Large $N$ expansion for normal and complex matrix ensembles

P. Wiegmann ∗ A. Zabrodin †

September 2003

Abstract

We present the first two leading terms of the $1/N$ (genus) expansion of the free energy for ensembles of normal and complex random matrices. The results are expressed through the support of eigenvalues (assumed to be a connected domain in the complex plane). In particular, the subleading (genus-1) term is given by the regularized determinant of the Laplace operator in the complementary domain with the Dirichlet boundary conditions. An explicit expression of the genus expansion through harmonic moments of the domain gives some new representations of the mathematical objects related to the Dirichlet boundary problem, conformal analysis and spectral geometry.

∗James Frank Institute and Enrico Fermi Institute of the University of Chicago, 5640 S.Ellis Avenue, Chicago, IL 60637, USA and Landau Institute for Theoretical Physics, Moscow, Russia
†Institute of Biochemical Physics, Kosygina str. 4, 119991 Moscow, Russia and ITEP, Bol. Chermushkinskaya str. 25, 117259 Moscow, Russia
1 Introduction

Ensembles of random matrices have numerous important applications in physics and mathematics ranging from energy levels of nuclei to number theory. An important information is encoded in the $1/N$ expansion ($N$ is the size of the matrix) of different expectation values in the ensemble. Many relevant references can be found in [1].

In this paper we discuss $1/N$-expansion in statistical ensembles of normal and complex matrices. A matrix $M$ is called normal if it commutes with its Hermitian conjugate: $[M, M^\dagger] = 0$, so both matrices can be diagonalized simultaneously. Eigenvalues of a normal matrix are complex numbers. The statistical weight

$$e^{\frac{1}{N} \text{tr} W(M)} d\mu(M)$$

of the normal matrix ensemble is specified by a potential function $W(M)$ (which depends on both $M$ and $M^\dagger$). Here $\hbar$ is a parameter, and the measure $d\mu$ of integration over normal matrices is induced from the flat metric on the space of all complex matrices.

Along the standard procedure of integration over angle variables [2], one passes to the joint probability distribution of eigenvalues $z_1, \ldots, z_N$. The partition function is then given by the integral

$$Z_N = \frac{1}{(2\pi^3 \hbar)^{N/2} N!} \int |\Delta_N(z_i)|^2 \prod_{j=1}^N e^{\frac{1}{N} \text{tr} W(z_j)} d^2 z_j$$

Here $\Delta_N(z_i) = \prod_{i>j}^N (z_i - z_j)$ is the Vandermonde determinant and $d^2 z \equiv dx dy$ for $z = x + iy$. The $N$-dependent normalization factor is put here for further convenience.

The model of normal matrices was introduced in [3]. This model is the particular $\beta = 1$ case of a more general one, referred to as 2D Coulomb gas with the joint probability distribution $|\Delta_N(z_i)|^{2\beta} \prod_{j=1}^N e^{\frac{1}{\beta} \text{tr} W(z_j)} d^2 z_j$.

For the potential of the form

$$W(z) = -z \bar{z} + V(z) + \overline{V(z)}$$

where $V(z)$ is an analytic function in some region of the complex plane (say, a polynomial), the normal matrix model is equivalent to the ensemble of all complex matrices with the same potential. It generalizes the gaussian Ginibre-Girko ensemble [4]. When passing to the integral over eigenvalues, the partition function for complex matrices differs from the one for normal matrices by a normalization factor only [2]. Both models are then reduced to the 2D Coulomb gas (with $\beta = 1$) in the external potential. Note also a formal similarity with the model of two Hermitian matrices. Its partition function is given by the same formula (1), with the potential (2), but $z_i$ and $\bar{z}_i$ are to be regarded as two independent real integration variables, with $d^2 z_i$ being understood as $dz_i d\bar{z}_i$.

It appears that the potential of the form (2) is most important for applications [5]. In the main part of the paper, we concentrate on this case, so one may, in this context, ignore the difference between the normal and complex ensembles, taking the 2D Coulomb gas partition function as a starting point. Physical applications of this model include the quantum Hall effect, the Saffman-Taylor viscous fingering and, conjecturally, more general growth problems which are mathematically described as a random evolution in
the moduli space of complex curves. Recently, the normal matrix model was shown [6] to be closely related to the matrix quantum mechanics, and, therefore, to the \( c = 1 \) string theory.

In addition to this it appears that the large \( N \) limit of the normal or complex random matrices admits a natural geometric interpretation relevant to the 2D inverse potential problem, the Dirichlet boundary problem and to spectral geometry of planar domains. In this paper we concentrate on calculation of the \( 1/N \) expansion of the free energy, \( F \propto \log Z_N \), and on its algebro-geometric meaning, leaving physical aspects for future publications.

The large \( N \) limit also implies the limit \( \hbar \to 0 \), so that \( \hbar N \) is finite and fixed. We prefer to work with the equivalent \( \hbar \)-expansion, rather than with the \( 1/N \) expansion, thus emphasizing its semiclassical nature. The free energy of the Hermitian, two-Hermitian, normal and complex matrix ensembles with the potential (2) has an \( \hbar \)-expansion of the form \( \log Z_N = \sum_{g \geq 0} \hbar^{2g-2} F_g \), where \( g \)-th term is associated with the contribution of diagrams with Euler characteristics \( 2 - 2g \), in the perturbative expansion of the free energy. Here we discuss the first two terms, \( F_0 \) and \( F_1 \):

\[
F = \hbar^2 \log Z_N = F_0 + \hbar^2 F_1 + O(\hbar^4) \tag{3}
\]

The leading term, \( F_0 \), is the contribution of planar diagrams, and \( F_1 \) is commonly referred to as genus 1 correction.

