Zeros of holomorphic functions in the unit disk and $\rho$-trigonometrically convex functions

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Received: date / Accepted: date

Abstract Let $M$ be a subharmonic function with Riesz measure $\mu_M$ on the unit disk $D$ in the complex plane $\mathbb{C}$. Let $f$ be a nonzero holomorphic function on $D$ such that $f$ vanishes on $Z \subset D$, and satisfies $|f| \leq \exp M$ on $D$. Then restrictions on the growth of $\mu_M$ near the boundary of $D$ imply certain restrictions on the distribution of $Z$. We give a quantitative study of this phenomenon in terms of special non-radial test functions constructed using $\rho$-trigonometrically convex functions.

Keywords Holomorphic function · zero set · subharmonic function · Riesz measure · uniqueness theorem · $\rho$-trigonometrically convex function

Mathematics Subject Classification (2000) Primary: 30C15 · 31A05; Secondary: 31A15

1 Introduction

We use our results [5]–[13] on zero subsets of holomorphic functions of one and several variables. The origins of our research including previous results of other authors are also described in sufficient detail in [5]–[13]. Earlier $\rho$-trigonometrically convex and its multidimensional versions, $\rho$-subspherical functions, were used to study zero sets of entire functions with constraints on their growth in the complex plane $\mathbb{C}$ [5, § 4], [7, Theorem 3.3.5] and in $\mathbb{C}^n$ [7, 4.2, Theorem 4.2.7], $1 < n \in \mathbb{N} := \{1, 2, \ldots\}$, of holomorphic functions on the punctured complex plane $\mathbb{C}_* := \mathbb{C} \setminus \{0\}$ [8, 5.1.6], as well as to study the distributions of Riesz measures $\mu := \frac{1}{2\pi} \Delta u$ for subharmonic

The work was supported by a grant of the Russian Science Foundation (project no. 18-11-00002, first author), by grants of the Russian Foundation of Basic Research (projects no. 16-01-00024, 18-51-06002, second author)

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functions \( u \) with constraints on their growth in \( \mathbb{C} \), and in \( \mathbb{R}^m \) \cite[§§ 2–3]{5}, \( 2 < m \in \mathbb{N} \), where \( \mathbb{R} \) is the real line in \( \mathbb{C} \), and \( \Delta \) is the Laplace operator acting in the sense of the theory of distributions. We use \( \rho \)-trigonometrically convex functions to study of zero sequences of holomorphic functions with restrictions to their growth in the unit disk

\[
D := \{ z = re^{i\theta} : 0 \leq r < 1, \, \theta \in \mathbb{R} \} \subset \mathbb{C}; \tag{1.1d}
\]

\[
\mathbb{R}_{++} = \mathbb{R} \cup \{ +\infty \}, \quad \mathbb{R}_{++} := \mathbb{R}_{++} \cup \mathbb{R}_{++}, \quad \mathbb{R}_+ := \mathbb{R} \setminus \{ 0 \}, \tag{1.1r}
\]

\[
\mathbb{R}^+ := \{ x \in \mathbb{R} : x \geq 0 \} \subset \mathbb{C}, \quad \mathbb{R}^+_+ := \mathbb{R}^+ \setminus \{ 0 \}. \tag{1.1+}
\]

Let \( S \subset \mathbb{C} \). By \( \text{Hol}(S) \), \( \text{har}(S) \), \( \text{sbh}(S) \), \( \text{sbh}(S) \) := \( \text{sbh}(S) - \text{sbh}(S) \), \( C^k(S) \) for \( k \in \mathbb{N} \cup \{ \infty \} \), we denote resp. the classes of holomorphic, harmonic, subharmonic, \( \delta \)-subharmonic \cite[3.1]{9}, \( \delta \)-subharmonically \( k \) times differentiable functions \( u \) on an open set \( \Omega \supseteq S \). But \( C(S) \) is the class of all continuous functions on \( S \). We denote the function identically equal to \( -\infty \) or \( +\infty \) on \( S \) by the same symbols \( -\infty \) or \( +\infty \), and

\[ \text{sbh}_r(S) := \text{sbh}(S) \setminus \{ -\infty \}, \quad \delta \text{-shh}_r(S) := \delta \text{-shh}(S) \setminus \{ \infty \}, \quad \text{Hol}_r(S) := \text{Hol}(S) \setminus \{ 0 \}, \]

where the symbol 0 is used to denote the number zero, the origin, zero vector, zero function, zero measure, etc. The positiveness is everywhere understood as \( \geq 0 \) according to the context.

