Transconducting transition for a dynamic boundary coupled to several Luttinger liquids

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We study a dynamic boundary, e.g. a mobile impurity, coupled to $N$ independent Tomonaga-Luttinger liquids (TLLs) each with interaction parameter $K$. We demonstrate that for $N \geq 2$ there is a quantum phase transition at $K \geq \frac{1}{2}$, where the TLL phases lock together at the particle position, resulting in a non-zero transconductance equal to $\frac{2}{\pi}$. The transition line terminates for strong coupling at $K = 1 - \frac{1}{N}$, consistent with results at large $N$. Another type of a dynamic boundary is a superconducting (or BEC) grain coupled to $N \geq 2$ TLLs, here the transition signals also the onset of a relevant Josephson coupling.

There is considerable interest in systems of $N$ independent one-dimensional Tomonaga-Luttinger liquids (TLLs) coupled via a dynamic boundary, e.g. a mobile impurity. This case has been realized in cold atom experiments employing a variety of impurity atoms, in either boson or fermion systems [12–20], focusing on the particle dynamics and response to an external force.

Quantum impurities in TLL have been extensively studied [4–7] and more recently to one-dimensional cold atom gases [8–11]. Quantum impurities in TLL have from polaronic effects in bulk baths [1–3], the approach to equilibrium [4–7] and more recently to one-dimensional cold atom gases [8–11]. Quantum impurities in TLL have been extensively studied [12–20], focusing on the particle dynamics and response to an external force.

A second type of dynamic boundary is realized by a superconducting grain, or a Bose-Einstein condensate (BEC), illustrated in the left pannel of Fig. 1. This can be realized with wires formed on LaAlO$_3$/SrTiO$_3$ nanostructures [21] or with carbon nanotubes. The latter system [22] has in fact shown a surprisingly large values of supercurrents. Of further interest are topological superconductors with Majorana islands coupled to TLLs via multiterminals. Theoretical studies [23,24] show the phenomena of inter-terminal conductance, with possible realizations in various experimental setups [26,27]. In dynamically coupled TLLs, as we show here, an analogous phenomenon, transconductance (see below), can occur even without Majorana states.

The case of an infinite number of TLLs was previously examined [25] and showed a phase transition in which the impurity can localize for a repulsive TLL. Understanding the finite $N$ case is thus important in view of the experimental realizations [22,26,27] and the theoretical studies [23,24]. Further motivation for studying $N > 1$ are studies of drag conductance in crossed carbon nanotubes [26,31].

In this Letter, using the renormalization group (RG) and duality methods akin to the study of quantum Brow-
Let us express the position $X$ in units of $(2\pi\rho_0)^{-1}$. The action of the particle becomes

$$S_{imp} = \frac{1}{2}M \int d\omega |X_\omega|^2 - g \int \sum_{i=1}^{N} \cos[X_\tau - 2\phi_i(X_\tau, \tau)]$$

where $M = M_0(2\pi\rho_0)^{-2}$, $g = \alpha_1 \rho_0 g_0$, and here and below $\int f(\omega) = \int \frac{d\omega}{2\pi} f(\omega)$, with $\int r = \int_0^2 dr$. We focus on the zero temperature $T = 1/\beta \to 0$ limit unless stated otherwise.

The theory defined by (3) is still highly non-linear in the particle position $X_\tau$ and difficult to treat exactly. However, we claim that an equivalent long time theory is obtained by

$$\cos[X_\tau - 2\phi_i(X_\tau, \tau)] \to \cos[X_\tau - 2\phi_i(0, \tau)]$$

This amounts to assume that the TLL correlations are dominated by time differences $|\tau - \tau'| \gg |X_\tau - X_{\tau'}|$, which is satisfied by the $X_\tau$ correlations that we find below. Since only the term $\phi_i(0, \tau)$ appears in the coupling, we can integrate the Bose fields at all other points $\phi_i(x \neq 0, \tau)$, leading to the well studied $\sim |\omega|$ term [37], hence the total action becomes

$$S = \frac{1}{2} \int \omega [M\omega^2|X_\omega|^2 + \sum_{i=1}^{N} \frac{2|\omega|}{\pi K} |\phi_i|^2]$$

$$- g \int \sum_{i=1}^{N} \cos[X_\tau - 2\phi_i(0, \tau)]$$

where $\phi_i(0, \tau) = \phi_i^\prime$ and its Fourier transform is $\phi_i^{\prime \prime}$. We now denote $B_i^\prime = 2\phi_i^\prime - X_\tau$ as the fields entering in the non-linear term and define $\tilde{X}_\omega$ in Fourier via

$$X_\omega = \tilde{X}_\omega - \frac{1}{N\omega} \sum_{i=1}^{N} B_i^\omega, \quad N_\omega = N + 2\pi MK|\omega|$$

where $N_\omega$ can be thought as an effective number of degrees of freedom. It is then easy to see that the action can be rewritten as a sum over two independent sectors, the field $\tilde{X}$ on one hand, and the $B_i$'s on the other, as

$$S = \frac{1}{2} \int \omega \{ |\omega| \sum_{i=1}^{N} B_i^\omega \tilde{X}_\omega + D_{i,j} B_i^\omega B_j^\omega \} - g \sum_{i=1}^{N} \int \cos B_i^\omega$$

$$D_{i,j}^{-1} = \frac{|\omega|}{2\pi K} (\delta_{i,j} - \frac{1}{N\omega}), \quad D_{i,j} = \frac{1}{2\pi K} \frac{2\pi K}{|\omega|} \delta_{i,j}$$

Hence one can first study the problem defined by the $B_i$ fields, and in a second stage obtain the position of the particle $X_\tau$ from (7) as the sum of two independent terms. This decomposition immediately leads to two exact bounds, first [38]

$$\langle |X_\omega|^2 \rangle \geq \langle |\tilde{X}_\omega|^2 \rangle = \frac{2\pi K}{\omega N\omega}$$

where $\langle \rangle$ denotes average over the action $S$. Furthermore

$$\langle \cos X_\omega \rangle \lesssim \langle \cos \tilde{X}_\omega \rangle \sim \left( \frac{4\pi^2 KM}{N\beta} \right)^{K/N} \beta \to \infty$$

Hence the finite $N$ behavior of $\langle \cos X_\omega \rangle$ differs from the $N \to \infty$ case [28] where it can be finite and then serve as an order parameter.

To 0-th order in $g$, with $\langle \rangle_0$ denoting an average w.r.t. $S_{g=0}$,

$$\langle \cos B_i^\omega \rangle_0 = e^{-\frac{1}{2} \int \omega (|\omega|^2 + \frac{1}{N\omega})} = 0$$
which is strongly irrelevant and cannot lead to an ordering of each individual $B_i$. Naively, one could conclude from power counting that the coupling $g$ is washed away by fluctuations, leading effectively to the $g = 0$ theory.

However, this is not the case, as we have found: although strongly irrelevant, the terms $g \cos(B_i^\tau - B_j^\tau)$ generate an effective coupling $g_2 \cos(B_i^\tau - B_j^\tau)$ between pairs of distinct fields. Indeed, the effective action evaluated to second order in perturbation theory in $g$ contains a term $\cos(B_i^\tau - B_j^\tau)$ multiplied by

$$g_2^2 \langle e^{iB_i^\tau - iB_j^\tau}\rangle_0 = g_2^2 e^{-\frac{4\pi}{\tau} |\tau - \tau|} \int_\mathbb{R} e^{\frac{2\pi i K}{\tau}}$$

and integrated over times. We note that the finite mass is crucial to provide a short time cutoff $\sim M$. The action involving the $B_i^\tau$ fields (denoted as $B_\omega$) can thus be replaced by the effective action

$$S_1 = \frac{1}{2} \int_\omega D_{i,j} \langle B_i^\tau B_j^\tau \rangle_{\omega} - g_2 \Lambda \sum_i \int_\tau e^{iV \cdot B_i^\tau}$$

(13)

where $g_2 \Lambda \sim M g_2^2$, $g_2$ is a running dimensionless coupling, and $\Lambda$ a high frequency cutoff with initial value $\sim M$. The vectors $V$ are $N$ dimensional, have one entry of $+1$, one of $-1$ and all other entries are $0$, i.e. $V \cdot B_\omega = B_i^\tau - B_j^\tau$ with $i \neq j$. Hence $V$ form the primitive unit cell of an $N - 1$ dimensional lattice that is perpendicular to the vector $(1,1,...,1)$ on a simple cubic $N$ dimensional lattice. This type of model appears in various contexts, e.g. the quantum Brownian motion in a periodic potential [35, 36]. To 2nd order the RG flow equation is (see [39])

$$\Lambda \partial_\Lambda g_2 = (1 - 2K)g_2 + \alpha(N - 2)g_2^2 + O(g_2^3)$$

(14)

where $\alpha = O(1) > 0$ is nonuniversal, depends on a smooth cutoff procedure. Note that the TLL parameter $K$ is not renormalized and does not flow. From (14) there is clearly a critical line for $K > 1/2$ and $N > 2$

$$g_2^* = \frac{2(K - \frac{1}{2})}{\alpha(N - 2)}$$

(15)

such that for $g_2 < g_2^*$ the RG flow is towards the Gaussian $g_2 = 0$ theory, while for $g_2 > g_2^*$, $g_2$ flows to strong coupling, signaling a phase where the relative fields $B_i^\tau$ lock together, in a way that we study below.

The $N=2$ case has a single nonlinear term $\sim \cos(B_1^\tau - B_2^\tau)$, equivalent to the static impurity problem [37, 40, 11] and has a vertical phase boundary at $K = \frac{1}{2}$ (Fig. 1). Going back to general $N$, we now study the $B_i^\tau$ correlations by adding a source term $\int_\omega |\omega| B_\omega \cdot A_{\omega}$ to the action (13) so that $\langle B_i^\tau B_j^\tau \rangle_{\omega} = \frac{1}{2} \int_\mathbb{R} e^{\frac{\delta^2 Z}{\delta A_{\omega}^\dagger \delta A_{\omega}}} \langle A_{\omega} \rangle_0$, where

$$Z_1 = \int DB \exp(-S_1)$$

is the partition sum in presence of the source. Before studying the general correlations, we note [39] an exact sum rule of the effective model (13)

$$\sum_{\omega} \langle B_\omega^i B_\omega^j \rangle = \frac{N_\omega}{(M \omega^2)}$$

We proceed to study the strong coupling fixed point by a duality transformation. The process is well known for the $N = 2$ case [37, 40], results are also stated for the quantum Brownian motion [33, 36], yet the extension to $N > 2$ of our case involves some subtleties. We perform first a change of variables so that the Gaussian part of $S_1$, Eq. (13), becomes diagonal,

$$C_\omega^i = B_\omega^i - \alpha_{\omega} \tilde{B}_\omega, \quad \alpha_{\omega} = 1 - \sqrt{1 - N/N_\omega}$$

(16)

where $\tilde{B}_\omega = \sum_i B_\omega^i/N$ and $\tilde{C}_\omega = \sum_i C_\omega^i/N = (1 - \alpha_{\omega}) \tilde{B}_\omega$. The action becomes

$$S_1 = \frac{1}{2} \int_\omega \left[ |\omega| C_\omega \cdot \tilde{C}_\omega - g_2 \Lambda \sum_i \int_\tau e^{iV \cdot C_\omega^i} \right] - \int_\omega |\omega| |C_\omega \cdot A_{\omega} + \frac{\alpha_{\omega}}{1 - \alpha_{\omega}} \tilde{C}_\omega \sum_i A_{i \omega}^i|$$

(17)

We consider next large $g_2$ where the trajectories of $C_\tau$ are dominated by instantons, i.e. a sequence of $n$ sharp jumps at consecutive times $\tau_1, \tau_2, ..., \tau_n$. The instantons shift $C_\tau$ between neighboring equivalent minima of the $g_2$ term by vectors chosen from a set $R_i$ such that $R_i \cdot V_j = \delta_{i,j}$. Hence $R_i$ form the reciprocal lattice to $V_i$. Each vector has one entry of $-1+1/N$ and all the rest are $1/N$, with norm $|R_i|^2 = 1 - 1/N$. The vectors $R_i$ are also orthogonal to $(1,1,1,...)$, however they do not form a primitive unit cell for $N > 3$ and then their lattice symmetry differs from that of the $V_i$. E.g., for $N = 3$ both $V_i, R_i$ form a 2D triangular lattice, however, for $N=4$ there are 12 vectors $V_i$ forming an fcc lattice while there are 8 vectors $R_i$ that form a bcc lattice.

