Optimum-width upward order-preserving poly-line drawings of trees

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Abstract An upward drawing of a tree is a drawing such that no parents are below their children. It is order-preserving if the edges to children appear in prescribed order around each vertex. Chan showed that any tree has an upward order-preserving drawing with width $O(\log n)$. In this paper, we consider upward order-preserving drawings where edges are allowed to have bends. We present a linear-time algorithm that finds such drawings with instance-optimal width, i.e., the width is the minimum possible for the input tree. We also briefly study order-preserving upward straight-line drawings, and show that some trees require larger width if drawings must additionally be straight-line.

1 Introduction

An ideal drawing of a tree [4] is one that is planar (no edges cross), strictly-upward (the curves from parents to children are strictly $y$-monotone), order-preserving (a given order of children is maintained in the drawing) and straight-line (edges are drawn straight-line segments). Chan [4] gave algorithms that achieve ideal drawings of area $O(n^{4\sqrt{\log n}})$ and width $O(2^{O(\sqrt{\log n})})$. He also briefly mentioned that a variant of the algorithm achieves width $O(\log n)$, and one can also achieve height $O(n)$ by adding one bend per edge. For binary trees, Garg and Rusu showed that $O(\log n)$ width and $O(n \log n)$ area can be achieved even for straight-line drawings [8]. See the recent overview paper by Frati and Di Battista [2] for many other related results.

Our results: This paper was motivated by the quest of finding ideal drawings for which the width is instance-optimal, i.e., tree $T$ is drawn with the smallest width that is possible for $T$. This problem remains unsolved. We here relax the restriction of straight-line drawings and instead study poly-line drawings where edges may have bends. We give a linear-time algorithm to find order-preserving strictly-upward planar poly-line drawings of trees that have optimal width. (For the rest of this paper, all drawings are required to be order-preserving, upward, and planar, and we will occasionally omit these quantifiers.) The approach is to

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1 Frati and Di Battista [2] asked later whether every tree has an upward order-preserving poly-line drawing of area $O(n \log n)$; the remark by Chan proves this.
first define a recursive rank-function $R(T)$, and show that this exactly describes the optimum width of a strictly-upward order-preserving drawing of $T$. Computing the rank can be done in linear time, and produces a rank-witness, which is enough information to construct the optimum-width drawing. We also briefly discuss straight-line drawings, and show that these sometimes require a larger width than poly-line drawings.

To our knowledge no previous paper addressed the issue of finding upward tree drawings with instance-optimal width. Alam et al.\cite{1} showed how to find upward tree drawings with instance-optimal height. In a companion paper \cite{3}, we also study upward drawings with instance-optimal width, focussing on the model of unordered (i.e., not necessarily order-preserving) drawing. As a byproduct, this paper also gives 2-approximation algorithms for the width of ideal drawings.

A few notations: Let $T$ be a tree rooted at node $u_r$, and assume that for every node the order of the children is fixed. Let the degree $\deg(u_r)$ of the root be the number of children of $u_r$. Let $c_1, \ldots, c_d$ be the children of the root, enumerated from left to right. We say that $c_i$ is “left of $c_j$” if $i \leq j$, and “strictly left of $c_j$” if $i < j$. Similarly define “right of”, “strictly right of”, “between” and “strictly between”. For any child $c_i$, let $T_{c_i}$ be the sub-tree rooted at child $c_i$.

We aim to find a poly-line drawing of $T$, which means that every edge is represented by a poly-line, i.e., a piecewise linear curve. All drawings in this paper require that vertices and bends of poly-lines have integral $x$-coordinate. The width of such a drawing is the smallest $W$ such that (after possible translation) all $x$-coordinates are between 1 and $W$. Column $X$ describes the vertical line with $x$-coordinate $X$. In some situations we analyze the height as well, and then require that all vertices and bends have integral $y$-coordinate and measure the height by the number of rows intersected by the drawing.

2 The rank-function

Key to our drawing algorithm is to express the width of drawings via another parameter that is defined from the structure of the tree alone. The rank-function $R(T)$ is defined recursively as follows:

**Definition 1.** Let $T$ be a tree and $c_1, \ldots, c_d$ be the children of the root from left to right. Define the rank $R(T)$ to be 1 if $T$ is a single-node tree, and to be the smallest value $W$ such that there exists a rank-$W$-witness for $T$ otherwise. Here, for given integer $W \geq 1$, a rank-$W$-witness for $T$ consists of

- a classification of $c_1, \ldots, c_d$ as either big or small,
- a coordinate $X$ which is an integer with $1 \leq X \leq W$, and
- an index of the vertical child, which is an index $v \in \{1, \ldots, d\}$ such that $c_v$ is a big child.

Such a rank-$W$-witness must satisfy the following rank-conditions:

**R1f** At most $X - 1$ big children are strictly left of $c_v$.
(R1r) At most $W - X$ big children are strictly right of $c_i$.

(R2f) Any small child $c_i$ with $i < v$ satisfies $R(T_{c_i}) \leq X - 1 - \ell_i$, where $\ell_i$ is the number of big children to the left of $c_i$.

(R2r) Any small child $c_i$ with $i > v$ satisfies $R(T_{c_i}) \leq W - X - r_i$, where $r_i$ is the number of big children to the right of $c_i$.

(R3) The ranks of the big children are dominated by a permutation of $\{1, \ldots, W\}$.

In other words, one can assign a rank-bound $\pi(c_i) \in \{1, \ldots, W\}$ to each big child $c_i$ such that for all $w \in \{W', \ldots, W\}$:

(C1) $T_{c_{\pi(w)}}$ has rank at most $w$, with equality for $w > W'$.

(C2) For any $i$ with $\sigma(w) < i < \sigma(w + 1)$, $T_{c_i}$ has rank at most $w - 1$.

Symmetrically, let a right-corner-$W$-witness consists of a number $1 \leq W' \leq W$ and a sequence $0 = \sigma(W) < \sigma(W') < \sigma(W' + 1) = d + 1$ such that for all $w \in \{W', \ldots, W\}$ child $c_{\sigma(w)}$ has rank at most $w$, and the children strictly between $c_{\sigma(w + 1)}$ and $c_{\sigma(w)}$ have rank at most $w - 1$. A corner-$W$-witness is a left-corner-$W$-witness or a right-corner-$W$-witness.

