Is There a Relationship between the Density of Primordial Black Holes in a Galaxy and the Rate of Cosmological Gamma-Ray Bursts?

Alexander Shatskiy *

*Astro Space Center, Lebedev Physical Institute, Russian Academy of Sciences, Russia
PACS numbers: 04.70.Bw, 98.70.Rz
DOI: 10.1134/S1063772906100015

Abstract
The rate of accretion of matter from a solar-type star onto a primordial black hole (PBH) that passes through it is calculated. The probability that a PBH is captured into an orbit around a star in a galaxy is found. The mean lifetime of the PBH in such an orbit and the rate of orbital captures of PBHs in the galaxy are calculated. It is shown that this rate does not depend on the mass of the PBH. This mechanism cannot make an appreciable contribution to the rate of observed gamma-ray bursts. The density of PBHs in the galaxy can reach a critical value - the density of the mass of dark matter in the galaxy.

1 INTRODUCTION
There have been several important developments in astronomy over the past two decades associated with the discovery of objects that are candidate black holes. Black holes with galactic masses ($10^6 \div 10^9 M_\odot$), stellar masses ($\sim 1 M_\odot$), and intermediate masses ($\sim 10^3 M_\odot$) have all been identified. Naturally, black holes with masses that are much less than a solar mass can be detected only indirectly (such as via gravitational microlensing). So-called primordial black holes (PBHs) are in this class. They can be formed only in the early stages of evolution of the Universe, since the natural evolution of a star does not permit the formation of black holes with masses less than a solar mass.

A black hole will unavoidably "devour" any matter in the immediate vicinity of its gravitational radius (more precisely, within about $3r_g$). Therefore, if a PBH is located inside a star, the life of the star will be shortened by some time. After the passage of this time, all the remaining matter of the star will unavoidably collapse into the black hole.

One consequence of the collapse of the star into such a PBH should be a powerful gamma-ray outburst, which is emitted by the remnants of the stellar material. It is possible that such gamma-ray bursts contribute to all the cosmological gamma-ray bursts that are observed in the Universe on a continual basis.

Knowing the rate of accretion of matter onto a PBH and the distribution of PBHs in space, we can estimate the rate of associated gamma-ray bursts and compare it with the observed rate, and also estimate the number of PBHs in a galaxy.

2 RATE OF ACCRETION OF A STAR ONTO A PBH
A model for accretion onto a black hole has been considered in detail in [1], [2], and we present here only the main conclusions of this theory.

We consider the hydrodynamical theory of accretion onto a black hole. The gravitational radius $r_g$ of the PBH with mass $m$ is assumed to be much smaller than the scale on which the self-gravitation of the star becomes stronger than the gravitation of the PBH. We denote this scale $R_{tg}$; note that it should be much smaller than the size of the star itself $R_\odot \approx 7 \cdot 10^{10}$ sm, (for a star of mass $M_\odot \approx 2 \cdot 10^{33}$ g).
We then have:

\[ r_g \ll R_{sg} \ll R_\odot, \quad R_{sg} = R_\odot \left( \frac{r_g}{R_g} \right)^{1/3}, \quad r_g = \frac{2Gm}{c^2}, \quad R_g = \frac{2GM_\odot}{c^2}. \]

(1)

The rate of accretion onto a PBH with mass \( m \) is equal to (the Bondi solution):

\[ \dot{m} = \frac{4\pi \lambda (Gm)^2 \rho_\odot}{c^3 \left( a^2 + \beta_p^2 \right)^{3/2}}. \]

(2)

Here, \( \lambda \) is a constant of order unity, \( c a \) is the sound speed in the star \( (c^2 a^2 = dP/d\rho_\odot) \), \( c \beta_p \) is the speed with which the PBH moves through the star (at the perigee of its orbit, \( a \ll \beta_p \)), \( \rho_\odot \) is the mean density of matter in the star. Thus, during its passage through the star, the PBH "digs a tunnel" in the stellar material with a cross-sectional area

\[ s = \frac{\lambda \pi r_g^2}{\beta_p (a^2 + \beta_p^2)^{3/2}} \approx \frac{\pi r_g^2}{\beta_p^2}. \]

(3)

The associated change in the star’s mass is

\[ \Delta m \approx \rho_\odot s R_\odot \approx m \cdot \frac{3r_g R_g}{4 \beta_p^4 R_\odot^2}. \]

(4)

During this accretion, the internal parameters of the star (dependence of the temperature and density on radius) can be taken during computations to be equal to the solar values from [3]. Questions related to the stationarity of the accretion are considered in Appendix 1.

