Communication with Crystal-Free Radios

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Abstract—We consider a communication channel where there is no common clock between the transmitter and the receiver. This is motivated by the recent interest in building system-on-chip radios for Internet of Things applications, which cannot rely on crystal oscillators for accurate timing. We identify two types of clock uncertainty in such systems: timing jitter, which occurs at a time scale faster than the communication duration (or equivalently blocklength); and clock drift, which occurs at a slower time scale. We study the zero-error capacity under both types of timing imperfections, and obtain optimal zero-error codes for some cases. Our results show that, as opposed to common practice, in the presence of clock drift it is highly suboptimal to try to learn and track the clock frequency at the receiver; rather, one can design codes that come close to the performance of perfectly synchronous communication systems without any clock synchronization at the receiver.

Index Terms—Asynchronous communication, clock drift, jitter, crystal-free radios

I. INTRODUCTION

The next exponential growth in connectivity is projected to be no longer in access between people but in connecting objects and machines in the age of Internet of Things (IoT). This is partly fueled by the emergence of tiny, low-cost wireless devices that combine communication, computation and sensing. These wireless devices are expected to form the fabric of smart technologies and cyberphysical systems, enabling a plethora of exciting applications: from in-body and personal health monitoring, to smart homes and transportation systems, to automation and monitoring in smart grids.

However, scaling wireless devices from billions to potentially trillions (as envisioned by some forecasts [2], [3]) requires orders of magnitude reduction in costs and often size, both of which are dominated by external components such as batteries and crystal oscillators. This has led to significant recent interest in building miniature radios that do not possess any external components [4]–[9]. For example, the ant-radios of [9] integrate a full wireless communication system, including the full transceiver, antenna, and clock, on a single CMOS chip of size 4.4 mm². A small crystal oscillator, on the other hand, is around 1.9 mm², which is about half the size of the entire system. In addition to reducing size and cost, eliminating external components is also desirable for eliminating the extra steps for integration, packaging, and assembly. In particular, the ant-radios of [9] use an on-chip low-power and low-accuracy 200 MHz ring oscillator to control the symbol rate instead of a crystal oscillator, and operate without a battery; they are instead powered remotely via wireless power transfer.

Compared to crystal resonator-based systems, ring oscillator systems experience greater jitter and drift, causing the clock frequency variation to lie within a ∼ 100 MHz range centered at 200 MHz. This is incompatible with many conventional communication schemes and poses a significant design challenge. Normally, the receiver employs a timing recovery mechanism such as an early-late gate [10] to extract the transmitter’s clock (or symbol rate). However, this is only possible when the transmitter’s clock is relatively stable.

In particular, consider transmission using pulse-position modulation (PPM), as done in [9] due to the energy efficiency of this modulation technique for wideband communication. In PPM, information is encoded in the position of a pulse transmitted in one of $M$ bins, where $M = 64$ in [9]. The bin duration is determined by the inaccurate ring oscillator, and can vary between 4 and 7 ns. Thus the uncertainty in the transmitter’s clock makes it impossible for the receiver to decode the received message. To overcome this problem, in [9] transmission begins with two extra pulses, transmitted back to back in two consecutive bins, and the frame size $M$ is restricted to 64. The receiver can learn the bin duration (and thus the transmitter’s clock) by measuring the time between the first two pulses, and subsequently decode the location of the third (information-bearing) pulse. Restricting the frame size limits the amount of accumulated jitter and prevents the transmitter and receiver clock from going out of sync during the course of transmission. This synchronization cost presents a significant burden on the transmitter, as the energy consumption of the transmitter is dominated by the transmitted energy, $2/3$ of which is now spent on synchronization.

The current paper provides a study of reliable communication in such systems, where there is no common clock between the transmitter and the receiver, from a fundamental perspective. Motivated by digital recording, communication...
without a synchronous clock has been considered in previous information theoretic literature [11]–[15], where these works model the absence of a common clock as timing jitter. For example in [11], which is most closely related to our work, jitter causes the transmitted signal to be arbitrarily “stretched” or “squeezed” in time by a varying factor during the course of communication. In other words, the “stretching” or “squeezing” occurs at a time scale faster than the duration of communication (or equivalently blocklength). In [9] however, the clock remains sufficiently stable during the course of the 64-PPM symbol. The real challenge is that each time the transmitter sends a 64-PPM symbol, it is encoded with an unknown (but stable) clock whose frequency can lie anywhere between 150 and 250 MHz. See Fig. 2.