**Figure 1.** Illustration for (left) a rank-$W$-witness and (right) a left-corner-$W$-witness.

Outline: We briefly outline our approach to finding optimum-width poly-line drawings. First, we show in Section 3 that from a left-corner-$W$-witness, we can easily construct a drawing of width $W$. A symmetric construction converts a right-corner-$W$-witness into a drawing of width $W$. Next, we show in Section 4 that from any (upward, order-preserving) drawing of width $W$ we can extract a rank-$W$-witness. Finally, to close the cycle, we show in Section 5 that any rank-$W$-witness implies the existence of a corner-$W$-witness. Hence the rank of a tree
equals the minimum width of an upward order-preserving drawing. The proof in Section 5 is constructive and in particular allows to test in linear time whether a corner-$W$-witness exists. Since the construction in Section 3 also takes linear time, this shows the following:

**Theorem 1.** For any tree $T$, we can find in linear time a planar strictly-upward order-preserving poly-line drawing that has optimum width.

Moreover, the root is placed in the top-left or top-right corner, and we can either choose to have linear height and at most 3 bends per edge, or to have at most 1 bend per edge.

We find it especially interesting that we can always assume the root to be in a corner without increasing width. Many previous tree-drawing algorithms (e.g. [5,4,8]) created drawings with the root in a corner, but proving, without going through rank-witnesses, that the root can be moved into a corner without increasing width seems daunting. Indeed, as we show in Section 6, this claim is not true for straight-line drawings.

3 From rank-witness to drawing

In this section, we use rank-witnesses to create drawings. We need the following theorem, whose lengthy proof is deferred to Section 5:

**Theorem 2.** Let $T$ be a tree with $n \geq 2$ nodes. If $R(T) = W$, then $T$ has a corner-$W$-witness.

**Lemma 1.** Any $n$-node tree $T$ has a planar strictly-upward order-preserving poly-line drawing of width $R(T)$ where the root has $x$-coordinate 1 or $W$.

Moreover, we can create such a drawing with at most 1 bend per edge. Alternatively, we can create such a drawing with at most 3 bends per edge and height at most $2n - 1$.

**Proof.** We proceed by induction on the height of $T$. The claim clearly holds if $T$ is a single node since $R(T) = 1$ and $T$ can be drawn with width 1 and height $1 = 2n - 1$. For the step, let $c_1, \ldots, c_d$ be the children of the root $u$, from left to right. Recursively find a drawing $\Gamma_{c_i}$ of $T_{c_i}$ with optimum width. If $R(T) = W$, there exists some rank-$W$-witness, and by Theorem 2 hence a corner-$W$-witness. We assume that this is a left-corner-$W$-witness; the construction is symmetric (and yields a drawing with $x$-coordinate $W$) if there is a right-corner-$W$-witness.

So we have a sequence $1 = \sigma(W) < \cdots < \sigma(W) < \sigma(W + 1) = d + 1$ (for some $1 \leq W' \leq W$) such that (C1) and (C2) hold. Declare a child to be *big* if its index is $\sigma(w)$ for some $1 \leq w \leq W$ and *small* otherwise.

Place the root at the top left corner. We place the children in two steps: first place the small children (and start poly-lines for the edges to big children), and then place the big children. See the figure below for an example.

**Step (1):** We parse the children in order $c_d, c_{d-1}, \ldots$
Presume that \( c_d, \ldots, c_{j+1} \) have already been handled for some \( 2 \leq j \leq d \), and \( Y \) is the lowest \( y \)-coordinate that has been used for them. Place a bend for \((u_r, c_j)\) in column 2 with \( y \)-coordinate \( Y-1 \). All edges \((u_r, c_k)\) with \( k > j \) received bends in column 2 at larger \( y \)-coordinate, so this respects the order of edges around \( u_r \).

Assume first that \( c_j \) is a small child, say \( \sigma(w-1) < j < \sigma(w) \) for some \( W' < w \leq W+1 \). Place \( \Gamma_{c_j} \) in rows \( Y-2 \) and below, and within columns \( 2, \ldots, w-1 \). This fits since by (C2) the rank of \( c_j \) is at most \( w-2 \), and so \( \Gamma_{c_j} \) occupies at most \( w-2 \) columns. We can connect \( c_j \) to the bend for edge \((u_r, c_j)\) with a straight-line segment since \( c_j \) is in the top row of \( \Gamma_{c_j} \), and hence one row below the bend.

Now assume that \( c_j \) is a big child, say \( j = \sigma(w) \) for some \( W' < w \leq W \). Place another bend for edge \((u_r, c_j)\) at point \((w, Y-1)\) and connect it horizontally to the bend at \((2, Y-1)\). Reserve the downward ray from this bend in column \( w \) for this edge; note that by construction no small child placed later will intersect this ray.

This continues until we are left with \( c_1 = c_{\sigma(W')} \). Assign the downward ray in column 1 from the root to \( c_1 \).

We have created some horizontal edges, and so the drawing, while upward, is not strictly-upward. We can make it strictly-upward by re-locating the second bend for each edge to a big child to one row below, i.e., within the ray reserved for that edge.

**Step (2):** At this point all drawings of small children are placed, and the edge to each big child \( c_{\sigma(w)} \) is routed up to a vertical downward ray in column \( w \) (resp. column 1 for \( w = W' \)). Place \( \Gamma_{c_{\sigma(W')}}, \Gamma_{c_{\sigma(W'+1)}}, \ldots, \Gamma_{c_{\sigma(W)}} \), in this order from top to bottom, below the drawing and flush left with column 1. For \( w \in \{W', \ldots, W\} \), since \( c_{\sigma(w)} \) has rank at most \( w \), its drawing will not intersect the rays to \( c_{\sigma(w+1)}, \ldots, c_{\sigma(W)} \). By inserting a bend (if needed) in the row just above \( c_{\sigma(w)} \), we can complete the drawing of \((u_r, c_{\sigma(w)})\).

