The Failure of the Ergodic Assumption

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The well established procedure of constructing phenomenological ensemble from a single long time series is investigated. It is determined that a time series generated by a simple Uhlenbeck-Ornstein Langevin equation is mean ergodic. However the probability ensemble average yields a variance that is different from that determined using the phenomenological ensemble (time average). We conclude that the latter ensemble is often neither stationary nor ergodic and consequently the probability ensemble averages can misrepresent the underlying dynamic process.

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Let us consider the OU stochastic process about which we know everything so as to test the ergodic assumption. Consider the OU Langevin equation
\[
\frac{dX(t)}{dt} = -\lambda X(t) + F(t)
\tag{1}
\]
where in a physics context \(X(t)\) is the one-dimensional velocity, the particle mass is set to one, \(\lambda\) is the dissipation parameter and \(F(t)\) is a random force. In other contexts the dynamic variable has been taken to be the voltage measured in an EEG record \cite{Uhlenbeck1945}, or any of a number of other interesting observables. The solution to the OU Langevin equation, as we show below, actually violates the stationarity assumption made by Wang and Uhlenbeck in 1945. To establish this violation we calculate the variance using the phenomenological ensemble averages constructed from a single historical trajectory of long but finite length \(T\) satisfies \(\langle X(t) \rangle\) for any \(g(x)\) other than linear as explicitly assumed in WU. It is probably obvious but it should be pointed out that first-order ergodicity is important in statistical physics because it guarantees that the microcanonical ensemble of Gibbs produces the correct statistical average in phase space for Hamiltonian systems \cite{Uhlenbeck1945}.

To answer the question regarding the validity of \(\langle X(t) \rangle\) for quantities higher than first-order consider the solution to the OU Langevin equation, which since the dynamic equation is linear has the same statistics as the random force. Assuming the random force is a zero-centered delta correlated Gaussian process, we know that the solution \(\langle X(t) \rangle\) is completely determined by the mean and variance. The solution with \(\langle X(t) \rangle = 0\) yields the average values \(\langle X^2(t) \rangle\) as \(t \to \infty\) is satisfied and consequently the solution to the OU Langevin equation is first-order ergodic.

Now let us consider the variance using the MTE averages consisting of an infinite number of trajectories all starting from \(X(0) = 0\) (without loss of generality) yielding
\[
\sigma^2(t) \equiv \langle X^2(t) \rangle_{\text{MTE}} - \langle X(t) \rangle_{\text{MTE}}^2 = \frac{\sigma_0^2}{2\lambda} [1 - e^{-2\lambda t}].
\tag{6}
\]
Note that \(\sigma^2(t)\) is also the variance obtained using the Gaussian solution to the Fokker-Planck equation for the ensemble probability distribution.

Now let us examine the single trajectory case. To create an ensemble of trajectories consider different portions of the single trajectory. Let \(X(t)\) \((X(0)=0)\) be a single trajectory of maximum finite length \(T\) and consider the set of trajectories for \(t \in [0, T - \tau]\)

\[
Z(t, \tau) = X(t + \tau) - X(t).
\tag{7}
\]
Using (2), we can write $X(t + \tau)$ as

$$X(t + \tau) = e^{-\lambda \tau} \left[ X(t) + \int_0^\tau F(t + t')e^{\lambda t'} dt' \right]$$

(8)

where the initial condition for the trajectory at $t + \tau$ is the time series at time $t$. Inserting (8) into (7), we obtain for $t \in [0, T - \tau]$

$$Z(t, \tau) = e^{-\lambda \tau} \int_0^\tau F(t + t')e^{\lambda t'} dt' + X(t) \left[ e^{-\lambda \tau} - 1 \right]$$

(9)

Note that the first term of the rhs of (9) is the solution of the OU Langevin equation (1) with $t = \tau$ and $X(0) = 0$, apart from the fact that in the integrand of (9) only the time in the range $[t, t + \tau]$ are considered for the function $F$ (instead of the range $(0, \tau)$ as in (4)). We now define the solution starting from zero to be

$$X_0(t, \tau) = e^{-\lambda \tau} \int_0^\tau F(t + t')e^{\lambda t'} dt'$$

(10)

so that the term $X(t)$ in (9) is equivalent to $X_0(0, t)$, and therefore with $t \in [0, T - \tau]$ (9) reduces to

$$Z(t, \tau) = X_0(t, \tau) + X_0(0, t) \left[ e^{-\lambda \tau} - 1 \right]$$

(11)

