ON DEGENERATE $q$-BERNOUlli POLYNOMIALS

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Abstract. In this paper, we introduce the degenerate Carlitz $q$-Bernoulli numbers and polynomials and give some interesting identities and properties of these numbers and polynomials which are derived from the generating functions and $p$-adic integral equations.

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume $|q-1|_p < p^{-\nu_p(q)}$ so that $q^x = \exp(x \log q)$ for $|x|_p < 1$. We use the notation $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$.

In [2], L. Carlitz considered $q$-Bernoulli numbers as follows:

$$(1.1) \quad \beta_0,q = 1, \quad q(q\beta_q + 1)^n - \beta_n,q = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing $\beta^n_q$ by $\beta_{n,q}$. The $q$-Bernoulli polynomials are defined by

$$(1.2) \quad \beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} \beta_{l,q} q^l [x]_q^{n-l} \quad (\text{see } [2, 8]).$$

In [4, 3], L. Carlitz defined the degenerate Bernoulli polynomials which are given by the generating function to be

$$(1.3) \quad \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1} (1+\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!} \quad (\text{see } [2, 5]).$$

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When \( x = 0, \beta_n(\lambda) = \beta_n(0|\lambda) \) are called the degenerate Bernoulli numbers. Note that \( \lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x) \), where \( B_n(x) \) are the ordinary Bernoulli polynomials (see [1-12]). Let \( UD(\mathbb{Z}_p) \) be the space of uniformly differentiable functions on \( \mathbb{Z}_p \). For \( f \in UD(\mathbb{Z}_p) \), the \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) is defined by

\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad (\text{see [8]}).
\]

The Carlitz’s \( q \)-Bernoulli polynomials can be represented by \( p \)-adic \( q \)-integrals on \( \mathbb{Z}_p \) as follows:

\[
\int_{\mathbb{Z}_p} [x + y]^n d\mu_q(y) = \beta_{n,q}(x) \quad (n \geq 0).
\]

Thus, by (1.4), we get

\[
\int_{\mathbb{Z}_p} e^{[x+y]t} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} \quad (\text{see [8]}).
\]

From (1.6), we can derive the following equation:

\[
\beta_{m,q}(x) = \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^j \frac{j+1}{j+1} t^n \quad (m \geq 0).
\]

In this paper, we introduce the degenerate Carlitz \( q \)-Bernoulli numbers and polynomials and give some interesting identities and properties of these numbers and polynomials which are derived from the generating functions and \( p \)-adic integral equations on \( \mathbb{Z}_p \).

2. Degenerate Carlitz \( q \)-Bernoulli numbers and polynomials

In this section, we assume that \( \lambda, t \in \mathbb{C}_p \) with \( 0 < |\lambda|_p \leq 1, |t|_p < p^{-\frac{1}{p-1}} \).

Then, as \( |\lambda|_p < p^{-\frac{1}{p-1}}, |\log(1 + \lambda t)|_p = |\lambda t|_p \) and hence \( |\frac{1}{\lambda} \log(1 + \lambda t)|_p = |t|_p < p^{-\frac{1}{p-1}} \) and now it makes sense to take the limit as \( \lambda \to 0 \).

In the viewpoint of (1.3), we consider the degenerate Carlitz \( q \)-Bernoulli polynomials which are given by the generating function to be

\[
\int_{\mathbb{Z}_p} (1 + \lambda t)^{[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x|\lambda) \frac{t^n}{n!}.
\]

When \( x = 0, \beta_{n,q}(\lambda) = \beta_{n,q}(0|\lambda) \) are called the degenerate Carlitz \( q \)-Bernoulli numbers.

Now, we observe that

\[
\int_{\mathbb{Z}_p} (1 + \lambda t)^{[x+y]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( \frac{[x+y]_q}{\lambda} \right) d\mu_q(y) \lambda^n t^n
\]

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left( \frac{[x+y]_q}{\lambda} \right) d\mu_q(y) \lambda^n t^n.
\]
where \( \frac{[x+y]_q}{\lambda} \) = \( \frac{[x+y]_q}{\lambda} \times \frac{[x+y]_q}{\lambda} - 1 \times \cdots \times \frac{[x+y]_q}{\lambda} - n + 1 \).

Now, we define \([x + y]_{n,\lambda}\) as \([x + y]_{0,\lambda} = 1\),

\[
[x + y]_{n,\lambda} = [x + y]_q ([x + y]_q - \lambda) \cdots ([x + y]_q - (n - 1)\lambda) \quad (n \geq 1).
\]

Therefore, by (2.1), (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
\int_{\mathbb{Z}_p} [x + y]_{n,\lambda} d\mu_q(y) = \beta_{n,q}(x|\lambda).
\]

Let \( S_1(n, m) \) be the *Stirling numbers of the first kind* which are defined by

\[
(x)_n = \sum_{l=0}^{n} S_1(n, l) x^l, \quad (n \geq 0).
\]

Then, by (2.2), we get

\[
\int_{\mathbb{Z}_p} \left[\frac{x + y}{\lambda}\right]^n \ d\mu_q(y) = \sum_{l=0}^{n} S_1(n, l) \lambda^{-l} \int_{\mathbb{Z}_p} \left[\frac{x + y}{\lambda}\right]^l d\mu_q(y)
\]

\[
= \sum_{l=0}^{n} S_1(n, l) \lambda^{-l} \beta_{l,q}(x).
\]

Therefore, by (2.2) and (2.4), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have

\[
\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} \beta_{l,q}(x).
\]

Note that \( \lim_{\lambda \to 0} \beta_{n,q}(x|\lambda) = \beta_{n,q}(x) \).

