Simple model of fractal networks formed by self-organized critical dynamics

Shogo Mizutaka
Department of Mathematics and Informatics, Ibaraki University, 2-1-1 Bunkyo, Mito, Japan 310-8512
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In this paper, a simple dynamical model in which fractal networks are formed by self-organized critical (SOC) dynamics is proposed; the proposed model consists of growth and collapse processes. It has been shown that SOC dynamics are realized by the combined processes in the model. Thus, the distributions of the cluster size and collapse size follow a power-law function in the stationary state. Moreover, through SOC dynamics, the networks become fractal in nature. The criticality of SOC dynamics is the same as the universality class of mean-field theory. The model explains the possibility that the fractal nature in complex networks emerges by SOC dynamics in a manner similar to the case with fractal objects embedded in a Euclidean space.

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I. INTRODUCTION

Networks consisting of elements and their interactions are observed over a wide range from nature to society. Most large-scale complex networks share some common structural properties, including the scale-free property, small-world or fractal nature, degree–degree correlation, and community structures [1, 2]. Based on the relation between the average path length and system size (i.e., the total number of nodes in a network), real-world networks are classified into two types, namely small-world and fractal networks. First, in the case of small-world networks, the distance between any two nodes is extremely small compared with the system size. More precisely, in small-world networks, the average path length increases only with the logarithm of the system size at most. In other words, if we cover a small-world network using boxes with a linear size of \( l_B \), the relation

\[
N_B(l_B) \propto e^{-l_B/l_0}
\]

is satisfied. Here, \( N_B(l_B) \) represents the minimum number of boxes required to cover a given network, while \( l_0 \) is the characteristic length of the network. Second, in contrast, fractal networks satisfy the following relation:

\[
N_B(l_B) \propto l_B^{-d_B},
\]

where \( d_B \) \(<\infty\) represents the fractal dimension. It is evident that Eq. (2) does not include a characteristic length. Because the small-world nature expressed in Eq. (1) corresponds to the case of \( d_B \to \infty \) in Eq. (2), these two concepts are conflicting.

Although it is known that the small-world structure in a network is formed by the generation of shortcut edges between long-range nodes, the origin of the fractality in networks is not yet understood well [1, 2]. However, it is known that most of the fractal objects embedded in a Euclidean space, such as coastlines and the branches of trees, emerge as a consequence that a dynamical system is spontaneously driven toward a critical point as an attractor and fluctuates near it [6–14]. We call the phenomenon self-organized criticality. In self-organized critical (SOC) dynamics, the characteristic scales of various quantities vanish, and their distributions obey a power-law function.

It is natural to expect that the fractal structures of networks are formed by SOC dynamics as is the case with fractal objects in a Euclidean space. In fact, it has recently been shown that the fractality of networks is formed by SOC dynamics [15]. The basic idea of the model proposed in Ref. [15] is to combine the growth of a network and its collapse by cascading overload failures, as argued in Ref. [10, 11]. In particular, the combined dynamics of the two percolation processes drive the network into a critical state; consequently, the network structure exhibits fractality in the stationary state. Moreover, the distribution of various quantities such as the cluster size, cascade size, and waiting time between two adjacent cascades obeys the power-law form. In the model, however, the criticality of the SOC dynamics is not clear owing to the complication of the model, which contains many predetermined parameters. Thus, this complication does not allow us to understand what factors change the criticality and what universality class is realized. It is important to clarify these points to better understand the fractal nature observed in networks and to establish fundamental knowledge in the field of network science. Therefore, in this paper, we propose a simple dynamical model combining growth and collapse processes in the network. We apply the Erdős-Rényi process [12] and cascading failure process based on the threshold model proposed by Watts [19] to the growth and collapse processes, respectively.

Our main result shows that SOC dynamics emerge by the combined dynamics of the growth and collapse processes, and the networks become fractal in the stationary state.

