Group Fairness for Knapsack Problems

Deval Patel  
IISc  
devalpatel@iisc.ac.in

Arindam Khan  
IISc  
arindamkhan@iisc.ac.in

Anand Louis  
IISc  
anandl@iisc.ac.in

Abstract

We study the knapsack problem with group fairness constraints. The input of the problem consists of a knapsack of bounded capacity and a set of items, each item belongs to a particular category and has an associated weight and value. The goal of this problem is to select a subset of items such that all categories are fairly represented, the total weight of the selected items does not exceed the capacity of the knapsack, and the total value is maximized. We study the fairness parameters such as the bounds on the total value of items from each category, the total weight of items from each category, and the total number of items from each category. We give approximation algorithms for these problems. These fairness notions could also be extended to the min-knapsack problem. The fair knapsack problems encompass various important problems, such as participatory budgeting, fair budget allocation, advertising.

1 Introduction

The knapsack (also known as max knapsack) problem is a classical packing problem. The input of the max-knapsack problem consists of a bounded capacity knapsack and a set of items, each having an associated weight and value. The objective is to select a subset of items such that the total weight of the selected items does not exceed the capacity of the knapsack and the total value is maximized. The min knapsack problem is an extensively studied variant, where the input consists of a set of items, each having an associated weight and value, along with the lower bound on the packing value. The goal of this problem is to select a subset of items such that the total value of the selected items is at least the given bound, and the total weight is minimized. Max knapsack and min knapsack problems represent the prototypical packing and covering problems, respectively.

From the practical viewpoint, the knapsack problem models many prominent industrial problems such as budgeting, cargo packing, resource allocation, assortment planning, etc. [KPP04]. The knapsack problem and its variants are special cases of many salient optimization problems, e.g., the generalized assignment problem, the packing and covering linear programs, etc. They are also key subproblems in the solution of more complex problems, such as the cutting stock problem [Van99].

In this work, we consider the notion of group fairness in knapsack problems. In this setting, each item belongs to a category, and our goal is to compute a subset of items such that each category is fairly represented, in addition to the total weight of the subset not exceeding the capacity, and the total value of the subset being maximized. We study various notions of group fairness, such as the number of items from each category, the total value of items from each category, and the total weight of items from each category.

In recent years, a considerable amount of research [TWR+19, JKMR16, ABC+19] has been focused on group fairness, i.e., to ensure that the algorithms are not biased towards or against any specific group in
the population. Here, the two key questions are: how to formalize the notion of fairness and how to design efficient algorithms that conform to these formalization. One such formalization, disparate impact (DI) doctrine [Cho17, CKLV17], posits that any group must have approximately equal or proportional representation in the solution provided by the algorithm. Using this doctrine, Bera et al. [BCFN19] introduced a notion of fairness in clustering problems where they deem a solution to be fair if it satisfies two properties: (a) restricted dominance, which upper bounds the fraction of items from a category, and (b) minority protection, which asserts a lower bound on the fraction of items from a category. They use LP based iterative rounding algorithm and the solution obtained by this algorithm might violate the fairness constraints by some additive amount. Similar group fairness notions as studied by Bera et al. [BCFN19] for clustering problem, have little been studied for resource allocation problems [BCH+18]. Resource allocation problems have been mostly studied under individual fairness and strategic viewpoint [SS18, BLMS19, KLK+18, Lus99, BCEZ19].

These group fairness notions seem to arise naturally in many practical applications. One such scenario is the case of a server serving multiple clients. These clients can be viewed as categories. The clients have a set of jobs to be processed by the server. Each job has some specific resource requirement, which can be viewed as the weight of the item, and a parameter denoting importance, which can be viewed as the associated value. Since the server has limited amount of available resources, which can be viewed as the capacity of the knapsack, it has to select a subset of jobs that can be processed using available resources. It is expected that the selected subset is fair for each client.

The knapsack problem with fairness constraints can also be used to model the problem of fair budget allocation in governing bodies. A government may have various project proposals related to different sectors such as agriculture, education, defense, etc. These sectors can be viewed as the categories. Each project has some cost and value (indicating social impact), which can be viewed as the associated weight and the value of an item, respectively. A government wants to allocate its budget, which can be viewed as the capacity of the knapsack, such that each sector is fairly addressed and the total social impact is maximized. The fair max-knapsack problem with group value constraint is well suited to model this scenario.

Similarly, we can also model the notion of participatory budgeting, which has been proposed recently to take into account the preferences of various stakeholders in the organizations’ budget [GKSA19, ALT18, ST17]. In this process, each stakeholder provides a subset of projects, which can be viewed as items, it prefers. The cost of a project could be viewed as the weight of an item. The number of stakeholders preferring a project could be viewed as the value of an item. An organization could be further divided into subparts, which can be viewed as the categories. The stakeholders associated with a particular subpart would have a preference for specific projects. Any budgeting which does not consider such preferences might lead to an allocation that discriminates against some subparts of the organization. The fair knapsack problems described in this work are well suited to model this scenario. There are many other applications of our problems in advertising [GP14], web search [IID+16], etc.

If all items have identical weights, then the knapsack problem with value lower bound considered in this paper, is a special case of the committee election problem considered by Bredereck et al. [BFI+17]. Chekuri and Kumar [CK04] studied maximum covering problems with group budget constraints. In this problem, we are given a collection of sets \( S = \{S_1, S_2, \ldots, S_m\} \) where each \( S_i \) is a subset of a given ground set \( X \). We are also given a partition of \( S \) into \( l \) groups. The goal is to pick \( k \) sets from \( S \) such that at most one set is selected from each group and the cardinality of their union is maximized. Chekuri and Kumar [CK04] show that the greedy algorithm gives constant factor approximation to this problem. This problem is analogous to one of our problem that has upper bound on number of items as fairness constraint.

There are also some practical heuristics for the knapsack problems (for e.g. [KPP04, Sel04], etc.), but they do not consider fairness constraints. It is not straightforward to extend the techniques in these heuristics
for knapsack problems with fairness constraints.

1.1 Our contributions

We study the following three notions of group fairness for the knapsack problem.

- **Bound on the number of items. (Problem 4)** Given a set of items, each belonging to one of $m$ categories, and a range for each category, the problem is to find a subset that maximizes the total value, such that the number of items from each category is in the given range, and the total weight of the subset is at most the capacity of the knapsack.

- **Bound on weight. (Problem 7)** Given a set of items, each belonging to one of $m$ categories, and upper and lower bounds for weight from each category, the problem is to find a subset that maximizes the total value, such that the total weight of items from each category is in between the given bounds, and the total weight of the subset is at most the capacity of the knapsack.

- **Bound on value. (Problem 5)** Given a set of items, each belonging to one of $m$ categories, and upper and lower bounds for value from each category, the problem is to find a subset that maximizes the total value, such that the total value of items from each category is in between the given bounds, and the total weight of the subset is at most the capacity of the knapsack.

We also study similar fairness notions in the min-knapsack problem which are defined analogously. We give approximation algorithms for these problems. For any input error parameter $\varepsilon > 0$, the running time of our algorithms are polynomial in the input size and $\frac{1}{\varepsilon}$. Our algorithms output a solution having the total value at least $(1 - \varepsilon)$ times the value of the optimal solution (for min-knapsack the weight is at most $(1 + \varepsilon)$ times optimal weight), but the solution produced by some of our algorithms might violate the fairness and/or the knapsack constraints by a small amount (multiplicative factor of $\pm \varepsilon$). For these cases with violations, we show that it is even NP-hard to obtain a feasible solution without violating any constraint. Thus violations are necessary for these cases. We summarize the results of our algorithms in Table 1.

| Knapsack Problem | Number of items from category | Total value from category | Total weight from category |
|------------------|--------------------------------|--------------------------|----------------------------|
| Max-Knapsack     | $(1 - \varepsilon, 0, 0)$     | $(1 - \varepsilon, \varepsilon, 0)$ | $(1, \varepsilon, \varepsilon)$ |
|                  | *Theorem 3.1*                  | *Theorem 3.4*            | *Theorem 3.7*              |
| Min-Knapsack     | $(1 + \varepsilon, 0, 0)$     | $(1, \varepsilon, \varepsilon)$ | $(1 + \varepsilon, \varepsilon, 0)$ |
|                  | *Theorem 4.1*                  | *Theorem 4.4*            | *Theorem 4.6*              |

Table 1: Each entry in the table is represented by a triplet. The first entry in the triplet indicates the approximation ratio achieved by our algorithm. The second entry in the triplet indicates the fractional amount by which the output of the algorithm might violate the fairness constraints. The third entry in the triplet indicates the fractional amount by which the output of the algorithm might violate the knapsack constraint. Reference to the theorem describing respective algorithm is given below the triplet.
**Proof overview:** There is a dynamic programming (DP) based algorithm to solve the classical knapsack problem that runs in pseudo-polynomial time. By rounding the values (or weights) such that the rounded value belongs to a small set (if the problem has \( n \) items to pack, then the value is rounded to the nearest multiple of \( \varepsilon/n \)), the DP technique will give an FPTAS \(^1\) [IK75, Law79]. At a high level, we also use multi-level DP. First, we compute bundles of different values of items from the same category. Next, we select one bundle from each category using a different DP table.

1.2 Related Work

**Max knapsack problem:** The classical knapsack problem was one of Karp’s 21 NP-complete problems. It is one of the fundamental problems in optimization and approximation algorithms. FPTAS for the knapsack problem is known [IK75, Law79]. PTAS\(^1\) for the multiple knapsack problem is also known [CK00, Jan12]. The knapsack problem has also been studied under the multidimensional setting [FC84, GGH\(^1\)+17]. Another variant of the knapsack problem is the submodular knapsack problem, where the value function is submodular [KST13, Svi04, LMNS09]. The knapsack problem has also been well-studied under an online setting [AKL19]. For different variants of the knapsack and related bin packing problems, we refer the readers to [CKPT17, Kha15].

**Min-knapsack problem:** The min-knapsack problem is a frequently encountered problem in the optimization. The problem admits an FPTAS [Csí91, PVRW85]. As the problem appears as a key substructure of numerous other optimization problems, the polyhedral study of this problem has led to the development of important tools, such as the knapsack cover inequalities and there is a rich connection of the problem with extension complexity and sum-of-squares hierarchy [BFHS17, KLM14, FMMS18].

**Class-constrained knapsack and fair knapsack:** To the best of our knowledge, the fairness notions for knapsack problems studied in this work have not been studied previously. The closest model to our model that has been studied before is the class-constrained multiple knapsack problem [ST00]. This problem restricts the maximum number of classes from which items could be packed in a knapsack. The algorithm of the class constrained knapsack problem [ST00] uses two levels of dynamic programming, and is similar in spirit to our algorithm.

Fairness notions for knapsack under multi-agent valuations have also been studied [FSTW17]. They have several approaches to aggregate voters’ preferences: individually best, diverse knapsack and fair knapsack. The main difference is that our model ensures fairness through constraints, while their model ensures fairness through objective function. The objective function used in their model is the Nash welfare function [Nas50]. The diverse knapsack model in [FSTW17] only ensures one representative item from each category, whereas our model allows different lower and upper bounds for different groups. We focus on approximation schemes for our problems, whereas [FSTW17] focused on the parameterized complexity and the complexity of the problem under some restricted special domains.

Ajay et al. [JLN\(^+\)+20] study some notion of group fairness for online/offline matching problem. Their notions of fairness are similar to ours in some respect.

---

\(^1\)A polynomial-time approximation scheme (PTAS) is an algorithm which takes an instance of an optimization problem and a parameter \( \varepsilon > 0 \) and, in polynomial time of input size for fixed \( \varepsilon \), produces a solution that is within a factor \( 1 + \varepsilon \) of being optimal (or \( 1 - \varepsilon \) for maximization problems).

A fully polynomial-time approximation scheme (FPTAS) is a PTAS with running time polynomial in both input size and \( \frac{1}{\varepsilon} \).
2 Preliminaries

Notations. Let the set \( \{1, 2, \ldots, r\} \) be denoted by \([r]\). Let \( m \) denote the number of categories. The category number is denoted from the set \([m]\). Each input item belongs to some category. Let \( V_i \) denote the set of all items from category \( i \), \( \forall i \in [m] \). For \( i \in [m] \), if \( V_i \) has \( k \) items, then \( V_i := [k] \), i.e. an item in \( V_i \) is represented by a number in \([k]\). Also, assume that \( w_j^{(i)} \) and \( v_j^{(i)} \) are the weight and the value of item \( j \in V_i \), respectively, \( \forall i \in [m] \). Also let \( \max\{|V_i| \mid i \in [m]\} = n \). Let \( N \) be the total number of items. By our notations, \( N \leq nm \).

All the algorithms presented in this paper use dynamic programming tables. The entry in the dynamic programming table might represent the total weight or value of a subset of items. The respective subsets can be obtained by maintaining the required references in the tables. Details of it are trivial and hence they are not discussed in this paper.

We will use following abbreviations for the problems discussed in this chapter.

Max knapsack problems: Abbreviations for max knapsack problems:
\( BN^{\text{max}} \): Bound (upper and lower both) on number of items in max knapsack (Problem 4).
\( BV^{\text{max}} \): Bound (upper and lower both) on value in max knapsack (Problem 5).
\( BW^{\text{max}} \): Bound (upper and lower both) on weight in max knapsack (Problem 7).

Min knapsack problems: Abbreviations for min knapsack problems:
\( BN^{\text{min}} \): Bound (upper and lower both) on number of items in min knapsack (Problem 9).
\( BV^{\text{min}} \): Bound (upper and lower both) on value in min knapsack (Problem 10).
\( BW^{\text{min}} \): Bound (upper and lower both) on weight in min knapsack (Problem 11).

Problem 1 (max-knapsack). Given a set of items, each with a weight and a value, along with a maximum weight capacity \( W \), find a subset of items maximizing the total value, such that the total weight of the subset is at most \( W \).

Problem 2 (min-knapsack). Given a set of items, each with a weight and a value, along with a lower bound \( M \), find a subset of items minimizing the total weight, such that the total value of the subset is at least \( M \).

Problem 3 (subset sum). Given a set \( I \) of non-negative rationals such that \( \sum_{a \in I} a = 1 \), find a subset \( S \subseteq I \) such that \( \sum_{a \in S} a = \frac{1}{2} \).

All of the above problems are known NP-hard problems.

3 Algorithms for fair max-knapsack

We study fair max-knapsack problems for various notions of fairness and give an algorithm for each of them. We consider various parameters for fairness such as the number of items from each category, bound on the total value of items, and the total weight of items from each category.

3.1 Fairness based on number of items

Problem 4 (\( BN^{\text{max}} \)). Given a set of items, each belonging to one of \( m \) categories, and numbers \( l^n_i \) and \( u^n_i \) for each category \( i \), \( \forall i \in [m] \), the problem is to find a subset that maximizes the total value, such that the
number of items from category $i$ is between $l_i^n$ and $u_i^n$, $\forall i \in [m]$, and the total weight of the subset is at most the capacity of the knapsack $B$.

**Theorem 3.1.** For any $\varepsilon > 0$, there exists a $(1 - \varepsilon)$-approximation algorithm for $B N^{max}$ (Problem 4) with running time $O\left(\frac{n N^3}{\varepsilon^2}\right)$.

