The static energy of a quark-antiquark pair from Laplacian eigenmodes

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Motivation

- calculate the static potential energy with high resolution
  - matching the lattice QCD potential with the perturbative potential to determine $\Lambda_{\overline{MS}}$ in Fourier space, e.g., Karbstein et al. (2014)
  - observation of string breaking in QCD, e.g., Bali et al. (2008), Bulava et al. (2019)

$\Rightarrow$ we have to work with off-axis separated quarks

- the spatial part of the Wilson loop has to go over stair-like paths through the lattice $\rightarrow$ not unique, computationally expensive

$\Rightarrow$ alternative operator which ensures gauge covariance of the quark-anti-quark $Q(\vec{x})\overline{Q}(\vec{y})$ trial state

- required gauge transformation behavior:

$$U^I_s(\vec{x}, \vec{y}) = G(\vec{x})U_s(\vec{x}, \vec{y})G^\dagger(\vec{y})$$
Consider the 3D covariant lattice Laplace operator:

\[
\Delta V = \frac{1}{a^2} \left[ U_x^\dagger (x - a, y, z) V(x - a, y, z) - 2V(x) + U_x(x) V(x + a, y, z) \\
+ U_y^\dagger (x, y - a, z) V(x, y - a, z) - 2V(x) + U_y(x) V(x, y + a, z) \\
+ U_z^\dagger (x, y, z - a) V(x, y, z - a) - 2V(x) + U_z(x) V(x, y, z + a) \right]
\]

transformation behavior: \( \Delta' = G(x) \Delta G^\dagger (y) \)

consider \( V(x) \) an eigenvector: \( \Delta V(x) = \lambda V(x) \)

\[
\Delta' V'(x) = \lambda V'(x) \\
G(x) \Delta G^\dagger (x) V'(x) = \lambda V'(x) \\
\Delta G^\dagger (x) V'(x) = \lambda G^\dagger (x) V'(x)
\]

\( V(x) \) and \( G^\dagger (x) V'(x) \) are members of the same eigen-space
Static $Q\bar{Q}$ pair, Wilson loop alternative...

- idea taken from Neitzel et al. (2016) SU(2)
- SU(3): the eigenvalues are in general non-degenerate
- spatial Wilson line: $U'_s(\vec{x}, \vec{y}) = G(\vec{x})U_s(\vec{x}, \vec{y})G^\dagger(\vec{y})$
  $$V'(\vec{x})V'^\dagger(\vec{y}) = G(\vec{x})V(\vec{x})V'^\dagger(\vec{y})G^\dagger(\vec{y})$$
- Wilson loop of size $(R = |\vec{x} - \vec{y}|) \times (T = |t_1 - t_0|)$
  $$W(R, T) = U_t(\vec{x}; t_0, t_1)U_s(\vec{x}, \vec{y}; t_1)U^\dagger_t(\vec{y}; t_0, t_1)U^\dagger_s(\vec{x}, \vec{y}; t_0)$$
  $$\rightarrow U_t(\vec{x}; t_0, t_1)V_j(\vec{x}, t_1)V'^\dagger_j(\vec{y}, t_1)U^\dagger_t(\vec{y}; t_0, t_1)V_i(\vec{y}, t_0)V^\dagger_i(\vec{x}, t_0)$$
Simulations

- $24^3 \times 48, \beta = 5.3, N_f = 2, \kappa = 0.13270, a = 0.0658$ fm
- (on-axis) Wilson loops from 4646 measurements, 4D HYP
- Laplace states from 1161 measurements, 20 APE (0.5) for Lanczos, 1 4D HYP for temporal Wilson lines
- static potential $aV(R) = \lim_{T \to \infty} \log\left[\frac{W(R, T)}{W(R, T + 1)}\right]$
- at $R = 12a$ the force between $Q\bar{Q}$ must vanish due to symmetry
- more eigenvectors gives earlier plateau and increase precision
Gaussian profiles

\[ W_{kl}(R, T) = \sum_{i,j}^{N_v} N_{kl}(\lambda_i, \lambda_j) \sum_{\vec{x}, t_0} \left\langle V_i^{\dagger}(\vec{x}, t_0) U_t(\vec{x}; t_0, t_1) V_j(\vec{x}, t_1) U_t^{\dagger}(\vec{y}; t_0, t_1) V_i(\vec{y}, t_0) \right\rangle \]

Gaussian profile functions:
\[ N_{kl}(\lambda_i, \lambda_j) = \exp(-\lambda_i^2/2\sigma_k^2) \exp(-\lambda_j^2/2\sigma_l^2) \] using 7 different \( \sigma \) values

⇒ prune \( W_{kl} \) using 3 most significant singular vectors \( u_i \) at \( t_0 \)

... improves stability and keeps useful operators

⇒ pruned ("optimal") profiles \( u_{i,j} \exp(-\lambda^2/\sigma_j^2) \)
Improved Results

