CENTRAL VALUES OF L-FUNCTIONS OVER CM FIELDS

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Abstract. In the paper we prove an explicit formula for the central values of certain Rankin L-functions. These L-functions are L-functions of Hilbert newforms over a totally real field F, twisted by unitary Hecke characters of a totally imaginary quadratic extension of F. The formula generalizes our former result on L-functions twisted by finite characters. It also eliminates the restriction on ramifications of Popa’s paper.

1. Introduction

In this paper we will prove an explicit formula for central values of certain Rankin L-functions. The setting is as follows. Let $f$ be a Hilbert newform over a totally real field $F$ with trivial central character and has weight $(2k_1, 2k_2, \cdots, 2k_d)$. Suppose the level of $f$ is $N$. Let $\chi$ be a unitary character on $\mathbb{A}_K^\times/K^\times \mathbb{A}_F^\times$ of conductor $c(\chi)$. Here $K$ is a totally imaginary quadratic extension of $F$, $\mathbb{A}_F^\times$ and $\mathbb{A}_K^\times$ are the idele groups of $F$ and $K$ respectively. Let $\pi$ and $\pi_\chi$ be the automorphic representations of $GL_2(\mathbb{A}_F)$ associated to $f$ and $\chi$ respectively, then we use
$L(s, \pi \times \pi_{\chi})$ to denote the Rankin-Selberg convolution of $L(s, \pi)$ and $L(s, \pi_{\chi})$. The $L$-function $L(s, \pi \times \pi_{\chi})$ has a functional equation of the form $L(s, \pi \times \pi_{\chi}) = \epsilon(s, \pi \times \pi_{\chi})L(1-s, \pi \times \pi_{\chi})$.

**Assumption.** We assume that $2$, $N$ and $d_{K/F}$ (the absolute relative discriminant of $K/F$) are co-prime to each other.

Under this assumption the sign $\epsilon(1/2, \pi_f \times \pi_{\chi})$ is given by $(-1)^{|\Sigma|}$, and $\Sigma$ is the following set of places of $F$

$$\Sigma = \Sigma_1 \cup \{v|N \text{ such that } \omega_v(N) = -1\},$$

where $\Sigma_1$ is a set of archimedean places determined by weights of $\chi$ and $f$ (see the following paragraph), $\omega$ is the quadratic character of $F^\times \backslash A^\times$ associated to $K/F$. In this paper we will find an explicit formula for $L(1/2, \pi \times \pi_{\chi})$ when the sign of the functional equation is $+1$, namely when $|\Sigma|$ is even.

To determine $\Sigma_1$ we assume that the character $\chi$ has the infinity type

$$\chi_{\infty} = \sum_{\tau \in \Phi} r_{\tau} \tau + \sum_{\rho} r_{\rho \tau} \rho \tau,$$

where $\Phi$ is a fixed CM type of $K$, $\rho$ is the complex conjugation. Since $\chi$ is unitary we have and assume $-r_{\tau} = r_{\rho \tau} > 0$. Let $r_v = r_{\rho \tau}$ if $v$ is the infinite place of $F$ under $\tau$, then $\Sigma_1$ is the set of $v$'s such that $r_v < k_v$. The rest subset of infinite places of $F$ is denoted by $\Sigma_2$.

If $|\Sigma|$ is even there is a unique (up-to isomorphism) quaternion algebra $B$ over $F$ such that $B$ is ramified exactly over $\Sigma$. The CM field $K$ can be embedded in $B$ and the embedding will be fixed from now on. It is well-known that there is exactly one irreducible automorphic representation $\pi_B$ of $B^\times \backslash A^\times$, such that $\pi_B$ and $\pi$ are related by Jacquet-Langlands correspondence.

We now state a short version of our main theorem. For the complete statement and notations see Theorem 2 in Section 4.

**Theorem 1.** The central value of $L(s, \pi \times \pi_{\chi})$ is given by

$$L(1/2, \pi \times \pi_{\chi}) = C \cdot \frac{(\varphi^*, \varphi^*)}{(\varphi_B^*, \varphi_B^*)} \cdot \left| \int_{K^\times \backslash A_F^\times} \varphi_B(t) \chi(t)^{-1} dt \right|^2,$$

where $C$ is an explicitly determined positive constant, $\varphi^*$ is a certain automorphic form in the representation space of $\pi$, and $\varphi_B^*$ is a certain automorphic form in the representation space of $\pi_B$.

Our paper is devoted to the proof of this theorem. When $\chi$ is a finite character we have obtained such a formula in [11] by using a geometric argument following [12] (initiated by Gross [2]). In [4], whose approach will be taken in our paper, a similar formula over real quadratic fields is proved under the assumption that $N$ is squarefree and $\chi$ is unramified. The goal of our paper is to obtain a similar result over CM fields and remove the restriction on conductors.

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1.1. Notations. We use $F$ to denote either a totally real number field or a non-archimedean local field, depending on the context. When $F$ is a non-archimedean local field, we write $\mathcal{O}$, $\mathcal{O}$, and $U$ to denote its fixed uniformizer, ring of integers and group of units respectively. The letter $K$ denotes either a totally imaginary quadratic extension of $F$ when $F$ is a number field, or a quadratic extension (split or not) of $F$ when $F$ is a local field. If $S$ is an algebraic object defined over $F$, for instance if $S$ is a vector field or an algebraic group over $F$, then we use $S_{\mathbb{A}}$ to denote its adelic points over the adele ring $\mathbb{A} = \mathbb{A}_F$. The space of Schwartz functions on a topological space $V$ is denoted by $S(V)$. For a subset $A$ of $V$ its characteristic function is denoted by $1_A$.

The notation $U_0(1)$ represents the standard maximal compact subgroup $GL_2(\hat{\mathcal{O}}_F)$ of $GL_2(\mathbb{A})$, while $U_0(N)$ denotes the matrices in $U_0(1)$ with lower left entries divisible by $N$. Similar notations hold in local situation. We denote by $Z$, $B$ and $N$ the center, the standard Borel and unipotent subgroup of $GL_2$ respectively. The notion $n(x)$ denotes a unipotent matrix with $x$ in the upper right. Sometimes we also denote $GL_2$ by $G$. We let $T_1$ be the diagonal subgroup of $GL_2$ with lower right entry equal to 1, and let $t(a)$ be the matrix in $T_1$ with $a$ in the upper left.

We fix an additive character $\psi$ of $\mathbb{A}/F$ by the formula $\psi(a) = \psi_Q(tr_{F/Q}(a))$, here $\psi_Q$ is the standard additive character of $\mathbb{A}_Q$ and $tr_{F/Q}$ denote the trace map from $\mathbb{A}/F$ to $\mathbb{A}_Q/Q$. Therefore the conductor $\delta$ of $\psi$ is the relative discriminant of $F/Q$. Locally we also fix an unramified additive character $\psi^0$ of $F_v$. The character $\psi^0$ and $\psi_v$ (the restriction of $\psi$ on $F_v$) are related by the identity $\psi(x) = \psi^0(\delta_v x)$, where $\delta_v$ is the discriminant of $F_v$ over $Q_v$.

We will use $N$ and $tr$ (with or without the subscript $V$) to denote the reduced norm and trace of elements in a quadratic or quaternion algebra $V/F$. We use $\omega_V$ to denote the quadratic character associated to a quadratic space $V$ and use $\omega$ especially to denote the quadratic character of the quadratic extension $K/F$. Finally we use $c(\pi)$ (or $c(\omega)$) to denote the conductor of the representation $\pi$ (or the character $\omega$).

2. Preliminaries

In this section we will first recall some facts on Weil representations of $SL_2$ and theta kernels. Then we will construct some special local Schwartz functions associated to these representations.

2.1. Weil representations and theta kernels. First let $F$ be a local field of characteristic 0. Let $\psi$ be a fixed nontrivial additive character of $F$. Let $V$ be a $2n$-dimensional space over $F$ with a non-degenerate quadratic form $q$. Let $GO(V)$ be the group of similitudes of the quadratic space $V$ defined by

$$GO(V) = \{\sigma \in GL_F(V) : q(\sigma v) = \nu(\sigma)q(v) \text{ with } \nu(\sigma) \in F^\times \text{ for } v \in V\},$$

where $\nu$ is called the similitude factor. We write

$$R(GO(V), GL_2) = \{(h, g) \in GO(V) \times GL_2 : \nu(h) = \det(g)\}.$$
Note that $R(GO(V),GL_2)$ is isomorphic to the semidirect product $GO(V) \ltimes SL_2$ by the map

$$(h,g) \mapsto \left( h, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right).$$

Since $\dim(V)$ is even, one can construct a Weil representation $r_q$ defines a representation of $SL_2(F)$ on the space $S(V)$ of Bruhat-Schwartz functions on $V$ by the following rule

$$r_q \left( \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right) f(x) = \psi(aq(x))f(x),$$

$$r_q \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) f(x) = |a|^n \omega_V(a)f(ax),$$

$$r_q \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) f(x) = \gamma \hat{f}(x),$$

where $\gamma$ is an eighth root of unity, $\omega_V$ is the quadratic character of $F^\times$ associated to the quadratic space $V$. The measure on $V$ is taken to be self-dual with respect to the Fourier transform $\hat{f}(x) = \int_V f(y)\psi(q(x,y))dy$, where $q(x,y) = q(x+y) - q(x) - q(y)$.

If $r' = r_\lambda$ is the Weil representation attached to $(V,\lambda q)$ then

$$\omega' = \omega, \quad \gamma' = \omega(\lambda)\gamma, \quad dx' = |\lambda|dx.$$

If $r'_q$ is the Weil representation attached to $(V,q)$ but with respect to $\psi'(x) = \psi(\delta x)$, then (2.1.1)

$$r'_q(g) = r(t(\delta)gt(\delta)^{-1}).$$

The representation $r_q$ can be extended to a representation $r_q$ of $R(GO(V),GL_2)$ on the same space $S(V)$ by letting

$$r_q(h,g)f(x) = L(h)r_q(g_1)f(x),$$

where $g_1 = \begin{pmatrix} 1 & 0 \\ 0 & \det(g)^{-1} \end{pmatrix} \cdot g \in SL_2(F)$ and $L(h)f(x) = |\nu(h)|^{-n}f(h^{-1}x)$.

In this paper we only consider two types of $V$. The first type is $V = K$ for a quadratic algebra $K$ over $F$ (split or not), equipped with a quadratic form $q(x) = \Lambda N_K$ for certain $\Lambda \in F$. The group $K^\times$ is a subgroup of index 2 of $GO(V)$ whose action on $K = V$ is given by multiplication. The similitude factor is given by $\nu(t) = N_K(t)$. The whole group $GO(V)$ is the semi-direct product of $K^\times$ and the involution group of $K/F$. The other type is $(V,q) = (B,N_B)$, where $B$ is a quaternion algebra over $F$. The similitude group $GO(V)$ is the semi-direct product of its connected identity component $GSO(V)$ and the standard involution of $B$. Also, one has $GSO(V) = B^\times \ltimes B^\times/F^\times$ through the map $(g_1,g_2) \mapsto g_1bg_2^{-1}$ for $b \in B$. Under this identification the similitude factor on $GSO(B)$ is given by $N_B(g_1g_2^{-1})$.

Now suppose $F$ is totally real number field. Let $(V,q)$ be an anisotropic quadratic space of dimension $2n$ over $F$. A similar global construction as the above gives the Weil representation of the group $R(GO(V_\mathbb{A}),GL_2(\mathbb{A}))$ for a fixed additive character $\psi$ of $\mathbb{A}/F$. 
The theta kernel on $R(GO(V_\Lambda), GL_2(\mathbb{A}))$ is defined by
\begin{equation}
\theta(h, g; \phi) = \sum_{x \in V_F} r_q(h, g)\phi(x)
\end{equation}
for any $\phi \in S(V_\Lambda)$.

There are three global quadratic spaces to be considered. The first one is $(K, \mathbb{N}_K)$ for the CM field $K$, and it corresponds to the construction of theta series associated to $\chi$, see Section 4.1. Another one is $(K, \Lambda \mathbb{N}_K)$ and corresponds to the construction of an Eisenstein series (Section 4.1). The third one is $(B, \mathbb{N}_B)$ for the quaternion algebra $B$ over $F$ and corresponds to the theta lift between $GSO(B)$ and $GL_2$ (Section 4.3).

2.2. Special vectors. For later applications we need to find some special vectors $\phi$ in the space $S(V_\Lambda)$ for various quadratic spaces $V$. We choose and fix a global embedding $K \to B$ such that $B = K + Kj$, where $j$ is an element in $B$ with $j^2 = -\Lambda \in \mathcal{O}_F$ and $j^t = -j$ ($t$ is an involution of $B$ with respect to the fixed embedding of $K$). As quadratic spaces over $F$ one has an orthogonal decomposition $(B, \mathbb{N}_B) = (K, \mathbb{N}_K) \oplus (K, \Lambda \mathbb{N}_K)$.

Let $K = F + F\iota$, where $\iota^2 = n_K$. So the quaternion algebra $B$ has a decomposition $B = F \oplus F\iota \oplus Fj \oplus Fk$, where $i^2 = n_K$, $j^2 = -\Lambda$, and $ij = -ji = k$. By the assumption we have $\epsilon(B_v) = (n_K, -\Lambda)_v$, where $(\cdot, \cdot)_v$ is the local Hilbert symbol of $K$ at $v$.

**Lemma 2.2.1.** Let $c = \text{Max}(v(c(\pi)), v(c(\pi^\chi))) = \text{Max}(v(N), v(D))$ with $D = c(\chi)^2c(\omega)$, then we can choose $j \in B$ such that $\Lambda = \mathbb{N}(j)$ falls in one of the following five cases for $m \in \mathbb{Z}_+$:

1. $c = 0$, with $v$ inert in $K$, then $v(\Lambda) = 2m$;
2. $c = 0$, with $v$ split in $K$, then $v(\Lambda) = m$;
3. $c = 2t + 1$, with $v$ inert in $K$, then $v(\Lambda) = 1$;
4. $c \geq 1$, $v$ is split in $K$, then $v(\Lambda) = m = 0, 1$;
5. $c = 1$, $v$ is ramified in $K$, then $v(\Lambda) = 0$.

