26 Results of hyperbolic partial differential equations in B-poly basis

Muhammad I Bhatti and Emilio Hinojosa
University of Texas Rio Grande Valley, Edinburg Texas, 78539, United States of America
E-mail: Muhammad.bhatti@utrgv.edu

Keywords: two variable differential equations, hyperbolic partial differential equations, generalized B-poly, partial differential equations

Abstract
A two-variable process to estimate results of Hyperbolic Partial Differentiation (HPD) equations in a B-Polynomial (B-Poly) bases is established. In the proposed process, a linear product of variable coefficients and B-Polys is manipulated to express the predicted solution of the HPD equation. The variable coefficients of the linear mixture in the results are concluded using Galerkin technique. The HPD equation is converted into a matrix which when inverted provided the unknown coefficients in the linear mixture of the solution. The anticipated solution is constructed from the variable coefficients and B-Poly basis set as a product with initial conditions implemented. Both the effectiveness and precision of the process depend on the number of B-Polys employed in the results and degree of the B-polys utilized in the linear sequence. The current process is applied to solve four linear examples of the HPD equations with various initial conditions. An excellent agreement was found between exact and estimated results and in some cases estimates were exact. The current process renders a higher degree of efficiency and accuracy for solving the HPD equations. The process can be easily employed in other fields such as physics and engineering to solve complex PDEs in two variables.

1. Introduction

For solving very complex problems in physics, engineering and computer science fields, the B-Polys are extremely useful [1, 2]. The B-Polys are smooth functions over an interval providing basis which could represent an arbitrary function to a needed correctness [1–3]. The analytic nature of B-Polys allows perfect integration and differentiation using symbolic programing language, such as Mathematica or Maple. In the prior years, the linear and nonlinear differential equations, as well as fractional-order differential equations, were solved using various procedures involving B-polys [4–12]. Recently, authors [1, 3, 13] have solved differential equations using a mixture of B-Poly basis and operational matrix techniques. The process was employed to solve differential equations in single variable. The present work is broadened to include two independent variables partial differential equations and HPD equations of two variables

In this paper, our aim is to broaden processes for solving hyperbolic partial differential equations in two variables (x, t) using a mixture of B-Poly basis, operational matrix, and Galerkin method. The process takes benefit of the continuous and the unitary property of the B-Polys [1]. The new development is successfully applied to partial differential equations in two variables on a closed interval [0, R]. The B-Polys definite integration matrix elements are converted into a big operational matrix providing flexibility to include initial and boundary conditions for the problems in hand. Note that the set of B-Polys formed a complete basis set of polynomials. The same set of B-Polys has been used to solve the partial differential equations. For example, the desired result of the partial differential equation may be expressed in terms of the B-Poly basis set, i.e.

\[ U(x, t) = \sum_{i=0}^{n} a_i(t) B_{i,n}(x), \]  

where \( a_i(t) \) is the \( i \)th variable coefficient of the linear mixture in equation (1). In equation (1), we impose the initial conditions on variable \( x \). The coefficients \( a_i(t) \) are a function of variable \( t \) and \( B_{i,n}(x) \) is \( n \)th degree polynomial.
B-Poly in variable x. Furthermore, the coefficients $a_i(t)$ can be developed in terms of the constant coefficients $b_i^k$ and the B-Polys as polynomials in t, such as,
\[
a_i(t) = \sum_{k=0}^{n} b_i^k B_{k,n}(t),
\]
over the interval $[0, T]$. In equation (2), initial conditions on variable t can be imposed. We present estimate solutions to several partial differential equations using a complete set of B-Polys of degree n, explaining the procedure and comparing the graphs of the results obtained in the following sections. To avoid repetition, we will provide $[1–3, 13]$ of our earlier work, where we have laid down the groundwork for generalization of a complete set of B-Poly basis sets that are employed to estimate solutions to a variety of differential equations. Graphs of B-Polys were also supplied to show unique characteristics of the B-Poly basis set to be utilized in estimating the solutions $[1, 13–15]$. The paper addresses HPD equation in two variables $(x, t)$. As mentioned above for the application process, we first integrate internal products with respect to x and second, integrate with respect to t to convert HPD equation into operational matrix with nonzero determinant which then inverted to determine the desired solution from equations (1) and (2). The initial conditions can be directly imposed on the both equations to start the process for determining the desired solution of the HPD. For one variable PDE, we have presented error analysis in many published papers $[1–3, 16, 17]$. Therefore, we shall provide converged solutions of the HPDEs with the number of polynomials used in both variables x and t in the following section.

2. Computation of results in B-Poly basis

A process to estimate solutions, $U(x, t)$, of the 2-Dimensional (2D) partial differential equations, considered a linear mixture of B-Polys in the variables x and t with initial condition $U(x, 0) = f(x)$ built-in is given as,
\[
U(x, t) = \sum_{i=0}^{n} a_i(t) B_i(x) + f(x)
\]
With the application of this mixture, equation (3), to a PDE, we convert it into a matrix formation and in terms of variables, $a_i(t)$. For the sake of simplicity, we shall drop subscript $n$ in the following sections. A second estimation to variable coefficients $a_i(t)$ is considered by expanding and imposing initial condition on the equation,
\[
a_i(t) = \sum_{i=0}^{n} b_i^k B_k(t).
\]
We employ Galerkin method $[18]$ and inversion of the operational matrix to solve the 2D partial differential equations. We also provide several examples to show successful applications of the proposed process for solving HPD equations.

