Efficient universal blind computation

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We give a cheat sensitive protocol for blind universal quantum computation that is efficient in terms of computational and communication resources: it allows one party to perform an arbitrary computation on a second party’s quantum computer without revealing either which computation is performed, or its input and output. The first party’s computational capabilities can be extremely limited: she must only be able to create and measure single-qubit superposition states. The second party is not required to use measurement-based quantum computation. The protocol requires the (optimal) exchange of $O(J \log_2(N))$ single-qubit states, where $J$ is the computational depth and $N$ is the number of qubits needed for the computation.

Blind computation allows one party (say Alice) who has limited computational power, to use the computational resources of another party (say Bob), without revealing which computation she performs, nor her input and output data. As one expects, arbitrary blind computation is impossible using a classical computer. Surprisingly, arbitrary blind computation is instead possible on a quantum computer, achieving unconditional security premised only on the correctness of quantum physics, similar to that achieved for quantum key distribution. Here we propose a blind universal computation protocol that is efficient in terms of communication between Alice and Bob. Differently from previous proposals, our scheme is based on a cheat-sensitive strategy: Alice can detect whether a dishonest Bob is trying to ascertain the computation she wishes to perform. Moreover, in contrast to the one-time-pad protocol of Childs, ours does not require the computational qubits to be exchanged between Alice and Bob and, in contrast to the BFK protocol, it does not require measurement-based computation, but is described in the circuit model. Our protocol requires only $O(J \log_2(N))$ qubits to be exchanged between Alice and Bob ($J$ the computation depth, $N$ the number of qubits required for the computation), achieving an exponential gain in $N$ in communication complexity over previous protocols, that require $O(NJ)$ communication overhead. As in the previous schemes, Alice’s computational capabilities can be extremely limited: she must only generate and measure single-qubit states in the computational $\{|0\rangle, |1\rangle\}$ or the complementary $\{|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}\}$ bases.

The main idea is that Alice communicates to Bob the gates that he needs to apply using a Quantum Private Query (QPQ)-inspired protocol: she encodes this information into a quantum register and she randomly intersperses her communication with decoys. Bob must apply the gates blindly and send back her register without extracting information from it. If he does try to extract the information, Alice can detect this from a single-qubit measurement of the decoy states she received back and she can interrupt her computation. In this way Bob can determine at most a constant number of steps of the computation before Alice has a high probability of detecting that he is cheating. Asymptotically in $J$, he thus obtains no information on her computation. Moreover, it is easy for Alice to hide both her input data (since the encoding of the input state is part of the computation) and her output data (since she can instruct Bob to make random flips of the output state bits prior to the final measurement). Finally, by adapting the approach proposed in our scheme can allow the computationally-limited Alice to test whether Bob is performing the computation requested.

The protocol:— Bob controls a quantum computing facility which includes a quantum memory $\mathcal{M}$ composed of $N$ qubits initialized in a fiducial state (say the vector $|0\rangle^\otimes N$) and a set $\mathcal{G}$ of $O(\text{poly}(N))$ universal gates he can apply to them. To fix the notation, we can assume for instance that $\mathcal{G}$ contains $G = N(N + 2)$ elements including a Hadamard and a $\pi/8$ gate for each qubit and a C-NOT for each (ordered) couple of qubits of $\mathcal{M}$. In this scenario, Alice can instruct him to perform an arbitrary computation by telling him which elements of $\mathcal{G}$ he must apply at each step of the computation. For example, Alice can send a number $n$ between 0 and $G - 1$, which Bob interprets in the following way: 0 to $N - 1$ means “act with a Hadamard gate on qubit $n$”, $N$ to $2N - 1$ means “act with a $\pi/8$ gate on qubit $n - N$”, and any other number means “act with a C-NOT gate using $n_1$ as control and $n_2$ as target”, where $n_1, n_2$ are such that $n = n_1 N + n_2 + 2N$. For this code, she needs a log$_2 G \simeq O(\log_2 N)$ bit register. She can thus instruct Bob to perform an arbitrary $J$-step computation by giving him $J \log_2 G \simeq O(J \log_2 N)$ bits. This communication cost is optimal, because a programmable quantum computer requires a program register of dimension at least as large as the number of possible computations that it can perform (since at each step Bob can apply...
one out of $G$ possible gates, in our case such number is indeed equal to $G^J$, which requires $J \log_2 G$ bits).