**Proof.** The algorithm is described in Algorithm 1. The algorithm starts by rounding the values of all items so that the rounded values lie in a small range (Step 2). Then it creates bundles having different cardinality and total rounded value from items in $V_i$, $\forall i \in [m]$. It uses the dynamic programming table $A$ for this (Step 3). After that the algorithm combines bundles from all categories to obtain the final solution using the dynamic programming table $B$ (Step 4). The following describes formal proof of the algorithm.

**Property of $A$.** We claim that $A(i, j, v, t)$, $\forall i \in [m], \forall t \in V_i \cup \{0\}$, $j \in V_i$, $\forall v \in \left[\frac{N^2}{\varepsilon}\right] \cup \{0\}$, denotes the weight of a minimum weight subset of cardinality $t$ from first $j$ items of $V_i$ having the total rounded value $v$. Let $S'$ be the subset satisfying above property for the entry $A(i, j, v, t)$, $\forall i \in [m]$, $\forall j \in V_i \setminus \{1\}$, $\forall t \in V_i \cup \{0\}$, $\forall v \in \left[\frac{N^2}{\varepsilon}\right] \cup \{0\}$.

- If $j \in S'$, then $A(i, j, v, t)$ is the sum of weight of $j$ ($w_{j}^{(i)}$) and the weight of minimum weight subset from first $j - 1$ items of $V_i$ having cardinality $t - 1$ and the rounded value $v - v_{j}^{(i)r}$ ($A(i, j - 1, v - v_{j}^{(i)r}, t - 1)$), which is equal to $A(i, j - 1, v - v_{j}^{(i)r}, t - 1) + w_{j}^{(i)}$.

- If $j \notin S'$, then $A(i, j, v, t)$ is equal to the weight of minimum weight subset from first $j - 1$ items of $V_i$ having cardinality $t$ and the rounded value $v$, which is equal to $A(i, j - 1, v, t)$.

The recursion in Step 3b captures both of above possibilities. The Step 3a initializes base cases for the recursion in Step 3b. The entry of $A$ also corresponds to the respective subset. The entry of $A$ could be $\infty$, which indicates no subset. We use the notation $A(i, j, v, t)$ to indicate both the entry and the subset.

**Property of $B$.** We claim that $B(i, v)$, $\forall i \in [m]$, $\forall v \in \left[\frac{N^2}{\varepsilon}\right] \cup \{0\}$, denotes the weight of a minimum weight subset of $\cup_{j=1}^{i} V_j$ having the total rounded value $v$, such that fairness constraints are satisfies for the subset for all the categories up to $i$. Let $S'$ be the subset of $\cup_{j=1}^{i} V_j$ satisfying above property for $B(i, v)$, for some $i \in [m] \setminus \{1\}$ and $v \in \left[\frac{N^2}{\varepsilon}\right] \cup \{0\}$. If $\sum_{j \in S'} v_{j}^{(i)r} = v_{r}$, then $B(i, v)$ is the sum of weights of minimum weight subset of $\cup_{j=1}^{i-1} V_j$ having the total rounded value $v - v_{r}$ that satisfies fairness condition for all categories up to $i - 1$ ($B(i - 1, v - v_{r})$), and the minimum weight subset of $V_i$ having total rounded value $v_{r}$ that satisfies fairness condition for category $i$ ($A(i, | V_i |, v_{r}, t)$) such that $l_{i}^{n} \leq t \leq u_{i}^{n}$). The recursion in Step 4b captures this for all possible $v_{r}$. The entry of $B$ also corresponds to the respective subset. The entry of $B$ could be $\infty$, which indicates no subset. We use the notation $B(i, v)$ to indicate both the entry and the subset.

By the property of $B$, the total rounded value of $S$ (Step 5) is at least the total rounded value of any optimal solution. By Theorem 3.2, we can show that the total value of $S$ is at least $(1 - \varepsilon)$ times the optimal solution.
Algorithm 1: Algorithm for $BN^{max}$ (Problem 4)

**Input:** The sets $V_i$ of items, $0 \leq l_i \leq u_i$ for $i \in [m]$, $B$ the capacity of knapsack and $\varepsilon > 0$.

**Output:** $S$ having the total value at least $1 - \varepsilon$ times the optimal value of $BN^{max}$ (Problem 4), such that $l_i \leq |S \cap V_i| \leq u_i$, $\forall i \in [m]$ and the total weight of $S$ is at most the knapsack weight $B$.

1. Remove all items $j \in V_i$, $\forall i \in [m]$ that do not come under any feasible solution. We can check whether an item comes under a feasible solution by checking the weight of least weight feasible subset containing the item is less than or equal to $B$. Let $v_{max}$ be the maximum value of the remaining items.

2. $\forall i \in [m], \forall j \in V_i$, let

$$v_j^{(i)} := \left\lfloor \frac{v_j}{\varepsilon v_{max}} \right\rfloor .$$

3. Let $A(i, j, v, t), \forall i \in [m], \forall j \in V_i, \forall t \in V_i \cup \{0\}, \forall v \in \left[\left\lceil \frac{N^2}{\varepsilon} \right\rceil \right] \cup \{0\}$, be the dynamic programming table constructed in the following way,

(a) $\forall i \in [m],
A(i, 1, v_1^{(i)}, 1) := w_1^{(i)}, A(i, 1, 0, 0) = 0, A(i, 1, v', 0) := \infty \text{ for } v' \in \left[\left\lceil \frac{N^2}{\varepsilon} \right\rceil \right] \setminus \{v_1^{(i)}\}$.

(b) $\forall i \in [m] \text{ and } \forall j \in V_i \setminus \{1\}, \forall t \in V_i \cup \{0\}, \forall v \in \left[\left\lceil \frac{N^2}{\varepsilon} \right\rceil \right],
\begin{align*}
\text{If } v < v_j^{(i)} \text{ or } t = 0, \text{ then } A(i, j, v, t) &= A(i, j - 1, v, t). \\
\text{Else}, \quad A(i, j, v, t) &= \min \left\{ A(i, j - 1, v - v_j^{(i)}, t - 1) + w_j^{(i)}; A(i, j - 1, v, t) \mid 0 \leq v_j^{(i)} \leq v \right\}.
\end{align*}$

4. Let $B(i, v), \forall i \in [m], \forall v \in \left[\left\lceil \frac{N^2}{\varepsilon} \right\rceil \right] \cup \{0\}$ be another the dynamic programming table.

(a) $\forall v \in \left[\left\lceil \frac{N^2}{\varepsilon} \right\rceil \right] \cup \{0\}$,

$$B(1, v) := \min \{A(1, V_1, v, t) \mid l_1 \leq t \leq u_1\}.$$

(b) For $i \in [m] \setminus \{1\}$ and $v \in \left[\left\lceil \frac{N^2}{\varepsilon} \right\rceil \right] \cup \{0\},$

$$B(i, v) := \min \{B(i - 1, v_r) + A(i, V_i, v - v_r, t) \mid l_i \leq t \leq u_i, v_r \leq v \}.$$

5. Output $S$ in the following way,

$$\arg\max_v \{B(m, v) \mid B(m, v) \leq B\}.$$

Running time analysis: The size of the table $A$ is $O\left(\frac{nN^3}{\varepsilon}\right)$, and to fill each entry in the table $A$, we require $O(1)$ time. The size of the table $B$ is $O\left(\frac{N^2}{\varepsilon}\right)$, and to fill each entry in the table $B$, we require $O\left(\frac{nN^3}{\varepsilon}\right)$ time. So, the total time required is $O\left(\frac{nN^3}{\varepsilon}\right)$.

Theorem 3.2. If $OPT$ is the optimal objective value of $BN^{max}$ (Problem 4), then the total value of a set returned by Algorithm 1 is at least $(1 - \varepsilon)OPT$.

Proof. Let $O \subseteq \cup_{i=1}^{m} V_i$ be the set of items in an optimal solution and $S \subseteq \cup_{i=1}^{m} V_i$ be the set of items selected by Algorithm 1. Since $S$ is an optimal solution of rounded value in Algorithm 1, the total rounded value of $S$ is more than the total rounded value of $O$.

$$
\sum_{i=1}^{m} \sum_{j \in V_i \cap O} v^{(i)}_{j} \leq \sum_{i=1}^{m} \sum_{j \in V_i \cap S} v^{(i)}_{j} \tag{1}
$$

Because of the rounding in Step 2, we have following inequalities $\forall i \in [m]$ and $\forall j \in V_i$,

$$
\frac{\varepsilon v_{\text{max}} v^{(i)}_{j}}{N} \leq v^{(i)}_{j} \leq \frac{\varepsilon v_{\text{max}} (v^{(i)}_{j} + 1)}{N} = \frac{\varepsilon v_{\text{max}} v^{(i)}_{j}}{N} + \frac{\varepsilon v_{\text{max}}}{N}. \tag{2}
$$

So we get,

$$
\sum_{i=1}^{m} \sum_{j \in V_i \cap O} v^{(i)}_{j} \leq \frac{\varepsilon v_{\text{max}}}{N} \left( \sum_{i=1}^{m} \sum_{j \in V_i \cap O} (v^{(i)}_{j} + 1) \right) \leq \frac{\varepsilon v_{\text{max}}}{N} \left( \sum_{i=1}^{m} \sum_{j \in V_i \cap O} v^{(i)}_{j} \right) + \varepsilon v_{\text{max}} = \frac{\varepsilon v_{\text{max}}}{N} \sum_{i=1}^{m} \sum_{j \in V_i \cap O} v^{(i)}_{j} + \varepsilon v_{\text{max}} \leq \left( \sum_{i=1}^{m} \sum_{j \in V_i \cap S} v^{(i)}_{j} \right) + \varepsilon v_{\text{max}} \tag{using Inequality 2}
$$

Since Step 1 of the Algorithm 1 discards all the items which doesn’t come in any feasible solution, we know that $v_{\text{max}} \leq OPT$. So,

$$
\left( \sum_{i=1}^{m} \sum_{j \in V_i \cap S} v^{(i)}_{j} \right) \geq (1 - \varepsilon)OPT. \tag{using Inequality 2}
$$

3.2 Fairness based on bound on value

Problem 5 ($BV_{\text{max}}$). Given a set of items $V$, each belonging to one of $m$ categories, a lower bound $l^v_i \geq 0$ and an upper bound $u^v_i$, for each category $i$, the goal is to find a subset that maximizes the total value, such that the total value of items from the category $i$ is in between $l^v_i$ and $u^v_i$, $\forall i \in [m]$, and the total weight of the subset is at most the knapsack capacity $B$. 


We prove that it is NP-hard to obtain the feasible solution of an instance of \( BV^{\max} \) (Problem 5).

**Theorem 3.3.** There is no polynomial time algorithm that outputs feasible solution of \( BV^{\max} \) (Problem 5), assuming \( \text{P} \neq \text{NP} \).

**Proof.** Given an instance of subset sum (Problem 3) with the set \( I \), we will construct an instance of \( BV^{\max} \) (Problem 5). Let \( m = 1, p_i = \frac{1}{2}, u_i = 1, B = \frac{1}{2} \). \( V_1 \) is the set of items in the instance of Problem 5. The items in \( V_1 \) correspond to the numbers in \( I \). An item corresponding to some \( a \in I \) has a value and a weight equal to \( a \). This proves that if we have an algorithm for \( BV^{\max} \) (Problem 5) that doesn’t violate the fairness constraint, then we can solve subset sum. But assuming \( \text{P} \neq \text{NP} \), this is not possible. \( \square \)

Theorem 3.3 implies that there does not exists polynomial time algorithm for \( BV^{\max} \) (Problem 5). We give here an algorithm for \( BV^{\max} \) (Problem 5) that might violates fairness constraints for a category by small amount.

**Theorem 3.4.** For any \( \varepsilon > 0 \), there exists an algorithm for \( BV^{\max} \) (Problem 5) that outputs a set \( S \) having the total value at least \( 1 - \varepsilon \) times the optimal value of \( BV^{\max} \) (Problem 5), such that \( (1 - \varepsilon)w_S \leq \sum_{v \in S \cap [n]} v \leq (1 + \varepsilon)w_S \), \( \forall k \in [m] \), and the total weight of \( S \) is at most \( B \). The running time of the algorithm is \( O \left( \frac{n^2 m^2 \log_3 \frac{1}{\varepsilon}}{\varepsilon} \right) \), where \( \min \left\{ v_{j}^{(i)} \mid i \in [m] \& j \in V_i \& v_{j}^{(i)} > 0 \right\} \), and \( \max \left\{ v_{j}^{(i)} \mid i \in [m] \& j \in V_i \right\} \).

Towards proving this theorem, we first study the following sub-problem.

**Problem 6.** Given \( v > 0, \varepsilon > 0 \) and a set \( V' = [n] \) of items, with item \( i \in V' \) having the value \( v_i' \) and the weight \( w_i' \), compute a subset \( S \subseteq V' \) minimizing \( \sum_{i \in S} w_i' \), such that \( (1 - \varepsilon)v \leq \sum_{i \in S} v_i' \leq (1 + \varepsilon)v \).

Problem 6 looks similar to the min-knapsack problem but it is different from it in the following way. The total value of output of min-knapsack problem is required to be more than the given lower bound, while the total value of the output of Problem 6 is required to be lying in a small range. We will use Algorithm 2 for Problem 6 to obtain bundles of items in \( V_i, \forall i \in [m] \), such that the total value of different bundles are in different required ranges.

**Theorem 3.5.** For any \( \varepsilon > 0 \) and \( v > 0 \), there exists an algorithm for Problem 6 that outputs a subset \( S \) having the total weight at most the optimal weight of Problem 6, and \( (1 - 4\varepsilon)v \leq \sum_{i \in S} v_i' \leq (1 + 4\varepsilon)v \). The running time of the algorithm is \( O \left( \frac{n^2}{\varepsilon} \right) \).

**Proof.** The algorithm for the theorem is described in Algorithm 2. We give the proof of correctness of the algorithm below.

We claim that the entry \( H(i, v', v''), \forall i \in [n], \forall v'' \in \left[ \frac{(1+2\varepsilon)n}{\varepsilon} \right] \cup \{0\} \), indicates the weight of the least weight subset from the first \( i \) items of \( V' \) having the total rounded value equal to \( v'' \). Let \( S' \) denote the subset satisfying above property for the entry \( H(i, v', v''), \forall i \in [n] \setminus \{1\}, \forall v'' \in \left[ \frac{(1+2\varepsilon)n}{\varepsilon} \right] \cup \{0\} \). We can have two cases as below.

- If \( i \in S' \), then \( H(i, v'', v''') \) is equal to the sum of the weight of \( i (w_i') \), and the weight of minimum weight subset from first \( i - 1 \) items having the total rounded value \( v'' - v''' (H(i - 1, v'' - v''')) \).
### Algorithm 2: Algorithm for Problem 6

**Input:** $v > 0, \varepsilon > 0$, a set $V' = [n]$ of items, with item $i \in V'$ having value $v_i'$ and weight $w_i'$.

**Output:** A subset $S \subseteq V'$ having total weight at most the optimal weight for Problem 6, such that $(1 - 4\varepsilon)v \leq \sum_{i \in S} v_i' \leq (1 + 4\varepsilon)v$.

