INFINITE-STEP STATIONARITY OF ROTOR WALK AND THE WIRED SPANNING FOREST

SWEE HONG CHAN

Abstract. We study rotor walk, a deterministic counterpart of the simple random walk, on infinite transient graphs. We show that the final rotor configuration of the rotor walk follows the law of the wired uniform spanning forest oriented toward infinity (OWUSF) measure when the initial rotor configuration is sampled from OWUSF. This result holds for all graphs for which each tree in the wired spanning forest has one single end almost surely. This answers a question posed in a previous work of the author (Chan 2018).

1. Introduction

In a rotor walk on a graph \( G := (V, E) \), each vertex of \( G \) has an arrow called a rotor that points toward a neighbor of the vertex; all of the rotors together constitute the rotor configuration of the walk. At each discrete time step, the walker changes the rotor of its current location by following a prescribed periodic sequence. After that, the walker moves to the location to which the new rotor is pointing.

Continuing on the previous work of the author [Cha18b], our initial rotor configuration will be sampled from the wired spanning forest oriented toward infinity measure: Let \( G \) be a simple connected graph that is locally finite and transient. Let \( Z_0 \supseteq Z_1 \supseteq \ldots \) be subsets of \( V \) such that each \( V \setminus Z_R \) is finite and \( \bigcap_{R \geq 0} Z_R = \emptyset \). Let \( G_R \) be obtained from \( G \) by identifying all vertices in \( Z_R \) to one new vertex \( z_R \), and let \( \mu_R \) be the uniform measure on spanning trees of \( G_R \) oriented toward \( z_R \). The wired spanning forest oriented toward infinity \( \overline{\text{WSF}} \) is the (unique) infinite volume limit of the measure. We will establish in this paper that this measure is an infinite-step stationary measure for rotor walk, in a manner to be made precise.

Date: October 1, 2019.

2010 Mathematics Subject Classification. 05C81, 82C20.

Key words and phrases. rotor walk, rotor-router, uniform spanning forest, wired spanning forest, stationary distribution, escape rate.

Department of Mathematics, University of California, Los Angeles. Partially supported by NSF grant DMS-1455272. Email: sweehong@math.ucla.edu,
Our starting point is the result of [Cha18b, Theorem 1.1] that, on average, the rotor walk visits each vertex at most as frequent as the simple random walk, i.e.,

\[ \mathbb{E}_{\rho \sim \text{WSF}}[u(\rho)(x)] \leq G(x) \quad \forall x \in V, \quad (1) \]

where \( u(\rho)(x) \) is the number of visits to \( x \) by the rotor walk with initial rotor configuration \( \rho \) sampled from \( \text{WSF} \), and \( G(x) \) is the expected number of visits to \( x \) by the simple random walk.

It follows from (1) that, when \( G \) is a transient graph, the aforementioned rotor walk visits each vertex only finitely many times a.s.. This implies that the sequence of rotor configurations \( (\rho_t)_{t \geq 0} \) at the \( t \)-th step of the walk converges pointwise (as \( t \to \infty \)) to a unique rotor configuration \( \sigma(\rho) \), which we refer to as the final rotor configuration of the rotor walk.

For finite graphs, it is a folk theorem that \( \sigma(\rho) \) has the same law as \( \rho \) when \( \rho \) is sampled from the uniform spanning tree measure (see [HLM+08, Lemma 3.11]). This naturally leads us to the following problem.

**Problem 1.1 (Infinite-step stationarity).** Let the initial rotor configuration be sampled from \( \text{WSF} \). Show that the final rotor configuration also follows the law of \( \text{WSF} \).

Somewhat surprisingly, the answer to Problem 1.1 can depend on the underlying graph \( G \). In [Cha18b], Problem 1.1 was answered positively for when \( G \) is the perfect \( b \)-ary tree \( (b \geq 2) \), but was answered negatively for when \( G \) is the perfect \( b \)-ary tree with an infinite path attached to its root. The case for the integer lattice \( \mathbb{Z}^d \) \( (d \geq 3) \) was posed as an open question in [Cha18b, Question 9.1].

In this paper we answer Problem 1.1 positively for a large family of graphs that include \( \mathbb{Z}^d \). An end in a tree is an equivalence classes of infinite paths under the relation where two paths are equal if they differ by only finitely many vertices.

**Theorem 1.2.** Let \( G \) be a simple connected graph that is locally finite and transient, and let \( \rho \) be sampled from \( \text{WSF} \). Suppose that

\[ \text{Every tree in } \text{WSF} \text{ has a single end a.s.} \quad (1\text{End}) \]

Then \( \sigma(\rho) \) follows the law of \( \text{WSF} \).

Transient graphs that satisfy (1End) include \( \mathbb{Z}^d \), vertex-transitive graphs, and graphs with reasonable isoperimetric profile (see [LMS08, LP16]). It is an open problem to determine if (1End) is a necessary condition for infinite-step stationarity.
Our proof of Theorem 1.2 uses techniques developed by Járai and Redig in [JR08] for studying recurrent sandpile. We study a new process where the rotor walk is now terminated upon hitting a fixed finite subset $W$ of $V$, and the initial rotor configuration is now sampled from the wired spanning forest $\overset{\rightarrow}{\text{WSF}}(W)$ oriented toward $W$. The addition of the termination rule is justified by [Cha18b, Theorem 6.2], and the modification of the initial rotor configuration is justified by [JR08, Lemma 7.5]. By coupling the original process with the new process, we show that Theorem 1.2 will follow from showing that the component of $W$ in $\overset{\rightarrow}{\text{WSF}}(W)$ is finite. The latter turns out to be a known consequence of (1End) [JR08, Proposition 7.11], and the proof is complete.

