Families of extensions of the Kantorovich-Rubinstein and Lipschitz norms

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Abstract
We propose a family of extensions of the Kantorovich-Rubinstein norm from the space of zero-charge countably additive measures on a compact metric space to the space of all countably additive measures, and a family of extensions of the Lipschitz norm from the quotient space of Lipschitz functions on a compact metric space to the space of all Lipschitz functions. These families are parameterized by $p, q \in [1, \infty]$, and if $p, q$ are Hölder conjugates, then the dual of the resulting $p$-Kantorovich space is isometrically isomorphic to the resulting $q$-Lipschitz space.

1 Introduction

Given a compact metric space $(X, d)$, the vector space $\mathcal{M}(X, 0)$ of countably additive measures $\mu$ on the Borel $\sigma$-algebra of $X$ such that $\mu(X) = 0$ can be normed by the Kantorovich-Rubinstein norm

$$\|\mu\|_{KR} = \inf_{\pi \in \mathcal{M}(X \times X) : \pi(X \times \cdot) - \pi(\cdot \times X) = \mu} \left\{ \int d(x, y) d\pi(x, y) \right\},$$

the theory of which was developed in [Kantorovich and Rubinstein, 1957] and [Kantorovich and Rubinstein, 1958]. See [Kantorovich and Akilov, 1982, Section VIII.4] or [Cobzas et al., 2019, Section 8.4] for summaries. The topological dual space can be identified with the quotient space of Lipschitz functions with respect to constant functions, or equivalently the space $\text{Lip}(X, x_0)$ of Lipschitz functions vanishing at an arbitrary base point $x_0 = X$, equipped with the Lipschitz norm

$$\|f\|_L = \sup_{x, y \in X, x \neq y} \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \right\},$$

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We propose to extend the Kantorovich-Rubinstein norm to the space \( \mathcal{M}(X) \) of all countably additive measures on \( X \) as

\[
\|\mu\|_{pK} = \inf_{\xi \in \mathcal{M}(X,0)} \left\{ (\|\xi\|^p_{KR} + \|\mu - \xi\|^p_{TV})^{\frac{1}{p}} \right\} \tag{1}
\]

with \( p \in [1, \infty] \) and \( \|\cdot\|_{TV} \) being the total variation norm, and to extend the Lipschitz norm to the space \( \text{Lip}(X) \) of all Lipschitz functions on \( X \) as

\[
\|f\|_{qL} = (\|f\|^q_L + \|f\|^q_\infty)^{\frac{1}{q}} \tag{2}
\]

with \( q \in [1, \infty] \) and \( \|\cdot\|_\infty \) being the sup norm. The limiting cases \( p, q = \infty \) are interpreted as usual. In the following, we will prove some properties of the resulting \( p \)-Kantorovich \( (\mathcal{M}(X), \|\cdot\|_{pK}) \) and \( q \)-Lipschitz \( (\text{Lip}(X), \|\cdot\|_{qL}) \) spaces, including that if \( p, q \) are Hölder conjugates, i.e. \( \frac{1}{p} + \frac{1}{q} = 1 \), then the topological dual of the \( p \)-Kantorovich space can be identified with the \( q \)-Lipschitz space.

The theory of the \( p = 1 \) and \( q = \infty \) case was developed by L. G. Hanin in a series of papers \cite{Hanin1992, Hanin1994, Hanin1997, Hanin1999}. See \cite{Cobzas2019} Section 8.5 for a summary. The case \( p = \infty \) and \( q = 1 \) was proposed in dual form in \cite{Chitescu2014}. Equivalence to the primal form was shown in \cite{Terjek2021}. For \( \mu, \nu \) probability measures, \( \|\mu - \nu\|_{\infty,K} \) is also known as the Fortet-Mourier distance.

\section{\( p \)-Kantorovich and \( q \)-Lipschitz norms}

The following propositions show that (1) and (2) define families of equivalent norms, with the spaces \( (\mathcal{M}(X), \|\cdot\|_{pK}) \) and \( (\text{Lip}(X), \|\cdot\|_{qL}) \) being incomplete and complete, respectively, while the pointwise product turns \( q \)-Lipschitz spaces into Gelfand algebras.

