Gale Duality and Free Resolutions of Ideals of Points

by

David Eisenbud and Sorin Popescu *

What is the shape of the free resolution of the ideal of a general set of points in \( \mathbb{P}^r \)? This question is central to the programme of connecting the geometry of point sets in projective space with the structure of the free resolutions of their ideals. There is a lower bound for the resolution computable from the (known) Hilbert function, and it seemed natural to conjecture that this lower bound would be achieved. This is the "Minimal Resolution Conjecture" (Lorenzini [1987], [1993]). Although the conjecture has been shown to hold in many cases, three examples discovered computationally by Frank-Olaf Schreyer in 1993 show that it fails in general. In this paper we shall describe a novel structure inside the free resolution of a set of points which accounts for the failure and provides a counterexample in \( \mathbb{P}^r \) for every \( r \geq 6, r \neq 9 \).

We begin by reviewing the conjecture and its status. Consider a set of \( \gamma \) points in the projective \( r \)-space over a field \( k \), say \( \Gamma \subset \mathbb{P}^r_k \). Let \( S = k[x_0, \ldots, x_r] \), let \( I_{\Gamma} \) be the homogeneous ideal of \( \Gamma \), let \( S_{\Gamma} \) denote the homogeneous coordinate ring of \( \Gamma \). Let

\[
F_\bullet : 0 \rightarrow F_{r-1} \rightarrow \ldots \rightarrow F_0 \rightarrow I_{\Gamma} \rightarrow 0
\]

be the minimal free resolution of \( I_{\Gamma} \), and define the associated (graded) betti numbers \( \beta_{ij} \) by the formula

\[
F_i = \bigoplus_j S(-j)^{\beta_{ij}}.
\]

The minimal free resolution conjecture can be formulated as follows:

**Minimal Resolution Conjecture.** If \( \Gamma \) is a general set of points in \( \mathbb{P}^r_k \) over an infinite field \( k \), then for any integers \( i, j \), at most one of \( \beta_{ij} \) and \( \beta_{i+1,j} \) is nonzero.

Given our knowledge of the Hilbert function of the general set of points (since \( \Gamma \) imposes independent conditions on forms of every degree) and the easy result that if \( I_{\Gamma} \) contains forms of degree \( d \), then \( \beta_{ij} = 0 \) for \( j > i + d \) that is, \( I_{\Gamma} \) is \((d+1)\)-regular, the minimal free resolution conjecture can be translated into an explicit formula for the \( \beta_{ij} \) (see §5 below).

The minimal resolution conjecture is known to be true in \( \mathbb{P}^2 \) (Gaeta [1951] and [1995], Geramita-Lorenzini [1989]), in \( \mathbb{P}^3 \) (Ballico-Geramita [1986]), in \( \mathbb{P}^4 \) (Walter [1995], Lauze [1996]), and in \( \mathbb{P}^n \) for \( n + 1 \leq \gamma \leq n + 4 \), or \( \gamma = \binom{n+2}{2} - n \) (Geramita-Lorenzini [1989], Cavaliere-Rossi-Valla [1991], Lorenzini [1993]). Its predictions about \( \beta_{r-1,j} \) are known to

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be true in general (Lauze [1995]). Most striking, the conjecture is known to hold whenever the number of points in \( \Gamma \) is sufficiently large compared to \( r \) (Hirschowitz-Simpson [1994]), where the bound given is \( \gamma > 6r^3 \log r \).

Schreyer discovered by computational experiments of a probabilistic nature that the following three cases give counterexamples to the conjecture: 11 points in \( P^6 \), 12 points in \( P^7 \), and 13 points in \( P^8 \). A few more such examples were discovered by computer search (Boij [1994], Beck-Kreuzer [1996]). Despite considerable effort, no-one was able to give a non computational treatment of these examples, nor to find any “explanation” of them, so that it was unclear whether they were unique accidents or part of a larger picture.

In this paper we give a geometric construction that gives rise to a subcomplex of the resolution of a general set of points. A consequence of our construction is the following, which includes all the examples that are currently known:

**Theorem 0.1.** For any integer \( r \geq 6, r \neq 9 \), there is an integer \( \gamma(r) \) such that the Minimal Resolution Conjecture fails for a set of \( \gamma(r) \) general points in \( P^r \). More explicitly, if \( s \) and \( k \) are (uniquely) defined by

\[
 r = \left( s + \frac{1}{2} \right) + k, \quad 0 \leq k \leq s,
\]

then we may take

\[
 \gamma(r) = r + s + 2 = \left( s + \frac{2}{2} \right) + k + 1.
\]

We do not know whether such examples exist in \( P^9 \); but Beck-Kreuzer [1996] have made computations showing that none occurs for 50 or fewer points.

Here is an outline of the ideas involved. Associated to any embedding \( \Gamma \subset P^r \) of a set of \( \gamma \) points (in sufficiently general position) in projective space is another embedding of the same set of points in projective \( (s := \gamma - r - 2) \)-space, called the Gale Transform of \( \Gamma \) (see §1). Call the image of the transformed embedding \( \Gamma' \subset P^s \); it is again a general set of points. We may identify the ambient space \( P^r \) of \( \Gamma \) with the space of lines in \( H^1(I_{\Gamma'}(1)) \). Using this identification, we relate the back ends of the resolutions of \( \Gamma \) and \( \Gamma' \) (see §1). Writing \( W := H^0(O_{P^r}(1)) \), and \( U := H^1(I_{\Gamma'}(2))^* \) we have a multiplication pairing

\[
 \mu : \ W \otimes U \longrightarrow H^1(I_{\Gamma'}(1))^* = H^0(O_{P^r}(1)).
\]

Associated to any such pairing is a complex built from a certain Koszul complex

\[
 E_\bullet(\mu) : \ldots \longrightarrow \wedge^3 W \otimes D_2 U \otimes O_{P^r}(-2) \longrightarrow \wedge^2 W \otimes U \otimes O_{P^r}(-1) \longrightarrow W \otimes O_{P^r},
\]

where \( D_l U \) denotes the \( l^{th} \) divided power of \( U \) (see §3). There is a natural map \( E_\bullet(\mu)(r + 2) \) into the dual \( F_\bullet^* \) of the free resolution of \( I_{\Gamma} \), regarded as a complex of sheaves (we sometimes regard \( E_\bullet(\mu) \) as a complex of free modules). When the map \( \mu \) satisfies a certain nondegeneracy condition, the complex \( E_\bullet(\mu) \) has a property we call linear exactness (see §2). In this situation the map \( E_\bullet(\mu)(r + 2) \longrightarrow F_\bullet^* \) is a monomorphism onto a direct summand, and this gives a lower bound for the betti numbers \( \beta_{ij} \) of \( I_{\Gamma} \) that is sometimes
in conflict with the conclusion of the Minimal Resolution Conjecture. One might say, in summary, that the failure of the conjecture for a set \( \Gamma \) comes from the failure of \( \Gamma' \) to impose independent conditions on forms of degree 2; not because \( \Gamma' \) isn’t sufficiently general, but because its degree is greater than the number of forms of degree 2 in \( \mathbf{P}^s \).

The heart of the paper, and by far the most difficult part, is the nondegeneracy of the pairing \( \mu \), established in \( \S 4 \). This nondegeneracy represents an open condition on the family of sets of points \( \Gamma \). Thus in order to prove that it holds for the general \( \Gamma \), it is enough to show that it holds for a special set of points \( \Gamma' \). We do this by specializing the points to lie on a curve \( C \). Under favorable circumstances, the nondegeneracy condition on \( \mu \) can be re-interpreted as a cohomology condition on a certain vector bundle on the curve \( C \); the argument has the flavor of Koszul cohomology. We complete the argument by specializing \( C \) to either a plane curve or to a curve with prescribed gonality (depending on the parity of \( s \)), and taking the points in such a way that the bundle in question decomposes into a direct sum of simpler bundles; even then the computation of cohomology involves some nonstandard ideas. For instance, in the plane curve case \( s = 4 \), we must show the following: Let \( C \) be a general plane curve of degree \( \geq 5 \) and let \( T \) be the restriction of the tangent bundle of the plane to \( C \). If \( L \) is a general line bundle on \( C \) of degree \( \text{genus}(C) - 1 \), then there are no “twisted endomorphisms” \( T \rightarrow L \otimes T \). We can actually prove the vanishing theorem in the plane curve case (Theorem 4.1) only in characteristic 0; but as Theorem 0.1 in full generality follows from the case of characteristic 0, this is no problem.

To see how all this works in the easiest interesting case, let \( \Gamma \) be a set of \( \gamma(6) = 11 \) general points in \( \mathbf{P}^6 \), Schreyer’s simplest example. With notation as in Theorem 0.1 we have \( s = 3 \). If we display the betti numbers \( \beta_{ij} \) associated with the resolution \( F_\bullet \) in a table in the style of the program Macaulay, the expected betti numbers, coming from the Minimal Resolution Conjecture, would be

| degree | 0 | 1 | 2 |
|--------|---|---|---|
|        | 1 | - | - |
|        | - | 17| 45|
|        | - | - | 25|
|        | - | - | 4 |

