Viscosity and thermodynamic properties of QGP in relativistic heavy ion collisions

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We study the viscosity and thermodynamic properties of QGP at RHIC by employing the recently extracted equilibrium distribution functions from two hot QCD equations of state of $O(g^5)$ and $O(g^6 \ln(1/g))$ respectively. After obtaining the temperature dependence of energy density, and entropy density, we focus our attention on the determination of shear viscosity for a rapidly expanding interacting plasma, as a function of temperature. We find that interactions significantly decrease the shear viscosity. They decrease the viscosity to entropy density ratio, $\eta/S$ as well.

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I. INTRODUCTION

Recent experimental results from RHIC reveal that the QGP produced in heavy ion collisions behaves like an almost perfect fluid with very low viscosity. In belying the earlier expectations that the deconfined phase would show nearly ideal behavior at temperatures close to $T_c$, the results from the flow measurements signal that the deconfined phase is strongly interacting. Lattice studies also predict that the equation of state for QGP is about 10 percent away from ideal EOS even at $T \sim 4T_c$. Therefore, studies based

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on the ideal EOS in the computations are certainly inadequate to address the physics in this
domain of QCD.

In an attempt to understand the flow measurement results, a variety of techniques have
been employed, including the employment of AdS/CFT correspondence. A strong result
from these studies is a lower bound on the viscosity to entropy density ratio given by \( \frac{\eta}{S} < \frac{1}{4\pi} \approx 0.08 \). Yet another approach is to study a strongly coupled classical plasma \[6\] which
again yields a value close to the lower bound mentioned above. There are two lattice results,
one by Meyer \[7\] who obtains a value 0.13 for \( \frac{\eta}{S} \) in pure gauge theory at \( T = 1.56T_c \). The
other result, also in pure gauge theory is due to Nakamura \[8\] who finds that \( \frac{\eta}{S} < 1 \).
Recall that the analyses \[9, 10\] based on \( v_2 \) measurements \[11\] arrive at values that vary
from 0.08 – 0.2. Results of other studies \[3, 4, 12, 13\] also yield numbers in the same range.
Interestingly, Asakawa et al \[14, 13, 16\] find that the ratio can take a value which is smaller
than the lower bound set by AdS/CFT studies, depending on the value assigned to the
transport parameter \( \hat{q}_R \).

It is noteworthy that while the perfect liquid picture implies a strongly interacting QGP
(sQGP), most of the estimates made above employ a perturbation about the ideal distri-
bution for quarks and gluons. While this could simplify matters, it is important to find
out what an EOS which includes the interactions would predict for \( \frac{\eta}{S} \). In doing so, one
could also gain an appreciation of the nature of interactions in sQGP. In this paper, we
study the predictions of EOS which are based on improved perturbative QCD \[17, 18\]. The
implications of lattice EOS will be taken up in a subsequent paper. Our study is based on
\[19, 20\] where it has been shown how the EOS may be adapted to study the properties of
QGP in heavy ion collisions. It is worth mentioning that the above mentioned works have
demonstrated the viability of the EOS since they yield reasonable values for the dissociation
temperatures for \( J/\Psi \) and \( \Upsilon \).

The paper is organized as follows. In Section (II), we introduce the EOS based on pQCD
and review the work contained in \[19, 20\]. We proceed to determine the the temperature de-
pendence of thermodynamic observables(energy density and entropy density) in Section(III).
In Section (IV), We obtain the expressions for anomalous and collisional contributions to
the parton shear viscosities, in the presence of interactions. We further study the behavior
of viscosity to entropy density(\( \eta/S \)) as a function of temperature for pure gauge theory
plasma, and compare the results with the ones obtained from an ideal gas distribution. In
II. HOT QCD EQUATIONS OF STATE AND THEIR QUASIPARTICLE DESCRIPTION

Recently Chandra et al. [19, 20] have considered two EOS based on pQCD, and developed a self-consistent method to recast them in terms of non-interacting/weakly interacting quasi particles with effective fugacities. Since the method is employed here, we briefly review the work.

