Quantifying Entanglement with Coherence

Neha Pathania · Tabish Qureshi

Abstract Quantifying entanglement is a work in progress which is important for the active field of quantum information and computation. A measure of bipartite pure state entanglement is proposed here, named entanglement coherence, which is essentially the normalized coherence of the entangled state in its Schmidt basis. Its value is 1 for maximally entangled states, and 0 for separable states, irrespective of the dimensionality of the Hilbert space. So a maximally entangled state is also the one which is maximally coherent in its Schmidt basis. Quantum entanglement and quantum coherence are thus intimately connected. Entanglement coherence turns out to be closely related to the unified entropy of the reduced state of one of the subsystems. Additionally it is shown that the entanglement coherence is closely connected to the Wigner-Yanase skew information of the reduced density operator of one of the subsystems, in an interesting way.

Keywords Quantum entanglement, Entanglement measures

1 Introduction

Much after the advent of quantum mechanics, it took a paper by Einstein, Podolsky and Rosen [1] to point out its highly nonclassical and nonlocal nature. The word entanglement came from the observation by Schrödinger that after two independent systems interact, they cannot generically be described individually by a state of their own [2]. Entanglement became an active subject of research because of the “spookiness” it implied [3]. Entanglement leads to certain effects which cannot be simulated by classical systems, a fact which firmly positions it as an essentially quantum phenomenon [4]. The demonstration of usefulness of entanglement in quantum cryptography [5], and quantum teleportation [6] led to the development of entanglement as a subject in itself [7]. Huge amount of work has been carried out in the development of the theory of entanglement. Three broad aspects, which the research on entanglement deals with, are [7] (i) detecting entanglement theoretically and experimentally, (ii) preventing and possibly reversing the loss of entanglement in various processes, and (iii) characterizing and quantifying entanglement. It is this last aspect which is the subject of the present investigation.

An essential ingredient in the quantification of entanglement is defining a measure of entanglement. A lot of research effort has already been made to that end [8, 9], and we can at best only mention some of the more popular measures in the passing. Entanglement cost, denoted by $E_C(\rho)$ is a measure of how efficiently one can convert...
a maximally entangled state to the given state $\rho$ [10–12]. Distillable entanglement, denoted by $D(\rho)$ is another measure which deals with the reverse of the process involved in entanglement cost [10]. It deals with how efficiently one can generate a maximally entangled state from a given state $\rho$. Apart from these two, there are a host of other entanglement measures based on a different approach to the problem. To name a few, entanglement of formation $E_F(\rho)$ [13], relative entropy of entanglement [14], logarithmic negativity [15]. We particularly mention a measure introduced by Hill and Wootters for two qubits, called concurrence [16], and its generalization to arbitrary Hilbert space dimension, called $I$-concurrence [17, 18], as we will discuss it in the context of the result of this work.

An aspect of all the entanglement measures, that we would like to draw attention to, is that while they yield value 0 for disentangled states, there is generally no fixed upper bound for maximally entangled states of arbitrary Hilbert space dimension. We feel that normalization is a desirable feature of a good entanglement measure because the measure should give an indication of how close a state is to a maximally entangled state, and also if a given state is maximally entangled or not. This is, of course, a matter of convention, and any entanglement measure can be appropriately normalized.

2 Coherence of an entangled state

Let us consider a bipartite system composed of two subsystems $a$ and $b$, where $a$ has the smaller Hilbert space of the two, in case they are unequal. Our main result is that for a bipartite entangled state $|\psi\rangle$, its entanglement coherence is the normalized quantum coherence of the entangled state in its Schmidt basis. It can also be represented in terms of the reduced density operator of one of two subsystems. If $\rho = \text{Tr}_b (|\psi\rangle \langle \psi|)$ is the reduced density operator for the subsystem $a$, for the bipartite entangled state $|\psi\rangle$, then its entanglement coherence is given by

$$C_E = \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - 1 \right].$$

where $n$ is the dimensionality of the Hilbert space of $a$.