**Conjectural shape of \( F_\bullet \).**

We have \( \dim(W) = 4 \). The set \( \Gamma' \) consists of 11 general points in \( \mathbf{P}^3 = \mathbf{P}(W) \), and since the space of quadrics in \( \mathbf{P}^3 \) is only 10-dimensional, \( \dim(U) = 1 \). Identifying the divided powers of \( U \) with the ground field, the complex \( E_\bullet(\mu) \) becomes

\[
E_\bullet(\mu) : 0 \longrightarrow \wedge^4 W \otimes \mathcal{O}_{\mathbf{P}^r}(-3) \longrightarrow \wedge^3 W \otimes \mathcal{O}_{\mathbf{P}^r}(-2) \longrightarrow \wedge^2 W \otimes \mathcal{O}_{\mathbf{P}^r}(-1) \longrightarrow W \otimes \mathcal{O}_{\mathbf{P}^r},
\]

which can be identified with the back end of (a twist of) the Koszul complex of a sequence of four linear forms on \( \mathbf{P}^6 \). The nondegeneracy condition on \( \mu \) becomes the condition that the complex \( E_\bullet(\mu) \) is exact (in this case exactness and linear exactness coincide). The nondegeneracy can be proved by a special argument in this case (see below), but our general method is the following: Since the condition on \( \Gamma \) (or, equivalently, on \( \Gamma' \)) is open, it suffices to prove the result after degenerating \( \Gamma' \) until it lies on a curve \( C \subset \mathbf{P}^s = \mathbf{P}^3 \).
In this case we take $C$ to be a general sextic curve in $\mathbb{P}^3$ of genus 3, and let $H$ denote its hyperplane class. Such a curve is projectively normal. By a Koszul homology argument we show that the nondegeneracy condition we need follows if we can show that

$$H^0(\wedge^2 M_H \otimes \mathcal{O}_C(K_C + \Gamma' - 2H)) = 0,$$

where $M_H$ denotes the rank 3 vector bundle that is the kernel of the evaluation map $H^0(\mathcal{O}_C(H)) \otimes \mathcal{O}_C \to \mathcal{O}_C(H)$, and $K_C$ is the canonical class of $C$. We may think of the line bundle $\mathcal{O}_C(L) := \mathcal{O}_C(K_C + \Gamma' - 2H)$ simply as a general line bundle of degree $(2g(C) - 2) + 11 - 2 \cdot 6 = 3$ on $C$. To prove the required vanishing, we degenerate $C$ to a curve of type $(2, 4)$ on a smooth quadric surface in $\mathbb{P}^3$ (so that $C$ is hyperelliptic), and $\mathcal{O}_C(H)$ to $\mathcal{O}_C(3H_0)$, where $\mathcal{O}_C(H_0)$ is induced by the class $(0, 1)$ on the quadric. The canonical series on $C$ is induced by $(0, 2)$ on the quadric, from which we easily see that $\mathcal{O}_C(3H_0)$ is nonspecial and that

$$M_{3H_0} \cong \bigoplus_{i=1}^3 \mathcal{O}_C(-H_0)$$

is a degeneration of $M_H$. Thus it suffices to show that

$$H^0((\wedge^2 M_{3H_0}) \otimes \mathcal{O}_C(L)) = \bigoplus_{i=1}^3 H^0(\mathcal{O}_C(L - 2H_0)) = 0.$$ But $\mathcal{O}_C(L - 2H_0)$ is a line bundle of degree $3 - 4 = -1$, so the result is now immediate. The same argument works for 12 or 13 points on $C$, giving the cases in $\mathbb{P}^7$ and $\mathbb{P}^8$ respectively. In the cases $r = 6$ and $r = 7$ (but already not for $r = 8$) the necessary nondegeneracy can be proved more simply: the pairing $\mu$ can be identified as the multiplication map of sections of certain line bundles on the curve $C$ (see §4) and as such is 1-generic (in the sense of Eisenbud [1988]). In general, the method works when $\dim U \leq 2$, and Kreuzer [1994] has proved the necessary 1-genericity in all cases, but the nondegeneracy we need does not follow from this when $\dim U > 2$.

As shown in the body of the paper, it follows that the complex $E_\bullet(\mu)(8)$ is a subcomplex of the dual $F_\bullet^*$ of the minimal free resolution of $I_{\Gamma}$. Equivalently, $F$ maps onto the complex $E_\bullet(\mu)(-8)^*$ (suitably shifted) which has betti display

| degree | 0     | 1     | 2     |
|--------|-------|-------|-------|
|        | -     |       |       |
|        | -     |       |       |
|        | -     |       | 1     |
|        |       |       | 4     |
|        |       |       | 6     |
|        |       |       | 4     |

$E_\bullet(\mu)(-8)^*$

Under these circumstances each betti number for $F_\bullet$ must be at least as big as the one for $E_\bullet(\mu)(-8)^*$, so we see that a lower bound for the size of $F_\bullet$ is given by the following betti diagram (we have indicated the differences inside boxes). In this case $r = 6$ (and also when $r = 7$, but already not when $r = 8$), computation shows that this diagram gives the
actual value of the $\beta_{ij}$, so that the theory here developed leads to an exact computation.

| degree | 0 | 1 | 2 |
|--------|---|---|---|
|        | 1 | - | - |
|        | 17| 46| 45|
|        | 25| 18| 4 |

**Actual shape of $F$.**

The four linear forms that enter the pairing $\mu$ in the case $r = 6$ have an amusing interpretation: They define a plane $\Pi$ in $\mathbb{P}^6$, which is distinguished just by the data of the 11 points; it is defined from the structure of the cohomology module of the ideal sheaf of the points in $\mathbb{P}^3$. The plane $\Pi \subset \mathbb{P}^6$ is spanned by (any) three points which together with the 11 initial ones form a collection of self-associated points in $\mathbb{P}^6$ (that is a set which is self-dual with respect to the Gale Transform). Charles Walter has pointed out to us that $\Pi$ could also be interpreted as the unique plane in $\mathbb{P}^6$ such that the projection of the 11 points from this plane into $\mathbb{P}^3$ is equivalent to the the Gale transform of the 11 points. (This latter characterization follows directly from the theory in §1.)

It is interesting to compare the case of points with that of curves. The minimal resolution conjecture for complete embeddings of large degree (compared with the genus) general curves was shown to be false by Schreyer [1983], and Green-Lazarsfeld [1988]; the failure comes essentially from the existence of special divisors on the curve, which give rise to rational normal scrolls containing the curve, and is quite different in character from the phenomena exhibited here. By contrast, no counterexamples to the appropriate minimal free resolution conjecture are known for ideals in a polynomial ring which are made from a generic vector space of forms of some degree $d$ plus all the forms of degree $d + 1$; these are the ideals that seem to be the most reasonable analogue of ideals of general sets of points. However, the problem is computationally difficult, and not many cases have been examined.

It is a pleasure to thank Mike Stillman, who joined us in discussions leading to some of the ideas in this paper, André Hirschowitz and Charles Walter, from whose ideas the exposition has benefitted, and Bob Friedman, who pointed out to us the beautiful paper of Raynaud [1982]. We are also grateful to Stillman and to Dave Bayer for the program Macaulay (Bayer-Stillman [1989–1996]) which has been extremely useful to us; without it we would probably have never been bold enough to guess the existence of the structure that we explain here. Finally, we are grateful to Mark Green: in earlier joint work the first author learned from him how useful maps on the cohomology of the ideal sheaves of points could be; this helped to spot the connection exploited in this paper.
§1. The Gale Transform

We first give a naive definition of the Gale transform of a set of points. Then we explain a more flexible view, in which the Gale transform is an involution — essentially Serre duality — on the set of linear series on a set of points. Finally, we exhibit a peculiar module which maps to the canonical module of a suitable set of points. This module has a natural interpretation in terms of the Gale transform. In the next section we will exhibit a subcomplex of the resolution of this module that is “responsible” for the failure of the minimal resolution conjecture.

**Definition.** Let \( k \) be a field, and write \( \mathbb{P}^r \) for \( \mathbb{P}_k^r \). Let \( \Gamma \subset \mathbb{P}^r \) be a set of \( \gamma \) labelled points such that every subset of \( \gamma - 1 \) of the points spans \( \mathbb{P}^r \). Choosing homogeneous coordinates for the points, we may write their coordinates in the form of a matrix \( G : k^{r+1} \rightarrow k^\Gamma \), and this matrix has rank \( r + 1 \). If we dualize this matrix and take the kernel, we get a matrix \( G' : k^{s+1} \rightarrow (k^\Gamma)^* \), where \( s + r + 2 = \gamma \). Since \( k^\Gamma \) has (up to scalars) a natural basis, consisting of functions vanishing at all but one point, we may identify \( k^\Gamma \) and \((k^\Gamma)^*\) in a way that is natural up to the choice of a diagonal matrix, and regard \( G' \) as being a map \( k^{s+1} \rightarrow k^\Gamma \). The rows of this matrix determine points in a set \( \Gamma' \) of labelled points in \( \mathbb{P}^s \), labelled by the same set as \( \Gamma \); the rows are all nonzero because of the condition that every subset of \( \gamma - 1 \) of the points spans \( \mathbb{P}^r \). The reader may check that \( \Gamma' \) is uniquely determined from \( \Gamma \subset \mathbb{P}^r \) up to the action of \( PGL(s + 1) \), and that it spans \( \mathbb{P}^s \). The set \( \Gamma' \) is called the (classical) Gale transform of \( \Gamma \).

The Gale transform has a long history, extending at least as far as Hesse’s thesis [1840]. It was studied (under the rubric “associated sets of points”) by Castelnuovo [1889] and later by A.B. Coble ([1915, 1916, 1917, 1922]), who discovered amazing geometric constructions and applied the Gale transform to the study of Theta-functions and Jacobians in the early part of this century. For a modern exposition with many extensions see Dolgachev and Ortland [1988]. The Gale transform has reappeared in many places. For example V. D. Goppa reinvented the idea in the context of coding theory. He proved that if \( \Gamma \) lies on a linearly normal curve \( C \subset \mathbb{P}^r \), then \( \Gamma' \) lies on a different embedding of the same curve (see Goppa, [1984]). The name “Gale transform” has become established by the very fruitful use of the idea (in a somewhat different form) in the study of convex polytopes and integer programming initiated by D. Gale in [1963]. We refer to the forthcoming paper Eisenbud-Popescu [1997] for more history and geometric constructions.

Recall that a linear series on a scheme \( X \) is a pair \((V, L)\) consisting of a line bundle \( L \) and a vector space \( V \) of global sections of \( L \). The Gale transform can be defined much more generally, as an involution on the space of linear series on a finite scheme \( \Gamma \). Of course it is somewhat pedantic to speak of line bundles and global sections on a finite scheme, since any such scheme is affine and every line bundle is trivial, but it has the same virtues as does the distinction between a vector space and its dual: this language will allow us to make definitions without any arbitrary choices.

