On the location of poles for the Ablowitz–Segur family of solutions to the second Painlevé equation

M Bertola

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W.,
Montréal, Québec, H3G 1M8, Canada
and
Centre de recherches mathématiques, Université de Montréal, Canada

E-mail: bertola@mathstat.concordia.ca

Received 7 September 2011, in final form 26 December 2011
Published 19 March 2012
Online at stacks.iop.org/Non/25/1179

1. Introduction and result

Since the introduction of the Painlevé equations and in recent years, due to their appearance
in many different models of mathematical physics, there has been a growing interest in the
‘Painlevé program’ of classifying solutions, connection formulae, etc (see the book [5] as a
prime example of this program).

The six Painlevé equations (and all the other equations with the Painlevé property) are such
that the only ‘movable’ singularities of the general solution1 are poles, whereas all the ‘nastier
singularities’ (essential singularities, branchpoints) are fixed (they depend on the particular
equation under scrutiny only). For example for the first two equations

\[
(\text{PI}) \quad u''(s) = 6u(s)^2 - s, \quad (\text{PII}) \quad u''(s) = su(s) + 2u(s)^3,
\]

(1.1)
each solution is meromorphic, with \( s = \infty \) being the essential singularity. Given a special
solution to a Painlevé equation, there is a certain interest in determining the location of its
poles; within this circle of ideas we may mention the work of [9] where it is shown that the
\textit{tritronquée} solution to Painlevé I has no poles in the sector \( |\arg(s)| < 4\pi/5 \) and \( |s| \) is
sufficiently large. Following this, and due to its relevance in the semiclassical asymptotics of

1 By ‘movable’ it is meant a singularity whose position depends on the member of the family, e.g. on the initial
values.
the Nonlinear Schrödinger equation, Dubrovin conjectured [3] that—in fact—the tritronquée is pole-free in the whole sector (not just the distal part). Asymptotic methods [10] and numerics based on Padé approximants [11] provide solid evidence that the aforementioned conjecture holds true.

The study of the distribution of poles for special solutions is in general an elusive task; in very special cases, using the correspondence with an appropriate Riemann–Hilbert problem, one may use certain ‘vanishing lemmas’ if the corresponding problem has sufficient symmetry (typically a Schwartz-reflection symmetry of some sort, see for example [1, 4]). These techniques allow one at best to prove that certain solutions are pole-free on the real axis, for example. When venturing into the complex plane of the independent variable, the required symmetry is lost and those methods cannot be applied.

Our more modest goal here is to study the pole distribution of the so-called Ablowitz–Segur family [12] of solutions to Painlevé II:

\[
\begin{align*}
    u''(s) &= su(s) + 2u^3(s), \quad (1.2) \\
    u(s) &\simeq \kappa \text{Ai}(s), \quad s \to +\infty, \quad \kappa \in \mathbb{C}. \quad (1.3)
\end{align*}
\]

A particular and very important example of this family corresponds to the Hastings–McLeod solution \(\kappa = 1\); this solution has the property [6] that it is pole-free for \textit{real values} of the variable \(s\); this property is maintained for \(|\kappa| < 1\) while for \(\kappa \in \mathbb{R} \setminus [-1, 1]\) the solution has poles on the real axis (but is still pole-free for sufficiently large positive \(s\)).

The Ablowitz–Segur family corresponds to purely imaginary solutions with behaviour as in (1.3) but \(\kappa \in i\mathbb{R}\); such solutions are oscillatory as \(s \to -\infty\) and decay as \(s \to +\infty\) and are also pole-free on the real axis.

In both the Hastings–McLeod and Ablowitz–Segur cases (and even more general solutions) a great number of results are known, including connection formulae and asymptotic forms of the solutions. In particular, the following is known (which we rephrase here in a form adapted for the context of this paper)\(^2\).