**Height-bound:** Observe that every row of the drawing contains the root, or intersects some drawing \( \Gamma_{c_i} \), or contains the first bend of the edge \((u_r, c_i)\) for some child \( c_i \). Hence the total height is at most \( 1 + \sum_{i=1}^{d} \) (height of \( \Gamma_{c_i} \)) + \( d \), which by induction is at most \( 1 + \sum_{i=1}^{d} (2n(T_{c_i}) - 1) + d = 2n - 1 \).

\[ \text{This bend can often be omitted, e.g. if } c_j \text{ is small and in the top left corner of } \Gamma_{c_j}, \]

\[ \text{but we show them in the figure for consistency.} \]
Reducing bends: Every edge from \( u \) to a small child is drawn with one bend. For a big child \( c_{\sigma(w)} \), the edge from \( u \) may have up to three bends. However, its poly-line consists of at most two \( x \)-monotone parts: from \( u \) to column \( w \), and from column \( w \) to \( c_{\sigma(w)} \). After subdividing at a point in column \( w \), we hence obtain a tree drawing where all edges are \( x \)-monotone. It is known [6,9] that such a drawing can be turned into a straight-line drawing without increasing the width. Neither of these references discusses whether strictly upward drawings remain strictly upward, but it is not hard to see that this can be done, essentially by “moving subtrees down” sufficiently far. We hence obtain a drawing with one bend per edge, at the cost of increasing the height.

4 From drawing to rank-witness

Lemma 2. If \( T \) has an upward order-preserving poly-line drawing of width \( W \), then \( R(T) \leq W \).

Moreover, if \( T \) is not a single node, then \( T \) has a rank-\( W \)-witness for which coordinate \( X \) equals the \( x \)-coordinate of the root.

Proof. If \( T \) is a single node then \( R(T) = 1 \leq W \) and the claim holds. So assume that the root \( u \) has children \( c_1, \ldots, c_d \) for some \( d \geq 1 \), and let \( X \) be the \( x \)-coordinate of \( u \). If there exists no edge that leaves \( u \) vertically, then modify \( \Gamma \) slightly as follows. Let \( c_i \) be the last child (in the order of children) that comes before the vertical ray downwards from \( u \). (If there is no such child, then instead take the first child that comes after that ray.) Re-route the edge \( (u, c_i) \) so that it goes vertically downward from \( u \) for a brief while, then has a bend, and then connects to where the old route crosses the column \( X−1 \) (respectively \( X+1 \)) for the first time. This adds no crossing and no width. So we may assume that one edge leaves \( u \) vertically; set \( c_v \) to be the corresponding child.

To classify each child \( c \) as big or small, we study the induced drawing of its subtree. Let \( \Gamma_c \) be the drawing of \( T_c \) induced by \( \Gamma \). Let \( \Gamma_c^+ \) be \( \Gamma_c \) together with the poly-line representing edge \( (u, c) \), but excluding the point of \( u \). We declare \( c \) to be big if \( \Gamma_c^+ \) contains a point in column \( X \); small otherwise. With this \( c_v \) is always a big child as desired.

The goal is to show that this classification as big/small, coordinate \( X \), and index \( v \) satisfies the conditions for a rank-\( W \)-witness.

Condition \( (R1\ell) \) and \( (R1r) \): We only prove \( (R1\ell) \) here; \( (R1r) \) is similar. Consider Figure 3(left). Let \( q \) be any point below \( u \) on the vertical segment of edge \( (u, c_v) \). Let \( c_i \) be any big child strictly left of \( c_v \). Since the drawing is order-preserving, edge \( (u, c_i) \) start towards \( x \)-coordinates less than \( X \). Since \( c_i \) is big, drawing \( \Gamma_c^+ \) contains a point with \( x \)-coordinate \( X \); let \( p_i \) be the topmost such point. Due to the vertical line-segment \( uq \), point \( p_i \) is below \( q \). Let \( P_i \) be the poly-line within \( \Gamma_c^+ \) that connects \( u \) to \( p_i \); this exists since \( \Gamma_c^+ \) is a drawing of a connected subtree. By choice of \( p_i \) all points in \( P_i \) have \( x \)-coordinate at most \( X \).
If there are \( k \) big children strictly left of \( c_w \) then we hence obtain \( k \) poly-lines \( P_1, \ldots, P_k \), which are disjoint except at \( u_r \) and reside within columns 1, \ldots, \( X \). They all bypass point \( q \) in the sense that they begin above \( q \) (in the same column) and end below \( q \) (in the same column). One can argue (details are in the appendix) that each poly-line requires a column distinct from the one containing \( q \) or used for the other poly-lines. Since point \( q \) and the poly-lines are all within columns 1, \ldots, \( X \), this shows \( k \leq X - 1 \) as desired.

**Figure 3.** (Left) \( k \) poly-lines bypass \( q \) within \( X \) columns, hence \( k + 1 \leq X \). (Middle) \( \ell_i \) poly-lines bypass \( P_i \) within \( X - 1 \) columns, hence \( \ell_i + R(T_{c_i}) \leq X - 1 \). (Right) \( W - w \) poly-lines bypass \( P_w \) within \( W \) columns, hence \( W - w + R(T_{c_{\sigma(w)}}) \leq W \).

**Conditions (R2̲ℓ) and (R2̲r):** We only prove (R2̲ℓ) here; (R2̲r) is similar. Fix one small child \( c_i \) left of \( c_w \). The goal is to find \( \ell_i \) poly-lines (one for each big child left of \( c_i \)) that bypass \( \Gamma_{c_i} \) in some sense. These poly-lines block \( \ell_i \) columns, leaving \( X - 1 - \ell_i \) columns for \( \Gamma_{c_i} \), hence \( R(T_{c_i}) \leq X - 1 - \ell_i \) by induction.