The rest of this paper is organized as follows. In Sec. [11] we briefly review the threshold model. Then, we introduce our proposed model. The results and their discussion are presented in Sec. [11]. Finally, Sec. [IV] includes
II. MODEL

A. Collapse Process: Threshold Model

Since we apply the threshold model proposed by Watts [19] to a collapse process in the model proposed in this paper, we briefly review the outline of this model in this subsection. It is noted that the cascading failure in the threshold model spreads along the nearest-neighbor nodes, which imitates the spread of rumors in the case of social networks and those of chain bankruptcies in financial networks, among others. The manner of the threshold model is as follows:

(i) Construct a network with $N$ nodes. Every node can only be in one of the two states (either active or inactive). All nodes are in the inactive state and have a threshold $\phi$.

(ii) Change the state of a randomly selected node to the active state.

(iii) Change the state of a node to be active if the ratio $\Phi_i$ of the number of active adjacent nodes to the degree of the selected node is larger than a given threshold value $\phi$. Repeat this update iteratively until no new active nodes are generated (i.e., a cascading process).

The threshold $\phi$ represents the amount of support a particular node needs from its nearest neighbors. The cascade size is defined as the number of active nodes after the cascading process. By demonstrating the relation between the threshold value $\phi$ and the characteristics of the network structures, such as the degree distribution and average degree, it has been clarified that a single activation of a node would cause a global cascade that spreads throughout networks under certain conditions [19]. In particular, for Erdős-Rényi random graphs, a phase diagram showing whether a global cascade occurs can be obtained using the generating function method for a plane formed by the average degree $\langle k \rangle$ and threshold value $\phi$. The previous results in Ref. [19] show that a phase border exists for $\langle k \rangle = 1$ and $\phi < 1/4$.

B. Present Model

In this paper, a dynamical model, which is the threshold model with a growth process, is proposed. The evolutionary manner of the present model is as follows:

(i) Prepare $N$ isolated nodes that can be only be in one of two states (either the active or inactive state). Each node is in the inactive state with a threshold $\phi$.

(ii) At each time step $t$, connect two randomly selected nodes that are not yet connected. Then, every node in the system changes its state to the active state with the probability $p$. If there exist active nodes in the system, go to Step (iii). Otherwise, repeat Step (ii).

(iii) Change the state of a node to active if the ratio $\Phi_i$ of the number of active adjacent nodes to the degree of the node is larger than the given threshold value $\phi$. Repeat this update iteratively until no new active nodes are generated. After completing the iterative renewals of node states, remove the edges of all active nodes and restore the states of these nodes to the inactive state.

(iv) Repeat Steps (ii)–(iii).

It is noted that the growth process is the evolution of random graphs in the Erdős-Rényi model [18]. The inverse of the probability $p$ would be the average lifetime of nodes if the collapse processes never occurred in the threshold model. For such an instability of each node, the network cannot reach a complete graph, in which every pair of nodes is connected.

III. RESULTS & DISCUSSION

It is interesting to clarify several properties of networks in the stationary state. Figure 1 shows the time evolu-
tion of the network size \( N(t) \) excluding isolated nodes, the size \( N_{LC}(t) \) of the largest cluster, and the cascade size \( S \) defined as the number of active nodes after completing the cascading process at each time step \( t \). As seen from both panels in Fig. 1 \( N(t) \) increases almost monotonically with the time \( t \) at the early stage and then attains the stationary state. However, the behaviors of several quantities in the stationary state in the two panels are entirely different. In the top panel corresponding to the result for \( \phi = 0.2 \), each quantity largely fluctuates. In contrast, for \( \phi = 0.3 \), the sizes \( N(t) \), \( N_{LC}(t) \), and \( S \) do not largely fluctuate, and the nodes with edges almost form a singly connected component. Furthermore, in Fig. 2 the time dependencies of the average degree for \( \phi = 0.2 \) and \( \phi = 0.3 \) are shown. In this figure, the average degree \( \langle k \rangle \) at time \( t \) is defined as the ratio of the total number of edges to the total number of nodes \( N \) in the system. For \( \phi = 0.2 \), the average degree fluctuates near \( \langle k \rangle = 1 \) after reaching the stationary state, while the average degree for \( \phi = 0.3 \) fluctuates near \( \langle k \rangle = 10 \). These differences imply that the behaviors of the cluster-size distributions for \( \phi = 0.2 \) and \( \phi = 0.3 \) change drastically.

