COMPLEXITY OF COUNTABLE CATEGORICITY IN FINITE LANGUAGES

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Abstract. We study complexity of the index set of countably categorical theories and Ehrenfeucht theories in finite languages.

S. Lempp and T. Slaman proved in [7] that indexes of decidable \(\omega\)-categorical theories form a \(\Pi^0_3\)-subset of the set of indexes of all computably enumerable theories. Moreover there is an infinite language so that the property of \(\omega\)-categoricity distinguishes a \(\Pi^0_3\)-complete subset of the set of indexes of computably enumerable theories of this language. Steffen Lempp asked the author if this could be done in a finite language. In this paper we give a positive answer (see Section 4). The crucial element of our proof is a theorem of Hrushovski on coding of \(\omega\)-categorical theories in finite languages (see [3], Section 7.4, pp. 353 - 355). Since we apply the method which was used in the proof of this theorem, we present all the details in Section 1. Sections 2 - 3 contain several other applications of this theorem. In particular in the very short Section 2 we give an example of a non-G-compact \(\omega\)-categorical theory in a finite language. In Section 3 we show that there is a finite language such that the indexes of Ehrenfeucht theories with exactly three countable models form a \(\Pi^1_1\)-hard set. Here we also use the idea of Section 4 of [7] where a similar statement is proved in the case of infinite languages.

The main results of the paper are available both for computability theorists and model theorists. The only place where a slightly advanced model-theoretical material appears is Section 2. On the other hand the argument applied in this section is very easy and all necessary preliminaries are presented.

1. Hrushovski on \(\omega\)-categorical structures and finite languages

The material of this section is based on Section 7.4 of [3], pp. 353 - 355 (and preliminary notes of W. Hodges). We also give some additional modifications and remarks.
Let $N$ be a structure in the language $L$ with a unary predicate $P$. For any family of relations $R$ on $P$ definable in $N$ over $\emptyset$ one may consider the structure $M = (P, R)$. We say that $M$ is a dense relativised reduct if the image of the homomorphism $\text{Aut}(N) \to \text{Aut}(M)$ (defined by restriction) is dense in $\text{Aut}(M)$.

Let $L$ be the language consisting of four unary symbols $P, Q, \lambda, \rho$, a two-ary symbol $H$ and a four-ary one $S$. We will consider only $L$-structures where $P$ and $Q$ define a partition of the basic sort and $\lambda, \rho$ and $H$ are defined on $Q$. Moreover when $S(a, b, c, d)$ holds we have that $a, c \in P$ and $b, d \in Q$.

Theorem 1. If $M_0$ is any countable $\omega$-categorical structure then there is a countable $\omega$-categorical $L$-structure $N$ such that $M_0$ is a dense relativised reduct of $N$. In particular $M_0$ is interpretable in $N$ over $\emptyset$.

For every set of sentences $\Phi$ axiomatising $\text{Th}(M_0)$ the theory $\text{Th}(N)$ is axiomatised by a set of axioms which is computable with respect to $\Phi$ and the Ryll-Nardzewski function of $\text{Th}(M_0)$.

Proof (E.Hrushovski). Let $M_0$ be any countable $\omega$-categorical structure in a language $L_0$. We remind the reader that the Ryll-Nardzewski function of an $\omega$-categorical theory $T$ assigns to any natural $n$ the number of $n$-types of $T$. So by the set $\Phi$ as in the formulation and by the Ryll-Nardzewski function of $\text{Th}(M_0)$ one can find an effective list of all pairwise non-equivalent formulas. Thus w.l.o.g. we may assume that $L_0$ is 1-sorted, relational and $M_0$ has quantifier elimination.

In fact we can suppose that $L_0 = \{ R_1, R_2, \ldots, R_n, \ldots \}$ where each $R_n$ describes a complete type in $M_0$ of arity not greater than $n$. We may also assume that for $m < n$ the arity of $R_m$ is not greater than the arity of $R_n$. We admit that tuples realising $R_n$ may have repeated coordinates.

We now use standard material about Fraïssé limits, see [2]. Note that the class of all finite substructures of $M_0$ (say $K_0$) has the joint embedding and the amalgamation properties. Moreover for every $n$ the number of finite substructures of size $n$ is finite (this is the place where we use the assumption that each $R_n$ describes a complete type).

Let us consider structures of the language $L \cup L_0$ which satisfy the property that all the relations $R_n$ are defined on $P$. For such a structure $M$ we call a tuple $(a_0, \ldots, a_{m-1}, c_0, \ldots, c_{n-1})$ of elements of $M$, an $n$-pair of arity $m$ if:
(1) $m \leq n$ and $M \models \bigwedge \{ P(a_i) : i < m \} \land \bigwedge \{ Q(c_j) : j < n \}$;

(2) the elements $c_i$ are pairwise distinct and $M \models H(c_i, c_j)$ iff $(j = i + 1) \mod(n)$;

(3) $M \models \lambda(c_i)$ iff $i = 0$ and $M \models \rho(c_i)$ iff $i = m - 1$;

(4) $M \models S(a_i, c_j, a_k, c_l)$ iff $a_i = a_j$.