1. Remove all items from $V'$ having value more than $(1 + \varepsilon)v$.
2. For each item $i \in V'$, let $v_i'' := \left\lfloor \frac{n v_i'}{\varepsilon v} \right\rfloor$.
3. Let $H(i, v'')$ for $v'' \in \left[\left\lfloor \frac{(1 + 2\varepsilon) n}{\varepsilon} \right\rfloor \right] \cup \{0\}$ and $i \in [n]$, be the dynamic programming table constructed in the following way.
   
   (a) $H(1, v_1'') = w_1'$. $H(1, 0) = 0$. $H(1, v'') = \infty$, $\forall v'' \in \left[\left\lfloor \frac{(1 + 2\varepsilon) n}{\varepsilon} \right\rfloor \right] \setminus \{v_1''\}$.
   
   (b) $\forall i \in [n] \setminus \{1\}$, $\forall v'' \in \left[\left\lfloor \frac{(1 + 2\varepsilon) n}{\varepsilon} \right\rfloor \right]$, 
   
   If $v'' < v_i''$, then $H(i, v'') = H(i - 1, v'').$
   
   Else,
   
   $$H(i, v'') := \min \{H(i - 1, v''), H(i - 1, v'' - v_i'') + w_i' \mid v_i'' \leq v'' \}.$$
4. Output the subset $S$ as follows,

   $$\min \left\{ H(n, v'') \mid \left\lfloor \frac{(1 - 2\varepsilon) n}{\varepsilon} \right\rfloor \leq v'' \leq \left\lfloor \frac{(1 + 2\varepsilon) n}{\varepsilon} \right\rfloor \right\}.$$

- If $i \notin S'$, then $H(i, v'')$ is equal to the weight of minimum weight subset from first $i - 1$ items having the total rounded value $v''$, i.e. $H(i, v'') = H(i - 1, v'')$.

The recursion in Step 3b captures both of above possibilities. Step 3a does necessary initialization for the recursion in Step 3b. Let $O$ denote the set of items in an optimal solution of Problem 6 and $S$ be the set output by Step 4 of Algorithm 2. By the construction of the table $H$, we have

$$\left\lfloor \frac{(1 - 2\varepsilon) n}{\varepsilon} \right\rfloor \leq \sum_{i \in S} v_i'' \leq \left\lfloor \frac{(1 + 2\varepsilon) n}{\varepsilon} \right\rfloor$$

$$\frac{(1 - 2\varepsilon) n}{\varepsilon} - 1 \leq \sum_{i \in S} v_i'' \leq \frac{(1 + 2\varepsilon) n}{\varepsilon} + 1.$$

Using the definition of $v_i''$ in Step 2, we get

$$(1 - 3\varepsilon)v - \frac{\varepsilon v}{n} \leq \sum_{i \in S} v_i' - \varepsilon v \leq (1 + 2\varepsilon)v + \frac{\varepsilon v}{n}$$

$$(1 - 4\varepsilon)v \leq \sum_{i \in S} v_i' \leq (1 + 4\varepsilon)v.$$
Now, we prove that \( \left\lfloor \frac{(1-2\varepsilon)n}{\varepsilon} \right\rfloor \leq \sum_{i \in O} v_i'' \leq \left\lceil \frac{(1+2\varepsilon)n}{\varepsilon} \right\rceil \). Since \( S \) is the least weight subset in the previous range, the above claim will imply that the total weight of \( S \) is at most the total weight of \( O \). We know that \( (1-\varepsilon)v \leq \sum_{i \in O} v_i' \leq (1+\varepsilon)v \). Using the definition of \( v_i'' \) in Step 2, we get
\[
\frac{n(1-\varepsilon)}{\varepsilon} - n \leq \sum_{i \in O} v_i'' \leq \frac{(1+\varepsilon)n}{\varepsilon} + n
\]
\[
\frac{n(1-2\varepsilon)}{\varepsilon} \leq \sum_{i \in O} v_i'' \leq \frac{(1+2\varepsilon)n}{\varepsilon}.
\]

Note that the algorithm returns \( \infty \) if it does not find a subset in Step 4. This is because of the initialization in Step 3a.

**Running time analysis.** The size of the table \( H \) is \( O\left(\frac{n^2}{\varepsilon}\right) \), and to fill each entry in the table \( H \), we require \( O(1) \) time. So, the total running time of the algorithm is \( O\left(\frac{n^2}{\varepsilon}\right) \).

**Proof of Theorem 3.4.** The algorithm for theorem is described in Algorithm 3. The algorithm creates bundles of items from \( V_i, \forall i \in [m] \), such that the total value of each bundle is in different ranges using Theorem 3.5 (Step 2). It stores these bundles in the table \( W \). After that the algorithm combines bundles from all categories to obtain the final solution using the dynamic programming table \( X \) (Step 3). The total value of each bundle is represented by some power of \( (1+\varepsilon') \) in the tables \( W \) and \( X \). The table \( W \) represents at least \( O\left(\frac{1}{(1+\varepsilon')}\right) \) fraction of the total value of any bundle (Step 2). So, the fraction of total value represented by the table \( X \) after combining the bundles from all categories in Step 3b is at least \( O\left(\frac{1}{(1+\varepsilon')}\right) \). This is at least \( O\left(\frac{1}{(1+\varepsilon')}\right) \) because of the choice of \( \varepsilon' \) in Step 1. We describe the formal proof of the algorithm below.

**Properties of \( W \).** We claim that the entry \( W(i,j), \forall i \in [m], j \in W_{\text{range}} \), indicates the weight of the subset of \( V_i \) that satisfies the two properties listed below. The entry of \( W \) also corresponds to respective subset. The entry of \( W \) could be \( \infty \), which indicates no subset. We use the notation \( W(i,j) \) to indicate both the entry and the subset.

1. If the entry \( W(i,j) \) is finite, then the total value of the corresponding subset is in between \( (1 - \frac{4}{10}\varepsilon) \) \( (1+\varepsilon')^j v_{\text{min}} \) and \( (1 + \frac{1}{10}\varepsilon) (1+\varepsilon')^j v_{\text{min}} \).
2. The total weight of the subset corresponding to \( W(i,j) \) is at most the total weight of any subset of \( V_i \) having the total value in between \( (1 - \frac{4}{10}\varepsilon) (1+\varepsilon')^j v_{\text{min}} \) and \( (1 + \frac{1}{10}\varepsilon) (1+\varepsilon')^j v_{\text{min}} \) (Since \( \varepsilon' < \frac{\varepsilon}{10} \), the statement is also true for any subset having the total value in between \( (1+\varepsilon')^j v_{\text{min}} \) and \( (1+\varepsilon')^j v_{\text{min}} \)).

The table \( W \) is created in Step 2 of the algorithm. This step uses Theorem 3.5. We get both above properties because of the guarantee of Theorem 3.5.
Algorithm 3: Algorithm for $BV^{\max}$ (Problem 5)

**Input:** The sets of items: $V_i$ for all $i \in [m]$, $0 \leq l_i \leq u_i$ for all $i \in [m]$, the knapsack capacity $B$ and $\varepsilon > 0$.

**Output:** $S$ having the total value at least $1 - \varepsilon$ times the optimal value of $BV^{\max}$ (Problem 5), such that $(1 - \varepsilon)l_i \leq \sum_{r \in S \cap V_i} v_r^{(i)} \leq (1 + \varepsilon)u_i, \forall i \in [m]$, and the total weight of $S$ is at most $B$.

1. Let $\varepsilon' = (1 + \varepsilon) \frac{m+1}{m+3} - 1$. Also let $W_{\text{range}} := \left[ \left\lceil \log_{1+\varepsilon'} \left( \frac{u_{\text{min}}^{\max}}{v_{\text{min}}} \right) \right\rceil \right] \cup \{0\}$ and $X_{\text{range}} := \left[ \left\lceil \log_{1+\varepsilon'} \left( \frac{N_{\text{min}}^{\max}}{v_{\text{min}}} \right) \right\rceil \right] \cup \{-m - 2, -m - 1, \ldots, -2, -1, 0\}$.

2. Let $W(i, j), \forall i \in [m], \forall j \in W_{\text{range}}$, be the table, where the entry $W(i, j)$ indicates the weight of a subset of $V_i$ that is obtained by Theorem 3.5 by setting $V_i$ as $V'$, $v_{\text{min}} (1 + \varepsilon')^j$ as $v$, and $\frac{\varepsilon}{10}$ as $\varepsilon$ in Theorem 3.5.

3. Let $X(i, j), \forall i \in [m], \forall j \in X_{\text{range}}$, be the DP table constructed as follows.

   (a) $X(1, j) := \min \left\{ \mathcal{W}(1, j'') \mid j'' \geq j \land j'' \in W_{\text{range}} \land (1 + \varepsilon')^j v_{\text{min}} \geq l_i \right\}.

   If the set satisfying above condition is empty, then set $X(1, j)$ to $\infty$.

   (b) $\forall i \in [m] \setminus \{1\}, \forall j' \in X_{\text{range}}$ and $\forall j'' \in W_{\text{range}}$, we have

   $$X(i, j) := \min \left\{ X(i - 1, j') + W(i, j'') \mid (1 + \varepsilon')^{j''} v_{\text{min}} \geq l_i \land (1 + \varepsilon')^{j''} v_{\text{min}} \geq l_i \right\} + (1 + \varepsilon')^{j''} v_{\text{min}} \leq u_i \land (1 + \varepsilon')^{j''} v_{\text{min}} \leq u_i \right\}.$$

   If the set satisfying above condition is empty, then set $X(i, j)$ to $\infty$.

4. Output the subset $S$ as follows,

   $$\arg\max_j \{ X(m, j) \mid X(m, j) \leq B \}.$$

**Properties of $\mathcal{X}$.** We claim that the entry $X(i, j), \forall i \in [m], \forall j \in X_{\text{range}}$, indicates the weight of a subset of $\cup_{k=1}^i V_k$ that satisfies the three properties mentioned below. The entry of $X$ also corresponds to the respective subset. The entry of $X$ could be $\infty$, which indicates no subset. We use the notation $X(i, j)$ to indicate both the entry and the subset.

1. If the entry $X(i, j)$ is finite, then $\sum_{k=1}^i \sum_{r \in X(i, j) \cap V_k} v_r^{(k)} \geq (1 - \frac{4}{10} \varepsilon')(1 + \varepsilon')^j v_{\text{min}}$.

2. If the entry $X(i, j)$ is finite, then $(1 - \frac{4}{10} \varepsilon')^{-1} l_i \leq \sum_{r \in X(i, j) \cap V_k} v_r^{(k)} \leq (1 + \frac{4}{10} \varepsilon') u_i$, $\forall k \in [i].$
3. For all $O' \subseteq \bigcup_{k=1}^{i} V_k$ having the total value at least $(1 + \varepsilon')^{j+i+2} v_{\min}$, and $l_i^v \leq \sum_{r \in O' \cap V_k} u_r^{(k)} \leq u_i^v$, $\forall k \in [i]$, the total weight of subset $X(i, j)$ is at most the total weight of $O'$.

In Property 3, any considered set $O'$ will have value at least $v_{\min}$. Thus in Property 3, $i + j + 2 \geq 0$, i.e. $j \geq (-m - 2)$. Hence, the minimum value in $X_{\text{range}}$ has been set to $(-m - 2)$ (Step 1).

Steps 3a and 3b of the algorithm chooses the subset $W(i, j'')$ that satisfies the inequalities $(1 + \varepsilon')^{j''+1} v_{\min} \geq l_i^v$ and $(1 + \varepsilon')^{j''} v_{\min} \leq u_i^v$. By Property 1 of $W$, the total value of $W(i, j'')$ is in between $(1 - \frac{4}{3} \varepsilon) (1 + \varepsilon')^{-1} l_i^v$ and $(1 + \frac{4}{3} \varepsilon) u_i^v$. This proves that the subset corresponding to finite entry $X(i, j)$, $\forall i \in [m]$, $\forall j \in X_{\text{range}}$, satisfies Property 2. The subset corresponding to finite entry $X(1, j)$, $\forall j \in X_{\text{range}}$, will satisfy Property 1. This is true because of Property 1 of $W$ and the condition $j'' \geq j$ in Step 3a for selecting subset $W(1, j'')$. Because of the condition $(1 + \varepsilon')^j \leq (1 + \varepsilon')^{j''} + (1 + \varepsilon')^{j''+1}$ in Step 3b of the algorithm and Property 1 of $W$, the subset corresponding to finite entry $X(i, j)$, $\forall i \in [m]\{1\}$, $\forall j \in X_{\text{range}}$, will satisfy Property 1.

We prove that the entries in the table $X$ will satisfy Property 3 by the induction. In the base case, the subset $X(1, j)$, $\forall j \in X_{\text{range}}$, satisfies Property 3 of $X$. We can prove this by the following argument. For any $O' \subseteq V_1$, which satisfies the fairness condition for category 1 having the total value in between $v_{\min} (1 + \varepsilon')^j$ and $v_{\min} (1 + \varepsilon')^{j+1}$, for some $j'' \in W_{\text{range}}$, the total weight of $W(1, j'')$ is less than or equal to the total weight of $O'$ (Property 2 of $W$). Also, the condition $j'' \geq j$ in Step 3a will be satisfied for all $j \in X_{\text{range}}$ such that the total weight of $O'$ is at least $v_{\min} (1 + \varepsilon')^{j+2}$. So, the subset $W(1, j'')$ is feasible for all such $j$ in Step 3a. For any $i \in [m]\{1\}$, assume the hypothesis that the subsets $X(i-1, j)$, $\forall j \in X_{\text{range}}$, satisfy Property 3. We will prove by induction that for any $O' \subseteq \bigcup_{k=1}^{i} V_k$ that satisfies the fairness conditions for all categories up to $i$ and for all $j \in X_{\text{range}}$ such that the total value of $O'$ is at least $(1 + \varepsilon')^{j+i+2} v_{\min}$, the subsets $X(i, j)$ satisfy Property 3. For any $j^* \in X_{\text{range}}$ that satisfies the following inequality:

$$
(1 + \varepsilon')^{j^*} v_{\min} \leq \sum_{k=1}^{i-1} \sum_{r \in O' \cap V_k} u_r^{(k)} \leq (1 + \varepsilon')^{j^*+1} v_{\min},
$$

the induction hypothesis implies the following:

$$
\sum_{k=1}^{i-1} \sum_{r \in X((i-1, j') \cap V_k)} w_r^{(k)} \leq \sum_{k=1}^{i-1} \sum_{r \in O' \cap V_k} w_r^{(k)},
$$

where $j' = j^* - i - 1$. Property 2 of $W$ implies that there exists a $j'' \in W_{\text{range}}$ such that,

$$
(1 + \varepsilon')^{j''} v_{\min} \leq \sum_{k \in V_i} w_k^{(i)} \leq (1 + \varepsilon')^{j''+1} v_{\min},
$$

and

$$
\sum_{r \in W(i, j'')} w_r^{(i)} \leq \sum_{r \in O' \cap V_i} w_r^{(i)}.
$$

Since $O'$ satisfies the fairness constraints for category $i$, $j''$ satisfies inequalities $(1 + \varepsilon')^{j''+1} v_{\min} \geq l_i^v$ and $(1 + \varepsilon')^{j''} v_{\min} \leq u_i^v$ in Step 3b. We claim that the inequality $(1 + \varepsilon')^j \leq (1 + \varepsilon')^{j''} + (1 + \varepsilon')^{j''+1}$ is satisfied in Step 3b for any $j \in X_{\text{range}}$, such that the total value of $O'$ is at least $v_{\min} (1 + \varepsilon')^{j+i+2}$. This is true because the maximum value of $O'$ is at most $v_{\min} (1 + \varepsilon')^{j+i+2}$ by Inequality 3 and
Inequality 5), which is more than \( v_{\text{min}} (1 + \varepsilon')^j + i + 2 \). So, the pair of subsets \( X(i - 1, j') \) and \( W(i, j'') \) is feasible in Step 3b for all such \( j \). Also, the total weight of union of \( X(i - 1, j') \) and \( W(i, j'') \) is at most the total weight of \( O' \) (Inequality 4 and Inequality 6). So Property 3 of \( X \) is satisfied for all such \( j \).

