A new model of the regular black hole in (2 + 1)−dimensions is introduced by considering an appropriate matter field as the energy-momentum tensor. First, we propose a novel model of d-dimensional energy density that in (2 + 1)−dimensions leads to the existence of an upper bound on the radius of the event horizon and a lower bound on the mass of the black hole which are motivated by the features of astrophysical black holes. According to these bounds, we introduce an admissible domain for the event horizon radius, depending on the metric parameters. After investigation of geometric properties of the obtained solutions, we study the thermal stability of the solution in the canonical ensemble and find that the regular black hole is thermally stable in the mentioned admissible domain. Besides, the Gibbs free energy is calculated to examine the global stability of the solution.

PACS numbers:

I. INTRODUCTION

Nowadays, the theory of General Relativity (GR) still well passes most gravitational tests. Without a doubt, the black hole is one of the most interesting predictions of Einstein's theory of GR. In recent years, these objects have been moved into the focus of observational and theoretical studies in gravity and cosmology due to the detection of gravitational waves emitted from the merger of two black holes [1–3] and the first picture of an event horizon by the event horizon telescope collaboration [4].

Roughly speaking, a special characteristic of the first known exact black hole solutions in GR is the existence of a region of spacetime in which the laws of physics break down [2]. This region, where just possible to enter while impossible to escape from in a classical point of view is delimited by the so-called event horizon (see a review [6] and references therein). Nevertheless, regarding the theory of GR and its black hole solutions, there are two theoretical difficulties: the singularity problem and the problem of its quantization. According to the singularity theorems, Penrose and Hawking showed that, mathematically, singularities are unavoidable in GR [7–11]. However, regarding the Hawking radiation of black holes, the existence of singularities is more complicated due to their relations with the information paradox. As a result, it is extensively believed that since these singularities are created by classical theories of gravity from the pure mathematical viewpoint, they are not physical objects and they cannot exist in nature, and therefore, non-singular solutions and their properties are well-motivated topics accordingly. For instance, new approaches to avoid singularities were suggested in some modified gravity theories [12–15] and nonlinear electrodynamics [16].

The main motivation for the appearance of regular solutions comes from the fact that most physicists and philosophers believe that singularities are not reasonable possibilities for physical manifestation in our real world. This opinion is supported by an indication that classical general relativity cannot be valid at all scales [17, 18], and therefore, at high energy scales, the quantum effects should be considered. As a consequence of loop quantum gravity, for instance, the pressure of quantum fluctuations may counterbalance the gravitational collapse of a typical supermassive star to avoid singularity formation. So, a dense central core can be formed inside a black hole whose functional density modeling would be interesting. In this way, the Sakharov’s [19] and Gliner’s [20] quantum arguments suggest that matter source with a de Sitter (dS) core at the center of the spacetime does not include the spacetime singularities. According to this idea, the first model of a regular (i.e non-singular) black hole was proposed by Bardeen [21]. In this model, the solution has a dS core and the singularity is avoided by considering a collapse of a charged matter with a charged matter core inside the black hole instead of its singularity. After that, a large class of regular black hole solutions was found based on the idea that singularity could be replaced by a regular distribution of matter [22–30]. It is worth mentioning that although the strong energy condition (SEC) may be violated by all regular black hole
solutions with spherical symmetry, the dominant energy condition (DEC), as well as weak energy condition (WEC), are satisfied in most cases. Regular black holes can be constructed in various circumstances. A particular kind of non-singular black holes in which the singularity is replaced by a dS core with regular geometry around the origin has been proposed by Nicolini et al. Dymnikova has found a different type of regular black holes with a dS core inside the horizon, smoothly connecting to a Schwarzschild as the outer geometry. Another interesting attempt is to construct regular black hole solutions including matter fields in the energy-momentum tensor (see for e.g., \cite{26,52,53}). Furthermore, models including nonlinear electrodynamics or scalar fields have been successful in the construction of non-singular solutions \cite{54,41}.

It is also possible to have regular black holes in lower-dimensional spacetime. Generally, models of gravity in \((2 + 1)\)-dimensions have become an active field of research in recent years. Due to the lack of the Newtonian limit and propagating degrees of freedom \cite{12}, the general idea was that Einstein gravity is a topological theory in \((2 + 1)\)-dimensional spacetime \cite{13} and there is no black holes in this geometry. However, the discovery of the famous BTZ black hole \cite{44,45} aroused great interest in the study on these objects \cite{46,52}. The main motivation to study the Einstein gravity in \((2 + 1)\)-dimensional spacetime is that it provides a simpler framework to understand many conceptual features of \((3 + 1)\)-dimensional Einstein gravity \cite{53}. In fact, the simplicity of the equations of motion in three dimensional geometry has led to being studied these models as a toy model to a better understanding of some problems of its \((3 + 1)\)-dimensional counterpart and assist us to comprehend some conceptual issues regarding the quantum gravity and string theory \cite{42,54}. Moreover, since the black hole properties at the quantum level in \((3 + 1)\)-dimensional gravity has remained as a mystery, the \((2 + 1)\)-dimensional black holes could provide a good relatively simple laboratory to examine the features of black holes in higher dimensions and to find the deeper insight into the general facets of black hole physics. Another major motivation is related to the AdS/CFT correspondence. The study of the near horizon of the \(3\)-dimensional black holes has helped us to explore some conceptual aspects of the AdS/CFT duality \cite{53}. In \cite{50} the authors have shown the coincidence between the quasinormal modes in this geometry and poles of the correlation function in the dual CFT which gives evidence for AdS\(_2\)/CFT\(_2\). Also, one dimensional holographic superconductors can be explored in the background of \((2 + 1)\)-dimensional black hole \cite{57,59}. An additional reason that motivates us to study the three dimensional gravity is that the investigation of the \((2 + 1)\)-dimensional black holes’ physical properties has enhanced our comprehension of the gravitational systems and their interactions in lower dimensions. Another motivate for examining this geometry is that all the characteristics of a configuration in a given \(d\)-dimensions are not necessarily transferred to its lower ones and a system in lower dimensions can exhibit different features. For instance, it is shown that the charge term of the lapse function in the Ricci flat \(d\)-dimensional Reissner-Nordström solutions is proportional to \(r^{-(2d-6)}\) that is a constant term in \((2 + 1)\)-dimensions. It indicates that higher-dimensional Reissner-Nordström black holes reduce to uncharged solutions in three dimensions. While we know that there is a logarithmic charge term in the lapse function of Einstein-Maxwell system in three dimensions.

The study of BTZ black holes in the noncommutative spacetime provides the possibility of the existence of gravitational Aharonov Bohm effect is a case in point \cite{61}. Also, the possibility of mimicking the BTZ black hole properties in higher dimensions has been studied in \cite{60,62}. Besides, the authors in \cite{49,50} have investigated the existence of the \((2 + 1)\)-dimensional solutions in the presence of the nonlinear electrodynamics. Three dimensional black holes not only exist in the context of Einstein’s gravity, but also the modified gravities such as Lifshitz gravity \cite{64}, dilatonic gravity \cite{46,53}, massive gravity \cite{66}, gravity’s rainbow \cite{67}, massive gravity’s rainbow \cite{68} have similar solutions.

In this paper, we propose a new model of the regular black hole in \(d\)-dimensions and then, as a special case, we investigate the geometric and thermodynamic properties of the \((2 + 1)\)-dimensional solution. To this end, the conventional approach is to start from the energy-momentum tensor of a given matter field and then to solve the field equation. On the other hand, it is feasible to take the opposite way that the regular black hole solution can be first constructed based on the requirement of divergence-free curvature and then, the corresponding energy-momentum or the corresponding matter field will be driven \cite{23,28,69,71}. Here, we will proceed according to latter approach.

