Fractal structure of the effective action in (quasi-) planar models with long-range interactions

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We derive the effective potential for composite fields in a class of (quasi-) planar models with long-range interactions. This class of models can be relevant for high temperature superconductors and graphite. The fractal structure of the effective potential is revealed and its physical interpretation is presented. It is argued that the multi-branched fractal structure of the potential reflects the presence of an infinite tower of excitonic bound states that occur as a result of the long-range interactions.

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For a long time, there has been a great interest in low dimensional quantum field theories. Typical examples are the 1 + 1 dimensional Gross-Neveu model and the 2 + 1 dimensional quantum electrodynamics (QED). They have been used as very instructive toy models of relativistic systems with complicated dynamics. In addition to that, it is quite remarkable that these same models mimic some intrinsic properties of the dynamics of non-Fermi liquids. In particular, numerous recent studies suggest that high-$T_c$ superconductors and graphite could be described well by a (quasi-) planar QED

The difficulties in studying strongly correlated systems such as high-$T_c$ superconductors and graphite might be related in part to the presence of long range Coulomb like interactions between quasiparticles. This limits considerably the use of many powerful techniques that used to work well in models with short-range interactions. One of such tools is the celebrated Ginzburg-Landau (GL) effective action. A method of the derivation of the GL action for composite operators in systems with short-range interactions has been known since the work of Gorkov. Its generalization to systems with a spontaneous symmetry breaking driven by long-range interactions is far from being straightforward. In relativistic field theories, a method of the derivation of the effective action in models with long-range interactions (driven by gauge fields) was elaborated in Ref. [5].

Recently, using the method of Ref. [5], the effective potential has been derived in the so-called reduced QED [6]. In such a model, while quasiparticles are confined to a 2-dimensional plane, the electromagnetic interaction between them is three dimensional in nature. The model is relevant for a class of condensed matter systems that include highly oriented pyrolytic graphite among others [6]. In this letter, we will reveal a remarkable property of the effective potential in reduced QED, not considered in Ref. [5]. It will be shown that the potential has a multi-branched fractal structure reflecting the presence of an infinite tower of excitonic bound states (a clear signature of long-range interactions). It is also important that, as was shown in Ref. [5], reduced QED is intimately connected with QED, relevant for cuprates [1,2,3]. In fact, as will be discussed below, the qualitative features of the effective potential in reduced QED are common for a wide class of models with long-range interactions, in particular, for QED.

We start from a general definition of the effective action in a theory in which the spontaneous symmetry breaking phenomenon is driven by the local composite order parameter $\langle 0|\bar{\psi}\psi|0\rangle$, where $\bar{\psi}$ is the Dirac spinor of the quasiparticle field and $\bar{\psi}$ is the (Dirac) conjugate spinor. Following the conventional way (see for example Ref. [10]), we introduce the generating functional $W(J)$ for the Green functions of the corresponding composite field through the path integral:

$$e^{iW(J)} = \int D\bar{\psi} D\psi \exp \left\{ i \int d^3x \left[ L_{qp}(x) \right. \right.$$}

where $J(x)$ is the source for composite field and $L_{qp}(x)$ is the lagrangian density of quasiparticles in the model at hand. Then, by definition, the effective action for the field $\sigma(x) \equiv -\langle 0|\bar{\psi}\psi(x)|0\rangle$ is given by the Legendre transform of the generating functional $W(J)$,

$$\Gamma(\sigma) = W(J) - \int d^3x J(x)\sigma(x),$$

where the external source $J(x)$ on the right hand side is expressed in terms of the field $\sigma(x)$ by inverting the relation