Consider Figure 3 (middle). Let \( p_i \) be the leftmost point of drawing \( \Gamma_{c_i}^+ \), breaking ties arbitrarily. Let \( q_i \) be the point where the initial line segment of \((u_r, c_i)\) intersects column \( X - 1 \); this must exist since edge \((u_r, c_i)\) leaves \( u_r \) vertically and \((u_r, c_i)\) must leave to the left of this. Let \( P_i \) be the poly-line from \( q_i \) to \( p_i \) within drawing \( \Gamma_{c_i}^+ \). Since \( c_i \) is small, \( P_i \) does not use column \( X \).

Let \( c_h \) be a big child to the left of \( c_i \) and let \( q_h \) be the point where the initial line segment of \((u_r, c_h)\) intersects column \( X - 1 \). Since the drawing is order-preserving, \( q_h \) is above \( q_i \). Since \( c_h \) is big, drawing \( \Gamma_{c_h}^+ \) intersects column \( X \), and in particular therefore has a line segment \( p_h p_h' \) with \( p_h \) in column \( X - 1 \) and \( p_h' \)
in column $X$. Since $p_h q'_h$ must not intersect $u_r q_r$, $p_h$ must be below $q_r$. Re-define $p_h$, if necessary, to be the topmost point below $q_r$ where $\Gamma^+_j$ intersects column $X - 1$. Let $P_h$ be the poly-line from $q_h$ to $p_h$ within $\Gamma^+_j$. By choice of $p_h$ and due to line segment $u_r q_r$, poly-line $P_h$ is within coordinates $1, \ldots, X - 1$.

Repeating this for all $\ell_i$ big children left of $c_i$ we obtain $\ell_i$ poly-lines that reside within $1, \ldots, X - 1$ and that bypass $P_i$ in the sense that they begin and end in column $X - 1$, with one end above $q_i$ and the other below $q_i$. Again one can show that these $\ell_i$ poly-lines each require one column in $\{1, \ldots, X - 1\}$ that does not intersect $P_i$. Therefore $P_i$ (and with it $\Gamma_{c_i}$) has width at most $X - 1 - \ell_i$, so $R(T_{c_i}) \leq X - 1 - \ell_i$ by induction.

**Condition (R3):** To verify this condition, we extract rank-bounds from drawing $\Gamma$ as follows. Let $p_w$ be the lowest point in column $X$ that is occupied by some element of $\Gamma$. Due to the vertical segment of edge $(u_r, c_w)$, point $p_w$ is not the locus of the root. Let $c_j$ be the child such that $\Gamma^+_j$ contains $p_w$; by definition $c_j$ is big. Set $\sigma(w) := j$ and $\pi(c_j) := W$. (Similarly as for corner-witnesses, we use $\sigma(w)$ for the index of a big child with rank at most $w$.)

Now repeat, but ignore all points that belong to $\Gamma^+_c$. Thus, presume we have found $\sigma(W), \sigma(W - 1), \ldots, \sigma(w + 1)$ already for some $w < W$. Let $p_w$ be the lowest point in column $X$ that is occupied by some element in $\Gamma$ but that does not belong to any of $\Gamma^+_c, \Gamma^+_c, \ldots, \Gamma^+_c$. If this point is at $u_r$, then stop: we have assigned a rank-bound to all big children. Else, let $c_j$ be the child such that $\Gamma^+_j$ contains $p_w$, set $\sigma(w) := j$ and $\pi(c_j) := w$, and repeat.

We must show that the chosen values are indeed rank-bounds, i.e., $R(T_{c_{\sigma(w)}}) \leq w$, for all $w$ where $\sigma(w)$ is defined. By induction it suffices to show that the width of $\Gamma_{c_{\sigma(w)}}$ is at most $w$. Consider Figure 3(right). Let $\hat{P}$ be the poly-line within $\Gamma_{c_{\sigma(w)}}$ that connects a leftmost and rightmost point of $\Gamma_{c_{\sigma(w)}}$. For any $j > w$, let $P_j$ be the poly-line that connects $u_r$ with point $p_j$ within $\Gamma_{c_{\sigma(j)}}$.

Poly-line $\hat{P}$ spans the width of $\Gamma_{c_{\sigma(w)}}$, and hence must cross column $X$, say at point $\hat{q}$. This point cannot be below $p_w$ due to choice of $p_w$ as the lowest point in column $X$ that is not in $\Gamma^+_c, \ldots, \Gamma^+_c$. For any $j > w$ point $p_j$ is below $p_w$ and hence also below $\hat{q}$. On the other hand $\hat{P}$ does not contain $u_r$ (since it resides within $\Gamma_{c_{\sigma(w)}}$, not $\Gamma^+_c$), and so $\hat{q}$ is below $u_r$.

We now have found $W - w$ poly-lines $P_{w + 1}, \ldots, P_W$ that bypass $\hat{P}$ in the sense that $P_j$ connects $u_r$ (a point above $\hat{q}$) with $p_j$ (a point below $\hat{q}$), and these poly-lines are vertex-disjoint from $\hat{P}$ and from each other except at $u_r$. Therefore each poly-line requires a column of its own that does not contain $\hat{P}$. Since there are $W - w$ such poly-lines, and the drawing of $T$ has width $W$, therefore $\hat{P}$ (and with it $\Gamma_{c_{\sigma(w)}}$) has width at most $w$.

This proves that the chosen classification, coordinate, and index indeed are a rank-$W$-witness, so $R(T) \leq W$ as desired. \hfill $\square$
5 Transforming rank-witnesses

The overall goal of this section is to show that any rank-$W$-witness gives rise to a left-corner-$W$-witness or a right-corner-$W$-witness. We first need to study some other results about ranks and rank-witnesses.

**Observation 1** The rank of a tree $T$ with $n \geq 2$ nodes is never smaller than the rank of any child of the root.

*Proof.* Assume for contradiction that $T$ has a rank-$W$-witness but $R(T_c) > W$ for some child $c$. Then $c$ cannot be big by (R3), but it also cannot be small by (R2ℓ) and (R2r). \hfill $\square$

**Observation 2** Assume that $T$ is a tree with $n \geq 2$ nodes and all children of the root have rank at most $W-1$. Then $T$ has a left-corner-$W$-witness.