The paper is organized as follows. In the next section, we introduce our proposed new model of the regular black hole and study the physical properties of the solution. Section III is devoted to the investigation of thermodynamics properties of the solution. We calculate thermodynamic quantities and examine the first law of thermodynamics. We also explore local/global thermodynamic stability in the canonical grand canonical ensemble. In the concluding remarks part, we give a summary and conclusion. Finally, in Appendix A we provide a brief introduction on the curvature singularity-free models of black holes.
II. NEW MODEL OF REGULAR BLACK HOLE AND ITS EXACT SOLUTION

As we mentioned before, there are various motivations to find appropriate solutions of Einstein field equations that describe regular black hole. In this section, we suggest a new model of energy density in\(d\)-dimensions, based on the inclusion of suitable matter field in the energy-momentum tensor, and show that it leads to an interesting\((2+1)\)-dimensional regular black hole solution.

Although one can introduce different ad hoc models of energy density to have a regular black hole, we should consider reasonable criteria to veto unphysical models. In order to have a well-defined physical solution, the suggested energy density must have a certain behavior at the origin as well as at infinity. Strictly speaking, the energy density must be positive, continuously differentiable and have a finite single maximum at the origin to avoid singularity. Besides, to guarantee a well-defined asymptotic behavior, the energy density should be a decreasing function of radial coordinate to vanish at spatial infinity, i.e. \(\rho = 0\) as \(r \to \infty\) \[36\] (see the appendix \(\text{A}\) for more details).

Keeping in mind the mentioned criteria, we can suggest the following energy density in \(d\)-dimensions

\[
\rho(r) = \frac{d - 1}{\Omega_{d-2}} \frac{k - 1}{2L^{d-2}} (1 + \frac{r^{d-1}}{2L^{d-2}M})^{-k},
\]

where \(k \geq 2\) is an integer number, the constant parameter \(L\) is a regulator with square dimension of length which should be positive to make sure the positivity of the energy density at the origin and \(M\) is the dimensionless mass parameter. We should mention that the demand for the absence of singularity in the energy density necessitates considering a positive definite mass parameter, \(M\). As is discussed in appendix \(\text{A}\) our proposed model of energy density can be considered as a generalization of the model that Estrada and Aros proposed for regular black holes.

Here, we are interested in studying the\((2+1)\)-dimensional solution and properties of the higher dimensional ones will be explored in future works. Based on Eq. \[4\], energy density in\((2+1)\)-dimensions will reduce to

\[
\rho(r) = \frac{k - 1}{2\pi L} \left(1 + \frac{r^2}{2LM}\right)^{-k},
\]

To derive an exact black hole solution corresponding to the above energy density, we conventionally start from the Einstein field equations which in\((2+1)\)-dimensions is given by

\[
G_{\mu\nu} + \Lambda \delta_{\mu\nu} = 8\pi G T_{\mu\nu},
\]

where \(G_{\mu\nu}\) and \(T_{\mu\nu}\) are, respectively, the Einstein tensor and a second rank symmetric tensor of the energy-momentum. Also, \(\Lambda = -l^{-2}\) is the cosmological constant which is related to the anti de Sitter (AdS) radius, \(l\), and \(G\) is dimensionless gravitational constant (we consider \(c = 1\) throughout this paper). We assume a spherically symmetric spacetime describing by the following metric ansatz

\[
g_{\mu\nu} = \text{diag}[-f(r), f(r)^{-1}, r^2].
\]

Here, we consider a three dimensional energy-momentum tensor describing a non-trivial anisotropic fluid with different radial and tangential pressures

\[
T_{\mu\nu} = \text{diag}[ -\rho(r), P_r(r), P_\phi(r) ],
\]

where it is easy to show that one has to consider \(P_\phi(r) = -\rho(r)\) due to the consistency with the Einstein field equations. In addition, taking into account Eq. \[5\] and solving the \(\phi\phi\) component of the Einstein field equation, one can obtain the functional form of tangential pressure

\[
P_\phi = \left(\frac{2kr^2}{2LM + r^2} - 1\right)\rho.
\]

We should note that although we suggest the energy density model, the functional form of the pressures do not depend on the metric function. In other words, we read the field equations \[3\] from right to left. Before obtaining the metric function, it is worth discussing the asymptotic behavior of energy density and pressures. It is easy to find that

\[
\lim_{r \to 0} \rho(r) &= - \lim_{r \to 0} P_r(r) = - \lim_{r \to 0} P_\phi = \frac{k - 1}{2\pi L},
\]

\[
\lim_{r \to \infty} \rho(r) &= - \lim_{r \to \infty} P_r(r) = - \lim_{r \to \infty} P_\phi = 0,
\]
where confirm that the nonzero components of the suggested energy-momentum tensor are finite values near the origin and they vanish at spatial infinity. So, the matter distribution does not affect the asymptotic behavior of the solution which is characterized by the cosmological constant. Besides, unlike the radial pressure (energy density) which is a smooth function of radial coordinate, tangential pressure enjoys a root at finite radius \( r = \sqrt{\frac{2LM}{k-1}} \) and an extremum point at \( r = \sqrt{\frac{6LM}{2k-1}} \).

Regarding the functional form of nonzero components of the energy-momentum tensor, we are in a position to obtain the metric function. For the line element \( \mathbf{I} \), we arrive at the following equation by combining the \( tt \) and \( rr \) components of the Einstein field equations

\[
\frac{1}{2r} \frac{df}{dr} + \Lambda = -8 \pi G \rho, \tag{7}
\]

with the following traditional ansatz

\[
f(r) = 1 - 8Gm(r) - \Lambda r^2, \tag{8}
\]

where the mass \( m(r) \) is related to the energy density as

\[
\frac{d}{dr} m(r) = 2\pi \rho. \tag{9}
\]

After a simple manipulation, the explicit form of the mass function for the proposed energy density of \( (2 + 1) \)-dimensional black hole can be rewritten as

\[
m(r) = 2\pi \int_0^r x\rho(x)dx = M \left[ 1 - \left( 1 + \frac{r^2}{2LM} \right)^{1-k} \right]. \tag{10}
\]

As it is expected, the mass function tends to its finite maximum value at infinity \( m(r) \rightarrow M \) as \( r \rightarrow \infty \) which confirms its well-defined asymptotic behavior. Now, by substituting \( \mathbf{10} \) into \( \mathbf{8} \), the solution will take the following analytical form

\[
f(r) = 1 - 8G M \left[ 1 - \left( 1 + \frac{r^2}{2LM} \right)^{1-k} \right] + \frac{r^2}{L}, \tag{11}
\]

which its asymptotic behavior is AdS

\[
f(r)|_{r \to \infty} = 1 - 8G M + \frac{r^2}{L^2} + \mathcal{O}(r^{2-2k}).
\]

In addition, one can find the regularity of the metric function near the origin as

\[
f(r)|_{r \to 0} = 1 + \left[ -\frac{4G (k-1)}{L} + \frac{1}{L^2} \right] r^2 + \frac{Gk (k-1)}{L^2M} r^4 + \mathcal{O}(r^6), \tag{12}
\]

which shows that our solution has no singularity at the origin. To study the situation of the core of our regular black hole solution, we should focus on the second term of Eq. \( \mathbf{12} \). Obviously, by choosing \( L = 4Gk^2(k-1) \), the black hole enjoys a flat core while for \( L < 4Gk^2(k-1) \) \( L > 4Gk^2(k-1) \) its core is dS [AdS].