If \( \Gamma \) is a Gorenstein scheme, finite over a field \( k \), and \( L \) is a line bundle on \( \Gamma \), then Serre duality provides a canonical “trace” \( \tau : H^0(K_\Gamma) \rightarrow k \) with the property that for any line bundle \( L \) on \( \Gamma \) the composition

\[
H^0(L) \otimes_k H^0(K_\Gamma \otimes L^{-1}) \rightarrow H^0(K_\Gamma) \rightarrow k.
\]
of \( \tau \) with the multiplication map gives a perfect pairing between \( H^0(L) \) and \( H^0(K_\Gamma \otimes L^{-1}) \). If \( V \subset H^0(L) \) is a subspace, then we write \( V^\perp \subset H^0(K_\Gamma \otimes L^{-1}) \) for the annihilator.

Using these ideas, we may define the Gale transform more generally:

**Definition.** Let \( k \) be a field, and let \( \Gamma \) be a Gorenstein scheme finite over \( k \). The Gale transform of a linear series \((V, L)\) on \( \Gamma \) is the linear series \((V^\perp, K_\Gamma \otimes L^{-1})\). (This is the natural definition of adjoint series in the zero-dimensional case.)

We recall that the “Veronese” linear series are defined by multiplication: If \( n \geq 1 \), then we write \( V^n \) for the image by multiplication of \( V^\otimes n \) in \( H^0(L^n) \). We set \( V^0 = k \subset H^0(\mathcal{O}_\Gamma) \), while for \( n < 0 \) we set \( V^n = 0 \) (again as a subset of \( H^0(L^n) \)).

The relation to the classical Gale transform is included in the following alternative description:

**Proposition 1.1.** Let \( k \) be a field, and let \( \Gamma \) be a Gorenstein scheme finite over \( k \). If \( r \geq 1 \) and the linear series \((V, L)\) defines an embedding of \( \Gamma \) in \( P^r = P(V) \) with ideal sheaf \( I_\Gamma \), then there are natural identifications \((V^n)^\perp = H^1(I_\Gamma(n))^* \). If further \( \Gamma \) is a reduced set of \( k \)-rational points and every subset of \( \gamma - 1 \) of the points of \( \Gamma \) spans \( P^r \), then the linear series \((V^\perp, K_\Gamma \otimes L^{-1})\) is base-point-free, and the image of \( \Gamma \) under the corresponding map is the classical Gale transform of \( \Gamma \).

**Proof.** Using Serre duality to identify \( H^0(K_\Gamma \otimes L^{-n}) \) with the dual of \( H^0(L^n) \), the space \((V^n)^\perp \) becomes the kernel of the map \( H^0(L^n)^* \rightarrow (V^n)^* \) dual to the inclusion. In the setting of the classical Gale transform we choose an identification of \( H^0(L) \) with \( k^r \), and the last statement of the Proposition follows. More generally, if \((V, L)\) defines an embedding of \( \Gamma \), then the exact sequence

\[
0 \rightarrow I_\Gamma \rightarrow \mathcal{O}_{P(V)} \rightarrow \mathcal{O}_\Gamma \rightarrow 0
\]

gives rise to an exact sequence

\[
\text{Sym}_n V = H^0(\mathcal{O}_{P(V)}(n)) \rightarrow H^0(L^n) \rightarrow H^1(I_\Gamma(n)) \rightarrow [0 = H^1(\mathcal{O}_{P(V)}(n))],
\]

which yields the identification \(((V^n)^\perp)^* = H^1(I_\Gamma(n))\) as required.

The next result gives a description of the \((V^n)^\perp\) that does not depend on the points being embedded:

**Proposition 1.2.** Suppose that \( \Gamma \) is a Gorenstein scheme, finite over \( k \), and let \((V, L)\) be a linear series on \( \Gamma \). For each integer \( n \) the product \( V \cdot (V^n)^\perp \) lies in \((V^{n-1})^\perp \). If \( n \neq 0 \), then \((V^n)^\perp\) is the largest subspace of \( H^0(K_\Gamma \otimes L^{-n}) \) that is multiplied by \( V \) into \((V^{n-1})^\perp \).

**Proof.** If \( n \leq 0 \) the result is trivial. If \( n > 0 \) and \( a \in H^0(K_\Gamma \otimes L^{-n}) \), then \( a \in (V^n)^\perp \) iff \( \tau(a \cdot V^n) = 0 \) iff \( \tau(aV \cdot V^{n-1}) = 0 \) iff \( aV \subset (V^{n-1})^\perp \).

By virtue of Proposition 1.2 we may regard \( \oplus_{n \in \mathbb{Z}} ((V^n)^\perp) \) as a graded \( k[V] \)-module, and with this structure we will call it \( \omega_{\Gamma, V} \). In the case where \( \Gamma \) is embedded in \( P(V) \), the following Corollary of Proposition 1.1 identifies this module with the canonical module of the affine cone over \( \Gamma \). Since the minimal free resolution of this canonical module is the dual of the minimal free resolution of \( k[V]/I_\Gamma \), this result provides the link with free resolutions:
Corollary 1.3. Let $\Gamma \subset \mathbb{P}(V) \cong \mathbb{P}^r$ be a zero-dimensional Gorenstein subscheme, and let $S = k[V]$ be the polynomial ring in $r + 1$ variables. Write $I_{\Gamma}$ for the homogeneous ideal of $\Gamma$. There is a natural isomorphism

$$\omega_{\Gamma,V} \cong \text{Ext}^{r-1}_S(I_{\Gamma}, S(-r - 1)).$$

Proof. By Serre duality, $\text{Ext}^{r-1}_S(I_{\Gamma}, S(-r - 1)) = \oplus_n (H^1(I_{\Gamma}(n))^*)$ as $S$-modules, the multiplication

$$V \otimes H^1(I_{\Gamma}(n))^* \longrightarrow H^1(I_{\Gamma}(n-1))^*$$

being the one induced by the multiplication $H^0(\mathcal{O}_{\mathbb{P}(V)}(1)) \otimes H^1(I_{\Gamma}(n-1)) \longrightarrow H^1(I_{\Gamma}(n))$. Proposition 1.1 identifies $H^1(I_{\Gamma}(n))^*$ with $(V^n)^\perp$ for $n > 0$, while the identification for $n \leq 0$ is trivial. The compatibility of these identifications with the multiplication maps follows from the same exact sequence as employed in the proof of Proposition 1.1. \qed

We now approach the fundamental construction to be studied in this paper. We write $S_{a,b}(W)$ for the Schur functor

$$S_{a,b}(W) := \text{Im}(\wedge^{a+1} W \otimes \text{Sym}_{b-1} W) \longrightarrow \wedge^a W \otimes \text{Sym}_b W)$$

$$= \ker(\wedge^a W \otimes \text{Sym}_b W \longrightarrow \wedge^{a-1} W \otimes \text{Sym}_{b+1} W).$$

For example, $S_{1,1}(W) = \wedge^2 W$, the inclusion into $\wedge^a W \otimes \text{Sym}_b W = W \otimes W$ being the diagonal map of the exterior algebra. The reader unfamiliar with Schur functors may avoid them at first by considering only this case, that is, taking $n = 2$ in the following result.

Theorem 1.4. Let $\Gamma$ be a Gorenstein scheme, finite over $k$. Let $(V, L)$ and $(W = V^\perp, K_{\Gamma} \otimes L^{-1})$ be dual linear series, and set $U := (W^n)^\perp$, with $n \geq 1$. The natural multiplication

$$\text{Sym}_{n-1} W \otimes U \longrightarrow W^{n-1} U \subset V$$

induces a map $\mu : \text{Sym}_{n-1} W \otimes U \longrightarrow V$ which in turn defines a map of free $k[V]$-modules

$$\delta : S_{1,n-1}(W) \otimes U \otimes k[V] \longrightarrow W \otimes k[V](1).$$

There is a unique map of $k[V]$-modules (coker $\delta$) $\longrightarrow \omega_{\Gamma,V}$ which extends the inclusion $W = (\omega_{\Gamma,V})_{-1} \subset \omega_{\Gamma,V}$.

Proof. The map $\delta$ is the composite

$$S_{1,n-1}(W) \otimes U \otimes k[V] \longrightarrow W \otimes \text{Sym}_{n-1} W \otimes U \otimes k[V] \overset{W \otimes \mu}{\longrightarrow} W \otimes k[V](1),$$

whereas the space of linear relations on $\omega_{\Gamma,V}$ can be identified with the vector space $N$ which is the kernel of the multiplication map $m : W \otimes V \longrightarrow H^0(K_{\Gamma})$. We must show that $N$ contains the relations on coker $\delta$, which are generated by the image of the composite

$$S_{1,n-1}(W) \otimes U \longrightarrow W \otimes \text{Sym}_{n-1} W \otimes U \overset{W \otimes \mu}{\longrightarrow} W \otimes V.$$
Now the natural multiplication map $\mu' : \text{Sym}_n W \otimes U \to H^0(K_\Gamma)$ fits in the diagram

\[
\begin{array}{cccc}
0 & \to & N & \to & W \otimes V & \overset{m}{\to} & W \cdot V & \hookrightarrow & H^0(K_\Gamma) \\
& & & & & & & & \\
& & & & & & & & \\
0 & \to & S_{1,n-1}(W) \otimes U & \leftarrow & W \otimes \text{Sym}_{n-1}(W) \otimes U & \to & \text{Sym}_n(W) \otimes U, \\
\end{array}
\]

which is commutative by the associativity of multiplication, and has exact rows by the definition of $N$ and $S_{1,n-1}$. Thus there is a vertical map induced on the left, which is the desired inclusion. ■

The significance of this result is that it gives a map of complexes from a complex $E_{r-1}^{-1}(\mu)(r+2)$ (described in §3) beginning with the map $\delta$ into the dual of the resolution of the ideal of the points. We shall see that under certain circumstances this map is an inclusion, and provides the subcomplex which “spoils” the Minimal Resolution Conjecture. The properties of this map will be the subject of §2. Of course Theorem 1.4 is vacuous if $U = (W^n)^\perp = 0$. By Proposition 1.1, if $\Gamma'$ is a set of points in $P^r = P(V)$ and $\Gamma'$ is its Gale transform, embedded in $P^s = P(W)$, then $U = H^1(I_{\Gamma'}(n))^*$; thus $U$ is nonzero iff $\Gamma'$ fails to impose independent conditions on forms of degree $n$. The analysis of the resolution of $I_\Gamma$ via the map $\delta$ will involve the geometry of the Gale transform $\Gamma'$.