Let \( \text{OPT} \) be the optimal value. Let \( j \) be the number in \( X_{\text{range}} \) such that

\[
(1 + \varepsilon')^{j+m+2} v_{\text{min}} \leq \text{OPT} \leq (1 + \varepsilon')^{j+m+3} v_{\text{min}}. \tag{7}
\]

If we apply Property 3 of \( X \) to any optimal solution, we get that the total weight of the subset \( X(m, j) \) is less than or equal to the total weight of an optimal subset. So, the subset \( X(m, j) \) is feasible in Step 4. By Property 1, the total value of \( X(m, j) \) is at least \( (1 - \frac{4}{10} \varepsilon) (1 + \varepsilon')^j v_{\text{min}} \). By Inequality 7, this value is at least

\[
\left(1 - \frac{4}{10} \varepsilon \right) \frac{\text{OPT}}{(1 + \varepsilon')^{m+3}} = \left(1 - \frac{4}{10} \varepsilon \right) \frac{\text{OPT}}{(1 + \varepsilon')^{m+3}} > \left(1 - \frac{4}{10} \varepsilon \right) (1 - \varepsilon) \text{OPT} > (1 - \varepsilon) \text{OPT}.
\]

If the entry \( X(m, j) \) is finite, Property 2 of \( X \) implies that the total value of items from \( V_i \cap X(m, j) \) is in between \( (1 - \frac{4}{10} \varepsilon) (1 + \varepsilon')^{-1} l_i^v \) and \( (1 + \varepsilon') u_i^v \) for all \( i \in [m] \). Since \( \varepsilon' \leq \frac{\varepsilon}{10} \), the total value of items from \( V_i \cap X(m, j) \) is at least \( (1 - \frac{4}{10} \varepsilon) (1 + \varepsilon')^{-1} l_i^v > (1 - \frac{4}{10} \varepsilon) (1 - \varepsilon) l_i^v \), and is at most \( (1 + \frac{\varepsilon}{10}) u_i^v < (1 + \varepsilon) u_i^v \).

**Running time analysis:** The size of the table \( W \) is \( O \left( m^2 \log_{1+\varepsilon} \left( \frac{N_{v_{\text{max}}}}{v_{\text{min}}} \right) \right) \), and to fill each entry of the table \( O \left( \frac{n^2 \varepsilon}{\varepsilon} \right) \) time is required. So, the total time required to build the table \( W \) is \( O \left( \frac{n^2 m^2 \log_{1+\varepsilon} \left( \frac{N_{v_{\text{max}}}}{v_{\text{min}}} \right)}{\varepsilon} \right) \).

The total time required to build the table \( X \) is \( O \left( m \log_{1+\varepsilon} \left( \frac{N_{v_{\text{max}}}}{v_{\text{min}}} \right) \right) = O \left( \frac{m^4 \log_{1+\varepsilon} \left( \frac{N_{v_{\text{max}}}}{v_{\text{min}}} \right)}{\varepsilon} \right) \). So, the total running time of the algorithm is \( O \left( \frac{n^2 m^4 \log_{1+\varepsilon} \left( \frac{N_{v_{\text{max}}}}{v_{\text{min}}} \right)}{\varepsilon} \right) \).

### 3.3 Fairness based on bound on weight

**Problem 7** (\( BW_{\text{max}} \)). Given a set of items, each belonging to one of \( m \) categories, a lower bound \( l_i^w \) and an upper bound \( u_i^w \) for each category \( i \), \( \forall i \in [m] \), the goal is to find a subset that maximizes the total value, such that the total weight of items from category \( i \) is in between \( l_i^w \) and \( u_i^w \), \( \forall i \in [m] \) and the total weight of the subset is at most the capacity of the knapsack \( B \).

We prove that it is NP-hard to obtain the feasible solution of an instance of \( BW_{\text{max}} \) (Problem 7).

**Theorem 3.6.** There is no polynomial time algorithm which outputs a feasible solution of \( BW_{\text{max}} \) (Problem 7), assuming \( P \neq NP \).

**Proof.** Given an instance of subset sum (Problem 3), we can construct an instance of \( BW_{\text{max}} \) (Problem 7) in following way. Let \( m = 1 \), \( l_1^w = \frac{1}{2} \), \( u_1^w = 1 \) and \( B = \frac{1}{2} \). The set \( V_1 \) contains items which correspond to the numbers in \( I \). An item corresponding to some \( a \in I \) has a value and a weight equal to \( a \). This proves that if we have an algorithm that outputs a feasible solution of \( BW_{\text{max}} \) (Problem 7) without violating any constraint, then we can solve subset sum (Problem 3). But assuming \( P \neq NP \), this is not possible.
Theorem 3.6 implies that there does not exist a polynomial time algorithm for $BW_{\text{max}}^{\text{max}}$ (Problem 7). We give here an algorithm for $BW_{\text{max}}^{\text{max}}$ (Problem 7) that might violate fairness constraints and knapsack constraint by small amount.

**Theorem 3.7.** For any $\varepsilon > 0$, there exists an algorithm for $BW_{\text{max}}^{\text{max}}$ (Problem 7) that outputs a solution $S$ whose objective value is at least the optimal value, and $(1 - \varepsilon)l^{(1)}_{i} \leq \sum_{j \in S \cap V_{i}} w_{i}^{(j)} \leq (1 + \varepsilon)u^{(1)}_{i}$, and $\sum_{i=1}^{l} \sum_{j \in S \cap V_{i}} w_{j}^{(i)} \leq (1 + \varepsilon)B$, $\forall i \in [m]$. The running time of the algorithm is $O\left(\frac{n^{2}m^{4}\log_{1+\varepsilon}^{3}(\frac{\theta}{\varepsilon \min})}{\varepsilon} \right)$, where $w_{\min} := \min \{ w_{j}^{(i)} \mid i \in [m] \& j \in V_{i} \& w_{j}^{(i)} > 0 \}$.

Towards proving this theorem, we first study the following sub-problem.

**Problem 8.** Given $w > 0$, $\varepsilon > 0$ and a set $V' := [n]$ of items, with item $i \in V'$ having the value $v_{i}'$ and the weight $w_{i}'$, compute a subset $S \subseteq V'$ maximizing $\sum_{i \in S} v_{i}'$, such that $(1 - \varepsilon)w \leq \sum_{i \in S} w_{i}' \leq (1 + \varepsilon)w$.

Above problem looks similar to the max-knapsack problem but it is different from it in the following way. The total weight of output subset in the max-knapsack problem is required to be less than or equal to the given bound, while the total weight of output subset in Problem 8 is required to be in small range. We could use the algorithm for Problem 8 to obtain bundles of items $V_{i}$, $\forall i \in [m]$, such that the total weight of different bundles are in different ranges.

**Theorem 3.8.** For any $\varepsilon > 0$ and $w > 0$, there exists an algorithm which outputs a subset $S$ having the total value at least the optimal value of Problem 8, and $(1 - 4\varepsilon)w \leq \sum_{i \in S} w_{i}' \leq (1 + 4\varepsilon)w$. The running time of the algorithm is $O\left(\frac{n^{2}}{\varepsilon} \right)$.

**Proof:** The algorithm for the theorem is described in Algorithm 4. The following describes the proof of correctness of Algorithm 4.

**Proof of Correctness.** We claim that the entry $H(i, w'')$, $\forall i \in [n], \forall w'' \in \left[\left\lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right\rceil \right] \cup \{0\}$, indicates the maximum value subset from the first $i$ items of $V'$ having the total rounded weight equal to $w''$. Let $S'$ denote the subset satisfying above property for the entry $H(i, w'')$, $\forall i \in [n] \setminus \{1\}, \forall w'' \in \left[\left\lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right\rceil \right] \cup \{0\}$.

- If $i \in S'$, then $H(i, w'')$ is the sum of the value of $i$ ($v_{i}'$), and the maximum value subset from first $i - 1$ items having the total rounded weight $w''' - w_{i}' (H(i - 1, w''' - w_{i}''))$.
- If $i \notin S'$, then $H(i, w'')$ is equal to the value of maximum value subset from first $i - 1$ items having the total rounded weight $w'''$, i.e. $H(i, w'') = H(i - 1, w''')$.

The recursion in Step 3b captures both of above possibilities. Step 3a does necessary initialization for the recursion in Step 3b.

We prove that the total weight of $S$ is in between $(1 - 4\varepsilon')w$ and $(1 + 4\varepsilon')w$. we have

$$\left\lceil \frac{(1 - 2\varepsilon)n}{\varepsilon} \right\rceil \leq \sum_{i \in S} w_{i}'' \leq \left\lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right\rceil$$

$$\frac{(1 - 2\varepsilon)n}{\varepsilon} - 1 \leq \sum_{i \in S} w_{i}'' \leq \frac{(1 + 2\varepsilon)n}{\varepsilon} + 1.$$
Algorithm 4: Algorithm for Problem 8

**Input:** $w > 0$, $\varepsilon > 0$ and a set $V' = [n]$ of items, with item $i \in V'$ having value $v_i'$ and weight $w_i'$. 

**Output:** A subset $S \subseteq V'$ having total value at least optimal weight of Problem 8, such that 

$$(1 - 4\varepsilon)w \leq \sum_{i \in S} w_i' \leq (1 + 4\varepsilon)w.$$ 

1. Remove all items from $V'$ which have weight more than $(1 + \varepsilon)w$.

2. For each item $i \in V'$, define 

$$w_i'' = \left \lceil \frac{nw_i'}{\varepsilon w} \right \rceil."$$

3. Let $H(i, w'')$ for $w'' \in \left[\left \lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right \rceil \cup \{0\} \right]$ and $i \in [n]$, be the dynamic programming table constructed in the following way.

   (a) $H(1, w''_1) = v_1'$, $H(1, 0) = 0$, $H(1, w'') = -\infty$, $\forall w'' \in \left[\left \lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right \rceil \setminus \{w''_1\} \right]$.

   (b) $\forall i \in [n] \setminus \{1\}$, $\forall w'' \in \left[\left \lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right \rceil \right]$,

   If $w'' < w_i''$, then $H(i, w'') = H(i - 1, w'')$.

   Else,

   $$H(i, w'') := \max \{H(i - 1, w''), H(i - 1, w'' - w_i'') + v_i' \mid w_i'' \leq w''\}.$$ 

4. Output the subset $S$ as follows,

$$\max \left \{ H(n, w'') \mid \left \lfloor \frac{(1 - 2\varepsilon)n}{\varepsilon} \right \rfloor \leq w'' \leq \left \lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right \rceil \right \}.$$ 

Using the definition of $w_i''$ in Step 2, we get

$$(1 - 2\varepsilon)w - \frac{\varepsilon w}{n} \leq \sum_{i \in S} w_i' + \varepsilon w \leq (1 + 3\varepsilon)w + \frac{\varepsilon w}{n},$$

which implies

$$(1 - 4\varepsilon)w \leq \sum_{i \in S} w_i' \leq (1 + 4\varepsilon)w.$$ 

Let $O$ denote the set of items in the optimal solution of Problem 8. Now, we prove that $\left \lfloor \frac{(1 - 2\varepsilon)n}{\varepsilon} \right \rfloor \leq \sum_{i \in O} w_i'' \leq \left \lceil \frac{(1 + 2\varepsilon)n}{\varepsilon} \right \rceil$. Since $S$ is the maximum value subset in the previous range, the above claim will imply that the total value of $S$ is at least the total value of $O$. We know that,

$$(1 - \varepsilon)w \leq \sum_{i \in O} w_i' \leq (1 + \varepsilon)w.$$
Using the definition of \( w_i'' \) in Step 2, we get
\[
\frac{n(1-\varepsilon)}{\varepsilon} - n \leq \sum_{i \in O} w_i'' \leq \frac{(1+\varepsilon)n}{\varepsilon} + n
\]
\[
\frac{n(1-2\varepsilon)}{\varepsilon} \leq \sum_{i \in O} w_i'' \leq \frac{(1+2\varepsilon)n}{\varepsilon}.
\]

Note that the algorithm returns \(-\infty\) if it does not find a subset in Step 4. This is because of the initialization in Step 3a.

**Running time analysis.** The size of the table \( \mathcal{H} \) is \( O\left(\frac{n^2}{\varepsilon}\right) \), and to fill each entry in the table \( \mathcal{H} \), we require \( O(1) \) time. So, the total running time of the algorithm is \( O\left(\frac{n^2}{\varepsilon}\right) \).

**Proof of Theorem 3.7.** The algorithm for theorem is described in Algorithm 5. The algorithm creates bundles of items from \( V_i, \forall i \in [m] \), such that the total weight of each bundle is in different ranges using Theorem 3.8 (Step 2). It stores these bundles in the table \( \mathcal{Y} \). After that the algorithm combines bundles from all categories to obtain the final solution using the dynamic programming table \( \mathcal{Z} \) (Step 3). The total weight of each bundle is represented by some power of \((1+\varepsilon')\) in the tables \( \mathcal{Y} \) and \( \mathcal{Z} \). The algorithm might over calculate at most \( O\left((1+\varepsilon')m\right) \) fraction of total weight in table \( \mathcal{Y} \) (Step 2). So, the total fraction of weight over calculated after combining bundles from all categories in Step 3b is at most \( O\left((1+\varepsilon')m\right) \). This is at most \( O\left((1+\varepsilon)\right) \) because of the choice of \( \varepsilon' \) in Step 1. We describe the formal proof of the algorithm below.

**Properties of \( \mathcal{Y} \).** We claim that the entry \( \mathcal{Y}(i,j), \forall i \in [m], \forall j \in \left[\log_{1+\varepsilon'}\left(\frac{B}{w_{\min}}\right)\right] + m + 2 \cup \{0\} \), indicates the value of a subset of \( V_i \) that satisfies the two properties listed below. The entry of \( \mathcal{Y} \) also corresponds to respective subset. The entry of \( \mathcal{Y} \) could be \(-\infty\), which indicates no subset. We use the notation \( \mathcal{Y}(i,j) \) to indicate both the entry and the subset.