**Proof.** We first show that one can remove all the even positive prime powers in $\Lambda$ at places where $c$ is nonzero. Let $p$ be a prime such that $p^{2t}|\Lambda$ for a maximal $t \geq 1$. By Chebotarev density theorem one can find an odd prime $q$ (which is coprime to every ramified place) such that $pq^{-1} = x\mathcal{O}_F$ with $x \in F$. If we let $j' = j/x^t$ and $\Lambda' = -(j')^2$, then the new $\Lambda'$ has the power less than 2 at the prime $p$. Obviously $K + Kj'$ equals $B$.

If $K/F$ is inert at a place $v$ (or equivalently, at a prime $p$), then $\epsilon(B_v) = (n_K, \Lambda)_v = (-1)^{v(\Lambda)}$. But by our construction we have $(-1)^c = \epsilon(B_v)$, so $v(\Lambda)$ and $c$ have the same parity. So we have got the first 4 cases.

Now we assume that $c = 1$ and $K_v/F_v$ is ramified. By the above argument we may also assume $v(\Lambda) = 0, 1$. By Chebotarev we find a good prime $\mathfrak{q}$ of $K$ and an $x \in K$ such that $x\mathcal{O}_K = \mathfrak{q}\mathfrak{P}^{-1}$, where $\mathfrak{P}$ is the prime ideal of $K$ which divides $p$ (the prime of $v$). If we take $j' = x^{-1}j$ and $\Lambda' = -j'^2 = -N_K(x^{-1})j^2$, then $B = K + Kj'$ and $v(\Lambda') = 0$. \hfill $\Box$
Note that the case $v(N) \neq 0$ corresponds to Cases 3 and 4, and the case $v(c(\chi)) \neq 0$ corresponds to Cases 3b and 4.

The theta series and Eisenstein series to be studied later on depend on the choice of $\phi \in S(V_\chi)$. There are three cases. The first one is the theta lift from $GSO(K) \backslash GSO(\mathbb{A}_K)$ to $G(F) \backslash G(\mathbb{A})$. The second one occurs in the definition of the Eisenstein series $f(s, g; \phi)$. The third one is the theta lift from $G(F) \backslash G(\mathbb{A})$ to $GSO(B) \backslash GSO(\mathbb{B}_K)$ (see Section 4).

**Nonarchimedean case.** Now we construct special Schwartz functions at a finite place $v$ according to the above three cases. Let $K = K_v$ and $F = F_v$.

**N1.** Let $V = K$ be the fixed quadratic extension of $F = F_v$ equipped with $\mathbb{N}_{K/F}$ as the quadratic form. If $v \not| c(\chi)$ then we take $\phi_1(x) = 1_{O_K}(x)$, if $v| c(\chi)$ then we take $\phi_1(x) = \chi(x)1_{O_K}$. Let $\phi_2 = S(K)$ be the characteristic function $1_L$ of a lattice $L$ constructed as follows (according to the five cases discussed in Lemma 2.2.1):

1. $L = \varpi^{-m}O_K$,
2. $L = O \oplus \varpi^{-m}O$,
3a and 3b: $L = \varpi^tO_K$,
4. $L = O \oplus \varpi^{c-m}O$, where $v(\Lambda) = m$ for $m = 0, 1$
5. $L = O_K$, where $v(\Lambda) = 0$.

Actually in Case 2, if we take $\varpi_K = (1, \varpi)$ be a uniformizer of $K = F \oplus F$ then $L = \varpi^{-m}O_K$, so Case 2 and Case 1 can be treated together. The remark also applies to Case 4, which can be treated similarly as Case 3.

Let $R = O_K + Lj$, a simple calculation shows the following.

**Lemma 2.2.2.** The lattice $R$ is an (Eichler) order of reduced discriminant $\varpi^c$ in $B$.

**N3.** Let $V = B$ be a quaternion algebra with its norm $\mathbb{N}_B$ as the quadratic form. Under the orthogonal decomposition $B = K + Kj$ we take $\phi(x_1 + x_2j) = \phi_1(x_1)\phi_2(x_2)$, where $\phi_1$ and $\phi_2$ are constructed in N1 and N2 respectively. So the function $\phi$ is given by

$$\phi(x) = \begin{cases} 1_R & \text{if } v \not| c(\chi), \\ \chi(x_1)1_{R^\times} & \text{if } v| c(\chi) \text{ and } x = x_1 + x_2j. \end{cases}$$

The last line is valid at $v| c(\chi)$ as we can write $Lj = O_Kj'$ such that $j'^2$ is a unit multiple of $c(\chi)^2$. In the second case we also denote $\phi$ by $\chi$, which is a character on $R^\times$.

If $K/F$ is ramified we let $\hat{R}$ denote the following maximal order of $M_2(F_v)$

$$\hat{R} = \{a + bj : a, b \in \delta^{1}_{K/F} \text{ such that } a - bu \in O_{K_v}\}, \quad (2.2.1)$$

where $\delta$ is the different of $K/F$, and $u \in U_K$ is such that $j^2 = N_Ku$. Here such an $u$ exists because $(n_K, j^2)_v = (n_K, -\Lambda)_v = 1$, i.e, $j^2$ is a norm in $K$. We write $\phi' = 1_{\hat{R}}$. 
Now back to the global setting. Let \( \psi \) be a fixed additive character of \( \mathbb{A}/F \) of conductor \( \delta \). We form a compact subgroup of \( B^x_{k_f} \)

\[
\hat{R}^x = \prod_{v \in \infty} R_v^x \prod_{v \in \omega} \hat{R}_v^x,
\]
and define a character \( \chi \) (by abuse of language) on \( \hat{R}^x \) by \( \chi(k) = \prod_{v \in \omega} \chi_v(k_v) \), where \( R_v, \hat{R}_v \) and \( \chi_v \) are defined as above. For \( x = \prod_{v \in \infty} x_v \in B_{k_f} \), we define

\[
\phi'_f(x) = \prod_{v \in \infty} \phi_v(x_v) \cdot \prod_{v \in \omega} \phi'_v(x_v).
\]

Now we determine the level structure of \( \phi'_f \) in the Weil representation \( r_B \) attached to \( (B, \mathbb{N}_B) \).

**Proposition 2.2.3** (Level structure). For \( \kappa \in t(\delta)^{-1}U_0(Nc(\chi)^2)t(\delta) \) and \( k_1, k_2 \in \hat{R}^x \) such that \( \det \kappa = \mathbb{N}_B(k_1k_2^{-1}) \)

\[
r_B((k_1, k_2), \kappa)\phi'_f = \chi(k_1^{-1}k_2)\phi'_f,
\]
where \( (k_1, k_2) \) is regarded as an element in \( \text{GSO}(B_{k_f}) = \mathbb{A}_f \backslash B^x_{k_f} \times B^x_{k_f} \).

**Proof.** As \( r_B((k_1, k_2), \kappa)\phi'(x) = r_B(\kappa_1)\phi'(k_1^{-1}xk_2) \) and \( \phi'(k_1^{-1}xk_2) = \chi(k_1^{-1}k_2)\phi' \), we are reduced to check

\[
r_B(\kappa)\phi' = \phi',
\]
for any \( \kappa \in t(\delta)^{-1}U_0(Nc(\chi)^2)t(\delta) \cap \text{SL}_2(\mathbb{A}_f) \). It suffices to check (2.2.2) locally. At a finite place \( v \), by (2.1.1) we may assume \( \psi_v \) is unramified and check (2.2.2) for generators of \( U_0(Nc(\chi)^2) \).

If \( v(Nc(\chi)^2) = 0 \), then \( \phi_v \) (or \( \phi'_v \)) is the characteristic function for a maximal \( \mathcal{O}_v \)-order of \( B_v \) and the check of (2.2.2) is simple (or see [10]).

We assume now \( v(Nc(\chi)^2) = c > 0 \).

**Lemma 2.2.4.** Let \( r_\chi \) as usual be the Weil representation associated to \( (K_v, \lambda_N) \). Assume \( K_v/F_v \) is unramified (split or not). If \( \phi_v = 1_{\mathcal{O}_v^\times} \mathcal{O}_v \) and \( v(\lambda) = M \), then \( r_\chi(\kappa)\phi_v = \phi_v \) for any \( \kappa \in U_0(\mathbb{w}^c) \cap \text{SL}_2(F_v) \).

If \( \phi_v = \chi(x)1_{\mathcal{O}_v^\times} \) for \( v(c(\chi)) \neq 0 \), then \( r_1(\kappa)\phi_v = \phi_v \) for any \( \kappa \in U_0(c(\chi)^2) \cap \text{SL}_2(F_v) \).

**Proof.** The subscript \( v \) will be suppressed during the proof. The group \( U_0(\mathbb{w}^c) \cap \text{SL}_2(F) \) is generated by matrices of the type

\[
l(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad m(u\mathbb{w}^c) = \begin{pmatrix} 1 & 0 \\ u\mathbb{w}^c & 0 \end{pmatrix},
\]
where \( a, u \in \mathcal{O}_f^\times, b \in \mathcal{O} \). It is clear that \( r_\chi(l(a))\phi = \phi \) and \( r_\chi(n(b))\phi = \phi \). For \( m(u\mathbb{w}^c) \) we note that

\[
m(u\mathbb{w}^c) = -wn(-u\mathbb{w}^c)w, \text{ for } w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Now \( r_\lambda(w)\phi = \gamma \hat{\phi} \), here \( \hat{\phi} \) is the Fourier transform with respect to the self-dual measure \( dx \). So \( \hat{\phi} = \text{const} \cdot 1_{\mathcal{M}_{K}K^*} \), where \( \mathcal{M}_{K} \) is the conjugate of \( \mathcal{M}_{K} \) over \( F \). It is easy to see \( r_\lambda(n(-u\omega^c)) \) fixes \( \hat{\phi} \), hence \( \gamma = 1 \) in our case

\[
\lambda(m(u\omega^c))\phi(x) = r_\lambda(w)\phi(x) = \phi(-x) = \phi(x)
\]

by the Fourier inversion formula.

See [4] Proposition 2.5.1 for the proof of the second statement.

Consider the following decomposition of \( r_B \) via \( (B_v, N_{B_v}) = (K_v, N_{K_v}) \oplus (K_v, \Lambda N_{K_v}) \)

\[
r_B(\kappa)\phi(x_1 + x_2j) = r_1(\kappa)\phi_1(x_1)r_\Lambda(\kappa)\phi_2(x_2),
\]

for \( \kappa \in U_0(\omega^c) \cap SL(2,F_v) \). The proof of Proposition 2.2.3 is completed by applying the lemma to both \( r_1(\kappa) \) and \( r_\Lambda(\kappa) \). □

**Archimedean case.** Assume that we have fixed the global decomposition of \( B = K + Kj \) induced by the chosen embedding of \( K \) into \( B \), such that \( N_B(j) = \Lambda \). At an archimedean place \( v \) there are two cases about the sign of \( \Lambda \). If \( k_v > r_v \), \( B_v \) is definite, then \( \Lambda_v \) has to be positive; if \( k_v \leq r_v \), \( B_v \) is split and \( \Lambda_v \) is negative. If we let \( j_v = \begin{pmatrix} 0 & \sqrt{\Lambda_v} \\ -\sqrt{\Lambda_v} & 0 \end{pmatrix} \) then we have the identification \( \mathbb{H} \cong K_v \oplus K_v j_v \) by

\[
\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \mapsto \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} + \begin{pmatrix} v/\sqrt{\Lambda_v} & 0 \\ 0 & \bar{v}/\sqrt{\Lambda_v} \end{pmatrix} j_v.
\]

If \( B_v \cong M_2(\mathbb{R}) \), and \( j_v = \begin{pmatrix} 0 & \sqrt{|\Lambda_v|} \\ \sqrt{|\Lambda_v|} & 0 \end{pmatrix} \), the identification becomes

\[
\begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix} \mapsto \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} + \begin{pmatrix} v/\sqrt{|\Lambda_v|} & 0 \\ 0 & \bar{v}/\sqrt{|\Lambda_v|} \end{pmatrix} j_v.
\]

In the following we will construct \( \phi \)'s on the spaces in the left side of the above identifications. In other words, we will construct Schwartz functions on the quadratic spaces \( \mathbb{C} \) with the standard norm \( N \) (or \( -N \), see Case A2), or \( \mathbb{H} \) and \( M_2(\mathbb{R}) \) with the standard reduced norms. See Remark 2.2.1 for the general case.

Now let the additive character \( \psi \) be defined by \( \psi_v(x) = e^{2\pi ix} \).

**A1.** Let \( V = \mathbb{C} \) be the quadratic space equipped with \( q(z) = x^2 + y^2 \) over \( F_v = \mathbb{R} \), where \( z = x + iy \in V \). Suppose the character \( \chi(z) \) is given by \( \frac{z}{\sqrt{r}} \), where \( r \) is a non-negative integer. Let us define \( \phi_1(z) = 2\sqrt{2r}e^{-2\pi|z|^2} \in S(\mathbb{C}, \chi) \), where \( S(\mathbb{C}, \chi) \) represents the \( \chi \)-isotypic subspace of the whole Schwartz space. Let \( X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) and \( X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) be two vectors in the Lie algebra \( sl_2 \) of \( SL_2(\mathbb{R}) \). In order to check the weights of \( \phi \)'s in the Weil representation we need to check the action of \( X_+ - X_- \) on these vectors.
The effects of these two vectors on \( S(\mathbb{C}) \) are well-known (see [10]):

\[
\begin{align*}
  r(X_+)\phi &= 2\pi i(z \cdot \bar{z})\phi, \\
  r(X_-)\phi &= -\frac{1}{2\pi i} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} \right)\phi.
\end{align*}
\]

It is easy to check that

\[
r(X_+ - X_-)\phi_1 = i(2r + 1)\phi_1,
\]

therefore \( \phi_1(z) = 2\bar{z}^2 e^{-2\pi|z|^2} \) or the corresponding theta lift has weight \( 2r + 1 \).

**A2.** For the Eisenstein series, there are two cases. The first case is when the quadratic space is \( \mathbb{C} \) with the standard norm \( q \). Then we take

\[
\phi_2(z) = p_l(4\pi|z|^2) e^{-2\pi|z|^2} \in S(\mathbb{C}, 1), \text{ where } l = k - r - 1 \text{ (when } k > r \text{) and } p_l(t) \text{ is the Laguerre polynomial of degree } l:
\]

\[
p_l(t) = \sum_{j=0}^{l} \binom{l}{j} \frac{(-t)^j}{j!}.
\]

Using the formulas for \( X_+ \) and \( X_- \) we can see that:

\[
r(X_+ - X_-)\phi_2 = i(2l + 1)\phi_2.
\]

This is because \( p_l \) satisfies the differential equation:

\[
p_l''(t) + (1 - t)p_l'(t) + lp_l = 0.
\]

Therefore the weight of the corresponding Eisenstein series is \( 2l + 1 = 2(k - r) - 1 \). Laguerre polynomials have very nice orthogonal properties:

\[
(2.2.3) \quad \int_0^{\infty} p_l(t)p_m(t)e^{-t}dt = \delta_{lm},
\]

where \( \delta_{lm} = 1 \) or \( 0 \) depending on \( l = m \) or not.

