Noisy Nonnegative Tucker Decomposition with Sparse Factors and Missing Data

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Abstract

Tensor decomposition is a powerful tool for extracting physically meaningful latent factors from multi-dimensional nonnegative data, and has been an increasing interest in a variety of fields such as image processing, machine learning, and computer vision. In this paper, we propose a sparse nonnegative Tucker decomposition and completion approach for the recovery of underlying nonnegative data under incompleted and generally noisy observations. Here the underlying nonnegative tensor data is decomposed into a core tensor and several factor matrices with all entries being nonnegative and the factor matrices being sparse. The loss function is derived by the maximum likelihood estimation of the noisy observations, and the $\ell_0$ norm is employed to enhance the sparsity of the factor matrices. We establish the error bound of the estimator of the proposed model under generic noise scenarios, which is then specified to the observations with additive Gaussian noise, additive Laplace noise, and Poisson observations, respectively. Our theoretical results are better than those by existing tensor-based or matrix-based methods. Moreover, the minimax lower bounds are shown to be matched with the derived upper bounds up to logarithmic factors. Numerical experiments on both synthetic data and real-world data sets demonstrate the superiority of the proposed method for nonnegative tensor data completion.

Key Words: Sparse nonnegative Tucker decomposition, maximum likelihood estimation, noisy observations, error bound

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1 Introduction

Tensors, also called multi-dimensional data, are high-order generalizations of vectors and matrices, and have a variety of applications including signal processing, machine learning, computer vision, and so on [27, 45]. The data required from real-world applications are usually represented as high-order tensors, for example, color images are third-order tensors, color videos are fourth-order tensors. In order to

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explore the intrinsic structure of tensor data, tensor decomposition is a powerful and effective approach to represent the data, and has been widely used in several fields [9, 43]. Besides, tensor decomposition can identify the fine details of tensor data and extract meaningful and explanatory information by a lower-dimensional set of latent factors, which is due to the fact that many high-dimensional data reside in a low-dimensional subspace [11].

Tensor data and latent factors are naturally nonnegative in several real-world applications such as images, video volumes, and text data. Nonnegative tensor decomposition can be employed to extract meaningful patterns by the decomposition formulation [32]. For example, many high-dimensional data, such as nonnegative hyperspectral images or video images, are factorized to find meaningful latent nonnegative components [39]. Besides dimension reduction, nonnegative tensor decomposition can be modeled and interpreted better by means of nonnegative and sparse components, which can achieve a unique additive parts-based representation.

In particular, when the order of a tensor is second, nonnegative tensor decomposition reduces to nonnegative matrix factorization (NMF), which has been attracted much attention in the past decades, see [15, 29] and references therein. The NMF, which requires the factors of the low-rank decomposition to be componentwise nonnegative, is capable of giving physically meaningful and more interpretable results, and has the ability of learning the local parts of objects [29]. There are plenty of real-world applications for NMF. For example, the NMF method can extract parts of faces, such as eyes, noses, and lips, in a series of facial images [41], identify topics in a set of documents [2], and extract materials and their proportions in hyperspectral images unmixing [34]. Moreover, sparse NMF was introduced further because it enhances the ability of NMF to learn a parts-based representation and produces more easily interpretable factors. For instance, in facial feature extraction, sparsity leads to more localized features and identifiable solutions, while fewer features are used to reconstruct each input image [15]. Besides, Hoyer [21] proposed a sparse NMF approach with sparse constraints to improve the decomposition for the observations with additive Gaussian noise, and designed a projected gradient descent algorithm to solve the resulting model. In theory, Soni et al. [46] proposed a noisy matrix completion method under a class of general noise models, and established the error bound of the estimator of their proposed model, which can reduce to sparse NMF and completion under nonnegative constraints. Moreover, Sambasivan et al. [42] showed that the error bound in [46] achieves minimax error rates up to multiplicative constants and logarithmic factors. However, for multi-dimensional data, the matrix based method may destroy the structure of a tensor via unfolding the tensor into a matrix.

For nonnegative tensor decomposition, the key issue is the decomposition formulation of a tensor. There are some popular decompositions for tensors such as CANDECOMP/PARAFAC (CP) decomposition [17], Tucker decomposition [48], tensor train decomposition [38], tensor decomposition via tensor-tensor product [23]. Due to the nonnegativity of a tensor, the nonnegative CP decomposition provides an interpretable, low tensor rank representation of the data and has been used in a variety of applications related to sparse image representation and image processing [8]. For example, Zhang et al. [53] employed a nonnegative CP decomposition for hyperspectral unmixing with the goal to identify materials and material abundance in the data and to compress the data. Moreover, Chi et al. [10] developed a Gauss-Seidel type algorithm for nonnegative CP decomposition, where the observed data were modeled by a Poisson distribution. Hong et al. [18] proposed a generalized CP low rank tensor decomposition that allows other loss functions besides squared error, and presented a gradient descent type algorithm to solve the resulting model, which can handle missing data by a similar approach in [1]. Besides, Jain et al. [22] proposed a CP decomposition and completion method with one sparse factor under a general class of noise models for third-order tensors, and established the error bound of the estimator of their proposed model, which was further derived to the error bound for the observations with additive Gaussian noise. The method in [22] can reduce to nonnegative CP decomposition with one sparse factor under nonnegative constraint of factor matrices. However, the previous methods need
to give the CP rank of a tensor in advance, which is NP-hard in general [16].

Based on the algebra framework of tensor-tensor product in [23], Soltani et al. [44] presented a nonnegative tensor decomposition method with one sparse factor for third-order tensors under additive Gaussian noise, which was constructed by a tensor dictionary prior from training data for tomographic image reconstruction. Furthermore, Newman et al. [37] proposed a sparse nonnegative tensor patch-based dictionary approach according to tensor-tensor product for image compression and deblurring, where the observations were corrupted by additive Gaussian noise. Recently, Zhang [55] proposed a sparse nonnegative tensor factorization and completion based on the algebra framework of tensor-tensor product for third-order tensors, where one factor tensor was sparse and the observations were corrupted by a general class of noise models. Then the error bound of the estimator in [55] was established, and the minimax lower bound was derived, which matched with the established upper bounds up to a logarithmic factor of the sizes of the underlying tensor. However, for $n$-dimensional $m$-th order tensors, there are $n^{m-2} n$-by-$n$ matrix singular value decompositions to be computed [23] and its computational cost may be quite high in the tubular singular value decomposition approach. For nonnegative TT decomposition, Lee et al. [30] proposed a hierarchical alternating least squares algorithm for feature extraction and clustering. Although the performance of nonnegative TT decomposition was better than that of standard nonnegative Tucker decomposition, the TT decomposition typically lacks interpretability and the efficient implementation depends on the tensor already being in the TT format [37].

Tucker decomposition is to decompose a given tensor into the product of a core tensor with smaller dimensions and a series of factor matrices. And the best low Tucker rank approximation of a tensor was discussed and studied in [13]. Due to the nonnegativity of tensor data, nonnegative Tucker decomposition (NTD) provides an additional multiway structured representation of tensor data, where all entries of the core tensor and factor matrices are nonnegative in Tucker decomposition [24, 39]. And NTD has a wide range of applications including image denoising and hyperspectral image restoration [3, 25, 32, 56]. Moreover, Mørup et al. [36] proposed two multiplicative update algorithms for sparse NTD with the observations corrupted by additive Gaussian noise and Poisson noise, respectively, which yields a parts-based representation and is a more interpretable decomposition as compared to NTD. Then Liu et al. [33] presented a fast and flexible algorithm for sparse NTD with a special core tensor based on columnwise coordinate descent, where the observations were corrupted by additive Gaussian noise and the factor matrices were nonnegative and sparse. Besides, Xu [52] developed an alternating proximal gradient algorithm for sparse NTD and completion with global convergence guarantee, where the observations were corrupted by additive Gaussian noise. Zhou et al. [56] developed a family of efficient first-order algorithms for sparse NTD by making use of low rank approximation, which can reduce the storage complexity and computing time. However, there is no theoretical result about the error bounds of the models for sparse NTD with missing values in the previous work. Also a general class of noise models is not discussed and studied for sparse NTD and completion in the existing literature.

In this paper, we propose a sparse NTD and completion approach for the observations with a general class of noise models. The loss function is derived by the maximum likelihood estimation of observations, and the nonnegativity for the entries of the core tensor and the factor matrices in Tucker decomposition are imposed, where the $\ell_0$ norm is used to characterise the sparsity of the factor matrices. Moreover, the error bound of the estimator of the proposed model under general noise distributions is established, and then the error bounds of the estimators for the special observations including additive Gaussian noise, additive Laplace noise, and Poisson observations are derived, respectively. Besides, the minimax lower bound of a general class of noisy observations is established, which matches to the upper bound up to a logarithmic factor. Then an alternating direction method of multipliers (ADMM) is designed to solve the resulting model. Numerical experiments on synthetic data and image data sets demonstrate the effectiveness of the proposed model compared with the matrix based method [46] and
the sparse nonnegative tensor factorization and completion via tensor-tensor product [55]. We summarize the existing literature and our method for sparse nonnegative tensor decomposition/factorization in Table 1.

Table 1: Comparisons of error bounds for sparse NTD and completion with existing methods based on different decompositions.

| Methods                        | Rank      | Order       | Noise type | Error bounds                                                                 |
|--------------------------------|-----------|-------------|------------|------------------------------------------------------------------------------|
| Matrix based method [46]       | matrix rank $r$ | second-order | general    | $O(r n_1 + \frac{1}{m} \log(\max\{n_1, n_2\}))$                          |
| Sparse nonnegative CP [22]     | CP rank $r$  | third-order | general    | $O(r n_2 + \frac{1}{m} \log(\max\{n_1, n_2, n_3, r\}))$                |
| Nonnegative TT [30]            | tensor train rank | third-order | Gaussian   | N/A                                                                           |
| Sparse NTF [55]                | tubal rank $r$ | third-order |            | $O(r n_1 + \frac{1}{m} \log(\max\{n_1, n_2\}))$                        |
| Sparse NTD (this paper)        | Tucker rank $(r_1, \ldots, r_d)$ | (d $\geq$ 3) | general    | $O(r_{d+1} + \frac{1}{m} \log(\max\{n_1, \ldots, n_d\}))$                |

The main contributions of this paper are summarized as follows.

(1) We propose a sparse NTD and completion approach for the recovery of underlying nonnegative tensor data under incomplete and generally noisy observations, where the underlying tensor has the Tucker decomposition form with nonnegativity and sparsity constraints. Moreover, the generic loss function is derived by the maximum likelihood estimation of the noisy observations.

(2) An error bound of the estimator of the proposed model is established for the general observation model, which is specified to some widely used noise models including additive Gaussian noise, additive Laplace noise, and Poisson observations. Moreover, the minimax lower bound of the observed model is also derived, which matches to the upper bound up to a logarithmic factor.

(3) An ADMM based algorithm is developed to solve the resulting model. Numerical experiments on synthetic data and real-world data sets show the superior performance of the proposed method over other comparison methods for sparse NTD and completion.

The remaining parts of this paper are organized as follows. In Section 2, some preliminaries about tensor Tucker decomposition and related information theoretical quality are provided. In Section 3, a sparse NTD and completion model is proposed under a general class of noisy observations. An upper error bound of the estimator of the proposed model is established in Section 4. Moreover, the error bounds of the estimators are further derived for the observations with special noise models. Section 5 is devoted to the minimax lower bound of the estimator. In Section 6, an ADMM based algorithm is developed to solve the resulting model. In Section 7, numerical experiments are conducted to demonstrate the effectiveness of the proposed model. Discussions and future work are given in Section 8. All proofs of theorems are provided in the appendices.

2 Preliminaries

2.1 Notation

Let $\mathbb{R}^n$ and $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ denote the $n$-dimensional Euclidean space with real numbers and the set of $n_1 \times n_2 \times \cdots \times n_d$ tensors with nonnegative real entries, respectively. Scalars, vectors, matrices, and tensors are represented by lowercase letters, lowercase boldface letters, uppercase letters, and capital Euler script letters, respectively, e.g., $x, \mathbf{x}, X, \mathcal{X}$. log refers to natural logarithm throughout this paper. $\lfloor x \rfloor$ and $\lceil x \rceil$ are the integer part of $x$ and smallest integer that is larger or equal to $x$, respectively. We denote $n_m = \max\{n_1, n_2, \ldots, n_d\}$. For any positive integer $n$, we let $[n] = \{1, 2, \ldots, n\}$. For an arbitrary set $\mathcal{C}$, $|\mathcal{C}|$ denotes the number of entries in $\mathcal{C}$. 


Let $X_{i_1 i_2 \ldots i_d}$ denote the $(i_1, i_2, \ldots, i_d)$th entry of a tensor $X \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$. The number of ways or dimensions of a tensor is called the order [27]. For an arbitrary vector $x \in \mathbb{R}^n$, $\|x\|$ and $\|x\|_1$ denote the Euclidean norm and the $\ell_1$ norm of $x$, respectively, where $\|x\| = \sum_{i=1}^n |x_i|$ and $x_i$ is the $i$th component of $x$, $i \in [n]$. $\|X\|_0$ denotes the number of nonzero entries of a matrix $X$. The tensor Frobenius norm of $X \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is defined as $\|X\|_F = \sqrt{\langle X, X \rangle}$, where the inner product of two same-sized tensors is defined as $\langle X, Y \rangle = \sum_{i_1, i_2, \ldots, i_d} X_{i_1 i_2 \ldots i_d} Y_{i_1 i_2 \ldots i_d}$. The tensor infinity norm of $X$, denoted by $\|X\|_\infty$, is defined as $\|X\|_\infty = \max_{i_1, i_2, \ldots, i_d} |X_{i_1 i_2 \ldots i_d}|$.

The mode-$i$ unfolding of a tensor $X \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$, denoted by $X(i)$, arranges the mode-$n$ fibers to be the columns of the resulting matrix, where the $(i_1, i_2, \ldots, i_d)$th entry of $X$ maps to matrix entry $(i_n, j)$ with

$$j = 1 + \sum_{k=1}^d (i_k - 1)J_k \text{ and } J_k = \prod_{l=1}^{k-1} n_l.$$  

Here a fiber is defined by fixing every index but one. The vectorization of a tensor $X$ is a vector, which is obtained by stacking all mode-$n$ fibers of $X$ and denoted by $\text{vec}(X)$.

Let $\mathcal{R}$ be a Euclidean space endowed with the Euclidean norm $\| \cdot \|$. For an arbitrary closed proper function $g : \mathcal{R} \to (\infty, +\infty]$, the proximal mapping of $g$ at $y$ is the operator given by

$$\text{Prox}_g(y) = \arg \min_{x \in \mathcal{R}} \left\{ g(x) + \frac{1}{2} \| x - y \|^2 \right\}.$$ 

### 2.2 Multilinear Operators

The Tucker decomposition of a $d$th-order tensor $X \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is given by [48]

$$X = C \times_1 A_1 \times_2 \cdots \times_d A_d,$$

where $C \in \mathbb{R}^{r_1 \times r_2 \times \cdots \times r_d}$ is called the core tensor, $A_i \in \mathbb{R}^{n_i \times r_i}$ are the factor matrices, $i \in [d]$, and $(r_1, r_2, \ldots, r_d)$ is called the Tucker rank of $X$. Here the $n$-mode product of a tensor $C$ by a matrix $A \in \mathbb{R}^{n \times r_d}$, denoted by $C \times_i A$, is a $r_1 \times \cdots \times r_i-1 \times n \times r_{i+1} \times \cdots \times r_d$-tensor whose entries are given by [13, Definition 3]

$$(C \times_i A)_{j_1 j_2 \cdots j_{i-1} k_{j_{i+1}} \cdots j_d} = \sum_{j_i = 1}^{r_i} C_{j_1 j_2 \cdots j_{i-1} j_{i+1} \cdots j_d} A_{k_{j_i}}.$$  

It is easy to verify that if $X = C \times_1 A_1 \times_2 \cdots \times_d A_d$, then

$$\text{vec}(X) = (\otimes_{i=d}^1 A_i)\text{vec}(C),$$  

where $\otimes_{i=d}^1 A_i := A_d \otimes A_{d-1} \otimes \cdots \otimes A_1$ and $A \otimes B$ denotes the Kronecker product of $A$ and $B$.