Since $R_\alpha$ are perpendicular to $(1,1,1,...)$ instanton trajectories do not describe the center of mass $\hat{C}_\omega$. Decoupling this center of mass is achieved by the shift $C_\omega = C_{\omega}^i - \hat{C}_{\omega}$, hence the Gaussian part in Eq. (17) decouples into $C_{\omega} \cdot C_{-\omega} = \hat{C}_{\omega} \cdot \hat{C}_{-\omega} + N|\hat{C}_{\omega}|^2$. The $C_{\omega}$ trajectory is described by $\hat{C}_\tau(\tau) = 2\pi \sum_i R_i \theta(\tau - \tau_i)$. The coupling with the source can be written as $\int_\omega |\omega| \hat{C}_\omega \cdot A_{i \omega} = i2\pi \alpha_{\omega} \cdot a(\tau_i)$ where $a_{\omega} = -\sin(\omega \theta_{\omega})$. The weight of each instanton is defined as $\Delta \Delta \sim e^{-S_{\text{inst}}}$ where $37, 10$ $S_{\text{inst}} \sim \sqrt{g_2}$. In the strong coupling limit $\Lambda = 0$, instantons are absent and the correlations become

$$\langle B_\omega^i B_{-\omega}^i \rangle = \frac{1}{(1 - \alpha_{\omega})^2} \langle |\hat{C}_{\omega}|^2 \rangle = \frac{N_\omega}{NM \omega^2}$$

(18)

so that all TLLs becomes equally coupled to each other. If $\Delta > 0$ the term $\frac{1}{2} \int_\omega \frac{2\pi i}{\pi K} C_\omega \cdot \hat{C}_{-\omega}$ produces, after integration over $\omega$, logarithmic interactions between instantons [39] which corresponds to the dual action,

$$S_2 = \frac{1}{2} \int_\omega K |\omega| |\theta(\omega)|^2 - \Delta \Delta \sum_i \int_\tau e^{iR_i \cdot (\theta(\tau) + 2\pi n(\tau))}$$

(19)

By shifting $\theta \rightarrow \theta(\tau) - 2\pi n(\tau)$ and taking a 2nd derivative in $\theta_{\omega}$ we obtain a relation between the $B_\omega$ and $\theta_{\omega}$.
In particular at the self-dual point from the impurity involves \[39\] e±i\delta_{i,j} - K^2\theta_{i}\theta_{j}$ (20)

The dual form allows for deriving the RG equation, using $|\mathbf{R}|^2 = 1 - 1/N$, to first order, 
\[
\Lambda \partial_\Lambda \Delta = 1 - \frac{1}{K}(1 - \frac{1}{N})\Delta
\] (21)

Hence the phase transition at strong coupling terminates at $K_c = 1 - \frac{1}{N}$. The next order in RG for $N=3$ is $\Delta^2$ (similarly to Eq. (14) in the dual coupling $g_2$), while for $N \geq 4$ there are no $\Delta^2$ terms since $\mathbf{R} \pm \mathbf{R}_d$ are all longer than $\mathbf{R}$ and are therefore irrelevant at the transition. The next order is then $O(\Delta^3)$, hence the critical line at strong coupling is $\Delta_c \sim \sqrt{\kappa_c - \kappa}$ with an infinite slope at $K_c$, for $N \geq 4$. This is similar to the $N \rightarrow \infty$ case \[28\] where $K_c = 1$ and $g_2^2 \sim 1/(1 - K)$. The various phase boundaries are illustrated in Fig. 1.

The $N = 3$ case is self-dual, i.e. we find a relation \[39\] between $(B^i_\alpha B^j_{\alpha})_K g_2$ and $(B^i_\alpha B^j_{\alpha})_K \rightarrow 3/2 \Delta$. In particular at the self-dual point $K = 1/\sqrt{3}$, $g_2 = \Delta$ on the critical line $(B^i_\alpha B^j_{\alpha})_K$ is exactly given by the average of its values for $g_2 = 0$ and for $\Delta = 0$ (Fig. 1).

We proceed now to identify the order parameter of our phase transition, i.e. the transconductance. The phenomenon of current in chain $i$ induced by a voltage on chain $j$ has been studied in the context of crossed nanotubes \[29\][34]. In our case transconductance is a spontaneous order parameter and not a mechanical junction as for the nanotubes. The usual experiment is a 2-probe type that for a single clean TLL yields \[12\][43] a conductance $\frac{e^2}{h}$ determined by the normal leads \[44\][47]. For our system of $N$ TLLs in the decoupled phase obviously $G_{i,j} = \frac{K}{\pi} \delta_{ij}$ while in the coupled phase the strong gener\nt coupling $\cos(\phi_i(0,\tau) - \phi_j(0,\tau))$ forces the currents $I_i = \phi_i(0,\tau)$ to be equal, $I_i = I$ with total dissipation $NI^2 \frac{\kappa}{2}$. We propose then that, with normal leads on each TLL, the resistance measured by a voltage in one wire is the sum of all individual resistances, hence

\[
G_{i,j} = \begin{cases} 
\frac{e^2}{h} \delta_{i,j} & g_2 = 0 \\
\frac{e^2}{h} \frac{1}{N} & \Delta = 0
\end{cases}
\] (22)

This implies that the transconductance exhibits a jump at the phase transition between these two values (Fig.1).

To substantiate this rationale, we consider first a "local conductance" for the response to a field applied on a length $L$ of a pure TLL \[37\]. The response function away from the impurity involves \[39\] $e^{i|\omega_i|\tau/\alpha}$, a constant in the DC limit. Hence $L \rightarrow 0$ can be taken, yielding

\[
G^{\text{local}}_{i,j}(\omega) = -\frac{e^2}{\pi h} i(\omega + i\delta)\langle \phi_i(\omega_n)\phi_j(-\omega_n)\rangle|_{\omega_n \rightarrow \omega + i\delta}
\]

In terms of the fields $B^i_{\omega}, \tilde{X}_{\omega}$ this becomes

\[
G^{\text{local}}_{i,j}(\omega) = \frac{e^2}{2\pi h} \omega ((B^i_\omega B^j_{\omega}) - \frac{1}{M\omega^2})
\] (23)

From the sum rule on the $B^i_{\omega}$ correlations we obtain the exact sum rule $\sum_i G^{\text{local}}_{i,j} = \frac{e^2}{h} K$. Using our results for the correlations, we obtain the DC local conductance at the fixed points and at the self dual point

\[
G^{\text{local}}_{i,j} = \begin{cases} 
\frac{e^2}{h} \frac{K\delta_{i,j}}{\pi} & g_2 = 0 \\
\frac{e^2}{h} \frac{K}{N} & \Delta = 0
\end{cases}
\] (24)

Along the transition line the conductance varies continuously: the correction to $G^{\text{local}}_{i,j}$ is proportional to $1 - N\delta_{i,j}$ with a positive prefactor $\sim (K - \frac{1}{2})^2$ near $g_2 = 0$ and a negative one $\sim -\frac{3}{2}(\frac{3}{2} - K)^2$ for $N = 3$ and $\sim -(1 - \frac{1}{N} - K)$ for $N \geq 4$ near $\Delta = 0$ \[39\] Eq. (49). The extension to the inhomogeneous case with normal leads is shown \[39\]: following ideas of the $N=1$ case \[45\][47] results in replacing $K \rightarrow 1$, yielding Eq. (22).

We have also considered an $N = 2$ case with two coupled LLs, one with normal leads and the other a homogeneous periodic TLL. We find \[39\] the conductance matrix $G_{i,j} = \frac{\pi}{2}$, which also follows from our rationale since the TLL loop by itself has vanishing resistance. A similar problem was considered in \[48\] (see comparison in \[39\]).

We consider next the realization of our model by a superconducting (or BEC) grain. The Josephson coupling to s wave pairs in each TLL involves \[37\] $\frac{1}{2} g_2 e^{i\pi}|X_r - i\pi\theta_{r,i}|$ where $X_r$ is now the superconducting phase of the grain and $\theta_{r,i}$ is the dual phase to $\phi_{r,i}$ in the charge sector of chain $i$. The action in terms of $\theta_{r,i}$ has the same form as in Eqs. (13) with $K \rightarrow 1/(2K_p)$ \[37\] and $\frac{1}{2\kappa}$ being the charging energy $E_c$ of the grain; hence the phase diagram is also given by Fig. 1 with the axis being $1/(2K_p)$. Thus, for $N = 2$ the phase boundary is at $K_p = 1$ and the strong coupled phase appears even for weakly attractive coupling $K_p > 1$. We note that the data \[22\] on single wall carbon nanotube, expected to have $N = 2$, shows with superconducting leads a surprisingly high supercurrent. If one of the leads contains a grain with not too small charging energy then our strong coupling phase, implying a strong Josephson coupling, would account for the data. For $N > 2$, possible for nanotube ropes \[22\], the phase boundary interpolates between $K_p = 1$ and $K_p = \frac{N}{N-1}$, allowing for a relevant Josephson coupling even in a range of repulsive interactions.

Finally, from the decomposition \[1\] and the sum rule for the $B^i_{\omega}$ correlations within the effective model Eq. (13), we obtain the fluctuations of $X_r$ as $\langle |X_r|^2 \rangle = \frac{1}{M\omega^2}$, i.e. they are not affected by the phase transition. Therefore $(X_r - X_r')^2 \sim |\tau - \tau'|$ justifies our assumption in deriving the action \[6\], i.e. that $|X_r - X_r'| \ll |\tau - \tau'|$.

We note that a finite impurity mass is essential for the derivation of our effective action \[8\], although it does not appear explicitly in the phase diagram. As seen from Eqs. (11,12) $1/M$ provides an upper limit on frequencies which implies an upper bound on temperature in a
possible experiment, $T^* \approx \frac{1}{\pi} = (2\pi\rho_0)^2/M_0$. For Cs atoms \[1\] and TLL density $\pi\rho_0 = 4.5\mu m^{-1}$ we find $T^* = 10^{-7} K$; increasing the TLL density or reducing $M_0$ increase the range of $T < T^*$. For the realization with a superconducting grain $T^* \approx E_c$ where $E_c \approx 1 - 10 K$ \[20\] is achievable in such devices. For $M \to \infty$ the problem reduces to a static impurity \[40\] with a phase transition at $K = 1$ that separates a conducting phase from an insulating phase and no transconductance.

In conclusion, we have found that a dynamic boundary, such as a mobile impurity or a superconducting (or BEC) grain, coupled to $N$ identical Luttinger Liquids induces a phase transition for all $N \geq 2$. The order parameter is the conductance matrix, in particular a large transconductance appear in the strong coupling phase. In the superconducting grain case the strong coupling phase is also identified by a strong Josephson coupling, relevant to a number of active experimental setups \[22\] \[20\] \[27\].

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SUPPLEMENTARY MATERIAL

We give here the details of the calculations and their applications, as described in the main text of the Letter.

I. RENORMALIZATION GROUP

We derive here the RG equation (14) of the main text. The 1st order RG is identified directly from Eq. (12) of the main text

$$g_2^R \Lambda' = g_2 \Lambda e^{-\int_\Lambda^\Lambda' \frac{2\pi K}{\omega} d\omega} = g_2 \Lambda (1 - 2K \frac{d\Lambda}{\Lambda})$$

$$\Rightarrow g_2^R = g_2 (1 + (1 - 2K) \frac{d\Lambda}{\Lambda}) \quad (25)$$

where $d\Lambda = \Lambda - \Lambda'$ is infinitesimal. To 2nd order the partition function is (subtracting the counterterm, i.e. the 2nd order correction from exponentiating Eq. (25))

$$Z^{(2)} = \frac{1}{2} (2g_2 \Lambda)^2 \int_{\tau, \tau'} \left< \sum_{i \neq j} \cos(B^i_\tau - B^j_\tau) \sum_{k \neq l} \cos(B^k_\tau - B^l_\tau) \right> \Lambda,$$

$$- \frac{1}{2} (2g_2 \Lambda)^2 e^{-4K \int_\Lambda^\Lambda' \frac{d\omega}{\omega} \sum_{i \neq j} \cos(B^i_\tau - B^j_\tau) \sum_{k \neq l} \cos(B^k_\tau - B^l_\tau)} \quad (26)$$

where $\left< ... \right>_{\Lambda'}$ means average over frequencies in the range $[\Lambda', \Lambda]$. Collect first the indices such that only one pair are equal, e.g. $l = j, k \neq i$. We need the average $\left< ... \right>_{\Lambda'}$ defined here in the frequency range $\Lambda', \Lambda$

$$\frac{1}{2} (g_2 \Lambda)^2 \int_{\tau, \tau'} \sum_{\pm} \left< e^{i(B^i_\tau - B^j_\tau) \pm i(B^j_\tau - B^k_\tau)} \right>_{\Lambda'} + h.c. \quad (27)$$

The average for the upper sign gives an exponent of

$$\frac{1}{2} \left< (B^i_\tau - B^j_\tau + B^j_\tau - B^k_\tau)^2 \right> = \int_{\Lambda'} \left< \frac{d\omega}{2\pi} [2D_{ij} - 2D_{i\neq j} + D_{i\neq j} e^{i\omega(\tau - \tau')} - D_{\neq j} e^{i\omega(\tau - \tau')} \right. - D_{\neq j} e^{i\omega(\tau - \tau')} + D_{i\neq j} e^{i\omega(\tau - \tau')} + c.c. \right] = \int_{\Lambda'} \left< \frac{d\omega}{2\pi} \left[ 4 \frac{2\pi K}{\omega} - 2 \frac{2\pi K}{\omega} \cos \omega(\tau - \tau') \right] \right.$$ 

Together with the counterterm in this group

$$Z^{(2a)} = (g_2 \Lambda)^2 \int_{\tau, \tau'} \cos(B^i_\tau - B^j_\tau + B^j_\tau - B^k_\tau) e^{-4K \int_\Lambda^\Lambda' \frac{d\omega}{\omega} \left[ e^{2K \int_\Lambda^\Lambda' \frac{2\pi K}{\omega} d\omega} - 1 \right]} \quad (29)$$

The $[...]$ factor is short range (at least after choosing a smooth cutoff, as done in the similar 2D sine-Gordon [49][51]) and is positive. We then replace $\tau' \rightarrow \tau$ in the integrand and the $\tau'$ integral becomes $\alpha \frac{d\Lambda}{\Lambda}$ where $\alpha = O(1) > 0$ is dimensionless and depends on the cutoff procedure. Hence a term $\sum_{i \neq j} \cos[B^i_\tau - B^j_\tau]$ is generated, and since there are $N - 2$ choices for $j$, the 2nd order RG yields Eq. (14) of the main text.

The lower sign in (27) generates a term $\tilde{g} \Lambda \int_\tau \cos[B^k_\tau + 2B^j_\tau] \frac{d\Lambda}{\Lambda}$ whose 1st order RG is

$$\tilde{g} \Lambda \left( \int_\tau \cos[B^k_\tau + 2B^j_\tau] \right)_{\Lambda} = e^{-\frac{1}{2} \int_\Lambda^\Lambda' (6D_{ij} - 2D_{i\neq j})} = e^{-6 \int_\Lambda^\Lambda' \frac{d\omega}{\omega} \frac{2\pi K}{\omega}}$$

(30)
The RG equation for $\hat{g}$ to 1st order is

$$\hat{g}^R = \hat{g}[1 + (1 - 6K)\frac{d\Lambda}{\Lambda}]$$  \hspace{1cm} (31)$$

Hence $\hat{g}$ is relevant only at $K < 1/6$, irrelevant for the phase transition of $g_2$.

