Numerically efficient version of the T-matrix method.

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Abstract
A version of the projection method for solving the scattering problem for acoustic and electromagnetic waves is proposed and shown to be more efficient numerically than the earlier ones.

1 Introduction
Consider the scattering problem:

\[ (\nabla^2 + k^2) u = 0 \text{ in } D^\prime := \mathbb{R}^2 \setminus D, \]
\[ u = 0 \text{ on } S := \partial D, \]
\[ u = e^{ik\alpha \cdot x} + v, \]

where \( D \) is a bounded domain in \( \mathbb{R}^2 \) with a piecewise smooth boundary \( S \), and the scattered field \( v \) has the following asymptotics:

\[ v = A(\alpha', \alpha, k) \frac{e^{ir}}{\sqrt{r}} + o \left( \frac{1}{\sqrt{r}} \right), \quad r = |x| \to \infty, \quad \frac{x}{r^2} = \alpha' \in S^1, \]

and \( S^1 \) is the unit sphere in \( \mathbb{R}^2 \). The coefficient \( A \) is called the scattering amplitude, \( k = \text{const} > 0 \) is fixed, and \( u := u(x, \alpha, k) \) is called the scattering solution.

Let \( g(x, y) := \frac{i}{4} H_0^{(1)}(k|x - y|) \). Take \( k = 1 \) in what follows without loss of generality. Then

\[ (\nabla^2 + 1) g = -\delta(x - y) \text{ in } \mathbb{R}^2, \quad g = c \frac{e^{iy|y|}}{\sqrt{|y|}} e^{-iy^0 x} + o \left( \frac{1}{\sqrt{|y|}} \right) \text{ as } |y| \to \infty, \quad y^0 := \frac{y}{|y|}, \]

\( c = \text{const} = \frac{i}{2\sqrt{2}\pi}. \) One has, using Green’s formula:

\[ u(x, \alpha) = e^{i\alpha \cdot x} - \int_S g(x, s) u_N(s', \alpha) ds', \quad x \in D^\prime, \quad k = 1, \]
where $u_N$ is the normal derivative of $u$ on $S$, $N$ is the exterior unit normal to $S$.

Taking $x \in D'$ to $S$ and using (1.2), one gets

$$Th := f, \quad h := u_N, \quad f := e^{i\alpha \cdot s}, \quad k = 1,$$

(1.7)

$$Th := \int_S g(s, s')h(s')ds'.$$

(1.8)

Equation (1.7) is the basic equation studied in this paper.

The T-matrix method is described in [1], [2] and analyzed in [1] mathematically.

The purpose of this paper is to give a version of this method for solving equation (1.7), and to analyze this version from the computational points of view. This is done in sections 2 and 3.

Let $H^\ell = H^\ell(S)$ be the Sobolev spaces, $\ell = 0, 1,$ and

$$Qh := \int_S \frac{1}{2\pi} \ln \frac{a}{r_{ss'}} h(s')ds',$$

(1.9)

where $a = \text{const} = \text{diam} \ D > 0$, so that $\inf_{s,s' \in S} \frac{a}{r_{ss'}} \geq 1$.

Write (1.7) as

$$Qh + Kh = f, \quad f \in H^1, \quad Kh := -\frac{\ln a}{2\pi} \int_S h(s')ds' + \int_S \left[ g(s, s') - \frac{1}{2\pi} \ln \frac{1}{r_{ss'}} \right] hdy$$

(1.10)

The operator $A = Q^{-1}K$ is compact in $H^1$, and the operator $Q$ is an isomorphism between $H^0$ and $H^1$ (see Lemmas 1.1 and 1.2 below). Moreover, $Q$ is a selfadjoint compact positive operator, in $H^0$: $(Qu, u) > 0$ if $u \neq 0$. In Remark 2.1 below we show a possible usage of (1.10). Let $\{\varphi_j\}_{j=1,2,...}$ be a Riesz basis of $H^0$, that is, every element $u \in H^0$ is uniquely representable as a convergent in $H^0$ series

$$u = \sum_{j=1}^\infty c_j \varphi_j,$$

(1.11)

and

$$m \sum_{j=1}^\infty |c_j|^2 \leq \|u\|_0^2 \leq M \sum_{j=1}^\infty |c_j|^2, \quad 0 < m \leq M, \quad m, M = \text{const} > 0.$$  

(1.12)

Let us prove the following Lemma 1.1:

**Lemma 1.1.** If $k^2$ is not a Dirichlet eigenvalue of the Laplacian in $D$ then the operator $T$ defined by (1.8) is an isomorphism of $H^0$ onto $H^1$.  

