A KOEBE DISTORTION THEOREM FOR QUASICONFORMAL MAPPINGS IN
THE HEISENBERG GROUP

TOMASZ ADAMOWICZ, KATRIN FÄSSLER, AND BEN WARHURST

ABSTRACT. We prove a Koebe distortion theorem for the average derivative of a qua-
siconformal mapping between domains in the sub-Riemannian Heisenberg group $\mathbb{H}^1$.
Several auxiliary properties of quasiconformal mappings between subdomains of $\mathbb{H}^1$ are
proven, including distortion of balls estimates and local BMO-estimates for the logarithm
of the Jacobian of a quasiconformal mapping. Applications of the Koebe theorem include
diameter bounds for images of curves, comparison of integrals of the average deriva-
tive and the operator norm of the horizontal differential, as well as the study of quasi-
conformal densities and metrics in domains in $\mathbb{H}^1$. The theorems are discussed for the
sub-Riemannian and the Korányi distances. This extends results due to Astala–Gehring,
Astala–Koskela, Koskela and Bonk–Koskela–Rohde.

CONTENTS

1. Introduction ........................................... 2
2. Definitions and preliminaries ....................... 4
  2.1. Modulus of curve families ................... 4
  2.2. The Heisenberg group .......................... 5
  2.3. Quasiconformal and quasisymmetric mappings 6
3. The Jacobian of a quasiconformal mapping ......... 9
  3.1. Higher integrability and reverse Hölder inequality 9
  3.2. $A_p$-weights and distortion of balls ...... 10
  3.3. Logarithm and BMO .......................... 15
4. The Koebe theorem .................................. 20
5. Applications ....................................... 26
  5.1. Diameter bounds for image curves ........ 26
  5.2. Comparison of the average derivative and the operator norm 29
  5.3. Quasiconformal metrics on domains in $\mathbb{H}^1$ 36
References .............................................. 41

Date: July 11, 2017.
2010 Mathematics Subject Classification. (Primary) 30L10 (Secondary) 30C65, 30F45.
K.F. was supported by the Swiss National Science Foundation through the grant 161299 'Intrinsic rectifi-
ability and mapping theory on the Heisenberg group'.
1. Introduction

The Koebe distortion theorem is a classical result in complex analysis which provides control of the absolute value of the complex derivative of a conformal function between domains in the complex plane. Various related results go under the name ‘Koebe distortion theorem’ in the literature. For the purpose of this introduction, we focus on the following statement, see Theorem 1.6 in [1].

**Theorem 1.1** (Koebe distortion theorem in the plane). If \( f : \Omega \to \Omega' \) is a conformal homeomorphism between simply connected domains \( \Omega, \Omega' \subset \mathbb{C} \), then

\[
\frac{1}{4} \frac{d(f(z), \partial \Omega')}{d(z, \partial \Omega)} \leq |f'(z)| \leq 4 \frac{d(f(z), \partial \Omega')}{d(z, \partial \Omega)}
\]

for all \( z \in \Omega \).

This follows from a distortion theorem for maps on the unit disk [37, Corollary 1.4] together with the Riemann mapping theorem. In dimensions \( n > 2 \), the situation changes significantly since according to the Liouville theorem, conformality is a rigid property. Therefore, if one is interested in an analog of Theorem 1.1 in higher dimensions, one should consider a larger class of mappings. Following this idea, K. Astala and F. Gehring established a version of Theorem 1.1 for quasiconformal maps in \( \mathbb{R}^n, n \geq 2 \), see Theorem 1.8 in [1]. Quasiconformal mappings are not necessarily differentiable everywhere, but they belong to the Sobolev class \( W^{1,n}_{loc} \). Consequently, Theorem 1.8 in [1] is formulated not for the pointwise derivative, but for a certain average derivative \( a_f \). This is a natural geometric quantity which, for \( n = 2 \) and \( f \) conformal, agrees with \( |f'(z)| \). Both \( a_f \) and the Koebe distortion theorem for \( a_f \) have found various applications, for instance in connection with the global distortion properties of a quasiconformal mapping [3], diameter bounds for images of curves [30], in the studies of conformal metrics [10] and more recently related to harmonic quasiconformal mappings [4]. We address counterparts of some of these results as well as their generalizations.

The goal of this note is to prove a Koebe distortion theorem for quasiconformal mappings in the Heisenberg group and to study several applications thereof. The Heisenberg group \( \mathbb{H}^1 \) endowed with a left-invariant sub-Riemannian metric \( d_s \) has played an important role as a testing ground and motivational example for the extension of the theory of quasiconformal maps from Euclidean to more abstract metric spaces. This development can be seen from a series of papers and notes [29, 22, 19, 23]. Given the role of the sub-Riemannian Heisenberg group in the development of the theory of quasiconformality, and the wealth of quasiconformal mappings which can be constructed in this particular space by methods described in [29, 15, 6, 7, 5], we consider \( \mathbb{H}^1 \) a natural non-Euclidean setting where it is worthwhile to study counterparts for \( a_f \) and Koebe’s theorem.

**Definition 1.2.** For a quasiconformal map \( f : \Omega \to \Omega' \) between domains \( \Omega, \Omega' \subset \mathbb{H}^1 \), we define

\[
a_f(x) := \exp \left( \frac{1}{4} (\log J_f)_B(x) \right)
\]

with \( B(x) := B(x, d(x, \partial \Omega)) \) and \( u_B := \frac{1}{mB} \int_B u \, dm \).

Here and in the following, \( B = B(x, r) \) denotes an open ball with center \( x \) and radius \( r > 0 \) with respect to a metric \( d \) which will depend on the context. Moreover, \( mB := B(x, mr) \). A domain is an open connected set. The constant 4 which appears in (1.3)
is unrelated to the factor 4 in Koebe’s distortion theorem, but instead agrees with the Hausdorff dimension of the sub-Riemannian Heisenberg group. The following is the main theorem of this paper.

**Theorem 1.4.** For every \( K \geq 1 \), there exists a constant \( 1 \leq c_K < \infty \) such that for every \( K \)-quasiconformal mapping \( f : \Omega \to \Omega' \) between domains in \( \mathbb{H}^1 \) with \( \Omega \subseteq \mathbb{H}^1 \), one has

\[
\frac{1}{c_K} \frac{d(f(x), \partial \Omega')}{d(x, \partial \Omega)} \leq a_f(x) \leq c_K \frac{d(f(x), \partial \Omega')}{d(x, \partial \Omega)}, \quad \text{for all } x \in \Omega. \tag{1.5}
\]

Before discussing our results in detail, let us notice that Theorem 1.4 is flexible with respect to the choice of the underlying distance in \( \mathbb{H}^1 \). In the Heisenberg group one often considers two bi-Lipschitz equivalent distances: the sub-Riemannian distance \( d_s \) and the Korányi distance \( d_{\mathbb{H}^1} \), see Section 2.2 for the definitions. Our results apply both to \( d = d_s \) and \( d = d_{\mathbb{H}^1} \). Since the two distances are bi-Lipschitz equivalent, a homeomorphism \( f : \Omega \to \Omega' \) is quasiconformal with respect to \( d_{\mathbb{H}^1} \) if and only if it is quasiconformal with respect to \( d_s \). More is true: as explained in [29, §1.1], one obtains the same class of \( K \)-quasiconformal mappings, \( K \geq 1 \), with respect to either metric. The definition of \( a_f \) as given in Definition 1.2 depends on the metric \( d \) used to define the ball \( B(x) = B(x, d(x, \partial \Omega)) \); let us momentarily denote \( a_f^{\mathbb{H}^1} \) and \( a_f^s \) to indicate dependence on \( d_{\mathbb{H}^1} \) or \( d_s \). Using Theorem 3.26, Theorem 3.32 and the proof of Lemma 3.19, we deduce by a similar argument as in the proof of Lemma 4.2 that for every \( K \geq 1 \), there exists a constant \( 0 < \Lambda_K < \infty \) such that

\[
\Lambda_K^{-1} a_f^{\mathbb{H}^1}(x) \leq a_f^s(x) \leq \Lambda_K a_f^{\mathbb{H}^1}(x), \quad \text{for all } x \in \Omega.
\]

It follows that once we have established Theorem 1.4 for either the Korányi or the sub-Riemannian distance, then it also holds for the other one.

**Structure of the paper. Novelty of results.** In Section 2 we introduce the most important notions used throughout this paper. We recall the modulus of curve families, some basic information about the Heisenberg group and discuss quasiconformal and quasisymmetric mappings in \( \mathbb{H}^1 \).

Section 3 is devoted to the equivalence of various characterizations of BMO spaces in \( \mathbb{H}^1 \) and the fact that \( \log J_f \in \text{BMO}(\Omega) \) for a quasiconformal map \( f : \Omega \to \Omega' \) between domains in \( \mathbb{H}^1 \). Many of the technical difficulties encountered in this discussion would not be present if \( \Omega \) was the entire space \( \mathbb{H}^1 \), and we consider it one of the contributions of the present paper to provide appropriate localizations of these results. In Proposition 3.3 we show fine quantitative estimates for the diameter and the measure of an image of a ball under a quasiconformal mapping of a domain. Similar estimates in the Euclidean setting, such as [38, Lemma 4] by M. Reimann, are proved using the modulus of Teichmüller rings, a method which is not available in our setting. Instead we use tools from analysis on metric measure spaces, for instance Loewner functions, which allow for generalizations of the argument to more abstract metric measure spaces. Among the results of Section 3.3, let us point out Theorem 3.32 showing that \( \log J_f \) belongs to BMO(\Omega). As far as we know, a direct proof of this result in the case \( \Omega \) is a domain, not the whole space, does not appear explicitly in the literature, even in the Euclidean setting (cf. [38, Remark 2]). One way to obtain the result is by first proving \( \log J_f \in \text{BMO}_{\text{loc}}(\Omega) \) and then using the identity \( \text{BMO}(\Omega) = \text{BMO}_{\text{loc}}(\Omega) \). This is the approach which we pursue here. In the case of the sub-Riemannian distance \( d_s \), the equality of BMO and \( \text{BMO}_{\text{loc}} \) goes back to
work of S. Buckley and O. Maasalo [12, 33]. We employ results by S. Staples [41] in order to deduce the corresponding identity for the Korányi distance $d_{\mathbb{H}^1}$ in place of $d_s$. This result does not follow directly from [12, 33] since $d_{\mathbb{H}^1}$ is not geodesic.

Section 4 is devoted to proving our main result, Theorem 1.4. The proof utilizes the auxiliary results established in Section 3 as well as other observations such as the distance estimate in Proposition 4.6. The latter extends [2, Lemma 5.15] from planar disks to arbitrary domains in $\mathbb{H}^1$. Certain tools available in the Euclidean setting, such as extension results for quasiconformal mappings or the Mori distortion theorem used in [2], are not available and a different approach is needed.

We conclude the paper with Section 5, in which we discuss various applications of Theorem 1.4, both for the sub-Riemannian and the Korányi distance. Coupled with ball estimates and covering arguments, the Koebe theorem yields quasiconformal versions of results established in [32] for quasisymmetries in an abstract setting. In Section 5.1 we extend a diameter estimate for images of curves under quasiconformal mappings by P. Koskela, [30, Lemma 2.6], to the setting of $\mathbb{H}^1$. Our result is slightly more flexible than the original lemma, since we allow for a quantitative control of the lengths of curves. By using an $\mathbb{H}^1$ version of the radial stretch mapping we show the sharpness of Proposition 5.1. Section 5.2 is devoted to proving the comparability relation between the $L^p$-operator norm of the differential of a quasiconformal mapping and the $L^p$-integral of $a_f$. The proof of this observation, see Theorem 5.10, employs several of the results proved in Sections 2–4. Furthermore, our proof requires a Harnack type-estimate for $a_f$, see Lemma 5.11, and a specific Whitney decomposition, Lemma 5.14. We consider such results of independent interest and trust that they will find more applications in geometric mapping theory on $\mathbb{H}^1$. Finally, in Section 5.3, we extend a result of Bonk–Koskela–Rohde [10] regarding conformal metrics and quasiconformal mappings on the unit ball to general domains in the Heisenberg group, see Proposition 5.26. We hope that quasiconformal densities and related metrics can be further investigated in the future.

Acknowledgements. We thank Pekka Koskela for bringing the article [3] to our attention. Part of the work on the present paper was done while K.F. visited IMPAN in October 2016 and while T.A. and B.W. visited the University of Fribourg in February 2017. The authors would like to thank the respective hosting institution for creating the scientific atmosphere and support.

2. Definitions and Preliminaries

The purpose of this section is to introduce key concepts used in this paper: modulus of curves, the Heisenberg group, and quasiconformal mappings (in the Heisenberg group). The definitions given here are standard, and a reader who is familiar with the subject may wish to go directly to Section 3.

2.1. Modulus of curve families. An important tool in this paper, and in the theory of quasiconformal mappings in general, is the modulus of curve families, discussed in detail for instance in monographs [35, 44].

By a curve in a metric space $(X, d)$ we mean a continuous map $\gamma : I \to X$ of an interval $I \subset \mathbb{R}$. A Borel function $\rho : X \to [0, +\infty]$ can be integrated with respect to arc length
along rectifiable curves. For a locally rectifiable curve $\gamma : I \to X$, we set
\[
\int_{\gamma} \rho \, ds = \sup_{\gamma'} \int_{\gamma'} \rho \, ds,
\]
where the supremum is taken over all rectifiable subcurves $\gamma'$ of $\gamma$.

**Definition 2.1.** Let $(X, d)$ be a metric space and let $\mu$ be a Borel measure on $X$. The admissible densities of a family $\Gamma$ of curves in $X$ are defined as
\[
\text{adm}(\Gamma) := \left\{ \rho : X \to [0, +\infty] \text{ Borel such that } \int_{\gamma} \rho \, ds \geq 1 \text{ for all } \gamma \in \Gamma \text{ locally rectifiable} \right\}.
\]
The $p$-modulus, $p \geq 1$, of $\Gamma$ is given by
\[
\text{mod}_p(\Gamma) := \inf \left\{ \int_X \rho^p \, d\mu : \rho \in \text{adm}(\Gamma) \right\}.
\]
We will often use the modulus of a family that consists of all curves in $X$ which connect two sets $E$ and $F$. We denote such a family by $\Gamma(E, F, X)$.

**Definition 2.2.** Let $(X, d)$ be a rectifiably connected metric space of Hausdorff dimension $Q$ and assume that $X$ is endowed with a locally finite Borel regular measure $\mu$ with dense support. Then $X$ is said to be a $(Q)$-Loewner space if for all $t \in (0, \infty)$ one has
\[
\psi(t) := \inf \left\{ \text{mod}_Q \Gamma(E, F, X) : \triangle(E, F) := \frac{\text{dist}(E, F)}{\min\{\text{diam}E, \text{diam}F\}} \leq t \right\} > 0, \tag{2.3}
\]
where the infimum is taken over disjoint nondegenerate continua $E$ and $F$ in $X$. We call the function $\psi$ the Loewner function of $(X, d, \mu)$.

### 2.2. The Heisenberg group.

The first Heisenberg group $\mathbb{H}^1$ is a noncommutative nilpotent Lie group homeomorphic to $\mathbb{R}^3$. It can be endowed with a left-invariant distance $d$ such that $(\mathbb{H}^1, d)$ does not biLipschitzly embed into any Euclidean space, yet exhibits a rich and interesting geometry. Comparable metrics on $\mathbb{H}^1$ can be obtained from different areas of mathematics such as control theory or complex hyperbolic geometry, where the Heisenberg group appears in a priori unrelated contexts. For an introduction to the subject, we refer the interested reader to the monograph [13].

Here we work with coordinates so that the group law on $\mathbb{H}^1$ reads
\[
(x, y, t)(x', y', t') = (x + x', y + y', t + t' - 2xy' + 2x'y).
\]
Using this group law, one can define a frame of left-invariant vector fields which agree with the standard basis at the origin:
\[
X := \partial_x + 2y\partial_t, \quad Y := \partial_y - 2x\partial_t, \quad T := \partial_t.
\]
The vector fields $X$ and $Y$, which are called horizontal, have a non-vanishing commutator $[X, Y] = -4T$. This ensures that any two points $p$ and $q$ in $\mathbb{H}^1$ can be connected by an absolutely continuous curve $\gamma : [0, 1] \to \mathbb{H}^1$ with the property that
\[
\dot{\gamma}(s) \in H_{\gamma(s)}, \quad \text{a.e. } s \in [0, 1], \text{ where } H_p := \text{span}\{X_p, Y_p\}.
\]
Such a $\gamma$ is called a horizontal curve. The sub-Riemannian distance $d_\gamma$ is defined by

$$d_\gamma(p,q) = \inf_\gamma \int_0^1 \sqrt{\gamma_1(s)^2 + \gamma_2(s)^2} \, ds,$$

where the infimum is taken over all horizontal curves $\gamma = (\gamma_1, \gamma_2, \gamma_3) : [0,1] \to \mathbb{H}^1$ that connect $p$ and $q$. It is well known that $d_\gamma$ defines a left-invariant metric on $\mathbb{H}^1$ which is homogeneous under the Heisenberg dilations $(\delta_\lambda)_{\lambda > 0}$, given by

$$\delta_\lambda : \mathbb{H}^1 \to \mathbb{H}^1, \quad \delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t).$$

Any two homogeneous left-invariant metrics on $\mathbb{H}^1$ are bi-Lipschitz equivalent, and it is often more convenient to work with a metric which is given by an explicitly computable formula, rather than to use $d_\gamma$. An example of such a metric is the Korányi distance, defined by

$$d_{\mathbb{H}^1}(p,q) = \|q^{-1}p\|_{\mathbb{H}^1}, \quad \text{where} \quad \|(x,y,t)\|_{\mathbb{H}^1} = \sqrt{4(x^2 + y^2)^2 + t^2}.$$

The length distance associated to $d_{\mathbb{H}^1}$ is exactly $d_\gamma$.

In addition to the metric structure, we endow the Heisenberg group with a bi-invariant Haar measure $m$ which is given by the Lebesgue measure on $\mathbb{R}^3$. We recall that this measure $m$ is Ahlfors 4-regular and satisfies the annular decay property. It agrees, up to a positive and finite multiplicative factor, with the 4-dimensional Hausdorff measure with respect to a left-invariant homogeneous distance on $\mathbb{H}^1$. Unless otherwise stated, “measurable” and “integrable” will in the following always mean “$m$ measurable” and “$m$ integrable”. We denote $m(A) := |A|$ for $A \subseteq \mathbb{H}^1$, and we write $\int f \, dm = \int f(x) \, dx$.

Equipped with $m$ and any homogeneous left-invariant distance, the Heisenberg group becomes a 4-Loewner space.

**Convention.** Whenever we discuss quantitative dependencies of parameters on certain constants, we will omit information that such constants may also depend on the data of the metric measure space $(\mathbb{H}^1, d_{\mathbb{H}^1}, m)$ or $(\mathbb{H}^1, d_\gamma, m)$. For instance, if we say that “a constant $C$ depends only on the distortion $K$ of the mapping”, the constant $C$ may depend also on the Loewner function, the quasiconvexity constant, etc.

As remarked in the introduction, Theorem 1.4 for $d_{\mathbb{H}^1}$ is equivalent to the analogous statement with respect to $d_\gamma$. The same holds true for the applications (Proposition 5.1, Theorem 5.10, Proposition 5.26). For auxiliary results needed in these proofs, we will always specify whether they hold with respect to $d_\gamma$, $d_{\mathbb{H}^1}$, or both.