$$\frac{\delta W}{\delta J(x)} = \sigma(x).$$

The effective action $\Gamma(\sigma)$ in Eq. (2) provides a natural framework for describing the low energy dynamics in the model at hand. It is common to expand this action in...
powers of space-time derivatives of the field \( \sigma \):
\[
\Gamma(\sigma) = \int d^3x \left[ -V(\sigma) + \frac{1}{2} Z^{\mu \nu}(\sigma) \partial_\mu \sigma \partial_\nu \sigma + \ldots \right],
\] (4)
where \( V(\sigma) \) is the effective potential. The ellipsis denote higher derivative terms as well as contributions of the Nambu-Goldstone bosons if the latter are required by the Goldstone theorem. By making use the definition in Eqs. 4 and 5, we derive the following convenient representation for the effective potential:
\[
V(\sigma) = -w(J) + J\sigma = \int^\sigma d\sigma J(\sigma),
\] (6)
where \( w(J) \equiv W(J)/V_{2+1} \), and \( V_{2+1} \) is the space-time volume. Thus, the crucial point for evaluating the equation turns into an equation of motion for the composite field \( \sigma(x) \). In a particular case of constant configurations, the equation reads \( dV/d\sigma = 0 \).

By keeping the external source on the right hand side of Eq. 5 nonzero but constant in space-time, we derive the following relation:
\[
\frac{\delta \Gamma}{\delta \sigma(x)} = -J(x).
\] (5)

In the limit of a vanishing external source, this equation becomes an equation of motion for the composite field \( \sigma(x) \). In a particular case of constant configurations, the equation reads \( dV/d\sigma = 0 \).

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We consider the case when the fermion fields carry an additional, “flavor”, index \( i = 1, 2, \ldots, N_f \). Then, the symmetry of the lagrangian \( \mathbb{S} \) is \( U(2N_f) \). Adding a bare quasiparticle gap (mass) term \( \Delta \) into the lagrangian density \( \mathbb{S} \) would reduce the \( U(2N_f) \) symmetry down to the \( U(N_f) \). Therefore the dynamical generation of the gap would lead to the spontaneous breakdown of the \( U(2N_f) \) down to the \( U(N_f) \times U(N_f) \).

Proceeding as in Ref. 6, we integrate out the bulk gauge bosons from the action. Then, up to relativistic corrections of order \( (v_F/e)^2 \), we are left with the following action of interacting planar quasiparticles:
\[
S_{qp} \sim \int dt d^2 \vec{r} L_0(t, \vec{r}) - \frac{1}{2} \int dt d^2 \vec{r} \int dt' d^2 \vec{r}' \bar{\psi}(t, \vec{r}) \times \gamma^0 \psi(t, \vec{r}) U_0(t - t', |\vec{r} - \vec{r}'|) \bar{\psi}(t', \vec{r}') \gamma^0 \psi(t', \vec{r}'),
\] (9)
where the bare potential \( U_0(t, |\vec{r}|) \) takes the following simple form:
\[
U_0(t, |\vec{r}|) = \frac{e^2}{\varepsilon_0} \int \frac{d^2\vec{k}}{(2\pi)^2} \exp\left(\frac{i\vec{k} \cdot \vec{r}}{|\vec{k}|}\right) \frac{2\pi}{|\vec{k}| + \Pi(\omega, |\vec{k}|)},
\] (10)
Note, however, that the polarization effects may considerably modify this bare Coulomb potential. Thus, after taking this into account, the interaction is given by
\[
U(t, |\vec{r}|) = \frac{e^2}{\varepsilon_0} \int \frac{d\omega}{2\pi} \int \frac{d^2\vec{k}}{2\pi} \frac{\exp\left(-i\omega t + i\vec{k} \cdot \vec{r}\right)}{|\vec{k}| + \Pi(\omega, |\vec{k}|)},
\] (11)
where the polarization function \( \Pi(\omega, |\vec{k}|) \) is proportional (with a factor of \( 2\pi/\varepsilon_0 \)) to the time component of the photon polarization tensor.