Solve generalized eigenvalue problems for

- correlation matrix of Wilson loops with 3 spatial smearing levels
- Laplace trial states $W_{kl}$ (or pruned version $\bar{W}_{ij} = u_i^\dagger W_{kl} u_j$)

\[ E_0(R,T) \]

\[ aV(R) \]

\[ \begin{align*}
\text{Wilson loops} & \quad \text{Laplace states} \\
\end{align*} \]

- on-axis Wilson loops for 3 spatial smearing levels (0,10,20HYP) are equally expensive as the calculation of 100 Laplacian eigenvectors and Laplace states with 3 Gaussian profiles including off-axis distances!
Fractional overlaps with the ground state

| $R$ | #\(\lambda = 1\) | 8   | 64  | 200 | Gauss | Wloop | HYP2 |
|-----|------------------|-----|-----|-----|-------|-------|------|
| 1   | 0.773(3)         | 0.945(1) | 0.970(1) | 0.982(1) | 0.993(1) | 0.921(1) | 0.983(1) |
| 2   | 0.747(4)         | 0.929(2) | 0.964(1) | 0.987(1) | 0.989(1) | 0.891(1) | 0.978(1) |
| 3   | 0.723(4)         | 0.878(2) | 0.984(2) | 0.986(1) | 0.988(1) | 0.867(1) | 0.972(2) |
| 4   | 0.726(5)         | 0.874(3) | 0.921(2) | 0.984(2) | 0.986(2) | 0.841(2) | 0.965(3) |
| 5   | 0.637(6)         | 0.871(4) | 0.979(3) | 0.982(3) | 0.983(3) | 0.813(2) | 0.956(5) |
| 6   | 0.629(6)         | 0.869(4) | 0.978(4) | 0.980(3) | 0.981(3) | 0.793(3) | 0.948(6) |
| 7   | 0.619(7)         | 0.869(5) | 0.977(4) | 0.979(4) | 0.987(4) | 0.772(3) | 0.934(7) |
| 8   | 0.598(8)         | 0.862(6) | 0.972(5) | 0.970(4) | 0.964(4) | 0.745(4) | 0.953(8) |
| 9   | 0.572(8)         | 0.857(6) | 0.960(5) | 0.934(4) | 0.963(3) | 0.708(4) | 0.947(9) |
| 10  | 0.540(9)         | 0.840(7) | 0.965(5) | 0.931(5) | 0.95(1)  | 0.671(5) | 0.94(1)  |
| 11  | 0.426(9)         | 0.807(7) | 0.943(6) | 0.93(1)  | 0.956(9) | 0.649(4) | 0.93(1)  |
| 12  | 0.33(7)          | 0.79(2)  | 0.94(1)  | 0.92(1)  | 0.95(1)  | 0.64(2)  | 0.92(1)  |

\(t\)-average over mass-plateau region of

\[
\frac{W(R, t)}{W(R, t_0)} \frac{\cosh \left( \left( \frac{T}{2} - t_0 \right) m_0 \right)}{\cosh \left( \left( \frac{T}{2} - t \right) m_0 \right)}
\]
Conclusions & Outlook

✓ alternative operator for a static quark-anti-quark pair based on Laplacian eigenmodes

✓ improved version (several eigenvectors weighted with Gaussian profiles) gives earlier plateau and better signal

✓ much higher resolution of the potential energy as off-axis distances basically come "for free"

✓ implementation of static-light (charm) correlator using "perambulators" $V(t_1)D^{-1}V(t_2)$ from distillation framework (quark field smearing via projection $\psi \rightarrow VV^\dagger\psi$), see Knechtli et. al (2022), ⇒ talk by J.-A. Urrea-Niño on hadron spectroscopy

❖ putting together building blocks for observation of string breaking in QCD (mixing matrix of static and light quark propagators)

❖ also working on hybrid static potentials (hybrid meson masses), instead of "gluonic handles" (excitations) use derivatives of $V$
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\[ Q(T) = \sum_{i,j} \sum_{\vec{x};t_0} \langle V_i^\dagger(\vec{x}, t_0)U_t(\vec{x}; t_0, t_1)V_j(\vec{x}, t_1) \rangle \]

- **SVD:** \( W = UDV^\dagger; \) \( U, V \) unitary, columns are orthonormal bases.
- **GEVP:** \( W(t)v_k = \rho_k W(t_0)v_k, \) \( \rho_k \) give effective energies
- \( u_i \) or \( v_k \) can be used to get 'optimal' profiles for energy states
Bonus: Static-light (charm) meson

\[ C(t) = - \sum_{t_0, i, j} \left\langle \text{tr}_d \{ [V_i^\dagger D^{-1} V_j](t_0 + t, t_0) P_+ \} \right\rangle \]

- with light (charm) perambulators \( V^\dagger D^{-1} V \)
- HYP smearing of temporal links in \( U_t \) removes free energy

R. Höllwieser, The static potential from Laplacian eigenmodes