GLOBAL PSEUDO-DIFFERENTIAL OPERATORS ON THE LIE GROUP $G = (-1, 1)^n$

DUVÁN CARDONA, ROLAND DUDUCHAVA, ARNE HENDRICKX, AND MICHAEL RUZHANSKY

Abstract. In this work we characterise the Hörmander classes $S^m_{\rho,\delta}(G, \text{Hör})$ on the open manifold $G = (-1, 1)^n$. We show that by endowing the open manifold $G = (-1, 1)^n$ with a group structure, the corresponding global Fourier analysis on the group allows one to define a global notion of symbol on the phase space $G \times \mathbb{R}^n$. Then, the class of pseudo-differential operators associated to the global Hörmander classes $S^m_{\rho,\delta}(G \times \mathbb{R}^n)$ recovers the Hörmander classes $S^m_{\rho,\delta}(G, \text{loc})$ defined by local coordinate systems. The analytic and qualitative properties of the classes $S^m_{\rho,\delta}(G \times \mathbb{R}^n)$ are presented in terms of the corresponding global symbols. In particular, $L^p$-Fefferman type estimates and Calderón-Vaillancourt theorems are analysed, as well as the spectral properties of the operators.

Contents

1. Introduction 2
2. Preliminaries 4
  2.1. Hörmander classes on open sets of $\mathbb{R}^n$ 4
  2.2. The group $G$ 5
  2.3. Function spaces on $G = (-1, 1)^n$ 6
  2.4. Global Fourier analysis on $G = (-1, 1)^n$ 8
  2.5. Sobolev and Bessel potential spaces on $G$ 9
3. Pseudo-differential operators on $G = (-1, 1)^n$ 12
  3.1. The global Hörmander classes 12
  3.2. The quantisation formula 12
  3.3. Composition of pseudo-differential operators 13
  3.4. The adjoint of a pseudo-differential operator 14
  3.5. Construction of parametrices 15
4. Boundedness properties for the Hörmander classes $\Psi^m_{\rho,\delta}(G \times \mathbb{R}^n)$ 17
  4.1. The $L^p$-theory: $L^p$-Fefferman theorem and $L^2$-Calderón-Vaillancourt theorem 17
  4.2. The sharp Gårding inequality and the Fefferman-Phong inequality 18

2010 Mathematics Subject Classification. 35S30, 42B20; Secondary 42B37, 42B35.

Key words and phrases. Pseudo-differential operators, Microlocal analysis, Index theory, $L^p$-Multipliers.

The authors are supported by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (Grant number 01M01021). Michael Ruzhansky is also supported by EPSRC grant EP/R003025/2. R. Duduchava is supported by the grant of the Shota Rustaveli Georgian National Science Foundation FR-19-676.
5. **Sharp spectral properties for the classes** $S_{\rho,\delta}^m(G \times \mathbb{R}^n)$ 19

5.1. The Gohberg lemma and compactness on $L^2(G)$ 19

5.2. The Atiyah-Singer-Fedosov index theorem on $G$ 20

6. **Fredholm properties and $L^p$-boundedness of $\Psi$DOs with non-classical symbols on** $G = (-1, 1)^n$ 23

6.1. Boundedness of pseudo-differential operators in the Bessel potential spaces 23

6.2. Fredholm properties of $\Psi$DOs 27

References 30

1. **Introduction**

The open set $G = (-1, 1)^n$ endowed with the operation $x + G y = (x + y)/(1 + x \cdot y)$ is a non-compact Lie group and in terms of the Fourier analysis associated to the group $(G, +_G)$, in this work we characterise the Hörmander classes $\Psi_{\rho,\delta}^m(G, \text{Hör})$ of pseudo-differential operators on $G$ defined by local coordinate systems. The approach described here can be extended for instance, to any star-shaped open sub-set $\tilde{G}$ of $\mathbb{R}^n$ in view if the natural diffeomorphism $G \cong \tilde{G}$. Indeed, in terms of the Fourier transform $\mathcal{F}_G$ on $G$, to any continuous linear operator $A$ on $C^\infty(G)$ we associate a global distribution $\sigma_A$ on $G \times \mathbb{R}^n$ in such a way that the quantisation of the symbol $\sigma_A$ gives the operator according to the formula

$$\forall f \in C^\infty(G), \quad Af = \mathcal{F}_G^{-1}[\sigma_A(x, \xi)[\mathcal{F}_G f]].$$

Then, when the distribution $\sigma_A$ agrees with a function on the phase space $G \times \mathbb{R}^n$, we analyse the properties of the operator $A$ in terms of the properties of the symbol $\sigma_A$. Summarising the results of this manuscript, we have investigated:

- Asymptotic expansions for the composition, the adjoint and the parametrices (of elliptic operators) for the global Hörmander classes $\Psi_{\rho,\delta}^m(G \times \mathbb{R}^n)$.
- The mapping properties of the classes $\Psi_{\rho,\delta}^m(G \times \mathbb{R}^n)$ on $L^p$-spaces on $G$. With $p \neq 2$ we prove a $L^p$-Fefferman type theorem, and with $p = 2$ we obtain the Calderón-Vaillancourt for these classes.
- We prove the corresponding Gohberg lemma for a suitable sub-class of the family $\Psi_{\rho,\delta}^0(G \times \mathbb{R}^n)$ and the characterisation of compact operators on $L^2(G)$ is established.
- We prove the Atiyah-Singer-Fedosov index formula in our setting. Indeed, we prove for the Shubin class of elliptic operators of order zero the index formula:

$$\text{ind}[A] = \frac{(n-1)!}{(-2\pi i)^n(2n-1)!} \int_{\partial B} \text{Tr}[a^{-1}(x, \xi)da(x, \xi)]^{2n-1}. \quad (1.2)$$

The left-hand side in 1.2 is the Fredholm index and the right hand side is the “winding number” of $a$.

- Other spectral properties for the pseudo-differential calculus on $G = (-1, 1)^n$.

In the case of the torus $\mathbb{T}^n = [-1, 1]^n$, $1 \sim -1$, the global characterisation of the Hörmander classes $\Psi_{\rho,\delta}^m(\mathbb{T}^n, \text{Hör})$ was done by MacLean in [48]. Also, an alternative proof for this fact was done in [60] using a periodisation technique compatible with
a global notion of symbol on the phase space $\mathbb{T}^n \times \mathbb{Z}^n$, (instead of the phase space $G \times \mathbb{R}^n = (-1,1)^n \times \mathbb{R}^n$). For the spectral and the analytical properties (and their applications) of the pseudo-differential calculus on the torus we refer the reader to [1], [3], [4], [5], [14], [15], [16], [24], [49], [51], [66], [61], and, mainly, the reference [60].

The construction of pseudo-differential operators using the Lie group approach as in this work is parallel to the pseudo-differential theories in [60], [38] where a global notion of symbol on the phase space $G \times \hat{G}$ has been consistently developed, with $G$ being a Lie group with a good Fourier analysis induced by its unitary dual $\hat{G}$. Even, generalising the global quantisation from the torus [61] to any compact Lie group as well as their applications many results were derived in the last years. Indeed, the applications of the global quantisation on compact Lie groups, its analytical and spectral properties as well as their applications for the analysis of PDE, index theorems, regularisation of traces and other aspects of the geometric and harmonic analysis can be found e.g. in [6], [7], [8], [9], [17], [18], [20], [19], [21], [10], [11], [12], [25], [26], [27], [28], [29], [30], [13], [23], [40], [53], [54], [46], [47], [57], [60], [58], [59], [65], [62], [63], [64] and in the extensive list of references of these works.

On the other hand, the Fourier transform $F_G$ and the convolution of functions $\varphi \ast_G \psi$ on the interval $G = (-1,1)$ (that corresponds to the one dimensional case; see below) were firstly defined by Petrov in [55, 56] by using the diffeomorphism $x : \mathbb{R} \to G$ and its inverse $t : G \to \mathbb{R}$. In these papers and in [70], the defined Fourier transform $F_G$ and convolution $\varphi \ast_G \psi$ were used for the investigation of convolution and differential equations, such as Prandtl, Tricomi, Lavrentjev-Bitsadze integral and integro-differential equations, Laplace-Beltrami equation on the sphere and some other equations from the Mathematical Physics. The authors in [55, 56, 70] did not utilise the group structure of $G = (-1,1)$. However, the group structure of $G = (-1,1)$ was used in [32] for the intrinsic definition of the Haar measure $d_G x$, the Fourier transform $F_G$ and the Fuchs-type differential operator $D$ (see below). These tools enabled one to define in [32] the Bessel potential-type spaces, to prove theorems on multipliers for convolution operators and derive more precise results for the above mentioned convolution equations and other applications to similar models arising from the mathematical physics.

This work will be dedicated to the consistent development of the global pseudo-differential calculus on $G = (0,1)$. In particular, the Hörmander classes on $G$ as defined in [45], will be characterised by this approach. This work is organised as follows:

- In Section 2 we present the basics on the Hörmander pseudo-differential calculus, we also present the topics related with the Fourier analysis on the group $G$, and the definition of the $L^p$-Sobolev spaces in this setting.
- In Section 3 we introduce the global classes of pseudo-differential operators on $G$, the corresponding quantisation formula and then we prove that these classes are stable under compositions, adjoints, and the construction of para-meres.
- In Section 4 we study the mapping properties of these classes, we establish the Calderón-Vaillancourt theorem, the Gårding inequality and the Fefferman-Phong inequality.
In Section 5 we investigate the spectral properties of the pseudo-differential calculus and in particular, the Atiyah-Singer-Fedosov theorem is proved in this setting.

Finally in Section 6 the Fredholmeness of the pseudo-differential operators is analysed as well as other mapping properties on $L^p$-Sobolev spaces. In this last section the results are derived from their corresponding analogues for Fourier multipliers.

2. Preliminaries

In this section we provide the preliminaries used in this work about the theory of pseudo-differential operators. This theory will be introduced for open subsets of the Euclidean space. To do this we follow Hörmander [45]. The Lie structure of $G = (-1,1)^n$ and its Fourier analysis are discussed here as well as, the function spaces on $G$ of interest for this work are defined. For the aspects about the Lie theory we follow [60].

2.1. Hörmander classes on open sets of $\mathbb{R}^n$. Let us introduce the Hörmander classes starting with the definition in the Euclidean setting.

**Definition 2.1 (Pseudo-differential operators on Euclidean open sets).** Let $U$ be an open subset of $\mathbb{R}^n$ such that $U \neq \emptyset$ and $U \neq \mathbb{R}^n$. We say that the “symbol” $a \in C^\infty(U \times \mathbb{R}^n, \mathbb{C})$ belongs to the Hörmander class of order $m$ and of $(\rho,\delta)$-type, denoted by $S^m_{\rho,\delta}(U, Hör)$, where $0 \leq \rho, \delta \leq 1$, if for every compact subset $K \subset U$ and for all $\alpha, \beta \in \mathbb{N}_0^n$, the symbol inequalities

$$\left| \partial^\beta_x \partial^\alpha_\xi a(x, \xi) \right| \leq C_{\alpha,\beta,K} (1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

(2.1)

hold true uniformly in $x \in K$ for all $\xi \in \mathbb{R}^n$. Then, a continuous linear operator $A : C^\infty_c(U) \to C^\infty(U)$ is a pseudo-differential operator of order $m$ of $(\rho,\delta)$-type, if there exists a symbol $a \in S^m_{\rho,\delta}(U, Hör)$ such that

$$\forall x \in \mathbb{R}^n, \quad Af(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} a(x, \xi) (\mathcal{F}_{\mathbb{R}^n} f)(\xi) d\xi,$$

for all $f \in C^\infty_c(U)$, where

$$\forall \xi \in \mathbb{R}^n, \quad (\mathcal{F}_{\mathbb{R}^n} f)(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx$$

is the Euclidean Fourier transform of $f$ at $\xi \in \mathbb{R}^n$, and $U$ is identified with $\mathbb{R}^n$.

**Remark 2.2 (Pseudo-differential operators on $\mathbb{R}^n$).** In the specific case where $U$ is the whole space $\mathbb{R}^n$, the conditions of the uniform estimates in (2.1) on compact subsets on $\mathbb{R}^n$ are removed and the following symbol estimates

$$\left| \partial^\beta_x \partial^\alpha_\xi a(x, \xi) \right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

(2.2)

for functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ define the symbol class $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n) := S^m_{\rho,\delta}(\mathbb{R}^n, Hör)$. 

2.2. The group $G$. Here we explain how we can endow $(-1, 1)^n$ with an intrinsic group structure. The general idea is that if we have a bijection from a group to a set, we can force this bijection to become an isomorphism by defining a group operation on the set by this condition.

Remark 2.3. Suppose $(G, \cdot)$ is an arbitrary group, $H$ is a set and $\theta : G \to H$ is a bijection. Then we can endow $H$ with a group structure such that $\theta$ becomes a group isomorphism between $G$ and $H$. Since $\theta$ is a bijection, we can write every element $h \in H$ as $h = \theta(g)$ for some $g \in G$. We define the group operation $\cdot_H$ on $H$ by

$$\theta(g_1) \cdot_H \theta(g_2) := \theta(g_1 \cdot g_2) \in H.$$ 

The neutral element in $H$ is $1_H := \theta(1)$, and one easily finds that

$$\theta(g) \cdot_H \theta(1) = \theta(g) = \theta(1) \cdot_H \theta(g)$$

and

$$\theta(g) \cdot_H \theta(g^{-1}) = \theta(1) = \theta(g^{-1}) \cdot_H \theta(g).$$

Similarly, the associativity of $\cdot_H$ follows from the associativity of $\cdot$ in $G$. We thus find indeed that $(H, \cdot_H)$ is a group and $\theta$ is an isomorphism by the definition of $\cdot_H$.

Remark 2.4 (The Lie group structure on $G = (-1, 1)^n$). We now apply Remark 2.3 to the particular case of the bijection

$$x : \mathbb{R}^n \to (-1, 1)^n : t \mapsto x(t) := (-\tanh t_1, \cdots, -\tanh t_n), \quad (2.3)$$

where

$$\tanh(t) := \frac{\sinh(t)}{\cosh(t)} = \frac{e^t - e^{-t}}{e^t + e^{-t}}, \quad t \in \mathbb{R}.$$ 

Note that the minus sign is only a matter of convention. We thus endow $G = (-1, 1)^n$ with the group operation

$$(-\tanh t_1) +_G (-\tanh t_2) := -\tanh(t_1 + t_2) = \frac{(-\tanh t_1) + (-\tanh t_2)}{1 + (-\tanh t_1)(-\tanh t_2)}.$$ 

Since we dispose of an addition formula for the hyperbolic tangent function, we can write the group operation more neatly by setting $x := -\tanh t_1$ and $y := -\tanh t_2$ so that we get

$$x +_G y = \frac{x + y}{1 + xy}.$$ 

This endows $G = (-1, 1)^n$ with a Lie group structure such that $x$ is an isomorphism, whose inverse is given by

$$t : (-1, 1)^n \to \mathbb{R}^n : x \mapsto t(x) := \left(\frac{1}{2} \ln \left(\frac{1 - x_1}{1 + x_1}\right), \cdots, \frac{1}{2} \ln \left(\frac{1 - x_n}{1 + x_n}\right)\right). \quad (2.4)$$

These isomorphisms $x$ and $t$ will play a very important role in the rest of this paper as they provide a canonical translation of properties of the Lie group $\mathbb{R}^n$ to the corresponding properties on $G = (-1, 1)^n$. Then, $G$ has now a natural structure of a Lie group.
2.3. **Function spaces on** $G = (-1, 1)^n$. We continue the idea that the isomorphisms $x$ and $t$ translate properties and concepts of $\mathbb{R}^n$ to $G$ by constructing function spaces on $G$. To this end it will be convenient to consider the pull-backs $t_*$ and $x_*$. 