The starting point in our derivation of the effective potential for the composite field \( \sigma \) is the gap equation in a model with a nonzero constant external source \( J \) [see Eq. 6]. By noticing that the external source term \( J\bar{\psi} \psi \) enters the action in exactly the same way as the quasiparticle bare mass term, we easily derive the corresponding gap equation 7,
\[
\Delta_p - J = \lambda \int \frac{dq d\Delta K(p, q)}{\sqrt{q^2 + (\Delta/\sqrt{q})}} + \frac{e^2}{2(e_0 v_F + e^2 N_f/4)},
\] (12)
where the approximate expression for the kernel \( K(p, q) \) is given by \( K(p, q) = \theta(p - q) + \theta(q - p)/q \). In the most important region of momenta \( |q| \gg \Delta/\sqrt{q} \), the pairing dynamics dominates, the only role of the term \( (\Delta/\sqrt{q})^2 \) in the denominator of the integrand on the right-hand side of Eq. 12 is to provide a cutoff in the
infrared region. Therefore, one can drop this term, instead introducing the explicit infrared cutoff $\Delta/v_F$ in the integral, where the gap $\Delta$ is defined from the condition $\Delta \equiv \Delta_\eta=\Delta/v_F$. This is the essence of the so called bifurcation approximation. As a result, we arrive at the following equation:

$$\Delta_p - J \propto \lambda \left( \int_{\Delta/v_F}^p \frac{dq}{p} \Delta_q + \int_{p}^{\Lambda} \frac{dq}{q} \Delta_q \right),$$

(13)

where $\Lambda \simeq \pi/a$ is the ultraviolet cutoff ($a$ is a lattice size). The last integral equation is equivalent to the differential equation,

$$p^2 \Delta_p'' + 2p \Delta_p' + \lambda \Delta_p = 0,$$

(14)

with appropriate boundary conditions. A non-trivial solution to the gap equation, satisfying the infrared boundary condition,

$$p^2 \Delta_p'|_{p=\Delta/v_F} = 0,$$

(15)

exists only for $\lambda \geq 1/4$, and it takes the following form:

$$\Delta_p = \frac{\Delta^{3/2}}{\sin(2\delta) \sqrt{v_F}} \sin \left( \frac{\nu}{2} \ln \frac{v_F \Lambda}{\Delta} + \delta \right),$$

(16)

where $\nu = \sqrt{4\lambda - 1}$ and $\delta = \arctan \nu$. The critical value $\lambda_c = 1/4$ corresponds to a continuous quantum phase transition.

In addition, the ultraviolet boundary condition,

$$J = (\Delta_p + p\Delta_p')|_{p=\Lambda},$$

(17)

produces the relation:

$$J = \frac{\Delta^{3/2}}{\sin(2\delta) \sqrt{v_F \Lambda}} \sin \left( \frac{\nu}{2} \ln \frac{v_F \Lambda}{\Delta} + 2\delta \right).$$

(18)

This relation determines $J$ as a function of the gap $\Delta$. If one supplies it with a similar representation for the composite field $\sigma(\Delta)$, expression (6) could be used directly to derive the effective potential,

$$V(\sigma(\Delta)) = \int d\Delta \frac{d\sigma(\Delta)}{d\Delta} J(\Delta).$$

(19)

As follows from the definition, the expression for $\sigma$ as a function of $\Delta$ is given by the trace of the quasiparticle propagator (multiplied by the factor $i$). Thus,

$$\sigma = -(\bar{\psi}\psi) = \frac{N_f}{\pi v_F} \int_0^\Lambda \frac{qdq}{\sqrt{q^2 + (\Delta_q/v_F)^2}}$$

$$= -\frac{N_f}{\pi \Lambda v_F} p^2 \Delta_p'|_{p=\Lambda}$$

$$= \frac{N_f \Delta^{3/2} \sqrt{\Lambda}}{\pi \Lambda v_F^2 \sin(2\delta)} \sin \left( \frac{\nu}{2} \ln \frac{v_F \Lambda}{\Delta} \right).$$

(20)

Note that the second line of this equation follows from Eq. (12) after differentiating its both sides with respect to momentum $p$ and substituting $p = \Lambda$.