1. If the entry \( \mathcal{Y}(i,j) \) is finite, then the total weight of the corresponding subset is in between \((1 - \frac{4}{10}\varepsilon)(1+\varepsilon')j \) \( w_{\min} \) and \((1 + \frac{4}{10}\varepsilon)(1+\varepsilon')j \) \( w_{\min} \).
2. The total value of the subset corresponding to \( \mathcal{Y}(i,j) \) is at least the total value of any subset of \( V_i \) having the total weight in between \((1 - \frac{2}{10}\varepsilon)(1+\varepsilon')j \) \( w_{\min} \) and \((1 + \frac{2}{10}\varepsilon)(1+\varepsilon')j \) \( w_{\min} \). (Since \( \varepsilon' < \frac{\varepsilon}{10} \), the statement is also true for any subset having the total weight in between \((1+\varepsilon')j \) \( w_{\min} \) and \((1+\varepsilon')j \) \( w_{\min} \)).

The table \( \mathcal{Y} \) is created in step 2 of the Algorithm 5. This step uses Theorem 3.8 for creation of \( \mathcal{Y} \). We get both the properties of \( \mathcal{Y} \) because of the guarantee of Theorem 3.8.

**Properties of \( \mathcal{Z} \).** We claim that the entry \( \mathcal{Z}(i,j), \forall i \in [m], j \in \left[\log_{1+\varepsilon'}\left(\frac{B}{w_{\min}}\right)\right] + m + 2 \cup \{0\} \), indicates the value of the subset of \( \bigcup_{k=1}^{i} V_k \) that satisfies the following three properties. The entry of \( \mathcal{Z} \) also corresponds to the respective subset. The entry of \( \mathcal{Z} \) could be \(-\infty\), which indicates no subset. We use the notation \( \mathcal{Z}(i,j) \) to indicate both the entry and the subset.
Algorithm 5: Algorithm for the $BW^\text{max}$ (Problem 7)

**Input:** The sets $V_i$ of items, $\forall i \in [m]$. $0 \leq l_i^w \leq u_i^w$ for $i \in [m]$, the knapsack capacity $B$ and $\varepsilon > 0$.

**Output:** $S$ having the total value at least the optimal value of $BW^\text{max}$ (Problem 7), such that $(1 - \varepsilon)l_i^w \leq \sum_{r \in S \cap V_i} w_r^{(i)} \leq (1 + \varepsilon)u_i^w$, $\forall i \in [m]$, and the total weight of $S$ is at most $(1 + \varepsilon)B$.

1. Let $\varepsilon' = (1 + \frac{\varepsilon}{10})^{\frac{1}{m+1}} - 1$.

2. Let $Y(i, j)$, $\forall i \in [m]$, $\forall j \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \cup \{0\}$ be the table, where the entry $Y(i, j)$ indicates the value of a subset of $V_i$ that is obtained by Theorem 3.8 by setting $V_i$ as $V'$, $w_{\text{min}} (1 + \varepsilon')^j$ as $w$ and $\frac{\varepsilon}{10}$ as $\varepsilon$ in Theorem 3.8.

3. Let $Z(i, j)$, $\forall i \in [m]$ and $\forall j \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \cup \{0\}$, be the DP table constructed as follows.

   (a)
   \[
   Z(1, j) := \max \left\{ Y(1, j''') \mid j''' \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \cup \{0\} \right. \\
   \left. \quad \& j \geq j''' \& (1 + \varepsilon')^j w_{\text{min}} \leq l_i^w \right\}.
   
   If the set satisfying above condition is empty, then set $Z(1, j)$ to $-\infty$.

   (b) $\forall i \in [m] \setminus \{1\}$ and any $j'$, $j''' \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \cup \{0\}$, we have
   
   \[
   Z(i, j) := \max \left\{ Z(i - 1, j') + Y(i, j''') \mid \left( 1 + \varepsilon' \right)^{j'''} w_{\text{min}} \geq l_i^w \& \left( 1 + \varepsilon' \right)^{j'} \geq \left( 1 + \varepsilon' \right)^{j''} + \left( 1 + \varepsilon' \right)^{j'''} w_{\text{min}} \leq u_i^w \right\}.
   
   If the set satisfying above condition is empty, then set $Z(i, j)$ to $-\infty$.

4. Output the subset $S$ as follows,

   \[
   \max \left\{ Z(m, j) \mid j \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \cup \{0\} \right\}.
   
1. If the entry $Z(i, j)$ is finite, then $\sum_{k=1}^{i} \sum_{r \in Z(i,j) \cap V_k} w_r^{(k)} \leq (1 + \frac{\varepsilon}{10}) \left( 1 + \varepsilon' \right)^j w_{\text{min}}$.

2. If the entry $Z(i, j)$ is finite, then $(1 - \frac{\varepsilon}{10})^{-1} l_k^w \leq \sum_{r \in V_k \cap Z(i,j)} w_r^{(k)} \leq (1 + \frac{\varepsilon}{10}) u_k^w$, $\forall k \in [i]$.

3. For all $O' \subseteq \bigcup_{k=1}^{i} V_k$ having the total weight at most $(1 + \varepsilon')^{i-2} w_{\text{min}}$, and $l_k^w \leq \sum_{r \in V_k \cap O'} w_r^{(k)} \leq u_k^w$, $\forall k \in [i]$, the total value of $Z(i, j)$ is at least the total value of $O'$.

We want that for any $O'$ having the total weight at most $B$, there exists some subset in $Z$ for which
Property 3 is applicable. This makes the inequality \((1 + \varepsilon)^j \min w \geq B\) in Property 3 should be satisfied for some \(j\). So, the maximum value of \(j\) is set to \(\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2.

Steps 3a and 3b of the algorithm chooses the subset \(Y(i, j'')\) that satisfies the inequalities \((1 + \varepsilon)^j \min w \geq l_i^w\) and \((1 + \varepsilon)^j \min w \leq u_i^w\). By Property 1 of \(Y\), the total weight of \(Y(i, j'')\) is in between \((1 - \frac{1}{2} \varepsilon)(1 + \varepsilon)^{-1}l_i^w\) and \((1 + \frac{1}{2} \varepsilon) u_i^w\). This proves that the subset corresponding to finite entry \(Z(i, j)\) satisfies Property 2. The subset corresponding to finite entry \(Z(1, j), \forall j \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\right] \cup \{0\}\), will satisfy Property 1. This is true because of Property 1 of \(Z\) and the condition \(j'' \leq j\) in Step 3a for selecting subset \(Z(1, j'')\). Because of the condition \((1 + \varepsilon)^j \geq (1 + \varepsilon)^j ' + (1 + \varepsilon)^j ''\) in Step 3b of the algorithm and Property 1 of \(Z\), the subset corresponding to finite entry \(Z(i, j), \forall i \in [m] \setminus \{1\}, \forall j \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\right] \cup \{0\}\), will satisfy Property 1.

We prove that the entries in the table \(Z\) will satisfy Property 3 by the induction. In the base case, the subset \(Z(1, j), \forall j \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\right] \cup \{0\}\), satisfies Property 3 of \(Z\). We can prove this by the following argument. For any \(O' \subseteq V_1\), which satisfies the fairness condition for category 1 having the total weight in between \(w_{\min} (1 + \varepsilon)^j \min w \) and \((1 + \varepsilon)^j \min w \), for some \(j'' \in \left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\) \(
\cup \{0\}\), the total value of \(Y(1, j'')\) is at least the total value of \(O'\) (Property 2 of \(Y\)). Also, the condition \(j'' \leq j\) in Step 3a will be satisfied for all \(j \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\right] \cup \{0\}\), such that the total weight of \(O'\) is at most \(w_{\min} (1 + \varepsilon)^j \min w \). So, the subset \(Y(1, j'')\) is feasible for all such \(j\) in Step 3a. For any \(i \in [m] \setminus \{1\}\), assume the hypothesis that the subsets \(Z(i - 1, j), \forall j \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\right] \cup \{0\}\), satisfy Property 3. We will prove by the induction that for any \(O' \subseteq \bigcup_{k=1}^i V_k\) that satisfies the fairness conditions for all categories up to \(i\) and for all \(j \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] + m + 2\right] \cup \{0\}\), such that the total weight of \(O'\) is at most \((1 + \varepsilon)^{j - i - 2} \min w\), the subset \(Z(i, j)\) satisfies Property 3. For any \(j^* \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] \cup \{0\}\) that satisfies the following inequality:

\[
(1 + \varepsilon^j)^* \min w \leq \sum_{k=1}^{i-1} \sum_{r \in O' \cap V_k} w^{(k)}_r \leq (1 + \varepsilon^j)^{i-1} \min w, \tag{8}
\]

the induction hypothesis implies the following:

\[
\sum_{k=1}^{i-1} \sum_{r \in Z(i - 1, j') \cap V_k} w^{(k)}_r \geq \sum_{k=1}^{i-1} \sum_{r \in O' \cap V_k} w^{(k)}_r, \tag{9}
\]

where \(j' = j^* + i + 2\). Property 2 of \(Y\) implies that there exists a \(j'' \in \left[\left[\log_{1 + \varepsilon} \left(\frac{B}{\min w}\right)\right] \cup \{0\}\) such that,

\[
(1 + \varepsilon^j)^j \min w \leq \sum_{k \in V_i \cap O'} w^{(i)}_k \leq (1 + \varepsilon^j)^{j+1} \min w, \tag{10}
\]

and

\[
\sum_{r \in Y(i, j'')} v^{(i)}_r \geq \sum_{r \in O' \cap V_i} v^{(i)}_r. \tag{11}
\]

Since \(O'\) satisfies the fairness constraints for category \(i\), \(j''\) satisfies inequalities \((1 + \varepsilon)^{j'' + 1} \min w \geq l_i^w\) and \((1 + \varepsilon)^{j''} \min w \leq u_i^w\) in Step 3b. We claim that the inequality \((1 + \varepsilon)^j \geq (1 + \varepsilon)^j' + (1 + \varepsilon)^j''\) is satisfied
in Step 3b for any \( j \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\min}} \right) \right] + m + 2 \cup \{0\} \), such that the total weight of \( O' \) is at most \( w_{\min}(1+\varepsilon')^{-i-2} \). This is true because the minimum weight of \( O' \) is at least \( w_{\min}(1+\varepsilon')^{-j-2} \). So, the pair of subsets \( Z(i-1, j') \) and \( Y(i, j'') \) is feasible in Step 3b for all such \( j \). Also, the total value of union of \( Z(i-1, j') \) and \( Y(i, j'') \) is at least the total value of \( O' \) (Inequality 9 and Inequality 11). So, Property 3 of \( Z \) is satisfied for all such \( j \).

By property 3 of \( Z \), there exists a subset \( Z(m, j) \), for some \( j \in \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\min}} \right) \right] + m + 2 \cup \{0\} \), that satisfies the condition in Property 3 for some optimal solution. This proves that the total value of \( S \) (Step 4) is at least the optimal value. By property 1 of \( Z \), the total weight of \( S \) (Step 4) is at most \((1 + \frac{1}{10}\varepsilon) (1 + \varepsilon')^{m+3} B \). By Step 1 of the algorithm, this number is \((1 + \frac{1}{10}\varepsilon) (1 + \frac{1}{10}\varepsilon) B < (1 + \varepsilon) B \).

If the entry \( Z(m, j) \) is finite, Property 2 of \( Z \) implies that the total weight of items from \( V_i \cap Z(m, j) \) is in between \((1 - \frac{1}{10}\varepsilon) (1 + \varepsilon')^{-1} l_i^w \) and \((1 + \frac{1}{10}\varepsilon) u_i^w \), \( \forall i \in [m] \). Since \( \varepsilon' \leq \frac{\varepsilon}{10} \), the total weight of items from \( V_i \cap Z(m, j) \) is at least \((1 - \frac{1}{10}\varepsilon) (1 + \frac{1}{10}\varepsilon) l_i^w > (1 - \varepsilon) l_i^w \), and is at most \((1 + \frac{1}{10}\varepsilon) u_i^w < (1 + \varepsilon) u_i^w \).

**Running time Analysis:** The size of the table \( Y \) is \( O \left( m^2 \log_{1+\varepsilon'} \left( \frac{B}{w_{\min}} \right) \right) \), and we require \( O \left( \frac{n^2}{\varepsilon} \right) \) time to fill each entry. So, the total time required to build the table \( Y \) is \( O \left( \frac{n^2 m^2 \log_{1+\varepsilon'} \left( \frac{B}{w_{\min}} \right)}{\varepsilon} \right) \). The total time required to build the table \( Z \) from the table \( Y \) is \( O \left( m^4 \log_{1+\varepsilon'} \left( \frac{B}{w_{\min}} \right) \right) \). So, the total running time is \( O \left( \frac{n^2 m^4 \log_{1+\varepsilon'} \left( \frac{B}{w_{\min}} \right)}{\varepsilon} \right) \). \( \square \)

4 **Fair min-knapsack**

The classical min-knapsack problem is to find the packing of minimum weight having the total value at least the given lower bound.

We will consider the same fairness notions in the min-knapsack as we have considered for the max-knapsack case. We give algorithms for these notions of fairness in the min-knapsack case in this section.

4.1 **Fairness based on number of items**

**Problem 9** \((B N^{min})\). Given a set of items, each belonging to one of \( m \) categories, and numbers \( l_i^n \) and \( u_i^n \) for category \( i \), \( \forall i \in [m] \), the problem is to find a subset that minimizes the total weight, such that the number of items from category \( i \) is between \( l_i^n \) and \( u_i^n \), \( \forall i \in [m] \), and the total value of the subset is at least the given value lower bound \( L \).

We prove the following theorem.

**Theorem 4.1.** For any \( \varepsilon > 0 \), there exists a \((1 + \varepsilon)\)-approximation algorithm for \(B N^{min} \) (Problem 9) with running time \( O \left( \frac{(1+\varepsilon)^2 n m^2 \log_{1+\varepsilon'} \left( \frac{N_{max}}{w_{\min}} \right)}{\varepsilon^2} \right) \), \( w_{\min} := \min \left\{ w_j^{(i)} \mid i \in [m] \& j \in V_i \& w_j^{(i)} > 0 \right\} \) and \( w_{\max} := \max \left\{ w_j^{(i)} \mid i \in [m] \& j \in V_i \& w_j^{(i)} > 0 \right\} \).

**Proof.** The algorithm approximately guesses the optimal weight in the beginning (Step 1). According to the guess, it rounds the weights of all items so that the rounded weights lie in a small range (Step 2). Then
it creates bundles of items from $V_i$, $\forall i \in [m]$, having different rounded weights and cardinality (Step 3). It uses the dynamic programming table $A$ for this. After that the algorithm combines the bundles from all categories to obtain the final solution using the dynamic programming table $B$ (Step 4). The algorithm is described in Algorithm 6. The following describes the proof of correctness of Algorithm 6.

**Property of $A$.** We claim that $A(i, j, w, t)$, $\forall i \in [m]$, $\forall t \in V_i \cup \{0\}$, $j \in V_i$, $\forall w \in \left\lceil \frac{N(1+2\epsilon)}{\epsilon} \right\rceil \cup \{0\}$, denotes the value of a maximum value subset of cardinality $t$ from first $j$ items of $V_i$ having total rounded weight $w$. Let $S'$ be the subset of $V_i$ satisfying above property for the entry $A(i, j, w, t)$, $\forall i \in [m]$, $\forall j \in V_i \setminus \{1\}$, $\forall t \in V_i \cup \{0\}$, $\forall w \in \left\lceil \frac{N(1+2\epsilon)}{\epsilon} \right\rceil \cup \{0\}$.