In this case we say that the $n$-pair $\vec{a} \vec{c}$ labels the tuple $\vec{a}$.

We now define a class $\mathcal{K}$ of finite $(L \cup L_0)$-structures as follows.

(i) In each structure of $\mathcal{K}$ all the relations $R_n$ are defined on $P$;

(ii) The $P$-part of any structure from $\mathcal{K}$ is isomorphic to a finite substructure of $M_0$;

(iii) For any $D \in \mathcal{K}$, any $n$ and any $n$-pair from $D$ labelling a tuple $\vec{a}$ we have $R_n(\vec{a})$.

It is obvious that $\mathcal{K}$ is closed under substructures and there is a function $f : \omega \to \omega$ so that for every $n$ the number of non-isomorphic structures of $\mathcal{K}$ of size $n$ is bounded by $f(n)$. The function $f$ is computable with respect to $\Phi$ and the Ryll-Nardzewski function.

**Lemma 2.** The class $\mathcal{K}$ has the amalgamation (and the joint embedding) property.

**Proof.** Let $D_1$ and $D_2$ be structures in $\mathcal{K}$ with intersection $C$. By induction it is enough to deal with the case where $|D_1 \setminus C| = |D_2 \setminus C| = 1$. Let $D_1 \setminus C = \{d_i\}$ and $d_1 \neq d_2$. There are three cases.

Case 1. $d_1$ and $d_2$ both satisfy $P$. Using that $M_0$ has quantifier elimination we amalgamate the $P$-parts of $D_1$ and $D_2$ remaining the $Q$-part and $S$ the same as before. By (4) there are no new $n$-pairs in the amalgam, for any $n$.

Case 2. $d_1$ and $d_2$ both satisfy $Q$. In this case we just take the free amalgamation (without any new tuples in relations). By (4) there are no new $n$-pairs in the amalgam, for any $n$.

Case 3. $d_1$ satisfies $P$ and $d_2$ satisfies $Q$. In this case we again take the free amalgamation and by (4) we again have that there are no new $n$-pairs in the amalgam, for any $n$. □

We now see that by Fraïssé’s theorem, the class $\mathcal{K}$ has a universal homogeneous (and $\omega$-categorical) structure $U$. In particular $\mathcal{K}/\sim$ coincides with $Age(U)$ (= collection of all types of finite substructures of $U$).
Since $M_0$ is the Fraïssé limit of the class of all $P$-parts of structures from $\mathcal{K}$, we see that the $P$-part of $U$ is isomorphic to $M_0$. Let $N$ be the reduct of $U$ to the language $L$. Note that $U$ (thus $M_0$) is definable in $N$. Indeed each $R_n$ is definable by the rule: $U \models R_n(\bar{a})$ if and only if there is an $n$-pair in $N$ which labels $\bar{a}$ (this follows from the fact that $\mathcal{K}$ contains an $n$-pair for such $\bar{a}$).

If two tuples $\bar{a}$ and $\bar{b}$ in $M_0$ realise the same type in $M_0$ they realise the same quantifier free type in $U$. So by quantifier elimination there is an automorphism of $U$ (and of $N$) which takes $\bar{a}$ to $\bar{b}$. This shows that $M_0$ is a dense relativised reduct of $N$.

To see the last statement of the theorem consider a set $\Phi$ axiomatising $Th(M_0)$. Thus the $P$-part of $U$ must satisfy $\Phi$ with respect to the relations $R_n$ defined in $N$ as above. The remaining axioms of $Th(N)$ (and of $Th(U)$) are just the axioms of the universal homogeneous structures of the corresponding class satisfying (i) - (iii) as above. □

Remark 3. The structure $U$ produced in the proof is axiomatised as follows.

**Axiomatation of $Th(U)$**

(a) all universal axioms forbidding finite substructures which cannot occur in $M_0$;

(b) all universal axioms stating property (iii) from the proof ;

(c) all $\exists$-axioms for finite substructures of $M_0$;

(d) all $\forall\exists$-axioms which realise the property of universal homogeneous structures that for any $\mathcal{K}$-structures $A < B$ with $A < U$ there is an $A$-embedding of $B$ into $U$.

Note that for every pair of natural numbers $n$ and $l$ the axioms of (a), (b) and (c) with at most $n$ quantifiers in the sublanguage of $L \cup L_0$ of arity $\leq l$ determine all $n$-element structures from $\mathcal{K}$ in this sublanguage. On the other hand by the Ryll-Nardzewski function of $Th(M_0)$ we can find the arity $l_n$ so that all $\mathcal{K}$-embeddings between structures of size $\leq n$ are determined by their relations of arity $\leq l_n$. Thus the axioms of (d) with at most $n$ quantifiers can be effectively found by the corresponding axioms (a - c) and the Ryll-Nardzewski function. Moreover there is an effective procedure which for every natural numbers $n$ produces all $\forall\exists$-sentences of $Th(U)$ with at most $n$ quantifiers, when one takes as the input the axioms of (a) and (c) of $U$ with at most $n$ quantifiers.
2. Finite language and non-G-compact theories