**Corollary 2.3.** For \( n \geq 0 \), we have

\[
\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{S_1(n, l)}{(1-q)^j} \binom{l}{j} (-1)^j q^j x^j + 1 \frac{\lambda^n}{l+1}.
\]

We observe that

\[
(1 + \lambda t)^{[x+y]_q} = e^\frac{[x+y]_q}{\lambda} \log(1 + \lambda t) = \sum_{n=0}^{\infty} \left( \frac{[x+y]_q}{\lambda} \right)^n \frac{1}{n!} (\log(1 + \lambda t))^n
\]

\[
= \sum_{m=0}^{\infty} \left( \frac{[x+y]_q}{\lambda} \right)^m \frac{1}{m!} \sum_{n=m}^{\infty} \frac{n!}{n!} S_1(n, m) \frac{\lambda^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) [x+y]_q^m \right) \frac{t^n}{n!}.
\]
Thus, by (2.5), we get
\[
\int_{\mathbb{R}_p} \left(1 + \lambda t \right)^{[x+y]_q} \, d\mu_q(y) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \int_{\mathbb{R}_p} [x+y]_q^m \, d\mu_q(x) \right) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n,m) \beta_{m,q}(x) \right) \frac{t^n}{n!}.
\]
Replacing \( t \) by \( \frac{1}{\lambda} (e^{\lambda t} - 1) \) in (2.1), we get
\[
\int_{\mathbb{R}_p} e^{[x+y]_q t} \, d\mu_q(y) = \sum_{m=0}^{\infty} \beta_{m,q}(x|\lambda) \frac{1}{m!} \left( e^{\lambda t} - 1 \right)^m \\
= \sum_{m=0}^{\infty} \beta_{m,q}(x|\lambda) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n,m) \frac{\lambda^n t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \beta_{m,q}(x|\lambda) \lambda^{n-m} S_2(n,m) \right) \frac{t^n}{n!},
\]
where \( S_2(n,m) \) are the Stirling numbers of the second kind.

We note that the left hand side of (2.6) is given by
\[
\int_{\mathbb{R}_p} e^{[x+y]_q t} \, d\mu_q(y) = \sum_{n=0}^{\infty} \int_{\mathbb{R}_p} [x+y]_q^n \, d\mu_q(y) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!}.
\]
Therefore, by (2.6) and (2.7), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
\beta_{n,q}(x) = \sum_{m=0}^{n} \beta_{m,q}(x|\lambda) \lambda^{n-m} S_2(n,m).
\]

Note that
\[
(1 + \lambda t)^{[x+y]_q} = (1 + \lambda t)^{[x]_q} (1 + \lambda t)^{[y]_q} \\
= \left( \sum_{m=0}^{\infty} [x]_q m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} [y]_q l \frac{t^l}{l!} \right) \\
= \left( \sum_{m=0}^{\infty} [x]_q m \frac{t^m}{m!} \right) \left( \sum_{k=0}^{\infty} \sum_{l=0}^{k} \lambda^{k-l} q^{l} [y]_q^l S_1(k,l) \frac{t^k}{k!} \right) \\
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!} \lambda^{k-n} q^{n-k} [x]_q^{n-k} [y]_q^k S_1(k,l) \frac{t^n}{n!}.
\]
Thus, by (2.8), we get

\[
\sum_{n=0}^{\infty} \beta_{n,q}(x|\lambda) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{l=0}^{k} [x]_{n-k,\lambda} \lambda^{k-l} q^{l \ell} \int_{\mathbb{Z}_p} \left[ y_1^q d\mu_q(y) S_1(k,l) \right] \left( \begin{array}{c} n \\ k \end{array} \right) \right) \frac{t^n}{n!}.
\]

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 0 \), we have

\[
\beta_{n,q}(x|\lambda) = \sum_{k=0}^{n} \sum_{l=0}^{k} \left( \begin{array}{c} n \\ k \end{array} \right) [x]_{n-k,\lambda} \lambda^{k-l} q^{l \ell} \beta_{k,l,q} S_1(k,l) \beta_{l,q}.
\]

For \( r \in \mathbb{N} \), we define the **degenerate Carlitz q-Bernoulli polynomials of order** \( r \) as follows:

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t) \frac{\left[ x_1 + \cdots + x_r + x \right]^n}{n!} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \beta^{(r)}_{n,q}(x|\lambda) \frac{t^n}{n!}.
\]