When \( N \) becomes large new macroscopic structures emerge. Invoking a physical analogy, one may say that the gas of eigenvalues segregates into “phases” with zero and non-zero density separated by a very narrow interface. The domain \( \mathcal{D} \) in the complex plane where the density is non-zero is called the support of eigenvalues (it may consist of several disconnected domains). The density at any point outside it is exponentially small as \( N \to \infty \).

The leading contribution to the free energy, the \( F_0 \) term in (3), is basically the Coulomb energy of particles confined in the domain \( \mathcal{D} \). For the potential of the form (2) it is the tau-function of curves introduced in [7]. It encodes solutions to archetypal problems of complex analysis and potential theory in planar domains.

Here we review these results and also compute the genus-1 correction to the free energy. The latter is identified with the free energy of a free bosonic field in the domain \( \mathcal{D}^c \) which is complementary to the support of eigenvalues, i.e., in the domain where the mean density vanishes:

\[
F_1 = -\frac{1}{2} \log \det(-\Delta_{\mathcal{D}^c}) \tag{4}
\]

Here \( \det(-\Delta_{\mathcal{D}^c}) \) is a properly regularized determinant of the Laplace operator in \( \mathcal{D}^c \) with Dirichlet boundary conditions. This suggests interesting links to spectral geometry of planar domains.

The genus expansion in the Hermitian random matrix model beyond the leading order has been obtained in the seminal paper [8]. In [9], the genus-1 correction was interpreted in terms of bosonic field theory on a hyperelliptic Riemann surface. The genus 1 correction to free energy of the model of two Hermitian matrices with polynomial potential was found only recently [10].
2 The planar large $N$ limit

In this section we briefly recall the large $N$ limit technique. This material is standard since early days of random matrix models (see., e.g., [11]). An appealing feature of the model of normal or complex matrices is a nice geometric interpretation and a direct relation to the inverse potential problem in two dimensions.

As was already mentioned, the parameter $h$ tends to zero simultaneously with $N \to \infty$ in such a way that $t_0 = N\bar{h}$ is kept finite and fixed. Using the Coulomb gas analogy, one may say that the leading contribution to the free energy is equal to the extremal value of the energy

$$\mathcal{E} = \sum_{i \neq j} \log |z_i - z_j| + \frac{1}{h} \sum_i W(z_i)$$

Equilibrium positions of charges are given by the extremum of the plasma energy: $\partial_{z_i} \mathcal{E} = \partial_{\bar{z}_i} \mathcal{E} = 0$.

Consider the 2D Coulomb potential $\Phi(z) = -\bar{h} \sum_i \log |z - z_i|^2$ created by the charges. Writing it as

$$\Phi(z) = -\int \log |z - \zeta|^2 \rho(z) \, d^2\zeta$$

where

$$\rho(z) = -\frac{1}{4\pi} \Delta \Phi(z) = \bar{h} \sum_i \delta^{(2)}(z - z_i)$$

is the microscopic density of eigenvalues (a sum of two-dimensional delta-functions), we assume that $\Phi$ in the limit can be treated as a continuous function. It is normalized as $\int \Delta \Phi(z) \, d^2z = -4\pi t_0$. Let $\Phi_0$ be this function for the equilibrium configuration of charges, then

$$\partial_{\bar{z}} (\Phi_0(z) - W(z)) = \partial_{\bar{z}} (\Phi_0(z) - W(z)) = 0$$

with the understanding that this equation holds only for $z$ belonging to a domain (or domains) where the density is nonzero. Applying $\partial_{\bar{z}}$ to the both sides, we see that the equilibrium density, $\rho_0(z)$, is equal to $-\frac{1}{4\pi} \Delta W(z)$ in some domain $D$ (the support of eigenvalues) and zero otherwise:

$$\rho_0(z) = \begin{cases} \sigma & z \in D \\ 0 & z \in D^c \end{cases} \quad \text{and} \quad \Phi_0(z) = -\int_D \log |z - \zeta|^2 \sigma \, d^2\zeta$$

Here $D^c = \mathbb{C} \setminus D$ is the domain complimentary to the support of eigenvalues and

$$\sigma = -\frac{1}{4\pi} \Delta W(z, \bar{z})$$

In this and in the next section we assume the special form of the potential (2). Then $\rho_0 = 1/\pi$ in the domain $D$.

The shape of $D$ is determined by the function $V(z)$. Let us assume, without loss of generality, that $0 \in D$ and parametrize $V(z)$ by Taylor coefficients at the origin:

$$V(z) = \sum_{k \geq 1} t_k z^k$$
The parameters $t_k$ (coupling constants of the matrix model) are in general complex numbers. Multiplying (7) by $z^{-k}$ and integrating over the boundary of $\mathcal{D}$, we conclude that the domain $\mathcal{D}$ is such that $-\pi k t_k$'s are moments of its complement, $\mathcal{D}^c$, with respect to the functions $z^{-k}$:

$$ t_k = -\frac{1}{\pi k} \int_{\partial \mathcal{D}} z^{-k} d\mathcal{D} = \frac{1}{2\pi i k} \oint_{\partial \mathcal{D}} z^{-k} z \bar{z} dz $$

(10)

Besides, from the normalization condition we know that the area of $\mathcal{D}$ is equal to $\pi t_0$.

To find the shape of the domain from its moments and area is the subject of the inverse potential problem. These data determine it uniquely, at least locally.

Here we assume that $\mathcal{D}$ is a connected domain. For example, in the potential $W = -\bar{z}z$ the eigenvalues uniformly fill the disk of radius $\sqrt{\bar{h}N}$. Small perturbations of the potential slightly disturb the circular shape.