**Remark.** There is a less invariant version of these ideas which is pleasingly direct: Again let $\Gamma$ be a Gorenstein scheme, finite over a field $k$, and let $O_\Gamma$ be the coordinate ring of $\Gamma$, a finite dimensional Gorenstein $k$-algebra. Suppose that $\Gamma$ is embedded in $P^r$. If we choose a hyperplane not meeting $\Gamma$, we may identify the line bundle $L = O_\Gamma(1)$ with $O_\Gamma$, and thus identify the linear series $(V = H^0(O_{P^r}(1)), L)$ with a subspace $V \subset O_\Gamma$. We also choose an identification of $O_\Gamma$ with $K_\Gamma$ (equivalently, we may choose a “trace” functional $\tau : O_\Gamma \to k$ not vanishing on any component of the socle of $O_\Gamma$) and consider the pairing on $O_\Gamma$ defined as the composition of this functional with multiplication. We may again define the powers $V^n$ and the spaces $W_n := (V^n)^\perp$, but this time they will all be subspaces of $O_\Gamma$. 

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§2. Linear exactness and linear rigidity

We shall use a property of certain complexes that we have not seen exploited before: we call it linear exactness. We give its definition in an abstract setting before plunging into the multilinear algebra necessary to define the complexes to which we will apply it.

Let $S$ be a graded ring with $S_0 = k$ a field, and let

$$E_\bullet : \ldots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow \ldots \longrightarrow E_n$$

be a linear complex in the sense that each $E_i$ is a free module generated in degree $i$, so that in particular the differentials are given by matrices of elements of $S_1$. We shall say that $E$ is linearly exact if, for all $i > n$, the homology $H_i(E_\bullet)$ is nonzero only in degrees $> i$, or equivalently if, in any matrix representing a differential of $E_\bullet$, the columns are linearly independent over $S_0$. The utility of this definition lies in the following result:

**Proposition 2.1.** Let $E_\bullet$ be a linearly exact linear complex as above, and let

$$F_\bullet : \ldots \longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow \ldots \longrightarrow F_n$$

be a graded minimal free resolution with $F_n$ generated in degrees $\geq n$. If $\alpha : E_\bullet \longrightarrow F_\bullet$ is a map of complexes such that $\alpha_n : E_n \longrightarrow F_n$ is an inclusion, then each map $\alpha_i : E_i \longrightarrow F_i$ is a split inclusion.

**Proof.** Any set of elements of degree $n$ in $F_n$ that are linearly independent over $S_0$ are part of a free basis. Thus $E_n$ maps to a direct summand of $F_n$. It follows from the hypothesis that for each $i$ the module $F_i$ is generated in degrees $\geq i$. By hypothesis, a free basis of $E_{n+1}$ maps to a set of linearly independent elements in $(E_n)_{n+1}$, which is a summand of $(F_n)_{n+1}$. Since $F_{n+1}$ is generated in degrees $\geq n+1$, the map $\alpha_{n+1}$ must take the basis of $E_{n+1}$ to a subset of a basis of $F_{n+1}$. Thus $\alpha_{n+1}$ is a split inclusion, and induction completes now the proof.

It often suffices to prove linear exactness at the first step:

**Lemma 2.2.** (Linear rigidity). Let $R = k[x_0, \ldots, x_r]$ be a polynomial ring, and let $M$ be an $R$-module generated in degrees $\geq 0$. Let $F_\bullet$ as above be a minimal free resolution of $M$. Suppose that $S$ as above is an $R$-algebra, and that

$$E_\bullet : \ldots \longrightarrow E_{i+1} \longrightarrow E_i \longrightarrow \ldots \longrightarrow E_0$$

as above is the linear part of $S \otimes_R F_\bullet$. If the homology $H_1(E_\bullet)$ is nonzero only in degrees $> 1$, then $E_\bullet$ is linearly exact.

**Proof.** We must show that if $\text{Tor}_1^R(S, M)_1 = 0$, then $\text{Tor}_i^R(S, M)_i = 0$ for each $i \geq 1$; a “linear rigidity” theorem for Tor. The proof of Auslander-Buchsbaum [1958] for the rigidity of Tor (reduction to the diagonal plus the rigidity of the Koszul complex) may easily be adapted.
In the theory above it actually suffices to suppose that $E_i$ is generated in degree $i$ just for $i > n$. Thus we may try to apply the theory to the Eagon-Northcott complex and so we see that the complex is linearly exact iff the minors are independent. This leads to the following:

**Problem:** Under what conditions are the $d \times d$ minors of an $e \times f$ matrix of linear forms linearly independent?

### 3. The complexes $E^m_\bullet(\mu)$

In this section we define the complexes whose linear exactness plays a role in our analysis of the resolutions of ideals of points. With appropriate choices, these complexes extend the map defined in Theorem 1.4 (for the moment only in the case $n = 2$) and thus admit a map to the dual of the free resolution of an ideal of points.

First we recall the notion of divided power. Let $U$ be a finitely generated free module over some ring. We write $D_lU$ for the $l$th divided power of $U$. It is convenient to define $D_lU$ as the dual of the $l$th symmetric power of the dual module, that is $D_lU = (\text{Sym}_l(U^*))^*$. What we shall use about $D_lU$ is that it has a “diagonal” map $D_{l+1}U \longrightarrow D_lU \otimes U$ which is the monomorphism dual to the surjective multiplication map $\text{Sym}_l(U^*) \otimes U^* \longrightarrow \text{Sym}_{l+1}(U^*)$. See for example Eisenbud [1995, Appendix 2] for the usual definition.

Suppose again that $S$ is a graded ring, with $S_0 = k$ a field, and that $U$ and $W$ are finite dimensional vector spaces over $k$. Let $\mu : W \otimes U \rightarrow S_1$ be a homomorphism.

For any integer $m$, and any integer $l \geq 0$ we define a free module

$$E^m_l(\mu) := \wedge^{l-m}W \otimes D_lU \otimes S(-l)$$

and a map

$$\delta^m_{l+1}(\mu) : E^m_{l+1}(\mu) \longrightarrow E^m_l(\mu),$$

which is the composite of the tensor product of the diagonal maps of the exterior and divided powers,

$$\wedge^{l+1-m}W \otimes D_{l+1}U \otimes S(-l-1) \longrightarrow \wedge^{l-m}W \otimes W \otimes D_lU \otimes U \otimes S(-l-1),$$

and the map induced by $\mu$

$$\wedge^{l-m}W \otimes W \otimes D_lU \otimes U \otimes S(-l-1) \longrightarrow \wedge^{l-m}W \otimes D_lU \otimes S(-l).$$

These maps form complexes of free $S$-modules

$$E^m_\bullet(\mu) : \ldots \longrightarrow E^m_{l+1}(\mu) \xrightarrow{\delta^m_{l+1}(\mu)} E^m_l(\mu) \longrightarrow \ldots \longrightarrow E^m_{m_+}(\mu)$$

where the term $E^m_l(\mu)$ is in position $l$, and $m_+$, which denotes the positive part of $m$, is equal to $m$ if $m \geq 0$ and to 0 if $m \leq 0$. The complex $E^m_\bullet(\mu)$ is the dual of the Eagon-Northcott complex resolving the maximal minors of $\mu$, whence the name $E$. As with the Eagon-Northcott complex, these complexes may be built up inductively:
Proposition 3.1. (Inductive Construction). With notation as above, suppose that

\[ 0 \longrightarrow W' \longrightarrow W \longrightarrow k \longrightarrow 0 \]

is an exact sequence, and let \( \mu' : W' \otimes U \longrightarrow S_1 \) denote the composition of \( \mu \) with the inclusion \( W' \otimes U \longrightarrow W \otimes U \). There is an exact sequence of complexes

\[ 0 \longrightarrow E^m_\bullet(\mu') \longrightarrow E^m_\bullet(\mu) \longrightarrow E^{m+1}_\bullet(\mu') \longrightarrow 0. \]

Proof. We use the exact sequence

\[ 0 \longrightarrow \wedge^{l-m} W' \longrightarrow \wedge^{l-m} W \longrightarrow \wedge^{l-m-1} W' \longrightarrow 0. \]

The commutativity of the necessary diagrams follows by straightforward computation.

Using Proposition 3.1 we can show that the complexes \( E_\bullet(\mu) \) satisfy the hypothesis of the linear rigidity lemma above.

Theorem 3.2. Let \( k \) be a field, and let \( W \) and \( U \) be finite dimensional vector spaces over \( k \). Let \( S = k[W \otimes U] \) be the symmetric algebra, and let \( \mu : W \otimes U \longrightarrow S(1) \) be the identity map.

a) The complex \( E^m_\bullet(\mu) \) is the linear part of a minimal free resolution.

b) For any integers \( m \) and \( i > m_+ \) the module \( H_i(E^m_\bullet(\mu)) \) is nonzero at most in degrees \( > i \).

Proof. An argument similar to that of Proposition 2.1 shows that parts a) and b) are equivalent.

To prove part b), we do induction on the rank \( w \) of \( W \). If \( w = 1 \) and \( m < 0 \) there is nothing to prove. If \( w = 1 \) and \( m \geq 0 \), then exactness follows from the fact that the diagonal map \( D_{m+1}U \longrightarrow D_m U \otimes U \) is a monomorphism.

Suppose now \( w > 1 \). Let \( W' \) be a codimension 1 subspace of \( W \), so that we have an exact sequence

\[ 0 \longrightarrow W' \longrightarrow W \longrightarrow k \longrightarrow 0. \]

Using the long exact sequence in homology coming from the inductive construction in Proposition 3.1, everything is clear except the cases where \( m \geq 0 \) and \( i = m+1 \). In this case the exact sequence of complexes has the form

\[
\begin{align*}
\text{Homological degree} : & \quad m + 1 & \quad m \\
E^{m+1}_\bullet(\mu') : & \quad \ldots \longrightarrow D_{m+1}U \otimes S(-m - 1) \\
E^m_\bullet(\mu) : & \quad \ldots \longrightarrow W \otimes D_{m+1}U \otimes S(-m - 1) \longrightarrow D_m U \otimes S(-m) \\
E^m_\bullet(\mu') : & \quad \ldots \longrightarrow W' \otimes D_{m+1}U \otimes S(-m - 1) \longrightarrow D_m U \otimes S(-m)
\end{align*}
\]

and we must show that the connecting homomorphism

\[ c : H_{m+1}(E^{m+1}_\bullet(\mu')) \longrightarrow H_m(E^m_\bullet(\mu')) \]
is a monomorphism in degree $m + 1$, which is the lowest degree present in $E_{m+1}^m(\mu') = D_{m+1}U \otimes S(-m - 1)$.