Here, we should examine the regularity of the solutions. To do so, one can, generally, consider some curvature invariants such as the Kretschmann scalar, Ricci square, Ricci scalar and Weyl square. It is worth mentioning that the Riemann tensor has six independent components corresponding to the Ricci tensor for a general three dimensional spacetime while for diagonal ansatz \( \mathbf{4} \), the nonzero components reduces to three. As a result, the Weyl square vanishes and we find that

\[
\text{Kretschmann scalar:}
\]

\[
\mathcal{K} = R_{abcd} R^{abcd} = f''^2(r) + 2 \left( \frac{f'}{r} \right)^2
\]

\[
= \frac{12}{L^4} - 96 (k-1) \frac{G}{L^2} \left[ 1 - \frac{r^2}{3LM} \left( k - \frac{3}{2} \right) \right] \left( 1 + \frac{r^2}{2LM} \right)^{-k-1} + 192 (k-1)^2 \frac{G^2}{L^2}
\]

\[
\times \left[ 1 - \frac{r^2}{3LM} \left( 2 \left( k - \frac{3}{2} \right) + \frac{r^2}{3LM} \left( k^2 - k + \frac{3}{4} \right) \right) \right] \left( 1 + \frac{r^2}{2LM} \right)^{-2k-2},
\]

\[
\text{Ricci square:}
\]

\[
R_{ij} R^{ij} = \frac{1}{L^4} \left( \frac{f'}{r} \right)^2 - \frac{12}{L^2} + 192 (k-1)^2 \frac{G^2}{L^2}
\]

\[
\times \left[ 1 - \frac{r^2}{3LM} \left( 2 \left( k - \frac{3}{2} \right) + \frac{r^2}{3LM} \left( k^2 - k + \frac{3}{4} \right) \right) \right] \left( 1 + \frac{r^2}{2LM} \right)^{-2k-2},
\]

\[
\text{Ricci scalar:}
\]

\[
R = -\frac{12}{L^4} + 96 (k-1) \frac{G}{L^2} \left[ 1 - \frac{r^2}{3LM} \left( k - \frac{3}{2} \right) \right] \left( 1 + \frac{r^2}{2LM} \right)^{-k-1} - 192 (k-1)^2 \frac{G^2}{L^2}
\]

\[
\times \left[ 1 - \frac{r^2}{3LM} \left( 2 \left( k - \frac{3}{2} \right) + \frac{r^2}{3LM} \left( k^2 - k + \frac{3}{4} \right) \right) \right] \left( 1 + \frac{r^2}{2LM} \right)^{-2k-2},
\]

\[
\text{Weyl square:}
\]

\[
F_{abcd} F^{abcd} = \frac{1}{L^4} \left( \frac{f'}{r} \right)^2 - \frac{12}{L^2} + 192 (k-1)^2 \frac{G^2}{L^2}
\]

\[
\times \left[ 1 - \frac{r^2}{3LM} \left( 2 \left( k - \frac{3}{2} \right) + \frac{r^2}{3LM} \left( k^2 - k + \frac{3}{4} \right) \right) \right] \left( 1 + \frac{r^2}{2LM} \right)^{-2k-2}.
\]
To study more precisely, we consider their behavior in the presence of large and small radii. No matter of the value of $k$, all curvature scalars have finite value at infinity

\[
K \big|_{r \to \infty} = \frac{12}{l^2} + \frac{\alpha(k) G M}{r^2} \left( \frac{L M}{l^2} \right)^{k-1} + O \left( \frac{1}{r^4} \right) \quad \text{(symmetry)},
\]

\[
\mathcal{R}^2 \big|_{r \to \infty} = \frac{12}{l^2} + \frac{\alpha(k) G M}{r^2} \left( \frac{L M}{l^2} \right)^{k-1} + O \left( \frac{1}{r^4} \right) \quad \text{(symmetry)},
\]

\[
\mathcal{R} \big|_{r \to \infty} = -\frac{6}{l^2} + \frac{\beta(k) G M}{r^2} \left( \frac{L M}{l^2} \right)^{k-1} + O \left( \frac{1}{r^4} \right),
\]

where $\alpha(k)$ and $\beta(k)$ are numbers which vary by changing the amount of $k$. Besides, when $r$ tends to zero, the mentioned invariants are given as

\[
K \big|_{r \to 0^+} = 192 \left( \frac{G(k-1)}{L} - \frac{1}{4 l^2} \right)^2 - \frac{320 G k(k-1)}{M L^2} \left( \frac{G(k-1)}{L} - \frac{1}{4 l^2} \right) r^2 + O \left( r^4 \right),
\]

\[
\mathcal{R}^2 \big|_{r \to 0^+} = 192 \left( \frac{G(k-1)}{L} - \frac{1}{4 l^2} \right)^2 - \frac{320 G k(k-1)}{M L^2} \left( \frac{G(k-1)}{L} - \frac{1}{4 l^2} \right) r^2 + O \left( r^4 \right),
\]

\[
\mathcal{R} \big|_{r \to 0^+} = 24 \left( \frac{G(k-1)}{L} - \frac{1}{4 l^2} \right)^2 - \frac{20 Gk(k-1)}{M L^2} r^2 + O \left( r^4 \right).
\]

To be more clear, we have provided table [I] As it is clear, none of the curvature invariants diverges neither at the origin nor other values of $r$. Furthermore, Fig. [I] shows the behavior of curvature invariants for other classes of metric parameters which confirms that they are free of divergencies as we expect for the regular black holes.

Before proceeding, it is worthwhile to study the behavior of the Ricci scalar by considering the structure of the black hole’s core. Investigating the behavior of this function near the origin indicates that depending on the values of the metric parameters, $\mathcal{R}$ can be zero, positive or negative at $r \to 0^+$. To make this point more clear, the behavior of the Ricci scalar with respect to $r$ is sketched for different values of $L$ and fixed values of the other parameters in Fig. [2] Since the asymptotic behavior of the solution is AdS, this function for the regular black hole with dS core enjoys a root at a finite radius and after having a local extremum, it tends to a constant value $(-6/l^2)$ at infinity.

**Table I: The values of curvature invariants for $k = 4$, $M = 0.17$, $G = l = 1$, $L = 3$**

| $r$  | $\mathcal{R}^2$ | $\mathcal{R}$ | $K$  |
|------|----------------|---------------|------|
| 0.0  | 108.00         | 18.00         | 108.00|
| 0.2  | 52.85          | 12.51         | 55.00|
| 0.4  | 6.09           | 2.55          | 17.82|
| 0.6  | 9.00           | -3.82         | 21.42|
| 0.8  | 14.45          | -6.10         | 20.64|
| 1.0  | 14.79          | -6.45         | 16.91|
| 1.2  | 13.85          | -6.40         | 14.46|
| 1.4  | 13.07          | -6.25         | 13.24|
| 1.6  | 12.60          | -6.14         | 12.64|
| 1.8  | 12.33          | -6.08         | 12.34|
| 2.0  | 12.19          | -6.04         | 12.19|
However, the Ricci scalar for the black hole whose core geometry is AdS (flat) is a smooth function of $r$ and its value starts from a negative point (zero) and tends to a constant value $(-6/l^2)$ at infinity. As a final comment, it is notable that the core discussion based on the behavior of curvature invariants near the origin is in agreement with what we mentioned before after Eq. (12).