**Definition 1** \cite[8.1, 2]{16} Let \( \rho \in \mathbb{R}^+_+ \). A \( 2\pi \)-periodic function \( h : \mathbb{R} \to \mathbb{R} \) is called a \( \rho \)-trigonometrically convex function if

\[
h(\theta) \leq \frac{\sin \rho (\theta_2 - \theta)}{\sin \rho (\theta_2 - \theta_1)} h(\theta_1) + \frac{\sin \rho (\theta - \theta_1)}{\sin \rho (\theta_2 - \theta_1)} h(\theta_2) \quad \text{for all } \theta \in (\theta_1, \theta_2) \tag{1.2}
\]

and for all \( \theta_1, \theta_2 \in \mathbb{R} \) such that \( 0 < \theta_2 - \theta_1 < \pi / \rho \). A function \( h : \mathbb{R} \to \mathbb{R} \) is a 0-trigonometrically convex function if \( h \equiv \text{const} \in \mathbb{R} \). Further, the class of all \( 2\pi \)-periodic \( \rho \)-trigonometrically convex function on \( \mathbb{R} \) is denoted as \( \rho \text{-trc} \),

\[
\rho \text{-trc}^+ := \{ h \in \rho \text{-trc} : h \geq 0 \text{ on } \mathbb{R} \}, \quad \text{trc} := \bigcup_{\rho \in \mathbb{R}^+_+} \rho \text{-trc}, \quad \rho \text{-trc}^+ := \bigcup_{\rho \in \mathbb{R}^+_+} \rho \text{-trc}^+. \tag{1.3}
\]

We recall some properties of \( 2\pi \)-periodic \( \rho \)-trigonometrically convex functions that can be found in the works \cite{15, 16, 14, 2, 17, 1}.

(i) If \( h \in \text{trc} \), then \( h \in C(\mathbb{R}) \).

(ii) If \( h \in \rho \text{-trc} \), then \( h^+ := \max \{ 0, h \} \in \rho \text{-trc}^+ \).

(iii) Let \( h \in C^2(\mathbb{R}) \) be a \( 2\pi \)-periodic function. \( h \in \rho \text{-trc} \) if and only if

\[
h''(\theta) + \rho^2 h(\theta) \geq 0 \quad \text{for all } \theta \in \mathbb{R}. \tag{1.4}
\]

(iv) A \( 2\pi \)-periodic continuous function \( h \) belongs to the class \( \rho \text{-trc} \) if and only if \( h'' + \rho^2 h \geq 0 \) in the sense of the distribution theory.

(v) A \( 2\pi \)-periodic continuous function \( h \) belongs to the class \( \rho \text{-trc} \) if and only if the function \( z = re^{i\theta} \mapsto h(\theta)r^\rho \) is subharmonic on \( \mathbb{C} \).

(vi) If \( h \in \rho \text{-trc}^+ \) and \( \rho \leq \rho' \in \mathbb{R}^+_+ \), then \( h \in \rho' \text{-trc}^+ \), i.e., \( \rho \text{-trc}^+ \subset \rho' \text{-trc}^+ \).

(vii) If a sequence of functions \( h_n \in \rho \text{-trc}^+ \), \( n \in \mathbb{N} \) is decreasing, then the function

\[
h := \lim_{n \to \infty} h_n \text{ belongs to the same class } \rho \text{-trc}^+.
\]
Example 1 Let $\rho \in \mathbb{R}^+$. The $2\pi$-periodic continuation of the function

$$h(\theta) := \begin{cases} \cos \rho \theta, & \text{if } |\theta| < \frac{\pi}{2\rho}, \\ 0, & \text{if } |\theta| \geq \frac{\pi}{2\rho}, \end{cases} \quad \theta \in (-\pi, \pi],$$

belong to the class $\rho$-trc$^+$. 

Example 2 Let $S \subset \mathbb{C}$ be a bounded subset. Then the support function $k_S(\theta) := \sup_{z \in S} \Re(\mathrm{e}^{-i\theta})$ of $S$ belongs to the class 1-trc. If $0 \in S$, then $k_S \in 1$-trc$^+$. For $\rho \in \mathbb{R}^+$, the $\rho$-support function of $\rho$-convex domain $S \subset \mathbb{C}$ belongs to the class $\rho$-trc, and this $\rho$-support function belongs to the class $\rho$-trc$^+$, when $0 \in S$ [1, Ch. VI, 2.2], [14, Ch. II, §3], [17, §9].

Example 3 Let $u$ be a subharmonic function on $\mathbb{C}$, and $\limsup_{z \to \infty} \frac{u(z)}{|z|^\rho} < +\infty$. Then its $\rho$-indicator function

$$h(\theta) := \limsup_{r \to +\infty} \frac{u(re^{i\theta})}{r^\rho}, \quad \theta \in \mathbb{R},$$

belongs to the class $\rho$-trc. See [15], [16], [14], [2], [17] for $u := \log |f|$ with an entire function $f$ on $\mathbb{C}$.