**Definition 2.5** (The pull-backs $t_*$ and $x_*$). The pull-back $t_* f$ of a (measurable) function $f : \mathbb{R}^n \to \mathbb{C}$, with $t$ as in (2.4), is defined via 

$$t_* f := f \circ t : G \to \mathbb{C}. \quad (2.5)$$

Similarly, if $f : G \to \mathbb{C}$ then $x_* f (t) := f(x(t))$.

**Remark 2.6.** These pull-backs will allow one to switch between function spaces on $\mathbb{R}^n$ and $G$ as we will show now.

If $f \in C^\infty(\mathbb{R}^n)$ then $t_* f : G = (-1, 1)^n \to \mathbb{C}$ is a smooth function as a composition of smooth functions. Conversely, for a smooth function $f : G \to \mathbb{C}$ we have that $x_* f$ is a smooth function on $\mathbb{R}^n$. Since $t_*$ and $x_*$ are inverse bijections, this establishes a canonical bijective correspondence given by $C^\infty(G) = t_*(C^\infty(\mathbb{R}^n))$.

**Definition 2.7** (The natural distance on $G$). Define the distance $d_G : G \times G \to [0, \infty)$ by $d_G(x, y) := |t(x) - t(y)|$. Note that this definition is chosen in such a way that $t : G \to \mathbb{R}^n$ and $x : \mathbb{R}^n \to G$ become isometries. As a consequence, this endows $G$ with a topology, which is homeomorphic with the Euclidean topology. For this topology it can easily be checked that we have a bijective correspondence between the spaces of compactly supported smooth functions on $\mathbb{R}^n$ and $G$ given by $C^\infty_c(G) = t_*(C^\infty_c(\mathbb{R}^n))$.

**Definition 2.8** (Canonical vector fields on $G$). Consider $f \in C^\infty(\mathbb{R}^n)$ and $g \in C^\infty(G)$. We use the expressions (2.4) and (2.3) to compute that 

$$\frac{\partial t_k}{\partial x_j}(x) = -\delta_{jk}\frac{1}{1 - x_k^2} \quad \text{and} \quad \frac{\partial x_k}{\partial t_j}(t) = -\delta_{jk}(1 - x_k^2),$$

where $\delta_{jk}$ is the Kronecker delta, which is 1 if $j = k$ and 0 otherwise. A change of variables then leads to the formulas 

$$\frac{\partial (t_* f)}{\partial x_j}(x) = \sum_{k=1}^n \frac{\partial f}{\partial t_k}(t(x)) \frac{\partial t_k}{\partial x_j}(x) = -\frac{1}{(1 - x_j^2)} \frac{\partial f}{\partial t_j}(t(x)) \quad (2.6)$$

and 

$$\frac{\partial (x_* g)}{\partial t_j}(t) = \sum_{k=1}^n \frac{\partial g}{\partial x_k}(x(t)) \frac{\partial x_k}{\partial t_j}(t) = -(1 - x_j^2) \frac{\partial g}{\partial x_j}(x(t)) \quad (2.7)$$

These equations motivate the introduction of the partial differential operators 

$$\mathcal{D}_{x_j} := -(1 - x_j^2) \frac{\partial}{\partial x_j}$$

for $1 \leq j \leq n$. Note that (2.6) can be rewritten as 

$$\mathcal{D}_{x_j} (t_* f)(x) = \frac{\partial f}{\partial t_j}(t(x)) = t_* \frac{\partial f}{\partial t_j}(x).$$

Since this identity is true for all $x \in G$ and all $f \in C^\infty(\mathbb{R}^n)$, we have the equality of operators 

$$t_* \frac{\partial}{\partial t_j} = \mathcal{D}_{x_j} t_* \quad (2.8)$$
This equality shows that the operators $\mathcal{D}_{x_j}$ for $1 \leq j \leq n$ are the natural partial differential operators on $G$ since $t_*$ in a sense transforms $\frac{\partial}{\partial t_j}$ to $\mathcal{D}_{x_j}$. Similarly, it follows from (2.7) that

$$x_*\mathcal{D}_{x_j} = \frac{\partial}{\partial t_j}x_*.$$  \hspace{1cm} (2.9)

**Definition 2.9** (Canonical partial differential operators on $G$). Before we proceed, let us introduce some common notation. For multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ we write

$$\partial_t^\alpha := \partial_{t_1}^{\alpha_1} \partial_{t_2}^{\alpha_2} \cdots \partial_{t_n}^{\alpha_n} \quad \text{and} \quad \mathcal{D}_x^\alpha := \mathcal{D}_{x_1}^{\alpha_1} \mathcal{D}_{x_2}^{\alpha_2} \cdots \mathcal{D}_{x_n}^{\alpha_n}.$$  

We denote the length of the multi-index $\alpha \in \mathbb{N}_0^n$ by

$$|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$  

Now we introduce the space of Schwartz functions on $G$.

**Remark 2.10** (The Schwartz class on $G$). Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then for every two multi-indices $\alpha, \beta \in \mathbb{N}_0^n$, we consider the seminorm

$$\sup_{t \in \mathbb{R}^n} \left| t^\alpha \partial_t^\beta f(t) \right| < \infty.$$  

Observe that we can rewrite this seminorm using (2.8) as

$$\sup_{t \in \mathbb{R}^n} \left| t^\alpha \partial_t^\beta f(t) \right| = \sup_{x \in G} \left| t(x)^\alpha \partial_t^\beta f(t(x)) \right| = \left( \frac{1}{2} \right)^{|\alpha|} \sup_{x \in G} \left( \ln \frac{1-x}{1+x} \right)^\alpha \mathcal{D}_x^2 t_* f(x).$$  

The latter expression can thus be used for the seminorms defining the Schwartz space $\mathcal{S}(G)$.

Note that $\ln \frac{1-x}{1+x}$ is asymptotically equivalent to $\ln(1-y^2)$ on $(-1,1)$ (with the topology induced by $-\tanh$) because

$$\frac{\ln \frac{1-x}{1+x}}{\ln(1-y^2)} \rightarrow 1.$$  

Hence, we may replace the factor $\left( \ln \frac{1-x}{1+x} \right)^\alpha$ in the expression for the seminorms by $[\ln(1-x^2)]^\alpha$. We thus define the Schwartz space on $G$ to be the space of all smooth functions $f$ on $G$ such that the seminorms

$$\sup_{x \in G} \left| [\ln(1-x^2)]^\alpha \mathcal{D}_x^2 f(x) \right| < \infty.$$  

Clearly, we also have that $\mathcal{S}(G) = t_*(\mathcal{S}(\mathbb{R}^n))$, as we expected.

**Remark 2.11** (Lebesgue spaces on $G$). Finally, we construct the Lebesgue spaces on $G$. Let $f \in L^p(\mathbb{R}^n)$ for $0 < p \leq \infty$. If $p < \infty$, we find using a change of variables that

$$\|f\|_{L^p(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} |f(t)|^p dt = \int_{(-1,1)^n} \frac{|f(t(x))|^p dx}{(1-x_1^2) \cdots (1-x_n^2)^{\frac{p}{2}}}. $$  

This shows that a natural measure on $G = (-1,1)^n$ is given by

$$d\mu_G(x) := \frac{dx}{(1-x_1^2)(1-x_2^2) \cdots (1-x_n^2)},$$
where $dx$ is the Lebesgue measure on $\mathbb{R}^n$. This indeed happens to be the Haar measure on $G$. Thus, $L^p(G) = t_*(L^p(\mathbb{R}^n))$, so that $t_*$ is an $L^p$-isometry, and the corresponding $L^p$-norm of an element $f \in L^p(G)$ is given by

$$\|f\|_{L^p(G)} = \left( \int_G |f(x)|^p \, d\mu_G(x) \right)^{\frac{1}{p}}.$$

Moreover, since $t_*$ is an $L^2$-isometry (or by a direct computation) we find for all $f, g \in L^2(\mathbb{R}^n)$ that

$$\langle f, g \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(t) \overline{g(t)} \, dt = \int_G f(t(x)) \overline{g(t(x))} \, d\mu_G(x) = \langle t_* f, t_* g \rangle_{L^2(G)}.$$  

(2.10)

For the case $p = \infty$ we similarly find that $L^\infty(G) = t_*(L^\infty(\mathbb{R}^n))$ with norm

$$\|f\|_{L^\infty(G)} = \text{ess sup}_{x \in G} |f(x)|.$$

Note that the inner product on $L^2(G)$ is given by

$$(u, v)_{L^2(G)} = \int_{(-1,1)^n} \frac{u(x)v(x) \, dx}{(1-x_1^2)(1-x_2^2) \cdots (1-x_n^2)}.$$  

(2.11)

2.4. Global Fourier analysis on $G = (-1,1)^n$. The Euclidean Fourier transform induces the group Fourier transform $F_G$ on $G$. We observe it in the following remark.

Remark 2.12. Let $f \in S(G)$. Then $x_* f \in S(\mathbb{R}^n)$ and we have

$$\mathcal{F}_{\mathbb{R}^n}[x_* f](\xi)$$

$$= \int_{\mathbb{R}^n} e^{-2\pi i t \xi} x_* f(t) \, dt = \int_{(-1,1)^n} \frac{e^{-2\pi i t \xi} f(x) \, dx}{(1-x_1^2)(1-x_2^2) \cdots (1-x_n^2)}$$

$$=: (F_G f)(2 \xi), \quad \xi \in \mathbb{R},$$

where we applied the change of variables $x = x(t)$. We thus find that

$$\mathcal{F}_{\mathbb{R}^n}[x_* f] \left( \frac{\xi}{2} \right) = \int_G \prod_{j=1}^n \left( \frac{1-x_j}{1+x_j} \right)^{-i\pi \xi_j} f(x) \, d\mu_G(x)$$

$$= \int_G \left( \frac{1-x}{1+x} \right)^{-i\pi \xi} f(x) \, d\mu_G(x),$$  

(2.12)

with the multilinear notation

$$(\frac{1-x}{1+x})^{-i\pi \xi} := \left( \frac{1-x_1}{1+x_1} \right)^{-i\pi \xi_1} \cdots \left( \frac{1-x_n}{1+x_n} \right)^{-i\pi \xi_n}.$$

Definition 2.13 (Group Fourier transform). In view of Remark 2.12, we define the Fourier transform on $G$ of a function $f \in S(G)$ by

$$\mathcal{F}_G[f](\xi) := \int_{\mathbb{R}^n} \left( \frac{1-x}{1+x} \right)^{-i\pi \xi} f(x) \, d\mu_G(x), \quad \xi \in \mathbb{R}^n.$$  

(2.13)
We can provide similar motivation for defining the inverse Fourier transform as ([32])

$$\mathcal{F}^{-1}_G[f](x) := \int_{\mathbb{R}^n} \left(1 - \frac{\xi}{1 + \xi}\right)^{i\pi x} f\left(\frac{\xi}{2}\right) d\xi, \quad x \in G.$$  \hspace{1cm} (2.14)

2.5. Sobolev and Bessel potential spaces on $G$. For a function of polynomial growth at infinity $|a(x)| \leq C|x|^N := C(1 + |x|^2)^{N/2}$ for some constant $C > 0$ and some integer $N \in \mathbb{N}_0$, by $a_{\mathbb{R}^n}(D)$ we denote the Fourier convolution operator on the Euclidean space $\mathbb{R}^n$, defined by

$$a_{\mathbb{R}^n}(D)\varphi(t) := \mathcal{F}^{-1}_{\mathbb{R}^n} a \mathcal{F}_{\mathbb{R}^n} \varphi(t), \quad t \in \mathbb{R}^n, \quad \varphi \in \mathcal{S}(\mathbb{R}^n),$$

and $a(\xi)$ is called its symbol (cf. [34], where one used a different notation $W^0 = a_{\mathbb{R}^n}(D)$). Let $\Delta_{\mathbb{R}^n}$ be the standard Laplacian on $\mathbb{R}^n$, given by

$$\Delta_{\mathbb{R}^n} = -\sum_{j=1}^n \partial^2_{x_j}. \quad (2.15)$$

The operator $\Delta_{\mathbb{R}^n}$ admits a self-adjoint extension on $L^2(\mathbb{R}^n)$.

**Definition 2.14** (Sobolev spaces on $\mathbb{R}^n$). The notation $H^s_p(\mathbb{R}^n)$ with $s \in \mathbb{R}$, $1 \leq p \leq \infty$, refers to the Bessel potential space on the Lie group (Euclidean space) $\mathbb{R}^n$, which represents the closure of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with respect of the norm

$$\|f\|_{H^s_p(\mathbb{R}^n)} := \|(1 - \Delta_{\mathbb{R}^n})^{s/2} f\|_{L^p(\mathbb{R}^n)} < \infty. \quad (2.16)$$

**Remark 2.15** (Sobolev spaces on $L^2(\mathbb{R}^n)$). We will apply the standard convention and for $p = 2$ use the notation $H^s(\mathbb{R}^n)$ for the Hilbert space $H^s_2(\mathbb{R}^n)$, dropping the subscript index $p = 2$. Due to the classical Parseval’s equality for $\mathbb{R}^n$ the formula

$$\|f\|_{H^s(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} |(1 + |\xi|^2)^{s/2} \mathcal{F}_{\mathbb{R}^n} f(\xi)|^2 d\xi\right)^{1/2} \quad (2.17)$$

provides an equivalent norm in the Hilbert space $H^s(\mathbb{R}^n)$.

**Remark 2.16**. It is well known that a partial derivative $(\partial^\alpha \varphi)(t) = (a^\alpha(D)\varphi)(t)$ is a convolution operator and its symbol is $a(\xi) := (-i\xi)^\alpha$ for arbitrary multi-index $\alpha \in \mathbb{N}^n$:

$$(\mathcal{F}_{\mathbb{R}^n} \partial^\alpha \varphi)(\xi) = (-i\xi)^\alpha (\mathcal{F}_{\mathbb{R}^n} \varphi)(\xi), \quad \xi \in \mathbb{R}^n. \quad (2.18)$$

**Remark 2.17** (Sobolev spaces of integer order). For an integer $m \in \mathbb{N}$ and an arbitrary $1 \leq p \leq \infty$ the Bessel potential space coincides with the Sobolev space $H^m_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$ and an equivalent norm is defined as follows (cf. [71])

$$\|f\|_{W^m_p(\mathbb{R}^n)} := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}\right)^{1/p} \quad (2.19)$$

with the usual ess sup-norm modification for $p = \infty$:

$$\|f\|_{W^m_\infty(\mathbb{R}^n)} := \sum_{|\alpha| \leq m} \text{ess sup}_{t \in \mathbb{R}^n} |\partial^\alpha f(t)|.$$

GLOBAL PSEUDO-DIFFERENTIAL OPERATORS ON $G = (-1, 1)^n$
Definition 2.18 (The class of $L^p(\mathbb{R}^n)$-multipliers). For $1 \leq p \leq \infty$ by $\mathcal{M}_p(\mathbb{R}^n)$ we denote the set of all symbols $a(\xi)$, $\xi \in \mathbb{R}^n$, for which the convolution operator $a_{\mathbb{R}^n}(D) : S(\mathbb{R}^n) \to S'(\mathbb{R}^n)$ extends to a bounded operator

$$a_{\mathbb{R}^n}(D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

Symbols in the class $\mathcal{M}_p(\mathbb{R}^n)$ are called $L^p$-multipliers.