Therefore, in this work, we distinguish between two types of clock uncertainty at the receiver: timing jitter, which can cause the transmitter’s clock to vary arbitrarily during the course of transmission; and clock drift, which occurs at a time scale much larger than the blocklength. The second can be modeled as a fixed but unknown clock. Timing jitter was studied in [11], where capacity was found and optimal codes were developed. However, the optimal codes for communication under clock drift is fundamentally different, and in this work we aim to develop codes that are optimal when both imperfections are present.

We show for example that when only clock drift is present, it is possible to code in such a way that the receiver never learns the exact clock frequency: by considering ratios of pulse positions instead of their absolute values, the clock cycle indeed does not play a part. This could be used to almost entirely eliminate the cost of synchronization in [9] (the extra two pulses used to convey the transmitter’s clock to the receiver). Indeed, we show that our scheme can improve from a rate of 6 bits per frame obtained by the 64-PPM scheme, to a rate of 10.76 bits per frame by encoding over ratios, nearly approaching the perfect synchronization upper bound which is 11.02 bits per frame.

II. CHANNEL MODEL

We consider multi-pulse PPM communication where the transmitter sends \( k \) short (0-width) pulses in \( M \) bins, where each pulse is located in one bin and information is encoded in the position of the pulses (or equivalently the occupied bins). Each of the \( \binom{M}{k} \) possible transmit signals can be represented as a binary sequence of length \( M \), where 1 indicates the presence of a pulse in the corresponding bin. Instead of this, however, in this paper we adopt an equivalent differential representation of the signals, where each one is represented by a vector of length \( k \), \( (X_1, \ldots, X_k) \), where \( X_i \) is the time (number of clock cycles, or number of bins) between the \( (i-1) \)-th and \( i \)-th pulses, which is also called the \( i \)-th run. Note that the first run \( X_1 \) is simply the bin of the first pulse (equivalently define \( X_0 = 0 \)). The runs \( X_i \) take values in the set \( \{1, \ldots, M\} \), and the vector must satisfy \( \sum_{i=1}^{k} X_i \leq M \), since there are exactly \( k \) pulses in the transmitted signal. Let the set of all such legitimate input vectors be denoted by \( \mathcal{X} \).

When \( k \) is clear from the context, we will use boldface \( \mathbf{X} \) as a shorthand for the vector \( (X_1, \ldots, X_k) \). In this paper we would like to study zero-error communication with vectors from \( \mathcal{X} \) in the presence of clock imperfections as we model next. While in this paper we only focus on communication with multi-pulse PPM (both for simplicity and because this is the modulation of choice for most low-energy systems), our model and results can be extended to allow general modulation techniques in the direction of [11] (e.g. pulse-code modulation).

Note that we will keep the blocklength \( M \) finite here as it is typically not a large number for systems of interest. We are interested in understanding the structure of optimal codes and the size of the optimal code for finite \( M \), rather than the behavior of capacity as \( M \) gets large. Moreover, the problem trivializes for \( M \to \infty \): if \( k \) remains finite, the rate is zero; on the other hand, if \( k \) grows with \( M \), then the first two pulses can be used to perfectly learn the clock as in [9] without any loss in the communication rate, and the problem becomes identical to one with perfect synchronization.

By using input vectors from \( \mathcal{X} \), our goal is to achieve zero-error communication under the presence of the following two types of clock imperfections:

**Clock drift:** The receiver observes the transmitted vector multiplied by an unknown fixed real number \( T \) that takes values in a closed interval \([T_1, T_2]\), for \( 0 < T_1 \leq T_2 \). We will also be interested in the case of unbounded clock drift, such that \( T \in [T_1, \infty) \). Hence the observed vector is \( T\mathbf{X} \), i.e. the observed run lengths are given by \( TX_i, i = 1, \ldots, k \). Note that this models the scenario where the receiver is unaware of the clock used by the transmitter (it only knows that it lies in a certain interval), but the transmitter’s clock remains stable during the transmission of the signal. This models variations of the transmitter’s clock frequency at a scale larger than the blocklength for communication (in a flavor similar to large scale fading in wireless systems [16]).

