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Essential curves in handlebodies and topological contractions

V. GRINES AND F. LAUDENBACH

Abstract. If $X$ is a compact set, a topological contraction is a self-embedding $f$ such that the intersection of the successive images $f^k(X)$, $k > 0$, consists of one point. In dimension 3, we prove that there are smooth topological contractions of the handlebodies of genus $\geq 2$ whose image is essential. Our proof is based on an easy criterion for a simple curve to be essential in a handlebody.

1. Introduction

For a compact set $X$ and a topological embedding $f : X \to X$, we shall say that $f$ is a topological contraction if $\cap_{k \geq 0} f^k(X)$ consists of one point. We shall show that such a contraction can be very complicated when $X$ is a 3-dimensional handlebody. Namely, we have the following result for which some more classical definitions will be recalled thereafter.

Theorem A. There exists a North-South diffeomorphism $f$ of the 3-sphere $S^3$ and a Heegaard decomposition $S^3 = P_- \cup P_+$ of genus $g \geq 2$ with the following properties:
1) $f|P_+$ is a topological contraction;
2) $f(P_+)$ is essential in $P_+$.

We shall limit ourselves to $g = 2$, since the generalization will be clear. We recall that a 3-dimensional handlebody of genus 2 is diffeomorphic to the regular neighborhood $P$ in $\mathbb{R}^3$ of the planar figure eight $\Gamma$. A compression disk of $P$ is a smooth embedded disk in $P$ whose boundary lies in $\partial P$ in which it is not homotopic to a point. Among the compression disks are the meridian disks $\pi^{-1}(x)$, where $x$ is a regular point$^1$ in $\Gamma$ and $\pi : P \to \Gamma$ is the regular neighborhood projection (that is, a submersion over the smooth part of $\Gamma$). A subset $X$ of $P_+$ is said to be essential in $P_+$ if it intersects every compression disk$^2$.

A diffeomorphism $f$ of $S^3$ is a North-South diffeomorphism if it has two fixed points only, one source $\alpha \in P_-$ and one sink $\omega \in P_+$, every other orbit going from $\alpha$ to $\omega$.

A Heegaard splitting of $S^3$ is made of an embedded surface dividing $S^3$ into two handlebodies. According to a famous theorem of F. Waldhausen such a decomposition is unique up to diffeomorphism$^3$ (hence up to isotopy after Cerf’s theorem $\pi_0(Diff_+ S^3) = 0$)$^4$. It is not

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$^1$Any point other than the center of the figure eight.

$^2$This definition goes back to Rolfsen’s book$^5$ p. 110.
hard to prove that the phenomenon mentioned in theorem A does not happen with a Heegaard splitting of genus 1: if $T$ is a solid torus and $f$ is a topological contraction of $T$, then there is a compression disk of $T$ avoiding $f(T)$.

The example which we are going to construct for proving theorem A is based on the next theorem, for which some more notation is introduced. Let $\Gamma_0 \subset \Gamma$ be a simple closed curve. There exists a solid torus $T \subset \mathbb{R}^3$ which contains $P$ and which is a tubular neighborhood of $\Gamma_0$. Let $i_0 : P \to T$ be this inclusion. We say that a simple curve is unknotted in $T$ if it bounds an embedded disk in $T$.

**Theorem B.** There exists an essential simple curve $C$ in $P$ such that $i_0(C)$ is unknotted in $T$.

Theorem B looks very easy as it is simple to draw a simple curve which intuitively satisfies its conclusion. Nevertheless, it appears that there are very few criteria for proving that a curve is essential in $P$. We are going to give one which is not algebraic in nature. Question: does there exist a topological algebraic tool which plays the same role.

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2. **Essential curves**

Our candidate for $C$ in Theorem B is pictured in figure 1.

![Figure 1](image)

It is clear that $i_0(C)$ is unknotted in $T$ (or, equivalently, in the complement of the vertical axis which is drawn on figure 1 and whose $T$ is a compact retract by isotopy deformation). Instead of proving that $C$ is essential in $P$, we are going to prove a stronger result. Clearly Proposition 1 below implies Theorem B.

$^3$Michel Boileau, Thomas Fiedler, John Guaschi and Claude Hayat.
Proposition 1. Let $p : \tilde{P} \to P$ be the universal cover of $P$ and let $\tilde{C}$ be the preimage $p^{-1}(C)$. Then $\tilde{C}$ is essential in $\tilde{P}$.