Remark 2.19. Note that $\mathcal{M}_p(\mathbb{R}^n)$ is a Banach algebra with the induced operator norm $\|a_{\mathbb{R}^n}(D)\|_{\mathcal{B}(L^p(\mathbb{R}^n))}$, because the composition of operators from this class has the property

$$a_{\mathbb{R}^n}(D)b_{\mathbb{R}^n}(D) = (ab)_{\mathbb{R}^n}(D).$$

For a function of polynomial growth $a(\xi)$, $\xi \in \mathbb{R}^n$, the convolution operator $a_G(\mathcal{D})$ on the group $G$ is defined in a standard way:

$$a_G(\mathcal{D})\varphi(x) := \mathcal{F}^{-1}a_{\mathcal{F}\mathcal{G}}\varphi(x), \quad x \in G = (-1, 1)^n, \quad \varphi \in S(G),$$

and $a(\xi)$ is called its symbol (cf. [32], where is used a different notation $W^0_{a,G} = a_G(\mathcal{D})$).

Remark 2.20 ($W^0_{a,G} = a_G(\mathcal{D})$). We adopt the notation $W^0_{a,G} = a_G(\mathcal{D})$ through this work. We emphasize the use of this notation in Section 6, specially if the reader is familiar with the references [33, 34].

Remark 2.21 (Partial derivatives on $G$). Note that according to (2.8) and (2.9) the counterpart of the derivative $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ on $\mathbb{R}^n$ is the derivative $\mathcal{D}_G^\alpha := \mathcal{D}_1^{\alpha_1} \cdots \mathcal{D}_n^{\alpha_n}$ on the Lie group $G$ and both of them are convolution operators with the same symbols $(-i\xi)^\alpha = (-i\xi_1)^{\alpha_1} \cdots (-i\xi_n)^{\alpha_n}$.

Remark 2.22 (The Laplacian on $G$). Note that in view of (2.9), the Laplacian on $G$ is given by

$$\Delta_G = -\sum_{j=1}^n \mathcal{D}_{x_j}^2,$$  \hspace{1cm} (2.19)

The operator $\Delta_G$ admits a self-adjoint extension on $L^2(G)$.

Definition 2.23 (Sobolev spaces on $G$). Arguing as in the Euclidean case, the notation $H^s_p(G)$ with $s \in \mathbb{R}$, $1 \leq p \leq \infty$ refers to the Bessel potential space on the Lie group $G$, where the norm is defined as follows (cf. (2.16)

$$\|f\|_{H^s_p(G)} := \|(1 - \Delta_G)^{\frac{s}{2}}f\|_{L^p(G)} < \infty.$$

Remark 2.24 (Sobolev spaces on $L^2(G)$). We follow the standard notation and we write $H^s(G)$ for the space $H^s_2(G)$ in the case $p = 2$. Due to the Parseval’s equality for the group $G$

$$(\mathcal{F}_G\varphi, \mathcal{F}_G\psi)_{L^2(G)} = \int_{\mathbb{R}^n} (\mathcal{F}_G\varphi)(\xi)(\overline{\mathcal{F}_G\psi})(\xi)d\xi = \pi^n \int_G \varphi(y)\overline{\psi(y)}d\mu_G y =: \pi^n (\varphi, \psi)_{L^2(G)}$$

the following

$$\|f\|_{H^s(G)} := \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\mathcal{F}_G f|(|\xi|)^2 d\xi\right)^{1/2}$$
defines an equivalent norm on $H^s(G)$.

**Remark 2.25** (Sobolev spaces of integer order). For an integer $s = m \in \mathbb{N}$ and arbitrary $1 \leq p \leq \infty$ the Bessel potential space coincides with the Sobolev space $H^m_p(G) = W^m_p(G)$ and an equivalent norm is defined as follows

$$
\|f\|_{W^m_p(G)} := \left( \sum_{|\alpha| \leq m} \|\mathcal{D}_G^\alpha f\|_{L^p(G)}^p \right)^{1/p},
$$

for $1 \leq p < \infty$, while for $p = \infty$ is used the usual ess sup-norm modification:

$$
\|f\|_{W^m_\infty(G)} := \sum_{|\alpha| \leq m} \text{ess sup}_{x \in \mathbb{R}^n} |\mathcal{D}_G^\alpha f(x)| .
$$

We remind that $\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2)$ denotes the space of all linear bounded operators $T : \mathcal{B}_1 \rightarrow \mathcal{B}_2$ between the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$.

The next Proposition 2.26 is proved in [32] for $n = 1$, but the proof for arbitrary $n$ is similar.

**Proposition 2.26.** For arbitrary $1 \leq p \leq \infty$ and $s \in \mathbb{R}$ a convolution operator

$$
a_{\mathbb{R}^n}(D) : H^s_p(\mathbb{R}^n) \rightarrow H^s_p(\mathbb{R}^n).
$$

is bounded if and only if the operator

$$
a_G(D) : H^s_p(G) \rightarrow H^s_p(G)
$$

is bounded and if and only if $a \in \mathcal{M}_p(\mathbb{R}^n)$ (i.e. $a$ is an $L^p$-multiplier).

**Remark 2.27.** In other words, multiplier classes for the spaces $H^s_p(\mathbb{R}^n)$ and $H^s_p(G)$ are independent of the parameter $s$ and both coincide with $\mathcal{M}_p(\mathbb{R}^n)$.

Based on formulae (2.18) and the isomorphism

$$
t_* : H^s_p(\mathbb{R}^n) \rightarrow H^s_p(G), \quad 1 \leq p \leq \infty, \quad s \in \mathbb{R},
$$

(2.21)

the following is proved (cf. [34, 71] for the case $H^s_p(\mathbb{R}^n) = W^m_p(\mathbb{R}^n)$).

**Proposition 2.28.** For arbitrary $1 < p < \infty$ and an integer $m \in \mathbb{N}_0$ the Bessel potential space $H^m_p(G)$ and the Sobolev space $W^m_p(G)$ have equivalent norms and are topologically isomorphic.

By applying the isomorphism (2.21) we can also justify the following propositions, proved in [71, § 2.4.2] for the Bessel potential space on the Euclidean space $\mathbb{R}^n$.

**Proposition 2.29.** Let $s_0, s_1, r_0, r_1 \in \mathbb{R}$, $1 \leq p_0, p_1, q_0, q_1 < \infty$, $0 < t < 1$ and

$$
1 = \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad 1 = \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad s = (1 - \theta)s_0 + \theta s_1, \quad r = (1 - \theta)r_0 + \theta r_1.
$$

If $A \in \mathcal{R}_j := \mathcal{B}(H^s_{p_0}(G), H^s_{q_1}(G))$, $j = 0, 1$, then $A$ is bounded between the interpolated spaces $A \in \mathcal{B} := \mathcal{B}(H^s_p(G), H^s_q(G))$ and the norm is estimated as follows

$$
\|A\|_{\mathcal{R}_j} \leq \|A\|_{\mathcal{B}_0}^{1 - \theta} \|A\|_{\mathcal{B}_1}^{\theta}.
$$

**Proposition 2.30** (see [34]). Let $s, r \in \mathbb{R}$, $1 \leq p \leq \infty$. The Bessel potential operator

$$
(1 - \Delta_G)^{\frac{r}{2}} : H^s_p(G) \rightarrow H^{s-r}_p(G).
$$

(2.22)
3. Pseudo-differential operators on $G = (-1, 1)^n$

3.1. The global Hörmander classes. Let $0 \leq \rho, \delta \leq 1$ and $m \in \mathbb{R}$. Our goal is to give a useful definition of (global) symbol classes on $G$. To this end we start from a symbol $a = a(t, \xi) \in S^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$.

**Remark 3.1** (Hörmander classes on $\mathbb{R}^n$ v.s. Hörmander classes on $G$). In the same spirit of Subsection 2.3 we consider $a^G(x, \xi) := a(t(x), \xi) \in C^\infty(G \times \mathbb{R}^n)$. This function satisfies for every $\alpha, \beta \in \mathbb{N}_0^n$ and every $(x, \xi) \in G \times \mathbb{R}^n$ the symbolic estimates

$$|D^\beta_x \partial^\alpha_\xi a^G(x, \xi)| = |(D^\beta_x \partial^\alpha_\xi a)(t(x), \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \quad (3.1)$$

Conversely, if a smooth function $a \in C^\infty(G \times \mathbb{R}^n)$ satisfies the symbolic estimates (3.1), then we obtain for $a^{R^n}(t, \xi) := a(x(t), \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ the inequalities

$$|\partial^\alpha_\xi a^{R^n}(t, \xi)| = |(\partial^\alpha_\xi a^{R^n})(x(t), \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}.$$

Hence, we define the symbol class $S^m_{\rho, \delta}(G \times \mathbb{R}^n)$ as the space of smooth functions $a \in C^\infty(G \times \mathbb{R}^n)$ that satisfy for all $(x, \xi) \in G \times \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{N}_0^n$ the symbolic estimates

$$|\partial^\alpha_\xi D^\beta_x a(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

where $C_{\alpha, \beta} > 0$ are some positive constants.

Moreover, it can easily be checked that $(a^G)^{R^n} = a$ for $a \in S^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$, and similarly $(a^{R^n})^G = a$ for $a \in S^m_{\rho, \delta}(G \times \mathbb{R}^n)$. Thus, there is a canonical bijection between $S^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $S^m_{\rho, \delta}(G \times \mathbb{R}^n)$.

3.2. The quantisation formula. Let $\sigma \in S^m_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $0 \leq \rho, \delta \leq 1$. The symbol $\sigma$ determines a pseudo-differential operator expressed as

$$\sigma(t, D)f(t) = \int_{\mathbb{R}^n} e^{2\pi i t \cdot \xi} \sigma(t, \xi) \mathcal{F}_{R^n} f(\xi) d\xi, \quad f \in S(\mathbb{R}^n).$$

We look for an expression of pseudo-differential operators on $G$ now. So, let $f \in S(G)$. Then $x_* f \in S(\mathbb{R}^n)$ so that $t_* \sigma(t, D)x_* f \in S(G)$. We compute an expression for this function for $y \in G$ with the help of (2.12):}

$$t_* [\sigma(t, D)(x_* f)](y) = \int_{\mathbb{R}^n} e^{2\pi i t(y) \cdot \xi} \sigma(t(y), \xi) \mathcal{F}_{R^n} [x_* f](\xi) d\xi = \int_{\mathbb{R}^n} \left(\frac{1-y}{1+y}\right)^{i\pi \xi} \sigma(y, \xi) \mathcal{F}_G f(\xi/2) d\xi \quad (3.2)$$

Hence, we define the pseudo-differential operator $\sigma(x, \mathcal{D})$ associated with the symbol $\sigma \in S^m_{\rho, \delta}(G \times \mathbb{R}^n)$ by

$$\sigma(x, \mathcal{D}) f(x) = \int_{\mathbb{R}^n} \left(\frac{1-x}{1+x}\right)^{i\pi \xi} \sigma(x, \xi) \mathcal{F}_G f \left(\frac{\xi}{2}\right) d\xi, \quad f \in S(G). \quad (3.3)$$
Note that the argument \( \xi/2 \) in (3.3) comes from the Fourier inversion formula (2.14). On the other hand, we denote by

\[
\Psi^{m}_{\rho,\delta}(G \times \mathbb{R}^n) = \{ \sigma(x, \mathcal{D}) : \sigma \in S^{m}_{\rho,\delta}(G \times \mathbb{R}^n) \} \tag{3.4}
\]

the family of (global) pseudo-differential operators on \( G \) with order \( m \) and of \((\rho, \delta)\)-type. Moreover, it follows from (3.2) that there is a bijective correspondence between pseudo-differential operators on \( \mathbb{R}^n \) and on \( G \) given by

\[
t_\ast \sigma(t, D)x_\ast = \sigma^{G}(x, \mathcal{D}). \tag{3.5}
\]

Note that this is not really surprising as this is the change of variables formula for pseudo-differential operators.

We summarise the interplay between the Hörmander classes on \( \mathbb{R}^n \) with the global ones on \( G \) in the following theorem.

**Corollary 3.2.** Let \( m \in \mathbb{R} \) and \( 0 \leq \delta, \rho \leq 1 \). Then, \( A : S(G) \to S(G) \) is a pseudo-differential operator in the class \( \Psi^{m}_{\rho,\delta}(G \times \mathbb{R}^n) \) if and only if there exists a pseudo-differential operator \( \tilde{A} : S(\mathbb{R}^n) \to S(\mathbb{R}^n) \) in the class \( \Psi^{m}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n) \) such that the following diagram

\[
\begin{array}{ccc}
S(\mathbb{R}^n) & \xrightarrow{\tilde{A}} & S(\mathbb{R}^n) \\
t_\ast \downarrow & & \downarrow t_\ast \\
S(G) & \xrightarrow{A} & S(G)
\end{array}
\tag{3.6}
\]

commutes. It means, for any \( f = f(t) \in S(\mathbb{R}^n) \), \( t_\ast \circ \tilde{A}f = At_\ast f \). Moreover, if \( \tilde{a} \) is the symbol of \( \tilde{A} \), then the symbol of \( a \) is given by

\[
a(x, \xi) \equiv \tilde{a}^{G}(x, \xi) = \tilde{a}(t(x), \xi),
\]

for all \( (x, \xi) \in G \times \mathbb{R}^n \).

**Proof.** The bijective correspondence (3.5) between \( \Psi^{m}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n) \) and \( \Psi^{m}_{\rho,\delta}(G \times \mathbb{R}^n) \) yields \( t_\ast \tilde{A}x_\ast = A, \) which is equivalent to \( t_\ast \tilde{A} = At_\ast \). This means that \( t_\ast \tilde{A}f = At_\ast f \) for all \( f \in S(\mathbb{R}^n) \). This bijective correspondence is indeed such that the symbols are related by \( a(x, \xi) = \tilde{a}^{G}(x, \xi) \). \(
\)

3.3. Composition of pseudo-differential operators. Here we study the composition of pseudo-differential operators. We show that the classes \( S^{m}_{\rho,\delta}(G \times \mathbb{R}^n) \) are stable under the composition of operators.

