Solutions of the reflection equation for the $U_q[G_2]$ vertex model.

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Abstract

We investigate the possible regular solutions of the boundary Yang-Baxter equation for the fundamental $U_q[G_2]$ vertex model. We find four distinct classes of reflection matrices such that half of them are diagonal while the other half are non-diagonal. The latter are parameterized by two continuous parameters but only one solution has all entries non-null. The non-diagonal solutions do not reduce to diagonal ones at any special limit of the free-parameters.

Keywords: Reflection Equation, K-matrices, $U_q[G_2]$

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1 Introduction

It is, by now, well known that integrable two-dimensional systems of statistical mechanics can be derived from the Yang-Baxter equation [1]. The respective Boltzmann weights can be related in a natural manner to the elements of a Yang-Baxter $\hat{R}$-matrix solution $\hat{R}_{ab}(x) \in C^N_a \otimes C^N_b$ satisfying the relation,

$$\hat{R}_{12}(x)\hat{R}_{23}(xy)\hat{R}_{12}(y) = \hat{R}_{23}(y)\hat{R}_{12}(xy)\hat{R}_{23}(x)$$

(1)

for arbitrary spectral parameters $x$ and $y$.

For an open statistical system not all possible types of boundaries are compatible with the Yang-Baxter condition (1). However, the integrability at the boundary may be assured when the boundary weights $K(x) \in C^N$ fulfill the so-called reflection equation [2, 3], which reads

$$\hat{R}_{12}(x/y)^{1\hat{K}(x)}\hat{R}_{12}(xy)^{1\hat{K}(y)} = \hat{K}(y)\hat{R}_{12}(xy)^{1\hat{K}(x)}\hat{R}_{12}(x/y)$$

(2)

where $1\hat{K}(x) = K(x) \otimes I_N$ and $I_N$ is the $N \times N$ identity matrix.

An important class of $\hat{R}$-matrices are those based on a quantum group $U_q[G]$ $q$-deformation of a classical Lie algebra $G$ [4]. For these models, from a given solution $K(x)$ of Eq.(2) one can, in principle, construct families of commuting transfer-matrix [3, 5]. This fact has motivated several authors, see for instance refs. [6, 7, 8], to search for $K$-matrices solutions associated to such quantum integrable models as well as for their elliptic extensions [9, 10]. More recently, attempts to classify all the $K$-matrices of the non-exceptional $U_q[G]$ vertex models either by direct analysis of Eq.(2) [11] or by others potentially systematic approaches [12, 13] have been discussed in the literature.

In spite of all these works not much is known about the structure of the $K$-matrices of the vertex models based on the quantum exceptional Lie algebras. In fact, even for the simplest such case, i.e. the fundamental $U_q[G_2]$ $\hat{R}$-matrix [14], only the diagonal solutions have been studied [15]. The purpose of this paper is to start to bridge this gap by presenting the complete reflection matrices associated to the minimal $U_q[G_2]$ vertex model.
This paper is organized as follows. In next section we describe the seven-dimensional $U_q[G_2]$ $\hat{R}$-matrix on the Weyl basis. This step makes it possible to adapt the method developed for non-exceptional models \cite{11} to deal with the exceptional $U_q[G_2]$ case. In section 3 we discuss what we hope to be the complete set of reflection matrices. We find two families of non-diagonal solutions and we confirm as well the special diagonal ones given before in ref.\cite{15}. We observe that the two types of diagonal $K$-matrices cannot be obtained as special limits of the non-diagonal ones. Section 4 is reserved for our conclusions. In Appendix A we present the explicit Boltzmann weights expressions of the $U_q[G_2]$ vertex model.

2 The $U_q[G_2]$ $R$-matrix

In this section we shall present the $U_q[G_2]$ $\hat{R}$-matrix in a suitable basis for the analysis of the reflection equation (2). In terms of the quantum group framework \cite{4}, this matrix can be written as linear combination of the $U_q[G_2]$ projectors operators $P_{V_\Lambda}$ \cite{14}:

$$\hat{R}(x) = \sum_{\Lambda=0,A_1,2A_2} \rho_\Lambda(x) P_{V_\Lambda}$$

(3)

where $V_\Lambda$ denotes the irreducible representations occurring in the seven-dimensional $U_q[G_2]$ decomposition $V_{A_2} \otimes V_{A_2} = V_0 \oplus V_{A_1} \oplus V_{A_2} \oplus V_{2A_2}$.

The weights $\rho_\Lambda(x)$ are functions of the quadratic Casimir element in an irrep with the highest weight. They satisfy the following relations,

$$\rho_{A_2}(x) = \frac{[12]_x}{[8]_x} \rho_0(x), \quad \rho_{2A_2}(x) = [2]_x[12]_x \rho_0(x), \quad \rho_{A_1}(x) = [12]_x \rho_0(x)$$

(4)

where $[a]_x = \frac{x^a - 1}{x - 1}$ and $\rho_0(x)$ is an arbitrary normalization.

A direct analysis of the reflection equation (2) is in principle easily made by representing the $K$-matrix $K(x)$ in terms of the Weyl basis. In this approach, the same has to be done for the $U_q[G_2]$ $\hat{R}$-matrix as well. This last step involves a considerable amount of additional work even when the projectors $P_{V_\Lambda}$ are known in terms of the so-called $q$-Wigner coefficients. Omitting here such technicalities, we find that the $\hat{R}$-matrix defined by Eqs.(3,4) can be rewritten in
terms of the following expression,

\[
\hat{R}(x) = \sum_{\alpha=1,\neq 4}^7 c_1(x)e_{\alpha\alpha} \otimes e_{\alpha\alpha} + \sum_{\alpha=1}^7 c_3(x)[e_{\alpha4} \otimes e_{4\alpha} + e_{4\alpha} \otimes e_{\alpha4}] + \sum_{\alpha,\beta=1}^7 c_{\alpha\beta}(x)e_{\alpha'\beta} \otimes e_{\alpha'\beta'}
\]