Here we present two interesting consequences of Theorem 1.2. The first consequence is a straight upgrade from (1), and can be derived as a direct corollary of Theorem 1.2 and [Cha18b, Theorem 1.1].

**Theorem 1.3.** Let $G$ be a simple connected graph that is locally finite, transient, and satisfies (1End). Then, for any $x \in V$,

$$
\mathbb{E}_{\rho \sim \overset{\rightarrow}{\text{WSF}}} [u(\rho)(x)] = \mathcal{G}(x).
$$

The second consequence is a formula for the escape rate of the rotor walk: Start the process with an initial environment $\rho$ and with $n$ particles initially located at a fixed vertex $a$. Each of these particles in turn performs rotor walk until it either returns to $a$ or escapes to infinity. Denote by $I(\rho, n)$ the number of particles that do not return to $a$. The escape rate is $\lim_{n \to \infty} I(\rho, n)/n$, if the limit exists.

The following is a direct corollary of Theorem 1.2 and [Cha18b, Theorem 1.5].

**Theorem 1.4.** Let $G$ be a simple connected graph that is locally finite, transient, and satisfies (1End). Then, for almost every $\rho$ sampled from $\overset{\rightarrow}{\text{WSF}},$

$$
\lim_{n \to \infty} \frac{I(\rho, n)}{n} = \alpha_G,
$$

where $\alpha_G$ is the probability that the simple random walk on $G$ never returns to the initial location. 

Theorem 1.2 is a straight upgrade from [Cha18b, Theorem 1.4], which derived the same conclusion but only for vertex-transitive graphs.

### 1.1. Related work

Rotor walk was first studied in [WLB96] as a model of mobile agents exploring a new territory, and in [PDDK96] as a model of self-organized criticality [BTW88]. It was rediscovered several times by researchers from different disciplines (e.g., [RSW98],...
We refer the reader to [HLM+08] for an excellent introduction and a detailed history of this subject.

The wired spanning forest was first constructed by Pemantle [Pem91] as an infinite volume limit of the uniform spanning tree of finite graphs. Other methods to construct the wired spanning forest include Wilson’s algorithm (see [Wil96, BLPS01]) that uses loop-erased random walks [Law80], and Aldous-Broder algorithm (see [Bro89, Ald90, Hut18]). We refer the reader to [LP16, Section 10] for a detailed discussion regarding this subject.

The phenomenon in Theorem 1.2 (where an object related to the wired spanning forest is invariant under the dynamics of the particle system) had previously been shown for sandpile [JR08] and (one-step) random walk with local memory [CGLL18]. These particle systems are notably all examples of abelian networks [BL16]. It is still unknown if all these results are consequences of a universal principle that belongs in the general framework of abelian networks.

Other rotor configurations for which the corresponding escape rate satisfy (2) had been constructed for trees by Angel and Holroyd in [AH11], for $\mathbb{Z}^d$ by He in [He14], and for all graphs by Chan in [Cha18a]. Note that the escape rate of a rotor walk with an arbitrary initial rotor configuration is at most equal to the escape rate of the simple random walk (see [HP10, Theorem 10]), and the former can be strictly smaller than the latter (see [AH12]).

Other aspects of rotor walk that had been studied in the literature include recurrence and transience [LL09, AH12, HMSH15], scaling limit [LP08, LP09], range [FLP16, HSH19, HS18], escape rate [FGLP14], and its performance in simulating the simple random walk [CDST06, CS06, DF09, HP10].

1.2. Outline of the paper. In Section 2 we review notations and definitions. In Section 3 we derive the technical lemmas that will be used to prove Theorem 1.2. In Section 4 we prove Theorem 1.2. In Section 5 we list some open problems.

2. Notations and definitions

Throughout this paper, we will denote by $G := (V, E)$ a connected simple (i.e., no loops or multiple edges) undirected graph that is locally finite (i.e., every vertex has finitely many edges). We will denote by $a$ the initial location of a rotor walk, which is a vertex of $G$. We will denote by $Z$ the sink of a rotor walk, which is a (possibly empty, possibly infinite) subset of $V$. 
2.1. **Rotor walk.** Each vertex \( x \in V \) is assigned a *local mechanism* \( \tau_x \), which is a bijection on the neighbors \( N(x) \) of \( x \). We assume that each local mechanism has one unique orbit (i.e., \( \{\tau_x^i(y) \mid i \geq 0\} = N(x) \) for any neighbor \( y \) of \( x \)). A *rotor configuration* of \( G \) is a function \( \rho : V \to V \) such that \( \rho(x) \) is a neighbor of \( x \) for any \( x \in V \).

Let \( a \) be a vertex of \( G \), let \( Z \) be a subset of \( V \), and let \( \rho \) be a rotor configuration of \( G \). The corresponding rotor walk \((X_t, \rho_t)_{t \geq 0} := (X_t(a, Z; \rho), \rho_t(a, Z; \rho))_{t \geq 0} \) is a sequence of vertices and rotor configurations defined recursively as follows.