This equation reveals that the statistics of the variable $Z(t, \tau)$ depends on the particular value of $t$ through the second term on the rhs. In fact the term $X_0(0, t)$ has zero mean and variance that depends on $t$ according to (6). Also notice that the first term of the above equation depends on $t$ but the "dependence" is a time translation of the variable $F$ in the integrand of (10). But the random force $F$ is stationary by assumption and thus $X_0(t, \tau)$ is statistically equivalent to $X_0(0, \tau)$. The function $Z(t, \tau)$ is the sum of two statistically independent Gaussian variables and therefore is itself a Gaussian variable with $\langle Z(t, \tau) \rangle_T = \langle Z(t, \tau) \rangle_{MTE} = 0$ and second-moment

$$\langle Z^2(t, \tau) \rangle_{MTE} = \sigma^2(\tau) + \left[ e^{-\lambda \tau} - 1 \right]^2 \sigma^2(t).$$

(12)

We use this variance to demonstrate that the phenomenological ensemble created is not equivalent to MTE because the trajectories $Z(t, \tau)$ are not statistically equivalent to those of $X(t)$. We show that $\sigma^2_{\text{straj}}$, the variance calculated with the ensemble $Z(t, \tau)$, is different from that calculated using the MTE (13).

To calculate the variance $\sigma^2_{\text{straj}}(\tau)$ using the phenomenological ensemble, we simply need to calculate the mean value (among the trajectories) using (12). Here the variance is calculated using the time average since only the time series is assumed to be available to us

$$\sigma^2_{\text{straj}}(\tau) = \frac{1}{T - \tau} \int_0^{T - \tau} dt \langle Z^2(t, \tau) \rangle_{MTE}.$$ 

(13)

Inserting the mean of the phenomenological second moment into the integrand yields

$$\sigma^2_{\text{straj}}(\tau) = \frac{1}{T - \tau} \int_0^{T - \tau} dt \left\{ \sigma^2(\tau) + \left[ e^{-\lambda \tau} - 1 \right]^2 \sigma^2(t) \right\}$$

(14)

and using (10) allows to write after integrating over time

$$\sigma^2_{\text{straj}}(\tau) = \sigma^2(\tau) + \left[ e^{-\lambda \tau} - 1 \right]^2 \times \left[ \frac{\sigma^2}{2\lambda} \right] \left[ 1 + \frac{1}{T - \tau} e^{-2\lambda(T - \tau)} - 1 \right]$$

(15)

Since $\lambda T \gg 1$ (because $T$ is the length of the record) we can write

$$\sigma^2_{\text{straj}}(\tau) \approx \sigma^2(\tau) + \left[ e^{-\lambda \tau} - 1 \right]^2 \left[ \frac{\sigma^2}{2\lambda} \right]$$

(16)

and substitution from (9) yields

$$\sigma^2_{\text{straj}}(\tau) \approx \frac{\sigma^2}{\lambda} (1 - e^{-\lambda \tau}).$$

(17)

The last expression shows that the phenomenological ensemble trajectories $Z(t, \tau)$ behave as a MTE with an effective dissipation that is half the dissipation rate in the Langevin equation generating the process, that is,

$$\lambda_{\text{eff}} = \lambda/2.$$ 

(18)

Consequently, it takes half as long for the MTE variance to decay as it does for the phenomenological variance of the time series.
Consider what has been established in this Letter. First a long time series is generated numerically from the U) Langevin equation. This long series is separated into a large number of equal length time series as prescribed by Wang and Uhlenbeck [11] to form a phenomenological ensemble of realizations of the UO process. This phenomenological ensemble of trajectories is shown to be mean ergodic. However when they are used to calculate the variance of the process anticipating that the same result will be obtained as that for a MTE average, the probability ensemble average, of the variance. The expected result turns out to be wrong. The dissipation parameter determined by the MTE variance is twice that obtained using the phenomenological ensemble. The inescapable conclusion is that the phenomenological prescription for calculating the average of any nonlinear function of the dynamic variable using time series is not equivalent to that using a MTE. The fact that \( g(x) \neq x \) implies that the ergodic assumption is violated by any finite time phenomenological ensemble. Given the simplicity of the dynamic model discussed here; it is mean ergodic; combined with the recent findings concerning the non-ergodicity of experimental data mentioned earlier, it is not unreasonable to conclude that the application of the ergodic assumption in general is probably unwarranted.

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