The following definitions and facts are partially taken from [1]. Let $C$ be a monster model of the theory $Th(C)$. For $\delta \in \{1, 2, ..., \omega\}$ let $E^\delta_L$ be the finest bounded $Aut(C)$-invariant equivalence relation on $\delta$-tuples (i.e. the cardinality of the set of equivalence classes is bounded). The classes of $E^\delta_L$ are called Lascar strong types. The relation $E^\delta_L$ can be characterized as follows: $(\bar{a}, \bar{b}) \in E^\delta_L$ if there are $\delta$-tuples $\bar{a}_0(= \bar{a}), \bar{a}_1, ..., \bar{a}_n(= \bar{b})$ such that each pair $\bar{a}_i, \bar{a}_{i+1}, 0 \leq i < n$, extends to an infinite indiscernible sequence. In this case denote by $d(\bar{a}, \bar{b})$ the minimal $n$ such that some $\bar{a}_0(= \bar{a}), \bar{a}_1, ..., \bar{a}_n(= \bar{b})$ are as above.

Let $E^\delta_{KP}$ be the finest bounded type-definable equivalence relation on $\delta$-tuples. Classes of this equivalence relation are called KP-strong types. The theory $Th(C)$ is called $G$-compact if $E^\delta_L = E^\delta_{KP}$ for all $\delta$. The first example of a non-G-compact theory was found in [1]. The first example of an $\omega$-categorical non-G-compact theory was found by the author in [4]. The following proposition is a straightforward application of Theorem 1.

**Proposition 4.** There is a countably categorical structure $N$ in a finite language such that $Th(N)$ is not $G$-compact.

**Proof.** Let $L$ be defined as in the proof of Theorem [1] Corollary 1.9(2) of [8] states that $G$-compactness is equivalent to existence of finite bound on the diameters of Lascar strong types. Let $M_0$ be an $\omega$-categorical structure which is not $G$-compact, see [4]. In [4] for every $n$ a pair $\bar{a}_n, \bar{b}_n$ of finite tuples of the same Lascar strong type is explicitly found so that $d(\bar{a}_n, \bar{b}_n) > n$.

Let $N$ be an $L$-structure, so that $M_0$ is a dense relativised reduct in $N$ defined by $P$. Then $Th(N)$ is not $G$-compact. Indeed for every $n$, the pair $\bar{a}_n, \bar{b}_n$ is of the same Lascar strong type and $d(\bar{a}_n, \bar{b}_n) > n$ with respect to the theory of $N$. To see this notice that if in $\bar{c}_0(= \bar{a}_n), \bar{c}_1, ..., \bar{c}_m(= \bar{b}_n)$ each $\bar{c}_i, \bar{c}_{i+1}$ extends to an indiscernible sequence in $Th(M_0)$, then this still holds in $Th(N)$ by density of the image of $Aut(N)$ in $Aut(M_0)$. On the other hand since $Aut(N) \leq Aut(M_0)$ on $P(M)$, we cannot find in $N$ such a sequence with $m \leq n$. □
3. Finite language and Ehrenfeucht theories

In this section we consider the situation where \( M_0 \) is obtained by an \( \omega \)-sequence of \( \omega \)-categorical expansions. We will see that under some natural assumptions the construction of Section 1 still works in this situation. Using this we will prove that there is a finite language such that the indexes of Ehrenfeucht theories with exactly three countable models form a \( \Pi^1_1 \)-hard set.

Let \( M_0 \) be a countable structure of a 1-sorted, relational language \( \mathcal{L}_0 = \{ R_1, R_2, \ldots, R_n, \ldots \} \). Suppose \( \mathcal{L}_0 = \bigcup_{i > 0} \mathcal{L}_i \), where for each \( i > 0 \), \( \mathcal{L}_i = \{ R_1, \ldots, R_i \} \) and the \( L_i \)-reduct of \( M_0 \) admits quantifier elimination (and thus \( \omega \)-categorical). We may assume that the arity of \( R_n \) is not greater than \( n \). Admitting \( R_n \) with repeated coordinates, we may also assume that for all \( m < n \) the arity of \( R_m \) is not greater than the arity of \( R_n \) and the arity of \( R_i \) is less than the arity of \( R_{i+1} \).

We now admit that \( M_0 \) is not \( \omega \)-categorical. On the other hand the theory of \( M_0 \) can be axiomatised as follows. For each \( i \) consider the \( L_i \)-reduct of \( M_0 \) and its age \( \text{Age}(M_0|L_i) \). Then this reduct is axiomatised by the standard axioms of a universal homogeneous structure (i.e. the versions of (a),(c),(d) from Remark 3 with respect to \( \text{Age}(M_0|L_i) \)). The collection of all systems of axioms of this kind gives an axiomatisation of \( \text{Th}(M_0) \).

Applying the proof of Theorem 1 we associate to each \( L_i \)-reduct of \( M_0 \), a class \( \mathcal{K}_i \) of \((\mathcal{L} \cup \mathcal{L}_i)\)-structures obtained by conditions (i)-(iii) from this proof. Since the \( L_i \)-reduct of \( M_0 \) has quantifier elimination, repeating the argument of Theorem 1 we obtain an \( \omega \)-categorical \((\mathcal{L} \cup \mathcal{L}_i)\)-structure \( U_i \) and the corresponding \( L \)-reduct \( N_i \) (since the language is finite, we do not need the assumption that each \( R_i \) describes a type). Notice that the construction forbids \( n \)-pairs for \( R_n \) of arity greater than the arity of \( L_i \).