**Proposition 1.** Given Hölder conjugates \( p, q \in [1, \infty] \), for any \( \mu \in \mathcal{M}(X) \) and \( f \in \text{Lip}(X) \), one has

\[
\left| \int f d\mu \right| \leq \|\mu\|_{pK} \|f\|_{qL}. \tag{3}
\]

**Proof.** For any \( f \in \text{Lip}(X) \), \( \mu \in \mathcal{M}(X) \) and \( \xi \in \mathcal{M}(X,0) \), one has

\[
\left| \int f d\mu \right| \leq \left| \int f d\xi \right| + \left| \int f d(\mu - \xi) \right| \leq \|f\|_L \|\xi\|_{KR} + \|f\|_\infty \|\mu - \xi\|_{TV}
\]

\[
\leq (\|f\|^q_L + \|f\|^q_\infty)^{\frac{1}{q}}(\|\xi\|^p_{KR} + \|\mu - \xi\|^p_{TV})^{\frac{1}{p}} = \|f\|_{qL}(\|\xi\|^p_{KR} + \|\mu - \xi\|^p_{TV})^{\frac{1}{p}}
\]

where the Hölder inequality for \( \mathbb{R}^2 \) was used. Since this holds for any \( \xi \), the proposition follows from the definition of \( \|\cdot\|_{pK} \). \hfill \square

**Proposition 2.** Given \( p, q \in [1, \infty] \), \( \|\cdot\|_{pK} \) is a norm on \( \mathcal{M}(X) \) and \( \|\cdot\|_{qL} \) is a norm on \( \text{Lip}(X) \).
Proof. Both functionals are clearly seminorms. By Proposition\[1\] they are separating, so that they are norms as well.

Proposition 3. Given $p_1, p_2 \in [1, \infty]$, the norms $\| \cdot \|_{p_1K}$ and $\| \cdot \|_{p_2K}$ on $\mathcal{M}(X)$ are equivalent. Given $q_1, q_2 \in [1, \infty]$, the norms $\| \cdot \|_{q_1L}$ and $\| \cdot \|_{q_2L}$ on $\text{Lip}(X)$ are equivalent.

Proof. Suppose that $p_1 \leq p_2$ and $q_1 \leq q_2$. One has
\[
2^{-\frac{1}{p_1}} (\| \xi \|_{KR}^p + \| \mu - \xi \|_{TV}^p)^{\frac{1}{p_1}} \leq \max \{ \| \xi \|_{KR}, \| \mu - \xi \|_{TV} \} \leq (\| \xi \|_{KR}^p + \| \mu - \xi \|_{TV}^p)^{\frac{1}{p_2}}
\]
for any $\mu \in \mathcal{M}(X)$ and $\xi \in \mathcal{M}(X, 0)$. Similarly, one has
\[
2^{-\frac{1}{q_1}} (\| f \|_{L}^q + \| f \|_{\infty}^q)^{\frac{1}{q_1}} \leq \max \{ \| f \|_{L}, \| f \|_{\infty} \} \leq (\| f \|_{L}^q + \| f \|_{\infty}^q)^{\frac{1}{q_2}}
\]
for any $f \in \text{Lip}(X)$.

Proposition 4. Given $p, q \in [1, \infty]$, the normed space $(\mathcal{M}(X), \| \cdot \|_{pK})$ is not complete if $X$ is an infinite set, while the normed space $(\text{Lip}(X), \| \cdot \|_{qL})$ is complete for any $X$, i.e. a Banach space.

Proof. The following is an adaptation of the proof of [Cobzaş et al., 2019, Theorem 8.4.7]. If $X$ is infinite, it has an accumulation point $x$. Let $(x_n)$ be a sequence in $X \setminus \{ x \}$ converging to $x$. Since $\| \delta_x - \delta_{x_n} \|_{pK} \leq d(x, x_n)$, one has $\lim_{n \to \infty} \delta_{x_n} = \delta_x$ with respect to the topology induced by $\| \cdot \|_{pK}$ and $\| \cdot \|_{TV}$, meaning that $\| \cdot \|_{TV}$ and $\| \cdot \|_{pK}$ are not equivalent norms. If $(\mathcal{M}(X), \| \cdot \|_{pK})$ was complete, the identity operator from $(\mathcal{M}(X), \| \cdot \|_{pK})$ to $(\mathcal{M}(X), \| \cdot \|_{TV})$ would be an isomorphism by the Banach isomorphism theorem, which would imply the equivalence of the norms $\| \cdot \|_{TV}$ and $\| \cdot \|_{pK}$.