2.3. **Quasiconformal and quasisymmetric mappings.** In this section we collect the relevant facts about quasiconformal mappings in the Heisenberg group. Quasiconformal maps can be defined primarily by three definitions, the metric, analytic and geometric definition, all of which are mutually and quantitatively equivalent on domains in spaces of locally $Q$-bounded geometry, even though the distortion factor need not be the same for each definition. The equivalence of these definitions is a central part of the general theory of quasiconformal maps and we refer the reader to [19, 24, 43] for details at a general level and [29] for the specific case of the Heisenberg group. An important feature to note is that the class of metrically defined quasiconformal maps is the same for any pair of bi-Lipschitz equivalent metrics with a quantitative control on the distortion. As remarked in the introduction, if the two metrics are the sub-Riemannian distance $d_\gamma$ and the Korányi metric $d_{\mathbb{H}^1}$, then one gets even the same class of $K$-quasiconformal maps.
Thus, in our context it often does not matter if we use the sub-Riemannian metric or the Korányi metric and so we leave the metric unspecified in the respective statements.

**Definition 2.4 (Metric definition).** If $f : \Omega \to f(\Omega) \subseteq \mathbb{H}^1$ is a homeomorphism of an open set $\Omega \subseteq \mathbb{H}^1$, then for all $p \in \Omega$ and all $r > 0$ we define

\[
L_f(p, r) := \sup\{d(f(p), f(q)) : q \in \Omega, d(p, q) \leq r\},
\]

\[
l_f(p, r) := \inf\{d(f(p), f(q)) : q \in \Omega, d(p, q) \geq r\}.
\]

\[
H_f(p) := \limsup_{r \to 0} \frac{L_f(p, r)}{l_f(p, r)}.
\]

The mapping $f$ is *quasiconformal* if $H_f$ is bounded on $\Omega$.

While metric quasiconformality is an infinitesimal property, *quasisymmetry* is a global and generally stronger condition.

**Definition 2.5 (Quasisymmetric definition).** If $\Omega$ is an open set in $\mathbb{H}^1$ and $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism, then we say that a homeomorphism $f : \Omega \to f(\Omega) \subseteq \mathbb{H}^1$ is $\eta$-*quasisymmetric* if

\[
\frac{d(f(p_1), f(p_2))}{d(f(p_1), f(p_3))} \leq \eta(t)
\]

for all triples of points $p_1, p_2, p_3 \in \Omega$ satisfying $d(p_1, p_2) \leq td(p_1, p_3)$. A map $f$ is said to be *quasisymmetric* if it is $\eta$-quasisymmetric for some $\eta$.

**Remark 2.7.** Definition 2.5 has an obvious local counterpart; we say that $f$ is *locally $(\eta)$-quasisymmetric* if for each $p \in \Omega$, condition (2.6) holds on some relative neighbourhood of $p$.

A quasiconformal map defined on all of $\mathbb{H}^1$ is quasisymmetric [22]. An analogous statement is not true in general for mappings defined on a subdomain of $\mathbb{H}^1$, but the metric definition still implies a local quasisymmetry condition. This goes back to [29, Proposition 22]. H. Heinonen remarked in [19] that one can obtain the following quantitative control on the neighbourhood where quasisymmetry holds.

**Theorem 2.8 (Heinonen, Koskela).** If $f$ is a quasiconformal homeomorphism of an open set $\Omega \subseteq (\mathbb{H}^1, d_\star)$ according to Definition 2.4, then there exists $\eta$ as in Definition 2.5, such that $f$ satisfies (2.6) for all triples $p, q_1, q_2 \in \Omega$ with $q_1, q_2 \in B(p, \frac{1}{t} \text{dist}(p, \partial \Omega))$.

By the triangle inequality, the “$p$-centered” quasisymmetry property in Theorem 2.8 implies quasisymmetry of $f$ on the sub-Riemannian ball $B(p, \frac{1}{t} \text{dist}(p, \partial \Omega))$. Following the terminology in [20, p.93], we call this fact an “egg yolk principle”. E. Soultanis and M. Williams [40] provided a proof for this principle in great generality, and with a quantitative control both on the size of the “egg yolk” and the $\eta$-function in the definition of local quasisymmetry. We include this statement as Proposition 4.4 for reference later in the text.

It was shown by Mostow that a quasiconformal map on a domain in $\mathbb{H}^1$ is *absolutely continuous on lines* (ACL), see the discussion in [29]. This property is defined analogously as the ACL property for mappings on open subsets of $\mathbb{R}^n$, but in terms of the fibrations given by the left invariant horizontal vector fields $X$ and $Y$ instead of lines parallel to the coordinate axes. In [36], P. Pansu showed that local quasisymmetry for a map $f$ on
an open subset of $\mathbb{H}^1$ implies further analytic features similar to those of quasiconformal mappings on domains in $\mathbb{R}^n$. First, for a quasiconformal map $f$ the horizontal derivatives $Xf(p)$ and $Yf(p)$ exist for almost every $p \in \Omega$. The second regularity property is Pansu differentiability ($P$-differentiability) at almost every $p \in \Omega$. Specifically this means that the parameterized family given by

$$q \to \delta_{1/s}(f(p)^{-1}f(p\delta_s(q)))$$

converges locally uniformly to a homomorphism $D_pf(p)$ of $\mathbb{H}^1$ as $s \to 0^+$. Here, $\delta_h(q)$ stands for a dilation by $h$ at point $q$.

We define the Jacobian of a quasiconformal map $f$ as the volume derivative

$$Jf(p) = \limsup_{r \to 0} \frac{|f(B(p,r))|}{|B(p,r)|}, \quad p \in \Omega.$$ 

According to Lebesgue’s differentiation theorem, the $\limsup$ can be replaced by $\lim$ in almost every point $p \in \Omega$. If a quasiconformal map $f = (f_1, f_2, f_3)$ is $P$-differentiable at $p$, then the homomorphism $D_pf(p)$ is given in a matrix form (with respect to the frame $\{X,Y,T\}$) by

$$\begin{pmatrix} Xf_1 & Yf_1 & 0 \\ Xf_2 & Yf_2 & 0 \\ 0 & 0 & Xf_1Yf_2 - Xf_2Yf_1 \end{pmatrix},$$

and $Jf(p) = \det D_Hf(p)^2$ where $D_Hf(p) = \begin{pmatrix} Xf_1 & Yf_1 \\ Xf_2 & Yf_2 \end{pmatrix}$.

The analytic definition of quasiconformal mappings can now be stated as follows.

**Definition 2.9** (Analytic definition). If $\Omega$ is an open set in $\mathbb{H}^1$, we say that a homeomorphism $f : \Omega \to f(\Omega) \subseteq \mathbb{H}^1$ is $K$-quasiconformal if it is ACL, Pansu differentiable at almost every $p \in \Omega$, and satisfies the following distortion condition: there exists $1 \leq K < \infty$ such that

$$\|D_Hf(p)\|^4 \leq KJf(p), \quad (2.10)$$

where

$$\|D_Hf(p)\| = \max\{|D_Hf(p)\xi| : \xi \in H_p, |\xi| = 1\}$$

and $|\cdot|$ is obtained from the inner product which makes $\{X,Y\}$ orthonormal. A map $f$ is said to be quasiconformal if it is $K$-quasiconformal for some $1 \leq K < \infty$.

Quasiconformal mappings are not only ACL, but also absolutely continuous in measure (Proposition 3 in [29]) with $Jf > 0$ almost everywhere on $\Omega$. The equivalence of Definitions 2.9 and 2.4 can be established via a geometric definition, see Heinonen [19] and Korányi–Reimann [29] for details and recall Definition 2.1 above.

**Definition 2.11** (Geometric definition). If $\Omega$ is an open set in $\mathbb{H}^1$, then we say that a homeomorphism $f : \Omega \to f(\Omega) \subseteq \mathbb{H}^1$ is quasiconformal, if there exists a constant $1 \leq L < \infty$ such that

$$\frac{1}{L} \text{mod}_4(\Gamma) \leq \text{mod}_4(f\Gamma) \leq L\text{mod}_4(\Gamma)$$

for every curve family $\Gamma$ in $\Omega$. 
As stated before, the metric, analytic and geometric definition of quasiconformality are all quantitatively equivalent in the Heisenberg setting. We have decided to define “$K$-quasiconformal” through the analytic distortion inequality (2.10). This is a matter of taste, but is convenient due to the following implications for a homeomorphism $f$:

1. $K$-quasiconformal $\iff$ metrically quasiconformal with $\text{esssup}_{p \in \Omega} H_f(p) \leq \sqrt{K}$,

2. $K$-quasiconformal $\implies$ geometrically quasiconformal with $L = K$.

Given the equivalence of the definitions presented above, the inverse of a quasiconformal mapping is quasiconformal. More precisely we have the following theorem due to Korányi and Reimann ([29], Proposition 20).

**Theorem 2.12.** The inverse of a $K$-quasiconformal mapping is $K$-quasiconformal.

3. **The Jacobian of a Quasiconformal Mapping**

It is well known that the Jacobian $J_f$ of a quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an $A_\infty$-weight and that $\log J_f$ is of bounded mean oscillation (BMO), see Sections 3.2 and 3.3 for the definitions. The situation is more subtle if one considers quasiconformal maps on a subdomain $\Omega \subset \mathbb{R}^n$. As shown in [21], it is not true in general that $J_f$ is an $A_\infty$-weight on $\Omega$, but even so $\log J_f$ lies in the (appropriately defined) space $\text{BMO}(\Omega)$.

The goal of this section is to show that the logarithm of the Jacobian of a quasiconformal map $f : \Omega \to \Omega'$ between domains in $\mathbb{H}$ belongs to $\text{BMO}(\Omega)$. The outline of the proof follows its Euclidean predecessors in [38, 39]. We emphasize that the main technical difficulty stems from the fact that we consider mappings which might be defined only on a subdomain of $\mathbb{H}$, and we work both with the sub-Riemannian distance and the Korányi metric. For mappings of the entire Heisenberg group, it is well known that $J_f$ is an $A_p$-weight for some $1 \leq p < \infty$, and an $A_\infty$-weight. This is a consequence of a ‘reverse Hölder inequality’ due to Korányi and Reimann (Theorem 3.1 below), see for instance the overview in Section 3 of [5]. In the latter paper, A. Austin considered weights in connection with the quasiconformal Jacobian problem on $\mathbb{H}$. This is a well known problem, which is open even in the Euclidean setting. The task is to characterize those nonnegative weights $w \in L^1_{\text{loc}}$ which are comparable to the Jacobian of a quasiconformal mapping. A large class of such weights has been identified by M. Bonk, J. Heinonen and E. Saksman in [9], and Austin established an analogous result in the Heisenberg group [5]. The goal of the present section is more elementary: we start from the Jacobian of a quasiconformal mapping and study its properties.

3.1. **Higher integrability and reverse Hölder inequality.** Quasiconformal mappings between domains in $\mathbb{H}$ satisfy a higher integrability property analogous to the one established by F. Gehring in $\mathbb{R}^n$. Specialized to the first Heisenberg group (endowed with the Korányi distance), Theorem G in [29] reads as follows.

**Theorem 3.1 (Korányi, Reimann).** For every $K \geq 1$, there exist constants $c, k > 0$ depending only on $K$ such that the Jacobian of a $K$-quasiconformal mapping $f : \Omega \to \Omega'$ between domains in $(\mathbb{H}, d_{\mathbb{H}})$ satisfies the inequality

$$\left( \frac{\fint_{B(x,r)} J_f^p \, dm}{d_{\mathbb{H}}(x, \partial \Omega)} \right)^{\frac{1}{p}} \leq \left( \frac{k}{4 + k - p} \right)^{\frac{1}{p}} \fint_{B(x,r)} J_f \, dm,$$

for all $p \in [4, 4 + k)$ and $3r \leq \frac{d_{\mathbb{H}}(x, \partial \Omega)}{c}$. 

In what follows we assume, as we may, that \( c > 1 \). The statement of Theorem G in [29] contains no mean value integrals, however the proofs of Propositions 19, 21 and 23 in [29] reveal that assertion (3.2) holds as stated above. The proof is based on a Heisenberg version of Gehring’s lemma and a reverse Hölder inequality for \( J_f \).

3.2. \( A_p \)-weights and distortion of balls. In this section we apply Theorem 3.1 in order to show that the Jacobian of a quasiconformal map satisfies an \( A_p \)-weight condition on balls which lie well inside the domain of the mapping, see Proposition 3.9 for the precise statement. This is one of the key ingredients in the proof of the BMO property of the Jacobian (Theorem 3.32).

3.2.1. Distortion of balls. For the proof of Proposition 3.9, we use the following auxiliary observation, a generalization of Lemma 4 in [38] to the setting of the Heisenberg group equipped with the Korányi distance. We seize the opportunity to state more detailed estimates than what is needed at this moment. The additional information will prove useful later for the applications in Section 5.

**Proposition 3.3.** Let \( f : \Omega \to \Omega' \) be a \( K \)-quasiconformal map between domains \( \Omega, \Omega' \subseteq (\mathbb{H}^1, d_{\mathbb{H}^1}) \). Then, for any constant \( c \geq 1 \), there exists a constant \( k > 1 \), depending only on \( K \), \( \beta \) and the data of \( \mathbb{H}^1 \), such that for all balls \( B' = B(y', r') \subseteq \Omega' \) satisfying \( 3kB' \subseteq \Omega' \), there is a ball \( B = B(f^{-1}(y'), r) \subseteq \Omega \) with the following properties:

1. \( cB \subseteq \Omega \)
2. \( B' \subseteq f(B) \subseteq kB' \) and \( \text{diam} f(B) \leq \text{dist}(f(B), \partial \Omega') \)
3. \( |f(B)| \leq k^4 |B'| \).

In particular, the claim holds for \( c \) as in the assertion of Theorem 3.1.

Before launching into the proof, let us discuss a useful consequence of this. Proposition 3.3 shows that for every \( \beta > 1 \), there exists a constant \( k > 1 \), depending only on \( \beta \) and \( K \) so that for every ball \( B' = B(y', r') \) with \( 3kB' \subseteq \Omega' \), we can find a ball \( B = B(x, r) \), \( x = f^{-1}(y') \) with \( f^{-1}(B') \subseteq B \) and \( \beta B \subseteq \Omega \). The condition \( \beta B \subseteq \Omega \) implies that

\[
\text{dist}(B, \partial \Omega) \geq d_{\mathbb{H}^1}(x, \partial \Omega) - r \geq (\beta - 1)r,
\]

hence \( \text{diam} B \leq 2r \leq 2 \text{dist}(B, \partial \Omega)/(\beta - 1) \), and so

\[
\text{diam} f^{-1}(B') \leq \text{diam} B \leq \frac{2 \text{dist}(B, \partial \Omega)}{\beta - 1} \leq \frac{2 \text{dist}(f^{-1}(B'), \partial \Omega)}{\beta - 1}.
\]

Moreover,

\[
d_{\mathbb{H}^1}(x, \partial \Omega) \leq r + \text{dist}(B, \partial \Omega) \leq \left( \frac{1}{\beta - 1} + 1 \right) \text{dist}(B, \partial \Omega) \leq \left( \frac{1}{\beta - 1} + 1 \right) \text{dist}(f^{-1}(B'), \partial \Omega).
\]

These observations yield the following corollary.

**Corollary 3.5.** Let \( g : U \to U' \) be a \( K \)-quasiconformal mapping between domains \( U, U' \subseteq (\mathbb{H}^1, d_{\mathbb{H}^1}) \). Then, for every \( \beta > 1 \), there exists a constant \( k > 1 \), depending only on \( \beta \) and \( K \) so that for every ball \( B = B(y, s) \) with \( 3kB \subseteq U \), one has

\[
\text{diam} g(B) \leq \frac{2d_{\mathbb{H}^1}(g(y), \partial U')}{\beta - 1}.
\]
and
\[ d_{\mathbb{H}^1}(g(y), \partial U') \leq \frac{\beta}{\beta - 1} \text{dist}(g(B), \partial U'). \]

Proof. This follows from the conclusion of (3.4) applied to \( f = g^{-1}, \Omega = U' \) and \( \Omega = U, \)
\( y' = y \) and \( r' = s. \) Observe that \( \text{dist}(g(B(y, r)), \partial U') \leq d_{\mathbb{H}^1}(g(y), \partial U'). \)

Our result in Proposition 3.3 differs from [38, Lemma 4] in two ways: first, we find balls that are far enough away from \( \partial \Omega \) with their distance quantified by the constant \( c. \)
Second, our proof of Proposition 3.3 employs Loewner functions, rather than the modulus of a Teichmüller ring, which is a Euclidean concept. This allows for generalizations of the argument and, in fact, estimates similar to the ones in Proposition 3.3 appear frequently in the theory of quasiconformal mappings on metric measure spaces.

Proof of Proposition 3.3. Without loss of generality we may assume that \( 0 \in \Omega, y' = 0, \) and \( f(0) = 0. \) This can be obtained by preliminary left translations, which by definition send balls to balls and preserve the class of \( K \)-quasiconformal maps under composition.

Denote by \( R' \subset \Omega \) the following Korányi ring domain \( R' = B(0, kr') \setminus B(0, r'), \) where \( k > 1 \) is a constant to be determined in the course of the proof. Let us consider the family \( \Gamma' \) consisting of all curves \( \gamma' \) joining \( \partial B(0, r') \) with \( \partial B(0, kr') \) such that \( |\gamma'| \in R' \) except for endpoints of \( \gamma'. \) By [28] we know that
\[ \text{mod}_4 \Gamma' = \omega_4 (\log k)^{-3}, \]
for a universal positive and finite constant \( \omega_4. \) Next, let \( R = f^{-1}(R') \subset \Omega \) be the image of the ring \( R \) under \( f^{-1}, \) and set
\[ s := \inf_{\|p\|_{\mathbb{H}^1} = kr'} \| f^{-1}(p) \|_{\mathbb{H}^1} \]
\[ r := \sup_{\|p\|_{\mathbb{H}^1} = r'} \| f^{-1}(p) \|_{\mathbb{H}^1}. \quad (3.6) \]
Note that with \( s \) and \( r \) as defined above, it holds that \( B(0, s) \subset f^{-1}(B(0, kr')) \) and \( f^{-1}(B') \subset B(0, r). \)

As we will see later, if \( r < s/c, \) then the image of \( B(0, r) \) is ‘roundish’ as desired. To arrive at the correct ratio of \( r \) and \( s, \) we employ the quasi-invariance of the modulus. To this end, we choose the following nontrivial continuum, that is, compact connected set containing more than one point: by the path-connectedness of the closed ball \( B(0, r') \) we may choose a curve \( \lambda \) connecting 0 to \( \partial B(0, r') \) inside \( B(0, r'). \) Namely, we choose to connect 0 to a point \( p \in \partial B(0, r') \) which realizes the supremum in the definition of \( r \) (since \( f \) is a homeomorphism and \( \partial B(0, r') \) is compact, such a point \( p \) exists). Then, upon letting \( C_0 := f^{-1}\lambda, \) we conclude that \( C_0 \) connects 0 and \( \partial B(0, r) \) and lies inside \( f^{-1}(B(0, r')). \)

Furthermore, we need to choose another continuum, denoted \( C_1, \) to be a closed connected set lying outside of \( f^{-1}(B(0, kr')) \) and starting at a point \( \partial B(0, s). \) Such a continuum is by construction disjoint from \( C_0. \) In addition, we require that \( C_1 \) is large enough so that \( \text{diam } C_1 > 2 \text{diam } C_0. \) The existence of such a continuum comes from the definition of \( s, \) continuity of \( f^{-1} \) and path-connectedness of open connected subsets of \( \mathbb{R}^3 \) (or \( \mathbb{H}^1 \)). More precisely, let \( q \in \partial B(0, kr') \) be a point which realizes the infimum in the
definition of $s$. Let $\delta$ be a curve in $\Omega'$ which joins $q$ and $\partial B(0, 2kr')$ (again, we invoke the path-connectedness of the closed annulus) and recall that by assumption $B'(0, 2kr') \subseteq \Omega'$. Then, let us consider the curve $f^{-1}\delta$. One of its endpoints is $f^{-1}(q)$ and lies on $\partial B(0, s)$. The other endpoint of this curve lies in the (path-connected) domain $\mathbb{H}^1 \setminus f^{-1}(B(0, kr'))$ and $f^{-1}\delta$ can thus be prolonged to become an arbitrarily long curve $C_1$, say such that $\text{diam} C_1 > 2 \text{diam} C_0$.

