PHASE-LOCKED TRAJECTORIES FOR DYNAMICAL SYSTEMS ON GRAPHS

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ABSTRACT. We prove a general result on the existence of periodic trajectories of systems of difference equations with finite state space which are phase-locked on certain components which correspond to cycles in the coupling structure. A main tool is the new notion of order-induced graph which is similar in spirit to a Lyapunov function. To develop a coherent theory we introduce the notion of dynamical systems on finite graphs and show that various existing neural networks, threshold networks, reaction-diffusion automata and Boolean monomial dynamical systems can be unified in one parametrized class of dynamical systems on graphs which we call threshold networks with refraction. For an explicit threshold network with refraction and for explicit cyclic automata networks we apply our main result to show the existence of phase-locked solutions on cycles.

1. Introduction. Dynamics of networks with discrete, i.e. finite, state space $S = \{0, 1, \ldots, n\}$ are an actual topic of research with applications e.g. in systems biology [9, 15]. The network coupling topology is encoded in the set of edges $E$ of a finite graph $G = (V,E)$, while vertices $i \in V$ represent the state variables $x_i \in S$. The dynamics of the network is described by an update rule which determines how the values of the state variables $x_i$, $i \in V$, at any given time $t \in \mathbb{N}$ or $\mathbb{Z}$, can be computed from the value of the state variables one time instance earlier, i.e. at time $t - 1$. On the one hand the dynamics seems simple because after a finite amount of time the values of each state variable repeat periodically and form so-called limit cycles or attractors. This is due to the fact that the finitely many state variables $x_i$, $i \in V$, can take only finitely many values $x_i \in S$ and hence the dynamics eventually becomes periodic. On the other hand, to determine the number of limit cycles, their length and the amount of time it takes for a specific initial state to reach a limit cycle turns out to be rather delicate, even for simple systems with a small number of state variables, because the complexity of the dynamical behavior typically scales exponentially with the size of the underlying graph.

Many results are published for networks with randomly chosen graph structures or random dynamics on given graphs [8, 13, 16, 17], a majority of papers addresses averaged properties if the size of the graph tends to infinity. At first glance it seems surprising that the important case of deterministic dynamics on explicitly given

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The set $M$ is called state space and $\varphi(\cdot) = \varphi(1, \cdot) : M \to M$ the corresponding time-1-map. Any map $\varphi : M \to M$ is the time-1-map of an induced dynamical system with discrete one-sided time $T = \mathbb{N}_0$ or two-sided time $T = \mathbb{Z}$ to be a map

$$\varphi : T \times M \to M$$

which satisfies the group property

$$\varphi(0, x) = x \quad \text{and} \quad \varphi(t, \varphi(s, x)) = \varphi(t + s, x) \quad \text{for all } t, s \in T, x \in M.$$ 

The set $M$ is called state space and $\varphi(\cdot) = \varphi(1, \cdot) : M \to M$ the corresponding time-1-map. Any map $\varphi : M \to M$ is the time-1-map of an induced dynamical system with discrete one-sided time $T = \mathbb{N}_0$ which is defined via composition or iteration as follows

$$\mathbb{N}_0 \times M \to M, \quad (t, x) \mapsto \varphi^t(x),$$

where $\varphi^0 = \text{id}$ and $\varphi^t = \varphi \circ \cdots \circ \varphi$. In this paper we describe dynamical systems via their time-1-map. For an $x \in X$ the trajectory $\text{traj}(x)$ through $x$ is defined as the function $\text{traj}(x) : \mathbb{N}_0 \to M, t \mapsto \varphi^t(x)$. An $x \in M$ is called a periodic point if there exists a period $p \in \mathbb{N}$ with $\varphi^p(x) = x$, if there exists no smaller $p \in \mathbb{N}$ with this property then $p$ is called minimal period of $x$. The set of all periodic points of $\varphi$ is denoted by $\text{Per}_M(\varphi)$, or simply $\text{Per}(\varphi)$.

We recall some basic definitions from graph theory, see e.g. [4]. A (directed) graph $G = (V, E)$ is a pair consisting of a finite set of vertices $V$ and a set of edges $E \subseteq V \times V$. The in-neighborhood $N^-(i)$ of a vertex $i$ is the set of all vertices $j$ such that $(j, i) \in E$, the elements in $N^+(i)$ are called predecessors of $i$. A walk in $G$ is a sequence of vertices $i_1, \ldots, i_q$ such that $(i_k, i_{k+1}) \in E$ for $k \in \{1, \ldots, q-1\}$, if the vertices are pairwise different, it is called a path. A cycle is a walk whose vertices are pairwise different apart from the first and last one, which coincide. A graph is called strongly connected, if for each pair of vertices $(i, j)$, $i \neq j$, there is a path from $i$ to $j$.

In the next section we define dynamical systems $\varphi$ on graphs $G = (V, E)$ with finite state space $M = S^V$, where $S^V$ denotes the set of functions from $V$ to $S$. For $i \in V$ let $\pi_i : S^V \to S$ be the projection on the $i$-th component and define $\varphi_i := \pi_i \circ \varphi$ and $\text{traj}_i := \pi_i \circ \text{traj}$. Because of the finiteness of $S^V$ there exists $t \in \mathbb{N}$ such that $\varphi^t(x) \in \text{Per}(\varphi)$ for every $x \in M$, i.e. every point is eventually periodic. We will focus on the dynamics on $\text{Per}(\varphi)$ which is invariant under $\varphi$, i.e. $\varphi(\text{Per}(\varphi)) \subseteq \text{Per}(\varphi)$, and we can therefore investigate the restriction $\varphi|_{\text{Per}(\varphi)} : \text{Per}(\varphi) \to \text{Per}(\varphi)$. Trajectories through periodic points $x \in \text{Per}(\varphi)$ can be trivially extended to periodic functions on $\mathbb{Z}$ and to simplify the discussion of periodic trajectories we define with a slight abuse of notation the trajectory of a periodic $x \in \text{Per}(\varphi)$ with period $p$ as

$$\text{traj}(x) : \mathbb{Z} \to M, \quad t \mapsto \text{traj}(x)(t) := \varphi^t \mod p(x).$$
This allows one to shift trajectories of periodic points to the left, as well as to the right. The right-shift on the set $M^Z$ of all functions or sequences from $Z$ to $M$ is defined as the dynamical system with time-1-map

$$\sigma : M^Z \to M^Z, \quad a \mapsto \sigma(a) \quad \text{with} \quad \sigma(a)(t) = a(t-1) \quad \text{for} \quad t \in Z.$$ 

Its inverse $\sigma^{-1}$ is the left-shift. Note that we will mainly work with the right shifts $\sigma = \sigma_M$ for $M = S$ or $M = S^V$, the choice of $M$ will be clear from the context. The mapping $\text{traj} : \text{Per}(\varphi) \to \text{traj}[\text{Per}(\varphi)] = \{\text{traj}(x) \mid x \in \text{Per}(\varphi)\}$ is a bijection with inverse $\text{traj}(x) \mapsto \text{traj}(x)(0) = x$. Moreover, the dynamical systems $\varphi |_{\text{Per}(\varphi)}$ and $\sigma^{-1}_{|_{\text{traj}[\text{Per}(\varphi)]}}$ are conjugate, i.e. for any $x \in \text{Per}(\varphi)$

$$\text{traj}(\varphi(x)) = \sigma^{-1}(\text{traj}(x))$$

and the following diagram commutes

$$\begin{array}{ccc}
\text{Per}(\varphi) & \xrightarrow{\varphi} & \text{Per}(\varphi) \\
\text{traj} \downarrow & & \downarrow \text{traj} \\
\text{traj}[\text{Per}(\varphi)] & \xrightarrow{\sigma^{-1}} & \text{traj}[\text{Per}(\varphi)]
\end{array}$$

