On The Formality Theorem for the Differential Graded Lie Algebra of Drinfeld

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Abstract
We discuss the differential graded Lie algebra (DGLA) of Drinfeld modeled on the tensor algebra $\bigotimes U\mathfrak{g}$ of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ over any field $\mathbb{K}$ of characteristic zero. We explicitly analyze the first obstruction to the existence of the formality quasi-isomorphism for this DGLA. Our analysis implies the formality of the DGLA $\bigotimes U\mathfrak{b}$ of Drinfeld associated to the two-dimensional Borel algebra $\mathfrak{b}$.

1 Introduction

Formality of a DG algebra (or equivalence of this DG algebra to its homology in the homotopy category) can be thought of as a “no-go” result which roughly means that the DG algebra is poor as a homotopy algebraic structure. On the other hand, formality of a DG algebra provides us with remarkable correspondences between homotopy invariant structures associated to this DG algebra and its homology. These correspondences give a lot of interesting applications to deformation theory [2], [4], [5], [6], [11], [15], [19] and [21].

In this paper we discuss a DGLA which governs the triangular deformations of Lie algebras proposed and classified in [7] by Drinfeld\(^{1}\). The total space of the DGLA is the tensor algebra $\bigotimes U\mathfrak{g}$ of the universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ and the DGLA structure is defined by formulas ((16),(17),(19)) in the next section. With these definitions it is clear that triangular deformations of a Lie algebra $\mathfrak{g}$ are in a bijective correspondence with Maurer-Cartan elements of the DGLA $\bigotimes U\mathfrak{g}[[h]]$. Equivalent triangular deformations correspond to equivalent Maurer-Cartan elements and vice-versa. For this reason we refer to the DGLA structure ((16),(17),(19)) on $\bigotimes U\mathfrak{g}$ as the DGLA of Drinfeld.

In paper [2] by Calaque it is shown that for any Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ there exists a quasi-isomorphism from the graded Lie algebra $\bigwedge \mathfrak{g}$ to the DGLA $\bigotimes U\mathfrak{g}$ of Drinfeld associated to $\mathfrak{g}$. This result, in particular, implies Drinfeld’s theorems about quantization of triangular $r$-matrices [7]. However, Calaque’s proof based on

\(^{1}\text{See also papers [1], [3], [16], and [17], in which explicit triangular deformations of various types of Lie algebras are discussed.}\)
Kontsevich’s quasi-isomorphism [15] and Dolgushev’s globalization technique [4] does not apply to any field of characteristic zero.

In our paper we show that there are no obstructions to the existence of the first non-trivial structure map of an $L_\infty$ quasi-isomorphism from the $\bigwedge \mathfrak{g}$ to $\bigotimes \mathcal{U}_g$. Our analysis implies the formality theorem for the DGLA $\bigotimes \mathcal{U}_b$ of Drinfeld associated to the two-dimensional Borel algebra $\mathfrak{b}$.

The organization of the paper is as follows. Section 2 is preliminary. In this section we introduce the notations and terminology we use in this paper. In section 3 we reduce the question of formality of the DGLA of Drinfeld to an analysis of admissible cocycles of the graded Lie algebra $\bigwedge \mathfrak{g}$ with values in adjoint representation. In the fourth section, we show that the 2-cocycles which appear as obstructions to the formality of the DGLA of Drinfeld are exact. Using this fact, we prove the formality theorem for the DGLA $\bigotimes \mathcal{U}_b$ associated to the two-dimensional Borel algebra $\mathfrak{b} \subset sl_2(\mathbb{K})$. In the concluding section we discuss recent results related to our question.

**Notation**

Throughout the paper, the underlying field $\mathbb{K}$ is assumed to be of characteristic zero. We use $\Sigma$ to denote the suspension of an graded vector space, namely, tensoring with $\mathbb{K}$ at degree 1. Given a cooperad $\mathcal{O}$, we use $F_{\mathcal{O}}(V)$ to denote the cofree coalgebra generated by $V$. Given a vector space $V$, we use $\text{Lie}(V)$, $\text{E}(V)$ and $S(V)$ to denote the free Lie algebra, the exterior algebra and the symmetric algebra generated by $V$, respectively. Moreover, we use $S^n(V)$ to denote the subspace of $S(V)$ consisting of degree $n$ elements. We also use $V^*$ for the dual vector space of $V$.

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**2 Preliminaries.**

In this section, we introduce the notion of $L_\infty$-algebra structures, the differential graded Lie algebra(DGLA) of Drinfeld $\bigotimes \mathcal{U}_b$, and give the motivation for our question.

Given a vector space $V$, we can construct $F_{\text{cocomm}}(\Sigma^{-1}V)$ as the vector space

$$F_{\text{cocomm}}(\Sigma^{-1}V) = \bigoplus_{n=1}^{\infty} S^n(\Sigma^{-1}V)$$

with comultiplication $\Delta : F_{\text{cocomm}}(\Sigma^{-1}V) \mapsto \bigotimes^2 F_{\text{cocomm}}(\Sigma^{-1}V)$ given by

$$\Delta(v_0) = 0, \quad \forall v_0 \in V$$
and
\[ \Delta(v_0 \ldots v_n) = \sum_{k=0}^{n-1} \sum_{\theta} \text{sign}(\theta) \frac{1}{(n-k)!(k+1)!} v_{\theta(0)} \ldots v_{\theta(k)} \otimes v_{\theta(k+1)} \ldots v_{\theta(n)} \]

where \( \theta \) is permutation of \( n + 1 \) elements and \( \text{sign}(\theta) \) is the sign of permutation.

**Definition 1** A coderivation of \( \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V) \) is a map
\[ d : \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V) \mapsto \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V) \] (2)
such that
\[ \Delta d(X) = (d \otimes I + I \otimes d) \Delta X, \quad X \in \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V). \] (3)

We use \( \text{Coder}(\mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V)) \) to denote the space of all coderivations of \( \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V) \).

We have the following proposition from [9] that describes \( \text{Coder}(\mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V)) \).

**Proposition 1** ([9] proposition 2.14) Let \( \xi \) be the projection of \( \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V) \) to \( V \).

