IMPROVED VARIABLE COEFFICIENT SQUARE FUNCTIONS AND LOCAL SMOOTHING OF FOURIER INTEGRAL OPERATORS

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Abstract. We establish certain square function estimates for a class of oscillatory integral operators with homogeneous phase functions. These results are employed to deduce a refinement of a previous result of Mockenhaupt Seeger and Sogge [24] on the local smoothing property for Fourier integral operators, which arise naturally in the study of wave equations on compact Riemannian manifolds. The proof is an adaptation of the bilinear approach of Tao and Vargas [30], and based on bilinear oscillatory integral estimates of Lee [21].

1. Introduction

1.1. Motivation and background. The purpose of this paper is to study the local smoothing property for a certain class of Fourier integral operators acting on locally $L^p$-integrable functions defined on paracompact manifolds, which in the terminology of [24] satisfy the cinematic curvature conditions. We obtain improvements upon the known $L^p \to L^p$ regularity results for these operators for relatively small $p$, which were established previously in [24].

Let $Z$ and $Y$ be smooth paracompact manifolds with dim $Z = n + 1$ and dim $Y = n \geq 2$, respectively. A Fourier integral operators $\mathcal{F} \in I_{\sigma}^{-\frac{1}{4}}(Z,Y;\mathcal{C})$ is said to satisfy the cinematic curvature condition in the terminology of [24] as follows. First of all, a Fourier integral operator $\mathcal{F}$ is determined globally by the canonical relation $\mathcal{C}$, which is a $2n+1$ dimensional closed homogeneous, conic Lagrangian submanifold of $T^*Z \setminus 0 \times T^*Y \setminus 0$ with respect to the symplectic form $d\zeta \wedge dz - d\eta \wedge dy$. Next, for a given $z_0 \in Z$, we consider the following diagram,

$$
\begin{array}{c}
\mathcal{C} \quad \Pi_{T^*Z} \\
\downarrow \Pi_Z \quad \downarrow \Pi_{T^*_0Z} \\
T^*Y \setminus 0 \quad Z \quad T^*_0Z \setminus 0
\end{array}
$$

where $\Pi_{T^*Y}$, $\Pi_{T^*_0Z}$ and $\Pi_Z$ are projections from $\mathcal{C}$ to $T^*Y \setminus 0$, $T^*_0Z \setminus 0$ and $Z$, respectively. The first part of the cinematic curvature condition is an assumption on the nondegeneracy of the first two projections in (1.1), requiring that both of them are submersions,

$$\text{rank } d\Pi_{T^*Y} = 2n, \quad (1.2)$$

$$\text{rank } d\Pi_Z = n + 1. \quad (1.3)$$

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The second part of this condition concerns the curvature properties of the image of $\Pi T^*_0 Z$ from $\mathcal{C}$, denoted by $\Gamma_{z_0} = \Pi T^*_0 Z(\mathcal{C})$, as an immersed hypersurfaces in the cotangent space $T^*_0 Z \setminus 0$. Then, as a consequence of (1.2)(1.3) and the homogeneity of $\mathcal{C}$, $\Gamma_{z_0}$ is a smooth conic $n-$dimensional hypersurface in $T^*_0 Z \setminus 0$. We shall impose the cone condition on $\mathcal{C}$ by requiring that for every $\zeta \in \Gamma_{z_0}$, there are $n - 1$ principal curvatures which do not vanish. We say thus $\mathcal{C}$ satisfies the cinematic curvature condition, if it satisfies (1.2)(1.3) and the cone condition.

One of the motivation on the study of this kind of operators may date back to the work of Stein [29], investigating the $L^p \to L^p$ boundness of maximal average operators on the sphere, which is in close connection to the pointwise convergence properties of wave equations and some of other topics in harmonic analysis on symmetric spaces. When $n \geq 3$, Stein proved the optimal $L^p \to L^p$ estimate in [29] for $\frac{n}{n-1} < p \leq \infty$, while the two dimensional case was left open due to impossibility of using the $L^2$—theory of Fourier transform. This problem in the two dimensional case was settled a couple of years later by Bourgain [4] and generalized to the variable coefficient setting by Sogge [27], where a primary version of local smoothing property for wave equations was employed in $(1 + 2)$ dimension to prove the $L^p \to L^p$ boundness of circular maximal averaging operators. A much more profound connection between the circular maximal functions and the local smoothing properties of wave equations was found and clarified later in [23], where the authors provided a much simplified proof of Bourgain’s circular maximal theorem based on a more sophisticated local smoothing property of 2D wave equations in Euclidean space, a greatly improved result in the constant variable setting compared to [27]. These results were generalized to an abstract theory concerning Fourier integral operators fulfilling the cinematic curvature condition later by Mockenhaupt, Seeger and Sogge [24] for all dimensions $n \geq 2$.

It is conjectured in [27] for the wave equation on Euclidean space, and in [24] for Fourier integral operators satisfying the cinematic curvature condition that for $p \geq 2n/(n - 1)$, there should always be an order $1/p$ local smoothing property for these operators, i.e., whenever $\sigma < -\frac{n-1}{2} + \frac{n}{p}$, the operators $\mathcal{F} \in I^{\sigma - \frac{1}{2}}(Z;Y;\mathcal{C})$ are bounded from $L^p_{\text{comp}}(Y)$ to $L^p_{\text{loc}}(Z)$.

This is referred as the local smoothing conjecture in the literature of modern Fourier analysis and has attracted extensive works of study. If this conjecture would have been proved, then it would imply positive answers to a number of open conjectures concerning fundamental problems of harmonic analysis and geometric measure theory, including (maximal) Bochner-Riesz means, Fourier restriction theorem, Kakeya and Nikodym maximal function estimates as well as Hausdorff dimensions of Besicovitch sets in all dimensions. See the works [5, 8, 31, 33, 18] and references therein.

Proving the sharp $L^p - L^p$ local smoothing estimates appears to be very difficult, even for the wave equation posed on Euclidean spaces. Using microlocal analysis and $L^2 \to L^p$ local smoothing, or rather Strichartz’s estimates in modern terminology, which turns out to be much easier to prove, Mockenhaupt, Seeger and Sogge [24] demonstrated certain $L^{q_n}$—square function inequality with $q_n := 2\frac{n+1}{n-1}$ for $n \geq 3$ (which coincides with the exponent of symmetric Strichartz’s estimate for wave equations) and established an $L^4$—square function inequality in dimension two by exploring orthogonality in circular directions via bilinear $L^2$ geometric approach. Combined with a variable coefficient version of Córdoba’s
Kakeya maximal operator along the direction of light rays, these square function estimates yielded certain non-sharp, $L^p \to L^p$ local smoothing estimates for Fourier integral operators of this kind.

Concerning the question of the constant coefficient wave equations, these local smoothing properties are known to be deduced from the cone multiplier estimates and there are subsequent improved results by many authors mainly on the $(1 + 2)$--dimensions: see Bourgain [7], the first improvement for the cone multiplier; Tao and Vargas [30], Garrigós and Seeger [14] using the bilinear method based on bilinear restriction estimates [34]; More recently, Lee [20] further improved the $L^4$ local smoothing estimate using $\ell^2$--decoupling inequality of Bourgain and Demeter [9]. All of these works are away from optimal with respect to the regularity and concern $L^p \to L^p$ local smoothing estimates for $p < q_n$.

The first sharp $L^p \to L^p$-local smoothing estimate was obtained by Wolff [32] in $(1 + 2)$--dimensions for $p > 74$ and was extended to the higher dimensional cases by Laba and Wolff [19]. The borderline for this range of $p$ was refreshed later by Garrigós, Schlag and Seeger [13] and ultimately improved down to the Strichartz exponent $p \geq q_n$ by Bourgain and Demeter [9] via their celebrated $\ell^2$--decoupling theorem. Lee and Vargas [22], using the Bourgain and Guth multilinear approach in [10] on the basis of the multilinear restriction theorem of Bennett, Carbery and Tao [3] obtained the sharp local smoothing estimates for $p = 3$.

Notice that none of the above works deals with possible improvements for abstract theory of [24] at the generality of Fourier integral operators within the framework set up at the very beginning of this section. For $p \geq q_n$, Beltran, Hickman and Sogge [2] established the sharp $L^p \to L^p$ local smoothing estimates for Fourier integral operators of this kind by extending Bourgain-Demeter’s decoupling inequality to the variable coefficient setting. In addition, they also construct examples to show the optimality of their results in odd dimensions at the level of such generality. In particular, one can not expect an order $1/p$ local smoothing estimates for all $p$ between $\frac{2n}{n-1}$ and $q_n$. It is conjectured in [2, 1] that if $n \geq 2$ is even, there should be optimal local smoothing estimates for Fourier integral operators satisfying the cinematic curvature conditions whenever $p \geq \frac{2(n+2)}{n}$. Furthermore, it is also conjectured in [2] that if one imposes a convexity condition on the cone $\Gamma_0$, with a requirement that $\Gamma_0$ always has $n - 1$ positive principal curvatures, then the optimal local smoothing property would be able to hold for these operators whenever $p \geq p_{n,+}$, with

$$p_{n,+} := \begin{cases} \frac{2(3n+1)}{3n-3}, & \text{if } n \text{ is odd,} \\ \frac{2(3n+2)}{3n-2}, & \text{if } n \text{ is even.} \end{cases}$$

When dimension $n = 2$, by interpolation with the trivial $L^2$--endpoint inequality, Beltran-Hickman-Sogge’s sharp $L^6$--estimate in [2] only recovers the $1/8$--result for the $L^4 \to L^4$ local smoothing estimate of Mockenhaupt, Seeger and Sogge [24].

To our knowledge, this is the best results so far at such a level of generality for the abstract theory of Fourier integral operators. In particular, when $n \geq 2$ and $p < q_n$, the above two conjectures formulated in [2, 1] are completely open and this paper is intended to provide some partial answers to them. Now, we state our main result.

**Theorem 1.** Let $Z$ and $Y$ be smooth paracompact manifolds of dimension 3 and 2 respectively. Suppose that $\mathcal{F} \in I^{\sigma-\frac{1}{4}}(Z,Y;\mathcal{C})$, is a Fourier integral operator of order $\sigma$, whose
canonical relation $C$ satisfies the cinematic curvature condition. Then the following estimate holds
$$\left\| \mathcal{F} f \right\|_{L^p_{\text{loc}}(Z)} \leq C \left\| f \right\|_{L^p_{\text{comp}}(Y)}, \quad \text{for all } \sigma < -\bar{s}_p + \varepsilon(p),$$  
(1.4)
where $2 \leq p \leq 6$, $\bar{s}_p = \frac{1}{2} - \frac{1}{p}$ and
$$\varepsilon(p) = \begin{cases} \frac{3}{10} - \frac{3}{8p}, & \text{if } \frac{10}{7} \leq p \leq 6, \\ \frac{3}{8} - \frac{3}{4p}, & \text{if } 2 \leq p \leq \frac{10}{3} \end{cases}$$  
(1.5)

It is natural to compare our result with the previous one in [24]. From (1.4), we obtain the $L^4 \to L^4$ estimate for $F$ whenever $\sigma < -\frac{3}{32}$, which brings in an improvement with respect to the regularity level of order $\frac{1}{32}$.

Now we turn to a couple of corollaries of Theorem 1. As in [1], the local smoothing theory of Fourier integral operators has many applications to various of interesting problems in harmonic analysis. In this paper, we include three examples of them below which follows immediately from Theorem 1 and we omit the proofs. One may consult the beautifully written self-contained article [1] for more details.

The first application concerns the maximal function estimate of Fourier integral operators. If we write $z = (x,t)$ and let $F_t f(x) = F f(x,t)$, then we have the following maximal theorem under the above assumptions in Theorem 1.

**Corollary 1.1.** Assume that $F \in I^{\sigma-1/4}(Z,Y; \mathcal{C})$ is a Fourier integral operator satisfying all the same conditions in Theorem 1. Then, if $I \subset \mathbb{R}$ is a compact interval and $Z = X \times I$ such that $X$ and $Y$ are assumed to be compact, then we have
$$\left\| \sup_{t \in I} |F_t f(x)| \right\|_{L^p(X)} \leq C \left\| f \right\|_{L^p(Y)},$$  
(1.6)
whenever $\sigma < -\bar{s}_p - \frac{1}{p} - \varepsilon(p)$ for all $2 \leq p \leq 6$.

Notice that it is conjectured in [24] that the maximal estimate (1.6) should hold as long as $\sigma < -\bar{s}_p$ for all $p \geq \frac{2n}{n-1}$. The sharp result for $p \geq 6$ is obtained in [2] and our result provides certain improvement of Corollary 6.3 in [24] for $p \leq 6$.

The second applications of Theorem 1 is related to the regularity properties of wave equations on smooth compact Riemannian manifold. Let $M$ be a smooth compact manifold without boundary of dimension $n$, equipped with a Riemannian metric $g$ and consider the Cauchy problem
$$\begin{cases} (\partial^2_t - \Delta_g) u(t,x) = 0, \; (t,x) \in \mathbb{R} \times M, \\ u(0,x) = f(x), \; \partial_t u(0,x) = h(x), \end{cases}$$  
(1.7)
where $\Delta_g$ is the Beltrami-Laplacian associated to a metric $g$. It is a well-known fact that the solution $u$ to this Cauchy problem can be written as
$$u(x,t) = \mathcal{F}_0 f(x,t) + \mathcal{F}_1 h(x,t),$$  
(1.8)
where $\mathcal{F}_j \in I^{-1/4}(M \times \mathbb{R}, M; \mathcal{C})$ with
$$\mathcal{C} = \left\{ (x,t,\xi,\tau,y,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = \pm \sqrt{ \sum g^{jk} \xi_j \xi_k } \right\},$$
where $\chi_t : T^*M \setminus 0 \times T^*M \setminus 0$ is given by flowing for time $t$ along the Hamilton vector field $H$ associated to $\sqrt{g^{jk}\xi_j \xi_k}$. As a consequence, the convexity condition is automatically verified by $\mathcal{C}$. The following result which improved Corollary 6.4 of [24] is readily deduced from Theorem 1.