Let $\mathcal{O} \subset \mathbb{C}$ be an open subset, and let

$$Z := \{z_k\}_{k=1,2,\ldots}, \quad z_k := r_k e^{i\theta_k} \in \mathcal{O}, \quad r_k := |z_k| \in \mathbb{R}^+, \quad \theta_k \in \arg z_k \subset \mathbb{R}, \quad (1.5)$$

be a sequence on $\mathcal{O}$ without limit points in $\mathcal{O}$. Some points $z_k$ can repeat. It is also possible that $Z = \emptyset$ is empty. We associate with each sequence $Z$ the integer-valued positive count measure $n_Z$ on $\mathcal{O}$ by setting

$$n_Z(S) := \sum_{z_k \in S} 1, \quad S \subset \mathcal{O}; \quad (1.6)$$

$n_Z(S)$ is the number of points $z_k$ lying in $S$. We denote by the same symbol as the sequence $Z$ the function $Z : z \mapsto n_Z(z)$, $z \in \mathcal{O}$, the divisor of the sequence $Z$. In particular, we have $\text{supp} Z := \text{supp} n_Z$ for the support $\text{supp} Z \subset D$ means that $\text{supp} Z \subset D$; $z \in Z$ (resp., $z \notin Z$) means the same as $z \in \text{supp} Z$ (resp., as $z \notin \text{supp} Z$).

Departing from the usual treatment of a sequence as a function of an integer or a positive integer variable we say that a sequence $Z$ coincides with a sequence $Z'$ or that they are equal (we write $Z = Z'$) if for the associated divisors we have $Z(z) \equiv Z'(z)$ for all $z \in \mathcal{O}$. In other words, we regard a point sequence as a representative of the equivalence class containing the sequences in $\mathcal{O}$ with equal divisors. An embedding $Z \subset Z'$ means that $Z(z) \leq Z'(z)$ for all $z \in \mathcal{O}$. See [7] in detail.

We denote by $\text{Zero}_f$ the zero sequence of the function $f \in \text{Hol}(\mathcal{O})$ in $\mathcal{O}$ numbered with multiplicities taken into account. Then [18, Theorem 3.7.8]

$$n_{\text{Zero}_f} = \frac{1}{2\pi} \Delta \log |f| \quad (1.7)$$

is the Riesz measure of function $\log |f| \in \sbh(\mathcal{O})$.  

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A function $f \in \text{Hol}_0(\mathcal{O})$ vanishes on $\mathcal{Z}$ if $\mathcal{Z} \subset \text{Zero}_f$ (we write $f(\mathcal{Z}) = 0$). The function $0 \in \text{Hol}(\mathcal{O})$ vanishes on any sequence $\mathcal{Z} \subset \mathcal{O}$.

For $r \in \mathbb{R}^+$ and $z \in \mathbb{C}$, we set $D(z, r) := \{z' \in \mathbb{C} : |z' - z| < r\}$ (i.e., $D(z, r)$ is an open disk of radius $r$ centered at $z$), $D(r) := D(0, r)$, $\bar{D}(z, r) := \{z' \in \mathbb{C} : |z' - z| \leq r\}$ (i.e., $D(z, r)$ is a closed disk of radius $r$ centered at $z$), $\bar{D}(r) := \bar{D}(0, r)$, $\bar{D}(0) := \{0\}$.

The class of all Borel real measures, i.e., charges, on a Borel subset $S \subset \mathbb{C}$ is denoted by $\text{Meas}(S)$, and $\text{Meas}^+(S) \subset \text{Meas}(S)$ is the subclass of all positive measures. For a charge $\mu \in \text{Meas}(S)$, we let $\mu^+ \mu^-$ and $|\mu| := \mu^+ + \mu^-$ resp. denote its upper, lower, and total variations.

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a bounded $2\pi$-periodic Borel function on $\mathbb{R}$; $\mu \in \text{Meas}(\mathbb{D})$. We define the radial counting function $\mu^{\text{rad}}(\cdot ; h)$ of charge $\mu$ with weight $h$ on $[0, 1)$ as [5, (3.1)], [6, (0.2)]

$$
\mu^{\text{rad}}(r; h) := \int_{D(r)} h(\arg z) \, d\mu(z), \quad r \in [0, 1).
$$

In particular, the function $\mu^{\text{rad}}(r) := \mu^{\text{rad}}(r; 1)$ with weight $h \equiv 1$ is the classical radial counting function of $\mu$. If $\mathcal{Z}$ is a sequence in $\mathcal{O}$, then [3], [5, (0.4)], [6, (0.2)]

$$
n^\mathcal{Z}_{2}^{\text{rad}}(r; h) := \sum_{|z_k| \leq r} h(\arg z_k), \quad r \in [0, 1).
$$

