A NON-LOCAL COUPLING MODEL INVOLVING THREE FRACTIONAL LAPLACIANS

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Abstract. In this article the authors study a non-local diffusion problem that involves three different fractional laplacian operators acting on two domains. Each domain has an associated operator that governs the diffusion on it, and the third operator serves as a coupling mechanism between the two of them. The model proposed is the gradient flow of a non-local energy functional. In the first part of the article we provide results about existence of solutions and the conservation of mass. The second part is devoted to study the asymptotic behaviour of the solutions of the problem when the two domains are a ball and its complementary. Fractional Sobolev inequalities in exterior domains are also provided.

1. Introduction

If one considers a non-local diffusion equation, probably one of the most famous and more deeply studied is the fractional heat equation, which can be written, formally, as

\[ u_t + (-\Delta)^r u = u_t + C_{N,r} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(y) - u(x)}{|x-y|^{N+2r}} dy = 0. \]

for an \( r \in (0,1) \), where \( N \) is the dimension of the space. This equation is naturally associated with the energy

\[ E(u) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))^2}{|x-y|^{N+2r}} dydx \]

in the sense that \( [1] \) is the \( L^2 \) gradient flow associated to \( E(u) \). This equation models non-local diffusion derived from Levi processes when the probability of particles jumping from point \( x \) to point \( y \) is given by the kernel \( |x-y|^{-N-2r} \), which is a symmetric but singular function.

One limitation of this model is that it considers the ambient space as uniform, so it is natural to think about a model where the ambient space produces a different diffusion depending on which part of it the particle is in. To this end, an obvious possibility is to

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Given the equation
\[
\begin{aligned}
  u_t(x, t) &= \alpha_s \int_{\Omega_s} \frac{u(y) - u(x)}{|x - y|^{N+2s}} dy, \quad x \in \Omega_s, \ t > 0, \\
  u_t(x, t) &= \alpha_r \int_{\Omega_r} \frac{u(y) - u(x)}{|x - y|^{N+2r}} dy, \quad x \in \Omega_r, \ t > 0, \\
  u(x, 0) &= u_0(x).
\end{aligned}
\]

for a couple of values \(r, s \in (0, 1)\) (the values \(\alpha_s\) and \(\alpha_r\) are just normalization constants) and \(\Omega_r \cup \Omega_s = \mathbb{R}^N\), but this is a naive approximation to the problem, since under this definition both domains (which we do not assume to be close to each other) are independent and thus the solutions of the equation must be studied separately by splitting the domain in two parts. No particle is allowed to have any information of what is happening in the domain it is not in. One possible way to solve this lack of intertwining is to “couple” the domains by considering “jumps” of the particles from \(\Omega_s\) to \(\Omega_r\) governed by a third non-local operator, another fractional laplacian. In other words,
\[
\begin{aligned}
  u_t(x, t) &= \alpha_s \int_{\Omega_s} \frac{u(y) - u(x)}{|x - y|^{N+2s}} dy + \alpha_c \int_{\Omega_s} \frac{u(y) - u(x)}{|x - y|^{N+2c}} dy, \quad x \in \Omega_s, \ t > 0, \\
  u_t(x, t) &= \alpha_r \int_{\Omega_r} \frac{u(y) - u(x)}{|x - y|^{N+2r}} dy + \alpha_c \int_{\Omega_s} \frac{u(y) - u(x)}{|x - y|^{N+2c}} dy, \quad x \in \Omega_r, \ t > 0, \\
  (u(x, 0) &= u_0(x).
\end{aligned}
\]

for \(r, s, c \in (0, 1)\). The reader must note that there are other possible options to couple this problem, see for example [11], but the reason why we chose this one is mainly because this equation is the gradient flow of the energy functional
\[
E(u) = \frac{\alpha_s}{2} \int_{\Omega_s} \int_{\Omega_s} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} dy dx + \frac{\alpha_r}{2} \int_{\Omega_r} \int_{\Omega_r} \frac{(u(x) - u(y))^2}{|x - y|^{N+2r}} dy dx + \alpha_c \int_{\Omega_s} \int_{\Omega_r} \frac{(u(x) - u(y))^2}{|x - y|^{N+2c}} dy dx,
\]
as we shall see in the next section. Other possible reason is that it is interesting to consider the case where the probability of jumping to your own domain is not the same as the probability of jumping to the other domain, hence the difference in the exponents of the integration kernels. This kind of coupling problems have already been studied in [11] in the case of a different coupling method and in [13] for different operators, the usual heat operator and another one given by a convolutions with a probability kernel.

The gradient flow structure of this problem already provides a certain \(L^2\) existence theory, but in order to study the problem in \(L^1\) we make use of semigroups.

**Theorem 1.1.** Given \(u_0 \in L^1(\Omega)\) there exists a unique solution in \(L^1(\Omega)\) to the problem (3) with initial datum \(u_0\) such that
\[
\|u(t_2)\|_{L^1(\Omega)} = \|u(t_1)\|_{L^1(\Omega)} \text{ for any } 0 \leq t_1 \leq t_2 < \infty.
\]
Keep in mind that depending on the initial datum the existing theory allows us to improve on the properties of the solutions, as we shall see.

Now once the existence of solutions of this problem is established properly, we would like to discuss a particular case on which the domains are a ball and its complementary, since we want to study the competition between diffusions when one domain is bounded and the other one is not and how this competition determines the shape of the solution for big times. We would expect the mass to accumulate in the unbounded domain and thus the solution must look like the solution of problem (1) with initial datum the Dirac’s delta and fractional exponent the one corresponding to the unbounded domain, but we will see that this is only true in a certain range of the exponents \( r, s \) and \( c \) satisfying \( r, s, c \in (0, 1) \) and

\[
2r - 2c \leq N.
\]

We will assume this assumption along the paper without to make it explicit each time when is needed.

In particular the result is as following.

**Theorem 1.2.** For any \( u_0 \in L^1(\mathbb{R}^N) \) there exists a positive constant \( C(r, N) \) such that the solution of system (3) satisfies

\[
\|u(t)\|_{L^p(\mathbb{R}^N)} \leq C(r, N)(t^{-\frac{N}{2r}(1-\frac{1}{p})} + t^{-\frac{N}{2\min\{r, s, c\}(1-\frac{1}{p})}})\|u_0\|_{L^1(\mathbb{R}^N)}, \quad \forall t > 0.
\]

The above behaviour near zero is almost optimal. In fact any \( \alpha \) such that

\[
\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2r}(1-\frac{1}{p})}\|u_0\|_{L^1(\mathbb{R}^N)}, \quad \forall t \in (0, 1)
\]
satisfies

\[
\alpha \leq \min\{r, s\}.
\]

The behaviour at infinity cannot be improved in any \( L^p \)-norm with \( 1 \leq p < \infty \) and this is obtained by the next result.

Moreover, we can be more precise about the limit profile when the time goes to infinity.

**Theorem 1.3.** For any \( u_0 \in L^1(\mathbb{R}^N) \) the solution of equation (3) with \( \Omega_s = B_1(0) \) and \( \Omega_r = B_1^c(0) \) satisfies

\[
\lim_{t \to \infty} \|u(t) - MK^r_t\|_{L^p(\mathbb{R}^N)}t^{\frac{N}{2r}(1-\frac{1}{p})} = 0
\]

for any \( 1 \leq p < \infty \), where \( MK^r_t \) is the solution of equation

\[
u_t + (-\Delta)^{r}u = 0 \quad \text{for all } x \in \mathbb{R}^N, \quad u(x, 0) = M\delta_0
\]

and \( \delta_0 \) is Dirac’s delta centered at \( x = 0 \) and \( M \) is the mass of the initial data.

The article is divided as follows. In Section 2 we address the issues of gradient flow structure, existence and conservation of mass. Section 3 focuses in the problem posed in the unit ball and its complementary and studies the \( L^p \) decay of the norms of the solution. Finally Section 4 deals with the large time behaviour of the solution in this last setting. There is an Appendix in Section 5 for some needed extra results.

\[1\]The optimality regarding the exponent \( c \) requires geometrical conditions that are being studied now.
2. Gradient flow structure and existence of solutions

Let us consider two different domains \( \Omega_s \) and \( \Omega_r \) such that \( \Omega_s \cup \Omega_r := \Omega \subseteq \mathbb{R}^N \) and the spaces

\[ X^c(\Omega_s, \Omega_r) := \left\{ u \in L^2(\Omega) : \int_{\Omega_s} \int_{\Omega_r} \frac{(u(x) - u(y))^2}{|x - y|^{N+2c}} \, dy \, dx < \infty \right\} \]

and

\[ \mathcal{H}(\Omega) := H^s(\Omega_s) \cap H^r(\Omega_r) \cap X^c(\Omega_s, \Omega_r) \]

where \( H^s \) is the usual s-fractional Sobolev space. The exponents satisfy \( s, r, c \in (0, 1) \). Now we take three constants \( \alpha_s, \alpha_r \) and \( \alpha_c \) depending on their subindexes and on the dimension \( N \) of the space and define, for every \( u \in L^2(\Omega) \), the energy

\[ E(u) = \frac{\alpha_s}{2} \int_{\Omega_s} \int_{\Omega_r} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \, dy \, dx + \frac{\alpha_r}{2} \int_{\Omega_r} \int_{\Omega_r} \frac{(u(x) - u(y))^2}{|x - y|^{N+2r}} \, dy \, dx + \alpha_c \int_{\Omega_s} \int_{\Omega_r} \frac{(u(x) - u(y))^2}{|x - y|^{N+2c}} \, dy \, dx. \]

(7)

if \( u \in \mathcal{H}(\Omega) \) and \( E(u) = \infty \) if not, with its associated Dirichlet form \( \mathcal{E} : \mathcal{H}(\Omega) \times \mathcal{H}(\Omega) \to \mathbb{R} \), defining with it the associated inner product \( \langle u, v \rangle := \mathcal{E}(u, v) \). This energy functional is proper, convex and lower semi-continuous, so following [10, 9.6.3, Thm 4] we define the operator that will be in the end the subdifferential as

\[ \mathcal{L}[u](x) := -\int_{\Omega} \left\{ \frac{u(y) - u(x)}{|x - y|^{N+2s}} \chi_{\Omega_s}(y) \chi_{\Omega_s}(x) + \frac{u(y) - u(x)}{|x - y|^{N+2r}} \chi_{\Omega_r}(y) \chi_{\Omega_r}(x) \right. \]

\[ + \frac{u(y) - u(x)}{|x - y|^{N+2c}} \chi_{\Omega_s}(y) \chi_{\Omega_r}(x) + \frac{u(y) - u(x)}{|x - y|^{N+2c}} \chi_{\Omega_r}(y) \chi_{\Omega_s}(x) \right\} \, dy \]

with domain \( D(\mathcal{L}) := \{ u \in \mathcal{H}(\Omega) : \mathcal{L}[u] \in L^2(\Omega) \} \) endorsed with the usual norm, taking into account the semi-norm \( |u|_{\mathcal{H}^c(\Omega)} := \mathcal{E}(u, u) \).

First we need to proof that \( D(\mathcal{L}) \subseteq D(\partial E) \) and to this end the only difficult point is to proof that chosen \( u \in D(\mathcal{L}) \) and defining \( v = \mathcal{L}[u] \), for every \( w \in \mathcal{H}(\Omega) \) we have that \( \mathcal{E}(v, w - u) \leq E[w] - E[u] \), but this is easy once we see that

\[ -\int_B \int_C \frac{u(y) - u(x)}{|x - y|^{N+2c}} f(x) \, dy \, dx = \int_C \int_B \frac{u(y) - u(x)}{|x - y|^{N+2c}} f(y) \, dy \, dx \]

(9)

which implies

\[ -\int_B \int_B \frac{u(y) - u(x)}{|x - y|^{N+2c}} f(x) \, dy \, dx = \frac{1}{2} \int_B \int_B \frac{u(y) - u(x)}{|x - y|^{N+2c}} (f(y) - f(x)) \, dy \, dx \]

(10)

whenever this integrals are well defined for general domains \( B, C \), a general function \( f \) and \( 0 < a < 1 \). In our case the chosen domain and the Cauchy-Schwarz Inequality ensure the integrals are well defined. It is also helpful to note that

\[ ab - b^2 = \frac{a^2 - b^2 - (a - b)^2}{2} \leq \frac{a^2 - b^2}{2} \]
for any real numbers $a, b$. The next and final part is to see that $D(L) \supseteq D(\partial E)$, so to this point we take a $f \in L^2(\Omega)$ and look for a minimizer of

$$J[w] = E[w] + \int_{\Omega} \frac{w^2(x)}{2} - f(x) \cdot w(x) \, dx$$

over $\mathcal{H}(\Omega)$, but this is precisely a function $u$ that satisfies $u + L[u] = f$ in $\Omega$, and clearly $u \in D(L)$ since $\|u\|_{D(L)} \leq 2\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}$. Consequently the range of $I + L$ is $L^2(\Omega)$ and this implies, see [Evans, 9.6.3, Thm 4], that $D(L) \supseteq D(\partial E)$.

