Kostka functions associated to complex reflection groups
and a conjecture of Finkelberg-Ionov

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Abstract. Kostka functions $K_{\lambda, \mu}^\pm(t)$, indexed by $r$-partitions $\lambda, \mu$ of $n$, are a generalization of Kostka polynomials $K_{\lambda, \mu}(t)$ indexed by partitions $\lambda, \mu$ of $n$. It is known that Kostka polynomials have an interpretation in terms of Lusztig’s partition function. Finkelberg and Ionov defined alternate functions $K_{\lambda, \mu}(t)$ by using an analogue of Lusztig’s partition function, and showed that $K_{\lambda, \mu}(t) \in \mathbb{Z}_{\geq 0}[t]$ for generic $\mu$ by making use of a coherent realization. They conjectured that $K_{\lambda, \mu}(t)$ coincide with $K_{\lambda, \mu}^-(t)$. In this paper, we show that their conjecture holds. We also discuss the multi-variable version, namely, $r$-variable Kostka functions $K_{\lambda, \mu}^\pm(t_1, \ldots, t_r)$.

1. Introduction

Let $K_{\lambda, \mu}(t) \in \mathbb{Z}[t]$ be Kostka polynomials indexed by partitions $\lambda, \mu$ of $n$. It is known by [M, III, Example 4] that Kostka polynomials have an interpretation in terms of Lusztig’s partition function. (Actually, Lusztig defined a partition function in [L1], and conjectured a formula on a $q$-analogue of the weight multiplicities for semisimple Lie algebras in terms of his partition function, as a generalization of the above result which corresponds to the case of type $A$. Soon after that the conjecture was proved by [K].) Let $\mathcal{P}_{n,r}$ be the set of $r$-tuples of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\sum_{i=1}^r |\lambda^{(i)}| = n$. In [S1, S2], as a generalization of the classical Kostka polynomials, Kostka functions $K_{\lambda, \mu}^\pm(t)$ attached to $\lambda, \mu \in \mathcal{P}_{n,r}$ were introduced. (In general, there exist two types, “+” and “−” types. If $r = 1$ or $2$, $K_{\lambda, \mu}^+(t) = K_{\lambda, \mu}^-(t)$. If $r = 1$, they coincide with the classical Kostka polynomials.) A priori, they are rational functions in $\mathbb{Q}(t)$, and the construction depends on the choice of a total order on $\mathcal{P}_{n,r}$. $K_{\lambda, \mu}^\pm(t)$ are called Kostka functions associated to complex reflection groups, or $r$-Kostka functions, in short (see [S1] for the relationship with the complex reflection group $S_n \rtimes (\mathbb{Z}/r\mathbb{Z})^n$).

In [FI], Finkelberg and Ionov introduced polynomials $K_{\lambda, \mu}(t) \in \mathbb{Z}[t]$ attached to $\lambda, \mu \in \mathcal{P}_{n,r}$, by using an analogue of Lusztig’s partition function on $GL_{mr}$, where we choose an integer $m$ such that the number of parts of $\lambda^{(i)}, \mu^{(i)}$ is smaller than $m$ for each $i$. They proved, in the case where $\mu$ is regular (see 7.6 for the precise definition), that $K_{\lambda, \mu}(t) \in \mathbb{Z}_{\geq 0}[t]$, by showing the higher cohomology vanishing $H^{i>0}(\mathcal{R}^-, \tilde{O}(\mu)) = 0$, where $\mathcal{R}^-$ is a certain $(GL_m)^r$-equivariant vector bundle over the flag variety $\mathcal{B}$ of $(GL_m)^r$, and $\tilde{O}(\mu)$ is the pull-back of the $(GL_m)^r$-equivariant ample line bundle $O(\mu)$ over $\mathcal{B}$ associated to $\mu \in \mathcal{P}_{n,r}$. (In fact, they proved the higher cohomology vanishing by showing the Frobenius splitting of $\mathcal{R}^-$. ) In turn, the higher cohomology vanishing for general $\mu$ was recently proved by Hu [H].
Hence the positivity property for $K_{\lambda,\mu}(t)$ now holds without any restriction. Their result is a natural generalization of the coherent realization of the classical Kostka polynomials due to Brylinski [B]. On the other hand, the vector bundle $\mathcal{X}$ is nothing but (a special case of) Lusztig’s iterated convolution diagram ([L2]) associated to the cyclic quiver of $r$-vertices. In this direction, Orr and Shimozono [OS] constructed a wider class of polynomials, as a generalization of $K_{\lambda,\mu}(t)$ of [FI], by making use of Lusztig’s iterated convolution diagram associated to arbitrary quivers.

Finkelberg and Ionov conjectured in [FI] that $K_{\lambda,\mu}(t)$ coincide with our $K_{\lambda,\mu}(t)$. More generally, they construct in [FI] polynomials $K_{\lambda,\mu}(t_1,\ldots,t_r) \in \mathbb{Z}[t_1,\ldots,t_r]$, a multi-variable version of $K_{\lambda,\mu}(t)$, by making use of Lusztig’s partition function. Those polynomials have a property that $K_{\lambda,\mu}(t_1,\ldots,t_r)$ coincides with $K_{\lambda,\mu}(t)$ if $t_1 = \cdots = t_r = t$. Inspired by their work, we generalize our $r$-Kostka functions to the multi-variable case. In the one-variable case, $K_{\lambda,\mu}(t)$ are defined as the coefficients of the expansion of Schur functions $s_\lambda(x)$ in terms of the Hall-Littlewood functions $P_{\mu}^\pm(x;t)$, where $x = (x^{(1)},\ldots,x^{(r)})$ are $r$ types of infinitely many variables $x^{(k)}_1, x^{(k)}_2, \ldots$. Hence we generalize the definition of Hall-Littlewood functions to the multi-parameter case $P_{\lambda}^\pm(x;t)$ with $t = (t_1,\ldots,t_r)$, and define $K_{\lambda,\mu}^\pm(t)$ as the coefficients of the expansion of $s_\lambda(x)$ in terms of $P_{\lambda}^\pm(x;t)$. Note that the construction of Hall-Littlewood functions depends on the choice of the total order which is compatible with the dominance order on $\mathscr{P}_{n,r}$, and they are symmetric functions with respect to $x^{(1)},\ldots,x^{(r)}$ with coefficients in $\mathbb{Q}(t)$. We show that both of $K_{\lambda,\mu}^\pm(t)$ have an interpretation in terms of an analogue of Lusztig’s partition function, and that $K_{\lambda,\mu}^-(t)$ coincides with their $K_{\lambda,\mu}(t)$, which proves their conjecture in a generalized form.

The main step for the proof is to establish a closed formula for Hall-Littlewood functions as given in [M, III, 2] for the classical case. By using this formula, one can show that Hall-Littlewood functions are actually independent of the choice of the total order, and are symmetric functions with coefficients in $\mathbb{Z}[t]$. This implies that $K_{\lambda,\mu}^\pm(t) \in \mathbb{Z}[t]$, and they are independent of the choice of the total order.

Note that in establishing the closed formula for Hall-Littlewood functions, the multi-variable setting is essential. Even if one is interested only in the one-variable case, our discussion does not work without multi-variable setting.

In December of 2015, Michael Finkelberg gave a lecture concerning their conjecture at the conference in Shanghai, Chongming Island. This work arose from his interesting talk there, and from his question on the stability of Kostka functions on the occasion of the conference at Besse-et-St-Anastaise in 2013. The author is very grateful for him for stimulating discussions.

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2. HALL-LITTLEWOOD FUNCTIONS WITH MULTI-PARAMETER

2.1. First we recall basic properties of Hall-Littlewood functions and Koszta polynomials in the original setting, following [M]. Let $\mathcal{P}_n$ be the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_i \geq 0$ such that $|\lambda| = \sum \lambda_i = n$. For a partition $\lambda$, the length $l(\lambda)$ of $\lambda$ is defined as the number of $\lambda_i$ such that $\lambda_i \neq 0$.

Let $\Lambda = \Lambda(y) = \bigoplus_{n \geq 0} \Lambda^n$ be the ring of symmetric functions over $\mathbb{Z}$ with respect to the variables $y = (y_1, y_2, \ldots)$, where $\Lambda^n$ denotes the free $\mathbb{Z}$-module of symmetric functions of degree $n$. For each $\lambda \in \mathcal{P}_n$, the Schur function $s_\lambda(y) \in \Lambda$ is defined as follows; first choose finitely many variables $y_1, \ldots, y_m$ such that $m \geq l(\lambda)$, and define the Schur polynomial $s_\lambda(y_1, \ldots, y_m) \in \mathbb{Z}[y_1, \ldots, y_m]$ by

$$s_\lambda(y_1, \ldots, y_m) = \det(y_i^{\lambda_j + m - j}) / \det(y_i^{m - j}).$$

$s_\lambda(y_1, \ldots, y_m)$ satisfies the stability property

$$s_\lambda(y_1, \ldots, y_m, y_{m+1})|_{y_{m+1}=0} = s_\lambda(y_1, \ldots, y_m),$$

and one can define $s_\lambda(y)$ by

$$s_\lambda(y) = \lim_{m \to \infty} s_\lambda(y_1, \ldots, y_m).$$

Then $\{s_\lambda | \lambda \in \mathcal{P}_n\}$ gives a $\mathbb{Z}$-basis of $\Lambda^n$.

2.2 We fix $m$, and consider a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_n$ such that $l(\lambda) \leq m$. We denote $\lambda$ as $\lambda = (0^{m_0}, 1^{m_1}, 2^{m_2}, \ldots)$, where $m_i = \#\{j | \lambda_j = i\}$ for $i = 0, 1, 2, \ldots$. Let $t$ be an indeterminate. We define a polynomial $v_\lambda(t) \in \mathbb{Z}[t]$ as follows; for each integer $k \geq 1$, we define $v_k(t)$ by

$$(2.2.1) \quad v_k(t) = \sum_{w \in S_k} t^{\ell(w)} = \prod_{i=1}^{k} \frac{1 - t^i}{1 - t},$$

where $\ell(w)$ is the length function of the symmetric group $S_k$ of degree $k$, and put $v_k(t) = 1$ for $k = 0$. Set

$$(2.2.2) \quad v_\lambda(t) = \prod_{i \geq 0} v_{m_i}(t).$$

The symmetric group $S_m$ acts on the set of variables $\{y_1, \ldots, y_m\}$ as permutations. For $l(\lambda) \leq m$, we define the Hall-Littlewood polynomial $P_\lambda(y_1, \ldots, y_m; t) \in \mathbb{Z}[y_1, \ldots, y_m; t]$ by
where we use the standard notation $y^\alpha = y_1^{\alpha_1}y_2^{\alpha_2}\cdots y_m^{\alpha_m}$ for $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$.

Let $\Lambda[t] = \mathbb{Z}[t] \otimes \mathbb{Z} \Lambda$ be the ring of symmetric functions with coefficients in $\mathbb{Z}[t]$. We have $\Lambda[t] = \bigoplus_{n \geq 0} \Lambda^n[t]$, where $\Lambda^n[t] = \mathbb{Z}[t] \otimes \mathbb{Z} \Lambda^n$. The Hall-Littlewood polynomial has the stability property, and one can define the Hall-Littlewood function $P_\lambda(y; t) = \sum_{\mu \in \mathcal{P}_n} K_{\lambda, \mu}(t)P_\mu(y; t)$.

2.3. For $\lambda, \mu \in \mathcal{P}_n$, the Kostka polynomials $K_{\lambda, \mu}(t) \in \mathbb{Z}[t]$ are defined by the formula

$$s_\lambda(y) = \sum_{\mu \in \mathcal{P}_n} K_{\lambda, \mu}(t)P_\mu(y; t).$$

We define a partial order $\xi \leq \eta$, the so-called dominance order, on $\mathbb{Z}^m$ by the condition, for $\xi = (\xi_1, \ldots, \xi_m), \eta = (\eta_1, \ldots, \eta_m) \in \mathbb{Z}^m$,

$$\sum_{i=1}^{k} \xi_i \leq \sum_{i=1}^{k} \eta_i \quad \text{for } k = 1, \ldots, m.$$

For each partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, we define an integer $n(\lambda)$ by

$$n(\lambda) = \sum_{i=1}^{m} (i - 1)\lambda_i.$$

It is known that $K_{\lambda, \mu}(t) = 0$ unless $\mu \leq \lambda$, and in which case, $K_{\lambda, \mu}(t)$ is monic of degree $n(\mu) - n(\lambda)$.

For any integer $s \geq 1$, we define a function $q_s(y_1, \ldots, y_m; t)$ by

$$q_s(y_1, \ldots, y_m; t) = (1 - t) \sum_{i=1}^{m} y_i^s \prod_{j \neq i} \frac{y_i - ty_j}{y_i - y_j},$$

and put $q_s = 1$ for $s = 0$. The generating function for $q_s$ is given as follows ([M, III, (2.10)]). Let $u$ be another indeterminate. Then we have

$$\sum_{s=0}^{\infty} q_s(y_1, \ldots, y_m; t)u^s = \prod_{i=1}^{m} \frac{1 - tuy_i}{1 - uy_i}.$$
For a partition $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathcal{P}_n$, we define a function $q_\lambda$ by

$$q_\lambda(y_1, \ldots, y_m; t) = \prod_{i=1}^{m} q_{\lambda_i}(y_1, \ldots, y_m; t).$$

By taking $m \mapsto \infty$, $q_\lambda(y_1, \ldots, y_m; t)$ defines $q_\lambda(y; t) \in \Lambda^n[t]$. Then $Q_\lambda$ has an expansion by $q_\mu$, $Q_\lambda = q_\lambda + \sum_{\mu > \lambda} a_{\lambda, \mu}(t) q_\mu$, with $a_{\lambda, \mu}(t) \in \mathbb{Z}[t]$. Hence $\{q_\lambda \mid \lambda \in \mathcal{P}_n\}$ gives a $Q(t)$-basis of $\Lambda^n_Q(t)$.

**2.4.** We fix an integer $r \geq 2$, and consider the $r$ types of variables $x = (x^{(1)}, \ldots, x^{(r)})$, where $x^{(k)}$ stands for the infinitely many variables $x_1^{(k)}, x_2^{(k)}, \ldots$. We consider the ring of symmetric functions $\Xi = \Xi(x) = \Lambda(x^{(1)}) \otimes \cdots \otimes \Lambda(x^{(r)})$, symmetric with respect to each variable $x^{(k)}$. We have $\Xi = \bigoplus_{n \geq 0} \Xi^n$, where $\Xi^n$ is the free $\mathbb{Z}$-module consisting of homogeneous symmetric functions of degree $n$.

Let $\mathcal{P}_{n,r}$ be the set of $r$-tuples of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$ such that $\sum_{k=1}^{r}|\lambda^{(k)}| = n$. For $\lambda \in \mathcal{P}_{n,r}$, we choose an integer $m$ such that $m \geq l(\lambda^{(k)})$ for any $k$, and consider the finitely many variables $\{x_i^{(k)} \mid 1 \leq k \leq r, 1 \leq i \leq m\}$. We prepare the index set

$$(2.4.1) \quad \mathcal{M} = \mathcal{M}(m) = \{(k, i) \mid 1 \leq k \leq r, 1 \leq i \leq m\},$$

and write $x_i^{(k)}$ as $x_{\nu}$ for $\nu = (k, i) \in \mathcal{M}$. We denote by $x_{\mathcal{M}}$ the set of variables $\{x_{\nu} \mid \nu \in \mathcal{M}\}$.

We define a polynomial $s_\lambda(x_{\mathcal{M}})$ by

$$s_\lambda(x_{\mathcal{M}}) = \prod_{k=1}^{r} s_{\lambda^{(k)}}(x_1^{(k)}, \ldots, x_m^{(k)}) \in \mathbb{Z}[x_{\mathcal{M}}].$$

$s_\lambda(x_{\mathcal{M}})$ has the stability property with respect to the operation $x_{m+1}^{(1)} = \cdots = x_{m+1}^{(r)} = 0$ in $\mathcal{M}(m+1)$, and by taking $m \mapsto \infty$, one can define $s_\lambda(x) \in \Xi$. Then $\{s_\lambda(x) \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a basis of $\Xi^n$.

For a partition $\lambda \in \mathcal{P}_n$, the monomial symmetric polynomial $m_\lambda(y_1, \ldots, y_m) \in \mathbb{Z}[y_1, \ldots, y_m]$ is defined for $m \geq l(\lambda)$. For $\lambda \in \mathcal{P}_{n,r}$, we define $m_\lambda(x_{\mathcal{M}})$ by

$$m_\lambda(x_{\mathcal{M}}) = \prod_{k=1}^{r} m_{\lambda^{(k)}}(x_1^{(k)}, \ldots, x_m^{(k)}) \in \mathbb{Z}[x_{\mathcal{M}}].$$

By taking $m \mapsto \infty$, one can define $m_\lambda(x) \in \Xi^n$, and $\{m_\lambda(x) \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a basis of $\Xi^n$.

**2.5.** For any integer $s \geq 1$ we define a function $q_{s, \pm}^{(k)}(x; t)$ (for $x = x_{\mathcal{M}}$) by

$$(2.5.1) \quad q_{s, \pm}^{(k)}(x; t) = \sum_{i=1}^{m} (x_i^{(k)})^{s-1} \frac{\prod_{j \geq 1} (x_i^{(k)} - tx_j^{(k+1)})}{\prod_{j \neq i} (x_i^{(k)} - x_j^{(k)})},$$
where we regard $k \in \mathbb{Z}/r\mathbb{Z}$, and put $q_{n,s}^{(k)}(x; t) = 1$ for $s = 0$.

Let $u$ be another indeterminate. As in the proof of [S1, Lemma 2.3], by using the Lagrange's interpolation formula

$$\prod_{i=1}^{m} \frac{1 - tu_{x_i}^{(k+1)}}{1 - u_{x_i}^{(k)}} = 1 + \sum_{i=1}^{m} \frac{u_{x_i}^{(k)} - tu_{x_i}^{(k+1)}}{1 - u_{x_i}^{(k)}} \prod_{j \neq i} \frac{x_j^{(k)} - tx_j^{(k+1)}}{x_i^{(k)} - x_j^{(k+1)}},$$

one can prove the formula

$$\sum_{s=0}^{\infty} q_{n,s}^{(k)}(x; t) u^s = \prod_{i=1}^{m} \frac{1 - tu_{x_i}^{(k+1)}}{1 - u_{x_i}^{(k)}}.$$  

(2.5.2)

It follows from (2.5.2) that $q_{n,s}^{(k)}(x; t) \in \mathbb{Z}[x, \mathcal{M}; t]$, and symmetric with respect to the variables $x^{(k)}, x^{(k+1)}$. Moreover, it satisfies the stability property.

Let $\Xi[t] = \mathbb{Z}[t] \otimes \mathbb{Z} \Xi$ be the ring of symmetric functions in $\Xi$ with coefficients in $\mathbb{Z}[t]$. Put $\Xi^n[t] = \mathbb{Z}[t] \otimes \mathbb{Z} \Xi^n$. More generally, we consider the multi-parameter case. Let $t = (t_1, \ldots, t_r)$ be $r$-parameters, and consider $\mathbb{Z}[t] = \mathbb{Z}[t_1, \ldots, t_r]$. Put $\Xi[t] = \mathbb{Z}[t] \otimes \mathbb{Z} \Xi$. We have $\Xi[t] = \bigoplus_{n \geq 0} \Xi^n[t]$, where $\Xi^n[t] = \mathbb{Z}[t] \otimes \mathbb{Z} \Xi^n$.

For $\lambda \in \mathcal{P}_{n,r}$, we define polynomials $q_{\lambda}^{(k)}(x, \mathcal{M}; t) \in \mathbb{Z}[x, \mathcal{M}; t] = \mathbb{Z}[x, \mathcal{M}; t_1, \ldots, t_r]$ by

$$q_{\lambda}^{(k)}(x, \mathcal{M}; t) = \prod_{k \in \mathbb{Z}/r\mathbb{Z}} \prod_{i=1}^{m} q_{\lambda_i^{(k)}, \pm}^{(k)}(x; t_{k-c}),$$  

(2.5.3)

where $c = 1$ for the “+”-case, and $c = 0$ for the “−”-case. Then $q_{\lambda}^{(k)}(x, \mathcal{M}; t)$ satisfies the stability condition, and one can define $q_{\lambda}^{(k)}(x; t) \in \Xi^n[t]$. Note that if $t_1 = \cdots = t_r = t$, $q_{\lambda}^{(k)}(x; t)$ coincides with $q_{\lambda}^{(k)}(x; t)$ defined in [S1, 2.4].

Put $\Xi^n_Q(t) = Q(t) \otimes \mathbb{Z} \Xi^n$, and $\Xi^n_Q(t) = Q(t) \otimes \mathbb{Z} \Xi^n$. It is known by [S1, (4.7.2)] that $\{q_{\lambda}^{(k)}(x; t) \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a $Q(t)$-basis of $\Xi^n_Q(t)$. The analogous fact holds also in the multi-parameter case.

