The Pohozaev identity for the fractional Laplacian

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(joint work with Joaquim Serra)
Outline of the talk

- The classical Pohozaev identity; applications
- The Dirichlet semilinear problem for the fractional Laplacian
- The Pohozaev identity for the fractional Laplacian
- Applications
- Sketch of the proof
The classical Pohozaev identity

Ω bounded Lipschitz domain,

\[
\begin{cases}
-\Delta u &= f(u) \quad \text{in } \Omega \\
    u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\]  

(1)

Theorem (Pohozaev)

\[
(2 - n) \int_{\Omega} u f(u) dx + 2n \int_{\Omega} F(u) dx = \int_{\partial \Omega} |\nabla u|^2 (x \cdot \nu) d\sigma
\]
Applications of the classical Pohozaev identity

\[(2 - n) \int_{\Omega} u f(u) \, dx + 2n \int_{\Omega} F(u) \, dx = \int_{\partial \Omega} |\nabla u|^2 (x \cdot \nu) \, d\sigma\]

- Nonexistence of solutions: critical exponent \(-\Delta u = u^{\frac{n+2}{n-2}}\)
- Ground states in \(\mathbb{R}^n\): monotonicity formulas, estimates
- Radial symmetry: proof of P.-L. Lions combining the Pohozaev identity with the isoperimetric inequality
- Stable solutions: uniqueness, \(H^1\) interior regularity
- etc.
Proof of the classical Pohozaev identity

First note that

\[ \Delta (x \cdot \nabla u) = 2\Delta u + x \cdot \nabla (\Delta u). \]

Then, integrating by parts twice and using that \( u \equiv 0 \) on \( \partial \Omega \), we obtain

\[
\int_\Omega (x \cdot \nabla u) \Delta u = 2 \int_\Omega u \Delta u + \int_\Omega u x \cdot \nabla (\Delta u) + \int_{\partial \Omega} (x \cdot \nabla u)(\nabla u \cdot \nu) d\sigma
\]

\[ = (2 - n) \int_\Omega u \Delta u - \int_\Omega (x \cdot \nabla u) \Delta u + \int_{\partial \Omega} |\nabla u|^2 (x \cdot \nu) d\sigma \]

We have used that \( \nabla u \cdot \nu = |\nabla u| \) on \( \partial \Omega \). Finally, since \( -\Delta u = f(u) \), then

\[
2 \int_\Omega (x \cdot \nabla u) \Delta u = -2 \int_\Omega x \cdot \nabla F(u) = 2n \int_\Omega F(u),
\]

and the identity follows.
The Dirichlet semilinear problem with \((-\Delta)^s\)

\(\Omega\) bounded \(C^{1,1}\) domain, \(\delta(x) := \text{dist}(x, \partial \Omega)\), \(f \in C^1\)

\[
\begin{cases}
(-\Delta)^s u = f(u) & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

**Theorem (X.R., J. Serra)**

1. \(u \in C^s(\mathbb{R}^n)\)
2. \(u/\delta^s \in C^\alpha(\overline{\Omega})\)
3. \([u]_{C^\beta(B_{\rho/2})} \leq C \rho^{s-\beta}\)
4. \([u/\delta^s]_{C^\beta(B_{\rho/2})} \leq C \rho^{\alpha-\beta}\)
The Pohozaev identity for the fractional Laplacian

\( \Omega \) bounded \( C^{1,1} \) domain,

\[
\begin{cases}
(\Delta)^s u &= f(u) \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

Theorem (X. R., J. Serra)

Denote \( \delta(x) := \text{dist}(x, \partial \Omega) \). Then \( u/\delta^s \in C^\alpha(\overline{\Omega}) \) and

\[
(2s - n) \int_\Omega uf(u)dx + 2n \int_\Omega F(u)dx = \Gamma(1 + s)^2 \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu)d\sigma,
\]

where \( \Gamma \) is the gamma function.
Corollary: nonexistence results

$\Omega$ bounded $C^{1,1}$ domain,

\[
\begin{cases}
(\Delta)^{s}u = f(u) \quad \text{in } \Omega \\
u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

Corollary

Assume that $\Omega$ is star-shaped and $F(t) < \frac{n-2s}{2n} t f(t)$ for all $t$. Then the problem admits no nontrivial solution.