The structure of the paper is as follows. In Section 2 we introduce the notion of a dynamical system on a graph. We focus on periodic dynamics and define local trajectories at a vertex and phase-locking of trajectories on subsets of all vertices. The section contains several introductory examples and establishes the relation between the minimal period of the trajectory and the minimal periods of all local trajectories in Lemma 2.5. Section 3 contains the main theoretical contributions. We introduce the notion of an order-induced graph in Definition 3.1 and show in a key Lemma 3.2 that phase-locking occurs on its cycles. The main result Theorem 3.4 shows that phase-locking occurs under testable and natural assumptions which are similar in spirit to the existence of a Lyapunov function: it is assumed that there exists an order on the space of all possible trajectories such that the order is compatible with the dynamics. We also discuss complete phase-locking on strongly connected graphs in Theorem 3.6. Section 4 is devoted to a discussion of various existing examples of neural networks, threshold networks, reaction-diffusion automata and Boolean monomial dynamical systems. We show that they can be unified in one parametrized class of dynamical systems on graphs (Example 4.3) which we call threshold networks with refraction in Definition 4.1. For an explicit threshold network with refraction 1 and threshold 1 we construct an order which is based on the lexicographic order of words and apply our main result to show the existence of phase-locked solutions on cycles of strongly connected graphs in Theorem 4.13. We get an upper bound on minimal periods for strongly connected graphs with no two disjoint cycles in Corollary 4.14. The notions double zero, difference sequence and minimal distance representation, which are relevant for the argumentation, are emphasized in Definitions 4.6, 4.8 and 4.10. Lemma 4.7 characterizes two-periodic trajectories. We close the section with an important Example 4.15 of a threshold network on a Hamiltonian graph with 17 vertices which has a trajectory with period 18, disproving the conjecture [2, Conjecture 1]. Section 5 is devoted to a class of dynamical systems on graphs with the cyclic group of order $n$ as the local state space and an update rule which sets a vertex to a certain activation level if the fraction of predecessors on that level reaches a threshold $\theta \in [0,1]$ (Definition 5.1).
We show that depending on the threshold, phase-locking might be inhibited (Example 5.2). For minimal and maximal threshold we construct an order based on the lexicographic or reverse lexicographic order which is compatible with the occurrence of jumps (Definition 5.3) and prove that on strongly connected graphs every non-constant and periodic trajectory gets phase-locked on a cycle (Theorem 5.5 and Theorem 5.8).

2. Dynamical systems on graphs. Although dynamical systems models with underlying graph structure are dealt with in many publications (see e.g. [10] and the references therein), there seems to be no common notation. We propose the following basic and unifying definition which is motivated by classical dynamical systems theory, see e.g. [7].

Definition 2.1 (Dynamical System on a Graph). Let $G = (V, E)$ be a directed graph and $S$ be a finite set. A map $\varphi : S^V \to S^V$ is called a (discrete) dynamical system on $G$ with state space $S^V$ and local state space $S$, if it respects the graph structure, i.e. each component $\varphi_i = \pi_i \circ \varphi : S^V \to S$ depends only on $i$ and its predecessors $j \in N^-(i)$ or, more formally,

$$\forall i \in V : \varphi_i \text{ depends on } j \Rightarrow j \in N^-(i) \cup \{i\}$$

where we define that

$$\varphi_i \text{ depends on } j \in V : \iff \exists a, b \in S^V : (\forall k \neq j : a_k = b_k) \land \varphi_i(a) \neq \varphi_i(b).$$

Example 2.2. Let $G = (V, E)$ be a finite graph. Then $\varphi : \{0, 1\}^V \to \{0, 1\}^V$ with

$$\varphi_i(x) := \begin{cases} 1 & \text{if } x_i = 0 \text{ and } |\{j \in N^-(i) : x_j = 1\}| \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

is a dynamical systems on $G$ with local state space $S = \{0, 1\}$. Each coordinate $\varphi_i$ corresponding to a vertex $i \in V$ depends only on the values of $x_i$ and $x_j$ for $j \in N^-(i)$. $\varphi$ is a simple neural network [1, 3, 14]. If the local state 0 is interpreted as inactive and 1 as active then the network dynamics could be described as follows: a vertex $i$ becomes active if it was inactive and if the number of active predecessors reaches the threshold 2.

Although a dynamical system $\varphi$ on a graph is defined only for one-sided time $T = \mathbb{N}_0$, it can be extended to a dynamical system with two-sided time $T = \mathbb{Z}$, if it is restricted to the set $\text{Per}(\varphi)$ of periodic points. In this paper we are interested in properties of periodic trajectories and hence we focus on periodic points or, equivalently, on the dynamics of $\varphi|_{\text{Per}(\varphi)}$.

Definition 2.3 (Local and phase-locked trajectories). Let $x \in \text{Per}(\varphi)$ and $a := \text{traj}(x)$.

(a) $a_i : \mathbb{Z} \to S$ is called local trajectory of $\varphi$ through $x$ at vertex $i \in V$.

(b) The trajectory $a$ is called phase-locked on $W \subseteq V$, if

$$\forall i, j \in W \exists t \in \mathbb{Z} : a_i = \sigma^t(a_j).$$

Example 2.4. Let $G = (V, E)$ be the graph with 10 vertices as depicted in Figure 1. Consider the dynamical system $\varphi$ from Example 2.2 on $G$. Figure 2 visualizes a 5-periodic trajectory $a = \text{traj}(x)$ of a periodic point $x \in \text{Per}(\varphi)$. At vertices $i$ and $j$ the local trajectories of $\varphi$ through $x$ are given by

$$a_i = (0, 0, 1, 0, 1) \quad \text{and} \quad a_j = (0, 1, 0, 0, 1),$$
where the overbar denotes periodic continuation. Note that

- the trajectory $a$ is phase-locked on the set of vertices $\{i, j\}$, since $a_i = \sigma^3(a_j)$,
- the trajectory $a$ is phase-locked on the set of vertices $V \setminus \{i, j\}$,

A local trajectory of $\varphi$ through an $x \in \text{Per}(\varphi)$, such that $x$ has minimal period $p$, is of course again $p$-periodic with respect to the right-shift $\sigma : S^2 \to S^2$, possibly with a minimal period which is smaller than $p$. The next lemma relates the minimal periods of all local trajectories.

**Lemma 2.5.** The minimal period of $x \in \text{Per}(\varphi)$ under $\varphi$ is the least common multiple of the minimal periods of all local trajectories through $x$.