The map sending \( d \in \text{Coder}(\mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V)) \) to \( \xi \circ d \in \text{Hom}(\mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V), V) \) is a vector space isomorphism between \( \text{Coder}(\mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V)) \) and \( \text{Hom}(\mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V), V) \).

As a result, a coderivation \( d \) of degree \( k \) is uniquely determined by a semi-infinite collection of multi-linear maps
\[ d_n : S^n(\Sigma^{-1}V) \mapsto \Sigma^{-k-1}V \]
where \( d_n \) is the restriction of \( \xi \circ d \) to \( S^n(\Sigma^{-1}V) \). For example,
\[ d(v) = d_1(v), \quad d(v_1, v_2) = d_2(v_1, v_2) + d_1(v_1) \otimes v_2 + (-1)^{(k_1-1)(k_2-1)}d_1(v_2) \otimes v_1 \]
\[ v \in V, \quad v_1 \in V^{k_1}, \quad v_2 \in V^{k_2}. \]

Combining definition 4.2.14 [10] and proposition 4.2.15 [10], we have the following definition:

**Definition 2** A graded vector space \( V \) is endowed with an \( L_\infty \)-algebra structure if on \( \mathbb{F}_{\text{cocomm}}(\Sigma^{-1}V) \), there is a coderivation \( Q \) of degree 1 such that
\[ Q^2 = 0. \] (4)

One can show that equation (4) is equivalent to a collection of quadratic relations on \( Q_n \). The lowest two of these relations are
\[ Q_1^2 = 0, \]
\[ Q_1(Q_2(v_1, v_2)) = -Q_2(Q_1(v_1), v_2) + (-1)^{k_1}Q_2(v_1, Q_1(v_2)). \]

**Example.** Any differential graded Lie algebra (DGLA) \( (\mathfrak{L}, d, [\cdot, \cdot]) \) is naturally an \( L_\infty \)-algebra with only two non-vanishing maps
\[ Q_1 = d : \mathfrak{L} \mapsto \mathfrak{L}, \]
\[ Q_2(h_1, h_2) = (-1)^{|h_1|}[h_1, h_2] : \otimes^2 \mathfrak{L} \mapsto \mathfrak{L}. \]
Definition 3 An $L_{\infty}$-morphism $F$ between $L_{\infty}$-algebras $V$ and $V^\square$ is a homomorphism between coalgebras $\mathbb{F}_{cocomm}(\Sigma^{-1}V)$ and $\mathbb{F}_{cocomm}(\Sigma^{-1}V^\square)$
\[
\Delta F(X) = (F \otimes F)(\Delta X), \quad X \in \mathbb{F}_{cocomm}(\Sigma^{-1}V)
\] such that
\[
Q^\square F(X) = F(QX), \quad X \in \mathbb{F}_{cocomm}(\Sigma^{-1}V).
\]

Similar to proposition 1, equation (5) implies that $F$ is uniquely determined by its composition with projection onto $V^\square$. That is, $F$ is defined by a collection of maps $F_n : S^n(\Sigma^{-1}V) \mapsto \Sigma^{-1}V^\square$.

For example,
\[
F(v) = F_1(v), \quad F(v_1, v_2) = F_2(v_1, v_2) + F_1(v_1) \cdot F_1(v_2).
\]

Equation (6) can also be expressed in terms of relations on $F_n$ and $Q_n$. These relations are more complicated for general $L_{\infty}$-algebras. The first example is that $F_1$ commutes with the differentials,
\[
F_1(Q_1 v) = Q^\square_1 F_1(v).
\]

Remark. For $L_{\infty}$-morphism between DGLAs, equation (6) takes a simpler structure since $Q_n$ is vanishing for all $n > 2$. Let $F$ be an $L_{\infty}$-morphism between DGLAs $(\mathfrak{L}, Q_1, Q_2)$ and $(\mathfrak{L}^\square, Q_1^\square, Q_2^\square)$, then equation (6) takes the following form:
\[
Q^\square_1(F_n(\gamma_1, \gamma_2, \ldots, \gamma_n)) - \sum_{i=1}^{n} (-1)^{k_1+\ldots+k_{i-1}+1-n} F_n(\gamma_1, \ldots, Q_1(\gamma_i), \ldots, \gamma_n) = \frac{1}{2} \sum_{k,l \geq 1, k+l = n} \frac{1}{k!l!} \sum_{\sigma \in S_n} \pm Q^\square_2(F_k(\gamma_{\sigma_1}, \ldots, \gamma_{\sigma_k}), F_l(\gamma_{\sigma_{k+1}}, \ldots, \gamma_{\sigma_{k+l}}))
\]
\[
\pm F_{n-1}(Q_2(\gamma_i, \gamma_j), \gamma_1, \ldots, \hat{\gamma}_i, \ldots, \hat{\gamma}_j, \ldots, \gamma_n), \quad \gamma_i \in \mathfrak{L}_1^{k_i}.
\]

It is the form of the equation, rather than the actual signs in the equation, that is important to us.

Equation (9) motivates the following natural definition.

Definition 4 A quasi-isomorphism $F$ between $L_{\infty}$-algebras $V$ and $V^\square$ is an $L_{\infty}$-morphism such that its first structure map, $F_1$, induces an isomorphism between the cohomology spaces $H^\bullet(V, Q_1)$ and $H^\bullet(V^\square, Q_1^\square)$.

Definition 5 An element $r \in \mathfrak{L}^1$ of a DGLA $\mathfrak{L}$ is a Maurer-Cartan element if it satisfies the following equation
\[
\text{dr} + \frac{1}{2}[r, r] = 0.
\]
Quasi-isomorphisms between DGLAs are of primary importance in deformation theory because a quasi-isomorphism $F$ from $L$ to $L\otimes$ gives a one-to-one correspondence between moduli spaces of Maurer-Cartan elements of $\h L[[h]]$ and $\h L\otimes[[h]]$. Given $F$, we define $\tilde{F}$ as follows:

$$\tilde{F} : hL[[h]] \mapsto hL\otimes[[h]]$$

$$\tilde{F}(r) = \sum_{n=1}^{\infty} \frac{1}{n!} F_n(r, r, \ldots, r),$$  \hspace{1cm} (12)

where $r$ is a Maurer-Cartan element of $hL[[h]]$.

