The application of the Lattice Boltzmann method to the one-dimensional modeling of blood flow in elastic vessels

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Abstract. We consider mass conservation and momentum equations for inviscid blood flow in elastic vessels. It is shown that the Lattice Boltzmann (LB) models for non-ideal gas in the limit of low Knudsen numbers are asymptotically equivalent to the blood flow equations. The equation of state for non-ideal gas becomes vessel’s pressure-area response when LB models are applied for blood motion modeling. Two test case problems are considered: a propagation of a sole nonlinear wave and pulse wave propagation in a stented vessel.

1. Introduction
The models of blood motion in cardiovascular system range from the 0-D lumped models, 1-D pulse propagation equations to 3D viscous flow equations [1]. In many cases the 3D approach based on the solution of the Navier-Stokes equations is too detailed while 0D lumped models are oversimplified and applicable only for the distal vasculature. In the case of the 1D models it is assumed that the radial velocity is negligible [2]. Then, integrating the Navier-Stokes equations over the radial variable the 1D nonlinear system of equations (depending only on one spatial variable, axial coordinate) for the luminal area change and the axial blood velocity is derived. The most popular method of solving of these equations is locally conservative Galerkin (LCG) method.

I will present an alternative way to model 1D blood dynamics based on the kinetic equations, namely, using the Lattice Boltzmann approach [3]. This method describes the motion of particles on Cartesian spatial lattice (advection part), the collision of the particles in spatial nodes is modeled by assuming that the velocity distribution of the particles tends to a local equilibrium state (trend to Gaussian distribution). The Lattice Boltzmann (LB) method correctly reproduces low-Mach incompressible flows like blood motion and can be used in the modeling of the cardiovascular network. The details about 3D modeling with the LB approach can be found in several papers [4, 5, 6, 7, 8].

We start with the one-dimensional model D1Q3 and show that there exists an interesting analogy between the hydrodynamics obtained from LB model at the limit of small Knudsen numbers (small times between collisions) and the blood flow equations in elastic vessels. More precisely, the LB hydrodynamics for D1Q3 model is equivalent to the 1D blood motion equations if one changes in the LB model the lattice sound velocity by the pulse propagation velocity and
the density by the luminal area. In this case the LB model describes a particular case of constant pulse propagation velocity though this case is nonlinear. The method can be generalized using the LB methods for non-ideal gases: the addition of fictitious force allows to model arbitrary area-pressure vessel responses.

The presented method is simpler than the finite-difference methods for the nonlinear 1D blood equations [9]. The method correctly describes the change in shape of the initial pulse wave and moreover can be applied for the assessment of the forward and the backward blood pressure-velocity wave (reflected wave) superposition. The blood velocity waveforms can be obtained for the different elastic area-to-pressure responses of the vessel wall, the wall viscoelastic effects can be also potentially implemented.

2. 1D blood motion equations and the Lattice Boltzmann method

Consider a long elastic vessel filled with incompressible fluid (blood). The conservation equations of mass and momentum in the elastic vessel are obtained from the Navier-Stokes equations by the integration over the cross-sectional spatial coordinate (radial variable). We assume that the radial component of the blood velocity is close to zero, and the velocity profile is flat. Then the following equations are obtained [9]

\[ \frac{\partial A}{\partial t} + \frac{\partial A u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \]

where \( \rho \) is the constant blood density (incompressible fluid), \( A, u \) is the luminal area, blood velocity respectively. Here \( x \) is the axial coordinate and \( p \) is the blood pressure. We mention that viscous forces are not included in Eqs (1).

It is obvious that one has three unknowns \( u, A, p \) and only the two governing equations. To close the system (1) one needs to introduce a pressure-area relation

\[ p = f(A), \]

where \( f \) is a some function, its form should be defined from the elastic properties of the considered vessel. For realistic vessels this dependence can be complicated and saturation effects are observed [10]. In practice the Laplace law is popular [1, 2].

From the pressure-area relation one can find the pulse-wave velocity using the formula

\[ c_{\text{pulse}}^2 = \frac{A}{\rho} \frac{\partial p}{\partial A}. \]

Using the pulse wave velocity definition Eqs (1)-(2) can be rewritten in the following equivalent form

\[ \frac{\partial A}{\partial t} + \frac{\partial A u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = -\frac{c_{\text{pulse}}^2(A)}{A} \frac{\partial A}{\partial x}. \]