We have $(H_{m+1}(E_{m+1}^m(\mu'))))_{m+1} = D_{m+1}(U)$. Let $f \in D_{m+1}(U)$, and write $\sum_i f_i \otimes f'_i \in U \otimes D_m(U)$ for the image of the diagonal map. If we write $W = \langle x \rangle \oplus W'$, and $\tilde{f}$ for the class of $f$ in the homology of $E_{m+1}^m(\mu')$, then we see by chasing the diagram that

$$c(\tilde{f}) = \sum_i (x \otimes f_i) \otimes f'_i \in (W \otimes U) \otimes D_m(U).$$

Since the differential of $E_{\bullet}^m(\mu')$ involves only $W'$, the homology module $H_m(E_{\bullet}^m(\mu'))$ subjects onto $(\langle x \rangle \otimes U) \otimes D_m(U) = U \otimes D_m(U)$, and we may recover $\sum_i f_i \otimes f'_i$ as the image of $c(\tilde{f})$. Since the diagonal map is a monomorphism on the divided powers, we are done.

A closer examination of the induction shows that the generic complexes in Theorem 3.2 actually are resolutions if $m \leq -w + 1$, but not otherwise; for example, if $m = 0$, $w = \dim W = 2$, and $\dim U := u > 2$, then the complex $E_{\bullet}^m(\mu)$ has the form

$$S^u(-1) \cong \wedge^2 W \otimes U \otimes S(-1) \longrightarrow W \otimes S = S^2,$$

and the free resolution of which this is the linear part is the Buchsbaum-Rim complex

$$\ldots \longrightarrow W^* \otimes (\wedge^3 S^u)(-3) \longrightarrow S^u(-1) \longrightarrow S^2.$$

The degree 2 relations $W^* \otimes (\wedge^3 S^u)(-3) \longrightarrow S^u(-1)$ are an expression of Cramer’s rule. See Eisenbud [1995, Appendix A2.6] for more information.

In our setting the map $\mu$ has a geometric origin, and we may use a technique similar to Green’s Koszul Homology to check the condition of linear exactness. The following is the result of this section that we shall use in the sequel:

**Corollary 3.3.** Let $C$ be a projective scheme over a field $k$, and let $H, L$ be Cartier divisors on $C$. Suppose that $\mathcal{O}_C(H)$ is generated by its global sections $W := H^0(\mathcal{O}_C(H))$, and let $M_H$ be the vector bundle on $C$ which is the kernel of the evaluation map $d_0 : \mathcal{O}_C \otimes W \longrightarrow \mathcal{O}_C(H)$. Set $U := H^0(\mathcal{O}_C(L))$, and $V := H^0(\mathcal{O}_C(H + L))$. Let $S = \text{Sym} V$ be the polynomial ring, and let $\mu : W \otimes U \longrightarrow V = S_1$ be the multiplication map. The complex $E_{\bullet}^1(\mu)$ is linearly exact if and only if $H^0(\wedge^2 M_H \otimes \mathcal{O}_C(L)) = 0$.

**Proof.** By the linear rigidity lemma it is enough to check linear exactness at the first step; that is, we must show that the induced map $\wedge^2 W \otimes U \longrightarrow W \otimes V$ is a monomorphism. For this purpose we use the Koszul complex built on the evaluation map $d_0$,

$$\ldots \longrightarrow \wedge^3 W \otimes \mathcal{O}_C(-3H) \overset{d_2}{\longrightarrow} \wedge^2 W \otimes \mathcal{O}_C(-2H) \overset{d_1}{\longrightarrow} W \otimes \mathcal{O}_C(-H) \overset{d_0}{\longrightarrow} \mathcal{O}_C \longrightarrow 0,$$

tensored with $\mathcal{O}_C(2H + L)$. Now $\ker d_i = \text{Im} d_{i+1} = \wedge^{i+1} M_H(-(i + 1)H)$, for all $i \geq 0$, so the claim of the lemma follows by taking global sections in the short exact sequence

$$0 \longrightarrow \wedge^2 M_H \otimes \mathcal{O}_C(L) \longrightarrow \wedge^2 W \otimes \mathcal{O}_C(L) \overset{d_1 \otimes \mathcal{O}_C(2H + L)}\longrightarrow W \otimes \mathcal{O}_C(H + L).$$

$\blacksquare$
Remark. We have made the restriction to projective schemes only to ensure the finite dimensionality of the spaces involved. This is actually unnecessary; the complexes $E^m_\bullet$ could have been developed for infinite dimensional spaces. We leave these things to the reader who can find an application . . .

§4. Subcomplexes of the resolution of $I_\Gamma$

We prove in this section the main result concerning resolutions of points: For a suitably chosen map $\mu$ the complex $E^{-1}_\bullet(\mu)$ defined above is linearly exact, and its dual is a subcomplex of the back end of the minimal free resolution of the ideal of the points.

**Theorem 4.1.** Let $V$ be an $(r+1)$-dimensional vector space over a field $k$ of characteristic 0, and let $\Gamma$ be a general set of $\gamma$ points in $\mathbb{P}(V)$. Let $W := V^\perp \subset H^0(K_\Gamma(-1))$, let $U := (W^2)^\perp \subset H^0(K_\Gamma^{-1}(2))$, and let $\mu : W \otimes U \longrightarrow V$ be the multiplication map. Set

$$r := \left(\frac{s+1}{2}\right) + t, \quad s \geq 2, \quad 0 \leq t \leq s, \quad \text{and} \quad \gamma := r + s + 2.$$

If $s$ is even suppose also $\text{char } k = 0$. The complex $E^{-1}_\bullet(\mu)(r+2)$ is a direct summand of the dual of the free resolution of the ideal of $\Gamma$.

**Remarks.** The given number of points in $\mathbb{P}^r = \mathbb{P}(V)$ is actually the largest number for which the construction is interesting; for smaller numbers there is still a nontrivial complex but it is only sometimes linearly exact. The restriction to characteristic 0 is most likely unnecessary, but is not important as our main Theorem 0.1 follows in all characteristics from the characteristic 0 case. The restriction comes only from the use of a theorem of Hartshorne and Gieseker on the semistability of symmetric powers of semistable vector bundles at the very end of the argument.

**Proof.** We shall show that the complex $E^{-1}_\bullet(\mu)$ is linearly exact. Since $s \geq 2$ we have $\gamma \leq (\frac{r+2}{2})$, so $\Gamma$ imposes independent conditions on quadrics and thus the homogeneous ideal $I_\Gamma$ is 3-regular. It follows that, with notation as in Theorem 1.4, $\omega_{\Gamma,V}$ is generated in degrees $\geq -1$. The dual of the free resolution of $I_\Gamma$ is (the beginning of) the minimal free resolution of $\omega_{\Gamma,V}(r+1)$. We will thus deduce Theorem 4.1 from Theorem 1.4 and Proposition 2.1, applied to the complex $E^{-1}_\bullet(\mu)$ and the minimal free resolution of $\omega_{\Gamma,V}(-1)$.

Our strategy for proving linear exactness is as follows. We wish to apply Corollary 3.3. To do this we must find a scheme $C$ such that $W$ may be interpreted as a space of sections generating a line bundle $O_C(H)$ and $U$ may be interpreted as the space of all sections of a line bundle $L$. It is most convenient to regard $\Gamma$ by its “other” embedding as the Gale transform $\Gamma'$, since there $W$ is the space of sections of the line bundle responsible for the embedding in $\mathbb{P}^s$, while $U$ may be identified with $H^1(I_{\Gamma'}(2))^*$. We cannot take $C = \Gamma'$ itself, however, because $U$ is not a complete linear series. Thus we need some higher-dimensional scheme on which $\Gamma'$ lies. Since the general set of points $\Gamma'$ does not (as far as we know) lie on any useful schemes of larger dimension, we will make a degeneration, using the (obvious) openness of the locus, in the space of maps $\mu : W \otimes U \longrightarrow V$, where $E^{-1}_\bullet(\mu)$ is linearly exact. We shall degenerate $\Gamma'$ to a set of points, lying on a convenient curve $C$. In doing this, we must keep the dimensions of $W$ and $U$ constant (since $V = W^\perp$, the constancy of its dimension is then automatic).
Since $\Gamma'$ is a general set of $\gamma > \binom{s+2}{2}$ points in $\mathbb{P}^s$, it lies on no quadrics, and this fact determines the dimension of $U$ as $h^1(I_{\Gamma'}(2))$. We may thus degenerate $\Gamma'$ to a general subset of a curve $C$ in $\mathbb{P}^s$ that lies on no quadrics (which we will again call $\Gamma'$).

In order to establish a simple relation between the cohomology of $I_{\Gamma'}$ and bundles on the curve we will require $C$ to be nonspecial and quadratically normal. Thus writing $H$ for the hyperplane class on $C$ and setting $d := \deg H$, $g := \text{genus } C$, we need

$$s + 1 = h^0(O_C(H)) = d + 1 - g$$
$$\binom{s + 2}{2} = h^0(O_C(2H)) = 2d + 1 - g,$$

which in turn yield $d = \binom{s+1}{2}$ and $g = \binom{s}{2}$.

It is easy to compute that the curve defined in $\mathbb{P}^s$ by the vanishing of the $3 \times 3$ minors of a general $3 \times (s + 1)$ matrix of linear forms $M$ has exactly the invariants required. From the existence of this curve $C$, and the openness of the desired properties, we see that we may take $C$ to be a general curve of genus $\binom{s}{2}$, embedded by the complete linear series associated to a general divisor $H$ of degree $\binom{s+1}{2}$ in $\mathbb{P}^s$. We will use this freedom to make further degenerations.