Consider next the case of two pairs of equal indices, i.e. $\cos(B^i_\tau - B^j_\tau + B^k_\tau - B^l_\tau)$. As above, there is a short range kernel that causes the argument of the cos to vanish; a next order expansion in $\tau - \tau'$ can yield $[\partial_\nu(B^i - B^j)]^2$ which is $\sim \omega^2$, hence negligible in comparison with $D^{-1} \sim |\omega|$. Finally, if all indices are distinct we need

$$\langle (B^i_\tau - B^j_\tau + B^k_\tau - B^l_\tau)^2 \rangle = \int_{\mathbb{N}} \frac{d\omega}{2\pi} [2D_{ii} - 2D_{ij \neq j} + D_{i \neq j} e^{i\omega(\tau - \tau')} - D_{i \neq j} e^{i\omega(\tau - \tau')} + D_{j \neq i} e^{i\omega(\tau - \tau')} + c.c.] = 4K \int_\mathbb{N} \frac{d\omega}{\omega}$$  \hspace{1cm} (32)$$

which is exactly canceled by the counterterm. Hence the only needed renormalization is Eq. (29), leading to Eq. (14)

of the main text.

The case $N = 2$ can be discussed separately. One can define $B^\pm_\tau = B^1_\tau \pm B^2_\tau$. The sum $B^+_\tau$ is Gaussian and decouples, while the action for the difference field becomes

$$S_{N=2} = \frac{1}{2} \int_{\mathbb{R}} \frac{|\omega|}{4\pi K} |B^\omega_\tau|^2 - g_2 \int_{\tau} \cos(B^\omega_\tau)$$  \hspace{1cm} (33)$$

This is the well studied static impurity problem [37, 40, 41], that has a phase transition at $K = \frac{1}{2}$, independent of $g_2$, i.e. with a vertical phase boundary.

II. CORRELATION FUNCTIONS

Notational remark: In the main text we use the shorthand notation $\langle B^i_\omega B^j_{-\nu} \rangle$ to denote $\lim_{\tau \to 0} \{T \langle B^i_\omega B^j_{-\nu} \rangle\}$ (with $\omega_n$ Matsubara frequencies) whereas the complete $T = 0$ correlation in continuum frequencies (denoted with a prime) is $\langle B^i_\omega B^j_{\nu}' \rangle = 2\pi \delta(\omega + \omega') \langle B^i_\omega B^j_{-\nu} \rangle$. In the Supplementary Material we use the form with Matsubara frequencies, though for brevity we denote $\omega_n \to \omega$.

In this section we introduce source terms, evaluate the $B^i_\omega$ correlations and derive the sum rule mentioned below Eq. (16) in the main text. Consider the effective action Eq. 13 in the main text and add a source $A_\tau$ (related to a vector potential)

$$S_1 = \frac{1}{2} \int_{\omega} D_{i \neq j}^{-1} B^i_\omega B^j_\tau - g_2 A \sum_{\nu}^V e^{iV \cdot B_\tau} - \int_{\omega} |\omega| |B_\omega \cdot A_{-\omega}|$$

$$= \frac{1}{2} \int_{\omega} \frac{|\omega|}{2\pi K} \{B_\omega \cdot B_{\nu} - \frac{1}{N_\nu} \sum_{i} B^i_\nu |^2 \} - g_2 A \sum_{\nu} \int_{\omega} e^{iV \cdot B_\tau} - \int_{\omega} |\omega| |B_\omega \cdot A_{-\omega}|$$  \hspace{1cm} (34)$$

The set of vectors $V$ are all $N$-component vectors with one entry +1, one entry −1 and all other entries 0. It is easy to see that they are all minimal norm vectors of the Bravais lattice generated by $N - 1$ basis vectors, labeled as $V_\nu$, $\nu = 1, \ldots, N - 1$ and which can be chosen as:

$$V_1 = (1, -1, 0, 0, \ldots) , \hspace{0.5cm} V_2 = (1, 0, -1, 0, \ldots) , \hspace{0.5cm} V_3 = (1, 0, 0, -1, \ldots) , \hspace{0.5cm} \ldots$$  \hspace{1cm} (35)$$

Their coordinates are denoted $(V_\nu)_i \equiv V^i_\nu$ with $V^i_\nu = \delta_{1,i} - \delta_{\nu+1,i}$, $i = 1, \ldots, N$. These vectors are all perpendicular to $(1,1,1,1,\ldots)$, hence the set $V_\nu$, $\nu = 1, \ldots, N - 1$, is a primitive unit cell of an $N$-1 dimensional space, a "hypertriangular" lattice [36], i.e. all lattice points of a simple cubic $N$ dimensional lattice that are perpendicular to $(1,1,1,1,\ldots)$. For $N = 3$ it is a 2D triangular lattice, for $N = 4$ these form tetrahedrons which are the primitive cell for an fcc lattice.

The partition sum of $S_1$ generates the correlation function needed later, using $\int_{\omega} = T \sum_{\omega_n}$,

$$\langle B^i_{\omega} B^j_{-\omega} \rangle = \frac{1}{Z_1} \frac{1}{\omega^2 T^2} \frac{\delta^2 Z_1}{\delta A_{\omega} \delta A_{\omega}} |_{A = 0}$$  \hspace{1cm} (36)$$
All correlations denoted $\langle \ldots \rangle$ here and below imply $A = 0$. An equivalent form of this correlation is obtained by writing the Gaussian terms as $\frac{|\omega|}{4\pi K}((B - 2\pi KA)\omega)^2 - \pi K|\omega ||A\omega|^2$ and shifting $B_{\omega} \rightarrow B_{\omega} + 2\pi KA_{\omega}$,

$$S_1 = \frac{1}{2} \int_{\omega} \frac{|\omega|}{2\pi K} \{B_{\omega} \cdot B_{\omega} - \frac{1}{N_{\omega}} \sum_i B_{\omega}^i + 2\pi KA_{\omega}^i \} - \pi K \int_{\omega} |\omega| |A\omega|^2$$

$$- g_2 \Lambda \sum_\mathcal{V} \int_{\tau} e^{i\mathcal{V} \cdot (B_{\tau} + 2\pi KA_{\tau})}$$

(37)

The partition sum $Z_1 = \int \mathcal{D}B e^{-S_1}$ involves a path integral $\int \mathcal{D}B$, i.e. integration on all periodic functions $B_{\tau}$ (with period $\beta$, where we study mostly the limit $\beta \rightarrow +\infty$). Therefore, from (36)

$$\langle B_{\omega}^i B_{\omega}^j \rangle = \frac{1}{Z_1 T \omega^2 \delta A_{\omega}} \int \mathcal{B} \{ \frac{|\omega|}{N_{\omega}} \sum_k (B_{\omega}^k + 2\pi KA_{\omega}^k) + 2\pi K|\omega|A_{\omega}^i \}$$

$$+ g_2 \Lambda \sum \frac{iV^i 2\pi K \int_{\tau} e^{i\omega \tau} e^{i\mathcal{V} \cdot (B_{\tau} + 2\pi KA_{\tau})}) e^{-S_1} \big|_{A=0} = \frac{2\pi K}{T |\omega|} \frac{1}{N_{\omega}} + \delta_{i,j}$$

$$+ \frac{1}{N_{\omega}^2} \sum_{k,l} \langle B_{\omega}^k B_{\omega}^l \rangle - (g_2 A)^2 \frac{2\pi K}{\omega^2} \sum_i V^i V^j \int_{\tau,\tau'} e^{i\omega(\tau-\tau')} \langle e^{iV \cdot B_{\tau} + iV \cdot B_{\tau'}} \rangle$$

$$- g_2 \Lambda \frac{(2\pi K)^2}{\omega^2} \sum_i V^i V^j \int_{\tau} \langle e^{iV \cdot B_{\tau}} \rangle$$

(38)

We use here $\int_{\tau} e^{-i\omega \tau} \langle e^{iV \cdot B_{\tau}} \sum_k B_{\omega}^k \rangle = 0$ which can be shown by using the variables $\bar{B}_i(\tau) = B_i(\tau) - \bar{B}(\tau)$ where $\bar{B}_{\tau} = \frac{1}{N} \sum_i B_{\tau}^i$; the Gaussian part of the action becomes

$$S_0 = \frac{1}{2} \int_{\omega} \sum_{i,j} \frac{|\omega|}{2\pi K} (\delta_{i,j} - \frac{1}{N_{\omega}})(\bar{B}_i + \bar{B}_{\omega})(\bar{B}_j + \bar{B}_{-\omega}) = \frac{1}{2} \int_{\omega} \frac{|\omega|}{2\pi K} \bar{B}_{\omega} \cdot \bar{B}_{-\omega} + \frac{1}{2} \int_{\omega} \frac{NM \omega^2}{N_{\omega}} |\bar{B}_{\omega}|^2$$

(39)

Since $e^{iV \cdot B_{\tau}} = e^{iV \cdot \bar{B}_{\tau}}$ is independent of $\bar{B}_{\tau}$ we have $\int_{\tau} e^{-i\omega \tau} \langle e^{iV \cdot B_{\tau}} \sum_k B_{\omega}^k \rangle \sim \langle \bar{B}_{-\omega} \rangle = 0$. Since $\sum_i V^i = 0$ we obtain from (38) an exact sum rule

$$\sum_i T(B_{\omega}^i B_{-\omega}^j) = \frac{2\pi K}{|\omega|} \frac{N}{N_{\omega}} + 1/(1 - \frac{N^2}{N_{\omega}^2}) = \frac{N_{\omega}}{M \omega^2}, \quad T(|\bar{B}_{\omega}|^2) = \frac{N_{\omega}}{N M \omega^2}$$

(40)

Hence the $g_2$ independent terms of $T(B_{\omega}^i B_{-\omega}^j)$ are

$$\frac{2\pi K}{|\omega|} \frac{1}{N_{\omega}} + \delta_{i,j} = \frac{2\pi K}{|\omega|} \delta_{i,j} + \frac{1}{M \omega^2} = D_{i,j}$$

which is consistent with the $g_2 = 0$ result. Hence finally we obtain the exact formula (for the effective model $S_1$)

$$\langle B_{\omega}^i B_{-\omega}^j \rangle = \frac{D_{i,j}}{T} - (g_2 \Lambda)^2 \frac{(2\pi K)^2}{\omega^2} \sum_i V^i V^j \int_{\tau,\tau'} e^{i\omega(\tau-\tau')} \langle e^{iV \cdot B_{\tau} + iV \cdot B_{\tau'}} \rangle$$

$$- g_2 \Lambda \frac{(2\pi K)^2}{\omega^2} \sum_i V^i V^j \int_{\tau} \langle e^{iV \cdot B_{\tau}} \rangle$$

(42)

We now use the latter form for a perturbative expansion of the correlation function in $g_2$. Consider first the last term of (42), to lowest order it has

$$\Lambda \int_{\tau} \langle e^{iV \cdot B_{\tau}} \rangle = \Lambda \int_{\tau} e^{-\frac{1}{2} \sum_k V^k} = \Lambda \int_{\tau} e^{-\frac{1}{2} \sum_k V^k} = \Lambda \int_{\tau} e^{-\frac{1}{2} \sum_k V^k} \int \frac{2\pi K}{|\omega|} \langle \bar{B}_{\omega} \rangle = \left( \frac{T}{\Lambda} \right)^{2K-1} K^{-\frac{1}{2}}$$

(43)

where $\langle \ldots \rangle_0$ is an average w.r.t. $S_1(g = 0, A = 0)$. We use temperature $T$ as a lower bound on the $\omega$ integral and $1/T$ as an upper bound on the $\tau$ integral. We note also that the factor $\frac{1}{M \omega^2}$ common to all $D_{kl}$ is canceled since $\sum_k V^k = 0$ and that the perturbation expansion is valid only for $K > \frac{1}{2}$.!
The next order of the last term of [42] combines with the 2nd term so that

\[
T\langle B^i_\nu B^j_\omega \rangle = D_{i,j} - (g_2\Lambda)^2 \frac{(2\pi K)^2}{\omega^2} \sum_V V^i V^j \int_{|\tau| > 1/\Lambda} (1 - e^{i\omega\tau})(\Lambda|\tau|)^{-4K}
\]

The average \(\langle (\mathbf{V} \cdot \mathbf{B}_r + i\mathbf{V}' \cdot \mathbf{B}_r' )^2 \rangle_0\) involves \(D_{ij}\) whose common term \(1/M\omega^2\) cancels since \(\sum_k V^k = 0\). Thus only diagonal \(D_{ii}\) survive with \(2\pi K/|\omega|\). Hence

\[
e^{-\frac{1}{2}((\mathbf{V} \cdot \mathbf{B}_r + i\mathbf{V}' \cdot \mathbf{B}_r')^2)_0} = e^{-\frac{1}{2} \sum_k [(V^k)^2 + (V'^k)^2 + 2V^k V'^k \cos\omega(\tau - \tau')] f_\omega \frac{2\pi K}{|\omega|}
\]

To cancel the infrared divergence, i.e. avoid the vanishing terms as in [43], only terms with \(\mathbf{V} = -\mathbf{V}'\) survive,

\[
\langle e^{i\mathbf{V} \cdot \mathbf{B}_r - i\mathbf{V}' \cdot \mathbf{B}_r'} \rangle_0 = e^{-\frac{1}{2} V^2 f_\omega \frac{4\pi K}{|\omega|}(\Lambda|\tau - \tau'|)^{-4K}} \quad |\tau - \tau'| > 1/\Lambda
\]

Hence to order \(O(g^2_2)\) we finally obtain,

\[
T\langle B^i_\nu B^j_\omega \rangle = D_{i,j} - (g_2\Lambda)^2 \frac{(2\pi K)^2}{\omega^2} \sum_V V^i V^j \int_{|\tau| > 1/\Lambda} (1 - e^{i\omega\tau})(\Lambda|\tau|)^{-4K}
\]

\[
= D_{i,j} - (g_2\Lambda)^2 \frac{(2\pi K)^2}{\omega^2} \sum_V V^i V^j \frac{2}{4K-1} (\frac{\omega}{\Lambda})^{4K-1}
\]

(44)

We can now use the previous RG study to express this result in terms of the renormalized coupling \(g^R\). For this purpose we consider \(\omega\) as the new cutoff \(\Lambda = \Lambda - d\Lambda\), hence \((\frac{\omega}{\Lambda})^{4K-2} \rightarrow 1 - (4K - 2) \frac{d\Lambda}{\Lambda}\). This identifies \(g^R_2\) to 1st order, using Eq. (25). Therefore near \(K = \frac{1}{2}\), we obtain to lowest order in \(g^R_2\),

\[
T\langle B^i_\nu B^j_\omega \rangle = D_{i,j} + (g^R_2)^2 \frac{4\pi^2}{|\omega|} (1 - N\delta_{i,j})
\]

(45)

where \(\sum_V V^i V^j = \sum_{a,b=1}^N (\delta_{ia} - \delta_{ib})(\delta_{ja} - \delta_{jb}) = 2(N\delta_{i,j} - 1)\). We can apply this along the critical line Eq. (15) in the main text. One sees that the correlation depends continuously on \(g^R_2\). For \(N > 2\) it yields a \(K\)-dependent correction, \(\sim (K - \frac{1}{2})^2 (1 - N\delta_{i,j})/|\omega|\) to the critical correlation function, \(T\langle B^i_\nu B^j_\nu \rangle\), originating from the \(g^R_2\) term.