Moreover, the mode-$n$ unfolding of $X$ is given by [27]

$$X(n) = A_n C(n)(\otimes_{i=d, i \neq n}^1 A_i)^T.$$  

### 2.3 Kullback-Leibler Divergence and Hellinger Affinity

Let $p_{x_1}(y)$ and $p_{x_2}(y)$ denote the probability density functions or probability mass functions of a real scalar random variable $y$ with corresponding parameters $x_1$ and $x_2$, respectively. The Kullback-Leibler (KL) divergence of $p_{x_1}(y)$ from $p_{x_2}(y)$ is denoted and defined as follows:

$$K(p_{x_1}(y) \| p_{x_2}(y)) = \mathbb{E}_{p_{x_1}} \left[ \log \frac{p_{x_1}(y)}{p_{x_2}(y)} \right].$$
The Hellinger affinity between \( p_{x_1}(y) \) and \( p_{x_2}(y) \) is denoted and defined as

\[
H(p_{x_1}(y) || p_{x_2}(y)) = \mathbb{E}_{p_{x_1}} \left[ \sqrt{ \frac{p_{x_2}(y)}{p_{x_1}(y)} } \right] = \mathbb{E}_{p_{x_2}} \left[ \sqrt{ \frac{p_{x_1}(y)}{p_{x_2}(y)} } \right].
\]

The joint probability distributions of random tensors, denoted by \( p_{X_1}(Y), p_{X_2}(Y) \), are the joint probability distributions of the vectorization of the tensors. Then the KL divergence of \( p_{X_1}(Y) \) from \( p_{X_2}(Y) \) is defined as

\[
K(p_{X_1}(Y) || p_{X_2}(Y)) := \sum_{i_1,i_2,\ldots,i_d} K(p_{(X_1)_{i_1 i_2 \ldots i_d}}(Y_{i_1 i_2 \ldots i_d}) || p_{(X_2)_{i_1 i_2 \ldots i_d}}(Y_{i_1 i_2 \ldots i_d})),
\]

and its Hellinger affinity is defined as

\[
H(p_{X_1}(Y) || p_{X_2}(Y)) := \prod_{i_1,i_2,\ldots,i_d} H(p_{(X_1)_{i_1 i_2 \ldots i_d}}(Y_{i_1 i_2 \ldots i_d}) || p_{(X_2)_{i_1 i_2 \ldots i_d}}(Y_{i_1 i_2 \ldots i_d})).
\]

### 3 Sparse NTD and Completion With Noisy Observations

Let \( X^* \in \mathbb{R}^{+}_{n_1 \times n_2 \times \cdots \times n_d} \) be an unknown nonnegative tensor with Tucker rank \((r_1, r_2, \ldots, r_d)\). Assume that \( X^* \) admits a following sparse nonnegative Tucker decomposition:

\[
X^* = C^* \times_1 A^*_1 \times_2 A^*_2 \cdots \times_d A^*_d,
\]

where \( C^* \in \mathbb{R}^{r_1 \times r_2 \times \cdots \times r_d} \) is the core tensor and \( A^*_i \in \mathbb{R}^{n_i \times r_i} \) are sparse factor matrices, \( i \in [d] \). Suppose that \( r_i \leq n_i, i \in [d] \). We also assume that each entries of \( X^*, C^*, A^*_i \) are bounded, i.e.,

\[
0 \leq A^*_i \leq \frac{c}{2}, \quad 0 \leq C^* \leq 1, \quad 0 \leq (A^*_i)_{lm} \leq a_i, \quad (l,m) \in [n_i] \times [r_i], i \in [d],
\]

where \( c, a_i > 0 \) are given constants. Here we use \( \frac{c}{2} \) for brevity in the subsequence analysis.

However, only a noisy and incompletely observed tensor \( X^* \) is available in practice. Let \( \Omega \subseteq [n_1] \times [n_2] \times \cdots \times [n_d] \) be a subset at which the entries of the observations \( Y \) are collected. Denote \( \mathcal{Y}_\Omega \subseteq \mathbb{R}^m \) to be a vector such that the entries of \( Y \) in the index set \( \Omega \) are vectorized by lexicographic order, where \( m \) is the number of the observed entries. Assume that \( n_1, n_2, \ldots, n_d \geq 2 \) and \( d \geq 3 \) throughout this paper. Let the location \( \Omega \) be generated according to an independent Bernoulli model with probability \( p = \frac{m}{n_1 n_2 \cdots n_d} \) (denoted by \( \text{Bern}(p) \)), i.e., each index \((i_1, i_2, \ldots, i_d)\) belongs to \( \Omega \) with probability \( p \), which is denoted as \( \Omega \sim \text{Bern}(p) \). Suppose that the entries of observations are conditionally independent, which implies that the overall likelihood is just the product of the likelihoods of each entry. Therefore, the joint probability density function or probability mass function of observations \( \mathcal{Y}_\Omega \) is given by

\[
p_{\mathcal{Y}_\Omega}(\mathcal{Y}_\Omega) := \prod_{(i_1, i_2, \ldots, i_d) \in \Omega} p_{X^*_{i_1 i_2 \cdots i_d}}(Y_{i_1 i_2 \cdots i_d}).
\]

By taking the negative logarithm of the probability distribution, we propose the following sparse NTD and completion model with nonnegative and bounded constraints:

\[
\min_{X \in \mathcal{Y}} \left\{ - \log p_{\mathcal{Y}_\Omega}(\mathcal{Y}_\Omega) + \sum_{i=1}^{d} \lambda_i \|A_i\|_0 \right\},
\]
where $\lambda_i > 0$ are the regularization parameters, $i \in [d]$, $\|A_i\|_0$ is employed to characterize the sparsity of the factor matrix $A_i$ in Tucker decomposition, and $\Upsilon$ is defined as

$$\Upsilon := \left\{ x = c_1 A_1 \cdots A_d : c_\in C, A_i \in B_i, i \in [d], 0 \leq x_{i_1, \ldots, i_d} \leq c, (i_1, i_2, \ldots, i_d) \in [n_1] \times [n_2] \times \cdots \times [n_d] \right\}. \tag{7}$$

Here $\Upsilon$ is a countable set of candidate estimates, and $C$ and $B_i$ are the nonnegative and bounded sets constructed as follows: First, let

$$\tau := 2^{\lceil \log_2(n_m^3) \rceil} \tag{8}$$

for a specified $\beta \geq 1$. Then, we construct $C$ to be the set of all nonnegative tensors $c \in \mathbb{R}^{r_1 \times r_2 \times \cdots \times r_d}$ whose entries are discretized to one of $\tau$ uniformly sized bins in the range $[0, 1]$, and $B_i$ to be the set of all nonnegative matrices $A_i \in \mathbb{R}^{n_i \times r_i}$ whose entries either take the value 0, or are discretized to one of $\tau$ uniformly sized bins in the range $[0, \alpha_i]$, $i \in [d]$.

**Remark 3.1** The model in (6) can handle the observations with general noise distributions, where one just needs to know the probability mass function or probability density function of the observations $Y_\Omega$. In particular, the noise models including additive Gaussian noise, additive Laplace noise, and Poisson observations will be discussed in detail in the next section.

**Remark 3.2** For the observations with additive Gaussian noise, Liu et al. [33] proposed a sparse NTD model with special core tensor $C$, which is effective for image compression. Moreover, Xu [52] proposed a gradient descent type algorithm for sparse NTD and completion with additive Gaussian noise, which used the relaxation of $\ell_0$ norm for each factor matrix. Besides, Mørup et al. [36] proposed two multiplicative update algorithms for sparse NTD, where additive Gaussian noise and Poisson observations were considered with full observations. However, there is no theoretical analysis of error bounds about these models in [33, 36, 52]. Only efficient algorithms were proposed and studied to solve their resulting sparse NTD models. Moreover, a unified framework for general loss functions is proposed in (6) for sparse NTD and completion, while the special noise type is considered in the existing literature, such as additive Gaussian noise [33, 52], Poisson observations [36].

**Remark 3.3** Recently, Zhang et al. [55] proposed a sparse nonnegative tensor factorization and completion model with general noise distributions based on the algebra framework of tensor-tensor product for third-order tensors, where the underlying tensor was decomposed into the tensor-tensor product of one sparse nonnegative tensor and one nonnegative tensor. The difference between model (6) and [55] is the factorization of the underlying tensor, where the tensor-tensor product with block circulant structure is used in [55] and the Tucker decomposition of the underlying tensor is utilized in model (6). And the nonnegative Tucker decomposition has more widely applications and explanatory information than the nonnegative tensor factorization based on tensor-tensor product, where the last one is mainly applied in X-ray CT imaging, image compression and deblurring based on tensor patch dictionary learning [37]. Besides, the model in [55] is only effective for third-order tensors, while the model in (6) can be employed to any order tensors and can extract more physically meaningful latent components by the sparse and nonnegative factors [56].

**Remark 3.4** Jain et al. [22] proposed a noisy tensor completion method based on CP decomposition, where one factor is sparse. The model in (6) can reduce to sparse nonnegative CP decomposition and completion when the core tensor is diagonal. However, the core tensor of Tucker decomposition is not diagonal in general in various real-world applications [56]. Moreover, Hong et al. [18] proposed a
generalized CP decomposition with a general loss function under various scenarios and designed a gradient descent algorithm to solve the resulting model, while there are no nonnegative and sparse constraints for the factor matrices, and there is no theoretical analysis of error bounds about the model in [18].

**Remark 3.5** In model (6), we only require that the factor matrices are sparse and do not impose additional sparsity constraints on the core tensor; which has been applied in feature extraction [56] and high-dimensional time series [50]. Furthermore, our model can be generalized to the model with the core tensor and factor matrices being sparse simultaneously.

## 4 Error Bounds

In this section, an upper error bound of the estimator of the sparse NTD and completion model in (6) is established under a general class of noise distributions, and then the upper error bounds of the estimators for the observations with special noise models are derived, including additive Gaussian noise, additive Laplace noise, and Poisson observations.

Let \( X^\lambda \) be a minimizer of (6). Next we state the main theorem about the error bound of the estimator in (6).

**Theorem 4.1** Let the sampling set \( \Omega \) be drawn from the independent Bernoulli model with probability \( p = \frac{m}{n_1 n_2 \cdots n_d} \), i.e., \( \Omega \sim \text{Bern}(p) \), and the joint probability density/mass function of \( Y_{i\Omega} \) be defined as (5). For any

\[
\lambda_i \geq 4(\beta + 2) \left( 1 + \frac{2\gamma}{3} \right) \log(n_m), \quad i \in [d],
\]

where \( \gamma \) is a constant satisfying

\[
\gamma \geq \max_{\lambda \in \Upsilon} \max_{i_1, i_2, \ldots, i_d} K \left( p_{X^\lambda(Y)}(Y_{i_1 i_2 \cdots i_d}) | p_{X^\lambda(Y)}(Y_{i_1 i_2 \cdots i_d}) \right),
\]

then

\[
\mathbb{E}_{\Omega, Y_{i\Omega}} \left[ -2 \log H(p_{X^\lambda(Y)} || p_{X^\lambda(Y)}(Y)) \right] \leq \frac{8\gamma \log(m)}{m} + 3 \min_{\lambda \in \Upsilon} \left\{ \frac{K(p_{X^\lambda(Y)} || p_{X^\lambda(Y)})}{n_1 n_2 \cdots n_d} \right\}
\]

\[
+ \left( \max_{i} \{ \lambda_i \} + \frac{8\gamma (\beta + 2) \log(n_m)}{3} \right) \left( \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^{d} ||A_i||_0}{m} \right)
\]

where the expectation is taken with regard to the joint distribution of \( \Omega \) and \( Y_{i\Omega} \).

Theorem 4.1 is the upper error bound of the estimator in (6) under a general class of noise distributions, which can be specified to some special noise distributions, such as additive Gaussian noise, additive Laplace noise, and Poisson observations. The key point for showing the upper error bound with special noise distributions is to specify the logarithmic Hellinger affinity \( \log H(p_{X^\lambda(Y)} || p_{X^\lambda(Y)}) \) and the minimum KL divergence \( \min_{\lambda \in \Upsilon} K(p_{X^\lambda(Y)} || p_{X^\lambda(Y)}) \).

From Theorem 4.1, we can see that the negative logarithmic Hellinger affinity with respect to the underlying tensor and the recovered tensor is on the order of \( O(\frac{r_1 r_2 \cdots r_d + \sum_{i=1}^{d} ||A_i||_0 \log(n_m)}{m}) \) if the KL divergence \( \min_{\lambda \in \Upsilon} K(p_{X^\lambda(Y)} || p_{X^\lambda(Y)}) \) is not too large and \( \lambda_i \) are fixed, where \( A_i \) are the factor matrices of \( X^\lambda \in \Upsilon \) in the Tucker decomposition. Therefore, if the Tucker rank is low, the error bound of the estimator in (6) will be small.
Remark 4.1 For the problem of sparse NTD with partial observations, the upper error bound is related to the degree of freedoms of the tensor, i.e., \( r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A_i \|_0 \). Moreover, it follows from Theorem 4.1 that the upper error bound decreases as the number of observed samples increases.

In the following subsections, we specify the noise models and establish the detailed upper error bounds based on Theorem 4.1. In the discretization levels (8), we choose

\[
\beta = 1 + \frac{\log \left( \frac{(2^{d+1} - 1) \sqrt{r_1 r_2 \cdots r_d a_1 a_2 \cdots a_d}}{c \sqrt{m}} + 1 \right)}{\log(n_m)},
\]

which implies that \( \beta \geq 1 \). For the regularization parameters \( \lambda_i \), we consider the specific choice

\[
\lambda_i = 4(\beta + 2) \left( 1 + \frac{2^\gamma}{3} \right) \log(n_m), \quad i \in [d],
\]

where \( \gamma \) satisfies (9) and is related to the maximum KL divergence between \( p_{\mathcal{Y}^{*}_{i_1 i_2 \cdots i_d}}(\mathcal{Y}^{*}_{i_1 i_2 \cdots i_d}) \) and \( p_{\mathcal{X}^{i_1 i_2 \cdots i_d}}(\mathcal{X}^{i_1 i_2 \cdots i_d}) \) for any \( \mathcal{X} \in \mathcal{X} \).

### 4.1 Additive Gaussian Noise

In this subsection, we establish the upper error bound of the estimator in (6) for the observations corrupted by additive Gaussian noise, which has been widely used in a variety of applications. In this case, the observation \( \mathcal{Y}^{*}_{i} \) is generated by

\[
\mathcal{Y}^{*}_{i_1 i_2 \cdots i_d} = \mathcal{X}^{*}_{i_1 i_2 \cdots i_d} + \mathcal{N}^{i_1 i_2 \cdots i_d}, \quad (i_1, i_2, \ldots, i_d) \in \Omega,
\]

where \( \mathcal{N}^{i_1 i_2 \cdots i_d} \) are independent zero-mean Gaussian noise with standard deviation \( \sigma > 0 \). The distribution of \( \mathcal{Y}^{*}_{i} \) obeys a multivariate Gaussian density with dimension \( |\Omega| \times |\Omega| \) whose mean and covariance matrix are \( \mathcal{X}^{*}_{i} \) and \( \sigma^2 I_{\Omega} \), respectively, where \( I_{\Omega} \) denotes the identity matrix with dimension \( |\Omega| \times |\Omega| \). Then the joint probability density function of \( \mathcal{Y}^{*}_{i} \) is given by

\[
p_{\mathcal{Y}^{*}_{i}}(\mathcal{Y}^{*}_{i}) = \frac{1}{(2\pi \sigma^2)^{|\Omega|/2}} \exp \left( -\frac{\| \mathcal{Y}^{*}_{i} - \mathcal{X}^{*}_{i} \|^2}{2\sigma^2} \right).
\]

Now we specify the upper error bound in Theorem 4.1 for the observations with additive Gaussian noise in the following theorem, where the joint probability density function of the observations is given by (12).

**Theorem 4.2** Let \( \beta \) and \( \lambda_i \) be defined as (10) and (11), \( i \in [d] \), where \( \gamma = \frac{c^2}{2 \sigma^2} \) in (11). Assume that the sampling set \( \Omega \sim \text{Bern}(p) \) with \( p = \frac{m}{n_1 n_2 \cdots n_d} \) and the joint probability density function of \( \mathcal{Y}^{*}_{i} \) is given by (12). Then the estimator in (6) satisfies

\[
\mathbb{E}_{\Omega, \mathcal{Y}^{*}} \left[ \| \mathcal{X} - \mathcal{X}^{*} \|^2_F \right] \leq \frac{22c^2 \log(m)}{m} \left( r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A_i \|_0 \right) \log(n_m) + 16(\beta + 2)(2c^2 + 3\sigma^2) \left( \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A_i \|_0}{m} \right) \log(n_m).
\]

By Theorem 4.2, we know that the upper error bound of the estimator of (6) for the observations with additive Gaussian noise is on the order of

\[
O \left( \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A_i \|_0}{m} \log(n_m) \right).
\]
Consequently, if the factor matrices are sparser, the error upper bound of the estimator is lower. Moreover, when \( m = n_1 \cdots n_d \) and \( \| A_i \|_0 = r_i n_i, \ i \in [d] \), which implies that \( A_i \) are dense, the error bound in Theorem 4.2 is just the error bound of the estimator of NTD with additive Gaussian noise, which has been widely used and investigated, see [11, 36] and references therein.

**Remark 4.2** For sparse NTD with additive Gaussian noise, Liu et al. [33] proposed a novel model with a special core tensor in the Tucker decomposition, where the \( \ell_0 \) norm is replaced by the \( \ell_1 \) norm for the factor matrices. In this case, Theorem 4.2 establishes the error bound of the model in [33] by employing the \( \ell_0 \) norm to characterize the sparsity of the factor matrices. Moreover, Xu et al. [52] proposed a sparse NTD and completion approach for the observations with additive Gaussian noise, which replaced the \( \ell_0 \) norm by the \( \ell_1 \) norm in (6). This implies that Theorem 4.2 established the error bound of the estimator of the model in [52] when the \( \ell_0 \) norm is utilized to measure the sparsity of the factor matrices.