The EOS which we label EOS2 [18] is given by

\[
P_{g^6 \ln(1/g)} = \frac{8\pi^2}{45\beta^4} \left\{ (1 + \frac{21N_f}{32}) - \frac{15}{4} (1 + \frac{5N_f}{12}) \alpha_s^\pi \right. \\
+ 30(1 + \frac{N_f}{6}) \left( \frac{\alpha_s}{\pi} \right)^2 + \left[ (237.2 + 15.97N_f) \\
- 0.413N_f^2 + \frac{135}{2} (1 - \frac{N_f}{6}) \ln(\frac{\alpha_s}{\pi}(1 + \frac{N_f}{6})) \right. \\
- \frac{165}{8} (1 + \frac{N_f}{12})(1 - \frac{2N_f}{33}) \ln[\tilde{\mu}_{MS}^\beta \frac{2\pi}{\alpha_s}] \left( \frac{\alpha_s}{\pi} \right)^2 \\
+ (1 + \frac{N_f}{6})^2 \left( - 799.2 - 21.99N_f - 1.926N_f^2 \right) \\
+ \frac{495}{2} (1 + \frac{N_f}{6})(1 + \frac{2N_f}{33}) \ln[\tilde{\mu}_{MS}^\beta \frac{2\pi}{\alpha_s}] \left( \frac{\alpha_s}{\pi} \right)^2 \\
+ \frac{8\pi^2}{45} T^4 \left[ 1134.8 + 65.89N_f + 7.653N_f^2 \right. \\
- \frac{1485}{2} \left( 1 + \frac{1}{6}N_f \right) \left( 1 - \frac{2}{33}N_f \right) \ln[\tilde{\mu}_{MS}^\beta \frac{2\pi}{\alpha_s}] \\
\left. \times \left( \frac{\alpha_s}{\pi} \right)^3 (\ln \frac{1}{\alpha_s} + \delta) \right\}.
\]

The other EOS [17], which we call EOS1, is of \(O(g^5)\), and is obtained from this equation by dropping the last term which has contributions of \(O(g^6(\ln(1/g) + \delta))\). The phenomenological parameter \(\delta\) is introduced in [18] to incorporate the undetermined contributions of \(O(g^6)\).

The above equation of state has several ambiguities, associated with the renormalization scale \((\mu_{\overline{MS}})\), the scale parameter \(\Lambda_T/\Lambda_{\overline{MS}}\) which occurs in the expression for the running copulping constant \(\alpha_s\), and the value of the phenomenological parameter \(\delta\). The ambiguity associated with \((\mu_{\overline{MS}})\) has been discussed well in literature and a popular way out is the BLM criterion due to Brodsky, Lepage and Mackenzie [23]. In this criterion, which is chosen
to make the highest power of $N_f$ vanish in the highest perturbative order, the value of $(\mu_{MS})$ is allowed to vary between $\pi T$ and $4\pi T$. In this paper we choose the renormalization scale $\mu_{MS} = 2.15\pi T \approx 6.752T$ close to the central value $2\pi T$. One feature of this particular choice is that all the contributions due to the logarithms containing $\mu_{MS}$ are very small. For the scale parameter $\Lambda_T$, we follow Huang and Lissia and set $\Lambda_T/\Lambda_{MS} = \exp(\gamma_E + 1/22)/4\pi \approx 0.148$, since with this choice, they find among other things that the coupling $g^2(T)$ is optimal for lattice perturbative calculations. The same value has also been employed by others. see e.g. [18, 25]. Finally, we set $\Lambda_{MS} = T_c$, which is close to the value $0.87T_c$ found by Gupta.

We turn our attention to the phenomenological parameter $\delta$. The optimal value of $\delta$ depends on the choice of the renormalization scale and the order in which the running coupling constant is determined. Blaizot, Iancu and Rebhan find that the optimal value is given by $\delta = 1/3$ if one employs the two loop running coupling constant while, the one loop running coupling constant yields $\delta$ in the range 0.7-0.9 [18, 20]. In this paper, we find that $\delta$ in the range 0.8 to 1.2 yields the best fit with the lattice results. The important point here is that once the phenomenological parameter $\delta$ is fixed by comparing EOS2 with lattice EOS, it can be employed to study the properties of QGP. As regards EOS1, we note that the matching with the lattice results has been found to be merely qualitative [20].

Let us briefly review the underlying idea and the findings of recent two papers [19, 20]. In Ref.[19], it has been shown that the interaction effects in EOS1 and EOS2 can be captured in terms of the effective fugacities $(z_g, z_q)$ for the quasi gluons and quarks. The effective fugacities are determined self-consistently order by order. The mapping has been found to be accurate up to about 5% error. Therefore, we expect an error of the same order for all the quantities which can directly be derived from the pressure for eg., the energy-density and the entropy density.

Interestingly, the temperature dependence of the screening length which is subsequently determined is seen to qualitatively agree with the lattice results of Zantow. In Ref.[20], the quasi-particle description developed in Ref.[19] has been combined with the formulation of the response function of QGP, and the dissociation temperatures for $J/\Psi$ and $\Upsilon$ have been estimated. These numbers are again reasonably close to the predictions of other theoretical works [30, 31]. This motivates us to further utilize EOS1 and EOS2 to study the behavior of thermodynamic quantities such as energy density, entropy density, and most
importantly, the transport parameters, shear viscosity $\eta$ and viscosity to entropy density ratio $\eta/S$, for the rapidly expanding plasma. In addressing this, we generalize the recent work of Asakawa, Müller and Bass\cite{14} on the transport properties of interacting QGP.

III. THERMODYNAMIC OBSERVABLES

Let us now turn our attention to study the behavior of thermodynamic observables. We consider energy density ($\epsilon$) first which brings out the physics of the quasi-particle description manifestly. We then determine the entropy density($S$), for both EOS1 and EOS2. We principally employ the method developed in \cite{19}.