It is easy to see that $\tilde{F}$ sends Maurer-Cartan elements to Maurer-Cartan elements since $F$ is quasi-isomorphism. We refer the readers to Chapter 2 of [6] for the proof that $\tilde{F}$ gives a bijection on moduli spaces of Maurer-Cartan elements.

**Definition 6** A DGLA $L$ is formal if there is a quasi-isomorphism from its cohomology $H^\bullet(L)$ to $L$.

**Definition 7** We say that a graded Lie algebra $L$ is intrinsically formal if any DGLA $L$ with $H(L) = L$ is formal.

As a result, formality theorem for a DGLA $L$ gives a one-to-one correspondence between moduli spaces of Maurer-Cartan elements of $hL[[h]]$ and $hL\otimes[[h]]$. Furthermore, in $hH^\bullet(L)[[h]]$, equation for Maurer-Cartan elements is simpler due to the vanishing differential. An element $r \in hH^\bullet(L)[[h]]$ is a Maurer-Cartan element if

$$[r, r] = 0.$$ \hspace{1cm} (13)

In our paper, the main object of our discussion is the DGLA $\otimes U_\g$ whose DGLA structure controls the quantization of triangular $r$-matrices proposed and described in paper [7] by Drinfeld.

**Definition 8** A triangular $r$-matrix of a Lie algebra $\g$ is an element $r \in \Lambda^2 \g$ such that

$$[r, r] = 0.$$ \hspace{1cm} (14)

We are going to define the DGLA structure of $\otimes U_\g$ as follows:

The symbol $\Delta$ denotes the standard comultiplication of $U_\g$, whose generators $I$ and $e \in \g$ are acted on by $\Delta$ as

$$\Delta I = I \otimes I,$$

$$\Delta e = e \otimes I + I \otimes e.$$ \hspace{1cm} (15)

and $\Delta$ is a homomorphism $U_\g \mapsto U_\g \otimes U_\g$.

Using the comultiplication (15), we define the following complex

$$\otimes U_\g = D = \bigoplus_{k \geq -1} D^k, \quad D^k = U_\g^{\otimes k+1}, \quad D^{-1} = \mathbb{K},$$ \hspace{1cm} (16)

with the differential $\delta : D^k \mapsto D^{k+1}$ given by the formula
\[ \delta \Phi = \Phi \otimes I - \sum_{l=0}^{k} (-1)^{k-l} \Delta_l \Phi + (-1)^{k} I \otimes \Phi, \quad \Phi \in D^k, \quad (17) \]

where

\[ \Delta_l = I \otimes \ldots I \otimes \underbrace{\Delta}_{\text{the } l-\text{th place}} \otimes I \ldots \otimes I. \]

In order to define a Lie algebra structure on the graded space (16), we define the following (non-associative) product

\[ \Phi_1 \circ \Phi_2 = \sum_{i=0}^{k_1} (-1)^{ik_2} \Delta_i \Phi_1(\Phi_2)_i, \quad \Phi_1 \in D^{k_1}, \quad \Phi_2 \in D^{k_2}, \quad (18) \]

where

\[ (\Phi)_i = I \otimes \ldots I \otimes \underbrace{\Phi}_{\text{the } i-\text{th place}} \otimes I \ldots \otimes I, \]

\[ \Delta_i^k = I \otimes \ldots I \otimes \underbrace{\Delta^k}_{\text{the } i-\text{th place}} \otimes I \ldots \otimes I, \]

and

\[ \Delta^k = (\underbrace{\Delta \otimes I \otimes \ldots \otimes I}_{k}) \Delta \ldots (\underbrace{\Delta \otimes I \otimes \ldots \otimes I}_{k-1}) \Delta, \quad k \geq 1, \]

\[ \Delta^0 = \text{Id}. \]

A Lie bracket between homogeneous elements \( \Phi_1 \in D^{k_1} \) and \( \Phi_2 \in D^{k_2} \) is defined as

\[ [\Phi_1, \Phi_2]_G = \Phi_1 \circ \Phi_2 - (-1)^{k_1 k_2} \Phi_2 \circ \Phi_1. \quad (19) \]

The respective Jacobi identity can be proved by direct computation.

It is easy to see that the differential (17) is an inner derivation of the Lie algebra (16), (19),

\[ \delta \Phi = [I \otimes I, \Phi]_G. \quad (20) \]

The Leibniz rule then follows from the Jacobi identity, and hence the space (16) is endowed with a DGLA structure.

We have the following PBW theorem:

**Theorem 1** There exists a filtered vector space isomorphism \( \sigma \) from \( U_\mathfrak{g} \) to \( S(\mathfrak{g}) \).

As a result, \( \sigma \) induces a filtered vector space isomorphism from \( \bigotimes U_\mathfrak{g} \) to \( \otimes S(\mathfrak{g}) \).

Notice that for a Lie algebra \( \mathfrak{g} \), \( E(\mathfrak{g}) \) has a natural graded Lie algebra structure induced by extending the Lie bracket of \( \mathfrak{g} \) by the Leibniz rule with respect to exterior product \( \wedge \)

\[ [\gamma, \gamma_1 \wedge \gamma_2] = [\gamma, \gamma_1] \wedge \gamma_2 + (-1)^{k(k_1+1)} \gamma_1 \wedge [\gamma, \gamma_2], \quad \gamma \in E^k(\mathfrak{g}), \quad \gamma_1 \in E^{k_1}(\mathfrak{g}). \quad (21) \]

We will use the same symbol for the Lie bracket in algebra \( \mathfrak{g} \) and the graded Lie bracket in algebra \( E(\mathfrak{g}) \) defined above.