We use the Loewner function $\psi : (0, +\infty) \to (0, +\infty)$ of $(\mathbb{H}^1, d_{\mathbb{H}^1}, m)$, as defined in (2.3). Note that $\psi$ is a decreasing function. Set $k > 1$ such that the following condition is satisfied:

$$K \omega_4(\log k)^{-3} < \min\{\psi(1/2), \psi(c/2)\}.$$  \hfill (3.7)

There are three distinct cases involving the scalars $r$ and $s$ which we now consider.

Case (i): $r < s/c$. Then $B := B(0, r)$ satisfies the claim. Indeed, since $B(0, r) \subseteq B(0, s/c)$, we have $cB(0, r) \subseteq B(0, s) \subseteq \Omega$ and part (1) of the assertion holds. By the definition of $r$ we have that $B' \subseteq f(B(0, r))$, moreover it holds that $f(B(0, r)) \subseteq f(B(0, s)) \subseteq B(0, kr')$.

This observation immediately implies parts (2) and (3) of the assertion.

The second case is $s/c \leq r$. If $c > 1$, we consider separately the following two subcases:
(ii) $s/c \leq r < s$ and (iii) $s \leq r$. We show that for the choice of parameter $k$ as in (3.7) these cases are not possible. For the proof let us first observe that for the curve family $\Gamma := f^{-1}(\Gamma')$ the modulus based definition of quasiconformality implies that

$$K \omega_4(\log k)^{-3} = K \text{mod}_4(\Gamma') \geq \text{mod}_4(\Gamma).$$

We observe that $\Gamma < \Gamma(C_0, C_1, \mathbb{H}^1)$, i.e. $\Gamma$ is minorized by $\Gamma(C_0, C_1, \mathbb{H}^1)$; every curve in $\Gamma(C_0, C_1, \mathbb{H}^1)$ has a subcurve which belongs to $\Gamma$. This, together with the definition of $\psi$ in (2.3) implies that

$$K \omega_4(\log k)^{-3} \geq \text{mod}_4(\Gamma) \geq \psi(\Delta(C_0, C_1)).$$  \hfill (3.8)

If $s/c \leq r < s$, then

$$\Delta(C_0, C_1) = \frac{\text{dist}(C_0, C_1)}{\min\{\text{diam} C_0, \text{diam} C_1\}} \leq \frac{s}{2r} \leq \frac{c}{2},$$

and, hence, $\psi(\Delta(C_0, C_1)) \geq \psi(c/2)$. Applying this estimate in (3.8) we reach a contradiction with the choice of $k$ in (3.7).

Finally, if $s \leq r$ then reasoning analogous to the previous case shows that

$$\Delta(C_0, C_1) \leq \frac{s}{2r} \leq \frac{1}{2},$$

and so $\psi(\Delta(C_0, C_1)) \geq \psi(1/2)$. This again contradicts the choice of $k$ as in (3.7). Therefore the discussion of the cases (ii) and (iii) is completed, and so is the proof of the proposition.

\[ \square \]

After this digression, we return to the main result of this section.
3.2.2. \(A_p\)-weights. With the preliminary ball distortion estimates at hand, we now prove a property for the Jacobian of quasiconformal mappings on subdomains of the Heisenberg group endowed with the Korányi distance.

**Proposition 3.9.** Let \( f : \Omega \to \Omega' \) be a \(K\)-quasiconformal map between domains in \((\mathbb{H}^1, d_{\mathbb{H}^1})\). Then there exists a constant \(C > 0\), depending on \(K\) only, such that the weight condition

\[
\frac{1}{|B'|} \int_B J_f(x)^{-\frac{4\alpha}{p}} f(x) \, dx \leq \frac{1}{|B|} \int_{f^{-1}(B)} J_f(x)^{-\frac{4\alpha}{p}} f(x) \, dx
\]

holds for all balls \(B\) in \(\Omega\) such that \(3kB \subset \Omega\), where \(k > 1\) denotes the constant from Proposition 3.3 (applied to \(f^{-1}\) and the constant \(c\) from Theorem 3.1 applied to \(f^{-1}\)) and \(p\) as in Theorem 3.1. In particular, \(k\) and the bound for \(p\) depend on \(K\) only.

**Proof.** If \(\Omega = \Omega' = \mathbb{H}^1\), the claim follows from Theorem 3.1 by estimates which are standard in the theory of Muckenhoupt weights, so we concentrate on the case where \(\Omega\) and \(\Omega'\) are strict subdomains in \(\mathbb{H}^1\). While the proof still follows largely the same steps of reasoning as given in the proof of Lemma 5 in [38] for a quasiconformal map from \(\mathbb{R}^n\) to \(\mathbb{R}^n\), we need to employ Proposition 3.3 since our result is localized to sets \(\Omega\) and \(\Omega'\).

Let \(c\) be the constant from Theorem 3.1 (applied to \(f^{-1}\)) and \(B' \subset \Omega'\) be any ball such that \(cB' \subset \Omega'\). Since \(J_{f^{-1}}(y) = J_f(f^{-1}(y))^{-1}\) for almost every \(y \in \Omega'\), Theorem 3.1 applied to \(f^{-1}\) shows that

\[
\left( \int_{B'} J_f(f^{-1}(y))^{-\frac{4\alpha}{p}} \, dy \right)^{\frac{1}{p}} \leq C \int_{B'} J_f(f^{-1}(y))^{-1} \, dy,
\]

for \(C = (k/(4 + k - p))^{1/p}\). We are now in a position to apply Proposition 3.3 for balls in \(\Omega'\) and \(f^{-1}\) with constant \(c\) as defined in the beginning of the proof. Hence, we find a constant \(k\) such that whenever the ball \(B \subset \Omega\) is such that \(3kB \subset \Omega\), then there exists a ball \(B' \subset \Omega'\) with the following properties:

\[
cB' \subset \Omega', \quad f^{-1}(B') \subset kB, \quad B \subset f^{-1}(B').
\]

In particular, (3.11) applies to such \(B'\). The inclusions (3.12) and the change of variable formula, see for instance Theorem 5.4(a) in [17], result in the following estimate:

\[
\left( \frac{1}{|B'|} \int_B J_f(x)^{-\frac{4\alpha}{p}} f(x) \, dx \right)^{\frac{1}{p}} \leq C \int_{f^{-1}(B')} J_f(x)^{-\frac{4\alpha}{p}} f(x) \, dx = \int_{B'} J_f(f^{-1}(y))^{-\frac{4\alpha}{p}} \, dy.
\]

This and (3.11) lead to the inequality:

\[
\left( \frac{1}{|B'|} \int_B J_f(x)^{-\frac{4\alpha}{p}} f(x) \, dx \right)^{\frac{1}{p}} \leq C \int_{B'} J_f(f^{-1}(y))^{-1} \, dy = C \frac{|f^{-1}(B')|}{|B'|}.
\]

Since \(|B'| = \int_{f^{-1}(B')} J_f(x) \, dx\), we obtain from (3.13) that

\[
\left( \frac{1}{|B'|} \int_B J_f(x)^{\frac{4\alpha}{p - 4}} \, dx \right)^{\frac{1}{p}} \leq C |f^{-1}(B')| \left( \frac{1}{|B'|} \int_{f^{-1}(B')} J_f(x) \, dx \right)^{\frac{4\alpha}{p - 4}}.
\]

Thus, by applying (3.12) and part (3) of Proposition 3.3, we arrive at the following inequality:

\[
\left( \frac{1}{|B'|} \int_B J_f(x)^{\frac{4\alpha}{p - 4}} \, dx \right)^{\frac{1}{p}} \leq C k^4 |B|^{\frac{4\alpha}{p - 4}} \left( \int_B J_f(x) \, dx \right)^{\frac{4\alpha}{p - 4}}.
\]
The above estimate immediately implies
\[
\left( \int_B J_f(x)^{\frac{4-p}{4}} \, dx \right)^{\frac{4}{p}} \leq Ck^A \left( \int_B J_f(x) \, dx \right)^{\frac{4-p}{4}}
\]
and hence
\[
\int_B J_f(x) \, dx \leq \left( Ck^A \right)^{\frac{p}{4-p}} \left( \int_B J_f(x)^{\frac{4-p}{4}} \, dx \right)^{\frac{4}{p}}.
\]
\[\square\]

Let us pause for a moment and interpret the statement of Proposition 3.9 in the context of Muckenhoupt $A_p$-weights. We recall the following definitions in metric measure spaces, cf. [27] and [16]. A weight in a metric measure space $(X,d,\mu)$ is a locally integrable function $w : X \to [0, +\infty]$.

**Definition 3.14.** A weight $w$ is said to be an $A_1$-weight if there exists a constant $C > 0$ such that for every ball $B$ in $X$, one has
\[
\int_B w \, d\mu \leq C \text{essinf}_B w.
\]

For $1 < p < \infty$, a weight $w$ is an $A_p$-weight if there exists a constant $C > 0$ such that for every ball $B$ in $X$, one has
\[
\left( \int_B w \, d\mu \right) \left( \int_B w^{-1/(p-1)} \right)^{p-1} \leq C.
\]

A weight $w$ is an $A_\infty$-weight if there exists a constant $C > 0$ such that for every ball $B$ in $X$, one has
\[
\left( \int_B w \, d\mu \right) \left( \exp \left( -\int_B \log w \, d\mu \right) \right) \leq C.
\]

Various definitions of $A_\infty$-weights are used concurrently in literature, and they are not all equivalent in an arbitrary metric space $(X,d,\mu)$. However, one has equivalence if $\mu$ is a Borel regular doubling measure with the so-called annular decay property. This has been proved by P. Shukla in his PhD thesis, see also [27, Theorem 3.1]. If $\mu$ is such a measure, then
\[
A_\infty(\mu) = \bigcup_{1 \leq p < \infty} A_p(\mu).
\]

(3.15)

Returning to Proposition 3.9, we see that the statement essentially says that $w = J_f$ is an $A_q$-weight in the sense of Definition 3.14 for $q = 1 + \frac{4}{p-4}$. This is not quite true, because the $A_q$-weight condition is verified only for those Korányi balls $B$ which lie well inside the domain $\Omega$ in the sense that $3kB \subset \Omega$. It is for this reason that one cannot conclude that $J_f$ is an $A_\infty$-weight if $f$ is defined on an arbitrarily given domain $\Omega$ in $\mathbb{H}^1$. This also means that one cannot directly conclude that $\log J_f \in \text{BMO}(\Omega)$, but as we will see in the next section, the latter statement still holds true.
3.3. **Logarithm and BMO.** A classical reference for BMO spaces on homogeneous groups is [18]. The goal of this section is to apply information on the Jacobian $J_f$ of a quasiconformal mapping (Proposition 3.9) to deduce that $\log J_f$ belongs to a certain local BMO space. A characterization of BMO spaces in doubling length metric spaces, dating back to S. Buckley, finally shows that $\log J_f$ belongs to the $BMO(\Omega)$-space defined with respect to the sub-Riemannian distance $d = d_s$. The definition of BMO-space also makes sense for $d$ equal to the Korányi (Heisenberg) distance $d_{\mathbb{H}^1}$, and eventually, we will show that all the respective spaces agree.

**Definition 3.16.** Let $\Omega$ be an open subset of $\mathbb{H}^1$. We say that a function $u \in L^1_{\text{loc}}(\Omega)$ belongs to $BMO(\Omega)$ if there is a constant $C$ such that

$$\int_B |u - u_B| \, dm \leq C, \quad \text{for every } d\text{-ball } B \subseteq \Omega.$$  

We say that $u \in L^1_{\text{loc}}(\Omega)$ belongs to the *local* $n$-BMO space $BMO_{n,\text{loc}}(\Omega)$ for $n \geq 1$ if there is a constant $C$ such that

$$\int_B |u - u_B| \, dm \leq C, \quad \text{for every } d\text{-ball } B \text{ with } nB \subseteq \Omega. \quad (3.17)$$  

We say that $u \in L^1_{\text{loc}}(\Omega)$ belongs to $BMO_{\text{loc}}(\Omega)$ if there is $n > 1$ such that $u \in BMO_{n,\text{loc}}(\Omega)$.

**Definition 3.18.** For a domain $\Omega \subseteq \mathbb{H}^1$ and a function $u \in BMO(\Omega)$, we define the BMO-norm as

$$\|u\|_* := \sup_B \int_B |u - u_B| \, dm,$$

where the supremum is taken over all balls $B \subseteq \Omega$.

The $BMO_{n,\text{loc}}$-norm for $u \in BMO_{n,\text{loc}}(\Omega)$ is defined analogously, with the difference that the supremum is taken only over those balls which satisfy $nB \subseteq \Omega$.

Note that the ‘BMO-norm’ is in fact only a seminorm since $\|u\|_* = 0$ for every $u$ which is constant.

The following lemma addresses some of the claims made in the introduction. In order to formulate it, for $n \geq 1$, let us denote by $BMO_{n,\text{loc}}(\Omega)$ the local $n$-BMO space with $\Omega$ equipped with the Korányi distance. Moreover, let $BMO^*_n(\Omega)$ stand for the local $n$-BMO space with $\Omega$ equipped with the (restricted) sub-Riemannian distance on $\mathbb{H}^1$.

**Lemma 3.19.** There exists a universal constant $1 \leq C < \infty$, such that for all open sets $\Omega \subseteq \mathbb{H}^1$, for all $n \geq 1$ and for all $u \in L^1_{\text{loc}}(\Omega)$, one has

$$\|u\|_{BMO^*_n(\Omega)} \leq C \|u\|_{BMO_{n,\text{loc}}(\Omega)}$$

and

$$\|u\|_{BMO^*_{\text{loc}}(\Omega)} \leq C \|u\|_{BMO^*_{n,\text{loc}}(\Omega)}.$$  

In particular, one has

$$BMO^*_n(\Omega) \subset BMO^*_{\text{loc}}(\Omega) \quad \text{and} \quad BMO^*_{n',\text{loc}}(\Omega) \subset BMO^*_{n,\text{loc}}(\Omega), \quad (3.20)$$

for all $n, n' \geq 1$.

---

1In our application, the constant $n$ will be determined by the proof. In the standard definition of $BMO_{\text{loc}}(\Omega)$ one would take $n = 2$, as for instance in [12].
Proof. Let us first recall the precise relation between $d_s$ and $d_{\mathbb{H}^1}$. Namely for all $p, q \in \mathbb{H}^1$ it holds that
\[
\frac{1}{\sqrt{\pi}} d_s(p, q) \leq d_{\mathbb{H}^1}(p, q) \leq d_s(p, q),
\]
see [8].
We prove the first inclusion in (3.20) and the estimate for the corresponding seminorms. Then, the proof for the remaining cases follows the same lines. Let us denote by $B = B_s(x, r) \subseteq \Omega$ a ball defined with respect to the sub-Riemannian distance such that $\sqrt{\pi} n B \subseteq \Omega$, for a given $n \geq 1$. By the above relation between distances, there exists a ball $B' \subseteq \Omega$, defined with respect to the Korányi distance such that $B \subseteq B'$ and $n B' \subseteq \Omega$, namely $B' = B_{\mathbb{H}^1}(x, r)$. We verify by direct computations that for any function $u \in BMO^1_{n,loc}(\Omega)$ it holds that
\[
|u_{B'} - u_B| = \left| u_{B'} - \frac{1}{|B|} \int_B u \, dm \right| = \frac{1}{|B|} \int_B |u_{B'} - u| \, dm \leq \frac{1}{|B|} \int_B |u_{B'} - u| \, dm \leq \frac{c}{|B'|} \int_{B'} |u_{B'} - u| \, dm,
\]
where $c = \pi^2$. From this estimate we infer the following inequality:
\[
\frac{1}{|B|} \int_B |u - u_B| \, dm \leq \frac{1}{|B|} \int_B |u - u_{B'}| + |u_B - u_{B'}| \, dm \leq \frac{c}{|B'|} \int_{B'} |u - u_{B'}| \, dm + \frac{c}{|B'|} \int_{B'} |u_{B'} - u| \, dm \leq \frac{2c}{|B'|} \int_{B'} |u - u_{B'}| \, dm.
\]
Applying this reasoning to all sub-Riemannian balls with $\sqrt{\pi} n B \subseteq \Omega$, it follows that $u \in BMO^1_{\sqrt{\pi},n,loc}(\Omega)$ provided that $u \in BMO^1_{n,loc}(\Omega)$, with the desired bound for the BMO-norm.

The second inclusion in (3.20) with the corresponding estimate for the BMO-norm follows the same reasoning with $B := B_{\mathbb{H}^1}(x, r)$, $B' := B_{\mathbb{H}}(x, \sqrt{\pi} r)$ and $n$ replaced by $n'$.

We will later see that $BMO^1(\Omega) = BMO^1_{\sqrt{\pi},loc}(\Omega)$ for all $n > 1$. Lemma 3.19 then implies $BMO^1_{\sqrt{\pi},loc}(\Omega) \subseteq BMO^1(\Omega)$, but the reverse inclusion does not follow directly from the lemma since this would require to consider arbitrary balls contained in $\Omega$. To discuss this, we follow Staples [41, Definition 2.1], and introduce the following definition.

**Definition 3.22.** Consider the metric measure space $(\mathbb{H}^1, d_s, m)$. A domain $D$ in this space is said to be an $L^1$-averaging domain if $m(D) < \infty$ and for some $n > 1$ we have
\[
\frac{1}{|D|} \int_D |u(x) - u_D| \, dm \leq C_{\text{ave}} \left( \sup_{B_s \subseteq D} \frac{1}{|B_s|} \int_{B_s} |u(x) - u_{B_s}| \, dm \right)
\]
for a constant $0 < C_{\text{ave}} < \infty$ which does not depend on $u$. The supremum in this definition is taken over all sub-Riemannian balls $B_s$ for which the enlarged ball $n B_s$ is contained in $D$. 

More generally, Staples defines $L^p$-averaging domains for $p \geq 1$ in the general setting of homogeneous spaces in the sense of Coifman and Weiss, see Section 2 in [41] for details, and Definition 3.22 is in fact a specific case sufficient for our purposes.

We wish to show that all Korányi balls $D$ in $\mathbb{H}^1$ are $L^1$-averaging domains with uniform constants $C_{ave}$ and $n$. To this end we will show that the unit ball is an $L^1$-averaging domain, and then conclude by left-invariance and homogeneity.

**Proposition 3.23.** The Korányi unit ball $D = B_{\mathbb{H}^1}(0,1)$ is an $L^1$-averaging domain.

**Proof.** The assertion follows by combining various results from the literature.

First, by Corollary 1 in [14] we have that Korányi balls are NTA domains (in fact X-NTA domains). And so such balls are, in particular, John domains.

Second, by Theorem 3.1 in [11], John domains in homogeneous metric spaces are Boman chain domains (cf. Definition 2.1 in [11]). We apply this to the homogeneous metric space $(\mathbb{H}^1, d_s, m)$ and the John domain $D = B_{\mathbb{H}^1}(0,1)$.

Finally, Theorem 4.2 in [41] implies that every domain which satisfies the Boman chain condition is an $L^p$-averaging domain for every $1 \leq p < \infty$, and thus, in particular, for $p = 1$. We note that the constant $C_{ave}$ of the averaging domain depends on the John constant and the John center. 

**Corollary 3.24.** Every Korányi ball $B_{\mathbb{H}^1}(p,r)$ is an $L^1$-averaging domain with the same constants $C_{ave}$ and $n$.