The following is an adaptation of the proof of [Cobzaş et al., 2019, Theorem 8.1.3]. Suppose that $(f_n)$ is a Cauchy sequence with respect to $\| \cdot \|_{qL}$. Then it is also Cauchy with respect to $\| \cdot \|_{\infty}$, hence converges uniformly to some bounded function $f : X \to \mathbb{R}$. Since it is also Cauchy with respect to $\| \cdot \|_{L}$, by [Cobzaş et al., 2019, Lemma 8.1.4] it also converges to the same $f$ with respect to $\| \cdot \|_{L}$, and $f$ is Lipschitz. Consequently, $(f_n)$ converges to $f$ with respect to both $\| \cdot \|_{\infty}$ and $\| \cdot \|_{L}$, hence with respect to $\| \cdot \|_{qL}$ as well, so that the space $(\text{Lip}(X), \| \cdot \|_{qL})$ is complete.

Corollary 5. Given $q \in [1, \infty]$, defining the product of $f, g \in \text{Lip}(X)$ as $fg(x) = f(x)g(x)$ turns $(\text{Lip}(X), \| \cdot \|_{qL})$ into a complete normed algebra whose product is continuous, i.e. a Gelfand algebra.

Proof. The result is known for $q = 1$ and $q = \infty$, see [Weaver, 2018, Section 7.1]. The result follows by the equivalence of $q$-Lipschitz norms by Proposition\[3\] and the completeness of $(\text{Lip}(X), \| \cdot \|_{qL})$ by Proposition\[4\].
3 Duality of $p$-Kantorovich and $q$-Lipschitz spaces

The following proposition is needed to prove duality.

**Proposition 6.** Given $p \in [1, \infty]$, the set of all measures with finite support is dense in $(\mathcal{M}(X), \|\cdot\|_{pK})$.

**Proof.** By [Hanin, 1999, Lemma 2], the set of all measures with finite support is dense in $(\mathcal{M}(X), \|\cdot\|_{1K})$. Since the norms $\|\cdot\|_{1K}$ and $\|\cdot\|_{pK}$ are equivalent for any $p \in [1, \infty]$ by Proposition 3, they generate the same topology, implying the proposition.

We are going to apply techniques of convex analysis to show that the dual of the $p$-Kantorovich space can be identified with the $q$-Lipschitz space if $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 7.** Given Hölder conjugates $p, q \in [1, \infty]$, for any $f \in \text{Lip}(X)$ the functional $u_f : (\mathcal{M}(X), \|\cdot\|_{pK}) \to \mathbb{R}$ defined by $u_f(\mu) = \int fd\mu$ is linear and continuous with $\|u_f\|_{pK}^* = \|f\|_{qL}$. Moreover, every continuous linear functional $v$ on $(\mathcal{M}(X), \|\cdot\|_{pK})$ is of the form $v(\mu) = u_f(\mu)$ for a uniquely determined function $f \in \text{Lip}(X)$ with $\|v\|_{pK}^* = \|f\|_{qL}$. Consequently, the mapping $(f \mapsto u_f)$ is an isometric isomorphism of $(\text{Lip}(X), \|\cdot\|_{qL})$ onto the topological dual $(\mathcal{M}(X), \|\cdot\|_{pK})^*$, i.e.

$$(\text{Lip}(X), \|\cdot\|_{qL}) \cong (\mathcal{M}(X), \|\cdot\|_{pK})^*. \quad (4)$$

**Proof.** It follows from [Hanin, 1999] that $u_f$ is a bounded and linear functional on $(\mathcal{M}(X), \|\cdot\|_{pK})$.

Consider the duality of $(\mathcal{M}(X), \|\cdot\|_{1K})$ and $(\text{Lip}(X), \|\cdot\|_{qL})$ given by [Hanin, 1999, Theorem 1]. For $p \in [1, \infty]$, let the indicators $t_p : (\mathcal{M}(X), \|\cdot\|_{1K}) \to \mathbb{R}$ be defined as

$$t_p(\mu) = \begin{cases} 0 & \text{if } \|\mu\|_{pK} \leq 1 \\ \infty & \text{otherwise} \end{cases}.$$ 

Their convex conjugates $t_p^* : (\text{Lip}(X), \|\cdot\|_{qL}) \to \mathbb{R}$ are defined as

$$t_p^*(f) = \sup_{\mu \in \mathcal{M}(X)} \left\{ \int fd\mu - t_p(\mu) \right\},$$

so that $t_p^*(f) = \sup_{\|\mu\|_{pK} \leq 1} \left\{ \int fd\mu \right\} = \|u_f\|_{pK}^*$ is exactly the dual norm of $u_f$. We claim that $t_p^*(f) = \|f\|_{qL}$, which would prove that the linear map $(f \mapsto u_f)$ is an isometry.