The binomial form of the genus formula suggests a plane curve of degree $s + 1$, and it is amusing to note that the determinantal curve just defined may be embedded in the plane by the line bundle that is the cokernel of the restriction of $M$ to the curve; in this planar embedding its equation is the determinant of the $(s + 1) \times (s + 1)$ matrix of linear forms in 3 variables which is adjoint to $M$. We shall use this construction implicitly later in the proof.

If $\Gamma'$ is a general divisor of degree $\gamma$ on a curve $C$ as above, then we can write $\mu$ as a map coming from bundles on $C$ as follows: Since $O_C(H)$ is nonspecial and the curve $C$ is projectively normal, the cohomology of the short exact sequences

$$0 \longrightarrow I_C(mH) \longrightarrow I_{\Gamma'}(mH) \longrightarrow O_C(mH - \Gamma') \longrightarrow 0,$$

together with Serre duality yield

$$H^1(I_{\Gamma'}(mH)) \cong H^1(O_C(mH - \Gamma')) \cong (H^0(O_C(K_C + \Gamma' - mH)))^*, \text{ for all } m \geq 1.$$

We now set $L := K_C + \Gamma' - 2H$, and we have $U = H^0(O_C(L))$ as required. Since $\gamma$ is greater than the genus of $C$, we may simply describe $L$ as the general divisor of degree $2g - 2 + \gamma - 2d = r - s \leq \binom{s+1}{2}$. With these identifications the pairing $\mu : W \otimes U \longrightarrow V$ becomes the multiplication

$$\mu : H^0(O_C(H)) \otimes H^0(O_C(L)) \longrightarrow H^0(O_C(L + H)).$$

By Corollary 3.3 it now suffices to prove for each $s \geq 2$ that $H^0(\wedge^2 M_H \otimes O_C(L)) = 0$ where

- $C$ is a general curve of genus $g := \binom{s}{2}$,
- $H$ is a general divisor on $C$ of degree $d := \binom{s+1}{2}$. 

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\[ L \text{ is a general divisor on } C \text{ of degree } l := \binom{s+1}{2}. \]

The standard method of proving such vanishing is by filtration and stability (see Green-Lazarsfeld [1986], Ein-Lazarsfeld [1992]) but it does not yield a strong enough result, and the stability of \( M_H \) would not be a strong enough condition, so instead we shall use further degenerations: Depending on the parity of \( s \) and the stability of \( M \) (Green-Lazarsfeld [1986], Ein-Lazarsfeld [1992]) but it does not yield a strong enough result, the dimension of \( H^0(\mathcal{O}_C(H)) \) is constant. Suppose that we can find, for each \( s \), a smooth curve \( C_0 \) of genus \( g \), a nonspecial divisor \( H \) of degree \( d \), and some divisor \( L \) of degree \( l \) on \( C \) such that \( H^0(\wedge^2 M_H \otimes \mathcal{O}_C(L)) = 0 \). Over a versal deformation of \( C_0 \) we may form the space of triples \( (C, H, L) \), where \( H, L \) are divisors of the given degrees. The base space of this versal deformation maps to the moduli space of curves of genus \( g \) and covers an open set therein. Thus the general curve \( C \), with general divisors \( H \) and \( L \) will have the properties required.

Assume now that \( s \) is odd. We let \( C_0 \) be a curve of type \( ((s + 1)/2, s + 1) \) on the smooth quadric \( Q \subset \mathbb{P}^3 \). Let \( \mathcal{N} \) be the restriction of \( \mathcal{O}_Q(0, 1) \) to \( C_0 \). Thus \( \mathcal{N} \) is a line bundle of degree \( (s + 1)/2 \) generated by global sections, and \( \mathcal{N}^{\oplus s} \) is a globally generated vector bundle of rank \( s \) and degree \( \binom{s+1}{2} \) on \( C_0 \). Let \( W^* \) be a general \( (s + 1) \)-dimensional subspace of sections \( W^* \subset H^0(\mathcal{N}^{\oplus s}) \). It is easy to see that \( W^* \) generates \( \mathcal{N}^{\oplus s} \), and so we define a line bundle \( \mathcal{O}_{C_0}(H) \) as the dual of the kernel of the natural evaluation

\[
0 \longrightarrow \mathcal{O}_{C_0}(-H) \longrightarrow W^* \otimes \mathcal{O}_{C_0} \longrightarrow \mathcal{N}^{\oplus s} \longrightarrow 0.
\]

With these choices \( \mathcal{O}_{C_0}(H) \) is a globally generated line bundle of the desired degree \( \binom{s+1}{2} \), and \( W \) maps naturally to \( H^0(\mathcal{O}_{C_0}(H)) \). In fact \( \mathcal{O}_{C_0}(H) = \det(\mathcal{N}^{\oplus s}) = \mathcal{N}^{\oplus s} = \mathcal{O}_{C_0}(0, s) \). Furthermore, since \( h^1(\mathcal{O}_Q(0, s)) = 0 \) and \( h^2(\mathcal{O}_Q(-\frac{(s+1)}{2}, -1)) = 0 \), taking cohomology of the exact sequence

\[
0 \longrightarrow \mathcal{O}_Q(-\frac{(s+1)}{2}, -1) \longrightarrow \mathcal{O}_Q(0, s) \longrightarrow \mathcal{O}_{C_0}(H) \longrightarrow 0
\]

we get \( h^1(\mathcal{O}_{C_0}(H)) = 0 \), that is \( \mathcal{O}_{C_0}(H) \) is non-special. Thus \( h^0(\mathcal{O}_{C_0}(H)) = \chi(\mathcal{O}_{C_0}(H)) = s + 1 \) and since \( h^0(\mathcal{N}^*) = 0 \) for degree reasons, we get that \( W = H^0(\mathcal{O}_{C_0}(H)) \), whence \( M_H \cong (\mathcal{N}^{\oplus s})^* \). To show that \( H^0(\wedge^2 M_H \otimes \mathcal{O}_C(L)) \), it is enough to show that \( H^0(\mathcal{N}^{\oplus s} \otimes \mathcal{N}^{\oplus s}) \) is zero. This is obviously true for a general \( L \) with \( \deg L = r - s \), since \( \deg(\mathcal{O}_C(L) \otimes \mathcal{N}^{\oplus s}) = r - s - 2 \leq \left( \binom{s}{2} - 1 \right) = g(C_0) - 1 \) by our initial hypothesis.

Consider the versal deformation of the curve \( C_0 \) and over it the space of triples \( (C, H, L) \) as above, where \( H \) is a divisor of degree \( d \) and \( L \) is a divisor of degree \( r - s \). The locus for which \( \mathcal{O}_C(H) \) defines an arithmetically normal embedding in \( \mathbb{P}^s \) is open and, as we have seen, non-empty. Furthermore, the vanishing of \( H^0(\wedge^2 M_H \otimes \mathcal{O}_C(L)) = 0 \) is an open condition on the collection of triples. Since the vanishing condition is satisfied on \( C_0 \), the same follows for the general curve.

Finally, consider the case where \( s \) is even. In order to produce a nonspecial divisor \( H \) with the desired properties in this case, we will degenerate further, letting \( H \) become special. Thus we must work with incomplete linear series.
Given a divisor $H$ on a curve $C$ and a space of global sections $W$ that generates $\mathcal{O}_C(H)$, we define $M_{W,H}$ to be the kernel of the natural evaluation map:

$$M_{W,H} := \ker(W \otimes \mathcal{O}_C \rightarrow \mathcal{O}_C(H)).$$

It now suffices, for each even $s$, to find: A curve $C$ of genus $g$, a divisor $H$ of degree $d$ on $C$, and a space of sections $W$ of dimension $s + 1$ of $\mathcal{O}_C(H)$ such that

- $H^0(\wedge^2 M_{W,H} \otimes \mathcal{O}_C(L)) = 0$ for the general divisor $L$ on $C$ of degree $r - s$, and
- The triple $(C, H, W)$ is a flat limit of triples for which $H$ is nonspecial (equivalently, where $W = H^0(\mathcal{O}_C(H))$).

A candidate is constructed for us by the following result:

**Proposition 4.2.** Let $s \geq 2$ be even. For any sufficiently general plane curve $C_0$ of degree $s + 1$ there exists a flat irreducible family of degree smooth plane curves $C_t$, with special fiber $C_0$ and general fiber $C_\eta$, a family of line bundles $H_t$ of degree $\binom{s+1}{2}$, and a family of spaces $W_t \subset H^0(\mathcal{O}_{C_t}(H_t))$ such that:

- $W_t$ generates $\mathcal{O}_{C_t}(H_t)$,
- $H_t$ is nonspecial, and
- $M_{H_0} := \ker(W_0 \otimes \mathcal{O}_{C_0} \rightarrow \mathcal{O}_{C_0}(H_0))$ is the direct sum of $s/2$ copies of the rank 2 vector bundle $M$ which is the kernel of the evaluation map $H^0(\mathcal{O}_{C_0}(1)) \otimes \mathcal{O}_{C_0} \rightarrow \mathcal{O}_{C_0}(1)$, where $\mathcal{O}_{C_0}(1)$ induces the planar embedding.

**Proof.** We shall construct the family of curves $C_t$ and the family of divisors $H_t$ by constructing the family of vector bundles $E_t := \ker(W_t \otimes \mathcal{O}_{C_t} \rightarrow \mathcal{O}_{C_t}(H_t))$. On the generic fiber, we use the following (old) observation: If $B : \mathcal{O}_{\mathbb{P}^2}^{s+1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{s+1}$ is an $(s + 1) \times (s + 1)$ matrix of linear forms in 3 variables such that $f := \det B \neq 0$, and such that the ideal $I_s(B)$ of $s \times s$-minors of $B$ contains a power of the irrelevant ideal, then $\mathcal{H} := \ker B$ is a line bundle on the degree $s + 1$ curve $\{f = 0\}$ with $h^0(\mathcal{H}) = s + 1$, $h^1(\mathcal{H}) = 0$ (and thus $\deg \mathcal{H} = \binom{s+1}{2}$) by Riemann-Roch. The vector bundle $M_{\mathcal{H}}$ is of course the image of $B$.