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Proof. This result is established in [1] so we only indicate the basic points of the proof. The assumption of Lemma 2.1 implies the injectivity of $T$: if $Th = 0$ then the function $w(x) := \int_S g(x, s')h(s')ds'$ solves the Dirichlet problem for the Helmholtz operator in $D$ and in $D'$, and satisfies the radiation condition at infinity. Thus $w = 0$ in $D$ by the assumption of Lemma 2.1, and $w = 0$ in $D'$ by lemma 1 in [1], p.25. Therefore $h = u_N^+ - u_N^- = u$, where we have used the jump relation for the normal derivative of the single-layer potentials. The operator $T : H^0 \to H^1$ is of Fredholm-type: it can be written as $T = Q+, K$ where $Q : H^0 \to H^1$ is an isomorphism and $K$ is compact as an operator from $H^0$ into $H^1$. The injectivity of $T$ together with its Fredholm property imply the conclusion of Lemma 1.1.

We assume throughout this paper that $k^2 = 1$ is not a Dirichlet eigenvalue of the Laplacian in $D$.

In the literature [2] one usually means by the T-matrix approach (in acoustic and electromagnetic wave scattering theory) a projection method for solving equations of the type (1.7) with the following choices of the basis functions: $\varphi_m = e^{im\theta}H_m(kr(\theta))$ or $\varphi_m = e^{im\theta}J_m(kr(\theta))$.

These choices lead to the following difficulties discussed in [1]: as the number $J$ of these functions grows: $J \to \infty$, the condition number of matrix $a_{ij}$ in (2.2) (see below) grows exponentially and depends strongly on the geometry of $S$. In contrast, in our version of the method, the condition number of $a_{ij}$ remains bounded as $J \to \infty$.

Lemma 1.2. The operator $Q^{-1}K$ is compact in $H^1$.

Proof. The kernel of the operator $K$, defined by (1.10), and its first derivatives are continuous functions of $s$ and $s'$ running through bounded sets, including the diagonal $s = s'$. By Lemma 1.1 the action of $Q^{-1}$ is equivalent (up to the terms preserving smoothness) to taking the first order derivatives. Therefore the conclusion of Lemma 2.1 follows.

Remark 1.1. Let us outline a possible way to use the splitting in equation (1.10). The idea is simple: $Q \geq c > 0$ is positive definite. Write (1.10) as

$$h + Ah = F, \quad F : Q^{-1}f.$$  (1.13)

Inverting $Q$ numerically is a relatively easy problem since $Q$ is positive definite. Using an orthonormal basis $\{\varphi_j\}$ of $H^0$, one can write the projection method for (1.13), namely:

$$h_J = \sum_{j=1}^J c_j \varphi_j,$$

$$h_i + \sum_{j=1}^J A_{ij} h_j = F_i; \quad 1 \leq i \leq J,$$  (1.14)

where $A_{ij} := (A\varphi_j, \varphi_i)$. 

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The matrix in (1.14) is $B_{ij} = \delta_{ij} + A_{ij}$. If $\|A\|_0 < 1$, then system (1.4) can be solved by iterations numerically efficiently. This happens if $ka << 1$, where $a = \text{diam } D$, but it may happen when the above condition does not hold.

