Game-theoretic Formulations of Sequential Nonparametric One- and Two-Sample Tests

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Abstract

We study the problem of designing consistent sequential one- and two-sample tests in a nonparametric setting. Guided by the principle of testing by betting, we reframe the task of constructing sequential tests into that of selecting payoff functions that maximize the wealth of a fictitious bettor, betting against the null in a repeated game. The resulting sequential test rejects the null when the bettor’s wealth process exceeds an appropriate threshold. We propose a general strategy for selecting payoff functions as predictable estimates of the witness function associated with the variational representation of some statistical distance measures, such as integral probability metrics (IPMs) and ϕ-divergences. Overall, this approach ensures that (i) the wealth process is a non-negative martingale under the null, thus allowing tight control over the type-I error, and (ii) it grows to infinity almost surely under the alternative, thus implying consistency. We accomplish this by designing composite e-processes that remain bounded in expectation under the null, but grow to infinity under the alternative. We instantiate the general test for some common distance metrics to obtain sequential versions of Kolmogorov-Smirnov (KS) test, χ²-test and kernel-MMD test, and empirically demonstrate their ability to adapt to the unknown hardness of the problem. The sequential testing framework constructed in this paper is versatile, and we end with a discussion on applying these ideas to two related problems: testing for higher-order stochastic dominance, and testing for symmetry.

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In this paper, we consider two fundamental problems in statistics; namely, (i) evaluating the goodness-of-fit of models from observed data or one-sample testing, and (ii) checking for homogeneity of samples drawn from two independent sources or two-sample testing. Prior works (with some exceptions discussed in Section 1.4) have mainly studied these problems either in the batch setting (or fixed sample-size setting) or in a sequential parametric setting. Batch tests run the risk of allocating too many (resp. too few) observations on easier (resp. harder) problem instances, whereas the strong parametric assumptions are often not satisfied in many practical tasks, thus limiting the applicability of parametric sequential tests. To address these issues, we propose a general strategy of designing consistent level $\alpha$ sequential nonparametric tests for both one- and two-sample testing problems. Our design strategy is motivated by the principle of testing by betting, recently elucidated by Shafer [2021]. In the context of hypothesis testing, this principle establishes the equivalence between gathering evidence against the null, and multiplying an initial wealth by a large factor by repeatedly betting on the observations with payoff functions bought for their expected value under the null (see Section 1.2 for a formal description of the betting game). Using this principle, the task of designing a sequential test can be restated as that of selecting a sequence of betting payoff functions that ensure the fastest rate of growth of the bettor’s wealth, as we describe next.

Consider a hypothesis testing problem with a null hypothesis $H_0$ and alternative $H_1$, and observations denoted by $Z_1, Z_2, \ldots$ lying in some space $\mathcal{Z}$. To test the null $H_0$, a bettor may place repeated bets on the outcomes $\{Z_t : t \geq 1\}$ starting with an initial wealth $K_0 > 0$. A single round of betting (say at time $t$) involves the following two steps. First, the bettor selects a payoff function $S_t : \mathcal{Z} \to [0, \infty)$, under the restriction that it ensures a fair bet if the null is true. Formally, this is imposed by requiring $S_t$ to satisfy
can be precisely characterized in terms of how closely the predictions on the quality of the predictions. In Section 3.3.1, we show that the performance of the sequential test stated informally, proceeds by repeatedly betting on the label of the next observation. Waudby-Smith and of a universal sequential probability assignment scheme for designing their sequential two-sample test that, two-sample tests with strong performance guarantees.

Furthermore, this connection also allows us to leverage the large body of work in the online learning function \( f \) of \( S_t \), we instead use data-driven predictable estimates \( Q \) of the (oracle) payoff function, since it depends on the true distribution shifted (to ensure conditional mean 1) version of \( S \). The restriction on the conditional expectation of the payoff functions implies that \( \{ S_t : t \geq 0 \} \) is a nonnegative martingale under the null, and hence \( K_t / K_0 \) is unlikely to take large values for any \( t \geq 1 \). On the other hand, when \( H_1 \) is true, there exist payoff functions, satisfying the above constraints, that can ensure exponential growth in the bettor’s wealth. The above discussion suggests a natural sequential test: reject the null if \( K_t \geq a \), for some appropriately selected threshold \( a \) based on the required confidence level \( \alpha \in (0, 1) \).

When testing simple hypotheses \( H_0 : Z_t \sim P \) and \( H_1 : Z_t \sim Q \) with \( P \) and \( Q \) known, an obvious choice of \( S_t \) is the likelihood ratio \( dQ/dP \). Indeed, with this choice of payoff functions, the resulting test reduces to the well-known Sequential Probability Ratio Test (SPRT) of Wald [1945], and hence also inherits its strong optimality properties (Wald and Wolfowitz [1948]. However, when dealing with cases where either one or both of \( H_0 \) and \( H_1 \) are composite and nonparametric (as is the case with one- and two-sample testing considered in this paper), there isn’t any obvious choice for the payoff functions. The primary conceptual contribution of this paper is to propose a general strategy of selecting appropriate payoff functions, that ensure consistency of the resulting sequential test when the alternative is known to deviate from the null in terms of some known statistical discrepancy or metric.

To summarize, the principle of testing by betting allows us to reduce the problem of designing good sequential tests to that of selecting a sequence of payoff functions \( S_t : Z \rightarrow (0, \infty) \) such that (i) \( S_t \) has a conditional mean 1 under the null, and (ii) \( S_t(Z_t) \) is likely to take large values under the alternative. Our proposed approach for selecting the payoff functions relies on using a class of statistical distance measures, namely integral probability metrics (IPMs) and \( \varphi \)-divergences, that admit certain forms of variational representation (details in Section 3.2). To get an overview of the idea, consider a one-sample testing problem with observations \( Z_1, Z_2, \ldots \), where \( Z_t \sim Q \) i.i.d. and \( H_0 : Q = P \) for known \( P \) and \( H_1 : Q \neq P \). With \( \mathcal{P}(Z) \) denoting the class of all probability distributions on \( Z \), choose a distance measure \( d_Q : \mathcal{P}(Z) \times \mathcal{P}(Z) \rightarrow [0, \infty] \), defined as

\[
d_Q(P, Q) := \sup_{g \in \mathcal{G}} [\mathbb{E}_{Z \sim P}[g(Z)] - \mathbb{E}_{Z \sim Q}[g(Z)]],
\]

where \( \mathcal{G} \) is some class of real-valued functions on \( Z \). The function \( g^\ast \) that achieves the supremum in (1) provides the maximum contrast between the two distributions \( P \) and \( Q \). This suggests that a suitable choice of the (oracle) payoff function is \( S^\ast = 1 + f^\ast \), where \( f^\ast \) is a possibly scaled (to ensure nonnegativity) and shifted (to ensure conditional mean 1) version of \( g^\ast \) to ensure nonnegativity of \( S^\ast \). We referred to \( S^\ast \) as the oracle payoff function, since it depends on the true distribution \( Q \) that is not known. To design practical tests, we instead use data-driven predictable estimates \( \{ f_t : t \geq 1 \} \) of \( f^\ast \) to form the payoff function \( \{ S_t = 1 + f_t : t \geq 1 \} \). Naturally, the performance of the resulting test under the alternative will depend on the quality of the predictions. In Section 3.3.1, we show that the performance of the sequential test can be precisely characterized in terms of how closely the predictions \( \{ f_t : t \geq 1 \} \) match the oracle payoff function \( f^\ast \). This observation is significant, as it establishes a formal connection between the performance in a sequential prediction game and the power of the resulting sequential test (see Proposition 5 and Proposition 6). Furthermore, this connection also allows us to leverage the large body of work in the online learning literature on the design of prediction algorithms and their regret analysis, to construct sequential one- and two-sample tests with strong performance guarantees.

We note that some of the ideas used in the testing framework developed in this paper have also been employed in some existing works in sequential analysis. For instance, Lheritier and Caizals [2018] made use of a universal sequential probability assignment scheme for designing their sequential two-sample test that, stated informally, proceeds by repeatedly betting on the label of the next observation. Waudby-Smith and

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1Throughout this paper, we overload the term ‘payoff function’ to refer to \( S_t \) as well as \( f_t \), where \( S_t(\cdot) = 1 + f_t(\cdot) \).
Ramdas (2020) adapted the principle of testing by betting to the task of testing and estimating bounded functions of random variables. More recently, Ramdas, Ruf, Larsson, and Koolen (2021) employed ideas from universal compression to design power one tests for testing the exchangeability of a binary sequence. The main distinguishing feature of our work is the general approach for selecting payoff functions based on certain statistical distance measures (detailed in Section 3) for defining the repeated betting game (see Definition 4) underlying the sequential test. Our tests constructed using this strategy satisfy stronger theoretical guarantees and also show better empirical performance in some experimental results. Furthermore, the generality of our framework implies that the same ideas can be used in designing sequential tests for several other testing problems, two of which are discussed in Appendix F.

Organization of the paper. In Section 1.1 we formally introduce the one- and two-sample testing problems, and also fix the notations used. Then, we explain the principle of testing by betting in Section 1.2 and discuss the main contributions of the paper in Section 1.3. We end the introduction with a discussion of the related works in this area. In Section 2, we describe some key ideas from related literature that form the building blocks of the design and analysis of our sequential tests. Section 3 contains a detailed description of our proposed framework of adapting the ideas from Section 1.2 to construct sequential one- and two-sample tests. In Section 4 and Section 5, we present instances of one- and two-sample tests designed based on the strategy outlined in Section 3. Finally, Section 6 contains the conclusion and discusses some extensions.

1.1 Problem Setup

We begin by formally defining one-sided sequential tests. The goal in one-sided sequential tests, motivated by the development of 'power one' tests by Darling and Robbins (1968), is to reject the null with minimum number of observations when the alternative is true while controlling the type-I error below a prescribed level \( \alpha \in (0, 1) \).

**Definition 1 (sequential-test).** A level-\( \alpha \) one-sided sequential test can be represented by a random stopping time \( \tau \) taking values in \( \{1, 2, \ldots\} \cup \{\infty\} \), and satisfying the condition

\[
P(\tau < \infty) \leq \alpha,
\]

under the null \( H_0 \). Here \( \tau \) denotes the time at which the null hypothesis is rejected, and sampling stops.

Next, we define the one- and two-sample hypothesis testing problems studied in this paper.

**Definition 2 (one-sample testing).** In the one-sample or goodness-of-fit testing problem, the observations \( Y_1, \ldots, Y_t, \ldots \) are assumed to be drawn i.i.d. from some distribution \( Q \). Given a target distribution \( P \), the goal is to test the null hypothesis that \( Q = P \) against the alternative that \( Q \neq P \); that is,

\[
H_0 : Y_t \sim Q \text{ i.i.d. and } Q = P, \quad \text{versus} \quad H_1 : Y_t \sim Q \text{ i.i.d. and } Q \neq P.
\]

The null hypothesis is simple, as it states that \( Q \) belongs to the singleton set \( \{P\} \), whereas the alternative \( Q \in \{Q' : Q' \neq P\} \) is composite.

**Definition 3 (two-sample testing).** In the two-sample testing problem (with paired observations), we observe a sequence \( (X_1, Y_1), \ldots, (X_t, Y_t), \ldots \) drawn i.i.d. from a product distribution \( P \times Q \), with both \( P \) and \( Q \) unknown. The goal again is to test the null hypothesis that \( P \) is equal to \( Q \), against the alternative that \( P \neq Q \).

\[
H_0 : (X_t, Y_t) \sim P \times Q \text{ i.i.d. and } Q = P, \quad \text{versus} \quad H_1 : (X_t, Y_t) \sim P \times Q \text{ i.i.d. and } Q \neq P.
\]

Unlike the one-sample case, here the null as well as the alternative hypotheses are composite.
Remark 1. In this paper, we study the two-sample testing problem with the paired observation model and i.i.d. observations, primarily to simplify the presentation and focus on the key ideas involved in designing the tests. However, we note that the same ideas are also applicable in much more general two-sample testing frameworks. In particular, we can relax the paired observation assumption as long as we see draws from both populations infinitely often. More importantly, our framework easily incorporates the case when the observations are drawn from time varying distributions $P_t$ and $Q_t$, as long as $P_t$ and $Q_t$ are predictable.

Notations. We use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote an underlying probability space over which all the random variables are defined. We assume that all the random variables considered are $\mathcal{X}$-valued, where $(\mathcal{X}, \mathcal{M})$ represents some measurable space. In most instances, $\mathcal{X}$ is either a finite set with $\mathcal{M} = 2^\mathcal{X}$, or $\mathbb{R}^m$ (for some $m \geq 1$) with $\mathcal{M}$ being the corresponding Borel sigma-algebra. We say an $\mathcal{X}$-valued random variable $Y$ has distribution $P$, if for any $E \in \mathcal{M}$ we have $\mathbb{P}(Y \in E) = P(E)$.

For the one-sample tests, we use $Y_1, Y_2, \ldots$ to represent the observations drawn i.i.d. according to an unknown distribution $Q$. The null hypothesis is $H_0 : Q = P$, for a known distribution $P$ on $(\mathcal{X}, \mathcal{M})$ while the alternative is $H_1 : Q \neq P$. For the two-sample tests, we use $(X_1, Y_1), (X_2, Y_2), \ldots$ to represent the i.i.d. observations drawn according to $P \times Q$, for two unknown distributions $P$ and $Q$ on $(\mathcal{X}, \mathcal{M})$. In this case, the null hypothesis is $H_0 : P = Q$ while the alternative is $P \neq Q$. In general, when discussing arbitrary hypothesis tests, we use $Z_1, Z_2, \ldots$ to represent the observations. For $t \geq 1$, we use $\mathcal{F}_t \subset \mathcal{F}$ to denote the sigma-algebra generated by the first $t$ observations $Z_1, \ldots, Z_t$; that is $\mathcal{F}_t = \sigma(Z_1, \ldots, Z_t)$ is the smallest sigma-algebra of subsets of $\Omega$ that makes $\{Z_i : 1 \leq i \leq t\}$ measurable. Thus, for the one-sample problems $\mathcal{F}_t$ represents $\sigma(Y_1, \ldots, Y_t)$, while for the two-sample tests $\mathcal{F}_t$ is $\sigma(X_1, Y_1, \ldots, X_t, Y_t)$. In both cases, $\mathcal{F}_0$ represents the trivial sigma-algebra $\{\Omega, \emptyset\}$.

If $P$ is a real-valued probability distribution, we use $F_P$ to denote its cumulative distribution functions (cdfs). Similarly, for $Y_1, \ldots, Y_t \sim P \text{ i.i.d.}$, we use $\hat{F}_t$ to denote the empirical cdf. When $P$ is a discrete distribution taking values in some set $\mathcal{X}$ with $|\mathcal{X}| = m$, we will use $p$ to denote the probability mass function (pmf) lying in $\Delta_m$. Here, we use $\Delta_m$ to represent the $m-1$-dimensional probability simplex; that is, $\Delta_m = \{(a_1, \ldots, a_m) \in [0,1]^m : \sum_{i=1}^m a_i = 1\}$.

For any compact and convex subset $E$ of $\mathbb{R}^m$ and any $x \in \mathbb{R}^m$, we use $\Pi_E(x)$ to denote the $(\ell_2\text{-norm})$ projection of $x$ onto $E$.

1.2 Testing by Betting

We now discuss the principle of testing by betting described by Shafer (2021). The fundamental idea is as follows (Shafer 2021 § 2): the claim that a random variable $Z$ is distributed according to $P_Z$ can be interpreted equivalently as the offer of a bet with any payoff function sold for its expected value under the distribution $P_Z$. In the rest of this section, we describe how this general principle can be adapted to design practical one- and two-sample tests.

Consider a full-information game involving three players, whom we refer to as Forecaster, Skeptic and Reality following Shafer and Vovk (2019) and Shafer (2021). The game begins with Forecaster claiming that the distribution of the $Z$-valued random variable $Z$, denoted by $P_Z$, belongs to some family of distributions $\mathcal{P}_0$. Skeptic, who has an initial wealth $\mathcal{K}_0$, then selects a payoff function $f : Z \to [-1, \infty)$ satisfying $\mathbb{E}_{Z \sim P_Z}[f(Z)] = 0$ for all $P_Z \in \mathcal{P}_0$ (assuming such an $f$ exists), and risks $\lambda \mathcal{K}_0$ on the outcome of $Z$ for some fraction $\lambda \in [0,1]$. Reality, then reveals a realization of the random variable $Z$, denoted by $z_1$, and Skeptic wins an amount $\lambda \mathcal{K}_0 f(z_1)$; that is, Skeptic’s total wealth after observing $z_1$ is $\mathcal{K}_0 + \lambda \mathcal{K}_0 f(z_1) = \mathcal{K}_0 (1 + \lambda f(z_1))$. If the payoff function chosen by Skeptic leads to a large profit, this can be interpreted as evidence against the claim made by Forecaster. This process may be continued over several realizations of the random variable $Z$, to define the following repeated game.
Definition 4 (Betting Protocol). Before the start of the game, Forecaster declares that the collection \( \{Z_t : t \geq 1\} \) taking values in \( Z \) is distributed i.i.d. according to some \( P_Z \in \mathcal{P}_0 \). Skeptic begins with an initial wealth \( K_0 = 1 \). Then, for \( t = 1, 2, \ldots, n \):

- **Skeptic** selects a function \( f_t : Z \to [-1, \infty) \) such that \( \mathbb{E}_{Z \sim P'} [f_t(Z) | \mathcal{F}_{t-1}] = 0 \) for all \( P' \in \mathcal{P}_0 \), and \( \mathcal{F}_{t-1} = \sigma(Z_1, \ldots, Z_{t-1}) \).
- **Skeptic** bets an amount \( \lambda_t \mathcal{K}_{t-1} \) for an \( \mathcal{F}_{t-1} \)-measurable \( \lambda_t \in [0, 1] \) on the next realization of \( Z \).
- **Reality** reveals the next realization \( Z_t \).
- The wealth of the **Skeptic** is updated as \( \mathcal{K}_t = \mathcal{K}_{t-1} + \lambda_t \mathcal{K}_{t-1} f_t(Z_t) = \mathcal{K}_{t-1} (1 + \lambda_t f_t(Z_t)) \).

Remark 2. As stated in Definition 4, in any round \( t \) of the betting game, **Skeptic** predicts two quantities: the value of the bet \( \lambda_t \in [0, 1] \) and the payoff function \( f_t : Z \to [-1, \infty) \). However, while updating the wealth process, these two quantities are combined, and hence, effectively **Skeptic** only needs to predict one function \( \widetilde{f}_t = \lambda_t f_t \) in each round. This decoupling of \( \widetilde{f}_t \) into \( \lambda_t \) and \( f_t \) is followed throughout this paper as it simplifies the presentation. With this convention, we can design consistent tests by using a simple plug-in strategy for predicting the functions \( \{f_t : t \geq 1\} \) along with a mixture or randomized strategy for \( \{\lambda_t : t \geq 1\} \). The alternative would be to use a randomized prediction strategy directly in the space of functions \( \{\widetilde{f}_t : t \geq 1\} \), that may be less intuitive.

Remark 3. For the rest of the paper, we will restrict our attention to the payoff functions \( f_t \) taking values in \([-1, 1\) instead of the more general \([-1, \infty)\). This is mainly because all the practical tests that we construct later in Section 3 and Section 5 are built on payoff functions satisfying this condition. This restriction also leads to a simpler path to lower-bounding the wealth process associated with the randomized betting strategies discussed in Section 2.1. Under this restriction, we can allow the bet value \( \lambda_t \) to take values in \([-1, 1\) instead of the earlier \([0, 1\). Here negative \( \lambda_t \) can be interpreted as **Skeptic** betting on \( f_t(Z_t) \) being negative, similar to the coin betting game of Orabona and Pal (2016).

Remark 4. In the one- and two-sample testing problems, the role of **Forecaster** is played by the null hypothesis, the role of **Skeptic** is played by the statistician and the role of **Reality** is played by the independent and identically distributed (i.i.d.) source, generating the observations. As mentioned earlier, one significant point of difference between the one- and two-sample problems is in the composition of the class \( \mathcal{P}_0 \) in the **Forecaster**'s claim (or equivalently in the definition of null hypothesis). In the one-sample case, \( \mathcal{P}_0 \) is a singleton set, corresponding to a simple null. On the other hand, in the two-sample case, the null hypothesis is composite, with \( \mathcal{P}_0 \) consisting of all product distributions \( P_X \times P_Y \) with \( P_X = P_Y \).

Remark 5. We note that the wealth process \( \{\mathcal{K}_t : t \geq 0\} \) resulting from the betting game of Definition 4 with an initial value \( \mathcal{K}_0 = 1 \), can be shown to satisfy the following property:

\[
\sup_{P \in \mathcal{P}_0} \sup_{\tau} \mathbb{E}_P [\mathcal{K}_\tau] \leq 1,
\]

where \( \tau \) is any stopping time adapted to \( \{\mathcal{F}_t : t \geq 0\} \). Here \( \mathbb{E}_P \) denotes the expectation when \( P \) is the true distribution generating the i.i.d. observations \( \{Z_t : t \geq 1\} \). The above inequality is the defining property of e-processes studied in recent works such as (Ramdas, Ruf, Larsson, and Koolen 2021; Wang and Ramdas 2020), and thus it means that if we can find \( \{f_t : t \geq 1\} \) satisfying the properties required in Definition 4, the resulting wealth process is an e-process. Hence, our general approach for designing such \( \{f_t : t \geq 1\} \) described in Section 3 can also be thought of as a method of constructing non-trivial \( \mathcal{P}_0 \)-safe e-processes that grow to infinity almost surely under the alternative.
To show how the ideas introduced above can be employed in practice, we now discuss an example involving a simple hypothesis testing problem. This example also illustrates how the different choices of the two design elements: the betting strategy (selecting the fraction $\lambda_t$ at time $t$) and the choice of pay-off functions ($f_t$) can impact the growth of Skeptic’s wealth (or equivalently the magnitude of evidence collected against Forecaster’s claim).

**Example 1.** Suppose Forecaster claims that the random variables $\{Y_t: t \geq 1\}$ are i.i.d. from the distribution $P = \text{Uniform}([-1, 1])$, while Skeptic believes that they are i.i.d. from a distribution $Q$, also supported on $[-1, 1]$, but with pdf $q(x) = 0.2I_{x \in [-1, 0]} + 0.8I_{x \in [0, 1]}$. To test Forecaster’s claim, Skeptic can play the repeated betting game of Definition 4. For simplicity, suppose that Skeptic plays the same $f_t$ and $\lambda_t$ for all $t \geq 1$ (this is only for illustrative purposes, and all our proposed strategies will adapt the bets and payoff functions based on the observations).

We consider two possible choices of the payoff functions, called payoff function 1 (or $f^{(1)}$) and payoff function 2 (or $f^{(2)}$) (shown in the top-right panel of Figure 1) as well as two choices of $\lambda$; namely $\lambda = 0.2$ and $\lambda = -0.2$ respectively. The performance of the four strategies corresponding the choice of payoff function from $\{f^{(1)}, f^{(2)}\}$ and bet from $\{-0.2, 0.2\}$ are shown in the bottom row of Figure 1. The main observations are as follows:

- **When the Forecaster is correct;** that is, when $P$ is the true distribution, none of the four strategies lead to a noticeable increase in Skeptic’s wealth.