As mentioned, EOS1 and EOS2 are mapped to the corresponding equilibrium functions with the quarks and the gluons possessing effective fugacities:

$$f_{eq}^{g/q} = \frac{1}{[z_{g/q}^{-1} \exp(\beta p) + 1]}$$

where all the interaction effects are captured in the fugacities $z_{g/q} = \exp \beta \mu_{g/q}$. The form of $z_{g/q}$ as a function of temperature is given in \cite{20}. With these distributions, it is straightforward to determine the thermodynamic quantities.

A. The energy-density

Notwithstanding appearances, the energy of the quasi-gluons and quasi-quarks in not merely given by the relation $E_p = p$. Rather, it should be determined from the fundamental thermodynamic relation between the energy density and the partition function, $\epsilon = -\partial_\beta \ln(Z)$. Substituting for the partition function in terms of quasi-gluons and quasi-quarks we obtain,

$$\epsilon_{q/g} = \frac{\nu_{g/q}}{8\pi^3} \int (p + T^2 \partial_T \ln(z_{g/q})) f_{eq}^{g/q},$$

where $\nu \equiv (\nu_g, \nu_q) = (2(N_c^2 - 1), 4N_cN_f)$. The modified dispersion relation reads,

$$E_p = p + T^2 \partial_T \ln(z_{g/q}).$$

After performing the momentum integration in Eq.3, we obtain the following expression for the energy-density:

$$\frac{\epsilon}{T^4} = \frac{\nu_g}{2\pi^2} 6 PolyLog[4, z_g] - \frac{\nu_q}{2\pi^2} 6 PolyLog[4, -z_q]$$
\frac{(\Delta_g + \Delta_q)}{T^4}, \quad (5)

where

\[ \Delta_g = T^2 \partial_T \ln(z_g) N_g \]
\[ \Delta_q = T^2 \partial_T \ln(z_q) N_q, \quad (6) \]

are the contributions from the quasi-gluons and quasi-quarks to the trace anomaly. The second term in the dispersion relation Eq.(4) may be thus looked upon as the anomalous component of the dispersion relation. The quantities \( N_g \) and \( N_q \) are the quasi-gluon and quasi-quark number densities and having the following form,

\[ N_g = \nu_g T^3 \frac{\pi^2}{3} PolyLog[3, z_g] \]
\[ N_q = -\nu_q T^3 \frac{\pi^2}{3} PolyLog[3, -z_q]. \quad (7) \]

![Graph showing behavior of energy density as a function of temperature](image)

**FIG. 1:** Behavior of the Energy density for pure gauge theory as a function of temperature

Recall that the corresponding ideal value \( \bar{\epsilon}/T^4 \) reads:

\[ \bar{\epsilon}/T^4 = (\nu_g + \frac{7}{8} \nu_q) \frac{\pi^2}{30} \quad (8) \]

The behavior of the energy density \( (\epsilon/T^4) \) as a function of temperature \( (T/T_c) \) is shown in Fig.1 for pure gauge theory and for full QCD with \( N_F = 2, 3 \) in Fig.2. From these plots it is clear that \( \epsilon/T^4 \) approaches the ideal value only asymptotically.

**B. The entropy-density**

We compute the entropy density for the interacting pure QCD as well the interacting quark-gluon plasma. This is again a straight forward exercise since we have the equilibrium
distribution function for the quasi-partons already in hand. The entropy density in terms of the grand canonical partition function reads:

\[
S = \frac{1}{V} \partial_T [T \ln(Z_g)] + \frac{1}{V} \partial_T [T \ln(Z_M)]
\]

\[
\ln(Z_g) = -V \nu_g \int \frac{d^3p}{2\pi^3} \ln(1 - z_g \exp(-\beta p))
\]

\[
\ln(Z_M) = \ln(Z_q)
= V \nu_q \int \frac{d^3p}{2\pi^3} \ln(1 + z_q \exp(-\beta p)).
\]

(9)

The expression for the gluonic and quark contributions to the entropy density are obtained as,

\[
S_g = \frac{\nu_g}{2\pi^2 \beta^3} \left( 8 \ \text{PolyLog}[4, z_g] \right)
\]
The total entropy can be obtained by adding the gluon and quark contributions, $S = S_g + S_q$. We have plotted the dimensionless quantity $S/T^3$ for pure gauge theory and full QCD ($N_f = 2, 3$), as a function of $T/T_c$ for EOS1 and EOS2. These are shown in Figs.(4) and (5) respectively. We shall utilize the expression for entropy density displayed in Eq. (10) for determining the viscosity to entropy density ratio. The corresponding ideal value for $S/T^3$ is given by,

$$\frac{S^I}{T^3} = (\nu_g + \frac{7}{8} \nu_q) \frac{2\pi^2}{45}.$$  

(11)
C. Comparison with lattice results

We compare the thermodynamic observables determined by employing this model with the lattice results obtained by Karsch [5]. The lattice results [42] are shown in Figs 3, and our results are displayed in Figs 1, 2 and 4-5. For 2- and 3-flavor lattice results, we consider Fig.14 of Ref. [5]. The agreement is overall good for EOS2, particularly beyond $2.5T_c$; the agreement is merely qualitative for EOS1. These findings are consistent with our earlier result [20] that EOS2 predictions for the temperature dependence of the Debye mass and the dissociation temperatures of heavy quark ground states are in broad agreement with the lattice values.