Therefore, by [10, 9.6.3, Thm 3], we say that this equation is provided by the gradient flow of the energy $E[u]$, and in this sense it motivates our study. Let us make use of Corollary 3.3.2 and Theorem 3.2.1 from [6] to state the existence result for our article.

Without entering in much detail, one can check, making use of (9) and (10), that $L$ is an $m$-accretive and self-adjoint operator, which provides the following.

**Theorem 2.1.** Given any $u_0 \in L^1(\Omega)$ there exists a unique solution $u$ of our problem in the sense of [6, Cor. 3.3.2]. This solution is given by the effect of a strongly continuous semigroup $S$, meaning that $u(\cdot, T) := S(t)u_0(\cdot)$, that preserves the mass: $\|u_0\|_{L^1(\Omega)} = \|u(\cdot, T)\|_{L^1(\Omega)}$ for any $T > 0$. Moreover, if $u_0 \in L^1(\Omega) \cap L^2(\Omega)$ this solution satisfies

$$\begin{cases} 
  u \in C([0, \infty) : L^2(\Omega)) \cap C((0, \infty) : D(L)) \cap C^1((0, \infty) : L^2(\Omega)); \\
  u'(t) = -L[u(t)] \text{ for all } t > 0; \\
  u(0) = u_0
\end{cases}$$

and in addition we have that

$$\|L[u(t)]\|_{L^2} \leq \frac{1}{\sqrt{2t}} \|u_0\|_{L^2},$$

$$\mathcal{E}(u(t), u(t)) = \langle -L[u(t)], u(t) \rangle \leq \frac{1}{2t} \|u_0\|_{L^2}^2$$

and, given $u_0 \in D(L)$,

$$\|L[u(t)]\|_{L^2}^2 \leq \frac{1}{2t} \langle L[u_0], u_0 \rangle.$$ 

**Proof.** With the aforementioned references, the only claim that is not trivial is the one regarding the conservation of mass. First of all, let us note that

$$\int_B \int_B \frac{u(y) - u(x)}{|x - y|^{N+2\delta}} \, dy \, dx = 0$$

for any domain $B$ and exponent $\delta \in (0, 1)$ whenever we can apply Fubini. Therefore it is not hard to check that for any $\varphi_0 \in C^\infty_0 \subseteq DL$ and $T > 0$ we have that

$$\int_{\Omega} \varphi_0(x) - \varphi(x, T) \, dx = \int_0^T \int_{\Omega} \varphi_t(x, t) \, dx \, dt = 0$$
where $\varphi$ is the unique solution of problem $[3]$ with initial datum $\varphi_0$. Let us then take a sequence of positive functions $\{\varphi_{0,n}\}$ with their respective solutions of our problem $\{\varphi_n\}$ such that

$$\|\varphi_{0,n}\|_{L^1(\Omega)} = \|\varphi_n(\cdot,T)\|_{L^1(\Omega)} \text{ for any } T > 0 \text{ and } \|\varphi_{0,n} - u_0\|_{L^1(\Omega)} \to 0 \text{ as } n \to \infty.$$ 

By [6, Thm. 3.4.4] we have that $\|u_0\|_{L^1(\Omega)} = \|u(\cdot,T)\|_{L^1(\Omega)} \geq 0$ for any $T > 0$. On the other hand since the functions $\varphi_n$ are positive and the operator is linear we have that

$$\|u_0\|_{L^1(\Omega)} - \|u(\cdot,T)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} - \|\varphi_{0,n}\|_{L^1(\Omega)} + \|\varphi_n(\cdot,T)\|_{L^1(\Omega)} - \|u(\cdot,T)\|_{L^1(\Omega)}$$

$$\leq \|u_0\|_{L^1(\Omega)} - \|\varphi_{0,n}\|_{L^1(\Omega)} + \|\varphi_n(\cdot,T) - u(\cdot,T)\|_{L^1(\Omega)} \leq 2\|u_0 - \varphi_{0,n}\|_{L^1(\Omega)}$$

and it is enough to make $n \to \infty$ to check that $\|u_0\|_{L^1(\Omega)} - \|u(\cdot,T)\|_{L^1(\Omega)} \leq 0$. □

3. The Problem in the Ball and the Complementary. $L^p$ Norm Decays

Consider now $\Omega_s = B_1(0)$, i.e., the ball of radius 1 centered at 0 (or any other ball), and $\Omega_r = B_1(0)$ its complementary, to which we will refer simply by $B, B^c$ or $B_R, B^c_R$ if the radius of the ball is of relevance. We emphasize that the method used here can be extended to more general exterior domains where GSN inequalities in Section 5.1 holds.

Using these HNS inequalities we prove Nash-like inequalities for exterior domains and use them to prove Theorem 1.2 regarding the long time decay of the solutions.

To simply the presentation for $0 < s < 1$ and $\Omega$ an open set, we will denote by $[f]_{s,\Omega}$ the following

$$[f]_{s,\Omega}^2 := \int_\Omega \int_\Omega \frac{(f(y) - f(x))^2}{|x - y|^{N + 2s}} dxdy.$$

Lemma 3.1. Let $N \geq 1, r \in (0,1)$. For any $f \in H^s(B^c) \cap L^1(B^c)$ the following holds

$$\|f\|_{L^2(B^c)} \leq C(r,N)\|f\|_{L^1(B^c)}^{\frac{2r}{N+2r}}[f]_{r,B^c}^{\frac{N}{N+2r}}. \tag{11}$$

Proof. For $N \geq 2$, so $N > 2r$ we use Hölder’s inequality and Lemma 5.3

$$\|f\|_{L^2(B^c)} \leq \|f\|_{L^1(B^c)}^{\frac{2r}{N+2r}}\|f\|_{L^2(B^c)}^{\frac{N}{N+2r}} \leq C(r,N)\|f\|_{L^1(B^c)}^{\frac{2r}{N+2r}}[f]_{s,B^c}^{\frac{N}{N+2r}}.$$

In dimension $N = 1$ we write $f = f_+ + f_-$ where $f_+$ and $f_-$ are the restrictions of $f$ to $(-\infty,-1)$ and $(1,\infty)$ respectively. Since

$$[f_+]_{r,(-\infty,-1)} + [f_+]_{r,(1,\infty)} \leq [f]_{r,\{|x|>1\}}^2,$$

it is sufficient to prove the above estimate only for $f_+$ and the corresponding interval $(1,\infty)$. After a translation of the interval to the origin we have to prove that for any $f \in H^s(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$ it holds

$$\|f\|_{L^2(\mathbb{R}_+)} \leq C(r,N)\|f\|_{L^1(\mathbb{R}_+)}^{\frac{2r}{N+2r}}[f]_{r,\mathbb{R}_+}^{\frac{N}{N+2r}}.$$
Let us consider \( f_{even} \), the even extension of \( f \). Using [9, Lemma 5.2] we obtain that \( f_{even} \in H^r(\mathbb{R}) \) and \([f_{even}]_{r,\mathbb{R}} \leq 2[f]_{r,\mathbb{R}}\). It is then sufficient to prove the inequality for functions defined on the whole line:

\[
\|f\|_{L^2(\mathbb{R})} \leq C(r, N)\|f\|_{L^{1,2r}(\mathbb{R})}^{2r/N} |f|_{r,\mathbb{R}}^N.
\]

This inequality holds in view of [1, Theorem 1.3] which translates inequalities for the Laplacian to fractional Laplacian with the corresponding power. \( \square \)

Let us define then the \( p \)-domain of the operator as

\[
D_p\mathcal{L}(\Omega) := \{ u \in L^p(\Omega) : \mathcal{L}[u] \in L^p(\Omega) \}.
\]

Note that clearly \( C^\infty_0(\Omega) \subset D_p\mathcal{L}(\Omega) \). From [4, Theorem 7.8] we see that given any \( u_0 \in D_p\mathcal{L}(\Omega) \subset L^p(\Omega) \) there exists a unique function \( u_p \in L^p(\Omega) \) such that

\[
\begin{cases}
  u_p \in C([0, \infty) : D_p\mathcal{L}(\Omega)) \cap C^1([0, \infty) : L^p(\Omega)); \\
  u'_p(t) = -\mathcal{L}[u_p(t)] \text{ for all } t \geq 0; \\
  u_p(0) = u_0.
\end{cases}
\]

and moreover,

\[
\|u_p(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)} \text{ and } \|u'_p(t)\|_{L^p(\Omega)} = \|\mathcal{L}[u_p(t)]\|_{L^p(\Omega)} \leq \|\mathcal{L}[u_0]\|_{L^p(\Omega)} \text{ for all } t \geq 0.
\]

If, in addition, \( u_0 \in L^1(\Omega) \cap D_p\mathcal{L}(\Omega) \) then \( u_p = u \text{ a.e.} \), the solution given by Theorem 2.1.

**Theorem 3.1. (Nash’s inequalities for exterior domains)** Let \( N \geq 1, r \in (0, 1) \) and \( p \in (1, \infty) \). For any \( f \in L^1(B^c) \) such that \( \mathcal{L}[f] \in L^p(B^c) \) the following holds

\[
\|f\|_{L^p(B^c)} \leq C(p, r, N)\|f\|_{L^1(B^c)}^{2r/N} |f|_{r,B^c}^N.
\]

**Proof.** First we observe that if \( \mathcal{L}[f] \in L^p(B^c) \) then \( |f|^{p/2} \in H^r(B^c) \). Indeed

\[
\int_{B^c} \int_{B^c} \frac{|f(y)^{p/2} - f(x)^{p/2}|^2}{|x - y|^{N+2r}} dxdy \leq \int_{B^c} \int_{B^c} \frac{(f(y) - f(x))(|f|^{p-2}f(y) - |f|^{p-2}f(x))}{|x - y|^{N+2r}} dxdy = \mathcal{E}(f, |f|^{p-2}f) = (\mathcal{L}f, |f|^{p-2}f).
\]

Since \( f \in D_p\mathcal{L}(B^c) \) it follows that \( \mathcal{L}f \) belongs to \( L^p(B^c) \) and \( |f|^{p-2}f \) to \( L^r(B^c) \). This shows that the term \([|f|^{p/2}]_{r,B^c}\) is well defined.

We distinguish two cases. The first one concerns the case when \( 2r < N \) and inequality (5.3) holds. The second one treat the remaining case \( N = 1 \) and \( r \in (0, 1) \).

Under the assumption \( 2r < N \) we use inequality (5.3) to obtain

\[
[[f]^{p/2}]_{r,B^c} \geq \|f\|_{p/2,B^c} = \|f\|_{L^{p/2}(B^c)}^{p/2}.
\]

Using interpolation we have

\[
\|f\|_{L^p(B^c)} \leq \|f\|_{p/2,B^c}^{\theta} \|f\|_{L^{1/2}(B^c)}^{1-\theta}.
\]
where
\[
\frac{1}{p} = \frac{\theta}{N} + \frac{1 - \theta}{1}, \quad i.e., \theta = \frac{N(p-1)}{N(p-1)+2r}.
\]
Putting together the last two inequalities we get
\[
\|f\|_{L^p(B)}^{p/2} \geq \left( \frac{\|f\|_p}{\|f\|_1} \right)^{\frac{p}{2}}
\]
and replacing the value of \( \theta \) we obtain the desired inequality.