In what follows, we need some functions associated with the domain $\mathcal{D}$, or rather with its complement, $\mathcal{D}^c$. The basic one is a univalent conformal map from the exterior of the unit disk onto the domain $\mathcal{D}^c$. Such a map exists by virtue of the Riemann mapping theorem. Let $U$ be the unit disk and $U^c$ its complement, i.e., the exterior of the unit disk. Consider the conformal map $z(w)$ from $U^c$ onto $\mathcal{D}^c$ normalized so that $z(\infty) = \infty$ and $r = \lim_{w \to \infty} z(w)/w$ is real, then the map is unique. In general, the Laurent expansion of the function $z(w)$ around infinity is

$$ z(w) = rw + \sum_{k \geq 0} u_k w^{-k} $$

(11)

The real number $r$ is called the (external) conformal radius of $\mathcal{D}$. Since the map is conformal, all zeros and poles of the derivative $z'(w) \equiv \partial_w z(w)$ are inside the unit circle. We also need the function $\bar{z}(w)$ given by the Laurent series (11) with complex conjugate coefficients and the Green function of the Dirichlet boundary problem in $\mathcal{D}^c$. In terms of the conformal map, the latter is given by the explicit formula

$$ G(z, z') = \log \left| \frac{w(z) - w(z')}{w(z)w(z') - 1} \right| $$

(12)

Here $w(z)$ is the conformal map from $\mathcal{D}^c$ onto $U^c$ inverse to the $z(w)$.

3 The leading term of the free energy

The leading contribution to the free energy is the value of the Coulomb energy (5) (multiplied by $\hbar^2$) for the extremal configuration of charges:

$$ F_0 = \int_{\mathcal{D}} \int_{\mathcal{D}} \log |z - z'| \sigma(z)\sigma(z')d^2z d^2z' + \int_{\mathcal{D}} W(z, \bar{z}) \sigma d^2z $$

The integrated version of the extremum condition (7) tells us that $\Phi_0(z) - W(z) = \text{const}$ for any $z \in \mathcal{D}$. The constant can be found from the same equality at $z = 0$, and we obtain $F_0$ as an explicit functional of the domain $\mathcal{D}$:

$$ F_0 = -\int_{\mathcal{D}} \int_{\mathcal{D}} \log \left| \frac{1}{z} - \frac{1}{z'} \right| \sigma(z)\sigma(z')d^2z d^2z' $$

(13)
For the special potential of the form (2), when $\sigma = 1$, the free energy is to be regarded as a function of $t_0$ and the coupling constants $t_k$.

Properties of $F_0$ immediately follow from known correlation functions of the model in the planar large $N$ limit. See [5, 7] for normal and complex matrices and [12] for similar results in the context of the Hermitian 2-matrix model. Some of these correlation functions previously appeared in studies of thermal fluctuations in classical confined Coulomb plasma [13]. Integrable structures associated with $F_0$ were studied in [14, 7, 15, 16]. Here is the list of main properties of $F_0$ for the most important case $\sigma = 1/\pi$.

- **1-st order derivatives:**
  \[
  \frac{\partial F_0}{\partial t_k} = \frac{1}{\pi} \int_D z^k d^2z, \quad k \geq 1, 
  \]
  \[
  \frac{\partial F_0}{\partial t_0} = \frac{1}{\pi} \int_D \log |z|^2 d^2z 
  \]
  can be combined in the generating formula
  \[
  D(z)F_0 = \frac{1}{\pi} \int_D \log |z^{-1} - \zeta^{-1}|^2 d^2\zeta, \quad z \in D^c, 
  \]
  where
  \[
  D(z) = \frac{\partial}{\partial t_0} + \sum_{k \geq 1} \frac{1}{k} \left( z^{-k} \frac{\partial}{\partial t_k} + z^{-k} \frac{\partial}{\partial t_k} \bar{z} \right) 
  \]
  Since the derivatives of $F_0$ with respect to the moments $t_k$ are moments of the complementary domain, this function formally solves the 2D inverse potential problem.

- **2-nd order derivatives:** for $z, z' \in D^c$ we have
  \[
  D(z)D(z')F_0 = 2G(z, z') - \log \left| \frac{1}{z} - \frac{1}{z'} \right|^2 
  \]
  where $G(z, z')$ is the Green function of the Dirichlet boundary problem in $D^c$ (12). Note that the logarithmic singularity of the Green function at $z = z'$ cancels by the second term in the right hand side. In a particular case when both $z, z'$ tend to infinity, we get a simple formula for the conformal radius:
  \[
  \partial^2 t_0 F_0 = 2 \log r 
  \]

- **3-d order derivatives.** The generating formula reads [15]:
  \[
  D(a)D(b)D(c)F_0 = -\frac{1}{2\pi} \oint_{\partial D} \partial_n G(a, \xi) \partial_n G(b, \xi) \partial_n G(c, \xi) d\xi | \quad (19)
  \]
  An important corollary of this formula and eq. (17) is the complete symmetry of the expression $D(a)G(b, c)$ with respect to all permutations of the points $a, b, c$. Another corollary of (19) is the following residue formula valid for $j, k, l \geq 0$ [16]:
  \[
  \frac{\partial^3 F_0}{\partial t_j \partial t_k \partial t_l} = \frac{1}{2\pi i} \oint_{|w|=1} h_j(w)h_k(w)h_l(w) \frac{dw}{z'(w)z'(w^{-1})} w 
  \]
Here $h_j(w)$ are polynomials in $w$ of degree $j$:

$$h_j(w) = w \frac{d}{dw} [(z^j(w)+)] \quad \text{for } j \geq 1 \text{ and } h_0(w) = 1,$$

where $(...)_+$ is the positive degree part of the Laurent series. The notation $z'(w^{-1})$ means the derivative $dz(u)/du$ taken at the point $u = w^{-1}$. This formula is especially useful when $z'(w)$ is a rational function, then the integral is reduced to a finite sum of residues. We use this below.