2 Solution of the basic equation

Let us look for an approximate solution to (1.7)

$$ h = \sum_{j=1}^{J} c_j \varphi_j, \quad (2.1) $$

where $\varphi_j$ is a Riesz basis of $H^0$, and

$$ \sum_{j=1}^{J} a_{ij} c_j = f_i \quad 1 \leq i \leq J, \quad (2.2) $$

where

$$ f_i := (f, T\varphi_i), \quad a_{ij} := (T\varphi_j, T\varphi_i)_0, \quad (2.3) $$

and $T$ is defined in (1.8). Since $T$ is injective, the elements $\{T\varphi_j\}$ are linearly independent. Therefore

$$ \det(a_{ij})_{1 \leq i,j \leq J} \neq 0 \quad \forall J = 1, 2, \ldots \quad (2.4) $$

**Definition 2.1.** A system $\{\psi_j\}$ is a Riesz basis of a Hilbert space $H$ if there is an isomorphism $B$ of $H$ onto $H$ such that $B\psi_j = e_j$, where $\{e_j\}$ is an orthonormal basis of $H$.

One gets system (2.2) by solving the following minimization problem:

$$ \| \sum_{j=1}^{J} c_j T\varphi_j - f \|_1 = \min. \quad (2.5) $$

Denote by $\{c_j^{(J)}\}_{1 \leq j \leq J}$ the unique solution to (2.2) or, equivalently, to (2.5).

**Lemma 2.1.** $\{T\varphi_j\}_{1 \leq j < \infty}$ is a Riesz basis of $H^1$ if the system $\{\varphi_j\}_{1 \leq j < \infty}$ is a Riesz basis of $H^0$.

**Proof.** Let $f \in H^1$ be an arbitrary element of $H^1$. Denote $T^{-1}f := h \in H^0$.

Since $\{\varphi_j\}$ is a basis of $H^0$, one has

$$ h = \sum_{j=1}^{\infty} c_j \varphi_j, \quad f = \sum_{j=1}^{\infty} c_j T\varphi_j. $$
If $\sum c_j T\varphi_j = 0$, then, applying the continuous operator $T^{-1}$, one gets $\sum_{j=1}^{\infty} c_j \varphi_j = 0$, so $c_j = 0$ for all $j$ since $\{\varphi_j\}_{1 \leq j < \infty}$ is a basis of $H^0$. We have proved that $\{T\varphi_j\}_{1 \leq j < \infty}$ is a basis of $H^1$.

Let us prove that if $\{\varphi_j\}_{1 \leq j < \infty}$ is a Riesz basis of $H^0$ then $\{T\varphi_j\}_{1 \leq j < \infty}$ is a Riesz basis of $H^1$, that is, there exists an isomorphism $B$ of $H^1$ onto $H^1$ such that $BT\varphi_j = e_j$,

$(e_j, e_i)_i = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

Let $\{e_j\}$ be an orthonormal basis of $H^1$. Define a linear operator $F : H^1 \to H^1$ by the formula:

$$F \sum_{j=1}^{\infty} c_j e_j := \sum_{j=1}^{\infty} c_j T\varphi_j,$$

in particular, $F e_j = T \varphi_j$. Let us prove that $F$ is an isomorphism of $H^1$ onto $H^1$. If this is proved, then $B := F^{-1}$, and Lemma 2.1 is proved.

Clearly $F$ is linear, is defined on all of $H^1$, and is continuous. Only the continuity of $F$ needs a proof.

Let $u_n \to u$ in $H^1$. Then $u_n = \sum_{j=1}^{\infty} c_j^{(n)} e_j$, $u = \sum_{j=1}^{\infty} c_j e_j$, $\sum_{j=1}^{\infty} |c_j^{(n)} - c_j|^2 \to 0$, $F u_n = \sum_{j=1}^{\infty} c_j^{(n)} T \varphi_j$, $F u = \sum_{j=1}^{\infty} c_j T \varphi_j$. Thus:

$$\|Fu_n - Fu\|_1^2 = \|\sum_{j=1}^{\infty} (c_j^{(n)} - c_j) T \varphi_j\|_1^2 =$$

$$\|T \sum_{j=1}^{\infty} (c_j^{(n)} - c_j) \varphi_j\|_1^2 \leq \|T\| \sum_{j=1}^{\infty} (c_j^{(n)} - c_j) \varphi_j \|_0 \leq$$

$$\|T\| \sum_{j=1}^{\infty} |c_j^{(n)} - c_j|^2 \to 0 \quad \text{as } n \to \infty,$$

where we have used the assumption that $\{\varphi_j\}_{1 \leq j < \infty}$ a Riesz basis of $H^0$.