Define \( X_0 := a \) and \( \rho_0 := \rho \); this indicates that \( a \) is the initial location of the walker, and \( \rho \) is the initial rotor configuration. At the \( t \)-th step of the walk, the rotor of the current location of the walker is incremented to point to the next vertex in the cyclic order specified by its local mechanism, and then the walker moves to the vertex specified by this new rotor, i.e.,

\[
\begin{align*}
\rho_{t+1}(x) &:= \begin{cases} 
\rho_t(x) & \text{if } x \neq X_t; \\
\tau_{X_t}(\rho_t(X_t)) & \text{if } x = X_t,
\end{cases} \\
X_{t+1} &:= \tau_{X_t}(\rho_t(X_t)).
\end{align*}
\]

The walk is immediately terminated if the walker reaches a vertex in the sink \( Z \). Note that this rule implies that the rotors of \( Z \) are inconsequential to the process. Therefore, we will not specify the value of these rotors (i.e., only \( \rho(x) \) for \( x \in V \setminus Z \) are given) when doing so will simplify the exposition.

The following lemma will be used in Section 3.3. An *oriented path* in \( \rho \) is a sequence of vertices \( x_0, \ldots, x_\ell \) such that \( \rho(x_i) = x_{i+1} \) for any \( i \in \{0, \ldots, \ell - 1\} \).

**Lemma 2.1.** For any \( t \geq 0 \) and any \( i \leq t \), there exists an oriented path in \( \rho_t \) that starts at \( X_i \) and ends at \( X_t \).

**Proof.** It suffices to prove that, for any \( t \geq 0 \) and any \( i < t \),

\[
\rho_t(X_i) \in \{X_{i+1}, X_{i+2}, \ldots, X_t\}. 
\]

We will prove (4) by induction on \( t \). First note that (4) is vacuously true for \( t = 0 \). Now suppose that \( t \geq 1 \); there are two possible scenarios:

- If \( X_i = X_{i-1} \), then it follows from (3) that \( \rho_t(X_i) = X_i \); or
- If \( X_i \neq X_{i-1} \), then

\[
\rho_t(X_i) = \rho_{t-1}(X_i) \in \{X_{i+1}, \ldots, X_{t-1}\},
\]

where the first equality is due to (3), and the second equality is due to the induction assumption.
In both cases we have that (4) is true, as desired. □

A rotor walk is transient if every vertex of $G$ is visited by the walker at most finitely many times, and is recurrent otherwise. One aspect of the rotor walk that we will study in this paper is the final rotor configuration of a transient walk, defined as follows.

**Definition 2.2 (Final rotor configuration).** The final rotor configuration $\sigma(\rho) := \sigma(a, \pi; \rho)$ of a transient rotor walk is given by

$$\sigma(\rho)(x) := \lim_{t \to \infty} \rho_t(x) \quad \forall x \in V.$$  

Note that $\sigma(\rho)$ is well defined as the sequence $(\rho_t(x))_{t\geq 0}$ is eventually constant by the assumption that the walk is transient.

2.2. **Oriented wired spanning forest.** All initial rotor configurations in this paper will be picked from spanning forests oriented toward $Z \subseteq V$, defined as follows.

**Definition 2.3 (Oriented spanning forest).** A $Z$-oriented spanning forest of $G$ is an oriented subgraph $F$ of $G$ such that

(i) Every vertex in $Z$ has outdegree 0 in $F$;
(ii) Every vertex in $G \setminus Z$ has outdegree 1 in $F$; and
(iii) $F$ contains no oriented cycles. △

Note that each $Z$-oriented spanning forest $F$ corresponds to a rotor configuration $\rho_F$, where for every $x \in V \setminus Z$, the state $\rho_F(x)$ is the out-neighbor of $x$ in $F$ (the state $\rho_F(x)$ for $x \in Z$ is inconsequential, as previously remarked). Due to this correspondence, we will treat $\rho$ both as a rotor configuration and as an oriented subgraph of $G$ interchangeably throughout this paper.

We denote by $\mathcal{SF}(Z)$ the set of $Z$-oriented spanning forests of $G$.

**Definition 2.4 (Oriented uniform spanning forest).** Suppose that $Z$ is a subset of $G$ such that $V \setminus Z$ is finite. The $Z$-oriented uniform spanning forest, denoted by $\overline{USF}(Z)$, is the uniform probability measure on $Z$-oriented spanning forests of $G$. △

Note that $\overline{USF}(Z)$ is well defined as there are only finitely many $Z$-oriented spanning forests by the assumption on $Z$.

We now describe how to pick $Z$-oriented forests uniformly at random when $G \setminus Z$ is not necessarily finite by using the exhaustion method.

Suppose that $Z$ is a finite subset of $V$. Throughout this paper, $Z_0 \supseteq Z_1 \supseteq \ldots$ will be a decreasing exhaustion of $Z$, which is an infinite
sequence of decreasing subsets of $V$ such that

the set $V \setminus Z_R$ is finite for all $R \geq 0$, and $\bigcap_{R \geq 0} Z_R = Z$. (DecEx)

**Definition 2.5 (Oriented wired uniform spanning forest).** Suppose that $G$ is a transient graph and $Z$ is a finite subset of $V$. The **$Z$-oriented wired uniform spanning forest** $\mathbf{WSF}(Z)$ is the (unique) probability measure on oriented subgraphs of $G$ such that, for any finite subset $B$ of oriented edges of $G$,

$$\mathbf{WSF}(Z)[B \subseteq F] = \lim_{R \to \infty} \mathbf{USF}(Z_R)[B \subseteq F_R],$$

where $F$ is an oriented subgraph of $G$ sampled from $\mathbf{WSF}(Z)$, and $F_R$ is a oriented subgraph of $G$ sampled from $\mathbf{USF}(Z_R)$. △

When the sink $Z$ is the empty set, we will omit $Z$ from the notation and write $\mathbf{WSF}$ instead.

