UNIFORM BOUNDS ON GROWTH IN O-MINIMAL STRUCTURES

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Abstract. We prove that a function definable with parameters in an o-minimal structure is bounded away from $\infty$ as its argument goes to $\infty$ by a function definable without parameters, and that this new function can be chosen independently of the parameters in the original function. This generalizes a result in [FM05]. Moreover, this remains true if the argument is taken to approach any element of the structure (or $\pm \infty$), and the function has limit any element of the structure (or $\pm \infty$).

1. Introduction

We begin with a special case of the main result of this paper.

Proposition 1.1. Let $M$ be an o-minimal expansion of a dense linear order $(M, <)$. Let $f : M^n \times M \rightarrow M$ be definable in $M$. Then there exist functions $g : M \rightarrow M$ and $h : M^n \rightarrow M$ definable in $M$ such that $f(x, t) \leq g(t)$ for all $x \in M^n$ and $t > h(x)$. Moreover, if $M'$ is the prime model containing the parameters used to define $f$, then $g$ and $h$ are defined over $M'$.

This was already known under the additional assumption that $M$ expands an ordered group; see 3.1 of [FM05], which uses [vdDM96, C.4] and [MS98]. Here, we remove the need for the group structure. Indeed, we show something stronger.

Theorem 1.2. Let $M$ be an o-minimal expansion of a dense linear order $(M, <)$. Let $f$ be an $n + 1$-ary $M$-definable function with domain $A \times M$ for some $A \subseteq M^n$. Suppose that, for some $b \in M \cup \{\infty\}$ and all $x \in A$, we have $\lim_{t \to b^-} f(x, t) = b$ and $f(x, t) < b$. Then there exist functions $g : M \rightarrow M$ and $h : A \rightarrow M$ definable in $M$ such that $h(A) < b$ and for $t \in (h(x), b)$, we have $g(t) \in [f(x, t), b]$. Moreover, if $M'$ is the prime model containing the parameters used to define $f$, then $g$ and $h$ are defined over $M'$.

If $M$ expands a field, then using the maps $1/(b - t)$ and $b - 1/t$ this theorem follows easily from 3.1 of [FM05]. When $b = \infty$ and $M$ expands an ordered group, this is essentially 3.1 of [FM05]. However, this result is new if $M$ does not expand a group, or if $M$ does not expand a field and $b \in M$.

Corollary 2.4 strengthens the theorem slightly, allowing $f$ to take any value as its limit, from either direction. Note that if Corollary 2.4 is applied in the case that $f$ is definable in the prime model of an o-minimal theory, this shows that any definable function is bounded as it approaches a limit by one definable in the prime model, assuming the limit is in the prime model or $\pm \infty$.

We use the terminology of [Tre05]: the definable 1-types in an o-minimal theory are called “principal.” To each principal type over a structure $M$ is associated a unique element $a \in M \cup \{\pm \infty\}$ to which it is “closest,” in the sense that no elements of $M$ lie between $a$ and any realization of the type. We say that a principal type

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Proof. If \( a \) is "principal above/below/near \( b \)," we write \( \langle a_1, \ldots, a_n \rangle \) to denote the tuple of length \( n \) having the element \( a_i \) as its \( i \)th component.

2. Results

Proof of Theorem 1.2. We first note that the theorem is equivalent to the following:

Claim 2.1. Let \( P \) be the prime model of the theory of \( M \), let \( b \in P \cup \{ \infty \} \), and let \( A \subseteq M^n \) be a \( \emptyset \)-definable set. Let \( f(x, t) : A \times M \to M \) be a \( \emptyset \)-definable function and \( a \in A \) a tuple, with \( \lim_{t \to b} f(a, t) = b \) and \( f(a, t) < b \) for all \( t < b \). Then there exists a \( \emptyset \)-definable function \( g : M \to M \) such that \( g(t) \in [f(a, t), b) \) for \( t \) sufficiently close to \( b \). Similarly, if the limit is taken as \( t \) approaches \( b \) from above and \( f \) approaches \( b \) from below, with \( b \in M \cup \{ -\infty \} \).

