A STOCHASTIC BENJAMIN-BONA-MAHONY TYPE EQUATION

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ABSTRACT. Considered herein is a particular nonlinear dispersive stochastic equation. It was introduced recently in [4], as a model describing surface water waves under location uncertainty. The corresponding noise term is introduced through a Hamiltonian formulation, which guarantees the energy conservation of the flow. Here the initial-value problem is studied.

1. Introduction

Consideration is given to the following Stratonovich one-dimensional BBM-type equation
\[ du = -\partial_x K(u + Ku^2) \, dt + \sum_j \gamma_j \partial_x (u + Ku)^2 \circ dW_j \] (1.1)
introduced in [4], as a model describing surface waves of a fluid layer. It is supplemented with the initial condition \( u(0) = u_0 \).

Equation (1.1) has a Hamiltonian structure with the energy
\[ \mathcal{H}(u) = \int_{\mathbb{R}} \left( \frac{1}{2} (K^{-1/2}u)^2 + \frac{1}{3} u^3 \right) \, dx. \] (1.2)

The Fourier multiplier operator \( K \), defined in the space of tempered distributions \( \mathcal{S}'(\mathbb{R}) \), has an even symbol of the form
\[ K(\xi) \simeq (1 + \xi^2)^{-\sigma_0} \] (1.3)
with \( \sigma_0 > 1/2 \). Expression (1.3) means that the symbol \( K(\xi) \) is bounded from below and above by RHS(1.3) multiplied by some positive constants. In other words the operator \( K \) essentially behaves as the Bessel potential of order \( 2\sigma_0 \), see [6]. The space variable is \( x \in \mathbb{R} \) and the time variable is \( t \geq 0 \). The unknown \( u \) is a real valued function of these variables and of the probability variable \( \omega \in \Omega \), representing the free surface elevation in the fluid layer. The scalar sequence \( \{\gamma_j\} \) satisfies the restriction \( \sum_j \gamma_j^2 < \infty \), and \( \{W_j\} \) is a sequence of independent scalar Brownian motions on a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\).

Model (1.1) was introduced in [4], where an attempt to extend an elegant Hamiltonian formulation of [1] to the stochastic setting was made. We will just briefly comment on the methodology of [4]. The white noise is firstly introduced via the stochastic transport theory presented in [8], which is based on splitting of fluid particle motion into smooth and random movements. Then it is restricted to a particular Stratonovich form in order to respect the energy conservation. In particular, it provides us with a model having multiplicative noise of Hamiltonian structure. Finally, a long wave approximation results in simplified models as (1.1), for example.

One may notice that after discarding the nonlinear terms in Equation (1.1), the details can be seen in [4], the corresponding linearised initial-value problem can be solved exactly with the help of the fundamental multiplier operator
\[ S(t,t_0) = \exp \left[ -\partial_x K(t - t_0) + \sum_j \gamma_j \partial_x (W_j(t) - W_j(t_0)) \right], \] (1.4)
where \( t_0, t \in \mathbb{R} \). Note that it can be factorised as \( S(t,t_0) = S(t - t_0)S_W(t_0), \) where \( S(t) = \exp(-\partial_x Kt) \) is a unitary semi-group and \( S_W \) containing all the randomness coming from the Wiener
process is unitary as well. They obviously commute as bounded differential operators. We recall that $S(t)$ is defined via the Fourier transform $\hat{S}(S(t)\psi) = \exp(-i\xi K(t)\psi)$ for any $\psi \in S'(\mathbb{R})$ and $\hat{\psi} = \hat{3}\hat{\psi}$. Similarly, $S_W(t, t_0)$ is defined by the line

$$S_W(t, t_0)\psi = \hat{3}^{-1} \left( \xi \mapsto \exp \left( i\xi \sum_j \gamma_j(W_j(t) - W_j(t_0)) \right) \hat{\psi}(\xi) \right).$$

It allows us to represent (1.1) in the Duhamel form

$$u(t) = S(t, 0) \left( u_0 + \int_0^t S(0, s)f(u(s))ds + \sum_j \gamma_j \int_0^t S(0, s)g(u(s))dW_j(s) \right), \quad (1.5)$$

where

$$f(u) = -\partial_x K^2 u^2 + \sum_j \gamma_j^2 \partial_x K(u \partial_x Ku^2)$$

and

$$g(u) = \partial_x Ku^2.$$ 

Existence and uniqueness of solution to Equation (1.5) is under consideration. It is worth to point out that both $S_W$ and the stochastic integral in (1.5) are well defined. Indeed, appealing to Doobs’ inequalities for the submartingale $\sum_{j=1}^{n+m} \gamma_j W_j$ and the Itô-Nisio theorem one can show that $\sum \gamma_j W_j$ converges uniformly in time almost surely, in probability and in $L^2$ sense. If the integrand of the stochastic integral in (1.5) is in some Sobolev space $H^\sigma(\mathbb{R})$ for each $s$ and a.e. $\omega$, then we can understand this sum of integrals as an integration with respect to a $Q$-Wiener process associated with a Hilbert space $H$ and a non-negative symmetric trace class operator $Q$ having eigenvalues $\gamma_j^2$ and eigenfunctions $e_j$ forming an orthonormal basis in $H$. Then the corresponding integrand is the unbounded linear operator between $H$ and $H^\sigma(\mathbb{R})$ that maps all $e_j$ to the same element of $H^\sigma(\mathbb{R})$, namely, to $S(0, s)g(u(s))$. In particular, it explains why we need the summability condition $\sum_j \gamma_j^2 < \infty$.

Before we formulate the main result it is left to introduce a notation as follows. By $C(0, T; H^\sigma(\mathbb{R}))$ we will note the space of continuous functions on [0, T] having values in $H^\sigma(\mathbb{R})$ with the usual supremum norm.

