Single-spin entanglement

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The date of receipt and acceptance will be inserted by the editor

Abstract We show that the operators and the quadrupole and Zeeman Hamiltonians for a spin $\frac{3}{2}$ can be represented in terms for a system of two coupling fictitious spins $\frac{1}{2}$ using the Kronecker product of Pauli matrices. Particularly, the quadrupole Hamiltonian which describes the interaction of the nuclear quadrupole moment with an electric field gradient is represented as the Hamiltonian of Ising model in a transverse selective magnetic field. The Zeeman Hamiltonian, which describes interaction of the nuclear spin with the external magnetic field, can be considered as the Hamiltonian of the Heisenberg model in a selective magnetic field. The total Hamiltonian can be interpreted as the Hamiltonian of 3D Heisenberg model in an inhomogeneous magnetic field applied along the $x$-axis. The representation of a single spin $\frac{3}{2}$ as a two-spin $\frac{1}{2}$ system allows us to study entanglement in the spin system. One of the features of the fictitious spin system is that, in both the pure and the mixed states, the concurrence tends to 0.5 with increase of applied magnetic field. The representation of a spin $\frac{3}{2}$ as a system of two coupling fictitious spins $\frac{1}{2}$ and possibility of formation of the entangled states in this system open a way to the application of a single spin $\frac{3}{2}$ in quantum computation.

Key words nuclear quadrupole interaction, Zeeman interaction, spin $\frac{3}{2}$, Ising model, Heisenberg model, fictitious spin $\frac{1}{2}$, entanglement

1 Introduction

Entanglement is a quantum mechanical phenomenon in which the quantum systems must be considered with reference to each other even they are spatially separated [1][2][3][4]. The paradoxical behaviour of spin particular
states, namely entangled state, has been primarily pointed out by Einstein, Podolsky and Rosen [5]. Today entanglement, as a long range quantum correlation between two or more quantum systems, is considered as a well-established concept in modern physics [6,7,8].

The unique properties of quantum entanglement and their important role in modern physics have stimulated intensive investigation of various quantum systems and search for measures and witnesses of entanglement [6,7,8]. In the last decade, the quantum entanglement has received much attention in studies involving quantum computing [9], quantum communication [10], and quantum metrology [11,12].

Since the first studies on quantum entanglement [13], the models consisting of spins $\frac{1}{2}$ have been intensively used as paradigms to describe a wide range of many-body entangled systems. It is very convenient to use spin models because each two-level system can be associated with a spin $\frac{1}{2}$ placed in a static magnetic field [14]. Many phenomena and systems in quantum physics can be described applying spin operator formalism because: (i) spin systems have a clear physical picture, (ii) they can be controlled by a resonance radiofrequency field, (iii) properties of the systems can be easily measured, for example, nuclear magnetization by the nuclear magnetic resonance (NMR) technique, and (iv) the spin $\frac{1}{2}$ systems can be relatively easily described theoretically, since spin $\frac{1}{2}$ operators are defined by simple operational transpositions and commutation rules.

To simplify the description of systems consisting of spins greater than $\frac{1}{2}$, various representations with projection operators [15,16] have been employed. In particular, it was shown that a nuclear spin $\frac{3}{2}$ can be represented by two spin subsystems which are described using the blocks consisting of Pauli spin $\frac{1}{2}$ matrices $2 \times 2$ [17,18,19]. An alternative way of using fictitious spins $\frac{1}{2}$ for describing a spin $\frac{3}{2}$ was proposed in [20,21]. The multiple-quantum spin dynamics in systems with spin 1 was studied using a fictitious spin of $\frac{1}{2}$, forming the SU(3) group [22]. However, all these attempts were limited to model where the spin operators and Hamiltonians for spin greater than $\frac{1}{2}$ were presented as a single fictitious spin [22] or as a system consisting of several non-coupled spins of $\frac{1}{2}$ [19]. Therefore, such an approach does not reflect realistic physics situation and processes and allow a consistent analysis of quantum phenomena, such as quantum entanglement, in the systems with a spin larger than $\frac{1}{2}$.

The same approach was used to implement the quantum gate using two qubits which are formed on the basis of a single quantum particle with spin $\frac{3}{2}$ [23,24] by applying technique of the nuclear magnetic resonance (NMR) with quadrupole splitting [23,24] and pure (without external magnetic fields) nuclear quadrupole resonance (NQR) [25,26,27,28]. This idea was confirmed in NMR experiments [24,28].

