Non-induced modular representations of cyclic groups

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ABSTRACT
We compute the ring of non-induced representations for a cyclic group, $C_n$, over an arbitrary field and show that it has rank $\varphi(n)$, where $\varphi$ is Euler’s totient function — independent of the characteristic of the field. Along the way, we obtain a “pick-a-number” trick; expressing an integer $n$ as a sum of products of $p$-adic digits of related integers.

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1. Introduction

Given a finite group $G$ and a field $k$ of characteristic $p \geq 0$, we may study the representation theory of $G$ over $k$ via the representation ring $R_{kG}$, whose elements are (isomorphism classes of) $kG$ modules, $\{V\}$. Here, and throughout, the representations of $G$ over $k$ will be identified with $kG$-modules. The addition and multiplication operations on the representation ring correspond to taking the direct sum and tensor product of modules. That is;

$\{V\} + \{U\} = \{V \oplus U\}$ and $\{V\} \cdot \{U\} = \{V \otimes_k U\}$.

Observe that the representation ring differs from the Grothendieck ring, as in the Grothendieck ring we have $\{V\} + \{U\} = \{W\}$ whenever there is a short exact sequence $0 \rightarrow V \rightarrow W \rightarrow U \rightarrow 0$. We shall abuse notation when working in the representation ring and write $V$ for the isomorphism class $\{V\}$.

Recall that if $H$ is a subgroup of $G$, and $V$ is a $kH$-module, then we may consider $V$ as a $kH$-module, simply by only considering the action of the elements in $kH$. This module will be denoted by $V \downarrow^G_H$, or often just $V \downarrow H$, and is called the restriction of $V$ to $H$. It is clear that if $K \leq H \leq G$, then

$V \downarrow^G_H \downarrow^K_H = V \downarrow^K_H$,

which is to say that restriction is transitive.

On the other hand, if $H \leq G$ and $V$ is a $kH$-module, then

$V \uparrow^G_H = kG \otimes_{kH} V$

is the $kG$-module induced from $H$ to $G$. Here, the $G$ action is given by left multiplication on the first factor.

We call this process induction and say that $V \uparrow^G_H$ is an induced module. Induction is also transitive, which
is to say if \( K \leq H \leq G \), then

\[
V \uparrow_{K}^{H} \downarrow_{H}^{G} = V \uparrow_{K}^{G}.
\]

Since \((V \oplus U) \uparrow_{H}^{G} \cong V \uparrow_{H}^{G} \oplus U \uparrow_{H}^{G}\), induction gives rise to a well defined (additive) group homomorphism of the representation rings, which we denote \( \text{Ind}_{H}^{G} : \mathcal{R}_{kH} \rightarrow \mathcal{R}_{kG} \).

**Lemma 1.** [6, Corollary 4.3.8 (4)] The image \( \text{Ind}_{H}^{G}(\mathcal{R}_{kH}) \) is an ideal in \( \mathcal{R}_{kG} \).

The quotient of the representation ring by this ideal gives a measure of the \( kG \)-modules which are not induced from \( kH \)-modules. We shall now turn our attention to the \( kG \)-modules which are not induced from any proper subgroup. That is, we study the quotient

\[
\mathcal{R}_{kG} / \sum_{H < G} \text{Ind}_{H}^{G} \mathcal{R}_{kH},
\]

which we shall refer to as the ring of non-induced representations. This was considered for cyclic groups by Srihari in [4], where the following is proved:

**Theorem 2.** Let \( G \) be a cyclic group of order \( n \) and \( k \) an algebraically closed field of characteristic 0. Then

\[
\frac{\mathcal{R}_{kG}}{\sum_{H < G} \text{Ind}_{H}^{G} \mathcal{R}_{kH}} \cong \mathbb{Z}[Y]/(\Phi_{n}(Y)),
\]

where \( \Phi_{n} \) is the \( n \)th cyclotomic polynomial. In particular,

\[
\text{rank} \left( \frac{\mathcal{R}_{kG}}{\sum_{H < G} \text{Ind}_{H}^{G} \mathcal{R}_{kH}} \right) = \varphi(n),
\]

where \( \varphi(n) \) is the number of integers less than \( n \) coprime to \( n \).

The main goal of this paper is to prove an analogous result over fields of positive characteristic. We first shall observe that the proof of Theorem 2 provided in [4] actually shows:

**Corollary 3.** Let \( G \) be a cyclic group of order \( n \) and \( k \) an algebraically closed field of characteristic \( p \) such that \( p \nmid n \). Then

\[
\frac{\mathcal{R}_{kG}}{\sum_{H < G} \text{Ind}_{H}^{G} \mathcal{R}_{kH}} \cong \mathbb{Z}[X]/(\Phi_{n}(X)).
\]

In particular the rank of the ring of non-induced representations is \( \varphi(n) \).

In Theorem 24 we will expand this result on cyclic groups to a field of arbitrary characteristic and see that the rank of the ring is unchanged at \( \varphi(n) \). Along the way we will discover a convenient, fractal-like basis (see Figure 1) of the representation ring which lifts to a basis of the quotient.