The limit in (5) exists and does not depend on the choice of the decreasing exhaustion of $Z$ if $G$ is transient. See [BLPS01, Theorem 5.1] for a proof when $Z = \emptyset$; the general case follows from a similar argument. Note that, if $G$ is recurrent, the limit in (5) can depend on the choice of the decreasing exhaustion. (However, only the orientation of $F$ is influenced by this choice; the underlying graph of $F$ remains unchanged!)

### 3. Proof of the Technical Lemmas

In this section, we derive three technical lemmas that will be used in our proof of Theorem 1.2.

#### 3.1. The first technical lemma.**

We need the following notations to state this lemma.

**Definition 3.1 (Odometer).** We denote by $u(a, Z; \rho)(x)$ the number of visits to $x \in V$ by the rotor walk with initial location $a$, sink $Z$, and initial rotor configuration $\rho$, i.e.,

$$u(a, Z; \rho)(x) := |\{t \geq 0 \mid X_t(a, Z; \rho) = x\}|.$$

For any $K \geq 0$ and any finite subset $W$ of $V$, we write

$$\mathcal{C}_{K,W}(a, Z) := \{ \rho \mid u(a, Z; \rho)(W) < K \},$$

the set of rotor configurations for which the corresponding rotor walk visits $W$ strictly less than $K$ times.

Recall the definition of decreasing exhaustion $(Z_R)_{R \geq 0}$ from Section 2.2. We now present the main lemma of this subsection, which gives a probabilistic upper bound for the odometer.
Lemma 3.2. Suppose that $G$ is a transient graph. Then, for any $\varepsilon > 0$ and any finite $W \subseteq V$, there exists $K := K(\varepsilon, G, a, W, Z)$ such that, for any $R \geq 0$,
\[
P[\rho_R \notin \mathcal{C}_{K,W}(a, Z_R)] \leq \frac{\varepsilon}{2},
\]
where $\rho_R$ is sampled from $\overline{\text{WSF}}(Z_R)$.

We will derive Lemma 3.2 as a consequence of the following lemma from [Cha18b]. We denote by $\mathcal{G}(a, Z)(x)$ the expected number of visits to $x$ by the simple random walk that starts at $x$ and terminated upon hitting $Z$.

Lemma 3.3 ([Cha18b, Proposition 3.4]). Suppose that $Z$ is a nonempty subset of $V$ such that $V \setminus Z$ is finite. Then, for any $x \in V$,
\[
\mathbb{E}[u(a, Z; \rho)(x)] = \mathcal{G}(a, Z)(x),
\]
where $\rho$ is sampled from $\overline{\text{WSF}}(Z)$.

Proof of Lemma 3.2. We have for any $K \geq 0$ that
\[
P[u(a, Z_R; \rho_R)(W) \geq K] \leq \frac{\mathbb{E}[u(a, Z_R; \rho_R)(W)]}{K} = \frac{\mathcal{G}(a, Z_R)(W)}{K}, \tag{6}
\]
where the inequality is due to Markov’s inequality, and the equality is due to Lemma 3.3. Now note that $\mathcal{G}(a, Z_R)(W)$ increases to $\mathcal{G}(a, Z)(W)$ as $R \to \infty$ by (DecEx). The latter is a finite number as $G$ is transient.

Now choose $K := \frac{2\mathcal{G}(a, Z)(W)}{\varepsilon}$. Substituting this value of $K$ to (6),
\[
P[u(a, Z_R; \rho_R)(W) \geq K] \leq \frac{\varepsilon \mathcal{G}(a, Z_R)(W)}{2 \mathcal{G}(a, Z)(W)} \leq \frac{\varepsilon}{2}.
\]
This proves the claim. \qed

3.2. The second technical lemma. We need the following notations to state this lemma. We denote by $d_G(\cdot, \cdot)$ the graph distance between two vertices in $G$.

Definition 3.4. For any $r \geq 0$ and any $W \subseteq V$, we denote by $\mathcal{E}_{r,W}(a, Z)$ the set of rotor configurations $\rho$ that satisfy
- there exists $t_1 \geq 0$ so that
  \[
d_G(W, X_{t_1}(a, Z; \rho)) \geq r; \quad \text{and}
\]
- there exists $t_2 > t_1$ so that
  \[
  X_{t_2}(a, Z; \rho) \in W.
  \]
That is to say, consider the rotor walk with initial location $a$, sink $Z$, and with initial rotor configuration $\rho$. Then $\rho$ is contained in $E_r(a, Z)$ if this rotor walk ends up visiting $W$ some time after visiting a point that is of (graph) distance $r$ from $W$. Understanding the event $E_{r,W}$ will be crucial in proving Theorem 1.2.