**III. DUALITY**

We derive here the duality between weak and strong coupling which is used in the main text. We follow ideas stated by Yi and Kane [35, 36]. We also derive explicitly the basis vectors, and the correlation functions given in the text.

In the following we introduce the basis of the reciprocal lattice, i.e. \(\mathbb{R}_\nu\) of vectors \(\mathbf{R}_\nu\) that satisfy \(\mathbf{R}_\nu \cdot \mathbf{V}_\nu = \delta_{\nu,\nu'}\) for \(\nu, \nu' = 1, ..., N - 1\). Define a matrix \(G_{i,\nu} = (\mathbf{V}_\nu)_i\), i.e. the \(i\)-th component of the vector \(\mathbf{V}_\nu\). This has \(N\) rows \((i = 1, 2, ..., N)\) and \(N-1\) columns \((\nu = 1, 2, ..., N - 1)\). Similarly, define a matrix \(R_{i,\nu} = (\mathbf{R}_\nu)_i\). By definition

\[
\sum_i R_{i,\nu} G_{i,\nu'} = \delta_{\nu,\nu'}, \quad \Rightarrow R^T G = 1
\]

(46)

A solution is:

\[
R^T = (G^T G)^{-1} G^T \quad \text{since} \quad R^T G = (G^T G)^{-1} G^T G = 1
\]

(47)

The norms are \(|\mathbf{R}_\nu|^2 = \sum_i R^2_{i,\nu} = (R^T R)_{\nu,\nu}\), so their sum satisfies \(\sum_i |\mathbf{R}_i|^2 = Tr RR^T = Tr G(G^T G)^{-1} G^T G = Tr (G^T G)^{-1}\). More explicitly, for \(N = 4\)

\[
G^T G = \begin{pmatrix}
1 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

\[
= 1 + \delta_{i,j} \quad \text{and} \quad (G^T G)^{-1} = \delta_{i,j} - \frac{1}{2}
\]

\[
R = G(G^T G)^{-1} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
The following is valid for all $N$: $G^T G = 1 + \delta_{i,j}$, $(G^T G)^{-1} = \delta_{i,j} - \frac{1}{N}$, $R_{i,\nu} = \frac{1}{N} - \delta_{i,\nu+1}$ and the norm of each $R_{\nu}$ is $|R_{\nu}|^2 = 1 - \frac{1}{N}$. The $R_{\nu}$ are also orthogonal to $(1,1,1,\ldots)$, however they are not a primitive unit cell for $N > 3$ and therefore their lattice symmetry differs from that of the $V_{\nu}$ (see discussion in the text). Note that for $N \to \infty$ it is an N-1 dimensional cubic lattice. Finally note, as mentioned in the text, that in this reciprocal lattice there are 2N vectors $R$ of minimal norm $|R_{\nu}|^2 = 1 - \frac{1}{N}$.

The duality relates $S_1$ to an action whose Gaussian part is diagonal. To apply duality we must change first the $B_{i,\tau}$ variables to $C_{i,\omega}$ such that $S_0$ is diagonal $\sim 1$, with $B_{\omega} = \sum_i B_{i,\omega}/N$,

$$C_{i,\omega} = B_{i,\omega} - \alpha_{\omega} \bar{B}_{\omega}, \quad \bar{C}_\omega = \sum_i C_{i,\omega}/N = (1 - \alpha_{\omega}) \bar{B}_{\omega}$$

$$\sum_{i,j}(\delta_{i,j} - \frac{1}{N\omega})B_{i,\omega}^*B_{j,\omega} = \sum_i (C_{i,\omega}^* + \alpha_{\omega} \bar{B}_{\omega})(C_{i,\omega} - \alpha_{\omega} \bar{B}_{\omega}) - \frac{N^2}{N\omega}|\bar{B}_{\omega}|^2$$

$$= \sum_i C_{i,\omega}^*C_{i,\omega} + N|\bar{B}_{\omega}|^2[\alpha_{\omega}^2 + 2\alpha_{\omega}(1 - \alpha_{\omega})] = \sum_i C_{i,\omega}^*C_{i,\omega}$$

$$\Rightarrow \alpha_{\omega} = 1 - \sqrt{1 - \frac{N}{N\omega}} \quad (48)$$

Note that $\text{Det}[\frac{\partial C_{\omega}}{\partial B_{\omega}}] = \text{Det}[\delta_{i,j} - \frac{\alpha_{\omega}}{N}] = 1 - \alpha_{\omega}$, hence this transformation is proper if $\alpha_{\omega} \neq 1$, i.e. $\omega \neq 0$; therefore $\omega = 0$ is used only as a limit. Using $\langle B_{i,\omega}^* \bar{B}_{\omega}\rangle = \langle |\bar{B}_{\omega}|^2 \rangle = \frac{N}{T\pi N\omega}$ it is useful to note

$$\langle C_{i,\omega}^*C_{j,\omega}\rangle = \langle B_{i,\omega}^*B_{j,\omega}\rangle + (\alpha_{\omega}^2 - 2\alpha_{\omega})\langle |\bar{B}_{\omega}|^2 \rangle = \langle B_{i,\omega}^*B_{j,\omega}\rangle - \frac{1}{T\pi \omega^2} \quad (49)$$

Using now $C_{i,\omega} = B_{i,\omega} - \frac{\alpha_{\omega}}{1 - \alpha_{\omega}} \bar{C}_\omega$ we obtain the source term in terms of $C_{i,\omega}$ and the action becomes

$$S_1 = \frac{1}{2} \int_\omega \frac{|\omega|}{2\pi K} \bar{C}_\omega \cdot \bar{C}_\omega - g_2 \Lambda \sum_{\nu} \int_{\tau} e^{iV} C_{\tau} - \int_\omega |\omega| C_{\omega} \cdot A_{-\omega} + \frac{\alpha_{\omega}}{1 - \alpha_{\omega}} \bar{C}_\omega \sum_i A_{i,\omega}^* \quad (50)$$

Consider now large $g_2$ where the trajectories of $C_{\tau}$ are dominated by instantons, i.e. a sequence of $n$ sharp jumps at consecutive times $\tau_1, \tau_2, \ldots, \tau_n$ that shift $C_{\tau}$ between neighboring equivalent minima by a vector $2\pi R_{\alpha}$, each chosen from the set of equivalent minimal length vectors (here the index $\alpha$ labels the instanton and should not be confused with $\nu$ which labels the basis vectors). Eventually there is a summation on all $n$ such that periodic boundary conditions (in imaginary time $\tau$) are maintained, i.e. $\sum_{\alpha} R_{\alpha} = 0$, in particular $n$ is even. Since all $R_{\alpha}$ are perpendicular to $(1,1,1,\ldots)$ the instanton trajectory describes only $\bar{C}_{i,\tau} = C_{i,\omega} - \bar{C}_\omega$. The action can be written as the sum of two independent parts,

$$S_1 = \frac{1}{2} \int_\omega \frac{|\omega|}{2\pi K} \bar{C}_\omega \cdot \bar{C}_\omega - g_2 \Lambda \sum_{\nu} \int_{\tau} e^{iV} C_{\tau} - \int_\omega |\omega| \bar{C}_\omega \cdot A_{-\omega}$$

$$+ \frac{1}{2} N \int_\omega \frac{|\omega|}{2\pi K} |\bar{C}_\omega|^2 - \frac{|\omega|}{1 - \alpha_{\omega}} \int_\omega \bar{C}_\omega \sum_i A_{i,\omega}^* \quad (51)$$

Note that evaluating directly the $\bar{C}_{i,\omega}$ part is cumbersome since these are dependent variables, yet this form is convenient for the instanton description.

$$\bar{C}(\tau) = 2\pi \sum_{\alpha} R_{\alpha} \theta(\tau - \tau_\alpha) = i2\pi \sum_{\alpha} R_{\alpha} \int_0^{\frac{\pi}{\omega}} \frac{e^{-i\omega(\tau - \tau_\alpha)}}{\omega + i\epsilon} \quad (52)$$

$$\bar{C}(\omega) = i2\pi \sum_{\alpha} R_{\alpha} \frac{e^{i\omega \tau_\alpha}}{\omega + i\epsilon}$$

$$\int_\omega |\omega| \bar{C}_{i,\omega} A_{i,\omega}^* = i2\pi \sum_{\alpha} R_{\alpha} \int_\omega |\omega| \frac{e^{i\omega \tau_\alpha}}{\omega + i\epsilon} A_{i,\omega}^* \equiv i2\pi R_{\alpha} \cdot a(\tau_\alpha)$$

$$a_\omega = \frac{|\omega|}{-\omega + i\epsilon} A_{\omega}, \quad a(\tau) = \int_\omega \frac{|\omega|}{\omega + i\epsilon} A_{-\omega} \quad (53)$$
The $i\epsilon$ is irrelevant since $|\omega|\delta(\omega) = 0$, hence $a(\tau)$ is real. The weight of each instanton is $\Lambda\Delta \sim e^{-S_{\text{instanton}}}$ with $S_{\text{instanton}} \sim \sqrt{2}$; $\Lambda$ is present so that $\Delta$ is dimensionless. The partition sum becomes (up to a constant $\sim \int_\omega \ln(1-|\omega|)$)

$$Z_1 = \tilde{Z} \sum_n \sum R_\alpha \int_{\tau_1 < \tau_2 < ... < \tau_n} (\Lambda\Delta)^n e^{-S_0^{(n)} + i2\pi \sum R_\alpha a(\tau_\alpha)}$$

$$\tilde{Z} = \int D\tilde{C}_\omega e^{-\frac{N}{2\pi} \sum C_\omega \cdot \bar{C}_\omega + \int_\omega \frac{|\omega|}{2\pi K} \sum C_\omega \cdot \bar{C}_\omega \sum A_i}$$

(54)

where $\tilde{Z}$ is the $\bar{C}_\tau$ dependent part and $S_0^{(n)}$ is the Gaussian part $\int_\omega \frac{|\omega|}{2\pi K} \sum C_\omega \cdot \bar{C}_\omega \omega$ with $n$ instantons inserted (a term $\sim \sum R_\alpha = 0$ by boundary condition is added),

$$S_0^{(n)} = \frac{1}{2} \int_\omega \frac{|\omega|}{2\pi K} \sum_{\alpha,\beta} 4\pi^2 R_\alpha \cdot R_\beta e^{i\omega(\tau_\alpha - \tau_\beta)} - \frac{1}{K} \sum_{\alpha \neq \beta} R_\alpha \cdot R_\beta \ln \Lambda|\tau_\alpha - \tau_\beta|$$

(55)

The partition sum is therefore, using $\int K|\omega| \frac{d\omega}{2\pi} = 2 \int_1^{\Lambda} \frac{d\omega}{2\pi} = \frac{1}{\pi} \ln \Lambda$ with a high frequency cutoff $\Lambda$,

$$Z_1 = \tilde{Z} \sum_{n=0}^{\infty} \sum_{\{R_\alpha\}} \int_{\tau_1, ... , \tau_n} (\Lambda\Delta)^n e^{\frac{N}{2\pi} \sum_{\alpha \neq \beta} R_\alpha \cdot R_\beta \ln \Lambda|\tau_\alpha - \tau_\beta| + i2\pi \sum R_\alpha a(\tau_\alpha)}$$

(56)

Using $B_\omega = C_\omega + \frac{\alpha_\omega}{1-\alpha_\omega} \tilde{C}_\omega = \bar{C}_\omega + \frac{\alpha_\omega}{1-\alpha_\omega} \tilde{C}_\omega$ we find that for $\Delta = 0$

$$\langle C_i^\omega C_j^\omega \rangle = \langle |\tilde{C}_\omega|^2 \rangle = \frac{2\pi K}{TN|\omega|}$$

$$\langle B_i^\omega B_j^\omega \rangle = \frac{1}{1-\alpha_\omega} \langle |\tilde{C}_\omega|^2 \rangle = \frac{N_\omega}{TN \omega^2}$$

(57)

This result can be obtained also with the original $B_i^\omega$ variables, the $C_i^\omega$ variables are needed only for the following duality.

