Linear instability and statistical laws of physics

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Abstract. – We show that a meaningful statistical description is possible in conservative and mixing systems with zero Lyapunov exponent in which the dynamical instability is only linear in time. More specifically, (i) the sensitivity to initial conditions is given by $\xi = [1 + (1 - q)\lambda_0 t]^{1/(1-q)}$ with $q = 0$; (ii) the statistical entropy $S_q = (1 - \sum p_i^q)/(q-1)$ ($S_1 = -\sum p_i \ln p_i$) in the infinitely fine graining limit (i.e., $W \equiv$ number of cells into which the phase space has been partitioned $\rightarrow \infty$), increases linearly with time only for $q = 0$; (iii) a nontrivial, $q$-generalized, Pesin-like identity is satisfied, namely the $\lim_{t \rightarrow \infty} \lim_{W \rightarrow \infty} S_0(t)/t = \max \{\lambda_0\}$. These facts (which are in analogy to the usual behaviour of strongly chaotic systems with $q = 1$), seem to open the door for a statistical description of conservative many-body nonlinear systems whose Lyapunov spectrum vanishes.

The possibility to derive macroscopic laws from the underlying microscopic deterministic dynamics crucially depends on whether the latter satisfies or not the randomness assumptions required by statistical mechanics. In this respect exponentially unstable systems appear precisely those deterministic random systems tacitly required by macroscopic laws of physics. In fact the exponentially unstable motion is called chaotic since almost all orbits, though deterministic, are unpredictable. This is a consequence of the Alekseev-Brudno theorem [1] in the algorithmic theory of dynamical systems, according to which the information $I(t)$ associated with a segment of trajectory of length $t$ is equal, asymptotically, to

$$\lim_{|t| \rightarrow \infty} \frac{I(t)}{|t|} = h$$

(1)
where $h$ is the so called KS (Kolmogorov-Sinai) entropy which is positive when the maximum Lyapunov exponent $\lambda$ is positive.

This means that in order to predict a new segment of a chaotic trajectory one needs an additional information proportional to the length of this segment and independent of the previous length of the trajectory. In this situation information cannot be extracted from the observation of past history of motion. This inherent randomness exhibited by many Newtonian systems provides hope that classical statistical mechanics can at last be rigorously derived without the use of additional *ad hoc* assumptions such as correlations decay and the like.

On the other hand one would like to know whether such a strong property, being sufficient, is also necessary. Certainly ergodicity only is not sufficient. Indeed it is known that ergodicity alone, by merely implying that time averages are equal to phase averages, is sufficient to justify equilibrium statistical mechanics, but does not lead to any approach to statistical equilibrium. For the latter, continuous spectrum of the motion and correlations decay is necessary. This property, in the ergodic theory of dynamical systems, is called *mixing* and provides statistical independence of different parts of a dynamical trajectory thus allowing a statistical description in terms of few macroscopic variables. Mixing is therefore a stronger property than ergodicity. However it does not require, in principle, exponential instability. Recently strong empirical evidence has been provided of mixing and diffusive behaviour in dynamical systems with linear instability (zero Lyapunov exponent) [2–6]. In addition, quite remarkably, normal heat transport, obeying the Fourier law, has been found in a quasi-1d gas with triangular scatterers [4] and in the 1d alternate masses hard point gas [6]. These results clearly demonstrate that linear instability can be sufficient for the derivation of macroscopic laws of physics.

Along these lines, a quite natural step forward is to analyze the connection between linear dynamical instability and the statistical entropy $S$. The latter, differently from the dynamical KS entropy, is a function of time and depends on the initial probability distribution (ensemble) for the state of the system. In [7] it was numerically shown that, for symplectic chaotic systems, the time evolution of the *phase-space averaged* statistical entropy of out-of-equilibrium initial ensembles displays a stage of linear increase with a slope that coincides with $h$ (and, through the Pesin equality, with the sum of positive Lyapunov exponents). It should be stressed that the above numerical result, though very interesting, is far from being rigorous. Actually, as pointed out also in a recent paper [8], such connection does not hold in systems with intermittent behavior or in systems which, though mixing and exponentially unstable, are characterized by different time scales in the relaxation process. This is the case of systems like the Sinai billiard or the Bunimovich stadium in which the local Lyapunov exponent is not constant.