**Corollary 1.2.** Let $u$ be the solution to the Cauchy problem (1.7). If $I \subset \mathbb{R}$ is a compact interval and $0 < \delta < \varepsilon(p)$ with $\varepsilon(p)$ is given by (1.5), then we have

$$
\|u\|_{L^p_{\alpha - \bar{s}_p + \delta}(M \times I)} \leq C \left( \|f\|_{L^p_{\alpha}(M)} + \|h\|_{L^p_{\alpha - 1}(M)} \right), \quad 2 \leq p \leq 6.
$$

The third application of our main theorem is about an averaging operator over smooth curves, a question raised by Sogge [27] and later extended to the higher dimensions over smooth hypersurface in $\mathbb{R}^n$ by Schlag and Sogge [26]. Let $\Sigma_{x,t} \subset \mathbb{R}^2$ be a smooth curve depending smoothly on the parameters $(x,t) \in \mathbb{R}^2 \times [1,2]$ and $d\sigma_{x,t}$ denotes the normalized Lebesgue measure on $\Sigma_{x,t}$. Following the notations in [27, 26], we may assume $\Sigma_{x,t} = \{y : \Phi(x,y) = t\}$ where $\Phi(x,y) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ such that its Monge-Ampere determinant is non-singular

$$
\det \begin{pmatrix}
0 & \frac{\partial \Phi}{\partial x} \\
\frac{\partial \Phi}{\partial y} & \frac{\partial^2 \Phi}{\partial x \partial y}
\end{pmatrix} \neq 0, \quad \text{when } \Phi(x,y) = t.
$$

This is referred to as Stein-Phong’s rotational curvature condition.

Define the averaging operator by

$$
Af(x,t) = \int_{\Sigma_{x,t}} f(y)a(x,y)d\sigma_{x,t}(y),
$$

where $a(x,y)$ is a smooth function with compact support in $\mathbb{R}^2 \times \mathbb{R}^2$.

If we let $\mathcal{F} : f(x) \mapsto Af(x,t)$, then $\mathcal{F}$ is a Fourier integral operator of order $-1/2$ with canonical relation given by

$$
\mathcal{C} = \{(x,t,\xi,\tau,y,\eta) : (x,\xi) = \chi_t(y,\eta), \tau = q(x,t,\xi)\},
$$

where $\chi_t$ is a local symplectomorphism, and the function $q$ is homogeneous of degree one in $\xi$ and smooth away from $\xi = 0$. Moreover, the rotational curvature condition holds if and only if

$$
\begin{cases}
q(x,\Phi(x,y),\Phi_x'(x,y)) \equiv 1 \\
\operatorname{corank} \ q''_{\xi\xi} \equiv 1.
\end{cases}
$$

In particular, the cinematic curvature condition is fulfilled and one has the following result by Theorem 1.

**Corollary 1.3.** Let $A_t$ be an averaging operator defined in (1.11) with $\Sigma_{x,t}$ satisfying the geometric conditions described as above. Then there exists a constant $C$ depending on $p$ such that if $f \in L^p(\mathbb{R}^2)$, we have

$$
\|Af\|_{L^p_{\gamma}(\mathbb{R}^{2+1})} \leq C \|f\|_{L^p(\mathbb{R}^2)}, \quad 2 \leq p \leq 6,
$$

for all $\gamma < -\bar{s}_p + \frac{1}{2} + \varepsilon(p)$. 


1.2. Square function estimates and an overview of the proof. Let us briefly recall the strategy in [24], which reduced Theorem 1 to a square function estimate for oscillatory integrals of Hörmander type. By interpolation with the trivial $L^2 \to L^2$ estimate and the sharp $L^6 \to L^6$ local smoothing estimate of [2],

$$\|\mathcal{F}f\|_{L^6_{\text{loc}}(Z)} \leq C \|f\|_{L^6_{\text{comp}}(Y)}, \text{ for all } \sigma < -\frac{1}{6},$$

(1.15)

one may reduce (1.4) to

$$\|\mathcal{F}f\|_{L^{10/3}_{\text{loc}}(Z)} \leq C \|f\|_{L^{10/3}_{\text{comp}}(Y)}, \text{ for all } \sigma < -\frac{1}{20}.$$  

(1.16)

Working microlocally, one may write an operator $\mathcal{F}$ in the class $I_{\sigma-1/4}(Z,Y;\mathcal{C})$ with $\mathcal{C}$ satisfying the cinematic curvature condition in an appropriate local coordinates as an oscillatory integrals

$$\mathcal{F}f(z) = \int e^{i\phi(z,\eta)} a(z, \eta) \hat{f}(\eta) \, d\eta,$$

where $a$ is a smooth symbol of order $\sigma$ supported in a specific conic subset of $T^*Z$ and the cinematic curvature condition is manifested by explicit analytic properties of phase function $\phi$, which is homogeneous one in $\eta$. In particular, we may assume that $a(z, \eta)$ vanishes unless $\eta$—is contained in a small neighborhood of the north pole $e_2 = (0,1)$. By the standard Littlewood-Paley decomposition and scaling argument, we are reduced to the study of an operator

$$T_\lambda(f)(z) := \int e^{i\lambda\phi(z,\eta)} a_\lambda(z, \eta) f(\eta) \, d\eta,$$

(1.17)

with $\lambda$ being taken sufficiently large and $a_\lambda$ being a smooth symbol of order zero after an appropriate normalization. Next, one would like to introduce an additional decomposition with respect to the $\eta$-variable in the angular direction. Specifically, we make a covering over the unit circle of the $\eta$-variable by sectors of radius $\approx \lambda^{-1/2}$. There is an associated decomposition of the oscillatory integral operator with respect to the angular directions

$$T_\lambda^\nu(f)(z) = \int e^{i\lambda\phi(z,\eta)} a_\lambda^\nu(z, \eta) f(\eta) \, d\eta,$$

(1.18)

so that $T_\lambda(f) = \sum_\nu T_\lambda^\nu(f)$. We shall return to this issue later with more details in the next section.

With the above preparation at hand, there are two crucial ingredients towards our main theorem in spirit of Mockenhaupt, Seeger and Sogge [23, 24]. The first one is a sharp $L^2$—estimate for variable coefficient versions of the Kakeya maximal function, which had appeared in the work of Córdoba [11]on Bochner-Riesz multiplier problems. Let $Z$ and $Y$ be as in the last subsection and for $(y, \xi) \in \Pi_{T^*Y}(\mathcal{C})$, we set

$$\gamma_{y,\xi} = \{z \in Z : (z, \zeta, y, \xi) \in \mathcal{C}, \text{ for some } \zeta\},$$

(1.19)

which is a smooth curve immersed in $Z$. Fix a smooth metric on $Z$ and define for $\delta > 0$ being a small number

$$R_{y,\xi}^\delta = \{z \in Z : \text{dist}(z, \gamma_{y,\xi}) < \delta\}.$$  

1 We refer to the next section for more details.
Let $\alpha \in C^\infty_0(Y \times Z)$ and put
\[ M_\delta g(y) = \sup_{\xi \in \Pi_{y, Y}^* (C)} \frac{1}{\text{Vol}(R^3_{y, \xi})} \left| \int_{R^3_{y, \xi}} \alpha(y, z) g(z) \, dz \right|. \]
Then we have
\[ \|M_\delta\|_{L^p \rightarrow L^p} \leq C \left( \log \frac{1}{\delta} \right)^{\frac{1}{2}}, \quad \text{for } 2 \leq p \leq \infty, \tag{1.20} \]
which is readily deduced from the following theorem established in [24] by interpolating with the trivial $L^\infty \rightarrow L^\infty$ estimate.

**Theorem 1.4.** Assume $Z$ and $Y$ are para-compact smooth manifold equipped with a smooth metric and $\dim Z = 3$, $\dim Y = 2$. Suppose that $\mathcal{C}$ satisfies the cinematic curvature condition. Then there exists a constant $C$ such that if $g \in L^2(Z)$, then
\[ \|M_\delta g\|_{L^2(Y)} \leq C \sqrt{\log \frac{1}{\delta}} \|g\|_{L^2(Z)}. \tag{1.21} \]

Aside from (1.20), the second ingredient is to show the following square function estimate in order to obtain Theorem 1. One may consult [24] for more details how our main theorem can be deduced from (1.20) and (1.22) below.

**Proposition 1.5.** Let $Z$ and $Y$ be smooth paracompact manifolds of dimension three and two respectively. Suppose that $\mathcal{F} \in I^{2-\frac{1}{4}}(Z, Y; \mathcal{C})$, is a Fourier integral operator of order $\sigma$, whose canonical relation $\mathcal{C}$ satisfies the cinematic curvature condition. Let $T_\lambda$ and $T_\nu$ be given by (1.17) and (1.18), then for $\lambda$ being sufficiently large, one has up to an error term which behaves like $O(\lambda^{-N})$ for arbitrarily large $N$
\[ \|T_\lambda(f)\|_{L^\infty(\mathbb{R}^3)} \lesssim_{\phi, N, \varepsilon} \lambda^\frac{1}{20} \left( \sum_\nu \|T_\nu^p(f)\|^2 \right)^{1/2} \|f\|_{L^\infty(\mathbb{R}^3)}. \tag{1.22} \]

The proof Proposition 1.5 will occupy the rest part of this paper and will be divided roughly into three parts. We first explore an equivalence of (1.22) with its bilinear version in Section 3 and then we dedicated Section 4 to the proof of this bilinear square-function estimate based on a bilinear oscillatory integral estimates (see Section 2). The implementation of bilinear oscillatory integral estimates calls for an application of the locally constant property which is allowed by the uncertainty principle. To this end, we need to introduce an additional decomposition along the radial directions cutting each sector into a union of blocks. Finally, in Section 6, we add up these blocks along the radial direction after the use of bilinear oscillatory integral estimates, by adapting a strategy of [2], which is an approximation argument via stability property of oscillatory integrals estimates in square function norms.

We end up this section with an explanation on the reason why we only focus on the two dimensional case. Indeed, the square function estimate (1.22) can be generalized verbatim to higher dimensions with $\dim Z = n + 1$ and $\dim Y = n \geq 3$ as can be seen in the last section. However, the intention of using the strategy of this paper to answer the question raised by Beltran, Hickman and Sogge in [2] on the local smoothing property of Fourier integral operators satisfying the cinematic curvature conditions when $p \leq q_n$ with or without the convexity assumption, is less satisfactory when $n \geq 3$. Indeed, one may check that the regularity result of higher dimensions derived by means of the method...
used in this paper is even worse than that obtained simply by interpolating the sharp $L^{q_0} \to L^{q_0}$ estimate in [2] and the trivial $L^2 \to L^2$ estimate. This is because the bilinear oscillatory integral estimates as well as $L^p-$ estimate for Kakeya maximal functions would have to afford an amount of loss of derivatives in dimensions higher than two.

However, this fact does not mean that the Mockenhaupt-Seeger-Sogge approach via “square functions ⊕ Kakeya” is not promising towards the resolution of the local smoothing conjecture, at least in the constant variable setting.

Notations. If $a$ and $b$ are two positive quantities, we write $a \lesssim b$ when there exists a constant $C > 0$ such that $a \leq Cb$ where the constant will be clear from the context. When the constant depends on some other quantity $M$, we emphasize the dependence by writing $a \lesssim_M b$. We will write $a \approx b$ when we have both $a \lesssim b$ and $b \lesssim a$. We will write $a \ll b$ (resp. $a \gg b$) if there exists a sufficiently large constant $C > 0$ such that $Ca \leq b$ (resp. $a \geq Cb$).

We adopt the notion of nature numbers $\mathbb{N} = \mathbb{Z} \cap [0, +\infty)$. For $\lambda \gg 1$, we use RapDec($\lambda$) to mean a quantity rapidly decreasing in $\lambda$. We use $a \lesssim b$ to mean $a \lesssim_{\varepsilon} \lambda^{\varepsilon}b$ for arbitrate $\varepsilon$.

Throughout this paper, $w_B$ is a rapidly decaying weights concentrated on the ball centered at $c(B)$ and of radius $r(B)$,

$$w_B(z) \lesssim \left(1 + \frac{|z - c(B)|}{r(B)}\right)^{-N}, \quad N \gg 1.$$

2. Preliminaries and Reductions

Given a point $(z_0, \zeta_0, y_0, \eta_0) \in T^*Z \setminus 0 \times T^*Y \setminus 0$, there exists a sufficiently small local conic coordinate patch around it, along with a smooth function $\phi(z, \eta)$ such that $\mathcal{E}$ is given by

$$\{(z, \phi'_z(z, \eta), \phi'_\eta(z, \eta), \eta) : \eta \in (\mathbb{R}^2 \setminus 0) \cap \Gamma_{\eta_0}\}$$

where $\Gamma_{\eta_0}$ denotes a conic neighborhood of $\eta_0$.