Because entanglement is an essential resource in current experimental implementations for quantum information processing, it will be very useful to study the conditions required to entangle qubits based on the states of a single spin $\frac{3}{2}$. Recently the conditions for quantum states of nuclei possessing
quadrupolar moment to be entangled were studied with presented a spin \( \frac{3}{2} \) as a set of two qubits, which is isomorphic to a qubit system consisting of two coupled spins \( \frac{1}{2} \) in our works \[27,29\]. It was obtained that entanglement can be achieved by applying an external magnetic field to spin- \( \frac{3}{2} \) nuclei in the electric field gradient (EFG) generated by charges in their surroundings. However, in our previous works the analysis of entanglement has not been based on explicit and confirmed representation of a spin \( \frac{3}{2} \) as a system of two spins \( \frac{1}{2} \). Such consideration, as shown below, does not affect the results obtained for quantum entanglements. One of the purposes of the present paper is elimination of the disadvantage and to develop a way based on explicit introduction of fictitious spins. This approach allows us to make clear the interaction between the fictitious spins and their interaction with a magnetic field, as well as possible their selective control. We consider a nucleus with spin \( \frac{1}{2} \) in an inhomogeneous electric field and an external magnetic field and develop the description of states of spin \( \frac{3}{2} \) by identifying it with a system of two spins \( \frac{1}{2} \) using the Kronecker product of the Pauli matrices. Thus, the particle possessing spin \( \frac{3}{2} \) can be considered as consisting of two parts and we study the entanglement of these parts.

The paper structure is the following. In the next section we explicitly show that the spin operators and the quadrupolar and Zeeman Hamiltonians for a spin \( \frac{3}{2} \) can be represented in the operator terms for a system of two coupling fictitious spins \( \frac{1}{2} \). In Section III, entanglement for the system of two fictitious spins in the pure and mixed states is studied. In the last section we conclude and discuss our results.

2 Decomposition of the Hamiltonian for spin \( \frac{3}{2} \) on the basis of the Pauli matrices

In a crystalline solid, the electric quadrupole moment, \( eQ \), of a nucleus possessing spin \( \frac{3}{2} \), interacts with the gradient of the electric field, \( \frac{\partial^2 V}{\partial x_i \partial x_j} \) \((x_i, x_j = x, y, z)\) generated by or the surrounding electrons or external charges of other nuclei. This interaction results in splitting of the energy levels which are separated by distances proportional to the quadrupole coupling constant \( \frac{e^2 Q}{h} \), where \( eQ = \frac{\partial^2 V}{\partial x_i \partial x_j} \) \( V \) is the potential of the electric field and \( e \) is the proton charge \[18,30\]. In the principal axis frame (PAF) with the \( z\)- and \( x\)-axes directed along the maximum and minimum of the electric field gradient (EFG), respectively, \( |V_{zz}| \geq |V_{yy}| \geq |V_{xx}| \), the EFG symmetric tensor is reduced to a diagonal form. The quadrupolar Hamiltonian, \( H_Q \), in the PAF takes the form (we used units where \( \hbar = 1 \)) \[18,30\]

\[
H_Q = \omega_Q \left[ 3I_z^2 - I^2 + \eta \left( I_x^2 - I_y^2 \right) \right],
\]

with the quadrupole frequency \( \omega_Q = \frac{e^2 Qn}{4I(2I-1)} \) and the asymmetry parameter \( \eta \) is defined as

\[
\eta = \frac{V_{yy} - V_{xx}}{2V_{zz}},
\]
and may vary between 0 and 1. $I_i$ ($i = x, y, z$) are the projections of the spin angular momentum operator $I$ on the $x$-, $y$-, and $z$-axes, respectively.

In the presence of an applied magnetic field $H_0$ directed along the $x$-axis of the PAF the Hamiltonian $\mathcal{H}$ of a nuclear spin can be written:

$$\mathcal{H} = \mathcal{H}_x + \mathcal{H}_Q,$$

where the Zeeman Hamiltonian $\mathcal{H}_x$ describes interaction of the nuclear spin with the magnetic field

$$\mathcal{H}_x = -\gamma H_0 I_x,$$

where $\gamma$ is the nuclear gyromagnetic ratio.