**Timing jitter:** On top of the slow clock drift, the transmitter’s clock experiences random jitter, i.e. variations at a scale faster than the blocklength (in a flavor similar to small scale fading in wireless systems [16]). We model this similarly to [11] by a strictly positive arbitrary process \( r(u) \), unknown to the transmitter nor to the receiver, such that \( r(u) \in [a, b] \) for some \( 0 < a \leq b < \infty \). This process represents the instantaneous deviation of the clock from its nominal frequency. If a pulse is transmitted at time \( t \), the receiver observes a pulse at time \( \int_t^{t+T} r(u)du \). Thus, the runs observed at the receiver are given by

\[
Y_i = \frac{\sum_{j=1}^{i} TX_j}{\sum_{j=1}^{i-1} TX_j} r(u)du, \quad i = 1, \ldots, k.
\]
III. OPTIMAL CODES

In this section we study optimal codes for the channel \((k, M, \xi, \gamma)\), for several special cases of interest. Denote an optimal code by \(C^*_{k,M,\xi,\gamma}\), where \(k, M\) should be understood from the context.

A. No Jitter (\(\xi = 1\)) and Unbounded Clock Drift (\(\gamma = \infty\))

The clock drift is \(T \in [T_1, \infty)\) for some \(T_1 > 0\), and we can let \(Z_i = 1\) without loss of generality. First, observe that if \(k = 1\), reliable communication is not possible. This is because upon transmitting \(X_1\), the output \(Y_1\) can be any number in \([T_1 X_1, \infty)\), and all input signals are indistinguishable.

Assume \(k \geq 2\). For an input vector \(x = (x_1, \ldots, x_k)\), define the ratio vector \(u = (u_2, \ldots, u_k)\) by

\[
u_i = \frac{x_i}{x_1}, \quad i = 2, \ldots, k.\]

**Lemma 1.** For a channel with \(k > 1\), \(\xi = 1\), and \(\gamma = \infty\), two vectors \(x\) and \(x'\) are distinguishable if and only if their ratio vectors \(u\) and \(u'\) are distinct.

**Proof.** Suppose \(\frac{x_i}{x_1} = \frac{x'_i}{x'_1}\) for \(i = 2, \ldots, k\). By letting \(T = T_1 x'_1\) and \(T' = T_1 x_1\), we see that (2) holds. For the other direction, suppose there exist \(T, T'\) s.t. \(T x_i = T' x'_i\) for \(i = 1, \ldots, k\). Dividing by \(T x_1\) or equivalently \(T' x'_1\) implies the appropriate ratio vectors are equal.

According to Lemma 1, we can construct an optimal code by taking the maximal number of input vectors with distinct ratio vectors.

**Theorem 1.** The following code is an optimal zero-error code for a channel with \(k > 1\), \(\xi = 1\), and \(\gamma = \infty\):

\[
C^*_{1,\infty} = \{ x \in \mathcal{X} : \gcd(x) = 1 \},
\]

where \(\gcd(x) = \gcd(x_1, \ldots, x_k)\) is the greatest common divisor of \((x_1, \ldots, x_k)\), i.e. it is the largest integer \(d\) s.t. \(d | x_i\) for all \(i = 1, \ldots, k\).

Any vector \(x\) can be divided by its \(\gcd\) to obtain a vector with the same ratios vector \(u\) and \(\gcd\). Therefore, this code is a maximal set of codewords with distinct ratio vectors. The receiver can decode by computing the ratios of the received signal \(\frac{Y_i}{Y_1} = \frac{T X_i}{T X_1} = U_i\). We note that there are \(\binom{M}{k}\) input vectors in \(\mathcal{X}\), so \(C^*_{1,\infty}\) can be constructed in \(O(M^k)\) time by an exhaustive search, which, to the best of our knowledge, is the best that can be done.

We can compare the optimal code here with the 64-PPM scheme in [9], by setting \(k = 2\) and \(M = 65\). While the scheme in [9] consisted of 3 pulses, the first one is used only to mark the beginning of a frame\(^3\) and hence will not be counted for the purpose of this comparison. Therefore discarding the first pulse in [9], the codebook contains codewords for which \(x_1 = 1\) and \(1 < x_2 \leq M - 1\), which has a rate of 6 bits per frame. Computing \(C^*_{1,\infty}\) from (3) yields \(R = 10.35\) bits per frame. In the next section we will see how this can be

\(^3\)While in our model we assume the receiver knows the exact starting time of communication, in practice there may be an unknown timing offset (caused by e.g. unknown time-of-flight). This adverse effect is not studied in this work.
improved even further by taking into account the bounds on the clock $T$.