- **When the Forecaster is wrong (and $Q$ is the true distribution),** the bottom left figure shows that it is possible for the Skeptic to achieve an exponentially growing wealth. Furthermore, for a fixed betting strategy ($\lambda = 0.2$), the actual growth rate is strongly influenced by the choice of the payoff functions. Since $f^{(2)}$ emphasizes the difference between $P$ and $Q$ more than $f^{(1)}$, it results in a larger growth rate.

- **Finally, the bottom right figure shows the importance of a good betting strategy.** The choice of $\lambda = -0.2$ assumes that the payoffs are more likely to take negative values, which is wrong when $Q$ is the true distribution. In this case, Skeptic’s wealth decays exponentially to zero.

As mentioned earlier, for simple null and alternative hypotheses, the natural choice of betting payoff function is the likelihood ratio. However, in the general case of composite hypotheses, there exists no such natural choice of $f_t$. Thus, in order to design practical sequential tests with composite hypotheses using this approach, we need to specify two things:

1. A **betting strategy** for selecting the fraction and sign of wealth to be wagered $\{\lambda_t: t \geq 1\}$.

2. A **prediction strategy** for selecting a sequence of good payoff functions $\{f_t: t \geq 1\}$.

For the betting strategy, we will employ a mixture-method based on ideas from the universal portfolio optimization literature as detailed in Section 2.1. In Section 3, we present an interpretable approach for designing payoff functions by employing class of statistical distance measures that admit a variational representation.

### 1.3 Summary of Contributions

We now describe the main contributions of our paper.

1. **Our first contribution is to describe a general framework for designing sequential nonparametric tests using the principle of testing by betting** outlined in Section 1.2. The details of this approach are in Section 3.
Wealth (log scale)

Two Probability Distribution Functions

Payoff Functions

Figure 1: The top-left figure shows the two distributions $P$ and $Q$ considered in Example 1, while the figure in the top-right panel shows the two possible payoff functions used by Skeptic. Since $\mathbb{E}_{Y \sim P} [f^{(i)}(Y)] = 0$ for $i = 1, 2$, both $f^{(1)}$ and $f^{(2)}$ are valid payoff functions for testing Forecaster’s claim that the observations $\{Y_t : t \geq 1\}$ are drawn i.i.d. from $P$. The two figures in the bottom row show the evolution of Skeptic’s wealth (averaged over 100 trials) with the four strategies (formed by the combination of the two payoff functions $f^{(1)}$ and $f^{(2)}$ with two constant betting strategies $\lambda_t = \lambda$ for all $t$, with $\lambda \in \{-0.2, 0.2\}$). If the true distribution is indeed $P$, then none of the strategies lead to a significant increase in Skeptic’s wealth. However, if the true distribution is $Q$, it is possible to obtain an exponential increase in wealth.

2. In Section 3.3.1, we obtain guarantees on the performance of the sequential tests resulting from the strategy mentioned in step 1. First, in Proposition 3, we show that the sequential tests control type-I error at the given level $\alpha$. This result uses the fact that the wealth process involved in defining the sequential is a nonnegative martingale by construction.

Our next set of results formalize the relation between the quality of our strategy of predicting the payoff function $\{f_t : t \geq 1\}$ and the power of the sequential test. In particular, we show the following:

- In Proposition 4, we observe that if the estimates ensure $\lim \inf_{n \to \infty} \left| \frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) \right| > 0$ on some sequence of observations $\{Z_t : t \geq 1\}$, then the test stops in finite time. Consequently, if this condition happens almost surely under the alternative, it implies that the sequential test has power one.
- A stronger notion of performance of the prediction strategy is the normalized regret, defined as $R_n := \frac{1}{n} \sum_{t=1}^{n} f^*(Z_t) - f_t(Z_t)$, where $Z_t$ denotes the observation in round $t$. In Proposition 4, we show that if the prediction strategy ensures that $R_n$ converges to zero uniformly over all sequences...
of observations \( \{Z_t : t \geq 1\} \), then the type-II error of the corresponding test converges to zero exponentially.

3. In Section 4, we consider the problem of one-sample testing, and design two sequential tests using the general strategy described earlier.

- Our first one-sample test is based on the Kolmogorov-Smirnov (KS) metric between two real valued distributions. For this test, we show the following two results. (i) In Corollary 1, we show that a simple plug-in (or greedy) prediction strategy based on empirical estimates of the unknown distribution is sufficient to achieve asymptotic consistency. (ii) Next, under the additional assumption that both the distributions are supported on \( \mathcal{X} = [0, 1] \), and that the target distribution \( P \) is uniform, we show in Proposition 7 that there exists a no-regret strategy for selecting payoff functions, hence implying an exponentially decaying type-II error. In the process, we obtain a general result about online learning with a continuum of experts and discontinuous losses that may be of independent interest.

- The second test is a sequential test for discrete distributions based on the \( \chi^2 \) distance. For this test, we show in Corollary 2 that a simple plug-in prediction strategy is sufficient to ensure asymptotic consistency. Furthermore, in Proposition 9 we show that a more involved strategy based on online projected gradient descent also ensures that the exponential decay of type-II error.

4. In Section 5, we consider the two-sample testing problem, and design two sequential tests based on the ideas of Section 3.

- The first test is a sequential two-sample KS test, for which we show in Corollary 3 that the plug-in prediction strategy results in an asymptotically consistent test.

- The second test is based on the kernel-MMD distance. For this test again, we show in Corollary 4 that the plug-in strategy results in an asymptotically consistent test. Furthermore, in Proposition 10, we show that by using a more complex prediction strategy motivated by an online linear optimization algorithm in Hilbert spaces proposed by Orabona and Pal (2016), we can ensure the exponential decay of the type-II error.

5. In Section 6, we conclude the paper and note that even though our focus in this paper is on one- and two-sample testing, the underlying ideas are much more general and can be easily extended to design sequential tests for several other problems. To illustrate this, we demonstrate how to design sequential tests for two problems: (i) testing for stochastic dominance of real-valued distributions, and (ii) testing for symmetry of real-valued distributions.

### 1.4 Related Work

The area of sequential hypothesis testing was initiated by Wald (1945), who proposed and analyzed the Sequential Probability Ratio Test (SPRT) for testing a simple null against a simple alternative. Wald and Wolfowitz (1948) established strong optimality properties of SPRT, and in particular, showed that the SPRT has the smallest expected sample-size among all tests (include fixed sample-size) that control the type-I and type-II errors below prescribed levels. Following Wald (1945), there has been a significant body of work on extending the SPRT to composite but parametric family of hypotheses. These extensions are based on different approaches such as invariant tests (Wijsman 1979; Lai 1981), generalized likelihood ratio tests (Lai 1988; Lai and Zhang 1994) or the mixture methods (Lai 1976). For a detailed overview of the literature on this topic, the reader is referred to (Ghosh and Sen 1991, Chapters 2 & 4).

Unlike the parametric case, the literature on sequential nonparametric tests is comparatively limited. Some early work in this area proposed sequential nonparametric two-sample tests under strong assumptions.
on the alternative, not satisfied under the general setting considered in this paper. For instance, Savage and Sethuraman (1966) presented a sequential rank test with Lehmann alternatives, where one of the distribution functions is a specified power of the other. For a more detailed discussion of early works in sequential nonparametric testing, see (Ghosh and Sen 1991, Chapter 14, § 2). More related to our paper, are the works such as (Darling and Robbins 1968; Howard and Ramdas 2022; Balsubramani and Ramdas 2016; Lhérétier and Cazals 2018; Manole and Ramdas 2021) that do not make strong simplifying assumptions on the alternative, and we discuss them in more details below.

Darling and Robbins (1968) considered several nonparametric one- and two-sample testing problems involving real-valued observations, and proposed sequential tests by combining fixed sample-size uniform deviation inequalities for the empirical distribution function with a union bound over time. Howard and Ramdas (2022) proposed a sequential Kolmogorov-Smirnov (KS) test by obtaining a tighter time-uniform deviation inequality for the empirical distribution functions. They followed a different approach than Darling and Robbins (1968), and used an inequality from the empirical process theory along with the peeling technique of dividing the set of natural numbers into disjoint intervals of exponentially increasing lengths. However, the sequential tests from both these works only work with real-valued observations (or more generally, observations in a totally ordered space) and cannot be applied in problems involving multivariate observations or observations in a discrete unordered set. The other tests discussed below address this issue.

Balsubramani and Ramdas (2016) derived a time-uniform empirical Bernstein inequality for random walks by building upon an earlier time-uniform martingale inequality of Balsubramani (2014). Using this result, they proposed a sequential nonparametric two-sample test by employing the linear-time kernel-MMD as the test statistic, and obtained theoretical guarantees on its power and expected stopping time. The original batch two-sample kernel-MMD test, proposed by Gretton, Borgwardt, Rasch, Schölkopf, and Smola (2012), uses a quadratic-time U-statistic as an empirical estimate of the squared MMD distance. The reliance of the sequential test of Balsubramani and Ramdas (2016) on the linear-time MMD statistic, while making the test computationally more efficient, also makes it less powerful than our proposed kernel-MMD test (in Section 5.2) and the tests of Lhérétier and Cazals (2018) and Manole and Ramdas (2021) discussed below. Figure 6 in Appendix E empirically shows the significant difference in the powers of the test of Balsubramani and Ramdas (2016) and our sequential kernel-MMD test.

Lhérétier and Cazals (2018) proposed a general approach to designing sequential nonparametric two-sample tests, by using sequentially learned probabilistic predictors of the labels indicating the population from which an observation was drawn. The authors identified sufficient conditions for the λ-consistency (a weaker notion of consistency, that informally requires the distributions to be well separated in terms of mutual information) of the resulting sequential test, and show that these conditions are satisfied when the predictors employed in defining the test are strongly pointwise consistent (Györfi, Kohler, Krzyżak, and Walk 2002, Definition 25.1). Compared to Lhérétier and Cazals (2018), our proposed approach leads to sequential tests with stronger performance guarantees. In particular, in Proposition 4, we identify sufficient conditions for our tests to be consistent (in the usual sense) and in Proposition 6, we identify sufficient conditions under which the type-II error of our tests decays at an exponential rate.

Manole and Ramdas (2021) propose a general technique for constructing confidence sequences for convex divergences between two probability distributions. Their approach relies on the key observation that the empirical divergence process is a reverse submartingale adapted to the exchangeable filtration. By instantiating the general confidence sequence for the special cases of the Kolmogorov-Smirnov metric (Manole and Ramdas 2021, § 4.1) and kernel-MMD distance (Manole and Ramdas 2021, § 4.2), the authors obtain consistent sequential nonparametric two-sample tests for both univariate and multivariate distributions. Unlike Manole and Ramdas (2021), our approach relies on constructing martingales (instead of reverse submartingales) from statistical distances with a variational definition. Hence, our resulting sequential tests are expected to be less conservative than those of Manole and Ramdas (2021) in rejecting the null. This intuition is verified in some numerical experiments in Section 5.2.2, where the stopping times, under the alternative, of our proposed
test are significantly smaller than those of Manole and Ramdas (2021).

2 Preliminaries

Our work is based on ideas drawn from different, seemingly unrelated, research areas. In this section, we recall (and in some cases, adapt) the key results and concepts from these different fields that will be used in designing our sequential tests.

2.1 Betting Strategies

In this paper, we will use a betting strategy, that we refer to as the mixture-method, based on the approach used in the universal portfolio optimization algorithm of Cover (1991). The main idea is as follows: consider the continuum of constant betting strategies that result in a wealth process \( \{K^\lambda_t : t \geq 1\} \) for every \( \lambda \in [-1,1] \). If our goal is to perform as well as the best constant betting strategy in hindsight, we can distribute our initial wealth \((K_0)\) among this continuum of constant betting strategies according to some distribution \(\nu\) on \([-1,1]\). Since the best constant strategy in hindsight after \(n\) rounds, say \(\lambda^*_n\), does exponentially better than strategies strictly bounded away from \(\lambda^*_n\), the average wealth is dominated by this term. This statement is made precise in Proposition 1.

First, we formally define the mixture-method.

Definition 5 (Mixture-method). The wealth process for the mixture-method of betting with probability density \(\nu\) on \([-1,1]\) and an initial wealth \(K_0\) is

\[
K_n = \int_{-1}^{1} K^\lambda_n \nu(\lambda) d\lambda, \quad \text{where} \quad K^\lambda_n := K_0 \prod_{t=1}^{n} (1 + \lambda f_t(Z_t)).
\]

Under mild conditions on the mixture distribution \(\nu\), we can show that the resulting wealth process grows nearly as fast as the best constant betting strategy in hindsight.

Proposition 1. Suppose the mixture density satisfies \(\nu(\lambda) \geq c_0\) for all \(\lambda \in [-1,1]\) for some \(c_0 > 0\). Then for any \(\lambda_{\max} \in (-1,1)\) and \(\Delta \lambda < 1 - \lambda_{\max}\) and \(\{f_t : t \geq 1\}\) with \(f_t(x) \geq -1\) for all \(x\), the wealth resulting from the mixture-method betting strategy after \(n\) rounds satisfies:

\[
K_n \geq \exp \left( n \left( \max_{\lambda \in [-\lambda_{\max}, \lambda_{\max}]} \frac{1}{n} \sum_{t=1}^{n} \log(1 + \lambda f_t(Z_t)) \right) - \frac{n \Delta \lambda}{1 - \lambda_{\max} - \Delta \lambda} \log \left( \frac{1}{c_0 \Delta \lambda} \right) \right). \quad (2)
\]

In particular, by setting \(\lambda_{\max} = 1/2\) and \(\Delta \lambda = 1/n\) for \(n \geq 4\), we get

\[
K_n \geq \exp \left( \frac{n}{8} \left( \frac{1}{n} \sum_{t=1}^{n} f_t(Y_t) \right)^2 - 4 - \log(n/c_0) \right) = \exp \left( \frac{n}{8} \left( \frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) \right)^2 - o(n) \right). \quad (3)
\]

The proof of this statement is in Appendix B.1 and it follows from two observations: (i) for any \(\bar{\lambda} \in (0,1)\), the mapping \(x \mapsto \log(1 + x)\) with \(x \in [-\bar{\lambda}, \bar{\lambda}]\) is \(1/(1 - \bar{\lambda})\)-Lipschitz, and (ii) \(\nu\) assigns a measure of at least \(c_0 \Delta \lambda\) to any interval of length \(\Delta \lambda\) containing the best constant bet. These two facts account for the second and third terms inside the exponent in (2).

Remark 6. In the practical implementation of the mixture-method, we replace the integral with a sum over values of \(\lambda\) in a finite uniform grid of points from \(-1\) to \(1\), with the end points excluded, and use the uniform distribution as the prior \(\nu\).
Remark 7. We note that there exist other betting strategies, such as potential-based strategies of Orabona and Pal (2016) or the Online Newton Step (ONS) strategy of Cutkosky and Orabona (2018), that can also be used in designing our tests. These methods can also guarantee results similar to those stated in Proposition 1 (in fact, with better constants in the $o(n)$ term). However, we restrict our attention to the mixture strategy of Definition 5 due to its simplicity, and the fact that in some preliminary comparisons the mixture-method outperformed the potential-based and ONS strategies, especially over shorter horizons. We leave a more thorough investigation of the effects of different betting schemes to future work.

2.2 Nonnegative Martingales and Ville’s Inequality

As we saw in Example 1, if Forecaster’s claim is true, then it is unlikely that Skeptic’s wealth can be multiplied by a large factor using any strategy. This intuition can be made formal by observing that for any valid sequence of payoff functions, the wealth process $\{K_t : t \geq 0\}$ is a nonnegative martingale if Forecaster’s claim is true (that is when the null hypothesis is true). Specifically, for the wealth process corresponding to payoff functions $\{f_t : t \geq 1\}$ and using the mixture-method for betting, we have the following under the null:

$$
E[K_t | F_{t-1}] = \int_0^1 K^\lambda_{t-1} (1 + \lambda f_t(Z_t)) \nu(\lambda) d\lambda | F_{t-1}
$$

$$
= \int_0^1 K^\lambda_{t-1} E[1 + \lambda f_t(Z_t) | F_{t-1}] \nu(\lambda) d\lambda
$$

(4)

$$
= \int_0^1 K^\lambda_{t-1} \nu(\lambda) d\lambda = K_{t-1}.
$$

(5)

The interchange of conditional expectation and integral to obtain (4) is justified by appealing to the conditional monotone convergence theorem, while (5) follows from the defining condition of a valid payoff function.

We now state a time-uniform inequality for nonnegative supermartingales, which implies that it is unlikely for the wealth process to take large values under the null hypothesis.

Fact 1 (Ville’s Inequality). Suppose $\{K_t : t \geq 0\}$ is a nonnegative supermartingale process adapted to a filtration $\{F_t : t \geq 0\}$. Then, we have the following for any $a > 0$:

$$
P(\exists t \geq 1 : K_t \geq a) \leq \frac{E[K_0]}{a}.
$$

This fact implies that for a given level $\alpha$, any wealth process constructed using a valid sequence of payoff functions will stay below the threshold $1/\alpha$ with probability at least $1 - \alpha$.

In the next subsection, we state some results about the asymptotic rate at which the optimal type-II error converges to zero in the non-sequential setting.

2.3 Optimal Error Exponent in Hypothesis Testing

For simple binary hypothesis tests, the following result (Cover and Thomas 2006, Theorem 11.8.3) characterizes the rate at which the probability of type-II error of the optimal level-$\alpha$ test vanishes asymptotically.

Fact 2 (Chernoff-Stein Lemma). Given $n$ i.i.d. the observations $Y_1, \ldots, Y_n$, consider the problem of testing $H_0 : Y_i \sim P$ versus $H_1 : Y_i \sim Q$ for known distributions $P$ and $Q$ with $0 < d_{KL}(P,Q) < \infty$. For a fixed $\alpha \in (0,1)$, let $\beta_n^*$ denote the type-II error for the optimal level-$\alpha$ test (i.e., the level-$\alpha$ test, with the smallest type-II error). Then we have,

$$
\lim_{n \to \infty} -\frac{1}{n} \log \beta_n^* = d_{KL}(P,Q).
$$

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This result tells us that the optimal probability of type-II error behaves roughly as $e^{-n d_{KL}(P,Q)}$ for the simple binary hypothesis testing problem with distributions $P$ and $Q$. The proof of this result relies on a more general result in the theory of large deviations, called Sanov’s theorem (Dembo [2010], Theorem 2.11) and (Cover and Thomas [2006], Theorem 11.4.1). In this paper, we will obtain similar guarantees on the exponent of the analogous quantity in the sequential setting, $P(\tau > n)$, when the alternative hypothesis is true.

2.4 Sequential Prediction

As we saw in the analysis of the mixture betting strategy in [3], the resulting wealth after $n$ rounds of bets can be lower-bounded by a depending exponentially on $(1/n \sum_{t=1}^{n} f_t(Z_t))^2$. Thus, any good strategy of selecting the payoff functions $\{f_t : t \geq 1\}$ should ensure that this term is as large as possible. If we restrict all the $f_t$’s to lie in some class $G$, then ideally, we would like the prediction strategy to perform comparable to the best constant function $f^*_n \in G$ in hindsight; that is, with the knowledge of the observations $Z_1, \ldots, Z_n$. This difference in performance can be captured by the notion of regret in the sequential prediction problem that we discuss next. We describe the sequential prediction problem in slightly abstract terms using general decision set $U$, and reward functions $r_t$.

**Definition 6 (Sequential-Prediction).** For $t = 1, 2, \ldots$:

- Player selects an action or decision $u_t \in U$.
- Nature (or adversary) reveals the next loss function $\ell_t : U \to \mathbb{R}$.
- Player obtains a reward $r_t(u_t)$ (or equivalently incurs a loss $\ell_t(u_t) = -r_t(u_t)$).

The goal of the player is to design a prediction strategy for selecting the actions $\{u_t : t \geq 1\}$ to ensure that the overall rewards gained (or equivalently, total losses incurred) is close to that of the best constant action in $U$ in hindsight. More specifically, the goal is to minimize the regret, defined as

$$R_n := \sup_{u \in U} \sum_{t=1}^{n} r_t(u) - \sum_{t=1}^{n} r_t(u_t).$$

A prediction strategy of selecting the actions $\{u_t : t \geq 1\}$ is called no-regret, is its normalized regret converges to zero; that is

$$\lim_{n \to \infty} \frac{R_n}{n} = 0.$$

To conclude this discussion, we now explain how these notions will be useful in our analysis of sequential one- and two-sample tests:

- In Section 3, we introduce a way of selecting the class $G$ from which the payoff functions should be selected. This is done by considering statistical distance measures that admit a variational representation.
- The class $G$ then serves as the decision set $U$ in our sequential prediction problem.
- Finally, we connect the regret incurred in this prediction problem to the performance of the statistical tests in Proposition [5] and Proposition [6]. In particular, we show that if $R_n$ of the prediction strategy converges to zero almost surely, then the resulting test is asymptotically consistent. Additionally, if $R_n$ converges to zero uniformly over all possible sequences of observations, we can further conclude that the power converges to 1 at an exponential rate.
3 General Approach

In this section, we describe the general approach used for designing sequential tests for both, one-sample and two-sample tests. To describe the idea in a unified manner, we represent the observed samples by \((Z_t)_{t \geq 1}\), with the understanding that \(Z_t = Y_t\) in the one-sample case, and \(Z_t = (X_t, Y_t)\) in the two-sample case.

Section 3.1 describes a sequential test in an idealized setting in which we are given an \(\text{oracle payoff function} f^*\) satisfying certain properties. Section 3.2 describes a principled approach to selecting the function \(f^*\), and Section 3.3 describes a practical test in which the oracle function \(f^*\) is not known, and must be estimated from the data available. The test described in Section 3.3 forms the abstract template of all the tests studied in the rest of the paper.

3.1 An Ideal Sequential Test

For two distributions \(P\) and \(Q\), suppose there exists a function \(f^*\) taking values in \([-1, 1]\), which may depend on both \(P\) and \(Q\), satisfying

\[
\mathbb{E}_{Z \sim P}[f^*(Z)] = 0, \quad \text{and} \quad \mathbb{E}_{Z \sim Q}[f^*(Z)] \neq 0.
\]

Assume, for now, that this \(\text{oracle function} f^*\) is known to us. Then, we can use \(f^*\) to instantiate a version of the betting game of Definition 4, where the payoff function at time \(t\) is \(f^*(Z_t)\). Now, if we employ the mixture betting strategy introduced in Definition 5 for selecting the bets, the resulting wealth process is defined as

\[
K_t^* = \int_0^1 \prod_{t=1}^n (1 + \lambda f^*(Z_t)) \nu(\lambda)d\lambda := \int_0^1 K^*_t \nu(\lambda)d\lambda.
\]

Recall that \(\nu\) represents the mixing density introduced in Definition 5. The above wealth process satisfies two important properties:

- Since \(f^*\) is selected to ensure \(\mathbb{E}[f^*(Z_t)|\mathcal{F}_{t-1}] = 0\) under the null, the process \(\{K_t^* : t \geq 1\}\) is a nonnegative martingale adapted to \(\{\mathcal{F}_t : t \geq 1\}\).