\[
+ \sum_{\alpha,\beta=1,\neq 4}^7 B_1(\alpha, \beta)[e_{\alpha\alpha} \otimes e_{\beta\beta} + e_{\beta\beta'} \otimes e_{\alpha'\alpha'}] + A_1(\alpha, \beta)[e_{\beta\beta} \otimes e_{\alpha\alpha} + e_{\alpha'\alpha'} \otimes e_{\beta'\beta'}]
\]

\[
+ \sum_{\alpha=1}^7 B_2(\alpha)[e_{\alpha\alpha} \otimes e_{44} + e_{44} \otimes e_{\alpha'\alpha'}] + A_2(\alpha)[e_{44} \otimes e_{\alpha\alpha} + e_{\alpha'\alpha'} \otimes e_{44}]
\]

\[
+ \sum_{\alpha,\beta=1,\neq 4}^7 C(\alpha, \beta)[e_{\alpha\beta} \otimes e_{\beta\alpha} + e_{\beta\alpha} \otimes e_{\alpha\beta} + e_{\alpha'\beta'} \otimes e_{\beta'\alpha'} + e_{\beta'\alpha'} \otimes e_{\alpha'\beta'}]
\]

\[
+ \sum_{k=1}^3 \sum_{\Gamma_k} \left\{ B_{[k]}[e_{\alpha'\beta'} \otimes e_{4\delta'} + e_{\beta'\alpha'} \otimes e_{\delta'4} + e_{\delta'4} \otimes e_{\beta'\alpha'} + e_{4\delta'} \otimes e_{\alpha'\beta'}] \right\}
\]

\[
+ A_{[k]}[e_{\alpha'\beta'} \otimes e_{4\delta'} + e_{\beta'\alpha'} \otimes e_{\delta'4} + e_{\delta'4} \otimes e_{\beta'\alpha'} + e_{4\delta'} \otimes e_{\alpha'\beta'}]ight\}
\]

(5)

where \(\alpha' = 8 - \alpha\) and the symbol \(\Gamma_k\) denotes the sum over a set of three indices \(\{(\alpha, \beta, \delta)\}\) such that \(\Gamma_1 = \{(1, 3, 2), (5, 2, 7), (6, 3, 7)\}\), \(\Gamma_2 = (1, 2, 3)\) and \(\Gamma_3 = \{(2, 1, 5), (3, 1, 6)\}\). We also recall that \(e_{\alpha,\beta}\) refers to the standard 7 \(\times\) 7 Weyl matrices.

The explicit expressions of all the weights appearing in expression (5) have been summarized in Appendix A. An advantage of the representation (5) is that it exhibits explicitly the two \(U(1)\) symmetries of the \(G_2\) Lie algebra. This property will be very useful in next section in order to classify the independent functional relations for the reflection matrices. We also note that the last term in Eq.(5) represents additional Boltzmann weights as compared with those present in the \(\hat{R}(x)\)-matrix of the non-exceptional vertex models [4].

We close this section by mentioning useful relations satisfied by the matrix \(R(x) = P\hat{R}(x)\), where \(P\) is the seven-dimensional permutator. Besides the standard properties of regularity and unitarity the matrix \(R(x)\) satisfies the so-called \(PT\)-symmetry given by

\[
P_{12}R_{12}(x)P_{12} = R_{12}^{t_{12}}(x)
\]

(6)

where the symbol \(t_k\) denotes the transposition in the space with index \(k\).
Yet another property is the crossing symmetry,

\[ R_{12}(x) = \frac{-x^3}{q^{18}} V_1 R_{12}^r(q^{12}/x) V_1^{-1} \]  

(7)

where \( V \) is an anti-diagonal matrix whose non-null elements are \( V_{17} = 1, V_{26} = -q, V_{35} = q^4, V_{44} = -q^5, V_{53} = q^6, V_{62} = -q^9, V_{71} = q^{10} \).

3 The \( U_q[G_2] \) \( K \)-matrices

The purpose here is to search for the complete set of regular \( K \)-matrices for the \( U_q[G_2] \) vertex model defined in the previous section. More specifically, we are interested in reflection matrices having the general form,

\[ K(x) = \sum_{\alpha,\beta=1}^7 k_{\alpha,\beta}(x) e_{\alpha,\beta} \]  

(8)

with the constraint \( k_{\alpha,\beta}(1) = \delta_{\alpha,\beta} \).

Direct substitution of the ansatz (8) and the \( R \)-matrix (5) in Eq.(2) give us several independent functional equations for the elements \( k_{\alpha,\beta}(x) \). In order to select and solve these equations we act on them with the operator \( \frac{d}{dy} \) and afterwards we take the \( y = 1 \) regular limit. Here we shall denote these resulting algebraic equations by the symbol \( E[i,j] \) where the index \([i,j]\) refer to the \( i \)th row and \( j \)th column of the original reflection equation (2). Note that such equations involve only the variable \( x \) and a number of free-parameters \( \omega_{\alpha,\beta} \) defined as

\[ \omega_{\alpha,\beta} = \frac{dk_{\alpha,\beta}(y)}{dy}\bigg|_{y=1} \]  

(9)

Our next step is to organize the algebraic equations \( E[i,j] \) in suitable blocks of relations \( B[i,j] \) involving related \( K \)-matrices elements. We find that such blocks are made by combining for a given pair \([i,j]\) the \( E[i,j], E[j,i], E[50-i,50-j] \) and \( E[50-j,50-i] \) set of relations. Considering that we are looking for general non-diagonal \( K \)-matrices we start our analysis by inspecting the blocks \( B[i,j] \) possessing only off-diagonal elements \( k_{\alpha,\beta}(x) \). The simplest ones are \( B[1,33], B[1,41], B[2,42], B[3,35], B[9,17] \) and \( B[12,20] \), providing us constraints to the
elements \( k_{\alpha,\alpha'}(x) \), which reads

\[
k_{\alpha,\alpha'}(x) = \frac{\omega_{\alpha,\alpha'} }{\omega_{1,7} } k_{1,7}(x)
\]  