**Proof.** Let $a = \text{traj}(x)$, $p$ be the minimal period of $a$ and $p_{\text{lcm}}$ the least common multiple of the minimal periods of the local trajectories $\{a_i \mid i \in V\}$. Then for all $i \in V$ we have

$$[\sigma^{p_{\text{lcm}}}(a)]_i = \sigma^{p_{\text{lcm}}}(a_i) = a_i,$$

i.e., $\sigma^{p_{\text{lcm}}}(a) = a$ and hence $p \leq p_{\text{lcm}}$. On the other hand,

$$\sigma^p(a_i) = [\sigma^p(a)]_i = a_i,$$

hence the minimal period of each local trajectory divides $p$ and consequently $p_{\text{lcm}}$ divides $p$. In particular $p_{\text{lcm}} \leq p$, proving that $p = p_{\text{lcm}}$. 

\[ \square \]
3. **Phase-locked trajectories on cycles.** Example 2.4 already indicates that the graph topology plays a crucial role for the occurrence of phase-locking of periodic trajectories. On the other hand, one cannot expect that phase-locking of a periodic trajectory typically occurs on the whole graph. Example 2.4 shows a periodic trajectory which is phase-locked only on a subset \( W \subseteq V \) of all vertices. It is not obvious how to determine reasonable vertex sets \( W \) on which phase-locking is likely to happen. One obstruction is the fact that a dynamical system \( \varphi \) on a graph \( G \) cannot be restricted to a subgraph with vertex set \( W \) because \( \varphi \) typically depends on all predecessors of vertices in \( W \). A tool to formally bypass this dynamical coupling structure is to utilize the conjugacy between the dynamical systems \( \varphi|_{\text{Per}(\varphi)} \) and \( \sigma|_{\text{traj}[\text{Per}(\varphi)]} \). The right-shift \( \sigma \) on \( \text{traj}[\text{Per}(\varphi)] \) encodes the full dynamics of \( \varphi \) but at the same time allows us to project the dynamics to single vertices or a set \( W \) of vertices. In the next definition, we elaborate on this idea to relate phase-locking of a periodic point to the underlying graph topology and introduce for each periodic trajectory a subgraph of \( G \) which is induced by a partial order \( \preceq \) on the space \( S^Z \) of all possible local trajectories and which is different from but similar in spirit to the notion of a Lyapunov function. We assume

\[(A) \ \text{Let } \varphi \text{ be a dynamical system on a finite graph } G = (V,E) \text{ with finite local state space } S \text{ and let } \preceq \text{ be a partial order on } S^Z \text{ which is invariant under the right-shift } \sigma : S^Z \to S^Z, \text{ i.e. } a \preceq b \iff \sigma(a) \preceq \sigma(b) \text{ for } a,b \in S^Z.

**Definition 3.1 (Order-induced subgraph).** Assume (A) and let \( x \in \text{Per}(\varphi) \), \( a := \text{traj}(x) \). The order-induced subgraph \( G_{\preceq} = G_{\preceq,x} \) is the graph \( G_{\preceq} = (V,E_{\preceq}) \) with the same set of vertices as \( G = (V,E) \) but only a subset of edges

\[E_{\preceq} = \{(j,i) \in E \mid \sigma(a_j) \preceq a_i\}.

The following key lemma shows that phase-locking occurs on cycles of order-induced subgraphs.

**Lemma 3.2.** Assume (A) and let \( x \in \text{Per}(\varphi) \), \( a := \text{traj}(x) \) and \( G_{\preceq} = G_{\preceq,x} \) be the order-induced subgraph. If \( i_0, \ldots, i_q = i_0 \) is a cycle in \( G_{\preceq} \), then

\[\forall k \in \{1, \ldots, q\} : \sigma(a_{i_{k-1}}) = a_{i_k} = \sigma^q(a_{i_k}).\]

**Proof.** Let \( p \) be the minimal period of \( x \). For each \( k \in \{1, \ldots, q\} \) we have

\[a_{i_k} \succeq \sigma(a_{i_{k-1}}) \]
\[\succeq \sigma^2(a_{i_{k-2} \mod q}) \succeq \ldots \succeq \sigma^q(a_{i_{k-q \mod q}}) = \sigma^q(a_{i_k}) \]
\[\succeq \sigma^{2q}(a_{i_k}) \succeq \ldots \succeq \sigma^{pq}(a_{i_k}) = a_{i_k}\]

and thus by antisymmetry of \( \preceq \)

\[a_{i_k} = \sigma(a_{i_{k-1}}) = \sigma^q(a_{i_k}).\]

\[\square\]

**Example 3.3.** Consider again Example 2.2. Note that

- each block of 5 consecutive states of the 5-periodic local trajectories \( a_i \) and \( a_j \) contains exactly 3 inactive and 2 active states,
- for every \( k \in V \setminus \{i,j\} \) each block of 5 consecutive states of the 5-periodic local trajectory \( a_k \) contains exactly 4 inactive and 1 active states,
- every cycle in \( G \) contains vertices in \( \{i,j\} \), as well as in \( V \setminus \{i,j\} \), and hence the trajectory \( a \) is not phase-locked on any of the cycles in \( G \).
As a consequence of Lemma 3.2, for the considered trajectory $a = \text{traj}(x)$ and any arbitrary partial order on $S^Z$ which is invariant under the right-shift, the order-induced graph $G_{\leq} = G_{\leq,x}$ contains no cycles.

We are now in a position to formulate our main theorem on the existence of phase-locked trajectories on cycles.

**Theorem 3.4** (Phase-locking on cycles). Assume (A) and let $x \in \text{Per}(\varphi)$, $a := \text{traj}(x)$. Assume that each vertex $i$ of $G$ has a predecessor $j$ such that $\sigma(a_j) \preceq a_i$. Then there exists a cycle $i_0, \ldots, i_q = i_0$ in $G$ on which the trajectory $a$ is phase-locked and, moreover, the local trajectories on the cycle are $q$-periodic and shifted along the cycle in the following sense

$$\forall k \in \{0, \ldots, q - 1\} : \sigma(a_{i_k}) = a_{i_{k+1}}.$$ 

**Proof.** (See Figure 3 for an illustration of the proof.) By assumption, every vertex of $G_{\leq} = G_{\leq,x}$ has a predecessor (in $G_{\leq}$). Let $i \in V$ be arbitrary and $i_1, \ldots, i_r = i$ a path of maximal length in $G_{\leq}$. By assumption $i_1$ also has a predecessor $i_0$ but, by maximality of the path, $i_0 = i_q$ for some $q \in \{1, \ldots, r\}$. Hence, $i_0, \ldots, i_q = i_0$ is a cycle in $G_{\leq}$ and the claim follows from Lemma 3.2. □

**Remark 3.5.** As can be seen from the proof of Theorem 3.4, for each $i \in V$ one can find a path $i_q, \ldots, i_r = i$ from $i_q$, located in a cycle where phase-locking occurs, to the vertex $i$, such that the local trajectories along the path are ordered in the sense that for each $k \in \{q, \ldots, r - 1\}$ one has $\sigma(a_{i_k}) \preceq a_{i_{k+1}}$.

The following theorem provides a sufficient condition for a trajectory to be phase-locked on the whole graph.

**Theorem 3.6** (Complete phase-locking). Assume (A) and that $G = (V, E)$ is strongly connected. Let $x \in \text{Per}(\varphi)$, $a := \text{traj}(x)$ and assume that for each edge $(j, i) \in E$ the relation $\sigma(a_j) \preceq a_i$ holds. Then the trajectory $a$ is phase-locked on all of $V$ and, moreover, $a$ is q-periodic, where $q$ is the period (see [5, Definition 4.5]) or loop number of $G$ (see [6]), i.e. the greatest common divisor of all cycle lengths of $G$.