Let $H : (\mathcal{M}(X, 0), \|\cdot\|_{KR}) \times (\mathcal{M}(X), \|\cdot\|_{1K}) \to \mathbb{R}^2$ be defined as

$$H(\xi_1, \xi_2) = (\|\xi_1\|_{KR}, \|\xi_2\|_{TV}).$$

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$G(x) = \begin{cases} 0 & \text{if } \|x\|_p \leq 1, \\ \infty & \text{otherwise} \end{cases},$$
i.e. \( G \) is the indicator of the unit ball of the \( l^p \) norm on \( \mathbb{R}^2 \), so that its conjugate is \( G^*(y) = \|y\|_q \), i.e. the \( l^q \) norm. With the usual ordering on \( \mathbb{R}^2 \), \( H \) is clearly convex while \( g \) is proper, convex and increasing. Then the mapping \( \varphi = g \circ H \) is convex. We are going to invoke [Zalinescu, 2002, Theorem 2.8.10]. In the notation of the theorem, \( D = Y_0 = \mathbb{R}^2 \) in our case, so that condition (vi) of the theorem clearly holds. This implies that the conjugate \( \varphi^* : (\text{Lip}(X, x_0), ||.||_L) \times (\text{Lip}(X), ||.||_{\infty L}) \to \mathbb{R} \) is

\[
\varphi^*(f_1, f_2) = \min_{y \in \mathbb{R}^2_+} \{(y_1 \|K_R + y_2 \|TV)^*(f_1, f_2) + \|y\|_q\}.
\]

By [Zalinescu, 2002, Theorem 2.3.1(v)], [Zalinescu, 2002, Theorem 2.3.1(viii)] and the well known conjugate relations

\[
(\xi_1 \to \|\xi_1\|_{KR})^* = \begin{cases} f_1 \to \begin{cases} 0 \text{ if } \|f_1\|_L \leq 1, \\ \infty \text{ otherwise} \end{cases} \\ \infty \text{ otherwise} \end{cases}
\]

and

\[
(\xi_2 \to \|\xi_2\|_{TV})^* = \begin{cases} f_2 \to \begin{cases} 0 \text{ if } \|f_2\|_{\infty} \leq 1, \\ \infty \text{ otherwise} \end{cases} \\ \infty \text{ otherwise} \end{cases}
\]

the conjugate of the mapping

\[
(x_1, x_2) \to y_1\|\xi_1\|_{KR} + y_2\|\xi_2\|_{TV}
\]

is the mapping

\[
(f_1, f_2) \to \begin{cases} 0 \text{ if } \|f_1\|_L \leq y_1 \text{ and } \|f_2\|_{\infty} \leq y_2, \\ \infty \text{ otherwise}, \end{cases}
\]

so that one has

\[
\varphi^*(f_1, f_2) = \min_{y \in \mathbb{R}^2_+, \|f_1\|_L \leq y_1, \|f_2\|_{\infty} \leq y_2} \{\|y\|_q\} = (\|f_1\|_L^q + \|f_2\|_{\infty L}^q)^{\frac{1}{q}}.
\]

Consider the linear map \( A \in L((\mathcal{M}(X, 0), \|\cdot\|_{KR}) \times (\mathcal{M}(X), \|\cdot\|_{1K}), (\mathcal{M}(X), \|\cdot\|_{1K}) \) defined as

\[
A(\xi_1, \xi_2) = \xi_1 + \xi_2,
\]

which is clearly bounded, hence continuous.