- Since the expected value of \(f^*\) under the alternative is non-zero, the term \(\lim_{t \to \infty} S_t^*/t\) converges almost surely to a non-zero value. This combined with the result of Proposition 1 implies that \(K_t^*\) grows exponentially fast under the alternative.

Based on the above two observations, we can define the following level-\(\alpha\) \(\text{oracle sequential test}^{14}\):

\[
\tau^* = \inf\{t \geq 1 : K_t^* \geq 1/\alpha\}. \quad (6)
\]

The choice of the rejection threshold \(1/\alpha\) is governed by the fact that the wealth process is a nonnegative martingale with initial value 1.

**Proposition 2.** Suppose the oracle payoff function \(f^*\) satisfies \(\mathbb{E}[f^*(Z_t)] = 0\) under \(H_0\), and \(\mathbb{E}[f^*(Z_t)] \neq 0\) under \(H_1\). Then, the test defined in (6) satisfies the following:

- Under \(H_0\) : \(\mathbb{P}(\tau^* < \infty) \leq \alpha\). \quad (7)
- Under \(H_1\) : \(\mathbb{P}(\tau^* < \infty) = 1\). \quad (8)

This result, proved in Appendix B.2, implies that given the knowledge of the oracle payoff function \(f^*\), we can easily define a level-\(\alpha\) consistent sequential nonparametric test. In the next subsection, we describe a general approach for selecting the payoff function \(f^*\).
3.2 Selecting an Appropriate $f^*$

The discussion in the previous section places some mild requirements on the function $f^*$ used in designing the oracle test [9]. Hence, for a given pair of distributions $P$ and $Q$, there exist many possible $f^*$ satisfying those conditions. As we saw in Example 1, the choice of payoff functions has a strong influence on the growth rate of the wealth process, and consequently, on the performance of the resulting sequential test. In this section, we describe a principled approach for selecting $f^*$ based on certain statistical distances.

Our starting point is to consider the variational definitions of two classes of statistical distance measures: namely integral probability metrics (IPMs) and $\varphi$-divergences. In particular, the IPM $d_G$ associated with a function class $\mathcal{G}$ is defined as

$$d_G(P,Q) = d_G(Q,P) := \sup_{f \in \mathcal{G}} \left| \int_X f dP - \int_X f dQ \right|.$$  

(9)

The definition above includes several important statistical distances including Wasserstein metric, kernel maximum mean discrepancy (MMD) and total variation metric.

For a convex function $\varphi : \mathcal{X} \to \mathbb{R}$ with $\varphi(1) = 0$, the $\varphi$-divergence between $Q$ and $P$ admits the following variational definition (Wu 2017, chapter 6)

$$d_{\varphi}(P,Q) = \sup_{f \in \mathcal{G}_\varphi} \mathbb{E}_{Z \sim Q} [f(Z)] - \mathbb{E}_{Z \sim P} [\varphi^*(f(Z))],$$

(10)

where $\mathcal{G}_\varphi$ consists of all measurable functions for the right-hand side of (10) is well-defined, and $\varphi^*(z) = \sup_{x \in \mathbb{R}} xz - \varphi(x)$ denotes the convex conjugate of $\varphi$. Several commonly used distance measures, such as KL-divergence ($\varphi(x) = x \log x$), Hellinger distance ($\varphi(x) = (\sqrt{x} - 1)^2$), $\chi^2$-distance ($\varphi(x) = (x - 1)^2$) and total variation ($\varphi(x) = |x - 1|$) can be obtained by suitable choices of the function $\varphi$. Note that total variation is the only distance that is simultaneously an IPM and a $\varphi$-divergence.

For both the classes of distance measure defined above, the function $g^*$ that achieves the supremum in their definition could be interpreted as a function that best highlights the differences between the two distributions $P$ and $Q$.

The variational definitions of the IPM (9) and $\varphi$-divergences (10) suggest the following natural way of selecting the function $f^*$.

- First choose a suitable distance measure, denoted by $d$, belonging to one of the two classes described in (9) and (10).
- Set $f^*$ equal to the function achieving in the supremum in the definition of $d$ among all uniformly bounded functions in $\mathcal{G}$ or $\mathcal{G}_\varphi$, with additional shifting and scaling (if needed) to ensure that (i) $\mathbb{E}_P[f^*(Z)] = 0$ and (ii) $f^*$ takes values in $[-1, 1]$.

To illustrate this design strategy, we return to the specific problem instance considered earlier in Example 1 and provide a formal justification why the payoff function $f^{(2)}$ was a better choice than $f^{(1)}$

Example 2. As in Example 1, we have the two distributions $P \sim \text{Uniform}([-1, 1])$ and $Q$ with a density $q$ such that $q(x) = 0.2 \mathbb{1}_{\{x \in [-1, 0)\}} + 0.8 \mathbb{1}_{\{x \in [0, 1]\}}$. Now, let us consider the IPM $d_G$ corresponding to the class of functions $\mathcal{G} = \{g : [-1, 1] \to [-b, b]\}$ for some $b > 0$. Computing $d_G(Q,P)$, we get

$$d_G(Q,P) = \max_{g \in \mathcal{G}} -0.3 \int_{-1}^{0} g(x) dx + 0.3 \int_{0}^{1} g(x) dx.$$  

The maximum in the above definition is achieved by the function $g_b^*(x) = b \left( \mathbb{1}_{\{x \in [0, 1]\}} - \mathbb{1}_{\{x \in [-1, 0)\}} \right)$. Finally, note that we have $\mathbb{E}_Y \sim P[g_b^*(Y)] = 0$ due to symmetry, and hence $f^*(x) = g_b^*(x) - \mathbb{E}_Y \sim P[g_b^*(Y)] = g^*(x)$ is a valid payoff function for testing Forecaster’s claim of $Y \sim P$.

The payoff function $f^{(2)}$ used in Example 1 as well as in Figure 1 corresponds to $g_b^*$ with $b = 0.6$. 

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3.3 A Practical Sequential Test

All nontrivial choices of the oracle payoff function $f^*$ introduced in the previous section depend on both $P$ and $Q$. In the one- and two-sample testing problems, one or both of these distributions are unknown. Thus, in order to design a practical test, we must employ a sequence of predictable estimates of $f^*$, denoted by $\{f_t : t \geq 1\}$ as we outline next.

**Definition 7 (Abstract-Sequential-Test).** Suppose $Z_1, Z_2, \ldots$ denote the samples observed either in the one-sample testing problem or the paired two-sample testing problem. Then proceed as follows:

- Construct a predictable empirical estimate, denoted by $f_t$, of the function $f^*$, using the observations $Z_1, \ldots, Z_{t-1}$. This function must satisfy the property $E[f_t(Z_t) | F_{t-1}] = 0$ under the null, and $F_{t-1} = \sigma(Z_1, \ldots, Z_{t-1})$.

- Observe $Z_t$ and update the wealth process for $t \geq 1$ as
  \[
  K_t = \int_0^1 \prod_{i=1}^t (1 + \lambda f_i(Z_i)) \nu(\lambda) d\lambda,
  \]
  with the assumption that $K_0 = 1$. Recall that $\nu$ is the mixture distribution introduced in Definition 5 satisfying the condition that $\nu(\lambda) \geq c_0$ for all $\lambda \in [-1, 1]$ for some $c_0 > 0$.

- Define the stopping time $\tau = \inf\{t \geq 1 : K_t \geq 1/\alpha\}$ for a given confidence level $\alpha \in (0, 1)$.

Clearly, the performance of this test will depend strongly upon how well the predictable estimates match the oracle payoff function. We derive the requirements on the predictions to ensure proper control of type-I and type-II errors in Section 3.3.1.

### 3.3.1 Performance Guarantees

We now present three results that formalize the requirements on the predictable estimates $\{f_t : t \geq 1\}$ to ensure control on type-I error and consistency of the sequential test. The first result, Proposition 3, shows that the test controls type-I error under $H_0$ if $f_t$ have zero mean (conditional on $F_{t-1}$) for all $t \geq 1$.

**Proposition 3.** Suppose the predictions $\{f_t : t \geq 1\}$ satisfy $E[f_t(Z_t) | F_{t-1}] = 0$ for all $t \geq 1$. Then we have the following:

under $H_0$, \[ P(\tau < \infty) \leq \alpha. \]

**Proof.** Note that under the condition on $f_t$ in the statement of Proposition 3, the wealth process $\{K_t : t \geq 0\}$ is a nonnegative martingale process adapted to the filtration $\{F_t : t \geq 0\}$. Hence, we have

\[ P(\tau < \infty) = P(\{\exists t \geq 0 : K_t \geq 1/\alpha\}) \leq \alpha E[K_0] = \alpha, \]

where (i) follows from an application of Ville’s inequality.\[ \square \]

Our next result, proved in Appendix B.3, identifies a sufficient condition on the real-valued sequence $\{f_t(Z_t) : t \geq 1\}$ in order to ensure that the test stops in finite time.

**Proposition 4.** Suppose $S_t = \sum_{i=1}^t f_i(Z_i)$. Then, for the test introduced in Definition 7

\[ \liminf_{t \to \infty} \left| \frac{S_t}{t} \right| > 0 \quad \text{implies} \quad \tau < \infty. \]
Note that the statement of the previous result is purely deterministic, and makes no assumptions on the distributions of the terms \( \{Z_t : t \geq 1\} \). However, this pathwise statement immediately allows us to conclude that
\[
P \left( \liminf_{t \to \infty} \left| \frac{S_t}{t} \right| > 0 \right) \leq P(\tau < \infty).
\]
This suggests that in order to characterize the power under the alternative, it suffices to analyze the behavior of the statistic \( S_t/t \). Our next result identifies sufficient conditions on the sequence of payoff functions \( \{f_t : t \geq 1\} \) that ensure that \( S_t/t \) converges to a non-zero value under the alternative with probability one.

**Proposition 5.** Suppose there exist a sequence of nonnegative real numbers \( \{\Delta f_t : t \geq 1\} \) converging to 0, and a sequence of events \( \{E_t \in F_{t-1} : t \geq 1\} \), with the properties:
\[
E_t = \{\|f^* - f_t\|_\infty \leq \Delta f_t\}, \quad \text{with} \quad \lim_{t \to \infty} \Delta f_t = 0, \quad \text{and} \quad \sum_{t=1}^{\infty} P(E_t^c) < \infty \quad \text{under } H_1.
\]
Then we have the following:
\[
\lim_{n \to \infty} \frac{S_n}{n} := \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f^*(Z_t) \neq 0.
\]

The proof of this statement is in Appendix B.4.

Next, we consider situations under which the type-II error of the sequential test converges to zero at an exponential rate. We will show that a sufficient condition for exponentially decaying type-II error is if the payoff functions \( \{f_t : t \geq 1\} \) achieve vanishing regret on an online prediction problem (described next).

**Definition 8.** For the general sequential test introduced in Definition 7, assume that the bets \( \{\lambda_t : t \geq 1\} \) are selected according to the strategy described in Definition 5. Suppose that the payoff functions must be selected from some family \( \mathcal{G} \), that is dependent on the choice of statistical distance measure \( d \). Then consider the following game:

For \( t = 1, 2, \ldots \):
- Predict \( f_t \) from \( \mathcal{G} \).
- Observe \( Z_t \).
- Obtain a reward \( f_t(Z_t) \).

The normalized regret associated with this prediction problem is defined as
\[
R_n := \sup_{f \in \mathcal{G}} \frac{1}{n} \left( \sum_{t=1}^{n} f(Z_t) - \sum_{t=1}^{n} f_t(Z_t) \right),
\]
where \( Z_t = Y_t \) for one-sample tests and \( Z_t = (X_t, Y_t) \) for two-sample tests.

The family of functions \( \mathcal{G} \) in the above prediction problem depends on the specific statistical distance used to define the test. In many instances, \( \mathcal{G} \) satisfies the following condition
\[
\sup_{f \in \mathcal{G}} \frac{1}{n} \sum_{t=1}^{n} f(Y_t) \geq C \times d(\hat{Q}_n, P) \quad \text{or} \quad \sup_{f \in \mathcal{G}} \frac{1}{n} \sum_{t=1}^{n} f(X_t, Y_t) \geq C \times d(\hat{Q}_n, \hat{P}_n),
\]
(12)
for one-sample and two-sample tests, respectively. Here $\hat{Q}_n$ (resp. $\hat{P}_n$) denotes the empirical distribution based on the samples $\{Y_t : 1 \leq t \leq n\}$ (resp. $\{X_t : 1 \leq t \leq n\}$). The term $C$ denotes a constant positive scaling factor, since we restrict the payoff functions to take values in the range $[-1, 1]$. For instance, in the case of one-sample testing with the Kolmogorov-Smirnov distance ($d_{KS}$), the function class $G$ is $\{1_{(-\infty, x]} - F_P(x) : x \in \mathbb{R}\}$. It can be checked that this $G$ satisfies the condition in (12) with an equality and $C = 1$.

Our next result shows that if the condition in (12) is satisfied and there exists a no-regret scheme of predicting $\{f_t : t \geq 1\}$, then the type-II error of the resulting sequential test converges exponentially to zero.

**Proposition 6.** Consider an instance of the general sequential-test introduced in Definition 7 based on some statistical distance measure $d$, for which:

- The mixture-method introduced in Definition 5 is used for betting;
- There exists a no-regret (for the prediction problem introduced in Definition 8) scheme for predicting $\{f_t : t \geq 1\}$;
- The condition in (12) is satisfied.

Then for the resulting sequential test, $\tau$, we have the following under the alternative:

$$\liminf_{n \to \infty} \frac{-1}{n} \log (P(\tau > n)) \geq \beta,$$

where the exponent $\beta$ is defined for one- and two-sample tests as follows:

- In the case of one-sample testing, the exponent $\beta$ is

  $$\beta = \sup_{\epsilon > 0} \inf_{P' \in \mathcal{P}_{\epsilon, d}(P)} d_{KL}(P', P), \quad \text{where} \quad \mathcal{P}_{\epsilon, d}(P) := \{P' \in \mathcal{P}(\mathcal{X}) : d(P', P) \leq \epsilon\}.$$

- In the case of two-sample testing,

  $$\beta = \sup_{\epsilon > 0} \inf_{(P', Q') \in \mathcal{P}_{\epsilon, d}^2} \frac{d_{KL}(P', P) + d_{KL}(Q', Q)}{2}, \quad \text{where} \quad \mathcal{P}_{\epsilon, d}^2 := \{(P', Q') \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) : d(Q', P') \leq \epsilon\}.$$

The proof of this statement is in Appendix B.5.1.

### 4 One-Sample Tests

Suppose $P$ and $Q$ denote two distributions on some space $\mathcal{X}$. When $\mathcal{X} = \mathbb{R}$, we will use $F_P$ and $F_Q$ to denote the cumulative distribution functions (CDFs) associated with $P$ and $Q$ respectively. In the one-sample testing problem, we assume that the target distribution $P$ is known, and given i.i.d. samples $Y_1, Y_2, \ldots \sim Q$, the goal is to test

$$H_0 : Q = P \quad \text{vs} \quad H_1 : Q \neq P.$$

In Section 4.1 we consider the case when $\mathcal{X} = \mathbb{R}$ and introduce a sequential test based on the Kolmogorov-Smirnov (KS) distance belonging to the IPM family. We first show in Corollary 1 that using a simple prediction strategy based on empirical estimates of the distribution results in an asymptotically consistent test. Next, in Proposition 7 we show that a more involved randomized prediction strategy can also ensure that the type-II error converges to zero exponentially.
In Section 4.2 we consider the case when the distributions are supported on a finite discrete set \( \mathcal{X} = \{ x_1, \ldots, x_m \} \), and proposed a sequential test based on the \( \chi^2 \) distance belonging to the family of \( \varphi \)-divergences. Again, for this test, we first show that a plug-in prediction strategy based on the empirical estimate of the data distribution is asymptotically consistent in Corollary 2. Finally, in Proposition 9 we show that an online projected gradient descent (PGS) based prediction strategy can also ensure that the type-II error converges to zero exponentially.

4.1 Sequential KS Test

Our first strategy is based on the Kolmogorov-Smirnov (KS) distance between two real valued distributions, \( P \) and \( Q \), that is defined as

\[
d_{KS}(P, Q) = \sup_{x \in \mathbb{R}} |F_P(x) - F_Q(x)| = \sup_{\phi \in \mathcal{G}_{KS}} \left| \int_{\mathbb{R}} f \, dP - \int_{\mathbb{R}} f \, dQ \right|,
\]

where \( \mathcal{G}_{KS} = \{ f_x : f_x(x) = 1_{\{x \leq x\}}, \ x \in \mathbb{R} \} \). Assuming, for now, that the supremum in (13) is achieved at a point \( u^* \) (not necessarily unique), the above definition suggests the function \( f^*(x) = 1_{\{x \leq u^*\}} - F_P(x) \) for constructing a sequential test.

Before describing the testing strategy, we introduce some additional notation. As mentioned earlier, \( F_P \) and \( F_Q \) denote the cdfs associated with the two distributions \( P \) and \( Q \) respectively. We will use \( \hat{F}_{Q,t-1} \) to denote the empirical estimate of \( F_Q \) based on samples \( Y_1, \ldots, Y_{t-1} \). Let \( G(x) := |F_Q(x) - F_P(x)| \) for \( x \in \mathbb{R} \), and denote its empirical counterpart at time \( t \) with \( \hat{G}_t = |\hat{F}_{t-1}(x) - F_P(x)| \). Let \( \hat{G}_t^* = \max_{x \in \mathcal{X}} \hat{G}_t \) and \( U_t = \{ x \in \mathcal{X} : \hat{G}_t(x) \geq \hat{G}_t^* - 2\Delta F_t \} \), where \( \Delta F_t = \sqrt{2\log(t)/t} \).

**Definition 9 (Sequential KS-Test).** For \( t = 2, 3, \ldots \), proceed as follows:

- Define \( u_t = \inf\{ u : u \in U_t \} \), and set \( f_t(y) = 1_{\{y \leq u_t\}} - F_P(u_t) \). Recall that \( U_t \) was introduced in the previous paragraph.
- Observe the next sample \( Y_t \sim Q \), and update the wealth process using the mixture-method from Definition 5, starting with \( K_0 = 1 \).
- If \( K_t \geq 1/\alpha \), then reject the null.

As a consequence of the general results of Section 3, we can establish the following properties of this test.

**Corollary 1.** Suppose \( H_0 : Q = P \) and \( H_1 : Q \neq P \). Then the following statements are true for the sequential KS-test defined in Definition 9.

\[
\begin{align}
\text{Under } H_0, \quad \mathbb{P}(\tau < \infty) &\leq \alpha, \quad (14) \\
\text{Under } H_1, \quad \mathbb{P}(\tau < \infty) &\leq 1. \quad (15)
\end{align}
\]

The proof of this statement is in Appendix C.1.

4.1.1 Exponent of Type-II error

We now show that, with a more complex strategy of predicting \( \{ f_t : t \geq 1 \} \), the type-II error of the resulting sequential test can be shown to converge exponentially to zero. In particular, we assume that \( \mathcal{X} = [0, 1] \) and that the target distribution is \( \text{Uniform}([0, 1]) \). This assumption is typical for testing goodness-of-fit for real-valued random variables as noted in (Lehmann, Romano, and Casella 2005 § 14.1). Furthermore, this easily generalizes to the case of all continuous target distributions \( P \), using the fact that \( F_P(Y) \sim \text{Uniform}([0, 1]) \) in this case.
Under these conditions, suppose we define two predictions \( f_t^+ \) and \( f_t^- \) as follows:

\[
\begin{align*}
  f_t^+(x) &= \mathbb{E}_{u_t \sim \pi_t^+} [\mathbb{1}_{\{x \leq u_t\}} - \mathbb{P}(u_t)] , \\
  f_t^-(x) &= \mathbb{E}_{u_t \sim \pi_t^-} [\mathbb{P}(u_t) - \mathbb{1}_{\{x \leq u_t\}}] \\
\end{align*}
\]

where

\[
\begin{align*}
  \pi_0^+(u) &= 1, \forall u \in \mathcal{X}, \quad \text{and} \\
  \pi_t^+(u) &= \pi_{t-1}^+(u) \exp \{ \eta_{t-1} (\mathbb{1}_{\{X_t \leq u\}} - \mathbb{P}(u)) \}, \\
  \pi_0^-(u) &= 1, \forall u \in \mathcal{X}, \quad \text{and} \\
  \pi_t^-(u) &= \pi_{t-1}^-(u) \exp \{-\eta_{t-1} (\mathbb{1}_{\{X_t \leq u\}} - \mathbb{P}(u))\},
\end{align*}
\]

for a positive nonincreasing sequence \( \{\eta_t : t \geq 0\} \). This strategy is referred to as the exponential weights (EW) or hedge strategy in the online learning literature. With these definitions, we can now define the wealth process:

\[
\begin{align*}
  \mathcal{K}_t &= \frac{1}{2} (\mathcal{K}_{t,+} + \mathcal{K}_{t,-}), \\
  \mathcal{K}_{t,\&} &= \int_0^t \prod_{i=1}^t (1 + \lambda f_t^\&(Y_i)) \nu(d\lambda), \quad \text{for } \& \in \{+, -, \}.
\end{align*}
\]

The two terms \( f_t^+ \) and \( f_t^- \) are needed to consider the two possibilities due to the absolute value in the definition of \( d_{KS} \). We now state the main result of this subsection, that characterizes the rate at which the type-II error of this test converges to zero.

**Proposition 7.** Suppose \( \mathcal{X} = [0, 1] \) and assume that the known target distribution \( P \) is \( \text{Unif}([0, 1]) \); that is, \( \mathbb{P}(u) = u \) for all \( u \in \mathcal{X} \). Then, for the strategy of predicting \( \{f_t : t \geq 1\} \) as described in (16), the type-II error of the resulting sequential KS test converges to zero exponentially under the alternative. In particular, we have the following under \( H_1 \),

\[
\lim_{n \to \infty} \frac{1}{n} \log \left( \mathbb{P} (\tau > n) \right) \geq d_{KL} (P, Q).
\]

The proof of this statement proceeds in two steps:

1. First, we show that there exists a no-regret prediction strategy for the instance of the general sequential-prediction problem corresponding to \( d_{KS} \). We show this by proving a more general result about prediction with discontinuous losses, stated below in Proposition 8, that may be of independent interest. Finally, by an application of Proposition 6, the existence of a no-regret strategy implies that the type-II error converges to zero exponentially fast.

2. Second, we show that the actual exponent for any alternative \( Q \) is \( d_{KL}(P, Q) \) if \( 0 < d_{KL}(P, Q) < \infty \). This relies on the facts that \( d_{KS} \) metrizes the weak topology on \( \mathcal{P}(\mathbb{R}) \) and that \( d_{KL}(\cdot, P) \) is lower-semicontinuous in the weak topology.