**Theorem 1.** Let $\sigma_0 > 1/2$ and $\sigma \geq \max\{\sigma_0, 1\}$. Then for any $F_0$-measurable $u_0 \in H^2(\Omega; H^\sigma(\mathbb{R})) \cap L^\infty(\Omega; H^{\sigma_0}(\mathbb{R}))$ with sufficiently small $L^\infty H^{\sigma_0}$-norm and any $T_0 > 0$ Equation (1.5) has a unique adapted solution $u \in L^2(\Omega; C(0, T_0; H^\sigma(\mathbb{R}))) \cap L^\infty(\Omega; C(0, T_0; H^{\sigma_0}(\mathbb{R}))).$ Moreover, $H(u(t)) = H(u_0)$ for each $t \in [0, T_0]$ almost surely on $\Omega$.

The conservation of energy (1.2) plays a crucial role in the proof. So it will be a bit more convenient to regard the energy norm defined by

$$\|u\|_H^2 = \frac{1}{2} \int_R (K^{-1/2}u)^2 dx$$

instead of the spatial $H^{\sigma_0}$-norm. They are obviously equivalent.

The proof is essentially based on the contraction mapping principle. We do not exploit much smoothing properties of the group $S(t, t_0)$, as for example is done in [2] for analysis of a stochastic nonlinear Schrödinger equation. It is enough to know that the absolute value of its symbol equals one, and that $S(t)$ is a unitary semigroup. However, in order to appeal to the fixed point theorem we have to truncate both deterministic $f$ and random $g$ nonlinearities. There are a couple of technical difficulties related to implementation of the energy conservation in our case. Firstly, for the truncated equation we can claim $H$-conservation only until a particular stopping time. Secondly, one can control $\|u\|_H$ with $H(u)$ only provided $\|u\|_H$ is small. These additional difficulties make us repeat the arguments of the last section in the paper iteratively in order to construct solution on the whole time interval $[0, T_0]$. 


As a final remark we point out that the noise in Equation (1.1) can be gathered in one dimensional \( \partial_x (u + Ku^2) \circ dB \) with the scalar Brownian motion \( B = \sum_j \gamma_j W_j \). However, this does not affect the proof below anyhow, so we continue to stick to the original formulation (1.1). In future works we are planning to extend it to \( \gamma_j \) being either Fourier multipliers or space-dependent coefficients.

2. Truncation

The Sobolev space \( H^\sigma(\mathbb{R}) \) consists of tempered distributions \( u \) having the finite square norm \( \|u\|_{H^\sigma}^2 = \int |\hat{u}(\xi)|^2 (1 + \xi^2)^\sigma d\xi < \infty \). Let \( \theta \in C_0^\infty(\mathbb{R}) \) with \( \sup \theta \in [-2, 2] \) being such that \( \theta(x) = 1 \) for \( x \in [-1, 1] \) and \( 0 \leq \theta(x) \leq 1 \) for \( x \in \mathbb{R} \). For \( R > 0 \) we introduce the cut off \( \theta_R(x) = \theta(x/R) \) and

\[
f_R(u) = \theta_R(\|u\|_{H^\sigma}) f(u), \quad g_R(u) = \theta_R(\|u\|_{H^\sigma}) g(u)
\]

that we substitute in (1.5) instead of \( f(u), g(u) \), respectively. The new \( R \)-regularisation of (1.5) reads as

\[
u(t) = S(t, t_0) \left( u(t_0) + \int_{t_0}^t S(t_0, s)f_R(u(s))ds + \sum_j \gamma_j \int_{t_0}^t S(t_0, s)g_R(u(s))dW_j(s) \right). \tag{2.1}
\]

In this section without loss of generality we can set \( t_0 = 0 \) and \( u(t_0) = u_0 \). We will vary time moments \( t_0 \) below in the next section. Equation (2.1) can be solved with a help of the contraction mapping principle in \( L^2(\Omega; C(0, T; H^\sigma(\mathbb{R})))). \)

**Proposition 1.** Let \( \sigma > 1/2, u_0 \in L^2(\Omega; H^\sigma(\mathbb{R})) \) be \( F_0 \)-measurable and \( T_0 > 0 \). Then (2.1) has a unique adapted solution \( u \in L^2(\Omega; C(0, T_0; H^\sigma(\mathbb{R}))). \) Moreover, it depends continuously on the initial data \( u_0 \).

**Proof.** We set \( \mathcal{T}u(t) = \text{RHS}(2.1) \). We will show that \( \mathcal{T} \) is a contraction mapping in \( X_T = L^2(\Omega; C(0, T; H^\sigma(\mathbb{R}))) \), provided \( T > 0 \) is sufficiently small, depending only on \( R \). Let \( u_1, u_2 \) be two adapted processes in \( X_T \). Firstly, one can notice that

\[
\|f_R(u_1) - f_R(u_2)\|_{H^\sigma} \leq C(1 + R)^2 \|u_1 - u_2\|_{H^\sigma},
\]

\[
\|g_R(u_1) - g_R(u_2)\|_{H^\sigma} \leq CR\|u_1 - u_2\|_{H^\sigma}.
\]