The structure of the cohomology space of the DGLA of Drinfeld is described by the following theorem:
Theorem 2 We have the following three statements:

(i) The cohomology space of the DGLA $D$ (16), (17), (19) is the vector space of the exterior products of $g$, $E(g)$, namely,

\[ H^k(D) = \wedge^{k+1} g, \quad k > -1; \quad H^{-1}(D) = \mathbb{K}. \]  

(ii) Let $f'$ be the map from $\otimes S(g)$ to $E(g)$ such that

\[ f'(g_1 \otimes \ldots \otimes g_k) = g_1 \wedge \ldots \wedge g_k \]  

where $g_i \in g$ and that $f(a_1 \otimes \ldots \otimes a_s) = 0$ where $a_j \in S(g)$ if for some $i$, $a_i \in S(g)$ has degree other than 1. Then,

\[ f = f' \circ \sigma \]  

is a quasi-isomorphism from $\otimes \mathcal{U}_g$ to $E(g)$ where $\sigma$ is the vector space isomorphism in theorem 1.

(iii) The Lie algebra structure of $E(g)$ induced by that of $D$ is the same as the one extended from bracket of $g$ by equation (21).

Proof. For the first statement, we may assume that $g$ is commutative since the definition of differential of $D$ uses only the comultiplication of $\mathcal{U}_g$ but not the multiplication of $\mathcal{U}_g$. Thus, $\mathcal{U}_g = S(g)$. We construct an isomorphism $i$ from the complex $(\otimes S(g), \Delta)$ to the Hochschild cochain complex $C^\bullet(S(g^*), \mathbb{K})$. Pick a basis \{g_1, \ldots g_n\} for $g$. We have

\[ S(g) = \mathbb{K}[g_1, \ldots, g_n] \]  

and

\[ S(g^*) = \mathbb{K}[g_1^*, \ldots, g_n^*]. \]  

Given $b \in \mathbb{K}[g_1^*, \ldots, g_n^*]$, we set

\[ i(g_{k_1} \ldots g_{k_s})(b) = \frac{\partial}{\partial g_{k_1}} \ldots \frac{\partial}{\partial g_{k_s}}(b)|_{g^*=0}. \]  

This defines a map $i : S(g) \rightarrow Hom(S(g^*), \mathbb{K})$ and we extend this map to

\[ i : \otimes S(g) \rightarrow C^\bullet(S(g^*), \mathbb{K}) \]  

by the formula

\[ i(a_1 \otimes \ldots \otimes a_k)(b_1 \otimes \ldots \otimes b_k) = i(a_1)(b_1) \times \ldots \times i(a_k)(b_k) \]  

where $a_i \in S(g)$ and $b_i \in S(g^*)$.

In order to compute $HH^\bullet(S(g^*), \mathbb{K})$, we use the following equation

\[ HH^\bullet(A, \mathbb{K}) = \text{Ext}^\bullet_{A \otimes A^{op}}(A, \mathbb{K}) \]  

where $A$ is an augmented algebra and acts on $\mathbb{K}$ by multiplying the image of the augmentation.
Let $A = S(g^*) = \mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[X]$. Then $A \otimes A = \mathbb{K}[X] \otimes \mathbb{K}[Y]$ and let $\theta^i$ be anti-commutative variables $1 \leq i \leq n$. We use the Koszul resolution $K^\bullet$ of $A$ to compute $\text{Ext}^\bullet_{A \otimes A}(A, \mathbb{K})$.

$$K^\bullet = \mathbb{K}[X, Y, \theta]$$

where $K^i$ is the module consisting of elements of degree $i$ in $\theta$ and with differential

$$\partial_K = \sum_j (x^j - y^j) \frac{\partial}{\partial \theta^j}.$$  

Direct computation shows that

$$\text{Hom}_{A \otimes A}(K^m, \mathbb{K}) = \text{Hom}_{\mathbb{K}}(E^m(g^*), \mathbb{K}) = E^m(g)$$

and that differential on $\text{Hom}_{A \otimes A}(K^\bullet, \mathbb{K})$ induced by $\partial_K$ is zero. This proves the first statement of the theorem.

For the second statement, we use the following quasi-isomorphism between Koszul resolution and bar resolution of $A$:

$$c_{i_1 \ldots i_k} a \otimes \theta^{i_1} \ldots \theta^{i_k} \otimes b \mapsto c_{i_1 \ldots i_k} a \otimes x^{i_1} \otimes \ldots \otimes x^{i_k} \otimes b$$

where $a \otimes b \in A \otimes A$. Notice that $\theta$s are anti-commutative variable and $x, y$ s are commutative variable. Let $f$ be the map defined in equation (24), and $h$

$$h : E(g) \mapsto \bigotimes U_g$$

be the natural map from $E(g)$ to $D$. It is easy to see that $f \circ h$ is the identity map. Furthermore, using the quasi-isomorphism (34), we can also show that $h \circ f$ is homotopic to the identity. Thus we can conclude that $f$ is a quasi-isomorphism. Similarly, we can proves (iii) using the maps $f$ and $h$. \hfill \Box

Since $H(\bigotimes U_g) = E(g)$ and a triangular $r$-matrix $r \in \wedge^2(g) = H^1(\bigotimes U_g)$ is also a Maurer-Cartan element, a quasi-isomorphism $F$ from $E(g)$ to $\bigotimes U_g$ induces a bijection of the moduli spaces of Maurer-Cartan elements of $hE(g)$ and $h(\bigotimes U_g)$ given by equation (12). Moreover, we can construct

$$T = I \otimes I + \tilde{F}(r).$$

Since $\tilde{F}(r)$ is a Maurer-Cartan element, $T$ satisfies the following equation:

$$T_{12,3}T_{12} = T_{1,23}T_{23}.$$  

The construction of an element $T$ satisfying equation (37) is called the quantization of triangular $r$-matrices [7].

### 3 Obstructions to the formality of $L_\infty$-Algebras

Given an $L_\infty$-algebra $(\mathfrak{L}, Q)$, its cohomology $H = H(\mathfrak{L})$ is naturally a DGLA with $Q^H_1 = 0$ and $Q^H_2([a], [b]) = [Q^H(a, b)]$ where $a, b \in \mathfrak{L}$ represent cohomology classes $[a]$ and $[b]$ in $H(\mathfrak{L})$. Denote this DGLA by $(H, Q^H)$. Let $(\mathfrak{h}, d)$ the DGLA of coderivation
of $\mathcal{F}_{\text{cocomm}}(\Sigma^{-1}H)$ with differential defined by commutator with $Q^H$. In this section we show that obstruction to the formality of $\mathcal{L}$ lies in $H^1(\mathfrak{h})$. A more general result is proven in section 4 of [13]. However, we decide to give a proof without referring to the technique of closed model category [18] and to give an explicit construction.