It remains to consider the case when \( N = 1 \). We use a trick that has been used previously in [3], namely we use the fact that \( r/2 < N = 1 \) and apply the previous inequality with \( r/2 \) instead of \( r \):
\[
\|f\|_{L^p(B)}^{p/(N-1)} \leq C(p, r, N)\|f\|_{L^p(B)}^{r/2} \|f\|_{L^1(B)} |f|_{r/2, B}^{N}\|f\|_{L^p(B)}^{r/2} \|f\|_{L^p(B)}^{r/2}.
\]
We claim that in any dimension
\[
(13) \quad \|v\|_{r, \Omega} \leq C(r)\|v\|_{L^2(\Omega)} |v|_{r, \Omega}^{1/2}.
\]
If we use this with \( v = |f|^{p/2} \) we get
\[
\|f\|_{L^p(B)}^{p/(N-1)} \leq C(p, r, N)\|f\|_{L^p(B)}^{r/2} \|f\|_{L^1(B)} |f|_{r/2, B}^{N}\|f\|_{L^p(B)}^{r/2} \|f\|_{L^p(B)}^{r/2}
\]
which after simplifying the \( L^p \)-norm from the right hand side is exactly our desired inequality.

It remains to prove claim (13). When \( \Omega = \mathbb{R}^N \) it is easily obtained using the Fourier transform. When \( \Omega \) is an arbitrary open set of \( \mathbb{R}^N \) we proceed as follow:

\[
[v]_{r, \Omega}^2 = \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^2}{|x-y|^{N+r}} \, dx \, dy
\]
\[
= \int \int_{x, y \in \Omega, |x-y|<\delta} \frac{|v(x) - v(y)|^2}{|x-y|^{N+r}} \, dx \, dy + \int \int_{x, y \in \Omega, |x-y|>\delta} \frac{|v(x) - v(y)|^2}{|x-y|^{N+r}} \, dx \, dy
\]
\[
\leq \delta^s \int \int_{x, y \in \Omega, |x-y|<\delta} \frac{|v(x) - v(y)|^2}{|x-y|^{N+2r}} \, dx \, dy + \int \int_{x, y \in \Omega, |x-y|>\delta} \frac{|v(x)|^2 + |v(y)|^2}{|x-y|^{N+r}} \, dx \, dy
\]
\[
\leq \delta^r [v]_{r, \Omega}^2 + 2 \int \int_{|x-y|>\delta} v^2(x) \frac{dy}{|x-y|^{N+r}} \, dx
\]
\[
\leq \delta^r [v]_{r, \Omega}^2 + C(r)\delta^{-r} \|v\|_{L^2(\Omega)}^2.
\]
Choosing \( \delta^r = \|v\|_r^2 \), the claim is proved.

**Lemma 3.2.** For any \( p \in (1, \infty) \) there exists a positive constant \( C(p, r, N) \) such that
\[
(14) \quad \|u\|_{L^p(\mathbb{R}^N)} \leq C(p, r, N)(E(|u|^{p/2}) + \|u_0\|_{L^1(B)}^{2r/p} (E(|u|^{p/2}))^{N/(p-1)+2r/p})
\]
holds for all \( u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \).
Proof. We will show the computations for $u_0 \in L^1(\mathbb{R}^N) \cap D_p\mathcal{L}(\mathbb{R}^N)$, and after that a simple density argument gives the desired result. We have that $E(u) = E_1(u) + E_2(u)$, $E(u, v) = (E(u + v) - E(u - v))/4$,

$$E_1(u) = \frac{\alpha_r}{2} \int_{B^c} \int_{B^c} \frac{(x) - u(y))^2}{|x - y|^{N+2r}} \, dydx + \alpha_c \int_{B^c} \int_{B^c} \frac{(x) - u(y))^2}{|x - y|^{N+2c}} \, dydx$$

and

$$E_2(u) = \frac{\alpha_r}{2} \int_{B^c} \int_{B^c} \frac{(x) - u(y))^2}{|x - y|^{N+2r}} \, dydx.$$

We estimate each of the above term. In the case of $E_1$ we estimate it from bellow in term of the $L^2$-norms of $u$. Recall the following elementary inequality

$$(a - b)^2 = a^2 - 2ab + b^2 \geq a^2 - (1 - \epsilon)a^2 - \frac{1}{1 - \epsilon}b^2 + b^2 = \epsilon(a^2 - \frac{1}{1 - \epsilon}b^2).$$

Choosing $\epsilon = 1/2$ we get

$$E_1(v) \geq \alpha_c \int_{B^c} \int_{B^c} \frac{(x) - u(y))^2}{|x - y|^{N+2c}} \, dydx \geq \alpha_c \int_{B^c} \int_{B^c} \frac{(x) - u(y))^2}{(1 + |x - y|)^{N+2c}} \, dydx$$

$$\geq \frac{\alpha_c}{2} \int_{B^c} \int_{B^c} \frac{1}{(1 + |x - y|)^{N+2c}} \left(v^2(x) - 2v^2(y)\right) \, dydx$$

$$= \frac{\alpha_c}{2} \int_{|x| < 1} v^2(x) \int_{|y| > 1} \left(1 + |x - y|\right)^{N+2c} - \alpha_c \int_{|y| > 1} v^2(y) \int_{|x| < 1} \left(1 + |x - y|\right)^{N+2c} \, dydx$$

$$\geq \alpha_c \int_{|x| < 1} v^2(x) \int_{|y| > 1} \left(2 + |y|\right)^{N+2c} - \alpha_c \int_{|y| > 1} v^2(y) \int_{|x| < 1} \, dx$$

$$= \frac{\alpha_c}{2} C(N, c) \int_{|x| < 1} v^2(x) \int_{|y| > 1} v^2(y) \, dy$$

$$\geq C(N, c) \left(\|v\|_{L^2(|x| < 1)}^2 - \|v\|_{L^2(|x| > 1)}^2\right).$$

Choosing $v = |u|^{p/2}$ we obtain that

$$\|u\|_{L^p(B^c)} \lesssim E_1(|u|^{p/2}) + \|u\|_{L^p(B^c)}^{p/2}$$

In the case of $E_2$ we use Nash inequality for exterior domains (12):

$$\left(\|u\|_{L^p(B^c)}^{p/2}\right)^{\frac{(N(p-1)+2r)}{N(p-1)}} \leq C(p, r, N)\|u\|_{L^1(B^c)}^{2r} \|u\|_{L^p(B^c)}^{2r/2}$$

It follows that the $L^p(\mathbb{R}^N)$ norm of $u$ satisfies

$$\|u\|_{L^p(\mathbb{R}^N)} \lesssim E_1(|u|^{p/2}) + \|u\|_{L^1(B^c)}^{2r} \left(E_2(|u|^{p/2})\right)^{\frac{N(p-1)}{N(p-1)+2r}}$$

$$\leq C(E(|u|^{p/2}) + \|u\|_{L^1(B^c)}^{2r} \left(E(|u|^{p/2})\right)^{\frac{N(p-1)}{N(p-1)+2r}}),$$

which finishes the proof. \qed
Proof of Theorem 1.2. Let us denote $\alpha = \min \{r, s, c\}$. A simple computation shows that

$$[v]_{\alpha, \mathbb{R}^N}^2 \leq C(\|v\|_{L^2(\mathbb{R}^N)}^2 + \mathcal{E}(v, v)), \quad \forall \ v \in \mathcal{H}(\mathbb{R}^N).$$

Using Nash inequalities for Fractional Laplacian (see for instance [1, Th. 1.3]) we get

$$\|v\|_{L^2(\mathbb{R}^N)}^{2 + 4\alpha N} \leq C\|v\|_{L^1(\mathbb{R}^N)}^{4\alpha N} \leq C\|v\|_{L^2(\mathbb{R}^N)}^{2 + 4\alpha N} + \mathcal{E}(v, v)).$$

In view of [5, Th. 2.1] we obtain that the semigroup satisfies

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \leq \frac{Be^t}{t^{\frac{\alpha}{2}}} \|u_0\|_{L^1(\mathbb{R}^N)} \quad \forall u_0 \in L^1(\mathbb{R}^N).$$

This estimates shows that the solution belongs to all the spaces $L^p(\mathbb{R}^N)$, $1 \leq p \leq \infty$, whenever the initial datum is in $L^1(\mathbb{R}^N)$. Using the $L^1$-contraction property one can obtain the the map $t \to \|S(t)\|_{L^1(\mathbb{R}^N)}$ is a decreasing function and then the above estimate can be improved for $t > 1$ but without obtaining a decay for large times

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \leq \frac{B'}{\min\{1, t\}^{\frac{\alpha}{2}}} \|u_0\|_{L^1(\mathbb{R}^N)} \quad \forall u_0 \in L^1(\mathbb{R}^N), \forall t > 0.$$  

This gives the desired estimate for small times $t < 1$.

Let us now prove that this blow-up at $t \downarrow 0$ cannot be improved. In fact [5, Th. 2.1] shows that estimate (15) is equivalent with the estimate

$$\|v\|_{L^2(\mathbb{R}^N)}^{2 + 4\alpha N} \leq C\|v\|_{L^1(\mathbb{R}^N)}^{4\alpha N} (\|v\|_{L^2(\mathbb{R}^N)}^2 + \mathcal{E}(v, v)).$$

We prove now that any $\alpha$ for which the above inequality holds satisfyes $\alpha \leq \min \{r, s\}$. (no me sale si pongo el $c$ tambien). Choose any function $\varphi \in C^\infty_c(\mathbb{R}^N)$ supported in the unit ball and define $\varphi_\varepsilon(x) = \varphi((x - x_0)/\varepsilon)$ for some point $x_0 \in \mathbb{R}^N$. It follows that $\|\varphi_\varepsilon\|_{L^p(\mathbb{R}^N)} \simeq \varepsilon^\frac{N}{p}$. Introducing this estimates in (17) we obtain that

$$\varepsilon^{\frac{N}{p}(2 + 4\alpha N)} \lesssim \varepsilon^{4\alpha N} (\varepsilon^N + \mathcal{E}(\varphi_\varepsilon, \varphi_\varepsilon))$$

and then

$$\varepsilon^{N-2\alpha} \lesssim \varepsilon^N + \mathcal{E}(\varphi_\varepsilon, \varphi_\varepsilon).$$

We now prove that choosing $x_0 \in B_1$ gives us that $\alpha \leq s$ while $x_0 \in B_1^c$ gives $\alpha \leq r$. The border case when $|x_0| = 1$ remains to be studied more careful.
Case I. Let us choose \( x_0 = 0 \) and \( \varphi \) supported in the unit ball. Then the support of \( \varphi_\varepsilon \) is contained in the ball \( B_\varepsilon(0) \) and \( [\varphi_\varepsilon]_{r,B_1} = 0 \). It follows that

\[
E(\varphi_\varepsilon, \varphi_\varepsilon) \leq \varepsilon^{N-2s}[\varphi]_{r,\mathbb{R}^N} + \int_{B^c} \int_{B^c} \frac{(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x-y|^{N+2c}} \, dx \, dy
\]

\[
\leq \varepsilon^{N-2s}[\varphi]_{r,\mathbb{R}^N} + \int_{B^c} \varphi_\varepsilon^2(x) \int_{B^c} \frac{1}{|x-y|^{N+2c}} \, dx \, dy
\]

\[
\leq \varepsilon^{N-2s}[\varphi]_{r,\mathbb{R}^N} + \int_{|x| \leq \varepsilon} \varphi_\varepsilon^2(x) \int_{|y| > 1} \frac{1}{|x-y|^{N+2c}} \, dx \, dy
\]

\[
\leq \varepsilon^{N-2s}[\varphi]_{r,\mathbb{R}^N} + \varepsilon^N \int_{|x| < 1} \varphi^2(x) \, dx.
\]

It implies that

\[
\varepsilon^{N-2\alpha} \lesssim \varepsilon^N + \varepsilon^{N-2s}, \quad \forall \varepsilon \in (0, 1),
\]

which clearly implies \( \alpha \leq s \).

Case II. Let now choose \( x_0 \) with \( |x_0| > 2 \). Thus supp \( \varphi_\varepsilon \subset B_\varepsilon(x_0) \subset B_1^c \) and \( [\varphi_\varepsilon]_{s,B_1} = 0 \).