**Lemma 2.6.** $\{q_{\lambda}^{(k)}(x; t) \mid \lambda \in \mathcal{P}_{n,r}\}$ gives a $Q(t)$-basis of $\Xi^n_Q(t)$.

**Proof.** Since $\{s_{\mu}(x) \mid \mu \in \mathcal{P}_{n,r}\}$ is a $\mathbb{Z}[t]$-basis of $\Xi^n[t]$ and $q_{\lambda}^{(k)}(t) \in \Xi^n[t]$, $q_{\lambda}^{(k)}(x; t)$ can be written as a linear combination of $s_{\mu}(x)$. Let $A(t) = (a_{\lambda, \mu}(t))$ be the corresponding matrix with $a_{\lambda, \mu}(t) \in \mathbb{Z}[t]$. Let $A(t)$ be the matrix obtained from $A(t)$ by putting $t_1 = \cdots = t_r = t$. Then $A(t)$ is a non-singular matrix by the above remark. Hence $A(t)$ is also non-singular, and $q_{\lambda}^{(k)}(x; t)$ gives a basis of $\Xi^n_Q(t)$.

$\square$
2.7. We consider two types of (infinitely many) variables \( x = (x^{(1)}, \ldots, x^{(r)}) \) and \( y = (y^{(1)}, \ldots, y^{(r)}) \), and put

\[
\Omega(x, y; t) = \prod_{k \in \mathbb{Z}/r} \prod_{i,j} \frac{1 - t_k x_i^{(k)} y_j^{(k+1)}}{1 - x_i^{(k)} y_j^{(k)}}.
\]

The following formula is a multi-parameter version of \([S1, (2.5.1)]\). The proof is done by an entirely similar way, and we omit it.

**Proposition 2.8.** Under the notation above, we have

\[
(2.8.1) \quad \Omega(x, y; t) = \sum_{\lambda} q_{\lambda}^+(x; t)m_{\lambda}(y) = \sum_{\lambda} m_{\lambda}(x)q_{\lambda}^-(y; t).
\]

**Remark 2.9.** In the one-parameter case, another expression of \( \Omega(x, y; t) \) involving power sum symmetric functions \( p_{\lambda}(x) \) was proved in \([S1, (2.5.2)]\). However, we don’t have a generalization of \((2.5.2)\) in \([S1]\) in the multi-parameter case.

2.10. We define a non-degenerate bilinear form \( \langle \cdot, \cdot \rangle : \Xi^n_Q(t) \times \Xi^n_Q(t) \to Q(t) \) by

\[
(2.10.1) \quad \langle q_{\lambda}^+(x; t), m_{\mu}(x) \rangle = \delta_{\lambda, \mu}
\]

for \( \lambda, \mu \in \mathcal{P}_{n,r} \). By using a similar argument as in \([M, I,4]\), \((2.8.1)\) implies that

\[
(2.10.2) \quad \langle m_{\lambda}(x), q_{\mu}^-(x; t) \rangle = \delta_{\lambda, \mu}.
\]

Let \( \mathcal{A} \) be the \( Q \)-subalgebra of \( Q(t) \) consisting of rational functions \( f/g \) such that \( g(0) \neq 0 \), where \( 0 = (0, \ldots, 0) \). Put \( \mathcal{A} \otimes_{\mathbb{Z}} \Xi^n = \Xi^n_{\mathcal{A}} \). Then \( \Xi^n_{\mathcal{A}}|_{t=0} = \Xi^n_Q \). By a similar argument as in \([M, I,4]\), one can show

\[
(2.10.3) \quad \Omega(x, y; 0) = \prod_{k \in \mathbb{Z}/r} \prod_{i,j} (1 - x_i^{(k)} y_j^{(k)})^{-1} = \sum_{\lambda} s_{\lambda}(x)s_{\lambda}(y).
\]

Hence if we define the symmetric bilinear form \( \langle \cdot, \cdot \rangle_0 \) on \( \Xi^n_Q \) by \( \langle s_{\lambda}, s_{\mu} \rangle_0 = \delta_{\lambda, \mu} \) for \( \lambda, \mu \in \mathcal{P}_{n,r} \), the restriction of \( \langle \cdot, \cdot \rangle \) on \( \Xi^n_{\mathcal{A}} \) gives rise to the form \( \langle \cdot, \cdot \rangle_0 \) on \( \Xi^n_Q \) by taking \( t \mapsto 0 \).

2.11. For \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r} \) with \( \lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}) \), we define \( c(\lambda) \in \mathbb{Z}_{\geq 0}^m \) by

\[
c(\lambda) = (\lambda_1^{(r)}, \ldots, \lambda_m^{(r)}).
\]
We define a partial order on $P_{n,r}$ by the condition, for $\lambda, \mu \in P_{n,r}$, $\lambda \leq \mu$ if $c(\lambda) \leq c(\mu)$ with respect to the dominance order on $Z^{n,r}$. The partial order $\lambda \leq \mu$ is called the dominance order on $P_{n,r}$.

In the remainder of this section, we fix a total order $\lambda \leq \mu$ on $P_{n,r}$ which is compatible with the dominance order $\lambda \leq \mu$.

By making use of the bilinear form $(\cdot, \cdot)$, we shall construct Hall-Littlewood functions with multi-parameter $P^\pm_\lambda(x; t)$. The following result is an analogue of [S1, Proposition 4.8].

Proposition 2.12. For each $\lambda \in P_{n,r}$, there exists a unique function $P^\pm_\lambda(x; t) \in \Xi_{n,r}$ satisfying the following properties.

(i) $P^\pm_\lambda(x; t)$ can be expressed as

$$P^\pm_\lambda(x; t) = s_\lambda(x) + \sum_{\mu \prec \lambda} u^\pm_{\lambda, \mu}(t)s_\mu(x),$$

where $u^\pm_{\lambda, \mu}(t) \in \mathcal{A}$.

(ii) $\langle P^+_\lambda, P^-_\mu \rangle = 0$ if $\lambda \neq \mu$.

(iii) $P^\pm_\lambda(x; 0) = s_\lambda(x)$.

Proof. We prove the proposition following the discussion in [S1, Remark 4.9]. We construct $P^\pm_\lambda(x; t)$ satisfying the properties (i), (ii), (iii) by induction on the total order $\leq$ on $P_{n,r}$. Let $\lambda_0 = (-; \ldots; -; (1^n))$. $\lambda_0$ is the minimum element in $P_{n,r}$ with respect to $\leq$, and so the minimum element with respect to $\preceq$. By (i), $P^\pm_{\lambda_0}(x; t)$ must coincide with $s_{\lambda_0}(x)$, which clearly satisfies (iii). Take $\lambda \in P_{n,r}$, and assume, for any $\lambda', \lambda'' \prec \lambda$, that $P^\pm_{\lambda'}, P^\pm_{\lambda''}$ satisfying (i), (ii) and (ii'): $\langle P^+_\lambda, P^-_{\lambda'} \rangle = 0$ for $\lambda'' \neq \lambda'$, was constructed. Note that the condition (i) for $P^\pm_\lambda$ is equivalent to the condition

$$(2.12.1) \quad P^\pm_\lambda = s_\lambda + \sum_{\lambda' \prec \lambda} d^\pm_{\lambda, \lambda'} P^\pm_{\lambda'}$$

with $d^\pm_{\lambda, \lambda'} \in \mathcal{A}$. By taking the inner product with $P^-_\mu (\mu \prec \lambda)$ in (2.12.1), we have a relation

$$\langle P^\pm_\lambda, P^-_\mu \rangle = \langle s_\lambda, P^-_\mu \rangle + d^+_{\lambda, \mu} \langle P^+_\lambda, P^-_\mu \rangle.$$  

By (iii) and by 2.10, $\langle P^+_\lambda, P^-_\mu \rangle |_{t=0} = \langle s_\lambda, s_\mu \rangle |_{t=0} = 1$. In particular, $\langle P^+_\lambda, P^-_\mu \rangle \neq 0$. Hence if we define $P^+_\lambda$ as in (2.12.1) with $d^+_{\lambda, \lambda'} = -\langle s_\lambda, P^-_{\lambda'} \rangle / \langle P^+_\lambda, P^-_\lambda \rangle |_{t=0} \in \mathcal{A}$, we have $\langle P^+_\lambda, P^-_\mu \rangle = 0$ for any $\mu \prec \lambda$. Since $\langle s_\lambda, P^-_\lambda \rangle |_{t=0} = \langle s_\lambda, s_\lambda \rangle |_{t=0} = 0$, we have $d^+_{\lambda, \lambda'}(0) = 0$. It follows that $P^+_\lambda(x; 0) = s_\lambda(x)$. A similar argument shows, if we define $P^-_\lambda$ as in (2.12.1) with $d^-_{\lambda, \lambda'} = -\langle P^+_{\lambda'}, s_\lambda \rangle / \langle P^+_\lambda, P^-_{\lambda'} \rangle |_{t=0} \in \mathcal{A}$, that $P^+_\lambda$ satisfies the required condition. Thus one can construct $P^\pm_\lambda$ satisfying (i), (ii), (iii). The uniqueness is clear from the construction. \qed

2.13. The discussion in the proof of Proposition 2.12 shows that $b_\lambda(t) = \langle P^+_\lambda, P^-_\lambda \rangle^{-1} \in Q(t) - \{0\}$. We define $Q^+_\lambda(x; t) \in \Xi_{n,r}$ by
Theorem 2.14.  Let $\varepsilon \in \{+,-\}$. For each $\lambda \in \mathcal{P}_{n,r}$, there exists a unique function $P^\varepsilon_\lambda(x;t)$ satisfying the following properties.

(i) $P^\varepsilon_\lambda(x;t)$ can be expressed as

$$P^\varepsilon_\lambda(x;t) = \sum_{\mu \geq \lambda} c_{\lambda,\mu}(t)q^\varepsilon_{\mu}(x;t),$$

where $c_{\lambda,\mu}(t) \in \mathbb{Q}(t)$ with $c_{\lambda,\lambda}(t) \neq 0$.

(ii) $P^\varepsilon_\lambda(x;t)$ can be expressed as

$$P^\varepsilon_\lambda(x;t) = \sum_{\mu \leq \lambda} u_{\lambda,\mu}(t)s_\mu(x),$$

where $u_{\lambda,\mu}(t) \in \mathbb{Q}(t)$ with $u_{\lambda,\lambda}(t) = 1$.

A similar property holds also for $Q^\varepsilon_\lambda(x;t)$ by replacing the condition for $c_{\lambda,\mu}(t), u_{\lambda,\mu}(t)$
by $c_{\lambda,\lambda}(t) = 1$ and $u_{\lambda,\lambda}(t) \neq 0$.

Proof. For bases $u = \{u_\lambda\}, v = \{v_\lambda\}$ of $\Xi^n_\mathbb{Q}(t)$, we denote by $M = M(u,v)$ the transition matrix $(m_{\lambda,\mu})$ of two bases, where $u_\lambda = \sum_{\mu \in \mathcal{P}_{n,r}} m_{\lambda,\mu}v_\mu$. Consider the bases $P^\pm = \{P^\pm_\lambda\}, q^\pm = \{q^\pm_\lambda\}, s = \{s_\lambda\}, m = \{m_\lambda\}$ of $\Xi^n_\mathbb{Q}(t)$, and put

$$A^\pm = M(q^\pm, P^\pm), \quad B^\pm = M(m, P^\pm).$$

We want to show that $A^\pm$ is upper triangular. By Proposition 2.12, $M(P^\pm, s)$ is lower unitriangular. On the other hand, since the total order $\preceq$ is compatible with the dominance order $\preceq$ on $\mathcal{P}_{n,r}$, $M(s,m)$ is lower unitriangular (the verification is reduced to the case where $r = 1$, in which case, it is well-known). Thus $B^\pm = \ldots$
$M(\pm s, m)$ is lower unitriangular, and $B_\pm$ is also lower unitriangular. Put $D_\pm = \lambda B_\pm A_\pm$. If we put $A_+ = (A^+_{\lambda, \mu}), B_- = (B^-_{\lambda, \mu})$, we have, by (2.8.1) and (2.13.3),

$$\sum_\nu q^+_{\mu}(x; t)m_\nu(y) = \sum_{\nu, \mu', \nu'} A^+_{\nu, \mu} B^-_{\nu, \mu'} P^+_{\mu}(x; t) P^-_{\mu'}(y; t)$$

$$= \sum_{\lambda} b_\lambda(t) P^+_{\lambda}(x; t) P^-_{\lambda}(y; t).$$

Since $P^+_{\mu}(x; t) P^-_{\mu'}(y; t)$ are linearly independent, this implies that $D_+ = \lambda B_- A_+$ is a diagonal matrix with $\lambda\mu$-entry $b_\lambda(t)$. Hence $A_+$ is upper triangular with $A^+_{\lambda\mu} = b_\lambda(t)$. A similar argument, by using the formula for $\sum m_\nu(x) q^-_{\nu}(y; t)$ in (2.8.1), shows that $A_-$ is upper triangular with $A^-_{\lambda\mu} = b_\lambda(t)$. Thus $P^\pm_{\lambda}(x; t)$ satisfies the conditions (i) and (ii).

Next we show the uniqueness of $P^\pm_{\lambda}$. Take $\varepsilon \in \{+, -\}$, and assume that $R$ satisfies the condition (i) and (ii) for $\varepsilon$. By (i) and (ii) for $P^\varepsilon_{\lambda}$, one can write as

$$s_\mu(x) = P^\varepsilon_{\mu}(x; t) + \sum_{\mu' \neq \mu} u_{\mu', \mu''} P^\varepsilon_{\mu''}(x; t),$$

$$q^\varepsilon_{\mu}(x; t) = \sum_{\mu' \geq \mu} c_{\mu', \mu''} P^\varepsilon_{\mu''}(x; t).$$

It follows that

$$R(x; t) = P^\varepsilon_{\lambda}(x; t) + \sum_{\mu'' < \lambda} u''_{\lambda, \mu''} P^\varepsilon_{\mu''}(x; t),$$

$$R(x; t) = \sum_{\mu'' \geq \lambda} c''_{\lambda, \mu''} P^\varepsilon_{\mu''}(x; t).$$

Hence $R = P^\varepsilon_{\lambda}$, and the uniqueness follows. This proves the theorem for $P^\varepsilon_{\lambda}(x; t)$.

The above discussion shows that $c_{\lambda, \lambda}(t)$ (for $P^\varepsilon_{\lambda}$) coincides with $b_\lambda(t)^{-1}$. Thus by multiplying $b_\lambda(t)$ on both sides of (i) and (ii), we obtain the corresponding formulas for $Q^\varepsilon_{\lambda}(x; t)$. The theorem is proved. $\square$

2.15. Since $\{P^\varepsilon_{\lambda}(x; t) \mid \lambda \in \mathcal{P}_{n,r}\}$ and $\{s_\lambda(x) \mid \lambda \in \mathcal{P}_{n,r}\}$ are bases of $\mathcal{E}_Q(t)$, there exist unique functions $K^\pm_{\lambda, \mu}(t) \in Q(t)$ satisfying the properties

$$(2.15.1) \quad s_\lambda(x) = \sum_{\mu \in \mathcal{P}_{n,r}} K^\pm_{\lambda, \mu}(t) P^\pm_{\mu}(x; t).$$

$K^\pm_{\lambda, \mu}(t) \in Q(t) = Q(t_1, \ldots, t_r)$ are called multi-variable Kostka functions. By definition, $K^\pm_{\lambda, \mu} = 0$ unless $\lambda \succeq \mu$. 
Put \( t_1 = (t, \ldots, t) \), and let \( \mathscr{A} \) be the \( \mathbb{Q} \)-subalgebra of \( \mathbb{Q}(t) \) consisting of rational functions \( f/g \) such that \( g(t_1) \neq 0 \). Put \( \Xi^n_{\mathscr{A}^1} = \mathscr{A} \otimes \mathbb{Z} \Xi^n \). We have \( \Xi^n_{\mathscr{A}^1} |_{t=t_1} = \Xi^n_{\mathbb{Q}}(t) \). We have

\[
\Omega(x, y; t)|_{t=t_1} = \Omega(x, y; t) = \prod_{k \in \mathbb{Z}/r} \prod_{i,j} \frac{1 - tx_i^{(k)} y_j^{(k+1)}}{1 - x_i^{(k)} y_j^{(k)}}.
\]

Thus if we define a bilinear form \( \langle \cdot, \cdot \rangle_1 \) on \( \Xi^n_{\mathbb{Q}}(t) \) by

\[
\langle q_{\lambda}(x; t), m_\mu(x) \rangle_1 = \delta_{\lambda, \mu},
\]

[S1, Proposition 2.5] implies that the restriction of \( \langle \cdot, \cdot \rangle \) on \( \Xi^n_{\mathscr{A}^1} \) induces the form \( \langle \cdot, \cdot \rangle_1 \) on \( \Xi^n_{\mathbb{Q}}(t) \) by putting \( t = t_1 \). By comparing [S1, Proposition 4.8] with Proposition 2.12, we see that

\[
P_\lambda^\pm(x; t_1) = P_\lambda^\pm(x; t), \quad Q_\lambda^\pm(x; t_1) = Q_\lambda^\pm(x; t),
\]

where \( P_\lambda^\pm(x; t), Q_\lambda^\pm(x; t) \) are Hall-Littlewood functions defined in [S1, Theorem 4.4]. In particular, we have

\[
K_{\lambda, \mu}^\pm(t, \ldots, t) = K_{\lambda, \mu}^\pm(t),
\]

where \( K_{\lambda, \mu}^\pm(t) \) is the Kostka function (with one variable) defined in [S1, 5.2].

**Remark 2.16.** In the one-variable case, a simple algorithm of computing Kostka functions \( K_{\lambda, \mu}^\pm(t) \) was given in [S1, Theorem 5.4] in connection with the representation theory of the complex reflection group \( S_n \ltimes (\mathbb{Z}/r \mathbb{Z})^n \). This formula is based on the formula (2.5.2) in [S1]. Since we don’t have an analogous formula for the multi-parameter case, we don’t know whether or not those Kostka functions \( K_{\lambda, \mu}^\pm(t) \) have a relationship with complex reflection groups as above.

### 3. Comparison of Hall-Littlewood functions for different \( r \)

#### 3.1. Let \( \mathcal{P}_{n,r}^a \) be the set of \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r} \) such that \( \lambda^{(k)} = \emptyset \) for \( k = 1, \ldots, a \). We identify \( \mathcal{P}_{n,r}^a \) with \( \mathcal{P}_{n,r-a} \) by \( \lambda \leftrightarrow \lambda' = (\lambda^{(a+1)}, \ldots, \lambda^{(r)}) \). We consider the variables \( x' = (x^{(a+1)}, \ldots, x^{(r)}) \) and \( t' = (t'_{a+1}, \ldots, t'_r) \). Assume that \( \lambda \in \mathcal{P}_{n,r}^a \). One can consider Hall-Littlewood functions \( P_\lambda^\pm(x'; t') \) and \( Q_\lambda^\pm(x'; t') \) associated to those data. In this section, we discuss the relationship between \( P_\lambda^\pm(x; t) \) and \( P_\lambda^\pm(x'; t') \), and also between \( Q_\lambda^\pm(x; t) \) and \( Q_\lambda^\pm(x'; t') \).

First consider the case where \( a = 1 \), and put \( x' = (x^{(2)}, \ldots, x^{(r)}) \). We denote by \( q_{s,\pm}^{(k)}(x'; t) \) the function of \( x' \) corresponding to the function \( q_{s,\pm}^{(k)}(x; t) \) of \( x \) (note that \( k \in \mathbb{Z}/(r-1)\mathbb{Z} \) for \( q_{s,\pm}^{(k)} \)). We have a lemma.
Lemma 3.2. \( q^{(k)}_{s,+}(x'; t_{k-1}) = q^{(k)}_{s,+}(x; t_{k-1}) \) for \( k = 3, \ldots, r \), and

(i) \( q^{(2)}_{s,+}(x'; t_1 t_r) = \sum_{s', s'' = s} (t_1)^{s''} q^{(2)}_{s',+}(x; t_1) q^{(1)}_{s'',+}(x; t_r) \).

(ii) \( q^{(k)}_{s,-}(x'; t_k) = q^{(k)}_{s,-}(x; t_k) \) for \( k = 2, \ldots, r - 1 \), and

(iii) \( q^{(r)}_{s,-}(x'; t_1 t_r) = \sum_{s', s'' = s} (t_r)^{s''} q^{(r)}_{s',-}(x; t_r) q^{(1)}_{s'',-}(x; t_1) \).