**Lemma 1** Given a quasi-isomorphism of complex $f$ form $H$ to $\mathcal{L}$, there exists an $L_\infty$ structure $\tilde{H}$ on the space $H$ such that $\tilde{H}$ is quasi-isomorphic to $\mathcal{L}$ and that the first structure map of the quasi-isomorphism is $f$.

**Proof.** We will prove the lemma by constructing a quasi-isomorphism $F$ from $(H[\lambda], \tilde{Q})$ to $(\mathcal{L}[\lambda], \tilde{Q}^H)$ where $\tilde{Q}$ and $\tilde{Q}^H$ on $\mathcal{L}[\lambda]$ and $H[\lambda]$ are

$$\tilde{Q}_n = \lambda^{n-1}Q_n,$$

and

$$\tilde{Q}^H_n = \lambda^{n-1}Q'_n$$

and $F$ have the following form:

$$\tilde{F}_n = \lambda^{n-1}F_n$$

and

$$\tilde{F}_1 = f.$$

We construct $\tilde{Q}_n^H$ and $\tilde{F}_n$ inductively as follows: For $n = 2$, notice that $f = F_1$ is quasi-isomorphism from $H$ to $\mathcal{L}$. As a result,

$$F_1(Q^H_2(\gamma_1, \gamma_2)) - Q_2(F_1(\gamma_1), F_1(\gamma_2))$$

is $Q_1$ exact. Hence, there exists $F_2$ such that

$$F_1(Q^H_2(\gamma_1, \gamma_2)) - Q_2(F_1(\gamma_1), F_1(\gamma_2)) = Q_1(F_2(\gamma_1, \gamma_2)).$$

(39)

Suppose that we have defined $\tilde{Q}_k^H$ and $\tilde{F}_k$ such that

$$\tilde{Q}\tilde{F} - \tilde{F}\tilde{Q}^H = 0 \mod \lambda^{n-1}$$

(40)

and that

$$\tilde{Q}^H\tilde{Q}^H = 0 \mod \lambda^n,$$

(41)

where $n > 3$. Notice that equation (40) implies that

$$\tilde{Q}(\tilde{Q}\tilde{F} - \tilde{F}\tilde{Q}^H) + (\tilde{Q}\tilde{F} - \tilde{F}\tilde{Q}^H)\tilde{Q}^H = 0 \ mod \lambda^n.$$

(42)

Let $\Phi_n$ be the sum of terms of degree $(n - 1)$ in $\lambda$ of $\xi(\tilde{Q}\tilde{F} - \tilde{F}\tilde{Q}^H)$ where $\xi$ is the projection of $\mathcal{F}_{\text{cocomm}}(\Sigma^{-1}H)$ to $H$. Equation (42) implies that

$$Q_1(\Phi_n) = 0.$$

(43)

As a result, there exists $\tilde{F}_n$ and $\tilde{Q}_n^H$ such that

$$\Phi_n + Q_1(\tilde{F}_n) = F_1(\tilde{Q}_n^H) = 0.$$
Let’s denote the change of modifying $\tilde{Q}^H$ and $\tilde{F}$ with $\tilde{Q}_n^H$ and $\tilde{F}_n$ by $\delta \tilde{Q}^H$ and $\delta \tilde{F}$. We have
\[
(\tilde{Q} \tilde{F} - \tilde{F} \tilde{Q}^H) + \tilde{Q} \delta \tilde{F} - \delta \tilde{F} \tilde{Q}^H - \tilde{F} \delta \tilde{Q}^H = 0 \mod \lambda^n.
\]
Furthermore, we have that
\[
\tilde{F}(\delta \tilde{Q}^H \tilde{Q}^H + \tilde{Q}^H + \delta \tilde{Q}^H)
= \tilde{Q} \tilde{F} \tilde{Q}^H - \tilde{F}(\tilde{Q}^H)^2 + \tilde{Q} \delta \tilde{F} \tilde{Q}^H + \tilde{Q} \tilde{F} \delta \tilde{Q}^H \mod \lambda^{n+1}
= -\tilde{F}(\tilde{Q}^H)^2 + \tilde{Q}(\lambda^n g) \mod \lambda^{n+1}
\]
where $g$ is of degree 0 or higher in $\lambda$ which shows that
\[
\xi \tilde{Q}(\lambda^n g) = Q_1((\lambda^n g) \mod \lambda^{n+1}).
\]
This shows that
\[
F_1(\xi((\delta \tilde{Q}^H \tilde{Q}^H + \tilde{Q}^H + \delta \tilde{Q}^H + (\tilde{Q}^H)^2))) + Q_1((\lambda^n g) = 0 \mod \lambda^{n+1}.
\]
Hence
\[
\delta \tilde{Q}^H \tilde{Q}^H + \tilde{Q}^H + \delta \tilde{Q}^H + (\tilde{Q}^H)^2 = 0 \mod \lambda^{n+1}.
\]
That is,
\[
\xi(\tilde{Q}^H + \delta \tilde{Q}^H)^2 = 0 \mod \lambda^{n+1},
\]
since $(\delta \tilde{Q}^H)^2$ is of degree $2n - 2 \geq n + 1$ in $\lambda$. This finishes the proof. \qed