Proof. We have the following description of $\tilde{P}$: it is a 3-ball with a Cantor set $E$ removed from its bounding 2-sphere. This Cantor set is the set of ends of $\tilde{P}$. A simple curve in $\partial\tilde{P}$ is not homotopic to zero if it divides $E$ into two non-empty parts. We get a fundamental domain $F$ for the action of $\pi_1(P)$ on $\tilde{P}$ by cutting $P$ along two non-parallel meridian disks $D_0$ and $D_1$. Here is a description of $\tilde{C} \cap F$ (see figure 2): $F$ is a 3-ball whose boundary consists of four disks $d_0, d'_0, d_1, d'_1$ and a punctured sphere $\partial_0 F$. We have $p(d_0) = p(d'_0) = D_0$ and $p(d_1) = p(d'_1) = D_1$. We have four strands in $\tilde{C} \cap F$: $\ell_1$ and $\ell_2$ joining $d_0$ and $d_1$, $\ell'_0$ (resp. $\ell'_1$) whose end points belong to $d'_0$ (resp. $d'_1$). Moreover $\ell'_i, i = 0, 1$, is linked with $\ell_j, j = 1, 2$, in the following sense: any embedded surface whose boundary is made of $\ell'_i$ and a simple arc in $d'_i$ intersects $\ell_j$ for $j = 1, 2$ (the algebraic intersection number is 1 for some choice of orientations).

Globally $\tilde{C}$ looks like an infinite Borromean chain: any finite number of components is unlinked. Suppose the contrary that $\tilde{C}$ is not essential and consider $\Delta$, a compression disk of $\tilde{P}$ avoiding $\tilde{C}$. We take it to be transversal to $\tilde{D} := p^{-1}(D_0 \cup D_1)$. Let $C$ be the finite family of curves (arcs or closed curves) in $\tilde{D} \cap \Delta$. An element $\gamma$ of $C$ is said to be innermost if $\gamma$ divides $\Delta$ into two domains, one of them being a disk $\delta$ whose interior contains no element of $C$. Take such an innermost element $\gamma$; its associated disk $\delta$ lies in $F$, up to a covering transformation, and divides $F$ into two balls $F_0$ and $F_1$.

Lemma 1. One of the balls, say $F_0$, avoids $\tilde{C}$.

Proof. Let us consider the case when $\gamma \subset d'_0$; say that $\ell'_0 \subset F_1$. The other cases are very similar. Let $\alpha = \delta \cap d'_0$. It is a simple arc dividing $d'_0$ into two parts. Both end points of $\ell'_0$ lie in the same part since $\delta$ avoids $\ell'_0$. They are joined by a simple arc $\alpha'$ disjoint from $\alpha$. Let $\delta'$ be an embedded disk bounded by $\ell'_0 \cup \alpha'$. This disk can be chosen disjoint from $\delta$. Indeed, if

\footnote{Take the universal cover of $\Gamma$ properly embedded in the hyperbolic plane and take a 3-dimensional thickening of it.}
\(\delta \cap \delta'\) is not empty, this intersection being transversal, by looking at an innermost intersection curve on \(\delta\) one finds an embedded 2-sphere \(S\) in the complement of \(\tilde{C}\) with one hemisphere in \(\delta\) and the other in \(\delta'\). As \(S\) bounds a 3-ball \(B_F\) in \(\text{int } F\), which hence is also disjoint from \(\tilde{C}\), there is an isotopy supported in a neighborhood of \(B_F\) whose effect on \(\delta'\) decreases the number of intersection curves with \(\delta\).

Once \(\delta \cap \delta'\) is empty, we have \(\delta' \subset F_1\). But \(\ell_1\) and \(\ell_2\) must intersect \(\delta'\). Hence we have \(\ell_1 \cup \ell_2 \subset F_1\). Similarly, we have \(\ell_1' \subset F_1\).

One checks easily that there is an isotopy of \(\Delta\), supported in a neighborhood of \(F_0\), till a new compression disk having fewer intersection curves with \(\tilde{D}\) than the cardinality of \(\mathcal{C}\). Repeating this process, we push \(\Delta\) into a fundamental domain, say \(F\). In that position we have \(\partial \Delta \subset \partial_0 F\).