**Remark 4.3** If we ignore the internal structure of a tensor and unfold the tensor into a matrix, we can compare the error bound in Theorem 4.2 with that in [46, Corollary 3], where the matrix based method in [46] can reduce to sparse nonnegative matrix factorization and completion with nonnegative and bounded constraints. For a given tensor \( X^* \) in the form of (3), without loss of generality, we unfold it along the first mode, i.e., \( X^*_{(1)} = A_1^*(C_{(1)}^* \otimes d A_1^*)^T \). In this case, one is capable of applying the matrix based method in [46] to \( (X^*_{(1)})^T \in \mathbb{R}^n, n \times n \), where the sparse factor is \( A_1^* \). Denote \( X^m \) to be the estimator by the matrix method, then the resulting error bound of the estimator of the matrix based method [46, Corollary 3] is

\[
\mathbb{E}_{\Omega, \mathcal{Y}_0} \left[ \| X^m - X^* \|_F^2 \right] \\
\leq \frac{70c^2 \log(m)}{n_1 n_2 \cdots n_d}
+ 8(3c^2 + 8c^2)(\beta_1 + 2) \log(\max\{n_1, n_2 \cdots n_d\})^r_1 n_2 \cdots n_d + \| A_1^* \|_0,
\]

where \( \beta_1 = \max\{1, 1 + \frac{\log(8r_1 a_1/c)}{\log(\max\{n_1, n_2 \cdots n_d\})} \} \). This implies that the order of the error bound of the estimator is \( O\left(\frac{n_2 \cdots n_d \| A_1^* \|_0}{m} \log(\max\{n_1, n_2 \cdots n_d\})\right) \). Now we compare it with the case that only nonnegative tensor decomposition is considered. In this case, the error bound of the tensor based method in [46] for the tensor with mode-1 unfolding is the order of \( O\left(\frac{n_1 n_2 \cdots n_d}{m} \log(\max\{n_1, n_2 \cdots n_d\})\right) \).

And the error bound in Theorem 4.2 is the order of \( O\left(\frac{r_1 n_2 \cdots r_d n_1}{m} \log(n_1)\right) \), and \( O\left(\frac{n_1 n_2 \cdots n_d}{m} \log(n_1)\right) \), respectively, where the error bound of the matrix based method is larger than that in Theorem 4.2 if \( r, d \) are much smaller than \( n \).

**Remark 4.4** We compare the error bound in Theorem 4.2 with that in [55, Proposition 4.1], which is based on the algebra framework of tensor-tensor product and is only effective for third-order tensors. In general, the two error bound results are not comparable directly since the sparse factors are different for the two factorizations. However, when the factors matrices in Tucker decomposition and the factor tensor in the algebra framework of tensor-tensor product are dense, the two error bounds can be comparable. For a third-order tensor with size \( n_1 \times n_2 \times n_3 \), the error bound of the estimator in Theorem 4.2 is \( O\left(\frac{r_1 n_2 r_3 n_1 + n_2 n_1}{m} \log(n_1)\right) \), while it is \( O\left(\frac{r_1 n_2 n_3}{m} \log(\max\{n_1, n_2\})\right) \) in [55], where \( r \) is the tubal rank of the underlying tensor. In this case, the error bound in Theorem 4.2 is smaller than that in [55] if \( r_i \) is much smaller than \( n_i \) and \( r \) is close to \( r_i \). In particular, if \( n_1 = n_2 = n_3 = n \) and
\( r_1 = r_2 = r_3 = r \), the error bound in Theorem 4.2 is \( O\left( \frac{r^3 + 3rn}{m} \log(n) \right) \), which is smaller than that (i.e., \( O\left( \frac{2rn^2}{m} \log(n) \right) \)) in [55] if \( r \leq 2n - 3 \) and \( n \geq 3 \). These conditions can be satisfied easily in real-world applications since \( n \) is generally large.

Remark 4.5 Jain et al. [22] proposed a noisy tensor completion model based on CP decomposition with a special sparse factor for an \( n_1 \times n_2 \times n_3 \) tensor, where the third factor matrix is sparse in CP decomposition. In particular, when the constraints are nonnegative, the model in [22] reduces to sparse nonnegative CP decomposition and completion. In this case, the error bound of the estimator of their model for the observations with additive Gaussian noise was

\[
\frac{E_{\Omega, Y_{\Omega}} \|X_{\text{cp}} - X^*\|_F^2}{n_1n_2n_3} = O\left( \left( \frac{n_1 + n_2}{m} \right)^r + \frac{\|C^*\|_0}{m} \log(\max\{n_1, n_2, n_3\}) \right),
\]

where \( X_{\text{cp}} \) is the estimator by [22], \( r \) is the CP rank of \( X^* \), and \( C^* \) is the third factor matrix in the CP decomposition of \( X^* \). However, (13) is hard to compare with the error bound in Theorem 4.2 directly since the CP rank of a tensor is not comparable to the Tucker rank of a tensor in general. Also the factor matrices are typically different between the CP and Tucker decompositions of a tensor. In particular, if the core tensor in Tucker decomposition is diagonal and \( r_1 = r_2 = r_3 = r \), the error bound in Theorem 4.2 is comparable with (13). In fact, the error bound in Theorem 4.2 is smaller than (13) if \( r \) is much smaller than \( n_i \).

### 4.2 Additive Laplace Noise

In this subsection, we establish the error bound of the estimator of model (6) for the observations with additive Laplace noise. In this case, the observation \( Y_{\Omega} \) is modeled by

\[
Y_{i_1 \cdots i_d} = X^*_{i_1 \cdots i_d} + \mathcal{N}_{i_1 \cdots i_d}, \quad (i_1, \ldots, i_d) \in \Omega,
\]

where \( \mathcal{N}_{i_1 \cdots i_d} \) are independent Laplace distributions with parameters \((0, \tau)\), \( \tau > 0 \), which is denoted by \( \text{Laplace}(0, \tau) \). The joint probability density function of \( Y_{\Omega} \) is given by

\[
p_{X_{\Omega}}(Y_{\Omega}) = \left( \frac{\tau}{2} \right)^|\Omega| \exp\left( -\frac{\|Y_{\Omega} - X^*_{\Omega}\|_1}{\tau} \right).
\]

Next we establish an upper error bound of the estimator of model (6) for the observations satisfying the noisy model (14).

**Theorem 4.3** Let \( \beta \) and \( \lambda_i \) be defined as (10) and (11), \( i \in [d] \), where \( \gamma = \frac{c^2}{2\tau^2} \). Assume that the sampling set \( \Omega \sim \text{Bern}(p) \) with \( p = \frac{m}{n_1n_2 \cdots n_d} \) and the joint probability density function of \( Y_{\Omega} \) is given by (15). Then the estimator of (6) satisfies

\[
\frac{E_{\Omega, Y_{\Omega}} \|X^\lambda - X^*\|_F^2}{n_1n_2 \cdots n_d} \leq \frac{11c^2(2\tau + c)^2 \log(m)}{2\tau^2 m} + 12 \left( 1 + \frac{2c^2}{3\tau^2} \right) (2\tau + c)^2 (\beta + 2) \log(\max_{n} r_1r_2 \cdots r_d + \sum_{i=1}^{d} A_{i}^*) \log(n_m).
\]

From Theorem 4.3, we know that the upper error bound of the estimator of model (6) is on the order of \( O\left( \frac{r_1r_2 \cdots r_d + \sum_{i=1}^{d} A_{i}^*}{m} \log(n_m) \right) \). Similar to the case of additive Gaussian noise, the error bound of the estimator of model (6) decreases as the number of observed samples increases. Moreover, if the factor matrices of Tucker decomposition are sparser, the error bound obtained by (6) is lower.
Remark 4.6 For the observations with additive Laplace noise, we compared the error bound in Theorem 4.3 with that of the matrix based method in [46, Corollary 5]. For a $d$th-order tensor $X^* \in \mathbb{R}_{+}^{n_1 \times \cdots \times n_d}$ with Tucker decomposition in (3), we unfold it into a matrix $X_{(1)}^* = A_1^*(C_{(1)}^* (\otimes_{i=2}^{d} A_i^*))^T \in \mathbb{R}_{+}^{n_1 \times (n_2 \cdots n_d)}$ along the first mode, where $A_1^*$ is sparse. The matrix based method in [46] is then applied to $(X_{(1)}^*)^T \in \mathbb{R}_{+}^{(n_2 \cdots n_d) \times n_1}$. And the error bound of the estimator in [46, Corollary 5] is on the order of $O\left(\frac{r_1 n_2 \cdots n_d + \|A_1^*\|_0}{m} \log(\max\{n_1, n_2 \cdots n_d\})\right)$. Therefore, if $r_i$ is much smaller than $n_i$, $i \in [d]$, and $d$ is not too large, the error bound of the estimator by model (6) is smaller than that of the matrix based method in [46]. In particular, if $r_1 = \cdots = r_d = r$ and $n_1 = \cdots = n_d = n$, the error bound of the estimator in Theorem 4.3 is the order of $O\left(\frac{r d^{d-1}}{m} + \|A_1^*\|_0 \log(n)\right)$, while the error bound of the estimator of the matrix based method in [46] is $O\left(\frac{n^{d-1}}{m} \log(d-1) \log(n)\right)$. This implies that the error bound of the estimator in Theorem 4.3 is smaller than that of the matrix based method in [46] as long as $r$ is much smaller than $n$ and $\|A_1^*\|_0$ is close to $\|A_1^*\|_0$, $i = 2, \ldots, d$.

Remark 4.7 Similar to the case for the observations with additive Gaussian noise, we compare the error bound in Theorem 4.3 with that of the sparse nonnegative tensor factorization method under the tensor-tensor product framework in [55], which is only effective for third-order tensors. Since the sparse factors are different between the sparse NTD and sparse nonnegative tensor factorization with tensor-tensor product in [55], it is difficult to compare the error bounds of the two methods directly. However, if the factor tensor is not sparse, the error bound of the estimator in [55, Proposition 4.2] is on the order of $O\left(\frac{r n_1 n_3 + r n_2 n_1 n_3}{m} \log(\max\{n_1, n_2, n_3\})\right)$, where $r$ is the tubal rank of the underlying tensor. The error upper bound of the estimator by model (6) is on the order of $O\left(\frac{r_1 r_2 n_3 + \sum_{i=1}^{d} r_i n_i}{m} \log(n_m)\right)$, which is smaller than that of [55] if $r$ is close to $r_i$ and $r_i$ is much smaller than $n_i$, $i = 1, 2, 3$.

4.3 Poisson Observations

In this subsection, we establish an upper error bound of the estimator for Poisson observations, where the observations are modeled as

$$\gamma_{i_1 \cdots i_d} = \text{Poisson}(\lambda_{i_1 \cdots i_d}^*), \quad (i_1, \ldots, i_d) \in \Omega.$$ 

Here $y = \text{Poisson}(x)$ denotes that $y$ obeys a Poisson distribution with parameter $x > 0$. The joint probability mass function of $\gamma_{\Omega}$ is given by

$$p_{X_{\Omega}^*}(\gamma_{\Omega}) = \prod_{(i_1, i_2, \ldots, i_d) \in \Omega} \frac{(\lambda_{i_1 i_2 \cdots i_d}^*)^{\gamma_{i_1 i_2 \cdots i_d}} \exp(-\lambda_{i_1 i_2 \cdots i_d}^*)}{\gamma_{i_1 i_2 \cdots i_d}!}.$$ 

(16)

Now we establish an upper error bound of the estimator of model (6), where the joint probability mass function of $\gamma_{\Omega}$ satisfies (16).

Theorem 4.4 Suppose that $\min_{i_1, i_2, \ldots, i_d} \lambda_{i_1 i_2 \cdots i_d}^* \geq \varrho$ and each entry of the tensor in the candidate set $\mathcal{Y}$ is also not smaller than $\varrho$, where $\varrho > 0$ is a given constant. Let $\beta$ and $\lambda_1$ be defined as (10) and (11), $i \in [d]$, where $\gamma = \frac{\varrho}{\varrho}$ in (11). Assume that the sampling set $\Omega \sim \text{Bern}(p)$ with $p = \frac{n}{n_1 n_2 \cdots n_d}$ and the joint probability mass function of $\gamma_{\Omega}$ is given by (16). Then the estimator in (6) satisfies

$$\mathbb{E}_{\Omega, \gamma_{\Omega}} \left[\left\|\lambda_{\Omega} - \lambda_{\Omega}^*\right\|_F^2 \right] \leq \frac{4 c_3^2 \log(m)}{n_1 n_2 \cdots n_d m} + 48c \left(1 + \frac{4c^2 \gamma}{3\varrho}\right) \beta + \frac{2}{m} \sum_{i=1}^{d} \|A_i^*\|_0 \log(n_m).$$

12
By Theorem 4.4, one can obtain that the upper error bound of the estimator of model (6) with Poisson observations is on the order of $O\left(\frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d \|A_i\|_0}{m} \log(n_m)\right)$. Moreover, the error bound will decrease as the number of observations increases. The lower bound of $\lambda^*$ will also influence the error bound of the estimator, and it will be difficult to recover $\lambda^*$ as $\rho$ is close to zero.

**Remark 4.8** In order to compare the error bound in Theorem 4.4 with that of the matrix based method in [46, Corollary 6], we need to unfold the underlying nonnegative tensor $\lambda^*$ ($n_1 \times \cdots \times n_d$) into a matrix, where the multi-linear structure of a tensor is ignored. Without loss of generality, assume that $\lambda^*$ is unfolded along the first mode, in which the resulting matrix is $\lambda^*(1) = \Lambda^*_1(C^*_1(\otimes_{i=1}^d \Lambda^*_i)^T) \in \mathbb{R}_{+}^{n_1 \times (n_2 \cdots n_d)}$. Then, by applying the matrix based method in [46] to $(\lambda^*(1))^T$, the error bound of the resulting estimator is on the order of $O\left(\frac{r_1 n_1 n_2 \cdots n_d + \|\Lambda^*_1\|_0}{m} \log(n_1 \cdot n_2 \cdots n_d)\right)$, which is larger than that of Theorem 4.4 if $r_i$ is much smaller than $n_i$, $i \in [d]$, and $d$ is not too large. In particular, when $r_1 = \cdots = r_d = r$, $n_1 = \cdots = n_d = n$, and $\|\Lambda^*_1\|_0 = \cdots = \|\Lambda^*_d\|_0$, the error bound of the estimator in [46, Corollary 6] is on the order of $O\left(\frac{r n_d - 1 + \|\Lambda^*_1\|_0}{m} (d - 1) \log(n)\right)$, and the error bound in Theorem 4.4 is $O\left(\frac{2^d + \|\Lambda^*_1\|_0}{m} \log(n)\right)$, which is smaller than that of [46] as long as $d \geq 3$ and $r$ is much smaller than $n$.

**Remark 4.9** Cao et al. [7] proposed a matrix completion method with the nuclear norm constraint for Poisson observations and established the error bound of the estimator of their proposed model. For higher-order tensor completion, without loss of generality, we unfolding the underlying tensor $\lambda^* \in \mathbb{R}_{+}^{n_1 \times n_2 \times \cdots \times n_d}$ into a matrix along the first mode, i.e., $\lambda^*(1)$. In this case, when $m \geq (n_1 + n_2 \cdots n_d) \log(n_1 \cdots n_d)$, the error bound of the estimator obtained by [7, Theorem 2] is

$$\frac{\|\lambda_{mt} - \lambda^*\|_F^2}{n_1 n_2 \cdots n_d} \leq C \sqrt{r_1 (n_1 + n_2 \cdots n_d)} \log(n_1 \cdots n_d)$$

(17)

with high probability, where $\lambda_{mt}$ is the estimator obtained by [7] and $C > 0$ is a constant. In particular, when $n_1 = \cdots = n_d = n$ and $r_1 = \cdots = r_d$, the upper bound in (17) reduces to $O(d \sqrt{\frac{r_1 (n_1 + n_d - 1)}{m}} \log(n))$, which is larger than $O(\frac{r_1 d + \|\lambda^*_1\|_0}{m} \log(n))$ if $r_1$ is much smaller than $n$ (e.g., $r_1 < \left(\frac{n-1}{d-1}\right)^{\frac{1}{d-1}}$). Moreover, the factor matrix $\Lambda_1$ is sparser, the error bound in Theorem 4.4 is lower, while the factor matrix $\Lambda_1$ has no influence on the error bound in [7, Theorem 2], $i \neq 1$.

**Remark 4.10** Similar to the observations with additive Laplace noise, we can only compare the error bound in Theorem 4.4 with that in [55, Proposition 4.3] for third-order tensors, which has smaller error bound than that of [54, Theorem 3.1] for low-rank tensor completion with Poisson observations. More detailed comparisons about the error bounds of the two methods can be referred to [54, Section IV.C]. We do not consider the sparse case since the factor tensor based on the algebra framework of tensor-tensor product is not comparable with the factor matrices in Tucker decomposition. In this case, the error bound of the estimator in [55, Proposition 4.3] is $O(\frac{r \sqrt{n_1 n_2 n_3} + \sum_{i=1}^3 m r_n}{m} \log(\max\{n_1, n_2\}))$, where $r$ is the tubal rank of the underlying tensor. By Theorem 4.4, we obtain that the error bound of the estimator of model (6) is on the order of $O\left(\frac{r_1 r_2 + \sum_{i=1}^3 r_n}{m} \log(\max\{n_1, n_2, n_3\})\right)$, which is smaller than that of [55] if $r$ is close to $r_i$ and $r_i$ is much smaller than $n_i$, $i = 1, 2, 3$.