By splitting $z = (x, t) \in \mathbb{R}^2 \times \mathbb{R}$ into space-time variables, where we put $z_0 = 0$ without loss of generality, any operator $\mathcal{F}$ in the class $I^{\sigma-1/4}(Z, Y; \mathcal{E})$ with $\mathcal{E}$ satisfying the cinematic curvature condition can be written in an appropriate local coordinates as a finite sum of oscillatory integrals

$$\mathcal{F} f(x, t) = \int_{\mathbb{R}^n} e^{i\phi(x, t, \eta)}b(x, t, \eta)\hat{f}(\eta) \, d\eta,$$

where $b$ is a smooth symbol of order $\sigma$. We may assume that the support of the map $z \to b(z, \eta)$ is contained in a ball $B(0, \varepsilon_0)$, with $\varepsilon_0 > 0$ being sufficiently small and $\eta \to b(z, \eta)$ is supported in a conic region $V_{\varepsilon_0}$, i.e.

$$b(x, t, \eta) = 0 \text{ if } \eta \notin V_{\varepsilon_0} := \{\xi = (\xi_1, \xi_2) : \mathbb{R}^2 \setminus 0 : |\xi_1| \leq \varepsilon_0 |\xi_2|\}.$$

Fix $\lambda \gg 1$ and $\beta \in C_\infty^\infty(\mathbb{R})$ which vanishes outside the interval $(1/4, 2)$ and equals one in $(1/2, 1)$. By standard Littlewood-Paley decomposition, (1.16) can be deduced from

$$\|\mathcal{F}_\lambda f\|_{L^{10}(\mathbb{R}^3)} \lesssim \lambda^{\frac{1}{10}} \|f\|_{L^{10}(\mathbb{R}^3)}, \quad (2.2)$$
Assume \( C \) and the angular direction, each of which is homogeneous of degree 0, such that spreading an angle \( \approx \) where \( \nu \) would have (a)

Let \( \{ \chi_\nu(a) \} \) be a series of smooth cutoff function associated with the decomposition in the angular direction, each of which is homogeneous of degree 0, such that \( \{ \chi_\nu \}_\nu \) forms a partition of unity on the unit circle and then extended homogeneously to \( \mathbb{R}^2 \setminus 0 \) such that

\[
\sum_{0 \leq \nu \leq N_\lambda} \chi_\nu(\eta) \equiv 1, \quad \forall \eta \in \mathbb{R}^2 \setminus 0,
\]

\[
|\partial^\alpha \chi_\nu(\eta)| \leq C_\alpha \lambda^{\frac{|\alpha|}{2}}, \quad \forall \alpha \in \mathbb{N}^2 \text{ if } |\alpha| = 1.
\]

Define

\[
T_\lambda f(z) = \int e^{i\lambda \phi(z,\eta)} a(z,\eta) f(\eta) \, d\eta, \quad T_\lambda^\nu f(z) = \int e^{i\lambda \phi(z,\eta)} a^\nu(z,\eta) f(\eta) \, d\eta,
\]

where \( a^\nu(z,\eta) = \chi_\nu(\eta) a(z,\eta) \). Then \( T_\lambda f = \sum_\nu T_\lambda^\nu f \). As explained in the first section, we would have (1.16) provided we could have proved the following square function estimate

\[
\|T_\lambda f\|_{L^{\frac{20}{19}}(\mathbb{R}^2)} \lesssim \lambda^{1/20} \left( \sum_\nu \|T_\lambda^\nu f\|^2 \right)^{1/2} + \text{RapDec} \|f\|_{L^{\infty}(\mathbb{R}^2)}.
\]

For technical reasons, we assume \( a \) is of the form \( a(z,\eta) = a_1(z) a_2(\eta) \), where

\[
a_1 \in C_c^\infty(B(0,\varepsilon_0)), \quad a_2 \in C_c^\infty(B(e_2,\varepsilon_0)).
\]

The general cases may be reduced to this special one via the following observation

\[
T_\lambda f(z) = \int_{\mathbb{R}^3} e^{i(z,\xi)} \left( \int_{\mathbb{R}^2} e^{i(\phi(\eta))} \hat{\psi}(\xi,\eta) f(\eta) \, d\eta \right) \, d\xi,
\]

and that \( \xi \mapsto \hat{\psi}(\xi,\eta) \) is a Schwartz function, where \( \psi(\xi) \) is a compactly supported smooth function and equals 1 on \( \text{supp}_z a \).

We may reformulate (1.2) (1.3) and the curvature condition as

\[
\text{H}_1 \quad \text{rank } \partial^2_{z\eta} \phi(z,\eta) = 2 \text{ for all } (z,\eta) \in \text{supp } a.
\]

\[
\text{H}_2 \quad \text{Define the Gauss map } G : \text{supp } a \to S^2 \text{ by } G(z,\eta) := \frac{G_0(z,\eta)}{|G_0(z,\eta)|} \text{ where }
\]

\[
G_0(x,\eta) := \partial_{\eta_1} \partial_{z_1} \phi(z,\eta) \wedge \partial_{\eta_2} \partial_{z_2} \phi(z,\eta).
\]

The curvature condition

\[
\text{rank } \partial^2_{\eta_0} (\partial_z \phi(z,\eta), G(z,\eta_0))|_{\eta_0 = \eta_0} = 1
\]

holds for all \( (z,\eta_0) \in \text{supp } a \).
The process of reduction in the above paragraphs is quite standard and we recommend the reader to consult the works \cite{1, 2, 23, 24} in the literature.

2.1. Normalization of the phase function. The conditions $H_1, H_2$ imply that there exists a special coordinate system, so that the phase function $\phi(z, \eta)$ can be written in a normalized form. More precisely, if we write $z = (x, t)$ and assume the normal vector $G(z, \eta)$ is parallel to the $t$-direction at $(0, e_2)$, then up to multiplying harmless factors to $T_\lambda f$ and $f$, we can write $\phi$ in this coordinate as

$$\phi(x, t, \eta) = (x, \eta) + \frac{t}{2} \partial_\eta \phi(0, e_2) \eta_1^2 / \eta_2 + \eta_2 \mathcal{E}(x, t, \eta_1 / \eta_2)$$

where $\eta = (\eta_1, \eta_2)$ and $\mathcal{E}(x, t, s)$ obeys

$$\mathcal{E}(x, t, s) = O\left((|x| + |t|)^2 s^2 + (|x| + |t|) |s|^3 \right).$$

An additional change of variables in $t$ allows us to assume $\partial_t \partial_{\eta_1} \phi(0, e_2) = 1$. Furthermore, for any given $N$ being sufficiently large, we may also assume the uniform bound of the higher order derivatives of the phase functions, analytically,

$$|\partial_\eta \phi(z, \eta)| \leq 1/2, \quad 0 \leq |\beta| \leq N, |\alpha| = 2, \text{ for all } (z, \eta) \in \text{supp } a.$$  

Otherwise, (2.11) can be guaranteed by replacing $\phi(z, \eta)$ by $A \phi(z/A, \eta)$ with $A > 0$ being sufficiently large depending on $N$ and $\phi$ along with its derivatives evaluated on the support of $a$. For more details, see Beltran, Hickman, Sogge \cite{2}, Lee \cite{21}, and the previous works of Bourgain \cite{6} and Hörmander \cite{17}.

Let $\Upsilon_{x,t}: \eta \to \partial_x \phi(x, t, \eta)$. If $\varepsilon_0$ is taken sufficiently small, $\Upsilon_{x,t}$ is a local diffeomorphism on $B(e_2, \varepsilon_0)$. If we denote by $\Psi_{x,t}(\xi) = \Upsilon_{x,t}^{-1}(\xi)$ the inverse map of $\Upsilon_{x,t}$, then clearly

$$\partial_x \phi(x, t, \Psi_{x,t}(\xi)) = \xi.$$  

Differentiating (2.12) with respect to $\xi$ on both sides yields

$$[\partial_{x, y} \phi](x, t, \Psi_{x,t}(\xi)) \partial_\xi \Psi_{x,t}(\xi) = \text{Id}.$$  

This manifests that

$$\det \partial_\xi \Psi_{x,t}(\xi) \neq 0, \quad \forall (x, t) \in B(0, \varepsilon_0), \quad \forall \xi \in \Upsilon_{x,t}(B(e_2, \varepsilon_0)).$$

Let

$$q(x, t, \xi) = \partial_t \phi(x, t, \Psi_{x,t}(\xi)).$$

Then, we have

$$\partial_t \phi(x, t, \eta) = q(x, t, \partial_x \phi(x, t, \eta)).$$

2.2. A bilinear estimate for oscillatory integrals. For $j = 1, 2$, we consider two oscillatory integral operators

$$W^j_{\lambda f}(z) = \int e^{i\lambda \phi_j(z, \eta)} a_j(z, \eta) f(\eta) \, d\eta, \quad z = (x, t) \in \mathbb{R}^2 \times \mathbb{R},$$

where $\phi_j$ satisfies $H_1, H_2$ and $a_j \in C^\infty_c(\mathbb{R}^3 \times \mathbb{R}^2)$. We shall use the following bilinear oscillatory integral estimates established in \cite{21}, which is an extension to the variable coefficient case of the bilinear adjoint restriction theorem of Wolff \cite{34}. Specifically, for each multi-index $\beta \in \mathbb{N}^3$, assuming that

$$|\partial_\xi \phi_j(z, \eta)| \leq A_\beta, \text{ for } 0 \leq |\beta| \leq N, j = 1, 2, \text{ for all } (z, \eta) \in \text{supp } a,$$
we have the following theorem.

**Theorem 2.1** ([21]). Let \( \phi_j \) with \( j = 1, 2 \) be smooth homogeneous functions of degree 1 with respect to \( \eta \), which satisfies conditions \( H_1, H_2 \). Assume that \( \partial_x \phi_j \neq 0 \) on the support of \( a_j \), which is small enough such that \( \phi_j \) satisfies (2.16). Suppose that

\[
\text{rank } \partial_{\eta_j}^2 q_j(x, t, \partial_x \phi_j(x, t, \eta^{(j)})) = 1
\]

on \( \text{supp } a_j \) and

\[
\left| \left( \frac{\partial_x \phi_j(z, \eta^{(j)})}{[\partial_x \phi_j(z, \eta^{(j)})]} \right) \cdot \partial_h q_1(z, \partial_x \phi_1(z, \eta^{(1)})) - \partial_h q_2(z, \partial_x \phi_2(z, \eta^{(2)})) \right| \geq c_0 > 0
\]

for \( j = 1, 2 \), whenever \( (z, \eta^{(1)}) \in \text{supp } a_1 \) and \( (z, \eta^{(2)}) \in \text{supp } a_2 \). Then for every \( p \geq \frac{5}{3} \), one has the bilinear estimate

\[
\| W_1^1 f W_1^2 g \|_{L^p(\mathbb{R}^{2+1})} \leq C(A_\beta, \varepsilon, c_0) \lambda^{-\frac{3}{2} + \varepsilon} \| f \|_{L^2(\mathbb{R}^2)} \| g \|_{L^2(\mathbb{R}^2)}.
\]

for a finite number of \( \beta \).

It remains to prove (2.6). To this end, we will employ in the following context the bilinear approach of [30, 14] and stability property which enables us to approximate the oscillatory integral operator by extension operator at an appropriate small scale.

### 3. Reduction to bilinear square-function estimate

We will convert (2.6) to its bilinear equivalent version which will be then established based on bilinear oscillatory estimate. One direction of the implication is a straightforward application of the Hölder’s inequality, the reverse direction is delicate.

**Proposition 3.1.** Assume \( \Omega, \Omega' \) are two sets consisting of \( \nu \) and \( \nu' \) respectively, which satisfy the following angular separation condition:

\[
\text{Ang}(\theta_\nu, \theta_{\nu'}) \approx 1, \text{ for every pair } (\nu, \nu') \in \Omega \times \Omega',
\]

where \( \text{Ang}(\theta_\nu, \theta_{\nu'}) \) measures the angle between \( \theta_\nu, \theta_{\nu'} \). Let \( T^\nu_\lambda \) be the operator defined in (1.17), with \( \phi \) satisfying \( H_1, H_2 \) and \( \alpha \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^2) \). If

\[
\left\| \sum_{\nu \in \Omega} T^\nu_\lambda g \sum_{\nu' \in \Omega'} T^{\nu'}_{\lambda'} h \right\|_{L^{5/3}(\mathbb{R}^{2+1})} \leq \lambda^{1/10} \left( \sum_{\nu \in \Omega} \left| T^\nu_\lambda g \right|^2 \right)^{1/2} \left( \sum_{\nu' \in \Omega'} \left| T^{\nu'}_{\lambda'} h \right|^2 \right)^{1/2}
\]

holds, up to a RapDec(\( \lambda \)) term, for all functions \( g, h \), where the implicit constant depends on the constant \( c_0 \) appearing in (2.20) and \( A_\beta \) for finite many \( \beta \)'s, then we have (2.6).

**Proof.** We first perform a Whitney type decomposition of the product sectors with respect to dyadic scales between \( \lambda^{-1/2} \) and 1, and then single out the contribution of bilinear forms corresponding to pairs of the generated subsectors for the off-diagonal part at each individual dyadic level by an orthogonality argument. In the second step, we use parabolic rescaling to reduce the bilinear forms at each dyadic scale to the situation in the statement of the proposition.
Step 1. Orthogonality argument. Let $j_0$ be the largest integer with $2^{j_0} \leq \lambda^{1/2}$. For each $j \in \mathbb{Z}$ with $|\log_2 \varepsilon_0| \leq j \leq j_0$, we denote by $\{\theta_{j,l}\}_l$ a collection of sectors of scale $\approx 2^{-j}$, where $l \in \{1, 2, \ldots, l_j\}$ for some $l_j \approx 2^j$, with the property that for each $l$, $\theta_{j,l}$ is a union of $\approx 2^{-j} \lambda^{1/2}$ many of consecutive $\theta_{j'}$’s as introduced in Section 2. Take a Whitney-type decomposition of $C(e_2, \varepsilon_0) \times C(e_2, \varepsilon_0)$

$$C(e_2, \varepsilon_0) \times C(e_2, \varepsilon_0) = \left( \bigcup_{j: \varepsilon_0^{-1} \leq 2^j \leq \varepsilon_0^{3/2} 2^{j_0}} \bigcup_{\theta_{j,l} \times \theta_{j,l'}} \right) \cup \left( \bigcup_{\text{Ang}(\theta_{j_0, l}, \theta_{j_0, l'}) \leq \varepsilon_0^{3/2} 2^{-j_0}} \theta_{j_0, l} \times \theta_{j_0, l'} \right),$$

where $\text{Ang}(\theta_{j,l}, \theta_{j,l'}) \approx 2^{-j}$ means $2^{-j(j+1)} \leq \text{Ang}(\theta_{j,l}, \theta_{j,l'}) \leq 2^{-(j-1)}$. For $1 \leq l \leq l_j$, we put $\chi_{j,l}(\eta) = \sum_{\nu: \theta_{\nu} \subset \theta_{j,l}} \chi_{\nu}(\eta)$, and set $a_{j,l}(x, t, \eta) = \chi_{j,l}(\eta)a(x, t, \eta)$. Define

$$(T_{\lambda}^j f)(x, t) = \int e^{i \lambda \phi(x, t, \eta)} a_{j,l}(x, t, \eta) f(\eta) \, d\eta. \quad (3.3)$$