Its adjoint \( A^* \in L((\text{Lip}(X), \|\cdot\|_{L} \times (\text{Lip}(X), \|\cdot\|_{L}) \times (\text{Lip}(X), \|\cdot\|_{\infty L})) \) is given by

\[
A^* f = (f - f(x_0), f).
\]

By [Zalinescu, 2002, Theorem 2.3.1(ix)], the conjugate of

\[
A \varphi(\mu) = \inf_{(\xi_1, \xi_2) \in (\mathcal{M}(X, 0), \|\cdot\|_{KR}) \times (\mathcal{M}(X), \|\cdot\|_{1K}) \mu = A(\xi_1, \xi_2)} \{\varphi(\xi_1, \xi_2)\} = \tau_p(\mu)
\]

is the mapping

\[
\varphi^* \circ A^*(f) = (\|f - f(x_0)\|_L^q + \|f\|_{\infty L}^q)^{\frac{1}{q}} = \|f\|_{qL},
\]

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proving the claim, so that \((f \to u_f)\) is an isometry.

To see that \((f \to u_f)\) is onto \(\text{Lip}(X)\), take any \(v \in (\mathcal{M}(X), \|\cdot\|_{pK})^*\) and set \(f(x) = v(\delta_x)\) for all \(x \in X\). One has \(|f(x)| \leq \|v\|_{pK} \|\delta_x\|_{pK} = \|v\|_{pK}\) and \(|f(x) - f(y)| \leq \|v\|_{pK} \|\delta_x - \delta_y\|_{pK} \leq \|v\|_{pK} d(x, y)\) for any \(x, y \in X\), so that \(f \in \text{Lip}(X)\). One has \(u_f(\delta_x) = v(\delta_x)\), so that by linearity, \(u_f(\mu) = v(\mu)\) for any \(\mu \in \mathcal{M}(X)\) with finite support. Since such measures are dense in \((\mathcal{M}(X), \|\cdot\|_{pK})\), by Proposition 6 one has \(u_f = v\), completing the proof.

We get the following dual representation of \(\|\cdot\|_{pK}\) as a corollary.

**Corollary 8.** Given \(\mu \in \mathcal{M}(X)\), one has the dual representation

\[
\|\mu\|_{pK} = \sup_{f \in \text{Lip}(X), \|f\|_{qL} \leq 1} \left\{ \int fd\mu \right\},
\]

and there exists \(f_\ast \in \text{Lip}(X)\) such that \(\|f_\ast\|_{qL} = 1\) and \(\int f_\ast d\mu = \|\mu\|_{pK}\).

**Proof.** This follows from Proposition 7 and the Hahn-Banach theorem.

### 4 Optimality conditions

The following proposition shows that the infimum in (1) is always attained.

**Proposition 9.** For any \(\mu \in \mathcal{M}(X)\), there exists \(\xi_\ast \in \mathcal{M}(X, 0)\) such that

\[
\|\mu\|_{pK} = (\|\xi_\ast\|_{pK}^p + \|\mu - \xi_\ast\|_{TV}^p)^{\frac{1}{p}}.
\]

**Proof.** The following is an adaptation of the proof of [Hanin, 1999] Proposition 6. Let \(\xi_n\) be a sequence in \(\mathcal{M}(X, 0)\) such that \(\|\mu\|_{pK} = \lim_{n \to \infty} (\|\xi_n\|_{pK}^p + \|\mu - \xi_n\|_{TV}^p)^{\frac{1}{p}}\). Without loss of generality one can assume that \(\|\xi_n\|_{TV} \leq R\) for all \(n\) for some constant \(R > 0\). By the weak-* compactness of the \(\|\cdot\|_{TV}\)-ball of radius \(R\) by [Cobzas et al., 2019] Theorem 8.4.25, up to a subsequence, \(\xi_n\) weak-* converges to some \(\xi_\ast \in \mathcal{M}(X, 0)\), so that \(\|\mu\|_{pK} \geq (a^p + b^p)^{\frac{1}{p}}\). For every \(n\) and any \(f \in \text{Lip}(X)\) with \(\|f\|_{L} \leq 1\) one has \(\left| \int fd\xi_n \right| \leq \|\xi_n\|_{pK}\), so that \(\left| \int fd\xi_\ast \right| \leq a\), hence \(\|\mu - \xi_\ast\|_{pK} \leq a\). Similarly, for every \(n\) and any \(f \in \text{Lip}(X)\) with \(\|f\|_{\infty} \leq 1\) one has \(\left| \int fd(\mu - \xi_n) \right| \leq \|\mu - \xi_\ast\|_{TV}\), so that \(\left| \int fd(\mu - \xi_\ast) \right| \leq b\), hence \(\|\mu - \xi_\ast\|_{TV} \leq b\). Thus \((\|\xi_\ast\|_{pK}^p + \|\mu - \xi_\ast\|_{TV}^p)^{\frac{1}{p}} \leq (a^p + b^p)^{\frac{1}{p}} \leq \|\mu\|_{pK}\), implying the proposition.