Consider now a dual action, defined on the lattice $R$ with a vector field $\theta_i(\tau)$

$$S_2 = \frac{1}{2} \int_\omega \frac{K|\omega|}{2\pi} \theta(\omega)^2 - \Lambda\Delta \sum_R \int_\tau e^{iR \cdot (\theta(\tau) + 2\pi a(\tau))}$$

(58)

The dual partition sum is expanded, the time integrals in $n$-th order can be ordered so that $\frac{1}{n!} \int_{\tau_1, ... , \tau_n} ... = \int_{\tau_1 < \tau_2 < ... < \tau_n}$, hence averaging on the $|\theta(\omega)|^2$ term yields

$$Z_2 = Z_0 \sum_{n=0}^{\infty} (\Lambda\Delta)^n \int_{\tau_1 < \tau_2 < ... < \tau_n} \langle e^{i\sum R_\alpha \theta(\tau_\alpha)} \rangle_0 e^{i2\pi \sum R_\alpha a(\tau_\alpha)}$$

(59)

where $Z_0 = \int D\theta e^{-\frac{1}{2} \int_\omega \frac{K|\omega|}{2\pi} \theta(\omega)^2}$. Each factor $e^{iR_\alpha \theta(\tau_\alpha)}$ must multiply a term with $R_\alpha \rightarrow -R_\alpha$, otherwise the average diverges and its exponent vanishes; hence $\sum R_\alpha = 0$ and a constant -1 can be added below. The Gaussian average involves

$$\frac{1}{2} \langle (\sum R_\alpha \cdot \theta(\tau_\alpha))^2 \rangle = \frac{1}{2} \sum_{\alpha,\beta} R_\alpha^i R_\beta^j \langle (\theta^i(\tau_\alpha) \theta^j(\tau_\beta))_0 = \frac{1}{2} \sum R_\alpha \cdot R_\beta \int_\omega \frac{e^{i\omega(\tau_\alpha - \tau_\beta)} - 1}{|\omega| K/2\pi}$$

(60)

so that

$$Z_2 = Z_0 \sum_{n=0}^{\infty} (\Lambda\Delta)^n \int_{\tau_1 < \tau_2 < ... < \tau_n} \langle e^{\frac{N}{2\pi} \sum R_\alpha \cdot R_\beta \ln \Lambda|\tau_\alpha - \tau_\beta| + i2\pi \sum R_\alpha a(\tau_\alpha)}$$

(61)

This is identical to $Z_1$, apart from the $\sim |\tilde{C}_\omega|^2$ term, i.e. $Z_1(A) = \tilde{Z}(A) Z_2(A) / Z_0$, displaying now the $A_\omega$ dependence. This proves the duality.
Let us now obtain the correlation functions. Since \( \langle \tilde{C}_i \tilde{C}_- \rangle = 0 \) the original \( B_i \) correlation is now, using \( \delta a_j(\tau)/\delta A^i_\omega = -T \text{sign} \omega e^{-i\omega \tau} \delta_{i,j} \),

\[
(B^\omega_i B^\omega_j) = \frac{1}{Z_1} \frac{1}{T^2 \omega^2} \delta^2 Z_1 |_{A=0} = \frac{1}{Z_2} \frac{1}{T^2 \omega^2} \delta^2 Z_2 |_{A=0} = \frac{1}{Z_2} \frac{1}{T^2 \omega^2} \delta^2 Z_2 |_{A=0}
\]

\[
= \frac{1}{(1 - \omega^2)} \langle |\tilde{C}_\omega|^2 \rangle + i \frac{2\pi \text{sign} \omega}{Z_2 T^2 \omega^2} \delta A^\omega_i |_{A=0} = N_\omega \frac{T N M \omega^2}{\omega^2} \sum R \Lambda \Delta \int_D \int R \langle e^{i R (\theta_\tau + 2 \pi a) R_j} e^{i \omega (\tau' - \tau)} \rangle + \Lambda \Delta \frac{(2\pi)^2}{\omega^2} \sum R R' \int \langle e^{i R \theta_\tau} \rangle
\]

This is an exact relation, which we now use for perturbative expansion. To 2nd order, similar to Eq. [44],

\[
\langle e^{i R (\theta_\tau - \theta_\tau')} \rangle = e^{-\frac{1}{2} R^2 f_\omega \frac{4\pi^2}{T^2} (1 - \cos \omega (\tau' - \tau))} = (\Lambda |\tau - \tau'| - \frac{\pi}{2} (1 - \frac{1}{N})) \Rightarrow
\]

\[
T(B^\omega_i B^\omega_j) = \frac{N_\omega}{N M \omega^2} + \frac{4\pi^2}{\omega^2} \Lambda \Delta \sum R R' \int \frac{1}{|\tau|} \left( \frac{\omega}{\Lambda} \right) \frac{\Delta (1 - \frac{1}{N})}{R^2} (\Delta R)^2
\]

The sum on \( R \) is performed using \((R_\omega)^i = \frac{1}{N} \delta_{i,a} \) as well as the \((R_\omega)^i \) vectors, hence \( \sum R R' = 2(\delta_{i,j} - \frac{1}{N}) \). From the critical behavior below [66] the correction term is \( \sim (K_c - K)^2 \) for \( N = 3 \) while it is \( \sim K_c - K \) for \( N \geq 4 \).

An alternative form of the action can be obtained by shifting in [55] \( \theta(\tau) \rightarrow \theta(\tau) - 2 \pi a(\tau) \) (\( a(\tau) \) should be also periodic) and using \( \delta a^\omega_i / \delta A^i_\omega = -\text{sign} \omega \delta_{i,j} \),

\[
S_2 = \frac{1}{2} \int_\omega \int_\omega |\theta(\omega) - 2 \pi a(\omega)|^2 - \Lambda \sum R \int \langle e^{i R \theta_\tau} \rangle
\]

\[
\langle B^\omega_i B^\omega_j \rangle = \frac{N_\omega}{T N M \omega^2} + \frac{K}{T N M} \int_\omega \delta A^\omega_i |_{A=0} = \frac{N_\omega}{T N M \omega^2} + \frac{2\pi K}{T |\omega|} \delta_{i,j} - K^2 \langle \theta^\omega_i \theta^\omega_j \rangle |_{A=0}
\]

For \( \Delta = 0 \) this indeed reproduces [57].

Before proceeding to self duality, we consider RG at small \( \Delta \). For each \( R \) term

\[
\Delta^R A' = \Delta \Lambda \langle e^{i R \theta_\tau} \rangle = \Delta \Lambda e^{-\frac{1}{2} \sum R R' \langle \theta_\tau \theta_\tau' \rangle} = \Delta \Lambda e^{-\frac{R^2}{2} \Lambda} = \Delta \Lambda (1 - \frac{R^2}{2K} \Lambda)
\]

The fixed point is at \( K_c = 1 - \frac{1}{N} \). For \( N \geq 4 \) there are no \( \Delta^2 \) terms since \( R + R' \) are longer minimal length vectors (unlike the \( V \) vectors) and are therefore less relevant. The next order is \( O(\Delta^3) \) with a \( 1/N \) coefficient for large \( N \) (Ref. [36] table I), hence the phase boundary is \( \Delta_c = \sqrt{N} \sqrt{\frac{1}{K_c}} \) approaching steeply the \( \Delta = 0 \) point.

### A. Self Duality

Here we derive the self-dual relations given in the main text. For \( N = 2, 3 \) the symmetry of the \( V \) and \( R \) lattices are the same, only the length of these vectors differ. For \( N = 2 \), \( ||V||/||R|| = \sqrt{2}/\sqrt{3} = 2 \). Therefore, rescaling
and using the sum rule (40) we obtain
\[ K = \sum_{\tau} e^{i\mathbf{\theta} \cdot \mathbf{\tau}/2} \Rightarrow \langle \theta^2_k \theta^2_{-\omega} \rangle = 4 \langle C^i_k C^j_{-\omega} \rangle K \rightarrow 1/(4K), g_2 \rightarrow \Delta = 0 \]

The self-dual point is at \( K = \frac{1}{2} \), \( \Delta = g_2 \) and the last relation determines the correlation at this point as the average of the \( g_2 = 0 \) (Eq. 41) and \( \Delta = 0 \) (Eq. 57) cases i.e. \( T \langle B^i_{\omega} B^j_{-\omega} \rangle_{1/(4K), \Delta} = \frac{1}{2} \langle D_{ij} + \frac{N_{ij}}{TM \omega^2} \rangle \). Since the transition line is vertical and the \( \Delta(g_2) \) relation is not known precisely this is not such a useful information.

For \( N=3 \) the R lattice is also triangular with \( ||V||/||R|| = \sqrt{2}/\sqrt{2/3} = \sqrt{3} \). Rescaling \( \theta(\tau) \rightarrow \sqrt{3} \theta \) and using both (49,65) yields
\[ S_{2N=3}^2 = \frac{1}{2} \int \frac{K[\omega]}{2\pi} |\theta^2_k - \Lambda \Delta \sum_{\tau} e^{i\mathbf{\theta} \cdot \mathbf{\tau}/2} \Rightarrow \langle \theta^2_k \theta^2_{-\omega} \rangle = 3 \langle C^i_k C^j_{-\omega} \rangle K \rightarrow 1/(3K), g_2 \rightarrow \Delta = 0 \]

The self-dual point is at \( K = \frac{1}{\sqrt{3}} \), \( \Delta = g_2 \) where the correlation becomes
\[ K = \frac{1}{\sqrt{3}} : \langle B^i_{\omega} B^j_{-\omega} \rangle = \frac{1}{2} \left( \frac{N_{ij}}{TM \omega^2} + \frac{2\pi K}{T|\omega|} \delta_{ij} + \frac{1}{TM \omega^2} \right) \]

which is Eq. (24) of the main text, and again is precisely the average of the \( g_2 = 0 \) (Eq. 42) and \( \Delta = 0 \) (Eq. 57) cases. The result is also consistent with the sum rule \( T \sum_i \langle B^i_{\omega} B^j_{-\omega} \rangle = \sum_i D_{ij} = \frac{N_{ij}}{TM \omega^2} \).

IV. CONDUCTANCE

In this section we derive the conductance of LL wires with normal leads, given in the text in Eq. (22). Our surmise is that the resistance in the strongly coupled phase is the sum of individual resistances. The reasoning is that an equal current \( i \) flows in all wires, hence the resistance \( R_i \) of each wire (actually a contact resistance) leads to a total dissipation equal to \( \sum_i i^2 R_i \). Since the voltage is applied on one wire its effective resistance is \( R_{eff} = \sum_i R_i \). The equality of all currents implies also that all components of the conductance matrix \( G_{ij} \) are equal. In the following we confirm this surmise in various geometries. We start, however, with deriving a local conductance, conceptually important for the following derivation, and given in the text in Eq. (24).

A. Local Conductance

Consider the local conductance [37], i.e. the current at \( x = 0 \) on chain \( i \) in response to a voltage in the range \((-L/2, L/2)\) on chain \( j \)
\[ G_{ij}^{local}(\omega) = -\frac{e^2}{2\pi \hbar} i(\omega + i\delta) \frac{1}{L} \int_{-L/2}^{L/2} dx' \langle \hat{\phi}_i(x = 0, \omega_n) \hat{\phi}_j(x', -\omega_n) \rangle |_{\omega_n \rightarrow \omega + i\delta} \]

As shown in the next subsection the response in the DC limit \( \omega \rightarrow 0 \) is \( x \), \( x' \) independent, hence we can take \( L \rightarrow 0 \) so that the conductance is expressed in terms of the local fields at \( x = 0 \), \( \phi_i(\omega_n) \), of our nonlinear problem. In terms of the fields \( B^i_{\omega} \)
\[ \phi^i_{\omega} = \frac{1}{2} (B^i_{\omega} + X_{\omega}) = \frac{1}{2} (B^i_{\omega} - \frac{1}{N_{\omega}} \sum_j B^j_{\omega} + \bar{X}_{\omega}) \]

and using the sum rule (40) we obtain
\[ G_{ij}^{local}(\omega) = \frac{e^2}{2\pi \hbar} \omega [T \langle B^i_{\omega} B^j_{-\omega} \rangle - \frac{1}{M_{\omega}}] |_{\omega_n \rightarrow \omega + i\delta} \]
Using the analytic continuation

$$\text{sign}(\omega_n) = \int \frac{d\epsilon}{\pi} \frac{i}{\epsilon + i\omega_n} \to \int \frac{d\epsilon}{\pi} \frac{i}{\epsilon + \omega + i\delta} = +1$$

(73)

and Eqs. 45 64], we obtain the conductance in the vicinity of the fixed lines ($g_2 = 0$ and $\Delta = 0$) associated to the two phases, as well as at the self dual point

$$g_2 \text{ small : } G_{i,j}^{\text{local}}(\omega) = \frac{e^2}{h}[K\delta_{i,j} + 2\pi(g_2 R)^2(1 - N\delta_{i,j})]$$

$$\Delta \text{ small : } G_{i,j}^{\text{local}}(\omega) = \frac{e^2}{N\hbar}[K - 4\pi(\Delta R)^2(1 - N\delta_{i,j})]$$

$$\text{self-dual : } G_{i,j}^{\text{local}}(\omega) = \frac{e^2}{2\hbar}K[\delta_{i,j} + \frac{1}{N}] \quad (N = 2, 3)$$

(74)

Inserting the RG fixed point values in these equations also give the leading corrections of the conductance along the phase transition line, as discussed in the text. The conductance matrix thus varies continuously along the phase transition line.

**B. Normal leads for N TLLs**

We consider here the strongly coupled phase of N TLLs with boundary conditions of normal leads on each TLL, represented by LLs with Luttinger parameter $K_L$ (eventually $K_L = 1$). In this geometry one can measure current and voltage at either of the external leads. For $N = 1$ the conductance is 45 15 52 $e^2/\pi K_L \to e^2/\pi$.