Correspondingly, we have

$$(T_{\lambda} f)(x, t)^2 = \sum_{(l,l') \in \Lambda_j} (T_{\lambda}^{j_0, l} f)(x, t)(T_{\lambda}^{j_0, l'} f)(x, t) \quad (3.4)$$

$$+ \sum_{j: \varepsilon_0^{-1} \leq 2^j \leq \varepsilon_0^{3/2} 2^{j_0}} \sum_{(l,l') \in \Lambda_j} (T_{\lambda}^{j, l} f)(x, t)(T_{\lambda}^{j, l'} f)(x, t), \quad (3.5)$$

where $\Lambda_j$ denotes the collection of $(l, l')$ indicating the sectors satisfying $\text{Ang}(\theta_{j,l}, \theta_{j,l'}) \approx 2^{-j}$ for $\varepsilon_0^{-1} \leq 2^j \leq \varepsilon_0^{3/2} 2^{j_0}$, and $\Lambda_{j_0}$ denotes the collection of $(l, l')$ indexing the sectors satisfying $\text{Ang}(\theta_{j_0, l}, \theta_{j_0, l'}) \leq \varepsilon_0^{3/2} 2^{-j_0}$. Therefore, by Minkowski’s inequality

$$\|T_{\lambda} f\|_{L^{10/3}_{x,t}}^2 = \| (T_{\lambda} f)^2 \|_{L^{5/3}_{x,t}} \leq \sum_{(l,l') \in \Lambda_{j_0}} \| (T_{\lambda}^{j_0, l} f)(T_{\lambda}^{j_0, l'} f) \|_{L^{5/3}_{x,t}} \quad (3.6)$$

$$+ \sum_{j: \varepsilon_0^{-1} \leq 2^j \leq \varepsilon_0^{3/2} 2^{j_0}} \sum_{(l,l') \in \Lambda_j} \| (T_{\lambda}^{j, l} f)(T_{\lambda}^{j, l'} f) \|_{L^{5/3}_{x,t}}, \quad (3.7)$$

For the first term (3.6), since each $\theta_{j_0, l}$ involves finitely many $\theta_{j'}$’s, by Schur’s test we have

$$\| \sum_{(l,l') \in \Lambda_{j_0}} (T_{\lambda}^{j_0, l} f)(T_{\lambda}^{j_0, l'} f) \|_{L^{5/3}(\mathbb{R}^{3+1}_{x,t})} \leq \left\| \sum_{\nu} |T_{\lambda}^\nu(f)(\nu)|^{1/2} \right\|_{L^{10/3}(\mathbb{R}^{3+1}_{x,t})}^2. \quad (3.8)$$

For (3.7), we will use the assumption (3.2). To this end, we will exploit the orthogonality property to prove, up to a RapDec($\lambda$) term

$$\| \sum_{(l,l') \in \Lambda_j} (T_{\lambda}^{j, l} f)(T_{\lambda}^{j, l'} f) \|_{L^{5/3}(\mathbb{R}^{3+1}_{x,t})} \approx \left( \sum_{(l,l') \in \Lambda_j} \left\| (T_{\lambda}^{j, l} f)(T_{\lambda}^{j, l'} f) \right\|_{L^{5/3}(\mathbb{R}^{3+1}_{x,t})}^{5/3} \right)^{3/5}. \quad (3.8)$$

The proof is essentially from [21]. For the sake of self-containedness, we sketch it below. Let $\psi(x)$ be a smooth function with its support contained in $B(0, \varepsilon_0)$ such that if we set
\( \psi_j^\mu(x) := \psi(2^j x - \mu) \), then we have
\[ \sum_{\mu \in \mathbb{Z}^2} \psi_j^\mu(x) \equiv 1. \tag{3.9} \]

Set \( \Phi(x, t, \eta, \eta') = \phi(x, t, \eta) + \phi(x, t, \eta') \) and define
\[
P_{\lambda, \mu}^{j,l'}(x, t, \eta, \eta') = \int e^{i\lambda \Phi(x, t, \eta, \eta')} d\eta \int e^{i\lambda \Phi(x, t, \eta, \eta')} d\eta' A_{\lambda, \mu}^{j,l'}(x, t, \eta, \eta') f(x) f(\eta') d\eta d\eta',
\]
where
\[
A_{\lambda, \mu}^{j,l'}(x, t, \eta, \eta') = a_{\lambda}^{j}(x, t, \eta) a_{\lambda}^{j,l'}(x, t, \eta').
\]

Denoting by \( r = |\eta|, r' = |\eta'| \) and by \( \kappa_j, \kappa_j' \) the center of \( \theta_{j,l} \) and \( \theta_{j,l'} \) respectively, in view of the support of \( \psi_j^\mu(x) \) and the cone condition, we have
\[
\partial_x (\lambda \Phi(x, t, \eta, \eta') - x\xi) = \lambda \partial_x \Phi(2^{-j} \mu, t, r\kappa_j, r'\kappa_j') - \xi + O(\lambda 2^{-j}).
\]

It follows from integration by parts that for \( \angle(\eta, \kappa_j) \leq \varepsilon_02^{-j} \) and \( \angle(\eta', \kappa_j') \leq \varepsilon_02^{-j} \) with \( (l, l') \in \Lambda_j \), the function \( \xi \to \mathcal{P}_{\lambda, \mu}^{j,l'}(\xi, t, \eta, \eta') \) decreases fast outside the rectangle
\[
\mathcal{R}_{\lambda, \mu}^{j,l'}(t) = \{ \xi \in \mathbb{R}^2 : |\xi - \lambda r\partial_x \Phi(2^{-j} \mu, t, \kappa_j)| \leq \lambda 2^{-j}, r \in (2-2\varepsilon_0, 2+2\varepsilon_0) \}.
\]

We define a smooth bump function \( R_{\lambda, \mu}^{j,l'}(\xi) \) associated with \( \lambda^\varepsilon \mathcal{R}_{\lambda, \mu}^{j,l'}(t) \), then, we have
\[
\left\| (\text{Id} - R_{\lambda, \mu}^{j,l'}(D_x, t)) \mathcal{A}_{\lambda, \mu}^{j,l'}(f)(t, \cdot) \right\|_{L^{10/3}(\mathbb{R}^2)} \lesssim \text{RapDec}(\lambda) \| f \|_{L^{10/3}(\mathbb{R}^2)}^2. \tag{3.10}
\]

Since the \( \xi \)-support of \( R_{\lambda, \mu}^{j,l'}(\xi) \) is contained in \( \lambda^\varepsilon \mathcal{R}_{\lambda, \mu}^{j,l'} \), by the nondegeneracy condition, we know for fixed \( \lambda^\varepsilon \mathcal{R}_{\lambda, \mu}^{j,l'} \), there are at most \( \approx \lambda^\varepsilon \) such sets intersecting with it with nonempty content. By using this finite-overlapping property we may conclude up to a fast decay term
\[
\left\| \sum_{(l, l') \in \Lambda_j} (T_{\lambda, \mu}^{j,l}(f))(T_{\lambda, \mu}^{j,l'}(f)) \right\|_{L^{5/3}(\mathbb{R}^{2+1}_{x,t})} \approx \left( \sum_{(l, l') \in \Lambda_j} \left\| (T_{\lambda, \mu}^{j,l}(f))(T_{\lambda, \mu}^{j,l'}(f)) \right\|_{L^{5/3}(\mathbb{R}^{2+1}_{x,t})}^{1/5} \right)^{5/3}. \tag{3.11}
\]

Going back to (3.8), we are reduced to show, up to a negligible term,
\[
\left\| (T_{\lambda, \mu}^{j,l}(f))(T_{\lambda, \mu}^{j,l'}(f)) \right\|_{L^{5/3}(\mathbb{R}^{2+1}_{x,t})} \lesssim (\lambda 2^{-j})^{1/3} \left\| \sum_{\nu, \theta_{j,l} \cap \theta_{j,l'}} |T_{\lambda, \nu}^{j,l}(f)|^2 \right\|_{L^{10/3}(\mathbb{R}^{2+1}_{x,t})} \left\| \sum_{\nu, \theta_{j,l} \cap \theta_{j,l'}} |T_{\lambda, \nu}^{j,l'}(f)|^2 \right\|_{L^{10/3}(\mathbb{R}^{2+1}_{x,t})}. \tag{3.12}
\]

Indeed, substituting (3.12) to (3.8), we obtain, up to a RapDec(\lambda) term, by using Cauchy-Schwarz’s inequality and \( \ell^2 \to \ell^{10/3} \)
\[
\left\| \sum_{(l, l') \in \Lambda_j} (T_{\lambda, \mu}^{j,l}(f))(T_{\lambda, \mu}^{j,l'}(f)) \right\|_{L^{5/3}(\mathbb{R}^{2+1}_{x,t})} \lesssim (\lambda 2^{-j})^{1/3} \left\| \sum_{\nu} |T_{\lambda, \nu}^{j,l}(f)|^2 \right\|_{L^{10/3}(\mathbb{R}^{2+1}_{x,t})}^2 \left\| \sum_{\nu} |T_{\lambda, \nu}^{j,l'}(f)|^2 \right\|_{L^{10/3}(\mathbb{R}^{2+1}_{x,t})}.
\]

Summing over \( j \), we obtain (2.6).
**Step 2. Parabolic rescaling.** In this step, we derive (3.12) from (3.2), where the angular separation condition can be verified via parabolic rescaling.

Let $\eta^{j,l}$ be the center of $\theta_{j,l}$ and set $\alpha_j = \eta_1^{j,l}/\eta_2^{j,l}$. Then for every $\eta \in \theta_{j,l}$, we have

$$\left| \frac{\eta_1}{\eta_2} - \alpha_j \right| \leq 2^{-j}.$$  

By change of variable $\eta_1 \to \alpha_j^t \eta_2 + \eta_1$, we may rotate the center of $\theta_{j,l}$ to the $e_2$ axis. Correspondingly, we will perform the change of variable in the space variable $x$.

$$\begin{cases}
x_1 + \alpha_j^t t = x_1^{(1)} \\
\alpha_j^t x_1 + x_2 + \frac{1}{2}(\alpha_j^t)^2 t = x_2^{(1)} \\
t = t^{(1)}
\end{cases}$$  

(3.13)

For convenience, we will use $x^{(1)}$ to denote $x^{(1)} := (x_1^{(1)}, x_2^{(1)})$. Since the transformation above is a diffeomorphism, we use $\Phi^{(1)}(x^{(1)}, t^{(1)})$ to denote the inverse map of (3.13). Under the new variable system, the phase function $\phi$ is transformed to

$$\phi^{(1)}(x^{(1)}, t^{(1)}, \eta) = \langle x^{(1)}, \eta \rangle + \frac{1}{2} t^{(1)} \frac{\eta_1^2}{\eta_2} + \eta_2 \mathcal{E}_1(x^{(1)}, t^{(1)}, \frac{\eta_1}{\eta_2} + \alpha_j^t)$$  

(3.14)

where

$$\mathcal{E}_1(x^{(1)}, t^{(1)}, \frac{\eta_1}{\eta_2} + \alpha_j^t) = \mathcal{E}\left(\Phi^{(1)}(x^{(1)}, t^{(1)}), \frac{\eta_1}{\eta_2} + \alpha_j^t\right).$$

Next, making Taylor’s expansion of $\mathcal{E}_1(x^{(1)}, t^{(1)}, \eta_1/\eta_2 + \alpha_j^t)$ as follows

$$\mathcal{E}_1(x^{(1)}, t^{(1)}, \eta_1/\eta_2 + \alpha_j^t) = \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_j^t) + \partial_s \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_j^t) \frac{\eta_1}{\eta_2}$$

$$+ \frac{1}{2} \partial^2_s \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_j^t) \left(\frac{\eta_1}{\eta_2}\right)^2$$

$$+ \frac{1}{2} \int_0^1 \partial^3_s \mathcal{E}_1(x^{(1)}, t^{(1)}, s, \frac{\eta_1}{\eta_2} + \alpha_j^t) \left(\frac{\eta_1}{\eta_2}\right)^3 (1-s)^2 ds$$

Making the change of variables

$$\begin{cases}
x_1^{(1)} + \partial_s \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_j^t) = x_1^{(2)} \\
x_2^{(1)} + \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_j^t) = x_2^{(2)} \\
\frac{1}{2} t^{(1)} + \frac{1}{2} \partial^2_s \mathcal{E}_1(x^{(1)}, t^{(1)}, \alpha_j^t) = \frac{1}{2} t^{(2)}
\end{cases}$$  

(3.15)

where as above, we use $\Phi^{(2)}(x^{(2)}, t^{(2)})$ to denote the inverse map of (3.15), the phase function of (3.14) is changed to

$$\phi^{(2)}(x^{(2)}, t^{(2)}, \eta) = \langle x^{(2)}, \eta \rangle + \frac{1}{2} t^{(2)} \eta_1^2/\eta_2 + \mathcal{E}_2(x^{(2)}, t^{(2)}, \eta_1/\eta_2),$$  

(3.16)

where

$$\mathcal{E}_2(x^{(2)}, t^{(2)}, \frac{\eta_1}{\eta_2}) = \frac{1}{2} \int_0^1 \partial^3_s \mathcal{E}\left(\Phi^{(1)} \circ \Phi^{(2)}(x^{(2)}, t^{(2)}), s, \frac{\eta_1}{\eta_2} + \alpha_j^t\right) \left(\frac{\eta_1}{\eta_2}\right)^3 (1-s)^2 ds,$$  