Indeed, \( H^\sigma(\mathbb{R}) \) poses an algebraic property for \( \sigma > 1/2 \) and \( \partial_x K \) is bounded in \( H^\sigma(\mathbb{R}) \). Then assuming \( \|u_1\|_{H^\sigma} \geq \|u_2\|_{H^\sigma} \) without loss of generality one deduces

\[
\|g_R(u_1) - g_R(u_2)\|_{H^\sigma} \leq C\|\theta_R(\|u_1\|_{H^\sigma})u_1^2 - \theta_R(\|u_2\|_{H^\sigma})u_2^2\|_{H^\sigma} \leq CR\|u_1 - u_2\|_{H^\sigma},
\]

where we have used the estimate \( |\theta_R(\|u_1\|_{H^\sigma}) - \theta_R(\|u_2\|_{H^\sigma})| \leq \|\theta'|_{L^\infty} R^{-1}\|u_1 - u_2\|_{H^\sigma} \) following obviously from the mean value theorem. The difference between \( f_R(u_1) \) and \( f_R(u_2) \) can be obtained in the same way. Thus

\[
\|\mathcal{T}u_1(t) - \mathcal{T}u_2(t)\|_{H^\sigma} \leq \left\| \int_0^t S(0, s)(f_R(u_1(s)) - f_R(u_2(s)))ds \right\|_{H^\sigma} + \left\| \sum_j \gamma_j \int_0^t S(0, s)(g_R(u_1(s)) - g_R(u_2(s)))dW_j(s) \right\|_{H^\sigma} = I + II.
\]

The first integral is estimated straightforwardly as

\[
I \leq \int_0^T \|f_R(u_1(s)) - f_R(u_2(s))\|_{H^\sigma} ds \leq C(1 + R)^2 T \|u_1 - u_2\|_{C(0, T; H^\sigma)}.
\]

The second one is estimated with the use of the Burkholder inequality [5] as

\[
\mathbb{E} \sup_{0 \leq t \leq T} II^2 \leq CE \int_0^T \|g_R(u_1(s)) - g_R(u_2(s))\|_{H^\sigma}^2 ds \leq CR^2 T \mathbb{E} \|u_1 - u_2\|_{C(0, T; H^\sigma)}^2.
\]
It is clear that time-continuity of $\mathcal{T}u_1, \mathcal{T}u_2$ follows from the factorisation $\mathcal{S} = SS_W$ and the estimate $\|S_W g_R(u)\|_{H^\theta} \leq CR^2$, so we have a stochastic convolution as in [5], Lemma 3.3. Thus

$$\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{X_T} \leq C \left( (1 + R)^2 T + R\sqrt{T} \right) \|u_1 - u_2\|_{X_T},$$

and so there exists a small $T$ depending only on $R$ such that $\mathcal{T}$ has a unique fixed point in $X_T$. Moreover, this estimate also gives us continuous dependence of solution in $X_T$ on the initial data $u_0 \in L^2(\Omega; H^\theta(\mathbb{R}))$, obviously. Clearly, the solution can be extended to the whole interval $[0, T_0]$.

The regularisation affects the energy conservation. Indeed, in the Itô differential form Equation (2.1) reads

$$du = \left( -\partial_x Ku + \frac{1}{2} \sum_j \gamma_j^2 \partial_x^2 u + f_R(u) + \sum_j \gamma_j^2 \partial_x g_R(u) \right) dt + \sum_j \gamma_j (\partial_x u + g_R(u)) dW_j, \quad (2.2)$$

and so applying the Itô formula to the energy functional $\mathcal{H}(u(t))$ defined by (1.2) with the use of (2.2), one can easily obtain

$$d\mathcal{H}(u) = \left( (\theta_R - 1) \int u^2 \partial_x Ku \, dx + \theta_R (\theta_R - 1) \sum_j \gamma_j^2 \int \left( \frac{1}{2} g(u) K^{-1} g(u) + ug^2(u) \right) dx \right) dt. \quad (2.3)$$

Indeed, assuming $\sigma \geq \sigma_0 + 2$ at first, we notice that the solution $u$ given by Proposition 1 solves Equation (2.2). Let us introduce the following notations

$$\Psi(t) dt + \Phi(t) dW = \Psi(t) dt + \sum_j \gamma_j \Phi(t) e_j dW_j = \text{RHS}(2.2).$$

Then Itô’s formula reads

$$\mathcal{H}(u(t)) = \mathcal{H}(u_0) + \int_0^t \partial_u \mathcal{H}(u(s)) \Psi(s) ds + \int_0^t \partial_u \mathcal{H}(u(s)) \Phi(s) dW(s) + \frac{1}{2} \int_0^t \text{tr} \partial^2_u \mathcal{H}(u(s))(\Phi(s), \Phi(s)) ds,$$

where the Fréchet derivatives are defined by

$$\partial_u \mathcal{H}(u) \phi = \int_{\mathbb{R}} \left( K^{-1/2} u K^{-1/2} \phi + u^2 \phi \right) dx,$$

$$\partial^2_u \mathcal{H}(u)(\phi, \psi) = \int_{\mathbb{R}} \left( K^{-1/2} \phi K^{-1/2} \psi + 2u \phi \psi \right) dx$$

at every $\phi, \psi \in H^{\sigma_0}(\mathbb{R})$. Substituting these expressions together with the definitions of $\Phi$ and $\Psi$ into the Itô’s formula one obtains (2.3). Let us, for example, calculate the stochastic integral

$$\int_0^t \partial_u \mathcal{H}(u(s)) \Phi(s) dW(s) = \sum_j \gamma_j \int_0^t \left( K^{-1/2} u K^{-1/2} + u^2 \right) (\partial_x u + \theta_R(\|u\|_{H^\theta}) \partial_x Ku^2) \, dx dW_j$$

that equals zero as one can see integrating by parts in the space integral. Similarly, one calculates the other two integrals in the Itô formula. Thus we have proved (2.3) for $\sigma \geq \sigma_0 + 2$. In order to lower the bound for $\sigma$, one would like to argue here by approximation of initial value $u_0$ via smooth functions and appeal to the continuous dependence on $u_0$, however, there is a problem here, since $\theta_R$ in (2.3) contains the dependence on $\sigma$. So even for a smooth initial data the corresponding solution lies a priori only in $H^\sigma$. This difficulty is overcome in the next statement, where we argue similar to [3].