\textbf{Theorem 1.1 (See theorem 11.1, 11.7 in [5]). For each }\(\kappa \in \mathbb{C}\)\text{ there exists a constant }\(R > 0\)\text{ such that the solutions of (1.2) with behavior (1.3) are pole-free in the region sector }\arg s \in [-\frac{\pi}{3}, \frac{\pi}{3}], \ |s| > R. \text{ In the special case }\kappa = 1\text{ (Hastings–McLeod solution) the region is }\arg(s) \in [-\frac{\pi}{3}, \frac{\pi}{3}] \cup \left[\frac{\pi}{3}, \frac{4\pi}{3}\right], \ |s| > R.\]

The question arises as to how large is the region in the complex \(s\)-plane where the solution remains free of poles; this question is—in principle—not easy to settle as the equation (1.2) has the Painlevé property, that is, its solutions have only poles as singularities, but their position depends on the initial data (\textit{movable singularities}). The result we shall prove is stated here:

\textbf{Theorem 1.2. The solutions to }\(u''(s) = su(s) + 2u^3(s)\)\text{ with the behavior }

\[
    u(s) \simeq \kappa \text{Ai}(s), \quad s \to +\infty, \quad \kappa \in \mathbb{C}
\]

\text{are pole-free in the whole region (the fractional power is the principal one) }

\[
    \Re(s^\frac{1}{\kappa}) > \frac{1}{2} \ln |\kappa|.
\]

\textit{In particular, if }\(|\kappa| = 1\text{ the region coincides with the sector }\arg s \in (-\pi/3, \pi/3).\]

\textbf{Remark 1.1. In general, the negative real axis is excluded since }\(s^\frac{1}{\kappa} \in i\mathbb{R}\text{ for }s \in -\mathbb{R}\) (no matter which determination of the fractional power); however, if \(|\kappa| < 1\) or if \(\kappa \in i\mathbb{R}\) it is known that there are no poles on the negative axis. See later remark 2.2. We can also replace inequality (1.5) by \(\Re(s^\frac{1}{\kappa}) \geq \frac{1}{2} \ln |\kappa|\), see remark 2.3.\)

\(^2\) The results in [5] are stated for even more general solutions of the general Painlevé II ODE \(u'' = su + 2u^3 + \alpha\) (our case being \(\alpha = 0\)). Much more detail is contained in the cited comprehensive book: some of those results appeared first in separate papers by the same authors and the specific references are contained therein.
2. Proof of theorem 1.2

The proof is based upon the following

**Proposition 2.1.** The solution of (1.2) with the asymptotics (1.3) is the second logarithmic derivative of the Fredholm determinant of the integral operator on $H := L^2(\gamma_+ \cup \gamma_-, |d\lambda|)$ with kernel

$$K_s(\lambda, \mu) := \frac{e^{i\theta_s(\lambda) - \theta_s(\mu)}}{2i\pi(\lambda - \mu)} \left[ \chi_+(\lambda)\chi_-(\mu) + \chi_+(\lambda)\chi_-(\mu) \right], \quad \theta_s(\lambda) := \frac{\lambda^3}{3} + s\lambda, \quad (2.1)$$

where $\chi_{\pm}(\lambda)$ denote the indicator functions of the sets $\gamma_{\pm} := \mathbb{R} \pm ic, c > 0$ (with $\gamma_+$ oriented from right to left and $\gamma_- = -\gamma_+$ oriented from left to right), as follows:

$$u(s)^2 = -\frac{d^2}{ds^2} \ln \det (\text{Id} - \kappa K_s). \quad (2.2)$$

The proof is contained in [2]; it essentially is a different representation of the Airy-operator on $L^2(\mathbb{R}, dx)$ with integral kernel

$$K_{Ai}(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} = \frac{i}{4\pi^2} \int_{\gamma_-} d\mu \int_{\gamma_+} d\lambda \frac{e^{i\theta_s(\lambda) - \theta_s(\mu)}}{\mu - \lambda}. \quad (2.3)$$