**Proof.** Denote $D_0 := B_{\mathbb{H}^1}(0,1)$. By Proposition 3.23, there exist $0 < C_0 < \infty$ and $n_0 > 1$ such that

$$
\frac{1}{|D_0|} \int_{D_0} |u(x) - u_{D_0}| \, dm \leq C_0 \left( \sup_{n_0 B_s \subset D_0} \frac{1}{|B_s|} \int_{B_s} |u(x) - u_{B_s}| \, dm \right). \tag{3.25}
$$

Consider now an arbitrary Korányi ball $D := B_{\mathbb{H}^1}(p,r)$. Recall that left translations, denoted $\tau_p$, have Jacobian determinant equal to 1, and dilations by $r$, denoted $\delta_r$, are diffeomorphisms with Jacobian $r^4$. This yields by the transformation formula that

$$
u_D := \frac{1}{|D|} \int_D u \, dm = \frac{1}{r^4 |D_0|} \int_{D_0} (u \circ \tau_p \circ \delta_r) r^4 \, dm = (u \circ \tau_p \circ \delta_r)_{D_0}.
$$

Hence, by (3.25),

$$
\frac{1}{|D|} \int_D |u(y) - u_D| \, dy = \frac{1}{r^4 |D_0|} \int_{D_0} |(u \circ \tau_p \circ \delta_r)(x) - u_D| r^4 \, dx
$$

$$
= \frac{1}{|D_0|} \int_{D_0} |(u \circ \tau_p \circ \delta_r)(x) - (u \circ \tau_p \circ \delta_r)_{D_0}| \, dx
$$

$$
= \frac{1}{|D_0|} \int_{D_0} |v(x) - v_{D_0}| \, dx
$$

$$
\leq C_0 \left( \sup_{n_0 B_s \subset D_0} \frac{1}{|B_s|} \int_{B_s} |v(x) - v_{B_s}| \, dm \right)
$$

$$
= C_0 \left( \sup_{n_0 B_s \subset D} \frac{1}{|\tau_p \delta_r B_s|} \int_{\tau_p \delta_r B_s} |u(x) - u_{\tau_p \delta_r B_s}| \, dm \right)
$$

$$
= C_0 \left( \sup_{n_0 B_s \subset D} \frac{1}{|B_s|} \int_{B_s} |u(x) - u_{B_s}| \, dm \right)
$$
for \( v := u \circ \tau_p \circ \delta_r \). \( \square \)

**Theorem 3.26.** For an open set \( \Omega \) in \( \mathbb{H}^1 \), one has

\[
\text{BMO}(\Omega) = \text{BMO}_{n,\text{loc}}(\Omega), \quad \text{for all } n > 1,
\]

(3.27)

with the BMO-norm \( \| \cdot \|_s \) controlled from above by the BMO
\n\text{BMO}_{n,\text{loc}} \)-norm in a quantitative way.

Equality between the spaces holds regardless whether both spaces are considered for balls with respect to the Korányi distance \( d_{\mathbb{H}^1} \), or in both definitions one uses balls with respect to the sub-Riemannian distance \( d_s \). Equality holds also if the spaces in (3.27) are considered with respect to different distances, that is, one with \( d_{\mathbb{H}^1} \), the other with \( d_s \), and in particular,

\[
\text{BMO}^s(\Omega) = \text{BMO}^{3\mathbb{H}^1}(\Omega).
\]

(3.28)

Finally, denoting by \( \| \cdot \|_{\mathbb{H}^1} \) and \( \| \cdot \|_s \) the BMO norms with respect to the \( d_{\mathbb{H}^1} \) and \( d_s \), respectively, there exist constants \( 0 < c_1 \leq c_2 < \infty \) such that

\[
c_1 \| \cdot \|_{\mathbb{H}^1} \leq \| \cdot \|_s \leq c_2 \| \cdot \|_{\mathbb{H}^1}.
\]

(3.29)

**Proof.** Let us first observe that the following inclusion holds trivially for any \( n > 1 \):

\[
\text{BMO}^{\mathbb{H}^1}(\Omega) \subseteq \text{BMO}^{n,\text{loc}}(\Omega)
\]

and the same inclusion is true for the BMO-spaces considered with respect to the sub-Riemannian distance.

Next, we note that

\[
\text{BMO}^s(\Omega) = \text{BMO}^s_{n,\text{loc}}(\Omega) \text{ for all } n > 1,
\]

(3.30)

where, as we have already noted in the discussion following the proof of Lemma 3.19, only the inclusion \( \| \cdot \|_s \geq \| \cdot \|_s \) is nontrivial. In the setting of doubling metric measure spaces with a length metric, the assertion is essentially proven in Theorem 0.3 in [12]. However, Buckley’s proof specifically considers balls \( B \) such that \( 2B \subset \Omega \) and requires significant effort in its adaptation to the more general condition \( nB \subset \Omega \) for some \( n > 2 \). Instead we follow the shorter and more transparent proof of Theorem 2.2 in [34]. Since \( \mathbb{H}^1 \) equipped with the sub-Riemannian distance \( d_s \) is a length space, Theorem 2.2 in [34] can now be applied with \( \lambda = n \), see the proof of [34, Theorem 2.2]. Since our reasoning is verbatim the same as in [34] for \( n = 2 \), let us focus only on the modifications required for general \( n \).

Consider a ball \( B(x_0, R) \), which is admissible for \( \| \cdot \|_s \) and a point \( x \in B(x_0, R) \). The key part of the proof is to construct a certain chain of balls \( B_i(x_i, r_i) \) in \( \Omega \) centered on a geodesic and connecting small enough balls \( B(x_0, r_0) \) and \( B(x, r_x) \) in such a way that for \( i = 1, \ldots, N - 1 \) one has \( r_i = \frac{R - d_{\mathbb{H}^1}(x_0, x_i)}{2^{i+1}} \). Then one shows that \( r_i = \alpha^{N-i} r_N \), with \( \alpha := \frac{2^{N+1}+1}{2^{N+1}-1} \). As in [34, (2.11)], we then obtain the following estimate for the length of the chain: \( N - 1 \leq c \log_{\alpha} \left( \frac{R}{R - d_{\mathbb{H}^1}(x_0, x)} \right) \). Thus, a counterpart of [34, (2.12)] reads: \( N - 1 \leq c \log_{\alpha} n^k \) if \( x \) is at a certain distance of \( x_0 \), as quantified by \( k \). The rest of the proof follows exactly as in [34]. This yields (3.30) with a quantitative control on the respective BMO-norms.

It follows from Lemma 3.19 and (3.30) that

\[
\text{BMO}^{\mathbb{H}^1}(\Omega) \subseteq \text{BMO}^{n,\text{loc}}(\Omega) \subseteq \text{BMO}^s_{n,\text{loc}}(\Omega) = \text{BMO}^s(\Omega)
\]
for all \( n > 1 \), with \( \| \cdot \|_n \) bounded from above by \( \| \cdot \|^{1/2} \) and thus in particular
\[
\text{BMO}^{1/2}(\Omega) \subseteq \text{BMO}^s(\Omega).
\]
The goal now is to show the reverse inclusion. To do so, we use the established fact that all Korányi balls are \( L^1 \)-averaging domains with uniform constants, cf. Corollary 3.24.

Let \( D_{\mathbb{H}}^1 \) denote a Korányi ball contained in \( \Omega \). Then
\[
\| u \|_{\text{BMO}^{1/2}(\Omega)} = \sup_{D_{\mathbb{H}}^1 \subseteq \Omega} \frac{1}{|D_{\mathbb{H}}^1|} \int_{D_{\mathbb{H}}^1} |u - u_{D_{\mathbb{H}}^1}| \, dm
\leq \sup_{D_{\mathbb{H}}^1 \subseteq \Omega} C_{\text{ave}} \left( \sup_{B_s \subset D_{\mathbb{H}}^1} \frac{1}{|B_s|} \int_{B_s} |u(x) - u_{B_s}| \, dm \right)
\leq \sup_{B_s \subset \Omega} \frac{1}{|B_s|} \int_{B_s} |u(x) - u_{B_s}| \, dm
= \| u \|_{\text{BMO}^s(\Omega)}.
\]
This concludes the proof of (3.28) and (3.29). Combined with these and (3.30), equality (3.27) for \( d_{\mathbb{H}}^1 \) is now a direct consequence of Lemma 3.19. 

Lemma 3 in [38] characterizes functions whose logarithm lies in \( \text{BMO}(\mathbb{R}^n) \) via an integral inequality. We generalize part of this result to functions defined on a domain \( \Omega \subset \mathbb{H}^1 \). Since in our discussion we only need the implication in one direction, and only for functions which arise as quasiconformal Jacobians, we state the result as follows.

**Proposition 3.31.** Let \( \Omega \) be a domain in \( \mathbb{H}^1 \) and \( f : \Omega \to f(\Omega) \subset \mathbb{H}^1 \) be a \( K \)-quasiconformal mapping. Then \( \log J_f \in \text{BMO}^{\mathbb{H}^1}(\Omega) \).

**Proof.** The proof is similar to the one for Lemma 3 in [38] and, therefore, we omit the details. We start from the inequality
\[
\int_B J_f \, dm \leq C \left( \int_B J_f^{(p-4)/4} \, dm \right)^{-\frac{4}{p-4}},
\]
which, by Proposition 3.9, holds for all balls \( B \) in \( \Omega \) such that \( 3kB \subset \Omega \), with \( p \) and \( C \) determined by Theorem 3.1. The crucial estimate, giving the \( \text{BMO}^{\mathbb{H}^1}_{\text{loc}, 3k} \)-norm bound, cf. [38, (2.7)], reads
\[
\int_B \left| \log J_f - \log J_f \right| \, dm \leq \frac{1}{s} \log(C^s + C^{-s})
\]
for \( s = \min\{1, (p-4)/4\} \), and the constant \( C \) depending on \( K \) and the data of the spaces, see the discussion of constants in Theorem 3.1 and Proposition 3.9. 

As a consequence of the above discussion, we deduce the main result of this section.

**Theorem 3.32.** The following holds both for the Korányi and the sub-Riemannian distance: Let \( f : \Omega \to \Omega' \) be a \( K \)-quasiconformal map between domains in \( \mathbb{H}^1 \). Then \( \log J_f \) belongs to \( \text{BMO}(\Omega) \) with a bound for \( \| \log J_f \|_s \) in terms of \( K \).

**Proof.** By Proposition 3.31 and its proof, we have that \( \log J_f \in \text{BMO}^{\mathbb{H}^1}_{\text{loc}}(\Omega) \) with the local \( \text{BMO} \)-norm depending on \( K \) only. Theorem 3.26 allows us to conclude that \( \log J_f \in \text{BMO}(\Omega) \) and to bound both its Korányi and the sub-Riemannian \( \text{BMO} \)-norm. 

4. The Koebe theorem

The main purpose of this section is to prove the key result of this work: the Koebe theorem for quasiconformal mappings between open sets in $\mathbb{H}^1$, see Theorem 1.4. Before providing the proof of this result we need further auxiliary observations.

The following lemma is a counterpart of [2, Lemma 5.10]. For our purposes it suffices to consider balls centered at one point, but we consider arbitrary domains in $\mathbb{H}^1$ instead of disks in $\mathbb{R}^2$. The proof goes the same way.

Lemma 4.1. The following statement holds both for the Korányi distance $d_{\mathbb{H}^1}$ and the sub-Riemannian distance $d_s$: Let $\Omega$ be a domain in $\mathbb{H}^1$ and let $u \in BMO(\Omega)$. Then for all balls $B_2 \subset B_1 \subset \Omega$ centered at a point $z \in \Omega$, one has

$$|u_{B_1} - u_{B_2}| \leq \frac{e}{2} \left( \log \frac{|B_1|}{|B_2|} + 1 \right) \|u\|_*.$$

Here and in the following, the logarithm is taken with respect to the basis $e$.

Proof. Let $B_1 = B(z, r_1)$ and $B_2 = B(z, r_2)$ be as in the formulation of the lemma. Suppose first that $|B_1| \leq e|B_2|$. Then

$$|u_{B_1} - u_{B_2}| = \frac{1}{|B_2|} \int_{B_2} (u - u_{B_1}) \, dm = \frac{1}{|B_2|} \int_{B_1 \setminus B_2} (u - u_{B_1}) \, dm,$$

where we have used the fact that $\int_{B_1} (u - u_{B_1}) \, dm = 0$. Hence

$$|u_{B_1} - u_{B_2}| \leq \frac{1}{2|B_2|} \int_{B_1} |u - u_{B_1}| \, dm \leq \frac{e}{2} \|u\|_*.$$

This concludes the proof in this first case. If instead $|B_1| > e|B_2|$, then there exists a smallest integer $k$ such that

$$e^{k-1} |B_2| < |B_1| \leq e^k |B_2|.$$

We can then choose balls $B(z, s_i)$ such that

$$B_2 \subset B(z, s_1) \subset \cdots \subset B(z, s_k) = B_1,$$

and

$$|B(z, s_1)| \leq e|B_2|, \quad |B(z, s_{j+1})| \leq e|B(z, s_j)|$$

for $j = 1, \ldots, k$.

Then, by the first part of the proof, and since

$$k - 1 + \log |B_2| \leq \log |B_1|,$$

it follows that

$$|u_{B_1} - u_{B_2}| \leq \frac{ke}{2} \|u\|_* \leq \frac{e}{2} \left( \log \frac{|B_1|}{|B_2|} + 1 \right) \|u\|_*.$$

\[\square\]

In the literature, the definition of the quantity $a_f(x)$ for a $K$-quasiconformal map $f : \Omega \to \Omega'$ between domains $\Omega, \Omega' \subseteq \mathbb{R}^n$ involves taking averages of $\log J_f$ over either $B(x, d(x, \partial\Omega))$ or over $B(x, d(x, \partial\Omega)/2)$. It turns out that the resulting quantities are comparable. In $\mathbb{H}^1$, it is for technical reasons sometimes more convenient to work with $B(x, d(x, \partial\Omega)/\lambda)$ for a number $1 < \lambda < \infty$ which depends only on $K$ (for instance in the proof of Proposition 4.6 below). The following lemma shows that this is possible.
Lemma 4.2. Take $d \in \{d_s, d_{H^1}\}$. Let $K \geq 1$ and let $L > 1$ be a number which depends only on $K$. Then there exists a constant $1 \leq C < \infty$, depending only on $K$, such that for every $K$-quasiconformal mapping $f : \Omega \to \Omega'$ between domains $\Omega, \Omega' \subseteq (H^1, d)$, one has
\[
C^{-1} \exp \left( \frac{1}{4} (\log J_f)(B_1) \right) \leq \exp \left( \frac{1}{4} (\log J_f)(B_2) \right) \leq C \exp \left( \frac{1}{4} (\log J_f)(B_1) \right),
\]
where
\[
B_1 := B(x, d(x, \partial \Omega)) \quad \text{and} \quad B_2 := B(x, d(x, \partial \Omega)/L).
\]

Proof. The proof rests on the following standard estimate (cf. Lemma 4.1) for $u \in BMO(\Omega)$:
\[
|u_{B_1} - u_{B_2}| \leq c(L)\|u\|_{\ast}, \quad (4.3)
\]
for some constant $c(L)$ depending on $L$ (and thus on $K$) via the ratio $|B_1|/|B_2|$. Here we have used the fact that the Haar measure $m$ is Ahlfors regular.

We apply this estimate to $u = \log J_f$. As discussed earlier, since $f$ is $K$-quasiconformal, $\log J_f \in BMO(\Omega)$ and $\|\log J_f\|_\ast$ can be bounded in terms of $K$, by Theorem 3.32. Coupled with (4.3), this shows that there exists a constant $C(K)$ so that
\[
\exp \left( \frac{1}{4} (\log J_f)(B_2) \right) \leq \exp \left( \frac{1}{4} (\log J_f)(B_1) + C(K) \right) = C \exp \left( \frac{1}{4} (\log J_f)(B_1) \right)
\]
and
\[
\exp \left( \frac{1}{4} (\log J_f)(B_1) \right) \leq \exp \left( \frac{1}{4} (\log J_f)(B_2) + C(K) \right) = C \exp \left( \frac{1}{4} (\log J_f)(B_2) \right)
\]
with $C := \exp(C(K))$. The statement of the lemma follows. \hfill \Box

Remark 4.5. As a special case of [40, Lemma 5.2], Proposition 4.4 holds a priori with respect to the sub-Riemannian distance $d = d_s$. Since $d_s$ is bi-Lipschitz equivalent to the Korányi distance, it follows immediately that an analogous statement (with quantitatively comparable quasisymmetry data) also holds for $d_{H^1}$, however for a different constant $\kappa$ which is still universal but might be larger than 1.

The egg yolk principle yields the following distance estimate, which is a counterpart for Lemma 5.15 in [2].

Proposition 4.6. For every $1 \leq K < \infty$, there exists a constant $\lambda \geq 1$ such that the following holds. If $0 < a < b$ and $f : \Omega \to \Omega'$ is a $K$-quasiconformal mapping between domains $\Omega, \Omega' \subseteq (H^1, d_{H^1})$, and $\|\log J_f\|_\ast \leq (2/c)a$, then for all $z_1, z_2 \in \Omega$ such that
\[
d_{H^1}(z_1, z_2) \leq \frac{1}{\lambda}d_{H^1}(z_1, \partial \Omega),
\]
one has
\[
d_{H^1}(f(z_1), f(z_2)) \leq c a f(z_1) d_{H^1}(z_1, \partial \Omega)^a d_{H^1}(z_1, z_2)^{1-a}, \quad (4.7)
\]
where the constant $c$ depends only on $K$ and the bound $b$ for $a$. 

The proof of Proposition 4.6 will show that \( c \) in the statement can be obtained as a monotone increasing function of \( b \). Also, for large values of \( a \), the conclusion in (4.7) is not very informative: assume for illustrative purposes that 
\[
d_{\bar{H}^1}(z_1, z_2) = \frac{1}{b} d_{\bar{H}^1}(z_1, \partial \Omega).
\]
Then the right-hand side of (4.7) is comparable to \( \lambda^{a-1} d_{\bar{H}^1}(z_1, \partial \Omega) \). If \( a \) becomes large, then also the multiplicative constant in this upper bound blows up. For these reasons, in our application we will be interested in having a small upper bound \( b \) for \( a \).

Proposition 4.6 can be proved along the lines of Lemma 5.15 in [2], although we consider now arbitrary subdomains of \( \mathbb{H}^1 \) instead of disks in \( \mathbb{R}^2 \). A significant difference in the proof arises from the fact that (i) we do not know whether the map \( f \) extends to a \( K_1(K) \)-quasiconformal map on the one point compactification of \( \mathbb{H}^1 \), and (ii) we do not have a Mori distortion theorem at our disposal. We compensate for this by resorting to the egg yolk principle (Proposition 4.4). This accounts for the presence of the constant \( \lambda \) in the statement of Proposition 4.6. Lemma 5.15 in [2] contains an analogous statement in \( \mathbb{R}^2 \) with \( \lambda = 1 \). The weaker formulation of Proposition 4.6 influences the proof of Koebe’s theorem, where now \( r_2 = r_1/(2\lambda) \) has to be used instead of \( r_2 = r_1/2 \). In light of Lemma 4.2, this change is immaterial. Moreover, working with arbitrary subdomains, rather than just disks or balls, has the advantage that we only have to define \( a_f \) and \( \| \cdot \|_s \) for one domain, namely \( \Omega \). Also note that if \( z \in B \subset \Omega \), then \( d(z, \partial B) \leq d(z, \partial \Omega) \), and for our purpose an estimate in terms of \( d(z, \partial \Omega) \) is sufficient.