3 CAPTURE OF A PBH IN A STELLAR ORBIT (ORBITAL CAPTURE)

Moving in a galaxy with virial speeds of the order of \( c \beta_\infty \approx 300 \text{ km/s} \ (\beta_\infty \approx 10^{-3}) \), a PBH has some probability to collide with stars. The cross section for such a collision \( S \) is \( \pi \) multiplied by the square of the maximum impact parameter at infinity, for which the PBH passes "within half" the radius of the star. The quantity \( S \) is determined by equating half the stellar radius and the distance from the star to the PBH at the perigee of its orbit for the motion of the PBH along a hyperbolic trajectory:

\[ S = \pi R_\odot^2 \left( \frac{1}{4} + \frac{R_g}{2R_\odot \beta_\infty^2} \right) \approx 2.4 \pi R_\odot^2. \]

(5)

Before the collision, the speed of the PBH is more than a third of the cosmic velocity for such a star. The speed at infinity \( \beta_\infty \) is related to the speed just before the collision (at perigee) by the formula

\[ \beta_p^2 = \beta_\infty^2 + \frac{R_g}{R_\odot}, \quad \rightarrow \quad \beta_p \approx 2.3 \beta_\infty. \]

(6)

Since momentum is conserved in collisions involving the PBH, its speed will decrease as its mass increases. Thus, its speed immediately after a collision will be lower than it was just before the collision. The collisions will continue until the PBH is captured into an orbit by a star, i.e., until the speed of the PBH turns out to be less than a third of the cosmic speed after a collision: \( \beta_3 = \sqrt{R_g/R_\odot} \approx 2.1 \beta_\infty. \)

Thus, the total change in the speed of the PBH required for orbital capture should be

\[ \Delta \beta = \beta_p - \beta_3 \approx 0.2 \beta_\infty. \]

(7)
In accordance with the conservation of momentum and expression (11), the change in speed during the collision process is determined by the formula

$$\Delta \beta_1 \approx \beta_p \frac{\Delta m}{m} \approx \frac{3 r_g R_g}{4 \beta_p^3 R_\odot^2} \approx \frac{r_g R_g}{16 R_\odot^2 \beta_\infty^3}.$$  

(8)

Accordingly, the number of collisions required for orbital capture is

$$k = \frac{\Delta \beta}{\Delta \beta_1} \approx 3.2 \frac{3 \beta_\infty^4 R_\odot^2}{R_g^2} \cdot \frac{R_g}{r_g} \approx 0.2 M_\odot. $$  

(9)

We can see that, in order for an orbital capture to occur as a result of the first collision, the mass of the PBH before the collision must

$$m \approx 0.2 M_\odot.$$  

In this case, the mass $m$ of the PBH will approximately double as a result of the collision.

4 RATE OF ORBITAL CAPTURES OF PBH IN A GALAXY

We can determine the mean time $T_1$ between collisions using the formula $S \beta_\infty c T_1 = V_0$, where $V_0 = R_0^3$ is the mean volume per star in the galaxy ($R_0 \approx 10$ light years $\approx 10^{19}$sm):

$$T_1 = \frac{V_0}{S \beta_\infty c} \approx 3.8 \cdot 10^{23} \text{years}.$$  

(10)

Thus, the time for the mean free path of a PBH (until its orbital capture) is $\tau = k \cdot T_1$. If the total number of uncaptured PBHs in the galaxy is $N_{bh}$, some number of them ($N_t$) will be captured by stars over a time $t$:

$$N_t = N_{bh} \cdot \left[1 - \exp(-t/\tau)\right] \approx N_{bh} \cdot t/\tau. $$  

(11)

The number of PBHs in the galaxy is given by the relation $N_{bh} = \kappa M_{tot}/m$, where $M_{tot} \approx 10^{46}$g is the total mass of the galaxy (assuming it is similar to the MilkyWay) and $\kappa$ is a coefficient determining the mass fraction of PBHs in the galaxy ($0 < \kappa < 1$). We can see that (11) does not depend on the mass of the PBH. The rate of orbital captures of PBHs in the galaxy is then

$$\frac{dN_t}{dt} = \dot{N}_t \approx \frac{2.4 \kappa c R_g}{V_0 \beta_\infty^3} \frac{2GM_{tot}}{c^2} \approx \kappa \cdot \frac{1}{10^6 \text{years}}.$$  

(12)

Thus, the rate of orbital capture can reach values of the order of one capture every million years in a MilkyWay-type galaxy.