## 5 Minimax Lower Bounds

In this section, we establish the minimax lower bound of an estimator for the observations satisfying (5). The accuracy of an estimator $\hat{\lambda}$ for estimating the true tensor $\lambda^*$ can be measured in terms of its
risk [47], which is defined as $\mathbb{E}_{\Omega, Y_0} [\| X - X^* \|_F^2]$. Now we consider a class of tensors with Tucker rank $r = (r_1, \ldots, r_d)$ satisfying (3), where each factor matrix in Tucker decomposition obeys $\| A_i \|_0 \leq s_i$, $i \in [d]$, and the amplitudes of each entry of the core tensor and factor matrices are not larger than 1 and $a_i$, respectively. In this case, we define the following set

$$\mathcal{L}(s, \mathbf{r}, \mathbf{a}) := \left\{ X = C \times A_1 \times \cdots \times A_d : C \in \mathbb{R}_+^{s_1 \times \cdots \times s_d}, A_i \in \mathbb{R}_{[0,1]}^{n_i \times r_i}, 0 \leq C_{i_1 \cdots i_d} \leq 1, \| A_i \|_0 \leq s_i, 0 \leq (A_i)_{lm} \leq a_i, (i_1, \ldots, i_d) \in [r_1] \times \cdots \times [r_d], (l, m) \in [n_i] \times [r_i], i \in [d] \right\}. \quad (18)$$

The worst-case performance of an estimator $\tilde{X}$ of $X^*$ is measured by the maximum risk on the set $\mathcal{L}(s, \mathbf{r}, \mathbf{a})$, which is defined as

$$\sup_{X^* \in \mathcal{L}(s, \mathbf{r}, \mathbf{a})} \mathbb{E}_{\Omega, Y_0} [\| \tilde{X} - X^* \|_F^2].$$

The estimator, which has the smallest maximum risk among all possible estimators, is said to achieve the minimax risk [47]. In this case, for the sparse NTD and completion problem, the minimax risk is defined as

$$\inf_{\tilde{X}} \sup_{X^* \in \mathcal{L}(s, \mathbf{r}, \mathbf{a})} \frac{\mathbb{E}_{\Omega, Y_0} [\| \tilde{X} - X^* \|_F^2]}{n_1 n_2 \cdots n_d}. \quad (19)$$

Next we will estimate the lower bound of (19), which is stated in the following theorem.

**Theorem 5.1** Suppose that the probability density function or probability mass function of any two entries of the noisy observations satisfies

$$K(\mathbb{P}_x, \mathbb{P}_y) \leq \frac{(x - y)^2}{2\mu}, \quad (20)$$

where $\mu > 0$ depends on the noise distribution. Assume that $r_i \leq s_i$ in the set $\mathcal{L}(s, \mathbf{r}, \mathbf{a})$ in (18), $i \in [d]$. Then, for the joint probability density function or probability mass function of the observations $Y_0$ obeying (5), there exist two constants $\tilde{\alpha}_m, \gamma_m > 0$ such that

$$\inf_{\tilde{X}} \sup_{X^* \in \mathcal{L}(s, \mathbf{r}, \mathbf{a})} \frac{\mathbb{E}_{\Omega, Y_0} [\| \tilde{X} - X^* \|_F^2]}{n_1 n_2 \cdots n_d} \geq \frac{\tilde{\alpha}_m}{2^{d+5(d+1)}} \min \left\{ \prod_{i=1}^d \Delta_i(s_i, n_i) a_i^2, \gamma_m^2 \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d s_i}{m} \right\},$$

where

$$\Delta_i(s_i, n_i) = \min \left\{ 1, \frac{s_i}{n_i} \right\}, \quad i \in [d]. \quad (21)$$

For a fixed $d$, Theorem 5.1 shows that the lower bound of the estimator satisfying (18) is on the order of $O\left( \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d s_i}{m} \right)$, which matches the upper bound in Theorem 4.1 up to a logarithmic factor. This demonstrates the upper bound in Theorem 4.1 is nearly optimal.
Remark 5.1 The proof of the minimax risk bound in Theorem 5.1 utilized the technique of the tools for matrix completion in [26, 42], see also [55]. The key issue of the proof is to construct the packing sets for the core tensors and sparse factor matrices, respectively. Then the Varshamov-Gilbert bound [47, Lemma 2.9] is used to determine the Frobenius norm of any two tensors in the set. Finally, one can get the minimax lower bound by the minimax analysis techniques in [47, Theorem 2.5].

Remark 5.2 Based on Theorem 5.1, for the explicit lower bounds of the minimax risk with special noise models in Section 4, we just need to specify $\mu$ in (20) except for Poisson observations. In particular, for the observations of sparse NTD and completion with additive Gaussian noise and additive Laplace noise, we can set $\mu = \sigma$ and $\mu = \tau$, respectively. For Poisson observations, we need each entry of the underlying tensor to be strictly positive in the construction of the packing set. Similar results can be found in [42, 55]. For brevity, we omit the detailed proof of the minimax risk here for sparse NTD and completion with Poisson observations.

6 ADMM Based Algorithm

In this section, we design an ADMM based algorithm [6] to solve problem (6). Notice that the feasible set $\mathcal{F}$ in (7) is discrete, which leads to the difficulty for the algorithm design. Similar to [22, 55], we drop the discrete assumption of the core tensor and factor matrices in order to use continuous optimization techniques, which may be justified by choosing a very large value of $\tau$ and by noting that continuous optimization algorithms use finite precision arithmetic when executed on a computer. Hence, we consider to solve the following problem:

$$\min_{A', C, A_i} - \log p_{X_1}(\mathcal{Y}_1) + \sum_{i=1}^{d} \lambda_i \|A_i\|_0$$

s.t. $A' = C \times_1 A_1 \times_2 \cdots \times_d A_d, 0 \leq A_{i_1i_2\ldots i_d} \leq c, (i_1, i_2, \ldots, i_d) \in [n_1] \times [n_2] \times \cdots \times [n_d]$, (22)

$$0 \leq c_{i_1i_2\ldots i_d} \leq 1, (i_1, i_2, \ldots, i_d) \in [r_1] \times [r_2] \times \cdots \times [r_d],$$

$$0 \leq (A_i)_{ml} \leq a_i, (m, l) \in [n_i] \times [r_i], i \in [d].$$

Note that the set $\mathcal{Y}$ in (7) and the constraint in (22) are different. Each entry of the core tensor $C$ in $\mathcal{Y}$ is taken from the set $\{0, \frac{1}{\tau}, \frac{2}{\tau}, \ldots, \frac{(\tau-1)}{\tau}, 1\}$, where $\tau$ is defined in (8). Moreover, each entry of the factor matrix $A_i$ in $\mathcal{Y}$ is taken from the set $\{0, a_i, \frac{2a_i}{\tau}, \ldots, \frac{(\tau-1)a_i}{\tau}, a_i\}, i \in [d]$. By relaxing the discrete assumption on the entries of the core tensor and factor matrices in $\mathcal{Y}$, we just consider each entry of the core tensor $C$ to be in $[0,1]$, i.e., $0 \leq C_{i_1i_2\ldots i_d} \leq 1, (i_1, i_2, \ldots, i_d) \in [r_1] \times [r_2] \times \cdots \times [r_d]$, and each entry of the factor matrix $A_i$ to be in $[0, a_i], i.e., 0 \leq (A_i)_{ml} \leq a_i, (m, l) \in [n_i] \times [r_i], i \in [d]$. In this case, the set $\mathcal{Y}$ in (7) is relaxed to the constraint in (22).

Let

$$\Xi_1 := \{X : 0 \leq X_{i_1i_2\ldots i_d} \leq c, (i_1, i_2, \ldots, i_d) \in [n_1] \times [n_2] \times \cdots \times [n_d]\},$$

$$\Xi_2 := \{C : 0 \leq C_{i_1i_2\ldots i_d} \leq 1, (i_1, i_2, \ldots, i_d) \in [r_1] \times [r_2] \times \cdots \times [r_d]\},$$

$$\Psi_i := \{A_i : 0 \leq (A_i)_{ml} \leq a_i, (m, l) \in [n_i] \times [r_i]\}, i \in [d].$$

For any set $\mathcal{C}$, let $\delta_{\mathcal{C}}(\cdot)$ denote the indicator function over $\mathcal{C}$, which is defined as

$$\delta_{\mathcal{C}}(x) = \begin{cases} 0, & \text{if } x \in \mathcal{C}, \\ +\infty, & \text{otherwise.} \end{cases}$$
By using the definition of indicator function and putting the bounded constraints with respect to $\mathcal{X}, \mathcal{C},$ and $A_i$ into the objective function, problem (22) can be rewritten as

$$
\min_{\mathcal{X}, \mathcal{C}, A_i} \quad -\log p_{\mathcal{X}^i}(\mathcal{Y}^i) + \sum_{i=1}^{d} \lambda_i \|A_i\|_0 + \delta_{\Xi_1}(\mathcal{X}) + \delta_{\Xi_2}(\mathcal{C}) + \sum_{i=1}^{d} \delta_{\Psi_i}(A_i)
$$

s.t. $\mathcal{X} = C \times_1 A_1 \cdots \times_d A_d.$

Note that minimizing the indicator functions $\delta_{\Xi_1}(\mathcal{X}) + \delta_{\Xi_2}(\mathcal{C}) + \sum_{i=1}^{d} \delta_{\Psi_i}(A_i)$ is equivalent to $\mathcal{X} \in \Xi_1, C \in \Xi_2, A_i \in \Psi_i, i \in [d]$. Therefore, models (22) and (23) are equivalent. The goal of using indicator functions in (23) is that there only exist equality constraints in (23) and then the ADMM can be applied to solve this model, where the ADMM is suitable to solve the optimization problem with equality constraints. Let $\mathcal{X} = Z, C = B, A_i = H_i, A_i = S_i, i \in [d]$. Then problem (23) is equivalent to

$$
\min \quad -\log p_{\mathcal{X}^i}(\mathcal{Y}^i) + \sum_{i=1}^{d} \lambda_i \|H_i\|_0 + \delta_{\Xi_1}(Z) + \delta_{\Xi_2}(B) + \sum_{i=1}^{d} \delta_{\Psi_i}(S_i)
$$

s.t. $\mathcal{X} = C \times_1 A_1 \cdots \times_d A_d, \mathcal{X} = Z, C = B, A_i = H_i, A_i = S_i, i \in [d].$

Models (23) and (24) are equivalent since only the variable substitution is used in (24). By introducing these variables in (24), each subproblem can be solved easily in the framework of ADMM. The augmented Lagrangian function associated with problem (24) is given by

$$
L(\mathcal{X}, \mathcal{C}, A_i, Z, B, H_i, S_i, T_i, M_i, N_i)
$$

$$
= -\log p_{\mathcal{X}^i}(\mathcal{Y}^i) + \sum_{i=1}^{d} \lambda_i \|H_i\|_0 + \delta_{\Xi_1}(Z) + \delta_{\Xi_2}(B) + \sum_{i=1}^{d} \delta_{\Psi_i}(S_i)
$$

$$
+ \langle T_1, \mathcal{X} - C \times_1 A_1 \cdots \times_d A_d \rangle + \langle T_2, \mathcal{X} - Z \rangle + \langle T_3, C - B \rangle
$$

$$
+ \sum_{i=1}^{d} \langle M_i, A_i - H_i \rangle + \sum_{i=1}^{d} \langle N_i, A_i - S_i \rangle
$$

$$
+ \frac{\beta_1}{2} \|\mathcal{X} - C \times_1 A_1 \cdots \times_d A_d\|_F^2 + \frac{\beta_2}{2} \|\mathcal{X} - Z\|_F^2 + \frac{\beta_3}{2} \|C - B\|_F^2
$$

$$
+ \sum_{i=1}^{d} \left( \frac{\rho_i}{2} \|A_i - H_i\|_F^2 + \frac{\alpha_i}{2} \|A_i - S_i\|_F^2 \right),
$$

where $T_1, T_2, T_3, M_i, N_i$ are the Lagrangian multipliers and $\beta_1, \beta_2, \beta_3, \rho_i, \alpha_i > 0$ are penalty pa-
The optimal solution of (20), Lemma 4.2.10. Then the optimal solution of (20) is given as follows:

\[
X^{k+1} = \arg\min_X L(X, C^k, A_i^k, Z^k, B_k, H_i^k, S_i^k, T_i^k, M_i^k, N_i^k),
\]
(25)

\[
C^{k+1} = \arg\min_C L(X^{k+1}, C, A_i^k, Z^k, B_k, H_i^k, S_i^k, T_i^k, M_i^k, N_i^k),
\]
(26)

\[
A_i^{k+1} = \arg\min_{A_i} L(X^{k+1}, C^{k+1}, A_i, Z^k, B_k, H_i^k, S_i^k, T_i^k, M_i^k, N_i^k),
\]
(27)

\[
Z^{k+1} = \arg\min_Z L(X^{k+1}, C^{k+1}, A_i^{k+1}, Z, B_k, H_i^k, S_i^k, T_i^k, M_i^k, N_i^k),
\]
(28)

\[
B^{k+1} = \arg\min_B L(X^{k+1}, C^{k+1}, A_i^{k+1}, Z^{k+1}, B, H_i^k, S_i^k, T_i^k, M_i^k, N_i^k),
\]
(29)

\[
H_i^{k+1} = \arg\min_{H_i} L(X^{k+1}, C^{k+1}, A_i^{k+1}, Z^{k+1}, B^{k+1}, H_i, S_i^k, T_i^k, M_i^k, N_i^k),
\]
(30)

\[
S_i^{k+1} = \arg\min_{S_i} L(X^{k+1}, C^{k+1}, A_i^{k+1}, Z^{k+1}, B^{k+1}, H_i^{k+1}, S_i, T_i^{k+1}, M_i^k, N_i^k),
\]
(31)

\[
T_1^{k+1} = T_1^k + \beta_1(X^{k+1} - C^{k+1} \times A_i^{k+1} \cdots A_d^{k+1}),
\]
(32)

\[
T_2^{k+1} = T_2^k + \beta_2(X^{k+1} - Z^{k+1}),
\]
(33)

\[
T_3^{k+1} = T_3^k + \beta_3(C^{k+1} - B^{k+1}),
\]
(34)

The main advantage of ADMM for solving (24) is that each subproblem has a closed-form solution. Next we give the explicit solutions of each subproblem in the ADMM. Let

\[
f(X) := -\log p_X(\mathcal{Y}_1).
\]

The optimal solution of (25) with respect to $X$ is given by

\[
X^{k+1} = \text{Prox}_{\frac{\beta_1}{\beta_1 + \beta_2}} f \left( \frac{1}{\beta_1 + \beta_2} \left( \beta_1 C^k \times A_1^k \cdots A_d^k - T_1^k + \beta_2 Z^k - T_2^k \right) \right).
\]
(35)

According to (1), problem (26) is equivalent to

\[
\min_C \frac{\beta_1}{2} \left\| \text{vec}(X^{k+1} + \frac{1}{\beta_1} T_1^k) - (\otimes_{i=d} A_i^k) \text{vec}(C) \right\|^2_F + \frac{\beta_3}{2} \left\| \text{vec}(C) - \text{vec}(B^k - \frac{1}{\beta_3} T_3^k) \right\|^2_F.
\]
(36)

Then the optimal solution of (36) is given by

\[
\text{vec}(C^{k+1}) = \left( \beta_1 (\otimes_{i=d} A_i^k)^T (\otimes_{i=d} A_i^k) + \beta_2 I \right)^{-1} q^k
\]
(37)

\[
= \left( \beta_1 (\otimes_{i=d} A_i^k)^T A_i^k + \beta_2 I \right)^{-1} q^k,
\]

where $q^k := (\otimes_{i=d} A_i^k)^T \text{vec}(\beta_1 X^{k+1} + T_1^k) + \text{vec}(\beta_3 B^k - T_3^k)$ and the second equality holds by [20, Lemma 4.2.10].

The problem (27) can be reformulated as

\[
\min_{A_i} \frac{\beta_1}{2} \left\| X^{k+1}_{(i)} - A_i R_{i}^k + \frac{1}{\beta_1} (T_1^k)_{(i)} \right\|^2_F + \frac{\rho_i}{2} \left\| A_i - H_i^k + \frac{1}{\rho_i} M_i^k \right\|^2_F
\]
(38)

where

\[
R_{i}^k := C^{k+1}_{(i)} (A_d^k \otimes \cdots \otimes A_{i+1}^k \otimes A_{i-1}^{k+1} \otimes \cdots \otimes A_1^{k+1})^T.
\]
(39)
The optimal solution of (38) is represented as
\[
A_i^{k+1} = \left( \left( \beta_1 X_i^{k+1} + (T_i^k)_i \right) (R_i^k)^T + \rho_i H_i^k - M_i^k \right) \left( \beta_1 R_i^k (R_i^k)^T + \rho_i I \right)^{-1}.
\] (40)

The optimal solution of (28) with respect to \( Z \) is given by
\[
Z^{k+1} = \arg \min_{Z} \|Z - (\lambda^{k+1} + \frac{1}{\beta_2} T_2^k) \|_F = P_{\Xi_1} \left( \lambda^{k+1} + \frac{1}{\beta_2} T_2^k \right),
\] (41)

where \( P_{\Xi_1}(\cdot) \) denotes the projection onto a given set \( \Xi_1 \). Similarly, the optimal solution of (29) with respect to \( B \) is given by
\[
B^{k+1} = \arg \min_{B} \|B - (C^{k+1} + \frac{1}{\beta_3} T_3^k) \|_F = P_{\Xi_2} \left( C^{k+1} + \frac{1}{\beta_3} T_3^k \right).
\] (42)

The optimal solutions of (30) and (31) are given by
\[
H_i^{k+1} = \arg \min_{H_i} \|H_i\|_0 + \frac{\rho_i}{2} \left\| H_i - \left( A_i^{k+1} + \frac{1}{\rho_i} M_i^k \right) \right\|_F^2 = \text{Prox}_{\frac{\rho_i}{2}\|\cdot\|_0} \left( A_i^{k+1} + \frac{1}{\rho_i} M_i^k \right)
\] (43)

and
\[
S_i^{k+1} = \arg \min_{S_i} \|S_i\|_0 + \frac{\alpha_i}{2} \left\| S_i - \left( A_i^{k+1} + \frac{1}{\alpha_i} N_i^k \right) \right\|_F^2 = \text{Prox}_{\frac{\alpha_i}{2}\|\cdot\|_0} \left( A_i^{k+1} + \frac{1}{\alpha_i} N_i^k \right).
\] (44)

Although the problem in (43) is nonconvex, the minimizer of (43) can be attained [4, Example 6.10]. Note that the proximal mapping of the \( \ell_0 \) norm of a matrix can be implemented in a point-wise manner [4, Theorem 6.6]. For any \( \lambda > 0 \), the proximal mapping of \( \ell_0 \) norm for the one-dimension case is given by (see [4, Example 6.10])
\[
\text{Prox}_{\lambda\|\cdot\|_0}(y) = \begin{cases} 
0, & \text{if } |y| < \sqrt{2\lambda}, \\
0, y, & \text{if } |y| = \sqrt{2\lambda}, \\
y, & \text{if } |y| > \sqrt{2\lambda}.
\end{cases}
\]

Therefore, by [4, Theorem 6.6], the minimizer of (43) is given by
\[
(H_i^{k+1})_{jt} = \begin{cases} 
0, & \text{if } |(A_i^{k+1} + \frac{1}{\rho_i} M_i^k)_{jt}| < \sqrt{\frac{2\lambda}{\rho_i}}, \\
0, (A_i^{k+1} + \frac{1}{\rho_i} M_i^k)_{jt}, & \text{if } |(A_i^{k+1} + \frac{1}{\rho_i} M_i^k)_{jt}| = \sqrt{\frac{2\lambda}{\rho_i}}, \\
(A_i^{k+1} + \frac{1}{\rho_i} M_i^k)_{jt}, & \text{if } |(A_i^{k+1} + \frac{1}{\rho_i} M_i^k)_{jt}| > \sqrt{\frac{2\lambda}{\rho_i}}.
\end{cases}
\] (45)

Now the ADMM for solving (24) is stated in Algorithm 1.