Now, we try to investigate other physical properties of the solution by studying the behavior of the obtained solution and looking for the horizons. In this regard, we have plotted the function $f(r)$ versus $r$ for different model parameters in Fig. 3. These figures show that, depending on the metric parameters, this solution could represent a black hole with two horizons or an extreme black hole with a degenerate root. In the case of positive definite $f(r)$, we have a regular horizonless spacetime which we are not interested in. Figure 3(a) indicates by increasing the value of the mass parameter (and fixed values of the other parameters), the number of horizons changes from one to two. Also, considering Fig. 3(b) and Fig. 3(c), we find that by decreasing the values of parameters $l$ and $k$, two horizons merge to a degenerate one creating an extremal solution.

To confirm our claims regarding the number of horizons, we calculate $f'(r)$ derivative’s roots. Since the number of function derivative’s roots represents the number of extrema of the function, if $f'(r)$ has more than one extrema, the number of horizons could be more than two. However, the existence of just one extremum represents that the maximum number of horizon will be two. In case of our solution, $f'(r)$ has three roots as

$$ r \big|_{f'=0} = 0, \pm \sqrt{2LM \left[ \frac{L}{4G l^2 (k-1)} \right]^{-1/k} - 1}, $$

(13)
where \( f' = \frac{\partial}{\partial r} f(r) \). Since there is only one acceptable extremum (real and positive), it can be ensured that the maximum number of horizons is two.

Elementary analysis regarding to roots of \( f(r) \) reveals a special mass

\[
M_{\text{ext}} = \frac{1}{8 \mathcal{G}} \left( 1 + \frac{L}{4 \mathcal{G} (k-1)^2} \left( \left[ 1 - \left( \frac{L}{4 \mathcal{G} (k-1)^2} \right)^{\frac{1}{k-1}} \right] k - 1 \right) \right)^{-1},
\]

such that for \( M \) larger than \( M_{\text{ext}} \), \( f(r) \) enjoys two simple roots at \( r = r_{\pm} \) and for \( M = M_{\text{ext}} \) (\( M < M_{\text{ext}} \)), \( f(r) \) has a degenerate zero at \( r = r_{\text{ext}} \) (no roots). Due to the complexity of the equations, it is difficult to find the horizon radius of extreme black hole for all values of \( k \) parameters. However, in the case of \( k = 2 \), mass and horizon radius of the solution will be obtained as follows

\[
M_{\text{ext}} \bigg|_{k=2} = \frac{1}{8 \mathcal{G}} \left( 1 + \frac{L}{4 \mathcal{G} L^{-1}} \left( 1 - 4 \sqrt{\mathcal{G} L^{-1}} \right) \right)^{-1},
\]

\[
r_{\text{ext}} \bigg|_{k=2} = \sqrt{\frac{l}{2 \sqrt{\mathcal{G} L^{-1}} - 1}}.
\]

Here, it is worth discussing the evolution of the mass parameter \( M \). To this end, we use Eq. (11) to derive the relation between the mass parameter and the horizon radius which for the case \( k = 2 \) will be as follows

\[
M(r_h)|_{k=2} = \frac{r^2}{2L} \left( \frac{4 \mathcal{G} L^2 r_h^2}{L (l^2 + r_h^2)} - 1 \right)^{-1},
\]

corresponding to the root of the equation \( f(r_h) = 0 \). To better understand this function, the evolution of the mass parameter as a function of the horizon radius is displayed in Fig. 4(a). As can be seen from this figure, there is a critical value of the mass parameter \( M_{\text{ext}} \), corresponding to the minimum value on the curve. At this point where the inner \((r_-)\) and outer horizon \((r_+)\) coincide, the solution meets the extreme black hole condition. However, the proposed regular black hole enjoys a pair of horizons when mass parameter exceeds critical mass \( M_{\text{ext}} \). Moreover, one can notice that

\[
\frac{dM}{dr_-} \leq 0, \quad \frac{dM}{dr_+} \geq 0,
\]

implying the fact that decreasing the value of \( r_+ \) results in decreasing the mass parameter. To investigate the effects of the parameter \( k \) on the evolution function of mass parameter \( M(r_h) \), we have provided Fig. 4(b) which shows that increasing the value of the \( k \) parameter leads to decreasing the value of \( M_{\text{ext}} \) and \( r_{\text{ext}} \) and, therefore, it causes the diagram to be inclined towards the origin.

In the following, we study some other properties of the proposed energy density. Regarding the behavior of the energy density with respect to \( r \), we calculate its derivative as

\[
\frac{dp}{dr} = - \frac{(k-1)kr}{2\pi L^2 M} \left( 1 + \frac{r^2}{2LM} \right)^{-k-1},
\]
FIGURE 4: Horizon mass-radius relation for $L = l = 1$ and $G = 1$

where shows that due to the positive values of $L$, the suggested function of energy density is strictly decreasing, having the maximum value at the origin and zero at infinity (see Fig. 5).

According to Fig. 4 we find that, as we expect, $\rho$ is finite at the origin. It is discussed the reason for the finiteness of the energy density at the origin in the appendix A. Also, increasing $k$ (when other metric parameters are fixed) can increase the amount of $\rho$ at $r = 0$ but keeps it finite. An important note is that $k$ is a finite parameter since when $k \to \infty$, the energy density vanishes ($\rho \to 0$). To investigate the behavior of the energy density on the event horizon, the value of $\rho$ should be calculated as a function of $r_+$ which will take the following form

$$\rho_+ = \frac{(k - 1) M}{\pi r_+^2} \left[ 1 - (1 - \eta)^{\frac{1}{k-1}} \right],$$  \hspace{1cm} (19)

where

$$\eta = \frac{1}{8 G M} \left( 1 + \frac{r_+^2}{l^2} \right).$$  \hspace{1cm} (20)

It should be mentioned that to arrive the above result we have used the following relation

FIGURE 5: Behavior of $\rho$ with respect to $r$ for $M = 0.2$ and $L = 3$
\[ L = \frac{r^2_+}{2M} \left\{ (1 - \eta)^{1 + \pi} - 1 \right\}^{-1}, \]  

(21)

to eliminate \( L \), which comes from the condition of the event horizon, i.e. \( f(r = r_+) = 0 \). To study more closely, we have provided Fig. 6 in which the behavior of \( \rho_+ \) in terms of \( r_+ \) is depicted. According to these figures (and also Eq. (19)), the energy density will be a positive real parameter if

\[ r^2_+ < r^2_{\text{max}} = (8M_G - 1)l^2, \]  

(22)

which means that there is an upper limit for the event horizon radius. Moreover, a lower limit will be placed on the mass parameter, i.e. \( M > \frac{1}{8G} \), since the event horizon must be a real parameter.

One of the interesting results of this paper is obtaining an upper limit on the event horizon radius as introduced in Eq. (22). The existence of the upper limit for the event horizon is in direct contradiction to classical black holes such as Schwarzschild and Reissner-Nordström whose radius of their event horizon is allowed to go to infinity. However, it does not make sense that the radius of the event horizon tends to infinity. Since there are some astrophysical black holes with an upper limit reported on their mass [72], it is expected that there will be an upper limit for their radius of event horizon, which is consistent with our result.

### III. THERMODYNAMICS

Black hole thermodynamics in AdS space is interesting from the AdS/CFT correspondence point of view that suggests the existence of a holographic duality between quantum gravity on AdS space and a certain Euclidean conformal field theory on its spacelike boundary. Besides, black hole remnant that may give a solution to the information paradox, can be considered in the context of black hole thermodynamics. In what follows, we study the thermodynamic feature of the obtained solution.