Proof of Proposition 4.6. Throughout the proof we will work with the Korányi distance \( d = d_{\bar{H}^1} \). The idea is to choose \( \lambda \) large enough so that for \( z_1 \) and \( z_2 \) as in the assumptions, we have that:

1. \( f \) is quasisymmetric on \( B(z_1, d(z_1, z_2)) \),
2. \( B(z_1, 2d(z_1, z_2)) \) is admissible for (3.10), i.e. for the \( A_p/(p-4) \)-weight condition for \( \log J_f \).

The requirements are satisfied under the following assumptions:

1. \( \lambda \geq (4\kappa + 1)/2 \), where \( \kappa > 0 \) is as in Remark 4.5 (egg yolk principle for \( d_{\bar{H}^1} \)),
2. \( \lambda \geq 3k \), where \( k \) is as in the admissibility condition for (3.10).

Let us choose \( \lambda \geq 1 \) as the smallest constant for which (1) and (2) hold. Such a \( \lambda \) is finite and depends only on \( K \). We set

\[
r_1 := d(z_1, \partial \Omega)/\lambda \quad \text{and} \quad r_2 := d(z_1, z_2)
\]

and denote \( B_i := B(z_1, r_i) \) for \( i \in \{1, 2\} \). Moreover, we write

\[
s_2 := \min_{d(z_1, z) = r_2} d(f(z), f(z_1)).
\]

Since \( f(B(z_1, d(z_1, z_2))) \) is \( H \)-quasisymmetric for a constant \( H \) which depends only on \( K \), we find

\[
d(f(z_1), f(z_2)) \leq Hs_2.
\]

This implies

\[
\left( \frac{d(f(z_1), f(z_2))}{d(z_1, z_2)} \right)^4 \leq \left( \frac{Hs_2}{r_2} \right)^4 \leq H^4 \frac{|f(B_2)|}{|B_2|}.
\]
In order to further estimate this from above, an analog of Lemma 5.14 in [2] would be useful. We concentrate on a manageable special case, which is sufficient for our application. Namely, using the same notation as above, we will prove that

\[
\left| \frac{f(B_2)}{|B_2|} \right| \leq C' \exp \left( \frac{1}{|B_2|} \int_{B_2} \log J_f \, dm \right),
\]

(4.9)

where \( C' \) is a constant depending only on \( K \). To show this, we consider the enlarged ball

\[
Q := 2B_2 := B(z_1, 2d(z_1, z_2)).
\]

By our choice of \( \lambda \), the ball \( Q \) is admissible for (3.10), and we deduce for a suitable constant \( c_1 > 0 \) (depending on \( K \)), that

\[
\left| \frac{f(Q)}{|Q|} \right| \leq c_1 \left( \frac{1}{|Q|} \int_Q J_f^{-(p-4)/4} \, dm \right)^{-4/(p-4)} \leq c_1 \exp \left( \frac{1}{|Q|} \int_Q \log J_f \, dm \right),
\]

(4.10)

where we have applied Jensen’s inequality to the convex function \( \varphi(x) = e^{-bx} \) for \( b = (p-4)/4 \) in the last step. The remaining steps to deduce (4.9) consist of a computations analogous to the proof of [2, Lemma 5.14]. First, we observe that

\[
\int_Q \log J_f \, dm = \frac{1}{2^4} \int_{B_2} \log J_f \, dm + \left( 1 - \frac{1}{2^4} \right) \int_{Q \setminus B_2} \log J_f \, dm
\]

and

\[
\int_{Q \setminus B_2} \log J_f \, dm \leq \log \left( \int_{Q \setminus B_2} J_f \, dm \right) = \log \left( \frac{2^4}{2^4 - 1} \int_Q J_f \, dm \right) \leq \log \left( \frac{2^4}{2^4 - 1} |f(Q)| \right).
\]

Inserting these estimates in (4.10), we find that

\[
\frac{1}{2^4} \log |f(Q)| \leq \log c_1 + \frac{1}{2^4} \int_{B_2} \log J_f \, dm + \frac{2^4 - 1}{2^4} \log \frac{2^4}{2^4 - 1},
\]

which yields (4.9) since \( |f(B_2)| \leq |f(Q)| \) and \( |Q| = 2^4|B_2| \). Combining (4.8) with (4.9), we deduce that

\[
\left( \frac{d(f(z_1), f(z_2))}{d(z_1, z_2)} \right)^4 \leq c_2 \exp \left( (\log J_f)_{B_2} \right)
\]

(4.11)

for a constant \( c_2 \) depending only on \( K \). The rest of the argument is similar to the proof of [2, Lemma 5.15], with the factor 1/2 replaced by 1/4. By Lemma 4.1 applied to \( u = \log J_f \), we have

\[
|\log J_f|_{B_1} - (\log J_f)_{B_2} | \leq \left( \log \frac{|B_1|}{|B_2|} + 1 \right) a = 4a \log \frac{r_1}{r_2} + a.
\]

Hence

\[
\frac{1}{4} (\log J_f)_{B_2} \leq \frac{1}{4} (\log J_f)_{B_1} + a \log \frac{r_1}{r_2} + \frac{b}{4}.
\]

Compared with (4.11), this shows that

\[
d(f(z_1), f(z_2)) \leq c_2^{1/4} \exp \left( \frac{1}{4} (\log J_f)_{B_2} \right) d(z_1, z_2)
\]

\[
\leq c_2^{1/4} \exp(b/4) \exp \left( \frac{1}{4} (\log J_f)_{B_1} \right) d(z_1, \partial \Omega)^a d(z_1, z_2)^{1-a},
\]

which concludes the proof of the proposition. \( \square \)
Finally we are in a position to prove the Koebe type theorem.

Proof of Theorem 1.4. As remarked in the introduction, it suffices to prove the theorem for the Korányi distance $d = d_{\mathbb{H}^1}$, as then the corresponding statement for $d_q$ follows.

Let us first observe that the assumption $\Omega \subseteq \mathbb{H}^1$ also implies that $\Omega' = f(\Omega) \subseteq \mathbb{H}^1$. Indeed, this is a consequence of the Liouville-type result for entire quasiconformal mappings. In the Euclidean setting such a theorem is well-known, see Theorems 17.3 and 17.4 in [44]. It has been generalized to the setting of $Q$-Loewner spaces and locally quasisymmetric embeddings, see Theorem 13.1 in [43]. In particular, we apply this result to $\mathbb{H}^1$ and upon recalling that quasiconformal mappings are locally quasisymmetric, conclude that $\Omega'$ is a proper subset of $\mathbb{H}^1$.

We fix an arbitrary point $x_1 \in \Omega$ and prove estimate (1.5) for $x = x_1$. To this end, we define

$$r_1 := d(x_1, \partial \Omega) \quad \text{and} \quad d_1 := d(f(x_1), \partial \Omega').$$

Note that both $r_1, d_1 \neq \infty$, as $\Omega, \Omega' \subseteq \mathbb{H}^1$. Set further

$$r_2 := r_1/m \quad \text{and} \quad d_2 := \max_{d(x_1,x) = r_2} d(f(x_1), f(x)),$$

where $m = 2\lambda$ with $\lambda$ as in Proposition 4.6. Let further $x_2 \in \Omega$ be a point which realizes the maximum in $d_2$, that is $d(x_1, x_2) = r_2$ and $d(f(x_1), f(x_2)) = d_2$.

We denote

$$B_1 := B(x_1, r_1) \quad \text{and} \quad B_2 := B(x_1, r_2).$$

The first step of the proof consists of finding a positive and finite constant $c_1 = c_1(K)$ such that

$$\frac{1}{c_1} \leq \frac{d_1}{d_2} \leq c_1. \quad (4.12)$$

This is done by comparing the modulus of suitable curve families. To this end it is convenient to use the Loewner function $\psi : (0, +\infty) \to (0, +\infty)$ of $(\mathbb{H}^1, d, m)$, as defined in (2.3). Recall that $t \leq t'$ implies $\psi(t) \geq \psi(t')$. According to (3.20) in [23], one can choose a number $t_0 > 0$, depending only on the data of $\mathbb{H}^1$, such that

$$\psi(t) \geq C' \log \frac{1}{t}, \quad \text{for all} \ t \in (0, t_0)$$

for a universal constant $C' > 0$.

We begin by proving the lower bound in (4.12). It suffices to do this for $0 < d_1/d_2 \leq t_0$. Let $E$ be a continuum connecting $f(x_1)$ to $f(x_2)$ inside $f(B(x_1, r_2))$. Since $f$ is a homeomorphism, one can take $E$ to be the $f$-image of a continuum connecting $x_1$ and $x_2$ inside $B(x_1, r_2)$.

The boundary $\partial \Omega'$ is closed and nonempty by assumption. The continuous function $h : \partial \Omega' \to \mathbb{R}, \ h(z) := d(f(x_1), z)$, assumes a global minimum (note that we may restrict $h$ to a compact subset of $\partial \Omega'$ since $h$ blows up as $d(z, f(x_1)) \to \infty$). From this we infer that there exists a point $z \in \partial \Omega'$ such that $d(f(x_1), z) = d_1$ and for every $n \in \mathbb{N}$ large enough so that $(1 - \frac{1}{n})r_1 > r_2$, one can find a continuum $F_n$ starting at $z$ and contained in $\mathbb{H}^1 \setminus f(B(x_1, (1 - \frac{1}{n})r_1))$ with $\text{diam}(F_n) \geq \text{diam}(E)$. That such a continuum $F_n$ can indeed be found is a consequence of the following reasoning: The definition of $r_1$ ensures that the closure of $B(x_1, (1 - \frac{1}{n})r_1)$ is contained inside the domain $\Omega$. As $f$ is a homeomorphism on $\Omega$ and Korányi spheres are topological spheres, it follows by the Jordan-Brouwer separation theorem that $S := f(\partial B(x_1, (1 - \frac{1}{n})r_1))$ is a topological
sphere, and hence \( \mathbb{H}^1 \setminus S \) consists exactly of two connected components, one bounded and one unbounded. The bounded component is \( f(B(x_1, (1 - \frac{1}{n})r_1)) \) and the other one \( \mathbb{H}^1 \setminus f(B(x_1, (1 - \frac{1}{n})r_1)) \). Since the component \( \mathbb{H}^1 \setminus f(B(x_1, (1 - \frac{1}{n})r_1)) \) is also open, it must be path connected (recall that \( \mathbb{H}^1 \) and \( \mathbb{R}^3 \) have the same topology). Now the point \( z \) lies in this unbounded component and hence we can find a continuum \( F_n \) which starts at \( z \), satisfies \( \text{diam}(F_n) \geq \text{diam}(E) \), and stays entirely inside \( \mathbb{H}^1 \setminus f(B(x_1, (1 - \frac{1}{n})r_1)) \).

Then, if \( \Gamma_n \) denotes the family of curves connecting the boundary components of the annulus \( B(x_1, (1 - \frac{1}{n})r_1) \setminus B(x_1, r_2) \), we find by the formula for the modulus of a ring domain [28] that

\[
K \omega_4 \log \left( (1 - \frac{1}{n})m \right)^{-3} = K \omega_4 \log \left( \frac{(1 - \frac{1}{n})r_1}{r_2} \right)^{-3} \geq \text{mod}_4(\Gamma_n) \geq \text{mod}_4(\Gamma(E, F_n, \mathbb{H}^1)) \geq \psi(\Delta(E, F_n)) \geq \psi\left( \frac{d_1}{d_2} \right) \geq C' \log \left( \frac{d_3}{d_4} \right).
\]

To justify the estimate \( \text{mod}_4(\Gamma_n) \geq \text{mod}_4(\Gamma(E, F_n, \mathbb{H}^1)) \), we have used the fact that every curve connecting \( E \) and \( F_n \) in \( \mathbb{H}^1 \) has a subcurve which belongs to \( f(\Gamma_n) \). Letting \( n \) tend to infinity, we obtain the desired lower bound in (4.12).

Let us now turn to the upper bound in (4.12). For this estimate, it is sufficient to consider the case \( d_2 < d_1 \). Let \( n \in \mathbb{N} \) be large enough so that \( d_2 < (1 - \frac{1}{n})d_1 \). Let \( \Gamma_n' \) be the family of curves connecting the boundary components of \( B(f(x_1), (1 - \frac{1}{n})d_1) \setminus B(f(x_1), d_2) \). We let \( E' \) be a continuum that connects \( x_1 \) to \( x_2 \) and is entirely contained inside \( B(x_1, r_2) \), except for the boundary point. Again by the Jordan-Brouwer separation theorem, \( \mathbb{H}^1 \setminus f^{-1}(B(f(x_1), (1 - \frac{1}{n})d_1)) \) is a path-connected component. The assumption \( d_1 > d_2 \) ensures that we can find a point \( x \in \partial B(x_1, r_1) \) which lies in this component, and thus we also find a continuum \( F_n' \) that starts at \( x \), stays entirely outside \( f^{-1}(B(f(x_1), (1 - \frac{1}{n})d_1)) \) and satisfies \( \text{diam}(F_n') \geq \text{diam}(E') \).

Similarly as before,

\[
\omega_4 \left( \log \left( \frac{(1 - \frac{1}{n})d_1}{d_2} \right) \right)^{-3} = \text{mod}_4(\Gamma_n') \geq K \text{mod}_4(f^{-1}(\Gamma_n')) \geq K \text{mod}_4(\Gamma(E', F_n', \mathbb{H}^1)) \geq K \psi(\Delta(E', F_n')) \geq K \psi(1/m).
\]

Letting \( n \) tend to infinity, this yields the upper bound in (4.12).

We are now able to deduce an upper bound for \( a_f(x_1) \), analogously to [1, (2.7)]. Indeed, since \( f(B_2) \subset B(f(x_1), d_2) \), we find by Jensen’s inequality and (4.12) that

\[
(\log J_f)_{B_2} \leq \log \left( \frac{|f(B_2)|}{|B_2|} \right) \leq 4 \log \frac{d_2}{r_2} \leq 4 \left( \log \frac{mc_1 d_1}{r_1} \right).
\]

Combining this with Theorem 3.32 and Lemma 4.1 (applied to \( u = J_f \)), we find

\[
(\log J_f)_{B_1} \leq 4 \left( \log \left( \frac{mc_1 d_1}{r_1} \right) + c_2 \right),
\]
for a constant $c_2$ which depends on $K$ only. Thus,

$$a_f(x_1) \leq \frac{mc_1d_1}{r_1} \exp(c_2).$$

(4.13)

The next step is to apply Proposition 4.6 for $z_1 = x_1$ and $z_2 = x_2$ in order to find a lower bound for $a_f(x_1)$. In this way we obtain, for constants $a$ and $c$ bounded in terms of $K$, that

$$d_2 = d(f(x_1), f(x_2)) \leq ca_f(x_1)d(x_1, \partial \Omega)^a d(x_1, x_2)^{1-a} \leq \left(\frac{1}{m}\right)^{1-a} ca_f(x_1)r_1.$$

By (4.12) we can bound $d_2$ from below by $d_1/c_1$, which yields the desired lower bound for $a_f(x_1)$ and thus, together with the upper bound in (4.13), concludes the proof.

\[\square\]

5. Applications

In this section we discuss applications of Theorem 1.4. Section contains analytic results regarding the horizontal derivative of a quasiconformal mapping. Results in the spirit of the ones in Sections 5.1 and 5.3 have been obtained by H. Len Ruth Jr. in his PhD thesis, [32, Section 3.7], for quasisymmetric mappings in a more abstract setting and for the quantity $d(f(\cdot), \partial \Omega')d(\cdot, \partial \Omega)$. By our version of the Koebe theorem, $d(f(\cdot), \partial \Omega')/d(\cdot, \partial \Omega)$ is comparable to $a_f$ for quasiconformal mappings between domains in $\mathbb{H}^1$. In this sense, Lemma 3.7.4 and Proposition 3.7.5 in [32] are quasisymmetric counterparts of our Propositions 5.1 and 5.26, respectively. Since a quasiconformal map $f$ on a subdomain $\Omega \subset \mathbb{H}^1$ is in general only locally quasisymmetric, our results do not follow directly from the ones in [32]. Potential localization arguments are complicated by the fact that Propositions 5.1 and 5.26 (part (2)), concern the global behavior of the mapping. For this reason, we give direct proofs in the quasiconformal category, which do not rely on the results in [32], but on similar proof arguments. The specific setting of the sub-Riemannian Heisenberg group allows us to illustrate the sharpness of Propositions 5.1 with an example and to formulate our results with less additional assumptions than in [32, Section 3.7]. In particular, Proposition 5.26 holds for arbitrary, not necessarily Ahlfors regular, domains.

5.1. Diameter bounds for image curves. If $\gamma : [a, b] \to \mathbb{R}^n$ is a rectifiable curve, and $f : \mathbb{R}^n \to \mathbb{R}^n$ a smooth mapping, then one has the following bound for the Euclidean length of the image curve:

$$\text{length}(f \circ \gamma) \leq \int_\gamma \|Df\| \, ds,$$

where $ds$ denotes integration with respect to an arc length, and $\| \cdot \|$ is the operator norm (maximal stretch) of a linear map. An inequality of this form continues to hold if $\mathbb{R}^n$ is replaced by a general metric space, $f$ is assumed to be a locally Lipschitz mapping of $X$, and instead of $\|Df\|$ one considers the pointwise lower Lipschitz constant, see [25, Lemma 6.2.6]. While quasiconformal mappings on nice enough metric spaces are absolutely continuous along almost every curve, one cannot hope to control for every rectifiable curve the length of its image in a similar manner. The reason is essentially that quasiconformality is not a condition on distances, but rather on ratios of distances.

However, it has been shown by Koskela in [30, Lemma 2.6], for quasiconformal mappings defined on a domain $\Omega$ in $\mathbb{R}^n$, that it is possible to control the diameter of $f \circ \gamma$ in
terms of \( \int_a \alpha f \, ds \) for all curves which are long enough in terms of their distance to the boundary of \( \Omega \). The goal of this section is to study similar estimates in the Heisenberg group. Koskela’s proof makes use of the Besicovitch covering theorem, which does not hold for the Korányi or the sub-Riemannian distance on \( \mathbb{H}^1 \). A possible approach would be to use one of the comparable distances with the Besicovitch covering property that were constructed in [31]. Instead, we will give below a direct proof using the basic \( 3r \)-covering theorem, which can be found for instance in [42, Theorem 2.1]. Our statement is slightly more flexible than the original version since we allow for a quantitative control of the lengths of curves in terms of a parameter \( \alpha \); this will prove useful later in applications.

**Proposition 5.1.** Let \( d \) denote either the Korányi or sub-Riemannian distance on \( \mathbb{H}^1 \). Let \( f : \Omega \rightarrow \Omega' \) be a \( K \)-quasiconformal mapping between domains in \( \mathbb{H}^1 \) with \( \Omega \neq \mathbb{H}^1 \). Then, for every \( \alpha \in (0,1] \) and for every rectifiable curve \( \gamma \) contained in \( \Omega \) with

\[
\text{length}(\gamma) \geq \alpha d(\gamma, \partial \Omega), \tag{5.2}
\]

one has

\[
\text{diam}(f \circ \gamma) \leq C \int_\gamma a_f \, ds
\]

for a constant \( C \) which depends only on \( \alpha, d \) and \( K \). Here \( \int ds \) denotes integration with respect to the \( d \)-length.

Recall that curves have the same length with respect to \( d_s \) and \( d_{\mathbb{H}^1} \). In particular, the two metrics generate the same class of rectifiable curves. It suffices to prove Proposition 5.1 for \( d = d_{\mathbb{H}^1} \). Since \( \frac{1}{\sqrt{\pi}} d_s \leq d_{\mathbb{H}^1} \leq d_s \) and \( a_f \simeq a_{f_{\mathbb{H}^1}} \), the claim for \( d_s \) follows from the result for \( d_{\mathbb{H}^1} \).