In this paper we address precisely the above problem, namely the connection between statistical entropy and dynamical instability, for a system which is linearly (instead of exponentially) unstable. It turns out that in order to recover results analogous to those relating to exponentially unstable systems (see e.g. [7]) a *generalization* of the definition of the classical entropic functional $S_{BG} = - \sum_{i=1}^{W} p_i \ln p_i$ (BG stands for *Boltzmann-Gibbs*, $\sum_{i=1}^{W} p_i = 1$) is needed. A convenient such generalization has been proposed in 1988 [10] as a basis for generalizing BG statistical mechanics (for reviews see [11,12]). This new functional is defined as

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \quad (q \in \mathbb{R}, \ S_1 = S_{BG}). \quad (2)$$

For equal probabilities $p_i$, we obtain $S_q = \ln_q W$, where $\ln_q x \equiv (x^{1-q} - 1)/(1-q)$ ($\ln_1 x = \ln x$).
If we extremize entropy (2) with appropriate constraints, we obtain [10] \( p \propto e_q^{-\beta q \mathcal{H}} \) for the canonical ensemble, where \( \beta_q \) is an inverse-temperature-like parameter, \( \mathcal{H} \) is the Hamiltonian, and \( e_q^x \) is the inverse function of \( \ln_q x \), called \( q \)-exponential, i.e., \( e_q^x \equiv [1 + (1 - q)x]^{1/(1-q)} \) \( (e^x = e^x) \). For systems for which BG statistical mechanics is valid, we have \( q = 1 \). For anomalous systems (e.g., for long-range interactions [11–14]), we expect \( q \) to be univocally determined by the (nonextensive) universality class to which the system belongs. For example, for the edge of chaos of unimodal maps it has been exactly shown [15], through a renormalization group approach, that: (i) In correspondence with the Feigenbaum attractor the sensitivity to initial conditions has an upper-bound envelope which is algebraic in time and precisely given by a \( q \)-exponential function for the specific value \( q = 0.2445 \ldots \) (this value is exactly deduced from the \( \alpha \) Feigenbaum’s universal constant); (ii) The connection between this power-law dynamical instability and a linear statistical entropy production-rate is obtained through the form \( (2) \), for the same value of \( q \). These results are available through a number of different methods (see [11, 12] and references therein for details).

In this paper we consider instead a conservative, linearly unstable system, defined by the area preserving triangle map \( z_{n+1} = T(z_n) \) [3] on a torus \( z = (x, y) \in [-1, 1) \times [-1, 1) \)

\[
\begin{align*}
y_{n+1} &= y_n + \alpha \text{sgn}x_n + \beta \quad \text{(mod 2)}, \\
x_{n+1} &= x_n + y_{n+1} \quad \text{(mod 2)},
\end{align*}
\]

where \( \text{sgn}x = \pm 1 \) is the sign of \( x \) and \( \alpha, \beta \) are two parameters \( (n = 0, 1, \ldots) \). The following facts and numerical results have been established [3]: For rational values of \( \alpha, \beta \) the system is pseudo-integrable, as the dynamics is confined on invariant curves. If \( \beta = 0 \) and \( \alpha \) is irrational, the dynamics is ergodic but the phase-space is filled very slowly (see also Ref. [9]), while for incommensurate irrational values of \( \alpha, \beta \) the dynamics is ergodic and mixing with dynamical correlation function decaying as \( t^{-3/2} \). We would like to stress here that, as shown in [3], the triangle map does not have any secondary time scales and that the exploration of the phase phase by a given orbit is arbitrarily close to that of a random model.