**Theorem 3.3.** Let \( 0 \leq \delta < \rho \leq 1 \). Let \( a \in S^{m_{1}}_{\rho,\delta}(G \times \mathbb{R}^n) \) and \( b \in S^{m_{2}}_{\rho,\delta}(G \times \mathbb{R}^n) \). Then there exists a symbol \( c \in S^{m_{1}+m_{2}}_{\rho,\delta}(G \times \mathbb{R}^n) \) such that \( c(x, \mathcal{D}) = a(x, \mathcal{D}) \circ b(x, \mathcal{D}) \). Moreover, we have the asymptotic formula

\[
\forall (x, \xi) \in G \times \mathbb{R}^n : c(x, \xi) \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial^{\alpha}_{\xi} a(x, \xi)) (\mathcal{D}^{\alpha}_{x} b(x, \xi)),
\]
in the sense that for any $N \in \mathbb{N}$,
\[
(x, \xi) \mapsto c(x, \xi) - \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a(x, \xi))(\mathbf{D}_x^\alpha b(x, \xi)) \in S^{m-(\rho-\delta)N}_{\rho, \delta}(G \times \mathbb{R}^n).
\]

Proof. Suppose $a \in S^{m_1}_{\rho, \delta}(G \times \mathbb{R}^n)$ and $b \in S^{m_2}_{\rho, \delta}(G \times \mathbb{R}^n)$. Then $a^{\mathbb{R}^n} \in S^{m_1}_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $b^{\mathbb{R}^n} \in S^{m_2}_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$. By the composition formula in $\mathbb{R}^n$ there exists a symbol $c \in S^{m_1+m_2}_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$ such that $c(t, D) = a^{\mathbb{R}^n}(t, D) \circ b^{\mathbb{R}^n}(t, D)$, and we have the asymptotic expansion
\[
c(t, \xi) \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a^{\mathbb{R}^n}(t, \xi))(\partial_\xi^\alpha b^{\mathbb{R}^n}(t, \xi)).
\] (3.7)

With the help of (3.5) we obtain
\[
c(x, \mathbf{D}) = t_* c(t, D)x_* = t_* a^{\mathbb{R}^n}(t, D)x_* b^{\mathbb{R}^n}(t, D)x_* = a(x, \mathbf{D}) \circ b(x, \mathbf{D}),
\]
where we have applied that $x_*$ and $t_*$ are each other’s inverse.

Let $N \in \mathbb{N}$. Using the formula $\sigma^G(x, \xi) = \sigma(t(x), \xi)$ for any $\sigma \in S^{m}_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$, we deduce from (3.7) that
\[
c^G(x, \xi) = \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha a^{\mathbb{R}^n}(x, \xi))(\partial_\xi^\alpha b^{\mathbb{R}^n}(x, \xi)) \in S^{m-(\rho-\delta)N}_{\rho, \delta}(G \times \mathbb{R}^n).
\]

This completes the proof. \hfill $\square$

3.4. **The adjoint of a pseudo-differential operator.** Here we study the adjoints of the pseudo-differential operators in the classes $S^{m}_{\rho, \delta}(G \times \mathbb{R}^n)$. We show that these classes are stable under adjoints.

**Theorem 3.4.** Let $0 \leq \delta < \rho \leq 1$. Let $a \in S^{m}_{\rho, \delta}(G \times \mathbb{R}^n)$. Then there exists a symbol $a^* \in S^{m}_{\rho, \delta}(G \times \mathbb{R}^n)$ such that $a^*(x, \mathbf{D}) = a(x, \mathbf{D})^*$. Moreover, we have the asymptotic formula
\[
\forall (x, \xi) \in G \times \mathbb{R}^n : a^*(x, \xi) \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \mathbf{D}_x^\alpha a(x, \xi),
\]
in the sense that for any $N \in \mathbb{N}$,
\[
(x, \xi) \mapsto a^*(x, \xi) - \sum_{|\alpha| < N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \mathbf{D}_x^\alpha a(x, \xi) \in S^{m-(\rho-\delta)N}_{\rho, \delta}(G \times \mathbb{R}^n).
\]

Proof. Let $a \in S^{m}_{\rho, \delta}(G \times \mathbb{R}^n)$. Then $a^{\mathbb{R}^n} \in S^{m}_{\rho, \delta}(\mathbb{R}^n \times \mathbb{R}^n)$ so that the adjoint formula in $\mathbb{R}^n$
\[
(a^{\mathbb{R}^n})^*(t, \xi) \sim \sum_{\alpha} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_\xi^\alpha a^{\mathbb{R}^n}(t, \xi).
\] (3.8)

Due to (3.5) it follows that
\[
a^{\mathbb{R}^n}(t, D) = x_* a(x, \mathbf{D})t_*.
\]

Let $a^* := [(a^{\mathbb{R}^n})^*]^G$. Using (2.10) we find on the one hand for every $f, g \in \mathcal{S}(\mathbb{R}^n)$ that
\[
\langle a^{\mathbb{R}^n}(t, D)^* f, g \rangle_{L^2(\mathbb{R}^n)} = \langle f, a^{\mathbb{R}^n}(t, D)g \rangle_{L^2(\mathbb{R}^n)} = \langle f, x_* a(x, \mathbf{D})t_* g \rangle_{L^2(\mathbb{R}^n)}
\]
exists the symbol is required. A symbol

Theorem 3.5. Let

Assume also that

Then, there exists

Proof. The idea is to find a symbol

while on the other

\[
\langle a^\ast (t, D) f, g \rangle_{L^2(\mathbb{R}^n)} = \left\langle (a^\ast)^\ast (t, D) f, g \right\rangle_{L^2(\mathbb{R}^n)} = \left\langle t_\ast \left( (a^\ast)^\ast (t, D) \right) x_\ast f, t_\ast g \right\rangle_{L^2(G)} = \langle a^\ast (x, D) f, t_\ast g \rangle_{L^2(G)}.
\]

Since \( t_\ast \) is a bijection from \( \mathcal{S}(\mathbb{R}^n) \) to \( \mathcal{S}(G) \), it follows that \( a^\ast (x, D) = a(x, D)^\ast \).

Applying the bijective correspondence between \( S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n) \) and \( S^m_{\rho,\delta}(G \times \mathbb{R}^n) \) to (3.8), we get

\[
a^\ast (x, \xi) - \sum_{|\alpha| < N} \frac{(2\pi)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha a(x, \xi) \in S^m_{\rho,\delta}(-(\rho-\delta)N)(G \times \mathbb{R}^n)
\]

for any \( N \in \mathbb{N} \), which proves the theorem. \( \square \)

3.5. Construction of parametrix. Now we will prove that the classes \( S^m_{\rho,\delta}(G \times \mathbb{R}^n) \) admit the construction of parametrixes. For this, the condition of ellipticity for the symbol is required. A symbol \( a \in S^m_{\rho,\delta}(G \times \mathbb{R}^n) \) is elliptic of order \( m \), if there exists \( R > 0 \), such that the inequality condition in (3.9) holds. For the corresponding operator \( A = a(x, D) \) we compute the parametrix in the next theorem.

**Theorem 3.5.** Let \( m \in \mathbb{R} \), and let \( 0 \leq \delta < \rho \leq 1 \). Let \( a = a(x, \xi) \in S^m_{\rho,\delta}(G \times \mathbb{R}^n) \). Assume also that \( a(x, \xi) \neq 0 \) for \( |\xi| \geq R \), for some \( R > 0 \), and it satisfies

\[
\inf_{(x, \xi) \in G \setminus \{ \xi \in \mathbb{R}^n : |\xi| \geq R \}} |a(x, \xi)| \geq C(1 + |\xi|)^m.
\]

Then, there exists \( B \in \Psi^{-m}_{\rho,\delta}(G \times \mathbb{R}^n) \), such that

\[
AB - I, BA - I \in \Psi^{-\infty}(G \times \mathbb{R}^n) := \bigcap_{s \in \mathbb{R}} \Psi^s(G \times \mathbb{R}^n).
\]

Moreover, the symbol \( \tau(x, \xi) \) of \( B \) satisfies the following asymptotic expansion

\[
\tau(x, \xi) \sim \sum_{N=0}^{\infty} \tau_N(x, \xi), \ (x, \xi) \in G \times \mathbb{R}^n,
\]

where \( \tau_N \in S^{-m-(\rho-\delta)N}_{\rho,\delta}(G \times \mathbb{R}^n) \) obeys to the recursive formula

\[
\tau_N(x, \xi) = -a(x, \xi)^{-1} \left( \sum_{k=0}^{N-1} \sum_{|\gamma|=N-k} (\partial_\xi^\gamma a(x, \xi)) (\partial_x^\gamma \tau_k(x, \xi)) \right), \ N \geq 1,
\]

with \( \tau_0(x, \xi) = a(x, \xi)^{-1} \).

**Proof.** The idea is to find a symbol \( \tau \) such that if \( I = AB \), then \( I - I \) is a smoothing operator, where

\[
\hat{\tau}(x, \xi) \sim \sum_{|\alpha|=0}^{\infty} (\partial_\xi^\alpha a(x, \xi)) (\partial_x^{(\alpha)} \tau(x, \xi)).
\]
Here \( \widehat{I}(x, \xi) \) denotes the symbol of \( \mathcal{I} \). The asymptotic expansion means that for every \( N \in \mathbb{N} \),

\[
\partial_\xi^\alpha \mathcal{D}_x^\alpha \left( \widehat{I}(x, \xi) - \sum_{|\alpha| \leq N} (\partial_\xi^\alpha a(x, \xi))(\mathcal{D}_x^\alpha \tau(x, \xi)) \right) \in S^{-\rho - \delta(N + 1) - \rho \ell + \delta|\beta|}(G \times \mathbb{R}^n),
\]

for every \( \alpha \in \mathbb{N}_0 \) of order \( \ell \in \mathbb{N}_0 \), where \( \tau \) is requested to satisfy the asymptotic expansion (3.10). So, formally we can write

\[
\widehat{I}(x, \xi) \sim \sum_{|\alpha| = 0}^\infty (\partial_\xi^\alpha a(x, \xi)) (\mathcal{D}_x^\alpha \tau(x, \xi)) = \sum_{|\alpha| = 0}^\infty \sum_{N=0}^\infty (\partial_\xi^\alpha a(x, \xi)) (\mathcal{D}_x^\alpha \tau_N(x, \xi)).
\]

Observe the fact that \( \tau_0 \in S^{-m}_{\rho, \delta}(G \times \mathbb{R}^n) \) follows from the hypothesis. Now, one can check easily that \( \tau_N \in S^{-m - (\rho - \delta)N}_{\rho, \delta}(G \times \mathbb{R}^n) \), for all \( N \geq 1 \) by using induction.

Consequently,

\[
\tau(x, \xi) - \sum_{j=0}^{N-1} \tau_j(x, \xi) \in S^{-m - (\rho - \delta)N}_{\rho, \delta}(G \times \mathbb{R}^n).
\]

This analysis allows us to deduce that

\[
\widehat{I}(x, \xi) - \sum_{k=0}^{N-1} \sum_{|\gamma| < N} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_k(x, \xi)) \in S^{-\rho - \delta N}_{\rho, \delta}(G \times \mathbb{R}^n).
\]

On the other hand, observe that

\[
\sum_{k=0}^{N-1} \sum_{|\gamma| < N} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_k(x, \xi))
\]

\[
= 1 + \sum_{k=1}^{N-1} \left( a(x, \xi) \tau_k(x, \xi) + \sum_{|\gamma| \leq N, |\gamma| \geq 1} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_k(x, \xi)) \right)
\]

\[
= 1 + \sum_{k=1}^{N-1} \left( a(x, \xi) \tau_k(x, \xi) + \sum_{|\gamma| = N-j, j < k} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_j(x, \xi)) \right)
\]

\[
+ \sum_{|\gamma| + j \geq N, |\gamma| < N, j < N} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_k(x, \xi))
\]

\[
= 1 + \sum_{|\gamma| + j \geq N, |\gamma| < N, j < N} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_j(x, \xi)),
\]

where we have used that

\[
a(x, \xi) \tau_k(x, \xi) + \sum_{|\gamma| = k-j, j < k} (\partial_\xi^\gamma a(x, \xi))(\mathcal{D}_x^\gamma \tau_j(x, \xi)) \equiv 0.
\]
in view of (3.11). Because, for \(|\gamma| + j \geq N\), \((\partial^j_x a(x, \xi)) (\mathcal{D}_x^\gamma \tau_k(x, \xi)) \in S_{\rho, \delta}^{-(\rho-\delta)N}(G \times \mathbb{R}^n)\), it follows that
\[
\sum_{k=0}^{N-1} \sum_{|\gamma| < N} (\partial^j_x a(x, \xi)) (\mathcal{D}_x^\gamma \tau_k(x, \xi)) - 1 \in S_{\rho, \delta}^{-(\rho-\delta)N}(G \times \mathbb{R}^n),
\]
and consequently, \(\hat{I}(x, \xi) - 1 \in S_{\rho, \delta}^{-(\rho-\delta)N}(G \times \mathbb{R}^n)\), for every \(N \in \mathbb{N}\). So, we have proved that \(AB - I \in S^{-\infty}(G \times \mathbb{R}^n)\). □

4. Boundedness properties for the Hörmander classes \(\Psi^m_{\rho, \delta}(G \times \mathbb{R}^n)\)

In this section we discuss the mapping properties of pseudo-differential operators even in the border line case \(0 \leq \rho \leq \delta \leq 1\), \(\delta \neq 1\).

4.1. The \(L^p\)-theory: \(L^p\)-Fefferman theorem and \(L^2\)-Calderón-Vaillancourt theorem. The following theorem provides sharp conditions for the \(L^p\)-boundedness of the pseudo-differential operators in the Hörmander classes \(\Psi^m_{\rho, \delta}(G \times \mathbb{R}^n)\).

**Theorem 4.1.** Let \(A : S(G) \to S(G)\) be a pseudo-differential operator with symbol \(\sigma \in S^{-m}_{\rho, \delta}(G \times \mathbb{R}^n)\), \(0 \leq \delta \leq \rho \leq 1\), \(\delta \neq 1\). Then, if
\[
m \geq m_p := n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|,
\]
where \(1 < p < \infty\), then \(A\) extends to a bounded operator from \(L^p(G)\) into \(L^p(G)\).

**Proof.** Let \(\tilde{A} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)\) be the continuous linear operator such that the following diagram
\[
\begin{array}{ccc}
S(\mathbb{R}^n) & \xrightarrow{\tilde{A}} & S(\mathbb{R}^n) \\
\downarrow t_* & & \downarrow t_* \\
S(G) & \xrightarrow{A} & S(G)
\end{array}
\]
commutes. Then, \(\tilde{A}\) is a pseudo-differential operator with symbol in the Hörmander class \(S_{\rho, \delta}^{-m}(\mathbb{R}^n \times \mathbb{R}^n)\) and in view of Fefferman \(L^p\)-theorem (see [36]), the order condition
\[
m \geq m_p := n(1 - \rho) \left| \frac{1}{p} - \frac{1}{2} \right|,
\]
implies that, for \(1 < p < \infty\),
\[
\tilde{A} : S(\mathbb{R}^n) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)
\]
extends to a bounded operator. In view of the topological density of \(S(\mathbb{R}^n)\) into \(L^p(\mathbb{R}^n)\), such an extension is unique and we can also denote it by \(\tilde{A} : L^p(\mathbb{R}^n) \to \).
\(L^p(\mathbb{R}^n)\). Now, there exists a unique operator \(\hat{A} : L^p(G) \to L^p(G)\) such that the following diagram

\[
\begin{array}{ccc}
L^p(\mathbb{R}^n) & \xrightarrow{\hat{A}} & L^p(\mathbb{R}^n) \\
\downarrow t_* & & \downarrow t_* \\
L^p(G) & \xrightarrow{\hat{A}} & L^p(G)
\end{array}
\]  

commutes. It is clear that \(\hat{A}|_{S(G)} = A\). So, we have proven that \(A\) admits a bounded extension on \(L^p(G)\). \(\square\)

With \(p = 2\), \(m = 0\) and \(0 \leq \delta \leq \rho \leq 1\), \(\delta \neq 1\), in Theorem 4.1 we obtain the following analogue of the Calderón-Vaillancourt theorem.