Consider a more general 2-dimensional HPDE of the form,
\[
\alpha \frac{dU(x, t)}{dt} + \beta \frac{dU(x, t)}{dx} + \gamma U(x, t) = 0,
\]
with constants $\alpha, \beta, \gamma$, and non-homogenous initial condition $U(x, 0) = f(x)$ at $t = 0$. Let’s substitute the desired solution of the equation (3) into equation (5) which returns,
\[
\sum_{i=0}^{n} \alpha a_i(t) B_i(x) + \sum_{i=0}^{n} \beta a_i B_i'(x) + \sum_{i=0}^{n} \gamma a_i(t) B_i(x) = -\beta f'(x).
\]
Here prime ($'$) and dot ($\cdot$) denote the differentials with respect to x and t, respectively. We may multiply the above equation with another B-Poly from the set and integrate it over the interval $[0, R]$ to generate,
\[
\sum_{i=0}^{n} \left[ \alpha a_i(t) \langle B_i(x)|B_i(x) \rangle + \beta a_i(t) \langle B_i'(x)|B_i(x) \rangle + \gamma a_i(t) \langle B_i(x)|B_i(x) \rangle \right] = -\beta \langle f'(x)|B_i(x) \rangle.
\]
Where matrix elements of the integrals of B-poly products are defined,
\[
M_{ij} = \langle B_i(x)|B_j(x) \rangle = \int_{0}^{R} B_i(x) B_j(x) dx,
N_{ij} = \langle B_i'(x)|B_j(x) \rangle \ and \ F_j = -\langle f'(x)|B_j(x) \rangle,
\]
we obtain the matrix equation,
\[ \sum_{i=0}^{n} [\alpha M_{ij} a_i(t) + \beta N_{ij} a_i(t) + \gamma M_{ij} a_i(t)] = F_j. \] (6)

In the next step, we build \( a_i(t) \) in terms of B-polys as presented in the equation (4), we attain,
\[ \sum_{k=1}^{n} \sum_{i=0}^{n} [\alpha M_{kj} b^k_i \hat{B}_k(t) + \beta N_{kj} b^k_i \hat{B}_k(t) + \gamma M_{kj} b^k_i \hat{B}_k(t)] = F_j. \]

Again, multiplying both sides of the above equation with another \( b^k_l \) from the set, and integrating it in the interval \( t \in [0, T] \), we obtain a simple representation in terms of the matrix elements,
\[ \sum_{k=0}^{n} \sum_{l=0}^{n} b^k_i [\alpha M_{kj} U_{k\ell}(t) + \beta N_{kj} V_{k\ell}(t) + \gamma M_{kj} V_{k\ell}(t)] = W_{k\ell}, l = j, \ldots, n, \] (7)

where the elements of matrices in the above equation (7) are defined by,
\[ U_{k\ell} = \langle \hat{B}_k(t) | B_{\ell}(t) \rangle = \int_{0}^{T} \hat{B}_k(t) B_{\ell}(t) dt, \quad V_{k\ell} = \langle \hat{B}_k(t) | V_{\ell}(t) \rangle, \quad W_{k\ell} = \langle \hat{F}_j | B_{\ell}(t) \rangle. \]

The precise solution of the equation (5) is found, [19],
\[ U(x, t) = g(\alpha x - \beta t) e^{-\gamma x}. \] (8)

The above exact result may be further streamlined subject to the initial conditions. We shall use the Galerkin Technique [18] to estimate the result to the partial differential equations (PDEs) in both variables \( (x, t) \). In the following section, we are going to utilize the proposed process to four examples as illustrations of how it works for assessing solutions of the HPD equations.

**Example 1.** We consider an equation which is attained presuming \( \alpha = 2, \beta = 1, \) and \( \gamma = 0 \) in the equation (5),
\[ 2 \frac{d}{dt} U(x, t) + \frac{d}{dx} U(x, t) = 0. \] (9)

Whose solution we are looking for in the intervals \( 0 \leq x \leq 2 \) and \( 0 \leq t \leq 2 \), with initial condition \( U(x, 0) = f(x) = x^3 \), at \( t = 0 \). An estimate solution to equation (9) may be written via the equation (3),
\[ U(x, t) = \sum_{i=0}^{n} a_i(t) B_i(x) + x^3, \quad n \geq 1. \] (10)

The \( a_i(t) \) is the \( i \)th variable coefficient in the development above equation (10). After substituting equation (10) into the equation (9), we shall come at the equation (6), that would now appear like,
\[ \sum_{i=0}^{n} [2 M_{ij} \hat{a}_i(t) + N_{ij} a_i(t)] = F_j. \] (11)