We start by writing a bipartite entangled state $|\psi\rangle$ in its Schmidt decomposed form [19]

$$|\psi\rangle = \sum_{i=1}^{n} \sqrt{p_i} |a_i\rangle |b_i\rangle,$$

where $|a_i\rangle , |b_i\rangle$ represent the basis states of the two sub-systems, respectively, $n$ is the dimensionality of the smaller Hilbert space of the two, and $\sqrt{p_i}$ are real and positive numbers called Schmidt coefficients. The full density operator for the state (2) is given by

$$\rho_F = |\psi\rangle \langle \psi| = \sum_{j,k=1}^{n} \sqrt{p_j p_k} |a_j\rangle \langle a_k| |b_j\rangle \langle b_k|.$$
in a slightly different way:
\[ C_E \equiv \frac{1}{n-1} \sum_{j \neq k} |(\rho_F)_{jk}| = \frac{1}{n-1} \sum_{j \neq k} \sqrt{p_j p_k}. \quad (4) \]

where \( n \) is chosen to be the dimension of the Hilbert space of \( a \), in order to normalize \( C_E \), since we are using a specific basis here. In the original definition, coherence is normalized by using \( n \) as the dimension of the Hilbert space of the full system [21,22]. If \( a \) and \( b \) are disentangled, only one Schmidt coefficient is nonzero. Consequently, for disentangled states \( C_E = 0 \). The maximally entangled state is generally defined as the state which has all Schmidt coefficients equal [9]. Thus \( \sqrt{p_i} = 1/\sqrt{n}, i = 1, n \) denotes a maximally entangled state. It is straightforward to see that for a maximally entangled state, \( C_E = \frac{1}{n-1} \sum_{j \neq k} \frac{1}{n} = 1 \). The reason for the choice of \( n \) in the prefactor \( \frac{1}{n-1} \) should be obvious now. A different choice would not give \( C_E = 1 \) for a maximally entangled state. In general, an entangled state will have \( C_E \) between 0 and 1. This looks like a good measure of entanglement, and is normalized too. We can now define entanglement coherence \( C_E \) as the normalized coherence of the entangled state in its Schmidt basis. It is an elegant measure of entanglement in that it is always bounded by 0 and 1, irrespective of the Hilbert space dimension, and is intimately connected to another fundamental property of quantum states, namely quantum coherence. It is rather satisfying to observe that there has been earlier work which goes in the reverse direction, namely in using entanglement to measure coherence [23]. Here we use coherence to measure entanglement. Since entanglement coherence arises from the Schmidt basis, it is obvious that it is applicable only to pure state entanglement, and not to mixed entanglement.

The reduced density operator for one of the subsystems is given by
\[
\rho = \text{Tr}_b (|\psi\rangle \langle \psi|) = \sum_{j=1}^{n} p_j |a_j\rangle \langle a_j| = \text{Tr}_a (|\psi\rangle \langle \psi|) = \sum_{j=1}^{n} p_j |b_j\rangle \langle b_j|, \quad (5)
\]

and the density matrix is the same for both. It would be nice to write \( C_E \) in a basis independent manner. For that we first notice that since \( \rho \), given by (5), is a diagonal matrix, \( \sqrt{\rho} \) can be simply written as
\[
\sqrt{\rho} = \sum_{j=1}^{n} \sqrt{p_j} |a_j\rangle \langle a_j|. \quad (6)
\]

Entanglement coherence \( C_E \), given by (4), can now be manipulated as follows
\[
C_E = \frac{1}{n-1} \sum_{j \neq k} \sqrt{p_j p_k} = \frac{1}{n-1} \left( \left[ \sum_{j} \sqrt{p_j} \right]^2 - \sum_{j} p_j \right) \\
= \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - \text{Tr} \rho \right] = \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - 1 \right]. \quad (7)
\]