**Proof.** By assumption, the order-induced subgraph $G_{\leq} = G_{\leq,x}$ equals $G$. Since $G$ is strongly connected, each edge $(i, j)$ in $E$ is contained in at least one cycle and therefore the trajectory is phase-locked on all of $V$ and thus by strong connectedness of $G$ on all of $V$. If $\ell$ is the length of a cycle in $G$, then the local trajectories of the vertices on this cycle are $\ell$-periodic and since all local trajectories have the same minimal period, the greatest common divisor of all cycle lengths is a period of $a$. □

**Figure 3.** Illustration of the proof of Theorem 3.4
4. Threshold networks with refraction. When McCulloch and Pitts introduced the term neural networks [19] in 1943, they formulated two main ingredients:

1. The activity of the neuron is an ‘all-or-none’ process.
2. A certain fixed number of synapses must be excited within the period of latent addition in order to excite a neuron at any time, and this number is independent of previous activity and position of the neuron.

Since then many different models and classes of neural networks have been investigated under different names, e.g. threshold networks [12], reaction-diffusion automata [11], Boolean monomial dynamical systems [6] or neuronal networks [14]. We introduce a class of dynamical systems on graphs which unifies several of these models.

Definition 4.1 (Threshold networks with refraction). Let \( G = (V, E) \) be a directed graph, \( \theta \in \mathbb{N} \) and \( r \in \mathbb{N}_0 \). The threshold network on \( G \) with refraction \( r \) and threshold \( \theta \) is the dynamical system \( \varphi_{G, \theta, r} = \varphi_{G, \theta, r} \) on \( G \) with local state space \( S = \{0, \ldots, \max(1, r)\} \) which is defined component-wise as

\[
[\varphi_{G, \theta, r}(x)]_i = \begin{cases} 
  x_i - 1 & \text{if } x_i \in \{1, \ldots, r\} \\
  \max(1, r) & \text{if } x_i \notin \{1, \ldots, r\} \text{ and } \{|j \in N^{-}(i) | x_j = \max(1, r)\} \geq \theta \\
  0 & \text{otherwise}
\end{cases}
\]

for vertices \( i \in V \) and \( x \in S^V \) (with the convention that \( \{1, \ldots, r\} = \emptyset \) for \( r = 0 \)).

Example 4.2. Consider again Example 2.4. Note that it is a threshold network \( \varphi_{G, \theta, r} \) with refraction \( r = 1 \) and threshold \( \theta = 2 \).

The following example deals with various existing network models.

Example 4.3. A careful look at the references [6, 11, 12, 14] shows that if the local state spaces of those models are relabeled to match the local state space \( S \), then they coincide with \( \varphi_{\theta, r} \) for a certain threshold \( \theta \) and refraction \( r \). In particular, \( \varphi_{\theta, r} \) provides a unifying notion to describe neural networks as in [1, 3, 14] and

- neural networks in [19] correspond to \( \varphi_{\theta, r} \) for \( \theta \in \mathbb{N} \) and \( r = 0 \),
- threshold networks in [12] correspond to \( \varphi_{\theta, r} \) for \( \theta \in \mathbb{N} \) and \( r = 0 \),
- reaction-diffusion automata in [11] correspond to \( \varphi_{\theta, r} \) for \( \theta \in \mathbb{N} \) and \( r \in \mathbb{N} \),
- Boolean monomial dynamical systems [6] correspond to \( \varphi_{\theta, r} \) for \( \theta = 1 \) and \( r = 0 \).

We deliberately simplified the presentation by assuming constant threshold \( \theta \) and refraction \( r \). Of course Definition 4.1 can be easily extended to allow the thresholds \( \theta_i \) and refractions \( r_i \) at vertex \( i \in V \) to depend on \( i \), thus unifying those cases in [6, 11, 12, 14] for unweighted graphs.

The following application of Theorem 3.6 provides a simple proof of the period length bound from [6, Theorem 4.10] on periodic points of Boolean monomial dynamical systems on strongly connected graphs.

Corollary 4.4. Assume that \( G \) is a strongly connected finite graph. Then the minimal period of an arbitrary periodic point of \( \varphi_{1,0} = \varphi_{G,1,0} \) divides the loop number of \( G \).
Proof. Apply Theorem 3.6 to the coordinate-wise order on the local trajectories, i.e., \( a \preceq b \iff \forall t \in \mathbb{Z} : a(t) \leq b(t) \) for any \( a, b \in \{0, 1\}^\mathbb{Z} \). \( \square \)

For the remainder of this section we focus on threshold networks with refraction 1 and threshold 1 on strongly connected graphs, i.e. we assume

(B) Let \( G = (V, E) \) be a strongly connected finite graph with \( |V| \geq 2 \) and let \( \varphi = \varphi_{G, \theta, r} \) be the threshold network with refraction \( r = 1 \) and threshold \( \theta = 1 \) on \( G \) with local state space \( S = \{0, 1\} \).

Remark 4.5. While the assumption that \( G \) is strongly connected can be relaxed, strongly connected graphs are the relevant cases. Each graph contains at least one strongly connected component \( C_{ini} \) whose vertices only have predecessors in \( C_{ini} \), so the dynamics of the vertices in \( C_{ini} \) does not depend on the dynamics on the rest of the graph. Notice, however, that \( C_{ini} \) might consist of only one vertex, whose local trajectory must then be constant equal to 0. Therefore one can derive the general case from the strongly connected one by first removing all vertices with constant local trajectory and then restricting to the strongly connected component \( C_{ini} \).

A key ingredient in the investigation of \( \varphi = \varphi_{G, 1, 1} \) is the occurrence of two consecutive 0s and their position in local trajectories of \( \varphi \).

Definition 4.6 (Double zeros). For \( a \in \{0, 1\}^\mathbb{Z} \) define the set of double zeros of \( a \) as \( Z(a) = \{ t \in \mathbb{Z} | a(t) = a(t + 1) = 0 \} \).

The following lemma shows that minimal period 2 implies complete phase-locking.

Lemma 4.7 (Characterization of minimal period 2). Assume (B) and let \( x \in \text{Per}(\varphi) \), \( a := \text{traj}(x) \). Then the following four statements are equivalent:

(i) \( x \) has minimal period 2,

(ii) \( \exists i \in V : a_i \) has minimal period 2,

(iii) \( \forall i \in V : Z(a_i) = \emptyset \),

(iv) \( \exists i \in V : Z(a_i) = \emptyset \).

In particular, each of the conditions (i)-(iv) implies that the trajectory \( a \) is phase-locked on all of \( V \).

Proof. First of all notice that by definition of \( \varphi \) there can be no two consecutive 1s in a local trajectory \( a_i \). Therefore local trajectories with no double zeros are alternating sequences of 0s and 1s, which are exactly the local trajectories with minimal period two. Hence (iii) \( \Rightarrow \) (i) and (ii) \( \Rightarrow \) (iv). The implication (i) \( \Rightarrow \) (ii) follows from Lemma 2.5.

It remains to show that (iv) implies (iii). Let \( i \) be a vertex such that \( Z(a_i) = \emptyset \) and let \( j \in V \) s.t. \( (i, j) \in E \). Assume \( Z(a_j) \neq \emptyset \) and let \( t_1, t_2 \) be two consecutive elements of \( Z(a_j) \). Then \( a_j|_{\{t_1+1, \ldots, t_2\}} \) is alternating, starting and ending with 0. Since \( a_i \) is also alternating, \( a_i(t_1 + 1) = a_i(t_2) \) and either \( a_i(t_1) = 1 \) or \( a_i(t_2) = 1 \). Therefore \( a_j(t_1 + 1) = 1 \) or \( a_j(t_2 + 1) = 1 \), contradicting \( \{t_1, t_2\} \subseteq Z(a_j) \). Thus \( Z(a_j) = \emptyset \). Since \( G \) is strongly connected this proves the assertion. \( \square \)

We now turn our attention to periodic points with arbitrary minimal period and introduce helpful tools for their study in the following two definitions.