Again, we make another approximation to the \( a_i(t) \) coefficients using the expansion provided in the equation (4). By replacing the equation (4) into equation (11), with a little bit of oversimplification, and from the variational property with respect to the coefficients, we may attain an expression given in the equation (7) which streamlines to the following expression,
\[ \sum_{k=0}^{n} \sum_{l=0}^{n} b^k_i [2 M_{kj} U_{k\ell}(t) + N_{kj} V_{k\ell}(t)] = W_{k\ell}, \quad j = l, \ldots, n. \] (12)

The matrix elements are clearly provided,
\[ M_{ij} = \int_{0}^{2} B_i(x) B_j(x) dx, \quad N_{ij} = \int_{0}^{2} B_i'(x) B_j(x) dx \]
\[ U_{k\ell} = \int_{0}^{2} \hat{B}_k(t) V_{\ell}(t) dt, \quad V_{k\ell} = \int_{0}^{2} \hat{B}_k(t) V_{\ell}(t) dt \]
\[ W_{k\ell} = F_j \int_{0}^{2} B_{\ell}(t) dt, \quad F_j = - \int_{0}^{2} f'(x) B_j(x) dx \]

This algorithm leads to an \((n + 1)^2\) by \((n + 1)^2\) system of equations \( AB = W \), in the unknown variables \( B_0, B_1, B_2, \ldots, B_n \) where the matrix A is given by,
Clearly, the HPD equation is converted into a large matrix, whose inverse provided specific values of the unknown coefficients $b_i$ of the linear mixture in the equation (4) by solving the $B = A^{-1}W$. Then the estimated solution is constructed from the product of these coefficients $a_i$ and B-Poly basis set, $B_x$ in the equation (10). Before solving the matrix equation $A B = W$, we also employ initial conditions on the matrix by deleting rows and columns of matrices $A$ and $W$ defined in the equation (12). The precise solution for the equation (9) is obtained analytically applying the initial condition via the equation (8) which is given,

$$U_{exact}(x, t) = \left(x - \frac{t}{2}\right)^3, \text{ when } t = x,$$

$$\text{we get } U(x) = \left(\frac{x}{2}\right)^3.$$  (14)

In this example both solutions, the exact solution in equation (14) and the estimate solution after ignoring small terms in equation (15), are the equivalent. To find the numerical solution of the differential equation (9), we only used $n = 3$ degree B-Polynomials. In figure 1, we submit a plot of the absolute difference between estimate and exact solutions supplied in the equations (14) and (15) at $t = x$ and at $t \neq x$. Both solutions are overlapping showing no noticeable differences. In the example 1, the absolute difference is so small that it reaches the order of $10^{-15}$ which is a clue that the estimated solution is in superb agreement with the exact one. This kind of precision was achieved only with $n = 3$ polynomials, see figure 1. Also, In figure 1 on the right, we have provided the 3D graphs of the exact and approximate solutions for comparisons. The analytic solution with $t = x$ is provided below:

$$U(x) = 0.125 \, x^3 \text{ at } t = x,$$  

$$U(x, t) = x^3 + t^2(1.8652 \times 10^{-14} + 0.7500 \times x + 9.9920 \times 10^{-15} x^2 - 1.3323 \times 10^{-15} x^3)$$

$$+ t^3(-0.1250 + 3.0309 \times 10^{-14} x - 7.6605 \times 10^{-15} x^2 - 3.6082 \times 10^{-16} x^3)$$

$$+ t(-1.0658 \times 10^{-14} + 6.3949 \times 10^{-15} x - 1.5000 x^2 + 1.9651 \times 10^{-14} x^3)$$  (15)

Results of different closed intervals for example $x \in [0, 2]$ and $t \in [0, 4]$ provided similar accuracy of the order of $10^{-14}$. The approximate solution to the equation (9) for different intervals is given in below in both variables. The 3D graph is also included in the figure 1 below:
To resolve the differential equation (9), only \( n = 9 \) degree B-Polynomials were used. For simplicity, in the following examples, we will consider the intervals of integration same for both variables \( x \) and \( t \).

Example 2. Let us consider another HPD equation with slightly different initial conditions. We substitute \( \alpha = 1, \beta = 1, \gamma = 0 \), and the initial condition in the general hyperbolic equation (5). So that the HPDE in two dimensions is given,

\[
\frac{2}{t} \frac{d U(x, t)}{dt} + \frac{d U(x, t)}{dx} = 0, \quad t \geq 0
\]

with the initial condition \( U(x, 0) = f(x) = \sin(x) \) at \( t = 0 \). The result of the equation (16) is sought in the intervals \( 0 \leq x \leq 2 \) and \( 0 \leq t \leq 2 \). We shall follow the same process as in Example 1 to determine the resolution of the equation (16) in the preferred regions, with the initial condition at \( t = 0 \). According to the equation (3), the estimated solution is,

\[
U(x, t) = \sum_{i=0}^{n} a_i(t) B_i(t) + \sin(x), \quad n \geq 1
\]

Substituting the equation (17) in terms of the basis set of B-Polys of degree \( n = 9 \), we will reach at the equation (12), as in Example 1, only the elements of column matrix \( W_I \) will change to,

\[
W_{I} = F_j \int_{0}^{2} B_j(t) dt, \quad F_j = - \int_{0}^{2} \cos(x) B_j(x) dx.
\]