**Proposition 2.** Let $\sigma_0 > 1/2$ and $\sigma \geq \max\{\sigma_0, 1\}$. Then (2.3) holds almost surely for $u$ satisfying Equation (2.1) given by Proposition 1.
Proof. The main idea is to cut off high frequencies of the differential operator $\partial_x$ in (2.2) as follows. Let $P_\lambda$ be a Fourier multiplier with the symbol $\theta_\lambda$, $\lambda > 0$. It is defined by the expression $\widehat{\theta_\lambda (P_\lambda \psi)} = \theta_\lambda \widehat{\psi}$. Now we consider instead of (2.2) the following regularisation
\[
du = \left(-\partial_x K u + \frac{1}{2} \sum_j \gamma_j^2 \partial_x^2 P^2 u + f_R(u) + \sum_j \gamma_j^2 \partial_x P_\lambda g_R(u)\right) dt + \sum_j \gamma_j (\partial_x P_\lambda u + g_R(u)) dW_j
\]
that has a strong solution. Indeed, it contains only bounded operators and the corresponding mild equation has exactly the same form as Equation (2.1) with $S^\lambda = S S^\lambda_W$ now instead of $S$, where
\[
S^\lambda_W = \exp \left[ \sum_j \gamma_j \partial_x P_\lambda (W_j(t) - W_j(t_0)) \right].
\]
So we can actually apply Proposition 1 to obtain $u = u_\lambda$ solving (2.4). Let $u = u_\infty$ stay for the solution of the original equation (2.1). Firstly, we will check that $u_\lambda \to u_\infty$ in $L^2(\Omega; L^2(0, T_0; H^s(\mathbb{R})))$ for any $\sigma > 1/2$ as $\lambda \to \infty$.

Let $0 \leq t \leq T \leq T_0$, where a positive small enough time moment $T$ is to be chosen below. Then
\[
\begin{aligned}
&\|u_\lambda(t) - u_\infty(t)\|_{H^s} = \left\|T^\lambda u_\lambda(t) - T^\infty u_\infty(t)\right\|_{H^s} \leq \left\|\left(S^\lambda(t, 0) - S^\infty(t, 0)\right) u_0\right\|_{H^s} \\
&+ \left\|\int_0^t \left(S^\lambda(t, s) - S^\infty(t, s)\right) f_R(u_\infty(s)) ds\right\|_{H^s} + \left\|\int_0^t S^\lambda(t, s) (f_R(u_\lambda(s)) - f_R(u_\infty(s))) ds\right\|_{H^s} \\
&+ \left\|\left(S^\lambda(t, 0) - S^\infty(t, 0)\right) \sum_j \gamma_j \int_0^t S^\infty(0, s) g_R(u_\infty(s)) dW_j(s)\right\|_{H^s} \\
&+ \left\|\sum_j \gamma_j \int_0^t \left(S^\lambda(0, s) - S^\infty(0, s)\right) g_R(u_\lambda(s)) dW_j(s)\right\|_{H^s} \\
&+ \left\|\sum_j \gamma_j \int_0^t S^\lambda(0, s) (g_R(u_\lambda(s)) - g_R(u_\infty(s))) dW_j(s)\right\|_{H^s} = I_1 + \ldots + I_6.
\end{aligned}
\]

The terms $I_3$ and $I_6$ are estimated exactly as the analogous integrals $I$ and $II$ in the proof of Proposition 1, namely,
\[
I_3 \leq C (1 + R)^2 \sqrt{T} \|u_\lambda - u_\infty\|_{L^2(0, T; H^s)}
\]
and
\[
\mathbb{E} \sup_{0 \leq t \leq T} I_6^2 \leq C \mathbb{E} \int_0^T \|g_R(u_\lambda(s)) - g_R(u_\infty(s))\|^2_{H^s} ds \leq CR^2 \mathbb{E} \|u_\lambda - u_\infty\|^2_{L^2(0, T; H^s)}.
\]

Thus
\[
\mathbb{E} \int_0^T (I_3^2 + I_6^2) dt \leq C ((1 + R)^4 T^2 + R^2 T) \mathbb{E} \|u_\lambda - u_\infty\|^2_{L^2(0, T; H^s)},
\]
and so there exists a small $T > 0$ depending only on $R$ such that
\[
\mathbb{E} \|u_\lambda - u_\infty\|^2_{L^2(0, T; H^s)} \leq C \mathbb{E} \int_0^T (I_1^2 + I_2^2 + I_4^2 + I_5^2) dt.
\]

One needs to show that the right hand side of this expression tends to zero when $\lambda \to \infty$. All these four integrals are treated similarly. Indeed, let us regard more closely the first one
\[
I_1^2 = \int \left| \exp \left(i\xi \theta_\lambda(\xi) \sum_j \gamma_j W_j(t)\right) - \exp \left(i\xi \sum_j \gamma_j W_j(t)\right) \right|^2 |\bar{u}_0(\xi)|^2 (1 + \xi^2)^\sigma d\xi
\]
that obviously tends to zero as \( \lambda \to \infty \) for a.e. \( \omega \) and any \( t \). Hence \( \mathbb{E} \int_0^T I_2^2 \, dt \to 0 \) by the dominated convergence theorem, since \( I_1 \leq 2 \|u_0\|_{H^\sigma} \). The integral of \( I_2^2 \) is estimated exactly in the same manner with the stochastic integral of \( S^\infty g_R(u_{\infty}) \) standing in place of \( u_0 \). The second integral