### 2. Cyclic \( p \)-groups

Throughout this section, let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( q = p^{\alpha} \) and denote by \( G = C_{q} \) the cyclic group of order \( q \) generated by element \( g \).

#### 2.1. The representation ring

**Lemma 4.** [1, Proposition 1.1] For \( k \) and \( G \) as above,

(i) there is a ring isomorphism \( kG \cong k[X]/(X^{q}) \), defined by sending \( g \) to \( 1 + X \),
Figure 1. The change of basis matrix, expressing \( V_i \) in terms of \( U_j \) over characteristic \( p = 3 \). Here \( i \) increases downwards along rows and \( j \) to the right along columns and a cell is filled if the coefficient is 1 and empty if it is 0. Notice, for example that \( V_i = U_j \) whenever \( i = ap^k \) with \( 1 \leq a < p \). This picture appears also in [5, Figure 2], where it shows the Weyl-Cartan matrix counting the decomposition multiplicities of simple \( \mathfrak{s}l_2 \) modules in Weyl \( \mathfrak{s}l_2 \) modules over characteristic \( p \).

(ii) under this isomorphism, a complete set of (pairwise non-isomorphic) indecomposable \( kG \)-modules is given by

\[
\{ V_r := k[X] / (X^r) : 1 \leq r \leq q \},
\]

(iii) the trivial module, \( V_1 \), is the unique irreducible \( kG \)-module.

To understand \( R_{kG} \), we first need to understand the structure of multiplication. In other words we need to understand the decomposition of the tensor product \( V_r \otimes V_s \) into indecomposable parts. The decomposition rule was known to Littlewood, and has been discussed by a number of authors. The following multiplication table is given in the lecture notes of Almkvist and Fossum [1] and is derived by Green [2]:

**Proposition 5.** For each \( k < \alpha \) and \( s \leq p^{k+1} \) we have the following decompositions. If \( s \leq p^k \), then

\[
V_{p^k-1} : V_s = (s - 1)V_{p^k} + V_{p^k-s},
\]

\[
V_{p^k+1} : V_s = (s - 1)V_{p^k} + V_{p^k+s}.
\]

Otherwise, if \( p^k < s \leq p^{k+1} \) then write \( s = s_0 p^k + s_1 \) with \( 0 \leq s_1 < p^k \). Then

\[
V_{p^k-1} : V_s = (s_1 - 1)V_{(s_0+1)p^k} + V_{(s_0+1)p^k-s_1} + (p^k - s_1 - 1)V_{s_0p^k}.
\]

If \( s_0 < p - 1 \), then

\[
V_{p^k+1} : V_s = (s_1 - 1)V_{(s_0+1)p^k} + V_{(s_0+1)p^k-s_1} + (p^k - s_1 - 1)V_{s_0p^k} + V_{s+p^k} + V_{s-p^k}
\]

while if \( s_0 = p - 1 \),

\[
V_{p^k+1} : V_s = (s_1 + 1)V_{p^k+1} + (p^k - s_1 - 1)V_{(p-1)p^k} + V_{(p-2)p^k+s_1},
\]
and if \( s_0 = p \) and \( s_1 = 0 \),

\[
V_{p^{k+1}} \cdot V_s = V_{p^k} + (p^k - 1)V_{p^{k+1}} + V_{(p-1)p^k}.
\]

This suggests the construction in \([1]\) which defines, for \( 0 \leq k < \alpha \), elements in \( \mathcal{R}_{kG} \) denoted \( \chi_k = V_{p^k} - V_{p^{k-1}} \) (where \( V_0 \) is interpreted as the zero representation — the zero in \( \mathcal{R}_{kG} \)) so that

\[
\chi_k \cdot V_s = \begin{cases} 
V_{s+p^k} - V_{p^{k-s}} & 1 \leq s \leq p^k \\
V_{s+p^k} + V_{s-p^k} & p^k < s < (p-1)p^k \\
V_{s-p^k} + 2V_{p^{k+1} - (s+p^k)} & (p-1)p^k \leq s < p^{k+1}
\end{cases}
\]  

(1)

In particular, if \( 1 < j < p \) then \( \chi_k \cdot V_{jp^k} = V_{(j-1)p^k} + V_{(j+1)p^k} \).

To complete the multiplication rule, Renaud \([3]\) gives a reduction theorem which allows us to express \( V_r \cdot V_s \) in terms of the tensor product of smaller modules:

**Theorem 6 (Reduction Theorem).** For \( 1 \leq r \leq s \leq p^{\beta+1} \), with \( r = r_0p^\beta + r_1, s = s_0p^\beta + s_1 \), where \( 1 \leq \beta < \alpha \), and \( 0 \leq (r_1, s_1) < p^\beta \):

\[
\begin{align*}
V_r \cdot V_s &= c_1 V_{p^\beta+1} + \left| r_1 - s_1 \right| \sum_{i=1}^{d_1} V_{(s_0-r_0+2i)p^\beta} + \max(0, r_1 - s_1) V_{(s_0-r_0)p^\beta} \\
&\quad + (p^\beta - s_1 - r_1) \sum_{i=1}^{d_2} V_{(s_0-r_0+2i-1)p^\beta} \\
&\quad + \sum_{j=1}^{l} a_j \left[ \sum_{i=1}^{d_1} V_{(s_0-r_0+2i)p^\beta+b_j} + V_{(s_0-r_0+2i)p^\beta-b_j} \right] + V_{(s_0-r_0)p^\beta+b_j},
\end{align*}
\]

where

\[
\begin{align*}
c_1 &= \begin{cases} 
0 & \text{if } r_0 + s_0 < p, \\
r + s - p^{\beta+1} & \text{if } r_0 + s_0 \geq p,
\end{cases} \\
d_1 &= \begin{cases} 
r_0 & \text{if } r_0 + s_0 < p, \\
p - s_0 - 1 & \text{if } r_0 + s_0 \geq p,
\end{cases} \\
d_2 &= \begin{cases} 
r_0 & \text{if } r_0 + s_0 < p, \\
p - s_0 & \text{if } r_0 + s_0 \geq p,
\end{cases}
\]

and \( V_{r_1} \cdot V_{s_1} = \sum_{j=1}^{l} a_j V_{b_j} \).

This allows a general tensor product to be reduced to the tensor product of smaller modules, which can be calculated via repeated applications of **Theorem 6**, and finally **Proposition 5**, or similar multiplication rules appearing in \([3]\).

To aid in exposing the structure of \( \mathcal{R}_{kG} \), we adjoin elements \( \mu_k^{\pm 1} \) for \( 0 \leq k < \alpha \) subject to \( \chi_k = \mu_k + \mu_k^{-1} \). If so, then for each \( 0 \leq s \leq p \), we have that \( \mu_k^s + \mu_k^{-s} = V_{p^k} - V_{p^k-1} \). With this setup Alkvis and Fossum are able to completely determine the structure of \( \mathcal{R}_{kG} \) by identifying \( \mathcal{R}_{kG} \) with a quotient of a polynomial ring. Before we can state their result, we will first state some identities involving the \( \chi_i \) and then define some families of polynomials.

**Lemma 7.** Let \( i < j \) and \( 0 < s < p \), then we have

\[
\chi_i^s = \sum_{v=0}^{\left\lfloor \frac{s+i}{p} \right\rfloor} \binom{s}{v} V_{(s-2v)p^i+1} - V_{(s-2v)p^i-1} + \begin{cases} 
(s/2) & \text{if } s \text{ even} \\
0 & \text{if } s \text{ odd}
\end{cases}
\]

and

\[
\chi_i \chi_j = V_{p^i+p^j+1} - V_{p^i-p^j-1} - V_{p^i+p^j-1} + V_{p^i-p^j+1}.
\]
Proof. The first fact can be verified by considering the expansion of \((\mu_i + \mu_i^{-1})^s\), while the second is obtained from applications of Proposition 5.

Now, still following [1], we define some families of polynomials. These are easier to state if we allow ourselves to use the language of quantum numbers which are briefly introduced here.

**Definition 8.** The quantum number, \([n]\) for \(n \in \mathbb{Z}\), are polynomials in \(\mathbb{Z}[X]\) that satisfy \([0] = 0, [1] = 1, [2] = X,\) and \([n] = [2](n - 1) - [n - 2]\). We shall write \([n]_x\) for the \(n\)th quantum number evaluated at \(X = x\).

The first quantum numbers are

| \(n\) | \([n]\) | \(n\) | \([n]\) |
|---|---|---|---|
| 0 | 0 | 4 | \(X^3 - 2X\) |
| 1 | 1 | 5 | \(X^4 - 3X^2 + 1\) |
| 2 | \(X\) | 6 | \(X^5 - 4X^3 + 3X\) |
| 3 | \(X^2 - 1\) | 7 | \(X^6 - 5X^4 + 6X^2 - 1\) |

Notice that the coefficients are such that \([n]_2 = n\). We can also write

\[
[n] = \sum_{i=0}^{[n]/2} (-1)^i \binom{n - 1 - i}{i} X^{n-1-2i}.
\]

In this formulation,

\[
[n + 1]_{q + q^{-1}} = q^n + q^{n-2} + \cdots + q^{2-n} + q^{-n}.
\]

Consider the polynomials in \(\mathbb{Z}[X_0, \ldots, X_{\alpha - 1}]\),

\[
F_j = (X_j - 2[p]x_{j-1} - 2[p - 1]x_{j-1}) [p]x_j,
\]

where \(1 \leq j < \alpha\), together with \(F_0 = (X_0 - 2)[p]x_0\). We are now ready to state a structure theorem for \(\mathcal{R}_{kG}\).