#### A. Conserved charge of the solution

In this subsection, we compute a conserved charge for our solution employing the method describing in [73]. According to [73], for any classical or quantum field theory on a general curved spacetime, the following quantity

\[ Q(t) := \int_{\Sigma_t} d^{d-1} \bar{x} \sqrt{\left| g \right|} J^0(t, \bar{x}), \]  

(23)

is conserved under the given time evolution, where \( \Sigma_t \) is a hypersurface or a time slice of the spacetime \( \Sigma \) at an arbitrarily fixed time \( t \), \( d \) is the dimension of the spacetime \( \Sigma \), and \( g \) denotes the determinant of \( g_{\mu\nu} \). Moreover, \( J^0 \) is the zero component of a covariantly conserved current \( J^\mu \), \( \nabla_\mu J^\mu = 0 \), where \( \nabla_\mu \) is the covariant derivative for
the metric $g_{\mu\nu}$. For a gravitational system with a Killing vector $\xi$ and a given energy-momentum tensor $T^\mu\nu$, the covariantly conserved current can be constructed as follows

$$ J^\mu = T^\mu\nu \xi^\nu. \quad (24) $$

One can easily prove the covariantly conservation of this current by using $\nabla_\mu T^\mu\nu = 0$ and $\nabla_\mu \xi^\nu + \nabla_\nu \xi_\mu = 0$. Therefore, the defined conserved charge will be a Noether charge corresponding to global symmetry of the system.

If $\xi^\mu$ is a Killing vector associated with the time translation, the conserved charge will be the total energy of the system

$$ E = \int_{\Sigma_t} d^{d-1}x \sqrt{|g|} T^0_0 \xi^0, \quad (25) $$

which will be in agreement with the standard definition of the energy in the flat background with $\xi^\mu = -\delta^\mu_0$.

Following this method and choosing $\xi^\mu = -\delta^\mu_0$, the total energy of the black hole corresponding to our model in $(2 + 1)$-dimensions will be as follows

$$ E = -2\pi \int_0^\infty dx x T^0_0 = 2\pi \lim_{r \to \infty} \int_0^r dx x \rho(x) = \lim_{r \to \infty} m(r) = M, \quad (26) $$

which confirms that the mass parameter $M$ represents the total energy of the black hole.

### B. First law of thermodynamics

Here, in order to study thermodynamic properties of the obtained regular black hole, we begin with calculating some thermodynamic quantities. As the first step, we focus on the entropy of the black hole which is equal to a quarter of the event horizon area since we are working in Einstein gravity [74, 75]

$$ S = \frac{1}{4} \int_0^{2\pi} \sqrt{g_{\phi\phi}} d\phi \bigg|_{r=r_+} = \frac{\pi r_+}{2}. \quad (27) $$

To investigate physical properties, we should determine temperature $T$ as the next step. One of the conventional methods of calculating $T$ is using the surface gravity ($\kappa$) interpretation and its relation to the Hawking temperature as

$$ T_H = \frac{\kappa}{2} = \frac{f'(r)}{4\pi} \bigg|_{r=r_+} = \frac{r_+}{2\pi l^2} - \frac{4}{\pi r_+} \frac{(k-1)M}{1-(1-\eta)\left(1-(1-\eta)\right)^{\frac{1}{k-1}}}, \quad (28) $$

where we have used (21) to eliminate $L$.

Taking the obtained entropy and temperature into account, we are in a position to examine the first law of thermodynamics. However, the first law of thermodynamics needs to modify for regular black holes due to the inconsistency between Bekenstein-Hawking area law and the conventional first law of black hole thermodynamics arising from the inclusion of the matter fields in the energy-momentum tensor [76]. Before proceeding and finding a structure of the first law of thermodynamics for regular black holes, it should be mentioned a point related to the dependence of the mass function on the mass parameter $M$. In fact, for spaces in which the asymptotic region is a proper limit, $m(r)$ should tend to a constant $M$ which is in direct proportion of the mass of the solution [36]. Therefore, $m(r, M) = 0$ and

$$ \frac{\partial m(r, M)}{\partial M} > 0, \quad (29) $$

for arbitrary values of $r$.

With the mentioned point in mind, we try to obtain the corrected form of the first law of thermodynamics for regular black holes by variation of the function $f(r, M)$ with respect to its parameters, i.e. $\delta f(r, M)|_{r=r_+} = 0$. However, since the transformation would be mapping a black hole into another black hole in the space of solutions, the function $f(r_+, M)$ must still vanishes under any transformation of the parameters. Therefore, $f(r_+, M) = 0$ and
$\delta f(r, M)\big|_{r=r_+} = 0$ are to be considered as constraints on the evolution along the space of parameters \[71, 78\]. Thus, from the variation of the function $f(r, M)$ we get

$$0 = \left. \frac{\partial f}{\partial r} dr \right|_{r=r_+} + \left. \frac{\partial f}{\partial M} dM. \right. \quad (30)$$

After some manipulations, one can find that the first law takes the following form

$$du = T dS, \quad (31)$$

where

$$du = \varpi \left. \frac{\partial m(r_+, M)}{\partial M} \right|_{r=r_+} dM.$$ 

It is worth mentioning that both terms, $du$ and $dS$, are local variables defined at the horizon. This is while the modification of the first law for regular black holes have also been investigated by the inclusion of an extra factor corresponding to an integration of the radial coordinate up to infinity \[70\].

Using (27) and (28) along with (31), we checked the modified first law of thermodynamics for regular black holes in the case of our model and one can confirm this law is satisfied.

C. Thermal stability

This subsection is devoted to the analysis of the black hole thermal stability making use of the canonical and grand canonical ensembles. Using the canonical ensemble method, one can explore the local stability and phase transition points of the black holes with regard to the sign/divergence points of the heat capacity. Global stability of the black holes, on the other hand, can be examined in the grand canonical ensemble by regarding the Gibbs free energy of the black holes. It is worth mentioning that the difference between the two types of stabilities lies in the fact that in global stability, a system in equilibrium with a thermodynamic reservoir is allowed to exchange energy with the reservoir while local stability is related to how the thermodynamical system responds to small variations of the thermodynamic parameters. In what follows we study the local and global stabilities of our proposed black hole in the canonical and grand canonical ensembles.

1. Local thermal stability in canonical ensemble

For the purpose of investigating the black hole local stability, we need to calculate the black hole heat capacity. In the canonical ensemble, the positivity of the heat capacity guarantees thermal stability of the solutions. It is notable that the temperature should be positive at the same time to ensure physical solutions.

In our case study, there is an additional condition that is related to the positivity of the energy density. As mentioned earlier, positive energy density impose an upper limit on the radius of the event horizon. It is known that the heat capacity is defined as

$$C = T \frac{dS}{dT}, \quad (32)$$

where for the obtained solution, it gets the following form

$$C = \frac{4\pi r_+ (k - 1) G M}{B(r_+)} \left[ 2 - (1 - \eta) \frac{1}{(k - 1)} - 8 G M k \eta \right], \quad (33)$$

where

$$B(r_+) = [8 G M (k + k \eta - 1) - 2k + 1] \left[ (1 - \eta) \frac{1}{(k - 1)} - 1 \right] + (8 G M \eta - 1) (1 - \eta) \frac{1}{(k - 1)}.$$

First, we consider the special case $k = 2$ to compare our result with [71]. It is notable that for the case $k = 2$ all relations are simpler which is another reason to focus on this value. For this special case, the heat capacity reduces to

$$C\big|_{k=2} = \frac{\pi r_+}{2} \left( 1 + \frac{8 G M \eta - 1}{8 G M - 1} \right) \left( 1 + 8 G M \eta - 1 \right) \left( 24 G M \eta - 1 \right). \quad (34)$$
To investigate thermal stability more precisely, we have provided Figs. 7-9. The behavior of temperature, heat capacity and energy density in terms of $r_+$ are depicted in these figures. Regarding the temperature and the heat capacity, we find similar behavior to that authors have done in Ref. [7]. However, we get more new results which we will discuss in what follows.