We will model these equations using the Lattice Boltzmann approach [3]. A simplest three-velocity model termed as D1Q3 is considered below. This model describes three populations of particles traveling on a 1D lattice with equal spacing \( c \) between the lattice nodes. During an each discrete time step \( \Delta t \) the particles at a node \( x \) can hop to adjacent nodes \( x \pm c \) or stay at the node \( x \). We define the corresponding concentrations of the particles as \( f_{\pm 1}(t, x) \) and \( f_0(t, x) \) respectively. The dynamics of \( f_{\pm 1}, f_0 \) obeys the following discrete equations in space and time

\[ f_{-1}(t + \Delta t, x - c\Delta t) - f_{-1}(t, x) = \frac{\Delta t}{\tau + \frac{1}{2}} \left( f_{-1}^{eq}(t, x) - f_{-1}(t, x) \right), \]
\[ f_0(t + \Delta t, x) - f_0(t, x) = \frac{\Delta t}{\tau + \frac{1}{2}} (f_0^e(t, x) - f_0(t, x)), \]
\[ f_{+1}(t + \Delta t, x + c\Delta t) - f_{+1}(t, x) = \frac{\Delta t}{\tau + \frac{1}{2}} (f_{+1}^e(t, x) - f_{+1}(t, x)), \]

where \( \tau \) is the relaxation time which can be considered as a free parameter and \( f_{0}^{eq}, f_{+1}^{eq} \) are the equilibrium states (analogs of the Maxwell distribution)

\[ f_{-1}^{eq}(t, x) = \frac{\rho(t, x)}{6} \left( 1 - 3 \frac{u(t, x)}{c} + 3 \frac{u(t, x)^2}{c^2} \right), \]
\[ f_0^{eq}(t, x) = \frac{4\rho(t, x)}{6} \left( 1 - 3 \frac{u(t, x)^2}{2c^2} \right), \]
\[ f_{+1}^{eq}(t, x) = \frac{\rho(t, x)}{6} \left( 1 + 3 \frac{u(t, x)}{c} + 3 \frac{u(t, x)^2}{c^2} \right), \]

where \( \rho, u \) are the density and flow velocity

\[ \rho(t, x) = f_{-1}(t, x) + f_0(t, x) + f_{+1}(t, x), \]
\[ \rho(t, x)u(t, x) = -f_{-1}(t, x)c + f_{+1}(t, x)c, \]

moreover, one can define the full energy of the lattice gas in every node by the formula

\[ \rho(t, x)(u(t, x)^2 + c_s^2) = (f_{-1}(t, x) + f_{+1}(t, x))c \]

and here \( c_s \) is the sound velocity of the lattice gas defined by

\[ c_s = \sqrt{\frac{T}{\rho}} c. \]

At the limit of small values of \( \tau \) the lattice gas can be considered as a continuous media since the time between collisions tends to zero (\( \tau \) as the relaxation time is proportional to the expected time between the collisions). In the continuous limit the considered LB model describes some hydrodynamics which can be obtained using the Chapman-Enskog expansion [11] on a small parameter (\( \tau \)). For the D1Q3 the following equations are obtained

\[ \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0, \quad \frac{\partial u}{\partial t} + \frac{\partial u^2/2}{\partial x} = -\frac{c_s^2}{\rho} \frac{\partial \rho}{\partial x} + \frac{\nu}{\rho} \left( \frac{\partial u}{\partial x} \right) + O(Ma^3) + O(\Delta t^2), \]

where \( O(Ma^3) \) is error term which appears due to the polynomial form of the local equilibrium on the bulk velocity (here \( Ma \) is Mach number), \( O(\Delta t^2) \) is space-time discretization error. If the flow is slow (\( Ma \sim 0 \)) and the time step is small then we can state that the LB model describes 1D hydrodynamics of the compressible gas.

Now if one changes the density \( \rho \) by the luminal area \( A \) and the sound velocity \( c_s \) by the pulse wave velocity \( c_{\text{pulse}} \) then Eqs (4) turn into Eqs (3) if the flow is slow, discretization error is negligible and viscous term can be omitted from consideration. These conditions are realistic due to a) \( A \) typical blood flow is slow, the blood velocity is over 0.4m/s, the pulse wave speed is over 4m/s, then \( Ma \sim 0.1 \) b) the viscous terms can be neglected in large vessels since the typical Reynolds number in aorta is close to 1000 or even greater c) the discretization error can be controlled by the choice of \( \Delta t \).
Note that the equations describe only the particular case of constant pulse velocity since $c_s$ is constant in the LB model. Indeed, the $D1Q3$ model describes isothermal flows with the equation of state $p = \rho c_s^2$ and if one changes $\rho$ by $A$ and $c_s$ by $c_{\text{pulse}}$ then we get area-pressure relation $p = A c_{\text{pulse}}^2$.

The generalization of the presented method for the arbitrary area-pressure relation can be performed using the extension of the Lattice Boltzmann method for non-ideal gases and arbitrary equations of state which results in general area-pressure relations in the our case [12, 13, 14].