Proof. We prove (i). The first statement is clear from the definition. Recall, by (2.5.2), that

\[
\prod_{i=1}^{m} \frac{1 - t u x_i^{(k-1)}}{1 - u x_i^{(k)}} = \sum_{s \geq 0} q^{(k)}_{s,+}(x; t) u^s.
\]

Since

\[
\prod_{i=1}^{m} \frac{1 - t_1 u x_i^{(1)}}{1 - u x_i^{(2)}} \prod_{i=1}^{m} \frac{1 - t_r t_1 u x_i^{(r)}}{1 - t_1 u x_i^{(1)}} = \prod_{i=1}^{m} \frac{1 - t_1 t_r u x_i^{(r)}}{1 - u x_i^{(2)}},
\]

we have

\[
\left( \sum_{s' \geq 0} q^{(2)}_{s',+}(x; t_1) u^{s'} \right) \left( \sum_{s'' \geq 0} q^{(1)}_{s'',+}(x; t_r)(t_1 u)^{s''} \right) = \sum_{s \geq 0} q^{(2)}_{s,+}(x'; t_1 t_r) u^s.
\]

Thus (i) holds. The proof for (ii) is similar. \( \square \)

3.3. By the map \( \lambda' = (\lambda^{(2)}, \ldots, \lambda^{(r)}) \mapsto \lambda = (-, \lambda^{(2)}, \ldots, \lambda^{(r)}) \), we can identify \( \mathcal{P}_{n,r-1} \) with the subset \( \mathcal{P}_{n,r}^1 \) of \( \mathcal{P}_{n,r} \). Since the dominance order on \( \mathcal{P}_{n,r} \) is compatible with the dominance order of \( \mathcal{P}_{n,r} \), one can choose a total order on \( \mathcal{P}_{n,r} \) compatible with the total order on \( \mathcal{P}_{n,r-1} \), namely, which satisfies the property if \( \lambda \in \mathcal{P}_{n,r}^1 \) and \( \mu \preceq \lambda \), then \( \mu \in \mathcal{P}_{n,r}^1 \). More generally, by considering the sequence \( \mathcal{P}_{n,1} \subset \mathcal{P}_{n,2} \subset \cdots \subset \mathcal{P}_{n,r} \), we can choose a total order on \( \mathcal{P}_{n,r} \) so that it is compatible with the total order on each subset \( \mathcal{P}_{n,k} \).

Assume that \( \lambda \in \mathcal{P}_{n,r}^1 \). Then \( \mu \in \mathcal{P}_{n,r}^1 \) for any \( \mu \preceq \lambda \), and \( s_{\mu}(x) = s_{\mu'}(x') \) is a function with respect to the variable \( x' = (x^{(2)}, \ldots, x^{(r)}) \). Since \( Q^\pm_\lambda \) is a linear combination of \( s_{\mu} \) with \( \mu \in \mathcal{P}_{n,r}^1 \), \( Q^\pm_\lambda \) is a function with respect to the variable \( x' \). We show the following.

Proposition 3.4. Assume that \( \lambda \in \mathcal{P}_{n,r}^1 \). Let \( Q^\pm_\lambda \) be the function defined with respect to \( x' = (x^{(2)}, \ldots, x^{(r)}) \), and \( \lambda' \in \mathcal{P}_{n,r-1} \), the element corresponding to \( \lambda \). Then

\[
Q^+_\lambda(x; t_1, \ldots, t_r) = Q^+_\lambda(x'; t_2, \ldots, t_{r-1}, t_1 t_r).
\]
Proof. Since $Q^\pm_{\lambda'}$ is written as a linear combination of $s_{\mu'}(x')$ with $\mu' \preceq \lambda'$, by our choice of the total order $\succeq$ on $\mathcal{P}_{n,r}$, it is written as a linear combination of $s_{\mu}(x)$ with $\mu \preceq \lambda$. Thus it is enough to show that $Q^\pm_{\lambda'}$ can be written as a linear combination of $q^\pm_{\mu}$ with $\mu \succeq \lambda$ such that the coefficient of $q^\pm_{\lambda}$ is equal to 1. We can write

$$Q^\pm_{\lambda'}(x'; t') = q^\pm_{\lambda'}(x'; t') + \sum_{\mu' \in \mathcal{P}_{n,r-1}} c_{\lambda', \mu'}(t') q^\pm_{\mu'}(x; t'),$$

where $t' = (t_2, \ldots, t_{r-1}, t_1 t_r)$. Here for $\mu' = (\mu(2), \ldots, \mu(r))$,

$$q^\pm_{\mu'}(x; t') = \prod_{i=1}^{m} q^\pm_{\mu_{(i)}, +}(x'; t_i t_r) \prod_{k=3}^{m} q^\pm_{\mu_{(k)}, +}(x; t_{k-1}).$$

$q^\pm_{\mu_{(i)}, +}(x'; t_i t_r)$ can be written as a linear combination of $q^\pm_{\nu_{(i)}, +}(x; t_i)$ by (3.2.1), where $s' \geq 0$ are such that $\mu_{(i)}(2) = s' + s''$. Hence $q^\pm_{\mu'}$ can be written as a linear combination of various $q^\pm_{\nu}$, where $\nu = (\nu(1), \ldots, \nu(r))$ are $r$-compositions such that $\nu_i^{(1)} + \nu_i^{(2)} = \mu_i^{(2)}$ for each $i$ and that $\nu_i^{(k)} = \mu_i^{(k)}$ for $k \geq 3$. Then clearly $\nu \succeq \mu$ for $\mu = (-, \mu(2), \ldots, \mu(r))$. If we denote by $\tilde{\nu}$ the $r$-partition obtained from $\nu$ by rearranging the order, then we have $\tilde{\nu} \succeq \nu$. This is true also for $q^\pm_{\lambda'}$. It follows that $Q^\pm_{\lambda'}$ is a linear combination of various $q^\pm_{\mu}$ for $\mu \in \mathcal{P}_{n,r}$ such that $\mu \succeq \lambda$. The term $q^\pm_{\lambda}$ comes only from $q^\pm_{\lambda'}$, and it is easily checked that the coefficient of $q^\pm_{\lambda}$ is equal to 1. This proves the proposition in the “+” case. The proof for the “−” case is done similarly by using (3.2.2).

As a corollary, we have the following result, which describes the relationship of Hall-Littlewood functions and Kostka functions for different $r$.

**Theorem 3.5.**

(i) Assume that $\lambda \in \mathcal{P}_{n,r}$ for $1 \leq a < r$. Let $Q^\pm_{\lambda'}$, $P^\pm_{\lambda'}$ be the functions defined with respect to $x' = (x^{(a+1)}, \ldots, x^{(r)})$, and $\lambda' \in \mathcal{P}_{n,r-a}$ the element corresponding to $\lambda$. Then

$$Q^\pm_{\lambda}(x; t_1, \ldots, t_r) = Q^\pm_{\lambda}(x', t_{a+1}, t_{a+2}, \ldots, t_{r-1}, t_a t_{a-1} \cdots t_1 t_r),$$

$$P^\pm_{\lambda}(x; t_1, \ldots, t_r) = P^\pm_{\lambda}(x', t_{a+1}, t_{a+2}, \ldots, t_{r-1}, t_a t_{a-1} \cdots t_1 t_r).$$

(ii) Let $\lambda, \mu \in \mathcal{P}_{n,r}$ be such that $\mu \preceq \lambda$. Assume that $\lambda \in \mathcal{P}_{n,a}$. Then $\mu \in \mathcal{P}_{n,r}$, and we have

$$K^\pm_{\lambda, \mu}(t_1, \ldots, t_r) = K^\pm_{\lambda', \mu'}(t_{a+1}, \ldots, t_{r-1}, t_a t_{a-1} \cdots t_1 t_r),$$

where $K^\pm_{\lambda', \mu'}$ is the Kostka function associated to $\lambda', \mu' \in \mathcal{P}_{n,r-a}$.
(iii) Assume that \( a = r - 1 \). For \( \lambda = (-, \ldots, -, \lambda^{(r)}) \in \mathcal{P}^{r-1}_{n,r} \), by setting \( t_1 = \cdots = t_r = t \), we have

\[
Q^\pm_{\lambda}(x; t) = Q_{\lambda^{(r)}}(x^{(r)}, t'), \\
P^\pm_{\lambda}(x; t) = P_{\lambda^{(r)}}(x^{(r)}, t'),
\]

where the left hand side is the one variable Hall-Littlewood functions associated to the \( r \)-partition \( \lambda \) (see (2.15.2)), and the right hand side is the classical Hall-Littlewood functions associated to the partition \( \lambda^{(r)} \).

(iv) Under the same assumption as in (iii), take \( \mu \in \mathcal{P}_{n,r} \) such that \( \mu \preceq \lambda \). Then \( \mu = (-, \ldots, -, \mu^{(r)}) \in \mathcal{P}^{r-1}_{n,r} \), and we have

\[
K^\pm_{\lambda, \mu}(t) = K_{\lambda^{(r)}, \mu^{(r)}}(t'),
\]

where the left hand side is the one-variable Kostka function associated to \( r \)-partitions (see (2.15.3)), and the right hand side is the classical Kostka polynomial associated to partitions.

\textbf{Proof.} The first formula of (i) follows from Proposition 3.4. In this formula, \( Q^\pm_{\lambda}(x; t) \) has an expansion in terms of Schur functions

\begin{equation}
Q^\pm_{\lambda}(x; t) = \sum_{\mu \preceq \lambda} u_{\lambda, \mu}(t) s_{\mu}(x).
\end{equation}

On the other hand, if \( \mu \preceq \lambda \) and \( \lambda \in \mathcal{P}^a_{n,r} \), then \( \mu \in \mathcal{P}^a_{n,r} \), and \( s_{\mu}(x) = s_{\mu'}(x') \).

Thus (3.5.1) also gives an expansion of \( Q^\pm_{\lambda'}(x'; t') \) in terms of Schur functions for \( \mu' \in \mathcal{P}_{n,r-a} \),

\begin{equation}
Q^\pm_{\lambda'}(x'; t') = \sum_{\mu' \preceq \lambda'} u'_{\lambda', \mu'}(t') s_{\mu'}(x')
\end{equation}

with \( u_{\lambda, \lambda}(t) = u'_{\lambda', \lambda'}(t') \), where \( t' = (t_{a+1}, t_{a+2}, \ldots, t_r, t, \ldots, t_1 t_{a-1} \cdots t_1 t_r) \). By Theorem 2.14, we have \( P^\pm_{\lambda}(x; t) = u_{\lambda, \lambda}(t)^{-1} Q^\pm_{\lambda}(x; t) \), \( P^\pm_{\lambda'}(x'; t') = u'_{\lambda', \lambda'}(t')^{-1} Q^\pm_{\lambda'}(x'; t') \). Thus we obtain the second formula of (i). Now (ii) is immediate from the second formula of (i). (iii) and (iv) are the special case of (i) and (ii). \( \square \)

\textbf{Remark 3.6.} In the case where \( r = 2 \), the formula (iv) was first proved by Achar-Henderson in [AH, Corollary 5.3] by a geometric method. After that a combinatorial proof of (iv) and the related formula (iii) for Hall-Littlewood functions (for \( r = 2 \)) were given in [LS, Proposition 1.11 and Corollary 1.12]. This argument also works for the general \( r \). In those discussions, the proof proceeds under the one-variable setting, namely under the setting where \( t_1 = t_2 = \cdots = t_r = t \). However, in order to describe the relationship among Kostka functions and Hall-Littlewood functions as in the setting of (i) and (ii), one needs to introduce multi-variable Kostka functions.
and Hall-Littlewood functions. Note that the proof of (iii) and (iv) here is much simpler than the discussion in [LS].

4. CLOSED FORMULA FOR HALL-LITTLEWOOD FUNCTIONS

4.1. In this section, we define a function $R^\pm_\lambda(x; t)$, as an analogue of the function $R_\lambda(x; t)$ in [M, III, 1], and show in this and next section that $Q^\pm_\lambda(x; t)$ has an explicit description in terms of $R^\pm_\lambda(x; t)$ under a mild restriction.

Let $\mathcal{M}$ be as in (2.4.1). We define a total order on $\mathcal{M}$ by

$$(1, 1) < (2, 1) < \cdots < (r, 1) < (1, 2) < (2, 2) < \cdots < (r, 2) < \cdots.$$ 

We fix $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{m, r}$ with $\lambda^{(k)} = (\lambda^{(k)}_1, \ldots, \lambda^{(k)}_m)$ for a common $m \geq 1$. Write $\lambda^{(k)}_i = \lambda_i$ and $x^{(k)}_i = x_i$ if $\nu = (k, i) \in \mathcal{M}$. Let $\nu_0 = (k_0, i_0) \in \mathcal{M}$ be the largest element such that $\lambda^{(k_0)}_{i_0} \neq 0$. We put $b(\nu) = k$ if $\nu = (k, i)$. We define a function $I^\pm_\nu(x; t)$ for $\nu = (k, i) \in \mathcal{M}$ by

$$(4.1.1) \quad I^\pm_\nu(x; t) = \begin{cases} \prod_{(k, i) < (k+1, j)} (x^{(k)}_i - t_{k-c} x^{(k+1)}_j) & \text{if } (k, i) \leq \nu_0 \text{ and } \lambda^{(k)}_i \neq 0, \\ \prod_{i<j} (x^{(k)}_i - t_{k-c} x^{(k+1)}_j) & \text{if } (k, i) \leq \nu_0 \text{ and } \lambda^{(k)}_i = 0, \\ \prod_{i<j} (x^{(k)}_i - t_k x^{(k)}_j) & \text{if } (k, i) > \nu_0, \end{cases}$$

where $c$ is as in (2.5.3). We regard $k \in \mathbb{Z}/r\mathbb{Z}$.

Let $S^r_m = S_m \times \cdots \times S_m$ ($r$-factors) be the permutation group of the variables $x = (x^{(1)}, \ldots, x^{(r)})$. We define a function $R^\pm_\lambda(x; t)$ by

$$(4.1.2) \quad R^\pm_\lambda(x; t) = \sum_{w \in S^r_m} w \left( \prod_{1 \leq k \leq r} \prod_{\nu \in \mathcal{M}} (x^{(k)}_i - x^{(k)}_j) \right)^{\varepsilon^\pm_1} \prod_{1 \leq k \leq r} \prod_{i<j} (x^{(k)}_i - x^{(k)}_j)^{\varepsilon^\pm_i},$$

where $\varepsilon^\pm = (\varepsilon^\pm_1, \ldots, \varepsilon^\pm_m) \in \mathbb{Z}^m$ is given by

$$\varepsilon^\pm_i = \begin{cases} 1 & \text{if } \lambda^{(1)}_i \neq 0 \text{ and } k = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$\varepsilon^\pm_i = \begin{cases} 1 & \text{if } \lambda^{(k)}_i \neq 0 \text{ and } k \neq r, \\ 0 & \text{otherwise}. \end{cases}$$
It follows from the definition that \( R^\pm_\lambda(x; t) \in \mathbb{Z}[x, \#; t] \) and that

\[(4.1.3) \quad R^\pm_\lambda(x; 0) = s_\lambda(x).\]

For \( \lambda \in \mathcal{P}_{n,r} \), define a subgroup \( S'_\lambda \) of \( S'_m \) by \( S'_\lambda = S'_{\lambda_1} \times \cdots \times S'_{\lambda_r} \), where \( \lambda'_i = \mathbb{Z} \{ 1 \leq i \leq m \mid (k, i) > \nu_0 \} \). We regard \( S'_{\lambda_i} \) as the permutation group of the set \( \{ x^{(k)}_i \mid (k, i) > \nu_0 \} \) fixing any variable \( x^{(k)}_i \) for \( (k, i) \leq \nu_0 \). We define a polynomial \( v'_\lambda(t) \in \mathbb{Z}[t] \) by

\[(4.1.4) \quad v'_\lambda(t) = \prod_{k=1}^r v'_{\lambda'_k}(t_k),\]

where \( v_i(t) \) is given as in (2.2.1).

### 4.2

We consider the special case where \( \lambda = ((s); -; \cdots; -) \) with \( s \geq 1 \). Then \( S'_\lambda = S_{m-1} \times S_{r-1} \), where \( S_{m-1} \) is the stabilizer of \( x^{(1)}_1 \). Hence \( v'_\lambda(t) = v_{m-1}(t_1) \prod_{k \geq 2} v_m(t_k) \). We have

\[
\prod_k (x^{(k)})^{\lambda(k)-\varepsilon^{(k)}} \prod_{\nu \in \#} I^{\pm}_\nu(x; t)
= (x^{(1)}_1)^{s-1} \prod_{j \geq 1} (x^{(1)}_1 - t_{1-c} x^{(1+1)}_j) \prod_{2 \leq i < j} (x^{(1)}_i - t_1 x^{(1)}_j) \prod_{k \geq 2} (x^{(k)}_i - t_k x^{(k)}_j).
\]

\( S'_\lambda \) stabilizes the factor

\[
(x^{(1)}_1)^{s-1} \prod_{j \geq 1} (x^{(1)}_1 - t_{1-c} x^{(1+1)}_j) / \prod_{1 < j} (x^{(1)}_1 - x^{(1)}_j)
\]

and by [M, p.207, (1.4)]

\[
\sum_{w' \in S'_\lambda} w' \left( \prod_{2 \leq i < j} x^{(1)}_i - x^{(1)}_j \prod_{k \geq 2} x^{(k)}_i - x^{(k)}_j \right) = v_{m-1}(t_1) \prod_{k \geq 2} v_m(t_k) = v'_\lambda(t).
\]

It follows that

\[
R^\pm_\lambda(x; t) = v'_\lambda(t) \sum_{w \in S_m/S_{m-1}} w \left( (x^{(1)}_1)^{s-1} (x^{(1)}_1 - t_{1-c} x^{(1+1)}_1) \prod_{1 < j} x^{(1)}_i - t_{1-c} x^{(1+1)}_j \right)
= v'_\lambda(t) \sum_{i=1}^m (x^{(1)}_i)^{s-1} (x^{(1)}_i - t_{1-c} x^{(1+1)}_i) \prod_{j \neq i} x^{(1)}_i - t_{1-c} x^{(1+1)}_j
= v'_\lambda(t) q^{(1)}_{s,\pm}(x; t).
\]

Hence we have
(4.2.1) Assume that $\lambda = (s; -; \cdots ; -)$. Then $R_\lambda^\pm(x; t) = v'_\lambda(t)q^\pm_\lambda(x; t)$.

The above computation can be generalized as follows. For $\lambda \in \mathcal{P}_{n,r}$, $S'_\lambda$ stabilizes the expression

$$
\prod_{1 \leq k \leq r} (x^{(k)})^\lambda(k) - \varepsilon^\pm k \prod_{\nu \leq \nu_0} I^\pm_\nu(x; t).
$$

Moreover, by [M, p.207, (1.4)],

$$
\sum_{w' \in S'_\lambda} w' \left( \prod_{(k,i) > \nu_0} i < j \frac{x^{(k)}_i - t_k x^{(k)}_j}{x^{(k)}_i - x^{(k)}_j} \right) = v'_\lambda(t).
$$

Thus we have an expression for $R_\lambda^\pm(x; t)$,

$$
R^\pm_\lambda(x; t) = v'_\lambda(t) \sum_{w \in S'_{m-1,S'_\lambda}} w \left( \prod_{1 \leq k \leq r} (x^{(k)})^\lambda(k) - \varepsilon^\pm k \prod_{(k,i) \leq \nu_0} x^{(k)}_i \prod_{(k,i) \leq \nu_0} \lambda^{(k)}_i - t_k x^{(k+1)}_j \right),
$$

where (*) is given by the first and the second condition in (4.1.1). It follows from (4.2.2) that

$$
R_\lambda^\pm(x; t) \in \mathbb{Z}[x_\xi; t] \text{ is divisible by } v'_\lambda(t). \quad \text{Further, } v'_\lambda(t)^{-1}R_\lambda^\pm(x; t) \in \mathbb{Z}[x_\xi; t] \text{ has the stability for the increase of variables } \{x^{(k)}_1, \ldots, x^{(k)}_m\} \rightarrow \{x^{(k)}_1, \ldots, x^{(k)}_{m+1}\}.
$$

In particular, by taking $m \rightarrow \infty$, we obtain $v'_\lambda(t)^{-1}R_\lambda^\pm \in \Xi^n[t]$.

We define a polynomial $\tilde{P}^\pm_\lambda(x; t) \in \mathbb{Z}[x_\xi; t]$ by

$$
\tilde{P}^\pm_\lambda(x; t) = v'_\lambda(t)^{-1}R^\pm_\lambda(x; t).
$$

(4.2.4) is an analogue of [M, III, (2.1)] in the classical case. But note that $v'_\lambda(t)$ is different from $v_\lambda(t)$ there. Let $p$ be the largest number such that $\lambda^{(k)}_0 \neq 0$ for $k = 1, \ldots, r-1$, and put $j_0 = \max\{\ell(\lambda^{(r)}) - p, 0\}$. We define a polynomial $\tilde{Q}^\pm_\lambda(x; t) \in \mathbb{Z}[x_\xi; t]$ by

$$
\tilde{Q}^\pm_\lambda(x; t) = (1 - t_0)^{j_0}v'_\lambda(t)^{-1}R^\pm_\lambda(x; t) = (1 - t_0)^{j_0}\tilde{P}^\pm_\lambda(x; t),
$$

where we put $t_0 = t_1 \cdots t_r$. By taking $m \rightarrow \infty$, $\tilde{P}^\pm_\lambda(x; t), \tilde{Q}^\pm_\lambda(x; t)$ determine symmetric functions $\tilde{P}^\pm_\lambda, \tilde{Q}^\pm_\lambda \in \Xi^n[t]$.