Now we want to construct a quasi-isomorphism from $H$ to $\tilde{H}$. Hinich uses the following inductive procedure to this construction. Let $H[\hbar]$ be the DGLA with
\[
Q_n^\hbar = \hbar^{n-1}Q_n^H.
\]
Suppose we have constructed a map $g : \mathbb{F}_{cocomm}(\Sigma^{-1}H[\hbar]) \mapsto \mathbb{F}_{cocomm}(\Sigma^{-1}\tilde{H}[\hbar])$ whose structure maps are
\[
g_n = \hbar^{n-1}\phi_n
\]
where $\phi_n$ are structure maps for $L_\infty$ morphism from $H$ to $\tilde{H}$ such that
\[
g^* : \mathbb{F}_{cocomm}(\Sigma^{-1}(H[\hbar]/(\hbar^n))) \mapsto \mathbb{F}_{cocomm}(\Sigma^{-1}(\tilde{H}[\hbar]/(\hbar^n)))
\]
is a quasi-isomorphism. Due to lemma 1, there exists a codifferential $Q'$ on $\mathbb{F}_{cocomm}(\Sigma^{-1}(H[\hbar]/(\hbar^{n+1})))$ such that
\[
g^* : (\mathbb{F}_{cocomm}(\Sigma^{-1}(H[\hbar]/(\hbar^{n+1}))), Q') \mapsto \tilde{F}_{cocomm}(\Sigma^{-1}(H[\hbar]/(\hbar^{n+1})))
\]
is a quasi-isomorphism. In fact, $Q_k^\hbar = Q_k^h$ for $k \leq n$ due to equation (48). Thus we conclude that $Q_{n+1}'$ viewed as a coderivation is a closed element of $H^1(\hbar)$ since $(Q')^2 = 0$. If $Q_{n+1}'$ is trivial in $H^1$, say $Q_{n+1}' = dz$, then we let $j$ be the $L_\infty$ morphism with two non-vanishing structure maps $j_1 = Id$ and $j_{n+1} = z$ then the maps $g' = g \circ j$ satisfies equation (47) and induces quasi-isomorphism from $H[\hbar]/(\hbar^{n+1})$ to $\tilde{H}[\hbar]/(\hbar^{n+1})$.

Combining this procedure with lemma 1, we prove the particular case of Theorem 4.2 in [13].

**Theorem 3** Given an $L_\infty$-algebra $(\mathfrak{L}, Q)$ with cohomology $(H, Q^H)$, let $\hbar$ be the DGLA of coderivation of coalgebra $\mathbb{F}_{cocomm}(\Sigma^{-1}H)$ with differential defined by commutator with $Q^H$. If $H^1(\hbar) = 0$ then $H$ is intrinsically formal.
4 Vanishing of the first obstruction to formality of $\otimes \mathcal{U}_g$

In this section, we use the proof of theorem 3 to show that the first obstruction to formality of $\otimes \mathcal{U}_g$ vanishes.

Direct computation shows that $H^1(\mathfrak{h})$ is nothing but the Lie algebra cohomology with the following differential

$$\partial\Phi(\gamma_0, \gamma_1 \ldots \gamma_{n+1}) = -\sum_{i=0}^{n+1} (-1)^i \gamma_i \Phi(\gamma_0, \gamma_1 \ldots \gamma_{i-1} \gamma_{i+1})$$

$$- \sum_{i=0}^{n} \sum_{j=i+1}^{n+1} (-1)^{i+j+k_i} \Phi([\gamma_i, \gamma_j], \gamma_0 \ldots \gamma_i \ldots \gamma_{j-1} \gamma_{j+1} \ldots \gamma_{n+1}),$$

where $\gamma_i$ has degree $k_i$ (Example 4.2.4 b [10]).

Definition 9 A cocycle $\Phi : \wedge^{n+1}E(\mathfrak{g}) \to E(\mathfrak{g})$ is admissible if any component of $\Phi(\gamma_0, \ldots, \gamma_n)$ is expressed in terms of components of $\gamma_i$ via the Lie bracket of $\mathfrak{g}$ and the coefficients depend only on the degrees of poly-vectors $\gamma_i$ and that the cochain is a cocycle for any Lie algebra $\mathfrak{g}$.

We have the following proposition for the cochains showing up as obstruction to the formality of $\otimes \mathcal{U}_g$.

Proposition 2 If one uses the pair of quasi-isomorphisms $f$ and $h$ defined in equation (24) and equation (35) then obstructions to the formality of $\otimes \mathcal{U}_g$ are represented by admissible $n$-cocycles of degree $2 - n$.

Proof. Let $h$ and $f$ be the quasi-isomorphisms defined in equation (24) and equation (35). We follow the construction in lemma 1 with $F_1 = h$. From equation (44) we know that the first non-vanishing $Q_n(\gamma_1, \ldots, \gamma_n)$ is a cocycle of $E(\mathfrak{g})$ representing the cohomological class of

$$\frac{1}{2} \sum_{k,l \geq 1, k+l=n} \frac{1}{k!l!} \sum_{\sigma \in S_n} \pm [F_k(\gamma_{\sigma_1}, \ldots, \gamma_{\sigma_k}), F_l(\gamma_{\sigma_{k+1}}, \ldots, \gamma_{\sigma_{k+l}})]G$$

$$- \sum_{i \neq j} \pm F_{n-1}([\gamma_i, \gamma_j], \gamma_1, \ldots, \hat{\gamma_i}, \ldots, \hat{\gamma_j}, \ldots, \gamma_n).$$

Let the above sum be $\zeta$. Consider the element $f(\zeta)$. It is a cocycle in $E(\mathfrak{g})$ and $h(f(\zeta))$ represents the cohomology class of $\zeta$. Thus, in equation (44) we may choose $Q_n$ to be $f(\zeta)$. Furthermore, from the construction in lemma 1, we see that $F_k$ can be chosen so that any component of the $f(\zeta)$ resulting from the construction is expressed in terms of components of $\gamma_i$ via the Lie bracket of $\mathfrak{g}$ and that the coefficient is independent of the particular choice component of $\gamma_i$. Thus, $Q_n$ is a admissible $(n - 1)$-cocycle of degree $2 - n$. $\square$

From the above proposition and theorem 3, we get:
Theorem 4 If all obstructions to formality of $\bigotimes U_{\text{Lie}(V)}$ are trivial for any finite dimensional vector space $V$, then for any finite dimensional Lie algebra $\mathfrak{g}$, $\bigotimes U_{\mathfrak{g}}$ is formal.

In the rest of this section, we prove that the first obstruction for the formality of $\bigotimes U_{\text{Lie}(V)}$ vanishes. Thus we prove that $\bigotimes U_{\mathfrak{b}}$ is formal where $\mathfrak{b}$ is a two-dimensional Lie algebra.