The exact solution of the equation (16) is,

\[
U_{\text{exact}}(x, t) = \sin\left(x - \frac{t}{2}\right), \quad \text{when} \ t = x, \ \text{we get} \ U(x) = \sin\left(\frac{x}{2}\right),
\]

which can be directly attained from the equation (8) using \( \alpha = 2, \beta = 1, \gamma = 0 \), and involving the initial condition \( f(x) = \sin(x) \). Also the estimated solution to the equation (16) is provided in equation (19) which is contrasted with the exact solution at \( t = x \). The 3D graph of both solutions (exact and estimated) are also supplied, and the contrast is shown in figure 2 on the right. A narrative of the absolute difference between the estimated and exact solutions at \( t = x \) is exhibited in the figure 2. The precision is of the order of \( 10^{-11} \) with only \( n = 9 \) degree B-Polys.

\[
U(x) = 0.5 x - 0.0208 x^3 + 0.0003 x^5 - 1.5501 \times 10^{-6} x^7 + 5.3823 \times 10^{-9} x^9 \\
- 1.2232 \times 10^{-11} x^{11} + 1.9603 \times 10^{-14} x^{13} - 2.3337 \times 10^{-17} x^{15} \\
+ 2.1450 \times 10^{-20} x^{17} - 1.5680 \times 10^{-23} x^{19} \\
\times t(-0.5 + 0.25x^2 - 0.0208x^4) + t^2(-0.125x) + t^4(0.0208) + \sin(x)
\]

To estimate the result of the differential equation (16), only \( n = 9 \) degree B-Polynomials were utilized. The full solution in terms of both variables \( x \) and \( t \) is provided below:
$U(x, t) = t^6(-9.5824 \times 10^{-9} + 4.4224 \times 10^{-8}x) + t^4(-4.5121 \times 10^{-9} - 0.125x
- 1.4593 \times 10^{-6}x^2 + 0.0208x^3 - 1.4117 \times 10^{-3}x^4 - 0.0010x^5
- 1.6643 \times 10^{-2}x^6 + 3.3430 \times 10^{-2}x^7 - 2.5473 \times 10^{-6}x^8)
+ t^3(1.1863 \times 10^{-9} + 1.3059 \times 10^{-7}x - 3.9801 \times 10^{-4}x^2)
+ t^2(0.0208 + 4.8643 \times 10^{-7}x - 0.0104x^2 + 9.4115 \times 10^{-6}x^3
+ 8.5177 \times 10^{-4}x^4 + 1.6644 \times 10^{-3}x^5 - 3.9001 \times 10^{-2}x^6
+ 3.3964 \times 10^{-6}x^7) + t(1.9497 \times 10^{-6} + 1.4860 \times 10^{-2}x
- 0.0447 \times 10^{-6}x^2 + 2.1227 \times 10^{-2}x^3 + 1.3059 \times 10^{-7}x^4
+ 0.0026 \times 3.5293 \times 10^{-7}x^2 - 4.2588 \times 10^{-10}x^3
- 1.0402 \times 10^{-2}x^4 + 2.9251 \times 10^{-3}x^5 - 2.9718 \times 10^{-6}x^6
+ 4.8752 \times 10^{-6}x^7 - 7.4926 \times 10^{-7}x^8 + 2.1294 \times 10^{-4}x^9
+ 7.0586 \times 10^{-7}x + 1.2774 \times 10^{-4}x^2 + 4.1609 \times 10^{-6}x^3
- 1.4626 \times 10^{-2}x^4 + 1.7831 \times 10^{-3}x^5 - 0.0500 \times 2.4115 \times 10^{-8}x
+ 0.2500 \times 1.9457 \times 10^{-7}x^2 + 0.0208 \times 1.1294 \times 10^{-10}x^3
+ 6.8142 \times 10^{-4}x^6 + 9.5106 \times 10^{-6}x^7 - 1.6715 \times 10^{-2}x^8
+ 1.1321 \times 10^{-6}x^9) + \sin(x)$

**Example 3.** Consider another more complicated HPD equation to show that the proposed process is extremely beneficial to estimating a solution to an anticipated accuracy. The hyperbolic differential equation is,

$$2 \frac{d}{dt} U(x, t) + 2 \frac{d}{dx} U(x, t) + U(x, t) = 0,$$  \hspace{1cm} (20)

with the initial condition $U(x, 0) = \sin(x)$ at $t = 0$. The equation (20) has an exact answer under the above initial condition which we achieved after substituting $\alpha = 3$, $\beta = 2$, and $\gamma = 1$ in the equation (8), $U(x, t) = \sin(x - \frac{2}{3}t)e^{-\frac{t}{3}}$. We utilized a basis set of 9 B-Polys to estimate the solution of the equation (20) and contrasted it with the exact result of this equation. The same process was used to seek the estimated solution as given in the Examples 1 and 2. The procedure provides matrix elements of the equation $A \cdot B = W$ as follows,

$$\sum_{k=0}^{n} \sum_{i=0}^{m} b_k[M_{ij}U_{ij} + 2N_{ij}V_{ij} + M_{ij}V_{ij}] = W_{ij}. \hspace{1cm} (21)$$