Any operator of a single spin $\frac{1}{2}$ can be presented as a combination of four $2 \times 2$ Hermitian matrices

$$\{\sigma_0, \sigma_x, \sigma_y, \sigma_z\},$$

where $\sigma_0 = \hat{1}$ is the identity operator and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli spin operators. This set of matrices forms an orthonormal basis, meaning that the trace of the product of any two different elements vanishes, while the trace of square of each element equals 2.

By analogy, any $4 \times 4$ matrix may be parametrized as a superposition of 16 direct products of four Hermitian matrices (5). The set of 16 operators

$$\sigma_i \otimes \sigma_j \quad \text{with } i, j = 0, x, y, z,$$

is orthonormal and complete for a spin $\frac{3}{2}$ in the same sense as a set of four Hermitian matrices (5) for a single spin $\frac{1}{2}$: the trace of the square of each operator equals 4 and the trace of the product of any two different matrices vanishes. As an example, three projections $I_i$ ($i = x, y, z$) of the spin angular momentum operator $I$ and the unit $4 \times 4$ matrix, $E$, can be presented by using set (6) in the following form

$$I_x = \frac{\sqrt{3}}{2} \sigma_0 \otimes \sigma_x + \frac{1}{2} \sigma_x \otimes \sigma_x + \frac{1}{2} \sigma_y \otimes \sigma_y,$$

$$I_y = \frac{\sqrt{3}}{2} \sigma_0 \otimes \sigma_y - \frac{1}{2} \sigma_x \otimes \sigma_y + \frac{1}{2} \sigma_y \otimes \sigma_x,$$

$$I_z = \sigma_z \otimes \sigma_0 + \frac{1}{2} \sigma_0 \otimes \sigma_z,$$

$$E = \sigma_0 \otimes \sigma_0.$$

Using (7), the quadrupolar Hamiltonian, $\mathcal{H}_Q$, and the Zeeman Hamiltonian, $\mathcal{H}_X$, can be rewritten in terms of the Pauli operators as

$$\mathcal{H}_Q = \omega Q \left(3\sigma_z \otimes \sigma_z + \frac{\sqrt{3}}{2} \eta \sigma_x \otimes \sigma_0\right),$$

(8)
\[ H_x = \omega_0 \left( \frac{\sqrt{3}}{2} \sigma_0 \otimes \sigma_x + \frac{1}{2} \sigma_x \otimes \sigma_x + \frac{1}{2} \sigma_y \otimes \sigma_y \right). \] (9)

Thus, the projections of angular momentum operator and the quadrupolar and Zeeman Hamiltonians for a spin \( \frac{1}{2} \) are represented in the operator terms for a system of two coupling spins \( \frac{1}{2} \).

The quadrupolar Hamiltonian (8) which describes the interaction of the nuclear quadrupole moment with EFG represents the Hamiltonian of the Ising model in a transverse selective magnetic field parallel to the \( x \)-axis. The constant of the spin interaction is \( 3\omega_Q \) and the strength of the magnetic field is \( \frac{\sqrt{3}}{2} \eta \omega_Q \). At \( \eta = 0 \) Hamiltonian (8) is reduced to the usual Ising Hamiltonian. Note, that even in the case with \( \eta = 0 \), Hamiltonian (8) cannot be represented as a combination of the \( \sigma_z \otimes \sigma_0 \) and \( \sigma_0 \otimes \sigma_z \). Therefore in the general case a spin \( \frac{3}{2} \) cannot be considered as a system of two non-coupled spins \( \frac{1}{2} \).

Hamiltonian (9), which describes the interaction of the nuclear spin \( \frac{3}{2} \) with the external magnetic field, can be considered as the Hamiltonian of a \( XY \) Heisenberg model in a selective magnetic field along the \( x \)-axis. The total Hamiltonian (3) can be interpreted as the Hamiltonian of 3D Heisenberg model with an inhomogeneous magnetic field applied along the \( x \)-axis

\[ H_H = J_x \sigma_x \otimes \sigma_x + J_y \sigma_y \otimes \sigma_y + J_z \sigma_z \otimes \sigma_z + h_{01} \sigma_x \otimes \sigma_0 + h_{02} \sigma_0 \otimes \sigma_x, \] (10)

where \( J_x = J_y = \frac{1}{2} \omega_0, J_z = 3\omega_Q, h_{01} = \frac{\sqrt{3}}{2} \eta \omega_Q \) and \( h_{02} = \frac{\sqrt{3}}{2} \omega_0 \). In contrast to the usual Heisenberg model, the constants of the spin interaction \( J_x \) and \( J_y \) depend on the external field, while the constant \( J_z \) is determined by the quadrupole interaction. The magnetic field acting on the first fictitious spin \( \frac{1}{2} \) does not depend on applied field and is determined only by the quadrupole interaction unlike the magnetic field acting the second spin depends on the applied field only.

