ON BMO AND CARLESON MEASURES ON RIEMANNIAN MANIFOLDS

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Abstract. Let $M$ be a Riemannian manifold with a metric such that the manifold is Ahlfors-regular and the Ricci curvature is bounded from below. We also assume a bound on the gradient of the heat kernel. We characterize BMO-functions $u : M \to \mathbb{R}$ by a Carleson measure condition of their $\sigma$-harmonic extension $U : M \times (0, \infty) \to \mathbb{R}$. We make crucial use of a $T(b)$ theorem proved in [11].

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1. Introduction

It is a classical result that in Euclidean space there is a relation between $BMO$-functions $u : \mathbb{R}^n \to \mathbb{R}$ and Carleson measures in $\mathbb{R}^{n+1}_+$. Precisely, the following statement can be found, e.g., in [14, IV, §4.3, Theorem 3, pp.159 and §4.4.3].
Theorem 1.1. Let $u \in C^\infty_c(\mathbb{R}^n)$ and denote by $U(x,t) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ the harmonic extension, i.e. the unique solution to the following equation:

\[
\begin{cases}
\Delta_x U \equiv (\partial_{tt} + \Delta_x)U = 0 & \text{in } \mathbb{R}^{n+1}_+ \\
U = u & \text{on } \mathbb{R}^n \times \{0\} \\
\lim_{|x,t| \to \infty} U(x,t) = 0
\end{cases}
\]

Then the following two $BMO$-seminorms are equivalent: The integral one

\[
[u]_{BMO} = \sup_{B \subset \mathbb{R}^n} |B|^{-1} \int_B |u - (u)_B|,
\]

and the Carleson-measure version

\[
[u]_{\tilde{BMO}} := \left( \sup_{B \subset \mathbb{R}^n} |B|^{-1} \int_{T(B)} t|\nabla_{(x,t)} U(x,t)|^2 \, dx \, dt \right)^{\frac{1}{2}}.
\]

Here, $(u)_B = |B|^{-1} \int_B u$ and $T(B)$ is the tent in $\mathbb{R}^n \times (0,\infty)$ over the ball $B$, namely if $B = B(x_0,r)$ then $T(B) = \{(x,t) : |x-x_0| < r-t\}$.

While relations between Carleson measures and certain extensions of functions have been extended to spaces of homogeneous type (see e.g. [10, 16, 7]), these extensions are usually of a potential type (with conditions on kernel decay). The main drawback is that these extensions in general do not satisfy an equation such as (1.1). On the other hand, in applications such as proving sharp commutator estimates, see [12], it is beneficial (and maybe even crucial) to have the extension satisfying certain PDEs such as (1.1) (or more generally satisfying a Dirichlet-to-Neumann principle [4]).

The aim of the present work is to prove an equivalence result like in the previous theorem involving a natural PDE-extension to the half-space in a rather general geometric framework. Let $(\mathcal{M}, g)$ be an $n$-dimensional smooth Riemannian manifold which is also Ahlfors regular, meaning that the measure of a ball of radius $r$ is (uniformly) comparable to $r^n$. By $\Delta_{\mathcal{M}}$ we denote the Laplace-Beltrami operator on $\mathcal{M}$. We equip $\mathcal{M}$ with the Carnot-Carathéodory metric $d$. Without further assumptions on the manifold it seems implausible that e.g. harmonic extensions satisfy a statement such as Theorem 1.1. In Theorem 1.3 we introduce such assumptions on the manifold and its heat kernel.

Besides the classical harmonic extension we also take $\sigma$-harmonic extensions into account. To define them we follow the semigroup representation (cf. [15] for the Euclidean analogue but as stated in the latter it extends to much more general contexts).
Let us clarify the differential operators that we use. Denote by \( d \) the exterior derivative and \( \ast \) the Hodge operator. We define the Laplace-Beltrami operator (or Laplace-de-Rham operator, which are the same for us since they act only on functions/0-forms) by \( \Delta = \Delta_M := \ast d \ast d \), and the gradient of a smooth function \( f \) by \( \nabla f = \nabla_M f = df \), (or depending on the context \( \nabla f = (df)^\sharp \)). With this setup, we have (with standard abuse of notation) \( |\nabla f|^2 = g(\nabla f, \nabla f) \) and

\[
\langle \nabla f, \nabla h \rangle_{L^2} = \int_M \ast df \wedge dh
\]
i.e. \( \|\nabla f\|_{L^2}^2 = \int_M g(\nabla f, \nabla f) \, dx = \int_M |\nabla f|^2 \, dx. \)

The \( \sigma \)-harmonic extension is defined as follows.

**Definition 1.2** \((\sigma\text{-Harmonic extension to } M \times (0, \infty))\). Let \( M \) be as above and \( 0 < \sigma < 1 \). For \( u \in C^\infty_c(M) \) the \( \sigma \)-harmonic extension \( U : M \times [0, \infty) \to \mathbb{R} \) is the solution to

\[
\begin{cases}
\Delta_M U + \frac{1-2\sigma}{t} \partial_t U + \partial_x U = 0 & \text{in } M \times (0, \infty) \\
U(x, 0) = u(x) & \text{in } M \\
\lim_{|t| \to \infty} U(x, t) = 0.
\end{cases}
\]

This solution is formally given by

\[
U(x, t) = \frac{1}{4\Gamma(\sigma)} t^{2\sigma} \int_0^\infty e^{s\Delta_M} u(x) e^{-\frac{s^2}{4t}} \frac{ds}{s^{1+\sigma}},
\]

and explicitly one has

\[
U(x, t) = \frac{1}{4\Gamma(\sigma)} \int_0^\infty \int_M p(x, y, s) u(y) \, dy \, t^{2\sigma} e^{-\frac{s^2}{4t}} \frac{ds}{s^{1+\sigma}},
\]

where \( p(x, y, s) \) is the heat kernel for \( M \), i.e.