The other case is when the quadratic space is \( (\mathbb{C}, -q) \). We let \( \phi_2(z) = p_l(4\pi|z|^2) e^{-2\pi|z|^2} \), where \( l = r - k \) (\( r \geq k \)). Now the formulas for \( r(X_+) \) and \( r(X_-) \) on \( S(\mathbb{C}) \) become:

\[
\begin{align*}
  r(X_+)\phi &= -2\pi i(z \cdot \bar{z})\phi, \\
  r(X_-)\phi &= \frac{1}{2\pi i} \left( \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial \bar{z}} \right)\phi.
\end{align*}
\]

It is easy to check that \( \phi_2 \) has weight \( 2(k - r) - 1 = 2k - 2r - 1 \).

**Remark 2.2.1.** In Case A2 (and Case A3 below) we have used the standard quadratic form on \( \mathbb{C} \), i.e, we have used \( j^2 = \pm 1 \). To get the corresponding \( \phi_2 \) for quadratic form \( \Lambda q(z) = \Lambda|z|^2 \) we only need to take \( \phi_2(z) = p_l(4\pi|z|^2) e^{-2\pi|\Lambda|z|^2} \). It is not hard to see that all the statements remain true because the map \( z \mapsto z/\sqrt{|\Lambda|} \) gives an isomorphism between \( (\mathbb{C}, q) \) and \( (\mathbb{C}, \Lambda q) \).
A3. First we let \( V = \mathbb{H} \), which is viewed as \( \mathbb{C} \oplus \mathbb{C} j \). Then we have on \( S(\mathbb{H}) \):
\[
    r(X_+ \phi) = 2\pi i (x\bar{x} + y\bar{y})\phi,
\]
and
\[
    r(X_- \phi) = -\frac{1}{2\pi i} \left( \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial \bar{x}} + \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial \bar{y}} \right) \phi.
\]
Here \( x \) and \( y \) are variables on the two copies of \( \mathbb{C} \) respectively. Let \( \phi(z) = \phi_1(x)\phi_2(y) \in S(\mathbb{H}) \), then \( \phi \) and thus the corresponding theta lift has weight \( 2r + 2l + 2 = 2k \) (which comes from a similar computation as in case A1 or A2).

Now we let \( V = M_2(\mathbb{R}) \), the real matrix algebra of order 2. It is also viewed as \( V = \mathbb{C} \oplus \mathbb{C} j \).

On \( S(M_2(\mathbb{R})) \) we have:
\[
    r(X_+ \phi) = 2\pi i (x\bar{x} - y\bar{y})\phi,
\]
and
\[
    r(X_- \phi) = -\frac{1}{2\pi i} \left( \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial \bar{x}} - \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial \bar{y}} \right) \phi.
\]
Let \( \phi \) still be the product of \( \phi_1 \) and \( \phi_2 \), then the corresponding theta lift has weight \( 2r - 2l = 2k \).

**Proposition 2.2.5** (Archimedean weight structure). For \( \kappa = e^{i\alpha} \in SL_2(\mathbb{R}) \), \( k_1 = e^{i\alpha_1}, k_2 = e^{i\alpha_2} \in K_v \subset B^\times_{\mathbb{R}} \) we have
\[
    r_B((k_1, k_2), \kappa, \phi) = e^{2ri(\alpha_1 - \alpha_2)} e^{2k_1} \phi.
\]

**Proof.** It is clear from the above discussions. \( \square \)

### 3. Rankin-Selberg Convolution

We will study the Rankin-Selberg convolution integral which involves certain Eisenstein series arising from a Weil representation. Here the Weil representation is associated to the quadratic space \((K, AN)\) and a fixed additive character \( \psi \). Let \( \delta \) be the conductor of \( \psi \), i.e., \( \psi(x) = \psi^0(\delta x) \) for an unramified additive character \( \psi^0 \) of \( \mathbb{A} \) (but not of \( \mathbb{A}/F \)). Let \( t(\delta) = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix} \). As before the quadratic character associated to \( K/F \) is denoted by \( \omega \). For \( g \in G(\mathbb{A}) \), \( s \in \mathbb{C} \), \( \phi \in S(V_\mathbb{A}) \) we define a function
\[
    f(s, g; \phi) = (r_\lambda(g_1)\phi)(0)|a_\delta(g)|^{2s-1}\det(g)^{-1/2}\omega^{-1}(\det(g)),
\]
where \( a_\delta(g) = |a/b|^{1/2} \) if \( g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \) for \( k' \) in the twisted maximal compact subgroup \( U' = t(\delta)^{-1}U t(\delta) \) of \( G(\mathbb{A}) \), where \( U \) is the standard maximal compact subgroup (as one has a twisted Iwasawa decomposition \( G(\mathbb{A}) = ANU' \)). The function \( f(s, g; \phi) \) belongs to the induced representation \( B(|s/2, \omega^{-1}| - s/2) \). The related Eisenstein series is given by
\[
    E(s, g; \phi) = \sum_{\gamma \in B(F) \cap SL_2(F)} f(s, \gamma g; \phi) = \sum_{\gamma \in B(F) \cap G(F)} f(s, \gamma g; \phi).
\]
This series converges for \( \Re(s) \gg 0 \) and has an analytic continuation to the whole \( s \)-plane.
If $a(g)$ denotes the function with respect to the standard $U$, then we have the following
\begin{equation}
(3.0.5) \quad a_S(g) = a(t(\delta^{-1})gt(\delta)).
\end{equation}

3.1. Non-archimedean case. Let $F = F_v$ be a nonarchimedean field. We first consider an unramified additive character $\psi^0$ of $F$. Let $\pi$ be an irreducible (infinite) representation of $G(F) = GL_2(F)$ of central character $\omega_\pi$. For every $W$ in the Whittaker model $W(\pi, \psi^0)$ of $\pi$ we define its Mellin transform by
\[
\Psi(s, W) = \int_{F^\times} W \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right) |t|^{s-1/2} d^\times t.
\]

A Whittaker function $W^0_\pi \in W(\pi, \psi^0)$ is called a Whittaker newform if it takes value 1 at the unit matrix and is right invariant under $U_1(c(\pi))$, where $c(\pi)$ denotes the conductor of $\pi$. By results of Casselman [1] and Zhang [12] it is known that the Whittaker newform exists uniquely and satisfies
\[
\Psi(s, W^0_\pi) = L(s, \pi).
\]

The Whittaker newform $W^0_\pi$ has a functional equation. Let $g_0 = \begin{pmatrix} 0 & 1 \\ -c(\pi) & 0 \end{pmatrix}$ and let $W^0_\pi$ be the Whittaker newform of the contragredient $\tilde{\pi}$, then (see [12])
\begin{equation}
(3.1.1) \quad W^0_\pi(gg_0) = W^0_\pi(g)\omega_\pi(\det g)\epsilon(\pi, \psi^0),
\end{equation}
where $\epsilon(\pi, \psi^0) = \epsilon(1/2, \pi, \psi^0)$ is the local $\epsilon$-factor of $\pi$ with respect to $\psi^0$.

Now we assume $\pi_1 = \pi_v$, $\pi_2 = \pi_{\chi, v}$ and let $W^0_i$ be the Whittaker newform of $\pi_i$ for $i = 1, 2$. Let $(K, AN_K)$ be a quadratic space (splitting or not) over $F$.

For $\phi_2 \in S(K)$ we define
\[
f^0(s, g; \phi_2) = r_\Lambda^0(g_1)\phi_2(0)|a(g)|^{2s-1}\det(g)^{-1/2}\omega(\det(g)),
\]
where $a(g) = |a/b|^{1/2}$ if $g = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k$ with $k$ in the standard maximal subgroup $U_0(1)$. The modified Rankin-Selberg convolution of $W^0_1$ and $W^0_2$ given by
\begin{equation}
(3.1.2) \quad W^+(s, W^0_1, W^0_2, \phi_2) = \int_{Z(F)N(F) \setminus G(F)^+} W^0_1(\epsilon g)W^0_2(g)f^0(s, g; \phi_2)dg,
\end{equation}
where $G(F)^+ = \{ g \in GL_2(F) : \det g \in N(K) \}$. Here the measure of $Z(F)N(F) \setminus G(F)^+$ is taken to be
\begin{equation}
(3.1.3) \quad dg = |t|^{-1}d^\times tdk
\end{equation}
under the decomposition $G(F)^+ = Z(F)N(F)T_1(F)U_0(1)^+$, where $d^\times t$ is the measure on $T_1(F) = F^\times$ such that $\mu(N_K(U_K)) = 1$, and $dk$ on $U_0(1)^+ = U_0(1) \cap G^+$ is such that $\mu(U_0(1)^+) = 1$. 

\[\]
**Proposition 3.1.1.** Let $c = \text{Max}(v(c(\pi_1)), v(c(\pi_2)))$. Let $W^0_1 \in W(\pi_1, \psi^0)$ and $W^0_2 \in W(\pi_2, \psi^0)$ be the Whittaker newforms of $\pi_1$ and $\pi_2$ respectively, then
\[
L(s, \pi_1 \times \pi_2) = \begin{cases} 
L(2s, \omega)\Psi^+(s, W^0_1, W^0_2, \phi_2) & \text{in cases (1) and (2)}, \\
\frac{L(1, \omega)}{\mu(U_0(\omega^c))} L(s, \pi_1 \times \pi_2) & \text{in cases (3), (4), (5)},
\end{cases}
\]
where $\omega$ as usual denotes the quadratic character of $F$ associated to $K/F$, and the five cases are divided according to Lemma 2.2.1.

**Proof.** Most of the claims are already proved in [4] (for squarefree $N$). We will supply a simpler proof for some cases. The method, which is first used in [12], can also treat non-squarefree $N$.

Cases 1, 2 and 5 can be proved using Corollary 3.1.4 Cases 1, 2 and 5 respectively. Since the proof is the same as the one used in Cases A1, A1’ and A3 of [4], so is omitted.

Case 3a occurs when $c = v(c(\pi_1)) = 2t + 1$ and $v(\Lambda) = 1$. As $W^0_1$ has the minimum level $U_1(\omega^c)$ and $\pi_1$ has conductor $\omega^c$, so
\[
\sum_{\gamma \in U_1(\omega^{-1})/U_1(\omega^c)} W^0_1(g\gamma) = 0.
\]
Then
\[(3.1.4) \quad \int W^0_1(\epsilon g)W(g)dg = \frac{1}{|U_1(\omega^{-1})/U_1(\omega^c)|} \sum_{\gamma \in U_1(\omega^{-1})/U_1(\omega^c)} \int W^0_1(\epsilon g)W(g\gamma^{-1})dg = 0
\]
for any function $W$ invariant under $U_1(\omega^{-1})$.

By Lemma 3.1.5 $f^0(s, k; \phi_2)$ is the sum of $(1 + |\omega|)1_{U_0(\omega^c)}$ and a function which is invariant under $U_0(\omega^{c-1})$, so
\[
\Psi^+(s, W^0_1, W^0_2, \phi_2) = (1 + |\omega|) \int_{T^1_F \times U_0(1)} W^0_1(tk)W^0_2(tk)1_{U_0(\omega^c)}|t|^{s-1}dkdt
= \frac{\mu(U_0(\omega^c))}{L(1, \omega)} \int_{T^1_F} W^0_1(tk)W^0_2(tk)1_{U_0(\omega^c)}|t|^{s-1}dt
= \frac{\mu(U_0(\omega^c))}{L(1, \omega)} L(s, \pi_1 \times \pi_2),
\]
where the last equality is a well-known fact. Since $U_0(\omega^c) = U_0(\omega^c)^+$ the proof of case 3a is complete.

Case 3b occurs when either $v(c(\pi_1)) = c = 2t$ or $v(c(\pi_2)) = c = 2t$. This case can be treated in the same way as case 3a.

Case 4 shows up when $v(c(\pi_2)) = c$. As $L(1, \omega)^{-1} = (1 - |\omega|)$ the above argument for Case 3a moves parallelly to this case (also using Lemma 3.1.5 for Case 4). $\Box$
Lemma 3.1.2. If $K/F$ is a quadratic field extension of different $\pi_K^C\mathcal{O}_K$, and $\alpha \in F^\times$, then

$$\int_{\pi_K^C\mathcal{O}_K} \psi^0(\alpha \overline{y}) d\overline{y} = \begin{cases} |\pi|^{ft} \mu(\mathcal{O}_K) & \text{if } v(\alpha) \geq -ft, \\ 0 & \text{if } -ft > v(\alpha) > -C - ft, \\ \gamma|\alpha|^{-1}\omega(\alpha)^{-1} & \text{if } v(\alpha) \leq -C - ft, \end{cases}$$

here $f$ is the residue degree of $K/F$.
If $K = F \oplus F$ is split, then

$$\int_{\pi^C\mathfrak{O} \times \pi^C\mathfrak{O}} \psi^0(\alpha xy) dx dy = \begin{cases} |\pi|^{s+t} & \text{if } v(\alpha) \geq -s - t, \\ |\alpha|^{-1} & \text{if } v(\alpha) \leq -s - t. \end{cases}$$

Proof. This is Lemma 2.5.2 in [4].