Lemma 5. (1) For any \( i < j \) the structures \( U_i \) and \( U_j \) satisfy the same axioms of the form (a) - (d) of Remark 3 where the language of the \( P \)-part is restricted to \( L_i \) and the number of variables of the \( Q \)-part is bounded by the arity of \( L_i \).

(2) The corresponding structures \( N_i \) and \( N_j \) satisfy the same sentences which are obtained by rewriting of the axioms of statement (1) as \( L \)-sentences (using the interpretation of \( U_i \) in \( N_i \)).
Proof. Let $m$ be the arity of $L_i$. To see statement (1) let us prove that the classes $K_i$ and $K_j$ consist of the same $(L \cup L_i)$-structures among those with the $Q$-part of size $\leq m$. The direction $j \to i$ is clear: the $(L \cup L_i)$-reduct of an $(L \cup L_j)$-structure of this form obviously satisfies the requirements (i) - (iii) corresponding to $K_j$ (and to $K_i$ too). To see the direction $i \to j$ note that the assumption that the size of the $Q$-part is not greater than $m$ implies that such a structure from $K_i$ has an expansion to an $(L \cup L_j)$-structure from $K_j$.

Now the case of axioms of the form (a),(b),(c) is easy. Consider case (d). Since the $L_i$-reduct of $M_0$ admits elimination of quantifiers, for any finite $L_j$-substructure $A < M_0$ and any embedding of the $L_i$-reduct of $A$ into any $B \in \text{Age}(M_0|L_i)$ there is an $L_j$-substructure of $M_0$ containing $A$ with the $L_i$-reduct isomorphic to $B$. This obviously implies that for any substructure $A' < U_j$ without $n$-pairs for arities greater than $\text{arity}(L_i)$, any embedding of the $(L \cup L_i)$-reduct of $A'$ into any $B' \in K_i$ can be realised as a substructure of $U_j$ containing $A'$ with the $(L \cup L_i)$-reduct isomorphic to $B'$. This proves (1).

Statement (2) follows from statement (1). □

We now additionally assume that $M_0$ is a generic structure with respect to the class $K_0$ of all finite $L_0$-substructures of $M_0$. This means that $K_0$ has the joint embedding and amalgamation properties (JEP and AP), $(K_0/ \equiv) = \text{Age}(M_0)$ and $M_0$ is a countable union of an increasing chain of structures from $K_0$ so that any isomorphism between finite substructures extends to an automorphism of $M_0$.

Let $\mathcal{K}$ be the class of all finite $(L_0 \cup L)$-structures satisfying the conditions (i)-(iii) with respect to $K_0$. In particular it obviously contains only countably many isomorphism types and the class $K_0$ appears as the class of all $P$-parts of $\mathcal{K}$. Applying the proof of Theorem 1 we see that $\mathcal{K}$ is closed under substructures and has the joint embedding and amalgamation properties. By Theorem 1.5 of [6] the class $\mathcal{K}$ has a unique (up to isomorphism) generic structure (i.e. a structure which is a countable union of an increasing chain of structures from $\mathcal{K}$ and satisfies axioms (a) - (d) of Remark 3). Note that this structure can be non-$\omega$-categorical.

Lemma 6. Under the circumstances of this section let $U$ be a generic $(L_0 \cup L_0)$-structure for $\mathcal{K}$ as above.
Then the $P$-part of $U$ is isomorphic to $M_0$. The structure $M_0$ is a dense relativised reduct of $U$.

Proof. The first statement is obvious. The second statement is an application of back-and-forth. □

It is worth noting here that for every $m$ the amalgamation of Theorem 1 preserves the subclass of $\mathcal{K}$ consisting of structures without $n$-pairs for arities greater than $m$ (for example structures with the size of the $Q$-part less than $m + 1$). If $m$ is the arity of the language $L_i$ then $(L_i \cup L)$-reducts of these structures form the Fraïssé class corresponding to the universal homogeneous structure $U_i$.

Proposition 7. (1) All axioms of $U_i$ of the form (a),(c),(d) of Remark \[5 also hold in $U$.

(2) The theory $Th(U)$ is model complete and is axiomatised by axioms of the form (b) of Remark \[5 together with the union of all axioms of the form (a),(c),(d) for all $Th(U_i)$.

(3) For any axiom $\phi$ of $Th(U)$ of the form (a)-(d) as in (2) there is a number $i$ so that $\phi$ holds in all $U_j$ for $j > i$.

Proof. (1) The case of axioms of the form (a),(c) is easy. Consider case (d). Since the $L_i$-reduct of $M_0$ admits elimination of quantifiers, for any substructure $A < M_0$ and any embedding of the $L_i$-reduct of $A$ into any $B \in Age(M_0|L_i)$ there is a substructure of $M_0$ containing $A$ with the $L_i$-reduct isomorphic to $B$. This obviously implies that for any substructure $A' < U$ without $n$-pairs of arity greater than $arity(L_i)$, any embedding of the $(L \cup L_i)$-reduct of $A'$ into any $B' \in \mathcal{K}_i$ can be realised as a substructure of $U$ containing $A'$ with the $(L \cup L_i)$-reduct isomorphic to $B'$. This proves (1).