It can be argued that the triangle map possesses essential features of bounce maps of polygonal billiards and 1d hardpoint gases [2, 3], namely the parabolic stability type in combination with decaying dynamical correlations, and as such represents a paradigmatic model for a larger class of systems. For the sake of definiteness, in the following we will fix the parameter values \( \alpha = [\frac{1}{2}(\sqrt{5} - 1) - e^{-1}] / 2, \beta = [\frac{1}{2}(\sqrt{5} - 1) + e^{-1}] / 2 \) as it has been done in [3], although it should be noticed that qualitatively identical results are obtained for other irrational parameter values. Fig. shows the mixing process of an ensemble of points initially localized inside a small square. The action of the map initially divides the area covered by the ensemble into different unconnected portions, each essentially stretched along a straight line. After a certain amount of time these portions overlap until a slow relaxation process to a complete mixing is observed.

In order to compute the dynamical evolution of the coarse grained statistical entropy \( S_q(n) \) we divide the phase space in \( W \equiv w \times w \) equal cells and consider an initial ensemble of \( N \) points randomly distributed over a single partition-cell. Let \( p_i(n) \) the fraction of orbits which after \( n \) time steps overlap with the cell of label \( i \). By means of Eq. we can then compute \( S_q(n) \). Typically, in order to get reliable results the statistics needs to be improved by averaging over many random initial cells [7]. For a chaotic system with dynamical entropy \( h \), one expects \( S_1(n) = hn \) in the limit \( W \to \infty \) and for sufficiently large \( n \) (i.e., after a possible initial transient) [7], whereas for a finite \( W \), \( S_1(n) \) grows linearly with \( n \) during an intermediate stage, before saturation. In contrast with such a behaviour, the time-evolution of the BG entropy associated to the triangle map is characterized by an intermediate stage
where the entropy $S_1$ grows logarithmically with time [16].

Our numerical simulations of $S_q(n)$ provide clear empirical evidence that, in analogy with the results for the edge of chaos of unimodal maps [15], there is only a specific value of $q \neq 1$ for which a linear entropy production rate is observed when $W$ is sufficiently large. Indeed, only for $q = 0$ we get

$$S_0(n) = K n,$$

with the numerical constant $K \simeq 1$. This production-rate regime is associated to the first stage of the mixing process (Fig. 1) which lasts infinitely long in the limit $W \to \infty$. We
remark that $S_0$ in Eq. (2) reduces to $W(n) - 1$, where $W(n)$ is the number of cells occupied by the ensemble at the iteration step $n$. In Fig. 2(a) we show the time evolution of $S_q$ for different values of the parameter $q$. As the analysis of the derivative confirms (Fig. 2(b)), only for $q = 0$ one gets a linear increase in time. Fig. 2(c) shows that the linear increase with slope $K \approx 1$ is reached, in the limit $W \to \infty$, from above. We remark that for a meaningful definition of the probabilities $p_i$ the condition $N \gg W(n)$ has to be fulfilled.

The above numerical result directly follows from the fact that the mixing process occurs essentially along straight lines. Indeed we have (1)

$$W(n) \sim \frac{|\Delta x(n)|}{l},$$

where $|\Delta x(n)|$ is the length of the segment that describes the maximum separation between points of the ensemble at time $n$, and $l$ is the size of a single cell partition. If we linearize map (3) then the growth of the tangent vector is given by

$$\Delta x_{n+1} = \Delta x_n + \Delta y_n \quad \text{(mod 2)},$$
$$\Delta y_{n+1} = \Delta y_n \quad \text{(mod 2)},$$

which leads to

$$|\Delta x(n)| \sim n|\Delta y_0|. \quad (7)$$

This linear increase of $|\Delta x(n)|$ implies the linear increase of $W(n)$ in (4). Notice that if the partition is not fine enough, relation (5) breaks down. This reflects in a faster growth of $S_0$ for small values of $W$, as observed in Fig. 4.