In view of the space dependence our duality method is not feasible, hence we replace the nonlinear coupling in the action Eq. (13) of the main text by a Gaussian one

$$\frac{1}{2} \mu \sum_{ij} \left[ \phi_i(0, \tau) - \phi_j(0, \tau) \right]^2.$$ 

This replacement is justified because (i) all $G_{ij}$ become equal, indicating equal currents as needed, (ii) the result is independent of the coupling $\mu$ in the limit $\omega_n/\mu \to 0$, (iii) the result reproduces the local conductance Eq. (74) in the formal limit $K_L \to K$. In the latter limit the TLL is extended without leads, corresponding to local conductance, which in turn incorporates the full nonlinearity of our system. We note that a similar replacement was also done in Ref. 48. The LL action, allowing for space dependence of $u(x), K(x)$, becomes

$$S = \frac{1}{2} \int_{\omega_n} \int_x \left\{ \sum_i \frac{\omega_n^2}{u(x)K(x)} |\phi_i(\omega_n, x)|^2 + \frac{u(x)}{K(x)} |\partial_x \phi_i(x)|^2 \right\} + \frac{1}{2} \mu \sum_{i \neq j} \delta(x)|\phi_i(\omega_n, x) - \phi_j(\omega_n, x)|^2$$

$$\equiv \int_{\omega_n} \int_x \phi(\omega_n, x)C_{i,j}^{-1}(\omega_n; x, x')\phi^*(\omega_n, x')$$

(75)

This identifies the retarded correlation $C_{i,j}^{-1}(\omega_n, x, x')$. We can then define the conductance as

$$G_{ij}(\omega_n, x, x') = \frac{e^2}{\pi \hbar} C_{i,j}(\omega_n, x, x')|_{\omega_n \to \omega + i\delta},$$

where the $x, x'$ dependence actually drops out in the DC limit (see below). Upon multiplication of $C_{i,k}^{-1}(\omega_n; x, x'')$ on the right by $C_{k,j}(\omega_n, x'', x')$ and summation on $k, x''$

$$\sum_k \left\{ \frac{\omega_n^2}{u(x)K(x)} - \partial_x \frac{u(x)}{K(x)} \partial_x + \mu(N - 1)\delta(x)\delta_{i,k} - \mu\delta(x)(1 - \delta_{i,k}) \right\} C_{k,j}(\omega_n, x, x') = \pi\delta(x - x')\delta_{i,j}$$

(76)

The symmetry among the wires and the common boundary conditions allow for all the diagonal $C_{ii}$ to be equal and also all off diagonal $C_{ij \neq j}$ to be equal. Hence two equations of motion

$$\frac{\omega_n^2}{u(x)K(x)} - \partial_x \frac{u(x)}{K(x)} \partial_x + M(N - 1)\delta(x)C_{ii} - M(N - 1)\delta(x)C_{i \neq j} = \pi\delta(x - x')$$

$$\frac{\omega_n^2}{u(x)K(x)} - \partial_x \frac{u(x)}{K(x)} \partial_x + M\delta(x)C_{i \neq j} - M\delta(x)C_{ii} = 0$$

(77)

The correlation functions need outgoing waves at the boundary as appropriate for normal wires. This also implies dissipation, e.g. at $x > \frac{1}{2}L$ and $\nu \equiv |\omega_n| \to -i(\omega + i\delta)$, then $e^{-\nu x/\hbar L} \to e^{i(\omega + i\delta)x/\hbar L}$. The solution can be written in
the following form

\[
C_{ii} = A e^{\nu x/u} \\
B e^{\nu x/u} + C e^{-\nu x/u} \\
B' e^{\nu x/u} + C' e^{-\nu x/u} \\
D e^{\nu x/u} + E e^{-\nu x/u} \\
D' e^{\nu x/u} + E' e^{-\nu x/u} \\
F e^{-\nu x/u} + G e^{\nu x/u} \\
C_{i\neq j} = A e^{\nu x/u} \\
B e^{\nu x/u} + C e^{-\nu x/u} \\
B' e^{\nu x/u} + C' e^{-\nu x/u} \\
D e^{\nu x/u} + E e^{-\nu x/u} \\
D' e^{\nu x/u} + E' e^{-\nu x/u} \\
F e^{-\nu x/u} + G e^{\nu x/u}
\]

where the coefficients are implicit functions of \(\nu, x', L\), which are determined by the conditions of continuity and derivative jumps at \(x = 0, x = x', x = \pm L/2\). This leads to a complicated expression which simplifies drastically in the limit \(\nu \to 0\) at fixed \(x, x', L\) leading to

\[
B + C \simeq A \approx \frac{\pi K L}{2 \nu N} + O(1) \\
\Rightarrow G_{ij} = \frac{e^2 K L}{\hbar N} \to \frac{e^2}{\hbar N}
\]

This result coincides with that in Eq. (28) in the main text on the \(\Delta = 0\) fixed line if \(K_L \to K\) instead of \(K_L \to 1\). This Eq. (24) was derived with the full nonlinear coupling using duality, hence this coincidence supports the replacement of the nonlinear coupling by a Gaussian term for the purpose of evaluating the DC conductance. The conductance at the fully coupled fixed point \(\Delta = 0\), with normal leads, is then \(G_{ij} = \frac{e^2}{\hbar N}\) and is consistent with our surmise.

It is also possible to calculate the non-local conductance, i.e. the response of the current at position \(x\) to a voltage at position \(x' \neq x\), using the above equations. A similar quantity was defined and obtained in Ref. [45] for a single LL, although calculated with a different method. Following Ref. [45] we take \(x = -\frac{1}{2} L\) and \(x' = \frac{1}{2} L\), so that resonances are expected at the eigenfrequencies of the wire, \(\frac{2\pi L}{u}\times\text{integer}\). This response is defined as

\[
G_{ij}^{\text{non-local}}(\omega, L) = \frac{e^2 \omega_n}{\pi^2 \hbar} C_{ij}(\omega_n; x = -\frac{L}{2}, x' = \frac{L}{2})|_{\omega_n = \omega + i\delta}
\]

Keeping \(L\) and \(\nu\) finite in the above calculation, and then taking \(L\) large with the product \(\nu L\) finite, we obtain

\[
G_{ij}^{\text{non-local}}(\omega, L) = \frac{e^2}{\hbar N} F(\omega L/u)
\]

where \(F(z)\) and its real part are periodic functions of \(z\)

\[
F(z) = \frac{2iKK_L^2}{(K^2 + K_L^2) \sin(z) + 2iK K_L \cos(z)} , \quad \Re F(z) = \frac{1}{(K^2 + K_L^2)^2 \tan(z) + \cos(z)}
\]

with \(F(z) = K_L - \frac{z^2(K^4 + K_L^4)}{4(K^2 K_L^2)} + O(z^3)\) at small \(z\). The function \(F(z)\) coincides with the result of Ref. [45] for \(K_L = 1\). The dependence on \(N\) of the non-local conductance in the coupled phase is thus very simple. While the DC limit is independent of \(K\) a measurement of the whole response \(F(z)\) can determine the interaction parameter \(K\).

C. One wire and one loop

Consider now \(N = 2\) with the same coupling between wires 1 and 2 as in Eq. (79), and assuming for generality that the 2nd wire has LL parameters \(u', K'\). Leads are attached only to wire 1 at \(x = \pm \frac{1}{2} L\) while wire 2 has uniform \(u', K'\) and uniform boundary conditions at \(x = \pm \frac{1}{2} L'\). The voltage is applied at \(|x| < \frac{1}{2} L\) and we assume that \(L \leq L'\),
a slightly simpler case. We extend the range that \( x' \) is needed to \( |x'| < \frac{1}{2}L' \) so that the contact points of wire 1 are included. The correlations \( C_{ij}(\omega_n, x, x') \) satisfy

\[
\left( \frac{\omega^2}{u(x)K(x)} - \frac{\partial^2}{uK} \right) \frac{\partial_x}{uK} + M\delta(x) = \frac{-M\delta(x)}{uK} - \frac{1}{4}uK \left( \frac{C_{11} C_{12}}{C_{21} C_{22}} \right) = \pi \delta (x - x') 1
\]

The correlation functions \( C_{11}(\omega_n, x, x') \) and \( C_{12}(\omega_n, x, x') \) obey, in the variable \( x \), the boundary conditions of wire 1 while \( C_{21}(\omega_n, x, x') \) and \( C_{22}(\omega_n, x, x') \) obey the periodic boundary conditions of wire 2. Therefore, the solutions have the form

\[
C_{11} = A_1e^{\nu x/uL} + B_1e^{\nu x/u} + C_1e^{-\nu x/u} + D_1e^{\nu x/u} + E_1e^{-\nu x/u} + F_1e^{-\nu x/uL} \quad x < -\frac{1}{2}L
\]

\[
B_1e^{\nu x/u} + C_1e^{-\nu x/u} + \frac{b_1 e^{\nu x/u} + c_1 e^{-\nu x/u}}{2} + \frac{d_1 e^{\nu x/u} + e_1 e^{-\nu x/u}}{2} \quad 0 < x < \frac{1}{2}L
\]

\[
X_1 e^{-\nu x/uL} \quad x > \frac{1}{2}L
\]

\[
C_{12} = A_1e^{\nu x/uL} + B_1e^{\nu x/u} + C_1e^{-\nu x/u} + D_1e^{\nu x/u} + E_1e^{-\nu x/u} + F_1e^{-\nu x/uL} \quad x < -\frac{1}{2}L
\]

\[
b_1 e^{\nu x/u} + c_1 e^{-\nu x/u} - \frac{b_1 e^{\nu x/u} + c_1 e^{-\nu x/u}}{2} \quad -\frac{1}{2}L < x < 0
\]

\[
d_1 e^{\nu x/u} + e_1 e^{-\nu x/u} \quad 0 < x < \frac{1}{2}L
\]

\[
f_1 e^{-\nu x/uL} \quad x > \frac{1}{2}L
\]

\[
C_{22} = B_2e^{\nu x/u'} + C_2e^{-\nu x/u'} + D_2e^{\nu x/u'} + E_2e^{-\nu x/u'} + F_2e^{-\nu x/u'L} \quad x > \frac{1}{2}L
\]

The various boundary conditions of continuity and jump of derivatives lead, after some algebra, to a conductance matrix in the DC limit \( \nu \to 0 \) equal to \( G_{ij} = \frac{e^2}{h} + O(\omega) \). This result is consistent with our surmise that the resistance of the coupled system equal to the sum of individual resistances. The periodic loop is ideal, hence its resistance vanishes, while the single wire with leads has the well known resistance \( \frac{e^2}{h} \) of \( \frac{e^2}{h} \).

We note that in Ref. \[48\] this geometry is considered in the main text, while the Gaussian coupling is slightly different, i.e. it is along the whole length \( L' \). In contrast to the result above they claim that the DC conductance is \( K \) dependent. However, in their supplement \[48\] they seem to use outgoing boundary conditions for the ideal loop, i.e. not periodic boundary conditions. Outgoing boundaries of an ideal LL wire imply a conductance of \( \frac{e^2}{h} K' \) (though it is not clear where dissipation originates; in any case it is not experimentally feasible) while the wire with boundaries has the usual \( \frac{e^2}{h} \). Summing up resistances, according to our surmise, gives \( G_{ij} = \frac{e^2}{h} \frac{1}{1 + 1/K'} \), which in fact we have also evaluated directly for these boundary conditions. The latter result is also given in Ref. \[48\] however, it does not correspond to the realistic geometry and boundary conditions.

V. 1/N EXPANSION

We study here the case of large \( N \) using a 1/N expansion. Starting from our model in the main text, Eqs. (1,3), we first derive an effective action to order \( g^4N \) (Eq. \[90\]), extending the leading order \( \sim g^2N \) case studied in Ref. \[28\]. We present here the mean field solution, valid near \( K = \frac{1}{2} \), and show that a phase transition survives for \( N \geq 4 \) (Eq. \[98\]). The approach is quite different from the one in the main text, since here we first integrate over all the degrees of freedom of the LL’s, seen as a bath, and study the effective action for \( X(\tau) \). While this expansion is valid for \( N \gg 1 \) it does capture some of the features studied in the main text, i.e. the presence of a phase transition for a finite \( N \).


A. effective action

Consider first the linear coupling to the density on chain $i$, i.e. $-g_0 \partial_x \phi_i(X_\tau, \tau)/\pi$ ($i = 1, ..., N$) which can be integrated exactly to yield the long wavelength impurity action $S_{long}$,

\[ \langle e^{-g_0 \int_0^\infty \partial_x \phi_i(X_\tau, \tau)/\pi} \rangle = e^{-S_{long}} \]

\[ S_{long} = -\frac{g_0^2}{\pi^2} \int_{\tau, \tau'} \langle \partial_x \phi_i(X_\tau, \tau) \partial_x \phi_i(X_{\tau'}, \tau') \rangle_0 = -\frac{g_0^2}{2\pi} \int_{\tau, \tau'} \frac{y_1^2 - (X(\tau) - X(\tau'))^2}{\frac{y_0^1}{\rho_0} (g_0^2 + (X(\tau) - X(\tau'))^2)} \]

(85)

where $\langle \ldots \rangle_0$ is an average on the LL action and \cite{37} $y_0 = \tau - \tau' + \alpha \text{sign}(\tau - \tau')$, $K$ is the cuttung parameter and $\alpha \sim 1/\rho_0$ is a cutoff. At long times the interaction is $\sim (\frac{y_0^1}{\rho_0})^2$, whose Fourier transform is $\sim \omega^3$, negligible relative to the mass term $M \omega^2$. This conclusion is valid for any $N \geq 1$.