(2) Let $U'$ and $U''$ satisfy axioms as in the formulation of (2). Then obviously the $(L_i \cup L)$-reducts of $U'$ and $U''$ satisfy the axioms of $Th(U_i)$ as in statement (1). In particular $P(U') \cong P(U'')$ in each $L_i$. Moreover if $U' < U''$, then by axioms (d) one can easily verify that this embedding is $\forall$-elementary. Thus $U'$ is an elementary substructure of $U''$ by a theorem od Robinson. It is also clear that $U$ is embeddable into any structure satisfying axioms as in (2).
(3) By Lemma 5 we see that for every sentence $\theta \in Th(U)$ of the form (a) - (d) of (2) there is a number $i$ such that for all $j > i$, $\theta$ holds in $U_j$. □

Some typical examples of Ehrenfeucht theories (i.e. with finitely many countable models) are build by the method of this section: the theory of all expansions of $(\mathbb{Q}, <)$ by infinite discrete sequences $c_1 < c_2 < ... < c_n < ...$, is Ehrenfeucht and can be easily presented in an appropriate $L_0$ as above.

**Proposition 8.** Under the circumstances of this section assume that $M_0$ is a generic structure with respect to the class $K_0$ of all finite substructures of $M_0$. Assume that $Th(M_0)$ is an Ehrenfeucht theory. Let $U$ be a generic $(L \cup L_0)$-structure for $K$ as above.

Then $Th(U)$ is also Ehrenfeucht.

**Proof.** Let $U', U''$ be countable models of $Th(U)$. Assume that the $P$-parts of $U'$ and $U''$ (say $M'$ and $M''$) are isomorphic. Identifying them let us show that $U'$ is isomorphic to $U''$. For this we fix a sequence of finite substructures $A_1 < A_2 < ... < A_i < ...$ so that $M' = \bigcup A_i$. Having enumerations of the $Q$-parts of $U'$ and $U''$ we build by back-and-forth, sequences $B'_1 < B'_2 < ... < B'_i < ...$ and $B''_1 < B''_2 < ... < B''_i < ...$ with $B'_i > A_i < B''_i$, $U' = \bigcup B'_i$ and $U'' = \bigcup B''_i$ so that $B'_i$ is isomorphic to $B''_i$ over $A_i$. By Proposition 7(2) using the fact that $U', U'' \models Th(U)$ we see that such sequences exist. □

We now prove that there is a finite language $L$ such that the set of Ehrenfeucht $L$-theories with exactly three models is $\Pi^1_1$-hard.

**Theorem 9.** There is a finite language $L$ such that for every $B \in \Pi^1_1$ there is a Turing reduction of $B$ to the set $3\text{Mod}_L$ of all indexes of decidable Ehrenfeucht $L$-theories with exactly three countable models.

**Proof.** Let $L$ be the language defined in Section 1. We use the idea of Section 4 of [7]. In particular we can reduce the theorem to the case when $B$ coincides with the index set $\text{NoPath}$ of the property of being a computable tree $\subseteq \omega^\omega$ having no infinite path. The Turing reduction of this set to $3\text{Mod}_L$ which will be built below, is a composition of the procedure described in [10] and [7], and the construction of
this section. The former one is as follows. Having an index \( e \) of a computable tree \( Tr_e \subset \omega^\omega \), R.Reed defines a complete decidable theory \( T_e \) of the language

\[
\langle \land, <_L, \leq_H, E^\eta_\xi, L^\eta_\xi, H_\eta, A_\eta, B_\eta, c_\eta \ (\eta, \xi \in Tr_e) \rangle,
\]

where \( \land \) is the function of the greatest lower bound of a tree, \( <_L \) is a Kleene-Brouwer ordering of this tree and \( \leq_H \) is a binary relation measuring 'heights' of nodes. Constants \( c_\eta, \eta \in Tr_e \), define embeddings of \( Tr_e \) into models of \( T_e \). The remaining relations are binary.

For each natural \( n \) define \( T_e|_n \) to be the restriction of \( T_e \) to the sublanguage corresponding to the indexes from the finite subtree \( Tr_e \cap n^{<n} \). The proof of Lemma 9 from \[10\] shows that \( T_e|_n \) admits effective quantifier elimination. Lemma 6 of \[10\] asserts that every quantifier-free formula of \( T_e|_n \) is equivalent to a Boolean combination of atomic formulas of the following form:

\[
\begin{align*}
    u \land v &= w \land z, \ u <_L w, \ u \land v \leq_H w \land z, \ E^\eta_\xi(u, w), \\
    L^\eta_\xi(u, w), \ H_\eta(u \land v, w), \ A_\eta(u \land v, w)
\end{align*}
\]

where \( u, v, w, z \) is either a variable or a constant in \( T_e|_n \). By Lemma 8 of \[10\] the corresponding Boolean combination can be found effectively. This implies that replacing the function \( \land \) by the first, third, sixth and seventh relations of the list above we transform the language of each \( T_e \) into an equivalent relational language. In particular we have that each \( T_e|_n \) is \( \omega \)-categorical.