**Remark 6.1** For the projections in (41), (42), and (44), the component of each entry keeps the same if it is in the set, otherwise projects to the nearest boundary. For instance, for the projection in (41),
\[
(Z^{k+1})_{i_1 i_2 \ldots i_d} = \left( P_{\Xi_1} \left( \lambda^{k+1} + \frac{1}{\beta_2} T_2^k \right) \right)_{i_1 i_2 \ldots i_d}
= \begin{cases} 
0, & \text{if } (\lambda^{k+1} + \frac{1}{\beta_2} T_2^k)_{i_1 i_2 \ldots i_d} < 0, \\
(\lambda^{k+1} + \frac{1}{\beta_2} T_2^k)_{i_1 i_2 \ldots i_d}, & \text{if } 0 \leq (\lambda^{k+1} + \frac{1}{\beta_2} T_2^k)_{i_1 i_2 \ldots i_d} \leq c, \\
c, & \text{if } (\lambda^{k+1} + \frac{1}{\beta_2} T_2^k)_{i_1 i_2 \ldots i_d} > c,
\end{cases}
\]

where \((i_1, i_2, \ldots, i_d) \in [n_1] \times [n_2] \times \cdots \times [n_d] \).
Algorithm 1 Alternating Direction Method of Multipliers for Solving Problem (24).

1: Initialization: $C^0, A^0_i, Z^0_i, B^0, T^0_1, T^0_2, T^0_3, M^0, N^0_i, i \in [d]$. Given parameters $\lambda_i, \beta_1, \beta_2, \rho_i, \alpha_i > 0, i \in [d]$.  
2: repeat  
3: \hspace{1em} Step 1. Compute $X^{k+1}$ by (35).  
4: \hspace{1em} Step 2. Update $C^{k+1}$ by (37).  
5: \hspace{1em} Step 3. Compute $A^{k+1}_i$ via (40), $i \in [d]$.  
6: \hspace{1em} Step 4. Compute $Z^{k+1}$ by (41).  
7: \hspace{1em} Step 5. Update $B^{k+1}$ according to (42).  
8: \hspace{1em} Step 6. Compute $H^{k+1}_i$ and $S^{k+1}_i$ by (45) and (44), respectively, $i \in [d]$.  
9: \hspace{1em} Step 7. Update $T^{k+1}_1, T^{k+1}_2, T^{k+1}_3, M^{k+1}_i, N^{k+1}_i$ by (32), (33) and (34), respectively.  
10: until A stopping condition is satisfied.

In the implementation of Algorithm 1, one needs to compute the proximal mapping of $\frac{1}{\beta_1 + \beta_2} f$ at $\mathcal{H}$, where

$$
\mathcal{H} := \frac{1}{\beta_1 + \beta_2} \left( \beta_1 C^k \times_1 A^k \cdots \times_d A^k - T^k_1 + \beta_2 A^k - T^k_2 \right) .
$$  

(46)

In particular, for the special noise observation models including additive Gaussian noise, additive Laplace noise, and Poisson observations, the proximal mapping in (35) is given in detail in [55], which is summarized as follows.

- Additive Gaussian noise: $X^{k+1} = \frac{1}{1 + \sigma^2 (\beta_1 + \beta_2)} \mathcal{P}_\Omega \left( \mathcal{Y} + \sigma^2 (\beta_1 + \beta_2) \mathcal{H} \right) + \mathcal{P}_\Omega \mathcal{H}$, where $\mathcal{P}_\Omega$ is the complementary set of $\Omega$ and $\mathcal{P}_\Omega$ is the projection operator onto the index $\Omega$ such that

$$
(\mathcal{P}_\Omega (A))_{i_1 i_2 \cdots i_d} = \begin{cases} A_{i_1 i_2 \cdots i_d}, & \text{if } (i_1, i_2, \cdots, i_d) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}
$$

- Additive Laplace noise: $X^{k+1} = \mathcal{P}_\Omega (\mathcal{Y} + \text{sign}(\mathcal{H} - \mathcal{Y}) \circ \max \{|\mathcal{H} - \mathcal{Y}| - \frac{1}{\sigma (\beta_1 + \beta_2)}, 0\}) + \mathcal{P}_\Omega \mathcal{H}$, where $\text{sign}(\cdot)$ and $\circ$ represents the signum function and the Hadamard point-wise product, respectively.

- Poisson observations:

$$
X^{k+1} = \frac{1}{2(\beta_1 + \beta_2)} \mathcal{P}_\Omega ((\beta_1 + \beta_2) \mathcal{H} - 1 + \sqrt{(\beta_1 + \beta_2) \mathcal{H} - 1}^2 + 4(\beta_1 + \beta_2) \mathcal{Y}) + \mathcal{P}_\Omega \mathcal{H},
$$

where $1$ represents an $n_1 \times n_2 \times \cdots \times n_d$ tensor with all entries being $1$, the square root and square are performed in the point-wise manner.

The computational complexity of ADMM at each iteration for solving (24) is given as follows. If the observations are chosen as additive Gaussian noise, additive Laplace noise, or Poisson observations, the computation cost of $X^{k+1}$ is on the order of $O(\sum_{i=1}^d (\prod_{i=1}^d n_i) \prod_{i=1}^d r_i)$. The computational cost of $C^{k+1}$ is $O(\sum_{i=1}^d n_i + \sum_{i=1}^d r_i + \sum_{i=1}^d (\prod_{i=1}^d n_i))$. In the implementation of $R^k_i$ in (39), we do not compute the Kronecker product directly. Let $K_i := C^{k+1} \times_1 A_1^{k+1} \cdots \times_{i-1} A_{i-1}^{k+1} \times_i A_i^{k+1} \times \cdots \times_d A_d^k$. Then $(K_i)_{(i)} = R^k_i$. Note that the computational cost of $K_i$ is [52, Appendix B]

$$
O \left( \sum_{j=1}^{i-1} \left( \prod_{t=1}^j n_t \right) \left( \prod_{t=j}^d r_t \right) + r_i \sum_{j=i+1}^{i-1} \left( \prod_{t=1}^j n_t \right) \sum_{j=i+1}^d \left( \prod_{t=j+1}^d n_t \right) \left( \prod_{t=j}^d r_t \right) \right) .
$$

(47)
For any \( i \in [d] \), the computational cost of each factor matrix \( A_i \) is the sum of (47) and \( O(r_in_1 \cdots n_d + r_i^2 + r_i^3) \). The computational costs of \( Z^{k+1} \) and \( B^{k+1} \) are \( O(\prod_{i=1}^d n_i) \) and \( O(\prod_{i=1}^d r_i) \), respectively. The computational costs of \( H_i^{k+1} \) and \( S_i^{k+1} \) are both \( O(n_ir_i) \). The computational costs of the multipliers \( T_1, T_2, T_3, M_i, N_i \) are \( O(\sum_{j=1}^d (\prod_{i=1}^j n_i)(\prod_{i=j}^d r_i)) \). Note that \( r_i \leq n_i, i \in [d] \), and the quantities of (47) and \( O(\sum_{j=1}^d (\prod_{i=1}^j n_i)(\prod_{i=j}^d n_i)) \) are similar, where the detailed discussions can be found in [52, Appendix B]. Therefore, the computational cost of ADMM in Algorithm 1 is

\[
O \left( \sum_{j=1}^d \left( \prod_{i=1}^j n_i \right) \left( \prod_{i=j}^d r_i \right) + \sum_{j=1}^d \left( \prod_{i=1}^j r_i \right) \left( \prod_{i=j}^d n_i \right) + \sum_{i=1}^d r_i^2 n_i + \prod_{i=1}^d r_i + \left( \prod_{i=1}^d n_i \right) \sum_{i=1}^d r_i \right).
\]

Remark 6.2 Notice that model (24) is nonconvex since the \( \ell_0 \) norm and the constraint \( X = C \times_1 A_1 \cdots \times_d A_d \) are both nonconvex. The convergence of ADMM cannot be guaranteed in general. Although great efforts have been made about the convergence of ADMM for solving nonconvex problems, the constraints and objective in (24) are both nonconvex, where the objective is not Lipschitz continuous for general loss functions mentioned in Section 4. Therefore, the existing literature about the convergence of ADMM for nonconvex problems (e.g., see [5, 19, 51]) cannot be applied to our model directly.

7 Numerical Experiments

In this section, we evaluate the effectiveness of the proposed sparse NTD and completion model (SNTDC) using both synthetic data and real image data. We compare SNTDC with a matrix based method in [46] (denoted by Matrix) and sparse nonnegative tensor factorization and completion with tensor-tensor product (SNTFTP) [55]. All experiments are performed in MATLAB 2020a on a computer with Intel Xeon W-2133 and 32 GB of RAM. In particular, the two comparison methods are introduced in detail below.

- A low-rank matrix factorization method with bounded constraints for a general class of noisy matrix completion tasks is proposed in [46], where the underlying matrix is approximated by the product of two matrices and one matrix is sparse. This approach can be reduced to sparse nonnegative matrix factorization and completion with specially bounded constraints on the matrices, where the corresponding model is given by

\[
\min_{X,A,B} - \log p_{X_\Omega}(Y_\Omega) + \lambda_1 \|B\|_0
\]

s.t. \( X = AB, 0 \leq X_{ij} \leq c_1, 0 \leq A_{ij} \leq 1, 0 \leq B_{ij} \leq b_1 \),

Here \( p_{X_\Omega}(Y_\Omega) \) is the same as that in (5) for the case of second-order tensors, and \( \lambda_1, c_1, b_1 > 0 \) are given constants. Then the error bounds for the estimator of the above model is established, which is applied to the special observations including additive Gaussian noise, additive Laplace noise, Poisson-distributed observations, and highly quantized observations.

- The SNTFTP in [55] is a sparse nonnegative tensor factorization method with partial and noisy observations for third-order tensors based on tensor-tensor product, where one factor tensor is sparse and all entries of the two factor tensors are nonnegative and bounded. More specifically, the SNTFTP model is given by

\[
\min_{X,A,B} - \log p_{X_\Omega}(Y_\Omega) + \lambda_2 \|B\|_0
\]

s.t. \( X = A \odot B, 0 \leq X_{ijk} \leq c_2, 0 \leq A_{ijk} \leq 1, 0 \leq B_{ijk} \leq b_2 \),

20
where \( A \diamond B \) denotes the tensor-tensor product of the two third-order tensors \( A, B \) defined in [23], and \( \lambda_2, c, b_2 > 0 \) are given constants. Besides, the error bound of the estimator of the SNTFTP model is established under general noise observations. And the minimax lower bound is shown to be matched with the established upper bound up to a logarithmic factor of the sizes of the underlying tensor.

Algorithm 1 will be stopped if the maximum number of iterations 300 is reached or the following condition is satisfied
\[
\frac{\|X^{k+1} - X^k\|_F}{\|X^k\|_F} \leq 10^{-4},
\]
where \( X^k \) is the \( k \)th iteration of Algorithm 1. For the initialization of Algorithm 1, we use a sequentially truncated high order singular value decomposition [49] on \( P_\Omega(Y) \) to get the initial values \( C^0, A^0_i, i \in [d] \), which can contain some information of \( \mathcal{X} \). For \( B^0 \), we choose it the same as \( C^0 \). We set \( Z^0 = T^0_1 = T^0_2 = T^0_3 = P_\Omega(Y) \) and \( M^0_i = N^0_i = A^0_i, i \in [d] \). For the regularization parameters \( \lambda_i \), we set \( \lambda_i \) to the same for any \( i \in [d] \) and choose them from the set \{5, 10, 50\} to get the best recovery performance. Moreover, \( \beta_i \) are set the same for different \( i \) and chosen from the set \{50, 70, 100, 250, 350, 450, 550\}.

For \( \epsilon_\ell \) and \( \rho_\ell \), we set them the same in each case for any \( i \in [d] \), and choose them from the set \{0.1, 0.01, 0.001\} to get the best recovery performance.

7.1 Synthetic Data
In this subsection, we test the synthetic data to demonstrate the effectiveness of the proposed method.

The synthetic tensor is generated as follows: The underlying tensor has the Tucker decomposition \( \mathcal{X} = C \times_1 A_1 \times_2 A_2 \cdots \times_d A_d \), where the nonnegative core tensor \( C \) is generated by MATLAB command \( \text{rand}([n_1, \ldots, n_d]) \), and the sparse nonnegative factor matrices \( A_i \) are generated by \( a_i \cdot \text{spmvrand}([n_i, r_i], v_i), i \in [d] \). Here \( v_i \) is the sparse ratio of \( A_i \), i.e., \( v_i = \frac{\|A_i\|_0}{n_i r_i} \), \( i \in [d] \). We set \( c = 2\|\mathcal{X}\|_\infty \). For these synthetic data, we test 10 trials in each case and the average result of these trials is the last result. The relative error is employed to measure the accuracy of different approaches, which is defined as \( \frac{\|\mathcal{X}_0 - \mathcal{X}^*\|_F}{\|\mathcal{X}^*\|_F} \). Here \( \mathcal{X}_0 \) and \( \mathcal{X}^* \) represent the recovered tensor and ground-truth tensor, respectively.

We first test the third-order tensors with \( n_1 = n_2 = n_3 = 100 \), whose Tucker rank is \( (5, 5, 5) \). For the observations with additive Gaussian noise and additive Laplace noise, we set \( \sigma^2 = 0.01 \) and \( \tau = 0.01 \), respectively. In Figure 1, we show the relative error versus sampling ratios of different methods for the observations with additive Gaussian noise, additive Laplace noise, and Poisson observations, respectively, where \( v_i = 0.3, i = 1, 2, 3 \). It can be observed that the relative errors of the matrix based method, SNTFTP, and SNTDC decrease as the sampling ratio increases. Moreover, the relative errors obtained by SNTDC are smaller than those obtained by the matrix based method and SNTFTP for the three kinds of observations.

Now we test a fourth-order tensor \( 50 \times 50 \times 50 \times 50 \) with Tucker rank \( (5, 5, 5, 5) \), where \( \sigma^2 = 0.01 \) for additive Gaussian noise, \( \tau = 0.01 \) for additive Laplace noise, and \( v_i = 0.3, i = 1, 2, 3 \). Since the SNTFTP is only effective for third-order tensors, we do not compare with it for fourth-order tensors in this experiment. The mean squared error (MSE) is used to measure the recovery accuracy of different methods, which is defined as \( \text{MSE} = \frac{\|\mathcal{X}_1 - \mathcal{X}^*\|_2^2}{n_1 \cdot n_2 \cdot n_3 \cdot n_4} \). Here \( \mathcal{X}_1 \) and \( \mathcal{X}^* \) denote the recovered tensor and the ground-truth tensor, respectively. Besides, we also compare the theoretical bounds of the matrix based method in [46] and SNTDC for different noise models, which are denoted by TB (Matrix) and TB (SNTDC), respectively. In Figure 2, we show the MSE on a logarithmic scale versus sampling ratio of the theoretical and testing results for the models with additive Gaussian noise, additive Laplace noise, and Poisson observations, respectively, where the sampling ratio is set from 0.1 to 0.6 with step
Figure 1: Relative error versus sampling ratio for synthetic tensors with size $100 \times 100 \times 100$ and Tucker rank $(5, 5, 5)$.

Figure 2: Plots of MSE on a logarithmic scale versus sampling ratio for synthetic tensors with size $50 \times 50 \times 50 \times 50$ and Tucker rank $(5, 5, 5, 5)$.

size 0.1. Naturally, in each case, the errors decrease as the sampling ratios increase. Furthermore, the theoretical bounds of SNTDC are lower than those of the matrix based method in [46] for the three different noise models. In the testing experiments, the errors of SNTDC are much smaller than those of the matrix based method for the observations with different noises. However, the theoretical bounds of SNTDC are larger than those of the experimental cases, which are similar for the matrix based method. The main reason for this phenomenon is that some inequalities are used to estimate the error bounds for both the matrix based method and SNTDC, which are not tight.