Trace being basis independent, one arrives at (1). Thus, for evaluating \( C_E \) one need not go about finding the Schmidt basis for the entangled state. One can find the
reduce density operator of one subsystem, and employ (1) to evaluate $C_E$. Notice that the reduced density operator for the subsystem $b$, 

$$
\rho' = \text{Tr}_a (|\psi\rangle \langle \psi|) = \sum_{j=1}^n p_j |b_j\rangle \langle b_j|, \quad (8)
$$

yields the same diagonal matrix in the basis $\{|b_i\rangle\}$, as $\rho$ yields in the basis $\{|a_i\rangle\}$. Then it follows that

$$
C_E = \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - 1 \right] = \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho'})^2 - 1 \right]. \quad (9)
$$

Notice that $C_E$ is also a normalized measure of the mixedness of the reduced density operator $\rho$. The purity of $\rho$ implies $\rho^2 = \rho$, which in turn implies

$$
\sqrt{\rho} = \rho. \quad (10)
$$

Taking trace of both sides, we arrive at the result $C_E \equiv \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - 1 \right] = 0$. For a maximally mixed $\rho$, $\text{Tr} \sqrt{\rho} = \sqrt{n}$, which leads to $C_E \equiv \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - 1 \right] = 1$.

Next we move on to finding the expression for $C_E$ in an arbitrary basis of the subsystem $a$.

$$
C_E = \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - 1 \right] = \frac{1}{n-1} \left[ (\text{Tr} \sqrt{\rho})^2 - \text{Tr} \rho \right]
$$

$$
= \frac{1}{n-1} \left[ \left( \sum_{i=1}^n \langle \psi_i | \sqrt{\rho} | \psi_i \rangle \right)^2 - \sum_{i,j=1}^n \langle \psi_i | \sqrt{\rho} | \psi_j \rangle \langle \psi_j | \sqrt{\rho} | \psi_i \rangle \right]
$$

$$
= \frac{1}{n-1} \sum_{i \neq j} \left[ \langle \psi_i | \sqrt{\rho} | \psi_j \rangle \langle \sqrt{\rho} | \psi_j \rangle - | \langle \psi_i | \sqrt{\rho} | \psi_j \rangle |^2 \right]. \quad (11)
$$

Thus we arrive at the following compact expression for entanglement coherence, which works for any basis

$$
C_E = \frac{1}{n-1} \sum_{j \neq k} \left( \sqrt{\rho}_{jj} \sqrt{\rho}_{kk} - |\sqrt{\rho}_{jk}|^2 \right). \quad (12)
$$

where $\sqrt{\rho}_{jk}$ are the elements of the square-root of the reduced density matrix in the chosen basis, and $n$ is the dimensionality of the Hilbert space of $a$. It is interesting to compare entanglement coherence with the well known entanglement measure “I-concurrence.” The square of I-concurrence is given by [17,18]

$$
E_a^2 = 2 \sum_{j \neq k} \left( \rho_{jj} \rho_{kk} - |\rho_{jk}|^2 \right) = 2 \left[ 1 - \text{Tr}(\rho^2) \right]. \quad (13)
$$

This expression can be compared with (12) and (1). The factor of 2 in (12) makes sure that the value of $E_a^2$ is 1 for two maximally entangled qubits. However, there is
no fixed value for maximally entangled states in arbitrary Hilbert space dimension. For the entangled state (3), square of I-concurrence is given by

\[ E_a^2 = 2 \sum_{j \neq k} p_j p_k = 2 \sum_{j \neq k} |(\rho_F)_{jk}|^2. \]  

(14)

Now it has been shown that for any density operator \( \rho \), the quantity \( \sum_{j \neq k} |\rho_{jk}|^2 \) cannot be a good measure of coherence, as there exist incoherent operations under which this quantity increases [20]. Here we are not concerned if this quantity is a good measure of coherence or not, but this result implies that \( E_a^2 \) given by (14) can increase under certain incoherent operations. In the light of this result, one may need to reassess if I-concurrence is a good entanglement measure, and under what kind of incoherent operations it can increase.