5 LIFETIME OF A PBH IN ORBIT AROUND A STAR

If the lifetime of a PBH in orbit is less than the lifetime of the star ($\sim 10^9$yrs), PBHs will not be accumulated in the galaxy.

The lifetime of a PBH in orbit is determined by the increase of its mass by a factor of two due to accretion. The corresponding number of revolutions $n$ of the PBH around the star is given by the relation

$$n = \frac{m}{\Delta m} \approx \frac{37 \beta_\infty^4 R_\odot^2}{R_g r_g} \approx \frac{2M_\odot}{m}. $$  

(13)

The period of revolution about the star $\tau_0$ is determined from the assumption that the major axis of the orbit is $\sim 0.5R_0$ (half the mean distance between stars):

$$\tau_0 \approx \pi R_0^{3/2}/(c \sqrt{R_g}) \approx 1.7 \cdot 10^8 \text{yrs}. $$  

(14)
We thus obtain for the lifetime in the orbit
\[ T_0 = n \tau_0 \approx \frac{117 \beta^4 \infty R^2 \circ}{r_g c} \cdot \left( \frac{R_g}{r_g} \right) \cdot \left( \frac{R_0}{R_g} \right)^{3/2} \approx 3.5 \cdot \frac{M_\odot}{m} \cdot 10^8 \text{years}. \] (15)

The masses of PBHs can be constrained from above by data available from gravitational microlensing observations: \( m < 0.1 M_\odot \) (see [4]). Therefore, in order for the time \( T_0 \) to be less than the lifetime of the Universe \( (\sim 15 \cdot 10^9 \text{yrs}) \), the mass of a PBH must lie in the very narrow range of \( 0.03 M_\odot \) to \( 0.1 M_\odot \). In this case, gamma-ray bursts induced by the collapse of a star into a PBH will occur from time to time in a galaxy.

However, even in this case, this mechanism will not make an appreciable contribution to the observed gamma-ray bursts in a galaxy, since the rate of such events (12) is much less than the mean rate of gamma-ray bursts in a galaxy \( \sim 0.001 \div 0.01 \text{ yrs}^{-1} \).

It follows from the above analysis that PBHs can comprise any mass fraction of the dark matter in a galaxy.

6 Appendixes

6.1 TIME FOR ESTABLISHING A QUASI-STATIONARY ACCRETION REGIME AS A PBH IMPACTS A STAR

The relative speed between the PBH and the particles in the stellar medium changes as a PBH moves through a star uniformly and linearly. Consequently, there are relative accelerations, which correspond to inertial forces. If we denote the coordinate of the PBH along its trajectory \( y = c \beta_p t \) and the minimum distance between the particles in the stellar medium and the PBH trajectory \( x \), this acceleration \( g \) can be expressed in terms of \( x \) and \( y \) as follows:
\[
|g| = \frac{(c \beta_p x)^2}{(x^2 + y^2)^{3/2}}.
\] (16)

Here, it is convenient to assign the initial value of \( y \) to be at a point that is closest to a particle of the medium. The condition for capture of a particle in the medium by the passing PBH is given by the inequality
\[
|g| < \frac{Gm}{x^2 + y^2}.
\] (17)

This corresponds to the dominance of the gravitational force over the inertial force, and must be satisfied at any moment in time. We thus obtain
\[
x < x_0 = \frac{r_g}{2 \beta^2 p}.
\] (18)

This qualitative expression approximately corresponds to the cross section \( s = \pi x_0^2 \) [see (3)].

The relaxation time for the stellar material during a collision with a PBH is determined (to order of magnitude) by the free-fall time for a particle to fall from a distance \( x_0 \) onto the PBH:
\[ t_0 = \frac{r_g}{3 \sqrt{2} c \beta_p^2}. \] (19)

We have for a mass of \( m < 0.1 M_\odot \) the time \( t_0 < \sim 10 \text{s} \). Thus, this time is much shorter than the time for the passage of the PBH through the internal layers of the star: \( \sim R_\odot / (c \beta_p) \) \( \sim 1000 \text{s} \). Consequently, the accretion process very rapidly becomes quasi-stationary.
6.2 STATISTICS OF THE MASS AND SPEED DISTRIBUTIONS OF PBHS IN THE UNIVERSE

The nature of PBHs in the early Universe (if indeed they exist) is not clear, and there are also no precise models describing their statistical distribution. Moreover, we cannot consider the early Universe to be in some kind of thermodynamic equilibrium. Therefore, the arguments presented below do not pretend to be the only valid description of the statistics of PBHs, and this has led us to present them in a separate appendix.