IV. VISCOSITY OF INTERACTING QUARK-GLUON PLASMA

A. A brief review

The determination of viscosity is not as straightforward an exercise as the determination of the thermodynamic observables. For, it requires modeling beyond the equilibrium properties, in terms of the collision terms and other transport parameters, and also the nature of the perturbation to the equilibrium distribution. We note here that for sQGP under consideration, the collision term is by no means easily determined since perturbative results involving lowest order contributions are by no means guaranteed to be reliable. A reliable non-perturbative collision term is even harder to obtain. We need to adopt methods that go beyond the determination of properties such as the energy density or the specific heat.

There are two ways to compute the transport parameters for QGP:(i) from quantum field theory by using the Kubo formula [2, 32] or (ii) from the semi-classical transport theory [13, 14, 32, 33]. To model the QGP produced in heavy ion collisions employing semi-classical transport theory, one needs to employ the Vlasov term which incorporates the dynamics of non-abelian color charges. We also need a reliable collision term, which is, as we have pointed out, difficult to determine. A collision term which has recently been computed by Arnold, Moore and Yaffe [32] by considering binary collisions at the tree level in the lowest order. The collision term so obtained is then utilized to estimate the shear viscosity for QGP [14, 32]. Further, Asakawa et. al [14] have included the Vlasov term for an ensemble of turbulent color fields, and determined the anomalous shear viscosity; this determination
does not require a collision term. This work generalizes the earlier work of Dupree [34] by including the nonabelian dynamics. Both these works are based on methods described in [33]. As mentioned earlier, studies based on AdS/CFT correspondence [2, 35] employ the Kubo formula and predict a lower bound of \((1/4\pi)\) for \(\eta/S\) for plasmas whose dynamics are governed by class of strongly coupled gauge theories. This speculation is supported in some studies based on transport theory [13, 16].

In a recent work [15, 16], it has been argued that one does not need to treat QGP as a strongly coupled plasma to understand low viscosity. The authors argue that the anomalous transport processes in the rapidly expanding QGP are actually responsible for the very low value of the shear viscosity, and not the binary collisions. On the other hand in a very recent work, Xu and Greiner [13] argue that the reason for a small value of \(\eta/S\) is mainly due to gluon bremsstrahlung contributions to the collision term.

In the present paper, we adopt the approach of Asakawa et al [14] and determine the contribution of both the collisional and the anomalous parts to the ratio \(\eta/S\). As in the earlier study [15], we find that the anomalous part dominates over the collisional contribution.

B. Determination of \(\eta\)

In this Section, we determine the viscosity of a rapidly expanding interacting QGP, as described by EOS1 and EOS2, and as represented by equivalent equilibrium distribution functions (section 2). Our procedure involves replacing the ideal gas distributions used in [14], by the ones obtained by us for EOS1 and EOS2. We find that all the assumptions made in Refs. [14, 15] will be applicable in the present case.

Let us first briefly outline the standard procedure of determining viscosity in transport theory [14, 33]. The shear viscosity of QGP in terms of parton occupation numbers can be obtained by comparing the microscopic definition of the stress tensor with the macroscopic definition of the viscous stress tensor. The microscopic definition of the stress tensor in terms of the distribution function is as follows:

\[
T_{ik} = \int \frac{d^3p}{(2\pi)^3 E_p} p_i p_k f(\vec{p}, \vec{r}). \tag{12}
\]

On the other hand the macroscopic expression for the viscous stress tensor reads:

\[
T_{ik} = P \delta_{ik} + \epsilon u_i u_k - 2\eta (\nabla u)_{ik} - \zeta \delta_{ik} \nabla \cdot \vec{u}, \tag{13}
\]
where $\eta$ is the shear viscosity, $\zeta$ is the bulk viscosity and $\nabla u_{ik}$ is the traceless, symmetrized velocity gradient,

$$(\nabla u)_{ij} = \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) - \frac{1}{3} \delta_{ij} \nabla \cdot \vec{u}.$$  \hfill (14)

To obtain the shear viscosity, we write the distribution function as

$$f(\vec{p}, \vec{r}) = \frac{1}{z g_f} \exp(-\beta E_p + f_1(\vec{p}, \vec{r})) \mp 1. \hfill (15)$$

Assuming that $f_1(\vec{p}, \vec{r})$ is a small perturbation to the equilibrium distribution, we expand $f(\vec{p}, \vec{r})$ and keep the linear order term in $f_1$; the following form of the distribution function is thus obtained:

$$f(\vec{p}, \vec{r}) = f_0(\vec{p}) + \delta f(\vec{p}, \vec{r}) = f_0(\vec{p}) [1 + f_1(\vec{p}, \vec{r})(1 \pm f_0(\vec{p}))], \hfill (16)$$

where $f_0 \equiv f_{eq}$ (see Eq.2).

