Algebraic stabilization of a $Q_1 - P_0$ Lagrangian formulation for frictional contact mechanics with hydraulically active fractures

Andrea Franceschini*, Nicola Castelletto, Joshua A. White, Hamdi A. Tchelepi

*Energy Resources Engineering, Stanford University, Stanford, United States
+Atmospheric, Earth, and Energy Division, Lawrence Livermore National Laboratory, United States

Abstract

Accurate numerical simulation of coupled fracture/fault deformation and fluid flow is crucial to the performance and safety assessment of many subsurface systems. In this work, we consider the discretization and enforcement of contact conditions at such surfaces. The bulk rock deformation is simulated using low-order continuous finite elements, while frictional contact conditions are imposed by means of a Lagrange multiplier method. We employ a cell-centered finite-volume scheme to solve the fracture fluid mass balance equation. From a modeling perspective, a convenient choice is to use a single grid for both mechanical and flow processes, with piecewise-constant interpolation of Lagrange multipliers, i.e., contact tractions and fluid pressure. Unfortunately, this combination of displacement and multiplier variables is not uniformly inf-sup stable, and therefore requires a stabilization technique. Starting from a macroelement analysis, we develop two algebraic stabilization approaches and compare them in terms of robustness and convergence rate. The proposed approaches are validated against challenging analytical two- and three-dimensional benchmarks to demonstrate accuracy and robustness. These benchmarks include both pure contact mechanics problems and well as problems with tightly-coupled fracture flow.

Keywords: Contact mechanics, Lagrange multipliers, Darcy fracture flow, stabilization

2010 MSC: 65N08, 65N12, 65N30

1. Introduction

To accurately simulate the geomechanical response of a subsurface system, such as an aquifer or reservoir, it is often important to model faults and fractures [1]. Phenomena such as micro-seismicity [2], fluid leakage [3], fault reactivation [4], and fracture propagation [5] depend strongly on coupled fluid-structure interaction. As a result, it often necessary to explicitly model complex hydromechanical behavior at these surfaces [6]. Experimental data confirms a strong dependence of fracture properties, like conductivity, on contact conditions [7–9]. The core of the modeling challenge is dealing with a lubricated frictional contact problem [10]. Specifically, we have fluid pressure acting as a normal forcing term on the surfaces of the fracture, while the conductivity of the fracture is a strong function of the effective aperture. This establishes a two-way and highly nonlinear coupling. A different but related challenge is to accurately model fracture propagation based on the resulting stress and strain fields near the fracture tip [11].

Many approaches have been developed over the years to solve the contact mechanics problem, and an extensive review is beyond the scope of this work. The largest difference between methods lies in how the discontinuity is fundamentally represented. A first class of models uses a Discrete Fracture Model (DFM), whereby a conformal computational grid is introduced, and the conservation equations are discretized using this grid. These methods have the advantage that standard discretization techniques may be directly applied. A potential drawback is that a conformal mesh for complex fracture networks may require distorted or excessively-refined meshes. With DFM-based models,
the discontinuity can be explicitly discretized using zero-thickness interface elements \[6\] \[12,14\]. In some models, thin layers of continuum finite elements with plastic behavior are used instead to mimic the fracture rheology \[15,18\].

A second class employs an embedding strategy, in which a continuum discretization scheme is enriched to capture discontinuities cutting through continuum elements. For example, Embedded Discrete Fracture Models (EDFM) introduce additional, local degrees of freedom to capture opening and sliding modes \[19,22\] while Extended Finite Elements Methods (XFEM) introduce global degrees of freedom for this purpose \[20,23,26\]. A third class discards explicit discontinuity surfaces altogether, and instead represents a fracture via a regularized (smooth) field using a continuum discretization. This latter class includes Phase Field and damage-mechanics-based approaches \[27,30\].

In this work, we represent fractures explicitly using a conforming discretization. For these elements, the Karush-Kuhn-Tucker (KKT) conditions \[31\] for normal impenetrability and frictional compatibility must be enforced. The two most common classes of methods used to fulfill these conditions are penalty \[13,14,32,33\] and Lagrange multiplier methods \[34,38\]. In penalty methods, constraints are satisfied in an approximate way using stiff “springs” connecting the two surfaces of a fracture, and no additional degree of freedom are introduced. However, penalty methods suffer from several drawbacks, including ill-conditioning of the resulting linear systems and a strong dependence of solution quality on the penalty parameters \[39\,40\]. In Lagrange multiplier-based methods, the KKT conditions are instead imposed directly. This advantage, however, comes at the cost of adding new global primary variables. Moreover, the resulting matrices have a generalized saddle-point structure that requires specialized solvers for efficiency and scalability \[41,43\]. Other widely used techniques include Nitsche’s method \[44,47\], regularized and augmented Lagrange multipliers \[48,51\] and mortar methods \[52,55\]. In particular, mortar methods were originally developed to allow matching of different regions in a domain decomposition framework. They provide useful flexibility in dealing with large deformations and non-conforming discretization of the contact area \[56,57\].

In this paper, we focus entirely on a Lagrange multiplier approach to impose the normal and frictional compatibility conditions on the fracture surfaces. The displacement field in the bulk is approximated by lowest-order continuous finite elements. Using lubrication theory \[7\], the fracture flow discretization relies on a cell-centered finite volume approach using a two-point flux approximation (TPFA) scheme \[58\] for the numerical flux. We assume that the rock matrix is impermeable, though an extension of the proposed strategy to handle a poroelastic bulk is clear. Motivated by the coupling between fracture contact and flow processes, we use piecewise-constant equal-order interpolation of Lagrange multipliers, i.e., contact tractions, and fluid pressure. This is the natural and convenient choice given the mixed finite-element / finite-volume scheme adopted. Unfortunately, this combination of displacement and contact traction/pressure variables is not uniformly inf-sup stable \[59\] Section 3.1] and requires the implementation of a stabilization methodology.

To rectify this deficiency, possible remedies include enriching the displacement space with locally supported bubble functions \[60,63\] or using a coarser mesh for the traction/pressure space \[64\]. Here, we instead introduce a stabilizing modification to the mass balance and constraint equations. We start by analyzing the pure contact mechanics problem, and explore several stabilization alternatives. First, we derive a local traction jump stabilization that depends on a stabilization parameter \(\alpha\) defined locally at the macroelement level, following the macroelement analysis \[65,68\] originally proposed for the Stokes equation. Second, to circumvent difficulties associated with the definition of \(\alpha\) in the presence of distorted grids and severe material heterogeneity, we develop an algebraic variant of the local traction jump stabilization technique and then generalize it to an algebraic global traction jump stabilization. Finally, we extend the stabilization procedure to problems that include fluid flow.

The paper is organized as follows. In Section 2, we present the general problem statement, providing the strong form of the governing equations. In Section 3, the discretized form and its linearization are provided. We also address the nonlinear optimization problem associated with the imposition of the KKT conditions. In Section 4, we describe three related stabilization strategies. We discuss the relative advantages of each, before settling on a global algebraic stabilization approach as our recommended strategy. We test the performance of this method on mechanical contact problems in Section 5 and hydromechanical contact problems in Section 6. We then conclude the paper with a few remarks regarding future work.

2. Problem statement

A fracture in a continuous medium is modeled as an internal discontinuity where additional constraints must be enforced. These constraints depend on the local state of the discontinuity—that is, whether the fracture is opening,
sliding, or sticking. Inside the fracture, a single-phase fluid is present and a mass balance equation—expressed at the Darcy scale—is solved for pressure. The momentum and mass balance equations are then coupled through the fluid pressure and fracture aperture. For simplicity of exposition, we assume that the matrix is impermeable. The extension to a poroelastic medium, however, is clearly of interest in practical applications and could be included in a more general formulation [69]. Similarly, extensions to multiphase flow and non-isothermal conditions may be required for some reservoir applications.

We consider an elastic, closed, polyhedral domain \( \Omega = \Omega \cup \partial \Omega \subset \mathbb{R}^3 \), with \( \Omega \) an open set, \( \partial \Omega \) its boundary, and \( n_\Omega \) its outer normal vector. As usual, the boundary is subdivided into two non-overlapping subsets where \( \partial \Omega = \partial \Omega_u \cup \partial \Omega_r \) such that \( \partial \Omega_u \cap \partial \Omega_r = \emptyset \), where Dirichlet and Neumann boundary conditions for displacement and traction are applied (see Fig. 1a). From a mathematical standpoint, a fracture is described as an internal boundary \( \Gamma_f \) embedded in \( \Omega \), consisting of two overlapping surfaces \( \Gamma_f^- \) and \( \Gamma_f^+ \) as shown in Fig. 1b. The fracture orientation is characterized by a unit vector orthogonal to the fracture plane. By convention, we choose \( n_f = n_f^- = n_f^+ \). On this lower-dimensional domain, the pressure field is defined. Let \( \Gamma_f^- = \Gamma_f^- \cup \partial \Gamma_f^- \) denote the closed domain occupied by the fracture, with \( \Gamma_f^- \) a two-dimensional (2D) open surface and \( \partial \Gamma_f^- \) a one-dimensional (1D) curve defining its boundary. The fracture boundary is subdivided into two non-overlapping subsets, \( \partial \Gamma_f^- = \partial \Gamma_{f,q} \cup \partial \Gamma_{f,p} \) such that \( \partial \Gamma_{f,q} \cap \partial \Gamma_{f,p} = \emptyset \), corresponding to the position of Dirichlet and Neumann boundary conditions for pressure and flux fields, respectively, as shown in Fig. 1c. We define \( m_f \) as the outer normal vector for \( \Gamma_f^- \). Finally, let \( T = (0, t_{\text{max}}) \) denote the time domain of interest.

We assume quasi-static conditions and infinitesimal strains. The fluid is taken to be incompressible, and we neglect body force and buoyancy effects. The strong form of initial boundary value problem (IBVP) can then be stated as [10, 70, 71]:

\[
\begin{align*}
-\text{div} \sigma(u) &= 0 & \text{in } \Omega \times T & \text{(linear momentum balance)}, \\
\dot{g}_N(u) + \text{div} q(u, p) &= q_s & \text{in } \Gamma_f \times T & \text{(mass balance)}, \\
\sigma(u) \cdot n_f - p n_f &= 0 & \text{on } \Gamma_f \times T & \text{(traction balance on the fracture)}, \\
u &= \bar{u} & \text{on } \partial \Omega_u \times T & \text{(prescribed boundary displacement)}, \\
\sigma(u) \cdot n &= \bar{i} & \text{on } \partial \Omega_r \times T & \text{(prescribed boundary traction)}, \\
p = \bar{p} & \text{on } \Gamma_{f,p} \times T & \text{(prescribed boundary pressure)}, \\
q \cdot m_f &= \bar{q} & \text{on } \partial \Gamma_{f,q} \times T & \text{(prescribed boundary flux)}, \\
u|_{t=0} &= \bar{u}_0 & \text{in } \Omega & \text{(initial displacement)}, \\
p|_{t=0} &= \bar{p}_0 & \text{in } \Gamma_f & \text{(initial pressure)},
\end{align*}
\]
subject to the constraints

\[ t_N = t \cdot n_f \leq 0 \quad \text{on } \Gamma_f \times T \quad \text{(normal contact conditions)}, \]

\[ g_N = [u] \cdot n_f \geq 0 \quad \text{on } \Gamma_f \times T, \]

\[ t_N g_N = 0 \quad \text{on } \Gamma_f \times T, \]

\[ \|t_T\|_2 - \tau_{\text{max}}(t_N) \leq 0 \quad \text{on } \Gamma_f \times T. \quad \text{(Coulomb frictional contact conditions)}, \]

\[ \dot{g}_T \cdot t_T - \tau_{\text{max}}(t_N) \|\dot{g}_T\|_2 = 0 \quad \text{on } \Gamma_f \times T. \]

Here, known boundary and initial conditions are given as \( \tilde{u} : \partial \Omega_0 \times T \rightarrow \mathbb{R}^3, \tilde{t} : \partial \Omega_\Gamma \times T \rightarrow \mathbb{R}^3, u_0 : \bar{\Omega} \rightarrow \mathbb{R}^3, q_f : \Gamma_f \times T \rightarrow \mathbb{R}, \rho : \partial \Gamma_f \times T \rightarrow \mathbb{R}, \dot{q} : \partial \Gamma_f \times T \rightarrow \mathbb{R}, \) and \( p_0 : \bar{\Gamma}_f \times T \rightarrow \mathbb{R}. \)

The following symbols, variables, and constitutive relationships are also introduced:

- \( \sigma(u) = C : \nabla^t u \) is the Cauchy stress tensor, which is related to the displacement vector \( u \) by the fourth-order elasticity tensor \( C, \) with \( \nabla^t \) the symmetric gradient operator;

- \( q(u, p) = - \frac{C(u)}{\mu} \nabla p \) is the fluid volumetric flux in the fracture domain—assuming laminar flow and validity of Darcy’s law [7]—with \( \nabla p \) the fluid pressure gradient, \( \mu \) the fluid viscosity (constant), and \( C_f \) the isotropic fracture hydraulic conductivity modeled as in [13]:

\[ C_f = C_{f0} + \frac{g_N}{12}. \]

Note that \( C_{f0} \) captures the conductivity associated with two irregular surfaces that are in contact [72]. This allows fluid to flow and pressure to propagate even if the fracture is nominally closed. From a physical viewpoint, the volume between two rough surfaces in contact is non-zero and fluid may infiltrate between asperities. For simplicity, here we assume a constant closed conductivity, though for some applications a normal-stress dependent model may be preferred.