The matrix elements of equation (21) are defined in the equation (7). The only change in the matrix $F_j$ is,

$$F_j = -\int_0^2 \cos(x)B_j(x) \, dx.$$

We note that in the above examples considered, the inverse of matrix A in the equation $A \cdot B = W$ is acquired after imposing the initial condition on the matrix A elements at $t = 0$ and $x = 0$. Finally, the equation $A \cdot B = W$ is resolved for the unknown coefficients to construct the anticipated solution of the differential equation (20). The final estimated solution to the equation (20) is provided in equation (22) over the intervals $0 \leq x \leq 2$ and $0 \leq t \leq 2$. Both the exact and the estimated results of the partial differential equation (20) were equated. The absolute difference between exact and estimated analytic solution of equation (20) over the ranges is exhibited in figure 3. The absolute discrepancy between the solutions is of the order of $10^{-11}$ with the usage of only $n = 9$ degree B-Poly basis. However, the anticipated accuracy of the numeric solution of the differential equation depends on the size of the basis set chosen and the degree of the polynomials. The larger the basis set, the better is the accuracy of the estimated solution. However, the drawback is that the larger the size of the matrix, the larger is the CPU time required to invert the matrix. The estimated solution of the equation (20) for $t = x$ is shown employing the set of $n = 9$ degree polynomials in the equation (22). We also present a 3D graph of the solution in the figure 3 on the right, which are overlapping.

$$U(x) = -0.6667x - 0.1111x^2 + 0.1790x^3 + 2.2030 \times 10^{-6}x^4 - 8.4753 \times 10^{-3}x^5
+ 2.1878 \times 10^{-5}x^6 + 1.9175 \times 10^{-4}x^7 + 3.4687 \times 10^{-6}x^8
- 4.0656 \times 10^{-6}x^9 + 3.1480 \times 10^{-7}x^{10} - 2.3201 \times 10^{-8}x^{11}
+ 4.8104 \times 10^{-9}x^{12} - 1.8274 \times 10^{-10}x^{13} - 8.2700 \times 10^{-11}x^{14}
- 1.9815 \times 10^{-13}x^{15} + 5.0809 \times 10^{-15}x^{16} + 2.3476 \times 10^{-14}x^{17}
+ 3.0402 \times 10^{-16}x^{18} + \sin(x) \hspace{1cm} (22)$$
To solve the differential equation (20), only \( n = 9 \) degree B-Polynomials were exploited. The full approximate solution in terms of both variables \( x \) and \( t \) is provided here.

\[
U(x, t) = t (-0.6667 - 0.3333 x + 0.3333 x^2 + 0.0556x^3 - 0.0278x^4 + 0.0028 x^5 + 0.0013x^6 - 0.00035x^7 + 5.1675 \times 10^{-5}x^8 + 1.0422 \times 10^{-5}x^9 - 1.7081 \times 10^{-7}x^{10}) + t^3(-0.0123 - 0.0036x + 0.0062x^2 + 6.3164 \times 10^{-4}x^3 - 5.4982 \times 10^{-5}x^4 - 9.4638 \times 10^{-5}x^5 + 1.2332 \times 10^{-5}x^6 + 3.3176 \times 10^{-7}x^7 - 4.7288 \times 10^{-7}x^8 - 1.5816 \times 10^{-9}x^9 + t^5(8.49109 \times 10^{-3} + 2.2629 \times 10^{-4}x - 4.7331 \times 10^{-5}x^2 - 3.3259 \times 10^{-5}x^3 + 2.7273 \times 10^{-6}x^4 + 1.269 \times 10^{-6}x^5 + 4.1096 \times 10^{-8}x^6 - 6.0505 \times 10^{-9}x^7 - 3.8576 \times 10^{-10}x^8 - 5.8558 \times 10^{-12}x^9) + t^7(1.64103 \times 10^{-6} - 1.6285 \times 10^{-6}x - 7.6756 \times 10^{-7}x^2 + 1.6234 \times 10^{-7}x^3 + 6.1308 \times 10^{-8}x^4 + 2.8908 \times 10^{-8}x^5 - 3.5597 \times 10^{-10}x^6 - 3.6431 \times 10^{-11}x^7 - 1.1130 \times 10^{-12}x^8 - 1.1089 \times 10^{-14}x^9) + t^9(-1.4845 \times 10^{-8} + 1.1387 \times 10^{-7}x + 1.4398 \times 10^{-8}x^2 - 1.28718 \times 10^{-8}x^3 - 2.41918 \times 10^{-9}x^4 - 2.0921 \times 10^{-11}x^5 + 1.8858 \times 10^{-11}x^6 - 1.3434 \times 10^{-12}x^7 + 3.4565 \times 10^{-14}x^8 + 3.0402 \times 10^{-16}x^9) + t^{11}(-2.6438 \times 10^{-5} - 2.4396 \times 10^{-5}x + 1.2006 \times 10^{-5}x^2 + 6.7251 \times 10^{-7}x^3 - 7.2374 \times 10^{-7}x^4 - 8.4726 \times 10^{-8}x^5 + 1.9274 \times 10^{-9}x^6 + 5.7452 \times 10^{-10}x^7 + 2.3230 \times 10^{-11}x^8 + 2.7765 \times 10^{-13}x^9) + t^{13}(0.0013 - 0.0014x - 6.6280 \times 10^{-4}x^2 + 2.5132 \times 10^{-4}x^3 + 4.2979 \times 10^{-5}x^4 - 9.2357 \times 10^{-6}x^5 - 1.1320 \times 10^{-6}x^6 + 2.8312 \times 10^{-8}x^7 + 5.0502 \times 10^{-9}x^8 + 1.0544 \times 10^{-10}x^9) + t^{15}(0.01235 + 0.0679x - 0.0062x^2 - 0.0113x^3 + 4.6301 \times 10^{-4}x^4 + 6.1347 \times 10^{-4}x^5 - 4.16579 \times 10^{-5}x^6 - 7.5944 \times 10^{-6}x^7 + 2.258 \times 10^{-7}x^8 + 1.8979 \times 10^{-8}x^9) + \sin(x)
\]