Next result describes the expansion of $R^\pm_\lambda(x; t)$ in terms of Schur functions. Note that in this formula, the total order $\preceq$ is replaced by the partial order $\leq$. 

Proposition 4.3. For a given $\lambda \in \mathcal{P}_{n,r}$, fix a set $\mathcal{M}$, and consider $R^\pm_\lambda(x;t) \in \mathbb{Z}[x,\mathcal{M};t]$. Then there exist polynomials $u^\pm_{\lambda,\mu}(t) \in \mathbb{Z}[t]$ such that

$$R^\pm_\lambda(x;t) = \sum_{\mu \leq \lambda} u^\pm_{\lambda,\mu}(t)s_\mu(x).$$

Moreover, $u^\pm_{\lambda,\mu}(0) = 0$ if $\mu \neq \lambda$, and $u^\pm_{\lambda,\lambda}(0) = 1$.

Proof. The product $\prod_{\nu \in \mathcal{M}} I^\pm_\nu(x;t)$ can be written as a sum of monomials

$$(4.3.1) \quad \prod_{\nu,\nu'} (x_\nu)^{r_{\nu,\nu'}} (-t_{\nu'}x_{\nu'})^{r'_{\nu,\nu}},$$

where $t_{\nu'} = \begin{cases} t_{b(\nu')-1+c} & \text{if } \nu \leq \nu_0, \\ t_{b(\nu')} & \text{if } \nu > \nu_0. \end{cases}$

(See 4.1 for the definition of $b(\nu)$.) Moreover, $(r_{\nu,\nu'})$ is an integral matrix indexed by $\mathcal{M}$ consisting of 0 and 1 satisfying the condition

$$(4.3.2) \quad r_{\nu,\nu'} + r'_{\nu',\nu} = \begin{cases} 1 & \text{if } \nu \leq \nu_0, \nu = (k,i), \nu' = (k \mp 1,j), \lambda_\nu \neq 0, \nu < \nu', \\ 1 & \text{if } \nu \leq \nu_0, \nu = (k,i), \nu' = (k \mp 1,j), \lambda_\nu = 0, i < j, \\ 1 & \text{if } \nu > \nu_0, \nu = (k,i), \nu' = (k,j), i < j, \\ 0 & \text{otherwise}, \end{cases}$$

and the matrices $(r_{\nu,\nu'})$ satisfying the above condition are in 1-1 correspondence with the above monomials. For a given matrix $(r_{\nu,\nu'})$ as above, we put

$$\beta^{(k)}_i = \lambda^{(k)}_i - \varepsilon^{(k)}_{i,\pm} + \sum_{\nu' \in \mathcal{M}} r_{\nu,\nu'}$$

for $\nu = (k,i)$. Let $\beta = (\beta^{(1)}, \ldots, \beta^{(r)})$ be the $r$-composition. Then $\beta$ produces the “Schur function” $a_\beta/a_\delta$, where $a_\beta = \sum_{w \in S_m} \varepsilon(w)w(x^\beta)$, and $\delta = (\delta^{(1)}, \ldots, \delta^{(r)})$ with $\delta^{(k)} = (m-1, \ldots, 1,0)$. By (4.1.2), $R^\pm_\lambda$ is a sum of $\prod_{\nu} (-t_{\nu'})^{r_{\nu',\nu}}a_\beta/a_\delta$ obtained from the matrix $(r_{\nu,\nu'})$. If the entries of the composition $\beta^{(k)}$ are not all distinct for some $k$, then $a_\beta/a_\delta = 0$. So we may assume that all the entries of $\beta^{(k)}$ are distinct for any $k$. By rearranging its entries in the decreasing order, one can write it as

$$\beta^{(k)}_{w_k(i)} = \mu^{(k)}_i + (m - i)$$

for some $w_k \in S_m$. Then $\mu = (\mu^{(k)}_i) \in \mathcal{P}_{n,r}$, and $a_\beta/a_\delta$ coincides with $\varepsilon(w)s_\mu(x)$ for $w = (w_1, \ldots, w_r) \in S^r_m$. We shall show that

$$(4.3.3) \quad \textbf{\mu} \leq \lambda.$$
Define the matrix \((s_{\nu,\nu'})\) by \(s_{\nu,\nu'} = r_{w(\nu),w(\nu')}\), where \(w(\nu) = (k, w_k(i))\) for \(\nu = (k, i) \in \mathcal{M}\). One can write as

\[
(4.3.4) \quad \mu_i^{(k)} + (m - i) = \lambda_{w_k(i)}^{(k)} - \varepsilon_{w_k(i),\pm}^{(k)} + \sum_{\nu' \in \mathcal{M}} s_{\nu,\nu'},
\]

where \(\nu = (k, i)\). We want to show that

\[
(4.3.5) \quad \sum_{k=1}^{r} \sum_{i=1}^{a} \mu_i^{(k)} + \sum_{k=1}^{a} \mu_{a+1}^{(k)} \leq \sum_{k=1}^{r} \sum_{i=1}^{a} \lambda_{w_k(i)}^{(k)} + \sum_{k=1}^{s} \lambda_{w_k(a+1)}^{(k)}
\]

for \(0 \leq a \leq m - 1\) and \(1 \leq s \leq r\). Note that (4.3.5) implies (4.3.3) since \(w(\lambda) \leq \lambda\) for any \(w \in S_r^m\). By (4.3.4), we have

\[
\begin{align*}
\sum_{k=1}^{r} \sum_{i=1}^{a} \mu_i^{(k)} + \sum_{k=1}^{a} \mu_{a+1}^{(k)} & = \sum_{k=1}^{r} \sum_{i=1}^{a} \lambda_{w_k(i)}^{(k)} - \varepsilon_{w_k(i),\pm}^{(k)} + \sum_{k=1}^{s} \lambda_{w_k(a+1)}^{(k)} - \varepsilon_{w_k(a+1),\pm}^{(k)} \\
& \quad - \sum_{k=1}^{r} \sum_{i=1}^{a} \mu_{a+1}^{(k)} - \sum_{k=1}^{s} \mu_{a+1}^{(k)} \\
& \quad + \sum_{\nu \in B, \nu' \in \mathcal{M}} s_{\nu,\nu'},
\end{align*}
\]

where

\[
B = \{(k, i) \in \mathcal{M} \mid 1 \leq k \leq r, 1 \leq i \leq a\} \cup \{(k, a+1) \mid 1 \leq k \leq s\}.
\]

Hence in order to show (4.3.5), it is enough to see that

\[
(4.3.6) \quad \sum_{\nu \in B, \nu' \in \mathcal{M}} s_{\nu,\nu'} \leq \sum_{k=1}^{r} \sum_{i=1}^{a} (m - i + \varepsilon_{w_k(i),\pm}^{(k)}) + \sum_{k=1}^{s} (m - (a + 1) + \varepsilon_{w_k(a+1),\pm}^{(k)})
\]

\[
= \sum_{k=1}^{s} \sum_{i=1}^{a+1} (m - i + \varepsilon_{w_k(i),\pm}^{(k)}) + \sum_{k=s+1}^{r} \sum_{i=1}^{a} (m - i + \varepsilon_{w_k(i),\pm}^{(k)}).
\]

Let \(w(B) = \{(k, w_k(i)) \mid (k, i) \in B\}\). Put \(w'_+ = (w_2, w_3, \ldots, w_r, w_1)\) and \(w'_- = (w_r, w_1, \ldots, w_{r-1})\). Define \(B'_+ \subset \mathcal{M}\) by \(w(B'_+) = w'_+(B)\). We have

\[
\sum_{\nu \in B, \nu' \in \mathcal{M}} s_{\nu,\nu'} = \sum_{\nu \in B, \nu' \in B'_+} s_{\nu,\nu'} + \sum_{\nu \in B, \nu' \in \mathcal{M} - B'_+} s_{\nu,\nu'}.
\]
For \( k = 1, \ldots, r \), let \( X_k^\pm = X_k' \cup X_k'' \) with
\[
X_k' = \{ (\nu, \nu') \in \mathcal{M}^2 \mid w(\nu) \leq \nu_0, b(\nu) = k, b(\nu') = k \mp 1 \},
\]
\[
X_k'' = \{ (\nu, \nu') \in \mathcal{M}^2 \mid w(\nu) > \nu_0, w(\nu') > \nu_0, b(\nu) = b(\nu') = k \},
\]
and put
\[
A_k^\pm = \sum_{(\nu, \nu') \in X_k^\pm} s_{\nu, \nu'},
\]
\[
B_k^\pm = \sum_{(\nu, \nu') \in X_k^\pm} s_{\nu, \nu'}.
\]
By (4.3.2) we have \( \sum_{k=1}^r (A_k^\pm + B_k^\pm) = \sum_{\nu \in \mathcal{B}, \nu' \in \mathcal{B}_\pm^\prime} s_{\nu, \nu'} \). By our choice of \( \mathcal{B} \) and \( \mathcal{B}_\pm \), the \( k \)-th row of \( w(\mathcal{B}) \) consists of \( \{(k, w_k(i)) \mid (k, i) \in \mathcal{B}\} \), and \( (k \mp 1) \)-row of \( w(\mathcal{B}_\pm') \) consists of \( \{(k \mp 1, w_k(i)) \mid (k, i) \in \mathcal{B}\} \). Assume that \( k > s \). It is easy to see that
\[
A_k^\pm \leq a(a + 1)/2 - \sum_{1 \leq i \leq a} \varepsilon_{w_k(i), \pm}^{(k)} - \frac{\#\{1 \leq i \leq a \mid \varepsilon_{w_k(i), \pm}^{(k)} = 0\}},
\]
\[
B_k^\pm \leq a(m - a).
\]
Hence
\[
A_k^\pm + B_k^\pm \leq ma - a(a + 1)/2 + \sum_{i=1}^a \varepsilon_{w_k(i), \pm}^{(k)}
\]
\[
= \sum_{i=1}^a (m - i + \varepsilon_{w_k(i), \pm}^{(k)}).
\]
A similar formula as (4.3.8) holds for the case where \( k \leq s \) by replacing \( a \) by \( a + 1 \). By summing up those formulas, we have
\[
\sum_{k=1}^r (A_k^\pm + B_k^\pm) \leq \sum_{k=1}^s \sum_{i=1}^{a+1} (m - i + \varepsilon_{w_k(i), \pm}^{(k)}) + \sum_{k=s+1}^r \sum_{i=1}^a (m - i + \varepsilon_{w_k(i), \pm}^{(k)}).
\]
Thus (4.3.6) holds, and so (4.3.3) follows. This proves the first assertion of the proposition. The second assertion follows from (4.1.3). \( \square \)

\textbf{4.4.} We shall determine the polynomial \( u_{\lambda, \lambda}^\pm(t) \). In the proof of Proposition 4.3, the equality \( \mu = \lambda \) holds if and only if \( w(\lambda) = \lambda \) and the equality holds for each \( k \) in the formulas (4.3.7), namely, \( s_{\nu, \nu'} = 1 \) for any \( s_{\nu, \nu'} \) appearing in the expression.
of $A_k^\pm, B_k^\pm$. It follows that

$$u_{\lambda, \lambda}^\pm(t) = \sum_{w \in S_\lambda} \varepsilon(w) \prod_{\nu'} (-t_{\nu'})^{s_{w^{-1}(\nu'), w^{-1}(\nu)}}.$$ 

Let $S'_\lambda$ be the subgroup of $S_m^r$ as given in 4.1. In order to obtain the equality for $B_k^\pm$ in (4.3.7) for any $a$, we must have $w \in S'_\lambda$. In that case, $s_{w^{-1}(\nu'), w^{-1}(\nu)} = 1$ only when the pair $(\nu, \nu')$ satisfies the condition that

$$\nu > \nu_0, \ \nu = (k, i), \ \nu' = (k, j), \ i < j, \ w^{-1}(i) > w^{-1}(j).$$

Since $t_{\nu'} = t_{b(\nu)}$ in this case, we have

$$(4.4.1) \quad u_{\lambda, \lambda}^\pm(t) = \prod_{k=1}^r \sum_{w_k \in S'_\lambda} t_{\ell(w_k)}^{\ell(w_k)} = v_{\lambda}(t),$$

where $\ell(w_k)$ is the length function of $S'_\lambda$.

By taking Proposition 4.3 and (4.4.1) into account, we have an expression

$$(4.4.2) \quad \tilde{Q}_\lambda^\pm(x; t) = s_\lambda(x) + \sum_{\mu < \lambda} w_{\lambda, \mu}^\pm(t)s_\mu(x)$$

with $w_{\lambda, \mu}^\pm(t) \in \mathbb{Z}[t]$.

5. Closed formula for Hall-Littlewood functions – continued

5.1. In this section, we discuss the expansion of $\tilde{Q}_\lambda^\pm$ in terms of $q_\mu^\pm$.

For given $1 \leq a \leq r$ and $m \geq 1$, we define a subset $\mathcal{M}^{(a)}$ of $\mathcal{M}$ by removing $(1, 1), \ldots, (a - 1, 1)$ from $\mathcal{M}$. Note that if $a = 1$, $\mathcal{M}^{(1)}$ coincides with the original $\mathcal{M}$. The total order on $\mathcal{M}^{(a)}$ is inherited from $\mathcal{M}$. We consider an $r$-partition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)})$, where $\lambda^{(k)} = (\lambda_2^{(k)}, \lambda_3^{(k)}, \ldots, \lambda_m^{(k)})$ for $k < a$ and $\lambda^{(k)} = (\lambda_1^{(k)}, \ldots, \lambda_m^{(k)})$ for $k \geq a$. Thus we have a bijective correspondence $\lambda_i^{(k)} \leftrightarrow (k, i) \in \mathcal{M}^{(a)}$. In this case, we say that $\lambda$ is compatible with $\mathcal{M}^{(a)}$. For such $\lambda$, one can construct a polynomial $R_\lambda^\pm(x; t)$ by extending the previous definition, where $x = \{x_i^{(k)} \mid (k, i) \in \mathcal{M}^{(a)}\}$. (Here we consider $S_m = S_m^{a-1} \times S_m^{r-a+1}$ as the permutation group of the variables $\{x_\nu \mid \nu \in \mathcal{M}^{(a)}\}$ for $m = (m - 1, \ldots, m - 1, m, \ldots, m)$.)

In the following discussion, by fixing $a$, we write $\mathcal{M}^{(a)}$ as $\mathcal{M}$. We discuss separately the “+”case and the “−”case. First consider the “−”case. Then $\tilde{Q}_\lambda^-$ is defined by using the formula (4.2.5). Let $b \leq r$ be the smallest integer such that $b \geq a$ and that $\lambda^{(b)} \neq \emptyset$, hence $\lambda_i^{(b)} \neq 0$. If such $b$ does not exist, i.e., $\lambda^{(a)} = \emptyset$ for any $r \geq a' \geq a$, we put $b = r$.

Let $\mu = (\mu^{(1)}, \ldots, \mu^{(r)})$ be the $r$-partition compatible with $\mathcal{M}^{(b+1)}$ obtained from $\lambda$ by removing $\{\lambda_i^{(a')} \mid a \leq a' \leq b\}$. Thus we have $\mu^{(k)} = (\lambda_2^{(k)}, \lambda_3^{(k)}, \ldots, \lambda_m^{(k)})$ for $1 \leq k \leq b$, and $\mu^{(k)} = \lambda^{(k)}$ otherwise. Put $\mathcal{M}' = \mathcal{M}^{(b+1)}$, and consider the
polynomial $\tilde{Q}_{\mu}^\dagger(x'; t)$ for $x' = (x_{\nu})_{\nu \in \mathcal{M}}$. (Note that if $b = r$, $\mathcal{M}'$ coincides with $\mathcal{M}^{(1)}$, but by replacing $m$ by $m - 1$.)

$m'$ is defined for $\mathcal{M}'$ similarly to $\mathcal{M}$ and we have $S_{m'} = S_m^b \times S_r^{b-r}$. Hence $S_{m}/S_{m'} \simeq [1, m]^{b-a+1}$, and we express the elements in $S_{m}/S_{m'}$ as $i = (i_a, i_{a+1}, \ldots, i_b) \in [1, m]^{b-a+1}$. For $i \in S_{m}/S_{m'}$, we denote by $Q_{\mu}^\dagger_{(i)}$ the polynomial obtained from $Q_{\mu}^\dagger_{(k)}$ by replacing the variables $x_{2}^{(k)}, \ldots, x_{m}^{(k)}$ by $x_{1}^{(k)}, \ldots, x_{i_k}^{(k)}, \ldots, x_{m}^{(k)}$ ($x_{i_k}^{(k)}$ is removed) for $a \leq k \leq b$, and leaving variables in other rows unchanged. We have the following lemma.

**Lemma 5.2.** Under the notation above, we have

$$
\tilde{Q}_{\lambda}^\dagger = \sum_{i \in S_{m}/S_{m'}} (x_{i_b}^{(b)})^{M_{(b)} - e_{i_b}^{(b)}} g_{i_1}^{(a)} Q_{\mu}^\dagger_{(i)},
$$

where $g_{i_1}^{(a)}$ for $i = (i_a, \ldots, i_b) \in [1, m]^{b-a+1}$ is given by

$$
g_{i_1}^{(a)}(x; t) = \frac{\left( \prod_{a \leq k < b} \prod_{j \geq 1 \atop j \neq i_{k+1}} (x_{i_k}^{(k)} - t_k x_{j}^{(k+1)}) \right) \prod_{j \geq 1} (x_{i_b}^{(b)} - t_b x_{j}^{(b+1)})}{\prod_{a \leq k \leq b} \prod_{j \neq i_k} (x_{i_k}^{(k)} - x_{j}^{(k)})}
$$

(5.2.1)

if $b < r$,

$$
g_{i_1}^{(a)}(x; t) = \frac{\left( \prod_{a \leq k < r} \prod_{j \geq 1 \atop j \neq i_{k+1}} (x_{i_k}^{(k)} - t_k x_{j}^{(k+1)}) \right) \prod_{j \geq 2} (x_{i_r}^{(r)} - t_r x_{j}^{(1)})}{\prod_{a \leq k \leq r} \prod_{j \neq i_k} (x_{i_k}^{(k)} - x_{j}^{(k)})}
$$

(5.2.2)

if $b = r$ and $a > 1$,

$$
g_{i_1}^{(a)}(x; t) = \frac{(1 - t_0) \left( \prod_{1 \leq k < r} \prod_{j \geq 1 \atop j \neq i_{k+1}} (x_{i_k}^{(k)} - t_k x_{j}^{(k+1)}) \right) \prod_{j \geq 2} (x_{i_r}^{(r)} - t_r x_{j}^{(1)})}{\prod_{1 \leq k \leq r} \prod_{j \neq i_k} (x_{i_k}^{(k)} - x_{j}^{(k)})}
$$

(5.2.3)

if $b = r$ and $a = 1$.

**Proof.** Here $S_{m'}$ is the stabilizer of the variables $x_{i}^{(k)}$ for $a \leq k \leq b$ in $S_{m}$. Note that $S_{\lambda} = S_{\mu}' \subset S_{m'}$. First assume that $b < r$. Let $i_0 = (1, \ldots, 1) \in [1, m]^{b-a+1}$. Since
\((x_1^{(b)})^\lambda - \varepsilon_1^{(b)}g_0^{(a)}\) is stable by \(S_{m'}\) we have

\[
(1 - t_0)^{j_0} \sum_{w \in S_{m'}/S_{m'}} w \left\{ \prod_{1 \leq k \leq r} (x^{(k)})^\lambda - \varepsilon^{(k)} \frac{I_r(x; t)}{\prod_{(k, i) \leq n_0} (x_i^{(k)} - x_j^{(k)})} \right\}
= (x_1^{(b)})^\lambda - \varepsilon_1^{(b)}g_0^{(a)} \tilde{Q}_\mu(x'; t).
\]

(Here \(j_0\) used in the definition of \(\tilde{Q}_\mu\) coincides with \(j_0\) used for \(\tilde{Q}_\mu^\lambda\).) Hence

\[
(5.2.4) \quad \tilde{Q}_\lambda(x; t) = \sum_{w \in S_m/S_{m'}} w \left\{ (x_1^{(b)})^\lambda - \varepsilon_1^{(b)}g_0^{(a)}Q_\mu(x'; t) \right\}.
\]

The lemma follows from this. Next assume that \(b = r\). Since \(\prod_{j \geq 2} (x_1^{(r)} - t_r x_j^{(1)})\) is stable by the action of \(S_{m-1} \times S_{m-1}\), \((x_1^{(r)})^\lambda - \varepsilon^{(r)}g_0^{(a)}\) is again stable by \(S_{m'}\). Then a similar argument works. Note that in the case of (5.2.2), \(j_0\) is common for \(\tilde{Q}_\lambda^\mu\) and for \(\tilde{Q}_\mu^\lambda\), but in the case of (5.2.3), the discrepancy for \(j_0\) between \(\tilde{Q}_\lambda^\mu\) and \(\tilde{Q}_\mu^\lambda\) occurs. The lemma is proved.