**Definition 3.5.** Suppose that $Z$ is a nonempty subset of $V$ such that $V \setminus Z$ is finite, $W$ is any subset of $V$, and $\overline{Z} := W \cup Z$. For any (not necessarily distinct) vertices $a_0, a_1, a_2, \ldots$ of $G$ and any rotor configuration $\rho$, we denote by $\xi_i := \xi_i(a_0, \ldots, a_{i-1}, \overline{Z}; \rho)$ the rotor configuration

$$
\xi_i := \begin{cases} 
\rho & \text{if } i = 0; \\
\sigma(a_{i-1}, \overline{Z}; \xi_{i-1}) & \text{if } i \geq 1,
\end{cases}
$$

(7)

where each $\xi_i$ is well defined as the corresponding rotor walk terminates in a finite time by the assumption on $Z$. △

Described in words, we let multiple walkers in turn perform a rotor walk that terminates upon hitting the enlarged sink $\overline{Z}$, and $\xi_i$ is the final rotor configuration after the $i$-th walker finishes its walk.

Recall that, for any subset $W$ of $V$, the neighbor set $N(W)$ of $W$ is the set of vertices of $G$ that are adjacent to a vertex in $W$. We now present the main lemma of this subsection.

**Lemma 3.6.** Suppose that $Z$ is a nonempty subset of $V$ such that $V \setminus Z$ is finite, $W$ is any subset of $V$, and $\overline{Z} := W \cup Z$. Suppose that $\rho$ is contained in $E_{r,W}(a, Z)$ for some $r \geq 0$. Then there exists $i \in \{0, 1, \ldots, u(a, Z; \rho) - 1\}$ and $a_1, \ldots, a_i \in N(W)$ such that $\xi_i(a_0, a_1, \ldots, a_{i-1}, \overline{Z}; \rho) \in E_{r,W}(a_i, \overline{Z})$.

Intuitively speaking, Lemma 3.6 allows us compare the original rotor walk to a new rotor walk with sink enlarged to include $W$. This will simplify the analysis of the event $E_{r,W}$ in Section 4.

**Proof of Lemma 3.6.** Let $k := u(a, Z; \rho)$, and let $t_1, t_2, \ldots, t_k$ be the time of visits to $W$ by the (original) rotor walk $\{(X_t(a, Z; \rho))_{t \geq 0}\}$, i.e.,

$$
t_i := \begin{cases} 
-1 & \text{if } i = 0; \\
\min\{ t > t_{i-1} \mid X_t(a, Z; \rho) \in W \} & \text{if } i \in \{1, \ldots, k\}.
\end{cases}
$$

We now make the following choice of $a_0, a_1, \ldots, a_k$:

$$
a_i = \begin{cases} 
a & \text{if } i = 0; \\
X_{t_{i+1}} & \text{if } i \in \{1, 2, \ldots, k\}.
\end{cases}
$$
Note that each \(a_1, a_2, \ldots, a_k\) are contained in \(N(W)\) by (3), as they are the vertices visited by the walker right after it reaches \(W\).

By making this choice, it now follows (from induction) that, for any \(i \in \{0, 1, \ldots, k-1\}\) and any \(s \in \{1, 2, \ldots, t_{i+1} - t_i\}\),
\[
X_s(a_i, \overline{Z}; \xi_i) = X_{t_i+s}(a, Z; \rho) \quad (8)
\]

Now suppose that \(\rho\) is contained in \(E_{r,W}(a, Z)\). This means that the original rotor walk \((X_t(a, Z; \rho))_{t \geq 0}\) ends up visiting \(W\) some time after visiting a point that is of distance \(r\) from \(W\). Suppose that this visit to \(W\) is the \(i+1\)-th visit to \(W\) (note that \(i < k\) as there are at most \(k\) visits). By (8), it follows that the same event (i.e., visiting \(W\) after visiting a point that is of distance \(r\) from \(W\)) also happens to the modified rotor walk \((X_s(a_i, \overline{Z}; \xi_i))_{s \geq 0}\), which is equivalent to \(\xi_i \in E_{r,W}(a_i, \overline{Z})\). This proves the lemma. \(\square\)

3.3. The third technical lemma. Throughout the rest of this paper, we will often require \(G\) to satisfy the assumption (1End), which we restate here for the convenience of the reader:

Every tree in \(\overrightarrow{WSF}(Z)\) has a single end a.s. \((1End)\)

We now state the main lemma of this subsection.

**Lemma 3.7.** Suppose that \(G\) is a transient graph for which (1End) holds, and \(Z\) is nonempty and finite. Then, for any \(\varepsilon > 0\), we have for any sufficiently large \(r := r(\varepsilon, G, Z)\) that
\[
\lim_{R \to \infty} \mathbb{P}[\rho_R \in E_{r,Z}(a, Z)] \leq \varepsilon,
\]
where \(\rho_R\) is sampled from \(\overrightarrow{WSF}(Z_R)\).

(Note that the choice of the constant \(r\) in Lemma 3.7 does not depend on the initial location \(a\)! Intuitively speaking, Lemma 3.7 says that \(E_{r,W}\) is a rare event with the right choice of the sink.

Our proof of Lemma 3.7 uses two technical lemmas from [JR08] and [HLM+08]; we restate them here for the convenience of the reader.

We denote by \(D_r\) the set of rotor configurations with an oriented path that starts at a point of distance \(r\) from \(Z\) and ends in \(Z\), i.e.,
\[
D_r := \{ \rho \mid \exists x \in V, \ell \geq 0 \text{ s.t. } d_G(x, Z) = r \text{ and } \rho^\ell(x) \in Z \}.
\]

**Lemma 3.8.** [JR08, Proposition 7.11] Suppose that \(G\) is a transient graph for which (1End) holds, and \(Z\) is nonempty and finite. Then, for any \(\varepsilon > 0\), we have for any sufficiently large \(r := r(\varepsilon, G, Z)\) that
\[
\lim_{R \to \infty} \mathbb{P}[\rho_R \in D_r] \leq \varepsilon,
\]
where $\rho_R$ is sampled from $\overline{\text{WSF}}(Z_R)$. □

Intuitively speaking, Lemma 3.8 says that a typical $Z_R$-oriented spanning forest does not have a very long oriented path that ends in $Z$. We remark that Lemma 3.8 was stated in [JR08] only for the case when $Z$ is a singleton, but the proof of the general case follows a similar argument.