\[
(1.2) \quad \begin{cases}
(\partial_t - \Delta_x) p(x, y, s) = 0 & \text{for all } x, y \in M \text{ and } s > 0 \\
p(x, y, 0) = \delta_{x,y}.
\end{cases}
\]

The previous definition is not explicitly stated in [15] but it is easy to check that the semi-group approach automatically carries over to such a geometric setting under very weak assumptions on the manifold, see Section A. For more information and properties about the heat kernel, see [8].
We define the following semi-norms: Let $U(x,t)$ be the $\sigma$-harmonic extension of $u$ to $\mathcal{M} \times (0, \infty)$. Denote the usual $\text{BMO}$-norm as

$$[u]_{\text{BMO}(\mathcal{M})} := \sup_{B \subset \mathcal{M}} |B|^{-1} \int_{B} |u - (u)_B|,$$

where the supremum is taken over balls $B$. Furthermore, we define a notion of $\text{BMO}$ in terms of the $\sigma$-harmonic extension and Carleson measures, namely

$$(1.3) \quad [u]_{\tilde{\text{BMO}}(\mathcal{M})} \equiv [u]_{\text{BMO}_{\sigma}(\mathcal{M})} := \left( \sup_{B \subset \mathcal{M}} |B|^{-1} \int_{T(B)} t |\nabla_{x,t} U(x,t)|^2 \, dx \, dt \right)^{1/2}.$$

Again, $T(B)$ is the tent in $\mathcal{M} \times (0, \infty)$ over the ball $B$, namely if $B = B(x_0, r)$ then $T(B) = \{(x,t) : d(x,x_0) < r-t\}$.

Our main result is the following:

**Theorem 1.3.** Let $\mathcal{M}$ be a complete path-connected and Ahlfors regular manifold without boundary, such that the Ricci curvature is bounded from below.

If moreover the heat kernel of $\mathcal{M}$ satisfies

$$\sup_{x,y \in \mathcal{M}} |\nabla_p(x,y,t)| \lesssim t^{-\frac{n+1}{2}},$$

then for any $0 < \sigma < 1$ the semi-norms of $\text{BMO}$ defined above are equivalent, i.e for any $u \in C_\infty(\mathcal{M})$ we have

$$[u]_{\text{BMO}(\mathcal{M})} \approx [u]_{\tilde{\text{BMO}}(\mathcal{M})}.$$

The remainder of this paper will be as follows: In Section 2 we introduce the notion of an admissible manifold, which is more general than the one in Theorem 1.3, but more complicated to check. In Section 3 we use the $T(b)$-theorem from [11] to obtain square function estimates. In Section 4 we prove Theorem 1.3. Computations concerning the $\sigma$-harmonic extensions are moved to the appendix, Section A.

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2. Admissible manifolds and heat kernel estimates

The proof of Theorem 1.3 is based on heat kernel estimates which allow to deduce a $T(b)$ theorem, see Section 3. We exhibit a large class of Riemannian manifolds for which our theorem applies. We first define a general setting in which our theory actually works.

**Definition 2.1 (Admissible Manifolds).** A manifold $\mathcal{M}$ is said to be admissible, if it is complete, path-connected, Ahlfors regular, without boundary and its heat kernel $p(x, y, t)$ satisfies the following conditions: For every $\sigma > 0$ there exists $\nu > 0$ such that for all $t > 0$, $x, y \in \mathcal{M}$:

\begin{align}
(2.1) & \int_0^\infty p(x, y, t^2 s) \left(1 + \frac{1}{s}\right) e^{-\frac{1}{4t^2}} \frac{ds}{s^{1+\sigma}} \lesssim \frac{t^{\nu}}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} ,

(2.2) & \int_0^\infty |\nabla_x p(x, y, t^2 s)| e^{-\frac{1}{4t^2}} \frac{ds}{s^{1+\sigma}} \lesssim \frac{t^{\nu-1}}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} ,

(2.3) & \int_0^\infty |\nabla_y p(x, y, t^2 s)| \left(1 + \frac{1}{s}\right) e^{-\frac{1}{4t^2}} \frac{ds}{s^{1+\sigma}} \lesssim \frac{t^{\nu-1}}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} ,

(2.4) & \int_0^\infty |\nabla_x \nabla_y p(x, y, t^2 s)| e^{-\frac{1}{4t^2}} \frac{ds}{s^{1+\sigma}} \lesssim \frac{t^{\nu-2}}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} .
\end{align}

The following lemma is providing a more treatable class of admissible manifolds.

**Lemma 2.2.** Let $\mathcal{M}$ be a complete path-connected and Ahlfors regular manifold without boundary. If the heat kernel satisfies

\begin{align}
(2.5) & p(x, y, t) \lesssim t^{-\frac{n}{2}} e^{-c\frac{d(x,y)^2}{t}} ,

(2.6) & |\nabla p(x, y, t)| \lesssim t^{-\frac{n+1}{2}} e^{-c\frac{d(x,y)^2}{t}} ,

(2.7) & |\nabla_x \nabla_y p(x, y, t)| \lesssim \frac{1}{t^{n+1}} e^{-c\frac{d(x,y)^2}{t}} ,
\end{align}

for some constant $c > 0$, then $\mathcal{M}$ is admissible with $\nu = 2\sigma$. 
Proof. The geometry of the manifold is the same as in Definition 2.1, so we only have to check the conditions (2.1) to (2.4).