Here and below, a reference mark over a symbol of (in)equality, inclusion, or more general binary relation, etc. means that this relation is somehow related to this reference. For $-\infty \leq r < R \leq +\infty$ always

$$
\int_r^R \ldots := \int_{(r, R)} \ldots
$$

A particular result of our investigation is the following

**Uniqueness Theorem** Let $M \in \text{sh}_0(\mathbb{D})$ be a subharmonic function with Riesz measure $\mu_M := \frac{1}{2\pi} \Delta M \in \text{Meas}^+(\mathbb{D})$, and let $Z^1 \{r_k e^{i\theta_k}\}_{k=1,2,\ldots} \subset \mathbb{D}$ be a sequence, $h \in \text{trc}^{+\infty}$, and $g : \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ be a convex function with $g(0) \equiv 0$. If a function $f \in \text{Hol}(\mathbb{D})$ vanishes on $\mathcal{Z}$, satisfies the inequality $|f| \leq \exp M$ on $\mathbb{D}$, and

$$
\int_{1/2}^1 g(2(1-t)) \, d\mu_M^{\text{rad}}(t; h) < +\infty, \quad (1.11M)
$$

but

$$
\sum_{1/2 < r_k < 1} g(1 - r_k) h(\theta_k) \underset{(1.5)}{=} +\infty, \quad (1.11Z)
$$

then $f$ is the zero function, i.e., $f \equiv 0$ on $\mathbb{D}$.
In the case \( M = 0 \) with \( \mu_M = 0 \), \( g(x) \equiv x, x \in \mathbb{R}^+ \), and \( h \equiv 1 \in 0\text{-}trc^+ \), the condition (1.12) contradicts the classical Blaschke condition \( \sum (1 - r_k) < +\infty \). So, the Nevanlinna theorem on the distribution of zeros of bounded holomorphic functions shows that our Uniqueness Theorem is accurate in this case.

By \( \text{const}_{a_1, a_2, \ldots} \in \mathbb{R} \) we denote constants that, in general, depend on \( a_1, a_2, \ldots \) and, unless otherwise specified, only on them; \( \text{const}^+ \geq 0 \).

**Main Theorem** Let \( M \in \delta\text{-}sbh(\mathbb{D}) \) and \( u \in \text{sbh}(\mathbb{D}) \) are functions, resp., with Riesz charge \( \mu_M := \frac{1}{2\pi} \Delta M \in \text{Meas}(\mathbb{D}) \) and with Riesz measure \( \mu_u := \frac{1}{2\pi} \Delta u \in \text{Meas}^+(\mathbb{D}) \).

Let \( \rho \in \mathbb{R}^+ \). If \( \rho \leq M \) on \( \mathbb{D} \), then there exists a constant \( C := \text{const}^+_{\rho, M, u} \geq 0 \) such that the inequality

\[
\int_{1/2}^1 g\left(\frac{1-t}{t}\right) d\mu_M^{\text{rad}}(t; h) \leq \int_{1/2}^1 g\left(\frac{1-t}{t}\right) d\mu_M^\ast(0; h) + C \quad (1.12)
\]

holds for any

- [g] convex function \( g: \mathbb{R}^+ \to \mathbb{R}^+ \) with \( g(0) = 0 \) and \( g(1) \leq 1 \).
- [h] \( 2\pi \text{-periodic } \rho\text{-trigonometrically convex function } h: \mathbb{R} \to [0, 1] \).

In particular, if \( Z \) is a sequence from (1.5) with \( \mathcal{O} := \mathbb{D} \), and there exists a function \( f \in \text{Hol}(\mathbb{D}) \), \( f(Z) = 0 \), satisfying the inequality \( |f| \leq \exp M \) on \( \mathbb{D} \), then there is a constant \( C := \text{const}^+_{\rho, M, Z} \) such that

\[
\sum_{1/2 < r_k < 1} g\left(\frac{1-r_k}{r_k}\right) h(t_k) \leq \int_{1/2}^1 g\left(\frac{1-t}{t}\right) d\mu_M^{\text{rad}}(t; h) + C \quad \text{for any } [g]–[h]. \quad (1.13)
\]

The cases \( u = M \) and \( M = \log |f|, Z = \text{Zero}_f \), show that the inequalities (1.12) and (1.13) uniform with respect to \([h]–[g]\) are optimal up to an additive constant \( C \).