To achieve blind computation, Alice must intersperse her instructions to Bob with decoy queries (we shall see in the following that this can be done with a linear overhead in terms of communication and size of the memory $\mathcal{M}$). Namely, at each computational step, she sends a register $A$ of $\log_2 G \simeq O(\log_2 N)$ qubits. This register contains either a plain instruction for Bob or, at random times, a quantum decoy. Plain instructions are encoded by preparing $A$ in a state $|n\rangle_A$ of the computational basis: for $n \in \{0, \cdots, G-1\}$, it indicates to Bob that the $n$-th element $U_n$ of the set $\mathcal{G}$ must be applied to the memory $\mathcal{M}$. In contrast, quantum decoys are prepared by creating superposition of instructions of the form $\sum_{n \in D} \eta_n |n\rangle_A$ where $D$ is a subset of $\{0, \cdots, G-1\}$ containing at least two elements, $\eta_n \in \{-1,1\}$ (for ease of notation, normalizations are dropped). To do so, as in the BB84 protocol \cite{BB84}, it is sufficient for Alice to initialize her register $A$ into factorized states where some of the qubits are in the computational basis $\{|0\rangle, |1\rangle\}$ while the others are in the basis $\{|+, -\rangle\}$ (a task she can achieve even with limited computational capabilities). For instance, (in the simplified case of two-qubit register) she can produce a quantum decoy where the instructions $|n = 0\rangle_A$ and $|n = 2\rangle_A$ are superimposed by sending the qubits of $A$ in the factorized state $|+, 0\rangle = |0, 0\rangle_A + |1, 0\rangle_A$ \cite{BB84}.

Since Alice is sending plain messages and decoys at random, Bob cannot perform even partial measurements in the computational basis without risking disrupting the coherence of Alice’s superpositions. Still he can use the register $A$ (without measuring it) as a quantum control to blindly trigger his operation on $\mathcal{M}$ e.g. by employing a qRAM (quantum Random Access Memory) \cite{qRAM}. Specifically, for each state $|\Psi\rangle_{\mathcal{M}}$ of $\mathcal{M}$ and $|\phi\rangle_A = \sum_n \alpha_n |n\rangle_A$ of $A$ he performs the control-unitary gate $U_{Bob} = \sum_n |n\rangle_A \otimes U_n$ which yields the mapping

$$|\phi\rangle_A \otimes |\Psi\rangle_{\mathcal{M}} \rightarrow \sum_n \alpha_n |n\rangle_A \otimes U_n |\Psi\rangle_{\mathcal{M}}. \quad (1)$$

After this operation, Bob sends the register $A$ back to Alice. Alice now checks her decoys to verify that Bob, when processing $A$, did not try to “read” its information content. There are two possible cases: her decoys can be unentangled from Bob’s qubits or they can be entangled.

Case (a): The first case happens whenever Alice’s decoy is a superposition of $n$, $n'$, $n''$, $n'''$, ..., such that

$$U_n |\Psi\rangle_{\mathcal{M}} = U_{n'} |\Psi\rangle_{\mathcal{M}} = U_{n''} |\Psi\rangle_{\mathcal{M}} = \cdots, \quad (2)$$