**Lemma 3.9 ([HLM+08, Lemma 3.11]).** Suppose that $Z$ is a nonempty subset of $V$ such that $V \setminus Z$ is finite. If $\rho$ is sampled from $\overline{\text{WSF}}(Z)$, then $\sigma(a, Z; \rho)$ follows the law of $\overline{\text{WSF}}(Z)$. □

Note that the rotor walk in Lemma 3.9 is transient a.s. because of the assumption on $Z$, and therefore the corresponding final rotor configuration is well defined a.s..

**Proof of Lemma 3.7.** Suppose that $\rho$ is a rotor configuration such that the rotor walk $(X_t(a, Z_R; \rho))_{t \geq 0}$ visits $Z$ some time after visiting a point $x$ that is of distance $r$ from $Z$. By Lemma 2.1, there exists an oriented path in $\sigma(a, Z_R; \rho)$ that starts at $x$ and ends in $Z$. That is to say,

$$\rho \in \mathcal{E}_{r,Z}(a, Z_R) \implies \sigma(a, Z_R; \rho) \in \mathcal{D}_r. \quad (9)$$

Let $\rho_R$ be a rotor configuration sampled from $\overline{\text{WSF}}(Z_R)$. Then

$$\mathbb{P}[\rho_R \in \mathcal{E}_{r,Z}(a, Z_R)] \leq \mathbb{P}[\sigma(a, Z; \rho_R) \in \mathcal{D}_r] = \mathbb{P}[\rho_R \in \mathcal{D}_r], \quad (10)$$

where the inequality is due to (9) and the equality is due to Lemma 3.9. The lemma now follows from (10) and Lemma 3.8. □

**4. Proof of Theorem 1.2**

In this section we present a proof of Theorem 1.2. We restate Theorem 1.2 here for the convenience of the reader. (Note that the sink $Z$ in Theorem 1.2 is equal to the empty set $\emptyset$.)

**Theorem 1.2.** Suppose that $G$ is a transient graph for which (1End) holds, and suppose that $\rho$ is sampled from $\overline{\text{WSF}}(\emptyset)$. Then the final rotor configuration $\sigma(a, \emptyset; \rho)$ follows the law of $\overline{\text{WSF}}(\emptyset)$.

Note that the rotor walk in Theorem 1.2 is transient a.s. (see [Cha18b, Theorem 1.1]), and therefore the corresponding final rotor configuration is well defined a.s..

We now build toward the proof of Theorem 1.2. Our first ingredient is the following lemma from [JR08]. Recall the definition of $\mathcal{E}_{r,W}(a, Z)$ from Definition 3.4.
Lemma 4.1 ([Cha18b, Theorem 6.2]). Suppose that $G$ is a transient graph and $Z = \emptyset$. TFAE:

- The configuration $\sigma(a, \emptyset; \rho)$ follows the law of $\bar{\WSF}(\emptyset)$ when $\rho$ is sampled from $\bar{\WSF}(\emptyset)$;
- For any $\varepsilon > 0$ and any nonempty finite subset $W$ of $V$, we have for any sufficiently large $r := r(\varepsilon, G, a, W)$ that
  \[
  \lim_{R \to \infty} \mathbb{P}[\rho_R \in E_{r,W}(a, Z_R)] \leq \varepsilon,
  \]
  (11)
  where $\rho_R$ is sampled from $\bar{\WSF}(Z_R)$.

Lemma 4.1 reduces proving Theorem 1.2 to checking that the event $E_{r,W}$ is very unlikely to occur.

Our second ingredient is the following technical lemma from [JR08]. A set $B$ of rotor configurations is $Z$-invariant if, for any $\rho, \rho'$,

\[\rho(x) = \rho'(x) \forall x \in V \text{ and } \rho \in B \implies \rho' \in B.\]

Lemma 4.2 ([JR08, Lemma 7.5]). Suppose that $G$ is a transient graph, $Z = \emptyset$, and $\overline{Z}$ is a nonempty finite subset of $V$. Then there exists $c := c(G, \overline{Z}) \geq 0$ such that, for any $R \geq 0$,

\[\mathbb{P}[\rho_R \in B_R] \leq c \mathbb{P}[\overline{\rho_R} \in B_R],\]

where $\rho_R$ is sampled from $\bar{\WSF}(Z_R)$, $\overline{\rho_R}$ is sampled from $\bar{\WSF}(\overline{Z_R})$, and $B_R$ is any $Z_R$-invariant set.

Intuitively speaking, Lemma 4.2 lets us change the initial rotor configuration to one that will interact nicely with the event $E_{r,W}$. We remark that Lemma 4.2 was stated in [JR08] only for the case when $Z$ is a singleton, but the proof of the general case follows a similar argument.

Our third, fourth, and fifth ingredient are Lemma 3.2, Lemma 3.6, and Lemma 3.7, respectively. We are now ready to present our proof of Theorem 1.2.