**Proof.** Let \( d = d_{\mathbb{H}^1} \). The following abbreviating notation will be used in this proof. For \( 0 < \chi < 1 \), we denote

\[
B_\chi(x) = B(x, \chi d(x, \partial \Omega))
\]

where \( x \in \Omega \).

We let \( k = k(2, K) \) be the constant given by Corollary 3.5 applied to \( \beta = 2 \). Then we fix \( \lambda \in (0,1) \) to be the largest number so that

\[
\lambda \leq \frac{3\alpha}{1 + \alpha} \quad \text{and} \quad 3k\lambda \leq \frac{1}{2}. \tag{5.3}
\]

By the second condition we have that \( 3kB_\lambda(x) \subset \Omega \) for all \( x \in \Omega \), which will be used to apply Corollary 3.5. Furthermore, since \( k > 1 \), we have \( \lambda \leq \frac{1}{6k} \leq \frac{1}{6} \). The use of the first condition in (5.3) will become clear later.

Consider now an arbitrary curve \( \gamma \) satisfying the assumptions of the proposition. For simplicity we continue to denote the trace of \( \gamma \) by the symbol \( \gamma \). Let \( B^\lambda \) be the family of all balls of the form \( B_{\lambda^{1/3}}(p) \) where \( p \in \gamma \). The balls in this family cover \( \gamma \). By compactness of \( \gamma \), we may without loss of generality assume that the family \( B^\lambda \) contains only finitely many balls.

This allows us to apply the \( 3r \)-covering lemma and select a (finite) disjointed subfamily \( \mathcal{F}^\lambda \subset B^\lambda \) so that the \( 3 \)-times enlarged balls in \( \mathcal{F}^\lambda \) cover \( \gamma \). More precisely, if we denote by
I the set of centers of the balls in $F^\lambda$, then we have

$$B_{\lambda/3}(p) \cap B_{\lambda/3}(q) = \emptyset \quad \text{for } p, q \in I, \, p \neq q \quad \text{and} \quad \gamma \subset \bigcup_{p \in I} B_\lambda(p).$$

Since $3kB_\lambda(p) \subset \Omega$ for all $p \in I$, we can apply Corollary 3.5 (with $\beta = 2$), and we have that

$$\text{diam}(f \circ \gamma) \leq \sum_{p \in I} \text{diam}(f(B_\lambda(p))) \leq 2 \sum_{p \in I} d(f(p), \partial \Omega').$$

Next we establish an estimate of $\int_\gamma a_f ds$ from below by a multiple of $\sum_{p \in I} d(f(p), \partial \Omega')$. To this end we employ the family of balls

$$\{B_{\lambda/3}(p) : p \in I\}.\quad (5.5)$$

We recall that these balls are disjoint, that is, each point on $\gamma$ is contained in at most one of them. We also note that the family does not necessarily cover $\gamma$, however since we are looking for a lower estimate of $\int_\gamma a_f ds$, and $a_f \geq 0$, this will not be a problem. Let $l(p) = \text{length}(\gamma \cap B_{\lambda/3}(p))$, then we claim that

$$l(p) \geq \frac{\lambda}{3} d(p, \partial \Omega). \quad (5.6)$$

We note that (5.6) is obvious if $\gamma$ exits $B_{\lambda/3}(p)$, however the assumption (5.2) implies that (5.6) is valid even if the entire curve is contained in $B_{\lambda/3}(p)$. Indeed, if $\gamma \subset B_{\lambda/3}(p)$, then

$$\alpha d(\gamma, \partial \Omega) \geq \alpha \left( d(p, \partial \Omega) - \frac{\lambda}{3} d(p, \partial \Omega) \right) \geq \frac{\lambda}{3} d(p, \partial \Omega) \quad (5.7)$$

where the second inequality is a consequence of the choice of $\lambda$ as in (5.3). The fact that $l(p) = \text{length}(\gamma) \geq \alpha d(\gamma, \partial \Omega)$, together with (5.7) proves (5.6) in this case.

Our desired estimate will result by approximating

$$\int_{\gamma \cap B_{\lambda/3}(p)} a_f ds$$

by $a_f(p)l(p)$, thus we require a constant $\tau$, depending on $K$ and the chosen metric only, such that $a_f(x) \geq \tau a_f(p)$ for all $x \in B_{\lambda/3}(p)$. To this end we observe that the second inequality in Corollary 3.5, applied to $g := f, \, U' = \Omega', \, B := B_{\lambda/3}(p)$ and $\beta = 2$, implies

$$d(f(x), \partial \Omega') \geq d(f(B_{\lambda/3}(p)), \partial \Omega') \geq \frac{1}{2} d(f(p), \partial \Omega')$$

for all $x \in B_{\lambda/3}(p)$. Moreover,

$$d(x, \partial \Omega) \leq d(x, p) + d(p, \partial \Omega) \leq \left( \frac{\lambda}{3} + 1 \right) d(p, \partial \Omega) \leq \frac{3}{2} d(p, \partial \Omega)$$

for all $x \in B_{\lambda/3}(p)$. Using the two inequalities above, together with Theorem 1.4, we have

$$a_f(x) \geq \frac{1}{c_K} \frac{d(f(x), \partial \Omega')}{d(x, \partial \Omega)} \geq \frac{1}{3c_K} \frac{d(f(p), \partial \Omega')}{d(p, \partial \Omega)} \geq \frac{1}{3c_K} a_f(p) \quad (5.8)$$

for all $x \in B_{\lambda/3}(p)$. Thus $\tau = \frac{1}{3c_K}$ is sufficient for our needs.
To finish the proof we observe that
\[
\int_\gamma a_f \, ds \geq \sum_{p \in I} \int_{\gamma \cap B_{3/4}(p)} a_f \, ds \geq \tau \sum_{p \in I} a_f(p) l(p) \geq \tau \sum_{p \in I} a_f(p) \frac{\lambda}{3} d(p, \partial \Omega)
\]
\[
\geq \frac{\tau \lambda}{cK} \sum_{p \in I} d(f(p), \partial \Omega).
\]
This estimate combined with (5.4) proves the claim with \(C = 3 \cdot 6c_K^3 / \lambda\).

\[\square\]

**Remark 5.9.** Analogously as in Euclidean spaces we can show that some assumption on the diameter of the curve \(\gamma\) is necessary in Proposition 5.1. We consider the Heisenberg radial stretch map \(f = f_k : \mathbb{H}^1 \to \mathbb{H}^1, 0 < k < 1\), discussed in [7]. This is a quasiconformal mapping which on the \((x, y)\)-plane agrees with the usual planar radial stretch map, that is \(f(z, 0) = (z|z|^{k-1}, 0)\), and which sends points with Korányi norm equal to \(r \geq 0\) onto points of Korányi norm \(r^k\). In light of Proposition 5.1, let us now consider the map \(f\) for \(k = 1/2\), restricted to the Korányi unit ball, \(\Omega = B(0, 1)\) and let \(\gamma\) be a line segment with length\((\gamma) = r \in (0, 1)\) on the \(x\)-axis emanating from 0 (note that restricted to the \(x\)-axis, the Korányi distance agrees with the Euclidean distance). Then \(f(\gamma)\) is again a line segment on the \(x\)-axis starting from 0, but with \(\text{diam}(f \circ \gamma) = \sqrt{r}\). For a fixed \(a\), we can choose \(r > 0\) small enough so that \(\gamma\) violates the assumption (5.2), and by letting \(r\) tend to 0, we will see that indeed the conclusion of Proposition 5.1 does not hold in this case since \(\sqrt{r} > r\) for small \(r\), yet \(\int_\gamma a_f \, ds \leq cr\) for a positive and finite constant \(c\) which does not depend on \(r\). To establish the last claim it suffices to observe that \(a_f(0) < \infty\) and that there exists \(r_0 > 0\) such that \(a_f(x) \leq c' a_f(0)\) for all \(x \in B(0, r_0)\) for a constant \(c'\) depending on \(K\), and in particular on \(c_K\) from Theorem 1.4. This can be seen by an argument as in (5.8). Therefore,

\[
\int_\gamma a_f \, ds \leq c' a_f(0) r \ll \sqrt{r} = \text{diam}(f \circ \gamma),
\]

for \(r < r_0\) small enough, which is impossible.

5.2. **Comparison of the average derivative and the operator norm.** As an application of the Koebe theorem for quasiconformal mappings in \(\mathbb{R}^n\), Astala and Koskela have shown that for a \(K\)-quasiconformal map \(f : \Omega \to \Omega'\), for \(\Omega, \Omega' \subseteq \mathbb{R}^n\), the following two integrals are comparable for \(p\) in a certain quantitatively given parameter range, see Theorem 3.4 in [3]:

\[
\int_{\Omega} \|Df(x)\|^p \, d\mathcal{L}^n(x) \quad \text{and} \quad \int_{\Omega} a_f(x)^p \, d\mathcal{L}^n(x).
\]

The main goal of this section is to prove the following counterpart of the aforementioned theorem, which is valid for both \(d = ds\) and \(d = d_{\mathbb{H}^1}\).

**Theorem 5.10.** Let \(f : \Omega \to \Omega'\) be a \(K\)-quasiconformal mapping between domains in \((\mathbb{H}^1, d)\) for some \(K \geq 1\). Moreover, denote by \(p = p(K) > 4\) a higher integrability exponent of the Jacobian of \(f\) as in Theorem 3.1. Then

\[
\frac{1}{c} \int_{\Omega} a_f(x)^q \, dm \leq \int_{\Omega} \|D_H f(x)\|^q \, dm \leq c \int_{\Omega} a_f(x)^q \, dm
\]

for all \(4 - p < q < p\), where \(c\) depends on \(K\) and \(q\).
Theorem 5.10 provides explicit bounds for the admissible exponents \( q \) by using the exponent \( p \) from Theorem 3.1 and Proposition 3.9. In particular, we obtain the lower bound \( q \geq -\epsilon \) for \( \epsilon = p - 4 \). Let us compare this to the expression

\[
\epsilon = \min\{p_0 - n, 1/(p_1 - 1)\}
\]

which is obtained for the \( \mathbb{R}^n \)-counterpart of this statement in [3, Theorem 3.4]. In this formula, \( p_0 \) denotes the exponent in Gehring’s reverse Hölder’s inequality for \( \|Df\| \), which corresponds to our parameter \( p \). The parameter \( p_1 \) is obtained from an estimate which is quantitatively equivalent to the Muckenhoupt weight condition for \( \|Df\|^n \) (equivalently, for \( J_f \)).

In the proof of Theorem 5.10 we employ a number of auxiliary results. The following key observation is of the independent interest. Namely, it shows that \( a_f \) as function of a point in the domain, satisfies a Harnack-type inequality. A similar estimate has already appeared in the proof of Proposition 5.1, but in the following we need a more general statement which we formulate as follows. Let us also remark that this result can be deduced from our Koebe theorem, Proposition 3.7.3 in [32] and an appropriate localization argument. Instead, we give a short direct proof using just the Koebe theorem and Corollary 3.5.

**Lemma 5.11.** Let \( f : \Omega \to \Omega' \) be a \( K \)-quasiconformal map between domains \( \Omega, \Omega' \subseteq (\mathbb{H}^1, d_{\mathbb{H}^1}) \). Suppose a ball \( B \subset \Omega \) satisfies the condition

\[
\text{diam } B \leq \lambda \text{dist}(B, \partial \Omega),
\]

where \( 0 < \lambda \leq 2/(3k + 1) \) for the constant \( k \) from Corollary 3.5 applied to \( \beta > 1 \). Then it holds

\[
a_f(x) \leq C a_f(y)
\]

for all \( x, y \in B \) and a constant \( C > 1 \) which depends only \( \beta, K, \) and the data of the space.

**Proof.** Throughout the proof we work in the metric \( d = d_{\mathbb{H}^1} \). Let \( B \subset \Omega \) be as above and let us fix \( x, y \in B \). Then, the Koebe theorem, see Theorem 1.4, implies

\[
a_f(x) \leq c_K \frac{d(f(x), \partial \Omega')}{d(x, \partial \Omega)} = c_K \frac{d(f(y), \partial \Omega')}{d(y, \partial \Omega)} \frac{d(f(x), \partial \Omega')}{d(f(y), \partial \Omega')} \frac{d(y, \partial \Omega)}{d(x, \partial \Omega)}
\]

\[
\leq c_K^2 a_f(y) \frac{d(f(x), \partial \Omega')}{d(f(y), \partial \Omega')} \frac{d(y, \partial \Omega)}{d(x, \partial \Omega)},
\]

so the task is reduced to bounding the quotient in the last expression.

Letting \( z \) be the center of the ball \( B \), we have that

\[
d(f(x), \partial \Omega') \leq d(f(x), f(z)) + d(f(z), \partial \Omega')
\]

\[
\leq \text{diam } f(B) + d(f(z), \partial \Omega')
\]

\[
\leq \left(\frac{2}{\beta - 1} + 1\right) d(f(z), \partial \Omega') \leq C' \text{dist}(f(B), \partial \Omega')
\]

with \( C' = \left(\frac{2}{\beta - 1} + 1\right) \frac{\beta}{\beta - 1} \). This follows from both parts of the assertion of Corollary 3.5 applied with \( \beta \). To justify the application of this corollary, we have to verify that \( 3kB \subset \Omega \) for the constant \( k \) associated to \( \beta \). This is indeed the case since the choice of \( \lambda \) ensures that

\[
(3k + 1) \frac{\text{diam } B}{2} \leq \text{dist}(B, \partial \Omega).
\]
Therefore, the relevant term in (5.13) can be bounded as follows:

\[
\frac{d(f(x), \partial \Omega')}{d(f(y), \partial \Omega')} \frac{d(y, \partial \Omega)}{d(x, \partial \Omega)} \leq 2C' \frac{\text{dist}(f(B), \partial \Omega')}{d(x, \partial \Omega)} \frac{d(y, \partial \Omega)}{d(x, \partial \Omega)} \leq 2C' \frac{d(y, \partial \Omega)}{d(x, \partial \Omega)}.
\]

To continue, we observe that (5.12) yields

\[
\frac{d(y, \partial \Omega)}{d(x, \partial \Omega)} \leq \frac{\text{diam}B + \text{dist}(B, \partial \Omega)}{\text{dist}(B, \partial \Omega)} \leq \lambda + 1,
\]

where the upper bound depends on \(\beta\) and \(K\) via the choice of \(\lambda\). Thus we can find a constant \(1 < C < \infty\) such that (5.13) reduces to \(a_f(x) \leq Ca_f(y)\), as desired. \(\square\)

We now move to proving the main result of this section. The proof proceeds similarly to the corresponding result for quasiconformal mappings in the Euclidean setting, see Theorem 3.4 in [3]. However, the proof in the Heisenberg setting involves several auxiliary observations for \(H^1\), proved above, such as Theorem 3.1, Proposition 3.3, Corollary 3.5, Proposition 3.9, Lemma 5.11 and the Koebe theorem (Theorem 1.4). Furthermore, the Whitney decomposition of the domain tailored for our construction must be carefully chosen in order to ensure all the restrictions imposed on balls in the course of the discussion of the aforementioned auxiliary results. This result holds both for the sub-Riemannian and the Korányi distance:

**Lemma 5.14 (Whitney decomposition).** Let \(\Omega \subseteq H^1\) be an open subset. For any \(\lambda \in (0, 1/2)\), there exists a countable collection \(C = \{B(x_i, r_i)\}\) of balls in \(\Omega\) such that

1. \(\Omega = \bigcup_i B(x_i, r_i)\)
2. \(\sum_i \chi_{B(x_i, 2r_i)} \leq C\),
3. \(\frac{\lambda}{4} \text{dist}(B, \partial \Omega) \leq \text{diam} B \leq \lambda \text{dist}(B, \partial \Omega)\),

for any ball \(B = B(x_i, r_i)\) in \(C\).

**Proof.** We fix a metric \(d \in \{d_s, d_{\mathbb{H}^1}\}\). Our goal is to find a collection of balls satisfying (1), (2), and

\[
(3') \quad c_1(\lambda) d(x_i, \partial \Omega) \leq r_i \leq c_2(\lambda) d(x_i, \partial \Omega).
\]

for \(c_1(\lambda) := \frac{\lambda}{4}\) and \(c_2(\lambda) := \frac{\lambda}{\lambda + 2}\). These constants have been chosen so that (3') implies (3). Indeed, assuming (3'), we find

\[
\frac{\lambda}{4} \text{dist}(B, \partial \Omega) \leq \frac{\lambda}{4} d(x_i, \partial \Omega) \leq 2r_i = \text{diam} B
\]

and

\[
\text{diam} B = 2r_i \leq \frac{2\lambda}{\lambda + 2} d(x_i, \partial \Omega) \leq \frac{2\lambda}{\lambda + 2} (\text{dist}(B, \partial \Omega) + r_i) = \frac{2\lambda}{\lambda + 2} (\text{dist}(B, \partial \Omega) + \frac{1}{2} \text{diam} B),
\]

\[
\frac{\lambda}{4} \text{dist}(B, \partial \Omega) \leq \frac{\lambda}{4} d(x_i, \partial \Omega) \leq 2r_i = \text{diam} B.
\]
which implies the right-hand side of (3). Thus it suffices to verify (1), (2) and (3’). We adapt the proof of Proposition 4.1.15 in [25] and for any \( k \in \mathbb{Z} \) define
\[
\mathcal{F}_k := \left\{ B \left( x, \frac{1}{2} \left( \frac{c_1(\lambda) + c_2(\lambda)}{5} \right) d(x, \partial \Omega) \right) : x \in \Omega \text{ and } 2^{k-1} \leq d(x, \partial \Omega) \leq 2^k \right\}.
\]
We apply the \( 5r \)-covering lemma to find a countable pairwise disjoint family of balls \( \mathcal{G}_k \subset \mathcal{F}_k \) so that a family of balls
\[
\mathcal{C} := \bigcup_{k \in \mathbb{Z}} \{ 5B : B \in \mathcal{G}_k \}
\]
satisfies assertion (1) of the lemma. By the definition of the radii \( r_i \) as the arithmetic averages of \( c_1(\lambda) \) and \( c_2(\lambda) \) for \( x_i \) that satisfy (3’). In order to show (2) we proceed as in [25], exploiting the doubling property of the metric. Suppose that there is \( x \in \Omega \) belonging to \( M \) balls of the form \( 2B_i \) for \( B \in \mathcal{C} \). We relabel the centers of these balls as \( x_1, \ldots, x_M \) in such a way that \( d(x_i, \partial \Omega) \leq d(x_1, \partial \Omega) \) for \( i = 1, \ldots, M \). The radii of the balls \( 2B_i \) are given by
\[
a(\lambda)d(x_i, \partial \Omega) := (c_1(\lambda) + c_2(\lambda))d(x_i, \partial \Omega).
\]
Note that the function \( a(\cdot) \) is increasing on \([0, 1/2]\). As \( x \) lies in the intersection of the balls \( 2B_i \) centered at \( x_i \), we find for all \( i = 1, \ldots, M \) that
\[
d(x_1, x_i) \leq a(\lambda)(d(x_1, \partial \Omega) + d(x_i, \partial \Omega))
\]
and hence
\[
d(x_i, \partial \Omega) \geq d(x_1, \partial \Omega) - d(x_1, x_i) \geq (1 - a(\lambda))d(x_1, \partial \Omega) - a(\lambda)d(x_1, \partial \Omega).
\]
This implies
\[
d(x_i, \partial \Omega) \geq \frac{1 - a(\lambda)}{1 + a(\lambda)}d(x_1, \partial \Omega).
\]
Moreover, for all \( i \) we have \( 2B_i \subset B(x_i, 3R_1) \), with \( 3R_1 = 3a(\lambda)d(x_1, \partial \Omega) \). If \( x_i \) and \( x_j \) are distinct centers of balls in the same family \( \mathcal{G}_k \) we have, by disjointedness of the balls in \( \mathcal{G}_k \), that
\[
d(x_i, x_j) \geq \frac{1}{5} a(\lambda) \min\{d(x_i, \partial \Omega), d(x_j, \partial \Omega)\} \geq \frac{1}{5} a(\lambda) \frac{1 - a(\lambda)}{1 + a(\lambda)}d(x_1, \partial \Omega).
\]
That is, such points form a \( \delta(\lambda) \)-separated set for
\[
\delta(\lambda) := \frac{1}{5} \frac{a(\lambda) 1 - a(\lambda)}{1 + a(\lambda)}d(x_1, \partial \Omega)
\]
and they are all included in a ball of radius \( R(\lambda) := 3a(\lambda)d(x_1, \partial \Omega) \). It is important for us to observe that \( \delta(\lambda)/R(\lambda) \) can be bounded from below by a strictly positive number which does not depend on \( \lambda \). This is the case since
\[
\frac{1 - a(\lambda)}{1 + a(\lambda)} \geq \frac{1 - a(1/2)}{1 + a(1/2)} > 0
\]
for all \( \lambda \in (0, 1/2) \). The doubling property (see Lemma 4.1.12 in [25]) gives us that at most \( N' \) of the balls \( 2B_i \) can have their centers in \( \mathcal{F}_k \) for a fixed \( k \), where \( N' \) is a constant which depends only on the doubling constant associated to the metric \( d \) and the universal lower bound for \( \delta(\lambda)/R(\lambda) \). Next we show that the centers of the balls in our family \( 2B_1, \ldots, 2B_M \) can lie in at most two different ‘layers’ \( \mathcal{F}_k \), which will provide the
desired universal upper bound for $M$. Indeed, assume that $x_1 \in \mathcal{F}_{k_1}$. Then, for every $i \in \{1, \ldots, M\}$, we find
\[ d(x_1, \partial \Omega) \geq d(x_i, \partial \Omega) \geq \frac{1 - a(\lambda)}{1 + a(\lambda)} d(x_1, \partial \Omega) \geq \frac{1 - a(1/2)}{1 + a(1/2)} d(x_1, \partial \Omega) > \frac{1}{2} d(x_1, \partial \Omega). \]
The last estimate finally explains our choice of the bound $\lambda < 1/2$. This estimate shows that all centers are contained in $\mathcal{F}_{k_1 - 1} \cup \mathcal{F}_{k_1}$ for some $k_1 \in \mathbb{Z}$. \hfill \Box