**Proof of Theorem 2** To show that this is a zero-error code, let $x, x' \in \mathcal{C}_{1,\infty}^*$ be two distinct codewords, and suppose their ratio vectors are equal:

$$\frac{x_i}{x_1} = \frac{x'_i}{x'_1}, \quad i = 2, \ldots, k.$$ 

Then necessarily $x_1 \neq x'_1$, otherwise the codewords are not distinct. Let $\frac{m}{n}$ be the reduced fraction of $\frac{x'_i}{x'_1}$, i.e. $\frac{m}{n} = \frac{x'_1}{x'_1}$ and $\gcd(m, n) = 1$. Then for each $i = 1, \ldots, k$:

$$x'_i = \frac{x'_1}{x'_1} x_i = \frac{m}{n} x_i,$$

and since $x'_i$ is an integer, $n$ must divide $x_i$ for all $i = 1, \ldots, k$. Since by assumption $\gcd(x) = 1$, we must have $n = 1$. This implies $x'_i = mx_i$ for all $i$, where $m$ is an integer greater than 1, which means $\gcd(x') = m > 1$. This is a contradiction since $x' \in \mathcal{C}_{1,\infty}^*$. Hence $x'_i \neq \frac{x'_1}{x'_1}$ for some $i$, and by Lemma 1 the code is zero-error.

Next, we claim that $\mathcal{C}_{1,\infty}^* \subseteq \mathcal{C}$ is optimal by showing that any other zero-error code $\mathcal{C}$ must have at most as many codewords as $\mathcal{C}_{1,\infty}^*$. To this end, construct the code $\tilde{\mathcal{C}}$ from $\mathcal{C}$ by modifying each codeword as follows:

$$\tilde{x}_i = \frac{x_i}{\gcd(x)}, \quad i = 1, \ldots, k,$$

or in short $\tilde{x} = \frac{x}{\gcd(x)}$. The new codewords all have $\gcd(\tilde{x}) = 1$, which implies $\mathcal{C} \subseteq \mathcal{C}_{1,\infty}$.

**Theorem 2.** An optimal zero-error code for a channel with $\xi = 1$ and $\gamma < \infty$ is given by

$$\mathcal{C}_{1,\gamma}^* = \bigcup_{x \in \mathcal{C}_{1,\infty}^*} \mathcal{L}_x^\gamma,$$

where $\mathcal{C}_{1,\infty}^*$ is given by (3).

In order to decode, the receiver first computes the ratios vector of the output. This uniquely identifies a vector $x$ with $\gcd(x) = 1$, or equivalently a set $\mathcal{L}_x^\gamma$. Then the correct codeword in $\mathcal{L}_x^\gamma$ can be decoded from any single run $Y_i$.

Equipped with this theorem, we compute the optimal code when the clock cycle is bounded between 4 and 7 ns, which are the actual system parameters in [9]. The clock drift parameter is $\gamma = 1.75$, which yields a rate of 10.76 bits per frame. It is interesting to note that, while this is an improvement over the code for $\gamma = \infty$, it is not particularly significant. Therefore, at least in this case, the fact that the clock drift is bounded does not provide a meaningful gain to capacity. Finally, note that the best rate that can be achieved, even without clock drift, is 11.02 bits per frame. This is obtained by the optimal code with $(\frac{M}{k}) = (\frac{55}{7})$ codewords.

**Proof of Theorem 2** From arguments made in the previous section and by the construction of $\mathcal{L}_x^\gamma$, it follows that $\mathcal{C}_{1,\gamma}^*$ is zero-error. To show that it is an optimal code, we take an arbitrary zero-error code $\mathcal{C}$ and construct another code $\tilde{\mathcal{C}}$.

Specifically, for each codeword $x \in \mathcal{C}$, let $d = \gcd(x)$ and consider the vector

$$\frac{x}{d} = (\frac{x_1}{d}, \ldots, \frac{x_k}{d}) \in \mathcal{C}_{1,\infty}^*.$$ 

Let $\tilde{d}$ be the largest integer such that $\tilde{d} \leq d$ and

$$\tilde{d} \frac{x}{d} = (\tilde{d} \frac{x_1}{d}, \ldots, \tilde{d} \frac{x_k}{d}) \in \mathcal{L}_{\tilde{x}/\tilde{d}}.$$ 

We map $x$ to $\tilde{x} = \frac{x}{\tilde{d}}$. The set of all vectors $\tilde{x}$ constitutes the new code $\tilde{\mathcal{C}}$.