Thus $F$ is a linear continuous, defined on all of $H^1$ operator.

Therefore $F$ is bounded.

Let us check that $F$ is injective: if $u \in H^1$, $u = \sum_{j=1}^{\infty} c_j e_j$, and $F u = 0$ then $\sum_{j=1}^{\infty} c_j T \varphi_j = 0$. Apply $T^{-1}$ and get $\sum_{j=1}^{\infty} c_j \varphi_j = 0$. Thus, $c_j = 0 \forall j$, since $\{\varphi_j\}$ is a basis. The injectivity of $F$ is proved.

To complete the proof one has to check that the range of $F$ is the whole space $H^1$. Let us do this. Take an arbitrary $f \in H^1$ and define $h := T^{-1} f \in H^0$. Let $h = \sum_{j=1}^{\infty} c_j \varphi_j$, then

$$Th = f = \sum_{j=-1}^{\infty} c_j T \varphi_j = F \sum_{j=1}^{\infty} c_j e_j.$$

Therefore $F$ is an isomorphism of $H^1$ onto $H^1$, and $\{T\varphi_j\}$ is a Riesz basis of $H^1$, as claimed. Lemma 2.1 is proved.
Let us summarize the proposed method for solving the basic equation (1.7):

**Step 1. Choose a Riesz basis** \( \{ \varphi_j \}_{1 \leq j < \infty} \) in \( H^0 = L^2(S) \).

We discuss this choice below.

**Step 2. Calculate the matrix entries** \( a_{ij} \) and the numbers \( f_i, \quad 1 \leq i, j \leq J \), where \( J \) is an a priori chosen integer.

**Step 3. Solve linear system** (2.2) **numerically.**

The matrix in (2.2) has condition number that remains bounded when \( J \) grows, as follows from Lemma 2.1.

Let us discuss the choice of the basis \( \{ \varphi_j \} \).

Assume that \( r = r(\theta) \) is the equation of \( S \) in the two-dimensional case that is, \( S \) is star-shaped. The element of the arc length of \( S \) is \( ds = \sqrt{r'^2(\theta) + r^2(\theta)} d\theta := a(\theta) d\theta \). Let \( S^1 \) denote the unit sphere \( S^1 := \{ x : x \in \mathbb{R}^2, |x| = 1 \} \). Choose

\[
\varphi_{co}(s) = \frac{1}{\sqrt{2\pi a(\theta)}}, \quad \varphi_{cm}(s) := \frac{\cos(m\theta)}{\sqrt{\pi a(\theta)}} \quad \varphi_{sm}(s) = \frac{\sin(m\theta)}{\sqrt{\pi a(\theta)}}, \quad m = 1, 2, \ldots
\]  

(2.6)

where \( s = (r(\theta), \theta) \).

Then

\[
\int_s \varphi_{cm}(s) \varphi_{sm'}(s) ds = \frac{1}{\pi} \int_0^{2\pi} \cos(m\theta) \sin(m'\theta) d\theta = 0,
\]

and

\[
\int_s \varphi_{cm} \varphi_{cm'} ds = \delta_{mm'}, \quad \int_s \varphi_{sm} \varphi_{sm'} ds = \delta_{mm'},
\]

so that \( \{ \varphi_m(s) \} \) is not only a Riesz basis of \( H^0 = L^2(S) \), but an orthonormal basis of \( H^0 \).

Similar construction holds in \( \mathbb{R}^3 \), where the normalized spherical harmonics \( Y_{\ell m}(\alpha) \) are used in place of \( \cos(m\theta) \) and \( \sin(m\theta) \), \( \alpha = (\theta, \varphi) \) is the unit vector in \( \mathbb{R}^3, \alpha \in S^2 \), \( S^2 \) is the unit sphere in \( \mathbb{R}^3 \).

**References**

[1] Ramm, A.G., Scattering by Obstacles, D. Reidel, Dordrecht, 1986.

[2] Varadan, V.K., Varadan, V.V., (editors), Acoustic, Electromagnetic and Elastic Wave Scattering- Focus on the T-matrix approach, Pergamon Press, New York, 1980.