(3.17)
Finally, by scaling $\eta_1 \to 2^{-j}\eta_1$, the corresponding map for the amplitude becomes

$$a_{2j,l}(x^{(2)}, t^{(2)}, \eta) = a(\Phi(1) \circ \Phi(2)(x^{(2)}, t^{(2)}), 2^{-j}\eta_1 + \alpha_j^l \eta_2, \eta_2) \chi_{j,l}(\alpha_j^l \eta_2 + 2^{-j}\eta_1, \eta_2)$$

$$a_{2j,l, l'}(x^{(2)}, t^{(2)}, \eta) = a(\Phi(1) \circ \Phi(2)(x^{(2)}, t^{(2)}), 2^{-j}\eta_1 + \alpha_j^l \eta_2, \eta_2) \chi_{j,l'}(\alpha_j^l \eta_2 + 2^{-j}\eta_1, \eta_2).$$

whose support in $\eta-$variable is contained respectively in

$$D_1 = \{ \eta : |\eta_1/\eta_2| \leq \varepsilon_0/4, |\eta_2 - 1| \leq \varepsilon_0 \},$$

$$D_2 = \{ \eta' : |\eta_1/\eta_2 - \alpha_j^{l'}| \leq \varepsilon_0/4, |\eta_2' - 1| \leq \varepsilon_0 \},$$

where $\alpha_j^{l'} = 2^j(\eta_1^{j,l'/} / \eta_2^{j,l'} - \eta_1^{j,l} / \eta_2^{j,l})$ and it is easy to find $|\alpha_j^{l,l'}| \geq \varepsilon_0$ and $|\alpha_j^{l,l'}|$ does not depend $j, l, l'$. Thus to prove \((3.12)\), it suffices to show

$$\|T_{\lambda}^{j,l} f T_{\lambda}^{j,l,l'} f\|_{L^{5/3}(\mathbb{R}^3)} \leq (\lambda^{-2j})^{1/10} \left( \sum_{\nu : \theta_\nu \subset \theta_{j,l}} |T_{\lambda}^{\nu} f|^2 \right)^{1/2} \left( \sum_{\nu : \theta_\nu \subset \theta_{j,l'}} |T_{\lambda}^{\nu} f|^2 \right)^{1/2} \left( \sum_{\nu : \theta_\nu \subset \theta_{j,l'}} |T_{\lambda}^{\nu} f|^2 \right)^{1/2},$$

where

$$T_{\lambda}^{j,l} f = \sum_{\nu : \theta_\nu \subset \theta_{j,l}} T_{\lambda}^{\nu} f, T_{\lambda}^{j,l,l'} f = \sum_{\nu : \theta_\nu \subset \theta_{j,l'}} T_{\lambda}^{\nu} f,$$

$$T_{\lambda}^{j,l} f(x^{(2)}, t^{(2)}) = \int e^{i\lambda \phi(3)(x^{(2)}, t^{(2)}, \eta)} a_{2j}^{(j,l)}(x^{(2)}, t^{(2)}, \eta) f(\eta) d\eta,$$

$$T_{\lambda}^{j,l,l'} f(x^{(2)}, t^{(2)}) = \int e^{i\lambda \phi(3)(x^{(2)}, t^{(2)}, \eta)} a_{2j}^{(j,l,l')} (x^{(2)}, t^{(2)}, \eta) f(\eta) d\eta,$$

$$T_{\lambda}^{\nu} f(x^{(2)}, t^{(2)}) = \int e^{i\lambda \phi(3)(x^{(2)}, t^{(2)}, \eta)} a_{2j}^{(\nu)} (x^{(2)}, t^{(2)}, \eta) f(\eta) d\eta,$$

$$a_{2j}^{(\nu)} (x^{(2)}, t^{(2)}, \eta) = a(\Phi(1) \circ \Phi(2)(x^{(2)}, t^{(2)}), 2^{-j}\eta_1 + \alpha_j^l \eta_2, \eta_2) \chi_{\nu}(\alpha_j^l \eta_2 + 2^{-j}\eta_1, \eta_2),$$

and the phase function reads

$$\phi(3)(x^{(2)}, t^{(2)}, \eta) = 2^{-j} x_1^{(2)} \eta_1 + x_2^{(2)} \eta_2 + 2^{-2j} t^{(2)} \eta_1^2 (2\eta_2)^{-1} + \eta_2 \mathcal{E}_2(x^{(2)}, t^{(2)}, 2^{-j}\eta_1/\eta_2).$$

(3.20)

Let $\{R_{\mu}\}_\mu$ be a collection of rectangles of sidelength $\sim 2^{-j} \times 2^{-2j}$ which forms a partition of a ball $B(0, \varepsilon_0) \subset \mathbb{R}^2$. For each $\mu$, we use $x_\mu$ to denote the center of $R_\mu$ and $R_{\mu,0}^\varepsilon_0$ to denote the region $R_\mu \times (-\varepsilon_0, \varepsilon_0)$. For $B(0, \varepsilon_0) \subset \mathbb{R}^3$, we have

$$B_{\varepsilon_0} := B(0, \varepsilon_0) \subset \bigcup_\mu R_{\mu,0}^{\varepsilon_0}. \quad (3.21)$$

Therefore it suffices to show for each $R_{\mu,0}^{\varepsilon_0}$

$$\|T_{\lambda}^{j,l} f T_{\lambda}^{j,l,l'} f\|_{L^{5/3}(R_{\mu,0}^{\varepsilon_0})} \leq (\lambda^{-2j})^{1/10} \left( \sum_{\nu : \theta_\nu \subset \theta_{j,l}} |T_{\lambda}^{\nu} f|^2 \right)^{1/2} \left( \sum_{\nu : \theta_\nu \subset \theta_{j,l'}} |T_{\lambda}^{\nu} f|^2 \right)^{1/2} \left( \sum_{\nu : \theta_\nu \subset \theta_{j,l'}} |T_{\lambda}^{\nu} f|^2 \right)^{1/2},$$

(3.22)

where the implicit constant is uniform with respect to $\mu$. Note that

$$\sum_\mu w_{R_{\mu,0}^{\varepsilon_0}} = w_{B_{\varepsilon_0}}, \quad (3.23)$$
The constant appearing in (3.19) will follow by squaring both sides of (3.22) and Cauchy-Schwarz inequality after summing over all \( \mu \)'s.

By changing variables \( x^{(2)} \rightarrow x_{\mu} + (2^{-j} x_1, 2^{-2j} x_2), t \rightarrow \tilde{t} \), it suffices to show

\[
\left\| \tilde{T}_{\lambda^2 - 2j} f \tilde{T}_{\lambda^2 - 2j} f \right\|_{L^2(B_{\infty})} \lesssim (\lambda^2 - 2j) \left( \sum_{\nu' : \theta_{\nu'} \subset \theta_{j,l}} \left\| \tilde{T}_{\lambda^2 - 2j} f_{\nu'}^j \right\|_{L^\infty(w_{B_{\infty}})} \right)^2 \left( \sum_{\nu' : \theta_{\nu'} \subset \theta_{j,l'}} \left\| \tilde{T}_{\lambda^2 - 2j} f_{\nu'}^{j'} \right\|_{L^\infty(w_{B_{\infty}})} \right)^2,
\]

where the phase function in \( \tilde{T}_{\lambda^2 - 2j} f \) and \( \tilde{T}_{\lambda^2 - 2j} f \) reads

\[
\tilde{\phi}^j(\tilde{x}, \tilde{t}, \eta) = \tilde{x}_\eta + \frac{1}{2} \tilde{t}_\eta^2 / \eta_2 + 2^{2j} \eta_2 E_2(x_{\mu} + (2^{-j} x_1, 2^{-2j} x_2), t, 2^{-j} \eta_1 / \eta_2),
\]

along with the amplitudes

\[
\tilde{a}^{j,l}(\tilde{x}, \tilde{t}, \eta) = a_{2}^{j,l}(x_{\mu} + (2^{-j} x_1, 2^{-2j} x_2), \tilde{t}, \eta),
\]

\[
\tilde{a}^{j,l,l'}(\tilde{x}, \tilde{t}, \eta) = a_{2}^{j,l,l'}(x_{\mu} + (2^{-j} x_1, 2^{-2j} x_2), \tilde{t}, \eta).
\]

The constant appearing in (2.20) is independent of \( j \) which will be clarified later and the finite many of \( A_{\beta} \)'s is uniformly bounded since the error term \( E_2 \) converges to

\[
\tilde{\phi}_\infty = x_\eta + \frac{t}{2} \eta_2
\]

in the sense that

\[
\left\| \partial_x^\beta \tilde{\phi}^j - \partial_x^\beta \phi_\infty \right\|_{L^\infty} \rightarrow 0, \text{ as } j \rightarrow \infty, |\beta| \leq N,
\]

which can be seen from formula (3.17). Using (3.2) with \( \lambda \) replaced by \( \lambda 2^{-2j} \) we will obtain (3.24).

4. Use of the bilinear oscillatory integral estimate

In this section, we use the bilinear oscillatory integral estimates to prove (3.2) and this will complete the proof of (2.6) due to Proposition 3.1. Let us define the rescaled function \( \phi^\lambda \) and \( a_\lambda \) as follows

\[
\phi^\lambda(z, \eta) := \lambda \phi\left( \frac{z}{\lambda}, \eta \right), \quad a_\lambda(z, \eta) := a\left( \frac{z}{\lambda}, \eta \right).
\]

Correspondingly, we define \( T_\lambda \) as

\[
(T_\lambda f)(z) := (T f)(z / \lambda) = \int e^{i \phi^\lambda(x, t, \eta)} a_\lambda(x, t, \eta) f(\eta) d\eta.
\]

We make angular decomposition as before to write \( T_\lambda f \) as

\[
T_\lambda f(z) = \sum_\nu T_{\lambda}^\nu f(z), \quad T_{\lambda}^\nu f(z) = \int e^{i \phi^\lambda(x, t, \eta)} a_\lambda^\nu(x, t, \eta) f(\eta) d\eta,
\]

where the associated amplitude \( a_\lambda^\nu \) for \( T_{\lambda}^\nu \) is to be \( a_\lambda^\nu(z, \eta) = a_\lambda(z, \eta) \chi_\nu(\eta) \). We rewrite (3.2) in the scaled version as follows
Proposition 4.1. Let \((\Omega, \Omega')\) satisfy the same angular separation condition in the sense of (3.1). Up to a negligible term, then we have

\[
\left\| \sum_{\nu \in \Omega} T_{\lambda}^\nu f \sum_{\nu' \in \Omega'} T_{\lambda}^\nu' f \right\|_{L^2_1(\mathbb{R}^{2+1})} \lesssim \lambda^\frac{1}{16} \left( \left( \sum_{\nu \in \Omega} | T_{\lambda}^\nu f |^2 \right)^{\frac{1}{2}} \left\| \sum_{\nu' \in \Omega'} | T_{\lambda}^\nu' f |^2 \right\|_{L^{10}_1(\mathbb{R}^{2+1})} \right)^{\frac{1}{2}} \tag{4.4}
\]

Definition 4.2 (locally constant property). For \(n \geq 1\), given a function \(F : \mathbb{R}^n \to [0, \infty)\), we say \(F\) satisfies the locally constant property if \(F(x) \approx F(y)\) whenever \(|x - y| \leq C_0 \rho\). Here the implicit constant in “\(\approx\)" could depend on the structure constant \(C_0\).

Let \(E\) be the extension operator given by

\[
Ef(x, t) := \int_{\mathbb{R}^2} e^{i(x \cdot \eta + \phi(\eta))} a_2(\eta) f(\eta) d\eta,
\]

where \(h(\eta)\) is a smooth function away from the origin and homogeneous of degree 1 with

\[
\text{rank } \partial^2_{\eta \eta} h = 1, \quad \text{for all } \eta \in \text{supp } a_2.
\]

For \(r \geq 1\), if \(f\) is supported in a \(r^{-1}\)-neighborhood of \(\eta_0 \in \text{supp } a_2\), then \(\hat{E} f\) is contained in a ball of radius \(r^{-1}\), by uncertainty principle, one may view \(|Ef|\) essentially as a constant at the scale of \(r\). However, this is not the case for the oscillatory operator \(T_{\lambda}\), since \(T_{\lambda} f\) is not necessarily compactly supported. One may nevertheless, up to phase rotation and a negligible term, recover the locally constant property.

Lemma 4.3 ([16]). Let \(T_{\lambda}\) be given by (4.2). There exists a smooth, rapidly decreasing function \(g : \mathbb{R}^3 \to [0, \infty)\) with the following property: \(\text{supp } \hat{g} \subset B(0, 1)\) such that if \(\varepsilon > 0\) and \(1 \leq r \leq \lambda^{1-\varepsilon},\) \(f\) is supported in a \(r^{-1}\)-cube centered at \(\bar{\eta}\), then

\[
e^{-i\phi(z, \eta)} T_{\lambda} f(z) = \left( \left[ e^{-i\phi(z, \eta)} T_{\lambda} \right] \ast g_r \right)(z) + \text{RapDec}(\lambda) \|f\|_{L^{10/3}(\mathbb{R}^2)}
\]

holds for all \(z \in \mathbb{R}^3\), where \(g_r(z) = r^{-3} g(z/r)\).

Remark 4.4. We further choose \(g\) to satisfy the locally constant property at the scale of 1. Consequently, one may view \(g_r\) as a constant at scale of \(r\).

To prove Proposition 4.1, we need further decompose the support of \(\eta \to a_2^\lambda(z, \eta)\) in the radial direction to obtain equally-spaced pieces so that we may exploit the locally constant property. Let \(\rho \in C_\infty^0(\mathbb{R}^2)\) satisfy

\[
\sum_{j \in \mathbb{Z}^2} \rho(\eta - j) \equiv 1, \quad \eta \in \mathbb{R}^2.
\]

Let \(\varepsilon > 0\) be small and \(Q = \{Q_k\}_k\) be a mesh of cubes of sidelength \(\lambda^{1/2-\varepsilon}\), which are centered at lattices belong to \(\lambda^{1/2-\varepsilon} \mathbb{Z}^{2+1}\) with sides parallel to the axis and form a tiling of \(\text{supp}_z a(\cdot, \eta)\). For each \(Q_k \in Q\), let \(z_k\) be the center of \(Q_k\) and set

\[
T_{\lambda, k}^\nu f(z) = \int e^{i\phi(z, \eta)} a_{\lambda, k}^\nu(z, \eta) f(\eta) d\eta,
\]

\[
a_{\lambda, k}^\nu(z, \eta) = a_2^\nu(z, \eta) \rho(\lambda^{1/2-\varepsilon/2} \partial_z \phi(z_k, \eta) - j).
\]
The support of $\eta \rightarrow a^{\nu,j}_{\lambda,k}(\cdot,\cdot)$ is contained in a cube of sidelength $\approx \lambda^{-1/2+\varepsilon/2}$ which is denoted by $\mathcal{D}^{\nu,j}_k$. Ultimately, we may write
\[
\mathcal{T}_\lambda f(z) \bigg|_{z \in \mathcal{Q}_k} = \sum_{\nu,j} \mathcal{T}^{\nu,j}_\lambda f(z), \quad \mathcal{T}_\lambda^\nu f(z) \bigg|_{z \in \mathcal{Q}_k} = \sum_j \mathcal{T}^{\nu,j}_\lambda f(z). \tag{4.9}
\]

Let $\eta^{\nu,j}_k$ be the center of $\mathcal{D}^{\nu,j}_k$ and fix $R = \lambda^{1/2-\varepsilon/2}$ in what follows. The key ingredient in the proof of Proposition 4.1 is the following discrete version of bilinear estimate.