- \( t = \sigma \cdot n_f^+ = -\sigma \cdot n_f^- = (t_N n_f + t_T) \) is the traction vector over \( \Gamma_f, \) with \( t_N \) and \( t_T = (t_{n_1} m_1 + t_{n_2} m_2) \) its normal and tangential component, respectively, with respect to the local reference system shown in Fig. [11];

- \( \tau_{\text{max}} = c - t_N \tan(\theta) \) is the limit value for the modulus of \( t_T \) according to the Coulomb criterion, with \( c \) and \( \theta \) the cohesion and friction angle, respectively;

- \( [\cdot] \) denotes the jump of a quantity across \( \Gamma_f, \) with \([u] = (u|_{\Gamma_f^+} - u|_{\Gamma_f^-}) = (g_N n_f + g_T)f \) the relative displacement across \( \Gamma_f, \) where \( g_N \) and \( g_T \) are normal and tangential components, respectively, and \( u|_{\Gamma_f^+} \) and \( u|_{\Gamma_f^-} \) are the restrictions of \( u \) on \( \Gamma_f^+ \) and \( \Gamma_f^-; \)

For additional details regarding the governing formulation, we refer the reader to [10, 70, 71]. Note that throughout this work, we will use subscripts \( N \) and \( T \) to identify the normal and tangential components of a vector quantity with respect to the discontinuity \( \Gamma_f. \) Specifically, given the vector \( v \in \mathbb{R}^3, \) we have

\[ v = v_N n_f + v_T, \]

\[ v_N = v \cdot n_f, \]

\[ v_T = (1 - n_f \otimes n_f) \cdot v, \]

with \( \otimes \) the dyadic product.

Note that we consider the static Coulomb law. Therefore, in a discrete setting the tangential velocity \( g_T \) in (1n) can be replaced with the tangential displacement increment \( \Delta g_T. \) This has to be done at every timestep if an implicit time-marching scheme is used. From now on, \( g_T \) will be replaced by \( \Delta_n g_T, \) where \( \Delta_n \) denotes the backward difference operator such that \( \Delta_n(\cdot) = (\cdot)_{n} - (\cdot)_{n-1} \) with the subscript \( n \) denoting the current discrete time level \( t_n. \) The Coulomb frictional contact conditions Eqs. (1m)–(1n) can then be rewritten as:

\[ \|t_T\|_2 < \tau_{\text{max}} \implies \Delta_n g_T = 0, \]
\[ ||t_{f,r}\|_2 = \tau_{\text{max}} \implies t_{f,r} = t_{f}^r(u, t, N_{h}) = \tau_{\text{max}}(t, N_{h}) \frac{\Delta_{t}G_{r}}{||\Delta_{t}G_{r}\|_2}. \] (4b)

In our framework, the fracture is explicitly modeled according to a Discrete Fracture Model [13, 23], with \( \Gamma_{f} \) encompassing the whole region where opening or contact may take place at any \( t \in \mathbb{T} \) [71]. That is, we assume \( \Gamma_{f} \) is fixed and does not propagate. The only unknown is then its partitioning into stick, slip and open patches at a given point in time, \( \Gamma_{f} = \Gamma_{f}^{\text{stick}} \cup \Gamma_{f}^{\text{slip}} \cup \Gamma_{f}^{\text{open}} \). These three modes are associated with behavior regimes, namely:

- **Stick mode** on \( \Gamma_{f}^{\text{stick}} \): the discontinuity is fully closed and compressed with the Coulomb criterion satisfied, i.e., \( t_{N} < 0 \) (Eq. 1) and \( ||t_{f}||_2 < \tau_{\text{max}}(t, N_{h}) \) (Eq. 1in). The three components of the traction are unknown and such that no relative movement is allowed between \( \Gamma_{f}^{+} \) and \( \Gamma_{f}^{-} \). The conductivity is constant and equal to \( C_{f,0} \). This implies linear, steady-state flow behavior within the stick portion.

- **Slip mode** on \( \Gamma_{f}^{\text{slip}} \): the discontinuity is compressed, but a slip displacement \( g_{f} \) between \( \Gamma_{f}^{+} \) and \( \Gamma_{f}^{-} \) is allowed. Only the normal traction component \( t_{N} \) is unknown, the tangential component having magnitude \( ||t_{f}^T||_2 = \tau_{\text{max}}(t, N_{h}) \) and direction collinear with \( \hat{g}_{f} \). The conductivity behavior is the same as for the stick mode, as no slip-induced dilation is modeled. The flow follows, as before, a linear, steady state model.

- **Open mode** on \( \Gamma_{f}^{\text{open}} \): \( \Gamma_{f}^{+} \) and \( \Gamma_{f}^{-} \) are not in contact and a free relative displacement \( [\mathbf{u}] \) is allowed for. The traction is known and equal to the zero vector in \( \mathbb{R}^{3} \). In this case, the conductivity is related to the opening (see Eq. (2)) and the fluid behavior can be described as a nonlinear transient flow, with linear storage and nonlinear conductivity.

The numerical strategy to resolve this partitioning is described in the next section.

3. Numerical model

3.1. Discrete formulation

We solve numerically the IBVP (1) using a mixed finite-element/finite-volume approach for the spatial discretization combined with a fully-implicit time-marching scheme. In particular, the contact mechanics problem is addressed using the saddle point formulation [10, 70, 71] where traction vectors acting on \( \Gamma_{f} \) are introduced as additional primary variables serving as Lagrange multipliers to enforce normal and frictional contact constraints. The simulation of the fluid flow through the discontinuity relies on a finite volume method [58].

We introduce a triangulation \( \mathcal{T} \) of the domain consisting of nonoverlapping hexahedral cells that conforms to the discontinuity surface, i.e., \( \overline{\Omega} = \bigcup_{\mathcal{T} \in \mathcal{T}} \bar{\overline{T}} \). Let us define \( \mathcal{T}_{f} \) as the set of quadrilateral faces \( \varphi \) in the triangulation such that \( \mathcal{T}_{f} = \bigcup_{\varphi \in \mathcal{T}_{f}} \varphi \). The edge normal unit vector to face \( \varphi_{K} \) in the discontinuity plane is denoted by \( \mathbf{m}_{K} \), with \( K \) the face global index. Let \( \mathcal{E}_{f} = (\mathcal{E}_{in} \cup \mathcal{E}_{slip} \cup \mathcal{E}_{ref}) \) be the set of edges belonging to faces defining \( \mathcal{T}_{f} \), with \( \mathcal{E}_{in} \) (respectively \( \mathcal{E}_{slip}, \mathcal{E}_{ref} \)) the set of edges included in \( \Gamma_{f}^{\text{stick}} \) (respectively \( \partial \mathcal{T}_{f}^{\text{r}}, \partial \mathcal{T}_{f}^{s} \)). An edge in \( \mathcal{E}_{in} \) shared by faces \( \varphi_{K} \) and \( \varphi_{L} \) is denoted as \( e_{K_L} \), with face global indices such that \( K < L \). Similarly, an edge in \( \mathcal{E}_{slip} \cup \mathcal{E}_{ref} \) belongs to a unique face \( \varphi_{K} \) and is simply denoted as \( e_{K} \). A unique orientation on \( \Gamma_{f}^{\text{stick}} \) is associated with each edge in \( \mathcal{E}_{in} \) through a unit vector \( \mathbf{m}_{e} \). We set \( \mathbf{m}_{e} = \mathbf{m}_{K} \) for any edge \( e_{K_L} \in \mathcal{E}_{in} \) and \( e_{K} \in \mathcal{E}_{slip} \cup \mathcal{E}_{ref} \). Finally, let \( \tilde{\mathbf{q}} \) denote a numerical flux approximating the volume flow rate per unit length through \( e \in \mathcal{E}_{f} \) such that \( \tilde{\mathbf{q}} \approx \int_{e} \mathbf{q}(u, p) \cdot \mathbf{m}_{e} \, dl \).

Discretizing the time interval \( \mathbb{T} \) into \( \mathcal{T}_{n} \) subintervals of size \( \Delta_{n} t = (t_{n} - t_{n-1}) \), the mesh-dependent fully discrete weak form of (1) is: find \( [u_{h}, r_{h}, p_{h}] \in \mathcal{U}^{n} \times \mathcal{M}^{n}(\mathcal{T}_{n}) \times \mathcal{P}^{n} \) such that for all \( [\eta, \mu, \chi] \in \mathcal{U}^{n} \times \mathcal{M}^{n}(\mathcal{T}_{n}) \times \mathcal{P}^{n} \)

\[
\mathcal{R}_{u} = \int_{\Omega} \nabla \cdot \sigma_{u} \, dV + \int_{\mathcal{T}_{f}} ||\eta_{f}|| (r_{h}^{f} - p_{h}^{f}) \, dA + \int_{\mathcal{T}_{f}} \eta_{f} \cdot r_{h}^{f} \, dA - \int_{\partial \mathcal{D}_{e}} \eta_{f} \cdot \tilde{t}_{f} \, dA = 0, \quad (5a)
\]

\[
\mathcal{R}_{r} = \int_{\mathcal{T}_{f}} (r_{h}^{f} - \mu_{\mathcal{T}}) \cdot \Delta_{t} G_{r} \, dA + \int_{\mathcal{T}_{f}} f_{r}^{h} \cdot \Delta_{t} G_{r} \, dA - \int_{\partial \mathcal{D}_{e}} \eta_{f} \cdot \tilde{t}_{f} \, dA \geq 0, \quad (5b)
\]

\[
\mathcal{R}_{p} = \int_{\mathcal{T}_{f}} \chi_{\mathcal{D}} \Delta_{t} G_{r} \, dA - \sum_{e_{\mathcal{D}} \in \mathcal{E}_{\mathcal{D}}} \left( \chi_{\mathcal{D}e_{\mathcal{D}}} - \chi_{\mathcal{D}e_{\mathcal{D}}}^0 \right) \tilde{\mathbf{q}}_{e_{\mathcal{D}}} + \sum_{e_{\mathcal{D}} \in \mathcal{E}_{\mathcal{D}}} \chi_{\mathcal{D}e_{\mathcal{D}}}^0 \tilde{\mathbf{q}}_{e_{\mathcal{D}}} \int_{\mathcal{T}_{f}} \chi_{\mathcal{D}e_{\mathcal{D}}} \, dA = 0, \quad (5c)
\]
with \( n \in \{1, 2, \ldots, n_T\} \). Here, \( \mathcal{U}^{t} \) and \( \mathcal{U}^{h}_t \) are finite dimensional affine and linear functional spaces, \( \mathcal{M}^h(\mathcal{O}_{\Omega_u}) \) is a solution dependent convex set \([59]\), and \( \mathcal{P}^h \) is a finite dimensional linear functional space. In particular,

\[
\mathcal{U}^t := \left\{ \eta \mid \eta \in \left[ C^0(\mathcal{O}_T) \right]^3, \eta = \bar{u} \text{ on } \partial \Omega_u, \eta|_{\tau} \in \mathcal{Q}_1(\tau) \forall \tau \in \mathcal{T} \right\},
\]

(6a)

\[
\mathcal{U}^{h}_t := \left\{ \eta \mid \eta \in \left[ C^0(\mathcal{O}_T) \right]^3, \eta = 0 \text{ on } \partial \Omega_u, \eta|_{\tau} \in \mathcal{Q}_1(\tau) \forall \tau \in \mathcal{T} \right\},
\]

(6b)

\[
\mathcal{M}^h(\mathcal{O}_{\Omega_u}) := \left\{ \mu \mid \mu \in \left[ L^2(\mathcal{O}_T) \right]^3, \mu|_{\varphi} \in \mathcal{P}_0(\varphi) \forall \varphi \in \mathcal{F}_f, \mu_{|\mathcal{E}_t} \leq 0, \|\mu_t\|_2 \leq \tau_{\text{max}}(\mathcal{O}_{\Omega_u}) \right\},
\]

(6c)

\[
\mathcal{P}^h := \left\{ x \mid x \in L^2(\mathcal{T}_f), x|_{\varphi} \in \mathcal{P}_0(\varphi) \forall \varphi \in \mathcal{F}_f \right\},
\]

(6d)

where \( C^0(\mathcal{O}_T) \) and \( L^2(\mathcal{O}_T) \) denote the space of continuous and square Lebesgue-integrable functions on \( \mathcal{O}_T \) and \( \Omega_T \). \( \mathcal{Q}_1(\tau) = \{ Q_1(\tau) \}^3 \), \( Q_1(\tau) = \text{span} \{1, x, y, z, xy, xz, yz\} \)-i.e., the mapping to \( \tau \) of the space of trilinear polynomials on the unit cube \([0, 1]^3 \) in \( \mathbb{R}^3 \), \( \mathcal{P}_0 = \{ P_0 \}^3 \), and \( \mathcal{P}_0 \) is the space of piecewise constant functions.