Lemma 3.1.3. Let $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(1)$ and $v(\Lambda) = M$. If $K/F$ is inert and $\phi_2 = 1_{\mathfrak{O}_K^C}$, then

$$f^0(s, k; \phi_2) = \begin{cases} 1 & \text{if } k \in U_0(\mathfrak{w}^{M+2\ell}), \\ |\Lambda c^{-1}\mathfrak{w}^2|\omega(\Lambda c^{-1}) & \text{otherwise.} \end{cases}$$

If $K = F \oplus F$ and $\phi_2 = 1_{\mathfrak{O}_K^C}$, then

$$f^0(s, k; \phi_2) = \begin{cases} 1 & \text{if } k \in U_0(\mathfrak{w}^{M+\ell}), \\ |\Lambda c^{-1}\mathfrak{w}^\ell| & \text{otherwise.} \end{cases}$$

If $K/F$ is ramified of ramification index 1 and $\phi_2 = 1_{\mathfrak{O}_K^{-M}}$, then

$$f^0(s, k; \phi_2) = \begin{cases} \omega(d) & \text{if } k \in U_0(\mathfrak{w}), \\ \gamma\mu(\mathcal{O}_K)|\mathfrak{w}^{-M}\Lambda c^{-1}|\omega(-\Lambda c^{-1}) & \text{otherwise.} \end{cases}$$

Proof. The claims can be proved through direct computations using explicit formulae in Section 2.1, Lemma 3.1.2 and the Bruhat decomposition

$$G = B \prod B \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} N.$$ 

More precisely, if $c \neq 0$ and $a \neq 0$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} c^{-1} & a \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & ba^{-1} + c^{-1}a^{-1} \\ 0 & 1 \end{pmatrix},$$

while if $c \neq 0$ and $a = 0$ we have

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & b \end{pmatrix}.$$ 

The detail is omitted. \qed
Corollary 3.1.4. According to the five cases of Proposition 3.1.1:

(1): \( \phi_2 = 1_{\mathcal{O} - m\mathcal{O}_K} \), then \( f^0(s, k; \phi_2) = 1 \);

(2): \( \phi_2 = 1_{\mathcal{O} \times 1_{\mathcal{O} - m\mathcal{O}}} \), then \( f^0(s, k; \phi_2) = 1 \);

(3): \( \phi_2 = 1_{\mathcal{O} - t\mathcal{O}_K} \), then

\[
f^0(s, k; \phi_2) = \begin{cases} 
1 & \text{if } k \in U_0(\varpi^c), \\
|\Delta c^{-1}| \omega(\Delta c^{-1}) & \text{otherwise};
\end{cases}
\]

(4): \( \phi_2 = 1_{\mathcal{O} \times 1_{\mathcal{O} - m\mathcal{O}}} \) for \( m = 0, 1 \), then

\[
f^0(s, k; \phi_2) = \begin{cases} 
1 & \text{if } k \in U_0(\varpi^c), \\
|\Delta c^{-1}| & \text{otherwise};
\end{cases}
\]

(5): \( \phi_2 = 1_{\mathcal{O}_K} \), then

\[
f^0(s, k, \phi_2) = \begin{cases} 
\omega(d) & \text{if } k \in U_0(\varpi), \\
\gamma \mu(\mathcal{O}_K)|\Lambda| \omega(-\Delta c^{-1}) & \text{otherwise}.
\end{cases}
\]

Here the norm \( | \cdot | \) is the norm in \( F \), and in case (5) we have assumed the residue characteristic of \( F \) is odd.

Therefore, \( f^0(s, g; \phi_2) \) is right invariant under \( U_0(\varpi^c) \) in the first 4 cases and is right invariant under \( U_0(\varpi^c)^+ \) in Case 5.

Lemma 3.1.5. Let \( k \in U_0(1) \). In Case 3, one has

\[
f^0(s, k; \varphi_2) = (1 + |\varpi|)1_{U_0(\varpi^{2t})} - |\varpi|(1 + |\varpi|)1_{U_0(\varpi^{2t-1})} + f(s, k; |\varpi|^2 1_{\varpi^{t-1}\mathcal{O}_K}).
\]

In Case 4, one has

\[
f^0(s, k; \varphi_2) = (1 - |\varpi|)1_{U_0(\varpi^{t})} + |\varpi| f^0(s, k; 1_{\mathcal{O} \times \varpi^{c-m-1}\mathcal{O}}).
\]

Proof. For Case 3 by Lemma 3.1.3

\[
f^0(s, k, 1_{\varpi^{t-1}\mathcal{O}_K} - |\varpi|^{-2} 1_{\varpi^{t}\mathcal{O}_K}) = (|\varpi| \omega(\varpi) - |\varpi|^2)1_{U_0(\varpi^{2t-1})} - (1 - |\varpi| \omega(\varpi))1_{U_0(\varpi^{2t})},
\]

hence the conclusion.

For Case 4 Lemma 3.1.3 again implies

\[
f^0(s, k; \varphi_2 - |\varpi| 1_{\mathcal{O} \times \varpi^{c-m-1}}) = (1 - |\varpi|)1_{U_0(\varpi^{c})},
\]

from which the result follows.

When \( v \) divides \( c(\chi)^2 = c(\pi_2) \), or equivalently in Cases 3b and 4, we need a stronger result similar to Proposition 2.5.1 in [12].
Proposition 3.1.6. Assume $0 \leq j \leq 2t - 1 = 2c(\pi_2) - 1$, then

$$
\Psi^+(s, \rho \begin{pmatrix} \varpi^{-j} & 0 \\ 0 & 1 \end{pmatrix} W_1^0, W_2^0; \phi_2) = |\varpi|^{j(s-1/2)} \alpha_j \frac{\mu(U_0(\varpi^c))}{L(1, \omega)} L(s, \pi_1 \times \pi_2),
$$

where $\alpha_j$ is defined by

$$
L(s, \pi_2) = \sum_{j=0}^{\infty} \alpha_j |\varpi|^{js} = 1.
$$

Proof. As $\pi_2$ has conductor $\varpi^{2t}$,

$$
\sum_{\gamma \in U_1(\varpi^{2t-1})/U_1(\varpi^{2t})} \rho(\gamma)W_2^0 = 0.
$$

If $j < 2t$, then $\rho(t(\varpi^{-j}))W_1^0$ is invariant under $U_1(\varpi^{2t-1})$. By Lemma 3.1.5 $f^0(s, k; \phi_2)$ is the sum of $L(1, \omega)^{-1} U_0(\varpi^c)$ and a function which is invariant under $U_0(\varpi^c)$. So, by (3.1.4) one has

$$
\Psi^+(s, \rho(t(\varpi^{-j}))W_1^0, W_2^0; \phi_2) = \frac{\mu(U_0(\varpi^c))}{L(1, \omega)} \int_{T^1} W_1^0(t(-a(\varpi^{-j}))W_2^0(t(a))|a|^{s-1} da^x
$$

$$
= |\varpi|^{j(s-1/2)} \alpha_j \frac{\mu(U_0(\varpi^c))}{L(1, \omega)} \int_{T^1} W_1^0(t(-a))W_2^0(t(a(\varpi^j)))|a|^{s-1} da^x
$$

$$
= |\varpi|^{j(s-1/2)} \alpha_j \frac{\mu(U_0(\varpi^c))}{L(1, \omega)} L(s, \pi_1 \times \pi_2),
$$

where we have used the fact that $W_2^0(t(a(\varpi^j))) = |\varpi|^{j/2} \alpha_j W_2^0(t(a))$ for any integral $a$. \hfill \Box

We assume now $j = 2t = c$. This time it is hard to prove a similar result above, but one can still obtain the information on the convolution at $s = 1/2$, which is sufficient for later applications. First, the functional equations (3.1.8) for $W_1^0$ and $W_2^0$ imply that (as $\pi_1 = \tilde{\pi}_1$, $\omega_{\pi_1} = 1$ and $\omega$ is unramified)

$$
W_1^0(gt(\varpi^{-2t})) = \epsilon(\pi_1, \psi^0)W_1^0\left(gt(\varpi^{-2t}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \epsilon(\pi_1, \psi^0)W_1^0\left(g \begin{pmatrix} 0 & 1 \\ -\varpi^{2t} & 0 \end{pmatrix}\right),
$$

$$
W_2^0\left(g \begin{pmatrix} 0 & 1 \\ -\varpi^{2t} & 0 \end{pmatrix}\right) = \epsilon(\pi_2, \psi^0)\tilde{W}_2^0(g)\omega(\det g) = \epsilon(\pi_2, \psi^0)W_2^0(g),
$$

where $\tilde{W}_2^0$ is the Whittaker newform for $\tilde{\pi}_2$. Write $g_0 = \begin{pmatrix} 0 & 1 \\ -\varpi^{2t} & 0 \end{pmatrix}$ and $C = \epsilon(\pi_1, \psi^0)\epsilon(\pi_2, \psi^0)$, then

$$
(3.1.5) \quad \Psi^+(s, \rho(t(\varpi^{-2t}))W_1^0, W_2^0; \phi_2)
$$

$$
= \epsilon(\pi_1, \psi^0)\Psi^+(s, \rho(g_0)W_1^0, W_2^0; \phi_2)
$$

$$
= C \int_{Z(F)N(F)/G(F)^+} W_1^0(\epsilon g)W_2^0(g)f^0(s, gg_0; \phi_2) dg.
$$
Lemma 3.1.7. Let $h \in K$ such that $N(h) = \det g_0 = \varpi^{2t}$ and define
\begin{equation}
\phi_2' = r_\Lambda^0(h, g_0) \phi_2,
\end{equation}
then
\begin{equation}
f^0(1/2, g; \phi_2') = f^0(1/2, gg_0; \phi_2),
\end{equation}
and $f^0(s, g; \phi_2')$ is right invariant under $U_0(1)$.

Proof. The identity (3.1.7) is easily verified through a simple calculation. To ease the computation we take $h = \varpi^t$ in Case 3b and take $h = (\varpi^{2t}, 1)$ in Case 4. By definition
\begin{equation}
r_\Lambda^0(h, g_0) \phi_2(x) = L(h)r_\Lambda^0(w)\phi_2(x).
\end{equation}
In Case 3b, $\phi_2 = 1_{\varpi^t O_K}$, since $v(\Lambda) = 0$ and $K/F$ is unramified, we get
\begin{equation}
r_\Lambda^0(w)\phi_2 = 1_{\varpi^{-t}O}.
\end{equation}
Hence $r_\Lambda^0(h, g_0) \phi_2 = L(h)1_{\varpi^{-t}O} = |\varpi|^t1_{O}$.

In Case 4, $K = F \oplus F$, $\phi_2 = 1_{O \times \varpi^{m-o}O}$ and $v(\Lambda) = m$, so
\begin{equation}
r_\Lambda^0(w)\phi_2 = |\Lambda|1_{\varpi^{-2t}O \times \varpi^{-m}O}.
\end{equation}
Therefore, $r_\Lambda^0(h, g_0) \phi_2 = L(h)1_{\varpi^{-2t}O \times \varpi^{-m}O} = |\Lambda|1_{O \times \varpi^{-m}O}$.

The claim now follows from Corollary 3.1.4 and Lemma 3.1.4. \hfill \Box

Proposition 3.1.8. We have
\begin{equation}
\Psi^+(s, W_1^0, W_2^0, \phi_2') = 0,
\end{equation}
Proof. As both $W_1^0$ and $f^0(s, g; \phi')$ are invariant under $U_0(1)$, thus the claim is clear by (3.1.4) and (3.1.5). \hfill \Box

We now assume that $\psi$ is a general additive character given by $\psi(x) = \psi_0(\delta x)$. Let $r_\Lambda$ be the Weil representation associated to $(K, \Lambda N)$ with respect to $\psi$. It is related to $r_\Lambda^0$ for $g_1 \in SL_2(F)$ through
\begin{equation}
r_\Lambda(g_1) = r_\Lambda^0(t(\delta)g_1t(\delta)^{-1}).
\end{equation}

Let $W_1(g) = W_1^0(t(\delta)gt(\delta^{-1}))$ and $W_2(g) = W_2^0(t(\delta)gt(\delta^{-1}))$, so they are in $\mathcal{W}(\pi_1, \psi)$ and $\mathcal{W}(\pi_2, \psi)$ respectively. The modified Rankin-Selberg convolution between $W_1$ and $W_2$ is given by
\begin{equation}
W^+(s, W_1, W_2, \phi_2) = \int_{Z(F)N(F) \backslash G^+(F)} W_1(eg)W_2(g)f(s, g; \phi_2)dg,
\end{equation}
where $f(s, g; \phi_2) = r_\Lambda(g_1)\phi_2(0)|\alpha_\delta(g)|^{2s-1}|\det(g)|^{-1/2}\omega(\det(g))$. Here, the measure $dg$ on $Z(F)N(F) \backslash G^+(F)$ is induced by that of (3.1.3) through the automorphism $g \rightarrow t(\delta^{-1})gt(\delta)$, and it is easy to see they are the same.

By (3.0.5) and (3.1.8)
\begin{equation}
f(s, t(\delta)^{-1}gt(\delta); \phi_2) = f^0(s, g; \phi_2).
\end{equation}
Proposition 3.1.9. Let $W_i(g) = W_i^0(t(\delta)gt(\delta^{-1}))$ for $i = 1, 2$. For $\Re(s) > 0$

(3.1.11) \[ \Psi^+(s, W_1, W_2, \phi_2) = \Psi^+(s, W_1^0, W_2^0, \phi_2). \]

For $1 \leq j < 2t = 2v(c(\chi))$

(3.1.12) \[ \Psi^+(s, \rho(t(\varpi^{-j}))W_1, W_2, \phi_2) = 0. \]

Proof. Let $g' = t(\delta)gt(\delta^{-1})$, then

\[
\Psi^+(s, W_1, W_2, \phi_2)
= \int_{Z(F)N(F)\setminus G^+(F)} W_1(\rho(t(\delta^{-1})g't(\delta)))W_2(t(\delta^{-1})g't(\delta))f(s, t(\delta^{-1})\gamma t(\delta); \phi_2)dg'
= \int_{Z(F)N(F)\setminus G^+(F)} W_1^0(\rho(\gamma)W_2^0(g')f^0(s, g'; \phi_2)dg',
\]

which is exactly the right hand side of (3.1.11).

The proof of (3.1.12) is similar and uses Proposition 3.1.6.