For example, for $f(u) = u^p$ we obtain nonexistence for $p \geq \frac{n+2s}{n-2s}$.

For positive solutions, this was done by [Fall-Weth, ’12] with moving planes.

Existence for subcritical $p$ by [Servadei-Valdinoci, ’12].
Proposition (X. R., J. Serra)

Assume

1. $\Omega$ bounded $C^{1,1}$ domain
2. $u \in C^s(\mathbb{R}^n)$, $u \equiv 0$ outside $\Omega$, $u/\delta^s \in C^\alpha(\Omega)$
3. Interior $C^\beta$ estimates for $u$ and $u/\delta^s$, $\beta < 1 + 2s$
4. $(-\Delta)^su$ is bounded in $\Omega$

Then

$$\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = \frac{2s - n}{2} \int_\Omega u(-\Delta)^s u - \frac{\Gamma(1 + s)^2}{2} \int_{\partial\Omega} \left(\frac{u}{\delta^s}\right)^2 (x \cdot \nu)$$
Main consequences

Changing the origin in our identity, we deduce the following

**Theorem (X. R., J. Serra)**

*Under the same hypotheses of the Proposition,*

\[
\int_{\Omega} (-\Delta)^s u \ v_{x_i} = - \int_{\Omega} u_{x_i} (-\Delta)^s v + \Gamma (1 + s)^2 \int_{\partial \Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} \nu_i
\]

It has a local boundary term!

Note the contrast with the nonlocal flux in the formula for \( \int_{\Omega} f(x, u) \)
Sketch of the Proof (Star-shaped domains)

1. \( u_\lambda(x) = u(\lambda x) \Rightarrow \)

\[
\int_\Omega (x \cdot \nabla u)(-\Delta)^s u = \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_\Omega u_\lambda(-\Delta)^s u
\]

2. \( \Omega \) star-shaped \( \Rightarrow \) \( u_\lambda \) vanishes outside \( \Omega \) for \( \lambda > 1 \) \( \Rightarrow \)

\[
\int_\Omega u_\lambda(-\Delta)^s u = \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u_\lambda(-\Delta)^{\frac{s}{2}} u
\]
\[
\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u \lambda (-\Delta)^{\frac{s}{2}} u = \lambda^s \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{s}{2}} u\right) (\lambda x) (-\Delta)^{\frac{s}{2}} u(x) \, dx
\]
\[
= \lambda^s \int_{\mathbb{R}^n} w(\lambda x) w(x) \, dx
\]
\[
= \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w(\lambda^{\frac{1}{2}} y) w(\lambda^{-\frac{1}{2}} y) \, dy
\]

where \( w = (-\Delta)^{\frac{s}{2}} u \). Therefore,

\[
\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = \frac{2s-n}{2} \int_{\mathbb{R}^n} w^2 + \frac{1}{2} \left. \frac{d}{d\lambda} \right|_{\lambda=1} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}
\]

where \( w_\lambda(x) = w(\lambda x) \).
\[ \int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} u \lambda (-\Delta)^{\frac{s}{2}} u = \lambda^s \int_{\mathbb{R}^n} ((-\Delta)^{\frac{s}{2}} u) (\lambda x) (-\Delta)^{\frac{s}{2}} u(x) \, dx \]
\[ = \lambda^s \int_{\mathbb{R}^n} w(\lambda x) w(x) \, dx \]
\[ = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w(\lambda^{\frac{1}{2}} y) w(\lambda^{-\frac{1}{2}} y) \, dy \]

where \( w = (-\Delta)^{\frac{s}{2}} u \). Therefore,

\[ \int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = \frac{2s-n}{2} \int_{\Omega} u (-\Delta)^s u + \frac{1}{2} \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \]

where \( w_\lambda(x) = w(\lambda x) \).
What about \( \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_{\lambda} w_{1/\lambda} \)?

\[
\mapsto \mathcal{I}(\varphi) = - \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx
\]

Important properties:

1. \( \mathcal{I}(\varphi) \geq 0 \) since

\[
\int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx \leq \left( \int_{\mathbb{R}^n} \varphi^2(\lambda x) \, dx \right)^{1/2} \left( \int_{\mathbb{R}^n} \varphi^2(x/\lambda) \, dx \right)^{1/2} = \int_{\mathbb{R}^n} \varphi^2
\]