**2 Subharmonic test functions and their role**

By \( C_\infty := C \cup \infty \) we denote the one-point Alexandroff compactification of \( C \). For a subset \( S \subset C_\infty \), \( \text{clos} S \), \( \text{int} S \), and \( \partial S \) are the closure, the interior, and the boundary of \( S \) in \( C_\infty \). A (sub)domain in \( C_\infty \) is an open connected subset in \( C_\infty \). Let \( S_0 \subset S \subset C_\infty \). If the closure \( \text{clos} S_0 \) is a compact subset of \( S \) in the topology induced on \( S \) from \( C_\infty \), then the set \( S_0 \) is the relatively compact subset of \( S \), and we write \( S_0 \subset S \). Let

\[
\emptyset \neq S \subset D \subset C_\infty, \quad \text{where } D \neq C_\infty \text{ is domain.} \quad (2.1)
\]

For a function \( v: D \setminus S \to \mathbb{R} \) we write

\[
\lim_{\partial D} v = 0, \quad \text{if } \lim_{D \ni z' \to z} v(z') = 0 \text{ for all } z \in \partial D. \quad (2.2)
\]

By definition, put

\[
\text{sbh}_0(D \setminus S) := \left\{ v \in \text{sbh}(D \setminus S): \lim_{\partial D} v \overset{(2.2)}{=} 0 \right\}, \quad (2.3a)
\]

\[
\text{sbh}_0^+(D \setminus S) := \left\{ v \in \text{sbh}_0(D \setminus S): v \geq 0 \text{ on } D \right\}. \quad (2.3+)\]
**Definition 2** ([9, Definition 1]) We say that a function \( v \in \text{sbh}_0^+ (D \setminus S) \) is a subharmonic test function on \( D \) outward \( S \) if the function \( v \) is bounded on \( D \setminus S \). The class of such functions \( w \) will be denoted by \( \text{sbh}_0^+ (D \setminus S ; \leq b) \). For \( b \in \mathbb{R}^+ \), put
\[
\text{sbh}_0^+ (D \setminus S ; \leq b) := \left\{ v \in \text{sbh}_0^+ (D \setminus S ; < +\infty) : \sup_{D \setminus S} v \leq b \right\}.
\]
(2.4)
Thus,
\[
\text{sbh}_0^+ (D \setminus S ; < +\infty) = \bigcup_{b \in \mathbb{R}^+} \text{sbh}_0^+ (D \setminus S ; \leq b).
\]
The main role will be played by the following

**Theorem A** ([9, Main Theorem] for \( \mathbb{C} \), see also [10, Main Theorem], [11]–[13]) Let \( M \in \delta - \text{sbh}_0 (D) \) be a \( \delta \)-subharmonic function with Riesz charge \( \mu_M = \frac{1}{2\pi} \Delta M \), and
\[
\emptyset \neq \text{int} S \subset S = \text{clos} S \subset D \subset \mathbb{C}_\infty \neq D.
\]
(2.5)
Then for any point \( z_0 \in \text{int} S \) with \( M(z_0) \in \mathbb{R} \), any number \( b \in \mathbb{R}^+ \) (1.1+), any regular for the Dirichlet Problem \([18, 4]\) domain \( \tilde{D} \subset \mathbb{C}_\infty \) with the Green function \( g^+_{\tilde{D}} (\cdot , z_0) \) with a pole at \( z_0 \) which satisfies the conditions \( S \in \tilde{D} \subset \tilde{D} \) and \( \mathbb{C}_\infty \setminus \text{clos} \tilde{D} \neq \emptyset \), any subharmonic function \( u \in \text{sbh}_0 (D) \) satisfying the inequality \( u \leq M \) on \( D \), and any subharmonic test function \( v \in \text{sbh}_0^+ (D \setminus S ; \leq b) \) the following inequality holds:
\[
\tilde{C} u(z_0) + \int_{D \setminus S} v d\mu_u \leq \int_{D \setminus S} v d\mu_M + \int_{D \setminus S} v d\mu + \tilde{C} \, \mathcal{C}_M,
\]
(2.6)
where \( \mu_u := \frac{1}{2\pi} \Delta u \) is the Riesz measure of the function \( u \),
\[
\tilde{C} := \text{const}^+_{\tilde{S},D,b} := \frac{b}{\inf_{z \in \tilde{D}} g^+_{\tilde{D}}(z, z_0)} > 0,
\]
(2.7)
and the value \( +\infty \) is possible for the constant
\[
\mathcal{C}_M := \int_{D \setminus \{z_0\}} g^+_{\tilde{D}} (\cdot , z_0) d\mu_M + \int_{D \setminus S} g^+_{D} (\cdot , z_0) d\mu_M + M^+ (z_0),
\]
(2.8)
but for \( \tilde{D} \subset D \) this is a certain constant \( \mathcal{C}_M \) \((2.8) \text{ const}^+_{\tilde{S},D,M,D} < +\infty \).

We use the following simplified version of Theorem A.