Let \( \{ \eta|_{\varphi} \}_{\varphi \in \mathcal{E}_t} \) be the standard vector nodal basis functions for the global finite element space of continuous piecewise-\([Q_1]\) functions associated with \( \mathcal{T} \), with \( \mathcal{N}_u \) and \( \mathcal{N}_u \) the set of indices of basis function vanishing on \( \partial \Omega_u \) and having support on \( \partial \Omega_u \), respectively. Note that \( \text{card}(\mathcal{N}_u) + \text{card}(\mathcal{N}_u) \) is equal to three times the number of vertices in \( \mathcal{T} \). Let \( \{ x_{\varphi} \}_{\varphi \in \mathcal{E}_t} \) be the characteristic function of the \( \tau \)-th face in \( \mathcal{F}_f \) such that \( x_{\varphi}(x) = 1 \), if \( x \in \varphi \), \( x(\tau) = 0 \), if \( x \notin \varphi \), and \( \mathcal{N}_p = \{ 1, \ldots, \text{card}(\mathcal{F}_f) \} \). Let \( \{ \mu|_{\varphi} \}_{\varphi \in \mathcal{E}_t} \) be the piecewise-constant vector basis for \( \mathcal{M}^h(\mathcal{O}_{\Omega_u}) \), with \( \mathcal{N}_p = \{ 1, \ldots, 3 \} \) - card(\mathcal{F}_f) \}-namely, the three basis functions \( x_{\varphi}(x) n_j(x), x_{\varphi}(x) m_j(x), \) and \( x_{\varphi}(x) m_j(x) \) associated to each face \( \varphi \in \mathcal{F}_f \). Discrete approximations for the displacement, traction, and pressure can then be expressed as

\[
\mathbf{u}_h(x) = \sum_{i \in \mathcal{N}_u} \eta_i(x) \mathbf{u}_{i,n} + \sum_{i \in \mathcal{N}_u} \eta_i(x) \mathbf{u}_{i,h}, \quad \mathbf{t}_h(x) = \sum_{j \in \mathcal{N}_p} \eta_j(x) \mathbf{t}_{j,n}, \quad \mathbf{p}_h(x) = \sum_{k \in \mathcal{N}_p} \eta_k(x) \mathbf{p}_{k,n}.
\]

The unknown nodal displacement components \( \{ u_{n} \} \), face-centered traction components \( \{ t_{n} \} \), and face-centered pressures \( \{ p_{n} \} \) at time level \( t_n \) are gathered in algebraic vectors \( \mathbf{u}_n, \mathbf{t}_n, \) and \( \mathbf{p}_n \). We emphasize that, at the right-hand side of the expression for \( \mathbf{u}_h \), the first sum represents an approximate displacement solution of the IBVP \([1]\) satisfying homogeneous prescribed displacement, whereas the second sum is the discrete extension by zero (to the degrees of freedom) of the boundary datum \( \bar{u} \) over \( \partial \Omega_u \). Hence, \( \{ \eta|_{\varphi} \}_{\varphi \in \mathcal{E}_t} \) is a basis for \( \mathcal{U}^{h}_t \). Finally, the numerical flux is computed through the classical two point flux approximation (TPFA) \([58]\):

\[
\tilde{\mathbf{u}}_h = \tilde{\mathbf{u}}_h^{t} = \begin{cases}
-T_{KL}(\mathbf{u}_h^{t})(p_{L,n} - p_{K,n}), & \text{if } \varepsilon \in \varepsilon_{K,L} \in \mathcal{E}_{\text{int}}, \\
-T_{KL}(\mathbf{u}_h^{t})(p_{R,n} - p_{K,n}), & \text{if } \varepsilon \in \varepsilon_{K,L} \in \mathcal{E}_{\text{p}}, \\
|\varepsilon| \tilde{\mathbf{q}}_{\varepsilon}, & \text{if } \varepsilon \in \varepsilon_{K,L} \in \mathcal{E}_{\text{q}},
\end{cases}
\]

(8)

where \( T_{KL} = (T_{KL} + T_{KL})/(T_{KL} + T_{L}) \) is the harmonic average of the one-sided transmissibility \( T_{KL} \) and \( T_{L} \), \( \tilde{\mathbf{q}}_{\varepsilon} \) and \( \tilde{\mathbf{q}}_{\varepsilon} \) are the mean values of prescribed boundary pressure and flux over \( \varepsilon \), and \( |\varepsilon| \) is the length of \( \varepsilon \). In particular, the one-sided transmissibility is expressed as the product of a nonlinear \( (C_{f,|\varphi|}) \) and a constant \( (\tilde{T}_\beta) \), respectively

\[
\mathbf{T}_\beta = C_{f,|\varphi|} \tilde{T}_\beta, \quad C_{f,|\varphi|} = C_{f,\beta} + \frac{1}{|\varphi|} \int_{\varphi} \frac{g_{\lambda,n}}{12} \mu \mathbf{A} \quad \tilde{T}_\beta = \frac{1}{\beta} \left| \mathbf{x}_e - \mathbf{x}_f \right| \mathbf{d}_\beta, \quad \beta = [K, L],
\]

(9)

with \( \mathbf{x}_e \) a collocation point associated with each edge in \( \mathcal{E}_f \) —used to impose point-wise pressure continuity between faces sharing that edge—and \( \mathbf{x}_f \) and \( |\varphi| \) the barycenter and area of face \( \varphi \). Notice that \( C_{f,|\varphi|} \) is the mean value of the fracture conductivity over \( \varphi \). The nonlinear dependence of the hydraulic conductivity on normal displacement jump makes the transition from a closed to open state even more challenging.

**Remark 1.** Different strategies can be used to choose the collocation point \( \mathbf{x}_e \). Following \([74]\), in our implementation we select \( \mathbf{x}_e \) as the intersection of the edge \( \varepsilon_{K,L} \) and the line connecting the barycenters of faces \( \varphi_e \) and \( \varphi_f \). For a boundary edge \( \varepsilon_K \in \mathcal{E}_p \cup \mathcal{E}_q, \mathbf{x}_e \) is chosen as the orthogonal projection of \( \mathbf{x}_K \).

**Remark 2.** With a slight abuse of notation, the reader may observe that we could have written quantities such as \( (p_{L} - p_{K}) \) using the jump notation \( [p]_{KL} \). We prefer the former as such quantities, stemming from the finite volume logic, are not actually the jump in the trace of a field at a surface. Practically speaking, however, the two are identical, and some may prefer the elegance of the jump notation.
3.2. Solution strategy

The discrete form of the IBVP (1) based on the weak form (5) consists of a nonlinear system of equations and inequalities in the unknowns $u$, $t$, and $p$. However, if we postulate that active contact regions $\Gamma_{\text{f},n}^{\text{stick}}$ and $\Gamma_{\text{f},n}^{\text{slip}}$ are known at $t_n$, the inequality (5c) can be replaced with a variational equality:

$$
\mathcal{R}_n = \int_{\Gamma_{\text{f},n}^{\text{stick}}} \mu N \mathbf{g} \cdot \mathbf{g}_n \, dA + \int_{\Gamma_{\text{f},n}^{\text{slip}}} \mu_T \cdot \mathbf{g}_T \, dA + \int_{\Gamma_{\text{f},n}^{\text{open}}} \mu_T \cdot (\mathbf{t}^n_{f,n} - \mathbf{t}^n_{f,n}) \, dA + \int_{\Gamma_{\text{f},n}^{\text{open}}} \mu \cdot \mathbf{t}^n \, dA = 0,
$$

where the four integrals correspond to the weak enforcement of the: (i) impenetrability condition on $\Gamma_{\text{f},n}^{\text{stick}} \cup \Gamma_{\text{f},n}^{\text{slip}}$, (ii) no slip condition on $\Gamma_{\text{f},n}^{\text{stick}}$, (iii) known tangential traction on $\Gamma_{\text{f},n}^{\text{slip}}$ given by Eq. (10), and (iv) zero traction condition on $\Gamma_{\text{f},n}^{\text{open}}$. Introducing expressions (7) and (8) into (5a), (10), and (5c) then allows for writing the discrete problem as a nonlinear system of equations that can be solved for the latest solution vectors $u_n$, $t_n$, and $p_n$,

$$
\begin{align*}
\mathbf{r}_n(u_n, t_n, p_n) &= 0, \\
\mathbf{r}_s(u_n, u_{n-1}, t_n) &= 0, \\
\mathbf{r}_p(u_n, u_{n-1}, p_n) &= 0,
\end{align*}
$$

with $u_{n-1}$ the known discrete displacement solution from the previous timestep. To advance one timestep, since the partition of $\Gamma_{\text{f},n}$ into $\Gamma_{\text{f},n}^{\text{stick}}$, $\Gamma_{\text{f},n}^{\text{slip}}$, and $\Gamma_{\text{f},n}^{\text{open}}$ at $t_n$ is of course not known a priori, an iterative procedure that includes a Newton method-based solver applied to residual equations (11) is needed.

In this work, we solve the contact problem using an active-set strategy, a numerical optimization technique employed in quadratic programming (75, 76). Algorithm 3.1 summarizes the sequence of steps of an active-set algorithm applied to the contact problem. From a practical viewpoint, first we assign an initial status (lines 2–6) to all the Lagrange multipliers and solve the discrete nonlinear problem (11). We highlight that there are just two states, active and inactive, for both normal and frictional contact. The initial state is the previous time step solution, whenever available, or the stick state, i.e., $\Gamma_{\text{f},0}^{\text{stick}} = \Gamma_f$, at the beginning of the simulation. Once the nonlinear problem is solved, the status of all Lagrange multipliers is checked and a new subdivision into stick, slip, and open portions is identified. If at least one multiplier changes status, we solve a new nonlinear system and check the consistency of the outcome again. The procedure stops when the starting subdivision of $\Gamma_f$ is consistent with the solution of the nonlinear problem. Figure 5 provides an illustration of the resulting nonlinear convergence profile, with three consistency checks required before the final accepted solution.

Remark 3. We note that there are no theoretical guarantees for the active-set convergence of the discrete version of the IBVP (1). It can occur that the exact solution may lie between two discrete solutions, i.e., two similar but different subdivisions of $\Gamma_f$. In these cases, one solution must be chosen. In this work, we select the one with the smaller internal energy.