Definition 4.8 (Difference sequence). A set \( A \subseteq 2^\mathbb{Z} \) is called \( p \)-periodic set of integers with period \( p \in \mathbb{N} \), if \( A + p = A \). The set of all non-empty periodic sets of integers is denoted by \( P_{2^\mathbb{Z}} \). For an \( A \in P_{2^\mathbb{Z}} \) let \( b : \mathbb{Z} \to \mathbb{Z} \) be the unique strictly
monotone two-sided sequence of integers with \( \{b(t) \mid t \in \mathbb{Z}\} = A, b(0) \geq 0 \) and \( b(-1) < 0 \). The difference sequence \( \Delta(A) : \mathbb{N} \to \mathbb{N} \) of \( A \) is defined by \( \Delta(A)(k) = b(k) - b(k-1) \) for \( k \in \mathbb{N} \).

**Example 4.9.** Let \( A = \{−3, 6, 7\} + 15\mathbb{Z} = \{..., −18, −9, −8, −3, 6, 7, 12, 21, 22, ...\} \). Then \( \Delta(A) = (7 − 6, 12 − 7, 21 − 12, 22 − 21, ...) = (1, 5, 9) \).

**Definition 4.10 (Minimal distance representation).** Let \( \prec_{\text{lex}} \) be the strict lexicographic order on \( \mathbb{N}^p \), i.e.

\[
\forall a, b \in \mathbb{N} \quad a \prec_{\text{lex}} b \iff \exists k \in \mathbb{N} : \langle a(k) < b(k) \text{ and } \forall \ell \in \{1, \ldots, k-1\} : a(\ell) = b(\ell) \rangle.
\]

We define the **minimal distance representation** \( R = R_{\prec_{\text{lex}}} : P_{\mathbb{Z}}^Z \to \mathbb{N}^p \) by

\[
R(A) = \min_{\prec_{\text{lex}}} \{ \Delta(A-k) \mid k \in \{0, \ldots, p_{\text{min}}(A)-1\} \cap A \},
\]

where \( p_{\text{min}}(A) \) denotes the minimal period \( p \in \mathbb{N} \) such that \( A + p = A \). The minimum \( R(A) \) exists, since \( \prec_{\text{lex}} \) is total. The minimal \( k \) for which \( \Delta(A-k) \) equals \( R(A) \) is denoted by

\[
m(A) := m_{\prec_{\text{lex}}}(A) := \min \{k \in \{0, \ldots, p_{\text{min}}(A)-1\} \cap A \mid \Delta(A-k) = R(A) \}.
\]

On the set of periodic, two-sided \( \{0, 1\} \)-sequences which have at least one double zero,

\[
P_{\mathbb{Z}}^{\{0,1\}} := \{a \in \text{Per}_{\mathbb{Z}}^{\{0,1\}}(\sigma) \mid Z(a) \neq \emptyset\},
\]

a strict partial order \( \prec \) is defined by

\[
a \prec b \iff R(Z(a)) \prec_{\text{lex}} R(Z(b)).
\]

The partial order \( \prec \) is shift-invariant (see Figure 4). The **induced partial order** \( \preceq \) on \( \{0, 1\}^Z \) is defined as \( a \preceq b \iff a \prec b \text{ or } a = b \).

**Example 4.11.** Consider the 21-periodic \( a = (a(0), \ldots, a(20)) \in P_{\mathbb{Z}}^{\{0,1\}} \) with

\[
\begin{array}{cccccccccccccccccccc}
a(\ell) & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
\ell & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20
\end{array}
\]

The set of double zeros of \( a \) and its corresponding difference sequence are

\[
Z(a) = \{6, 17, 18\} + 21\mathbb{Z} \text{ and } \Delta(Z(a)) = (11, 1, 9).
\]

The minimal distance representation of \( Z(a) \) is \( R(Z(a)) = \Delta(Z(a) - k) \) with \( k = m(Z(a)) = 17 \) and it equals

\[
R(Z(a)) = (1, 9, 11) \text{.}
\]

The statement (c) of the following lemma shows that \( \varphi = \varphi_{G,1,1} \) satisfies the assumptions of Theorem 3.4 on phase-locking on cycles.
Lemma 4.12. Assume (B) and let $x \in \text{Per}(\varphi)$, $a := \text{traj}(x)$. Then the following statements hold:

(a) For $t \in Z(a_i)$ there is $j \in N^-(i)$ with $t \in Z(\sigma(a_j))$.
(b) For all $j \in N^-(i)$ with $m(Z(a_i)) \in Z(\sigma(a_j))$ we have $\sigma(a_j) \leq a_i$.
(c) Each vertex $i \in V$ has a predecessor $j \in V$ such that $\sigma(a_j) \leq a_i$.

Proof. (a) Let $t \in Z(a_i)$. By definition of $Z$ we have $a_i(t + 1) = a_i(t) = a_j(t) = 0$ for all $j \in N^-(i)$. Now either $a_i(t - 1) = 0$ and therefore $a_j(t - 1) = 0$ for all $j \in N^-(i)$ or $a_i(t - 1) = 1$ but then there must be $j' \in N^-(i)$ with $a_{j'}(t - 2) = 1$ and thus $a_{j'}(t - 1) = 0$. Altogether this implies the existence of $j' \in N^-(i)$ with $t - 1 \in Z(a_j)$.

(b) Figure 5 shows an illustration of the proof. Define $b_i = \sigma^{-m(Z(a_i))}(a_i)$ and $b_j = \sigma^{-m(Z(a_j))}(a_j)$. By the assumptions $0 \in Z(b_j)$ and $0 \in Z(b_i)$.

If $\Delta(Z(b_j)) = \Delta(Z(b_i))$, then $b_i = b_j$ and hence $a_i = \sigma(a_j)$. Otherwise either (i) $\Delta(Z(b_j)) < \Delta(Z(b_i))$ or (ii) $\Delta(Z(b_j)) > \Delta(Z(b_i))$. In case (i)

$$R(Z(\sigma(a_j))) = R(Z(b_j)) \leq \Delta(Z(b_j)) < \Delta(Z(b_i)) = R(Z(a_i))$$

and thus $\sigma(a_j) < a_i$. In case (ii) there is $k \in \mathbb{N}$ such that $[\Delta(Z(b_j))](k) > [\Delta(Z(b_i))](k)$ and $[\Delta(Z(b_j))](\ell) = [\Delta(Z(b_i))](\ell)$ for all $\ell \in \{1, \ldots, k - 1\}$. Define

$$t_1 := \sum_{\ell=1}^{k-1} [\Delta(Z(b_j))](\ell) \quad \text{and} \quad t_2 := t_1 + [\Delta(Z(b_i))](k).$$

Since $0 \in Z(b_j)$ and $0 \in Z(b_i)$ this implies that $t_1 \in Z(b_j) \cap Z(b_i)$, $t_2 \in Z(b_i)$ and $Z(b_j) \cap \{t_1 + 1, \ldots, t_2\} = \emptyset$. Thus

$$\forall s \in \{0, \ldots, t_2\}: b_j(s) = b_i(s).$$

Since $b_j(t_2) = b_i(t_2) = 0$ and $t_2 \notin Z(b_j)$ we have $b_j(t_2 + 1) = 1$. By the definition of $b_i$ and $b_j$ this is equivalent to $a_i(m(Z(a_i)) + t_2) = 0$ and $a_j(m(Z(a_j)) + t_2) = 1$. Hence we have $a_i(m(Z(a_i)) + t_2 + 1) = 1$, contradicting $m(Z(a_i)) + t_2 \in Z(a_i) = Z(b_i) + m(Z(a_i))$.