Theorem 5 The first obstructions to formality of $\bigotimes U_{\text{Lie}(V)}$ are trivial.

Proof. First we recall that first obstructions are admissible 2-cocycles of degree $-1$. Let the first obstruction to be

$$\Phi(\gamma_0, \gamma_1, \gamma_2)$$

which is a summation of terms which are expressed as wedge products of components of the polyvectors $\gamma_i$ via three bracket operations. Three bracket operations can involve four to six components of $\gamma_i$. We start from analyzing terms with the least number of components involved.

From now on, we will assume $\gamma_i$ to be polyvectors given by

$$\gamma_0 = x_0 \wedge x_1 \wedge \ldots \wedge x_p,$$

$$\gamma_1 = y_0 \wedge y_1 \wedge \ldots \wedge y_q,$$

$$\gamma_2 = z_0 \wedge z_1 \wedge \ldots \wedge z_r,$$

$$\gamma_3 = w_0 \wedge w_1 \wedge \ldots \wedge w_s.$$  

Up to permutation of polyvectors and the indices, there are four types of terms involving four components of $\gamma_i$. They are

$$[[[x_0, y_0], x_1], x_2] \wedge x_3 \ldots \wedge x_p \wedge y_1 \ldots \wedge y_q \wedge \gamma_2$$

$$[[[x_0, y_0], x_1], y_1] \wedge x_2 \ldots \wedge x_p \wedge y_2 \ldots \wedge y_q \wedge \gamma_2$$

$$[[[x_0, y_0], x_1], z_0] \wedge x_2 \ldots \wedge x_p \wedge y_1 \ldots \wedge y_q \wedge z_1 \ldots \wedge z_r$$

$$[[[x_0, y_0], z_0], x_1] \wedge x_2 \ldots \wedge x_p \wedge y_1 \ldots \wedge y_q \wedge z_1 \ldots \wedge z_r$$

We will show that in the cocycle, terms of the types (57) and (58) do not appear and terms of types (59) and (60) can be killed by adding coboundary terms.

First we consider terms in $\partial \Phi$ with brackets involving $x_0$, $x_1$, $x_2$, $y_0$, $w_0$. There are four sums containing such terms. Two of them are coming from the sums $[\Phi(\gamma_0, \gamma_1, \gamma_2), \gamma_3]$ and $[\Phi(\gamma_0, \gamma_2, \gamma_3), \gamma_1]$ and two of them are from the sums $\Phi(\gamma_0, \gamma_1, \gamma_2)$ and $\Phi(\gamma_0, \gamma_2, \gamma_3)$. Furthermore, we can notice that the first two are antisymmetric when we exchange $x_0$ with $x_1$ but the last two are symmetric. Thus in a cocycle, the contribution of terms involving $x_0$, $x_1$, $x_2$, $y_0$, $w_0$ from the first two sums must be zero. That is we have

$$\alpha_1[[[x_0, y_0], x_1], x_2, w_0] + \alpha_2[[[x_0, w_0], x_1], x_2, y_0] = 0,$$
where \( \alpha_1 \) and \( \alpha_2 \) are coefficients of the terms in the first two sums respectively. However, \( \mathfrak{g} \) is free Lie algebra and thus

\[
\alpha_1 = \alpha_2 = 0,
\]  

which shows that terms of the type (57) do not appear.

Secondly, we consider terms in \( \partial \Phi \) with brackets involving \( x_0, x_1, y_0, y_1 \) and \( w_0 \) which are antisymmetric in the pairs \( (x_0, x_1) \) and \( (y_0, y_1) \). In, \( \partial \Phi \), there is only one such term from the sum \( [\Phi(\gamma_0, \gamma_1, \gamma_2), \gamma_3] \) and thus in a cocycle \( \Phi(\gamma_0, \gamma_1, \gamma_2) \), terms of type (58) do not appear.

Finally, we will show that terms of types (59) and (60) can be killed by adding coboundary terms \( \partial \phi(\gamma_0, \gamma_1, \gamma_2) \) where

\[
\phi(\gamma_0, \gamma_1) = \sum a_{i,j,k}[[x_i, y_j], x_k] \wedge \tilde{\gamma}_0 \wedge \tilde{\gamma}_1
\]  

in which \( \tilde{\gamma}_0 \) and \( \tilde{\gamma}_1 \) are obtained from the original polyvectors by removing the components appeared in front.

We can choose \( a_{i,j,k} \) so that the terms of the type (59) are killed by adding \( \partial \phi(\gamma_0, \gamma_1, \gamma_2) \). That is, in \( (\Phi - \partial \phi)(\gamma_0, \gamma_1, \gamma_2) \), the only possible terms involving four components in the brackets are the fourth type. Now, we consider terms in \( \partial \Phi \) with brackets involving \( x_0, x_1, y_0, z_0 \) and \( w_0 \) and are antisymmetric when permuting \( x_0 \) with \( x_1 \). There are three such terms from the sums \( [\Phi(\gamma_0, \gamma_1, \gamma_2), \gamma_3] \) and cyclic permutation of 1, 2, 3.

That \( \Phi \) is a cocycle implies

\[
\alpha_{y,z}[[[x_0, y_0], x_1], z_0, w_0] + \alpha_{x,w}[[[x_0, z_0], x_1], w_0, y_0] + \alpha_{y,w}[[[x_0, y_0], x_1], w_0, z_0] = 0.
\]  

Again, \( \mathfrak{g} \) is free Lie algebra and thus all three \( \alpha \)'s above are zero. As a result, up to a coboundary, terms with four components in brackets are zero.

Next we consider terms with five components in brackets. Up to permutation of polyvectors and the indices, there are two types of terms

\[
[[x_0, y_0], z_0] \wedge [\ast, \ast] \wedge \tilde{\gamma}_0 \wedge \tilde{\gamma}_1 \wedge \tilde{\gamma}_2
\]  

\[
[[x_i, y_j], x_k] \wedge [\ast, \ast] \wedge \tilde{\gamma}_0 \wedge \tilde{\gamma}_1 \wedge \tilde{\gamma}_2
\]

where \( \tilde{\gamma}_i \) is obtained from original polyvectors by removing the components appeared in front.