Let us consider a system of a larger number of PBHs in a spatial volume $V$. We will consider this volume to be sufficiently large so that we can apply a statistical analysis to the PBHs. Their masses are bounded from below by the relation $m_{\min} = E_{\min}/c^2$ because of the condition for the Hawking evaporation of black holes.

Let us assume that the PBHs have a canonical Gibbs distribution [5]. Thus, the probability density for the existence of a PBH with a mass $m$ and a spatial velocity $v$ is

$$w = c_1 \exp \left\{ -\frac{y\alpha}{\sqrt{1-\beta^2}} \right\}, \quad (20)$$

where $y \equiv mc^2/E_{\min}$ is the dimensionless rest energy of a PBH, $\alpha \equiv E_{\min}/T$ is the dimensionless inverse temperature of the system, $\beta = v/c$ the speed of the PBHs in units of the speed of light, and $c_1$ is a normalization constant.

The normalization condition is

$$\int_1^{\infty} dy \int_0^1 4\pi \beta^2 \ d\beta \ w \int_0^V d^3r = 1. \quad (21)$$

We now introduce a number of functions that depend on $\alpha$:

$$I_n(\alpha) = \int_0^1 4\pi \beta^2 \ d\beta \left[ \sqrt{1-\beta^2} \right]^n \exp \left\{ -\frac{\alpha}{\sqrt{1-\beta^2}} \right\}, \quad I'_n(\alpha) = -I_{n-1}(\alpha). \quad (22)$$

The condition (21) then takes the form

$$c_1 VI_1 = \alpha. \quad (23)$$

We now introduce the mean energy of the PBH:

$$U = \left\langle \frac{y}{\sqrt{1-\beta^2}} \right\rangle = \int_1^{\infty} dy \int_0^1 4\pi \beta^2 \ d\beta \frac{y}{\sqrt{1-\beta^2}} c_1 V \exp \left\{ -\frac{\alpha y}{\sqrt{1-\beta^2}} \right\}. \quad (24)$$

After integrating over $y$, this expression can be written in the form

$$U \alpha^2 = c_1 V \left[ I_0 \alpha + I_1 \right]. \quad (25)$$

The energy and entropy of any system are additive quantities, equal to the sums of the contributions from their individual parts. Black holes should not be an exception to this behavior. Therefore, the total energy and entropy of the system will have the forms

$$U_{\text{tot}} = \int_1^{\infty} dy \int_0^1 4\pi \beta^2 \ d\beta N \frac{y}{\sqrt{1-\beta^2}} c_1 V \exp \left\{ -\frac{\alpha y}{\sqrt{1-\beta^2}} \right\} = NU. \quad (26)$$

$$S_{\text{tot}} = \int_1^{\infty} dy \int_0^1 4\pi \beta^2 \ d\beta N S_0 c_1 V \exp \left\{ -\frac{\alpha y}{\sqrt{1-\beta^2}} \right\} = N \langle S_0 \rangle. \quad (27)$$
where \( N \) is the total number of PBHs in the system, \( S_0 = c_2 y^2 \) is the entropy of a PBH with mass \( m \), and \( c_2 \) is a dimensionless coefficient \[4\]:

\[
c_2 = \frac{4\pi G m_{\text{min}}^2}{\hbar c} \gg 1 \quad \text{at} \quad m_{\text{min}} \gg 10^{-5} \text{g}. \tag{28}\]

Thus, \( S_{\text{tot}} = \langle S_0 \rangle U_{\text{tot}}/U \). Using the thermodynamic expression for the temperature, \( 1/T = dS_{\text{tot}}/dE_{\text{tot}} \) (see \[5\]), we obtain

\[
\alpha = \frac{dS_{\text{tot}}}{dU_{\text{tot}}} = \int_1^\infty dy \int_0^1 4\pi \beta^2 d\beta \frac{c_1 V c_2}{U} y^2 \exp \left\{ -\frac{\alpha y}{\sqrt{1 - \beta^2}} \right\}. \tag{29}\]

Integrating over \( y \), we obtain

\[
U \alpha^4 = c_2 c_1 V \left[ 2I_3 + 2\alpha I_2 + \alpha^2 I_1 \right]. \tag{30}\]