**Remark 5.3.** A similar formula was proved in Lemma 3.3 in [S1]. But in our definition of \(R^\pm_\lambda(x; t)\), we can not take \(b = a\) if \(\lambda_1^{(a)} = 0\) for \(a < r\) since the product \(\prod_{j \geq 2} (x_1^{(a)} - t_a x_j^{(a+1)})\) is not stable by the action of \(S_{m-1} \times S_m\).

**5.4.** We consider the “+” case. Let \(\cal{M} = \cal{M}^{(a)}\) be as in 5.1. In this case, we assume that \(\lambda_2^{(1)} \neq 0\). Then \(\tilde{Q}_\lambda^+\) is defined as in (4.2.5). Put \(\cal{M}' = \cal{M}^{(a+1)}\), and consider \(m'\) with respect to \(\cal{M}'\). We have \(S_{m'} = S_{m-1}^a \times S_{r-a}^r, \) and \(S_m/S_{m'} \simeq [1, m]\). Put, for \(i = 1, \ldots, m\),

\[
g_{i,+}^{(a)}(t) = \prod_{(a-1, j) \in \cal{M}} (x_1^{(a)} - t_{a-1} x_j^{(a-1)})/ \prod_{j \neq i} (x_1^{(a)} - x_j^{(a)}).
\]

Here the condition \((a-1, j) \in \cal{M}\) is given by \(j \geq 2\) if \(a > 1\) and \(j \geq 1\) if \(a = 1\). Then the product \(\prod_{j \geq 2} (x_1^{(a)} - t_{a-1} x_j^{(a-1)})\) (resp. \(\prod_{j \geq 1} (x_1^{(a)} - t_r x_j^{(r)})\)) for \(a > 1\) (resp. \(a = 1\)) is stable by the action of \(S_{m'}\). (Note that the condition \(j \geq 1\) in the \(a = 1\) case comes from the assumption that \(\lambda_1^{(1)} \neq 0\). If \(\lambda_1^{(1)} = 0\), we need the condition \(j \geq 2\), in which case, the product is not stable by \(S_{m'}\).)

Now \(\mu\) is defined as in 5.1, and \(\tilde{Q}_{\lambda}^{[i,+]}\) can be defined for \(i = 1, \ldots, m\) (apply 5.1 to the case where \(b = a\)). The following formula can be proved in a similar way as in Lemma 5.2 (Note, by the condition \(\lambda_2^{(1)} \neq 0\), that \(j_0\) is common for \(\lambda\) and \(\mu\)).

**Lemma 5.5.** Assume that \(\lambda_2^{(1)} \neq 0\). Then we have

\[
\tilde{Q}_\lambda^+ = \sum_{i=1}^m (x_i^{(a)})^\lambda - \varepsilon_i^{(a)} g_{i,+}^{(a)} \tilde{Q}_{\lambda}^{[i,+]}.
\]
The following lemmas will be used in later discussions.

**Lemma 5.6.** Assume that $a < b$. Consider the variables $x^{(k)}$ for $a \leq k \leq b$, and denote the action of $S_m$ on the variable $x^{(k)}$ by $S^{(k)}_m$. Then we have

\[(5.6.1) \quad \sum_{w \in S^{(a)}_m \times \cdots \times S^{(b-1)}_m} w\left( \prod_{a \leq k < b} \prod_{2 \leq j \leq m} \frac{x^{(k)}_1 - t_k x^{(k+1)}_j}{x^{(k)}_1 - x^{(k)}_j} \right) = |S_{m-1}|^{b-a}.\]

**Proof.** If we write $S^{(a)}_m \times \cdots \times S^{(b-1)}_m$ as $S^{b-a}$, the left hand side of (5.6.1) is equal to

\[
\sum_{w \in S^{b-a}_m} \epsilon(w) w\left( \prod_{a \leq k < b} \prod_{j \geq 2} (x^{(k)}_1 - t_k x^{(k+1)}_j) \prod_{2 \leq i < j} (x^{(k)}_i - x^{(k)}_j) \right)
\]

But the sum in the numerator is an alternating polynomial with respect to the variables $x^{(k)}$ for $a \leq k < b$, and so is divisible by $\prod_{a \leq k < b} \prod_{i < j} (x^{(k)}_i - x^{(k)}_j)$. Hence by comparing the degrees as polynomials with respect to the variables $x^{(k)}_1$ for a fixed $k$, the last formula is equal to

\[
\frac{1}{\prod_{a \leq k < b} \prod_{i < j} (x^{(k)}_i - x^{(k)}_j)} \sum_{w \in S^{b-a}_m} \epsilon(w) w\left( \prod_{a \leq k < b} (x^{(k)}_1)^{m-1} \prod_{2 \leq i < j} (x^{(k)}_i - x^{(k)}_j) \right),
\]

where the numerator coincides with

\[
\prod_{a \leq k < b} \sum_{w \in S^{(k)}_m} \epsilon(w) w\left( (x^{(k)}_1)^{m-1} \prod_{2 \leq i < j} (x^{(k)}_i - x^{(k)}_j) \right).
\]

If we put $x^{(k)}_1 = y_i, S^{(k)}_m = S_m$, we have

\[
\sum_{w \in S_m} \epsilon(w) w\left( y_1^{m-1} \prod_{2 \leq i < j \leq m} (y_i - y_j) \right)
\]

\[
= \sum_{w \in S_m} \epsilon(w) w \sum_{w' \in S_{m-1}} \epsilon(w') w'(y_1^{m-1}y_2^{m-2} \cdots y_{m-1})
\]

\[
= |S_{m-1}| \sum_{w \in S_m} \epsilon(w) w(y_1^{m-1}y_2^{m-2} \cdots y_{m-1})
\]

\[
= |S_{m-1}| \prod_{i < j} (y_i - y_j).
\]
The lemma is proved.

\[ \text{Lemma 5.7.} \] Consider three types of variables \( x_i, y_i, z_i \) for \( i = 1, \ldots, m \). Then the following identity holds.

\[
\sum_{w \in S_m} w \left( \frac{\prod_{j \geq 2} (x_1 - t_1 y_j) \prod_{j \geq 2} (z_1 - t_3 x_j)}{\prod_{j \geq 2} (x_1 - x_j)} \right) = |S_{m-1}| \prod_{j \geq 2} (z_1 - t_1 t_3 y_j),
\]

where \( S_m \) acts on the variables \( x_1, \ldots, x_m \) as permutations.

**Proof.** We define an operator \( \mathcal{A}_x \) on the variables \( x_1, \ldots, x_m \) by

\[
\mathcal{A}_x = \prod_{i<j} (x_i - x_j)^{-1} \sum_{w \in S_m} \varepsilon(w) w.
\]

Then the left hand side of (5.7.1) can be written as

\[
\mathcal{A}_x \left( \prod_{j \geq 2} (x_1 - t_1 y_j) \prod_{j \geq 2} (z_1 - t_3 x_j) \prod_{2 \leq i<j} (x_i - x_j) \right).
\]

We can write

\[
\prod_{j \geq 2} (x_1 - t_1 y_j) = \sum_{k=0}^{m-1} (-t_1)^k x_1^{m-1-k} \sum_{I \subset [2,m], |I|=k} y_I,
\]

\[
\prod_{j \geq 2} (z_1 - t_3 x_j) = \sum_{\ell=0}^{m-1} (-t_3)^\ell z_1^{m-1-\ell} \sum_{J \subset [2,m], |J|=\ell} x_J,
\]

\[
\prod_{j \geq 2} (z_1 - t_1 t_3 y_j) = \sum_{k=0}^{m-1} (-t_1 t_3)^k z_1^{m-1-k} \sum_{J \subset [2,m], |J|=k} y_J,
\]

where \( y_I = y_{i_1} \cdots y_{i_k} \) for \( I = \{i_1, \ldots, i_k\} \), and similarly for \( x_J \).

Put \( F = \prod_{j \geq 2} (x_1 - t_1 y_j) \prod_{j \geq 2} (z_1 - t_3 x_j) \). Then

\[
F = \sum_{k=0}^{m-1} \sum_{\ell=0}^{m-1} \sum_{I \subset [2,m]} \sum_{J \subset [2,m]} (-t_1)^k x_1^{m-1-k} y_I x_J (-t_3)^\ell z_1^{m-1-\ell}.
\]

We compute, for a fixed \( k, \ell \),

\[
\sum_{J \subset [2,m], |J|=\ell} x_J x_1^{m-1-k} \prod_{2 \leq i<j} (x_i - x_j)
\]
\[ \sum_{J \subseteq [2,m]} \sum_{\sigma \in S_{m-1}} \mathcal{A}(\sigma) x_{\sigma(2)}^{m-2} x_{\sigma(3)}^{m-3} \cdots x_{\sigma(m-1)}^{m-1}, \]

where \( S_{m-1} \) is the stabilizer of \( x_1 \) in \( S_m \). We apply the operator \( \mathcal{A} \) for each monomial \( X = x_j x_1^{m-1-k} x_{\sigma(2)}^{m-2} \cdots x_{\sigma(m-1)}^{m-1} \) determined by the choice of \( J \) and \( \sigma \). If \( k > \ell \), then \( \mathcal{A}(X) = 0 \) by the degree reason. So assume that \( k \leq \ell \). We write

\[ x_1^{m-1-k} x_{\sigma(2)}^{m-2} x_{\sigma(3)}^{m-3} \cdots x_{\sigma(m-1)}^{m-1} = x_{\sigma(2)}^{m-2} \cdots x_{\sigma(k+1)}^{m-2} x_{\sigma(k+2)}^{m-2} \cdots x_{\sigma(m-1)}^{m-1}. \]

Note that if \( a_1, \ldots, a_m \) are not all distinct in the monomial \( X = x_1^{a_1} \cdots x_m^{a_m} \), then again \( \mathcal{A}(X) = 0 \). It follows that, if \( \mathcal{A}(X) \neq 0 \), then \( X \) must have the form

\[ X = x_{\sigma(2)}^{m-2} \cdots x_{\sigma(k+1)}^{m-2} x_{\sigma(k+2)}^{m-2} \cdots x_{\sigma(m-2)}, \]

namely, \( \ell = k \) and \( J = \{ \sigma(2), \ldots, \sigma(k+1) \} \). Write \( X \) as \( x_{r(1)}^{m-1} x_{r(2)}^{m-2} \cdots x_{r(m-1)}^{m-1} \) for \( r \in S_m \). Then we have

\[ \mathcal{A}(X) = \varepsilon(r) = (-1)^k \varepsilon(\sigma). \]

For a fixed \( J \), the number of such \( \sigma \) is equal to \( k! \times (m-1-k)! \). The number of the choices of \( J \) is \( \binom{m-1}{k} \). Hence

\[ \mathcal{A} \left( \sum_{J \subseteq [2,m]} \sum_{|J|=k} (t_1 t_3)^k x_j x_1^{m-1-k} \prod_{2 \leq i < j} (x_i - x_j) \right) = (m-1)! (-t_1 t_3)^k. \]

It follows that

\[
|S_{m-1}|^{-1} \mathcal{A} \left( F \prod_{2 \leq i < j} (x_i - x_j) \right) = \sum_{k=0}^{m-1} \sum_{J \subseteq [2,m]} (-t_1 t_3)^k z_1^{m-1-k} y_j \]

\[ = \prod_{j \geq 2} (z_1 - t_1 t_3 y_j). \]

This proves (5.7.1). The lemma is proved. \( \square \)

**5.8.** Recall the definition of \( q_{s,r}^{(k)}(x; t) \) in (2.5.1). In the "-" case with \( k = r \), we define a function \( \tilde{q}_{s,-}^{(r)}(x; t) \) by

\[ \tilde{q}_{s,-}^{(r)}(x; t) = \sum_{1 \leq i \leq m} (x_i^{(r)})^s \prod_{j \geq 2} (x_i^{(r)} - t x_j^{(1)}) \prod_{j \neq i} (x_i^{(r)} - x_j^{(r)}). \]
for $s \geq 0$. If $s \geq 1$, we have $q_{s,-}(x; t)|_{x_1^{(1)}=0} = \tilde{q}_{s,-}(x; t)$. We note that $\tilde{q}_{s,-}^{(r)} = 1$ if $s = 0$. In fact,

$$
\sum_{1 \leq i \leq m} \prod_{j \neq i} (x_i^{(r)} - x_j^{(1)}) = |S_m|^{-1} \sum_{w \in S_m} w\left(\frac{\prod_{j \geq 2} (x_1^{(r)} - t x_j^{(1)})}{\prod_{i=1}^m (1 - u x_i^{(1)})}\right) = 1.
$$

The last equality follows from Lemma 5.6 by applying it to the case where $b - a = 1$. Thus $q_{s,-}(x; t)|_{x_1^{(1)}=0} = \tilde{q}_{s,-}(x; t)$, and $\tilde{q}_{s,-}^{(r)} \in \mathbb{Z}[x; t]$ for $s \geq 0$. By using the generating function of $q_{s,-}^{(r)}(x; t)$ in (2.5.2), we have

$$(5.8.2) \quad \sum_{s \geq 0} \tilde{q}_{s,-}^{(r)}(x; t) u^s = \prod_{i=2}^m (1 - t u x_i^{(1)}) / \prod_{i=1}^m (1 - u x_i^{(r)}).$$

In the “+” case with $k > 1$, we define $\tilde{q}_{s,+}^{(k)}(x; t)$ by

$$(5.8.3) \quad \tilde{q}_{s,+}^{(k)}(x; t) = \sum_{1 \leq i \leq m} (x_i^{(k)})^s \frac{\prod_{j \geq 2} (x_i^{(k)} - t x_j^{(k-1)})}{\prod_{j \neq i} (x_i^{(k)} - x_j^{(k)})}$$

for $s \geq 0$. As in the “−” case, we see that $q_{s,+}^{(k)}(x; t)|_{x_1^{(k-1)}=0} = \tilde{q}_{s,+}^{(k)}(x; t)$, and that the generating function of $\tilde{q}_{s,+}^{(k)}$ is given by

$$(5.8.4) \quad \sum_{s \geq 0} \tilde{q}_{s,+}^{(k)}(x; t) u^s = \prod_{i=2}^m (1 - t u x_i^{(k-1)}) / \prod_{i=1}^m (1 - u x_i^{(k)}).$$

In the “−” case, we need the following.

**Proposition 5.9.** Assume that $a \leq b \leq r$, and put $d = b - a + 1$. Set

$$G_s^{a,b}(x; t) = \sum_{i \in [1, m]^d} (x^{(b)}_{i_0})^{s-\varepsilon} q_{i_0}^{(a)}(x; t),$$

where $\varepsilon = 1$ if $b < r$ and $\varepsilon = 0$ if $b = r$. Furthermore assume that $s \geq 1$ if $b < r$. Then we have

$$G_s^{a,b}(x; t) = \begin{cases} q_{s,-}^{(b)}(x; t_b) & \text{if } b < r, \\ \tilde{q}_{s,-}^{(r)}(x; t_r) & \text{if } b = r \text{ and } a > 1, \\ q_s(x^{(r)}; t_0) & \text{if } b = r \text{ and } a = 1, \end{cases}$$

where $t_0 = t_1 \cdots t_r$, and $q_s(x^{(r)}; t_0)$ is the original $q$-function given in (2.3.4) with respect to the variables $x^{(r)}$. 
Proof. First assume that $b < r$. We denote by $S^{(k)}_m$ the action of $S_m$ on the variables $x^{(k)}$.

\[
G_s^{a,b}(x; t) = \sum_{s \in [1,m]^d} (x^{(b)}_s)^{s-1} \frac{\prod_{a \leq b} \prod_{j \geq 1} (x^{(k)}_{i_k} - t_k x_j^{(k+1)}) \cdot \prod_{j \geq 2} (x^{(b)}_1 - t_b x_j^{(b+1)})}{\prod_{a \leq b} \prod_{j \neq i_k} (x^{(k)}_{i_k} - x^{(k)}_j)}
\]

\[
= |S_{m-1}|^{-d} \sum_{w \in S_m^{(a)} \times \cdots \times S_m^{(b)}} w \left( \frac{\prod_{a \leq b} \prod_{j \geq 1} (x^{(k)}_1 - t_k x_j^{(k+1)}) \cdot (x^{(b)}_1)^{s-1} \prod_{j \geq 2} (x^{(b)}_1 - t_b x_j^{(b+1)})}{\prod_{a \leq b} \prod_{j \geq 2} (x^{(k)}_1 - x^{(k)}_j)} \right)
\]

\[
= |S_{m-1}|^{-d} \sum_{w' \in S_m^{(b)}} w' \left( A \frac{(x^{(b)}_1)^{s-1} \prod_{j \geq 1} (x^{(b)}_1 - t_b x_j^{(b+1)})}{\prod_{j \geq 2} (x^{(b)}_1 - x^{(b)}_j)} \right),
\]

where

\[
A = \sum_{w \in S_m^{(a)} \times \cdots \times S_m^{(b-1)}} w \left( \frac{\prod_{a \leq b} \prod_{j \geq 2} (x^{(k)}_1 - t_k x_j^{(k+1)})}{\prod_{a \leq b} \prod_{j \geq 2} (x^{(k)}_1 - x^{(k)}_j)} \right).
\]

By Lemma 5.6, we have $A = |S_{m-1}|^{b-a}$. Since

\[
\sum_{w' \in S_m^{(b)}} w' \left( \frac{(x^{(b)}_1)^{s-1} \prod_{j \geq 1} (x^{(b)}_1 - t_b x_j^{(b+1)})}{\prod_{j \geq 2} (x^{(b)}_1 - x^{(b)}_j)} \right) = |S_{m-1}|q_s^{(b)}(x; t_b),
\]

we obtain the required formula. A similar argument works also for the case where $b = r$ and $a > 1$. In that case, the formula in the last step is given by

\[
\sum_{w' \in S_m^{(r)}} w' \left( \frac{(x^{(r)}_1)^{s} \prod_{j \geq 2} (x^{(r)}_1 - t_r x_j^{(1)})}{\prod_{j \geq 2} (x^{(r)}_1 - x^{(r)}_j)} \right) = |S_{m-1}|q_s^{(r)}(x; t_r).
\]

Thus the assertion holds.

Finally consider the case where $b = r$ and $a = 1$. In this case, we have

\[
(1 - t_0)^{-1} G_s^{1,r}(x; t) = \sum_{w \in S_m^r} w \left( (x^{(r)}_1)^s \prod_{1 \leq k < r} \prod_{j \geq 2} \frac{x^{(k)}_1 - t_k x_j^{(k+1)}}{x^{(k)}_1 - x^{(k)}_j} \prod_{j \geq 2} \frac{x^{(r)}_1 - t_r x_j^{(1)}}{x^{(r)}_1 - x^{(r)}_j} \right).
\]
Remark 5.10. Let

\[ H \]

Hence the assertion holds. The proposition is proved.

Thus by repeating this procedure, we see that

\[
(1 - t_0)^{-1} G^{1,r}_s(x; t) = |S_{m-1}|^{-1} \sum_{w \in S_m} w \left( (x_1^{(r)})^s \prod_{2 \leq k < r} \prod_{j \geq 2} \frac{x_1^{(r)} - t_k x_j^{(k+1)}}{x_1^{(k)} - x_j^{(k)}} \prod_{j \geq 2} \frac{x_1^{(r)} - t_1 t_r x_j^{(r)}}{x_1^{(r)} - x_j^{(r)}} \right).
\]

Hence the assertion holds. The proposition is proved.