In this case the cross term is small similar to the previous case

\[
\int_{B^c} \int_{B^c} \frac{(\varphi_\varepsilon(x) - \varphi_\varepsilon(y))^2}{|x-y|^{N+2c}} \, dx \, dy = \int_{B^c} \varphi_\varepsilon^2(y) \int_{B^c} \frac{1}{|x-y|^{N+2c}} \, dx \, dy
\]

\[
= \int_{|x-x_0| < \varepsilon} \varphi_\varepsilon^2(y) \int_{|x| < 1} \frac{1}{|x-y|^{N+2c}} \, dx \, dy \leq \varepsilon^N \int_{|z| < 1} \varphi^2(z) \, dz.
\]

It follows that

\[
E(\varphi_\varepsilon, \varphi_\varepsilon) \leq \varepsilon^{N-2s}[\varphi]_{r,\mathbb{R}^N} + \varepsilon^N \int_{|z| < 1} \varphi^2(z) \, dz.
\]

Similar as in the first case we get \( \alpha \leq r \).

We now improve (16) for larger times. We use inequality (14) with \( p = 2 \). It gives us that

\[
\|u\|_{L^2(\mathbb{R}^N)}^2 \leq C(E(u) + \|u\|_{L^1(\mathbb{R}^N)}^{\frac{4r}{r+2_N}} E(u)^{\frac{N}{r+2_N}}).
\]

For functions \( u \) with \( \|u\|_{L^1(\mathbb{R}^N)} \leq 1 \) we get \( \|u\|_{L^2(\mathbb{R}^N)}^2 \leq h(E(u)), \ h(t) = C(t + t^{\frac{N}{r+2_N}}) \). This shows that \( E(u) \geq h^{-1}(\|u\|_{L^2(\mathbb{R}^N)}) \). When restricting to the class of functions \( u \) with its \( L^2 \) norm less than a constant \( a \) it means we are looking to the behaviour of \( h^{-1}(t) \) near the origin. This means that

\[
E(u, u) \geq h^{-1}(\|u\|_{L^2(\mathbb{R}^N)}) \geq c(\|u\|_{L^2(\mathbb{R}^N)})^{\frac{N+2_N}{N}}, \quad \forall \|u\|_{L^2(\mathbb{R}^N)} \leq a.
\]

We are in the framework of [7, Prop. III.2, Th. III.3] with \( \theta(t) = t^{1+\frac{N}{r+2_N}} \). Indeed, from the previous \( L^1 - L^\infty \) property we know that there exists a positive time \( t_0 \) such that \( \|S(t_0)\|_{1,2} = a \) or \( \|S(t_0)\|_{1,\infty} = a^2 < \infty \). It follows that we have directly the right decay of the semigroup for any \( t > t_0 \):

\[
\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \leq \frac{B}{t^{\frac{N}{r+2_N}}} \|u_0\|_{L^1(\mathbb{R}^N)} \quad \forall u_0 \in L^1(\mathbb{R}^N), \forall t > t_0.
\]
where \( t_0 \) depends only on \( a \) above. Toggether (16) and (18) give the desired result.

\[ \square \]

4. The asymptotic expansion of solutions

In this section our main goal is to prove Theorem 1.3. The proof is quite technical and we will split it through various subsections. It is based on the method of rescaled solutions [15, 18].

Given a solution \( u \) of problem (3) we consider

\[ u_{\lambda}(x, t) = \lambda^N u(\lambda x, \lambda^2 t) \]

which is a solution to

\[
\begin{aligned}
\left\{ 
\begin{align*}
\alpha_s \lambda^{2r-2s} & \int_{B_{1/\lambda}} \frac{u_{\lambda}(y) - u_{\lambda}(x)}{|x - y|^{N+2s}} dy + \alpha_c \lambda^{2r-2c} & \int_{B_{1/\lambda}} \frac{u_{\lambda}(y) - u_{\lambda}(x)}{|x - y|^{N+2c}} dy, & x \in B_{1/\lambda}, t > 0, \\
\alpha_r & \int_{|y|>1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N+2r}} dy + \alpha_c \lambda^{2r-2c} & \int_{|y|<1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N+2c}} dy, & |x| > 1/\lambda
\end{align*}
\right.
\end{aligned}
\]

\( (\mathcal{L}_\lambda \varphi)(x) = \)

\[
\begin{aligned}
\left\{ 
\begin{align*}
\alpha_s \lambda^{2r-2s} & \int_{|y|<1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N+2s}} dy + \alpha_c \lambda^{2r-2c} & \int_{|y|>1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N+2c}} dy, & |x| < 1/\lambda, \\
\alpha_r & \int_{|y|>1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N+2r}} dy + \alpha_c \lambda^{2r-2c} & \int_{|y|<1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N+2c}} dy, & |x| > 1/\lambda
\end{align*}
\right.
\end{aligned}
\]

This section is divided as follows: first we obtain various estimates for operator \( \mathcal{L}_\lambda \) and for the bilinear form associated with it \( \mathcal{E}_\lambda(\varphi, \varphi) = (-\mathcal{L}_\lambda \varphi, \varphi) \). We use them to obtain uniform estimates for the family of rescaled solutions \( (u_{\lambda})_{\lambda>0} \) and use them to prove the compactness of considered family which will converge to a profile \( U_M \). We will characterize the profile \( U_M \) to be the unique solution of the fractional heat equation \( U_t + (-\Delta)^r U = 0 \) with initial data \( M\delta_0 \) taken in the sense of measures [2]. Let us recall that in [2, Th. 9.1] the authors show that for any initial measure \( \mu_0 \in \mathcal{M}_s(\mathbb{R}^N) \), the space of locally finite Radon measures satisfying

\[ \int_{\mathbb{R}^N} (1 + |x|)^{-(N+2s)} d|\mu|(x) < \infty, \]

there exist a unique very weak solution in the following sense:
Definition 4.1. We say that $u$ is a very weak solution of the equation (1) if i) $u \in L^1_{\text{loc}}(0,T : L^1(\mathbb{R}^N, (1 + |x|)^{-(N + 2s)}))$, ii) it satisfies the equality

$$\int_0^T \int_{\mathbb{R}^N} u(t,x) \partial_t \psi(t,x) dx dt = \int_0^T \int_{\mathbb{R}^N} u(t,x)(-\Delta)^s \psi(t,x) dx dt$$

for all $\psi \in C_0^\infty((0,\infty) \times \mathbb{R}^N)$ and iii) its trace is $\mu_0$ in the following sense

$$\int_{\mathbb{R}^N} \psi dx = \lim_{t \to 0^+} \int_{\mathbb{R}^N} u(t,x) \psi(x) dx, \quad \text{for all } \psi \in C_0(\mathbb{R}^N).$$

Moreover, the solutions is given by the representation formula

$$U(t, x) = \int_{\mathbb{R}^N} K_t^s(x - y) d\mu_0(y).$$

Finally we conclude the proof of Theorem 1.3.

4.1. Estimates for the operator $L_\lambda$. We include here various lemmas used along the paper.

Lemma 4.1. Let $\rho > 0$ and $1/\lambda < \rho/2$. For any $\varphi \in W^{2,\infty}(\mathbb{R}^d)$ such that $\varphi(x) = \varphi(0)$ in the ball $B_\rho(0)$ the following holds:

$$\| (L_\lambda \varphi)(x) \| \lesssim \begin{cases} \lambda^{2r-2c} \rho^{-2a} \| \varphi \|_{L^\infty(\mathbb{R}^N)}, & |x| < 1/\lambda, \\ \rho^{2-2r} \| D^2 \varphi \|_{L^\infty(\mathbb{R}^N)} + \rho^{-2r} \| \varphi \|_{L^\infty(\mathbb{R}^N)}, & 1/\lambda < |x| < \rho, \\ \rho^{2-2r} \| D^2 \varphi \|_{L^\infty(\mathbb{R}^N)} + (\rho^{-2r} + \lambda^{2r-2c-N} \rho^{-N-2a}) \| \varphi \|_{L^\infty(\mathbb{R}^N)}, & |x| > \rho. \end{cases}$$

Proof. The main idea is to use the second derivative of $\varphi$ where $x$ and $y$ are close and the fact that in the ball of radius $\rho$ the first derivative of $\varphi$ vanishes.

Let us first consider the case $|x| < 1/\lambda$. Since $1/\lambda < \rho/2$ and $\varphi$ is constant in the ball $B_\rho(0)$ the first term in $(L_\lambda \varphi)(x)$ vanishes. For the second term we use the fact that $x$ and $y$ are separated

$$\int_{|y| > 1/\lambda} \frac{|\varphi(y) - \varphi(x)|}{|x - y|^{N+2c}} dy = \int_{|y| > \rho} \frac{|\varphi(y) - \varphi(x)|}{|x - y|^{N+2r}} dy \leq 2 \| \varphi \|_{L^\infty(\mathbb{R}^d)} \int_{|y| > \rho} \frac{1}{|y - x|^{N+2c}} dy \lesssim \| \varphi \|_{L^\infty(\mathbb{R}^d)} \int_{|y-x| > \rho/2} \frac{1}{|y - x|^{N+2c}} dy \simeq \| \varphi \|_{L^\infty(\mathbb{R}^d)} \int_{\rho/2}^{\infty} s^{-1-2a} \sim \rho^{-2a} \| \varphi \|_{L^\infty(\mathbb{R}^d)}.$$
For the first term we use the fact that the gradient of \( \varphi \) vanishes at the point \( x \). Hence 
\[
\varphi(y) - \varphi(x) = (y - x) \nabla \varphi(x) + |y - x|^2 O(||D^2 \varphi||_{L^\infty(\mathbb{R}^N)}).
\]
We obtain that
\[
|\langle \mathcal{L}_\lambda \varphi \rangle(x)| \lesssim ||D^2 \varphi||_{L^\infty(\mathbb{R}^N)} \int_{|y| < 2 \rho} \frac{dy}{|x - y|^{N + 2r - 2}} + ||\varphi||_{L^\infty(\mathbb{R}^N)} \int_{\frac{1}{2} \rho < |y|} \frac{1}{|x - y|^{N + 2r}} dy
\]
\[
\lesssim ||D^2 \varphi||_{L^\infty(\mathbb{R}^N)} \int_{|z| < 3 \rho} \frac{dz}{|z|^{N + 2r - 2}} + ||\varphi||_{L^\infty(\mathbb{R}^N)} \int_{\frac{1}{2} \rho < |z|} \frac{dz}{|z|^{N + 2r}}
\]
\[
\lesssim \rho^{2 - 2r} ||D^2 \varphi||_{L^\infty(\mathbb{R}^N)} + \rho^{-2r} ||\varphi||_{L^\infty(\mathbb{R}^N)}.
\]
Let us take an \( |x| > \rho \). Here we split the operator in three parts:
\[
\langle \mathcal{L}_\lambda \varphi \rangle(x) = \int_{1/\lambda < |y| < \rho/2} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N + 2r}} dy + \int_{\rho/2 < |y|} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N + 2r}} dy + \lambda^{2r - 2c} \int_{|y| < 1/\lambda} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N + 2c}}.
\]
In the case of \( A \) we use the fact that we can put a ball centered at \( x \) and of radius \( \rho/2 \) in the domain of integration \( |y| > 1/\lambda \). Hence
\[
A = \int_{1/\lambda < |y|, |x - y| < \rho/2} \frac{\varphi(y) - \varphi(x) - (y - x) \nabla \varphi(x)}{|x - y|^{N + 2r}} dy + \int_{1/\lambda < |y|, |x - y| > \rho/2} \frac{\varphi(y) - \varphi(x)}{|x - y|^{N + 2r}} dy
\]
\[
\lesssim ||D^2 \varphi||_{L^\infty(\mathbb{R}^N)} \int_{|y - x| < \rho/2} \frac{dy}{|x - y|^{N + 2r - 2}} + ||\varphi||_{L^\infty(\mathbb{R}^N)} \int_{|y - x| > \rho/2} \frac{dy}{|x - y|^{N + 2r - 2}}
\]
\[
\lesssim ||D^2 \varphi||_{L^\infty(\mathbb{R}^N)} \int_{|z| < \rho/2} \frac{dz}{|z|^{N + 2r - 2}} + ||\varphi||_{L^\infty(\mathbb{R}^N)} \int_{|z| > \rho/2} \frac{dz}{|z|^{N + 2r}}
\]
\[
\simeq \rho^{2 - 2r} ||D^2 \varphi||_{L^\infty(\mathbb{R}^N)} + \rho^{-2r} ||\varphi||_{L^\infty(\mathbb{R}^N)}.
\]
For the second term we use that \( |x - y| > \rho - 1/\lambda > \rho/2 \). It satisfies
\[
|B| \lesssim \lambda^{2r - 2c} ||\varphi||_{L^\infty(\mathbb{R}^N)} \lambda^{-N} \rho^{-N - 2c}.
\]
The proof is now finished. \( \square \)

Let us consider a function \( \psi \in C^\infty(\mathbb{R}^N) \) that vanishes identically in the unit ball, \( \psi(x) \equiv 1 \) for \( |x| > 2 \) and \( 0 \leq \psi \leq 1 \). Set \( \psi_R(x) = \psi(x/R) \). As a consequence of the above lemma we obtain the following result for \( \psi_R \).