The theorem implies Claim 2.1 since the fact that \( g \) bounds \( f \) transfers to elementary extensions. Inversely, if the theorem failed, then we could add the parameters needed to define \( f \) to the language, so that \( f \) would become \( \emptyset \)-definable, and then by compactness we could find \( a \) in an elementary extension of \( P \) such that the claim failed. (Note that \( b \) is definable from the parameters used to define \( f \).) Therefore, we prove Claim 2.1.

Notation 2.2. Let \( w \) be an element in some elementary extension of \( M \) that realizes the principal type below \( b \) over \( M \). In other words, \( w \) is infinitesimally close to \( b \) with respect to \( M \). It is easy to see that all \( M \)-definable functions extend to this elementary extension, and that if \( \varphi \) is any \( \emptyset \)-definable (respectively \( \emptyset \)-definable) predicate, \( \varphi(w) \) holds if and only if \( \varphi(t) \) holds for all \( t \) in some interval \( (c, b) \), with \( c \in P \) (respectively \( c \in M \)). Thus, whenever we write \( \varphi(w) \), the reader should understand this as equivalent to "\( \varphi(t) \) for all \( t \) in some interval with right endpoint \( b \) and left endpoint definable over the same parameters used to define \( \varphi \)."

We go by induction on the length of \( a \), simultaneously for all o-minimal structures, all \( \emptyset \)-definable functions, and all tuples of appropriate length. Let \( f(x, t) \) and \( a \) satisfy the conditions of Claim 2.1 for some \( b \). If \( a = \langle a_1, \ldots, a_n \rangle \) with \( n \geq 1 \), we can add constants for \( a_1, \ldots, a_{n-1} \) to the language and use induction for the cases of \( n = 1 \) and 1 to prove the claim. Thus, we may suppose that \( a \) is a singleton. If \( a \in P \), then the claim is trivial, so suppose not.

We can use regular cell decomposition [vdD98, 2.19(2)] to ensure that \( f \) is monotone in \( x \) and increasing in \( t \) on its two-dimensional domain cell, \( C \), which we can take to be

\[
\{ \langle x, t \rangle \mid x \in (d_1, d_2) \land k(x) < t < b \},
\]

for some \( \emptyset \)-definable monotone function \( k \) and \( d_1, d_2 \in P \cup \{ \pm \infty \} \) (with \( d_1 < a < d_2 \)). We may also require that \( f(C) < b \).

The case where \( f(x, w) \) is constant in \( x \) at \( a \) is easy by standard o-minimality arguments, since then the value \( f(a, w) \) is definable from \( w \) without using \( a \). Thus, we may suppose that \( f(x, t) \) is non-constant in \( x \) at \( a \), for all \( t \in (k(a), b) \). Without loss of generality, assume that \( f \) is increasing in \( x \) on \( C \).

If \( \text{tp}(a) \) is not principal below \( d_2 \), then we can choose \( a' \in P \) with \( a < a' < d_2 \). Then \( f(a', t) > f(a, t) \) for \( t \in (\max\{k(a), k(a')\}, b) \), and so we are done. Thus, we may suppose that \( \text{tp}(a) \) is principal below \( d_2 \).

The proof relies on the following claim.

Claim 2.3. Let \( p \in S_1(\emptyset) \) be the principal type below \( b \). If there is no \( \emptyset \)-definable map between \( \text{tp}(a) \) and \( p \), then Claim 2.1 holds.

Proof. If \( k(a) \models p \), then \( k \) is the desired map between \( \text{tp}(a) \) and \( p \). Thus, we can assume that \( k(a) < c \) for some \( c \in P \) with \( c < b \). Increasing \( d_1 \) if necessary, we may
Corollary 2.4. Theorem 1.2 holds when $t = f$ in both coordinates, \(\langle x, t \rangle \in C\). Now consider the formula

\[ \varphi(t) := \sup \{ f(x, t) | x \in (d_1, d_2) \} = b. \]

First, suppose that \(\varphi(w)\) does not hold. Then, for any \(t\) sufficiently close to \(b\),

\[ \sup \{ f(x, t) | x \in (d_1, d_2) \} < b. \]