Although it is difficult to say which entanglement measure is better than the others, it may be useful to compare the behavior of entanglement coherence with other measures for some example entangled states. Let us consider the following bipartite entangled state of two qutrits

\[ |\Psi_1\rangle = \sqrt{\frac{2}{3}} \sin \theta |+_1+_2 + \sqrt{\frac{2}{3}} \cos \theta |0_10_2 + \sqrt{\frac{1}{3}} |-_1-_2, \]  

(15)

where \(|+_\rangle, |0\rangle, |-_\rangle\) are orthonormal states of a qutrit, and \(0 \leq \theta \leq \pi/2\). The state is maximally entangled for \( \theta = \pi/4 \). Another state that can be considered is

\[ |\Psi_2\rangle = \sqrt{\frac{x}{3}} |+_1+_2 + \sqrt{\frac{x}{3}} |0_10_2 + \sqrt{1 - \frac{2x}{3}} |-_1-_2, \]  

(16)

where \(0 \leq x \leq 1\). For \(x = 0\) the state is disentangled, and for \(x = 1\) it is maximally entangled. Entanglement coherence, I-concurrence and entropy of entanglement are plotted in Fig. 1.

One may wonder if there is a reason why coherence in the Schmidt basis, of all the joint basis sets, gives a good measure of entanglement. Here we try to explore this
question. Consider a joint basis for the two subsystems, \( \{|\alpha_i\rangle \otimes |\beta_j\rangle\} \). Coherence of the full state \( \rho_F = |\Psi\rangle\langle\Psi| \) in this basis can be written as

\[
C_{\alpha\beta}(\rho_F) = \sum_{i \neq i'} \sum_{j \neq j'} |\langle \alpha_i | \rho_F | \alpha_{i'} \beta_j \rangle|^2 + \sum_{i \neq i'} \sum_{j \neq j'} |\langle \alpha_i | \rho_F | \alpha_{i'} \beta_{j'} \rangle|^2
\]

\[
+ \sum_{i \neq i'} \sum_{j \neq j'} |\langle \alpha_i | \rho_F | \alpha_{i'} \beta_{j'} \rangle|^2 \]

\[
= \sum_{i \neq i'} |\langle \alpha_i | \rho | \alpha_{i'} \rangle|^2 + \sum_{j \neq j'} |\langle \beta_j | \rho' | \beta_{j'} \rangle|^2 + \sum_{i \neq i'} \sum_{j \neq j'} |\langle \alpha_i | \rho_F | \alpha_{i'} \beta_{j'} \rangle|^2, \quad (17)
\]

where \( \rho \) and \( \rho' \) are the reduced density operators of the subsystems \( a \) and \( b \), respectively. For a joint basis in which both \( \rho \) and \( \rho' \) are diagonal, the first two terms in the above expression are zero. Schmidt basis is precisely that basis. In addition to that, in the third sum in the above expression, all terms where \( i \neq j \) and \( i' \neq j' \) are zero, in the Schmidt basis. So, it may be that, of all the joint basis sets, the Schmidt basis gives the minimum coherence for the entangled state.

### 3 Connection with Unified Entropy

For a given density operator \( \rho \), the unified entropy, introduced by Hu and Ye [24], is defined as

\[
S^s_r(\rho) = \frac{1}{(1 - r)s} \left[ (\text{Tr}(\rho^r))^s - 1 \right], \quad (18)
\]

where \( r > 0, r \neq 1 \), and \( s \neq 0 \). It is a family of entropies that yields various quantum entropies as particular or limiting cases. From (1), one can see that if \( \rho \) is the reduced density operator for one of the subsystems of an entangled state, the entanglement coherence of the entangled state is related to the unified entropy of one of the subsystems as

\[
C_E = \frac{1}{n-1} S^2_{r/2}(\rho) = \frac{1}{n-1} \left[ (\text{Tr}(\rho^{r/2}))^2 - 1 \right]. \quad (19)
\]

The reduced state of one of the two subsystems is expected to have a non-zero von Neumann entropy, if the state of the system is entangled. However, it is interesting to observe that the coherence of the entangled state in its Schmidt basis is virtually the same as a particular case of the unified entropy of the reduced state of one of the subsystems. Hu and Ye have presented several results for various properties of unified entropy [24], many of which will apply to entanglement coherence too.