**Algorithm 3.1 Active-set strategy**

1: $\ell \leftarrow 0$ ★ initialize iteration count
2: if $t_n \geq t_0$ then
3: $\Gamma_{f,n}^{r} = \Gamma_{f,(n-1)}^{r}$ ★ set Lagrange multipliers to previous timestep status
4: else
5: $\Gamma_{f,n}^{r} = \Gamma_{f,n}^{\text{stick}}$ ★ set initial discontinuity status to stick
6: end if
7: repeat
8: $\ell \leftarrow \ell + 1$ ★ update active set iteration count
9: solve nonlinear problem (11) for $\Gamma_{f,n}^{r,(\ell-1)}$ status
10: set stick, slip, and open portions for $\Gamma_{f,n}^{r}$ ★ update status using last solution
11: until ($\Gamma_{f,n}^{r,\text{stick}} = \Gamma_{f,n}^{r,(\ell-1),\text{stick}}$ and $\Gamma_{f,n}^{r,\text{slip}} = \Gamma_{f,n}^{r,(\ell-1),\text{slip}}$ and $\Gamma_{f,n}^{r,\text{open}} = \Gamma_{f,n}^{r,(\ell-1),\text{open}}$)
In Algorithm 3.1, Newton’s method is used to drive the norm of the combined residual vector below a specified relative tolerance (line 9). To better identify each contribution associated with the contact process in the linearized problem at every iteration \( \ell \) of the active-set algorithm, we further partition the vector containing the unknown traction degrees of freedom as \( \mathbf{r}_ \ell^{(k)} = (\mathbf{r}_{N,\ell}^{(k)}, \mathbf{r}_{T,\ell}^{(k)}, \mathbf{r}_{O,\ell}^{(k)}) \). Subscripts \( S \) and \( O \) denote traction degrees of freedom belonging to \( \Gamma_{f,n}^{\text{stick}} \) and \( \Gamma_{f,n}^{\text{open}} \), respectively. Subscripts \( N \) and \( T \) identify normal and tangential traction degrees of freedom on \( \Gamma_{f,n}^{\text{slip}} \). The set of indices of traction basis functions \( N_i \) is also consistently expressed as union of disjoint sets such that \( N_i = N_i^S \cup N_i^T \cup N_i^O \). The solution of the nonlinear problem (11) is then computed as follows. Given an initial guess for displacement \((\mathbf{u}_n^{(0)})\), traction \((\mathbf{t}_{\beta,\ell}^{(0)}, \beta \in \{S, N, T, O\})\), and pressure \((\mathbf{p}_{\beta,\ell}^{(0)})\) vectors, for \( k = 0, 1, \ldots \) until convergence

\[
\begin{align*}
\text{Solve} \quad [A_{\text{uu}} & A_{\text{us}} A_{\text{uN}} A_{\text{uT}} A_{\text{uO}} A_{\text{up}}]_{n}^{(k)} [\delta \mathbf{u}]_n = -[\mathbf{r}_u]_n^{(k)} \\
\text{Set} \quad [\mathbf{u}]_n^{(k+1)} = [\mathbf{u}]_n^{(k)} + [\delta \mathbf{u}]_n \\
& [\mathbf{t}_s]_n [\mathbf{t}_T]_n [\mathbf{t}_N]_n [\mathbf{t}_O]_n [\mathbf{p}]_n
\end{align*}
\]

The linearized system (12) is assembled in the standard way from elementary contributions. The global expressions for the residual block vectors read:

\[
\begin{align*}
[r_u]_n &= \int_{\Omega} \nabla \mathbf{u}_n : \mathbf{e}_\Omega + \int_{\Gamma_f} \| \mathbf{e}_\Omega \| (\mathbf{t}_{\beta,\ell}^{(k)} - \mathbf{t}_{\beta,n}^{(k)}) \cdot \mathbf{e}_\Omega \, dA + \int_{\Gamma_f} \mathbf{e}_\Omega \cdot \mathbf{t}_{\beta,\ell}^{(k)} \, dA - \int_{\partial \Omega_\nu} \mathbf{e}_\Omega \cdot \mathbf{t}_{\beta,n} \, dA \quad \forall i \in N_u, \quad (14a) \\
[r_S]_n &= \int_{\Gamma_{f,n}^{\text{stick}}} \mathbf{e}_{\Gamma_{f,n}^{\text{stick}}} \cdot \mathbf{n}_{\Gamma_{f,n}^{\text{stick}}} \, dA + \int_{\Gamma_{f,n}^{\text{open}}} \mathbf{e}_{\Gamma_{f,n}^{\text{open}}} \cdot \mathbf{n}_{\Gamma_{f,n}^{\text{open}}} \, dA \quad \forall i \in N_i^S, \quad (14b) \\
[r_N]_n &= \int_{\Gamma_{f,n}^{\text{slip}}} \mathbf{e}_{\Gamma_{f,n}^{\text{slip}}} \cdot \mathbf{n}_{\Gamma_{f,n}^{\text{slip}}} \, dA \quad \forall i \in N_i^N, \quad (14c)
\end{align*}
\]
\[ [r_T]_i = \int_{t_{a,i}}^{t_{b,i}} \mu \cdot (\dot{r}_T^i - \dot{r}_n^i) \, dA \quad \forall i \in N_T^T, \quad (14d) \]

\[ [r_O]_i = \int_{t_{a,i}}^{t_{b,i}} \mu \cdot r_n^i \, dA \quad \forall i \in N_T^O, \quad (14e) \]

\[ [r_P]_i = \int_{t_{a,i}}^{t_{b,i}} \chi_i \frac{\Delta_n G_{TI}}{\Delta_T} \, dA - \sum_{s_{x \in E_{x,s}}} (\chi_i \phi_x - \chi_i \phi_x) \dot{q}_n^s + \sum_{s_{x \in E_{x,s}}} \chi_i \phi_x \dot{q}_n^s + \int_{t_T} \chi_i q_{z, n} \, dA \quad \forall i \in N_p. \quad (14f) \]

The global expressions for the sub-matrices appearing in the Jacobian matrix read:

\[ [A_{au}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial u_j} = \int_{t_T} \nabla \cdot \eta \cdot C \cdot \nabla \eta_j \, dV \quad \forall (i, j) \in N_u \times N_u, \quad (15a) \]

\[ [A_{ag}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial t_j} = \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA + \int_{t_T} \nabla \cdot \eta \cdot \mu_j T \, dA \quad \forall (i, j) \in N_u \times N_T^\beta, \beta = \{S, N, T, O\} \]

\[ [A_{ap}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial p_j} = - \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA \quad \forall (i, j) \in N_u \times N_p. \quad (15b) \]

\[ [A_{as}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial u_j} = \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA + \int_{t_T} \nabla \cdot \eta \cdot \mu_j T \, dA \quad \forall (i, j) \in N_s^T \times N_u, \quad (15d) \]

\[ [A_{Au}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial u_j} = \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA \quad \forall (i, j) \in N_s^T \times N_u, \quad (15e) \]

\[ [A_{Ah}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial t_j} = \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA + \int_{t_T} \nabla \cdot \eta \cdot \mu_j T \, dA \quad \forall (i, j) \in N_s^T \times N_T^\beta, \beta = \{S, N, T, O\} \]

\[ [A_{Ah}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial p_j} = \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA - \int_{t_T} \nabla \cdot \eta \cdot \mu_j T \, dA \quad \forall (i, j) \in N_s^T \times N_T^\beta, \beta = \{S, N, T, O\} \]

\[ [A_{Ao}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial u_j} = \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA + \int_{t_T} \nabla \cdot \eta \cdot \mu_j T \, dA \quad \forall (i, j) \in N_p \times N_u, \quad (15j) \]

\[ [A_{Ap}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial u_j} = \frac{1}{\Delta_T} \int_{t_T} \nabla \cdot \eta \cdot \mu_j n \, dA - \sum_{s_{x \in E_{x,s}}} (\chi \phi_x - \chi \phi_x) \left( \frac{\partial \phi_x}{\partial u_j} \right)_{t_T} + \sum_{s_{x \in E_{x,s}}} \chi \phi_x \left( \frac{\partial \phi_x}{\partial u_j} \right)_{t_T} \quad \forall (i, j) \in N_p \times N_u, \quad (15j) \]

\[ [A_{Pp}]_{ij} = \frac{\partial [r_{A,i,j}]}{\partial p_j} = - \sum_{s_{x \in E_{x,s}}} (\chi \phi_x - \chi \phi_x) \left( \frac{\partial \phi_x}{\partial p_j} \right)_{t_T} (\chi \phi_x - \chi \phi_x) - \sum_{s_{x \in E_{x,s}}} \chi \phi_x \left( \frac{\partial \phi_x}{\partial p_j} \right)_{t_T} \chi \phi_x \quad \forall (i, j) \in N_p \times N_p. \]

with the partial derivatives expanded as:

\[ \frac{\partial T}{\partial \Delta(g_T)} = \tau_{\max}(t_n^k) \left[ \frac{\Delta(g_T)}{\Delta(g_T)} \right]_{t_n^k}^{k}, \quad (16a) \]

\[ \frac{\partial T}{\partial N} = - \tan(\theta) \frac{\Delta(g_T)}{\Delta(g_T)} \left[ \frac{\Delta(g_T)}{\Delta(g_T)} \right]_{t_n^k}^{k}, \quad (16b) \]
Remark 4. In the formulation proposed in [77], the Jacobian system is assembled in its reduced form with matrices $K$ and $B$ as presented in Section 3.1. We first elaborate the proposed stabilization procedures considering the simpler stick-contact problem, in the absence of frictional slip or fluid flow. The extension to more sophisticated physics will follow seamlessly at the end of the section.

The focus of this work is on the nonlinear algorithm and stabilization only. Thus, the linear solution step however, and is the subject of future research.

4. Stabilization

In this section we explore a family of stabilization techniques to enable the successful use of the spatial discretization presented in Section 3.1. We first elaborate the proposed stabilization procedures considering the simpler stick-contact problem, in the absence of frictional slip or fluid flow. The extension to more sophisticated physics will then follow seamlessly at the end of the section.

To clearly highlight the source of instability, consider a fracture entirely in stick mode, $\Gamma_f = \Gamma_f^{\text{stick}}$. In this case, system (17) reduces to

$$
\begin{bmatrix}
A_{uu} + B_{uu} & A_{uS} & A_{uN} + B_{uN} & A_{uO} \\
A_{Su} & 0 & 0 & 0 \\
A_{Nu} & 0 & 0 & 0 \\
A_{pu} & 0 & 0 & A_{pp}
\end{bmatrix}
\begin{bmatrix}
\delta u \\
\delta s \\
\delta n \\
\delta p
\end{bmatrix}
= -
\begin{bmatrix}
\begin{bmatrix}
\partial^2 \Gamma_T \mu \\
\partial \mu \delta T
\end{bmatrix}^{(k)} \\
\begin{bmatrix}
\partial^2 \Gamma_S \mu \\
\partial \mu \delta S
\end{bmatrix}^{(k)} \\
\begin{bmatrix}
\partial^2 \Gamma_N \mu \\
\partial \mu \delta N
\end{bmatrix}^{(k)} \\
\begin{bmatrix}
\partial^2 \Gamma_O \mu \\
\partial \mu \delta O
\end{bmatrix}^{(k)}
\end{bmatrix}
+ 
\begin{bmatrix}
A_{uN} & A_{uO} \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
r_n \\
r_S \\
r_N \\
r_O
\end{bmatrix}
+ 
\begin{bmatrix}
A_{uN} & A_{uO} \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
r_n \\
r_S \\
r_N \\
r_O
\end{bmatrix}
,$$

(17)

with $B_{uu} = -A_{uu}A_{TT}^{-1}A_{Tn}$ and $B_{uN} = -A_{uu}A_{TN}^{-1}A_{TN}$.

Remark 5. The focus of this work is on the nonlinear algorithm and stabilization only. Thus, the linear solution step in (12) is carried out using a direct solver. The design of a scalable solver for this linear system is clearly nontrivial, however, and is the subject of future research.

4. Stabilization

In this section we explore a family of stabilization techniques to enable the successful use of the spatial discretization presented in Section 3.1. We first elaborate the proposed stabilization procedures considering the simpler stick-contact problem, in the absence of frictional slip or fluid flow. The extension to more sophisticated physics will then follow seamlessly at the end of the section.

To clearly highlight the source of instability, consider a fracture entirely in stick mode, $\Gamma_f = \Gamma_f^{\text{stick}}$. In this case, system (17) reduces to

$$
\begin{bmatrix}
K & C \\
C^T & 0
\end{bmatrix}
\begin{bmatrix}
\delta u \\
\delta s
\end{bmatrix}
= -
\begin{bmatrix}
r_n \\
r_S
\end{bmatrix},
$$

(19)

with $K = A_{uu}$ and $C = A_{uS}$. For this saddle point system, stability requires the spaces $\mathcal{U}_0$ and $\mathcal{M}^c(\Gamma_f)$ satisfy the discrete inf-sup condition [78]

$$
\inf_{\mu \in \mathcal{M}^c(\Gamma_f)} \sup_{\eta \in \mathcal{U}_0} \frac{b(\mu, \eta)}{\|\mu\|_{\mathcal{M}^c(\Gamma_f)} \|\eta\|_{\mathcal{U}_0}} \geq \gamma > 0,
$$

(20)

with the bilinear form

$$
b(\mu, \eta) = \int_{\Gamma_f} \mu \cdot \eta \|dA = \int_{\Gamma_f} \mu_N \|\eta_N\| dA + \int_{\Gamma_f} \mu_T \cdot \|\eta_T\| dA,
$$

(21)

10
and \( \gamma \) independent of the mesh size used in the discretization. Unfortunately, the \( Q_1 \)-displacement/\( P_0 \)-Lagrange multiplier interpolation does not uniformly satisfy (20), leading to unstable approximations [59, Section 3.1].