- **Dispersionless Hirota equations.** The function $F_0$ obeys an infinite number of nonlinear differential equations which are combined into the integrable hierarchy of dispersionless Hirota’s equations. See [7, 15] for details.

- **WDVV equations.** Suppose $V(z)$ is a polynomial of $m$-th degree, i.e., $t_k = 0$ for all $k > m$. On this subspace of parameters, $F_0$ obeys the system of Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations

$$F_i F_j^{-1} F_k = F_k F_j^{-1} F_i \quad \text{for all } 0 \leq i, j, k \leq m - 1 \quad (21)$$

where $F_i$ is the $m$ by $m$ matrix with matrix elements $(F_i)_{jk} = \frac{\partial^3 F_i}{\partial t_i \partial t_j \partial t_k}$. See [16] for details.

To conclude: $F_0$ is a “master function” which generates objects of complex analysis in planar simply-connected domains. The full free energy of the matrix ensemble, $F$, may be regarded as its “quantization”.

### 4 The genus 1 correction to the free energy

#### 4.1 The result for $F_1$

We now describe the result for the genus-1 correction $F_1$. We start with the special potential (2). Then $F_1$ is expressed entirely in terms of the metric on the $U^c$ induced from the standard flat metric on the $z$-plane by the conformal map: $dz \, d\bar{z} = e^{2\phi(w)} \, dw \, d\bar{w}$. Here

$$\phi(w) = \log |z'(w)| \quad (22)$$

and $z(w)$ is the conformal map $U^c \to D^c \quad (11)$. The derivation of this formula and its extension to a general potential is outlined in Section 5.

We found that

$$F_1 = -\frac{1}{24\pi} \int_{|w|=1} (\phi \partial_n \phi + 2\phi) |dw| \quad (23)$$

Here $\partial_n$ is the normal derivative, with the normal vector pointing outside the unit circle. The derivation of this formula is outlined, for a more general model, in Section 5.

Since $\phi(w)$ is harmonic in $U^c$, we may rewrite the r.h.s. of (23) as

$$F_1 = \frac{1}{24\pi} \int_{|w|>1} |\nabla \phi|^2 \, dw - \frac{1}{12\pi} \int_{|w|=1} \phi \, |dw|$$
Here we recognize the formula for the regularized determinant of the Laplace operator $\Delta_{D^c} = 4\partial_z\partial_{\bar{z}}$ in $D^c$ with Dirichlet conditions on the boundary. The first term is the bulk contribution first found by Polyakov [17] (for a metric induced by a conformal map it reduces to a boundary integral), while the second term, computed in [18], is a net boundary term (see also Section 1 of [19]):

$$F_1 = -\frac{1}{2} \log \det (-\Delta_{D^c})$$

(24)

In the particular case $W(z) = -z\bar{z}$ we get $F_1 = -\frac{1}{12} \log t_0$ that coincides with the result of [20] obtained by a direct calculation.

The appearance of elements of quantum field theory in a curved space is not accidental. A field-theoretical derivation of this result will be given elsewhere.

4.2 Rational case

Before explaining the origin of the explicit formula for $F_1$ we write it in yet another suggestive form. Consider a domain such that $z'(w)$ is a rational function:

$$z'(w) = r \prod_{i=0}^{m-1} \frac{w-a_i}{w-b_i}$$

All the points $a_i$ and $b_i$ must be inside the unit circle, otherwise the map $z(w)$ is not conformal. On the unit circle we have $|dw| = \frac{du}{w}$ and $\phi(w) = \frac{1}{2}(\log z'(w) + \log \bar{z}'(w^{-1}))$, where the first and the second term (the Schwarz reflection) are analytic outside and inside it, respectively. (Recall that our notation $z'(w^{-1})$ means $d\bar{z}(u)/du$ at the point $u = w^{-1}$.) Plugging this into (23), we get:

$$F_1 = -\frac{1}{24\pi i} \oint_{|w|=1} \log z'(w) \left[ \frac{1}{2} \partial_w \log z'(w) + \frac{1}{w} \right] dw -$$

$$-\frac{1}{24\pi i} \oint_{|w|=1} \log z'(w^{-1}) \frac{dw}{w} - \frac{1}{48\pi i} \oint_{|w|=1} \log z'(w^{-1}) \frac{z''(w)}{z'(w)} dw$$

The integrals can be calculated by taking residues either outside or inside the unit circle. The poles are at $\infty$, at 0, and at the points $a_i$ and $b_i$. The result is

$$F_1 = -\frac{1}{24} \left( \log r^4 + \sum_{z'(a_i)=0} \log \bar{z}'(a_i^{-1}) - \sum_{z'(b_i)=\infty} \log \bar{z}'(b_i^{-1}) \right)$$

(25)

If the potential $V(z)$ is polynomial, $V(z) = \sum_{k=1}^{m} t_k z^k$, i.e., $t_k = 0$ as $k > m$ for some $m > 0$, then the series for the conformal map $z(w)$ truncates: $z(w) = rw + \sum_{l=0}^{m-1} u_l w^{-l}$ and

$$z'(w) = r \prod_{i=0}^{m-1} (1 - a_i w^{-1})$$

is a polynomial in $w^{-1}$ (all poles $b_i$ of $z'(w)$ merge at the origin). Then the last sum in (25) becomes $m \log r$ and the formula (25) gives

$$F_1 = -\frac{1}{24} \log \left( r^4 \prod_{z'(a_j)=0} \frac{\bar{z}'(a_j^{-1})}{r} \right) = -\frac{1}{24} \log \left( r^4 \prod_{i,j=0}^{m-1} \frac{1}{a_i a_j} \right)$$

(26)
This formula is essentially identical to the genus-1 correction to the free energy of the Hermitian 2-matrix model with a polynomial potential recently computed by Eynard [10].

### 4.3 Determinant representation of $F_1$ for polynomial potentials

For polynomial potentials the genus-1 correction enjoys an interesting determinant representation.