Example 4. The final example we consider is that of a hyperbolic differential equation,

\[
3 \frac{d^2 U(x, t)}{dt^2} + 2 \frac{d U(x, t)}{dx} + 8 U(x, t) = 0,
\]

with initial condition \( U(x, 0) = \cos(x) \). The equation (23) is obtained by exchange of \( \alpha = 3, \beta = 2 \), and \( \gamma = 8 \) in the general equation (5). The precise solution subjected to the initial condition \( U(x, 0) = \cos(x) \) is,

\[
U(x, t) = \cos(3x - 2t) e^{-4t},
\]

Expanding the solution of the equation (23) in the estimated form via the equation (3),

\[
U(x, t) = \sum_{i=0}^{n} a_i(t)B_i(x) + \cos(x).
\]
Substituting equation (25) into the equation (23) generates an equation of the form of the equation (6). Furthermore, the coefficients \( a_i(t) \) in the equation (6) are developed in terms of B-Polys as a function of \( t \) in order to determine the coefficients of the equation (4). This process leads to an articulation provided in the equation (21). All the matrix elements remain the same as in the Example 3, except the elements of \( W_f \) which are furnished in the matrix,

\[
W_f = F \int_{0}^{1} B_f(x) \, dx, \quad \text{and where} \quad F = \int_{0}^{1} \sin(x)B_f(x) \, dx.
\] (26)

The coefficients in the equation (25) were attained by employing the Galerkin method [18] and inverting the matrix \( A \). The values of these coefficients were manipulated to determine the coefficients \( a_i(t) \) in terms of the B-Polys as a function of \( t \) and the final result to the equation (23) was analyzed from the equation (25). Only \( n = 9 \), B-Polys of degree 9 were used to estimate the solution of the equation (25). The errors were of the order of \( 10^{-12} \) between the exact and the estimated solutions of the equation (23).

In figure 4 on left, a narrative of the absolute discrepancy between the estimated and the exact results is shown when \( t = x \) is substituted in the solutions. The absolute error, the difference between the two solutions improved as the number of B-Polys were increased from 2 through 9-degree polynomials, systematically. A 3D graph is also supplied in the figure 4 on the right to show the comparison of both solutions. Again, in this example we have shown the validity of the process employed to the two-dimensional HPD equations. The estimated solution of the equation (23) for \( t = x \) is submitted using the set of \( n = 9 \) degree polynomials in the equation (27).

\[
U(x) = -0.3333 x + 0.5000 x^2 + 0.0123 x^3 - 0.0437 x^4 + 0.00014 x^5 + 0.001 x^6 + 3.4590 \times 10^{-7} x^7 - 2.5591 \times 10^{-5} x^8 + 4.2254 \times 10^{-7} x^9 + 1.5033 \times 10^{-7} x^{10} + 1.3902 \times 10^{-8} x^{11} + 1.3342 \times 10^{-9} x^{12} - 1.1149 \times 10^{-9} x^{13} - 8.4884 \times 10^{-13} x^{14} + 2.0058 \times 10^{-11} x^{15} + 1.6700 \times 10^{-12} x^{16} + 4.9942 \times 10^{-12} x^{17} + 5.114 \times 10^{-16} x^{18} + \cos(x)
\] (27)

To solve the differential equation (23), only \( n = 9 \) degree B-Polynomials were employed. The full estimated solution in terms of both variables \( x \) and \( t \) is provided here.