**Proposition 4.5.** Let $Q_k \in \mathcal{Q}$ be defined as above and $(\nu,\nu') \in \Omega \times \Omega'$ satisfy the angular separation condition (3.1). Then, we have
\[
\left\| \sum_{\nu,j} e^{i\phi^{\nu}(z,\eta^{\nu,j}_k)} c^{\nu,j} \sum_{\nu',j'} e^{i\phi^{\nu'}(z,\eta^{\nu',j'}_k)} c^{\nu',j'} \right\|_{L^{5/3}(Q_k)} \lesssim \lambda \left( \sum_{\nu,j} |c^{\nu,j}|^2 \right)^{\frac{1}{2}} \left( \sum_{\nu',j'} |c^{\nu',j'}|^2 \right)^{\frac{1}{2}}. \tag{4.10}
\]

**Proof.** Without loss of generality, we may assume $z_k = 0$ and normalise the phase function by setting $\psi^{\lambda}(z,\eta) = \psi^{\lambda}(z,\eta) - \psi^{\lambda}(0,\eta)$. Let
\[
b^{\nu,j}_k(z,\eta) = \left( \int_{\mathcal{D}^{\nu,j}_k} e^{i\psi^{\lambda}(z,\eta) - \psi^{\lambda}(z,\eta^{\nu,j}_k))} |\eta\rangle \right)^{-1} \chi_{\mathcal{D}^{\nu,j}_k}(\eta), \tag{4.11}
\]
where $\chi_{\mathcal{D}^{\nu,j}_k}$ denotes the characteristic function of $\mathcal{D}^{\nu,j}_k$. It is easy to see for $\lambda$ sufficiently large\(^2\)
\[
|b^{\nu,j}_k(z,\eta)| \lesssim R^2, \quad |\partial_2 b^{\nu,j}_k(z,\eta)| \lesssim R^{2-|\alpha|}, \quad \forall z \in Q_k. \tag{4.12}
\]

We may evaluate the left side of (4.10) by
\[
\left\| \left( \int e^{i\psi^{\lambda}(z,\eta)} \sum_{\nu,j} b^{\nu,j}_k(z,\eta) e^{i\nu,j} \text{d}\eta \right) \left( \int e^{i\psi^{\lambda}(z,\eta')} \sum_{\nu',j'} b^{\nu',j'}_k(z,\eta') e^{i\nu',j'} \text{d}\eta' \right) \right\|_{L^{5/3}(Q_k)}. \tag{4.13}
\]

Let
\[
B_k(z,\eta,\eta') = \sum_{\nu,j} \sum_{\nu',j'} e^{i\nu,j} b^{\nu,j}_k(z,\eta) e^{i\nu',j'} b^{\nu',j'}_k(z,\eta'). \tag{4.14}
\]

By the fundamental theorem of calculus, we may write
\[
B_k(z,\eta,\eta') = B_k(0,\eta,\eta') + \int_0^{x_1} \frac{\partial B_k}{\partial u_1}((u_1,0,0),\eta,\eta') \text{d}u_1 + \cdots + \int_0^{x_1} \int_0^{x_2} \int_0^{x_3} \frac{\partial^3 B_k}{\partial u_1 \partial u_2 \partial u_3}((u,\eta,\eta') \text{d}u. \tag{4.15}
\]

We take $B_k(0,\eta,\eta')$ as an example for the other terms can be handled in a similar way.

The explicit formula of $B_k(0,\eta,\eta')$ reads
\[
B_k(0,\eta,\eta') = \sum_{\nu,j} e^{i\nu,j} b^{\nu,j}_k(0,\eta) \sum_{\nu',j'} e^{i\nu',j'} b^{\nu',j'}_k(0,\eta'), \tag{4.16}
\]
and we are led to estimating
\[
\left\| \left( \int e^{i\psi^{\lambda}(z,\eta)} \sum_{\nu,j} e^{i\nu,j} b^{\nu,j}_k(0,\eta) \text{d}\eta \right) \left( \int e^{i\psi^{\lambda}(z,\eta')} \sum_{\nu',j'} e^{i\nu',j'} b^{\nu',j'}_k(0,\eta') \text{d}\eta' \right) \right\|_{L^{5/3}}. \tag{4.17}
\]

\(^2\)See the proof of Lemma 4.6 in [24] for a similar fact.
According to Theorem 2.1, (4.17) would be dominated as
\[ \left\| \sum_{\nu,j} e^{\nu z} b_k^{\nu,j} (0, \cdot) \right\|_{L^2} \left\| \sum_{\nu',j'} e^{\nu' z} b_k^{\nu',j'} (0, \cdot) \right\|_{L^2} \lesssim C \lambda \left( \sum_{\nu,j} |e^{\nu z}|^2 \right)^{\frac{1}{2}} \left( \sum_{\nu',j'} |e^{\nu' z}|^2 \right)^{\frac{1}{2}}, \]
provided the phase functions \( \psi^\lambda(z, \eta), \psi^\lambda(z, \eta') \) fulfill the condition (2.20).

To see this is the case, and hence complete the proof of (4.10), we resort to the angular separation condition (3.1). In fact, after changing variables \( z \to \lambda z \), it suffices to show that if \( \eta, \eta' \) satisfy (3.1), then (2.20) holds for \( \psi(z, \eta) \) and \( \psi(z, \eta') \) where
\[ \psi(x, t, \eta) = \phi(x, t, \eta) - \phi(0, \eta), \quad (4.18) \]
with \( \phi(x, t, \eta) \) of the form (2.9), i.e.
\[ \phi(x, t, \eta) = \langle x, \eta \rangle + \frac{t}{2} (\eta_1^2 / \eta_2) + \eta_2 \mathcal{E}(x, t, \eta_1 / \eta_2), \quad (4.19) \]
where the error term \( \mathcal{E}(z, s) \) satisfies
\[ \mathcal{E}(z, s) = O((|x| + |t|) s^3). \quad (4.20) \]
Furthermore, we may neglect \( \phi(0, \eta) \) since it is independent of \( (x, t) \).

To guarantee (2.20), we need to remove the dependence on \( \varepsilon_0 \) of the scale of angular separation through an additional angular transformation with respect to the \( \eta \)-variable. Without loss of generality, we may assume \( \text{supp} \eta a(z, \cdot) \) contains \( e_2 \), since the general case follows by repeating the argument in the proof of Proposition 3.1.

By changing of variable \( \eta_1 \to \varepsilon_0 \eta_1 \), we are reduced to the following property for \( \eta, \eta' \), which are contained in
\[ \mathcal{D}_1 = \left\{ \eta = (\eta_1, \eta_2) \in \mathbb{R}^2 : \left| \frac{\eta_1}{\eta_2} \right| \leq \frac{1}{10}, |\eta_2 - 1| \leq \varepsilon_0 \right\}, \]
\[ \mathcal{D}_2 = \left\{ \eta' = (\eta_1', \eta_2') \in \mathbb{R}^2 : \left| \frac{\eta_1'}{\eta_2'} - \frac{1}{2} \right| \leq \frac{1}{10}, |\eta_2' - 1| \leq \varepsilon_0 \right\}, \]
respectively. Within the above setting, the phase function now becomes
\[ \phi(x, t, \eta) = \varepsilon_0 x_1 \eta_1 + x_2 \eta_2 + \frac{t}{2} \varepsilon_0^2 \eta_1^2 / \eta_2 + \eta_2 \mathcal{E}(x, t, \varepsilon_0 \eta_1 / \eta_2). \quad (4.21) \]

Let \( \{ R_\mu \}_\mu \) be a collection of pairwise disjoint rectangles of sidelength \( \varepsilon_0 \times \varepsilon_0^2 \), covering \( B(0, \varepsilon_0) \subset \mathbb{R}^2 \). For each \( \mu \), we use \( x_\mu \) to denote the center of \( R_\mu \). For simplicity, we use \( R_\mu^{\varepsilon_0} \) to denote the rectangle \( R_\mu \times (-\varepsilon_0, \varepsilon_0) \).

Changing variables \( x \to x_\mu + (\varepsilon_0 x_1, \varepsilon_0^2 x_2) \), replacing \( \lambda \) with \( \varepsilon_0^2 \lambda \), and after neglecting harmless terms, we have (4.21) of the following form in new variables
\[ \phi(x, t, \eta) = x \cdot \eta + \frac{t}{2} \eta_1^2 / \eta_2 + \varepsilon_0^2 \eta_2 \mathcal{E}(x_\mu + (\varepsilon_0 x_1, \varepsilon_0^2 x_2), t, \varepsilon_0 \eta_1 / \eta_2). \quad (4.22) \]
Taking \((x_u + (\varepsilon_0 x_1, \varepsilon_0^2 x_2), t) \in B(0, 2\varepsilon_0) \subset \mathbb{R}^3\) into account, by direct calculation for \(\varepsilon_0\) small enough, we have
\[
\begin{align*}
\nabla_x \phi(x, t, \eta) &= \eta + O(\varepsilon_0 |\eta|^2) + O(\varepsilon_0 |\eta|^3), \\
\partial_{x,\eta}^2 \phi(x, t, \eta) &= \text{Id} + O(\varepsilon_0 |\eta|), \\
\partial_t \phi(x, t, \eta) &= \frac{1}{2} \eta_1^2/\eta_2 + O(\varepsilon_0 |\eta|^2) + O(\varepsilon_0 |\eta|^3).
\end{align*}
\]
From (2.16), we have
\[
\nabla_\eta \partial_t \phi(x, t, \eta) = (\nabla_\eta q)(x, t, \partial_x \phi(x, t, \eta)) \partial_{x,\eta}^2 \phi(x, t, \eta),
\]
and hence
\[
(\nabla_\eta q)(x, t, \partial_x \phi(x, t, \eta)) = \left(\eta_1/\eta_2, -\eta_1^2/(2\eta_2^3) \right) + O(\varepsilon_0 |\eta|).
\]
Therefore, by choosing \(\varepsilon_0\) sufficiently small, we have for \(\eta \in D_1, \eta' \in D_2\)
\[
\text{Left hand side of (2.20)} \approx |\eta_1/\eta_2 - \eta_1'/\eta_2'|^2 + O(\varepsilon_0) \approx 1. \tag{4.23}
\]
This verifies (2.20).

Now we turn to the proof of Proposition 4.1. For given \(R = \lambda^{1/2 - \varepsilon/2}\), \(\varrho_R\) has the locally constant property at scale of \(R\). For \(z \in Q_k\), we denote by
\[
\mathcal{H}_\lambda^{\nu,j} f(z) = e^{-i\phi_\lambda(z, \eta_{k}\nu')} \mathcal{F}_{\lambda,k}^{\nu,j} f(z).
\]
Based on Lemma 4.3 and the compactness of \(\text{supp}_{\eta} a_{\lambda,k}^{\nu,j}(z, \cdot)\) in (4.9), we start with proving the following estimate
\[
\sum_{Q_k \in \mathcal{Q}} \left\| \sum_{\nu,j} e^{i\phi_\lambda(z, \eta_{k}\nu')} (\mathcal{H}_\lambda^{\nu,j} f) \ast \varrho_R \sum_{\nu', j'} e^{i\phi_\lambda(z, \eta_{k}'\nu')} (\mathcal{H}_\lambda^{\nu',j'} f) \ast \varrho_R \right\|_{L^2(Q_k)} \tag{4.24}
\]
\[
\lesssim \lambda^{10} \left( \sum_{Q_k \in \mathcal{Q}} \left( \sum_{\nu,j} \left| \mathcal{F}_{\lambda,k}^{\nu,j} f \right|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \left( \sum_{Q_k \in \mathcal{Q}} \left( \sum_{\nu', j'} \left| \mathcal{F}_{\lambda,k}^{\nu',j'} f \right|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} \left( \frac{10}{L_{w_k}(w_{Q_k})} \right)^{\frac{1}{2}}.
\]
Indeed, by Minkowski’s inequality and the locally constant property at scale \(R\) enjoyed by \(\varrho_R\), we have
\[
\left\| \int \int \varrho_R(z - y) \varrho_R(z - y') \sum_{\nu, \nu', j, j'} e^{i\phi_\lambda(z, \eta_{k}\nu')} \mathcal{H}_\lambda^{\nu,j} f(y) e^{i\phi_\lambda(z, \eta_{k}'\nu')} \mathcal{H}_\lambda^{\nu',j'} f(y') \ dy dy' \right\|_{L^2(Q_k)}
\]
is bounded by
\[
\int \int \left\| \sum_{\nu,j} e^{i\phi_\lambda(z, \eta_{k}\nu')} \mathcal{H}_\lambda^{\nu,j} f(y) \sum_{\nu', j'} e^{i\phi_\lambda(z, \eta_{k}'\nu')} \mathcal{H}_\lambda^{\nu',j'} f(y') \right\|_{L^2(Q_k)} \times \varrho_R(z - y) \varrho_R(z - y') \ dy dy',
\]
whenever \(\tilde{z} \in Q_k\). We use Proposition 4.5 to obtain the following bound of the above formula,
\[
\lambda \int \int \left( \sum_{\nu,j} \left| \mathcal{F}_{\lambda,k}^{\nu,j} f(z - y) \right|^2 \right)^{\frac{1}{2}} \left( \sum_{\nu', j'} \left| \mathcal{F}_{\lambda,k}^{\nu',j'} f(z - y') \right|^2 \right)^{\frac{1}{2}} \varrho_R(y) \varrho_R(y') \ dy dy'. \tag{4.25}
\]
Thus, it remains to prove (4.25) can be controlled by
\[
\lambda^{\frac{1}{10}} \int \left\| \left( \sum_{\nu, j} |\mathcal{T}_{\lambda, k}^{\nu, j} f(z - y)|^2 \right)^{\frac{1}{2}} \right\|_{L^5_{|z|} (Q_k)} \varrho_R(y) dy \\
\times \int \left\| \left( \sum_{\nu', j'} |\mathcal{T}_{\lambda, k}^{\nu', j'} f(z - y')|^2 \right)^{\frac{1}{2}} \right\|_{L^5_{|z|} (Q_k)} \varrho_R(y') dy'.
\] (4.26)

By Hölder’s inequality, we have
\[
\int \left\| \left( \sum_{\nu, j} |\mathcal{T}_{\lambda, k}^{\nu, j} f(z - y)|^2 \right)^{\frac{1}{2}} \right\|_{L^{10/3} (w_{Q_k})} \varrho_R(y) dy \lesssim \left( \int \left( \sum_{\nu, j} |\mathcal{T}_{\lambda, k}^{\nu, j} f(z)|^2 \right)^{\frac{5}{3}} w_{Q_k} (z) dz \right)^{\frac{3}{10}},
\]
where we have used the following fact,
\[
\int_{\mathbb{R}^3} w_{Q_k}(z + y) \varrho_R(y) dy \lesssim w_{Q_k}(z).
\]
Summing over \( Q_k \in \mathcal{Q} \) and applying Cauchy-Schwarz’s inequality, we obtain (4.24).