*Proof.* Set $W' = W$, and $\sigma(W) = 1$. Since any child has rank at most $W-1$, both (C1) and (C2) hold. \hfill $\square$

In consequence, the rank of a tree with $n \geq 2$ nodes is always $W$ or $W+1$, where $W$ is the maximum rank of a child.

**Observation 3** If $T$ has a left-corner-$W$-witness, then it also has a rank-$W$-witness with $X = 1$ and $v = 1$.

*Proof.* Declare $c_{\sigma(W')}, \ldots, c_{\sigma(W)}$ to be big and all other children to be small. Set $X = 1$ and $v = 1$ ($c_1 = c_{\sigma(W')}$ is big as required) and verify all conditions. (R3) holds by setting $\pi(c_{\sigma(w)}) = w$ for $w \in \{W', \ldots, W\}$; this works by (C1). (R1ℓ) and (R2ℓ) hold trivially since $v = 1$. (R1r) holds since there $W' \geq 1 = X$ and there are $W - W'$ big children right of the vertical child $c_1 = c_{\sigma(W')}$. To show (R2r), fix any index $i$ of a small child and let $w$ be such that $\sigma(w-1) < i < \sigma(w)$. We know that $c_i$ has rank at most $w - 2$ by (C2). Also $r_i = W - w + 1$ since $c_i$ has big children $c_{\sigma(w)}, \ldots, c_{\sigma(W)}$ to its right. Hence the rank of $c_i$ is at most $w - 2 = W - 1 - (W - w + 1) = W - X - r_i$. \hfill $\square$

We first show that for any rank-$W$-witness, we can “push” the root into one of the corners.

**Lemma 3.** Let $T$ be a tree. If $W := R(T) \geq 2$, then $T$ has a rank-$W$-witness with $X = 1$ and $v = 1$, or with $X = W$ and $v = d$.

*Proof.* If all children have rank at most $W - 1$, then this holds by Observation 2 and 3, so assume some child $c_m$ has rank $W$. Fix any rank-$W$-witness of $T$, and assume $1 < X < W$ for its coordinate, otherwise we are done.

By (R2ℓ) and (R2r), any small child has rank at most $X - 1 \leq W - 2$ or $W - X \leq W - 2$. Therefore any child of rank $W - 1$ or $W$ must be big. By (R3) therefore we have no child other than $c_m$ with rank $W$, and at most one child $c_s$ with rank $W - 1$. 

Assume that \( c_s \) either doesn’t exist or is strictly right of \( c_m \). Create a rank-
W-witness using \( X = 1 \) and \( v = 1 \) and declaring \( c_1 \) and \( c_m \) to be big and all
other children to be small. Verify the conditions for this new witness as follows.
(R3) holds since we have at most two big children, and only one of them has
rank \( W \). (R1\(T \)) and (R2\(T \)) hold trivially since \( v = 1 \). (R1r) holds since at most
\( 1 \leq W - 1 \) big children are right of \( c_1 \). (R2r) holds for \( i > m \) since then \( r_i = 0 \)
and \( c_i \) has rank at most \( W - 1 \). It also holds for \( 1 < i < m \) since then \( r_i = 1 \)
and \( c_i \) has rank at most \( W - 2 \) since \( c_s \) (if it exists) is strictly right of \( c_m \).

This creates a rank-W-witness with \( X = 1 \) if \( c_s \) doesn’t exist or is strictly
right of \( c_m \). If \( c_s \) is strictly left of \( c_m \), then similarly we can create a rank-W-
with \( X = W \) and \( v = d \).

\[ \square \]

Now we show that the other conditions on corner-witnesses can be achieved,
possibly after re-defining which children are big. The lemma also gives an algo-
rithm that finds a witness by parsing the the ranks of the children.

**Lemma 4.** Presume \( T \) is a tree with \( n \geq 2 \) nodes and the ranks of the children
of the root are known. For any \( W \geq 1 \), we can test in \( O(d) \) time whether \( T \) has
a rank-W-witness with \( X = 1 \) and \( v = 1 \). Moreover, if \( T \) has such a rank-W-
witness, then the algorithm returns a left-corner-W-witness.

**Proof.** We operate under the assumption that some rank-W-witness with \( X = 1 \)
and \( v = 1 \) exists, and with the following algorithm either find a contradiction,
or obtain the required sequence that proves the existence of a left-corner-W-
witness. Let \( c_{\sigma(W)} \) be the child with rank \( W \). Note that \( c_{\sigma(W)} \) must be big since
it is either \( c_1 \) (which must be big) or it is to the right of the vertical child \( c_1 \) and
(R2r) allows only width \( W - 1 \) for children that are small. We iterate this process,
say for some \( k \geq 0 \) we have found \( \sigma(W - k) < \sigma(W - k + 1) < \cdots < \sigma(W) \) such
that \( c_{\sigma(w)} \) has rank \( w \) and must be big for all \( w \in \{W - k, \ldots, W\} \).

The iteration continues if some child strictly left of \( c_{\sigma(W - k)} \) has rank \( W - k - 1 \)
or higher. Set \( \sigma(W - k - 1) < \sigma(W - k) \) to be the largest index of such a child.
Then \( c_{\sigma(W - k - 1)} \) has \( k + 1 \) siblings strictly to the right that are big. Therefore
\( c_{\sigma(W - k - 1)} \) must be big, because (R2r) allows rank at most \( W - X - (k + 1) =
W - k - 2 \) for a small child that has \( k + 1 \) big children to its right. If (R3)
holds, then at least one of the \( k + 2 \) big children \( c_{\sigma(W - k - 1)} \), \ldots, \( c_{\sigma(W)} \) must have
rank-bound \( W - k - 1 \), and this can only be \( c_{\sigma(W - k - 1)} \) since the others have
higher rank. So if \( c_{\sigma(W - k - 1)} \) has rank \( W - k \) or higher then no rank-W-witness
with \( X = 1 \) and \( v = 1 \) exists and we can stop. Else iterate the process.

