Superpotential for novel symmetry beyond shape invariance

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I. INTRODUCTION

Supersymmetry formulated nearly four decades ago is a landmark contribution of Witten [1] to understand interesting energy relations between two nearly identical partner Hamiltonians [2,13]

\[ H^{(±)} = p^2 + W^2 \mp \frac{dW}{dx} \] (1)

connected with energy conditions as

\[ E_{n+1}^{(±)} = E_n^{(±)}, \quad E_0^{(−)} = 0. \] (2)

Later on Gendenshtein [2] proposed an elegant way of calculating energy \( E_n^{(−)} \) on some potentials satisfying the condition

\[ V_+(x, \lambda) = V_-(x, \beta) + R(\beta), \quad \beta = f(\lambda) \] (3)

can directly yield energy level \( E_n^{(−)} \) of \( H^{(−)} \) as

\[ E_n^{(−)} = \sum R(\beta). \] (4)

In fact, this excellent idea works well only on some potentials [2,10]. Further a list of model potentials are reflected in [2,6]. A common feature on these models, where shape invariance (SI) remains valid is "continuous" nature of superpotential (W). However, Bougie, Gangopadhyaya and Mallow (BGM) [7] suggested that Coulomb like model W can also be handled using the SI. Authors have also given a model one-dimensional potential. By curiosity we notice that when the (BGM) is combined with SUSY model of Marques, Negini and da Silva (MND) [6], we explore non-validity of SUSY conditions. This brings us a curiosity to study new symmetry that can emerge from the model of BGM and MND. Hence, in this communication we propose a new type of discontinuity in superpotential (W) and study its spectral properties with appropriate mathematical development associated with examples. A common model superpotential satisfying the above relation [8] is \( W_1 = \lambda x \), it is easy to see that

\[ V_1^{(1)}(x, \lambda) = V_1^{(−)}(x, -\lambda) + 2\lambda \] (5)

\[ E_n^{(−)} = 2\lambda n. \] (6)

In this model of superpotential both SUSY and shape invariance remain valid. However, Bougie, Gangopadhyaya and Mallow [7] proposed a model superpotential as

\[ W_2 = wx - \frac{a}{x} + \left[ \frac{2wx}{(wx^2 + 2a - 1)} - \frac{2wx}{(wx^2 + 2a + 1)} \right] \] (7)

and claimed a few interesting natures connecting to Euler equation. However, neglecting the extra term we have

\[ W_2 \sim wx - \frac{a}{x} = x - \frac{1}{x}, \quad \text{for } w = a = 1. \] (8)

Here SUSY remains invalid and also shape invariance is no longer useful in releasing energy \( E_n^{(−)} \) because it is practically impossible to visualize

\[ V_2^{(1)} = V_2^{(−)} + f(\beta). \] (9)

This simple superpotential nature has been reflected in Fig.4 which also displays the natures of other superpotential \( W_2 - W_3 \) simultaneously. Apart from this, a new model superpotential was proposed by Marques, Negreni and Da Silva [4] as

\[ W_3 = \lambda x|x| = x|x|, \quad \text{for } \lambda = 1. \] (10)

In this model SUSY remains valid but SI keeps invalid. Stimulated from \( W_2 \) and \( W_3 \), we may suggest

\[ W_4 = x|x| - \lambda|x| = x|x| - \frac{|x|}{x}. \] (11)
FIG. 1: (Color online) Plots of superpotentials $W_i (i=1, 2, 3, 4, 5, 6)$. Both SUSY and SI for $W_1$ are valid, SUSY for $W_3$ is valid but its SI is invalid. Both SUSY and SI of the remaining are invalid.

Here neither SUSY nor shape invariance remains valid. Below we present a few energy levels of Hamiltonian generated by the superpotential $W_4$ as shown in Table I

$$H^{(-)} = p^2 + W^2 - \frac{dW}{dx}. \quad (12)$$

| $n$ | $E_n^{(-)}$ |
|-----|-------------|
| 0   | -0.333 8    |
| 1   | 0.553 1     |
| 2   | 3.821 7     |
| 3   | 6.961 7     |

The corresponding wave functions are displayed in Fig. 2.

The plan of this work is organized as follows. In Section III, we will propose a new symmetry model and verify their difference is a constant. Some concluding remarks are given in Section III.

II. NOVEL SYMMETRY $E_n^{(+)} - E_n^{(-)} = 2$

Here, we suggest a new model on superpotential as

$$W_5 = x - \lambda \frac{|x|}{x} \quad (13)$$

whose nature is also reflected in Fig. 1.

Further, Hamiltonians generated from the above new model are

$$H^\pm = p^2 + x^2 + \lambda^2 - 2\lambda |x| \pm 1. \quad (14)$$

The corresponding SUSY potentials satisfy the relation

$$V_+(x, \lambda) = V_-(x, \lambda) + 2 \quad (15)$$

Hence using shape invariance condition one can easily verify that

$$E_n^{(-)} \neq 2n. \quad (16)$$

In other words, shape invariance method fails to address the correct energy levels of

$$H^{(-)} = p^2 + x^2 - 2|x|, \quad \text{for } \lambda = 1, \quad (17)$$

as reflected in Table III. The wave functions with respect to this superpotential $W_5$ are shown in Fig. 3.

TABLE II: Energy levels of novel symmetry related to $W_5$ ($\lambda = 1$)

| $n$ | $E_n^{(-)}$ |
|-----|-------------|
| 0   | -0.381 0    |
| 1   | 0.468 4     |
| 2   | 2.000 0     |
| 3   | 3.395 0     |

Similarly, another model for the superpotential can be written as

$$W_6 = x + e^{-|x|/x}. \quad (18)$$

In this case, the above superpotential $W$ must be an odd function of $x$ in order to justify the well-behaved nature of wave functions of the Hamiltonians

$$H^{\pm} = p^2 + x^2 + e^{-2|x|/x} + 2xe^{-|x|/x} \pm 1. \quad (19)$$

In this case, we also find superpotential approach is invalid

$$E_n^{(-)} \neq 2n \quad (20)$$
Proof: $E_{n}^{(+)} - E_{n}^{(-)} = 2$

Let $E_{n}$ be the energy of Hamiltonian

\[ H = p^2 + W^2 \tag{21} \]

then energy of

\[ H^{(+)} = p^2 + W^2 + \frac{dW}{dx} \]

becomes $E_{n}^{(+)} = E_{n} + 1$. Similarly, the spectrum of the following Hamiltonian

\[ H^{(-)} = p^2 + W^2 - \frac{dW}{dx} \tag{23} \]

becomes $E_{n}^{(-)} = E_{n} - 1$. Hence, it is easy to equate and see that

\[ E_{n}^{(+)} - E_{n}^{(-)} = 2 \tag{24} \]

However, in general it should be

\[ E_{n}^{(+)} - E_{n}^{(-)} = 2\lambda \tag{25} \]

where $\lambda$ is the multiplicative constant of linear term in superpotential $W_{5,6}$.

III. CONCLUDING REMARKS

In conclusion, new symmetry operator has no relation with shape invariance. We compare the nature of $W_{5,6}$ with others in the same fig. Further the computed results in tables have been cross checked using calculations involving MATLAB [14,15].

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$n$ & $E_{n}^{(-)}$ \\
\hline
0 & 0.001 3 \\
1 & 1.725 2 \\
2 & 2.105 7 \\
3 & 4.023 5 \\
\hline
\end{tabular}
\caption{Energy levels $E_{n}^{(-)}$ generated from $W_{6}$.}
\end{table}

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