Therefore, Hamiltonian (3) for a spin \( \frac{3}{2} \) is represented as the Hamiltonian (10) which describes the system of two fictitious coupling spins \( \frac{1}{2} \) in an inhomogeneous magnetic field. This reformulation allows us interpreted the obtained results as a Hamiltonian of system of two coupled fictitious spins \( \frac{1}{2} \). The coupled constants depend on the external field, which allows us to control the strength of the coupling between the fictitious spins. The control of the strength of the interaction between the spins makes it possible, on the one hand, to reduce the execution time of quantum gates in the implementation of logic gates, and on the other hand, to regulate the decoherence processes in such a spin system.

The energy levels of the spin system with Hamiltonian (8) are degenerate, but the degeneracy is removed in the presence of an applied magnetic field directed along the \( x \)-axis (9) and results in differences in the resonance frequencies of the fictitious spins: \( \Omega_1 = \sqrt{3} \eta \omega_Q \) for the first fictitious spin and \( \Omega_2 = \sqrt{3} \omega_0 \) for the second fictitious spin. Therefore,
an efficient way to manipulate the fictitious spins is to irradiate the spin system with selective radio frequency fields directed along the $z$-axis. The Hamiltonian parts describing the acting on the first and the second spins are $H_{r.f}^{(1)} = \gamma H_1 I_z \cos \Omega_1 t$ and $H_{r.f}^{(2)} = \gamma H_2 I_z \cos \Omega_2 t$, respectively. Here $H_1$ and $H_2$ are the strengths of the first and second radio frequency fields.

3 Entanglement in a system of fictitious spins

3.1 Pure state

The representation of a spin $\frac{3}{2}$ as two fictitious coupling spins $\frac{1}{2}$ allows us to investigate entanglement using the methods developed for spin $\frac{1}{2}$ systems [3,4]. The energy levels $E_m$ of Hamiltonian (3) are determined by a solution of $H |\Psi_m\rangle = E_m |\Psi_m\rangle$, where $|\Psi_m\rangle$ are eigenfunctions, $m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.

The energy levels are

$$E_{\pm \frac{3}{2}} = \frac{1}{2} \left( -\omega_0 \mp \sqrt{4\omega_0^2 - 6\omega_0\omega_Q (2 - \eta) + \omega_Q^2 (12 + \eta^2)} \right),$$

$$E_{\pm \frac{1}{2}} = \frac{1}{2} \left( \omega_0 \mp \sqrt{4\omega_0^2 + 6\omega_0\omega_Q (2 - \eta) + \omega_Q^2 (12 + \eta^2)} \right) \quad (11)$$

Thus, for a spin $\frac{3}{2}$ in an external field, $\omega_0 \neq 0$, the states are non-degenerate. The energy level of $E_{+ \frac{3}{2}}$ corresponds to the ground state described by the wave function

$$|\Phi_{\frac{3}{2}}\rangle = \begin{bmatrix} a_- \\ b_+ \\ b_- \\ a_+ \end{bmatrix}, \quad (12)$$

where

$$a_\pm = \pm \frac{1}{d} \sqrt{3} (\alpha - \eta),$$

$$b_\pm = \pm \frac{1}{d} \left( 6 + \alpha + \sqrt{36 + 4\alpha (3 + \alpha) - 6\alpha\eta + 3\eta^2} \right),$$

$$d = \sqrt{6 (\alpha - \eta)^2 + 2 \left( 6 + \alpha + \sqrt{36 + 4\alpha (3 + \alpha) - 6\alpha\eta + 3\eta^2} \right)^2}.$$ 

Here $\alpha = \frac{\omega_0}{\omega_Q}$ is the normalized external magnetic field.