\[
\mathbb{E} \int_0^T I_2^2 \, dt \leq T \mathbb{E} \int_0^T \int_0^T \left\| \left( S^\lambda(t, s) - S^\infty(t, s) \right) f_R(u_{\infty}(s)) \right\|_{H^\sigma}^2 \, ds \, dt \to 0
\]

due to the Burkholder inequality and the dominated convergence theorem, since \( \| \ldots \|_{H^\sigma} \leq CR^2(1 + R)^4 \). Finally, the last integral

\[
\mathbb{E} \int_0^T I_3^2 \, dt \leq T \mathbb{E} \sup_{t \in [0, T]} I_3^2 \leq C T \mathbb{E} \int_0^T \left( \left( S^\lambda(0, s) - S^\infty(0, s) \right) g_R(u_{\infty}(s)) \right)_{H^\sigma}^2 \, ds \to 0
\]

due to the Burkholder inequality and the dominated convergence theorem, since \( \| \ldots \|_{H^\sigma} \leq CR^4 \).

Repeating this argument iteratively on subintervals of \([0, T_0] \) of the size \( T \) one obtains that \( u_\lambda \to u_\infty \) in \( L^2(\Omega \times [0, T_0]; H^\sigma(\mathbb{R})) \).

Let us calculate each term in the Itô formula for \( u = u_\lambda \). As we shall see the corresponding stochastic integral is not zero, and moreover, it is difficult to pass to the limit \( \lambda \to \infty \) treating the stochastic part. So instead of \( \mathcal{H} \) we consider at first a sequence \( \mathcal{H}_n, n \in \mathbb{N} \), with the cubic term being cut off in the following way

\[
\mathcal{H}_n(u) = \|u\|_{\mathcal{H}}^2 + \frac{1}{3} \theta_n \left( \|u\|_{\mathcal{H}}^2 \right) \int u^3 \, dx,
\]

that clearly tends to \( \mathcal{H}(u) \) almost surely at any fixed time moment. The corresponding Fréchet derivatives are defined by

\[
\partial_u \mathcal{H}_n(u) \phi = \int_\mathbb{R} \left[ \left( 1 + \frac{1}{3} \theta_n' \left( \|u\|_{\mathcal{H}}^2 \right) \int u^3 \, dy \right) K^{-1/2} u K^{-1/2} \phi + \theta_n \left( \|u\|_{\mathcal{H}}^2 \right) u^2 \phi \right] \, dx,
\]

\[
\partial_u^2 \mathcal{H}_n(u)(\phi, \psi) = \int_\mathbb{R} \left[ \left( 1 + \frac{1}{3} \theta_n' \left( \|u\|_{\mathcal{H}}^2 \right) \int u^3 \, dx \right) K^{-1/2} \phi K^{-1/2} \psi + 2 \theta_n \left( \|u\|_{\mathcal{H}}^2 \right) u \phi \psi \right] \, dx
\]

\[
+ \theta_n' \left( \|u\|_{\mathcal{H}}^2 \right) \int u^2 \phi dx \int K^{-1/2} u K^{-1/2} \psi dy
\]

\[
+ \frac{1}{3} \theta_n'' \left( \|u\|_{\mathcal{H}}^2 \right) \int u^3 dx \int K^{-1/2} u K^{-1/2} \phi dy \int K^{-1/2} u K^{-1/2} \psi dz
\]

at every \( \phi, \psi \in H^\sigma_0(\mathbb{R}) \). Substituting it to the stochastic integral one obtains the following expression that can be simplified by integration by parts

\[
\int_0^t \partial_u \mathcal{H}_n(u(s)) \Phi(s) \, dW(s) = \sum_j \gamma_j \int_0^t \int_\mathbb{R} \left[ \left( 1 + \frac{1}{3} \theta_n' \left( \|u\|_{\mathcal{H}}^2 \right) \int u^3 \, dy \right) K^{-1/2} u K^{-1/2} + \theta_n \left( \|u\|_{\mathcal{H}}^2 \right) u^2 \right]
\]

\[
(\partial_x P_\lambda u + \theta_R(\|u\|_{H^\sigma}) \partial_x K u^2) \, dx \, dW_j = \sum_j \gamma_j \int_0^t \theta_n \left( \|u\|_{\mathcal{H}}^2 \right) \int \int_\mathbb{R} u^2 \partial_x P_\lambda u u \, dx \, dW_j,
\]