(10)

By the same token, the blocks \( B[1,47], B[1,48] \) and \( B[8,38] \) lead us to determine the following elements,

\[
k_{1,5}(x) = \Gamma(x) \left( a_1(x) c_{11}(x) \omega_{1,5} - c_2(x) c_{13}(x) \omega_{3,7} \right) \frac{k_{1,7}(x)}{\omega_{1,7}}
\]  

(11)

\[
k_{5,1}(x) = \Gamma(x) \left( a_1(x) c_{11}(x) \omega_{5,1} - c_2(x) c_{13}(x) \omega_{7,3} \right) \frac{k_{7,1}(x)}{\omega_{7,1}}
\]  

(12)

\[
k_{7,3}(x) = \Gamma(x) \left( b_1(x) c_{11}(x) \omega_{7,3} - c_2(x) c_{31}(x) \omega_{5,1} \right) \frac{k_{7,1}(x)}{\omega_{7,1}}
\]  

(13)

\[
k_{3,7}(x) = \Gamma(x) \left( b_1(x) c_{11}(x) \omega_{3,7} - c_2(x) c_{31}(x) \omega_{1,5} \right) \frac{k_{1,7}(x)}{\omega_{1,7}}
\]  

(14)

\[
k_{16}(x) = \Gamma(x) \left( a_1(x) c_{11}(x) \omega_{1,6} - c_2(x) c_{12}(x) \omega_{2,7} \right) \frac{k_{1,7}(x)}{\omega_{1,7}}
\]  

(15)

\[
k_{61}(x) = \Gamma(x) \left( a_1(x) c_{11}(x) \omega_{6,1} - c_2(x) c_{12}(x) \omega_{7,2} \right) \frac{k_{7,1}(x)}{\omega_{7,1}}
\]  

(16)

\[
k_{27}(x) = \Gamma(x) \left( b_1(x) c_{11}(x) \omega_{2,7} - c_2(x) c_{21}(x) \omega_{1,6} \right) \frac{k_{1,7}(x)}{\omega_{1,7}}
\]  

(17)

\[
k_{72}(x) = \Gamma(x) \left( b_1(x) c_{11}(x) \omega_{7,2} - c_2(x) c_{21}(x) \omega_{6,1} \right) \frac{k_{7,1}(x)}{\omega_{7,1}}
\]  

(18)

\[
k_{23}(x) = \Gamma(x) \left( a_1(x) c_{11}(x) \omega_{2,3} - c_2(x) c_{25}(x) \omega_{5,6} \right) \frac{k_{1,7}(x)}{\omega_{1,7}}
\]  

(19)

\[
k_{32}(x) = \Gamma(x) \left( a_1(x) c_{11}(x) \omega_{3,2} - c_2(x) c_{25}(x) \omega_{6,5} \right) \frac{k_{7,1}(x)}{\omega_{7,1}}
\]  

(20)

\[
k_{56}(x) = \Gamma(x) \left( b_1(x) c_{11}(x) \omega_{5,6} - c_2(x) c_{52}(x) \omega_{2,3} \right) \frac{k_{1,7}(x)}{\omega_{1,7}}
\]  

(21)

\[
k_{65}(x) = \Gamma(x) \left( b_1(x) c_{11}(x) \omega_{6,5} - c_2(x) c_{52}(x) \omega_{3,2} \right) \frac{k_{7,1}(x)}{\omega_{7,1}}
\]  

(22)

where we have chosen the entry \( k_{1,7}(x) \) as an overall normalization. We also have made use of the identities \( c_{12}(x)c_{21}(x) = c_{13}(x)c_{31}(x) = c_{25}(x)c_{52}(x) \) in order to define the common function,

\[
\Gamma(x) = \frac{c_1(x) c_{11}(x) - c_2(x) c_4(x)}{c_{11}(x) b_1(x) a_1(x) - c_2^2(x) c_{12}(x) c_{21}(x)}
\]  

(23)

5
At this point we turn our attention to the diagonal $B[i, i]$ blocks. Each of them involves two distinct equations and after some cumbersome manipulations we find the relation,

$$\omega_{\beta, \alpha} k_{\alpha, \beta}(x) = \omega_{\alpha, \beta} k_{\beta, \alpha}(x), \text{ for } \alpha \neq \beta$$  \hspace{1cm} (24)

provided that the following constraints between the coefficients $\omega_{\alpha, \beta}$ are satisfied,

$$\omega_{\alpha, \beta} \omega_{\alpha', \beta'} = \omega_{\beta, \alpha} \omega_{\beta', \alpha'}, \text{ for } \beta \neq \alpha, \alpha'$$  \hspace{1cm} (25)

The last off-diagonal elements we need to determine are the entries $k_{4, \beta}(x)$ for $\beta > 4$ and $k_{\alpha, 4}(x)$ for $\alpha < 4$. It turns out that they can be fixed by using the blocks $B[1, 17]$, $B[9, 33]$, $B[2, 20]$, $B[3, 12]$ and $B[8, 35]$. The final results are

$$k_{1, 4}(x) = \frac{\omega_{1, 4}}{\omega_{2, 3}} k_{2, 3}(x), \quad k_{2, 4}(x) = \frac{\omega_{2, 4}}{\omega_{1, 5}} k_{1, 5}(x), \quad k_{3, 4}(x) = \frac{\omega_{3, 4}}{\omega_{1, 6}} k_{1, 6}(x)$$  \hspace{1cm} (26)

$$k_{4, 5}(x) = \frac{\omega_{4, 5}}{\omega_{2, 7}} k_{2, 7}(x), \quad k_{4, 6}(x) = \frac{\omega_{4, 6}}{\omega_{3, 7}} k_{3, 7}(x), \quad k_{4, 7}(x) = \frac{\omega_{4, 7}}{\omega_{5, 6}} k_{5, 6}(x)$$  \hspace{1cm} (27)