As for (2.1): Using the change of variables, we see that

\[
\int_0^\infty p(x, y, t^2 s) \left(1 + \frac{1}{s}\right) e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}} = t^{2+2\sigma-2} \int_0^\infty p(x, y, s) \left(1 + \frac{t^2}{s}\right) e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}} \lesssim t^{2\sigma} \int_0^\infty \left(1 + \frac{t^2}{s}\right) e^{-\frac{\text{d}(x, y)^2}{4s}} e^{-\frac{t^2}{4s}} \frac{ds}{s^{\frac{1}{2}+\sigma}}.
\]

In the last step we employed (2.5). Set \(\Lambda := d(x, y)^2 + t^2\), and making the change of variables \(s \mapsto \frac{\Lambda}{s}\), we see

\[
\ldots = t^{2\sigma} \int_0^\infty \left(1 + \frac{t^2}{\Lambda s}\right) e^{-\frac{\text{d}(x, y)^2}{4s}} \frac{ds}{s^{\frac{1}{2}+\sigma}} \lesssim t^{2\sigma} \int_0^\infty \left(1 + \frac{t^2}{s}\right) e^{-\frac{s}{4s}} \frac{ds}{s^{\frac{1}{2}+\sigma}}.
\]

The integral can be estimated by a multiple of \(\Gamma\left(\frac{n}{2} + \sigma\right) + \Gamma\left(\frac{n}{2} + \sigma + 1\right)\), so it is finite. This shows (2.1).

As for (2.2) and for (2.3): It suffices to show (2.3), since in our setting \(\nabla_x p(x, y, t) = \nabla_y p(x, y, t)\) by the symmetry of the heat kernel. Using the gradient estimate (2.6) we deduce

\[
\int_0^\infty |\nabla_y p(x, y, t^2 s)| \left(1 + \frac{1}{s}\right) e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}} = t^{2+2\sigma-2} \int_0^\infty |\nabla_y p(x, y, s)| \left(1 + \frac{t^2}{s}\right) e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}} \lesssim t^{2\sigma} \int_0^\infty \left(1 + \frac{t^2}{s}\right) e^{-\frac{\text{d}(x, y)^2}{s}} e^{-\frac{t^2}{4s}} \frac{ds}{s^{\frac{n+1}{2}+1+\sigma}}.
\]

As in (2.1), setting \(\Lambda := d(x, y)^2 + t^2\), and making the change of variables \(s \mapsto \frac{\Lambda}{s}\), we see

\[
\ldots = t^{2\sigma} \int_0^\infty \left(1 + \frac{t^2}{\Lambda s}\right) e^{-\frac{\text{d}(x, y)^2}{s}} \frac{ds}{s^{\frac{n+1}{2}+\sigma}} \lesssim t^{2\sigma} \int_0^\infty \left(1 + \frac{1}{s}\right) e^{-\frac{\Lambda}{4s}} \frac{ds}{s^{\frac{n+1}{2}+1+\sigma}}.
\]

The integral can be estimated by a multiple of \(\Gamma\left(\frac{n+1}{2} + \sigma\right) + \Gamma\left(\frac{n+1}{2} + \sigma + 1\right)\), so it is finite. Moreover, since \(\Lambda \geq t^2\), we get in the end

\[
\int_0^\infty |\nabla_y p(x, y, t^2 s)| \left(1 + \frac{1}{s}\right) e^{-\frac{1}{4s}} \frac{ds}{s^{1+\sigma}} \lesssim t^{2\sigma-1} \frac{\Lambda^{\frac{n+1}{2}}}{\Lambda^{\frac{n+1}{2}}}.
\]

This shows (2.3), and hence also (2.2).
As for (2.4): We proceed as in (2.1) to (2.3). Using the estimate (2.7), we get
\[ \int_0^\infty |\nabla_x \nabla_y p(x, y, t^2 s)| e^{-\frac{1}{4} s} \frac{ds}{s^{1+\sigma}} = t^{2+2\sigma-2} \int_0^\infty |\nabla_x \nabla_y p(x, y, s)| e^{-\frac{t^2}{4} s} \frac{ds}{s^{1+\sigma}} \lesssim t^{2\sigma} \int_0^\infty e^{-c \frac{d(x, y)^2}{s}} e^{-\frac{t^2}{4} s} \frac{ds}{s^{\frac{n}{2}+1+\sigma}}. \]

As before, setting \( \Lambda := d(x, y)^2 + t^2 \), and making the change of variables \( s \mapsto \Lambda s \), we see
\[ \ldots = t^{2\sigma} \int_0^\infty e^{-\frac{t^2}{4} s} \frac{ds}{s^{\frac{n}{2}+1+\sigma}} \Lambda^{\frac{n+2}{2}+\sigma} \lesssim \frac{t^{2\sigma} \Lambda^{\frac{n+2}{2}+\sigma}}{\Lambda^2} \int_0^\infty e^{-\frac{t^2}{4} s} \frac{ds}{s^{\frac{n}{2}+1+\sigma}}. \]
The integral can be rewritten as a multiple of \( \Gamma(\frac{n+2}{2}+\sigma) \), so it is finite. Moreover, since \( \Lambda \geq t^2 \), we get in the end
\[ \int_0^\infty |\nabla_x \nabla_y p(x, y, t^2 s)| e^{-\frac{1}{4} s} \frac{ds}{s^{1+\sigma}} \lesssim \frac{t^{2\sigma-2}}{\Lambda^\frac{n+2}{2}}. \]
This finishes the proof. \( \square \)

**Corollary 2.3.** Let \( \mathcal{M} \) be as in Theorem 1.3. Then \( \mathcal{M} \) is admissible.

**Proof.** The statement follows from known zero-order and first-order bounds on the heat kernel on \( \mathcal{M} \) which we recall below.

By the curvature assumption we have (2.5), see [13, Corollary 3.1], see also [3, Theorem 2.34].

