lambda_0^a)\) in \(\mathcal{M}^{(V^s, V_0), ss}\). Recall that we associate two Hermitian spaces
\[
\tilde{V} = \text{Hom}_{\mathcal{O}_k}(V_0, V), \quad \tilde{V}' = \text{Hom}_{\mathcal{O}_k}(E^a, A^p) \otimes \mathcal{O}_K
\]
with Hermitian forms \(h(\_\_\_)\) and \(h'(\_\_\_)\) (see equation (1.53)) respectively. By Lemma 5.2, these two Hermitian spaces are isomorphic at all finite places except for \(p\). At the infinite place, \(h'(\_\_\_)\) is positive definite while \(h(\_\_\_)\) is of signature \((n-1, 1)\).
A choice of $\eta_0$ and $\eta_0'$ as in equation (6.1) determines an isomorphism
\[ \tilde{V} \otimes A_p^0 \cong \text{Hom}_{k \otimes \mathbb{Z}_p}(T^p(E^0), T^p(A^0)). \]
For $x \in \tilde{V}'$, let
\[ \mathcal{X} = \eta'' \circ x \circ (\eta_0')^{-1} \in \text{Hom}_{k \otimes \mathbb{Z}_p}(V_0(A_p^0), V(A_p^0)), \]
and let
\[ \mathcal{X} \in \text{Hom}_{\mathbb{Z}_p \otimes \mathbb{Z}_p}(\mathcal{X}_0, \mathcal{X}) = V \]
be the corresponding quasi-homomorphism between the underlying $p$-divisible groups. Note that $I^V(A_p^0) \times I^{V_0}(A_p^0)$ acts isometrically on the space $\text{Hom}_{k \otimes \mathbb{Z}_p}(V_0(A_p^0), V(A_p^0))$. Similarly the group $I^V(Q_p) \times I^{V_0}(Q_p)$ acts isometrically on $V$. The maps $x \mapsto \mathcal{X}$ and $x \mapsto \mathcal{X}$ are $I^V(Q) \times I^{V_0}(Q)$-equivariant.

**Lemma 6.2.** $\tilde{V} \in \mathcal{R}(n-1,1)(k)$ and $\tilde{V}' \in \mathcal{R}(n,0)(k)$. They are isomorphic at all finite place except for $p$.

**Proof.** Since $V_0 \in \mathcal{R}(1,0)$ and $V \in \mathcal{R}(n-1,1)$, $\tilde{V} \in \mathcal{R}(n-1,1)$. Since $\tilde{V} \otimes A_p^0 \cong \text{Hom}_{k \otimes \mathbb{Z}_p}(T^p(E^0), T^p(A^0))$, it contains a self dual lattice. By Corollary 3.1, $\tilde{V}' \otimes Q_p$ also contains a self dual lattice. As $\tilde{V}'$ is positive definite, it is in $\mathcal{R}(n,0)(k)$. The fact that $\tilde{V} \in \mathcal{R}(n-1,1)(k)$ and $\tilde{V}' \in \mathcal{R}(n,0)(k)$ are isomorphic at all finite place except for $p$ follows from Proposition 6.2.

Recall that we have defined special cycles $Z(x)$ in definition 5.3. The following proposition is the analogue of Proposition 6.3 of [KR2].

**Proposition 6.2.** Fix a base point $(A^0, \iota^0, \lambda^0; E^0, \iota_0^0, \lambda_0^0)$ in $\mathcal{M}(V^1, V_0)^{ss}$. For $S \in \operatorname{Nil}_{\mathbb{Z}_p}$, define $\operatorname{Inc}_p(T; V^1, V_0)(S)$, the incidence set inside
\[ (N \times N_0)(S) \times (G^V(A_p^0)/K_{V'}^p, p \times G^{V_0}(A_p^0)/K_{0'}^p) \times \tilde{V}'(Q)^m \]
to be the subset of collections $(\xi, \xi_0, gK^{V^1-p}, g_0K_0^p; x^o)$ determined by the following incidence relations
\begin{enumerate}
    \item $h'(x^o, x^o') = T$
    \item $g^{-1} \circ x^o \circ \eta_0 \in \text{Hom}_{\mathbb{Z}_p \otimes \mathbb{Z}_p}(T^p(E^0), T^p(A^0)).$
    \item $(\xi, \xi_0) \in Z(x^o)(S) \subset (N \times N_0)(S).$
\end{enumerate}
Then $\operatorname{Inc}_p(T; V^1, V_0)(S)$ is the set of $S$-points of the formal scheme
\[ \operatorname{Inc}_p(T; V^1, V_0)(S) = \bigsqcup_{(gK^{V^1-p}, g_0K_0^p)} \bigcup \mathcal{Z}(x^o), \]
where $(gK^{V^1-p}, g_0K_0^p)$ runs over $G^V(A_p^0)/K_{V'}^p \times G^{V_0}(A_p^0)/K_{0'}^p$, and $x^o \in \tilde{V}'(Q)^m$ runs over the set of $m$-tuples satisfying conditions (1) and (2). Moreover there is an isomorphism of formal stacks over $\mathcal{O}_H$,
\[ (I^V(Q) \times I^{V_0}(Q)) \setminus \operatorname{Inc}_p(T; V^1, V_0) \sim \tilde{Z}_p(T)(V^1, V_0)^{ss}. \]