The important point to be noted is that while deriving the transport coefficients, one assumes a slow variation of the particle distribution so that the deviation from the equilibrium distribution is homogeneous in space and proportional to the gradients of the equilibrium parameters. Employing the standard approach [14,33], we write

$$f_1(\vec{p}, \vec{r}) = \frac{\hat{\Delta}(p)}{E_p T^2} p_i p_j (\nabla u)_{ij}, \hfill (17)$$

where the dimensionless function $\hat{\Delta}(p)$ measures the deviation from the equilibrium configuration. Since $\eta$ is a Lorentz scalar, it may be evaluated conveniently in the local rest frame. For a boost invariant longitudinal flow, $(\nabla u)_{ij} = \frac{1}{3} \tau \text{diag}(-1,-1,2)$ in the local rest frame, and $f_1(p)$ takes the form

$$f_1(\vec{p}) = -\frac{\hat{\Delta}(p)}{E_p T^2 \tau} \left( p^2 - \frac{p_z^2}{3} \right), \hfill (18)$$

where $\tau$ is the proper time ($\tau = \sqrt{t^2 - z^2}$).

The expression for $\eta$ is then obtained as

$$\eta = \frac{-\beta}{15} \int \frac{d^3 p}{8\pi^3 E_p^2} \hat{\Delta}(p) \frac{\partial f_{eq}}{\partial E_p}, \hfill (19)$$
entirely in terms of the unknown function $\tilde{\Delta}(p)$. $E_p$ is the particle energy.

We adopt the ansatz in \[14\] and take the form of $\tilde{\Delta}(p)$ to be 

$$\tilde{\Delta}(p) = A|p|/T; A \equiv A_g, A_q. \quad (20)$$

To appreciate the ansatz better, we note that the dispersion relation in Eq.[14] gets modified, in the presence of the perturbation $f_1$ as given by

$$E_{\text{eff}}(p) = E_p - \frac{pA}{E_p T^3 \tau} \left(p^2 - \frac{p^2}{3}\right). \quad (21)$$

The velocity is given by

$$\vec{v}_p = \partial_{\vec{p}} E_{\text{eff}}(\vec{p})$$

$$\vec{v}_p = \hat{p} - \frac{A}{T^3 \tau} (p_z \hat{k} - \frac{p}{3} \hat{p}) \quad (22)$$

From this expression for $\vec{v}_p$, it is clear that the perturbation leads to different velocities in transverse and longitudinal directions. This introduces manifest anisotropy in the system.

We determine $\tilde{\Delta}(p)$ by the variational method by minimizing the linearized transport equation\[14, 33\] with a Vlasov term and a collision term computed by Arnold et al\[32\].

The factor $A$ is yet undetermined. To fix its value, we minimize the quadratic functional\[14, 15,\]

$$W[\tilde{f}_1] = \int \frac{d^3p}{8\pi^3} \tilde{f}_1(\vec{p}) \left[ v^\mu \partial_{x^\mu} f(\vec{p}) + \frac{1}{2} (-\nabla_p \cdot D \cdot \nabla_p \delta \tilde{f}(\vec{p}) + I[f_1(\vec{p})] \right], \quad (23)$$

where the first term gives the drift, the second represents the diffusive Vlasov dynamics, and the last term is the collision integral. The expression in parenthesis is just the transport equation satisfied by $f_1(\vec{p})$ after averaging over the color fields \[14\]. The equilibrium distribution functions modify the results obtained in \[14\] by rendering the coefficients $A_q, A_g$ dependent on temperature and the coupling constant. After performing the momentum integrals, the drift, diffusive and the collisional terms in the quadratic functional Eq.[23] acquire finally the form

$$\tilde{W}_D[\tilde{f}_1] = -\frac{32|\nabla u|^2}{3 \pi^2} T^2 \left[(N_c^2 - 1) I_g A_g + N_c N_f I_q A_q \right],$$

$$\tilde{W}_V[\tilde{f}_1] = \frac{16|\nabla u|^2}{5 \pi^2 T} g^2 \langle B^2 \rangle \tau m \left[ N_c I_4 A_g^2 + N_f I_q^4 A_q^2 \right],$$