Note that extending the set of relations we can eliminate constants \( c_\eta \) from our language. Admitting empty relations we may assume that all \( T_e \) have the same language (where \( \omega^{<\omega} \) is the set of indexes). Admitting repeated coordinates we may assume that this language \( L_0 = \{ R_1, ..., R_i, ... \} \) satisfies the assumptions of the beginning of the section and each sublanguage \( L_n \) of the presentation \( L_0 = \bigcup_{i>0} L_i \) corresponds to \( T_e|_n \).

We now apply Lemma 5 to all \( T_e|_n \). Since each \( T_e|_n \) is computably axiomatisable uniformly in \( e \) and \( n \), we obtain an effective enumeration of computable axiomatisations of \( L \)-expansions of all \( T_e|_n \) (with \( T_e|_n \) on the \( P \)-part). For each \( e \) taking the axioms which hold in almost all \( L \)-expansions of \( T_e|_n \) we obtain by Lemma 5(1) a computable axiomatisation of a theory of \( L \)-expansions of \( T_e \).
When $T_e$ is an Ehrenfeucht theory with exactly three models (i.e. $e \in NoPath$), the prime model of $T_e$ is generic with respect to its age. Applying Proposition 7 to $T_e$ and all $T_e|_n$ we obtain a generic $(L_0 \cup L)$-structure such that its theory is computably axiomatised as above. This theory has exactly three countable models by Proposition 8.

When we take $L$-reducts of all $T_e$ and the corresponding computable axiomatisations we obtain a computable enumeration of $L$-theories which gives the reduction as in the formulation of the theorem. □

Remark 10. In the proof above we used Proposition 7 in order to obtain a complete $L$-expansions of Ehrenfeuch $T_e$’s. We cannot apply it in the case when $T_e$ does not have an appropriate generic model, for example when the corresponding $Tr_e$ has continuum many paths. Nevertheless the author hopes that the proof can be modified so that the reduction as above also shows that the set of all $L$-theories with continuum many models is $\Sigma^1_1$-hard. In the case of infinite languages this is shown in Section 4 of [7].

4. Coding $\omega$-categorical theories

The main theorem of this section improves the corresponding result of [7] (where the authors do not demand that the language is finite). It is worth noting that the author together with Barbara Majcher-Iwanow have found some other improvements in [5].

**Theorem 11.** There is a finite language $L$ such that the property of $\omega$-categoricity distinguishes a $\Pi^0_3$-complete subset of the set of all decidable complete $L$-theories.

**Proof.** In the formulation of the theorem $L$ is the language defined in Section 1. It is shown in [7] that the property of $\omega$-categoricity is $\Pi^0_3$. The proof of $\Pi^0_3$-completeness in the case of $L$ is based on Theorem 1, Section 3 and the idea of Section 2 of [7]. The latter one will be presented in some special form, the result of a fusion with some ideas from [9].

Let us fix the standard enumeration $p_n$ of prime numbers and a Gödel 1-1-enumeration of the set of pairs $(i, j)$. Let $a(x)$ be a computable increasing function
from \( \omega \) to \( \omega \setminus \{0, 1, 2\} \) so that if natural numbers \( x_1 < x_2 \) enumerate pairs \( \langle i_1, j_1 \rangle \) and \( \langle i_2, j_2 \rangle \) then \( p_{i_1}a(x_1) < p_{i_2}a(x_2) \).

Let \( L_E \) consist of \( 2p_n \)-ary relational symbols \( E_n, n \in \omega \), and \( T_E \) be the \( \forall \exists \)-theory of the universal homogeneous structure of the universal theory saying that each \( E_n \) is an equivalence relation on the set of \( p_n \)-tuples which does not depend on the order of tuples and such that all \( p_n \)-tuples with at least one repeated coordinate lie in one isolated \( E_n \)-class (Remark 4.2.1 in [9]). It is worth mentioning here that the joint embedding property and the amalgamation property are easily verified by an appropriate version of free amalgamation (modulo transitivity of \( E_n \)-s). Note also that \( T_E \) is \( \omega \)-categorical and decidable.

We now define an auxiliary language \( L_{ESP} \). We firstly extend \( L_E \) by countably many sorts \( S_n, n \in \omega \). Start with a countable model \( M_E \models T_E \) and take the expansion of \( M_E \) to the language \( L_E \cup \{ S_1, \ldots, S_n, \ldots \} \cup \{ \pi_1, \ldots, \pi_n, \ldots \} \), where each \( S_n \) is interpreted by the non-diagonal elements of \( M^{p_n}/E_n \) and \( \pi_n \) by the corresponding projection. To define \( L_{ESP} \) we extend \( L_E \cup \{ S_1, \ldots, S_n, \ldots \} \cup \{ \pi_1, \ldots, \pi_n, \ldots \} \) by an \( \omega \)-sequence of relations \( P_m, m \in \omega \), with the following properties. If \( m \) is the Gödel number of the pair \( \langle n, i \rangle \) then we interpret \( P_m \) as a subset of the diagonal of \( S^{a(m)}_n \).