First we consider terms in \( \partial \Phi \) of the form:

\[
[[[x_0, y_0], z_0], w_0] \wedge [\ast, \ast'] \wedge \tilde{\gamma}_0 \wedge \tilde{\gamma}_1 \wedge \tilde{\gamma}_2 \wedge \tilde{\gamma}_3.
\]

In general, we may assume \( \ast = x_1 \), there are at most three terms in \( \partial \Phi \) of the above form and is symmetric when exchanging \( x_0 \) and \( x_1 \) which are coming from the sum \( \Phi([\gamma_0, \gamma_1], \gamma_2, \gamma_3) \) and cyclic permutation of (1,2,3). Thus we have

\[
\beta_{x,y,z}[[[x_0, y_0], z_0], w_0] + \beta_{x,z,w}[[[x_0, z_0], w_0, y_0] + \beta_{x,y,w}[[[x_0, y_0], w_0], z_0] = 0,
\]  

\[13\]
which implies \( \beta \) are all zero since \( \mathfrak{g} \) is free Lie algebra. As a result, the terms of type (65) do not appear in \( \Phi - \partial \phi \).

For the terms of type (66), if one of the \( \ast \) or \( \ast' \) is from the polyvector \( \gamma_0 \) or \( \gamma_1 \) then we can use exactly the same argument as above to show that such terms do not appear in cocycles. Thus we only need to consider the term

\[
[[x_0, y_0], x_1] \wedge [z_0, z_1] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2.
\]

In \( \partial \Phi \) we consider terms of the form

\[
[[[x_0, y_0], x_1], w_0] \wedge [z_0, z_1] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3,
\]

which are symmetric with respect to \( x_0 \) and \( x_1 \). There are at most two such terms from the sums \( \Phi([\gamma_0, \gamma_1], \gamma_2, \gamma_3) \) and \( \Phi([\gamma_0, \gamma_3], \gamma_2, \gamma_1) \). And for a coboundary, we have

\[
a[[[x_0, y_0], w_0], x_1] + b [[[x_0, w_0], x_0], x_1] = 0,
\]

which implies \( a \) and \( b \) are zero and we finish the proof of the statement that up to a coboundary, terms with five components in brackets do not appear in the cocycle.

Finally we consider terms with six components in brackets. Up to permutation of polyvectors and the indices, there are four possible types of terms

\[
[x_0, y_0] \wedge [x_1, z_0] \wedge [y_1, z_1] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2
\]

\[
[x_0, x_1] \wedge [x_2, y_0] \wedge [y_0, z_0] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2
\]

\[
[x_0, x_1] \wedge [x_2, y_0] \wedge [x_3, z_0] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2
\]

\[
[x_0, x_1] \wedge [y_0, y_1] \wedge [z_0, z_1] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2
\]

Notice that in the first three of the above terms, there exists one bracket such that for each component in the bracket, the polyvector of that components have another components in another bracket. \([x_1, z_0], [x_2, y_0] \) and \([x_0, x_1] \) are examples of such brackets in the three terms respectively.

Let’s first consider terms of the form

\[
[x_0, y_0] \wedge [[x_1, z_0], w_0] \wedge \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3
\]

in \( \partial \Phi \) which is anti-symmetric with respect to the pair \( (x_0, x_1) \) and \( (z_0, z_1) \). There is only one such term from the sum \( \Phi(\gamma_0, \gamma_1, \gamma_2, \gamma_3) \) and thus for a cocycle, the terms of type (72) are vanishing.

Using exactly the same argument, we can show that terms of type (73) and (74) are vanishing.

For terms of type (75), it is clear from equation (51) that such terms does not show up as obstructions in our construction using the pair of quasi-isomorphism \( f \) and \( h \) ((24),(35)). Thus, we have shown that the first obstruction is trivial. \( \square \)

As we mentioned in previous section, if we can prove all obstructions are trivial, then we prove formality theorem. However, we may notice that the result for every two-cocycle implies the formality theorem for two dimensional Lie algebra since we have the following:
Lemma 2 For a two-dimensional Lie algebra $g$, all admissible $n$-cochains of degree $1 - n$ for $n > 2$ are vanishing.

Proof. We know that any non-abelian two-dimensional Lie algebra, up to isomorphism, is $b$, the Borel subalgebra of $sl_2(K)$, namely

$$[e_1, e_2] = 2e_1.$$ (77)

Also notice that the space $E(g)$ is spanned by $e_1, e_2, e_1 \wedge e_2$ and thus, for an $n$-cochain, we have at most $n + 1$ components of $e_2$. However, we may notice that for each bracket, we must have a component of $e_2$ or the bracket will otherwise be zero. For a non-zero $n$-cochain of degree $1 - n$, we have $2n - 1$ bracket. As a result, or $n > 2$ the number of brackets is more than the possible number of components of $e_2$. Therefore, we prove that all $n$-cochains $n > 2$ are vanishing. $\Box$

Thus we have proved the following:

Theorem 6 For the two-dimensional Borel Lie algebra $b$, $\bigotimes U_b$ is formal.

5 Concluding remarks

We would like to mention that the construction of the cohomology classes of $E(g)$ which appear as obstructions to the formality depends on the choice of the quasi-isomorphism (24). Therefore, these classes are not homotopy invariant. However, the classes can be used to introduce a partially defined products, so-called Massey-Lie products [12] which are homotopy invariant.

Unfortunately, the Massey-Lie products do not allow us to formulate a criterion of formality of a DG algebraic structure. Recently, D. Kaledin [14] found a good refinement of the notion of Massey-Lie products which allowed him to formulate the criterion of formality for DG (associative) algebras.

While this manuscript was in preparation paper [11] by G. Halbout appeared on arxiv.org. In this paper the author uses Tamarkin’s technique [13], [20] to establish the formality of the DGLA of Drinfeld over any field of characteristic zero. In [11] G. Halbout also used this result to get a partial answer to Drinfeld’s question [8] about the quantization of coboundary Lie bialgebras. It would be interesting to further develop this technique and get a complete answer to this question.

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