(e.g. Alice, through the register $|n\rangle_A + |n'\rangle_A$, instructs Bob to apply a $\pi/8$ gate to a superposition of two qubits of $\mathcal{M}$ which she knows are initially in $|0\rangle$). In this case, if Bob follows the protocol, the final state of $A$ and $\mathcal{M}$ is factorized and the $A$ register is unchanged, see Eq. \cite{factorization}. In contrast, if Bob had measured the register $A$ (or had entangled it with an ancilla), the superposition will be corrupted. Alice can exploit this to monitor Bob: she measures the register he sent back (single-qubit measurements suffice) and checks that the results match the qubits she had sent Bob. If they do not match, the superposition of one of her decoys was collapsed: she knows that Bob is trying to find out the values encoded in her registers and she stops the computation. Otherwise, she can assume that Bob has not obtained information on which computation she instructed him to perform. Alice can easily enforce Eq. \cite{security} by devoting a (random, secret) subset of Bob’s memory $\mathcal{M}$ to doing trivial operations on decoy states (e.g. keeping some qubits in that subset in a state $|0\rangle$ on which decoys composed of $\pi/8$ gates and C-NOTs act trivially, and other qubits in eigenstates of the Hadamard, on which Hadamard decoys act trivially).

Case (b): The second case happens whenever Eq. \cite{security} is not satisfied and the register $A$ becomes entangled with Bob’s qubits [e.g. this could happen if Alice instructed him to apply a Hadamard gate to an equally weighted superposition of the two qubits initially in $|0\rangle$]. In this case she may disentangle $A$ from $\mathcal{M}$ by sending to Bob a new instruction which undoes the previous transformation, and then she can measure the qubits of $A$ and check whether they match with her original decoy. For most universal gate sets this can be easily done without requiring nontrivial quantum processing by Alice: when $U_n = U_n^\dag$ [e.g. for Hadamards and C-NOTs] she just needs to send back to Bob the same register $A$ she had received from him, using it as a subsequent instruction of the computation. For the $\pi/8$ gates the same result is obtained with 8 consecutive iterations. The need of bouncing back the same register multiple times does not weaken the security of the scheme. In fact, from Bob’s point of view this is just equivalent to him seeing such states for a longer time: any coherent cheating strategy he can apply to the successive iterations of such states is equivalent to a strategy that he applies the first time he sees them. In other words, he does not gain any advantage from the fact that Alice is sending them multiple times. Bob may become entangled with Alice’s register during the protocol, but, importantly, he must be disentangled before Alice measures: he cannot retain any information by the time the computation ends.

Summarizing, if Alice knows that the qubits she received back from Bob are factorized from his computation qubits, she measures them and checks whether they match the decoy state she had sent him. If, instead, she does not know it, she bounces back to Bob the register $A$ (either twice or 8 times depending on the gates involved) and then she measures it. If her measurements disagree with the state she had originally sent him, she is certain that Bob is trying to extract information from her queries. The protocol is summarized in table \cite{summary}.

**Security analysis:** The protocol security is based on the fact that Bob does not know whether each single reg-
Bob initializes his qubits in $|0\rangle^N$.

2. The $j$th computation step: Alice sends Bob a register $A$ of $O(\log_2 N)$ qubits. It (randomly) either contains the qubit to which a gate is to be applied, e.g. $|3\rangle_A$ means “apply the Hadamard gate to qubit #3”, or it contains a decoy (e.g. $|n\rangle_A + |n\rangle_A$).

3. Bob uses Alice’s register to establish to which qubits to apply the gates of the universal set: e.g. Bob’s action $U_{Bob}|3\rangle_A|\Psi\rangle_M$ applies the Hadamard gate to Bob’s qubit #3 (here $|\Psi\rangle_M$ represents the global state of Bob’s qubits). If the register contains a decoy, he will apply the gates to a superposition of registers.

4. Bob sends the register $A$ back to Alice.

5. If Alice knows that the register $A$ is unentangled from Bob’s qubits, she measures it [case (a), see text]. Otherwise she sends it back to Bob as one of the successive instructions until it becomes unentangled, and then measures it [case (b)]. If the measurement result matches the state she had initially prepared, she proceeds to the next step of the computation through point 2, otherwise she halts the computation.