The details of these steps are in Appendix C.2. Next, we present the analysis of the EW strategy in a more general version of the problem involved in characterizing the exponent of the sequential KS test above.

**Proposition 8.** Consider an instance of the sequential prediction of Definition 6 with a decision set \( \mathcal{U} = [0, 1]^m \) for some \( m \geq 1 \), and the reward function \( \tau_t = g_t - h_t \) taking values in \( [0, B] \) (or equivalently loss functions \( \ell_t = B - g_t + h_t \)) and satisfying the following properties:

- \( g_t \) is piece-wise continuous with the number of pieces uniformly bounded by some \( M < \infty \) for all \( t \geq 1 \). More specifically, \( g_t(x) = \sum_{i=1}^M \mathbb{1}_{\{u \in E_{i,t}\}} g_i(u) \) for a partition \( \{E_{i,t} : 1 \leq i \leq M\} \) of \( \mathcal{X} \) consisting of connected sets and continuous functions \( \{g_{i,t} : 1 \leq i \leq t\} \). Furthermore, assume that \( g_t \) is nondecreasing with respect to the usual coordinate-wise partial ordering \( \preceq \) on \( \mathbb{R}^d \). That is, if \( u_1 \preceq u_2 \), then \( g_t(u_1) \leq g_t(u_2) \) for all \( t \geq 1 \).
• The functions \( \{h_t : t \geq 1\} \) are all nonnegative uniformly continuous with the same modulus of continuity \( \omega \). Furthermore, there exist a sequence of real numbers \( \{\delta_t : t \geq 1\} \) with \( \lim_{t \to \infty} \delta_t = 0 \) satisfying:

\[
\lim_{t \to \infty} \omega(\delta_t) = 0, \quad \text{and} \quad \lim_{n \to \infty} \frac{\log(1/\delta_t)}{t^a} = 0, \quad \text{for some } a \leq 1.
\]

• Suppose \( U^- \subset U \) denote the set of minimal points; that is \( x \in U^- \) if for all \( x' \) comparable to \( x \), we have \( x \leq x' \). Similarly, let \( U^+ \) denote the set of maximal points in \( U \). Then we have \( r_t(x^-) = r_t(x^+) = 0 \), for all \( x^- \in U^- \) and \( x^+ \in U^+ \).

Then, there exists a sequence of step sizes \( \{\eta_t : t \geq 1\} \), such that the resulting EW strategy predicting \( u_t \sim \pi_t \in \mathcal{P}(U) \), with \( \pi_t(u) \propto \exp\left(\sum_{i=1}^{t-1} \eta_t \ell_i(u)\right) \), is asymptotically no-regret. That is,

\[
\lim_{n \to \infty} \frac{1}{n} \left( \sup_{u \in U} r_t(u) - \sum_{t=1}^{n} \mathbb{E}_{u_t \sim \pi_t}[r_t(u_t)] \right) = 0.
\]

The proof of this statement is in Appendix C.3.

**Remark 8.** The prediction problem analyzed in Proposition 8 has two interesting features: the decision set \( U \) is a continuum, and the losses \( \ell_t \) are allowed to be discontinuous in every round. Prior works, such as Krichene, Balandat, Tomlin, and Bayen (2015), on the regret analysis of EW strategy for prediction with continuous decision sets have mainly focused on the case of Lipschitz continuous losses \( \ell_t \). This is because, without additional structural assumptions, even the simplest prediction problems with discontinuous losses involving piecewise continuous functions with one discontinuity are not learnable (that is, incur linear regret), as shown by Cohen-Addad and Kanade (2017). Some recent results dealing with discontinuous losses, such as Balcan, Dick, and Vitercik (2018) and Sharma, Balcan, and Dick (2020) proved that a sufficient condition for learnability is to impose a dispersion condition on the sequence of loss functions. Informally, this condition allows only \( o(n) \) of the loss functions \( \{\ell_t : 1 \leq t \leq n\} \) to be discontinuous at any \( n \) within a small region of the input space. The losses \( \ell_t = 1_{t \leq X_t} - F_P(\cdot) \) in the prediction problem for \( d_{KS} \) are not required to satisfy this dispersion condition for the result of Proposition 8 to hold. Thus, the class of problems analyzed in Proposition 8 provide a non-trivial example of online learning problems with discontinuous losses that do not satisfy the dispersion condition, but still admit a no-regret strategy.

### 4.1.2 Experiments

**Implementation Details.** To make the practical implementation of the sequential KS test computationally feasible (especially for sample size greater than \( 10^4 \)), we make the following simplification in defining \( f_t \). To obtain \( u_t \) (introduced in Definition 8) at any time \( t \), we first choose a grid \( X_t \) of points between the largest and the smallest values among \( \{Y_{i} : 1 \leq i \leq t - 1\} \). We limit the maximum size of the grid to 5000, and then define \( u_t = \arg\max_{u \in X_t} [\hat{F}_{Q,t-1}(u) - F_P(u)] \). For the betting strategy, we used the mixture method as described in Section 2.1 with \( \nu \) being the uniform mixture over a grid of 100 points between -0.9 and 0.9.

To benchmark the performance of our proposed test, we used the standard fixed-sample size or batch version of the one-sample KS test, the sequential tests of Darling and Robbins (1968) and Howard and Ramdas (2022). We refer to these tests by their abbreviations Batch, DR and HR respectively in the figures. The details of their implementation are in Appendix E.

**Experiment Setup.** We set \( Q = N(1 + \epsilon, 1) \) and \( P = N(1, 1) \) where \( N(\mu, \sigma^2) \) denotes the Normal distribution with mean \( \mu \) and variance \( \sigma^2 \). We then ran 500 trials of the sequential test under the null as well as under the alternative and estimated the type-I error, type-II error and the distribution of the stopping times. Next, ran 500 trials each of the batch KS test for several values of the sample size, and obtained the
type-I and type-II errors for all those sample sizes. Finally, we repeated this experiment for different choices of the parameter $\epsilon$, and plot the results in Figure 2. Some additional figures are also plotted in Figure 7 in Appendix E.

Figure 2: The figures plot the performance of different tests on the one-sample testing problem with target distribution $P \sim N(0,1)$ and alternative distribution $Q \sim N(\epsilon,1)$. (Left) The figure shows the power and type-I error curves of our betting based sequential one-sample KS test (Betting) along with several baselines for $\epsilon = 0.4$: the batch one sample KS test (Batch) and the sequential one-sample KS tests of Howard and Ramdas (2022) and Darling and Robbins (1968), denoted by HR and DR respectively. The shaded histograms represent the distribution of the stopping time of the sequential tests over 500 trials under the alternative. The dashed vertical line denotes the average of the stopping times over these 500 trials. (Right) The figure on the right plots the power curve for our sequential KS test (solid curves) and the power curve of the batch one-sample KS test (dashed curves) for different values of $\epsilon$.

4.2 Sequential Chi-Squared Test

In this section, we consider the problem of testing discrete distributions with finite support. In particular, we assume that $\mathcal{X} = \{x_1, \ldots, x_m\}$ for some $m < \infty$. Any distribution $P$ over this set $\mathcal{X}$ can be equivalently represented by its probability mass function $p : \mathcal{X} \mapsto [0,1]$, with $\sum_{x \in \mathcal{X}} p(x) = 1$. We will use $\Delta_m$, the $m-1$ dimensional probability simplex, to denote the set of all possible pmfs on $\mathcal{X}$.

Given two distributions $P$ and $Q$ with pmfs $p$ and $q$ respectively, we can define the $\chi^2$-distance between them as follows:

$$d_{\chi^2}(Q, P) = \mathbb{E}_{Y \sim P} \left[ \left( \frac{q(Y)}{p(Y)} - 1 \right)^2 \right] = \sum_{y \in \mathcal{X}} \frac{1}{p(y)} (p(y) - q(y))^2.$$  

Note that $d_{\chi^2}$ is a $\varphi$-divergence (or $f$-divergence) with $\varphi(x) = (x - 1)^2$, and hence admits the following variational definition:

$$d_{\chi^2}(Q, P) = \sup_{g : \mathcal{X} \to \mathbb{R}} \mathbb{E}_{Y \sim Q} [g(Y)] - \mathbb{E}_{Y \sim P} [\varphi^*(g(Y))], \text{ where } \varphi^*(y) = \sup_{x \in \mathbb{R}} (xy - \varphi(x)) = y + y^2/4.$$  

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On further simplification, we obtain

\[ d_{\chi^2}(Q, P) = \sup_{g: \mathcal{X} \to \mathbb{R}} \mathbb{E}_{Y \sim Q}[g(Y)] - \mathbb{E}_{Y \sim P} \left[ \left( \frac{g(Y)}{2} + 1 \right)^2 - 1 \right]. \]

\[ \Rightarrow d_{\chi^2}(Q, P) = \sup_{g: \mathcal{X} \to \mathbb{R}} \mathbb{E}_{Y \sim Q} \left[ 2 \left( \frac{g(Y)}{2} + 1 \right) - 2 \right] \mathbb{E}_{Y \sim P} \left[ \left( \frac{g(Y)}{2} + 1 \right)^2 - 1 \right]. \]

Finally, introducing the notation \( h(x) = \frac{g(x)}{2} + 1 \), we obtain

\[ d_{\chi^2}(Q, P) = \sup_{h: \mathcal{X} \to \mathbb{R}} 2\mathbb{E}_{Y \sim Q} [h(Y)] - \mathbb{E}_{Y \sim P} [h^2(Y)] - 1, \]

\[ \Rightarrow d_{\chi^2}(Q, P) = \sup_{h: \mathcal{X} \to \mathbb{R}} \mathbb{E}_{Y \sim Q} \left[ 2h(Y) - h^2(Y) \frac{p(Y)}{q(Y)} \right] - 1. \quad (17) \]

The last term in (17) is optimized by \( h^*(x) = \frac{g(x)}{p(x)} \) under the assumption that \( p(x) > 0 \) for all \( x \in \mathcal{X} \).

Following the general strategy outlined in Section 3.3, we can appropriately scale and shift the function \( h^* \) to obtain the oracle payoff function \( f^* \) taking values in \([-1, 1]\). In particular, we define

\[ f^*(x) = \frac{1}{C_P} \left( h^*(x) - \sum_{x'} p(x') h^*(x') \right) = \frac{h^*(x) - 1}{C_P}, \quad \text{with} \quad C_P = \frac{1}{\min_x p(x)} - 1, \]

and build our sequential \( \chi^2 \) test using this function, as described next.

**Definition 10.** At \( t = 0 \), set \( K_t = 1 \). For \( t = 1, 3, \ldots \), proceed as follows:

- Set \( f_t(x) = \frac{1}{C_P} (q_{t-1}(x)/p(x) - 1) \), where
  \[ q_{t-1}(x) = \frac{1}{t-1} \sum_{i=1}^{t-1} \mathbb{1}_{\{Y_i = x\}}. \]

- Observe the next realization \( Y_t \) and update the wealth process \( K_t \) following the mixture method of Definition 8.

- Check for the stopping condition \( \tau := \inf\{ t \geq 1 : K_t \geq 1/\alpha, \ \text{or} \ Y_t \notin \mathcal{X} \} \).

The test defined above uses a simple plug-in strategy for predicting the payoff functions. Our first result, proved in Appendix C.4, states that the resulting sequential \( \chi^2 \) test with this plug-in strategy is asymptotically consistent and also controls the type-I error at the desired level.

**Corollary 2.** Suppose \( P \) is a known discrete distribution with finite support \( \mathcal{X} \), and \( Q \) is another distribution with possibly larger support. Let \( H_0 : Q = P \) and \( H_1 : Q \neq P \). Then the following statements are true for the test defined in Definition 10 based on samples \( Y_1, Y_2, \ldots \), drawn i.i.d. according to \( Q \):

\[ \text{Under } H_0, \quad \mathbb{P} (\tau < \infty) \leq \alpha, \quad (18) \]

\[ \text{Under } H_1, \quad \mathbb{P} (\tau < \infty) = 1. \quad (19) \]

This simple plug-in test is easy to implement and adapts to the hardness of a testing problem as shown later in Section 4.2.2. Next, we analyze a computationally more expensive version of the test of Definition 10 for which we can obtain stronger performance guarantees.
4.2.1 Exponent of Type-II error

If it is known that the alternative distribution $Q$ is also supported on the set $\mathcal{X}$, we can propose an alternative scheme for selecting the payoff functions, for which we can ensure exponentially decaying type-II error.

More specifically, under this assumption consider the modification of the test of Definition 10, that selects $f_t(x) = \frac{1}{C_p}(q_t(x)/p(x) - 1)$, where $q_t$ is defined as:

$$q_t = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right), \quad \text{where} \quad m = |\mathcal{X}|,$$

$$q_t = \Pi_{\Delta_m} \left( q_{t-1} + \frac{1}{t} \frac{1}{C_p} p(X_{t-1}) e_{X_t} \right) \quad \text{for} \ t \geq 2, \quad (20)$$

where $\Pi_{\Delta_m}$ denotes the projection onto the probability simplex $\Delta_m = \{ (a_1, \ldots, a_m) \in \mathbb{R}^m : a_i \geq 0, \ \text{and} \ \sum_{j=1}^m a_i = 1 \}$, and $e_x = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Delta_m$ is the pmf vector that places all the mass on $x \in \mathcal{X}$. The definition of $q_t$ in (20) is the update rule of the online projected gradient descent scheme (see Definition 15 in Appendix C.5 for a formal definition) with a linear reward function $r_t(\tilde{q}) = (1/C_p)\langle \tilde{q}, w_t \rangle$ and $w_t = e_{X_t}/.p$ where we use ./ to denote the element-wise division.

Plugging the $q_t$ defined by the scheme in (20) in the test defined in Definition 10, we get a modified sequential $\chi^2$ test, for which we can show the following result.

**Proposition 9.** Under the alternative, the sequential $\chi^2$ test with the prediction strategy described in (20) satisfies the following:

$$\liminf_{n \to \infty} -\frac{1}{n} \log P(\tau > n) \geq d_{\text{KL}}(P,Q).$$

The proof of this statement is Appendix C.5. This result shows that the error exponent of type-II error of our sequential $\chi^2$ test matches the optimal exponent in the simple hypothesis testing as implied by the Chernoff-Stein lemma (Fact 2), but with a composite alternative.

4.2.2 Experiments

**Implementation Details.** We implement the exact version of the sequential test defined in Definition 10 using the empirical pmf to construct the betting payoff function. For updating the wealth process, we use the mixture method over a grid of 100 points from 0 to 1 with uniform weights.

We compare the performance of our sequential test with the fixed-sample-size or batch version of the $\chi^2$-test. In our implementation, we used the statistic $T_n = \sum_{j=1}^m (N_{j,n} - np_j)^2$, where $N_{j,n} = \sum_{i=1}^n 1_{\{Y_i = x_j\}}$ denotes the number of observations equal to $x_j$. The limiting distribution of $T_n$ is the $\chi^2$-distribution with $m - 1$ degrees of freedom, and we use the $1 - \alpha$ quantile of this asymptotic distribution as the threshold for rejecting the null.

**Experiment Setup.** We set $\mathcal{X} = \{1, 2, \ldots, m\}$ for $m = 10$, and fix the target distribution $P$ to be the uniform distribution on $\mathcal{X}$. To construct the alternative distribution $Q = Q(j, \epsilon)$, we fix $j \leq \lfloor m/2 \rfloor$ and proceed in two steps: (1) First, we randomly select $\mathcal{X}_+ \subset \mathcal{X}$ with $|\mathcal{X}_+| = j$, and set $q(x) = p(x) + \epsilon$ for $x \in \mathcal{X}_+$ and some $\epsilon \in (0, 1/m)$. (2) Then, we randomly select $\mathcal{X}_- \subset (\mathcal{X} \setminus \mathcal{X}_+)$ with $|\mathcal{X}_-| = j$, and set $q(x) = p(x) - \epsilon$ for $x \in \mathcal{X}_-$. Thus, the parameters $j$ and $\epsilon$ control the difficulty of the testing problem.

We fix $j = 2$, and compare the performance of our proposed sequential $\chi^2$ test with the usual batch version for various choices of $\epsilon$. The results are shown in Figure 3. Some additional results are also plotted in Figure 7 in Appendix C.4.
Figure 3: The figure compares the performance of our sequential $\chi^2$ test and batch $\chi^2$ test on a one sample testing problem, where the observations $Y_1, Y_2, \ldots$ are i.i.d. $Q$. With $X = \{1, 2, \ldots, 10\}$, the target distribution $P$ is the uniform distribution on $X$, and the alternative is $Q = Q(2, \epsilon)$ (defined in Section 4.2.2). In both the figures, $\alpha$ is set to 0.05.

(Lef) The figure plots the power and type-I error curves of our betting based $\chi^2$ test (Betting), and the usual batch $\chi^2$ test (Batch) with $\epsilon = 0.004$. The shaded histogram represents the distribution of the stopping time of our proposed sequential test in 500 trials under the alternative, and the dashed vertical line denotes the average of the stopping times over these 500 trials.

(Right) The figure on the right plots the power curve for our sequential $\chi^2$ test (solid curves) and the power curve of the batch one-sample $\chi^2$ test (dashed curves) for different values of $\epsilon$.

5 Two-Sample Tests

We now move on to the problem of two-sample testing. As mentioned earlier, here we observe paired realizations $\{(X_t, Y_t) : t \geq 1\}$ drawn i.i.d. from a product distribution $P \times Q$. The goal is to test

$H_0 : P = Q$ versus $H_1 : P \neq Q$.

Following the general framework described in Section 4 we present two instances of the abstract test introduced in Section 4 for the two-sample testing problem: (1) sequential two sample KS test in Section 5.1 and (2) sequential kernel-MMD test in Section 5.2.

5.1 Sequential Two-Sample KS Test

We will use $\hat{F}_{Q,t-1}$ and $\hat{F}_{P,t-1}$ to denote the empirical cdfs constructed using the samples $Y_1, \ldots, Y_{t-1}$ and $X_1, \ldots, X_{t-1}$ respectively. Similar to Section 4.1 we define $G(x) := |F_Q(x) - F_P(x)|$ for $x \in \mathbb{R}$, and denote its empirical counterpart at time $t$ with $\hat{G}_t = |\hat{F}_{Q,t-1}(x) - \hat{F}_{P,t-1}(x)|$. Let $\hat{G}_t^* = \max_{x \in \mathcal{X}} \hat{G}_t$ and $U_t = \{x \in \mathcal{X} : \hat{G}_t(x) \geq \hat{G}_t^* - 4\Delta F_t\}$, where $\Delta F_t = \sqrt{2\log(t)/t}$. With these terms defined, we can now formally describe the sequential two-sample KS test.

Definition 11 (Sequential Two-Sample KS-Test). Set $\mathcal{K}_t = 1$ at $t = 0$. For $t = 1, 2, 3, \ldots$, proceed as follows:

- Define $u_t = \inf\{u : u \in U_t\}$, and set $f_t(x, y) = 1_{\{y \leq u_t\}} - 1_{\{x \leq u_t\}}$.
• Observe the next pair $X_t, Y_t \sim P \times Q$, and update the wealth process following the mixture-method in Definition 5:

$$K_t = \int_{-1}^{1} \prod_{i=1}^{t} (1 + \lambda f_i(X_i, Y_i)) \nu(\lambda) d\lambda.$$  

• If $K_t \geq 1/\alpha$, then reject the null.

Having defined the test, we can again obtain the guarantees on the type-I and type-II error by appealing to Proposition 3 and Proposition 4.

**Corollary 3.** Suppose $H_0 : Q = P$ and $H_1 : Q \neq P$. Then the following statements are true for the sequential KS-test defined in Definition 11:

- Under $H_0$, $\mathbb{P}(\tau < \infty) \leq \alpha$,
- Under $H_1$, $\mathbb{P}(\tau < \infty) = 1$.

The proof of this result follows along the same lines as the one-sample case. We present an outline of the main steps in Appendix D.1 but omit the details to avoid repetition.

**Remark 9.** In the one-sample case, we were able to show, in Proposition 8, that the type-II error of the sequential one-sample KS test can be made to decay at an exponential rate when the target distribution $P$ was uniform over the set $[0, 1]$. More generally, this extends to any continuous distribution over $\mathbb{R}$ that could be transformed into a $\text{Uniform}([0, 1])$ distribution by applying its cdf $F_P$. Showing an analogous result for the two-sample KS test is not possible because the loss functions involved in the corresponding prediction problem, $\{\ell_t(u) = \mathbb{1}_{X_t \leq u} - \mathbb{1}_{Y_t \leq u} : t \geq 1\}$, do not have the structure required by Proposition 8. In fact, for prediction problems with such discontinuous loss functions, Cohen-Addad and Kanade (2017) proved that the worst case regret incurred by any strategy must be $\Omega(\sqrt{n})$.

### 5.1.1 Experiments

**Implementation Details.** We followed the same implementation strategy for the two-sample KS test as we did for the one-sample case. More specifically, we used the plug-in approach introduced in Definition 11 and selected the point $u_t$ by maximizing the absolute difference between the empirical cdfs $\hat{F}_{Q,t-1}$ and $\hat{F}_{P,t-1}$ over points in a grid of size $M = 5000$.

As in the one-sample case, we used the standard batch version of the two-sample KS test along with the sequential tests of Darling and Robbins (1968), Lhéritier and Cazals (2018), and Manole and Ramdas (2021) to benchmark our sequential test. The details of their implementation are in Appendix E.

**Experiment Setup.** We compare the performance of the sequential two-sample KS test with the batch version on testing Normal means. In particular, we set $P \sim \mathcal{N}(1, 1)$ and $Q \sim \mathcal{N}(1+\epsilon, 1)$. For different values of $\epsilon$, we show the variation of the power of the tests with sample size in Figure 4.

### 5.2 Sequential Two-Sample Kernel MMD Test

Another common metric between distributions, that belongs to the IPM family, is the kernel maximum mean discrepancy (MMD), defined as

$$d_{\text{MMD}}(P, Q) = \sup_{\|f\|_{\mathcal{K}} \leq 1} \mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{Y \sim Q}[f(Y)].$$
Figure 4: The figures plot the performance of different tests on the one-sample testing problem with target distribution $P \sim \mathcal{N}(0, 1)$ and alternative distribution $Q \sim \mathcal{N}(\epsilon, 1)$.

(Left) The figure on the left plots the power and type-I error curves of our betting based sequential one-sample KS test (Betting) along with the following baselines: the batch two-sample KS test (Batch) and the sequential two-sample KS tests of Howard and Ramdas [2022], Darling and Robbins [1968] and Lhéritier and Cazals [2018], denoted by HR, DR and LC respectively. The shaded histograms represent the distribution of the stopping time of the sequential tests over 500 trials under the alternative. The dashed vertical line denotes the average of the stopping times over these 500 trials.