As we describe strategies to fix this deficiency, it is useful to apply them to an illustrative example for comparison purposes. We will use the 2D, plain strain problem shown in Fig. 3. The size of the domain is \( 8 \times 20 \) m and the fracture has a dip of \( 10^\circ \). Three material regions are considered, characterized by Young’s modulus values \( E_1 = 3 \) GPa, \( E_2 = 15 \) GPa, and \( E_3 = (E_1 + E_2)/2 \) GPa, and a homogeneous Poisson’s ratio \( \nu = 0.25 \). The horizontal and vertical loads are 2 MPa \( \cdot \) m and 4 MPa \( \cdot \) m, respectively. We compute solutions on a base grid with \( 16 \times 40 \) elements, as well as on an anisotropically refined one with \( 16 \times 200 \) elements. The resulting traction components orthogonal and parallel to the fracture are identified as \( t_N \) and \( t_T \), respectively. We also compute a reference solution using a highly refined elastic model without a discontinuity (since pure stick conditions are assumed). In the remainder of the section, we will refer to this numerical solution as the continuous one. For each stabilized solution, we compute two integral relative differences with respect to the continuous one, \( E_N \) and \( E_T \), for the two components of the traction. In Fig. 4, results using the \( Q_1 - P_0 \) interpolation without any stabilization are shown. We observe that the traction solution on average coincides with the continuous one, but it exhibits substantial checkerboard oscillations.

We now consider three possible strategies to fix this issue:

1. Analytic macroelement stabilization
2. Algebraic macroelement stabilization
3. Algebraic global stabilization

As we will see, the first method makes significant assumptions regarding the grid topology and material heterogeneity, while the latter methods provide greater generality.

4.1. Analytic macroelement stabilization

We first explore stabilizing the discretization using a macroelement approach [79]. The method is most easily described using a 2D reference macroelement as shown in Fig. 5a. This macroelement is formed by two interface elements and four quadrilaterals, two for each side of the fracture. To create a patch test, the ten displacement degrees of freedom (DOFs) located on the boundary are fixed. The objective is to derive a stabilization that produces a
for the traction normal- and tangential-component, respectively, confirming the numerical results shown in Fig. 4. Performing an eigen-decomposition, its complete set of eigenpairs with $S$ well-posed problem on an individual patch. Such a stabilization then implies well-posedness on a grid consisting of stabilized macroelements.

We can achieve this by investigating the kernel modes of the Schur complement for the Lagrange multipliers. For example, for the macroelement of Fig. 5, there are 4 displacement DOFs—the $x$- and y-components for innermost nodes, above and below the fracture—and 4 traction DOFs—one normal and one tangential component for each element on $\Gamma_f$. Assuming the ordering $u = \begin{bmatrix} u_N^x, u_N^y, u_N^x, u_N^y \end{bmatrix}$, $t = \begin{bmatrix} t_N^x, t_N^y, t_N^x, t_N^y \end{bmatrix}$ for the unknowns, the explicit expressions of $K$ and $C$ are:

$$K = \frac{E(1-\nu)}{3(1+\nu)(1-2\nu)} \begin{bmatrix} K_2 & 0 \\ 0 & K_2 \end{bmatrix}, \quad K_2 = \begin{bmatrix} (1-2\nu) \frac{E}{h_x} + \frac{2h_y}{h_x} & \frac{2h_y}{h_x} \\ \frac{2h_y}{h_x} & (1-2\nu) \frac{E}{h_y} + \frac{2h_x}{h_y} \end{bmatrix}, \quad C = \frac{h_x}{2} \begin{bmatrix} R_2 & R_2 \\ R_2 & -R_2 \end{bmatrix},$$

(22)

with $E$ the Young’s modulus, $\nu$ the Poisson’s ratio, $h_x$ and $h_y$ the mesh size in the $x$- and $y$-direction, respectively, and $R_2$ a $2 \times 2$ rotation matrix from the local to global reference system. In the example of Fig. 5, $R_2$ reads

$$R_2 = \begin{bmatrix} n_f \cdot e_x & m_f \cdot e_x \\ n_f \cdot e_y & m_f \cdot e_y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

(23)

with $\{e_x, e_y\}$ the standard Euclidean basis in $\mathbb{R}^2$. By definition, the Schur complement $S \in \mathbb{R}^{4 \times 4}$ is

$$S = -C^T K^{-1} C = -\left(\frac{h_x}{2}\right)^2 \frac{6(1+\nu)(1-2\nu)}{E(1-\nu)} \begin{bmatrix} K_2^{-1} & 0 \\ 0 & K_2^{-1} \end{bmatrix}.$$

(24)

Performing an eigen-decomposition, its complete set of eigenpairs $\{\lambda_i, v_i\}_{i=1}^4$ is

$$\begin{align*}
\{\lambda_1, v_1\} &= \begin{bmatrix} 0, 1 \\ -1, 0 \end{bmatrix}, \\
\{\lambda_2, v_2\} &= \begin{bmatrix} 0, 1 \\ 1, 0 \end{bmatrix}, \\
\{\lambda_3, v_3\} &= \begin{bmatrix} -\frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \frac{3h_x^2}{h_y (1-2\nu) + 2h_y h_x} , 1 \\ h_y (1-2\nu) + 2h_y h_x , 0 \end{bmatrix}, \\
\{\lambda_4, v_4\} &= \begin{bmatrix} -\frac{(1+\nu)(1-2\nu)}{E(1-\nu)} \frac{3h_y^2}{h_x (1-2\nu) + 2h_x h_y} , 1 \\ h_x (1-2\nu) + 2h_x h_y , 0 \end{bmatrix}.
\end{align*}$$

(25a)

We observe that $S$ has rank 2. Eigenvectors $v_1$ and $v_3$ span the kernel of $S$ and represent the checkerboard mode for the traction normal- and tangential-component, respectively, confirming the numerical results shown in Fig. 4. Conversely, the component-wise constant eigenvectors $v_3$ and $v_4$ span the column space of $S$. 

12
The null eigenvectors $v_1$ and $v_2$ are the source of the macroelement instability and need to be removed from the null space of $S$. To do so, we introduce the symmetric and positive semi-definite matrix $H^* = VV^T$, with $V$ the matrix having columns $v_1$ and $v_2$. By construction, $v_1$ and $v_2$ are eigenvectors of $H^*$, both corresponding to the eigenvalue 2, which span the range of $H^*$. Also, $v_3$ and $v_4$ are now a basis for the kernel of $H^*$. Let $H = \alpha H^*$ be a scaled matrix, with $\alpha$ a scalar stabilization constant having units of squared length per pressure, i.e. the same units as the eigenvalues of $S$. The system to be solved is modified as

$$\begin{bmatrix} K & C \\ C^T & -H \end{bmatrix} \begin{bmatrix} \delta u \\ \delta t \end{bmatrix} = - \begin{bmatrix} r_u \\ r_S - H_S t \end{bmatrix},$$

where a stabilizing contribution now replaces the zero block of the original matrix. The resulting modified Schur complement is then $\tilde{S} = -C^T K^{-1} C - \alpha H^*$. The eigenpairs of $\tilde{S}$ are the same as those of $S$ with the only difference that $v_1$ and $v_2$ are now associated with the same non zero eigenvalue $2\alpha$. The scaling constant $\alpha$ is chosen in such a way that, in a homogeneous case, the eigenvalues of $\tilde{S}$ are bounded between $\min(\lambda_3, \lambda_4)$ and $\max(\lambda_3, \lambda_4)$, so that the matrix conditioning does not depend on the stabilization constant. For a regular Cartesian grid with $h_x = h_y = h$, from Eq. (25b) we can write

$$\lambda = \lambda_3 = \lambda_4 = - \frac{(1 + \nu)(1 - 2\nu)}{E} \frac{3h^2}{(3 - 4\nu)},$$

thus, the “optimal” scaling factor simply reads:

$$\alpha^* = \frac{1 + \nu}{E} \frac{3}{2} \frac{h^2}{(3 - 4\nu)}.$$

In case of geometric anisotropy, $h$ can be computed as the average length of the interface elements composing the macroelement.

We emphasize that the stabilization matrix $H$ represents an example of a minimal stabilization operator [80] that does not pollute the physical eigenpairs. It requires, however, explicit knowledge of the eigenvectors associated with non-zero eigenvalues. For the macroelement of Fig. 5a, $H$ can also be interpreted as a macroelement stabilization matrix for the $\alpha$-weighted inter-element traction (component-wise) jump since the following relationship holds true

$$t^T H t = \alpha t^T H^* t = \alpha \left( t_N^{(\phi_1)} - t_N^{(\phi_2)} \right)^2 + \left( t_T^{(\phi_1)} - t_T^{(\phi_2)} \right)^2,$$
Applying this stabilization to the 2D test case (Fig. 3) yields a substantial reduction of the oscillations observed in the unstabilized case as shown in Fig. 6, with the errors $E_N = 1.32 \cdot 10^{-1}$, $E_T = 1.36 \cdot 10^{-1}$ and $E_N = 7.36 \cdot 10^{-2}$, $E_T = 1.55 \cdot 10^{-1}$ for the base and the refined grids, respectively.

Remark 6. The 2D reference macroelement (Fig. 5a) consists of four equal rectangular elements—hence, the stiffness matrix $K$, see Eq. (22), is diagonal—with the normal $n_f$ aligned with the $y$-axis. Consequently, normal- and tangential-component of the traction are decoupled as the eigenvectors defined in Eqs. (25a)-(25b) reveal. This is not the case in a more general geometry where the two traction components are typically related.

Extension to 3D is based on the reference macroelement shown in Fig. 5b, which consists of four quadrilateral interface elements and eight hexahedral elements. The derivation follows the same steps discussed above for the 2D case. We omit the computations and simply report the main results. The Schur complement $S \in \mathbb{R}^{12 \times 12}$ has rank three.

![Figure 6: Numerical results for example in Fig. 3 using the analytic macroelement stabilization. First row: base grid, last row: vertically refined grid.](image)

![Figure 7: Three different modes that need to be stabilized for the 3D case. Each of them applies to the three components (one normal and two tangential) of the traction vector, thus the kernel size is 9. Note that the remaining mode is a linear combination of these three, thus it is part of the same space, the Schur complement kernel.](image)
Figure 8: General interface element patches for a 2D (a) and a 3D (b) macroelement, respectively. Squares (□) indicate the barycenter of the interface elements. Quantities $\tilde{\ell}_i, i \in \{1, 2\}$ and $A_j, j \in \{1, \ldots, 4\}$ denote the length and area, respectively, associated to the common vertex (○).

with column space spanned by three component-wise constant eigenvectors. There are nine spurious traction modes that need to be stabilized. A convenient basis for the kernel of $S$ is shown in Fig. 7 and corresponds to checkerboard-like modes for each component of the traction vector with respect to three internal edges of the interface element patch. Note that the choice of the three edges is arbitrary. The optimal $\alpha$ value to obtain a stabilization contribution that falls in the already present lower/upper spectral Schur complement bounds is:

$$\alpha^* = \frac{(1 + \nu)(1 - 2\nu)}{E} \frac{3h^3}{(16 - 24\nu)}.$$  \quad (30)

As for the 2D case, this value is exact in case of $h_x = h_y = h_z = h$ and constant physical properties. In general, $h$ can be computed as the cubic root of the average volume of the elements surrounding the fracture and sharing a face.

### 4.2. Algebraic macroelement stabilization

The scaling factor $\alpha$ in the macroelement approach above is a scalar value collecting both mechanical and geometric information. Its definition can be quite difficult, especially for 3D problems with distorted grids and material heterogeneity. In this section, we propose an algebraic alternative for computing the stabilization matrix at the macroelement level that circumvents the need for introducing the factor $\alpha$. The development is based on the following observation: $\alpha$ is fundamentally needed to scale matrix $H^*$ introduced in Sec. 4.1 so that the spectrum of the stabilized Schur complement $\tilde{S}$ is bounded between the smallest and largest nonzero eigenvalues of $S$. Thus, whenever we provide a different but admissible scaling, $\alpha$ can be avoided.