**Proof.** The proof is similar to that of Proposition 6.3 of [KR2]. The point is that a quasi-homomorphism between two Abelian varieties $E$ and $A$ is an actual homomorphism if and only if it map $T^p(E)$ to $T^p(A)$ (instead of just $T^p(A)^0$) and it induces a homomorphism from the the underlying $p$-divisible group $X(E)$ of $E$ to $X(A)$. We leave the details to the reader. \qed
For a Hermitian $\mathcal{O}_k$ lattice $L$ and $T \in \text{Herm}_m(\mathcal{O}_k)_{>0}$, define
$$\Omega(T, L) = \{x \in L^m \mid (x, x) = T\}.$$\
$|\Omega(T, L)|$ is always finite since $T$ is positive definite.

**Corollary 6.1.** Let $T \in \text{Herm}_m(\mathcal{O}_k)_{>0}$ such that $\text{Diff}_0(T)$ is empty. $T$ determines a Hermitian space $V_T$. Let $V^{(p)} \in \mathcal{R}(1, n-1)(k)$ such that $\text{inv}(V^{(p)}) = \text{inv}(V_T)$ for all primes $\ell \neq p$ as described in part (iii) of Proposition 5.2. Suppose $V_0 \in \mathcal{R}(1,0)(k)_{\text{str.sim.}}, V \in \mathcal{R}(1, n-1)(k)_{\text{str.sim.}},$ and $\text{Hom}_k(V_0, V) = V^{(p)}$. Let $L$ be a self dual lattice of $V, V^* = (V, [L])$ and $L_0$ be the unique self dual lattice of $V_0$. Then there exists a unique $G_1V_T$-genus of self dual lattices $[\tilde{L}]$ of $V_T$ such that the $G_1^T(\mathbb{A}_n^p)$-genus of $\tilde{L}' \otimes \mathcal{O}_k L_0$ is the same with that of $L$. In other words, there exists $g \in G_1^T(\mathbb{A}_n^p)$ such that
\begin{equation}
g \cdot (\tilde{L}' \otimes \mathcal{O}_k L_0 \otimes \mathbb{Z}_p) = L \otimes \mathbb{Z}_p.
\end{equation}

Then $Z_p(T)^{(V^*, V_0), \text{ss}}$ is not empty if and only if there exists a lattice $\tilde{L}'' \in [\tilde{L}]$ such that
$$|\Omega(T, \tilde{L}'')| > 0.$$\

**Remark 6.3.** By Proposition 5.1 the above condition on $\tilde{L}'$ is satisfied if and only if the $G_1^T(\mathbb{Q}_p)$-genus of $\tilde{L}' \otimes \mathcal{O}_k L_0$ is the same with that of $L$.

**Proof.** By Proposition 5.1 there is a unique $G_1^T(\mathbb{Q}_p)$-genus of lattices that is self dual since $p \neq 2$. The existence and uniqueness of such self dual $[\tilde{L}]$ as described in equation (6.3) now follows from Proposition 5.1 and its corollary.