It is important to ascertain whether SOC dynamics emerge in the behaviors depicted in Fig. 1. Thus, in order to confirm this, we numerically investigate the distribution functions \( n_s \) and \( P(S) \) of the cluster size \( s \) and cascade size \( S \), respectively. The top panel in Fig. 3 shows the distribution \( n_s \) of the cluster size \( s \) averaged over the system after it has attained the stationary state for various values of the threshold \( \phi \). For \( \phi < 1/4 \), the distribution \( n_s \) obeys the following power law:

\[ n_s \sim s^{-\tau}. \quad (3) \]

This implies that networks remain near the critical point of the percolation transition by the combined dynamics. Moreover, the slope of each result for \( \phi < 1/4 \) is parallel to the guide line (dashed light-blue line), which indicates the critical exponent \( \tau = 5/2 \) for \( \phi < 1/4 \). This result indicates that the exponent \( \tau \) does not depend on the threshold \( \phi \). In addition, the value of the exponent corresponds to that of the exponent characterizing the cluster distribution in the Erdős-Rényi random graph at criticality. Furthermore, the behavior of the average degree depicted in Fig. 2(a) implies that a network formed in the present model attains the critical point in the Erdős-Rényi random graph. In contrast, for \( \phi \geq 1/4 \), the distribution \( n_s \) decays exponentially, which indicates that below the threshold \( \phi_{ch} = 1/4 \), the distribution of \( n_s \) obeys the abovementioned power law, which is consistent with the previous result for the threshold model [19] described in Sec. II A.

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**FIG. 2.** Time dependence of the average degree \( \langle k \rangle \) for (a) \( \phi = 0.2 \) and (b) \( \phi = 0.3 \). The horizontal dashed lines in both panels are for \( \langle k \rangle = 1 \), which represents the condition of the percolation transition in the Erdős-Rényi random graph. The results are obtained under the conditions \( N = 10^4 \) and \( p = 10^{-5} \).

**FIG. 3.** Distribution functions of (a) \( n_s \) of the cluster size \( s \) and (b) \( P(S) \) of the avalanche size \( S \) for various values of the threshold \( \phi \). Although the results for \( \phi < 1/4 \) are plotted on a double-logarithmic scale (left and lower axes), the results for \( \phi \geq 1/4 \) are plotted on a semilogarithmic scale (left and upper axes) in both panels. The distributions \( n_s \) and \( P(S) \) are obtained from the data until \( t = 10^6 \) after attaining the stationary state. The plots are vertically shifted for clarity. The slopes of the guide lines (dashed light-blue lines) are 5/3 and 3/2 in (a) and (b), respectively. The other parameters are set as follows: \( N = 10^4 \) and \( p = 10^{-5} \).
FIG. 4. $N_B(l_B)$ for the giant components in networks in the stationary state under the conditions (a) $\phi = 0, 0.2, 0.249$ and (b) $\phi = 0.25$ and $0.4$. The longitudinal axis indicates $N_B(l_B)/N_B(1)$ averaged over $10^4$ realizations of networks. In panel (a), the plots present the results for $\phi = 0, 0.2, 0.249$ from top to bottom, respectively. The bottom-most panel shows the results for the giant components of an Erdős–Rényi random graph with $N = 10^4$ and $\langle k \rangle = 1.0$, i.e., at criticality. The dashed line with a slope of $-2$ represents the fractal dimension of the Erdős–Rényi random graph.