Assuming that up to a \( \text{RapDec}(\lambda) \)–term, one may add up the blocks of square functions along radial directions
\[
\left\| \left( \sum_{\nu, j} |\mathcal{T}_{\lambda, k}^{\nu, j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^{10/3} (w_{Q_k})} \lesssim \left\| \left( \sum_{\nu} |\mathcal{T}_{\lambda}^{\nu} f|^2 \right)^{\frac{1}{2}} \right\|_{L^{10/3} (w_{Q_k})} + \text{RapDec}(\lambda) \| f \|_{L^\infty_{|\zeta|} (\mathbb{R}^3)},
\] (4.27)
we obtain (4.4) and this completes the proof of Proposition 4.1. Thus, it remains to prove (4.27) which will be achieved in the next section.

5. Adding up blocks along radial directions

This section is devoted to showing
\[
\left\| \left( \sum_{\nu, j} |\mathcal{T}_{\lambda, k}^{\nu, j} f|^2 \right)^{\frac{1}{2}} \right\|_{L^{10} (w_{Q_k})} \lesssim \left\| \left( \sum_{\nu} |\mathcal{T}_{\lambda}^{\nu} f|^2 \right)^{\frac{1}{2}} \right\|_{L^{10} (w_{Q_k})} + \text{RapDec}(\lambda) \| f \|_{L^\infty_{|\zeta|} (\mathbb{R}^3)},
\] (5.1)

The main idea is to effectively approximate \( \mathcal{T}_{\lambda} \) by an extension operator \( E \) at suitable small spatial scale.

Assume \( \delta > 0 \) and \( 1 < K \lambda^{1/2 - \delta} \). By taking Taylor expansion of \( \phi^\lambda \) around the point \( \tilde{z} \) and changing variables: \( \eta \rightarrow \Psi^\lambda (\tilde{z}, \eta) := (\tilde{z}/\lambda, \eta), \) we have
\[
\mathcal{T}_{\lambda} f(z) = \int_{\mathbb{R}^2} e^{i((z, -\partial_z \phi^\lambda (\tilde{z}, \Psi^\lambda (\tilde{z}, \eta)))+\frac{1}{2} (z, -\partial_z \phi^\lambda (\tilde{z}, \eta))} a_{\lambda, \tilde{z}} (z, \eta) f_{\tilde{z}} (\eta) d\eta, \quad \text{for} \ |z - \tilde{z}| \leqslant K, \] (5.2)
where
\[
f_{\tilde{z}} := e^{i\phi^\lambda (\tilde{z}, \Psi^\lambda (\tilde{z}, \cdot))} f \circ \Psi^\lambda (\tilde{z}, \cdot),
\]
\[
a_{\lambda, \tilde{z}} (z, \eta) = a_{\lambda} (z, \Psi^\lambda (\tilde{z}, \eta)) |\text{det} \partial_\eta \Psi^\lambda (\tilde{z}, \eta)|,
\]
and for \( |v| \leqslant K \),
\[
\varepsilon_{\lambda}^z (v, \eta) = \frac{1}{\lambda} \int_0^1 (1 - s) \langle (\partial_{\tilde{z}^2} \phi^\lambda ((\tilde{z} + sv)/\lambda, \Psi^\lambda (\tilde{z}, \eta))v, v \rangle ds.
\] (5.3)
Owing to (2.11), we have for $\lambda \gg 1$

$$\sup_{(v,\eta) \in B(0,K) \times \text{supp}_\rho a_{t,\hat{\xi}}} |\partial_\eta \hat{\xi}^\beta (v, \eta)| \leq 1, \text{ for } |v| \leq K,$$

(5.4)

where $\beta \in \mathbb{N}^2$ and $|\beta| \leq N$.

In view of (2.12), we obtain

$$\langle z, \partial_\xi \phi^\lambda (\tilde{z}, \Psi^\lambda (\tilde{z}, \eta)) \rangle = x\eta + th_\lambda (\eta),$$

(5.5)

where $h_\lambda (\eta) := (\partial_\eta \phi^\lambda ) (\tilde{z}, \Psi^\lambda (\tilde{z}, \eta))$.

Since we assume $a(z, \eta) = a_1(z)a_2(\eta)$, up to the negligible influence of space variable, heuristically, we may approximate $\mathcal{T}_\lambda$ by extension operators $E_{\tilde{z}}$

$$E_{\tilde{z}}g(z) := \int_{\mathbb{R}^d} e^{i(x\eta + th_\lambda (\eta))} a_{2,\tilde{z}} (\eta) g(\eta) d\eta,$$

(5.6)

in a sufficiently small neighborhood of $\tilde{z}$, where

$$a_{2,\tilde{z}} (\eta) = a_2 (\Psi^\lambda (\tilde{z}, \eta)) |\det \partial_\eta \Psi^\lambda (\tilde{z}, \eta)|.$$

It is obvious that $h_\lambda (\eta)$ is homogeneous of degree 1 and satisfying

$$\text{rank } \partial_{\eta \eta} h_\lambda (\eta) = 1, \text{ for all } \eta \in \text{supp } a_{2,\tilde{z}}.$$

Due to the compactness of the support of $a$ and (2.13), we may assume the nonvanishing eigenvalue of $\partial_{\eta \eta} h_\lambda (\eta)$ is comparable to 1 independent of $\tilde{z}$.

To show (5.1), we shall need the following two lemmas.

**Lemma 5.1.** Let $z_k$ be the center of $Q_k$ and $E_{\tilde{z}_k}^{\nu, j}$ be an extension operator defined by

$$E_{\tilde{z}_k}^{\nu, j} g(z) := \int_{\mathbb{R}^d} e^{i(x\eta + th_{\tilde{z}_k} (\eta))} \rho(\lambda^{1/2-\varepsilon/2} \eta - j) a_{2,\tilde{z}_k} (\eta) g(\eta) d\eta$$

(5.8)

with $\rho$ satisfying (4.8) and

$$a_{2,\tilde{z}_k} (\eta) = a_2 (\Psi^\lambda (z_k, \eta)) |\det \partial_\eta \Psi^\lambda (z_k, \eta)|.$$

Then we have

$$\left\| \left( \sum_{\nu, j} |E_{\tilde{z}_k}^{\nu, j} g|^2 \right)^{1/2} \right\|_{L^p (w_{Q_0})} \lesssim \left\| \left( \sum_{\nu} |E_{\tilde{z}_k}^{\nu} g|^2 \right)^{1/2} \right\|_{L^p (w_{Q_0})} + \text{RapDec}(\lambda) \| g \|_{L^{1/p}}. $$

(5.9)

The following lemma shows that when localized in a relatively small region, $\mathcal{T}_\lambda$ is comparable to $E_{\tilde{z}}$ in a suitable sense. A slightly weaker version of this lemma appeared in the work [2], which is applicable to the decoupling norm but is not sufficient to handle square-function estimates. It is for this reason that we need a pointwise refinement of stability lemma as below.

**Lemma 5.2.** Let $0 < \delta \leq 1/2$ and $1 \leq K \leq \lambda^{1/2-\delta}$. Then for any $N$ given by (2.11),

$$|\mathcal{T}_\lambda f(\tilde{z} + v)| \leq |E_{\tilde{z}} f_{\tilde{z}} (v)| + \frac{3}{\pi} N \sum_{l \in \mathbb{Z}^d \setminus \{0\}} |l|^{-N} |E_{\tilde{z}} (f_{\tilde{z}} e^{i(4\pi l, \cdot)} (v))|$$

(10.5)

$$|E_{\tilde{z}} f_{\tilde{z}} (v)| \leq |\mathcal{T}_\lambda f(\tilde{z} + v)| + \frac{3}{\pi} N \sum_{l \in \mathbb{Z}^d \setminus \{0\}} |l|^{-N} |\mathcal{T}_\lambda [e^{i(4\pi l, (\partial_\xi \phi)^\lambda) (\tilde{z}, \cdot)} f] (\tilde{z} + v)|,$$

(11.5)

whenever $|v| \leq K$. 

We postpone the proof of Lemma 5.1 and Lemma 5.2 in the next two subsections. Let us continue the proof of (5.1). Let \( z_k \) be the center of \( Q_k \), therefore by (5.10) we have

\[
\left\| \left( \sum_{\nu,j} |\mathcal{T}_{\lambda,k}^\nu f|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_k})} \\
\lesssim N \sum_{l \in \mathbb{Z}^2} (1 + 4\pi|l|)^{-N} \left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j}(f_{z_k} e^{i(4\pi l \cdot z)})|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} + \text{RapDec}(\lambda) \| f \|_{L^{10}} \tag{5.12}
\]

It should be noted that the cube \( Q_0 \) appearing in the last inequality of (5.10) may be slightly larger than the original one.

Now using the estimate obtained in (5.9), up to a negligible error term, we have

\[
\sum_{l \in \mathbb{Z}^2} (1 + 4\pi|l|)^{-N} \left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j}(f_{z_k} e^{i(4\pi l \cdot z)})|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} \\
\lesssim \sum_{l \in \mathbb{Z}^2} (1 + 4\pi|l|)^{-N} \left\| \left( \sum_{\nu} |E_{z_k}^{\nu}(f_{z_k} e^{i(4\pi l \cdot z)})|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} \\
\lesssim \sum_{l \in \mathbb{Z}^2} (1 + 4\pi|l|)^{-N} \left\| \left( \sum_{\nu} |E_{z_k}^{\nu} f_{z_k}|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0,1})},
\]

where \( w_{Q_0,1}(z) = w_{Q_0}((4\pi l, 0) + z) \).

The last inequality we have used the fact the extension operator \( E_{z} \) is invariant under translation transformation

\[
E_{\tilde{z}}[e^{i(4\pi l \cdot z)} g](x, t) = E_{\tilde{z}} g(x + 4\pi l, t). \tag{5.13}
\]

Since

\[
\sum_{l \in \mathbb{Z}^2} (1 + 4\pi|l|)^{-N} w_{Q_0,1}(z) \lesssim w_{Q_0}(z), \tag{5.14}
\]

we obtain that

\[
\left\| \left( \sum_{\nu,j} |\mathcal{T}_{\lambda,k}^\nu f|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} \lesssim \left\| \left( \sum_{\nu} |E_{z_k}^{\nu} f_{z_k}|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} + \text{RapDec}(\lambda) \| f \|_{L^{10}}. \tag{5.15}
\]

To finish the proof, it suffices to replace \( E_{z_k}^{\nu} f_{z_k} \) by its variable coefficient counterpart.

From (5.11) and Minkowski’s inequality we have

\[
\left\| \left( \sum_{\nu} |E_{z_k}^{\nu} f_{z_k}|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} \\
\leq \left\| \left( \sum_{\nu} |E_{z_k}^{\nu} f_{z_k}|^2 \right)^{\frac{1}{2}} \chi_{\{ |z| \leq \lambda r + \frac{r}{2} \}} \right\|_{L^{10}(w_{Q_0})} + \left\| \left( \sum_{\nu} |E_{z_k}^{\nu} f_{z_k}|^2 \right)^{\frac{1}{2}} \chi_{\{ |z| \geq \lambda r + \frac{r}{2} \}} \right\|_{L^{10}(w_{Q_0})} \\
\lesssim N \sum_{l \in \mathbb{Z}^2} (1 + 4\pi|l|)^{-N} \left\| \left( \sum_{\nu} |\mathcal{T}_{\lambda}^{\nu}(e^{i(4\pi l \cdot \partial_x \phi(z_k, \cdot)} f)(z_k + v)|^2 \right)^{\frac{1}{2}} \right\|_{L^{10}(w_{Q_0})} + \text{RapDec}(\lambda) \| f \|_{L^{10}}.
\]
Note that when \( l = 0 \), that is what we desire, thus it remains to control the error term. By (5.10),
\[
\sum_{l \in \mathbb{Z}^2 \setminus \{0\}} \left( \frac{\pi |l|}{3} \right)^{-N} \left\| \left( \sum_{\nu} |\mathcal{F}_\lambda f| \right)^{\frac{1}{2}} \right\|_{L^p(w_{Q_0})} \\
\leq \sum_{l \in \mathbb{Z}^2 \setminus \{0\}} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \left( \frac{\pi |l|}{3} \right)^{-N} \left( \frac{\pi |k|}{3} \right)^{-N} \left\| \left( \sum_{\nu} |E_{z_k} f_{z_k}| \right)^{\frac{1}{2}} \right\|_{L^p(w_{Q_0}((4\pi(l+k),0)+))} \\
+ \sum_{l \in \mathbb{Z}^2 \setminus \{0\}} \left( \frac{\pi |l|}{3} \right)^{-N} \left\| \left( \sum_{\nu} |E_{z_k} f_{z_k}| \right)^{\frac{1}{2}} \right\|_{L^p(w_{Q_0}((4\pi l,0)+))} \\
\leq \frac{1}{2} \left\| \left( \sum_{\nu} |E_{z_k} f_{z_k}|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w_{Q_0})},
\]
the last inequality can be ensured by presetting \( N \) sufficiently large. Therefore we combine the above estimate together
\[
\left\| \left( \sum_{\nu} |E_{z_k} f_{z_k}|^2 \right)^{\frac{1}{2}} \right\|_{L^4(w_{Q_0})} \\
\leq \left\| \left( \sum_{\nu} |\mathcal{F}_\lambda f|^2 \right)^{\frac{1}{2}} \right\|_{L^4(w_{Q_0})} + \frac{1}{2} \left\| \left( \sum_{\nu} |E_{z_k} f_{z_k}|^2 \right)^{\frac{1}{2}} \right\|_{L^4(w_{Q_0})} + \text{RapDec}(\lambda) \|f\|_{L^4}.
\]
The term \( \|\sum_{\nu} |E_{z_k} f_{z_k}|^2\|_{L^4(w_{Q_0})} \) appearing in the right hand side can be absorbed in the left hand side, then we complete the proof.

5.1. Proof of Lemma 5.1. We shall need the following two lemmas. The first one is due to Rubio de Francia [25], which handles the square function estimate for equally-spaced cubes in frequency space.

