The SWITCH test for discriminating quantum evolutions

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Abstract
We study different quantum circuits that can discriminate between two arbitrary quantum evolution operators. These circuits can be used to check whether two quantum operators are equal or not and to estimate a fidelity measure telling how close the operators are. This operator comparison is related to the SWAP test for discriminating two quantum states. In terms of their practical realization, we comment possible laboratory implementations with light along the same lines of recent experimental realizations of quantum superpositions of causal orders exploiting the different degrees of freedom of photons. We also discuss hardware efficient realizations for noisy intermediate scale quantum computers. Finally, we comment potential applications to the discrimination of quantum communication channels and to the search for simpler quantum circuits in quantum compilers.

Keywords: unitary discrimination, channel discrimination, SWAP test, NISQ computing

1. Introduction: state comparison and the SWAP test

Distinguishing between two objects is a most fundamental task in both quantum and classical information theory. For instance, discriminating two quantum channels is a key problem of quantum information [1–14]. In this work, we describe a test for comparing two quantum systems and different ways to carry out that comparison.

Quantum information processing tasks are conventionally described as quantum circuits. Using quantum gates as building blocks, the evolution from an initial state is represented by a circuit. Quantum superpositions of states provide the intrinsic parallelism absent in computations performed using classical means. Nevertheless, quantum theory also permits

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the superposition of quantum operations [15]. One implication of this property of quantum systems is that it allows for a relaxation of the notion of a predefined causal order. Such a dynamic causal structure can be fundamental, for instance, in the physical description of quantum gravity [16]. Recently, this view of a quantum theory without a definite causal structure has opened a new way to study quantum computation. The simplest example of this new form of quantum computation is a quantum switch where a qubit controls the causal order of a quantum circuit composed of two cascaded systems [17–19].

The superposition of evolutions can be simulated in a conventional quantum circuit by expanding the Hilbert space dimension with an ancillary control qubit. There exists a recent optical implementations of such a system, where the superposition of gate orders is created using additional degrees of freedom of the photon [20–22].

The SWITCH test described in this work also relies on this type of simulation of the superposition of quantum evolutions and can be thought of as being based on a simplified version of a quantum switch [23, 24]. Even though it does not imply any modification of the causal order, it bears some resemblance with those recent proposals that explore the causal structure of quantum physics.

These quantum switches can provide advanced communication channels with enhanced capacity and improved behavior against noise [25–31].

Our suggestions for various SWITCH test circuits and implementations are closely related to the SWAP test. Originally proposed for quantum fingerprinting applications [32], the SWAP test permits one to verify whether two quantum states are equal or not. It has also been shown to be equivalent to the Hong–Ou–Mandel effect [33, 34]. This test is based on a quantum controlled SWAP gate. Depending on the state of the control qubit, |0⟩ or |1⟩, it respectively leaves unchanged the input states |ϕ⟩|ψ⟩ or swaps them producing the output |ψ⟩|ϕ⟩.

The quantum circuit implementing the SWAP test is shown in figure 1. The aim is to test whether quantum states |ψ⟩ and |ϕ⟩ are equal or not. Before the measurement of the ancillary qubit, the resulting quantum state is given by

$$\frac{1}{2} \left[ |0⟩ (|ϕ⟩|ψ⟩ + |ψ⟩|ϕ⟩) + |1⟩ (|ϕ⟩|ψ⟩ − |ψ⟩|ϕ⟩) \right]. \quad (1)$$

The SWAP test is passed if the measurement of the ancillary qubit gives 0 and fails otherwise. If the two states are equal, |ψ⟩ = |ϕ⟩, the test is always passed. When the states are different, there is a finite probability P of still passing the test that depends on the overlap of the two states |⟨ϕ|ψ⟩|^2

$$P = \frac{1}{2} \left( 1 + |⟨ϕ|ψ⟩|^2 \right). \quad (2)$$

Even for very similar states with a large overlap, the probability of two states that are different passing the test becomes exponentially small in the number of repetitions of the test and the equality can be verified with any required confidence if a sufficiently large number of copies
of the states are available. These results can be easily extended to mixed states. For two mixed states with density matrices $\rho$ and $\sigma$, the probability of passing the SWAP test is $\frac{1 + tr(\rho \sigma)}{2}$ [35].