Indeed it is shown in [2, theorem 3.1] that (in the present notation)

$$\det \left( \text{Id}_{L^2(\mathbb{R} \cup |\gamma_+|)} - \kappa K_s \right) = \det \left( \text{Id}_{L^2([s, \infty))} - \kappa^2 K_{Ai} \right)_{[s, \infty)}, \quad (2.4)$$

where the right-hand side is the Fredholm determinant that appears in the spacing distributions of the Gaussian unitary ensemble (GUE) of random matrices [13]

$$F_2(s; \kappa) := \det \left( \text{Id}_{L^2([s, \infty))} - \kappa^2 K_{Ai} \right) \quad (2.5)$$

and it is related to the Ablowitz–Segur family (1.2, 1.3) by

$$u(s; \kappa)^2 := -\frac{d^2}{ds^2} \ln F_2(s; \kappa). \quad (2.6)$$

The simple idea behind the proof of identity (2.4) is that of re-expressing its restriction to the semi-interval $[s, \infty)$ in Fourier space. The advantage is that, while on one hand it is unclear how to extend the representation (2.5) to complex values of $s$, on the other hand the kernel $K_s$ of proposition 2.1 depends analytically on $s$.

**Remark 2.1.** The determinant of the above operator is independent of the details of the contours $\gamma_{\pm}$; the usual choice for $\gamma_+$ would be any (smooth) contour in the upper half plane extending to infinity in such a way that $\|e^{\theta_s}\| = O(|\lambda|^{-\infty})$. Typically, this contour extends along directions $\arg \lambda = \pi \div 5\pi \div 6$, but we can use $\mathbb{R} + ic, c > 0$ thanks to the simple computation

$$\Re[(ix + ic)^3] = -3cx^2 + c^3. \quad (2.7)$$

**Corollary 2.1.** The poles of the solutions $u(s)$ mentioned in proposition 2.1 coincide with the zeroes of the Fredholm determinant $\det(\text{Id} - \kappa K_s)$ as a function of $s$.

---

3 Relative to loc. cit. we have performed an overall rotation $\lambda \mapsto i\lambda$ so that the phase function that was $\theta(\lambda) := (\lambda^3/3) - x\lambda$ becomes the one used in the present paper $\theta_s(\lambda) := (\lambda^3/3) + x\lambda$ (with an overall factor of $-i$). Therefore the contours that were denoted $\gamma_L, \gamma_R$ (for ‘left’ and ‘right’) have been rotated to $\gamma_+$ and $\gamma_-$, respectively.
It follows at once that if the operator norm of $\kappa K_s$ can be estimated above by unity in certain regions of the $s$-plane, the Fredholm determinant is therefore nonvanishing and the corresponding solution is pole-free; and this is precisely what we set out to accomplish below.

We shall follow [2] and represent the operator $K_s$ as a block-antidiagonal operator relative to the natural splitting $L^2(\gamma_+ \cup \gamma_-) = L^2(\gamma_+) \oplus L^2(\gamma_-)$, and then use the following identity of Fredholm determinants

$$\det \left[ \text{Id}_{L^2(\gamma_+ \cup \gamma_-)} - \kappa \begin{bmatrix} 0 & G \\ F & 0 \end{bmatrix} \right] = \det \left[ \text{Id}_{L^2(\gamma_-)} - \kappa^2 G \circ F \right], \tag{2.8}$$

where the two operators $F, G$ are defined as follows:

$$L^2(\gamma_-) \xrightarrow{F} L^2(\gamma_+), \tag{2.9}$$

$$(F g)(\lambda) := e^{-\frac{i}{2} \theta_s(\lambda)} \int_{\mathbb{R}-ic} \frac{d\mu}{2\pi i} \frac{e^{\frac{i}{2} \theta_s(\mu)} g(\mu)}{\mu - \lambda}. \tag{2.10}$$

$$(G h)(\mu) := e^{\frac{i}{2} \theta_s(\mu)} \int_{\mathbb{R}+ic} \frac{d\lambda}{2\pi i} \frac{e^{-\frac{i}{2} \theta_s(\lambda)} h(\lambda)}{\mu - \lambda}. \tag{2.11}$$