If none of the iterations disproves the existence of a witness, then we stop
at some value \( W^* \) with \( 1 \leq W^* \leq W \) such that such that no child \( c_i \) strictly
left of \( c_{\sigma(W^*)} \) has rank \( W^* - 1 \) or higher. We now have a sequence \( \sigma(W^*) <
\sigma(W^* + 1) < \cdots < \sigma(W) \), and for all \( W^* \leq w \leq W \) child \( c_{\sigma(w)} \) has rank \( w \).
Further, any child \( c_i \) strictly left of \( c_{\sigma(w)} \) is either \( c_{\sigma(w - 1)} \) or has rank at most
\( w - 2 \). If \( \sigma(W^*) = 1 \), then setting \( W' = W^* \) gives the desired left-corner-W-
witness. If \( \sigma(W^*) > 1 \), then setting \( W' = W^* - 1 \) and \( \sigma(W') = 1 \) gives the
desired left-corner-W-witness since \( c_1 \) then has rank at most \( W^* - 2 < W' \).
The algorithm hence either finds a left-corner-$W$-witness or proves that no rank-$W$-witness with $X = 1$ and $v = 1$ exists. It consists of scanning all children right-to-left, and hence takes $O(d)$ time.

Inspecting the proof, we can see the following necessary condition that will be useful later.

**Corollary 1.** Let $T$ be a tree with $n \geq 2$ nodes. If the ranks of the children (ordered from left to right) contain a subsequence $W', W', W'+1, W'+2, \ldots, W-1, W$ for some $1 \leq W' \leq W$, then $T$ has no left-corner-$W$-witness.

With this, we have all ingredients for the main results about witnesses.

**Theorem 2.** Let $T$ be a tree with $n \geq 2$ nodes. If $R(T) = W$, then $T$ has a corner-$W$-witness.

*Proof.* If $R(T) = W$, then $T$ has a rank-$W$-witness. Using Lemma 3, we can convert this into a rank-$W$-witness that has $X = 1$ and $v = 1$, or $X = W$ and $v = d$. If $X = 1$, then using the algorithm in Lemma 4 we can hence find a left-corner-$W$-witness. If $X = W$ then with a symmetric algorithm we can find a right-corner-$W$-witness. □

**Theorem 3.** For any tree $T$, $R(T)$ can be computed in linear time. In the same time we can also find a corner-witness for the optimum width for each rooted subtree with at least 2 nodes.

*Proof.* If $T$ has one node, then $R(T) = 1$ and we are done. So assume $n \geq 2$ and we have already recursively computed ranks and corner-witnesses for the children. Let $W$ be the maximal rank among the children. Run the algorithm in Lemma 4 to test whether $T$ has a left-corner-$W$-witness. Run a symmetric algorithm to test whether $T$ has a right-corner-$W$-witness. If one of them succeeds, then $R(T) = W$ and we have found the corner-witness. Otherwise $R(T) \geq W + 1$, and we know $R(T) \leq W + 1$ and can find the left-corner-$(W+1)$-witness using Observation 2.

This computation takes $O(\text{deg}(v))$ time each node $v$ and hence $O(n)$ time total. □

With this, all ingredients for Theorem 1 have been assembled and the theorem holds. We also note that our proof shows that for order-preserving poly-line drawings, it makes no difference for the width whether we demand upward or strictly-upward drawings. The extraction of the rank-$W$-witness from a drawing (Lemma 2) works even if the drawing has horizontal edges, while the construction of the drawing (Lemma 1) creates strictly-upward drawings.

### 6 Straight-line drawings?

We showed that the rank exactly describes the optimum width of poly-line upward order-preserving drawings. A natural question is whether this also describes the optimum width of ideal drawings where additionally we require edges to be straight-line. The answer is “no”.
Theorem 4. There exists a quaternary tree $T$ that has a planar strictly-upward order-preserving poly-line drawing of width 2, but no ideal drawing of width 2.

Proof. Consider the tree $T$ in Figure 4(a). Clearly it has a poly-line drawing with width 2. Observe that $u_3$ has rank 2 since $u_3$ has two children of rank 1. Therefore the rank-sequence of children of $u_2$ contains 1,1,2 as a subsequence, proving that $u_2$ has no left-corner-2-witness by Corollary 1. Likewise $u_1$ has no left-corner-2-witness since $u_2$ has rank 2. By Lemma 4 therefore $u_i$ (for $i = 1, 2$) does not have a rank-2-witness with $X = 1$. By Lemma 2 therefore no drawing of $T_{u_i}$ of width 2 has $u_i$ in column 1.

Fix an arbitrary upward order-preserving drawing $\Gamma$ of $T$ of width 2. For $i = 1, 2$, the induced drawing of $T_{u_i}$ has also width 2, and by the above $u_i$ must be drawn in column 2. This drawing cannot be straight-line, else $u_1u_2$ would be vertical, making it impossible to draw the rightmost child of $u_1$ while preserving the order. So any such drawing of width 2 contains bends. $\square$

If we replace any leaf in $T$ with a subtree that requires width $W - 1$ (e.g. a binary tree of height $W - 1$), then much the same proof shows that this tree has a poly-line drawing of width $W$, but no straight-line drawing of width $W$.

In summary: The optimal width for straight-line order-preserving drawings is sometimes larger than for poly-line order-preserving drawings. One can show that the two optimum widths are within a factor of 2, and coincide for ternary trees; see [3].

![Figure 4](image-url)

**Figure 4.** (a) A tree that cannot be drawn straight-line with the same width. (b) A tree that cannot be drawn straight-line with the root at the corner and the same width. (c) and (d): A tree where ordered drawings require nearly twice as much width as unordered drawings.
Nevertheless, might there be a similar algorithm to compute optimum-width straight-line drawings? This question remains open, but we can show that one key ingredient will fail: There do not always exist optimum-width drawings where the root is at a corner.

**Theorem 5.** There exists a tree $T$ that has an upward order-preserving straight-line drawing of width 3, but in any such drawing the root has to be in the middle column.