**Proof of Theorem 5.10.** Throughout the proof we work with the metric $d = d_{H^1}$; the corresponding result for $d_s$ can be deduced from the final statement for $d_{H^1}$. Let $\lambda \in (0, \frac{1}{2})$ be the largest number for which the following conditions are satisfied:

1. $\lambda \leq 2/(3k + 1)$ where $k$ is\(^2\) as in Corollary 3.5 applied to $g := f$ and $\beta = 2$,
2. $\lambda \leq 2/(3c)$, where $c$ is as in Theorem 3.1 applied to $f$,
3. $\lambda \leq 2/(3k)$, where $k$ is as in Proposition 3.9 applied to $f$.

These conditions are such that $\lambda$ is a positive constant depending on $K$, and every ball $B := B(x_0, r) \subset \Omega$ with $\text{diam} B \leq \lambda \text{dist}(B, \partial \Omega)$ satisfies the assumptions of the following results (applied to the map $f$):

1. Corollary 3.5 (ball distortion) and Lemma 5.11 (Harnack-type inequality for $a_f$) for $\beta = 2$,
2. Theorem 3.1 (higher integrability),
3. Proposition 3.9 (weight property of the Jacobian).

We will prove a statement for such balls which in addition satisfy a lower bound on the diameter:
\[
\frac{\lambda}{4} \text{dist}(B, \partial \Omega) \leq \text{diam } B \leq \lambda \text{dist}(B, \partial \Omega). \tag{5.15}
\]

Our first step is to obtain a double inequality comparing $a_f(x_0)$ to a mean value of the appropriate power of $\|D_H f\|$ over the ball $B$.

By quasiconformality of $f$, condition (5.15), Corollary 3.5 applied to $B$, $g := f$ and $\beta = 2$, and Theorem 1.4, we obtain that
\[
\int_{B} \|D_H f\|^4 \, dm \leq K \frac{|f(B)|}{|B|} \lambda^4 \text{diam}(f(B))^4 \leq K \frac{(2d(f(x_0), \partial \Omega')^4)}{(\lambda^4 \text{dist}(B, \partial \Omega))^4} \leq K \frac{(2d(f(x_0), \partial \Omega')^4)}{(\lambda^4 \text{diam}(f(B))^4)} \leq K \frac{4^4(\lambda + 2)4^4 c_k^4 a_f^4(x_0)}{\lambda^4}.
\]

Thus, denoting $C'(K) := \frac{K^{1/4}(\lambda + 2)c_k}{\lambda}$, we find that
\[
\int_{B} \|D_H f\|^4 \, dm \leq C'(K) a_f^4(x_0). \tag{5.16}
\]

We emphasize that $C'(K)$ depends on $K$ only.

In order to obtain a similar estimate for $a_f(x_0)$ from above, we appeal to the reasoning similar to the proof of Lemma 4.2. Namely, let $B_1 := B(x_0, d(x_0, \partial \Omega))$. Recall that by the\(^2\)The exact value of $\beta$ is not essential, any constant larger than 1 which depends at most on $K$ would work.
discussion in Section 3 we have that \( \log J_f \in BMO(\Omega) \) and \( \| \log J_f \|_* \) can be bounded in terms of \( K \). Consequently

\[
| (\log J_f)_{B_1} - (\log J_f)_B | \leq C \| \log J_f \|_* ,
\]

(5.17)

for a constant \( C \) depending on \( K \) via the ratio

\[
\frac{|B_1|}{|B|} = \frac{d(x_0, \partial \Omega)^4}{r^4} \leq \frac{(r + \text{dist}(B, \partial \Omega))^4}{r^4} \leq \left( 1 + \frac{8}{\lambda} \right)^4 ,
\]

see (5.15). Therefore, since \( \| \log J_f \|_* \) can be bounded by a constant in terms of \( K \), we have

\[
a_f(x_0)^4 = \exp \left( (\log J_f)_{B_1} \right) \leq \exp \left( (\log J_f)_B + C \| \log J_f \|_* \right)
\]

\[\leq C(K) \int_B J_f \, dm \leq C(K) \int_B \| D_H f \|^4 \, dm \]

(5.18)

for a constant \( C(K) \) which depends only on \( K \). Here, we have used the Jensen inequality for the convex function \( e^t \) and the Hadamard inequality in order to estimate \( J_f = (\det D_H f)^2 \) in terms of \( \| D_H f \|^4 \).

So far we have shown that \( a_f(x_0) \) is comparable to the average of \( \| D_H f \|^4 \) over \( B = B(x_0, r) \). The next goal is to replace “4” by a different power. Starting from (5.18), we apply the Hölder inequality, the Gehring-type estimate in Theorem 3.1 (with exponent \( p > 4 \)) together with Proposition 3.9 to arrive at the following estimates:

\[
a_f(x_0) \leq C(K) \left( \int_B \| D_H f \|^4 \, dm \right)^{\frac{1}{4}} \leq C(K) \left( \int_B \| D_H f \|^p \, dm \right)^{\frac{1}{p}} \leq C(K) \left( \int_B J_f \, dm \right)^{\frac{1}{2}} \leq c(K) \left( \int_B \| D_H f \|^{4-p} \, dm \right)^{\frac{1}{4-p}} .
\]

(5.19)

As in the proof of [3, Theorem 3.4] we recall the following inequality for \( g \in L^1(B) \) and \( \epsilon > 0 \), whose proof is a direct consequence of the Hölder and the Jensen inequalities:

\[
\left( \int_B \frac{1}{|g|^\epsilon} \, dm \right)^{-\frac{1}{\epsilon}} \leq \int_B |g| \, dm .
\]

(5.20)

This estimate applied for \( g := \| D_H f \| \) and \( \epsilon := p - 4 \), together with the Hölder inequality, gives the following:

\[
\left( \int_B \| D_H f \|^{4-p} \, dm \right)^{\frac{1}{4-p}} \leq \int_B \| D_H f \| \, dm \leq \left( \int_B \| D_H f \|^4 \, dm \right)^{\frac{1}{4}} .
\]

(5.21)

This combined with (5.16) results in a lower integral estimate for \( a_f(x_0) \) in terms of \( \| D_H f \|^{4-p} \):

\[
\left( \int_B \| D_H f \|^{4-p} \, dm \right)^{\frac{1}{4-p}} \leq C'(K) a_f(x_0) .
\]

(5.22)
At this stage we apply Lemma 5.11 (a Harnack-type inequality) together with estimate (5.19) (for \(0 < q < p\)) or estimate (5.22) (for \(4 - p < q < 0\)) and obtain that
\[
\int_B a_f^q \, dm \leq C^q \int_B a_f(x_0)^q \, dm = C^q a_f(x_0)^q \leq C(K) \left( \int_B \|DHf\|^{4-p} \, dm \right)^{\frac{q}{4-p}},
\]
where the constant \(C(K)\) arises as a product of \(q\)-th powers of the constants \(C\) in Lemma 5.11 (for \(\beta = 2\)) and \(c(K)\) in (5.19) (or \(C'(K)\) (5.22), depending on the sign of \(q\)). We wish to estimate the above integral further from above.

We consider three cases: (1) \(4 - p < q < 0\), (2) \(0 < q < 1\), (3) \(1 \leq q < p\). In the first case, the Hölder inequality gives us that
\[
\left( \int_B \frac{dm}{\|DHf\|^{-q}} \right)^{\frac{1}{q}} \leq \left( \int_B \frac{dm}{\|DHf\|\|DHf\|^q} \right)^{\frac{1}{q}}.
\]
In the second case a direct application of (5.20) for \(g := \|DHf\|^q\) and \(\epsilon := (p-4)/q\) results in the following estimate:
\[
\left( \int_B \|DHf\|^{4-p} \, dm \right)^{\frac{q}{4-p}} \leq \int_B \|DHf\|^q \, dm.
\]
Finally, in the third case we apply (5.20) and the Hölder inequality to obtain the estimate (5.23). Therefore, as a consequence of the above case analysis, we get
\[
\int_B a_f^q \, dm \leq c(K) \int_B \|DHf\|^q \, dm.
\]
By the analogous estimates we obtain the lower bound for the mean value of \(a_f^q\) over \(B\). Thus, it holds that
\[
\frac{1}{c(K)} \int_B a_f^q \, dm \leq \int_B \|DHf\|^q \, dm \leq c(K) \int_B a_f^q \, dm.
\]
In the last step we apply the Whitney decomposition argument and show that \(\Omega\) can be expressed as a union of balls with controlled overlap satisfying (5.15). That this is indeed the case, follows from Lemma 5.14. \(\square\)

As in the Euclidean case we have the following consequence of Theorem 5.10, cf. Corollary 3.5 in [3].

**Corollary 5.24.** Let \(f : \Omega \to \Omega'\) be a \(K\)-quasiconformal mapping between domains in \(\mathbb{H}^1\) for some \(K \geq 1\). Then for all \(4 - p < q < p\) with \(p = p(K) > 4\) and \(c\) depending on \(K\) and \(q\), it holds that
\[
\frac{1}{c} \int_{\Omega'} a_{f^{-1}}(x)^{4-q} \, d\mu \leq \int_{\Omega} a_f(x)^q \, d\mu \leq c \int_{\Omega'} a_{f^{-1}}(x)^{4-q} \, d\mu.
\]
**Proof.** The proof follows the same lines as the proof of the corresponding result, Corollary 3.5 in [3], and is based on the change of variable formula, see e.g. Theorem 5.4(a) in [17] and the proof of Proposition 3.9. \(\square\)
5.3. Quasiconformal metrics on domains in $H^1$. In [10], the authors study quasiconformal metrics (more precisely, densities) defined on the unit ball in $\mathbb{R}^n$. The motivating example for their considerations is that a conformal map from the planar unit disc into $\mathbb{C}$ is, up to post-compositions with isometries, uniquely determined by the absolute value of its derivative. This suggests to think of the latter as a ‘density’ on the unit disk, and it motivates the following axiomatic definition.

Let $B = B(0,1) \subset \mathbb{R}^n$ be the unit ball in the Euclidean space $\mathbb{R}^n$. Let further $\varrho : B \to (0, \infty)$ be a strictly positive continuous function (called a density). Additionally, we require $\varrho$ to satisfy the following conditions, cf. Section 1 in [10]:

1. (Harnack-type inequality.) There exists a constant $\lambda \in (0,1)$ such that $\varrho(x) \approx \varrho(y)$, for all $x, y \in B(z, \lambda d(z, \partial B))$ for $z \in B$.

2. (Upper Ahlfors condition with respect to $d_\varrho$.) There exists a constant $A > 0$ such that
\[
\mu_\varrho(B_\varrho(x,r)) := \int_{B_\varrho(x,r)} \varrho^n(y) \, dm(y) \leq Ar^n, \quad \text{for all } x \in B, r > 0. \tag{5.25}
\]

Here $B_\varrho(x,r)$ stands for an open ball with respect to the length metric $d_\varrho(a,b) := \inf_\gamma l_\varrho(\gamma)$ with weighted length $l_\varrho(\gamma) = \int_\gamma \varrho \, ds$ and curves $\gamma$ joining $a,b \in B$ within $B$.

It turns out, see [10], that these simple conditions imposed on a density function are enough to infer several interesting geometric properties of distances defined via such densities. Among the examples of such densities studied in [10, Section 2.4], is
\[
\varrho := \left( \int_{B(x,\text{dist}(x,\partial B))} J_f \, dm \right)^{\frac{1}{n}},
\]
where $f : \mathbb{H}^n \to \Omega$ is a $K$-quasiconformal mapping from the unit ball in $\mathbb{R}^n$ into a domain $\Omega \subset \mathbb{R}^n$. The purpose of this section is to show a counterpart of this observation for quasiconformal mappings between domains in $\mathbb{H}^1$. The results of this paper allow us to move beyond the setting of mappings from a unit ball and study more general domains in $\mathbb{H}^1$. Since the sub-Riemannian distance $d_s$ is the length distance associated to the Korányi metric $d_{\text{Kor}}$, we obtain the same distance $d_\varrho$ with respect to either metric for any density $\varrho$.

The following holds both for $d = d_s$ and $d = d_{\text{Kor}}$, and the length element $ds$ in the definition of $l_\varrho$ taken with respect to the distance $d$:

**Proposition 5.26.** Let $f : \Omega \to \Omega'$ be a $K$-quasiconformal map between domains $\Omega, \Omega' \subseteq \mathbb{H}^1$. Then the function $a_f$ possesses the following properties:

1. There exists a constant $\lambda \in (0,1)$ such that for all balls $B \subset \Omega$ satisfying $\text{diam} B \leq \lambda \text{dist}(B, \partial \Omega)$, it holds $a_f(x) \approx a_f(y)$ for all $x,y \in B$,

with the equivalence constants depending on $K$ and the data of the space.

2. The upper Ahlfors regularity holds for the measure $\mu_\varrho$ as in (5.25) with $\varrho = a_f$, $n = 4$ and constants depending on $K$ and the data of the space.
Proof. Assertion (1) for \( d = d_{\mathbb{H}^1} \) follows from Lemma 5.11 applied to a fixed universal constant \( \beta \). Since \( a_f^2 \) and \( a_f^4 \) are comparable, it is clear that the respective statement also holds for \( d = d_s \), with a possibly smaller (but uniformly controlled) \( \lambda \).

In order to prove the second assertion for \( d = d_s \) (and a posteriori for \( d = d_{\mathbb{H}^1} \)) we follow the steps of the proof of the corresponding property for quasiconformal mappings from a unit ball in \( \mathbb{R}^n \) into \( \mathbb{R}^n \), see [10, Section 2.4]. Let \( x \in \Omega \) and \( r > 0 \). We consider two cases. Suppose that \( r \leq (c(K)a_f(x))d_s(x, \partial \Omega) \) for a constant \( 0 < c(K) < 1 \) depending only on \( K \) and to be determined later. Let \( \lambda > 0 \) be the constant from Lemma 5.11 associated to, say, \( \beta = 2 \). We have the following inclusion of sub-Riemannian balls

\[
B_s \left( x, \frac{(\lambda/\sqrt{\pi})r}{(\lambda/\sqrt{\pi} + 2)c(K)a_f(x)} \right) \subseteq B_s \left( x, \frac{\lambda/\sqrt{\pi}}{\lambda/\sqrt{\pi} + 2} \cdot d_s(x, \partial \Omega) \right).
\]

Here the radius of the smaller ball has been chosen so that it is included in a ball which satisfies the assumption of Lemma 5.11, so that a Harnack-type inequality for \( a_f \) is valid on that ball. Note that the factor \( \sqrt{\pi} \) appears since Lemma 5.11 has been formulated for the Korányi distance. The latter is bi-Lipschitz equivalent to the sub-Riemannian distance, but the value of the constant \( \lambda \) in the assumption changes accordingly (see (3.21)). Note that also the precise value of the constant \( C \) in Lemma 5.11 will change since we compute \( a_f \) with respect to \( d_s \) rather than \( d_{\mathbb{H}^1} \), yet this plays no role in the following.

Consider now \( z \in B_g(x, r) \). By definition,

\[
d_g(x, z) = \inf_{\gamma_{xz}} \int_{\gamma_{xz}} a_f ds = \inf_{\gamma_{xz}} \int_{\gamma_{xz}} \frac{a_f(x)}{a_f(x)} ds
\]

where \( \gamma_{xz} \) is an arbitrary (rectifiable) curve joining \( x \) and \( z \) within \( \Omega \). The plan is to apply Lemma 5.11 in order to bound this quantity from below by \( \frac{1}{c(K)a_f(x)}d_s(x, z) \), for a positive and finite constant \( C \), which depends only on \( K \). To justify the application of Lemma 5.11, it suffices to ensure that we can consider curves \( \gamma_{xz} \) which stay inside the sub-Riemannian ball \( B_s(x, \frac{(\lambda/\sqrt{\pi})r}{(\lambda/\sqrt{\pi} + 2)c(K)a_f(x)}) \). Let us explain why this is the case. First, since \( z \in B_g(x, r) \), there exists a rectifiable curve \( \gamma_{xz} \) which connects \( x \) to \( z \) and satisfies

\[
\int_{\gamma_{xz}} a_f ds < r.
\]

In the definition of \( d_g(x, z) \) we can restrict the infimum to curves satisfying (5.27). Assume that such a curve \( \gamma_{xz} \) exits \( B_s(x, \frac{(\lambda/\sqrt{\pi})r}{(\lambda/\sqrt{\pi} + 2)c(K)a_f(x)}) \). Then, by connectedness, there must exist a (first) point \( w \) on the trace of \( \gamma_{xz} \) with

\[
w \in \partial B_s \left( x, \frac{\lambda/\sqrt{\pi}r}{(\lambda/\sqrt{\pi} + 2)c(K)a_f(x)} \right).
\]

We denote by \( \gamma_{xw} \) the subcurve of \( \gamma_{xz} \) which connects \( x \) and \( w \) inside \( B_s(x, \frac{\lambda/\sqrt{\pi}r}{(\lambda/\sqrt{\pi} + 2)c(K)a_f(x)}) \). Since \( a_f \) is a positive function, we find

\[
\int_{\gamma_{xz}} a_f ds \geq \int_{\gamma_{xw}} a_f ds \geq \frac{1}{c_f(x)} \int_{\gamma_{xw}} ds \geq \frac{1}{c_f(x)} d_s(x, w) = \frac{\lambda/\sqrt{\pi}r}{(\lambda/\sqrt{\pi} + 2) \cdot c \cdot c(K)}.
\]
The constant $C$ from Lemma 5.11 is larger than 1, so we may choose $0 < c(K) < 1$ such that
\[ c(K) < \frac{\lambda/\sqrt{\pi}}{C \cdot (\lambda/\sqrt{\pi} + 2)}, \] (5.28)
which leads to a contradiction to the assumption $\int_{\gamma_{xz}} a_f \, ds < r$. With this choice of $c(K)$, we may restrict the curves in the definition of $d_y(x, z)$ to those curves $\gamma_{xz}$ along which the Harnack inequality for $a_f$ is valid, and we find
\[ \frac{1}{C} a_f(x) d_s(x, z) \leq d_y(x, z) < r. \]
In particular we have for our choice of $c(K)$ that
\[ B_y(x, r) \subseteq B_s \left( x, C \frac{r}{a_f(x)} \right) \subseteq B_s \left( x, \frac{\lambda/\sqrt{\pi}}{\lambda/\sqrt{\pi} + 2} d_s(x, \partial \Omega) \right), \]
for $0 < r \leq c_K a_f(x) d_s(x, \partial \Omega)$. Since the Harnack inequality from Lemma 5.11 is valid on $B_y(x, r)$, we find
\[ \mu_e(B_y(x, r)) = \int_{B_y(x, r)} a_f(y)^3 \, dy \leq C^4 a_f(x)^4 m \left( B_y(x, r) \right). \]
Thus, we obtain:
\[ \mu_e(B_y(x, r)) \leq C^4 a_f(x)^4 m \left( B_y(x, r) \right) \leq C^4 a_f(x)^4 m \left( B_s \left( x, C \frac{r}{a_f(x)} \right) \right) \leq C^8 r^4 \]
and the proposition is proven in this case.