(c) By (a) there is a predecessor $j$ of $i$ with $m(Z(a_i)) \in Z(\sigma(a_j))$ and by (b) we have $\sigma(a_j) \leq a_i$. $\Box$

Theorem 3.4 in combination with Lemma 4.12(c) implies the following main result on threshold networks with refraction 1 and threshold 1, which corresponds to [2, Theorem 5].

Theorem 4.13 (Phase-locking of $\varphi_{G,1,1}$ on cycles of strongly connected graphs). Let $G$ be a strongly connected graph with at least two vertices, $\varphi = \varphi_{G,1,1}$, $x \in \text{Per}(\varphi)$ and $a := \text{traj}(x)$. Then there exists a cycle $i_0, \ldots, i_q = i_0$ in $G$ on which the
trajectory $a$ is phase-locked and, moreover, the local trajectories on the cycle are $q$-periodic and shifted along the cycle in the following sense

$$\forall k \in \{0, \ldots, q - 1\} : \sigma(a_{i_k}) = a_{i_{k+1}}.$$  

The following corollary provides an upper bound on minimal periods for $\varphi_{G,1,1}$ if $G$ has no two disjoint cycles, which can also be found in [2, Corollary 20].

**Corollary 4.14** (Sufficient condition for upper bound on minimal periods). *Let $G = (V,E)$ be a strongly connected graph which contains no two disjoint cycles. Then the minimal period of every periodic point of $\varphi_{G,1,1}$ is less or equal to the number $|V|$ of vertices.*

**Proof.** If $|V| \in \{0,1,2\}$, then the theorem is obvious. Assume that there is a periodic point $x \in \text{Per}(\varphi_{G,1,1})$ whose trajectory $a = \text{traj}(x)$ has minimal period $> 2$. By Lemma 4.7 no local trajectory is 2-periodic and by Theorem 4.13 $V$ contains a cycle with vertex set $C$ such that $a$ is phase-locked on $C$. Then $G - C$, the graph obtained from $G$ by removing the vertices in $C$ and all edges adjacent to them, is acyclic. Let $N$ be the set of vertices whose local trajectory is not $|C|$-periodic. Since $N \subseteq V \setminus C$ and $G$ does not contain a cycle disjoint from $C$, there must be a vertex $i$ in $N$ which has only predecessors in $V \setminus N$. But then the local trajectories of all of its predecessors have period $|C|$. Let $t \in Z(a_i)$, then $a_j(t + |C|) = a_j(t) = 0$ for all $j \in N^-(i)$. So $a_i(t + 1) = a_i(t + 1 + |C|) = 0$ and therefore $a_i$ has period $|C|$. Thus $N$ is empty and $x$ has period $|C| \leq |V|$. \qed

The following example shows that if the graph $G$ has more than two disjoint cycles, then $\varphi_{G,1,1}$ might exhibit periodic orbits whose period is larger than the number of vertices, even if the graph is Hamiltonian, i.e. it contains a cycle containing all vertices. Note that this is in contrast to the conjecture in [1]. Since the graph in the example contains only 17 vertices, it also provides a counterexample to the conjecture [2, Conjecture 1].

**Example 4.15.** Consider the threshold network with refraction $\varphi = \varphi_{G,1,1}$ on the graph $G = (V,E)$ with 17 vertices as depicted in Figure 6. Let $x \in \{0,1\}^V$ denote the state which is indicated in Figure 6. A computation of its trajectory $a = \text{traj}(x)$ shows that $x \in \text{Per}(\varphi)$ with minimal period 18.

Based on this explicit example, one can construct for each $m \in \mathbb{N}_0$ a Hamiltonian graph $G_m$ with $5(2m + 3) + 2$ vertices, such that $\varphi_{G_m,1,1}$ has periodic points with minimal period larger or equal to $6(2m + 3)$. More explicitly, the graph $G_m$
consists of a cycle \( i_0, i_1, \ldots, i_{3(2m+3)} = i_0 \), a cycle \( j_0, j_1, \ldots, j_{2(2m+3)} = j_0 \) and two additional nodes \( k, k' \) connected as shown in Figure 6, i.e. the edge set of \( G \) is \( \{(i_1, k), (j_1, k), (k, i_0), (i_1, k'), (j_1, k'), (k', j_0)\} \cup \{(i_{\ell+1}, i_\ell) \mid \ell \in \{0, \ldots, 6m+8\}\} \cup \{(j_{\ell+1}, j_\ell) \mid \ell \in \{0, \ldots, 4m+5\}\} \). Notice that both cycles are indexed against their direction as the figure shows. To get an initial state with period length \( 6(2m+3) \), take \( a, b \in \text{Per}_{(a,1)}(\sigma) \) with no two consecutive 1s, \( Z(a) = \{1, 2+2m, 5+4m\} + (9+6m)\mathbb{Z} \) and \( Z(b) = \{1, 2+2m\} + (6+4m)\mathbb{Z} \). Define \( x_k = x_{k'} = 1 \), \( x_{i_\ell} = a(\ell) \) for \( \ell \in \{1, \ldots, 6m+9\} \) and \( x_{j_\ell} = b(\ell) \) for \( \ell \in \{1, \ldots, 4m+6\} \).

5. Cyclic automata networks. We introduce a class of dynamical systems on graphs with the cyclic group \( \mathbb{Z}/n\mathbb{Z} \) of order \( n \) as local state space, with the property that a vertex \( i \) increases its state from \( x_i \) to \( x_i + 1 \) (mod \( n \)) whenever at least a certain fixed fraction of its predecessors are in state \( x_i + 1 \).

**Definition 5.1.** Let \( G = (V, E) \) be a directed graph, \( \theta \in [0, 1] \) and \( n \in \mathbb{N}, n \geq 2 \). The cyclic automata network on \( G \) with cycle length \( n \) and threshold \( \theta \) is the dynamical system \( \psi_{\theta, n} = \psi_{G, \theta, n} \) on \( G \) with local state space \( S = \mathbb{Z}/n\mathbb{Z} \) which is defined component-wise as

\[
[\psi_{G, \theta, n}(x)]_i := \begin{cases} 
  x_i + 1 & \text{if } |\mathcal{F}(i)| \geq \max(1, \theta|N^-(i)|) \\
  x_i & \text{otherwise}
\end{cases}
\]

for vertices \( i \in V \) and \( x \in S^V \) where \( \mathcal{F}(i) := \{j \in N^-(i) \mid x_j = x_i + 1\} \).

If the threshold \( \theta = 0 \) then a vertex \( i \) increases its state from \( x_i \) to \( x_i + 1 \) if at least one predecessor is in state \( x_i + 1 \). This subclass of cyclic automata networks \( \psi_{G, 0, n} \) is considered in [18].