**Lemma 4.2.** For any \( R > 2 \) and \( \lambda < 1 \) the following holds:

\[
|\langle \mathcal{L}_\lambda \psi_R \rangle(x)| \lesssim \begin{cases} 
\lambda^{2r - 2c} R^{-2c} ||\psi||_{L^\infty(\mathbb{R}^N)}, & |x| < 1/\lambda, \\
R^{-2r} ||D^2 \psi||_{L^\infty(\mathbb{R}^N)} + R^{-2r} ||\psi||_{L^\infty(\mathbb{R}^N)}, & \frac{1}{\lambda} < |x| < \rho, \\
R^{-2r} ||D^2 \psi||_{L^\infty(\mathbb{R}^N)} + R^{-2r} (1 + (\lambda R)^{2r - 2c - N}) ||\psi||_{L^\infty(\mathbb{R}^N)}, & |x| > \rho.
\end{cases}
\]

**Proof.** We apply Lemma 4.1 with \( \rho = R \) and \( 1/\lambda < 1 < \rho/2 \). The case \( |x| < 1/\lambda \) follows immediately. When \( 1/\lambda < |x| < \rho \) we use the definition of \( \psi_R \) to obtain that
\[ \|D^2\psi_R\|_{L^\infty(\mathbb{R}^N)} = R^{-2}\|D^2\psi\|_{L^\infty(\mathbb{R}^N)} \] and \[ |(L\lambda\psi_R)(x)| \lesssim R^{-2r}(\|D^2\psi\|_{L^\infty(\mathbb{R}^N)} + \|\psi\|_{L^\infty(\mathbb{R}^N)}). \]

Similar arguments for \(|x| > \rho\) show that
\[ |(L\lambda\psi_R)(x)| \lesssim R^{-2r}\|D^2\psi\|_{L^\infty(\mathbb{R}^N)} + R^{-2r}(1 + (\lambda R)^{2r-2c-N})\|\psi\|_{L^\infty(\mathbb{R}^N)}. \]

Under the assumption that \(2r - 2c \leq N\) we obtain an estimate for any \(1/\lambda < |x| < \rho:\)
\[ |(L\lambda\psi_R)(x)| \lesssim R^{-2r}\|\psi\|_{W^{2,\infty}(\mathbb{R}^N)}. \]
This is the same as when the fractional Laplacian of order \(r\), \((-\Delta)^r\) acts on the rescaled function \(\psi_R\) defined above.

We now give few estimates for the energy \(E_\lambda\) associated with \(u_\lambda:\)
\begin{equation}
E_\lambda(\varphi, \varphi) = \lambda^{2r-2s} \iint_{|x|,|y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2s}} + \iint_{|x|,|y| > 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2r}} + \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2c}}
\end{equation}

\[ \lambda^{2r-2c} \iint_{|x| < 1/\lambda < |y|} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2c}} \]

**Lemma 4.3.** Let us denote \(m = \max\{r, s, c\}\). For any \(\varphi \in H^m(\mathbb{R}^N)\) (smooth enough) the following holds for any \(\lambda > 0:\)
\[ E_\lambda(\varphi, \varphi) \leq (\lambda^{2c-2m} + \lambda^{2s-2m})[\varphi]_{m, B_{2\lambda}(0)}^2 + [\varphi]_{r, B_{1/\lambda}}^2 + (\lambda^{-N} + \lambda^{2r-2c-N}) \int_{\mathbb{R}^N} \frac{(\varphi(y) - \varphi(0))^2}{|y|^{N+2c}} dy. \]

**Proof.** For the first term we use the fact that both \(x\) and \(y\) are in the ball \(B_{1/\lambda}(0)\):
\[ \lambda^{2r-2s} \iint_{|x|,|y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2s}} = \lambda^{2r-2s} \iint_{|x|,|y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2|x-y|^{2m-2s}}{|x-y|^{N+2m}} \]
\[ \leq \lambda^{2r-2m} \iint_{|x|,|y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2m}} = \lambda^{2r-2m}[\varphi]_{m, B_{1/\lambda}(0)}^2. \]

It remains to estimate only the last term. We split it in two parts: one in which \(x\) and \(y\) are close enough where we can use the previous arguments and a second one where \(x\) and \(y\) are separated. Indeed
\[ \lambda^{2r-2c} \iint_{|x| < 1/\lambda < |y|} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2c}} \]
\[ = \lambda^{2r-2c} \iint_{|x| < 1/\lambda < |y| < 2/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2c}} + \lambda^{2r-2c} \iint_{|x| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2c}} \]

In the case of \(A\) the same arguments as above shows that
\[ |A| \leq \lambda^{2r-2m}[\varphi]_{m, B_{2\lambda}(0)}. \]
In the case of $B$ we use the introduce the value of $\varphi$ at $x = 0$ since for $\lambda$ large enough $\varphi(x)$ and $\varphi(y)$ are close to that value. We obtain

$$|B| \leq \lambda^{2r - 2c} \int_{|x| < 1/\lambda} (\varphi(x) - \varphi(0))^2 \frac{dy}{|x - y|^{N + 2c}} + \lambda^{2r - 2c} \int_{2\lambda < |y|} (\varphi(y) - \varphi(0))^2 \frac{dy}{|x - y|^{N + 2c}}$$

$$= B_1 + B_2.$$

Using that $|x| < 1/\lambda < 2/\lambda < |y|$ we have $|y - x| > |y| - |x| \geq |y|/2$. Thus

$$B_1 \leq \lambda^{2r - 2c} \int_{|x| < 1/\lambda} (\varphi(x) - \varphi(0))^2 dx \int_{|y| > 2/\lambda} \frac{dy}{|y|^{N + 2c}} = \lambda^{2r} \int_{|x| < 1/\lambda} (\varphi(x) - \varphi(0))^2 dx$$

$$\leq \lambda^{-N} \int_{|x| < 1/\lambda} \frac{(\varphi(x) - \varphi(0))^2}{|x - y|^{N + 2r}} dx.$$ 

In the case of $B_2$ the same argument gives us

$$B_2 \leq \lambda^{2r - 2c - N} \int_{2\lambda < |y|} \frac{(\varphi(y) - \varphi(0))^2}{|y|^{N + 2c}} dy.$$ 

The proof is now finished. We emphasize that $B_2$ cannot be improved in the case when $y \simeq R$ is large and $x \neq 0$. \hfill \square

**Lemma 4.4.** Let $2r - 2c \leq N$. For any $\rho > 0$ and $\lambda > 2/\rho$ the following holds

(23) \hspace{1cm} \mathcal{E}_\lambda(\varphi, \varphi) \leq C(\rho)\|\varphi\|^2_{H^1(|x| > \rho)}, \hspace{0.5cm} \forall \varphi \in C_c^\infty(|x| > \rho).$

**Proof.** Recall that

$$\mathcal{E}_\lambda(\varphi, \varphi) = \lambda^{2r - 2s} \int_{|x|, |y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x - y|^{N + 2s}} dxdy + \int_{|x|, |y| > 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x - y|^{N + 2r}} dxdy$$

$$+ \lambda^{2r - 2c} \int_{|x| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x - y|^{N + 2c}} dx = I_1 + I_2 + I_3.$$

Let us consider $\varphi \in C_c^\infty(|x| > \rho)$. We prove that $\mathcal{E}_\lambda(\varphi, \varphi) \leq C\|\varphi\|^2_{H^1(\mathbb{R}^N)}$. When $\lambda > 1/\rho$ the support of $\varphi$ gives us that the first term vanishes. We emphasize that this term can be estimated without the support assumption. Indeed, let us choose $s \leq m < 1$:

$$I_1 \leq \lambda^{2r - 2s} \int_{|x|, |y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x - y|^{N + 2m}} |x - y|^{2m - 2s} dxdy$$

$$\leq \lambda^{2r - 2m} \int_{|x|, |y| < 1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x - y|^{N + 2m}} dxdy = \lambda^{2r - 2m} \|\varphi_m\|^2_{L^2(\mathbb{R}^N)} \leq \lambda^{2r - 2m} \|\varphi\|^2_{H^1(\mathbb{R}^N)}.$$
For second one we use the integrals in the whole space \( \mathbb{R}^d \):
\[ I_2 \leq \int_{|x|,|y|>1/\lambda} \frac{(\varphi(y) - \varphi(x))^2}{|x-y|^{N+2c}} \, dx \, dy \leq [\varphi]^2_{L^r} \leq \|\varphi\|_{H^1(\mathbb{R}^N)}^2. \]

It remains to analyze \( I_3 \). Again we use that \( \varphi(x) \) for \( |x| < 1/\lambda \), so
\[ I_3 = \lambda^{2r-2c} \int_{|y|<\lambda} \frac{\varphi^2(y)}{|x-y|^{N+2c}} \, dx \, dy = \lambda^{2r-2c} \int_{|y|<\lambda} \frac{\varphi^2(y)}{|x-y|^{N+2c}} \, dx \, dy. \]
\[ = \lambda^{2r-2c} \int_{|y|<\lambda} \varphi^2(y) \frac{dx}{|x-y|^{N+2c}} \, dy. \]
For \( \lambda > 2/\rho \) we get \( |x-y| \geq |y| - |x| > \rho - 1/\lambda > \rho/2 \) and since the integral on \( x \) is over a set of measure \( \lambda^{-N} \) we get
\[ I_3 \leq \rho^{-N-2c} \lambda^{2r-2c-N} \int_{|y|>\rho} \varphi^2(y) \, dy \leq \lambda^{2r-2c-N} \|\varphi\|_{H^1(\mathbb{R}^N)}. \]
This finishes the proof. \( \square \)

4.2. Uniform estimates for the rescaled solutions. We now obtain uniform estimates for the rescaled solutions \( u_\lambda = \lambda^N u(\lambda^r t, \lambda x) \). We assume that \( 2r - 2c \leq N \).