(1) Suppose $Z_p(T)^{(V^*, V_0), \text{ss}}$ is not empty. Let $(A, i, \lambda, E, \iota, \lambda_0, x) \in Z_p(T)^{(V^*, V_0), \text{ss}}(\mathbb{F})$. By definition we have
$$h'(x, x) = T.$$\
Hence
$$\tilde{V}' = \text{Hom}_{\mathcal{O}_k}(E, A) \otimes \mathbb{Q} \cong V_T.$$\

It suffice to show that
$$[\text{Hom}_{\mathcal{O}_k}(E, A)] = [\tilde{L}].$$

Since $T^p(E) \cong L_0$ (resp. $T^p(A) \cong L$) and
$$\text{Hom}_{\mathcal{O}_k}(E, A) \otimes \mathbb{Z}_p = \text{Hom}_{k(\mathbb{Q}_p)}(T^p(E), T^p(A)),$$
the fact that
$$\text{Hom}_{\mathcal{O}_k}(E, A) \otimes \mathbb{Z}_p = \tilde{L}' \otimes \mathbb{Z}_p$$
as Hermitian lattices follows from Proposition 5.2 and the definition of $\tilde{L}'$. Let $X(A)$ (resp. $X(E)$) be the underlying $p$-divisible group of $A$ (resp. $E$), and $M(A)$ (resp. $M(E)$) be its covariant Dieudonné module. We also know that,
$$\text{Hom}_{\mathcal{O}_k}(E, A) \otimes \mathbb{Z}_p \cong \text{Hom}_{\mathcal{O}_k}(X(E), X(A)).$$

Moreover this isomorphism is an isomorphism as Hermitian lattices by equation (3.2) and equation (5.3). By Corollary 3.1 $\text{Hom}_{\mathcal{O}_k}(X(E), X(A))$ is a self dual lattice in $V_T \otimes \mathbb{Q}_p$. By Lemma 5.1 $\tilde{L}'$ and $\text{Hom}_{\mathcal{O}_k}(E, A)$ are in the same $G_1^T(\mathbb{A}_n^p)$-genus. We have proved that $[\text{Hom}_{\mathcal{O}_k}(E, A)] = [\tilde{L}].$

(2) Conversely, assume that there is a lattice $\tilde{L}''$ in the genus of $\tilde{L}'$ such that there is a quasi-homomorphism $x \in (\tilde{L}'')^m$ such that $h'(x, x) = T$. Choose the
any prime \( \ell = \ell_0, \lambda_0, E_0, L_0, \lambda_0^2 \) in \( \mathcal{M}(V^d, V_0) \) as in Proposition 6.2 and recall that 
\[ V_T = \tilde{V} = \text{Hom}_{\mathcal{O}_k}(E^0, A^0) \otimes \mathbb{Q}. \]
By definition of \( \tilde{L} \) we can assume 
\[ \tilde{L} \otimes_{\mathcal{O}_k} L_0 \otimes \mathbb{T} = g_1 \cdot (L \otimes \mathbb{T}), \]
for some \( g_1 \in G^V_L(\mathbb{A}) \).

By definition of \( \tilde{L} \), there is a \( g' \in G^V_L(\mathbb{A}) \) such that 
\[ g' \cdot (\tilde{L} \otimes \mathbb{T}) = \tilde{L}' \otimes \mathbb{T}. \]
Since \( V' \otimes \mathbb{A} \) is \( \operatorname{Hom}(V_0, V) \otimes \mathbb{A} \), we can write \( g = g' \otimes \text{Id} \) for some \( g \in G^V_L(\mathbb{A}) \).

Thus 
\[ g_1^{-1}g^{-1}(\tilde{L} \otimes_{\mathcal{O}_k} L_0 \otimes \mathbb{T}) = L \otimes \mathbb{T}, \]
and 
\[ (\eta)^{-1}((g_1^{-1} \otimes \text{Id})(g^{-1} \otimes \text{Id})\mathbb{X}) \circ \eta_0^\mathbb{X} \in \text{Hom}_{\mathcal{O}_k \otimes \mathbb{T}}(T^p(E), T^p(A)). \]
On the other hand by Theorem 3.9 \( Z(\mathbb{X}) \) is nonempty.

Hence by the Proposition 6.2 \( Z_p(T)(V^d, V_0), ss \) is nonempty.

Let 
\[ Z(T)[\frac{1}{2}] = Z(T) \times_{\text{Spec} \mathcal{O}_k} \text{Spec} \mathcal{O}_k[\frac{1}{2}]. \]

**Theorem 6.4.** For \( T \in \text{Herm}_n(\mathcal{O}_k)_{>0} \), suppose \( \text{Diff}_0(T) \) is empty.

(i) \( Z(T)[\frac{1}{2}] \) is nonempty if and only if there is a self-dual lattice \( \tilde{L}' \) of \( V_T \) such that \( \Omega(T, \tilde{L}') \neq \emptyset \). Furthermore, if \( Z(T)[\frac{1}{2}] \) is non-empty, then for all \( p | \Delta \) such that \( p \neq 2 \), \( Z_p(T) \) is nonempty.