**Corollary 4.2.** Let \(A : S(G) \to S(G)\) be a pseudo-differential operator with symbol \(\sigma \in S_{\rho,\delta}^m(G \times \mathbb{R}^n)\), \(0 \leq \delta \leq \rho \leq 1\), \(\delta \neq 1\). Then, \(A\) extends to a bounded operator from \(L^2(G)\) into \(L^2(G)\).

### 4.2. The sharp Gårding inequality and the Fefferman-Phong inequality.

In this section we discuss the lower bounds for operators.

**Theorem 4.3.** Let \(A : S(G) \to S(G)\) be a pseudo-differential operator with symbol \(a \in S_{\rho,\delta}^m(G \times \mathbb{R}^n)\) such that \(a(x, \xi) \geq 0\) for all \((x, \xi) \in G \times \mathbb{R}^n\). Then the following sharp Gårding inequality

\[
\forall u \in S(G), \quad \text{Re}(a(x, D)u, u)_{L^2} \geq -C\|u\|_{H^{m-(\rho-\delta)/2}(G)}
\]  

(4.3)

holds. Moreover, if \(\sigma \in S_{1,0}^2(G \times \mathbb{R}^n)\) and \(\sigma(x, \xi) \geq 0\) for all \((x, \xi) \in G \times \mathbb{R}^n\), then the Fefferman-Phong inequality

\[
\forall u \in S(G), \quad \text{Re}(a(x, D)u, u)_{L^2} \geq -C\|u\|_{L^2(G)}
\]  

(4.4)

remains valid.

**Proof.** Let \(\tilde{A} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)\) be the continuous linear operator such that the following diagram

\[
\begin{array}{ccc}
S(\mathbb{R}^n) & \xrightarrow{\tilde{A}} & S(\mathbb{R}^n) \\
\downarrow t_* & & \downarrow t_* \\
S(G) & \xrightarrow{A} & S(G)
\end{array}
\]  

(4.5)

commutes. Then, \(\tilde{A}\) is a pseudo-differential operator with the symbol in the Hörmander class \(S_{\rho,\delta}^{-m}(\mathbb{R}^n \times \mathbb{R}^n)\) and in view of the sharp Gårding inequality for the classes
Using that any \( u \in \mathcal{S}(G) \) can be written in a unique way as \( \tilde{u} = (t_*)^{-1}u \), and that \( t_* : L^2(\mathbb{R}^n) \to L^2(G) \), \( (t_*)^{-1} : L^2(G) \to L^2(\mathbb{R}^n) \) are isometries of Hilbert spaces, we have for \( u \in \mathcal{S}(G) \) the identity of inner products

\[
\text{Re}(\tilde{A}(t_*)^{-1}u, (t_*)^{-1}u)_{L^2(\mathbb{R}^n)} = \text{Re}((t_*)^{-1}t_* \tilde{A}(t_*)^{-1}u, (t_*)^{-1}u)_{L^2(\mathbb{R}^n)} = \text{Re}((t_*)^{-1}Au, (t_*)^{-1}u)_{L^2(\mathbb{R}^n)} = \text{Re}(Au, u)_{L^2(G)}.
\]

Since

\[
\|u\|_{H^m-\rho,\delta(G)} = \|\tilde{u}\|_{H^m-\rho,\delta(\mathbb{R}^n)},
\]

we conclude that

\[
\text{Re}(Au, u)_{L^2(G)} \geq -C\|u\|_{H^m-\rho,\delta(G)}
\]
as desired. On the other hand, if \( m = 2 \), and \( (\rho, \delta) = (1, 0) \), the Fefferman-Phong inequality (see [37]) gives the lower bound

\[
\forall \tilde{u} \in \mathcal{S}(\mathbb{R}^n), \quad \text{Re}(\tilde{A}\tilde{u}, \tilde{u})_{L^2(\mathbb{R}^n)} \geq -C\|\tilde{u}\|_{L^2(\mathbb{R}^n)}.
\]

By following the analysis in the first part of the proof, and the fact that \( \|u\|_{L^2(G)} = \|\tilde{u}\|_{L^2(\mathbb{R}^n)} \) implies the following Fefferman-Phong inequality

\[
\forall u \in \mathcal{S}(G), \quad \text{Re}(Au, u)_{L^2(G)} \geq -C\|u\|_{L^2(G)}.
\]

The proof is complete. \( \square \)

5. Sharp spectral properties for the classes \( S^m_{\rho,\delta}(G \times \mathbb{R}^n) \)

In this section we discuss the spectral properties of operators. We investigate the compactness and the Fredholmness of the operators.

5.1. The Gohberg lemma and compactness on \( L^2(G) \). In the following theorem we characterise the \( L^2(G) \)-compactness of a sub-class of operators in \( \Psi^0_{1,0}(G \times \mathbb{R}^n) \). To do so, we compute the distance of a pseudo-differential operator \( A \) to the set \( \mathcal{C}(L^2(G)) \) of all compact operator on \( L^2(G) \) and we prove that this distance inequality is sharp.

**Theorem 5.1.** Let \( A : \mathcal{S}(G) \to \mathcal{S}(G) \) be a pseudo-differential operator with symbol \( \sigma : G \times \mathbb{R}^n \to \mathbb{C} \) satisfying the symbol inequalities

\[
|\mathcal{D}^\alpha_x \mathcal{D}^\beta_\xi \sigma(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{-|\alpha|}(1 + |\ln(1 + x^2)|)^{-|\beta|}.
\]

Then \( A \in \Psi^0(G \times \mathbb{R}^n) \) and its essential norm has the following estimate from below

\[
\|A\|_{\mathcal{B}(L^2(G))} := \inf_{K \in \mathcal{C}(L^2(G))} \|A - K\| \geq d := \limsup_{|\xi| \to \infty, |x| \to 1} |\sigma(x, \xi)|.
\]

Moreover, \( A \) is compact, \( A \in \mathcal{C}(L^2(G)) \), if and only if \( d = 0 \).
Proof. That $A \in \Psi_{1,0}^0(G \times \mathbb{R}^n)$, can be easily proved. Indeed, it is consequence of the inequality
\begin{equation}
|\mathfrak{D}_x^\beta \delta^{\alpha} \sigma(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha|} (1 + |\ln(1 - x^2)|)^{-|\beta|} \lesssim (1 + |\xi|)^{-|\alpha|}.
\end{equation}
Let $\tilde{A} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ be the continuous linear operator such that the following diagram
\begin{equation}
\begin{array}{cccc}
\mathcal{S}(\mathbb{R}^n) & \xrightarrow{\tilde{A}} & \mathcal{S}(\mathbb{R}^n) \\
\downarrow t_* & & & \downarrow t_* \\
\mathcal{S}(G) & \xrightarrow{\tilde{A}} & \mathcal{S}(G)
\end{array}
\end{equation}
commutes. Denote by $\tilde{a}(t, \xi)$ the symbol of $\tilde{A}$. Then, in terms of $\tilde{a}$, (5.1) is equivalent to the following symbol estimates
\begin{equation}
|\partial^{\beta}_t \partial^\alpha_\xi \tilde{a}(t, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|)^{-|\alpha|} (1 + |t|)^{-|\beta|}.
\end{equation}
Then, in view of the Gohberg lemma proved by Grushin [42], for any compact operator $\tilde{K} \in \mathcal{C}(L^2(\mathbb{R}^n))$
\begin{equation}
\|\tilde{A} - \tilde{K}\|_{\mathcal{S}(\mathbb{R}^n)} \geq d := \limsup_{|\xi| \to \infty, |t| \to \infty} |\tilde{a}(t, \xi)|.
\end{equation}
Now, any compact operator $K \in \mathcal{C}(L^2(G))$ is the pull-back by $t_* : L^2(\mathbb{R}^n) \to L^2(G)$ of compact operator $\tilde{K} \in \mathcal{C}(L^2(\mathbb{R}^n))$, that is $K = t_*(\tilde{K})$, and then
\begin{equation}
\|A - K\|_{\mathcal{S}(L^2(G))} = \|\tilde{A} - \tilde{K}\|_{\mathcal{S}(L^2(\mathbb{R}^n))} \geq d := \limsup_{|\xi| \to \infty, |t| \to \infty} |\tilde{a}(t, \xi)| = \limsup_{|\xi| \to \infty, |x| \to 1} |\sigma(x, t)|.
\end{equation}
The previous inequality proves the first part of Theorem 5.1 by taking the infimum over all compact operators $K \in \mathcal{C}(L^2(G))$. On the other hand, it was proved by Molahajloo in Theorem 1.4 of [50], that $\tilde{A}$ is compact on $L^2(G)$, if and only if $d := \limsup_{|\xi| \to \infty, |t| \to \infty} |\tilde{a}(t, \xi)| = 0$. Since $A$ is compact on $L^2(G)$ if and only if $\tilde{A}$ is compact on $L^2(\mathbb{R}^n)$, then we have that $A$ is compact on $L^2(G)$ if and only if $d = 0$. The proof of Theorem 5.1 is complete. \qed

5.2. The Atiyah-Singer-Fedosov index theorem on $G$. In this section we compute the index for the elliptic pseudo-differential operators with symbols in the subclass
\begin{equation}
\Sigma_{1,0}^0(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu})
\end{equation}
of $\Sigma_{1,0}^0(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu})$ defined as follows.
For $\nu = 1$, $a^G \in \Sigma_{1,0}^0(G \times \mathbb{R}^n) = \Sigma_{1,0}^0(G \times \mathbb{R}^n; \mathbb{C})$, if it satisfies for every $\alpha, \beta \in \mathbb{N}_0^n$ and every $(x, \xi) \in G \times \mathbb{R}^n$ the symbolic estimates
\begin{equation}
|\mathfrak{D}_x^\beta \delta^{\alpha} a^G(x, \xi)| = |(\partial^\beta_\xi \partial^\alpha_x a)(t(x), \xi)| \leq C_{\alpha, \beta} (1 + |\ln(1 - x^2)| + |\xi|)^{-|\rho| + \delta|\beta|}.
\end{equation}
Conversely, if a smooth function \( a \in C^\infty(G \times \mathbb{R}^n) \) satisfies the symbolic estimates (5.9), then we obtain for \( a^G_n(t, \xi) := a(x(t), \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) the inequalities
\[
\left| \partial_t \partial_\xi a^G_n(t, \xi) \right| = \left| (\partial_x^\beta \partial_\xi^\alpha a)(x(t), \xi) \right| \leq C_{\alpha, \beta}(1 + |t| + |\xi|)^{-|\rho| + |\delta| + |\beta|}. \tag{5.10}
\]
So the class \( \Sigma^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) = \{ a^G_n \text{ satisfying (5.10)} \} \) is the Shubin class of order zero (see e.g. [38, Page 459]). So, by following this nomenclature, we call to (5.8) the Shubin class of order zero on \( G \).

For \( \nu \in \mathbb{N} \) with \( \nu \geq 2 \), the class of matrix-symbols \( S^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \) is the class of functions \( a \in C^\infty(G \times \mathbb{R}^n, \mathbb{C}^{\nu \times \nu}) \) whose entries are symbols in the class \( S^0_{1,0}(G \times \mathbb{R}^n) \). So, to a matrix-symbol \( a = (a_{ij})_{i,j=1}^\nu \in S^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \) we associate the matrix-valued operator
\[
A \equiv \text{Op}(a) := (\text{Op}(a_{ij}))_{i,j=1}^\nu.
\]

We denote by
\[
\Psi^0_{\rho,\delta}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) = \{ \sigma(x, D) : \sigma \in S^m_{\rho,\delta}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \} \tag{5.11}
\]
the family of (global) matrix pseudo-differential operators on \( G \) with order \( m \) and of \((\rho, \delta)\)-type. It is clear that if
\[
\Sigma^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) = \{ a = (a_{ij})_{i,j=1}^\nu : a_{ij} \in \Sigma^0_{1,0}(G \times \mathbb{R}^n) \},
\]
then
\[
\Xi^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) = \{ \text{Op}(a) : a = (a_{ij})_{i,j=1}^\nu \in \Sigma^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \} \tag{5.12}
\]
is a subset of the operator class \( \Psi^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \).

In general the class of operators \( \Xi^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \) is a good sub-class of \( \Psi^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \) if one is thinking in generalising several of the results that are available for pseudo-differential operators on compact manifolds. In particular, if as in this section one wants to compute an Atiyah-Singer-Fedosov type formula. These formulas give topological data in terms of the symbols of the operators. Indeed, in 1974 B. Fedosov suggested in [35] the analytic index formula for an elliptic pseudo-differential operator, representing the index by the “winding” number of the symbol. The following result is of Atiyah-Singer-Fedosov type. We prove, in particular, that the analytical index of an elliptic operator of the class \( \Psi^0_{\rho,\delta}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \) agrees with the “winding” number of its global symbol.

**Theorem 5.2.** Let \( A : \mathcal{S}(G, \mathbb{C}^{\nu}) \rightarrow \mathcal{S}(G, \mathbb{C}^{\nu}) \) be a pseudo-differential operator with symbol \( a \in \Sigma^0_{1,0}(G \times \mathbb{R}^n; \mathbb{C}^{\nu \times \nu}) \) such that for any \((x, \xi)\) outside of a ball \( B \) in \( G \times \mathbb{R}^n \), \( a(x, \xi) \in \text{End}[^n \mathbb{C}] \) is invertible. Then \( A \) is a Fredholm operator on \( L^2(G, \mathbb{C}^{\nu}) \) with index
\[
\text{ind}[A] = -\frac{(n-1)!}{(-2\pi i)^n(2n-1)!} \int_{\partial B} \text{Tr}[a^{-1}(x, \xi) d\sigma(x, \xi)]^{2n-1} \tag{5.13}
\]
where \( G \times \mathbb{R}^n \) is positively oriented by the volume form \( dx_1 \wedge d\xi_1 \wedge \cdots \wedge dx_n \wedge d\xi_n \).

**Proof.** We start by observing that \( A \) is bounded on \( L^2(G) \) in view of the Calderón-Vaillancourt theorem in the form of Corollary 4.2. Now, let us prove that \( A \) is
Fredholm on $L^2(G)$ and let us compute its index. Let $\tilde{A} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ be the continuous linear operator such that the following diagram

\[
\begin{array}{ccc}
\mathcal{S}(\mathbb{R}^n) & \xrightarrow{\tilde{A}} & \mathcal{S}(\mathbb{R}^n) \\
\downarrow t_* & & \downarrow t_* \\
\mathcal{S}(G) & \xrightarrow{A} & \mathcal{S}(G)
\end{array}
\]

(5.14)

commutes. Denote by $\tilde{a}(t, \xi)$ the symbol of $\tilde{A}$. In view of the Index theorem in $\mathbb{R}^n$ by Hörmander (see [45, Page 215]), since $A : \mathcal{S}(\mathbb{R}^n, \mathbb{C}^\nu) \rightarrow \mathcal{S}(\mathbb{R}^n, \mathbb{C}^\nu)$ is a pseudodifferential operator such that for any $(t, \xi)$ outside of a ball $\tilde{B}$ of $\mathbb{R}^n \times \mathbb{R}^n$, $\tilde{a}(x, \xi) \in \text{End}[\mathbb{C}^\nu]$ is invertible, then $\tilde{A}$ is a Fredholm operator on $L^2(\mathbb{R}^n, \mathbb{C}^\nu)$ with the index

\[
\text{ind}[\tilde{A}] = -\frac{(n-1)!}{(-2\pi i)^n(2n-1)!} \int_{\partial \tilde{B}} \text{Tr}[\tilde{a}^{-1}(t, \xi)d\tilde{a}(t, \xi)]^{2n-1},
\]

(5.15)

where $\mathbb{R}^n \times \mathbb{R}^n$ is oriented by the volume form $dt_1 \wedge d\xi_1 \wedge \cdots \wedge dt_n \wedge d\xi_n$. The topology in $G$ induced by the mapping $t : G \rightarrow \mathbb{R}^n$, implies that

$B = t^{-1}[\tilde{B}]$

is a ball with respect to the distance $d_G(x, y) = |t(x) - t(y)|$, and then

$\partial B = t^{-1}[\partial \tilde{B}]$.