In order to get the most convenient representation for the potential, we introduce a new parameter $\Delta_0$ which denotes the dynamical gap in the case of a vanishing external source. It is determined from Eq. (13) at $J = 0$:

$$\Delta_0 = \Lambda v_F \exp \left[ -\frac{2\pi - 4\delta}{\nu} \right].$$

(21)

Therefore the critical coupling $\lambda_c = 1/4$ corresponds to a continuous phase transition of infinite order.

By making use of the last equation, we obtain a rather convenient representation of the functions $J(\Delta)$ and $\sigma(\Delta)$:

$$J(\Delta) = -\frac{\Delta^{3/2}}{\sin(2\delta) \sqrt{v_F \Lambda}} \sin \left( \frac{\nu}{2} \ln \frac{\Delta_0}{\Delta} \right),$$

(22)

$$\sigma(\Delta) = -\frac{N_f \Delta^{3/2} \sqrt{\Lambda}}{\pi \nu \Lambda v_F^2 \sin(2\delta)} \sin \left( \frac{\nu}{2} \ln \frac{\Delta_0}{\Delta} - 2\delta \right).$$

(23)

Finally, by substituting these into Eq. (19) and performing the integration on the right hand side, we arrive at the following parametric form of the effective potential:

$$V(\sigma) = \frac{N_f}{2\pi \nu} \left[ \frac{1 - \nu^2}{2\nu^2} \left[ 1 - \cos \left( \frac{\nu}{2} \ln \frac{\Delta_0}{\Delta} \right) \right] \right.$$

$$+ \frac{1}{\nu} \sin \left( \nu \ln \frac{\Delta_0}{\Delta} \right) - \frac{1}{3} \right\} +$$

$$\left. \frac{4N_f \Delta^{3/2} \sqrt{\Lambda}}{\pi \nu (1 + \nu^2) v_F^2} \cos \left( \frac{\nu}{2} \ln \frac{\Delta_0}{\Delta} \right) \right.$$

$$\left. + \frac{1 - \nu^2}{2\nu} \sin \left( \nu \ln \frac{\Delta_0}{\Delta} \right) \right].$$

(24)

(25)

Around the global minimum, $\sigma_0 = \sigma(\Delta_0)$, the potential is approximated as

$$V(\sigma) = \frac{\pi (1 + \nu^2) v_F \sigma^2}{48 N_f \Lambda} \left( \ln \frac{\sigma}{\sigma_0} - \frac{1}{2} \right).$$

(26)

This expression may suggest that there is nothing unusual about the potential given by the parametric representation in Eqs. (24) and (25). A closer look, however, reveals a rather rich, fractal, structure of $V(\sigma)$ with infinite number of branches near the origin $\sigma = 0$ (see Fig. 11 and discussion below).

The extrema of the potential $V(\sigma)$ are determined by the equation $dV/d\sigma = J(\Delta) = 0$. By solving this equation, we obtain an infinite set of solutions for $\Delta$:

$$\Delta^{(n)}_{\min} = \Delta_0 \exp \left( -\frac{2\pi n}{\nu} \right), \quad n = 0, 1, \ldots,$$

(27)
The corresponding values of $\sigma(\Delta_{\text{min}}^{(n)})$ are determined by Eq. (29). It is easy to check that the second derivative of the potential calculated at every extremal point in Eq. (27) is positive, $d^2V/d\sigma^2 > 0$. Therefore, these extrema are local minima. The values of the potential at these minima $\sigma(\Delta_{\text{min}}^{(n)})$ are calculated to be

$$V(\Delta_{\text{min}}^{(n)}) = -\frac{N_f}{6\pi v_F} (\Delta_{\text{min}}^{(n)})^3 < 0.$$  \hspace{1cm} (28)

We see that the solution with $n = 0$ gives the largest value of the fermion gap and the lowest value of the potential. Therefore, it corresponds to the global minimum of $V(\sigma)$, or in other words, to the most stable (ground) state in the model at hand.

![Diagram](image)

FIG. 1: The schematic fractal structure of the effective potential $V(\sigma)$ that appears as the consequence of the long range interaction.