The identity (2.8) holds because (as proven in [2]) both $F, G$ are of trace-class in $L^2(\gamma_+ \oplus L^2(\gamma_-)$. It is also clear that $F$ and $G$ (between the respective spaces) have the same norms by the symmetry; we shall now make a very rough (but sufficient) estimate of their norms, from which the (simple) proof of theorem 1.2 shall follow at once.

We shall consider $F$ for definiteness, and we start by observing that it is the composition of three operators:

- the multiplication by $e^{\frac{i}{2} \theta_s(\mu)}$ on $L^2(\mathbb{R} - ic)$;
- the Cauchy operator from $L^2(\mathbb{R} + ic)$ to $L^2(\mathbb{R} - ic)$;
- the multiplication by $e^{-\frac{i}{2} \theta_s(\mu)}$ on $L^2(\mathbb{R} + ic)$.

The norm of the Cauchy operator

$$C : L^2(\mathbb{R} - ic) \mapsto L^2(\mathbb{R} + ic) \tag{2.12}$$

is promptly shown to be unity. To see this fact, let $\mu = x + ic$ and $\lambda = y - ic$, so that the above operator can be written out simply as an operator of $L^2(\mathbb{R}, dx)$,

$$(C f)(y) := \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f(x)}{x - y - 2ic} \, dx. \tag{2.13}$$

Since this is a convolution operator with the function $\varphi(x) = (1/2\pi)(-2ic - x)$, its representation in Fourier transform\(^5\) is the multiplication operator by $\sqrt{2\pi} \hat{\varphi}(t)$ where the Fourier transform of the function $\varphi(x)$ is easily computed with the aid of the residue theorem to be

$$\hat{\varphi}(t) = \frac{1}{\sqrt{2\pi}} e^{-2ic} \chi_{\mathbb{R}_+}(t). \tag{2.14}$$

Since the norm of a multiplication operator is its $L^\infty$ norm and the Fourier transform is unitary, we conclude that $\|C\| = 1$.

Thus the norm of $F$ (and similarly $G$) is bounded by the $L^\infty$ norm of the multiplication operator; to achieve the optimal estimate we must choose $c$ appropriately so that the line $\mathbb{R} + ic$

\(^4\) We abuse the notation and denote the operator by the same symbol as its integral kernel.

\(^5\) We use the convention $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itx} f(x) \, dx$.  

passes through the saddle point of the phase function \( \theta_j = (\lambda_j^3/3) + s \lambda_j \). Elementary calculations give that the saddle point is \( \lambda_j := \pm i \sqrt{s} \), \( \argin(s) \in (-\pi, \pi) \). We thus need to choose \( c_0 := \sqrt{s} = \sqrt{|\arg(s)/2|} \). We leave to the reader the elementary verification that
\[
\max_{\lambda \in \mathbb{R} - i c_0} |e^{\frac{1}{2} \theta_0(\lambda)}| = e^{-\frac{1}{2} |\arg(s)|}, \quad \max_{\mu \in \mathbb{R} + i c_0} |e^{-\frac{1}{2} \theta_0(\mu)}| = e^{-\frac{1}{2} |\arg(s)|},
\]
and that the maxima are taken on at the saddle points only. It follows thus that
\[
\|F\| \leq \|e^{\frac{1}{2} \theta_0(\lambda)}\|_{L^\infty(\mathbb{R} - i c_0)} \|e^{\frac{1}{2} \theta_0(\mu)}\|_{L^\infty(\mathbb{R} + i c_0)} = e^{-\frac{1}{2} |\arg(s)|}.
\]
A completely parallel computation shows that
\[
\|G\| \leq e^{-\frac{1}{2} |\arg(s)|},
\]
and hence
\[
\|\kappa^2 G \circ F\| \leq |\kappa|^2 e^{-\frac{1}{2} |\arg(s)|}.
\]