7.2 Image Data

In this subsection, we test the Swimmer dataset\(^1\) ($32 \times 32 \times 256$) [14] and the Columbia Object Image Library (COIL-100)\(^2\) for sparse nonnegative Tucker decomposition and completion. For the Swimmer dataset, the size of each image is $32 \times 32$ and there are 256 swimmer images. Similar to [52], the Tucker rank of the Swimmer dataset is set to $(24, 20, 20)$ in Algorithm 1. For the COIL-100, which contains 100 objects, we resize each image to $64 \times 64 \times 3$ and choose first 50 images for the fifty-sixth object. In this case, the size of the resulting tensor is $64 \times 64 \times 3 \times 50$, whose Tucker rank is set to $(15, 15, 3, 10)$ in Algorithm 1.

In Figure 3, we show the relative error versus sampling ratio of different methods for the obser-\(^1\)https://stodden.net/Papers.html
\(^2\)https://www.cs.columbia.edu/CAVE/software/softlib/coil-100.php
vations with additive Gaussian noise, additive Laplace noise, and Poisson observations, respectively, where $\sigma^2 = 0.01$ for additive Gaussian noise and $\tau = 0.01$ for additive Laplace noise. It can be seen from this figure that the relative errors of the SNTDC are smaller than those of the matrix based method and SNTFTP for different sampling ratios. Moreover, the performance of SNTFTP is better than that of the matrix based method in terms of relative errors for the three kinds of observations. And the relative errors of the matrix based method, SNTFTP, and SNTDC decrease as the number of samples increases for the observations with additive Gaussian noise, additive Laplace noise, and Poisson observations.

Since the SNTFTP is only effective for third-order tensors, we stack all images of the COIL-100 based on the color channels into a third-order tensors along the frontal slices for the SNTFTP, where the size of the resulting tensor is $64 \times 64 \times 150$. In Figure 4, we display the relative error versus sampling ratio of different methods for the COIL-100 dataset, where $\sigma^2 = 0.05$ for additive Gaussian noise and $\tau = 0.05$ for additive Laplace noise. We can see that the relative errors obtained by the matrix based method, SNTFTP, and SNTDC decrease as the sampling ratio increases for different noise observations. And the relative errors obtained by SNTDC are smaller than those obtained by the matrix based method and SNTFTP. Moreover, the SNTFTP performs better than the matrix based method in terms of relative error for the observations with additive Gaussian noise, additive Laplace noise, and Poisson observations.

8 Conclusions and Future Work

In this paper, we have investigated the problem of sparse NTD and completion with a general class of noise observations, where the underlying tensor is decomposed into a core tensor and several factor
matrices with nonnegativity and sparsity constraints. Moreover, the loss function is derived by the maximum likelihood estimation of the noisy observations. And the error bound of the estimator of the SNTDC model is established under a class of noise distributions. Then the error bounds are specified to some widely used noise models including additive Gaussian noise, additive Laplace noise, and Poisson observations. Besides, the minimax lower bound of the observed model is derived, which matches to the upper bound up to a logarithmic factor. Numerical experiments on synthetic data and real-world data sets demonstrate the superior performance of the proposed sparse NTD and completion model compared with other methods.

In the experiments, we need to set the Tucker rank of the underlying tensor in advance, which is unknown in general for real-world data sets. An interesting direction for future work on this problem is to update the Tucker rank adaptively. Besides, in the sparse NTD and completion model, the \( \ell_0 \) norm is employed to characterise the sparsity of the factor matrices. However, the \( \ell_0 \) norm is nonconvex and it is challenge to get the unique and global optimal solution. Therefore, it is also of great interest to replace the \( \ell_0 \) norm by the \( \ell_1 \) norm for the factor matrices and then analyze the error bound of the corresponding model.

The convergence of Algorithm 1 cannot be guaranteed in general since our model in (24) is non-convex and multi-block. Future work is devoted to establishing the convergence of ADMM for our nonconvex multi-block model. Moreover, due to effectiveness and explanatory information of orthogonal constraints on the factor matrices for nonnegative Tucker decomposition (c.f. [39]), it will also be of great interest to extend the error bound of our model to that of orthogonal nonnegative Tucker decomposition under general loss functions.

Appendix A. Proof of Theorem 4.1

At the beginning, the following lemma is first recorded, which plays a vital role for the proof of Theorem 4.1.

**Lemma 8.1** Let \( \mathcal{Y} \) be a countable collection of candidate reconstructions \( \mathcal{X} \) of \( \mathcal{X}^* \) and its penalty \( \text{pen}(\mathcal{X}) \geq 1 \) satisfying \( \sum_{\mathcal{X} \in \mathcal{T}} 2^{-\text{pen}(\mathcal{X})} \leq 1 \). For any integer \( 2^d \leq m \leq n_1 n_2 \cdots n_d \), let \( \Omega \sim \text{Bern}(p) \). Moreover, the joint probability density/mass function of the corresponding observations follows

\[
p_{\mathcal{X}^*}(\Omega) = \prod_{(i,j,k) \in \Omega} p_{\mathcal{X}^*_{ijk}}(Y_{ijk}),
\]

which are assumed to be conditionally independent on the given \( \Omega \). Let \( \gamma \) be a constant satisfying

\[
\gamma \geq \max_{\mathcal{X} \in \mathcal{T}} \max_{i_1, i_2, \ldots, i_d} K \left( p_{\mathcal{X}^*_{i_1 i_2 \cdots i_d}}(Y_{i_1 i_2 \cdots i_d}) \right).
\]

Consider the following complexity penalized maximum likelihood estimator

\[
\mathcal{X}^\mu = \arg \min_{\mathcal{X} \in \mathcal{T}} \left\{ -\log p_{\mathcal{X}^*} (\Omega, \mathcal{Y}) + \mu \cdot \text{pen}(\mathcal{X}) \right\}.
\]

Then for any \( \mu \geq 2 \left( 1 + \frac{2n}{3} \right) \log(2) \), one has

\[
E_{\Omega, \mathcal{Y}_\Omega} \left[ -2 \log H(p_{\mathcal{X}^\mu}(\mathcal{Y}), p_{\mathcal{X}^*}(\mathcal{Y})) \right] \leq 3 \cdot \min_{\mathcal{X} \in \mathcal{T}} \left\{ \frac{K(p_{\mathcal{X}^*} || p_{\mathcal{X}})}{n_1 n_2 \cdots n_d} + \left( \mu + \frac{4 \gamma \log(2)}{3} \right) \frac{\text{pen}(\mathcal{X})}{m} \right\} + \frac{8 \gamma \log(m)}{m},
\]

where the expectation is taken with respect to the joint distribution of \( \Omega \) and \( \mathcal{Y}_\Omega \).
The proof of Lemma 8.1 can be obtained easily based on the matrix case [46, Lemma 8] and [31], see also for the cases of CP decomposition in [22, Lemma 1] and tensor factorization via tensor-tensor product in [55, Lemma 1.1]. The three steps of proof in [46, Lemma 8] are giving the “good” sample set characteristics, a conditional error guarantee and some simple conditioning arguments, which are in point-wise manners in fact for the KL divergence, negative logarithmic Hellinger affinity, and maximum likelihood estimation. As a consequence, we can extend them to the tensor case with Tucker decomposition easily. Here we omit the details.

The proof of Theorem 4.1 follows the line of the proof of [46, Theorem 1], see also the proofs of [40, Theorem 3] and [55, Theorem 4.1]. The main technique of this proof is the well-known Kraft-McMillan inequality [28, 35]. The penalty of the underlying tensor \( \mathcal{X} \) is constructed by the codes of the core tensors and sparse factor matrices, where the underlying tensor has the Tucker decomposition form with the core tensor being nonnegative and the factor matrices being nonnegative and sparse. Next we return to the proof of Theorem 4.1.

Based on the result in Lemma 8.1, one should define the penalties \( \text{pen}(\mathcal{X}) \geq 1 \) in the candidate reconstructions \( \mathcal{X} \) of \( \mathcal{X}^* \) such that the penalties \( \mathcal{X} \) in \( \Upsilon \) satisfy

\[
\sum_{\mathcal{X} \in \Upsilon} 2^{-\text{pen}(\mathcal{X})} \leq 1. \tag{50}
\]

The condition (50) is the well-known Kraft-McMillan inequality for coding elements of \( \Upsilon \) with an alphabet of size 2, which is satisfied automatically if we choose the penalties to be code lengths for the unique decodable binary code for the elements \( \mathcal{X} \in \Upsilon \) [28, 35], see also [12, Section 5]. This also gives the constructions of the penalties.

Let \( \Upsilon_1 := \{ \mathcal{X} = \mathcal{C} \times_1 A_1 \cdots \times_d A_d : \mathcal{C} \in \mathcal{C}, A_i \in \mathcal{B}_i \} \), where \( \mathcal{C} \) and \( \mathcal{B}_i, i \in [d] \) are the same as those constructed in Section 3. Note that \( \Upsilon_1 \subseteq \Upsilon \) by the construction of \( \Upsilon_1 \). Now we consider the discretized core tensor \( \mathcal{C} \in \mathcal{C} \) and sparse factor matrices \( A_i \in \mathcal{B}_i, i \in [d] \).

1. We encode the amplitude of each element of \( \mathcal{C} \) using \( \log_2(\tau) \) bits, where \( \tau \) is defined as (8). Then a total of \( r_1 r_2 \cdots r_d \log_2(\tau) \) bits are used to encode the core tensor \( \mathcal{C} \).

2. Let \( \zeta_i := 2^{[\log_2(r_i n_i)]}, i \in [d] \). We encode each nonzero element of \( A_i \) using \( \log_2(\zeta_i) \) to denote the location and \( \log_2(\tau) \) bits for its amplitude. In this case, a total of \( \| A_i \|_0 (\log_2(\tau) + \log_2(\zeta_i)) \) bits is used to encode the sparse factor matrix \( A_i \).

3. Finally, we can assign each \( \mathcal{X} \in \Upsilon_1 \) whose code length satisfies

\[
\text{pen}(\mathcal{X}) = r_1 r_2 \cdots r_d \log_2(\tau) + \sum_{i=1}^d \| A_i \|_0 (\log_2(\tau) + \log_2(\zeta_i)).
\]

By the constructions, we know that such codes are uniquely decodable, which implies that \( \sum_{\mathcal{X} \in \Upsilon_1} 2^{-\text{pen}(\mathcal{X})} \leq 1 \) [28, 35]. Since \( \Upsilon \subseteq \Upsilon_1 \), one has

\[
\sum_{\mathcal{X} \in \Upsilon} 2^{-\text{pen}(\mathcal{X})} \leq \sum_{\mathcal{X} \in \Upsilon_1} 2^{-\text{pen}(\mathcal{X})} \leq 1.
\]

Consequently, by Lemma 8.1, we get that the estimator \( \mathcal{X}^\mu \) in the following

\[
\mathcal{X}^\mu \in \arg \min_{\mathcal{X} \in \Upsilon} \left\{ - \log p_{X_\Omega}(Y_\Omega) + \mu \cdot \text{pen}(\mathcal{X}) \right\}
\]

\[
= \arg \min_{\mathcal{X} \in \Upsilon} \left\{ - \log p_{X_\Omega}(Y_\Omega) + \mu \sum_{i=1}^d \| A_i \|_0 (\log_2(\tau) + \log_2(\zeta_i)) \right\}, \tag{51}
\]
satisfies
\[
\mathbb{E}_{\Omega, \eta_{\Omega}} \left[ -2 \log H(p_{X^*}, p_{X^*}) \right] \leq \frac{8\gamma \log(m)}{m} + 3 \cdot \min_{\lambda \in \mathcal{T}} \left\{ \frac{K(p_{X^*}(\mathcal{Y}) \| p_{X}(\mathcal{Y}))}{n_1 n_2 \cdots n_d} \right\}
\]
\[
+ \left( \mu + \frac{4\gamma \log(2)}{3} \right) r_1 r_2 \cdots r_d \log(\tau) + \frac{\sum d_i \| A_i \|_0 (\log(\tau) + \log(\zeta_i))}{m},
\]
where \( \mu \geq 2 \left( 1 + \frac{22}{21} \right) \log(2) \) and \( \gamma \) satisfies (48).

Let \( \lambda_i = \mu(\log_2(\tau) + \log_2(\zeta_i)), i \in [d] \). Note that \( r_i \leq n_i \) and
\[
\log_2(\tau) + \log_2(\zeta_i) \leq (\beta + 2) \frac{\log(n_m)}{\log(2)}.
\]
Therefore, for any \( \lambda_i \geq 4(\beta + 2)(1 + \frac{22}{21}) \log(n_m), i \in [d] \), the estimator of
\[
\mathcal{X}^\lambda \in \arg \min_{\mathcal{X} \in \mathcal{T}} \left\{ -\log p_{X^*}(\mathcal{Y}^\lambda) + \sum_{i=1}^d \lambda_i \| A_i \|_0 \right\}
\]
satisfies
\[
\mathbb{E}_{\Omega, \eta_{\Omega}} \left[ -2 \log H(p_{X^*}(\mathcal{Y}), p_{X^*}(\mathcal{Y})) \right] \leq \frac{8\gamma \log(m)}{m} + 3 \cdot \min_{\lambda \in \mathcal{T}} \left\{ \frac{K(p_{X^*}(\mathcal{Y}) \| p_{X}(\mathcal{Y}))}{n_1 n_2 \cdots n_d} \right\}
\]
\[
+ \left( \max_i \lambda_i \right) + \frac{8\gamma (\beta + 2) \log(n_m)}{3} \left( r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A_i \|_0 \right) m,
\]
where the inequality holds by (52). This completes the proof. \( \square \)

**Appendix B. Proof of Theorem 4.2**

First, we establish an upper bound of the tensor infinity norm between the underlying tensor \( \mathcal{X}^* \) and its closest surrogate in \( \mathcal{Y} \), where \( \mathcal{X}^* = C^* \times_1 A_1^* \times_2 A_2^* \cdots \times_d A_d^* \), and \( C^*, A_i^* \in [d] \) are defined as (4). The following estimate will be useful in the sequel.

**Lemma 8.2** Let \( \mathcal{X}_s^* = C^* \times_1 A_1^* \cdots \times_d A_d^* \), where the entries of \( \mathcal{X}^* \) are the closest discretized surrogates of the entries of \( C^* \) in \( \mathcal{Y} \), and the entries of \( A_i^* \) are the closest discretized surrogates of the entries of \( A_i^* \) in \( \mathcal{Y} \), \( i \in [d] \). Then it holds that
\[
\| \mathcal{X}_s^* - \mathcal{X}^* \|_\infty \leq \frac{2d+1}{\tau} \left( \prod_{i=1}^d a_i r_i \right),
\]
where \( \tau \) is defined as (8).

**Proof.** Let \( C^* = C^* + \Delta C \) and \( A_i^* = A_i^* + \Delta A_i, i \in [d] \). By the definition of \( C^* \) and \( A_i^* \), \( i \in [d] \) in (4), we get that \( \| \Delta C \|_\infty \leq \frac{1}{\tau-1} \) and \( \| \Delta A_i \|_\infty \leq \frac{a_i}{\tau-1} \). It follows from (1) that
\[
\mathcal{X}_s^* - \mathcal{X}^* = (\otimes_{i=1}^d (A_i^* + \Delta A_i)) \text{vec}(C^* + \Delta C) - (\otimes_{i=1}^d A_i^*) \text{vec}(C^*)
\]
\[
= (\otimes_{i=1}^d (A_i^* + \Delta A_i)) \text{vec}(C^*) + (\otimes_{i=1}^d (A_i^* + \Delta A_i))(\Delta C) - (\otimes_{i=1}^d A_i^*) \text{vec}(C^*).
\]
Furthermore, we have
\[
\bigotimes_{i=1}^d (A_i^* + \Delta A_i)
= A_d^* \otimes \cdots \otimes A_1^* + \Delta A_d \otimes A_{d-1}^* \otimes \cdots \otimes A_1^* + \cdots + A_d^* \otimes \cdots \otimes A_2^* \otimes \Delta A_1
\]
\[
+ \Delta A_d \otimes \Delta A_{d-1} \otimes A_{d-3}^* \otimes \cdots \otimes A_1^* + \cdots + A_d^* \otimes \cdots \otimes A_3^* \otimes \Delta A_2 \otimes \Delta A_1
\]
\[
+ \cdots + \Delta A_d \otimes \cdots \otimes \Delta A_2 \otimes \Delta A_1 + \cdots + A_d^* \otimes \Delta A_{d-1} \otimes \cdots \otimes \Delta A_1
\]
\[
+ \Delta A_d \otimes \cdots \otimes \Delta A_1.
\]
Since \(0 \leq (A_i^*)_{lm} \leq a_i\) and \(0 \leq C_{i_1 i_2 \ldots i_d} \leq 1\), \(\forall (l, m) \in [n_i] \times [r_i], (i_1, i_2, \ldots, i_d) \in [r_1] \times [r_2] \times \cdots \times [r_d], i \in [d]\), we have that
\[
\| (A_d^* \otimes \cdots \otimes A_1^*) \text{vec}(C^*) \|_{\infty} \leq \prod_{i=1}^d (a_i r_i),
\]
\[
\| (\Delta A_d \otimes A_{d-1}^* \otimes \cdots \otimes A_1^* + \cdots + A_d^* \otimes \cdots \otimes A_2^* \otimes \Delta A_1) \text{vec}(C^*) \|_{\infty} \leq \frac{C_d^1}{\tau - 1} \prod_{i=1}^d (a_i r_i),
\]
\[
\vdots
\]
\[
\| (A_d^* \otimes \Delta A_{d-1} \otimes \cdots \otimes \Delta A_1) \text{vec}(C^*) \|_{\infty} \leq \frac{C_{d-1}^d}{(\tau - 1)^d} \prod_{i=1}^d (a_i r_i),
\]
\[
\| (\Delta A_d \otimes \cdots \otimes \Delta A_1) \text{vec}(C^*) \|_{\infty} \leq \frac{1}{(\tau - 1)^d} \prod_{i=1}^d (a_i r_i),
\]
which yields that
\[
\| (\otimes_{i=d}^1 (A_i^* + \Delta A_i)) \text{vec}(C^*) - (\otimes_{i=d}^1 A_i^*) \text{vec}(C^*) \|_{\infty} \leq \left(1 + \frac{1}{\tau - 1} \right) \prod_{i=1}^d (a_i r_i) - \prod_{i=1}^d (a_i r_i). \tag{56}
\]
Similarly, by \(\| \Delta C \|_{\infty} \leq \frac{1}{\tau - 1}\), we can easily get that
\[
\| (\otimes_{i=d}^1 (A_i^* + \Delta A_i))(\Delta C) \|_{\infty} \leq \frac{1}{\tau - 1} \left(1 + \frac{1}{\tau - 1} \right) \prod_{i=1}^d (a_i r_i). \tag{57}
\]
Note that
\[
\left(1 + \frac{1}{\tau - 1} \right)^{d+1} - 1 = C_{d+1}^1 \frac{1}{\tau - 1} + C_{d+1}^2 \frac{1}{(\tau - 1)^2} + \cdots + C_{d+1}^{d+1} \frac{1}{(\tau - 1)^{d+1}}
\]
\[
= \frac{1}{\tau - 1} \left(C_{d+1}^1 + C_{d+1}^2 \frac{1}{\tau - 1} + \cdots + C_{d+1}^{d+1} \frac{1}{(\tau - 1)^d} \right) \tag{58}
\]
\[
\leq \frac{1}{\tau - 1} \left(C_{d+1}^1 + C_{d+1}^2 + \cdots + C_{d+1}^{d+1} \right) = \frac{2^{d+1} - 1}{\tau - 1},
\]
where the inequality holds by $\tau \geq 2$. Therefore, combining (54), (55), (56) and (57), we obtain that
\[
\|X^*_s - X^*\|_\infty \leq \left(1 + \frac{1}{\tau - 1}\right) \prod_{i=1}^{d} (a_i r_i) - \prod_{i=1}^{d} (a_i r_i) + \frac{1}{\tau - 1} \left(1 + \frac{1}{\tau - 1}\right) \prod_{i=1}^{d} (a_i r_i)
\]
\[
= \left(1 + \frac{1}{\tau - 1}\right)^{d+1} \prod_{i=1}^{d} (a_i r_i)
\]
\[
\leq \frac{2^{d+1} - 1}{\tau - 1} \prod_{i=1}^{d} (a_i r_i),
\]
where the last inequality follows from (58). This furnishes the desired statement. \[\square\]