where \( u = u_\lambda \). We will show that this integral tends to zero as \( \lambda \to \infty \). That is exactly the place where we need the cut off \( \theta_n \). Applying some algebraic manipulations to the space integral and the
Burkholder inequality to the stochastic integral, one deduces the estimate
\[
\mathbb{E} \sup_{0 \leq t \leq T_0} \left| \int_0^t \partial_u \mathcal{H}_n(u(s)) \Phi(s) dW(s) \right|^2 \leq C \mathbb{E} \int_0^{T_0} \theta_n^2 \left( \left\| u_{\lambda}(t) \right\|^2_{\mathcal{H}} \right) \left( \int_{\mathbb{R}} u_{\lambda}(t) \partial_x (P_{\lambda} - 1) u_{\lambda}(t) dx \right)^2 \ dt 
\]
\[
\leq C \mathbb{E} \int_0^{T_0} \theta_n^2 \left( \left\| u_{\lambda}(t) \right\|^2_{\mathcal{H}} \right) \left\| u_{\lambda}(t) \right\|^4_{\mathcal{H}} \left( \left\| (P_{\lambda} - 1) u_{\infty}(t) \right\|^2_{H^{1/2}} + \left\| (P_{\lambda} - 1) (u_{\lambda}(t) - u_{\infty}(t)) \right\|^2_{H^{1/2}} \right) \ dt 
\]
\[
\leq C \mathbb{E} \int_0^{T_0} \left( \left\| (P_{\lambda} - 1) u_{\infty}(t) \right\|^2_{H^{1/2}} + \left\| (u_{\lambda}(t) - u_{\infty}(t)) \right\|^2_{H^{1/2}} \right) \ dt \to 0 
\]
as \lambda \to 0 \ for \ each \ fixed \ n \in \mathbb{N}. \ Note \ that \ the \ use \ of \ the \ functional \ \mathcal{H}_n \ instead \ of \ \mathcal{H} \ is \ important \ here. \ Similarly, \ we \ calculate \ the \ rest \ two \ terms \ in \ the \ Itô \ formula
\[
\partial_u \mathcal{H}_n(u) \Phi + \frac{1}{2} \mathrm{tr} \partial_u^2 \mathcal{H}(u)(\Phi, \Phi) = (\theta_R - \theta_n) \int u^2 \partial_x K u \ dx + \theta_n \theta_R (\theta_R - 1) \sum_j \gamma_j^2 \int u g^2(u) \ dx 
\]
\[
+ \frac{\theta_R (\theta_R - 1)}{2} \sum_j \int g(u) K^{-1} g(u) \ dx + \frac{\theta_n}{2} \sum_j \int (u^2 \partial_x^2 P_{\lambda}^2 u + 2u \partial_x P_{\lambda} u^2) \ dx 
\]
\[
+ \sum_j \int u^3 \ dy \left( \frac{\theta_R - 1}{2} \sum_j \int g(u) K^{-1} g(u) \ dx - \int u g(u) \ dx \right) = J_1 + \ldots + J_6, 
\]
where as above \( u = u_{\lambda} \). \ One \ can \ prove \ that \ for \ a.e. \ \omega \in \Omega \ and \ t \in [0, T_0] \ the \ first \ three \ terms \ \( J_1 + J_2 + J_3 \) \ tend \ to \ the \ integrand \ of \ the \ right \ hand \ side \ of \ Expression \ (2.3) \ in \ the \ subsequent \ limits, \ firstly, \ as \ \lambda \to \infty \ and \ then \ as \ n \to \infty. \ Both \ \( J_4 \) \ and \ \( J_5 \) \ tend \ to \ zero \ as \ \lambda \to \infty. \ Meanwhile \ the \ last \ term \ \( J_6 \) \ stays \ bounded \ by \ \( C/n \), \ and \ so \ \lim_{n \to \infty} \lim_{\lambda \to \infty} J_6 = 0. \ Let \ us \ show, \ for \ example, \ that \ \( J_4 \to 0 \) \ which \ is \ the \ most \ troublesome \ term \ in \ the \ sum, \ since \ here \ is \ the \ only \ place \ in \ the \ paper \ where \ we \ make \ use \ of \ the \ fact \ \sigma \geq 1. \ The \ rest \ are \ treated \ similarly \ without \ this \ additional \ restriction. \ Indeed,
\[
J_4 \leq C \left( \int (u \partial_x P_{\lambda} u - P_{\lambda}(u \partial_x u)) (P_{\lambda} - 1) \partial_x u dx \right) \leq C \left\| u_{\lambda} \right\|^2_{H^1} \left( \left\| (P_{\lambda} - 1) u_{\infty} \right\|^2_{H^1} + \left\| u_{\lambda} - u_{\infty} \right\|^2_{H^1} \right)
\]
that obviously tends to zero as \( \lambda \to \infty. \) \ This \ concludes \ the \ proof.

\[ \square \]

At this stage one cannot claim the energy conservation yet, so we will prove a weaker result that will be sharpened later. \ Note \ that \ there \ exists \ \( C_{\mathcal{H}} > 0 \) \ such \ that
\[
\left\| u \right\|^2_{\mathcal{H}} (1 - C_{\mathcal{H}} \left\| u \right\|_{\mathcal{H}}) \leq \mathcal{H}(u) \leq \left\| u \right\|^2_{\mathcal{H}} (1 + C_{\mathcal{H}} \left\| u \right\|_{\mathcal{H}}), \tag{2.5}
\]
following from the well-known embedding \( H^{\sigma_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \), \ recall \ that \ \( \sigma_0 > 1/2 \).

**Lemma 1.** \ There \ exists \ a \ constant \ \( T_1 > 0 \) \ independent \ of \ \omega \ such \ that \ if \ \( u \) \ solving \ Equation \ (2.1) \ has \ \left\| u \right\|_{\mathcal{H}} \leq \frac{1}{2C_{\mathcal{H}}} \) \ on \ some \ interval \ \([0, \tau] \) \ then \ \( \mathcal{H}(u) \leq 2\mathcal{H}(u(0)) \) \ on \ \([0, T_1 \wedge \tau] \).

**Proof.** \ At first one can notice \ that \ as \ long \ as \ \left\| u \right\|_{\mathcal{H}} \ stays \ bounded \ by \ \( (2C_{\mathcal{H}})^{-1} \), \ we \ have
\[
\frac{1}{2} \left\| u \right\|^2_{\mathcal{H}} \leq \mathcal{H}(u) \leq \frac{3}{2} \left\| u \right\|^2_{\mathcal{H}}.
\]
Moreover, \ one \ can \ as \ well \ easily \ deduce \ from \ (2.3) \ the \ following \ bound
\[
\mathcal{H}(u(t)) \leq \mathcal{H}(u(0)) + C \int_0^t \mathcal{H}(u(s)) \, ds,
\]
and \ so \ the \ proof \ is \ concluded \ by \ Grönwall’s \ lemma.

\[ \square \]
3. Proof of the main result

We construct a solution $u$ of \((1.5)\) iteratively on the intervals \([0, T_1], [T_1, 2T_1]\) and so on. Here the interval size $T_1$ is defined by Lemma \[1]\footnote{Staying under the assumptions of Theorem \[1]\footnote{where we subsequently set $t_0 = 0, T_1, 2T_1, \ldots$. We define the stopping times
$$
\tau_m = \tau_m^{t_0} = \inf \{t \in [0, T_0] : \|u_m(t)\|_{H^\sigma} > m\}
$$
with the agreement $\inf \emptyset = T_0$. Starting with $t_0 = 0$ we firstly show the following result.