Thus, if the right side of inequality (2.18) is bounded by 1 the operator \( \text{Id} - \kappa^2 G \circ F \) is invertible and thus the Fredholm determinant is nonzero. The condition \( |\kappa|^2 e^{-\frac{1}{2} |\arg(s)|} < 1 \) is precisely the condition of theorem 1.2, which is thus proved.

**Remark 2.2.** The operator \( G \circ F \) is basically the Fourier transform of the kernel \( \chi_{(s, \infty)}(x)K_{\text{Ai}}(x, y) \); as such it is shown in [13] that its norm (for \( s \in \mathbb{R} \), of course!) is strictly less than one and hence the Fredholm determinant (2.8) cannot vanish if \( |\kappa| \leq 1 \). Moreover, it is well known that \( K_{\text{Ai}} \) on \( L^2(\mathbb{R}) \) is a positive operator and hence for \( \kappa \in i \mathbb{R} \) we also have that the determinant of \( \text{Id} + |\kappa|^2 K_{\text{Ai}} \) cannot vanish.

**Remark 2.3.** Since the operator \( G \circ F \) is trace-class, it has only discrete spectrum. Tracing the inequalities about the norms we see that when \( c > 0 \) the Cauchy operator has only the null-space and then has only continuous spectrum, because in Fourier space it is a multiplication operator by the function \( e^{-2\pi i t} \), \( t > 0 \) (nowhere constant). It then follows that the spectrum of \( G \circ F \) is strictly bounded by the value in (2.18) (the maximum eigenvalue cannot achieve the bound since it is at the boundary of the essential spectra of the various operators involved). Therefore we can improve the statement of theorem 1.2 by replacing the strict inequality by \( \geq \).

**Remark 2.4.** The kernel of proposition 2.1 is an ‘integrable kernel’ in the sense of [8] and it is thus immediately related to Riemann–Hilbert problem for a \( 2 \times 2 \) matrix; this problem turns out to be the very standard RHP in the isomonodromic approach to Painlevé II, see for example [7]. Thus the estimate of the kernel \( K_t \) is nothing but an estimate of the norm underlying singular-integral operator associated with said RHP. We could have phrased this paper in that language but we opted for the current presentation in the interest of brevity because it does not require introducing the setup of any Riemann–Hilbert problem.

3. Comments and conclusion

Although very simple, the result of theorem 1.2 seems not to be fully appreciated in the literature, to the knowledge of the author. It is also clear from the numeric evidence in figure 1 that the result is almost optimal; one may improve it only by a more careful estimate of the operators involved, which seems to me a rather difficult task.

We would also like to briefly comment on the special member \( \kappa = 1 \), the so-called Hastings–McLeod solution [6]; again numerical evidence strongly suggests that there are no poles also in the whole sector \( \argin(s) \in [2\pi/3, 4\pi/3] \) as remarked in the introduction (see the results of [5] summarized in theorem 1.1), this is known for sufficiently large \( |s| \) but only conjectured (and observed numerically) for \( |s| \) finite [11]. We can offer an informal argument...
Figure 1. The poles of the Ablowitz-Segur solution with various indicated values of $\kappa$ and the boundary of the pole-free region as per theorem 1.2. The numerics have been produced using the algorithm explained in [11] based on Padé approximants to the solution. It also shows that the poles on the left tend to minus infinity as $\kappa$ tends to the critical value $\kappa = 1$ (of course this is only a visual cue, not any proof).
below that indicates that one is not to expect an estimate bounding the spectrum of $K_s$ within an a priori domain that excludes the point 1.