Remark 5.10. Let \( \lambda = (\lambda_1, \ldots, \lambda_m) \) be a partition, and \( Q_\lambda(y; t) \) the original Hall-Littlewood function. In this case it is known by [M, III, 2.14] that

\[
Q_\lambda(y; t) = \sum_{i=1}^{m} y_i^{\lambda_i} g_i Q_{\mu_i}^{[i]}(y; t)
\]

with

\[
g_i = (1 - t) \prod_{j \neq i} \frac{y_i - t y_j}{y_i - y_j},
\]

where \( \mu = (\mu_2, \ldots, \mu_m) \) and \( Q_{\mu_i}^{[i]} \) is defined similarly to 5.1. Now assume that \( \lambda = (-, \ldots, -, \lambda^{(r)}) \in P_{n,r} \). We compare the formula (5.10.1) with Lemma 5.2. Then by using a similar argument as in the proof of the third formula in Proposition 5.9, one can show, by induction on \( m \), that \( \widetilde{Q}_\lambda(x; t) \) coincides with \( Q_{\lambda^{(r)}}(x^{(r)}; t_0) \) for \( t_0 = t_1 \cdots t_r \). By Theorem 3.5 we know that \( \widetilde{Q}_\lambda(x; t) = Q_{\lambda^{(r)}}(x^{(r)}; t_0) \). It follows, for a special case \( \lambda = (-, \ldots, -, \lambda^{(r)}) \), that we obtain

\[
\widetilde{Q}_\lambda(x; t) = Q_{\lambda}(x; t).
\]

5.11. Based on the discussion in 5.8, we define functions \( \Psi^{(k,i)}_-(u), \Psi^{(k)}_+(u) \) associated to \( M \) as follows. Set \( \Delta = \Delta(\lambda) = \{(k, i) \in M \mid \lambda_i^{(k)} \neq 0\} \). Let \( \Delta_0 \) be the subset of \( \Delta \) consisting of \( (r, i) \) such that \( \lambda_i^{(k)} = 0 \) for \( 1 \leq k < r \), and set \( \Delta_1 = \Delta - \Delta_0 \).

In the “−” case, for \( (k, i) \in \Delta_1 \), set

\[
\Psi^{(k,i)}_-(u) = \prod_{i \leq j \atop (k+1,j) \in M} (1 - t_k u x_j^{(k+1)}) \prod_{i \leq j \atop (k,i) \in M} (1 - u x_j^{(k)}).
\]
Also set, for \((r, i) \in \Delta_0\) and \(t_0 = t_1 \cdots t_r\),

\[
\Psi_{(r,i)}(u) = \prod_{t \leq j \atop (r,i) \in \mathcal{M}} \frac{\prod_{t \leq j \atop (r,i) \in \mathcal{M}} (1 - t_0ux_j^{(r)})}{\prod_{t \leq j \atop (r,i) \in \mathcal{M}} (1 - ux_j^{(r)})}.
\]

In the “+” case, for \(1 \leq k \leq r\), set

\[
\Psi^{(k)}_{+}(u) = \prod_{(k-1,i) \in \mathcal{M}} \frac{\prod_{(k-1,i) \in \mathcal{M}} (1 - t_{k-1}ux_{k-1}^{(k-1)})}{\prod_{(k,i) \in \mathcal{M}} (1 - ux_i^{(k)})}.
\]
Then we have

\[
\Psi^{(k,1)}_{-}(u) = \begin{cases} 
\sum_{s=0}^{\infty} q_{s, -}^{(k)}(x; t_{k}) u^{s} & \text{if } (k, 1) \in \Delta_{1} \text{ and } k < r, \\
\sum_{s=0}^{\infty} \tilde{q}_{s, -}^{(r)}(x; t_{r}) u^{s} & \text{if } (k, 1) \in \Delta_{1}, k = r \text{ and } a > 1, \\
\sum_{s=0}^{\infty} q_{s}(x^{(r)}; t_{0}) u^{s} & \text{if } (k, 1) \in \Delta_{0}, k = r, 
\end{cases}
\]

\[
\Psi^{(k)}_{+}(u) = \begin{cases} 
\sum_{s=0}^{\infty} q_{s, +}^{(1)}(x; t_{r}) u^{s} & \text{if } k = 1 \text{ and } a = 1, \\
\sum_{s=0}^{\infty} \tilde{q}_{s, +}^{(k)}(x; t_{k-1}) u^{s} & \text{if } 1 < a \leq k \leq r.
\end{cases}
\]

We introduce infinitely many variables $u_{1}^{(k)}, u_{2}^{(k)}, \ldots$ for $1 \leq k \leq r$. We consider the set $\hat{\mathcal{M}} = \hat{\mathcal{M}}^{(a)}$ by letting $m \mapsto \infty$, and give the total order on $\hat{\mathcal{M}}$ inherited from $\mathcal{M}$. We define functions $\Phi_{\pm}(u)$ with multi-variables $u = \{u_{i}^{(k)} \mid (k, i) \in \hat{\mathcal{M}}\}$ by

\[
\Phi_{-}(u) = \prod_{(k,i) \in \Delta} \prod_{j \geq i} [\Psi^{(k,i)}_{-}(u_{j}^{(k)})]
\times \frac{\prod_{i < j, (k,i) \in \Delta_{1}} (1 - (u_{i}^{(k)})^{-1} u_{j}^{(k)})}{\prod_{(k,i), (k-1,i) \in \Delta_{1}} (1 - t_{k-1}(u_{i}^{(k)})^{-1} u_{j}^{(k-1)})} \prod_{1 \leq j < i \in \Delta_{0}} \frac{1 - (u_{i}^{(r)})^{-1} u_{j}^{(r)}}{1 - t_{0}(u_{i}^{(r)})^{-1} u_{j}^{(r)}},
\]

\[
\Phi_{+}(u) = \prod_{(k,i) \in \hat{\mathcal{M}}} \Psi^{(k,i)}_{+}(u_{j}^{(k)}) \prod_{i < j, (k,i) < (k+1,i)} \frac{1 - (u_{i}^{(k)})^{-1} u_{j}^{(k)}}{1 - t_{k}(u_{i}^{(k)})^{-1} u_{j}^{(k+1)}},
\]

Recall that $\nu_{0} = (k_{0}, i_{0})$ (see 4.1). We consider the following condition on $\lambda \in \mathcal{P}_{n,r}$:

(A) : $\lambda_{i_{0}}^{(1)} \neq 0$.

We shall prove the following result.
Proposition 5.12. In the “+” case, assume that $\lambda$ satisfies the condition (A). In the “−” case, give no assumption. Then $\tilde{Q}_\lambda^r(x; t)$ coincides with the coefficient of $u^\lambda = \prod_{k,i} (u_i^{(k)})^{\lambda_i^{(k)}}$ in the function $\Phi_{\pm}(u)$.

Proof. First we consider the “−” case. We assume that either $a > 1$ or $a = 1$ and $\lambda$ is not of the form $(-, \ldots, -, \lambda^{(r)})$. Let $b \leq r$ be the smallest integer $b \geq a$ such that $\lambda^{(b)} \neq \emptyset$. We follow the notation in 5.1. First assume that such $b$ exists. Hence $(b, 1) \in \Delta_1$. Let $i_0 = (1, \ldots, 1) \in [1, m]^d$ be as before, and put $u' = u - \{u_i^{(k)} \mid a \leq k \leq b\}$. The function $\Phi_{\pm[b]}(u')$ is defined similarly to $\Phi_{\pm}(u)$, by replacing $u$ and $x_{\#}$ by $u'$ and $x_{\#'}$. For each $i = (i_a, \ldots, i_b) \in [1, m]^d$ with $d = b - a + 1$, we denote by $\Phi_{\pm[i]}(u)$ the function obtained from $\Phi_{\pm[b]}(u')$ by replacing the variables $x_{2}^{(k)}, \ldots, x_{m}^{(k)}$ by $x_{i_1}^{(k)}, \ldots, x_{i_b}^{(k)}$ (where $x_{i_b}^{(k)}$ is removed) for $a \leq k \leq b$. Then by (5.11.6), we have

\begin{equation}
\Phi_{\pm[b]}(u') = \Phi_{\pm}(u') \frac{\prod_{j \geq 2} 1 - x_{i_b}^{(b)} u_j^{(b)}}{\prod_{(s)} 1 - t_{b-1} x_{i_b}^{(b)} u_{\ell}^{(b-1)}},
\end{equation}

where the condition (*) is that $(b - 1, 1) \in \Delta_1$ and $(b - 1, \ell) \in \mathcal{M}'$ (this occurs only when $a = b$). Moreover, $\Phi_{\pm}(u')$ is defined as the product of factors in $\Phi_{\pm}(u)$ not containing $u_1^{(b)}$. Let $\mu$ be as in 5.1. By induction hypothesis, we may assume that $\tilde{Q}_{\mu^{-[b]}}$ is obtained as the coefficient of $u^\mu$ in the function $\Phi_{\pm[b]}(u')$. Thus a similar result holds also for $\tilde{Q}_{\mu^{-[i]}}$. Combining it with Lemma 5.2, we see that $\tilde{Q}_\lambda(x; t)$ is the coefficient of $u^\lambda$ in

$$
\sum_{s \geq 0} (u_1^{(b)})^s \sum_{i \in [1, m]^d} (x_{i_b}^{(b)})^{s - \varepsilon_{i, b}} g_{i -} \Phi_{\pm[i]}(u'),
$$

where $\varepsilon_{i, b} = 1$ (resp. $\varepsilon_{i, b} = 0$) if $b < r$ (resp. $b = r$). Now by (5.12.1) this expression is equal to

\begin{equation}
\Phi_{\pm}(u') \sum_{s \geq 0} (u_1^{(b)})^s \sum_{i \in [1, m]^d} (x_{i_b}^{(b)})^{s - \varepsilon_{i, b}} g_{i -} \prod_{j \geq 2} 1 - x_{i_b}^{(b)} u_j^{(b)} \prod_{(s)} 1 - t_{b-1} x_{i_b}^{(b)} u_{\ell}^{(b-1)}.
\end{equation}

We expand the products in (5.12.2) as a power series in $x_{i_b}^{(b)}$,

$$
\frac{\prod_{j \geq 2} 1 - x_{i_b}^{(b)} u_j^{(b)}}{\prod_{(s)} 1 - t_{b-1} x_{i_b}^{(b)} u_{\ell}^{(b-1)}} = \sum_{p \geq 0} f_p(u'; t)(x_{i_b}^{(b)})^p,
$$

where $f_p(u'; t)$ is a polynomial in $u', t$. Write the expression (5.12.2) as $\Phi_{\pm}(u') Z$. Now assume that $b < r$. Substituting the above expansion into $Z$, we have

$$
Z = \sum_{s \geq 0} (u_1^{(b)})^s \sum_{p \geq 0} f_p \sum_{i \in [1, m]^d} (x_{i_b}^{(b)})^{s - 1 + p} g_{i -}^{(a)}
$$
\[
\sum_{s,p \geq 0} (u_1^{(b)})^s q_s^{(b)}_{s+p,-} f_p = \sum_{j \geq 0} (u_1^{(b)})^j q_j^{(b)}_{j,-} \sum_{p=0}^j f_p (u_1^{(b)})^{-p},
\]

where the second identity follows from Proposition 5.9. By using (5.11.4), the positive degree part of \( Z \) with respect to \( u_1^{(b)} \) coincides with that of

\[
\Psi_{-}^{(b,1)} (u_1^{(b)}) \sum_{p \geq 0} f_p (u_1^{(b)})^{-p} = \Psi_{-}^{(b,1)} (u_1^{(b)}) \prod_{j \geq 2} \frac{1 - (u_1^{(b)})^{-1} u_j^{(b)}}{1 - t_{b-1} (u_1^{(b)})^{-1} u_\ell^{(b-1)}}.
\]

Thus \( \tilde{Q}_\lambda \) is the coefficient of \( u^\lambda \) in

\[
\Psi_{-}^{(b,1)} (u_1^{(b)}) \prod_{j \geq 2} \frac{1 - (u_1^{(b)})^{-1} u_j^{(b)}}{1 - t_{b-1} (u_1^{(b)})^{-1} u_\ell^{(b-1)}} \cdot \Phi_{-} (u') = \Phi_{-} (u)
\]
as asserted.

If \( b = r \), in the last expression of \( Z \), \( q_j^{(b,-)} \) should be replaced by \( \tilde{q}_j^{(b,-)} \) by Proposition 5.9. By using (5.11.4), we obtain the required formula in this case. If such \( b \) does not exist, i.e., \( \lambda^{(b)} = \emptyset \), still a similar formula as (5.12.1) holds, but we must replace the numerator by 1. The remaining argument is the same as above.

Next consider the case where \( \lambda = (-, \ldots, -, \lambda^{(r)}) \) and \( a = 1 \), namely, \( (r, 1) \in \Delta_0 \). In this case, the formula (5.12.1) is replaced by

\[
\Phi_{-}^{[\lambda]} (u') = \Phi_{-} (u') \prod_{j \geq 2} \frac{1 - x_j^{(r)} u_j^{(r)}}{1 - t_{0} x_1^{(r)} u_j^{(r)}} = \Phi_{-} (u') Z.
\]

Then a similar computation as above shows that the positive degree part of \( Z \) with respect to \( u_1^{(r)} \) coincides with that of

\[
\Psi_{-}^{(r,1)} (u_1^{(r)}) \prod_{j \geq 2} \frac{1 - (u_1^{(r)})^{-1} u_j^{(r)}}{1 - t_{0} (u_1^{(r)})^{-1} u_\ell^{(r)}}.
\]

Hence the assertion holds by a similar argument as above.

Finally we consider the “+”-case. The formula corresponding to (5.12.1) is given as

\[
\Phi_{+}^{[\lambda]} (u') = \Phi_{+} (u') \prod_{j \geq 2} \frac{1 - x_j^{(a)} u_j^{(a)}}{1 - t_{a+1} x_i^{(a)} u_\ell^{(a+1)}},
\]
where the condition (**) is that \((a + 1, \ell) \in \mathcal{M}'\), namely, \(\ell \geq 1\) if \(a < r\) and \(\ell \geq 2\) if \(a = r\). Then a similar argument works by using Lemma 5.5 instead of Lemma 5.2. The proposition is proved. \(\square\)

5.13. Returning to the original setting, we consider \(\mathcal{M} = \mathcal{M}^{(1)}\). Let \(\beta = (\beta^{(1)}, \ldots, \beta^{(r)})\) be an \(r\)-composition such that \(\sum_i |\beta^{(i)}| = n\). We write \(\beta_i^{(k)} = \beta_\nu\) if \(\nu = (k, i) \in \mathcal{M}\), and identify \(\beta = (\beta_i^{(k)})\) with an element \((\beta_\nu) \in \mathbb{Z}^\mathcal{M}\). For \(\nu \neq \nu' \in \mathcal{M}\), we define an operator \(R_{\nu,\nu'} : \mathbb{Z}^\mathcal{M} \rightarrow \mathbb{Z}^\mathcal{M}\) as follows; for \(\beta = (\beta_\xi) \in \mathbb{Z}^\mathcal{M}\), \(R_{\nu,\nu'}(\beta) = (\beta_\xi')\) is given by

\[
\beta_\nu' = \beta_\nu + 1, \quad \beta_{\nu'}' = \beta_{\nu'} - 1,
\]

and \(\beta_\xi' = \beta_\xi\) for \(\xi \neq \nu, \nu'\). A raising operator is defined as a product of various \(R_{\nu,\nu'}\) for \(\nu < \nu'\).

For each \(r\)-composition \(\beta = (\beta_i^{(k)})\), one can define \(q_{\beta}^\pm(x; t)\) by generalizing the definition (2.5.3). We also extend the definition of \(q_{\beta}^\pm\) to the case where \(\beta \in \mathbb{Z}^\mathcal{M}\), by putting \(q_{\beta}^\pm = 0\) if some \(\beta_i^{(k)} \in \mathbb{Z}_{<0}\). Then the action of the raising operator \(R\) on the functions \(q_{\beta}^\pm\) is defined by \(R(q_{\beta}^\pm) = q_{\beta'}^\pm\) with \(\beta' = R(\beta)\). As a corollary to Proposition 5.12 we have the following.

Corollary 5.14. Under the same assumption as in Proposition 5.12, for each \(\lambda \in \mathcal{P}_{n,r}\), \(Q_{\lambda}^\pm\) is expressed as

\[
Q_{\lambda}^- = \left( \prod_{\nu = (k, i)} \frac{1 - R_{\nu,\nu'}}{1 - t_{k', R_{\nu,\nu'}}} \prod_{\nu < \nu'} \frac{1 - R_{\nu', \nu}}{1 - t_{0, R_{\nu', \nu}}} \right) q_{\lambda}^-,
\]

(5.14.1)

\[
Q_{\lambda}^+ = \left( \prod_{\nu < \nu'} \frac{1}{1 - t_{b(\nu') - b(\nu) + 1}} \prod_{b(\nu') = b(\nu)} \frac{1 - R_{\nu,\nu'}}{1 - t_{b(\nu) R_{\nu,\nu'}}} \right) q_{\lambda}^+.
\]

(5.14.2)

In particular, \(Q_{\lambda}^\pm \in \mathbb{Z}[x; t]\), and is expressed as

\[
Q_{\lambda}^\pm(x; t) = q_{\lambda}^\pm(x; t) + \sum_{\mu \in \mathcal{P}_{n,r}, \mu > \lambda} c_{\lambda,\mu}(t) q_{\mu}^\pm(x; t)
\]

(5.14.3)

with \(c_{\lambda,\mu}(t) \in \mathbb{Z}[t]\).

Proof. First consider the “+” case. By applying (5.11.4) for the case \(\mathcal{M} = \mathcal{M}^{(1)}\), we have \(\prod_{(k, i) \in \Delta_1} \prod_{j \geq i} \Psi^{(k)}(u^{(k)}_j) = \sum_{\beta} q_{\beta} \mathbf{u}^\beta\), where \(\beta = (\beta^{(1)}, \ldots, \beta^{(r)})\) runs over all \(r\)-compositions such that \(\beta_i^{(k)} = 0\) if \((k, i) \notin \Delta_1\). Moreover, \(\prod_{(r, i) \in \Delta_0} \prod_{j \geq i} \Psi^{(r)}(u^{(r)}_j) = \sum_{\beta} q_{\beta} (u^{(r)})^\beta\) where \(\beta\) runs over all the compositions in \(\mathbb{Z}_{\geq 0}^r\) (\(j_0\) is as in (4.2.4)) and
$q_\beta = q_\beta(x^{(r)}; t_0)$ is defined similarly to $q_\beta^-$ under the notation of (5.11.4). Then the coefficient of $u^\lambda$ in $\Phi_-(u)$ is equal to that of $u^\lambda$ in

$$
\prod_{i<j, (k,i) \in \Delta_1} (1 - (u_i^{(k)})^{-1}u_j^{(k)}) \prod_{(k,i), (k,i) \in \Delta_1} (1 - t_{k-1}(u_i^{(k)})^{-1}u_k^{(k-1)}) \prod_{i<j} \frac{1 - (u_i^{(r)})^{-1}u_j^{(r)}}{1 - t_0(u_i^{(r)})^{-1}u_j^{(r)}} \sum_\beta q_\beta u^\beta,
$$

where $q_\beta^-(x; t)$ should be replaced by $q_{\beta^{(r)}}(x^{(r)}; t_0)$ if $\beta = (-, \ldots, -, \beta^{(r)})$ Thus the coefficient of $u^\lambda$ is equal to

$$
\left( \prod_{\nu = (k,i)}^{\nu^\prime = (k^\prime,i)} \prod_{\nu^\prime = (k^\prime,i) \in \Delta_1} (1 - R_{\nu,\nu^\prime}) \prod_{\nu^\prime \in \Delta_0} \frac{1 - R_{\nu,\nu^\prime}}{1 - t_0R_{\nu,\nu^\prime}} \right) q_\lambda^-. 
$$

Hence (5.14.1) holds. The “+” case is dealt with similarly, and we obtain (5.14.2). It follows that $\tilde{Q}_\lambda^\pm$ is a sum of $Rq_\lambda^\pm = q_{R\lambda}^\pm$ for various raising operators $R$. $\mu^\prime = R\lambda$ is an $r$-composition if $Rq_\lambda \neq 0$, and in that case we have $\mu^\prime \geq \lambda$. Let $\mu$ be the $r$-partition obtained from $\mu^\prime$ by permuting the parts of $\mu^\prime$. Then $\mu \geq \mu^\prime$, and so $\mu \geq \lambda$. We have $Rq_\lambda^\pm = q_\mu^\pm$. Thus $\tilde{Q}_\lambda^\pm$ can be written as a linear combination of $q_\mu^\pm$ with $\mu \geq \lambda$. The equality $\mu = \lambda$ holds only when $R = \text{id}$. Hence (5.14.3) holds.

By using the characterization of Hall-Littlewood functions in Theorem 2.14, we have the following result.

**Theorem 5.15.** Assume that $\lambda$ satisfies the condition (A) in 5.11 in the “+” case. Give no assumption in the “-” case. Then we have

$$
\tilde{P}_\lambda^\pm(x; t) = P_\lambda^\pm(x; t) \quad \tilde{Q}_\lambda^\pm(x; t) = Q_\lambda^\pm(x; t).
$$

**Proof.** The second formula follows from Proposition 4.3 and Corollary 5.14. Then the first formula follows from (4.4.2). \qed

**Remark 5.16.** In the “+” case, the inductive argument as in the proof of Proposition 5.12 does not work if (A) is not satisfied. Although the definition of $\tilde{Q}_\lambda^+$ makes sense even in that case, this function does not coincide with $Q_\lambda^+$ in general. In next section, we discuss the excluded case, and give the closed formula for $Q_\lambda^+$ without assuming (A).