**Lemma 2.** For a.e. \(\omega \in \Omega\), any $m \in \mathbb{N}$ and each $t \in [0, \tau]$ with \(\tau(\omega) = \min \{\tau_m(\omega), \tau_{m+1}(\omega)\}\), it holds true that $u_m(t) = u_{m+1}(t)$.

**Proof.** We define

$$
\tilde{u}_i(t) = \begin{cases} u_i(t) & \text{if } t \in [0, \tau] \\
S(t, \tau) u_i(\tau) & \text{if } t \in [\tau, T_0]
\end{cases}, \quad i = m, m + 1.
$$

At first we will show that $\tilde{u}_m$ and $\tilde{u}_{m+1}$ coincide in $X_T$ provided $T$ is sufficiently small. Then we will finish the proof by an iteration procedure. The difference of these functions has the form

$$
\tilde{u}_{m+1}(t) - \tilde{u}_m(t) = S(t, 0) \int_0^{t \wedge \tau} S(0, s) (f(\tilde{u}_{m+1}(s)) - f(\tilde{u}_m(s))) \, ds + S(t, 0) \sum_j \gamma_j \int_0^{t \wedge \tau} S(0, s) (g(\tilde{u}_{m+1}(s)) - g(\tilde{u}_m(s))) \, dW_j(s),
$$

where the stochastic integral is estimated via

$$
\mathbb{E} \sup_{0 \leq t \leq T} \left\| S_W(t, 0) \sum_j \gamma_j \int_0^t S(t - s) \chi_{s \leq \tau}(s) S_W(0, s) (g(\tilde{u}_{m+1}(s)) - g(\tilde{u}_m(s))) \, dW_j(s) \right\|_{H^\sigma}^2
\leq C \mathbb{E} \int_0^T \chi_{s \leq \tau}(s) \| S_W(0, s) (g(\tilde{u}_{m+1}(s)) - g(\tilde{u}_m(s))) \|_{H^\sigma}^2 \, ds
\leq C \mathbb{E} \int_0^T \chi_{s \leq \tau}(s) \left( \| \tilde{u}_{m+1}(s) \| + \| \tilde{u}_m(s) \|_{H^\sigma} \right)^2 \| \tilde{u}_{m+1}(s) - \tilde{u}_m(s) \|_{H^\sigma}^2 \, ds
\leq C (2m + 1)^2 T \mathbb{E} \sup_{[0, T]} \| \tilde{u}_{m+1} - \tilde{u}_m \|_{H^\sigma}^2
$$

with the help of the Burkholder inequality for convolution with the unitary group $S$, see \cite{[5]} Lemma 3.3. The first integral is estimated more straightforwardly, notice a similar argument employed to $I$ in the proof of Proposition \[1]\footnote{and so one obtains}

$$
\| \tilde{u}_{m+1} - \tilde{u}_m \|_{X_T} \leq C(m) \sqrt{T} \| \tilde{u}_{m+1} - \tilde{u}_m \|_{X_T}.
$$

Hence $\tilde{u}_{m+1} = \tilde{u}_m$ on $[0, T]$ for a.e. $\omega \in \Omega$ provided $T$ is chosen sufficiently small depending only on $m$. Thus we can iterate this procedure to show that $\tilde{u}_{m+1} = \tilde{u}_m$ on the whole interval $[0, T_0]$, which concludes the proof of the lemma.

Our goal is to bound $\|u_m\|_{L^2(C(0, T_1); H^\sigma)}$ by a constant independent of $m \in \mathbb{N}$, and so we will need to estimate $\|f(u_m)\|_{H^\sigma}, \|g(u_m)\|_{H^\sigma}$, in particular. This can be easily done with the help of

$$
\|\phi \psi\|_{H^\sigma} \leq C(\sigma, \sigma_0) (\|\phi\|_{H^\sigma} \|\psi\|_{H^{\sigma_0}} + \|\phi\|_{H^{\sigma_0}} \|\psi\|_{H^\sigma})
$$

being true for any $\sigma \geq 0$ and $\sigma_0 > 1/2$, see for example \cite{[7]} Estimate (3.12)].

For a.e. $\omega \in \Omega$ and any $m \in \mathbb{N}$, $t \in [0, T_0]$ we have

$$
\|u_m(t)\|_{H^\sigma} \leq \|u_0\|_{H^\sigma} + \int_0^t \|f(u_m(s))\|_{H^\sigma} \, ds + \left\| \sum_j \gamma_j \int_0^t S(0, s) g_m(u_m(s)) \, dW_j(s) \right\|_{H^\sigma},
$$
where $\|f(u_m(s))\|_{H^\sigma} \leq C \left( \|u_m(s)\|_{H^{\sigma_0}} + \|u_m(s)\|^2_{H^{\sigma_0}} \right) \|u_m(s)\|_{H^\sigma}$. Now taking into account that $\|S(0, s)g_m(u_m(s))\|_{H^\sigma} \leq C \|u_m(s)\|_{H^{\sigma_0}} \|u_m(s)\|_{H^\sigma}$, the stochastic integral can be estimated by the Burkholder inequality, and so we obtain for any $0 < T \leq T_0$ the following inequality