(ii) Assume \( p | \Delta \) and \( p \neq 2 \). Suppose \( \Delta = \epsilon p \) with \( \epsilon \in \mathbb{Z}_p^\times \). Let \( \eta \in W(\mathbb{F}_p) \) such that \( \eta \cdot \eta^p = \epsilon^{-1} \). Choose any \( \delta \in \mathbb{Z}_p^\times \setminus \mathbb{Z}_p^\times \). Then \( Z_p(T) \) is equidimensional of dimension \( \frac{1}{2} t(L_p) \) when it is nonempty. Here \( L_p \) is the rank \( n \) free \( \mathcal{O}_k \)-module whose hermitian inner product is defined by the matrix \( -\epsilon^{-1} \delta^2 T \) and \( t(L_p) \) is defined in Theorem 3.9. In particular, this dimension only depends on \( p \) and \( T \).

**Proof.** Proof of (i)

First recall that by Proposition 5.3 \( Z(T) \) is supported on the super singular locus over the ramified primes and 
\[ Z_p(T)^{(V^d, ss)} = \bigsqcup_{V_0 \in \mathcal{R}(1, 0)_{(k)/\text{str.sim.}}} \bigsqcup_{V \in \mathcal{R}(n-1, 1)_{(k)/\text{str.sim.}}} \bigsqcup_{\text{Hom}_k(V_0, V) = V^{(p)}} Z_p(T)^{(V, V_0), ss}, \]
where \( V^{(p)} \) is described in Proposition 5.3. So if \( Z(T)[\frac{1}{2}] \neq \emptyset \), then \( Z_p(T)^{(V, V_0), ss} \neq \emptyset \) for some \( p | \Delta \) (\( p \neq 2 \)) and \( V_0 \in \mathcal{R}(1, 0)_{(k)/\text{str.sim.}}, V \in \mathcal{R}(n-1, 1)_{(k)/\text{str.sim.}} \) such that \( \text{Hom}_k(V_0, V) = V^{(p)} \).

Corollary 6.1 then implies that there is a lattice \( \tilde{L}' \) of \( V_T = \tilde{V} = \text{Hom}_{\mathcal{O}_k}(E, A) \otimes \mathbb{Q} \) such that \( \Omega(T, \tilde{L}') \neq \emptyset \).

Conversely if there is a self-dual lattice \( \tilde{L}' \) of \( V_T \) such that \( \Omega(T, \tilde{L}') \neq \emptyset \). Fix any prime \( p \) such that \( p | \Delta \), \( p \neq 2 \) and any \( V_0 \in \mathcal{R}(1, 0)_{(k)/\text{str.sim.}}, V \in \mathcal{R}(n-1, 1)_{(k)/\text{str.sim.}} \) such that \( \text{Hom}_k(V_0, V) = V^{(p)} \). Since \( p \neq 2 \), by Proposition 5.1 and its corollary, there is a lattice \( L \) of \( V \) such that 

1. \( L \otimes_{\mathcal{O}_k} L_0 \otimes \mathbb{T} = L \otimes \mathbb{T} \).
2. \( L \) is self-dual.
The $G_V^1(A_f)$-genus $[L]$ of such lattices is unique. By Corollary 6.1 we have
\[ Z_p(T)(V^\sharp, V_0),ss \neq \emptyset. \]

Proof of (ii): By Proposition 6.2 there is an identification of formal schemes
\[ (I^V(Q) \times I^{V_0}(Q)) \backslash \text{Inc}_p(T; V^\sharp, V_0) \sim \hat{Z}_p(T)(V^\sharp, V_0),ss, \]
where
\[ \text{Inc}_p(T; V^\sharp, V_0)(S) = \bigsqcup_{(gK^V,p,g_0K^0_p)} \bigcup Z(x_0^\circ). \]

By Lemma 5.2 and the definition of $x_0^\circ$, we know that the Lattice
\[ (x_0^\circ(1_0), x_0^\circ(1_0)) = T(1_0, 1_0). \]
By the calculation of Remark 2.1 we know that \((1_0, 1_0) = -\epsilon^{-1}\delta^2\). The dimension of $Z_p(T)(V^\sharp, V_0),ss$ then follows from Theorem 3.9.

\[ \square \]

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