2. \( \psi \) smooth \( \Rightarrow \) \( \mathcal{I}(\psi) = 0 \)

3. If \( \mathcal{I}(\psi) = 0 \) \( \Rightarrow \) \( \mathcal{I}(\varphi + \psi) = \mathcal{I}(\varphi) \)
What about \( \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_{\lambda} w_{1/\lambda} \)?

\[ \sim \quad I(\varphi) = - \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx \]

Important properties:

1. \( I(\varphi) \geq 0 \) since

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2. \( \psi \) smooth \( \Rightarrow \ I(\psi) = 0 \)

3. If \( I(\psi) = 0 \) \( \Rightarrow \ I(\varphi + \psi) = I(\varphi) \)
What about $\frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w\lambda w_{1/\lambda}$?

$\implies I(\varphi) = -\frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx$

Important properties:

1. $I(\varphi) \geq 0$ since

$$\int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx \leq \left( \int_{\mathbb{R}^n} \varphi^2(\lambda x) \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \varphi^2(x/\lambda) \, dx \right)^{\frac{1}{2}} = \int_{\mathbb{R}^n} \varphi^2$$

2. $\psi$ smooth $\implies I(\psi) = 0$

3. If $I(\psi) = 0 \implies I(\varphi + \psi) = I(\varphi)$
What about \( \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_{\lambda} w_{1/\lambda} \)?

\[ \sim \quad \mathcal{I}(\varphi) = - \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx \]

Important properties:

1. \( \mathcal{I}(\varphi) \geq 0 \) since

\[ \int_{\mathbb{R}^n} \varphi(\lambda x) \varphi(x/\lambda) \, dx \leq \left( \int_{\mathbb{R}^n} \varphi^2(\lambda x) \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \varphi^2(x/\lambda) \, dx \right)^{\frac{1}{2}} = \int_{\mathbb{R}^n} \varphi^2 \]

2. \( \psi \) smooth \( \Rightarrow \quad \mathcal{I}(\psi) = 0 \)

3. If \( \mathcal{I}(\psi) = 0 \) \( \Rightarrow \quad \mathcal{I}(\varphi + \psi) = \mathcal{I}(\varphi) \)
What about \( \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \)?

We want to compute:

\[
\mathcal{I}(w) = -\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda}
\]

Reduce to a \( 1-D \) calculation

Use “star-shaped” \((t, z)\)-coordinates

\[
x = tz, \quad z \in \partial\Omega, \quad t > 0
\]

\[
\frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} = \frac{d}{d\lambda}\bigg|_{\lambda=1^+} \int_{\partial\Omega} (z \cdot \nu) d\sigma(z) \int_0^\infty t^{n-1} w(\lambda tz) w\left(\frac{tz}{\lambda}\right) dt
\]
What about \[ \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \]?

We want to compute:

\[ \mathcal{I}(w) = -\frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} \]

Reduce to a \(1 - D\) calculation

Use “star-shaped” \((t, z)\)-coordinates

\[ x = t z, \quad z \in \partial \Omega, \quad t > 0 \]

\[ \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} = \int_{\partial \Omega} (z \cdot \nu) d\sigma(z) \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{0}^{\infty} t^{n-1} w(\lambda t z) w \left( \frac{t z}{\lambda} \right) dt \]
What do we know about \( w = (-\Delta)^{s/2} u \)?

**Proposition (X. R., J. Serra)**

Fix \( z \in \partial \Omega \). Then,

\[
w(tz) = (-\Delta)^{s/2} u(tz) = c_1 \left\{ \log^+ |t - 1| + c_2 \chi_{(0,1)}(t) \right\} \frac{u}{\delta^s}(z) + h(t)
\]

where

\[
\left. \frac{d}{d \lambda} \right|_{\lambda=1^+} \int_0^{\infty} t^{n-1} h(\lambda t) h \left( \frac{t}{\lambda} \right) dt = 0
\]

\[
c_1 = \frac{\Gamma(1 + s) \sin \left( \frac{\pi s}{2} \right)}{\pi}, \quad \text{and} \quad c_2 = \frac{\pi}{\tan \left( \frac{\pi s}{2} \right)}
\]
\[ w(tz) = c_1 \left\{ \log^- |t - 1| + c_2 \chi_{(0,1)}(t) \right\} \frac{u}{\delta_s}(z) + h(t) \]