By [6, Theorem 4.9] the assumption on the heat kernel together with (2.5) implies
\[ |\nabla p(x, y, t)| \lesssim t^{-\frac{n+1}{2}} \left( 1 + \frac{d^2(x, y)}{t} \right) e^{-c \frac{d(x, y)^2}{t}}. \]
Since \( (1+|x|) e^{-|x|} \leq C e^{-\frac{1}{2}|x|} \) this readily implies (2.6). Recall that the heat kernel is symmetric, so the gradient estimate holds both for \( \nabla_x \) and \( \nabla_y \).

For (2.7) we use the semi-group property of the heat kernel, i.e.
\[ p(x, y, 2t) = \int_{\mathcal{M}} p(x, z, t) p(y, z, t) \, dz. \]
Keep in mind, that if $a \leq b + c$, then $a^2 \leq 2(b^2 + c^2)$. So using Hölder’s inequality, we arrive at

\begin{equation}
(2.8)
e^{\frac{d(x,y)^2}{At}} |\nabla_x \nabla_y p(x, y, 2t)| \leq \int_{\mathcal{M}} e^{\frac{d(x, z)^2}{At}} |\nabla_x p(x, z, t)| e^{\frac{2d(y,z)^2}{At}} |\nabla_y p(y, z, t)| \, dz
\leq \left( \int_{\mathcal{M}} e^{\frac{d(x, z)^2}{At}} |\nabla_x p(x, z, t)|^2 \, dz \right)^{\frac{1}{2}} \left( \int_{\mathcal{M}} e^{\frac{2d(y,z)^2}{At}} |\nabla_y p(y, z, t)|^2 \, dz \right)^{\frac{1}{2}}
\leq \left( \int_{\mathcal{M}} e^{\frac{d(x, z)^2}{At}} \frac{1}{t^{n+1}} e^{-c\frac{d(x, z)^2}{t}} \, dz \right)^{\frac{1}{2}}
\times \left( \int_{\mathcal{M}} e^{\frac{2d(y,z)^2}{At}} \frac{1}{t^{n+1}} e^{-c\frac{d(y,z)^2}{t}} \, dz \right)^{\frac{1}{2}}
= \frac{1}{t^{n+1}} \int_{\mathcal{M}} e^{\frac{d(x, z)^2}{At}} e^{-c\frac{d(x, z)^2}{t}} \, dz.
\end{equation}

We now choose $A$ so big, such that $\frac{4}{A} < c$ in the equation above, which means we look for an estimate of the form

\[
\int_{\mathcal{M}} e^{-\frac{d(x, z)^2}{t}} \, dz \lesssim t^n,
\]

where $c > 0$. Let $B_0 = B(x, \sqrt{t})$ and let $B_k = B(x, 2^k \sqrt{t}) \setminus B(x, 2^{k-1} \sqrt{t})$. Then we have $\mathcal{M} = \cup B_k$. Furthermore, it holds

\[
\int_{B_0} e^{-\frac{d(x, z)^2}{t}} \, dz \lesssim |B_0| \lesssim t^n
\]

by the Ahlfors regularity. Moreover, it holds

\[
\int_{B_k} e^{-\frac{d(x, z)^2}{t}} \, dz \lesssim 2^{nk} e^{-c2^{k-1}} t^\frac{n}{2}
\]

again by the Ahlfors regularity. Together, we have

\[
\int_{\mathcal{M}} e^{-\frac{d(x, z)^2}{t}} \, dz \lesssim \sum_{k=0}^{\infty} \int_{B_k} e^{-\frac{d(x, z)^2}{t}} \, dz \lesssim t^n \sum_{k=0}^{\infty} 2^{nk} e^{-c2^{k-1}} \lesssim t^n.
\]

This estimate together with (2.8) gives (2.7) as desired. \hfill \Box

**Remark 2.4.** The previous results, thanks to [2], extend straightforwardly to Lie groups with polynomial volume (notice that these are spaces of homogeneous type in the sense of Coifman and Weiss [5]).
3. \(T(b)\)-Theorem and square function estimates

We use the following important version of the local \(T(b)\)-theorem on manifolds, which is proven in much greater generality in [11, Theorem 3.7.]. It allows us to pass from local estimates in small balls of \(\mathcal{M}\) (i.e. essentially the Euclidean space) to global estimates.

**Theorem 3.1.** Let \(\mathcal{M}\) be an admissible manifold and let \(T\) be an operator, acting on functions \(f : \mathcal{M} \to \mathbb{R}\) via

\[
Tf(x, t) := \int_{\mathcal{M}} \kappa(x, y, t) f(y) \, dy,
\]

where \(\kappa : \mathcal{M} \times \mathcal{M} \times (0, \infty) \to \mathbb{R}\) is integrable and satisfies

\[
\int_{\mathcal{M}} \kappa(x, y, t) \, dy = 0 \quad \text{for all } x \in \mathcal{M}, \ t > 0,
\]

\begin{align*}
(3.1) \quad |\kappa(x, y, t)| & \lesssim \frac{t^\nu}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} \quad \text{for all } x, y \in \mathcal{M}, \ t > 0 \\
(3.2) \quad \frac{|\kappa(x, y_1, t) - \kappa(x, y_2, t)|}{d(y_1, y_2)} & \lesssim \frac{t^{\nu - 1}}{(d(x, y_1)^2 + t^2)^{\frac{n+\nu}{2}}} \quad \text{for all } d(y_1, y_2) \leq \frac{1}{2}(d(x, y_1)^2 + t^2)^{\frac{1}{2}}.
\end{align*}