2. Comparison of unitary evolutions: the SWITCH test

Taking the SWAP test as our starting point, we can derive some tests to compare quantum operators. Figure 2 shows a basic SWITCH test that helps us to understand the possibilities and limitations of our approach. This is a modified version of the Hadamard test used in quantum algorithms for the calculation of Jones polynomials [36, 37] or in different quantum machine learning tasks [38, 39].

Here, and in the rest of the paper, the quantum circuits with a single line indicate a qubit (a quantum system with two levels). General states, like $|\psi\rangle$ or $|\phi\rangle$, are assumed to be qudits (quantum systems with $d$ levels) unless there is a number of qubits specified. In that case, we will use $n$ to count the number of qubits that make up the total system of dimension $d = 2^n$. In the circuits, lines with multiple qubits appear as a crossed line.

In the circuit of figure 2, for an arbitrary input state $|\phi\rangle$ and $U_1$ and $U_2$ the unitary evolutions being tested, we define the output of the controlled SWITCH $\{U_1, U_2\}$ gate as $U_1|\phi\rangle$ if the control qubit is $|0\rangle$ and $U_2|\phi\rangle$ if it is $|1\rangle$. For the input superposition in the control qubit we get a superposition of the two evolutions $U_1$ and $U_2$.

Before the measurement of the ancillary qubit, the resulting quantum state is given by

$$|\varphi\rangle = \frac{1}{2} \left( |0\rangle (U_1 |\phi\rangle + U_2 |\phi\rangle) + |1\rangle (U_1 |\phi\rangle - U_2 |\phi\rangle) \right).$$

(3)

A circuit diagram of a possible implementation, when the two operators $U_1$ and $U_2$ can be made subject to a quantum control, is shown in figure 3. A simple deployment of this system for an optical system could follow the same lines demonstrated in [20] with the control qubit realized spatially and the quantum operations acting on the polarization degree of freedom of the same photon. The system can be implemented using a Mach–Zehnder interferometer as shown in figure 4. This kind of interferometric comparison setup was already proposed in [40] as a way to tell apart two unknown evolutions for two-dimensional systems. In the test, the photon is sent through a 50/50 beam splitter that creates the spatial qubit implementing the first Hadamard gate in the control channel in figure 3. State $|0\rangle$ is obtained if it is reflected at the beamsplitter and $|1\rangle$ if it is transmitted. The two spatial paths are associated with two different operations on the photon polarization $U_1$ and $U_2$. At the output, the two paths recombine coherently at a second 50/50 beamsplitter that implements the second Hadamard gate in figure 3. The measurement of the photon and the determination of the output path completes the SWITCH test for the quantum operations $U_1$ and $U_2$.

This system can compare qubits encoded in the polarization degree of freedom, but we can have access to higher-dimensional systems if we take advantage of other degrees of freedom of the photon, like its orbital angular momentum. As long as all the degrees of freedom are encoded into a single photon, the setup can compare the alternative evolutions, in this case in a larger Hilbert space.

The SWITCH test is passed if the measurement of the ancillary qubit gives 0 and fails otherwise. If the two systems are equal, $U_1 = U_2$, the measurement result is necessarily 0 and the test is always passed. When the systems are different, there is a finite probability $P$ of still passing the test that is dependent on the input state $|\phi\rangle$. The probability of passing the test can be written in terms of the trace of the product of the measurement projector and
the density matrix representation of the state \( |\varphi\rangle \) in equation (3), \( \text{tr}(\langle 0| \langle \varphi| (\varphi) \rangle) \), which gives

\[
P = \frac{1}{2} \left( 1 + \text{Re} \{ \text{tr}(U_2 \rho U_1^\dagger) \} \right)
\]

or

\[
P = \frac{1}{2} \left( 1 + \text{Re} \{ \langle \varphi| U_2^\dagger U_2 |\varphi\rangle \} \right)
\]

for pure states.

A failure of the test, under ideal conditions, confirms that the two systems \( U_1 \) and \( U_2 \) are certainly distinct. Passing the test gives some probability that the systems are identical. This probability can be optimized by the repetition of the test a sufficient number of times, but it requires a careful strategy.