Because the index is a spectral invariant under unitary transformations, we have that

$\text{ind}[\tilde{A}] = \text{ind}[A]$.

Now, let us compute $\text{ind}[A]$ in terms of $a$. For this, note that

\[
\tilde{a}^{-1}(t, \xi)d\tilde{a}(t, \xi) = a^{-1}(t, \xi) \sum_{j=1}^{n} \frac{\partial a(t, \xi)}{\partial t_j} dt_j + a^{-1}(t, \xi) \sum_{j=1}^{n} \frac{\partial a(t, \xi)}{\partial \xi_j} d\xi_j
\]

\[
= a(x, \xi)^{-1} \sum_{j=1}^{n} (-1 - x_j^2) \frac{\partial a(x, \xi)}{\partial x_j} dt_j + a^{-1}(x, \xi) \sum_{j=1}^{n} \frac{\partial a(x, \xi)}{\partial \xi_j} d\xi_j
\]

\[
= a(x, \xi)^{-1} \sum_{j=1}^{n} (-1 - x_j^2) \frac{\partial a(x, \xi)}{\partial x_j} \times \frac{dx_j}{(-1 - x_j^2)}
\]

\[
+ a^{-1}(x, \xi) \sum_{j=1}^{n} \frac{\partial a(x, \xi)}{\partial \xi_j} d\xi_j
\]

\[
= a(x, \xi)^{-1} \sum_{j=1}^{n} \frac{\partial a(x, \xi)}{\partial x_j} dx_j + a^{-1}(x, \xi) \sum_{j=1}^{n} \frac{\partial a(x, \xi)}{\partial \xi_j} d\xi_j
\]

\[
= a(x, \xi)^{-1} da(x, \xi).
\]
In consequence, with \( \tilde{C} = \partial \tilde{B} \) and \( C = \partial B \) we have,

\[
\int_{\tilde{C}} \text{Tr}[\tilde{a}^{-1}(t, \xi) da(t, \xi)]^{2n-1} = \int_{C} \text{Tr}[a^{-1}(x, \xi) da(x, \xi)]^{2n-1}.
\]

The proof is complete. \( \square \)

Remark 5.3. Note that in (5.13), \( \omega = a^{-1} da \) is a one form on \( G \times \mathbb{R}^{n} \) with coefficients taking values in \( \nu \times \nu \) matrices. If \( n = \nu = 1 \), the right-hand side of the equality (5.13) is the winding number of \( a \) considered as a mapping from \( \partial B \) into \( \mathbb{C} \setminus \{0\} \). Then, in this case the index formula is

\[
\text{ind}(A) = -\frac{1}{2\pi i} \int_{\partial B} \frac{da}{a},
\]

and we re-obtain the index formula by F. Noether [52].

6. Fredholm properties and \( L^{p} \)-boundedness of \( \Psi \)DOs with non-classical symbols on \( G = (-1, 1)^{n} \)

In this section we analyse the Fredholmness and the \( L^{p} \)-boundeness of pseudo-differential operators by using result of the same type for Fourier multipliers on the group.

6.1. Boundedness of pseudo-differential operators in the Bessel potential spaces. In the present subsection we will expose conditions on a symbol \( a(x, \xi) \) which ensure the boundedness of the corresponding pseudo-differential operator \( a(x, D) \) in (3.3) in the Bessel potential spaces

\[
a(x, D) : H^{s}_{p}(G) \rightarrow H^{s-r}_{p}(G)
\]

We start with convolution operators \( a_{\mathbb{R}^{n}}(D) \) and \( a_{G}(D) \), which are pseudo-differential, but independent of the variable \( x \).

Definition 6.1 (Polytopes). Consider polytope domain \( \Omega \) in \( \mathbb{R}^{n} \) which represents an open domain, intersection of finite number of affine transformed half spaces

\[
\bigcap_{j=1}^{N} \{ \xi \in \mathbb{R}^{n} : \langle V^{j}, \xi - \eta^{j} \rangle > 0, \ N \geq n + 1 \} \neq \emptyset, \ V^{1}, \ldots, V^{N}, \eta^{1}, \ldots, \eta^{n} \in \mathbb{R}^{n}.
\]

Polytope domains include cones open to infinity and having empty intersection with a sufficiently large polytope domain, containing 0. Examples of polytopes in \( \mathbb{R}^{3} \) are given in Figure 1.

Definition 6.2 (Piecewise constant functions). We call a function \( a(\xi), \xi \in \mathbb{R}^{n} \), piecewise-constant if \( \mathbb{R}^{n} \) is divided into a finite number of polytopes and \( a(\xi) \) is constant on each of these polytops. \( \text{PC}(\mathbb{R}^{n}) \) denotes the algebra of piecewise-constant functions.

Lemma 6.3. Any piecewise constant function belongs to the multiplier class \( \text{PC}(\mathbb{R}^{n}) \subset \widetilde{M}_{p}(\mathbb{R}^{n}) \subset M_{p}(\mathbb{R}^{n}) \) for all \( 1 < p < \infty \).
Proof. For the proof we consider the following steps:

- Step 1: We define a polysingular integral operator $S_{G,k}$. We recall some identities between the operator $S_{G,k}$ and the partial Hilbert transforms.
- Step 2. We define the class of affine transformations on $\mathbb{R}^n$. These transformations are applied to the operator and an identity involving the symbol $a(\xi)$ with the symbol of partial Hilbert transforms allow us to conclude the proof.

So we continue the proof under this strategy.

- Step 1: The Cauchy polysingular integral operators

$$S_{G,k}\varphi(x) := \frac{1}{\pi} \int_{-1}^{1} \varphi(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n) \frac{dy_k}{1 + x_k \circ (-y_k)} \ln \frac{1}{1 - x_k \circ (-y_k)} \bigg( \frac{1}{1 - y_k^2} \bigg),$$

where $x = (x_1, \ldots, x_n)^\top \in G$, are bounded in $L^p(G)$ for all $1 < p < \infty$ and all $k \in \mathbb{N}$ because their counterpart on the Euclidean space $\mathbb{R}^n$

$$S_{R,k}\psi(t) := x^*S_{G,k}t^*\psi(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi(t_1, \ldots, t_{k-1}, \tau_k, t_{k+1}, \ldots, t_n) d\tau_k \bigg( \frac{1}{\tau_k - t_k} \bigg),$$

are partial Hilbert transforms (which are polysingular operators) and are bounded in $L^p(\mathbb{R}^n)$ (see, e.g. [33]). These operators are, obviously, of convolution type

$$S_{R,k} = W_{R,S_k}^0, \quad S_{G,k} = W_{G,S_k}^0, \quad k = 1, \ldots, n,$$

and the symbol of both of them is (cf. [34, Lemma 1.35, p.23] $S_k(\xi) = -\text{sign} \xi, \xi = (\xi_1, \ldots, \xi_n)^\top \in \mathbb{R}^n$.)
Therefore, the operators
\[ \frac{1}{2} [I \pm S_{G,k}] = W^0_{G, \mathcal{H}_k}, \quad \mathcal{H}_k^\pm (\xi) := \frac{1}{2} (1 \pm \text{sign} \, \xi_k) \quad k = 1, \ldots, n \]
are bounded in the space \( L^p(G) \) and \( \mathcal{H}_k^\pm \in \mathcal{M}_p(\mathbb{R}^n) \) are multipliers for all \( 1 < p < \infty \) and all \( k = 1, \ldots, n \). Note, that, \( \mathcal{H}_k(\xi) \) are the characteristic functions of the coordinate half spaces \( \Pi_k^\pm := \{ \xi \in \mathbb{R}^n : \pm \xi_k > 0 \}, \quad k = 1, \ldots, n \).

\begin{itemize}
  \item Step 2. On the other hand, for any vector \( \eta \in \mathbb{R}^n \), any affine transformation
    \[ \sigma \in \text{AT}(\mathbb{R}^n) := \{ \sigma : \mathbb{R}^n \to \mathbb{R}^n : \sigma^\top = \sigma^{-1} \} \]
    and any multiplier \( a \in \mathcal{M}_p(\mathbb{R}^n) \) we have the inclusion \( \sigma_* a, V_\eta a \in \mathcal{M}_p(\mathbb{R}^n) \), where \( \sigma_* u(\xi) := u(\sigma \xi) \) and \( V_\eta a(\xi) = a(\xi - \eta) \).

    Indeed, note first that
    \[ \sigma_* : H^s_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n), \quad e^{\pm i \sigma t} I : H^s_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n) \]
    are isometric isomorphisms of the spaces. Then the claimed inclusions \( \sigma_* a, V_\eta a \in \mathcal{M}_p(\mathbb{R}^n) \) follow from the equalities:
    \[ (\sigma_* a)(D)\sigma_*^\top \phi(t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i \sigma t \cdot \xi} a(\xi) d\xi \int_{\mathbb{R}^n} e^{i \xi \cdot \eta} \sigma_*^\top \phi(y) dy = (\sigma_* a)(D)\phi(t), \]
    \[ e^{-i \eta \cdot \xi} a_{\mathbb{R}^n}(D) e^{i \eta \cdot \xi} \phi(x) = \int_{\mathbb{R}^n} e^{-i \eta \cdot (\xi + t)} a(\xi) d\xi \int_{\mathbb{R}^n} e^{i (\xi - \eta) \cdot \eta} \phi(y) dy = a_{\mathbb{R}^n}(D - \eta I). \]

    Therefore, the characteristic function \( \mathcal{H}_{\Pi(\sigma, \eta)} \) of any half space \( \Pi(\sigma, \eta) \), situated from one side of any hyperplane \( \sigma \Pi^1_+ - \eta, \sigma \in \text{AT}(\mathbb{R}^n), \eta \in \mathbb{R}^n \), is a multiplier \( \mathcal{H}_{\Pi(\sigma, \eta)} \in \mathcal{M}_p(\mathbb{R}^n) \).

    Any piecewise constant function \( a \in \text{PC}(\mathbb{R}^n) \) is represented in the form
    \[ a(\xi) = \sum_{k=1}^N d_k \prod_{j=1}^M \mathcal{H}_k^\pm (\sigma_{jk} \xi - \eta_{jk}), \quad d_k \in \mathbb{C}, \ \eta_{jk} \in \mathbb{R}^n, \ j, k = 1, \ldots, n, \]
    where \( \sigma_{jk} \in \text{AT}(\mathbb{R}^n), j = 1, \ldots, M, k = 1, \ldots, N \). Then \( a \in \mathcal{M}_p(\mathbb{R}^n) \), because the multiplier class \( \mathcal{M}_p(\mathbb{R}^n) \) is an algebra \( W^0_{G,\beta} W^0_{G,\gamma} = W^0_{G,\beta + \gamma} \).

The proof of Lemma (6.3) is complete. \( \square \)

Now we analyse the boundedness of pseudo-differential operators using the norm of \( L^p \)-Fourier multipliers.

**Theorem 6.4.** Let \( 1 < p < \infty \) and let \( m \in \mathbb{N} \), and assume that a symbol \( a(x, \xi) \) satisfies the following hypothesis:

\begin{enumerate}
  \item[A] \( a(\beta + \gamma) \in C(\mathbb{R}^n, \mathcal{M}_p(\mathbb{R}^n)) \), \( \beta, \gamma \in \mathbb{N}_0^n \), \( |\beta| \leq 1 \), \( |\gamma| \leq m \);
  \item[B] \( M_{\kappa, \gamma} := \int_{\mathbb{R}^k} \left\| \frac{\partial^k a^\kappa_\gamma(y_1, \ldots, y_k, 0 \ldots, 0, \cdot)}{\partial y_1 \ldots \partial y_k} \right\|_{\mathcal{M}_p(\mathbb{R}^n)} dy_1 \ldots dy_k < \infty, \)  
\end{enumerate}

where
\[ a^\kappa_\gamma(y, \xi) := a(\gamma)(\kappa(y), \xi) \]
and $|\gamma| \leq m$. Then the pseudo-differential operators

$$
a_{\mathbb{R}^n}(x, D) : H^s_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n),
$$

$$
a_G(x, \mathcal{D}) : H^s_p(G) \to H^s_p(G)
$$

are bounded provided that

$$-m - 1 + \frac{1}{p} < s < m + \frac{1}{p},$$

and the corresponding operators norms satisfy the estimate

$$
\|a_{\mathbb{R}^n}(x, D)\|_{\mathcal{B}(H^s_p(\mathbb{R}^n))} = \|a_G(x, D)\|_{\mathcal{B}(H^s_p(G))} \leq C_{s,p} \sup_{\kappa, \gamma} \sum_{|\gamma| \leq m} M_{\kappa, \gamma},
$$

where the constant $C_{s,p} < \infty$ are depending only on the parameters $s$, $p$ and $\mathcal{B}(\mathcal{B})$ denotes the algebra of bounded operators in the Banach space $\mathcal{B}$.

**Proof.** The continuity of the operator $a_{\mathbb{R}^n}(x, D)$ in (6.4) is proved by E. Shargorodsky in [69, Theorem 5.1].

The continuity of the operator $a_G(x, \mathcal{D})$ in (6.5) is proved similarly to Theorem 4.1 and Theorem 5.2 above, based on the continuity of the operator $a_{\mathbb{R}^n}(x, D)$ in (6.4). □

**Definition 6.5** (The class $PC_p(\mathbb{R}^n)$). Let $PC_p(\mathbb{R}^n)$ denote the subalgebra of $\mathcal{M}_p(\mathbb{R}^n)$, obtained by closing the algebra $PC(\mathbb{R}^n)$ of piecewise constant functions on polytope domains in the norm of the multiplier algebra $\mathcal{M}_p(\mathbb{R}^n)$.

From Lemma 6.3 and Theorem 6.4 it follows that we have the following property.

**Corollary 6.6.** Let $1 < p < \infty$, $m \in \mathbb{N}$ and assume that $a = a(x, \xi)$ satisfies the following symbol conditions

$$
a_{(\beta + \gamma)} \in C(\mathbb{R}^n, PC_p(\mathbb{R}^n)), \quad \beta, \gamma \in \mathbb{N}_0^n, \quad |\beta| \leq 1, \quad |\gamma| \leq m;
$$

If the condition (6.3) holds, then the pseudo-differential operator $a_{\mathbb{R}^n}(t, D)$ in (6.6) and the pseudo-differential operator $a_G(x, \mathcal{D})$ in (6.8) are bounded and the inequality (6.6) holds.