We will show that for strongly connected graphs \( G \) and maximal threshold \( \theta = 1 \) or minimal threshold \( \theta = 0 \) the dynamical system \( \psi_{G, \theta, n} \) exhibits a cycle on which the local trajectories are phase-locked. The following example shows that for \( \frac{1}{3} < \theta \leq \frac{1}{2} \) there might exist periodic orbits on cycles which are not phase-locked.

**Example 5.2.** Choose a threshold \( \theta \in \left(\frac{1}{2}, \frac{2}{3}\right] \) and consider the cyclic automata network \( \psi = \psi_{G, \theta, 2} \) on the strongly connected graph \( G = (V, E) \) as depicted in Figure 7. Let \( x \in (\mathbb{Z}/2\mathbb{Z})^V \) denote the state where all vertices except vertex 1 and 11 have state 0, as indicated in Figure 7 and the first line of the table in Figure 8. A computation of its trajectory \( a = \text{traj}(x) \) shows that \( x \in \text{Per}(\psi) \) with minimal period 10.

Note that there is no phase-locked cycle for this trajectory, since the vertices in \( \{1, \ldots, 4\} \cup \{16, \ldots, 19\} \) are active 2-times during a period of 10 time steps and all other vertices are only active once during a period but each cycle in the graph contains vertices from both sets.

**Minimal threshold.** Consider \( \psi_{G, 0, n} \) with the lowest possible threshold \( \theta = 0 \). To define a partial order on the local trajectories we consider those times at which a local trajectory increases, which we call jumps.

**Definition 5.3 (Jumps).** For \( a \in (\mathbb{Z}/n\mathbb{Z})^2 \) we define the set of jumps of \( a \) as \( J(a) = \{ t \in \mathbb{Z} \mid a(t+1) = a(t) + 1 \} \).

Note that \( J(a) \) is in the set \( P_{2\mathbb{Z}} \) of all non-empty periodic sets of integers, as defined in Definition 4.8, if and only if \( a \) jumps periodically.
To define a partial order on periodically jumping local trajectories, i.e. those \( a \in (\mathbb{Z}/n\mathbb{Z})^2 \) with \( J(a) \in P_{2\mathbb{Z}} \), we use the difference sequence \( \Delta(A) : \mathbb{N} \to \mathbb{N} \) from Definition 4.8 for \( A = J(a) \in P_{2\mathbb{Z}} \).

In contrast to Section 4, however, we do not utilize the strict lexicographic order \( \prec_{\text{lex}} \) and double zeros as in Definition 4.10, but the strict reverse lexicographic order \( \prec_{\text{rev}} = \prec_{\text{lex}}^{-1} \) and jumps. Furthermore the construction uses the corresponding minimal distance representation \( R_{\prec_{\text{rev}}} \) as introduced in (1) in Definition 4.10 with \( \prec_{\text{lex}} \) replaced by \( \prec_{\text{rev}} = \prec_{\text{lex}}^{-1} \) and the function \( m_{\prec_{\text{rev}}} \) as introduced in (2). We then define

\[
a \prec b :\Leftrightarrow R_{\prec_{\text{rev}}}(J(a)) \prec_{\text{rev}} R_{\prec_{\text{rev}}}(J(b)).
\]

The induced partial order on \( (\mathbb{Z}/n\mathbb{Z})^2 \) is defined by \( a \preceq b :\Leftrightarrow (a \prec b \text{ or } a = b) \). This partial order is again shift-invariant.

The following lemma shows that \( \psi = \psi_{G,0,n} \) satisfies the assumptions of Theorem 3.4 on phase-locking on cycles.
Lemma 5.4. Let $G = (V, E)$ be a strongly connected graph, $\psi = \psi_{G,0,n}$, $x \in \text{Per}(\psi)$ and $a = \text{traj}(x)$. If $a$ is not constant, then the following statements hold:

(a) For all $j \in N^{-}(i)$ with $m_{\text{rev},(J(a_i))}(J(a_i)) \in J(\sigma(a_j))$ and $a_j(m_{\text{rev},(J(a_i))} - 1) = a_i(m_{\text{rev},(J(a_i))})$ we have $\sigma(a_j) \leq a_i$.

(b) Each vertex $i \in V$ has a predecessor $j \in V$ such that $\sigma(a_j) \leq a_i$.

Proof. First note, that if the local trajectory $a_i$ of a vertex $i$ is not constant, then the local trajectory of each of its successors $j$ cannot be constant either, since $x$ is a periodic state and thus $a_i$ must assume all values in $\mathbb{Z}/n\mathbb{Z}$ and so must $a_j$. Since $G$ is strongly connected, this implies that $J(a_i) \in P_{\mathbb{Z}}$ for all $i \in V$.

(a) We define $b_j = \sigma^{-m_{\text{rev},(J(a_i))}}(a_i)$ and $b_j = \sigma^{-m_{\text{rev},(J(a_i))}+1}(a_j)$. Then either (i) $\Delta(J(b_j)) \leq \Delta(J(b_i))$, or (ii) $\Delta(J(b_j)) > \Delta(J(b_i))$. In case (i), similarly as in the proof of Lemma 4.12(b) case (i), we get $\sigma(a_j) \leq a_i$. In case (ii) there exists $k \in \mathbb{N}$ such that $[\Delta(J(b_j))](k) < [\Delta(J(b_i))](k)$ and $[\Delta(J(b_j))](t) = [\Delta(J(b_j))](t)$ for all $t \in \{1, \ldots, k-1\}$. Similarly as in the proof of Lemma 4.12(b) case (ii), we define

$$t_1 := \sum_{k=1}^{k-1} [\Delta(J(b_j))](\ell) \quad \text{and} \quad t_2 := t_1 + [\Delta(J(b_j))](k).$$

Since $b_i(0) = b_j(0)$ we also have $b_i(t_1 + 1) = b_j(t_1 + 1) = b_j(t_2) = b_i(t_2 + 1)$ and $b_j(t_2 + 1) = b_j(t_2) + 1 = a_j(m_{\text{rev}}(J(a_i)) + t_2) + a_j(m_{\text{rev}}(J(a_i)) + t_2 - 1) + 1$. Therefore by the definition of $\psi$ one has $b_i(t_2 + 1) = a_i(m_{\text{rev}}(J(a_i)) + t_2 + 1) = a_j(m_{\text{rev}}(J(a_i)) + t_2) + 1 = b_i(t_2) + 1$ implying $t_2 \in J(b_i)$. This contradicts $[\Delta(J(b_j))](k) < [\Delta(J(b_i))](k)$.

(b) Let $t = m_{\text{rev}}(J(a_i))$. By the definition of $\psi$ there must be a predecessor $j$ of $i$ with $a_j(t) = a_i(t) + 1$. Let $t_1$ be the next point of time after $t - 1$ (i.e. $t_1 \geq t$) at which $a_j$ jumps. By definition of $\psi$ there is $s \in \{t+1, \ldots, t_1+1\} \cap J(a_i)$. Let $t_2$ be the previous point in time before $t$ at which $a_j$ jumps. Either $t_2 < t - 1$ and therefore $[R(J(a_i))](t) > [R(J(a_i))](1)$ implying $\sigma(a_j) < a_i$, or $t_2 = t - 1$ and thus $\sigma(a_j) \leq a_i$ by (a).