where $I_g = I_g(\vec{p})$ and $I_q = I_q(\vec{p})$, representing the color-dependent integrals.
\[
\tilde{W}_C[\tilde{f}_1] = \frac{[\nabla u]^2 T^2}{2} (N_c^2 - 1) g^4 \log(g^{-1})
\times \left[ \frac{7}{24 \pi^2} (2N_c + N_f) \left( N_c \frac{I^g_2 A_g^2}{z_g} + N_f \frac{I^q_2 A_q^2}{z_q} \right) + N_f \frac{I^q_2}{z_q} A_q^2 \right]
\times \left[ N_f \frac{I^g_2}{z_g} A_g^2 \right]
+ N_f N_c \frac{I^g}{2 \pi^3} (N_c^2 - 1) \left( z_g \frac{I^g_4 + I^q_4}{z_g + z_q} (A_g - A_q)^2 \right), \tag{24}
\]

where

\[
I^g_n = \text{PolyLog}[n, z_g^{-1}],
\]
\[
I^q_n = -\text{PolyLog}[n, -z_q^{-1}]. \tag{25}
\]

The function PolyLog\([n, a]\) has the series representation

\[
\text{PolyLog}[n, a] = \sum_{k=0}^{\infty} \frac{a^k}{k^n}. \tag{26}
\]

These expressions reduce to the ones obtained in [14], if we put \( z_q, z_g = 1 \), corresponding to ideal quark and gluon distributions.

\textbf{C. The anomalous and collisional viscosities}

Let us now turn our attention to determine the analytic expressions for anomalous and collisional contributions to the shear viscosity, which are determined respectively by the diffusive Vlasov and the collision terms in Eq.(24). To determine either of them, one utilizes Eq.(19) along with Eq.(24) by following exactly the path taken in Ref. [14].

By inserting Eq.(19) in Eq.(20) and performing the momentum integration, we obtain the following expression for viscosity \( \eta \):

\[
\eta = \frac{8}{\pi^2} \beta^{-3} \left[ (N_c^2 - 1) I^g_5 A_g + N_c N_f I^q_5 A_q \right]. \tag{27}
\]

The minimization of the functional \( \tilde{W}[\tilde{f}_1]\) (Eq.(24)) leads to the following matrix equation for the column vector \( A = (A_g, A_q) \):

\[
\left( \tilde{a}_A + \tilde{a}_C \right) A = \tilde{r}, \tag{28}
\]
where, the column vector $\tilde{r}$ and the matrices $\tilde{a}_A$ and $\tilde{a}_C$ are given by

$$\tilde{r} = \frac{32}{3\pi^2} \begin{pmatrix} (N_c^2 - 1)I^g_5 \\ N_c N_f I^q_5 \end{pmatrix}$$  \hspace{1cm} (29)$$

$$\tilde{a}_A = \frac{32}{5\pi^2} \frac{g^2(B^2) \tau_m}{T^3} \begin{pmatrix} N_c I^q_4 & 0 \\ 0 & N_f I^q_4 \end{pmatrix}$$  \hspace{1cm} (30)$$

$$\tilde{a}_C = \frac{7}{24\pi^2} (2N_c + N_f) C_g \begin{pmatrix} N_c I^q_4 & 0 \\ 0 & I^q_4 N_f \end{pmatrix}$$

$$+ \frac{N_f N_c (N_c^2 - 1)}{2\pi^3} C_g \frac{z_g}{z_q + z_g} \times (I^q_4 + I^g_4) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$  \hspace{1cm} (31)$$

where $C_g = (N_c^2 - 1)g^4\log(g^{-1})$. This leads to the following expressions for the anomalous and collisional contributions to the shear viscosity,

$$\tilde{\eta}_A = 3 \frac{\beta^{-3}}{4} \tilde{r} \cdot (\tilde{a}_A^{-1}) \cdot \tilde{r}$$

$$\tilde{\eta}_C = 3 \frac{\beta^{-3}}{4} \tilde{r} \cdot (\tilde{a}_C^{-1}) \cdot \tilde{r}$$  \hspace{1cm} (32)$$

By employing the additivity of rates, the expression for the total viscosity is obtained as

$$\frac{1}{\eta} = \frac{1}{\eta_c} + \frac{1}{\eta_A}$$  \hspace{1cm} (33)$$

It is clear from Eqs. (29, 30, 31 and 32) that $\eta_c \sim \frac{1}{g^4 \ln(1/g)}$. In the weak coupling limit ($g << 1$), at which the hot EOS are also valid, the total viscosity in Eq. (33) will be dominated by the anomalous component. Therefore for a weakly coupled QGP $\eta \approx \eta_A$. Hence we confine our attention to anomalous shear viscosity and study it’s behavior with temperature.