We now reached a point in which all the above considerations can be collected together. Substituting the determined off-diagonal elements back to the reflection equation (2) we find out that the parameter $\omega_{2, 7}$ has to satisfy the following polynomial equation,

$$\omega_{2, 7}(\omega_{2, 7} - \omega_{1, 6}q^{-5})(\omega_{2, 7} + \omega_{1, 6}q^{-5})(\omega_{2, 7} + \omega_{1, 6}q^{-8}) = 0$$  \hspace{1cm} (28)

The values $\omega_{2, 7} = 0, -\omega_{1, 6}q^{-8}$ lead us to solutions whose non-null elements are only the diagonal $k_{\alpha, \alpha}(x)$ and the anti-diagonal $k_{\alpha, \alpha'}$ entries. This branch gives origin to two distinct classes of solution depending on whether $\omega_{1, 7} = 0$ or $\omega_{1, 7} \neq 0$. By setting $\omega_{1, 7} = 0$ we get the two diagonal solutions found previously in ref.[15], which in current notation reads,

$$K^{(1)}(x) = I_7$$  \hspace{1cm} (29)

$$K^{(2)}(x) = \text{Diag}(1, 1, x\frac{q x + \epsilon}{q + \epsilon x}, x\frac{q x + \epsilon}{q + \epsilon x}, x\frac{q x + \epsilon}{q + \epsilon x}, x^2, x^2)$$  \hspace{1cm} (30)

where $\epsilon = \pm 1$ is a discrete parameter.
On the other hand, for $\omega_{1,7} \neq 0$ we obtain the following novel non-diagonal solution

$$K^{(3)}(x) = \begin{pmatrix}
    k_{1,1}^{(3)} & k_{1,7}^{(3)}(x) \\
    k_{2,2}^{(3)} & k_{3,3}^{(3)} & k_{3,5}^{(3)} \\
    k_{3,3}^{(3)} & k_{3,5}^{(3)} & k_{4,4}^{(3)} \\
    k_{5,5}^{(3)} & k_{5,5}^{(3)} & k_{6,6}^{(3)} \\
    k_{6,2}^{(3)} & k_{7,1}^{(3)} \\
    k_{7,1}^{(3)} & k_{7,7}^{(3)}
\end{pmatrix}$$

(31)

where the diagonal entries are given by,

$$k_{1,1}^{(3)} = k_{2,2}^{(3)} = k_{3,3}^{(3)} = \frac{1}{\omega_{1,7}} \frac{2\omega_{1,7} x^2 - 1}{x^2 - 1} k_{1,7}^{(3)}(x)$$

(32)

$$k_{4,4}^{(3)} = \frac{1}{\omega_{1,7}} \frac{q^2 - x^2}{q^2 - 1} \frac{2}{x^2 - 1} k_{1,7}^{(3)}(x)$$

(33)

$$k_{5,5}^{(3)} = k_{6,6}^{(3)} = k_{7,7}^{(3)} = \frac{1}{2} \frac{2\omega_{1,7} x^2 - 1}{x^2 - 1} k_{1,7}^{(3)}(x)$$

(34)

while the off-diagonal elements are,

$$k_{2,6}^{(3)} = \frac{\omega_{2,6}}{\omega_{1,7}} k_{1,7}^{(3)}(x)$$

(35)

$$k_{3,5}^{(3)} = -\frac{1}{\omega_{2,6}} \left( \frac{2q}{q^2 - 1} \right) k_{1,7}^{(3)}(x)$$

(36)

$$k_{5,5}^{(3)} = -\frac{\omega_{2,6}}{\omega_{1,7}^2} \left( \frac{2q}{q^2 - 1} \right) k_{1,7}^{(3)}(x)$$

(37)

$$k_{6,2}^{(3)} = \frac{1}{\omega_{1,7} \omega_{2,6}} \left( \frac{2q}{q^2 - 1} \right)^2 k_{1,7}^{(3)}(x)$$

(38)

$$k_{7,1}^{(3)} = \frac{1}{\omega_{1,7}^2} \left( \frac{2q}{q^2 - 1} \right)^2 k_{1,7}^{(3)}(x)$$

(39)

The remaining branch $\omega_{2,7} = \epsilon \omega_{1,6} q^{-5}$ is the most complicated one since it leads us to $K$-matrices with all non-null elements. Denoting the entries of this complete matrix by $k_{\alpha,\beta}^{(4)}(x)$ we find, after elaborated algebraic manipulations, that their expressions can be defined as
follows. The diagonal entries are,

$$k_{1,1}^{(4)}(x) = \mathcal{F}^{(\epsilon)}(2, 2) \left( \frac{q^{10} (q^8 - 1) (1 - \epsilon x)}{(x^2 - 1)(q^5 - \epsilon x)} - \frac{q^9 (1 + q) (q^8 + \epsilon x)(1 + q^4)}{(q^3 + 1)(x^2 - 1)(q^5 - \epsilon x)} \right) k_{1,1}^{(4)}(x)$$  \hspace{1cm} (40)

$$k_{2,2}^{(4)}(x) = k_{1,1}^{(4)}(x) - \mathcal{F}^{(\epsilon)}(2, 2) \frac{q^{10}(q^3 + 1)}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (41)

$$k_{3,3}^{(4)}(x) = k_{1,1}^{(4)}(x) + \mathcal{F}^{(\epsilon)}(2, 2) \frac{q^{10}(q^6 - 1)}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (42)

$$k_{4,4}^{(4)}(x) = \mathcal{F}^{(\epsilon)}(2, 2) \left( \frac{q^{12}(q^2 - x^2)}{(q^3 + 1)(x^2 - 1)} - \frac{q^9(q^9 + 1)(\epsilon qx + 1)}{(x^2 - 1)(q^5 - \epsilon x)} \right) k_{1,1}^{(4)}(x)$$ (43)