**Theorem 4.1.** For any \( u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \) the rescaled solutions \( u_\lambda(t,x) \) satisfy the following uniform estimates:

i) \[ \|u_\lambda(t)\|_{L^p(\mathbb{R}^N)} \leq C(p, r, N, \|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^p(\mathbb{R}^N)}) t^{-\frac{N}{2} \left(1 - \frac{1}{p}\right)}, \quad \forall \, t > 0, \]

ii) if \( p = 2 \), \[ \mathcal{E}_\lambda(u_\lambda(t), u_\lambda(t)) \leq C(r, N, \|u_0\|_{L^1(\mathbb{R}^N)}, \|u_0\|_{L^2(\mathbb{R}^N)}) t^{-\left(1 + \frac{N}{2}\right)}, \quad \forall \, t > 0. \]

**Proof.** The first property follows from the definition of the rescaled solutions \( u_\lambda \), the decay properties in Theorem 1.2 and the \( L^p \)-contractivity of the semigroup:
\[ \|u(t)\|_{L^p(\mathbb{R}^N)} \leq \min\{\|u_0\|_{L^p(\mathbb{R}^N)}, C(p, r, N) t^{-\frac{1}{2} \left(1 - \frac{1}{p}\right)} \|u_0\|_{L^1(\mathbb{R}^N)}\} \]
\[ \lesssim (1 + t)^{-\frac{1}{2} \left(1 - \frac{1}{p}\right)}. \]

For the second one we now use Theorem 2.1 and the decay of the \( L^2 \) norm to obtain that
\[ \mathcal{E}(u(t), u(t)) \leq \frac{\|u(t/2)\|_{L^2(\mathbb{R}^N)}^2}{t} \leq t^{-1} \min\{\|u_0\|_{L^p(\mathbb{R}^N)}, C(p, r, N) t^{-\frac{1}{2} \left(1 - \frac{1}{p}\right)} \|u_0\|_{L^1(\mathbb{R}^N)}\} \]
\[ \lesssim (1 + t)^{-1 - \frac{1}{2} \left(1 - \frac{1}{p}\right)}. \]

Definition of \( \mathcal{E} \) and \( \mathcal{E}_\lambda \) gives us that
\[ \mathcal{E}_\lambda(u_\lambda(t), u_\lambda(t)) = \lambda^{-2r-N} \mathcal{E}(u(\lambda^{2r} t), u(\lambda^{2r} t)) \leq t^{-\left(1 + \frac{N}{2}\right)} \|u_0\|_{L^1(\mathbb{R}^N)}^2, \]
which finishes the proof. \( \square \)
Lemma 4.5. For any \( u_0 \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \) and \( \rho > 0 \) there exists a \( \lambda(\rho) \) such that for any \( T > \tau > 0 \)
\[
||\partial_t u_\lambda||_{L^2([\tau,T]; H^{-1}(B_{\rho}^c))} \leq C(\tau)
\]
uniformly for \( \lambda > \lambda(\rho) \).

Proof. Let us take \( \psi \in C_c^\infty(B_{\rho}^c) \). We have that
\[
||\partial_t u_\lambda||_{H^{-1}(B_{\rho}^c)} = \sup <\partial_t u_\lambda, \psi >_{H^{-1},H^1_0} \text{ for } ||\psi||_{H^1_0(B_{\rho}^c)} \leq 1.
\]
By Theorem 2.1 \( \partial_t u_\lambda(t) \in L^2(\mathbb{R}^N) \) for any \( t > 0 \) thus we have that this duality product can be expressed as
\[
<\partial_t u_\lambda, \psi >_{H^{-1},H^1_0} = \int_{B_{\rho}^c} \partial_t u_\lambda \psi = \int_{\mathbb{R}^N} \partial_t u_\lambda \psi = \int_{\mathbb{R}^N} \mathcal{L} u_\lambda \psi = -\mathcal{E}_\lambda(u_\lambda, \psi).
\]
From Lemma 4.4 we have \( \mathcal{E}_\lambda(\psi, \psi) \leq c(\rho) \) for all \( \lambda > 2/\rho \) and this provides that
\[
\mathcal{E}_\lambda(u_\lambda, \psi) \leq \mathcal{E}_\lambda(u_\lambda, u_\lambda)^{1/2} \mathcal{E}_\lambda(\psi, \psi)^{1/2} \leq \mathcal{E}_\lambda(u_\lambda, u_\lambda)^{1/2}
\]
which in view of (25) means that
\[
\int_\tau^T ||\partial_t u_\lambda||_{H^{-1}(B_{\rho}^c)}^2 \leq \int_\tau^T \mathcal{E}_\lambda(u_\lambda, u_\lambda) \leq ||u_0||_{L^1(\mathbb{R}^N)}^2 \int_\tau^T t^{-1-\frac{N}{2}} \leq C(\tau),
\]
which finishes the proof.

Lemma 4.6. There exists a constant \( C > 0 \) such that for any \( R > 0 \)
\[
\int_{|x|>2R} |u_\lambda(x,t)| \, dx \leq \int_{|x|>R} u_0(x) \, dx + C \left( \frac{1}{R^2c} + \frac{M}{R^{2r}} \right) t
\]
holds uniformly for all \( \lambda > 1 \), where \( M \) is the mass of the solution.

Proof. By the comparison principle it is sufficient to consider nonnegative solutions. Let \( \psi \in C^\infty(\mathbb{R}^N) \) such that \( \psi(x) \equiv 0 \) for \( |x| < 1 \), \( \psi(x) \equiv 1 \) for \( |x| > 2 \) and \( 0 \leq \psi \leq 1 \). Set \( \psi_R(x) = \psi(x/R) \). Multiplying the equation satisfied by \( u_\lambda \) by \( \psi_R \) and integrating in time we get
\[
\int_{|x|>2R} u_\lambda \, dx \leq \int_{\mathbb{R}^N} u_\lambda \psi_R \, dx = \int_{\mathbb{R}^N} u_{0,\lambda} \psi_R \, dx + \int_0^t <\mathcal{L} u_\lambda, \psi_R> \, dt
\]
but
\[
\int_{\mathbb{R}^N} u_{0,\lambda} \psi_R \, dx \leq \chi^N \int_{|x|>R} u_0(\lambda x) \, dx = \int_{|x|>\lambda R} u_0(x) \, dx \leq \int_{|x|>R} u_0(x) \, dx
\]
given \( \lambda > 1 \), so we focus now on the last integral. We split it on two parts, say
\[
\int_0^t \int_{|x|<1/\lambda} \mathcal{L} u_\lambda \psi_R \, dx \, dt + \int_0^t \int_{|x|>1/\lambda} \mathcal{L} u_\lambda \psi_R \, dx \, dt.
\]
Let us start with $I$. Applying Lemma 4.2 and Hölder’s inequality we get

$$|I| \leq \|u_\lambda(t)\|_{L^{1+\varepsilon}(B_{t/\lambda})} \cdot \|L_\lambda^\psi R\|_{L^{1+\varepsilon\ast}(B_{t/\lambda})} \leq \|u_\lambda(t)\|_{L^{1+\varepsilon}(B_{t/\lambda})} \lambda^{2r-2c-N/(1+\varepsilon)\ast} R^{-2c}$$

(26) \hspace{1cm} \leq t^{-\frac{N}{2r}(1-\frac{1}{1+\varepsilon})} \lambda^{2r-2c-N/(1+\varepsilon)\ast} R^{-2c}

where $(1+\varepsilon)^\ast = \varepsilon/(1+\varepsilon)$, the conjugate exponent. Now we can integrate by parts to obtain that

$$\int_0^t |I| \leq \lambda^{2r-2c-N/(1+\varepsilon)\ast} R^{-2c} t^{1-\frac{N}{2r}+\varepsilon}$$

provided that $1 > \frac{N}{2r+\varepsilon}$, so we can finish this part by choosing $\varepsilon$ such that

$$\frac{2r-2c}{N} \leq \frac{\varepsilon}{1+\varepsilon} < \frac{2r}{N}$$

and this is always possible if $2r-2c < N$. We have shown that

$$\int_0^t I \ dt \leq R^{-2c} \varepsilon(r,c,N) t$$

where $\varepsilon(r,c,N)$ is a constant dependant of said parameters.

For the second integral the mass conservation of $u_\lambda$ and the estimates for $|L_\lambda^\psi R|$ from Lemma 4.2 provide

$$\int_0^t |II| \leq \int_0^t \int_{|x|>1/\lambda} u_\lambda |L_\lambda^\psi R| \ dx \ dt \leq \frac{Mt}{R^2}$$

where $M$ is the mass of the solution. This finishes the proof.

\[ \Box \]

4.3. Compactness of the family $\{u_\lambda\}$. In this section we will show several results about the family $\{u_\lambda\}$ that will allow us study the convergence of this family to a certain function $U$. Since the decay obtained in Theorem 1.2 is not uniform for small and large time we need first to assume that the initial data $u_0$ belongs to the space $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. This assumption will be assumed until the final step of this subsection where it will be dropped and the result of Theorem 1.3 is obtained by using a density argument.

**Step I. Convergence in** $C_{\text{loc}}((0,\infty), L^2_{\text{loc}}(B^c_{\rho}))$. Let us now choose $0 < \tau < T < \infty$ and $\rho > 0$. Using estimate (25) we obtain that

$$\|u_\lambda\|_{L^\infty((\tau,T),H^r(B^c_{\rho}))} \leq C(\tau) = \tau^{-\left(\frac{N}{2r}+\frac{1}{2}\right)}.$$ (27)

Using also that by Lemma 4.5 for any $\lambda > \lambda(\rho)$ we have a uniform bound for the time derivative

$$\|\partial_t u_\lambda\|_{L^2((\tau,T):H^{-1}(B^c_{\rho}))} \leq C(\tau)$$

we can apply Aubin-Lions-Simon compactness argument to obtain that, up to a subsequence,

$$u_\lambda \to U$$ in $C([\tau,T], L^2_{\text{loc}}(B^c_{\rho}))$.

By a diagonal argument we obtain the desired property.
Step II. Convergence in $C_{\text{loc}}((0,\infty), L^2_{\text{loc}}(\mathbb{R}^N))$. Let us choose an $R > 0$. We prove that $u_\lambda$ converges to $U$ in $C_{\text{loc}}((0,\infty), L^2(B_R))$.

For any $\rho < R$ we have
$$\int_{B_R} |u_\lambda(t) - U(t)|^2 dx \leq \int_{B_\rho} |u_\lambda(t)|^2 + |U(t)|^2 dx + \int_{\rho < |x| < R} |u_\lambda(t) - U(t)|^2 dx.$$ 

By the previous lemma the second integral goes to 0, so let us focus on the first one. By Hölder’s inequality and estimate (24)
$$\int_{B_\rho} |u_\lambda(t)|^2 \leq \|u_\lambda(t)\|^2_{L^4(\mathbb{R}^N)} \cdot \rho^{N/2} \leq t^{-\frac{N}{4}(1-\frac{1}{r})}\rho^{N/2}.$$

On the other hand, in view of estimate (24) for each $t > 0$, $u_\lambda(t) \rightharpoonup U(t)$ in $L^2(\mathbb{R}^N)$. Moreover estimate (24) transfers to $U$:
$$\|U(t)\|_{L^2(\mathbb{R}^N)} \leq Ct^{-\frac{N}{4}(1-\frac{1}{r})}.$$

Hence for any $\varepsilon > 0$ we can choose a small enough $\rho$ such that
$$\int_{B_\rho} (|u_\lambda(t,x)|^2 + |U(t,x)|^2) dx \leq C(t)\rho^{N/2} < \frac{\varepsilon}{2}.$$

For this $\rho$ fixed we choose $\lambda > \lambda(\rho)$ such that
$$\int_{\rho < |x| < R} |u_\lambda(t) - U(t)|^2 dx < \frac{\varepsilon}{2},$$

proving the desired result.

Step III. Convergence in $C_{\text{loc}}((0,\infty), L^1(\mathbb{R}^N))$. The convergence in $C((\tau, T), L^1_{\text{loc}}(\mathbb{R}^N))$ follows from the previous step. The tail control proved in Lemma 4.6 shows that this convergence is also in $C((\tau, T), L^1(\mathbb{R}^N))$. It means that for each $t > 0$, $u_\lambda(t) \rightharpoonup U(t)$ in $L^1(\mathbb{R}^N)$. This shows that in particular $U(t)$ conserves the mass along the time:
$$\int_{\mathbb{R}^N} U(t,x) dx = M, \quad \forall t > 0.$$

Step IV. Regularity of the profile $U$. Estimate (27) gives us that $u_\lambda(t) \rightharpoonup U(t)$ in $H^r(B^c_\rho)$ and
$$\|U\|_{L^\infty((\tau, T), H^r(B^c_\rho))} \leq C(\tau).$$

We will prove that in fact $U \in L^\infty((\tau, T) : H^r(\mathbb{R}^N))$.