(Right) The figure on the right plots the power curve for our sequential KS test (solid curves) and the power curve of the batch one-sample KS test (dashed curves) for different values of $\epsilon$.

where $\|f\|_K \leq 1$ is the unit ball of the RKHS $\mathcal{H}_K$ formed by a known positive-definite kernel $K$, with feature map $\phi: \mathcal{X} \to \mathcal{F}$. The mean map of a distribution $P$ is a function in the RKHS (or a mapping from distributions over $\mathcal{X}$ to the RKHS) that is given by

$$\mu_P := E_P[\phi(X)].$$

When $P \neq Q$, the “witness” function $g^*$ that achieves the supremum in $d_{\text{MMD}}(P, Q)$ (i.e. witnesses the difference between $P, Q$) is simply given by

$$g^* := \frac{\mu_P - \mu_Q}{\|\mu_P - \mu_Q\|_K}.$$ 

meaning that $d_{\text{MMD}}(P, Q) = E_{X \sim P}[g^*(X)] - E_{Y \sim Q}[g^*(Y)]$. This motivates the following definition of a betting payoff function $f^*: \mathcal{X} \times \mathcal{X} \to \mathbb{R}$:

$$f^*(x, y) = \frac{1}{2B} (g^*(x) - g^*(y)) = \frac{1}{2B} ((g^*, \phi(x) - \phi(y))) , \text{ where}$$

$$B = \sup_{x' \in \mathcal{X}} \sqrt{K(x', x')}.$$ 

From the second equality in (21), and Cauchy-Schwarz inequality, we observe that

$$|f^*(x, y)| \leq \|g^*\|_K \|\phi(x) - \phi(y)\|_K \leq 1 \times (\|\phi(x)\|_K + \|\phi(y)\|_K) \leq 2B,$$

and thus the scaling ensures that $f^*$ takes values in the range $[-1, 1]$. This definition of $f^*$ also implies that if the null hypothesis is true, then $E[f^*(X, Y)] = 0$; thus ensuring that $f^*$ is indeed a valid payoff function.
We can construct empirical estimates of the mean maps and the witness function \( g^* \) at any time \( t \), based on observations \((X_1, Y_1), \ldots, (X_{t-1}, Y_{t-1})\) as follows:

\[
\hat{\mu}_P := \frac{1}{t-1} \sum_{i=1}^{t-1} \phi(X_i), \quad \text{and} \quad \hat{\mu}_Q := \frac{1}{t-1} \sum_{i=1}^{t-1} \phi(Y_i),
\]

\[
\hat{g}_t := \frac{\hat{\mu}_P - \hat{\mu}_Q}{\|\hat{\mu}_P - \hat{\mu}_Q\|_K} = \frac{\sum_{i=1}^{t-1} K(X_i, x) - K(Y_i, x)}{\sqrt{\sum_{i,j \leq t-1} K(X_i, X_j) + K(Y_i, Y_j) - K(X_j, Y_i) - K(X_i, Y_j)}}.
\]

Following the above discussion, we can now describe our sequential kernel MMD test.

**Definition 12 (Sequential Kernel MMD Test).** At \( t = 1 \), set \( \lambda_i = 0 \). For \( t = 2, 3, \ldots \), proceed as follows:

- Based on the first \( t - 1 \) samples, define the empirical mean map \( \hat{\mu}_{P,t} \) and \( \hat{\mu}_{P,t} \) as defined in (23) and use them to define the plug-in witness function \( \hat{g}_t \) according to (24).
- Use the plug-in witness function to define the betting payoff function \( f_t \) at time \( t \) as follows:

\[
f_t(x, y) = \frac{1}{2B} (\hat{g}_t(x) - \hat{g}_t(y)),
\]

where \( B \) was introduced in (22).

- Observe the next pair \( Z_t = (X_t, Y_t) \sim P \times Q \), and update the wealth process according to the mixture method in Definition 11. In this case, it suffices to select a mixture distribution over \([0, 1]\) instead of \([-1, 1]\) as we know that the \( E_P[g^*(X)] - E_Q[g^*(Y)] \geq 0 \) under the null.

\[
K_t = \int_0^1 \prod_{i=1}^t \left( 1 + \lambda f_t(X_t, Y_t) \right) \nu(\lambda)d\lambda.
\]

- Check the stopping condition: \( K_t \geq 1/\alpha \) for the given confidence level \( \alpha \in (0, 1) \).

Note that \( f_t(X_t, Y_t) \) defined in (25) conditioned on \( F_{t-1} = \sigma(X_1, Y_1, \ldots, X_{t-1}, Y_{t-1}) \) is symmetric, mean zero random variable under the null. Furthermore, we can also construct the required sequence of events under which the requirements of Proposition 5 for asymptotic consistency are satisfied.

**Corollary 4.** The following statements are true for the sequential kernel MMD test introduced in Definition 12.

- Under \( H_0 \), \( P\left( \tau < \infty \right) \leq \alpha \),
- Under \( H_1 \), \( P\left( \tau < \infty \right) = 1 \).

The proof of the consistency follows by identifying appropriate events \( \{E_t : t \geq 1\} \) required for applying Proposition 4. The details are in Appendix D.2.

### 5.2.1 Type-II Error Exponent

To characterize the exponent of the type-II error of our sequential kernel-MMD test by applying Proposition 4, we need to show that there exists a no-regret prediction strategy for the following problem.

**Definition 13 (prediction-MMD).** Let \( \mathcal{G} = \{g \in \mathcal{H}_K : \|g\|_K \leq 1/(2B)\} \) denote the \( 1/(2B) \)-norm ball in the RKHS associated with a positive definite kernel \( K \) with \( \sup_{x \in X} K(x, x) \leq B^2 \). Then, for \( t = 1, \ldots, n \):
• Predict a function $g_t \in \mathcal{G}$.
• Observe the next pair of observations $(X_t, Y_t)$.
• Incur a loss $\ell_t(g_t) := g_t(X_t) - g_t(Y_t) = (g_t, K(X_t, \cdot) - K(Y_t, \cdot))$.

The (normalized) regret for this problem is defined as

$$R_n = \max_{g \in \mathcal{G}} \frac{1}{n} \sum_{t=1}^{n} g(X_t) - g(Y_t) - \frac{1}{n} \sum_{t=1}^{n} g_t(X_t) - g_t(Y_t)$$

$$= \frac{1}{2B} d_{MMD}(\hat{Q}_n, \hat{P}_n) - \frac{1}{n} \sum_{t=1}^{n} g_t(X_t) - g_t(Y_t).$$

Since the loss functions $\{\ell_t(\cdot) : t \geq 1\}$ in this case are linear in the prediction $g_t$, we can employ results from Online Linear Optimization (OLO) in Hilbert spaces to construct a no-regret prediction strategy for the problem in Definition 13. In particular, by using the KT-potential based strategy for OLO in Hilbert spaces proposed by Orabona and Pal (2016) for predicting the functions $\{f_t : t \geq 1\}$, we can show that the resulting normalized regret $R_n$ is $o(n)$ uniformly over all sequences of observations. This, in turn, implies that the type-II error of the sequential kernel-MMD test converges to zero at an exponential rate. Before formally stating the result, we first present the notion of KT potential, introduced by Orabona and Pal (2016).

For $t \in \mathbb{N}$, and $x \in (-t-1, t-1)$, we define the Krichevsky-Trofimov potential (or KT potential) function as

$$V_t(x) = \frac{x^t}{\pi t!} \Gamma \left( \frac{t+1+x}{2} \right) \Gamma \left( \frac{t+1-x}{2} \right),$$

(26)

where $\Gamma(\cdot)$ denotes the Gamma function. We can now state the main result of this section.

**Proposition 10.** Suppose we predict $f_t(X_t, Y_t) = g_t(X_t) - g_t(Y_t)$ as follows in the test defined in Definition 13:

$$g_t = \frac{1}{2B \|W_{t-1}\|_K} \left( 1 + \sum_{i=1}^{t-1} g_i(X_i) - g_i(Y_i) \right),$$

(27)

$$W_t := \hat{\mu}_{t,P} - \hat{\mu}_{t,Q} = \frac{1}{t-1} \sum_{i=1}^{t-1} K(X_i, \cdot) - K(Y_i, \cdot), \quad \text{and} \quad v_t := \frac{V_t(\|W_{t-1}\|_K+1)}{V_t(\|W_{t-1}\|_K+1) + V_t(\|W_{t-1}\|_K-1)},$$

where $V_t$ denotes the KT-potential introduced in [26]. Then, for the resulting sequential test with this prediction strategy satisfies the following under the alternative:

$$\liminf_{n \to \infty} -\frac{1}{n} \log (\mathbb{P}(\tau > n)) \geq \inf_{P' \in \mathcal{P}(X)} \frac{1}{2} \left( d_{KL}(P', P) + d_{KL}(P', Q) \right).$$

This result follows by combining the existing regret bounds on the prediction problem of Definition 13 with the sufficient conditions required for exponential decay of type-II error derived earlier in Proposition 10. The details are in Appendix D.3.

### 5.2.2 Experiments

**Implementation details.** In the experiments, we consider the RKHS associated with the Gaussian Kernel, defined as $K(x, y) = \exp \left( \frac{\|x-y\|^2}{2\sigma^2} \right)$. We fixed the bandwidth $b$ equal to 1.0 for all the experiments, and implemented the plug-in version of the sequential kernel-MMD test of Definition 12.
We used the fixed-sample-size or batch version of the two-sample kernel-MMD test, along with the sequential tests of Manole and Ramdas (2021) and Lhéritier and Cazals (2018), to benchmark the performance of our sequential test. The details of the implementation of these tests are in Appendix E.

**Experiment Setup.** We consider two-sample testing problems with multi-variate normal distributions in $m = 5$ dimensions. In particular, we set $P \sim N(0, I_m)$ where $0$ denotes the all zeros vector in $\mathbb{R}^m$ and $I_m$ is the corresponding identity matrix. Under the null, we then have $Q = P$, whereas under the alternative, we set $Q \sim N(\epsilon 1, I_m)$ where $1$ is the all-ones vector in $\mathbb{R}^m$ and $\epsilon > 0$ is a parameter characterizing the hardness of the problem. The performance of our proposed sequential kernel-MMD test and the batch kernel-MMD test for different choices of the parameter $\epsilon$ is shown in Figure 5.

![Figure 5: The figure compares the performance of different sequential and batch kernel-MMD tests on a two sample testing problem, where the observations $(X_1, Y_1), (X_2, Y_2), \ldots$ are i.i.d. $P \times Q$. In our experiments, we assume $P \sim N(0, I_m)$ with $m = 5$. The distribution $Q$ equals $P$ under the null, while under the alternative we have $Q \sim N(\epsilon 1, I_m)$. In both the figures, $\alpha$ is set to 0.05. (Left) The figure on the left plots the power and type-I error curves of our betting based sequential kernel-MMD test (Betting) along with the following baselines: the batch two-sample Kernel-MMD test (Batch) and the sequential two-sample KS tests of Manole and Ramdas (2021) and Lhéritier and Cazals (2018), denoted by MR and LC respectively. The shaded histograms represent the distribution of the stopping time of the sequential tests over 500 trials under the alternative. The dashed vertical line denotes the average of the stopping times over these 500 trials. (Right) The figure on the right plots the power curves for our sequential kernel-MMD test (solid curves) and the quadratic-time batch kernel-MMD test (dashed curves) for different values of $\epsilon$.](image)

### 6 Conclusion and Future Work

In this paper, we described a general strategy of constructing (one-sided) sequential tests for one- and two-sampling testing problems. The fundamental idea underlying our approach is the principle of testing by betting, which motivates a game-theoretic formulation of the problems of sequential testing. Within this framework, constructing sequential tests reduces to designing strategies for selecting values of bets and the payoff functions for a sequence of betting games.

While our focus in this paper was on the problems of one- and two-sample testing, the general design principles are easily transferable to other problems as well. Two such problem instances are:

1. **Testing for stochastic dominance.** Here, the $\mathcal{X} = [0, 1]$-valued observations $Y_1, Y_2, \ldots$ are assumed to
be i.i.d. according to an unknown $Q$. For a given distribution $P$, the goal is to test

$$H_0 : Q = P, \quad \text{versus} \quad H_1 : P \preceq_k Q,$$

where \(\preceq_k\) denotes \(k^{th}\) order stochastic dominance (see Appendix F for definition).

2. Testing for symmetry. One way to formulate this problem is to consider real valued observations $Y_1, Y_2, \ldots$ distributed i.i.d. according to a distribution $Q$ with a cdf $F_Q$. The goal is to test

$$H_0 : F_Q(y) + F_Q(-y) = 1 \forall y \in \mathbb{R}, \quad \text{versus} \quad H_1 = H_{1,+} \cup H_{1,-}, \quad \text{where}$$

$$H_{1,+} : \sup_{y \in \mathbb{R}} F_Q(y) + F_Q(-y) > 1, \quad H_{1,-} : \inf_{y \in \mathbb{R}} F_Q(y) + F_Q(-y) < 1.$$

For both these problems, we can design sequential tests based on the ideas introduced in this paper. The details are presented in Appendix F. A detailed investigation about the power and optimality of these tests is an interesting topic for future work.

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A Additional Background

A.1 Connections to Universal Portfolio Optimization

The betting game introduced in Definition 4 can also be interpreted within the framework of portfolio optimization. In particular, introduce the change of variables,

\[
\mu_t = \frac{1 + \lambda_t}{2}, \quad \text{and} \quad s_t = \frac{1 + f_t(Y_t)}{2}.
\]

Note that both \( \mu_t \) and \( s_t \) belong to \([0, 1]\) for all \( t \). Introduce \( M_t = (\mu_t, 1 - \mu_t) \in \Delta_2 = \{(a, 1-a) : a \in [0, 1]\} \) and \( S_t = (s_t, 1 - s_t) \in \Delta_2 \) and let \( (M_t, S_t) = M_t^T S_t \) denote the usual inner product in \( \mathbb{R}^2 \). Then, we observe the following:

\[
1 + \lambda_t f_t(Y_t) = 1 + \lambda_t (2s_t - 1) = (1 - \lambda_t + 2\lambda_t s_t) = (1 - \lambda_t + (1 + \lambda_t - (1 - \lambda_t)) s_t)
\]

\[
= 2 \left( \frac{1 - \lambda_t}{2} (1 - s_t) + \frac{1 + \lambda_t}{2} s_t \right) = 2(\mu_t s_t + (1 - \mu_t)(1 - s_t))
\]

\[
= 2\langle M_t, S_t \rangle. \tag{28}
\]

From (28), we conclude that the wealth accrued after \( n \) rounds of the game from Definition 4 is

\[
K_n = \prod_{t=1}^{n} (1 + \lambda_t f_t(Y_t)) = 2^n \prod_{t=1}^{n} \langle M_t, S_t \rangle := 2^n \tilde{K}_n.
\]

The term \( \tilde{K}_n \) is the total wealth after investing in a portfolio of 2 stocks over a period of \( n \) days. More formally, at the beginning of the \( t \)th day, the investor allocates a \( \mu_t \) fraction of the total wealth to the first stock, and a \( 1 - \mu_t \) fraction to the second. At the end of the day, the multiplicative return from stock 1 is \( s_t \) and that from stock 2 is \( 1 - s_t \).

A.2 Large Deviation Results

The following result is used in showing the exponential rate at which the term \( \mathbb{P}(\tau > n) \) decays to zero under the alternative for the one-sample chi-squared test with finite \( X \) in Proposition 9, and for the sequential KS test in Proposition 7.

**Fact 3 (Sanov’s Theorem).** Suppose \( \mathcal{X} \) is a Polish space (that is, a complete separable metric space) and let \( \mathcal{P}(\mathcal{X}) \) denote the set of probability distributions on \( \mathcal{X} \). Suppose \( Q \in \mathcal{P}(\mathcal{X}) \) and let \( Y_1, \ldots, Y_n \sim Q \) denote \( n \) i.i.d. samples from \( Q \). For any subset \( E \) of \( \mathcal{P} \), we then have the following:

\[
\limsup_{n \to \infty} -\frac{1}{n} \log \left( \mathbb{P} \left( \hat{Q}_n \in E \right) \right) \leq \inf_{P' \in E^\circ} d_{KL}(P', Q),
\]

\[
\liminf_{n \to \infty} -\frac{1}{n} \log \left( \mathbb{P} \left( \hat{Q}_n \in E \right) \right) \geq \inf_{P' \in \overline{E}} d_{KL}(P', Q).
\]

Here \( \hat{Q}_n \) denotes the empirical measure constructed using \( Y_1, \ldots, Y_n \) and \( E \) and \( E^\circ \) denote, respectively, the closure and interior of \( E \) with respect to the weak topology on \( \mathcal{P} \).

To obtain a similar in the case of kernel-MMD two-sample test, we require an extension of the usual Sanov’s theorem to deal with observations drawn from a pair of distributions. We next state this extension of Sanov’s theorem which was derived by Zhu, Chen, Chen, and Yang (2021).
For any $\lambda$ corresponding wealth. That is, the following notations:

\[
\lambda_n \in \arg\max_{\lambda \in L} \sum_{t=1}^{n} \left( 1 + \lambda f_t(Y_t) \right), \quad \text{and} \quad K^*_n = \prod_{t=1}^{n} \left( 1 + \lambda^*_n f_t(Y_t) \right).
\]

For any $\lambda$ such that $|\lambda - \lambda_n^*| \leq \Delta \lambda$ and for any $x \in [-1, 1]$, note that we have by the mean-value theorem:

\[
|\log(1 + \lambda x) - \log(1 + \lambda_n^* x)| \leq \sup_{\lambda \in L} \frac{x}{1 + \lambda x} |\lambda - \lambda_n^*| \leq \frac{\Delta \lambda}{1 - \lambda_{\max} - \Delta \lambda}. \quad (29)
\]

The last inequality in the above display follows by noting that $|x/(1 + \lambda x)| \leq 1/(1 - \lambda)$. Now for any $\lambda$ with $|\lambda_n^* - \lambda| \leq \Delta \lambda$, (29) implies that

\[
K^{\lambda}_n = \exp \left( \sum_{t=1}^{n} \log \left( 1 + \lambda f_t(Y_t) \right) \right) \\
\geq \exp \left( \sum_{t=1}^{n} \log \left( 1 + \lambda_n^* f_t(Y_t) \right) - \frac{\Delta \lambda}{1 - \lambda_{\max} - \Delta \lambda} \right) \\
= K^*_n \exp \left( \frac{-n\Delta \lambda}{1 - \lambda_{\max} - \Delta \lambda} \right).
\]

Finally, using the condition on the density in the hypothesis of this proposition, we get

\[
K_n = \int_{-1}^{1} K^{\lambda}_n \nu(d\lambda) \geq (c_0 \Delta \lambda) K^*_n \exp \left( \frac{-n\Delta \lambda}{1 - \lambda_{\max} - \Delta \lambda} \right) \\
= K^*_n \exp \left( \frac{-n\Delta \lambda}{1 - \lambda_{\max} - \Delta \lambda} - \log \left( \frac{1}{c_0 \Delta \lambda} \right) \right).
\]

This completes the proof of (2).
Proof of (3). First, by setting \( \lambda_{\text{max}} = 0.5 \) and \( \Delta \lambda = 1/n \), we get that
\[
K_n \geq K_n^* \exp \left( -4 - \log(n/c_0) \right), \quad \text{for } n \geq 4.
\]
It remains to show that
\[
K_n^* = \max_{\lambda \in [0,1/2]} K_n^\lambda \geq \exp \left( \frac{1}{2} \left( \frac{1}{n} \sum_{t=1}^{n} f_t(Y_t) \right)^2 \right).
\]
For this, we use the fact that \( \log(1 + \lambda x) \geq \lambda x - \frac{\lambda^2 x^2}{2} \) for all \( \lambda \in [-1/2, 1/2] \) and \( x \in [-1, 1] \). Hence,
\[
K_n^* \geq \exp \left( \frac{1}{8} \left( \frac{1}{n} \sum_{t=1}^{n} f_t(Y_t) \right)^2 \right).
\]
(30)
The last inequality follows by noting that there exists a value of \( \lambda \in [-1/2, 1/2] \) that ensures a wealth above the RHS in (30). In particular, with \( A = \sum_{t=1}^{n} f_t(Y_t) \) and \( B = \sum_{t=1}^{n} f_t^2(Y_t) \), an appropriate value of \( \lambda \) is \( A/(2(|A| + B)) \), as used in (Cutkosky and Orabona 2018, Theorem 1).

B.2 Proof of Proposition 2

Proof of (7). First, we note that if the null is true, then the process \( \{K_t^* : t \geq 1\} \) is a nonnegative martingale.

\[
E[K_t^* | \mathcal{F}_{t-1}] = K_{t-1}^* E[1 + \lambda_t f_t(Z_t) | \mathcal{F}_{t-1}] = K_{t-1}^*.
\]
The second equality in the display above follows from the assumption that \( \lambda_t f_t(Z_t) \) have conditional mean zero under the null.

Note that \( K_0^* = 1 \), and if \( K_{t-1}^* > 0 \) it implies that \( K_t^* \geq 0 \) since \( 1 + \lambda_t f^*(Z_t) \geq 0 \). By induction, this means that \( \{K_t^* : t \geq 1\} \) is a nonnegative martingale process. Hence, an application of Ville’s inequality implies
\[
P_0 (\exists t \geq 1 : K_t^* \geq 1/\alpha) \leq \alpha,
\]
as required.

Proof of (8). To establish that \( \tau < \infty \) almost surely under the alternative, it suffices to show that
\[
\liminf_{n \to \infty} \frac{\log(K_n^*)}{n} (a.s.) > 0.
\]
This is sufficient to imply that \( \tau < \infty \) almost surely, because the stopping criterion is that \( \frac{1}{t} \log(K_t^*) \geq \frac{1}{t} \log(1/\alpha) \) and the RHS converges to 0 as \( t \to \infty \).

So, to complete the proof, we first observe that Proposition 1 implies that
\[
\frac{\log K_n^*}{n} \geq \frac{1}{n} \left( \frac{1}{2} \left( \frac{1}{n} \sum_{t=1}^{n} f_t^*(Z_t) \right)^2 - 4 - \log \left( \frac{n}{c_0} \right) \right).
\]
Now, taking the limit as $n$ goes to $\infty$, and using the continuity of the mapping $z \mapsto z^2$, we get
\[
\liminf_{n \to \infty} \frac{\log (K_n^*)}{n} \geq \frac{1}{8} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f^*(Z_t) \right)^2 - \lim_{n \to \infty} \frac{1}{n} \left( 4 + \log \left( \frac{n}{c_0} \right) \right) \\
= \frac{1}{8} (E[f^*(Z_1)])^2 > 0,
\]
where the last inequality uses the fact that $E[f^*(Z)] \neq 0$ under the alternative. This concludes the proof.