There are some clear limitations in the SWITCH test we have seen so far. For instance, it can happen that \( U_1 |\varphi\rangle = U_2 |\varphi\rangle \) and a repeated use of the state \( |\varphi\rangle \) would always pass the test, even though the operators are different. A clear example for two qubit states are the CNOT operation, \( \text{CNOT}|x\rangle|y\rangle = |x\rangle|x+y \mod 2\rangle \), and the identity \( (U_1 = \text{CNOT}, U_2 = I) \). Both unitaries
would leave a state $|\phi\rangle = |00\rangle$ unchanged. Therefore, as opposed to the SWAP test, a repeated evaluation of the SWITCH test with the same input state is not a judicious option.

We need an appropriate collection of states $\{\{\phi_i\}\}$ to be used in multiple test repetitions. For instance, we could use a basis spanning the Hilbert space on which the $U_j$ gates act. If the operators are different, there will be at least one input state of the basis for which the output states are different and there is a probability greater than 0 of failing the test. This is too costly.

A good alternative is sampling random input states. If we choose uniformly at random a unitary $U_r$ (sampling from the Haar measure), we can apply it on a fixed initial state, for instance setting all the qubits to $|0\rangle$. The result is a random initial state.

For a large enough number of repetitions of the SWITCH test, different unitaries $U_1$ and $U_2$ will eventually fail the test. We can show (see section ‘$E[\text{tr}(U_1^\dagger V_0^\dagger V U_2)]_{\text{Haar}}$’ in the appendix) that the average probability of passing the test with random input states becomes

$$E[P]_{\text{Haar}} = \frac{1 + \Re \{\langle U_1, U_2 \rangle \}}{2}$$

(6)

where $\langle U_1, U_2 \rangle = \text{tr}(U_1^\dagger U_2)$ is the Frobenius inner product of the two evolutions and $d$ the dimension of the Hilbert space of the input state on which the operators act. This average gives a good estimation of how close the two operators are.

For $U_1 = U_2$, the test is always passed. For different evolutions, the average probability of passing the test will, in general, be close to 1/2 for large Hilbert spaces unless the operators are particularly close to each other.

The random Haar unitaries we need can be generated from classical random distributions [41] and then converted into quantum circuits, but that can require a large number of elementary quantum gates. In the noisy intermediate scale quantum computers that are available today [42], we prefer quantum circuits of a small depth, i.e. the total number of consecutive elementary gates should be as small as possible. In longer circuits the noise at each gate can compound to the point of making the resulting output states unusable.

There are economical approximations to random Haar matrices. For all the functions we will need to average in this paper, we can use two-designs to obtain the same averages as the full Haar measure [43, 44] using only a finite set of gates to generate the random states. We can further simplify the testing by using approximate designs [45, 46], which are enough for our purposes.

The number of states we need to generate to obtain an exact average can grow exponentially with the size of the state space with direct methods like full process tomography or generating all the states in a two-design. Usually, we will take a Monte Carlo approach assuming that, for a large enough number of properly sampled random states, the finite average approximates the Haar average for a large enough number of samples. If we use an approximate method, we can approximate the Haar average with a reasonable number of states. For instance, the approximate two-designs in [46] can be used to approximate the Haar averages of fidelity functions like the ones we are interested in with an error $\epsilon > 0$ after $O(\log 1/\epsilon)$ evaluations. For a noisy quantum computer this is probably enough. The noise will likely be more important than any improvement in the measurement precision. There are also explicit compact constructions to produce pseudorandom quantum states with few gates [47] which can give an efficient Monte Carlo SWITCH test for our evolutions $U_1$ and $U_2$.

For a less formal approach, we can try any of the parametrized circuits used in quantum machine learning to explore the state space. These circuits generate a trial state, or ansatz $|\psi\rangle_{\tilde{\theta}}$ that depends on a short list of classical parameters $\tilde{\theta} = (\theta_1, \ldots, \theta_k)$ [38]. A good ansatz circuit needs to balance the ability to cover as much of the Hilbert space as possible while keeping a low number of stages so that the noise cannot build up to a dangerous level. For the
proposed use for pseudorandom state generation, reasonable \textit{ansätze} should be good options to provide random sampling. However, like in most machine learning approaches, this would be a heuristic method and the estimates should be taken as an orientation to the actual values. If we only need to have a qualitative idea of whether the evolutions are similar or not this might be enough.