6. **Closed formula for Hall-Littlewood functions – “+” case**

6.1. Take $\lambda \in \mathcal{P}_{n,r}$ and consider $\mathcal{M} = \mathcal{M}^{(1)}$ associated to $\lambda$. Recall that $\nu_0 = (k_0, i_0)$. We define a sequence of integers $m_1 \leq m_2 \leq \cdots \leq m_r = i_0$ inductively
by $m_1 = \ell(\lambda^{(1)})$, $m_k = \max(m_{k-1}, \ell(\lambda^{(k)}))$ for $k \geq 2$. We define a function $I^{(k)}(x; t)$ as follows:

$$I^{(k)}(x; t) = \prod_{(k,i) \leq \nu_0} \prod_{i<j} (x_i^{(k)} - t_{k-1}x_j^{(k-1)})$$

for $k = 2, \ldots, r$, and $I^{(1)} = J_1J_2 \cdots J_r$, where

$$J_a(x; t) = \prod_{i=m_a+1}^{m_a} \prod_{i \leq j} (x_i^{(a)} - t_a \cdots t_{a-1}t_{a-2}x_j^{(r)})$$

if $m_{a-1} < m_a$, and $J_a(x; t) = 1$ if $m_{a-1} = m_a$. (By convention, put $m_0 = 0$.) Moreover for $k = 1, \ldots, r$, put

$$I_0^{(k)}(x; t) = \prod_{(k,i) > \nu_0} \prod_{i<j} (x_i^{(k)} - t_kx_j^{(k)}).$$

We define a function $R^*_x(x; t)$ by

$$R^*_x(x; t) = \sum_{w \in S'_m} w \left( \prod_{1 \leq k \leq r} \prod_{1 \leq k \leq r} \prod_{i<j} x_i^{(k)} - x_j^{(k)} \right) .$$

Here $\varepsilon^{(k)} = (\varepsilon_1^{(k)}, \ldots, \varepsilon_m^{(k)})$, where $\varepsilon_i^{(k)} = 1$ for $m_{k-1} < i \leq m_k$ and $\varepsilon_i^{(k)} = 0$ otherwise.

Let $v'_x(t)$ be as in (4.1.4). By a similar discussion as in the proof of (4.2.2), we obtain a formula

$$(6.1.1) \quad R^*_x(x; t) = v'_x(t) \sum_{w \in S'_m/S'_x} w \left( \prod_{1 \leq k \leq r} \prod_{(k,i) \leq \nu_0} x_i^{(k)} - x_j^{(k)} \right) .$$

Thus $R^*_x(x; t)$ is a polynomial in $\mathbb{Z}[x_1, \ldots, t]$, and is divisible by $v'_x(t)$. Moreover, $v'_x(t)^{-1}R^*_x(x; t)$ satisfies the stability property for $m \mapsto \infty$. We note that $R^*_x(x; t)$ satisfies a similar property as in Proposition 4.3, namely,

**Proposition 6.2.** There exist polynomials $u_{x, \mu}(t) \in \mathbb{Z}[t]$ such that

$$R^*_x(x; t) = \sum_{\mu} u_{x, \mu}(t)s_{\mu}(x).$$

**Proof.** We use the same notation as in the proof of Proposition 4.3 (here we consider the “+” case), in particular, let $w = (w_1, \ldots, w_r) \in S'_m$ be as defined there. In the “+”-case in 4.3, we modify the definition of $X'_1$ as follows ($X'_k$ for $k \neq 1$ are...
unchanged). \( X'_1 = \prod_{a=1}^r Y_a \), where

\[
Y_a = \{ (\nu, \nu') \in \mathcal{M}^2 \mid w(\nu) = (a, i), m_{a-1} + 1 \leq i \leq m_a, b(\nu') = r \}
\]

Let \( \mathcal{M}_k \) (resp. \( \mathcal{B}_k \)) be the k-th row of \( \mathcal{M} \) (resp. \( \mathcal{B} \)). We define a subset \( \mathcal{B}_{r,a} \) of \( \mathcal{M}_r \) by \( w_a(\mathcal{B}_a) = w_r(\mathcal{B}_{r,a}) \). Put

\[
A_{1,a}^+ = \sum_{(\nu, \nu') \in Y_a} s_{\nu, \nu'},
\]

\[
B_{1,a}^+ = \sum_{(\nu, \nu') \in Y_a} s_{\nu, \nu'}.
\]

We put \( A_1^+ = \prod_{a=1}^r A_{1,a}^+ \), \( B_1^+ = \prod_{a=1}^r B_{1,a}^+ \). Then a similar argument as in the proof of Proposition 4.3 works by replacing \( A_1^+, B_1^+ \) there by the current version. Thus the proposition follows. \( \square \)

6.3. Let \( \lambda \in \mathcal{P}_{n,r} \). We define a polynomial \( f_\lambda(t) \) by

\[
(6.3.1) \quad f_\lambda(t) = \prod_{i=1}^{r-1} t_i^{A_i},
\]

where \( A_i = (m - i_0) + (m - i_0 + 1) + \cdots + (m - m_i - 1) \). We define a function \( Q_\lambda^+(x; t) \) by

\[
Q_\lambda^+(x; t) = v_\lambda(t)^{-1} f_\lambda(t)^{-1} R_\lambda^+(x; t).
\]

We show the following result.

**Theorem 6.4.** Let \( \lambda \in \mathcal{P}_{n,r} \) be an arbitrary \( r \)-partition of \( n \).

(i) \( Q_\lambda^+(x; t) \) coincides with \( Q_\lambda^+(x; t) \).

(ii) \( Q_\lambda^+(x; t) \) can be expressed as

\[
Q_\lambda^+(x; t) = \prod_{\nu < \nu'} \frac{\prod_{b(\nu) = b(\nu')} 1 - R_{\nu, \nu'}}{\prod_{b(\nu) = b(\nu') + 1} 1 - t_{b(\nu)} R_{\nu, \nu'} q_\lambda^+}.
\]

(iii) \( P_\lambda^+(x; t) = (1 - t_0)^{-j_0} Q_\lambda^+(x; t) \).

6.5. We prove the theorem in 6.10 after some preliminaries. Assume that \( \lambda = (-, \lambda^{(2)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r}^1 \) and put \( \mu = (\lambda^{(2)}, \ldots, \lambda^{(r)}) \). We consider the function \( Q_{\mu}^{(x')}(x'; t') \), defined similarly to \( Q_{\lambda}^+(x; t) \), but by replacing \( x \) by \( x' = (x^{(2)}, \ldots, x^{(r)}) \) and \( t \) by \( t' = (t_2, \ldots, t_{r-1}, t_1 t_r) \). We have a lemma.

**Lemma 6.6.** Assume that \( \lambda \in \mathcal{P}_{n,r}^1 \). Under the notation as above, we have

\[
Q_{\lambda}^+(x; t) = Q_{\mu}^{(x')} (x'; t').
\]
Proof. Since $m_1 = 0$, we have $J_1 = 1$. Hence

\begin{align}
(6.6.1) & \quad v'_\lambda(t) - R^t_\lambda(x; t) \\
& = \sum_{w'\in S_m'/S_{\mu'}} w' \left\{ \sum_{w\in S_m^{(1)}/S_{\lambda(1)}} w \left( \prod_{i<j} x_i^{(2)} - t_1 x_j^{(1)} \right) \prod_{k=2}^r (x^{(k)})^{\lambda(k) - \varepsilon(k)} I^{(k)}(x'; t') \right\},
\end{align}

where $I^{(k)}(x'; t')$ is defined by replacing $r$ by $r - 1$ with respect to the variables $x'$, and $S_m' = S_m^{(2)} \times \cdots \times S_m^{(r)}$. We note that

\begin{align}
(6.6.2) & \quad \sum_{w\in S_m^{(1)}/S_{\lambda(1)}} w \left( \prod_{i<j} x_i^{(2)} - t_1 x_j^{(1)} \right) = r_1^{A_1}.
\end{align}

In fact, let $S_0$ be the subgroup of $S_m$ which stabilizes $1, \ldots, i_0$. Since $\lambda^{(1)} = \emptyset$, $S_{\lambda^{(1)}}' = S_0$. Let $X$ be the left hand side of (6.6.2). We use the notation $x_i^{(1)} = y_i, x_i^{(2)} = z_i$. $S_m$ acts only on $y_i$ variables. Then

\begin{align}
X &= |S_0|^{-1} \sum_{w\in S_m} w \left\{ \prod_{i<j} \frac{\prod_{i<i_0} (z_i - t_1 y_j) \prod_{i>i_0} (y_i - y_j)}{y_i - y_j} \right\} \\
& = |S_0|^{-1} \prod_{i<j} (y_i - y_j)^{-1} \sum_{w\in S_m} \varepsilon(w) \left( \prod_{i<i_0} (z_i - t_1 y_j) \prod_{i>i_0} (y_i - y_j) \right).
\end{align}

Since $\sum_{w\in S_m} \varepsilon(w) \prod_{i<i_0} (z_i - t_1 y_j) \prod_{i>i_0} (y_i - y_j)$ is an alternating polynomial with respect to $y_i$, non-zero contribution only comes from the term

\begin{align}
\prod_{i<i_0} (-t_1 y_j) \prod_{i>i_0} (y_i - y_j)
& = (-t_1)^{A_1} (y_m y_{m-1} \cdots y_{i_0+1})^{i_0} (y_{i_0})^{i_0-1} (y_{i_0-1})^{i_0-2} \cdots y_2 \prod_{i>i_0} (y_i - y_j).
\end{align}

Since the last product can be written as

\begin{align}
\sum_{w'\in S_0} \varepsilon(w') w' ((y_{i_0+1})^{m-i_0-1} (y_{i_0+2})^{m-i_0-2} \cdots y_{m-1}),
\end{align}

we have

\begin{align}
X &= (-t_1)^{A_1} \prod_{i<i_0} (y_i - y_j)^{-1}
\end{align}
Thus (6.6.2) holds.

By (6.6.1) and (6.6.2), we have $v_\lambda'(t)^{-1}R_\lambda^R(x; t) = t_1^{-1}v_\mu'(t')^{-1}R_\mu(t')$. Since $f_\lambda(t) = t_1^{-1}f_\mu(t')$, we obtain the lemma. \hfill \Box

### 6.7.

We consider the special case where $m_1 = \cdots = m_{r-1} = 0$ and $m_r = i_0$. In this case,

$$I^{(1)} = J_r = \prod_{1 \leq i \leq i_0} \prod_{1 \leq i < j} (x_i^{(r)} - t_0 x_j^{(r)}) = (1 - t_0)^{i_0} \prod_{1 \leq i \leq i_0} x_i^{(r)} \prod_{i < j} (x_i^{(r)} - t_0 x_j^{(r)}),$$

where $t_0 = t_1 \cdots t_r$. We write $\lambda = (-; \cdots; ; \mu)$ with $\mu \in \mathcal{P}_n$. By a similar computation as in the proof of Lemma 6.6 (note that $f_\lambda(t) = 1$), we have

$$Q_\lambda^R(x; t) = v_\lambda'(t)^{-1}R_\lambda^R(x; t) = (1 - t_0)^{i_0}v_\lambda'(t_0)^{-1}R_\mu(x^{(r)}; t_0),$$

where $R_\mu$ is the function defined in [M, III, 1]. Under the notation in [M, III, 2], we have

$$R_\mu(x^{(r)}; t_0) = v_\mu(t_0)P_\mu(x^{(r)}; t_0) = v_\mu(t_0)b_\mu(t_0)^{-1}Q_\mu(x^{(r)}; t_0),$$

where $P_\mu, Q_\mu$ are classical Hall-Littlewood functions associated to the partition $\mu$. Since $v_\mu(t_0) = v_\mu(t_0)b_\mu(t_0)/(1 - t_0)^{i_0}$ (see [M, III, 2], note that $v_\lambda'(t_0) = v_{n-i_0}(t_0)$), we have the following.

**Lemma 6.8.** Assume that $\lambda \in \mathcal{P}_{n,r}^{r-1}$. Then

(6.8.1) \quad $Q_\lambda^R(x; t) = Q_\lambda(x^{(r)}; t_1 \cdots t_r).$

Next we show the following proposition.

**Proposition 6.9.** Assume that Theorem 6.4 holds for $r-1$. Then for any $\lambda \in \mathcal{P}_{n,r}^1$, we have

(6.9.1) \quad $Q_\lambda^+(x; t) = \prod_{\nu \prec \lambda} \frac{1 - R_{\nu, \mu}}{\prod_{b(\nu') = b(\nu) + 1} 1 - t_{b(\nu)} R_{\nu, \mu} q_\lambda^+}.$

**Proof.** Write $\lambda = (-, \lambda^{(2)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r}^1$ and $\lambda' = (\lambda^{(2)}, \ldots, \lambda^{(r)}) \in \mathcal{P}_{n,r-1}$. By Proposition 3.4, we have $Q_\lambda^R(x; t) = Q_\lambda^+(x', t')$ with $t' = (t_2, \ldots, t_{r-1}, t_1 t_r).$ By
appplying Theorem 6.4 (ii) for $Q_{X}^{r}$, we have an expression

$$Q_{X}^{r+}(x'; t') = \prod_{\nu < \nu'} \frac{\prod_{b(\nu') = b(\nu'')} 1 - R'_{\nu, \nu'}}{\prod_{b(\nu') = b(\nu) + 1} 1 - t'_{b(\nu')} R'_{\nu, \nu'}} q_{\chi}^{r+}. \tag{6.9.2}$$

where $t'_{b(\nu)}$ is defined with respect to $t'$, and $R'_{\nu, \nu'}$ is the raising operator with respect to $\mathcal{M}' = \{(k, i) \mid 2 \leq k \leq r, 1 \leq i \leq m\}$. In particular $R'_{\nu, \nu'} = R_{\nu, \nu'}$ if $b(\nu) = b(\nu')$ or $b(\nu') = b(\nu) + 1$ with $b(\nu') \neq r$. Assume that $b(\nu) = r$ and $b(\nu') = 2$. Write $\nu = (r, i), \nu' = (2, j)$. In the computation below, we omit the sign “+”. By Lemma 3.2, $q_{j}^{(2)}(x; t_{1} t_{r}) = \sum_{j_{1} + j_{2} = j} t_{1}^{j_{1}} q_{j_{2}}^{(2)}(x; t_{1}) q_{j}^{(1)}(x; t_{r})$. For $a \geq 1$, we denote by $(t R'_{\nu, \nu'})^{[a]}$ the operator $(t R'_{\nu, \nu'})^{[a]} R_{\nu, \nu'}(t R'_{\nu, \nu'})^{[a - 1]} \cdots (t R'_{\nu, \nu'})^{[1]}$. Then the action of $(t_{1} t_{r} R'_{\nu, \nu'})^{[a]}$ on $q_{j}^{(2)}(x)$ can be written as

$$(t_{1} t_{r} R'_{\nu, \nu'})^{[a]}(q_{j}^{(2)}(x)) = (t_{1} t_{r})^{a} q_{j}^{(2)}(x) = (t_{1} t_{r})^{a} \sum_{j' + j'' = j} (t_{1})^{j''} q_{j'}^{(2)}(x) q_{j''}^{(1)}$$

$$= \sum_{j_{1} + j_{2} = j} \sum_{0 \leq b \leq j_{2}} (t_{1})^{j_{1} + b} t_{r}^{a} q_{j_{2}}^{(2)}(x) q_{j_{1} - a + b} q_{j''}^{(1)}$$

$$= \sum_{j_{1} + j_{2} = j} \sum_{0 \leq b \leq j_{2}} (t_{1} R_{j_{1} - a, j_{2}}^{(1)})^{[b]}(t_{r} R_{j_{1}, j_{2}}^{(1)})^{[a]}(t_{r}^{j_{1}} q_{j_{2}}^{(2)} q_{j_{1} - a + b} q_{j''}^{(1)})$$

$$= \sum_{j_{1} + j_{2} = j} \sum_{0 \leq b \leq j_{2}} (t_{1} R_{j_{1} - a, j_{2}}^{(1)})^{[b]}(t_{r} R_{j_{1}, j_{2}}^{(1)})^{[a]}(t_{1} R_{0,j}^{(1)})^{[j_{1}]} q_{j_{2}}^{(2)} q_{j''}^{(1)} q_{j_{1} - a + b} q_{j''}^{(1)}$$

where $R_{j_{1}, j_{2}}^{(1)} = R_{r, i}, (1, j_{1})$ and $R_{j_{1}, j_{2}}^{(1)} = R_{(1, j_{1}), (2, j_{2})}$. We understand that $R_{0,j}^{(1)}$ is the raising operator which sends $q_{0}^{(1)} = 1$ to $q_{1}^{(1)}$ and $q_{j_{2}}^{(2)}$ to $q_{j_{2} + 1}^{(2)}$. It follows that for any $\lambda \in \mathscr{P}_{n,r}$,

$$\prod_{\nu < \nu'} (1 - t_{1} t_{r} R'_{\nu, \nu'})^{-1} q_{\chi}^{r+}(x; t') = \prod_{\nu < \nu'} (1 - R_{\nu, \nu'}) \prod_{\nu < \nu'} (1 - t_{1} R_{\nu, \nu'})^{-1} \prod_{\nu < \nu'} (1 - t_{r} R_{\nu, \nu'})^{-1} q_{\chi}(x; t).$$

Note that since $\lambda^{(1)} = \emptyset$, the operator $\prod_{b(\nu') = b(\nu) + 1}(1 - R_{\nu, \nu'})$ acts trivially. The proposition follows from this formula. \hfill \Box

6.10. We are now ready to prove Theorem 6.4. Assume that $\lambda^{(1)} \neq \emptyset$. Let $\mathcal{M} = \mathcal{M}^{(1)}$ in the notation of 5.1, and consider $\mathcal{M}^{(a)}$ for $a = 1, \ldots, r$. Then a similar formula as in Lemma 5.5 holds for $Q_{X}^{a}$. Repeating this procedure for $a = 1$ to $r$, one can replace $\mathcal{M}$ by $\mathcal{M}'$ which is defined by replacing $m$ by $m - 1$. By induction, we may assume that the statements (i), (ii) of the theorem hold for the corresponding
function \( Q^{\mu}_{\mu} \) on \( \mathcal{M}' \). In particular, \( Q^{\mu}_{\mu} \) coincides with \( Q^{n}_{\mu} \), and has an expression in terms of raising operators (assertion (ii) of the theorem). This is equivalent to saying that \( Q^{\mu}_{\mu} \) can be expressed as a coefficient of \( u^{n} \) in the function \( \Phi_{+}(u) \) given in Proposition 5.12. Then a similar argument as in the proof of Proposition 5.12 works for \( Q^{\mu}_{\mu} \) and \( Q^{\mu}_{\mu} \) can be expressed as the coefficient of \( u^{\mu} \) in \( \Phi_{+}(u) \), in other words, \( Q^{\mu}_{\mu} \) has an expression in terms of raising operators as in (ii) of the theorem (see Corollary 5.14). Since \( Q^{\mu}_{\mu} \) satisfies the condition for the expansion by Schur functions (Lemma 6.2), we see that \( Q^{\mu}_{\mu} = Q^{\mu}_{\mu} \) by Theorem 2.14. Hence (i), (ii) holds in this case.

Next assume that \( \lambda^{(1)} = \emptyset \), i.e., \( \lambda \in \mathcal{P}_{n,r}^{+} \). In this case, by Proposition 3.4, we have \( Q_{\lambda}(x; t) = Q^{\mu}_{\mu}(x'; t') \) with \( \mu = (\lambda^{(2)}, \ldots, \lambda^{(r)}) \) and \( t' = (t_{2}, \ldots, t_{r-1}, t_{1}t_{r}) \). We also have \( Q_{\lambda}(x; t) = Q^{\mu}_{\mu}(x'; t') \) by Lemma 6.6. By induction, we assume that the theorem holds for \( Q^{\mu}_{\mu} \). Hence \( Q^{\mu}_{\mu}(x', t') = Q^{\mu}_{\mu}(x'; t') \). This implies that \( Q^{\mu}_{\mu}(x; t) = Q^{\mu}_{\mu}(x; t) \), which proves (i). Then (ii) holds for \( Q^{\mu}_{\mu}(x; t) \) by Proposition 6.9.

It remains to consider the case where \( \lambda \in \mathcal{P}_{n,r}^{-} \), namely \( \lambda = (\ldots, -, \lambda^{(r)}) \). In this case, by Theorem 3.5 and Lemma 6.8, we have \( Q_{\lambda}(x; t) = Q^{\mu}_{\mu}(x; t) = Q_{\lambda}(x^{(r)}; t_{1} \cdots t_{r}) \). Hence (i) holds. By [M, III, (215)], \( Q_{\lambda}(x^{(r)}; t_{1} \cdots t_{r}) \) has an expression by raising operators. Then by a similar argument as in the proof of Proposition 6.9, one sees that \( Q^{\mu}_{\mu} \) has an expression by raising operators as in (ii) of the theorem.