- If $j \in S'$, then $A(i, j, w, t)$ is the sum of value of $j$ ($v_j^{(i)}$) and the value of maximum value subset from first $j - 1$ items of $V_i$ having cardinality $t - 1$ and the rounded weight $w - w_j^{(i)}$, ($A(i, j - 1, w - w_j^{(i)}, t - 1)$), which is equal to $A(i, j - 1, w - w_j^{(i)}, t - 1) + v_j^{(i)}$.
- If $j \notin S'$, then $A(i, j, w, t)$ is equal to the value of maximum value subset from first $j - 1$ items of $V_i$ having cardinality $t$ and the rounded weight $w$, which is equal to $A(i, j - 1, w, t)$.

The recursion in Step 3b captures both of above possibilities. The Step 3a initializes base case for the recursion in Step 3b. The entry of $A$ also corresponds to respective subset. The entry of $A$ could be $-\infty$, which indicates no subset. We use the notation $A(i, j, w, t)$ to indicate both the entry and the subset.

**Property of $B$.** We claim that $B(i, w)$, $\forall i \in [m]$, $\forall w \in \left\lceil \frac{N(1+2\epsilon)}{\epsilon} \right\rceil \cup \{0\}$, denotes the value of a maximum value subset of $\bigcup_{j=2}^i V_j$ having the total rounded weight $w$, such that the fairness constraints are satisfies for all categories up to $i$. Let $S'$ be the subset of $\bigcup_{j=1}^i V_j$ satisfying above property for the entry $B(i, w)$, $\forall i \in [m] \setminus \{1\}$, $\forall w \in \left\lceil \frac{N(1+2\epsilon)}{\epsilon} \right\rceil \cup \{0\}$. If $\sum_{j \in S' \setminus \{i\}} w_j^{(i)} = w_r$, then $B(i, w)$ is sum of values of maximum value subset of $\bigcup_{j=1}^{i-1} V_j$ having the total rounded weight $w - w_r$ that satisfies fairness condition for all categories up to $i - 1$ ($B(i - 1, w - w_r)$), and the maximum value subset of $V_i$ having the total rounded weight $w_r$ that satisfies fairness condition for category $i$ ($A(i, |V_i|, w_r, t)$) such that $l_i^r \leq t \leq u_i^r$. The recursion in Step 4b captures this for all possible $w_r$. The entry of $B$ also corresponds to respective subset. The entry of $B$ could be $-\infty$, which indicates no subset. We use the notation $B(i, w)$ to indicate both the entry and the subset.

If $\text{OPT}$ is the weight of the optimal solution of $BN_{\text{min}}^{\text{min}}$ (Problem 9), then we can guess $b_{\text{opt}}$ such that $b_{\text{opt}} \leq \text{OPT} \leq (1+\epsilon)b_{\text{opt}}$, in time $O\left(\log_{1+\epsilon} \left(\frac{Nw_{\text{max}}}{w_{\text{min}}}\right)\right)$. This is true because of the inequality $w_{\text{min}} \leq \text{OPT} \leq Nw_{\text{max}}$. $S$ (Step 5) will be fair for all categories because of the property of $B$. Also, $S$ will satisfy the knapsack value lower bound constraint because of the condition in the Step 5. Theorem 4.2 proves this theorem.

**Running time analysis:** There are $O\left(\log_{1+\epsilon} \left(\frac{Nw_{\text{max}}}{w_{\text{min}}}\right)\right)$ possible values of $b_{\text{opt}}$ in Step 1. The size of the table $A$ is $O\left(\frac{(1+\epsilon)mN^2}{\epsilon}\right)$, and we require $O(1)$ time to fill each entry in it. The size of the table $B$ is $O\left(\frac{(1+\epsilon)mN}{\epsilon}\right)$. We require $O\left(\frac{(1+\epsilon)mN}{\epsilon}\right)$ time to fill each entry of the table $B$. So, the total time required by the algorithm is $O\left(\frac{(1+\epsilon)^2n^3m^3}{\epsilon^2} \log_{1+\epsilon} \left(\frac{Nw_{\text{max}}}{w_{\text{min}}}\right)\right)$. 

\[ \square \]
Algorithm 6: Algorithm for $BN^{\min}$ (Problem 9)

**Input:** The sets $V_i$ of items, $\forall i \in [m]$, $0 \leq l_i^n \leq u_i^n$ for $i \in [m]$, and $\epsilon > 0$, and value lower bound $L$

**Output:** $S$ having the total weight at most $1 + \epsilon$ times the weight of the optimal solution of Problem 9, such that $l_i^n \leq |S \cap V_i| \leq u_i^n \forall i \in [m]$ and the total value of of $S$ is at least the given bound $L$.

1. If $OPT$ is the weight of the optimal solution, then we can guess $b_{opt}$ such that
   
   $b_{opt} \leq OPT \leq (1 + \epsilon)b_{opt}$.

2. $\forall i \in [m]$ and $\forall j \in V_i$, let $w^{(i)}_j := \left\lfloor \frac{w^{(i)}_j \cdot N}{\epsilon b_{opt}} \right\rfloor$.

3. Let $A(i, j, w, t), \forall i \in [m], \forall j \in V_i, \forall t \in V_i \cup \{0\}, \forall w \in \left[\left\lfloor \frac{N(1+2\epsilon)}{\epsilon} \right\rfloor \right] \cup \{0\}$, be the dynamic programming table constructed in the following way,

   (a) $\forall i \in [m], A(i, 1, w^{(i)}_1, 1) = v^{(i)}_1, A(i, 1, 0, 0) = 0, A(i, 1, w, .) = -\infty, \forall w \in \left[\left\lfloor \frac{N(1+2\epsilon)}{\epsilon} \right\rfloor \right] \setminus \{w^{(i)}_1\}$
   
   (b) $\forall i \in [m]$ and $\forall j \in V_i \setminus \{1\}, \forall t \in V_i \cup \{0\}, \forall w \in \left[\left\lfloor \frac{N(1+2\epsilon)}{\epsilon} \right\rfloor \right] \cup \{0\}$.

   If $w < w^{(i)}_j$ or $t = 0$, then $A(i, j, w, t) = A(i, j - 1, w, t)$.

   Else ,

   $A(i, j, w, t) := \max \left\{ A(i, j - 1, w - w^{(i)}_j, t - 1) + v^{(i)}_j, A(i, j - 1, w, t) \right\}$

   $\forall 0 \leq w^{(i)}_j \leq w \& w^{(i)}_j \in \left[\left\lfloor \frac{N(1+2\epsilon)}{\epsilon} \right\rfloor \right] \cup \{0\}$.

4. Let $B(i, w), \forall i \in [m], \forall w \in \left[\left\lfloor \frac{N(1+2\epsilon)}{\epsilon} \right\rfloor \right] \cup \{0\}$ be another dynamic programming table.

   (a)

   $B(1, w) := \max \{ A(1, |V_1|, w, t) | l_1^n \leq t \leq u_1^n \}$.

   (b) For $i \in [m] \setminus \{1\}$,

   $B(i, w) := \max \{ B(i - 1, w_r) + A(i, |V_i|, w - w_r, t) | l_i^n \leq t \leq u_i^n \}$

   $w_r \leq w \& w_r \in \left[\left\lfloor \frac{N(1+2\epsilon)}{\epsilon} \right\rfloor \right] \cup \{0\}$.

5. Output the subset $S$ as follows,

   $\arg\min_w \{ B(m, w) | B(m, w) \geq L \}$.
Theorem 4.2. If $\text{OPT}$ is the weight corresponding to the optimal solution of $\text{BN}^{\text{min}}$ (Problem 9) and $b_{\text{opt}}$ satisfies the inequality $b_{\text{opt}} \leq \text{OPT} \leq (1 + \varepsilon)b_{\text{opt}}$, then the total weight of set returned in Step 5 of Algorithm 6 is at most $(1 + \varepsilon)\text{OPT}$.

Proof. Let $O \subseteq \bigcup_{i=1}^{m} V_i$ be the set of items in the optimal solution, and $S \subseteq \bigcup_{i=1}^{m} V_i$ be the set of items selected by Step 5 of Algorithm 6. The total value of $O$ and $S$ is at least $L$ (Step 5). Because of the rounding in Step 2 and the inequality $\text{OPT} \leq (1 + \varepsilon)b_{\text{opt}}$, the total rounded weight of $O$ is in range $\left[\left\lceil \frac{N(1+2\varepsilon)}{\varepsilon} \right\rceil \right] \cup \{0\}$. Since $S$ is the subset with total value at least $L$ having minimum rounded weight in the previous range (Step 5), the total rounded weight of $S$ is at most the total rounded weight of $O$.

\[
\sum_{i=1}^{m} \sum_{j \in V_i \cap O} w_{j}^{(i)} \geq \sum_{i=1}^{m} \sum_{j \in V_i \cap S} w_{j}^{(i)}. \tag{12}
\]

As per the Step 2 of Algorithm 6, for all $i \in [m]$ and $j \in V_i$,

\[
\frac{\varepsilon b_{\text{opt}}(w_{j}^{(i)}) - 1}{N} = \frac{\varepsilon b_{\text{opt}}w_{j}^{(i)}}{N} \leq w_{j}^{(i)} \leq \frac{\varepsilon b_{\text{opt}}(w_{j}^{(i)})}{N}. \tag{13}
\]

So we get,

\[
\sum_{i=1}^{m} \sum_{j \in V_i \cap O} w_{j}^{(i)} \geq \frac{\varepsilon b_{\text{opt}}}{N} \left( \sum_{i=1}^{m} \sum_{j \in V_i \cap O} (w_{j}^{(i)}) - 1 \right) \tag{Inequality 13}
\]

\[
\geq \frac{\varepsilon b_{\text{opt}}}{N} \left( \sum_{i=1}^{m} \sum_{j \in V_i \cap O} w_{j}^{(i)} \right) - \varepsilon b_{\text{opt}} \left( \sum_{i=1}^{l} \sum_{j \in V_i \cap S} 1 \leq N \right) \tag{Inequality 12}
\]

\[
\geq \frac{\varepsilon b_{\text{opt}}}{N} \left( \sum_{i=1}^{m} \sum_{j \in V_i \cap S} w_{j}^{(i)} \right) - \varepsilon b_{\text{opt}} \tag{Inequality 13}
\]

If the guess of $b_{\text{opt}}$ is correct, then we have $b_{\text{opt}} \leq \text{OPT}$. So, $\left( \sum_{i=1}^{m} \sum_{j \in V_i \cap S} w_{j}^{(i)} \right) \leq (1 + \varepsilon)\text{OPT}$. \qed

4.2 Fairness based on bound on value

Problem 10 ($\text{BV}^{\text{min}}$). Given a set of items, each belonging to one of $m$ categories, and numbers $l_{i}^{u}$ and $u_{i}^{u}$ for category $i$, $\forall i \in [m]$, the problem is to find a subset that minimizes the total weight, such that the total value of items from category $i$ is between $l_{i}^{u}$ and $u_{i}^{u}$, $\forall i \in [m]$, and the total value of the subset is at least the given value lower bound $L$.  

23
We prove that it is NP-hard to obtain a feasible solution of an instance of $BV_{min}$ (Problem 10).

**Theorem 4.3.** There is no polynomial time algorithm which outputs a feasible solution of $BV_{min}$ (Problem 10), assuming $P \neq NP$.

**Proof.** Given an instance of subset sum (Problem 3), we can construct an instance of $BV_{min}$ (Problem 10) in the following way. Let $m = 1$, $l^*_1 = 0$, $u^*_1 = \frac{1}{8}$ and $L = \frac{1}{2}$. The set $V_1$ contains items which correspond to the numbers in $I$. An item corresponding to some $a \in I$ has a value and a weight equal to $a$. This proves that if we have an algorithm that outputs a feasible solution of $BV_{min}$ (Problem 10), then we can solve subset sum (Problem 3). But assuming $P \neq NP$, this is not possible. □

Theorem 4.3 implies that there does not exists polynomial time algorithm for $BV_{min}$ (Problem 10). We give here an algorithm for $BV_{min}$ (Problem 10) that might violates fairness constraints and knapsack constraint by a small amount.

**Theorem 4.4.** For any $\varepsilon > 0$, there exists an algorithm for $BV_{min}$ (Problem 10) that outputs a set $S$ having the total weight at most the optimal weight of $BV_{min}$ (Problem 10), such that $(1 - \varepsilon)l^*_k \leq \sum_{i \in S \cap V_k} v_i(k) \leq (1 + \varepsilon)u^*_k$, $\forall k \in [m]$, and the total value of items in $S$ is at least $(1 - \varepsilon)L$. The running time of the algorithm is $O \left( \frac{n^2 m^4 \log^2 (\frac{1}{\varepsilon} \frac{v_{max}}{v_{min}})}{\varepsilon} \right)$, where $v_{min} := \min \{ v_j(i) \mid i \in [m] \land j \in V_i \land v_j(i) > 0 \}$, and $v_{max} := \max \{ v_j(i) \mid i \in [m] \land j \in V_i \}$.

**Proof.** The algorithm for the theorem is described in Algorithm 7. The algorithm creates bundles of items from $V_i$, $\forall i \in [m]$, such that the total value of each bundle is in different ranges using Theorem 3.5 (Step 2). It stores these bundles in the table $W$. After that the algorithm combines bundles from all categories to obtain the final solution using the dynamic programming table $X$ (Step 3). The total value of a bundle is represented by some power of $(1 + \varepsilon')$ in the tables $W$ and $X$. The table $W$ represents at least $O \left( \frac{1}{(1 + \varepsilon')} \right)$ fraction of the total value of any bundle (Step 2). So, the fraction of total value represented by the table $X$ after combining the bundles from all categories in Step 3b is at least $O \left( \frac{1}{(1 + \varepsilon')} \right)$. This is at least $O \left( \frac{1}{(1 + \varepsilon)} \right)$ because of the choice of $\varepsilon'$ in Step 1. We describe the formal proof of the algorithm below.

The tables $W$ (Step 2) and $X$ (Step 3) are same as the tables created by the algorithm (Algorithm 3) in Theorem 3.4. As proved in Theorem 3.4, the entries in these tables satisfy following properties.

**Properties of $W$.** The entry $W(i, j)$, $\forall i \in [m]$, $j \in W_{range}$, indicates the weight of the subset of $V_i$ that satisfies the two properties listed below. The entry of $W$ also corresponds to respective subset. The entry of $W$ could be $\infty$, which indicates no subset. We use the notation $W(i, j)$ to indicate both the entry and the subset.

1. If the entry $W(i, j)$ is finite, then the total value of the corresponding subset is in between $(1 - \frac{4}{10^2} \varepsilon) (1 + \varepsilon')^j v_{min}$ and $(1 + \frac{4}{10^2} \varepsilon) (1 + \varepsilon')^j v_{min}$.