### 4 Connection with skew information

In 1963 Wigner and Yanase introduced the concept of skew information, which is believed to quantify the quantum part of the uncertainty of an observable in a mixed state [25]. For a mixed state, the usual measure of uncertainty, the variance, incorporates both quantum and classical uncertainty. The skew information of an observable \( A \), in a state \( \rho \) is defined as [25]

\[
I(\rho, A) := -\frac{1}{2} \text{Tr} [\sqrt{\rho}, A]^2 = \text{Tr} \rho A^2 - \text{Tr} \sqrt{\rho} A \sqrt{\rho} A, \quad (20)
\]
where the square brackets denote the commutator. The skew information is useful when $\rho$ is mixed. For a pure state ($\sqrt{\rho} = \rho$) the skew information reduces to the usual variance $\langle A^2 \rangle - \langle A \rangle^2$. Skew information has emerged as an interesting and useful tool \[26,27\].

4.1 Basis optimization

For mixed states, one may want to know what is the maximum coherence of the state, for any basis. To investigate the basis dependence, one can define a coherence based on the Wigner-Yanase skew information, specific to a basis. Then one can ask which basis maximizes it \[28\]. Given a basis $\{|k\rangle\}$ for the subsystem $a$, we can calculate the skew information for a projection operator $|k\rangle \langle k|$ as

$$I(\rho, |k\rangle \langle k|) = -\frac{1}{2} \text{Tr} [\sqrt{\rho}, |k\rangle \langle k|]$$

(21)

where $\rho$ is the reduced density operator of (say) the subsystem $a$. We define a skew-information based coherence as

$$C_I = \sum_{k=1}^{n} I(\rho, |k\rangle \langle k|),$$

(22)

where the sum goes over all states of the basis. Because of the sum, this quantity depends on the whole basis. The skew information normally depends on one particular observable. Here we can consider the reduced density operator of our entangled state $\rho$. So, this quantity can be treated as a basis dependent measure of coherence of the state $\rho$. Next we ask, which basis maximizes this coherence, i.e., maximizes the sum of skew informations. This skew-information based coherence can be written as \[28\]

$$\max_{\{\{k\}\}} C_I = \max_{\{\{k\}\}} \frac{1}{n} \left( \sum_{k=1}^{n} \left( \text{Tr}(\rho(|k\rangle \langle k|)^2) - \text{Tr}(\sqrt{\rho}|k\rangle \langle k|\sqrt{\rho}|k\rangle \langle k|) \right) \right)$$

$$= \max_{\{\{k\}\}} \frac{1}{n} \left( \sum_{k=1}^{n} \left( |k\rangle \langle k| \rho \langle k| - \langle k| \sqrt{\rho} |k\rangle \langle k| \sqrt{\rho} \right) \right)$$

$$= 1 - \min_{\{\{k\}\}} \frac{1}{n} \left( \sum_{k=1}^{n} \langle k| \sqrt{\rho} |k\rangle \right)^2$$

$$= 1 - \frac{1}{n} \left( \sum_{i=1}^{n} \sqrt{p_i} \right)^2$$

(23)

where $p_i$ are the eigenvalues of $\rho$. Since $\sum_{i=1}^{n} \sqrt{p_i} = \text{Tr} \sqrt{\rho}$, the above result can written as

$$\max_{\{\{k\}\}} C_I = 1 - \frac{1}{n} \left( \text{Tr} \sqrt{\rho} \right)^2 = \frac{n-1}{n} (1 - C_E)$$

$$C_E = 1 - \frac{n}{n-1} \max_{\{\{k\}\}} C_I.$$  

(24)
This result says that the entanglement coherence of an entangled state is closely related to optimal skew-information based coherence of the reduced state of the subsystem $a$. Intuitively, stronger the entanglement between the two subsystems, smaller will be any measure of coherence of the reduced state of one of the subsystems. The skew-information based coherence, of one of the subsystems, has a closer connection with the entanglement of the composite system.