$$k_{5,5}^{(4)}(x) = x^2 k_{1,1}^{(4)}(x) + \mathcal{F}^{(\epsilon)}(2, 2) \frac{q^{12}(q^6 - 1)}{q^5 - \epsilon x} \epsilon x k_{1,1}^{(4)}(x)$$ (44)

$$k_{6,6}^{(4)}(u) = x^2 k_{1,1}^{(4)}(x) + \mathcal{F}^{(\epsilon)}(2, 2) \frac{q^{15}(q^3 + 1)}{q^5 - \epsilon x} \epsilon x k_{1,1}^{(4)}(x)$$ (45)

$$k_{7,7}^{(4)}(x) = x^2 k_{1,1}^{(4)}(x)$$ (46)

The structure of the off-diagonal entries shall be described in terms of the elements of each row separately. For the first row we find,

$$k_{12}^{(4)}(x) = -\mathcal{F}^{(\epsilon)}(1, 2) \frac{q^9}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (47)

$$k_{13}^{(4)}(x) = -\mathcal{F}^{(\epsilon)}(2, 1) \frac{q^9}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (48)

$$k_{14}^{(4)}(x) = \sqrt{q^2(1 + q^2)}\mathcal{F}^{(\epsilon)}(1, 1) \frac{q^6}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (49)

$$k_{15}^{(4)}(x) = -\mathcal{F}^{(\epsilon)}(0, 1) \frac{q^6}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (50)

$$k_{16}^{(4)}(x) = -\mathcal{F}^{(\epsilon)}(1, 0) \frac{q^6}{q^5 - \epsilon x} k_{1,1}^{(4)}(x)$$  \hspace{1cm} (51)
The second row is,

\begin{align*}
k^{(4)}_{21}(x) &= F^{(\epsilon)}(3, 2) \frac{q^{14}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (52) \\
k^{(4)}_{23}(x) &= -F^{(\epsilon)}(3, 1) \frac{q^{13}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (53) \\
k^{(4)}_{24}(x) &= \sqrt{q^2 (1 + q^2)} F^{(\epsilon)}(2, 1) \frac{q^{10}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (54) \\
k^{(4)}_{25}(x) &= -F^{(\epsilon)}(1, 1) \frac{q^{10}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (55) \\
k^{(4)}_{26}(x) &= -F^{(\epsilon)}(2, 0) \frac{q^5}{q^3 + 1} k^{(4)}_{1,7}(x), \quad (56) \\
k^{(4)}_{27}(x) &= -F^{(\epsilon)}(1, 0) \frac{q}{q^5 - \epsilon x} \epsilon x k^{(4)}_{1,7}(x). \quad (57)
\end{align*}

The third row is,

\begin{align*}
k^{(4)}_{31}(x) &= -F^{(\epsilon)}(2, 3) \frac{q^{17}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (58) \\
k^{(4)}_{32}(x) &= F^{(\epsilon)}(1, 3) \frac{q^{16}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (59) \\
k^{(4)}_{34}(x) &= -\sqrt{q^2 (1 + q^2)} F^{(\epsilon)}(1, 2) \frac{q^{13}}{q^5 - \epsilon x} k^{(4)}_{1,7}(x), \quad (60) \\
k^{(4)}_{35}(x) &= F^{(\epsilon)}(0, 2) \frac{q^8}{q^3 + 1} k^{(4)}_{1,7}(x), \quad (61) \\
k^{(4)}_{36}(x) &= -F^{(\epsilon)}(1, 1) \frac{q^5}{q^5 - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \quad (62) \\
k^{(4)}_{37}(x) &= F^{(\epsilon)}(0, 1) \frac{q^4}{q^5 - \epsilon x} \epsilon x k^{(4)}_{1,7}(x). \quad (63)
\end{align*}

The fourth row is,
\[ k_{41}^{(4)}(x) = -\sqrt{q^{-2}(1 + q^2)F^{(\epsilon)}(3, 3)} \frac{q^{21}}{q^5 - \epsilon x} k_{1,7}^{(4)}(x), \]  
(64)

\[ k_{42}^{(4)}(x) = \sqrt{q^2(1 + q^2)F^{(\epsilon)}(2, 3)} \frac{q^{18}}{q^5 - \epsilon x} k_{1,7}^{(4)}(x), \]  
(65)

\[ k_{43}^{(4)}(x) = \sqrt{q^2(1 + q^2)F^{(\epsilon)}(3, 2)} \frac{q^{18}}{q^5 - \epsilon x} k_{1,7}^{(4)}(x), \]  
(66)

\[ k_{45}^{(4)}(x) = -\sqrt{q^2(1 + q^2)F^{(\epsilon)}(1, 2)} \frac{q^7}{q^5 - \epsilon x} \epsilon x k_{1,7}^{(4)}(x), \]  
(67)

\[ k_{46}^{(4)}(x) = -\sqrt{q^2(1 + q^2)F^{(\epsilon)}(2, 1)} \frac{q^7}{q^5 - \epsilon x} \epsilon x k_{1,7}^{(4)}(x), \]  
(68)

\[ k_{47}^{(4)}(x) = \sqrt{q^2(1 + q^2)F^{(\epsilon)}(1, 1)} \frac{q^6}{q^5 - \epsilon x} \epsilon x k_{1,7}^{(4)}(x). \]  
(69)