Clearly $\tilde{\mathcal{C}} \subseteq \mathcal{C}_{1,\gamma}^*$. It remains to show $|\tilde{\mathcal{C}}| = |\mathcal{C}|$, i.e. no two codewords in $\tilde{\mathcal{C}}$ map to the same codeword in $\tilde{\mathcal{C}}$. For this purpose, let $x, x' \in \mathcal{C}$ be two distinct codewords, and assume they map to the same codeword $\tilde{x} \in \tilde{\mathcal{C}}$. Let $d = \gcd(x)$ and $d' = \gcd(x')$. First, notice that necessarily $\frac{x}{d} = \frac{x'}{d'}$, otherwise they cannot map to the same $\tilde{x}$. Denote $\tilde{x} = \frac{x}{\tilde{d}} = \frac{x'}{\tilde{d}}$. Then, we have $\tilde{x} = \frac{x}{\tilde{d}}$, where $\tilde{d}$ is the largest integer $\tilde{d} \leq d$ and $\tilde{d} \leq d'$ s.t. $\tilde{d} \frac{x}{d} \in \mathcal{L}_{\tilde{x}/\tilde{d}}$. Since $x, x'$ are distinct, we can assume without loss of generality $d < d'$. By construction of $\mathcal{L}_{\tilde{x}/\tilde{d}}$, we must have $\frac{\tilde{d}}{\tilde{d}} \leq \gamma$, otherwise there must be another integer $q \leq \tilde{d}$ s.t. $q \frac{x}{d} \in \mathcal{L}_{\tilde{x}/\tilde{d}}$ and $\frac{q}{\tilde{d}} > \gamma$, in contradiction to the fact that $\tilde{d}$ is the largest such integer with $\tilde{d} \leq d'$. Along with the inequality $\tilde{d} \leq d$, it follows that $\frac{d}{d} \leq \gamma$. This, in turn, implies that $x, x'$ are indistinguishable, which contradicts the assumption that $\mathcal{C}$ is zero-error.
C. Jitter (\(\xi > 1\)) and No Clock Drift (\(\gamma = 1\))

When \(\gamma = 1\), the problem reduces to the one studied in \([11]\). Nevertheless, we provide here the code construction and proofs for completeness.

**Lemma 2.** For a channel with \(\xi > 1\) and \(\gamma = 1\), two input vectors \(x, x'\) are distinguishable if and only if there is an index \(1 \leq i \leq k\) such that \(x_i/x'_i > \xi\) or \(x'_i/x_i > \xi\).

**Proof.** We can assume without loss of generality that \(T = 1\). Then, two input vectors \(x, x'\) are indistinguishable if and only if there exist \(Z_1, \ldots, Z_k, Z'_1, \ldots, Z'_k \in [a, b]\) such that \(Z_i x_i = Z'_i x'_i\), or equivalently \(\frac{z_i}{z'_i} = \frac{x_i}{x'_i}\), for every \(i = 1, \ldots, k\). This, in turn, holds if and only if \(\xi^{-1} \leq \frac{x_i}{x'_i} \leq \xi\) for all \(i\), completing the proof.

Next, we show that this is an optimal code by modifying an arbitrary zero-error code \(C\). Specifically, we construct the code \(C\) in a similar manner to the proof of Theorem 4. Specifically, we construct the code \(C\) by mapping each codeword \(x \in \mathcal{C}\) to the codeword \(\hat{x} = (x_1, \ldots, x_k)\), where \(\hat{x}\) is the largest element in \(\mathcal{L}_i\) such that \(\hat{x}_i \leq x_i\).

Clearly \(C \subset \mathcal{C}_i\), therefore \(\mathcal{C}_i \subset \mathcal{C}\). Then it remains to show that we do not lose anything by modifying \(C\) to \(\hat{C}\), i.e. no two codewords in \(\hat{C}\) map to the same codeword in \(\hat{C}\). This will imply \(C\) has at most as many codewords as \(C\), which will conclude the proof that \(C\) is an optimal code.