To apply the methods of entanglement investigation, developed for spin-$\frac{1}{2}$ systems, we first map the Hilbert space for a spin $\frac{3}{2}$, which is four-dimensional, onto the Hilbert space for two fictitious spins $\frac{1}{2}$ according to Eqs. (8) - (10): $|\Phi_{\frac{3}{2}}\rangle = |\psi\rangle$. The most general wave function represented in terms of the standard basis for two fictitious spins has the form [32]:

$$|\psi\rangle = a_- |00\rangle + b_+ |01\rangle + b_- |10\rangle + a_+ |11\rangle \quad (13)$$
Concurrence $C$, a measure of entanglement, is determined by the expression \[ C = 2|a_+a_- - b_+b_-|. \] (14)

Applying formulas (14) to wave function (12), we determine the concurrence in the ground state of 3-dimensional Heisenberg model (10) which is isomorphic to the ground state of a single nuclear spin $\frac{3}{2}$ in an inhomogeneous magnetic and electric fields

\[ C = \frac{6 + \alpha}{\sqrt{36 + 4\alpha(3 + \alpha) - 6\alpha\eta + 3\eta^2}}. \] (15)

Dependence of concurrence $C$ on an external magnetic field $\alpha$ and asymmetry parameter $\eta$ is shown in Fig. 1. The concurrence decreases with an increase of the external field and slowly depends on the asymmetry parameter. At $\eta = 0$ and low magnetic field ($\alpha \ll 1$) the maximum concurrence is 1; with the increase of the asymmetry parameter the maximum is shifted to higher magnetic field and it is observed at $\alpha \simeq 1$ if $\eta = 1$. In a high magnetic field $\alpha >> 1$, $C = 0.5$. This differs from the results of the Ising and Heisenberg models [34] which predict drop of the concurrence to zero at high fields.

In a high magnetic field $\alpha >> 1$, $\omega_0 = 0$. This differs from the results of the Ising and Heisenberg models [34] which predict drop of the concurrence to zero at high fields. The difference can be explained by the fact that, in the considered system, the external field increases coupling of the fictitious spins and acts only one spin (see (10)); the spins do not become completely aligned along the field direction. However, Eq. (15) is not valid for strictly zero magnetic field, $\omega_0 = 0$, in which limit it gives the concurrence of $\left(\frac{\omega_0^2}{k_B} + 1\right)^{-\frac{1}{2}}$. At precisely $\omega_0 = 0$, pure NQR, no entanglement is present (the eigenstates are degenerate as for the usual Ising Hamiltonian without any magnetic field, where entanglement is absent [35]). Similarly to the Ising model, the entanglement jumps from zero (at $\omega_0 = 0$) to a finite value even for an infinitesimal increase of a magnetic field, indicating the quantum phase transition. Note, that when the external magnetic field is applied along direction the $z$-axis, the entanglement between spins is absent [35,37].

### 3.2 Mixed state

In real experiments temperature is finite, and a spin-$\frac{3}{2}$ system is in a mixed quantum state. The system described by Hamiltonian (10) in the thermodynamic equilibrium is characterized by the density matrix

\[ \rho = Z^{-1} \exp \left( -\frac{\hat{H}_H}{k_BT} \right). \] (16)

where $T$ is the spin temperature, $k_B$ is the Boltzmann constant, $Z = Tr \left[ \exp \left( -\frac{\hat{H}_H}{k_BT} \right) \right]$ is the partition function.
To quantify the entanglement of the system of two fictitious spins, we will use the concurrence $C_T$ defined by the following expression [38]:

$$C_T = \max \left\{ 0, \lambda_1 - \sum_{j=2}^{4} \lambda_j \right\}$$

(17)

where $\lambda_1 = \max \{ \lambda_j \}$ and $\lambda_j$ ($j = 1, 2, 3, 4$) are the square roots of the eigenvalues of the matrix

$$R = \rho (\sigma_y \otimes \sigma_y) \bar{\rho} (\sigma_y \otimes \sigma_y)$$

(18)

where $\bar{\rho}$ is the complex conjugation of the density matrix (16).