4.2 Quantum uncertainty

In the following we connect the entanglement coherence to another quantity based on the skew information, which has been studied earlier. Now, the skew information given by (20) is dependent on the observable $A$. If one is looking for an inherent property of the state, which is not tied down to one observable, one can carry out the following procedure. If the Hilbert space of the system is $n$-dimensional, the set of all observables form another Hilbert space which is $n^2$–dimensional, if one defines the inner product of two observables $A$ and $B$ as $\langle A, B \rangle := \text{Tr} AB$. One can define a basis of observables on this $n^2$–dimensional Hilbert space as $X_1, X_2, X_3, \ldots, X_{n^2}$. A quantum uncertainty, for a mixed state $\rho$, can then be defined as the sum of the skew informations of all these $n^2$ observables [29]

$$Q(\rho) := \sum_{i=1}^{n^2} I(\rho, X_i). \quad (25)$$

It can be shown that $Q(\rho)$ is independent of the particular basis $\{X_i\}$. Consequently this quantum uncertainty for a mixed state can be evaluated to yield [29]

$$Q(\rho) = \sum_{i=1}^{n^2} I(\rho, X_i) = n - (\text{Tr} \sqrt{\rho})^2. \quad (26)$$

Using (1) and (26), one then finds

$$C_E = 1 - \frac{1}{n-1} Q(\rho). \quad (27)$$

Thus one finds that the entanglement coherence of an entangled state is closely connected to the skew information based quantum uncertainty of the reduced density operator of one of the subsystems.

In general, the uncertainty of a mixed state has two quite different origins. One is classical mixing, and the other is quantum randomness. The quantum uncertainty $Q(\rho)$ is supposed to represent the latter. If two subsystems are maximally entangled, the reduced state of one of the two is not expected to have any quantum part of uncertainty. The subsystem will look like a classically mixed state. On the other extreme, if the two subsystems are disentangled, the reduced state of one of the two will be pure, and will have maximal quantum uncertainty, and no classical mixedness. The relation (27) quantifies this connection between entanglement and the quantum uncertainty of one of the subsystems.
5 Discussion and Conclusion

One might ask if the measure, entanglement coherence, which is defined for pure entangled states, can be extended to mixed entangled states. One way to extend the definition is by a convex roof construction [7]

\[
C_m(\rho) = \min_{\{q_k, |\Psi_k\rangle\}} \sum_k q_k C_E(|\Psi_k\rangle),
\]

where \(\{q_k, |\Psi_k\rangle\}\) is a decomposition of the mixed state density operator \(\rho\) to pure states \(\rho = \sum_k q_k |\Psi_k\rangle\langle \Psi_k|\), and \(C_E(|\Psi_k\rangle)\) is the entanglement coherence corresponding to the pure entangled state \(|\Psi_k\rangle\). A method of evaluating such convex roof entanglement measures was recently proposed [30].

In conclusion, we have shown that normalized coherence of an entangled bipartite state, in the Schmidt basis, can be considered a good measure of the degree of entanglement of the two systems. It uncovers an interesting connection between and entanglement and coherence, which are separately studied interesting properties of quantum systems. The normalized coherence of an entangled state is also interestingly connected to the quantum uncertainty and optimal coherence of the reduced density operator of one of the subsystems, defined through Wigner-Yanase skew information. It is also intimately connected to the unified entropy. Ramifications of these connections may be explored further.

Acknowledgments

Neha Pathania acknowledges financial support from the Department of Science and Technology, through the Inspire Fellowship (registration code IF180414).