Let \(x, x' \in \mathcal{C}\) be two distinct codewords, and assume they are mapped to the same codeword \(\hat{x} \in \hat{C}\). Therefore for every \(i \in \{1, \ldots, k\}\), the element \(\hat{x}_i \in \mathcal{L}_i\) is the maximal such that \(\hat{x}_i \leq x_i\) and \(\hat{x}_i \leq x'_i\). By construction of \(\mathcal{L}_i\), it follows that \(x_i, x'_i \leq [\xi \hat{x}_i + 1]\), which implies \(\hat{x}_i \leq x_i, x'_i \leq \xi \hat{x}_i\). Hence, \(\xi^{-1} \leq \frac{x_i}{x'_i} \leq \xi\) for every \(1 \leq i \leq k\), and it follows from Lemma 2 that \(x\) and \(x'\) are indistinguishable, which is a contradiction to the assumption that \(C\) is zero-error.

The nature of jitter requires different coding and decoding techniques as compared to clock drift. When only clock drift is present, the receiver needs to wait for the entire signal before it can decode (which is done by computing the ratios vector). When there is jitter but no clock drift, i.e. the clock cycle is known exactly at the receiver, the receiver can decode each run independently, and does not have to wait for the entire output vector. As will be seen in the following section, this poses an interesting challenge when both jitter and clock drift occur.

D. Jitter (\(\xi > 1\)) and Unbounded Clock Drift (\(\gamma = \infty\))

We solve this only for the case of \(k = 2\). In the following lemma, we state a necessary and sufficient condition for two input vectors to be distinguishable at the receiver.

**Lemma 3.** A pair of input vectors \((x_1, x_2)\) and \((x'_1, x'_2)\) are distinguishable for a channel with \(k = 2, \xi > 1\), and \(\gamma = \infty\), if and only if \(\frac{\xi a}{x_1} > \xi^2 \frac{a}{x_1} \) or \(\frac{\xi b}{x_2} > \xi^2 \frac{b}{x_2}\).

Intuitively, when the clock drift is unbounded, distinguishable vectors must have distinct ratios \(\frac{a}{x_1}\), hence vectors can be equally represented by their appropriate ratio. However, when jitter corrupts the signal, the numerator and the denominator can “stretch” or “squeeze” independently, by a factor \(\xi\) each. Together, the ratio can change by up to a factor of \(\xi^2\).

**Proof of Lemma 3.** We will show that if two vectors satisfy \(\frac{\xi a}{x_1} \leq \xi^2 \frac{a}{x_1}\) and \(\frac{\xi b}{x_2} \leq \xi^2 \frac{b}{x_2}\) then they are indistinguishable. For this purpose, we need to find \(T, T' \in [T_1, \infty)\) and \(Z_1, Z_2, Z'_1, Z'_2 \in [a, b]\) where \(a, b, T_1 > 0\) and \(a/b = \xi\), such that \(\mathcal{Z}\) holds, i.e. \(T Z_1 x_1 = T' Z'_1 x'_1\) and \(T Z_2 x_2 = T' Z'_2 x'_2\).

Observe that \(\xi^{-2} \leq \frac{z_1}{z'_1} \leq \xi^2\). Let \(Z_1, Z_2, Z'_1, Z'_2\) be such that \(Z_1 Z'_2 = x_1 x'_1 < x_1 x'_2\); these exist since

\[\xi^{-2} \leq \frac{z_1}{z'_1} \leq \xi^2.\]

Having fixed \(Z_1, Z_2, Z'_1, Z'_2\), find \(T, T'\) such that \(\frac{T}{T'} = \frac{Z_1 x_1}{Z'_1 x'_1}\). This is possible since the ratio \(\frac{T}{T'}\) can take any positive number. Now, we have

\[\frac{T' Z'_2 x'_2}{T Z_2 x_2} = \frac{Z_1 x_1 Z_2 x'_2}{Z'_1 x'_1 Z_2 x_2} = 1,\]

implying that \((x_1, x_2)\) and \((x'_1, x'_2)\) are indistinguishable.

The other direction, namely that if \((x_1, x_2)\) and \((x'_1, x'_2)\) are indistinguishable then \(\xi^{-2} \leq \frac{z_1}{z'_1} \leq \xi^2\), follows by repeating the previous arguments in the reverse direction.