**Theorem B** Under the agreements \((2.1)\), and \((2.5)\), let \( M \in \delta - \text{sbh}_0 (D) \) be a function with Riesz charge \( \mu_M \in \text{Meas}(D) \). Then, for any function \( u \in \text{sbh}_0 (D) \) with Riesz measure \( \mu_u \) satisfying the inequality \( u \leq M \) on \( D \), we have the inequality
\[
\int_{D \setminus S} v d\mu_u \leq \int_{D \setminus S} v d\mu_M + C \text{ for all } v \in \text{sbh}_0^+ (D \setminus S ; \leq 1),
\]
(2.9)
where a constant \( C := \text{const}^+_{D,S,M} \in \mathbb{R}^+ \) is independent of \( v \in \text{sbh}_0^+ (D \setminus S ; \leq 1) \).
Proof There exists always a point \( z_0 \in \text{int} S \) and \( r_0 \in \mathbb{R}^+ \) such that \([9, 3.1]\)

\[
D(z_0, r_0) \subset \text{int} S, \quad u(z_0) \neq -\infty, \quad M(z_0) \neq \pm \infty, \quad \int_{D(z_0, r_0)} \log |z - z_0| d\mu < +\infty. \tag{2.10}
\]

There is always a regular for the Dirichlet Problem domain \( \tilde{D} \) such that \( \text{int} S \subset \tilde{D} \subset D \) \([18, 4]\). The choice of such point \( z_0 \) and such domain \( D \) is predetermined solely by sets \( S, D \). We choose \( b := 1 \). Thus, \( C^{(2.7)} \) const \( p, S \in \mathbb{R}^+ \) is a constant depending only on \( S \) and \( D \). In view of (2.10), the constant \( C_M \) \((2.8)\) \(\tilde{C}_M \) \((2.10)\) const \( D, S, M \in \mathbb{R}^+ \) is depending only on \( D, S, M \). Hence the constant

\[
C := |\tilde{C}u(z_0)| + |\mu_M| (\tilde{D} \setminus S) + \tilde{C}C_M \geq -\tilde{C}u(z_0) + \int_{D \setminus S} v d\mu - \tilde{C}C_M,
\]

deeps only on \( D, S, u, M \), i.e., \( C = \text{const}_{D, S, u, M} \in \mathbb{R}^+ \). So, (2.9) follows from (2.6).

A method of constructing subharmonic test functions on \( D \) outward \( D(r) \) by means of \( \rho \)-trigonometrically convex positive functions is given by the following

**Proposition 1** Let \( h \in \rho^{\text{trc}} \) be a \( 2\pi \)-periodic \( \rho \)-trigonometrically convex positive function, and let \( g: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a convex function with \( g(0) = 0 \). We set

\[
\frac{1}{2} \leq r_0 := \max \left\{ \frac{1}{2}, 1 - \frac{1}{\rho^2} \right\} < 1. \tag{2.11}
\]

Then the function

\[
z := re^{i\theta} \mapsto g \left( \frac{1 - r}{r} \right) h(\theta), \quad r \in (0, 1), \quad \theta \in \mathbb{R}, \quad z \in D \setminus \{0\}, \tag{2.12}
\]

belongs to the class (see (2.4))

\[
\text{shh} + (\mathbb{D} \setminus \mathcal{D}(r_0); \leq b_0), \quad \text{where} \quad b_0 := g \left( \frac{1 - r_0}{r_0} \right) \max h(\theta). \tag{2.13}
\]

**Proof** We use the properties (i)–(vii) of \( \rho \)-trigonometrically convex functions.

There is a decreasing sequence of convex positive functions \( g_n \xrightarrow{n \to \infty} g \) on \( S \) such that \( g_n(0) = 0 \) and \( g_n \in C^2(\mathbb{R}^+) \), \( n \in \mathbb{N} \). There is also a sequence of \( 2\pi \)-periodic \( \rho \)-trigonometrically convex positive functions \( h_n \xrightarrow{n \to \infty} h \) ([5, Proposition 1.4], [2, Theorem 51]) such that \( h_n \in C^2(\mathbb{R}^+) \), \( n \in \mathbb{N} \). The limit of each decreasing sequence of positive subharmonic functions is a subharmonic positive function. Therefore it suffices to prove the subharmonicity of the function (2.12) outward \( \mathcal{D}(r_0) \) for the case \( h \in C^2(\mathbb{R}) \) and \( g \in C^2(\mathbb{R}^+) \). The calculation of the Laplace operator of the function (2.12) in polar coordinates \((r, \theta)\) gives

\[
\Delta \left( g \left( \frac{1 - r}{r} \right) h(\theta) \right) = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \left( g \left( \frac{1 - r}{r} \right) h(\theta) \right)
\]

\[
= \left( g'' \left( 1 - \frac{1}{r^2} \right) \frac{1}{r^2} + \frac{1}{r^2} g' \left( 1 - \frac{1}{r^2} \right) \right) h(\theta) + \frac{1}{r^2} g' \left( 1 - \frac{1}{r^2} \right) h''(\theta). \tag{2.14}
\]
Each convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$, $g \in C^2(\mathbb{R}^+)$, with $g(0) = 0$ has the following properties:

$g'' \geq 0$, $g'(x) \geq \frac{g(x)}{x}$ for all $x \in \mathbb{R}^+$, and $g \in C(\mathbb{R}^+)$ is increasing.  \hspace{1cm} (2.15)

It follows from (2.14), (2.15), (1.4) that, for $h \in \rho \text{-trc}^+ \cap C^2(\mathbb{R})$,

$$
\Delta \left( g \left( \frac{1-r}{r} \right) h(\theta) \right) \\
\geq \left( \frac{g''(1/r - 1)}{r^2} + \frac{1}{r^2(1-r)} g(1/r - 1) \right) h(\theta) + \frac{1}{r^2} g(1/r - 1) h''(\theta)
$$

$$
\geq \left( \frac{1}{r^2} \left( \frac{1}{1-r} - \rho^2 \right) g(1/r - 1) h(\theta) \right) \text{ for all } r \in \mathbb{R}^+, \theta \in \mathbb{R}. \hspace{1cm} (2.16)
$$

If $r \geq r_\rho$, then the right-hand side of the inequalities (2.16) is positive. Therefore the function (2.12) is subharmonic on $\mathbb{D} \setminus \mathbb{D}(r_\rho)$. Obviously, the function (2.12) is positive, since the functions $h \in \rho \text{-trc}^+$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ are positive, and

$$
g(0) = 0 \implies \lim_{0<r<0} g \left( \frac{1-r}{r} \right) \overset{(2.15)}{=} 0 \implies \lim_{1<r<1} g \left( \frac{1-r}{r} \right) h(\theta) \overset{(2.12)}{=} 0. \hspace{1cm} (2.17)
$$

Besides, in view of (2.15), we have

$$
g \left( \frac{1-r}{r} \right) \max \limits_{\theta} h(\theta) \overset{(2.11)}{\leq} g \left( \frac{1-r_\rho}{r_\rho} \right) \max \limits_{\theta} h(\theta) \overset{(2.13)}{=} b_\rho \text{ for all } r \in (r_\rho, 1).
$$

So, by Definition 2, in view of (2.17), the function (2.12) belongs to the class (2.13).

3 Proofs of main results

**Proof (of Main Theorem)** Let $0 \leq \rho \leq \sqrt{2}$. Then $r_\rho \overset{(2.14)}{=} 1/2$. By Proposition 1, the function (2.12) belong to the class $\text{sbh}_0^\rho (\mathbb{D} \setminus \mathbb{D}(1/2); \leq 1)$, since $g(1) \leq 1$ and $\max \limits_{\theta} h(\theta) \leq 1$ under the conditions $[g]-[h]$ of Main Theorem. Hence, by Theorem B, there exists a constant $C = \text{const}^\nu_{M,a}$ such that the inequality (2.9) holds for any function $\nu$ of the form (2.12). Thus, we obtain

$$
\int_{\mathbb{D}(1/2)} g \left( \frac{1-t_\lambda}{t} \right) h(\theta) d\mu(\theta) \overset{(2.9)}{\leq} C \text{ for any } [g]-[h]. \hspace{1cm} (3.1)
$$

**Lemma 1** ([3], [5]–[7]) Let $f$ be a continuous function on $(r, 1) \subset (0, 1)$, $\mu \in \text{Meas} \mathbb{D}$. Under the conditions before (1.8) we have the equality

$$
\int_{\mathbb{D}(1/2)} f(t) k(\theta) d\mu(\theta) \overset{(1.10)}{=} \int_{r}^{1} f(t) d\mu^\text{rad}(t; k). \hspace{1cm} (3.2)
$$
By Lemma 1, we get from (3.1) the conclusion (1.12) of Main Theorem for \( \rho \leq \sqrt{2} \).

Consider now the case \( \rho > \sqrt{2} \), i.e., \( r^2 \simeq 1 - 1/\rho^2 > 1/2 \). By Proposition 1, the function (2.12) belongs to the class (2.13), and

\[
\operatorname{shb}^+ \left( D \setminus \overline{D}(r_p) \right) \subset b_p \ \subset \operatorname{shb}^+ \left( D \setminus \overline{D}(r_p) \right) \leq 1
\]

since \( g(1) \leq 1 \) and \( \max \theta h(\theta) \leq 1 \) for \([g]-[h] \), and

\[
b_p \leq g \left( \frac{1 - (1 - \rho^{-2})}{1 - \rho^{-2}} \right) \max \theta h(\theta) \leq g(1) \max \theta h(\theta) \leq 1, \quad \text{when} \ \rho > \sqrt{2}.
\]