The individual expressions for the gluon and quark contribution to the anomalous viscosity from Eq. (32) are obtained to be

$$\eta_A^q = 40 \beta^{-6} \frac{(N_c^2 - 1)^2(I^g_5)^2}{3\pi^2 g^2(B^2) \tau_m N_c I^q_4}$$

$$\eta_C^q = 40 \beta^{-6} \frac{N_c^2 (I^q_5)^2}{3\pi^2 g^2(B^2) \tau_m N_f I^q_4}.$$

$$\eta_A = \frac{40 \beta^{-6} (N_c^2 - 1)^2(I^g_5)^2}{3\pi^2 g^2(B^2) \tau_m N_c I^q_4}$$.  \hspace{1cm} (34)$$
The total anomalous shear viscosity is obtained by summing up these two contributions, \( \eta_A = \eta_A^g + \eta_A^q \). We note that the above expressions are valid for a purely magnetic plasma. For the case when both chromo-electric and chromo-magnetic fields are present in the turbulent phase, and all their components are of equal size, the expressions for viscosity can be obtained simply by the replacement \( < B^2 > \tau_m \rightarrow \frac{4}{3} (< E^2 > + < B^2 >) \tau_m \).

Accordingly, we rewrite Eq.34 as

\[
\eta_A^g(z_g) = \frac{10\beta - 6}{\pi^2 g^2 (E^2 + B^2) \tau_m} \frac{(N_c^2 - 1)^2 (I_5^g)^2}{N_c I_4^g} \\
\eta_A^q(z_q) = \frac{10\beta - 6}{\pi^2 g^2 (E^2 + B^2) \tau_m} \frac{N_f^2 (I_5^q)^2}{N_f I_4^q}.
\]

(35)

We pause to compare the viscosities with their ideal values. Recall that the contribution from ideal distribution functions are obtained by setting \( z_g = 1 \) and \( z_q = 1 \) in Eqs.34 and 35 as expected they match with the expressions of Asakawa et al. \[14\]. Thus, the expressions for the relative viscosities (anomalous) are read off as

\[
\mathcal{R}_g \equiv \frac{\eta_A^g(z_g)}{\eta_A^g} = \frac{\zeta(4)}{\zeta(5)^2} \frac{(I_5^g)^2}{I_4^g} \\
\mathcal{R}_q \equiv \frac{\eta_A^q(z_q)}{\eta_A^q} = \frac{56 \zeta(4)}{225 \zeta(5)^2} \frac{(I_5^q)^2}{I_4^q}.
\]

(36)

The behavior of \( \mathcal{R}_g \) and \( \mathcal{R}_q \) as a function of temperature for EOS1 and EOS2 are shown in

![Graph showing the behavior of the relative viscosity](image)

FIG. 6: Behavior of the relative viscosity for pure gauge theory as a function of temperature.

Figs.(7) and (8). Clearly incorporation of interaction effects in the EOS further reduces the viscosities. While the gluon viscosity can reduce up to \( \sim 30\% \), the fall in the quark viscosity can be as steep as \( 80\% \), indicating a more ideal fluid like behaviour.
FIG. 7: Behavior of the quark contribution to the viscosity in 2- and 3-flavor QCD as a function of temperature

D. Viscosity to entropy density ratio

A determination of the absolute values of viscosity requires further a knowledge of the quenching parameter $\hat{q}_R$ [16], which is defined as the rate of growth of the transverse momentum fluctuation of a fast parton in an ensemble of turbulent color fields [16]. In turn, $\hat{q}_R$ is given by

$$\hat{q}_R = \frac{8\pi\alpha_s N_c}{3(N_c^2 - 1)} \langle E^2 + B^2 \rangle \tau_m,$$

(37)

in terms of the total energy density and an appropriate relaxation time $\tau$ [14].

One can combine this with the expression for the anomalous viscosity of gluons and obtain the relation as follows,

$$\eta_A^g(z_g) = \frac{20T^6_c}{3\hat{q}_R^2} \left( N_c^2 - 1 \right) \left( \frac{T}{T_c} \right)^6 \left( \frac{I_g^5}{I_g^4} \right)^2.$$

(38)

Estimates for $\hat{q}_R$ are available for the gluonic case only. In this case, $\hat{q}_R$ is estimated from the data by various approaches [36]. Studies within the framework of the twist expansion by fitting the experimental data on hadron suppression in the most central Au-Au collisions [37, 38] yield values in the range $1 - 2GeV^2/fm$ for the gluon quenching parameter. On the other hand, an eikonal approach [39, 40] estimates it to be roughly ten times larger than the twist estimates, in the range $10 - 30GeV^2/fm$. The expression for the gluonic $\eta/S$ is obtained as

$$\frac{\eta}{S} = \frac{20T^3_c(N_c^2 - 1)}{3\hat{q}_R} \left( \frac{T}{T_c} \right)^3 \frac{I_g^5}{I_g^4} \left( \frac{I_g^5}{I_g^4} \right)^2 \left[ \nu_g \left( 4I_g^5 - \ln(z_g)I_g^5 \right) + \frac{\pi^2 \Delta_T}{T^4} \right].$$

(39)

It is clear that the estimates for $\eta/S$ inherit the uncertainty in $\hat{q}_R$, up to an order of magnitude. For purposes of concreteness, we choose the QCD transition temperature($T_c$) to be
We have plotted Eq.39 as a function of temperature for EOS1 and EOS2 in Fig.8 and 9, with respective values $\hat{q}_R = 1, 10 GeV^2/fm$. We see that in the latter case, the ratio can fall significantly below the AdS/CFT bound $\frac{1}{4\pi} \sim 0.08$ even at $3T_c$, but it may not be reliable since the large value of $(\hat{q}_R = 10 GeV^2/fm)$ which we have employed may not be accommodated within weak coupling framework [16] which we consider in the present paper.