Let \( T_{ESP} \) be the \( L_{ESP} \)-theory axiomatized by \( T_E \) together with the natural axioms for all \( \pi_n \) and \( P_m \) as above.

Having a structure \( M \models T_{ESP} \) (which is an expansion of \( M_E \)) we now build another expansion \( M^* \) of \( M_E \) (in the 1-sorted language). For each relational symbol \( P_m \) of the sort \( S^{a(m)}_n \) we add a new relational symbol \( P^*_m \) on \( M^{a(m)p_n}_E \) interpreted in the following way:

\[
M^* \models P^*_m(\bar{a}_1, \ldots, \bar{a}_{a(m)}) \iff M \models P_m(\pi_n(\bar{a}_1), \ldots, \pi_n(\bar{a}_{a(m)})).
\]

It is clear that \( M^* \) and \( M \) are bi-interpretable.

By \( T^*_{ESP} \) we denote the theory of all \( M^* \) with \( M \models T_{ESP} \). Let \( L_0 \) be the corresponding language. Then \( M_E \) is the \( L_E \)-reduct of any countable \( M^* \models T^*_{ESP} \).

It is clear that \( T^*_{ESP} \) is axiomatized by the \( \forall \exists \)-axioms of \( T_E \), \( \forall \)-axioms of \( E_n \)-invariantness of all \( P^*_m \) and \( \forall \)-axioms that every \( P_m \) is a subset of an appropriate diagonal. Moreover for every natural \( l \) we have \( \leq 1 \) relations of arity \( l \) in \( L_0 \) and the function of arities of \( P^*_m \) is increasing. Admitting empty relations (say \( R_j \)) we may think that for every natural number \( l > 0 \) the language \( L_0 \) contains
exactly one relation of arity \( l \). In particular \( L_0 \) satisfies basic requirements on \( L_0 \) from Section 3. We present \( L_0 \) as the union of a sequence of finite languages \( L_1 \subset L_2 \subset \ldots \subset L_m \subset \ldots \) of arities \( l_1 < l_2 < \ldots < l_m < \ldots \) where \( L_m \) consists of all relations of arity \( \leq p_n a(m) \) (\( = l_m \)) with \( n \) to be the first coordinate of the pair enumerated by \( m \). Note that when \( m \) codes a pair \( (n, j) \) the relation \( E_n \) is also in \( L_m \).

For every \( m \in \{ 1, \ldots, i, \ldots, \omega \} \) and a finite set \( D \) of indexes of relations \( P_i^* \) of arity \( \leq l_m \) we consider the class \( K_D \) of all finite substructures of models of \( T_{ESP}^* \) satisfying the property that all \( P_i^* \) with \( i \not\in D \), are empty. It is clear that for any natural number \( k \) the number of structures of \( K_D \) of size \( k \) is finite. We will also denote \( K_{\omega, D} := K_D \). When \( m < \omega \) we define \( K_{m, D} \) as the class of all reducts of \( K_D \) to the sublanguage \( L_m \).

By an appropriate version of free amalgamation we see that \( K_{m, D} \) has the joint embedding property and the amalgamation property. Let \( M_{m, D} \) be the corresponding universal homogeneous structure and let \( T_{m, D}^* \) be the theory of \( M_{m, D} \). It follows from \( T_{\omega, D}^* \) that \( T_{ESP}^* \subset T_{\omega, D}^* \) and for every \( n \) the family of all \( P_i^* \), with \( i \in D \) coding some \( (n, j) \), freely generates a Boolean algebra of infinite subsets of the sort \( S_n \) (we may interpret such \( P_i^* \) as a unary predicate on \( S_n \)).

By the definition of the class \( K_D \) we see that for any \( t < m \) and any two finite sets \( D' \) and \( D'' \) satisfying

\[
D' \cap \{ 0, \ldots, l_t \} = D'' \cap \{ 0, \ldots, l_t \}
\]

the reducts of \( M_{m, D'} \) and \( M_{m, D''} \) to \( L_t \) are isomorphic.

Let us apply the construction of Theorem 1 to \( M_{m, D} \). Then we obtain the \((L_m \cup L)\)-structure \( U_{m, D} \) and the corresponding \( L \)-reduct \( N_{m, D} \), where \( L \) is the language as in Theorem 1. It follows from the proof of that theorem that in the situation of the previous paragraph the structures \( U_{m, D'} \) and \( U_{m, D''} \) satisfy the same axioms of the form (a) - (d) of Remark 3 where the language of the \( P \)-part is restricted to \( L_t \) and the number of variables of the \( Q \)-part is bounded by \( l_t \). When we rewrite these axioms as \( L \)-sentences (using the corresponding definition of the relations of \( L_m \)) we obtain that \( N_{m, D'} \) and \( N_{m, D''} \) satisfy the same axioms of this kind.

1\( L_{ESP} \)-predicates corresponding to \( P_i^* \)
Let \( \varphi(x, y) \) be a universal computable function, i.e. \( \varphi(e, x) = \varphi_e(x) \). Find a computable function \( \rho \) (with \( \text{Dom}(\rho) = \omega \)) enumerating \( \text{Dom}(\varphi(y, z), x) \), i.e. the set of all triples \( \langle e, n, x \rangle \) with \( x \in W_{\varphi_e(n)} \).