**Theorem 9.** [1, Proposition 1.6] The map \(\mathbb{Z}[X_0, \ldots, X_{\alpha - 1}] \rightarrow \mathcal{R}_{kG}\) defined by \(X_i \mapsto \chi_i\) induces a ring isomorphism

\[
\mathbb{Z}[X_0, \ldots, X_{\alpha - 1}] / (F_0, F_1, \ldots, F_{\alpha - 1}) \cong \mathcal{R}_{kG}.
\]

### 2.2. Induced representations

The subgroups of \(G\) are all cyclic \(p\) groups generated by some power of \(g\). Consider \(H = \langle g^{p^{\alpha - \beta}} \rangle\), the subgroup of \(G\) of order \(p^\beta\). The group algebra \(kH \subseteq kG\) is identified, under the isomorphism in Lemma 4, with \(k[X^{p^{\alpha - \beta}}] / (X^{p^\beta})\). Its indecomposable representations are again each of the form \(W_i = k[X^{p^{\alpha - \beta}}] / (X^{ip^{\alpha - \beta}})\) for \(1 \leq i < p^\beta\). Inducing to obtain a \(kG\)-module we get:

**Lemma 10.** We have an isomorphism of \(kG\)-modules

\[
W_i \uparrow^G_H \cong V_{rp^{\alpha - \beta}}.
\]

Proof. Note \(kG \otimes_{kH} W_r\) is cyclic as a \(kG\)-module, generated by \(e \otimes 1\) and has dimension \(rp^{\alpha - \beta}\). □
Thus \( \text{Ind}^G_H \cdot \mathcal{R}_{kH} \) consists of all \( kG \)-modules \( V_i \) such that \( i \) is divisible by \( p^\beta \). Hence, since \( G \) has a unique maximal subgroup, and induction is transitive, \( \sum_{H < G} \text{Ind}^G_H \cdot \mathcal{R}_{kH} \) consists of all the \( kG \)-modules \( V_i \) such that \( i \) is divisible by \( p \). We have thus shown:

**Proposition 11.** Let \( G = C_q \) the cyclic group of order \( q = p^\alpha \) and let \( k \) be a field of characteristic \( p \). Then

\[
\sum_{H < G} \text{Ind}^G_H \cdot \mathcal{R}_{kH} = \langle \{ V_i : p \mid i \} \rangle \mathbb{Z}.
\]

To describe the ring of non-induced representations,

\[
\mathcal{R}_{kG} \sum_{H < G} \text{Ind}^G_H \cdot \mathcal{R}_{kH},
\]

via the isomorphism in Theorem 9 we would aim to write the \( kG \)-modules \( V_i \) where \( p \mid i \) in terms of the \( \chi_i \). Instead, we shall change basis and give an alternative description of the ideal \( \sum_{H < G} \text{Ind}^G_H \cdot \mathcal{R}_{kH} \) in terms of modules which are easier to describe by polynomials in the \( \chi_i \).

### 2.3. Change of basis

Our new basis for \( \mathcal{R}_{kG} \) shall use the language of quantum numbers and cousins. Observe, from Proposition 5, that tensoring with \( V_2 \) satisfies a similar relation to the quantum numbers.

**Lemma 12.** For \( r < p \),

\[
V_r \cdot V_2 = V_{r+1} + V_{r-1}.
\]

Moreover,

**Lemma 13.** For \( 0 \leq k \) and \( a \in \{0, 1, \ldots, p - 1\} \),

\[
[a]_{\chi_k} [p]_{\chi_{k-1}} [p]_{\chi_{k-2}} \cdots [p]_{\chi_0} = V_{ap^k}.
\]

**Proof.** The case \( a = 0 \) is trivial. By Lemma 12, for \( r < p \), we know that \( \chi_0 \cdot V_r = V_2 \cdot V_r = V_{r+1} + V_{r-1} \). Since this is the defining relation for \( [r]_{\chi_0} \) the result is shown for \( k = 0 \). Equivalently it holds for \( k = 1 \) and \( a = 1 \). We now induct on \( k \), with the key observation that, for \( 1 \leq a \leq p \) and \( 1 \leq s \leq p^k \),

\[
[a]_{\chi_k} \cdot V_s = \left( V_{(a-1)p^k+s} - V_{(a-1)p^k-s} \right) + \left( V_{(a-3)p^k+s} - V_{(a-3)p^k-s} \right) + \cdots + \begin{cases} V_{p^k+s} - V_{p^k-s} & a \text{ even} \\ V_s & a \text{ odd} \end{cases}
\]

from which the desired result follows by the substitution \( s = p^k \).

To prove Eq. (3) we induct on \( a \). The base case of \( a = 1 \) is trivial, and the case \( a = 2 \) is covered by the first line of Eq. (1). Now, assuming Eq. (3) holds for \( a - 1 < p \), we can apply the second line of Eq. (1) to obtain

\[
[2]_{\chi_k} [a-1]_{\chi_k} \cdot V_s = \left( V_{ap^k+s} + V_{(a-2)p^k+s} - V_{ap^k-s} - V_{(a-2)p^k-s} \right) + \left( V_{(a-2)p^k+s} + V_{(a-4)p^k+s} - V_{(a-2)p^k-s} - V_{(a-4)p^k-s} \right) + \cdots + \begin{cases} V_{2p^k+s} + V_s - V_s & a \text{ even} \\ V_{p^k+s} - V_{p^k-s} & a \text{ odd} \end{cases}
\]

from which we can subtract \( [a-2]_{\chi_k} \cdot V_s \) to obtain Eq. (3). \( \square \)
This relationship motivates defining a new basis.