$$\mathbb{E} \sup_{t \in [0, T]} \|u_m(t)\|^2_{H^\sigma} \leq \mathbb{E} \|u_0\|^2_{H^\sigma} + C \mathbb{E} \int_0^T \left( \|u_m(t)\|^2_{H^{\sigma_0}} + \|u_m(t)\|^4_{H^{\sigma_0}} \right) \|u_m(t)\|^2_{H^\sigma} dt, \quad (3.2)$$

where $C$ depends only on $\sigma_0, \sigma, T_0, \sum \gamma_j^2$. This inequality we will use iteratively on the intervals $[0, T_0 \land kT_1]$, $k \in \mathbb{N}$, with $T_1$ found in Lemma 1. Let $\|u_0\|_{H} \leq (5C_H)^{-1}$ a.e. on $\Omega$. Consider the following stopping time

$$T_2^m = \inf \{t \in [0, T_0] : \|u_m(t)\|_{H} > (2C_H)^{-1} \}.$$

Then a.e. $T_1 \leq T_2^m$. Indeed, assuming the contrary $T_1 > T_2^m$ one can deduce from (2.5) and Lemma 1 that

$$\|u_m(T_2^m)\|_{H} \leq \sqrt{2H(u_m(T_2^m))} \leq 2 \sqrt{H(u_0)} \leq 2 \sqrt{1 + C_H \|u_0\|_{H}} \|u_0\|_{H} \leq \sqrt{\frac{24}{125} C_H^{-1} < (2C_H)^{-1}},$$

which contradicts the definition of the stopping time $T_2^m$ due to continuity of $\|u_m\|_{H}$. As a result $\|u_m\|_{H}$ stays bounded by $(2C_H)^{-1}$ on the interval $[0, T_1]$ for a.e. $\omega$, and this simplifies (3.2) in the following way

$$\mathbb{E} \sup_{t \in [0, T]} \|u_m(t)\|^2_{H^\sigma} \leq \mathbb{E} \|u_0\|^2_{H^\sigma} + C \int_0^T \mathbb{E} \sup_{s \in [0, t]} \|u_m(s)\|^2_{H^\sigma} dt$$

holding true for any $0 < T \leq T_1$. Hence by Grönwall’s lemma we obtain

$$\|u_m\|^2_{L^2C([0, T_1]; H^\sigma)} \leq 3 \|u_0\|^2_{L^2H^\sigma} e^{CT_1} = M,$$

where $M$ does not depend on $m \in \mathbb{N}$. Hence

$$\mathbb{P}(\tau_m \geq T_1) = \mathbb{P}\left( \|u_m\|_{C([0, T_1]; H^\sigma)} \leq m \right) \geq 1 - \frac{1}{m^2} \mathbb{E} \|u_m\|^2_{C([0, T_1]; H^\sigma)} \geq 1 - \frac{M}{m^2},$$

and so $[0, T_1] \subset \cup_{m \in \mathbb{N}} [0, \tau_m(\omega)]$ for a.e. $\omega \in \Omega$. Thus we can define $u$ on $[0, T_1]$ by assigning $u = u_m$ on $[0, T_1]$. This is obviously a solution of (1.5) on $[0, T_1]$ satisfying $dH(u) = 0$ and $\|u\|_{H} < (2C_H)^{-1}$ for a.e. $\omega \in \Omega$.

Now one can repeat the argument on $[T_1, 2T_1]$ by constructing new solutions $u_m$ of Equation (2.1) with the initial data $u(T_1)$ given at the time moment $t_0 = T_1$. The stopping times $\tau_m$ are defined by (3.11) with $t_0 = T_1$. The fact that $\|u_m\|_{H}$ does not exceed the level $(2C_H)^{-1}$, is guaranteed by the energy conservation, namely by $H(u(T_1)) = H(u_0)$ in the same manner as above. The rest is similar, and so we get a solution on $[T_1, 2T_1]$ with the constant energy equalled $H(u_0)$. After several repetitions of the argument we construct a solution on $[0, T_0]$.

It remains to prove the uniqueness. Let $u_1, u_2 \in L^2(\Omega; C([0, T_0]; H^\sigma(\mathbb{R})))$ solve Equation (1.5). For $R > 0$ we introduce

$$\tau_R = \inf \left\{ t \in [0, T_0] : \max_{i=1, 2} \|u_i(t)\|_{H^\sigma} > R \right\}.$$

Clearly, for a.e. $\omega \in \Omega$ both $u_1$ and $u_2$ are solutions of (2.1) on $[0, \tau_R]$. By Proposition 1 it holds true that $u_1 = u_2$ on $[0, \tau_R]$ for a.e. $\omega \in \Omega$. Taking $R \in \mathbb{N}$ and exploiting the time-continuity of $u_1, u_2$ one obtains $u_1 = u_2$ on $[0, \lim_{R \to \infty} \tau_R]$ for a.e. $\omega \in \Omega$. Now from sub-additivity and Chebyshev’s inequality we deduce

$$\mathbb{P}(\tau_R \geq T_0) = \mathbb{P}\left( \max_{i=1, 2} \|u_i\|_{C([0, T_0]; H^\sigma)} \leq R \right) \geq 1 - \frac{1}{R^2} \mathbb{E} \left( \|u_1\|^2_{C([0, T_0]; H^\sigma)} + \|u_2\|^2_{C([0, T_0]; H^\sigma)} \right) \to 1$$

as $R \to \infty$, proving $u_1 = u_2$ on $[0, T_0]$. This concludes the proof of Theorem 1.

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