2. The total weight of the subset corresponding to $W(i, j)$ is at most the total weight of any subset of $V_i$ having the total value in between $(1 - \frac{4}{10^2} \varepsilon) (1 + \varepsilon')^j v_{min}$ and $(1 + \frac{4}{10^2} \varepsilon) (1 + \varepsilon')^j v_{min}$ (Since $\varepsilon' < \frac{\varepsilon}{10^2}$, the statement is also true for any subset having the total value in between $(1 + \varepsilon')^j v_{min}$ and $(1 + \varepsilon')^j v_{min}$).
**Algorithm 7: Algorithm for the $BV_{\min}$ (Problem 10)**

**Input:** The sets $V_i$ of items, $\forall i \in [m]$, $0 \leq l_i^v \leq u_i^v$, $\forall i \in [m]$, the value lower bound $L$ and $\varepsilon > 0$.

**Output:** The subset $S$ of items, having the total weight at most the optimal weight of $BV_{\min}$ (Problem 10), such that $(1 - \varepsilon)l_i^v \leq \sum_{r \in S \cap V_i} (1 + \varepsilon)u_i^v$, $\forall i \in [m]$, and the total value of $S$ is at least $(1 - \varepsilon) L$.

1. Let $\varepsilon' = \left(1 + \frac{\varepsilon}{10}\right)^{1/m} - 1$. Also let $W_{\text{range}} := \left[\left[\log_{1+\varepsilon'}\left(\frac{N_{\text{max}}}{v_{\min}}\right)\right]\right] \cup \{0\}$ and $X_{\text{range}} := \left[\left[\log_{1+\varepsilon'}\left(\frac{N_{\text{max}}}{v_{\min}}\right)\right]\right] \cup \{-m - 2, -m - 3, \ldots, -2, -1, 0\}$.

2. Let $W(i, j), \forall i \in [m], \forall j \in W_{\text{range}}$ be the table where the entry $W(i, j)$ indicates the weight of a subset of $V_i$ that is obtained by Theorem 3.5 by setting $V_i$ as $V'$, $v_{\min} (1 + \varepsilon')^j$ as $v$ and $\frac{\varepsilon}{10}$ as $\varepsilon$ in Theorem 3.5.

3. Let $X(i, j), \forall i \in [m], \forall j \in X_{\text{range}}$, be the DP table constructed as follows.

(a) 
\[ X(1, j) := \min \left\{ W(1, j'') \mid j'' \geq j \& j'' \in W_{\text{range}} \& \left(1 + \varepsilon\right)^{j''} v_{\min} \geq l_i^v \& \left(1 + \varepsilon\right)^{j''} v_{\min} \leq u_i^v \right\}. \]

If the set satisfying above condition is empty, then set $X(1, j)$ to $\infty$.

(b) $\forall i \in [m] \setminus \{1\}, \forall j' \in X_{\text{range}}$ and $\forall j'' \in W_{\text{range}}$, we have

\[ X(i, j) := \min \left\{ X(i - 1, j') + W(i, j'') \mid \left(1 + \varepsilon\right)^{j''} v_{\min} \geq l_i^v \& \left(1 + \varepsilon\right)^{j''} v_{\min} \leq u_i^v \& \left(1 + \varepsilon\right)^{j'} \leq \left(1 + \varepsilon\right)^j + \left(1 + \varepsilon\right)^j'' \right\}. \]

If the set satisfying above condition is empty, then set $X(i, j)$ to $\infty$.

4. Output the subset $S$ as follows,

\[ \min \left\{ X(m, j) \mid v_{\min} (1 + \varepsilon)^{j+3} \geq L \right\}. \]

**Properties of $X$**. The entry $X(i, j), \forall i \in [m], \forall j \in X_{\text{range}}$, indicates the weight of a subset of $\bigcup_{k=1}^i V_k$ that satisfies the three properties mentioned below. The entry of $X$ also corresponds to the respective subset. The entry of $X$ could be $\infty$, which indicates no subset. We use the notation $X(i, j)$ to indicate both the entry and the subset.

1. If the entry $X(i, j)$ is finite, then $\sum_{k=1}^i \sum_{r \in X(i, j) \cap V_k} u_r^{(k)} \geq \left(1 - \frac{4}{10}\varepsilon\right) \left(1 + \varepsilon\right)^j v_{\min}$.

2. If the entry $X(i, j)$ is finite, then $\left(1 - \frac{4}{10}\varepsilon\right) \left(1 + \varepsilon\right)^{-1} l_i^v \leq \sum_{r \in X(i, j) \cap V_k} u_r^{(k)} \leq \left(1 + \frac{4}{10}\varepsilon\right) u_i^v, \forall k \in [i]$. 

25
3. For all $O' \subseteq \bigcup_{k=1}^{\beta} V_k$ having the total value at least $(1 + \varepsilon')^{j+i+2} v_{\text{min}}$, and $u_{k}^{(i)} \leq \sum_{r \in O' \cap V_k} u_{r}^{(k)} \leq u_{k}^{(i)}$, for all $k \in [i]$, the total weight of subset $X(i, j)$ is at most the total weight of $O'$.

The total value of any optimal subset is at least $L$. Because of the condition $v_{\text{min}} (1 + \varepsilon)^{j+m+3} \geq L$ in Step 4, we can apply Property 3 of $X$ to any optimal subset and $S$. This proves that the total weight of $S$ is at most the total weight of an optimal solution. By Property 1 of $X$ and the condition $v_{\text{min}} (1 + \varepsilon)^{j+m+3} \geq L$ in Step 4, the total value of $S$ is at least $(1 - \frac{4}{10} \varepsilon) (1 + \varepsilon')^{-m-3} L$. By Step 1 of the algorithm, this number is at least

$$\left(1 - \frac{4}{10} \varepsilon\right) \left(1 + \frac{1}{10} \varepsilon\right)^{-1} L > \left(1 - \frac{4}{10} \varepsilon\right) \left(1 - \frac{1}{10} \varepsilon\right) L > (1 - \varepsilon)L.$$\[\]

If the entry $X(m, j)$ is finite, Property 2 of $X$ implies that the total value of items from $V_i \cap X(m, j)$ is in between $(1 - \frac{4}{10} \varepsilon) (1 + \varepsilon')^{-1} l_i^{v}$ and $(1 + \frac{4}{10} \varepsilon) u_i^{v}$ for all $i \in [m]$. Since $\varepsilon' \leq \frac{\varepsilon}{10}$, the total value of items from $V_i \cap X(m, j)$ is at least $(1 - \frac{4}{10} \varepsilon) (1 + \frac{4}{10} \varepsilon) (1 - \frac{1}{10} \varepsilon) l_i^{v} > (1 - \varepsilon l_i^{v}$, and is at most $(1 + \frac{4}{10} \varepsilon) u_i^{v} < (1 + \varepsilon) u_i^{v}$.

The algorithm is similar to the algorithm (Algorithm 3) in Theorem 3.4. The only difference is the output step (Step 4). The running time analysis of Algorithm 3 is also applicable to this algorithm.

\[\]

4.3 Fairness based on bound on weight

**Problem 11** $(BW_{\text{min}})$. Given a set of items, each belonging to one of $m$ categories, and a lower bound $l_i^{w}$ and an upper bound $u_i^{w}$ such that $0 \leq l_i^{w} \leq u_i^{w}$ for each category $i \in [m]$, the goal is to find a subset that minimizes the total weight, such that the total weight of items from each category $i$ is in between the given bounds $l_i^{w}$ and $u_i^{w}$, $\forall i \in [m]$, and the total value of the subset is at least the given bound $L$.

We prove that it is NP-hard to obtain the feasible solution of an instance of $BW_{\text{min}}$ (Problem 11).

**Theorem 4.5.** There is no polynomial time algorithm which outputs a feasible solution of $BW_{\text{min}}$ (Problem 11), assuming $P \neq \text{NP}$.

**Proof.** Given an instance of subset sum (Problem 3), we can construct an instance of $BW_{\text{min}}$ (Problem 11) in following way. Let $m = 1$, $l_1^{w} = \frac{1}{2}$, $u_1^{w} = 1$ and $L = \frac{1}{2}$. The set $V_1$ contains items which correspond to the numbers in $I$. An item corresponding to some $a \in I$ has a value and a weight equal to $a$. This proves that if we have an algorithm that outputs a feasible solution of $BW_{\text{min}}$ (Problem 11), then we can solve subset sum (Problem 3). But assuming $P \neq \text{NP}$, this is not possible.

Theorem 4.3 implies that there does not exists polynomial time algorithm for $BW_{\text{min}}$ (Problem 11).

We give here an algorithm for $BW_{\text{min}}$ (Problem 11) that might violates fairness constraints for a category by a small amount.

**Theorem 4.6.** For any $\varepsilon > 0$, there exists an algorithm for $BW_{\text{min}}$ (Problem 11) that outputs a solution $S$ whose total weight is at most $1 + \varepsilon$ times of the total weight of an optimal solution, and $(1 - \varepsilon) l_i^{w} \leq \sum_{j \in V_i \cap S} w_j^{(i)} \leq (1 + \varepsilon) u_i^{w}$, $\forall i \in [m]$, and the total value of $S$ is at least $L$. The running time of the algorithm is $O\left(\frac{n^2 m^4 \log(1 + \varepsilon)}{\varepsilon} \frac{\text{w}_{\text{min}}}{\text{w}_{\text{min}}}\right)$. Here $w_{\text{min}} := \min\{w_j^{(i)} | i \in [m] \land j \in V_i \land w_j^{(i)} > 0\}$ and $W = \sum_{i=1}^{m} \sum_{j \in V_j} w_j^{(i)}$.  

26
Proof. The algorithm for theorem is described in Algorithm 8. The algorithm creates bundles of items from 
\( V_i, \forall i \in [m], \) such that the total weight of each bundle is in different ranges using Theorem 3.8 (Step 2). It 
stores these bundles in the table \( Y. \) After that the algorithm combines bundles from all categories to obtain 
the final solution using the dynamic programming table \( Z \) (Step 3). The total weight of each bundle is 
represented by some power of \((1 + \varepsilon')\) in the tables \( Y \) and \( Z. \) The algorithm might over calculate at most 
\( O((1 + \varepsilon')) \) fraction of total weight in table \( Y \) (Step 2). So, the total fraction of weight over calculated 
after combining bundles from all categories in Step 3b is at most \( O((1 + \varepsilon')^m). \) This is at most \( O((1 + \varepsilon)) \) 
because of the choice of \( \varepsilon' \) in Step 1. We describe the formal proof of the algorithm below.

The tables \( Y \) (Step 2) and \( Z \) (Step 3) are same as the tables created by the algorithm (Algorithm 5) in 
Theorem 3.7. As proved in Theorem 3.7, the entries in these tables satisfy following properties.

Properties of \( Y. \) The entry \( Y(i, j), \forall i \in [m], \forall j \in \left[ \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \right] \cup \{0\}, \) indicates the 
value of a subset of \( V_i \) that satisfies the two properties listed below. The entry of \( Y \) also corresponds 
to respective subset. The entry of \( Y \) could be \( -\infty, \) which indicates no subset. We use the notation \( Y(i, j) \) to 
indicate both the entry and the subset.

1. If the entry \( Y(i, j) \) is finite, then the total weight of the corresponding subset is in between \( (1 - \frac{1}{10}\varepsilon) (1 + \varepsilon')^j w_{\text{min}} \) and \( (1 + \frac{1}{10}\varepsilon) (1 + \varepsilon')^j w_{\text{min}}. \)

2. The total value of the subset corresponding to \( Y(i, j) \) is at least the total value of any subset of 
\( V_i \) having the total weight in between \( (1 - \frac{1}{10}\varepsilon) (1 + \varepsilon')^j w_{\text{min}} \) and \( (1 + \frac{1}{10}\varepsilon) (1 + \varepsilon')^j w_{\text{min}}. \) (Since 
\( \varepsilon' < \frac{\varepsilon}{10}, \) the statement is also true for any subset having the total weight in between \( (1 + \varepsilon')^j w_{\text{min}} \) 
and \( (1 + \varepsilon')^j w_{\text{min}}). \)

Properties of \( Z. \) The entry \( Z(i, j), \forall i \in [m], j \in \left[ \left[ \log_{1+\varepsilon'} \left( \frac{B}{w_{\text{min}}} \right) \right] + m + 2 \right] \cup \{0\}, \) indicates the 
value of the subset of \( \bigcup_{k=1}^i V_k \) that satisfies the following three properties. The entry of \( Z \) also corresponds 
to the respective subset. The entry of \( Z \) could be \( -\infty, \) which indicates no subset. We use the notation \( Z(i, j) \) to 
indicate both the entry and the subset.

1. If the entry \( Z(i, j) \) is finite, then \( \sum_{k=1}^i \sum_{r \in Z(i, j) \cap V_k} w_r^{(k)} \leq (1 + \frac{1}{10}\varepsilon) (1 + \varepsilon')^j w_{\text{min}}. \)

2. If the entry \( Z(i, j) \) is finite, then \( (1 - \frac{4}{10}\varepsilon) (1 + \varepsilon')^{-1} l_k^w \leq \sum_{r \in V_k \cap Z(i, j)} w_r^{(k)} \leq (1 + \frac{4}{10}\varepsilon) u_k^w, \) 
\( \forall k \in [i]. \)

3. For all \( O' \subseteq \bigcup_{k=1}^i V_k \) having the total weight at most \( (1 + \varepsilon')^{j-i-2} w_{\text{min}}, \) and \( l_k^w \leq \sum_{r \in V_k \cap O'} w_r^{(k)} \leq u_k^w, \forall k \in [i], \) the total value of \( Z(i, j) \) is at least the total value of \( O'. \)

Let OPT be the weight of an optimal solution and \( j \in \left[ \left[ \log_{1+\varepsilon'} \left( \frac{W}{w_{\text{min}}} \right) \right] + m + 2 \right] \cup \{0\}, \) be the 
number such that,

\[
 w_{\text{min}} (1 + \varepsilon')^{j-m-3} \leq \text{OPT} \leq w_{\text{min}} (1 + \varepsilon')^{j-m-2}. \tag{14}
\]

So we can apply Property 3 of \( Z \) to an optimal set and \( Z(m, j). \) So, the total value of \( Z(m, j) \) is at 
least the total value of an optimal set, which is more than \( L. \) So, the subset \( Z(m, j) \) is feasible in Step 4 of 
the algorithm. If \( S \) in Step 4 corresponds to some \( j' \in \left[ \left[ \log_{1+\varepsilon'} \left( \frac{W}{w_{\text{min}}} \right) \right] + m + 2 \right] \cup \{0\}, \) then \( j' \leq j. \)

The total weight of \( S \) at most
Algorithm 8: Algorithm for the $BW_{\min}$ (Problem 11)

**Input:** The sets $V_i$ of items, $\forall i \in [m]$. $0 \leq l_i^w \leq u_i^w$ for $i \in [m]$. The value lower bound $L$ and $\varepsilon > 0$.

**Output:** The subset $S$ having the total weight at most $1 + \varepsilon$ times the weight of the optimal solution of $BW_{\min}$ (Problem 11), such that $(1 - \varepsilon)l_i^w \leq \sum_{r \in S \cap V_i} \leq (1 + \varepsilon)u_i^w$, $\forall i \in [m]$, and the total value of $S$ is at least $L$.