**Definition 14.** Let \( r < q = p^\alpha \) and write \( r - 1 = \sum_{i=0}^{\alpha-1} r_ip^i \) with \( r_i \in \{0, 1, \ldots, p - 1\} \). Set

\[
U_r := \prod_{i=0}^{\alpha-1} [r_i + 1]_{\chi_i}.
\]

Of course, each \( U_j \in \mathcal{R}_{kG} \) can be written in terms of the indecomposable modules \( V_i \) using repeated applications of **Theorem 9** and **Proposition 5**. The largest indecomposable appearing in this expression for \( U_r \) comes from the term \( \prod_{i=0}^{\alpha-1} \chi_i^{r_i} \). This largest module appearing in this term corresponds to the largest indecomposable appearing in this expression for \( U_r \). In particular, the set \( \{U_j : 0 < j < q \} \) is a basis for \( \mathcal{R}_{kG} \) and the change-of-basis matrix is lower triangular (see **Figure 1**).

We have a ring homomorphism from \( \mathcal{R}_{kG} \) to \( \mathbb{Z} \) simply by taking dimensions. As \( \chi_i \) is the difference of the indecomposable modules \( V_{p_i+1} \) and \( V_{p_i-1} \), this homomorphism sends \( \chi_i \) to 2. In particular, the image of \( U_r \) under the dimension homomorphism can be realized by evaluating the polynomials at \( \chi_i = 2 \). As observed earlier, the quantum polynomials are such that \([r]_2 = r\), thus the “dimension” of \( U_r \) is \( \prod_{i=0}^{\alpha-1} (r_i + 1) \).

**Example 15.** Let \( p = 5, \alpha = 3 \). We then have that

\[
U_{12} = [1]_{\chi_3}[3]_{\chi_1}[2]_{\chi_0} = (1)(\chi_3^2 - 1)(\chi_0)
\]

\[
= (V_{11} - V_9 + 1) \cdot V_2 = V_{12} - V_8 + V_2,
\]

where the third equality follows from the identity \( \chi_3^2 = V_{2p+1} + V_{2p-1} + 2 \), and the final equality follows from **Proposition 5**, which shows \( V_{11} \cdot V_2 = V_{12} + V_{10} \) and \( V_3 \cdot V_2 = V_{10} + V_8 \). Observe that if \( p = 5 \) then \( U_{12} = V_{12} - V_8 + V_2 \) for any \( \alpha \geq 2 \) as the factors \([1]_{\chi_i} \) for \( i \geq 2 \) make no contribution. Observe that the image of \( U_{12} \) under the dimension map is \( 12 - 8 + 2 = (1) \cdot (3) \cdot (2) = 6 \).

In fact, we are able to give a closed form for the \( V_r \) in terms of the \( U_j \). See **Figure 1** for a visual representation of this proposition. In order to describe this change of basis, we will introduce a few number theoretic concepts.

**Definition 16.** Let \( n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_0 \) be the \( p \)-adic expansion of \( n \). We define the cousins of \( n \) as

\[
\text{cous}(n) = \{n_k p^k \pm n_{k-1} p^{k-1} \pm \cdots \pm n_0\},
\]

and for \( \alpha > k \), the \( \alpha \)-anti-cousins of \( n \) as

\[
\text{acous}_\alpha(n) = \{j : p^\alpha - n \in \text{cous}(p^\alpha - j)\}.
\]

Note that the cousins of \( n \) are at most \( n \), and hence the anti-cousins of \( n \) are also at most \( n \). Now, if \( j \leq p^\alpha \), then

\[
\text{cous}(p^\alpha+1 - j) = \text{cous}((p - 1)p^\alpha + (p^\alpha - j)) = (p - 1)p^\alpha \pm \text{cous}(p^\alpha - j).
\]

Hence if \( p^\alpha+1 - r \in \text{cous}(p^\alpha+1 - j) \) then \( p^\alpha - r \in \text{cous}(p^\alpha - j) \). However, the reverse inclusion also holds, so in fact \( \text{acous}_{\alpha+1}(r) = \text{acous}_\alpha(r) \). Hence we may talk about the anti-cousins of \( r \).
We can now state and prove the change of basis theorem.

**Proposition 17.** We have that

\[ V_r = \sum_{j \in \text{acous } r} U_j. \]

**Proof.** We note that \( U_{ap^k} = V_{ap} \) by Lemma 13. Further

\[
\text{acous } (ap^k) = \text{acous}_{k+1} (ap^k)
\]

\[
= \left\{ j : p^{k+1} - j \in \text{cous } (p - a)p^k \right\}
\]

\[
= \left\{ j : p^{k+1} - j \in (p^{k+1} - ap^k) \right\}
\]

\[
= \{ ap^k \}
\]

and thus the result holds for \( r = ap^k \).