These sampling strategies can be taken to the interferometric setups we have discussed for optical channels. There are efficient optical systems that generate random Haar evolutions\cite{48} as well as configurable integrated optical circuits that can be adjusted on demand\cite{49–52}.

Apart from the input state problem, the test is dependent on a phase reference. For instance, testing \( U \) against \( e^{i\alpha}U \) for \( \alpha = \frac{\pi}{2} \) would always give a negative result even though the two operators would produce states that are equal up to a global phase and, as such, indistinguishable. However, using controlled gates introduces a phase reference. The control qubit allows one to distinguish relative phases that would be unmeasurable if we only had the systems under test. This is, in fact, the expected behavior for interferometers, which are used to find phase differences that serve as indirect measurements of other magnitudes, such as changes in length. Our system is equivalent to an interferometer and must show a phase dependence, which is associated to the real part in equation (5).

The dependence on the phase can be somewhat countered if the test is repeated for \( U_1, U_2 \) and a phase-shifted version of \( U_1, iU_1, \) and \( U_2 \). The second test permits one to estimate \( \text{Im}\{ \langle \phi | U_1^* U_2 | \phi \rangle \} \). If we combine both tests, we can reconstruct an estimate of \( |\langle \phi | U_1^* U_2 | \phi \rangle|^2 \). Of course, we would need to repeat the measurement for a suitable set of test states \( \{|u_i\rangle\} \) and we would lose the desirable property that a negative tells us for sure the systems are different when there appear phase shifts.

In order to remedy these shortcomings, we can modify the SWAP test as shown in figure 5. In the repeated operation of the circuit, the two inputs are set to equal values. For an input density matrix \( \rho \) that is a linear combination of matrices of the form \( |\phi\rangle \langle \phi| \), the SWAP test succeeds with a probability

\[
P = \frac{1 + \text{tr} \left( U_2 \rho U_1^* U_1 \rho U_1^* \right)}{2}
\]

or

\[
P = \frac{1}{2} \left( 1 + |\langle \phi | U_1^* U_2 | \phi \rangle|^2 \right).
\]

for pure state inputs of the form \( |\phi\rangle \langle \phi| \).
We can compute again the average probability of passing this test for random Haar input states where we generate two copies of the same state for each of the two inputs of the circuit. The result (see section ‘E[⟨0| V† U1 V|0⟩ ⟨0| V† U2 V|0⟩]_{\text{Haar}}’ in the appendix) is:

\[
E[P]_{\text{Haar}} = \frac{d + |\text{tr}(U_1^\dagger U_2)|^2}{d(d + 1)}.
\]  

This probability is proportional to the squared absolute value of the Frobenius inner product of the operators and gives an idea of how one unitary ‘projects’ into the other. As required, when \( U_1 = U_2 \), the test is passed for any input state.

3. Equivalent circuits and possible implementations

There are different configurations that give equivalent results for operator comparison. For instance, the alternative SWITCH test circuit of figure 6 is equivalent to the modified SWAP test. Even though the general evolution is different from that of the circuit in figure 5, the outputs are identical when input states \( |\psi\rangle = |\psi\rangle \) are used.

The cost of this approach is doubling all the qubits except for the control qubit. However, this modification allows for a reduction in the number of gates in the test. Usually a controlled unitary requires multiple two qubit gates and there are different decomposition strategies. This is a general problem in the Hadamard test of figure 2 and derived measurements and there have been various attempts to reduce the depth of these circuits [53–56]. In the SWAP test this problem can be circumvented. It is possible to perform a SWAP test without a controlled SWAP. The circuit in figure 7 shows the destructive SWAP test presented in [34] and later rediscovered by a machine learning algorithm [57]. The additional cost of this test is performing some simple classical computations on the measurement results. The test reduces to performing a Bell measurement on each corresponding pair of qubits of the states to be compared and then computing the parity of the bitwise AND of the obtained binary sequences (assigning the 0 bit to a \( |0\rangle \) result and 1 to \( |1\rangle \)). The whole cost, apart from the unavoidable gates for \( U_1 \) and \( U_2 \), is adding \( n \) Hadamard gates and \( n \) CNOT gates. Classically, we just need simple AND operations and counting the number of 1s in the resulting binary sequence (a series of modulo 2 additions). In noisy implementations, this depth reduction can compensate for the additional number of qubits.