Finally, we show (iii). We already know by (4.2.5) and Theorem 5.15 that \( Q_{\lambda} = (1-t_{0})^{j_{0}}P_{\lambda} \). Then (iii) follows from (2.13.1). The theorem is proved.

Combining Theorem 5.15 and Theorem 6.4, we have the following result.

**Theorem 6.11.** Let \( P^{\pm}_{\lambda}(x; t), Q^{\pm}_{\lambda}(x; t) \in \Xi_{Q}(t) \) be the Hall-Littlewood functions.

(i) \( P^{\pm}_{\lambda}(x; t), Q^{\pm}_{\lambda}(x; t) \in \Xi_{Q}[t] \).

(ii) \( Q^{\pm}_{\lambda}(x; t) = (1-t_{1} \cdots t_{r})^{j_{0}}P^{\pm}_{\lambda}(x; t) \), where \( j_{0} \) is as in 4.2.

(iii) \( P^{\pm}_{\lambda}(x; t) \) and \( Q^{\pm}_{\lambda}(x; t) \) are characterized by the property as in Theorem 2.14, but the total order \( \preceq \) can be replaced by the dominance order \( \leq \), and the coefficients \( c_{\lambda, \mu}(t), u_{\lambda, \mu}(t) \in \mathbb{Z}[t] \). In particular, \( P^{\pm}_{\lambda}, Q^{\pm}_{\lambda} \) are determined independently from the choice of the total order.

(iv) \( \{P^{\pm}_{\lambda} \mid \lambda \in \mathcal{P}_{n,r} \} \) give rise to \( \mathbb{Z}[t] \)-bases of \( \Xi_{Q}[t] \). We have \( K^{\pm}_{\lambda, \mu}(t) \in \mathbb{Z}[t] \).

**Proof.** (ii) follows from Theorem 5.15 and Theorem 6.4. By (4.2.4) and (4.2.5), \( \tilde{P}_{\lambda}^{-}(x; t), \tilde{Q}_{\lambda}^{-}(x; t) \in \mathbb{Z}[x; \mathfrak{g}, t] \). Thus \( P^{-}_{\lambda}, Q^{-}_{\lambda} \in \Xi_{Q}[t] \) by Theorem 5.15. By Theorem 6.4, \( Q^{-}_{\lambda} \) has an expression by raising operators. Thus \( Q^{-}_{\lambda} \in \Xi_{Q}[t] \). Hence (i) holds. Then (iii) follows from Proposition 4.3 and Corollary 5.14 in the “−” case, and from Proposition 6.2 and Theorem 6.4 in the “+” case. (iv) follows from (iii). \( \square \)

7. A conjecture of Finkelberg-Ionov

7.1 For two basis \( u = \{u_{\lambda}\}, v = \{v_{\mu}\} \) on the \( Q(t) \)-space \( \Xi_{Q}(t) \), we denote by \( M(u, v) = (M_{\lambda, \mu}) \) the transition matrix between \( u \) and \( v \) as in the proof of Theorem 2.14. Recall the non-degenerate bilinear form \( \langle , \rangle \) on \( \Xi_{Q}[t] \) introduced in 2.10, which satisfies the properties.
\[
\langle q_\lambda^+, m_\mu \rangle = \langle m_\lambda, q^-_\mu \rangle = \delta_{\lambda, \mu}, \\
\langle P_\lambda^+, Q^-_\mu \rangle = \langle Q_\lambda^+, P^-_\mu \rangle = \delta_{\lambda, \mu}.
\]

For a matrix \( M \), let \( M^* \) be the matrix \( t^*M^{-1} \). We denote the basis \( \{ P^\pm_\lambda | \lambda \in \mathcal{P}_{n,r} \} \) of \( \Xi_Q^n(t) \) by \( P^\pm \), and similarly define \( Q^\pm, s, m, \) with respect to \( Q^\pm_\lambda, s_\lambda, m_\lambda, \) respectively. Then we have

\[
M(Q^\pm, q_\pm) = M(P^\mp, m)^* = M(P^\mp, s)^* M(s, m)^*.
\]

Since \( M(s, P^+) = K(t) = (K^\mp_{\lambda, \mu}(t)) \), we have \( M(P^\mp, s)^* = t^*(K(t)^\mp) \). The Kostka number \( K_{\lambda, \mu} \) for \( \lambda, \mu \in \mathcal{P}_{n,r} \) is defined by \( K_{\lambda, \mu} = \prod_i K_{\lambda(i), \mu(i)} \) if \( |\lambda(i)| = |\mu(i)| \) for each \( i \), and \( K_{\lambda, \mu} = 0 \) otherwise. Put \( K = (K_{\lambda, \mu}) \). We know that \( M(s, m)^* = K^* = M(s, h) \), and by \( [M, I, (3.4')] \)

\[
s_\lambda = \prod_{\nu < \nu'} \prod_{b(\nu') = b(\nu)} (1 - R_{\nu, \nu'}) h_\lambda,
\]

where \( h_\lambda \) is the complete symmetric function defined similarly to \( s_\lambda, m_\lambda \). Hence the operation \( \prod_{\nu < \nu'} (1 - R_{\nu, \nu'}) \) on the basis \( \{ h_\mu \} \) corresponds to the matrix operation \( K^* \). Moreover, the matrix operation \( M(P^\mp, s)^* \) on the basis \( \{ s_\mu \} \) coincides with the matrix operation \( M(Q^\pm, q_\pm) \) on the basis \( \{ h_\mu \} \). In particular, if we write \( M(Q^\pm, q_\pm) = (c_{\lambda, \mu}^\pm) \), we have

\[
\sum_\mu K^\pm_{\mu, \lambda}(t) s_\mu = \sum_\mu c_{\lambda, \mu}^\pm h_\mu
\]

for each \( \lambda \). By Corollary 5.14 and Theorem 6.4, we have an expression for \( Q^\pm_\lambda \) such as (5.14.1), (5.14.2). Hence, for a fixed \( \lambda \), we have

\[
(7.1.1) \quad \sum_\mu c_{\lambda, \mu}^+ h_\mu = \prod_{\nu < \nu'} \prod_{b(\nu') = b(\nu) + 1} (1 - R_{\nu, \nu'}) h_\lambda \\
= \prod_{\nu < \nu'} \prod_{b(\nu') = b(\nu) + 1} (1 - t_{b(\nu)} R_{\nu, \nu'})^{-1} s_\lambda,
\]

\[
(7.1.2) \quad \sum_\mu c_{\lambda, \mu}^- h_\mu = \left( \prod_{\nu = (k, i)} \prod_{\nu' = (k', j)} (1 - t_{k'} R_{\nu, \nu'}) \right) \left( \prod_{\nu < \nu'} \prod_{b(\nu') = b(\nu) + 1} \frac{1 - R_{\nu, \nu'}}{1 - t_0 R_{\nu, \nu'}} \right) h_\lambda \\
= \prod_{\nu = (k, i)} \prod_{\nu' = (k', j), \nu < \nu'} (1 - t_{k'} R_{\nu, \nu'})^{-1} \prod_{\nu < \nu'} (1 - t_0 R_{\nu, \nu'})^{-1} s_\lambda.
\]
It follows that the right hand sides of (7.1.1) and (7.1.2) coincide with $\sum_\mu K_{\mu,\lambda}^+(t)s_\mu$.

7.2. We keep the notation for $\mathcal{M} = \mathcal{M}^{(1)}$. Put $M = rm$. The $n$-function $n(\xi)$ for $\xi \in \mathcal{P}_M$ can be extended to any composition $\xi = (\xi_i)_{1 \leq i \leq M} \in \mathbb{Z}_M^{\geq 0}$ by $n(\xi) = \sum_{i=1}^M (i - 1)\xi_i$. Recall the $a$-function on $\mathcal{P}_{n,r}$ ([S1]),

$$a(\lambda) = r \cdot n(\lambda) + |\lambda^{(2)}| + 2|\lambda^{(3)}| + \cdots + (r - 1)|\lambda^{(r)}|,$$

where $n(\lambda) = n(\lambda^{(1)}) + \cdots + n(\lambda^{(r)})$. Let $c(\lambda)$ be the composition of $M$ associated to $\lambda$ as in 2.11. We note that

$$(7.2.1) \quad n(c(\lambda)) = a(\lambda).$$

In fact

$$n(c(\lambda)) = \sum_{k=1}^r \sum_{i=1}^m ((i - 1)r + (k - 1))\lambda_i^{(k)} = \sum_{k=1}^r r \cdot n(\lambda^{(k)}) + \sum_{k=1}^r (k - 1)|\lambda^{(k)}| = a(\lambda).$$

Let $\langle , \rangle$ be the standard inner product on $\mathbb{Z}^M$, and put $\delta = (M-1, M-2, \ldots, 0) \in \mathbb{Z}^M$. Take $\lambda, \mu \in \mathcal{P}_{n,r}$. Then we have

$$(7.2.2) \quad \langle c(\lambda) + \delta, \delta \rangle - \langle c(\mu) + \delta, \delta \rangle = a(\mu) - a(\lambda).$$

In fact, if we put $M = (M-1, M-1, \ldots, M-1) \in \mathbb{Z}^M$, then $\langle c(\lambda), M - \delta \rangle = n(c(\lambda)) = a(\lambda)$. Also we have $\langle c(\lambda), M \rangle = \langle c(\mu), M \rangle$ since $c(\lambda), c(\mu)$ are compositions of $M$. Hence

$$a(\mu) - a(\lambda) = \langle c(\mu), M - \delta \rangle - \langle c(\lambda), M - \delta \rangle = \langle c(\lambda), \delta \rangle - \langle c(\mu), \delta \rangle.$$

Thus (7.2.2) holds.

Let $\varepsilon_1, \ldots, \varepsilon_M$ be the standard basis of $\mathbb{Z}^M$. We denote by $R^+$ the set of positive roots of type $A_{M-1}$, namely $R^+ = \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq M \}$. We identify $\mathbb{Z}^M$ with $\mathbb{Z}^\mathcal{M}$ by the given total order, and denote $\xi = (\xi_i) \in \mathbb{Z}^M$ as $\xi = (\xi_\nu)$ with $\nu \in \mathcal{M}$. For any $\xi = (\xi_i) \in \mathbb{Z}^M$ such that $\sum \xi_i = 0$, we define a function $L_{\pm}(\xi; t)$ by

$$(7.2.3) \quad L_{\pm}(\xi; t) = \sum_{(m_\gamma)} \prod_{v \in \mathcal{M}} t_{b(v) - c}^m.$$
where \( c \) is as in (2.5.3), and \((m_\gamma)\) runs over all the choices such that \( \xi = \sum_{\gamma \in R^+} m_\gamma \gamma \) with \( m_\gamma \geq 0 \), and that \( \gamma = \varepsilon_\nu - \varepsilon_\nu' \) for \( \nu < \nu' \) with the condition

\[(7.2.4) \quad b(\nu') = b(\nu) \mp 1.\]

Note that \( L_\pm(\xi; t) \neq 0 \) only when \( \xi = \sum_i \eta_i (\varepsilon_i - \varepsilon_{i+1}) \) with \( \eta_i \geq 0 \) satisfying the condition (7.2.4). We have \( L_-(\xi; t) \) is monic of degree \( \sum_i \eta_i = \langle \xi, \delta \rangle \) (see [M, III, 6, Ex. 4]), and \( \deg L_+(\xi; t) < \langle \xi, \delta \rangle \) if \( r \geq 3 \).

In the “+”-case, we define a function \( L_+^\mu(\xi; t) \) (depending on the choice of \( \mu \)) by

\[(7.2.5) \quad L_+^\mu(\xi; t) = \sum_{(m_\gamma)} \left( \prod_{\gamma \in A_1} t_{k-1}^{m_\gamma} \prod_{\gamma \in A_0} t_0^{m_\gamma} \right),\]

where \((m_\gamma)\) runs over all the choices such that \( \xi = \sum_{\gamma \in R^+} m_\gamma \gamma \) with \( m_\gamma \geq 0 \), and that \( \gamma = \varepsilon_\nu - \varepsilon_\nu' \) for \( \nu < \nu' \) with the condition

\[(7.2.6) \quad A_1 : \nu = (k, i) \in \Delta_1, (k - 1, i) \in \Delta_1, \nu' = (k - 1, j), \quad A_0 : \nu \in \Delta_0, b(\nu) = b(\nu') = r.\]

(here \( \Delta = \Delta(\mu) \) is the subset of \( \mathcal{M} \) determined as in 5.11). We have the following result.

**Theorem 7.3.** Let \( \lambda, \mu \in \mathcal{P}_{n, r} \). Under the natural embedding \( S_m^r \subset S_M \), we have

\[(7.3.1) \quad K_{\lambda, \mu}(t) = \sum_{w \in S_m^r} \varepsilon(w) L_-(w^{-1}(c(\lambda) + \delta) - (c(\mu) + \delta); t),\]

\[(7.3.2) \quad K_{\lambda, \mu}^+(t) = \sum_{w \in S_m^r} \varepsilon(w) L_+^\mu(w^{-1}(c(\lambda) + \delta) - (c(\mu) + \delta); t).\]

In particular, \( K_{\lambda, \mu}(t) \) is monic of degree \( a(\mu) - a(\lambda) \), and \( \deg K_{\lambda, \mu}^+ < a(\mu) - a(\lambda) \) if \( r \geq 3 \).

**Proof.** By (7.1.1), \( K_{\lambda, \mu}(t) \) is the coefficient of \( s_\lambda \) in

\[\prod_{\nu < \nu'} \prod_{b(\nu') = b(\nu) + 1} (1 + t_{b(\nu')} R_{\nu, \nu'} + t_{b(\nu')}^2 R_{\nu, \nu'}^2 + \cdots) s_\mu,\]

hence is the coefficient of \( x^{\lambda+\delta_1} \) in

\[\sum_{w \in S_m^r} \varepsilon(w) \prod_{(m_\gamma)} \prod_{\gamma} t_{b(\nu)}^{m_\gamma} x^{w(\mu + \delta_1 + \sum m_\gamma)},\]

where \( \delta_1 = (\delta^{(1)}, \ldots, \delta^{(r)}) \) with \( \delta^{(k)} = (m - 1, \ldots, 0) \) and \((m_\gamma)\) are given as in (7.2.3) and (7.2.5). We now consider the change of variables \( \{ x_{i}^{(k)} : 1 \leq k \leq r, 1 \leq i \leq m \} \) to \( \{ y_j : 1 \leq j \leq M \} \) by the assignment \( x_{i}^{(k)} \mapsto y_{i-1+r+k} \). Then the above coefficient
coincides with the coefficient of $y^{c(\lambda)+\delta}$ in
\[
\sum_{w \in S_n^+} \varepsilon(w) \prod \gamma \left[ t_{(m_\gamma)}^{m_\gamma} y^{w(c(\mu)+\delta)+\sum m_\gamma} \right].
\]

This proves (7.3.1). (7.3.2) is proved in a similar way by using (7.1.2).

We have
\[
\langle w^{-1}(c(\lambda)+\delta)-(c(\mu)+\delta), \delta \rangle = \langle c(\lambda)+\delta, w(\delta) \rangle - \langle c(\mu)+\delta, \delta \rangle \\
\leq \langle c(\lambda)+\delta, \delta \rangle - \langle c(\mu)+\delta, \delta \rangle \\
= a(\mu) - a(\lambda).
\]

The last step follows from (7.2.2). The equality holds only when $w = 1$. Note that $\lambda$ is obtained from $\mu$ by applying the raising operator $R_{\nu, \nu'}$ with $b(\nu') = b(\nu) + 1$, $c(\lambda) - c(\mu)$ can be written as a sum of $\gamma \in R^{+}$ satisfying (7.2.4). Hence $L_{-}(c(\lambda) - c(\mu))$ is monic of degree $a(\mu) - a(\lambda)$, and the same is true for $K_{\lambda, \mu}^{-}(t)$.

Finally consider the degree of $K_{\lambda, \mu}^{+}(t)$. In this case, for $\nu = (r, i), \nu' = (r, i+1)$, $\gamma = \varepsilon_{\nu} - \varepsilon_{\nu'}$ can be written as
\[
\gamma = (\varepsilon_{(1,i)} - \varepsilon_{(1,i+1)}) + (\varepsilon_{(1,i+1)} - \varepsilon_{(2,i+1)}) + \cdots + (\varepsilon_{(r-1,i+1)} - \varepsilon_{(r,i+1)}).
\]

Then the computation of $L_{+}^{\mu}(\xi; t)$, which involves the monomials with respect to $t_{1}, \ldots, t_{r}$ and $t_{0}$, is reduced to the computation of $L_{+}(\xi; t)$, which involves only $t_{1}, \ldots, t_{r}$. Since $\deg L_{+}(\xi, t) < \langle \xi, \delta \rangle$ (for $r \geq 3$), the assertion holds. \(\square\)

7.4. In the “+” case, we consider a special situation where the function $L_{r}^{\mu}$ can be described easily. We put the following condition for $\mu \in P_{n,r}$:

(B) $\ell(\lambda(k)) = i_{0}$ for $k = 1, \ldots, r$.

If $\mu$ satisfies the condition (B), then $\Delta_{0} = \emptyset$ and $\Delta_{1} = M$ for $\Delta = \Delta(\mu)$. In this case, $L_{+}^{\mu}(\xi; t)$ coincides with $L_{+}(\xi; t)$. Hence as a corollary of Theorem 7.3 (ii), we have the following.

Corollary 7.5. Assume that $\mu$ satisfies the condition (B). Then we have
\[
K_{\lambda, \mu}^{+}(t) = \sum_{w \in S_n^+} \varepsilon(w) L_{+}(w^{-1}(c(\lambda)+\delta)-(c(\mu)+\delta); t).
\]

7.6. In [FI], Finkelberg and Ionov defined the multi-variable Kostka polynomials $K_{\lambda, \mu}(t)$ by using the Lusztig’s partition function ([L1]) as defined in (7.2.3) for “−” case, which is exactly the formula in the right hand side of (7.3.1). They conjectured (in the case where $t = t_{1} = \cdots = t_{r}$) that $K_{\lambda, \mu}(t)$ coincides with our $K_{\lambda, \mu}^{-}(t)$. Theorem 7.3 gives an affirmative answer to their conjecture (for the multi-variable case). Following [FI], we say that $\mu \in P_{n,r}$ is regular if $\mu_{1}^{(k)} > \mu_{2}^{(k)} > \cdots > \mu_{n}^{(k)}$ for $k = 1, \ldots, r$. They proved in [FI], in the case where $\mu$ is regular, that
$K_{\lambda,\mu}(t) \in \mathbb{Z}_{\geq 0}[t]$ by making use of the higher cohomology vanishing of a certain vector bundle over the flag variety of $(GL_m)^r$. Recently Hu [H] proved the higher cohomology vanishing for arbitrary $\mu$, hence the positivity property of $K_{\lambda,\mu}(t)$ now holds without any restriction.

Combined with Theorem 7.3, we have

**Proposition 7.7.** $K_{\lambda,\mu}^-(t) \in \mathbb{Z}_{\geq 0}[t]$.

**Remark 7.8.** The last statement in Theorem 7.3 and Proposition 7.7 give an answer to the conjecture proposed in [S1, Conjecture 5.5] at least for the "$-$" case.

7.9. Let $\theta = (\theta^{(1)}, \ldots, \theta^{(r)})$ be an $r$-partition, where $\theta^{(k)} = (\theta_1, \theta_2, \ldots, \theta_m)$ for $k = 1, \ldots, r$ (independent of $k$). Let $\lambda, \mu \in \mathcal{P}_{n,r}$. Then $\lambda + \theta, \mu + \theta \in \mathcal{P}_{n',r}$ for some $n'$. As a corollary of Theorem 7.3 and Corollary 7.5, we have the following result, which was conjectured by Finkelberg (for the "$-$" case).

**Corollary 7.10.** Let $\lambda, \mu \in \mathcal{P}_{n,r}$. Assume that $\theta_1 \gg \theta_2 \gg \cdots \gg \theta_m > 0$. Then $K_{\lambda+\theta,\mu+\theta}(t)$ has a stable value, independent of the choice of $\theta$, and we have

$$K_{\lambda+\theta,\mu+\theta}^\pm(t) = L^\pm(c(\lambda) - c(\mu); t).$$

**Proof.** The value $K_{\lambda+\theta,\mu+\theta}^\pm(t)$ can be expressed by the formula in Theorem 7.3 and Corollary 7.5. (Note that in the "$+$" case, (B) holds for $\mu + \theta$ since $\theta_m > 0$.) By our assumption $\theta_1 \gg \theta_2 \gg \cdots \gg \theta_m$, the non-zero contribution only occurs in the case where $w = 1$. Hence

$$K_{\lambda+\theta,\mu+\theta}^\pm(t) = L^\pm(c(\lambda + \theta) - c(\mu + \theta); t) = L^\pm(c(\lambda) - c(\mu); t).$$

The corollary is proved. \hfill $\square$

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