References

1. A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. 47, 777-780 (1935).
2. E. Schrödinger, Die gegenwärtige Situation in der Quantenmechanik Naturwiss. 23, 807 (1935).
3. J.S. Bell, On the Einstein Podolsky Rosen paradox. Physics 1, 195 (1964).
4. D. Paneru, E. Cohen, R. Fickler, R.W. Boyd, E. Karimi, Entanglement: quantum or classical? Rep. Prog. Phys. 83, 064001 (2020).
5. A.K. Ekert, Quantum cryptography based on Bell’s theorem. Phys. Rev. Lett. 67, 661 (1991).
6. C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W.K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels. Phys. Rev. Lett. 70, 1895 (1993).
7. A.K. Ekert, Quantum cryptography based on Bell’s theorem. Phys. Rev. Lett. 67, 661 (1991).
8. D. Bruß, Characterizing Entanglement. J. Math. Phys. 43, 4237 (2002).
9. Entanglement measures, M.B. Plenio, S.S. Virmani In: D. Bruß, G. Leuchs (eds) Quantum Information. Wiley-VCH Verlag (2016).
10. C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, W.K. Wootters, Mixed-state entanglement and quantum error correction. Phys. Rev. A 54, 3824 (1996).
11. P.M. Hayden, M. Horodecki, and B.M. Terhal, The asymptotic entanglement cost of preparing a quantum state. J. Phys. A 34, 6891 (2001).
12. X. Wang, M.M. Wilde, Cost of quantum entanglement simplified. Phys. Rev. Lett. 125, 040502 (2020).
13. W.K. Wootters, Entanglement of formation of an arbitrary state of two qubits. *Phys. Rev. Lett.* **80**, 2245 (1998).

14. V. Vedral, M.B. Plenio, M.A. Rippin, P.L. Knight, Quantifying entanglement. *Phys. Rev. Lett.* **78**, 2275 (1997).

15. M. B. Plenio, Logarithmic negativity: a full entanglement monotone that is not convex. *Phys. Rev. Lett.* **95**, 090503 (2005).

16. S. Hill, W.K. Wootters, Entanglement of a pair of quantum bits. *Phys. Rev. Lett.* **78**, 5022 (1997).

17. P. Rungta, V. Buzek, C.M. Caves, M. Hillery, G.J. Milburn, Universal state inversion and concurrence in arbitrary dimensions. *Phys. Rev. A* **64**, 042315 (2001).

18. V.S. Bhaskara, P.K. Panigrahi, Generalized concurrence measure for faithful quantification of multiparticle pure state entanglement using Lagrange’s identity and wedge product. *Quantum Inf. Process.* **16**, 118 (2017).

19. E. Schmidt, Zur Theorie der linearen und nichtlinearen Integralgleichungen. *Math. Ann.* **63**, 433 (1907).

20. T. Baumgratz, M. Cramer, M. B. Plenio, Quantifying coherence. *Phys. Rev. Lett.* **113**, 140401 (2014).

21. T. Qureshi, Coherence, interference and visibility. *Quanta* **8**, 24 (2019).

22. M.N. Bera, T. Qureshi, M.A. Siddiqui, A.K. Pati, Duality of quantum coherence and path distinguishability. *Phys. Rev. A* **92**, 012118 (2015).

23. A. Streltsov, U. Singh, H.S. Dhar, M.N. Bera, G. Adesso, Measuring quantum coherence with entanglement. *Phys. Rev. Lett.* **115**, 020403 (2015).

24. X. Hu, Z. Ye, Generalized quantum entropy *J. Math. Phys.*, **47**, 023502 (2006).

25. E. P. Wigner and M. M. Yanase, Information contents of distributions. *Proc. Nat. Acad. Sci. USA*, **49**, 910 (1963).

26. S. Luo, Wigner-Yanase skew information and uncertainty relations. *Phys. Rev. Lett.*, **91**, 180403 (2003).

27. M. Banik, P. Deb, S. Bhattacharya, Wigner–Yanase skew information and entanglement generation in quantum measurement. *Quantum Inf. Process.* **16**, 97 (2017).

28. C-s. Yu, S-r. Yang, B-q. Guo, Total quantum coherence and its applications. *Quantum Inf. Process.*, **15**, 3773 (2016).

29. S. Luo, Quantum uncertainty of mixed states based on skew information. *Phys. Rev. A*, **73**, 022324 (2006).

30. G. Tóth, T. Moroder, O. Gühne, Evaluating convex roof entanglement measures. *Phys. Rev. Lett.*, **114**, 160501 (2015).