Again \(\Delta\) divides \(F\) into two balls and one of them, \(F_0\), avoids \(\tilde{C}\). This proves that \(\partial \Delta\) bounds a disk in \(\partial_0 F\), namely \(F_0 \cap \partial_0 F\). Hence \(\Delta\) is not a compression disk.

**Remark.** We used local linking information (namely, linking of strands in a fundamental domain of the universal covering space) which, as in this example, follows from usual linking numbers and we got a global result. This method looks very efficient. The general criterion is the following, where we use the same notation as above.

**Criterion.** Let \(C\) be any simple closed curve in \(P\). We assume that there is no embedded disk \(\delta\) in \(F\) satisfying:

1) the boundary of \(\delta\) is made of two arcs \(\alpha\) and \(\beta\), where \(\alpha\) is an arc in \(\tilde{D}\) and \(\beta\) is an arc in \(\partial \tilde{P} \cap F\);

2) \(\delta\) non trivially separates the components of \(\tilde{C} \cap F\) (both components of \(F \setminus \delta\) meet \(\tilde{C}\)).

Then \(C\) is essential in \(P\).

### 3. Proof of Theorem A

We recall the embedding \(i_0 : P \to \text{int } T\). We start with a curve \(C\) in \(P\) which meets the conclusion of Theorem B. We equip it with its 0-normal framing (a section in this framing is not linked with \(C\) in \(\mathbb{R}^3\)) and we choose an embedding \(j_0 : T \to P\) whose image is a tubular neighborhood of \(C\). Let \(B\) be a small ball in \(\text{int } T\). As \(C\) is unknotted in \(T\), there is an ambient isotopy, supported in \(\text{int } T\), deforming \(i_0\) to \(i_1 : P \to \text{int } T\) such that \(i_1 \circ j_0(T)\) is a standard small solid torus in \(B\). One half of the desired Heegaard splitting of genus 2 will be given by \(P_+ := i_1(P)\). At the present time \(f\) is only defined on \(T\) by \(f := i_1 \circ j_0 : T \to \text{int } T\). If we compose \(i_1\) with a sufficiently strong contraction of \(B\) into itself, then \(f\) is a contraction in the metric sense. Hence \(\cap_{k>0} f^k(T)\) consists of one point.

Choose a round ball \(B'\) containing \(T\) in its interior. Since \(f|T\) is isotopic to the inclusion \(T \hookrightarrow \mathbb{R}^3\), \(f\) extends as a diffeomorphism \(B' \to B\), and further as a diffeomorphism \(S^3 \to S^3\). We are free to choose \(f : S^3 \setminus B' \to S^3 \setminus B\) as we like. If we compose \(f^{-1}\) with a strong contraction of \(S^3 \setminus B'\), the intersection \(\cap_{k} f^{-k}(S^3 \setminus B')\) consists of one point. We now have a North-South diffeomorphism \(f\) of \(S^3\) which induces a topological contraction of \(T\). Since
\( f(T) \subset \text{int } P_+ \subset P_+ \subset \text{int } T, \) \( f \) also induces a topological contraction of \( P_+ \).

It remains to prove that \( f(P_+) \) is essential in \( P_+ \). We know that \( i_1(C) \) is essential in \( P_+ \). As a consequence, any compression disk \( \Delta \) of \( P_+ \) crosses \( f(T) \). We can take \( \Delta \) to be transversal to \( f(\partial T) \) such that no intersection curve is null-homotopic in \( f(\partial T) \). Let \( \gamma \) be an intersection curve which is \textit{innermost} in \( \Delta \) and let \( \delta \) be the disk that \( \gamma \) bounds in \( \Delta \).

**Lemma 2.** We have \( \delta \subset f(T) \).

**Proof.** If not, we have \( \delta \subset P_+ \setminus f(\text{int } T) \) and the simple curve \( \gamma \) in \( f(\partial T) \) is unlinked with the core \( i_1(C) \). Therefore, up to isotopy in \( f(\partial T) \), it is a section of the 0-framing. In that case, \( i_1(C) \) itself bounds an embedded disk in \( P_+ \). This is impossible, as \( i_1(C) \) is essential in \( P_+ \). \( \square \)

Therefore \( \delta \) is a compression disk of the solid torus \( f(T) \). But \( P_+ = i_1(P) \), like \( P \) itself, is essential in \( T \). Hence \( f(P_+) \) is essential in \( f(T) \) and \( \delta \) must cross \( f(P_+) \). \( \square \)

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