Figure 7 shows that all three functions are positive in the interval $r_{\text{min}} < r_+ < r_{\text{max}}$, where $r_{\text{min}}$ is the radius in which temperature (heat capacity) is starting to get the positive values. Besides, for $r_+ > r_{\text{max}}$ the energy density will be negative which is unphysical. Therefore, if the radius of the event horizon is in the mentioned interval, the black hole is thermally stable and meets the necessary criteria for viable solutions. This admissible domain can be altered by changing the metric parameters. According to Fig. 8 increasing the values of the mass parameter leads to increase the values of both $r_{\text{min}}$ and $r_{\text{max}}$ and makes the admissible domain larger. In addition, Fig. 9 represents that increasing $l$ leads to increasing both special limits ($r_{\text{min}}$ and $r_{\text{max}}$) and hence, the variation of this parameter can also change the stability interval. To elucidate, the results for a range of changes of $M$ and $l$ are displayed in Table II. As one can see from this table, by increasing the values of $M$ (left panel) and $l$ (middle table), the admissible domain increases.

The existence of the stability range for black holes in this model does not occur only in case $k = 2$. To better investigate this claim, the behavior of the heat capacity, temperature and $\rho_+$ for different values of the parameter $k$ are depicted in Fig. 10. According to these figures, we find that for each value of parameter $k$, there is an admissible domain for the black hole which decreases with increasing the value of $k$. To clarify, one can see Table III (right panel). According to this table, by increasing the value of $k$, the amount of $r_{\text{min}}$ also increases while the amount of $r_{\text{max}}$ is independent of the parameter $k$ and hence, the admissible domain gets smaller. It should be noted that due to the imaginary values of the heat capacity and the temperature as well as the energy density at the horizon, the
corresponding plots cannot be continued much further after $r_{\text{max}}$ and that’s why they are interrupted.

It is worthwhile to investigate the possibility of phase transition of the proposed solution by studying the divergencies of the heat capacity. Our calculations show that for the allowed region of the mass parameter, i.e. $M > \frac{G}{l^2}$, the heat capacity does not diverge in the admissible domain of the black hole. This claim is supported by the numerical results mentioned in the fourth column of the table II. To find these results, we have used Eq. (33) to calculate the radius in which function of $C$ diverges ($r_{\text{div}}$) for different values of free parameters $M$, $k$ and $l$. As can be seen from these tables, all the radii ($r_{\text{div}}$) are imaginary which means that the heat capacity never diverges for the black holes whose radius of the event horizon is in the admissible domain. It should be mentioned that the divergencies of the heat capacity for the real values of $r_+$ will occur if the mass parameter is out of the allowed region. For more clarifications, we have plotted the function $C$ together with the energy density at the horizon versus $r_+$ in Fig. 11. Based on this figure, for the real value of radius in which the heat capacity diverges, the energy density is negative and hence, for the allowed region of the mass parameter and the energy density, the solution is stable and divergencies do not appear.

2. Global thermal stability in grand canonical ensemble

The idea of studying the black hole global stability was first suggested by Hawking and Page [79]. According to their suggestion, the global stability of the black holes can be explored with the help of a useful thermodynamic quantity, i.e. Gibbs free energy.

Generally, the behavior of the Gibbs free energy particularly as a function of thermodynamic parameters such as temperature, which was first classified by Paul Ehrenfest, can indicate a phase transition and some of the thermodynamic properties of different phases [81]. Under Ehrenfest’s scheme, phase transition occurs when Gibbs free energy or at least one of its derivatives with respect to one of its variables is discontinuous. More clearly, phase transitions
TABLE II: Admissible domain (AD) for $G = 1$: $l = 1$, $k = 5$ (left table), $M = 0.17$, $k = 5$ (middle table) and $M = 0.5$, $l = 1$ (right table)

| $M$   | $r_{\text{min}}$ | $r_{\text{max}}$ | $\Delta r_{\text{div}}$ | $l$   | $r_{\text{min}}$ | $r_{\text{max}}$ | $\Delta r_{\text{div}}$ | $k$   | $r_{\text{min}}$ | $r_{\text{max}}$ | $\Delta r_{\text{div}}$ |
|-------|------------------|------------------|------------------|-------|------------------|------------------|------------------|-------|------------------|------------------|------------------|
| 0.17  | 0.48             | 0.60             | 0.12             | 1.0   | 0.48             | 0.60             | 0.12             | 3.0   | 1.09             | 1.73             | 0.64             |
| 0.18  | 0.52             | 0.66             | 0.14             | 1.2   | 0.58             | 0.72             | 0.14             | 4.12  | 1.73             | 0.61             | 0.70             |
| 0.19  | 0.56             | 0.72             | 0.16             | 1.4   | 0.67             | 0.84             | 0.17             | 5.14  | 1.73             | 0.59             | 0.72             |
| 0.20  | 0.60             | 0.77             | 0.17             | 1.6   | 0.77             | 0.96             | 0.19             | 6.16  | 1.73             | 0.57             | 0.49             |
| 0.21  | 0.63             | 0.82             | 0.19             | 1.8   | 0.86             | 1.08             | 0.22             | 7.17  | 1.73             | 0.56             | 0.74             |
| 0.22  | 0.67             | 0.87             | 0.20             | 2.0   | 0.96             | 1.20             | 0.24             | 8.17  | 1.73             | 0.56             | 0.75             |
| 0.23  | 0.69             | 0.92             | 0.23             | 2.2   | 1.06             | 1.32             | 0.26             | 9.18  | 1.73             | 0.55             | 0.75             |
| 0.24  | 0.72             | 0.96             | 0.24             | 2.4   | 1.16             | 1.44             | 0.28             | 10.18 | 1.73             | 0.55             | 0.76             |
| 0.25  | 0.74             | 1.00             | 0.26             | 2.6   | 1.25             | 1.56             | 0.31             | 11.18 | 1.73             | 0.55             | 0.76             |
| 0.26  | 0.77             | 1.04             | 0.27             | 2.8   | 1.34             | 1.68             | 0.34             | 12.19 | 1.73             | 0.54             | 0.76             |

FIG. 11: Behavior of $C$ and $\rho_+$ respect to $r_+$ for $k = 2$, $l = G = 1$ and $M = 1/12$

are labeled by the lowest derivative of the Gibbs free energy which is discontinuous at the transition. The existence of discontinuity in the first derivative of the Gibbs free energy with respect to some thermodynamic parameters indicates first-order phase transitions [81]. In the case of second-order phase transitions, Gibbs function and its first derivative are continuous while the second derivative of the Gibbs free energy meets discontinuity [81]. According to the Ehrenfest classification, third, fourth, and higher-order phase transitions could in principle occur.

The black hole Gibbs free energy in the grand canonical ensemble, in terms of the mass, temperature and entropy is given by

$$G = M - TS.$$  \hfill (35)

It is well-known that the black hole with positive temperature is globally stable provided that its Gibbs free energy is positive.