Finally, we would like to show some possible optical implementations of these two-state SWITCH tests. We assume the probe states are qubits encoded into single photons using their different degrees of freedom such as polarization, frequency or orbital angular momentum. Figure 8 shows an optical setup implementing the circuit of figure 5.

The comparison comes from the quantum interference of single photons at a balanced beamsplitter in the Hong–Ou–Mandel experiment [58] which, for input states with density matrices \( \rho \) and \( \sigma \), gives a probability \( P = \frac{1 - \text{tr}(\rho \sigma)}{2} \) of finding a coincidence. Two photons in the same state will bunch and always come out together, which gives a natural way to implement a SWAP test [34]. If we can produce any state on demand and have two operations we want to compare, the presented optical setup implements a SWITCH test.
Figure 6. Quantum circuit implementation of a two-state SWITCH gate test.

Figure 7. Quantum circuit implementation of a destructive two-state SWITCH test. The circuit complexity besides the unitaries $U_1$ and $U_2$ can be reduced to a measurement circuit with two stages and light classical processing.

Figure 8. Optical implementation of a two-state SWITCH test with Hong–Ou–Mandel interference. The test fails if we find a coincidence (both detectors measure a photon at the same time).
4. Applications

The main application of the circuits and procedures described is the comparison of quantum channels [1–14], particularly for optical quantum communication channels.

Apart from channel analysis, comparing quantum evolutions can help in the design of better quantum hardware. In current quantum computers the qubits and gates are noisy and suffer from the effects of decoherence. The SWITCH test can be used to compare implementations of the same gate or gate sequence in order to identify which configurations offer a better performance (as long as there is one good reference). Apart from benchmarking, the SWITCH test can be used in the design of efficient quantum circuits. The first direct application is in quantum compilers [59]. In most of the current architectures of intermediate scale noisy quantum computers, not all the interactions are available. For instance, not every pair of qubits is connected with two qubit gates. These computers need efficient compilers: tools that translate the idealized, high-level, quantum circuits to the computer-specific gates that can be actually implemented in the computer at hand.

While there are general compiling methods, like the decomposition given by the Solovay–Kitaev theorem [60], there is a constant need for efficient compilers. One alternative is using machine learning to identify which circuits producing the same output have less gates or are best suited for a particular machine [61]. The SWITCH test circuits offer a perfect application for an experimental search where, instead of simulating the gates, the objective function to be maximized comes from measurements. The function subject to optimization would be the estimated distance between a desired unitary $U_1$ with a known, but maybe too cumbersome, gate sequence and a parametrized quantum circuit that can be described with a few classical parameters. While this does not seem efficient for real-time compilers, it can be a good way to optimize common circuits, like adders, or the quantum Fourier transform, and give compact circuit alternatives to useful tasks, similar to the machine learning rediscovery of the destructive SWAP test in [57].

5. Discussion

We have shown how different SWITCH tests can estimate how close two quantum evolution operators are. The one-state test has several limitations, but can be readily realized in interferometric experiments with one photon. The two-state tests are related to the SWAP test used to compare quantum states. We have given two alternative quantum circuits implementing this test, one of which can be implemented in a quantum optics lab using a beamsplitter in a Hong–Ou–Mandel setup. This expands previous optical setups that can compare quantum processes using entanglement [62]. We have also commented different sampling strategies to obtain meaningful averages. The required random state generation can be performed with simple quantum circuits which could be built with the current noisy intermediate scale quantum computers. In optical setups, there are also known constructions that allow one to prepare the needed input states. Combined with the recent advances in integrated photonics, this offers a compact way to compare to two optical quantum communication channels and to study the hardware of physical realizations of quantum computers.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).
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Appendix

In our tests, we want to consider the average of functions \( f(V) : V \rightarrow \mathbb{C} \) acting on unitary matrices \( V \) chosen at random with respect to the Haar measure. This average is given by the integral

\[
E[f(V)]_{\text{Haar}} = \int_{V \in \mu_V} f(V) d\mu_V,
\]

where \( \mu_V \) is the Haar measure for the unitary group we are interested in.