We already have, by (25), that
$$\int_{B^c_\rho} \int_{B^c_\rho} \frac{(U(t,x) - U(t,y))^2}{|x-y|^{N+2r}} dxdy \leq \|u_0\|_{L^1(\mathbb{R}^N)}^2 t^{-\frac{N}{2}-1},$$

for any $\rho > 0$. We can define now the family
$$w_\rho(x,y) := \frac{(U(t,x) - U(t,y))^2}{|x-y|^{N+2r}} \chi_{B^c_\rho}(x) \chi_{B^c_\rho}(y)$$

2It is continuous in time? or only bounded
which satisfies that \(0 \leq w_\rho(x, y) \leq w_\rho'(x, y)\) for any \(\rho > \rho'\) and as \(\rho \to 0\),

\[
w_\rho(x, y) \to \frac{(U(t, x) - U(t, y))^2}{|x - y|^{N + 2r}} \text{ almost everywhere in } \mathbb{R}^N \times \mathbb{R}^N
\]

so by the monotone convergence theorem and the fact, that by (28), \(U \in L^2(\mathbb{R}^N)\) we obtain that \(U(t)\) belongs \(H^r(\mathbb{R}^N)\) and

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(U(t, x) - U(t, y))^2}{|x - y|^{N + 2r}} \, dx \, dy \leq \|u_0\|^2_{L^1(\mathbb{R}^N)} t^{-\frac{N}{2r} - 1}.
\]

**Step V. Equation satisfied by the limit \(U\).**

 revisar las constantes y la prueba. probamos mucho mas de lo necesario :-) pero tenemos que ponerlo claro que esto implica lo de la definicion (4.1)

 We now show that the limit function \(U\) is solution in the sense of definition (4.1) of the equation

(31) \[U_t + (-\Delta)^r U = 0 \text{ for } x \in \mathbb{R}^N, \quad U_0(x) := U(x, 0) = M\delta_0\]

where \(M\) is the mass of the solution \(u\). The results of \([2]\) show that \(U_M = MK_t^r\) the classical kernel of the fractional heat equation. check the renormalization constants

**Lemma 4.7.** The limit function \(U \in C((0, \infty), L^1(\mathbb{R}^N))\) satisfies

\[
\int_{\mathbb{R}^N} U(x, t) \varphi(x) \, dx - \int_{\mathbb{R}^N} U(x, \tau) \varphi(x) \, dx = \int_{\tau}^{t} \int_{\mathbb{R}^N} \frac{U(x) - U(y)}{|x - y|^{N + 2r}} (\varphi(x) - \varphi(y)) \, dx \, dy
\]

for any \(\varphi \in C^\infty_c(\mathbb{R}^N)\) and \(0 < s < t < \infty\).

**Remark 1.** This shows that in particular \(U_t(t) + (-\Delta)^r U(t) = 0\), for all \(t > 0\).

**Proof.** Let us multiply our equation by \(\varphi\) to obtain, for any \(0 < \tau < t < \infty\),

\[
\int_{\mathbb{R}^N} u_\lambda(x, t) \varphi(x) \, dx - \int_{\mathbb{R}^N} u_\lambda(x, \tau) \varphi(x) \, dx = \int_{\tau}^{t} \int_{\mathbb{R}^N} \mathcal{L}_\lambda(u_\lambda) \varphi(x) \, dx.
\]

By the results shown in the previous section the left-hand side of this equality converges to

\[
\int_{\mathbb{R}^N} U(x, t) \varphi(x) \, dx - \int_{\mathbb{R}^N} U(x, \tau) \varphi(x) \, dx
\]

so let us focus in the right-hand side, which we divide in three parts. First,

\[
I_1 := \chi_{[2^{r-2s}]} \int_{\tau}^{t} \int_{\mathbb{R}^N} \frac{u_\lambda(x) - u_\lambda(y)}{|x - y|^{N + 2s}} (\varphi(x) - \varphi(y)) \, dx \, dy
\]
but by Hölder’s inequality

\[
I_1 \leq \left[ \lambda^{2r-2s} \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \left( \frac{u_\lambda(x) - u_\lambda(y)}{|x-y|^{N+2s}} \right)^2 \right]^{1/2} \left[ \lambda^{2r-2s} \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \left( \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2s}} \right)^2 \right]^{1/2} \\
\leq C(\tau) \left[ \lambda^{2r-2-N} \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \frac{\|\varphi\|^2_\infty}{|x-y|^{N+2s-2}} \right]^{1/2}
\]

and this last integral is finite, so \( I_1 \to 0 \) as \( \lambda \to \infty \). Now,

\[
I_2 := \lambda^{2r-2c} \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \frac{u_\lambda(x) - u_\lambda(y)}{|x-y|^{N+2c}} \left( \varphi(x) - \varphi(y) \right).
\]

Again we split it in two parts and the one with \( u_\lambda \) is bounded by a constant \( C(\tau) \), so let us focus in the second one. Indeed,

\[
\lambda^{2r-2c} \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \frac{\left( \varphi(x) - \varphi(y) \right)^2}{|x-y|^{N+2c}} \leq t \lambda^{2r-2c} \left[ \int_{B_{1/\lambda}} \int_{1/\lambda < |y| < 1} \frac{\left( \varphi(x) - \varphi(y) \right)^2}{|x-y|^{N+2c}} \right]_A \\
+ t \lambda^{2r-2c} \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \frac{\left( \varphi(x) - \varphi(y) \right)^2}{|x-y|^{N+2c}}.
\]

One hand we have that

\[
A \leq t \lambda^{2r-2c} \|\varphi'\|^2_\infty \int_{B_{1/\lambda}} \int_{|z| < 1/\lambda} \frac{1}{|z|^{N+2c-2}} \leq C t \|\varphi'\|^2_\infty \lambda^{2r-2c-N}
\]

for a certain positive constant \( C \). On the other hand, and by similar arguments

\[
B \leq t \lambda^{2r-2c} \|\varphi\|^2_\infty \int_{B_{1/\lambda}} \int_{|z| > 1/\lambda} \frac{1}{|z|^{N+2c}} \leq C t \|\varphi\|^2_\infty \lambda^{2r-2c-N}.
\]

In both cases the quantities go to 0 when \( \lambda \) grows, provided that \( 2r - 2c - N < 0 \). Let us focus now on

\[
I_3 := \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \frac{u_\lambda(x) - u_\lambda(y)}{|x-y|^{N+2r}} \left( \varphi(x) - \varphi(y) \right) = t \int_\tau^t \int_{B_{1/\lambda}} \int_{B_{1/\lambda}} \frac{\varphi(x) - \varphi(y)}{|x-y|^{N+2r}}.
\]

Now since \( \varphi \in C^\infty_c(\mathbb{R}^N) \subset H^r(\mathbb{R}^N) \) we can define a function \( \psi \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \) by

\[
\psi(x,y) := \frac{\varphi(x) - \varphi(y)}{|x-y|^{(N+2r)/2}},
\]

and

\[
f_\lambda := \frac{(u_\lambda(x) - u_\lambda(y))}{|x-y|^{(N+2r)/2}} \chi_{B_{1/\lambda}}(x) \chi_{B_{1/\lambda}}(y).
\]
Clearly, by (27) we have that the family \( f \) is uniformly bounded in \( L^2(\mathbb{R}^N \times \mathbb{R}^N) \) and so there exists a \( f \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \) such that \( f_f \rightarrow f \) and therefore
\[
< f_f, \psi >_{L^2} \rightarrow < f, \psi >_{L^2}
\]
which will prove our claim once we demonstrate that
\[
f = \frac{(U(x) - U(y))}{|x - y|^{(N+2r)/2}}, \text{ a.e. in } \mathbb{R}^N \times \mathbb{R}^N.
\]

For a bounded domain \( \Omega \), \( u_\lambda(x) \rightarrow U(x) \) in \( L^2(\Omega) \) it means that, up to a subsequence, \( u_\lambda(x) \rightarrow U(x) \) almost everywhere in \( \Omega \), which means that \( f_\lambda \rightarrow (U(x) - U(y))|x - y|^{-(N+2r)/2} \) a.e. in \( \Omega \). Since we also have that \( \|f_\lambda\|_{L^2(\Omega \times \Omega)} \leq \|f\|_{L^2(\Omega \times \Omega)} < \infty \), the family \( f_\lambda \) is uniformly bounded and so we have (see for example [19, Prop. 5.4.6, p. 106]) that \( f_\lambda \rightarrow (U(x) - U(y))|x - y|^{-(N+2r)/2} \) in \( L^2(\Omega \times \Omega) \), which means precisely that \( f = (U(x) - U(y))|x - y|^{-(N+2r)/2} \) in \( \Omega \times \Omega \). Since \( \Omega \) was arbitrary, we are done. \( \square \)

Next we focus on the initial data.

**Lemma 4.8.** For any \( \varphi \in BC(\mathbb{R}^N) \)
\[
(32) \quad \lim_{t \to 0} \int_{\mathbb{R}^N} U(t, x) \varphi(x) dx = M \varphi(0).
\]

**Remark 2.** In fact in order to use the uniqueness results of [2] it is sufficient to consider the above limit only for functions \( \varphi \in C_0(\mathbb{R}^N) \), i.e. functions that vanishes at infinity.

**Proof.** Let us first consider a smooth function \( \varphi \) that is constant in a ball centered at the origin \( B_\rho(0) \). Multiplying equation (19) by \( \varphi \) and integrating in the space variable gives us
\[
\int_{\mathbb{R}^N} u_\lambda(t, x) \varphi(x) dx - \int_{\mathbb{R}^d} u_0,\lambda(x) \varphi(x) dx = \int_0^t \int_{\mathbb{R}^N} u_\lambda(s, x)(\mathcal{L}_\lambda \varphi)(x) dx ds
\]
\[
= \int_0^t \int_{|x| < 1/\lambda} u_\lambda(s, x)(\mathcal{L}_\lambda \varphi)(x) dx ds + \int_0^t \int_{|x| > 1/\lambda} u_\lambda(s, x)(\mathcal{L}_\lambda \varphi)(x) dx ds
\]
\[
:= I_1 + I_2.
\]
For \( \lambda \geq \max\{1, 2/\rho\} \) we can apply Lemma 4.4 to estimate the second term
\[
|I_2| \lesssim M(t) \left( \rho^{-2r} \|D^2 \varphi\|_{L^\infty(\mathbb{R}^d)} + (\rho^{-2r} + \rho^{-N-2\alpha}) \|\varphi\|_{L^\infty(\mathbb{R}^N)} \right).
\]
In the case of \( I_1 \) we proceed as in the tail control. Let \( \varepsilon > 0 \) be a small parameter that will be chosen latter. We use the first estimates in Lemma 4.4
\[
|I_1| \leq \|\mathcal{L}_\lambda \varphi\|_{L^{1+\varepsilon}(\mathbb{R}^N)} \int_0^t \|u_\lambda(s)\|_{L^{1+\varepsilon}(\mathbb{R}^N)} ds
\]
\[
\lesssim \lambda^{2r-2\alpha} \rho^{-2\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^N)} \lambda^{-N(1+\varepsilon)} \int_0^t s^{-\frac{N\varepsilon}{2r(1+\varepsilon)}} ds
\]
\[
\lesssim \rho^{-2\alpha} \|\varphi\|_{L^\infty(\mathbb{R}^N)} \lambda^{2r-2\alpha-N\varepsilon} t^{1-\frac{N\varepsilon}{2r(1+\varepsilon)}},
\]

provided that $\frac{N\varepsilon}{2r(1+\varepsilon)} < 1$. Under the assumption $2r - 2a < N$ we can choose $\varepsilon = \varepsilon(r, a, N) > 0$ such that $2r - 2a \leq \frac{N\varepsilon}{1+\varepsilon}$ and $\frac{N\varepsilon}{2r(1+\varepsilon)} < 1$. Thus there exists a positive constant $\alpha(r, a, N)$ such that

$$|I_1| \leq \rho^{-2a}\|\varphi\|_{L^\infty(\mathbb{R}^N)} t^{\alpha(r, a, N)}.$$ 

It follows that for large enough $\lambda$ we have

$$\left| \int_{\mathbb{R}^N} u_\lambda(t, x)\varphi(x)dx - \int_{\mathbb{R}^d} u_{0,\lambda}(x)\varphi(x)dx \right| \leq C(\varphi)(t + t^{\alpha(r, a, N)}).$$

Letting $\lambda \to \infty$ we obtain the same property for the limit point $U$:

$$(33) \quad \left| \int_{\mathbb{R}^N} U(t, x)\varphi(x)dx - M\varphi(0) \right| \leq C(\varphi)(t + t^{\alpha(r, a, N)}).$$

This shows that (32) holds for functions $\varphi \in W^{2, \infty}(\mathbb{R}^d)$ which are locally constants near the origin.