Let us consider a general patch of interface elements in a non regular macroelement (Fig. 8). The only topological assumption we make is that the fracture elements within an individual macroelement are co-linear in 2D or co-planar in 3D. For the two dimensional case of Fig. 8a, matrix $C$ given in Eq. (22) reads

$$C = \begin{bmatrix} \tilde{\ell}_1 R_2 & \tilde{\ell}_2 R_2 \\ -\tilde{\ell}_1 R_2 & -\tilde{\ell}_2 R_2 \end{bmatrix},$$  \quad (31)

where $\tilde{\ell}_1$ and $\tilde{\ell}_2$ are the interface element lengths associated to the vertex in common between elements $\varphi_1$ and $\varphi_2$, i.e. the integral of the standard hat function associated to that vertex over $\varphi_1$ and $\varphi_2$, respectively. The rows of the following matrix $\tilde{C}$, formed from $C$ swapping block columns and changing sign to the first block column,

$$\tilde{C} = \begin{bmatrix} -\tilde{\ell}_2 R_2 & \tilde{\ell}_1 R_2 \\ \tilde{\ell}_2 R_2 & -\tilde{\ell}_1 R_2 \end{bmatrix},$$  \quad (32)

are orthogonal by construction to $C$, i.e., $CC^T = 0$. Thus, columns of $\tilde{C}^T$ belong to the kernel of the Schur complement $S = -C^T K^{-1}C$. Also, $\tilde{C}$ has rank 2, the number of modes that are known to require the stabilization. Hence, $C^T \tilde{C}$ is a stabilizing contribution to the Schur complement. It has to be scaled, but from the observation that $\tilde{C}$ has the same entries of $C$, except for the order (they are swapped on a Lagrange multiplier base) and the sign, it is natural to use the inverse of the stiffness matrix in (22), i.e. $K^{-1}$, to scale it. Indeed, a scaling that incorporates element size and material
properties information is needed and $K$ collects both of them. Numerical tests show that the inverse of its diagonal, denoted as $D_K \in \mathbb{R}^{4\times4}$, is enough. Therefore, a macroelement stabilization matrix can be defined as $H = C^T D^{-1} C$.

Using the same notation of Sec. 4.1, such a stabilization can also be expressed in the following compact form:

$$H = V D V^T, \quad V = \begin{bmatrix} -t_2 R_1^T \cr t_2 R_2^T \end{bmatrix}, \quad D = D_{K,N_1}^{-1} + D_{K,N_2}^{-1}, \quad (33)$$

with $D \in \mathbb{R}^{2\times2}$ a diagonal matrix given by the sum of the inverse of the diagonal portion of the stiffness matrix associated with the two nodes (on both the discontinuity surfaces) shared by the two interface elements forming the macroelement (i.e., $N_1^1$ and $N_2^1$ in Fig. 5a). Note that whenever applied to a regular grid, with $h_x = h_y$, and homogeneous material properties, this stabilization reduces to the $\alpha$-based method of the previous section.

To compare the $\alpha$-based and algebraic macroelement stabilizations, in Fig. 9 we report the behavior of the conditioning number $\kappa(S) = \lambda_{\text{max}}(S)/\lambda_{\text{min}}(S)$ of the stabilized Schur complement for different macroelement configurations. The $x$-axis is normalized with respect to the “optimal” value according to Eq. (28), the dot is the conditioning arising from the $\alpha$-based stabilization and the dashed line is the conditioning of the algebraic stabilization. From the left to the right, we have three cases:

- Regular grid, with homogeneous material properties. It can be observed that the two techniques offer the same optimal result.
- Stretched grid (Fig. 9) with $h_x, t_1 = 5h_x, t_1$. The material is still homogeneous. In this case, the stabilization based on $\alpha$ is not able to get the minimal conditioning, while the algebraic one is.
- Regular grid (Fig. 9), with heterogeneity in the Young’s modulus: $E_{t_1} = E_{t_2} = 5E_{t_1} = 5E_{t_2}$. Again, the stabilization that uses a scalar value is not able to provide the minimal conditioning, while the algebraic one is.

Observing the plots in Fig. 9, it can be noticed that the conditioning has a unique minimum when the grid is regular, but there is a set of minima in the middle case, with a stretched grid. The former occurrence implies that the two eigenvalues $\lambda_3$ and $\lambda_4$ are the same, while in the latter that they are different. This finding is consistent with the analytical definition of the Schur complement eigenvalues (see Eqs. 25b), where the aspect ratio $h_x/h_y$ is a parameter. Using a more refined definition of $\alpha^*$, considering also the geometric anisotropy, would improve the conditioning number of the $\alpha$-based $\tilde{S}$ for this specific case. In general, however, as assumptions regarding mesh geometry and material properties are relaxed, an algebraic approach is increasingly appealing in its simplicity.

To further test this approach, we focus again on the model problem depicted in Fig. 3. Numerical results are provided in Fig. 10 with the errors $E_N = 1.33 \times 10^{-4}$, $E_T = 1.35 \times 10^{-4}$ and $E_N = 7.49 \times 10^{-2}$, $E_T = 1.63 \times 10^{-1}$ for the base and the refined grids, respectively. Comparing this technique with the $\alpha$-based approach, we observe quite similar performance.

We conclude this section by discussing the extension to 3D problems. Matrix $C$ for a 3D macroelement reads

$$C = \begin{bmatrix} \tilde{\alpha}_1 R_3 & \tilde{\alpha}_2 R_3 & \tilde{\alpha}_3 R_3 & \tilde{\alpha}_4 R_3 \\ -\tilde{\alpha}_1 R_3 & -\tilde{\alpha}_2 R_3 & -\tilde{\alpha}_3 R_3 & -\tilde{\alpha}_4 R_3 \end{bmatrix}, \quad (34)$$
where the areas $\tilde{\mathcal{A}}_i, i = \{1, \ldots, 4\}$, are the counterpart of lengths $\tilde{\ell}_1$ and $\tilde{\ell}_2$ (Fig. 8b), and $R_3$ is a 3D rotation matrix. Using the same arguments as in the 2D case, matrix $H$ can be obtained by stabilizing the traction jumps across internal edges of the macroelement as $H = \sum_{i=1}^{4} V_i D V_i^T$, where $D = D_{N_1}^{-1} - D_{N_2}^{-1} \in \mathbb{R}^{3 \times 3}$ and

$$
V_1 = \begin{bmatrix} -\tilde{\mathcal{A}}_2 R_1^T \\ \tilde{\mathcal{A}}_1 R_3^T \\ 0 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ -\tilde{\mathcal{A}}_2 R_1^T \\ \tilde{\mathcal{A}}_1 R_3^T \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 \\ 0 \\ -\tilde{\mathcal{A}}_3 R_1^T \\ \tilde{\mathcal{A}}_2 R_3^T \\ 0 \end{bmatrix}, \quad V_4 = \begin{bmatrix} \tilde{\mathcal{A}}_3 R_1^T \\ 0 \\ 0 \\ -\tilde{\mathcal{A}}_1 R_3^T \end{bmatrix}.
$$

(Remark 7) No explicit assembly process in needed in the construction of $V$ and $D$ both in 2D and 3D. A local gathering from the global stiffness and coupling matrix is enough to extract the required blocks.

4.3. Algebraic global stabilization

In this section, we extend the algebraic approach of Sec. 4.2 to overcome the key limitation of any macroelement-based approach, the topological restriction that the mesh be partitioned into macroelements in the first place. As previously stated, the key point is to stabilize the jump between two interface elements. Rather than doing this for macroelement-internal edge alone, we now simply stabilize all possible jumps between each pair of interface elements.

To do so, it is sufficient to compute the matrix $\tilde{C}_{loc}$ for all degrees-of-freedom shared between two elements discretizing the fracture, gather the local matrix $K_{loc}$, and then assemble individual $H_{loc}$ contributions into a global stabilization matrix $H$. Algorithm 4.1 describes the approach for the 2D case—see Figure 11—using a Matlab-style notation. Note that the proposed algorithm assumes, implicitly, that fractures are planar. Also recall that multiple DOFs are associated to each geometric object (nodes and faces) and thus the gather/scatter indexing refers to vector sets of component indices.
allowing for frictional sliding and fluid flow. In particular, all that is required is to modify the original system (12) to

4.4. Inclusion of friction and fluid flow

The methods described for the stick problem form the building blocks for stabilizing the full physical model, allowing for frictional sliding and fluid flow. In particular, all that is required is to modify the original system (12) to
Figure 12: Numerical results for example in Fig. 3 using the algebraic global stabilization. First row: base grid, last row: vertically refined grid.

Figure 13: Comparison of different stabilization techniques for a refined regular grid. From the left to the right: analytic macroelement, algebraic macroelement, and algebraic global stabilization approach.
its stabilized version,

$$
\begin{bmatrix}
A_{uu} & A_{uS} & A_{uN} & A_{uO} & A_{up} & f^{(k)} \\
A_{Su} & -H_{SS} & -H_{SN} & 0 & 0 & 0 \\
A_{Nu} & -H_{NS} & -H_{NN} & 0 & 0 & 0 \\
A_{pu} & 0 & 0 & 0 & A_{pp} + \frac{1}{\Delta t} H_{pp} \delta p & 0 \\
\end{bmatrix}
\begin{bmatrix}
\delta u \\
\delta t_S \\
\delta t_N \\
\delta t_O \\
\end{bmatrix}
= \begin{bmatrix}
\delta p \\
-H_{SS} t_S - H_{SN} t_N \\
-H_{NS} t_S - H_{NN} t_N \\
0 \\
\end{bmatrix} + \begin{bmatrix}
r_u \\
r_s - H_{SS} t_S - H_{SN} t_N \\
r_N - H_{NS} t_S - H_{NN} t_N \\
r_O \\
\end{bmatrix} + \begin{bmatrix}
ar_p + \frac{1}{\Delta t} H_{pp} p \\
\end{bmatrix},
$$

where five stabilizing sub-matrices $H_{ij}$ have been added to particular components of the multiphysics problem. In particular, the traction components $t_S$ and $t_N$ and the pressure field $p$ must all receive stabilizing contributions. Stability is primarily provided by the matrices $H_{SS}$, $H_{NN}$, and $H_{pp}$, which are assembled on macroelements using the same procedures as described in the previous section, but now applied to different traction and pressure components as warranted. The off-diagonal terms $H_{SN}$ and $H_{NS}$ are necessary to capture contributions from stabilized edges between elements in a “mixed” state—that is, where an element in a stick state is adjacent to one in frictional sliding state. We observe that $t_T$—i.e., the friction component of the traction on $\Gamma_{\text{slip}}$—is a function of the relative displacement rate $\Delta g_T$ and therefore does not need to be stabilized. Similarly, for $\Gamma_{\text{open}}$, there is no need for stabilization, as the traction $t_O$ is $a$ priori known to be zero. As before, because the matrices $A_{TT}$ and $A_{OO}$ are diagonal, a Schur-complement reduction can be performed to eliminate $t_T$ and $t_O$. The reduced system will closely resemble Eq. (17), but with the stabilizing blocks above included in the appropriate places.

5. Analytical benchmarks for contact mechanics without fluid flow

In this section, we validate the formulation, the discretization, and the stabilization strategies for pure contact mechanics problems, without fluid flow.

5.1. Analytical benchmark for shear behavior: Constant solution

This analytical benchmark was originally proposed in [84]. The representation of the 2D model domain is reported in Fig. 14a, where the lower boundary is $y$-constrained, the circled corner is constrained also in the $x$ direction and on the upper boundary a uniform displacement is imposed ($\bar{u}_y = 0.10$ m). The material is homogeneous with elastic parameters $E = 5000$ MPa and $\gamma = 0.25$. Coulomb’s frictional parameters are $c = 0$ and $\theta = 5.71^\circ$, such that the friction coefficient is 0.1. The solution is a constant sliding on the fracture of value $g_T = \frac{\|g_T\|}{\Delta t} = 0.1 \sqrt{2} \approx 0.1414$ m.

We highlight that the simulation is carried out on a 3D mesh (Fig. 14b), because in a 2D setting the direction of the shear vector is known; thus, the Coulomb frictional contact condition is not needed and the shear behavior is
linear. To test the nonlinearity introduced by the dependency of the direction on the relative displacement rate, when
the failure condition is matched and the fracture slides, we need to work in a 3D setting. To match a 2D analytical
solution, we respect the plane strain assumption and build the domain extruding the surface shown in Fig. 14a by
0.5 m and z-constraining the two surfaces parallel to the x-y plane.

From Figs. 14c-15a, it is clear that the model is able to match the analytic linear solution even with a small number
of elements (4680 nodes, 3610 hexahedra and 95 quadrilaterals for the fracture). We report also the convergence
profile of the nonlinear solution algorithm 3.1 in Fig. 15b. In this specific case, after the elastic step, i.e., the first
solution with $\Gamma_f = \Gamma_{\text{stick}}$, the final configuration of $\Gamma_f$ is obtained and no other outer loop iterations are required.