Now, suppose the equation holds for \( s < r \), and write \( r = p^\beta + r' \) for some \( \beta \) such that \( r < p^{\beta+1} \).

Note that we may permit \( r' > p^\beta \). In fact we will take two cases:

1. **Case** \( r' \leq p^\beta \): If \( r' = 0 \) or \( r' = p^\beta \) we are done as \( r = p^\beta \) or \( r = 2p^\beta \) which is handled above. Otherwise, by induction \( V_{r'} \) can be written as a sum of \( U_j \) with \( j < r' \). Moreover we can write, using Eq. (1)

\[
V_r = \chi_{\beta} \cdot V_{r'} + V_{p^\beta - r'}
\]

\[
= \sum_{j \in \text{acous } (r')} \chi_{\beta} U_j + \sum_{j \in \text{acous } (p^\beta - r')} U_j
\]

\[
= \sum_{j \in \text{acous } (r')} U_{p^\beta + j} + \sum_{j \in \text{acous } (p^\beta - r')} U_j.
\]

Here we have used the fact that \( \chi_{\beta} U_j = [2]_{\beta} U_j = U_{p^\beta + j} \) for all \( j < p^\beta \). The result will follow given the claim that

(a) if \( r' < p^\beta \), then \( \text{acous } (r' + p^\beta) = (\text{acous } (r') + p^\beta) \sqcup \text{acous } (p^\beta - r') \)

2. **Case** \( p^\beta < r' \leq (p - 1)p^\beta \): If \( r' = (p - 1)p^\beta \) then \( r = p^{\beta+1} \) and we are done by the base case. Otherwise, again, using the second case of Eq. (1),

\[
V_r = \chi_{\beta} \cdot V_{r'} - V_{r' - p^\beta}
\]

\[
= \sum_{j \in \text{acous } (r')} (U_{p^\beta + j} + U_{j - p^\beta}) - \sum_{j \in \text{acous } (r' - p^\beta)} U_j
\]

and again the claims below will show that most terms cancel and those that remain are what is desired:

(b) if \( p^\beta < r' < (p - 1)p^\beta \), then \( \text{acous } (r') - p^\beta = \text{acous } (r' - p^\beta) \), and
(c) if \( p^\beta < r' < (p - 1)p^\beta \), then \( \text{acous } (r') + p^\beta = \text{acous } (r' + p^\beta) = \text{acous } (r) \)

We thus need to show the three claims, which is an exercise in dealing with cousins and anti-cousins.

(a) Firstly note that the union is disjoint as the first set is larger than \( p^\beta \) and the second set is smaller. Then notice that

\[
\begin{align*}
j \in \text{acous } (p^\beta - r') & \iff r' \in \text{cous } (p^\beta - j) \\
& \iff p^{\beta+1} - r \in (p - 1)p^\beta + \text{cous } (p^\beta - j)
\end{align*}
\]
\[ \implies p^{\beta+1} - r \in \text{cous}(p^{\beta+1} - j) \]
\[ \iff j \in \text{acous}(r), \]

where we have used the fact that \( \text{cous}(p^{\beta+1} - j) = \text{cous}((p - 1)p^{\beta} + (p^{\beta} - j)) \) which can be written as \((p - 1)p^{\beta} \pm \text{cous}(p^{\beta} - j)\). If \( j < p^{\beta} \) then

\[ j \in \text{acous}(r) \iff (p - 1)p^{\beta} - r' \in \text{cous}((p - 1)p^{\beta} + (p^{\beta} - j)) \]
\[ \implies (p - 1)p^{\beta} - r' \in (p - 1)p^{\beta} - \text{cous}(p^{\beta} - j) \]
\[ \implies r' \in \text{cous}(p^{\beta} - j) \]
\[ \iff j \in \text{acous}(p^{\beta} - r'), \]

and we have shown that \( \text{acous}(r' + p^{\beta}) \cap \{0, 1, \ldots, p^{\beta}\} = \text{acous}(p^{\beta} - r'). \)

On the other hand, if

\[ j \in \text{acous}(r') + p^{\beta} \iff p^{\beta+1} + p^{\beta} - r \in \text{cous}\left(p^{\beta+1} + p^{\beta} - j \right) \]
\[ \iff p^{\beta+1} + p^{\beta} - r \in \text{cous}\left((p - 1)p^{\beta} + (2p^{\beta} - j) \right) \]
\[ \implies p^{\beta+1} - r \in (p - 2)p^{\beta} \pm \text{cous}(2p^{\beta} - j) \]
\[ \implies p^{\beta+1} - r \in \text{cous}(p^{\beta+1} - j) \]
\[ \iff j \in \text{acous}(r), \]

and if \( p^{\beta} < j < 2p^{\beta} \) then

\[ j \in \text{acous}(r) \iff p^{\beta+1} - p^{\beta} - r' \in \text{cous}(p^{\beta+1} - j) \]
\[ \iff p^{\beta+1} - p^{\beta} - r' \in \text{cous}\left((p - 1)p^{\beta} - (j - p^{\beta}) \right) \]
\[ \iff (p - 1)p^{\beta} - r' \in (p - 1)p^{\beta} \pm \text{cous}(j - p^{\beta}) \]
\[ \iff p^{\beta+1} - r' \in p^{\beta+1} \pm \text{cous}(j - p^{\beta}) \]
\[ \iff p^{\beta+1} - r' \in \text{cous}(p^{\beta+1} - (p^{\beta} - j)) \]
\[ \iff j \in \text{acous}(r') + p^{\beta}. \]