Proposition 3.1.10. For $j = 2v(c(\chi)) > 0$ and $C = \epsilon(\pi_1, \psi^0)\epsilon(\pi_2, \psi^0)$

(3.1.13) \[ \Psi^+(s, \rho(t(\varpi^{-j}))W_1, W_2, \phi_2) = C \int_{Z(F)N(F)\setminus G(F)^+} W_1(\rho(\gamma)W_2(\gamma)f(s, \gamma t(\delta); \phi_2)dg,
\]

and for $\phi_2'$ in (3.1.6)

(3.1.14) \[ f(1/2, \gamma t(\delta^{-1})\gamma t(\delta); \phi_2) = f(1/2, \gamma; \phi_2'). \]

Moreover,

(3.1.15) \[ \Psi^+(s, W_1, W_2, \phi_2') = 0. \]

Proof. Let $g' = t(\delta)gt(\delta^{-1})$. The left side of (3.1.13) becomes

\[
\int_{Z(F)N(F)\setminus G(F)^+} W_1(\rho(\gamma)W_2(\gamma)f(s, \gamma t(\delta); \phi_2)dg
= \int_{Z(F)N(F)\setminus G(F)^+} W_1^0(\rho(\gamma)W_2^0(\gamma)f^0(s, \gamma'; \phi_2)dg'
= C \int_{Z(F)N(F)\setminus G(F)^+} W_1^0(\rho(\gamma)W_2^0(\gamma)f^0(s, \gamma' \gamma t(\delta); \phi_2)dg',
\]

where in the last step we have used (3.1.5). The right side of (3.1.13) becomes

\[
C \int_{Z(F)N(F)\setminus G(F)^+} W_1(\rho(\gamma)W_2(\gamma)f(s, \gamma t(\delta); \phi_2)dg
= C \int_{Z(F)N(F)\setminus G(F)^+} W_1^0(\rho(\gamma)W_2^0(\gamma)f^0(s, \gamma' t(\delta); \phi_2)dg'
= C \int_{Z(F)N(F)\setminus G(F)^+} W_1^0(\rho(\gamma)W_2^0(\gamma)f^0(s, \gamma' \gamma t(\delta); \phi_2)dg'.
\]
By (3.1.7)

\[ f(1/2, gt(\delta^{-1})g_0 t(\delta); \phi_2) = f^0(1/2, t(\delta)gt(\delta^{-1})g_0 t(\delta)t(\delta^{-1}); \phi_2) = f^0(1/2, t(\delta)gt(\delta^{-1}); \phi'_2). \]

The proof of (3.1.15) is the same as that of (3.1.11). \( \square \)

Over a finite place \( v | c(\chi) \) we now pick a special vector in \( W(\pi_1, \psi) \). The space \( W(\pi_1, \psi) \) is equipped with an inner product, denoted by \( , \). Let \( W_1 \in W(\pi_1, \psi) \) be the Whittaker function defined in Proposition 3.1.9. By newform theory (see Lemma 3.1.11 below for a proof in our twisted situation), the subspace of vectors in \( W(\pi_1, \psi) \) which are invariant under \( t(\delta^{-1})U_1(c(\chi)^2)t(\delta) \) is generated by \( \rho(t(\varpi^{-j}))W_1 \) for \( j = 0, \cdots, 2v(\chi) \).

We define \( W_1^* \) to be a vector invariant under \( t(\delta^{-1})U_1(c(\chi)^2)t(\delta) \), such that
1. \( (W_1^*, W_1 - W_1^*) = 0 \),
2. \( W_1^*, \rho \left( \begin{array}{cc} \varpi^{-j} & 0 \\ 0 & 1 \end{array} \right) W_1 \) = 0 for \( 0 < j \leq v(c) \).

Such a vector is called a local quasi-newform in [12]. Note that \( W_1^* \) is unique and does not depend on the choice of inner product.

**Lemma 3.1.11.** We have \( W_1^*(1) = 1 \).

**Proof.** Let \( \tilde{W}_1^* = W_1^*(t(\delta^{-1})g_0 t(\delta)) \), so \( \tilde{W}_1^* \in W(\pi_1, \psi^0) \) and is invariant under \( U_1(c(\chi)^2) \). By newform theory ([1] [12]) \( \tilde{W}_1^* \) is of the form

\[ \tilde{W}_1^* = \sum_{j=0}^{2t} c_j \rho(t(\varpi^{-j}))W_1^0. \]

Conjugating back by \( t(\delta) \)

\[ W_1^* = \sum_{j=0}^{2t} c_j \rho(t(\varpi^{-j}))W_1. \]

So

\[ W_1^*(1) = \sum_{j=0}^{2t} c_j W_1^0(t(\varpi^{-j})) = c_0, \]

and

\[ (W_1^*, W_1^*) = (W_1^*, c_0 W_1 + \sum_{j \neq 0} c_j \rho(t(\varpi^{-j}))W_1)) = c_0(W_1^*, W_1), \]

therefore \( W_1^*(1) = c_0 = 1 \). \( \square \)
3.2. Archimedean case. Assume now that \( F = \mathbb{R} \) and let \( \psi \) be the standard additive character of \( F \) defined by \( \psi(x) = e^{2\pi ix} \). We write \( \pi_1 = \pi_v \) and \( \pi_2 = \pi_{\chi,v} \).

The representation \( \pi_1 \) is a discrete series of weight \( 2k = 2k_v \) and \( \pi_2 \) is a discrete series of weight \( 2r + 1 = 2r_v + 1 \). Let \( \pi \) be a discrete series of lowest weight \( k \), then \( W^0 \in W(\pi, \psi) \) is called a standard Whittaker function of weight \( k \) if its weight is \( k \), \( W^0(t(a)) = 0 \) for \( a < 0 \) and

\[
\int_{\mathbb{R}} W^0(t(a)) a^{s-1/2} d^\times a = L(s, \pi).
\]

Such a Whittaker function satisfies

\[
W^0(t(a)) = \begin{cases} 2a^{k/2} e^{-2\pi a} & \text{if } a > 0, \\ 0 & \text{if } a < 0. \end{cases}
\]

Similarly, we say \( W^0 \) is a standard Whittaker function of weight \(-k\) if its weight is \(-k\), and

\[
W^0(t(a)) = \begin{cases} 2|a|^{k/2} e^{2\pi a} & \text{if } a < 0, \\ 0 & \text{if } a > 0. \end{cases}
\]

The definition of \( L(s, \pi) \) will be recalled during the proof of the following proposition.

Locally at \( v \) (we drop \( v \) occasionally in the rest of this section):

\[
f(s, g; \phi_2) = r_\Lambda(g_1) \phi_2(0)|a(g)|^{2s-1}|\det(g)|^{-1/2} \omega(\det(g)),
\]

where \( \phi_2 \) is defined in Remark 2.2.1. Note that \( \phi_2(0) = 1 \), so \( f(s, 1, \phi_2) = 1 \). In the following we take \( l = k - r - 1 \) if \( k > r \) and \( l = r - k \) if \( r \geq k \). As in Section 1 we write \( \Sigma_1 \) for the set of infinite \( v \) with \( k_v > r_v \) and \( \Sigma_2 \) for the set of \( v \) with \( r_v \geq k_v > 0 \).

**Lemma 3.2.1** (Barnes’ Lemma). Assume \( f_1, f_2 \) are smooth functions on \( \mathbb{R}_+ \), such that

\[
\int_{\mathbb{R}_+} f_1(a) |a|^{s-1/2} d^\times a = G_1(s + r_1) G_1(s + r_2)
\]

\[
\int_{\mathbb{R}_+} f_2(a) |a|^{s-1/2} d^\times a = G_1(s + t_1) G_1(s + t_2),
\]

then

\[
\int_{\mathbb{R}_+} f_1(a) f_2(a) |a|^{s-1/2} d^\times a = \frac{\prod_{i,j=1}^2 G_1(s + r_i + t_j)}{G_1(2s + r_1 + r_2 + t_1 + t_2)}.
\]

**Proposition 3.2.2.** Let \( W_1 \) be the standard Whittaker function of weight \(-2k\) in \( \pi_1 \) and let \( W_2 \) be the standard Whittaker function of weight \( 2r \) in \( \pi_2 \). Then

\[
L(s, \pi_1 \times \pi_2) = \begin{cases} 2^{s+k+r+1/2} G_2(s + k - r - 1/2) \cdot \Psi^+(s, W_1, W_2, \phi_2), & \text{if } v \in \Sigma_1, \\ 2^{s+k+r+1/2} G_2(s + r - k + 1/2) \cdot \Psi^+(s, W_1, W_2, \phi_2), & \text{if } v \in \Sigma_2. \end{cases}
\]

Here the measure is taken such that the total measure of \( SO_2(\mathbb{R}) \) is 1.
Proof. By the Iwasawa decomposition and invariance of the triple product under $SO_2(\mathbb{R})$ from the weight computation
\[
\Psi^+(s) = \Psi^+(s, W_1, W_2, \phi_2) = \int_{Z(\mathbb{R})N(\mathbb{R})\backslash G(\mathbb{R})^+} W_1(eg)W_2(g)f(s, g; \phi_2)dg
\]
\[
= \int_{\mathbb{R}^+} W_1 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) W_2 \left( \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right) |a|^{s-1}f(s, 1, \phi_2)d^*a.
\]
Recall that
\[
G_1(s) = \pi^{s/2}\Gamma(s/2), \quad G_2(s) = 2(2\pi)^{-s}\Gamma(s).
\]
If $\nu$ is in $\Sigma_1$, i.e. $k > r$, then the local $L$-factors are given by
\[
L(s, \pi_1) = G_2(s + k - \frac{1}{2}), \quad L(s, \pi_2) = G_2(s + r),
\]
and
\[
L(s, \pi_1 \times \pi_2) = G_2(s + k + r - 1/2) \cdot G_2(s + k - r - 1/2)
\]
Now we let $r_1 = k - 1/2$, $r_2 = k + 1/2$, $t_1 = r$, $t_2 = r + 1$ and apply the Barnes’ formula:
\[
\Psi(s) = \frac{G_1(s + k + r - 1/2)G_1(s + k + r + 1/2)G_1(s + k + r + 1/2)}{G_1(2s + 2k + 2r + 1)}.
\]
So the difference between $\Psi$ and the Rankin-Selberg convolution is given by
\[
L(s, \pi_1 \times \pi_2) = \Psi(s) \cdot \frac{G_2(s + k - r - 1/2)G_1(2s + 2k + 2r + 1)}{G_2(s + k + r + 1/2)}
\]
\[
= 2^{s+k+r+1/2}G_2(s + k - r - 1/2) \cdot \Psi(s),
\]
where we have used the formula
\[
G_1(2s) = 2^{s-1}G_2(s) = 2^{s-1}G_1(s)G_1(s + 1).
\]
The case when $\nu \in \Sigma_2$ can be treated exactly in the same way. □

3.3. Global case. Let $\pi_1 = \pi = \prod_v \pi_v$ and $\pi_2 = \pi_\chi = \prod_v \pi_{\chi, v}$. We take the measure on $G(\mathbb{A})^+$ to be the product of the local ones used in previous sections, and define
\[
W = \prod_v W_{1, v}, \quad W^* = \prod_{v \notin \chi} W_{1, v} \prod_{v \in \chi} W_{1, v}^*, \quad W_\chi = \prod_v W_{2, v},
\]
where these local Whittaker functions are given in Sections 3.1 and 3.2. The corresponding automorphic forms on $G(\mathbb{A})$ are constructed as follows
\[
(3.3.1) \quad \varphi(g) = \sum_{\xi \in F^*} W \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right),
\]
\[
(3.3.2) \quad \varphi^*(g) = \sum_{\xi \in F^*} W^* \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right),
\]
(3.3.3) \[ \varphi_{\chi}(g) = C_{\chi}(g) + \sum_{\xi \in F^\times} W_{\chi} \left( \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g \right), \]

where \( C_{\chi}(g) \) is the (possibly zero) constant term of \( \varphi_{\chi} \). For a nonempty set \( T \) of places dividing \( v \) we let \( g_0^T = \prod_{v \in T} g_{0,v} \), where \( g_{0,v} = \begin{pmatrix} 0 & 1 \\ -c(\chi_v)^2 & 0 \end{pmatrix} \). We also write

\[ \phi_2 = \prod_v \phi_{2,v}, \quad \phi'_{2,T} = \prod_{v \not\in T} \phi_{2,v} \prod_{v \in T} \phi'_{2,v}, \]

where \( \phi_{2,v} \in S(K_v) \) is given in Propositions 3.1.1 and 3.2.2, and \( \phi'_{2,v} \) is defined in Lemma 3.1.7. The Eisenstein series associated to \( \phi_2 \) is defined by

(3.3.4) \[ E(s, g; \phi_2) = \sum_{\gamma \in B(F) \backslash G(F)} f(s, \gamma g; \phi_2), \]

and the Eisenstein series attached to \( \phi'_{2} \) is given by

(3.3.5) \[ E(s, g; \phi'_{2,T}) = \sum_{\gamma \in B(F) \backslash G(F)} f(s, \gamma g; \phi'_{2,T}), \]

where \( f(s, g; \phi) = r_A(g_1) \phi(0)|a_\delta(g)|^{2s-1} \det g^{-1/2} \omega(\det g) \) for any \( \phi \in S(K_h) \). Both Eisenstein series have analytic continuation over the whole \( s \)-plane. By (3.1.14) and analytic continuation

(3.3.6) \[ E(1/2, g t (\delta^{-1}) g_0^T t(\delta); \phi_2) = E(1/2, g; \phi'_{2,T}). \]

**Lemma 3.3.1.** Let \( a|c(\chi)^2 = c(\pi^\chi) \) be a non-unit integral idele and let \( \varphi_a = \rho(t(a^{-1}))\varphi \), then

(3.3.7) \[ \int_{Z(k)G(F) \backslash G(k)} \varphi_a(g) \varphi_{\chi}(g) E(1/2, g; \phi_2) dg = 0. \]

**Proof.** We write \( W_a = \rho(t(a^{-1}))W \). If \( v(a) < c(\pi^\chi) \) for all \( v \), then

\[ \int_{Z(k)G(F) \backslash G(k)} \varphi_a(g) \varphi_{\chi}(g) E(s, g; \phi_2) dg = \int_{Z(k)N(k) \backslash G(k)} W_a(\epsilon g) W_{\chi}(g) f(s, g; \phi_2) dg = 0. \]

If \( v(a) = v(c(\pi^\chi)) \) for \( v \in T \), then by Proposition 3.1.9

\[ \int_{Z(k)G(F) \backslash G(k)} \varphi_a(g) \varphi_{\chi}(g) E(1/2, g; \phi_2) dg = \int_{Z(k)N(k) \backslash G(k)} \varphi_a(g) \varphi_{\chi}(g) E(1/2, g; \phi'_{2,T}) dg, \]

which is zero by (3.1.15) through analytic continuation. \( \square \)

We say \( \psi \in \pi \) is holomorphic (or anti-holomorphic) if its Whittaker function

\[ W_{\psi}(g) = \int_{F \backslash \Lambda} \psi \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx \]
satisfies
\[ W_\psi \left( \begin{pmatrix} a_\infty a_f & 0 \\ 0 & 1 \end{pmatrix} \right) = \hat{\psi}(a_f) W_\infty(t(a_\infty)) \] (or \( \hat{\psi}(a_f) W_\infty(t(a_\infty)) \)),

where \( \hat{\psi} \) is a function of finite ideles \( a_f \), \( W_\infty = \prod_{v|\infty} W_v \) (or \( W_\infty \)) is the standard holomorphic (or anti-holomorphic) Whittaker function of weight \((2k_1, \cdots, 2k_d)\) (or \((-2k_1, \cdots, -2k_d)\)) defined in Section 3.2. The number \( \hat{\psi}(a_f) \) is called the \( a_f \)-th Fourier coefficient of \( \psi \).