Set

$$D_m := \det \left( \frac{\partial^3 F_0}{\partial t_0 \partial t_j \partial t_k} \right)_{0 \leq j, k \leq m-1}$$

Using the residue formula (20) we compute:

$$D_m = \frac{1}{(2\pi i)^m} \oint_{|w_0|=1} \frac{dw_0}{w_0} \cdots \oint_{|w_{m-1}|=1} \frac{dw_{m-1}}{w_{m-1}} \det [h_j(w_j)h_k(w_j)] \prod_{l=0}^{m-1} z'(w_l)z'(w_l^{-1})$$

(27)

Clearly, the determinant in the numerator can be substituted by $\frac{1}{m} \det^2(h_j(w_k))$ and $\det [h_j(w_k)] = (m-1)! r^{m-1} \Delta_m(w_i)$, where $\Delta_m(w_i)$ is the Vandermonde determinant.

Each integral in (27) is given by the sum of residues at the points $a_i$ inside the unit circle (the residues at $w_i = 0$ vanish). Computing the residues and summing over all permutations of the points $a_i$, we get:

$$D_m = (-1)^{\frac{1}{2}m(m-1)((m-1)!)^2} r^{m(m-3)} \prod_{i,j} a_j^{m-1} \prod_{i,j} \frac{1}{(1-a_ja_i)}$$

(28)

As is seen from (10), the last non-zero coefficient of $V(z)$ equals $t_m = \frac{6m-3}{m(m-1)}$. (We regard it as a fixed parameter.) Therefore, $\prod_{i=1}^n a_i = (-1)^m m(m-1)r^{m-2}t_m$, and we represent $F_1$ (26) in the form

$$F_1 = \frac{1}{24} \log D_m - \frac{1}{12}(m^2-3m+3) \log r - \frac{1}{24}(m-1) \log t_m + \text{const}$$

(29)

where const is a numerical constant. Recalling (18), we see that $F_1$, for models with polynomial potentials of degree $m$, is expressed through derivatives of $F_0$:

$$F_1 = \frac{1}{24} \log \det_{m \times m} \left( \frac{\partial^3 F_0}{\partial t_0 \partial t_j \partial t_k} \right) - \frac{1}{24}(m^2-3m+3) \frac{\partial^2 F_0}{\partial t_0^2} - \frac{1}{24}(m-1) \log t_m + \text{const}$$

(30)

where $j, k$ run from 0 to $m-1$.

Similar determinant formulas are known for genus-1 corrections to free energy in topological field theories [21].

### 5 $F_1$ from loop equation

The standard (and powerful) method to obtain $1/N$-expansions in matrix models is to use invariance of the partition function under changes of matrix integration variables. In
the 2D Coulomb gas formalism, this reduces to invariance of the partition function (1) under diffeomorphisms

\[ z_i \rightarrow z_i + \epsilon(z_i, \bar{z}_i), \quad \bar{z}_i \rightarrow \bar{z}_i + \bar{\epsilon}(z_i, \bar{z}_i) \]

The invariance of the partition function in the first order in \( \epsilon \) results in the identity

\[ \sum_i \int \partial z_i \left( \epsilon(z_i, \bar{z}_i) e^\epsilon \right) \prod_j d^2 z_j = 0 \]  

(31)

for any function \( \epsilon(z, \bar{z}) \). One may read it as Ward identity obeyed by correlation functions of the model. For historical reasons, it is called the loop equation. Since correlation functions are variational derivatives of the free energy with respect to the potential, the loop equation is an implicit functional relation for the free energy.

### 5.1 Loop equation in general normal matrix model

A closed loop equation does not emerge for the special potential (2). It can be written only for the ensemble of normal matrices with a general potential \( W \) in (1). Let it be of the form

\[ W(z) = -z\bar{z} + V(z) + \sqrt{V(z)} + U(z) \]

where \( U \) is only assumed to have a regular Taylor expansion around the origin starting from cubic terms.

Choosing \( \epsilon(z_i, \bar{z}_i) = (z - z_i)^{-1} \) and plugging it into (31) with \( \mathcal{E} \) given in (5), one is able to rewrite (31) as a relation between correlation functions of the field

\[ \Phi(z) = -\hbar \text{tr} \log \left( (z - M)(\bar{z} - M^\dagger) \right) = -\hbar \sum_i \log |z - z_i|^2 \]

or \( \partial \Phi(z) = -\hbar \text{tr} (z - M)^{-1} \) (here and below \( \partial \equiv \partial_z \)). Note that \( \partial \Phi(z) \) is trace of the resolvent of the matrix \( M \) and \( \Delta \Phi(z) = -4\pi \rho(z) \), where \( \rho \) is the density of eigenvalues.

After some simple rearrangings, the loop equation following from (31) acquires the form

\[ \frac{1}{2\pi} \int \frac{\partial W(\zeta) \langle \Delta \Phi(\zeta) \rangle}{z - \zeta} d^2 \zeta = \langle (\partial \Phi(z))^2 \rangle + \hbar \langle \partial^2 \Phi(z) \rangle \]  

(32)

(For any symmetric function \( f(\{z_i\}) \), the correlation function \( \langle f \rangle \) is defined as the integral \( \int f(\{z_i\}) |\Delta N(z_i)|^2 \prod_j e^{\frac{1}{\hbar} W(z_j)} d^2 z_j \) with a normalization factor such that \( \langle 1 \rangle = 1 \)) This relation is exact for any finite \( N \). Supplemented by the relation

\[ \langle \partial \Phi(z) \rangle = -\frac{t_0}{z} + \partial_z D(z) F \]  

(33)

(also exact) which directly follows from the definitions of the free energy and the field \( \Phi \), the loop equation allows one to find the \( \hbar \)-expansion of the free energy.
5.2 Expanding the loop equation