**Proof.** Construct the tree $T$ by giving four children $c_1, c_2, c_3, c_4$ to the root. $T_{c_1}$ and $T_{c_4}$ are single nodes. $T_{c_2}$ is a subtree that requires width 2, and in any width-2 drawing the root must be in the top-left corner. $T_{c_3}$ is a subtree that requires width 3, and in any width-3 drawing the root must be in the top-right corner. Such trees can be found easily, using variants of subtree $T_{u_2}$ from the previous proof.

Figure 4(b) shows $T$, and a straight-line drawing with width 3. Presume we had a straight-line drawing of $T$ of width 3 where the root $u_r$ is in the top left corner. Since $T_{c_3}$ requires width 3, it contains a point $p_3$ in column 1. The poly-line from $u_r$ to $p_3$ blocks $T_{c_2}$ from using column 3, so $T_{c_2}$ must be drawn with width 2 and hence $c_2$ is in column 1. Now the straight-line segment $u_r c_2$ is vertical and $c_1$ cannot be drawn. Likewise, if $u_r$ is in the top right corner, then (since $c_3$ must be in column 3) the straight-line segment $u_r c_3$ prevents $c_4$ from being drawn. Thus the root cannot be in a corner. \[ \square \]

The tree in this counter-example has degree 4; we note that for any ternary tree straight-line ordered drawings with optimum width and the root in a corner exist [3].

### 7 Ordered vs. unordered drawings

In a companion paper [3], we gave algorithms to find unordered upward drawings of optimum width. We also showed that for quaternary trees, it makes no difference to the optimum width as to whether we demand ordered or unordered drawings (presuming that bends are allowed). However, as we will see now, if vertices have 5 children, then demanding an ordered drawing may increase the width-requirement significantly.

**Theorem 6.** For any $i \geq 1$, there exists a tree $T_i$ with degree 5 that has an unordered upward drawing of width $i$, but any ordered upward drawing requires width at least $2i - 1$.

**Proof.** $T_1$ is a single node, which can be drawn with width 1 and requires width at least $1 = 2 \cdot 1 - 1$.

\[ \text{This result is proved here, and not in [3], since its proof requires the techniques of rank-witnesses.}\]
For $i \geq 2$, tree $T_i$ consists of a node with degree 5 for which children $c_1, c_2, c_4, c_5$ are roots of $T_{i-1}$. Child $c_3$ has two children, each of which is the root of $T_{i-1}$. See Fig. 1(c) and (d), which also illustrates how to obtain an unordered drawing of $T_i$ with width $i$.

We show that $R(T_i) \geq 2i - 1$. Clearly this holds for $T_1$, so assume we know that $R(T_{i-1}) \geq 2i - 3$. Since $c_3$ has two children with rank $2i - 3$, $T_{c_3}$ has rank at least $2i - 2$. Therefore the rank-sequence of children contains $2i - 3, 2i - 3, 2i - 2$, which by Corollary 1 means that $T_i$ has no left-corner-$(2i - 2)$-witness. Likewise the rank-sequence $2i - 2, 2i - 3, 2i - 3$ means that $T_i$ has no right-corner-$(2i - 2)$-witness. By Theorem 2 therefore $T_i$ has no rank-$(2i - 2)$-witness and $R(T_i) \geq 2i - 1$ as desired.

One can show that this gap in the width between ordered and unordered drawings is maximal: Any tree that has an unordered upward drawing of width $W$ also has an ordered upward drawing of width $2W - 1$ [3].

8 Conclusion

In this paper, we gave a linear-time algorithm that finds for any tree a planar strictly-upward poly-line drawing that respects the given order of the children at all nodes, and has optimal width among all such drawings. We also gave an example that showed that straight-line drawings of this kind require strictly larger width for some trees. Many open problems remain:

- Can we compute straight-line upward order-preserving drawings of optimum width?
- Can we find tree drawings that have optimal area, or is this NP-hard? (The question could be asked for many different types of drawings, such as order-preserving or not, or straight-line or not, upward or not.)
- Can we at least prove the conjecture in [2] that every tree has a strictly-upward straight-line order-preserving drawing of area $O(n \log n)$? The best currently known bound is $O(\Delta n \log n)$ [3].

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A Bypassing poly-lines

In the proof of Lemma 2 we repeatedly used that some set of poly-lines bypasses another poly-line, and therefore each of them requires a column of its own. This is quite intuitive: many lower-bound arguments for planar graph drawing use arguments where so-called “nested cycles” each require two additional columns (see e.g. [7]). However, the argument is non-trivial for poly-lines since they are open-ended curves and hence do not separate the drawing of the rest from the “outside”, except under the special conditions that we called bypassing.

We previously described three different situations for bypassing, but one easily checks that the following definition encompasses them all:

**Definition 3.** Let \( \hat{P}, P_1, \ldots, P_k \) be a set of poly-lines that are disjoint except that ends of \( P_1, \ldots, P_k \) may coincide. We say that \( P_1, \ldots, P_k \) bypass \( \hat{P} \) if there exists a point \( \hat{q} \) in \( \hat{P} \) such that poly-line \( P_i \) begins at a point above \( \hat{q} \) and ends at a point below \( \hat{q} \), for \( i = 1, \ldots, k \).

Here, a point above[below] \( \hat{q} \) means a point with the same \( x \)-coordinate as \( \hat{q} \) and with \( y \)-coordinate strictly larger[smaller] than the one of \( \hat{q} \).

Recall that for poly-lines the endpoints and all bends must have integral \( x \)-coordinates, and that we measure the width of a set of poly-lines by the minimum number of consecutive columns that contain them. Let \( x_{\min}(P) \) and \( x_{\max}(P) \) be the minimum and maximum \( x \)-coordinate of points in poly-line \( P \).

**Lemma 5.** Let \( P_1, \ldots, P_k \) be a set of poly-lines that bypass a poly-line \( \hat{P} \). If these poly-lines all reside within columns \( 1, \ldots, W \), then

\[
W \geq (x_{\max}(\hat{P}) - x_{\min}(\hat{P}) + 1) + k
\]

In other words, every bypassing poly-line requires one additional column beyond the width occupied by \( \hat{P} \).