For any natural \( e, s \) we define a finite set \( D^*_e \) of codes \( m \leq l_s \) of all pairs \( \langle n, k \rangle \) such that

\[
(\exists x)(\rho(k) = \langle e, n, x \rangle \land (\forall k' < k)(\rho(k') \neq \langle e, n, x \rangle)).
\]

Let \( T_e \) and \( T_e^* \) be the \( L_{\text{ESP}} \)-theory and the corresponding 1-sorted version (containing \( T_{\text{ESP}}^* \)) such that for all natural \( s \) the reduct of \( T_e^* \) to \( L_s \) coincides with the corresponding reduct of \( T_{s,D^*_e}^* \). Since for any \( s < t \) we have \( D^*_e \cap \{0, ..., l_s\} = D^*_e \), the definition of \( T_e \) and \( T_e^* \) is correct. It is clear that both \( T_e \) and \( T_e^* \) are axiomatisable by computable sets of axioms uniformly in \( e \). Since for each \( s \) the reduct of \( T_e^* \) as above is \( \omega \)-categorical, the theories \( T_e \) and \( T_e^* \) are complete. Thus \( T_e \) and the corresponding theory \( T_e^* \) are decidable uniformly in \( e \). It is worth noting that for each \( m \) the \( L_m \)-reduct of \( T_e^* \) admits elimination of quantifiers (it is of the form \( T_m^* \) as above). Moreover, the class \( \bigcup_l \mathcal{K}_l,D^*_e \) considered as a class of \( L_0 \)-structures where almost all \( P_m^* \) are empty, is a countable class with JEP and AP. It is clear that \( T_e^* \) is the theory of the corresponding universal homogeneous structure \( M^*_e \).

Applying Proposition 7 to \( M^*_e \) and all \( M_{l,D^*_e} \) we obtain \( (L_0 \cup L) \)-structures \( U_e \) and their approximations \( U_{l,D^*_e} \) (and \( N_{l,D^*_e} \)), which for \( l \to \infty \) give a computable axiomatisation of the complete \( L \)-theory \( T_e^L \) of the corresponding \( L \)-reducts \( N_e \).

By Remark 3 applied to all \( U_{l,D^*_e} \) (with decidable theories), this axiomatisation (the corresponding decidability of \( T_e^L \)) can be found by an effective uniform in \( e \) procedure.

We now fix a Gödel coding of the language \( L \), and identify decidable complete \( L \)-theories with computable functions from \( \{\text{sgn}(\varphi_e(x)) : e \in \omega\} \) realising the corresponding characteristic functions (by \( \text{sgn}(x) \) we denote the function which is equal to 1 for all non-zero numbers and \( \text{sgn}(0) = 0 \)). We want to prove that the set of all natural numbers \( e \) satisfying the relation

"\( \text{sgn}(\varphi_e(x)) \) codes a decidable \( \omega \)-categorical theory"

is \( \Pi^0_3 \)-complete.

Fix a Turing machine \( \kappa(x, y) \) which decides when for a pair \( d, e \) the number \( d \) codes a sentence which belongs to \( T_e^L \) (in this case \( \kappa(d, e) = 1 \)). The following
procedure defines a computable function $\xi(z)$ and a computably enumerable set $Z$. At step $e$ we take the Turing machine for $\text{sgn}(\varphi_e(x))$ and check if any replacement of some parameter $e'$ in that program by a variable $y$ makes it the Turing machine $\kappa(x, y)$. If this happens we put $e$ into $Z$ and define $e' := \xi(e)$. As a result we obtain a computably enumerable set $Z$ and a computable function $\xi$ with $\text{Dom}(\xi) \supset Z$ and $\text{Rng}(\xi) = \omega$ such that for every $e \in Z$ the function $\text{sgn}(\varphi_e(x))$ is computed by the machine $\kappa(x, \xi(e))$ (for $T^L_{\xi(e)}$).

By Ryll-Nardzewski’s theorem the $L_{ESP}$-theory $T_{\xi(e)}$ is $\omega$-categorical if and only if all $W_{\varphi_{\xi(e)}(n)}$ are finite (i.e. the set of 1-types (pairwise non-equivalent Boolean combinations of $P_n$) of each $S_n$ is finite). If we consider the corresponding $T^L_{f(e)}$, then this property remains true.

Since for any Turing machine computing $\varphi_{e'}(x)$ we can effectively find a Turing machine deciding $T^L_{e'}$ (i.e. in fact we can find $\text{sgn}(\varphi_{e'}(x))$ with $\xi(e) = e'$), we see that the $\Pi^0_3$-set $\{e' : \forall n(W_{\varphi_{\xi(e)}}(n) \text{ finite})\}$ is reducible to $\{e : \text{sgn}(\varphi_{e'}(x)) \text{ codes an} \ \omega\text{-categorical} \ L \text{-theory}\}$. Since the former one is $\Pi^0_3$-complete (see [7]) we have the theorem. □

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