Let us now show that property (32) holds for all $\varphi \in BC(\mathbb{R}^N)$. Indeed by Lemma 4.6 and the strong convergence (check that it is for any $\varphi \in BC(\mathbb{R}^N)$) there exists a sequence of functions $\varphi_n \in W^{2, \infty}(\mathbb{R}^d)$ which are locally constant in a neighborhood of the origin such that $\varphi_n \to \varphi$ uniformly on compact sets and $\|\varphi_n\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^N)}$.

For completeness we prefer to write the full argument here even it is a standard procedure. Let us choose $R$ large enough that will be fixed latter. We write

$$\int_{\mathbb{R}^N} U(t, x)\varphi(x)dx - M\varphi(0) = \int_{|x|>2R} U(t, x)(\varphi(x) - \varphi_n(x))dx$$

$$- M(\varphi(0) - \varphi_n(0)) + \int_{|x|<2R} U(t, x)(\varphi(x) - \varphi_n(x))dx$$

$$+ \int_{\mathbb{R}^N} U(t, x)\varphi_n(x)dx - M\varphi_n(0)$$

$$= I + II + III.$$

Lemma 4.6 and the strong convergence (check that it is for any $t$ not only for a.e. $t$) in $L^1(\mathbb{R}^N)$ of $u_\lambda(t)$ toward $U(t)$ show that limit point $U$ satisfies the same tail control

$$\int_{|x|>2R} |U(x, t)| dx \leq \int_{|x|>R} |u_0(x)| dx + C\left(\frac{1}{R^{2c}} + \frac{M}{R^{2r}}\right)t$$

It implies that given any $\varepsilon > 0$ the first term satisfies

$$|I| \leq 2\|\varphi\|_{L^\infty(\mathbb{R}^N)} \int_{|x|>2R} |U(t, x)|dx \leq 2\|\varphi\|_{L^\infty(\mathbb{R}^N)} \left(\int_{|x|>R} |u_0(x)| dx + C\left(\frac{1}{R^{2c}} + \frac{M}{R^{2r}}\right)\right) < \varepsilon.$$ 

for $t < 1$ and $R > R(\varepsilon, u_0)$. For this large $R$ we can choose an $n$ large enough such that $II$ is small. Indeed using the uniform convergence of $\varphi_n$ toward $\varphi$ in the ball of radius $2R$ we get for large $n$ that

$$|II| \leq M\|\varphi_n - \varphi\|_{L^\infty(|x|<2R)} < \varepsilon.$$
For this $n$ large enough we apply estimate \[33\] so we can choose a small $t$ such that $|III| < \varepsilon$. It follows that $|I + II + III| < 3\varepsilon$ for small enough $t$ which proves the desired estimate \[32\] for $U$.

**Final Step and proof of Theorem 1.3** We first observe that it is sufficient to prove the result for initial data in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. For given $u_0 \in L^1(\mathbb{R}^N)$ we choose a sequence $u_{0n} \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ such that $u_{0n} \to u_0$ in $L^1(\mathbb{R}^N)$. For $t > 1$ using the decay in Theorem 6 we get

$$t^{N(1-\frac{1}{p})}\|u(t) - MK_r^t\|_{L^p(\mathbb{R}^N)} \leq t^{N(1-\frac{1}{p})}\|u(t) - u_n(t)\|_{L^p(\mathbb{R}^N)} + t^{N(1-\frac{1}{p})}\|u_n(t) - MK_r^t\|_{L^p(\mathbb{R}^N)}$$

$$\leq \|u_0 - u_{0n}\|_{L^1(\mathbb{R}^N)} + t^{N(1-\frac{1}{p})}\|u_n(t) - MK_r^t\|_{L^p(\mathbb{R}^N)}.$$ 

Given an $\varepsilon > 0$ we first choose an large $n$ such that the first term in the right hand side is less than $\varepsilon$ and then for any large time $t$ we obtain the desired result.

In the case of initial data in $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ all the estimates in the previous steps holds and then we can use the convergence in $L^1(\mathbb{R}^N)$ obtained in Step III to finish the proof. From Step III we know that for each $t > 0$ we have that up to a subsequence $u_\lambda(t) \to U(t)$ in $L^1(\mathbb{R}^N)$. Since in Step V we uniquely identified the profile $U = MK_r^t$ it means that the convergence holds for the whole sequence not only for a subsequence.

The $L^1$ convergence of $u_\lambda(1)$ towards $U(1)$ shows the desired property in $L^1$:

$$\|u(t) - U_M(t)\|_{L^1(\mathbb{R}^N)} \to 0.$$ 

For $1 < p < \infty$ we use the decay in $L^{2p}(\mathbb{R}^N)$ norm of the solution and of the function $U_M$ (\[14\] Lemma 2.2) and Hölder interpolation with exponent $\alpha = 2(p-1)/(2p-1)$:

$$\|u(t) - U_M(t)\|_{L^p(\mathbb{R}^N)} \leq \|u(t) - U_M(t)\|_{L^1(\mathbb{R}^N)}^{1-\alpha}\|u(t) - U_M(t)\|_{L^{2p}(\mathbb{R}^N)}^{\alpha} \leq o(1)t^{-\frac{N}{2}(1-\frac{1}{p})}.$$ 

5. **Appendix**

We first prove an approximation argument.

**Lemma 5.1.** For any function $\varphi \in BC(\mathbb{R}^N)$ there exists a sequence of approximation functions $\varphi_n \in W^{2,\infty}(\mathbb{R}^N)$ such that $\varphi_n$ are constant in a neighborhood of the origin and satisfy:

i) $\|\varphi_n\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^N)}$;

ii) $\varphi_n$ converges to $\varphi$ uniformly on compact sets.

**Proof.** Let us choose a sequence of mollifiers $(\rho_n)_{n \geq 1}$ as in \[14\] Ch. 4.4, p. 108 and $\psi_n = \rho_n*\varphi$. It follows that $\psi_n$ are smooth enough, $\|\psi_n\|_{L^\infty(\mathbb{R}^N)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^N)}$ and $\psi_n \to \varphi$ uniformly on compact sets \[14\] Prop. 4.2.1, Ch. 4, p. 108. It remains to make the approximation to be locally constant near origin. To do that we choose a function $\theta \in C_c^\infty(\mathbb{R}^N)$, $0 \leq \theta \leq 1$, such that $\theta(x) \equiv 1$ in $|x| < 1$ and $\theta(x) \equiv 0$ in $|x| > 2$ and set $\theta_\rho(x) = \theta(x/\rho)$. We consider

$$\varphi_{n,\rho}(x) = \psi_n(0)\theta_\rho(x) + \psi_n(x)(1 - \theta_\rho(x)).$$
It follows that $\varphi_{n,\rho}$ is smooth and $\varphi_{n,\rho}(x) = \psi_n(0)$ for $|x| < \rho$. Since $0 \leq \theta_{\rho} \leq 1$ it follows that the first property it is satisfied. For the second property let us observe that the difference between $\varphi_{n,\rho}$ and $\varphi$ satisfies

$$
|\varphi_{n,\rho}(x) - \varphi(x)| \leq |\psi_n(x) - \varphi(x)| + |\theta_{\rho}(x)||\psi_n(x) - \psi_n(0)|.
$$

Let us choose a compact set $K$. For any $\varepsilon > 0$ we choose an $N_\varepsilon$ such that for any $n \geq N_\varepsilon$, $|\psi_n(x) - \psi(x)| \leq \varepsilon$ for any $x \in K$. For each such $n \geq n_\varepsilon$ function $\psi_n$ being continuous we can choose $\rho_n$ small enough such that $|\psi_n(x) - \psi(0)| \leq \varepsilon$ for all $|x| < 2\rho_n$. Then $|\varphi_{n,\rho}(x) - \varphi(x)| \leq 2\varepsilon$ and setting $\varphi_n = \varphi_{n,\rho_n}$ we finish the proof. □

5.1. GNS inequalities for the fractional laplacian in exterior domains.

**Lemma 5.2.** Let $\Omega = \{x \in \mathbb{R}^n : |x| > 1\}$ and fix $x \in \Omega$. Let $E \subset \Omega$ a nonempty measurable set with finite measure. Then for any $1 < p < \infty$

$$
(34) \quad \int_{\Omega \setminus E} \frac{dy}{|x - y|^{N + ps}} \geq C(n, p, s, \Omega)|E|^{-ps/n}.
$$

**Remark 3.** When $\Omega$ is the whole space $\mathbb{R}^N$ this has been proved in [17, Lemma A.1]. The results of this lemma remains true for a larger class of exterior domains $\Omega$ but the analysis of such inequalities on general exterior domains will be done elsewhere.

**Proof.** Let us consider $\rho$ which we will fix later. We have
Let $\rho = \rho(x, E, \Omega)$ such that $|\Omega \cap B_\rho(x)| = |E|$. This is possible since $x \in \Omega$, $\Omega$ is an open unbounded set so the map $\rho \in (0, \infty) \mapsto |\Omega \cap B_\rho(x)| \in (0, \infty)$ is onto. Then

$$|(\Omega \setminus E) \cap B_\rho(x)| = |\Omega \cap B_\rho(x)| - |E \cap B_\rho(x)|$$

$$= |E| - |E \cap B_\rho(x)| = |E \cap (\Omega \setminus B_\rho(x))|.$$ 

Hence

$$\int_{\Omega \setminus E} \frac{dy}{|x - y|^{N+ps}} \geq \int_{E \cap (\Omega \setminus B_\rho(x))} \frac{dy}{\rho^{N+ps}} + \int_{(\Omega \setminus E) \cap (\Omega \setminus B_\rho(x))} \frac{dy}{|x - y|^{N+ps}}$$

$$\geq \int_{E \cap (\Omega \setminus B_\rho(x))} \frac{dy}{|x - y|^{N+ps}} + \int_{(\Omega \setminus E) \cap (\Omega \setminus B_\rho(x))} \frac{dy}{|x - y|^{N+ps}}$$

$$= \int_{\Omega \setminus B_\rho(x)} \frac{dy}{|x - y|^{N+ps}}.$$
Since \( x \notin B \), for any \( \rho > 0 \) the intersection \( \Omega \cap B_\rho(x) \) contains always half of the ball \( B_r(x) \) with \( r > \rho \) (see Figure 2). \(^3\) We obtain that \( |E| \geq |B_\rho|/2 = C(n)\rho^n \) and then
\[
\int_{\Omega \setminus E} \frac{dy}{|x - y|^{N + ps}} \geq \int_\rho^\infty \int_{S^+_{n-1}} \frac{\rho^{n-1}}{\rho^{n+sp}} d\sigma dr = c(n, \Omega) \frac{1}{\rho^{sp}} \geq C(n, \Omega) |E|^{-sp/n}
\]
which finishes the proof. \( \square \)

**Lemma 5.3.** Let \( s \in (0, 1) \) and \( 2s < N \). Then for any \( f \in H^s(|x| > 1) \) we have
\[
(35) \quad \|f\|_{L^2(B_\rho)} \leq C(N, s)[f]_{s, B_\rho}.
\]

**Remark 4.** When \( s = 1 \) Gagliardo-Niremberg-Sobolev inequalities in locally Lipschitz exterior domains have been considered in [8, Lemma 3.1] and [12, Th. II.6.1, p. 88]. A proof in the whole space \( \mathbb{R}^N \) without using Fourier analysis tools has been done in [16, Theorem 7.1].

**Proof.** Let us denote \( A_k = \{x \in \Omega : |f| > 2^k\} \) for any integer \( k \). Repeating the arguments in the Appendix of [16] we prove the estimate for functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) bounded measurable and compactly supported in \( \mathbb{R}^n \). Since \( \Omega \) has bounded Lipshitz boundary we have that \( C^\infty_c(\mathbb{R}^N) \) is dense in \( H^s(\Omega) \) [9]. By density we obtain the desired inequality. \( \square \)

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\(^3\)If \( \Omega \) has the "uniform" cone property we can always put in the intersection an infinite cone and use the integral on that cone.
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