Now we return to the proof of Theorem 4.2. Recall that the negative logarithmic Hellinger affinity and KL divergence for additive Gaussian noise are (see, e.g., [46, 55])
\[
- \log(H(p_{X_n^{(s)}}(Y^{(s)}), p_{X_{n+1}^{(s)}}(Y^{(s)}))) = \frac{(X_{12}^* - (X^*)_{12})^2}{8\sigma^2}
\]
and
\[
K(p_{X_n^{(s)}}(Y^{(s)}), p_{X_{n+1}^{(s)}}(Y^{(s)})) = \frac{(X_{12}^* - (X^*)_{12})^2}{2\sigma^2}.
\]
Therefore, we obtain that
\[
-2\log H(p_{X_n}(Y), p_{X_{n+1}}(Y)) = \frac{\|X^\lambda - X^*\|_F^2}{4\sigma^2}.
\]
Since $X^\lambda \in \mathcal{Y}$ and $0 \leq X^*_1 \leq \frac{\sigma}{\sqrt{2m}}$, we can take $\gamma = \frac{\sigma^2}{2m}$ in (48). Therefore, it follows from Theorem 4.1 that
\[
\mathbb{E}_{X_n \in \mathcal{X}} \left[ \frac{\|X^\lambda - X^*\|_F^2}{m} \right] \leq \frac{16c^2 \log(m)}{m} + 12\sigma^2 \cdot \min_{X^\lambda \in \mathcal{Y}} \left\{ \frac{K(p_{X_n^{(s)}}(Y), p_{X_n^{(s)}}(Y))}{n_1 n_2 \cdots n_d} \right\}
\]
\[
+ \left(\max_i \{\lambda_i\} + \frac{4c^2 (\beta + 2) \log(n_m)}{3\sigma^2} \right) \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^{d} \|A_i\|}{m}.
\]
By the definition of $\beta$ in (10), we know that
\[
\tau = 2^\left[\log_2(n_m)^\beta\right] \geq 2^{2\log_2(n_m)} \geq 2^{\log_2(n_m) + \log_2 \left(\frac{(2^{d+1} - 1)\sqrt{d\lambda r_1 r_2 \cdots r_d a_1 a_2 \cdots a_d}}{c}\right)}
\]
\[
\geq \frac{(2^{d+1} - 1)\sqrt{d\lambda r_1 r_2 \cdots r_d a_1 a_2 \cdots a_d}}{c} + 1.
\]
For any $X^s$ constructed in Lemma 8.2, we have
\[
\|X^s - X^*\|_\infty \leq \frac{c}{\sqrt{d\lambda m}} \leq \frac{c^2}{2},
\]
where the last inequality follows from $d, n_m \geq 2$. Therefore, $\|X^s\|_\infty \leq \|X^s\|_\infty + \frac{c^2}{2} \leq c$, which implies $X^s \in \mathcal{Y}$. As a consequence, we get that
\[
\min_{X^\lambda \in \mathcal{X}} \left\{ \frac{K(p_{X_n^{(s)}}(Y), p_{X_n^{(s)}}(Y))}{n_1 n_2 \cdots n_d} \right\} = \min_{X^\lambda \in \mathcal{Y}} \left\{ \frac{\|X^\lambda - X^*\|_F^2}{2\sigma^2 n_1 n_2 \cdots n_d} \right\}
\]
\[
\leq \frac{\|X^s - X^*\|_F^2}{2\sigma^2 n_1 n_2 \cdots n_d} \leq \frac{\|X^s - X^*\|_\infty^2}{2\sigma^2}
\]
\[
\leq \frac{c^2}{2\sigma^2 d\lambda m} \leq \frac{c^2}{2\sigma^2 m}.
\]
where the third inequality holds by (62) and Lemma 8.2, and the last inequality holds by the fact that $m \leq d n_m$.

Moreover, by the construction of $A^*_i$ in Lemma 8.2, we know that $\| A^*_i \|_0 = \| A^*_i \|_0, \forall i \in [d]$. Therefore, plugging (63) and (11) into (61), we have

$$
\mathbb{E}_{\Omega, \mathcal{Y}_0} \left[ \| \mathcal{X}^\lambda - \mathcal{X}^* \|_F^2 \right] \leq \frac{22 c^2 \log(m)}{m} n_1 n_2 \cdots n_d + 16(\beta + 2)(2c^2 + 3\sigma^2) \left( \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A^*_i \|_0}{m} \right) \log(n_m),
$$

which leads to the desired conclusion. \qed

**Appendix C. Proof of Theorem 4.3**

**Proof.** For the observations with additive Laplace noise, by [55, Lemma 3.1], we know that

$$
K \left( p_{\mathcal{X}^\lambda_{i_1 i_2 \cdots i_d}}(\mathcal{Y}_{i_1 i_2 \cdots i_d}) \| p_{\mathcal{X}^*_{i_1 i_2 \cdots i_d}}(\mathcal{Y}_{i_1 i_2 \cdots i_d}) \right) = \frac{|\mathcal{X}_{i_1 i_2 \cdots i_d}^\lambda - \mathcal{X}_{i_1 i_2 \cdots i_d}^*|}{\tau} \geq 1 - \exp \left( - \frac{|\mathcal{X}_{i_1 i_2 \cdots i_d}^\lambda - \mathcal{X}_{i_1 i_2 \cdots i_d}^*|}{\tau} \right) \leq \left( \frac{\mathcal{X}_{i_1 i_2 \cdots i_d}^\lambda - \mathcal{X}_{i_1 i_2 \cdots i_d}^*}{2\tau} \right)^2,
$$

where the inequality follows from the fact that $e^{-x} \leq 1 - x + \frac{x^2}{2}$ for any $x > 0$. Therefore, we can choose $\gamma = \frac{c^2}{2\tau}$ in (48).

Let $f(t) := \log(1 + \frac{t}{2\tau})$, where $t \in [0, c]$. Then by Taylor’s expansion, we get that

$$
f(t) = f(0) + f'(0)t + \frac{f''(\xi)}{2} t^2,
$$

where $\xi \in [0, c]$. By [55, Lemma 3.1], we obtain that

$$
-2 \log(\mathcal{H}(p_{\mathcal{X}^\lambda_{i_1 i_2 \cdots i_d}}(\mathcal{Y}_{i_1 i_2 \cdots i_d}), p_{\mathcal{X}^*_{i_1 i_2 \cdots i_d}}(\mathcal{Y}_{i_1 i_2 \cdots i_d}))) = \frac{|\mathcal{X}_{i_1 i_2 \cdots i_d}^\lambda - \mathcal{X}_{i_1 i_2 \cdots i_d}^*|}{\tau} \geq \frac{(\mathcal{X}_{i_1 i_2 \cdots i_d}^\lambda - \mathcal{X}_{i_1 i_2 \cdots i_d}^*)^2}{(2\tau + c)^2},
$$

where the inequality holds by letting $t = |\mathcal{X}_{i_1 i_2 \cdots i_d}^\lambda - \mathcal{X}_{i_1 i_2 \cdots i_d}^*|$ and the fact that $f''(\xi) \leq -\frac{1}{(2\tau + c)^2}$ in (64). Therefore, by Theorem 4.1, we can deduce

$$
\mathbb{E}_{\Omega, \mathcal{Y}_0} \left[ \| \mathcal{X}^\lambda - \mathcal{X}^* \|_F^2 \right] \leq \frac{8c^2 (2\tau + c)^2 \log(m)}{2\tau^2 m} + 3(2\tau + c)^2 \min_{\lambda \in \mathbb{F}} \left\{ \frac{K(p_{\mathcal{X}^\lambda}(\mathcal{Y}) \| p_{\mathcal{X}}(\mathcal{Y}))}{n_1 n_2 \cdots n_d} \right\} \left( \max_i \lambda_i \right) + \frac{4c^2 (\beta + 2) \log(n_m)}{3\tau^2} \left( \frac{r_1 r_2 \cdots r_d + \sum_{i=1}^d \| A_i \|_0}{m} \right),
$$

\(65\).
Then by a similar argument of (63) in the proof of Theorem 4.2, we know that

\[
\min_{\lambda} \left\{ K(p_{X^*}(Y)||p_X(Y)) \right\} \leq \min_{\lambda} \left\{ \frac{\|X - X^*\|_F^2}{2\tau^2n_1n_2 \cdots n_d} \right\} \\
\leq \frac{\|X_s - X^*\|_F^2}{2\tau^2n_1n_2 \cdots n_d} \leq \frac{\|X_s - X^*\|_F^2}{2\tau^2} \leq \frac{c^2}{2\tau^2m},
\]

(66)

where \(X^s\) is defined in Lemma 8.2. By the construction of \(A^*_i\) in Lemma 8.2, we know that \(\|A^*_i\|_0 = \|A^*_i\|_0, \forall i \in [d]\). As a consequence, plugging (66) and (11) into (65), the estimator of (6) satisfies

\[
\mathbb{E}_{\Omega, Y_0} \left[ \|\lambda - X^*\|_F^2 \right] \leq \frac{11c^2(2\tau + c)^2 \log(m)}{2\tau^2m} \\
+ 12 \left( 1 + \frac{2c^2}{3\tau^2} \right) (2\tau + c)^2 \left( \beta + 2 \right) \log(n_m) \frac{\sum^d_{i=1} \|A^*_i\|_0}{m},
\]

which is the desired statement. \(\square\)

**Appendix D. Proof of Theorem 4.4**

**Proof.** By [7, Lemma 8], we obtain that the KL divergence of Poisson observations is

\[
K \left( p_{X_{1i2\cdots id}}(Y_{1i2\cdots id}) || p_{X_{1i2\cdots id}}(Y_{1i2\cdots id}) \right) \leq \frac{1}{X_{1i2\cdots id}}(X_{1i2\cdots id} - X^*_{1i2\cdots id})^2 \\
\leq \frac{1}{\gamma}(X_{1i2\cdots id} - X^*_{1i2\cdots id})^2.
\]

Consequently, we can choose \(\gamma = \frac{c^2}{\rho^2}\) in (11). Moreover, according to the proof of [40, Appendix IV], we have

\[
-2\log(H(p_{X_{1i2\cdots id}}(Y_{1i2\cdots id}), p(X^*_{1i2\cdots id}))) = \left( \sqrt{X^*_{1i2\cdots id}} - \sqrt{X_{1i2\cdots id}} \right)^2 \\
\geq \frac{1}{4c} \left( X_{1i2\cdots id} - X^*_{1i2\cdots id} \right)^2,
\]

where the inequality holds by

\[
(X_{1i2\cdots id} - X^*_{1i2\cdots id})^2 = \left( \sqrt{X^*_{1i2\cdots id}} - \sqrt{X_{1i2\cdots id}} \left( \sqrt{X^*_{1i2\cdots id}} + \sqrt{X_{1i2\cdots id}} \right)^2 \\
\leq 4c \left( \sqrt{X^*_{1i2\cdots id}} - \sqrt{X_{1i2\cdots id}} \right)^2.
\]

Hence, by Theorem 4.1, we get that

\[
\mathbb{E}_{\Omega, Y_0} \left[ \|\lambda - X^*\|_F^2 \right] \leq \frac{32c^3 \log(m)}{m\rho} + 12c \cdot \min_{\lambda} \left\{ K(p_{X^*}(Y)||p_X(Y)) \right\} \\
+ \left( \max_i \{\lambda_i\} \right) + \frac{8c^2 (\beta + 2) \log(n_m)}{3\rho} \frac{r_1r_2 \cdots r_d + \sum^d_{i=1} \|A_i\|_0}{m}. \tag{67}
\]
Let $\mathcal{X}$ be defined as in Lemma 8.2. Then by a similar argument of (63), we get that
\[
\min_{\mathcal{X} \in \mathcal{I}} \left\{ \frac{K(p_{\mathcal{X}}(\mathcal{Y}))}{n_1 n_2 \cdots n_d} \right\} \leq \min_{\mathcal{X} \in \mathcal{I}} \left\{ \frac{\|\mathcal{X} - \mathcal{X}^*\|_F^2}{\varrho n_1 n_2 \cdots n_d} \right\}
\leq \frac{\|\mathcal{X} - \mathcal{X}^*\|_F^2}{\varrho n_1 n_2 \cdots n_d} \leq \frac{\|\mathcal{X}^s - \mathcal{X}^*\|_F^2}{\varrho} \leq \frac{c^2}{\varrho n_1 n_2 \cdots n_d}. \tag{68}
\]

Notice that $\|A_i^s\|_0 = \|A_i^s\|_0, \forall i \in [d]$. By combining (11), (67), and (68), after some rearrangements, we obtain that the estimator of (6) satisfies
\[
\mathbb{E}_{\Omega, \mathcal{Y}_i} \left[ \|\mathcal{X}^\lambda - \mathcal{X}^*\|_F^2 \right] \leq \frac{44c^3 \log(m)}{\varrho n_1 n_2 \cdots n_d} + 48c \left( 1 + \frac{4c^2}{3\varrho} \right) (\beta + 2) \log(n_m) r_1 r_2 \cdots r_d + \sum_{i=1}^d \|A_i^s\|_0.
\]
This concludes the proof. \hfill \square

### Appendix E. Proof of Theorem 5.1

Let
\[
\mathcal{L} = \{ \mathcal{X} = C \times_1 A_1 \cdots \times_d A_d : C \in \mathcal{D}, A_i \in \mathcal{X}_i, i \in [d] \},
\]
where
\[
\mathcal{D} = \{ C \in \mathbb{R}_{+}^{r_1 \times r_2 \times \cdots \times r_d} : C_{i_1 \cdots i_d} \in \{0, 1, c_0\}, (i_1, \ldots, i_d) \in [n_1] \times \cdots \times [n_d] \}
\]
with
\[
c_0 = \min \left\{ 1, \frac{\gamma c \mu}{\prod_{i=1}^d (a_i \sqrt{\Delta_i(s_i, n_i)})} \sqrt{\frac{r_1 r_2 \cdots r_d}{m}} \right\}, \tag{69}
\]
and
\[
\mathcal{X}_i = \{ A_i \in \mathbb{R}_{+}^{n_i \times r_i} : (A_i)_{jk} \in \{0, a_i, b_i\}, ||A_i||_0 \leq s_i, (j, k) \in [n_i] \times [r_i] \}
\]
with
\[
b_i = \min \left\{ a_i, \frac{\gamma_i \mu}{\prod_{j \neq i} a_j} \sqrt{\frac{s_i}{\Delta_i(s_j, n_j)}} \right\}, i \in [d].
\]
We will specify $\gamma_c$, $\gamma_i$, $i \in [d]$ in detail later. Therefore, by the construction of $\mathcal{L}$, we know that $\mathcal{L} \subseteq \mathcal{L}(s, r, a)$. Next we consider to construct the packing sets of the core tensor and factor matrices, respectively.