### B.3 Proof of Proposition 4

We proceed similar to the proof of (8) in Appendix B.2, and show that the condition on $S_n/n$ implies
\[
\liminf_{n \to \infty} \frac{1}{n} \log (K_n) > 0.
\]
This is sufficient to imply that $\tau < \infty$ for this sequence of predictions and observations, because the stopping criterion is that $\frac{1}{n} \log (K_t) > \frac{1}{n} \log (1/\alpha)$ and the RHS is bounded away from 0 as $t \to \infty$. More specifically, we observe that for the betting strategy defined in Definition 5, we have
\[
\liminf_{n \to \infty} \frac{1}{n} \log (K_n) \geq \liminf_{n \to \infty} \left( \frac{1}{8} \left( \frac{S_n}{n} \right)^2 - \frac{1}{n} \left( 4 + \log \left( \frac{n}{c_0} \right) \right) \right) \\
= \frac{1}{8} \left( \liminf_{n \to \infty} \left| \frac{S_n}{n} \right| \right)^2 \\
> 0.
\]
The last inequality in the above display follows from the assumption on $S_n$. This completes the proof.

### B.4 Proof of Proposition 5

Introduce the notation $E[f^*(Z)] = A$ under the alternative. Then, to prove the $\lim_{n \to \infty} S_n/n = A$, it suffices to show that
\[
\liminf_{n \to \infty} \frac{S_n}{n} \geq A, \quad \text{and} \quad \limsup_{n \to \infty} \frac{S_n}{n} \leq A.
\] (31) (32)
We now present the proof of (31). Since (32) follows in an almost identical manner, we omit its proof.

First we note that the condition (11) on the events $\{E_t : t \geq 1\}$ implies that
\[
P(\cap_{n=1}^{\infty} \cup_{t=n}^{\infty} E_t^c) = 0, \quad \text{under the alternative},
\]
by an application of the Borel-Cantelli Lemma. As a result of this, we have
\[
\liminf_{n \to \infty} \frac{1}{n} S_n = \liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) \overset{(a.s.)}{=} \liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) \mathbb{1}_{\{E_t\}} := S_n.
\]
Thus, for the rest of the proof, we can focus on the quantity \( \tilde{S}_n \). Note that the statement in (31) is equivalent to saying that \( \liminf_{n \to \infty} \frac{\tilde{S}_n}{n} \geq A - \delta \) for any \( \delta > 0 \). So, let’s fix an arbitrary \( \delta > 0 \), and define \( n_\delta = \min\{t \geq 1 : \Delta f_t < \delta \} \). Note that \( n_\delta < \infty \) because of the assumption that \( \lim_{t \to \infty} \Delta f_t = 0 \). Then, we have the following for any \( n > n_\delta \):

\[
\frac{\tilde{S}_n}{n} \geq \frac{1}{n} \left( \sum_{t=1}^{n} f^*(Z_t) \mathbb{1}_{\{E_t\}} - \sum_{t=1}^{n_\delta} \|f^* - f_t\|_\infty - \sum_{t=n_\delta + 1}^{n} \|f^* - f_t\|_\infty \right) \\
\geq \frac{1}{n} \left( \sum_{t=1}^{n} f^*(Z_t) \mathbb{1}_{\{E_t\}} - n_\delta \left( \max_{1 \leq t \leq n_\delta} \Delta f_t \right) - \delta \left( n - n_\delta \right) \right) \\
\geq \frac{1}{n} \left( \sum_{t=1}^{n} f^*(Z_t) \mathbb{1}_{\{E_t\}} - n_\delta (1 - \delta) - \delta n \right)
\]

By taking \( n \) to infinity, we get

\[
\liminf_{n \to \infty} \frac{\tilde{S}_n}{n} = \liminf_{n \to \infty} \frac{\tilde{S}_n}{n} \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f^*(Z_t) \mathbb{1}_{\{E_t\}} - (1 - \delta) \limsup_{n \to \infty} \frac{n_\delta}{n} - \delta
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f^*(Z_t) - \delta
\]

\[
= A - \delta.
\]

Since \( \delta > 0 \) was arbitrary, this completes the proof of the statement (31). The proof of (32) follows in an entirely analogous manner, but by upper bounding \( \tilde{S}_n/n \) and then taking the lim sup.

### B.5 Proof of Proposition 6

The vanishing regret condition on the predicted payoff functions implies that the wealth process grows approximately as \( e^{nC^2d^2(\hat{Q}_n,P)} \) in the one-sample tests and \( e^{nC^2d^2(\hat{Q}_n,\hat{P}_n)} \) in the two-sample problem.

**Lemma 1.** Under the conditions of Proposition 6, the wealth process satisfies for \( n \geq 4 \):

\[
K_n \geq \exp \left( \frac{1}{8} \left( nD_n - 2nR_n \right) - 5 \log n \right),
\]

where \( D_n = C^2d^2(\hat{Q}_n,P) \) in the one-sample testing problem and \( D_n = C^2d^2(\hat{Q}_n,\hat{P}_n) \) in the two-sample testing problem.

Using this lemma (proved at the end of this subsection in Appendix [B.5.1]), we can bound the probability \( P(\tau > n) \) for both one- and two-sample problems.

\[
P(\tau > n) \leq P \left( \log K_n < \log \left( \frac{1}{\alpha} \right) \right)
\]

\[
\leq P \left( \frac{n}{8} \left( D_n - 2R_n \right) - 5 \log n < \log \left( \frac{1}{\alpha} \right) \right)
\]

\[
= P \left( \frac{D_n}{C^2d^2} < \frac{8}{C^2n} \left( \log \left( \frac{1}{\alpha} \right) + \frac{nR_n}{8} + 5 \log n \right) \right)
\]

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Now, by the assumption of the vanishing regret, we note that for any $\epsilon > 0$, there exists a $n_\epsilon < \infty$ such that for all $n \geq n_\epsilon$, we have

$$\frac{8}{C^2 n} \left( \log \left( \frac{1}{\alpha} \right) + \frac{n R_n}{8} + 5 \log n \right) \leq \epsilon^2.$$  

**One-sample case.** In this case, $D_n = C^2 d^2(\hat{Q}_n, P)$, and hence we have

$$\mathbb{P}(\tau > n) \leq \mathbb{P}(d(\hat{Q}_n, P) \leq \epsilon) = \mathbb{P}(\hat{Q}_n \in \mathcal{P}_{\epsilon,d}(P)),$$

where $\mathcal{P}_{\epsilon,d}(P) = \{P' \in \mathcal{P}(X) : d(P', P) \leq \epsilon\}$. Finally, an application of Sanov’s theorem implies the following result for any $\epsilon > 0$

$$\liminf_{n \to \infty} -\frac{1}{n} \log (\mathbb{P}(\tau > n)) \geq \inf_{P' \in \mathcal{P}_{\epsilon}} d_{KL}(P', P)$$

as required.

**Two-Sample Case.** In this case, $D_n = C^2 d^2(\hat{Q}_n, \hat{P}_n)$, and hence we have

$$\mathbb{P}(\tau > n) \leq \mathbb{P}(d(\hat{Q}_n, \hat{P}_n) \leq \epsilon) = \mathbb{P}(\hat{P}_n, \hat{Q}_n) \in \mathcal{P}_{\epsilon,d}^2,$$

where $\mathcal{P}_{\epsilon,d}^2 = \{(P', Q') \in \mathcal{P}(X) \times \mathcal{P}(X) : d(Q', P') \leq \epsilon\}$. Finally, an application of Sanov’s theorem implies the following result for any $\epsilon > 0$

$$\liminf_{n \to \infty} -\frac{1}{n} \log (\mathbb{P}(\tau > n)) \geq \inf_{(P', Q') \in \mathcal{P}_{\epsilon,d}^2} \frac{1}{2} \left( d_{KL}(P', P) + d_{KL}(Q', Q) \right)$$

as required.

**B.5.1 Proof of Lemma 1**

By definition of regret, we have the following:

$$\frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) \geq \left( \frac{1}{n} \sup_{f \in G} f(Z_t) \right) - R_n \geq CD_n - R_n,$$

where the last inequality uses the condition that \[12\] holds. Recall that $D_n = C^2 d^2 \left( \hat{Q}_n, P \right)$ in the one-sample testing problem and $D_n = C^2 d^2(\hat{Q}_n, \hat{P}_n)$ in the two-sample testing problem.

Squaring both sides of the inequality, we get

$$\left( \frac{1}{n} \sum_{t=1}^{n} f_t(Z_t) \right)^2 \geq C^2 D_n^2 + R_n^2 - 2CD_n R_n$$

$$\geq C^2 D_n^2 - 2R_n \left( \sup_{f \in G} \frac{1}{n} \sum_{t=1}^{n} f(Z_t) \right) \geq C^2 D_n^2 - 2R_n.$$

The second inequality in the above display follows by another application of the condition in \[12\].

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C Deferred Proofs from Section 4

C.1 Proof of Corollary 1

Proof of [14]. Note that the betting function \( f_t(\cdot) \) used in this test has zero mean under the null: \( \mathbb{E}[f_t(Y_t)] = \mathbb{E}[ \mathbb{1}_{Y_t \leq u_t} ] - F_P(u_t) = 0 \). The second equality uses the fact that \( u_t \) is predictable (i.e., \( \mathcal{F}_{t-1} \) measurable). This implies that the wealth process \( \{ \mathcal{K}_t : t \geq 1 \} \) is a nonnegative martingale with initial value equal to 1 under the null. The result then follows from an application of Ville’s inequality.

Proof of [15]. We know from Proposition 3 that it suffices to show that \( \lim \inf_{n \to \infty} |S_n/n| > 0 \) for this to hold. Throughout the proof, we assume that \( d_{KS}(P, Q) = \mathbb{E}_Q[f^*(Y)] \) for \( f^* \) of the form \( \mathbb{1}_{\{ \cdot \leq u^* \}} - F_P(u^*) \).

The other case, when \( f^* \) is of the form \( F_P(u^*) - \mathbb{1}_{\{ \cdot \leq u^* \}} \) follows in an identical manner and we omit the details.

Now, for this test, we have the following:

\[
\lim_{n \to \infty} \frac{S_n^*}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f^*(Y_t) \left( \equiv \mathbb{E}[f^*(Z)] \right) = d_{KS}(P, Q) > 0.
\]

We will use this to prove [15], by showing that for any \( 0 < \delta < d_{KS}(P, Q) \), we have \( \lim \inf_{n \to \infty} S_n/n > d_{KS}(P, Q) - \delta > 0 \).

We now introduce the events \( \mathcal{E}_t := \{ \| \hat{F}_{1:n-1} - F_Q \|_\infty \leq \sqrt{\log(2|t|)/t} := \Delta F_t \} \) for \( t \geq 1 \), and observe the following.

\[
\mathbb{P}(\mathcal{E}_t) \leq \frac{1}{t^2} \Rightarrow \sum_{t=1}^{\infty} \mathbb{P}(\mathcal{E}_t) < \infty \Rightarrow \mathbb{P}(\mathcal{E}_t \text{ i.o.}) = \mathbb{P}(\cap_{n \geq 1} \cup_{t=n}^{\infty} \mathcal{E}_t) \equiv 0.
\]

In the above display, (a) follows from an application of the DKW inequality, and (b) follows from an application of the first Borel-Cantelli Lemma.

Define \( U^* = \arg \max_{x \in \mathbb{R}} G(x) = \| F_Q(x) - F_P(x) \|, \ u^* = \min \{ x \in U^* \} \), and \( f^*(y) = \mathbb{1}_{\{ y \leq u^* \}} - F_P(u^*) \). Note that if \( \max_{y \in \mathbb{R}} | F_Q(y) - F_P(y) | \) is positive, then \( |U^*| \) must be finite and hence \( u^* \) is well-defined.

To proceed, we need to consider two cases, depending on whether the set \( U^* = \arg \max_{x \in \mathbb{R}} | F_P(x) - F_Q(x) | \) is empty or not.

Case 1: \( U^* \) is not empty. We first present a lemma (proof in Appendix 1) that states that \( u_t \) approaches the optimal point \( u^* \) from below under the event \( \mathcal{E}_t \).

Lemma 2. With \( u_t \) as defined in Definition 4, \( \mathcal{E}_t = \{ \| \hat{F}_{1:n-1} - F_Q \|_\infty \leq \Delta F_t \} \) and assuming that the set \( U^* = \arg \max_{x \in \mathbb{R}} | F_P(x) - F_Q(x) | \) is nonempty, we have the following:

\[
U_t \supseteq U^* \text{ under event } \mathcal{E}_t,
\]

\[
u_t \mathbb{1}_{\mathcal{E}_t} \leq u^* \mathbb{1}_{\mathcal{E}_t} \text{ and } \lim_{t \to \infty} (u_t - u^*) \mathbb{1}_{\mathcal{E}_t} \equiv 0.
\]

Next, for any \( 0 < \delta < d_{KS}(P, Q) \), introduce the set \( U_\delta = \{ u \in \mathbb{R} : F_Q(u) - F_P(u) \geq d_{KS}(P, Q) - \delta \} \) and define \( u^*_\delta = \inf \{ u : u \in U_\delta \} \). Furthermore, from the statement of Lemma 2, we know that there exists a random time \( t_0 < \infty \) (almost surely) such that for all \( t \geq t_0 \), we have \( \mathbb{1}_{\mathcal{E}_t}(u_t - u^*_\delta) \geq 0 \). Combining these
two statements, we can now observe the following for \( n > t_0 \):

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f_t(Y_t) = \liminf_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} f_t(Y_t) \Pi_{\{\varepsilon_t\}} = \liminf_{n \to \infty} \frac{1}{n} \left( \sum_{t=0}^{t_0} f_t(Y_t) + \sum_{t=t_0+1}^{n} f_t(Y_t) \right) \Pi_{\{\varepsilon_t\}}
\]

\[
\geq \liminf_{n \to \infty} \left( - \frac{t_0}{n} + \frac{1}{n} \sum_{t=t_0+1}^{n} (\Pi_{\{\varepsilon_t\}} - F_P(u^*)) \Pi_{\{\varepsilon_t\}} \right)
\]

\[
\geq \liminf_{n \to \infty} \left( - \frac{2t_0}{n} + \frac{1}{n} \sum_{t=1}^{n} (\Pi_{\{Y_1 \leq u_\delta\}} - F_P(u^*)) \Pi_{\{\varepsilon_t\}} \right)
\]

\[
= F_Q(u_\delta) - F_P(u^*) \geq F_Q(u^*) - F_P(u^*) - \delta = d_{KS}(P,Q) - \delta > 0.
\]

In the above display, \([33]\) uses the fact that by definition of the time \( t_0 \) and the event \( \mathcal{E}_t \), we know that \( u_\delta \leq u_t \leq u^* \) under the event \( \mathcal{E}_t \) for \( t \geq t_0 \). Hence, it follows \( \Pi_{\{Y_1 \leq u_\delta\}} \leq \Pi_{\{Y_1 \leq u_t\}} \) and \( -F_P(u_t) \leq -F_P(u^*) \). Similarly, the inequality \([34]\) uses the fact that for all \( t \leq t_0 \), we can simply lower bound \( \Pi_{\{Y_1 \leq u_\delta\}} - F_P(u^*) \) with \(-1\).

**Case 2: \( U^* \) is empty.** When \( U^* \) is empty, it means that the supremum in the definition of \( d_{KS}(P,Q) \) is not achieved. In this case, we can redefine \( u^* \) as follows:

\[
v_n := \inf \{ x : G(x) = |F_Q(x) - F_P(x)| \geq 1/n \}, \quad \text{and} \quad u^* = \lim_{n \to \infty} v_n.
\]

Note that the inf in the definition of the term \( v_n \) in \([35]\) is achieved because of the right-continuity of the function \( G(\cdot) = |F_Q(\cdot) - F_P(\cdot)| \).

With \( u^* \) defined, we note that the oracle payoff function \( f^* \) now becomes \( \Pi_{\{x < u^*\}} - F_P^-(u^*) \), where \( F_P^- \) denotes the left-side limit of \( F_P \). Again, recall that in this proof, we are assuming that \( d_{KS}(P,Q) = F_Q(u^*) - F_P(u^*) \). The other case, when \( F_P \) is larger than \( F_Q \) near \( u^* \), follows in an entirely analogous manner.

Now, with this new definition of \( u^* \) and \( f^* \), we can again proceed as in the previous case and observe that

- At any time \( t \), under the event \( \mathcal{E}_t \), the set \( U_t \) contains \( u^* \). Hence, \( u_t \leq u^* \) (under \( \mathcal{E}_t \)) and, \( \lim_{t \to \infty}(u^* - u_t) \Pi_{\{\varepsilon_t\}} \to 0 \) almost surely.
- For any \( \delta > 0 \), there exist \( \delta_1 > 0 \) and \( t_0 < \infty \) such that \( 0 < (u_t - u^*) \Pi_{\{\varepsilon_t\}} < \delta_1 \) for all \( t \geq t_0 \) and \( F_Q(u) - F_P^- (u) \geq d_{KS}(P,Q) - \delta \) for all \( u \in (u^* - \delta_1, u^*) \).

Based on these two observations, we again get the required statement that

\[
\liminf_{n \to \infty} \frac{S_n}{n} > d_{KS} - \delta,
\]

as needed.

**C.1.1 Proof of Lemma 2**

Suppose \( u \in U^* := \arg\max_{x \in \mathbb{R}} |F_Q(x) - F_P(x)| \). Also, recall the notations \( G(x) = |F_Q(x) - F_P(x)| \), 
\( \tilde{G}_t(x) = |F_{Q,t-1}(x) - F_P(x)| \), \( \hat{G}_t = \sup_{x \in \mathbb{R}} \tilde{G}_t(x) \) and \( U_t = \{ x \in \mathbb{R} : \hat{G}_t(x) \geq \hat{G}_t - 2\Delta F_t \} \).

Then we have the following at time \( t \) under the event \( \mathcal{E}_t \):

\[
\max_{x \in \mathbb{R}} |\tilde{F}_{Q,t-1}(x) - F_P(x)| \leq \max_{x \in \mathbb{R}} |F_Q(x) - F_P(x)| + \Delta F_t
\]

\[
= |F_Q(u) - F_P(u)| + \Delta F_t
\]

\[
\leq |\tilde{F}_{Q,t-1}(u) - F_P(u)| + 2\Delta F_t,
\]

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where the two inequalities use the fact that under $\mathcal{E}_t$, we have $\|\hat{F}_{Q,t-1} - F_Q\|_\infty < \Delta F_t$. As a result of the above chain, we have that $|\hat{F}_{Q,t-1}(u) - F_P(u)| \geq \max_{x \in \mathbb{R}} |\hat{F}_{Q,t-1}(x) - F_P(x)| - 2\Delta F_t$. This implies that $u \in U_t$, and hence $U^* \subset U_t$ under $\mathcal{E}_t$ as required.

The second result that $u_t \leq u^*$ under the event $\mathcal{E}_t$ follows immediately from the inclusion $U^* \subset U_t$ under $\mathcal{E}_t$ proved above.

We now show the final statement that $(u_t - u^*)\mathbb{I}_{\{\mathcal{E}_t\}} \xrightarrow{a.s.} 0$. Since $U^*$ is nonempty, it must be a finite set and hence $u^* = \min\{u : u \in U^*\}$ is well-defined. Let $u < u^*$ be any point in $\mathbb{R}$. Then, by definition $G(u) = |F_Q(u) - F_P(u)| < |F_Q(u^*) - F_P(u^*)| = G(u^*)$. Suppose $t$ is large enough to ensure that $4\Delta F_t < G(u^*) - G(u)$. Then we have the following:

$$
\max_{x \in \mathbb{R}} \hat{G}_t(x) - 2\Delta F_t \geq \hat{G}_t(u^*) - 2\Delta F_t \geq G(u^*) - 3\Delta F_t
$$

\[(a)\] $G(u) + \Delta F_t \geq \hat{G}_t(u),
\]

where (a) uses the fact that $G(u^*) - G(u) > 4\Delta F_t$. Thus, for all $t$ large enough, the point $u \notin U_t$ under the event $\mathcal{E}_t$. Thus, for every $\delta > 0$, there exists a $t_\delta$ such that for all $t \geq t_\delta$, we have

$$
0 \leq (u^* - u_t)\mathbb{I}_{\{\mathcal{E}_t\}} \leq \delta.
$$

This implies the required result that $(u^* - u_t)\mathbb{I}_{\{\mathcal{E}_t\}} \to 0$.

### C.2 Proof of Proposition 7

For a given sequence of observations $Y_1, \ldots, Y_n$, let $\hat{Q}_n$ and $\hat{F}_{Q,n}$ denote the resulting empirical distribution and empirical CDF respectively. Then, the KS distance between $\hat{Q}_n$ and $P \sim \text{Uniform } ([0, 1])$ is defined as,

$$
d_{\text{KS}}\left(\hat{Q}_n, P\right) := \max \left\{ \sup_{u \in \mathcal{X}} \hat{F}_n(u) - u, \sup_{u \in \mathcal{X}} u - \hat{F}_n(u) \right\},
$$

where we used the fact that $F_P(u) = u$ for uniform distributions. Our goal is to design a strategy of predicting $\{f_t : t \geq 1\}$ such that $S_n/n = \frac{1}{n} \sum_{t=1}^n f_t(Y_t)$ is close to $d_{\text{KS}}(\hat{Q}_n, P)$. To deal with these two possibilities in the definition of $d_{\text{KS}}(\hat{Q}_n, P)$ we use a hedged wealth process $K_t = \frac{1}{2} (K_{t,+} + K_{t,-})$ as defined in Section 4.1.1.

For the rest of this proof, we assume that the sequence of observations $Y_1, \ldots$ are such that the KS distance is equal to $\sup_{u \in \mathcal{X}} \hat{F}_n(u) - u$. The proof of the other case follows in an entirely analogous manner.

Now, to obtain the exponent, we will appeal to Proposition 6. In particular, we define the following prediction problem:

**Definition 14 (KS1 Prediction Problem).** For $t = 1, 2, \ldots$:

- Predict a randomized $u_t$ distributed according to $\pi_t^+$ and predict $f_t^+(y) = \mathbb{E}_{u_t \sim \pi_t^+} \left[ \mathbb{I}_{\{y \leq u_t\}} - u_t \right]$ (recall that we have assumed $F_P(u) = u$).
- Observe $Y_t \in \mathcal{X}$.
- Incur a loss $\ell(\pi_t, Y_t) := 1 - f_t^+(Y_t) = 1 - \mathbb{E}_{u_t \sim \pi_t^+} \left[ \mathbb{I}_{\{y \leq u_t\}} - u_t \right]$.

If the predicted $u_t$ is deterministic (i.e., $\pi_t^+$ is delta function), we will use the notation $\ell(u_t, Y_t)$ to denote the loss $1 - \mathbb{I}_{\{y \leq u_t\}} + u_t$. Note that the loss is bounded and lies in the range $[0, 2]$ for all $t$. Finally, we can
define the normalized regret for this prediction problem as
\[
R_n = \frac{1}{n} \left( \sum_{t=1}^{n} \ell(\pi_t^+, Y_t) - \inf_{u \in \mathcal{X}} \sum_{t=1}^{n} \ell(u, Y_t) \right) = \sup_{u \in \mathcal{X}} \left( \hat{F}_n(u) - u \right) - \frac{1}{n} \sum_{t=1}^{n} f_t^+(Y_t)
\]

= \text{d}_{\text{KS}} \left( \hat{Q}_n, P \right) - \frac{1}{n} \sum_{t=1}^{n} f_t^+(Y_t).