**Remark 6.7.** By applying the well known inequality

$$
\|a\|_{\mathcal{M}_p(\mathbb{R}^n)} := \|W^0_{a, G}\|_{L^p(G)} = \|W^0_{a, \mathbb{R}^n}\|_{L^p(\mathbb{R}^n)} \geq \|a\|_{\mathcal{M}_2(\mathbb{R}^n)} = \|a\|_{L^\infty(\mathbb{R}^n)}
$$

(cf. [43]), it can easily be checked, that any function $a(\xi)$ from $PC_p(\mathbb{R}^n)$ have radial limits at any point including the infinity $\xi \in \mathbb{R}^n_*$ (the spherical compactification of $\mathbb{R}^n$ at the infinity), where limits are taken along all beams emerging from $\xi \in \mathbb{R}^n_*:

$$a(\xi, \omega) := \lim_{\lambda \to 0} a(\xi + \lambda \omega), \quad a(\infty, \omega) := \lim_{\lambda \to \infty} a(\lambda \omega)
$$

(6.9)

for all $\xi \in \mathbb{R}^n, \quad \omega \in \mathbb{B}_1(0), \quad |\omega| = 1.

where $\mathbb{B}_1(0)$ is the unit sphere centered at 0.
6.2. Fredholm properties of \( \Psi \)DOs. Here we prove a theorem on Fredholm properties of \( \Psi \)DOs with non-classical symbols on the group \( G \). First we investigate a particular case of convolution operators. We denote by \( \mathbb{R}_n \) the spherical compactification of \( \mathbb{R}^n \) at the infinity.

**Theorem 6.8.** Let \( a(\xi) \) be a complex valued matrix symbol \( a \in \text{PC}_{p}^{m \times m}(\mathbb{R}^n) \). The convolution operators

\[
a_{\mathbb{R}^n}(D) = W^0_{a,\mathbb{R}^n} : H^s_p(\mathbb{R}^n) \to H^s_p(\mathbb{R}^n),
\]

\[
a_G(\mathcal{Q}) = W^0_{a,G} : H^s_p(G) \to H^s_p(G)
\]

are Fredholm if and only if the symbol is elliptic

\[
\inf_{(\xi,\omega) \in \mathbb{R}_n^* \times B_1(0)} |a(\xi,\omega)| > 0.
\]

Moreover, if the ellipticity holds, the operators in (6.10) and in (6.11) are invertible for all \( p \in (1, \infty) \) and all \( s \in \mathbb{R} \) and the inverse operators read, respectively, \( W^0_{a^{-1},\mathbb{R}^n} \) and \( W^0_{a^{-1},G} \).

**Proof.** We divide the proof in the following steps.

- Step 1. We apply the Gohberg-Krupnik’s localization method.
- Step 2. We introduce a class of localising operators \( \Delta_{(\xi_0,\omega_0)} \).
- We locally approximate the symbol \( a(\xi) \) with the elements of the class \( \Delta_{(\xi_0,\omega_0)} \) and we conclude the proof.

Let us start with the Gohberg-Krupnik’s localization method.

- Step 1. We will apply Gohberg-Krupnik’s localization method, described in [41, Ch. 5], [34, § 1.7], [68] and [2]. Let us consider the set \( \mathbb{R}_n^* \times B_1(0) \), where \( \mathbb{R}_n^* : = \mathbb{R}_n \cup \{\infty\} \) denotes the one point compactifications of the space \( \mathbb{R}^n \). We cover the set \( \mathbb{R}_n^* \times B_1(0) \) by polytope cones, defined as follows. Neighbourhoods of a point \( (\xi_0,\omega_0) \), \( \xi_0 \neq \infty, \omega_0 \in B_1(0) \), is a part of polytope cone with the vertex at \( \xi_0 \), inside a small polytope domain containing \( \xi_0 \) and containing the point \( \xi_0 + \varepsilon \omega_0 \) for some small \( \varepsilon > 0 \). Neighbourhoods of a point \( (\infty,\omega_0) \), \( \omega_0 \in B_1(0) \), is a part of polytope cone with the vertex at \( 0 \), open to infinity, outside a big polytope neighbourhood of \( 0 \) and containing the point \( R\omega_0 \) for some large \( R > 0 \).
- The localizing class \( \Delta_{(\xi_0,\omega_0)} \), \( (\xi_0,\omega_0) \in \mathbb{R}_n^* \times B_1(0) \), comprizes operators \( W^0_{h,G} \), with symbols \( h(\xi) \) which are characteristic functions of all kind of non-empty polytope neighbourhoods of \( (\xi_0,\omega_0) \). It is clear, that the system of localizing classes \( \{\Delta_{(\xi_0,\omega_0)}\} \) \( (\xi_0,\omega_0) \in \mathbb{R}_n^* \times B_1(0) \) are covering in the algebra of linear bounded operators

\[
\mathcal{B}(H^s_p(\mathbb{R}^n)).
\]

Indeed, from any collection of operators \( \left\{ W^0_{h(\xi_0,\omega_0),G} \right\} \) \( (\xi_0,\omega_0) \in \mathbb{R}_n^* \times B_1(0) \) we can select a finite number of operators \( \left\{ W^0_{h_k,G} \right\} \) \( k = 1 \) such that the sum

\[
\sum_{k=1}^{n} W^0_{h_k,G} = W^0_{h_0,G}, \quad h_0(\xi) := \sum_{k=1}^{n} h_k(\xi)
\]
is invertible \((h_0 \in \text{PC}^{m \times m}_p(\mathbb{R}^n)\) is elliptic) and the inverse reads \(W_{h_0^{-1},G}^0\), because \(h_0^{-1} \in \text{PC}^{m \times m}_p(\mathbb{R}^n)\).

Obviously, operators from \(\Delta_{\xi_0,\omega_0}\) commute with convolutions \(W_{h,G}^0, b \in \text{PC}^{m \times m}_p(\mathbb{R}^n)\), because they are all convolutions.

Using the well known Hörmander’s inequality (cf. [43])

\[
\|a\|_{\mathfrak{M}(\mathbb{R}^n)} = \|a\|_{\mathfrak{M}(\mathbb{R}^n)} = \|W_{a,G}^0\|_{L^p(G)} = \|W_{a,\mathbb{R}^n}^0\|_{L^p(\mathbb{R}^n)} \\
\leq \left[\sup_{\xi \in \mathbb{R}^n} |a(\xi)|\right]^\theta \|a\|_{\mathfrak{M}(\mathbb{R}^n)}^{1-\theta},
\]

(6.12)

\(p \neq 2, \quad p \in (r,r'), \quad p' = \frac{p}{p-1}, \quad \theta = \frac{2 |r-p|}{p |r-2|},\)

• we prove the local equivalence

\[
W_{a,G}^{0,\Delta_{\xi_0,\omega_0}} W_{a(\xi_0,\omega_0),G}^{0} = a(\xi_0,\omega_0)I, \quad (\xi_0,\omega_0) \in \mathbb{R}_+^n \times B_1(0),
\]

(see (6.9) for \(a(\xi_0,\omega_0)\)) which means that

\[
\inf_{W_{a,G}^{0,\Delta(\xi_0,\omega_0)}} \|W_{h,G}^0 [W_{a,G}^0 - a(\xi_0,\omega_0)I] L^p(G)\| \\
= \inf_{W_{h,G}^{0,\Delta(\xi_0,\omega_0)}} \|W_{h[G(a-a(\xi_0,\omega_0))]}^0 L^p(G)\| = 0.
\]

According to the main theorem on localization operators, \(W_{a,G}^0\) is Fredholm if and only if the local representatives (which are in our case multiplication by constants \(a(\xi_0,\omega_0)I\)) are locally invertible for all \((\xi_0,\omega_0) \in \mathbb{R}_+^n \times B_1(0)\): There exists operator \(R(\xi_0,\omega_0)\) and an element of the localizing class \(W_{h,G}^0 \in \Delta_{\xi_0,\omega_0}\) such that the equalities hold:

\[
W_{h,G}^0 a(\xi_0,\omega_0) R(\xi_0,\omega_0) = R(\xi_0,\omega_0) a(\xi_0,\omega_0) W_{h,G}^0 = W_{h,G}^0 \quad \forall (\xi_0,\omega_0) \in \mathbb{R}_+^n \times B_1(0).
\]

But the local invertibility of the multiplication by a constant \(a(\xi_0,\omega_0)I\) coincides with its global invertibility and coincides with the ellipticity condition at the point \(a(\xi_0,\omega_0) \neq 0\). Thus, ellipticity means Fredholness, but since for an elliptic symbol \(a(\xi)\) the inverse is also a multiplier \(a^{-1} \in \text{PC}^{m \times m}_p(\mathbb{R}^n)\), the inverse to \(W_{a,G}^0\) is \(W_{a^{-1},G}^0\). A similar property holds for the operator \(W_{a,\mathbb{R}^n}^0\).

\[\square\]

**Definition 6.9** (A class of homogeneous of order zero symbols). For a function \(a(x,\xi)\) satisfying conditions (6.8), we define the following limit functions

\[
a_{\infty}(x,\xi) := \lim_{\lambda \to \infty} a(x,\lambda \xi)
\]

which are homogeneous of order 0 in \(\xi\): \(a_{\infty}(x,\theta \xi) = a_{\infty}(x,\xi)\) for all \(\theta > 0, x \in \mathbb{R}^n, \xi \in \mathbb{R}^n\).

Prior we formulate and prove the main theorem of the present section, we expose the result about local operators, due to V.N. Semenyuta and A.V. Kozak and exposed with the proof in [2, Proposition 5.7] and in [68, Definition 2.5, Theorem 2.5].
**Definition 6.10** (Local type operators). For any $s \in \mathbb{R}$, let $\mathcal{C}(H^s_p(G))$ be the family of compact operators on $H^s_p(G)$. An operator $A \in \mathcal{B}(H^s_p(G))$ is of local type if for all $v_1, v_2 \in C^\infty(G)$ such that $\text{supp} \ v_1 \cap \text{supp} \ v_2 = \emptyset$, the localisation operator $v_1 Av_2 I$ is a compact operator on $H^s_p(G)$, that is $v_1 Av_2 I \in \mathcal{C}(H^s_p(G))$. The class of local type operators on $\mathcal{B}(H^s_p(G))$ will be denoted by $\mathcal{B}(LH^s_p(G))$.

In the following proposition the class of local type operators is characterised by a commutator criterion.

**Proposition 6.11.** Operator $A \in \mathcal{B}(LH^s_p(G))$ is of local type if and only if the commutator $[A, bI] := AbI - bA$ is compact $[A, bI] \in \mathcal{C}(LH^s_p(G, d\mu_G))$ for all smooth functions $b \in C^\infty(G)$, where $G$ is the one point compactifications of the Lie group $G$, namely, $G = G \cup \{\infty\}$.

**Definition 6.12.** Let $C^0_p(G, PC_p(\mathbb{R}^n))$ denote the class of symbols of ΨDOs which are closure (in the norm of $\mathcal{B}(L^p(G))$) of operators

$$
\sum_{k=1}^n a_k(x)W^0_{g_k,G}, \quad a_k \in C(G), \ g_k \in PC_p(\mathbb{R}^n), k = 1, 2, \ldots, N,
$$

when $n = 1, 2, \ldots$ is not fixed.

**Remark 6.13.** Note that, due to Proposition 6.11, the commutator of the operators $a(x, \mathcal{D}), b(x, \mathcal{D})$ with symbols from the class $C^0_p(G, PC_p(\mathbb{R}^n))$, is compact

$$
a(x, \mathcal{D})b(x, \mathcal{D}) - b(x, \mathcal{D})a(x, \mathcal{D}) \in \mathcal{C}(L^p(G)).
$$

Indeed, for this we only need to check that the commutator $[aW^0_{b,G}, W^0_{b,G}aI] := aW^0_{b,G} - W^0_{b,G}aI$, $a \in \Delta_{\mathcal{D}_0}$, $b \in PC_p(\mathbb{R}^n)$ is compact. It is easy to ascertain that $W^0_{b,G}$ are operators of local type: $v_1 W^0_{b,G}v_2 I \in \mathcal{C}(LH^s_p(G, d\mu_G))$ for any functions $v_1$ and $v_2$ with disjoint supports, because its kernel $k(x, y)$ is $C^\infty$-smooth and uniformly bounded. Then, due to Proposition 6.11, the commutator is also compact $[aW^0_{b,G}, W^0_{b,G}aI] \in \mathcal{C}(LH^s_p(G, d\mu_G))$.

**Theorem 6.14.** Let $1 < p < \infty$ and let a matrix symbol $a = a(x, \xi) : G \times \mathbb{R}^n \to \mathbb{C}^{m \times m}$ belong to the class $C^0_p(G, PC_p(\mathbb{R}^n))$.

The ΨDO

$$
a_G(x, \mathcal{D}) : L^p(G) \to L^p(G)
$$

is Fredholm if and only if the symbol is elliptic

$$
\inf_{(x, \xi, \omega) \in G \times \mathbb{R}^n \times \mathbb{B}_1(0)} |a(x, \xi, \omega)| > 0.
$$

**Proof.** We will apply again Gohberg-Krupnik’s localization method (see Theorem 6.8). Let us define localizing class $\Delta_{(x_0, \xi_0, \omega_0)}$ at the point $(x_0, \xi_0, \omega_0) \in \mathbb{R}^n \times \mathbb{B}_1(0)$, consisting of operators $gW^0_{h,G}$, where $|g(x)| \leq 1$ is a smooth function, equal 1 in some neighbourhood of $x_0 \in G$ and $h(\xi)$ is described in the proof of Theorem 6.8.
We can prove invertibility of the factor-class \([a_G(x, \mathcal{D})]\) in the Kalkin factor-algebra of linear bounded operators \(\mathcal{B}(L^p(G))/\mathcal{C}(L^p(G))\), because invertibility of the class in the Kalkin algebra is equivalent to the Fredholmness of the operator in the algebra of linear bounded operators \(\mathcal{B}(L^p(G))\).

Since operators from localizing classes \(\Delta_{x_0,\xi_0,\omega_0}\) have the form (6.16), their commutators with \(a_G(x, \mathcal{D})\) are compact (cf. (6.15)), i.e. the corresponding classes commute in the Kalkin algebra.

It is clear, that the system of localizing classes \(\{\Delta_{x_0,\xi_0,\omega_0}\}_{(x_0,\xi_0,\omega_0)\in G\times \mathbb{R}_1^n \times \mathbb{R}_1(0)}\) is covering in the Kalkin algebra \(\mathcal{B}(L^p(G))/\mathcal{C}(L^p(G))\). We prove easily the local equivalences:

\[
[a_G(x, \mathcal{D})]^{\Delta_{(x_0,\xi_0,\omega_0)}} \sim \left[ W_{a(x_0,\xi_0,\omega_0),G}^0 \right] = [a(x_0,\xi_0,\omega_0)I], \quad (x_0,\xi_0,\omega_0) \in G \times \mathbb{R}_1^n \times \mathbb{R}_1(0).
\]

According the main theorem on localization operator \(a_G(x, \mathcal{D})\) is Fredholm if and only if the local representatives are locally invertible for all \(x_0 \in G\). But the local invertibility of the class of multiplication by a constant \([a(x_0,\xi_0,\omega_0)I]\) coincides with its global invertibility and coincides with the ellipticity condition at the point \(a(x_0,\xi_0,\omega_0) \neq 0\). Thus, ellipticity means Fredholmness.

**Remark 6.15.** Observe that similarly to \(C^0_p(G, \mathbf{PC}_p(\mathbb{R}^n))\), one defines the class of symbols \(C^0_p(\mathbb{R}^n, \mathbf{PC}_p(\mathbb{R}^n))\) by taking coefficients \(a_k \in C(\mathbb{R}^n)\) in (6.14) and taking the approximation in the space \(\mathcal{B}(L^p(\mathbb{R}^n))\).