5.2. Single crack under compression

The second example is a single crack in a 2D infinite domain under a constant uniaxial compression. The bench-
mark geometry is described in detail by [85] and reproduced in Fig. 16a. Being $E$ and $\nu$ the linear elastic parameters,
$\sigma$ the compressive stress, $\psi$ the fracture inclination, $2b$ its length and $\theta$ the friction angle for Coulomb’s criterion (with
zero cohesion), the analytical solution provides the normal traction on the fracture and the sliding on it:

$$t_N = -\sigma \sin^2 \psi \quad \text{and} \quad g_T = \left\| \mathbf{g}_T \right\|_2 = \frac{4(1 - \nu^2)}{E} \left( \sigma \sin \psi (\cos \psi - \sin \psi \tan \theta) \right) \sqrt{b^2 - (b - \xi)^2},$$

respectively, where $0 \leq \xi \leq 2b$ is a local coordinate on the fracture. A plane strain status is assumed. For the
simulation, we used the following values: $E = 25000$ MPa and $\nu = 0.25$, the friction angle is $\theta = 30^\circ$, the fracture is
tilted by $\psi = 20^\circ$ and extends for $2b = 2$ m, and $|\sigma : (\mathbf{e}_x \otimes \mathbf{e}_y)| = 100$ MPa.
The model is discretized with different resolutions in the x-y plane, from 5K to 68K quadrilaterals and from 108 to 432 interface elements, for the coarsest and the finest grid, respectively. The final 3D domain is obtained by assuming a plane strain status. The less refined computational domain is shown in Fig. 16b, where a zoom on the mesh around the fracture is also provided. The boundary conditions for $u_x$ and $u_y$ are set in order to respect the symmetry of the expected solution (see Fig. 16b). The two faces parallel to the x-y plane are constrained in the z direction, because of the plane strain assumption.

Comparisons between numerical and analytical solution are provided for the frictional traction (Fig. 17a) and the relative displacement (Fig. 17b). It can be noticed that the computed displacement is in good agreement with the expected one everywhere, while the traction is quite different very close to the fracture tip, where some oscillations are observed. Refining the mesh, these oscillations are still present, but always closer to the fracture tip. From a physical viewpoint, these can be explained as a locking phenomenon [84]. Given the imposed external load, the fracture tries to slip, but the elements close to the tip are not allow for because of the tip itself. To accommodate the expected behavior, the fracture has to open near the tip, even if there is a compressive normal traction on the crack. With a less refined mesh, the interface element with a fixed edge on the tip opens, yielding a positive normal traction. In agreement with these considerations, the rightmost and leftmost traction values are much lower than the average, tending to zero, i.e., the near-tip elements tend to open.

Using the previous example, where an analytical solution for displacement and traction field on the fracture is available, we study the convergence rate, i.e., the error dependence on the mesh size. First of all, we observe that the oscillations on $t_N$ (see Fig. 17a) will prevent any traction error norm from convergence, thus, as done in [86], to compute a meaningful norm we neglect the extreme portions of the domain, i.e., the traction related norms are computed on the central 90% of the fracture trace. In Fig. 18, we show the convergence of the two error norms, on the...
normal Lagrange multiplier (Fig. 18a) and on the sliding component of the displacement (Fig. 18b). While the first one is a properly defined norm on the domain \( \Gamma_f \), where \( l \) is defined, the latter one is just the norm of a continuous field projected on a surface, indeed \( u \) is defined on \( \Omega \) but the norm is computed on \( \Gamma_f \) only. The order of convergence \( m \) is slightly larger than 1 for the Lagrange multiplier norm, while it is close to 1 for the projection of the displacement. There is no well-defined value for the convergence rate of the mixed finite elements space used in this work. The rate 1.11 obtained here is in agreement with the theoretical results given in \([59\text{, Section 4}]\).

5.3. Zipper crack problem

To complete the validation of the model, we use the line crack problem as described in \([87]\). The zipper crack case is also known as Griffith problem \([83]\) and consists of an infinite plane (\(x-y\) plane) with a single linear crack of length \(2l\) in the \(x\)-direction. Inside the fracture, there is a fluid with a given pressure \(p(x)\). The assumption is of plane strain. Fig. 19a represents a sketch of the setup. The analytical solution provide the opening of the fracture for every location \(x\) and the stress in the continuous medium on the \(x\)-direction (\(y = 0\)), for \(x > l\). The far-field stress orthogonal to the fracture is \(\sigma_0\). We impose a constant pressure \(p_0\) only on one part of the fracture, in such a way that the tip closes smoothly. The pressurized length ends at \(x_0\), with:

\[
x_0 = l\sin\frac{\pi\sigma_0}{p_0}.
\]

Introducing \(q_1 = \sqrt{l^2 - x_0^2}, q_2 = \sqrt{l^2 - x^2}\) and \(q_3 = \sqrt{x_0^2 - x^2}\), the analytical solutions for opening and stress in \(y\) direction are \([87]\):

\[
g_n(x) = \frac{2(1 - \nu^2)}{\pi E}p_0 \left(4x_0\log\frac{q_1 + q_2}{q_3} + x\log\frac{l^2 - 2q_1q_2x_0x + l^2x^2 - 2x_0^2x^2}{l^2x_0^2 + 2q_1q_2x_0x + l^2x^2 - 2x_0^2x^2}\right), \quad 0 \leq x \leq l,
\]

\[
\sigma_j(x, 0) = -p_0 + \frac{2}{\pi}\left(p_0\text{arctan}\frac{x_0}{x_0q_3}\right), \quad x \geq l.
\]

In Fig. 19b we show the computational domain used to reproduce this solution. The 3D domain, whose size is \(150 \times 300 \times 0.3\) m, is discretized with 17772 nodes, 13050 hexahedra and 228 interface finite elements. The objective is to simulate a fracture length of \(l = 10\) m, but we discretize a longer fracture, \(l_1 = 15\) m, to verify if the remaining length \(l_1 - l = 5\) m remains closed. We need a large domain to attenuate the boundary effect as the analytical solution is obtained for an infinite domain. It is \(10 \times 20\) times the actual fracture length. Referring to Fig. 19b the only constrained boundary is the one intersecting the fracture, where symmetric conditions are imposed. The only fixed point is the one highlighted in the figure. The 3D domain is obtained extruding a 2D domain, and the faces parallel to the \(x-y\) plane are \(z\)-constrained, to fulfill the conditions of the assumed plane strain state. Regarding material properties, we have \(E = 25\) GPa and \(\nu = 0.25\). Fluid pressure and far-field stress are \(p_0 = 15\) MPa and \(\sigma_0 = 10\) MPa. With the chosen set of parameters, we have \(\lim_{x \to l} \sigma_j(x, 0) = -15 + \frac{20}{\pi}\text{arctan}\frac{2}{\sqrt{3}} \approx -6.816\) MPa, thus there is a discontinuity in the stress field at the fracture tip.

In Figs. 20a, 20b we report the comparison between the outcomes of the numerical model and the analytical solutions. It can be notice, for \(x > l\), the fracture is closed and the transition is smooth, as predicted by the theoretical
solution. The stress behavior is very similar to the analytical one, except very close to the tip, where the numerical model shows a smaller jump between the fracture and the continuous material. Nevertheless, we have a smooth transition between the solutions computed with two different discretizations—on a portion of the closed length interface elements are used, while on the other classical finite elements are employed. Overall, we can see a good agreement, with an integral relative error of 1.0% for the displacements and 2.5% for the stress. For the error evaluation related to the stress, we neglected the near-tip portion of the domain, i.e., we considered only $x > 5/4\ell = 12.5$ m.

As a concluding remark for this analytical benchmark, Fig. 20c shows the convergence rate for the $L_2$-norm of the error on the fracture aperture $g_N(x)$ only, as the stress field is unbounded close to the tip, so unsuited for such a test. As expected, the order of convergence is around 1 because of the discontinuity in the stress field, indeed, the asymptotic behavior is almost always lost whenever there are singularities in the solution [89]. The coarser mesh has $1K$ finite elements and 60 interface finite elements, while the finest has $194K$ and 900 finite and interface finite elements, respectively.

6. Analytical benchmarks for contact mechanics with fluid flow

To verify the numerical model for the full IBVP (1), we use two classic analytical solutions: (i) the Kristianovic-Geerstma-deKlerk (KGD) problem [14, 87, 90–94] and (ii) the penny-shaped crack problem [14, 87, 93, 95–97]. The first test case is mainly a 2D problem, while the latter is a real 3D case.

6.1. KGD problem

We consider a 2D hydraulic fracture propagation assuming plane strain conditions. The medium is isotropic, homogeneous, impermeable, and is fully described using a linear elastic model. An incompressible Newtonian fluid is injected from a fixed point, at a constant rate $Q_0$. The fracture propagates in the direction that is orthogonal to the maximum principal direction of the stress tensor in the surrounding medium. In Fig. 21a, we represent the set-up of the problem and introduce the quantities of interest:

- $g_N(x,t)$: the fracture opening for any time and position $x \leq \ell$;
- $p(x,t)$: the net fluid pressure inside the fracture for any time and position $x \leq \ell$;
- $l(t)$: the fracture half-length.

The analytical solution is provided in terms of $g_N(x,t)$, $p(x,t)$ and $l(t)$, but the complete expressions require the definition of some dimensionless quantities [93]: (i) an opening $\Omega$, (ii) a net pressure $\Pi$ and (iii) a fracture length $\gamma$. Given these dimensionless variables, the quantities of interest become:

$$l(t) = \gamma L(t), \quad g_N(x,t) = \varepsilon(t)L(t)\Omega(\xi), \quad p(x,t) = \varepsilon(t)E'\Pi(\xi),$$  

Figure 20: Zipper crack problem: (a) comparison between numerical and analytical solution for fracture opening; (b) comparison between numerical and analytical solution for $\sigma_y$-stress in the continuous domain; (c) convergence rate for fracture opening with the order of convergence $m$ computed through a least-square linear interpolation. In (b), the gray diamond ($\phi$) indicate where the discretization with interface elements ends and classical finite elements are used.
where $\xi = x/l$ is the similarity variable, i.e., a dimensionless fracture coordinate. For the zero toughness case, we have [92]:

$$
\varepsilon(t) = \left( \frac{\mu'}{E' t} \right)^{\frac{1}{3}}, \quad L(t) = Q_0^{1/2} \left( \frac{E'}{\mu'} \right)^{\frac{1}{3}} t^{2/3}, \quad \gamma = \frac{1}{\left(2 \int_0^\infty \frac{\Pi d\xi}{\Omega} \right)^{1/2}}. \quad (42a)
$$

In [42], we introduced $\Omega = \frac{L}{t}$, $\mu' = 12\mu_t$, with $\mu_t$ the fluid viscosity, and $E' = \frac{E}{1-\nu}$, i.e., the plane strain modulus. The two functions $\Omega(\xi)$ and $\Pi(\xi)$, called self-similar fracture opening and self-similar net fluid pressure, respectively, are approximated through polynomial series, based on the Gegenbauer polynomials [98] for the former one, while the latter uses Euler’s beta functions and Gauss’s hyper-geometric functions [98]. The full expression of these two dimensionless functions, with the numerical coefficients of the series expansion, are provided in [92]. We emphasize the fact that $\lim_{\xi \to -\infty} \Pi = -\infty$, thus, the analytical solution predicts an infinite pressure at the fracture tip. This nonphysical result is due to the assumption that the fluid reaches the tip and fill every empty space at the same density, without allowing for cavitation. Note that both the fluid density and the initial stress regime do not affect any quantity of interest.

In Fig. 21B we show the mesh used for the simulation. It is composed by 10209 nodes, 6600 hexahedra and 76 interface elements. The domain is $300 \times 600 \times 8$ m, with a fracture of 150 m and an average element size of $h_{x} \approx 4$ m. We highlight that currently our model does not handle fracture propagation, but we know a priori the fracture trajectory and can pre-discretize a surface of sufficient length. In some sense this is then a fracture “reactivation” problem. To minimize the boundary effects, we imposed the symmetric condition on the face parallel to the y axis and intersecting the fracture and $z$-constrained the two surfaces parallel to the $x$-$y$ plane, to reflect the plane strain assumption. The 2D node highlighted in Fig. 21B that corresponds to a column of nodes in 3D, is the only one that is $y$-constrained. The material parameters are $E = 30$ GPa and $\nu = 0.25$. The fracture frictional behavior is governed by Coulomb’s criterion, characterized by $\theta = 30^\circ$ and zero cohesion. The fluid viscosity is $\mu_t = 10^{-9}$ MPa $\cdot$ s. The injection rate and the confining stress are $Q_0 = 6 \cdot 10^{-3} \text{ (m}^3/\text{s})/$m and $\sigma_0 = 10$ MPa, respectively. Finally, according to [72], the conductivity initial value (see Eq. (2)) is $C_{f,0} = 10 \text{ mD} \cdot \text{m} = 9.87 \cdot 10^{-15}$ m$^2$ $\cdot$ m. The simulated time is 100 s, with $\Delta t = 1$ s.