The $\hbar$-expansion of the free energy for the general normal matrix model is more complicated than the one discussed in the previous sections. It contains all powers of $\hbar$, not only even:

$$h^2 \log Z_N = F_0 + h F_{1/2} + h^2 F_1 + O(h^3)$$

so it hardly has a direct topological interpretation. Accordingly, $\hbar$-expansions of mean values and other correlation functions are expansions in $\hbar$ rather than $\hbar^2$. In particular,

$$\langle \Phi(z) \rangle = \Phi_0(z) + h \Phi_{1/2}(z) + h^2 \Phi_1(z) + O(h^3)$$

We proceed by expanding the loop equation in powers of $\hbar$. In the leading order, the second term in the r.h.s. vanishes, and $\bar{z}$-derivative of the both sides gives:

$$\left( \partial W(z) - \partial \Phi_0(z) \right) \Delta \Phi_0(z) = 0$$

This just means that for $z \in \mathbb{D}$ the extremum condition (7) is satisfied and $\Delta \Phi_0(z) = 0$ otherwise. Inside $\mathbb{D}$, the leading term of the mean density, $\rho_0(z)$, is given by $\rho_0(z) = \sigma(z)$, where $\sigma(z)$ is defined in (8). (Note that the function $\sigma$ is defined by this formula everywhere in the complex plane, and does not depend on the shape of $\mathbb{D}$, while $\rho_0$ coincides with $\sigma$ in $\mathbb{D}$ and equals 0 in $\mathbb{D}^c$.) For potentials of the form (2), $\sigma(z) = 1/\pi$.

Being developed into a series in $\hbar$, the loop equation gives an iterative procedure to find the coefficients $\Phi_i(z)$. We need the following results on the correlation functions for the general normal matrix ensemble (see [5]):

$$\langle \partial \Phi(z) \rangle = \int_{\mathbb{D}} \frac{\sigma(\zeta) d^2 \zeta}{\zeta - z} + O(h)$$

$$\langle \Phi(z_1) \Phi(z_2) \rangle_{\text{conn}} = 2h^2 \left( G(z_1, z_2) - G(z_1, \infty) - G(\infty, z_2) - \log \frac{|z_1 - z_2|}{r} \right) + O(h^3)$$

where the connected correlation function is defined as $\langle fg \rangle_{\text{conn}} = \langle fg \rangle - \langle f \rangle \langle g \rangle$. Note that the function (38) has no singularity at coinciding points $z_1, z_2 \in \mathbb{D}^c$. Merging the points, we get:

$$\langle (\partial \Phi(z))^2 \rangle_{\text{conn}} = \frac{h^2}{6} \{w; z\} + O(h^3)$$

where

$$\{w; z\} = \frac{w''(z)}{w'(z)} - \frac{3}{2} \left( \frac{w''(z)}{w'(z)} \right)^2$$

is the Schwarzian derivative of the conformal map $w(z)$.

After these preparations, further steps are straightforward. Terms of order $h$ and $h^2$ of the loop equation give:

$$\frac{1}{2\pi} \int L(z, \zeta) \Delta \Phi_{1/2}(\zeta) d^2 \zeta = -\partial^2 \Phi_0(z)$$

$$\frac{1}{2\pi} \int L(z, \zeta) \Delta \Phi_1(\zeta) d^2 \zeta = - \left[ (\partial \Phi_{1/2}(z))^2 + \partial^2 \Phi_{1/2}(z) \right] - \frac{1}{6} \{w; z\}$$
where the kernel of the integral operator in the l.h.s. is

\[ L(z, \zeta) = \frac{\partial W(\zeta) - \partial \Phi_0(z)}{\zeta - z} \]  

(41)

It should be noted that the \( h \)-expansion of the loop equation may break down for \( z \in D \). This is mainly because the correlator \( \langle \Phi(z)\Phi(z') \rangle \), when the two points are close to each other and belong to the support of eigenvalues, is not given by eq. (38). At the same time, for our purpose we need this correlator just on very small distances, when the two points merge. Naively, for \( z, z' \in D \) the correlator diverges as \( z' \to z \). This means that its short-distance behaviour is in fact of a different order in \( h \) and must be calculated separately. Fortunately, this problem can be avoided by restricting the equations to \( D^c \), where no divergency emerges on any scale and one may think that the short-distance behaviour of correlation functions is still given by eq. (38). (However, we understand that this argument is not rigorous and need to be justified by an actual calculation of correlation functions at small scales.) Hereafter, \( z \) in (40) is assumed to be outside the support of eigenvalues, i.e., the equations should be solved for \( z \in D^c \). In this region the functions \( \Phi_k(z) \) are harmonic.

From (33) we see that

\[ \partial_z D(z) F_{1/2} = \partial_z \Phi_{1/2}(z) , \quad \partial_z D(z) F_1 = \partial_z \Phi_1(z) \]  

(42)

The strategy is to find \( \Phi_k \)'s from (40) and then “to integrate" them to get \( F_k \)'s, i.e., to find a functional \( F_k \) such that it obeys (42). A general method to find the “derivative” \( D(z) \) of any proper functional of the domain \( D^c \) is proposed in [15].

An important remark is in order. Suppose we restrict ourselves to the class of models with potentials of the form (2) (i.e., with \( \sigma(z) = 1/\pi \)), like in previous sections. Applying \( \partial_z D(z) \) to the functional (23), that is \( F_1 \) in this case, we obtain a wrong answer for \( \partial_2 \Phi_1(z) \), which does not obey the loop equation (40)! This seemingly contradicts eqs. (42) and so explains why one has to deal with the arbitrary potential. The matter is simply that there are functionals such that they vanish for potentials with \( \sigma(z) = 1/\pi \) but their “derivatives”, \( \partial_2 D(z) \), do not. They do contribute to \( \Phi_1 \) and restore the right answer.