Theorem 3.4 in combination with Lemma 5.4 implies the following main result on cyclic automata networks with minimal threshold.

**Theorem 5.5** (Phase-locking of $\psi_{G,0,n}$ on cycles). Let $G$ be a strongly connected graph, $\psi = \psi_{G,0,n}$, $x \in \text{Per}(\psi)$ and $a = \text{traj}(x)$ which is not constant. Then there exists a cycle $i_0, \ldots, i_q = i_0$ in $G$ on which the trajectory $a$ is phase-locked and the local trajectories on the cycle are $q$-periodic and shifted along the cycle in the following sense

$$\forall k \in \{0, \ldots, q-1\}: \sigma(a_{i_k}) = a_{i_{k+1}}.$$

**Remark 5.6.** In general, the statement of Theorem 5.5 does not hold if the graph $G$ is not strongly connected as the following example for $\psi_{G,0,2}$ shows:

$$G := \begin{array}{c}
1 \\
\longrightarrow \\
2 \\
\longrightarrow \\
3
\end{array}$$

For $x = (0,0,1)$, the local trajectory for $x$ under $\psi_{G,0,2}$ of vertex 2 is alternating between 0 and 1 and is in particular not constant, but the graph does not even contain a cycle.
**Maximal threshold.** Consider \( \psi_{G,1,n} \) with the largest possible threshold \( \theta = 1 \).
To define the partial order on the local trajectories we use the jumps defined in Definition 5.3 together with the strict lexicographic order \( \prec_{\text{lex}} \) and corresponding minimal distance representation \( R \) and function \( m \) as in Definition 4.10. The strict partial order on jumping local trajectories, i.e. those \( a \in (\mathbb{Z}/n\mathbb{Z})^2 \) with \( J(a) \in \mathcal{P}_{2^\infty}, \) is then given by

\[
a < b : \iff R(J(a)) \prec_{\text{lex}} R(J(b)).
\]

The induced partial order on \( (\mathbb{Z}/n\mathbb{Z})^2 \) is defined by \( a \preceq b : \iff (a < b \text{ or } a = b) \).

The following lemma shows that \( \psi = \psi_{G,1,n} \) satisfies the assumptions of Theorem 3.4 on phase-locking on cycles.

**Lemma 5.7.** Let \( G = (V,E) \) be a strongly connected graph, \( \psi = \psi_{G,1,n}, x \in \text{Per}(\psi) \) and \( a = \text{traj}(x) \). If \( a \) is not constant, then the following statements hold:

(a) For \( t \in J(a_i) \), there is \( j \in N^-(i) \) with \( t \in J(\sigma(a_j)) \).

(b) For all \( j \in N^-(i) \) with \( m(J(a_j)) \in J(\sigma(a_j)) \) we have \( \sigma(a_j) \leq a_i \).

(c) Each vertex \( i \in V \) has a predecessor \( j \in V \) such that \( \sigma(a_j) \leq a_i \).

**Proof.**
(a) If \( t \in J(a_i) \) then \( a_{i}(t + 1) = a_{i}(t) + 1 = a_{j}(t) \) for all \( j \in N^-(i) \) by definition of \( \psi \). Assume for all \( j \in N^- (i) \) we had \( a_j(t) = a_j(t - 1) = 1 + a_j(t) \) then \( a_i(t - 1) = a_i(t) \) and \( a_i(t - 1) = 1 + a_i(t - 1) \). But this would imply \( a_i(t) = a_{j}(t - 1) - 1 + a_i(t) \). Therefore there is at least one \( j \in N^-(i) \) with \( a_j(t) = a_j(t - 1) + 1 \) and thus \( t \in J(\sigma(a_j)) \).

(b) We define \( b_j = \sigma^{-m(J(a_j))}(a_j) \) and \( b_j = \sigma^{-m(J(a_j))}(a_j) \). Then either (i) \( \Delta(J(b_j)) \leq \Delta(J(b_i)) \), or (ii) \( \Delta(J(b_j)) > \Delta(J(b_i)) \). In case (i), similarly as in the proof of Lemma 4.12(b) case (i), we get \( \sigma(a_j) \leq a_i \). In case (ii) there exists \( k \in \mathbb{N} \) such that \( \Delta(J(b_j))(k) > \Delta(J(b_i))(k) \) and \( \Delta(J(b_j))(\ell) = \Delta(J(b_i))(\ell) \) for all \( \ell \in \{1, \ldots, k - 1\} \). Similarly as in the proof of Lemma 4.12(b) case (ii), we define

\[
t_1 := \sum_{\ell = 1}^{k - 1} \Delta(J(b_j))(\ell) \quad \text{and} \quad t_2 := t_1 + \Delta(J(b_i))(k).
\]

With this definition we get \( b_j(t_1 + 1) = b_j(t_2) = b_j(t_2 + 1) = b_i(t_2 + 1) = b_i(t_2 + 1) = b_i(t_2 + 1) = b_i(t_2 + 1) + 1 \), implying \( b_i(t_2) = b_i(t_2) + 1 \), which yields a contradiction.

(c) Since \( a \) is not constant, the local trajectory of at least one vertex is not constant. Therefore none of the local trajectories of its predecessors is constant and thus all local trajectories are not constant, since \( G \) is strongly connected. Hence \( J(a_i) \neq \emptyset \). By (a) there is a predecessor \( j \) of \( i \) with \( m(J(a_j)) \in J(\sigma(a_j)) \) and by (b) we have \( \sigma(a_j) \leq a_i \).

\( \square \)

Theorem 3.4 in combination with Lemma 5.7 implies the following main result on cyclic automata networks with maximal threshold.

**Theorem 5.8** (**Phase-locking of \( \psi_{G,1,n} \) on cycles**). Let \( G \) be a strongly connected graph, \( \psi = \psi_{G,1,n}, x \in \text{Per}(\psi) \) and \( a = \text{traj}(x) \) which is not constant. Then there exists a cycle \( i_0, \ldots, i_q = i_0 \) in \( G \) on which the trajectory \( a \) is phase-locked and the local trajectories on the cycle are \( q \)-periodic and shifted along the cycle in the following sense

\[
\forall k \in \{0, \ldots, q - 1\} : \sigma(a_{i_k}) = a_{i_{k+1}}.
\]
6. Conclusion. Network models with refraction and excitation threshold received a considerable amount of attention in theoretical, as well as application-oriented publications. We develop basic notions for a theory of dynamical systems on graphs which allows for the use of tools from graph theory in combination with dynamical systems methods. A key idea which is similar to Lyapunov functions is encoded in the notion of order-induced graph. It turns out that the lack of structure in the finite state space can partly be compensated by introducing a partial order on the space of potential trajectories such that the order is compatible with the dynamics. We believe that this theoretical approach might be useful to extend the results in this paper to broader classes of applications. A main application which we address are phase-locked trajectories on cycles for general dynamical systems on graphs. We apply this general result to explicit threshold networks and cyclic automata networks. For both classes of dynamical systems we discuss different parameter values and construct taylor-made orders which are based on the lexicographic or reverse lexicographic order of words. Similar as for Lyapunov functions, it is not obvious how to construct an appropriate order, this depends crucially on the dynamics of the underlying system and the topology of the graph. We develop methods to address questions on upper bounds of minimal periods and provide an example which shows that the assumptions for these bounds are sharp in the sense that more than two disjoint cycles in a graph may cause the existence of trajectories with higher periods, in contrast to a known conjecture.

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