The fifth row is,

\[ k_{51}^{(4)}(x) = -F^{(\epsilon)}(4, 3) \frac{q^{24}}{q^5 - \epsilon x} k_{1,7}^{(4)}(x), \]  
(70)

\[ k_{52}^{(4)}(x) = F^{(\epsilon)}(3, 3) \frac{q^{23}}{q^5 - \epsilon x} k_{1,7}^{(4)}(x), \]  
(71)

\[ k_{53}^{(4)}(x) = F^{(\epsilon)}(4, 2) \frac{q^{18}}{q^3 + 1} k_{1,7}^{(4)}(x), \]  
(72)

\[ k_{54}^{(4)}(x) = \sqrt{(1 + q^2)q^2F^{(\epsilon)}(3, 2)} \frac{q^{12}}{q^5 - \epsilon x} \epsilon x k_{1,7}^{(4)}(x), \]  
(73)

\[ k_{56}^{(4)}(x) = -F^{(\epsilon)}(3, 1) \frac{q^{12}}{q^5 - \epsilon x} \epsilon x k_{1,7}^{(4)}(x), \]  
(74)

\[ k_{57}^{(4)}(x) = F^{(\epsilon)}(2, 1) \frac{q^{11}}{q^5 - \epsilon x} \epsilon x k_{1,7}^{(4)}(x). \]  
(75)

The sixth row is,
\[
k^{(4)}_{61}(x) = F^{(e)}(3, 4) \frac{q^{27}}{q^{5} - \epsilon x} k^{(4)}_{1,7}(x), \\
k^{(4)}_{62}(x) = -F^{(e)}(2, 4) \frac{q^{21}}{q^{3} + 1} k^{(4)}_{1,7}(x), \\
k^{(4)}_{63}(x) = F^{(e)}(3, 3) \frac{q^{18}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{64}(x) = -\sqrt{1+q^{2}} q^{2} F^{(e)}(2, 3) \frac{q^{15}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{65}(x) = F^{(e)}(1, 3) \frac{q^{15}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{66}(x) = -F^{(e)}(1, 2) \frac{q^{14}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{71}(x) = F^{(e)}(4, 4) \frac{q^{26}}{q^{3} + 1} k^{(4)}_{1,7}(x), \\
k^{(4)}_{72}(x) = F^{(e)}(3, 4) \frac{q^{22}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{73}(x) = F^{(e)}(4, 3) \frac{q^{22}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{74}(x) = -\sqrt{1+q^{2}} q^{2} F^{(e)}(3, 3) \frac{q^{21}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{75}(x) = F^{(e)}(2, 3) \frac{q^{19}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x), \\
k^{(4)}_{76}(x) = F^{(e)}(3, 2) \frac{q^{19}}{q^{5} - \epsilon x} \epsilon x k^{(4)}_{1,7}(x),
\]

where the auxiliary function \(F^{(e)}(n, m)\) is defined by,

\[
F^{(e)}(n, m) = \frac{2^{n+m} (q^{3} + 1)^{n+m+1}}{(\epsilon + q)^{n+m} (\epsilon + q^{4})^{n+m} (\epsilon + q^{8})^{n+m} \omega_{1,2}^{n} \omega_{1,3}^{m}}.
\]

Note that this latter solution contains an additional discrete parameter besides the two continuous \(\omega_{1,2}\) and \(\omega_{1,3}\) variables. We also observe that a striking feature of our results is that the non-diagonal solutions do not reduce to any of the admissible diagonal matrices \(K^{(1)}(x)\)
and $K^{(2)}(x)$. This means that the four possibilities discussed above are indeed the distinct types of $K$-matrices of the $U_q[G_2]$ model. The latter feature should be contrasted with the findings for the non-exceptional vertex models [11] in which reductions of general non-diagonal matrices guide us always to some particular diagonal solution. It remains to be seen whether such peculiarity is special to the $U_q[G_2]$ symmetry or is valid for all exceptional vertex model as well.

Finally, equipped with the four reflection matrices $K^{(l)}(x)$, an integrable model with open boundary condition can be obtained through the double-row transfer matrix formulated by Sklyanin [3], namely

$$t^{(l,m)}(x) = \text{Tr}_a \left[ \frac{a^{(m)}}{K_+^{(m)}(x)} T(x) \frac{a^{(l)}}{K^{(l)}(x)} T^{-1}(1/x) \right] \quad \text{for} \ l, m = 1, \cdots, 4$$

where $T(x) = R_{aL}(x) \cdots R_{a1}(x)$ is the standard monodromy matrix of the corresponding closed chain with $L$ sites. The matrix $K^{(m)}_+(x)$ is automatically determined [5] from $K^{(m)}(x)$ with the help of the crossing property (7),

$$K^{(m)}_+(x) = \left[ K^{(m)}(q^{12}/x) \right]^t\ V^t\ V$$

4 Conclusions

In this paper we have been able to classify the possible families of reflection matrices associated to the fundamental $U_q[G_2]$ vertex model. We find that there exists four different classes of $K$-matrices. Two of them are the diagonal solutions studied before [15] and the remaining ones are new non-diagonal $K$-matrices with continuous free parameters. The first type of non-diagonal solution is made by combining only the diagonal and the anti-diagonal elements while the second one consists of matrix whose all entries are non-null and has an extra discrete parameter. Interesting enough, these non-diagonal solutions do not possess reduction to particular diagonal matrices for any values of the free parameters at disposal. We stress that this peculiarity does not occur in the $K$-matrices of the vertex models based on the non-exceptional Lie algebras [11].
We hope that our results will prompt further lines of investigations. One possibility is to explore the Weyl representation of the $U_q[G_2]$ $\hat{R}$-matrix to establish, at least for diagonal boundaries, the diagonalization of the transfer-matrix (89) from a first principle framework such as the quantum inverse scattering method [16]. This will give us an opportunity to check certain assumptions, usually denominated “doubling hypotheses” [17], used to determine the corresponding Bethe ansatz solution [15].