### 5.3 Free energy of the general model

Skipping further details, we present the results for the general model of normal matrices.

The answer for \( F_0 \) is familiar [5]. It is given by (13). The first correction, \( F_{1/2} \), is

\[ F_{1/2} = - \int_D \sigma(z) \log \sqrt{\pi \sigma(z)} \, d^2 z \]  

(43)

To write down the full answer for \( F_1 \) in a compact form, we need to introduce, along with the \( \phi(w) \) (22), another function,

\[ \chi(z) = \log \sqrt{\pi \sigma(z)} \]  

(44)
and the function $\chi^H(z)$ defined in the domain $\mathbb{D}^c$. It is a harmonic function in $\mathbb{D}^c$ with the boundary value $\chi(z)$. The function $\chi^H$ is the solution to the Dirichlet boundary problem: $\chi^H(z) = -\frac{1}{2}\int_{\partial \mathbb{D}} \partial_n G(z, \xi) \chi(\xi) |d\xi|$. The explicit formula for $F_1$ reads:

\[
F_1 = \frac{1}{24\pi} \left[ \int_{|w|>1} |\nabla (\phi + \chi)|^2 d^2w - 2 \oint_{|w|=1} (\phi + \chi) |dw| - \int_{\mathbb{C}} |\nabla \chi|^2 d^2w \right] + \frac{1}{8\pi} \left[ \int_{\mathbb{D}} |\nabla \chi|^2 d^2z - \oint_{\partial \mathbb{D}} \chi \partial_n \chi^H |dz| - \frac{1}{2} \int_{\mathbb{D}} \Delta \chi d^2z \right]
\]

(45)

where $\chi$ in the first three integrals is treated as a function of $w$ through $\chi = \chi(z(w))$.

The r.h.s. of this formula is naturally decomposed into two parts having completely different nature, the “quantum” and “classical” parts: $F_1 = F_1^{(q)} + F_1^{(cl)}$. The (most interesting) quantum part can be again represented through the regularized determinant of the Laplace operator in $\mathbb{D}^c$ with Dirichlet boundary conditions. However, now the Laplacian should be taken in conformal metric with the conformal factor $e^{2\chi(z)}$. Equivalently, on the exterior of the unit circle the Laplacian should be taken in the metric with the conformal factor $e^{2\chi(z) + 2\chi(z(w))}$; we see that $\phi$ and $\chi$ do enter as the sum $\phi + \chi$ in the first line. More precisely, the formula for regularized determinants of Laplace operators in domains with boundary known in the literature (eq. (4.42) in [18]) allows us to identify

\[
F_1^{(q)} = \frac{1}{2} \log \frac{\det (-e^{-2\chi} \Delta_{\mathbb{D}^c})}{\det (-e^{-2\chi} \Delta_{\mathbb{D}^c})}
\]

(46)

The classical part comes from “classical” (though of order $\hbar$) corrections to the shape of the support of eigenvalues, which always exist unless $\sigma(z)$ is a constant (see below). It is essentially given by $F_1^{(cl)} = \lim_{\hbar \to 0} \frac{1}{2\hbar} \langle \text{tr} \chi(M) \rangle_{\text{conn}}$.

Different terms of the $\hbar$-expansion gain a clear interpretation in terms of the collective field theory of the normal matrix model, in the spirit of [22]. In this context, it is natural to start with the general Coulomb gas model with arbitrary $\beta$. The generalized loop equation

\[
\frac{1}{2\pi} \int_{z - \zeta} \frac{\partial W(\zeta)}{\zeta} \langle \Delta \Phi(\zeta) \rangle d^2\zeta = \beta \langle (\partial \Phi(z))^2 \rangle + (2 - \beta) \hbar \langle \partial^2 \Phi(z) \rangle
\]

(47)

can be understood as the conformal Ward identity for the collective theory. This allows one to find the effective action in the form

\[
S = S_0 + S_1
\]

\[
S_0 = \beta \int \rho(z) \log |z - \zeta| \rho(\zeta) d^2zd^2\zeta + \int W(z) \rho(z) d^2z
\]

(48)

\[
S_1 = \left( 1 - \frac{\beta}{2} \right) \hbar \int \rho(z) \log \rho(z) d^2z
\]

The second term, $S_2$, is a combination of the short range part $-\frac{\beta}{2} \rho \log \rho$ and the entropy $\rho \log \rho$. (See [23, 24], where similar actions for unitary and Hermitian matrix ensembles were discussed.)
This action suggests to rearrange the $\hbar$-expansion of the free energy (34) and write it in the “topological” form $F = \sum_{g \geq 0} \hbar^{2g} F_g$, where each term has its own expansion

$$F_g = F_g^{(0)} + \sum_{n \geq 1} \hbar^n F_g^{(n)}$$

(49)

Here $\hbar_\beta \equiv (2 - \beta)\hbar$ is regarded as an independent parameter. The equilibrium density of charges, $\rho_0$, is determined by $\delta S/\delta \rho = 0$ which leads to the Liouville-like equation

$$-\frac{\hbar_\beta}{8\pi} \Delta \log \rho_0(z) + \beta \rho_0(z) = \sigma(z)$$

(50)
in the bulk. For $\beta \neq 2$ the first term generates corrections to the shape of the support of eigenvalues. The classical free energy is $F_0 = F_0^{(0)} + \hbar_\beta F_0^{(1)} + \hbar^2_\beta F_0^{(2)} + O(\hbar^3_\beta)$. In particular we see that $F_{1/2}$ given in (43) is in fact $F_0^{(1)}$ while the “classical” part $F_1^{(c)}$ of (45) is $F_0^{(2)}$. The “quantum” part is then $F_1^{(q)} = F_1^{(0)}$.