1. Let $\varepsilon' = (1 + \varepsilon)^{\frac{1}{m+1}} - 1$.

2. Let $Y(i, j), \forall i \in [m], \forall j \in \left(\log_{1+\varepsilon'} \left(\frac{W}{w_{\min}}\right) + m + 2\right) \cap \{0\}$, be the table, where the entry $Y(i, j)$ indicates the value of a subset of $V_i$ that is obtained by Theorem 3.8 by setting $V_i$ as $V', w_{\min} (1 + \varepsilon')^j$ as $w$ and $\frac{\varepsilon}{m}$ as $\varepsilon$ of Theorem 3.8.

3. Let $Z(i, j), \forall i \in [m]$ and $\forall j \in \left(\log_{1+\varepsilon'} \left(\frac{W}{w_{\min}}\right) + m + 2\right) \cap \{0\}$, be the DP table constructed as follows.

   (a) 
   \[
   Z(1, j) := \max \left\{ Y(1, j') \mid j' \in \left(\log_{1+\varepsilon'} \left(\frac{W}{w_{\min}}\right) + m + 2\right) \cap \{0\}, j \geq j' \land
   \left(1 + \varepsilon'\right)^{j'+1} w_{\min} \geq l_i^w \land
   \left(1 + \varepsilon'\right)^j w_{\min} \leq u_i^w \right\}.
   \]

   If the set satisfying above condition is empty, then set $Z(1, j)$ to $-\infty$.

   (b) For all $i \in [m] \setminus \{1\}$ and $j', j'' \in \left(\log_{1+\varepsilon'} \left(\frac{W}{w_{\min}}\right) + m + 2\right) \cap \{0\}$,
   \[
   Z(i, j) := \max \left\{ Z(i - 1, j') + Y(i, j'') \mid (1 + \varepsilon')^{j'+1} w_{\min} \geq l_i^w \land
   \left(1 + \varepsilon'\right)^j \geq (1 + \varepsilon')^{j'} \land
   \left(1 + \varepsilon'\right)^j w_{\min} \leq u_i^w \right\}.
   \]

   If the set satisfying above condition is empty, then set $Z(i, j)$ to $-\infty$.

4. Output the subset $S$ as follows,
\[
\arg\min_{j} \left\{ Z(m, j) \mid j \in \left(\log_{1+\varepsilon'} \left(\frac{W}{w_{\min}}\right) + m + 2\right) \cap \{0\}, Z(m, j) \geq L \right\}.
\]
\[ \leq \left(1 + \frac{4}{10}\varepsilon\right)^{m+3}\text{OPT} \quad \text{ (Inequality 14)} \]
\[ \leq \left(1 + \frac{4}{10}\varepsilon\right)\left(1 + \frac{1}{10}\varepsilon\right)\text{OPT} \quad \text{(Step 1 of the algorithm)} \]
\[ < (1 + \varepsilon)\text{OPT}. \]

If the entry \( Z(m, j) \) is finite, Property 2 of \( Z \) implies that the total weight of items from \( V_i \cap Z(m, j) \) is in between \((1 - \frac{4}{10}\varepsilon)(1 + \varepsilon')^{-1}l_i^u\) and \((1 + \frac{4}{10}\varepsilon)u_i^w\), \( \forall i \in [m] \). Since \( \varepsilon' \leq \frac{\varepsilon}{10} \), the total weight of items from \( V_i \cap X(m, j) \) is at least \((1 - \frac{4}{10}\varepsilon)(1 + \frac{\varepsilon}{10})^{-1}l_i^u > (1 - \frac{4}{10}\varepsilon)(1 - \varepsilon)l_i^u > (1 - \varepsilon)l_i^u \), and is at most \((1 + \frac{4}{10}\varepsilon)u_i^w < (1 + \varepsilon)u_i^w\).

The algorithm is similar to the algorithm (Algorithm 5) in Theorem 3.7. The only differences are the output step (Step 4) and the size of tables \( \mathcal{Y} \) and \( Z \). So, the running time analysis of algorithm is similar to the running time of Algorithm 5.

**Running time Analysis:** The size of the table \( \mathcal{Y} \) is \( O \left( m^2 \log(1+\varepsilon) \left( \frac{W}{\min w} \right) \right) \), and we require \( O(n^2 \varepsilon) \) time to fill each entry. So, the total time required to build the table \( \mathcal{Y} \) is \( O \left( \frac{n^2 m^2 \log(1+\varepsilon)}{\varepsilon} \left( \frac{W}{\min w} \right) \right) \). The total time required to build the table \( Z \) from the table \( \mathcal{Y} \) is \( O \left( m^4 \log^3(1+\varepsilon) \left( \frac{W}{\min w} \right) \right) \). So, the total running time is \( O \left( \frac{n^2 m^4 \log^3(1+\varepsilon)}{\varepsilon} \left( \frac{W}{\min w} \right) \right) \).

**5 Conclusion**

In this paper, we studied various fairness notions for knapsack problems. Studying fairness notions in related problems such as multiple knapsack problem [Jan12], multidimensional knapsack problem [FC84], submodular knapsack problem [LMNS09], is an interesting open problem.

**Acknowledgements.** AL was supported in part by SERB Award ECR/2017/003296 and a Pratiksha Trust Young Investigator Award.

**References**

[ABC+19] Junaid Ali, Mahmoudreza Babaei, Abhijnan Chakraborty, Baharan Mirzasoleiman, Krishna P. Gummadi, and Adish Singla. On the fairness of time-critical influence maximization in social networks. ArXiv, abs/1905.06618, 2019.

[AKL19] Susanne Albers, Arindam Khan, and Leon Ladewig. Improved online algorithms for knapsack and GAP in the random order model. In Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2019, September 20-22, 2019, Massachusetts Institute of Technology, Cambridge, MA, USA, pages 22:1–22:23, 2019.
[ALT18] Haris Aziz, Barton E. Lee, and Nimrod Talmon. Proportionally representative participatory budgeting: Axioms and algorithms. In Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS ’18, pages 23–31, Richland, SC, 2018. International Foundation for Autonomous Agents and Multiagent Systems.

[BCEZ19] Nawal Benabbou, Mithun Chakraborty, Edith Elkind, and Yair Zick. Fairness towards groups of agents in the allocation of indivisible items. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019, pages 95–101, 2019.

[BCFN19] Suman Kalyan Bera, Deeparnab Chakrabarty, Nicolas Flores, and Maryam Negahbani. Fair algorithms for clustering. In Advances in Neural Information Processing Systems 32: Annual Conference on Neural Information Processing Systems, NIPS 2019, 8-14 December 2019, Vancouver, BC, Canada, pages 4955–4966, 2019.

[BCH+18] Nawal Benabbou, Mithun Chakraborty, Xuan-Vinh Ho, Jakub Sliwinski, and Yair Zick. Diversity constraints in public housing allocation. In Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems, AAMAS ’18, page 973–981, Richland, SC, 2018. International Foundation for Autonomous Agents and Multiagent Systems.

[BFHS17] Abbas Bazzi, Samuel Fiorini, Sangxia Huang, and Ola Svensson. Small extended formulation for knapsack cover inequalities from monotone circuits. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, pages 2326–2341, 2017.

[BFI+17] Robert Bredereck, Piotr Faliszewski, Ayumi Igarashi, Martin Lackner, and Piotr Skowron. Multiwinner elections with diversity constraints. In AAAI, 2017.

[BLMS19] Xiaohui Bei, Xinhang Lu, Pasin Manurangsi, and Warut Suksompong. The price of fairness for indivisible goods. In Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019, pages 81–87, 2019.

[Cho17] Alexandra Chouldechova. Fair prediction with disparate impact: A study of bias in recidivism prediction instruments. Big Data, 5(2):153–163, 2017.

[CK00] Chandra Chekuri and Sanjeev Khanna. A ptas for the multiple knapsack problem. In Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’00, pages 213–222, Philadelphia, PA, USA, 2000. Society for Industrial and Applied Mathematics.

[CK04] Chandra Chekuri and Amit Kumar. Maximum coverage problem with group budget constraints and applications. In APPROX-RANDOM, 2004.

[CKLV17] Flavio Chierichetti, Ravi Kumar, Silvio Lattanzi, and Sergei Vassilvitskii. Fair clustering through fairlets. In Advances in Neural Information Processing Systems 30: Annual Conference on Neural Information Processing Systems (NIPS) 2017, 4-9 December 2017, Long Beach, CA, USA, pages 5029–5037, 2017.

[CKPT17] Henrik I. Christensen, Arindam Khan, Sebastian Pokutta, and Prasad Tetali. Approximation and online algorithms for multidimensional bin packing: A survey. Computer Science Review, 24:63–79, 2017.
[Csir91] János Csirik. Heuristics for the 0-1 min-knapsack problem. *Acta Cybernetica*, 10(1-2):15–20, 1991.

[FC84] A.M. Frieze and M.R.B. Clarke. Approximation algorithms for the m-dimensional 0–1 knapsack problem: Worst-case and probabilistic analyses. *European Journal of Operational Research*, 15(1):100 – 109, 1984.

[FMMS18] Yuri Faenza, Igor Malinovic, Monaldo Mastrolilli, and Ola Svensson. On bounded pitch inequalities for the min-knapsack polytope. In *Combinatorial Optimization - 5th International Symposium, ISCO 2018, Marrakesh, Morocco, April 11-13, 2018, Revised Selected Papers*, pages 170–182, 2018.

[FSTW17] Till Fluschnik, Piotr Skowron, Mervin Triphaus, and Kai Wilker. Fair knapsack. *Proceedings of the AAAI Conference on Artificial Intelligence*, 33, 11 2017.

[GFMW17] Waldo Gálvez, Fabrizio Grandoni, Sandy Heydrich, Salvatore Ingala, Arindam Khan, and Andreas Wiese. Approximating geometric knapsack via L-packings. In *58th IEEE Annual Symposium on Foundations of Computer Science, FOCS 2017, Berkeley, CA, USA, October 15-17, 2017*, pages 260–271, 2017.

[GKSA19] Ashish Goel, Anilesh K. Krishnaswamy, Sukolsak Sakshuwong, and Tanja Aitamurto. Knapsack voting for participatory budgeting. *ACM Trans. Econ. Comput.*, 7(2):8:1–8:27, July 2019.

[GP14] Sreenivas Gollapudi and Debmalya Panigrahi. Fair allocation in online markets. In *Proceedings of the 23rd ACM International Conference on Conference on Information and Knowledge Management*, pages 1179–1188, 2014.

[IID+16] Yusuke Inoue, Takhiru Imada, Syunya Doi, Lei Chen, Takehito Utsuro, and Yasuhide Kawada. Selecting web search results of diverse contents with search engine suggests and a topic model. In *2016 30th International Conference on Advanced Information Networking and Applications Workshops (WAINA)*, pages 455–460. IEEE, 2016.

[IK75] Oscar H. Ibarra and Chul E. Kim. Fast approximation algorithms for the knapsack and sum of subset problems. *Journal of the ACM (JACM)*, 22(4):463–468, 1975.

[Jan12] Klaus Jansen. A fast approximation scheme for the multiple knapsack problem. In Mária Bieliková, Gerhard Friedrich, Georg Gottlob, Stefan Katzenbeisser, and György Turán, editors, *SOFSEM 2012: Theory and Practice of Computer Science*, pages 313–324, Berlin, Heidelberg, 2012. Springer Berlin Heidelberg.

[JKMR16] Matthew Joseph, Michael Kearns, Jamie H. Morgenstern, and Aaron Roth. Fairness in learning: Classic and contextual bandits. In D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, *Advances in Neural Information Processing Systems 29 (NIPS)*, pages 325–333. Curran Associates, Inc., 2016.

[JLN+20] Ajay Saju Jacob, Anand Louis, Meghana Nasre, Prajakta Nimbhorkar, and Govind S. Sankar. Matchings with fairness constraints: Online and offline algorithms. *Manuscript 2020*, 2020.

[Kha15] Arindam Khan. *Approximation algorithms for multidimensional bin packing*. PhD thesis, Georgia Institute of Technology, 2015.
[KLK⁺18] Jalal Khamse-Ashari, Ioannis Lambadaris, George Kesidis, Bhuvan Urgaonkar, and Yiqiang Q. Zhao. An efficient and fair multi-resource allocation mechanism for heterogeneous servers. *IEEE Trans. Parallel Distrib. Syst.*, 29(12):2686–2699, 2018.

[KLM14] Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. On the lasserre/sum-of-squares hierarchy with knapsack covering inequalities. *CoRR*, abs/1407.1746, 2014.

[KPP04] Hans Kellerer, Ulrich Pferschy, and David Pisinger. *Knapsack problems*. Springer, 2004.

[KST13] Ariel Kulik, Hadas Shachnai, and Tami Tamir. Approximations for monotone and nonmonotone submodular maximization with knapsack constraints. *Math. Oper. Res.*, 38(4):729–739, November 2013.

[Law79] Eugene L Lawler. Fast approximation algorithms for knapsack problems. *Mathematics of Operations Research*, 4(4):339–356, 1979.

[LMNS09] Jon Lee, Vahab S. Mirrokni, Viswanath Nagarajan, and Maxim Sviridenko. Non-monotone submodular maximization under matroid and knapsack constraints. *CoRR*, abs/0902.0353, 2009.

[Lus99] Hanan Luss. On equitable resource allocation problems: A lexicographic minimax approach. *Operations Research*, 47(3):361–378, 1999.

[Nas50] John Nash. The bargaining problem. *Econometrica*, 18(2):155–162, 1950.

[PVRW85] Manfred W Padberg, Tony J Van Roy, and Laurence A Wolsey. Valid linear inequalities for fixed charge problems. *Operations Research*, 33(4):842–861, 1985.

[Sel04] Meinolf Sellmann. The practice of approximated consistency for knapsack constraints. In Deborah L. McGuinness and George Ferguson, editors, *Proceedings of the Nineteenth National Conference on Artificial Intelligence, Sixteenth Conference on Innovative Applications of Artificial Intelligence, July 25-29, 2004, San Jose, California, USA*, pages 179–184. AAAI Press / The MIT Press, 2004.

[SS18] Erel Segal-Halevi and Warut Suksompong. Democratic fair allocation of indivisible goods. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018, July 13-19, 2018, Stockholm, Sweden*, pages 482–488, 2018.

[ST00] Hadas Shachnai and Tami Tamir. Polynomial time approximation schemes for class-constrained packing problem. In *Approximation Algorithms for Combinatorial Optimization, Third International Workshop, APPROX 2000, Saarbrücken, Germany, September 5-8, 2000, Proceedings*, pages 238–249, 2000.

[ST17] Ehud Shapiro and Nimrod Talmon. A democratically-optimal budgeting algorithm. *CoRR*, abs/1709.05839, 2017.

[Svi04] Maxim Sviridenko. A note on maximizing a submodular set function subject to a knapsack constraint. *Oper. Res. Lett.*, 32(1):41–43, January 2004.

[TWR⁺19] Alan Tsang, Bryan Wilder, Eric Rice, Milind Tambe, and Yair Zick. Group-fairness in influence maximization. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019*, pages 5997–6005, 2019.
François Vanderbeck. Computational study of a column generation algorithm for bin packing and cutting stock problems. *Mathematical Programming*, 86(3):565–594, Dec 1999.