Let us now consider the case $r \geq c(K) a_f(x) d_s(x, \partial \Omega)$. Then, by Theorem 1.4 we have $r \geq c(K)/c_K d_s(f(x), \partial Y)$. We will use this estimate below.

First, by Lemma 5.14 let us decompose $\Omega$ as a union of balls satisfying the Whitney condition (5.15) for $d_s$ and $\lambda \in (0, \frac{1}{4})$ the largest number, possibly different from the first part of the proof, for which the following conditions are satisfied
\[ \lambda < \lambda/(1 - \lambda) \leq 1/(4k + 1) \] (5.29)
\[ \sqrt{\pi} \lambda \leq 2/(3k + 1), \] (5.30)
where $\kappa$ is as in Proposition 4.4 (egg yolk principle) and $k$ is as in Corollary 3.5 applied to $g := f$ and $\beta = 2$. The value of $\lambda$ depends only on $K$. The first condition, (5.29), is to ensure quasisymmetry of $f$ on all the relevant balls which will appear later in the proof. The second condition, (5.30), is to guarantee that every ball in the constructed Whitney decomposition satisfies the assumptions of Lemma 5.11 applied to map $f$. Note that the constant $\sqrt{\pi}$ is present because Lemma 5.11 has originally been formulated for $d_{\|1}$ instead of $d_s$. Since the balls in the respective metric satisfy $B_s(x, r) \subseteq B_{\|1}(x, \sqrt{\pi} r)$, this ensures that we can from now on work with the sub-Riemannian distance $d_s$, which we denote for simplicity by $d$.

Let $C_s$ be the collection of those sub-Riemannian balls $B$ in the chosen Whitney decomposition for which $B \cap B_y(x, r) \neq \emptyset$. Then, we claim that
\[ f \left( \bigcup_{B \in C_s} B \right) \subseteq B_s(f(x), cr), \] (5.31)
for some constant $c > 0$, which can be bounded from above in terms of $K$.

In order to show (5.31), let us consider $y \in B$ for $B \in C_x$ and discuss separately the two cases: (i) $y \in B_0(x, r)$ and (ii) $y \in B \setminus B_0(x, r)$. In the first case, by the definition of $d_\gamma$, there exists a rectifiable curve $\gamma$ joining $x$ and $y$ with $l_\gamma(\gamma) = \int_\gamma a_f(s)ds < r$. Let $\alpha := \frac{1}{4c+1}$, where $\kappa$ is a constant in the egg yolk principle, Proposition 4.4. If $d(x, y) \geq \alpha d(x, \partial \Omega)$, then $\text{length}(\gamma) \geq \alpha d(\gamma, \partial \Omega)$ and so Proposition 5.1 allows us to conclude the following estimate:

$$d(f(x), f(y)) \leq \text{diam } f(\gamma) \leq C \int_\gamma a_f(s)ds < Cr.$$  
From this

$$f\left(B \cap B_0(x, r) \cap \{y : d(x, y) \geq \alpha d(x, \partial \Omega)\}\right) \subseteq B_s(f(x), cr)$$
with $c \geq C$ follows, which is a first step towards the proof of (5.31).

If $d(x, y) < \alpha d(x, \partial \Omega)$, we will invoke Proposition 4.4, which we may by our choice of $\alpha$. Applied to $f$ and $\Omega$, this shows that there is a constant $H$, depending only on $K$, such that $f$ is $H$-quasisymmetric when restricted to $B(x, \frac{d(x, \partial \Omega)}{4c+1}) = B(x, \alpha d(x, \partial \Omega))$.

For $t > 0$ and $x_0 \in \Omega$, set

$$L_f(x_0, t) := \sup_{\{z \in B(x, \frac{d(x, \partial \Omega)}{4c+1}) : d(x_0, z) \leq t\}} d(f(x_0), f(z)),$$

$$l_f(x_0, t) := \inf_{\{z \in B(x, \frac{d(x, \partial \Omega)}{4c+1}) : d(x_0, z) \geq t\}} d(f(x_0), f(z)).$$

With this notation, it holds that

$$d(f(x), f(y)) \leq L_f(x, \alpha d(x, \partial \Omega)) \leq Hl_f(x, \alpha d(x, \partial \Omega))$$

$$\leq H \frac{cK}{c(K)}.$$ 
In the last step we use the assumption that $r \geq c(K)/cKd(f(x), \partial \Omega')$. Altogether we have shown that

$$f(B \cap B_0(x, r)) \subseteq B_s(f(x), cr)$$
holds with $c \geq \max\{C, He_K/c(K)\}$. This concludes the discussion of case (i) regarding (5.31).

For (ii), suppose that $y \in B \setminus B_0(x, r)$ for some ball $B \in C_x$. Then, by the definition of $C_x$, there is $z \in B \cap B_0(x, r)$ and it holds that

$$d(f(x), f(y)) \leq d(f(x), f(z)) + d(f(z), f(y)).$$

The first term on the right-hand side above can be estimated by the reasoning of the previous case, since in particular $z \in B_0(x, r)$. In order to estimate the second term, we proceed as follows. Let $x_B$ be the center of $B$. Then, by Whitney condition (5.15) for $\lambda$ we observe that

$$d(x_B, z) \leq \text{diam } B \leq \lambda \text{dist}(B, \partial \Omega) \leq \lambda d(z, \partial \Omega).$$

Thus, $x_B \in B(z, \lambda d(z, \partial \Omega))$. Using this observation together with the definition of Whitney-type decomposition (5.15) with balls satisfying condition (5.29), we see that the conclusion of the egg yolk principle holds on

$$B(x_B, \lambda d(x_B, \partial \Omega)) \subseteq B(x_B, \text{diam } B) \subseteq B.$$
and on
\[ B(x_B, d(x_B, \partial \Omega)/(4\kappa + 1)) \supseteq B(z, \lambda d(z, \partial \Omega)). \]
Thus, exploiting the quasisymmetry property of \( f \) on the respective balls, we get
\[
d(f(z), f(y)) \leq d(f(z), f(x_B)) + d(f(x_B), f(y))
\leq 2L_f(x_B, \text{diam } B) \leq 2Hl_f(x_B, \text{diam } B)
\leq 2Hd(f(x_B), \partial \Omega') \leq 2H(d(f(x_B), f(z)) + d(f(z), \partial \Omega'))
\leq 2H(L_f(z, \lambda d(z, \partial \Omega)) + d(f(z), \partial \Omega')) \leq 2H(l_f(z, \lambda d(z, \partial \Omega')) + d(f(z), \partial \Omega'))
\leq 2H(H + 1)d(f(z), \partial \Omega') \leq 2H(H + 1) (d(f(z), f(x)) + d(f(x), \partial \Omega'))
\leq 2H(H + 1) \left( \max \{C, H^{-c_{K'}}} + c(K)/c_k \right) r.
\]
In the last step we appeal to the previously discussed case (as \( z \in B_\delta(x, r) \)) and use the assumption that \( r \geq c(K)/c_{K'}d(f(x), \partial \Omega') \). This completes the proof of this case and the whole claim (5.31), as well. Let us remark that the last estimate is similar to the proof of Proposition 3.7.5, pg. 70 in [32].

In order complete the proof of the proposition we observe that Lemma 5.11 together with the Jensen inequality for the exponential function and Lemma 4.2, applied to a suitable \( L \) depending on \( \lambda \), allow us to infer that for \( B \in C_2 \) it holds that
\[
\int_B a_f(y)^4 \, dm(y) \leq C' \int_B J_f(y) \, dm(y)
\]
for a suitable constant \( C' \geq 1 \) which depends only on \( K \). (This works analogously as in the proof of Theorem 5.10). Therefore,
\[
\mu_\varrho(B_\delta(x, r)) = \int_{B_\delta(x, r)} a_f(y)^4 \, dm(y) \leq C' \sum_{B \in C_2} \int_B J_f(y) \, dm(y) \leq Cm(B_\delta(f(x), cr))
\]
\[
\leq C''r^4,
\]
by (5.31) and the controlled overlap in the Whitney decomposition. Here the constants \( C \) and \( C'' \) depend only on \( K \). This completes the proof of the second assertion and the proof of the proposition.

Proposition 5.26 shows the upper Ahlfors regularity of \( \mu_\varrho \). More can be said under additional assumptions on \( \Omega \). Let us recall that a domain \( \Omega \subset \mathbb{H}^1 \) equipped with the sub-Riemannian distance \( d_s \) is called \( L \)-quasiconvex if any two points \( x, y \in \Omega \) can be joined by a curve \( \gamma \) such that its trace \( |\gamma| \) is in \( \Omega \) and length(\( \gamma \)) \( \leq Ld_s(x, y) \). Among examples of such domains let us mention uniform domains, half space and compact \( C^{1, \alpha} \)-domains. For further examples of quasiconvex domains in \( \mathbb{H}^1 \), see [26] and the references therein.

**Proposition 5.33.** Let \( f : \Omega \rightarrow \Omega' \) be a \( K \)-quasiconformal map from a quasiconvex domain \( \Omega \neq \mathbb{H}^1 \) onto a domain \( \Omega' \neq \mathbb{H}^1 \). Then, there exist constants \( 0 < c_1 < c_2 < \infty \) and \( 0 < c(K) < 1 \) such that for all \( x \in \Omega \) and all \( 0 < r < c(K)a_f(x)d_s(x, \partial \Omega) \) one has
\[
\quad c_1 r^4 \leq \mu_\varrho(B(x, r)) \leq c_2 r^4.
\]
**Proof.** Let us assume that \( \Omega \) is \( L \)-quasiconvex for some constant \( L \geq 1 \). The upper bound for \( \mu_\varrho(B(x, r)) \) has already been established in the first part of the proof of Proposition 5.26 (and this holds in fact for any \( r > 0 \)). Let us therefore consider the lower bound.
If \( c(K) \) is chosen as in the proof of Proposition 5.26, that is, as in (5.28), then we know already that the Harnack inequality for \( a_f \) holds on \( B_s(x, r/(Ca_f(x)L)) \). Thus, for all points \( z \) in this ball, we find
\[
d_\gamma(x, z) = \inf_{\gamma_{xz} \subset \Omega} \int_{\gamma_{xz}} a_f \, ds \leq Ca_f(x) \inf_{\gamma_{xz} \subset \Omega} \int_{\gamma_{xz}} ds \leq Ca_f(x)Ld_s(x, z),
\]
where we have used in the last step the assumption that \( \Omega \) is \( L \)-quasiconvex. The above estimate shows that
\[
B_s(x, \frac{r}{Ca_f(x)L}) \subseteq B_\gamma(x, r)
\]
and hence
\[
m(B(0, 1)) r^4 = \frac{a_f(x)^4}{C^4} m \left( B_s \left( x, \frac{r}{Ca_f(x)L} \right) \right) \leq \int_{B_s(x, r/(Ca_f(x)L))} a_f^4 \, dm \leq \mu_\gamma(B_\gamma(x, r)),
\]
which concludes the proof.

\[\square\]

REFERENCES

[1] K. Astala and F. W. Gehring. Quasiconformal analogues of theorems of Koebe and Hardy-Littlewood. Michigan Math. J., 32(1):99–107, 1985.
[2] K. Astala and F. W. Gehring. Injectivity, the BMO norm and the universal Teichmüller space. J. Analyse Math., 46:16–57, 1986.
[3] K. Astala and P. Koskela. Quasiconformal mappings and global integrability of the derivative. J. Anal. Math., 57:203–220, 1991.
[4] K. Astala and V. Manojlović. On Pavlovic’s theorem in space. Potential Anal., 43(3):361–370, 2015.
[5] A. D. Austin. Logarithmic potentials and quasiconformal flows on the Heisenberg group. arXiv:1701.04163, 2017.
[6] Z. M. Balogh. Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group. J. Anal. Math., 83:289–312, 2001.
[7] Z. M. Balogh, K. Fässler, and I.D. Platis. Modulus method and radial stretch map in the Heisenberg group. Ann. Acad. Sci. Fenn., Math., 38(1):149–180, 2013.
[8] A. Bellaïche. The tangent space in sub-Riemannian geometry. Sub-Riemannian Geometry, Eds. A. Bellaïche and J.J. Risler. Birkhäuser, Basel, 1996.
[9] M. Bonk, J. Heinonen, and E. Saksman. Logarithmic potentials, quasiconformal flows, and \( Q \)-curvature. Duke Math. J., 142(2):197–239, 2008.
[10] M. Bonk, P. Koskela, and S. Rohde. Conformal metrics on the unit ball in Euclidean space. Proc. London Math. Soc. (3), 77(3):635–664, 1998.
[11] S. Buckley, P. Koskela, and G. Lu. Boman equals John. In XVIIIth Rolf Nevanlinna Colloquium (Joensuu, 1995), pages 91–99. de Gruyter, Berlin, 1996.
[12] S. M. Buckley. Inequalities of John-Nirenberg type in doubling spaces. J. Anal. Math., 79:215–240, 1999.
[13] L. Capogna, D. Danielli, S. D. Pauls, and J. Tyson. An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem. Progress in Mathematics. Birkhäuser Basel, 2007.
[14] L. Capogna and N. Garofalo. Boundary behavior of nonnegative solutions of subelliptic equations in NTA domains for Carnot-Carathéodory metrics. J. Fourier Anal. Appl., 4(4-5):403–432, 1998.
[15] L. Capogna and P. Tang. Uniform domains and quasiconformal mappings on the Heisenberg group. Manuscr. Math., 86(3):267–281, 1995.
[16] R. R. Coifman and G. Weiss. Analyse harmonique non-commutative sur certains espaces homogènes. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. Étude de certaines intégrales singulières.
[17] N. S. Dairbekov. Mappings with bounded distortion on Heisenberg groups. Sibirsk. Mat. Zh., 41(3):567–590, 2000.
[18] G. B. Folland and E. M. Stein. Hardy spaces on homogeneous groups, volume 28 of Mathematical Notes. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.
[19] J. Heinonen. Calculus on Carnot groups. Fall school in analysis (Jyväskylä, 1994), pp.1-32.
[20] J. Heinonen. Lectures on Analysis on Metric Spaces. Universitext. Springer New York, 2001.
[21] J. Heinonen and P. Koskela. A_p-condition for the Jacobian of a quasiconformal mapping. Proc. Amer. Math. Soc., 120(2):535–543, 1994.
[22] J. Heinonen and P. Koskela. Definitions of quasiconformality. Invent. Math., 120(1):61–79, 1995.
[23] J. Heinonen and P. Koskela. Quasiconformal maps in metric spaces with controlled geometry. Acta Math., 181(1):1–61, 1998.
[24] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson. Sobolev classes of Banach space-valued functions and quasiconformal mappings. J. Anal. Math., 85:87–139, 2001.
[25] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson. Sobolev spaces on metric measure spaces, volume 27 of New Mathematical Monographs. Cambridge University Press, Cambridge, 2015. An approach based on upper gradients.
[26] D. A. Herron, A. Lukyanenko, and J. T. Tyson. Quasiconvexity in the Heisenberg group. arXiv:1609.07749, preprint, 2017.
[27] J. Kinnunen and P. Shukla. Gehring’s lemma and reverse Hölder classes on metric measure spaces. Comput. Methods Funct. Theory, 14(2-3):295–314, 2014.
[28] A. Korányi and H. M. Reimann. Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group. Bull. Sci. Math., II. Sér., 111:3–21, 1987.
[29] A. Korányi and H. M. Reimann. Foundations for the theory of quasiconformal mappings on the Heisenberg group. Adv. Math., 111(1):1–87, 1995.
[30] P. Koskela. An inverse Sobolev lemma. Rev. Mat. Iberoamericana, 10(1):123–141, 1994.
[31] E. Le Donne and S. Rigot. Besicovitch Covering Property for homogeneous distances in the Heisenberg groups. J. Eur. Math. Soc. (JEMS) 6, pages 1589–1617, 2016.
[32] H. Len Ruth Jr. Conformal Densities and Deformations of Uniform Loewner Metric Spaces. PhD thesis, University of Cincinnati, 2008.
[33] O. E. Maasalo. Global integrability of p-superharmonic functions on metric spaces. Journal d’Analyse Mathématique, 106(1):191–207, 2008.
[34] O. E. Maasalo. Global integrability of p-superharmonic functions on metric spaces. J. Anal. Math., 106:191–207, 2008.
[35] O. Martio, V. Ryazanov, U. Srebro, and E. Yakubov. Moduli in Modern Mapping Theory. Springer Monographs in Mathematics. Springer New York, 2008.
[36] P. Pansu. Métriques de carnot-carathéodory et quasiisométries des espaces symétriques de rang un. Ann. of Math., 129(1):1–60, 1989.
[37] C. Pommerenke and G. Jensen. Univalent functions. Studia mathematica. Vandenhoeck und Ruprecht, 1975.
[38] H. M. Reimann. Functions of bounded mean oscillation and quasiconformal mappings. Comment. Math. Helv., 49:260–276, 1974.
[39] H. M. Reimann and T. Rychener. Funktionen beschränkter mittlerer Oszillation. Lecture Notes in Mathematics, Vol. 487. Springer-Verlag, Berlin-New York, 1975.
[40] E. Soultanis and M. Williams. Distortion of quasiconformal maps in terms of the quasihyperbolic metric. J. Math. Anal. Appl., 402(2):626–634, 2013.
[41] S. G. Staples. L_p-averaging domains in homogeneous spaces. J. Math. Anal. Appl., 317(2):550–564, 2006.
[42] X. Tolsa. Analytic capacity, the Cauchy transform, and non-homogeneous Calderón-Zygmund theory, volume 307 of Progress in Mathematics. Birkhäuser/Springer, Cham, 2014.
[43] J. T. Tyson. Metric and geometric quasiconformality in Ahlfors regular Loewner spaces. Conform. Geom. Dyn., 5:21–73, 2001.
[44] J. Väisälä. Lectures on n-dimensional quasiconformal mappings. Lecture Notes in Mathematics, Vol. 229. Springer-Verlag, Berlin-New York, 1971.