Proof of Theorem 1.2. Let $Z = \emptyset$, let $\varepsilon > 0$, and let $W$ be an arbitrary nonempty finite subset of $V$. It suffices to check that (11) holds under the assumptions of Theorem 1.2.

Let $\rho_R$ be sampled from $\bar{\WSF}$. Let $K := K(\varepsilon, G, a, W)$ be as in Lemma 3.2. We have for any $r \geq 0$ that

\[
\begin{align*}
\mathbb{P}[\rho_R \in E_{r,W}(a, Z_R)] &\leq \mathbb{P}[\rho_R \notin C_{K,W}(a, Z_R)] + \mathbb{P}[\rho_R \in E_{r,W}(a, Z_R) \cap C_{K,W}(a, Z_R)] \\
&\leq \frac{\varepsilon}{2} + \mathbb{P}[\rho_R \in E_{r,W}(a, Z_R) \cap C_{K,W}(a, Z_R)],
\end{align*}
\]

(12)
where the second inequality is due to Lemma 3.2.

Write $\overline{Z}_R := W \cup Z_R$. We have by Lemma 3.6 that
\[
\mathbb{P}[\rho_R \in \mathcal{E}_{r,W}(a, \overline{Z}_R) \cap \mathcal{C}_{K,W}(a, \overline{Z}_R)] 
\leq \sum_{i=0}^{K-1} \sum_{a_1, \ldots, a_i \in N(W)} \mathbb{P}[\xi_i(a, \ldots, a_{i-1}, \overline{Z}_R; \rho_R) \in \mathcal{E}_{r,W}(a, \overline{Z}_R)].
\] (13)

Let $c := c(G, W)$ be as in Lemma 4.2. We have by Lemma 4.2 that
\[
\mathbb{P}[\xi_i(a, \ldots, a_{i-1}, \overline{Z}_R; \rho_R) \in \mathcal{E}_{r,W}(a, \overline{Z}_R)] 
\leq c \mathbb{P}[\xi_i(a, \ldots, a_{i-1}, \overline{Z}_R; \overline{\rho}_R) \in \mathcal{E}_{r,W}(a, \overline{Z}_R)],
\] (14)

where $\overline{\rho}_R$ is sampled from $\overline{\text{WSF}}(\overline{Z}_R)$.

Now note that $\xi_i(a, \ldots, a_{i-1}, \overline{Z}_R; \overline{\rho}_R)$ follows the law of $\overline{\text{WSF}}(\overline{Z}_R)$ by (7) and Lemma 3.9, which implies that
\[
\mathbb{P}[\xi_i(a, \ldots, a_{i-1}, \overline{Z}_R; \overline{\rho}_R) \in \mathcal{E}_{r,W}(a, \overline{Z}_R)] = \mathbb{P}[\overline{\rho}_R \in \mathcal{E}_{r,W}(a, \overline{Z}_R)].
\] (15)

Now let $L := 2c \sum_{i=0}^{K-1} |N(W)|^i$, and choose $r = r(\varepsilon/L, G, W)$ to be as in Lemma 3.7. It follows from Lemma 3.7 that
\[
\mathbb{P}[\overline{\rho}_R \in \mathcal{E}_{r,W}(a, \overline{Z}_R)] \leq \frac{\varepsilon}{L}.
\] (16)

Combining (12), (13), (14), (15), and (16) together, we get
\[
\mathbb{P}[\rho_R \in \mathcal{E}_{r,W}(a, \overline{Z}_R)] \leq \frac{\varepsilon}{2} + c \sum_{i=0}^{K-1} \sum_{a_1, \ldots, a_i \in N(W)} \frac{\varepsilon}{L} = \varepsilon.
\] (17)

The theorem now follows from (17) and Lemma 4.1. \qed

Finally, we remark that the following strengthened version of Theorem 1.2 can be proved by using a similar argument.

**Theorem 4.3.** Suppose that $G$ is a transient graph for which (1End) holds and $Z$ is finite. Then the final rotor configuration $\sigma(a, Z; \rho)$ follows the law of $\overline{\text{WSF}}(Z)$ when $\rho$ is sampled from $\overline{\text{WSF}}(Z)$. \qed

5. Some open questions

We conclude with a few natural questions:

1. Is $\overline{\text{WSF}}$ the unique infinite-step stationary measure for rotor walk on $\mathbb{Z}^d$ for $d \geq 3$? Does there exist an infinite-step stationary measure for rotor walk on $\mathbb{Z}^2$?
2. Is (1End) a necessary condition for $\overline{\text{WSF}}$ to be an infinite-step stationary measure for rotor walk?
(3) Choose the initial rotor $\rho(x)$ for $x \in \mathbb{Z}^d$ independently and uniformly at random from the outgoing edges of $x$. What is the escape rate of this rotor walk?

ACKNOWLEDGEMENT

The author would like to thank Lionel Levine and Yuval Peres for their advising throughout the whole project, Ander Holroyd for inspiring discussions, and Yuwen Wang for proofreading. Part of this work was done when the author was visiting the Theory Group at Microsoft Research, Redmond and when the author was a graduate student in Cornell University.

REFERENCES

[AH11] O. Angel and A. Holroyd, Rotor walks on general trees, *SIAM J. Discrete Math.* 25 (2011), 423–446.

[AH12] O. Angel and A. Holroyd, Recurrent rotor-router configurations, *J. Comb.* 3 (2012), 185–194.

[Ald90] D. Aldous, The random walk construction of uniform spanning trees and uniform labelled trees, *SIAM J. Discrete Math.* 3 (1990), 450–465.