For the special fiber, we proceed differently. Recall that $\widetilde{M} := \Omega_{\mathbb{P}^2}(1)$ is the image of the middle Koszul map

$$\rho : \wedge^2 H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^2},$$

induced by the $3 \times 3$ generic skew-symmetric matrix over the ring $S = \text{Sym}(H^0(\mathcal{O}_{\mathbb{P}^2}(1))) \cong k[x, y, z]$. If $C_0$ is any plane curve, then the bundle $M$ defined in the Proposition is simply $\widetilde{M}|_{C_0}$. We wish to define a general matrix of linear forms whose image is $\widetilde{M}$. For this (and for later purposes) the idea of a “generalized submatrix” of a matrix will be useful: by a generalized $p \times q$ submatrix of a matrix $C$ we mean simply a composition $PCQ$, where $P$ and $Q$ are scalar matrices, $P$ has $p$ rows, and $Q$ has $q$ columns. Generalized rows or columns of $C$ are generalized submatrices with one row or one column, respectively.

Now let $A$ be a (sufficiently general) generalized $(s + 1) \times (s + 1)$-submatrix of a $(3s/2) \times (3s/2)$-matrix inducing $\rho^{\oplus \frac{s}{2}}$. Notice that $\det A = 0$ since the module $\Omega_{\mathbb{P}^2}(1)\oplus \frac{s}{2}$ has only rank $s$. Let $B$ be a general $(s + 1) \times (s + 1)$-matrix with linear entries, and set $A_t := A + tB$. For $t \neq 0$, we set $f_t := \det A_t$, and for $t = 0$ we take $f_0$ to be the “limit”

$$f_0 := \lim_{t \to 0} \frac{\det(A + t \cdot B)}{t} = \sum_{i,j=1}^{s+1} b_{ij} |A_{ij}|,$$

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where \(b_{ij}\) are the entries of \(B\), while \(|A_{ij}|\) denotes the (signed) minor of \(A\) obtained by deleting row \(i\) and column \(j\).

**Proposition 4.3.** Set \(m = s/2\). For each generalized \((s + 1) \times (s + 1)\) submatrix \(A\) of \(\rho^{\oplus m}\), the ideal of \(s \times s\) minors of \(A\) may be written in the form \(I_s(A) = I_{m-1}(K_1) \cdot I_{m-1}(K_2) \cdot (x, y, z)^2\) for matrices of linear forms \(K_1, K_2\) of sizes \(m \times (m - 1)\) and \((m - 1) \times m\), respectively. Each pair of matrices \(K_1\) and \(K_2\) with linear entries arises for some generalized submatrix \(A\). Thus, for a general choice of \(A\), the ideal \(I_s(A)\) is a nonsaturated ideal of a reduced set of points, and the equation \(\{f = 0\}\) defines a general plane curve \(C\) of degree \((s + 1)\).

**Proof of Proposition 4.3.** To compute \(I_s(A)\) we make use of a special case of the structure theorem for finite free resolutions of Buchsbaum-Eisenbud [1974]. Consider the resolution obtained by taking the direct sum of \(m\) copies of the Koszul complex in 3 variables:

\[
0 \rightarrow S^m(-2) \xrightarrow{\kappa^*(-1)^{\oplus m}} S^{3m}(-1) \xrightarrow{\rho^{\oplus m}} S^{3m} \xrightarrow{\kappa^{\oplus m}} S^m(1) \rightarrow 0;
\]

to simplify notation, write it as

\[
0 \rightarrow F_3 \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0.
\]

Let \(r_i := \text{rank}(f_i)\), so that in particular \(\text{rank}(F_i) = r_i + r_{i+1}\), for all \(1 \leq i \leq 3\).

The structure theorem asserts the commutativity (up to a sign) of the diagram:

\[
\begin{align*}
\wedge^{r_3} F_2^* &= \wedge^{r_2} F_2 \xrightarrow{\wedge^{r_2} f_2} \wedge^{r_2} F_1 = \wedge^{r_1} F_1^* \\
\wedge^{r_3} F_3^* &\cong R \cong \wedge^{r_1} F_0^*
\end{align*}
\]

In other words, any minor of order \(r_2\) of \(f_2\) may be expressed as the product of complementary minors of orders \(r_1\) and \(r_3\) of \(f_1\) and \(f_3\), respectively.

The choice of the matrix \(A\) involves the choice of \(s + 1 = 2m + 1\) generalized rows and columns of the matrix defining \(\rho^{\oplus m}\), hence the choice of \(m - 1\) complementary columns of \(\kappa\) and \(m - 1\) complementary rows of \(\kappa^*\), respectively. We denote by \(K_1\) and \(K_2\) the \(m \times (m - 1)\) and \((m - 1) \times m\)-submatrices of \(\kappa\) and \(\kappa^*(-1)\) distinguished in this way.

Because of the structure of \(\kappa\), any \(m \times (m - 1)\) matrix with linear entries in \(S\) can be obtained as \(K_1\) through the appropriate choice of \((m - 1)\) generalized columns of \(\kappa\), and similarly for \(K_2\). In particular, \(K_i\) may be chosen to make \(I_{m-1}(K_i)\) be the ideal of any sufficiently general set of \(\binom{m}{2}\) points in the plane. Diagram (*) expresses the \((2m) \times (2m)\)-minors \(A_{ij}\) of \(A\) as the products of \(m \times m\) minors of \(\kappa\) and \(\kappa^*\) that contain \(K_1\) and \(K_2\), respectively. An \(m \times m\)-minor of \(\kappa\) containing \(K_1\) is a linear combination of the \((m - 1) \times (m - 1)\) minors of \(K_1\), with coefficients the elements of an arbitrary generalized
column of $\kappa$. Again because of the structure of $\kappa$ this column can be taken to be an arbitrary column of linear forms in $S$. Thus the ideal of $m \times m$-minors of $\kappa$ containing $K_1$ is $I_{m-1}(K_1) \cdot (x, y, z)$. As similar remarks hold for $K_2$, we have proven the first part of the Proposition.

If the choice of the generalized submatrix $A$ is general, then $K_1$ and $K_2$ will be general matrices of linear forms, hence their ideals of minors will be reduced ideals of distinct general sets of points in the plane, and the ideal $I_s(A) = I_{m-1}(K_1) \cdot I_{m-1}(K_2) \cdot (x, y, z)^2$ will be a nonsaturated ideal of the union of these two sets of points, as claimed.

Varying the matrix $B$ we obtain for $f$ any form of degree $(s+1)$ in $I_s(A)(x, y, z) = I_{m-1}(K_1) \cdot I_{m-1}(K_2) \cdot (x, y, z)^3$. Since $\binom{s+3}{2} > 2\binom{m}{2}$ the general curve $C_0$ of degree $s+1$ through two general sets of $\binom{m}{2}$ points in the plane is a general plane curve of degree $s+1$, concluding the argument.

Completion of the proof of Proposition 4.2. Let $C_t$ be defined by the equation $\{f_t = 0\}$, and let $E_t$ be the image of the restriction to $C_t$ of the morphism induced by the matrix $A + t \cdot B$. Let $H^{-1}_t$ be the kernel of the restriction of the matrix $(A + Bt)^*$ to $C_t$; we write the dual in the form $O_C(H_t) = H_t$, for a family of divisors $H_t$ (defined, for example, by the family of sections that are the images of the first basis vector of the target free module of $A + Bt$). Part $a)$ of the Proposition now follows from the definitions; part $b)$ follows from the remark at the beginning of the proof; and part $c)$ follows from the form of the matrix $A = A + 0 \cdot B$.

Continuation of the proof of Theorem 4.1. We adopt the notation of Proposition 4.2, but for simplicity we now set $C = C_0$. By Proposition 4.2, $C$ may be chosen to be a general plane curve of degree $s+1$. It suffices to show that $H^0(\wedge^2(M \otimes \mathcal{O}_C(L)) = 0$, where $L$ is a general divisor of degree $r - s$, and for this it is enough to show that both $H^0(\wedge^2 M \otimes \mathcal{O}_C(L)) = 0$, and $H^0(M \otimes M \otimes \mathcal{O}_C(L)) = 0$.

Two remarks will make the plausibility of this conclusion clear. First, $r \geq \binom{s+1}{2}$ so $\deg(L) = r - s \geq \binom{s}{2}$, and $g = \binom{s}{2}$ is the genus of $C$. Thus $\mathcal{O}_C(L)$ is a general line bundle in the Picard variety of $C$. Second, $\deg(M) = -(s+1)$, so $\chi(\wedge^2 M \otimes \mathcal{O}_C(L))$ and $\chi(M \otimes M \otimes \mathcal{O}_C(L))$ are both $\leq 0$. Thus each of the desired vanishings has the form: $H^0(F \otimes \mathcal{O}_C(L')) = 0$ with $F$ a vector bundle on $C$ with $\chi(F) = 0$, and $L'$ a general divisor of degree $\leq 0$. This condition obviously implies that the bundle $F$ must be semistable, and indeed Raynaud [1982] shows that the condition is equivalent to semistability when rank $F \leq 2$, and also when rank $F = 3$ on a general curve of a given genus. In fact his argument proves a little more:

**Theorem 4.4 (Raynaud).** Let $C$ be a general plane curve of any degree $\geq 3$. A vector bundle $F$ of rank $\leq 3$ on $C$ with $\chi(F) = 0$ is semistable iff $H^0(F \otimes \mathcal{O}_C(L')) = 0$ for the general line bundle $\mathcal{O}_C(L')$ of degree $0$ on $C$.

Discussion of Theorem 4.4. Raynaud [1982, §2] enunciates the result for general curves (not planar). However, his proof shows that if we replace “vector bundle” by “torsion-free sheaf”, then the truth of the Theorem for $C$ defines an open set in the moduli of stable curves. Furthermore, his proof shows that this open set includes every irreducible rational curve of arithmetic genus $g$, having exactly $g$ ordinary nodes. Since the the general map
from \( \mathbb{P}^1 \) into the plane has as image a curve with only ordinary nodes as singularities, these facts imply that the Theorem holds for a general plane curve.

The first of the necessary vanishings is immediate from the remarks above: Since a general line bundle of degree \( \leq g - 1 \) has no sections \( H^0(\wedge^2 M \otimes \mathcal{O}_C(L)) = H^0(\mathcal{O}_C(L - N)) = 0 \), where \( N \) is the divisor of the intersection of \( C \) with a line.