On the other hand, in the former case $\hat{q}_R = 1 GeV^2/fm$ the value of $\eta/S$ does not violate the AdS/CFT bound, although it is quite close to it near $2T_c$. It appears that the violation of bound, which can occur at $\hat{q}_R > |1 GeV^2/fm$ will be marginal near $2T_c$.

As expected the ratio increases with increasing temperature. Interestingly, unlike other thermodynamic variables, the ratio is not sensitive to the EOS employed.

![Graph](https://via.placeholder.com/150)

**FIG. 8:** Viscosity to entropy density ratio for pure gauge theory for $\hat{q}_R = 1 GeV^2/fm$

We now establish the connection between our results and that of Asakawa et al [14]. The expression for the ratio $\eta/S$ in Eq.39 reduces to that of Asakawa et al if we set $z_g = 1$ and employ the ansatz, $\bar{\Delta}(p) = A_{g/q}p/T$ for the anisotropy parameter [43]. The corresponding expression in this limit reads,

$$\frac{\eta}{S} = \frac{20T_c^3(N_c^2 - 1)}{\nu_{g} \hat{q}_R} \left(\frac{T}{T_c}\right)^3 \frac{\zeta(5)^2}{4\zeta(4)}.$$  \hspace{1cm} (40)

Let us normalize the viscosity to entropy ratios for EOS1 and EOS2 wrt the ideal values. We have plotted the relative ratios, which we denote by $R_{\eta}$, as a function of temperature, in Fig.10, which shows the effects of interactions in $\eta/S$. From Fig.10, we see that $(R_{\eta})$ is less than unity, approaching the ideal value asymptotically. Interestingly, the EOS2 values are closer to the ideal case, differing by about 3\% near $2T_c$.

Finally, note that the expression for $\eta/S$ in Eq.40 is identical to the expression used in [16] except for a numerical factor of $O(1)$. This discrepancy arises because we consider both
the diffusive Vlasov and the collision terms in the transport equation, while the analysis of [16] neglects the collision term.

![Graph](image)

**FIG. 9:** Viscosity to entropy density ratio for pure gauge theory for $\hat{q}_R = 10 GeV^2/fm$

![Graph](image)

**FIG. 10:** Behavior of $R_\eta$ as a function of temperature in the case of pure gauge theory for EOS1 and EOS2. Note that $R_\eta$ scales with $T/T_c$.

V. CONCLUSIONS AND OUTLOOK

In conclusion, we find that hot QCD EOS corresponding to interactions of $O(g^5)$ and $O(g^6 \ln(1/g))$ can significantly impact the values of the thermodynamic observables such as the energy density. The viscosity and the ratio $\eta/S$, which we have studied as functions of temperature, get reduced by approximately 7% for EOS1 and 4% for EOS2 near $2T_c$ in contrast to their ideal counterparts. We found that the value of $\eta/S$ for $\hat{q}_R = 1 GeV^2/fm$ near $2T_c$ is closer to the lower bound $1/4\pi$ placed on $\eta/S$ by AdS/CFT studies. Further, the choice $\hat{q}_R \sim 10 GeV^2/fm$ is difficult to accommodate within the weak perturbative
framework and hence the violation of the AdS/CFT bound may not represent the factual situation. The choice $\hat{q}_R \sim 2 GeV^2/fm$ does lead to a violation near $2T_c$, but only marginally so. In short, the findings in the present work strengthen the near perfect fluid picture of the hot and dense matter created in relativistic heavy ion collisions. This analysis has been rendered possible because of the mapping of interacting partons to non-interacting quasi partons with effective fugacities $[19, 20]$. While the present study points definitively to the importance of interaction effects, it is by no means complete, because of inherent uncertainties in the estimates of the gluonic quenching parameter, and an absence of the knowledge of the quenching parameter for the quarks. The EOSs which we study are also perturbative. It should be of great interest to employ the lattice EOS $[5, 41]$. This will be taken up separately.

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[43] In writing Eq.40 we have employed the following results: \( \text{PolyLog}[4, z_g = 1] = \zeta(4) = \pi^4/90 \) and \( \text{PolyLog}[5, z_g = 1] = \zeta(5) \) and that \( S_g = 4 \nu_g \zeta(4) T^3/\pi^2 \). Eq.40 follows from Eqs.(6.32 and 6.33) in Ref.[14] by employig the form of \( S_g \) given above.