We observe that

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \lambda t) \frac{\left[ x_1 + \cdots + x_r + x \right]^n}{n!} d\mu_q(x_1) \cdots d\mu_q(x_r)
\]

\[
= \sum_{m=0}^{\infty} \lambda^{-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r + x]^m d\mu_q(x_1) \cdots d\mu_q(x_r) \frac{1}{m!} (\log(1 + \lambda t))^m
\]

\[
= \sum_{m=0}^{\infty} \beta^{(r)}_{m,q}(x) \lambda^{-m} \sum_{n=m}^{\infty} S_1(n,m) \frac{\lambda^n}{n!} \frac{t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{-m} \beta^{(r)}_{m,q}(x) S_1(n,m) \right) \frac{t^n}{n!},
\]

where \( \beta^{(r)}_{m,q}(x) \) are the Carlitz q-Bernoulli polynomials of order \( r \).

Therefore, by (2.10) and (2.11), we obtain the following theorem.

**Theorem 2.6.** For \( n \geq 0 \), we have

\[
\beta_{n,q}^{(r)}(x|\lambda) = \sum_{m=0}^{n} \lambda^{-m} \beta^{(r)}_{m,q}(x) S_1(n,m).
\]
Replacing \( t \) by \( \frac{1}{\lambda} (e^{\lambda t} - 1) \) in (2.10), we have

\[
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} e^{[x_1 + \cdots + x_r + \varepsilon]t} d\mu_q(x_1) \cdots d\mu_q(x_r)
\]

\[
= \sum_{m=0}^{\infty} \beta_{m,q}^{(r)}(x|\lambda)\frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m
\]

(2.12)

\[
= \sum_{m=0}^{\infty} \beta_{m,q}^{(r)}(x|\lambda)\lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} \beta_{m,q}^{(r)}(x|\lambda) S_2(n, m) \right) \frac{t^n}{n!}.
\]

The left hand side of (2.12) is given by

(2.13)

\[
\int_{\mathbb{Z}} \cdots \int_{\mathbb{Z}} e^{[x_1 + \cdots + x_r + \varepsilon]t} d\mu_q(x_1) \cdots d\mu_q(x_r) = \sum_{n=0}^{\infty} \beta_{n,q}^{(r)}(x) \frac{t^n}{n!}.
\]

By comparing the coefficients on the both sides of (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.7.** For \( n \geq 0 \), we have

\[
\beta_{n,q}^{(r)}(x) = \sum_{m=0}^{n} \lambda^{n-m} S_2(n, m) \beta_{m,q}^{(r)}(x|\lambda).
\]

We recall that

\[
\int_{\mathbb{Z}} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x
\]

\[
= \lim_{N \to \infty} \frac{1}{[d^N]_q} \sum_{x=0}^{d^N-1} f(x) q^x,
\]

where \( d \in \mathbb{N} \) and \( f \in UD(\mathbb{Z}_p) \).

Now, we observe that

(2.14) \[
\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} \int_{\mathbb{Z}} [x + y]_q^l d\mu_q(y),
\]

and

(2.15) \[
\int_{\mathbb{Z}} [x + y]_q^l d\mu_q(y) = \frac{1}{[m]_q^l} \sum_{i=0}^{m-1} q^i [m]_q^l \int_{\mathbb{Z}} \left[ \frac{x + i}{m} + y \right]_q^l d\mu_q(y)
\]

\[
= [m]_q^{l-1} \sum_{i=0}^{m-1} q^i \beta_{l,q}^{(n)} \left( \frac{x + i}{m} \right),
\]

where \( l \in \mathbb{Z}_{\geq 0} \) and \( m \in \mathbb{N} \).
Therefore, by (2.14) and (2.15), we obtain the following theorem.

**Theorem 2.8.** For $n \geq N \geq 0$, $m \in \mathbb{N}$, we have

$$\beta_{n,q}(x|\lambda) = \sum_{l=0}^{n} \sum_{i=0}^{m-1} S_1(n,l) \lambda^{n-l} |m|_q^{l-1} q^i \beta_{l,q}^m \left( \frac{x+i}{m} \right).$$

From (1.4), we note that

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0),$$

where $f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}$.

By (2.16), we get

$$q\beta_{n,q}(x+1|\lambda) - \beta_{n,q}(x|\lambda) = (q-1)\lambda^n \left( \frac{[x]_q}{\lambda} \right)_n + \sum_{l=1}^{n} S_1(n,l) \lambda^{n-l} [x]_q^{l-1} q^x,$$

where $n \in \mathbb{N}$.

Therefore, by (2.17), we obtain the following theorem.

**Theorem 2.9.** For $n \geq 0$, we have

$$q\beta_{n,q}(x+1|\lambda) - \beta_{n,q}(x|\lambda) = (q-1)\lambda^n \left( \frac{[x]_q}{\lambda} \right)_n + \sum_{l=1}^{n} S_1(n,l) \lambda^{n-l} [x]_q^{l-1} q^x.$$

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