**Proposition 3.3.2.** For any anti-holomorphic form \( \psi \in \pi \) of level \( t(\delta^{-1})U_0(Nc(\chi)^2)t(\delta) \)
\[ (\psi, \varphi^*) = \hat{\psi}(1)(\varphi^*, \varphi^*). \]

**Proof.** By newform theory such a \( \psi \) has the form
\[ \psi = c_0 \varphi + \sum_{a|c(\chi)^2} c_a \varphi_a, \]
where \( a \) is a nontrivial class modulo local units. As \( W_a(1) = 0 \) for nontrivial \( a \), so \( \hat{\psi}(1) = c_0 \varphi(1) = c_0 \), and
\[ (\psi, \varphi^*) = \hat{\psi}(1)(\varphi^*, \varphi^*) = \hat{\psi}(1)(\varphi^*, \varphi^*). \]

Here we have used the facts that \( (\varphi^*, \varphi_a) = 0 \) for non-unit \( a|c(\pi_\chi) \), and \( (\varphi^*, \varphi^*) = (\varphi^*, \varphi) \) (from the local definition). \( \square \)

**Proposition 3.3.3.** Let us retain the above notations, then
\begin{equation}
L(1/2, \pi \times \pi_\chi) = \frac{M \cdot L_f(1, \omega)^2|S|}{\mu(ND)^+} \int_{Z(\mathcal{A})G(F)^+ \backslash G(\mathcal{A})^+} \varphi^*(g) \varphi_\chi(g) E(1/2, g; \phi_2) dg,
\end{equation}
where \( L_f(1, \omega) \) denotes the finite part of the \( L \)-function, \( D = c(\chi)^2 c(\omega) \), \( \mu(ND)^+ = \mu(U_0(ND)^+) \), \( S \) is the set of finite places dividing \( c(\omega) \), and \( M \) is given by
\begin{equation}
M = \prod_{v|\infty} 2^{k_v + r_v + 1} \prod_{v \in \Sigma_1} G_2(k_v - r_v) \prod_{v \in \Sigma_2} G_2(1 + r_v - k_v).
\end{equation}

**Proof.** By Lemma 3.1.11 and newform theory we know that \( \varphi^* = \varphi + \sum_{a|c(\chi)^2} c_a \varphi_a \), where the sum is over non-unit integral \( a \) modulo the local units. Now (3.3.8) follows from Proposition 3.1.9, Proposition 3.2.2 and Lemma 3.3.1. \( \square \)

### 4. Theta correspondence and main formula

In this section we first realize the terms \( \varphi_\chi \) and \( E(1/2, g; \phi_2) \) in (3.3.3) as theta lifts. Then we apply a seesaw duality to express the central value as a double torus integral. At last we use the Shimizu correspondence to determine the integrand explicitly. The main central value formula is therefore obtained.
4.1. Theta series and Eisenstein series. Let $K$ as before be the quadratic CM extension of $F$. Let $\phi_1 = \prod_v \phi_{1,v} \in S(K_v)$, where $\phi_{1,v}$ are defined in Section 2.2. The theta lift of $\chi$ from $GSO(K_v)$ to $G(\mathbb{A}_F^+)$ is defined by

\[
\theta(g,\chi;\phi_1) = \int_{SO(K)\backslash SO(\mathbb{A}_K)} \theta(th,g;\phi_1)\chi(th)dt,
\]

where $h$ is any element of $GSO(\mathbb{A}_K) = \mathbb{A}_K^\times$ such that $N_K(h) = \det(g)$, and $\theta(h,g;\phi_1)$ is the theta kernel defined in (2.1.2) attached to $(K,N_K)$. We normalize the measure on $SO(\mathbb{A}_K) = \mathbb{A}_K^1$ such that $K_v^1 \cap U_{K_v}$ is of measure 1 for any finite place $v$, and $K_v^1 = SO(2)$ is of measure 1 for any archimedean $v$. The total measure of $SO(K)\backslash SO(\mathbb{A}_K)$ is $2L_f(1,\omega)$ (Waldspurger [9] Section 1.5).

**Proposition 4.1.1.** Let $\phi_2 \in S(K_v)$ be a Schwartz function in $S(K_v)$ defined by

\[
\phi_{1,v}(z) = \begin{cases} 1_{\mathcal{O}_{K_v}}(z) & \text{if } v \text{ is finite and } \chi_v \text{ is unramified}, \\ \chi_v(z)1_{\mathcal{O}_{K_v}^*}(z) & \text{if } \chi_v \text{ is ramified}, \\ 2z^{2r_v}e^{-2\pi(|z|^2)} & \text{if } v \text{ is archimedean}. \end{cases}
\]

(1) Let $U_0(D)_v^+ = U_0(D) \cap G(F_v)^+$ and let $\delta_v$ be the conductor of $\psi$ at a finite place $v$. For $k' \in t(\delta_v)^{-1}U_0(D)_v^+t(\delta_v)$ and $g \in G(\mathbb{A}_F^+)$ we have

\[
\theta(\chi, gk';\phi_1) = \theta(\chi, g;\phi_1)
\]

if $v$ is unramified in $K$, and

\[
\theta(\chi, gk';\phi_1) = \omega(k')\theta(\chi, g;\phi_1)
\]

if $v$ is ramified in $K$. Here in the last identity $\omega(k') = \omega(d)$ if $k' = t(\delta_v^{-1})\begin{pmatrix} a & b \\ c & d \end{pmatrix}t(\delta_v)$. Recall that $t(\delta) = \begin{pmatrix} \delta & 0 \\ 0 & 1 \end{pmatrix}$.

At an archimedean $v$ the function $\theta(\chi, g;\phi_1)$ has weight $2r_v + 1$.

(2) For all places $v$ not dividing $D$ and $g \in G(F_v)^+$, the local Whittaker function $W_v(\chi, g;\phi_1)$ of $\theta(\chi, g;\phi_1)$ equals $W_{\chi,v}(g)$ (defined in Section 3). At $v|D$ we have $W_v(\chi, g;\phi_1) = W_{\chi,v}(g)$ for $g \in T_1(F_v)^+$.

**Proof.** (1) The conclusion is already proved in [4] Proposition 4.1.2 for a finite $v$. The weight at an infinite $v$ is given in Proposition 2.2.5.

(2) By [4] Proposition 4.1.2 we only need to check $g = t(a) = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ for $a > 0$ at an archimedean place $v$.

We quote a formula for the Whittaker function of $\theta(\chi, g;\phi)$ (see [6])

\[
W_v(\chi, g;\phi_1) = \int_{\mathbb{C}^1} L(h)r_1(g_1)\phi_1(\sigma^{-1})\chi(\sigma h)d\sigma,
\]

where $h \in \mathbb{C}$ is any number such that $N(h) = \det(g)$. We will simply take $h = \sqrt{a}$.
As \( \phi_1(z) = 2(x - iy)^2e^{-2\pi(x^2+y^2)} \), so we have

\[
W_{e}(\chi, t(a); \phi_1) = a^{1/2} \int_{\mathbb{C}^1} \phi_1(at^{-1}h^{-1})\chi(th)dt
= a^{1/2} \int_{\mathbb{R}/2\pi\mathbb{Z}} 2(\sqrt{ae^{-i\theta}})^2 e^{-2\pi a}e^{-2r\theta}da
= 2a^{(2r+1)/2}e^{-2\pi a},
\]

which is exactly \( W_{\chi,v}(t(a)) \).

Let \( \theta(h, g; \phi_2) \) be the theta kernel attached to the quadratic space \((K, \Lambda N_K)\) (Section 2.1). The theta lift of 1 from \(GSO(K)\) to \(G(\mathbb{A})^+\) is given by

\[
I(g; \phi_2) = \theta(1, g; \phi_2) = \int_{SO(K)\backslash SO(K)} \theta(th, g; \phi_2)dt,
\]

where \( h \in \mathbb{A}_K^\times \) with \( N_K(h) = \det g \), and the measure on \( SO(K)\backslash SO(K) \) is the same as that used in defining the theta series \( \theta(g, \chi; \phi_1) \).

**Proposition 4.1.2 (Siegel-Weil).** For any \( g \in G(\mathbb{A})^+ \) one has

\[
E(1/2, g; \phi_2) = L_f(1, \omega)^{-1}I(g; \phi_2).
\]

**Proof.** For \( g \in SL_2(\mathbb{A}) \) and the Eisenstein series defined through the standard maximal compact subgroup, the above formula is proved in Proposition 31 of [8]. At \( s = 1/2 \) the standard Eisenstein series and our twisted one coincide as \( |a(g)|^{2s-1} = |a_3(g)|^{2s-1} \) for \( s = 1/2 \).

The extension to \( G(\mathbb{A})^+ \) now follows from Theorem 4.2 of [3].

**Proposition 4.1.3.** Let us retain the notations, then

\[
(4.1.2) \quad L(1/2, \pi \times \pi_\chi) = \frac{2|S|M}{\mu(ND)^+} \int_{Z(\Lambda)G(F)^+\backslash G(\mathbb{A})^+} \varphi^*(g)\theta(\chi, g; \phi_1)I(g; \phi_2)dg.
\]

**Proof.** The same argument as [4] Proposition 4.3.1 shows the following identity for \( \Re(s) \gg 0 \)

\[
\int_{Z(\Lambda)G(F)^+\backslash G(\mathbb{A})^+} \varphi'(g)\varphi_\chi(g)E(s, g; \phi_2)dg = \int_{Z(\Lambda)G(F)^+\backslash G(\mathbb{A})^+} \varphi'(g)\theta(\chi, g; \phi_1)E(s, g; \phi_2)dg,
\]

for any automorphic form \( \varphi' \) in \( \pi \). By analytic continuation and Propositions 3.3.3

\[
L(1/2, \pi \times \pi_\chi) = \frac{2|S|ML_f(1, \omega)}{\mu(ND)^+} \int_{Z(\Lambda)G(F)^+\backslash G(\mathbb{A})^+} \varphi^*(g)\theta(\chi, g; \phi_1)E(1/2, g; \phi_2)dg.
\]

Now Proposition 4.1.2 gives the desired formula. \( \square \)
4.2. **Seesaw identity.** We fix an embedding of $K$ into $B$ and $j \in B$ as in Section 2.2, such that $B = K + Kj$ with $j^2 = -\Lambda$. Therefore $(B, N_B) = (K, N_K) \oplus (K_j, N_K) = (K, N) \oplus (K, \Lambda N)$.

In Proposition 4.1.3 we find that the central value is an integral which involves two theta lifts $\theta(\chi; g; \phi_1)$ and $I(g; \phi_2)$. To place these theta lifts together we use the following seesaw dual pair ([5]):

\[
\begin{array}{ccc}
R(G(\mathbb{A})^+) \times G(\mathbb{A})^+ & \overset{GSO(B)}{\longrightarrow} & GSO(K_{\mathbb{A}}) \\
G(\mathbb{A})^+ & \overset{R(GSO(K_{\mathbb{A}}) \times GSO(K_{\mathbb{A}}, j))}{\longrightarrow} & R(GSO(K_{\mathbb{A}}) \times GSO(K_{\mathbb{A}, j}))
\end{array}
\]

Here the left vertical map is the diagonal embedding, and the right vertical one is the natural embedding given by regarding $(\mu, \nu)$ as the similitude

\[x + yj \mapsto \mu(x) + \nu(y)j \in B^2_{\mathbb{A}}.\]

Let $r_1, r_\Lambda$ and $r_B$ be the Weil representations defined on $R(GSO(K_{\mathbb{A}}) \times G(\mathbb{A})^+)$, $R(GSO(K_{\mathbb{A}, j}) \times G(\mathbb{A})^+)$, and on $R(GSO(B_{\mathbb{A}}) \times G(\mathbb{A})^+)$ respectively (Section 2.1).