Then,

\[
\left( \int_\mathcal{M} \int_0^\infty |Tf(x, t)|^2 \frac{dt}{t} \, dx \right)^{\frac{1}{2}} \lesssim \left( \int_\mathcal{M} |f(x)|^2 \, dx \right)^{\frac{1}{2}}.
\]

**Proof.** All conditions in [11, Theorem 3.7.] are satisfied, once we confirm 1.,2.,3.: We choose a smooth decomposition of unity \(b_Q \in C_c^\infty(\mathcal{M})\) each supported within a coordinate patch of \(\mathcal{M}\) and constantly one in a small Whitney cube \(Q\). Then it suffices to show that for some \(\alpha \in (0, 1)\) the following holds for any \(\eta \in C_c^\infty(B(x_0, r))\):

\[
(3.3) \quad \int_{B(x_0, r)} \int_0^r |T\eta(x, t)|^2 \frac{dt}{t} \, dx \lesssim r^{n+2\alpha}[\eta]_{C^\alpha}^2.
\]
But observe that because of \( \int_M \kappa(x, y, t) \, dy = 0 \) by assumption (and \( \kappa \) is integrable by the assumptions as well)

\[
|T \eta(x, t)| = \left| \int_M \kappa(x, y, t) (\eta(y) - \eta(x)) \, dy \right|
\leq [\eta]_{C^\alpha} \int_M \frac{t^n}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} \, d(x, y)^\alpha \, dy
\]

\[
= [\eta]_{C^\alpha} t^{-n+\alpha} \int_M \frac{t^n}{(d(x, y)^2 + 1)^{\frac{n+\nu}{2}}} \, dy
\]

which holds for any \( \alpha \in [0, 1] \). Now,

\[
\int_M \frac{(d(x, y)/t)^\alpha}{(d(x, y)^2 + 1)^\frac{n+\nu}{2}} \, dy = \sum_{k=1}^{\infty} \int_{B(x, 2^k t) \setminus B(x, 2^{k-1} t)} \frac{(d(x, y)/t)^\alpha}{(d(x, y)^2 + 1)^\frac{n+\nu}{2}} \, dy
\]

\[
+ \int_{B(x, t)} \frac{(d(x, y)/t)^\alpha}{(d(x, y)^2 + 1)^\frac{n+\nu}{2}} \, dy.
\]

Observe that

\[
\int_{B(x, t)} \frac{(d(x, y)/t)^\alpha}{(d(x, y)^2 + 1)^\frac{n+\nu}{2}} \, dy \lesssim \int_{B(x, t)} 1 \, dy \lesssim t^n
\]

and

\[
\int_{B(x, 2^k t) \setminus B(x, 2^{k-1} t)} \frac{(d(x, y)/t)^\alpha}{(d(x, y)^2 + 1)^\frac{n+\nu}{2}} \, dy \lesssim (2^k t)^n \frac{2^{ak}}{(2^{2k} + 1)^\frac{n+\nu}{2}} \lesssim t^n 2^{k(\alpha - \nu)}.
\]

We conclude that for \( \alpha < \nu \),

\[
|T \eta(x, t)| \lesssim [\eta]_{C^\alpha} t^\alpha.
\]

This implies (3.3).

\[\square\]

The main point is that an admissible manifold allows for the \( T(b) \)-theorem to be applied to the \( \sigma \)-harmonic extension.
As a corollary we obtain a result which is essentially a square function estimate.

**Proposition 3.2.** Let $\mathcal{M}$ be admissible and $0 < \sigma < 1$. Let $U$ be the $\sigma$-harmonic extension of $u \in C^\infty_c(\mathcal{M})$, then

\begin{align*}
(3.4) & \int_{\mathcal{M} \times (0, \infty)} t | \partial_t U(x, t)|^2 \, dx \, dt \lesssim \|u\|_{L^2(\mathcal{M})}^2, \\
(3.5) & \int_{\mathcal{M} \times (0, \infty)} t | \nabla_x U(x, t)|^2 \, dx \, dt \lesssim \|u\|_{L^2(\mathcal{M})}^2.
\end{align*}

**Proof.** Let $p(x, y, s)$ be the heat kernel for $\mathcal{M}$. Then

\[
U(x, t) := \frac{1}{4^\sigma \Gamma(\sigma)} \int_0^\infty \int_{\mathcal{M}} p(x, y, s) u(y) \, dy \, t^{2\sigma} e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}}.
\]

Regarding (3.4), we have

\[
t \partial_t U(x, t) = \frac{1}{4^\sigma \Gamma(\sigma)} \int_0^\infty \int_{\mathcal{M}} p(x, y, s) u(y) \, dy \, t^{2\sigma} e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}}.
\]

Thus, for

\[
k(x, y, t) := \frac{1}{4^\sigma \Gamma(\sigma)} \int_0^\infty \int_{\mathcal{M}} p(x, y, s) \, t \, \partial_t \left(t^{2\sigma} e^{-\frac{t^2}{4s}}\right) \, dy \, \frac{ds}{s^{1+\sigma}}.
\]

and

\[
(3.6) \quad Tu(x, t) := \int_{\mathcal{M}} k(x, y, t) u(y) \, dy,
\]

we have

\[
\int_{\mathcal{M} \times (0, \infty)} t | \partial_t U(x, t)|^2 \, dx \, dt = \int_{\mathcal{M}} \int_0^\infty |Tu(x, t)|^2 \frac{dt}{t} \, dx.
\]

Since for $u$ constant we know that $U(x, t)$ is also constant (see Appendix A), one has $t \partial_t U \equiv 0$. We conclude that

\[
\int_{\mathcal{M}} k(x, y, t) \, dy = 0 \quad \text{for all} \ x \in \mathcal{M}, \ t > 0.
\]