Hence, by Theorem B, there is a constant \( C' = \operatorname{const}^+_{S,M,\mu} = \operatorname{const}^+_{p,M,\mu} \) for \( S := \overline{D}(r_p) \) such that the inequality (2.9) holds for any function \( v \) of the form (2.12). So, we get

\[
\int_{\overline{D}(r_p) \setminus \overline{D}(1/2)} g \left( \frac{1 - t}{t} \right) h(\theta) \, d\mu_\theta(te^{i\theta}) \leq \int_{\overline{D}(r_p) \setminus \overline{D}(1/2)} g \left( \frac{1 - t}{t} \right) h(\theta) \, d\mu_M(te^{i\theta}) + C' \ \text{for any} \ \ [g]-[h]. \quad (3.3)
\]

It is easy to see that there are constants \( C'' := \operatorname{const}^+_{p,M}, C''' := \operatorname{const}^+_{\rho,M} \) such that

\[
\int_{D(r_p) \setminus \overline{D}(1/2)} g \left( \frac{1 - t}{t} \right) h(\theta) \, d\mu_\theta(te^{i\theta}) \leq \mu_\theta(D(r_p) \setminus \overline{D}(1/2)) \leq C',
\]

\[
\left| \int_{D(r_p) \setminus \overline{D}(1/2)} g \left( \frac{1 - t}{t} \right) h(\theta) \, d\mu_M(te^{i\theta}) \right| \leq |\mu_M|(D(r_p) \setminus \overline{D}(1/2)) \leq C''.
\]

Hence, in view of (3.3), we obtain (3.1) with \( C := C' + C'' + C''' = \operatorname{const}^+_{p,M,\mu} \) for any \([g]-[h]\). By Lemma 1, we again obtain from (3.1) the conclusion (1.12) of Main Theorem already for the case \( \rho > \sqrt{2} \).

Let \( Z \) be a sequence from (1.5) with \( D := \mathbb{D} \), and let \( f \in \text{Hol}_\mu(\mathbb{D}) \) be a function that vanishes on the sequence \( Z \subset \text{Zero}_\theta \) and satisfies the inequality \( u := \log |f| \leq M \). By the conclusion (1.12) of Main Theorem, there exists a constant \( C := \operatorname{const}^+_{p,M,f} \) such that we have (1.12) for \( u := \log |f| \). Here the choice of the function \( f \) is predetermined solely by the sequence \( Z \) and function \( M \). So, \( C = \operatorname{const}^+_{p,M,Z} \) and, in view of the (in)equalities

\[
\sum_{1/2 < r_k < 1} g \left( \frac{1 - r_k}{r_k} \right) h(\theta_k) \leq \int_{1/2}^1 g \left( \frac{1 - t}{t} \right) d\mu_\theta^\text{rad}(t; h) \]

\[
\leq \int_{1/2}^1 g \left( \frac{1 - t}{t} \right) d\mu_\theta^\text{rad}(t; h) \leq \int_{1/2}^1 g \left( \frac{1 - t}{t} \right) d\mu_\theta^\text{rad}(t; h) \quad \text{for} \ u := \log |f|,
\]

the inequality (1.13) follows from (1.12).
Proof (of Uniqueness Theorem) Without loss of generality, we can assume that \( h \not\equiv 0 \), i.e., \( h_0 := \max_\theta h(\theta) > 0 \), and \( g(1) > 0 \). If \( f \in \text{Hol}_s(\mathbb{D}) \), i.e., \( f \neq 0 \), \( f'(Z) = 0 \), and \( |f| \leq \exp M \) on \( \mathbb{D} \), then, by Main Theorem, we have

\[
\frac{1}{g(1)h_0} \sum_{1/2 < r_k < 1} g(1 - r_k)h(\theta_k) \leq \sum_{1/2 < r_k < 1} \frac{1}{g(1)} g\left(\frac{1 - r_k}{r_k}\right) \frac{1}{h_0}h(\theta_k) \leq \int_{1/2}^1 \frac{1}{g(1)} g\left(\frac{1 - t}{t}\right) d\mu_M(t; h/h_0) \leq \int_{1/2}^1 g(2(1 - t)) d\mu_M(t; h) < +\infty.
\]

So, if \( f \neq 0 \), then the latter contradicts the condition (1.11Z).

Acknowledgements The authors thank the organizers of International Conferences “Complex Analysis and Related Topics 2018” (April 23–27, 2018, Euler International Mathematical Institute, St. Petersburg, Russia) and “XXVII St.Petersburg Summer Meeting in Mathematical Analysis” (August 6–11, 2018, St. Petersburg, Russia) for the invitation and for the opportunity to report the results related to the content of this article.

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