Lemma 5.3. Let \( \{O_k\}_k \) be a collection of equally spaced cubes, and let \( \varphi_k(\xi) = \varphi(\xi - \xi_k) \) be the bump function adapted to \( O_k \), where \( \xi_k \) denotes the center of \( O_k \). Then for any function \( f \) we have the pointwise estimate
\[
\left( \sum_k |\hat{\varphi}_k * f|^2 \right)^{1/2} \leq C(\varphi)(M\|f\|^2)^{1/2}
\]
(5.16)
where \( M \) denotes the Hardy-Littlewood maximal function, and \( C(\varphi) \) depends only on the dimension and finitely many of the derivatives of \( \varphi \) which is associated with the unit cube.

The next lemma about the vector-valued maximal function estimate is due to Fefferman-Stein [12] (see also [15]).

Lemma 5.4. Let \( 1 < r, p < \infty \), \( \{f_k\}_k \) be a sequence of functions, then
\[
\left\| \left( \sum_k |M_{f_k}|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)} \leq C_n A_{p,r} \left\| \left( \sum_k |f_k|^r \right)^{1/r} \right\|_{L^p(\mathbb{R}^n)},
\]
(5.17)
where \( A_{p,r} = \frac{r}{r-1} \left( \frac{p}{p-1} \right) \).
The crux of the problem is that the weight function $w_{Q_0}$ is not an $A_p$ weight. Specifically, the following estimate for the maximal operator

$$\|Mf\|_{L^p(w_{Q_0})} \leq C\|f\|_{L^p(w_{Q_0})}, \quad 1 < p < \infty, \quad (5.18)$$

fails. In order to overcome this failure and preserve some kind of localized property, we need to make a series of localization reduction.

Indeed

$$\left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j}g|^2 \right)^{\frac{1}{2}} \right\|_{L^{\frac{10}{3}}(w_{Q_0})} \leq \left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j}g|^2 \right)^{\frac{1}{4}} \chi_{\{|x| \leq \lambda^{1/2-\varepsilon/4}\}} \right\|_{L^{\frac{10}{3}}(w_{Q_0})} + \left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j}g|^2 \right)^{\frac{1}{4}} \chi_{\{|x| > \lambda^{1/2-\varepsilon/4}\}} \right\|_{L^{10/3}(w_{B_0})}. \quad (5.19)$$

Due to the fast decay of the weight $w_{Q_0}$ away from $|z| \geq \lambda^{1/2-\varepsilon/4}$, it suffices to consider

$$\left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j}g|^2 \right)^{\frac{1}{4}} \chi_{\{|x| \leq \lambda^{1/2-\varepsilon/4}\}} \right\|_{L^{10/3}(w_{Q_0})}. \quad (5.19)$$

Freeze $t_0$ and note that

$$E_{z_k}^{\nu,j}g(x, t_0) = \int e^{i(x, \eta)} \rho(\lambda \frac{1}{2} - \frac{\varepsilon}{2} \eta - j)(E_{z_k}^{\nu})^\wedge(\eta, t_0) d\eta \quad (5.20)$$

where $E_{z_k}^{\nu}g(x, t_0)$ is defined by

$$E_{z_k}^{\nu}g(x, t_0) := \int_{\mathbb{R}^d} e^{i(x, \eta) + t_0 h_{z_k}(\eta)} a_{z_k}^{\nu}(\eta) g(\eta) d\eta. \quad (5.21)$$

We further decompose

$$E_{z_k}^{\nu}g(x, t_0) = \chi_{\{|x| \leq \lambda^{1/2-\varepsilon/4}\}}(x) E_{z_k}^{\nu}g(x, t_0) + \chi_{\{|x| > \lambda^{1/2-\varepsilon/4}\}}(x) E_{z_k}^{\nu}g(x, t_0). \quad (5.22)$$

It remains to estimate

$$\int e^{i(x, \eta)} \rho(\lambda \frac{1}{2} - \frac{\varepsilon}{2} \eta - j)(\chi_{\{|x| \leq \lambda^{1/2-\varepsilon/4}\}}(\cdot) E_{z_k}^{\nu})^\wedge(\eta, t_0) d\eta. \quad (5.23)$$

In fact for $|x| \leq \lambda^{1/2-\varepsilon/4}$, the contribution of the second term in (5.22) to (5.20) can be controlled by

$$\int \frac{1}{\lambda^{1-\varepsilon}} \hat{\rho}(x-y)(\chi_{\{|x| > \lambda^{1/2-\varepsilon/4}\}}(y) E_{z_k}^{\nu}g(y) dy \lesssim \text{RapDec}(\lambda) \|g\|_{L^{10/3}}. \quad (5.24)$$

We continue the estimate of (5.23), by Lemma 5.3

$$\left( \sum_{\nu,j} \left| \int e^{i(x, \eta)} \rho(\lambda \frac{1}{2} - \frac{\varepsilon}{2} \eta - j)(\chi_{\{|x| \leq \lambda^{1/2-\varepsilon/4}\}}(\cdot) E_{z_k}^{\nu})^\wedge(\eta, t_0) d\eta \right|^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{\nu} M \left[ \chi_{\{|x| \leq \lambda^{1/2-\varepsilon/4}\}}(\cdot) E_{z_k}^{\nu}g(\cdot, t_0) \right]^2 \right)^{\frac{1}{2}}. \quad (5.25)$$
Now unfreezing $t_0$, after plugging the estimate (5.25) into (5.19) and using Lemma 5.4, we obtain
\[
\left\| \left( \sum_{\nu,j} |E_{z_k}^{\nu,j} g'(\cdot)|^2 \right) \right\|_{L^{10/3}(wQ_0)} \leq \left\| \left( \sum_{\nu} M \left[ |\chi_{\{|x| \leq \lambda^k + \frac{\lambda}{2} \}} E_{z_k}^{\nu} g'(\cdot, t_0) \right]^2 \right) \right\|_{L^{10/3}(wQ_0)} \leq \left\| \left( \sum_{\nu} |E_{z_k}^{\nu} g'(\cdot)|^2 \right) \right\|_{L^{10/3}(wQ_0)}
\]

(5.26)

**Remark 5.5.** In the last inequality, additional $\lambda^k$ appears in the course of using Fefferman-Stein’s square function estimate. We actually use the following inequality which can be deduced directly from Lemma 5.4
\[
\left\| \sum_m M g_m \right\|_{L^{10/3}(\mathbb{R}^3)} \lesssim \varepsilon \left\| \sum_m |g_m| \right\|_{L^{10/3}(\mathbb{R}^3)}
\]
where $\#\{m\} = N$ and arbitrary $\varepsilon > 0$.

In fact, by Hölder’s inequality and Lemma 5.4, we have
\[
\left\| \sum_m M g_m \right\|_{L^{10/3}(\mathbb{R}^3)} \leq \left\| \left( \sum_m |M g_m|^r \right)^{1/r} \right\|_{L^{10/3}(\mathbb{R}^3)} \left( \#\{m\} \right)^{1/r'} \lesssim_r \left\| \left( \sum_m |g_m|^r \right)^{1/r} \right\|_{L^{10/3}(\mathbb{R}^3)} \left( \#\{m\} \right)^{1/r'} \lesssim_r \left\| \sum_m |g_m| \right\|_{L^{10/3}(\mathbb{R}^3)} \left( \#\{m\} \right)^{1/r'}
\]
We may choose $r'$ sufficiently large such that $\frac{1}{r'} \leq \varepsilon$.

5.2. **Proof of Lemma 5.2.**

Proof. Noting that $\text{supp}_\eta \partial_\alpha \delta_{\lambda, \varepsilon}(\cdot, \cdot) \subset B(e_2, \varepsilon_0)$ where $\varepsilon_0$ can be chosen small if necessary, we may replace $f$ with $f \psi$, where $\psi$ is a smooth function that equals to 1 on $B(e_2, \frac{1}{100})$ and vanishing outside of $B(e_2, \frac{1}{5})$ such that
\[
|\partial_\eta^\alpha \psi(\eta)| \leq 6^N, \quad \text{for} \ \alpha \in \mathbb{N}^2, \ 1 \leq |\alpha| \leq N.
\]
By performing a Fourier expansion of $e^{i\varepsilon \lambda(z, \cdot)} \psi(\eta)$ in $\eta$ variable, one may write
\[
e^{i\varepsilon \lambda(z, \cdot)} \psi(\eta) = \sum_{l \in \mathbb{Z}^2} b_l(v) e^{i(4\pi l, \eta)},
\]
where
\[
b_l(v) = 2 \int_{Q(e_2, 1/4)} e^{-i(4\pi l, \eta)} e^{i\varepsilon \lambda(z, \cdot)} \psi(\eta) d\eta
\]
and
\[
Q(e_2, 1/4) \text{ denotes the cube centered at } e_2 \text{ with sidelength 1/2.}
\]
By (5.4), it is easy to show $|b_0(v)| \leq 1$. Integration by parts show that
\[
|b_l(v)| \leq 12^N (4\pi |l|)^{-N} \quad \text{whenever } |v| \leq K, l \neq (0, 0),
\]
which leads to (5.10).
For the reverse direction, one may write
\[ E_\tilde{z} f_\tilde{z}(v) = \int_{\mathbb{R}^2} e^{i\phi_\lambda(z + v, \Psi^\lambda(\tilde{z}, \eta))} e^{-i\varepsilon \lambda(\tilde{z}, \eta)} a_{2, \tilde{z}}(\eta) f \circ \Psi^\lambda(\tilde{z}, \eta) d\eta. \] 
(5.31)
Performing the Fourier expansion of \( e^{-i\varepsilon \lambda(\tilde{z}, \eta)} \) in \( \eta \) and reversing the change of variables \( \eta \to \Psi^\lambda(\tilde{z}, \eta) \), we have (5.11).

6. Comments on higher dimensional cases

The results in Proposition 4.1 can be generalized to higher dimensions with an additional convexity assumption on the phase function which becomes superfluous in 2+1 dimensions since there exists only one non-vanishing eigenvalue. This assumption essentially makes the separation condition (2.20) justifiable.

As the situation described in \( \mathbb{R}^{2+1} \), let \( n \geq 3 \), \( a(z, \eta) \in C^\infty_c(\mathbb{R}^n \times \mathbb{R}^{n+1}) \) with compact support contained in \( B(0, \varepsilon_0) \times B(e_n, \varepsilon_0) \). Assume \( C(e_n, \varepsilon_0) := B(e_n, \varepsilon_0) \cap \mathbb{S}^{n-1} \) and make angular decomposition with respect to the \( \eta \)-variable by cutting \( C(e_n, \varepsilon_0) \) into \( N_\lambda \approx \lambda^{(n-1)/2} \) many caps \( \{ \theta_\nu : 1 \leq \nu \leq N_\lambda \} \), each cap \( \theta_\nu \) extends \( \approx \varepsilon_0 \lambda^{-(n-1)/2} \). We denote by \( \kappa_\nu \in \mathbb{S}^{n-1} \) the center of \( \theta_\nu \).

Let \( \{ \chi_\nu(\eta) \} \) be a family of smooth cutoff function associated with the decomposition in the angular direction, each of which is homogeneous of degree 0, such that \( \{ \chi_\nu \} \) forms a partition of unity on the unit circle and then extended homogeneously to \( \mathbb{R}^n \setminus 0 \) such that
\[
\left\{ \begin{array}{l}
\sum_{0 < \nu < N_\lambda} \chi_\nu(\eta) \equiv 1, \quad \forall \eta \in \mathbb{R}^n \setminus 0, \\
|\partial^\alpha \chi_\nu(\eta)| \leq C_\alpha \lambda^{\frac{|\alpha|}{2}}, \quad \forall \alpha \text{ if } |\eta| = 1.
\end{array} \right.
\]
Define
\[
T_\lambda f = \int e^{i\phi_\lambda(z, \eta)} a(z, \eta) f(\eta) d\eta = \sum_\nu T_\lambda^\nu f, \\
T_\lambda^\nu f(z) = \int e^{i\phi_\lambda(z, \eta)} a^\nu(z, \eta) f(\eta) d\eta,
\]
(6.1)
where \( a^\nu(z, \eta) = \chi_\nu(\eta)a(z, \eta) \).

Let \( n \geq 3 \), by carrying over the approach in the proof of Proposition 3.1, one may obtain, under the similar condition of that in 3.1 and the convexity condition
\[
\| \sum_{\nu \in \Omega} T_\lambda^\nu g \sum_{\nu' \in \Omega'} T_\lambda^\nu' h \|_{L^{\frac{n+3}{n+1}}(\mathbb{R}^{n+1})} \lesssim_{\phi, \varepsilon} \lambda^{\frac{n-1}{2(n+1)}} \left( \sum_{\nu \in \Omega} |T_\lambda^\nu g|^2 \right)^{1/2} L^{\frac{2(n+3)}{n+1}}(\mathbb{R}^{n+1}) \left( \sum_{\nu' \in \Omega'} |T_\lambda^\nu' h|^2 \right)^{1/2} L^{\frac{2(n+3)}{n+1}}(\mathbb{R}^{n+1}),
\]
(6.2)
which implies the square function estimate
\[
\| T_\lambda f \|_{L^{\frac{2(n+3)}{n+1}}(\mathbb{R}^{n+1})} \lesssim_\lambda \lambda^{\frac{n-1}{2(n+1)}} \left( \sum_{\nu} |T_\lambda^\nu f|^2 \right)^{1/2} L^{\frac{2(n+3)}{n+1}}(\mathbb{R}^{n+1})
\]
(6.3)
up to a RapDec(\( \lambda \)) term.

Unfortunately, we are unable to obtain a better result than interpolation between the sharp \( L^{q_0} \) estimates of [2] with the \( L^2 \) estimate. One of the reason responsible for this...
shortage is due to the poor knowledge on \( L^p \rightarrow L^p \) estimates for the variable coefficient version of Kakeya type maximal function in the light ray directions

\[
\|M_\delta\|_{L^p \rightarrow L^p} \leq C \max \left\{ \left(\log \frac{1}{\delta}\right)^{\frac{1}{2}}, \delta^{-\frac{n-2}{p}} \right\}, \quad \text{for } 2 \leq p \leq \infty, \quad (6.4)
\]

when \( n \geq 3 \). The \( L^p \)–Kakeya maximal function estimate for \( p > 2 \) is known also for its profundity and difficulty in the literature, for which we refer to [28]. In conclusion, it seems very difficult to adopt the bilinear method used in this paper to refine the result in [24] for \( p \leq q_n \) when \( n \geq 3 \).

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