Our main concern is therefore the coupling to the oscillatory term of the density Eq. (2) in the main text by expanding in $g_0$ and averaging on $\phi_i(X_\tau, \tau)$ with known LL correlations \cite{37}, i.e.

\[ \langle e^{i 2 \phi_i(X_\tau, \tau) - 2 i \phi_i(X_\tau, \tau')} \rangle_0 = \left( \frac{\alpha^2}{y_0^2 + (X(\tau) - X(\tau'))^2} \right)^K \delta_{i,j} \]

(86)

Already at this stage we note that we expect at long times $|X_\tau - X_{\tau'}| \ll |\tau - \tau'|$, as confirmed by the derived phase diagram \cite{28}. Hence we neglect the $X_\tau$ terms in the correlation \cite{80}. The expansion in $g = \alpha_1 \rho_0 g_0$, equivalent to a $1/N$ expansion, then yields

\[ e^{-\frac{1}{2} M \int_{\tau} X(\tau)^2 \langle e^{-g_0 \sum_i \phi_i(X_{\tau_i}, \tau) + h.c.} \rangle_0} = e^{-S_{eff}} \]

\[ S_{eff} = \frac{1}{2} M \int_{\tau} \dot{X}(\tau)^2 - g^2 N \Lambda^{-2K} \sum_{i,j} \int_{\tau_1, \tau_2} \langle [\dot{X}(\tau_1) + \phi_i(\tau_1)] + h.c. \rangle \langle [\dot{X}(\tau_2) - 2 \phi_i(\tau_2)] + h.c. \rangle_0 = 1 \frac{g^4}{4} \sum_{i_1, i_2} \int_{\tau_1, \tau_2} \langle [\dot{X}(\tau_1) - \phi_i(\tau_1)] + h.c. \rangle \langle [\dot{X}(\tau_2) - 2 \phi_i(\tau_2)] + h.c. \rangle_0 + O(g^6N) \]

where $\langle \ldots \rangle_0$ is a cumulant average, subtracting the next order of the $g^2$ term. In the last term we choose 2 out of 4 phases(4!/4 choices) to be positive and define them as 1,3, (including c.c. terms), defining $\tau_{ij} = \tau_i - \tau_j$,

\[ S_{eff} = \frac{1}{2} M \int_{\tau} \dot{X}(\tau)^2 - g^2 N \Lambda^{-2K} \sum_{i,j} \int_{\tau_1, \tau_2} \cos[X(\tau_1) - X(\tau_2)] |\tau_{12}|^{2K} + \frac{1}{4} g^4 N \Lambda^{-4K} \sum_{i_1, i_2} \int_{\tau_1, \tau_2} \cos[X(\tau_1) - X(\tau_2) + \phi_1(\tau_1) + \phi_1(\tau_2)] \langle e^{2i \phi_1(\tau_1) - \phi_1(\tau_2)} \rangle_0 = 2K \ln |\tau_{12}| + \ln |\tau_{13}| + \ln |\tau_{14}| + \ln |\tau_{23}| + \ln |\tau_{24}| + \ln |\tau_{14}| \]

(88)

The second term $O(g^2N)$ is a classical long range XY model which was studied in Ref. \cite{28}. We symmetrize the 3rd term, for later convenience, i.e. replace $|\tau_{12}|^{2K} = \frac{1}{2} |\tau_{12}|^{2K} + |\tau_{14}|^{2K}$. Next we choose $i_1 = i_2 = 1, 2, ..., N$ and $i_3 = i_4 = 1, 2, ..., N$ but $i_3 \neq i_4$ giving (in the last one term the like the last one, similarly with $i_1 = i_4, i_2 = i_3$, hence $-\frac{1}{2} g^4 |N(N-1) - N^2| = \frac{1}{2} g^4 N$. Remaining is the term $i_1 = i_2 = i_3 = i_4$ giving $g^4 N$. Using Ref. \cite{37} Eq. C.37

\[ \langle e^{2i \phi_1(\tau_1) - \phi_1(\tau_2)} \rangle_0 \approx e^{-2K |\tau_{24} - \tau_{14}| + \ln |\tau_{23}| + \ln |\tau_{24}| + \ln |\tau_{13}|} \]

(89)

our final form is (with $\tau_{1} \rightarrow \tau_{i} \Lambda$)

\[ S_{eff} = \frac{1}{2} M \int_{\tau} \dot{X}(\tau)^2 - g^2 N \Lambda^{-2K} \sum_{i,j} \int_{\tau_1, \tau_2} \cos[X(\tau_1) - X(\tau_2)] \frac{1}{|\tau_{12}|^{2K}} - \frac{1}{4} g^4 N \Lambda^{-4K} \times \]

\[ \int_{\tau_1, \tau_2} \cos[X(\tau_1) - X(\tau_2) + \phi_1(\tau_1) + \phi_1(\tau_2)] \left( \frac{1}{|\tau_{12}|^{2K}} - \frac{1}{|\tau_{13}|^{2K}} - \frac{1}{|\tau_{14}|^{2K}} \right) \]

(90)

Note that the $\cos[X(\tau_1) - X(\tau_2) + \phi_1(\tau_1) + \phi_1(\tau_2)]$ term cannot be decomposed, in general, to a cos product since the $\tau$ integrals are mixed. Note also that if the pair $\tau_1, \tau_2$ is separated by a large time $\bar{T}$ from the pair $\tau_3, \tau_4$ with $\bar{T} \gg |\tau_1 - \tau_2|, |\tau_3 - \tau_4|$ then $\tau_{13}, \tau_{23}, \tau_{24}, \tau_{13} \rightarrow \bar{T}$ and the first 2 terms in the last line of (90) cancel while the last term is small $\sim 1/\bar{T}^2$; similarly when the pair $\tau_1, \tau_2$ is well separated from the pair $\tau_2, \tau_3$, then the 1st and 3rd term cancel and the result is again $\sim 1/\bar{T}^2$. Hence well separated pairs are already included in the 1st term of $S_2$ and cancel in the 2nd order cumulant.
B. Expansion near $K = \frac{1}{2}$

Near $K = \frac{1}{2}$ the interactions in the effective action have a very long range and mean field theory is justified. We reproduce first the $N\eta^2$ result [28]. Replace each factor $(e^{iX(\tau)}) = h$ as a complex order parameter, i.e. $e^{iX(\tau_1) - iX(\tau_2)} \rightarrow he^{-iX(\tau_2)} + h^*e^{iX(\tau_1)}$. We now introduce the dimensionless coupling constant, $\eta = 2\pi g^2 N\Lambda^{-2}$, in terms of which

$$S^{(1)}_{eff} = \frac{1}{2}M \int \frac{d^4x}{(2\pi)^2} X^2 - \frac{\eta A^{2-2K}}{2\pi} \left( h^* \int_{\tau_1,\tau_2} e^{-iX(\tau_2)} \frac{1}{|\tau_{12}|^{2K}} + h \int_{\tau_1,\tau_2} e^{iX(\tau_1)} \frac{1}{|\tau_{12}|^{2K}} \right)$$

(91)

Expand $e^{-S^{(1)}_{eff}}$ to have linear term in $h$, the critical $\eta$ is the solution of

$$1 = \langle e^{iX(\tau_0)} \rangle_h / h = \frac{\eta A^{2-2K}}{2\pi} \int_{\tau_1,\tau_2} \frac{\langle e^{iX(\tau_0) - iX(\tau_2)} \rangle_0}{|\tau_{12}|^{2K}} = \frac{\eta A^{2-2K}}{2\pi} \int_{\tau_1,\tau_2} \frac{e^{-|\tau_{12}|/2M}}{|\tau_{12}|^{2K}} = \frac{8M\eta A}{\pi(2K - 1)}$$

(92)

taking the short time cutoff of $\tau_{12}$ as $1/\Lambda$.

The next order with the dimensionless $\eta_2 = \frac{1}{4}g^4 N\Lambda^{-4}$ becomes

$$S_{eff} = S^{(1)}_{eff} + \frac{\eta_2 A^{4-4K}}{2\pi} \int_{\tau_1,\ldots,\tau_4} \left[ he^{-iX(\tau_2) + iX(\tau_3) - iX(\tau_4)} + h^* e^{iX(\tau_1) + iX(\tau_3) - iX(\tau_4)} ight]$$

$$+ he^{iX(\tau_1) - iX(\tau_2) - iX(\tau_4)} + h^* e^{iX(\tau_1) - iX(\tau_2) + iX(\tau_3)} + h.c.] \left\{ \frac{\tau_{13}\tau_{24}}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{14}\tau_{23}|^{2K}} \right\}$$

(93)

Expand $e^{-S_{eff}}$ to have linear term in $h$, only half of the 8 terms contribute,

$$1 = \langle e^{iX(\tau_0)} \rangle_h / h \big|_{h=0} = \frac{8M\eta A}{2\pi(2K - 1)} + \frac{\eta_2 A^{4-4K}}{2\pi} \int_{\tau_1,\ldots,\tau_4} \left[ e^{iX(\tau_0) - iX(\tau_2) + iX(\tau_3) - iX(\tau_4)} ight]$$

$$+ e^{iX(\tau_0) - iX(\tau_1) + iX(\tau_3) + iX(\tau_4)} + e^{iX(\tau_0) - iX(\tau_1) - iX(\tau_2) - iX(\tau_4)} + e^{iX(\tau_0) - iX(\tau_1) + iX(\tau_2) - X(\tau_3)}$$

$$\left\{ \frac{\tau_{13}\tau_{24}}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{14}\tau_{23}|^{2K}} \right\}$$

(94)

All 4 terms are identical by interchanging variables: in 2nd term 1$\leftrightarrow$2, 3$\leftrightarrow$4, in 3rd term 1$\leftrightarrow$3, in 4th term 1$\rightarrow$2, 3$\rightarrow$4, 2$\rightarrow$3, 4$\rightarrow$1.

Performing the Gaussian averages,

$$1 = \frac{8M\eta A^{2-2K}}{2\pi(2K - 1)} + \eta_2 A^{4-4K} \int_{\tau_1,\ldots,\tau_4} e^{-\frac{1}{2M}|\tau_0 - \tau_2| - |\tau_0 - \tau_3| + |\tau_0 - \tau_4| + |\tau_2 - \tau_3| - |\tau_2 - \tau_4| + |\tau_3 - \tau_4|}$$

$$\left\{ \frac{\tau_{13}\tau_{24}}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{14}\tau_{23}|^{2K}} \right\}$$

(95)

Shifting all $\tau_i \rightarrow \tau_i + \tau_0$ eliminates $\tau_0$, as expected. One can also shift by one of the $\tau_i$ to eliminate it in the power factors and then integrate it exactly in the exponent. The most efficient shift is by $\tau_3$ so as to keep the symmetry under exchange of $\tau_2, \tau_4$, hence shift $\tau_1 \rightarrow \tau_1 + \tau_3$, $\tau_2 \rightarrow \tau_2 + \tau_3$, $\tau_4 \rightarrow \tau_4 + \tau_3$, furthermore shift all $\tau_i \rightarrow 2M\tau_i$ and finally $\tau_3 \rightarrow -\tau_3$,

$$1 = \frac{8\eta A M}{2\pi(2K - 1)} + \eta_2 (2MA)^{4-4K} I(K)$$

(96)

$$I(K) = \int_{\tau_1,\tau_2,\tau_4} e^{-|\tau_2 + \tau_3 - \tau_4|} \left\{ \frac{\tau_{13}\tau_{24}}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{12}\tau_{43}|^{2K}} - \frac{1}{|\tau_{14}\tau_{23}|^{2K}} \right\} \int_{\tau_3} e^{-|\tau_3 - \tau_2| + |\tau_3 - \tau_4|}$$

The subsection below details how the leading form at $K = \frac{1}{2}$ is evaluated. The critical $\eta$, using $\eta_2 = \frac{\sigma^2}{(2\pi)^2 4N}$, near $K = \frac{1}{2}$, is the solution of

$$1 = \frac{4\eta A M}{2\pi(K - \frac{1}{2})} - \frac{16\eta^2}{4(2\pi)^2 N(K - \frac{1}{2})^2 (2MA)^2}$$

(97)
which is solved by \( \eta_c \)

\[
\eta_c = \frac{K - \frac{1}{3}MA}{4} \cdot \frac{\pi N}{4} \left[ 1 - \sqrt{1 - \frac{4}{N}} \right]
\]  

(98)

For large \( N \) the solution is \( \eta_c = \frac{K - \frac{1}{3}MA}{4} \left( 1 + \frac{1}{N} + O(1/N^2) \right) \) consistent with Eq. (92); Eq. (97) has in fact a second solution with \( 1 + \sqrt{1 - \frac{4}{N}} \) which for large \( N \) has a high slope \( \eta_c \sim N \), hence is probably nonphysical.

Our main result is that the phase transition survives, at least near \( K = \frac{1}{2} \), for \( N \geq 4 \). Hence the large \( N \) expansion has some correspondence with the phase diagram as found in the main text.