**Proof.** We proceed by induction on \( W \), with an inner induction on the total number of bends in poly-lines \( P_1, \ldots, P_k \). Clearly \( W \geq x_{\max}(\hat{P}) - x_{\min}(\hat{P}) + 1 \) since \( \hat{P} \) alone occupies this many columns. In the base case, \( W = x_{\max}(\hat{P}) - x_{\min}(\hat{P}) + 1 \), which means that poly-line \( \hat{P} \) extends from leftmost to rightmost
column. Therefore $\hat{P}$ separates all points above $\hat{q}$ from points below $\hat{q}$. This implies that no poly-line $P_1$ exists since $P_1$ is disjoint from $\hat{P}$ and hence cannot cross it. Thus, $k = 0$ and the claim holds.

For the induction step $W > x_{\text{max}}(\hat{P}) - x_{\text{min}}(\hat{P}) + 1$, so $\hat{P}$ does not span all columns. Say $x_{\text{max}}(P) < W$, so $\hat{P}$ is within columns $1, \ldots, W-1$. We have cases.

In the first case, at most one of $P_1, \ldots, P_k$ intersects column $W$. Say this poly-line (if one exists) is $P_k$. Then $P_1, \ldots, P_{k-1}$ all reside within columns $1, \ldots, W-1$, as does $\hat{P}$. By induction therefore $W - 1 \geq x_{\text{max}}(\hat{P}) - x_{\text{min}}(\hat{P}) + 1 + (k - 1)$, which proves the claim.

In the second case, some poly-line $P_i$ contains three or more points in the column $X$ that contains $\hat{q}$. Then some strict sub-poly-line of $P_i$ connects a point in column $X$ above $\hat{q}$ with a point in column $X$ below $\hat{q}$. We can shorten $P_i$ to this smaller poly-line without affect the conditions on bypassing. This removes at least one bend from $P_i$ and the claim holds by induction.

![Figure 5](image.png) Bypassing poly-lines require extra columns. (Left) Pruning a path that intersects column $X$ three times. (Right) Finding a $K_4$-minor if none of the previous cases applies.

Finally we argue that one of the above cases must apply. Assume for contradiction that two poly-lines, say $P_{k-1}$ and $P_k$, both contain a point in column $W$. Observe that $X < W$, since column $X$ must intersect $\hat{P}$ due to point $\hat{q}$, but $x_{\text{max}}(P) < W$. Since the second case does not apply, each $P_i$ (for $i = k-1, k$) stays strictly right of $X$ except at its endpoints. Hence $P_i$ starts at point $q_i$ in column $X$ above $\hat{q}$, connects to a point $r_i$ in column $W$, and then returns to point $p_i$ below $\hat{q}$ in column $X$, all the while staying within $X+1, \ldots, W$ except at the ends. One can observe that this is impossible without a crossing. Formally one proves this by creating an outer-planar drawing of a $K_4$-minor as follows: Consider the drawing induced by $P_k$ and $P_{k-1}$. Connect the points in column $X$ with vertical edges in order, and add a new vertex $z$ in column $W+1$ adjacent to $r_k$ and $r_{k-1}$. See also Figure 5. This clearly maintains planarity and all vertices remain on the outer-face. Since $q_k$ and $q_{k-1}$ are strictly above $\hat{q}$ while $p_k$ and $p_{k-1}$ are strictly below, not all points with $x$-coordinate $X$ can coincide. Since $P_{k-1}$ and $P_k$ are disjoint (except perhaps at their ends), points $r_k$ and $r_{k-1}$
cannot coincide. So this indeed gives an outer-planar drawing of a minor of \( K_4 \), which is impossible. So one of the above cases must apply, and the claim holds by induction. \( \square \)

B Bounds on the rank

The algorithm implicit in Lemma 1 draws trees upward and order-preserving with optimal width, but how big is this width? We know \( R(T) \in O(\log n) \) from Chan’s work \(^4\). The complete binary tree has \( R(T) \geq \log(n + 1) \), so asymptotically this is tight. We now show that the lower bound is in fact tight up to a small additive constant.

Lemma 6. Any \( n \)-node tree \( T \) has \( R(T) \leq \log n + 1 \).

Proof. Let \( N(W) \) be the minimum number of nodes in a tree that has rank \( W \). We aim to show that \( N(W) \geq 2^{W-1} \); this proves the claim.

Clearly \( N(1) \geq 1 = 2^0 \), so the claim holds for \( W = 1 \). Assume it holds for all values up to \( W \), and let \( T \) be a node-minimal tree that has rank \( W + 1 \). No child of \( T \) can have rank \( W + 1 \) by minimality of \( T \), so the ranks of the children belong to \( \{1, \ldots, W\} \). Let \( w \leq W \) be the largest value such that root does not have exactly one child with rank \( w \). (Hence there might be zero or at least 2 children with rank \( w \).)

Assume first that \( T \) has no child of rank \( w \), and exactly one child each of rank \( w+1, \ldots, W \). Following the computation of a left-corner-\( W \)-witness (Lemma 4), one sees that this algorithm will stop with success at some \( W^* \geq w + 1 \), so \( R(T) \leq W^* \), a contradiction. So there must be at least two children of rank \( w \).

The subtree of the child with rank \( i \) has at least \( N(i) \) nodes, so \( N(W + 1) = |T| \geq N(W) + N(W - 1) + \cdots + N(w + 1) + 2 \cdot N(w) \), and by induction therefore \( N(W + 1) \geq 2^{W-1} + 2^{W-2} + \cdots + 2^w + 2 \cdot 2^{w-1} = 2^W \) as desired. \( \square \)

We note here that the bound is not tight (for example, we can add a ‘+1’ in the final inequality, since we did not count the root). By distinguishing a large number of cases we have been able to show that \( N(W) \geq \frac{3}{2}2^{W-1} \). We suspect that in fact \( N(W) \geq 2^W - 1 \), but the enormous work to prove this does not seem worth the minor improvement in the bound on \( R(T) \).