For this problem, we will now use the following exponentially weighted prediction strategy
\[
\pi_t(u) \propto \pi_{t-1}(u) \exp (-\eta_t \ell_t(u)), \quad \text{for all } u \in [0, 1],
\]
with \( \eta_t = 1/\sqrt{t} \).

We state the regret bound in the following lemma, which follows as a direct consequence of Proposition 8 for this specific choices of \( \eta_t \) stated above.

**Lemma 3.** For the prediction problem of Definition 14, the exponentially weighted strategy described in (36) incurs a regret
\[
R_n = d_{\text{KS}} \left( \hat{Q}_n, P \right) - \frac{1}{n} \sum_{t=1}^{n} f_t(Y_t) = \mathcal{O} \left( \sqrt{\frac{\log n}{n}} \right).
\]

Proof. The result follows by plugging \( \eta_t = 1/\sqrt{t} \) and \( \delta_t = 1/\sqrt{t} \) for all \( t \geq 1 \) in (38) in the proof of Proposition 8.

Now that we have a no-regret prediction scheme, we can appeal to Proposition 6 to conclude that the type-II error of the sequential KS test (with the EW scheme used for predicting \( f_t \)) converges to zero exponentially. That is, we have under the alternative
\[
\lim \inf_{n \to \infty} -\frac{1}{n} \log \left( \mathbb{P} (\tau > n) \right) = \sup_{\epsilon > 0} \inf_{P' \in \mathcal{P}_\epsilon \cdot d_{\text{KS}}(P)} d_{\text{KL}}(P', Q) := \beta_{\text{KS}}.
\]

Finally, we observe that the exponent \( \beta_{\text{KS}} \) above is actually equal to \( d_{\text{KL}}(P, Q) \).

**Lemma 4.** Under the assumptions of Proposition 7, the exponent \( \beta_{\text{KS}} \) introduced in (37) is equal to \( d_{\text{KL}}(P, Q) \).

Proof. This statement follows in two steps:

- As mentioned in (Gibbs and Su 2002, Theorem 6), the KS metric metrizes the weak convergence if the limiting distribution (in our case \( \hat{P} \)) is absolutely continuous w.r.t. the Lebesgue measure on \( \mathbb{R} \). As a result of this, the set \( \mathcal{P}_\epsilon \cdot d_{\text{KS}}(P) := \{ P' \in \mathcal{P}(\mathcal{X}) : d_{\text{KS}}(P', P) < \epsilon \} \) is an open set in the weak topology.
- Since \( P \in \mathcal{P}_\epsilon \cdot d_{\text{KS}}(P) \), the projection \( \inf_{P' \in \mathcal{P}_\epsilon \cdot d_{\text{KS}}(P)} d_{\text{KL}}(P', Q) \) is always upper bounded by \( d_{\text{KL}}(P, Q) \). The result \( \beta_{\text{KS}} = d_{\text{KL}}(P, Q) \) then follows by appealing to the fact that KL-divergence is lower-semicontinuous in the weak topology.
C.3 Proof of Proposition 8

To prove this proposition, we first recall an intermediate result in the analysis of the exponentially weighted prediction algorithm over a continuum of actions. Since it is easier to work with losses, we introduce $\ell_t(u) := B - r_t(u)$.

**Fact 5** (Lemma 1 of Krichene, Balandat, Tomlin, and Bayen (2015)). Introduce the notations $u^*_t \in \arg \max_{u \in U_t} \sum_{i=1}^{t} r_i(u) = \arg \min_{u \in U} \sum_{i=1}^{t} \ell_i(u)$ and $L_t(u) = \sum_{i=1}^{t} \ell_i(u)$. Then the normalized regret, $R_n = R_n/\kappa$, incurred by the exponentially weighted scheme satisfies

$$R_n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{u_i \sim \pi_i} [\ell_t(u_i)] - \frac{1}{n} \min_{u \in U_t} L_n(u) \leq \frac{1}{2n} \sum_{i=1}^{n} \eta_t + \frac{1}{n} \left( \xi(\eta_t, L_n) - L_n(u^*_n) \right),$$

where $\xi(\eta, g) := \frac{1}{\eta} \log \left( \int_{\mathcal{X}} \exp(-\eta g(u)) \, du \right)$.

Next, we show that for all $t \geq 2$, we can find subsets of the decision set, denoted by $I_t$, that contain $u^*_t$ and also have a non-trivial volume.

**Lemma 5.** For all $t \geq 2$, there exists an $I_t \subset \mathcal{X}$ satisfying the following properties:

- $u^*_t \in I_t$, with $\mathrm{vol}(I_t) = \Omega(\delta_t^{m})$
- $\sup_{u \in I_t} |L_t(u) - L_t(u^*_t)| = c_t \leq T \omega(\delta_t)$.

Recall that $\omega(\cdot)$ denotes the modulus of continuity of the continuous part of the loss function, and the sequence \{\delta_t : t \geq 1\} were introduced in the statement of Proposition 8.

The proof of this statement is given in Appendix C.3.1 at the end of this section.

Having guaranteed the existence of subsets \{I_t : t \geq 1\} of $\mathcal{U}$, we can now proceed in a manner analogous to the proof of (Krichene, Balandat, Tomlin, and Bayen 2015 Lemma 2). In particular, we consider the term $\xi(\eta_t, L_t) - L_t(u^*_t)$.

$$\begin{align*}
\xi(\eta_t, L_t) &= -\frac{1}{\eta_t} \log \left( \int_{\mathcal{X}} \exp(-\eta_t L_t(u)) \, du \right) \\
&\leq -\frac{1}{\eta_t} \log \left( \int_{I_t} \exp(-\eta_t L_t(u)) \, du \right) \\
&\leq -\frac{1}{\eta_t} \log \left( \int_{I_t} \exp(-\eta_t (L_t(u^*_t) + c_t)) \, du \right) \\
&= L_t(u^*_t) + c_t + \frac{1}{\eta_t} \log \left( \frac{1}{\mathrm{vol}(I_t)} \right) \\
&= L_t(u^*_t) + c_t + \frac{1}{\eta_t} \log \left( \frac{1}{\mathrm{vol}(I_t)} \right) \\
&= L_t(u^*_t) + T \omega(\delta_t) + \frac{m}{\eta_t} \log \left( \frac{1}{\delta_t} \right) + O(1).
\end{align*}$$

The last equality in the above display uses the bounds on $c_t = \sup_{u \in I_t} |L_t(u) - L_t(u^*_t)|$ and $\mathrm{vol}(I_t)$ guaranteed by Lemma 5.
Combining this with Fact 5, we get

\[ R_n \leq \frac{1}{2n} \sum_{i=1}^{n} \eta_i + \frac{1}{n} \left( n \omega(\delta_n) + \frac{m}{\eta_n} \log \left( \frac{1}{\delta_n} \right) + O(1) \right) \]

\[ = \frac{1}{2n} \sum_{i=1}^{n} \eta_i + \frac{m}{n \eta_n} \log \left( \frac{1}{\delta_n} \right) + \omega(\delta_n) + O \left( \frac{1}{n} \right) \]

\[ = (i) \frac{1}{2n} \sum_{i=1}^{n} \eta_i + \frac{m}{n \eta_n} \log \left( \frac{1}{\delta_n} \right) + o(1). \] (38)

The equality (i) above uses the assumption that \( \omega(\delta_n) = o(1) \). Finally, we employ the assumption that there exists \( 0 < a < 1 \) such that \( \log(1/\delta_n)/n^a = o(1) \) and set \( \eta_i = t^{-1-\alpha} \). This implies the following:

\[ \sum_{i=1}^{n} \eta_i \leq \int_{1}^{n} t^{-(1-\alpha)} dt = \frac{n^a}{a} \quad \text{and} \quad \frac{m}{nm} \log \left( \frac{1}{\delta_n} \right) = \frac{m}{n^a} \log \left( \frac{1}{\delta_n} \right) = o(1). \]

Plugging this into (38), we get

\[ R_n = \frac{1}{2a} n^{-(1-\alpha)} + o(1), \quad \text{or} \quad R_n = o(1) \]

as required.

C.3.1 Proof of Lemma 5

Introduce the notations \( J_t(u) = \sum_{i=1}^{t} g_i(u), \quad H_t(u) = \sum_{i=1}^{t} h_i(u) \) and \( A_t = J_t(u^*_t) - H_t(u^*_t) = \sup_{u \in U} J_t(u) - H_t(u) = tB - \inf_{u \in U} L_t(u) \). Note that, by the assumptions on \( \{g_t, h_t : t \geq 1\} \), we know that \( A_t \geq 0 \) for all \( t \geq 1 \). Finally, we will also use the notation \( C(u, \delta) \) to represent the set \( \{u' \in U : u \preceq u', \quad \text{and} \quad \|u - u'\| \leq \delta\} \cap U \).

We now consider the following cases:

- If \( A_t = 0 \), then it means that \( \cup^{t-1} \cup U \subset \arg \max_{u \in U} J_t(u) - H_t(u) \). Let us select \( u^*_t = 0 \), and define \( I_t = C(0, \delta_t) = U \cap \bigotimes_{j=1}^{m} [0, 0 + \delta_t] \). Then, we observe the following:
  - Due to the uniform continuity assumption on the functions \( \{h_i : i \geq 1\} \), we have that \( \sup_{u \in I_t} |H_t(u) - H_t(0)| \leq t \omega(\delta_t) \).
  - By the assumption that \( A_t = 0 \), we have \( J_t(u) - H_t(u) \leq 0 \) for all \( u \in I_t \) and by the monotonicity of \( J_t \), we know that \( J_t(u) \geq 0 \) for all \( u \in I_t \). Together, these facts imply that
    \[ c_t := \sup_{u \in I_t} |L_t(u) - L_t(u^*_t)| \leq t \omega(\delta_t). \]

- If \( A_t \leq t \omega(\delta_t) \) and \( \text{Vol} \{C(u^*_t, \delta_t)\} < \delta_t^m \). In this case, we define \( I_t = C(u^*_t, \delta_t) \cap C(0, \delta_t) \). By construction, we have \( \text{vol}(I_t) = \Omega(\delta_t^m) \). Furthermore, we also have \( c_t \leq 2t \omega(\delta_t) \). This is due to the following observations:
  - By assumption, we have \( |L_t(u) - L_t(u^*)_t| \leq t \omega(\delta_t) \) for all \( u \in C(u^*_t, \delta_t) \). This follows from the uniform continuity of \( H_t \) and the monotonicity of \( J_t \).
  - By an argument similar to the case when \( A_t = 0 \), we have \( |L_t(u) - L_t(0)| \leq t \omega(\delta_t) \) for all \( u \in C(0, \delta_t) \).
Combining these two statements, we get that \( \sup_{u \in I_t} |L_t(u) - L_t(u_t^*)| \leq 2t \omega(\delta_t) \).

- Finally, we consider the case in which \( A_t > t \omega(\delta_t) \). Due to the uniform continuity of \( H_t \) and the monotonicity of \( J_t \), this is only possible if the maximum in the definition of \( A_t \) is achieved at a point \( u_t^* \) satisfying \( \|u_t^* - U^+\| \geq \delta_t \). Hence, we can define \( I_t = C(u_t^*, \delta_t) \) and note that \( \text{vol}(I_t) = \Omega(\delta_t^m) \). Next, we observe that

\[
 c_t = \sup_{u \in I_t} |L_t(u) - L_t(u_t^*)| = \sup_{u \in I_t} |J_t(u) - J_t(u_t^*) + H_t(u_t^*) - H_t(u)| \\
\leq \left| H_t(u_t^*) - H_t(u) \right| \\
\leq t \omega(\delta_t).
\]

The inequality (i) in the display above relies on the following argument. Due to the definition of \( u_t^* \), we must have \( L_t(u) - L_t(u_t^*) \leq 0 \). Hence, the supremum in the definition of \( c_t \) will be achieved at a point where \( L_t(u) - L_t(u_t^*) = J_t(u) - J_t(u_t^*) + L_t(u_t^*) - L_t(u) \) is most negative. Furthermore, by the construction of \( I_t \), any point \( u \in I_t \) satisfies \( u_t^* \leq u \) and that implies \( J_t(u) \geq J_t(u_t^*) \) due to the monotonicity assumption on \( q_t \). Hence, removing the \( J_t(u) - J_t(u_t^*) \) term only makes the absolute value larger; justifying the inequality (i).

To conclude, we have shown that for all \( t \geq 1 \), there always exists an \( I_t \subset \mathcal{U} \) such that

1. \( u_t^* \in I_t \) such that \( L_t(u_t^*) = J_t(u_t^*) - H_t(u_t^*) \geq L_t(u) \) for all \( u \in \mathcal{U} \),
2. \( \text{vol}(I_t) = \Omega(\delta_t^m) \), and
3. \( \sup_{u \in I_t} |L_t(u) - L_t(u_t^*)| \leq 2t \omega(\delta_t) \).

This completes the proof.

### C.4 Proof of Corollary 2

**Proof of (18).** This follows from the fact that the wealth process introduced in Definition 10 is a martingale under the null. In particular,

\[
\mathbb{E}_0[K_t|\mathcal{F}_{t-1}] \stackrel{(a)}{=} \mathbb{K}_{t-1}\mathbb{E}_0 \left[ (1 + \lambda_t f_t(X_t)) + \frac{1}{\alpha} \mathbb{1}_{\{X_t \notin X^*\}} | \mathcal{F}_{t-1} \right] \\
= \mathbb{K}_{t-1} \left( 1 + \lambda_t \mathbb{E}_0 \left[ f_t(X_t) | \mathcal{F}_{t-1} \right] \right) \\
= \mathbb{K}_{t-1} \left( 1 + \lambda_t \mathbb{E}_0 \left[ \tilde{q}_{t-1}(X_t)/p(X_t) - 1 \right] | \mathcal{F}_{t-1} \right) \\
= \mathbb{K}_{t-1} \left( 1 + \frac{\lambda_t}{C_P} \sum_{x \notin \mathcal{X}} p(x) \left( \frac{\tilde{q}_{t-1}(x)}{p(x)} - 1 \right) \right) = 0.
\]

In the above display, (a) uses the fact that \( Q = P \) under the null, and hence \( \mathbb{1}_{\{X_t \notin X^*\}} \) is a zero probability event.

**Proof of (19).** We first consider the case in which \( Q(\mathcal{X}) < 1 \).

\[
\mathbb{P}_1(\tau = \infty) = \mathbb{P}_1 \left( \bigcap_{t=1}^\infty \left( \{ K_{t-1} < 1/\alpha \} \cap \{ X_t \notin \mathcal{X} \} \right) \right) \\
\leq \mathbb{P}_1 \left( \bigcap_{t=1}^\infty \{ X_t \notin \mathcal{X} \} \right) \\
= \lim_{t \to \infty} (1 - Q(\mathcal{X}))^t = 0.
\]
For the case when $Q(\mathcal{X}) = 1$, we appeal to Proposition 3. In particular, we introduce the event $\mathcal{E}_t$ for $t \geq 1$ as follows,

$$\mathcal{E}_t = \{|\hat{q}_{t-1}(x) - q(x)| \leq \Delta q_{x,t}, \text{ for all } x \in \mathcal{X}\}, \quad \text{and}$$

$$P_1(\mathcal{E}_t) \geq 1 - \frac{1}{t^2}, \quad \Rightarrow \sum_{t=1}^{\infty} P_1(\mathcal{E}_t') < \infty.$$

Thus it remains to show that $\lim_{t \to \infty} \log (1 + \lambda_t f_t(X_t)) - \log (1 + \lambda^* f^*(X_t)) 1_{\{\mathcal{E}_t\}} = 0$ almost surely.

This result follows from the fact that $\lim_{t \to \infty} \Delta q_{t,x} = 0$, which implies that, $\lim_{t \to \infty} f_t(X_t) = f^*(X_t)$ and $\lim_{t \to \infty} \lambda_t = \lambda^*$ almost surely.

C.5 Proof of Proposition 9

First, we note that as a result of Proposition 1, the wealth at time $n \geq 2$ satisfies:

$$K_n \geq \exp \left( \frac{n^{8C_2P}}{8C_2P} \left( \frac{1}{n} \sum_{t=1}^{n} q_t(X_t) \right) \right)^2 - 5 \log n.$$

Our next lemma relates the first term inside the exponent in the above display to the chi-squared distance between the empirical distribution $\hat{Q}_n$ and the target distribution $P$.

**Lemma 6.** For the choice of $q_t$ as described in (20), the following is true:

$$\left( \frac{1}{n} \sum_{t=1}^{n} q_t(X_t) \right)^2 \geq d^2_{\chi^2}(\hat{Q}_n, P) - C \frac{1}{\sqrt{n}},$$

for some constant $C$ depending on $m = |\mathcal{X}|$.

The proof of this statement is deferred to the end of this subsection Appendix C.5.1.

We now use Lemma 6 to observe that

$$K_n \geq \exp \left( \frac{n^{8C_2P}}{8C_2P} d^2_{\chi^2}(\hat{Q}_n, P) - C \frac{1}{\sqrt{n}} \right),$$

where $C_1 := \sup_{n \in \mathbb{N}} \frac{8C_2P}{\sqrt{n}} + \frac{\log n}{\sqrt{n}} < \infty$.

Now, for a given value of $\alpha \in (0, 1)$, we introduce $C_2 := \sup_{n \in \mathbb{N}} \left( 8C_2^2P (C_1 + \log(1/\alpha)/\sqrt{n}) \right)$, and obtain the following:

$$\{\tau > n\} = \bigcap_{t=1}^{n} \{K_t < 1/\alpha\} \subset \{\log K_n < \log (1/\alpha)\} \subset \left\{ \frac{n}{8C_2^2P} d^2_{\chi^2}(\hat{Q}_n, P) - C_1 \sqrt{n} < \log(1/\alpha) \right\} \subset \left\{ d^2_{\chi^2}(\hat{Q}_n, P) < 8C_2^2P \left( C_1 + \frac{\log(1/\alpha)}{\sqrt{n}} \right) \frac{1}{\sqrt{n}} \right\} \subset \left\{ d^2_{\chi^2}(\hat{Q}_n, P) < C_2/\sqrt{n} \right\} \subset \left\{ d^2_{TV}(\hat{Q}_n, P) < C_2/\sqrt{n} \right\}. $$
For a fixed value of \( \epsilon_1 > 0 \), assume that \( n \) is large enough to ensure \( C_2/\sqrt{n} < \epsilon_1^4 \). Introduce the notation \( \mathcal{P}_{\epsilon_1} = \{ P' : d_{TV}(P', P) \leq \epsilon_1 \} \), and observe that we have

\[
\{ \tau > n \} \subset \{ d_{TV}(\hat{Q}_n, P) < \epsilon_1 \} = \{ \hat{Q}_n \in \mathcal{P}_{\epsilon_1} \}.
\]

Now, by an application of Sanov’s theorem, we get

\[
\lim \inf_{n \to \infty} -\frac{1}{n} \log (P_1(\tau > n)) \geq \lim \inf_{n \to \infty} -\frac{1}{n} \log \left( \mathbb{P}_1 \left( \hat{Q}_n \in \mathcal{P}_{\epsilon_1} \right) \right) \geq \inf_{P' \in \mathcal{P}_{\epsilon_1}} d_{KL}(P', P).
\]

The final step of the proof is to observe the following:

- The total variation distance metrizes the weak topology of probability measures on \( \mathcal{X} \) when \( |\mathcal{X}| \) is finite (Gibbs and Su 2002, Theorem 2).
- The KL-divergence is lower-semicontinuous w.r.t. the weak topology on the space of probability measures on \( \mathcal{X} \) (Van Erven and Harremos 2014, Theorem 19).

As a result of these two statements, for any given \( \epsilon > 0 \), there exists an \( \epsilon_1 > 0 \) such that

\[
\inf_{P' \in \mathcal{P}_{\epsilon_1}} d_{KL}(P', Q) \geq d_{KL}(P, Q) - \epsilon.
\]

Hence, we obtain the statement

\[
\lim \inf_{n \to \infty} -\frac{1}{n} \log (P_1(\tau > n)) \geq d_{KL}(P, Q) - \epsilon.
\]

This completes the proof since \( \epsilon > 0 \) is arbitrary.

### C.5.1 Proof of Lemma 6

To simplify the proof, we introduce the following notation:

\[
A_n = \frac{1}{n} \sum_{t=1}^{n} q_t(X_t) - 1,
\]

\[
B_n = \frac{1}{n} \sum_{t=1}^{n} q_t(X_t) - 1 = d_{x^2} \left( \frac{\hat{Q}_n}{\hat{P}}, P \right) = \frac{1}{n} \sum_{x \in \mathcal{X}} n_x \left( \frac{n_x}{np(x)} - 1 \right), \quad \text{where} \quad n_x = \sum_{t=1}^{n} 1 \{ X_t = x \}, \quad x \in \mathcal{X}.
\]

Now we consider two cases.

**Case I:** \( B_n < A_n \). In this case, we obtain the trivial inequality

\[
A_n \geq B_n \Rightarrow A_n^2 \geq B_n^2 - \frac{1}{\sqrt{n}}
\]

for any \( C > 0 \).

**Case I:** \( B_n \geq A_n \). First, we note that

\[
A_n^2 = (B_n - (B_n - A_n))^2 = B_n^2 + (B_n - A_n)^2 - 2B_n (B_n - A_n) \geq B_n^2 - 2C_P (B_n - A_n),
\]

where the last inequality uses the fact that \( (B_n - A_n)^2 \geq 0 \) and \( B_n \leq C_P \). Therefore, to complete the proof, it suffices to show that \( B_n - A_n = O(1/\sqrt{n}) \).

Consider the following online prediction problem...
Definition 15. For \( t = 1, 2, \ldots, n \):

- Player predicts a distribution \( s_t \in \Delta_m \) where \( m = |\mathcal{X}| \) and \( \Delta_m \) is the \( m - 1 \) dimensional probability simplex.

- Nature reveals \( X_t \in \mathcal{X} \).

- Player incurs a loss \( \ell_t(s_t) = -\frac{s_t(X_t)}{p(X_t)} = -\langle s_t, w_t \rangle \) where \( w_t = e_{X_t}/p \) (recall that ./ denotes a coordinate-wise division of two vectors).

Note that the loss is a linear (hence also convex) function of the prediction \( s_t \), and hence this game is an online convex optimization problem. The regret after \( n \) rounds of this game, with respect to the best constant action in hindsight, is defined as

\[
R_n := -\sum_{t=1}^{n} \frac{s_t(X_t)}{p(X_t)} - \inf_{r \in \Delta_m} -\sum_{t=1}^{n} \frac{r(X_t)}{p(X_t)}.
\]

In the game introduced in Definition 15, the gradient of the loss at time \( t - 1 \) with respect to the prediction \( s_{t-1} \) can be observed to be

\[
\nabla \ell_{t-1}(s_{t-1}) = \nabla \left( -\frac{\langle e_{X_{t-1}}, s_{t-1} \rangle}{p(X_t)} \right) = -\frac{s_{t-1}(X_t)}{p(X_t)} e_{X_{t-1}}.
\]

This then implies that the prediction strategy \( \{q_t : t \geq 1\} \) introduced in (20) follows the projected online gradient descent strategy with step sizes equal to \( \sqrt{m/t} \). For this strategy, we recall the following bound on its regret.