There are two common computation tricks that will be useful. In both cases we consider two Hilbert spaces of dimension \( d \), \( \mathcal{H}_1^d \) and \( \mathcal{H}_2^d \), and the joint product space \( \mathcal{H} = \mathcal{H}_1^d \times \mathcal{H}_2^d \).

First, we notice that, for the trace operation and two unitary matrices \( A, B \) acting on states in \( \mathcal{H} \),

\[
\text{tr} (AB) = \text{tr} (\text{SWAP}(A \otimes B)),
\]

where the SWAP operation acts on the halves corresponding to each individual space, \( \mathcal{H}_1^d \) and \( \mathcal{H}_2^d \), and \( \otimes \) is the usual tensor product.

This SWAP trick [63] can be easily verified by writing the tensor product as a sum of the weights for each matrix position so that

\[
\text{tr} (\text{SWAP}(A \otimes B)) = \text{tr} \left( \text{SWAP} \sum_{ijkl} A_{ij} B_{kl} |i⟩⟨j| \otimes |k⟩⟨l| \right)
= \sum_{ijkl} A_{ij} B_{kl} \text{tr} (|k⟩⟨j| \otimes |i⟩⟨l|)
= \sum_{ijkl} A_{ij} B_{kl} \delta_{kl} \delta_{il}
= \sum_{ik} A_{ik} B_{ki} = \text{tr} (AB).
\]

We will also consider the Haar average for the tensor product of a random unitary matrix and its inverse (see corollary 3.5 in [64]):

\[
\int_{V \in \mu_V} V \otimes V^\dagger d\mu_V = \frac{\text{SWAP}}{d^2},
\]

where SWAP is the \( 2d \times 2d \) SWAP operation acting on each half of the larger space.

\[
E[\text{tr} (U_1^\dagger V^\dagger \rho_0 V U_2)]_{\text{Haar}}
\]

We will compare the unitary matrices \( U_1 \) and \( U_2 \) using random initial states generated from the fixed state \( |0⟩ \) with density matrix \( \rho_0 = |0⟩⟨0| \) under a random unitary \( V \).
First, we use the SWAP trick of equation (11) to notice that
\[
\text{tr} \left( U_2 V \rho_0 V^\dagger U_1^\dagger \right) = \text{tr} \left( \text{SWAP} \left( \rho_0 \otimes U_1^\dagger U_2 \right) \right).
\]
(14)

The average for random input states can be calculated using equation (13). We find that:
\[
E_{\text{Haar}} \left[ \text{tr} \left( U_2 V \rho_0 V^\dagger U_1^\dagger \right) \right] = \frac{1}{d} \text{tr} \left( \rho_0 \text{tr} \left( U_1^\dagger U_2 \right) \right)
\]
for our pure state input \( \rho_0 \) of trace 1.

\[
E_{\text{Haar}} \left[ \langle 0 | V^\dagger U_1^\dagger U_2 V | 0 \rangle \langle 0 | V^\dagger U_2^\dagger U_1 V | 0 \rangle \right] = \frac{1}{d} \text{tr} \left( U_1^\dagger U_2 \right)^2,
\]
(15)

for unitaries \( U_1 \) and \( U_2 \):
\[
E_{\text{Haar}} \left[ \langle 0 | V^\dagger U_1^\dagger U_2 V | 0 \rangle \langle 0 | V^\dagger U_2^\dagger U_1 V | 0 \rangle \right] = \frac{d}{d+1} \text{tr} \left( U_1^\dagger U_2 \right)^2.
\]
(17)

For \( U_1 = U_2 \), \( \text{tr} \left( U_1^\dagger U_2 \right)^2 = d^2 \) and the test succeeds with probability one.

While we have considered unitaries, this result uses equation (16), which is valid for completely general quantum transformations, showing the SWITCH test can be used to compare any arbitrary pair of quantum channels.

The result from equation (17) can also be deduced using proposition 3.9 of [64] and using the SWAP and the integration tricks in a similar fashion to what was done in equation (15).

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