We take $\phi \in S(B_{\mathbb{A}}) = S(K_{\mathbb{A}}) \otimes S(K_{\mathbb{A}, j})$ to be $\phi(x_1 \oplus x_2 j) = \phi_1(x_1)\phi_2(x_2)$, where $\phi_1$ and $\phi_2$ are given in Proposition 4.1.3, so

\[r_B[(h_1, h_2), g]|(x_1 \oplus x_2 j) = r_1(h_1, g)\phi_1(x_1)r_2(h_2, g)\phi_2(x_2).\]

where $(h_1, h_2) \in R(GSO(K_{\mathbb{A}}) \times GSO(K_{\mathbb{A}, j}))$ with similitude factors $\nu(h_1) = \nu(h_2) = \det(g)$. Consequently one has a decomposition for the theta kernels

\[
\theta_B((h_1, h_2), g; \phi) = \theta(h_1, g; \phi_1)\theta(h_2, g; \phi_2).
\]

Before going anywhere further we first prove a general seesaw identity for the above seesaw dual pair. Let $F_1$ and $F_2$ be two cuspidal forms on $Z(\mathbb{A})G(F) \setminus G(\mathbb{A})^+$ and $Z(\mathbb{A})H(F) \setminus H(\mathbb{A})$ respectively, here we write $H = R(GSO(K) \times GSO(K, j))$ for short. We define theta lifts

\[
\theta(F_1, h; \phi) = \int_{SL_2(F) \setminus SL_2(\mathbb{A})} \theta_B(h, g_1 g; \phi)F_1(g_1 g)dg_1,
\]

\[
\theta(F_2, h; \phi) = \int_{H_1(F) \setminus H_1(\mathbb{A})} \theta_B(h_1 h; g_1 \phi)F_2(h_1 h)dh_1,
\]

where $\nu(h) = \det(g)$,

**Lemma 4.2.1** (Seesaw identity). We have

\[
\int_{Z(\mathbb{A})G(F) \setminus G(\mathbb{A})^+} \theta(F_2, g; \phi)F_1(g)dg = \int_{Z(\mathbb{A})H(F) \setminus H(\mathbb{A})} \theta(F_1, h; \phi)F_2(h)dh.
\]
Proof. Let $S$ be the compact group $F^+ \mathbb{A}_{\infty} \backslash \mathbb{A}^+$, where $F^+ = \mathbb{N}(K^\times)$, $\mathbb{A}^+ = \mathbb{N}(\mathbb{A}_K^\times)$ and $\mathbb{A}_{\infty}^+ = \mathbb{N}(K_{\infty}^\times)$. Using the maps $\text{det}$ of $G$ and $\nu$ of $H$ we have

$$1 \to SL_2(F) \backslash SL_2(\mathbb{A}) \to \mathbb{A}_{\infty}^+ G(F)^+ \backslash G(\mathbb{A})^+ \to S \to 1,$$

and

$$1 \to H_1(F) \backslash H(\mathbb{A}) \to \mathbb{A}_{\infty}^+ H(F) \backslash H(\mathbb{A}) \to S \to 1.$$

We also have

$$1 \to \mathbb{A}_{\infty}^+ F^\times \mathbb{A}^\times \to \mathbb{A}_{\infty}^+ G(F)^+ \backslash G(\mathbb{A})^+ \to Z(\mathbb{A}) G(F)^+ \backslash G(\mathbb{A})^+ \to 1,$$

and

$$1 \to \mathbb{A}_{\infty}^+ F^\times \mathbb{A}^\times \to \mathbb{A}_{\infty}^+ H(F) \backslash H(\mathbb{A}) \to Z(\mathbb{A}) H(F) \backslash H(\mathbb{A}) \to 1.$$

Here the measure $ds$ on $S$ is the product of local measures on $K_v^+$ such that $\mathbb{N}_K(O_{K,v}^\times)$ has volume 1. The measure on $\mathbb{A}_{\infty}^+ F^\times \mathbb{A}^\times$ is the product of local measures on $F_v^\times$ such that $O_v^\times$ has volume 1. The measures on $SL_2(\mathbb{A})$ and $H_1(\mathbb{A})$ are induced from the exact sequences. Since $H_1(F) \backslash H(\mathbb{A}) = [SO(K) \backslash SO(K_{\mathbb{A}})]^2$, the measure thus chosen on $SO(K) \backslash SO(K_{\mathbb{A}})$ is the same as the one in Section 4.1. If we write $\mu = \mu(\mathbb{A}_{\infty}^+ F^\times \mathbb{A}^\times)$, then the left hand side of (4.2.3) equals

$$\int_{Z(\mathbb{A}) G(F)^+ \backslash G(\mathbb{A})^+} \theta(F_2, g; \phi) F_1(g) dg$$

$$= \frac{1}{\mu} \int_{\mathbb{A}_{\infty}^+ G(F)^+ \backslash G(\mathbb{A})^+} \theta(F_2, g; \phi) F_1(g) dg$$

$$= \frac{1}{\mu} \int S \int_{SL_2(F) \backslash SL_2(\mathbb{A})} \theta(F_2, g_1 s; \phi) F_1(g_1 s) dg_1 ds,$$

and the right hand side of (4.2.3) becomes

$$\frac{1}{\mu} \int S \int_{H_1(F) \backslash H(\mathbb{A})} \theta(F_1, h_1 s; \phi) F_2(h_1 s) dh_1 ds.$$

Now (4.2.3) follows by Fubini theorem and the definition of theta lifts. \hfill \Box

Now we take $F_1 = \varphi^*$ and $F_2(h) = \chi(h) = \chi(t_1)$ for $h = (t_1, t_2)$, then by (4.2.2) and (4.2.3) the integral in (4.1.2) becomes

$$(4.2.4) \quad \int_{Z(\mathbb{A}) G(F)^+ \backslash G(\mathbb{A})^+} \varphi^*(g) \theta(\chi, g; \phi_1) I(g; \phi_2) dg = \int_{Z(\mathbb{A}) H(F) \backslash H(\mathbb{A})} \theta(\varphi^*, h; \phi) \chi(h) dh.$$

To obtain a better form of the integral we use the following identification:

$$(4.2.5) \quad \begin{array}{c}
\text{GSO}(B_{\mathbb{A}}) \xleftarrow{\sim} B_{\mathbb{A}}^\times \times B_{\mathbb{A}}^\times / \mathbb{A}^\times \\
\uparrow \hspace{1cm} \uparrow \\
H(\mathbb{A}) \xrightarrow{\sim} K_{\mathbb{A}}^\times \times K_{\mathbb{A}}^\times / \mathbb{A}^\times
\end{array}$$
Here, the top isomorphism is given by \((g_1, g_2)b = g_1b g_2^{-1}\) for \(g_1, g_2 \in B^x\) and \(b \in B\); the left vertical arrow is the one given in the above, and the right one is induced by the fixed embedding. So the bottom one sends \((t_1, t_2) \in K_h^x \times K_h^x\) to \((t_1 t_2^{-1}, t_1 t_2^{-1}) = (h_1, h_2) \in R(GSO(\hat{\mathfrak{a}}) \times GSO(K_hj))\).

Under the above identification the right side of (4.2.4) becomes

\[
\int_{Z(\mathfrak{a})H(F)\setminus H(\mathfrak{a})} \theta(\varphi^*, h; \phi) \chi(h) dh = \int_{(A^x K_h^x \backslash K_h^x)^2} \theta(\varphi^*, (t_1, t_2); \phi) \chi(t_1 t_2^{-1}) dt_1 dt_2.
\]

**Proposition 4.2.2.** The central critical value of \(L(s, \pi \times \pi_{\chi})\) is given by

\[
L(1/2, \pi_f \times \pi_{\chi}) = \frac{2|S| M}{\mu(D)^{1+}} \int_{(A^x K_h^x \backslash K_h^x)^2} \theta(\varphi^*, (t_1, t_2); \phi) \chi(t_1 t_2^{-1}) dt_1 dt_2,
\]

where \(M\) and \(D\) are given in Proposition 3.3.2.

**4.3. Main formula.** In this section we first show that the theta lift \(\theta(\varphi^*, \sigma; \phi')\) on \(A^x \setminus B_h^x \times B_h^x\) decomposes as a product of two automorphic forms on \(B_h^x / B_h^x A^x\) (here and later on we always identify \(GSO(B)\) and \(B^x \times B^x / F^x\)). Then we use the decomposition to derive the main central value formula.

**Lemma 4.3.1.** The automorphic form \(\theta(\varphi^*, x, y; \phi)\), regarded as a form on \(B_h^x \times B_h^x\), has the following level (or weight) structures:

1. \(\theta(\varphi^*, xk_1, yk_2; \phi') = \chi(k_1^{-1} k_2)\theta(\varphi^*, x, y; \phi')\) for \(k_1, k_2 \in R^x\),
2. \(\theta(\varphi^*, xk, yk\beta; \phi) = e^{2ri(\alpha - \beta)} \theta(\varphi^*, x, y; \phi)\) for \(k_\alpha = e^{\alpha}\). See Section 2.2 for the definition of \(\hat{R}^x\) and \(\chi\) on it.

**Proof.** The theta kernel is given by

\[
\theta_B((x, y), g; \phi') = \sum_{b \in B(F)} r_B((x, y), g)) \phi'(b) = \sum_{b \in B(F)} |N_B(x^{-1} y)| r_B(g_1) \phi'(x^{-1} b y),
\]

where \((x, y) \in B_h^x\) such that \(\nu(x, y) = N_B(xy^{-1}) = \det g\). Now the claims follow from Propositions 2.2.3 and 2.2.5. \(\square\)

**Proposition 4.3.2.** We have

\[
\theta(\varphi^*, x, y; \phi') = C \varphi^B(x) \cdot \varphi^B(y),
\]

where \(C\) is certain constant to be determined later in Theorem 2, \(\varphi^B\) is an automorphic form in the automorphic representation \(\pi^B\), which is determined up-to a constant multiple by the following level structures:

1. \(\varphi^B\) has weight \(2r_v\) at an archimedean place \(v\),
2. the action of \(k \in \hat{R}^x\) is given by

\[
\varphi^B(xk) = \chi(k) \varphi^B(x).
\]
Proof. In [6] it was shown that $\theta(\phi, x, y; \psi)$ is in the product of $\pi^B \otimes \check{\pi}^B = \pi^B \otimes \pi^B$, where $\check{\pi}^B$ is the contragredient of $\pi^B$. By Theorem 2.4.3 of [12] level structures (1) and (2) determine an automorphic form $\varphi^B$ in $\pi^B$ uniquely (up-to a constant multiple). The statement is now clear by Lemma 4.3.1.

\begin{proposition}
The central value is given by
\begin{equation}
L(1/2, \pi \times \pi) = \frac{M}{\mu(Nc(\chi)^2)^+}\left| \int_{A^x K^x \setminus A^x_K} \varphi^B \chi^{-1}(t)dt \right|^2.
\end{equation}
\end{proposition}

where $M$ is given in (3.3.9), and $C, \varphi^B$ are given in Proposition 4.3.2.

Proof. By Proposition 4.3.2 it suffices to show
\begin{equation}
\int_{(A^x K^x \setminus K^x_K)^2} \theta(\varphi^*, (t_1, t_2); \psi)\chi(t_1 t_2^{-1})dt_1dt_2
\end{equation}
\begin{equation}
= \frac{2|S|}{\mu(c(\omega)^+} \int_{(A^x K^x \setminus K^x_K)^2} \theta(\varphi^*, (t_1, t_2); \psi)\chi(t_1 t_2^{-1})dt_1dt_2.
\end{equation}

But this is proved in Theorem 5.3.9 of [4].

We now determine the constant $C$ in Proposition 4.3.3. The method used here is inspired by [10], but is simpler.

Let $\varphi'$ be the theta lift of $\varphi^B(x)\varphi^B(y)$ from $GSO(B_K)$ to $G(\mathbb{A})^+$ with respect to $\psi' \in S(B_K)$ (here $G(\mathbb{A})^+$ denotes the matrices with determinants in $N_B(B^x_K)$). In other words ([6] or [10])
\begin{equation}
\varphi'(g) = \int_{A^x B^x \setminus B^x_K} \int_{B^x \setminus B^x_K} \theta_B[(yx \sigma, y), g; \psi'] \varphi^B(y \sigma) \varphi^B(y)dxdy
\end{equation}

with $\sigma \in B^x_K$ such that $N(\sigma) = \det(g)$. The measure on $GSO(B_K) \cong PB^x_K \times B^x_K$ is normalized such that the following adjoint identity holds
\begin{equation}
(\varphi', \overline{\varphi}) = (\varphi^B \varphi^B, \theta(\cdot, \varphi^*, \psi'))_{GSO(B)} = C(\varphi^B \varphi^B, \varphi^B \varphi^B).
\end{equation}

Shimizu [6] showed that $\varphi'$ is a cuspidal form in $\pi$ (after extending $\varphi'$ to $G(\mathbb{A})$ by left-invariance under $G(F)$). By Proposition 2.2.5 and Proposition 2.2.3 the form $\varphi'$ has weight $(2k_1, \cdots, 2k_d)$ and level $t(\delta)^{-1}U_0(Nc(c)^2)t(\delta)$, so $\overline{\varphi'}$ is anti-holomorphic of weight $(-2k_1, \cdots, -2k_d)$. Proposition 3.3.2 implies
\begin{equation}
(\varphi', \overline{\varphi}) = (\varphi^*, \varphi') = \overline{\varphi'}(1)(\varphi^*, \varphi^*).
\end{equation}

The Whittaker function of $\overline{\varphi'}$ is related to that of $\varphi'$ by $W_{\overline{\varphi'}}(g) = \overline{W_{\varphi'}(\epsilon g)}$, where $\epsilon = t(-1) \in G(\mathbb{A})$, so $\overline{\varphi'}(1) = \overline{\varphi'}(1)$ and
\begin{equation}
(\varphi', \overline{\varphi}) = \overline{\varphi'}(1)(\varphi^*, \varphi^*).
\end{equation}
To compute the first Fourier coefficient of \( \varphi' \) we note that the Whittaker function of \( \varphi' \) is given by \((6) [10]\)

\[
W_{\varphi'}(g) = \int_{B^\times \setminus B_\mathbb{A}^\times} L(x\sigma, 1)r_B(g_1)\phi'(1)\varphi^B(yx\sigma)dx\overline{\varphi^B(y)}dy.
\]

**Proposition 4.3.4.** Let

\[W(g, y) = \int_{B\mathbb{A}\setminus B_\mathbb{A}^\times} L(x\sigma, 1)r_B(g_1)\phi'(1)\varphi^B(yx\sigma)dx,
\]

then for \( g = t(a_\infty 1_f) \)

\[W(g, y) = \mu(\hat{R}^1)M' W_\infty(t(a_\infty))\varphi^B(y),\]

where \( W_\infty \) is the standard Whittaker function of weight \((2k_1, \ldots, 2k_d)\), \( \mu(\hat{R}^1) \) is the measure of \( \hat{R}^1 \subset B_\mathbb{A}^1 \), and

\[
M' = \prod_{v|\infty} 2(4\pi)^{k_v - r_v - 1} \frac{(k_v + r_v - 1)!}{(2k_v - 1)!}.
\]

**Proof.** At a finite place \( v \nmid c(\chi) \)

\[
\int_{B_{v}^\times} L(x, 1)r_B(1)\phi'(1)\varphi^B(yx_v)dx_v
\]

\[= \int_{B_{v}^\times} \phi'(x_v^{-1})\varphi^B(yx_v)dx_v = \int_{R_v^1} \varphi^B(yx_v)dx_v = \mu(R_v^1)\varphi^B(y)
\]
as \( \varphi^B \) is invariant under \( R_v^\times \) (here \( R_v \) may be \( \hat{R}_v \)). At a place \( v|c(\chi) \)

\[
\int_{B_{v}^\times} L(x, 1)r_B(1)\phi'(1)\varphi^B(yx_v)dx_v
\]

\[= \int_{B_{v}^\times} \phi'(x_v^{-1})\varphi^B(yx_v)dx_v = \int_{R_v^1} \chi(x_v)^{-1} \varphi^B(yx_v)dx_v = \mu(R_v^1)\varphi^B(y)
\]
as \( \varphi^B \) is \( \chi \)-isotypic under the action of \( R_v^\times \).

We assume now \( v \) is an archimedean place. Because \( \varphi' \) has the lowest weight \( 2k_v \) at each archimedean place \( v \) it must be holomorphic over \( v \) and \( W_{\varphi'}(t(a_v)) = W_\infty(t(a_v)) = 0 \) for \( a_v < 0 \). Therefore we assume that \( g = t(a_v) \) for \( a_v > 0 \) from now on.