A similar result to Theorem 6.14 holds for the operator

\[ a_{\mathbb{R}^n}(x, D) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \quad (6.17) \]

with a symbol from the class \(C^0_p(\mathbb{R}^n, \mathbf{PC}_p(\mathbb{R}^n))\).

For the class of symbols \(C^{s,t}_p(G, \mathbf{PC}_p^s(\mathbb{R}^n))\) (and \(C^{s,t}_p(\mathbb{R}^n, \mathbf{PC}_p^s(\mathbb{R}^n))\)) obtained by closing the set of \(\Psi DOs\) (6.14) with smooth coefficients \(a_k \in C^\infty\) and symbols \((1 + \xi^2)^{-t}d_k \in \mathbf{PC}_p(\mathbb{R}^n)\) in the norm of \(\mathcal{B}(H^s_p(G, d\mu_G), H^{s-t}_p(G, d\mu_G))\) (in the norm of \(\mathcal{B}(H^s_p(\mathbb{R}^n), H^{s-t}_p(\mathbb{R}^n))\)), a theorem similar to Theorem 6.14 can be proved for the operator

\[
a_G(x, \mathcal{D}) : H^s_p(G) \to H^{s-t}_p(G) \quad \text{(for } a_{\mathbb{R}^n}(x, D) : H^s_p(\mathbb{R}^n) \to H^{s-t}_p(\mathbb{R}^n)).
\]

For the proof we first lift the operators to the space setting \(L^p \to L^p\) with the help of Bessel potentials and then apply Theorem 6.14.

**References**

1. Agranovich, M. S. Spectral properties of elliptic pseudodifferential operators on a closed curve Funct. Anal. Appl. 13, 279-281 (1971).
2. Böttcher, A. Krupnik, N. Silbermann, B. A general look at local principles with special emphasis on the norm computation aspect. Integr. Equ. Oper. Theory 11, 455-479 (1988).
3. Cardona, D. Estimativos \(L^2\) para una clase de operadores pseudodiferenciales definidos en el toro Rev. Integr. Temas Mat. 31(2), 147–152 (2013).
4. Cardona, D. Weak type \((1, 1)\) bounds for a class of periodic pseudodifferential operators. J. Pseudo-Differ. Oper. Appl. 5(4), 507–515, (2014).
5. Cardona, D. On the boundedness of periodic pseudo-differential operators, Monat. Math., 185(2), pp. 189–206, (2017).
6. Cardona, D. Besov continuity of pseudo-differential operators on compact Lie groups revisited. C. R. Math. Acad. Sci. Paris Vol. 355(5), pp. 533–537, (2017).
7. Cardona, D. Nuclear pseudo-differential operators in Besov spaces on compact Lie groups. J. Fourier Anal. Appl. 23(5), pp. 1238–1262, (2017).
8. Cardona, D. Continuity of pseudo-differential operators on Besov spaces on compact homogeneous manifolds, J. Pseudo-Differ. Oper. Appl. 9(4), pp. 861–880, (2018).
9. Cardona, D. Del Corral, C. The Dixmier trace and the non-commutative residue for multipliers on compact manifolds. In: Georgiev V. Ozawa T. Ruzhansky M. Wirth J. (eds) Advances in Harmonic Analysis and Partial Differential Equations. Trends in Mathematics. Birkhäuser, Cham.
10. Cardona, D. Delgado, J. Ruzhansky, M. Dixmier traces, Wodzicki residues, and determinants on compact Lie groups: the paradigm of the global quantisation. preprint.
11. Cardona, D. Delgado, J. Ruzhansky, M. Drift diffusion equations with fractional diffusion on compact Lie groups. to appear in J. Evol. Equ. arXiv:2205.02320.
12. Cardona, D. Delgado, J. Ruzhansky, M. Determinants and Plemelj-Smithies formulas, Monatsh. Math. arXiv:2012.13216.
13. Cardona, D. Delgado, J. Ruzhansky, M. $L^p$-bounds for pseudo-differential operators on graded Lie groups. J. Geom. Anal. Vol. 31, 11603-11647, (2021).
14. D. Cardona, V. Kumar, $L^p$-boundedness and $L^p$-nuclearity of multilinear pseudo-differential operators on $\mathbb{Z}^n$ and the torus $\mathbb{T}^n$. J. Fourier Anal. Appl. Vol. 25 (6), 2973–3017, (2019).
15. Cardona, D. Messiouene, R. Senoussaoui, A. Periodic Fourier integral operators in $L^p$ spaces. C. R. Math. Acad. Sci. Paris. 355(5), 547–553, 2021.
16. Cardona, D. Messiouene, R. Senoussaoui, A. $L^p$-bounds for Fourier integral operators on the torus. to appear in Complex Var. Elliptic Equ. arXiv:1807.09892.
17. Cardona, D. Ruzhansky, M. Multipliers for Besov spaces on graded Lie groups. C. R. Math. Acad. Sci. Paris. 355(4), pp. 400–405, (2017).
18. Cardona, D. Ruzhansky, M. Boundedness of pseudo-differential operators in subelliptic Sobolev and Besov spaces on compact Lie groups. arXiv:1901.06825.
19. Cardona, D. Ruzhansky, M. Subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups. submitted. arXiv:2008.09651.
20. Cardona, D. Ruzhansky, M. Fourier multipliers for Triebel-Lizorkin spaces on compact Lie groups. Collect. Math. Vol. 73, 477–504, (2022). arXiv:2101.12314.
21. Cardona, D. Ruzhansky, M. Björk-Sjölin condition for strongly singular convolution operators on graded Lie groups, Math. Z. arXiv:2205.03456.
22. Coquand, T. Stolzenberg, G. The Wiener lemma and certain of its generalizations. Bull. Amer. Math. Soc. (N.S.) 24, no. 1, 1–9, (1991).
23. Dasgupta, A. Ruzhansky, M. The Gohberg lemma, compactness, and essential spectrum of operators on compact Lie groups. J. Anal. Math. 128, pp. 179–190, (2016).
24. Delgado, J.: $L^p$ bounds for pseudo-differential operators on the torus Operators Theory, advances and applications. 231, 103-116 (2012).
25. Delgado, J. Ruzhansky M. $L^p$-bounds for pseudo-differential operators on compact Lie groups, J. Inst. Math. Jussieu, 18, no. 3, pp. 531–559, (2019).
26. Delgado, J. Ruzhansky, M. Fourier multipliers, symbols, and nuclearity on compact manifolds. J. Anal. Math. 135, no. 2, pp. 757–800, (2018).
27. Delgado, J. Ruzhansky, M. Schatten classes and traces on compact groups. Math. Res. Lett. 24, no. 4, pp. 979–1003, (2017).
28. Delgado, J. Ruzhansky, M. Kernel and symbol criteria for Schatten classes and $r$-nuclearity on compact manifolds. C. R. Math. Acad. Sci. Paris 352, no. 10, pp. 779–784, (2014).
29. Delgado, J. Ruzhansky, M. $L^p$-nuclearity, traces, and Grothendieck-Lidskii formula on compact Lie groups. J. Math. Pures Appl. (9) 102, no. 1, pp. 153–172, (2014).
30. De Moraes, W. A. A. Regularity of solutions to a Vekua-type equation on compact Lie groups. Ann. Mat. Pura Appl. DOI: 10.1007/s10231-021-01120-7.
31. Deza, A. Deza, A. Guan, Z. et al. Distance between vertices of lattice polytopes. Optim Lett 14, 309–326 (2020).
32. Duduchava, R. Convolution equations on the Lie group $(-1, 1)$. arXiv:2208.08765.
33. Duduchava, R. On bisingular integral operators with discontinuous coefficients, Matematicheskiĭ Sbornik 101, 4, 584-609, 1976 (Russian); Translated in English: Mathematics USSR, Sbornik 30, 515-537, (1976).
34. Duduchava, R. Integral equations with fixed singularities. Teubner, Leipzig, (1979).
35. Fedosov, B. V. Analytic formulas for the index of elliptic operators, Transactions of Moscow Mathematical Society 30, 159-241, (1974).
36. Fefferman, C. $L^p$-bounds for pseudo-differential operators, Israel J. Math. 14, 413–417, (1973).
37. Fefferman, C. Phong, D. H. On Positivity of Pseudo-Differential Operators, Proceedings Nat. Acad. Sci. USA, 75, 4673–4674, (1978).
38. Fischer, V. Ruzhansky, M. Quantization on nilpotent Lie groups, Progress in Mathematics, Vol. 314, Birkhäuser, xiii+557pp, 2016.
39. Folland, G. Stein, E. Hardy Spaces on Homogeneous Groups, Princeton University Press, Princeton, N.J. 1982.
40. Garetto, C. Ruzhansky, M. Wave equation for sum of squares on compact Lie groups, J. Differential Equations. 258, 4324–4347, (2015).
41. Gohberg, I. Krupnik, N. One-Dimensional Linear Singular Integral Equations, I-II, vol. 53-54 of Operator Theory, Advances and Applications, Birkhäuser Verlag, Basel, (1979).
42. Grushin, V. V. Pseudodifferential operators on Rn with bounded symbols. Funct. Anal. Appl. 4, 202–212, (1970).
43. Hörmander, L. Estimates for translation invariant operators in $L_p$ spaces, Acta Mathematica, 93–140, (1960).
44. Hörmander, L. Pseudo-differential operators and non-elliptic boundary problems, Ann. of Math. 83(2), 129–209, (1966).
45. Hörmander, L. The Analysis of the linear partial differential operators Vol. III. Springer-Verlag, (1985).
46. Mântoiu, M. Ruzhansky, M. Quantizations on nilpotent Lie groups and algebras having flat coadjoint orbits. J. Geom. Anal. 29, no. 3, pp. 2823–2861, (2019).
47. Mântoiu, M. Ruzhansky, M. Pseudo-differential operators, Wigner transform and Weyl systems on type I locally compact groups. Doc. Math. 22, pp. 1539–1592, (2017).
48. Mclean, W. M. Local and Global description of periodic pseudo-differential operators, Math. Nachr. 150, 151-161, (1991).
49. Molahajloo, S. A characterization of compact pseudo-differential operators on $S^1$ Oper. Theory Adv. Appl. Birkhäuser/Springer Basel AG, Basel. 213, 25-29 (2011).
50. Molahajloo, S. A characterization of compact SG pseudo-differential operators on $L^2(R^n)$. Math. Model. Nat. Phenom. 9(5), 239-243, (2014).
51. Molahajloo, S. Wong, M. W. Ellipticity, Fredholmness and spectral invariance of pseudo-differential operators on $S^1$. J. Pseudo-Differ. Oper. Appl. 1 183-205 (2010)
52. Noether, F. ¨Uber eine Klasse singul¨arer Integralgleichungen. Math. Ann. 82, 42-63. (1921)
53. Nursultanov, E. Ruzhansky, M. Tikhonov, S. Nikolskii inequality and functional classes on compact Lie groups. Funct. Anal. Appl. 49, pp. 226–229, (2015).
54. Nursultanov, E. Ruzhansky, M. Tikhonov S. Nikolskii inequality and Besov, Triebel-Lizorkin, Wiener and Beurling spaces on compact homogeneous manifolds, Ann. Sc. Norm. Super. Pisa Cl. Sci. XVI, pp. 981–1017, (2016).
55. Petrov, V. E. Integral transform on a segment, Problemy Matemat. Analiza 31 (2005), 67–95; English transl. J. Math. Sci. 132(4), 451-481, (2006).
56. Petrov V. E. The generalized singular Tricomi equation as a convolution equation, Dokl. Ros. Akad. Nauk 411 (2006), no. 2, 1–5; English transl. Dokl. Math. 74(3), 901–905.
GLOBAL PSEUDO-DIFFERENTIAL OPERATORS ON G = (−1, 1)^n

57. Rodriguez Torijano, C. A. Ruzhansky, M. Subelliptic wave equations with log-Lipschitz coefficients. arXiv:2007.09396.
58. Ruzhansky, M. Tokmagambetov, N. Nonharmonic analysis of boundary value problems, Int. Math. Res. Notices, (12), 3548–3615, (2016).
59. Ruzhansky, M. Tokmagambetov, N. Nonharmonic analysis of boundary value problems without WZ condition, Math. Model. Nat. Phenom., 12, 115–140, (2017).
60. Ruzhansky, M. Turunen, V. Pseudo-differential Operators and Symmetries: Background Analysis and Advanced Topics Birkhäuser-Verlag, Basel, 2010.
61. M. Ruzhansky, V. Turunen, Quantization of pseudo Differential operators on the torus, J. Fourier. Annal. Appl, Vol. 16, pp. 943–982, Birkhäuser Verlag, Basel, (2010).
62. Ruzhansky, M. Turunen, V. Global quantization of pseudo-differential operators on compact Lie groups, SU(2) and 3-sphere, Int. Math. Res. Not. IMRN. 11, pp. 2439–2496, (2013).
63. Ruzhansky, M. Wirth, J. Global functional calculus for operators on compact Lie groups, J. Funct. Anal. 267, 144–172, (2014).
64. Ruzhansky, M. Wirth, J. L^p Fourier multipliers on compact Lie groups, Math. Z. 280, pp. 621–642, (2015).
65. Ruzhansky, M. Turunen, V. Wirth J. Hörmander class of pseudo-differential operators on compact Lie groups and global hypoellipticity, J. Fourier Anal. Appl. 20, pp. 476–499, (2014).
66. Turunen, V. Vainikko, G. On symbol analysis of periodic pseudodifferential operators, Z. Anal. Anwendungen, 17, pp. 9–22, (1998).
67. Simonenko, I. A new general method of investigating linear operator equations of singular integral equation type. I. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 29, 1965, 567-586.
68. Simonenko, I. B. and Chin Ngok Min: The local method in the theory of one-dimensional singular integral equations with piecewise continuous coefficients. Izdo Rostov-on-Don State University, (Russian), (1986).
69. Shargorodsky, E. Some remarks on the boundedness of pseudodifferential operators Math. Nachr., 183, 229-2731, (1997).
70. Suslina T. A. Petrov V.E. Regularity of the solution of the Prandtl equation. arXiv:2008.06715.
71. Triebel H. Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam 1978, 2-nd edition, Johann Ambrosius Barth Verlag, Heidelberg–Leipzig, (1995).

DUVÁN CARDONA:
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
GHENT UNIVERSITY, BELGIUM
E-mail address duvanc306@gmail.com, duvan.cardonasanchez@ugent.be

ROLAND DUDUCHAVA:
INSTITUTE OF MATHEMATICS
THE UNIVERSITY OF GEORGIA
TBILISI, GEORGIA
AND
A. RAZMADZE MATHEMATICAL INSTITUTE
TBILISI STATE UNIVERSITY
GEORGIA
E-mail address roldud@gmail.com, r.duduchava@ug.edu.ge

ARNE HENDRICKX
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
GHENT UNIVERSITY, BELGIUM
E-mail address: arnhendr.Hendrickx@UGent.be

MICHAEL RUZHANSKY:
DEPARTMENT OF MATHEMATICS: ANALYSIS, LOGIC AND DISCRETE MATHEMATICS
GHENT UNIVERSITY, BELGIUM
AND
School of Mathematical Sciences
Queen Mary University of London
United Kingdom
E-mail address michael.ruzhansky@ugent.be, m.ruzhansky@qmul.ac.uk