In the bottom panel of Fig. 3, the distribution $P(S)$ of the avalanche size $S$ is shown. The behavior of the distribution $P(S)$ of the avalanche size for $\phi$ is similar to that of $n_s$. For $\phi < 1/4$, the distribution $P(S)$ also follows a power-law form,

$$P(S) \sim S^{-\alpha},$$

with the exponent $\alpha = 3/2$. The behavior of the dynamics in the present model is similar to that in the forest fire model proposed by Drossel and Schwabl [10]. The activation rate $p$ of each node and the addition of a new edge for each time step in our model correspond to the lightning probability and growing probability of a tree in the forest fire model, respectively. In particular, in the case of $\phi = 0$, our model is equivalent to a random neighbor treatment for the forest fire model [22].

The reason why the exponent $\alpha = 3/2$ is as follows. A node in a cluster changes a state from inactive to active with the probability $p$. Then, the expectation of the occurrence of a state change with the probability $p$ in an $s$-cluster is proportional to $psn_s$. Because the size of the cascade is equal to the size of cluster for $\phi = 0$, the exponents $\alpha$ and $\alpha$ satisfy the relation $\alpha = \alpha - 1$. This relation holds even for $\phi = 0.2$ and $0.249$ if the system is sufficiently large.

Finally, we study the properties of network structures in the stationary state. If the network has a fractal nature, the relation defined by Eq. (4) is satisfied. We estimate the minimum number of subgraphs $N_B(l_B)$ for the largest components using the compact-box-burning algorithm [20]. Figures (b) (a) and (b) show the linear size $l_B$ dependence of $N_B(l_B)/N_B(1)$ for different values of the threshold $\phi$, where $N_B(1)$ is the number of nodes belonging to the largest component. In panel (a), we also plot the results of a fractal analysis for the giant component of Erdős–Rényi random graphs with the average degree $\langle k \rangle = 1$, i.e., at criticality. As shown in Fig. 4, $N_B(l_B)$ for $\phi < 1/4$ satisfies the relation given by Eq. (2). This result indicates that the fractality in networks appears by SOC dynamics in a manner analogous with the case of a Euclidean space. In addition, as seen from Fig. 4(a), in the plots for the present model with $\phi < 1/4$ and Erdős–Rényi random model, the tails are parallel, which indicates that the fractality of networks with $\phi < 1/4$ corresponds to that of the Erdős–Rényi random graphs at criticality. The slightly difference between the analysis (dashed guide line) and present simulated results is based on the finite-size effect that the simulated results experience. Therefore, we conclude that networks governed by the present SOC dynamics are driven towards the percolation transition point for Erdős–Rényi random graphs, which, in turn, indicates that the criticality of the present model is the same as the universality class of mean-field theory.

IV. CONCLUSION

In this paper, we proposed a simple dynamical model in which fractal networks are formed by SOC dynamics; in particular, our model consists of the Erdős–Rényi process [15] and a cascading failure process based on the threshold model [19]. In the proposed model, the distributions of the cluster size and cascade size follow a power-law function in the stationary state. Moreover, through SOC dynamics, networks in the stationary state become fractal. Thus, this indicates that fractality in networks would emerge by SOC dynamics, as is the case with fractal objects embedded in a Euclidean space. In addition, our simulation results show that the universality class of the self-organized criticality in the developed model corresponds to that of the Erdős–Rényi random graph. In particular, for the model parameter $\phi = 0$, the present model corresponds to the mean-field treatment for the forest fire model [22].

The combined dynamics realizing SOC dynamics do not have to be the Erdős–Rényi process and the cascading failure process based on the threshold model. The SOC dynamics leading to fractality in networks would be realized in different scenarios. For instance, a simple and possible candidate of the collapse process is the spread of diseases based on the susceptible-infected-recovered model. Considering that the universality class of the SOC dynamics in the present model is the same as that of the Erdős–Rényi random graph, the universality class might be related to a percolation process, i.e., a growth process. In order to clarify the diversity of fractal networks, it is important to understand whether growth processes change the universality class.
Furthermore, most of the fractal networks observed in the real world, such as the World Wide Web or metabolic networks, also have the scale-free property. Recent work has highlighted that such networks possess long-range repulsive correlations in nodes with similar degrees. It is interesting to clarify whether fractal networks with the scale-free property are formed by SOC dynamics, which explain the long-range degree correlations if the former is true.

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