\[
U(x, t) = t(-0.3333 + 0.6666 x + 0.1667 x^2 - 0.1111 x^3 - 0.0139 x^4 + 0.0055 x^5 + 4.7358 \times 10^{-4} x^6 - 1.3862 \times 10^{-4} x^7 - 6.3152 \times 10^{-6} x^8 + 1.7240 \times 10^{-6} x^9 + t(-0.1667 - 0.2222 x + 0.0833 x^2 + 0.0370 x^3 - 0.00693 x^4 - 0.00183 x^5 - 0.00123 x^6 - 0.00083 x^7 - 0.00050 x^8 - 0.00021 x^9 + 1.3895 \times 10^{-6} x^{10} + 9.9155 \times 10^{-7} x^{11} + 4.0766 \times 10^{-6} x^{12} - 1.6169 \times 10^{-6} x^{13} - 1.3392 \times 10^{-7} x^{14} - 2.6605 \times 10^{-8} x^{15} + t^2(2.2406 - 5 \times 8.6208 \times 10^{-5} x^{16} - 1.1067 \times 10^{-5} x^{17} + 1.3628 \times 10^{-5} x^{18} + 8.7271 \times 10^{-6} x^{19} - 2.6396 \times 10^{-7} x^{20} - 2.6283 \times 10^{-7} x^{21} - 2.4669 \times 10^{-8} x^{22} + 8.5023 \times 10^{-10} x^{23} - 9.8505 \times 10^{-12} x^{24}) + t^3(-1.7547 \times 10^{-6} + 1.7167 \times 10^{-6} x + 7.6125 \times 10^{-7} x^2 + 3.1042 \times 10^{-7} x^3 - 3.1053 \times 10^{-8} x^4 - 1.7725 \times 10^{-8} x^5 - 2.0325 \times 10^{-9} x^6 + 1.0041 \times 10^{-10} x^7 - 2.2555 \times 10^{-12} x^8 - 1.8652 \times 10^{-14} x^9 + t^4(1.2966 \times 10^{-7} + 2.4218 \times 10^{-8} x - 5.8362 \times 10^{-9} x^2 - 8.7093 \times 10^{-9} x^3 + 3.3734 \times 10^{-9} x^4 + 9.0003 \times 10^{-10} x^5 + 8.1697 \times 10^{-11} x^6 + 3.4564 \times 10^{-12} x^7 + 6.8594 \times 10^{-14} x^8 + 5.1139 \times 10^{-16} x^9) + t^5(-3.1649 \times 10^{-9} + 2.6104 \times 10^{-9} x + 1.8144 \times 10^{-9} x^2 - 4.0460 \times 10^{-6} x^3 - 3.3892 \times 10^{-7} x^4 + 1.5531 \times 10^{-7} x^5 + 2.8446 \times 10^{-8} x^6 + 1.8045 \times 10^{-9} x^7 + 4.8622 \times 10^{-11} x^8 + 4.6704 \times 10^{-13} x^9) + t^6(-0.00141 - 0.00130 x + 0.00708 x^2 - 5.6912 \times 10^{-5} x^3 - 9.8825 \times 10^{-5} x^4 + 1.2366 \times 10^{-5} x^5 + 2.5062 \times 10^{-6} x^6 + 1.2120 \times 10^{-6} x^7 + 1.7756 \times 10^{-10} x^8) + t^7(0.0679 - 0.0124 x - 0.0339 x^2 + 0.0020 x^3 + 0.00285 x^4 - 0.000113 x^5 + 8.9306 \times 10^{-5} x^6 + 2.8852 \times 10^{-6} x^7 + 1.0324 \times 10^{-6} x^8 + 3.1926 \times 10^{-8} x^9) + \cos(x)
\]

We present error analysis for the numerical and analytical results of the HPD equation. For a typical calculation, the desired accuracy, of the results, depends on the number of B-polys to form our trial functions for the approximate solution obtained by using Galerkin method. We would like to show convergence of results as well as absolute error for example 4 presented above. Similar error analysis can be repeated for the other examples presented in the paper. The absolute error between the approximate and the exact solution are presented in 4 graphs for \( n = 3, 6, 9, \) and 12 in figures 5–8, respectively. We are going to show that as the B-polys
are increased systematically in the calculations, the absolute error decreases and our approximate solution to the HPD equation gets closer to the exact solution as mentioned in [1, 2, 17]. Normally, the results obtained by using the procedure presented in this paper showed that the desired agreement with exact solutions started to converge when \( n = 9 \) or higher. However, the accuracy of the solutions keeps improving beyond \( n = 9 \) degree of B-polys on the intervals of integration. The calculation does not require any grid over the closed intervals chosen. In figure 5, we used number of B-polys \( n = 3 \), the calculated absolute error is of the order of \( 1 \times 10^{-2} \). In the graph of figure 6, we set \( n = 6 \), the absolute error shown is of the order of \( 1 \times 10^{-4} \). Similarly, as we increase the number of B-polys to \( n = 9 \), the absolute error decreases to \( 6 \times 10^{-8} \) as shown in the 3D graph of figure 7. To show that the method provided higher accuracy for \( n = 12 \), we have plotted absolute error as function of both variables \((x, t)\) as shown in figure 8 and the final error is of the order of \( 10^{-11} \). This clearly indicates that method is providing converged results after appropriate number of B-polys. However, there is a price to pay for higher degree of polynomials to be included that is the CPU time for the calculations go higher as the size of the matrix to be inverted increased significantly.