The inspection of the numerics (for example at the top right frame in figure 1 ($\kappa = 0.3i$)), implies that the eigenvalues of the operator $K_s(\lambda, \mu)$ are not confined within the unit disk as $s$ ranges in the (left) sector $\arg s \in [2\pi/3, 4\pi/3]$. To see this recall that the poles that are clearly visible in the left sector of the indicated picture correspond to at least one eigenvalue (possibly with multiplicity) taking the value $1/\kappa$, which is larger than one in modulus.

On the other hand for real $s \in \mathbb{R}$ the operator $K_s$ has the same spectrum of the Airy kernel because it is unitarily equivalent to it: the results of [13] imply that all the eigenvalues $\lambda_j(s)$ are (simple and) confined in the interval $[0, 1)$. Therefore, by continuity, there is a region around the negative axis where all the eigenvalues are confined within the unit disk; on the other hand, the aforementioned numerical evidence indicates that these eigenvalues exit the unit disk as $s$ ranges in the left sector. In view of the above, one could expect that there are values of $s$ in the left sector at which some of the eigenvalues equal one: the fact that this is not the case for $|s|$ large is implied by theorem 1.1, and for arbitrary $|s|$ is suggested by the numerics. In conclusion, a proof of the absence of poles in the whole sector cannot follow from simple estimates directly on the norm of the operator $K_s$ alone.

Acknowledgments

The author would like to thank A Tovbis for illuminating discussions. The work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

[1] Ablowitz M J and Fokas A S 2003 Complex Variables: Introduction and Applications (Cambridge Texts in Applied Mathematics) 2nd edn (Cambridge: Cambridge University Press)
[2] Bertola M and Cafasso M 2011 The transition between the gap probabilities from the Pearcey to the Airy process—a Riemann–Hilbert approach Int. Math. Res. Not. at press
[3] Dubrovin B, Grava T and Klein C 2009 On universality of critical behavior in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the tritronquée solution to the Painlevé-I equation J. Nonlinear Sci. 19 57–94
[4] Fokas A S and Zhou X 1992 On the solvability of Painlevé II and IV Commun. Math. Phys. 144 601–22
[5] Fokas A S, Its A R, Kapaev A A and Novokshenov V Yu 2006 Painlevé Transcendents (Mathematical Surveys and Monographs) vol 128 (Providence, RI: American Mathematical Society)
[6] Hastings S P and McLeod J B 1980 A boundary value problem associated with the second Painlevé transcendent and the Korteweg–de Vries equation Arch. Ration. Mech. Anal. 73 31–51
[7] Its A R and Kapaev A A 2002 The Nonlinear Steepest Descent Approach to the Asymptotics of the Second Painlevé Transcendent in the Complex Domain (Progress in Mathematical Physics vol 23) (Boston, MA: Birkhäuser Boston)
[8] Its A R, Izergin A G, Korepin V E and Slavnov N A 1990 Differential equations for quantum correlation functions (Proc. Conf. on Yang–Baxter Equations, Conformal Invariance and Integrability in Statistical Mechanics and Field Theory) Internat. J. Modern Phys. B 4 1003–37
[9] Kapaev A A 2004 Quasi-linear Stokes phenomenon for the Painlevé first equation J. Phys. A: Math. Gen. 37 11149–67
[10] Masoero D 2010 Poles of intégrale tritronquée and anharmonic oscillators. A WKB approach J. Phys. A: Math. Theor. 43 095201
[11] Novokshenov V Yu 2009 Padé approximations for Painlevé I and II transcendents Teoret. Mat. Fiz. 159 515–26 (in Russian)
Novokshenov V Yu 2009 Theoret. Mat. Phys. 159 853–62 (Engl. transl.)
[12] Segur H and Ablowitz M J 1981 Asymptotic solutions of nonlinear evolution equations and a painlevé transcendent Physica. D 3 165–84
[13] Tracy C A and Widom H 1994 Level-spacing distributions and the Airy kernel Commun. Math. Phys. 159 151–74