For the second vanishing, from the exact sequence

\[
0 \rightarrow \wedge^2 M \rightarrow M \otimes M \rightarrow \text{Sym}^2 M \rightarrow 0,
\]

together with the first vanishing result above, it suffices to show that \( H^0(\text{Sym}^2(M) \otimes \mathcal{O}_C(L)) = 0 \), and this puts us in the case of a bundle \( F = \text{Sym}^2(M) \otimes \mathcal{O}_C(L) \) of rank 3. An easy degree computation shows that \( \chi(F) \leq 0 \). We may now invoke Theorem 4.4 to conclude the argument if we can show that \( F \) is semistable, and by Hartshorne [1971], Gieseker [1979] it suffices to show that \( M \) itself is semistable. As \( M \) differs from the restriction to \( C \) of the tangent bundle of the projective plane only by twisting by a line bundle, the following elementary result completes the proof of Theorem 4.1:

**Proposition 4.5.** Let \( T = T_{\mathbb{P}^2} \) be the tangent bundle of the projective plane. If \( C \) is a smooth plane curve of degree \( m \geq 3 \), then \( T|_C \) is stable.

**Proof.** Suppose \( Q \) is a line bundle quotient of \( T|_C \). Since \( \text{deg}(T|_C) = 3m \), we must show that \( \text{deg}(Q) > 3m/2 \). But \( T(-1) \) is globally generated, so either \( Q(-1) = \mathcal{O}_C \), or \( Q(-1) \) defines a base point free linear series. Let \( e := \text{deg}(Q) - m \) be the degree of \( Q(-1) \); we must show that \( e > m/2 \).

Assume first that \( Q(-1) = \mathcal{O}_C \). Restricting the presentation of \( T \) to \( C \) we obtain maps

\[
\mathcal{O}_C(-1) \rightarrow \mathcal{O}_C^2 \rightarrow T|_C(-1) \rightarrow Q(-1) = \mathcal{O}_C
\]

with composition 0, where the last two maps are surjective. Since the restrictions of linear forms on \( \mathbb{P}^2 \) are still linearly independent on \( C \), this is a contradiction.

Thus we may suppose that \( Q(-1) \) defines a base point free linear series of degree \( e \). It follows at once that \( e \geq m - 1 > m/2 \). For the reader’s convenience we give the elementary proof: Since the genus of \( C \) is positive, we must have \( e \geq 2 \). If \( m = 3 \), then this is the desired result. On the other hand, if \( m \geq 4 \), then \( m - 2 < \binom{m - 1}{2} \), the genus of \( C \), so if \( e \leq m - 2 \) then \( Q(-1) \) is special. In other words the points in a divisor \( D \) in the linear series represented by \( Q \) impose dependent conditions on the canonical series \( \mathcal{O}_C(K_C) = \mathcal{O}_C(m - 3) \). But any finite scheme of length \( \leq m - 2 \) in the plane imposes independent conditions on forms of degree \( m - 3 \).
§5. Numerology of resolutions and failure of the Minimal Rank Conjecture.

In this section we derive first a lower bound for the graded betti numbers of the homogeneous coordinate ring of a scheme $\Gamma$ of $\gamma$ general points in $\mathbb{P}^r$, and then prove Theorem 0.1, providing counterexamples to the Minimal Resolution Conjecture. When the lower bound is achieved, we shall say that $\Gamma$ has expected betti numbers. It is well-known how to do the computation (it appears explicitly in the Queen’s University thesis of Anna Lorenzini as well as in Lorenzini [1987], [1993]), but for the reader’s convenience, and because we need details in a certain special case, we spell it out. Since all we use about $\Gamma$ is its Hilbert function, the same computation would work for any subscheme finite over $k$ imposing “as many conditions as possible” on forms of each degree; we call such a subscheme sufficiently general. We shall use the following elementary facts:

a) If $S_\Gamma$ is the homogeneous coordinate ring of a finite scheme $\Gamma$ of points in $\mathbb{P}^r$, and $S$ is the homogeneous coordinate ring of $\mathbb{P}^r$, and $x$ is a linear form not vanishing on any point in the support of $\Gamma$, then the graded betti numbers of $S_\Gamma$ as an $S$-module are the same as the graded betti numbers of $S_\Gamma/xS_\Gamma$ as an $R := S/x$-module.

b) Write $R = k[x_1, \ldots, x_r]$, set $m = (x_1, \ldots, x_r)$, and let $d$ be the largest integer such that $\gamma \geq \binom{r+d-1}{d-1}$. In other words, assuming $\Gamma$ is sufficiently general, $d$ is the smallest degree of a form contained in the homogeneous ideal $I_\Gamma$. We may write $S_\Gamma/xS_\Gamma$ in the form $R/I$ where $m^{d+1} \subset I \subseteq m^d$.

c) The module $R/I$ may be obtained as a “jump deformation” of the module $R/m^d \oplus m^d/I$. Thus the resolution of $R/I$ has all graded betti numbers $\geq$ the betti numbers in the resolution of $R/m^d \oplus m^d/I$.

We get the desired estimates by putting these things together with a knowledge of the free resolution of $R/m^d$, which may for example be described as an Eagon-Northcott complex (see Eisenbud [1995]). We write $\tilde{\beta}_{i,j}$ for the expected dimensions of the Koszul homology, and $\{n\}_+$ for $\max(n, 0)$. We set $\binom{r}{k} = 0$ for $k > r$.

**Proposition 5.1.** Let $\Gamma$ be a finite sufficiently general subscheme of $\mathbb{P}^r$ having degree $\gamma$ with

$$\binom{r + d - 1}{d - 1} \leq \gamma < \binom{r + d}{d},$$

and set $a := \gamma - \binom{r+d-1}{d-1} \geq 0$. The Koszul homology dimensions

$$\beta_{i,j}(I) = \dim_k(\text{Tor}_i^S(I_\Gamma, k))$$

in the “interesting” range $0 \leq i \leq r - 1$ satisfy:

a) $\beta_{i,j} = 0$ unless $j = i + d$ or $j = i + d + 1$;

b) $\binom{d+i-1}{i} \binom{r+d-1}{d+i} \beta_{i,i+d} \beta_{i,i+d+1} \geq \beta_{i,i+d} = \left\{ \binom{d+i-1}{i} \binom{r+d-1}{d+i} - a \binom{r}{i} \right\}_+ \bigg( \begin{array}{c} d+i-1 \\ i \end{array} \right) \left( \begin{array}{c} r+d-1 \\ d+i \end{array} \right) - a \binom{r}{i} \bigg\}_+$

$$a \binom{r}{i+1} \geq \beta_{i,i+d+1} \geq \beta_{i,i+d+1} = \left\{ a \binom{r}{i+1} - \binom{d+i}{i+1} \binom{r+d-1}{d+i+1} \right\}_+.$$
For simplicity, and because it is the case we shall use, we now specialize to the case where \( d = 2 \), so that the ideal of \( \Gamma \) is generated by quadrics and cubics.

**Corollary 5.2.** Let \( \Gamma \) be a finite sufficiently general subscheme of \( \mathbb{P}^r \) having degree \( \gamma \) with \( r + 1 \leq \gamma < \binom{r+2}{2} \). The expected dimensions of the Koszul homology of \( \Gamma \) are:

\[
\tilde{\beta}_{i,i+2} = \left\{ (i+1) \binom{r+2}{i+2} - \gamma \binom{r}{i} \right\}_+, \quad \tilde{\beta}_{i,i+3} = \left\{ \gamma \binom{r}{i+1} - (i+2) \binom{r+2}{i+3} \right\}_+, \quad i = 0, r - 1
\]

In particular \( \tilde{\beta}_{i,i+2} \neq 0 \) iff \( i < \frac{(r+2)(r+1)}{\gamma} - 2 \). Furthermore \( \tilde{\beta}_{i,i+3} \neq 0 \) iff \( i \geq \frac{(r+2)(r+1)}{\gamma} - 3 \).

**Proof.** Arithmetic, starting from the previous result.

Thus the "expected" shape minimal free resolution of \( \mathcal{I}_\Gamma \) is

| degree | 0 | 1 | 2 |
|--------|---|---|---|
|        | 1 | * | * |
|        | * | ? | * |
|        | * | * | * |
|        |   |   |   |
|        |   |   |   |

(where not both of the "?"s in the above display are non-zero!)

As a an easy corollary of Theorem 4.1 on linear exactness we obtain now the result announced in the introduction.

**Proof of Theorem 0.1.** For \( r \) and \( s \) in the given range the complex \( E_{s-1}^{-1}(\mu) \) defined at the beginning of section §4 is linearly exact. Moreover, the twisted complex \( E_{s-1}^{-1}(\mu)(r+2) \) maps monomorphically onto a direct summand of the dual of the minimal free resolution of \( \mathcal{I}_\Gamma \).

On the other hand, Corollary 5.2 gives as expected graded betti number \( \tilde{\beta}_{(r-s-1),(r-s+2)} \) for \( \mathcal{I}_\Gamma \)

\[
\tilde{\beta}_{(r-s-1),(r-s+2)} = \left\{ \frac{2k+4-s^2+s}{s^2-s+2k+4} \binom{s+1}{s}, \binom{2k+4}{k} \right\}_+, \quad \text{whereas the last (i.e., the } s\text{-th) syzygy module in the complex } E_{s-1}^{-1}(\mu) \text{ has rank}
\]

\[
\text{rank } E_{s-1}^{-1}(\mu) = \binom{s+k}{k}
\]

The theorem follows since

\[
s^2-s+2k+4 \geq s^2-s > 0 \quad \text{and} \quad 2k+4-s^2+s \leq 3s+4-s^2 \leq 0, \quad \text{for all } s \geq 4, 0 \leq k \leq s,
\]

while

\[
\frac{(2k-2)}{2k+10} \cdot \binom{k+6}{k+3} < \binom{k+3}{3}
\]

only for \( r = k + 6 \in \{6, 7, 8\} \).
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Author Addresses:
David Eisenbud
Department of Mathematics, Brandeis University, Waltham MA 02254
eisenbud@math.brandeis.edu

Sorin Popescu
Department of Mathematics, Brandeis University, Waltham MA 02254
current address:
Department of Mathematics, Columbia University, New York, NY 10027
psorin@math.columbia.edu