**Definite case.** Let \( B_v = \mathbb{H} \) be the Hamiltonian quaternion and take \( \sigma = a^{1/2} \in \mathbb{H} \). The integral \( W(g, y) \) is a Hecke operator on \( \varphi^B \) at place \( v \). We will compute its eigenvalues using the model of \( \pi \) that is given by matrix coefficients \( \{t_{r,s}^l\} \) on \( SU(2) \), where \( l = k - 1 \) and \( s \) varies (see Appendix A). Because the weight of \( \varphi^B \) at \( v \) is \( 2r \) the vector \( t_{r,s}^l \) is the one in this model that
corresponds to $\varphi^B$. One has

$$W(g, y) = \int_{SU(2)} a \phi(e^{x^2}) \varphi^B(y x \sigma) dx$$

$$= \int_{SU(2)} a \cdot 2(a^{1/2} u)^{2r} p_{k-r-1}(4\pi a |v|^2) e^{-2\pi a |u|^2 |v|^2} t^{k-1}_{rr}(x) dx \cdot \varphi^B(y)$$

$$= 2a^{r+1} e^{-2\pi a} \int_{SU(2)} u^{2r} p_{k-r-1}(4\pi a |v|^2) t^{k-1}_{rr}(x) dx \cdot \varphi^B(y)$$

(4.3.8)  

$$= 2a^{r+1} e^{-2\pi a} \int_0^\pi \left( \frac{\cos \theta + 1}{2} \right)^r p_{k-r-1}(2\pi a (1 - \cos \theta)) P^{k-1}_{2r} \sin \theta \theta \cdot \varphi^B(y)$$

Here in the above identity we have used $(l = 1 - k)$

$$P^l_{rr}(z) = \frac{(-1)^{l-r}}{2^l} \frac{1}{(l+r)!} (1+z)^r \frac{d^{l+r}}{dz^{l+r}} [(1+z)^{l-r} (1-z)^{l-r}],$$

and the fact that $t^{l}_{rr}(1) = P^l_{rr}(1) = 1$. The coefficient of the highest degree term of $P^l_{rr}(z)$ is

$$\frac{(2l)!}{2^l (l+r)! (l-r)!} = \frac{(2k-2)!}{2^k (k+r-1)! (k-r-1)!},$$

while the coefficient of the highest degree term (in $\cos \theta$) of $\left( \frac{\cos \theta + 1}{2} \right)^r p_{k-r-1}(2\pi a (1 - \cos \theta))$ is given by

$$\frac{(2\pi a)^{k-r-1}}{2^r (k-r-1)!}.$$ 

Therefore, using the orthogonality of Legendre polynomials (A.0.13) we can see that the integral in (4.3.8) is given by

$$2(4\pi)^{k-r-1} a^{k-r-1} \frac{(k+r-1)!}{(2k-1)!}.$$ 

Hence after putting $v$ back in we get

(4.3.9)  

$$W_v(g, y) = 4 \cdot 4^{k_v-r_v-1} \pi^{k_v-r_v-1} a^{k_v} \frac{(k_v+r_v-1)!}{(2k_v-1)!} e^{-2\pi a} \varphi^B(y)$$

$$= 2(4\pi)^{k_v-r_v-1} \frac{(k_v+r_v-1)!}{(2k_v-1)!} W_\infty(t(a_v)) \varphi^B(y).$$

Indefinite case. We assume first that $k_v > 0$. We will occasionally drop the subscript $v$ in the following.

This time we use the model generated by matrix elements $t_{rr}^{-k}$ on the group $SL_2(\mathbb{R})$. The vector in the model that corresponds to $\varphi^B$ is also $t^l_{rr}$ by weight consideration. Let $g = t(a)$ ($a > 0$) and $\sigma = \sqrt{a}$. The function $W(g, y)$ becomes:

$$W(g, y) = 2a^{r+1} \int_{SL_2(\mathbb{R})} u^{2r} p_{r-k}(4\pi a |v|^2) e^{-2\pi a |u|^2 |v|^2} t^{-k}_{rr}(x) dx \cdot \varphi^B(y).$$

Using Euler angles we obtain (Appendix A):

$$W(g, y) = 2a^{r+1} \int_{\mathbb{R}^+} (\cosh \frac{t}{2})^{2r} p_{r-k}(4\pi a \sinh^2 \frac{t}{2}) e^{-2\pi a \cosh \frac{t}{2}} \varphi^B_{rr}^{-k}(\cosh t) \sinh t dt,$$
and we have

\[ q_{rr}^{-k}(\cosh t) = (\cosh \frac{t}{2})^{-2r} P_{r-k}^{(0,-2r)}(\cosh t). \]

Note that \( t_{rr}^{-k}(1) = 1 \), thus the above integral becomes

\[ W(g, y) = 2a^{r+1} \int_1^\infty p_{r-k}(2\pi a(t-1)) e^{-2\pi at} P_{r-k}^{(0,-2r)}(t+1) dt \cdot \varphi^B(y). \]

Recall that the Jacobi polynomial \( P_{r-k}^{(0,-2r)}(z) \) is given by:

\[ \frac{\Gamma(r - k + 1)}{(r - k)!} F(k - r, -k - r + 1; 1; \frac{1 - z}{2}), \]

and \( F(\alpha_1, \alpha_2; \beta; z) = \sum_{n=0}^\infty \frac{(\alpha_1)_n(\alpha_2)_n}{n!(\beta)_n} z^n \). So we get:

\[ W(g, y) = 2a^{r+1} e^{-2\pi a} \int_{\mathbb{R}^+} p_{r-k}(2\pi at) \sum_{n=0}^{r-k} \frac{(k - r)_n(1 - k - r)_n}{(n!)^2} \left( \frac{-t}{2} \right)^n e^{-2\pi at} dt. \]

By the orthogonality relation of Laguerre polynomials (2.2.3) we can see the integral equals:

\[ 2a^{r+1} e^{-2\pi a} \frac{(k + r - 1) \cdots (2k)}{(r - k)!} \int_{\mathbb{R}^+} p_{r-k}(2\pi at) \left( \frac{-t}{2} \right)^{r-k} e^{-2\pi at} dt \]

\[ = 2a^{r+1} e^{-2\pi a} \frac{(k + r - 1) \cdots (2k)}{2^{r-k}(r - k)!} \cdot \frac{(r - k)!}{(2\pi a)^{r-k+1}} \]

\[ = 2(4\pi)^{k-r-1}(k + r - 1) \cdots (2k) 2a^k e^{-2\pi a}. \]

Putting \( v \) back and we obtain

\[ W_v(g, y) = 2(4\pi)^{k_v-r_v-1} \frac{(k_v + r_v - 1)!}{(2k_v - 1)!} W_{\infty}(t(a_v)) \varphi^B(y). \]

Proposition 4.3.4 now follows from (4.3.6), (4.3.7), (4.3.9) and (4.3.10).

We now come to the final central value formula.

**Theorem 2.** The central value of \( L(s, \pi \times \pi_\chi) \) is given by

\[ L(1/2, \pi \times \pi_\chi) = M M' \frac{\mu(\hat{R})}{\mu(Nc(\chi)^2)} \cdot \frac{\langle \varphi^*, \varphi^* \rangle}{\langle \varphi^B, \varphi^B \rangle} \int_{\hat{K} \times \hat{K} \setminus \hat{K}} \varphi^B(t) t^{-1} dt \]

where \( M \) is given by

\[ M = \prod_{v|\infty} 2^{k_v+r_v+1} \prod_{v \in \Sigma_1} G_2(k_v - r_v) \prod_{v \in \Sigma_2} G_2(r_v - k_v + 1), \]

\( M' \) is given by

\[ M' = \prod_{v|\infty} 2(4\pi)^{k_v-r_v-1} \frac{(k_v + r_v - 1)!}{(2k_v - 1)!}, \]
\( \mu(\tilde{R}^1) \) is the measure of \( \tilde{R}^1 = \tilde{R}^\times \cap B_{hf}^1 \) (Section 2.2) and \( \mu(Nc(\chi)^2)^+ \) is the measure of \( U_0(Nc(\chi)^2)^+ \).

**Proof.** From Proposition 4.3.4

\( \tilde{\varphi}'(1) = \mu(\tilde{R}^1)M'(\varphi^B, \varphi^B) \),

so by (4.3.2) and (4.3.3)

\[
C = \frac{\tilde{\varphi}'(1)(\varphi^*, \varphi^*)}{(\varphi^B, \varphi^B)^2} = \frac{\mu(\tilde{R}^1)M'(\varphi^*, \varphi^*)}{(\varphi^B, \varphi^B)}. 
\]

Now the formula is clear from Proposition 4.3.3. \( \square \)

Note that the level of \( \varphi^* \) is (the twist of) \( Nc(\chi)^2 \), so is the level of \( \varphi^B \). In a subsequent paper we will lower the level to \( N \), which is more suitable for applications.

**Appendix A. Models of representations at infinity**

In the next section we need to compute eigenvalues of certain archimedean Hecke operators which are defined as integrals on \( B_v^1 \). Here \( v \) is an archimedean place, and we will drop \( v \) throughout this section. So \( B^1 \) is either \( SU_2(\mathbb{C}) \) or \( SL_2(\mathbb{R}) \) depending on whether \( B \) is the quaternion algebra or the matrix algebra. To compute the eigenvalues we can restrict the irreducible representation \( \pi \) of \( B^\times \) to \( B^1 \) (still denoted by \( \pi \)), and calculate the eigenvalues on the restricted representation. This process will give the same eigenvalues.

Now we describe a convenient model for the representation \( \pi \) of \( B^1 \). It realizes \( \pi \) as a subspace of harmonic functions on the Lie group \( B^1 \). These functions are actually the matrix coefficients of \( \pi \). The facts recorded here are well-known and can be found, for instance, in Chapters 6 and 7 of [7]. We start with \( B^1 = SL_2(\mathbb{R}) \).

First, the group \( SL_2(\mathbb{R}) \) can be realized as a subgroup of \( GL_2(\mathbb{C}) \) and any \( g \in SL_2(\mathbb{R}) \) is parametrized by three Euler angles

\[
0 \leq \phi < 2\pi, \ 0 < t < \infty, \ -2\pi \leq \psi < 2\pi,
\]

such that

\[
(A.0.11) \quad g = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}.
\]

The measure of \( SL_2(\mathbb{R}) \) is given by

\[
\frac{1}{8\pi^2} \sinh t \, d\theta \, d\psi \, dt.
\]

So the total measure of \( SL_2(\mathbb{R}) \) is one. The torus \( SO(2) \) is embedded in \( SL_2(\mathbb{R}) \) by mapping \( e^{i\alpha} \) to

\[
\begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix}.
\]

Jacobi polynomials are defined by

\[
P_n^{(\alpha, \beta)}(z) = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} F\left( -n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - z}{2} \right)
\]
which is a polynomial of degree in , here \( \alpha \in \mathbb{Z}_{\geq 0}, n \in \mathbb{Z}_{+} \) and 

\[
F(\alpha, b; \alpha; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (b)_n}{n! (\alpha)_n} z^n
\]
is a hypergeometric series, where \((\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)\).

Assume \( \pi \) is the discrete series of lowest weight \(-2l\) with \( l \) a negative integer. Let the matrix elements of \( \pi \) be given by 

\[
t^l_{mn}(g) = (T^l v_m, v_n),
\]
where \((, )\) is an invariant Hermitian form and 

\( v_m \) is a weight \( m \) vector of norm 1. So 

\[
t^l_{mn}(g) = e^{-i(m\phi + n\psi)} \mathfrak{F}^l_{mn}(\cosh t),
\]
under the decomposition (A.0.11), then

\[
\mathfrak{F}^l_{mn}(\cosh t) = (\sinh \frac{t}{2})^{m-n}(\cosh \frac{t}{2})^{m+n} P^l_{m-n,m+n}(\cosh t),
\]
where \( n \leq m \leq l < 0 \), for other cases we use the symmetric relations. Another symmetric relation is

\[
\mathfrak{F}^l_{mn}(\cosh t) = \mathfrak{F}^l_{-m,-n}(\cosh t).
\]

If \( \pi \) is the discrete series representation of \( SL_2(\mathbb{R}) \) of lowest weight \(-2l\), then for every fixed \( m \leq l \) it has a model generated by the matrix coefficients \( t^l_{mn}(g) \), where \( n \) runs through all integer numbers less than or equal to \( l \).

Similar facts hold for \( SU(2) \). The group \( SU(2) \) can also be regarded as a subgroup of \( GL_2(\mathbb{C}) \) and every element \( g \in SU(2) \) has an Euler angle parametrization

\[
g = \begin{pmatrix} e^{i\phi/2} & 0 \\
0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} \\
i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\
0 & e^{-i\psi/2} \end{pmatrix},
\]
where

\[
0 \leq \phi < 2\pi, \ 0 \leq \theta \leq \pi, \ -2\pi \leq \psi < 2\pi.
\]
The measure is chosen such that the total measure of \( SU(2) \) is 1. We fix the same embedding of \( SO(2) \) into \( SU(2) \). The matrix coefficients of an irreducible representation \( \pi \) of \( SU(2) \) with dimension \( 2l+1 \) have the form

\[
t^l_{mn}(g) = i^{m-n} e^{-i(m\phi + n\psi)} P^l_{mn}(\cos \theta),
\]
where \( P^l_{mn} \) is a Legendre polynomial and \(|m|, |n| \leq l\). For a fixed \( n \), the space generated by \( t^l_{mn} \), where \( m = -l, \cdots, l \), is a model of \( \pi \) (under the right regular action). We have the following Rodrigues formula for \( P^l_{mn}(z) \):

\[
P^l_{mn}(z) = \frac{(-1)^{l-m}}{2^l} \left[ \frac{(l+m)!}{(l-n)!(l+n)!(l-m)!} \right]^{1/2} \times (1+z)^{-(m+n)/2}(1-z)^{(n-m)/2} \frac{d^{l-m}}{dz^{l-m}} [(1-z)^l - (1+z)^l].
\]
The Legendre polynomials have nice orthogonal properties:

\[
\int_{-1}^{1} P^l_{mn}(z) P^r_{st}(z) dz = \frac{2}{2l+1} \delta_{lr} \delta_{ms} \delta_{nt},
\]
where \( \delta_{ab} = 1 \) or 0 depending on \( a = b \) or not.
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