It remains to establish the estimates (3.1) and (3.2), then the claim follows from Theorem 3.1.
We estimate
\[|\kappa(x, y, t)| \lesssim t^{2\sigma} \int_0^\infty p(x, y, s) |\sigma - \frac{t^2}{4s}| e^{-\frac{s^2}{4s}} \frac{ds}{s^{1+\sigma}}.\]

Since \(M\) was assumed to be admissible, we can use (2.1) (after the transformation \(s \mapsto t^2 s\)) to conclude (3.1). In order to show (3.2), we use the mean value theorem to rewrite
\[
\frac{|\kappa(x, y_1, t) - \kappa(x, y_2, t)|}{d(y_1, y_2)} = |\nabla_y \kappa(x, y, t)|
\lesssim t^{2\sigma} \int_0^\infty |\nabla_y p(x, y, s)| |\sigma - \frac{t^2}{4s}| e^{-\frac{s^2}{4s}} \frac{ds}{s^{1+\sigma}}.
\]

Again, since \(M\) was assumed to be admissible, we can use (2.3) to deduce (3.2).

In order to derive the second estimate (3.5), we argue similarly for a slightly different kernel \(\kappa\). By the representation formula for \(U\), we can rewrite
\[
t \nabla_x U(x, t) = \int_M \nabla_x p(x, y, s) u(y) dy t^{2\sigma} e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}},
\]
so we define the operator
\[
Tu(x, t) = \int_M \kappa(x, y, t) u(y) dy
\]
with the kernel
\[
\kappa(x, y, t) := t^{2\sigma+1} \int_0^\infty \nabla_x p(x, y, s) e^{-\frac{s^2}{4s}} \frac{ds}{s^{1+\sigma}},
\]
\[
= t \int_0^\infty \nabla_x p(x, y, t^2 s) e^{-\frac{s^2}{4s}} \frac{ds}{s^{1+\sigma}}.
\]

Then again it holds \(\int \kappa = 0\) and
\[
\int_{M \times (0, \infty)} t |\nabla_x U(x, t)|^2 \, dx \, dt = \int_M \int_0^\infty |Tu(x, t)|^2 \frac{dt}{t} \, dx.
\]
So it suffices to show the estimates (3.1) and (3.2). This follows analogously to the first case by the admissibility of \(M\) and the mean value theorem.

\[\square\]

**Corollary 3.3.** Let \(M\) be admissible and \(0 < \sigma < 1\). Let \(U\) denote the \(\sigma\)-harmonic extension of \(u\), then
\[
Tu(x, t) := t \nabla_{(x, t)} U(x, t)
\]
satisfies the conditions of Theorem 3.1. In particular, the following estimate holds for all functions $u \in C_c^\infty(M)$:

$$\int_M \int_{(0,\infty)} t |\nabla_{(x,t)} U(x,t)|^2 \, dt \, dx \lesssim \|u\|^2_{L^2(M)}.$$  

**Proof.** Follows immediately from Proposition 3.2. \qed

4. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 consists in proving two directions. The easier one is Proposition 4.1, the more difficult one is Proposition 4.2.

**Proposition 4.1.** Let $\mathcal{M}$ be admissible and $0 < \sigma < 1$. Let $U$ be the $\sigma$-harmonic extension of $u \in C_c^\infty(M)$, then

$$\sup_B |B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U(x,t)|^2 t \, dx \, dt \lesssim [u]_{BMO(M)}^2.$$  

**Proof.** We extend the argument from [14, IV, §4.3, pp.158f].

Fix any ball $B \subset M$, and denote by $B^*$ the ball with twice the radius. We decompose

$$\nabla_{(x,t)} U = \nabla_{(x,t)} U_1 + \nabla_{(x,t)} U_2 + \nabla_{(x,t)} U_3,$$

where $U_i$ is the $\sigma$-harmonic extension of $u_i$, respectively, given as

$$u_1 := \chi_{B^*}(u - (u)_{B^*}),$$

$$u_2 := (1 - \chi_{B^*})(u - (u)_{B^*}),$$

$$u_3 := (u)_{B^*}.$$  

Observe that $U_3$ is constant and thus $\nabla U_3 = 0$. Moreover,

$$|B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U_1(x,t)|^2 t \, dx \, dt \leq |B|^{-1} \int_{\mathcal{M} \times (0,\infty)} |\nabla_{(x,t)} U_1(x,t)|^2 t \, dx \, dt.$$  

In view of Corollary 3.3,

$$|B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U_1(x,t)|^2 t \, dx \, dt \lesssim |B|^{-1} \left( \int_{B^*} |u - (u)_{B^*}|^2 \right).$$

Since $\mathcal{M}$ is supposed to be Ahlfors-regular, $|B| \approx |B^*|$ with uniform constants, one has by John-Nirenberg inequality [1, (5.8)], see also [5],

$$|B^*|^{-1} \left( \int_{B^*} |u - (u)_{B^*}|^2 \right) \lesssim [u]_{BMO}^2.$$
This implies,

$$|B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U_1(x, t)|^2 \, t \, dx \, dt \lesssim [u]_{BMO}^2.$$ 

It remains to estimate $U_2$. As in the proof of Proposition 3.2,

$$|B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U_2(x, t)|^2 \, t \, dx \, dt = |B|^{-1} \int_{T(B)} \left| \int_{\mathcal{M}} \kappa(x, y, t) u_2(y) \, dy \right|^2 \, dx \, \frac{dt}{t}.$$ 

In view of (3.1), for some given $\nu > 0,$

$$|B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U_2(x, t)|^2 \, t \, dx \, dt \lesssim |B|^{-1} \int_{T(B)} \int_{\mathcal{M}} \frac{t^\nu}{\nu} |u_2(y)| \, dy \, dx \, \frac{dt}{t}$$

We denote with $B_k$, $k \in \mathbb{N}$, the ball concentric around $B$ but with radius $2^k$ times the radius of $B$.