One of the possible approaches to generalize the equation of state is an addition of a fictitious force term in the LB model [12]

$$f_{-1}(t + \Delta t, x - c \Delta t) - f_{-1}(t, x) = \frac{1}{\tau + \frac{k}{2}} \left( f_{-1}^{eq}(t, x) - f_{-1}(t, x) \right) + F_{-1},$$

$$f_{0}(t + \Delta t, x) - f_{0}(t, x) = \frac{1}{\tau + \frac{k}{2}} \left( f_{0}^{eq}(t, x) - f_{0}(t, x) \right) + F_{0},$$

$$f_{+1}(t + \Delta t, x + c \Delta t) - f_{+1}(t, x) = \frac{1}{\tau + \frac{k}{2}} \left( f_{+1}^{eq}(t, x) - f_{+1}(t, x) \right) + F_{+1},$$

where

$$F_{0} = 0, \quad F_{\pm 1} = \pm 2 c \frac{c}{6} \frac{\partial}{\partial x} \Phi, \quad \Phi \equiv \sqrt{A c_{\text{pulse}}^2 - P(A)}$$

and $P(A)$ is the desired area-pressure relation. The stability conditions for the presented method [13] are the follows

$$\left( \frac{\Delta x}{\Delta t} \right)^2 \geq \max \left\{ \frac{3 P(A)}{A}, 1 + \frac{\partial P(A)}{\partial A} \right\}.$$  

### 3. Test problems: nonlinear forward sole wave propagation, pulse wave propagation in a stented vessel

Now we consider two examples.

The first problem is nonlinear sole forward pulse wave propagation [15]. We again consider the equations (1)-(2) in the semi-infinite interval $x \geq 0$. We apply the following area-pressure relation

$$p = p_0 + \frac{1}{n D_0} (A^n - A_0^n),$$

where $D_0$ is some constant.

For a forward traveling wave one has the relation between the wave velocity $u$ and the luminal area $A$ [15]

$$u = \int_{A_0}^{A} \frac{dz}{\sqrt{\rho D_0 / z^n z}},$$

then the equations (1)-(2) reduce to a differential equation for $A$

$$\frac{\partial A}{\partial t} + \frac{1}{\sqrt{\rho D_0}} \left( \left( 1 + \frac{2}{n} \right) A^{n/2} - \frac{2}{n} A_0^{n/2} \right) \frac{\partial A}{\partial x} = 0.$$  

We supplement the equation (7) with the symmetric triangle-shaped initial-boundary condition at $x = 0$ for $t \in [0, T_0]$, where $T_0$ is one heartbeat

$$A(t, x)|_{x=0} = A_0 + at, \quad t \in [0, t_0),$$

$$A(t, x)|_{x=0} = A_0 + a(2t_0 - t), \quad t \in [t_0, 2t_0].$$
Figure 1. Two pressure profiles at $t = 0.35$. Initial pulse wave was symmetrical at $t = 0$. Both methods – Lattice Boltzmann (boxes) and analytical solution (circles) show very closed results.

$$A(t,x)\big|_{x=0} = A_0, \quad t \in (2t_0, T_0]$$

(10)

and

$$A(t,x)\big|_{t=0} = A_0, \quad x > 0.$$  

(11)

We consider the case $n = 1/2$ in (5) which corresponds to the Laplace law. The analytical solutions to this problem are obtained in [15]. For the sake of brevity, we do not give them here but they will be used for benchmarking. We solve $D1Q3$ model with external force, applying the initial-boundary conditions (8)-(11) and also using (6) at the left boundary; then compare the results with the analytical solutions at different moments of time. The results show very good agreement.

The second problem is the wave propagation in a vessel with varying material properties. We consider a vessel of a length $1m$. We assume that the values of the linearized pressure pulse wave velocity which is given by the formula $\sqrt{A_0^n/\rho D_0}$, where $A_0$ is undisturbed vessel area are different in the considered region. We assume that in the interval $x \in [0.2m, 0.3m]$ the linearized pulse velocity equals $10m/s$, while in the rest part of the tube the linearized pulse velocity has a value of $4m/s$ (typical physiological value). The increase of the pulse wave velocity is observed due to the smaller values of the distensibility $D_0$ in the interior region in comparison with the outer part of the vessel. This can happen if the internal region is stented and has increased stiffness.

We consider the dynamics of a half-sinusoidal pressure pulse wave starting at the point $x = 0$. Assume that at the points $x = 0.15m$ (0.05m before the stiff region) and at $x = 0.35m$ (0.05m after) pressure recording devices are placed. The amplitude of the pressure pulse wave at $x = 0.15$ is greater than the amplitude at $x = 0.35m$ (Fig. 2). This feature is an effect of the wave reflection at the boundary where the vessel changes its elastic properties. This fact is well known in the fluid dynamics [16]. Moreover, pressure oscillations are observed which can
Figure 2. Two relative pressure profiles ($P_s/P_d$ where $P_s$ systolic pressure, $P_d$ is the diastolic pressure) at the two spatial points: before the stented region ($x = 0.15m$, large pressure amplitudes) and after stented region ($x = 0.35m$). The wave with larger amplitude is related to the point $x = 0.15m$. The increase in amplitude is a result of the reflected wave. The further oscillations appear due to the multiple wave reflections in the stiff region.

be explained by multiple wave reflections in the interior stiff region. This result is in a good agreement with the similar simulations presented in [2].

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