6. At the end of the computation, Bob measures the computation qubits and reveals the computation result (possibly encrypted, see text).

| TABLE I: Scheme of the protocol. |
|----------------------------------|

The probability that Bob can cheat for $j$ computational steps without being detected by Alice decreases exponentially as $p^{-\gamma j}$, where $\gamma$ is the average fraction of instructions that are decoys and $p$ is the probability of being detected on a single decoy. So on average he can cheat only for a constant number of steps before triggering Alice’s cheat detection. He will thus be able to obtain information only on a constant fraction of Alice’s gates before she stops the protocol: asymptotically in $J$ he obtains no information on Alice’s computation. In contrast to the cheat-sensitivity of the QPQ or of the protocol of [1], a cheating Bob cannot obtain the full information here (strong cheat-sensitivity).

Resource accounting:— We now give a resource-accounting of the protocol. As stated above, this protocol is optimal in terms of the number of exchanged bits of information between Alice and Bob, as a universal quantum computer cannot have a software register of less than $O(J \log_2 N)$ bits. The blind protocol simply requires these to be qubits instead of bits and requires a small overhead composed by the decoys and the final one-time-pad encoding (see below). The total communication complexity is then still $O(J \log_2 N)$ qubits. The running time overhead will be linear (a constant fraction $\gamma$ of the operations will be decoy operations), so the algorithm running time will be $O((1 + \gamma)J)$. The fraction $\gamma$ can be chosen arbitrarily small for $J \to \infty$ as (ignoring logarithmic corrections) it can scale as $J^{-1/2}$, see [23]. The overhead in terms of qubits is linear: we need a constant fraction $J$ of qubits to be devoted to Alice’s decoy operations, so that the qubit cost goes from $N$ to $N(1 + \gamma)$. In terms of gates, there is no overhead with respect to what is necessary for a universal programmable quantum computer [18]. The only difference is that the software is encoded in quantum bits (Alice’s register) instead of a sequence of classical numbers. This means that Bob needs controlled-swaps that are controlled by a quantum register instead of a classical register that would be sufficient for a programmable quantum computer. Summarizing, except for logarithmic or constant corrections, the blind computation protocol proposed here does not require a significant computational or communication overhead over what is necessary for a universal programmable quantum computer: the only substantial difference is that the software (Alice’s registers) is encoded in qubits instead of bits.

Add-ons:— In addition to guaranteeing the privacy of Alice’s computation, our protocol ensures also privacy of the input and output data. The protection of the input is a trivial consequence of the fact that the preparation of the initial state is included in the algorithm [remember that Bob’s quantum computer starts from a fiducial state e.g. $|0\rangle^N$]. To protect the output Alice can instruct Bob to perform random bit flips on the computation qubits in the same basis on which he will perform the computation’s final measurement. This means that Bob’s outcome will be randomized with a one-time-pad of which only Alice has the key: it will be secure from anyone else.
In the protocol described up to now Alice cannot ascertain whether Bob is indeed performing the computation she has requested. Even a non-cheating Bob could still be uncooperative and perform a different computation. Interestingly, even though Alice has limited computational capabilities, she can still check that Bob is cooperating using the ideas of “interacting proofs” described in [16] and adapted to blind computation in [3, 7, 11]. The basic idea is very simple: hidden in her computation, Alice places some trap qubits. Since Bob does not know the position of the traps, he will flip the trap qubits with high probability if he does not follow Alice’s instructions. It is also possible to use quantum error correction codes to increase the probability that an uncooperative Bob flips the trap [3, 7, 11, 16].

Conclusions:— We presented a scheme for performing universal blind quantum computation where both Alice’s algorithm and her input-output data are hidden from anyone else. It is efficient in terms of communication and computational resources. It is a cheat-sensitive scheme: if Bob tries to extract information from Alice’s registers or to perform a different computation, she can find it out and stop the protocol before he gains a significant fraction of the information on the computation.

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