Our calculation shows that the concurrence slowly depends on the asymmetry parameter (Fig. 2). Fig. 3 shows dependences of the concurrence $C_T$ in the mixed state on the normalized inverse temperature, $\beta = \frac{e^{\frac{\epsilon Q(q - \eta)}{2J(2I - 1)kB T}}}{4(2I - 1)kB T}$, and the normalized external magnetic field $\alpha$ at $\eta = 0.14$. The system of fictitious spins is in the separable state without applying external magnetic field ($\omega_0 = 0$) at any temperature. At applying sufficiently high magnetic field the entangled state is observed; decrease in the field leads to a sudden disappearance of the entangled state. The dependence of the critical inverse temperature $\beta_C$, which segregates the separable and entangled states, on the magnetic field is presented in Fig. 4. At low magnetic fields $\alpha < 0.1$ the critical temperature $T_C^{-1}/\beta_C$ sharply decreases with an increase of the field. At high magnetic fields $\alpha >> 1$ the dependence of the critical inverse temperature $\beta_C$ on the magnetic field is well approximated by $\alpha \beta_C = 0.85$. The concurrence monotonically grows with an increase in $\beta$ above the critical value (Fig. 5) while the dependence of the concurrence on magnetic field possesses the maximum (Fig. 6). The maximum grows and moves to area of lower magnetic field with an increase in $\beta$. At high fields $\alpha >> 1$, the concurrence tends to 0.5. The limit is independent of temperature and equals to the concurrence limit for the system in the pure state at high fields.

4 Discussion and conclusions

According to the quantum mechanics rules, the mathematical description of a system consisting of two particles is realized using the Kronecker product of the operators of individual particles. We have explicitly shown that the spin operators and the quadrupolar and Zeeman Hamiltonians for a spin $\frac{3}{2}$ can be represented in the operator terms for a system of two coupling fictitious spins $\frac{1}{2}$ using the Kronecker product of the Pauli matrices. Thus, the particle possessing spin $\frac{3}{2}$ can be considered as consisting of two parts and we study the entanglement of these parts.

The representation of a single spin $\frac{3}{2}$ as a system of two fictitious spins $\frac{1}{2}$ allowed us to study entanglement in the spin system. One of the features of the fictitious spin system is that in both the pure and the mixed states the concurrence tends to 0.5 in a high magnetic field $\alpha >> 1$. 
The calculation for $^{63}Cu$ in the five-coordinated copper ion site of $YBa_2Cu_3O_{7-\delta}$ at $\alpha = 1$, $\eta = 0.14$ and $\epsilon_{Qzz} = 62.8$ MHz \cite{30}, gives that the concurrence appears at $\beta = 0.24$ (Fig. 5). This $\beta$ value corresponds to temperature $T \approx 2$ mK. It has been shown \cite{40,41,36,43,39} that, for the XY and dipolar coupling spin-$\frac{1}{2}$ systems entanglement appears at very low temperatures $T \sim 0.3 \div 0.5$ $\mu K$. This value is four orders smaller than the value estimated by us for a quadrupole system.

The representation of a spin $\frac{3}{2}$ as a system of two coupling fictitious spins $\frac{1}{2}$ and possibility of formation of the entangled states in quadrupole systems open a way to the application of a single spin $\frac{3}{2}$ in quantum computation.

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Caption figures

Fig. 1 The dependence of concurrence $C$ on the normalized external magnetic field $\alpha$ and $\eta$ (pure state).

Fig. 2 Concurrence $C_T$ in the mixed state as a function of asymmetry parameter $\eta$ at $\alpha =0.5$ (a): $\beta =1$ (solid black); $\beta =2$ (green dashed); $\beta =3$ (blue dotted); $\beta =4$ (red dot-dashed) and at $\beta =2$ (b): $\alpha =0.5$ (solid black); $\alpha =1$ (green dashed); $\alpha =2$ (blue dotted); $\alpha =3$ (red dot-dashed).

Fig. 3 The dependence of concurrence $C_T$ in the mixed state on plane $\beta$ and $\alpha$ at $\eta=0.14$.

Fig. 4 The phase diagram. The line presents boundary between the entangled and separated states in the plane $\beta C - \alpha$.

Fig. 5 Concurrence $C_T$ in the mixed state as a function of inverse temperature $\beta$ at $\eta=0.14$: $\alpha =1$ (solid black); $\alpha =2$ (green dashed); $\alpha =3$ (blue dotted); $\alpha =4$ (red dot-dashed).

Fig. 6 Concurrence $C_T$ in the mixed state as a function of magnetic field $\alpha$ at $\eta=0.14$: $\beta =1$ (solid black); $\beta =2$ (green dashed); $\beta =3$ (blue dotted); $\beta =4$ (red dot-dashed).
Entagled state