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Appendix A: The Boltzmann weights

In this Appendix we are going to describe all the Boltzmann weights appearing in Eq.(5). In order to do that we find convenient to introduce an auxiliary function $\varphi(\alpha, \beta)$ defined by,

$$\varphi(\alpha, \beta) = \begin{cases} 
  \left[\frac{\beta}{2}\right] - \alpha + 4\delta_{4,\beta} & \text{for } \alpha < \beta, \alpha < \beta' \\
  \left[\frac{\beta}{2}\right] - \beta' + 4\delta_{\alpha',4} & \text{for } \alpha < \beta, \alpha > \beta' \\
  \beta - \left[\frac{\alpha}{2}\right] - 4\delta_{\alpha,4} & \text{for } \alpha > \beta, \alpha' > \beta \\
  \alpha' - \left[\frac{\beta'}{2}\right] - 4\delta_{4,\beta'} & \text{for } \alpha > \beta, \alpha' < \beta 
\end{cases} \tag{A.1}$$

where $\left[\frac{\beta}{2}\right]$ denotes the integer part of $\frac{\beta}{2}$.

We begin by listing the weights $A_1(\alpha, \beta)$, $A_2(\alpha)$ and $A_{[k]}$ as well as $B_1(\alpha, \beta)$, $B_2(\alpha)$ and $B_{[k]}$. They are given by,

$$A_1(\alpha, \beta) = \begin{cases} 
  a_7(x) & \text{for } \varphi(\alpha, \beta) = -1 \\
  a_1(x) & \text{for } \varphi(\alpha, \beta) = 0 \\
  a_6(x) & \text{for } \varphi(\alpha, \beta) = 1, 2
\end{cases}, \quad A_2(\alpha) = \begin{cases} 
  a_4(x) & \text{for } \varphi(\alpha, 4) = 5 \\
  a_8(x) & \text{for } \varphi(\alpha, 4) \neq 5
\end{cases} \tag{A.2}$$

$$A_{[1]} = a_2(x), \quad A_{[2]} = a_3(x), \quad A_{[3]} = a_5(x) \tag{A.3}$$
\[ B_1(\alpha, \beta) = \begin{cases} b_7(x) & \text{for } \varphi(\alpha, \beta) = -1 \\ b_1(x) & \text{for } \varphi(\alpha, \beta) = 0 \\ b_6(x) & \text{for } \varphi(\alpha, \beta) = 1, 2 \end{cases} \]

\[ B_{[1]} = b_2(x), \quad B_{[2]} = b_3(x), \quad B_{[3]} = b_5(x) \]

while the weight \( C(\alpha, \beta) \) is,

\[ C(\alpha, \beta) = \begin{cases} c_2(x) & \text{for } \varphi(\alpha, \beta) = 0 \\ c_4(x) & \text{for } \varphi(\alpha, \beta) \neq 0 \end{cases} \]

The functions \( a_\alpha(x) \) and \( b_\alpha(x) \) entering in the above expressions are,

\[ a_1(u) = (q^2 - 1) (q^8 - x) (q^{12} - x), \quad b_1(x) = xa_1(x), \]

\[ a_2(u) = -q^7 \sqrt{1 + q^{-2}} (q^2 - 1) (q^{12} - x) (x - 1), \quad b_2(x) = \frac{x}{q^5} a_2(x), \]

\[ a_3(u) = q^4 \sqrt{1 + q^{-2}} (q^2 - 1) (q^{12} - x) (x - 1), \quad b_3(x) = x a_3(x), \]

\[ a_5(u) = q^5 \sqrt{1 + q^{-2}} (q^2 - 1) (q^{12} - x) (x - 1), \quad b_5(x) = \frac{x}{q^3} a_5(x), \]

\[ a_4(x) = (q^4 - 1) (q^{12} - x) f_1(x), \quad b_4(x) = \frac{xf_2(x)}{f_1(x)} a_4(x), \]

\[ a_6(x) = (q^2 - 1) (q^{12} - x) g_1(x), \quad b_6(x) = \frac{xg_2(x)}{g_1(x)} a_6(x), \]

\[ a_7(x) = (q^2 - 1) (q^{12} - x) g_2(x), \quad b_7(x) = \frac{xg_1(x)}{g_2(x)} a_7(x), \]

\[ a_8(x) = (q^4 - 1) (q^{12} - x) f_2(x), \quad b_8(x) = \frac{xf_1(x)}{f_2(x)} a_8(x), \]

where

\[ f_1(x) = q^2(1 - q^2 + q^4) - x, \]

\[ f_2(x) = q^6 - (1 - q^2 + q^4) x, \]

\[ g_1(x) = q^2(1 + q^2 + q^6) - (1 + q^2 + q^4) x, \]

\[ g_2(x) = q^4(1 + q^2 + q^4) - (1 + q^4 + q^6) x. \]
Next, the weights $c_\alpha(x)$ are,

\begin{align*}
c_1(x) &= (q^2 - x)(q^8 - x)(q^{12} - x), \\
c_2(x) &= -q(q^8 - x)(q^{12} - x)(x - 1), \\
c_3(x) &= -q^2(q^6 - x)(q^{12} - x)(x - 1), \\
c_4(x) &= -q^3(q^4 - x)(q^{12} - x)(x - 1) \tag{A.19}
\end{align*}

while the $c_{\alpha\beta}(x)$ weights are given by

\[
c_{\alpha\beta}(x) = \begin{cases} 
-q^4(q^4 - x)(q^{10} - x)(x - 1), & (\alpha = \beta, \alpha \neq 4) \\
(q^6 - x)[x^2 + (1 + q^4)(q^4 - 1)^2(q^2 - 1)^2 - q^6]x + q^{14}, & (\alpha = \beta = 4) \\
(-1)^{\alpha + \beta}(q^2 - 1)q^{16+\bar{5}_\beta}(x-1)F_{\varphi(\alpha,\beta)}, & (\alpha < \beta, \beta \neq \alpha') \\
(-1)^{\alpha + \beta}(q^2 - 1)q^{5-\bar{5}_\beta}(x-1)xF_{\varphi(\alpha,\beta)}, & (\alpha > \beta, \beta \neq \alpha')
\end{cases} \tag{A.23}
\]