### C. Details of expansion near \( K = \frac{1}{2} \)

We evaluate here the critical line for the mean field transition near \( K = \frac{1}{2} \) starting from Eq. (96). When integral limits are not specified they correspond to \(-\infty, \infty\). The \( \tau_3 \) integral needs to be done for each of the 6 orderings of \( 0, \tau_2, \tau_4 \), however, the symmetry \( \tau_2 \leftrightarrow \tau_4 \) and overall sign change of all \( \tau_i \) yield that the 4 orderings \((0, \tau_2, \tau_4), (\tau_1, \tau_2, 0), (\tau_2, \tau_4, 0)\) are the same and also the 2 orderings \((\tau_2, 0, \tau_4), (\tau_4, 0, \tau_2)\) are the same. Consider first \( 0 < \tau_2 < \tau_4 \)

\[
\int_{\tau_3} e^{-|\tau_3 - \tau_2| + |\tau_3 - |\tau_3 - \tau_4|} = \int_{0}^{\infty} e^{\tau_3 - \tau_2 + \tau_3 - \tau_4} + \int_{0}^{\tau_2} e^{\tau_3 - \tau_2 + \tau_3 - \tau_4} + \int_{\tau_2}^{\tau_4} e^{-\tau_3 + \tau_2 + \tau_3 - \tau_4} + \int_{\tau_4}^{\infty} e^{-\tau_3 + \tau_2 + \tau_3 - \tau_4} = \\
\frac{2}{3} e^{-\tau_2 - \tau_4} - \frac{2}{3} e^{2\tau_2 - \tau_4} + 2e^{\tau_2}
\]  

(99)

The additional exponent in this range is \( e^{-2\tau_2} \). This yields then 3 terms, \( I_1, I_2, I_3 \) respectively, so that

\[
I(K) = 4(I_1 + I_2 + I_3) + 2(I_4 + I_5 + I_6)
\]  

(100)

with \( I_4, I_5, I_6 \) are defined below for the \((\tau_2, 0, \tau_4)\) ordering.

\[
I_1 = \frac{2}{3} \int_{\tau_1} \int_{0 < \tau_2 < \tau_4} e^{-3\tau_2 - \tau_4} \left\{ \frac{\tau_1 \tau_2}{\tau_1 \tau_2 \tau_4} \right\}^{2K} - \frac{1}{|\tau_1 \tau_2 \tau_4|^{2K}} - \frac{1}{|\tau_1 \tau_2 \tau_4|^{2K}} = O\left( \frac{1}{2K-1} \right)
\]  

(101)

It turns out that all the integrals below converge at short times. The \( \tau_2, \tau_4 \) integrals converge, while the \( \tau_1 \) integral give a single \( \frac{1}{2K-i} \) factor. Below we find terms that diverge more strongly at \( K \to \frac{1}{2} \), so this one can be neglected. Next is

\[
I_2 = -\frac{2}{3} \int_{\tau_1} \int_{0 < \tau_2 < \tau_4} e^{-\tau_4} \left\{ \frac{\tau_1 \tau_2}{\tau_1 \tau_2 \tau_4} \right\}^{2K} - \frac{1}{|\tau_1 \tau_2 \tau_4|^{2K}} - \frac{1}{|\tau_1 \tau_2 \tau_4|^{2K}} = O\left( \frac{1}{2K-1} \right)
\]  

(102)

Since \( \tau_2 < \tau_4 \) both these integrals converge and the result can again be neglected. Next is

\[
I_3 = 2 \int_{\tau_1} \int_{0 < \tau_2 < \tau_4} e^{-\tau_2} \left\{ \frac{\tau_1 \tau_2}{\tau_1 \tau_2 \tau_4} \right\}^{2K} - \frac{1}{|\tau_1 \tau_2 \tau_4|^{2K}} - \frac{1}{|\tau_1 \tau_2 \tau_4|^{2K}}
\]  

(103)

Subdivide \( I_3 \) in various regions of \( \tau_1 \) integral, i.e. \( I_3 = I_{31} + I_{32} + I_{33} + I_{34} \). Starting with \( I_{31} \) in the range \( 0 < \tau_2 < \tau_4 < \tau_1 \), with the variables \( x = \frac{\tau_1 - \tau_2}{\tau_2}, y = \frac{\tau_1 - \tau_4}{\tau_2} \) so that \( 0 < x, y < \infty \) we obtain

\[
I_{31} = 2 \int_{0}^{\infty} dx \int_{0}^{\infty} dy \left\{ \frac{y(1 + x + y)}{x(x + y)(1 + y)} \right\}^{2K} - \frac{1}{|x(1 + y)(1 + y)|^{2K}} - \frac{1}{x^{2K}}
\]  

(104)

The \( \tau_2 \) integral for \( K \to 1/2 \) gives 1. For the \( x, y \) integrals, for \( K \to \frac{1}{2} \) use the series

\[
(1 - \alpha)^{2K} = 1 - 2K\alpha + 2K(2K - 1) \sum_{n=2}^{\infty} \frac{(n - 2)!}{n!} \alpha^n \quad |\alpha| < 1
\]  

(105)
with \( \alpha = \frac{x}{(x+y)(1+y)} < 1 \). The \( n = 0 \) term cancels the divergent last term of \( I_{31} \). The \( n = 1 \) term can be integrated (using Mathematica) leading to \( -\frac{1}{4(K-\frac{1}{2})^2} + O(K - \frac{1}{2})^{-1} \), while the 2nd term of \( I_{31} \) yields \( \frac{-1}{8(K-\frac{1}{2})^2} + O(K - \frac{1}{2})^{-1} \). Hence the leading term has

\[
I_{31} = -\frac{3}{4(K-\frac{1}{2})^2} + O\left(\frac{1}{K-\frac{1}{2}}\right) \tag{106}
\]

The next part of \( I_{3} \) is \( I_{32} \) with \( 0 < \tau_2 < \tau_1 < \tau_4 \) and with the variables \( x = \frac{x - \tau_2}{\tau_2} \), \( y = \frac{y - \tau_2}{\tau_2} \) we obtain

\[
I_{32} = 2 \int_0^\infty d\tau_2 \frac{e^{-\tau_2}}{\tau_2^4} 2 \int_0^\infty dx \int_0^\infty dy \left\{ \left( \frac{(1+y)(x+y)}{xy(1+x+y)} \right)^{2K} \frac{1}{y(1+x+y)^{2K}} \right\} \tag{107}
\]

\( I_{32}(K = \frac{1}{2}) = 0 \), however the derivative, using the expansion \( x^{2K} = x + x(2K-1) \ln x + O(2K-1)^2 \) diverges, hence \( I_{32} \) may be discontinuous, and actually it is. The \( y < 1 \) range can be shown numerically to be less divergent while the \( y > 1 \) range can be evaluated with the expansion Eq. (105) and \( \alpha = -\frac{x}{y(1+x+y)} \). Again only the \( n = 1 \) term and the 2nd term of \( I_{32} \) contribute to the leading divergence with

\[
I_{32} = +\frac{1}{4(K-\frac{1}{2})^2} + O\left(\frac{1}{K-\frac{1}{2}}\right) \tag{108}
\]

The next part is

\[
I_{33} = 2 \int_{0 < \tau_1 < \tau_2 < \tau_4} e^{-\tau_2} \left\{ \left( \frac{\tau_1 \tau_2}{\tau_2^2 \tau_4 \tau_1^2} \right)^{2K} \frac{1}{\tau_1 \tau_2 \tau_4^{2K}} \right\} \tag{109}
\]

Mathematica v8.1 manages to evaluate this directly, with the result \( I_{33} \sim 1/(K-\frac{1}{2}) \), which is confirmed by expansions similar to those of \( I_{31} \). The final part of \( I_{3} \) is \( I_{34} \) with \( \tau_1 < 0 < \tau_2 < \tau_4 \). Defining \( x = \frac{\tau_2}{\tau_2}, y = \frac{\tau_4 - \tau_1}{\tau_2} \) so that \( 1 < x < y < \infty \),

\[
I_{34} = 2 \int_0^\infty d\tau_2 \frac{e^{-\tau_2}}{\tau_2^4 - \frac{1}{2}} \int_1^\infty dy \int_1^y dx \left\{ \left( \frac{(1-x)(x-y)}{xy(1-x+y)} \right)^{2K} \frac{1}{y(1-x+y)^{2K}} \right\} \tag{110}
\]

Using the expansion (105) with \( \alpha = \frac{1}{x(1-x+y)} \) where again the \( n = 1 \) term and the 2nd term of \( I_{34} \) dominate

\[
I_{34} = -\frac{3}{2(K-\frac{1}{2})^2} + O\left(\frac{1}{(K-\frac{1}{2})}\right) \tag{111}
\]

Collecting all \( I_{3} \) terms we obtain,

\[
I_{3} = -\frac{2}{(K-\frac{1}{2})^2} + O\left(\frac{1}{(K-\frac{1}{2})}\right) \tag{112}
\]

Consider next the range \( \tau_2 < 0 < \tau_4 \).

\[
\int_{\tau_3}^{\tau_4} e^{-|\tau_3 - \tau_2| + \tau_3 - |\tau_3 - \tau_4|} = \int_{-\infty}^{\tau_2} e^{-\tau_3 + \tau_3 - \tau_3 - \tau_4 + \tau_3} + \int_{\tau_2}^{0} e^{-\tau_3 + \tau_2 - \tau_3 - \tau_4 + \tau_3} + \int_{0}^{\tau_4} e^{-\tau_3 + \tau_2 + \tau_3 - \tau_4 + \tau_3} + \int_{\tau_4}^{\infty} e^{-\tau_3 + \tau_2 + \tau_3 - \tau_4 + \tau_3} \tag{113}
\]

The additional exponent in this range =1. This yields then 3 terms \( I_4, I_5, I_6 \) for the integral \( I(K) \):

\[
I_4 = 2 \int_{\tau_1} \int_{\tau_2 < 0 < \tau_4} e^{-\tau_4} \left\{ \left( \frac{\tau_1 \tau_2}{\tau_2^2 \tau_4 \tau_1^2} \right)^{2K} \frac{1}{\tau_1 \tau_2 \tau_4^{2K}} \right\} \tag{114}
\]

Subdivide this into 4 ranges of \( \tau_1 \) so that \( I_4 = I_{41} + I_{42} + I_{43} + I_{44} \), considering first \( I_{41} \) for \( \tau_2 < 0 < \tau_4 < \tau_1 \) with variables \( x = \frac{\tau_2 - \tau_4}{\tau_4}, y = -\frac{\tau_4}{\tau_4} \) we obtain

\[
I_{41} = 2 \int_0^\infty d\tau_4 \frac{e^{-\tau_4}}{\tau_2^4 - \frac{1}{2}} \int_0^\infty dx \int_0^\infty dy \left\{ \left( \frac{(1+x)(1+y)}{xy(1+x+y)} \right)^{2K} \frac{1}{(1+x+y)^{2K}} \right\} \tag{115}
\]
\( I_{41}(K = \frac{1}{2}) = 0 \), yet as \( I_{32} \) it is discontinuous. Using (105) with \( \alpha = -\frac{1+x+y}{xy} \) and Mathematica we obtain
\[
I_{41} = + \frac{1}{2(K - \frac{1}{2})^2} + O(\frac{1}{K - \frac{1}{2}})
\]

(116)

The next \( \tau_1 \) interval has
\[
I_{42} = 2 \int_{\tau_2 < 0 < \tau_1 < \tau_4} e^{-\tau_4} \left\{ \left( \frac{\tau_1 \tau_2}{\tau_21 \tau_41 (-\tau_2) \tau_4} \right)^{2K} \frac{1}{(\tau_21 \tau_41)^{2K}} - \frac{1}{(\tau_1 \tau_41)^{2K}} \right\}
\]

(117)

An expansion as in Eq. (105) leads to \( \sim \frac{1}{(K - \frac{1}{2})} \), hence \( I_{42} \) can be neglected. The next \( \tau_1 \) range for \( I_{43} \) has \( \tau_2 < \tau_1 < 0 < \tau_4 \). Defining \( x = \frac{\tau_2 - \tau_1}{\tau_4} \), \( y = \frac{\tau_4}{\tau_4} \) yields
\[
I_{43} = 2 \int_{\tau_4}^{\infty} d\tau_4 \frac{e^{-\tau_4}}{\tau_4^2} \int_{\tau_4}^{\infty} dx \int_{0}^{\infty} dy \left\{ \left( \frac{y(1+x+y)}{x(1+y)(x+y)} \right)^{2K} \frac{1}{(x)^{2K}} - \frac{1}{((1+y)(x+y))^{2K}} \right\}
\]

(118)

Using (105) with \( \alpha = \frac{2}{1+y(1+y)} < 1 \) to cancel the divergent 2nd term of \( I_{43} \) leads to
\[
I_{43} = \frac{-3}{4(K - \frac{1}{2})^2} + O(1/(K - \frac{1}{2}))
\]

(119)

The last interval of \( I_4 \) is
\[
I_{44} = 2 \int_{\tau_1 < \tau_2 < 0 < \tau_4} e^{-\tau_4} \left\{ \left( \frac{\tau_1}{\tau_21 \tau_41 (-\tau_2) \tau_4} \right)^{2K} \frac{1}{(\tau_21 \tau_41)^{2K}} - \frac{1}{(\tau_1 \tau_41)^{2K}} \right\}
\]

(120)

\( I_{44}(K = 1/2) = 0 \), however it is also discontinuous. Using variables \( x = \frac{\tau_2 - \tau_1}{\tau_4} \), \( y = \frac{\tau_4}{\tau_4} \)
\[
I_{44} = 2 \int_{\tau_4}^{\infty} d\tau_4 \frac{e^{-\tau_4}}{\tau_4^2} \int_{\tau_4}^{\infty} dx \int_{0}^{\infty} dy \left\{ \left( \frac{x+y}{x(1+y)(x+y)} \right)^{2K} \frac{1}{(x)^{2K}} - \frac{1}{((1+y)(x+y))^{2K}} \right\}
\]

(121)

and with Mathematica we find
\[
I_{44} = \frac{1}{4(K - \frac{1}{2})^2} + O(\frac{1}{K - \frac{1}{2}})
\]

(122)

Collecting all terms of \( I_4 \)
\[
I_4 = 0 + O(1/(K - \frac{1}{2}))
\]

(123)

The next term is
\[
I_5 = -2 \int_{\tau_1}^{\tau_2} \int_{\tau_2 < 0 < \tau_4} e^{\tau_2 - \tau_4} \left\{ \frac{\tau_1 \tau_21 \tau_41 \tau_2}{\tau_21 \tau_41 \tau_21 \tau_4} \right]^{2K} \frac{1}{(\tau_21 \tau_41 \tau_21 \tau_4)^{2K}} - \frac{1}{(\tau_21 \tau_41 \tau_21 \tau_4)^{2K}} \right\} = O(\frac{1}{2K - 1})
\]

(124)

since both \( \tau_2, \tau_4 \) exponentially converge, only \( \tau_1 \) gives \( \sim 1/(2K - 1) \). Next one is
\[
I_6 = 2 \int_{\tau_1}^{\tau_2} \int_{\tau_2 < 0 < \tau_4} e^{\tau_2} \left\{ \frac{\tau_1 \tau_21 \tau_41 \tau_2}{\tau_21 \tau_41 \tau_21 \tau_4} \right]^{2K} \frac{1}{(\tau_21 \tau_41 \tau_21 \tau_4)^{2K}} - \frac{1}{(\tau_21 \tau_41 \tau_21 \tau_4)^{2K}} \right\} = I_4
\]

(125)

by \( \tau_2 \to -\tau_4, \tau_4 \to -\tau_2, \tau_1 \to -\tau_1 \). Finally, our integral \( I(K) \) in Eq. (96) is
\[
I(K) = 4(I_1 + I_2 + I_3) + 2(I_4 + I_5 + I_6) = -\frac{8}{(K - \frac{1}{2})^2} + O(1/(K - \frac{1}{2}))
\]

(126)

This leads to the equation for the critical \( \eta_c \), Eq. (98).