Since distinguishable codewords must have distinct ratios (whether jitter is present or not), we can, without loss of generality, take only codewords for which \(\gcd(x_1, x_2) = 1\).
Hence we can construct an optimal code for this channel by taking a subset of the optimal code for the channel without jitter, that is $C_{\xi,\infty}^* \subseteq C_{1,\infty}^*$. Since the ratios $\frac{x_1}{x_2}$ of all codewords in $C_{1,\infty}^*$ are distinct, we define the following set of fractions:

$$U = \left\{ \frac{x_1}{x_2} : (x_1, x_2) \in C_{1,\infty}^* \right\}.$$ 

There is a one-to-one mapping between $U$ and $C_{\xi,\infty}^*$. Using the set $U$, we construct an optimal code $C_{\xi,\infty}^*$ by means of the following algorithm:

1. Start with $C_{\xi,\infty}^* = \{(M - 1, 1)\}$ and let $u_1 = \frac{1}{M-1}$, which is the smallest element in $U$.
2. Given $u_{i-1}$, consider the set of all elements $u \in U$ s.t. $u > \xi^2 u_{i-1}$, or in other words, the set $U \cap (\xi^2 u_{i-1}, \infty)$, where $(\xi^2 u_{i-1}, \infty)$ denotes an open interval. If the set is empty, stop the construction. Otherwise, let $(x_1, x_2) \in \mathcal{X}$ be the single vector s.t. $\gcd(x_1, x_2) = 1$ and $\frac{x_1}{x_2}$ is the smallest element in $U \cap (\xi^2 u_{i-1}, \infty)$. Set $u_i = \frac{x_1}{x_2}$, add $(x_1, x_2)$ to $C_{\xi,\infty}^*$, and repeat this step.

This construction, while similar to the constructions $L^k_\infty$ and $L^1_\infty$ from the previous sections, operates on $U$ which is a set of fractions, rather than on vectors or elements of vectors. It is somewhat surprising that, given the different structure of $U$ as compared to the set of vectors, this construction is indeed optimal, as stated formally in the following theorem.

**Theorem 4.** The code $C_{\xi,\infty}^*$ obtained by the above construction is an optimal code for a channel with $k = 2$, $\xi > 1$, and $\gamma = \infty$.

**Proof.** By Lemma 3, this code is zero-error. To show that it is optimal, take any zero-error code $C$ and construct the code $\tilde{C}$ as follows: for every codeword $(x_1, x_2) \in C$, let $(\tilde{x}_1, \tilde{x}_2)$ be the codeword in $C_{\xi,\infty}^*$ with the largest ratio $\frac{x_1}{x_2}$ such that $\frac{x_1}{x_2} \leq \frac{\tilde{x}_1}{\tilde{x}_2}$. Clearly $C \subseteq C_{\xi,\infty}^*$ thus it remains to show that no two codewords in $C$ map to the same codeword $(\tilde{x}_1, \tilde{x}_2) \in \tilde{C}$. Since $C$ is zero-error, $(x_1, x_2)$ and $(\tilde{x}_1, \tilde{x}_2)$ are distinguishable. Hence, by Lemma 3, we can assume without loss of generality $\xi^2 \frac{\tilde{x}_1}{\tilde{x}_2} \leq \xi^2 \frac{x_1}{x_2}$. By definition of $\tilde{C}$, we have $\frac{\tilde{x}_1}{\tilde{x}_2} \leq \frac{x_1}{x_2}$. It follows that $\xi^2 \frac{\tilde{x}_1}{\tilde{x}_2} \leq \xi^2 \frac{x_1}{x_2} < \frac{x_1}{x_2}$. Then, since $\frac{x_1}{x_2} \in U$, we have in particular $\frac{x_1}{x_2} \in U \cap (\xi^2 \frac{x_1}{x_2}, \infty)$. Since $(\tilde{x}_1, \tilde{x}_2) \in C_{\xi,\infty}^*$, there must be a codeword $(\tilde{x}_1, \tilde{x}_2) \in C_{\xi,\infty}^*$ such that $\gcd(\tilde{x}_1, \tilde{x}_2) = 1$ and $\xi^2 \frac{x_1}{x_2} < \frac{x_1}{x_2}$. This contradicts the assumption that $(\tilde{x}_1, \tilde{x}_2)$ is the codeword with the largest ratio $\frac{x_1}{x_2}$. Hence, the code $\tilde{C}$ is optimal.