(b) Suppose that \( j < (p - 2)p^{\beta} \) and write \( j = j_0p^{\beta} + j_1 \) for \( 0 \leq j_1 < p^{\beta} \) and hence \( 0 < j_0 < p - 2 \). Then

\[ j \in \text{acous}(r' - p^{\beta}) \iff p^{\beta+1} + p^{\beta} - r' \in \text{cous}(p^{\beta+1} - j) \]
\[ \iff p^{\beta+1} + p^{\beta} - r' \in \text{cous}\left((p - 1 - j_0)p^{\beta} + (p^{\beta} - j_1) \right) \]
\[ \iff p^{\beta+1} + p^{\beta} - r' \in (p - 1 - j_0)p^{\beta} \pm \text{cous}(p^{\beta} - j_1) \]
\[ \iff p^{\beta+1} - r' \in (p - 2 - j_0)p^{\beta} \pm \text{cous}(p^{\beta} - j_1) \]
\[ \iff p^{\beta+1} - r' \in \text{cous}(p^{\beta+1} - j - p^{\beta}) \]
\[ \iff j \in \text{acous}(r') - p^{\beta}. \]

(c) Write \( j = j_0p^{\beta} + j_1 \) for \( 0 \leq j_1 < p^{\beta} \) and \( 0 < j_0 < p - 1 \). Then

\[ j \in \text{acous}(r') + p^{\beta} \iff p^{\beta+1} + p^{\beta} - r \in \text{cous}(p^{\beta+1} + p^{\beta} - j) \]
\[ \iff p^{\beta+1} + p^{\beta} - r \in \text{cous}\left((p - j_0)p^{\beta} + (p^{\beta} - j_1) \right) \]
\[ \iff p^{\beta+1} + p^{\beta} - r \in (p - j_0)p^{\beta} \pm \text{cous}(p^{\beta} - j_1) \]
\[ \iff p^{\beta+1} - r \in (p - j_0 - 1)p^{\beta} \pm \text{cous}(p^{\beta} - j_1) \]
\[ \iff p^{\beta+1} - r \in \text{cous}(p^{\beta+1} - j) \]
\[ \iff j \in \text{acous}(r). \]
In particular, note that the $V_r$ are multiplicity-free in the $U_j$. We can alternatively express $V_r = \mathcal{U}_r^\alpha$ where we define $\mathcal{U}_r^\alpha$ as below.

**Definition 18.** Let $r < q = p^\alpha$ and let $\beta \leq \alpha$. Write $r = mp^\beta + j$ for $j < p^\beta$. Set $\mathcal{U}_r^0 = U_r$ and define

$$\mathcal{U}_r^\beta = \begin{cases} \mathcal{U}_{mp^\beta + j}^{\beta - 1} + \mathcal{U}_{mp^\beta - j}^{\beta - 1} & p \nmid m \\ \mathcal{U}_{rp^\beta - 1} & \text{else} \end{cases}$$

**Example 19.** For example, if $p = 5$ and $\alpha = 3$ then $V_{62} = \mathcal{U}_{62}^3 = \mathcal{U}_{62}^2$ where

$$\mathcal{U}_{62}^2 = \mathcal{U}_{62}^1 + \mathcal{U}_{38}^1 = U_{62} + U_{58} + U_{38} + U_{32}.$$

**Remark 20.** The aforementioned dimension map enables us to play a number theoretic game. Indeed, recall this map sends $V_r \mapsto r$ and $U_j \rightarrow$ some product of its digits plus one.

Thus select a prime $p$ and natural number $n$. Compute all of the $U_j$ appearing in $\mathcal{U}_n^\beta$ for $\beta$ such that $p^\beta > n$. Let all the appearing $j$ be collected in a set $J$. Then for each such $j \in J$, write out the $p$-adic digits of $j - 1$ as $(j_0, \ldots, j_k)$. Finally,

$$n = \sum_{j \in J} (j_0 + 1)(j_1 + 1) \cdots (j_k + 1).$$

This is reminiscent of the “pick-a-number” trick played by schoolchildren. In fact, we may relax the condition that $p$ is prime in Definition 18 and the trick still works, in particular we may use the usual base 10 expansion. In this situation, however, we lose the representation theoretic interpretation of this fact.