Fact 6. The regret incurred by the strategy \( \{q_t : t \geq 1\} \) described in (20) for the prediction problem introduced in Definition 15 satisfies

\[
R_n \leq C' \sqrt{n},
\]

for a constant \( 0 < C' < \infty \) depending on \( m \).

To conclude the proof, we show that the statement of Fact 6 implies our required conclusion on the term \( B_n - A_n \).

\[
C' \sqrt{n} \geq R_n = -\sum_{t=1}^{n} \frac{q_t(X_t)}{p(X_t)} - \inf_{r \in \Delta_m} -\sum_{t=1}^{n} \frac{r(X_t)}{p(X_t)}
\]

\[
= \sup_{r \in \Delta_m} \sum_{t=1}^{n} \frac{r(X_t)}{p(X_t)} - \sum_{t=1}^{n} \frac{q_t(X_t)}{p(X_t)}
\]

\[
= \left( \sup_{r \in \Delta_m} \sum_{t=1}^{n} \frac{r(X_t)}{p(X_t)} - n \right) - \left( \sum_{t=1}^{n} \frac{q_t(X_t)}{p(X_t)} - n \right)
\]

\[
\geq \left( \sum_{t=1}^{n} \frac{\hat{q}_t(X_t)}{p(X_t)} - n \right) - \left( \sum_{t=1}^{n} \frac{q_t(X_t)}{p(X_t)} - n \right)
\]

\[
= n (B_n - A_n).
\]

Thus, we observe that \( B_n - A_n \leq C'/\sqrt{n} \). Plugging this in (39) and defining the constant \( C = C' \times C_P \), we get

\[
A_n^2 \geq B_n^2 - \frac{C}{\sqrt{n}}
\]

as required.
D Deferred Proofs from Section 5

D.1 Proof of Corollary 3

The steps in the proof of Corollary 3 mirror exactly, the steps in the proof of the corresponding one-sample result, Corollary 1. First, we note that the oracle payoff function, \( f^* \), can be of the form \( f^*(x, y) = \mathbb{1}_{\{x < u^*\}} - \mathbb{1}_{\{y < u^*\}} \) or \( f^*(x, y) = \mathbb{1}_{\{y \leq u^*\}} - \mathbb{1}_{\{x \leq u^*\}} \), due to the absolute value in the definition of \( d_{KS} \). Without loss of generality, we consider the former case, as the latter can be handled in an entirely analogous manner. Now, define the event \( \mathcal{E}_t = \{\|\hat{F}_{Q,t} - F_Q\|_\infty \leq \Delta F_t, \|\hat{F}_{P,t} - F_P\|_\infty \leq \Delta F_t\} \), and note that by two applications of DKW inequality and a union bound, we have \( \mathbb{P}(\mathcal{E}_t^c) \leq 2/t^2 \). Clearly, this implies that \( \sum_{t=1}^{\infty} \mathbb{P}(\mathcal{E}_t^c) < \infty \), and thus \( \mathbb{P}(\mathcal{E}_t \ i.o.) = 1 \). Next, define the set \( U^* = \{x \in \mathbb{R} : F_P(x) - F_Q(x) = d_{KS}(P, Q)\} \), and consider the two cases: \( U^* \neq \emptyset \) and \( U^* = \emptyset \). When \( U^* \) is not empty, we can define \( u^* \) as the infimum of all elements in \( U^* \) and then proceed exactly as in Lemma 2 to show that the \( u_t \) defined in Definition 1 converges to \( u^* \) from below. In the other case, when \( U^* \) is empty, we can define \( u^* \) as in Equation (35) and observe that \( f^* \) in this case becomes \( f^*(x, y) = \mathbb{1}_{\{x < u^*\}} - \mathbb{1}_{\{y < u^*\}} \). Again, following the same steps as in Appendix C.1 we can show that \( u_t \) converge to \( u^* \) from below. This completes the proof.

D.2 Proof of Corollary 4

The first statement on the control of type-I error follows from the fact that under the null, \( \mathbb{E}[f_t(Z_t)|\mathcal{F}_{t-1}] = 0 \) by construction.

To prove that under the alternative, the sequential test rejects the null almost surely, we appeal to Proposition 1. In particular, we show the following

**Lemma 7.** For any \( t \geq 1 \), there exists an event \( \mathcal{E}_t \in \mathcal{F}_{t-1} \) and a real number \( \Delta f_t \in [0, 1] \) such that

\[
\mathcal{E}_t \subset \{\|f_t - f^*\|_\infty \leq \Delta f_t\}, \quad \text{with} \quad \lim_{t \to \infty} \Delta f_t = 0, \quad \text{and} \quad \mathbb{P}(\mathcal{E}_t^c) \leq \frac{3}{t^2}.
\]

**Proof.** We define \( \mathcal{E}_t = \cap_{i=1}^3 \mathcal{E}_{t,i} \) with

\[
\mathcal{E}_{t,1} = \{d_{MMD}(\hat{Q}_{t-1}, \hat{P}_{t-1}) - d_{MMD}(Q, P) \leq \delta_t\}, \quad \delta_t := \frac{B \left(2 + \sqrt{2\log(t-1)}\right)}{\sqrt{t} - 1},
\]

\[
\mathcal{E}_{t,2} = \{\|\mu_{Q,t} - \mu_Q\|_\infty \leq \gamma_t\}, \quad \gamma_t := \frac{1}{\sqrt{t}} \left(2B + \sqrt{2\log(2t^2)}\right).
\]

\[
\mathcal{E}_{t,3} = \{\|\mu_{P,t} - \mu_P\|_\infty \leq \gamma_t\}, \quad \text{and}
\]

\[
\Delta f_t := \begin{cases} 1, & \text{if } t \leq t_0 = \min\{t : 2\delta_t < \|\mu_P - \mu_Q\|_K\} \\ 4\gamma_t(1 + 2\delta_t/\|\mu_P - \mu_Q\|_K), & \text{for } t > t_0 \end{cases}
\]

First we note that \( \mathbb{P}(\mathcal{E}_{t,1}^c) \leq 1/t^2 \) as a consequence of the concentration result for the (biased) MMD estimator as shown in (Gretton, Borgwardt, Rasch, Schölkopf, and Smola 2012, Theorem 7).

For the other two events, we will bound their probability in terms of the Rademacher complexity of the function class \( \mathcal{G} := \{K(x, \cdot)/B : x \in \mathcal{X}\} \), defined as

\[
\mathcal{R}_n := \sup_{X^n \in \mathcal{X}^n} \mathbb{E}_{\sigma^n} \left[ \sup_{x \in \mathcal{X}} \frac{1}{Bn} \sum_{t=1}^{n} \sigma_t K(x, X_i) \right],
\]

where \( \sigma^n \sim \text{Uniform}\{\{-1, 1\}^n\} \). Now, by an application of the uniform convergence results for bounded function classes (Wainwright 2019, Theorem 2.10), we have that with probability \( 1 - 1/t^2 \) each, we have the
The regret incurred by the prediction scheme introduced in Fact 7.

Hence, it also satisfies the following regret bound, proved by Orabona and Pal (2016)

\[
\|\hat{\mu}_{P,t} - \mu_P\|_\infty \leq 2R_t + \sqrt{\frac{2 \log(2t^2)}{t}}, \quad \text{and}
\|
\hat{\mu}_{Q,t} - \mu_Q\|_\infty \leq 2R_t + \sqrt{\frac{2 \log(2t^2)}{t}}.
\]

To complete the proof, it remains to show that \( R_t \leq \frac{B}{\sqrt{t}} \). We proceed as follows:

\[
R_t = \sup_{X^t} \mathbb{E}_{\sigma^t} \left[ \sup_x \frac{1}{Bt} \sum_{i=1}^t \sigma_i K(x, X_i) \right] = \sup_{X^t} \mathbb{E}_{\sigma^t} \left[ \sup_x \frac{1}{t} \left( \frac{K(x, \cdot)}{B}, \sum_{i=1}^t \sigma_i K(X_i, \cdot) \right) \right]
\]

\[
\leq \sup_{X^t} \mathbb{E}_{\sigma^t} \left[ \left( \sum_{i=1}^t \sigma_i K(X_i, \cdot), \sum_{i=1}^t \sigma_i K(X_i, \cdot) \right)^{1/2} \right]
\]

\[
= \frac{1}{t} \sup_{X^t} \left( \mathbb{E}_{\sigma^t} \left[ \left( \sum_{i=1}^t \sigma_i K(X_i, \cdot), \sum_{i=1}^t \sigma_i K(X_i, \cdot) \right) \right] \right)^{1/2} = \frac{1}{t} \sup_{X^t} \left( \mathbb{E}_{\sigma^t} \left[ \sum_{i,j=1}^t \sigma_i \sigma_j K(X_i, X_j) \right] \right)^{1/2}
\]

\[
= \frac{1}{t} \sup_{X^t} \left( \mathbb{E}_{\sigma^t} \left[ \sum_{i=1}^t \sigma_i^2 K(X_i, X_i) \right] \right)^{1/2} = \frac{B}{\sqrt{t}}.
\]

\[\square\]

D.3 Proof of Proposition 10

We begin by noting that the prediction scheme used in (21) for the problem defined in Definition 13 is a special instance of the more general scheme for online optimization proposed by Orabona and Pal (2016). Hence, it also satisfies the following regret bound, proved by Orabona and Pal (2016).

**Fact 7.** The regret incurred by the prediction scheme introduced in (27) for the problem of Definition 13 satisfies

\[
R_n = O \left( \frac{1}{B} \sqrt{\frac{1}{n} \log \left( \frac{n}{B^2} \right)} \right).
\]

Next, by combining the vanishing regret condition with Proposition 5, we get the following inequality under the alternative

\[
\liminf_{n \to \infty} \frac{1}{n} \log (\mathbb{P}(\tau > n)) \geq \sup_{e > 0} \frac{\inf_{P^*, Q^*} \mathcal{P}_{e,d_{\text{MMD}}}^2}{\mathbb{E}} \frac{d_{\text{KL}}(P^*, P) + d_{\text{KL}}(Q^*, Q)}{2}, \quad \text{where} \quad \mathcal{P}_{e,d_{\text{MMD}}}^2 := \{(P^*, Q^*) \in \mathcal{P}(X) \times \mathcal{P}(X) : d_{\text{MMD}}(Q^*, P^*) \leq \epsilon\}.
\]

Finally, we note that the exponent in (40) is lower bounded by \( \beta^* := \inf_{P \in \mathcal{P}(X)} (1/2) (d_{\text{KL}}(P^*, P) + d_{\text{KL}}(P^*, Q)) \). This can be concluded by an application of (Zhu, Chen, Chen, and Yang 2021; Theorem 9). The derivation of this result exploits the fact that the kernel-MMD distance metrizes the weak topology over the space of probability distributions.
E Additional Experimental Results

Implementation details of other tests. In the experiments, we used the following tests to compare the performance of our proposed betting based sequential tests.

- **Batch-KS1.** This test uses the statistic $T_n = \sqrt{n} d_{KS}(\hat{Q}_n, P)$ and rejects the null when $T_n$ exceeds $s_{n, 1-\alpha}$, where $s_{n, 1-\alpha}$ is the $(1-\alpha)$ quantile of $T_n$. In our experiments, we used the implementation of KS test in `scipy.stats.kstest` module. For sample size smaller than 1000, we used the exact computation of the threshold, while for larger sample sizes we used the asymptotic $(1-\alpha)$ quantile as the threshold.

- **Batch-KS2.** This test uses the statistic $T_n = \sqrt{n} d_{KS}(\hat{Q}_n, \hat{P}_n)$ and rejects the null when $T_n$ exceeds a threshold $s_{n, 1-\alpha}$. In our experiments, we used the implementation of two-sample KS test in `scipy.stats.ks_2samp` module, and used the exact $1-\alpha$ quantile of $T_n$ for $n \leq 1000$ and the asymptotic $(1-\alpha)$ quantile for larger $n$ values.

- **Batch-$\chi^2$.** In our implementation, we used the statistic $T_n = \sum_{j=1}^m \frac{(N_{j,n} - np_j)^2}{np_j}$, where $N_{j,n} = \sum_{1=1}^n \mathbb{1}_{(Y_i = x_j)}$ denotes the number of observations equal to $x_j$. The limiting distribution of $T_n$ is the $\chi^2$-distribution with $m-1$ degrees of freedom, and we use the $(1-\alpha)$ quantile of this asymptotic distribution as the threshold for rejecting the null.

- **Batch-Kernel-MMD.** This test uses the statistic $T_n = d_{MMD}(\hat{P}, \hat{Q}_n)$ and rejects the null when $T_n$ exceeds a threshold $s_{1-\alpha}$. We used the bootstrap resampling over the aggregated data with 1000 draws, as described by Gretton, Borgwardt, Rasch, Schölkopf, and Smola (2012), to estimate the threshold.

- **LC.** We implemented the version of the sequential test of Lhéririer and Cazals (2018) with Bayesian model averaging (BMA) and switching as described in (Lhéririer and Cazals 2018 Appendix B). Given observations $X_t \sim P$ and $Y_t \sim Q$, this test proceeds by assigning labels 0 to the $X$-observations and 1 to the $Y$-observations. Then, the two sample testing problem can be reduced to testing the independence of the label $L$ (0 or 1) given the observation $Z$ ($X$ or $Y$). To predict the probability of the next label, we can use any probabilistic predictor $Q_t(\cdot|Z_t^1, L_t^{t-1})$. This test rejects the null when $T_n = \prod_{t=1}^n Q_t(L_t|Z_t^1, L_t^{t-1})/\mathbb{P}_0(L_t)$ is smaller than $\alpha$. In our implementation, we used the BMA approach using $k$-nearest-neighbor regressors with time varying $k = t^p$ and $p \in \{0.3, 0.5, 0.7, 0.9\}$. We also used model switching with a uniform prior distribution. For details on BMA and model switching used in this test, see (Lhéririer and Cazals 2018, Appendix B).

- **HR.** We used the one- and two-sample KS tests based on the result in (Howard and Ramdas 2022 Theorem 2). In particular, for the one-sample case, the HR test rejects the null when $d_{KS}(\hat{F}_{Q,t}, \hat{F}_P)$ exceeded the threshold $\tau_t = 0.85\sqrt{\log \log (1 + \log t)} + C_\alpha$, with $C_\alpha = (4/5)\log(1612/\alpha)$.

The same theorem also gives us a sequential two-sample test that rejects the null if $d_{KS}(\hat{F}_{Q,t}, \hat{F}_{P,t})$ exceeds the threshold $\tau_t = 1.70\sqrt{\log \log (1 + \log t)} + C_{\alpha/2}$ with $C_{\alpha/2} = (4/5)\log(3224/\alpha)$.

- **DR.** For the one-sample KS test, we followed the result of (Darling and Robbins 1968, Theorem 4.1) and the recommended threshold in (Darling and Robbins 1968 Remark (d)) to reject the null at time $t$ if $d_{KS}(\hat{F}_{Q,t}, F_P)$ exceeds the threshold $\tau_t = \sqrt{(t+1)/2}\log t + \log \left(\frac{4\sqrt{2}}{\alpha}\right)$.

Using the same result, we can define the two-sample test that rejects the null if $d_{KS}(\hat{Q}_t, \hat{P}_t)$ exceeds $\tau_t = 2\sqrt{(t+1)/2}\log t + \log \left(\frac{8\sqrt{2}}{\alpha}\right)$.
• **MR-MMD.** Following the discussion in (Manole and Ramdas 2021 § 4.2), the two-sample kernel-MMD test rejects the null when \(d_{\text{MMD}}(\hat{Q}_t, \hat{P}_t)\) exceeds the threshold \(\sqrt{2B + 4B\sqrt{\frac{2}{t}} (\log \ell(\log_2 t) + \log(1/\alpha))}\), where the function \(\ell(\cdot)\) is the same as in the previous item.

• **BR.** This two-sample test was introduced by Balsubramani and Ramdas (2016), and is based on the linear-time statistic \(T_t = \sum_{i=1}^{\lfloor t/2 \rfloor} h_i\), where \(h_i = K(X_{2i-1}, X_{2i}) + K(Y_{2i-1}, Y_{2i}) - K(X_{2i-1}, Y_{2i}) - K(X_{2i}, Y_{2i-1})\). At any time \(t\), this test rejects the null if \(T_t\) exceeds the threshold
\[
1.1 \left( \log(1/\alpha) + \sqrt{2\hat{V}_n + \log \log \left( \frac{\hat{V}_t}{\alpha} \right)} \right)
\] where \(\hat{V}_t := \sum_{i=1}^{\lfloor t/2 \rfloor} h_i^2\).

**Comparison with Balsubramani and Ramdas (2016).** In Figure 6, we compare the performance of the sequential kernel-MMD test of Balsubramani and Ramdas (2016), denoted by BR, with our betting based sequential kernel-MMD test of Section 5.2. Note that the BR test relies on a linear-time MMD statistic, and hence is computationally much more efficient than our proposed test that uses the quadratic time MMD statistic. However, this computational gain comes at the cost of reduced power, as can be seen in the figure.

The experiment setup used is the same as described in Section 5.2.2. We consider a two-sample testing problem with paired observations. Under the null, both the distributions are \(N(0, I_m)\) with \(m = 5\), while under the alternative, one of the distributions is \(N(\epsilon I, I_m)\). From Figure 6 we can see that there is a significant difference between the power of the BR test, and the betting based test proposed in this paper.

![Figure 6: Comparison of the performance of our betting based sequential test and the sequential MMD test of Balsubramani (2014), based on the linear-time MMD statistic.](image)

**Additional experimental results.** We show some additional results comparing the performance of our proposed sequential tests with the relevant baselines in Figure 7.

**F Extensions**

In this section, we discuss how the ideas developed in this paper can be used for designing tests for stochastic dominance and symmetry.

**F.1 Testing for Stochastic Dominance**

We work in the same setting as Barrett and Donald (2003), and consider real valued distributions supported on the interval \(X = [0, 1]\). Given probability distributions \(P\) and \(Q\) on \(X\), the \(k^{th}\) order (weak) stochastic
dominance of $Q$ over $P$ can be stated in terms of integral operators $\mathcal{I}_k$, defined below as

$$\mathcal{I}_1(z, P) = F_P(z),$$
and

$$\mathcal{I}_k(z, P) = \int_0^z \mathcal{I}_{k-1}(t, P)dt, \quad \text{for } k \geq 2.$$ 

**Definition 16 (Stochastic-Dominance).** For two real valued distributions $P$ and $Q$ supported on $\mathcal{X} \subset \mathbb{R}$, we say that $P \preceq_k Q$, if $\mathcal{I}_k(z, Q) \leq \mathcal{I}_k(z, P)$ for all $z \in \mathcal{X}$.

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For simplicity, we present a test for the case of \( k = 2 \). In the batch setting, Barrett and Donald (2003) suggested a KS-type test based on the following statistic:

\[
T_t = \sqrt{\frac{t}{2}} \sup_{z \in \mathcal{X}} \left( \mathcal{I}_2 \left( z, \hat{F}_{P,t} \right) - \mathcal{I}_2 \left( z, \hat{F}_{Q,t} \right) \right). \tag{41}
\]

On expanding the definition of \( \mathcal{I}_2 \) in (41), we see that

\[
\sqrt{\frac{2}{t}} T_t = \sup_{z \in \mathcal{X}} \left( \frac{1}{t} \sum_{i=1}^{t} \mathbb{1}_{\{X_i \leq z\}} (z - X_i) - \frac{1}{t} \sum_{i=1}^{t} \mathbb{1}_{\{Y_i \leq z\}} (z - Y_i) \right)
\]

\[
= \sup_{g_{z} \in \mathcal{G}} \mathbb{E}_{X \sim \hat{P}_t} [g_{z}(X)] - \mathbb{E}_{Y \sim \hat{Q}_t} [g_{z}(Y)], \tag{42}
\]

where \( \mathcal{G} \) denote the class of functions \( \{g_{z}(\cdot) = \mathbb{1}_{\{\cdot \leq z\}} (z - \cdot) : z \in \mathcal{X}\} \). This suggests the following level-\( \alpha \) sequential test for second order dominance.

**Definition 17 (Stochastic-Dominance-Test).** Set \( K_0 = 1 \). For \( t = 1, 2, \ldots \), proceed as follows:

- Construct the empirical cdfs \( \hat{F}_{Q,t}^{-1} \) and \( \hat{F}_{P,t}^{-1} \) based on the initial \( t - 1 \) observations \( \{(X_i, Y_i) : 1 \leq i \leq t - 1\} \).
- With the function \( g_t \) denoting the maximizer in (42), define the payoff function \( f_t(x, y) = g_t(x) - g_t(y) \).
- Observe the next pair \( (X_t, Y_t) \) and update the wealth process using the mixture-method

\[
K_t = \int_0^1 \prod_{i=1}^{t} (1 + \lambda f_t(X_i, Y_i)) \nu(\lambda) d\lambda.
\]

- Check for the stopping condition \( K_t \geq 1/\alpha \).

**F.2 Testing for Symmetry**

We work in a simplified version of the setting considered by Chatterjee and Sen (1973). Suppose \( Y_1, Y_2, \ldots \) are drawn i.i.d. from some unknown real-valued and continuous distribution \( Q \). We say that \( Q \) is symmetric about 0 if for all \( x \geq 0 \), we have \( F_Q(x) = 1 - F_Q(-x) \).

In the batch setting, Chatterjee and Sen (1973) suggested a test based on the following KS-type statistic

\[
D_t = \sup_{z \in \mathbb{R}} |\hat{F}_{Q,t}(z) - (1 - \hat{F}_{Q,t}(-z))|.
\]

This can also be rewritten as

\[
D_t = \sup_{g_z \in \mathcal{G}} |\mathbb{E}_{Y \sim \hat{Q}_t} [g_z(Y)] - 1|, \tag{43}
\]

where \( \mathcal{G} = \{g_z(\cdot) = \mathbb{1}_{\{\cdot \leq z\}} - \mathbb{1}_{\{\cdot \geq -z\}} : z \in \mathbb{R}\} \). We can use this to define the following level-\( \alpha \) sequential test for symmetry.

**Definition 18 (Symmetry-Test).** Set \( K_0 = 1 \). For \( t = 1, 2, \ldots \), proceed as follows:

- Construct the empirical cdf \( \hat{F}_{Q,t-1} \) based on the initial \( t - 1 \) observations \( \{Y_i : 1 \leq i \leq t - 1\} \).
- With the function \( g_t \) denoting the maximizer in (43), define the payoff function \( f_t(y) = g_t(y) - 1 \).
• Get the next observation $Y_t$ and update the wealth process using the mixture-method

$$K_t = \int_{-1}^{1} \prod_{i=1}^{t} (1 + \lambda f_i(Y_i)) \nu(\lambda) d\lambda.$$

• Check for the stopping condition $K_t \geq 1/\alpha$. 