Since $\text{supp } u_2 \subset \mathcal{M} \setminus B^*$, we find that for any $x \in B$

$$\int_{\mathcal{M}} \frac{t^\nu}{(d(x, y)^2 + t^2)^\frac{n+\nu}{2}} |u_2(y)| \, dy \lesssim \sum_{k=0}^\infty \frac{t^\nu}{((2^k r)^2 + t^2)^\frac{n+\nu}{2}} \int_{B_{k+1} \setminus B_k} |u(y) - (u)_B| \, dy.$$ 

By triangle inequality and a telescoping sum,

$$\int_{B_{k+1} \setminus B_k} |u(y) - (u)_B| \, dy \leq \int_{B_{k+1}} |u(y) - (u)_{B_{k+1}}| \, dy + \sum_{i=0}^k |(u)_{B_i} - (u)_{B_{i+1}}|.$$ 

Using the doubling property of the measure and the definition of BMO, we have

$$\int_{B_{k+1} \setminus B_k} |u(y) - (u)_B| \, dy \lesssim |B_k| (k + 1) [u]_{BMO}.$$ 

Consequently,

$$\int_{\mathcal{M}} \frac{t^\nu}{(d(x, y)^2 + t^2)^\frac{n+\nu}{2}} |u_2(y)| \, dy \lesssim \sum_{k=0}^\infty \frac{t^\nu}{((2^k r)^2 + t^2)^\frac{n+\nu}{2}} |B_k| (k + 1) [u]_{BMO}.$$ 

This implies

$$|B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U_2(x, t)|^2 \, t \, dx \, dt \lesssim A(r) [u]_{BMO},$$

where

$$A(r) = \int_0^r \sum_{k=0}^\infty \frac{t^\nu}{((2^k r)^2 + t^2)^\frac{n+\nu}{2}} (2^k r)^n (k + 1) \, \frac{dt}{t}.$$
By a substitution $t \mapsto t/r$ we see that $A(r) = A(1)$, and

$$A(1) \leq \left( \int_0^1 t^{\nu-1} \, dt \right) \cdot \left( \sum_{k=0}^{\infty} 2^{-\nu k} (k + 1) \right) = C_\nu < \infty.$$  

This concludes the proof of Proposition 4.1. □

**Proposition 4.2.** Let $M$ be admissible and $0 < \sigma < 1$. Let $U$ be the $\sigma$-harmonic extension of $u \in C^\infty_c(M)$, then

$$[u]_{BMO(M)}^2 \lesssim \sup_B |B|^{-1} \int_{T(B)} |\nabla_{(x,t)} U(x,t)|^2 \, t \, dx \, dt$$

One technical ingredient in the proof of Proposition 4.2 is the following observation (cf. [14, (40) p.163]).

**Lemma 4.3.** Let $\Phi, U : M \times (0, \infty) \to \mathbb{R}$ be the $\sigma-$harmonic extension of $\varphi$ and $u$, respectively. Then

$$\left| \int_M u \varphi \right| \lesssim \int_{M \times (0,\infty)} t |\partial_t \Phi(x,t)||\partial_t U(x,t)| \, dx \, dt + \int_{M \times (0,\infty)} t |\nabla_x \Phi(x,t)||\nabla_x U(x,t)| \, dx \, dt$$

**Proof.** By integration by parts and the decay as $t \to \infty$ (see Appendix A), we have for every $x \in M$:

$$u(x) \varphi(x) = \frac{1}{2\sigma} \int_0^\infty t^{2\sigma} \partial_t \left( t^{1-2\sigma} \partial_t \left( \Phi(x,t) U(x,t) \right) \right) \, dt.$$  

Since $U$ is the $\sigma$-harmonic extension,

$$\partial_t \left( t^{1-2\sigma} \partial_t U(x,t) \right) = -t^{1-2\sigma} \Delta_x U(x,t)$$

and likewise for $\Phi$ after integration by parts

$$\int_M u(x) \varphi(x) \, dx = \frac{1}{\sigma} \int_{M \times (0,\infty)} t \partial_t U(x,t) \partial_t \Phi(x,t) \, dx \, dt + \frac{1}{\sigma} \int_0^\infty t \langle \nabla_x U(\cdot,t), \nabla_x \Phi(\cdot,t) \rangle_{L^2(M)} \, dt,$$

from which the claim follows immediately. □

The following is proved in [14, IV, §4.3, pp.162, Proposition]. It is stated there in $\mathbb{R}^n$, but the proof easily extends almost verbatim to Ahlfors regular spaces.
Lemma 4.4. Let $\mathcal{M}$ be an Ahlfors-regular manifold and let $F, G : \mathcal{M} \times (0, \infty) \to \mathbb{R}$ be measurable functions. Then,

$$\int_{\mathcal{M} \times (0, \infty)} t |F(x, t)| |G(x, t)| \, dx \, dt$$

$$\lesssim \left( \int_{\mathcal{M}} \left( \int_{d(x,y)<t} |F(y, t)|^2 \frac{dy \, dt}{t^{n-1}} \right)^{\frac{1}{2}} \, dx \right) \left( \sup_B |B|^{-1} \int_{T(B)} |G(x, t)|^2 t \, dx \, dt \right)^{\frac{1}{2}}$$

Proof of Proposition 4.2. Again, we essentially can follow Stein’s book, namely [14, IV, §4.3, pp.163f].