**Case I.** We construct the following set
\[
\mathcal{L}_c = \{ \mathcal{X} = C \times_1 A_1 \cdots \times_d A_d : C \in \mathcal{D}_0 \},
\]
where $\mathcal{D}_0$ is defined as
\[
\mathcal{D}_0 = \{ C \in \mathbb{R}_{+}^{r_1 \times r_2 \times \cdots \times r_d} : C_{i_1 \cdots i_d} \in \{0, c_0\}, (i_1, \ldots, i_d) \in [n_1] \times \cdots \times [n_d] \} \tag{70}
\]
and
\[
A_i = a_i \begin{pmatrix} I_{r_i} \\ \vdots \\ I_{r_i} \\ 0_{r_i} \end{pmatrix} \in \mathbb{R}_{+}^{n_i \times r_i} \text{ with } I_{r_i} \in \mathbb{R}^{r_i \times r_i}, \ i \in [d].
\]
There are \( \left\lfloor \frac{s_i \wedge n_i}{r_i} \right\rfloor \) blocks identity matrix \( I_{r_i} \) in \( A_i \) and \( 0_{r_i} \in \mathbb{R}^{(n_i-r_i) \times r_i} \) is a zero matrix. Consequently, \( \mathcal{L}_C \subseteq \mathcal{L} \).

For any \( \mathcal{X} \in \mathcal{L}_C \), by (2), we have

\[
\mathcal{X}(1) = A_1 C(1) (A_d \otimes A_{d-1} \otimes \cdots \otimes A_2)^T
\]

\[
= a_1 \begin{pmatrix}
C(1) \\
\vdots \\
C(1) \\
0_{r_{12}}
\end{pmatrix} (A_d \otimes A_{d-1} \otimes \cdots \otimes A_2)^T
\]

where \( 0_{r_{12}} \in \mathbb{R}^{(n_1-r_1) \times (r_2 \cdots r_d)} \) is a zero matrix. By some simple calculations, we get

\[
\|\mathcal{X}\|^2_F = \|\mathcal{X}(1)\|^2_F = (a_1 \cdots a_d)^2 \|C(1)\|^2_F \prod_{i=1}^{d} \left\lfloor \frac{s_i \wedge n_i}{r_i} \right\rfloor .
\]

(71)

Note that the entries of \( C \) only takes 0 or \( c_0 \) by the construction of \( \mathcal{D}_0 \) in (70). By the Varshamov-Gilbert bound [47, Lemma 2.9], we know that there exists a subset \( \mathcal{L}_C^0 \subseteq \mathcal{L}_C \) such that

\[
|\mathcal{L}_C^0| \geq 2^{r_1 \cdots r_d / 8} + 1,
\]

(72)

and for any \( \mathcal{X}_1, \mathcal{X}_2 \in \mathcal{L}_C^0, \)

\[
\|\mathcal{X}_1 - \mathcal{X}_2\|^2_F = \|\mathcal{X}_1(1) - \mathcal{X}_2(1)\|^2_F
\]

\[
\geq \frac{r_1 \cdots r_d}{8} \left( \prod_{i=1}^{d} \left\lfloor \frac{s_i \wedge n_i}{r_i} \right\rfloor \right) (c_0 a_1 \cdots a_d)^2
\]

\[
\geq \frac{n_1 n_2 \cdots n_d}{2^{d+3}} \left( \prod_{i=1}^{d} \Delta_i(s_i, n_i) \right) (c_0 a_1 \cdots a_d)^2
\]

\[
= \frac{n_1 n_2 \cdots n_d}{2^{d+3}} \min \left\{ \prod_{i=1}^{d} a_i^2 \Delta_i(s_i, n_i), \gamma_c^2 \frac{r_1 r_2 \cdots r_d}{m} \right\},
\]

where the second inequality follows from the fact that \( |x| \geq \frac{x}{2} \) for any \( x \geq 1, \Delta_i(s_i, n_i) \) is defined as (21), and the last equality follows from the definition of \( c_0 \) in (69).

For an arbitrary tensor \( \mathcal{X} \in \mathcal{L}_C^0 \), we have that

\[
K(\mathbb{P}, \mathcal{X}, \mathbb{P}_0) \leq \frac{m}{n_1 n_2 \cdots n_d} \left( \frac{1}{2 \mu^2} \right) \sum_{i_1, i_2, \ldots, i_d} |\mathcal{X}_{i_1,i_2 \cdots i_d}|^2
\]

\[
\leq \frac{m}{n_1 n_2 \cdots n_d} \left( \frac{1}{2 \mu^2} \right) c_0^2 \left( \prod_{i=1}^{d} r_i a_i^2 \left\lfloor \frac{s_i \wedge n_i}{r_i} \right\rfloor \right)
\]

\[
\leq \frac{m}{2 \mu^2} \min \left\{ \prod_{i=1}^{d} a_i^2 \Delta_i(s_i, n_i), \gamma_c^2 \frac{r_1 r_2 \cdots r_d}{m} \right\}
\]

\[
\leq \frac{\gamma_c^2 r_1 \cdots r_d}{2} \leq 4 \gamma_c^2 \log_2(|\mathcal{L}_C^0|) - 1,
\]

where the first inequality holds by (20) and \( \Omega \sim \text{Bern}(p) \) with \( p = \frac{m}{n_1 n_2 \cdots n_d} \), the second inequality holds by (71) and \( \mathcal{L}_C^0 \subseteq \mathcal{L}_C \), the third inequality holds by \( |x| \leq x \) for any \( x > 0 \) and the definition of
\( c_0 \) in (69), and the last inequality holds by (72). Consequently, summing up the inequality in (73) for all \( \mathcal{X} \in \mathfrak{L}^0_c \) except for \( \mathcal{X} = 0 \), we deduce

\[
\frac{1}{|\mathfrak{L}^0_c| - 1} \sum_{\mathcal{X} \in \mathfrak{L}^0_c} K(\mathbb{P}_\mathcal{X}, \mathbb{P}_0) \leq \alpha \log(|\mathfrak{L}^0_c| - 1)
\]

where \( \gamma_c := \frac{\sqrt{\alpha \log(2)}}{2} \) with \( \alpha \in (0, 1/8) \). Hence, by [47, Theorem 2.5], we have

\[
\inf \sup_{\mathcal{X} \in \mathfrak{L}^0_c} \mathbb{P} \left( \frac{\|\hat{X} - \mathcal{X}\|^2}{n_1 \cdots n_d} \geq \frac{1}{2d+4} \min \left\{ \prod_{i=1}^{d} \alpha_i^2 \Delta_i(s_i, n_i), \gamma_c^2 \mu^2 \frac{r_1 r_2 \cdots r_d}{m} \right\} \right) 
\geq \inf \sup_{\mathcal{X} \in \mathfrak{L}^0_c} \mathbb{P} \left( \frac{\|\hat{X} - \mathcal{X}\|^2}{n_1 \cdots n_d} \geq \frac{1}{2d+4} \min \left\{ \prod_{i=1}^{d} \alpha_i^2 \Delta_i(s_i, n_i), \gamma_c^2 \mu^2 \frac{r_1 r_2 \cdots r_d}{m} \right\} \right) \geq \tilde{\alpha},
\]

where \( \tilde{\alpha} \) is defined as

\[
\tilde{\alpha} := \frac{\sqrt{|\mathfrak{L}^0_c| - 1}}{1 + \sqrt{|\mathfrak{L}^0_c| - 1}} \left( 1 - 2\alpha - \sqrt{\frac{2\alpha}{\log(|\mathfrak{L}^0_c| - 1)}} \right).
\]

**Case II.** Now we consider the packing set about the factor matrices. For a fixed \( i \), then for any \( j \neq i \), we let

\[
A_j = a_j \begin{pmatrix} I_{r_j} \\ \vdots \\ I_{r_j} \\ 0_j \end{pmatrix} \in \mathbb{R}^{n_j \times r_j},
\]

where there are \( \lfloor \frac{s_j \wedge n_j}{r_j} \rfloor \) blocks identity matrix \( I_{r_j} \) in \( A_j \) and \( O_j \) is an \( (n_j - r_j \lfloor \frac{s_j \wedge n_j}{r_j} \rfloor) \times r_j \) zero matrix. Let

\[
\tilde{A}_i := \left\{ A_i = (A_{i'}, 0_{i'}) \in \mathbb{R}^{n_i \times r_i} : A_{i'} \in \mathbb{R}^{n_i \times r_i'} \text{ with } (A_{i'})_{k_j} \in \{0, b_i\}, 0_{i'} \in \mathbb{R}^{n_i \times (r_i - r_i')} \right\},
\]

where \( r_i' := \lfloor \frac{s_i}{n_i} \rfloor \), there are at most \( s_i \) nonzero entries in \( A_{i'} \), and \( 0_{i'} \) is an \( n_i \times (r_i - r_i') \) zero matrix. Now we construct the following set

\[
\mathfrak{L}_{A_i} = \left\{ \mathcal{X} = \mathcal{C} \times_1 A_1 \cdots \times_{i-1} A_{i-1} \times_i A_i \times_{i+1} A_{i+1} \cdots \times_d A_d : A_i \in \tilde{A}_i \right\},
\]

where the mode-\( i \) unfolding of \( \mathcal{C} \) is defined as

\[
\mathcal{C}_{(i)} = \left( I_{r_i'} \quad I_{r_i'} \quad \cdots \quad I_{r_i'} \quad 0_{1_{i'}'} \quad 0_{2_{i'}'} \quad \cdots \quad 0_{2_{i'}'} \quad 0_{3_{i'}'} \right).
\]

Here there are \( \lfloor \frac{r_j \times r_i'}{r_i} \rfloor \) blocks identity matrices \( I_{r_i'} \in \mathbb{R}^{r_i' \times r_i'} \) in \( \mathcal{C}_{(i)} \), and

\[
0_{1_{i'}'} \in \mathbb{R}^{r_i' \times \lfloor \frac{r_j \times r_i'}{r_i} \rfloor}, 0_{2_{i'}'} \in \mathbb{R}^{(r_i - r_i') \times r_i'}, 0_{3_{i'}'} \in \mathbb{R}^{(r_i - r_i') \times \lfloor \frac{r_j \times r_i'}{r_i} \rfloor}.
\]

33
are zero matrices. From this construction, we know that $L_{A_i} \subseteq L$. Therefore, for any $X \in \mathcal{L}_{A_i}$, we have

$$X = A_i C_i (A_d \otimes \cdots \otimes A_{i+1} \otimes A_{i-1} \otimes \cdots \otimes A_1)^T$$

$$= (A_{i_1} \cdots A_{i_r} 0_{ni_r} (A_d \otimes \cdots \otimes A_{i+1} \otimes A_{i-1} \otimes \cdots \otimes A_1)^T,$$

where the first equality follows from (2) and $0_{ni_r} \in \mathbb{R}^{n_1 \times (\Pi_j r_j - r_i \Pi_j r_j)}$ is a zero matrix. Hence, we can obtain through some simple calculations that

$$\|X\|_2^2 = \|X(i)\|_2^2 = \left(\prod_{j \neq i} \left(\frac{s_j \wedge n_j}{r_j}ight) a_j^2\right) \|A_{i'}\|_F^2. \quad (75)$$

Notice that the entries of $A_{i'}$ only take 0 or $b_i$. By the Varshamov-Gilbert bound [47, Lemma 2.9], we know that there exists $\mathcal{L}^0_{A_i} \subseteq \mathcal{L}_{A_i}$ such that

$$|\mathcal{L}^0_{A_i}| \geq 2^{n_i/8} + 1, \quad (76)$$

and for any $X_1, X_2 \in \mathcal{L}^0_{A_i}$,

$$\|X_1 - X_2\|_2^2 \geq \frac{s_i}{8} \left(\frac{\prod_{j \neq i} r_j}{r_i} \right) \prod_{j \neq i} \left(\frac{s_j \wedge n_j}{r_j}ight) a_j^2$$

$$\geq \frac{s_i b_i^2}{2^{d+3r_i}} \prod_{j \neq i} (s_j \wedge n_j) a_j^2$$

$$\geq \frac{n_1 \cdots n_d}{2^{d+4}} \left(\prod_{j=1}^d \Delta_j(s_j, n_j)\right) \left(\prod_{j \neq i} a_j^2\right) \min\left\{a_i^2, \frac{\gamma_i^2 \mu^2}{(\Pi_{j \neq i} a_j^2) \Pi_{j=1}^d \Delta_j(s_j, n_j)} \left(\frac{s_i}{m}\right)\right\}$$

$$= \frac{n_1 \cdots n_d}{2^{d+4}} \min\left\{\prod_{j=1}^d \Delta_j(s_j, n_j) a_j^2, \gamma_i^2 \mu^2 \left(\frac{s_i}{m}\right)\right\}, \quad (77)$$

where the second inequality holds by $|x| \geq \frac{x}{2}$ for any $x \geq 1$ and the third inequality holds by $x/\lceil x \rceil \geq \frac{1}{2} \min\{x, 1\}$ for any $x > 0$.

In addition, for any $X \in \mathcal{L}^0_{A_i}$, we have that

$$K(P_X, P_0) \leq \frac{m}{n_1 n_2 \cdots n_d} \left(\frac{1}{2\mu^2}\right) \sum_{i_1, i_2, \ldots, i_d} |X_{i_1 i_2 \cdots i_d}|^2$$

$$\leq \frac{m}{n_1 n_2 \cdots n_d} \left(\frac{1}{2\mu^2}\right) \left[\prod_{j \neq i} \frac{r_j}{r_i}\right] \prod_{j \neq i} \left(\frac{s_j \wedge n_j}{r_j}ight) a_j^2 (n_i r_i' \wedge s_i) b_i^2$$

$$\leq \frac{m}{2\mu^2} \left(\prod_{j=1}^d \Delta_j(s_j, n_j)\right) \left(\prod_{j \neq i} a_j^2\right) b_i^2$$

$$\leq \frac{m}{2\mu^2} \min\left\{\prod_{j=1}^d a_j^2 \Delta_j(s_j, n_j), \gamma_i^2 \mu^2 s_i/m\right\}$$

$$\leq \frac{\gamma_i^2 s_i}{2} \leq 4\gamma_i^2 \log_2(|\mathcal{L}^0_{A_i}| - 1), \quad (78)$$

34
where the first inequality holds by (20) and $\Omega \sim \text{Bern}(p)$ with $p = \frac{m}{n_1 n_2 \cdots n_d}$, the second inequality holds by (75) and $\|A_{ij}\|_F^2 \leq (n_i r_i^j \wedge s_i) b_i^j$, the third inequality holds by the fact that $|x| \leq x$ for any $x > 0$, and the last inequality holds by (76). Let $\gamma_i = \sqrt{\alpha_i \log(2)} / 2$ with $\alpha_i \in (0, 1 \frac{1}{8})$. Summing up the inequality in (78) for all $0 \neq X \in \mathcal{L}_A$ yields

$$
\frac{1}{|\mathcal{L}_A| - 1} \sum_{X \in \mathcal{L}_A} K(\mathbb{P}_X, \mathbb{P}_0) \leq \alpha_i \log(|\mathcal{L}_A| - 1).
$$

It follows from [47, Theorem 2.5] that

$$
\inf_{X} \sup_{X^* \in \mathcal{L}(s, r, a)} \mathbb{P} \left( \frac{\|\tilde{X} - X^*\|_F^2}{n_1 \cdots n_d} \geq \frac{1}{2d+5} \min \left\{ \prod_{j=1}^{d} \Delta_j(s_j, n_j)a_j^2, \frac{\gamma_i^2 \mu^2 (s_i)}{m} \right\} \right) \\
\geq \inf_{X} \sup_{X^* \in \mathcal{L}_A} \mathbb{P} \left( \frac{\|\tilde{X} - X^*\|_F^2}{n_1 \cdots n_d} \geq \frac{1}{2d+5} \min \left\{ \prod_{j=1}^{d} \Delta_j(s_j, n_j)a_j^2, \frac{\gamma_i^2 \mu^2 (s_i)}{m} \right\} \right) \\
\geq \tilde{\alpha}_i,
$$

where

$$
\tilde{\alpha}_i = \frac{\sqrt{|\mathcal{L}_A| - 1}}{1 + \sqrt{|\mathcal{L}_A| - 1}} \left( 1 - 2\alpha_i - \sqrt{\frac{2\alpha_i}{\log(|\mathcal{L}_A| - 1)}} \right).
$$

Let

$$
\eta := \min \left\{ \prod_{j=1}^{d} \Delta_j(s_j, n_j)a_j^2, \frac{\gamma_i^2 \mu^2 (s_i)}{m} \right\},
$$

where $\gamma_m = \min\{\gamma_c, \gamma_1, \ldots, \gamma_d\}$. Combining (74) with (79), we deduce that

$$
\inf_{X} \sup_{X^* \in \mathcal{L}(s, r, a)} \mathbb{P} \left( \frac{\|\tilde{X} - X^*\|_F^2}{n_1 \cdots n_d} \geq \frac{1}{2d+5(d+1)} \eta \right) \geq \tilde{\alpha}_m,
$$

where $\tilde{\alpha}_m = \min\{\tilde{\alpha}_i, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_d\} \in (0, 1)$. Therefore, by Markov’s inequality, we get that

$$
\inf_{X} \sup_{X^* \in \mathcal{L}(s, r, a)} \frac{\mathbb{E}_{\Omega, Y\sim \mathcal{L}(s, r, a)} \|\tilde{X} - X^*\|_F^2}{n_1 \cdots n_d} \\
\geq \frac{\tilde{\alpha}_m}{2d+5(d+1)} \min \left\{ \prod_{i=1}^{d} \Delta_i(s_i, n_i)a_i^2, \frac{\gamma_m^2 \mu^2 (r_1 r_2 \cdots r_d + \sum_{i=1}^{d} s_i)}{m} \right\},
$$

which concludes the proof. \hfill \Box

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