[BL16] B. Bond and L. Levine, Abelian networks I. Foundations and examples, *SIAM J. Discrete Math.* 30 (2016), 856–874.

[BLPS01] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, Uniform spanning forests, *Ann. Probab.* 29 (2001), 1–65.

[Bro89] A. Broder, Generating random spanning trees, in *Proc. 30th SFCS*, IEEE Comp. Soc., Washington, DC (1989), 442–447.

[BTW88] P. Bak, C. Tang, and K. Wiesenfeld, Self-organized criticality, *Phys. Rev. A* 38 (1988), 364–374.

[CDST06] J. Cooper, B. Doerr, J. Spencer, and G. Tardos, Deterministic random walks, in *Proc. 8th ALENEX*, SIAM, Philadelphia (2006), 185–197.

[CGLL18] S. H. Chan, L. Greco, L. Levine, and P. Lü, Random walks with local memory, preprint (2018), 29 pp., arXiv:1809.04710.

[Cha18a] S. H. Chan, A rotor configuration with maximum escape rate, preprint (2018), 6 pp., arXiv:1810.12784.

[Cha18b] S. H. Chan, Rotor walks on transient graphs and the wired spanning forest, preprint (2018), 32 pp., arXiv:1809.09790.

[CS06] J. Cooper and J. Spencer, Simulating a random walk with constant error, *Combin. Probab. Comput.* 15 (2006), 815–822.

[DF09] B. Doerr and T. Friedrich, Deterministic random walks on the two-dimensional grid, *Combin. Probab. Comput.* 18 (2009), 123–144.

[DTW03] I. Dumitriu, P. Tetali, and P. Winkler, On playing golf with two balls, *SIAM J. Discrete Math.* 16 (2003), 604–615.

[FGLP14] L. Florescu, S. Ganguly, L. Levine, and Y. Peres, Escape rates for rotor walks in $\mathbb{Z}^d$, *SIAM J. Discrete Math.* 28 (2014), 323–334.

[FLP16] L. Florescu, L. Levine, and Y. Peres, The range of a rotor walk, *Amer. Math. Monthly* 123 (2016), 627–642.
D. He, A rotor configuration in $\mathbb{Z}^d$ where Schramm’s bound of escape rates attains, preprint (2014), 20 pp., arXiv:1405.3400.

A. Holroyd, L. Levine, K. Mészáros, Y. Peres, J. Propp, and D. Wilson, Chip-firing and rotor-routing on directed graphs, in *In and out of equilibrium. 2*, Progr. Probab. 60, Birkhäuser, Basel (2008), 331–364.

W. Huss, S. Müller, and E. Sava-Huss, Rotor-routing on Galton-Watson trees, Electron. Commun. Probab. 20 (2015), no. 49, 12.

W. Huss and E. Sava-Huss, A law of large numbers for the range of rotor walks on periodic trees, preprint (2018), 29 pp., arXiv:1805.11983.

W. Huss and E. Sava-Huss, Range and Speed of Rotor Walks on Trees, to appear in *J. Theoret. Probab.* (2019).

T. Hutchcroft, Interlacements and the wired uniform spanning forest, Ann. Probab. 46 (2018), 1170–1200.

A. Járai and F. Redig, Infinite volume limit of the abelian sandpile model in dimensions $d \geq 3$, Probab. Theory Related Fields 141 (2008), 181–212.

G. Lawler, A self-avoiding random walk, Duke Math. J. 47 (1980), 655–693.

I. Landau and L. Levine, The rotor-router model on regular trees, J. Combin. Theory Ser. A 116 (2009), 421–433.

R. Lyons, B. Morris, and O. Schramm, Ends in uniform spanning forests, Electron. J. Probab. 13 (2008), no. 58, 1702–1725.

L. Levine and Y. Peres, Spherical asymptotics for the rotor-router model in $\mathbb{Z}$, Indiana Univ. Math. J. 57 (2008), 431–449.

L. Levine and Y. Peres, Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile, Potential Anal. 30 (2009), 1–27.

R. Lyons and Y. Peres, *Probability on trees and networks*, Camb. Ser. Stat. Probab. Math. 42, Cambridge University Press, New York (2016), xv+699.

V. Priezzhev, D. Dhar, A. Dhar, and S. Krishnamurthy, Eulerian walkers as a model of self-organized criticality, Phys. Rev. Lett. 77 (1996), 5079–5082.

R. Pemantle, Choosing a spanning tree for the integer lattice uniformly, Ann. Probab. 19 (1991), 1559–1574.

J. Propp, Random walk and random aggregation, derandomized, online lecture (2003), Link.

Y. Rabani, A. Sinclair, and R. Wanka, Local Divergence of Markov Chains and the Analysis of Iterative Load-Balancing Schemes, in *Proc. 39th FOCS*, IEEE Comp. Soc., Washington, DC (1998), 694–708.

D. Wilson, Generating random spanning trees more quickly than the cover time, in *Proc. 28th STOC*, ACM, New York (1996), 296–303.

I. Wagner, M. Lindenbaum, and A. Bruckstein, Smell as a computational resource—a lesson we can learn from the ant, in *Israel Symposium on Theory of Computing and Systems*, IEEE Comput. Soc. Press, Los Alamitos (1996), 219–230.
Swee Hong Chan, Department of Mathematics, UCLA, Los Angeles California 90095. Webpage: http://www.math.ucla.edu/~sweehong/