From Lemma 4.3 and Lemma 4.4 for $F = \nabla_{(x,t)} \Phi$ and $G = \nabla_{(x,t)} U$ we obtain (using the notation (1.3))

$$\left| \int_{\mathcal{M}} u \varphi \right| \lesssim \left( \int_{\mathcal{M}} \left( \int_{d(x,y)<t} |\nabla \Phi(y, t)|^2 \frac{dy \, dt}{t^{n-1}} \right)^{\frac{1}{2}} \, dx \right) [u]_{BMO}.$$

We can conclude once we show that

$$\left( \int_{\mathcal{M}} \left( \int_{d(x,y)<t} |\nabla \Phi(y, t)|^2 \frac{dy \, dt}{t^{n-1}} \right)^{\frac{1}{2}} \, dx \right) \lesssim \| \varphi \|_{H^1},$$

where $H^1$ is the Hardy space. Indeed, the claim then follows in view of the duality of Hardy spaces and BMO [1, (7.154)].

To obtain (4.1) we use an extrapolation result [11, Theorem 6.18.], which essentially states that a suitable operator, if it is bounded from $L^2$ to $L^2$, can be extended to a bounded operator from the Hardy space into $L^1$. To apply this result, first observe that from Fubini we have

$$\left\| x \mapsto \left( \int_{d(x,y)<t} |\nabla \Phi(y, t)|^2 \frac{dy \, dt}{t^{n-1}} \right)^{\frac{1}{2}} \right\|^2_{L^2(\mathcal{M})}$$

$$= \int_0^\infty \int_{\mathcal{M}} t^{1-n} \chi_{d(x,y)<t} |\nabla \Phi(y, t)|^2 \, dx \, dy \, dt$$

$$\lesssim \int_0^\infty \int_{\mathcal{M}} t |\nabla \Phi(y, t)|^2 \, dy \, dt.$$
Appendix A. Computations for the $\sigma$-harmonic extension

We recall our definition for the extension $U$, motivated by [15]. Given $u \in C_c^\infty(M)$ and $0 < \sigma < 1$, the $\sigma$-harmonic extension $U : M \times [0, \infty) \to \mathbb{R}$ is formally given by

$$U(x, t) := \frac{1}{4^\sigma \Gamma(\sigma)} t^{2\sigma} \int_0^\infty e^{s \Delta_M u(x)} e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}}.$$  

More explicitly, one has

$$U(x, t) := \frac{1}{4^\sigma \Gamma(\sigma)} \int_0^\infty \int_M p(x, y, s) u(y) \, dy \, t^{2\sigma} e^{-\frac{t^2}{4s}} \frac{ds}{s^{1+\sigma}},$$

where $p(x, y, s)$ is the heat kernel for $M$. Furthermore, $U$ is the smooth (in the interior) solution of

$$\Delta_M U + \frac{1-2\sigma}{t} \partial_t U + \partial_{tt} U = 0 \quad \text{in } M \times (0, \infty)$$

$$U(x, 0) = u(x) \quad \text{in } M$$

$$\lim_{|x, t| \to \infty} U(x, t) = 0.$$

The most important property for us is that constants are extended by constants: our manifold being assumed to be Ahlfors regular, it is stochastically complete, i.e.

$$\int_M p(x, y, s) \, dy = 1 \quad \text{for all } x \in M, \ s > 0,$$

see [9, Theorem 1]. Now using the representation formula, we deduce that for constant $u : M \to \mathbb{R}$

$$U(x, t) = \frac{u}{\Gamma(\sigma)} \int_0^\infty \int_M p(x, y, \frac{t^2}{4s}) \, dy \, e^{-s \sigma -1} \, ds = u.$$

One can also check, that the representation formula solves the PDE (A.1). Using the second line in the representation formula and that $p$ solves the heat equation (1.2) one sees that

$$\partial_t U(x, t) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_M (\partial_t p)(x, y, \frac{t^2}{4s}) u(y) \, dy \, e^{-s \sigma -1} \left[\frac{t}{2} s^{\sigma -2}\right] ds$$

$$\partial_{tt} U(x, t) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_M (\partial_t p)(x, y, \frac{t^2}{4s}) u(y) \, dy \, e^{-s \sigma -1} \left[-\frac{1}{2} s^{\sigma -2} - s^{\sigma -1} - (1-\sigma) s^{\sigma -2}\right] ds$$

$$\Delta_x U(x, t) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \int_M (\partial_t p)(x, y, \frac{t^2}{4s}) u(y) \, dy \, e^{-s \sigma -1} \left[s^{\sigma -1}\right] ds.$$
Comparing the square brackets shows that $U$ indeed solves the PDE (A.1).

Moreover, since the heat kernel is an approximation of the identity, we see that $U$ has also the correct boundary data. Even more, as $t \to \infty$, we have $U \to 0$. This follows from the admissibility of $\mathcal{M}$, i.e. given $u \in C_c^\infty(\mathcal{M})$, we can compute

$$|U(x, t)| \lesssim t^{2\sigma} \int_{\mathcal{M}} |u(y)| \int_0^\infty p(x, y, s) e^{-\frac{s^2}{4t}} \frac{ds}{s^{1+\sigma}} dy$$

$$= \int_{\mathcal{M}} |u(y)| \int_0^\infty p(x, y, t^2s) e^{-\frac{s}{4t}} \frac{ds}{s^{1+\sigma}} dy$$

$$\lesssim \int_{\mathcal{M}} |u(y)| \frac{t^\nu}{(d(x, y)^2 + t^2)^{\frac{n+\nu}{2}}} dy$$

$$\leq \frac{1}{t^n} \int_{\mathcal{M}} |u(y)| dy.$$

With an analogue computation, we see that

$$|\partial_t U(x, t)| \lesssim t^{(n+1)} \int_{\mathcal{M}} |u(y)| dy.$$

This concludes our discussion about the heat kernel.

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