Equations (23), (25) and (30) can be used to find \( \alpha \) and \( U \), e.g., by multiplying (25) by \( \alpha^2 \) and substituting this into (30). We differentiate the resulting expression with respect to \( \alpha \) to obtain

\[
\alpha^2 I_{-1} - \alpha(c_2 + 2)I_0 - 2I_1 = 0 \tag{31}\]

This is a transcendental equation in \( \alpha \). However, if \( c_2 \gg 1 \), then \( \alpha \gg 1 \). We will use this fact and multiply the function \( I_n \) by the factor \( \exp\{\alpha\}/(4\pi) \). Denote the new functions \( \tilde{I}_n \):

\[
\tilde{I}_n(\alpha) = \int_0^1 \beta^2 d\beta \left[ \frac{1}{\sqrt{1 - \beta^2}} \right]^n \exp \left\{ \alpha \left( 1 - \frac{1}{\sqrt{1 - \beta^2}} \right) \right\} \approx \exp\{-\alpha\beta_0^2/2\} \cdot \beta_0^3 \cdot \left( 1 - \frac{\beta_0^2 n}{2} \right). \tag{32}\]

In \( \tilde{I}_n \), the integral is already rapidly cut off by the exponential when \( \beta_0^2 \approx 2\kappa/\alpha \ll 1 \), where \( \kappa \) is a coefficient of order 10. Thus, the functions obtained can be expanded in a series in \( \beta_0 \), keeping only the first term. The function \( \tilde{I}_n \) preserves the form of (31). Substituting the asymptotic (32) into (31) yields

\[
\alpha^2 - \alpha(c_2 + 2 - \kappa) - 2 + 2\kappa/\alpha = 0. \tag{33}\]

We find in the main approximation

\[
\alpha \approx c_2, \quad U \approx 1 + 5/(4\alpha) \approx 1. \tag{34}\]

Hence, the temperature of the Gibbs distribution of PBHs is of order the Hawking temperature \( (T_H) \) for \( m_{\text{min}} \):

\[
T \equiv \frac{E_{\text{min}}}{\alpha c_2} \approx \frac{E_{\text{min}}}{4\pi G m_{\text{min}}} = \frac{\hbar c^3}{4\pi G m_{\text{min}}} = 2T_H = 2, 4 \cdot \left( \frac{10^{26}\text{g}}{m_{\text{min}}} \right) K^\circ \tag{35}\]

It follows that the bulk of PBHs have very narrow ranges of their mass (from \( m_{\text{min}} \) to \( m_{\text{min}}(1 + 1/\alpha) \)) and velocity:

\[
\text{from 0 to } \beta_0 \sim \frac{10^{-4}\text{g}}{m_{\text{min}}} \ll \beta_\infty \ll 1. \tag{36}\]

The probability density for the PBHs falls off exponentially outside these intervals. With such a low value for \( \beta_0 \), its contribution to the motion of the PBH can be neglected compared to the virial speeds in the galaxy, which are determined by the gravitation of all its bodies.

Thus, to a high degree of accuracy, this model for PBHs represents an ideal gas with particles of mass \( m_{\text{min}} \) and velocities of the order of the virial velocity in the galaxy.
ACKNOWLEDGMENTS

The author thanks N.S. Kardashev, V.N. Lukash, B.V. Komberg, D.A. Kompaneits, and other participants in seminars at which this work was discussed, refined, and augmented.

The work was supported by the Program "Non-stationary Phenomena in Astronomy, 2005" (no. NSh-1653.2003.2) and the Russian Foundation for Basic Research (project nos. 05-02-17377, 05-02-16987-a).

References

[1] H. Bondi, Mon. Not. R. Astron. Soc. 112, 195 (1952).

[2] S. L. Shapiro and S. A. Teukolsky, Black Holes, White Dwarfs, and Neutron Stars: the Physics of Compact Objects (Wiley, New York, 1983; Mir, Moscow, 1985), Part 2.

[3] Current Problems in Stellar Physics and Evolution, Ed. by A. G. Masevich (Nauka, Moscow, 1989) [in Russian].

[4] Black Holes: the Membrane Paradigm, Ed. by K. S. Thorne, R. H. Price, and D. A. Macdonald (Yale Univ. Press, New Haven, 1986, Mir, Moscow, 1988).

[5] L.D. Landau and E.M. Lifshitz, Course of Theoretical Physics, Vol. 5: Statistical Physics (Nauka, Moscow, 1995; Pergamon, Oxford, 1980), Part 1.

[6] A. F. Zakharov, astro-ph/0403619 (2004).