The collective field theory approach to the normal and complex matrix ensembles will be presented elsewhere.

Acknowledgments

We thank A.Cappelli, L.Chekhov, B.Dubrovin, V.Kazakov, I.Kostov, Yu.Makeenko, A.Marshakov, M.Mineev-Weinstein and R.Theoarescu for useful discussions. P.W. would like to thank INFN and the University of Florence, Italy, for hospitality. A.Z. is grateful to B.Julia for opportunity to present these results at the Les Houches Spring School in March 2003. P. W. was supported by the NSF MRSEC Program under DMR-0213745, NSF DMR-0220198 and Humboldt foundation. The work of A.Z. was supported in part by RFBR grant 03-02-17373, by grant for support of scientific schools NSh-1999.2003.2 and by Federal Program of the Russian ministry of industry, science and technology 40.052.1.1.1112. The work of both authors was partially supported by the NATO grant PST.CLG.978817.

References

[1] P.J.Forrester, N.C.Snaith and J.J.M.Verbaarschot, Developments in random matrix theory, J. Phys. A: Math. Gen. 36 (2003) R1-R10, an introductory review for the special issue “Random Matrix Theory”

[2] M.L.Mehta, Random matrices, 2nd edition, Academic Press, NY, 1991

[3] L.-L.Chau and Y.Yu, Phys. Lett. 167A (1992) 452; L.-L.Chau and O.Zaboronsky, Commun. Math. Phys. 196 (1998) 203-247, e-print archive: hep-th/9711091

[4] J.Ginibre, J. Math. Phys. 6 (1965) 440; V.Girko, Theor. Prob. Appl. 29 (1985) 694

[5] P.Wiegmann and A.Zabrodin, J. Phys. A: Math. Gen. 36 (2003) 3411-3424; e-print archive: hep-th/0210159; A.Zabrodin, New applications of non-hermitian random matrices, to be published, e-print archive: cond-mat/0210331
[6] S.Alexandrov, V.Kazakov and I.Kostov, *2D string theory as normal matrix model*, e-print archive: hep-th/0302106

[7] I.Kostov, I.Krichever, M.Mineev-Weinstein, P.Wiegmann and A.Zabrodin, *τ-function for analytic curves*, in: Random Matrices and Their Applications (MSRI publications, vol. 40), ed. P.Bleher and A.Its (Cambridge: Cambridge Academic Press), 285-299; e-print archive: hep-th/0005259

[8] J.Ambjorn, L.Chekhov, C.F.Kristjansen and Yu.Makeenko, Nucl. Phys. B404 (1993) 127-172; Erratum: ibid. B449 (1995) 681, e-print archive: hep-th/9302014

[9] I.Kostov, *Conformal Field Theory Techniques in Random Matrix models*, e-print archive: hep-th/9907060

[10] B.Eynard, *Large N expansion of the 2-matrix model*, JHEP 0301 (2003) 051, e-print archive: hep-th/0210047; B.Eynard, *Large N expansion of the 2-matrix model, multicut case*, e-print archive: hep-th/0307052

[11] E.Brezin, C.Itzykson, G.Parisi and J.B.Zuber, Commun. Math. Phys. 59 (1978) 35-51

[12] M.Bertola, *Free energy of the two-matrix model/dToda tau-function*, e-print archive: hep-th/0306184

[13] A.Alastuey and B.Jancovici, J. Stat. Phys. 34 (1984) 557; B.Jancovici, J. Stat. Phys. 80 (1995) 445; P.J.Forrester, Phys. Rep. 301 (1998) 235-270

[14] M.Mineev-Weinstein, P.Wiegmann and A.Zabrodin, Phys. Rev. Lett. 84 (2000) 5106-5109, e-print archive: nlin.SI/0001007; P.Wiegmann and A.Zabrodin, Commun. Math. Phys. 213 (2000) 523-538, e-print archive: hep-th/9909147

[15] A.Marshakov, P.Wiegmann and A.Zabrodin, Commun. Math. Phys. 227 (2002) 131-153; e-print archive: hep-th/0109048

[16] A.Boyarsky, A.Marshakov, O.Ruchayskiy, P.Wiegmann and A.Zabrodin, Phys. Lett. B515 (2001) 483-492; e-print archive: hep-th/0105260

[17] A.Polyakov, Phys. Lett. B103 (1981) 207-210

[18] O.Alvarez, Nucl. Phys. B216 (1983) 125-184; P.Di Vecchia, B.Durhuus, P.Olesen and J.Petersen, Nucl.Phys. B207 (1982) 77; J.Ambjorn, B.Durhuus, J.Frölich and P.Orland, Nucl.Phys. B270 (1986) 457

[19] B.Osgood, R.Phillips and P.Sarnak, J. Func. Anal. 80 (1988) 148-211

[20] P. Di Francesco, M.Gaudin, C.Itzykson and F.Lesage, Int. J. Mod. Phys. A9 (1994) 4257-4351

[21] B.Dubrovin and Y.Zhang, Commun. Math. Phys. 198 (1998) 311-361, e-print archive: hep-th/9712232
[22] A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511; A. Jevicki and B. Sakita, Nucl. Phys. B185 (1981) 89; A. Jevicki, Collective field theory and Schwinger-Dyson equations in matrix models, preprint Brown-HET-777, Proceedings of the meeting “Symmetries, quarks and strings” held at the City College of New York, Oct. 1-2, 1990

[23] F. Dyson, Statistical theory of the energy levels of complex systems, part II, J. Math. Phys. 3 (1962) 157

[24] O. Lechtenfeld, Semiclassical approach to finite-N matrix models, e-print archive: hep-th/9112045; S. Ben-Menahem, $D = 0$ matrix model as conjugate field theory, preprint SLAC-PUB-5377, February 1992