\[
\frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_{\lambda} w_{1/\lambda} = \int_{\partial\Omega} (z \cdot \nu) d\sigma(z) \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_0^\infty t^{n-1} w(\lambda t)w \left( \frac{tz}{\lambda} \right) dt
\]

\[
= \int_{\partial\Omega} (z \cdot \nu) d\sigma(z) \left( \frac{u}{\delta_s(z)} \right)^2 \int_0^\infty t^{n-1} \phi_s(\lambda t)\phi_s \left( \frac{t}{\lambda} \right) dt
\]

\[
= \int_{\partial\Omega} (z \cdot \nu) d\sigma(z) \left( \frac{u}{\delta_s(z)} \right)^2 C(s)
\]
Summarising...

\[ w(tz) = \phi_s(t) \frac{u}{\delta^s}(z) + h(t) \]

where \( \phi_s(t) = c_1 \{ \log^- |t - 1| + c_2 \chi_{(0,1)}(t) \} \)

\[
\frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_{\mathbb{R}^n} w_\lambda w_{1/\lambda} = \int_{\partial\Omega} (z \cdot \nu) d\sigma(z) \frac{d}{d\lambda} \bigg|_{\lambda=1^+} \int_0^\infty t^{n-1} w(\lambda tz) w \left( \frac{tz}{\lambda} \right) dt
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\]

\[
= \int_{\partial\Omega} (z \cdot \nu) d\sigma(z) \left( \frac{u}{\delta^s}(z) \right)^2 C(s)
\]
And if the domain is not star-shaped...

Key observations:

1. Pohozaev identity is quadratic in $u$ and it “comes from a bilinear identity”

$$\int_{\Omega}(x \cdot \nabla u)(-\Delta)^s u = \frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s u - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu)$$

$$\int_{\Omega}(x \cdot \nabla u)(-\Delta)^s v + \int_{\Omega}(x \cdot \nabla v)(-\Delta)^s u =$$

$$\frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s v + \frac{2s-n}{2} \int_{\Omega} v(-\Delta)^s u - \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu)$$

2. every $C^{1,1}$ domain is locally star-shaped

3. the bilinear identity holds easily when $u$ and $v$ have disjoint support
And if the domain is not star-shaped...

Key observations:

1. Pohozaev identity is quadratic in $u$ and it “comes from a bilinear identity”

$$\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u = \frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s u - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu)$$

$$\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s v + \int_{\Omega} (x \cdot \nabla v)(-\Delta)^s u = \frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s v + \frac{2s-n}{2} \int_{\Omega} v(-\Delta)^s u - \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu)$$

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Key observations:

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\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s u = \frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s u - \frac{\Gamma(1+s)^2}{2} \int_{\partial\Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu)
$$

$$
\int_{\Omega} (x \cdot \nabla u)(-\Delta)^s v + \int_{\Omega} (x \cdot \nabla v)(-\Delta)^s u =
$$

$$
\frac{2s-n}{2} \int_{\Omega} u(-\Delta)^s v + \frac{2s-n}{2} \int_{\Omega} v(-\Delta)^s u - \Gamma(1+s)^2 \int_{\partial\Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu)
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Key observations:

1. Pohozaev identity is quadratic in $u$ and it “comes from a bilinear identity”

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\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s u = \frac{2s-n}{2} \int_{\Omega} u (-\Delta)^s u - \frac{\Gamma(1+s)^2}{2} \int_{\partial \Omega} \left( \frac{u}{\delta^s} \right)^2 (x \cdot \nu)
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\int_{\Omega} (x \cdot \nabla u) (-\Delta)^s v + \int_{\Omega} (x \cdot \nabla v) (-\Delta)^s u =
\]

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\frac{2s-n}{2} \int_{\Omega} u (-\Delta)^s v + \frac{2s-n}{2} \int_{\Omega} v (-\Delta)^s u - \Gamma(1+s)^2 \int_{\partial \Omega} \frac{u}{\delta^s} \frac{v}{\delta^s} (x \cdot \nu)
\]

2. every $C^{1,1}$ domain is locally star-shaped

3. the bilinear identity holds easily when $u$ and $v$ have disjoint support
The end

Thank you!