3. Results and discussions

In this article, we have given a broad description of the 2D algorithm to show how B-Polynomial basis may be employed to provide highly accurate results of the 2D Hyperbolic Partial Differential Equations (HPDEs). To describe the application of this process, we laid down a 2D groundwork and utilized it to solve 4 examples of the HPD equations with various initial conditions enforced on the results. In each of the four examples worked out, we were capable to contrast the exact and the estimated solutions achieved using the Galerkin method [18] in two variables \((x, t)\), and an agreement was discovered to desired accuracy of \( 10^{-11} \) as exhibited in figures 1–4. We have noticed that by increasing the number of B-Polys in the estimated solutions, increase the precision of the results [1]. All computations and analytic integrations over the intervals were conducted utilizing Wolfram Mathematica’s symbolic program version 11 [20]. Comparisons between the exact and the estimated solutions

---

**Figure 4.** A narrative of absolute difference between exact and estimated solutions is depicted on the left for \( t = x \). A 3D graph \( U(x, t) \) of both exact and estimated solutions is given on the right.

**Figure 5.** Plot of the absolute error between approximate and exact solutions of example 4 with \( n = 3 \) B-polys.
were shown in the 2D and 3D graphs in figures 1–4. In each case the precision of the results was increased, with boosting the number of B-polys in the basis set. Furthermore, errors were also contrasted with the exact results of the PDEs when parameter (t) was set equal to x and precision was shown to better than the order of $10^{-11}$ for all examples considered.

The current process, which utilizes the continuous B-Polynomials, may offer great potential for solving linear and nonlinear examples of 2D problems in other disciplines, particularly in physics. More recently, several authors [16, 17, 20–22] have used B-polys techniques to construct operational matrix to solve variety of differential equations in one dimensional variable. We have, for the first time, successfully extended the technique to solve HPDEs in two variables. We had already published results of KdV equation and nonlinear
Burger equations in a B-polynomial basis. Our method worked very well for solving both equations using operational matrix scheme [16, 17]. The only difference is that we had discretized the time variable using fourth order Runge-Kutta [17] method while spatial variable was expanded in terms of B-polynomial basis. First, we would like to have present valuable procedure published. Our plan is to publish separately some examples of nonlinear PDEs such as Burger equation by implementing the current method. All calculations were performed using symbolic Mathematica Code [20]. Typically, the CPU time used to perform all the calculations for each example is about 32 s, except example 1, CPU time used was about 2.5.

In this article, we have shown that the process, based on the B-Poly technique, will give appropriate results for 2D partial differential equations after the initial conditions are enforced on the operational matrix. It is a powerful tool that we may utilize to surmount the difficulties associated with complex system of differential equations where there are no exact solutions available, particularly in two variable differential equations. It has also established to be effective in producing precise results and could be quickly executed in various disciplines.

ORCID iDs
Muhammad I Bhatti © https://orcid.org/0000-0002-0162-2624

References
[1] Bhatti M and Bracken P 2007 Solution of differential equations in a Bernstein polynomial basis J. Comp. and Applied Math 205 272–80
[2] Bhatti M I 2014 Solution of Fractional Harmonic Oscillator in a Fractional B-poly Basis 28–13
[3] Bhatti M, Bracken P, Dimakis N and Herrera A 2018 Solution of mathematical model for gas solubility using fractional-order bhatti polynomials J. Phys. Commun. 2 085013
[4] Gejji V D and Jafari H 2007 Appl. Math. Comput. 189 541–8
[5] Sweilam N H, Khader M M and Al-Bar R F 2007 Phys. Lett. A 371 26–33
[6] Hashim I, Abdulaziz O and Momani S 2009 Commun. Nonlinear Sci. Numer. Simulat. 14 674–84
[7] Isik R, Güney Z and Sezer M 2012 J. Differ. Equ. Appl. 18 357–74
[8] Saadatmandi A and Dehghan M 2010 Compt. Math. Appl. 59 1326–36
[9] Yüzbası S and Math A 2013 Comput. 219 6328–43
[10] Yousefi S A and Behrozifar M 2010 Operational matrices of Bernstein polynomials and their applications Int. J. Syst. Sci. 41 709–16
[11] Caputo M 1967 J. Ry. Aust. Soc. 13 529–39
[12] Polubny I 1999 Fractional Differential Equations (New York: Academic)
[13] Bhatti M I and Studies Theor A 2009 Phys. 3 451–60
[14] Debnath L 2003 Int. J. Theor. Appl. 6 119–55
[15] Erdelyi A et al 1955 Higher Transcendental Functions vol 3 (New York: McGraw-Hill)
[16] Bhatta D D and Bhatti M I 2006 Numerical solution of KdV equation using modified Bernstein polynomials Appl. Math. Comput. 174 1255–68
[17] Bhatti M I and Bhatta D D 2006 Numerical solutions of Burgers’ equation in a B-polynomial basis Phys. Scr. 73 539–44
[18] Fletcher W A 1984 Computational Galerkin Methods (New York, NY: Springer)
[19] Alinhac S 2009 Hyperbolic Partial Differential Equations (Berlin: Springer)
[20] Wolfram Research Inc. Mathematica (Champaign, IL, USA: 100 Trade Center Drive)
[21] Singh A K, Singh V K and Singh O P 2009 The Bernstein operational matrix of integration Appl. Math. Sci. 3 2427–36
[22] Farouki R T 2000 Legendre-Bernstein basis transformations J. Comput. Appl. Math. 119 143–60