Figs. 22A, 22B show the outcomes of the model in terms of opening and pressure for two different time steps, i.e., at half ($t = 50$ s) and at the end of the simulation ($t = 100$ s). The continuous line is the analytical solution. Overall, there is a good agreement, for both the aperture (with an integral relative error of 2.4% for $t = 100$ s) and the pressure (with an integral relative error of 2.2% at the same time step). At time $t = 100$ s, the pressure is slightly different close to the tip, where the theoretical behavior tends to $-\infty$. We emphasize that the analytical solution for the pressure is constrained in an integral sense by the propagation criterion, being

$$
\int_0^1 \frac{\Pi}{\sqrt{1 - \xi^2}} d\xi = 0. \quad (43)
$$

In our model, we use a Dirichlet boundary condition on the pressure value at the fracture end, that is not the fracture “tip”, where $p = 0$ is imposed. To compare the two behaviors, our outcome in terms of pressure is shifted by a
constant value. Finally, in Fig. 22c we report the fracture length as function of time. The model is able to predict quite accurately the fracture length, with an average relative error of 1.2%.

6.2. Penny-shaped crack

The problem consists on an axisymmetric hydraulic fracture in an infinite medium, that is isotropic, homogeneous, impermeable and with a linear elastic behavior. From the center of the fracture, an incompressible Newtonian fluid is injected, at a constant rate \( Q_0 \). The fracture propagation does not depend on the far-field stress status and, as in the KGD example, it is enough to solve for the net fluid pressure. In Fig. 23a we represent the setup for the problem. The quantities of interest are the same as before, except the fracture half-length, that is now substitute by \( R(t) \), i.e., the fracture radius.

The behavior in space and time, represented by \( g_N(x,t) \), \( p(x,t) \) and \( L(t) \), is provided by the analytical solution through some dimensionless quantities \[93\]: (i) an opening \( \Omega \), (ii) a net pressure \( \Pi \) and (iii) a fracture radius \( \gamma \). The main quantities can be expressed from these dimensionless functions as:

\[
R(t) = \gamma L(t), \quad g_N(x,t) = \varepsilon(t)L(t)\Omega(\rho), \quad p(x,t) = \varepsilon(t)E'\Pi(\rho),
\]

where the similarity variable \( \rho = r/R \) is the dimensionless fracture coordinate. For the zero toughness case, we have \[96\]:

\[
\varepsilon(t) = \left( \frac{\mu'}{E't} \right)^{1/3}, \quad L(t) = Q_0^{1/3} \left( \frac{E'}{\mu'} \right)^{1/3} r^{4/9}, \quad \gamma = \frac{1}{2\pi} \left( \int_0^1 \frac{\overline{\Omega}(\rho) d\rho}{1/3} \right). \tag{45a}
\]

In \[45\], we introduced \( \overline{\Omega} = \frac{\Omega}{\gamma}, \mu' = 12\mu \) and \( E' = \frac{E}{\gamma} \) as in the KDG study. The self-similar fracture opening \( \overline{\Omega}(\rho) \) and the self-similar net fluid pressure \( \Pi(\rho) \) are expressed as sum of a general and a particular solution. The first
respects the governing equations, while the second represents the inlet asymptotic behavior. According to [96], the chosen basis function is a combination of Jacobi polynomials of arbitrary order in the interval \([0, 1]\). In this reference, there is the complete expression for the two self-similarity functions. In the current work, we use just the first order expansion for both \(\bar{\Omega}(\rho)\) and \(\Pi(\rho)\). The pressure solution is unbounded, being \(\lim_{\rho \to 0} \Pi = +\infty\) and \(\lim_{\rho \to 1} \Pi = -\infty\), and the model can struggle in the approximation of this nonphysical values happening at the extremes of the domain.

As for the KGD solution, we highlight that the analytical solutions are not affected by neither the fluid density nor the initial stress regime.

The computational domain simulates one fourth of the problem, composed by 16061 nodes, 13896 hexahedra and 640 interface elements, as shown in Figs. [23a][23c]. The global size is \(100 \times 100 \times 100\) m, with a fracture radius of about 60 m and an average element area of 2.4 m\(^2\). The discretized fracture surface is large enough to allow the propagation of the fracture without geometric constraints. The symmetric boundary conditions are imposed on the two symmetry planes, i.e., the boundaries parallel to \(x\) and \(y\) axes containing the well. The other two faces parallel to \(x\) and \(y\) axes are \(z\)-constrained. The material parameters are \(E = 30\) GPa and \(\nu = 0.25\). The fracture frictional behavior is governed by Coulomb’s criterion, characterized by \(\theta = 30^\circ\) and zero cohesion. The fluid viscosity is \(\mu = 10^{-9}\) MPa \(\cdot\) s. The injection rate and the confining stress are \(Q_0 = 3 \cdot 10^{-2}\) m\(^3\)/s and \(\sigma_0 = 10\) MPa, respectively. Finally, we use the same value of initial conductivity as in the KGD example, i.e., \(C_f,0 = 10\) mD \(\cdot\) m = 9.87 \(\cdot\) \(10^{-15}\) m\(^2\) \(\cdot\) m. The simulated time is 100 s, with \(\Delta t = 1\) s. We emphasize that the proposed mesh is not really suited for a TPFA finite volumes solution scheme, nevertheless, the results prove to be accurate enough to verify the analytical solution.

For two different simulated times, i.e., at half \((t = 50\) s\) and at the end of the simulation \((t = 100\) s\), we show the opening and pressure profiles for the \(\gamma = 0\) fracture trace in Figs. [24a][24b] where the continuous line is the analytical solution. Overall, there is a quite good agreement, for both the aperture (with an integral relative error of 2.8\% for \(t = 100\) s) and the pressure (with an integral relative error of 1.9\% at the same time step). The pressure is slightly different close to the extremes of the domain, i.e., \(\rho = 0\) and \(\rho = 1\), because the theoretical behavior diverges, tending to \(\pm\infty\). As in the KGD case, the propagation criterion provides an integral constraint for the pressure analytical solution, being

\[
\int_0^1 \frac{\Pi}{\sqrt{1 - \rho^2}} \rho \, d\rho = 0. \tag{46}
\]

Using a simple Dirichlet boundary condition on the pressure value at the fracture maximum radius, where \(p = 0\) is imposed, we need to shift our outcome for the sake of comparison. Finally, Fig. [24c] represents the fracture radius as function of time. The model is able to predict quite accurately the fracture length, with an average relative error of 1.5\%.

7. Conclusions

In this work, we have presented a stabilized displacement-Lagrange multiplier-pressure formulation for quasi-static contact mechanics coupled with fracture fluid flow. Our discretization is based on a finite element method for the contact mechanics subproblem combined with a finite volume scheme for the flow subproblem. The global nonlinear problem is solved using an active set strategy. We employ lowest-order continuous finite elements for the

![Figure 24: Results for the penny-shaped crack simulation. From the left to the right: fracture opening, fluid pressure and fracture radius.](image-url)
displacement field, piecewise constant functions for both Lagrange multiplier (traction) and pressure field, and a linear two-point flux approximation for intercell numerical fluxes on the discontinuity surface.

The mixed space we adopt does not automatically satisfy the discrete inf-sup stability condition. To stabilize the formulation, we began by revisiting the well-known macroelement approach, originally proposed for the Stokes equation. We then developed two new algebraic strategies that work under less restrictive assumptions while providing good performance. The resulting approach, both without and with fluid flow in the fracture, has been benchmarked against analytical solutions to validate its behavior in case of normal and frictional activation, as well as in two- and three-dimensional fracture reactivation.

Future developments will deal with the simulation of fluid flow in the matrix coupled with the structural mechanics problem through porosity variation and fracture flow through leak-off. Networks of fractures, for which the computation of the hydraulic conductivity requires particular care, will also be investigated. Finally, a key step to enable high-performance computing applications is the design of an efficient preconditioning strategy for the particular Jacobian linear systems encountered here.

Acknowledgements

Funding was provided by TOTAL S.A. through the FC-MAELSTROM project. The authors wish to thank Randolph Settgast for helpful discussions. Portions of this work were performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07-NA27344.

References

[1] H. Fossen, Structural geology, Cambridge University Press, 2016.
[2] T. I. Urbancic, C.-I. Trifu, R. P. Young, Microseismicity derived fault-planes and their relationship to focal mechanism, stress inversion, and geologic data, Geophys. Res. Lett. 20 (22) (1993) 2475–2478. doi:10.1029/93GL02937
[3] B. Dockrill, Z. K. Shipton, Structural controls on leakage from a natural CO2 geologic storage site: Central Utah, USA, J. Struct. Geol. 32 (11) (2010) 1768–1782. doi:10.1016/j.jsg.2010.01.009
[4] A. Morris, D. A. Ferrill, B. Henderson, Slip-tendency analysis and fault reactivation, Geology 24 (3) (1996) 275–278. doi:10.1130/0091-7613(1996)024<0275:STAFA>2.3.CO;2
[5] X. Zhang, M. J. Thiercelin, R. G. Jeffrey, Effects of frictional geological discontinuities on hydraulic fracture propagation, in: SPE hydraulic fracturing technology conference, Society of Petroleum Engineers, 2007, pp. 1–11. doi:10.2118/106111-MS
[6] M. Ferronato, G. Gambolati, C. Janna, P. Teatini, Numerical modelling of regional faults in land subsidence prediction above gas/oil reservoirs, International journal for numerical and Anal. Methods in geomechanics 32 (6) (2008) 633–657. doi:10.1002/nag.640
[7] P. A. Witherspoon, J. S. Y. Wang, K. Iwai, J. E. Gale, Validity of cubic law for fluid flow in a deformable rock fracture, Water Resour. Res. 16 (6) (1980) 1016–1024. doi:10.1029/WR016i006p01016
[8] R. W. Zimmermann, G. S. Bodvarsson, Hydraulic conductivity of rock fractures, Transp. Porous Media 23 (1) (1996) 1–30. doi:10.1007/BF00145263
[9] X. Zhang, J. Chai, Y. Qin, J. Cao, C. Cao, Experimental Study on Seepage and Stress of Single-fracture Radiation Flow, KSCE J. Civ. Eng. 23 (3) (2019) 1132–1140. doi:10.1007/s12205-019-1519-7
[10] P. Wriggers, Computational Contact Mechanics, 2nd Edition, Springer-Verlag Berlin Heidelberg, 2006. doi:10.1007/978-3-540-32609-0
[11] A. Dahi-Taleghani, J. E. Olson, Numerical modeling of multistranded-hydraulic-fracture propagation: accounting for the interaction between induced and natural fractures, SPE J. 16 (03) (2011) 575–581. doi:10.2118/124884-PA
[12] R. E. Goodman, R. L. Taylor, T. L. Brekke, A model for the mechanics of jointed rock, Journal of Soil Mechanics & Foundations Div. 94 (1968) 637–659.
[13] T. A. Garigov, M. Karimi-Fard, H. A. Tchelepi, Discrete fracture model for coupled flow and geomechanics, Comput. Geosci. 20 (1) (2016) 149–160. doi:10.1007/s10596-015-9564-2
[14] R. R. Settgast, P. Fu, S. D. C. Walsh, J. A. White, C. Annavarapu, F. J. Ryerson, A fully coupled method for massively parallel simulation of hydraulically driven fractures in 3-dimensions, Int. J. Numer. Anal. Methods Geomech. 41 (5) (2017) 627–653. doi:10.1002/nag.2567
[15] J. Rutqvist, Y.-S. Wu, C.-F. Tsang, G. Bodvarsson, A modeling approach for analysis of coupled multiphase fluid flow, heat transfer, and deformation in fractured porous rock, Int. J. Rock Mech. Min. Sci. 39 (4) (2002) 429–442. doi:10.1016/S1569-5046(02)00022-9
[16] J. Rutqvist, J. T. Birkholzer, C.-F. Tsang, Coupled reservoir–geomechanical analysis of the potential for tensile and shear failure associated with CO2 injection in multilayered reservoir–caprock systems, Int. J. Rock Mech. Min. Sci. 45 (2) (2008) 132–143. doi:10.1016/j.ijrmms.2007.04.006
[17] P.-Z. Pan, J. Rutqvist, X.-T. Feng, F. Yan, An approach for modeling rock discontinuous mechanical behavior under multiphase fluid flow conditions, Rock Mech. Rock Eng. 47 (2) (2014) 589–603. doi:10.1007/s00603-013-0428-1