where the auxiliary index $\bar{\alpha}$ is defined as,

\[
\bar{1} = 0, \bar{2} = 3, \bar{3} = 6, \bar{4} = 8, \bar{5} = 10, \bar{6} = 13, \bar{7} = 16 \tag{A.24}
\]

and the functions $F_{\varphi(\alpha,\beta)}$ are

\begin{align*}
F_0 &= q^4 - x \tag{A.25} \\
F_5 &= F_{-3} = F_{-4} = \frac{1}{q}(1 + q^2)\left[q^6(1 - q^2 + q^4) - x\right] \tag{A.26} \\
F_{-5} &= F_3 = F_4 = \frac{1}{q^2}(1 + q^2)\left[q^{10} - (1 - q^2 + q^4)x\right] \tag{A.27} \\
F_1 &= F_2 = \frac{1}{q^3}[q^8(1 + q^4 + q^6) - (1 + q^2 + q^4)x] \tag{A.28} \\
F_{-1} &= F_{-2} = \frac{1}{q^5}[q^{10}(1 + q^2 + q^4) - (1 + q^2 + q^6)x] \tag{A.29}
\end{align*}
Finally, the remaining weights $c_{\alpha,\alpha'}(x)$ are given by

\begin{align*}
  c_{17}(x) &= (q^2 - 1)(q^8 - 1)[q^8(1 + q^4) - (q^8 + 1)x] & (A.30) \\
  c_{26}(x) &= (q^2 - 1)[q^{10}(q^{10} - 1) - q^6(q^2 - 1)(q^4 + 1)x - (q^6 - 1)x^2] & (A.31) \\
  c_{35}(x) &= (q^2 - 1)[q^{10}(q^4 - 1) + q^8(q^8 - 1)x - (q^{12} - 1)x^2] & (A.32) \\
  c_{53}(x) &= (q^2 - 1)[q^8(q^{12} - 1) - q^4(q^8 - 1)x - (q^4 - 1)x^2]x & (A.33) \\
  c_{62}(x) &= (q^2 - 1)[q^{14}(q^6 - 1) + q^8(q^2 - 1)(q^4 + 1)x - (q^{10} - 1)x^2]x & (A.34) \\
  c_{71}(x) &= (q^2 - 1)(q^8 - 1)[q^4(q^8 + 1) - (q^4 + 1)x]x^2 & (A.35)
\end{align*}
References

[1] R.J. Baxter, “Exactly Solved Models in Statistical Mechanics”, Academic Press, New York, 1982.

[2] I. Cherednik, Theor.Math.Phys. 61 (1984) 35.

[3] E.K. Sklyanin, J. Phys. A: Math. Gen. 21 (1988) 2375

[4] M. Jimbo, Commun.Math.Phys. 102 (1986) 247; V.V. Bazhanov, Phys.Lett.B 159 (1985) 321.

[5] L. Mezincescu and R.I. Nepomechie, J. Phys. A : Math. Gen. 24 (1991) L17; Int. J. Mod. Phys. A7 (1991) 5231; Nucl.Phys.B 72 (1992) 597.

[6] H.J. de Vega and A. Gonzalez-Ruiz, J. Phys. A : Math. Gen. 26 (1993) L519

[7] J. Abad and M. Rios, Phys.Lett.B 352 (1995) 92; M.T. Batchelor, V. Fridkin, A. Kuniba, Y.K. Zhou, Phys.Lett.B 376 (1996) 266; T. Inami, S. Odake and Y-Z. Zhang, Nucl. Phys. B47 (1996) 419;

[8] A. Lima-Santos, Nucl.Phys.B 558 (1999); C.-X. Liu, G.-X. Ju, S.-K. Wang, K. Wu, J.Phys.A:Math.Gen. 32 (1999) 3505; M.J. Martins and X.-W. Guan, Nucl.Phys.B 583 (2000) 721; Y. Yamada, Phys.Lett.A 298 (2002) 350

[9] B.Y. Hou and R.H. Yue, Phys. Lett. A 183 (1993) 169; T. Inami and H. Konno, J. Phys. A : Math. Gen. 27 (1994) L913; R.E. Behrend, P.A. Pearce, D.L. O’Brien, J. Stat. Phys. 84 (1996) 1

[10] C. Ahn and W.M. Koo, Nucl. Phys. B468 (1996) 461; Y.K. Zhou, Nucl. Phys. B 468 (1996) 504; R.E. Behrend and P.A. Pearce, Int.J.Mod.Phys.B 11 (1997) 2833; H. Fan, B.Y. Hou, G.L. Li and K.J. Shi, Phys.Lett.A 250 (1998) 79

[11] A. Lima-Santos, Nucl.Phys.B 612 (2001) 446; Nucl.Phys.B 644 (2002) 568; Nucl.Phys.B 654 (2003) 466; A. Lima-Santos and R. Malara, Nucl.Phys.B 675 (2003) 661

17
[12] G.W. Delius and R.I. Nepomechie, J.Phys.A:Math.Gen. A 35 (2002) L341; R.I. Nepomechie, Lett.Math.Phys. 62 (2002) 83; G.W. Delius and A. George, Lett.Math.Phys. 62 (2002) 211

[13] W-L. Yang and Y-Z. Zhang, J.H.E.Physics 12 (2004) art.No.019

[14] A. Kuniba, J.Phys.A: Math.Gen. 23 (1990) 1349

[15] C.M. Yung and M.T. Batchelor, Phys.Lett.A 198 (1995) 395

[16] V.E. Korepin, G. Izergin and N.M. Bogoliubov, “Quantum Inverse Scattering Method and Correlation Functions”, Cambridge Univ. Press, Cambridge, 1993.

[17] S. Artz, L. Mezincescu and R. Nepomechie, J.Phys.A:Math.Gen. 28 (1995) 5131.