It is natural to expect that the potential also has maxima lying between those minima. However, the situation is more subtle: while, as a function of parameter $\Delta$, the potential does have maxima, it does not have them as a function of $\sigma$. Instead, the potential $V(\sigma)$ has turning points there, see Fig. 1. Indeed, the first derivative of the potential $V(\sigma)$ has turning points at the following maxima:

$$\Delta_{\text{min}}^{(n)} = \Delta_0 \exp \left( -\frac{2\pi}{\nu} - \frac{2}{\nu} \arctan \left( \frac{7 - \nu^2}{3 - 3\nu^2} \right) \right),$$  \hspace{1cm} (29)

(with $n = 0, 1, \ldots$). At the same time, the first derivative of $\sigma$-field with respect to $\Delta$ is also zero at these points. As a result, the derivative of the potential with respect to $\sigma$, i.e., $dV/d\sigma \equiv \partial V/\partial \sigma$, is nonzero there.

From our analysis, we see that the potential $V(\sigma)$, defined parametrically in Eqs. (24) and (25), is a multi-branched and multi-valued function of $\sigma$. The location of the branching (turning) points are determined by the values of the gap parameter given in Eq. (29). Each branch of the effective potential has a local minimum at $\sigma = \sigma(\Delta_{\text{min}}^{(n)})$. We also notice that, while the locations of the branching points converge to $\sigma = 0$ in the limit $n \to \infty$, the shape of all higher branches (after an overall scaling transformation) resembles the shape of the first branch. Therefore, the potential around the origin exhibits a fractal structure. This is shown schematically in Fig. 1.

We will now demonstrate that the fractal structure of the potential reflects a rich spectrum of excitonic composites in this model. It is well known (see, for example, Ref. 10) that, because of the Ward identities for currents connected with spontaneously broken symmetries, the gap equation coincides with the Bethe-Salpeter equation for corresponding gapless Nambu-Goldstone composites (gapless excitons in the present case). Therefore, the infinite number of the solutions $\Delta_{\text{min}}^{(n)}$ in Eq. (27) for the gap implies that there are gapless excitons in each of the vacua corresponding to different values of $n$. The genuine, stable, vacuum is of course that with $n = 0$. Let us show that the excitonic composites, which are gapless in the false vacuum with $n > 0$, become gapped in the genuine vacuum with $n = 0$. Indeed, the transition from a false vacuum (with $n > 0$) to the genuine one corresponds to increasing the fermion gap, $\Delta_{\text{min}}^{(n)} \to \Delta_{\text{min}}^{(0)} = \Delta_0$, without changing the dynamics. Therefore, as a result of this increase of the gap of their constituents, these excitonic composites should become gapped. Thus we conclude that the fractal structure of the potential reflects the presence of an infinite tower of gapped excitons that are radial excitations of the gapless Nambu-Goldstone composites. It is clear that this effect is intimately connected with the long-range interactions in this model.

A confirmation of this point comes from studying the effective potential at finite temperature $T$ and/or chemical potential $\mu$. Because of the Debye screening, both $T$ and $\mu$ lead to a dynamical infrared cutoff for interactions. Our studies show that the (dis-)appearance of the higher order branches is very sensitive to the values of temperature and the chemical potential. In particular, all branches with $n > n_0$ disappear at a temperature in the region $\frac{1}{2}\Delta_{\text{min}}^{(n_0+1)} \lesssim T \lesssim \frac{1}{2}\Delta_{\text{min}}^{(n_0)}$, or at a chemical potential in the region $\sqrt{2}\Delta_{\text{min}}^{(n_0+1)} < \mu \leq \sqrt{2}\Delta_{\text{min}}^{(n_0)}$. This observation reflects the process of “melting” of those bound states whose binding energy is of order $T$ (or $\mu$) or less.

It is also clear that this picture should be quite general and valid for a wide class of models with long-range interactions. In particular, we have found that a similar fractal structure takes place in the effective action of QED$_{2+1}$ relevant for cuprates.

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