It is well known that for any \(\xi \in \mathcal{M}(X, 0)\) there exists \(\pi_\ast \in \mathcal{M}(X \times X)\) such that \(\pi_\ast(\cdot \times \cdot) - \pi_\ast(\cdot \times X) = \xi\) and \(\|\xi\|_{pK} = \left( \int d(x, y)d\pi_\ast(x, y) \right)^{\frac{1}{p}}\), i.e.,

\[
\|\mu\|_{pK} = \left( \left( \int d(x, y)d\pi_\ast(x, y) \right)^{\frac{1}{p}} + \|\xi_\ast\|_{TV}^p \right)^{\frac{1}{p}}.
\]
while by Proposition 8 there exists an optimal \( f_* \in \text{Lip}(X) \) such that \( \|\mu\|_{pK} = \int f_* d\mu \) and \( \|f\|_{qL} = 1 \). The following proposition characterizes such optimal variables, generalizing [Hanin, 1999] Proposition 7.

**Proposition 10.** Given \( \mu \in \mathcal{M}(X) \), the measures \( \xi_\epsilon \in \mathcal{M}(X,0) \), \( \pi_* \in \mathcal{M}(X \times X) \) with \( \pi_\epsilon(X \times \cdot) - \pi_\epsilon(\cdot \times X) = \xi_\epsilon \) are optimal if and only if there exists a \( f_* \in \text{Lip}(X) \) such that the conditions

(i) \( \|f_*\|_{qL} = 1 \),

(ii) \( \|f_*\|_L\|\xi_\epsilon\|_K R + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV} = (\|\xi_\epsilon\|_{K R}^p + \|\mu - \xi_\epsilon\|_{TV}^p)^{\frac{1}{p}} \),

(iii) \( f_\epsilon(x) - f_\epsilon(y) = \|f_*\|_Ld(x,y) \) if \( (x,y) \in \text{supp}(\pi_\epsilon) \) and

(iv) \( f_\epsilon(x) = \pm\|f_*\|_\infty \) if \( x \in \text{supp}(\mu - \xi_\epsilon) \)

are satisfied. In this case, \( f_* \) is optimal, i.e. \( \int f_* d\mu = \|\mu\|_{pK} \).

Proof. Let \( f_* \in \text{Lip}(X) \) be a function satisfying the above conditions. Then one has

\[
\|\mu\|_{pK} \geq \int f_* d\mu = \int f_* d\xi_* + \int f_* d(\mu - \xi_*)
\]

\[
= \int f_* (x) - f_* (y) d\pi_\epsilon (x,y) + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV}
\]

\[
= \|f_*\|_L \int d(x,y) d\pi_\epsilon (x,y) + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV}
\]

\[
\geq \|f_*\|_L\|\xi_\epsilon\|_K R + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV}
\]

\[
= (\|\xi_\epsilon\|_{K R}^p + \|\mu - \xi_\epsilon\|_{TV}^p)^{\frac{1}{p}} \geq \|\mu\|_{pK},
\]

so that the conditions are sufficient.

Now let \( x_\epsilon, \pi_* \) and \( f_* \) be optimal. Clearly (i) is satisfied. On the other hand, one has

\[
\|\mu\|_{pK} = \int f_* d\mu = \int f_* d\xi_* + \int f_* d(\mu - \xi_*)
\]

\[
\leq \int f_* (x) - f_* (y) d\pi_\epsilon (x,y) + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV}
\]

\[
\leq \|f_*\|_L \int d(x,y) d\pi_\epsilon (x,y) + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV}
\]

\[
= \|f_*\|_L\|\xi_\epsilon\|_K R + \|f_*\|_\infty\|\mu - \xi_\epsilon\|_{TV}
\]

\[
\leq (\|f\|_L^p + \|f\|_\infty^p)^{\frac{1}{p}} (\|\xi_\epsilon\|_{K R}^p + \|\mu - \xi_\epsilon\|_{TV}^p)^{\frac{1}{p}} = \|\mu\|_{pK},
\]

implying (ii), (iii) and (iv). This shows that the conditions are necessary as well. \( \square \)
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