**E. Jitter ($\xi > 1$) and Bounded Clock Drift ($\gamma < \infty$)**

For this case, which is the most general, it is difficult to obtain exact characterization of the optimal code. Nevertheless, we provide an achievable zero-error code for the case of $k = 2$.

Recall the construction of $L^k_\infty$ from Section III-B, defined for a tuple $x$ with $\gcd(x) = 1$. Here, since each element $Y_i = T Z_i x_i$ can change by a factor of $\gamma \xi$, we take the same construction but with parameter $\gamma \xi$, namely $L^{\gamma \xi}_\infty$. In the following theorem we build a (possibly suboptimal) zero-error code using these sets and the optimal code for $\gamma = \infty$ developed in the previous section.

**Theorem 5.** The following code is zero-error for a channel with $k = 2$, $\xi > 1$, and $\gamma < \infty$:

$$C_{\xi,\gamma} = \bigcup_{x \in C_{\xi,\infty}^*} L^x_\gamma,$$  

where $L^x_\gamma$ is defined in Section III-B and $C_{\xi,\infty}^*$ is defined in Section III-D.

**Proof.** By construction of the codebook $C_{\xi,\infty}^*$ and from the arguments of the previous section, it is clear that codewords from different $L^x_\gamma$ are distinguishable (for any $\gamma$). After decoding the ratio $\frac{x_1}{x_2}$ at the receiver, it then needs to decode one of the runs, say $x_1$. That is, we need to show that there are no $T, T' \in [T_1, T_2]$ and $Z_1, Z_1' \in [a, b]$ such that $T Z_1 = T' Z_1' x_1' x_2'$ for two distinct codewords, or equivalently $\frac{x_1'}{x_2'} = \frac{\gamma x_1}{x_2}$. By construction of $L^x_\gamma$, we must have $\frac{x_1'}{x_2'} \geq \gamma \xi$ or $\frac{x_1'}{x_2'} < \gamma^{-1} \xi^{-1}$. On the other hand, we have

$$\frac{x_1}{x_2} < T_1 a / T_2 b \leq T_1' Z_1 / T_1 a = \gamma \xi.$$ 

Therefore all codewords in $C_{\xi,\gamma}$ are distinguishable.

**IV. Numerical Results**

Fig. 4 shows the zero-error capacity without jitter ($\xi = 1$) as a function of the clock drift ratio $\gamma$. It can be seen that the decrease in rate incurred by clock drift is rather small.

Fig. 5 shows the zero-error capacity and capacity lower bound as a function of jitter, for $\gamma = 1$ (no clock drift), $\gamma = 7/4$ (as in [9]), and $\gamma = \infty$ (unbounded clock drift). Note that in general, a zero-error code designed for jitter $\xi$ will be zero-error for any $\xi'$ such that $\xi' < \xi$. Hence the zero-error capacity should be a decreasing function of $\xi$. This does not hold for the lower bound in Theorem 5 since the construction there is not necessarily optimal. However, we can obtain a tighter lower bound for a given jitter parameter $\xi$ by using the largest codebook out of all the codebooks for $\xi' \geq \xi$.

$$R_{\xi,\gamma} = \max_{\xi' \geq \xi} \log |C_{\xi',\gamma}|,$$

which is depicted by the dashed curve.

Finally, Fig. 6 and Fig. 7 show the capacity as a function of the frame size $M$, without clock drift and with unbounded clock drift, for $k = 2$ and $k = 3$ pulses, respectively. This is compared to the naive scheme of [9], where the first pulse is used to learn the clock cycle duration $T$, and the remaining $k - 1$ pulses can be allocated freely in the remaining $M - 1$ bins, yielding a rate of $\log(M - 1)$.

**V. Conclusion**

We introduced a model for communication with crystal-free radios, which includes two components of clock uncertainty: jitter and clock drift. The effects of slow clock drift suggest a new approach to designing codes for this type of radios. In particular, we show that estimating the clock cycle at the
receiver may be suboptimal, and characterize the optimal code by considering ratios of runs, which are unaffected by clock drift.

When both jitter and clock drift are present, we find the capacity or achievable rate for a number of special cases. Characterizing the capacity and optimal zero-error codes for the case of general \((k, M, \xi, \gamma)\) is the subject of ongoing research.

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