Next, we discuss the connection between the statistical entropy and the sensitivity to initial conditions defined as

$$\xi(n) \equiv \frac{|\Delta x(n)|}{|\Delta x(0)|}. \quad (8)$$

From the linearized map (6) we see that the sensitivity to initial conditions for the triangle map casts into the $q$-exponential form provided by the afore-mentioned generalization of the BG formalism (see e.g. [11, 12]):

$$\xi = [1 + (1 - q)\lambda_0 n]^{1/(1-q)}, \quad \text{with } q = 0 \text{ and } \lambda_0 = |\sin \theta_0|, \theta_0 \text{ being the polar angle in the } (x, y)-\text{plane of the initial vector } (|\Delta y_0| = |\sin \theta_0| |\Delta x(0)|).$$

Taking the upper-bound of the sensitivity to initial conditions we get

$$K = \max \{\lambda_0\} = 1. \quad (9)$$

Eq. (9) is in some sense a generalization of the Pesin equality to a system with zero Lyapunov exponent, in fact it relates the dynamical instability to an entropic quantity. In spite of its simplicity this is a non trivial result (analogous to that exhibited in [15]) If, on the other hand, one considers the average value of $\xi(t)$ over an ensemble of uniformly distributed initial directions $\theta_0$ (due to ergodicity this coincides with the long-time-average), one has $K = (\pi/2)\langle \lambda_0 \rangle$, since $\langle \lambda_0 \rangle = \langle |\sin \theta_0| \rangle = 2/\pi$.

In conclusion, while positivity of Lyapunov exponents is sufficient for a meaningful statistical description (the BG statistical mechanics), it might be not necessary. Indeed, we have illustrated, for a conservative, mixing and ergodic nonlinear dynamical system, that the use of the more general entropy $S_q$ (with the value $q = 0$ for this case) provides a satisfactory frame for handling nonlinear dynamical systems whose maximal Lyapunov exponent vanishes. In

$^{(1)}$The correction factor due to the orientation of the stretched ensemble with respect to the partition grid becomes negligible as $|\Delta x(n)| \gg l$. 


Fig. 2 - Time-evolution of the statistical entropy $S_q$ for different values of $q$. The phase space has been divided into $W = 4000 \times 4000$ equal cells of size $l = 5 \times 10^{-4}$ and the initial ensemble is characterized by $N = 10^3$ points randomly distributed inside a partition-square. Curves are the result of an average over 100 different initial squares randomly chosen in phase space. The analysis of the derivative of $S_q$ in (b) shows that only for $q = 0$ a linear behaviour is obtained. In fact, a linear regression provides $S'_0(n) = 1.029 \times 10^2 + 1.997$ with a correlation coefficient $R = 0.99993$. (c) shows that the linear growth for $S_0$ is reached from above, in the limit $W \to \infty$. 
particular, we have shown that (the upper bound of) the coefficient $\lambda_q$ of the sensitivity to the initial conditions coincides with the entropy production per unit time, in total analogy with the Pesin theorem for standard chaotic systems. These results suggest that a thermostatistical approach of such systems is possible. Indeed, the structure that we have exhibited here for the time dependence of $S_q$, is totally analogous to the one that has been recently exhibited [17] for the $N$-dependence of $S_q$, where $N$ is the number of elements of a many-body system. When the number of nonzero-probability states of the system increases as a power of $N$ (instead of exponentially with $N$ as usually), a special value of $q$ below unity exists such that $S_q$ is extensive. In other words, $S_q$ asymptotically increases linearly with $N$, whereas $S_{BG}$ does not.

In this paper we have considered an abstract model of dynamical systems. However the same features of linear instability together with a nice diffusive behaviour are exhibited by more realistic systems like billiards in irrational polygons and one dimensional systems of unequal masses hard point particles. On one hand this fact generalizes the validity of our conclusions. On the other hand all these systems belong to the extreme case of linear instability leading to the value $q = 0$. Certainly it would be much more interesting to find dynamical systems leading to $0 < q < 1$. Natural candidates are systems with power law instability $t^\alpha$ with $\alpha > 1$. We are not aware of any conservative system with such property and with nice mixing behaviour. It would be highly interesting to explore such possibility.

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