Regarding the above-mentioned point, one can use Eqs. (27) and (28) to obtain the functional form of the Gibbs free energy with the following relation

$$G = M + 2G M(k - 1) \left[ 1 - (1 - \eta)^{\frac{-1}{\sqrt{k - 1}}} \right] + \frac{1}{4} \left[ 1 - k \left( 1 + \frac{r^2}{l^2} \right) \right].$$  \hfill (36)

Due to the complexity of the above function, it is not possible to study its behavior analytically and, therefore, we use the numerical solution to analyze its behavior. Examination of a wide range of parameters shows that black holes with the event horizon in the interval $r_{\text{min}} < r_+ < r_{\text{max}}$ are globally stable owing to the strictly decreasing behavior.
FIG. 12: Behavior of Gibbs free energy versus $r_+$ (left) and $T$ (right) for $G = l = 1$ and $k = 2$

FIG. 13: Behavior of $G'$ and $G''$ versus $r_+$ for $G = 1$, $M = 1/5$, $l = 3$ and $k = 2$

of the Gibbs free energy functions. For instance, the behavior of Gibbs free energy in terms of $r_+$ and $T$ for $k = 2$ are sketched in Fig. 12 for different values of the mass parameter.

To ensure the above result, we calculate the first and second derivative of the Gibbs free energy which will be as follows

$$G' = \frac{dG}{dr_+} = \frac{kr}{2l^2} \left[ \left( 1 - \frac{1}{8GM} \left[ 1 + \frac{r_+^2}{l^2} \right] \right)^{\frac{1}{k-1}} - 1 \right], \quad (37)$$

$$G'' = \frac{d^2G}{dr_+^2} = \frac{k}{2l^2} \left\{ \left( 1 - \frac{1}{8GM} \left[ 1 + \frac{r_+^2}{l^2} \left( k + 1 \right) \right] \right) \left( 1 - \frac{1}{8GM} \left[ 1 + \frac{r_+^2}{l^2} \right] \right)^{-\frac{k-2}{8-1}} - 1 \right\}. \quad (38)$$

We try to analyze the behavior of the above functions using the numerical solution. The choice of different sets of parameters indicate that the first and second derivatives of Gibbs free energy are smooth functions with respect to $r_+$ which as an example, the result of a set of parameters is shown in Fig. 13. In the same way, one can show that the
higher derivatives of $G$ with respect to $r_+$ also behave similarly and they are smooth functions. The smoothness of the Gibbs free energy along with its derivatives guarantee the global stability of the black hole.

Before ending this section, it is worth studying the energy condition of the proposed black hole. To this end, we recall that the singularity theorems of Hawking and Penrose establish the relation between the appearance of singularities inside the black holes and the validity of the SEC \[82\]. However, avoidance of Hawking and Penrose singularity theorem to construct regular black holes was first explained by Borde in 1997 \[83\]. Following Borde’s theorem, Mars, Martin-Prats and Senovilla considered the spherically symmetric and static spacetime and proved that if these spacetimes are regular at the origin and satisfy the SEC, they cannot include any black hole region in GR \[84\]. Therefore, reversing the singularity theorem of Hawking and Penrose leads to the conclusion that “regular black holes violate the SEC somewhere inside the horizon” \[85\]. It is interesting to note that the violation of the SEC inside the event horizon received a simple formulation in terms of the Tolman mass which is considered as a clear criterion to evaluate the degree of such violation \[31\].

To study the energy condition for our suggested model in the obtained admissible domain, we make use of Eq. (5) and we receive the following results \[86\]

\[
\begin{align*}
SEC & = \rho + P_r + P_\Phi \geq 0, \\
NEC_{1,2} & \equiv WEC_{1,2} = \rho + P_{r,\Phi} \geq 0, \\
DEC_3 & \equiv WEC_3 = \rho \geq 0, \\
DEC_{1,2} & = \rho - P_{r,\Phi} \geq 0.
\end{align*}
\]

According to Eqs. (2) and (6), one can easily find that just for areas with radii larger than $\sqrt{\frac{2LM}{k-1}}$, the SEC is satisfied that actually corresponds to the region in which the tangential pressure is positive. Moreover, due to the positive values of the mass parameter and $L$, the WEC is met everywhere. This is while, regarding the DEC, our calculations show that it is violated in some areas. In fact, for regions with $r_{\text{min}} < r_+ < \sqrt{\frac{2LM}{k-1}}$, DEC is satisfied while for the rest of the admissible domain, $\sqrt{\frac{2LM}{k-1}} < r_+ < r_{\text{max}}$, DEC will be violated. To be more clear, we have provided Fig.14 in which the behavior of the combination of the energy-momentum tensor components (various energy conditions) in terms of $r$ are depicted.

FIG. 14: Energy conditions for $G = l = L = 1$, $M = 2$ and $k = 2$. 
IV. CONCLUDING REMARKS

In this paper, we proposed a new model of regular black hole based on considering a new model of energy density in \( d \)-dimensions that follows requirements mentioned in reference [36]. This suggested function is strictly decreasing, having the maximum value at the origin, which leads to avoiding the central singularity, and zero at infinity. In this paper we focused on the \((2 + 1)\)-dimensional solutions and studying the properties of the solutions in the higher dimensions was left for future works. Studying the behavior of the proposed energy density on the event horizon showed that there is an upper limit on the radius of the event horizon of such black holes which is completely compatible with the condition of the black holes whose mass is finite. Moreover, we understood that to have a real value for the radius of the event horizon, the mass parameter of the black hole must be larger than \( \frac{1}{8G} \).

Regarding the obtained solution, we found that, depending on the metric parameters, this solution could represent a black hole with two horizons or an extreme black hole. By selecting a certain value of \( L \) one could get a flat, dS or AdS core. Also, the thermodynamics of the proposed solution was studied and the first law of black hole thermodynamics was checked.

Next, we studied the thermal stability in the canonical ensemble. In this regard, in addition to the positivity of the temperature and heat capacity, our proposed model also required an additional condition that is related to the positivity of the energy density. To be more accurate, the behavior of temperature, heat capacity and energy density were studied for different values of the model parameters. We concluded that for the stable black holes the radius of the event horizon should be selected in the interval \( r_{\text{min}} < r_+ < r_{\text{max}} \). The critical behavior of the obtained solution was investigated by studying the divergencies of the heat capacity. We found that the radius in which the function of \( C \) diverges \( (r_{\text{div}}) \) is not in the admissible domain of the black hole. Also, we investigated the global stability of the solution by studying the Gibbs free energy. Our investigation showed that the black holes with the radius of the event horizon in the mentioned interval are globally stable and hence, we conjecture they do not experience any kind of classical thermal phase transitions. In the end, the energy conditions were checked for this solution. We concluded that although the WEC is met everywhere, the SEC, as well as the DEC, will be violated for a part of the admissible domain of the event horizon radius.

It is interesting to investigate the dynamic stability of the solutions and analyze quasi-normal modes. Besides, it is nice to study the causal structure of the obtained solution via the possible Penrose diagrams. Moreover, one can examine the effect of perturbations on the Cauchy horizon stability. Also, more geometrical/topological investigations and looking for topological defects can be regarded. All these interesting suggestions can be addressed in independent work.

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Appendix A: A review on some regular black hole models

In this appendix, we try to review some efforts that have been done to propose curvature singularity-free models of black holes.

Curvature singularities of GR could be understood as points where every classical theory of gravity does not make sense. However, the important point is that classical theories of gravity cannot be valid at all scales \([87]\) and, therefore, describing nature at some scales such as Planck scales requires a new theory of gravity. In the context of these new theories, such as string theory or Loop Quantum Gravity (LQG), the problem of singularity can be solved by considering quantum effects. For instance, the results of LQG indicate that quantum gravity fluctuations produce enough pressure to counteract the gravitational effect before the matter reaches the Planck density. In connection with black holes, this scenario suggests that the gravitational collapse terminates before a singularity is formed. This process leads to the formation of a dense central core whose density is of the order of the Planck density. These objects are called Planck stars which exist within a black hole’s event horizon \([88]\). It is necessary to mention that since the starting point for the emergence of the quantum-gravitational effects is controlled by energy density and not by size, a Planck star is calculated to be much larger than the Planck scale \([88]\).