**Example 19 (continued).** Continuing from our example above where $p = 5$, $\alpha = 3$, and $n = 62$ then $J = \{62, 58, 38, 32\}$. We then write

$$\begin{align*}
62 - 1 &= 2 \cdot 5^2 + 2 \cdot 5^1 + 1 \cdot 5^0 = 221_5 \\
58 - 1 &= 2 \cdot 5^2 + 1 \cdot 5^1 + 2 \cdot 5^0 = 212_5 \\
38 - 1 &= 1 \cdot 5^2 + 2 \cdot 5^1 + 2 \cdot 5^0 = 122_5 \\
32 - 1 &= 1 \cdot 5^2 + 1 \cdot 5^1 + 1 \cdot 5^0 = 111_5.
\end{align*}$$

Observe that

$$62 = (3)(3)(2) + (3)(2)(3) + (2)(3)(3) + (2)(2)(2),$$

as claimed.

The purpose of this basis, apart from Theorem 9, is that it allows us to write down the ideal of induced modules very simply.

**Proposition 21.** The ideal of induced modules is principal. To be exact,

$$\sum_{H < G} \text{Ind}_H^G \mathcal{R}_{KH} = \langle U_j : p \mid j \rangle_\mathbb{Z} = \langle U_p \rangle.$$

**Proof.** Let $I = \sum_{H < G} \text{Ind}_H^G \mathcal{R}_{KH}$. Note $U_p = V_p \in I$ and each $U_{mp} = U_{(m-1)p+1} \cdot U_p$ by definition. Hence

$$\langle U_j : p \mid j \rangle_\mathbb{Z} \subseteq \langle U_p \rangle \subseteq I = \langle V_j : p \mid j \rangle_\mathbb{Z}$$

where the last equality is by Proposition 11. But these have the same rank and hence we have equality. \(\square\)
We are thus able to give an explicit structure to the non-induced representation ring, analogous to Theorem 2.

**Corollary 22.** The ring of non-induced representations of $C_q$ over a field of characteristic $p$ is isomorphic to

$$\frac{\mathbb{Z}[X_0, \ldots, X_{\alpha-1}]}{(p)_{X_0}, F_1, \ldots, F_{\alpha-1})}$$

and thus has rank $\varphi(q)$.

### 3. General cyclic groups

In general a cyclic group is of the form $G := C_n = C_m \times C_q$, where $q = p^\alpha$ and $n = m \cdot q$ with $p \nmid m$. Let $k$ be an algebraically closed field of characteristic $p$. It is well known that the representation ring $\mathbb{R}_{kC_n}$ is the tensor product of the representation rings $\mathbb{R}_{kC_m}$ and $\mathbb{R}_{kC_q}$, so our task is to understand the ideal generated by induced representations. Subgroups of $G$ are of the form $H = H_1 \times H_2$ where $H_1 \leq C_m$ and $H_2 \leq C_q$. In particular, $H_1 = C_{m'}$ for some $m' | m$ and $H_2 = C_{p^\beta}$ for some $\beta \leq \alpha$.

An indecomposable $kH$-module $N$ is of the form $N_1 \otimes_k N_2$ where $N_i$ is a $kH_i$ module. Inducing we get $N \uparrow^G_H = N_1 \uparrow^{C_m}_{H_1} \otimes_k N_2 \uparrow^{C_q}_{H_2}$. In particular:

**Proposition 23.** Let $G$ be a cyclic group and $k$ a field of characteristic $p$. Suppose $G$ is of order $n = mq$, where $q = p^\alpha$ and $m \nmid p$. Let $\{V'_i : 0 < i < m\}$ be the complete set of indecomposable $kC_m$-modules and let $\{V_i : 0 < i < q\}$ be the complete set of indecomposable $kC_q$-modules. Then

$$\sum_{H < G} \text{Ind}^G_H \mathbb{R}_{kH} = \langle V'_i \otimes_k V_j : 0 < i < m, 0 < j < q, p | j \text{ or } i | m \rangle,$$

which is the ideal generated by $\sum_{H < C_m} \text{Ind}^{C_m}_H \mathbb{R}_{kH}$ and $\sum_{H < C_q} \text{Ind}^{C_q}_H \mathbb{R}_{kH}$.

It follows then that:

**Theorem 24.** Let $G$ be a cyclic group of order $n = mq$, where $q = p^\alpha$ and $m \nmid p$. The ring of non-induced representations is isomorphic to

$$\mathbb{Z}[Y] \otimes_{\mathbb{Z}[\Phi_m(Y)]} \frac{\mathbb{Z}[X_0, \ldots, X_{\alpha-1}]}{(p)_{X_0}, F_1, \ldots, F_{\alpha-1})},$$

or equivalently

$$\mathbb{Z}[X_0, \ldots, X_{\alpha-1}, Y] \otimes_{\mathbb{Z}[\Phi_m(Y), [p]_{X_0}, F_1, \ldots, F_{\alpha-1})},$$

In particular,

$$\text{rank} \left( \frac{\mathbb{R}_{kG}}{\sum_{H < G} \text{Ind}^G_H \mathbb{R}_{kH}} \right) = \varphi(m) \times \varphi(p^{\alpha}) = \varphi(n).$$

Note that the ranks of both the ring of non-induced representations and the ideal of induced representations are independent of the characteristic of the field, even though these sets may differ from field to field.

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