Let \(z(t)\) be this (uniformly \(t\)-definable) supremum. Then \(z(t) \in [f(a, t), b]\), and so Claim 2.1 holds. Thus, the case that remains to consider is when \(\varphi(w)\) does hold. We can then fix \(t_0 \in (c, b)\) with \(t_0 \in P\) such that \(\varphi(t_0)\) holds, and we have a \(\emptyset\)-definable map, \(f(x, t_0)\). We show \(f(a, t_0) \models p\). For any \(e \in P\) with \(e < b\), we can find \(r \in (d_1, d_2) \cap P\) such that \(f(r, t_0) \in (e, b)\), by \(\varphi(t_0)\). Since \(r < a\) (else \(a\) would not be principal below \(d_2\)) and \(f(x, t_0)\) is increasing in \(x\), we have \(f(a, t_0) > f(r, t_0) > e\). Thus, \(f(a, t_0) \models p\), witnessing the \(\emptyset\)-definable map between \(\text{tp}(a)\) and \(p\).

We now complete the proof of Claim 2.1. By Claim 2.3 we can assume that \(\text{tp}(a)\) is principal below \(b\). Then the domain cell \(C\) has the form

\[ \{ \langle x, t \rangle | x \in (d_1, b) \land k(x) < t < b \}. \]

If \(k(w) \geq f(a, w)\), then we are done, so we may assume that \(f(a, w) > k(w)\). Then we may increase \(d_1\) and suppose that for any \(x \in (d_1, b)\) we have \(f(x, w) > k(w)\), as well as \(f(x, w) > w\). Fix \(e \in P\) with \(e \in (d_1, b)\). We have \(f(e, w) > k(w)\). Then \(\langle w, f(e, w) \rangle \in C\). For any \(t \in \langle a, b \rangle\), since \(f(e, t) > t\) and \(f\) is increasing in both coordinates, \(f(t, f(e, t)) > f(a, t)\). So we are done, since \(f(t, f(e, t))\) is \(\emptyset\)-definable and \(f(t, f(e, t)) \in (f(a, t), b)\) for \(t\) sufficiently close to \(b\) — namely, for \(t \in \langle a, b \rangle\).

Corollary 2.4. Theorem 1.2 holds when \(\lim_{t \to b^+} f(x, t) = c\), with \(c \in M \cup \{ \pm \infty \}\) and \(f\) approaching \(c\) from either direction, with \(g\) and \(h\) now definable over the prime model containing \(c\) and the parameters defining \(f\).

Proof. Suppose that the limit is taken as \(t\) approaches \(b\) from below and that \(f(x, t)\) approaches \(c\) from above. The other cases are similar. Let \(P\) be the prime model of the statement. Choose \(a \in A \cap P^n\). Let \(\psi(t)\) denote the inverse of \(f(a, t)\), so \(\psi\) is \(\emptyset\)-definable. Then for any \(x \in M^n\), the limit of \(\psi(f(x, t))\) is \(b\) as \(t\) goes to \(b\), and this value approaches \(b\) from below. By Theorem 1.2 there are \(\emptyset\)-definable \(\tilde{g}\) and \(\tilde{h}\) with \(\tilde{g}(t) \in [\psi(f(x, t), b)\) for \(t \in (\tilde{h}(x), b)\). Then for \(t \in (\tilde{h}(x), b)\), we have \(f(a, \tilde{g}(t)) \in (c, f(x, t)]\). Since \(f(a, \tilde{g}(t))\) is still \(\emptyset\)-definable, \(f(a, \tilde{g}(t))\) is the desired function \(g\), and \(\tilde{h}\) is the desired function \(h\).

References

[FM05] Harvey Friedman and Chris Miller. Expansions of o-minimal structures by fast sequences. J. Symbolic Logic, 70(2):410–418, 2005.
[MS98] Chris Miller and Sergei Starchenko. A growth dichotomy for o-minimal expansions of ordered groups. Trans. Amer. Math. Soc., 350(9):3505–3521, September 1998.
[Tre05] Marcus Tressl. Model completeness of o-minimal structures expanded by Dedekind cuts. J. Symbolic Logic, 70(1):29–60, March 2005.
[vdD98] Lou van den Dries. tame Topology and O-minimal Structures. Cambridge University Press, 1998.
[vdDM96] Lou van den Dries and Chris Miller. Geometric categories and o-minimal structures. Duke Math. J., 84(2):497–539, August 1996.