Black holes whose inside contain a dense core instead of a singularity could be regarded as regular (or non-singular) black holes. In practice, regular black holes can be studied as a geometry that recovers a standard black hole solution at distances sufficiently far from the core while whose center can be treated as a manifold. The general idea for providing a suitable model to describe these types of black holes is changing the mass parameter into a radial mass
function such that near the origin the mass function behaves in a way that singularity disappears and the solution would be regular. More clearly, the regularity of the energy density at the center in $d$ dimensional spacetime requires the mass function $m(r)$ to vanish as $r^{d-1}$ when $r$ tends to zero. In this way, the first idea was proposed by Sakharov and Gline which based on their suggestion singularities could be avoided by a non-singular dS core, with the equation of state $p = -\rho$ \cite{15, 20}. Following this idea, the first regular black holes solution was proposed by Bardeen in which there are horizons but no singularity and close to the origin, solution meets a dS geometry \cite{21}. The Bardeen model is described by the following metric

$$ds^2 = -\left(1 - \frac{2mr^2}{(r^2 + L^2)^{3/2}}\right)dt^2 + \left(1 - \frac{2mr^2}{(r^2 + L^2)^{3/2}}\right)^{-1} dr^2 + r^2 d\Omega^2,$$  \hspace{1cm} (A1)$$

where $L$ has the role of a regulator to avoid the presence of a singularity. Calculation of the curvature invariants shows that this model characterizes the regular spacetime. To investigate the physical interpretation for parameters $m$ and $L$, we can examine the asymptotic behavior of the metric which will be as follows

$$g_{tt} = -1 + 2m/r - 3mL^2/r^3 + O(1/r^5).$$  \hspace{1cm} (A2)$$

Since the second term goes as $1/r$ the parameter $m$ will be associated with the mass of the configuration. However, the next term changes as $1/r^3$ and thus we are not allowed to relate the parameter $L$ with some kind of charge like, for instance, in the Reissner–Nordström solution. A physical source associated with Bardeen’s solution was clarified nearly thirty years later, when Ayon-Beato and Garcia \cite{89} successfully interpreted Bardeen’s black hole in the context of nonlinear electrodynamics and found that $L$ can be interpreted as the monopole charge of a self-gravitating magnetic field described by nonlinear electrodynamics.

As it was mentioned before, considering the quantum gravity corrections can lead to the removal of the curvature singularities existing in the standard black hole geometries. However, since the theory of quantum gravity is not available, regular black holes issues can be regarded as phenomenological toy models in order to explore possible ways to solve the problem of singularity. In fact, quantum gravity corrections can be imitated by introducing an anisotropic fluid that must satisfy a set of conditions. It means that introducing an anisotropic fluid that strongly concentrates at the origin could have the same results to eliminate the singularity. The logic behind this statement is that geometry whose source is the mentioned anisotropic fluid could effectively arise from a low energy limit of quantum gravity, as a solution to Einstein’s equations modified by quantum theory. Therefore, it is feasible to construct non-singular black hole solutions including matter fields in the energy-momentum tensor. In this regard, to provide an appropriate model of energy density which leads to a regular black hole solution, the general conditions must be satisfied by the energy density which will be discussed in what follows.

Here, for simplicity, we impose a highly symmetric geometry and consider only the static case. A $d$—dimensional static spherical symmetric geometry in Schwarzschild coordinates can be described by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2_{d-2}.$$  \hspace{1cm} (A3)$$

Moreover, an anisotropic fluid living in this spacetime, considering the symmetries of the geometry, must have the following form

$$T^\mu_\nu = \text{diag}(-\rho, p_r, p_\theta, p_\phi, ...),$$  \hspace{1cm} (A4)$$

where $\rho = -p_r$, due to the consistency with the Einstein field equations. It is notable that the negative value of the radial component of the pressure can explain the regularity of the solution in the sense that it can impede collapse by weakening the gravitational field \cite{90}. Besides, the conservation law $\nabla_\mu T^{\mu \nu} = 0$ implies that

$$p_\theta = \frac{r}{d-2} \frac{d}{dr} p_r + p_r.$$  \hspace{1cm} (A5)$$

Now, to examine the general behavior of the energy density, it is convenient to define the mass function as follows \cite{36}

$$m(r) = \Omega_{d-2} \int_0^r \rho(x)x^{d-2} dx.$$  \hspace{1cm} (A6)$$

In order to have a well-defined physical solution, the suggested energy density must meet the requirements described in what follows \cite{36}:

1. Although singularities could be replaced by regular regions including the matter that might violate the SEC, the WEC must be satisfied and therefore the energy density must be positive.
2. $\rho$ must be a continuously differentiable function to avoid singularities. This point requires that mass function ($m(r)$), is a positive monotonically increasing function, i.e. if $r_1 \geq r_2$ then $m(r_1) \geq m(r_2)$, which vanishes at the origin.

3. To guarantee a well posed asymptotic behavior, the energy density should be a decreasing function of radial coordinate to vanish at spatial infinity. In fact, $\rho(r)$ must be such that $m(r)$ tends to a constant $M$ which is proportional to the mass of the solution, i.e.

$$\lim_{r \to \infty} \rho(r) = 0,$$
$$\lim_{r \to \infty} m(r) = M,$$

which means that

$$\lim_{r \to \infty} \frac{d}{dr} m(r) = 0.$$  \hspace{1cm} (A7)

4. In order to intimate the quantum gravitational effects, the energy density $\rho$ must have a single maximum at the origin ($r = 0$) and rapidly decreases away from the center which yields the following condition

$$m(r)|_{r \approx 0} \approx K r^{d-1},$$  \hspace{1cm} (A9)

where $K$ is a positive constant and proportional to $\rho(0)$. The finiteness of $\rho(0)$ guarantees absence of singularity at $r = 0$. Moreover, $\rho$ must be such that there is a radious $r = R$ in which $m(R) \approx M$ and $\frac{d}{dr} m(r)|_{r = R} \approx 0$. Generally speaking, $R$ could be in the interval $\ell_P \ll R \ll r_+$ for large masses where $\ell_P$ and $r_+$ stand for Planck length and horizon radius, respectively. However, this condition cannot be applied to configuration whose mass is within the range of Planck scale.

Regarding the mentioned criteria, Estrada and Aros proposed a $d-$dimensional model of energy density to describe a non-singular black hole as follows

$$\rho(r) = \frac{d-1}{\Omega_{d-2}} \frac{L^{d-2} M^2}{(L^{d-2} M + r^{d-1})^2},$$  \hspace{1cm} (A10)

where in 4–dimensions, it reduces to the Hayward metric, a minimal model of energy density to describe Planck stars. It should be noted that, here, $L$ plays the role of the regulator to prevent the formation of singularity. However, the above relation for the energy density is not the most general form that meets all the mentioned criteria. Therefore, motivated by the model suggested by Estrada and Aros, we propose the following energy density

$$\rho(r) = \frac{d-1}{\Omega_{d-2}} \frac{k-1}{2 L^{d-2}} (1 + \frac{r^{d-1}}{2 L^{d-2} M})^{-k},$$  \hspace{1cm} (A11)

where $k \geq 2$ is an integer number. Our suggestion could be viewed as a generalization of the model was put forward by Estrada and Aros which for $k = 2$ and $d = 4$ coincides with the Hayward model.

Finally, it should be noted that although the above energy density is inspired by a quantum gravity model, it can effectively be used as a classical model to describe regular black holes.

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